ON RADON TRANSFORMS BETWEEN LINES AND HYPERPLANES

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Abstract. We obtain new inversion formulas for the Radon transform and its dual between lines and hyperplanes in $\mathbb{R}^n$. The Radon transform in this setting is non-injective and the consideration is restricted to the so-called quasi-radial functions that are constant on symmetric clusters of lines. For the corresponding dual transform, which is injective, explicit inversion formulas are obtained both in the symmetric case and in full generality. The main tools are the Funk transform on the sphere, the Radon-John $d$-plane transform in $\mathbb{R}^n$, the Grassmannian modification of the Kelvin transform, and the Erdélyi-Kober fractional integrals.

1. Introduction

Let $\mathcal{L}$ and $\mathcal{H}$ be the manifolds of all non-oriented lines $\ell$ and all non-oriented hyperplanes $h$ in $\mathbb{R}^n$, respectively. In the present article we consider the Radon-like transform that takes functions on $\mathcal{L}$ to functions on $\mathcal{H}$. We also consider the corresponding dual transform acting in the opposite direction. Both transforms are defined by the integrals

\begin{equation}
(Rf)(h) = \int_{\ell \subset h} f(\ell) \, d_h \ell, \quad (R^*\varphi)(\ell) = \int_{h \supset \ell} \varphi(h) \, d_\ell h,
\end{equation}

where the integration is performed with respect to the corresponding canonical measures. Our aim is to obtain explicit inversion formulas for $R$ and $R^*$ in the cases when these operators are injective.

The manifolds $\mathcal{L}$ and $\mathcal{H}$ are important representatives of the more general vector bundles $G(n, k)$ over the Grassmann manifolds $G_{n,k}$ of $k$-dimensional linear subspaces of $\mathbb{R}^n$. Elements of $G(n, k)$ are non-oriented $k$-dimensional affine planes in $\mathbb{R}^n$. The corresponding Radon

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transforms $R_{k,k'}$, $R^*_{k,k'}$ that take functions on $G(n,k)$ to functions on $G(n,k')$, $1 \leq k < k' \leq n - 1$, and backwards, were considered by Gonzalez and Kakehi [5, 6] who studied these operators on smooth functions in the group-theoretic terms. The paper [5] contains an explicit inversion formula for $R_{k,k'}$ in the case of $k' - k$ even. This formula was obtained by applying the Fourier transform over fibers and using the corresponding inversion formula for compact Grassmannians due to Kakehi [10]. We also mention the paper [4] by Gonzalez that contains the range description of the plane-to-line transform for smooth functions on $\mathbb{R}^3$.

Inversion formulas of different kind for both $R_{k,k'}$ and $R^*_{k,k'}$ in Lebesgue spaces were obtained by the first-named co-author [11] who reduced the problem to the compact case with the aid of a certain analogue of the stereographic projection. The dimensions $k$ and $k'$ in [11] can be of arbitrary parity and the corresponding inversion formula for the compact Grassmannians was borrowed from Grinberg and Rubin [7].

One should also mention the work by Strichartz [18], who developed harmonic analysis on Grassmannian bundles, and a series of publications related to integral geometric problems on compact Grassmannians; see, e.g., [1, 2, 7, 10, 13, 17, 19, 20] and references therein. The boundedness of the operators $R_{k,k'}$ and $R^*_{k,k'}$ in $L^p$ spaces with power weights was studied in [15].

It is a challenging open problem to find an alternative approach to inversion formulas for $R_{k,k'}$ and $R^*_{k,k'}$ that would be straightforward (without stereographic projection), available for all admissible $k$ and $k'$, and applicable to large classes of functions (not only to $C^\infty$ and rapidly decreasing ones). Here one should take into account that for $1 \leq k < k' \leq n - 1$, the operators $R_{k,k'}$ and $R^*_{k,k'}$ are injective if and only if $k + k' \leq n - 1$ and $k + k' \geq n - 1$, respectively; see [11, Propositions 1.3 and 1.4] for precise statements. Because $\dim G(n,k) = (n-k)(k+1)$, the condition $k + k' \leq n - 1$ is equivalent to $\dim G(n,k) \leq \dim G(n,k')$.

In the present article we give a solution to the aforementioned inversion problem in the case $1 = k < k' = n - 1$. This simple case is less technical and reflects basic features of Radon transforms on Grassmannian bundles.

The paper is organized as follows. Section 2 contains necessary preliminaries. It is worth noting that planes in $\mathbb{R}^n$ can be parametrized in different ways and our main concern in this section is to choose suitable parametrizations.

Section 3 is devoted to the operator $R \equiv R_{1,n-1}$. Because

$$\dim G(n,1) = 2n - 2 > n = \dim G(n,n-1),$$
this operator is non-injective. The situation changes if we restrict \( R \) to functions \( f \) satisfying additional symmetry. Specifically, we assume \( f \) to be constant on symmetric clusters of parallel lines in every direction. We call such functions \( \text{quasi-radial} \). It will be shown that if \( f \) is quasi-radial, then \( Rf \) is a tensor product of the classical Funk transform on the sphere \([3, 9, 16]\) and a one-dimensional fractional integration operator of the Erdélyi-Kober type \([16]\). Both can be explicitly inverted. The main result of this section is presented by Theorem 3.2.

In Section 4 we obtain explicit inversion formulas for the dual transform \( R^* \equiv R^*_{1,n-1} \). Here we consider three different approaches. The first one deals with even locally integrable functions \( \varphi \) on \( \mathcal{H} \) such that \( \varphi(h) = \varphi(-h) \) for all \( h \in \mathcal{H} \), and relies on averaging over the line clusters. The second approach covers the general case and invokes a Grassmannian modification of the Kelvin transform introduced in \([11]\). Here \( \varphi \) is assumed to be continuous with prescribed behavior at the origin and at infinity or belonging to some weighted \( L^p \)-space. The third approach is based on a certain alternative parametrization of line and hyperplanes. Main results of this section are presented by Theorems 4.3, 4.11, 4.13, and 4.17.

The methods of the present article can be generalized to the Radon transforms \( R_{k,k'} \) and \( R^*_{k,k'} \) for arbitrary \( 0 < k < k' < n \). We plan to consider this case in the forthcoming publication.

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2. Preliminaries

2.1. Notation. We recall that \( G(n, k) \) is the affine Grassmann manifold of non-oriented \( k \)-dimensional planes in \( \mathbb{R}^n \), \( 0 < k < n \); \( G_{n,k} \) is the compact Grassmann manifold of \( k \)-dimensional linear subspaces of \( \mathbb{R}^n \). If \( \xi \in G_{n,k} \), then \( \xi^\perp \) denotes the orthogonal complement of \( \xi \) in \( \mathbb{R}^n \). Each element \( \tau \) of \( G(n, k) \) is parameterized by the pair \( (\xi, u) \), where \( \xi \in G_{n,k} \) and \( u \in \xi^\perp \). In the following \( |\tau| \) denotes the Euclidean distance from the origin to \( \tau \equiv \tau(\xi, u) \) so that \( |\tau| = |u| \) is the Euclidean norm of \( u \). We write \( C_{0}(G(n, k)) \) for the space of all continuous functions \( f \) on \( G(n, k) \) satisfying \( \lim_{|\tau| \to \infty} f(\tau) = 0 \). We also set

\[
C_{\mu}(G(n, k)) = \{ f \in C(G(n, k)) : f(\tau) = O(|\tau|^{-\mu}) \},
\]

\[
C_{\mu}(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) : f(x) = O(|x|^{-\mu}) \},
\]
where \( C(G(n,k)) \) and \( C(\mathbb{R}^n) \) are the space of continuous functions on \( G(n,k) \) and \( \mathbb{R}^n \), respectively.

The manifold \( G(n,k) \) will be endowed with the product measure \( d\tau = d\xi du \), where \( d\xi \) is the \( O(n) \)-invariant probability measure on \( G_{n,k} \) and \( du \) is the usual volume element on \( \xi^\perp \).

For \( k' > k \) and \( \eta \in G_{n,k'} \), we denote by \( G_k(\eta) \) the Grassmann manifold of all \( k \)-dimensional linear subspaces of \( \eta \). In the following, \( S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \} \) is the unit sphere in \( \mathbb{R}^n \). For \( \theta \in S^{n-1} \), \( \{ \theta \} \) denotes the one-dimensional linear subspace spanned by \( \theta \), \( d\theta \) stands for the surface element on \( S^{n-1} \), and \( \sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2) \) is the surface area of \( S^{n-1} \). We set \( d_n \theta = d\theta/\sigma_{n-1} \) for the normalized surface element on \( S^{n-1} \).

We write \( e_1, \ldots, e_n \) for the coordinate unit vectors in \( \mathbb{R}^n \). Given \( 1 \leq k < k' < n \), the following notations are used for the coordinate planes:

\[
\mathbb{R}^k = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k, \quad \mathbb{R}^{k'} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{k'}, \quad \mathbb{R}^{n-k} = \mathbb{R}e_{k+1} \oplus \cdots \oplus \mathbb{R}e_n.
\]

### 2.2. Radon Transforms on Affine Grassmannians.

Let \( G(n,k) \) and \( G(n,k') \) be a pair of affine Grassmann manifolds of non-oriented \( k \)-planes \( \tau \) and \( k' \)-planes \( \zeta \) in \( \mathbb{R}^n \), respectively; \( 1 \leq k < k' \leq n - 1 \). We write

\[
\tau \equiv \tau(\xi, u) \in G(n,k), \quad \zeta \equiv \zeta(\eta, v) \in G(n,k'),
\]

where \( \xi \in G_{n,k}, \ u \in \xi^\perp, \ \eta \in G_{n,k'}, \ v \in \eta^\perp \). The Radon transform of a function \( f(\tau) \) on \( G(n,k) \) is a function \( (R_{k,k'}f)(\zeta) \) on \( G(n,k') \) defined by

\[
(R_{k,k'}f)(\zeta) = \int_{\tau \subset \zeta} f(\tau) d\xi \tau = \int_{\xi \subset \eta} \int_{\xi^\perp \cap \eta} f(\xi, v+x) dx.
\]  

(2.2)

Here \( d\xi \tau \) denotes the probability measure on the Grassmannian \( G_k(\eta) \).

This transform integrates \( f(\tau) \) over all \( k \)-planes \( \tau \) in the \( k' \)-plane \( \zeta \).

The dual Radon transform of a function \( \varphi(\zeta) \) on \( G(n,k') \) is a function \( (R^*_{k,k'}\varphi)(\tau) \equiv (R^*_{k,k'}\varphi)(\xi, u) \) on \( G(n,k) \) defined by

\[
(R^*_{k,k'}\varphi)(\tau) = \int_{\zeta \supset \tau} \varphi(\zeta) d\eta \zeta = \int_{\eta \supset \xi} \varphi(\eta + u) d\xi \eta
\]  

(2.3)

Here \( \text{Pr}_{\eta \perp} u \) is the orthogonal projection of \( u \ (\in \xi^\perp) \) onto \( \eta^\perp (\subset \xi^\perp) \), \( d\xi \eta \) is the relevant probability measure. This transform integrates \( \varphi(\zeta) \) over all \( k' \)-planes \( \zeta \) containing the \( k \)-plane \( \tau \). In order to give (2.3)
precise meaning, we choose an orthogonal transformation \( g_\xi \in O(n) \) so that \( g_\xi \mathbb{R}^k = \xi \), and let \( O(n - k) \) be the subgroup of \( O(n) \) that consists of orthogonal transformations preserving the coordinate plane \( \mathbb{R}^{n-k} \). Then (2.3) means
\[
(R_{k,k'}^* \varphi)(\tau) \equiv (R_{k,k'}^* \varphi)(\xi, u) = \int_{O(n-k)} \varphi(g_\xi \rho \mathbb{R}^{k'} + u) \, d\rho. \tag{2.4}
\]

**Proposition 2.1.** [11, Lemma 2.1] The equality
\[
\int_{G(n,k')} (R_{k,k'} f)(\zeta) \varphi(\zeta) \, d\zeta = \int_{G(n,k)} f(\tau) (R_{k,k'}^* \varphi)(\tau) \, d\tau \tag{2.5}
\]
holds provided that at least one of these integrals exists in the Lebesgue sense (i.e., it is finite if \( f \) and \( \varphi \) are replaced by \(|f|\) and \(|\varphi|\), respectively).

**Proposition 2.2.**

(i) If \( f \in L^p(G(n,k)) \), \( 1 \leq p < (n-k)/(k' - k) \), then \((R_{k,k'} f)(\zeta)\) is finite for almost all \( \zeta \in G(n,k') \). If \( f \in C_\mu(G(n,k)) \), \( \mu > k' - k \), then \((R_{k,k'} f)(\zeta)\) is finite for all \( \zeta \in G(n,k') \). The conditions \( p < (n-k)/(k' - k) \) and \( \mu > k' - k \) are sharp.

(ii) The dual transform \((R_{k,k'}^* \varphi)(\tau)\) is finite a.e. on \( G(n,k) \) for every locally integrable function \( \varphi \).

The statement (i) is proved in [11, Corollary 2.6]. The statement (ii) follows from the equality
\[
\int_{|\tau|<a} (R_{k,k'}^* \varphi)(\tau) \, d\tau = \text{const} \int_{|\zeta|<a} \varphi(\zeta) \left( a^2 - |\zeta|^2 \right)^{(k'-k)/2} \, d\zeta \tag{2.6}
\]
which is a particular case of the formula (2.19) from [11].

**Proposition 2.3.** [11, Lemma 2.3] For \( \tau \in G(n,k) \) and \( \zeta \in G(n,k') \), let \( r = |\tau|, s = |\zeta| \). If \( f(\tau) = f_0(r) \) and \( \varphi(\zeta) = \varphi_0(s) \), then
\[
(R_{k,k'} f)(\zeta) = \sigma_{k'-k-1} \int_s^\infty f_0(r) (r^2 - s^2)^{(k'-k)/2-1} \, r \, dr, \tag{2.7}
\]
\[
(R_{k,k'}^* \varphi)(\tau) = \frac{\sigma_{k'-k-1} \sigma_{n-k'-1}}{\sigma_{n-k-1} \tau^{n-k-2}} \int_0^r \varphi_0(s) (r^2 - s^2)^{(k'-k)/2-1} s^{n-k'-1} \, ds, \tag{2.8}
\]
provided that the corresponding integrals exist in the Lebesgue sense.
2.3. Alternative Parametrization of Lines and Hyperplanes. In this subsection we consider the case when the Radon transform $R$ takes a function $f$ on $\mathcal{L} = G(n, 1)$ to a function $Rf$ on $\mathcal{H} = G(n, n - 1)$. For the future purposes we parametrize the manifolds $\mathcal{L}$ and $\mathcal{H}$ in a slightly different way in comparison with subsection 2.2. Specifically, let

$$\tilde{\mathcal{L}} = \left\{ (\omega, u) : \omega \in S^{n-1}, u \in \mathbb{R}^n, u \perp \omega \right\}. \quad (2.9)$$

Every line $\ell$ has the form $\ell = \{\omega\} + u$, where $(\omega, u) \in \tilde{\mathcal{L}}$, $\{\omega\} = \text{span}(\omega)$. Setting $\ell = \ell(\omega, u)$, every function $f$ on $\mathcal{L}$ can be regarded as a function $\tilde{f}(\omega, u)$ on $\tilde{\mathcal{L}}$ satisfying $\tilde{f}(\omega, u) = \tilde{f}(-\omega, u)$. We equip $\tilde{\mathcal{L}}$ with the product measure $d\omega du$, where $d\omega$ is the normalized surface measure on $S^{n-1}$ and $du$ is the Euclidean volume element on $\omega^\perp$. Then, for $f \in L^1(\mathcal{L})$,

$$\int \mathcal{L} f(\ell) d\ell = \int \tilde{f}(\omega, u) d\omega du = \int_{S^{n-1}} d\omega \int_{\omega^\perp} \tilde{f}(\omega, u) du, \quad (2.10)$$

where the integral on the left-hand side has the same meaning as in subsection 2.2.

For the hyperplane case $h \in \mathcal{H} = G(n, n - 1)$, in parallel with the parametrization $h = h(\eta, v)$, where $\eta \in G_{n,n-1}$ and $v \in \eta^\perp$ (cf. (2.1)), we set $h = h(\theta, t) = \{x \in \mathbb{R}^n : x \cdot \theta = t\}$, where $\theta \in S^{n-1}, t \in \mathbb{R}$. Let

$$\tilde{\mathcal{H}} = \{(\theta, t) : \theta \in S^{n-1}, t \in \mathbb{R}\}.$$ 

Every function $\varphi$ on $\mathcal{H}$ can be thought of as a function $\tilde{\varphi}(\theta, t) = \varphi(\theta^\perp, t\theta)$ on the cylinder $\tilde{\mathcal{H}}$, so that $\tilde{\varphi}(\theta, t) = \tilde{\varphi}(-\theta, -t)$. Endowing $\tilde{\mathcal{H}}$ with the measure $d\theta dt$, we get

$$\int \mathcal{H} \varphi(h) dh \equiv \int_{G_{n,n-1}} d\eta \int_{\eta^\perp} \varphi(\eta, v) dv = \int_{O(n)} d\gamma \int_{\mathbb{R}} \varphi(\gamma e_n^\perp, t\gamma e_n) dt = \int_{S^{n-1}} d\theta \int_{\mathbb{R}} \varphi(\theta^\perp, t\theta) dt = \int_{\tilde{\mathcal{H}}} \tilde{\varphi}(\theta, t) d\theta dt. \quad (2.11)$$

Abusing notation, we identify

$$\mathcal{L} \equiv \tilde{\mathcal{L}}, \quad \mathcal{H} \equiv \tilde{\mathcal{H}}, \quad f \equiv \tilde{f}, \quad \varphi \equiv \tilde{\varphi}. \quad (2.12)$$

According to (2.2) and the identification (2.12), the Radon transform (2.2) can be written in the new parametrization as

$$(Rf)(\theta, t) = \int_{S^{n-1}} d\theta \int_{\omega^\perp} f(\omega, t\theta + x) dx, \quad (\theta, t) \in \tilde{\mathcal{H}}, \quad (2.13)$$
(set \( \eta = \theta^\perp, v = t\theta, \xi = \{\omega\} \)). Here \( d\theta \omega \) is the probability measure on \( \mathbb{S}^{n-1} \cap \theta^\perp \) that is invariant under orthogonal transformations leaving \( \theta \) fixed. Similarly, (2.3) yields
\[
(R^* \varphi)(\omega, u) = \int_{\mathbb{S}^{n-1} \cap \omega^\perp} \varphi(\theta, \theta \cdot u) \, d\omega \theta, \quad (\omega, u) \in \tilde{L}, \tag{2.14}
\]
(use the equality \( \Pr_{\{\theta\}} u = \theta \theta^T u \), where \( \theta \) is interpreted as a matrix with one column and \( \theta^T \) is its transpose). By Proposition 2.1 and equalities (2.10) and (2.11),
\[
\int_{\tilde{H}} (Rf)(\theta, t) \varphi(\theta, t) \, d\omega \theta \, dt = \int_{\tilde{L}} f(\omega, u) (R^* \varphi)(\omega, u) \, d\omega \theta du \tag{2.15}
\]
provided that at least one of these integrals exists in the Lebesgue sense.

3. **Inversion of the Radon Transform of Quasi-Radial Functions**

As we mentioned in Introduction, the Radon transform that takes functions on \( \mathcal{L} \) to functions on \( \mathcal{H} \) is noninjective because \( \dim \mathcal{L} > \dim \mathcal{H} \). To remedy the situation, we need to reduce the dimension of the source space or increase the dimension of the target space. Of course, the geometrical meaning of the problem should remain unchanged.

Below we pursue the first approach. Suppose that the value of \( f \) at \( \ell = \ell(\omega, u) \) depends only on \( \omega \) and \( |u| \), that is, \( f(\omega, \cdot) \) is constant on the set of all lines equidistant from the central line \( \{\omega\} \). We call such a set a line cluster and denote
\[
\text{cl}(\omega, r) = \{\ell(\omega, u) \in \mathcal{L} : |u| = r\}, \quad \omega \in \mathbb{S}^{n-1}, \quad r > 0. \tag{3.1}
\]
A line function \( f \) is called quasi-radial if it is constant on all clusters (3.1), that is, \( f(\omega, u) = f_0(\omega, |u|) \) for some function \( f_0 \) and all or almost all \((\omega, u) \in \tilde{L} \). A more restrictive, radial case, when \( f(\omega, u) = f_0(|u|) \), that is, \( f(\ell) \) depends only on the distance from the origin to \( \ell \), was studied in [11]; cf. (2.7). The set of all clusters (3.1) has the same dimension \( n \), as the set \( \mathcal{H} \) of all hyperplanes. Hence it is natural to expect that the restriction of the Radon transform (2.13) to quasi-radial functions is injective.
Lemma 3.1. If \( f \) is quasi-radial, \( f(\omega, u) = f_0(\omega, |u|) \), then
\[
(Rf)(\theta, t) = \sigma_{n-3} \int_{|t|}^{\infty} \frac{(r^2 - t^2)^{(n-4)/2}}{r} r dr \int_{S^{n-1}} f_0(\omega, r) d\theta \omega
\]  
(3.2)
provided that the integral on the right-hand side exists in the Lebesgue sense.

Proof. Passing to polar coordinates in (2.13), we obtain
\[
(Rf)(\theta, t) = \sigma_{n-3} \int_{\mathbb{S}^{n-1} \cap \theta^\perp} d\theta \omega \int_{0}^{\infty} f_0(\omega, \sqrt{s^2 + t^2}) s^{n-3} ds.
\]
This coincides with (3.2). \( \square \)

The formula (3.2) is a generalization of (2.7) for \( k = 1 \) and \( k' = n-1 \). It shows that on quasi-radial functions, \((Rf)(\theta, t)\) is a constant multiple of the tensor product of the classical Funk transform
\[
(F\psi)(\theta) = \int_{\mathbb{S}^{n-1} \cap \theta^\perp} \psi(\omega) d\theta \omega, \quad \theta \in \mathbb{S}^{n-1},
\]  
(3.3)
and the fractional integration operator of the Erdélyi-Kober type
\[
(I_{\alpha}^{n/2-1}\chi)(t) = \frac{2}{\Gamma(\alpha)} \int_{0}^{\infty} \frac{(r^2 - t^2)^{\alpha-1}}{r} \chi(r) r dr, \quad t > 0,
\]  
(3.4)
with \( \alpha = n/2 - 1 \). Both operators were studied systematically in [16, Sections 5.1 and 2.6.2]. Thus, for \( t > 0 \), we can write
\[
(Rf)(\theta, t) = \pi^{n/2-1} (\tilde{R} f_0)(\theta, t), \quad \tilde{R} = I_{\alpha/2}^{n/2-1} \otimes F,
\]  
(3.5)
where \( F \) acts in the \( \omega \)-variable and \( I_{\alpha/2}^{n/2-1} \) in the \( r \)-variable, as in (3.2).

By Lemma 2.42 from [16, p. 65] and the existence results for the Funk transform (see, e.g., [16, p. 281]), the integral (3.2) is absolutely convergent for almost all \((\theta, t) \in \mathcal{H}\) provided that
\[
\int_{\mathbb{S}^{n-1}} d\omega \int_{a}^{\infty} |f_0(\omega, r)| r^{n-3} dr < \infty \quad \forall a > 0,
\]  
(3.6)
where the exponent \( n - 3 \) in (3.6) is exact.

A variety of inversion formulas for \( F \) and \( I_{\alpha/2}^{n/2-1} \) can be found in [16, subsections 5.1.6-5.1.8, 2.6.2]. For example, the following statement is an immediate consequence of the formulas (2.6.23), (2.6.25), and (5.1.96) from [16].
Theorem 3.2. Let \( \varphi = Rf, f(\omega, u) = f_0(\omega, |u|) \), where \( f_0 \) satisfies (3.6). Then

\[
f_0(\omega, r) = \pi^{1-n/2} (\tilde{R}^{-1} \varphi)(\omega, r), \quad \tilde{R}^{-1} = \mathcal{D}_{-2}^{n/2-1} \otimes F^{-1}. \tag{3.7}
\]

The Erdélyi-Kober derivative \( \mathcal{D}_{-2}^{n/2-1} \) is defined by the following formulas:

\[
\mathcal{D}_{-2}^{n/2-1} \chi = (-D)^{n/2-1} \chi, \quad D = \frac{1}{2r} \frac{d}{dr}, \tag{3.8}
\]

if \( n \) is even, and

\[
\mathcal{D}_{-2}^{n/2-1} \chi = r (-D)^{(n-1)/2} r^{n-2} l_{-2}^{1/2} r^{1-n} \chi, \tag{3.9}
\]

if \( n \) is odd, where the powers of \( r \) stand for the corresponding multiplication operators. Furthermore,

\[
(F^{-1} \psi)(\omega) = \lim_{t \to 1} \left( \frac{1}{2t} \frac{\partial}{\partial t} \right)^{n-2} \left[ \frac{2}{(n-3)!} \int_0^t (t^2 - s^2)^{n/2-2} \Phi_\omega(s) s^{n-2} ds \right], \tag{3.10}
\]

\[
\Phi_\omega(s) = \int_{S^{n-1} \cap \omega} \psi(s \xi + \sqrt{1-s^2} \omega) d_\omega \xi, \quad -1 \leq s \leq 1.
\]

The limit in (3.10) is understood in the \( L^1(S^{n-1}) \)-norm.

Remark 3.3. If the function \( f \) in Theorem 3.2 is smooth, then (3.10) can be replaced by the corresponding expression in terms of the Beltrami-Laplace operator on \( S^{n-1} \); see [11, Theorem 5.37]. If \( f \) is a radial function, i.e., \( f(\omega, u) = f_0(|u|) \), the spherical component \( F \) in (3.5) and (3.7) disappears and we simply have \( \tilde{R}^{-1} = \mathcal{D}_{-2}^{n/2-1} \). The last formula agrees with [11, Lemma 2.3].

4. Inversion of the Dual Transform

4.1. The Dual Transform of Even Functions. In this subsection we confine to hyperplane functions \( \varphi \) with the property

\[
\varphi(h) = \varphi(-h) \quad \forall h \in \mathcal{H}. \tag{4.1}
\]

A pair \((h, -h)\) of hyperplanes is a natural counterpart of the line cluster (3.1). It is convenient to take the dual transform \( R^* \) in the form (2.14), namely,

\[
(R^* \varphi)(\omega, u) = \int_{S^{n-1} \cap \omega} \varphi(\theta, \theta \cdot u) d_\omega \theta, \tag{4.2}
\]
where \( \omega \in \mathbb{S}^{n-1} \) and \( u \in \omega^\perp \). In this notation, the function \( \varphi \) on \( \mathcal{H} \) is identified with a function on the cylinder \( \tilde{\mathcal{H}} = \{ (\theta, t) : \theta \in \mathbb{S}^{n-1}, t \in \mathbb{R} \} \) satisfying
\[
\varphi(\theta, t) = \varphi(-\theta, -t) \quad \forall (\theta, t) \in \tilde{\mathcal{H}}. \tag{4.3}
\]
Moreover, if \( \varphi \) is even in the sense of (4.1), we additionally have
\[
\varphi(\theta, t) = \varphi(\theta, -t) = \varphi(-\theta, t) \quad \forall (\theta, t) \in \tilde{\mathcal{H}}. \tag{4.4}
\]

**Remark 4.1.** We pay attention to the following interesting fact. Unlike the Radon transform \( R \) that takes quasi-radial functions on \( \mathcal{L} \) to even functions on \( \mathcal{H} \) (cf. Lemma 3.1 and (4.4)), the dual transform \( R^* \) of an even function on \( \mathcal{H} \) is not necessarily a quasi-radial function on \( \mathcal{L} \). Let, for example, \( n = 3, k = 1 \), and choose \( \varphi(\theta, t) = |t\theta|^2 \), where \( \theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{S}^2 \) and \( t \in \mathbb{R} \). Obviously, \( \varphi \) satisfies (4.4). By (4.2),
\[
(R^* \varphi)(e_1, e_2) = \frac{1}{2\pi} \int_{\mathbb{S}^2 \cap e_1^\perp} \varphi(\theta, \theta \cdot e_2) d\theta.
\]
To evaluate this integral, we set
\[
\theta = \gamma(\alpha) e_2, \quad \gamma(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}, \quad 0 \leq \alpha \leq 2\pi.
\]
Then
\[
(R^* \varphi)(e_1, e_2) = \frac{1}{2\pi} \int_0^{2\pi} \varphi \left( \begin{bmatrix} 0 \\ \cos \alpha \\ -\sin \alpha \end{bmatrix}, \cos \alpha \right) d\alpha
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \cos^2 \alpha d\alpha = \frac{1}{2}.
\]
Similarly,
\[
(R^* \varphi)(e_1, e_3) = \frac{1}{2\pi} \int_0^{2\pi} |\sin \alpha \cos \alpha| d\alpha = \frac{1}{\pi}.
\]
Thus \( (R^* \varphi)(e_1, e_2) \neq (R^* \varphi)(e_1, e_3) \) which means that \( R^* \varphi \) is not quasi-radial.

Now we proceed to reconstruction of \( \varphi(h) \equiv \varphi(\theta, t) \) from \( (R^* \varphi)(\omega, u) \) assuming that \( \varphi \) enjoys the symmetry (4.1) (or (4.4)). Suppose \( u \neq 0 \)
and apply the slice integration formula from [16, formula (A.11.12)] to the cross-section $S^{n-1} \cap \omega^\perp$. Owing to (4.3), we can write (4.2) as

\[
(R^* \varphi)(\omega, u) = c \int_0^1 \tilde{d}t \int_{(S^{n-1} \cap \omega^\perp) \cap \tilde{u}^\perp} \varphi(\sqrt{1 - t^2} \sigma + t \tilde{u}, tr) d_* \sigma, \tag{4.5}
\]

where

\[
c = \frac{2 \sigma_{n-3}}{\sigma_{n-2}}, \quad \tilde{d}t = \left(1 - t^2\right)^{(n-4)/2} dt, \quad r = |u|, \quad \tilde{u} = u/|u|,
\]

and $d_* \sigma$ denotes the probability measure on the $(n - 3)$-dimensional sphere $(S^{n-1} \cap \omega^\perp) \cap \tilde{u}^\perp$.

Let $O(\omega^\perp) \subset O(n)$ be the stationary subgroup of the vector $\omega$. For a function $f = f(\omega, u)$ on $\mathcal{L}$ (i.e., $\omega \in S^{n-1}, u \in \omega^\perp$), we introduce the mean value operator

\[
(M_{\omega} f)(r) = \int_{O(\omega^\perp)} f(\omega, r \rho \tilde{u}) d\rho, \quad r > 0,
\]

that averages $f \equiv f(\ell)$ over the cluster $\text{cl}(\omega, r)$.

**Lemma 4.2.** If $\varphi$ satisfies (4.4), then, for $\omega \in S^{n-1}, r > 0$, and $c = 2 \sigma_{n-3}/\sigma_{n-2}$,

\[
(M_{\omega} R^* \varphi)(r) = \frac{c}{r^{n-3}} \int_0^r (r^2 - t^2)^{(n-4)/2} dt \int_{S^{n-1} \cap \omega^\perp} \varphi(\theta, t) d\theta \tag{4.6}
\]

provided that the integral on the right-hand side exists in the Lebesgue sense.

**Proof.** Changing the order of integration and using (4.5), we obtain

\[
(M_{\omega} R^* \varphi)(r) = \int_{O(\omega^\perp)} (R^* \varphi)(\omega, r \rho \tilde{u}) d\rho
\]

\[
= c \int_0^1 \tilde{d}t \int_{O(\omega^\perp)} d\rho \int_{(S^{n-1} \cap \omega^\perp) \cap (\rho \tilde{u})^\perp} \varphi(\sqrt{1 - t^2} \sigma + t \rho \tilde{u}, tr) d_* \sigma.
\]
The latter can be transformed as follows.

\[
(M_\omega R^* \varphi)(r) = c \int_0^1 \int_{\partial(\mathbb{R}^{n-1} \cap \omega^\perp)} \varphi(\sqrt{1 - t^2 \rho^2} + t \rho \tilde{u}, tr) \, d\rho \, dr
\]

\[
= c \int_0^1 \int_{\partial(\mathbb{R}^{n-1} \cap \omega^\perp)} \varphi(\rho(\sqrt{1 - t^2 \theta} + t \tilde{u}), tr) \, d\rho
d\theta
\]

\[
= \frac{c}{r^{n-3}} \int_0^r (r^2 - t^2)^{(n-4)/2} \int_{\partial(\mathbb{R}^{n-1} \cap \omega^\perp)} \varphi(\tilde{\theta}, t) \, d\tilde{\theta}.
\]

This gives (4.6). □

The right-hand side of (4.6) is a constant multiple of the tensor product (up to weight factors) of the Funk transform (3.3) and the left-sided modification of the Erdélyi-Kober operator (3.4). Specifically, let

\[
(I_n^{\alpha, 2} \chi)(t) = \frac{2}{\Gamma(\alpha)} \int_0^r (r^2 - t^2)^{\alpha - 1} \chi(t) \, t \, dt,
\]

\[
\alpha > 0;
\]

(4.7)

cf. [16, formula (2.6.8)]. Then (4.6) can be written as

\[
(M_\omega R^* \varphi)(r) = c_1 r^{3-n} ([I_n^{n/2-1} \chi] \otimes F)(\omega, r),
\]

\[
c_1 = \frac{\Gamma(n - 1) / 2}{\pi^{1/2}}.
\]

Here $F$ acts in the $\theta$-argument of $\varphi$ and $I_n^{n/2-1}$ acts in the $t$-argument, as in (4.6). Note that, owing to (4.4), the operator $F$ is injective.

The integral $(I_n^{\alpha, 2} \chi)(t)$ is absolutely convergent for almost all $r > 0$ whenever $t \chi(t)$ is a locally integrable function on $\mathbb{R}_+$; see [16, p. 65]. It follows that (4.6) holds for every locally integrable even function $\varphi$ on $\mathcal{H}$. The inversion procedure for the operator $I_n^{\alpha}$ is described in [16, p. 67]. In particular,

\[
D_n^{n/2-1} \chi = D^{n/2-1} \chi, \quad D = \frac{1}{2r} \frac{d}{dr},
\]

(4.8)

if $n$ is even, and

\[
D_n^{n/2-1} \chi = D^{(n-1)/2} J_{n/2}^{1/2} \chi,
\]

(4.9)

if $n$ is odd.
Thus an even hyperplane function $\varphi$ can be explicitly reconstructed from the line function $R^*\varphi$ as follows.

**Theorem 4.3.** Let $\varphi$ be a locally integrable function on $\mathcal{H}$ satisfying (4.4), and let
$$\Phi(\omega, r) = \frac{r^{n-3}n!^{1/2}}{\Gamma(n - 1)/2} (\mathcal{M}_\omega R^*\varphi)(r), \quad \omega \in S^{n-1}, \quad r > 0.$$  

Then for $t > 0$,
$$\varphi(\theta, t) = ([F^{-1} \otimes tD_{+,2}^{n/2-1}]\Phi)(\theta, t), \quad (4.10)$$
where the inverse Funk transform $F^{-1}$ acts in the $\omega$-argument of $\Phi$ by the formula (3.10) and the Erdélyi-Kober derivative $D_{+,2}^{n/2-1}$, defined by (4.8)-(4.9), acts in the $r$-argument of $\Phi$.

**Remark 4.4.** As in Remark 3.3, if $\varphi$ is smooth, then (3.10) can be replaced by the corresponding expression in terms of the Beltrami-Laplace operator. If $\varphi$ is radial, i.e., $\varphi(\theta, t) = \varphi_0(|t|)$, then the spherical component $F^{-1}$ in (4.10) disappears.

**4.2. The General Case. The Kelvin-Type Transform.** It is known [11, Section 5] that the Radon transform for a pair of affine Grassmannians and the corresponding dual transform can be expressed one through another by making use of a certain Kelvin-type transformation. Below we recall this construction.

Let $\tau \in G(n, k)$ be a $k$-plane in $\mathbb{R}^n$ not passing through the origin and parameterized by $\tau = \tau(\xi, u)$, where $\xi \in G_{n,k}$, $u \in \xi^\perp$, $u \neq 0$. Let $\{\xi, u\} = \text{span}(\xi, u) \in G_{n,k+1}$ be the smallest linear subspace containing $\tau$ and let $\{\xi, u\}^\perp \in G_{n,n-k-1}$ be the orthogonal complement of $\{\xi, u\}$ in $\mathbb{R}^n$. To every $k$-plane $\tau = \tau(\xi, u)$ with $u \neq 0$ we associate an $(n-k-1)$-dimensional plane $\tilde{\tau} = \tilde{\tau}(\tilde{\xi}, \tilde{u})$ defined by
$$\tilde{\tau} = \tilde{\tau}(\tilde{\xi}, \tilde{u}), \quad \text{where} \quad \tilde{\xi} = \{\xi, u\}^\perp, \quad \tilde{u} = -\frac{u}{|u|^2}, \quad (4.11)$$
so that $\tilde{u} \in \tilde{\xi}^\perp$ and $|\tilde{u}| = |u|^{-1}$ or $|\tilde{\tau}| = |\tau|^{-1}$ (we recall that $|\tau|$ denotes the Euclidean distance from the origin to the plane $\tau$).

The map $\nu : \tau \to \tilde{\tau}$ is called a quasi-orthogonal inversion map. It can be regarded as a Grassmannian modification of the Kelvin transform $x \to x/|x|^2$ for $x \in \mathbb{R}^n \setminus \{0\}$.

One can readily see that $\nu$ is an involution, i.e., $\nu = \nu^{-1}$.

By analogy with (4.11), $\nu$ acts from $G(n, k')$ to $G(n, n - k' - 1)$, so that if $\zeta = \zeta(\eta, v)$ with $v \neq 0$, then $\nu(\zeta) = \tilde{\zeta}$, where
$$\tilde{\zeta} \equiv \tilde{\zeta}(\tilde{\eta}, \tilde{v}), \quad \tilde{\eta} = \{\eta, v\}^\perp, \quad \tilde{v} = -\frac{v}{|v|^2}. \quad (4.12)$$
Example 4.5. If $k = 0$ and $\tau = x \in \mathbb{R}^n \setminus \{0\}$, then $\nu(x)$ is a hyperplane orthogonal to the vector $x$ and passing through the point $-x/|x|^2$.

Example 4.6. If $k = 1$ and $\tau = \ell$ is a line not passing through the origin, that is, $\ell = \ell(\omega, u) = \{\omega\} + u$, $\omega \in S^{n-1}$, $u \in \omega^\perp \setminus \{0\}$, then $\nu(\ell)$ is an $(n-2)$-dimensional plane

$$
\pi = \pi(\{\omega, u\}^\perp, -u/|u|^2) = \{\omega, u\}^\perp - u/|u|^2.
$$

Example 4.7. If $k' = n - 1$ and $\zeta = h(\theta, t) = \{x \in \mathbb{R}^n : x \cdot \theta = t\}$ is a hyperplane in $\mathbb{R}^n$ with $t \neq 0$, then $\tilde{\zeta} = -\theta/t$ is a point in $\mathbb{R}^n$.

Definition 4.8. Let $R : f(\tau) \to (Rf)(\zeta)$ be the Radon transform taking functions on $G(n, k)$ to functions on $G(n, k')$, $k' > k$. If $\bar{\tau} = \nu(\tau)$, and $\bar{\zeta} = \nu(\zeta)$, then the associated Radon transform $\bar{R} : \bar{f}(\bar{\zeta}) \to (\bar{Rf})(\bar{\tau})$ from functions on $G(n, n-k'-1)$ to functions on $G(n, n-k-1)$ is called quasi-orthogonal to $R$.

Theorem 4.9. [11, Theorem 5.5] Let $0 \leq k < k' < n$. For a function $\varphi$ on $G(n, k')$, we denote

$$
(A\varphi)(\tilde{\zeta}) = |\tilde{\zeta}|^{k-n}\varphi(\nu^{-1}(\tilde{\zeta})), \quad \tilde{\zeta} \in G(n, n-k'-1).
$$

(i) The following relation holds

$$
\int_{G(n,k')} \frac{\varphi(\zeta) \, d\zeta}{(1 + |\zeta|^2)^{(k+1)/2}} = \frac{\sigma_{n-k'-1}}{\sigma_{k'}} \int_{G(n,n-k'-1)} \frac{(A\varphi)(\tilde{\zeta}) \, d\tilde{\zeta}}{(1 + |\tilde{\zeta}|^2)^{(k+1)/2}}
$$

provided that either side of this equality exists in the Lebesgue sense.

(ii) If at least one of the integrals in (4.14) is finite, then

$$
(R^*\varphi)(\tau) = c |\tau|^{k-n}(\bar{R}A\varphi)(\nu(\tau)), \quad c = \frac{\sigma_{n-k'-1}}{\sigma_{n-k-1}}.
$$

Let us consider the line-to-hyperplane case $1 = k < k' = n - 1$ in more detail. By Examples 4.6 and 4.7,

$$
(R^*\varphi)(\omega, u) = \frac{2}{|u| \sigma_{n-2}} (\bar{R}A\varphi)(\pi), \quad \pi = \pi(\{\omega, u\}^\perp, -u/|u|^2),
$$

where

$$
(A\varphi)(x) = |x|^{1-n}\varphi(\nu^{-1}(x))
$$

and $\bar{R}$ is the Radon-John $(n-2)$-plane transform in $\mathbb{R}^n$ [3, 9, 12]. We write (4.16) as

$$
(\bar{R}A\varphi)(\pi) = \frac{\sigma_{n-2}}{2|\pi|} (R^*\varphi)(\nu^{-1}(\pi)), \quad \pi \in G(n, n-2), \quad 0 \notin \pi.
$$
Inverting \( \tilde{R} \) by one of the known inversion methods (see, e.g., [9, 12, 14]) and using (4.17), we formally obtain

\[ \varphi(h) = |h|^{1-n}(\tilde{R}^{-1}\Phi)(\nu(h)), \quad h \in \mathcal{H}, \]  

(4.19)

where

\[ \Phi(\pi) = \frac{\sigma_{n-2}}{2|\pi|} (R^*\varphi)(\nu^{-1}(\pi)). \]  

(4.20)

The equality

\[ \int_{\mathcal{H}} \frac{\varphi(h)}{1 + |h|^2} dh = \frac{2}{\sigma_{n-1}} \int_{\mathbb{R}^n} (A\varphi)(x) \frac{dx}{1 + |x|^2} \]  

(4.21)

which is a particular case of (4.14), allows us to choose the suitable class of functions \( \varphi \). Indeed, by (4.21) and (4.18), the invertibility of \( R^* \) on functions \( \varphi : \mathcal{H} \to \mathbb{C} \) satisfying

\[ \int_{\mathcal{H}} |\varphi(h)| \frac{dh}{1 + |h|^2} < \infty \]  

(4.22)

is equivalent to the invertibility of \( \tilde{R} \) on functions \( \tilde{\varphi} : \mathbb{R}^n \to \mathbb{C} \) satisfying

\[ \int_{\mathbb{R}^n} |\tilde{\varphi}(x)| \frac{dx}{1 + |x|^2} < \infty. \]  

(4.23)

We also note that by Theorem 3.2 from [14], the condition (4.23) guarantees the existence of \((R\tilde{\varphi})(\pi)\) for almost all \( \pi \) (this condition is necessary on nonnegative radial functions \( \tilde{\varphi} \)).

For the sake of simplicity, we restrict our consideration to the following two subclasses of hyperplane functions satisfying (4.22).

For \( 1 \leq p < \infty \), we denote

\[ \tilde{L}^p(\mathcal{H}) = \left\{ \varphi : \int_{\mathcal{H}} |h|^{(n-1)(p-1)-2} |\varphi(h)|^p \, dh < \infty \right\}. \]  

(4.24)

For \( \mu > 0 \), let \( \tilde{C}_\mu(\mathcal{H}) \) be the space of all functions \( \varphi \) which are continuous on the set of all hyperplanes not passing through the origin and satisfy the following condition:

\[ \begin{cases}  
|h|^{n-1-\mu} \varphi(h) = O(1) \quad \text{if } |h| \to 0, \\
|h|^{n-1} \varphi(h) \to c = \text{const} \quad \text{if } |h| \to \infty. 
\end{cases} \]  

(4.25)

The space \( \tilde{C}_\mu(\mathcal{H}) \) is an antipodal modification of the space \( C_\mu(\mathcal{H}) = \{ f \in C(\mathcal{H}) : f(\tau) = O(|\tau|^{-n}) \} \). The reason for the above definitions is explained by the following lemma.
Lemma 4.10.

(i) If \( \varphi \in \tilde{L}^p(H) \) with \( 1 \leq p < n/(n-2) \), then \( \varphi \) satisfies (4.22). The relations \( \varphi \in L^p(H) \) and \( A\varphi \in L^p(\mathbb{R}^n) \) are equivalent.

(ii) If \( \varphi \in \tilde{C}_\mu(H) \) with \( \mu > n - 2 \), then \( \varphi \) satisfies (4.22). The relations \( \varphi \in \tilde{C}_\mu(H) \) and \( A\varphi \in C_\mu(\mathbb{R}^n) \) are equivalent.

Proof. (i) The first statement follows by Hölder’s inequality. To prove the second statement, we observe that

\[
||A\varphi||_p^p = \int_{\mathbb{R}^n} |x|^{(1-n)p} |\varphi(\nu^{-1}(x))|^p \, dx = \int_{\mathbb{R}^n} \frac{(A\psi)(x) \, dx}{1 + |x|^2},
\]

where

\[
(A\psi)(x) = |x|^{1-n} \psi(\nu^{-1}(x)), \quad \psi(h) = |h|^{(n-1)(p-1)}|\varphi(h)|^p(1+1/|h|^2).
\]

Hence, by (4.21),

\[
||A\varphi||_p^p = \frac{\sigma_{n-1}}{2} \int_{\mathbb{R}^n} \psi(h) \, dh = \frac{\sigma_{n-1}}{2} \int_{\mathbb{R}^n} |h|^{(n-1)(p-1)-2} |\varphi(h)|^p \, dh,
\]

as desired.

(ii) Both statements can be easily checked straightforward.

Now we can formulate the inversion result for \( R^* \). Following [14, Section 3], we introduce the mean value operator

\[
(R_x^* \Phi)(r) = \int_{SO(n)} \Phi(\gamma \mathbb{R}^{n-2} + x + r\gamma e_n) \, d\gamma, \quad r > 0. \tag{4.26}
\]

Theorem 4.11.

(i) The function \( \varphi \in \tilde{L}^p(H) \), \( 1 \leq p < n/(n-2) \), can be reconstructed from \( (R^*\varphi)(\ell) \), \( \ell \in \mathcal{L} \), by the formula (4.19) in which \( \tilde{R}^{-1} \) denotes the inverse Radon-John \((n-2)\)-plane transform in \( \mathbb{R}^n \) that can be represented in different forms. For example, Theorem 3.5 from [14] yields

\[
(\tilde{R}^{-1}_x \Phi)(x) = \lim_{r \to 0} r^{1-n/2} (\mathcal{D}^{n/2-1}_{-2} \tilde{R}^*_x \Phi)(r), \tag{4.27}
\]

where the limit is understood in the \( L^p \)-norm and the Erdélyi-Kober derivative \( \mathcal{D}^{n/2-1}_{-2} \) is defined by (3.8) and (3.9).

(ii) If \( \varphi \in C_\mu(H) \) with \( \mu > n - 2 \), then (4.27) holds with the limit interpreted in the sup-norm.

This theorem follows immediately from (4.18) and Lemma 4.10 if we apply Theorems 3.5 and 3.4 from [14].
Remark 4.12. The conditions $1 \leq p < n/(n - 2)$ and $\mu > n - 2$ in Theorem 4.11 are not only of technical nature. In fact, they are necessary for the existence of $R^* \varphi$. Let, for instance,

$$\varphi_p(h) = |h|^{-n}(2 + 1/|h|)^{-n/p}[\log(2 + 1/|h|)]^{-1}.$$  \hspace{1cm} (4.28)

One can readily check that this function belongs to $\tilde{L}^p(\mathcal{H})$ for any $p > 1$. However, if $p \geq n/(n - 2)$, then $\varphi_p$ is not locally integrable and $R^* \varphi_p \equiv \infty$. The latter can be easily seen if we apply (2.8) with $k = 1$, $k' = n - 1$. The same function with $p = n/\mu$, belonging to $\tilde{C}_\mu(\mathcal{H})$, gives the necessity of the condition $\mu > n - 2$.

Another inversion formula for $R^* \varphi$ can be obtained if we invert $\tilde{R}$ by making use of Theorem 5.4 from [12]. Let

$$\kappa_\ell = \int_0^\infty (1 - e^{-t})^\ell t^{-n/2} dt = \begin{cases} \Gamma(1 - n/2) \sum_{j=1}^\ell (\ell)_j (-1)^j j^{n/2 - 1} & \text{if } n \text{ is odd}, \\
\frac{(-1)^{n/2}}{(n/2 - 1)!} \sum_{j=1}^\ell (\ell)_j (-1)^j j^{n/2 - 1} \log j & \text{if } n \text{ is even}. \end{cases}$$

**Theorem 4.13.** Suppose that $\Phi$ and $\tilde{R}^*_\nu \Phi$ are defined by (4.20) and (4.26), respectively. If $\varphi \in \tilde{L}^p(\mathcal{H})$, $1 \leq p < n/(n - 2)$, then for any $\ell > n/2 - 1$,

$$\varphi(h) = \frac{\pi^{1-n/2}}{\kappa_\ell |h|^{n-1}} \int_0^\infty \left[ \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j (\tilde{R}^*_\nu(\Phi)(\sqrt{jr}) \right] \frac{dr}{r^{n/2}}, \hspace{1cm} (4.29)$$

where $\int_0^\infty \lim_{\epsilon \to 0} \int_0^\infty \right. \text{ in the a.e. sense. If } \varphi \in \tilde{C}_\mu(\mathcal{H}), \mu > n - 2,$

this limit is uniform on every compact subset of $\mathcal{H}$ not containing hyperplanes through the origin.

**Remark 4.14.** The right-hand side of (4.29) resembles Marchaud’s fractional derivative of order $n/2 - 1$ of the function $(\tilde{R}^*_\nu(\Phi))(r)$ evaluated at $r = 0$; cf. [16, pp. 56, 122]. For $n = 3$, (4.29) has an especially simple form

$$\varphi(h) = \frac{1}{\pi |h|^2} \int_0^\infty \left[ (\tilde{R}^*_\nu(\Phi))(0) - (\tilde{R}^*_\nu(\Phi))(r) \right] \frac{dr}{r^2}. \hspace{1cm} (4.30)$$
Remark 4.15. It would be natural to obtain an explicit inversion formula for \( R^* \varphi \) under the general assumption (4.22). This problem is equivalent to inversion of the Radon-John \((n - 2)\)-plane transform \( \tilde{R} \) on the space of all functions satisfying (4.23). The last problem can be solved in the sense of distributions, but we cannot give a reference where the desired formula is obtained in the pointwise sense.

Remark 4.16. A function \( \varphi \) can be reconstructed from \( R^* \varphi \) in a different way if we change the parametrization of the set \( \mathcal{L} \) of all lines. For example, fix any \( x \in \mathbb{R}^n \) and set

\[
(R^*_1 \varphi)(\omega, x) = \int_{\mathbb{S}^{n-1} \cap \omega^\perp} \varphi(\theta, \theta \cdot x) \, d\omega \theta,
\]

so that

\[
(R^*_1 \varphi)(\omega, x) = (R^* \varphi)(\omega, \text{Pr}_{\omega^\perp} x).
\]

For the sake of simplicity we assume that \( \varphi \) is smooth. The operator \( R^*_1 \) averages \( \varphi \) over all hyperplanes containing the line through the point \( x \) in the direction of \( \omega \). The function \( \tilde{\varphi}_x(\theta) = \varphi(\theta, \theta \cdot x) \) is even and can be reconstructed from \( (R^*_1 \varphi)(\omega, x) \) using the inverse Funk transform:

\[
\tilde{\varphi}_x(\theta) = F^{-1}[(R^*_1 \varphi)(\cdot, x)](\theta) = F^{-1}[(R^* \varphi)(\cdot, \text{Pr}_{(\cdot) \perp} x)](\theta).
\]

It particular, for \( x = t\theta \), (4.33) yields \( \tilde{\varphi}_{t\theta}(\theta) = \varphi(\theta, t \cdot \theta) = \varphi(\theta, t) \). This gives the following

**Theorem 4.17.** An even smooth function \( \varphi \) on \( \mathbb{S}^{n-1} \times \mathbb{R} \) can be reconstructed from the dual Radon transform \( f(\omega, u) = (R^* \varphi)(\omega, u) \) (see (4.2)) by the formula

\[
\varphi(\theta, t) = (F^{-1}[f(\cdot, \text{Pr}_{(\cdot) \perp} x)](\theta))\big|_{x = t\theta}.
\]

A simple inversion formula in this theorem can be explained by the fact that the parametrization of the set of all lines in (4.31) is redundant. Indeed, the dimension of the set of all pairs \( (\omega, x) \) in (4.31) is \( 2n - 1 \), whereas the dimension of \( \mathcal{L} = G(n, 1) \) is \( 2n - 2 \).

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