SUPERCONGRUENCES INVOLVING PRODUCTS OF TWO BINOMIAL COEFFICIENTS MODULO $p^4$

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Abstract. In this paper, we mainly prove a congruence conjecture of Z.-W. Sun [13]: Let $p > 5$ be a prime. Then

$$\sum_{k=(p+1)/2}^{p-1} \binom{2k}{k}^2 \equiv -\frac{21}{2} H_{p-1} \pmod{p^4},$$

where $H_n$ denotes the $n$-th harmonic number.

1. Introduction

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2),$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \quad (n = 0, 1, 2, \ldots).$$

Let $m > 0$ and let $(a_1, a_2, \ldots, a_m) \in (\mathbb{N})^m = \mathbb{N} \times \mathbb{N} \cdots \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. For any $n \geq m$, we define the alternating multiple harmonic sum as

$$H(a_1, a_2, \ldots, a_m; n) = \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n} \prod_{i=1}^{m} \frac{\text{sign}(a_i)^{k_i}}{k_i^{\left|a_i\right|}}.$$

The integers $m$ and $\sum_{i=1}^{m} |a_i|$ are respectively the depth and the weight of the harmonic sum. As a matter of convenience, we remember $H(1; n)$ as $H_n$. We know several non-alternating harmonic sums modulo a power of a prime as follows:

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(i). ([2]) for $a, r > 0$ and for any prime $p > ar + 2$

$$H(\{a\}; p - 1) \equiv \begin{cases} \left(-1\right)^{r} \frac{a(2r + 1)}{2(a - 1)} B_{p - ar - 2} \pmod{p^3} & \text{if } ar \text{ is odd}, \\ \left(-1\right)^{r-1} \frac{a}{ar+1} B_{p - ar - 1} \pmod{p^2} & \text{if } ar \text{ is even}. \end{cases}$$

(ii). ([8]) for a positive integer $a$ and for any prime $p > a + 2$, we have

$$H\left(a; \frac{p - 1}{2}\right) \equiv \begin{cases} -2q_p(2) \pmod{p} & a=1, \\ -2^{p-2} B_{p - a} \pmod{p} & \text{if } a > 1 \text{ is odd}, \\ \frac{a(2^{a+1} - 1)}{2(a+1)} B_{p - a - 1} \pmod{p^2} & \text{if } a \text{ is even}, \end{cases}$$

where $q_p(a) = (a^{p-1} - 1)/p$ stands for the Fermat quotient.

(iii). ([2]) For $a, b > 0$ with $a + b$ odd and for any prime $p > a + b + 1$, we have

$$H(a, b; p - 1) \equiv \frac{(-1)^b}{a+b} \left(\frac{a+b}{a}\right) B_{p - a - b} \pmod{p}.$$

(iv). ([3, Lemma 1]) if $a, b$ are positive integers and $a + b$ is odd, then for any prime $p > a + b$,

$$H\left(a, b; \frac{p - 1}{2}\right) \equiv B_{p - a - b} \left(\frac{a+b}{a+b}\right) \left(-1\right)^b \left(\frac{a+b}{a}\right) + 2^{a+b} - 2 \pmod{p}.$$

(v). ([15, Corollary 2.3]) Let $a \in \mathbb{Z}_{\geq 0}$ and $p \geq a + 2$ be a prime. Then

$$H(-a; p - 1) \equiv \begin{cases} -2^{\left(\frac{p-2}{2}\right)} B_{p - a} \pmod{p} & \text{if } a \text{ is odd}, \\ \frac{a(2^{a+1} - 1)}{a+1} B_{p - a - 1} \pmod{p^2} & \text{if } a \text{ is even}. \end{cases}$$

(vi). ([15, Theorem 3.1]) Let $a, b \in \mathbb{N}$ and $p \geq a + b + 2$ be a prime. If $a + b$ is odd then we have

$$H(-a, b; p - 1) \equiv H(a, -b; p - 1) \equiv \frac{1 - 2^{p-a-b}}{a+b} B_{p - a - b} \pmod{p}.$$

(vii). ([15, Propositions 6.3 and 7.3, (116)]) Let $p > 5$ be a prime and

$$X := \frac{B_{p-3}}{p-3} - \frac{B_{2p-4}}{4p-8}.$$

Then

$$H(-4; p - 1) \equiv H(2, 2; p - 1) \equiv H(1, 3; p - 1) \equiv 0 \pmod{p},$$

$$H(2, -1; p - 1) \equiv -\frac{3}{2} X - \frac{7}{6} pq_p(2) B_{p - 3} + pH(1, -3; p - 1) \pmod{p^2},$$

$$-\frac{1}{2} H(1, 2; p - 1) \equiv \frac{1}{2} H(2, 1; p - 1) \equiv H(-3; p - 1) \equiv -2H(1, -2; p - 1) \equiv 3X \pmod{p^2}.$$
(viii). ([8, Theorems 5.1 and 5.2, Remark 5.1]) Let \( p > 3 \) be a prime. Then

\[
-\frac{2}{p} H_{p-1} \equiv H(2; p - 1) \equiv \frac{2}{7} H \left( \frac{p - 1}{2} \right) \equiv -4pX \pmod{p^3},
\]

\[
H \left( 3; \frac{p - 1}{2} \right) \equiv 12X \pmod{p^2}.
\]

Throughout the paper, \( p \) always stands for a prime, and \( \mathbb{Z}_p \) denotes the set of \( p \)-adic integers, and for \( a \in \mathbb{Z}_p \), let \( \langle a \rangle_p \in \{0, 1, \ldots, p - 1\} \) be given by \( \langle a \rangle_p \equiv a \pmod{p} \).

In [14], Tauraso proved that for any prime \( p > 5 \),

\[
\sum_{k=1}^{p-1} \binom{2k}{k} \binom{a}{k} \left( -\binom{a}{k} \right)^2 \equiv -2H_{p-1} \pmod{p^3}, \tag{1.1}
\]

and in [13], Sun proved that

\[
\sum_{k=\frac{p+1}{2}}^{p-1} \binom{2k}{k} \binom{a}{k} \left( -\binom{a}{k} \right)^2 \equiv \frac{7}{2} p^2 B_{p-3} \pmod{p^3}.
\]

Tauraso [14] also obtained the following congruence: Let \( p > 5 \) be a prime, \( a \in \mathbb{Z}_p \) and \( t = (a - \langle a \rangle_p)/p \). Then

\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \left( -\binom{a}{k} - 1 \right) \equiv -2H_{(a)_p} + 2ptH(2; \langle a \rangle_p) \pmod{p^2}.
\]

Sun [10] generalised Tauraso’s result to

\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \left( -\binom{a}{k} - 1 \right)
\equiv -\frac{2}{3} p^2 tB_{p-3} - 2H_{(a)_p} + 2ptH(2; \langle a \rangle_p) + 2p^2 tH(3; \langle a \rangle_p) \pmod{p^3}.
\]

Motivated by the above, we generalised Z.-H. Sun’s result and confirm a conjecture of Z.-W. Sun [12]:

**Theorem 1.1.** Let \( p > 5 \) be a prime, \( a \in \mathbb{Z}_p \) and \( t := (a - \langle a \rangle_p)/p \). Then

\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \left( -\binom{a}{k} - a \right) \equiv 4p^2 tX - 2H_{(a)_p} + 2ptH(2; \langle a \rangle_p) + 2p^2 tH(3; \langle a \rangle_p)
- 2p^3 t(2t^2 + 4t + 1)H(4; \langle a \rangle_p) + 4p^3 t(t + 1) \sum_{k=1}^{\langle a \rangle_p} \frac{H_k}{k^3} \pmod{p^4},
\]
and if \( \langle a \rangle_p \leq (p - 1)/2 \),
\[
\sum_{k=1}^{p-1} \frac{1}{k} \binom{a}{k} \binom{-1 - a}{k} \equiv -12p^2t^2X + 14p^2tX - 2H_{\langle a \rangle_p} + 4ptH(2; \langle a \rangle_p) - 6p^2t^2H(3; \langle a \rangle_p)
\]
\[+ 8p^3t^3H(4; \langle a \rangle_p) + 4p^2t \sum_{k=1}^{\langle a \rangle_p} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{2j - 1} - 8p^3t^2 \sum_{k=1}^{\langle a \rangle_p} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{(2j - 1)^2} \pmod{p^4}.\]

Furthermore,
\[
\sum_{k=1}^{p-1} \frac{(2k)^2}{k16^k} \equiv -2H_{\frac{p-1}{2}} - p^3 \sum_{k=1}^{\frac{p-1}{2}} H_k \pmod{p^4}. \tag{1.2}
\]
\[
\sum_{k=\frac{p-1}{2}}^{p-1} \frac{(2k)^2}{k16^k} \equiv -\frac{21}{2}H_{p-1} \pmod{p^4}. \tag{1.3}
\]

**Remark 1.1.** (1.3) was conjectured by Z.-W. Sun [12, Conjecture 1.1].

In 2008, Z.-H. Sun [9, Remark 4.1] proved that for any prime \( p > 3 \),
\[
\sum_{k=1}^{p-1} \frac{2k}{k3} \equiv -\frac{1}{3}g_p(2) - \frac{7}{24}B_{p-3} \pmod{p}.
\]

Tauraso and Zhao [15, (79)] reproved the above congruence in 2010, Mattarei and Tauraso [5] also reproved this congruence in 2013. And Tauraso and Zhao [15, Remark 7.7] said that they not able to express
\[
H(1, 1, -1; p-1) \pmod{p^2}
\]
explicitly, but they found that it is equivalent to determining
\[
\sum_{k=1}^{p-1} \frac{2k}{k3} \pmod{p^2}.
\]

In this paper, we give \( \sum_{k=1}^{p-1} \frac{2k}{k3} \pmod{p^2} \).

**Theorem 1.2.** For any prime \( p > 5 \), we have
\[
\sum_{k=1}^{p-1} \frac{2k}{k3} \equiv -\frac{1}{3}g_p(2) + \frac{7}{4}X + \frac{5}{12}q_{p^2}(2) + \frac{7p}{6}g_p(2)B_{p-3} - \frac{3p}{8} \sum_{k=1}^{\frac{p-1}{2}} \frac{H_k}{k3} \pmod{p^2}.
\]

We are going to prove Theorem 1.1 in Section 2. Section 3 is devoted to proving Theorem 1.2. Our proofs make use of some combinatorial identities which were found by the package **Sigma** [6] via the software **Mathematica**. The proof of Theorem 1.1 is somewhat difficult and
complex because it is rather convoluted. We are not able to know the exact result of
\[ \sum_{k=1}^{p-1} \frac{H_k}{k^3} \equiv H\left(1, 3; \frac{p-1}{2}\right) \equiv 4q_p(2)B_{p-3} - H\left(3, 1; \frac{p-1}{2}\right) \pmod{p}, \]
so we just keep it in the congruences of above and following.

2. Proof of Theorem 1.1

Lemma 2.1. Let \( p > 5 \) be a prime, and let \( t \in \mathbb{Z}_p \). If \( k \in \{1, 2, \ldots, p-1\} \), then
\[
\left(\begin{array}{c}
pt + k - 1 \\
p - 1
\end{array}\right) \left(\begin{array}{c}
-pt - k - 1 \\
p - 1
\end{array}\right) \equiv \frac{p^2 t (t+1)}{k^2} \left(1 + 2pH_k - \frac{p}{k} - \frac{2pt}{k^2}\right) \pmod{p^4}.
\]
If \( k \in \{1, 2, \ldots, (p-1)/2\} \), then
\[
\left(\begin{array}{c}
pt + k - 1 \\
\frac{p-1}{2}
\end{array}\right) \equiv \frac{pt}{k} \left(1 - \frac{pt}{k} + 2p \sum_{j=1}^{k} \frac{1}{2j - 1} + \frac{p^2 t^2}{k^2}\right) \pmod{p^4}.
\]

Proof. It is easy to check that
\[
\left(\begin{array}{c}
pt + k - 1 \\
p - 1
\end{array}\right) = \frac{(pt + k - 1) \cdots (pt + 1)(pt - 1) \cdots (pt + k - p + 1)}{(p - 1)!} \equiv \frac{pt(k - 1)! (1 + ptH_{k-1})(-1)^{p-1-k} (p - 1 - k)! (1 - ptH_{p-1-k})}{(p - 1)!} \equiv \frac{pt}{k} \left(1 + pH_k - \frac{pt}{k}\right) \pmod{p^3},
\]
and by (i), we have
\[
\left(\begin{array}{c}
-pt - k - 1 \\
p - 1
\end{array}\right) = \frac{(pt + k + 1) \cdots (pt + p - 1)p(t + 1)(pt + p + 1) \cdots (pt + p + k - 1)}{(p - 1)!} \equiv \frac{p(t + 1)(p - 1)! (1 + pt(H_{p-1} - H_k))(k - 1)! (1 + p(t + 1)H_{k-1})}{k!(p - 1)!} \pmod{p^3}.
\]
\[
\equiv \frac{p(t + 1)}{k} \left(1 + pH_{k-1} - \frac{pt}{k}\right) \pmod{p^3},
\]
hence
\[
\left(\frac{pt + k - 1}{p - 1}\right) \left(-\frac{pt - k - 1}{p - 1}\right) \equiv \frac{p^2 t(t + 1)}{k^2} \left(1 + 2pH_k - \frac{p}{k} - \frac{2pt}{k}\right) \pmod{p^4}.
\]
Similarly,
\[
\left(\frac{pt + k - 1}{\frac{p-1}{2}}\right) = \frac{(pt + k - 1) \cdots (pt + 1)pt(pt - 1) \cdots (pt + k - \frac{p-1}{2})}{(\frac{p-1}{2})!}
\equiv pt(k - 1)!(1 + ptH_{k-1} + p^2 t^2 H(1, 1; k - 1))(-1)^{\frac{p-1}{2} - k}(\frac{p-1}{2} - k)!
\times (1 - ptH_{\frac{p-1}{2} - k} + p^2 t^2 H(1, 1; \frac{p-1}{2} - k))
\equiv pt(-1)^{\frac{p-1}{2} - k}k(\frac{p-1}{2})\left(1 + ptH_{k-1} - ptH_{\frac{p-1}{2} - k} - p^2 t^2 H_{k-1}H_{\frac{p-1}{2} - k} + p^2 t^2 H(1, 1; k - 1) + p^2 t^2 H(1, 1; \frac{p-1}{2} - k)\right) \pmod{p^4}
\]
and
\[
\left(-\frac{pt - k - 1}{\frac{p-1}{2}}\right) = \frac{(-1)^{\frac{p-1}{2}}(pt + k + 1) \cdots (pt + k + \frac{p-1}{2})}{(\frac{p-1}{2})!}
\equiv (-1)^{\frac{p-1}{2}}\left(\frac{p-1}{2} + k\right)\left(1 + ptH_{\frac{p-1}{2} + k} - ptH_k - p^2 t^2 H_kH_{\frac{p-1}{2} + k} + p^2 t^2 H_k^2 - p^2 t^2 H(1, 1; k) + p^2 t^2 H(1, 1; \frac{p-1}{2} + k)\right) \pmod{p^4},
\]
Hence by the fact that \(H_{p-1-k} \equiv H_k (\mod p)\), \(H(2; p - 1 - k) \equiv -H(2; k) (\mod p)\) and
\[
H(1, 1; k) - H(1, 1; k - 1) = \frac{1}{2}(H_k^2 - H_{k-1}^2 - H(2; k) + H(2; k - 1)) = \frac{H_k}{k} - \frac{1}{k^2}
\]
we have
\[
\left(\frac{pt + k - 1}{\frac{p-1}{2}}\right) \left(-\frac{pt - k - 1}{\frac{p-1}{2}}\right)
\equiv pt(-1)^k(\frac{p-1}{2} + k)k(\frac{p-1}{2})\left(1 - \frac{pt}{k} + \frac{p^2 t^2}{k^2} + ptH_{\frac{p-1}{2} + k} - ptH_{\frac{p-1}{2} - k}\right) \pmod{p^4}
In view of [11, (3.2)] and [12, Lemma 4.2], we have
\[ (-1)^k \binom{\frac{p-1}{2} + k}{k} \binom{\frac{p-1}{2}}{k} \left( 1 - \frac{p}{4} (H_{\frac{p-1}{2} + k} - H_{\frac{p-1}{2}}) \right) \equiv \frac{(2k)^2}{16^k} \pmod{p^4} \]
and
\[ \left( \frac{\frac{p-1}{2}}{k} \right)^2 16^k \equiv 1 - 2p \sum_{j=1}^{k} \frac{1}{2j - 1} + 2p^2 \left( \sum_{j=1}^{k} \frac{1}{2j - 1} \right)^2 - p^2 \sum_{j=1}^{k} \frac{1}{(2j - 1)^2} \pmod{p^3}. \]
Thus by the first congruence in [11, pp 16], we have
\[ \frac{(-1)^k \binom{\frac{p-1}{2} + k}{k}}{k\left( \frac{\frac{p-1}{2}}{k} \right)^2} = \frac{(-1)^k \binom{\frac{p-1}{2} + k}{k} \left( \frac{\frac{p-1}{2}}{k} \right)}{k\left( \frac{\frac{p-1}{2}}{k} \right)^2} \]
\[ \equiv 1 + 2p \sum_{j=1}^{k} \frac{1}{2j - 1} + 2p^2 \left( \sum_{j=1}^{k} \frac{1}{2j - 1} \right)^2 \pmod{p^3}. \]
Therefore by the first congruence in [11, pp 16] again, we immediately obtain that
\[ \binom{pt + k - 1}{\frac{p-1}{2}} \binom{-pt - k - 1}{\frac{p-1}{2}} \equiv \frac{pt}{k} \left( 1 - \frac{pt}{k} \right) + 2p \sum_{j=1}^{k} \frac{1}{2j - 1} + \frac{p^2 t^2}{k^2} \]
\[ + 2p^2 \left( \sum_{j=1}^{k} \frac{1}{2j - 1} \right)^2 - \frac{2p^2 t}{k} \sum_{j=1}^{k} \frac{1}{2j - 1} - 4p^2 t \sum_{j=1}^{k} \frac{1}{(2j - 1)^2} \pmod{p^4}. \]
These complete the proof of Lemma 2.1. \[ \square \]

**Proof of Theorem 1.1.** Set \( S_n(a) = \sum_{k=1}^{n} \frac{1}{k} \binom{a}{k} (-1)^k \binom{a - 1}{k} \), then in view of [10, (2.1)], we have \( S_n(a) - S_n(a - 1) = -\frac{2}{a} + \frac{a - 1}{a} (-1)^{a - 1} \). Thus, by Lemma 2.1, we have
\[ S_{p-1}(a) - S_{p-1}(a - \langle a \rangle_p) = \sum_{k=0}^{\langle a \rangle_p - 1} (S_{p-1}(a - k) - S_{p-1}(a - k - 1)) \]
\[ = 2 \sum_{k=0}^{\langle a \rangle_p - 1} \left( \frac{-1}{a - k} + \frac{1}{a - k} \binom{a - k - 1}{p - 1} \binom{-a + k - 1}{p - 1} \right) \]
\[ = 2 \sum_{k=1}^{\langle a \rangle_p} \left( \frac{-1}{pt + k} + \frac{1}{pt + k} \binom{pt + k - 1}{p - 1} \binom{-pt - k - 1}{p - 1} \right) \]
\[ = 2 \sum_{k=1}^{\langle a \rangle_p} \frac{-1}{(pt + k)^3} + 2p^2 t (t + 1) \left( H(3; \langle a \rangle_p) - (3pt + p) H(4; \langle a \rangle_p) \right) \]
Similarly, by Lemma 2.1, we have

\[ S_{p - 1}(a) - S_{p - 1}(a - \langle a \rangle_p) = \sum_{k=0}^{(a)_p} (S_{p - 1}(a - k) - S_{p - 1}(a - k - 1)) \]

\[ = 2 \sum_{k=0}^{(a)_p} \left( \frac{-1}{a - k} + \frac{1}{a - k} \left( \frac{a - k - 1}{p - 1} \right) \left( \frac{-a + k - 1}{p - 1} \right) \right) \]

\[ = 2 \sum_{k=1}^{(a)_p} \left( \frac{-1}{pt + k} + \frac{1}{pt + k} \left( \frac{pt + k - 1}{p - 1} \right) \left( \frac{-pt - k - 1}{p - 1} \right) \right) \]

\[ \equiv 2 \sum_{k=1}^{(a)_p} \frac{-1}{pt + k} + 2ptH(2; \langle a \rangle_p) - 4p^2t^2H(3; \langle a \rangle_p) + 6p^3t^3H(4; \langle a \rangle_p) \]

\[ + 4p^2t \sum_{k=1}^{(a)_p} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{2j - 1} - 8p^3t^2 \sum_{k=1}^{(a)_p} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{2j - 1} \]

\[ + 4p^3t \sum_{k=1}^{(a)_p} \frac{1}{k^2} \left( \sum_{j=1}^{k} \frac{1}{2j - 1} \right)^2 - 8p^3t^2 \sum_{k=1}^{(a)_p} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j - 1)^2} \] (mod \( p^4 \)).

\[ (2.2) \]

**Lemma 2.2.** Let \( p > 5 \) be a prime, \( t = \frac{a - \langle a \rangle_p}{p} \in \mathbb{Z}_p \) and set \( X \) as above in (vii). Then

\[ S_{p - 1}(a - \langle a \rangle_p) = S_{p - 1}(pt) \equiv 4p^2tX \) (mod \( p^4 \)),

\[ S_{p - 1}(a - \langle a \rangle_p) = S_{p - 1}(pt) \equiv -12p^2t^2X + 14p^2tX \) (mod \( p^4 \)).

**Proof.** It is easy to see that

\[ \begin{pmatrix} pt \\ k \end{pmatrix} \begin{pmatrix} -1 - pt \\ k \end{pmatrix} = \frac{pt(pt + k)(-1)^{k} \prod_{j=1}^{k-1} (p^2t^2 - j^2)}{k!^2} \]

\[ \equiv - \frac{pt}{k} \left( 1 + \frac{pt}{k} \right) (1 - p^2t^2H(2; k - 1)) \) (mod \( p^4 \)).

Thus, by (i), (vii) and (viii), we have

\[ S_{p - 1}(pt) = \sum_{k=1}^{p - 1} \frac{1}{k} \begin{pmatrix} pt \\ k \end{pmatrix} \begin{pmatrix} -1 - pt \\ k \end{pmatrix} \]
\[ \equiv -ptH(2; p - 1) - p^2 t^2 H(3; p - 1) + p^3 t^3 H(2, 2; p - 1) \]
\[ \equiv 4p^2 tX \pmod{p^4}. \]

Similarly, by the identity \(2H(2, 2; \frac{p-1}{2}) = H(2; \frac{p-1}{2})^2 - H(4; \frac{p-1}{2})\), (ii) and (viii), we have
\[ S_{\frac{p-1}{2}}(pt) \]
\[ \equiv -ptH \left(2; \frac{p-1}{2} \right) - p^2 t^2 H \left(3; \frac{p-1}{2} \right) + p^3 t^3 H \left(2, 2; \frac{p-1}{2} \right) \]
\[ \equiv -12p^2 t^2 X + 14p^2 tX \pmod{p^4}. \]

These prove Lemma 2.2. \(\square\)

**Lemma 2.3.** For all prime \(p > 5\), \(h_{31} := H(3, 1; \frac{p-1}{2})\), and set \(X\) as above. Then
\[ \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{2j - 1} \equiv -\frac{21}{2} X + 2pq(2)B_{p-3} - \frac{p}{2} h_{31} \pmod{p^2}, \] (2.3)
\[ \sum_{k=1}^{\frac{p-1}{2}} \frac{H_k}{k^3} \equiv 4q(2)B_{p-3} - h_{31} \pmod{p}, \] (2.4)
\[ \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{(2j - 1)^2} \]
\[ \equiv \frac{21}{4} X - pq(2)B_{p-3} + pH(1, -3; p - 1) + \frac{p}{4} h_{31} \pmod{p^2}. \] (2.5)

Proof. It is easy to see that
\[ \sum_{k=1}^{\frac{p-1}{2}} \frac{H_{2k}}{k^2} = 2 \sum_{k=1}^{\frac{p-1}{2}} \frac{(1 + (-1)^k) H_k}{k^2} \]
= \(2(H(1, 2; p - 1) + H(3; p - 1) + H(1, -2; p - 1) + H(-3; p - 1))\).

So in view of (i) and (vii), we have
\[ \sum_{k=1}^{\frac{p-1}{2}} \frac{H_{2k}}{k^2} \equiv -9X \pmod{p^2}. \]

It is easy to see that
\[ H \left(1, 2; \frac{p-1}{2} \right) + H \left(3; \frac{p-1}{2} \right) = \sum_{k=1}^{\frac{p-1}{2}} \frac{H_k}{k^2} = \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{j} \]
\[
= \sum_{j=1}^{p-1} \frac{1}{j} \sum_{k=j}^{p-1} \frac{1}{k^2} = H_{p-1} \left( 2; \frac{p-1}{2} \right) - H \left( 2, 1; \frac{p-1}{2} \right).
\]

This with (ii) and (viii) yields that
\[
H \left( 1, 2; \frac{p-1}{2} \right) + H \left( 2, 1; \frac{p-1}{2} \right)
= H_{p-1} \left( 2; \frac{p-1}{2} \right) - H \left( 3; \frac{p-1}{2} \right)
\equiv -\frac{14}{3} pq_p(2) B_{p-3} - 12X \pmod{p^2}.
\]

In view of [15, (112)-(115)], we have
\[
H \left( 1, 2; \frac{p-1}{2} \right) - H \left( 2, 1; \frac{p-1}{2} \right)
\equiv -6X - \frac{10}{3} pq_p(2) B_{p-3} + 2ph_{31} \pmod{p^2}.
\]

Thus,
\[
H \left( 1, 2; \frac{p-1}{2} \right) \equiv -9X - 4pq_p(2) B_{p-3} + ph_{31} \pmod{p^2}
\]
and
\[
H \left( 2, 1; \frac{p-1}{2} \right) \equiv -3X - \frac{2}{3} pq_p(2) B_{p-3} - ph_{31} \pmod{p^2}. \tag{2.6}
\]

So by (viii), we have
\[
\sum_{k=1}^{p-1} \frac{H_k}{k^2} = H \left( 1, 2; \frac{p-1}{2} \right) + H \left( 3; \frac{p-1}{2} \right)
\equiv 3X - 4pq_p(2) B_{p-3} + ph_{31} \pmod{p^2}.
\]

Hence
\[
\sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{2j-1} = \sum_{k=1}^{p-1} \frac{H_{2k} - \frac{1}{2} H_k}{k^2}
\equiv -\frac{21}{2} X + 2pq_p(2) B_{p-3} - \frac{p}{2} h_{31} \pmod{p^2}.
\]

Similarly,
\[
\sum_{k=1}^{p-1} \frac{H_k}{k^3} = H_{p-1} \left( 3; \frac{p-1}{2} \right) - H \left( 3, 1; \frac{p-1}{2} \right)
\]
\[ \equiv 4q_p(2)B_{p-3} - h_{31} \pmod{p} \]

and

\[
\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = \sum_{k=1}^{p-1} \frac{1 + (-1)^k}{k} \frac{H(2; k) - 1}{p} \sum_{k=1}^{p-1} \frac{H(2; k)}{k}
\]

\[ = H(2, 1; p - 1) + H(3; p - 1) + H(2, -1; p - 1) + H(-3; p - 1)
\]

\[ - \frac{1}{4} H\left(2, 1; \frac{p-1}{2}\right) - \frac{1}{4} H\left(3; \frac{p-1}{2}\right). \]

By (2.6), (i), (vii) and (viii), we have

\[
\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \equiv \frac{21}{4} X - pq_p(2)B_{p-3} + pH(1, -3; p - 1) + \frac{p}{4} h_{31} \pmod{p^2}. \]

These complete the proof of Lemma 2.3. □

**Lemma 2.4.** Let \( p > 5 \) be a prime. Then

\[ H\left(1, 3; \frac{p-1}{2}\right) \equiv 4H(1, -3; p - 1) \pmod{p}. \]

**Proof.** In view of [1, p. 804, 23.1.4-7] or [15, (7)], and by Fermat’s Little Theorem, we have

\[
H\left(1, 3; \frac{p-1}{2}\right) = \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{j} \equiv \sum_{k=1}^{p-1} \frac{1}{k^3} \sum_{j=1}^{k-1} j^{p-2}
\]

\[ = \sum_{r=0}^{p-2} \binom{p-1}{r} \frac{B_r}{p-1} \sum_{k=1}^{p-1} k^{p-4-r} \equiv \sum_{r=0}^{p-6} \binom{p-1}{r} \frac{B_r}{p-1} \sum_{k=1}^{p-1} k^{p-4-r}
\]

\[ + \sum_{r=p-5}^{p-2} \binom{p-1}{r} \frac{B_r}{p-1} \sum_{k=1}^{p-1} k^{p-4-r} \pmod{p}. \]

By Fermat’s Little Theorem again, (ii) and \( B_n = 0 \) for all odd \( n > 1 \), we have

\[
\sum_{r=0}^{p-6} \binom{p-1}{r} \frac{B_r}{p-1} \sum_{k=1}^{p-1} k^{p-4-r} \equiv \sum_{r=0}^{p-6} (-1)^r + 1 B_r \sum_{k=1}^{p-1} \frac{1}{r+3}
\]

\[ \equiv \sum_{r=0}^{p-6} (-1)^r B_r \frac{2^{r+3} - 2}{r+3} \equiv \sum_{r=0}^{p-6} \frac{2^{r+3} - 2}{r+3} B_{p-3-r} \pmod{p}. \]
and since $p - 2 > 1$ and $p - 4 > 1$ are odd,

$$
\sum_{r=p-5}^{p-2} \binom{p-1}{r} \frac{B_r}{p-1} \sum_{k=1}^{p-1} k^{p-4-r}
= \frac{p-1}{p-3} B_{p-3} \frac{H_{p-1}}{r} + \frac{p-1}{p-5} B_{p-5} \sum_{k=1}^{p-1} k
\equiv 2q_p(2)B_{p-3} + \frac{1}{8}B_{p-5} \pmod{p}.
$$

Thus, with $B_2 = 1/6$, we have

$$
H \left( 1, 3; \frac{p-1}{2} \right) \equiv \sum_{r=0}^{p-6} B_r \frac{2^{r+3}-2}{r+3} B_{p-3-r} + 2q_p(2)B_{p-3} + \frac{1}{8}B_{p-5}
\equiv 2 \sum_{r=2}^{p-3} \frac{1 - 2^{p-1-r}}{r} B_r B_{p-3-r} + 2q_p(2)B_{p-3} \pmod{p}.
$$

In view of [15, pp. 14], we have the following congruences modulo $p$,

$$
H(-1, 3; p - 1) \equiv 2 \sum_{r=2}^{p-3} (1 - 2^r)(1 - 2^{p-3-r}) \frac{B_r B_{p-3-r}}{r} - 2q_p(2)B_{p-3}
\equiv 2 \sum_{r=2}^{p-3} 1 - 2^r B_r B_{p-3-r} + \frac{1}{2} \sum_{r=2}^{p-3} 1 - 2^{p-1-r} \frac{B_r B_{p-3-r}}{r} - 2q_p(2)B_{p-3}
$$

and

$$
H(1, -3; p - 1) \equiv -2 \sum_{r=2}^{p-3} \frac{1 - 2^r}{r} B_r B_{p-3-r} + 2q_p(2)B_{p-3}.
$$

These with [15, (55)] yield that

$$
H \left( 1, 3; \frac{p-1}{2} \right) \equiv 4 \left( H(-1, 3; p - 1) + H(1, -3; p - 1) \right) + 2q_p(2)B_{p-3}
\equiv 4H(1, -3; p - 1) \pmod{p}.
$$

This proves Lemma 2.4. \qed
Lemma 2.5. For any prime $p > 5$, we have the following modulo $p$

\[ \sum_{k=1}^{p-1} \frac{1}{k^2} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 + \sum_{k=1}^{p-1} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{2j-1} + \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \equiv 0. \]

Proof. We can find and prove the following identity by \text{Sigma} [6],

\[ \sum_{k=1}^{n} \frac{(-4)^k}{k^2 \binom{2k}{k}} \sum_{j=1}^{k} \frac{1}{2j-1} = -2 \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}. \]

Set $n = (p - 1)/2$ in the above identity and by [12, Lemma 4.2], we have

\[ -2 \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = \sum_{k=1}^{p-1} \frac{(-4)^k}{k^2 \binom{2k}{k}} \sum_{j=1}^{k} \frac{1}{2j-1} \]

\[ \equiv \sum_{k=1}^{p-1} \frac{1}{k^2} \left( 1 - p \sum_{j=1}^{k} \frac{1}{2j-1} \right) \sum_{j=1}^{k} \frac{1}{2j-1} \]

\[ = \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{2j-1} - p \sum_{k=1}^{p-1} \frac{1}{k^2} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 \pmod{p^2}. \]

This, with (2.3) and (2.5) yields that

\[ \sum_{k=1}^{p-1} \frac{1}{k^2} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 \equiv 2H(1, -3; p - 1) \pmod{p}. \] (2.7)

It is easy to see that

\[ \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} = 2H(2; 2; p - 1) + 2H(4; p - 1) + 2H(2, -2; p - 1) \]

\[ + 2H(-4; p - 1) - \frac{1}{4} H \left( 2, 2; \frac{p-1}{2} \right) - \frac{1}{4} H \left( 4, \frac{p-1}{2} \right). \]

So in view of (i), (ii), (vii), [15, (60)] and

\[ H \left( 2, 2; \frac{p-1}{2} \right) = \frac{1}{2} \left( H \left( 2; \frac{p-1}{2} \right)^2 - H \left( 4; \frac{p-1}{2} \right) \right) \equiv 0 \pmod{p}, \]

we have

\[ \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \equiv -2H(-2; 2; p - 1) \pmod{p}. \] (2.8)
Similarly, by (i), (ii) and (vii), we have
\[
\sum_{k=1}^{\frac{p-1}{2}} \sum_{j=1}^{k} \frac{1}{2j-1} = 4H(1, 3; p-1) + 4H(4; p-1) + 4H(1, -3; p-1) \\
+ 4H(-4; p-1) - \frac{1}{2} H \left( 1, 3; \frac{p-1}{2} \right) - \frac{1}{2} H \left( 4; \frac{p-1}{2} \right) \\
\equiv 4H(1, -3; p-1) - \frac{1}{2} H \left( 1, 3; \frac{p-1}{2} \right) \pmod{p}.
\]
(2.9)

These, with Lemma 2.4, [15, (61) and (62)] yield that
\[
\sum_{k=1}^{\frac{p-1}{2}} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 + \sum_{k=1}^{\frac{p-1}{2}} \sum_{j=1}^{k} \frac{1}{(2j-1)^2} + \sum_{k=1}^{\frac{p-1}{2}} \sum_{j=1}^{k} \frac{1}{2j-1} \\
\equiv 2H(1, -3; p-1) - 2H(-2, 2; p-1) + 4H(1, -3; p-1) - \frac{1}{2} H \left( 1, 3; \frac{p-1}{2} \right) \\
\equiv -2H(-2, 2; p-1) + 4H(1, -3; p-1) \equiv 0 \pmod{p}.
\]
Now the proof of Lemma 2.5 is complete.

In view of (2.1), (2.2) and Lemma 2.2, we have
\[
S_{p-1}(a) \equiv 4p^2 t X + 2 \sum_{k=1}^{\langle a \rangle_p} \frac{-1}{pt+k} + 2p^2 t(t+1) \left( H(3; \langle a \rangle_p) \\
- (3pt+p)H(4; \langle a \rangle_p) + 2p \sum_{k=1}^{\langle a \rangle_p} \frac{H_k}{k^3} \right) \pmod{p^4}.
\]
and
\[
S_{\frac{p-1}{2}}(a) \\
\equiv -12p^2 t^2 X + 14p^2 t X + 2 \sum_{k=1}^{\langle a \rangle_p} \frac{-1}{pt+k} + 2pt H(2; \langle a \rangle_p) - 4p^2 t^2 H(3; \langle a \rangle_p) \\
+ 6p^3 t^3 H(4; \langle a \rangle_p) + 4p^2 t \sum_{k=1}^{\langle a \rangle_p} 1 \sum_{j=1}^{k} \frac{1}{2j-1} - 8p^3 t^2 \sum_{k=1}^{\langle a \rangle_p} 1 \sum_{j=1}^{k} \frac{1}{2j-1} \\
+ 4p^3 t \sum_{k=1}^{\langle a \rangle_p} \frac{1}{k^2} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 - 8p^3 t^2 \sum_{k=1}^{\langle a \rangle_p} 1 \sum_{j=1}^{k} \frac{1}{(2j-1)^2} \pmod{p^4}.
\]
Set $a = -1/2$, then $t = -1/2$, thus by (ii), (viii), Lemmas 2.3 and 2.5, we have

$$\sum_{k=1}^{p-1} \frac{(2k)^2}{k16^k} = S_{p-1} \left( -\frac{1}{2} \right) \equiv -2H_{\frac{p-1}{2}} - p^3 \sum_{k=1}^{\frac{p-1}{2}} \frac{H_k}{k^3} \pmod{p^4}$$

and

$$\sum_{k=\frac{p-1}{2}}^{p-1} \frac{(2k)^2}{k16^k} = S_{p-1} \left( -\frac{1}{2} \right) - S_{\frac{p-1}{2}} \left( -\frac{1}{2} \right) \equiv -21p^2 X + 2p^3 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2} \left( \sum_{j=1}^{k} \frac{1}{2j-1} \right)^2 + 2p^3 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}$$

$$+ 2p^3 \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{2j-1} \equiv -21p^2 X \equiv -\frac{21}{2} H_{p-1} \pmod{p^4}.$$
It is easy to check that
\[
\sum_{k=1}^{n} \frac{(\binom{n}{k}(-4)^k)}{k^2(2k)} = -2 \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{2j-1}.
\]

It is easy to check that
\[
-2 \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{2j-1} = H(1, 1; \frac{p-1}{2}) + H(2; \frac{p-1}{2}) - 2H(1, 1; p-1) - 2H(2; p-1) - 2H(1, -1; p-1) - 2H(-2; p-1).
\]
By [12, Lemma 4.2], we have
\[
\sum_{k=1}^{\nu+1} \frac{(\frac{-1}{k})}{k^2 \binom{2k}{k}}(-4)^k \equiv \sum_{k=1}^{\nu+1} \frac{1}{k^2} \left(1 - p \sum_{j=1}^{k} \frac{1}{2j-1} + \frac{p^2}{2} \left(\frac{1}{j+1} + \frac{1}{j-1}\right)\right)^2 \mod (p^3).
\]

This, with the identity \( H(1, 1; n) = \frac{1}{2}(H_n^2 - H(2; n)) \) yields that
\[
\frac{p^2}{2} \left(\sum_{k=1}^{\nu+1} \frac{1}{k^2} \left(\sum_{j=1}^{k} \frac{1}{2j-1}\right)^2 + \sum_{k=1}^{\nu+1} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{2j-1} + \sum_{k=1}^{\nu+1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}\right)
\equiv \frac{1}{2} H_{\nu+1}^2 - \frac{1}{2} H \left(2; \frac{p-1}{2}\right) - H_{p-1}^2 - H(2; p-1) - 2H(-2; p-1)
\]
\[
-2H(1, -1; p-1) + p \sum_{k=1}^{\nu+1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{2j-1} + \frac{p^2}{2} \sum_{k=1}^{\nu+1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}
\]
\[
+ \frac{p^2}{2} \sum_{k=1}^{\nu+1} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{2j-1} \mod (p^3).
\]

By (viii), we have
\[
H(-2; p-1) = \sum_{k=1}^{\nu-1} \frac{(-1)^k + 1}{k^2} - H(2; p-1)
= \frac{1}{2} H \left(2; \frac{p-1}{2}\right) - H(2; p-1) \equiv -3pX \mod (p^3).
\]

These, with (i), (viii), (2.8)-(2.9), Lemma 2.3 and [15, (59),(62)] yield that
\[
\frac{p^2}{2} \left(\sum_{k=1}^{\nu+1} \frac{1}{k^2} \left(\sum_{j=1}^{k} \frac{1}{2j-1}\right)^2 + \sum_{k=1}^{\nu+1} \frac{1}{k^3} \sum_{j=1}^{k} \frac{1}{2j-1} + \sum_{k=1}^{\nu+1} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{(2j-1)^2}\right)
\equiv \frac{1}{2} H_{\nu+1}^2 + \frac{13}{2} pX - 2H(1, -1; p-1) + p^2q_{p}(2)B_{p-3} - \frac{1}{4} p^2 h_{31}
\]
\[
-2p^2 H(1, -3; p-1) \mod (p^3).
\]

(3.1)

It is easy to see that
\[
-2q_{p}(2) = \sum_{k=1}^{\nu-1} \frac{(-1)^k}{k} \left(\frac{p-1}{k}\right) \equiv \sum_{k=1}^{\nu-1} \frac{(-1)^k}{k} \left(1 - pH(1; k-1)\right)
\]
\[ + p^2 H(1, 1; k - 1) - p^3 H(1, 1, 1; k - 1) \]
\[ = H_{p-1} - H_{p-1} - pH(1, -1; p - 1) + p^2 H(1, 1, -1; p - 1) \]
\[ - p^3 H(1, 1, 1, -1; p - 1) \pmod{p^4}. \]

In view of [15, (87) and Remark 7.7], we have

\[ H(1, 1, -1; p - 1) \equiv \sum_{k=1}^{p-1} \frac{2^k}{k^3} + pH(1, 1, 1, -1; p - 1) - \frac{7}{12} pq_p(2) B_{p-3} \]
\[ + pH(1, -3; p - 1) \pmod{p^2}. \]

These, with (viii) yield that

\[ H(1, -1; p - 1) \equiv \frac{H_{p-1}}{2} + \frac{2q_p(2)}{p} + p \sum_{k=1}^{p-1} \frac{2^k}{k^3} - \frac{7}{12} p^2 q_p(2) B_{p-3} \]
\[ + p^2 H(1, -3; p - 1) - 2pX \pmod{p^3}. \]

Thus, in view of (3.1) and Lemma 2.5, we have

\[ \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv \frac{1}{4p} H_{p-1}^2 + \frac{21}{4} X - \frac{H_{p-1}^2 + 2q_p(2)}{p^2} + \frac{13}{12} pq_p(2) B_{p-3} \]
\[ - 2pH(1, -3; p - 1) - \frac{1}{8} ph_{31} \pmod{p^2}. \]

Therefore, by Lemmas 2.3-2.4 and 3.1, we immediately obtain the desired result

\[ \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p^3(2) + \frac{7}{4} X + \frac{5}{12} pq_p^4(2) + \frac{7p}{6} q_p(2) B_{p-3} - \frac{3p}{8} \sum_{k=1}^{p-1} \frac{H_k}{k^3} \pmod{p^2}. \]

Now the proof of Theorem 1.2 is complete. \( \square \)

References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, (1972) Dover, New York.

[2] M. E. Hoffman, Quasi-symmetric functions and mod p multiple harmonic sums, Kyushu. J. Math. 69 (2015), 345–366.

[3] Kh. Hessami Pilehrood, T. Hessami Pilehrood and R. Tauraso, Congruences concerning Jacobi polynomials and Apéry-like formulae, Int. J. Number Theory 8 (2012), no.7, 1789–1811.

[4] K. Ireland, M. Rosen, A classical Introduction to Modern Number Theory, Springer, New York, 1982, pp.239–248.

[5] S. Mattarei and R. Tauraso, Congruences for central binomial sums and finite polylogarithms, J. Number Theory 133 (2013), 131–157.
[6] C. Schneider, *Symbolic summation assists combinatorics*, Sém. Lothar. Combin. 56 (2007), Article B56b.

[7] Z.-H. Sun, *Congruences for Bernoulli numbers and Bernoulli polynomials*, Discrete Math. 163 (1997), 153–163.

[8] Z.-H. Sun, *congruences concerning Bernoulli numbers and Bernoulli Polynomials*, Discrete Appl. Math. 105 (2000), 193–223.

[9] Z.-H. Sun, *Congruences involving Bernoulli and Euler numbers*, J. Number Theory 128 (2008), no.2, 280–312.

[10] Z.-H. Sun, *Super congruences concerning Bernoulli polynomials*, Int. J. Number Theory 11 (2015), no.8, 2393–2404.

[11] Z.-W. Sun, *Super congruences and Euler numbers*, Sci. China Math. 54 (2011), 2509–2535.

[12] Z.-W. Sun, *A new series for $\pi^3$ and related congruences*, Internat. J. Math. 26 (2015), no.8, 1550055 (23 pages).

[13] Z.-W. Sun, *p-adic congruences motivated by series*, J. number Theory 134 (2014), no.1, 181–196.

[14] R. Tauraso, *Supercongruences for a truncated hypergeometric series*, Integers. 12 (2012), #A45, 12pp (electrobic).

[15] R. Tauraso and J.Q. Zhao, *Congruences of alternating multiple harmonic sums*, J. Combin. Number Theory 2 (2010), 129–159.

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