INTRODUCTION TO ANDERSON T-MOTIVES: A SURVEY

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Abstract. This is a survey on Anderson t-motives — the functional field analogs of abelian varieties with multiplication by an imaginary quadratic field. We define their lattices, the group $H^1$, their tensor products and the duality functor. Some examples of explicit calculations are given, some elementary research problems are stated.

Drinfeld in [D] defined new objects — elliptic modules, now called Drinfeld modules. Anderson in [A86] defined their high-dimensional generalizations — Anderson t-motives. Their theory forms — in some sense — a parallel world to the classical theory of abelian varieties (see Section 6, Table 2), like the non-Euclidean geometry (Bolyai, Gauss, Lobachevsky) forms a parallel world to the Euclidean geometry. Namely, Anderson t-motives can be considered as analogs of abelian varieties with MIQF (multiplication by imaginary quadratic fields), see 1.8. This analogy is a source of numerous research papers: a scientist considers a theorem of the theory of abelian varieties and proves its analog for the theory of Anderson t-motives (this is not a routine activity, because the analogy is not complete).

A detailed introduction to the subject can be found in [G]. Nevertheless, all these three sources [D], [A86], [G] can be too difficult for the beginners. One of the reasons of this fact is the subject itself. Namely, important objects of the theory are elements $T$ and $\theta$, see below. Their roles are close one to another, there is a map $\iota$ sending $T$ to $\theta$, until now some authors confuse them, identifying them by this map. Really, it is necessary to distinguish carefully these objects. This was not made satisfactorily at earlier stages of development of the theory.

The purpose of the present survey is to give an elementary introduction to the subject. We indicate the most important ideas omitting technical details. Also, we give explicit examples. Further, we give a detailed coordinate description of the objects of the principal exact sequence (9.11), namely of $E(M)$ — the t-module associated to Anderson t-motive $M$ of dimension $n$ (see below for a definition of $n$), and its Lie group $\text{Lie}(M)$. It turns out that $E(M)$ is isomorphic to $\mathbb{C}_\infty^n$ (here $\mathbb{C}_\infty$ is a finite characteristic analog of $\mathbb{C}$, see Section 3), but this isomorphism is not canonical for $n > 1$: it depends on a choice of a basis of $M$ over $\mathbb{C}_\infty\{\tau\}$, see 5.2.2. Particularly, there is no well-defined $c \cdot x$, where $c \in \mathbb{C}_\infty$, $x \in E(M)$. From another side, there exists a canonical isomorphism $\text{Lie}(M) \to \mathbb{C}_\infty^n$.

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These facts are not emphasized in both [A86], [G], although a non-canonical coordinate description of $E(M)$ (i.e. an isomorphism $E(M) \to \mathbb{C}^n$) is used essentially in the proofs of theorems. This can cause confusion for beginners. The present survey contains a detailed exposition of this subject (Sections 6.2 and 9).

From a formal point of view, Anderson t-motives are simple objects: they are modules (having some specific properties (5.2.1 – 5.2.3)) over a ring of non-commutative polynomials in two variables over a field. But in order to understand that they are really analogs of abelian varieties, it is necessary to plunge into the theory.

The generalizations of the theory of Anderson t-motives are very deep and complicated. Nevertheless, until now there exist elementary problems that are not solved yet. Another purpose of this survey is to indicate some “down-to-earth” problems that can be a research subject for the beginners. For example, calculation of $h^1, h_1$ of some Anderson t-motives is a long, but easy problem that will definitely give a result (continuation of [GL21], [EGL]). See Section 9.12. Other elementary research subject is described in Section 12.1.

Let us describe the analogy between Anderson t-motives and abelian varieties in more details. First of all, while abelian varieties depend on one discrete parameter — their dimension $g$, Anderson t-motives depend on two parameters — dimension $n$ and rank $r$. It turns out that Anderson t-motives are analogs not of general abelian varieties, but of abelian varieties with MIQF of signature $(n, r - n)$, see 1.8 below and [GL09]. Surprisingly, this fact is not emphasized in most survey papers on the subject.

Continuing the analogy, we have: many objects attached to abelian varieties, for example Tate modules, Galois action on them, lattices, modular curves, L-functions etc., also can be attached to Anderson t-motives. Nevertheless, this analogy is far to be complete. For example, there is no functional equation for L-functions of Anderson t-motives; notion of their algebraic rank is not known yet; 1 - 1 correspondence between Anderson t-motives and lattices is known only for Drinfeld modules (see [D]) and for some other cases (9.4; Theorem 9.5).

The paper is organized as follows. In Section 1 we give briefly some properties of abelian varieties over number fields, in order to show the analogy. Particularly, in Section 1.8 we give a definition of abelian varieties with MIQF. In Section 2 we show how these properties are modified for abelian varieties over global functional fields. In Section 3 we define $\mathbb{F}_q[\theta]$, $\mathbb{F}_q(\theta)$, $\mathbb{R}_\infty$, $\mathbb{C}_\infty$ — analogs of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the finite characteristic case, and we consider an explicit calculation in $\mathbb{C}_\infty$. In Section 4 we define $\mathbb{F}_q[\theta]$-lattices in vector spaces over $\mathbb{C}_\infty$, and their Siegel matrices.

Section 5 gives a definition and properties of Anderson t-motives, considered as modules over Anderson ring, having some specific properties. We give explicit formulas for them and for Drinfeld modules as their particular cases; we define their tensor product. In Section 6 we consider t-modules $E(M)$ associated to Anderson t-motives $M$ (historically, it was the first definition). We consider identifications of $E(M)$ with $\mathbb{C}_\infty^n$ — there is no canonical identification unless $n = 1$. In Section 7 we give briefly some properties of reductions of Anderson t-motives. In Section 8 we define lattices of Drinfeld modules, and we give a detailed explicit calculation
of the lattice map for the Carlitz module. In Section 9 we define lattices of Anderson t-motives having \( N = 0 \), we formulate the main theorem of Anderson for uniformizable t-motives and define some objects that are used for the proof of this theorem. Particularly, we consider the space \( \text{Lie}(M) \) and show that — unlike \( E(M) \) — it has a canonical structure of \( \mathbb{C}_\infty \)-vector space. Finally, we define homology and cohomology groups of an Anderson t-motive.

In Section 10 we define duality for Anderson t-motives, as a particular case of the Hom functor (case \( N = 0 \)). We formulate the main theorem: the lattice of the dual t-motive is the dual of the lattice of the initial t-motive. Section 11 extends the results of Sections 9, 10 to the case of t-motives having \( N \neq 0 \). The main technical tool used for this case is a Hodge-Pink structure ([P], [HJ]).

In Section 12 we give a definition of a L-function of an Anderson t-motive (only one type of two existing types of L-functions), and we give its explicit calculation for the Carlitz module over \( \mathbb{F}_2 \). Finally, in Section 13 we consider some generalizations and modifications of Anderson t-motives.

Many important subjects are not considered in this short survey. We do not consider problems related to parametrizations of Drinfeld modules and their generalizations, hence we do not consider analogs of Eichler-Shimura theorem on reductions of Hecke correspondences on modular curves for the functional field case, and all further theory leading to proofs of Langlands conjectures.

1. ABELIAN VARIETIES IN CHARACTERISTIC 0.

Here we describe briefly main objects related to abelian varieties over number fields.

An abelian variety \( A \) over \( \mathbb{C} \) of dimension \( g \) is \( \mathbb{C}^g/L \), where \( L = \mathbb{Z}^{2g} \) is a lattice in \( \mathbb{C}^g \) satisfying the Riemann condition:

\[ \exists H = B + i\Omega \quad \text{an hermitian form on} \quad \mathbb{C}^g \quad (\text{here} \quad B, \Omega \quad \text{are its real and imaginary parts}) \quad \text{such that} \]

1.1.1. \( H \) is a positively defined hermitian form;

1.1.2. \( \Omega|_L \in \mathbb{Z} \), i.e. for \( u, v \in L \subset \mathbb{Z}^{2g} \) we have \( \Omega(u,v) \in \mathbb{Z} \).

There exists a basis \( \{e\} = \{e_1, \ldots, e_g, e_{g+1}, \ldots, e_{2g}\} \) of \( L \) over \( \mathbb{Z} \) such that the matrix of \( \Omega \) in this basis is

\[
\begin{pmatrix}
0 & D \\
-D & 0
\end{pmatrix}
\]

(entries are \( g \times g \)-blocks) where \( D = \text{diag} \begin{pmatrix} d_1, d_2, \ldots, d_g \end{pmatrix} \) is a diagonal \( g \times g \)-matrix with integer positive entries satisfying \( d_1 \mid d_2 \mid \ldots \mid d_g \). If all \( d_i \) are 1 then \( A \) (more exactly, a pair \( \{A, H\} \) ) is called a principally polarized variety; for simplicity, we shall consider only them.

There exists a matrix \( S \in M_{g \times g}(\mathbb{C}) \) such that

\[
\begin{pmatrix}
e_{g+1} \\
\vdots \\
e_{2g}
\end{pmatrix} = S \begin{pmatrix} e_1 \\
e_g
\end{pmatrix} \quad \text{(equality in} \quad \mathbb{C}^g) .
\]

\( S \) is called a Siegel matrix of \( A \) (and of the basis \( \{e\} \) ).

Conditions 1.1.1, 1.1.2 are equivalent to (case of principally polarized varieties):

1.2. \( S \) is symmetric, and \( \text{Im}(S) \) is positively defined.
The set of Siegel matrices is denoted by $\mathcal{H}_g$ (the Siegel upper half plane).

1.3. The symplectic group $Sp_{2g}(\mathbb{Z})$ acts on $\mathcal{H}_g$. Action of $Sp_{2g}(\mathbb{Z})$ corresponds to a change of basis of $L$ over $\mathbb{Z}$.

Two lattices are called equivalent (notation: $L_1 \sim L_2$) if there exists a $\mathbb{C}$-linear map $\varphi : \mathbb{C}^g \to \mathbb{C}^g$ such that $\varphi(L_1) = L_2$. Equivalent lattices have the same Siegel matrices (in appropriate bases).

**Theorem 1.4.** There exists a 1–1 equivalence between the set of abelian varieties (up to isomorphism) and the set of $L$ satisfying the Riemann condition, up to equivalence.

**Corollary.** The set of principally polarized abelian varieties (up to isomorphism) is isomorphic to $\mathcal{H}_g/Sp_{2g}(\mathbb{Z})$.

Particularly, the dimension of the moduli space of abelian varieties of dimension $g$ is $(g+1)^2 = (g+1)g$, because $S$ is symmetric.

1.5. Tate modules. Let $A_n$ be the group of $n$-torsion points of an abelian variety $A$. We have $A_n = (\mathbb{Z}/n)^{2g}$. Let $A$ be defined over $\mathbb{Q}$. In this case $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ — the Galois group of $\mathbb{Q}$ — acts on $A_n$.

Let $l$ be a prime. The Tate module of $A$ is defined as follows: $T_l(A) := \lim_{\longrightarrow} A_{l^n}$. We have $T_l(A) = \mathbb{Z}^{2g}_l$. $\text{Gal}(\mathbb{Q})$ acts on $T_l(A)$.

Let $p$ be a prime. We tell that $A$ has a good reduction at $p$ if there exists a system of equations defining $A$ such that all coefficients of these equations belong to $\mathbb{Q}$, are $p$-integer and after reduction of these coefficients modulo $p$, we get a system of equations defining an abelian variety over $\mathbb{F}_p$ (this is only a rough definition; for an exact definition we should distinguish forms of $A$ over $\mathbb{Q}$ and to prove that the reduction does not depend on a choice of a system of equations defining $A$).

If $A$ has a good reduction at $p$ then the Frobenius automorphism $Fr(p)$ acts on $T_l(A)$ (for simplicity, we consider the case $p \neq l$). It is defined up to a conjugation, and its characteristic polynomial $P$ is uniquely defined.

**Theorem 1.6. (A. Weil):** $P \in \mathbb{Z}[X]$, it does not depend on $l$, and its roots $\alpha_{p,1}, \ldots, \alpha_{p,2g} \in \mathbb{C}$ have properties:

\[
\alpha_i \alpha_{g+i} = p; \quad |\alpha_i| = p^{1/2}.
\]

1.7. Reduction. We need the following definition. Let $A$ be an abelian variety of dimension $g$ over $\overline{\mathbb{F}}_p$, and let $A_p$ be the group of its closed points of order $p$. We have $A_p = (\mathbb{Z}/p)^{g_0}$ where $0 \leq g_0 \leq g$. $A$ is called ordinary if $g_0 = g$, i.e. $A_p = (\mathbb{Z}/p)^g$. An equivalent definition: let $A_p$ be the group scheme of $p$-torsion of $A$, i.e. the kernel of multiplication by $p$. $A$ is ordinary iff $A_p = (\mu_p)^g \oplus (\mathbb{Z}/p)^g$ as a group scheme, where $\mu_p$ is the group scheme $\text{Spec} \overline{\mathbb{F}}_p[x]/(x^p - 1)$.

Here we consider only two properties concerning reduction of abelian varieties. We consider an abelian variety $\tilde{A}$ of dimension $g$ defined over $\mathbb{Q}$ such that it has a good reduction $\tilde{A}$ at a prime $p$, and this $\tilde{A}$ is ordinary. We have a reduction map on points of order $p$: $A_p \to \tilde{A}_p$. 4
Theorem 1.7.1. It is surjective, hence its kernel has dimension \( g \) over \( \mathbb{F}_p \). This kernel is an isotropic subspace of \( A_p \) with respect to a skew form on it (coming from the above \( \Omega \)).

Theorem 1.7.2. For any \( k = 0, \ldots, g \) the dimension of the moduli space of abelian varieties of dimension \( g \) over \( \overline{\mathbb{F}}_p \) whose \( g_0 \) is \( \leq k \) is \( \binom{k+1}{2} = \binom{k+1}{k} \).

Particularly, ”almost all” abelian varieties over \( \overline{\mathbb{F}}_p \) are ordinary; there are only finitely many abelian varieties having \( g_0 = 0 \).

1.8. Abelian varieties with MIQF.

Since Anderson t-motives are analogs of abelian varieties with MIQF, we recall briefly their definition and properties. Let \( K \) be an imaginary quadratic field, \( A \) an abelian variety of dimension \( r \) such that there exists an inclusion \( K \hookrightarrow \text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q} \) (we fix this inclusion).

Rings of endomorphisms of abelian varieties are described for example in [Sh]. Our case of \( A \) with MIQF is the type IV of [Sh], Proposition 1, p. 153. For this case (for generic \( A \)), in notations of [Sh], we have \( F = \mathbb{Q}, F_0 = K \), i.e. the degrees \( g, q \) of [Sh], 2.1, p. 155, are both 1. Further, \( m \) of [Sh], (7), p. 156 is \( r \).

A description of the action of \( K \) on \( A \) is the following. Let \( V = \mathbb{C}^r \) and \( A = V/L \) as above. The field \( K \) acts on \( V \). The space \( V \) is the sum of two subspaces: \( V = V^+ \oplus V^- \) where the action of \( K \) on \( V^+ \) is the ”direct” one (i.e. simply the multiplication by the corresponding element of \( K \)), while the action of \( K \) on \( V^- \) is the ”conjugate” one: for \( x \in K, v \in V^- \) we have \( x(v) = \bar{x} \cdot v \) (here \( x(v) \) is the action of \( x \) on \( v \), bar means the complex conjugation, and \( \bar{x} \cdot v \) is the multiplication in the vector space \( V \)).

Let \( n := \dim V^+ \), hence \( \dim V^- = r - n \). The pair \((n, r - n)\) is called the signature of \( A \), it is \((r_1, s_1)\) of [Sh], (8), p. 156.

We do not give a definition of a Siegel matrix for the case of these \( A \) (see [Sh] or, more generally, [De] for a general definition). We only indicate that for our case of \( A \) with MIQF its Siegel matrix is a \( n \times (r - n) \) complex matrix \( S \) satisfying (see [Sh], 2.6, p. 162)

\[
I_n - SS^t \text{ is positively defined} \quad (1.8.1)
\]

1.8.2. Particularly, the dimension of the moduli space of abelian varieties with MIQF of signature \((n, r - n)\) is \( n(r - n) \), because the condition (1.8.1) does not impose algebraic relations on entries of \( S \).

We can associate a reductive group over \( \mathbb{Q} \) to any type of abelian varieties with a fixed endomorphism ring (see [De] for a much more general situation). For abelian varieties with MIQF of signature \((n, r - n)\) this group is \( GU(n, r - n) \).

We do not give here a definition of ordinariness of reduction of abelian varieties with MIQF over \( \overline{\mathbb{F}}_q \). We indicate only that if \( A \) is ordinary as a variety with MIQF then it is not ordinary as a variety obtained from \( A \) by forgetting the MIQF-structure (unless \( n = r - n \)). Further, like for the general abelian varieties, ”almost all” \( A \) with MIQF are ordinary as a variety with MIQF. In this case the kernel of the reduction map of Theorem 1.7.1 is \( n \) (if \( n \geq r - n \)).

1.8.3. Finally, we indicate an amusing construction of a lattice for an abelian variety with MIQF. This lattice is an analog of a lattice of an Anderson t-motive.
Namely, we shall see (Section 9) that if $M$ is an uniformizable Anderson t-motive of rank $r$ and dimension $n$ then we can associate it a lattice in $\mathbb{C}_\infty^n$ of dimension $r$ over $\mathbb{F}_q[\theta]$ — the functional analog of $\mathbb{Z}$. Therefore, we can expect that if $A$ is an abelian variety with MIQF of signature $(n, r - n)$ then we can associate it a ”lattice” in $\mathbb{C}^n$. This is really so! This ”lattice” is an $O_K$-module of rank $r$ in $\mathbb{C}^n$ having some properties. See [GL09], Theorem 2.6 for the exact statement; there is a 1 – 1 correspondence between abelian varieties with MIQF and such ”lattices”.

This is a rare example of ”an analogy to the opposite direction”: we consider a construction in the theory of Anderson t-motives, and we find its analog in the theory of abelian varieties with MIQF.

2. ABELIAN VARIETIES IN CHARACTERISTIC $p$.

Let $q$ be a power of a prime $p$, $\mathbb{F}_q$ a finite field of order $q$. Let $\theta$ be an abstract transcendent element. The analog of $\mathbb{Z} \subset \mathbb{Q}$ is $\mathbb{F}_q[\theta] \subset \mathbb{F}_q(\theta)$. Let $\overline{\mathbb{F}}_q(\theta)$ be an algebraic closure of $\mathbb{F}_q(\theta)$.

Let $A$ be an abelian variety over $\overline{\mathbb{F}}_q(\theta)$ of dimension $g$. There is no analog of the above formula $A = \mathbb{C}^g/L$, but we have

$$A_n = (\mathbb{Z}/n)^{2g} \text{ if } (n, p) = 1 \text{ (as earlier)},$$

and hence $T_l(A) = \mathbb{Z}_{l}^{2g} (l \neq p)$.

If $A$ is defined over $\mathbb{F}_q(\theta)$, then $\text{Gal}(\mathbb{F}_q(\theta))$ acts on $T_l(A)$, and we have an analog of the Weil Theorem.

Hence, we have the following table for abelian varieties:

| Abelian varieties over $\mathbb{Q}$ | $\mathbb{Z}_{l}^{2g}$ | $\text{Gal}(\mathbb{Q})$ |
| Abelian varieties over $\mathbb{F}_q(\theta)$ | $\mathbb{Z}_{l}^{2g}$ | $\text{Gal}(\mathbb{F}_q(\theta))$ |

3. INITIAL RINGS AND FIELDS

For the characteristic 0 we have inclusions of rings and fields: $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Let us consider their analogs in characteristic $p$. As it was mentioned above, the analog of $\mathbb{Z} \subset \mathbb{Q}$ is $\mathbb{F}_q[\theta] \subset \mathbb{F}_q(\theta)$.

Valuations of $\mathbb{Q}$ are $v_p$ where $p$ is prime, and $v_\infty$ the Archimedean valuation.

Valuations of $\mathbb{F}_q(\theta)$ have a similar description. Let $P \in \mathbb{F}_q[\theta]$ be an irreducible polynomial. It defines a valuation $v_P$ on $\mathbb{F}_q(\theta)$. The analog of $v_\infty$ on $\mathbb{F}_q(\theta)$ (it has the same notation $v_\infty$) is the valuation ”minus degree”: for $S \in \mathbb{F}_q(\theta)$ we have $v_\infty(S) := -\text{degree}(S)$. Equivalently, this is the order of zero of a function at infinity; also $v_\infty$ can be defined as the only valuation satisfying $v_\infty(\theta) = -1$. Unlike $v_\infty$ for the number field case, $v_\infty$ for the functional field case is a non-archimedean valuation.

Valuation $v_\infty$ defines a topology in $\mathbb{F}_q(\theta)$. Later we shall consider only this valuation and its topology. We have in it: $\theta^{-n} \to 0$ for $n \to +\infty$. 

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We have \( \mathbb{R}_\infty = \mathbb{F}_q(\langle \theta^{-1} \rangle) \) is the completion of \( \mathbb{F}_q(\theta) \). Like in the number field case, \( \mathbb{F}_q[\theta] \) is discrete in \( \mathbb{R}_\infty \), and the quotient \( \mathbb{R}_\infty / \mathbb{F}_q[\theta] = \theta^{-1} \mathbb{F}_q[\langle \theta^{-1} \rangle] \) is compact.

Finally, \( \mathbb{C}_\infty := \hat{\mathbb{R}}_\infty \) is the completion of the algebraic closure of \( \mathbb{R}_\infty \). It is complete by definition, and algebraically closed (see [G], Proposition 2.1). \( \mathbb{R}_\infty \subset \mathbb{C}_\infty \) is a characteristic \( p \) analog of \( \mathbb{R} \subset \mathbb{C} \).

**Remark 3.1.** Let us consider a field formed by convergent series generated by rational powers of \( \theta^{-1} \), with coefficients in \( \overline{\mathbb{F}}_p \). More exactly, let \( \alpha_1 < \alpha_2 < \alpha_3 < \ldots \) be a sequence of rational numbers tending to \( +\infty \), and let \( c_i \in \overline{\mathbb{F}}_p \) be coefficients.

The series \( \sum_{i=0}^{\infty} c_i \theta^{-\alpha_i} \) form a field denoted by \( \mathbb{C}_{\infty,s} \). We have \( \mathbb{R}_\infty \subset \mathbb{C}_{\infty,s} \subset \mathbb{C}_{\infty} \).

The reader can think that \( \mathbb{C}_{\infty,s} = \mathbb{C}_{\infty} \), but this is wrong. A well-known example of \( r \in \mathbb{C}_{\infty} - \mathbb{C}_{\infty,s} \) (see [GL21], Remark 4.3) is a root to the equation

\[
x^2 + x + \theta^2 = 0 \quad (3.2)
\]

(here \( q = 2 \)). Really, formally we have

\[
r = \theta + \theta^\frac{1}{2} + \theta^\frac{1}{4} + \theta^\frac{1}{8} + \ldots \quad (3.3)
\]

but this series \( \not\in \mathbb{C}_{\infty,s} \), because \( -\frac{1}{2^n} \) does not tend to \( +\infty \) as \( n \to \infty \). The \( n \)-th approximation \( r_{n,i} \) to \( r \) (here \( i = 1, 2 \) : there are two roots to (3.2), it is separable) is given by the formula

\[
r_{n,i} = \theta + \theta^\frac{1}{2} + \theta^\frac{1}{4} + \theta^\frac{1}{8} + \ldots + \theta^\frac{1}{2\pi} + \delta_{in}
\]

We have: \( \delta_{in} \) is a root to

\[
y^2 + y + \theta^\frac{1}{2\pi} = 0
\]

and hence both \( \delta_{1n}, \delta_{2n} \) have \( v_\infty(\delta_{1n}) = v_\infty(\delta_{2n}) = -\frac{1}{2n+1} \). This shows once again that the series (3.3) does not converge to \( r \).

This phenomenon plays an important role in explicit calculations, see for example [GL21], proof of 4.1, 4.2. It shows that not all equations can be solved by a method of consecutive approximations.

### 4. LATTICES IN FUNCTIONAL FIELD

The dimension of \( \mathbb{C} \) over \( \mathbb{R} \) is 2, hence a \( 2g \)-dimensional lattice \( L \) over \( \mathbb{Z} \) in \( \mathbb{C}^g \) is complete: the quotient \( \mathbb{C}^g / L \) is compact, and a basis of \( L \) over \( \mathbb{Z} \) is also a basis of \( \mathbb{C}^g \) over \( \mathbb{R} \).

Unlike the number field case, the dimension of \( \mathbb{C}_\infty \) over \( \mathbb{R}_\infty \) is infinite (and moreover of cardinality continuum), hence all lattices in \( \mathbb{C}_{\infty,n} \) are “incomplete”.

**Definition 4.1.** Let \( L = \mathbb{F}_q[\theta]^r \) and \( L \subset \mathbb{C}_{\infty,n} \). \( L \) is called a lattice if:

1. \( L \) generate all \( \mathbb{C}_{\infty,n} \) over \( \mathbb{C}_{\infty} \);

2. \( L \) generate a space of dimension \( r \) over \( \mathbb{R}_\infty \) (i.e. elements of a basis of \( L \) over \( \mathbb{F}_q[\theta] \) are linearly independent over \( \mathbb{R}_\infty \)).

We see that the pair \( (L, \mathbb{C}_{\infty,n}) \) has two discrete parameters: \( r \) and \( n \). Their analogs in the number field case are \( 2g \), resp. \( g \).
Remark. It is meaningful to consider incomplete lattices in the number field case as well. For example, we have the exponential map

\[ 0 \to L \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 0 \]

where \( L \) is a 1-dimensional lattice in \( \mathbb{C} \).

The notion of equivalence of lattices in \( \mathbb{C}_\infty^n \) is the same as in the number field case.

The definition of a Siegel matrix for the functional field case is also the same as the one for the number field case. Let \( e_* := \{ e_1, \ldots, e_n, e_{n+1}, \ldots, e_r \}^t \) (here and below \( t \) means transposition) be a basis of \( L \) over \( \mathbb{F}_q[\theta] \) such that \( e_1, \ldots, e_n \) is a basis of \( \mathbb{C}_\infty^n \) over \( \mathbb{C}_\infty \). Hence, there exists a matrix \( S = S_{ij} \in M_{(r-n) \times n}(\mathbb{C}_\infty) \) such that

\[
\begin{pmatrix}
e_{n+1} \\
\vdots \\
e_r
\end{pmatrix} = S \begin{pmatrix} e_1 \\
\vdots \\
e_n
\end{pmatrix}, \text{i.e. } \forall j = 1, \ldots, r - n \text{ we have } e_{n+j} = \sum_{i=1}^n S_{ji} e_i.
\]

It is called a Siegel matrix of \( L \) in the basis \( \{ e_* \} \).

Let us consider a functional field case analog of (1.3). Let \( e'_* := \{ e'_1, \ldots, e'_n \}^t \) be another basis of \( L \) over \( \mathbb{F}_q[\theta] \) and \( g \in GL_r(\mathbb{F}_q[\theta]) \) be the matrix of the change of basis from \( e_* \) to \( e'_* \) (we use the agreement \( e'_* = g \cdot e_* \), where \( e_* \), \( e'_* \) are considered as column matrices of size \( r \times 1 \)).

Unlike the number field case, it can happen that \( e'_1, \ldots, e'_n \) is not a \( \mathbb{C}_\infty \)-basis of \( \mathbb{C}_\infty^n \). In this case, a Siegel matrix for \( e'_* \) does not exist. If \( e'_1, \ldots, e'_n \) is a \( \mathbb{C}_\infty \)-basis of \( \mathbb{C}_\infty^n \) then \( S' \) — the Siegel matrix of \( L \) in the basis \( \{ e_* \} \) — is related with \( S \) by the same formula as in the number field case. Namely, we consider a block form of \( g \) \( g = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{pmatrix} \) where blocks \( \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22} \) are of sizes respectively \( n \times n, n \times (r - n), (r - n) \times n, (r - n) \times (r - n) \). We have:

\[
S' = (\gamma_{21} + \gamma_{22} S)(\gamma_{11} + \gamma_{12} S)^{-1} \tag{4.2}
\]

Condition \( \gamma_{11} + \gamma_{12} S \in GL_n(\mathbb{C}_\infty) \) is equivalent to the condition that \( e'_1, \ldots, e'_n \) form a \( \mathbb{C}_\infty \)-basis of \( \mathbb{C}^n_\infty \).

We see that the difference with the number field case is the following. First, there is no notion of polarization, i.e. there is no conditions like symmetry imposed on \( S \) — instead of the symplectic group \( GSp \) we have the linear group \( GL \). Second, formula 4.2 defines not the action of \( GL_r(\mathbb{F}_q[\theta]) \) on the set of Siegel matrices, but only an ”almost action”: if \( \gamma_{11} + \gamma_{12} S \notin GL_n(\mathbb{C}_\infty) \) then the action is not defined.

We see that really we have an analogy with the case of abelian varieties with MIQF. We have the same size of Siegel matrices and the same reductive group acting on them.

5. ANDERSON T-MOTIVES

Let \( \mathbb{C}_\infty[T, \tau] \) be a ring of non-commutative polynomials in two variables \( T, \tau \) with the following relations (here \( a \in \mathbb{C}_\infty \)):

\[
aT = Ta; \quad \tau T = T \tau; \quad \tau a = a^q \tau \text{ (and hence } \tau^k a = a^{q^k} \tau^k) \tag{5.1}
\]
It is called the Anderson ring. \( \mathbb{C}_\infty [T, \tau] \) has subrings \( \mathbb{C}_\infty [T], \mathbb{C}_\infty \{\tau\} \).

**Definition 5.2.** Anderson t-motive \( M \) is a left \( \mathbb{C}_\infty [T, \tau] \)-module satisfying conditions:

5.2.1. \( M \) as a \( \mathbb{C}_\infty [T] \)-module is free of finite dimension (denoted by \( r \));
5.2.2. \( M \) as a \( \mathbb{C}_\infty \{\tau\} \)-module is free of finite dimension (denoted by \( n \));
5.2.3. The action of \( T - \theta \) on \( M/\tau M \) is nilpotent.

Homomorphisms of Anderson t-motives are module homomorphisms.

Numbers \( r \), resp. \( n \) are called the rank (resp. dimension) of \( M \). We shall use by default these notations \( r, n \).

For generalizations of the notion of Anderson t-motive see Section 13. In most generalizations \( M \) is considered as a \( \mathbb{C}_\infty [T] \)-module with a skew \( \tau \)-action, i.e. for \( m \in M \) we consider \( \tau(m) \) instead of \( \tau m \); skew action means that \( \tau(am) = a^q \tau(m) \) (here \( a \in \mathbb{C}_\infty \)), according (5.1).

**Definition 5.3.** Drinfeld module\(^2\) is an Anderson t-motive of dimension \( n = 1 \).

**Example 5.4.** Let \( M \) be a Drinfeld module, and \( \{e\} = e_1 \) the only element of a basis of \( M \) over \( \mathbb{C}_\infty \{\tau\} \). This means that any \( m \in M \) can be uniquely written

\[
m = (c_0 + c_1 \tau + c_2 \tau^2 + \cdots + c_{k-1} \tau^{k-1} + c_k \tau^k)e\tag{5.5}
\]

where \( c_i \in \mathbb{C}_\infty \).

To define a left \( \mathbb{C}_\infty [T, \tau] \)-module structure on \( M \), it is sufficient to define the element \( Te \in M \). (5.5) implies that there exist \( a_0, \ldots, a_r \in \mathbb{C}_\infty \), \( a_r \neq 0 \), such that

\[
Te = (a_0 + a_1 \tau + a_2 \tau^2 + \cdots + a_{r-1} \tau^{r-1} + a_r \tau^r)e\tag{5.6}
\]

Hence, elements \( a_0, \ldots, a_r \) define \( M \) uniquely. Condition (5.2.3) implies that

\[
a_0 = \theta\tag{5.7}
\]

**Exercise 5.8.** \( r \) is the rank of \( M \), elements

\( e, \tau e, \tau^2 e, \ldots, \tau^{r-1} e \)

form a basis of \( M \) over \( \mathbb{C}_\infty [T] \).

We have an analog of (5.6) for Anderson t-motives. Let \( \{e\} = \{e_1, \ldots, e_n\}^t \) be a basis of \( M \) over \( \mathbb{C}_\infty \{\tau\} \) considered as a matrix column. Instead of \( a_i \) of (5.6), we have \( n \times n \) matrices \( A_i \) with entries in \( \mathbb{C}_\infty \). (5.6) becomes a matrix equality

\[
T\{e\} = (A_0 + A_1 \tau + A_2 \tau^2 + \cdots + A_{k-1} \tau^{k-1} + A_k \tau^k)\{e\}
\]

\[
= A_0\{e\} + A_1 \tau\{e\} + A_2 \tau^2\{e\} + \cdots + A_{k-1} \tau^{k-1}\{e\} + A_k \tau^k\{e\}\tag{5.9}
\]

\(^2\)See Remark 6.1 why similar objects have different names — modules and motives.
Condition (5.2.3) implies that $N := A_0 - \theta I_n$ is a nilpotent matrix. Anderson t-motives having $N = 0 \iff A_0 = \theta I_n$ are more simple objects than Anderson t-motives having $N \neq 0$.

We denote $A := \sum_{i=0}^k A_i \tau^i \in M_{n \times n}(\mathbb{C}_\infty\{\tau\})$, Clearly $A, A_i$ depend on a choice of basis $\{e\}$.

An analog of Exercise 5.8 for this case is

Exercise 5.10. If $\det A_k \neq 0$ then elements $\tau^i e_j$, $i = 0, \ldots, k-1$, $j = 1, \ldots, n$ form a basis of $M$ over $\mathbb{C}_\infty[T]$.

Hence if $\det A_k \neq 0$ then $r = kn$. We see that for interesting examples of Anderson t-motives we should have $\det A_k = 0$.

5.11. We can consider an Anderson t-motive $M$ not as a $\mathbb{C}_\infty\{\tau\}$-module with $T$-action, but as a $\mathbb{C}_\infty[T]$-module with $\tau$-action. Moreover, this type of consideration is used more frequently in applications. Let $\{f\} = \{f_1, \ldots, f_r\}^t$ be a basis of $M$ over $\mathbb{C}_\infty[T]$ considered as a matrix column. There exists a matrix $Q \in M_{r \times r} \mathbb{C}_\infty[T]$ defining the multiplication by $\tau$, namely

$$\tau\{f\} = Q\{f\}$$

Exercise 5.12. For a Drinfeld module $M$ defined by (5.6) such that $a_r = 1$, the matrix $Q$ is the following:

$$Q = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
T - \theta & -a_1 & -a_2 & -a_3 & \ldots & -a_{r-2} & -a_{r-1}
\end{pmatrix}$$

For an Anderson t-motive given by (5.9) such that $A_k = I_n$, the matrix $Q$ is analogous, all entries are $n \times n$-blocks.

Example 5.13. Let us consider a particular case of (5.9), namely an Anderson t-motive (denoted by $M(A)$) given by the formula

$$T\{e\} = \theta\{e\} + A\tau\{e\} + \tau^2\{e\} \quad (5.13.1)$$

i.e. we have $A_0 = \theta I_n$, $N = 0$, $A_1 = A$, $A_2 = I_n$. It has $r = 2n$, it is a close analog of abelian varieties. These Anderson t-motives $M(A)$ are studied in [GL17], [GL21], [EGL].

5.14. Change of basis. First, we consider the change of basis for a Drinfeld module $M$. Let $\{e\} = e_1$ be from 5.4 — the only element of a basis of $M$ over $\mathbb{C}_\infty\{\tau\}$. Let $c \in \mathbb{C}_\infty^*$ and $c' = c^{-1} e$ be the only element of another basis of $M$ over $\mathbb{C}_\infty\{\tau\}$. (6.6) becomes (because $\tau^k c = c^q \tau^k$)

$$cTe' = c\theta e' + c^q a_1 \tau e' + c^q a_2 \tau^2 e' + \cdots + c^q a_{r-1} \tau^{r-1} e' + c^{q^r - 1} a_r \tau^r e' \quad (5.14.1),$$

or

$$Te' = \theta e' + c^{q-1} a_1 \tau e' + c^{q-1} a_2 \tau^2 e' + \cdots + c^{q^{r-1} - 1} a_{r-1} \tau^{r-1} e' + c^{q^r - 1} a_r \tau^r e' \quad (5.14.2)$$
Hence, coefficients \( \{ \theta, c^q a_1, c^{q^2} a_2, \ldots, c^{q^{r-1}} a_r \} \) define the same Drinfeld module as coefficients \( \{ \theta, a_1, \ldots, a_r \} \) (both are coefficients of (5.6)). Particularly, any Drinfeld module can be defined by an equation (5.6) with \( a_r = 1 \).

**Corollary 5.14.3.** Two Drinfeld modules of rank two defined by the equations

\[
Te = \theta e + a_1 \tau e + \tau^2 e
\]

\[
Te = \theta e + a_2 \tau e + \tau^2 e
\]

are isomorphic iff \( a_2 = \beta a_1 \) where \( \beta^{q+1} = 1 \). Really, we can choose the above \( c \) as \( c = \beta^{1/(q-1)} \); we have \( e^q - 1 \).

**Corollary 5.14.4.** The moduli space of Drinfeld modules of rank \( r \) has dimension \( r - 1 \).

Really, a Drinfeld module of rank \( r \) is defined by \( r \) parameters \( a_1, \ldots, a_r \), and their equivalence is defined by one above parameter \( c \).

For Anderson t-motives the notion of isomorphism is more complicated, because a matrix of change of basis from \( e_* \) to \( e' \) belongs to \( GL_n(\mathbb{C}_\infty \{ \tau \}) \). For \( n = 1 \) it is simply \( \mathbb{C}_\infty \), but for \( n > 1 \) \( GL_n(\mathbb{C}_\infty \{ \tau \}) \) is a "doubly non-abelian" group (the first "non-abelianity" comes from \( GL_n \), the second one from non-abelianity of \( \mathbb{C}_\infty \)).

If a matrix \( C \) of change of basis has constant coefficients, i.e. \( C \in GL_n(\mathbb{C}_\infty) \subset GL_n(\mathbb{C}_\infty \{ \tau \}) \) then (5.9) becomes (a calculation similar to (5.14.2), case \( A_0 = \theta I_n \); here \( C^{(m)} \) is a matrix obtained by elevation of all entries of \( C \) to the \( q^m \)-th degree):

\[
T\{e'\} = \theta I_n \{e'\} + C^{-1} A_1 C^{(1)} \tau \{e'\} + C^{-1} A_2 C^{(2)} \tau^2 \{e'\} + \ldots \\
+ C^{-1} A_{k-1} C^{(k-1)} \tau^{k-1} \{e'\} + C^{-1} A_k C^{(k)} \tau^k \{e'\}
\]

(5.14.5)

According a theorem of Lang that \( H^1(\mathbb{F}_q, GL_n(\mathbb{C}_\infty)) = 1 \) we get that if \( \det A_k \neq 0 \) then \( \exists C \in GL_n(\mathbb{C}_\infty) \) such that \( C^{-1} A_k C^{(k)} = I_n \), i.e. in this case any Anderson t-motive of this type can be defined by an equation (5.9) with \( A_k = I_n \).

It can happen that \( C \in GL_n(\mathbb{C}_\infty \{ \tau \}) - GL_n(\mathbb{C}_\infty) \). This phenomenon occurs even for \( M \) defined by (5.13.1). More exactly, there exist two matrices \( A_1, A_2 \) such that the Anderson t-motives \( M(A_1) \), resp. \( M(A_2) \) defined by (5.13.1) are isomorphic, and the matrix \( C \) of a change of basis cannot be chosen in \( GL_n(\mathbb{C}_\infty) \), but only in \( GL_n(\mathbb{C}_\infty \{ \tau \}) - GL_n(\mathbb{C}_\infty) \). See [GL17] for examples.

**5.14.6.** Moreover, according the knowledge of the authors, we have no algorithm to check whether two Anderson t-motives \( M_1, M_2 \) defined by (5.9), case \( A_k = A'_k = I_n \), are isomorphic, or not. For the case of \( C \in GL_n(\mathbb{C}_\infty) \) this algorithm clearly exists, because in this case we have \( C = C^{(k)} \), i.e. \( C \in GL_n(\mathbb{F}_q^r) \); there exists only finitely many such \( C \). But really \( C \in GL_n(\mathbb{C}_\infty \{ \tau \}) \), so we have an infinite problem. A natural way to solve such problems is finding an invariant of \( M \). A lattice (see below) could be such invariant, but we do not know whether two non-isomorphic Anderson t-motives can have isomorphic lattices, or not.

An analog of Corollary 5.14.4 for Anderson t-motives holds not for all of them, but only for an important class of t-motives called pure t-motives. We do not give their definition (all Anderson t-motives defined by (5.9) such that \( \det A_k \neq 0 \) are
pure; particularly, all Drinfeld modules are pure). See [G], Definition 5.5.2. We have:

**Theorem 5.15.** ([H], Theorem 3.2). The dimension of the moduli space of pure t-motives of dimension $n$ and rank $r$ is $n(r - n)$.

We see that this dimension coincides with the dimension of the moduli space of abelian varieties with MIQF, of dimension $r$ and signature $(n, r - n)$, and with the set of Siegel matrices of lattices of dimension $r$ in $\mathbb{C}^n$, see Section 4.

**Example 5.16.** Case $r = 1$. The only Drinfeld module of rank 1 satisfies

$$Te = \theta e + \tau e$$  \hspace{1cm} (5.16.1)

It is called the Carlitz module. It is denoted by $\mathcal{C}$.

**Example 5.16.2.** Forms of Drinfeld module of rank 1.

Let $P \in \mathbb{F}_q(\theta)^*$. We consider a Drinfeld module of rank 1 (denoted by $\mathcal{C}_P$) defined by the equation

$$Te = \theta e + P\tau e$$

It is defined over $\mathbb{F}_q(\theta)$ (we do not give here the exact definition of ”defined over” and (see below) ”isomorphic over”, it is clear). The above arguments show that $\mathcal{C}$ is isomorphic to $\mathcal{C}_P$ over $\mathbb{C}_\infty$. But they are not isomorphic over $\mathbb{F}_q(\theta)$ unless $P^{\frac{1}{r-1}} \in \mathbb{F}_q(\theta)^*$.

Properties of $\mathcal{C}_P$ can differ from the properties of $\mathcal{C}$. See, for example, [GL16] for their $L$-functions.

Clearly forms exist for other Anderson t-motives, not only for $\mathcal{C}$.

**5.17. Tensor products.** Let $M_1, M_2$ be Anderson t-motives. We can consider their tensor product $M_1 \otimes_{\mathbb{C}_\infty[T]} M_2$ over $\mathbb{C}_\infty[T]$. It is clear that it is free of dimension $r_1r_2$ over $\mathbb{C}_\infty[T]$. Let us define the $\tau$-action on $M_1 \otimes_{\mathbb{C}_\infty[T]} M_2$ by the formula

$$\tau(m_1 \otimes m_2) := \tau(m_1) \otimes \tau(m_2)$$  \hspace{1cm} (5.17.1)

**Theorem 5.17.1a.** Let $M_1, M_2$ be pure t-motives. Then (5.2.2), (5.2.3) hold for $M_1 \otimes_{\mathbb{C}_\infty[T]} M_2$ with this $\tau$-action, hence it is an Anderson t-motive. It is also pure. Formula for $n$ of $M_1 \otimes M_2$ ([G], 5.7.2, (3)):

$$n = n_1r_2 + n_2r_1$$

The same is true for those t-motives which can be written as appropriate extensions of pure t-motives. They are called mixed in [HJ], Definition 3.5b. See [HJ], Example 3.9 where it is indicated which should be a direction of arrows of an exact sequence of pure t-motives in order to get a mixed t-motive.

**Remark.** The nilpotent operator $N$ of $M_1 \otimes M_2$ is not 0 even if $N$ of both $M_1, M_2$ are 0.
Analogously, we can consider multiple tensor products, symmetric and external tensor powers.

**Example 5.17.2.** Let us consider \( \mathcal{C}^\otimes n \) — the \( n \)-th tensor power of the Carlitz module. For the \( i \)-th factor of \( \mathcal{C}^\otimes n \) we denote \( e \) from (5.16.1) by \( e_i \), hence the only element of a basis of \( \mathcal{C}^\otimes n \) over \( \mathbb{C}_\infty[T] \) is \( e_1 \otimes e_2 \otimes \ldots \otimes e_n \). We denote \( e_1 \otimes e_2 \otimes \ldots \otimes e_n \) by \( e \), and for \( i = 0, \ldots, n-1 \) we denote \( (T - \theta)^i e_1 \) by \( e_{i+1} \). According (5.16.1) and (5.17.1), we have \( \tau(e_1) = (T - \theta)^n e_1 \).

Recall that \( \varepsilon_{ij} \) denotes a matrix whose \((i,j)\)-th element is 1 and all other elements are 0.

**Exercise.** Elements \( e_1, e_2, \ldots, e_n \) form a basis of \( \mathcal{C}^\otimes n \) over \( \mathbb{C}_\infty \{\tau\} \). The multiplication by \( T \) in this basis is defined by the formula

\[
T(e_\ast)^t = (\theta I_n + N)(e_\ast)^t + \varepsilon_{n1} \tau(e_\ast)^t
\]

where

\[
N = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

**Example 5.17.3.** Let \( M_1, M_2 \) be Drinfeld modules of ranks \( r_1, r_2 \). A description of \( M_1 \otimes M_2 \) is the following. Let \( e_1, e_2 \) be \( e \) from (5.6) for \( M_1, M_2 \) respectively. We have: elements

\[
e_1 \otimes e_2, e_1 \otimes \tau e_2, e_1 \otimes \tau^2 e_2, \ldots, e_1 \otimes \tau^{r-1} e_2,
\]

\[
\tau e_1 \otimes e_2, \tau^2 e_2 \otimes e_2, \ldots, \tau^{r_1-1} e_1 \otimes e_2, \quad (T - \theta)(e_1 \otimes e_2)
\]

are a \( \mathbb{C}_\infty \{\tau\} \)-basis of \( M_1 \otimes M_2 \). Equality \( T(e_1 \otimes e_2) = \theta(e_1 \otimes e_2) + (T - \theta)(e_1 \otimes e_2) \) shows that for this basis we have \( N = \varepsilon_{1, r_1+r_2} \) (details are left as an exercise for the reader).

**5.18. Exterior powers of Drinfeld modules.**

There exists the only case when a tensor power of Anderson t-motives having \( N = 0 \) is also a t-motive having \( N = 0 \). These are exterior powers of Drinfeld modules (and their duals, see Section 10). Namely, if \( M \) is a Drinfeld module of rank \( r \) then its \( k \)-th exterior power \( \lambda^k(M) \) is an Anderson t-motive of rank \( \binom{r}{k} \) dimension \( \binom{r}{k-1} \) having \( N = 0 \). For a proof see for example [GL09]; the corresponding fact for lattices (see Sections 8, 9) is an exercise.

According the analogy between Anderson t-motives and abelian varieties with MIQF, we have the following proposition. Let \( A \) be an abelian variety with MIQF of dimension \( r \) and signature \((1, r - 1)\). There exists an abelian variety with MIQF (denoted by \( \lambda^k(A) \)) of dimension \( \binom{r}{k} \) and signature \( \left( \binom{r}{k-1}, \binom{r-1}{k} \right) \) such that the lattice of \( \lambda^k(A) \) mentioned in (1.8.3) is obtained from the lattice of \( A \) using the same construction which gives the lattice of \( \lambda^k(M) \) starting from the lattice of \( M \).\(^3\)

\(^3\)Prof. Claire Voisin told to the second author that experts in abelian varieties knew the construction of \( \lambda^k(A) \) for an abelian variety with MIQF \( A \); apparently, it is not published. But she did not know that this construction (which looks quite artificial for the case of abelian varieties with MIQF) comes from a very natural construction for Drinfeld modules.
5.19. It is natural to consider not only the tensor product of t-motives, but also their Hom over $\mathbb{C}_\infty[T]$. But here we have a difficulty. The standard action of $\tau$ on Hom group is the following. Let $\varphi \in \text{Hom}(M_1, M_2)$. By definition, we have

$$\tau(\varphi)(m_1) := \tau(\varphi(\tau^{-1}(m_1))) \quad (5.19.1)$$

But $\tau^{-1}$ is not necessarily defined.

A way to get rid from this difficulty is to use a version of a definition of t-motives. See, for example, a definition of [HJ] (Section 11 and 13.3). For this definition Hom always exists, because (see 13.3) $\tau$ is an isomorphism.

Another way is to consider not all t-motives. For example, let $M$ be a t-motive having $N = 0$. It turns out that for many $M$ its dual

$$M' := \text{Hom}(M, \mathfrak{C}) \quad (5.19.2)$$

exists as a t-motive according definition 5.2; the action of $\tau$ is defined by (5.19.1)). See Section 10 for its properties; here we indicate only that its dimension is $r - n$.

If $N$ satisfies $N^m = 0$ then for many such $M$ there exists

$$M'^m := \text{Hom}(M, \mathfrak{C}^m) \quad (5.19.3)$$

which is called the $m$-th dual of $M$. See the end of Section 11 for its properties.

6. **Action of $\mathbb{F}_q[T]$ on $\mathbb{C}_\infty^n$ associated to an Anderson t-motive.**

First, we consider a case of a Drinfeld module $M$ defined by (5.6). Associated is an action of $\mathbb{F}_q[T]$ on $\mathbb{C}_\infty^n$. The actions of the elements $a \in \mathbb{F}_q \subset \mathbb{F}_q[T]$, resp. $T \in \mathbb{F}_q[T]$ are given by the formulas (here $x \in \mathbb{C}_\infty$):

$$a(x) = ax, \text{ resp. } T(x) = \theta x + a_1x^q + a_2x^{q^2} + \cdots + a_rx^{q^r}$$

The definition of the action of other elements of $\mathbb{F}_q[T]$ is defined by the formulas:

$$(P_1 + P_2)(x) = P_1(x) + P_2(x), \quad P_1P_2(x) = P_1(P_2(x)) \quad \text{for example, } T^2(x) = T(T(x))$$

It is easy to see that we have a formula: if $P \in \mathbb{F}_q[T]$ and

$$Pe = \left( b_0 + b_1 \tau + b_2 \tau^2 + \cdots + b_{k-1} \tau^{k-1} + b_k \tau^k \right) e \text{ in } M \text{ (here } b_i \in \mathbb{C}_\infty),$$

then $P(x) = b_0x + b_1x^q + b_2x^{q^2} + \cdots + b_kx^{q^k}$.

**Example.** Let $\mathfrak{C}$ be the Carlitz module. Its action:

$$T(x) = \theta x + x^q, \quad T^2(x) = T(\theta x + x^q) = \theta^2 x + (\theta + \theta^q)x^q + x^{q^2}, \quad (T^2 + T)(x) = \left( \theta^2 + \theta \right) x + (\theta^q + \theta + 1)x^q + x^{q^2}$$

This action defines a structure of a $\mathbb{F}_q[T]$-module on $\mathbb{C}_\infty$, because the formulas

$$P(x_1 + x_2) = P(x_1) + P(x_2), \quad P(ax) = aP(x) \text{ where } a \in \mathbb{F}_q, \text{ trivially hold (here } P \in \mathbb{F}_q[T]).$$

This module is denoted by $E(M)$.

**Remark 6.1.** Initially, Drinfeld gave in [D] exactly this definition, this explains the word ”module” in the terminology ”Drinfeld module”. This definition
of reductions.  

4.4.1). These Drinfeld modules in finite characteristic are useful for a description of non-triviality. Condition [G], 4.4.2, (1) is (5.7) of the present paper, and [G], 4.4.2, (2) is a condition of non-triviality.

Really, Goss considers a slightly more general case. First, his \( A \) can be a ring bigger than \( \mathbb{F}_q[T] \).

Second, \( \iota \) can have a non-trivial kernel \( \phi \) (a finite characteristic case, see [G], 4.4.1). These Drinfeld modules in finite characteristic are useful for a description of reductions.

Let us define \( E(M) \) for an Anderson t-motive \( M \) (see [G], proof of 5.6.3). Let us consider \( \mathbb{C}_\infty \) as a \( \mathbb{C}_\infty \{ \tau \} \)-module defined by the formula \( \tau(x) = x^q \) where \( x \in \mathbb{C}_\infty \).

**Definition 6.2A.** \( E(M) = \text{Hom}_{\mathbb{C}_\infty \{ \tau \}}(M, \mathbb{C}_\infty) \).

The functor \( M \mapsto E(M) \) is contravariant. According a general formalism of homological algebra, \( E(M) \) is a module over \( \mathbb{F}_q[\tau] \) — the center of \( \mathbb{C}_\infty \{ \tau \} \). Particularly, there is no canonical structure of \( \mathbb{C}_\infty \)-module on \( E(M) \).

Further, since \( M \) is a \( \mathbb{C}_\infty [T] \)-module, \( E(M) \) is a module over \( \mathbb{F}_q[T] \). Namely, for \( \phi : M \to \mathbb{C}_\infty \) we have \( T\phi(m) := \phi(Tm) \).

**6.2B.** Nevertheless, it is convenient to have a coordinate description of \( E(M) \) and an explicit description of the action of \( T \) on \( E(M) \). We fix the following notation: \( \{ e \} \) will mean a basis of \( M \) over \( \mathbb{C}_\infty \{ \tau \} \), written as vector column: \( \{ e \} = (e_1, \ldots, e_n)^t \).

We use notations of (5.9) for the action of \( T \) on \( \{ e \} \).

For \( x \in E(M) \) we denote \( x(e_i) := x_i \) (\( x \) is considered as a map from \( M \) to \( \mathbb{C}_\infty \)). The column vector \( \{ x \} = \{ x_1, \ldots, x_n \}^t \in \mathbb{C}_\infty^n \) is denoted by \( i_{\{ e \}}(x) \), i.e. there is an isomorphism \( i_{\{ e \}} : E(M) \to \mathbb{C}_\infty^n \) (coordinates on \( E(M) \)). The action of \( T \) on \( E(M) \) in these coordinates becomes the following action on \( \mathbb{C}_\infty^n \) via \( i_{\{ e \}} \) (here \( \{ x \} = i_{\{ e \}}(x) \)):

\[
i_{\{ e \}}(T(x)) = A_0 \{ x \} + A_1 \{ x \}^{(1)} + A_2 \{ x \}^{(2)} + \cdots + A_{k-1} \{ x \}^{(k-1)} + A_k \{ x \}^{(k)} \quad (6.2)
\]

Here, as above for matrices, \( \{ x \}^{(i)} := \{ x_1^q, \ldots, x_n^q \}^t \), and \( A_i \) are from (5.9). This formula is checked immediately.

For \( \{ x \} = (x_1, \ldots, x_n)^t \in \mathbb{C}_\infty^n \) we denote

\[
A_{\mathbb{C}_\infty}(\{ x \}) := A_0 \{ x \} + A_1 \{ x \}^{(1)} + \cdots + A_k \{ x \}^{(k)} \quad (6.2.1)
\]

(and analogously \( W_{\mathbb{C}_\infty}(\{ x \}) \) for \( W \), see 6.2C below). Hence, in this notation (6.2) becomes \( T(\{ x \}) := A_{\mathbb{C}_\infty}(\{ x \}) \).
Equivalently, (6.2) is a commutative diagram:

\[
\begin{array}{ccc}
E(M) & \overset{i(e)}{\rightarrow} & \mathbb{C}_\infty^n \\
T \downarrow & & \downarrow A_{\mathbb{C}_\infty} \\
E(M) & \overset{i(e)}{\rightarrow} & \mathbb{C}_\infty^n
\end{array}
\]

6.2C. Let us consider the change of basis. Let \( \{e'\} = (e'_1, \ldots, e'_n)^t \) be another basis of \( M \) over \( \mathbb{C}_\infty\{\tau\} \) and \( W = \sum_{i=0}^k W_i \tau^i \in GL_n(\mathbb{C}_\infty\{\tau\}) \) (here \( W_i \in GL_n(\mathbb{C}_\infty) \)) the matrix of change of basis:

\[ \{e'\} = W_0\{e\} + W_1 \tau(\{e\}) + \cdots + W_k \tau^k(\{e\}) \]

\( i_{\{e\}}, i_{\{e'\}} \) enter in the following commutative diagram:

\[
\begin{array}{ccc}
E(M) & \overset{i_{\{e\}}}{\rightarrow} & \mathbb{C}_\infty^n \\
\text{id} \downarrow & & \downarrow W_{\mathbb{C}_\infty} \\
E(M) & \overset{i_{\{e'\}}}{\rightarrow} & \mathbb{C}_\infty^n
\end{array}
\]  

(6.2D)

Remark. For \( n > 1 \) there is no canonical structure of \( \mathbb{C}_\infty \)-module on \( E(M) \). Really, the only way to try to define this structure is the following. Let \( x \in E(M) \) and \( c \in \mathbb{C}_\infty \). We define \( cx \) via \( i_{\{e\}} \), namely \( cx := cx_{\{e\}} = i_{\{e\}}^{-1}(c \cdot i_{\{e\}}(x)) \). But if \( \exists \ i > 0 \) such that \( W_i \neq 0 \) then \( cx_{\{e\}} \neq cx_{\{e'\}} \).

For \( n = 1 \) (Drinfeld modules) we have \( W \in GL_1(\mathbb{C}_\infty\{\tau\}) = GL_1(\mathbb{C}_\infty) \), hence for \( n = 1 \) there exists a canonical structure of \( \mathbb{C}_\infty \)-module on \( E(M) \).

6.2E. Also, we can give an invariant definition of \( E(M) \) in coordinates. Let \( \mathfrak{B} \) be the set of \( \mathbb{C}_\infty\{\tau\} \)-bases of \( M \) (a principal homogeneous space over \( GL_n(\mathbb{C}_\infty\{\tau\}) \)). \( E(M) \) is the quotient set of \( \mathbb{C}_\infty^n \times \mathfrak{B} \) by the equivalence relation coming from (6.2D): a pair \( (\{x\}, \{e\}) \) is equivalent to a pair \( (\{x'\}, \{e'\}) \) iff \( \{x'\} = W_{\mathbb{C}_\infty}(\{x\}) \).

6.3. Torsion points and Tate modules. Let \( M \) be an Anderson t-motive and \( P \in \mathbb{F}_q[T] \). We define \( M_P \) — the set of \( P \)-torsion points of \( M \) — as follows, see [G], Proposition 5.6.3. It is the following subset of \( E(M) \) (not of \( M \) itself!)

\[
M_P := \{ x \in E(M) \mid P(x) = 0 \}.
\]

Choosing \( \{e\} \) and identifying \( E(M) \) with \( \mathbb{C}_\infty^n \) via \( i_{\{e\}} \), we can consider \( M_P \) as a subset of \( \mathbb{C}_\infty^n \).

6.3.1. Example: Let \( P = T \) be the simplest irreducible polynomial, and \( M \) a Drinfeld module defined by (5.6). Then \( M_T \subset \mathbb{C}_\infty \), it is a set of the roots of the following polynomial:

\[
\theta x + a_1 x^q + a_2 x^{q^2} + \cdots + a_r x^{q^r}
\]

This is a \( \mathbb{F}_q \)-vector space of dimension \( r \) — a phenomenon that never occurs in characteristic 0!
For any Anderson t-motive $M$ we have: $M_T$ is an abelian group, moreover, a $\mathbb{F}_q$-module of dimension $r$. Analogously, for any $P \in \mathbb{F}_q[T]$ we have: $M_P$ is a free $\mathbb{F}_q[T]/P$-module of dimension $r$, see [G], Corollary 5.6.4.

6.3.2. Definition. Let $\mathcal{L}$ be a monic irreducible polynomial in $\mathbb{F}_q[T]$, i.e. a finite place of $\mathbb{F}_q(T)$. The $\mathcal{L}$-Tate module of $M$ is:

$$T_\mathcal{L}(M) := \underset{k \to \infty}{\lim} M_{\mathcal{L}^k}$$

Namely, $x \in T_\mathcal{L}(M)$ is a sequence $(\ldots, x_3, x_2, x_1, x_0 = 0)$, $x_i \in \mathbb{C}_\infty^n$ such that $\mathcal{L}(x_i) = x_{i-1}$.

We have: $T_\mathcal{L}(M) = (\mathbb{F}_q[\theta]\mathcal{L})^r$ (here $\mathbb{F}_q[\theta]\mathcal{L}$ is the $\mathcal{L}$-adic completion of $\mathbb{F}_q[\theta]$).

Let $M$ be defined over a field $K \supset \mathbb{F}_q(T)$ (or, more exactly, we consider $M$ over $K$ with a fixed $K$-structure). The absolute Galois group $\text{Gal}(K)$ acts on $T_\mathcal{L}(M)$ for any $\mathcal{L}$. An analog of the Weil theorem holds for it. See [G], Section 4.10 for details.

Hence, we have the following table for Tate modules and Galois groups of our objects. Problem: define an object giving the fourth line!

**Table 2**

| Tate module              | Galois group |
|-------------------------|--------------|
| Abelian varieties over $\mathbb{Q}$ | $\mathbb{Z}_l^{2g}$ | $\text{Gal}(\mathbb{Q})$ |
| Abelian varieties over $\mathbb{F}_q(\theta)$ | $\mathbb{Z}_l^{2g}$ | $\text{Gal}(\mathbb{F}_q(\theta))$ |
| Anderson t-motives over $\mathbb{F}_q(\theta)$ | $\mathbb{F}_q[\theta]\mathcal{L}$ | $\text{Gal}(\mathbb{F}_q(\theta))$ |
| $\ldots$                | $\mathbb{F}_q[\theta]\mathcal{L}$ | $\text{Gal}(\mathbb{Q})$ |

7. REDUCTIONS.

Let $\mathfrak{P}$ be a prime of $\mathbb{F}_q[\theta]$ and $M$ an Anderson t-motive such that all entries of $A_i$ from (5.9) belong to $\mathbb{F}_q[\theta]$ and are $\mathfrak{P}$-integer. Let us consider the equation obtained from (5.9) by reduction of all entries of $A_i$ modulo $\mathfrak{P}$. The obtained object is called the reduction of $M$ modulo $\mathfrak{P}$. See [G], (4.10) for the details, the notion of a good reduction, of an ordinary reduction etc. Roughly speaking, $M$ has a good reduction at $\mathfrak{P}$ if the mayor coefficient of (5.9) ”does not loose its rank” after reduction (i.e. the reduced t-motive has the same $r$ as $M$ itself).

To understand a notion of ordinary reduction we consider the action of $\iota^{-1}(\mathfrak{P})$ on $\mathbb{C}_\infty^n$ defined in Section 6, where $\mathfrak{P}$ is considered as an element of $\mathbb{F}_q[\theta]$ and $\iota : \mathbb{F}_q[T] \to \mathbb{F}_q[\theta]$ is from 6.1a, i.e. $\iota(T) = \theta$. For example, if $\mathfrak{P} = \theta$ then this action is defined by (6.2). We consider the minor coefficient of the formula of this action; it is $\mathfrak{P}I_n +$ a nilpotent operator. Hence, its reduction at $\mathfrak{P}$ is nilpotent. Roughly speaking, a good reduction of $M$ is ordinary if only this minor coefficient becomes nilpotent after the reduction, i.e. the next coefficient ”does not loose its rank” after reduction. We do not give here an exact definition.
7.1. **Theorem.** Let $M$ have a good ordinary reduction at $\mathfrak{p}$ (denoted by $\tilde{M}$). Then the set of closed points of $\tilde{M}_\mathfrak{p}$ is $(\mathbb{F}_q[\theta]/\mathfrak{p})^r - n$, and hence the dimension over $\mathbb{F}_q[\theta]/\mathfrak{p}$ of the kernel of the reduction map $M_\mathfrak{p} \rightarrow \tilde{M}_\mathfrak{p}$ is $n$.

Caution: We see that it is important do not confuse $T$ and $\theta$ in the above formulas. Really, reduction is at $P \in \mathbb{F}_q[\theta]$ while $P$-torsion points are defined for $\iota^{-1}(P) \in \mathbb{F}_q[T]$. We must use the map $\iota$.

We see that the behavior of Anderson t-motives under reduction is analogous to the behavior of abelian varieties with MIQF.

8. **LATTEICES OF DRINFELD MODULES**

Let $M$ be a Drinfeld module defined by (5.6). Let us consider the below diagram where $exp_M(z) = z + c_1 z^q + c_2 z^{q^2} + c_3 z^{q^3} + \ldots$ is a map making it commutative (here $c_i \in \mathbb{C}_\infty$ are indefinite coefficients), the left vertical arrow is simply multiplication by $\theta$, and the right vertical arrow is the action of $T$ on $\mathbb{C}_\infty$ from 6.1:

$$
\begin{array}{ccc}
\mathbb{C}_\infty & \xrightarrow{\exp_M} & \mathbb{C}_\infty \\
\theta \downarrow & & \downarrow z \mapsto T(z) \\
\mathbb{C}_\infty & \xrightarrow{\exp_M} & \mathbb{C}_\infty
\end{array}
$$

**Theorem 8.1.1 (Drinfeld).** For any Drinfeld module $M$ the numbers $c_i$ can be found consecutively, they exist and are unique.

**Example 8.2.** Finding of $c_i$ for the case of the Carlitz module $\mathcal{C}$. We have:

$$Te = (\theta + \tau)e,$$

hence $T(z) = \theta z + z^q$.

In the below diagram the inner square is (8.1) for $\mathcal{C}$, the external square shows the images of any $z \in \mathbb{C}_\infty$ (upper left) under the arrows of the inner square.

$$
\begin{array}{ccc}
z & \xrightarrow{\exp_\mathcal{C}} & z + c_1 z^q + c_2 z^{q^2} + c_3 z^{q^3} + \ldots \\
\downarrow & & \downarrow \\
\mathbb{C}_\infty & \xrightarrow{\exp_\mathcal{C}} & \mathbb{C}_\infty
\end{array}
$$

$$
\begin{array}{ccc}
w & \mapsto & \theta w + w^q \\
\downarrow & & \downarrow \\
\mathbb{C}_\infty & \xrightarrow{\exp_\mathcal{C}} & \mathbb{C}_\infty
\end{array}
$$

$$
\begin{array}{ccc}
\theta z & \xrightarrow{\exp_\mathcal{C}} & u_1 = \theta(z + c_1 z^q + c_2 z^{q^2} + c_3 z^{q^3} + \ldots) \\
& & + (z + c_1 z^q + c_2 z^{q^2} + c_3 z^{q^3} + \ldots)^q \\
& & = \theta z + c_1 (\theta z)^q + c_2 (\theta z)^{q^2} + c_3 (\theta z)^{q^3} + \ldots \\
\theta z & \xrightarrow{\exp_\mathcal{C}} & u_2 = \theta z + c_1 z^q + c_2 z^{q^2} + c_3 z^{q^3} + \ldots \\
\end{array}
$$

$u_1$ is $T(exp_\mathcal{C}(z))$, and $u_2$ is $exp_\mathcal{C}(\theta z)$, hence they are equal. We get an equality of power series:

$$
\begin{align*}
\theta z & + \theta c_1 z^q + \theta c_2 z^{q^2} + \theta c_3 z^{q^3} + \theta c_4 z^{q^4} + \ldots \\
+ & \\
z^q & + c_1^q z^{q^2} + c_2^q z^{q^3} + c_3^q z^{q^4} + \ldots \\
= & \\
\theta z & + \theta^q c_1 z^q + \theta^q c_2 z^{q^2} + \theta^q c_3 z^{q^3} + \theta^q c_4 z^{q^4} + \ldots
\end{align*}
$$

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We get a system of equations satisfied by $c_i$:

$$\begin{align*}
\theta c_1 + 1 &= \theta^q c_1; \quad c_1 = \frac{1}{\theta^q - \theta} \\
\theta c_2 + c_1^q &= \theta^q^2 c_2; \quad c_2 = \frac{1}{(\theta^q^2 - \theta^q)(\theta^q^2 - \theta)} \\
\theta c_3 + c_2^q &= \theta^q^3 c_3; \quad c_3 = \frac{1}{(\theta^q^3 - \theta^q^2)(\theta^q^3 - \theta^q)(\theta^q^3 - \theta)}
\end{align*}$$

etc. Further, $v_\infty(c_i) = iq^i$ (see Section 3 for $v_\infty$), hence $c_i \to 0$. Moreover, $\forall z \in \mathbb{C}_\infty$ we have $v_\infty(c_i z q^i) \to +\infty$, hence $exp(z)$ converges for all $z \in \mathbb{C}_\infty$.

**8.3. Theorem (Drinfeld).** 1. For all Drinfeld modules $M$ the function $exp_M(z)$ converges for all $z \in \mathbb{C}_\infty$, it is surjective, and its kernel is a lattice in $\mathbb{C}_\infty$ of rank $r$.

2. Let us denote the above lattice by $L(M)$. The map $M \mapsto L(M)$ is a $1 - 1$ correspondence between the set of Drinfeld modules, up to isomorphism, and the set of lattices in $\mathbb{C}_\infty$, up to equivalence.

**Idea of the proof:** Let $L \subset \mathbb{C}_\infty$ be a lattice. We associate it the following function $\wp_L : \mathbb{C}_\infty \to \mathbb{C}_\infty$ (here $L' := L - 0$):

$$\wp_L(z) = z \prod_{\omega \in L'} \left(1 - \frac{z}{\omega}\right)$$

We have: $L = L(M)$ iff $exp_M = \wp_L$.

This theorem permits us to describe the moduli space of Drinfeld modules in terms of lattices. Practically, it is the quotient space of the set of Siegel matrices of size $1 \times r - 1$ by the action of $GR_r(F_q[\theta])$ defined in Section 4. We do not consider these subjects in the present survey.

**9. Lie(M), PRINCIPAL EXACT SEQUENCE AND A LATTICE ASSOCIATED TO AN ANDERSON t-MOTIVE**

According the general formalism of Lie groups, we can define $Lie(E(M))$, or simply $Lie(M)$, as follows. We use notations of 6.2A, and we define $Lie(M)$ like in 6.2E. Namely, for any $\{e\}$ there exists an isomorphism $j_{\{e\}} : Lie(M) \to \mathbb{C}_\infty^n$, and the condition of concordance is that the following diagram is commutative:

$$\begin{array}{ccc}
Lie(M) & \xrightarrow{j_{\{e\}}} & \mathbb{C}_\infty^n \\
\downarrow{id} & & \downarrow{W_0} \\
Lie(M) & \xrightarrow{j_{\{e'\}}} & \mathbb{C}_\infty^n
\end{array}$$

Namely (notations of 6.2E) $Lie(M)$ is the quotient set of $\mathbb{C}_\infty^n \times \mathfrak{B}$ by the equivalence relation coming from (9.1): a pair $(\{x\}, \{e\})$ is equivalent to a pair $(\{x'\}, \{e'\})$ iff $\{x'\} = W_0(\{x\})$. 

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Since \( W_0 \in GL_n (\mathbb{C}_\infty) \) we get that \( \text{Lie}(M) \) has a canonical structure of \( \mathbb{C}_\infty \)-vector space: for \( x \in \text{Lie}(M) \), \( c \in \mathbb{C}_\infty \) we have \( c \cdot x := j_{\{e\} \!}^{-1}(c \cdot j_{\{e\}}(x)) \); it does not depend on \( \{e\} \).

Moreover, \( \text{Lie}(M) \) is a \( \mathbb{C}_\infty[T] \)-module. The multiplication by \( T \) is defined by the commutative diagram (A\(_0\) is from 5.9):

\[
\begin{array}{ccc}
\text{Lie}(M) & \xrightarrow{j_{\{e\}}} & \mathbb{C}_\infty^n \\
T & \downarrow & \downarrow A_0 \\
\text{Lie}(M) & \xrightarrow{j_{\{e\}}} & \mathbb{C}_\infty^n 
\end{array}
\] (9.2)

Clearly \( A_0 \) depends on \( \{e\} \). We have \( A_0 = \theta + N \) where \( N \) (also depending on \( \{e\} \)) is nilpotent.

There exists a canonical map \( \text{exp} : \text{Lie}(M) \rightarrow E(M) \) making the following diagram commutative:

\[
\begin{array}{ccc}
\text{Lie}(M) & \xrightarrow{\text{exp}} & E(M) \\
T & \downarrow & \downarrow T \\
\text{Lie}(M) & \xrightarrow{\text{exp}} & E(M) 
\end{array}
\] (9.3)

To prove its existence, we consider \( \text{exp} \) in coordinates. Namely, for any \( \{e\} \) there exists a map \( \text{exp}_{\{e\}} : \mathbb{C}_\infty^n \rightarrow \mathbb{C}_\infty^n \) making the following diagram commutative:

\[
\begin{array}{ccc}
\mathbb{C}_\infty^n & \xrightarrow{\text{exp}_{\{e\}}} & \mathbb{C}_\infty^n \\
A_0 & \downarrow & \downarrow A_{\mathbb{C}_\infty^n} \\
\mathbb{C}_\infty^n & \xrightarrow{\text{exp}_{\{e\}}} & \mathbb{C}_\infty^n 
\end{array}
\] (9.4)

An equivalent form of (9.4) is the following (for the case \( N = 0 \)):

\[
\begin{array}{ccc}
\mathbb{C}_\infty^n & \xrightarrow{\text{exp}_{\{e\}}} & \mathbb{C}_\infty^n \\
\theta & \downarrow & \downarrow z \mapsto T(z) \\
\mathbb{C}_\infty^n & \xrightarrow{\text{exp}_{\{e\}}} & \mathbb{C}_\infty^n 
\end{array}
\] (9.4a)

The map \( \text{exp}_{\{e\}} \) is given by the formula: For \( z \in \mathbb{C}_\infty^n \) we have

\[
\text{exp}_{\{e\}}(z) = z + C_1 z^q + C_2 z^{q^2} + C_3 z^{q^3} + \ldots
\]

where \( C_i = C_i(\{e\}) \in M_n(\mathbb{C}_\infty) \), and \( C_0 = \text{id} \) for all \( \{e\} \). The matrices \( C_i(\{e\}) \) can be found consecutively, see [A], (2.1.4) (see also Example 8.2). For all \( M \) and \( \{e\} \) the map \( \text{exp}_{\{e\}} \) converges for all \( z \in \mathbb{C}_\infty^n \).
exp and \( \exp_{\{e\}} \) enter in the following commutative diagram:

\[
\begin{array}{ccc}
\text{Lie}(M) & \xrightarrow{\exp} & E(M) \\
j_{\{e\}} & \downarrow & i_{\{e\}} \\
\mathbb{C}^n & \xrightarrow{\exp_{\{e\}}} & \mathbb{C}^n \\
\end{array}
\] (9.5)

Hence, diagrams (9.3), (9.4), (9.5) are parts of the following commutative diagram:

\[
\begin{array}{ccc}
\text{Lie}(M) & \xrightarrow{\exp} & E(M) \\
j_{\{e\}} & \downarrow & i_{\{e\}} \\
T & \xrightarrow{\downarrow} & \mathbb{C}^n \\
j_{\{e\}} & \downarrow & i_{\{e\}} \\
\text{Lie}(M) & \xrightarrow{\exp} & E(M) \\
\downarrow & \xrightarrow{A_0} & \downarrow \\
\mathbb{C}^n & \xrightarrow{\exp_{\{e\}}} & \mathbb{C}^n \\
\end{array}
\] (9.6)

**Principal exact sequence.**

Let us consider an exact sequence

\[0 \to \mathbb{F}_q[T] \to \mathbb{F}_q((T^{-1})) \to T^{-1} \cdot \mathbb{F}_q[[T^{-1}]] \to 0\]

We denote its terms by \( A, K, K/A \) respectively, and we define a topology on \( K, K/A \) by the condition that \( T^{-i} \) tends to 0 as \( i \) tends to \( +\infty \); \( A \) is discrete in this topology. Further, we consider an exact sequence

\[0 \to \text{Hom}_{\mathbb{F}_q}^\text{cont}(K/A, \mathbb{C}_\infty) \to \text{Hom}_{\mathbb{F}_q}^\text{cont}(K, \mathbb{C}_\infty) \to \text{Hom}_{\mathbb{F}_q}^\text{cont}(A, \mathbb{C}_\infty) \to 0\]

and we denote its terms by \( Z_1, Z_2, Z_3 \) respectively.

\( Z_1, Z_2, Z_3 \) have a natural structure of \( \mathbb{C}_\infty[T, \tau] \)-modules, see [A], below the Lemma 2.6.4, and [G], below Definition 5.9.21.

**9.8.** We have \( Z_1 = \mathbb{C}_\infty\{T\} \) is a subset of \( \mathbb{C}_\infty[[T]] \) formed by power series \( \sum_{i=0}^{\infty} a_i T^i \) (here and below \( a_i \in \mathbb{C}_\infty \)), such that \( a_i \to 0 \) as \( i \to \infty \) (or, equivalently, \( v_\infty(a_i) \to +\infty \)). The \( \mathbb{C}_\infty[T, \tau] \)-module structure on \( Z_1 \) is defined as follows: the multiplication by \( T \) is simply a multiplication of a power series by \( T \), and

\[\tau \cdot \sum_{i=0}^{\infty} a_i T^i := \sum_{i=0}^{\infty} a_i q^i T^i.\]

\( Z_2 \) is the set of power series \( \sum_{i=-k}^{\infty} a_i T^i \) (\( k \geq 0 \) is any number), all other conditions are the same;

\( Z_3 \) is the set of polynomials in \( T^{-1} \), with the free term 0, i.e. \( Z_3 = \{ \sum_{i=-k}^{1} a_i T^i \} \); condition \( a_i \to 0 \) does not exist for this case, the \( \mathbb{C}_\infty[T, \tau] \)-module structure is the same, the product \( T \cdot T^{-1} = 0 \).
The isomorphism \( Z_1 \to \mathbb{C}_\infty \{T\} \) is given as follows: if \( \sum_{i=0}^{\infty} a_i T^i \in \mathbb{C}_\infty \{T\} \) then the corresponding \( \varphi : K/A \to \mathbb{C}_\infty \) is defined by the formula \( \varphi(T^{-i}) = a_{i-1} \).

Now we consider two exact sequences

\[
0 \to \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1) \to \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_2) \to \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_3) \tag{9.9}
\]

\[
0 \to \text{Hom}_{\mathbb{F}_q[T]}(K/A, E(M)) \to \text{Hom}_{\mathbb{F}_q[T]}(K, E(M)) \to \text{Hom}_{\mathbb{F}_q[T]}(A, E(M)) \tag{9.10}
\]

Let \( L = L(M) \) be the kernel of \( \exp \), hence we have an exact sequence

\[
0 \to L(M) \xrightarrow{\delta} \text{Lie}(M) \xrightarrow{\exp} E(M) \tag{9.11}
\]

**Theorem 9.11a** (Anderson): Sequences (9.9) – (9.11) are canonically isomorphic. \( L \) is a free \( \mathbb{F}_q[T] \)-module of rank \( \leq r \), i.e. it is a lattice in \( \text{Lie}(M) \). The rank of \( L \) is \( r \) iff \( \exp \) is surjective.

Hence, not always \( \exp_M \) is surjective, this is an important phenomenon that does not occur for the case of Drinfeld modules. We do not know an analog of the formula (8.4) for the case \( n > 1 \). t-motives \( M \) such that \( \exp \) is surjective are called uniformizable.

\( L \) is denoted by \( H_1(M) \) as well, and the rank of \( L \) is denoted by \( h_1(M) \).

We refer to the proof of Theorem 9.11a to [A], [G]. Here we give definitions of some objects that are used for the proof. First, we give explicit formulas (in coordinates) of isomorphisms between terms of (9.9) – (9.11). Since \( A = \mathbb{F}_q[T] \) we have \( \text{Hom}_{\mathbb{F}_q[T]}(A, E(M)) = E \). Let us describe an isomorphism \( \delta : \text{Hom}_{\mathbb{F}_q[T]}(K, E(M)) \to \text{Lie}(M) \).

Let \( \varphi \in \text{Hom}_{\mathbb{F}_q[T]}(K, E(M)) \). We denote \( x_i := \varphi(T^{-i}) \in E(M) \). We have \( x_i \) tend to 0 as \( i \) tends to +\( \infty \). Since \( \exp \) is a local isomorphism near 0, for sufficiently large \( i \) the elements \( \exp^{-1}(x_i) \in \text{Lie}(M) \) are well-defined. We define

\[
\delta(\varphi) := T^i(\exp^{-1}(x_i)) \in \text{Lie}(M)
\]

It does not depend on \( i \).

Also, for any \( \{e\} \) we have: \( j_e(\delta(\varphi)) = A_0^i(\exp_{\{e\}}^{-1}(i_{\{e\}}(x_i))) \) where \( A_0 \) is from (9.2).

The inverse map \( \delta^{-1} : \text{Lie}(M) \to \text{Hom}_{\mathbb{F}_q[T]}(K, E(M)) \) is defined as follows. Let \( x \in \text{Lie}(M) \). We have \( [\delta^{-1}(x)](T^i) = \exp(T^i(x)) \).

In order to prove that (9.9) and (9.10) are isomorphic, we consider an exact sequence

\[
0 \to \text{Hom}(M \times K/A, \mathbb{C}_\infty) \to \text{Hom}(M \times K, \mathbb{C}_\infty) \to \text{Hom}(M \times A, \mathbb{C}_\infty) \tag{9.12}
\]

where for \( f \in M, c \in \mathbb{C}_\infty, k \in K/A, or K, or A, any \( \varphi \) belonging to these Hom’s must satisfy

\[
\varphi(\tau f, k) = \varphi(f, k)^q; \quad \varphi(T f, k) = \varphi(f, Tk); \quad \varphi(cf, k) = c \cdot \varphi(f, k)^q
\]
We denote the isomorphism from $\text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$ to $L(M)$ by $v$, and the isomorphism from $\text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$ to $\text{Hom}_{\mathbb{F}_q[T]}(K/A, E(M))$ by $w$.

**9.12a.** Let us describe explicitly $\zeta \circ v : \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1) \to \text{Lie}(M)$. Let $\varphi \in \text{Hom}_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$ and $f \in M$. We denote $\varphi(f) \in Z_1$ by $\sum_{i=0}^{\infty} a_i T^i$ and we denote $[\varphi(T^{-i})] \in E(M)$ by $x_i$. Because of isomorphisms of the first terms of (9.9), (9.12), (9.10), and because of (9.8), we get that the image of $\varphi$ in $\text{Hom}_{\mathbb{F}_q[T]}(K/A, E(M))$ satisfies $x_i(f) = a_{i-1}$.

Since $\zeta \circ v(\varphi) = T^i(exp^{-1}(x_i))$ for all sufficiently large $i$, we have

$$[exp(T^{-i}(\zeta \circ v(\varphi)))](f) = a_{i-1} \quad (9.12b)$$

Let now $x \in \text{Lie}(M)$ and $\{e\}$ as above. We denote $[exp(T^{-i}(x))\{e\}]$ by $a_{+i} = (a_{i1}, \ldots, a_{ni})$. Let us find $i_{\{e\}}(x)$. We consider a commutative diagram

$$
\begin{align*}
\text{Lie}(M) & \xrightarrow{exp} E(M) & \mathbb{C}_\infty^n & \xrightarrow{exp A_{\{e\}}} & \mathbb{C}_\infty^n \\
T^i \downarrow & \quad \downarrow T^i & A_0^i \downarrow & \quad \downarrow A_0^i & \quad (9.13)
\end{align*}
$$

$$
\begin{align*}
\text{Lie}(M) & \xrightarrow{exp} E(M) & \mathbb{C}_\infty^n & \xrightarrow{exp A_{\{e\}}} & \mathbb{C}_\infty^n \\
\text{Lie}(M) & \xrightarrow{exp} E(M) & \mathbb{C}_\infty^n & \xrightarrow{exp A_{\{e\}}} & \mathbb{C}_\infty^n
\end{align*}
$$

together with the arrows (not drawn) $j_{\{e\}} : \text{Lie}(M) \to \mathbb{C}_\infty^n$ (two arrows) and $i_{\{e\}} : E(M) \to \mathbb{C}_\infty^n$ (two arrows); this is (9.6) for $T^i$ instead of $T$.

We consider $T^{-i}(x)$ in the upper $\text{Lie}(M)$, and we apply the diagram chasing to it. We have $j_{\{e\}}(x)$ is its image in the down-left $\mathbb{C}_\infty^n$. Further, $[exp(T^{-i}(x))\{e\}] = i_{\{e\}} \circ exp \circ T^{-i}(x) = a_{+i}$. We get that $j_{\{e\}}(x) = A^i_0(exp^{-1}_{\{e\}}(a_{+i}))$. For $i \to +\infty$ we have $a_{+i} \to 0$. Since $exp^{-1}_{\{e\}}$ is id near 0, we get that (recall that $A_0$, $A_{\infty}$ depend on $\{e\}$)

$$i_{\{e\}}(x) = \lim_{i \to +\infty} A^i_0(a_{+i}). \quad (9.13.1)$$

For $x \in \text{Lie}(M)$ and $f \in M$ there exists a canonically defined element $\partial_x(f) \in \mathbb{C}_\infty$, see [A], 3.3.2. Let us give an explicit formula for it. First, we describe the set $\text{Hom}(E, G_a)$ in coordinates (i.e. let a basis $\{e\}$ be fixed). We begin with the formula $\text{Hom}(G_a, G_a) = \bigcup_{\mu=0}^{\infty} \mathbb{C}^{\mu+1}_{\infty}$: if $c = \{c_0, \ldots, c_\mu\} \in \mathbb{C}^{\mu+1}_{\infty}$ and $x \in G_a = \mathbb{C}_\infty$ then

$$c(x) = \sum_{i=0}^{\mu} c_i x^q^i$$

For a basis $\{e\}$ we have a coordinate map $\xi_{\{e\}} : \text{Hom}(E, G_a) \to \bigcup_{\mu=0}^{\infty} \mathbb{C}^{\mu+1}_{\infty}$. We denote an element of $\mathbb{C}^{\mu+1}_{\infty}$ as $\{c_{ij}\} = \{c_{i0}, \ldots, c_{i\mu}\} \in \mathbb{C}^{\mu+1}_{\infty}$ where $i = 1, \ldots, n$, $j = 0, \ldots, \mu$. The map $\xi_{\{e\}}$ is defined as follows. Let $x \in E(M)$ and
For some \( k \) we have \( \mathfrak{t}_e(\varphi) = \{c_{ij}\} \in \mathbb{C}^{n(\mu+1)} \) iff \( \forall x \in E(M) \) holds

\[
\varphi(x) = \sum_{j=0}^{\mu} \sum_{i=1}^{n} c_{ij} x_i^{q_j}.
\] (9.14)

The diagram of concordance of \( \mathfrak{t}_e, \mathfrak{t}_e' \) is the following (\( W \) is from 6.2C):

\[
\begin{array}{ccc}
\text{Hom}(E, G_a) & \xrightarrow{\mathfrak{t}_e} & \bigcup_{\mu=0}^{\infty} \mathbb{C}^{n(\mu+1)} \\
\text{id} & \downarrow & \downarrow W^{-1} \\
\text{Hom}(E, G_a) & \xrightarrow{\mathfrak{t}_e'} & \bigcup_{\mu=0}^{\infty} \mathbb{C}^{n(\mu+1)}
\end{array}
\] (9.15)

where the action of \( GL_n(\mathbb{C}\{\tau\}) \) on \( \bigcup_{\mu=0}^{\infty} \mathbb{C}^{n(\mu+1)} \) is the following. Let \( \{c_{ij}\} = \{c_{00}, \ldots, c_{\mu\mu}\} \in \mathbb{C}^{n(\mu+1)} \) as above and \( W \in GL_n(\mathbb{C}\{\tau\}) \) from 6.2C. The action is from the right:

\[
\{c_{ij}\}^W := \{c_{00}W_0; c_{01}W_1 + c_{11}W_0^{(1)}; c_{02}W_2 + c_{12}W_1^{(1)} + c_{22}W_0^{(2)}; \ldots\}
\]

For any \( \gamma \in \text{Hom}(E, G_a) \) and \( \lambda \in \text{Lie}(M) \) the number \( \partial_\lambda(\gamma) \in \mathbb{C}_\infty \) is canonically defined as follows. Let \( \{e\} \) be as above, \( \mathfrak{t}_e(\gamma) = \{c_{ij}\} \in \bigcup_{\mu=0}^{\infty} \mathbb{C}^{n(\mu+1)}, \mathfrak{i}_e(\lambda) = \{x\} \). We let \( \partial_\lambda(\gamma) := \sum_{i=1}^{n} c_{0i}x_i \). (9.1) and (9.15) show that it does not depend on \( \{e\} \).

Finally, let \( f = \sum_{i=1}^{n} a_i e_i \in M \), where \( \forall i \) we have \( a_i = \sum_j a_{ij} \tau_j, a_{ij} \in \mathbb{C}_\infty \). \( f \) defines an element \( \gamma_f \) of \( \text{Hom}(E, G_a) \) such that \( \mathfrak{t}_e(\gamma_f) = \{a_{ij}\} \). Hence, \( \partial_\lambda(f) \in \mathbb{C}_\infty \) is defined as

\[
\sum_{i=1}^{n} a_{0i}x_i
\] (9.16)

Let \( \varphi \in L(M), f \in M, a_i \) be as in 9.12a. Let \( \varphi(f) = \sum_{i=0}^{\infty} a_i T_i \in \mathbb{C}\{T\} \), i.e. \( a_i \to 0 \). We consider the \( \theta \)-shift of \( \sum_{i=0}^{\infty} a_i T_i \):

\[
\sum_{i=0}^{\infty} a_i T_i = \sum_{i=1}^{k} \frac{z_i}{N_i} + \sum_{i=0}^{\infty} y_i N_i
\]

for some \( k \). We use notations \( z_i = z_i(\varphi, f) \).

**Theorem 9.18.** ([A], Th. 3.3.2). We have \( k = m \) and for \( i \geq 0 \) we have

\[
\partial_{N_i(\zeta(\varphi))}(f) = -z_{i+1}
\] (9.18.1)
**Sketch of the proof.** First, we consider the case \( i = 0 \). Let \( f = \{ e_\ast \} \) (a basis). We have \( \partial \zeta(\varphi)(e_\ast) = j_e(\zeta(f)) \) (vector column). Let us consider the column series

\[
\varphi(e_\ast) = \sum_{i=0}^{\infty} a_{i\ast} T^i
\]

(here \( a_{i\ast} \) are column vectors). According (9.12b), we have

\[
\exp(T^{-(i+1)}(\zeta(\varphi))(\{e_\ast\})) = a_{i\ast}
\]

Let us consider (9.5). We denote \( \beta = i_e \circ \exp = \exp \circ j_e \). 9.18.1 for \( f = \) one of \( e_i \) is a corollary of a formula that for any \( x \in \text{Lie}(M) \) we have

\[
z_1(\sum_k \beta(T^{-(k+1)}(x))T^k) = -j_e(x) \quad (9.20)
\]

Let us prove 9.20. We have

\[
\beta(T^{-k}(x)) \approx (\theta + N_e)^{-k}(\beta(x)) \quad (9.21)
\]

The difference has no influence on \( z_1 \). Formally, \( N_e \) in 9.21 is not necessarily the same \( N \) that we use in \( \theta \)-shift. According (9.21), we have: (9.20) is reduced to the equality: for all \( x \in \mathbb{C}_\infty^n \), for all nilpotent operator \( \bar{N} \) on \( \mathbb{C}_\infty^n \) we have

\[
z_1(\sum_k (\theta + \bar{N})^{-(k+1)}(x)T^k) = -x \quad (9.22)
\]

This is because

\[
-\sum_{k=0}^{\infty} (\theta + \bar{N})^{-(k+1)}T^k = (T - \theta)^{-1} + \sum_{k=1}^{\infty} \bar{N}^k(T - \theta)^{-k} \quad (9.23)
\]

(this is [A], (3.3.5)). Proof of (9.23): both sides are (up to the sign)

\[
\sum_{k, j=0}^{\infty} (-1)^k \binom{k+j}{k} \theta^{-(k+j+1)} \bar{N}^k T^j
\]

Therefore, 9.18.1 is proved for \( f = \) one of \( e_i \). By linearity, it is sufficient to prove that \( z_1(\varphi, \tau e_i) = 0 \). Since \( \varphi \) commute with \( \tau \), we have (to finish).

If 9.18 holds for \( i = 0 \) then it holds for \( i = 1 \). Proof: Let

\[
l(f) = \frac{z_2}{N^2} + \frac{z_1}{N} + ...
\]

then

\[
Tl(f) = T(\frac{z_2}{N^2} + \frac{z_1}{N} + ...) = \frac{\theta z_2}{N^2} + \frac{\theta z_1 + z_2}{N} + ...
\]

Further, \( N\zeta(l) = \zeta(Tl) - \theta \zeta(l) \) and

\[
\partial_{N(\zeta(\varphi))}(f) = \partial_{\zeta(T\varphi)}(f) - \theta \cdot \partial_{\zeta(\varphi)}(f)
\]
If 9.18 holds for \( i = 1 \) then \( \partial_{\zeta(T\varphi)}(f) = \theta z_1 + z_2 \) and \( \partial_{\zeta(\varphi)}(f) = z_1 \), hence the result.

Proof by induction, from 4 to 1. We have \( \partial_{N^{i+1}(\zeta(\varphi))})(f) = z_{-(i+1)} \). Let us prove that \( \partial_{N^{i+1}(\zeta(\varphi))}(f) = z_{-(i+2)} \). We have \( N^{i+1}(\zeta(\varphi)) = T(N^i(\zeta(\varphi))) - \theta N^i(\zeta(\varphi)) \), hence

\[
\partial_{N^{i+1}(\zeta(\varphi))}(f) = \partial_{N^i(\zeta(T(\varphi)))}(f) - \theta \cdot \partial_{N^i(\zeta(\varphi))}(f)
\]

By induction supposition, we have \( \partial_{N^i(\zeta(T(\varphi)))}(f)z_{-(i+1)}(T(\varphi)), \partial_{N^i(\zeta(\varphi))}(f)z_{-(i+1)} \).

9A. Other results.

Let us consider \( M(A) \) from 5.13. If all entries of \( A \) are sufficiently small then \( M(A) \) is uniformizable. An explicit estimate is given in \([GL17]\) (end of page 383 and Proposition 2): if \( v_\infty \) of all entries of \( A \) is \( > \frac{q}{q^2-1} \) then \( M(A) \) is uniformizable. Clearly this estimate is too weak and can be improved, but we do not know the best result. The same situation is for \( M \) defined by (5.9).

We saw in Section 4 that a Siegel matrix of a lattice of rank \( r \) in \( \mathbb{C}_\infty^n \) is of size \( (r - n) \times n \), like a Siegel matrix of an abelian variety with MIQF.

For \( n > 1 \) it is not known what is an analog of the Theorem 8.3, (2). Theorem 5.15 indicates that we should consider only pure t-motives, i.e. we have

**Open problem.** Let us consider the lattice map \( M \mapsto L(M) \) from the set of pure uniformizable t-motives to the set of lattices. Is it true that its image is open, and its fibre at a generic point is discrete?\(^4\)

9A.1. The main result of \([GL17]\) gives evidence that maybe the lattice map is 1 – 1 near a distinguished point. This result is the following. We consider, from one side, Anderson t-motives given by (5.13.1) such that the matrix \( A \) is in a neighborhood of 0. From another side, we consider lattices whose Siegel matrix is in a neighborhood of \( \omega I_n \) (here \( \omega \in \mathbb{F}_q^2 - \mathbb{F}_q \); a Siegel matrix of an Anderson t-motive given by (5.13.1) with \( A = 0 \) is exactly \( \omega I_n \)). We prove that in a neighborhood of \( A = 0 \) we have a 1 – 1 correspondence between t-motives and lattices (the main difficulty is to check that we have the same action of automorphism groups of t-motives, from one side, and of lattices, from another side).

The obtained result is not too strong, because the size of a neighborhood where we have a 1 – 1 correspondence, depends on a ”degree” of elements of the above automorphism groups. It is desirable to find a ”universal” neighborhood. This is a subject of further research. Nevertheless, it is proved unconditionally that the lattice map is a surjection in a neighborhood of a Siegel matrix \( \omega I_n \).

There is another result that the lattice map is almost 1 – 1. Namely, for \( n = r - 1 \) the duality theory (see Section 10) gives us an immediate corollary of Theorem 8.3:

**Theorem 9A.2.** ([GL07], Corollary 8.4). All pure t-motives of rank \( r \) and dimension \( r - 1 \) over \( \mathbb{C}_\infty \) are uniformizable. There is a 1 – 1 correspondence between their set, and the set of lattices of rank \( r \) in \( \mathbb{C}_\infty^{r-1} \) having dual.

\(^4\)The second author made some calculations for non-pure uniformizable t-motives having \( r = 5 \), \( n = 2 \). Apparently there exists a 7-dimensional irreducible component in their moduli space. Since the set of lattices having \( r = 5 \), \( n = 2 \) is 6-dimensional, this would imply that the lattice map has a fibre of dimension 1. Unfortunately, because of 5.14.6, it is hardly to prove that t-motives of this 7-dimensional set are really non-isomorphic.
See 10.1 for a definition of dual lattice. Not all lattices of rank \( r \) in \( \mathbb{C}^{-1}_{r} \) have dual, but almost all, i.e. even in this simple case the correspondence is not strictly 1 – 1, but only an “almost 1 – 1”.

We can interpret \( L(M) \) as follows. Let \( M\{T\} := \text{Hom}_{\mathbb{C}_{\infty}[T]}(M, \mathbb{C}_{\infty}\{T\}) \). We have: \( \tau \) acts on \( M\{T\} \) by the standard action of operator on Hom group, i.e. \( (\tau(\phi))(m) := \tau(\phi(\tau^{-1}(m))) \) (we neglect here that \( \tau^{-1}(m) \) maybe does not exist — this is a model example). We have

\[
L(M) = \text{Hom}_{\mathbb{C}_{\infty}[T,\tau]}(M, \mathbb{C}_{\infty}\{T\}) = M\{T\}^\tau = (\text{Hom}_{\mathbb{C}_{\infty}[T]}(M, \mathbb{C}_{\infty}\{T\}))^\tau
\]

We denote \( L(M) \) by \( H_1(M) \). Now we can consider the tensor product instead of Hom, i.e. we define

\[
M\{T\} := M \otimes_{\mathbb{C}_{\infty}[T]} \mathbb{C}_{\infty}\{T\}
\]

We have: \( \tau \) acts on \( M\{T\} \) by the standard formula of the action of an operator on tensor products (i.e. \( \tau(a \otimes b) = \tau(a) \otimes \tau(b) \)), and we define

\[
H^1(M) := M\{T\}^\tau
\]

\( H^1(M) \) is also free \( \mathbb{F}_q[T] \)-module, its rank is denoted by \( h^1(M) \). Like for the case of \( H_1(M) \), we have \( h^1(M) \leq r \).

There exists a canonical pairing

\[
\pi : H_1(M) \otimes_{\mathbb{F}_q[T]} H^1(M) \to \mathbb{F}_q[T]
\]

(9A.2.1)

Really, let \( \varphi : M \to \mathbb{C}_{\infty}\{T\} \) be in \( H_1(M) \) and \( \alpha = \sum_i m_i \otimes v_i \), where \( m_i \in M \), \( v_i \in \mathbb{C}_{\infty}\{T\} \), be in \( H^1(M) \). By definition, \( \pi(\varphi, \alpha) := \sum_i \varphi(m_i) \cdot v_i \in \mathbb{F}_q[T] \) (because it belongs to \( \mathbb{C}_{\infty}\{T\}^\tau = \mathbb{F}_q[T] \)).

9A.3. An important technical tool is a notion of a scattering matrix (first defined in [A86], Section 3). Let \( \{l_*\} = \{l_1, \ldots, l_r\} \) be a basis of \( L(M) \) over \( \mathbb{F}_q[T] \), and let \( \{f\} = \{f_1, \ldots, f_r\}^t \) be a basis of \( M \) over \( \mathbb{C}_{\infty}[T] \) (see 5.11). Considering \( l_i \) as an element of \( \text{Hom}(M, \mathbb{C}_{\infty}\{T\}) \) we define a matrix \( \Psi \in M_r(\mathbb{C}_{\infty}\{T\}) \) whose \((i, j)\)-th entry is \( l_i(f_j) \in \mathbb{C}_{\infty}\{T\} \). It is called a scattering matrix of \( M \) with respect to the bases \( \{l_*\}, \{f_*\} \). It satisfies a relation

\[
\Psi Q^t = \Psi^{(1)}
\]

where \( Q \) is from 5.11. This relation follows immediately from the fact that elements of \( L(M) \) are \( \tau \)-invariant maps.

There exists a stronger form of Theorem 9.3:

**Theorem 9A.4.** (Anderson, [A86]; [G], 5.9.14). \( h^1(M) = r \iff h_1(M) = r \iff \text{exp}_M \) is surjective. In this case \( \pi \) is perfect over \( \mathbb{F}_q[T] \).

**Remark.** It is natural to expect that always \( h^1(M) = h_1(M) \). Really, this is wrong: for a \( t \)-motive \( M(A) \) of Example 5.13 where \( q = 2 \) and \( A = \begin{pmatrix} \theta & \theta^6 \\ \theta^{-2} & 0 \end{pmatrix} \) we have \( h^1(M(A)) = 0 \), \( h_1(M(A)) = 1 \) ([GL21], Section 4).
Example 9A.5. Explicit calculation of $H^1(\mathcal{C})$. Recall that $\mathcal{C}$ is the Carlitz module. Let as above $e$ be the only element of a basis of $\mathcal{C}$ over $\mathbb{C}_\infty[T]$, it satisfies $\tau e = (T - \theta)e$. Elements of $\mathcal{C}\{T\}$ have the form

$$(d_0 + d_1 T + d_2 T^2 + ...)e$$

(9A.5.1)

where $d_i \in \mathbb{C}_\infty$, $d_i \to 0$. Condition that the element (9A.5.1) is $\tau$-stable is:

$$\tau(d_0 + d_1 T + d_2 T^2 + ...)e = (d_0 + d_1 T + d_2 T^2 + ...)e$$

(9A.5.2)

We have

$$\tau(d_0 + d_1 T + d_2 T^2 + ...)e = (d_0' + d_1' T + d_2' T^2 + ...)\tau e = (d_0' + d_1' T + d_2' T^2 + ...)(T - \theta)e$$

hence (9A.5.2) means

$$(d_0' + d_1' T + d_2' T^2 + ...)(T - \theta) = d_0 + d_1 T + d_2 T^2 + ...$$

(9A.5.3)

Multiplying we get a system of equations

$$\theta d_0' + d_0 = 0$$

$$\theta d_1' + d_1 - d_0' = 0$$

$$\theta d_2' + d_2 - d_1' = 0$$

(9A.5.4)

etc. Solving them consecutively we get that there exists a solution satisfying $v_\infty(d_i) = \frac{2^i}{q-1} \to +\infty$. This means that $\dim H^1(\mathcal{C}) = 1$, as it should be, according the following

**Corollary 9A.6.** For a Drinfeld module $M$ of rank $r$ we have $h^1(M) = h_1(M) = r$ (because all Drinfeld modules are uniformizable).

An example of a non-uniformizable t-motive is given in [G], 5.9.9. The proof of its non-uniformizability given in [G] is "artificial". The same example is treated in [EGL] by a direct calculation, see [EGL], Example 6.3. It is shown that the example of [G], 5.9.9 is, in some sense, the "simplest" non-uniformizable t-motive.

Examples of calculation of $h^1$, $h_1$ of some other t-motives are given in [EGL], [GL21]. These are explicit calculations similar to the above (9.10.3), (9.10.4), but more complicated. For example, formulas [GL21], (3.9) permit us to find $h^1(M)$ for any $M(A)$ of the form (5.13.1), case $n = 2$. Paper [EGL] gives a complete answer to the problem of finding of $h^1(M)$, $h_1(M)$ for $M(A)$ of the form (5.13.1) where

$$A = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$ 

**9A.7.** There is a natural problem to continue calculations of [EGL], to extend them to all $M$ of the form (5.13.1). Advance in this activity is a good (and easy) research problem for students. It can be compared with the problems of catalogizations of all stars in the sky in astronomy, or of all species of insects in biology: it is necessary to consider a lot (probably infinite quantity) of cases. Like any biologist who comes to Manaus jungle (surrounding the home university of the second author) will find, without doubt, some new species of insects, any mathematician
who will consider new types of matrices $A$ will get new results in calculation of $h^1; h_1$ of the corresponding $M(A)$.

10. DUALITY

Let $M$ be an Anderson t-motive. Its dual $M'$ is defined by (5.19.2) (for the case of Drinfeld modules the definition of dual was given in [T], for any t-motive in [GL07]. See also [HJ]).

There exists a notion of duality for lattices. The simplest definition is the following. Let $L \subset \mathbb{C}_\infty^n$ be a lattice of rank $r$ and $S$ its Siegel matrix. Its dual lattice is denoted by $L'$.

**Definition 10.1.** $L'$ is a lattice having a Siegel matrix $S'$. Hence, it is a lattice of rank $r$ in $\mathbb{C}^{r-n}$.

It is easy to show that this definition is well-defined (i.e. does not depend on a choice of Siegel matrix). There exists also an invariant definition ([GL07], Definition 2.3).

Let $M$ be a t-motive having the dual $M'$. Let $\{f\} = \{f_1, \ldots, f_r\}^t$ be a basis of $M$ over $\mathbb{C}_\infty[T]$ (see 5.11). It defines the dual basis $\{f'\} = \{f'_1, \ldots, f'_r\}^t$ of $M'$ over $\mathbb{C}_\infty[T]$ (here and below the dual bases and some other objects are defined up to multiplication by elements of $\mathbb{F}_q^\ast$).

Properties of $M'$ are the following:

10.2.1. The matrix $Q'$ of $M'$ (see 5.11) in $\{f'\}$ is $(T - \theta)(Q^t)^{-1}$ where $Q$ is the matrix for $M$ over $\{f\}$.

10.2.2. Dimension of $M'$ is $r - n$ (the rank is the same, i.e. $r$).

10.2.3. If $M$ is uniformizable then $M'$ is uniformizable, and $L(M') = (L(M))'$ (see [GL07] for the case of $M$ having $N = 0$. For the case $N \neq 0$ see Section 11 below).

10.2.4. For all $M$ (maybe non-uniformizable) we have $H^1(M) = H_1(M'), H_1(M) = H^1(M')$ (functorial isomorphisms). (9.7) implies that there are pairings $H^1(M) \otimes H^1(M') \to \mathbb{F}_q[T], H_1(M) \otimes H_1(M') \to \mathbb{F}_q[T]$. They are perfect if $M, M'$ are uniformizable.

Particularly, if $M$ is uniformizable and $\{l_*\}$ is a basis of $L(M)$, then there exists the dual basis $\{l'_*\}$ of $L(M')$.

We need an element $\Xi = \sum_{i=0}^{\infty} a_i T^i \in \mathbb{C}_\infty\{T\}$ satisfying

$$\Xi = (T - \theta) \sum_{i=0}^{\infty} a_i^q T^i, \quad \lim_{i \to \infty} a_i = 0, \quad |a_0| > |a_i| \quad \forall i > 0$$

We have $\Xi = c(1 - \theta^{-1}T)(1 - \theta^{-q}T)(1 - \theta^{-q^2}T)(1 - \theta^{-q^3}T)\ldots$ where $c = (-\theta)^{-\frac{1}{q-1}}$ (see [G], p. 171, (*) and p. 172, line 1; $\Xi$ is defined up to multiplication by $\mathbb{F}_q^\ast$).

10.2.5. Let $\Psi$ be the scattering matrix of $M$ with respect to the bases $\{l_*\}, \{f_*\}$ and $\Psi'$ the scattering matrix of $M'$ with respect to the bases $\{l'_*\}, \{f'_*\}$. We have $\Psi' = \Xi^{-1}(\Psi^t)^{-1}$. 

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All these formulas, except 10.2.3, are straightforward. To prove 10.2.3, we need to use a relation between $\Psi$ and a Siegel matrix of $L(M)$ (see [GL07], (5.4.5) and the second above line; these formulas appear non-explicitly in [A86]) and then to use 10.2.5.

Let us explain the notations of [GL07], (5.4.5). It turns out that an important (and non-expected) technical tool of the proof is the notion of $\theta$-shift. Let $\mathcal{Y} = \sum_{i=0}^{\infty} y_i T^i \in \mathbb{C}_{\infty}(T)$ be a series. We substitute $T = N + \theta$ in this formula (here $N$ is an abstract symbol). For some (clearly not for all) $\mathcal{Y}$, as a result of this substitution, we get $\sum_{j=-\kappa}^{\infty} z_{-j} N^j \in \mathbb{C}_{\infty}((T))$ for some $\kappa \geq 0$, $z_* \in \mathbb{C}_{\infty}$. In this case, we denote this series $\sum_{j=-\kappa}^{\infty} z_{-j} N^j$ by $\mathcal{Y}_N$.

The same construction can be applied to series with matrix coefficients. Namely, for a scattering matrix $\Psi$ we denote by $\Psi_N$ the result of application of $\theta$-shift to all entries of $\Psi$. If $M$ has $N = 0$ then for its $\Psi_N$ we have $\kappa = 1$. We denote by $D_1$ the matrix coefficient at the $N^{-1}$-th term of $\Psi_N$, and by $D_{11}$, $D_{12}$ some its submatrices. These submatrices enter in the mentioned above relation between $\Psi$ and a Siegel matrix of $L(M)$.

11. Case $N \neq 0$.

Here we consider the case $N \neq 0$. Let $M$ be an Anderson $t$-motivic. Recall (see beginning of Section 5) that $T - \theta$ is nilpotent on $M/\tau M$. We can consider $T$ on $E(M)$ and $\text{Lie}(M)$ as well. On $\text{Lie}(M)$ we have $T = N + \theta I_n$ where $N$ is nilpotent. Let $\mathfrak{m}$ be the minimal number such that $N^\mathfrak{m} = 0$. Diagram (9.1) remains the same, but the multiplication by $\theta$ (the left vertical arrow) is replaced by the operator $T = N + \theta I_n$. We need to modify a definition of a lattice as follows.

Let $\mathfrak{Y} = \mathbb{C}_\infty^n$ be any vector space and $T = N + \theta I_n$ any operator on $\mathfrak{Y}$, where $N$ is nilpotent. This $\mathfrak{Y}$ can be considered as a $\mathbb{F}_q[T]$-module. A lattice $L$ of rank $r$ in $\mathfrak{Y}$ is a $\mathbb{F}_q[T]$-submodule of $\mathfrak{Y}$ which is isomorphic to $(\mathbb{F}_q[T])^r$ as a $\mathbb{F}_q[T]$-module such that analogs of 4.1.1, 4.1.2 hold, namely

11.1. $\mathbb{F}_q[[T]]$-envelope of $L$ (clearly the action of $\mathbb{F}_q[[T]]$ on $\mathfrak{Y}$ is defined) is isomorphic to $\mathbb{R}_{\infty}^r$ (i.e. elements of a basis of $L$ over $\mathbb{F}_q[T]$ are linearly independent over $\mathbb{F}_q[[T]]$) and

11.2. $\mathbb{C}_\infty$-envelope of $L$ is $\mathfrak{Y}$.

Let us consider the case of uniformizable $M$, and let as earlier $L(M)$ be the kernel of $exp_M$. We have: $L(M)$ is a lattice of $\mathfrak{Y} = \text{Lie}(M)$ in the above meaning.

The next important object is due to Pink [P], see also [GL07], [GL18], [HJ]. There is a natural inclusion $\mathbb{C}_\infty[T] \hookrightarrow \mathbb{C}_\infty[[N]]$: $T \mapsto N + \theta$. It defines an inclusion $\mathbb{F}_q[T] \hookrightarrow \mathbb{C}_\infty[[N]]$. Hence, we can consider $L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]]$. An inclusion $L(M) \hookrightarrow \text{Lie}(M)$ defines a map $L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \to \text{Lie}(M)$ which is a surjection because of (11.2). Its kernel is denoted by $q = q(M)$. It is an invariant of $M$ which is more "thin" than $L(M)$ for the case $N \neq 0$. Hence, we have an exact sequence

$$0 \to q \to L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \to \text{Lie}(M) \to 0 \quad (11.3)$$
We have: \( \mathbb{C}_\infty[[N]] \) is a local Dedekind ring, the theory of finitely generated modules over these rings is very simple. \( L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \) is free of rank \( r \) over \( \mathbb{C}_\infty[[N]] \), and \( \text{Lie}(M) \) is isomorphic to \( \mathbb{C}_\infty^n \). (11.3) gives us that \( \mathfrak{q} \) is isomorphic to \( \mathbb{C}_\infty[[N]]^r \) as a \( \mathbb{C}_\infty[[N]] \)-module, i.e. it is a lattice in \( L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \). It is checked immediately that \( \mathfrak{q} \supset N^m(L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]]) \).

**Remark.** We can define an abstract Hodge-Pink structure as follows (see, for example, [HJ]). Let \( V \) be any free \( \mathbb{F}_q[T] \)-module\(^5\) of rank \( r \) and \( \mathfrak{q} \subset V \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \) a \( \mathbb{C}_\infty[[N]] \)-sublattice of rank \( r \). A homomorphism between two such objects \( (V_1, \mathfrak{q}_1) \), \( (V_2, \mathfrak{q}_2) \) is a \( \mathbb{F}_q[T] \)-linear \( V_1 \rightarrow V_2 \) inducing a map \( \mathfrak{q}_1 \rightarrow \mathfrak{q}_2 \).

Really, a definition of Hodge-Pink structure of [HJ] is a version of the above one. First, they consider a weight filtration on \( V \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \). Let \( \mathfrak{q} \) be any free \( \mathbb{C}_\infty[[N]] \)-module but only is commensurable with it.

\( \{ \mathfrak{q} = q_M, L(M) \} \) from (11.3) are called the Hodge-Pink structure associated to \( M \) (recall that \( M \) is uniformizable).

In order to describe explicitly the inclusion \( \mathfrak{q} \subset V \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \), we use an analog of a Siegel matrix. Namely, for the case \( m = 1 \) we get exactly a Siegel matrix. Really, let \( e_1, \ldots, e_r \) be a basis of \( V \) over \( \mathbb{F}_q[T] \) such that images of \( e_1, \ldots, e_n \) form a basis of \( V \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]]/\mathfrak{q} \). There exists a matrix \( S = \{ s_{ij} \} \) such that elements

\[
N e_1, \ldots, N e_n, \quad e_{n+i} - \sum_{j=1}^n S_{ij} e_j, \quad \text{where} \quad i = 1, \ldots, r - n
\]

form a \( \mathbb{C}_\infty[[N]] \)-basis of \( \mathfrak{q} \). Clearly this \( S \) is exactly a Siegel matrix of \( L(M) \) if \( \mathfrak{q} \) comes from \( L(M) \), as explained above.

If \( m > 1 \) then an analog of Siegel matrix is a family of matrices called a Siegel object, it is defined for example in [GL18], (3.15). Roughly speaking, we consider the \( \mathbb{C}_\infty \)-vector spaces

\[
((N^i \cdot V \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]]) \cap \mathfrak{q})/((N^{i+1} \cdot V \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]]) \cap \mathfrak{q}), \quad i = 0, \ldots, m - 1,
\]

their bases, lifts of these bases to \( \mathfrak{q} \) etc, see [GL18] for details. The family of these matrices is parametrized by integer points of a tetrahedron, i.e. it depends on 3 integer parameters. These matrices are denoted by \( S_{uvyz} \) in [GL18], where \( u, v, y, z \) are integer parameters.\(^6\)

Let \( M \) be an Anderson t-motive and \( \mathfrak{q} \subset L(M) \otimes_{\mathbb{F}_q[T]} \mathbb{C}_\infty[[N]] \) its Hodge-Pink structure. Like for the case \( N = 0 \), there is a formula for a Siegel object of \( \mathfrak{q} \)

---

\(^5\)Do not confuse this \( V \) and the above \( \mathfrak{V} \), they play different roles.

\(^6\)This 4-parameter notation is used for convenience for a proof of duality theorem. Really, always \( v = u - 1 \), i.e. practically we have 3 parameters.
defined in terms of \( \theta \)-shift of a scattering matrix \( \Psi \) of \( M \). Namely, we have

\[
\Psi_N = \sum_{u=-m}^{\infty} D_u N^u
\]

where \( D_1, \ldots, D_m \) are matrix coefficients at negative powers of \( N \). Formula (3.38) of [GL18] gives relations between some submatrices of \( D_i \) and matrices of Siegel object of \( \mathfrak{q}(M) \).

There are natural definitions of the tensor product and Hom of Hodge-Pink structures. Namely, let

\[
PH_1 = \{ q_1 \subset V_1 \otimes \mathbb{C}_\infty[[N]] \}, \quad PH_2 = \{ q_2 \subset V_2 \otimes \mathbb{C}_\infty[[N]] \}
\]

be two Hodge-Pink structures. Their tensor product is

\[
q_1 \otimes \mathbb{C}_\infty[[N]] q_2 \subset (V_1 \otimes V_2) \otimes \mathbb{C}_\infty[[N]].
\]

The definition of Hom is similar, see [HJ] (it is defined only for Hodge-Pink structures in the meaning of [HJ]).

Let us give the definition of the \( m \)-dual structure — an important particular case of Hom. We denote by \( \mathcal{E}^m \) (the \( m \)-th tensor power of the Carlitz Hodge-Pink structure) a structure \( N^m \mathbb{C}_\infty[[N]] \subset \mathbb{C}_\infty[[N]] \).

The Hodge-Pink structure Hom\((HP_1, \mathcal{E}^m)\) — the \( m \)-dual of \( HP_1 \) — is defined as follows. Its \( V \) is \( V_1' := \text{Hom}(V_1, \mathbb{F}_q[T]) \) and

\[
q' = \{ \varphi : V_1 \otimes \mathbb{C}_\infty[[N]] \to \mathbb{C}_\infty[[\theta]] \mid \varphi(q) \subset N^m \mathbb{C}_\infty[[N]] \}
\]

**Theorem.** The functor of Hodge-Pink structure from uniformizable Anderson \( t \)-motives to Hodge-Pink structures commutes with tensor products and Hom’s.

For tensor products this was proved by Anderson in 2000 (non-published), for tensor products of \( t \)-motives having \( N = 0 \) in [GL07], for the general tensor products and Hom’s in [HJ] (under a restriction that they are mixed, see [HJ], 3.5b). For the case of duals and \( N = 0 \) the theorem was proved in [GL07], for \( m \)-duals, case \( N \neq 0 \), in [GL18] by means of explicit calculation of Siegel objects of \( L(M) \) and of \( L(M^m) \) (recall that \( M^m \) is the \( m \)-dual of \( M \), see 5.19.3).

**12. L-FUNCTIONS.**

We give here the definition of the simplest version of L-function of \( t \)-motives. Let \( M \) be defined over \( \mathbb{F}_q(\theta) \). Let \( Q \in M_{r \times r}(\mathbb{F}_q(\theta)[T]) \) be from Section 5 (the matrix of multiplication by \( \tau \) in a \( \mathbb{F}_q(\theta)[T] \)-basis of \( M \)). Let \( \mathfrak{P} \) be an irreducible polynomial in \( \mathbb{F}_q[\theta] \). Let us assume that \( M \) has a good reduction at \( \mathfrak{P} \). This implies that there exists a \( \mathbb{F}_q(\theta)[T] \)-basis of \( M \) such that all entries of \( Q \) are integer at \( \mathfrak{P} \). The set of bad primes is denoted by \( S \).

We need the following notation. For \( a \in (\mathbb{F}_q(\theta)/\mathfrak{P})[T] \), \( a = \sum c_i T^i \) where \( c_i \in \mathbb{F}_q(\theta)/\mathfrak{P} \), we denote \( a^{(k)} := \sum c_i^k T^i \), for a matrix \( A = (a_{ij}) \in M_{r \times r}(\mathbb{F}_q(\theta)[T]) \) \( A^{(k)} := (a_{ij}^{(k)}) \) and \( A^{[k]} := A^{(k-1)} \cdots A^{(1)} \cdot A \).
The local $\mathfrak{P}$-factor $L_{\mathfrak{P}}(M, U)$ is defined as follows ($\mathfrak{P} \not\in S$). Let $d$ be the degree of $\mathfrak{P}$ and $\tilde{Q} \in M_{r \times r}((\mathbb{F}_q[\theta]/\mathfrak{P})[T])$ the reduction of $Q$ at $\mathfrak{P}$. We have:

$$L_{\mathfrak{P}}(M, U) := \det(I_r - \tilde{Q}^{[d]} U^d)^{-1} \in \mathbb{F}_q[T][[U^d]]$$

(because obviously $\det(I_r - \tilde{Q}^{[d]} U) \in \mathbb{F}_q[T, U]$ and does not depend on a $\mathbb{F}_q(\theta)[T]$-basis of $M$);

$$L_S(M, U) := \prod_{\mathfrak{P} \not\in S} L_{\mathfrak{P}}(M, U) \in \mathbb{F}_q[T][[U]]$$

**Example:** the Carlitz module over $\mathbb{F}_q$. We have $Q = T - \theta$, $S = \emptyset$. Let $\mathfrak{P} = a_0 + a_1 \theta + a_2 \theta^2 + \ldots + a_d \theta^d$, where $a_i \in \mathbb{F}_q$.

We have $\tilde{Q}^{[d]} = a_0 + a_1 T + a_2 T^2 + \ldots + a_d T^d$ (exercise for the reader).

Hence, the local $\mathfrak{P}$-factor of $L(\mathfrak{C}, U)$ is $(1 - (a_0 + a_1 T + a_2 T^2 + \ldots + a_d T^d) U^d)^{-1} \in \mathbb{F}_q[T][[U]]$. For example, for $q = 2$ we have a table of $L_{\mathfrak{P}}(\mathfrak{C}, U)$ for the first small $\mathfrak{P}$:

| $\mathfrak{P}$ | $L_{\mathfrak{P}}(\mathfrak{C}, U)$ |
|---------------|----------------------------------|
| $\theta$     | $1 + T U + T^2 U^2 + T^3 U^3 + \ldots$ |
| $\theta + 1$ | $1 + (T + 1) U + (T^2 + 1) U^2 + (T + 1)^3 U^3 + \ldots$ |
| $\theta^2 + \theta + 1$ | $1 + (T^2 + T + 1) U^2 + (T^2 + T + 1)^2 U^4 + \ldots$ |
| $\theta^3 + \theta + 1$ | $1 + (T^3 + T + 1) U^3 + (T^3 + T + 1)^2 U^6 + \ldots$ |

and $L(\mathfrak{C}, U)$ is their product.

**Theorem.** For $q = 2$ we have $L(\mathfrak{C}, U) = 1 + U$, for $q > 2$ we have $L(\mathfrak{C}, U) = 1$.

Proof is an exercise for the reader.

There exists a formula for $L(M, U)$ for a slightly other object — a sheaf $\mathcal{F}$ on a curve $X$ instead of a $\mathbb{C}_\infty[T]$-module $M$, see Section 13. This formula (a version of the Lefschetz trace formula) gives us $L(\mathcal{F}, U)$ in terms of Frobenius action on $H^0(X, \mathcal{F}), H^1(X, \mathcal{F})$. A statement of this formula is given in [L], p. 2603, or in [GL16], (3.4). For a proof see [A00] (the original proof), or [B12], Section 9.

**12.1.** As an example, we can apply this formula to twists of Carlitz modules. The corresponding theory is developed in [GL16], [GLZ22]. A generalization of this theory to other types of t-motives (for example, Drinfeld modules) is a research subject for beginners. It is much simpler than other research subjects related to Drinfeld modules.

Another type of $L$-function of t-motives is defined in [G], Section 8. We shall not give details here.

**13. Generalizations of Anderson t-motives.**

Most generalizations use a description of an Anderson t-motive as a $\mathbb{C}_\infty[T]$-module with a $\tau$-action. Usually it is not convenient to consider $\tau$ as a skew map (i.e. $\tau(am) = a^\sigma \tau(m)$), hence most definitions use a formalism of a tensor product by a Frobenius map. Namely, let $A$ be a ring (it can be $\mathbb{C}_\infty[T]$ or a similar ring)
and \( \varphi : A \to A \) an isomorphism called a Frobenius isomorphism. For example, for \( A = \mathbb{C}_\infty[T] \) and \( C = \sum_{i=0}^\infty c_i T^i \) we define \( \varphi(C) = \sum_{i=0}^\infty c_i^q T^i \).

For a left \( A \)-module \( M \) we can consider an \( A \)-module \( \sigma^*(M) := M \otimes_A A \) where the tensor product is taken with respect to the map \( \varphi^{-1} : A \to A \).

For \( m \in M \) we denote the element \( m \otimes 1 \in \sigma^*(M) \) by \( \bar{m} \), and multiplication of elements of \( \sigma^*(M) \) by elements of \( A \) will be denoted by asterisk, in order do not confuse it with multiplication of elements of \( M \) by elements of \( A \). All elements of \( \sigma^*(M) \) are of the form \( \bar{m} \) for some \( m \in M \). There is a formula \( a \ast \bar{m} = \varphi(a) \cdot \bar{m} \).

This means that a skew map \( \tau : M \to M \) that enters in the definition of an Anderson t-motive can be considered as an \( A \)-module map (denoted by \( \tau \) as well) \( \tau : \sigma^*(M) \to M \), defined by the formula \( \tau(\bar{m}) = \tau(m) \).

If \( M \) is free over \( A \) of dimension \( r \) then an analog of Condition 5.2.1 holds automatically. Analogs of Conditions 5.2.2, 5.2.3 are either omitted, or formulated by some other manner.

**Example 13.1.** Let \( A = \mathbb{F}_q(T) \), \( \varphi(a) = a^q \) for \( a \in \mathbb{F}_q \), \( \varphi(T) = T \). Let \( M = V \) a finite dimensional vector space over \( A \). By definition, a \( \varphi \)-space is a bijective map \( \tau : \sigma^*(V) \to V \).

**Example 13.2.** Let \( A = \mathbb{F}_q((T)) \), and let \( \varphi, V \) be the same. A bijective map \( \tau : \sigma^*(V) \to V \) is called a Dieudonné module.\(^7\)

We see that for these objects analogs of Conditions 5.2.2, 5.2.3 are omitted.

**13.3.** A definition of an object generalizing Anderson t-motives is given in [HJ], Definition 3.1 (it is called an \( A \)-motive in [HJ]). Namely (we give a slightly simplified version), an \( A \)-motive is a pair \((M, \tau)\) where \( M \) is a free \( \mathbb{C}_\infty[T] \)-module of dimension \( r \) and \( \tau : \sigma^*(M)[N^{-1}] \to M[N^{-1}] \) is an isomorphism of \( \mathbb{C}_\infty[T][N^{-1}] \)-modules (here \( N = T - \theta \in \mathbb{C}_\infty[T] \)).

We see that here it is required that \( \tau \) is an isomorphism. An Anderson t-motive in the meaning of the present paper is an effective \( A \)-motive, see [HJ], Definition 3.1c. They are defined by the condition that \( \tau \) comes from a \( \mathbb{C}_\infty[T] \)-homomorphism \( \sigma^*(M) \to M \).

This definition has an advantage that Hom of \( A \)-motives is always defined, and is an \( A \)-motive.

A natural generalization of a module over a ring is a sheaf over a scheme. Hence, instead of the ring \( \mathbb{C}_\infty[T] \) (affine line) we consider any (projective) curve \( X \) over \( \mathbb{C}_\infty \) (or its subfields if we consider Anderson t-motives over fields), and instead of \( M \) as a free \( \mathbb{C}_\infty[T] \)-module we consider a locally free sheaf \( F \) on \( X \). A typical example that should be kept in mind is \( X = P^1(\mathbb{C}_\infty) \).

An analog of the above map \( \varphi : A \to A \) is a scheme automorphism of \( X \) denoted by \( \varphi \) as well. For example, if \( X = P^1(\mathbb{C}_\infty) \) and \( a \in \mathbb{C}_\infty \cup \infty = P^1(\mathbb{C}_\infty) \) then the action of \( \varphi \) on \( a \) is the Frobenius action: \( \varphi(a) = a^q \). On functions, we have \( \varphi(T) = T \). Analogously to the case of modules, we define \( \sigma^*(F) := F \otimes_X A \) where the tensor product is taken with respect to the map \( \varphi^{-1} : X \to X \).

\(^7\) In order to define localizations of a \( \varphi \)-space at places of \( \mathbb{F}_q(T) \) it is necessary to use a version of the present definition of Dieudonné module.
Let us consider analogs of a map \( \tau : \sigma^*(F) \to F \). We need to prescribe its behavior not only at the point \( \theta \in X \) (i.e. an analog of 5.2.3), but also its behavior at a point on \( X \) which is an analog of \( \infty \in P^1(\mathbb{C}_\infty) \) (this point is denoted by \( \infty \) as well). It turns out that we should consider not one map \( \tau \), but a diagram of maps.

There are various versions of the definitions. The simplest of them is

**Definition** ([D87]; notations of [G], p. 191, (*)). A right \( F \)-sheaf is a diagram

\[
\sigma^*(F_0) \xrightarrow{\alpha} F_1, \quad F_0 \xrightarrow{\beta} F_1
\]

where \( F_0, F_1 \) are locally free sheaves on \( X \), \( \alpha \) and \( \beta \) are inclusions such that the supports of their cokernels consist of one point.

The support of \( \text{Coker} (\alpha) \) is called the zero of \( F \)-sheaf (it is an analog of \( \theta \) of 5.2.3), and the support of \( \text{Coker} (\beta) \) is called the pole of \( F \)-sheaf (it is an analog of \( \infty \in P^1(\mathbb{C}_\infty) \)).

Really, Drinfeld considers the case \( \dim \text{Coker} (\alpha) = \dim \text{Coker} (\beta) = 1 \), i.e. analog of the case \( n = 1 \), because he did not need an analog of Anderson t-motives. Also, he considers a relative case, i.e. a case when the whole diagram is over a base scheme \( S \).

The notions of \( F \)-sheaf and its generalizations were introduced in order to prove Langlands conjecture for \( GL_2 \) (Drinfeld), \( GL_r \) (L. Lafforgue) et al. Nevertheless, they can be used for solutions of much more elementary problems.

As an elementary example of application of the notion of \( F \)-sheaf, we can mention a theorem of existence of an Anderson t-motive having a complete multiplication over a field \( \mathfrak{K} \) with a given complete multiplication type. The idea is to consider a curve \( X \) corresponding to \( \mathfrak{K} \) and its Picard variety \( \text{Pic}_0(\mathfrak{X}) \). The map \( fr - Id \) is an algebraic isogeny on \( \text{Pic}_0(\mathfrak{X}) \), hence it is surjective. A complete multiplication type defines a divisor on \( X \) and hence an element of \( \text{Pic}_0(\mathfrak{X}) \). Its preimage with respect to the map \( fr - Id \) defines a sheaf corresponding to a desirable t-motive of complete multiplication. See [GL07], Theorem 12.6 for details.\(^9\)

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\(^8\)Complete multiplication of Anderson t-motives is an analog of the complex multiplication of abelian varieties. See [A87] for an analog of the main theorem of complex multiplication for this case.

\(^9\)This proof was indicated to the authors by V. Drinfeld.
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