On the automorphism group of the $m$-coloured random graph

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Abstract

Let $R_m$ be the (unique) universal homogeneous $m$-edge-coloured countable complete graph ($m \geq 2$), and $G_m$ its group of colour-preserving automorphisms. The group $G_m$ was shown to be simple by John Truss. We examine the automorphism group of $G_m$, and show that it is the group of permutations of $R_m$ which induce permutations on the colours, and hence an extension of $G_m$ by the symmetric group of degree $m$. We show further that the extension splits if and only if $m$ is odd, and in the case where $m$ is even and not divisible by 8 we find the smallest supplement for $G_m$ in its automorphism group.

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1 Introduction

Fix an integer $m \geq 2$, and let $R_m$ be the unique homogeneous universal $m$-edge-colouring of the countable complete graph (see Truss [6]). (Universality means that any $m$-edge-coloured finite or countable complete graph is embeddable in $R_m$, and homogeneity means that every colour-preserving isomorphism between finite subgraphs extends to an automorphism of $R_m$. The uniqueness is a special case of Fraïssé’s theory of countable homogeneous structures. The graph $R_m$ is the ‘random $m$-edge-coloured complete graph’: that is, we colour edges independently at random, we obtain $R_m$ with probability 1. More relevant to us is the fact that the isomorphism class of $R_m$ is residual in the set of all $m$-coloured complete graphs on a fixed countable vertex set. See [1] for discussion.)

Let $\text{Aut}(R_m)$ be the group of permutations of the vertex set fixing all the colours. Truss [6] showed that $\text{Aut}(R_m)$ is a simple group.

For any permutation $\pi$ of the set of colours, let $R_m^\pi$ be the graph obtained by applying $\pi$ to the colours. Then $R_m^\pi$ is universal and homogeneous, and hence isomorphic to $R_m$. This means that, if $\text{Aut}^*(R_m)$ is the group of permutations of the vertex set which induce permutations of the colours, then $\text{Aut}^*(R_m)$ induces the symmetric group $\text{Sym}(m)$ on the colours; so $\text{Aut}^*(R_m)$ is an extension of $\text{Aut}(R_m)$ by $\text{Sym}(m)$.

The first question we consider here is: when does this extension split? That is, when is there a complement for $\text{Aut}(R_m)$ in $\text{Aut}^*(R_m)$ (a subgroup of $\text{Aut}^*(R_m)$ isomorphic to $\text{Sym}(m)$ which permutes the colours)? We also show that $\text{Aut}^*(R_m)$ is the automorphism group of the simple group $\text{Aut}(R_m)$ (so that the outer automorphism group of this group is $\text{Sym}(m)$).

Theorem 1 The group $\text{Aut}^*(R_m)$ splits over $\text{Aut}(R_m)$ if and only if $m$ is odd.

Theorem 2 The automorphism group of $\text{Aut}(R_m)$ is $\text{Aut}^*(R_m)$.

2 Proof of Theorem 1

We show first that the extension does not split if $m$ is even. Suppose that a complement exists, and let $s$ be an element of this complement acting as $(1,2)(3,4)\cdots(m-1,m)$ on the colours. Then $s$ maps the subgraph with
colours 1, 3, ..., \(m - 1\) to its complement. But this is impossible, since the edge joining points in a 2-cycle of \(s\) has its colour fixed.

Now suppose that \(m\) is odd; we are going to construct a complement.

First, we show that there exists a function \(f\) from pairs of distinct elements of Sym\((m)\) to \(\{1, \ldots, m\}\) satisfying

\[
\begin{align*}
&\bullet f(x, y) = f(y, x) \text{ for all } x \neq y; \\
&\bullet f(xg, yg) = f(x, y)^g \text{ for all } x \neq y \text{ and all } g.
\end{align*}
\]

To do this, we first define \(f(1, y)\) for \(y \neq 1\) arbitrarily subject to the condition \(f(1, x^{-1}) = f(1, x)^{x^{-1}}\). Note that this condition requires \(f(1, s)^s = f(1, s)\) whenever \(s\) is an involution; but this is possible, since any involution has a fixed point (as \(m\) is odd). Then we extend to all pairs by defining \(f(x, y) = f(1, yx^{-1})^x\). A little thought shows that no conflict arises.

Now we take a countable set of vertices, and let Sym\((m)\) act semiregularly on it. Each orbit is naturally identified with Sym\((m)\); we let \(x_i\) denote the element identified with \(x\) in the \(i\)th orbit, as \(i \in \mathbb{N}\) (where orbits are indexed by natural numbers). Then we colour the edges within each orbit by giving \(\{x_i, y_i\}\) the colour \(f(x, y)\). For edges between orbits \(i\) and \(j\), with \(i < j\), we colour \(\{x_i, 1_j\}\) arbitrarily, and then give \(\{y_i, z_j\}\) the image of the colour of \(\{(yz^{-1})_i, 1_j\}\) under \(z\).

Clearly the group Sym\((m)\) permutes the colours of the edges consistently, the same way as it permutes \(\{1, \ldots, m\}\).

Next we show that a residual set of the coloured graphs we obtain are isomorphic to \(R_m\). We have to show that, given \(m\) finite disjoint sets of vertices, say \(U_1, \ldots, U_m\), the set of graphs containing a vertex \(v\) joined by edges of colour \(i\) to all vertices in \(U_i\) (for \(i = 1, \ldots, m\)) is open and dense. The openness is clear. To see that it is dense, note that the \(m\) finite sets are contained in the union of a finite number of orbits (say those with index less than \(N\)); then, for any \(i \geq N\), we are free to choose the colours of the edges joining these vertices to \(1_i\) arbitrarily.

Now by construction, the group Sym\((m)\) we have constructed meets Aut\((R_m)\) in the identity; so it is the required complement.

How close can we get when \(m\) is even? The construction in the second part can easily be modified to show that, if there is a group \(G\) which acts as Sym\((m)\) on the set \(\{1, \ldots, m\}\), in such a way that all involutions in \(G\) have fixed points on \(\{1, \ldots, m\}\), then \(G\) is a supplement for Aut\((R_m)\) in
$\text{Aut}^\ast(R_m)$ (that is, $G, \text{Aut}(R_m) = \text{Aut}^\ast(R_m)$), and $G \cap \text{Aut}(R_m)$ is the kernel of the action of $G$ on $\{1, \ldots, m\}$. We simply replace $\text{Sym}(m)$ by $G$ in the construction, and in place of $f(xg, yg) = f(x, y)^g$ we require that $f(xg, yg) = f(x, y)^{\phi g}$, where $\phi$ is the action of $G$ on $\{1, \ldots, m\}$.

If $m$ is even but not a multiple of 8, then there is a double cover of $\text{Sym}(m)$, for $m$ even, in which the fixed-point-free involutions lift to elements of order 4. (There are two double covers of $\text{Sym}(n)$ for $n \geq 4$, described in [3] Chapter 2] and called there $\tilde{S}_m$ and $\hat{S}_m$. In $\tilde{S}_m$, the product of $r$ disjoint transpositions lifts to an element of order 4 if and only if $r \equiv 1$ or $2$ mod 4, while in $\hat{S}_m$, the condition is that $r \equiv 2$ or $3$ mod 4.) This shows that there is a supplement meeting $\text{Aut}(R_m)$ in a group of order 2 for $m$ even but not divisible by 8.

What happens in the remaining case, when $m$ is a multiple of 8? Is there a finite supplement, and what is the smallest such?

3 Proof of Theorem 2

Since $\text{Aut}(R_m)$ is primitive and not regular, its centraliser in the symmetric group is trivial; so $\text{Aut}^\ast(R_m)$ acts faithfully on $\text{Aut}(R_m)$ by conjugation. We have to show that there are no further automorphisms.

A permutation group $G$ of countable degree is said to have the small index property if any subgroup $H$ satisfying $|G : H| < 2^\aleph_0$ contains the pointwise stabiliser of a finite set; it has the strong small index property if any subgroup $H$ satisfying $|G : H| < 2^\aleph_0$ lies between the pointwise and setwise stabiliser of a finite set.

Step 1 $R_m$ has the strong small index property.

This is proved by a simple modification of the arguments for the case $m = 2$. The small index property is proved by Hodges et al. [3], using a result of Hrushovski [5]; the strong version is a simple extension due to Cameron [2].

Hrushovski showed that any finite graph $X$ can be embedded into a finite graph $Z$ such that all isomorphisms between subgraphs of $X$ extend to automorphisms of $Z$. Moreover, the graph $Z$ is vertex-, edge- and nonedge-transitive. He uses this to construct a generic countable sequence of automorphisms of $R$. To extend this to $R_m$ is comparatively straightforward. It is necessary to work with $(m-1)$-edge-coloured graphs (regarding the $m$th
Now the arguments of Hodges et al. and Cameron go through essentially unchanged.

**Step 2** Since $\text{Aut}(R_m)$ acts primitively on the vertex set, with permutation rank $m+1$, the vertex stabilisers are maximal subgroups of countable index with $m+1$ double cosets. Moreover, any further subgroup of countable index has more than $m + 1$ double cosets.

For let $H$ be a maximal subgroup of countable index. By the strong SIP, $H$ is the stabiliser of a $k$-set $X$. If $g$ maps $X$ to a disjoint $k$-set, then $HgH$ determines the colours of the edges between $X$ and $X^g$, up to permutations of these two sets. By universality, there are at least $m^{k^2}/(k!)^2$ such double cosets. Now it is not hard to prove that $m^{k^2}/(k!)^2 > m$ for $k \geq 2$. Hence we must have $k = 1$.

**Step 3** It follows that any automorphism permutes the vertex stabilisers among themselves, so is induced by a permutation of the vertices which normalises $\text{Aut}(R_m)$. To finish the proof, we show that the normaliser of $\text{Aut}(R_m)$ in the symmetric group is $\text{Aut}^*(R_m)$.

This is straightforward. A vertex permutation which normalises $\text{Aut}(R_m)$ must permute among themselves the $\text{Aut}(R_m)$-orbits on pairs of vertices, that is, the colour classes; so it belongs to $\text{Aut}^*(R_m)$.

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