Uniqueness of Petrov type D spatially inhomogeneous irrotational silent models

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Abstract

The consistency of the constraint with the evolution equations for spatially inhomogeneous and irrotational silent (SIIS) models of Petrov type I, demands that the former are preserved along the timelike congruence represented by the velocity of the dust fluid, leading to new non-trivial constraints. This fact has been used to conjecture that the resulting models correspond to the spatially homogeneous (SH) models of Bianchi type I, at least for the case where the cosmological constant vanish. By exploiting the full set of the constraint equations as expressed in the 1+3 covariant formalism and using elements from the theory of the spacelike congruences, we provide a direct and simple proof of this conjecture for vacuum and dust fluid models, which shows that the Szekeres family of solutions represents the most general class of SIIS models. The suggested procedure also shows that, the uniqueness of the SIIS of the Petrov type D is not, in general, affected by the presence of a non-zero pressure fluid. Therefore, in order to allow a broader class of Petrov type I solutions apart from the SH models of Bianchi type I, one should consider more general “silent” configurations by relaxing the vanishing of the vorticity and the magnetic part of the Weyl tensor but maintaining their “silence” properties i.e. the vanishing of the curls of $E_{ab}, H_{ab}$ and the pressure $p$. 
1 Introduction

The discovery of the Szekeres family of solutions of the Einstein’s Field Equations (EFE), initiated a continuous research effort to understand the richness of the structure of this interesting class of inhomogeneous models [1]-[6]. The main reason for this interest was that, due to the inhomogeneity condition, these solutions can in principle describe either expanding or collapsing matter distributions leading for example to a deeper insight into the formation mechanism of the local structure inhomogeneities due to the amplification of small density fluctuations in the early universe [7].

Nevertheless, it emerged that Szekeres models were “hiding” a sufficient number of crucial, in some sense, properties in their intrinsic structure. At a geometrical level, it was proved immediately after their discovery that they admit no isometries i.e. there are no non-trivial Killing Vector Fields (KVF$s$) admitted by the spacetime manifold. Therefore one could reasonably expect that the absence of KVF$s$ should make Szekeres models quite general within the class of inhomogeneous models. However, the well known kinematical properties of the vanishing of the acceleration and the vorticity of the fluid velocity (or, in the case of the vacuum subfamily of models, the timelike vector field normal to the spatial hypersurfaces \( t = \text{const.} \)), indicate that these models are more special than expected. In addition, Barnes and Rowlingson have shown that, apart from they being algebraically special (namely of Petrov type D), the magnetic part \( H_{ab} \) of the Weyl tensor vanishes [8]. Consequently Szekeres models belong to a wider family of models satisfying \( H_{ab} = 0 \) which can then be of the Petrov type I, D or O. This family contains, as special cases, many well-known and important solutions like the standard Friedmann-Lemaître (FL) cosmological model, the Spatially Homogeneous (SH) Bianchi type I models, as well as other cosmological or astrophysical solutions of the EFE with sound physical interest [9].

Furthermore this result opened the possibility that these models may represent the exact solution analogues of the inhomogeneous first order scalar perturbations of the FL model, since it has been shown that the magnetic part of the Weyl tensor does not contain scalar contributions at the linear regime [10]. Although it has been proved that this is not true at second (or higher) order perturbations, the undertaken analysis for models satisfying the conditions:

\[
p = 0 = q^a, \quad \pi_{ab} = 0, \quad \dot{u}_a = 0, \quad \omega_{ab} = 0, \quad H_{ab} \equiv \frac{1}{2} \epsilon_{abc}^{\ \ d} C_{efbd} u^c u^d = 0.
\]  

(1.1)

has uncovered a striking (and, from a mathematical point of view, very often desirable) feature of the complete set of the EFE: the decoupling of the spatial divergence and curl equations (or equivalently the constraints i.e. the fully projected derivatives normal to the timelike vector field \( u^a \)) from the evolution equations of the kinematical and dynamical variables [11]-[14]. As a result, there is no physical mechanism for gravity wave propagation \( (H_{ab} = 0) \) or the presence of sound waves \( (p = 0) \) between any nearby timelike curves, thus leading to the name silent models. Furthermore, due to the above mentioned decoupling, the associated equations for the physical variables are reduced to a set of first-order ordinary differential equations (odes) which has been studied using methods from the theory of dynamical systems with a view to analyzing the asymptotic behaviour of either expanding or collapsing collisionless matter configurations [15],[16].

However, one must complement the above discussion by examining whether a suitable set of initial data evolves consistently, which will ensure that Spatially Inhomogeneous Irrotational Silent (SIIS) models retain their generality within the class of inhomogeneous solutions of the EFEs. This is equivalent to demanding that the spatial divergence and curl equations (encoded in the set of the initial data), are consistent with the evolution equations hence, they are preserved identically along the timelike congruence \( u^a \) without imposing new geometrical, kinematical or dynamical restrictions. Although it has been stated that the constraints are generally consistent [17],[18] however, it has been demonstrated that this is not the case [19]. In particular, the consistency analysis for the SIIS of the Petrov type I has shown that the constraints, coming from the silent conditions (1.1), are not trivial and after repeated propagation along \( u^a \), we expect to obtain an infinite chain of non-trivial algebraic relations which allows one to conjecture that there are no SIIS models of the Petrov type I [20, 21, 22, 23].

Clearly, if we assume that the conjecture is true, the Szekeres family of solutions keeps a privileged
position within the SIIS models due to its geometrical and dynamical characteristics and can be exploited in order to clarify some open issues in spatially inhomogeneous models. For example, the consistency of the Petrov type D models together with their non-symmetry property (i.e. the non-existence of isometries), indicate that in the case of the Petrov type I models the new constraints must be related with the existence of some type of symmetry. This expectation can be justified by the presence of the non-trivial algebraic (consistency) relations in the Petrov type I models which necessary lead to a reduction of the allowed models, similarly to the effect by the existence of a geometric symmetry. On the other hand, the qualitative study of the SIIS models shows that, if the Szekeres family of solutions is unique, we can restrict the dynamical systems analysis only to the Petrov type D invariant set [25]. Among others, this provides a way to reveal the geometric nature and to describe the dynamical role of the equilibrium points of the associated state space and examine to what extent the structure of the latter has similarities with the corresponding state space of the SH models or the inhomogeneous models with one spatial degree of freedom [25]-[39].

The above discussion initiated a study towards either proving the conjecture or, at least, presenting results that enforce its validity. For example, it has been shown the uniqueness of the vacuum Szekeres sub-family within the SIIS models and the non existence of Petrov type I dust models admitting a $G_3$ group of isometries, provided in both cases that the cosmological constant $\Lambda$ vanishes [23, 24],[40]. On the other hand, the presence of a cosmological constant (i.e. the $\Lambda$–term generalization of the spatially inhomogeneous and irrotational models studied in [41]) cancels, in some way, the dynamical effects of the Petrov type I conditions, therefore under certain circumstances the conjecture would not hold and a broader family of models exists [42].

The purpose of this article is to revisit the consistency problem in vacuum and pressureless SIIS models and provide an alternative method to resolve it, using the time propagation of the constraints only up to third order and avoiding the appearance of complicated algebraic relations between the dynamical variables. The suggested approach is used to provide a conclusive answer to the consistency conjecture in the affirmative, in a transparent and fully covariant manner. Our method heavily relies on the use of the theory of spacelike congruences in conjunction with the 1+3 covariant formalism as applied in relativistic cosmology and astrophysics [43]-[47]. The paper is organized as follows: Section 2 is devoted to present the basic elements of the theory of timelike and spacelike congruences and subsequently the 1+3 and 1+1+2 covariant description of the SIIS models, in order to express the resultant constraints in terms of the irreducible kinematical quantities of the spacelike congruences. The main result of the paper is given in Section 3 where we prove the uniqueness of the Petrov type D family within the class of SIIS models. This is achieved by showing that models of the Petrov type I always admit a three dimensional Abelian group of isometries with spacelike orbits, and thus reduce to the subclass of SH models of the Bianchi type I [48]. We should note that the above uniqueness result is still valid when we allow the presence of a non-zero pressure fluid configuration or a non-vanishing cosmological constant, provided in both cases that the extrinsic curvature of the spacelike hypersurfaces is not degenerated or, more generally, the shear eigenvalues are not restricted to be proportional to the expansion of the timelike congruence. Finally in Section 4 we draw our conclusions and discuss the ways of how the approach presented in the present article can be extended to check the broadness of more general silent models.

Throughout this paper, the following conventions have been used: the pair $(\mathcal{M}, g)$ denotes the space-time manifold endowed with a Lorentzian metric of signature $(-,+,+)$, spacetime indices are denoted by lower case Latin letters $a, b, \ldots = 0, 1, 2, 3$, spacelike eigenvalue indices are denoted by lower case Greek letters $\alpha, \beta, \ldots = 1, 2, 3$ and we have used geometrised units such that $8\pi G/c^2 = 1 = c$.

2 Covariant description of SIIS models

2.1 Elements from the theory of timelike congruences

The 1+3 covariant formalism [47] starts by introducing a unit timelike vector field $u^a$ ($u^a u_a = -1$) which is identified with the average velocity of matter in the Universe, thus representing the congruence of worldlines of the so-called fundamental (preferred) observers. The existence of the timelike vector field
\( u^a \) generates, at any spacetime event, a unique splitting of all the geometrical, kinematical and dynamical (tensorial) quantities into temporal and spatial parts using the projection tensor: 

\[
h_{ab} \equiv g_{ab} + u_a u_b, \quad h_a^c h_{cb} = h_{ab}, \quad h_a^b h_b^a = 3, \quad h_{ab} u^b = 0. \tag{2.1}
\]

Essentially, \( h_{ab} \) represents the intrinsic metric of the 3-dimensional rest spaces of the observers \( u^a \), even if these 3-spaces do not constitute, in general, an integrable submanifold \( S \) of \( M \). The external geometric structure of the submanifold \( S \) can be efficiently described in terms of the irreducible kinematical parts, coming from the 1+3 decomposition of the first covariant derivatives of \( u^a \) according to:

\[
u_{a;b} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} - \dot{u}_a u_b \tag{2.2}
\]

where \( \theta = u_{a;b} h^{ab} \) is the overall volume expansion (or contraction) rate, \( \sigma_{ab} = h_a^k h_b^l \left[ u_{(k;l)} - \frac{\theta}{3} h_{kl} \right] \) is the shear tensor describing the rate of distortion of \( S \) in different directions (i.e. the change of its shape), \( \omega_{ab} = h_a^k h_b^l u_{[k;l]} \) is the vorticity tensor and \( \dot{u}_a = u_{a;b} u^b \) is the four-acceleration vector field. Moreover, since \( u^a \) corresponds to the average velocity of the matter fluid, the above kinematical quantities have also a direct physical interpretation. Their dynamical behaviour can be studied using the Ricci identity:

\[
2u_{a;[bc]} = R_{dabc} u^d \tag{2.3}
\]

and taking into account the EFE.

For the case under consideration the fluid pressure vanishes and the EFE are given by:

\[
R_{ab} = \rho \left( u_a u_b + h_{ab} \right) + \Lambda g_{ab} \tag{2.4}
\]

where \( \rho \) is the energy density of the matter fluid and \( \Lambda \) is the cosmological constant.

The complete 1+3 decomposition of (2.3) into temporal and spatial parts, leads to *evolution* and *constraint* equations of the kinematical quantities. Because the timelike congruence \( u^a \) is irrotational \( (\omega_{ab} = 0) \) and geodesic \( (\dot{u}_a = 0) \) and using the “silent” conditions (1.1), the Ricci identity reduces to two evolution equations for \( \theta \) and \( \sigma_{ab} \) and two constraint (divergence and curl) equations for the shear tensor [47]:

\[
\dot{\theta} = -\frac{1}{3} \theta^2 - \sigma_{ab} \sigma^{ab} - \frac{1}{2} \rho + \Lambda \tag{2.5}
\]

\[
\dot{\sigma}_{ab} = \frac{2}{3} \theta \sigma_{ab} - \sigma_a^c \sigma_{cb} - E_{ab} + \frac{1}{3} (\sigma_{cd} \sigma^{cd} ) h_{ab} \tag{2.6}
\]

\[
h_{ac} \sigma_{bc}^{k;:k} = \frac{2}{3} h_a^k \theta_k \tag{2.7}
\]

\[
\dot{h}_{a(b)}^{(a k)} m_s \sigma_{n l;m} u_s = 0 \tag{2.8}
\]

where

\[
E_{ab} \equiv C_{acbd} u^c u^d \tag{2.9}
\]

is the electric part of the Weyl conformal tensor and we have used the notation:

\[
\dot{K}_{a...} \equiv K_{a...;k} u^k \tag{2.10}
\]

for the time derivative of any scalar or tensorial quantity.

We should note that the shear-curl constraint (2.8) is the result of imposing the silent conditions (1.1). In order to obtain a closed set of equations one must also use the Bianchi identities

\[
R_{ab[cd;e]} = 0. \tag{2.11}
\]
The associated 1+3 decomposition of (2.11) gives the energy and momentum conservation equations plus two evolution and two constraints equations for the electric and magnetic part of the Weyl tensor which for the case of a SIIS model take the form:

\[
\dot{E}_{ab} = -\frac{1}{2}\rho\sigma_{ab} - \theta E_{ab} + 3E_{(a}e_{b)c} - (E_{cd}\sigma^{cd}) h_{ab}
\]

(2.12)

\[
\dot{\rho} = -\rho\theta
\]

(2.13)

\[
\dot{h}_{ac}E^{kc}_{\cdot ;k} = \frac{1}{3}h_{a}^{k}\rho_{,k}
\]

(2.14)

\[
h^{\ell(a}E_{\ell\cdot mn}u_{s}) = 0
\]

(2.15)

\[
e^{abcd}E_{bk}\sigma_{c}^{k}u_{d} = 0.
\]

(2.16)

Similarly, the electric-curl constraint (2.15) follows from the silent conditions (1.1). The above closed system of equations completely describes the dynamics and can be used in order to determine the broadness of the family of SIIS models. This depends on the consistency of the constraints (2.8) and (2.15) with the set of evolution equations (2.5)-(2.6) and (2.12)-(2.13). Although equations (2.8) and (2.15) appear similar they have a completely different origin and one should expect that their consistency demands will be independent. Eventually this is indeed the case since it has been shown that the propagation of (2.8) does not generate further conditions and evolves consistently along the worldlines of the fundamental observers [21]. On the other hand propagation of (2.15) leads to a set of non-trivially satisfied equations, indicating that equation (2.15) represents a new constraint. It has been pointed out that the different character of (2.15) is a reminiscent of the implications of the fact that the vanishing of the evolution equation of a dynamical variable leads to a new integrability condition [22]. As we shall demonstrate in the next section, equation (2.15) is not, in general, consistent with the evolution equations (2.6) and (2.12)-(2.13) and represents the compact form of a set of “hidden” geometric restrictions.

A direct consequence of equation (2.16) is that the shear \(\sigma_{ab}\) and the electric part \(E_{ab}\) tensors commute which implies that they have a common eigenframe. As a result, if \(\{x^{a}, y^{a}, z^{a}\}\) is a set of three mutually orthogonal and unit spacelike vector fields that constitute the spatial eigenframe of \(\sigma_{ab}\) and \(E_{ab}\), we may write:

\[
E_{ab} = E_{1}x_{a}x_{b} + E_{2}y_{a}y_{b} + E_{3}z_{a}z_{b}
\]

(2.17)

\[
\sigma_{ab} = \sigma_{1}x_{a}x_{b} + \sigma_{2}y_{a}y_{b} + \sigma_{3}z_{a}z_{b}
\]

(2.18)

where \(E_{\alpha}, \sigma_{\alpha}\) are the associated eigenvalues satisfying the trace-free conditions:

\[
\sum_{\alpha} E_{\alpha} = \sum_{\alpha} \sigma_{\alpha} = 0.
\]

(2.19)

Furthermore it has been shown that each of \(\{x^{a}, y^{a}, z^{a}\}\) is hypersurface orthogonal [8] i.e.

\[
x_{[a}x_{b]c} = y_{[a}y_{b]c} = z_{[a}z_{b]c} = 0.
\]

(2.20)

which implies that \(\dot{x}_{a} = \dot{y}_{a} = \dot{z}_{a} = 0\) and local coordinates can be found such that the metric can be written:

\[
ds^{2} = g_{\alpha\beta}dx^{\alpha}dx^{\beta} = -dt^{2} + e^{2A}dx^{2} + e^{2B}dy^{2} + e^{2\Gamma}dz^{2}
\]

(2.21)

where \(A(t, x, y, z), B(t, x, y, z), \Gamma(t, x, y, z)\) are smooth functions of their arguments.
2.2 Elements from the theory of spacelike congruences

In many situations with clear geometrical or physical importance, apart from the existence of a preferred timelike congruence, there also may exist a preferred spacelike direction \( x^a \) \((x^a x_a = 1)\) representing an intrinsic geometrical or physical feature of the corresponding model (for example, due to the existence of a spacelike KVF or the presence of a magnetic field). Consequently, a further 1+2 splitting of the 3-dimensional space naturally arises leading to the concept of the 1+1+2 decomposition of the spacetime manifold.

In complete analogy with the 1+3 decomposition, the starting point is to introduce the projection tensor:

\[
p_{ab}(x) = g_{ab} + u_a u_b - x_a x_b = h_{ab} - x_a x_b \quad (2.22)
\]

\[
p_a^c p_{cb} = p_{ab}, \quad p_a^b p_b^a = 2, \quad p_{ab} u^b = 0 = p_{ab} x^b \quad (2.23)
\]

which is identified with the associated metric of the 2-dimensional space \( \mathcal{X} \) (the so-called *screen space*) normal to each vector field of the pair \( \{ u^a, x^a \} \) at any spacetime event. Then the geometric structure of \( \mathcal{X} \) is studied by decomposing, into irreducible kinematical parts, the first covariant derivatives of the spacelike vector field \( x^a \) according to \([43]-[46]\):

\[
x_{a:b} = T_{ab}(x) + \frac{\mathcal{E}_x}{2} p_{ab} + \mathcal{R}_{ab}(x) + \hat{x}_a u_b + p_b \hat{x}_c u_a + [2\omega_{ab} x^c - N_b(x)] u_a \quad (2.24)
\]

where:

\[
\mathcal{E}_x = x_{a;b} p^{ab} = x^a_{;a} + \hat{x}^a u_a \quad (2.25)
\]

\[
T_{ab}(x) = p_a^k p_b^l [x_{(k;l)} - \frac{1}{2} \mathcal{E}_x p_{kl}], \quad T_{ab} p^{ab} = 0 \quad (2.26)
\]

\[
\mathcal{R}_{ab}(x) = p_a^k p_b^l x_{[k;l]} \quad (2.27)
\]

\[
N_a = p_a^k \left( \hat{x}_k - u_k \right) \quad (2.28)
\]

are the rate of the surface expansion, the rate of shear tensor, the rotation tensor and the Greenberg vector field of the spacelike congruence \( x^a \) respectively and we have used the notation:

\[
\hat{K}_{a...k} x^k = \hat{K}_{a...k} x^k \quad (2.29)
\]

for the directional derivative along the vector field \( x^a \) of any scalar or tensorial quantity.

The geometrical meaning of each of the defined kinematical quantities of the spacelike congruence is similar to that of the corresponding quantities of \( u^a \), carrying information on the (overall or in different directions) distortion of \( \mathcal{X} \) as measured by the fundamental observers \( u^a \). The new ingredient is the appearance of the Greenberg vector \( N_a \) which is of crucial importance in the theory of spacelike congruences and represents the “coupling” mechanism of the dynamical behaviour of the model in directions normal and parallel to the screen space \( \mathcal{X} \). For example, the equation \( N^a = 0 \) implies that the pair of vector fields \( \{ u^a, x^a \} \) generates a 2-dimensional integrable submanifold of \( \mathcal{M} \) and the spacelike congruence \( x^a \) is “comoving” (“frozen-in”) along the worldlines of the fundamental observers \( u^a \). In addition this fact ensures that \( T_{ab}(x) \) and \( \mathcal{R}_{ab}(x) \) lie in the screen space and the unit vector fields \( \{ x^a, u^a \} \) are orthogonal at any spacetime instant.

The incorporation of the EFE is achieved by applying (2.3) and (2.11) to the unit vector field \( x^a \). This leads to propagation (along \( x^a \)) and “constraint” (lying on the screen space) equations which describe the influence of the gravitational field in the spatial variation of the kinematical and dynamical variables of the timelike and spacelike congruences. In the next section we shall derive the complete set of the dynamical equations associated with the orthonormal tetrad \( \{ u^a, x^a, y^a, z^a \} \) in order to analyze and express the evolution and constraint equations (2.6)-(2.8) and (2.12)-(2.15) in terms of the kinematical quantities of the spacelike congruences with a view to reveal the special structure of the new constraint (2.15).
2.3 SIIS models and spacelike congruences

From equations (2.20) and after appropriate projections, it follows that each of the eigenvectors \( \{x^a, y^a, z^a\} \) has the following properties:

\[
\begin{align*}
\dot{x}_a &= N_a(x) = 0 = R_{ab}(x) \quad (2.30) \\
\dot{y}_a &= N_a(y) = 0 = R_{ab}(y) \quad (2.31) \\
\dot{z}_a &= N_a(z) = 0 = R_{ab}(z) \quad (2.32)
\end{align*}
\]

which implies that each generator of the spacelike congruence is comoving with the fundamental observers and the corresponding screen spaces are integrable submanifolds of \( \mathcal{M} \). Projecting equation (2.8) or (2.15) with \( x^a x^b, y^a y^b \) and \( z^a z^b \) we obtain:

\[
\begin{align*}
T_{ab}(x) &= \alpha (y_a y_b - z_a z_b) \quad (2.33) \\
T_{ab}(y) &= \beta (z_a z_b - x_a x_b) \quad (2.34) \\
T_{ab}(z) &= \gamma (x_a x_b - y_a y_b) \quad (2.35)
\end{align*}
\]

where \( \alpha, \beta, \gamma \) are the eigenvalues of the associated shear tensors of the spacelike vector fields \( x^a, y^a, z^a \) respectively.

Furthermore projecting (2.8) and (2.15) along \( y^a z^b, z^a x^b, x^a y^b \), the shear and electric part constraints can be expressed as spatial variations of the corresponding eigenvalues along the individual spacelike curve:

\[
\begin{align*}
(\sigma_3 - \sigma_2)^* &= -\frac{\mathcal{E}_x}{2} (\sigma_3 - \sigma_2) - 3\alpha \sigma_1 \quad (2.36) \\
(\sigma_1 - \sigma_3)' &= -\frac{\mathcal{E}_y}{2} (\sigma_1 - \sigma_3) - 3\beta \sigma_2 \quad (2.37) \\
(\sigma_2 - \sigma_1)' &= -\frac{\mathcal{E}_z}{2} (\sigma_2 - \sigma_1) - 3\gamma \sigma_3 \quad (2.38) \\
(E_3 - E_2)^* &= -\frac{\mathcal{E}_x}{2} (E_3 - E_2) - 3\alpha E_1 \quad (2.39) \\
(E_1 - E_3)' &= -\frac{\mathcal{E}_y}{2} (E_1 - E_3) - 3\beta E_2 \quad (2.40) \\
(E_2 - E_1)' &= -\frac{\mathcal{E}_z}{2} (E_2 - E_1) - 3\gamma E_3 \quad (2.41)
\end{align*}
\]

where we have introduced the notations:

\[
K^\prime_{a...} \equiv K_{a...} y^k, \quad (K_{a...})^\prime \equiv K_{a...} z^k.
\]

The advantage of decomposing the constraints along the specific spacelike curve is immediately apparent from equations (2.36)-(2.41): the associated spatial variations of the shear and electric part eigenvalues have been partially decoupled and expressed in terms of the corresponding kinematical variables of the spacelike congruences. Obviously, this decoupling process is the result of the conditions (2.30)-(2.32).

The above set of equations must be augmented with the evolution equations (2.6), (2.12), (2.13) and the divergence equations (2.7), (2.14):

\[
\begin{align*}
\dot{\sigma}_a &= -\frac{2}{3}\theta \sigma_a - (\sigma_a)^2 - E_a + \frac{1}{3} \sum_\alpha (\sigma_\alpha)^2 \quad (2.43) \\
\dot{E}_a &= -\frac{1}{2}\rho \sigma_a - \theta E_a + 3\sigma_a E_a - \sum_\alpha E_a \sigma_\alpha \quad (2.44) \\
\dot{\rho} &= -\rho \theta. \quad (2.45)
\end{align*}
\]
\[ \dot{\rho} = 3 \left[ \dot{E}_1 + \frac{3\xi_x}{2} E_1 - \alpha (E_2 - E_3) \right], \quad \dot{\theta} = \frac{3}{2} \left[ \dot{\sigma}_1 + \frac{3\xi_x}{2} \sigma_1 - \alpha (\sigma_2 - \sigma_3) \right] \]  

\[ \rho' = 3 \left[ E'_2 + \frac{3\xi_y}{2} E_2 - \beta (E_3 - E_1) \right], \quad \theta' = \frac{3}{2} \left[ \sigma'_2 + \frac{3\xi_y}{2} \sigma_2 - \beta (\sigma_3 - \sigma_1) \right] \]  

\[ \dot{\rho} = 3 \left[ \dot{E}_3 + \frac{3\xi_z}{2} E_4 - \gamma (E_1 - E_2) \right], \quad \dot{\theta} = \frac{3}{2} \left[ \dot{\sigma}_3 + \frac{3\xi_z}{2} \sigma_3 - \gamma (\sigma_1 - \sigma_2) \right]. \]  

Using a similar splitting of the Ricci identities for each spacelike vector field of the orthonormal triad \( \{x^a, y^a, z^a\} \), we obtain evolution (along \( u^a \)), propagation (along the individual spacelike vector field) and “constraint” (lying in the associated screen space) equations of the kinematical quantities of the spacelike congruences which we give them below.

The evolution and the “constraint” equations arise from the temporal and the spatial projections of the trace part of the Ricci identity respectively:

\[
\begin{align*}
\dot{\xi}_x &= \frac{1}{2} \xi_x \sigma_1 - \frac{1}{3} \xi_x \theta - \alpha (\sigma_2 - \sigma_3), \\
\dot{\xi}_y &= \frac{1}{2} \xi_y \sigma_2 - \frac{1}{3} \xi_y \theta - \beta (\sigma_3 - \sigma_1), \\
\dot{\xi}_z &= \frac{1}{2} \xi_z \sigma_3 - \frac{1}{3} \xi_z \theta - \gamma (\sigma_1 - \sigma_2).
\end{align*}
\]

(2.49) (2.50) (2.51)

\[
\begin{align*}
(\frac{\dot{\xi}_x}{2} - \alpha)' &= 2\alpha \left( \frac{\xi_y}{2} + \beta \right), \\
(\frac{\xi_y}{2} + \beta)' &= -2\beta \left( \frac{\xi_x}{2} - \alpha \right), \\
(\frac{\dot{\xi}_z}{2} - \gamma)' &= 2\gamma \left( \frac{\xi_y}{2} + \beta \right).
\end{align*}
\]

(2.52) (2.53) (2.54)

It is interesting to note that the evolution equations (2.49)-(2.51) imply for example \( \alpha = 0 \) \( \Leftrightarrow \xi_x = 0 \) for a Petrov type I model. In addition, by taking appropriate linear combinations of the “constraint” equations (2.52)-(2.54) this set is essentially equivalent with the Jacobi identities or the twist-free property \( R = 0 \) of the spacelike congruences.

The propagation equations of the expansion and the shear of the spacelike curves follow from the trace and trace free part of the Ricci identity [44] which in the case of the SSII models take the form:

\[
\begin{align*}
(\xi_x)' &= -\frac{\xi_x^2}{2} - 2\alpha^2 - \left( E_1 + \frac{2\rho}{3} \right) + \left( \sigma_1 + \frac{\theta}{3} \right) \left( \frac{2\theta}{3} - \sigma_1 \right) \\
&\quad - \left( \frac{\xi_x}{2} + \gamma \right)' - \left( \frac{\xi_x}{2} + \gamma \right) \xi_x - \left( \frac{\xi_y}{2} - \beta \right)' - \left( \frac{\xi_y}{2} - \beta \right) \xi_y \\
(\xi_y)' &= -\frac{\xi_y^2}{2} - 2\beta^2 - \left( E_2 + \frac{2\rho}{3} \right) + \left( \sigma_2 + \frac{\theta}{3} \right) \left( \frac{2\theta}{3} - \sigma_2 \right) \\
&\quad - \left( \frac{\xi_x}{2} + \alpha \right)' - \left( \frac{\xi_x}{2} + \alpha \right) \xi_x - \left( \frac{\xi_z}{2} - \gamma \right)' - \left( \frac{\xi_z}{2} - \gamma \right) \xi_z \\
(\xi_z)' &= -\frac{\xi_z^2}{2} - 2\gamma^2 - \left( E_3 + \frac{2\rho}{3} \right) + \left( \sigma_3 + \frac{\theta}{3} \right) \left( \frac{2\theta}{3} - \sigma_3 \right) \\
&\quad - \left( \frac{\xi_y}{2} + \beta \right)' - \left( \frac{\xi_y}{2} + \beta \right) \xi_y - \left( \frac{\xi_x}{2} - \alpha \right)' - \left( \frac{\xi_x}{2} - \alpha \right) \xi_x.
\end{align*}
\]

(2.55) (2.56) (2.57)
\[ (\alpha)^* = -\alpha \xi - \frac{1}{2} \left( \frac{\xi_y}{2} - \beta \right)' + \left( \frac{\xi_y}{2} - \beta \right) \beta + \frac{1}{2} \left( \frac{\xi_x}{2} + \gamma \right) \gamma \]
\[ - \frac{E_1 + 2E_2}{2} + \frac{1}{2} \left( \sigma_1 + 2\sigma_2 \right) \left( \sigma_1 + \theta_3 \right) \]  
(2.58)

\[ (\beta)' = -\beta \xi - \frac{1}{2} \left( \frac{\xi_z}{2} - \gamma \right)' + \left( \frac{\xi_z}{2} - \gamma \right) \gamma + \frac{1}{2} \left( \frac{\xi_x}{2} + \alpha \right) \alpha \]
\[ - \frac{E_2 + 2E_3}{2} + \frac{1}{2} \left( \sigma_2 + 2\sigma_4 \right) \left( \sigma_2 + \theta_3 \right) \]  
(2.59)

\[ (\gamma)' = -\gamma \xi - \frac{1}{2} \left( \frac{\xi_x}{2} - \alpha \right)' + \left( \frac{\xi_x}{2} - \alpha \right) \alpha + \frac{1}{2} \left( \frac{\xi_y}{2} + \beta \right)' + \left( \frac{\xi_y}{2} + \beta \right) \beta \]
\[ - \frac{E_3 + 2E_1}{2} + \frac{1}{2} \left( \sigma_3 + 2\sigma_4 \right) \left( \sigma_3 + \theta_3 \right) \]  
(2.60)

The set of equations (2.36)-(2.41) and (2.43)-(2.60) completely characterizes the dynamics of SIIS models and it will be used in order to prove the conjecture by showing that vacuum and non-vacuum silent models of Petrov type I always admit a three dimensional group of isometries thus reducing to the subclass of SH models of the Bianchi type I.

Finally it should be remarked that there exists a direct correspondence between the description of SIIS models with the theory of spacelike congruences and the Orthonormal Frame (ONF) approach [47, 49]. For example, the spatial curvature quantities \( n_{12}, n_{31}, n_{23}, a_1, a_2, a_3 \) and the kinematical quantities of the spacelike congruences are related via:

\[ n_{23} = -\alpha, \quad n_{31} = -\beta, \quad n_{12} = -\gamma \]  
(2.61)

\[ a_1 = -\frac{\xi_x}{2}, \quad a_2 = -\frac{\xi_y}{2}, \quad a_3 = -\frac{\xi_z}{2} \]  
(2.62)

whereas the set (2.36)-(2.41) and (2.43)-(2.60) is the covariant version of the tetrad equations presented in [22]. As we shall see in the next section, this fact (already emphasized in [22]) makes clear the covariant property of the incompatibility of the constraints (2.36)-(2.41) with the evolution equations. At the same time reveals in a transparent way, the geometric nature of (2.36)-(2.41) since they induce, through the kinematical variables of the spacelike congruences, a set of symmetry constraints leading to the subclass of SH models of Bianchi type I.

3 Uniqueness of the Szekeres solutions within the family of SIIS models

As we have mentioned in the Introduction, the vanishing of the magnetic part of the Weyl tensor gives rise to a set of constraints which in the 1+1+2 covariant formalism takes the form of equations (2.36)-(2.41). In order to be consistent with the evolution equations (2.43)-(2.45), they must not generate a new set of constraints after their repeated propagation along the timelike congruence \( u^a \). The outcome of this procedure has led to the proof of the consistency of Petrov type D models and that type I models develop a chain of new highly non-linear and non-trivial algebraic constraints that need to be solved [20, 22, 23].

However, it will be enlightening if we choose an alternative direction to deal with the consistency checking, in the spirit of the notion of geometric symmetries. The concept of a geometric symmetry or collineation admitted by the spacetime manifold, is closely related to that of a transformation that leaves a specific geometric object invariant (up to a conformal factor) along the integral lines of the generating vector field. Although there exists (and can be defined) a sufficiently large number of geometric
symmetries, they have, in general, the drawback of imposing severe restrictions on the geometry and the dynamics leading, in most cases, either to physically unsound models or models with very special properties (see for example [48] for SH models of Bianchi type I). In order to demonstrate this fact, let us consider the case of a spacelike Conformal Vector Field (CVF) \( X^a = X x^a \) which is defined according to the relation:

\[
\mathcal{L}_X g_{ab} = 2\psi g_{ab}
\]  

(3.1)

where \( \psi \) is the scale amplification or dilation of the spacetime metric.

In terms of the kinematical variables of the spacelike congruence the conformal equation (3.1) can be shown to be equivalent to the conditions [46]:

\[
\mathcal{T}_{ab}(x) = 0, \quad (\ln X)_a = \frac{1}{2} \mathcal{E}_x x_a - \dot{x}_a
\]

\[
\dot{x}_a u^a = -\frac{1}{2} \mathcal{E}_x, \quad N_a = -2\omega_{ab} x^b
\]

(3.2)

and the conformal factor is then given by \( \psi = \frac{1}{2} X \mathcal{E}_x = X \).

Furthermore, the coupling of the geometry with the matter fields, through the EFE, implies that there is a mutual influence between any geometrical or dynamical constraint with subsequent restrictions in the structure of the corresponding models. For example, in the case of an irrotational and geodesic fluid flow the conditions (3.2) become:

\[
\mathcal{T}_{ab}(x) = 0, \quad (\ln X)_a = -\dot{x}_a
\]

\[
\mathcal{E}_x = 0, \quad N_a = 0, \quad \psi = 0 = X
\]

(3.3)

i.e. the spacelike congruence is necessarily comoving with the observers \( u^a \) and the CVF reduces to a KVF. Using the covariant description of the Petrov type D models in terms of the kinematical quantities of the spacelike congruences (see below), this in particular implies that the Szekeres solutions cannot admit spacelike CVF parallel to any of the eigenvectors of the shear (or the Weyl electric part) tensor.

The consistency of the Petrov type D models along with their non-symmetry property (non-existence of isometries), allows one to speculate about the existence of some type of link between the consistency conditions and the presence of a geometric symmetry in the case of the Petrov type I models. This expectation can be justified by the fact that the new algebraic constraints in Petrov type I models are coming from the time propagation of the initial integrability conditions (2.39)-(2.41). In addition, they involve the kinematical and dynamical variables and necessarily lead to a restriction of the underlying geometric structure of the corresponding models. On the other hand the existence of any geometric symmetry is controlled by a set of integrability conditions which, after repeated covariant differentiation, also lead to a system of algebraic relations. Consequently one may argue that the presence of the non-trivial constraints (2.39)-(2.41) (together with the rest of conditions imposed by (1.1)) in SIIS models of Petrov type I can be loosely interpreted as corresponding to some type of symmetry restriction.² In the following we shall prove that this is the case and the corresponding symmetry constraints are equivalent to the Killing conditions (3.3).

The above arguments and the conclusions they lead to, can be summarized in the following:

---

¹Even if we assume that at most one, in the whole set of algebraic relations, is linearly independent this means that one of the kinematical or dynamical variables of the model can be expressed as a function of the other, thus leading to an associated restriction of their broadness.

²The interpretation of the induced constraints in terms of a set of symmetry conditions indicates a possible connection between the full set of EFE and the existence of a specific symmetry. For a subclass of SH vacuum and non-titled perfect fluid models, this connection has been established showing that a specific symmetry may play a role in the invariant description of general cosmological models [50].
**Theorem** There do not exist vacuum and non-vacuum SIIS models of the Petrov type I with vanishing cosmological constant

Before we proceed with the proof of the Theorem, it will be useful to covariantly characterize the SIIS models of the Petrov type D (S_D) in terms of the kinematical variables of the spacelike congruences. Assuming that $E_1 = E_2 \Leftrightarrow \sigma_1 = \sigma_2$, equation (2.38) (or (2.41)) implies that $\gamma = 0 \Leftrightarrow T_{ab}(z) = 0 \Leftrightarrow p^k_a(z)p^l_b(z)z_{k:l} = \frac{\Xi}{2} p_{ab}(z)$. Therefore SIIS models of the Petrov type D are described by the conditions:

$$E_1 = E_2 \Leftrightarrow \sigma_1 = \sigma_2, \quad T_{ab}(z) = 0, \quad (3.4)$$

It is interesting to note that under the conditions (3.4) the complete set of equations (2.36)-(2.41) and (2.43)-(2.51) is consistent under repeated propagation along the timelike congruence $u^a$. Furthermore, using the local coordinate form of the SIIS metric (2.21) and choosing $x^a = e^{-A}\delta^a_z$, $y^a = e^{-B}\delta^a_y$, $z^a = e^{-\Gamma}\delta^a_z$ we can verify that the conditions (3.4) follow from the assumption $A = B + H$ where the function $H$ satisfies $\dot{H} = \ddot{H} = 0$.

**Proof**

Consider the case of Petrov type I models (S_I) which implies that $E_\alpha \neq E_\beta$ and $\sigma_\alpha \neq \sigma_\beta$ for every $\alpha, \beta = 1, 2, 3$. Using the Ricci identity (2.3) it is straightforward to show the following commutation relations for every scalar quantity $S$ (see also equation (A8) in [21]):

$$\left(\hat{S}\right)\dot{\gamma} = \left(\hat{S}\right)^\gamma - \hat{S} \left(\sigma_1 + \frac{\theta}{3}\right) \quad (3.5)$$

$$\left(S\right)\dot{\gamma} = \left(\hat{S}\right)^\gamma - \hat{S} \left(\sigma_2 + \frac{\theta}{3}\right) \quad (3.6)$$

$$\left(\hat{S}\right)\dot{\gamma} = \left(\hat{S}\right)^\gamma - \hat{S} \left(\sigma_3 + \frac{\theta}{3}\right) \quad (3.7)$$

In order to analyzing the consistency of the constraints (2.39)-(2.44), we propagate them along $u^a$ and use the commutation relations (3.5)-(3.7) and the set of equations (2.43)-(2.44) and (2.46)-(2.51). As expected, the dynamical conditions (2.39)-(2.41) are not identically satisfied and they lead to the following new constraints:

$$6 \left[ E_2 (\sigma_3 - \sigma_2) + \hat{E}_2 (E_3 - E_2) \right] = 3\mathcal{E}_x \left[ E_2 (2\sigma_2 - \sigma_3) - E_3 \sigma_2 \right]$$

$$+ 2\alpha \left[ E_2 (2\sigma_2 - 11\sigma_3) - E_3 (11\sigma_2 + 16\sigma_3) \right] \quad (3.8)$$

$$6 \left[ E'_3 (\sigma_3 - \sigma_1) + \sigma'_3 (E_3 - E_1) \right] = 3\mathcal{E}_y \left[ -E_3 (2\sigma_3 - \sigma_1) + E_1 \sigma_3 \right]$$

$$+ 2\beta \left[ -E_3 (2\sigma_3 - 11\sigma_1) + E_1 (11\sigma_3 + 16\sigma_1) \right] \quad (3.9)$$

$$6 \left[ \hat{E}_1 (\sigma_2 - \sigma_1) + \hat{\sigma}_1 (E_2 - E_1) \right] = 3\mathcal{E}_z \left[ E_1 (2\sigma_1 - \sigma_2) - E_2 \sigma_1 \right]$$

$$+ 2\gamma \left[ E_1 (2\sigma_1 - 11\sigma_2) - E_2 (11\sigma_1 + 16\sigma_2) \right] \quad (3.10)$$

Clearly, equations (3.8)-(3.10) represent thecovariant form of the constraints (67)-(69) given in [22]. Moreover, the assumption of a Petrov type D model (for example $E_1 = E_2 \Leftrightarrow \sigma_1 = \sigma_2$) implies that $\gamma = 0$ and the rest of equations are trivially satisfied.
Solving equations (3.8)-(3.10) with respect to $E_2$, $E_3$, $\dot{E}_1$, a second temporal propagation leads to the following expressions for the spatial derivatives $\dot{\sigma}_2$, $\dot{\sigma}_3$ and $\dot{\sigma}_1$:

$$
\dot{\sigma}_2 \frac{A_1}{A_2}, \quad \dot{\sigma}_3 = \frac{B_1}{B_2}, \quad \dot{\sigma}_1 = \frac{C_1}{C_2}
$$

where:

$$
A_1 = -2E_2^2 [3\xi_x \sigma_2 + 2\alpha (\sigma_2 - 10\sigma_3)] + E_2 [4E_3 [3\xi_x \sigma_2 + 2\alpha (\sigma_2 + 8\sigma_3)] + 3 (\sigma_3 - \sigma_2) \left[ \xi_x (10\sigma_2^3 + 4\sigma_2 \sigma_3 - 5\sigma_3^3) + 2\alpha (14\sigma_2^2 + 20\sigma_2 \sigma_3 + 11\sigma_3^2) \right] - 2E_3^2 [3\xi_x \sigma_2 + 2\alpha (19\sigma_2 + 8\sigma_3)] - 3E_3 (\xi_x + 2\alpha) (\sigma_2 - \sigma_3) (2\sigma_2^2 - 7\sigma_2 \sigma_3 - 4\sigma_3^2) - \rho (\sigma_3 - \sigma_2) [3\xi_x \sigma_2 (\sigma_2 - \sigma_3) + 2\alpha (\sigma_2^2 - 11\sigma_2 \sigma_3 - 8\sigma_3^2)]
$$

$$
A_2 = 6 [2E_2^2 - E_2 [4E_3 + 3 (\sigma_3 - \sigma_2) (2\sigma_2 + \sigma_3)] + 2E_3 [3\xi_x (\sigma_2 - \sigma_3) (2\sigma_2 + \sigma_3) - \rho (\sigma_2 - \sigma_3)^2]]
$$

$$
B_1 = -2E_2^2 [3\xi_y \sigma_3 + 2\beta (\sigma_3 - 10\sigma_1)] + E_3 [4E_1 [3\xi_y \sigma_3 + 2\beta (\sigma_3 + 8\sigma_1)] + 3 (\sigma_3 - \sigma_1) \left[ \xi_y (10\sigma_3^3 + 4\sigma_3 \sigma_1 - 5\sigma_1^3) + 2\beta (14\sigma_3^2 + 20\sigma_3 \sigma_1 + 11\sigma_1^2) \right] - 2E_2^2 [3\xi_y \sigma_3 + 2\beta (19\sigma_3 + 8\sigma_1)] - 3E_2 (\xi_y + 2\beta) (\sigma_3 - \sigma_1) (2\sigma_3^2 - 7\sigma_3 \sigma_1 - 4\sigma_1^2) - \rho (\sigma_1 - \sigma_3) [3\xi_y \sigma_3 (\sigma_3 - \sigma_1) + 2\beta (\sigma_3^2 - 11\sigma_3 \sigma_1 - 8\sigma_1^2)]
$$

$$
B_2 = 6 [2E_3^2 - E_3 [4E_1 + 3 (\sigma_3 - \sigma_1) (2\sigma_3 + \sigma_1)] + 2E_2^2 + 3E_1 (\sigma_3 - \sigma_2) (2\sigma_3 + \sigma_1) - \rho (\sigma_3 - \sigma_1)^2]]
$$

$$
C_1 = -2E_2^2 [3\xi_x \sigma_1 + 2\gamma (\sigma_1 - 10\sigma_2)] + E_1 [4E_2 [3\xi_x \sigma_1 + 2\gamma (\sigma_1 + 8\sigma_2)] + 3 (\sigma_2 - \sigma_1) \left[ \xi_x (10\sigma_1^3 + 4\sigma_1 \sigma_2 - 5\sigma_2^3) + 2\gamma (14\sigma_1^2 + 20\sigma_1 \sigma_2 + 11\sigma_2^2) \right] - 2E_2^2 [3\xi_x \sigma_1 + 2\gamma (19\sigma_1 + 8\sigma_2)] - 3E_2 (\xi_x + 2\gamma) (\sigma_1 - \sigma_2) (2\sigma_1^2 - 7\sigma_1 \sigma_2 - 4\sigma_2^2) - \rho (\sigma_2 - \sigma_1) [3\xi_x \sigma_1 (\sigma_1 - \sigma_2) + 2\gamma (\sigma_1^2 - 11\sigma_1 \sigma_2 - 8\sigma_2^2)]
$$

$$
C_2 = 6 [2E_1^2 - E_1 [4E_2 + 3 (\sigma_2 - \sigma_1) (2\sigma_1 + \sigma_2)] + 2E_2^2 + 3E_2 (\sigma_2 - \sigma_1) (2\sigma_2 + \sigma_1) - \rho (\sigma_2 - \sigma_1)^2]].
$$

The above equations correspond to the covariant form of the 1+3 orthonormal frame expressions (72)-(73) of [22].
In the reduction to the Petrov type D models i.e. as one takes the “limit” \( S_I \to S_D \) by successively substituting \( \gamma = 0 \) and \( E_3 = -2E_2, \sigma_3 = -2\sigma_2 \), the new constraints must lead smoothly to identically satisfied relations. Therefore the consistency (\( \Leftrightarrow \) existence) of SIIS Petrov type I models implies that, in this “limit”, the constraints (3.11) (and also equations (3.8)-(3.10)) will be reduced to the corresponding constraints of the Petrov type D models. Indeed we can verify that for \( S_I \to S_D \), equations (3.8)-(3.10) and (3.11) lead to the set (2.36)-(2.41) when we specialize it to \( S_D \) models.

On the other hand the vanishing of the polynomials \( A, B, C \) in the equations (3.11) gives relations between the kinematical variables of the spacelike congruences of the form \( \alpha \sim \mathcal{E}_x, \beta \sim \mathcal{E}_y, \gamma \sim \mathcal{E}_z \). At the “limit” \( S_I \to S_D \) these relations reduce to \( \alpha = \mathcal{E}_x/2, \beta = -\mathcal{E}_y/2, \gamma = 0 \). We observe that the third equation reproduces the Petrov type D property \( \gamma = 0 \). The rest of the “limiting” values suggest that in this case, if a Petrov type I model exists it should be reduced (in that “limit”) to the subclass of Petrov type D models satisfying \( \alpha = \mathcal{E}_x/2 \) and \( \beta = -\mathcal{E}_y/2 \). In fact we can easily show that this case corresponds to the Locally Rotationally Symmetric (LRS) models of Ellis class II [47, 51].

A further propagation of (3.11) and the use of (2.43)-(2.45), (2.49)-(2.51) and (3.5)-(3.7) give algebraic relations of the form:

\[
\begin{align*}
\alpha &= \mathcal{E}_x f_1(E_2, E_3, \theta, \rho, \sigma_2, \sigma_3) \quad (3.12) \\
\beta &= \mathcal{E}_y f_2(E_1, E_3, \theta, \rho, \sigma_1, \sigma_3) \quad (3.13) \\
\gamma &= \mathcal{E}_z f_3(E_1, E_2, \theta, \rho, \sigma_1, \sigma_2) \quad (3.14)
\end{align*}
\]

where \( f_1, f_2, f_3 \) are rational functions in which both the numerators and denominators are fourth order polynomials with respect to the factor \( E_2 - E_1 \) and their corresponding coefficients are fifth order polynomials in \( \sigma_2 - \sigma_1 \).

As expected, the propagation of equations (3.11) is also not identically satisfied which means that (3.12)-(3.14) represent another set of non-trivial constraints. Proceeding in a similar way as before, we take the “limit” \( S_I \to S_D \) and expect to obtain identities since, at this level of differentiation, there are no corresponding equations for the Petrov type D models. However, it can be shown that (with an obvious abuse of notation):

\[
\lim_{S_I \to S_D} f_1 = \frac{1}{2}, \quad \lim_{S_I \to S_D} f_2 = -\frac{1}{2}, \quad \lim_{S_I \to S_D} f_3 = 0. \quad (3.15)
\]

At this stage, the values of the “limits” (3.15) can be understood due to the specific structure of the rational functions (3.12)-(3.14). In particular, the linear parts of the polynomial expressions in \( f_1 \) and \( f_2 \) are proportional to each other with coefficients \( 1/2 \) and \( -1/2 \) respectively. The ratio of the associated linear parts in \( f_3 \) scales as \( (E_3 + 2E_2)(\sigma_3 + 2\sigma_2) \) which implies that \( f_3 \to 0 \) for \( S_I \to S_D \). Moreover, the indeterminate cases where either the two pairs of numerators and denominators or the linear parts of \( f_1, f_2 \) vanish, lead to algebraic relations \( \theta = \theta(\sigma_1, ...) \). However, it can be easily verified that none of these solutions is consistent with the evolution equations (2.5)-(2.6) and (2.12)-(2.13) except if \( \Lambda \neq 0 \), which in turn implies the existence of SIIS models of the Petrov type I with non-vanishing cosmological constant [42]. In the Appendix, we give the solutions of the indeterminate cases and for illustration purposes, we demonstrate the inconsistency of one of these solutions when the cosmological constant vanishes.

Assuming that \( \mathcal{E}_x, \mathcal{E}_y, \mathcal{E}_z \neq 0 \), from the evolution equations (2.49)-(2.51) and for every Petrov type I model, we obtain the following equations:

\[
\begin{align*}
(f_1)' &= \left( \frac{\alpha}{\mathcal{E}_x} \right)' = (1 - 4f_1^2)(\sigma_3 - \sigma_2)/4 \Rightarrow \left( \ln \frac{2f_1 + 1}{2f_1 - 1} \right)' = \sigma_3 - \sigma_2 \quad (3.16) \\
(f_2)' &= \left( \frac{\beta}{\mathcal{E}_y} \right)' = (1 - 4f_2^2)(\sigma_1 - \sigma_3)/4 \Rightarrow \left( \ln \frac{2f_2 + 1}{2f_2 - 1} \right)' = \sigma_1 - \sigma_3 \quad (3.17) \\
(f_3)' &= \left( \frac{\gamma}{\mathcal{E}_z} \right)' = (1 - 4f_3^2)(\sigma_2 - \sigma_1)/4 \Rightarrow \left( \ln \frac{2f_3 + 1}{2f_3 - 1} \right)' = \sigma_2 - \sigma_1. \quad (3.18)
\end{align*}
\]
It should be noted that in the last equations, we have calculated the time derivatives of the functions \(f_\alpha\) by using the evolution equations (2.49)-(2.51). In this sense, the initial constraints have been time propagated only up to third order, avoiding to compute the derivatives directly from the highly complicated expressions (3.12)-(3.14).

Expressing equations (3.16)-(3.18) in (local) coordinate form, a straightforward calculation yields:

\[
\begin{align*}
 f_1 &= \frac{e^{\Gamma+D_1(x,y,z)} \mp e^B}{2(e^{\Gamma+D_1(x,y,z)} \pm e^B)} \\
 f_2 &= \frac{e^{A+D_2(x,y,z)} \mp e^\Gamma}{2(e^{A+D_2(x,y,z)} \pm e^\Gamma)} \\
 f_3 &= \frac{e^{B+D_3(x,y,z)} \mp e^A}{2(e^{B+D_3(x,y,z)} \pm e^A)}
\end{align*}
\]  

(3.19)–(3.21)

where \(D_\alpha(x, y, z)\) are smooth functions of the spatial coordinates.

Equations (3.19)-(3.21) imply that for \(S_I \rightarrow S_D \Rightarrow A = B + H\) we have \(f_3 = 0 \Rightarrow D_3(x, y, z) = H(x, y)\). However \(f_1 \neq 1/2\) and \(f_2 \neq -1/2\) for any smooth functions \(A, B, D_1, D_2\). This contradiction shows that the \(x, y\) expansion rates must vanish \(\mathcal{E}_x = \mathcal{E}_y = 0\) which by means of equations (2.49)-(2.50) implies that \(\alpha = \beta = 0\). Hence, due to the Killing conditions (3.3), there exists a (necessarily) Abelian group of isometries acting on 2-dimensional spacelike orbits. Furthermore, equations (2.30) and (2.31) imply that each of the spacelike KVF's is hypersurface orthogonal, therefore the SIIS models of Petrov type I are reduced to the subclass of diagonal \(G_2\) models [39]. Using a similar procedure we can further show that \(\mathcal{E}_z = \gamma = 0\). Consequently, from equations (3.3) or, equivalently, the correspondence (2.61)-(2.62), it follows that the Petrov type I model always admits three independent and commuting spacelike KVF's and the resulting silent model being SH of the Bianchi type I.

It should be remarked that the above methodology cannot be applied when: a) two \(^3\) of the functions \(f_\alpha\) are equal to \(\pm 1/2\) or b) when \(f_1 = f_2 = 0\) and \(f_3 = \pm 1/2\) (a special class of the diagonal \(G_2\) family of perfect models). Nonetheless, both cases are incompatible with the full set of the dynamical equations. In particular, in the diagonal \(G_2\) class of models, satisfying \(f_3 = \pm 1/2 \Leftrightarrow \gamma = \pm \mathcal{E}_z/2\), equations (2.55)-(2.60) imply that

\[
E_1 = -\frac{\theta^2 - 3\theta \sigma_1 - 3(\rho + 3\sigma_1 \sigma_2 + 3\sigma_2^2)}{9}, \quad E_2 = \frac{2\theta^2 + 3\theta \sigma_2 - 6\rho - 9\sigma_2^2}{9}.
\]  

(3.22)

However, from (3.38), (2.41) and (2.48) it follows that the quantities \(w_1 = \sigma_2 + \theta/3\) and \(w_2 = E_2 + \rho/6\) are spatially homogeneous. Using the commutation relation (3.7), we find that the algebraic equation \((\dot{w}_2) = 0\) has as a solution \(\sigma_2 = \sigma_2(w_1, w_2, \sigma_1)\). Substituting in the initial relation \(f_3 = \pm 1/2\) we finally compute \(\sigma_1 = \sigma_1(w_1, w_2)\) and \(\sigma_2 = \sigma_2(w_1, w_2)\) i.e. the shear eigenvalues are also spatial homogeneous which in turn implies that \(\mathcal{E}_z = 0 = \gamma\). Using a similar method we can easily show that the conditions \(f_1 = 1/2\) and \(f_2 = -1/2\) lead also to \(\mathcal{E}_x = \mathcal{E}_y = 0 = \alpha = \beta\) and should be excluded.

We conclude this section by noticing that the above “limiting process” is always possible within the SIIS solutions of the EFE due to their specific geometrical and dynamical behaviour. In particular, let us assume that a Petrov type I model exists and is completely disconnected (in the spirit of the above reduction) from the Petrov type D models (for example, due to the occurrence of denominators of the form \((E_2 - E_1)\) etc.). Then, this must be true at any point of the spacetime manifold or, strictly speaking, during the time evolution of the model. Nevertheless, it has been shown that (apart from a set of measure zero) for initial states within the Petrov type I invariant set, the orbits approach at the asymptotic regimes (i.e. at early or late times), either a Petrov type D model or a vacuum SH model of Bianchi type I [15, 16, 25]. The exceptional initial states are treated similarly. From equations (3.16)-(3.18) it turns out that \(f_\alpha\) are monotone functions of the time coordinate \(t\) or the dimensionless time variable \(\tau\) which is defined according to \(dt/d\tau = 3/\theta\). In addition and taking into account the relations

\(^3\)Using the Ricci identities (2.52)-(2.54) for the spatial triad \(\{x^a, y^a, z^a\}\) it can be shown that e.g. \(\alpha = \mathcal{E}_z/2\) implies \(\mathcal{E}_z = 0 = \alpha\) or \(\beta = -\mathcal{E}_y/2\).
(3.16)-(3.18), the functions \( f_\alpha \) satisfy \( f_\alpha \to \pm 1/2 \) as \( \tau \to \pm \infty \) which due to equations (2.52)-(2.54) and the previous discussion implies that \( \mathcal{E}_x = \mathcal{E}_y = \mathcal{E}_z = 0 \) and the model will approach a SH model of Bianchi type I.

4 Discussion

In the present article we have undertaken a detailed covariant analysis of the question regarding the broadness of vacuum and non-vacuum SIIS models which is equivalent to the consistency of the spatial div and curl equations with the evolution equations. By exploiting the basic tools from the theory of the spacelike congruences and the fact that the shear \( \sigma_{ab} \) and the electric part \( E_{ab} \) share a common spatial eigenframe, we have expressed the constraints (2.36)-(2.41) in terms of the spacelike irreducible kinematical variables. These expressions make clear the significant role that the spacelike expansions and shear magnitudes play in the sense that they influence the overall spatial variation of the kinematical and dynamical variables (and vice versa) of the corresponding models. Therefore in order to have a transparent and complete picture of the dynamics of SIIS models, we have incorporated the associated time and spatial propagation equations (2.49)-(2.60) of the spacelike kinematical quantities.

Obviously, the suggested procedure has strong similarities with the 1+3 ONF approach and the basic steps in analyzing the SIIS are essentially the same, which shows the covariant nature of the incompatibility of the constraints (2.36)-(2.41) with the evolution equations. However, the approach followed here using elements from the theory of spacelike congruences, has allowed us to interpret the implications of the constraints (2.36)-(2.41) in a geometrical manner by expressing them as a set of conditions that must be satisfied by the kinematical quantities of the family of the spacelike curves.

It is worth noticing that, since the spacelike vector fields have been chosen in a unique way (as eigenvectors of the shear tensor and the electric part of the Weyl tensor), these conditions represent invariant relations. This fact facilitated the analytical study of the induced constraints in a (local) coordinate system and allowed us to show that the SH models of Bianchi type I are the only compatible Petrov type I spacetimes within the family of silent models.

We note that, the present approach permits us to draw further conclusions for more general spatially inhomogeneous configurations of the Petrov type I. For example, even when we allow the presence of a fluid with non-zero pressure (losing in that way the “silence” property), Petrov type D models are still the only permissible. This can be justified by observing that the pressure is necessary spatially homogeneous [5] (because \( \dot{x}^a = \dot{y}^a = \dot{z}^a = 0 \)), hence when we apply the commutation relations (3.5)-(3.7) in the evolution of the eigenvalues of the electric part \( E_{ab} \) (into which the fluid pressure enters) the form of the new constraints (3.8)-(3.10) is not affected. In addition, the fact that we didn’t use equation (2.5), shows that the uniqueness of Petrov type D models is also independent, in general, from the presence of a non-zero cosmological constant. Nonetheless, an exception to this conclusion is when the eigenvalues of the shear tensor \( \sigma_{ab} \) are constrained to be proportional to \( \theta \), due to the appearance of indeterminate expressions. In this case, a non-zero pressure or the cosmological constant play a key role in the existence of a broader class of SIIS solutions apart from the SH models of Bianchi type I [42]. Finally we expect that this approach can be used in order to check how large is the family of SIIS models by relaxing the vanishing of the vorticity and the magnetic part of the Weyl tensor but maintaining their “silence” properties i.e. the vanishing of the curls of \( E_{ab}, H_{ab} \) and the pressure \( p \).

Appendix

In this Appendix, we present the solutions of the indeterminate cases i.e. when either the two pairs of numerators and denominators or the linear parts of \( f_1, f_2 \) vanish. In particular, solving the algebraic
On the other hand, the vanishing of the linear parts of $f_1, f_2$ give the following solution:

$$\theta = -3 \sigma_2, \quad \rho = \frac{2 E_2 (E_2 - 3 \sigma_2^2)}{\sigma_2^4}.$$  \hspace{1cm} (4.4)

As we have mentioned in Section 3, none of the above solutions are consistent with the evolution equations when $\Lambda = 0$. For illustration purposes, we show the inconsistency of the solution (4.2). Propagating the second relation of (4.2) and using the evolution equations (2.5) and (2.43)-(2.45) we get the algebraic equation:

$$E_1^2 - 8 E_1 E_2 + 16 E_2^2 - 9 \Lambda \sigma_1^2 = 0$$

with solutions $E_1 = 4 E_2 \pm 3 \sqrt{\Lambda} \sigma_1$. Obviously, these solutions are valid only for non-zero (and positive) cosmological constant. Similarly, we can easily show the inconsistency for the rest solutions for the case $\Lambda = 0$.

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\[^{4}\text{We note that we have ignored the solutions } \sigma_1 = 0 \text{ and } \sigma_1 + \sigma_2 = 0 \text{ since they lead to unphysical models (} \rho < 0).\]
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