Perturbed Einstein field equations using Maple

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Abstract

We obtain the perturbed components of affine connection and Ricci tensor using algebraic computation. Naturally, the perturbed Einstein field equations for the vacuum can be written. The method can be used to obtain perturbed equations of the superior order.

Key words -tensorial perturbations, algebraic computation.

1 Introduction

The description of the observed universe is considered an homogeneous and isotropic background with inhomogeneities. For large scale the smoothness of the universe is described by the Friedmann model, while the small scale structure are considered as small deviations from homogeneity. These small deviations can be treated in the Newtonian or General Relativity framework. The first procedure is applicable to the scales where the effects of the space-time curvature can be negligible. Although, the Newtonian analysis provides insight into the behavior, the involved scales are smaller than the Hubble radius.

The relativistic treatment of cosmological perturbations is necessary since that all the relevant scales involved stay outside of the horizon at early time.

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Therefore, there is a narrow relation among the cosmological perturbations in the framework of the general relativity and structure formation [1].

In another hand, a new branch in astronomy can emerge in a next future, gravitational radiation astronomy. The gravitational radiation is considered as tensorial perturbations produced from the cosmological background oscillations or by pulsation of astrophysical objects (black hole, strange stars, neutron stars)[2], [3], [4], [5], [6]. So, the perturbative analyses is important for the study of large scale structure in the universe (scalar perturbation) and stellar objects (tensorial perturbation).

Our interest in this work is to obtain the perturbed field equations taking into account small perturbations in the metric tensor, using algebraic computation. More explicitly, Maple software and GRtensor package. With the help of algebraic computation we can not only to make a check up of the published perturbed equations, but also to obtain new results to superior order than one.

The perturbed Einstein field equations have been deduced in the literature in several occasions [5], [6], [7], [8], [9], [10], [11]. Generally, to obtain these equations expend a considerable quantity of work. We give a maple worksheet that calculates the perturbed tensorial components of the Ricci tensor. Consequently, this enable us to write the perturbed field equations for the vacuum.

\section{Basic equations}

The background considered is homogeneous and isotropic, given by [5]

\[ ds^2 = -dt^2 + R(t)^2(dx_1^2 + dx_2^2 + dx_3^2). \]  

(1)

Consequently, the non vanishing affine connections are

\[ \Gamma^t_{ij} = R(t)\dot{R}(t)\delta_{ij} \]  

(2)

\[ \Gamma^i_{tj} = \frac{R(t)}{\dot{R}(t)}\delta_{ij}, \]

where the dot means time derivative and the latin indices assumes the values 1,2,3.

The disturbance in the metric is defined by \( \hat{g}_{\mu\nu} = h_{\mu\nu} \), with \( h_{\mu\nu} \) small. We use a hat to denote the perturbations. So, the perturbed affine connections
are given by \[5\]

\[
\hat{\Gamma}_{jk}^i = \frac{1}{2R(t)^2} \{ h_{ij;k} + h_{ik;j} - h_{jk;i} \}, \\
\hat{\Gamma}_{jk}^t = \frac{-1}{2} \{ h_{tj;k} + h_{tk;j} - h_{jk;t} \}, \\
\hat{\Gamma}_{ij}^t = \frac{1}{2R(t)^2} \{ h_{it;j} + h_{ij;t} - h_{tj;i} \}.
\]

The semicolon denotes the covariant derivative. The perturbed Ricci tensor can be obtained using Palatini formula:

\[
\hat{R}_{\mu\kappa} = (\hat{\Gamma}_{\mu\lambda})_{;\kappa} - (\hat{\Gamma}_{\mu\kappa})_{;\lambda}.
\] (3)

Consequently, the perturbed Einstein field equations \[5\] can be written as:

\[
\hat{R}_{\mu\nu} = -8\pi G \hat{S}_{\mu\nu},
\] (4)

where

\[
S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\lambda}^\lambda,
\] (5)

and \(T_{\mu\nu}\) is the energy-momentum tensor.

Generally, to obtain the perturbed high side of the equation (4) is straightforward and this do not expend a lot quantity of time. Otherwise, the same do not happens with the left hand where it is necessary a considerable quantity of attention and time.

### 3 Using maple and the grtensor

We use the GRtensor package version 1.77 (R4) and maple release VII for the calculus. In this section we summarize the commands that we must insert in the prompt of maple to obtain the perturbed tensorial quantities, namely:

\[
> \text{grtw();}
\]
This first input run the Grtensor package.

\[
> \text{makeg(cod)};
\]
The last input is necessary to give a name for the spacetime metric (I call by cod) of the background, that we will define in the next lines.

Now we must provide the data of the background.
Do you wish enter a:
1) metric \([g(dn,dn)]\),
2) line element \([ds]\),
3) non-holonomic basis \([e(1)...e(n)]\), or
4) NP tetrad \([l,n,m,mbar]\)?

> 1;

Enter with the coordinates

\[ \{t,x1,x2,x3\} \]

Is the metric
1) Diagonal, or 2) Symmetric?

> 1;

Now, define the components of the metric tensor

Enter \(g[t,t]\):

> -1;

Enter \(g[x1,x1]\):

> \(R(t)^2\);

Enter \(g[x2,x2]\):

> \(R(t)^2\);

Enter \(g[x3,x3]\):

> \(R(t)^2\);

In this point the GRtensor inquires about the existence of the complex variables, if you do not have complex variables in the spacetime than write \(\{\}\) and press ENTER.

The next step given us the last opportunity in this maple section to modify the background data.

0) Use the metric WITHOUT saving it,
1) Save the metric as it is,
2) Correct an element of the metric,
3) Re-enter the metric,
4) Add/change constraint equations,
5) Add a text description, or
6) Abandon this metric and return to Maple.

> 0;

Now, the definition of the background is finish.

\(\text{grcalc}(\text{Einstein})\);

Note that, before we defining the perturbed quantities we calculate the components of the Einstein tensor for the background in the last input. With
this procedure we have calculated all important tensorial quantities of order zero, for any eventuality.

The first step is to define the perturbations in the metric tensor.

\[ g_{a\ b} := h_{11}(t, x_1, x_2, x_3) \delta_{a\ x_1} \delta_{b\ x_1} + h_{12}(t, x_1, x_2, x_3) \delta_{a\ x_1} \delta_{b\ x_2} + h_{13}(t, x_1, x_2, x_3) \delta_{a\ x_1} \delta_{b\ x_3} + h_{12}(t, x_1, x_2, x_3) \delta_{a\ x_2} \delta_{b\ x_1} + h_{13}(t, x_1, x_2, x_3) \delta_{a\ x_2} \delta_{b\ x_3} + h_{23}(t, x_1, x_2, x_3) \delta_{a\ x_3} \delta_{b\ x_2} + h_{13}(t, x_1, x_2, x_3) \delta_{a\ x_3} \delta_{b\ x_1} + h_{23}(t, x_1, x_2, x_3) \delta_{a\ x_3} \delta_{b\ x_2} + h_{33}(t, x_1, x_2, x_3) \delta_{a\ x_3} \delta_{b\ x_3} \]

The next input define the perturbed components of the affine connection

\[ \Gamma^a_{\ bc} := -g^{\ cd} g_{a\ d} \mathrm{Chr}_{\ bc} \]

Finally, we define the perturbed components of the Ricci tensor.

\[ R_{a\ b} := \Gamma^e_{\ a\ b\ e} - \Gamma^e_{\ a\ e\ b} \]

In the definitions of perturbations in the affine connection and in the Ricci tensor components we use \( \Gamma^a_{\ bc} \) and \( R_{ab} \), respectively. In the other sections we maintain the perturbations denoted by a hat.

4 Perturbed field equations

In this section we write the results obtained for the perturbated quantities.

4.1 The metric perturbations

\[
\begin{pmatrix}
0 & h_{11}(t, x_1, x_2, x_3) & h_{12}(t, x_1, x_2, x_3) & h_{13}(t, x_1, x_2, x_3) & 0 \\
0 & h_{12}(t, x_1, x_2, x_3) & h_{22}(t, x_1, x_2, x_3) & h_{23}(t, x_1, x_2, x_3) & 0 \\
0 & h_{13}(t, x_1, x_2, x_3) & h_{23}(t, x_1, x_2, x_3) & h_{33}(t, x_1, x_2, x_3) & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
4.2 Perturbed affine connections

The components of the perturbed affine connections different from zero are given below. Note that we do not write the perturbed components that can be obtained by symmetry properties.

\[
\begin{align*}
\hat{\Gamma}_{x_1x_1}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{11} \right) \\
\hat{\Gamma}_{x_1x_2}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{12} \right) \\
\hat{\Gamma}_{x_1x_3}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{13} \right) \\
\hat{\Gamma}_{x_2x_1}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{21} \right) \\
\hat{\Gamma}_{x_2x_2}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{22} \right) \\
\hat{\Gamma}_{x_2x_3}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{23} \right) \\
\hat{\Gamma}_{x_3x_1}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{31} \right) \\
\hat{\Gamma}_{x_3x_2}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{32} \right) \\
\hat{\Gamma}_{x_3x_3}^t &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{33} \right)
\end{align*}
\]

\[
\begin{align*}
\hat{\Gamma}_{x_1x_1}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{11} \right) \\
\hat{\Gamma}_{x_1x_2}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{12} \right) \\
\hat{\Gamma}_{x_1x_3}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{13} \right) \\
\hat{\Gamma}_{x_2x_1}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{21} \right) \\
\hat{\Gamma}_{x_2x_2}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{22} \right) \\
\hat{\Gamma}_{x_2x_3}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{23} \right) \\
\hat{\Gamma}_{x_3x_1}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{31} \right) \\
\hat{\Gamma}_{x_3x_2}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{32} \right) \\
\hat{\Gamma}_{x_3x_3}^{x_1} &= \frac{1}{2} \left( \frac{\partial}{\partial x_1} h_{33} \right)
\end{align*}
\]

\[
\begin{align*}
\hat{\Gamma}_{x_1x_1}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{11} \right) \\
\hat{\Gamma}_{x_1x_2}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{12} \right) \\
\hat{\Gamma}_{x_1x_3}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{13} \right) \\
\hat{\Gamma}_{x_2x_1}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{21} \right) \\
\hat{\Gamma}_{x_2x_2}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{22} \right) \\
\hat{\Gamma}_{x_2x_3}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{23} \right) \\
\hat{\Gamma}_{x_3x_1}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{31} \right) \\
\hat{\Gamma}_{x_3x_2}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{32} \right) \\
\hat{\Gamma}_{x_3x_3}^{x_3} &= \frac{1}{2} \left( \frac{\partial}{\partial x_3} h_{33} \right)
\end{align*}
\]

\[
\begin{align*}
\hat{\Gamma}_{x_1x_1}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{11} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{11} \right) R(t) - 2 h_{11} \frac{\partial}{\partial t} h_{11} \frac{\partial}{\partial t} h_{11} \\
\hat{\Gamma}_{x_1x_2}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{12} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{12} \right) R(t) - 2 h_{12} \frac{\partial}{\partial t} h_{12} \frac{\partial}{\partial t} h_{12} \\
\hat{\Gamma}_{x_1x_3}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{13} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{13} \right) R(t) - 2 h_{13} \frac{\partial}{\partial t} h_{13} \frac{\partial}{\partial t} h_{13} \\
\hat{\Gamma}_{x_2x_1}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{21} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{21} \right) R(t) - 2 h_{21} \frac{\partial}{\partial t} h_{21} \frac{\partial}{\partial t} h_{21} \\
\hat{\Gamma}_{x_2x_2}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{22} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{22} \right) R(t) - 2 h_{22} \frac{\partial}{\partial t} h_{22} \frac{\partial}{\partial t} h_{22} \\
\hat{\Gamma}_{x_2x_3}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{23} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{23} \right) R(t) - 2 h_{23} \frac{\partial}{\partial t} h_{23} \frac{\partial}{\partial t} h_{23} \\
\hat{\Gamma}_{x_3x_1}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{31} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{31} \right) R(t) - 2 h_{31} \frac{\partial}{\partial t} h_{31} \frac{\partial}{\partial t} h_{31} \\
\hat{\Gamma}_{x_3x_2}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{32} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{32} \right) R(t) - 2 h_{32} \frac{\partial}{\partial t} h_{32} \frac{\partial}{\partial t} h_{32} \\
\hat{\Gamma}_{x_3x_3}^{t_1} &= \frac{1}{2} \left( \frac{\partial}{\partial t} h_{33} \right) + \frac{1}{2} R(t) \left( \frac{\partial}{\partial t} h_{33} \right) R(t) - 2 h_{33} \frac{\partial}{\partial t} h_{33} \frac{\partial}{\partial t} h_{33}
\end{align*}
\]
\[
\begin{align*}
\tilde{\Gamma}_{x_1 x_1}^{x_3} &= -\frac{1}{2} \left[ -\frac{2}{R(t)^2} (\frac{\partial}{\partial x_1} h_{11}) + (\frac{\partial}{\partial x_2} h_{11}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{11}) (\frac{\partial}{\partial t} R(t)) + 2 R(t) h_{11} (\frac{\partial^2}{\partial t^2} R(t)) \\
&- 2 h_{11} (\frac{\partial}{\partial t} R(t))^2 - (\frac{\partial^2}{\partial t^2} h_{22}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{22}) (\frac{\partial}{\partial t} R(t)) \\
&+ 2 R(t) h_{22} (\frac{\partial^2}{\partial t^2} R(t)) - 2 h_{22} (\frac{\partial}{\partial t} R(t))^2 - (\frac{\partial^2}{\partial t^2} h_{33}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{33}) (\frac{\partial}{\partial t} R(t)) \\
&+ 2 R(t) h_{33} (\frac{\partial^2}{\partial t^2} R(t)) - 2 h_{33} (\frac{\partial}{\partial t} R(t))^2 \right].
\end{align*}
\]

\[
\begin{align*}
\tilde{\Gamma}_{x_2 x_2}^{x_3} &= -\frac{1}{2} \left[ -\frac{2}{R(t)^2} (\frac{\partial}{\partial x_2} h_{11}) + (\frac{\partial}{\partial x_3} h_{11}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{11}) (\frac{\partial}{\partial t} R(t)) + 2 R(t) h_{11} (\frac{\partial^2}{\partial t^2} R(t)) \\
&- 2 h_{11} (\frac{\partial}{\partial t} R(t))^2 - (\frac{\partial^2}{\partial t^2} h_{22}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{22}) (\frac{\partial}{\partial t} R(t)) \\
&+ 2 R(t) h_{22} (\frac{\partial^2}{\partial t^2} R(t)) - 2 h_{22} (\frac{\partial}{\partial t} R(t))^2 - (\frac{\partial^2}{\partial t^2} h_{33}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{33}) (\frac{\partial}{\partial t} R(t)) \\
&+ 2 R(t) h_{33} (\frac{\partial^2}{\partial t^2} R(t)) - 2 h_{33} (\frac{\partial}{\partial t} R(t))^2 \right].
\end{align*}
\]

4.3 Perturbed Ricci components

The perturbed Ricci components different from zero are:

\[
\begin{align*}
\hat{R}_{xx} &= -\frac{1}{2R(t)^4} \left[ -\frac{2}{R(t)^2} (\frac{\partial^2}{\partial x_1^2} h_{11}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{11}) (\frac{\partial}{\partial t} R(t)) + 2 R(t) h_{11} (\frac{\partial^2}{\partial t^2} R(t)) \\
&- 2 h_{11} (\frac{\partial}{\partial t} R(t))^2 - (\frac{\partial^2}{\partial t^2} h_{22}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{22}) (\frac{\partial}{\partial t} R(t)) \\
&+ 2 R(t) h_{22} (\frac{\partial^2}{\partial t^2} R(t)) - 2 h_{22} (\frac{\partial}{\partial t} R(t))^2 - (\frac{\partial^2}{\partial t^2} h_{33}) R(t)^2 + 2 R(t) (\frac{\partial}{\partial t} h_{33}) (\frac{\partial}{\partial t} R(t)) \\
&+ 2 R(t) h_{33} (\frac{\partial^2}{\partial t^2} R(t)) - 2 h_{33} (\frac{\partial}{\partial t} R(t))^2 \right].
\end{align*}
\]
\begin{align*}
\hat{R}_{x_1 x_1} &= -\frac{1}{2R(t)^2} \left[ 2 h_{11} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{11} \right) R(t^2) - \frac{\partial}{\partial t} h_{22} + R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \right] \\
\hat{R}_{x_1 x_1} &= -\frac{1}{2R(t)^2} \left[ 2 h_{11} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{11} \right) R(t^2) - \frac{\partial}{\partial t} h_{22} + R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \right] \\
\hat{R}_{x_2 x_2} &= -\frac{1}{2R(t)^2} \left[ 2 h_{22} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{22} \right) R(t^2) - \frac{\partial}{\partial t} h_{11} \right] \\
\hat{R}_{x_3 x_3} &= -\frac{1}{2R(t)^2} \left[ 2 h_{23} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{23} \right) R(t^2) - \frac{\partial}{\partial t} h_{22} + R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \right] \\
\hat{R}_{x_3 x_3} &= -\frac{1}{2R(t)^2} \left[ 2 h_{33} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{33} \right) R(t^2) - \frac{\partial}{\partial t} h_{22} + R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \right] \\
\hat{R}_{x_2 x_2} &= -\frac{1}{2R(t)^2} \left[ 2 h_{22} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{22} \right) R(t^2) - \frac{\partial}{\partial t} h_{11} \right] \\
\hat{R}_{x_3 x_3} &= -\frac{1}{2R(t)^2} \left[ 2 h_{23} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{23} \right) R(t^2) - \frac{\partial}{\partial t} h_{22} + R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \right] \\
\hat{R}_{x_1 x_1} &= -\frac{1}{2R(t)^2} \left[ 2 h_{11} \frac{\partial}{\partial t} (R(t))^2 + \left( \frac{\partial}{\partial t} h_{11} \right) R(t^2) - \frac{\partial}{\partial t} h_{22} + R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \right].
\end{align*}
\[
+ R(t) \left( \frac{\partial}{\partial t} h_{22} \right) \left( \frac{\partial}{\partial t} R \right) - 2 h_{22} \left( \frac{\partial}{\partial t} R(t) \right)^2 + 2 \left( \frac{\partial^2}{\partial x_3 \partial x_2} h_{23} \right) - \left( \frac{\partial^2}{\partial x_2^2} h_{33} \right) \right].
\]

5 Conclusions and final remarks

The perturbed Einstein field equations have been deduced in the literature in several occasions. Generally, to obtain these equations expand a considerable quantity of time. This work intent give an introductory worksheet that calculates the perturbed tensorial components of the Ricci tensor. Consequently, this enable us to write the perturbed field equations in vacuum \((\hat{R}_{\mu \nu} = 0)\).

Comparing the perturbed components of the Ricci tensor and of the affine connections in this work with the published in Weinberg’s book \([5]\), note that they are identical.

This procedure can be extended to study perturbations of superior orders and gauge invariant perturbations. Taking into account the vacuum or perturbations in the energy momentum tensor.

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