Continuous-time block-monotone Markov chains and their block-augmented truncations

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Abstract

This paper considers continuous-time block-monotone Markov chains (BMMCs) and their block-augmented truncations. We first introduce the block monotonicity and block-wise dominance relation for continuous-time Markov chains, and then provide some fundamental results on the two notions. Using these results, we show that the stationary distribution vectors obtained by the block-augmented truncation converge to the stationary distribution vector of the original BMMC. We also show that the last-column-block-augmented truncation (LC-block-augmented truncation) provides the best (in a certain sense) approximation to the stationary distribution vector of a BMMC among all the block-augmented truncations. Furthermore, we present computable upper bounds for the total variation distance between the stationary distribution vectors of a Markov chain and its LC-block-augmented truncation, under the assumption that the original Markov chain itself may not be block-monotone but is block-wise dominated by a BMMC with exponential ergodicity. Finally, we apply the obtained bounds to a queue with a batch Markovian arrival process and state-dependent departure rates.

Keywords: Block-monotone Markov chain; Block-augmented truncation; Total-variation-distance error bound; GI/G/1-type Markov chain; Level-dependent QBD (LD-QBD); Exponential ergodicity

Mathematics Subject Classification: 60J27; 60J22

1 Introduction

This paper considers continuous-time block-structured Markov chains characterized by an infinite number of block matrices, such as GI/G/1-type Markov chains (including M/G/1- and GI/M/1-type ones) [4, 14, 23] and level-dependent quasi-birth-and-death processes (LD-QBDs) [14]. It is, in general, difficult to calculate the stationary distribution vectors of such Markov chains. A simple and practical solution to this problem is to
adopt the augmented northwest-corner truncations of the infinitesimal generators (resp. the transition probability matrices) in order to compute the stationary distribution vectors of continuous-time (resp. discrete-time) Markov chains \[8, 9\]. Naturally, the stationary distribution vector obtained by the augmented northwest-corner truncation is an approximation to the original stationary distribution vector. Therefore, it is important to estimate the error caused by the augmented northwest-corner truncation.

In fact, such error estimation is facilitated by using the (stochastic) monotonicity of Markov chains (see, e.g., \[7\]). Indeed, it is shown \[8\] Theorem 1] that the last-column-augmented northwest-corner truncation (last-column-augmented truncation, for short) yields the best (in a certain sense) approximation to the stationary distribution vector of a discrete-time monotone Markov chain. In addition, there have been some studies on the total-variation-distance error bound, i.e., upper bound for the total variation distance between the stationary distribution vectors of the original Markov chain and its last-column-augmented truncation. Tweedie \[25\] assumed that the original Markov chain is monotone and geometrically ergodic, and then derived a computable total-variation-distance error bound. Liu \[16\] presented such a bound, assuming the monotonicity and polynomial ergodicity of the original Markov chain. On the other hand, without the monotonicity, Hervé and Ledoux \[10\] developed a total-variation-distance error bound for the stationary distribution vector obtained approximately by the last-column-augmented truncation of a geometrically ergodic Markov chain, though the bound includes the second largest eigenvalue of the truncated and augmented transition probability matrix. Therefore, Hervé and Ledoux \[10\]’s bound is not easy to compute, compared with the bounds presented by Tweedie \[25\] and Liu \[16\].

We have seen that the monotonicity is useful for the error estimation of the augmented northwest-corner truncations. However, the monotonicity is a somewhat strong restriction on block-structured Markov chains. Thus, Li and Zhao \[15\] introduced the block monotonicity of discrete-time block-structured Markov chains. The block monotonicity is an extension of the monotonicity to block-structured Markov chains. Li and Zhao \[15\] also proved (see Theorem 3.6 therein) that the last-column-block-augmented northwest-corner truncation (LC-block-augmented truncation, for short) yields the best approximation to the stationary distribution vector of the block-monotone Markov chain (BMMC) among all the block-augmented northwest-corner truncations (called block-augmented truncations, for short). Masuyama \[19, 20\] presented computable upper bounds for the total variation distance between the stationary distribution vectors of the original BMMC and its LC-block-augmented truncation in the cases where the original BMMC satisfies the geometric and subgeometric drift conditions. The bounds presented in \[19, 20\] are the generalization of those in \[25, 16\].

The existing results reviewed above are established for discrete-time BMMCs. These results can be applied to continuous-time Markov chains with bounded infinitesimal generators by the uniformization technique \[24\] Section 4.5.2]. As for the continuous-time case, Zeifman et al. \[29\] presented an error bound for the periodic stationary distribu-
tion obtained by the truncation of a periodic and exponentially weakly ergodic non-time-
homogeneous birth-and-death process with bounded transition rates (see also [27, 28]).
Hart and Tweedie [9] provided some sets of conditions, under which the stationary distri-
bution vectors of the augmented northwest-corner truncations of a continuous-time mono-
tone Markov chain converge to the stationary distribution vector of the original Markov
chain.

In this paper, we consider continuous-time block-structured Markov chains with possi-
ibly unbounded infinitesimal generators. We first provide fundamental results on the block
monotonicity and block-wise dominance relation for continuous-time block-structured
Markov chains. Next, we present the definition of the block-augmented truncation and
LC-block-augmented truncation of continuous-time block-structured Markov chains. We
then prove that the LC-block-augmented truncation of a BMMC is the best among all the
block-augmented truncations of the BMMC. We also present computable total-variation-
distance error bounds for the stationary distribution vector obtained approximately by the
LC-block-augmented truncation of a block-structured Markov chain, under the assump-
tion that the original Markov chain is block-wise dominated by a BMMC with expo-
nential ergodicity. Finally, we apply the obtained bounds to the queue length process in a
queueing model with a batch Markovian arrival process (BMAP) [17] and state-dependent
departure rates.

The rest of this paper is divided into five sections. Section 2 introduces basic defini-
tions and notation. Section 3 provides fundamental results associated with continuous-
time BMMCs. Section 4 discusses the block-augmented truncations. Section 5 presents
error bounds for the stationary distribution vector obtained by the LC-block-augmented
truncation. Section 6 applies the error bounds to a queueing model.

2 Basic definitions and notation

Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\} = \{1, 2, 3, \ldots\}$ and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Furthermore,
let $\mathbb{Z}_{\leq N} = \{0, 1, \ldots, N\}$ and $\mathbb{F}_{\leq N} = \mathbb{Z}_{\leq N} \times \mathbb{D}$ for $N \in \overline{\mathbb{N}}$, where $\mathbb{D} = \{1, 2, \ldots, d\} \subset \mathbb{N}$.
Note here that $\mathbb{Z}_{\leq \infty} = \mathbb{Z}_+$. For simplicity, we write $\mathbb{F}$ for $\mathbb{F}_{\leq \infty}$ and $(k, i; \ell, j)$ for ordered pair $((k, i), (\ell, j))$.

We define $I_d$ as the $d \times d$ identity matrix. We may write $I$ for the identity matrix when
its order is clear from the context. We also define $O$ as the zero matrix. Furthermore, let $T_d := T_d^{\leq \infty}$ denote

$$T_d = \begin{pmatrix}
I_d & O & O & O & \cdots \\
I_d & I_d & O & O & \cdots \\
I_d & I_d & I_d & O & \cdots \\
I_d & I_d & I_d & I_d & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

and $T_d^{\leq N}$, $N \in \mathbb{Z}_+$, denote the $|\mathbb{F}_{\leq N}| \times |\mathbb{F}_{\leq N}|$ northwest-corner truncation of $T_d$, where
| · | represents the cardinality of the set between the vertical bars. It is easy to see that

\[ T_d^{-1} = \begin{pmatrix} I_d & O & O & O & \cdots \\ -I_d & I_d & O & O & \cdots \\ O & -I_d & I_d & O & \cdots \\ O & O & -I_d & I_d & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \]

and that \((T_d^{\leq N})^{-1}\) is equal to the \(|\mathbb{P}^{\leq N}| \times |\mathbb{P}^{\leq N}|\) northwest-corner truncation of \(T_d^{-1}\).

We now introduce the block monotonicity and block-wise dominance relation for probability vectors and stochastic matrices, and provide the definition of block-increasing column vectors. To this end, we suppose \(N \in \mathbb{N}\). We then define \(\mu = (\mu(k,i))_{(k,i) \in \mathbb{F}^{\leq N}}\) and \(\eta = (\eta(k,i))_{(k,i) \in \mathbb{F}^{\leq N}}\) as arbitrary probability vectors with block size \(d\). We also define \(P = (p(k,i;\ell,j))_{(k,i), (\ell,j) \in \mathbb{F}^{\leq N}}\) and \(\tilde{P} = (\tilde{p}(k,i;\ell,j))_{(k,i), (\ell,j) \in \mathbb{F}^{\leq N}}\) as arbitrary stochastic matrices with block size \(d\).

**Definition 2.1** The probability vector \(\mu\) is said to be block-wise dominated by the probability vector \(\eta\) (denoted by \(\mu \prec_d \eta\)) if \(\mu T^{\leq N} \leq \eta T^{\leq N}\).

**Definition 2.2 (Definition 1.1 and Proposition 2.1, [19])** The stochastic matrix \(P\) and Markov chains characterized by \(P\) are said to be block-monotone with block size \(d\) if \((T_d^{\leq N})^{-1} PT_d^{\leq N} \geq O\), or equivalently, if

\[
\sum_{m=\ell}^{N} p(k,i;m,j) \leq \sum_{m=\ell}^{N} p(k+1,i;m,j), \quad (k,i) \in \mathbb{F}^{\leq N-1}, \quad (\ell,j) \in \mathbb{F}^{\leq N}.
\]

The set of block-monotone stochastic matrices with block size \(d\) is denoted by \(\text{BM}_d\).

**Definition 2.3** The stochastic matrix \(P\) is said to be block-wise dominated by the stochastic matrix \(\tilde{P}\) (denoted by \(P \prec_d \tilde{P}\)) if \(PT_d^{\leq N} \leq \tilde{PT}_d^{\leq N}\).

**Definition 2.4 (Definition 2.1, [15])** A column vector \(f = (f(k,i))_{(k,i) \in \mathbb{F}^{\leq N}}\) is said to be block-increasing with block size \(d\) if \((T_d^{\leq N})^{-1} f \geq 0\), i.e., \(f(k,i) \leq f(k+1,i)\) for all \((k,i) \in \mathbb{Z}_+^{\leq N-1} \times \mathbb{D}\). The set of column vectors block-increasing with block size \(d\) by \(\text{Bl}_d\).

Finally, we present a basic result on block-monotone stochastic matrices.

**Proposition 2.1 (Proposition 2.2, [19])** The following are equivalent:

(a) \(P \in \text{BM}_d\);

(b) \(\mu P \prec_d \eta P\) for any two probability vectors \(\mu\) and \(\eta\) such that \(\mu \prec_d \eta\); and

(c) \(P f \in \text{Bl}_d\) for any \(f \in \text{Bl}_d\).
3 Block-monotone continuous-time Markov chains

In this section, we first provide the basic assumption and characterization of a continuous-time block-structured Markov chain. We then describe the block monotonicity and block-wise dominance relation for the infinitesimal generators of continuous-time block-structured Markov chains. We also present some fundamental results on the block monotonicity and block-wise dominance relation.

3.1 Block-structured Markov chains

Let \( \{(X_t, J_t); t \geq 0\} \) denote a continuous-time Markov chain with state space \( \mathbb{F} \leq N \), where \( N \in \mathbb{N} \). Let \( P^{(t)} = (p^{(t)}(k, i; \ell, j))_{(k, i), (\ell, j) \in \mathbb{F} \leq N} \) denote the transition matrix function of \( \{(X_t, J_t); t \geq 0\} \), i.e.,

\[
p^{(t)}(k, i; \ell, j) = P(X_t = \ell, J_t = j \mid X_0 = k, J_0 = i), \quad t \geq 0.
\]

(3.1)

It is known that 
\[
P^{(t+s)} = P^{(t)} P^{(s)} = P^{(s)} P^{(t)}, \quad t, s \geq 0,
\]

(3.2)

which is called the Chapman-Kolmogorov equation [5, Chapter 8, Section 2.1].

We assume that \( P^{(t)} \) is continuous (or standard), i.e., \( \lim_{t \downarrow 0} P^{(t)} = I \) (see [5, Chapter 8, Section 2.2] and [1, Definition at p. 5]). It then follows from [1, Section 1.2, Proposition 2.2] that, for all \( (k, i) \in \mathbb{F} \leq N \), \( q_{(k, i)} := \lim_{t \downarrow 0}(1 - p^{(t)}(k, i; k, i)) / t \geq 0 \) exists. Although \( q_{(k, i)} \) is possibly infinite, we assume in what follows that 

\[
q_{(k, i)} < \infty \quad \text{for all } (k, i) \in \mathbb{F} \leq N,
\]

that is, all the states in the state space \( \mathbb{F} \leq N \) are stable [1, Definition at p. 9]. Thus, it follows from [1, Section 1.2, Proposition 2.4 and Corollary 2.5] that \( P^{(t)} \) satisfies the Kolmogorov forward differential equation (3.3) and the backward differential equation (3.4):

\[
\frac{d}{dt} P^{(t)} = P^{(t)} Q,
\]

(3.3)

\[
\frac{d}{dt} P^{(t)} = Q P^{(t)},
\]

(3.4)

where \( Q := (q(k, i; \ell, j))_{(k, i), (\ell, j) \in \mathbb{F} \leq N} \) is a matrix whose elements are all finite, which is given by

\[
Q = \lim_{t \downarrow 0} \frac{P^{(t)} - I}{t}.
\]

(3.5)

Note here that \( q_{(k, i)} = |Q(k, i; k, i)| < \infty \) for all \( (k, i) \in \mathbb{F} \leq N \). The matrix \( Q \) is called the infinitesimal generator [5, Chapter 8, Definition 2.3] of the Markov chain \( \{(X_t, J_t); t \geq 0\} \) and the transition matrix function \( P^{(t)} \). In general, the infinitesimal generator is a diagonally dominant matrix with nonpositive diagonal and nonnegative off-diagonal elements.
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Such a matrix is referred to as the $q$-matrix \cite[Definitions at p. 13 and p. 64]{ref1}.

For further discussion, we assume that $Q$ is conservative \cite[Definition at p. 13]{ref1}, i.e.,

$$Qe = 0,$$

where $e$ is a column vector of 1’s.

In the rest of this paper, we proceed under Assumption\ref{assumption:3.1} below, which is a summary of the assumptions made above.

**Assumption 3.1** (i) $P^{(t)}$ is continuous; and (ii) $Q$ is stable and conservative.

We now mention an important notion for infinitesimal generators (or $q$-matrices). The infinitesimal generator $Q$ is said to be regular (or non-explosive) if the equation

$$Qx = \gamma x, \quad 0 \leq x \leq e$$

has no nontrivial solution for some (and thus all) $\gamma > 0$ (see \cite[Chapter 8, Theorem 4.4]{ref5} and \cite[Section 2.2, Theorem 2.7]{ref1}). In fact, $Q$ is regular if and only if $\{(X_t, J_t); t \geq 0\}$ is a regular-jump process \cite[Chapter 8, Definition 2.5]{ref5}. Furthermore, if $Q$ is regular, then Assumption\ref{assumption:3.1} holds \cite[Chapter 8, Definition 2.4 and Theorem 3.4]{ref5} and thus $P^{(t)}e = e$ for all $t \geq 0$ and $\{P^{(t)}\}$ is the unique solution of both equations \eqref{eq:equations3.3} and \eqref{eq:equations3.4} \cite[Corollary 2.5 and Theorems 2.2 and 2.7 of Section 2.2 and Definition at p. 81]{ref1}.

**Remark 3.1** If $Q$ is bounded \cite[Section 4.5.2]{ref24}, then

$$P^{(t)} = \sum_{m=0}^{\infty} \frac{(Qt)^m}{m!} = \exp\{Qt\}.$$

Finally, we introduce the definition of a stationary distribution vector (or stationary distribution) of the Markov chain $\{(X_t, J_t)\}$.

**Definition 3.1** Let $\pi = (\pi(k, i))_{(k, i) \in F \times N}$ denote a probability vector such that

$$\pi P^{(t)} = \pi \quad \text{for all } t \geq 0.$$

The vector $\pi$ is called a stationary distribution vector (or stationary distribution) of the Markov chain $\{(X_t, J_t)\}$ and the transition matrix function $P^{(t)}$ (see \cite[Definition at pp. 159–160]{ref1}).

**Remark 3.2** Suppose that the Markov chain $\{(X_t, J_t)\}$ and thus $Q$ are irreducible. It then holds that $\{(X_t, J_t)\}$ and $Q$ are positive recurrent if and only if there exists a stationary distribution vector of $\{(X_t, J_t)\}$ and $P^{(t)}$ \cite[Section 5.1, Proposition 1.7]{ref1}. Furthermore,
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it is known [1, Section 5.4, Theorem 4.5] that if \{ (X_t, J_t) \} is ergodic (i.e., irreducible and positive recurrent) then its stationary distribution vector \( \pi \) satisfies

\[
\pi Q = 0, \quad (3.8)
\]

\[
\lim_{t \to \infty} P^{(t)} = e \pi, \quad (3.9)
\]

and (3.9) implies that \( \pi \) is the unique stationary distribution vector.

**Remark 3.3** If \( Q \) is ergodic, then \( Q \) is regular. Indeed, suppose that \( Q \) is ergodic but is not regular. Under this assumption, the equation (3.7) has a nontrivial solution \( x \geq 0, \neq 0 \) for some \( \gamma > 0 \). Pre-multiplying (3.7) by \( \pi \) and using (3.8), we have \( 0 = \gamma (\pi x) > 0 \), which yields a contradiction. Consequently, the ergodicity of \( Q \) implies that \( Q \) is regular.

### 3.2 Block monotonicity and block-wise dominance for infinitesimal generators

In this subsection, we present the fundamental results on the block monotonicity and block-wise dominance relation for \( |F| \times |F| \)-infinitesimal generators, where \( N \in \mathbb{N} \).

To this end, we introduce another Markov chain \( \{ (\tilde{X}_t, \tilde{J}_t); t \geq 0 \} \) with state space \( \mathbb{F}^{\leq N} \) and infinitesimal generator \( \tilde{Q} := \left( \tilde{q}(k, i; \ell, j) \right)_{(k, i), (\ell, j) \in \mathbb{F}^{\leq N}} \). We then define \( \tilde{P}^{(t)} := \left( \tilde{p}^{(t)}(k; i, \ell, j) \right)_{(k, i), (\ell, j) \in \mathbb{F}^{\leq N}} \) as the transition matrix function of the Markov chain \( \{ (\tilde{X}_t, \tilde{J}_t) \} \).

By definition,

\[
\tilde{p}^{(t)}(k; i, \ell, j) = P(\tilde{X}_t = \ell, \tilde{J}_t = j | \tilde{X}_0 = k, \tilde{J}_0 = i),
\]

for \( t \geq 0 \) and \((k; i, \ell, j) \in \mathbb{F}^{\leq N} \times \mathbb{F}^{\leq N} \). In addition, we define \( \mathbb{B}^{\leq N} \), \( N \in \mathbb{N} \), as

\[
\mathbb{B}^{\leq N} = (\mathbb{F}^{\leq N} \times \mathbb{F}^{\leq N}) \setminus \{(k, i; k, i); (k, i) \in \mathbb{F}^{\leq N}\},
\]

and write \( \mathbb{B} \) for \( \mathbb{B}^{\leq \infty} \).

We now provide the definition of the block monotonicity and block-wise dominance.

**Definition 3.2** The infinitesimal generator \( Q \) and Markov chains characterized by \( Q \) are said to be block-monotone with block size \( d \) (denoted by \( Q \in \text{BM}_d \)) if all the off-diagonal elements of \( (T^{\leq N}_d)^{-1} QT^{\leq N}_d \) are nonnegative, i.e.,

\[
\sum_{m=\ell}^{N} q(k - 1, i; m, j) \leq \sum_{m=\ell}^{N} q(k, i; m, j), \quad (k, i; \ell, j) \in \mathbb{B}^{\leq N} \text{ with } k \in \mathbb{N}.
\]

**Definition 3.3** The infinitesimal generator \( Q \) is said to be block-wise dominated by \( \tilde{Q} \) (denoted by \( Q \prec_d \tilde{Q} \)) if \( QT^{\leq N}_d \leq \tilde{Q} T^{\leq N}_d \).

In what follows, we present five lemmas: Lemmas 3.1–3.5. For the respective lemmas, we give the proofs in the case where \( N = \infty \) only, which can be applied to the case where \( N < \infty \), with minor modifications.
Lemma 3.1 If $Q \in \text{BM}_d$, then (a) $\xi(i, j) := \sum_{\ell \in \mathbb{Z}^+} q(k, i; \ell, j)$ is constant with respect to $k \in \mathbb{Z}^{<N}$; and (b) $\{J_t; t \geq 0\}$ is a Markov chain with state space $\mathbb{D}$ and infinitesimal generator $\Xi = (\xi(i, j))_{i, j \in \mathbb{D}}$.

Proof. We first prove statement (a). Since $Q \in \text{BM}_d$ (see Definition 3.2), we have

$$\sum_{\ell = 0}^{\infty} q(k, i; \ell, j) \leq \sum_{\ell = 0}^{\infty} q(k + 1, i; \ell, j) \quad \text{for all } (k, i) \in \mathbb{F} \text{ and } j \in \mathbb{D}.$$  

Combining this and (3.6) yields

$$0 = \sum_{(i, j) \in \mathbb{F}} q(k, i; \ell, j) \leq \sum_{(i, j) \in \mathbb{F}} q(k + 1, i; \ell, j) = 0 \quad \text{for all } (k, i) \in \mathbb{F},$$

which implies that, for each $(i, j) \in \mathbb{D}^2$, $\sum_{\ell = 0}^{\infty} q(k, i; \ell, j)$ is constant with respect to $k \in \mathbb{Z}^+$, i.e.,

$$\xi(i, j) = \sum_{\ell = 0}^{\infty} q(k, i; \ell, j) \quad \text{for all } (k, i) \in \mathbb{F} \text{ and } j \in \mathbb{D}. \quad (3.10)$$

Therefore, statement (a) holds.

Next, we prove statement (b). Let $p_{k}^{(t)}(i, j) = P(J_t = j \mid X_0 = k, J_0 = i)$ for $k \in \mathbb{Z}^+$ and $i, j \in \mathbb{D}$. It then follows from (3.1) that

$$p_{k}^{(t)}(i, j) = \sum_{\ell = 0}^{\infty} p^{(t)}(k, i; \ell, j), \quad k \in \mathbb{Z}^+, \ i, j \in \mathbb{D}. \quad (3.11)$$

Since $P^{(t+s)} = P^{(s)}P^{(t)}$ (see (3.2)) and $P^{(t)}e \leq e$, we have

$$\sum_{(i, j) \in \mathbb{F}} \frac{|p^{(t+s)}(k, i; \ell, j) - p^{(t)}(k, i; \ell, j)|}{s} \leq \frac{1 - p^{(s)}(k, i; k, i)}{s} p^{(t)}(k, i; \ell, j) + \sum_{(\ell', j') \in \mathbb{F}_s \setminus \{(k, i)\}} \frac{p^{(s)}(k, i; \ell', j')}{s} p^{(t)}(\ell', j'; \ell, j) \leq \frac{1 - p^{(s)}(k, i; k, i)}{s} + \sum_{(\ell', j') \in \mathbb{F}_s \setminus \{(k, i)\}} \frac{p^{(s)}(k, i; \ell', j')}{s} \leq \frac{2\{1 - p^{(s)}(k, i; k, i)\}}{s} \leq 2|q(k, i; k, i)|, \quad t \geq 0, \ s > 0, \ (k, i) \in \mathbb{F}, \quad (3.12)$$

where the last inequality holds due to [6] Theorem II.3.1. It also follows from (3.11) and (3.12) that, for $t \geq 0, s > 0, k \in \mathbb{Z}^+$ and $i, j \in \mathbb{D}$,

$$\frac{|p_{k}^{(t+s)}(i, j) - p_{k}^{(t)}(i, j)|}{s} \leq \sum_{(i, j) \in \mathbb{F}} \frac{|p^{(t+s)}(k, i; \ell, j) - p^{(t)}(k, i; \ell, j)|}{s} \leq 2q(k, i; k, i).$$
Thus, combining (3.3), (3.11) and the dominated convergence theorem yields, for $t \geq 0$, $k \in \mathbb{Z}_+$ and $i, j \in \mathcal{D}$,

$$
\frac{d}{dt}p_k^{(t)}(i, j) = \lim_{s \downarrow 0} \frac{p_k^{(t+s)}(i, j) - p_k^{(t)}(i, j)}{s} \\
= \sum_{\ell=0}^{\infty} \lim_{s \downarrow 0} \frac{p_k^{(t+s)}(k; \ell, j) - p_k^{(t)}(k; \ell, j)}{s} \\
= \sum_{\ell=0}^{\infty} \sum_{(\ell', j') \in \mathcal{F}} p_k^{(t)}(k; i, j') q(\ell', j'; \ell, j) \\
= \sum_{(\ell', j') \in \mathcal{F}} p_k^{(t)}(k; i, j') \sum_{\ell=0}^{\infty} q(\ell', j'; \ell, j).
$$

Substituting (3.10) and (3.11) into the above equation, we obtain

$$
\frac{d}{dt}p_k^{(t)}(i, j) = \sum_{j' \in \mathcal{D}} \sum_{\ell \in \mathbb{Z}_+} p_k^{(t)}(k; i, \ell, j') q(\ell, j'; j, j) \\
= \sum_{j' \in \mathcal{D}} p_k^{(t)}(i, j') \xi(j', j), \quad t \geq 0, \; k \in \mathbb{Z}_+, \; i, j \in \mathcal{D}.
$$

Therefore,

$$
P(J_t = j \mid X_0 = k, J_0 = i) = p_k^{(t)}(i, j) = \left[\exp\{\Xi t\}\right]_{i,j}, \quad t \geq 0, \; k \in \mathbb{Z}_+, \; i, j \in \mathcal{D},
$$

where $[\cdot]_{i,j}$ denotes the $(i, j)$th element of the $|\mathcal{D}| \times |\mathcal{D}|$ matrix in the square brackets. In addition, from (3.13), we have

$$
P(J_t = j \mid J_0 = i) = \sum_{k=0}^{\infty} \left[\exp\{\Xi t\}\right]_{i,j} P(X_0 = k \mid J_0 = i) \\
= \left[\exp\{\Xi t\}\right]_{i,j}, \quad t \geq 0, \; i, j \in \mathcal{D}.
$$

Note here that $\Xi$ is a conservative $q$-matrix (i.e., $\Xi e = 0$), because $\Xi$ satisfies (3.10) and $Q$ is the infinitesimal generator of the Markov chain $\{(X_t, J_t)\}$. As a result, (3.14) shows that $\{J(t); t \geq 0\}$ is a Markov chain with state space $\mathcal{D}$ and infinitesimal generator $\Xi$. □

**Lemma 3.2** If $Q$ is regular, then the following are equivalent: (a) $Q \in \text{BM}_d$; and (b) $P^{(t)} \in \text{BM}_d$, i.e., $(T_{d}^{\leq N})^{-1}P^{(t)}T_{d}^{\leq N} \geq O$ for all $t \geq 0$.

**Proof.** Before the proof of this lemma, we introduce some symbols. Fix $n \in \mathbb{N}$ arbitrarily and let $t_n = \inf\{t \geq 0 : X_t \geq n\}$. Since the Markov chain $\{(X_t, J_t)\}$ is regular,

$$
P(\lim_{n \to \infty} t_n = \infty) = 1.
$$
Thus, we define \( \{ (X_t^{\leq n}, J_t^{\leq n}) ; t \geq 0 \} \) as a Markov chain with state space \( \mathbb{F}^{\leq n} \) such that
\[
X_t^{\leq n} = \begin{cases} 
X_t, & 0 \leq t < t_n, \\
n, & t \geq t_n,
\end{cases} \quad J_t^{\leq n} = J_t, \quad t \geq 0. \tag{3.16}
\]

We also define \( Q^{\leq n} = (q^{\leq n}(k; i, \ell, j))_{(k, i), (\ell, j) \in \mathbb{F}^{\leq n}} \) as the infinitesimal generator of \( \{ (X_t^{\leq n}, J_t^{\leq n}) \} \). It then follows that, for \( i, j \in \mathbb{B} \),
\[
q^{\leq n}(k; i, \ell, j) = \begin{cases} 
q(k; i, \ell, j), & k, \ell \in \mathbb{Z}^{\leq n-1}_+, \\
\sum_{m=1}^{\infty} q(k; i, m, j), & k \in \mathbb{Z}^{\leq n-1}_+, \ell = n, \\
\sum_{m=0}^{\infty} q(n; i, m, j) = \xi(i, j), & k = \ell = n, \\
0, & \text{otherwise}.
\end{cases} \tag{3.17}
\]

Furthermore, let \( p^{\leq n; (t)}(k; i, \ell, j) \) denote
\[
\begin{align*}
p^{\leq n; (t)}(k; i, \ell, j) &= P(X_t^{\leq n} = \ell, J_t^{\leq n} = j \mid X_0^{\leq n} = k, J_0^{\leq n} = i), \\
&= \sum_{m=\ell}^{\infty} q(k-1; i, m, j) - \sum_{m=\ell}^{\infty} q(k; i, m, j), \quad (k, i, \ell, j) \in \mathbb{B} \text{ with } k \in \mathbb{N}.
\end{align*} \tag{3.19}
\]

From (3.17) and (3.19), we also have
\[
\sum_{m=\ell}^{n} q^{\leq n}(k-1; i, m, j) \\
= \sum_{m=\ell}^{\infty} q(k-1; i, m, j) - \sum_{m=\ell}^{\infty} q(k; i, m, j) \\
= \sum_{m=\ell}^{n} q^{\leq n}(k; i, m, j), \quad (k, i, \ell, j) \in \mathbb{B}^{\leq n} \text{ with } k \in \mathbb{N}. \tag{3.20}
\]

The inequality (3.20) implies that all the off-diagonal elements of \( (T_d^{\leq n})^{-1}Q^{\leq n}T_d^{\leq n} \) are nonnegative. Thus, we can choose \( \sigma_n \in (0, \infty) \) such that \( (T_d^{\leq n})^{-1}(I + \sigma_n^{-1}Q^{\leq n})T_d^{\leq n} \geq O \), which yields
\[
(T_d^{\leq n})^{-1}\exp\{Q^{\leq n}t\}T_d^{\leq n} = \sum_{m=0}^{\infty} e^{-\sigma_nt} \frac{(\sigma_n t)^m}{m!} (T_d^{\leq n})^{-1}(I + \sigma_n^{-1}Q^{\leq n})^m T_d^{\leq n} \geq O, \quad t \geq 0. \tag{3.21}
\]
It follows from (3.21) that, for \((k, i) \in \mathbb{F}^{\leq n-1}\) and \((\ell, j) \in \mathbb{F}^{\leq n}\),
\[
\sum_{m=\ell}^{n} \left\{ p^{\leq m}(t)(k + 1, i; m, j) - p^{\leq m}(t)(k, i; m, j) \right\} \geq 0, \quad t \geq 0. \tag{3.22}
\]

It also follows from (3.15) and (3.16) that, for any fixed \(T > 0\), the process \(\{X_t^{\leq n}, J_t^{\leq n}; 0 \leq t < T\}\) converges to the process \(\{X_t, J_t; 0 \leq t \leq T\}\) with probability one (w.p.1) as \(n \to \infty\) and thus, for each \((k, i; m, j) \in \mathbb{F}^2\),
\[
\lim_{n \to \infty} p^{\leq n}(t)(k, i; m, j) = p(k, i; m, j) \quad \text{for all } t \in [0, T). \tag{3.23}
\]

Applying the dominated convergence theorem to (3.22) and using (3.23) yield
\[
\sum_{m=\ell}^{\infty} \left\{ p^{(t)}(k + 1, i; m, j) - p^{(t)}(k, i; m, j) \right\} \geq 0, \quad 0 \leq t \leq T; \ (k, i; \ell, j) \in \mathbb{F}^2,
\]
where \(T > 0\) is arbitrarily fixed. Note here that
\[
\sum_{m=\ell}^{\infty} p^{(t)}(0, i; m, j) \geq 0, \quad t \geq 0, \ i \in \mathbb{D}, \ (\ell, j) \in \mathbb{F}.
\]

As a result, we obtain \(T^{-1}_d P^{(t)} T_d \geq \mathbf{O}\) for all \(t \geq 0\).

Next, we prove that statement (b) implies statement (a). To this end, we consider the limit \(\lim_{t \downarrow 0} T^{-1}_d (P^{(t)} - I) T_d / t\), that is,
\[
\lim_{t \downarrow 0} \sum_{m=\ell}^{\infty} \frac{p^{(t)}(0, i; m, j) - \chi(0, i)(m, j)}{t}, \quad (0, i; \ell, j) \in \mathbb{F}^2,
\]
and
\[
\lim_{t \downarrow 0} \sum_{m=\ell}^{\infty} \left[ \frac{p^{(t)}(k, i; m, j) - \chi(k, i)(m, j)}{t} \right. \\
- \left. \frac{p^{(t)}(k - 1, i; m, j) - \chi(k - 1, i)(m, j)}{t} \right], \quad (k, i; \ell, j) \in \mathbb{F}^2 \text{ with } k \in \mathbb{N},
\]
where \(\chi(k, i)(\ell, j), (k, i; \ell, j) \in \mathbb{F}^2\), is given by
\[
\chi(k, i)(\ell, j) = \begin{cases} 1, & (k, i) = (\ell, j), \\ 0, & (k, i) \neq (\ell, j). \end{cases}
\]

For all \((k, i) \in \mathbb{F}\), we have
\[
\sum_{(m, j) \in \mathbb{F}} \left| p^{(t)}(k, i; m, j) - \chi(k, i)(m, j) \right| \\
\leq 1 - p^{(t)}(k, k; i) + \sum_{(m, j) \in \mathbb{F} \setminus \{(k, i)\}} p^{(t)}(k, i; m, j) \\
\leq 2 \{1 - p^{(t)}(k, i; k, i)\} \leq 2t \left| q(k, i; k, i) \right|,
\]
where the last inequality follows from [6, Theorem II.3.1]. Therefore, using dominated convergence theorem and (3.5), we obtain
\[
\lim_{t \to 0} \frac{T_d^{-1}(P^{(t)} - I)T_d}{t} = T_d^{-1}QT_d.
\]
Note here that
\[T_d^{-1}dP^{(t)}T_d \geq O\] (due to statement (b)), which implies that all the off-diagonal elements of \(T_d^{-1}QT_d\) are nonnegative, i.e., \(Q \in BM_d\). Consequently, statement (a) holds.

We now make the following assumption, in addition to Assumption 3.1.

**Assumption 3.2** Suppose that \(Q \prec_d \tilde{Q}\) and either \(Q \in BM_d\) or \(\tilde{Q} \in BM_d\).

**Lemma 3.3** Suppose that Assumption 3.2 holds. It then holds that
\[
\xi(\ell, j) = \sum_{k \in \mathbb{Z}^{<N}_+} q(k; i, \ell, j) = \sum_{k \in \mathbb{Z}^{<N}_+} \tilde{q}(k; i, \ell, j), \quad k \in \mathbb{Z}^{<N}_+, i, j \in \mathbb{D},
\]
which is constant with respect to \(k\). Furthermore, \(\Xi = (\xi(i, j))_{i, j \in \mathbb{D}}\) is the common infinitesimal generator of the Markov chains \(\{J_t; t \geq 0\}\) and \(\{\tilde{J}_t; t \geq 0\}\).

**Proof.** It follows from \(QT_d \leq \tilde{Q}T_d\) (see Definition 3.3) that
\[
\sum_{\ell=0}^{\infty} q(k; i, \ell, j) \leq \sum_{\ell=0}^{\infty} \tilde{q}(k; i, \ell, j), \quad k \in \mathbb{Z}^{<N}_+, i, j \in \mathbb{D}.
\]
Using this inequality, \(Qe = 0\) (due to Assumption 3.1) and \(\tilde{Q}e \leq 0\) (see [1, Section 1.2, Proposition 2.6]), we have
\[
0 = \sum_{\ell=0}^{\infty} \sum_{j \in \mathbb{D}} q(k; i, \ell, j) \leq \sum_{\ell=0}^{\infty} \sum_{j \in \mathbb{D}} \tilde{q}(k; i, \ell, j) \leq 0, \quad k \in \mathbb{Z}^{<N}_+, i \in \mathbb{D},
\]
which leads to
\[
\sum_{\ell=0}^{\infty} q(k; i, \ell, j) = \sum_{\ell=0}^{\infty} \tilde{q}(k; i, \ell, j), \quad k \in \mathbb{Z}^{<N}_+, i, j \in \mathbb{D}. \tag{3.24}
\]
Furthermore, it follows from Lemma 3.1 (a) and either \(Q \in BM_d\) or \(\tilde{Q} \in BM_d\) that, for each \((i, j) \in \mathbb{D}\), either of \(\sum_{\ell=0}^{\infty} q(k; i, \ell, j)\) and \(\sum_{\ell=0}^{\infty} \tilde{q}(k; i, \ell, j)\) is constant with respect to \(k \in \mathbb{Z}^{<N}_+\). As a result, both sides of (3.24) are constant with respect to \(k\). The remaining statement is immediate from Lemma 3.1 (b).

**Lemma 3.4** Suppose that Assumption 3.2 holds. Furthermore, if \(\tilde{Q}\) is regular, then

(a) \(Q\) is regular; and
(b) \(P(t) \prec_d \tilde{P}(t)\) for all \(t \geq 0\).

**Remark 3.4** Lemma [3.4](a) (b) is proved by using Lemma [3.4](a), and the latter is proved based on Lemma [A.2](a) (where \(N\) is assumed to be finite). Lemma [A.2](a) is proved without Lemma [3.4](a) or (b) whereas Lemma [A.2](b) (where \(N\) is possibly infinite) is proved by Lemma [3.4](b). For details, see the proof of Lemma [A.2] in Appendix A.

**Proof of Lemma 3.4.** We first provide some preliminaries to the proof of statement (a). Recall here that \(\{(X_t^{\leq n}, J_t^{\leq n}) ; t \geq 0\}\) is derived from \(\{(X_t, J_t) ; t \geq 0\}\), as shown in (3.16). Similarly, we define \(\{(\tilde{X}_t^{\leq n}, \tilde{J}_t^{\leq n}) ; t \geq 0\}\) as a Markov chain with state space \(\mathbb{F}^{\leq n}\) such that

\[
\tilde{X}_t^{\leq n} = \begin{cases} 
\tilde{X}_t, & 0 \leq t < \tilde{t}_n, \\
q, & t \geq \tilde{t}_n,
\end{cases} \quad \tilde{J}_t^{\leq n} = \tilde{J}_t, \quad t \geq 0,
\]

(3.25)

where \(\tilde{t}_n = \inf \{ t \geq 0 : \tilde{X}_t \geq n \}\). Let \(\tilde{P}^{\leq n;(t)}(k; i, \ell, j) = (\tilde{p}^{\leq n;(t)}(k; i, \ell, j))_{(k,i),(\ell,j) \in \mathbb{F}}, t \geq 0\), and \(\tilde{Q}^{\leq n} = (\tilde{q}^{\leq n}(k, i; \ell, j))_{(k,i),(\ell,j) \in \mathbb{F}}\) denote the transition matrix function and infinitesimal generator, respectively, of the Markov chain \(\{(\tilde{X}_t^{\leq n}, \tilde{J}_t^{\leq n})\}\), i.e.,

\[
\tilde{p}^{\leq n;(t)}(k; i, \ell, j) = P(\tilde{X}_t^{\leq n} = \ell, \tilde{J}_t^{\leq n} = j | \tilde{X}_0^{\leq n} = k, \tilde{J}_0^{\leq n} = i), \quad t \geq 0,
\]

(3.26)

\[
\tilde{q}^{\leq n}(k, i; \ell, j) = \lim_{t \downarrow 0} \frac{\tilde{p}^{\leq n;(t)}(k; i, \ell, j) - \chi(k,i)(\ell,j)}{t}.
\]

(3.27)

It then follows from (3.25) and Lemma [3.3] that, for \(i, j \in \mathbb{D}\),

\[
\tilde{q}^{\leq n}(k, i; \ell, j) = \begin{cases} 
\tilde{q}(k, i; \ell, j), & k, \ell \in \mathbb{Z}^{\leq n-1}, \\
\sum_{m=0}^{\infty} \tilde{q}(k, i; m, j), & k \in \mathbb{Z}^{n-1}, \ell = n, \\
\sum_{m=0}^{\infty} \tilde{q}(n, i; m, j) = \xi(i,j), & k = \ell = n, \\
0, & \text{otherwise.}
\end{cases}
\]

(3.27)

Using (3.17), (3.27) and \(QT_d \leq \tilde{Q}T_d\), we have

\[
Q^{\leq n} T_d^{\leq n} \leq \tilde{Q}^{\leq n} T_d^{\leq n}.
\]

(3.28)

Note here that \(Q \in \text{BM}_d\) (resp. \(\tilde{Q} \in \text{BM}_d\)) implies \(Q^{\leq n} \in \text{BM}_d\) (resp. \(\tilde{Q}^{\leq n} \in \text{BM}_d\)). Therefore, according toLemma [A.2](a), we assume, without loss of generality, that

\[
X_t^{\leq n} \leq \tilde{X}_t^{\leq n}, \quad J_t^{\leq n} = \tilde{J}_t^{\leq n} \quad \text{for all} \ t > 0,
\]

(3.29)

given that \(X_0^{\leq n} \leq \tilde{X}_0^{\leq n}\) and \(J_0^{\leq n} = \tilde{J}_0^{\leq n}\).

We now prove statement (a) by contradiction. To this end, we suppose that \(Q\) is not regular. Thus, there exist some \(T_\infty \in (0, \infty)\) and \((k_0, i_0) \in \mathbb{F}\) such that

\[
P(\cap_{n \geq k_0} \{ t_n \leq T_\infty \} | X_0 = k_0, J_0 = i_0) > 0.
\]

(3.30)
In addition, it follows from (3.16), (3.25) and (3.29) that if \( X_0 = \tilde{X}_0 = k_0 \) and \( J_0 = \tilde{J}_0 = i_0 \) then
\[
X_0^\leq n = X_0^\leq n = k_0, \quad J_0^\leq n = J_0^\leq n = i_0, \quad n > k_0,
\]
and
\[
\cap_{n > k_0} \{ t_n \leq T_\infty \} \Rightarrow \cap_{n > k_0} \{ X_t^\leq n \geq n \text{ for all } t \geq T_\infty \} \\
\Rightarrow \cap_{n > k_0} \{ \tilde{X}_t^\leq n \geq n \text{ for all } t \geq T_\infty \} \\
\Rightarrow \cap_{n > k_0} \{ \tilde{t}_n \leq T_\infty \}.
\]
Combining these and (3.30) yields
\[
P(\cap_{n > k_0} \{ \tilde{t}_n \leq T_\infty \} | \tilde{X}_0 = k_0, \tilde{J}_0 = i_0) > 0,
\]
which is inconsistent with the assumption that \( \tilde{Q} \) is regular. As a result, \( Q \) must be regular.

Next, we prove statement (b). According to statement (a), the two infinitesimal generators \( Q \) and \( \tilde{Q} \) are regular and thus
\[
Q e = \tilde{Q} e = 0, \quad (3.31)
\]
\[
P(\lim_{n \to \infty} t_n = \infty) = P(\lim_{n \to \infty} \tilde{t}_n = \infty) = 1. \quad (3.32)
\]
It follows from (3.17), (3.27) and (3.31) that \( I + \varsigma_n^{-1} Q^\leq n \) and \( I + \varsigma_n^{-1} \tilde{Q}^\leq n \) are stochastic, where
\[
\varsigma_n = \max_{(k,i) \in \mathbb{P}^\leq n} \max \left( |q^\leq n(k,i;\ell,j)|, |\tilde{q}^\leq n(k,i;\ell,j)| \right) < \infty.
\]
It also follows from (3.28) and [19, Proposition 2.3 (b)] that
\[
(I + \varsigma_n^{-1} Q^\leq n)^m T_d^\leq n \leq (I + \varsigma_n^{-1} \tilde{Q}^\leq n)^m T_d^\leq n, \quad m \in \mathbb{Z}^+_*,
\]
and thus, for \( t \geq 0 \),
\[
\exp\{Q^\leq nt\} T_d^\leq n = \sum_{m=0}^{\infty} \frac{e^{-ct}(\varsigma t)^m}{m!} (I + \varsigma_n^{-1} Q^\leq n)^m T_d^\leq n \leq \sum_{m=0}^{\infty} \frac{e^{-ct}(\varsigma t)^m}{m!} (I + \varsigma_n^{-1} \tilde{Q}^\leq n)^m T_d^\leq n = \exp\{\tilde{Q}^\leq nt\} T_d^\leq n. \quad (3.33)
\]
By definition (see (3.18) and (3.26)), \( p^\leq n;:\ell(t)(k,i;\ell,j) \) and \( \tilde{p}^\leq n;:\ell(t)(k,i;\ell,j) \) are equal to the \( (k,i;\ell,j) \)th elements of \( \exp\{Q^\leq nt\} \) and \( \exp\{\tilde{Q}^\leq nt\} \), respectively. Therefore, from (3.33), we have, for \( t \geq 0 \) and \( (k,i;\ell,j) \in \mathbb{P}^\leq n \times \mathbb{P}^\leq n \),
\[
\sum_{m=\ell}^{n} \{ \tilde{p}^\leq n;:\ell(t)(k,i;m,j) - p^\leq n;:\ell(t)(k,i;m,j) \} \geq 0. \quad (3.34)
\]
Furthermore, combining (3.16), (3.25) and (3.32), we obtain, for any fixed $T > 0$,

\[\lim_{n \to \infty} p^{<n;(t)}(k, i; \ell, j) = p(t)(k, i; \ell, j), \quad t \in [0, T), \ (k, i; \ell, j) \in \mathbb{F}^2, \]  
\[\lim_{n \to \infty} \tilde{p}^{<n;(t)}(k, i; \ell, j) = \tilde{p}(t)(k, i; \ell, j), \quad t \in [0, T), \ (k, i; \ell, j) \in \mathbb{F}^2. \]  

(3.35) (3.36)

Applying (3.35), (3.36) and the dominated convergence theorem to (3.34) yields

\[\sum_{m=\ell}^{\infty} \{ \tilde{p}(t)(k, i; m, j) - p(t)(k, i; m, j) \} \geq 0, \quad t \in [0, T), \ (k, i; \ell, j) \in \mathbb{F}^2. \]  

(3.37)

Letting $T \to \infty$ in the above inequality, we have $P^{(t)}T_d \leq \tilde{P}^{(t)}T_d$ for all $t \geq 0$. \hfill \Box

**Lemma 3.5** Suppose that Assumption 3.2 holds. Furthermore, suppose that $\tilde{Q}$ is regular and irreducible. Under these conditions, the following are true:

(a) If $\tilde{Q}$ is recurrent, then $Q$ has exactly one recurrent communicating class $C \subseteq \mathbb{F}^{<N}$ that includes the states $\{(0, i); i \in \mathbb{D}\}$, which is reachable from all the other states w.p.1.

(b) Furthermore, if $\tilde{Q}$ is positive recurrent, then the unique communicating class $C$ is positive recurrent and $\pi \prec_d \tilde{\pi}$, where $\pi := (\pi(k, i))_{(k, i)\in\mathbb{F}^{<N}}$ and $\tilde{\pi} := (\tilde{\pi}(k, i))_{(k, i)\in\mathbb{F}^{<N}}$ are the unique stationary distribution vectors of $Q$ and $\tilde{Q}$, respectively.

**Remark 3.5** An irreducible infinitesimal generator of a finite order is ergodic [5, Theorems 3.3 and 5.2 and Definitions 5.1 and 5.2] and thus is regular (see Remark 3.3). Therefore, if Assumption 3.2 holds for $N < \infty$ and $\tilde{Q}$ is irreducible, then statement (b) of Lemma 3.5 is true.

**Proof of Lemma 3.5** We first prove statement (a). To this end, we assume, without loss of generality, that the two Markov chains $\{(X_t, J_t); t \geq 0\}$ and $\{\tilde{(X}_t, \tilde{J}_t); t \geq 0\}$ are pathwise ordered as follows (see Lemma A.2):

\[X_t \leq \tilde{X}_t, \quad J_t = \tilde{J}_t \quad \text{for all} \ t \geq 0. \]  

(3.37)

It follows from (3.37), together with the irreducibility and recurrence of $\{\tilde{(X}_t, \tilde{J}_t)\}$, that $\{(X_t, J_t)\}$ can reach any state in $\{(0, i); i \in \mathbb{D}\}$ from all the states in the state space $\mathbb{F}$ w.p.1. Therefore, $Q$ has exactly one recurrent communicating class $C \subseteq \mathbb{F}$ such that $C \supseteq \{(0, i); i \in \mathbb{D}\}$.

Next, we prove statement (b). For this purpose, we additionally assume that $\tilde{Q}$ and thus $\{(\tilde{X}_t, \tilde{J}_t)\}$ are positive recurrent (i.e., ergodic), under which $\tilde{Q}$ has the unique stationary distribution vector $\tilde{\pi}$ (see Remark 3.2) and

\[\lim_{t \to \infty} \tilde{P}^{(t)} = e\tilde{\pi}. \]  

(3.38)
Furthermore, the ergodicity of \( \tilde{Q} \) and (3.37) imply that the mean first passage time of \( \{(X_t, J_t)\} \) to each state in \( \{(0, i); i \in D\} \) is finite for any given initial state, which leads to the result that the unique communicating class \( C \) of \( Q \) is positive recurrent. Therefore, it follows from [6, Theorems II.10.1 and II.10.2] and [1, Section 5.4, Theorem 4.5] that \( Q \) has the unique stationary distribution vector \( \pi \) and

\[
\lim_{t \to \infty} P^{(t)} = e \pi. \tag{3.39}
\]

It also follows from Lemma 3.4 (b) that \( P^{(t)} T_d \leq \tilde{P}^{(t)} T_d \) for \( t \geq 0 \). From this inequality together with (3.38), (3.39) and the dominated convergence theorem, we obtain \( e \pi T_d \leq e \tilde{\pi} T_d \) and thus \( \pi T_d \leq \tilde{\pi} T_d \). □

4 Block-augmented truncations

In this section, we discuss the block-augmented truncation of infinite-order block-structured infinitesimal generators. Thus, we assume that Assumption 3.1 holds for \( N = \infty \), i.e., \( Q \) is an \( |F| \times |F| \) stable and conservative infinitesimal generator.

We begin with the definition of the block-augmented truncation of \( Q \).

**Definition 4.1** Let \( (n)Q_* = (n)q_*(k, i; \ell, j) \) denote an infinitesimal generator such that, for \( i, j \in D \),

\[
\begin{align*}
(n)q_*(k, i; \ell, j) & \geq q(k, i; \ell, j), & k \in \mathbb{Z}_+, 0 \leq \ell \leq n, \\
(n)q_*(k, i; k, j) & = q(k, i; k, j), & k = \ell \geq n + 1, \\
(n)q_*(k, i; \ell, j) & = 0, & k \in \mathbb{Z}_+, \ell \geq n + 1, \ell \neq k, \\
\sum_{\ell=0}^{\infty} (n)q_*(k, i; \ell, j) & = \sum_{\ell=0}^{\infty} q(k, i; \ell, j), & k \in \mathbb{Z}_+.
\end{align*}
\]

The infinitesimal generator \( (n)Q_* \) is called a block-augmented northwest-corner truncation (block-augmented truncation, for short) of \( Q \).

Clearly, \( (n)Q_* \) has the following form:

\[
(n)Q_* = \begin{pmatrix}
(n)Q_*^{\leq n} & O & O & O & \cdots \\
* & * & O & O & \cdots \\
* & O & * & O & \cdots \\
* & O & O & * & \cdots \\
* & O & O & O & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \tag{4.1}
\]

where \( (n)Q_*^{\leq n} \) denotes the \( |F|^{\leq n} \times |F|^{\leq n} \) northwest-corner of \( (n)Q_* \). It may seem more reasonable to define \( (n)Q_*^{\leq n} \) as a block-augmented truncation of \( Q \), instead of \( (n)Q_* \).
Nevertheless, we adopt \((n)Q\) in order to perform algebraic operations on the original infinitesimal generator and its block-augmented truncation.

For further discussion, we assume that \(Q\) is irreducible, under which we present some fundamental results on the block-augmented truncation.

**Lemma 4.1** If \(Q\) is irreducible, then \((n)Q_*\) has no closed communicating classes in \(F^{>n} := F \setminus F^{\leq n}\).

**Proof.** We assume that there exists a closed communicating class \(C\) in \(F^{>n}\). Since \((n)Q_*\) is block-diagonal in \(F^{>n}\), the closed communicating class \(C\) must be within a set \(\{(k, i); i \in D\}\) for some \(k \geq n + 1\), which implies that the principal submatrix \((n)q_\ast(k, i; \ell, j))_{(k, i), (\ell, j) \in C}\) of \((n)Q_*\) is a conservative infinitesimal generator. From this result and Definition 4.1 of \((n)Q_*\), we have

\[
\sum_{(\ell, j) \in C} q(k, i; \ell, j) = \sum_{(\ell, j) \in C} (n)q_\ast(k, i; \ell, j) = 0, \quad (k, i) \in C. \tag{4.2}
\]

Note here that the whole matrices \(Q\) and \((n)Q_*\) are also conservative infinitesimal generators, i.e., for \((k, i) \in F\),

\[
q(k, i; \ell, j) \geq 0, \quad (n)q_\ast(k, i; \ell, j) \geq 0, \quad (\ell, j) \in F \setminus \{(k, i)\}; \tag{4.3}
\]

\[
\sum_{(\ell, j) \in F} q(k, i; \ell, j) = \sum_{(\ell, j) \in F} (n)q_\ast(k, i; \ell, j) = 0. \tag{4.4}
\]

It follows from (4.2)–(4.4) that

\[
q(k, i; \ell, j) = (n)q_\ast(k, i; \ell, j) = 0 \quad \text{for all } (k, i) \in C \text{ and } (\ell, j) \in F \setminus C.
\]

Therefore, the original Markov chain \(\{(X(t), J(t))\}\) with infinitesimal generator \(Q\) cannot move out of \(C \subset F^{>n}\). This contradicts to the irreducibility of \(\{(X(t), J(t))\}\). As a result, \((n)Q_*\) has no closed communicating classes in \(F^{>n}\). \qed

Lemma 4.1 shows that any closed communicating class of \((n)Q_*\) is finite (because it is in the finite set \(F^{\leq n}\)) and thus is positive recurrent due to the combination of [1, Section 5.1, Proposition 1.4] and [13, Theorem 4.8]. Therefore, it follows from [1, Section 5.4, Theorem 4.5] that \((n)Q_*\) has at least one stationary distribution vector.

We now have the following result.

**Lemma 4.2** Suppose that \(Q\) is irreducible. Let \((n)\pi_\ast := ((n)\pi_\ast(k, i))_{(k, i) \in F}\) denote an arbitrary stationary distribution vector of \((n)Q_*\). It then holds that

\[
(n)\pi_\ast(k, i) = 0 \quad \text{for all } (k, i) \in F^{>n}. \tag{4.5}
\]

**Proof.** By definition,

\[
(n)\pi_\ast(n)Q_* = 0.
\]
Thus, from (4.1) and Definition 4.1, we have
\[(n)\pi_*(k) (n)Q_*(k; k) = 0 \quad \text{for all } k \geq n + 1,\]
where \((n)\pi_*(k) = (\pi_*(k, i))_{i \in \mathbb{D}}\) and \((n)Q_*(k; k) = (q(k, i; k, j))_{i, j \in \mathbb{D}}\). Note here that \((n)Q_*(k; k), k \geq n + 1,\) is the infinitesimal generator \((q\text{-matrix})\) of a Markov chain restricted to the set of states \(\{(k, i); i \in \mathbb{D}\} \subset \mathbb{F}^{n}\). Note also that all the states in \(\mathbb{F}^{n}\) are transient, as shown in Lemma 4.1. Therefore, for all \(k \geq n + 1,\) \((n)Q_*(k; k)\) is nonsingular [5, Section 8.6.2]. Post-multiplying both sides of (4.6) by \((n)Q_*(k; k)^{-1}\), we obtain \((n)\pi_*(k) = 0\) for all \(n \geq k + 1,\) i.e., (4.5) holds. \(\square\)

It follows from Assumption 3.1 and Definition 4.1 that \((n)Q_\ast\) is stable and conservative. In addition, the irreducibility of \(Q\) makes \((n)Q_\ast\) regular, as stated in the following lemma.

**Lemma 4.3** If \(Q\) is irreducible, then \((n)Q_\ast\) is regular.

**Proof.** We assume that \((n)Q_\ast\) is not regular, i.e., the equation
\[(n)Q_\ast x = \gamma x, \quad 0 \leq x \leq e\] (4.7)
has a nontrivial solution for some \(\gamma > 0\) (see (3.7)). Let \(\tilde{x} := (\tilde{x}(k, i))_{(k, i) \in \mathbb{F}} \geq 0, \neq 0\) denote such a solution. It then follows from (4.1) and (4.7) that
\[\left[(n)Q_\ast (k, k) - \gamma I\right] \tilde{x}(k) = 0, \quad k \geq n + 1,\]
where \(\tilde{x}(k) = (\tilde{x}(k, i))_{i \in \mathbb{D}}\) for \(k \in \mathbb{Z}_+\). Since \((n)Q_\ast (k, k)\) is a \(q\text{-matrix},\) the matrix \((n)Q_\ast (k, k) - \gamma I\) is nonsingular and thus \(\tilde{x}(k) = 0\) for all \(k \geq n + 1.\) Therefore, \(\tilde{x}(k', i') > 0\) for some \((k', i') \in \mathbb{F}^{n}\) due to \(\tilde{x} \neq 0.\)

Recall that \(\mathbb{F}^{n}\) is a closed set of the states of \((n)Q_\ast\) (see (4.1)). Thus, there exists a closed communicating class including the state \((k', i') \in \mathbb{F}^{n}\), which implies that there exists a stationary distribution vector \((n)\tilde{\pi}_\ast := (\tilde{\pi}_\ast (k, i))_{(k, i) \in \mathbb{F}}\) of \((n)Q_\ast\) such that \((n)\tilde{\pi}_\ast (k', i') > 0\) [1, Section 5.4, Theorem 4.5]. Therefore, \((n)\tilde{\pi}_\ast \tilde{x} \geq (n)\tilde{\pi}_\ast (k', i') \tilde{x}(k', i') > 0.\) On the other hand, pre-multiplying both sides of (4.7) by \((n)\tilde{\pi}_\ast,\) we have \(0 = \gamma \cdot (n)\tilde{\pi}_\ast \tilde{x} > 0,\) which is a contradiction. As a result, the assumption at the beginning is denied, i.e., \((n)Q_\ast\) is regular. \(\square\)

We consider two special cases of the block-augmented truncation. Let \((n)q_n = (n)q_n(k, i; l, j))_{(k, i), (l, j) \in \mathbb{D}}\) denote an infinitesimal generator such that, for \(i, j \in \mathbb{D},\)
\[(n)q_n(k, i; l, j) = \begin{cases} q(k, i; l, j), & k \in \mathbb{Z}_+, \ 0 \leq l \leq n - 1, \\ q(k, i; n, j) + \sum_{m > n, m \neq k} q(k, i; m, j), & k \in \mathbb{Z}_+, \ l = n, \\ q(k, i; k, j), & k = \ell \geq n + 1, \\ 0, & \text{otherwise.} \end{cases}\] (4.8)
Let $\pi_n = (\pi_n(k, i; \ell, j))_{(k, i), (\ell, j) \in \mathbb{F}}$ denote an infinitesimal generator such that, for $i, j \in \mathbb{D}$, 

$$
(n)q_0(k, i; \ell, j) = \begin{cases}
q(k, i; 0, j) + \sum_{m > n, m \neq k} q(k, i; m, j), & k \in \mathbb{Z}_+, \ell = 0, \\
q(k, i; \ell, j), & k \in \mathbb{Z}_+, 1 \leq \ell \leq n, \\
q(k, i; k, j), & k = \ell \geq n + 1, \\
0, & \text{otherwise}.
\end{cases}
$$

We refer to $(n)Q_n$ as the last-column-block-augmented northwest-corner truncation (LC-block-augmented truncation, for short) of $Q$. We also refer to $(n)Q_0$ as the first-column-block-augmented northwest-corner truncation (FC-block-augmented truncation, for short) of $Q$.

Let $(n)\pi_n := ((n)\pi_n(k, i))_{(k, i) \in \mathbb{F}}$ and $(n)\pi_0 := ((n)\pi_0(k, i))_{(k, i) \in \mathbb{F}}$ denote the stationary distribution vectors of $(n)Q_n$ and $(n)Q_0$, respectively. It then follows from Lemma 4.2 that 

$$(n)\pi_n(k, i) = (n)\pi_0(k, i) = 0 \quad \text{for all } (k, i) \in \mathbb{F}^>,$$

The following theorem is a generalization of [15, Theorem 3.6].

**Theorem 4.1** If $Q$ is ergodic (i.e., irreducible and positive recurrent) and $Q \in \text{BM}_d$, then the following are true:

(a) An arbitrary block-augmented truncation $(n)Q$ has the unique stationary distribution vector $(n)\pi$ and 

$$(n)\pi_0 \prec_d (n)\pi \prec_d (n)\pi_n \prec_d \pi, \quad n \in \mathbb{N}.
$$

(b) As $n \to \infty$, $\{(n)\pi_n\}$ converges to $\pi$ elementwise, i.e., 

$$
\lim_{n \to \infty} (n)\pi_n = \pi.
$$

**Proof.** We first prove statement (a). Definition 4.1 (4.8) and (4.9) imply that 

$$
(n)Q_0 \prec_d (n)Q \prec_d (n)Q_n \prec_d Q, \quad n \in \mathbb{N},
$$

which leads to 

$$
(n)Q_0 \prec_d (n)Q_n \prec_d Q, \quad (n)Q \prec_d (n)Q_n \prec_d Q, \quad n \in \mathbb{N}.
$$

Note here that $Q \in \text{BM}_d$ implies $(n)Q_n \in \text{BM}_d$ for all $n \in \mathbb{N}$. It follows from these results and Lemma 3.5 (b) that $(n)\pi_0$, $(n)\pi$, and $(n)\pi_n$ are the unique stationary distribution vectors of $(n)Q_0$, $(n)Q$ and $(n)Q_n$, respectively, and 

$$(n)\pi_0 \prec_d (n)\pi_n \prec_d \pi, \quad (n)\pi \prec_d (n)\pi_n \prec_d \pi, \quad n \in \mathbb{N}.$$
Therefore, it remains to prove that \((n)\pi_0 \prec_d (n)\pi_*\) for \(n \in \mathbb{N}\). To this end, we define \((n)Q_0^{\leq n}\) and \((n)Q_*^{\leq n}\) as the \(|F^{\leq n}| \times |F^{\leq n}|\) north-west-corner truncations of \((n)Q_0\) and \((n)Q_*\), respectively. We also define \((n)\pi_0^{\leq n}\) and \((n)\pi_*^{\leq n}\) as the unique stationary distribution vectors of \((n)Q_0^{\leq n}\) and \((n)Q_*^{\leq n}\), respectively. Lemma 4.2 implies that \((n)\pi_0^{\leq n}\) and \((n)\pi_*^{\leq n}\) are the unique stationary distribution vectors of \((n)Q_0^{\leq n}\) and \((n)Q_*^{\leq n}\), respectively. Note here that \((n)Q_0^{\leq n}\) is the subinvariant probability vector of \((n)Q_*^{\leq n}\) for \(n \in \mathbb{N}\). Thus, proceeding as in the derivation of (3.33), we can readily show that, for \(n \in \mathbb{N}\),

\[
\exp\{nQ_0^{\leq n}t\}T_d^{\leq n} \exp\{nQ_*^{\leq n}t\}T_d^{\leq n}, \quad t \geq 0.
\]

Letting \(t \to \infty\) in the above inequality, we have

\[
e \cdot (n)\pi_0^{\leq n}T_d^{\leq n} \leq e \cdot (n)\pi_*^{\leq n}T_d^{\leq n}, \quad n \in \mathbb{N},
\]

which shows that \((n)\pi_0 \prec_d (n)\pi_*\) for \(n \in \mathbb{N}\). Consequently, statement (a) has been proved.

Next, we prove statement (b). It follows from statement (a) that, for \(n \in \mathbb{N}\), \((n)\pi_* \prec_d \pi_*\), that is,

\[
\sum_{k=\ell}^\infty (n)\pi_*^{\leq n}(k, i) \leq \sum_{k=\ell}^\infty \pi(k, i), \quad (\ell, j) \in F.
\]

Therefore, for any \(\varepsilon \in (0, 1)\), there exists some \(k_{\varepsilon} \in \mathbb{Z}_+\) such that

\[
\sum_{(k, i) \in F > k_{\varepsilon}} (n)\pi_*^{\leq n}(k, i) < \varepsilon \quad \text{for all} \ n \in \mathbb{N},
\]

which shows that \((n)\Pi_* := \{(n)\pi_*; n \in \mathbb{N}\}\) is tight and thus relatively compact (see, e.g., [3] Theorem 5.1), i.e., there exists a convergent subsequence of \((n)\Pi_*\).

Let \(\{(n_m)\pi_*; m \in \mathbb{Z}_+\}\) denote an arbitrary convergent subsequence of \((n)\Pi_*\) such that

\[
\lim_{m \to \infty} (n_m)\pi_* = \pi_*\quad (4.12)
\]

where \(\pi_* := (\pi_*(k, i))_{(k, i) \in F}\) is a probability vector. Furthermore, let \((n)P_*^{(t)} := ((n)P_*^{(t)}(k, i; \ell, j))_{(k, i), (\ell, j) \in F}\) denote the transition matrix function of the infinitesimal generator \((n)Q_*\). By definition,

\[
(n_m)\pi_* (n_m)P_*^{(t)} = (n_m)\pi_*\quad t \geq 0. \quad (4.13)
\]

It also follows from [9] Theorem 2.1 and Remark 2.2] that

\[
\lim_{n \to \infty} (n)P_*^{(t)} = P^{(t)}, \quad t \geq 0. \quad (4.14)
\]

Applying Fatou’s lemma, (4.12) and (4.14) to (4.13), we have

\[
\pi_* P^{(t)} \leq \pi_*, \quad t \geq 0, \quad (4.15)
\]

which shows that \(\pi_*\) is the subinvariant probability vector of \(P^{(t)}\). Recall here that \(Q\) is ergodic. The ergodicity of \(Q\) together with (4.15) implies that \(\pi_*\) is the unique probability
vector satisfying $\pi_s, P^{(t)} = \pi_s$ [1 Section 5.2, Theorem 2.8]. Therefore, $\pi_s = \pi$ (see Definition [3.1] and Remark [3.2]). This result and [3 Corollary of Theorem 5.1] yield

$$\lim_{n \to \infty} (n)\pi_s = \pi.$$ 

As a result, we have (4.11).

Theorem 4.1 shows that $\{(n)\pi_n\}, \{(n)\pi_0\}$ and $\{(n)\pi_*\}$ can be approximations to $\pi$. In addition, it follows from (4.10) that, for all $m \in \mathbb{Z}^+$,

$$0 \leq m \sum_{k=0}^m \sum_{i \in D} \{(n)\pi_n(k, i) - \pi(k, i)\} \leq m \sum_{k=0}^m \sum_{i \in D} \{(n)\pi_0(k, i) - \pi(k, i)\} \leq m \sum_{k=0}^m \sum_{i \in D} \{(n)\pi_0(k, i) - \pi(k, i)\}. \quad (4.16)$$

Therefore, we can say that the LC- (resp. FC-) block-augmented truncation of $Q \in \text{BM}_d$ is the best (resp. worst) among all the block-augmented truncations of $Q$ in the sense shown in (4.16).

5 Error bounds for last-column-block-augmented truncations

In the previous section, we already have shown that the stationary distribution vector $\{(n)\pi_n\}$ of the LC-block-augmented truncation $(n)Q_n$ is the best approximation to the stationary distribution vector $\pi$ of $Q \in \text{BM}_d$. In this section, we do not necessarily assume $Q \in \text{BM}_d$, but assume that $Q$ is block-wise dominated by another generator $\tilde{Q} \in \text{BM}_d$, which is possibly equal to $Q$. We then present upper bounds for the total variation distance between $(n)\pi_n$ and $\pi$, i.e.,

$$\| (n)\pi_n - \pi \| := \sum_{(k, i) \in F} |(n)\pi_n(k, i) - \pi(k, i)|,$$

where $\| \cdot \|$ denotes the total variation norm.

We begin with the introduction of some definitions and assumptions for the subsequent discussion. For any $1 \times |\mathbb{F}|$ vector $x = (x(k, i))_{(k, i) \in \mathbb{F}}$, let $\| x \|_v$ denote

$$\| x \|_v := \sup_{|g| \leq v} \left| \sum_{(k, i) \in \mathbb{F}} x(k, i) g(k, i) \right| = \sup_{0 \leq g \leq v} \sum_{(k, i) \in \mathbb{F}} |x(k, i)| g(k, i),$$

where $|g|$ is a column vector obtained by taking the absolute value of each element of $g$. The quantity $\| x \|_v$ is called the $v$-norm of $x$. Note here that $\| \cdot \|_e = \| \cdot \|$, i.e., the $e$-norm is equivalent to the total variation norm. Let $\mathbb{I}\{ \cdot \}$ denote a function that takes value one
if the statement in the braces is true and takes value zero otherwise. For \( K \in \mathbb{Z}_+ \), let \( 1_K = (1_K(k, i))_{(k, i) \in \mathcal{F}} \) denote a column vector such that

\[
1_K(k, i) = \begin{cases} 
1, & (k, i) \in \mathcal{F}^\leq K, \\
0, & (k, i) \in \mathcal{F}^> K.
\end{cases}
\]

Furthermore, let \( \{(n)X_t, (n)J_t; t \geq 0\} \) denote a Markov chain with infinitesimal generator \((n)Q_n\) and let \((n)p_n^{(t)}(k, i) = ((n)p_n^{(t)}(k, i; \ell, j))_{(\ell, j) \in \mathcal{F}}\) denote a probability vector such that

\[
(n)p_n^{(t)}(k, i) = \mathbb{P}(n)X_t = \ell, (n)J_t = j \mid (n)X_0 = k, (n)J_0 = i.
\]

It follows from (4.8) that

\[
(n)p_n^{(t)}(k, i; \ell, j) = 0, \quad t \geq 0, \quad (k, i) \in \mathcal{F}^\leq n, \quad (\ell, j) \in \mathcal{F}^> n. \tag{5.1}
\]

Finally, we make the following assumptions.

**Assumption 5.1** (i) \( Q \prec_d \bar{Q} \); (ii) \( \bar{Q} \in \text{BM}_d \) and \( \bar{Q} \) is irreducible.

**Assumption 5.2** There exist some constants \( c, b \in (0, \infty) \) and column vector \( \mathbf{v} = (v(k, i))_{(k, i) \in \mathcal{F}} \in \mathcal{B}l_d \) with \( \mathbf{v} \geq \mathbf{e} \) such that

\[
\bar{Q}\mathbf{v} \leq -c\mathbf{v} + b\mathbf{1}_d. \tag{5.2}
\]

Clearly, Assumption 5.1 implies Assumption 3.2. In addition, Assumption 5.2, together with Assumption 5.1, implies that \( \bar{Q} \) is exponentially ergodic [21, Theorem 20.3.2]. Thus, it follows from Lemma 3.5 that \( \bar{Q} \) and \( Q \) have the unique stationary distribution vectors \( \bar{\pi} \) and \( \pi \), respectively, such that \( \pi \prec_d \bar{\pi} \). We now define \( \varphi = (\varphi(i))_{i \in \mathbb{D}} \) as a \( 1 \times d \) probability vector such that \( \varphi(i) = \sum_{k=0}^{\infty} \pi(k, i) \) for \( i \in \mathbb{D} \). It then follows from Lemma 3.3 that \( \varphi \) is the common stationary distribution vector of the Markov chains \( \{J_t\} \) and \( \{\bar{J}_t\} \) and

\[
\varphi(i) = \sum_{k=0}^{\infty} \pi(k, i) = \sum_{k=0}^{\infty} \bar{\pi}(k, i), \quad i \in \mathbb{D}. \tag{5.3}
\]

In what follows, we estimate \( \| (n)\pi_n - \pi \| \). By the triangular inequality,

\[
\| (n)\pi_n - \pi \| \leq \| (n)p_n^{(t)}(0, \varphi) - \pi \| + \| (n)p_n^{(t)}(0, \varphi) - (n)\pi_n \| + \| (n)p_n^{(t)}(0, \varphi) - (n)p_n^{(t)}(0, \varphi) \|, \quad n \in \mathbb{N}, \ t \geq 0, \tag{5.4}
\]

where \( p_n^{(t)}(k, i) = (p_n^{(t)}(k, i; \ell, j))_{(\ell, j) \in \mathcal{F}} \) and, for any function \( \varphi \) on \( \mathcal{F} \),

\[
\varphi(k, \varphi) = \sum_{i \in \mathbb{D}} \varphi(i)\varphi(k, i), \quad k \in \mathbb{Z}_+.
\]

The following lemma provides upper bounds for the first and second terms in the right hand side of (5.4).
Lemma 5.1 If Assumptions [5.7] and [5.2] hold, then the following inequalities hold for all \( k \in \mathbb{Z}_+ \) and \( t \geq 0 \):
\[
\left\| \mathbf{p}^{(t)}(k, \varnothing) - \pi \right\|_v \leq 2e^{-ct} [v(k, \varnothing)(1 - 1_0(k, \varnothing)) + b/c],
\]
and, for \( n \in \mathbb{N} \),
\[
\left\| (n)\mathbf{p}^{(t)}_n(k, \varnothing) - (n)\pi_n \right\|_v \leq 2e^{-ct} [v(k, \varnothing)(1 - 1_0(k, \varnothing)) + b/c].
\]

**Proof.** We first prove (5.5). For this purpose, we provide definitions and notation. Let \((X, J)\) and \((\tilde{X}, \tilde{J})\) denote two random vectors on a probability space \((\Omega, \mathcal{F}, P)\) such that
\[
P(X = k, J = i) = \pi(k, i), \quad (k, i) \in \mathcal{F},
\]
\[
P(\tilde{X} = k, \tilde{J} = i) = \tilde{\pi}(k, i), \quad (k, i) \in \mathcal{F}.
\]
It follows from (5.3) and \( \pi \prec_d \tilde{\pi} \) that
\[
P(J = i) = \sum_{\ell=0}^{\infty} \pi(\ell, i) = \varnothing(i) = \sum_{\ell=0}^{\infty} \tilde{\pi}(\ell, i) = P(\tilde{J} = i), \quad i \in \mathcal{D},
\]
\[
P(X > k \mid J = i) = \frac{\sum_{\ell=k+1}^{\infty} \pi(\ell, i)}{\varnothing(i)} \leq \frac{\sum_{\ell=k+1}^{\infty} \tilde{\pi}(\ell, i)}{\varnothing(i)} = P(\tilde{X} > k \mid \tilde{J} = i), \quad k \in \mathbb{Z}_+.
\]
Thus, we assume, without loss of generality, that \( X \leq \tilde{X} \) and \( J = \tilde{J} \) [22 Theorem 1.2.4].

We now define \( \{(\tilde{X}^{(h)}_t, \tilde{J}^{(h)}_t); t \geq 0\}, h \in \{0, 1, 2\} \), as Markov chains with infinitesimal generator \( \tilde{Q} \) on the probability space \((\Omega, \mathcal{F}, P)\) such that
\[
(\tilde{X}^{(0)}_0, \tilde{J}^{(0)}_0) = (0, \tilde{J}), \quad (\tilde{X}^{(1)}_0, \tilde{J}^{(1)}_0) = (k, \tilde{J}), \quad (\tilde{X}^{(2)}_0, \tilde{J}^{(2)}_0) = (\tilde{X}, \tilde{J}).
\]
We also define \( \{(X^{(h)}_t, J^{(h)}_t); t \geq 0\}, h \in \{0, 1, 2\} \), as Markov chains with infinitesimal generator \( Q \) on the probability space \((\Omega, \mathcal{F}, P)\) such that
\[
(X^{(0)}_0, J^{(0)}_0) = (0, J), \quad (X^{(1)}_0, J^{(1)}_0) = (k, J), \quad (X^{(2)}_0, J^{(2)}_0) = (X, J).
\]
Recall here that \( Q \prec_d \tilde{Q} \) and \( \tilde{Q} \in \text{BM}_d \). Therefore, according to Lemmas [A.1] and [A.2], we assume that, for each \( h \in \{0, 1, 2\} \),
\[
X^{(h)}_t \leq \tilde{X}^{(h)}_t, \quad J^{(h)}_t = \tilde{J}^{(h)}_t \quad \text{for all } t \geq 0,
\]
and that
\[
\tilde{X}^{(0)}_t \leq \tilde{X}^{(1)}_t, \quad \tilde{X}^{(0)}_t \leq \tilde{X}^{(2)}_t, \quad \tilde{J}^{(0)}_t = \tilde{J}^{(1)}_t = \tilde{J}^{(2)}_t \quad \text{for all } t \geq 0.
\]
For the Markov chains \(\{(X_t^{(h)}, J_t^{(h)})\}\)'s, \(h \in \{0, 1, 2\}\), we introduce two coupling times \(T^{(1)}\) and \(T^{(2)}\) as follows:

\[
T^{(1)} = \inf\{t \geq 0 : X_t^{(1)} = X_0^{(0)}\},
\]
\[
T^{(2)} = \inf\{t \geq 0 : X_t^{(2)} = X_0^{(0)}\}.
\]

We then assume, without loss of generality (see Remark A.1), that

\[
X_t^{(1)} = X_0^{(0)} \quad \text{for all } t \geq T^{(1)},
\]
\[
X_t^{(2)} = X_0^{(0)} \quad \text{for all } t \geq T^{(2)}.
\]

For convenience, we also introduce the following notation:

\[
E_{(k,i)}[\cdot | \cdot] = E[\cdot | X_0 = k, J_0 = i],
\]
\[
E_{(k,i);(0,j)}[\cdot | \cdot] = E[\cdot | (X_0^{(h)}, J_0^{(h)}) = (k, i), (X_0^{(0)}, J_0^{(0)}) = (0, j)],
\]

where \(h \in \{1, 2\}\), \((k, i) \in \mathbb{F}\) and \(j \in \mathbb{D}\).

We are now ready to prove \((5.5)\). Proceeding as in the derivation of \([19, \text{Eq. (3.18)}]\) (see also \([18, \text{Eq. (3.6)}]\)), we have, for \(|g| \leq v\),

\[
|\mathbf{p}^{(t)}(k, \varpi)g - \pi g| \leq E\left[ E_{(k,j);(0,J)}[v(X_t^{(1)}, J_t^{(1)}) \cdot \mathbb{I}_{\{T^{(1)}>t\}}]\right]
+ E\left[ E_{(k,j);(0,J)}[v(X_t^{(0)}, J_t^{(0)}) \cdot \mathbb{I}_{\{T^{(1)}>t\}}]\right]
+ E\left[ E_{(X,J);(0,j)}[v(X_t^{(2)}, J_t^{(2)}) \cdot \mathbb{I}_{\{T^{(2)}>t\}}]\right]
+ E\left[ E_{(X,J);(0,j)}[v(X_t^{(0)}, J_t^{(0)}) \cdot \mathbb{I}_{\{T^{(2)}>t\}}]\right].
\]  \hspace{1cm} (5.9)

Combining \((5.9)\) with \((5.7)\), \((5.8)\) and \(v \in \text{Bil}_d\), we obtain, for \(|g| \leq v\),

\[
|\mathbf{p}^{(t)}(k, \varpi)g - \pi g| \leq 2E\left[ E_{(k,j);(0,J)}[v(\tilde{X}_t^{(1)}, \tilde{J}_t^{(1)}) \cdot \mathbb{I}_{\{T^{(1)}>t\}}]\right]
+ 2E\left[ E_{(X,J);(0,j)}[v(\tilde{X}_t^{(2)}, \tilde{J}_t^{(2)}) \cdot \mathbb{I}_{\{T^{(2)}>t\}}]\right].
\]  \hspace{1cm} (5.10)

Furthermore, it follows from \((5.7)\) and \((5.8)\) that, for each \(h \in \{1, 2\}\), \(\{\tilde{X}_t^{(h)} = 0\}\) implies \(\{X_t^{(h)} = X_0^{(0)} = 0\}\), which leads to \(T^{(h)} \leq \inf\{t \geq 0 : \tilde{X}_t^{(h)} = 0\}\). Therefore, \((5.10)\) yields

\[
\|\mathbf{p}^{(t)}(k, \varpi) - \pi\|_v \leq 2E\left[ E_{(k,j)}[v(\tilde{X}_t, \tilde{J}_t) \cdot \mathbb{I}_{\{\tilde{\tau}_0>t\}}]\right]
+ 2E\left[ E_{(X,J)}[v(\tilde{X}_t, \tilde{J}_t) \cdot \mathbb{I}_{\{\tilde{\tau}_0>t\}}]\right],
\]  \hspace{1cm} (5.11)

where \(\tilde{\tau}_0 = \inf\{t \geq 0 : \tilde{X}_t = 0\}\).

Let \(M_t = e^t v(\tilde{X}_t, \tilde{J}_t) \mathbb{I}_{\{\tilde{\tau}_0>t\}}\) for \(t \geq 0\). It is shown in Lemma B.1 that \(\{M_t\}\) is a supermartingale. Let \(\{\theta_t; \nu \in \mathbb{Z}_+\}\) denote a sequence of stopping times for \(\{M_t; t \geq 0\}\).
such that \( 0 \leq \theta_1 \leq \theta_2 \leq \cdots \) and \( \lim_{\nu \to \infty} \theta_\nu = \infty \). It then follows that, for any \( u \geq 0 \), \( \min(u, \theta_\nu) \) is a stopping time for \( \{ M_t; t \in \mathbb{Z}_+ \} \). Therefore, using Doob’s optional sampling theorem (see, e.g., [26, Section 10.10]), we have \( E_{(k,i)}[M_{\min(t,\theta_\nu)}] \leq E_{(k,i)}[M_0] \), i.e.,

\[
E_{(k,i)}[e^{c \min(t,\theta_\nu)} v(\tilde{X}_{\min(t,\theta_\nu)}, \tilde{J}_{\min(t,\theta_\nu)}) II_{\{\tilde{\tau}_0 > \min(t,\theta_\nu)\}}] \leq v(k,i)(1 - 1_0(k,i)),
\]

for \((k,i) \in \mathbb{F}\). Letting \( \nu \to \infty \) in the above inequality and using Fatou’s lemma, we obtain

\[
E_{(k,i)}[v(\tilde{X}_{t}, \tilde{J}_t) II_{\{\tilde{\tau}_0 > t\}}] \leq e^{-ct} v(k,i)(1 - 1_0(k,i)), \quad (k,i) \in \mathbb{F}, \quad (5.12)
\]

and thus

\[
E\left[ E_{(k,j)}[v(\tilde{X}_{t}, \tilde{J}_t) II_{\{\tilde{\tau}_0 > t\}}]\right] = \sum_{i \in \mathcal{D}} \varpi(i) E_{(k,i)}[v(\tilde{X}_{t}, \tilde{J}_t) II_{\{\tilde{\tau}_0 > t\}}]
\leq e^{-ct} v(k, \varpi)(1 - 1_0(k, \varpi)), \quad k \in \mathbb{Z}_+,
\]

where we use \( 1_0(k,i) = 1_0(k, \varpi) \) for all \( i \in \mathcal{D} \). Furthermore, pre-multiplying both sides of \((5.2)\) by \( \tilde{\pi} \), we have \( \tilde{\pi} v \leq b/c \). Combining this and \((5.12)\) yields

\[
E\left[ E_{(\tilde{X}, \tilde{J})} [v(\tilde{X}_{t}, \tilde{J}_t) : II_{\{\tilde{\tau}_0 > t\}}]\right] \leq e^{-ct} \sum_{(k,i) \in \mathbb{F}} \pi(k,i) v(k,i) \leq e^{-ct} \frac{b}{c}. \quad (5.14)
\]

Substituting \((5.13)\) and \((5.14)\) into \((5.11)\), we obtain \((5.5)\).

Next, we prove \((5.6)\). Let \( (n)\tilde{Q}_n := ((n)\tilde{\varrho}_n(k,i; \ell,j))_{(k,i), (\ell,j) \in \mathbb{F}} \) denote the LC-block-augmented truncation of \( \tilde{Q} \), which is defined in a similar way to \((4.8)\). Since \( Q \prec_d \tilde{Q} \) and \( \tilde{Q} \in \text{BM}_d \), we have \( (n)\tilde{Q}_n \in \text{BM}_d \) and

\[
(n)\tilde{Q}_n \prec_d (n)\tilde{Q}_n \prec_d \tilde{Q}.
\]

It follows from \( (n)\tilde{Q}_n \prec_d \tilde{Q}, \nu \in \text{Bl}_d \) and \((5.2)\) that

\[
(n)\tilde{Q}_n \nu \leq \tilde{Q} \nu \leq -cv + b1_0. \quad (5.15)
\]

Therefore, we can readily prove \((5.6)\) by replacing \( \tilde{Q} \) and \( Q \) with \( (n)\tilde{Q}_n \) and \( (n)Q_n \), respectively, in the proof of \((5.5)\). The details are omitted.

According to Lemma \(5.1\), it remains to estimate the third term in the right hand side of \((5.4)\) in order to derive an upper bound for \( \| (n)\pi_n - \pi \| \). By doing this, we obtain the following theorem.

**Theorem 5.1** If Assumptions \(5.1\) and \(5.2\) hold, then, for all \( n \in \mathbb{N} \) and \( t \geq 0 \),

\[
\| (n)\pi_n - \pi \| \leq \frac{b}{c} \left( 4e^{-ct} + 2t \sum_{j \in \mathcal{D}} \frac{|(n)\tilde{\varrho}(n,j;n,j)|}{v(n,j)} \right).
\]

(5.16)
Remark 5.1 It is easy to see that, for each \( n \in \mathbb{N} \), the right hand side of (5.16) takes the minimum value at \( t = c^{-1}t^*(n) \), where

\[
t^*(n) = \max \left\{ -\log \left( \frac{1}{2c} \sum_{j \in D} |\tilde{q}(n, j; n, j)| \right), 0 \right\}, \quad n \in \mathbb{N}. \tag{5.17}
\]

Substituting (5.17) into (5.16) yields

\[
\|(n)\pi_n - \pi\| \leq \frac{4b}{c} (t^*(n) + 1) \exp \{ -t^*(n) \}, \quad n \in \mathbb{N}. \tag{5.18}
\]

If \( \lim_{n \to \infty} \sum_{j \in D} |\tilde{q}(n, j; n, j)| / v(n, j) = 0 \) and thus \( \lim_{n \to \infty} t^*(n) = \infty \), then the right hand side of (5.18) converges to zero as \( n \to \infty \). A similar discussion is found in [25] (see Eq. (50) therein).

Proof of Theorem 5.1 Letting \( k = 0 \) and \( v = e \) in (5.5) and (5.6) (see Lemma 5.1) and substituting the result into (5.4), we have

\[
\|(n)\pi_n - \pi\| \leq \frac{4be^{-ct}}{c} + \|(n)p_n^{(t)}(0, \varpi) - p^{(t)}(0, \varpi)\|.
\]

Therefore, it suffices to prove that, for all \( n \in \mathbb{N} \) and \( t \geq 0 \),

\[
\|(n)p_n^{(t)}(0, \varpi) - p^{(t)}(0, \varpi)\| \leq \frac{2tb}{c} \sum_{j \in D} |\tilde{q}(n, j; n, j)| / v(n, j). \tag{5.19}
\]

Note here that all the off-diagonal elements of \( Q \) are nonnegative. Using this fact, (5.1) and Lemma B.2, we have

\[
\|(n)p_n^{(t)}(0, \varpi) - p^{(t)}(0, \varpi)\|
\leq \sum_{i \in D} \varpi(i) \|(n)p_n^{(t)}(0, i) - p^{(t)}(0, i)\|
\leq 2 \int_0^t \sum_{(\ell, j) \in F} \left( \sum_{i \in D} \varpi(i) p_n^{(u)}(0, i; \ell, j) \right) du \sum_{(\ell', j') \in F^>n} |q(\ell, j; \ell', j')|
\leq 2 \int_0^t \sum_{(\ell, j) \in F^>n} \left( \sum_{i \in D} \varpi(i) p_n^{(u)}(0, i; \ell, j) \right) du \sum_{(\ell', j') \in F^>n} q(\ell, j; \ell', j'). \tag{5.20}
\]

In addition, \( Q \approx_d \tilde{Q} \) yields

\[
\sum_{(\ell', j') \in F^>n} q(\ell, j; \ell', j') \leq \sum_{(\ell', j') \in F^>n} \tilde{q}(\ell, j; \ell', j'), \quad (\ell, i) \in F^<=n.
\]
Substituting this inequality into (5.20) and using (5.1), we obtain

\[
\left\| (n)p_n^{(t)}(0, \varpi) - p^{(t)}(0, \varpi) \right\|
\leq 2 \int_0^t \sum_{(\ell, j) \in F} \left( \sum_{i \in D} \varpi(i) (n)p_n^{(u)}(0, i; \ell, j) \right) du \sum_{(\ell', j') \in F > n} q(\ell, j; \ell', j')
\]

\[
= 2 \int_0^t \sum_{(\ell, j) \in F} \left( \sum_{i \in D} \varpi(i) (n)p_n^{(u)}(0, i; \ell, j) \right) du \sum_{(\ell', j') \in F > n} q(\ell, j; \ell', j')
\]

\[
= 2 \int_0^t (\varpi, 0, 0, \ldots, (n)p_n^{(u)}) a_n du,
\] (5.21)

where \(a_n := (a_n(\ell, j))(\ell, j) \in F \) is a column vector such that

\[
a_n(\ell, j) = \begin{cases} 
\sum_{(\ell', j') \in F > n} q(\ell, j; \ell', j'), & (\ell, j) \in F \leq n, \\
\sum_{(\ell', j') \in F > n} q(n, j; \ell', j'), & (\ell, j) \in F > n.
\end{cases}
\] (5.22)

It follows from (5.22) and \(\tilde{Q} \in BM_d\) that \(a_n \in Bl_d\). It also follows from (5.22) and the definition of \((n)\tilde{Q}_n\) that, for \((\ell, j) \in F \leq n\),

\[
\sum_{j' \in D} (n)\tilde{q}_n(\ell, j; n, j') = \sum_{j' \in D} \tilde{q}(\ell, j; n, j') + \sum_{(\ell', j') \in F > n} \tilde{q}(\ell, j; \ell', j')
\]

\[
= \sum_{j' \in D} \tilde{q}(\ell, j; n, j') + a_n(\ell, j),
\]

which leads to

\[
a_n(\ell, j) = \sum_{j' \in D} \left( (n)\tilde{q}_n(\ell, j; n, j') - \tilde{q}(\ell, j; n, j') \right), \quad (\ell, j) \in F \leq n.
\] (5.23)

We now define \(\tilde{\pi}_n := (\tilde{\pi}_n(k, i))(k, i) \in F\) as the stationary distribution vector of \((n)\tilde{Q}_n\). It then follows from Lemma 4.2 and the irreducibility of \(Q\) that

\[
(n)\tilde{\pi}_n(k, i) = 0, \quad (k, i) \in F > n.
\] (5.24)

It also follows from \(Q \prec_d \tilde{Q} \in BM_d\) that \((n)\tilde{Q}_n \in BM_d\) and \((n)Q_n \prec_d (n)\tilde{Q}_n \prec_d \tilde{Q}\). Thus, Lemmas 3.2, 3.3 and 3.4 imply that

\[
\sum_{k=0}^\infty (n)\tilde{\pi}_n(k, i) = \sum_{k=0}^\infty \tilde{\pi}(k, i) = \varpi(i), \quad i \in D,
\] (5.25)

\[
(n)p_n^{(t)} \prec_d (n)\tilde{p}_n^{(t)} \in BM_d, \quad t \geq 0,
\] (5.26)

where \((n)p_n^{(t)}\) and \((n)\tilde{p}_n^{(t)}\) are the transition matrix functions of the infinitesimal generators \((n)Q_n\) and \((n)\tilde{Q}_n\), respectively. Using (5.26) and \(a_n \in Bl_d\), we have

\[
(n)p_n^{(t)} a_n \leq (n)\tilde{p}_n^{(t)} a_n, \quad t \geq 0.
\]
Applying this inequality to \((5.21)\) yields
\[
\| \langle n \rangle P_n^{(t)}(0, \varpi) - P^{(t)}(0, \varpi) \| \leq 2 \int_{0}^{t} \langle \varpi, 0, 0, \ldots \rangle \langle n \rangle \tilde{P}_n^{(a)} a_n du. \tag{5.27}
\]

Note here that \((5.25)\) leads to \((\varpi, 0, 0, \ldots) \prec_d \langle n \rangle \tilde{\pi}_n\). Combining this relation with \((n)\tilde{P}_n^{(t)} \in BM_d\) (see Proposition 2.1), we have, for \(t \geq 0\),
\[
(\varpi, 0, 0, \ldots) \langle n \rangle \tilde{P}_n^{(t)} \prec_d (n)\tilde{\pi}_n \langle n \rangle \tilde{P}_n^{(t)} = (n)\tilde{\pi}_n, \tag{5.28}
\]
where the last equality follows from \((n)\tilde{\pi}_n \langle n \rangle \tilde{P}_n^{(t)} = (n)\tilde{\pi}_n\). Furthermore, using \((5.28)\) and \(a_n \in B_{1, d}\), we obtain
\[
(\varpi, 0, 0, \ldots) \langle n \rangle \tilde{P}_n^{(t)} a_n \leq (n)\tilde{\pi}_n a_n, \quad t \geq 0.
\]

Substituting this into \((5.27)\) results in
\[
\| \langle n \rangle P_n^{(t)}(0, \varpi) - P^{(t)}(0, \varpi) \| \leq 2t \langle n \rangle \tilde{\pi}_n a_n, \quad n \in \mathbb{N}, t \geq 0. \tag{5.29}
\]

In what follows, we estimate \((n)\tilde{\pi}_n a_n\). From \((5.23)\), \((5.24)\) and \((n)\tilde{\pi}_n \langle n \rangle \tilde{Q}_n = 0\), we have
\[
\langle n \rangle \tilde{\pi}_n a_n = \sum_{(\ell, j) \in \mathbb{F}^{\ll n}} \langle n \rangle \tilde{\pi}_n(\ell, j) \sum_{j' \in \mathbb{D}} \left( \langle n \rangle \tilde{q}_n(\ell, j; n, j') - \tilde{q}(\ell, j; n, j') \right)
\]
\[
= \sum_{(\ell, j) \in \mathbb{F}^{\ll n}} \langle n \rangle \tilde{\pi}_n(\ell, j) \sum_{j' \in \mathbb{D}} (-\tilde{q}(\ell, j; n, j'))
\]
\[
= \sum_{j' \in \mathbb{D}} \sum_{(\ell, j) \in \mathbb{F}^{\ll n}} \langle n \rangle \tilde{\pi}_n(\ell, j) \left( -\tilde{q}(\ell, j; n, j') \right). \tag{5.30}
\]

By definition, \(-\tilde{q}(\ell, j; n, j') \leq 0\) for \((\ell, j) \in \mathbb{F}^{\ll n} \setminus \{(n, j')\}\) and \(-\tilde{q}(n, j'; n, j') \geq 0\). Therefore,
\[
\sum_{(\ell, j) \in \mathbb{F}^{\ll n}} \langle n \rangle \tilde{\pi}_n(\ell, j) \left( -\tilde{q}(\ell, j; n, j') \right) \leq (n)\tilde{\pi}_n(n, j') \tilde{q}(n, j'; n, j'), \quad j' \in \mathbb{D}.
\]

Applying this to \((5.30)\), we have
\[
\langle n \rangle \tilde{\pi}_n a_n \leq \sum_{j' \in \mathbb{D}} (n)\tilde{\pi}_n(n, j') |\tilde{q}(n, j'; n, j')|. \tag{5.31}
\]

Pre-multiplying both sides of \((5.13)\) by \((n)\tilde{\pi}_n\), we obtain \((n)\tilde{\pi}_n v \leq b/c\), which leads to
\[
\langle n \rangle \tilde{\pi}_n(n, j') \leq \frac{b}{c} \cdot \frac{1}{v(n, j')}, \quad j' \in \mathbb{D}. \tag{5.32}
\]

Using \((5.31)\) and \((5.32)\), we have
\[
\langle n \rangle \tilde{\pi}_n a_n \leq \frac{b}{c} \sum_{j' \in \mathbb{D}} \frac{|\tilde{q}(n, j'; n, j')|}{v(n, j')}.
\]
Substituting this inequality into (5.29) yields (5.19).

From Theorem 5.1 and Remark 5.1, we obtain Corollary 5.1 below, where the drift condition (5.2) for $\tilde{Q}$ is weakened whereas the set of states $\{(0, i); i \in D\}$ is assumed to be reachable directly from each state in $F \subseteq K$ with a sufficiently large $K$.

**Corollary 5.1** Suppose that Assumptions 5.1 and 5.2 hold and there exist some $c', b' \in (0, \infty)$, $K \in \mathbb{Z}_+$ and column vector $v' := (v'(k, i)) (k, i) \in F \subseteq B_1$ with $v' \geq e$ such that

$$\tilde{Q}v' \leq -c'v' + b'1_K.$$

(5.33)

Let $\tilde{Q}(k; \ell) = (\tilde{q}(k, i; \ell, j)) (i, j) \in D$ for $k, \ell \in \mathbb{Z}_+$, and suppose that $\tilde{Q}(K; 0)e > 0$.

Under these conditions, the bound (5.18), together with (5.17), holds for all $n \in \mathbb{N}$, where $c, b, B \in (0, \infty)$ are constants such that

$$c = \frac{c'}{1 + B},$$

(5.34)

$$b e \geq b'e - B\tilde{Q}(0; 0)e,$$

(5.35)

$$B \cdot \tilde{Q}(K; 0)e \geq b'e,$$

(5.36)

and block-wise increasing $v = (v(k, i)) (k, i) \in F$ is given by

$$v(k, i) = \begin{cases} v'(0, i), & k = 0, i \in D; \\ v'(k, i) + B, & k \in \mathbb{N}, i \in D. \end{cases}$$

(5.37)

**Proof.** It suffices to prove that (5.2) holds for $c, b, B \in (0, \infty)$ and $v \in B_1$ such that (5.34)–(5.37) are satisfied. We begin with the estimation of $\sum_{\ell=0}^{\infty} \tilde{Q}(0; \ell)v(\ell)$. It follows from (5.37) and $\sum_{\ell=0}^{\infty} \tilde{Q}(k; \ell)e = 0$ for all $k \in \mathbb{Z}_+$ that

$$\sum_{\ell=0}^{\infty} \tilde{Q}(k; \ell)v(\ell) = \sum_{\ell=1}^{\infty} \tilde{Q}(k; \ell)(v'(\ell) + Be) + \tilde{Q}(k; 0)v'(0)$$

$$= \sum_{\ell=0}^{\infty} \tilde{Q}(k; \ell)v'(\ell) - B\tilde{Q}(k; 0)e, \quad k \in \mathbb{Z}_+. \quad (5.38)$$

Substituting $k = 0$ into (5.38), and applying (5.33) and $v'(0) = v(0)$ to the resulting equation, we have

$$\sum_{\ell=0}^{\infty} \tilde{Q}(0; \ell)v(\ell) \leq -c'v(0) + b'e - B\tilde{Q}(0; 0)e$$

$$\leq -c'v(0) + be \leq -c0v(0) + be,$$

(5.39)
where the last two inequalities follow from (5.35) and \(c' \geq c\) (due to (5.34)).

Next we consider \(\sum_{\ell=0}^{\infty} \bar{Q}(k; \ell)v(\ell)\) for \(k \in \{1, 2, \ldots, K\}\). It follows from \(\bar{Q} \in BM_d\) that \(\bar{Q}(k; 0)e \geq \bar{Q}(K; 0)e\) for \(k = 1, 2, \ldots, K\). Incorporating this inequality and (5.33) into (5.38) for \(k \in \{1, 2, \ldots, K\}\), we obtain
\[
\sum_{\ell=0}^{\infty} \bar{Q}(k; \ell)v(\ell) \leq \sum_{\ell=0}^{\infty} \bar{Q}(k; \ell)v'(\ell) - B\bar{Q}(K; 0)e \\
\leq -c'v'(k) + b'e - B\bar{Q}(K; 0)e \\
\leq -c'v'(k), \quad k = 1, 2, \ldots, K, \tag{5.40}
\]
where the last inequality holds due to (5.36). Note here that (5.34), (5.37) and \(v' \geq e\) imply
\[
cv(k) = cv'(k) + Be \leq c'v'(k), \quad k \in \mathbb{N}, \tag{5.41}
\]
from which and (5.40) we have
\[
\sum_{\ell=0}^{\infty} \bar{Q}(k; \ell)v(\ell) \leq -cv(k), \quad k = 1, 2, \ldots, K. \tag{5.42}
\]

Finally, we estimate \(\sum_{\ell=0}^{\infty} \bar{Q}(k; \ell)v(\ell)\) for \(k \geq K + 1\). Substituting (5.33) into (5.38) for \(k \geq K + 1\) and using (5.41) and \(\bar{Q}(k; 0)e \geq 0\), we obtain
\[
\sum_{\ell=0}^{\infty} \bar{Q}(k; \ell)v(\ell) \leq -c'v'(k) - B\bar{Q}(k; 0)e \\
\leq -cv(k) - B\bar{Q}(k; 0)e \\
\leq -cv(k), \quad k = K + 1, K + 2, \ldots. \tag{5.43}
\]
As a result, combining (5.39), (5.42) and (5.43), we have (5.2). \(\square\)

6 Applications

In this section, we demonstrate the applicability of the error bounds presented in Section 5. To this end, we consider a queue with a batch Markovian arrival process (BMAP) [17] and level-dependent departure rates.

We first describe the BMAP. The BMAP is controlled by an irreducible continuous-time Markov chain \(\{J(t); t \geq 0\}\) with a finite state space \(D = \{1, 2, \ldots, d\}\), which is called the background Markov chain. Let \(N(t), t \geq 0\), denote the total number of arrivals in time interval \((0, t]\), where \(N(0) = 0\). We assume that \(\{(N(t), J(t)); t \geq 0\}\) is a continuous-time Markov chain with state space \(F = \mathbb{Z}_+ \times D\) and conservative infinitesimal
generator $Q_{\text{BMAP}}$ given by

$$
Q_{\text{BMAP}} = \begin{pmatrix}
D(0) & D(1) & D(2) & D(3) & \cdots \\
O & D(0) & D(1) & D(2) & \cdots \\
O & O & D(0) & D(1) & \cdots \\
O & O & O & D(0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
$$

where $D(k) := (D_{i,j}(k))_{i,j \in \mathbb{D}}, k \in \mathbb{N}$, is an $d \times d$ nonnegative matrix and $D(0) := (D_{i,j}(0))_{i,j \in \mathbb{D}}$ is a $d \times d q$-matrix. It then follows that, for $t \geq 0$ and $\Delta t \geq 0$,

$$
P(N(t + \Delta t) - N(t) = k, J(t + \Delta t) = j \mid J(t) = i) = \left\{ \begin{array}{ll}
1 + D_{i,i}(0)\Delta t + o(\Delta t), & k = 0, i = j \in \mathbb{D}, \\
D_{i,j}(0)\Delta t + o(\Delta t), & k = 0, i \neq j, i,j \in \mathbb{D}, \\
D_{i,j}(k)\Delta t + o(\Delta t), & k \in \mathbb{N}, i,j \in \mathbb{D},
\end{array} \right. \quad (6.1)
$$

where $f(x) = o(g(x))$ represents $\lim_{x \to 0} |f(x)|/|g(x)| = 0$. According to (6.1), the BMAP is characterized by $\{D(k); k \in \mathbb{Z}_+\}$ and thus is denoted by BMAP $\{D(k); k \in \mathbb{Z}_+\}$.

It should be noted that the infinitesimal generator of the background Markov chain $\{J(t); t \geq 0\}$ is given by $D := \sum_{k=0}^{\infty} D(k)$, which is irreducible and conservative. We define $\eta$ as the stationary distribution vector of $D$ and define $\lambda$ as the arrival rate of BMAP $\{D(k); k \in \mathbb{Z}_+\}$, i.e.,

$$
\lambda = \eta \sum_{k=1}^{\infty} kD(k)e. \quad (6.2)
$$

To avoid triviality, we assume that $\lambda \in (0, \infty)$.

Let $\hat{D}(z) = \sum_{k=0}^{\infty} z^k D(k)$ and

$$
r_D = \sup \left\{ z \geq 0; \sum_{k=0}^{\infty} z^k D(k) \text{ is finite} \right\}.
$$

We then assume the following.

**Assumption 6.1** $r_D > 1$.

**Remark 6.1** Assumption [6.1] holds if $\{D(k); k \in \mathbb{N}\}$ is light-tailed, i.e., $D(k) \leq r^{-k} \Lambda$ for some $r > 1$ and $d \times d$ finite nonnegative matrix $\Lambda$.

For further discussion, we define $\hat{E}(z), 0 \leq z < r_D$, as

$$
\hat{E}(z) = I + \frac{\hat{D}(z)}{\max_{j \in \mathbb{D}} |D_{j,j}(0)|} \geq O. \quad (6.3)
$$
It follows from \cite{11} Theorem 8.3.1 that there exists nonnegative vectors \( \eta(z) = (\eta(z,j))_{j \in D} \) and \( u(z) = (u(z,j))_{j \in D} \) such that, for \( 0 \leq z < r_D \),

\[
\eta(z)\hat{E}(z) = \delta_E(z)\eta(z), \quad \hat{E}(z)u(z) = \delta_E(z)u(z), \\
\eta(z)u(z) = 1, \quad u(z) \geq e,
\]

(6.4) \hspace{1cm} (6.5)

where \( \delta_E(z) \) denotes the spectral radius of \( \hat{E}(z) \). Since \( D = \hat{D}(1) \) is an irreducible infinitesimal generator, the nonnegative matrix \( \hat{E}(z) \) is also irreducible for all \( 0 < z < r_D \), which implies that, for \( 0 < z < r_D \), \( \delta_E(z) \) is the Perron-Frobenius eigenvalue of \( \hat{E}(z) \) \cite{11} Theorem 8.4.4.

We now define \( \delta_D(z), 0 \leq z < r_D \), as

\[
\delta_D(z) = (\delta_E(z) - 1) \max_{j \in D} |D_{j,j}(0)|,
\]

where \( \delta_D(z) \) is increasing and convex because so is \( \delta_E(z) \) \cite{12}. It follows from (6.3) and (6.4) that

\[
\eta(z)\hat{D}(z) = \delta_D(z)\eta(z), \quad \hat{D}(z)u(z) = \delta_D(z)u(z).
\]

(6.6)

From (6.5) and (6.6), we have

\[
\delta_D(z) = \eta(z)\hat{D}(z)u(z).
\]

(6.7)

Note here that \( \delta_D(z), 0 < z < r_D \), is a simple eigenvalue of \( \hat{D}(z) \). In addition, Assumption [6.1] shows that \( \hat{D}(z) \) is analytic in a neighborhood of the point \( z = 1 \). Therefore, \( \delta_D(z), \eta(z) \) and \( u(z) \) are analytic at \( z = 1 \) \cite{2} Theorem 2.1. Differentiating (6.7) at \( z = 1 \) and using \( \eta(1) = \eta \) and \( u = e \), we obtain

\[
\delta_D'(1) = \eta\hat{D}'(1)e = \lambda,
\]

(6.8)

where the last equality holds due to (6.2). Note also that \( \delta_D(1) = 0 \) because \( D \) is an irreducible and conservative infinitesimal generator.

Next, we explain the queueing model considered in the section. The system consists of an infinite buffer and a possibly infinite number of servers (the number of servers is not specified for flexibility). Customers arrive at the system according to BMAP \( \{D(k); k \in \mathbb{Z}_+\} \). When there are \( k \) customers in the system at time \( t \), one of them leaves the system, independently of the other customers, in time interval \( (t, t+\Delta t] \) with probability \( \mu(k)\Delta t + o(\Delta t) \), where \( \mu(0) = 0 \) and \( \mu(k) \geq 0 \) for \( k \in \mathbb{N} \). Note here that the departure of a customer is caused by the completion of its service or the impatience with waiting for the service. In addition, the system can suffer from disasters, which can be regarded as negative customers that remove all the customers in the system including themselves on their arrivals. Disasters arrive at the system according to a Poisson process with rate \( \psi \geq 0 \), which is independent of the arrival and departure processes of (ordinary) customers.
We now define $L(t), t \geq 0,$ as the queue length, i.e., the number of customers in the systems at time $t.$ It then follows that the joint process $\{(L(t), J(t)); t \geq 0\}$ of the queue length and the background state is a continuous-time Markov chain with state space $\mathbb{F}$ and infinitesimal generator $Q = (q(k,i;\ell,j))_{(k,i),(\ell,j)\in\mathbb{F}}$ given by

$$Q = \begin{pmatrix}
D(0) & D(1) & D(2) & D(3) & \cdots \\
\psi I & D(0) - \bar{\mu}(1) I & D(1) & D(2) & \cdots \\
\psi I & \mu(2) I & D(0) - \bar{\mu}(2) I & D(1) & \cdots \\
\vdots & \vdots & \mu(3) I & D(0) - \bar{\mu}(3) I & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}, \quad (6.9)$$

where $\bar{\mu}(k) = \psi + \mu(k)$ for $k \in \mathbb{N}.$ It is easy to see that $Q \in \text{BM}_d.$

**Remark 6.2** Suppose that $\psi = 0$ and $\bar{\mu}(k) = \mu(k) = \mu + (k - 1)\mu'$ for $k \in \mathbb{N},$ where $\mu, \mu' \in (0, \infty).$ In this case, $Q$ is the infinitesimal generator of the joint process of a BMAP/M/1 queue with impatient customers and no disasters, where service times follow an exponential distribution with rate $\mu$ and “patient times” in queue are independent and identically distributed (i.i.d.) exponentially with rate $\mu'.$ In addition, if $\mu = \mu',$ then $Q$ is the infinitesimal generator of the joint process of a BMAP/M/$\infty$ queue with service rate $\mu.$

In what follows, we consider two cases: (a) no disasters occur; and (b) disasters can occur.

### 6.1 Case where no disasters occur

We make the following assumption.

**Assumption 6.2** (i) $Q$ is irreducible; (ii) $\psi = 0;$ and (iii) $\inf_{k \in \mathbb{N}} \mu(k) > \lambda.$

Let $G(z), 0 < z < r_D,$ denote

$$G(z) = \inf_{k \in \mathbb{N}} \mu(k) \cdot (1 - z^{-1}) - \delta_D(z), \quad 0 < z < r_D.$$

Recall here that $\delta_D(1) = 0, \delta_D'(1) = \lambda$ (see (6.8)) and $\delta_D(z)$ is increasing and convex for $z \in (0, r_D).$ It follows from these facts and Assumption 6.2 that $G(z)$ is continuous for $z \in (0, r_D)$ and

$$G(1) = 0, \quad G'(1) = \inf_{k \in \mathbb{N}} \mu(k) - \lambda > 0,$$

which show that there exists some $\beta > 1$ such that

$$c := \inf_{k \in \mathbb{N}} \mu(k)(1 - \beta^{-1}) - \delta_D(\beta) > 0. \quad (6.10)$$
Let $Q(k; \ell) = (q(k, i; \ell, j))_{i,j \in D}$ for $k, \ell \in \mathbb{Z}_+$. Let $v(k) = \beta^k u(\beta)$ for $k \in \mathbb{Z}_+$. From (6.6), (6.9) and $\psi = 0$, we then have

$$
\sum_{\ell=0}^{\infty} Q(0; \ell) v(\ell) = \sum_{\ell=0}^{\infty} D(\ell) v(\ell) = \hat{D}(\beta) u(\beta) = \delta_D(\beta) u(\beta)
$$

$$
= -cv(0) + (c + \delta_D(\beta)) u(\beta)
$$

$$
\leq -cv(0) + be, \quad (6.11)
$$

where

$$
b = (c + \delta_D(\beta)) \max_{j \in D} u(\beta, j). \quad (6.12)
$$

In addition, from (6.6), (6.10), (6.9) and $\psi = 0$, we have, for $k \in \mathbb{N}$,

$$
\sum_{\ell=0}^{\infty} Q(k; \ell) v(\ell) = \sum_{\ell=0}^{\infty} D(\ell) v(l + k) + \mu(k) v(k - 1) - \mu(k) v(k)
$$

$$
= \hat{D}(\beta) \beta^k u(\beta) - \mu(k)(1 - \beta^{-1}) \beta^k u(\beta)
$$

$$
= [\delta_D(\beta) - \mu(k)(1 - \beta^{-1})] \beta^k u(\beta)
$$

$$
\leq -cv(k). \quad (6.13)
$$

The inequalities (6.11) and (6.13) imply that Assumptions 5.1 and 5.2 hold for $\tilde{Q} := Q$, $c \in (0, \infty)$ and $b \in (0, \infty)$ given by (6.9), (6.10) and (6.12), respectively. As a result, it follows from Theorem 5.1 and Remark 5.1 that

$$
\| (n) \pi_n - \pi \| \leq \frac{4b}{c} (t_1^*(n) + 1) \exp \{-t_1^*(n)\}, \quad n \in \mathbb{N},
$$

where

$$
t_1^*(n) = \max \left\{ -\log \left( \frac{1}{2c} \sum_{j \in D} \frac{\mu(n) + |D_{j,j}(0)|}{u(\beta, j)} \beta^{-n} \right), 0 \right\}, \quad n \in \mathbb{N}.
$$

6.2 Case where disasters can occur

Instead of Assumption 6.2, we assume the following.

**Assumption 6.3** (i) $Q$ is irreducible; (ii) $\psi > 0$; and there exists some $K \in \mathbb{Z}_+$ such that

$$
c' := \inf_{k \geq K+1} \left\{ \mu(k)(1 - \beta^{-1}) + \psi(1 - \beta^{-k}) - \delta_D(\beta) \right\} > 0. \quad (6.14)
$$

**Remark 6.3** Assumption 6.3 holds if

$$
\liminf_{k \to \infty} \mu(k) > \frac{\delta_D(\beta) - \psi}{1 - \beta^{-1}}.
$$
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Let \( v'(k) = \beta^k u(\beta) \) for \( k \in \mathbb{Z}_+ \) and

\[
b' = \max_{0 \leq k \leq K} \left\{ [c' + \delta_D(\beta) - \mu(k)(1 - \beta^{-1}) - \psi(1 - \beta^{-k})] \beta^k \right\} \times \max_{j \in \mathbb{D}} u(\beta, j). \tag{6.15}\]

Proceeding as in the derivation of (6.11) and (6.13), we have

\[
\sum_{\ell=0}^{\infty} Q(0; \ell)v'(\ell) = -c'v'(0) + (c' + \delta_D(\beta))u(\beta), \tag{6.16}\]

and, for \( k \in \mathbb{N} \),

\[
\sum_{\ell=0}^{\infty} Q(k; \ell)v'(\ell) = [\delta_D(\beta) - \mu(k)(1 - \beta^{-1})] \beta^k u(\beta) + \psi v'(0) - \psi v'(k)
\]

\[
= [\delta_D(\beta) - \mu(k)(1 - \beta^{-1}) - \psi(1 - \beta^{-k})] \beta^k u(\beta)
\]

\[
= -c'v'(k) + [c' + \delta_D(\beta) - \mu(k)(1 - \beta^{-1}) - \psi(1 - \beta^{-k})] \beta^k u(\beta). \tag{6.17}\]

Applying (6.14) and (6.15) to (6.16) and (6.17) yields

\[
\sum_{\ell=0}^{\infty} Q(k; \ell)v'(\ell) \leq -c'v'(k) + b' e, \quad k = 0, 1, \ldots, K, \tag{6.18}\]

\[
\sum_{\ell=0}^{\infty} Q(k; \ell)v'(\ell) \leq -c'v'(k), \quad k = K + 1, K + 2, \ldots. \tag{6.19}\]

It follows from (6.18) and (6.19) that (5.33) in Corollary 5.1 holds for \( \tilde{Q} := Q, c' \in (0, \infty) \) and \( b' \in (0, \infty) \) given by (6.9), (6.14) and (6.15), respectively, where \( K \in \mathbb{Z}_+ \) is fixed such that (6.14) holds. Note here that \( \tilde{Q}(K, 0)e = Q(K, 0)e = \psi e > 0 \). Thus, according to (5.36), fix \( B = b' \psi^{-1} \). Furthermore, according to (5.34), (5.35) and (5.37), fix

\[
c = \frac{c'}{1 + b' \psi^{-1}},
\]

\[
b = b' \left( 1 - \psi^{-1} \min_{i \in \mathbb{D}} \sum_{j \in \mathbb{D}} D_{i,j}(0) \right),
\]

\[
v(k,i) = \begin{cases} v'(0,i), & k = 0, i \in \mathbb{D}, \\
v'(k,i) + b' \psi^{-1}, & k \in \mathbb{N}, i \in \mathbb{D}. \end{cases}
\]

Note also that

\[
\tilde{q}(n,j;n,j) = q(n,j;n,j) = -\tilde{\mu}(n) + D_{i,j}(0)
\]

\[
= -\psi - \mu(n) - |D_{i,j}(0)|.
\]
Consequently, if follows from Corollary 5.1 that, for \( n \in \mathbb{N} \),
\[
\| (n) \pi_n - \pi \| \leq \frac{4b'(1+b'\psi^{-1})}{c'} \left( 1 - \psi^{-1} \min_{i \in \mathcal{D}} \sum_{j \in \mathcal{D}} D_{i,j}(0) \right) 
\times (t_2^*(n) + 1) \exp \{-t_2^*(n)\},
\]
where \( t_2^*(n) \), \( n \in \mathbb{N} \), is given by
\[
t_2^*(n) = \max \left\{ - \log \left( \frac{1 + b'\psi^{-1}}{2c'} \sum_{j \in \mathcal{D}} \frac{\psi + \mu(n) + |D_{j,j}(0)|(1 + b'\psi^{-1} \beta^{-n})}{u(\beta, j)} \right), 0 \right\}.
\]

### A Pathwise ordering

This appendix presents two lemmas on the pathwise ordering associated with BMMCs. For this purpose, we consider two Markov chains \( \{(X_t, J_t); t \geq 0\} \) and \( \{\{\tilde{X}_t, \tilde{J}_t\}; t \geq 0\} \) with infinitesimal generators \( Q = (q(k, i; \ell, j))_{(k,i),(\ell,j) \in \mathbb{F} \times \mathbb{N}} \) and \( \tilde{Q} = (\tilde{q}(k, i; \ell, j))_{(k,i),(\ell,j) \in \mathbb{F} \times \mathbb{N}} \), respectively, which have the same state space \( \mathbb{F} \times \mathbb{N} \). In what follows, we do not necessarily assume that \( Q \) and \( \tilde{Q} \) are ergodic.

**Lemma A.1 (Pathwise ordering of a BMMC)** If \( Q \) is regular and \( Q \in \text{BM}_d \), then there exist two regular-jump Markov chains \( \{(X'_t, J'_t); t \geq 0\} \) and \( \{(X''_t, J''_t); t \geq 0\} \) with infinitesimal generator \( Q \) on a common probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that
\[
X'_t(\omega) \leq X''_t(\omega), \quad J'_t(\omega) = J''_t(\omega) \quad \text{for all } t > 0,
\]
for any \( \omega \in \Omega \) with \( X'_0(\omega) \leq X''_0(\omega) \) and \( J'_0(\omega) = J''_0(\omega) \).

**Proof.** We fix \( \delta > 0 \) arbitrarily. It follows from Lemma 3.2 that \( P^{(\delta)} \in \text{BM} \). Therefore, according to [19, Lemma A.1], we can construct two discrete-time Markov chains \( \{(Y'_\delta, H'_\delta); \nu \in \mathbb{Z}_+\} \) and \( \{(Y''_\delta, H''_\delta); \nu \in \mathbb{Z}_+\} \) with transition probability matrix \( P^{(\delta)} \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that, for any \( \omega \in \Omega \) with \( Y'_{\delta,0}(\omega) \leq Y''_{\delta,0}(\omega) \) and \( H'_{\delta,0}(\omega) = H''_{\delta,0}(\omega) \),
\[
Y'_{\delta,\nu}(\omega) \leq Y''_{\delta,\nu}(\omega), \quad H'_{\delta,\nu}(\omega) = H''_{\delta,\nu}(\omega) \quad \text{for all } \nu \in \mathbb{N}.
\]

Using the two Markov chains, we define two stochastic processes \( \{(X'_\delta, J'_\delta); t \geq 0\} \) and \( \{(X''_\delta, J''_\delta); t \geq 0\} \) on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) as follows:
\[
X'_\delta = Y'_\delta, \quad J'_\delta = H'_\delta, \quad \nu \delta \leq t < (\nu + 1) \delta, \quad \nu \in \mathbb{Z}_+,
\]
\[
X''_\delta = Y''_\delta, \quad J''_\delta = H''_\delta, \quad \nu \delta \leq t < (\nu + 1) \delta, \quad \nu \in \mathbb{Z}_+.
\]

It then holds that, for any \( \omega \in \Omega \) with \( X'_{\delta,0}(\omega) \leq X''_{\delta,0}(\omega) \) and \( J'_{\delta,0}(\omega) = J''_{\delta,0}(\omega) \),
\[
X'_\delta(\omega) \leq X''_\delta(\omega), \quad J'_\delta(\omega) = J''_\delta(\omega) \quad \text{for all } t > 0. \tag{A.1}
\]
We also define
\[ \mathcal{G}'_{\delta,s} = \bigcup_{\alpha \geq \delta} \mathcal{F}'_{\alpha,s}, \quad \mathcal{G}''_{\delta,s} = \bigcup_{\alpha \geq \delta} \mathcal{F}''_{\alpha,s}, \quad \delta > 0, \]
where \( \mathcal{F}'_{\alpha,s} \) and \( \mathcal{F}''_{\alpha,s}, \alpha > 0, \) are the \( \sigma \)-algebras generated by \( \{(X^r_{\alpha,u}, J^r_{\alpha,u}); 0 \leq u \leq s\} \) and \( \{(X'^r_{\alpha,u}, J'^r_{\alpha,u}); 0 \leq u \leq s\} \), respectively. Note here that \( \{(X^r_{\delta,t}, J^r_{\delta,t}); t \geq 0\} \) is a semi-Markov process with the embedded Markov chain \( \{(X^r_{\delta,t}, J^r_{\delta,t}); t \geq 0\} = (Y^r_{\delta,t}, H^r_{\delta,t}); \nu \in \mathbb{Z}_+ \). Thus, for \( s \in [\nu \delta, (\nu + 1) \delta), t \in [(\nu + 1) \delta, (\nu + 2) \delta) \) and \( \nu \in \mathbb{Z}_+ \),
\[
\mathbb{P}(X'_{\delta,t} = \ell, J'_{\delta,t} = j \mid X'_s = k, J'_s = i, \mathcal{G}'_{\delta,s}) = \mathbb{P}(Y'_{\delta,t} = \ell, H'_{\delta,t} = j \mid Y'_s = k, H'_s = i) = p^{(\delta)}(k, i; \ell, j), \quad (k, i; \ell, j) \in \mathbb{F}^2. \tag{A.2}
\]
It follows from (A.2) and (3.2) that, for all \( t \geq s \geq 0 \),
\[
\mathbb{P}(X''_{\delta,t} = \ell, J''_{\delta,t} = j \mid X''_s = k, J''_s = i, \mathcal{G}''_{\delta,s}) = p^{((n_{\delta,t} - n_{\delta,s})\delta)}(k, i; \ell, j), \quad (k, i; \ell, j) \in \mathbb{F}^2, \tag{A.3}
\]
where \( n_{\delta,u} = \sup\{n \in \mathbb{Z}_+; n \delta \leq u\} \) for \( u \geq 0 \). Similarly, for all \( t \geq s \geq 0 \),
\[
\mathbb{P}(X'^{''}_{\delta,t} = \ell, J'^{''}_{\delta,t} = j \mid X'^{''}_s = k, J'^{''}_s = i, \mathcal{G}'^{''}_{\delta,s}) = p^{((n_{\delta,t} - n_{\delta,s})\delta)}(k, i; \ell, j), \quad (k, i; \ell, j) \in \mathbb{F}^2. \tag{A.4}
\]
We now define
\[
\{(X'_t, J'_t); t \geq 0\} = \lim_{\delta \downarrow 0}\{(X^r_{\delta,t}, J^r_{\delta,t}); t \geq 0\}, \quad \{(X''_t, J''_t); t \geq 0\} = \lim_{\delta \downarrow 0}\{(X'^{''}_{\delta,t}, J'^{''}_{\delta,t}); t \geq 0\}.
\]
It then follows from (A.1) that, for any \( \omega \in \Omega \) with \( X'_0(\omega) \leq X''_0(\omega) \) and \( J'_0(\omega) = J''_0(\omega), \)
\[
X'_t(\omega) \leq X''_t(\omega), \quad J'_t(\omega) = J''_t(\omega) \quad \text{for all } t > 0.
\]
Recall here that \( Q \) is regular and thus \( \lim_{\delta \downarrow 0} \mathbb{P}^{(\delta)} = I \) (see subsection 3.1). Therefore, we have
\[
\lim_{\delta \downarrow 0} \mathbb{P}^{((n_{\delta,t} - n_{\delta,s})\delta)} = \mathbb{P}^{(t-s)}, \quad 0 \leq s \leq t. \tag{A.5}
\]
It follows from (A.3)–(A.5) and the continuity of probability \([5 \text{ Chapter 1, Theorem 1.1}] \) that, for all \( t \geq s \geq 0 \) and \( (k, i; \ell, j) \in \mathbb{F}^2 \),
\[
\mathbb{P}(X'_t = \ell, J'_t = j \mid X'_s = k, J'_s = i, \cup_{\delta > 0} \mathcal{G}'_{\delta,s}) = \mathbb{P}^{(t-s)}(k, i; \ell, j),
\mathbb{P}(X''_t = \ell, J''_t = j \mid X''_s = k, J''_s = i, \cup_{\delta > 0} \mathcal{G}'^{''}_{\delta,s}) = \mathbb{P}^{(t-s)}(k, i; \ell, j).
\]
As a result, \( \{(X'_t, J'_t)\} \) and \( \{(X''_t, J''_t)\} \) are regular-jump Markov chains characterized by the common transition matrix function \( \mathbb{P}^{(t)} \) with the regular infinitesimal generator \( Q \). The proof is completed. \( \square \)
Lemma A.2 (Pathwise ordering from the block-wise dominance relation) Suppose that $Q \prec_d \tilde{Q}$ and either $Q \in \text{BM}_d$ or $\tilde{Q} \in \text{BM}_d$. Under these conditions, the following are true:

(a) If $N < \infty$, i.e., $Q$ and $\tilde{Q}$ are finite infinitesimal generators, then there exist two regular-jump Markov chains $\{(X'_t, J'_t); t \geq 0\}$ and $\{(\tilde{X}'_t, \tilde{J}'_t); t \geq 0\}$ with infinitesimal generators $Q$ and $\tilde{Q}$, respectively, on a common probability space $(\mathcal{P}, \mathcal{F}, \Omega)$ such that

$$X'_t(\omega) \leq \tilde{X}'_t(\omega), \quad J'_t(\omega) = \tilde{J}'_t(\omega) \quad \text{for all } t > 0,$$

for any $\omega \in \Omega$ with $X'_0(\omega) \leq \tilde{X}'_0(\omega)$ and $J'_0(\omega) = \tilde{J}'_0(\omega)$.

(b) If $\tilde{Q}$ is regular, then statement (a) holds without $N < \infty$.

Proof of Lemma A.2 We first prove statement (a). Since $Q$ and $\tilde{Q}$ are finite infinitesimal generators, we can readily show that

$$\exp\{Qt\} \prec_d \exp\{\tilde{Q}t\},$$

in a way similar to the derivation of (3.33). Thus, we have $P^{(t)} \prec_d \tilde{P}^{(t)}$. Furthermore, since either $Q \in \text{BM}_d$ or $\tilde{Q} \in \text{BM}_d$, it follows from Lemma 3.2 that either $P^{(t)} \in \text{BM}_d$ or $\tilde{P}^{(t)} \in \text{BM}_d$ for all $t \geq 0$.

We now fix $\delta > 0$ arbitrarily. According to [19] Lemma A.2, we can construct two discrete-time Markov chains $\{(Y'_{\delta,\nu}, H'_{\delta,\nu}); \nu \in \mathbb{Z}_+\}$ and $\{(\tilde{Y}'_{\delta,\nu}, \tilde{H}'_{\delta,\nu}); \nu \in \mathbb{Z}_+\}$ with transition probability matrices $P^{(\delta)}$ and $\tilde{P}^{(\delta)}$, respectively, on the common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that, for any $\omega \in \Omega$ with $Y'_{\delta,0}(\omega) \leq \tilde{Y}'_{\delta,0}(\omega)$ and $H'_{\delta,0}(\omega) = \tilde{H}'_{\delta,0}(\omega)$,

$$Y'_{\delta,\nu}(\omega) \leq \tilde{Y}'_{\delta,\nu}(\omega), \quad H'_{\delta,\nu}(\omega) = \tilde{H}'_{\delta,\nu}(\omega) \quad \text{for all } \nu \in \mathbb{N}.$$

We then define two stochastic processes $\{(X'_{\delta,t}, J'_{\delta,t}); t \geq 0\}$ and $\{(\tilde{X}'_{\delta,t}, \tilde{J}'_{\delta,t}); t \geq 0\}$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ as follows:

$$X'_{\delta,t} = Y'_{\delta,\nu}, \quad J'_{\delta,t} = H'_{\delta,\nu}; \quad \nu \delta \leq t < (\nu+1)\delta, \nu \in \mathbb{Z}_+,$$

$$\tilde{X}'_{\delta,t} = \tilde{Y}'_{\delta,\nu}, \quad \tilde{J}'_{\delta,t} = \tilde{H}'_{\delta,\nu}; \quad \nu \delta \leq t < (\nu+1)\delta, \nu \in \mathbb{Z}_+.$$

By definition, for any $\omega \in \Omega$ with $X'_{\delta,0}(\omega) \leq \tilde{X}'_{\delta,0}(\omega)$ and $J'_{\delta,0}(\omega) = \tilde{J}'_{\delta,0}(\omega)$,

$$X'_{\delta,t}(\omega) \leq \tilde{X}'_{\delta,t}(\omega), \quad J'_{\delta,t}(\omega) = \tilde{J}'_{\delta,t}(\omega) \quad \text{for all } t > 0.$$

Let

$$\{(X'_t, J'_t); t \geq 0\} = \lim_{\delta \downarrow 0}\{(X'_{\delta,t}, J'_{\delta,t}); t \geq 0\},$$

$$\{(\tilde{X}'_t, \tilde{J}'_t); t \geq 0\} = \lim_{\delta \downarrow 0}\{(\tilde{X}'_{\delta,t}, \tilde{J}'_{\delta,t}); t \geq 0\}.$$
Clearly, for any $\omega \in \Omega$ with $X'_0(\omega) \leq \tilde{X}'_0(\omega)$ and $J'_0(\omega) = \tilde{J}'_0(\omega)$,

$$X'_t(\omega) \leq \tilde{X}'_t(\omega), \quad J'_t(\omega) = \tilde{J}'_t(\omega) \quad \text{for all } t > 0.$$ 

In addition, proceeding as in the proof of Lemma A.1, we can readily prove that \{(X'_t, J'_t)\} and \{(\tilde{X}'_t, \tilde{J}'_t)\} are regular-jump Markov chains with infinitesimal generators $Q$ and $\tilde{Q}$, respectively. As a result, statement (a) is proved.

As for statement (b), it follows from Lemma 3.4 (b) that $P^{(t)} \prec_d \tilde{P}^{(t)}$ for all $t \geq 0$. Therefore, we can prove statement (b) in the same way as the proof of statement (a). The details are omitted.

**Remark A.1** The pathwise-ordered continuous-time Markov chains that appear in Lemmas A.1 and A.2 can be generated via their respective skeletons (see the proofs of the lemmas), which are constructed in the way described in the proofs of Lemmas A.1 and A.2 of [19]. As shown in those proofs, the pathwise-ordered discrete-time Markov chains therein are defined by the update functions $F^{-1}(u \mid k, i, j)$ and $\tilde{F}^{-1}(u \mid k, i, j)$ unique to the respective transition probability matrices, together with common sequences of i.i.d. uniform random variables. Therefore, any pair of such pathwise-ordered Markov chains with a common transition probability matrix has the first-meeting-lasts-forever property, that is, the pathwise-ordered Markov chains run together (i.e., their trajectories coincide) forever after their first meeting time. As a result, we can assume that the first-meeting-lasts-forever property holds for pathwise-ordered continuous-time Markov chains with a common infinitesimal generator, which originate from Lemmas A.1 and A.2 in this paper.

**B Basic lemmas**

This appendix presents basic lemmas, which are used in Section 5.

**Lemma B.1** Suppose that $\tilde{Q}$ is ergodic. If Assumption 5.2 holds, then

$$\{M_t := e^{ct} v(\tilde{X}_t, \tilde{J}_t)I_{(\tilde{\tau}_0 > t)}; t \geq 0\}$$

is supermartingale, where $\tilde{\tau}_0 = \inf\{t \geq 0 : \tilde{X}_t = 0\}$.

**Proof.** Since \{(\tilde{X}_t, \tilde{J}_t)\} is a time-homogeneous Markov chain, it suffices to prove that $E_{(k,i)}[M_t] \leq v(k, i)$ for all $t \geq 0$ and $(k, i) \in \mathbb{F}$. Note that if $(\tilde{X}_0, \tilde{J}_0) = (0, i)$ then $\tilde{\tau}_0 = 0$ and thus

$$E_{(0,i)}[M_t] = 0 < 1 \leq v(0, i), \quad t \geq 0, \; i \in \mathbb{D},$$

where the last inequality is due to $v \geq e$.

In what follows, we prove that

$$E_{(k,i)}[M_t] \leq v(k, i), \quad t \geq 0, \; (k, i) \in \mathbb{N} \times \mathbb{D}. \quad (B.1)$$
Let $\tilde{t}_n = \inf\{t \geq 0 : \tilde{X}_t \geq n\}$ for $n \in \mathbb{N}$. It then follows from the ergodicity of $\tilde{Q}$ that $P(\lim_{n \to \infty} \tilde{t}_n = \infty) = 1$ and $\{\tilde{t}_n; n \in \mathbb{N}\}$ is a nondecreasing sequence. Thus, using the monotone convergence theorem, we obtain

$$E_{(k,i)} \left[ v(\tilde{X}_t, \tilde{J}_t) I_{\{\tilde{t}_n > t\}} \right] = \lim_{n \to \infty} E_{(k,i)} \left[ v(\tilde{X}_t, \tilde{J}_t) I_{\{\tilde{t}_0 > t, \tilde{t}_{n+1} > t\}} \right] = \lim_{n \to \infty} g_t^{[1,n]}(k,i), \quad t \geq 0, \ (k,i) \in \mathbb{N} \times \mathbb{D}. \quad (B.2)$$

where, for $(k,i) \in \mathbb{F}^{[1,n]} := \{1, 2, \ldots, n\} \times \mathbb{D}$,

$$g_t^{[1,n]}(k,i) = E_{(k,i)} \left[ v(\tilde{X}_t, \tilde{J}_t) I_{\{\tilde{t}_0 > t, \tilde{t}_{n+1} > t\}} \right], \quad t \geq 0.$$

It then follows that column vector $g_t^{[1,n]} := (g_t^{[1,n]}(k,i))_{(k,i) \in \mathbb{F}^{[1,n]}}$ satisfies

$$g_t^{[1,n]} = \exp\{\tilde{Q}^{[1,n]} t\} v^{[1,n]}, \quad t \geq 0, \quad (B.3)$$

where $\tilde{Q}^{[1,n]} = (\tilde{q}(k,i; \ell,j))_{(k,i),(\ell,j) \in \mathbb{F}^{[1,n]}}$ and $v^{[1,n]} = (v(k,i))_{(k,i) \in \mathbb{F}^{[1,n]}}$. Furthermore, it follows from (5.2) that $\tilde{Q}_+ v^{[1,n]} \leq -fv^{[1,n]}$. Using this inequality and (B.3), we have

$$g_t^{[1,n]} = \exp\{\tilde{Q}^{[1,n]} t\} v^{[1,n]} \leq e^{-ct}v^{[1,n]}, \quad t \geq 0.$$

Substituting this inequality into (B.2) yields

$$E_{(k,i)} \left[ v(\tilde{X}_t, \tilde{J}_t) I_{\{\tilde{t}_0 > t\}} \right] \leq e^{-ct}v(k,i), \quad t \geq 0, \ (k,i) \in \mathbb{N} \times \mathbb{D},$$

which shows that (B.1) holds. \hfill \Box

**Lemma B.2** If Assumption 3.1 holds and $Q$ is irreducible, then, for all $n \in \mathbb{N}$, $t \geq 0$ and $(k,i) \in \mathbb{F}^{\leq n}$,

$$\| (n)\mathbf{P}_n^{(t)}(k,i) - \mathbf{P}^{(t)}(k,i) \| \leq 2 \int_0^t \sum_{(\ell,j) \in \mathbb{F}} (n)\mathbf{P}_n^{(u)}(k,i; \ell,j) \times \sum_{(\ell',j') \in \mathbb{F}^{\geq n}} |q(\ell,j; \ell',j')| du, \quad (B.4)$$

where $\mathbf{P}^{(t)}(k,i) = (p^{(t)}(k,i; \ell,j))_{(\ell,j) \in \mathbb{F}}$ and $(n)\mathbf{P}_n^{(t)}(k,i) = ((n)\mathbf{P}_n(k,i; \ell,j))_{(\ell,j) \in \mathbb{F}}$.

**Proof.** For $n \in \mathbb{N}$, $u \geq 0$ and $(k,i) \in \mathbb{F}$, let $f_n^{(u)}(k,i)$ denote

$$f_n^{(u)}(k,i) = \| (n)\mathbf{P}_n^{(u)}(k,i) - \mathbf{P}^{(u)}(k,i) \| = \sum_{(\ell,j) \in \mathbb{F}} |(n)\mathbf{P}_n^{(u)}(k,i; \ell,j) - p^{(u)}(k,i; \ell,j)|. \quad (B.5)$$
From the Chapman-Kolmogorov equation, we have
\[
\begin{align*}
(n)P_{n}^{(u+\delta)} - P^{(u+\delta)} &= (n)P_{n}^{(u)}(n)P_{n}^{(\delta)} - P^{(u)}P^{(\delta)} \\
&= (n)P_{n}^{(u)}(n)P_{n}^{(\delta)} - (n)P_{n}^{(u)}P^{(\delta)} + (n)P_{n}^{(u)} - P^{(u)}P^{(\delta)}.
\end{align*}
\]

Thus, we obtain, for \(\delta > 0\) and \((k, i) \in \mathbb{F}^n\),
\[
\begin{align*}
f_{n}^{(u+\delta)}(k, i) &\leq \sum_{(\ell, j) \in \mathbb{F}} (n)P_{n}^{(u)}(k; i, \ell, j) \left\| (n)P_{n}^{(\delta)}(\ell, j) - P^{(\delta)}(\ell, j) \right\| \\
&\quad + \sum_{(\ell, j) \in \mathbb{F}} \left| (n)P_{n}^{(u)}(k; i, \ell, j) - p^{(u)}(k; i, \ell, j) \right| \sum_{(\ell', j') \in \mathbb{F}} p^{(\delta)}(\ell, j; \ell', j') \\
&= \sum_{(\ell, j) \in \mathbb{F}^n} (n)P_{n}^{(u)}(k; i, \ell, j) \left\| (n)P_{n}^{(\delta)}(\ell, j) - P^{(\delta)}(\ell, j) \right\| + f_{n}^{(u)}(k, i), \quad (\text{B.6})
\end{align*}
\]

where the last equality holds due to (5.1), (B.5) and \(\sum_{(\ell', j') \in \mathbb{F}} p^{(\delta)}(\ell, j; \ell', j') = 1\). Note here that
\[
(n)P_{n}^{(0)}(\ell, j; \ell', j') = p^{(0)}(\ell, j; \ell', j') = \chi_{(\ell, j)}(\ell', j'), \quad (\ell, j; \ell', j') \in \mathbb{F}^2.
\]

This equation and the triangle inequality yield, for \((\ell, j) \in \mathbb{F}\),
\[
\begin{align*}
\left\| (n)P_{n}^{(\delta)}(\ell, j) - P^{(\delta)}(\ell, j) \right\| &\leq \left\| p^{(\delta)}(\ell, j) - P^{(0)}(\ell, j) \right\| + \left\| (n)P_{n}^{(\delta)}(\ell, j) - (n)P_{n}^{(0)}(\ell, j) \right\| \\
&= 2 \left\{ 1 - p^{(\delta)}(\ell, j; \ell, j) \right\} + 2 \left\{ 1 - (n)P_{n}^{(\delta)}(\ell, j; \ell, j) \right\}.
\end{align*}
\]

It follows from Assumption 3.1 and Lemma 4.3 that both \(Q\) and \((n)Q_{n}\) are stable and conservative and their respective transition matrix functions are continuous. Thus, we have [6, Theorem II.3.1]
\[
\begin{align*}
1 - p^{(\delta)}(\ell, j; \ell, j) &\leq \delta |q(\ell, j; \ell, j)|, \quad (\ell, j) \in \mathbb{F}, \\
1 - (n)P_{n}^{(\delta)}(\ell, j; \ell, j) &\leq \delta \left| (n)q_{n}(\ell, j; \ell, j) \right|, \quad (\ell, j) \in \mathbb{F}.
\end{align*}
\]

Applying these inequalities to (B.7) yields
\[
\begin{align*}
\left\| (n)P_{n}^{(\delta)}(\ell, j) - P^{(\delta)}(\ell, j) \right\| &\leq 2\delta \cdot \left\{ |q(\ell, j; \ell, j)| + \left| (n)q_{n}(\ell, j; \ell, j) \right| \right\} \\
&\leq 4\delta \cdot |q(\ell, j; \ell, j)|,
\end{align*}
\]

where the last inequality follows from (4.8). Note here that
\[
\begin{align*}
\lim_{\delta \downarrow 0} \frac{(n)P_{n}^{(\delta)}(\ell, j; \ell', j') - \chi_{(\ell, j)}(\ell', j')}{\delta} &= (n)q_{n}^{(\delta)}(\ell, j; \ell', j'), \quad (\ell, j; \ell', j') \in \mathbb{F}^2, \\
\lim_{\delta \downarrow 0} \frac{p^{(\delta)}(\ell, j; \ell', j') - \chi_{(\ell, j)}(\ell', j')}{\delta} &= q^{(\delta)}(\ell, j; \ell', j'), \quad (\ell, j; \ell', j') \in \mathbb{F}^2.
\end{align*}
\]
Therefore, using the dominated convergence theorem, we obtain
\[
\lim_{\delta \to 0} \left\| \left( n \right) p^{(\delta)}_n(\ell, j) - p^{(\delta)}(\ell, j) \right\| = \left\| \left( n \right) q_n(\ell, j) - q(\ell, j) \right\|, \quad (\ell, j) \in \mathbb{F}, \quad (B.8)
\]
where \( (n)q_n(k, i) = (n)q_n(k; \ell, j))_{(\ell, j) \in \mathbb{F}} \) and \( q(k, i) = (q(k; \ell, j))_{(\ell, j) \in \mathbb{F}} \) for \( (k, i) \in \mathbb{F} \). Combining (B.6) and (B.8), we have
\[
\frac{\partial}{\partial u} f_n^{(u)}(k, i) \leq \sum_{(\ell, j) \in \mathbb{F}^{\leq n}} (n)p^{(u)}_n(k; \ell, j) \left\| (n)q_n(\ell, j) - q(\ell, j) \right\|. \quad (B.9)
\]
From (4.8), we also have
\[
\left\| (n)q_n(\ell, j) - q(\ell, j) \right\| = 2 \sum_{(\ell', j') \in \mathbb{F}^{> n}} |q(\ell, j; \ell', j')|, \quad (\ell, j) \in \mathbb{F}^{\leq n}.
\]
Substituting this into (B.9) and using (5.1) yield, for \( (k, i) \in \mathbb{F}^{\leq n} \),
\[
\frac{\partial}{\partial u} f_n^{(u)}(k, i) \leq 2 \sum_{(\ell, j) \in \mathbb{F}^{\leq n}} (n)p^{(u)}_n(k; \ell, j) \sum_{(\ell', j') \in \mathbb{F}^{> n}} |q(\ell, j; \ell', j')| = 2 \sum_{(\ell, j) \in \mathbb{F}^{\leq n}} (n)p^{(u)}_n(k; \ell, j) \sum_{(\ell', j') \in \mathbb{F}^{> n}} |q(\ell, j; \ell', j')|. \quad (B.10)
\]
Note here that \( f_n^{(0)}(k, i) = \sum_{(\ell, j) \in \mathbb{F}} |x(k, i)(\ell, j) - x(k, i)(\ell, j)| = 0 \). As a result, integrating both sides of (B.10) with respect to \( u \) from 0 to \( t \), we obtain (B.4). \( \square \)

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