Representation of Crystallographic Subperiodic Groups by Geometric Algebra

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Abstract We explain how following the representation of 3D crystallographic space groups in geometric algebra it is further possible to similarly represent the 162 so-called subperiodic groups of crystallography in geometric algebra. We construct a new compact geometric algebra group representation symbol, which allows to read off the complete set of geometric algebra generators. For clarity we moreover state explicitly what generators are chosen. The group symbols are based on the representation of point groups in geometric algebra by versors (Clifford group, Lipschitz elements).

1 Introduction

The 3D crystallographic space groups [1] have been successfully represented [3][2] in geometric algebra [4]. Following this an interactive 3D visualization has been created [6][5]. But for crystallographers the subperiodic space groups [7] in 2D and 3D with only one or two degrees of freedom for translations are also of great interest.

Fig. 1 Regular polygons ($p = 1, 2, 3, 4, 6$) and point group generating vectors $a, b$ subtending angles $\pi/p$ shifted to center.
Fig. 2 Generating vectors $a, b$ of oblique and rectangular cells for 2D frieze groups.

Fig. 3 From left to right: Triclinic, monoclinic inclined, monoclinic orthogonal, orthorhombic, and tetragonal cell vectors $a, b, c$ for rod and layer groups.

Fig. 4 Generating vectors $a, b, c$ of trigonal (left), hexagonal (center) and hexagonally centered (right, Bravais symbol: $H$ or $h$) cells for 3D rod and layer groups.

Table 1 Geometric and international notation for 2D point groups.

| Crystal      | Oblique | Rectangular | Trigonal | Square | Hexagonal |
|--------------|---------|-------------|----------|--------|-----------|
| geometric    | 1 2     | 1 2         | 3 3      | 4 4    | 6 6       |
| international| 1 2     | m mm        | 3m 3     | 4m 4   | 6m 6      |

**Clifford geometric algebra.** Clifford’s associative geometric product of two vectors simply adds the inner product to the outer product of Grassmann

$$ab = a \cdot b + a \wedge b.$$  \hspace{1cm} (1)

This allows to write the reflection of a vector $x$ at a hyperplane through the origin with normal $a$ as
Table 2  Geometric 3D point group symbols \( \{2\} \) and generators with \( \theta_{ab} = \pi/p, \theta_{bc} = \pi/q, \theta_{ac} = \pi/2, p, q \in \{1, 2, 3, 4, 6\} \).

| Symbol | 1 | p ≠ 1 | b | pq | b̅q | p̅q | p̅ | b̅p |
|--------|---|-------|---|----|-----|-----|----|-----|
| Generators | a | a, b | a, b, c | ab, c | a | ab | bc | abc |

Table 3  Table of frieze groups. Group number (col. 1), intern. frieze group notation \( \{7\} \) (col. 2), related intern. 3D space group numbers \( \{1\} \) (col. 3), and notation \( \{1\} \) (col. 4), geometric 3D space group notation \( \{3\} \) (col. 5), related intern. 2D space group numbers \( \{1\} \) (col. 6), and notation \( \{1\} \) (col. 7), related geometric 2D space group notation \( \{3\} \) (col. 8), geometric frieze group notation (col. 9), geometric algebra frieze group versor generators (col. 10). The pure translation generator \( T_a \) is omitted.

Frieze Intern. 3D Intern. Geom. 2D Intern. Geom. Geom. Frieze Group
| Group # Notat. SG# 3D SGN 3D SGN SG# 2D SGN 2D SGN Notat. Generators |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Oblique |
| F1  | 1 | 1 | P1 | P̅Τ | 1 | p̅ | p | a | a \( b \) |
| F2  | 211 | 3 | P2 | P̅Σ | 2 | p2 | p2 | a \( b \) |
| Rectangular |
| F3  | 1\text{m}1 | 6 | Pm | P1 | 3 pm(p1m1) | p1 | a | a | a \( b \) |
| F4  | 11m | 6 | Pm | P1 | 3 pm(p1m1) | p1 | a | a | a \( b \) |
| F5  | 11g | 7 | Pc | P_1 | 4 pq(p1g1) | p_q | a | a \( b \) |
| F6  | 2\text{mm} | 25 | P\text{mm}2 | P2 | 6 p2mm | p2 | a | a \( b \) |
| F7  | 2mg | 28 | P\text{ma}2 | P2\_a | 7 p2mg | p2 \( a \) | a \( b \) |

\[
x' = -a^{-1}xa, \quad a^{-1} = \frac{a}{a^2}.
\]

The composition of two reflections at hyperplanes whose normal vectors \( a, b \) subtend the angle \( \alpha/2 \) yields a rotation around the intersection of the two hyperplanes by \( \alpha \)

\[
x' = (ab)^{-1}xab, \quad (ab)^{-1} = b^{-1}a^{-1}.
\]

In general the geometric product of \( k \) normal vectors (the versor \( S \)) corresponds to the composition of reflections to all symmetry transformations \( \{2\} \) of 2D and 3D crystal cell point groups

\[
x' = (-1)^kS^{-1}xS.
\]

**Point groups.** 2D point groups are generated (cf. Table \( \{1\} \)) by multiplying vectors selected \( \{2\} \) as in Fig. \( \{1\} \). For example the hexagonal point group is given by multiplying its two generating vectors \( a, b \)

\[
6 = \{a, b, R = ab, R^2, R^3, R^4, R^5, R^6 = -1, aR^2, bR^2, aR^4, bR^4\}.
\]
Table 4: Table of triclinic, monoclinic and orthorhombic rod groups. The pure translator $T_c$ is omitted.

| Rod Group # | Intern. Notat. | 3D Space Intern. Group # | 3D SGN | Geom. Group # | 3D SGN | Geom. Notat. | Rod Group Generators |
|-------------|----------------|---------------------------|--------|---------------|--------|--------------|----------------------|
| Triclinic   |                |                           |        |               |        |              |                      |
| $R_1$       | $\not{1}$     | 1                         | $P1$   | $P\bar{1}$    | $\not{1}$ | $a$          |                      |
| $R_2$       | $\not{2}$     | 2                         | $P\bar{1}$ | $P_{22}$ | $\not{2}$ | $a \wedge b \wedge c$ |                      |
| Monoclinic/inclined | | | | | | | |
| $R_3$       | $\not{211}$   | 3                         | $P112$ | $P2$ | $\not{2}$ | $b \wedge c$ |                      |
| $R_4$       | $\not{m11}$   | 6                         | $Pm$   | $P1$ | $\not{1}$ | $a$ |                      |
| $R_5$       | $\not{c11}$   | 7                         | $Pc$   | $P1$ | $\not{1}$ | $aT_c^{1/2}$ |                      |
| $R_6$       | $\not{2/m11}$ | 10                        | $P2/m$ | $P22$ | $\not{2}$ | $a, b \wedge c$ |                      |
| $R_7$       | $\not{2/c11}$ | 13                        | $P2/c$ | $P_{22}$ | $\not{2}$ | $aT_c^{1/2}, b \wedge c$ |                      |
| Monoclinic/orthogonal | | | | | | | |
| $R_8$       | $\not{112}$   | 3                         | $P112$ | $P2$ | $\not{2}$ | $a \wedge b$ |                      |
| $R_9$       | $\not{112_1}$ | 4                         | $P2_1$ | $P_{21}$ | $\not{2}$ | $(a \wedge b)T_c^{1/2}$ |                      |
| $R_{10}$    | $\not{11m}$   | 6                         | $Pm$   | $P1$ | $\not{1}$ | $c$ |                      |
| $R_{11}$    | $\not{112/m}$ | 10                        | $P2/m$ | $P22$ | $\not{2}$ | $a \wedge b, c$ |                      |
| $R_{12}$    | $\not{112_1/m}$ | 11                       | $P2_1/m$ | $P_{21}$ | $\not{2}$ | $(a \wedge b)T_c^{1/2}, c$ |                      |
| Orthorhombic |                |                           |        |               |        |              |                      |
| $R_{13}$    | $\not{222}$   | 16                        | $P222$ | $P_{22}$ | $\not{2}$ | $ab, bc$ |                      |
| $R_{14}$    | $\not{222_1}$ | 17                        | $P22_1$ | $P_{22}$ | $\not{2}$ | $abT_c^{1/2}, bc$ |                      |
| $R_{15}$    | $\not{nn2}$   | 25                        | $Pmm2$ | $P2$ | $\not{2}$ | $a, b$ |                      |
| $R_{16}$    | $\not{cc2}$   | 27                        | $Pcc2$ | $P_{2c}$ | $\not{2}$ | $aT_c^{1/2}, bT_c^{1/2}$ |                      |
| $R_{17}$    | $\not{mc2_1}$ | 26                        | $Pmc2_1$ | $P_{2c}$ | $\not{2}$ | $a, bT_c^{1/2}$ |                      |
| $R_{18}$    | $\not{2mm}$   | 25                        | $Pmm2$ | $P2$ | $\not{2}$ | $b, c$ |                      |
| $R_{19}$    | $\not{2cm}$   | 28                        | $Pma2$ | $P_{2a}$ | $\not{2}$ | $bT_c^{1/2}, c$ |                      |
| $R_{20}$    | $\not{mmm}$   | 47                        | $Pmmm$ | $P_{22}$ | $\not{2}$ | $a, b, c$ |                      |
| $R_{21}$    | $\not{ccm}$   | 49                        | $Pccm$ | $P_{22}$ | $\not{2}$ | $aT_c^{1/2}, bT_c^{1/2}, c$ |                      |
| $R_{22}$    | $\not{mcm}$   | 51                        | $Pmma$ | $P_{22}$ | $\not{2}$ | $a, bT_c^{1/2}, c$ |                      |

The rotation subgroups are denoted with bars, e.g. $\not{6}$. The selection of three vectors $a, b, c$ from each crystal cell [2, 5] for generating (cf. Table 2) 3D point groups are indicated in Figs. 3 and 4.

**Space groups.** The smooth composition with translations is best done in the conformal model [4] of Euclidean space (in the GA of $\mathbb{R}^{4,1}$). A plane can be described by the vector

$$m = p - d e_\infty,$$  

(6)
Table 5 Table of tetragonal and trigonal rod groups. The pure translator $T_r$ is omitted.

| Rod Group # | Intern. Notat. | 3D Space | Intern. Group # | Geom. 3D SGN | Geom. Notat. | Rod Group Generators |
|-------------|----------------|----------|-----------------|--------------|--------------|----------------------|
### Tetragonal

| $R_{23}$ | $\not\in\mathbb{Z}$ | 75 | $P4$ | $P4$ | $\not\in\mathbb{Z}$ | $ab$ |
| $R_{24}$ | $\not\in\mathbb{Z}$ | 76 | $P4_1$ | $P4_1$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}$ |
| $R_{25}$ | $\not\in\mathbb{Z}$ | 77 | $P4_2$ | $P4_2$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}$ |
| $R_{26}$ | $\not\in\mathbb{Z}$ | 78 | $P4_3$ | $P4_3$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}$ |
| $R_{27}$ | $\not\in\mathbb{Z}$ | 81 | $P4$ | $P4 \overline{2}$ | $\not\in\mathbb{Z}$ | $abc$ |
| $R_{28}$ | $\not\in\mathbb{Z}$ | 83 | $P4/m$ | $P4_2$ | $\not\in\mathbb{Z}$ | $ab,c$ |
| $R_{29}$ | $\not\in\mathbb{Z}$ | 84 | $P4_2/m$ | $P4_2$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}, c$ |
| $R_{30}$ | $\not\in\mathbb{Z}$ | 89 | $P422$ | $P422$ | $\not\in\mathbb{Z}$ | $ab, bc$ |
| $R_{31}$ | $\not\in\mathbb{Z}$ | 91 | $P4_122$ | $P4_122$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}, bc$ |
| $R_{32}$ | $\not\in\mathbb{Z}$ | 93 | $P4_222$ | $P4_222$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}, bc$ |
| $R_{33}$ | $\not\in\mathbb{Z}$ | 95 | $P4_322$ | $P4_322$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}, bc$ |
| $R_{34}$ | $\not\in\mathbb{Z}$ | 99 | $P4mm$ | $P4$ | $\not\in\mathbb{Z}$ | $a,b$ |
| $R_{35}$ | $\not\in\mathbb{Z}$ | 101 | $P4_2cm$ | $P4$ | $\not\in\mathbb{Z}$ | $aT_r \frac{1}{2}, b$ |
| $R_{36}$ | $\not\in\mathbb{Z}$ | 103 | $P4cc$ | $P4_1$ | $\not\in\mathbb{Z}$ | $aT_r \frac{1}{2}, bc$ |
| $R_{37}$ | $\not\in\mathbb{Z}$ | 115 | $P4_2m2$ | $P4_2$ | $\not\in\mathbb{Z}$ | $a,b,c$ |
| $R_{38}$ | $\not\in\mathbb{Z}$ | 116 | $P4_2c2$ | $P4_2$ | $\not\in\mathbb{Z}$ | $aT_r \frac{1}{2}, bc$ |
| $R_{39}$ | $\not\in\mathbb{Z}$ | 123 | $P4/mmm$ | $P4_2$ | $\not\in\mathbb{Z}$ | $a,b,c$ |
| $R_{40}$ | $\not\in\mathbb{Z}$ | 124 | $P4_1/mcc$ | $P4_1$ | $\not\in\mathbb{Z}$ | $aT_r \frac{1}{2}, bT_r \frac{1}{2}, c$ |
| $R_{41}$ | $\not\in\mathbb{Z}$ | 131 | $P4_2/mmc$ | $P4_2$ | $\not\in\mathbb{Z}$ | $a,bT_r \frac{1}{2}, c$ |
### Trigonal

| $R_{42}$ | $\not\in\mathbb{Z}$ | 143 | $P3$ | $P3$ | $\not\in\mathbb{Z}$ | $ab$ |
| $R_{43}$ | $\not\in\mathbb{Z}$ | 144 | $P3_1$ | $P3_1$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}$ |
| $R_{44}$ | $\not\in\mathbb{Z}$ | 145 | $P3_2$ | $P3_2$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}$ |
| $R_{45}$ | $\not\in\mathbb{Z}$ | 147 | $P3$ | $P3 \overline{2}$ | $\not\in\mathbb{Z}$ | $abc$ |
| $R_{46}$ | $\not\in\mathbb{Z}$ | 149 | $P312$ | $P3_2$ | $\not\in\mathbb{Z}$ | $ab, bc$ |
| $R_{47}$ | $\not\in\mathbb{Z}$ | 151 | $P3_12$ | $P3_12$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}, bc$ |
| $R_{48}$ | $\not\in\mathbb{Z}$ | 153 | $P3_12$ | $P3_12$ | $\not\in\mathbb{Z}$ | $abT_r \frac{1}{2}, bc$ |
| $R_{49}$ | $\not\in\mathbb{Z}$ | 156 | $P3m1$ | $P3$ | $\not\in\mathbb{Z}$ | $a,b$ |
| $R_{50}$ | $\not\in\mathbb{Z}$ | 158 | $P3c1$ | $P3_1$ | $\not\in\mathbb{Z}$ | $aT_r \frac{1}{2}, bc$ |
| $R_{51}$ | $\not\in\mathbb{Z}$ | 162 | $P31m$ | $P3_1m$ | $\not\in\mathbb{Z}$ | $a,b, c$ |
| $R_{52}$ | $\not\in\mathbb{Z}$ | 163 | $P31c$ | $P3_1c$ | $\not\in\mathbb{Z}$ | $aT_r \frac{1}{2}, bc$ |
\[
\rho = m' m X m m' = T^{-1} X T, \quad T_i = 1 + \frac{1}{2} t e_{\rho}.
\]

Reflection at two non-parallel planes \(m, m'\) yields the rotation around the \(m, m'\)-intersection by twice the angle subtended by \(m, m'\). Applying these techniques one can compactly tabulate geometric space group symbols and generators [2], and consequently visualize them [5].
Table 7 Table of triclinic and monoclinic layer groups. The pure translators $T_a, T_b$ are omitted.

| Layer Group # | Intern. Notat. | 3D SGN | Intern. Geom. | Geom. Layer Group | Generators |
|---------------|---------------|--------|---------------|-------------------|------------|
| L$_1$         | p1            | 1      | P$^1$         | $p^1$             | a $\land$ b $\land$ c |
| L$_2$         | p1            | 2      | P$^1$         | $p^2$             | a $\land$ b $\land$ c |

Monoclinic/oblique

| L$_3$         | p112          | 3      | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_4$         | p11m          | 6      | Pm            | $p^1$             | a $\land$ b $\land$ c |
| L$_5$         | p11a          | 7      | P$^1$         | $p^2$             | a $\land$ b $\land$ c |
| L$_6$         | p112/m        | 10     | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_7$         | p112/a        | 13     | P$^2$         | $p^2$             | a $\land$ b $\land$ c |

Monoclinic/rectangular

| L$_8$         | p211          | 3      | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_9$         | p21,11        | 4      | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_{10}$      | c211          | 5      | A2$^1$        | c $\land$ b $\land$ c | |
| L$_{11}$      | pm11          | 6      | Pm            | $p^1$             | a $\land$ b $\land$ c |
| L$_{12}$      | pb11          | 7      | P$^1$         | $p^2$             | a $\land$ b $\land$ c |
| L$_{13}$      | cm11          | 8      | Cm            | $p^2$             | a $\land$ b $\land$ c |
| L$_{14}$      | p2/m11        | 10     | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_{15}$      | p21/m11       | 11     | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_{16}$      | p2/b11        | 13     | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_{17}$      | p21/b11       | 14     | P$^2$         | $p^2$             | a $\land$ b $\land$ c |
| L$_{18}$      | c2/m11        | 12     | C2$^1$        | c $\land$ b $\land$ c | |

2 Subperiodic groups represented in Clifford geometric algebra

Compared to [3] we have introduced dots: If one or two dots occur between the Bravais symbol (a, p, c) and index 1, the vector b or c, respectively, is present in the generator list. If one or two dots appear between the Bravais symbol and the index 2 (without or with bar), then the vectors b, c or a, c, respectively, are present in the generator list.

In agreement [3] the indexes a, b, c, n (and g for frieze groups) in first, second or third position after the Bravais symbol indicate that the reflections a, b, c (in this order) become glide reflections. Index n indicates diagonal glides. The dots also serve as position indicators. For example rod group 5: $\alpha c 1$ has glide reflection $aT_c^{1/2}$, rod group 19: $\alpha c 2$ has $bT_c^{1/2}$, and layer group 39: $p_b 2a 2n$ has $aT_b^{1/2}$, $bT_a^{1/2}$ and $cT_{a+b}^{1/2}$.

The notation $p_{\pi}$ indicates a right handed screw rotation of $2\pi/n$ around the $\pi$-axis, with pitch $T_i^{1/n}$ where i is the shortest lattice translation vector parallel to the
Table 8 Table of orthorhombic/rectangular layer groups. The pure translators $T_a, T_b$ are omitted.

| Layer Group # | Intern. Notat. | 3D Space Group # | Intern. Geom. Notat. | Geom. Notat. | Layer Group Generators |
|---------------|----------------|------------------|----------------------|--------------|------------------------|
| $L_{20}$      | $p222$         | 16               | $P222$               | $P222$       | $ab, bc$               |
| $L_{22}$      | $p2_12_2$     | 18               | $P2_12_2$            | $P222$       | $ab, bcT_a^{1/2}$      |
| $L_{23}$      | $p2_12_2$     | 21               | $C222$               | $P2_12_2$    | $bcT_a^{1/2}, acT_b^{1/2}$ |
| $L_{24}$      | $Pmn2$        | 25               | $P2$                 | $P2$         | $a, b$                 |
| $L_{25}$      | $Pma2$        | 28               | $P2_a$               | $P2_a$       | $a, bT_a^{1/2}$        |
| $L_{26}$      | $Pbc2$        | 27               | $P2_c$               | $P2_c$       | $aT_b^{1/2}, cT_a^{1/2}$ |
| $L_{27}$      | $Pma2$        | 28               | $P2_a$               | $P2_a$       | $a, cT_a^{1/2}$        |
| $L_{28}$      | $Pmn2n$       | 31               | $P2_n$               | $P2_n$       | $a, cT_a^{1/2}$        |
| $L_{29}$      | $Pca2_1$      | 29               | $P2_c$               | $P2_c$       | $aT_b^{1/2}, cT_a^{1/2}$ |
| $L_{30}$      | $Pbc2$        | 30               | $P2_b$               | $P2_b$       | $aT_b^{1/2}, cT_a^{1/2}$ |
| $L_{31}$      | $Aem2$        | 39               | $A2$                 | $A2$         | $a, cT_a^{1/2}$        |
| $L_{32}$      | $Pmnm$        | 47               | $P22$                | $P22$        | $a, b, c$              |
| $L_{33}$      | $Pccm$        | 49               | $P2_12_2$            | $P2_12_2$    | $aT_b^{1/2}, cT_a^{1/2}$ |
| $L_{34}$      | $Pba2_1$      | 50               | $P2_12_2$            | $P2_12_2$    | $aT_b^{1/2}, cT_a^{1/2}$ |
| $L_{35}$      | $Pmmn$        | 51               | $P2_22_2$            | $P2_22_2$    | $a, bT_a^{1/2}$        |
| $L_{36}$      | $Pmmm$        | 53               | $P2_22_2$            | $P2_22_2$    | $a, b, cT_a^{1/2}$     |
| $L_{37}$      | $Pmmm$        | 55               | $P2_22_2$            | $P2_22_2$    | $a, b, cT_a^{1/2}$     |
| $L_{38}$      | $Pbcn$        | 54               | $P2_22_2$            | $P2_22_2$    | $aT_b^{1/2}, cT_a$     |
| $L_{39}$      | $Pba2_1$      | 57               | $P2_22_2$            | $P2_22_2$    | $aT_b^{1/2}, cT_a$     |
| $L_{40}$      | $Pmmn$        | 59               | $P2_22_2$            | $P2_22_2$    | $a, b, cT_a^{1/2}$     |
| $L_{41}$      | $Pmmm$        | 65               | $P2_22_2$            | $P2_22_2$    | $a, b, cT_a^{1/2}$     |
| $L_{42}$      | $C222$        | 67               | $C222$               | $C222$       | $a, b, cT_a^{1/2}$     |

axis, in the screw direction. For example the layer group 21: $p2_22_2$ has the screw generators $bcT_a^{1/2}$ and $acT_b^{1/2}$.

Frieze groups. Figure 2 shows the generating vectors $a, b$ of oblique and rectangular cells for 2D frieze groups. The only translation direction is $a$. Table 3 lists the seven frieze groups with new geometric symbols and generators.

Rod groups. Figure 3 shows the generating vectors $a, b, c$ of triclinic, monoclinic, orthorhombic and tetragonal cells for 3D rod and layer groups. Figure 4 shows the
| Layer Group Notat. | Layer Group Notat. | Layer Group Notat. | Layer Group Notat. | Layer Group Notat. |
|------------------|------------------|------------------|------------------|------------------|
| $L_{48}$        | $p4$             | 75               | $P4$             | $p4$             |
| $L_{50}$        | $p4$             | 81               | $P4$             | $p4$             |
| $L_{51}$        | $p4/m$           | 83               | $P4/m$           | $p4/m$           |
| $L_{52}$        | $p4/n$           | 85               | $P4/n$           | $p4/n$           |
| $L_{53}$        | $p42$            | 89               | $P42$            | $p42$            |
| $L_{54}$        | $p42_1$          | 90               | $P42_1$          | $p42_1$          |
| $L_{55}$        | $p4mm$           | 99               | $P4mm$           | $p4mm$           |
| $L_{56}$        | $p4bm$           | 100              | $P4bm$           | $p4bm$           |
| $L_{57}$        | $p42m$           | 111              | $P42m$           | $p42m$           |
| $L_{58}$        | $p42_2$          | 113              | $P42_2$          | $p42_2$          |
| $L_{59}$        | $p4m2$           | 115              | $P4m2$           | $p4m2$           |
| $L_{60}$        | $p4_21$          | 117              | $P4_21$          | $p4_21$          |
| $L_{61}$        | $p4/mmm$         | 123              | $P4/mmm$         | $p4/mmm$         |
| $L_{62}$        | $p4/nbm$         | 125              | $P4/nbm$         | $p4/nbm$         |
| $L_{63}$        | $p4/mnb$         | 127              | $P4/mnb$         | $p4/mnb$         |
| $L_{64}$        | $p4/nmm$         | 129              | $P4/nmm$         | $p4/nmm$         |

### Trigonal/hexagonal Groups

| Layer Group Notat. | Layer Group Notat. | Layer Group Notat. | Layer Group Notat. | Layer Group Notat. |
|------------------|------------------|------------------|------------------|------------------|
| $L_{65}$        | $p3$             | 143              | $P3$             | $p3$             |
| $L_{66}$        | $p3$             | 147              | $P3$             | $p3$             |
| $L_{67}$        | $p312$           | 149              | $P312$           | $p312$           |
| $L_{68}$        | $p311$           | 150              | $P311$           | $p311$           |
| $L_{69}$        | $p3m1$           | 156              | $P3m1$           | $p3m1$           |
| $L_{70}$        | $p3m1$           | 157              | $P3m1$           | $p3m1$           |
| $L_{71}$        | $p31m$           | 162              | $P31m$           | $p31m$           |
| $L_{72}$        | $p31m$           | 164              | $P31m$           | $p31m$           |

### Hexagonal/hexagonal Groups

| Layer Group Notat. | Layer Group Notat. | Layer Group Notat. | Layer Group Notat. | Layer Group Notat. |
|------------------|------------------|------------------|------------------|------------------|
| $L_{73}$        | $p6$             | 168              | $P6$             | $p6$             |
| $L_{74}$        | $p6$             | 174              | $P6$             | $p6$             |
| $L_{75}$        | $p6/m$           | 175              | $P6/m$           | $p6/m$           |
| $L_{76}$        | $p62$            | 177              | $P62$            | $p62$            |
| $L_{77}$        | $p62m$           | 183              | $P62m$           | $p62m$           |
| $L_{78}$        | $p6mm$           | 187              | $P6mm$           | $p6mm$           |
| $L_{79}$        | $p62mm$          | 189              | $P62mm$          | $p62mm$          |
| $L_{80}$        | $p6/mmm$         | 191              | $P6/mmm$         | $p6/mmm$         |

*Table 9* Table of tetragonal, trigonal and hexagonal layer groups. $L_{37}$, $L_{58}$ and $L_{71}$ use special vector notation. The pure translators $T_{e}, T_{o}$ are omitted.
same for trigonal and hexagonal cells. For rod groups the only translation direction is \( c \). Tables 4, 5, 6 list the 75 rod groups with new geometric symbols and generators: Rod group number (col. 1), intern. rod group notation [7] (col. 2), related intern. 3D space group numbers [1] (col. 3), and notation [1] (col. 4), related geometric 3D space group notation [3] (col. 5), geometric rod group notation (col. 6), geometric algebra generators (col. 7).

**Layer groups.** For layer groups the two translation directions are \( a, b \). Tables 7, 8, and 9 list the 80 3D layer groups with new geometric symbols and generators: Layer group number (col. 1), intern. layer group notation [7] (col. 2), related intern. 3D space group numbers [1] (col. 3), and notation [1] (col. 4), related geometric 3D space group notation [3] (col. 5), geometric layer group notation (col. 6), geometric algebra generators (col. 7). The layer groups are classified according to their 3D crystal system/2D Bravais system. The monoclinic/oblique(rectangular) system corresponds to the monoclinic/orthogonal(inclined) system of Fig. 3. Figure 4 shows the hexagonally centered cell with Bravais symbols \( H \) (space group) and \( h \) (layer group).

**Conclusion** We have devised a new Clifford geometric algebra representation for the 162 subperiodic space groups using versors (Clifford group, Lipschitz elements). In the future this may be extended to magnetic subperiodic space groups, the sign of the generators may achieve that. We expect that the present work forms a suitable foundation for interactive visualization software of subperiodic space groups [5]. Fig. 5 shows how the rod groups 13: \( \bar{p}222 \) and 14: \( \bar{p}2_{1}22 \), and the layer group 11: \( p1 \) and might be visualized in the future, based on [5].

![Fig. 5](image_url) How a future subperiodic space group viewer software might depict rod groups 13: \( \bar{p}222 \) and 14: \( \bar{p}2_{1}22 \), and the layer group 11: \( p1 \), based on [5].

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