Relative Arbitrage Opportunities with Interactions among $N$ investors

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Abstract

The relative arbitrage portfolio outperforms a benchmark portfolio over a given time-horizon with probability one. With market price of risk processes depending on the market portfolio and investors, this paper analyzes the multi-agent optimization of relative arbitrage opportunities in the coupled system of market and wealth dynamics. We construct a well-posed market dynamical system of McKean-Vlasov type under an empirical measure of investors, where each investor seeks for relative arbitrage with respect to a benchmark dependent on market and all the agents. We show the conditions to guarantee relative arbitrage opportunities among competitive investors through the Fichera drift. Under mild conditions, we derive the optimal strategies for investors and the unique Nash equilibrium that depends on the smallest nonnegative solution of a Cauchy problem.

1 Introduction

Market participants usually compare the performance of an investment strategy with a benchmark index. Among different metrics and tools for capturing opportunities that outperform a benchmark portfolio, relative arbitrage established in Stochastic Portfolio Theory (SPT), see Fernholz [8], is of special interest to investment and portfolio management. However, market dynamics is constantly influenced by large investing entities where complicated interactions occur among them. We need a market model that captures these behaviors and develops a multi-agent optimization framework. To better describe and analyze the market based on SPT, this paper investigates the following questions: How do we capture the competitive behaviors of participants in the financial market? With additional information on these investors, how do we improve the market model and make portfolio suggestions? We aim to develop the optimization scheme for portfolio managers or asset management entities in a realistic market environment. This scheme would provide the information structure (for example, feedback from capitalization processes, wealth processes, or agent’s preference profile, etc.) that is required for effective portfolio strategy and the corresponding optimal investments.

The relative arbitrage problem first defined in SPT considers generating a strategy that outperforms a benchmark portfolio almost surely at the end of a certain time span and looks for the highest relative return. It shows in [10] that relative arbitrage can exist in equity markets that resemble actual markets and that relative arbitrage results from market diversity, a condition that prevents the concentration of all market capital into a single stock. Specific examples of the market, including the stabilized volatility model, in which a relative arbitrage opportunity exists, are introduced in [9]. To relax the assumptions about the behavior of the market imposed in the SPT, [27] considers relative arbitrage in regulated markets where dividends and the merging and splitting of companies are taken into account. Our model arises from the pioneering work of Fernholz and Karatzas [5], which characterizes the best possible relative arbitrage with respect to the market portfolio, and derives non-anticipative investment strategies of the best arbitrage in a Markovian setting. The best arbitrage opportunity is further analyzed in [6] in a market with Knightian uncertainty. The smallest proportion of the initial market capitalization is described as the min-max value of a zero-sum stochastic game between the investor and the market. Further investigation of the exploitation of relative arbitrage opportunities has been carried out in [2, 11, 25, 26]. Assuming the market is diverse and sufficiently volatile, the functionally generated portfolios introduced in SPT are a tool to construct portfolios with favored return characteristics.

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The optimization problem from the functionally generated portfolio point of view is handled in [32]. The papers [22] and [31] connect relative arbitrage with information theory and optimal transport problems. The robust optimization perspective is studied in [16, 17, 18] in terms of the asymptotic growth rate and model uncertainty.

Our work focuses on the multi-agent optimization theory for relative arbitrage opportunities. This paper forms a stochastic differential game system of the equity market, where investors aim to pursue the optimal strategies to outperform the market index and peer investors. To our knowledge, this is the first paper to study Stochastic Portfolio Theory with interactions between market and investors.

We define a benchmark process $\mathcal{V}(\cdot) := \delta \cdot X(\cdot) + (1 - \delta) \mathcal{V}(\cdot)$, where $X(\cdot)$ is the total market capitalization, $\mathcal{V}(\cdot)$ is the average wealth of all investors, $\delta \in [0, 1]$ is a given constant weight. An investor $\ell$ achieves relative arbitrage if his/her logarithmic terminal wealth can outperform the logarithmic terminal benchmark by a personalized preference level $c_{\ell}$ given at time 0.

The first question raised in this paper is: Among noncooperative agents, how does one achieve the best strategy to achieve the relative arbitrage opportunities? The optimal arbitrage for an investor $\ell$ is formulated as $u^\ell(T)$ in Definition 3.3. Under some market conditions, we tackle the optimal arbitrage problem over $[0, T]$ by solving the sub-problems (3.18) starting from every $0 \leq t \leq T$. That is, investors would obtain the optimal initial investment amount at time $t \in [0, T)$, characterized as the smallest nonnegative solution of a Cauchy problem (3.22)-(3.23), and achieve relative arbitrage at the terminal time $T$. We show in Proposition 3.4 the existence of relative arbitrage using the Fichera drift method [13]. In the meantime, the conditions in Proposition 3.2 generally, investors are allowed to strictly outperform the benchmark by a bit, while the average of the whole group’s preference indices needs to be controlled.

The next question arises: Is it possible for every investor to have optimal arbitrage in the market $\mathcal{M}$? If so, what are their optimal strategies? We characterize the optimal wealth that can be achieved by the unique Nash equilibrium of the finite-population game. We elaborate on this point in Definition 4.2. Theorems 4.1 search for a set of optimal strategies motivated by functionally generated portfolios to achieve Nash equilibrium. We derive the optimal strategy profile as a fixed-point problem over the path space of strategies and show that the Nash equilibrium exists. We show the uniqueness of Nash equilibrium in Theorem 4.2 through a fixed point problem over the path space of optimal arbitrage quantity instead.

To conclude, we establish the relative arbitrage problem in the $N$-investor regime. We clarify the interaction among market and investors through the strong solution of the McKean-Vlasov system and the relative arbitrage objectives. We show the optimal arbitrage as the smallest nonnegative solution of a Cauchy problem. The existence of relative arbitrage opportunities is guaranteed under the Fichera drift condition on the auxiliary market coefficients, when $\delta > 0$. So, direct interaction with the market is indispensable. We construct the $N$-player game involving the interactions of the market and investors, and derive the optimal strategies as the unique Nash equilibrium through the empirical measure of wealth processes. The strategy that achieves Nash equilibrium is in closed-loop feedback form, as the market structure takes the feedback from trading volumes. The uniqueness of the Nash equilibrium depends on the weight $\delta$ in the relative arbitrage benchmark and the average preference level of the investors. The results of Nash equilibrium suggest that the time horizon of the problem and the scale of interactions in the market are important. This also motivates future studies of the relative arbitrage problem with mean-field games, cooperative investors, short-term arbitrage opportunities, and a more general relative arbitrage objective.

Organization of this Paper. This paper is structured as follows. Section 2 introduces the market with $N$ investors as a well-posed interacting particle system. Section 3 discusses the relative arbitrage problem and market price of risk processes in a multi-investor formulation. The conditions to obtain relative arbitrage opportunities are explained in detail. In Section 4, we set up $N$-player games among investors where the optimization of relative arbitrage is determined by Nash equilibrium. We discuss the existence and uniqueness of the Nash equilibrium, and provide an example inspired by the volatility-stabilized market model. Finally, we include additional proofs and theoretical supports of the model in Appendices A-B.

2 The Market Model

We consider an equity market and focus on the market behavior and a group of investors in this market. The number of investors that we include is large enough to affect the market as a whole. However, this group of interest is only part of the entire market. The rest of the market participants do not influence the market and can be viewed as exogenous.
2.1 Capitalizations, wealth and portfolios

For a given finite time horizon \([0, T]\), an admissible market model \(\mathcal{M}\) we use in this paper consists of a given \(n\) dimensional standard Brownian motion \(W(\cdot) := (W_1(\cdot), \ldots, W_n(\cdot))'\) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The filtration \(\mathbb{F}\) represents the “flow of information” in the market driven by the Brownian motion, that is, \(\mathbb{F} = \{\mathcal{F}^W(t)\}_{0 \leq t \leq \infty}\) and \(\mathcal{F}^W(t) := \sigma(W(s); 0 < s < t)_{0 \leq t \leq \infty}\) with \(\mathcal{F}^W(0) := \{\emptyset, \Omega\}, \text{mod} \, \mathbb{P}\). All local martingales and supermartingales are with respect to the filtration \(\mathbb{F}\) if not written specifically.

There are \(n\) risky assets (stocks) with prices per share \(X(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))'\) driven by \(n\) independent Brownian motions. We define a factor \(\mathcal{Y}(\cdot) := (\mathcal{Y}_1(\cdot), \ldots, \mathcal{Y}_n(\cdot))'\), each of which represents the aggregated feedback effect of investors on each asset capitalization. We will specify this term shortly. Here, ‘\(\cdot\)’ stands for the transpose of matrices. We assume \(\beta(\cdot), \sigma(\cdot), \gamma(\cdot)\) and \(\tau(\cdot)\) are defined on \(\mathbb{R}_+^n \times \mathbb{R}_+^n\), are time homogeneous and the process \((\mathcal{X}(t), \mathcal{Y}(t)), t \geq 0\) in Definition 2.1 is Markovian. That is, it follows the system of stochastic differential equations below: for \(t \in [0, T]\

\[
dX_i(t) = X_i(t)(\beta_i(\mathcal{X}(t), \mathcal{Y}(t))dt + \sum_{k=1}^{n} \sigma_{ik}(\mathcal{X}(t), \mathcal{Y}(t))dW_k(t), \quad i = 1, \ldots, n, \tag{2.1}
\]

with initial condition \(X_i(0) = x_i\). The coefficients \(\beta\) and \(\sigma\) depend on the capitalization \(\mathcal{X}\) and the factor \(\mathcal{Y}\). The factor models have been considered in the Capital Asst Pricing Models to assess the risk and return of the market. For example, in the Fama-French three factor model, the factors are market excess return, outperformance of small companies against big companies and out-performance of high market-to-book ratio companies against low market-to-book ratio companies.

Since equity capitalization moves in continuous time, based on the supply and demand for stock shares, we consider the average capital invested as a factor in the capitalization processes. Each investor \(\ell\) invests the proportion \(\pi^\ell(t)\) of current wealth \(V^\ell(t)\) in the \(i\)th stock at each time \(t\). We define \(\mathcal{Y}(t) := (\mathcal{Y}_1(t), \ldots, \mathcal{Y}_n(t))'\), \(t \geq 0\). In the paper, the factor we consider is an interaction term among the investors. In particular, we consider specifically the average trading volume

\[
\mathcal{Y}_i(t) := \frac{1}{N} \sum_{\ell=1}^{N} V^\ell(t) \pi^\ell_i(t), \quad \mathcal{Y}_i(0) := y_{0,i}, \tag{2.2}
\]

where for each \(\ell = 1, \ldots, N\), \(\pi^\ell_i(\cdot)\) is fixed in Section 2.3, and satisfy the following Definition 2.1 without concern on the interactions among investors. Later, we address the market model to depend on the actions of investors, where \(\pi^\ell_i(\cdot)\) involves a fixed point problem in Section 4, for \(i = 1, \ldots, n\), \(\ell = 1, \ldots, N\). The average trading volume \(\mathcal{Y}_i(t)\) invested by the \(N\) players on stock \(i\) is assumed to follow an Itô diffusion process

\[
\mathcal{Y}_i(t) = y_{0,i} + \int_{0}^{t} \gamma_i(\mathcal{X}(r), \mathcal{Y}(r))dr + \int_{0}^{t} \sum_{k=1}^{n} \tau_{ik}(\mathcal{X}(r), \mathcal{Y}(r))dW_k(r), \quad t \in (0, T] \text{ for an arbitrary } T > 0, \tag{2.3}
\]

and the initial value \(\mathcal{Y}_i(0) := y_{0,i}\), \(i = 1, \ldots, n\).

In this paper, we assume that \(\dim(W(t)) = \dim(\mathcal{X}(t)) = n\), that is, we have exactly as many randomness sources as there are stocks on the market \(\mathcal{M}\). The dimension \(n\) is chosen to be large enough to avoid unnecessary dependencies among the stocks we define. Here, \(\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))' : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}^n\) as the mean rates of return for \(n\) stocks and \(\sigma(\cdot) = (\sigma_{ik}(\cdot))_{n \times n} : \Omega \to \text{GL}(n)\) as volatilities are assumed to be invertible, \(\mathbb{F}\)-progressively measurable in which \(\text{GL}(n)\) is the space of \(n \times n\) invertible real matrices. For simplicity, denote \(\omega(\cdot) := (\mathcal{X}(t), \mathcal{Y}(t))\). To satisfy the integrability condition, we assume that for any \(T > 0\),

\[
\sum_{i=1}^{n} \int_{0}^{T} \left(|\beta_i(\omega)| + \alpha_{ii}(\omega)\right)dt < \infty,
\]

where \(\alpha(\cdot) := \sigma(\cdot)\sigma'(\cdot)\), and its \(i, j\) element \(\alpha_{ij}(\cdot)\) is the covariance process between the logarithms of \(X_i\) and \(X_j\) for \(1 \leq i, j \leq n\). The market \(\mathcal{M}\) is hence a complete market. We assume \(\gamma(\cdot)\) and \(\tau(\cdot)\) satisfy that for any \(T > 0\),

\[
\sum_{i=1}^{n} \int_{0}^{T} \left(|\gamma_i(\omega)| + \psi_{ii}(\omega)\right)dt < \infty,
\]
where \( \psi(\cdot) := \tau(\cdot)\tau'(\cdot) \).

In this model, there are \( N \) small investors, “small” is in the sense that each individual of these \( N \) investors has very little influence on the overall system.

**Definition 2.1** (Investment strategy).  
(1) An \( \mathbb{F} \)-progressively measurable \( n \)-dimensional process \( \pi \) is called an admissible investment strategy if

\[
\int_0^T ((\pi(t))'\beta(\omega_t) + (\pi(t))'\alpha(\omega_t)\pi(t))dt < \infty, \quad T \in (0, \infty), \text{ a.e.} \tag{2.4}
\]

The strategy here is a self-financing portfolio, since wealth at any time is obtained by trading the initial wealth according to the strategy \( \pi(\cdot) \). We denote the admissible set of the investment strategy process of one investor by \( \mathcal{A} \). In the remainder of the paper, we only consider optimizing the strategy processes in the admissible set \( \mathcal{A} \).

(2) An investor uses the proportion \( \pi_i(\cdot) \) of current wealth \( V^\ell(\cdot) \) to invest in the stock \( i \). The proportion \( \pi_0 = 1 - \sum_{i=1}^n \pi_i(\cdot) \) is on the money market. A special case of the admissible strategy is when \( \pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_n(\cdot))' \) is a portfolio, i.e., it takes values in the set

\[
\Delta_n := \{ \pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n | \pi_1 + \ldots + \pi_n = 1 \}.
\]

From now on, we add the superscript \( \ell \) to the strategy \( \pi(\cdot) \) defined above to distinguish the strategies of different investors. The dynamics of the wealth process \( V^\ell(\cdot) \) of an individual investor \( \ell \), invested in the stock market, is determined by

\[
\frac{dV^\ell(t)}{V^\ell(t)} = \sum_{i=1}^n \pi^\ell_i(t)\frac{dX_i(t)}{X_i(t)}, \quad V^\ell(0) = v^\ell_0. \tag{2.5}
\]

In the rest of the paper, we simplify the notation as

\[
X_i(t)\beta_i(t) = b_i(\mathcal{X}(t), \mathcal{Y}(t)),
\]

\[
X_i(t)\sigma_{ik}(t) = s_{ik}(\mathcal{X}(t), \mathcal{Y}(t)), \quad \sum_{k=1}^n s_{ik}(t)s_{jk}(t) = a_{ij}(\mathcal{X}(t), \mathcal{Y}(t)).
\]

**Assumption 1.**  
\( a. \) Assume the Lipschitz continuity and linear growth condition are satisfied with Borel measurable mappings \( b(\mathbf{x}, \mathbf{y}), s(\mathbf{x}, \mathbf{y}), \gamma(\mathbf{x}, \mathbf{y}), \tau(\mathbf{x}, \mathbf{y}) \). For simplicity, we specify the conditions for \( b(\mathbf{x}, \mathbf{y}) \) and \( s(\mathbf{x}, \mathbf{y}) \) below, but the conditions for the coefficients \( \gamma(\cdot) \) and \( \tau(\cdot) \) of the trading volume processes \( \mathcal{Y}(\cdot) \) can be written in the same vein. That is, there exists a constant \( C_L, C_G \in (0, \infty) \) that is independent of \( t \in [0, T] \), such that

\[
|b(\mathbf{x}, \mathbf{y}) - b(\mathbf{x}', \mathbf{y}')| + |s(\mathbf{x}, \mathbf{y}) - s(\mathbf{x}', \mathbf{y}')| \leq C_L \left( |\mathbf{x} - \mathbf{x}'| + |\mathbf{y} - \mathbf{y}'| \right),
\]

\[
\|b(\mathbf{x}, \mathbf{y})\| + \|s(\mathbf{x}, \mathbf{y})\| \leq C_G (1 + \|\mathbf{x}\| + \|\mathbf{y}\|),
\]

for any \( (\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \).

\( b. \) The matrix \( \tau(\cdot) := (\tau_{ik}(\cdot))_{1 \leq i, k \leq n} \) is nondegenerate. The market price of the risk process \( \theta : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) exists and is square integrable. That is, there exists an \( \mathbb{F} \)-progressively measurable process such that for any \( t \in [0, \infty) \),

\[
\sigma(\omega_t)\theta(\omega_t) = \beta(\omega_t), \quad \tau(\omega_t)\theta(\omega_t) = \gamma(\omega_t), \tag{2.6}
\]

\[
\mathbb{P} \left( \int_0^T ||\theta(\omega_t)||^2dt < \infty, \forall T \in (0, \infty) \right) = 1,
\]

where \( \omega_t := (\mathcal{X}(t), \mathcal{Y}(t)) \).

In the scope of a complete market, Assumption 1b. shows that the price of risk process \( \theta(t) \) governs both the risk premium per unit volatility of stocks and trading volumes, since the market is simultaneously defined by the stocks and the investors. The group of investors we consider in this paper influences the stock capitalization through the trading volumes driven by the same \( W(\cdot) \). Thus, it does not bring an extra risk factor to the market. We extend the above market model to a general interacting particle system in Appendix D.
3 Optimization of relative arbitrage in finite systems

We first recall the definition of relative arbitrage in Stochastic Portfolio Theory.

Definition 3.1 (Relative Arbitrage). Given two investment strategies \( \pi(\cdot) \) and \( \rho(\cdot) \), with the same initial capital \( V^\pi(0) = V^\rho(0) = 1 \), we shall say that \( \pi(\cdot) \) represents an arbitrage opportunity relative to \( \rho(\cdot) \) over the time horizon \([0, T]\), with a given \( T > 0 \), if

\[
\mathbb{P}(V^\pi(T) \geq V^\rho(T)) = 1 \quad \text{and} \quad \mathbb{P}(V^\pi(T) > V^\rho(T)) > 0.
\]

We use the total capitalization

\[ X(t) = X_1(t) + \ldots + X_n(t), \quad t \in (0, T]; \quad X(0) = x_0 := x_1 + \cdots + x_n \tag{3.1} \]

to represent the capitalization of the entire market. The market portfolio \( m \) amounts to the ownership of the entire market by investing in proportion to the market weight of each stock,

\[ m_i(t) = m_i^m(t) := \frac{X_i(t)}{X(t)}, \quad i = 1, \ldots, n, \quad t \geq 0. \]

Consider the wealth process \( V^m(\cdot) \) generated by the market portfolio. Let \( V^m(0) = v_0 \),

\[
\frac{dV^m(t)}{V^m(t)} = \sum_{i=1}^{n} m_i^m(t) \cdot \frac{dX_i(t)}{X_i(t)} = \frac{dX(t)}{X(t)}, \quad t \geq 0.
\]

In the following sections, instead of treating the reference strategy as the market portfolio as in \([5]\), we consider arbitrage opportunities relative to a modified benchmark strategy in Definition 3.2 and Proposition 3.1. Without loss of generality, assume that \( X(0) = V^m(0) \) from now on.

3.1 Benchmark of the market and investors

In general, the performance of a portfolio is measured with respect to the market portfolio and other factors. For example, asset managers care about not only absolute performance compared to the market index, but also relative performance with respect to all collegiate managers - they try to exploit strategies that achieve an arbitrage relative to market and peer investors. Next, we define the benchmark of the overall performance.

Definition 3.2 (Benchmark). A relative arbitrage benchmark \( \mathcal{V}(T), T \in (0, \infty) \), which is the weighted average of the performance of the market portfolio and the average portfolio of \( N \) investors, is defined as

\[
\mathcal{V}(T) = \delta \cdot X(T) + (1 - \delta) \cdot \overline{V}(T), \tag{3.2}
\]

where

\[
\overline{V}(\cdot) := \frac{1}{N} \sum_{\ell=1}^{N} V^\ell(\cdot)
\]

is the average amount of wealth processes \( V^\ell(\cdot), \ell = 1, \ldots, N. \delta \in (0, 1) \) is a given constant weight. \( X(T) \) is the wealth of the market portfolio in (3.1).

We assume that each investor measures the logarithmic ratio of his or her own wealth at time \( T \) to the benchmark in (3.2), and searches for a strategy with which the logarithmic ratio is, almost surely, above a personal level of preference. For \( \ell = 1, \ldots, N. \) we denote the investment preference of investor \( \ell \) by \( c_\ell \), a real number given at \( t = 0 \). Note that \( c_\ell \) is an investor-specific constant and thus might be different among the individuals \( \ell = 1, \ldots, N. \) An arbitrary investor \( \ell \) tries to achieve

\[
\log \frac{V^\ell(T)}{\mathcal{V}(T)} \geq c_\ell, \quad \text{a.s.} \quad \text{or equivalently,} \quad V^\ell(T) \geq e^{c_\ell} \mathcal{V}(T), \quad \text{a.s.} \tag{3.3}
\]

Thus \( \mathcal{V}(T) \) is the benchmark and an investor \( \ell \) aims to match \( e^{c_\ell} \mathcal{V}(T) \) based on their preferences. For simplicity, we define

\[
e^{c} := \frac{1}{N} \sum_{\ell=1}^{N} e^{c_\ell}.
\]

The following theorem shows that the benchmark \( \mathcal{V} \) is a valid wealth process.
Proposition 3.1. Benchmark $\mathcal{V}(t) = \delta X(t) + (1 - \delta)\mathcal{Y}(T)$ in (3.2) can be generated from a strategy $\Pi(\cdot) := (\Pi_1(\cdot), \ldots, \Pi_n(\cdot)) \in \mathcal{A}$, 

$$\Pi_i(t) = \frac{1}{\mathcal{V}(t)} \left( \delta X_i(t) + (1 - \delta)\mathcal{Y}_i(t) \right),$$

where $\mathcal{Y}_i(t)$ is defined in (2.3).

Proof. To show $\mathcal{V}(t)$ is a wealth process generated by a strategy, we use (2.5), (2.3), (3.4) to get

$$\frac{d\mathcal{V}(t)}{\mathcal{V}(t)} = \frac{1}{\mathcal{V}(t)} \left( \delta dX(t) + \frac{1}{N}(1 - \delta) \sum_{i=1}^{N} V^{\ell} \pi_i(t) \frac{dX_i(t)}{X_i(t)} \right) = \sum_{i=1}^{N} \frac{\Pi_i(t) dX_i(t)}{X_i(t)}, \quad \text{for } t \in (0, T],$$

and

$$\mathcal{V}(0) = \delta X(0) + \frac{1 - \delta}{N} \sum_{i=1}^{N} v^\ell,$$

where

$$\Pi_i(t) = \frac{\delta X(t)}{\mathcal{V}(t)} m_i(t) + \frac{(1 - \delta)}{N\mathcal{V}(t)} \sum_{i=1}^{N} V^{\ell} \pi_i(t) = \frac{\delta X_i(t) + (1 - \delta)\mathcal{Y}_i(t)}{\mathcal{V}(t)}.$$

Further computations show that $\Pi_i(t)$ satisfies the self-financing condition (2.4). $\Pi \in \mathcal{A}$ since $\sum_{i=1}^{N} \Pi_i(t) = 1$ and $0 < \Pi_i(t) < 1$, for any $t \in [0, T]$, $i = 1, \ldots, n.$ \hfill $\Box$

3.2 Optimization in relative arbitrage

Now, we shall answer the first question posed in the Introduction. What is the best strategy to achieve relative arbitrage over $[0, T]$ for investor $\ell = 1, \ldots, N$, with the fixed portfolios of the rest of the investors? We adapt an idea in [5] for the optimal relative arbitrage with respect to the market. Using the optimal strategy $\pi^\ell$, the investor $\ell$ will start with the least amount of initial capital (or initial cost) relative to $\mathcal{V}(0)$, in order to match or exceed the benchmark $e^{c^\ell}\mathcal{V}(T)$ at the terminal time $T$.

Definition 3.3 (Optimal arbitrage among agents). Given the market dynamics $X(0)$, the initial wealth of the other investors $v^{-\ell} := (v^1(\cdot), \ldots, v^{\ell-1}(\cdot), v^{\ell+1}(\cdot), \ldots, v^N(\cdot))$, and the admissible portfolios

$$\pi^{-\ell}(\cdot) := (\pi^1(\cdot), \ldots, \pi^{\ell-1}(\cdot), \pi^{\ell+1}(\cdot), \ldots, \pi^N(\cdot)), \quad \text{(3.5)}$$

where each portfolio $\pi^k \in \mathcal{A}$, $k = 1, \ldots, \ell - 1, \ell + 1, \ldots, N$. In the market system (2.1)-(2.3), for every $\ell = 1, \ldots, N$, the investor $\ell$ pursues the optimal arbitrage characterized by the smallest initial relative wealth

$$v^\ell(T) = \inf \left\{ w^\ell \in (0, \infty) \bigg| \text{there exists } \pi^\ell(\cdot) \in \mathcal{A} \text{ such that } v^\ell = w^\ell\mathcal{V}(0), \mathcal{V}^{v^\ell, \pi^\ell}(T) \geq e^{c^\ell} \cdot \mathcal{V}(T) \right\} \quad \text{(3.6)}$$

and the relative arbitrage portfolio $\{\pi^\ell(t)\}_{0 \leq t \leq T}$ that achieves such smallest initial relative wealth $v^\ell(T)$.

Note that in Definition 3.3, the optimal strategy vector $\pi^\ell$ is not necessarily uniquely determined. Thus, the investor $\ell$ searches for his or her smallest initial wealth relative to the benchmark $\mathcal{V}(0)$, and the optimal strategy $\pi^\ell(\cdot)$ as a pair. The initial benchmark $\mathcal{V}(0)$ is fixed once the smallest initial wealth of the investor $\ell$ is determined, since $v^{-\ell}$ and $\pi^{-\ell}(\cdot)$ are fixed.

The following proposition gives the conditions of $c^\ell$ and $\delta$ to achieve relative arbitrage opportunities. We show that direct interaction with the market is important.

Proposition 3.2. We show the following properties of relative arbitrage opportunities with respect to $c^\ell$ and $\delta$.

1. If every investor achieves relative arbitrage opportunity in the sense of (3.3), then we must have

$$\frac{(1 - \delta)}{N} \sum_{\ell=1}^{N} e^{c^\ell} < 1. \quad \text{(3.7)}$$

2. There is no relative arbitrage opportunity when $\delta = 0$ and $c^\ell \geq 0$ for every $\ell$. 


3. When $0 < \delta \leq 1$, we have

$$V(0) = \frac{\delta x_0}{1 - \frac{1}{\delta} \sum_{k=1}^{N} u^k(T)}, \quad \text{thus} \quad v^\ell = \frac{u^\ell(T)\delta x_0}{1 - \frac{1}{\delta} \sum_{k=1}^{N} u^k(T)}. \quad (3.8)$$

4. If $c_\ell = c > 0$, for $\ell = 1, \ldots, N$, then

$$V(0) = \frac{\delta x_0}{1 - (1 - \delta)u(T)}. \quad (3.9)$$

Thus, in this paper, we mainly discuss relative arbitrages with $0 < \delta \leq 1$. We show the proofs and outline the finance interpretation in the following.

**Proof.** 1. Since everyone follows $V^\ell(T) \geq e^{\ell\delta}V(T)$, we sum up this expression for $\ell = 1, \ldots, N$ to get an inequality of $\sum_{\ell=1}^{N} V^\ell(t)/N$, and (3.7) follows immediately. We easily see that if all investors adopt the market portfolio $m(\cdot)$, then any $c_\ell \leq 0$, $\ell = 1, \ldots, N$ is a valid level of satisfaction. Here, (3.7) tells us that some of $c_\ell$’s can be small positive numbers.

2. Next, by Definition 3.1, if we have

$$c_\ell \leq \log \left( \frac{V^\ell(T)}{V^N(T)} \right) = \log \left( \frac{V^\ell(T)}{(\delta X^N(t) + (1 - \delta)V(T))} \right), \quad \ell = 1, \ldots, N,$$

then the relative arbitrage exists in the sense of (3.3).

From (3.6), we get $v^\ell = u^\ell(T)V(0)$, for $\ell = 1, \ldots, N$. When $\delta = 0$, $c_\ell \geq 0$ for every $\ell$, then (3.7) is violated, and there is no relative arbitrage opportunity. This means that in an extremely competitive group, where every investor solely wants to outperform one another in the market, the relative arbitrage is not possible.

3. Let $1$ be the $n \times 1$ column vector that has all $n$ elements equal to one. When $\delta > 0$, by Definition 3.3,

$$u^\ell(T, x, y)\mathcal{V}(0) = v^\ell_0, \quad (x, y) := (\mathcal{V}(0), \mathcal{Y}(0)), \quad (3.10)$$

where

$$\mathcal{V}(0) = \delta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \sum_{\ell=1}^{N} v^\ell_0\pi_\ell(0) = \delta \sum_{i=1}^{n} x_i + (1 - \delta) \sum_{i=1}^{n} y_i = \delta x \cdot 1 + (1 - \delta)y \cdot 1. \quad (3.11)$$

By (3.10), it follows that $y \cdot 1 = \frac{1}{N} \sum_{\ell=1}^{N} u^\ell(T, x, y)[\delta x \cdot 1 + (1 - \delta)y \cdot 1]$, so

$$y \cdot 1 = \frac{1}{N} \delta x \cdot 1 \sum_{\ell=1}^{N} u^\ell(T, x, y) \left[ 1 - (1 - \delta) \frac{1}{N} \sum_{\ell=1}^{N} u^\ell(T, x, y) \right]^{-1}. \quad (3.12)$$

Further computation based on (3.10)-(3.12) shows that (3.8) follows.

4. When $c_\ell = c$, every investor has the same optimal arbitrage objective $u(\cdot)$, and (3.9) follows from (3.8).

We define the deflator based on the market price of the risk process $L(t)$ as

$$dL(t) = -\theta(t)L(t)dW_t, \quad t \geq 0. \quad (3.13)$$

Equivalently,

$$L(t) := \exp \left\{ -\int_{0}^{t} \theta(s)dW(s) - \frac{1}{2} \int_{0}^{t} ||\theta(s)||^2 ds \right\}, \quad 0 \leq t < \infty.$$

The market is endowed with the existence of a local martingale $L$ with $E[L(T)] \leq 1$ under Assumption 1b.. The discounted processes $\hat{V}^\ell(\cdot) := V^\ell(\cdot)L(\cdot)$, and $\hat{X}(\cdot) := X(\cdot)L(\cdot)$. $\hat{V}^\ell(\cdot)$ admits

$$d\hat{V}^\ell(t) = dV^\ell(t)L(t) = \hat{V}^\ell(t)(\pi^\ell(t)\sigma(t) - \theta^\ell(t))dW(t); \quad \hat{V}^\ell(0) = \tilde{\nu}_\ell, \quad \ell = 1, \ldots, N, \quad (3.14)$$
d\(\hat{X}_i(t)\) = \(\hat{X}_i(t) \sum_{k=1}^{n} (\sigma_{ik}(t) - \theta_k(t))dW_k(t)\); \(\hat{X}_i(0) = x_i\), \(i = 1, \ldots, n\),

d\(\hat{X}(t) = \hat{X}(t) \sum_{k=1}^{n} \mathbf{m}_i(t)\sigma_{ik}(t) - \theta_k(t)dW_k(t)\); \(\hat{X}(0) = x\). \quad (3.15)

**Remark 1.** With Assumption 1b, assume that the market \(M\) has bounded variance (Appendix A). Denote \(v := \frac{1}{N} \sum_{\ell=1}^{N} v^\ell\). On \([0, T]\), given the existence of relative arbitrage in the sense of (3.3) and Definition 3.2, the process \(L(\cdot)\) is a strict local martingale, i.e., \(\mathbb{E}[L(T)] < 1\).

This can be proved by contradiction, assuming that \(L(T)\) is a martingale. We generalize Proposition 6.1 in [7] where the case \(N = 1\) is studied. As \(u^\ell(T) = \frac{v^\ell}{\tilde{v}(0)} = e^{c\ell} \) for some \(\ell\), it holds that

\[c_\ell \geq \log v_\ell - \log(\delta x_0 + (1 - \delta)\tilde{v}).\] \quad (3.16)

By Girsanov theorem, \(Q_{x,y}(A) := \mathbb{E}^x \mathbb{I}[L(T) = 1]A\), \(A \in \mathcal{F}_T\) defines a probability measure that is equivalent to \(\mathbb{P}\). We can show that \(\Delta^\ell(t) := V^\ell(t) - e^{c\ell}(\delta X(t) + (1 - \delta)\frac{1}{N} \sum_{\ell=1}^{N} V^\ell(t))\) is a martingale under \(Q_{x,y}\). Hence, the martingale property of \(\Delta^\ell(t)\) and (3.16) suggests that \(\mathbb{E}^x \mathbb{E}^{\tilde{V}}[\Delta^\ell(0)] = \mathbb{E}^x \mathbb{E}^{\tilde{V}}[\Delta^\ell(t)] = v_\ell - e^{c\ell}\delta x_0 - e^{c\ell}(1 - \delta)\tilde{v} \leq 0\), contradicting to Definition 3.1 of the relative arbitrage. Thus, the process \(L(\cdot)\) is a strict local martingale.

### 3.3 The modified subproblems

We define \(u^\ell : (0, \infty) \times (0, \infty)^n \rightarrow (0, \infty)\) from the processes \((X(\cdot), \mathcal{Y}(\cdot))\) starting at \((x, y) \in (0, \infty)^n \times (0, \infty)^n\), and write the terminal values

\[u^\ell(T) := u^\ell(T, x, y); \quad \ell = 1, \ldots, N.\]

The following proposition characterizes one’s best relative arbitrage opportunities by the customized benchmark \(e^{c\ell} \cdot \mathcal{Y}(T)\), for any \(\ell = 1, \ldots, N\). \(T\) is a fixed real number and \(N\) is a fixed natural number.

We shall assume that \(\mathbb{P} = \mathbb{P}^{x,y} = \mathbb{P}^{W}\), where \(\mathbb{P}^{x,y}\) is the filtration generated by the \(\sigma\)-fields \(\{\sigma(X(s), \mathcal{Y}(s); 0 < s < t), t \geq 0\}\). In general, the trading strategy does not necessarily have to be measurable with respect to \(\mathbb{P}^{x,y}\). An example of a single investor case is presented in [25, Example 6]. In Section 4, the trading strategies that are generated in \(\mathbb{P}^{x,y}\) is the optimal strategy which we will derive explicitly. Then \(\mathbb{F}^{W} = \mathbb{F}^{x,y}\) is a natural relation from the structure of the optimal strategy we consider, and is required for the derivation of the martingale representation results below.

**Proposition 3.3.** Under Assumption 1, for each fixed \(\ell = 1, \ldots, N\), assume \(u^\ell(t, x, y) \in C^{1,3,3}\), and \(\pi^\ell\) is the optimal strategy that achieves \(u^\ell(t, x, y)\) in Definition 3.3. \(^1\) With the given admissible portfolio \(\pi^{-\ell}(\cdot)\) defined in (3.5), and the initial values \((x, y)\), \(u^\ell(T)\) in (3.6) can be derived as \(e^{c\ell} \mathcal{Y}(T)\)’s discounted expected values over \(\mathbb{P}^{x,y}\),

\[u^\ell(T, x, y) = e^{c\ell} \mathbb{E}^{x,y}[\mathcal{Y}(T)|L(T)] / \mathcal{V}(0).\]

This result is essential for the PDE characterization of the objective \(u^\ell(T)\) in Section 3.4. \(\mathbb{E}^{x,y}[\cdot] := \mathbb{E}[\cdot|(X(0), \mathcal{Y}(0)) = (x, y)]\). The proof is based on the supermartingale property of \(\tilde{\mathcal{Y}}(\cdot)\) and the martingale representation theorem; see Appendix B for details of the proof.

Denote \(\tau := T - t\). By Assumption 1b and the Markovian market setup, we have the Markovian property of \(u^\ell(\cdot)\),

\[u^\ell(\tau, x, y) = e^{c\ell} \frac{\mathbb{E}[\mathcal{Y}(T)|L(T)]|\mathcal{F}_\tau]}{\mathcal{V}(t)L(t)}.\] \quad (3.17)

We can understand \(u^\ell(T - t)\) as the initial optimal arbitrage quantity to start at \(t \in [0, T]\) such that we match or exceed the benchmark portfolio at terminal time \(T\), that is,

\[u^\ell(T - t) = \inf \left\{ \omega^\ell \in (0, \infty) \mid \text{there exists } \bar{\pi}^\ell(\cdot) \in \mathcal{A} \text{ such that } u^\ell = \omega^\ell \bar{\mathcal{Y}}(t), \bar{\mathcal{Y}}^\ell(T) \geq e^{c\ell} \cdot \bar{\mathcal{Y}}(T) \right\}\] \quad (3.18)

for \(0 \leq t \leq T\) and \(\ell = 1, \ldots, N\).

\(^1\)The choice of \(u^\ell(x, y) \in C^{1,3,3}(0, \infty)^n \times (0, \infty)^n\) guarantees that the optimal strategy \(\pi^\ell\) is in the admissible set \(\mathcal{A}\), which is explained later in Appendix (C.2).
At every time \( t \), each investor optimizes \( \hat{\pi}^t(\cdot) \) from \( t \) to \( T \), in order to get the optimal quantity as defined in (3.18). In other words, here \( \hat{V}^v, \pi^t(T) \) is generated by the admissible portfolio \( \{\pi^t(s), s \geq t\} \) starting from time \( t \geq 0 \). That is, we consider \( \hat{\pi}^t(s) \), \( t \leq s \leq T \) so that the corresponding portfolio value \( \hat{V}^t(s) \), \( t \leq s \leq T \) satisfies
\[
\hat{V}^t(t) = 0
\]
and
\[
\hat{V}^t(t) = \hat{V}^t(s) \cdot \frac{dX_i(s)}{X_i(s)}; \quad t \leq s \leq T,
\]
\[
\hat{V}(s) = \hat{\theta} \cdot X(s) + (1 - \hat{\theta}) \cdot \frac{1}{N} \sum_{i=1}^{N} \hat{V}^i(s), \quad t \leq s \leq T.
\]

In the next section, we characterize \( u^t(\tau, x, y) \) as a solution of a Cauchy problem.

3.4 PDE characterization of the best relative arbitrage

We use the notation \( D_i \) and \( D_{ij} \) for the partial and second partial derivative with respect to the \( i \)-th or the \( i \)-th and \( j \)-th variables in \( X(\cdot) \), respectively; \( D_p \) and \( D_{pq} \) for the first and second partial derivative in \( Y(\cdot) \), \( D_{pi} \) is the cross derivative in \( X(\cdot) \) and \( Y(\cdot) \).

**Assumption 2.** There exists a function \( H : \mathbb{R}^n_+ \times \mathbb{R}_+^n \to \mathbb{R} \) of class \( C^2 \), such that
\[
b(x, y) = 2a(x, y)D_x H(x, y), \quad \gamma(x, y) = 2\psi(x, y)D_y H(x, y),
\]
i.e., \( b_i(\cdot) = \sum_{j=1}^{n} a_{ij}(\cdot)D_j H(\cdot), \quad \gamma_i(\cdot) = \sum_{q=1}^{n} \psi_{pq}(\cdot)D_q H(\cdot) \) in component wise for \( i, p = 1, \ldots, n \).

We provide here an example that satisfies the assumption above.

**Example 3.1.** In real financial markets, the leverage effect suggests that the smaller stocks tend to have greater volatility than the larger stocks. Similarly, lower trading volumes in a stock are often associated with higher volatility in trading. Hence, we consider
\[
b_i(x, y) = x_1 + \ldots + x_n, \quad a_{ij}(x, y) = x_i(x_1 + \ldots + x_n)\delta_{ij} y_i,
\]
\[
\gamma_i(x, y) = y_1 + \ldots + y_n, \quad \psi_{ij}(x, y) = y_i(y_1 + \ldots + y_n)\delta_{ij} x_i,
\]
where \( \delta_{ij} = 1 \), when \( i = j \); and \( \delta_{ij} = 0 \) otherwise, when \( i \neq j \). Then,
\[
H(x, y) = \sum_{i=1}^{n} (\log x_i + \log y_i).
\]

The generator for the process \( (X(\cdot), Y(\cdot)) \) can be written as
\[
\mathcal{L}f := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x, y)\left[\frac{1}{2} D_{ij} f + 2D_i f D_j H(x, y)\right] + \sum_{p=1}^{n} \sum_{q=1}^{n} \psi_{pq}(x, y)\left[\frac{1}{2} D_{pq} f + 2D_p f D_q H(x, y)\right]
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{n} (s\tau')_{ip}(x, y)D_{ip} f + \frac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{n} (s\tau')_{ip}(x, y)D_{ip} f,
\]
where \( (s\tau')_{ip}(x, y) = (s\tau')_{ip}(x, y) = \sum_{k=1}^{K} s_{ik}(x, y)\tau_{pk}(x, y) \). By the definition of \( \theta(\cdot) \) in (2.6) and Assumption 2,
\[
\theta(x, y) = s'(x, y)D_x H(x, y) + \tau'(x, y)D_y H(x, y).
\]

Then it follows from (3.19) and Itô’s lemma applying on \( H(\cdot) \) that
\[
L(t) = \exp \left\{ -\int_{0}^{t} \theta'(s)dw(s) - \frac{1}{2} \int_{0}^{t} ||\theta(s)||^2 ds \right\}
\]
\[
= \exp \left\{ -H(X(t), Y(t)) + H(x, y) - \int_{0}^{t} (k(X(s), Y(s)) + \tilde{k}(X(s), Y(s)))ds \right\},
\]

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where

$$k(x, y) := -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x, y) [D_{ij}^2 H(x, y) + 3D_i H(x, y) D_j H(x, y)],$$

$$\tilde{k}(x, y) := -\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \psi_{pq}(x, y) [D_{pq}^2 H(x, y) + 3D_p H(x, y) D_q H(x, y)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{n} (s_{i}^T)_{ip} D_i H(x, y) D_p H(x, y)$$

for $(x, y) \in (0, \infty)^n \times (0, \infty)^n$. Thus, (3.17) can be written as

$$u^\ell(\tau, x, y) = e^{\ell \tau} \frac{G(\tau, x, y)}{g(x, y)},$$

where, by (3.11),

$$g(x, y) := \mathcal{Y}(0) e^{-H(x, y)} = (\delta x \cdot 1 + (1 - \delta) y \cdot 1) e^{-H(x, y)},$$

$$G(T, x, y) := \mathbb{E}^{x,y}[g(\mathcal{X}(T), \mathcal{X}(T)) e^{-\int_0^T \kappa(\mathcal{X}(t)) + \tilde{k}(\mathcal{X}(t)) dt}].$$

**Assumption 3.** The function $G(\cdot) \in C^{1,3,3}((0, \infty) \times (0, \infty)^n \times (0, \infty)^n)$ yields the following dynamics by Feynman-Kac formula,

$$\frac{\partial G}{\partial \tau}(\tau, x, y) = \mathcal{L}G(\tau, x, y) - (k(x, y) + \tilde{k}(x, y)) G(\tau, x, y), \quad (\tau, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^n,$n

$$G(0, x, y) = g(x, y), \quad (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n.$$n

By [24], this assumption is satisfied if we in addition that $k(\cdot)$ is $\alpha$-Hölder continuous with some $\alpha \in (0, 1]$, uniformly in compact subsets of $\mathbb{R}_+^n \times \mathbb{R}_+^n$, $\ell = 1, \ldots, N$; $k(x, y)$ is bounded from below, $\gamma(\cdot)$ and $\tau(\cdot)$ are continuously differentiable on $(0, \infty)^n \times (0, \infty)^n$, and satisfy the growth condition

$$\|\gamma(x, y)\| + \|\tau(x, y)\| \leq M(1 + \|x\| + \|y\|).$$

Under Assumption 3, $u^\ell(\tau, x, y) \in C^{1,3,3}((0, \infty) \times (0, \infty)^n \times (0, \infty)^n)$ is bounded on $K \times (0, \infty)^n \times (0, \infty)^n$ for each compact $K \subset (0, \infty)$. For simplicity, we write $u^\ell(t)$ in place of $u^\ell(t, x, y)$. After the calculations (Appendix B) based on (3.17) and (3.21), $u^\ell(\cdot)$ follows a Cauchy problem

$$\frac{\partial u^\ell(\tau, x, y)}{\partial \tau} = A u^\ell(\tau, x, y), \quad \tau \in (0, \infty), \quad (x, y) \in (0, \infty)^n \times (0, \infty)^n,$n

$$u^\ell(0, x, y) = e^{\ell \tau}, \quad (x, y) \in (0, \infty)^n \times (0, \infty)^n,$n

where

$$A u^\ell(\tau, x, y) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x, y) \left( D_{ij}^2 u^\ell(\tau, x, y) + \frac{2 \delta D_i u^\ell(\tau, x, y)}{\delta x \cdot 1 + (1 - \delta) y \cdot 1} \right)$$

$$+ \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \psi_{pq}(x, y) \left( D_{pq}^2 u^\ell(\tau, x, y) + \frac{2 (1 - \delta) D_p u^\ell(\tau, x, y)}{\delta x \cdot 1 + (1 - \delta) y \cdot 1} \right)$$

$$+ \sum_{i=1}^{n} \sum_{p=1}^{n} (s_{i}^T)_{ip} D_i u^\ell(\tau, x, y)$$

$$+ \sum_{i=1}^{n} \sum_{p=1}^{n} (s_{i}^T)_{ip} D_p u^\ell(\tau, x, y).$$

We emphasize that (3.22) is entirely determined by the volatility structure of $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$. As a result, when the drift $\gamma(\cdot)$ and volatility term $\tau(\cdot)$ in (2.3) is given, (3.22) - (3.24) are satisfied.

**Theorem 3.1.** Under Assumption 1-3, the function $u^\ell : [0, \infty) \times (0, \infty)^n \times (0, \infty)^n \rightarrow (0, 1]$ is the smallest nonnegative continuous function, of class $C^2$ on $(0, \infty) \times (0, \infty)^n$, that satisfies $u^\ell(0, \cdot) \equiv e^{\ell \tau}$ and

$$\frac{\partial u^\ell(\tau, x, y)}{\partial \tau} \geq A u^\ell(\tau, x, y),$$

where $A(\cdot)$ follows (3.24).

Proof of this theorem can be found in Appendix B.
3.5 Existence of Relative Arbitrage

The Cauchy problem (3.22)-(3.23) admits a trivial solution \( u^\ell(\tau, x, y) \equiv e^{ct} \). In this section we discuss the conditions for the existence of relative arbitrage – we need \( u^\ell(\tau, x, y) \) to take values less than \( e^{ct} \), indicating that the uniqueness of the Cauchy problem fails.

For an admissible market model \( \mathcal{M} \), we introduce the normalized reciprocal of the deflated total market capitalization (3.4) and solve it using \( L(t) \) in (3.13). Starting with \( V(0) \), it holds
\[
d(V(t)L(t)) = V(t)L(t)(\Pi'(t)\sigma(t) - \dot{\theta}(t))dW(t),
\]
where \( \Pi(\cdot) \) is the corresponding portfolio of wealth \( V(t) \). We define \( \dot{\theta}(\cdot) := \theta(\cdot) - \sigma'(\cdot)\Pi(\cdot) \), and \( \tilde{W}(\cdot) := W(\cdot) + \int_0^t \dot{\theta}(t)dt \), then
\[
\Lambda(t) := V(0)/L(t)V(t) = \exp\left\{ \int_0^t \dot{\theta}(s)d\tilde{W}(s) - \frac{1}{2} \int_0^t ||\dot{\theta}(s)||^2 ds \right\}, \quad t \in [0, T].
\]

We also introduce the stopping time of \( \Lambda(t) \) touching zero
\[
\mathcal{T} := \inf\{t \geq 0|\Lambda(t) = 0\} = \inf\{t \geq 0|L(t)V(t) = \infty\}.
\]

If \( L(t)V(t) \) is not a true martingale, then \( L(T)V(T) \) is not a Radon-Nikodym derivative, and the process \( \tilde{W}(\cdot) \) is not necessarily a Brownian motion under an equivalent local martingale measure. Following the route suggested by [5] and [25], there exists a probability measure \( Q \) on \( (\Omega, \mathcal{F}) \), such that \( \mathbb{P} \) is locally absolutely continuous with respect to \( Q \): \( \mathbb{P} \ll Q \), \( \Lambda(T) \) is a \( Q \)-martingale, and \( d\mathbb{P} = \Lambda(T)dQ \) holds on each \( \mathcal{F}_T, T \in (0, \infty) \). Through the Föllmer exit measure [12] we can characterize the solution of the Cauchy problem \( u^\ell(t) \) to the maximal \( Q \)-probability of the supermartingale \( L(T)V(T) \) staying in \((0, \infty)^n \) at any time \( t \in [0, T] \).

\[
u^\ell(T) = e^{ct}\mathbb{E}^Q[L(T)V(T)]/V(0) = e^{ct}\mathbb{E}^Q\left[\frac{1}{\Lambda(T)}1_{(T \geq T)}\right] = e^{ct}Q(T > t), \quad \forall T \in [0, \infty).
\]

**Definition 3.4** (Auxiliary process and the Fichera drift). We define the following

1. The auxiliary process \( \zeta = (\zeta_1, \ldots, \zeta_{2n}) \) is defined as
\[
d\zeta_i(\cdot) = \hat{b}_i(\zeta(\cdot))dt + \sum_{k=1}^n \hat{\sigma}_{ik}(\zeta(\cdot))dW_k, \quad \zeta(0) = \zeta_0, \quad i = 1, \ldots, 2n,
\]

where
\[
\hat{b}_i(x, y) = \begin{cases} \sum_{j=1}^n \frac{1}{(1-\delta)(1-\delta')} \delta y \psi_{ij}(x, y) \delta' y' \psi_{ij'}(x, y') & \text{if } i = 1, \ldots, n, \\ \sum_{j=1}^n \frac{1}{(1-\delta)(1-\delta')} \delta y \psi_{ij}(x, y) \delta' y' \psi_{ij'}(x, y') & \text{if } i = n+1, \ldots, 2n,
\end{cases} \]
\[
\hat{\sigma}_{ik}(x, y) = \begin{cases} s_{ik}(x, y) & \text{if } i, k = 1, \ldots, n, \\ r_{ik}(x, y) & \text{if } i = n+1, \ldots, 2n, \quad k = 1, \ldots, n,
\end{cases}
\]
\[
\text{for } i = 1, \ldots, 2n, \quad (x, y) \in (0, \infty)^n \times (0, \infty)^n.
\]

**Remark 2.** The auxiliary market portfolio \( \zeta \) takes into account the interactions of investors. The induced measure \( Q \) corresponds to a drift change, in which the auxiliary market portfolio cannot be outperformed as it satisfies numéraire property in this auxiliary system.

The maximal probability term is also related to the containment probability in the stochastic control literature. In [29], an optimization problem is introduced to maximize the probability that the state trajectories remain in a bounded region over a given finite time horizon. This idea and the techniques of stochastic control have been used in [6] to consider the uncertainty of the optimal arbitrage problem (for one investor).
Define $\mathcal{O}^{2n}$ as the boundary of $[0, \infty)^{2n}$. We construct the auxiliary process to be a $([0, \infty)^{2n}\setminus\{0\})$-valued process. In the following, we show the condition for this process not to hit the limit of $[0, \infty)^{2n}$ during $[0, T]$.

**Assumption 4.** The system of $\zeta(\cdot)$ admits a weak, unique in distribution solution with values in $[0, \infty)^n \times [0, \infty)^n/\{0\}$, with given $(x,y)$.

The sufficient conditions for the above assumption are that $\hat{h}(\cdot)$ is locally Lipschitz and Assumption 1, where the Lipschitz continuity can be relaxed to the local Lipschitz continuity (C.1), and we can show that the covariance term in $\zeta(\cdot)$ also satisfies the growth condition.

**Proposition 3.4.** Under Assumption 1-4, suppose that the functions $\hat{\sigma}_{ik}(\cdot)$ are continuously differentiable on $(0, \infty)^{2n}$; that the matrix $\hat{\sigma}(\cdot)$ degenerates on $\mathcal{O}^{2n}$; and that the Fichera drifts for the process $\zeta(\cdot)$ can be extended by continuity on $[0, \infty)^{2n}$. For an investor $\ell$, if $f_i(\cdot) \geq 0$ holds on each face of the orthant, then $u^i(\cdot, \cdot) \equiv e^{c_i}$, and there is no arbitrage with respect to the market portfolio on any time horizon. If $f_i(\cdot) < 0$ on each face $\{x_i = 0\}$, $i = 1, \ldots, n$ and $\{|y_i = 0\}$, $i = n+1, \ldots, 2n$ of the orthant, then $u^i(\cdot, \cdot) < e^{c_i}$ and arbitrage with respect to the market portfolio exists, on every time horizon $[0, T]$ with $T \in (0, \infty)$.

**Proof.** Fix $\zeta_0$ in Definition 3.4 as $z := (x, y)$. As $\zeta(\cdot)$ is Markovian, we use a similar approach as in Theorem 2 of [5] to connect $u^i(T, z)$ to the probability of the first hitting time of an auxiliary process to touch $\mathcal{O}^{2n}$. We denote $\mathcal{T} := \inf\{t \geq 0|\zeta(t) \in \mathcal{O}^{2n}\}$ as the first hitting time of the auxiliary process $\zeta(\cdot)$ to $\mathcal{O}^{2n}$. By using the nondegeneracy condition of $a_{ij}$, and (3.26), we get that starting from $z$,

$$u^i(T, z) = e^{c_i}Q_z[T > T], \quad (T, z) \in [0, \infty) \times [0, \infty)^{2n}.$$  

For the first claim, if $f_i(x, y) \geq 0$ (in (3.28)), we only need to show the probability $Q_z[T > T] = 1$, for $(T, x, y) \in [0, \infty) \times [0, \infty)^n \times [0, \infty)^n$. Denote a bounded and connected $C^3$ boundary $\tilde{G} \subset \bigcup_{i=1}^{2n-1} \{z \in \mathbb{R}^2; z_i < 0, \|z\| < R\} =: G_R$, and $R > \|\zeta_0\|$. Then from Theorem 9.4.1 (or Corollary 9.4.2) of [13], under Assumption 1, since

$$\sum_{i=1}^{n} \left( \hat{b}_i(z) - \frac{1}{2} \sum_{j=1}^{n} D_j \hat{a}_{ij}(z) \right) n_i \leq 0,$$

in which $n = (n_1, \ldots, n_{2n})$ is the outward normal vector at $(x, y)$ to $\mathcal{O}^{2n}$, the boundary $\mathcal{O}^{2n} \cap \tilde{G}$ is an obstacle outside of $G_R$, i.e., $\mathcal{G} := B_R(0)/G_R$, where $B_R(0)$ is the open ball centered at the origin with radius $R$. The Fichera vector field points toward the domain interior at the boundary. Let $R \to \infty$, the limit $\mathcal{O}^{2n}$ is not attainable by $\zeta(z)$ almost surely for $z \in [0, \infty)^{2n}$.

If $f_i(\cdot) < 0$ on each face $\{z_i = 0\}$, $i = 1, \ldots, 2n$, then

$$\sum_{i=1}^{n} \left( \hat{b}_i(z) - \frac{1}{2} \sum_{j=1}^{n} D_j \hat{a}_{ij}(z) \right) n_i \geq 0,$$

and the Fichera drift at $\mathcal{O}^{2n}$ points toward the exterior of $[0, \infty)^{2n}$. It is equivalent to showing that $Q_z[T > T] < 1$, for $(T, x, y) \in [0, \infty) \times [0, \infty)^n \times [0, \infty)^n$, we only need to show $Q_z[T < T] > 0$, i.e., the boundary $\{z_i = 0\}$, $i = 1, \ldots, 2n$, is attainable by $\zeta(\cdot)$.

As discussed in Chapter 11 Problems 7-8, Chapter 13 in [13] and Theorem 2 in [23], every point $z_0 \in \partial \mathcal{G}$ is a regular point, which means for every fixed $\delta$, every $z_0 \in \partial \mathcal{G}$,

$$\lim_{\mathcal{G} \to z_0, \mathcal{G} \in \partial \mathcal{G}} Q_{\mathcal{G}}(\tau^\mathcal{G} < \infty, \|\zeta(\tau^\mathcal{G}) - z_0\| < \delta) = 1,$$

(3.29)

where $\tau^\mathcal{G}$ is the exit time from $\tilde{G} := B_R(0)/G_R$. Define $\Sigma := \bigcup_{i=1}^{2n} \{z \in \mathbb{R}^2n : z_i = 0\} \bigcap \mathcal{G}$. Thus, the Fichera drift is negative and the diffusion coefficient degenerates in $\Sigma$. The other part of the boundary is $\Sigma_2 := \Sigma^c \cap \partial \mathcal{G}$, where the diffusion coefficient is non-degenerate.

Consider $z_0 \in \Sigma$, there exists $\delta$ such that $B_\mathcal{G}(z_0) := \bigcap_{i=1}^{2n} \{z \in \mathbb{R}^2n : z_i > 0\} \bigcap B_\delta(z_0) \subset \mathcal{G}$. Then by (3.29), there exists $\eta > 0$ such that whenever $\|x - z_0\| \leq \eta$, $x \in \mathcal{G}$, we have

$$Q_{\mathcal{G}}(\tau^\mathcal{G} < \infty, \|\zeta(\tau^\mathcal{G}) - z_0\| < \delta) > \frac{1}{2}.$$
• For \( ||z - z_0|| \leq \eta \), the probability of reaching the boundary \( \Sigma \) is positive, i.e.,
\[
Q_z(\tau^g < \infty, \zeta(\tau^g) \in \Sigma) > 0.
\]

• For \( ||z - z_0|| > \eta \),
\[
\inf_{\xi \in A} Q_z(\tau^g < \infty, \zeta(\tau^g) \in B_{\delta}^+(z_0)) > \frac{1}{2},
\]
where
\[
A := \bigcup_{i=1}^{2n} \{z \in \mathbb{R}^{2n} : z_i > 0, ||z - z_0|| = \eta \}.
\]

We have \( \{z\} \cap A = \emptyset \), and we choose a subset \( \mathcal{H} \subset \mathcal{G} \) that contains \( \{z\} \cup A \). Define \( \tau_* := \inf\{t > 0 : \zeta(t) \notin \mathcal{H} \setminus A\} \), that is, \( \tau_* \) is the exit time of \( \zeta(\cdot) \) from the bounded domain \( \mathcal{H} \setminus A \). \( \tau_* < \infty \) almost surely since the diffusion coefficients are non-degenerate in the interior of \( \mathcal{G} \), and the drift coefficients are finite. Take a continuous sample path \( \omega_* \) such that \( \omega_*(0) = z \), \( \omega_*(\tau_*) = r \in A \), and \( \omega_*(s) \notin A \) for \( 0 \leq s < \tau_* \). Consider an \( \epsilon \)-neighborhood \( N_{\epsilon, \omega_*} \) of \( \omega_* \in \mathcal{C}(\mathcal{G}) \),
\[
N_{\epsilon, \omega_*} = \{\omega \in \mathcal{C}(\mathcal{G}) : \omega(0) = z, ||\omega - \omega_*|| < \epsilon, \omega(\tau_*) = r \} \subset \{\omega \in \Omega : \zeta(\tau_*(\omega), \omega) \in A\},
\]
where \( ||\cdot|| \) is the supremum norm \( ||\omega_1 - \omega_2|| = \sup_{0 \leq s \leq \tau_*} |\omega_1(s) - \omega_2(s)| \), \( \omega_1, \omega_2 \in \mathcal{C}(\mathcal{G}) \). By the support theorem [28, Lemma 3.1, Exercise 6.7.5],
\[
Q_z(N_{\epsilon, \omega_*}) > 0, \quad \text{and hence } Q_z(\zeta(\tau_*) \in A, \tau_* < \infty) > 0.
\]

Therefore, we have
\[
Q_z(\zeta(\tau^g) \in \Sigma, \tau^g < \infty) \geq Q_z(\zeta(\tau^g) \in B_{\delta}^+(z_0), \tau^g < \infty) \\
\geq \mathbb{E}_z[Q_z(\zeta(\tau^g) \in B_{\delta}^+(z_0), \tau^g < \infty, \zeta(\tau_*) \in A, \tau_* < \infty | \mathcal{F}_{\tau_*})] \\
= \mathbb{E}_z[Q_{\zeta(\tau_*)}(\zeta(\tau^g) \in B_{\delta}^+(z_0), \tau^g < \infty) \cdot 1(\zeta(\tau_*) \in A, \tau_* < \infty)] \\
\geq \mathbb{E}_z[\inf_{\xi \in A} Q_z(\zeta(\tau^g) \in B_{\delta}^+(z_0), \tau^g < \infty) \cdot 1(\zeta(\tau_*) \in A, \tau_* < \infty)] \\
\geq \frac{1}{2} Q_z(\zeta(\tau_*) \in A, \tau_* < \infty) > 0.
\]

The equality in the above expressions is from the strong Markov property of \( \zeta(\cdot) \).

Since \( R \) is arbitrary, the process \( \zeta(\cdot) \) reaches the set \( \bigcup_{i=1}^{2n} \{z_i = 0\} \) with positive probability. In conclusion, \( u^\ell(\cdot, \cdot) < e^{e^x} \) when Fichera drift \( f_i(\cdot) < 0 \) on each face of \( O^{2n} \).

Therefore, the investor \( \ell \) can find relative arbitrage opportunities with a unique \( u^\ell(\cdot) \), the minimal solution of (3.25) given \( f_i(\cdot) < 0 \) on each face of \( O^{2n} \). In the next section, we derive the corresponding optimal strategy process \( \pi^\ell \), for each \( \ell = 1, \ldots, N \).

4 The N player game

The stock capitalizations and investors’ wealth are coupled as the strategies adopted by the group of competitive investors contribute to the change of the trading volume of each stock, and thus to the change of stock capitalizations. This situation presents fixed-point problems on the path space of the strategies and motivates the formulation of a game. We will discuss the formulation in this section. The formulation of the game and the concept of Nash equilibrium provide us with a comprehensive understanding of the actions of investors and the dynamics of the market.

In the previous sections, we focus on the relative arbitrage opportunity of an investor. Now, we seek to answer the second question in the introduction: If there exists an optimal strategy as formulated in the last section, is it possible for all \( N \) investors to follow it?

We model the investors as participants in a \( N \)-player game and search for optimal strategies in the \( N \)-player game. We are interested in the existence and uniqueness of Nash equilibrium, since this informs us of the
incentives that investors face to change their strategies, as well as the challenges as a rational investor to predict the market. If Nash equilibrium does not exist, investors cannot find such stable relative arbitrage opportunities in a fixed amount of time. If there is more than one Nash equilibrium, it is more difficult for decision-making and predictions in the market. We may need to resort to other schemes, such as collaborations, to achieve an optimization. This will be discussed in Section 5.

The following sections are discussed under Assumptions 1-4.

4.1 Construction of Nash equilibrium

The solution concept of this $N$ player game is the Nash equilibrium. In this spirit, assuming that the others have already chosen their own strategies, a typical player seeks the best response to all the other players, which amounts to the solution of an optimal control problem to minimize the expected cost $u^\ell$. Specifically, when investor $\ell$ assumes the wealth of other players are fixed, they wish to take the solution of (3.22) and (3.23) as their wealth to begin with so that

$$V^\ell(T) = e^{c_x} V(T) = \delta \cdot e^{c_x} X(T) + (1 - \delta) \cdot e^{c_x} \frac{1}{N} \sum_{\ell=1}^{N} V^\ell(T).$$

Thus, we define the cost functional

$$J^\ell(\pi) := \inf \left\{ \omega^\ell > 0 \mid V^\omega^\ell V(0), \pi^\ell (T) \geq e^{c_x} V(T) \right\},$$

for all admissible strategy profiles $\pi(\cdot) = (\pi^1(\cdot), \ldots, \pi^N(\cdot)) \in \mathcal{A}^{(N)}$. The influence of $\pi^k$, $k \neq \ell$, is implicitly defined in the wealth $V(\cdot)$ and benchmark $\mathcal{V}(\cdot)$.

We give the definition of Nash equilibrium over the entire time horizon $[0, T]$.

Definition 4.1 (Nash Equilibrium). The strategy profile $\pi^\ast(\cdot) = (\pi_1^\ast(\cdot), \ldots, \pi^N(\cdot))$, where $\pi^{\ast\ell} = (\pi_1^{\ast\ell}, \ldots, \pi^N_{\ell^*}) \in \mathcal{A}$, $\ell = 1, \ldots, N$ are admissible strategies in Definition 2.1, is a Nash equilibrium over $[0, T]$, if, for every $\pi^\ell \in \mathcal{A}$,

$$J^\ell(\pi^{\ast\ell}, \pi^{-\ell\ast}) \leq J^\ell(\pi^{\ell\ast}, \pi^{-\ell\ast}), \quad \ell = 1, \ldots, N,$$

(4.2)

and $\pi^{-\ell}$ is the subset of the strategic profile without investor $\ell$, $\pi^{-\ell}(\cdot) = (\pi^1(\cdot), \ldots, \pi^{\ell-1}(\cdot), \pi^{\ell+1}(\cdot), \ldots, \pi^N(\cdot))$. For $t \geq 0$, we define the empirical measures of the corresponding wealth $(V^{\ell\ast}(t))_{\ell=1,\ldots,N} \in \mathbb{R}_+$ of investor $\ell$ with the Nash equilibrium strategy $\pi^{\ell\ast}$, given the initial measure $\mu_0 \in \mathcal{P}_2(\mathbb{R}_+)$ by

$$\mu^\ast_t := \frac{1}{N} \sum_{\ell=1}^{N} \omega_{V^{\ell\ast}(t)},$$

where $\delta_x$ is the Dirac delta mass at $x \in \mathbb{R}_+$. We write the Nash equilibrium as $(\pi^\ast, \mu^\ast)$, where $\pi^\ast \in C([0, T], \mathcal{A}^N)$, $\mu^\ast \in \mathcal{P}_2(C([0, T], \mathbb{R}_+))$.

Each individual aims to minimize the relative amount of initial capital so that one can match or exceed the benchmark at terminal time. From the previous section it follows that $\inf_{\pi^\ell \in A} J^\ell(\pi) = u^{\ast}(T)$, where $u^{\ast}$ is the minimum nonnegative solution of (3.25).

Investors focus more on changes in the wealth processes of other investors than changes in their strategies, because two distinct strategy processes can lead to identical wealth at the same time $T$. Therefore, the uniqueness-in-distribution of wealth is more important. We give the following notion of the uniqueness of the Nash equilibrium in a $N$-player game in terms of wealth processes.

Definition 4.2 (Uniqueness). Consider two sets of Nash equilibrium strategies $\pi_a$ and $\pi_b$ such that (4.2) holds. The corresponding empirical measure flows $\mu_a := (\mu^a_t)_{t \in [0,T]}$, $\mu_b := (\mu^b_t)_{t \in [0,T]}$ are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the same initial law $\mu_0 \in \mathcal{P}_2(\mathbb{R}_+)$. We say that the Nash equilibrium is unique if the measure flows $\mu_a$ and $\mu_b$ are indistinguishable, that is,

$$\mathbb{P}[\mu_a = \mu_b] = 1.$$
In general, the market $\mathcal{X} \subset C([0, T], \mathbb{R}^n_+)$ and investors’ wealth $\mathbf{V} \subset C([0, T], \mathbb{R}^N_+)$ interact through two mean-field interaction terms, the joint empirical measure of wealth and strategies $\nu$ in Definition D.1 and the empirical measure of wealth. In the special case (4.1), the latter becomes the empirical mean of wealth. Denote the interactions $\mu$ and $\nu$ in the Nash equilibrium of games of $N$ players by $\mu^\ast$ and $\nu^\ast$. Generally, the uniqueness of $\mu^\ast$ is a less restricted condition than the uniqueness of $\nu^\ast$. If there is a unique optimal $\nu^\ast$ in the sense of Definition 4.1, then it implies that its marginal distribution $\mu^\ast$ is the unique Nash equilibrium defined by Definition 4.2. However, the converse is not true: A unique $\mu^\ast$ does not necessarily give a unique optimal $\nu^\ast$.

For each $\ell$, solving the corresponding $\{\pi^\ell(t)\}_{t=1,\ldots,n}$ of the unique $V^\ell(t)$ relies on the Malliavin calculus and the solution can be written by different stochastic processes. Thus, there could be multiple possible quantities of the optimal measure $\nu^\ast$, and multiple solutions of the strategy profile $\pi^\ast(\cdot)$ that generated the unique $V^\ast$ or $\mu^\ast$ for $\ell = 1, \ldots, N$. See the general setup of the joint empirical measure $\nu$ and the general particle system in Appendix D.

### 4.2 Optimal arbitrage opportunities and the corresponding strategies in $N$-player game

All $N$ players have the same goal of competing with the market and other participants to pursue relative arbitrage opportunities. Essentially, the players pursue the optimal strategy in the Nash equilibrium in order to reach the optimal initial amount of investment for each player. If the Nash equilibrium solution satisfies the condition about the Fichera drift in Proposition 3.4, then there is a relative arbitrage opportunity for each investor.

#### 4.2.1 Searching Nash equilibrium

We next derive Nash equilibrium solution and the corresponding optimal strategies following the above searching procedure. Here we assume the Markovian structure of the admissible strategies, that is, the strategy that an investor adopts at time $t$ is in the form of $\pi^\ell(t) = \phi^\ell(X, Y)|_{\langle X, Y \rangle = \langle X^\ell(t), Y^\ell(t) \rangle}$.

We specify the methodology to find the optimal path in the sense of the equilibrium of $N$ players in the following. Notice that we consider the $N$ player game in a dynamic programming fashion, where we solve the subproblems specified in Section 3.3 for every $t \in [0, T]$. We solve the corresponding strategies and ultimately determine the optimal initial wealth to achieve relative arbitrage as defined in Definition 3.3.

Solving the Nash equilibrium takes the following steps:

1. Suppose we start with a given set of control processes $\pi := (\pi_1, \ldots, \pi_N)$, where $\pi^\ell(\cdot)$ is of the form $\phi^\ell(X^\ell(\cdot), Y^\ell(\cdot))$, $\ell = 1, \ldots, N$.
   
   (a) Solve the $N$-particle system (2.1)-(2.3), whose coefficients are determined by Itô’s formula of the function $\phi^\ell(X^\ell(\cdot), Y^\ell(\cdot))$, $\ell = 1, \ldots, N$. The detail is included in (C.3).
   
   (b) Solve $u^\ell(T-t) := \inf_{\pi \in \pi(N)} J^\ell(\pi)$ through the nonnegative minimal solution of the linear PDE similar to (3.22), with $u^\ell(0, x, y) = e^{c \ell}$, $\ell = 1, \ldots, N$.

2. With the solution $\{u^\ell(T-t)\}_{t \in [0, T], \ell = 1, \ldots, N}$, determine the corresponding optimal control $\hat{\pi}$. Thus, we can find a map $\Phi$ such that $\hat{\pi} = \Phi(\pi)$. We specify the existence of such $\Phi(\cdot)$ in Theorem 4.1. Note that the fixed point mapping $\Phi(\cdot)$ is generally not unique, as explained in Remark 3.

3. If there exists $\hat{\pi}$ such that $\pi^\ast = \Phi(\pi^\ast)$, then $(\pi^\ast, \mu^\ast)$ is the Nash equilibrium, where $\mu^\ast := \frac{1}{N} \sum_{\ell=1}^N \delta_{\langle \pi^\ast, \mu^\ast \rangle}$, where $\pi^\ast$ is specified through (3.8).

#### 4.2.2 Fixed point problem

We derive the following expression that explains an explicit relationship of the optimal strategy with market capitalization, trading volume, and the portfolio of the benchmark strategy.

**Theorem 4.1** (Fixed point problem). Consider $\delta \in (0, 1)$. Assume $u(\cdot) \in C^{1,3,3}([0, T] \times \mathbb{R}^n_+ \times \mathbb{R}^N_+)$. Under Assumption 1-4, the Nash equilibrium $(\pi^\ast, \mu^\ast) \in C([0, T], \mathbb{R}^n \times \mathbb{R}^N_+) \times \mathcal{P}(C([0, T], \mathbb{R}^N_+))$ exists, it is given by the
optimal Markovian strategy and the corresponding empirical measure through the fixed point problem \( \pi^* = \Phi(\pi^*) \).

Specifically, for every \( \ell = 1, \ldots, N \), the strategies \( \pi^{-\ell} \) are fixed, so at each time \( t \),

\[
\Phi(\pi^\ell)(t) = \mathcal{X}(t)D_{x_\ell} \log u^\ell,\pi^\ell(T - t, x, y) + \tau(t, x, y)\sigma^{-1}(x, y)D_y \log u^\ell,\pi^\ell(T - t, x, y) + \delta X(t)\frac{\mathcal{V}(t)}{\mathcal{V}(t)} - \left( \frac{1 - \delta}{N\mathcal{V}(t)} \sum_{k \neq \ell} V^{k,\pi}(t)\pi^k(t) + \frac{1 - \delta}{N\mathcal{V}(t)} V^{\ell,\pi}(t)\pi^\ell(t) \right).
\]

(4.3)

In particular,

\[
\pi^\ell_0(t) = X_i(t)D_{x_i} \log u^\ell,\pi^\ell(T - t, x, y) + \tau_i(t, x, y)\sigma^{-1}(x, y)D_y \log u^\ell,\pi^\ell(T - t, x, y) \bigg|_{(x, y) = (\mathcal{X}(t), \mathcal{Y}(t))} + \Pi^\ell_0(t).
\]

(4.4)

At terminal time \( t = T \), \( \pi^\ell(T) = \Pi^\ell(T) \). \( \{u^\ell(T, x, y)\}_{\ell = 1, \ldots, N} \) reads as the value function of the \( N \)-player game under the equilibrium.

We use the notation \( u^\ell,\pi^\ell(\cdot) \) to emphasize that the coefficients of the Cauchy problem depend on \( \pi \). In particular, \( \tau(\cdot) \) can be determined by Itô’s formula of the function \( \phi(t, x, y) \). Similarly, \( V^{k,\pi}(t) \) is generated from \( \pi^k(t) \), \( \Pi^\ell(t) \) is the benchmark portfolio that replicates \( \mathcal{V}(t) \) in Proposition 3.1 at equilibrium.

**Corollary 4.1.** Consider that we search for the best strategies only to outperform the market \((\delta = 1)\). Under Assumption 1-4, the optimal Markovian strategy at Nash equilibrium is given by the solution of the following fixed point problem

\[
\pi^\ell_0(t) = X_i(t)D_{x_i} \log u^\ell,\pi^\ell(T - t, x, y) + \tau_i(t, x, y)\sigma^{-1}(x, y)D_y \log u^\ell,\pi^\ell(T - t, x, y) \bigg|_{(x, y) = (\mathcal{X}(t), \mathcal{Y}(t))} + \Pi^\ell_0(t),
\]

for \( t \in [0, T] \). In particular, \( \pi^\ell_0(T) = \Pi^\ell_0(T) \).

When the benchmark for relative arbitrage concerns only the peers \((\delta = 0)\), the relative arbitrage opportunity does not exist.

**Proof of Theorem 4.1.** 1. The first step is to find the best response map \( \Phi(\cdot) \). Consider a given set of admissible strategies \( \pi := (\pi^1, \ldots, \pi^N) \), where \( \pi^k = \{\pi^k(t)\}_{0 \leq t \leq T} \), \( k = 1, \ldots, N \). Solve the \( N \)-particle system \((2.1) - (2.3)\). The solution \((\mathcal{X}, \mathcal{Y})\) gives \( u^\ell(T) = \inf_{\pi^k} J^\ell(\pi) \) that is uniquely determined by the smallest nonnegative solution of \((3.25)\). We get the corresponding optimal control \( \pi^* \). We find a map \( \Phi \) such that \( \pi^* = \Phi(\pi) \) through this sequence of individual optimization problems.

In particular, a player \( \ell \) reacts to changes in the market (generally speaking, there are interactions related to the empirical distribution \( \mu \) and \( \nu \)) and adopts the best response \( \Phi(\pi) \) that achieves

\[
V^{\ell,\pi}(\cdot) = \mathcal{V}(\cdot)u^\ell(\cdot).
\]

(4.5)

The Markovian property \((3.17)\) implies the deflated wealth process

\[
\hat{V}^{\ell,\pi}(t) := V^{\ell,\pi}(t)L(t) = \mathbb{E}\left[V(T)L(T) \mid \mathcal{F}_t \right]
\]

(4.6)

is a martingale. As a result, from \((4.6)\), the \( dt \) terms in \( d\hat{V}^{\ell,\pi}(t) = d(V^{\ell}(t)L(t)u^\ell(T - t)) \) will vanish, namely,

\[
\hat{V}^{\ell,\pi}(t) = \hat{V}^{\ell,\pi}(0) + \sum_{k=1}^N \int_0^t \hat{V}^{\ell,\pi}(s)B_k(T - s, \mathcal{X}(s), \mathcal{Y}(s))dW_k(s), \ 0 \leq t \leq T,
\]

(4.7)

where for \( \rho = T - t, t \in [0, T] \),

\[
B_k(\rho, x, y) = \sum_{i=1}^n \sigma_{ik}(x, y)x_iD_{x_i} \log u^\ell(\rho, x, y) + \sum_{m=1}^n \tau_{mk}(x, y)D_m \log u^\ell(\rho, x, y) + \sum_{i=1}^n \frac{\delta X(t)}{\mathcal{V}(t)} \left( \frac{x_i}{\sum_{i=1}^n x_i} \sigma_{ik}(t) \right) - \theta_k(x, y) \right) \bigg|_{(x, y) = (\mathcal{X}(t), \mathcal{Y}(t))}.
\]

Hence, the best response gives the strategy that replicates \((4.7)\). That is, assume that all controls \( \pi^k(\cdot), \ k \neq \ell \) are chosen, and player \( \ell \) will choose the optimal strategy by comparing the general formula \( V^{\ell} \) in \((3.14)\) and \( V^{\ell,\pi} \) in \((4.7)\). Thus, we derive that \((4.3)\) holds.
2. The second step is to find the fixed-point solution of the best response map $\Phi(\cdot)$. With a fixed set of control processes $\{\pi^\ell(t)\}_{0 \leq t \leq T}$, we solve $u^\ell_{t-1}$ for $t \in [0, T]$, and expect that the optimal strategy $\pi^\ell$ will coincide with the fixed $\pi^\ell(\cdot)$ for each $\ell$. Every player $k = 1, \ldots, N$ acts homogeneously, so that their wealth follows $V^\ell(\cdot) = V(\cdot)u^\ell(T - \cdot)$.

With an optimal strategy $\pi^{\ell*}(\cdot)$, player $\ell$ replicates their wealth $V^\ell(\cdot)$ by $e^{\cdot\cdot\cdot V^N(\cdot)}$. Hence, for a candidate portfolio vector $(\pi^1(\cdot), \ldots, \pi^N(\cdot))$ to be a Nash equilibrium, we need the $\pi^\ell(\cdot)$ to be identical to $\pi^{\ell*}(\cdot)$, for $\ell = 1, \ldots, N$. That is, we solve for the fixed point problem for $\pi^{\ell*}$

$$
\pi^{\ell*}(t) = X_i(t)D_i \log u^\ell(T - t, x, y) + \tau_i(x, y)\sigma^{-1}(x, y)D_k \log u^\ell(T - t, x, y)
$$

$$
+ \frac{\delta X(t)}{V(t)}m_i(t) + \frac{(1 - \delta)}{N} \sum_{\ell=1}^N V^{\ell*}(t)\pi^{\ell*}(t), \quad \text{(4.8)}
$$

where $V^\ell(t)$ is generated from $\pi^{\ell*}(t)$.

From (4.8), we derive (4.4) which explains the explicit relationship that the optimal strategy consists of market, trading volume and portfolio of the benchmark strategy. As shown in (C.2), $\phi(t, x, y) \in C^{1, 2}([0, T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n)$. So $\phi(\phi)(t, x, y)$ needs to be $C^{1, 2}([0, T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n)$, which contains the first order derivative of $u(\cdot)$ with respect to $(x, y)$. Thus, the second and the third order derivative of $u(\cdot)$ with respect to $(x, y)$ is continuous. $\Phi$ is a continuous mapping from $C([0, T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n; \mathcal{A})$ to $C([0, T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n; \mathcal{A})$. So, the fixed point solution $\hat{\pi}$ exists, such that $\hat{\pi} = \Phi(\hat{\pi})$, then $\pi^* := \frac{1}{N} \sum_{i=1}^N V^{\ell*}(\cdot)$ is the Nash equilibrium.

Although the result in Theorem 4.1 provides the fixed-point condition and an elegant form of optimal strategies with economic interpretations and connections with functionally generated portfolios, it is not an explicit solution of optimal strategies. One can ensure the existence of their relative arbitrage opportunities. To this end, we provide the following example.

**Example 4.1.** We construct the stock capitalization coefficients using the similar idea in volatility-stabilized market models ([9]). The main characteristics of volatility-stabilized market models are the leverage effect, where the rate of return and volatility have a negative correlation with the stock capitalization relative to the market $\{m_i(t)\}_{i=1}^n$. Smaller stocks tend to have higher volatility than larger stocks. The coefficients $\beta(\cdot)$ and $\sigma(\cdot)$ in $\mathcal{M}$ are set to the following specific forms that agree with these market behaviors. For $1 \leq i, j \leq n$, with infinite number of investors,

$$
\beta_i(t) = \frac{C_x}{m_i(t)Y_i(t)}, \quad a_{ij} = \frac{X_i(t)}{Y_j(t)}X(t)\delta_{ij}, \quad \text{(4.9)}
$$

where $\delta_{ij} = 1$, when $i = j$; and $\delta_{ij} = 0$ otherwise, when $i \neq j$. $C_x$ is a given nonnegative constant. $m(\cdot)$ is the market portfolio.

Let $\delta = \frac{1}{2}$, and consider a simplified market structure with $dY_i = y_i dt$, where $Y_0$ is defined as $Y_0 = \frac{-\frac{\delta_0}{2} - \sum_{k=1}^n w^k(t)}{\frac{\delta_0}{2}}$, and $y_0 = \frac{-\frac{\delta_0}{2} - \sum_{k=1}^n w^k(t)}{\frac{\delta_0}{2}}$. Then the diffusion coefficient $\tau(x, y) = 0$, for any $(x, y) \in (0, \infty) \times (0, \infty)$. Hence, we adapt Assumption 2 to the existence of a function $H : \mathbb{R}_+^n \rightarrow \mathbb{R}$ of class $C^2$, such that $b(x, y) = a(x, y)D_H(x)$. Thus, further computation shows that

$$
D_iH(x) := \frac{b_i(x, y)}{a_{ii}(x, y)} = \frac{X_i(t)\beta_i(x, y)}{a_{ii}(x, y)} = \frac{C_x}{x_i},
$$

$$
k(x, y) := -\sum_{i=1}^n \sum_{j=1}^n \frac{a_{ij}(x, y)}{2} [D_{ij}^2 H(x) + D_i H(x, y) D_j H(x)] = 0,
$$

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and the market price of risk follows $\theta_i(\mathcal{X}(t), \mathcal{Y}(t)) = \frac{\sigma_i'(\mathcal{X}(t), \mathcal{Y}(t))D_t H(\mathcal{X}(t))}{C_x(X(t))^\lambda_\ell(X_i(t)Y_i(t))^{\lambda_\ell}}$; 

$i = 1, \ldots, n$, with $H(x) = C_x \sum_{i=1}^n \log x_i$. $L(t) = \prod_{j=1}^t \frac{x_j}{X_j(t)}$. $\hat{X}(t) = X(t) \prod_{j=1}^t \frac{x_j}{X_j(t)}$.

For $i = 1, \ldots, n$, the Fichera drift follows 

$$f_i(x, y) = \frac{a_{ii}(x, y)}{x \cdot 1 + y \cdot 1} \frac{1}{2} D_i a_{ii}(x, y)$$

$$= \frac{x_i}{y_{0,i}} \left( \frac{x_0}{x \cdot 1 + y \cdot 1} - \frac{1}{2} \frac{x_0}{y_{0,i}} \right)$$

$$= \frac{x_i}{y_{0,i}} \left( \frac{x_0}{x \cdot 1 + y \cdot 1} - \frac{1}{2} \right)$$

and $f_i(x, y) = 0$, for $i = n + 1, \ldots, 2n$. Since $\frac{x_0}{x \cdot 1 + y \cdot 1} - \frac{1}{2} < \frac{x_0}{y_{0,i}}$, we have $f_i(x, y) \leq 0$. This tells us that $\hat{\mathcal{Y}}(\cdot)$ and $\hat{\mathcal{X}}(\cdot)$ are strict local martingales. Note that the corresponding auxiliary process $\zeta(\cdot)$ is modified to take values in $\mathbb{R}^n$, whose values take the values of the first $n$ coordinates in the original process in Definition 3.4.

We can now simplify (4.8). With $u(T - t, x, y) = u(T - t)$ it holds 

$$\pi_i^*(t) = 1 - (1 - m_i(t)) \frac{\delta X(t)}{\mathcal{V}(t)} + X_i(t)D_i \log u(T - t) + \frac{1 - \delta}{\mathcal{U}^2} m_i(t) \sum_{k=1}^N V_k(t)D_i \log u_k(T - t)$$

$$= 1 - \frac{1}{m_i(t)} - (1 - \delta) X_i(t) \frac{1}{N} \sum_{k=1}^N u_k(T - t) D_i \log u_k(T - t) + X_i(t)D_i \log u(T - t)$$

$$= \frac{m_i(t)}{1 - \frac{1}{N} \sum_{k=1}^N u_k(T - t) D_i \log u_k(T - t)} \frac{1}{\mathcal{U}(t)(1 - (1 - \delta) e^x)} \frac{X(T)}{X_i(t) \cdot \mathcal{Y}(t)} \frac{\mathcal{V}(T)}{x_{1}(t) \cdots x_{n}(t)} \frac{\mathcal{F}_t}{\mathcal{F}_t}$$

where for $\ell = 1, \ldots, N$, by (3.17) and (3.20), 

$$\mathcal{U}(T - t) = e^{x_{1}(t)} \cdots x_{n}(t) \left[ \frac{\mathcal{V}(T)}{x_{1}(t) \cdots x_{n}(t)} \left| \mathcal{F}_t \right. \right]$$

As a special case, if $\delta = 1$, $\tau(x, y) = 0$, for any $(x, y) \in (0, \infty)^n \times (0, \infty)^n$, then 

$$\pi_i^*(t) = X_i(t)D_i \log u(T - t, x, y) + m_i(t).$$

This optimal strategy appears to be of a form similar to the result in [9]. In fact, we recover the model in [9], if $N = 1$ and there is no interaction term $\mathcal{Y}(\cdot)$ in the market model. In our case, $D_i \log u(T - t, x, y)$ contains the interaction term $y$, which influences the coefficients in the associated Cauchy problem of $u(T - t, x, y)$ in (3.24). If the instantaneous growth rates and volatilities are of the form 

$$(b_i, \gamma_{ik}, \tau_{ik})(\mathcal{X}(t), \mathcal{Y}(t)) = (\bar{b}_i, \bar{\gamma}_{ik}, \bar{\tau}_{ik})(\mathbf{m}(t), n(t)),$$

where $\mathbf{m}(\cdot)$ is the market portfolio and $\mathbf{n}(\cdot) := (\mathbf{m}(\cdot), \ldots, \mathbf{n}(\cdot))$ are the relative weights of the volume of trading, i.e., $\mathbf{n}(\cdot) := \frac{Y(\cdot)}{Y(\cdot)}$, $Y(\cdot) := \sum_{i=1}^n \mathcal{Y}_i(\cdot)$. Then, 

$$X_i(t)D_i \log u^\lambda_{\ell}(T - t, x, y) = \mathbf{m}_i \left( D_m \log \tilde{U}(T - t, m(t), n(t)) - \sum_{j=1}^n m_j D_{m_j} \log \tilde{U}(T - t, m(t), n(t)) \right).$$

where $\tilde{U}(\cdot) \in C^{1,3}((0, \infty) \times \Delta_n \times \Delta_n)$. In this case, $\sum_{i=1}^n \pi_i^*(t) = 1$, for $t \in [0, T]$, $\ell = 1, \ldots, N$.

If $c_\ell = c$, $\tau(x, y) = 0$, for any $(x, y) \in (0, \infty)^n \times (0, \infty)^n$, then 

$$\pi_i^*(t) = \frac{X_i(t)}{1 - (1 - \delta) u(T - t, x, y)} D_i \log u(T - t, x, y) + m_i(t).$$
The above special example with a simplified trading volume process shows an explicit solution to the optimal arbitrage problem. However, \( \gamma(\cdot) \) and \( \tau(\cdot) \) must be consistent with what the Nash equilibrium entails. If we restrict the strategy functions as formulated in (4.10), verifying the consistency of \( \gamma(\cdot) \) and \( \tau(\cdot) \) with the Nash equilibrium involves solving a fixed-point problem for \( \pi(\cdot) \). To ensure this consistency, achieving the uniqueness of \( \mu^* \), i.e., Nash equilibrium, is sufficient. This is a weaker requirement compared to the uniqueness of optimal strategies, as elaborated at the end of Section 4.1.

**Remark 3.** Note that Theorem 4.1 provides one example of the optimal strategy. This structure of the optimal strategy is reminiscent to functional generated portfolios, as studied in [33], which yields practical implementations and data-driven methods for stochastic portfolio theory. We adopt this particular structure in Example 4.1. However, there may be other strategies that lead to the same optimal empirical wealth distribution \( \mu^* \). As shown in Definition 4.2 and the subsequent statements, it is challenging and unnecessary to search for all the possible searches for a set of optimal strategies to achieve Nash equilibrium. The natural question is whether \( \hat{C} \) provides one example of the optimal strategy. This structure of the optimal strategies, as elaborated at the end of Section 4.1.1, is sufficient. This is a weaker requirement compared to the uniqueness of optimal strategies, as elaborated at the end of Section 4.1.

**4.3 The uniqueness of Nash equilibrium**

Theorem 4.1 searches for a set of optimal strategies to achieve Nash equilibrium. The natural question is whether the \( N \)-player game has a unique Nash equilibrium, since the unique Nash equilibrium leads to the unique optimal strategy for investors in the form of (4.4). To derive the uniqueness result, we consider the fixed point problem regarding the optimal arbitrage quantity: With a given set of \( \{(u^1, \ldots, u^N)(T-t, x, y)\}_{t \in [0,T]} \), investors solve optimal strategies (4.4), and we solve the drift and diffusion coefficients of the coupled system \( \mathcal{X}(t), \mathcal{Y}(t) \) through optimal strategies. Then we expect that the optimal arbitrage quantity based on this updated system coincides with the given \( \{(u^1, \ldots, u^N)(T-t, x, y)\}_{t \in [0,T]} \) we started with. In particular, at the Nash equilibrium, (4.5) holds and

\[
u^\ell(T-t, x, y) = e^{c_\ell} \mathbb{E}^x \mathbb{Y}[\mathbb{V}(T-t)L(T-t)] / \mathbb{V}(0) = \delta \epsilon^{c_\ell} \mathbb{E}^x \mathbb{Y} \left[ \frac{\hat{X}(T-t)}{1 - \frac{1}{N} \delta \epsilon U(t, x, y) \sum_{k=1}^N e^{c_k}} \right],
\]

where we define the corresponding common factor

\[U(T-t, \mathcal{X}(t), \mathcal{Y}(t)) = u^k(T-t, \mathcal{X}(t), \mathcal{Y}(t))/e^{c_k}, \quad \text{for every } k = 1, \ldots, N.\]

Here, \( \hat{X}(T-\cdot) \) depends on \( U(\cdot) \) through (3.15).

To this end, we first summarize the procedure to arrive at a fixed-point solution in the space of the paths of \( \{u^\ell(T-t, \mathcal{X}(t), \mathcal{Y}(t))\}_{t \in [0,T]} \) for every \( \ell = 1, \ldots, N \) in the following chart.

\[
\begin{array}{cccc}
\{u^\ell(T-\cdot, x, y)\}_{\ell=1}^N & \xrightarrow{\text{Theorem 4.1}} & \{\phi^\ell(\cdot)\}_{\ell=1}^N & \xrightarrow{(C.3)} \{b, \sigma, \gamma, \tau(\cdot)\} \\
& & \xrightarrow{(2.1)-(2.3)} & (\mathcal{X}, \mathcal{Y})
\end{array}
\]

Figure 1: The formulation of the fixed-point problems. Note that this chart works for the fixed-point problem on the space of the paths of strategies as well, i.e., \( \pi = \Phi(\pi) \) in Section 4.2.1-4.2.2, if we start the flow from \( \{\phi^\ell(x, y)\}_{\ell=1}^N \).

Next, we provide the sufficient condition for the unique Nash equilibrium as the empirical measure \( \mu^* \). We consider the optimal arbitrage quantity on the path space. Let \( \mathcal{U} = C^{1,3,3}([0,T] \times \mathbb{R}^+_x \times \mathbb{R}^+_y; \mathbb{R}_+^+) \) denote the set of continuous \( \mathbb{R}_+ \)-valued functions equipped with the supremum norm \( \| \cdot \| \)

\[\|u\|_{\mathcal{U}} = \sup_{\tau \in [0,T], (x, y) \in \mathbb{R}^+_x \times \mathbb{R}^+_y} |u(\tau, x, y)|.\]
Proposition 4.1. When the optimal strategy is the fixed point solution of (4.4), there exists a fixed point operator $\mathcal{G}$ such that $\mathcal{G}(U) = U$, where $U := (U_t)_{t \in [0,T]} \in \mathcal{U}$. If the fixed point solution $U$ is unique, then Nash equilibrium $\mu^* \in \mathcal{P}_2(C([0,T], \mathbb{R}_+))$ is unique in the sense of Definition 4.2.

Proof. We first set up the fixed-point problem of $u(\cdot)$.

Define $f : \mathcal{U} \to C([0, T], \mathbb{R})$ as following:

$$f(u) = \left[ 1 - \frac{1 - \delta}{N} \sum_{k=1}^{N} e^{\circ k} \right]^{-1}. \quad (4.13)$$

Then, it holds,

$$\sup_{\tau \in [0,T], (x,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n} f(u)(\tau, x, y) = \left[ 1 - \frac{1 - \delta}{N} \sum_{k=1}^{N} e^{\circ k} \right]^{-1} =: C_f,$$

Define an operator $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ by

$$[\mathcal{F} \ast U](T - t, x, y) = \mathbb{V}^{-1}(0) \mathbb{E}^{x,y} \left[ \frac{1}{1 - \frac{1 - \delta}{N} U(t, x, y) \sum_{k=1}^{N} e^{\circ k}} \right] = \mathbb{V}^{-1}(0) \mathbb{E}^{x,y} \left[ \frac{1}{1 - \frac{1 - \delta}{N} U(t, x, y) \sum_{k=1}^{N} e^{\circ k}} \right],$$

for every $t \in [0, T]$, where $U := (U_t)_{t \in [0,T]} \in \mathcal{U}$, $f$ is defined in (4.13). Consider a mapping $\mathcal{I} : \mathcal{U} \to \mathcal{U}$, such that $\mathcal{I}(U)(t, x, y) = U(T - t, x, y)$, for every $t \in [0, T]$. In particular, $\mathcal{I}(\cdot)$ maps the optimal arbitrage quantity for the time horizon $[0, T]$ to the optimal arbitrage quantity for the time horizon $[0, t]$. Note that this is different from the subproblems in (3.18), since the time horizon there is $[t, T]$, for every $t \in [0, T]$. We have

$$\mathcal{I}(\mathbb{E}^{x,y} [\mathcal{X}](t))(T - t, x, y) = \mathbb{E}^{x,y} [\mathcal{X}(t)].$$

This mapping $\mathcal{I}$ is continuous and bounded in $\mathcal{U}$. The boundedness is immediately followed by the bounded nature of $U(\cdot)$. Let $U_m(\cdot)$ be a sequence of functions in $\mathcal{U}$ that converges uniformly to a function $U \in \mathcal{U}$ on $[0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$ as $m \to \infty$. Then for each $t \in [0, T]$,

$$\lim_{m \to \infty} \mathcal{I}(U_m)(t) = \lim_{m \to \infty} U_m(T - t) = U(T - t) = \mathcal{I}(U)(t).$$

Thus, $\mathcal{I}$ is continuous and hence, $\mathcal{F} \ast \mathcal{I}(U)(t) = \mathcal{F}(U)(T - t)$ for $t \in [0, T]$. At Nash equilibrium, (4.11) leads to the fixed-point condition

$$\mathcal{G}(U)(T - t, x, y) = [\mathcal{F} \ast \mathcal{I}(U)](T - t, x, y) = U(T - t, x, y) \quad (4.14)$$

for every $t \in [0, T], (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Since $\mathcal{U}$ is a convex and closed set, the fixed-point solution exists by the Brouwer fixed-point theorem. Let $u$ be the unique solution of (4.14), then the optimal wealth processes are uniquely determined by

$$V^t(t) = \frac{u^t(T - t, X(t), Y(t)) \delta X(t)}{1 - (1 - \delta) \sum_{k=1}^{N} u^t(T - t, X(t), Y(t))}. \quad (4.15)$$

Hence, from (4.15), if the fixed point solution $U \in \mathcal{U}$ is unique, the Nash equilibrium is unique in the sense of Definition 4.2.

Remark 4. We use the Cauchy problem to derive the existence of relative arbitrage opportunities with explicit market coefficients $(b, \sigma, \gamma, \tau)(\cdot)$; whereas here the market coefficients $(b, \sigma, \gamma, \tau)(\cdot)$ need to be solved at Nash equilibrium. The discussion of Nash equilibrium does not directly result in the attainment of relative arbitrage opportunities. At Nash equilibrium, relative arbitrage opportunities are accessible to all investors, or the opportunities are universally unattainable.
We make the following assumption for the uniqueness of the Nash equilibrium.

**Assumption 5.** For \( t \in [0, T] \), \( \tau := T - t \), denote the deflated market capitalization in (3.15) as \( \hat{X}^u(\tau) \) when the common factor in (4.12) is \( u(\tau, \cdot, \cdot) \). We assume that for every \( u, v \in \mathcal{U}, t \in [0, T] \), it satisfies

\[
\sup_{\tau \in [0, T]} \mathbb{E}^{x,y}[\hat{X}^u(\tau) - \hat{X}^v(\tau)] < M||u - v||_U
\]

for some constant \( M > 0 \).

One special case is when \( \mathcal{X}(t) \) is of the form

\[
dX_i(t) = X_i(t)(\beta_i(\mathcal{X}(t))dt + \sum_{k=1}^{n} \sigma_{ik}(\mathcal{X}(t))dW_k(t)), \quad i = 1, \ldots, n.
\]

That is, the dynamics of the stock capitalization is not influenced by the trading volume of the investors. Hence, Assumption 5 is satisfied, as the market is not influenced by investors, while the wealth processes of the investors are influenced by their empirical mean of the trading volume. Then (4.15) suggests that if \( \{u^\ell(\cdot)\}_{\ell=1,\ldots,N} \) is unique, then we get the unique Nash equilibrium \( \mu \).

**Theorem 4.2** (Uniqueness of Nash equilibrium). Under Assumption 1-5, consider the subproblems \( u^\ell(T - t) \) at every \( t \in [0, T] \). Starting at time \( t \in [0, T] \), take \( u, v \in (0, 1) \) as the different values of the initial relative arbitrage quantity of investor \( \ell \), as defined in (3.17). Nash equilibrium \((\pi^*, \mu^*)\) is unique when

\[
\frac{1 - \delta^2}{\delta} \epsilon \in (0, 1), \quad M < x_0 \frac{\delta + \epsilon \delta^2 - 1}{1 - (1 - \delta)\epsilon},
\]

(4.16)

\( M \) is the constant in Assumption 5.

**Proof.**

1. Denote \( \hat{M} := MCF\left(x_0 + \frac{1 - \delta - \epsilon}{\delta}v_0 - \frac{M}{\delta x_0}\right)^{-1} = \frac{M}{\delta x_0} \). \( \lambda := \sup_{x,y} \mathbb{E}^{x,y}\left[\hat{X}(T)\right] / \mathcal{V}(0) \). By triangle’s inequality, for \( u, v \in \mathcal{U} \),

\[
\|F \ast I(u) - F \ast I(v)\|_U = \sup_{\tau \in [0, T] \times (x,y) \in \mathbb{R}_+^2} |F \ast I(u(\tau, x, y)) - F \ast I(v(\tau, x, y))|
\]

\[
\leq \sup_{\tau \times x,y} \mathbb{E}^{x,y}\left[\hat{X}^u(\tau) - f(u) - f(v)\right] / \mathcal{V}(0) + \hat{M} \sup_{\tau \times x,y} \mathbb{E}^{x,y}\left[\hat{X}^u(\tau) - \hat{X}^v(\tau)\right] / \mathcal{V}(0) + \hat{M} \|u - v\|_U
\]

\[
\leq (\lambda L + \hat{M}) \|u - v\|_U,
\]

where the second inequality is derived from the local Lipschitz continuity of \( f \). \( L := \sup_{u \in [0, 1]} |f'(u)| \leq \frac{1 - \delta}{(1 - (1 - \delta)\epsilon)\epsilon} \). Since \( \mathbb{E}\left[\hat{X}(t)\right] < x_0 \) from the supermartingale property proved in Proposition 3.3, we have

\[
\lambda := \sup_{\tau \times x,y} \mathbb{E}^{x,y}\left[\hat{X}(\tau)\right] / \mathcal{V}(0) = \sup_{\tau \times x,y} \frac{\mathbb{E}^{x,y}\left[\hat{X}(\tau)\right]}{\mathcal{V}(0)} \frac{X(0)}{\mathcal{V}(0)} < \frac{X(0)}{\mathcal{V}(0)} = 1 - \frac{1 - \delta}{\delta} \epsilon.
\]

Combining these quantities, we get

\[
\frac{\lambda L}{\delta} \leq 1 - \frac{1 - \delta}{\delta} \epsilon < 1.
\]

2. To show the contraction property of the operator \( F \ast I \), we need \( \lambda L + \hat{M} < 1 \) to hold. This is equivalent to the following conditions

\[
\epsilon(1 - \delta) < 1,
\]

and

\[
\frac{1}{\delta} \left(1 - \frac{(1 - \delta)}{(1 - \delta)\epsilon}\right) + \frac{M}{\delta x_0} < 1.
\]
Further computation yields that (4.16) is a sufficient condition that every participant achieves the unique Nash equilibrium as derived in Proposition 3.2. Therefore, for all the functions \( u, v \in \mathcal{U} \), we conclude the contraction condition that there exists a \( 0 \leq k < 1 \) such that
\[
\| \mathcal{F} \ast \mathcal{I}(u) - \mathcal{F} \ast \mathcal{I}(v) \|_{\mathcal{U}} \leq k \| u - v \|_{\mathcal{U}}.
\]

Then by the Banach Fixed Point Theorem, the solution \( u \) in (4.14) is unique.

Remark 5. With the conditions of unique Nash equilibrium and the Fichera drift condition on the market coefficients, investors can outperform their benchmark and achieve optimal arbitrage at the unique Nash equilibrium. Consider Example 4.1, with a sufficiently small \( T \) and a suitable value for \( C_x \) in (4.9), we can satisfy the bound on \( M \) in (4.16).

Remark 6. We use (4.10) to show a counterexample for the two-fund separation theorem ([20]). That is, the optimal strategy of the form (4.8) for \( \delta \in (\delta_-, \delta_+) \) is not a linear combination of the optimal strategy in the form (4.8) for \( \delta_- \) and \( \delta_+ \). We know that a simultaneous relative arbitrage opportunity is not possible when \( \delta = 0 \). When \( \delta \) approaches zero, the conditions for achieving a unique Nash equilibrium are violated, adding uncertainties and difficulties to solve an optimal strategy for competitive investors.

5 Discussions and Future work

In this paper we discuss the optimization of the relative arbitrage problem with \( N \) competitive investors. We derive the optimal strategy in Nash equilibrium under the information structure of a feedback form. As the market dynamics receives the feedback of the trading volumes, a deviation from the optimal strategy of one investor would give rise to the change in stock capitalization through the average trading volume of investors. Some interesting follow-up questions include considering the relative arbitrage problem with cooperative investors and short-term arbitrage opportunities.

The other direction is to consider the numerical solution of the Cauchy problem or the construction of the corresponding functionally generated portfolio, which helps in the implementation of the optimal strategy. The numerical scheme for the single investor case is studied in [33] through the Bessel process, which is closely related to the stock capitalizations. As the mean-field game regime in [34] can be a good approximation for a finite-player game, one promising direction is to extend the numerical scheme for the \( N \)-player game we consider in this paper when \( N \) is large.

In terms of the supply and demand mechanism, we characterize the supply and demand of the market through the stochastic differential equations of stock capitalization and the trading volume of the agents. The drift and diffusion coefficients of these two stochastic processes interact with each other and thus lead to a coupled market system. To better understand the supply and demand mechanism, we may consider an environment with model uncertainty to relax the structural restriction of the SDE systems in the paper.

References

[1] R. F. Bass, E. A. Perkins, Degenerate stochastic differential equations with Hölder continuous coefficients and super-Markov chains. Trans. Amer. Math. Soc. 355, 373-405, (2003).
[2] E. Bayraktar, Y.-J. Huang, Q. Song, Outperforming the market portfolio with a given probability. Ann. Appl. Probab. 22(4), 1465-1494 (2012).
[3] R. Carmona, Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications. SIAM Book Series in Financial Mathematics 1 (2016).
[4] R. Carmona, F. Delarue, Probabilistic Theory of Mean Field Games with Applications I: Mean Field Games with Common Noise and Master Equations. Volume 84 of Probability Theory and Stochastic Modelling, Springer, 2018.
[5] D. Fernholz, I. Karatzas, On Optimal Arbitrage. Ann. Appl. Probab., 20 1179-1204 (2010).
D. Fernholz, I. Karatzas, *Optimal Arbitrage under model uncertainty*. Ann. Appl. Probab, Vol. 21, No.6, 2191-2225 (2011).

R. Fernholz, I. Karatzas, *Stochastic portfolio theory: A survey*. In *Handbook of Numerical Analysis, Mathematical Modeling and Numerical Methods in Finance* (A. Bensoussan, ed.). 89-168 (2009).

R. Fernholz, *Stochastic Portfolio Theory, volume 48 of Applications of Mathematics (New York)*. Springer-Verlag, New York. Stochastic Modelling and Applied Probability. (2002).

R. Fernholz, I. Karatzas, *Relative arbitrage in volatility-stabilized markets*. Ann. Finance 1, 149-177 (2005).

R. Fernholz, I. Karatzas, C. Kardaras, *Diversity and relative arbitrage in equity market*. Finance & Stochastics 9, 1-27 (2005).

R. Fernholz, I. Karatzas, J. Ruf, *Volatility and arbitrage*. Ann. Appl. Probab. 28 (1) 378-417 (2018).

H. Föllmer, *The exit measure of a supermartingale*. Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete 21, 154-166 (1972).

A. Friedman, *Stochastic Differential Equations and Applications*. Vol. I, Vol. 28 of Probability and Mathematical Statistics, Academic Press, New York (1975).

O. Guéant, J.M. Lasry, and P.L. Lions, *Mean field games and applications*. In R. Carmona et al., editor, Paris Princeton Lectures in Mathematical Finance IV, volume 2003 of Lecture Notes in Mathematics. Springer Verlag (2010).

D. Heath, M. Schweizer, *Martingales versus pdes in finance: an equivalence result with examples*. Journal of Applied Probability, 37(4):947–957, 2000.

D. Itkin, B. Koch, M. Larsson, J. Teichmann. *Ergodic robust maximization of asymptotic growth under stochastic volatility*. arXiv preprint arXiv:2211.15628, 2022.

D. Itkin, M. Larsson, *Open markets and hybrid Jacobi processes*. arXiv preprint arXiv:2110.14046, 2021.

D. Itkin, M. Larsson, *Robust asymptotic growth in stochastic portfolio theory under long-only constraints*. Mathematical Finance, 32(1):114–171, 2022.

F. Russo and P. Vallois, *Stochastic calculus via regularizations*. volume 11. Springer Nature, 2022.

J. Tobin, *Liquidity preference as behavior towards risk*. The review of economic studies. 25(2):65–86, 1958.

B. K. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*. Springer, Berlin. 6th edition. ISBN 9783642143946. (2010).

S. Pal, T.K.L. Wong, *The geometry of relative arbitrage*. Math. Financ. Econ. 10, 263-293 (2016).

M. Pinsky, *A note on degenerate diffusion processes*. Theor. Probability. Appl. 14, 502-506 (1969).

E. Platen, and D. Heath, *A Benchmark Approach to Quantitative Finance*. Springer, Berlin, 2006. MR2267213

J. Ruf, *Optimal Trading Strategies Under Arbitrage*. PhD thesis, Columbia University, New York, USA (2011).

J. Ruf, *Hedging under Arbitrage*. Math. Financ. 23, 297-317 (2013).

W. Strong, J.-P. Fouque, *Diversity and arbitrage in a regulatory breakup model*. Ann Finance 7, 349–374 (2011).

D. W. Stroock, S. R. S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Probability Theory, Berkeley, Calif., University of California Press, pp. 333–359 (1972).
[29] L. J. Van Mellaert, and P. Dorato, *Numerical solution of an optimal control problem with a probability criterion*. IEEE Transactions on Automatic Control AC-17 543-546 (1972).

[30] C. Villani, *Topics in optimal transportation*. Volume 58. American Mathematical Soc., 2021.

[31] T.K.L. Wong, *Information geometry in portfolio theory*. In Frank Nielsen (Ed.), Geometric Structures of Information, Springer (2019).

[32] T.K.L. Wong, *Optimization of relative arbitrage*. Ann. Finance 11 345–382 (2015).

[33] T. Yang, *Topics in relative arbitrage, stochastic games and high-dimensional PDEs*. Ph.D. Dissertation, University of California, Santa Barbara (2021).

[34] N. T. Yang, T. Ichiba, *Relative arbitrage opportunities in an extended mean field system*. arXiv preprint arXiv:2311.02690, 2023.
Appendices

A Market dynamics and conditions

This section recalls some properties of the market which are related to the existence of relative arbitrage.

**Definition A.1** (Non-degeneracy and bounded variance). A market is a family $\mathcal{M} = \{X_1, \ldots, X_n\}$ of $n$ stocks, each of which is defined as in (2.1), such that the matrix $\alpha(t)$ is nonsingular for every $t \in [0, \infty)$, a.s. The market $\mathcal{M}$ is called nondegenerate if there exists a number $\epsilon > 0$ such that for $x \in \mathbb{R}^n$

$$P(\|x\|^2 \geq \epsilon \|x\|^2, \forall t \in [0, \infty)) = 1,$$

The market $\mathcal{M}$ has bounded variance from above, if there exists a number $M > 0$ such that for $x \in \mathbb{R}^n$

$$P(\|x\|^2 \leq M \|x\|^2, \forall t \in [0, \infty)) = 1,$$

We restate the non-degeneracy in a specific form in Assumption 1 in order to show the existence of relative arbitrage.

B Proofs

**Proof of Theorem 3.1.** We first show some main steps of computing (3.24). Plugging (3.17) in the above equations set and using the Markovian property of $g(\cdot)$ gives

$$\frac{\partial u^f(t, x, y)}{\partial t} g(x, y) = \mathcal{L}(u^f(t, x, y)g(x, y)) - (k(x, y) + \tilde{k}(x, y))u^f(t, x, y)g(x, y).$$

By expanding the above, it follows

$$\frac{\partial u^f(t)}{\partial t} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, y) \left( D^2_{ij} u^f(t) + 2D_i u^f(t) \frac{D_j g(x, y)}{g(x, y)} + u^f(t) \frac{D_{ij} g(x, y)}{g(x, y)} \right)$$

$$+ 2 \sum_{i,j=1}^{n} a_{ij}(x, y) \left( D_i u^f(t) + u^f(t) \frac{D_j g(x, y)}{g(x, y)} \right) D_j H(x, y)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \left[ D^2_{ij} H(x, y) + 3D_i H(x, y) D_j H(x, y) \right] u^f(t)$$

$$+ \frac{1}{2} \sum_{p,q=1}^{n} \psi_{pq}(x, y) \left( D^2_{pq} u^f(t) + 2D_p u^f(t) \frac{D_q g(x, y)}{g(x, y)} + u^f(t) \frac{D_{pq} g(x, y)}{g(x, y)} \right)$$

$$+ 2 \sum_{p,q=1}^{n} \psi_{pq}(x, y) \left( D_p u^f(t) + u^f(t) \frac{D_q g(x, y)}{g(x, y)} \right) D_q H(x, y)$$

$$+ \frac{1}{2} \sum_{p,q=1}^{n} \psi_{pq} \left[ D^2_{pq} I(y) + 3D_p H(x, y) D_q H(x, y) \right] u^f(t)$$

$$+ \sum_{i,p=1}^{n} (sT^T)_{ip}(x, y) \left( D^2_{ip} u^f(t) + D_i u^f(t) \frac{D_p g(x, y)}{g(x, y)} + D_p u^f(t) \frac{D_i g(x, y)}{g(x, y)} + u^f(t) \frac{D_{ip} g(x, y)}{g(x, y)} \right)$$

$$- \sum_{i,p=1}^{n} (sT^T)_{ip}(x, y) D_i H(x, y) D_p H(x, y) u^f(t).$$

We can simplify this equation with the following computations.

By (3.11), and the definition of $g(\cdot)$,

$$\frac{D_i g(x, y)}{g(x, y)} = -D_i H(x, y) + \frac{\delta}{\delta x \cdot 1 + (1 - \delta) y \cdot 1}, \quad \frac{D_p g(x, y)}{g(x, y)} = -D_p H(x, y) + \frac{1 - \delta}{\delta x \cdot 1 + (1 - \delta) y \cdot 1}.$$
Th second order derivative with respect to \( x \) is

\[
\frac{D_{ij}g(x, y)}{g(x, y)} = \frac{\delta(D_i H(x, y) + D_j H(x, y))}{\delta x \cdot 1 + (1 - \delta) y \cdot 1} - \frac{D^2 H(x, y) + D_i H(x, y)D_j H(x, y)}{y},
\]

and the counterparts of second order derivative \( \frac{D_{ij}g(x, y)}{g(x, y)} \) and \( \frac{D_{ij}g(x, y)}{g(x, y)} \) can be derived in the same vein. As a result, when the drift term \( \gamma(\cdot) \) and volatility term \( \tau(\cdot) \) in (2.3) is given, (3.22) - (3.24) are satisfied.

Suppose that a solution of (3.25) and (3.23) is \( \bar{\omega}^\ell : C^2((0, \infty) \times (0, \infty)^n \times (0, \infty)^n) \to (0, \infty), \) \( \bar{\omega}^\ell(0) = e^{c\ell} \). Define \( \tilde{N}(t) := \tilde{w}^\ell(T - t, \mathcal{X}_t, \mathcal{Y}_t)\mathcal{V}(t)L(t), 0 \leq t \leq T. \) We solve

\[
\frac{d\tilde{N}(t)}{\tilde{N}(t)} = \bar{\omega}^\ell(T - t, \mathcal{X}_t, \mathcal{Y}_t) + (\Pi(t)\sigma(t) - \theta(t))(1 + \frac{\bar{\omega}^\ell(T - t, \mathcal{X}_t, \mathcal{Y}_t)}{\bar{\omega}^\ell(T - t, \mathcal{X}_t, \mathcal{Y}_t)})dW(t)
\]

by using the inequality (3.25). We get that the \( dt \) terms in \( d\tilde{N}(t)/\tilde{N}(t) \) is always no greater than 0. \( \tilde{N}(t) \) is a positive local supermartingale. Thus, \( \tilde{N}(t) \) is a supermartingale.

Hence, \( \tilde{N}(0) = \tilde{w}^\ell(T, \mathcal{X}_0, \mathcal{Y}_0)\mathcal{V}(0) \geq \mathbb{E}[\tilde{N}(t)] = \mathbb{E}^P[e^{c\ell}\mathcal{V}(T)L(T)] \) holds for every \( (T, \mathcal{X}, \mathcal{Y}) \in (0, \infty) \times (0, \infty)^n \times (0, \infty)^n \). Then \( \tilde{w}^\ell(T, \mathcal{X}, \mathcal{Y}) \geq \mathbb{E}^P[e^{c\ell}\mathcal{V}(T)L(T)]/\mathcal{V}(0) = u^\ell(T, \mathcal{X}, \mathcal{Y}). \)

**Proof of Proposition 3.3.** From Itô’s formula, the discounted process \( \hat{\mathcal{V}}^\ell(\cdot) \) admits

\[
d\hat{\mathcal{V}}^\ell(t) = \hat{\mathcal{V}}^\ell(t)(\pi^\ell(t)\sigma(t) - \theta(t))dW(t); \quad \hat{\mathcal{V}}^\ell(0) = \hat{\pi}^\ell.
\]
\( \hat{\mathcal{V}}^\ell(\cdot) \) is a positive local martingale as \( \mathbb{E}[L(T)] \leq 1. \) Thus \( \hat{\mathcal{V}}^\ell(\cdot) \) is a supermartingale, and we get that for an arbitrary \( \omega^\ell \) in (3.6),

\[
u^\ell\mathcal{V}(0) = \hat{\pi}^\ell \geq \mathbb{E}[\hat{\mathcal{V}}^\ell(T)] \geq \mathbb{E} \left[ \tilde{X}(T) \delta e^{c\ell} + L(T)(1 - \delta)e^{c\ell}\mathcal{V}(T) \right] := p^\ell.
\]

Hence, \( u^\ell(T) \geq \frac{\rho^\ell}{\mathcal{V}(0)}, \ell = 1, \ldots, N. \)

To prove the opposite direction \( u^\ell(T) \leq \frac{\rho^\ell}{\mathcal{V}(0)} \), we use the martingale representation theorem (Theorem 4.3.4, [21]) to find

\[
\omega^\ell(T) := \mathbb{E}[e^{c\ell}\mathcal{V}(T)L(T)|\mathcal{F}_t] = e^{c\ell} \int_t^T \tilde{p}^\ell(s)dW(s) + p^\ell, \quad 0 \leq t \leq T,
\]

where \( \tilde{p} : [0, T] \times \Omega \to \mathbb{R}^n \) is \( \mathbb{F} \)-progressively measurable and almost surely square-integrable. Next, construct a wealth process \( \mathcal{V}_\ell(\cdot) = U^\ell(\cdot)/L(\cdot) \), it satisfies \( \mathcal{V}_\ell(0) = p^\ell, \mathcal{V}_\ell(T) = e^{c\ell}\mathcal{V}(T) \). Use the trading strategy \( h^\ell_\ell(\cdot) \) in (3.14), where

\[
h^\ell_\ell(\cdot) = \frac{1}{L(\cdot)\mathcal{V}_\ell(\cdot)}\alpha^{-1}(\cdot)[\sigma(\cdot) + U^\ell(\cdot)\theta(\cdot)].
\]

It follows that \( h^\ell_\ell(\cdot) \in \mathbb{A} \) replicates \( V^\ell_0(\cdot) \), i.e., \( V^h^\ell_\ell(T) = e^{c\ell}\mathcal{V}(T) \) a.s., with \( V^h^\ell_\ell(0) = p^\ell \). Consequently, from

\[
\omega^\ell/\mathcal{V}(0) \in \{ \omega \geq 0 \mid \text{there exists } \pi^\ell \in \mathbb{A}, \text{given } \pi^{-\ell}(\cdot) \in \mathbb{A}^{N-1}, \text{ such that } V^{\omega\mathcal{V}(0), \pi^\ell} \geq e^{c\ell}\mathcal{V}(T) \},
\]

it follows \( p^\ell/\mathcal{V}(0) \geq u^\ell(T) \). We therefore conclude \( u^\ell(T) = \mathbb{E}[e^{c\ell}\mathcal{V}(T)L(T)]/\mathcal{V}(0), \) for \( \ell = 1, \ldots, N. \)

**Proof of Corollary 4.1.** If we search for the best strategies only to outperform the market \( (\delta = 1) \) then the last term in (4.4) boils down to the market portfolio. When the benchmark for relative arbitrage concerns only the peers \( (\delta = 0) \), the relative arbitrage opportunity does not exist, as mentioned in Proposition 3.2.

**C Derivations related to fixed point problem**

We use this section to clarify the fixed-point mapping in Figure 1.
To generalize Assumption 1, we instead assume that the coefficients $\gamma(\cdot)$ and $\tau(\cdot)$ of the trading volume processes $Y(t)$ are local Lipschitz continuous with respect to $(x, y)$ uniformly with respect to time $t$ on every compact interval, i.e., for any $T, M > 0$, there exists $k > 0$ such that for any $x, x', y, y' \in \mathbb{R}^n$,

$$\sup_{t \in [0, T]} \left( |g(t, x, y) - g(t, x', y')| + |\tau(t, x, y) - \tau(t, x', y')| \right) \leq k (|x - x'| + |y - y'|), \quad |x|, |x'|, |y|, |y'| < M. \tag{C.1}$$

We can make sure the well-posedness of the market system of $(X(\cdot), Y(\cdot))$ by [19, Theorem 12.3, Lemma 12.4]. The regularity assumption of $u(\cdot)$ in Theorem 4.1 indicates that $\phi(\cdot) \in C^{1,2,2}([0, T] \times (0, \infty)^n \times (0, \infty)^n)$, where $\phi(t, x, y) : [0, T] \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathcal{A}$, $\ell = 1, \ldots, N$. In addition, $u(\cdot)$ is generalized to the solution of a Cauchy problem with time-inhomogeneous coefficients.

$$dY(t) = \frac{1}{N} \sum_{\ell=1}^N d\left[V^\ell(t)\phi^\ell(t, X(t), Y(t))\right] = Y(t) + \sum_{k=1}^N \tau_k(t, X(t), Y(t))dW_k(t),$$

where the coefficients $\gamma(\cdot)$ and $\tau(\cdot)$ can be determined by Itô's formula on $\phi^\ell(x, y) : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathcal{A}$, i.e.,

$$d[V^\ell(t)\phi^\ell(t, x, y)] = V^\ell(t) \left( \phi^\ell(t, x, y) + D_t \phi^\ell(t, x, y) + \frac{1}{2} \sum_{p,q=1}^{2n} \hat{a}_{pq}(x, y) \epsilon_{pq}^2 \phi^\ell(t, x, y) \right) dt + \sum_{p=1}^n \beta_p(x, y) D_p \phi^\ell(t, x, y) dt$$

$$+ \sum_{p=1}^{2n} \gamma_{p-n}(t, x, y) D_p \phi^\ell(t, x, y) dt + \sum_{p=1}^n \hat{\sigma}_p(x, y) D_p \phi^\ell(t, x, y) dW_p(t). \tag{C.2}$$

or for the time homogeneous case,

$$d[V^\ell(t)\phi^\ell(x, y)] = V^\ell(t) \left( \phi^\ell(x, y) + D_t \phi^\ell(x, y) + \frac{1}{2} \sum_{p=1}^{2n} \hat{a}_{pq}(x, y) \epsilon_{pq}^2 \phi^\ell(x, y) \right) dt$$

$$+ \sum_{p=1}^n \beta_p(x, y) D_p \phi^\ell(x, y) dt + \sum_{p=n+1}^{2n} \gamma_{p-n}(x, y) D_p \phi^\ell(x, y) dt$$

$$+ \sum_{p=1}^n \hat{\sigma}_p(x, y) D_p \phi^\ell(x, y) dW_p(t). \tag{C.3}$$

$\hat{\sigma}(\cdot)$ and $\hat{a}(\cdot)$ are defined in (3.27). $D_p \phi^\ell(t, x, y) := \frac{\partial \phi^\ell(t, x, y)}{\partial x_p}$, for $p = 1, \ldots, n$; $D_p \phi^\ell(t, x, y) := \frac{\partial \phi^\ell(t, x, y)}{\partial y_p-n}$, for $p = n+1, \ldots, 2n$. Thus, by equating the drift terms to $\gamma(\cdot)$, and equating the diffusion terms to $\tau(\cdot)$, we get the fixed point condition of $\phi(t, x, y)$, and we can specify the coefficients $\tau(\cdot)$ and $\gamma(\cdot)$ from the fixed point solution $\phi(\cdot)$.

### D General finite dynamical system

We use interacting particle models to describe the market. For fixed $N$, we model the $N$ investors as $N$ particles, where the particles have a common source of noise $W$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. For any metric space $(X, d)$, $P(X)$ denotes the space of probability measures on $X$ endowed with the topology of the weak convergence. $P_p(X)$ is the subspace of $P(X)$ of the probability measures of order $p$, that is, $\mu \in P_p(X)$ if $\int_X d(x, x_0)^p \mu(dx) < \infty$, where $x_0 \in X$ is an arbitrary reference point. For $p \geq 1$, $\mu, \nu \in P_p(X)$, the $p$-Wasserstein metric on $P_p(X)$ is defined by

$$W_p(\nu_1, \nu_2)^p := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int_{X \times X} d(x, y)^p \pi(dx, dy),$$
where \( d \) is the underlying metric on the space. \( \Pi(\nu_1, \nu_2) \) is the set of Borel probability measures \( \pi \) on \( \mathbb{X} \times \mathbb{X} \) with the first marginal \( \nu_1 \) and the second marginal \( \nu_2 \). Specifically, \( \pi(A \times \mathbb{X}) = \nu_1(A) \) and \( \pi(\mathbb{X} \times A) = \nu_2(A) \) for every Borel subset \( A \subset \mathbb{X} \).

Let \( C([0, T]; \mathbb{R}^{d_0}) \) be the space of continuous functions from \([0, T]\) to \( \mathbb{R}^{d_0} \). In this paper, we often take \( \mathbb{X} = \mathbb{R}^{d_0} \) when considering a real-valued random variable or take \( \mathbb{X} \) as the path space \( \mathbb{X} = C([0, T]; \mathbb{R}^{d_0}) \) with metric \( d \) as the supremum norm for a process, where a fixed number \( d_0 \) will be specified later. \( \mathcal{P}_p(\mathbb{R}^{d_0}) \) equipped with the Wasserstein distance \( \mathcal{W}_p \) is a complete separable metric space, since \( \mathbb{R}^{d_0} \) is complete and separable. See [30] for more details on Wasserstein metric and its properties.

We first define the empirical measure in the finite-particle system that we use in this paper.

**Definition D.1** (Empirical measure in the finite \( N \)-particle system). Consider \( \mathcal{F} \)-measurable \( C([0, T]; \mathbb{R}^+) \times C([0, T]; \mathcal{A}) \)-valued random variables \((V^\ell, \pi^\ell)\) for every investor \( \ell = 1, \ldots, N \). We define the empirical measure of \((V^\ell, \pi^\ell)\) as \( \nu \in \mathcal{P}_2(C([0, T], \mathbb{R}^+) \times C([0, T], \mathcal{A})) \) to be \( \mathcal{P}_2(C([0, T], \mathbb{R}^+) \times \mathbb{R}^n) \), whose time-\( t \) marginal is

\[
\nu_t := \frac{1}{N} \sum_{\ell=1}^N \delta_{(V^\ell(t), \pi^\ell(t))}, \quad t \geq 0,
\]

where \( \delta_x \) is the Dirac delta mass at \( x \in \mathbb{R}^+ \times \mathbb{R}^n_+ \). Thus, for any Borel set \( A \subset \mathbb{R}^+ \times \mathbb{R}^n_+ \),

\[
\nu_t(A) = \frac{1}{N} \sum_{\ell=1}^N \delta_{(V^\ell(t), \pi^\ell(t))}(A) = \frac{1}{N} \cdot \#\{\ell \leq N : (V^\ell(t), \pi^\ell(t)) \in A\},
\]

where \( \#\{\cdot\} \) represents the cardinality of the set. In particular, the weighted average vector \( \mathcal{V} \) defined in (2.2) is given by \( \mathcal{V}(t) = \int_{\mathbb{R}_+ \times \mathbb{R}^n_+} x y \nu_t(dx \times dy) \), \( t \geq 0 \), where \( x \) represents wealth \( V^\ell(t) \) and \( y \) represents the strategies defined in the admissible set \( \pi^\ell(t) \in \mathcal{A} \).

Denote \( \mathcal{X}(t) = (X_1(t), \ldots, X_n(t)) \), \( \mathcal{V}_t = (V^1(t), \ldots, V^N(t)) \) for \( t \geq 0 \). For a fixed \( N \), with \( \nu^N_t \) in Definition D.1 that generalizes \( \mathcal{V}(t) \), we can generalize the \((n+N)\)-dimensional system as

\[
dX_i(t) = X_i(t) \beta_i(t, \mathcal{X}(t), \nu_t) dt + \sum_{k=1}^n X_i(t) \sigma_{ik}(t, \mathcal{X}(t), \nu_t) dW_k(t); \quad \mathcal{X}(0) = x_0
\]  
(D.1)

for \( i = 1, \ldots, n \), and for \( \ell = 1, \ldots, N \),

\[
dV^\ell(t) = V^\ell(t) \left( \sum_{i=1}^n \pi^\ell_i(t) \beta_i(t, \mathcal{X}(t), \nu_t) dt + \sum_{i=1}^n \sum_{k=1}^n \pi^\ell_i(t) \sigma_{ik}(t, \mathcal{X}(t), \nu_t) dW_k(t) \right); \quad V^\ell(0) = v^\ell.
\]  
(D.2)

On the filtered probability space \( (\mathcal{O}, \mathcal{F}, \mathbb{P}) \), we call

\[
(\mathcal{X}, \mathcal{V}, \nu, W) \in \mathcal{P}_2(C([0, T], \mathbb{R}^+_N), C([0, T], \mathbb{R}^+_N), \mathcal{P}_2(C([0, T], \mathbb{R}_+ \times \mathbb{R}^n_N), C([0, T], \mathbb{R}^n_N))
\]
a strong solution of the conditional McKean-Vlasov system (D.1)-(D.2) with respect to the filtration generated by the fixed \( n \)-dimensional Brownian motion \( W(\cdot) \), with initial condition \((\mathcal{X}_0, \mathcal{V}_0, \nu_0)\), if \((X_t)_{t \in [0, T]}\) has continuous sample paths, satisfies \( \mathbb{P} \)-almost surely

\[
X_i(t) = X_i(0) + \int_0^t X_i(s) \beta_i(s, \mathcal{X}(s), \nu_s) ds + \sum_{k=1}^n \int_0^t X_i(s) \sigma_{ik}(s, \mathcal{X}(s), \nu_s) dW_k(t), \quad t \in [0, T]
\]

and is adapted to the smallest complete filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \) in which \( X_0 \) is \( \mathcal{F}_0 \)-measurable and \( W \) is \( \mathbb{F} \)-adapted. The system (2.1) and (2.3) is a special case of the above.

We make the following assumptions to ensure that the system (D.1)-(D.2) is well-posed. In the following, \( | \cdot | \) denotes the Euclidean norm of vector \( \mathbb{R}^d \) and the Frobenius norm of matrix \( \mathbb{R}^{d \times n} \), \( d = n \) or \( N \) in particular. Also, let us define

\[
b_i(t, x, \nu) := x_i \beta_i(t, x, \nu), \quad s_{ik}(t, x, \nu) := x_i \sigma_{ik}(t, x, \nu); \quad 1 \leq i \leq n, t \geq 0, x \in \mathbb{R}^n, \nu \in \mathcal{P}.\]

(D.3)
Assumption 6.  

a. Assume the Lipschitz continuity and linear growth condition are satisfied with Borel measurable mappings $b_i(t, x, \nu)$, $s_{ik}(t, x, \nu)$ in (D.3) from $[0, T] \times \mathbb{R}^n_+ \times \mathcal{P}_2(\mathbb{R}_+ \times \mathbb{R}^n_+)$ to $\mathbb{R}^n$. That is, there exists a constant $C_1, C_2 \in (0, \infty)$ that is independent of $t \in [0, T]$, such that

$$|b(t, x, \nu) - b(t, \tilde{x}, \tilde{\nu})| + |s(t, x, \nu) - s(t, \tilde{x}, \tilde{\nu})| \leq C_1 ||x - \tilde{x}|| + W_2(\nu, \tilde{\nu}),$$

$$|x_\beta(t, x, \nu) + |x_\sigma(t, x, \nu)| \leq C_2 (1 + |x| + M_2(\nu)),$$

and $a_{ij}()$ satisfy the nondegeneracy condition, i.e., if there exists a number $\epsilon > 0$ such that

$$a_{ij}(x, \nu) \geq \epsilon(|x|^2 + M_2^2(\nu)), (x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+.$$

where

$$M_2(\nu) = \left( \int_{C([0, T], \mathbb{R}^n_+ \times \mathbb{R}^n_+)} |x|^2 d\nu(x) \right)^{1/2}; \quad \nu \in \mathcal{P}_2(\mathbb{R}_+ \times \mathbb{A}).$$

b. Assume the following Lipschitz continuity and boundedness, i.e., there exist constants $C_2, B \in (0, \infty)$ such that

$$|v_\beta(t, x, \nu) - \tilde{v}_\beta(t, \tilde{x}, \tilde{\nu})| + |v_\sigma(t, x, \nu) - \tilde{v}_\sigma(t, \tilde{x}, \tilde{\nu})| \leq C_2 ||x - \tilde{x}|| + n|v - \tilde{v}| + W_2(\nu, \tilde{\nu}),$$

$$|v_\beta(t, x, \nu) + |v_\sigma(t, x, \nu)| \leq B,$$

for every $v, \tilde{v} \in \mathbb{R}_+, t \in [0, T]; x, \tilde{x} \in \mathbb{R}^n_+ ; \nu, \tilde{\nu} \in \mathcal{P}_2(\mathbb{R}_+ \times \mathbb{R}^n_+)$.

c. Let $n$ be fixed and $\ell = 1, \ldots, N$. We assume the strategies adopted by investors are closed loop feedback controls of the wealth processes and are Lipschitz continuous in their variables, i.e., there exists a bounded mapping $\phi^\ell : \mathbb{R}^n_+ \to \mathbb{A}$ such that $\pi^\ell(t) = \phi^\ell(V_t)$, $|\phi^\ell(\cdot)| < M$, and

$$|\phi^\ell(v) - \phi^\ell(\tilde{v})| \leq nC_3|v - \tilde{v}|$$

for every $v, \tilde{v} \in \mathbb{R}^n_+$.

Remark 7. We choose to present the well-posedness of the market system by assuming a general structure in the coefficients including the market capitalization $\mathcal{X}(\cdot)$, wealth $\mathcal{V}(\cdot)$, and empirical measure $\nu$, with closed-loop strategies. In later sections, we will focus on the market coefficients as functions of market capitalization $\mathcal{X}(\cdot)$ and empirical mean $\mathcal{Y}(\cdot)$ instead, and the assumptions can be adapted accordingly.

If participants adopt open-loop strategies, the interaction with other players and the market is much limited, as their wealth depends on their own open loop control and a small part of the trading volume. This condition is desirable to impose for trading strategies, as the strategies will not be too volatile in the face of small changes in wealth. Furthermore, the Lipschitz continuity of market coefficients and strategy functions is common in the literature on mathematical finance and stochastic games, for example, in [4, Lemma 3.3].

Theorem D.1. Assume that the stock capitalization vector $x_0$ at time 0 has a finite second moment, that is, $E|x_0|^2 < \infty$, and is independent of the Brownian motion $W(\cdot)$. Under Assumptions 6, the $(n + N)$-dimensional SDE system (D.1)-(D.2) admits a unique strong solution for any given number of stocks, and any given number of investors $N$.

Proof. We restrict the discussion to the time-homogeneous case, whereas the inhomogeneous case can be proved in the same fashion. Rewrite the system as a $(n + N)$-dimensional SDE system:

$$d \left( \begin{array}{c} \mathcal{X}_t \\ \mathcal{V}_t \end{array} \right) := f(\mathcal{X}(t), \mathcal{V}(t), \nu_t)dt + g(\mathcal{X}(t), \mathcal{V}(t), \nu_t)dW_t,$$

where $f(\mathcal{X}(t), \mathcal{V}(t), \nu_t) := (f_1(\cdot), \ldots, f_{n+N}(\cdot))$, $f_i(\cdot) := X_i(t) \beta_i(\cdot)$ for $i = 1, \ldots, n$, $f_j(\cdot) := V_j^{t-n} \pi_j^{t-n} (\cdot)$ for $j = n + 1, \ldots, n + N$; Similarly, $g(\mathcal{X}(t), \mathcal{V}(t), \nu_t) := (g_1(\cdot), \ldots, g_{n+N}(\cdot))$, $g_i(\cdot) := X_i(t) \sigma_i(\cdot)$ for $i = 1, \ldots, n$ and $g_j(\cdot) := V_j^{t-n} \pi_j^{t-n} (\cdot)$ for $j = n + 1, \ldots, n + N$.

Let us consider a closed-loop strategy $\pi^\ell_t := \phi^\ell(V_t)$. Define a mapping $L_N : \mathbb{R}^n_+ \to \mathcal{P}_2(C([0, T], \mathbb{R}_+ \times \mathbb{R}^n_+))$

$$L_N(\mathcal{V}(t)) = \frac{1}{N} \sum_{\ell=1}^N \delta_{\mathcal{V}_t(\cdot), \phi^\ell(\mathcal{V}(t))} = \nu_t.$$
and define $F : \mathbb{R}^{N+n}_+ \to \mathbb{R}^{N+n}, G : \mathbb{R}^{N+n}_+ \to \mathbb{R}^{N+n} \times \mathbb{R}^n$, with

$$F(\mathbf{X}(t), \mathbf{V}(t)) = f(\mathbf{X}(t), \mathbf{V}(t), L_N(\mathbf{V}(t))); \quad G(\mathbf{X}(t), \mathbf{V}(t)) = g(\mathbf{X}(t), \mathbf{V}(t), L_N(\mathbf{V}(t))).$$

Write $(x, v) = (x_1, \ldots, x_n, v^1, \ldots, v^N)$ and $(y, u) = (y_1, \ldots, y_m, u^1, \ldots, u^N)$ for two pairs of random values of $(\mathbf{X}(\cdot), \mathbf{V}(\cdot))$. Denote the empirical measure $\tilde{\pi}$ induced by the joint distribution of random variable $u$ and $v$, i.e.,

$$\tilde{\pi} = \frac{1}{N} \sum_{\ell=1}^N \delta_{(u^\ell, v^\ell)}.$$

It is a coupling of the function $L_N(v)$ and $L_N(u)$. From the definition of the Wasserstein distance, we have

$$W_2^2(L_N(v), L_N(u)) \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} |(v, \phi(v)) - (u, \phi(u))|^2 \tilde{\pi}(dv, du) \leq \frac{1}{N} \sum_{\ell=1}^N |(v^\ell, \phi^\ell(v)) - (u^\ell, \phi^\ell(u))|^2 \leq \left( \frac{1}{N} + n^2 C_3^2 \right) |v - u|^2. \tag{D.4}$$

For every $(x, v)$ and $(y, u)$ in $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, and for every $\ell$, by (D.4), the Cauchy-Schwartz inequality, and Lipschitz condition of $\beta_i$ and $\phi^\ell$, we have

$$|F(x, v) - F(y, u)|^2 \leq \sum_{i=1}^n |b_i(x, L_N(v)) - b_i(y, L_N(u))|^2 + \sum_{\ell=1}^N |v^\ell \phi^\ell(v) \beta(x, L_N(v)) - u^\ell \phi^\ell(u) \beta(y, L_N(u))|^2 \leq 2C_1^2|x - y|^2 + 6W_2^2(L_N(v), L_N(u)) + 2n^2 B^2 C_3^2 |v - u|^2,$$

where $L_m^2 = \max\{2C_1^2 + 6nM^2C_3^2, 2C_1^2(\frac{1}{N} + n^2 C_3^2) + 6M^2C_3^2(n^2 + 1 + n^2 C_3^2 N) + 2n^2 B^2 C_3^2 N\}$. The second inequality follows from the triangle inequality, uniform boundedness and Lipschitz condition of $\beta_i$ and $\phi^\ell$,

$$|v^\ell \phi^\ell(v) \beta(x, L_N(v)) - u^\ell \phi^\ell(u) \beta(y, L_N(u))|^2 \leq 2|\phi^\ell(v)^\ell \beta(x, L_N(v)) - u^\ell \beta(y, L_N(u))|^2 + 2B^2 |\phi^\ell(v) - \phi^\ell(u)|^2 \leq 6C_3^2 |x - y|^2 + 2n^2 |v^\ell - u^\ell|^2 + W_2^2(L_N(v), L_N(u)) + 2n^2 B^2 C_3^2 |v - u|^2.$$

Thus we get the Lipschitz continuity of $F(\cdot)$. In the same vein, we conclude the Lipschitz continuity of $G(\cdot)$. Thus according to the existence and uniqueness conditions of McKean-Vlasov dynamics in [3], the system (D.1)-(D.2) is well-defined. \qed