An effective field theory for coupled-channel scattering

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The problem of describing low-energy two-body scattering for systems with two open channels with different thresholds is addressed in the context of an effective field theory. In particular, the problem where the threshold is unnaturally small and the cross section at low energy is unnaturally large is considered. It is shown that the lowest-order point coupling associated with the mixing of the channels scales as \( \Lambda^{-2} \) rather than \( \Lambda^{-1} \) (the scaling of the same-channel coupling and the scaling in a single-channel case) where \( \Lambda \) is the ultraviolet cutoff. The renormalization of the theory at lowest order is given explicitly. The treatment of higher orders is straightforward. The potential implications for systems with deep open channels are discussed.

I. INTRODUCTION

During the past decade methods of effective field theory (EFT) have been developed for problems of interest to low-energy nuclear physics. The underlying idea is based on the separation of scales. If a problem has widely disparate scales and one is interested in low-energy phenomena, then the details of the short-distance physics are unimportant — all that matters is its effects on the long-range phenomena. Formally, EFT replaces these short-distance interactions with contact interactions containing an arbitrary number of derivatives. Power counting justifies truncating the interaction Lagrangian and allows an a priori estimate of errors at a given order. An archetypical example of a pure EFT treatment of relevance in nuclear and molecular physics is the EFT of short-range forces. This formalism is useful when the low-energy two-body cross section is much larger than what is expected based on the underlying scales. When — as in the two-nucleon system — there is a shallow real or virtual bound state, there is also a large scale separation between the scattering length and all other scales in the problem. A power counting then requires that the lowest-order contact interaction be iterated to all orders while other effects can be treated perturbatively. Ultimately, as far as two-body scattering is concerned, this approach essentially reproduces the effective range expansion. The true power of the method becomes clear when more bodies and/or external probes (e.g., electroweak currents) are coupled to the two-body system. For example, recent success has been achieved in the systematic description of three-nucleon systems, and promising avenues exist for the exploration of many-nucleon systems (see, e.g., Ref. 1).

The purpose of the present paper is to develop an EFT treatment for the nonrelativistic two-body scattering problems that have more that one open channel with different thresholds. The scattering in both channels can be treated as being unnaturally large at low energy. Our purpose in developing this expansion is partly formal: the renormalization of such a problem is characteristically different from that in the single-channel case, and it is interesting to construct explicitly the renormalized amplitudes. In addition, our analysis may be of practical use in a few specialized cases of nucleon-nucleus scattering. Finally, it provides some insight into EFT treatments of a very general class of problems of importance to both nuclear and hadronic physics, namely, the scattering of two bodies when there are open channels even at zero kinetic energy.

II. A COUPLED-CHANNEL SCATTERING

To set up the problem consider a case of a nucleon \( N \) scattering off of a nucleus \( A \). We assume that the nucleus has a low-lying excited state \( A^* \) with the excitation energy much smaller than a typical scale of the interaction between \( N \) and \( A \). In addition, the system has an unnaturally-large cross section at low energy. The scattering reaction of a nucleon \( N \) off a nucleus \( A \) has two channels:

\[
\text{channel I} : \quad N A \rightarrow N A \\
\text{channel II} : \quad N A \rightarrow N^* A^* \quad (2.1)
\]

Channel I corresponds to an elastic, and channel II to an inelastic scattering. We assume for simplicity that the scattering in both channels occurs in an s wave in a singlet spin and isospin state. However, we allow the
outgoing nucleon in channel II to be in a different state, in order to explicitly account for different charge states.

The formal development of this theory is designed to closely follow the case of the pionless EFT and as such may be viewed as a generalization of the effective range expansion. However, the kinematics is quite different in the present case and the generalized effective range expansion will be qualitatively different from the usual one in the case of a single channel. Formally, we develop an effective field theory which is valid for the case where there are two unnaturally-light scales in the problem. One of them is the threshold of the second channel of the scattering reaction. This threshold is given by the excitation energy of $A^*$,

$$\Delta = m_{A^*} - m_A + m_{N'} - m_N,$$  \hspace{1cm} (2.2)

where $m_A$, $m_{A^*}$, $m_N$ and $m_{N'}$ are the masses of the nucleus $A$, its excited state $A^*$ and the two nucleon states, respectively. The present treatment is applicable when

$$\Delta \ll \frac{M^2}{2\mu_1} \sim \frac{M^2}{2\mu_2},$$  \hspace{1cm} (2.3)

where $M \sim 1/R$ is the typical scale describing the underlying interactions of short range $R$, and $\mu_1 = m_N m_{A^*}/(m_N + m_{A^*})$ and $\mu_2 = m_{N'} m_{A^*}/(m_{N'} + m_{A^*})$ are the reduced masses for the $N A$ and $N' A^*$ systems, respectively. We want to study the scattering at an energy $E$ comparable to $\Delta$. We take the scattering cross section at such low energy to be very large compared to $4\pi R^2$. The second unnaturally small quantity is the inverse scattering length, which generates the very large scattering cross section.

As in a single-channel scattering, observables are obtained from the on-shell $T$-matrix (or the physical scattering amplitude). The $T$-matrix for the scattering with two open channels has a form of a two-by-two square matrix. Diagonal entries of this matrix describe elastic scattering while the two off-diagonal entries correspond to the inelastic reactions. The scattering is considered here in the center-of-mass frame of the initial $N A$ system.

In close analogy to single-channel scattering the coupled-channel $T$-matrix is a solution of a Lippmann-Schwinger-like equation,

$$T_{ij}(\vec{p}, \vec{q}) = -V_{ij}(\vec{p}, \vec{q}) + \int \frac{d^3k}{(2\pi)^3} V_{ir}(\vec{p}, \vec{k}) G_{rs}(|\vec{p}|, |\vec{k}|) T_{sj}(\vec{k}, \vec{q}),$$  \hspace{1cm} (2.4)

where $\vec{p}$ and $\vec{q}$ are the relative momenta of the initial and final particles in the center-of-mass frame. The interaction potential $V_{ij}(\vec{p}, \vec{q})$, and the coupled channel two-body non-relativistic Green function are two-by-two matrices. The latter is a diagonal matrix; neglecting terms of higher order in momenta over large masses,

$$G(|\vec{p}|, |\vec{k}|) = \begin{pmatrix} \frac{2\mu_1}{p^2-k^2+i\epsilon} & 0 \\ 0 & \frac{2\mu_2}{p^2-2\mu_2\Delta-k^2+i\epsilon} \end{pmatrix},$$  \hspace{1cm} (2.5)

where $\Delta$ is the kinematical threshold for the channel II defined in Eq. (2.2). Note that in the present regime the threshold $\Delta$ is positive—channel II is energetically above channel I. In section 11 we will consider a case when $\Delta$ is negative—channel II is below channel I.

The diagonal elements of $G(|\vec{p}|, |\vec{k}|)$, $G_{11}$ and $G_{22}$, describe off-shell propagation of $N A$ and $N' A^*$ systems, respectively. These Green functions can be obtained from the non-interacting part of the total Lagrange density,

$$\mathcal{L}(x) = \Psi_A^\dagger(x) \left( i \partial_0 + \frac{\nabla^2}{2m_A} \right) \Psi_A(x) + \Psi_{N'}^\dagger(x) \left( i \partial_0 - (m_{N'} - m_N) + \frac{\nabla^2}{2m_{N'}} \right) \Psi_{N'}(x)$$

$$+ \Psi_N^\dagger(x) \left( i \partial_0 + \frac{\nabla^2}{2m_N} \right) \Psi_N(x) + \Psi_{A^*}^\dagger(x) \left( i \partial_0 - (m_{A^*} - m_A) + \frac{\nabla^2}{2m_{A^*}} \right) \Psi_{A^*}(x) + \ldots$$

$$+ \mathcal{L}_{\text{int}}(x),$$  \hspace{1cm} (2.6)

where the field $\Psi_A^\dagger(x)$ ($\Psi_A(x)$) creates (destroys) a particle of type $\alpha$. The ellipsis in Eq. (2.6) stand for the relativistic corrections. Eq. (2.6) gives Eq. (2.5) apart from higher-order terms, since in leading order $\mu_2 = \mu_1$.

To obtain the $T$-matrix from Eq. (2.4) we need to know the form of the coupled-channel interaction potential $V(\vec{p}, \vec{q})$. It describes the interactions between the incoming nucleon $N$ and the nucleons inside the nucleus $A$. In general, this is a many-body problem. However, as long as the kinetic energy is small so that the wavelength of the nucleon $N$ is much larger than the range of the interaction, the effects of this interaction on the low-energy scattering observables can be represented by contact interactions between a nucleon and the nucleus.
as a whole. Since the threshold $\Delta$ is of order of the kinetic energy $E$, each nuclear state has to be considered as an explicit degree of freedom.

The effective potential, $V(\vec{p}, \vec{q})$, is given by an infinite sum of contact interactions, which in momentum space has the form

$$V_{ij}(\vec{p}, \vec{q}) = C^{ij}_0 + C^{ij}_2 (\vec{p}^2 + \vec{q}^2) + \ldots ,$$  \hspace{1cm} (2.7)

where ellipsis stand for terms with higher powers of momenta $[17]$. The constant coefficients $C^{ij}_0$, $C^{ij}_2$, etc., are

$$\mathcal{L}_{\text{int}}(x) = -c_{11} \Psi_N^\dagger \Psi_A^\dagger \Psi_N \Psi_A - c_{22} \Psi_N^\dagger \Psi_A^\dagger \Psi_N \Psi_A + \ldots ,$$  \hspace{1cm} (2.8)

where the last term represents the coupling between the two channels, and the ellipsis stand for terms containing derivatives.

The existence of an infinite number of singular interactions in Eq. (2.7) requires one to introduce a power counting, which justifies a truncation in the number of interaction terms that must be kept up to a given order in an expansion in powers of $E/M$. In addition, as in any field theory, a renormalization scheme is necessary to relate parameters in the Lagrangian to observables, and render the latter independent of the cutoff.

In the case of a single-channel scattering at low energies with short-range forces, this situation has been thoroughly investigated and a well-defined EFT exists $[2, 3]$. The power-counting scheme depends on whether the scattering length is natural or unnatural; in the latter case, it relies on the fact that other scattering observables, such as effective range and shape terms in the effective range expansion, are natural, as one would naively expect when the underlying interactions are short ranged. The large scattering length usually indicates the presence of a shallow real or virtual bound state. In this case one “bare” constant $c_0(\Lambda)$ such as to reproduce the $T$-matrix at leading order the coupled-channel $T$-matrix is obtained by iterating the first term in Eq. (2.7) — the constant matrix $C_0$. The rest of the terms are expected to be treated perturbatively according to the mass dimension of the constants $C_{2n}$. Note that while such a power counting seems to be a quite logical generalization of the power counting in the single-channel case, it is not totally obvious how to implement the renormalization for the present case. One principal result of this paper is to demonstrate how this can be accomplished.

The success of the power counting for single-channel scattering is ultimately connected to an ability to “fine tune” one “bare” constant $c_0(\Lambda)$ such as to reproduce the scattering length $a_s$, which can be much larger than the typical scale $R \sim 1/M$ of the underlying short-range interactions. The single-channel $s$-wave $T$-matrix at leading order has the following form (in the center-of-mass frame of two particles with reduced mass $\mu$ and the initial relative momentum $\vec{p}$) $[2]$, 

$$\frac{1}{T} = \frac{\mu}{2\pi} \left\{ -\frac{2\pi}{\mu c_0(\Lambda)} - \frac{2A}{\pi} - ip + \mathcal{O} \left( \frac{p^2}{A} \right) \right\} = \frac{\mu}{2\pi} \left\{ -\frac{1}{a_s} - ip + \mathcal{O} \left( \frac{p^2}{A} \right) \right\} .$$  \hspace{1cm} (2.10)

The right-hand side of Eq. (2.10) corresponds to the effective range expansion in which only the first term — the
scattering length— is reproduced; interactions with more derivatives account for the other effective range parameters. As follows from Eq. (2.10), in order to obtain a \( \Lambda \)-independent on-shell \( T \)-matrix at leading order, the bare coupling constant \( c_0 \) has to have the following dependence on the cutoff scale \( \Lambda \),

\[
\frac{1}{c_0(\Lambda)} = \frac{\mu}{2\pi} \left\{ \frac{1}{a_s} - \frac{2\Lambda}{\pi} \right\}.
\] (2.11)

The above equation determines the renormalization-group flow of the coupling \( c_0 \).

In the present case of coupled-channel scattering, the leading-order interaction contains three bare coupling constants, Eq. (2.4). Since at this order the coupled-channel potential \( V(\vec{p}, \vec{q}) \) (Eq. (2.7)) is momentum independent, the coupled-channel Lippmann-Schwinger equation, Eq. (2.11), can be solved analytically to all orders in the coupling constants \( C_{ij}^\Lambda \). The solution is,

\[
T_s = -(1 - C_0 G^\Lambda)^{-1} C_0,
\] (2.12)

where the cutoff-dependent matrix \( G^\Lambda \) is

\[
G^\Lambda = \begin{pmatrix}
-\frac{\mu \Lambda}{\pi^2} - \frac{\mu_1}{2\pi} ip & -\frac{\mu_2 \Lambda}{\pi^2} - \frac{\mu_1}{2\pi} i \sqrt{\vec{p}^2 - 2\mu_2 \Delta} \\
0 & 0
\end{pmatrix}.
\] (2.13)

Note that, due to the particularly simple form of the coupled-channel potential at leading order, the on-shell \( s \)-wave \( T \)-matrix (or the physical coupled-channel scattering amplitude) is a function of the energy \( E = |\vec{p}|^2/2\mu_1 \) only (for a fixed threshold \( \Delta \)). As in the single-channel case, the \( \Lambda \)-dependent terms come from the linearly divergent part of integrals in Eq. (2.11) regularized using a momentum cutoff \( \Lambda \). We have already neglected terms that depend on inverse powers of the cutoff and are analytic in \( E \); these terms cannot be considered separately from the higher-derivative terms in Eq. (2.11), which are also analytic in \( E \). Using Eqs. (2.10) and (2.11), the explicit form of the coupled-channel \( T \)-matrix at leading order is

\[
T_s = -\frac{1}{1 - (c_{11} G_{11}^\Lambda + c_{22} G_{22}^\Lambda) - (c_{12}^2 - c_{11} c_{22}) G_{11}^\Lambda G_{22}^\Lambda} \begin{pmatrix}
c_{11} + (c_{12}^2 - c_{11} c_{22}) G_{22}^\Lambda & c_{12}
c_{12} & c_{22} + (c_{12}^2 - c_{11} c_{22}) G_{11}^\Lambda
\end{pmatrix}.
\] (2.14)

As written, the \( T \)-matrix in Eq. (2.14) contains \( \Lambda \)-dependent terms. As in the single-channel case (Eq. (2.10)), one can expect to obtain cutoff independent \( T \)-matrix elements because the bare coupling constants \( c_{ij} \) depend on \( \Lambda \). Note that in addition to linearly-divergent terms \( (G_{11}^\Lambda \) and \( G_{12}^\Lambda \) separately), the \( T \)-matrix elements in Eq. (2.14) contain a product that scales as \( \Lambda^2 - G_{11}^\Lambda G_{22}^\Lambda \). Moreover, the form in which this product appears is non-trivial.

In order to determine the cutoff dependence of the coupling constants \( c_{11}, c_{22} \) and \( c_{12} \), it is convenient to consider the inverse of the \( T \)-matrix given in Eq. (2.11),

\[
T_s^{-1} = \begin{pmatrix}
c_{12} & \frac{c_{11} \Lambda}{\pi^2} - \frac{\mu_1}{2\pi} \frac{1}{a_{11}} \\
\frac{c_{12}}{c_{11}^2 - c_{11} c_{22}} & \frac{c_{11} \Lambda}{\pi^2} + \frac{\mu_2}{2\pi} \frac{1}{a_{12}}
\end{pmatrix}.
\] (2.15)

which is the coupled-channel analog of the first part of the equation (2.10).

The requirement that the \( T \)-matrix be cutoff-independent (at a given order in an EFT expansion) leads to

\[
\frac{c_{22}}{c_{11}^2 - c_{11} c_{22}} + \frac{\mu_1 \Lambda}{\pi^2} = \frac{\mu_1}{2\pi a_{11}}
\]

\[
\frac{c_{11}}{c_{11}^2 - c_{11} c_{22}} + \frac{\mu_2 \Lambda}{\pi^2} = \frac{\mu_2}{2\pi a_{12}}
\]

where \( a_{11}, a_{22} \) and \( a_{12} \) are constants, the constants of proportionality having been chosen for further convenience. The system of coupled equations in Eq. (2.10) can be solved to obtain the \( \Lambda \)-scaling of \( c_{11}, c_{22} \) and \( c_{12} \),

\[
\frac{c_{12}}{c_{11} c_{22} - c_{12}^2} = \frac{\sqrt{\mu_1 \mu_2}}{2\pi a_{12}}.
\] (2.16)
\[
\frac{1}{c_{11}(\Lambda)} = \frac{\mu_1}{2\pi} \left\{ \frac{1}{a_{11}} - \frac{2\Lambda}{\pi} - \frac{1}{a_{12}^2} - \frac{2\Delta}{\pi} \right\} = \frac{\mu_1}{2\pi} \left\{ \frac{1}{a_{11}} - \frac{2\Lambda}{\pi} + \mathcal{O}\left(\frac{1}{a_{12}^2}\right) \right\},
\]
\[
\frac{1}{c_{22}(\Lambda)} = \frac{\mu_2}{2\pi} \left\{ \frac{1}{a_{22}} - \frac{2\Lambda}{\pi} - \frac{1}{a_{12}^2} - \frac{2\Delta}{\pi} \right\} = \frac{\mu_2}{2\pi} \left\{ \frac{1}{a_{22}} - \frac{2\Lambda}{\pi} + \mathcal{O}\left(\frac{1}{a_{12}^2}\right) \right\},
\]
\[
\frac{1}{c_{12}(\Lambda)} = \frac{\sqrt{\mu_1\mu_2}}{2\pi} a_{12} \left\{ \frac{1}{a_{11}} - \frac{2\Lambda}{\pi} \right\} \left\{ \frac{1}{a_{22}} - \frac{2\Lambda}{\pi} \right\} = \frac{\sqrt{\mu_1\mu_2}}{2\pi} a_{12} \left\{ \frac{1}{a_{11}} - \frac{2\Lambda}{\pi} \right\} \left\{ \frac{1}{a_{22}} - \frac{2\Lambda}{\pi} \right\}.
\]

Note, the $\Lambda$-dependence of the coupling constants $c_{11}$ and $c_{22}$ is the same up to terms of order $1/\Lambda$ as the $\Lambda$-scaling of the coupling constant $c_0$ in the single-channel case, Eq. (2.11). The $\Lambda$-scaling of $c_{12}$, on the other hand, is qualitatively different: it scales as $\Lambda^{-2}$.

Using Eq. (2.17) in Eq. (2.15) we obtain the cutoff-independent form for the inverse of the coupled-channel $T$-matrix,

\[
T_s^{-1} = \begin{pmatrix}
-\frac{\mu_1}{2\pi} \left( \frac{1}{a_{11}} + ip \right) & \frac{\sqrt{\mu_1\mu_2}}{2\pi a_{12}} \\
\frac{\sqrt{\mu_1\mu_2}}{2\pi a_{12}} & -\frac{\mu_2}{2\pi} \left( \frac{1}{a_{22}} + i\sqrt{p^2 - 2\mu_2 \Delta} \right)
\end{pmatrix},
\]

which depends only on the physical constants $a_{11}$, $a_{22}$ and $a_{12}$. Thus, the coupled-channel $T$-matrix at leading order is,

\[
T_s = \left( \frac{1}{a_{12}} - \left( \frac{1}{a_{11}} + ip \right) \left( \frac{1}{a_{22}} + i\sqrt{p^2 - 2\mu_2 \Delta} \right)^{-1} \left( \frac{2\pi}{\mu_1} \left( \frac{1}{a_{22}} + i\sqrt{p^2 - 2\mu_2 \Delta} \right) \frac{2\pi}{\sqrt{\mu_1\mu_2}} \frac{1}{a_{12}} \right) \right)^{-1},
\]

where $p = \sqrt{2\mu_1 E}$ with $E$ being the center-of-mass energy in channel I.

Equation (2.19) represents the leading term in the generalization effective range expansion of the coupled-channel $s$-wave $T$-matrix for a system with short-range interactions. It depends on three constants — $a_{11}$, $a_{22}$ and $a_{12}$— that play the same role as the scattering length in the single-channel case. Higher-order terms can be obtained in similar fashion. Likewise, more channels can be included. We see that the EFT reproduces the standard multi-channel generalization of the effective range expansion in the single-channel case, Eq. (2.10). As expected, the existence of the low-energy threshold (given by $\Delta$) for the second reaction channel (channel II) modifies the scattering in the initial channel — channel I. Even though Eq. (2.18) is an obvious generalization of the single-channel case, it has some noteworthy features.

When the kinetic energy $E < \Delta$, channel II is closed. Thus, we expect that the expansion of $T_{s11}$ (describing the scattering in channel I) in powers of the relative momentum $\bar{p}$ takes the same form as the effective range expansion in the single-channel case, Eq. (2.10). Indeed, expanding the inverse of the element $T_{s11}$ in Eq. (2.19) in powers of $p^2/\mu_2 \Delta$, one obtains

\[
\frac{1}{T_{s11}} = \frac{\mu_1}{2\pi} \left\{ -\frac{1}{a_{\text{eff}}} + \frac{1}{2} \text{re} \ p^2 + \mathcal{O}\left(\frac{p^4}{a_{12}^2(\mu_2 \Delta)^{5/2}}\right) \right\},
\]

where the effective range parameters are

\[
a_{\text{eff}} = \frac{a_{11} a_{12} (1 - a_{22} \sqrt{\mu_2 \Delta})}{a_{12}^2 (1 - a_{22} \sqrt{2\mu_2 \Delta}) - a_{11} a_{22}},
\]

\[
\text{re} = -\frac{1}{\sqrt{2\mu_2 \Delta}} \left( \frac{a_{22}}{a_{12} (1 - a_{22} \sqrt{2\mu_2 \Delta})} \right)^2.
\]

The above expansion has indeed the form of the effective range expansion. Despite channel II being closed ($E < \Delta$), the fact that the threshold $\Delta$ is much smaller than the scale $M$ of the short-range interactions changes the scaling of the effective range parameters. While the effective range for the single-channel case — or, effectively, for coupled channels when $\sqrt{\mu_2 \Delta} \sim M$ — is of order $1/M$, here it is of order $1/\sqrt{\mu_2 \Delta} \gg 1/M$. The
same is true for other effective range parameters. From the perspective of channel I alone, all effective range parameters are large.

The off-diagonal elements of the coupled-channel $T$-matrix, Eq. (2.19), are not identically zero even for the case when $E < \Delta$. This, however, does not mean that there is a non-zero cross section for the scattering into channel II. The reason is that the asymptotic wave function in channel II is, up to a constant,

$$\frac{1}{r} e^{-i r \sqrt{p^2 - 2 \mu_2 \Delta}},$$

which gives zero contribution to the outgoing flux when $p^2 < 2 \mu_2 \Delta$ and, as a result, the scattering into this channel has zero cross section.

As the energy $E$ increases and approaches the kinematical threshold for channel II from below, the $i \sqrt{p^2 - 2 \mu_2 \Delta}$’s in Eq. (2.19) —which are real— decrease, and above the threshold they acquire an imaginary part. At $p_t = \sqrt{2 \mu_2 \Delta}$, the amplitude for channel I —given by $T_{11}$— is continuous, but its derivatives are not. The derivatives at threshold explode as $1/\sqrt{-p - p_t}$ from below, and as $1/\sqrt{p - p_t}$ from above. The sign of the derivatives is governed by

$$\lambda = \left( \frac{1}{a_{11}} - \frac{a_{22}}{a_{12}}^2 \right) \frac{1}{\sqrt{2 \mu_2 \Delta}}.$$

The sign of the derivative of the real (imaginary) part of the amplitude is that of $\lambda^2 - 1$ ($-\lambda$) below, and $\lambda (\lambda^2 - 1)$ above threshold. As a consequence, the amplitude exhibits a cusp and a rounded step [9]. Fig. 1 illustrates this behavior for the particular combination $\lambda < 0$ and $\lambda^2 - 1 > 0$.

III. OPTICAL POTENTIAL AND EFT

Optical potentials —potentials with real and imaginary parts—are usually introduced to describe scattering in which the flux in the sector explicitly considered is not conserved. This occurs when there are open channels that are not explicitly described by the theory. A well-known example in hadronic physics is $p \bar{p}$ annihilation at low energies [10, 11]. It is clearly of interest to develop an effective-field-theory treatment for such cases. Presumably, the generic logic underlying EFT’s power counting applies. The subtle issue concerns whether one can integrate out a channel that is separated by a large mass scale but is open, that is, represents asymptotic states that are accessible even at low energy. Certainly, if one does this one can no longer impose unitarity. Thus it is plausible that the form of an EFT in such a regime would be that of a usual EFT with usual power counting but with complex coefficients [18]. We also expect that the result would be an expansion analogous to the effective range expansion, with complex parameters; indeed, such expansion already exists [14]. If this can be established, then the EFT approach can be usefully extended to a wide array of new systems.

One way to test this idea is in the context of a solvable model. Consider, for example, the coupled-channel scattering discussed in section II but with negative $\Delta$. Assume furthermore that $\sqrt{\mu_2 |\Delta|}$ is large compared to the other low scales in the problem ($1/a_{ij}$ and $p$) but remains small compared to the underlying scale $M$, which justifies the treatment in the previous section. In this case we have a simple example of such an EFT and we can test how it works.

To see how the coupled-channel calculation goes with these kinematics, we only need to slightly modify the treatment. In section II we discussed an EFT treatment of the low-energy scattering with two open channels for which the threshold $\Delta$ was positive so that the coupled channel was above the initial channel —channel I. Yet, the treatment is valid also when the second channel is
below the initial channel. In fact the form of the $T$-matrix in Eq. (2.19) is the same with just a single difference — the threshold $\Delta$ is negative. The renormalization is not affected by this change. Thus the scattering in channel I is given by $T_{a11}$, Eq. (2.19). While channel II is now open for any non-zero initial energy, $T_{a11}$ can still be expanded in powers of $E/|\Delta|$ when $E < |\Delta|$. The first three terms in such an expansion are, again,

$$
\frac{1}{T_{a11}} = \frac{\mu_1}{2\pi} \left\{ -\frac{1}{a_{\text{eff}}} + \frac{1}{2} r_{\text{eff}} p^2 - i p + O \left( \frac{p^4}{a_{12}^2 (\mu_2 |\Delta|)^{3/2}} \right) \right\},
$$

(3.1)

but now the expansion parameters are given by

$$
a_{\text{eff}} \equiv \frac{a_{11} a_{12}^2 (1 + i a_{22} \sqrt{2 \mu_2 |\Delta|})}{a_{12}^2 (1 + i a_{22} \sqrt{2 \mu_2 |\Delta|}) - a_{11} a_{22}},
$$

$$
r_{\text{eff}} \equiv \frac{i}{\sqrt{2 \mu_2 |\Delta|}} \left( \frac{a_{22}}{a_{12} (1 + i a_{22} \sqrt{2 \mu_2 |\Delta|})} \right)^2.
$$

(3.2)

The expansion in Eq. (3.1) has the form of the effective range expansion with complex coefficients that reflect the scales in the problem. While the effective scattering length is approximately real and large, $a_{\text{eff}} \simeq a_{11}$, the effective range is approximately imaginary and small, $r_{\text{eff}} \simeq i/(a_{12}^2 (2 \mu_2 |\Delta|)^{3/2})$.

We expect, therefore, that the EFT developed here can be generalized to describe other problems with open channels well below the scale explicitly included in the EFT. It remains an open question how relevant such a treatment would be for $p\bar{p}$ scattering at low energy. The obvious complication is the presence of many open channels containing mesons in the final states. One, however, can exploit the scale separation between the elastic and charge-exchange channels (with only nucleons and anti-nucleons in the final states), and the channels containing mesons. As far as the EFT description of $p\bar{p} \to p\bar{p}$ and $p\bar{p} \to n\bar{n}$ channels is concerned, the rest of the channels can perhaps be described by a single effective channel separated by a threshold $\Delta$.

IV. SUMMARY

We have constructed an effective field theory for the low-energy coupled-channel scattering with short-range forces. The EFT is valid for the scattering with an unnaturally-large cross section and an unnaturally-small threshold between channels. At leading order, the $s$-wave $T$-matrix in the two-channel case is given in Eq. (2.19) and is determined by three momentum-independent contact interactions shown in Eq. (2.9). This $T$-matrix is expressed in terms of three renormalized quantities — $a_{11}$, $a_{22}$ and $a_{12}$ — that are analogous to the scattering length in the single-channel case. The treatment here can be readily extended to include higher orders and more channels [19].

In addition, we have also discussed the case where the second channel is energetically below the initial channel. In this case the scattering in the initial channel can be described by a generalized effective range expansion with complex coefficients, Eqs. (3.1) and (3.2).

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[17] Only terms that contribute to s-wave scattering are shown.

[18] Such an EFT has been used, for example, to describe the annihilation decay rates of positronium and heavy quarkonium and three-atom recombination into an atom and a deep dimer.

[19] After the present paper was completed, the authors became aware of the discussions of coupled-channel scattering in Ref. 12—where the same Eqs. 24 and 25 were obtained—and in Ref. 13—where the ∆∆ channel was explicitly calculated in S-wave N N scattering. Our analysis is, however, considerably more detailed. We thank B.R. Holstein for communication on the subject and encouragement.