Spanning closed walks with bounded maximum degrees of graphs on surfaces

Morteza Hasanvand

Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

Abstract

Gao and Richter (1994) showed that every 3-connected graph which embeds on the plane or the projective plane has a spanning closed walk meeting each vertex at most 2 times. Brunet, Ellingham, Gao, Metzlar, and Richter (1995) extended this result to the torus and Klein bottle. Sanders and Zhao (2001) obtained a sharp result for higher surfaces by proving that every 3-connected graph embeddable on a surface with Euler characteristic \( \chi \leq -46 \) admits a spanning closed walk meeting each vertex at most \( \left\lceil \frac{6 - 2\chi}{3} \right\rceil \) times. In this paper, we develop these results to the remaining surfaces with Euler characteristic \( \chi \leq 0 \).

Keywords:
Graphs on a surface; spanning walk; spanning trail; connectivity.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let \( G \) be a graph. The vertex set, the edge set, and the number of components of \( G \) are denoted by \( V(G) \), \( E(G) \), and \( \omega(G) \), respectively. For a vertex set \( S \) of \( G \), we denote by \( e_G(S) \) the number of edges of \( G \) with both ends in \( S \). Also, \( S \) is called an independent set, if there is no edge of \( G \) connecting vertices in \( S \). A minor of \( G \) refers to a graph \( R \) which can obtained from \( G \) by contracting some vertex-disjoint connected subgraphs of \( G \). A \( k \)-walk (\( k \)-trail) in a graph refers to a spanning closed walk (trail) meeting each vertex at most \( k \) times. In this paper, we assume that all walks use each edge at most two times. Note that 1-walks and
1-trails of a graph are equivalent to Hamilton cycles. A graph $G$ is called $m$-tree-connected, if it has $m$ edge-disjoint spanning trees. It was known that the vertex set of any graph $G$ can be expressed uniquely (up to order) as a disjoint union of vertex sets of some induced $m$-tree-connected subgraphs [2]. These subgraphs are called the $m$-tree-connected components of $G$. For a graph $G$, we define $\Omega(G) = |P| - \frac{1}{2}e_G(P)$, in which $P$ is the unique partition of $V(G)$ obtained from the 2-tree-connected components of $G$ and $e_G(P)$ denotes the number of edges of $G$ joining different parts of $P$.

In 1956 Tutte [18] showed that every 4-connected plane graph admits a Hamilton cycle. Later, Gao and Richter (1994) [4] proved that every 3-connected graph which embeds on the plane or the projective plane admits a 2-walk, which was conjectured by Jackson and Wormald (1990) [10]. In 1995 Brunet, Ellingham, Gao, Metzlar, and Richter [1] extended this result to the torus and Klein bottle, and also proposed the following conjecture. This conjecture is verified in [3, 15] with linear bounds on $k_\chi$ (but not sharp).

**Conjecture 1.1.** ([1]) For every integer $\chi$, there is a positive integer $k_\chi$ such that every 3-connected graph which embeds on a surface with Euler characteristic $\chi$ admits a $k_\chi$-walk.

In 2001 Sander and Zhao [16] obtained a sharp bound on $k_\chi$ for surfaces with small enough Euler characteristic and established the following theorem. In Section 3, we develop this result to the surfaces with Euler characteristic $\chi \leq 0$. Our proof is based on a recent result in [9] and inspired by some methods that introduced in [13, 16].

**Theorem 1.2.** ([16]) Every 3-connected graph $G$ embeddable on a surface with Euler characteristic $\chi \leq -46$ has a $\left\lceil \frac{6-2\chi}{3} \right\rceil$-walk.

In 1994 Gao and Wormald [5] investigated trails in triangulations and derived the following theorem. They also deduced that every 5-connected triangulation in the double torus with representativity at least 6 admits a 4-trail. In Section 4, we conjecture that for each $\chi$, there is a positive integer $k_\chi$ such that every 4-connected graph which embeds on a surface with Euler characteristic $\chi$ admits a $k_\chi$-trail, and verify it for 5-connected graphs. As a consequence, we deduce that every 5-connected graph which embeds on the double torus (not necessarily triangulation) admits a 3-trail.

**Theorem 1.3.** ([5]) All triangulations in the projective plane, the torus and the Klein bottle have 4-trails.
2 Minor minimal 3-connected graphs having no \( k \)-walks

The following theorem is inspired by Theorem 5 in [13] and provides a common improvement for Lemmas 3.1 and 3.2 in [16]. We will apply it in the next section.

**Theorem 2.1.** Let \( G \) be a 3-connected graph and let \( k \) be an integer with \( k \geq 3 \). If \( G \) has no \( k \)-walks, then \( G \) contains a minor 3-connected bipartite graph \( R \) with the bipartition \( (X,Y) \) with the following properties:

1. \( R \) has no \( k \)-walks.
2. \( X = \{v \in V(G) : d_R(v) \geq k + 1\} \).
3. \( Y = \{v \in V(G) : d_R(v) = 3\} \).

Let \( G \) be a 3-connected graph. For an edge \( e \in E(G) \), define \( G/e \) to be the graph obtained from \( G \) by contracting \( e \). An edge \( e \) is said to be contractible if \( G/e \) is still 3-connected. For proving Theorem 2.1, we require the following two lemmas, which the first one is well-known.

**Lemma 2.2.** (See Halin [7]) Let \( G \) be a 3-connected graph except for \( K_4 \). Then every vertex of degree 3 in \( G \) is incident with a contractible edge.

**Lemma 2.3.** (Halin [7, 8]) Let \( G \) be a minimally 3-connected graph and define \( V_3(G) = \{v \in V(G) : d_G(v) = 3\} \). Then the following statements hold:

(i) \( V_3(G) \neq \emptyset \).

(ii) Every edge connecting two vertices in \( V(G) \setminus V_3(G) \) is contractible.

(iii) The graph obtained by contracting any edge connecting two vertices in \( V(G) \setminus V_3(G) \) is also minimally 3-connected.

(iv) Every cycle of \( G \) contains at least two vertices of \( V_3(G) \).

**Proof of Theorem 2.1.** Let \( G \) be a counterexample with the minimum \( |V(G)| \). We may assume that \( G \) is minimally 3-connected and also \( |V(G)| \geq 5 \). We here prove the following claim which was essentially shown by Sanders and Zhao in [16, Lemma 3.2].
Claim 1. \( \{ v \in V(G) : 3 \leq d_G(v) \leq k \} \) is independent.

Suppose otherwise that there is an edge \( xy \in E(G) \) such that \( d_G(x) \leq k \) and \( d_G(y) \leq k \). If all edges incident with \( x \) or \( y \) are not contractible, then by Lemma 2.2, we must have \( d_G(x) > 3 \) and \( d_G(y) > 3 \), which contradicts Lemma 2.3 (ii). Thus we may assume that there is a contractible edge incident with \( y \), say \( yz \) (possibly \( z = x \)). By the minimality of \( G \), the graph \( G/yz \) has a \( k \)-walk \( W \). Define \( H \) to be the graph with the vertex set \( V(G) \) having the same edges of \( W \) by considering multiplicity of each edge. First, assume \( d_H(y) > 0 \). If both of \( d_H(y) \) and \( d_H(z) \) are odd, define \( H' \) to be the graph obtained from \( H \) by adding a copy of \( yz \); otherwise, define \( H' \) to be the graph obtained from \( H \) by adding two copies of \( yz \). Next, assume \( d_H(y) = 0 \). In this case, define \( H' \) to be the graph obtained from \( H \) by adding two copies of \( xy \). It is not difficult to check that \( H' \) is an Eulerian graph with \( \Delta(H') \leq 2k \). Hence \( G \) admits a \( k \)-walk, which is a contradiction.

We now prove the next claim.

Claim 2. \( \{ v \in V(G) : d_G(v) \geq 4 \} \) is independent.

Let \( P = x_0x_1 \ldots x_l \) be a maximal path in the subgraph of \( G \) induced by \( \{ v \in V(G) : d_G(v) \geq 4 \} \). By Lemma 2.3 (iv), this subgraph is a forest. Also, \( x_0 \) and \( x_l \) have degree one in it and \( x_ix_j \not\in E(G) \), for any \( 0 \leq i < j \leq l \) and \( j \neq i + 1 \). By applying Lemma 2.3 (iii) repeatedly, one can conclude that \( G/P \) is 3-connected. By the minimality of \( G \), the graph \( G/P \) has a \( k \)-walk \( W \). Define \( H \) to be the graph with the vertex set \( V(G) \) having the same edges of \( W \) by considering multiplicity of each edge. Since \( x_0 \) and \( x_l \) have degree one in \( P \), there are two edges \( x_0y_1, x_ly_2 \in E(G) \) such that \( y_1, y_2 \in Y \). Note that \( \sum_{0 \leq i \leq l} d_H(x_i) \leq 2k \). Let \( x_j \) be a vertex of \( V(P) \) with the maximum \( d_H(x_j) \). If \( d_H(x_j) \leq 2k - 3 \), define \( H' \) to be the graph obtained from \( H \) by adding some of the edges of a copy of \( P \) and adding another new copy of \( P \) such that \( H' \) forms an Eulerian graph. Otherwise, if \( d_H(x_j) \geq 2k - 2 \), define \( H' \) to be the graph obtained from \( H \) by adding a copy of the paths \( y_1x_0Px_{i-1} \) (if \( i \neq 1 \)) and \( x_{i+1}Px_iy_2 \) (if \( i \neq l \)) and adding some of the edges of \( y_1Py_2 \) such that \( H' \) forms an Eulerian graph. According to the construction, it is not difficult to check \( \Delta(H') \leq 2k + 1 \). Hence \( G \) admits a \( k \)-walk, which is a contradiction again.

By the above-mentioned claims, \( G \) is a bipartite graph with the bipartition \((X,Y)\) in which \( X = \{ v \in V(G) : d_R(v) \geq k + 1 \} \) and \( Y = \{ v \in V(G) : d_R(v) = 3 \} \). By taking \( R = G \), we derive that \( G \) is not a counterexample and so the proof is completed.
3 3-connected graphs on surfaces

We shall below develop Theorem 1.2 as mentioned in the Abstract. For this purpose, we recall the following recent result from [9] that guarantees the existence of walks with bounded maximum degrees on specified independent vertex set.

Lemma 3.1. ([9]) Let $G$ be a connected graph with the independent set $X \subseteq V(G)$ and let $k$ be a positive integer. Then $G$ contains an spanning closed walk meeting each $v \in X$ at most $k$ times, if for every $S \subseteq X$, at least one of the following conditions holds:

1. $\omega(G \setminus S) \leq (k - \frac{1}{2})|S| + 1$.
2. $G$ contains an spanning closed walk meeting each $v \in S$ at most $k$ times.

Now, we are in a position to prove the main result of this paper.

Theorem 3.2. Every 3-connected graph $G$ embeddable on a surface with Euler characteristic $\chi \leq 0$ admits a $\left\lceil \frac{6 - 2\chi}{3} \right\rceil$-walk.

Proof. We may assume that $\chi < 0$, as the assertion was already proved in [1] for the surfaces with Euler characteristic zero (namely the torus and Klein bottle). By Theorem 2.1, we may also assume that $G$ is a bipartite graph with the bipartition $(X, Y)$ such that $X = \{v \in V(G) : d_G(v) \geq k + 1\}$ and $Y = \{v \in V(G) : d_G(v) = 3\}$, where $k = \left\lceil \frac{6 - 2\chi}{3} \right\rceil$. Let $S \subseteq X$ be a vertex cut of $G$ so that $|S| \geq 3$. Define $H$ to be the bipartite simple graph obtained from $G$ by contracting any component of $G \setminus S$ such that one partite set is $S$. Note that $H$ can be embedded on the surface as $G$ is embedded. Since $G$ is 3-connected, the minimum degree of $H$ is at least 3. Since $H$ is triangle-free, by Euler’s formula, it is easy to check that $|E(H)| \leq 2|V(G)| - 2\chi$. Thus

$$3\omega(G \setminus S) \leq |E(H)| \leq 2|V(H)| - 2\chi = 2(|S| + \omega(G \setminus S)) - 2\chi,$$

and so

$$\omega(G \setminus S) \leq 2|S| - 2\chi \leq (2 + \frac{-2\chi - 3/2}{3})|S| + 3/2 \leq (\left\lceil \frac{6 - 2\chi}{3} \right\rceil - 1/2)|S| + 3/2.$$

If the equalities hold, the we must have $|S| = 3$ and $\omega(G \setminus S) = 6 - 2\chi$, and also every component of $G \setminus S$ has exactly 3 neighbours in $S$. In this case, we shall show that $G$ has a spanning closed walk meeting each $v \in S$ at most $k$ times. Since $G$ is 3-connected, there is a cycle $C$ of $G$ containing all
three vertices of $S$. Since $S$ is independent, the cycle $C$ contains at least one vertex of exactly three components of $G \setminus S$. For the remaining $3 - 2\chi$ components of $G \setminus S$, join every of them to $C$ such that each vertex in $S$ has degree at most $\lceil (3 - 2\chi)/3 \rceil + 2$. Finally, add the edges of any component of $G \setminus S$ to this new graph. Obviously, the resulting spanning subgraph of $G$ has a spanning closed walk meeting each $v \in S$ at most $k$ times and so does $G$. Therefore, by Lemma 3.1, the graph $G$ have a spanning closed walk meeting each $v \in X$ at most $k$ times. Since $k \geq 3$ and every vertex of $Y$ in $G$ has degree 3, this walk forms a $k$-walk for $G$. Hence the theorem holds.

4 Graphs with higher connectivity

As we already mentioned, Tutte [18] proved that every 4-connected planar graph admits a Hamilton cycle. Thomas and Yu [17] extended this result to the projective plane and Grünbaum [6] and independently Nash-Williams [12] conjectured that Tutte’s result could be developed to the torus. Motivated by Conjecture 1.1 and these results, we now propose the following conjecture. By considering the complete bipartite graph $K_{4, -\chi+4}$ (which can be embedded on a surface with Euler characteristic $\chi$), if this conjecture would be true, then we must have $k\chi \geq \lceil \frac{6 - 3\chi}{4} \rceil$.

Conjecture 4.1. For every integer $\chi$, there is a positive integer $k_\chi$ such that every 4-connected graph which embeds on a surface with Euler characteristic $\chi$ admits a $k_\chi$-trail.

We shall below prove Conjecture 4.1 for 5-connected graphs. For this purpose, we require the following recent result which gives a sufficient condition for the existence of $k$-trails.

Lemma 4.2.([9]) Let $G$ be a graph and let $k$ be a positive integer. If for all $S \subseteq V(G)$,
$$\Omega(G \setminus S) \leq (k - \frac{1}{2})|S| + 1 - \frac{1}{2}e_G(S),$$
then $G$ admits a $k$-trail.

Now, we are ready to prove the main result of this section.

Theorem 4.3. Every 5-connected graph $G$ which embeds on a surface with Euler characteristic $\chi \leq 0$ admits a $\lceil \frac{6 - 3\chi}{4} \rceil$-trail.

Proof. Let $S \subseteq V(G)$ with $\Omega(G \setminus S) > 1$. For each vertex $v$, we have $\Omega(G - v) = 1$, because the graph $G - v$ is 2-tree-connected [11, 19]. Thus $|S| \geq 2$. Define $H$ to be the simple graph
obtained form $G$ by contracting the 2-tree-connected components of $G \setminus S$. Set $S' = V(H) \setminus S$. Note that $|V(H)| = |S| + |S'| \geq 3$ and $H$ can be embedded on the surface as $G$ is embedded. By Euler’s formula, it is easy to check that $|E(H)| \leq 3|V(H)| - 3\chi$. Note also that there is no a pair of edges of $G$ joining two different 2-tree-connected components of $G \setminus S$. Since $G$ is 5-connected, the minimum degree of $H$ is at least 5. Thus

$$5|S'| \leq |E(H)| - e_H(S) + e_H(S'),$$

and so

$$5|S'| - e_H(S') \leq |E(H)| - e_H(S) \leq 3|V(H)| - 3\chi - e_H(S) = 3(|S'| + |S|) - 3\chi - e_G(S).$$

Therefore,

$$\Omega(G \setminus S) = |S'| - \frac{1}{2}e_H(S') \leq \frac{3}{2}|S| + \frac{-3\chi}{2} - \frac{1}{2}e_G(S).$$

and so

$$\Omega(G \setminus S) \leq \left(\frac{3}{2} + \frac{-3\chi/2 - 1}{2}\right)|S| + \frac{1}{2}e_G(S) \leq \left(\left\lceil \frac{6 - 3\chi}{4} \right\rceil - \frac{1}{2}\right)|S| + \frac{1}{2}e_G(S).$$

Hence the theorem follows from Lemma 4.2 with $k = \left\lceil \frac{6 - 3\chi}{4} \right\rceil$. \hfill \Box

The following result improves Corollary 2 in [5], which says that every 5-connected triangulation in the double torus with representativity at least 6 admits a 4-trail. Note that the Euler characteristic of the double torus is $-2$.

**Corollary 4.4.** Every 5-connected graph $G$ which embeds on the double torus admits a 3-trail.

**References**

[1] R. Brunet, M. N. Ellingham, Z. Gao, A. Metzlar, and R. B. Richter, Spanning planar subgraphs of graphs in the torus and Klein bottle, J. Combin. Theory Ser. B, 65 (1995), pp. 7–22.

[2] P.A. Catlin, Double cycle covers and the Petersen graph, J. Graph Theory 13 (1989), 465–483.

[3] M. N. Ellingham, Spanning paths, cycles, trees and walks for graphs on surfaces, Congr. Numer., 115 (1996), pp. 55–90.

[4] Z. Gao and R. B. Richter, 2-walks in circuit graphs, J. Combin. Theory Ser. B, 62 (1994), pp. 259–267.
[5] Z. Gao and N. C. Wormald, Spanning Eulerian subgraphs of bounded degree in triangulations, Graphs Combin., 10 (1994), pp. 123–131.

[6] B. Grünbaum, Polytopes, graphs, and complexes, Bull. Amer. Math. Soc., 76 (1970), pp. 1131–1201.

[7] R. Halin, A theorem on $n$-connected graphs, J. Combinatorial Theory, 7 (1969), pp. 150–154.

[8] R. Halin, Untersuchungen über minimale $n$-fach zusammenhängende Graphen, Math. Ann., 182 (1969), pp. 175–188.

[9] M. Hasanvand, Spanning trees and spanning Eulerian subgraphs with small degrees. II, https://arxiv.org/abs/1702.06203

[10] B. Jackson and N. C. Wormald, $k$-walks of graphs, Australas. J. Combin., 2 (1990), pp. 135–146.

[11] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc., 36 (1961), pp. 445–450.

[12] C. St. J. A. Nash-Williams, Unexplored and semi-explored territories in graph theory, (1973), pp. 149–186.

[13] K. Ota and K. Ozeki, Spanning trees in 3-connected $K_{3,t}$-minor-free graphs, J. Combin. Theory Ser. B, 102 (2012), pp. 1179–1188.

[14] K. Ozeki, Spanning trees with bounded maximum degrees of graphs on surfaces, SIAM J. Discrete Math. 27 (2013), pp. 422–435.

[15] D. P. Sanders and Y. Zhao, On 2-connected spanning subgraphs with low maximum degree, J. Combin. Theory Ser. B, 74 (1998), pp. 64–86.

[16] D. P. Sanders and Y. Zhao, On spanning trees and walks of low maximum degree, J. Graph Theory, 36 (2001), pp. 67–74.

[17] R. Thomas and X. Yu, 4-connected projective-planar graphs are Hamiltonian, J. Combin. Theory Ser. B, 62 (1994), pp. 114–132.

[18] W. T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc., 82 (1956), pp. 99–116.

[19] W. T. Tutte, On the problem of decomposing a graph into $n$ connected factors, J. London Math. Soc., 36 (1961), pp. 221–230.