Gauge equivalent universes in 5d Kaluza-Klein theory

Gyeong Yun Jun, Pyung Seong Kwon *
Department of Physics, Kyungsung University, Pusan 608-736, Korea

Abstract

We examine in the framework of 5d Kaluza-Klein theory the gauge equivalence of \( x^5 \)-dependent cosmological solutions each of which describes in the 4d sector an arbitrarily evolving isotropic, homogeneous universe with some pure gauge. We find that (1) within a certain time scale \( \tau_c \) (which is characterized by the compactification radius \( R_c \)) any arbitrarily evolving 4d universe is allowed to exist by field equations, and these 4d universes with appropriate pure gauges are all gauge equivalent as long as they are of the same topology. (2) Outside \( \tau_c \) the gauge equivalence disappears and the evolution of the universe is fixed by field equations.

PACS number:0450

Keywords: Kaluza-Klein; gauge; \( x^5 \)-dependent; cosmology

*E-mail:bskwon@star.ks.ac.kr
Caused by brane world scenarios[1] the 5d cosmology has been severely modified during the past several years[3]. The general feature of the new scenarios is that the scale of extra dimensions is not so small as expected by the traditional Kaluza-Klein theories, and the dependence of metric components on these extra dimensions becomes important. In this paper we will examine a 5d cosmology which is conventional in the sense that the theory possesses a \( U(1) \) gauge symmetry at the massless level\(^2\), but where metric components have \( x^5 \)-dependence.

Recently, there has been an argument[4] in 5d Kaluza-Klein theory that an evolving universe may be related with a static universe by a gauge transformation. The authors have used the simplest \( x^5 \)-dependent cosmological solution to the vacuum Einstein equation to show that the time degree of freedom of an evolving universe can be absorbed by a gauge transformation into the fifth dimension, and consequently the evolving universe turns into a static one. The solution considered in ref.[4] took the form of the Tolman metric

\[
d s^2 = -d t^2 + R^2(t, x^5) d \Omega_k^2 + e^{\mu(t, x^5)} (dx^5)^2
\]  

with

\[
R^2(t, x^5) = R_0^2 + \alpha^2 f_0 [x^5 + (t - t_0)/\alpha]^2,
\]

and \( e^{\mu(t, x^5)} \) being related with \( R^2(t, x^5) \) by the equation

\[
e^{\mu} = \frac{R^2}{\alpha^2} = \frac{\alpha^2 f_0^2 [x^5 + (t - t_0)/\alpha]^2}{R_0^2 + \alpha^2 f_0 [x^5 + (t - t_0)/\alpha]^2},
\]

where \( d \Omega_k^2 \) is the metric of the 3d volume with constant curvature \( k = 1, 0, -1 \), and \( R_0, f_0 \) and \( \alpha \) are all (integral) constants, and in particular

\[
\alpha = \pm (k + f_0)^{1/2}.
\]

The solution (1) reduces to a static solution of the form

\[
d s^2 = -d t^2 + \tilde{R}^2(\tilde{x}^5) d \Omega_k^2 + e^{\tilde{\mu}(\tilde{x}^5)} [d \tilde{x}^5 + \kappa \tilde{A}_0(t) dt]^2
\]

once we perform a gauge transformation

\[
x^5 \rightarrow \tilde{x}^5 = x^5 + (t - t_0)/\alpha,
\]

\(^{1}\)For the traditional 5d cosmological solutions, see ref.[2, 5]

\(^{2}\)Thus, in this paper, the extra dimension is an analogue of an internal Calabi-Yau space, rather than a \( S^1/Z_2 \) orbifold

\(^{3}\)In this paper we will use the same notations as those in ref.[4]; i.e., the “prime” denotes the \( x^5 \)-derivative, while “overdot” the time-derivative etc.
which shows that the time degree of freedom of an evolving universe can be gauged away by a U(1) gauge transformation, and in compensation for this a pure gauge comes into being. But here we should notice that the fifth coordinates $x^5$ and $\tilde{x}^5$ are both compact variables whose principal values can not exceed $\pm 2\pi R_c$ (where $R_c$ represents the compactification radius of the fifth dimension), while $t$ is a noncompact variable which can be arbitrarily large. This means that the relation in eq.(6) fails to hold once $|(t - t_0)/\alpha|$ exceeds the value $2\pi R_c$, and the solutions in eqs.(1) and (5) are not gauge equivalent anymore in the region $(t - t_0)/\alpha > 2\pi R_c$. This is remarkable because it indicates the possibility that any two different 4d universes may be entirely equivalent at the early stage of evolution within a certain time scale characterized by $R_c$.

Let us consider a metric

$$ds^2 = -N^2(t, x^5)dt^2 + R^2(t, x^5)d\Omega^2_k + e^{\mu(t, x^5)}[dx^5 + N^5(t)dt]^2,$$

(7)

where $N(t, x^5)$ is the lapse function which has been introduced for the time being, and $N^5(t)$ is the fifth (and the only non-vanishing) component of the shift vector $N^A(t)$. With $N(t, x^5) = 1$ and $N^5(t) = 0$ the metric (7) reduces to the Tolman metric. The action on the other hand is given by the Hilbert-Einstein action

$$I_g = -\frac{1}{2\kappa} \int d^5x \sqrt{-g} \sqrt{R}$$

= $\frac{1}{2\kappa} \int dt d^4x N \sqrt{g} \left( R + K_{\mu\nu}K^{\mu\nu} - K^2 \right) + \text{surface terms}$

(9)

$$\equiv \int dt d^4x L_g + \text{surface terms}$$

(10)

plus perhaps some matter action

$$I_m = \int d^5x L_m$$

(11)

which has not been given in a definite form. In eq.(9), $R$ and $K_{\mu\nu}$ are the Ricci scalar and the second fundamental form of the 4d spacelike hypersurface, and they are given by

$$\frac{4}{R} = \frac{6k}{R^2} - e^{-\mu}[6\frac{R''}{R} + 6(\frac{R'}{R})^2 - 3\mu'(\frac{R'}{R})],$$

(12)

$$K_{\mu\nu} = \frac{1}{2N}[\partial_\mu g_{\nu\mu} - (\nabla_\mu N_\nu + \nabla_\nu N_\mu)], \quad (\mu, \nu = 1, 2, 3, 5; \partial_0 \equiv \partial_t).$$

(13)

where $\nabla_\mu$ denotes the covariant derivative associated with $g_{\mu\nu}$ which is induced on the hypersurface. The action in (9) can be put into a canonical form by introducing
canonical momenta $\pi^{\mu\nu}$ which are defined by

$$
\pi^{\mu\nu} \equiv \frac{\partial L_g}{\partial (\partial_0 g^{\mu\nu})} = -\frac{\sqrt{\frac{4}{g}}}{2\kappa} (K^{\mu\nu} - \frac{4}{g}g^{\mu\nu}K).
$$

For a given metric (7) the non-vanishing components of $\pi^{\mu\nu}$ are

$$
\pi^{ij} = \sqrt{\frac{4}{g}}\frac{2}{N}D_0R + \frac{1}{2N}(D_0\mu)\frac{4}{g}g^{ij}, \quad (i, j = 1, 2, 3),
$$

(15)

$$
\pi^{55} = \sqrt{\frac{4}{g}}\frac{3}{2\kappa}D_0R e^{-\mu},
$$

(16)

where the derivative $D_0$ is defined by $D_0 \equiv \partial_0 - N^5\partial_5$. With these $\pi^{\mu\nu}$ the Lagrangian density $L_g$ is then written as

$$
L_g = \pi^{\mu\nu}\partial_0\delta^{\mu\nu} - (N\mathcal{H} + N_5\mathcal{H}^5),
$$

(17)

where

$$
\mathcal{H} \equiv -\frac{2\kappa}{\sqrt{g}}(\pi_{\mu\nu}\pi^{\mu\nu} - \frac{1}{3}\pi^2) + \frac{\sqrt{\frac{4}{g}}}{2\kappa} R
$$

(18)

and

$$
\mathcal{H}^5 \equiv -2\nabla_\nu\pi^{5\nu}
$$

(19)

are related with the Hamiltonian $H$ by the equation

$$
H = \int d^4x(N\mathcal{H} + N_5\mathcal{H}^5).
$$

(20)

In the presence of matter fields (i.e. for $\mathcal{L}_m \neq 0$) the field equations obtained by varying $N$ and $N_5$ are the Hamiltonian constraint

$$
\mathcal{H} = N^2\sqrt{\frac{4}{g}T^{00}},
$$

(21)

and the momentum constraint

$$
\mathcal{H}^5 = -N\sqrt{\frac{4}{g}(T^{05} + N_5T^{00})},
$$

(22)

where $T^{AB}$ are expectation values of the stress-energy tensor of 5d matter fields.

Having found constraint equations we now set $N \equiv 1$ and $N_5 \equiv \kappa A_0$; namely, we are considering a metric

$$
\mathcal{L}_g = -dt^2 + R^2(t, x^5)d\Omega_k^2 + \epsilon^{\mu(t,x^5)}[dx^5 + \kappa A_0(t)dt]^2.
$$

(23)
Upon this setting the Hamiltonian constraint in eq.(18) can be recast into more suggestive form:

\[
\mathcal{H} = \sqrt{\frac{g}{\kappa}} G_{00} + \sqrt{\frac{g}{\rho}} \rho_A,
\]

where

\[
G_{00} = 3\left(\frac{\dot{R}}{R}\right)^2 + \frac{3}{2} \frac{\ddot{R}}{R} + \frac{1}{2} \frac{4}{R}
\]

is the 00-component of the 5d Einstein tensor derived from the Tolman metric (1), and

\[
\rho_A = \frac{1}{2\kappa} [\kappa A_0 H_1 + (\kappa A_0)^2 H_2]
\]

with

\[
H_1 = -12\left(\frac{\dot{R}}{R}\right)\left(\frac{R'}{R}\right) - 3 \frac{\dot{R}}{R} \mu' - 3 \frac{R'}{R} \mu,
\]

\[
H_2 = 6\left(\frac{R'}{R}\right)^2 + 3 \frac{R'}{R} \mu'
\]

is the energy density associated with the pure gauge \(A_0\). This is quite surprising. Being a physically non-observable quantity a pure gauge essentially does not contribute to the energy (or Lagrangian) of spacetime due to the vanishing of the field strength. Indeed a pure gauge does not play any role in ordinary 4d theories of spacetime. This, however, is not true anymore in the 5d theory under discussion. Eq.(24) shows that part of the energy is engaged in the dynamics of spacetime, but the rest is stored in \(\rho_A\) in the form of a pure gauge. \(\rho_A\) manifests itself once the metric components have \(x^5\)-dependence.

To find the solution to the constraint equations (21) and (22), \(T^{AB}\) must be definitely given. In our discussion we will focus our attention on the case \(T^{AB} = 0\) because it is not only simple, but it is of particular interest in the context of the discussion in ref.\[4\]. As for the solution to field equations we consider an ansatz of the form

\[
R^2(t, x^5) = R_0^2 + \alpha^2 f_0(x^5 + \kappa \xi(t))^2,
\]

\[
e_{\mu(t, x^5)}^2 = \frac{R'^2}{\alpha^2} = \frac{\alpha^2 f_0^2 (x^5 + \kappa \xi(t))^2}{R_0^2 + \alpha^2 f_0 (x^5 + \kappa \xi(t))^2},
\]

which is obviously a generalization of eqs.(2) and (3), and where both \(R^2\) and \(e^\mu\) are expressed in terms of a single function \(\xi(t)\), meaning that the dynamics of the universe is entirely described by \(\xi(t)\) alone. The function \(\xi(t)\) is of course to be determined.
by field equations for a given (in our case, zero) matter distribution. However, \( \xi(t) \) is subject to the gauge transformation; \( \xi(t) \) experiences a transformation

\[
\xi(t) \rightarrow \tilde{\xi}(t) = \xi(t) - \Lambda(t)
\]

under the gauge transformation

\[
x^5 \rightarrow \tilde{x}^5 = x^5 + \kappa \Lambda(t),
\]

\[
A_0(t) \rightarrow \tilde{A}_0(t) = A_0(t) - \dot{\Lambda}(t).
\]

Since the gauge parameter \( \Lambda(t) \) is totally arbitrary eq.(31) implies that the gauge transformation relates two arbitrarily different universes with dynamics described, respectively, by \( \tilde{\xi}(t) \) and \( \xi(t) \). In the 4d sector these correspond to two arbitrarily evolving 4d isotropic universes with scales described by two arbitrary functions \( R(t, x^5) \) and \( \tilde{R}(t, \tilde{x}^5) \). Indeed, using (24) and (25) one can convert the Hamiltonian constraint (21) into the Einstein equation for a homogeneous, isotropic 4d cosmology

\[
3 \left( \frac{\dot{R}}{R} \right)^2 + 3k \frac{\dot{R}}{R^2} = \kappa \frac{4}{4} \rho_{\text{eff}},
\]

where the 4d effective source \( \frac{4}{4} \rho_{\text{eff}} \) is given by\(^4\)

\[
\frac{4}{4} \rho_{\text{eff}} = -\frac{3}{2\kappa} \frac{\dot{R}}{R} + \frac{3}{\kappa} \alpha^2 - \rho_A.
\]

The first term in eq.(35) is a conventional term, typical of ordinary 5d cosmology\(^5\), while the remaining two terms are new terms that appear only when the metric components have \( x^5 \)-dependence. Now it is important to note that \( \frac{4}{4} \rho_{\text{eff}} \) contains the energy density \( \rho_A \), whose value essentially depends on \( A_0 \). Since \( A_0 \) is subject to the gauge transformation (33) this implies that \( \frac{4}{4} \rho_{\text{eff}} \) can have arbitrary values depending on \( \Lambda(t) \). Thus, any two 4d universes with arbitrarily different 4d sources can be related by the gauge transformation (32) and (33). As discussed before the gauge transformation (32) is valid only within a certain time scale characterized by \( R_c \); to be precise, for the gauge transformation (32) the time scale is given by \( |\kappa \Lambda(t)| \sim \pi R_c \) (we will call this scale \( \tau_c \)). Outside \( \tau_c \), the transformation (32), and consequently the relation (31) are broken down and the gauge equivalence between two universes disappears. Further, in

\(^4\)Note that the relation \( e^\mu = R'^2 / \alpha^2 \) has been used to obtain eq.(35). We also have set \( T^{00} = 0 \); thus, all the gauge equivalent 5d universes considered in this paper have the same (i.e., zero) 5d energy density.
the region sufficiently far from $\tau_c$ (i.e. for $\kappa \xi \gg \pi R_c$) it is convenient to introduce a new time variable $\tilde{t}$ which is defined by

$$x^5 + \kappa \xi(t) \equiv \kappa \xi(\tilde{t}). \quad (36)$$

In that region $\tilde{t}$ approximates $t$, and $x^5$-dependencies of the metric components can be neglected.

Turning back to field equations one can show that $\mathcal{H}^5$ in (19) identically vanishes upon substituting (29) and (30):

$$\mathcal{H}^5 = 0, \quad (37)$$

that is, the momentum constraint is automatically satisfied by the given ansatz. However, $\mathcal{H}$ and $\rho_A$ in eqs.(18) and (26) do not vanish identically upon substituting (29) and (30); they are calculated to give

$$\mathcal{H} = 3 f_0 \sqrt{\frac{4}{\kappa}} \frac{\sqrt{g}}{R^2} \{ \kappa^2 \alpha^2 (A_0 - \dot{\xi})^2 - 1 \},$$

$$\sqrt{\frac{4}{\kappa}} \frac{\sqrt{g}}{R^2} \rho_A = 3 \kappa \alpha^2 f_0 \frac{\sqrt{g}}{R^2} [A_0 (A_0 - 2 \dot{\xi})]. \quad (39)$$

In eq.(39), $\rho_A$ vanishes not only for $A_0 = 0$, it also vanishes when $A_0 = 2 \dot{\xi}$. Also with $A_0 = 0$ the solution to the Hamiltonian constraint $\mathcal{H} = 0$ coincides with the solution described by (2) and (3) as it should be. However, it is important to note that the equation $\mathcal{H} = 0$ does not generally determine $A_0$ and $\xi$ separately; it only determines the combination $A_0 - \dot{\xi}$, which is gauge invariant under the combined transformations (31) and (33). In fact, within $\tau_c$, $A_0$ and $\xi$ are not determined even by field equations as can be seen in the followings. The Lagrangian density calculated from eq.(9) takes the form

$$\mathcal{L}_g = 3 f_0 \frac{\sqrt{g}}{R^2} \{ \kappa^2 \alpha^2 (A_0 - \dot{\xi})^2 + 1 \},$$

but one can verify that the above $\mathcal{L}_g$ is simply a sum of total derivative terms\(^5\). Thus the variation of the action $I_g$ always vanishes for any $\xi$ and $A_0$:

$$\delta \xi I_g = \delta A_0 I_g = 0, \quad (41)$$

which means that the field equations are trivially satisfied by any $\xi$ and $A_0$, and any universe described by $(\xi(t), A_0(t))$ is allowed within $\tau_c$. This supports our conjecture that the gauge equivalent universes are equally allowed in this region.

\(^5\)One can use the relation $e^\nu = R^2/\alpha^2$ in eq.(30) to show that $\sqrt{g}/R^2 = \partial_\nu [\sqrt{g} R e^{-\nu}/2\alpha^2 R]$. 

6
What about outside $\tau_c$ then? Since $x^5$-dependencies of $R^2$ and $e^\mu$ disappear in this region (see eq.(36)) all the covariant derivatives $D_0$ in $\pi^{\mu\nu}$ (or $K^{\mu\nu}$) are replaced by ordinary derivatives $\partial_0$, and therefore the Lagrangian (or the action) does not include $A_0$ anymore. In this case the action in fact takes the same form as the conventional action associated with the standard 5d Robertson-Walker metric, so the solution that minimize the action is expected to be [4, 5].

$$ds^2 = -dt^2 + [R_0^2 - k(t - t_0)^2]d\Omega_2^2 + \frac{(t - t_0)^2}{[R_0^2 - k(t - t_0)^2]}[dx^5 + \kappa A_0(t)dt]^2,$$  \hspace{1cm} (42)

where $A_0(t)$ is arbitrary$^6$. From this we see that the pure gauge can exist even in the region outside $\tau_c$. However, in this region the pure gauge does not play any physically important role (actually, it is insensible) because it manifests itself always through the covariant derivative $D_0 = \partial_0 - \kappa A_0 \partial_5$, but the metric components are $x^5$-independent there. In fact, $dx^5 + \kappa A_0(t)dt$ in (42) can always be replaced simply by $dx^5$ by an appropriate coordinate transformation, then we recover the ordinary 5d Robertson-Walker metric representing a radiation-dominated universe.

The above discussion may be extended to the general case. For instance, the ansatz in (29) and (30) may not be relevant to the case of $T^{AB} \neq 0$. Recall that it is just a generalization of the solution to vacuum($T^{AB} = 0$) field equations. The most general ansatz for $R^2$ and $e^\mu$ would be in fact of the form

$$R^2(t, x^5) = R^2(x^5 + \kappa \xi(t)),$$  \hspace{1cm} (43)

$$e^\mu(t,x^5) = \frac{R^2}{\alpha^2},$$  \hspace{1cm} (44)

and one can verify that the momentum constraint $\mathcal{H}^5$ identically vanishes as before upon substituting (43) and (44):

$$\mathcal{H}^5 = 0.$$  \hspace{1cm} (45)

This equation agrees with (22) when

$$N^5 = \kappa A_0 = -T^{05}/T^{00},$$  \hspace{1cm} (46)

which suggests that the pure gauge $A_0$ can be interpreted (in a 5d sense) as a momentum density (normalized by $\kappa T^{00}$) along the fifth direction. The Hamiltonian constraint on the other hand takes the form

$$\mathcal{H} = \frac{3}{\kappa^2 R^2} \frac{\sqrt{g}}{4} \frac{R^2}{2} (A_0 - \dot{\xi})^2 (R^2)^\prime - f_0],$$  \hspace{1cm} (47)

$^6A_0(t)$ is arbitrary because it does not appear in the action, and the equation of motion for $A_0(t)$ does not exist.
which is obviously a generalization of eq. (38), and still the constraint equation (21) does not determine $A_0$ and $\xi$ separately\(^7\). Finally, The Lagrangian density $\mathcal{L}_g$ (i.e., the generalization of (40)) is calculated to give

$$\mathcal{L}_g = \frac{3\sqrt{\frac{4}{\kappa}g}}{R^2} \left(\frac{\kappa^2}{2} (A_0 - \dot{\xi})^2 (R^2)' + f_0\right),$$

(48)

and one can verify that the terms in eq. (48) only contribute to surface terms of the action $I_g$ as before\(^8\). Thus, $A_0$ and $\xi$ are not fixed by field equations in this general case either.

**Summary**

We have examined in the category of 5d Kaluza-Klein theory a gauge equivalence of $x^5$-dependent solutions each of which describes in the 4d sector an arbitrarily evolving isotropic, homogeneous universe with a zero, or non-zero pure gauge. The main points we have observed are: (1) Within a certain time scale $\tau_c$ (which is characterized by the compactification radius $R_c$) any arbitrarily evolving 4d universe with an appropriate pure gauge is allowed to exist by field equations and these isotropic, homogeneous universes are all gauge equivalent as long as they are of the same topology. (2) In this case the pure gauge $A_0$ plays a role of the 4d effective matter source as does the dynamics of the fifth dimension\(^9\). A pure gauge has its own energy density which manifests itself when the metric components have $x^5$-dependence. (3) Outside $\tau_c$ the evolution of the universe is set by field equations. In particular, for $T^{AB} = 0$ the only allowed state of universe is a radiation-dominated universe with an arbitrary pure gauge which can be removed by a coordinate transformation.

The above result naturally leads us to a certain conjecture which perhaps makes our point more clear. The suggested conjecture is that it may be totally meaningless in 5d Kaluza-Klein theory (or even in any higher-dimensional theory) to distinguish one 4d universe from another (as long as they are of the same topology) within a certain time region $\tau_c$ which is expected to be of order of the compactification scale, because there is no known gauge-fixing mechanism to select a preferred universe among the infinite number of gauge equivalent universes. The physics in such a region is always subtle and complicated due to the existence of such as an initial singularity, or a

\(^7\)In the case of $T^{00} \neq 0$, the quantity $\sqrt{T^{00}}$ must be taken to be gauge invariant in order for $\mathcal{H}$ to be gauge invariant.

\(^8\)In addition to the equation in footnote 5, eq. (44) also implies that $\sqrt{g}(R^2)' / R^2 = \partial_5 [\sqrt{g} R'/ R]$.\(^9\)

\(^9\)It is well-known that the dynamics of the fifth dimension acts as a 4d effective radiation source. See, for instance, ref. [5].
quantum fluctuation etc. The above conjecture may provide a possibility of avoiding such difficulties. For instance, we can avoid the initial singularity by dealing with a gauge equivalent static universe with no initial singularity, instead of dealing with an evolving universe with initial singularity. This amounts to say that the initial singularity is alleviated due to the existence of the internal dimension(s) to the extent of its size. In short, the above conjecture tells that within $\tau_c$ we do not need to consider detailed dynamics or matter contents of 4d universes. We may simply ignore them!

**Acknowledgments**

This research was supported by the Kyungsung University research grants in 2002.

**References**

[1] A. Lukas, Burt A. Ovrut, K.S. Stelle and D. Waldram, Phys.Rev. **D59** (1999) 086001 ; N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 429** (1998) 263 ; Phys. Rev. **D 59** (1999) 086004 ; R. Sundrum, Phys. Rev. **D 59** (1999) 085009

[2] A. Chodos and S. Detweiler, Phys. Rev. **D 21** (1980) 2167 ; K. Behrndt and S. Förste, Phys. Lett. **B 320** (1994) 253 ; Nucl. Phys. **B 430** (1994) 441 ; F. Larsen and F. Wilczek, Phys. Rev. **D 55** (1997) 4591

[3] See for instance, A. Lukas, Burt A. Ovrut and D. Waldram, Phys.Rev. **D60** (1999) 086001 ; C. Csáki et al., Phys.Lett. **B462** (1999) 34 ; P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. **B 565** (2000) 269 ; C. Deffayet, Phys.Lett. **B502** (2001) 199

[4] G. Y. Jun and P. S. Kwon, Phys. Lett. **B 500** (2001) 209

[5] R. A. Matzner and A. Mezzacappa, Phys. Rev. **D 32** (1985) 3114