Matrix factorizations and Cohomological Field Theories

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Abstract

We give a purely algebraic construction of a cohomological field theory associated with a quasihomogeneous isolated hypersurface singularity $W$ and a subgroup $G$ of the diagonal group of symmetries of $W$. This theory can be viewed as an analogue of the Gromov-Witten theory for an orbifoldized Landau-Ginzburg model for $W/G$. The main geometric ingredient for our construction is provided by the moduli of curves with $W$-structures introduced by Fan, Jarvis and Ruan. We construct certain matrix factorizations on the products of these moduli stacks with affine spaces which play a role similar to that of the virtual fundamental classes in the Gromov-Witten theory. These matrix factorizations are used to produce functors from the categories of equivariant matrix factorizations to the derived categories of coherent sheaves on the Deligne-Mumford moduli stacks of stable curves. The structure maps of our cohomological field theory are then obtained by passing to the induced maps on Hochschild homology. We prove that for simple singularities a specialization of our theory gives the cohomological field theory constructed by Fan, Jarvis and Ruan using analytic tools.

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Introduction

The notion of a cohomological field theory was introduced in [34] (see also [41]) to axiomatize properties of quantum cohomology and Gromov-Witten invariants and to provide a basis for formulating some mathematical aspects of mirror symmetry. Recall that a (complete) cohomological field theory (CohFT) is an algebraic structure on a vector space \( \mathcal{H} \) (called the state space of the theory) with a collection of operations indexed by homology classes of the Deligne-Mumford moduli spaces \( \overline{M}_{g,n} \) of stable curves with marked points.
The main example of a CohFT is provided by the Gromov-Witten theory associated with a smooth projective variety $X$ (or more generally, with a compact symplectic manifold). In this context mirror symmetry can be viewed as an isomorphism of CohFTs originating from two different geometric inputs (the so called A- and B-models, where the GW-theory corresponds to the A-model and the B-model is related to deformations of complex structures of $X$). The part of the CohFT data corresponding to genus zero curves can be described as a formal Frobenius manifold structure, which leads in the case of GW-theory to the notion of quantum cohomology, a certain deformation of the cohomology ring of $X$. Through Frobenius manifolds CohFTs are related to integrable hierarchies of systems of partial differential equations (see [9]).

The first example of CohFTs besides GW-theory was provided by the theory of $r$-spin curves constructed in [60], [61], [23], [48] and [47] (see also [42]). The corresponding Frobenius manifolds are isomorphic to the ones constructed by Saito for simple singularities of type $A_{r-1}$, and the corresponding integrable hierarchies are the Gelfand-Dickey hierarchies (see [23], [12]).

Starting from the work of Givental [20] it was realized (see [12], [56]) that an arbitrary generically semisimple Frobenius manifold extends to a unique CohFT. This construction can be applied to Saito's Frobenius manifold of any quasihomogeneous isolated singularity $w$ to give a CohFT for such a singularity, which corresponds to the B-side of the Landau-Ginzburg model related to $w$ (see [39]). A CohFT corresponding to the A-side for these LG-models was recently constructed by Fan, Jarvis and Ruan in [14], [15]. Their construction is based on the study of a certain PDE over coverings of moduli spaces of stable curves (that generalize the moduli spaces of $r$-spin curves). More precisely, they construct a virtual fundamental cycle on the moduli space of solutions of this PDE corresponding to a linear perturbation of the potential $w$. The dependence of this virtual class on the perturbation is governed by the state space of the theory which is given by the orbifolded Milnor ring of $(w,G)$ (see [59], [28]), where $G$ is a finite group of diagonal symmetries of $w$. One of the main results of [14] is that in the case of ADE simple singularities this CohFT is isomorphic to Givental's CohFT associated with Saito's Frobenius manifold of the dual singularity (which is the same singularity for the series $E$ and for series $D$ with non-maximal symmetry group). Using the work of Frenkel-Givental-Milanov (see [21], [17]) it is shown in [14] that the associated total potential function is a $\tau$-function of the corresponding Kac-Wakimoto hierarchy. Another recent result on the mirror symmetry for the Landau-Ginzburg models is due to Krawitz [35] who established the isomorphism between Frobenius algebras associated with A- and B-side of the dual invertible quasi-homogeneous potentials.

In this paper we present a purely algebraic (and perhaps more general) version of the FJR-theory. The main role in our construction is played by matrix factorizations. These are generalizations of complexes obtained by replacing the condition $d^2 = 0$ with $d^2 = w$ (see [11], [5]). Matrix factorizations appeared in physics in connection with open-closed topological string theories (see [24], [25], [26]). They are also related to an important invariant of the potential $w$, the singularity category (see [43], [54], [40]). The category of matrix factorizations for an isolated singularity fits naturally into the framework of noncommutative geometry developed from the point of view of dg-categories or $A_\infty$-algebras (see [10], [27]).
Our main construction gives a CohFT whose state space is built from the Hochschild homology of the dg-categories of equivariant matrix factorizations associated with a quasi-homogeneous isolated singularity \( w \) and a finite group of symmetries \( G \). This CohFT carries a priori more information than the FJR-theory: it is a CohFT with coefficients in the representation ring \( R \) of \( G \). Conjecturally, the reduced theory obtained by the specialization \( R \to \mathbb{C} \) is equivalent to the FJR-theory. We show that this is true for all simple singularities.

Let \( w(x_1, \ldots, x_n) \) be a quasihomogeneous polynomial with an isolated singularity at the origin. We fix the degrees of quasihomogeneity \( d_j = \deg(x_j) \) and we set \( q_j = d_j/d \), where \( d = \deg(w) \). We denote the Milnor ring of \( w \) by \( A_w = \mathbb{C}[x_1, \ldots, x_n]/(\partial_1 w, \ldots, \partial_n w) \) and set \( H(w) = A_w \otimes (dx_1 \wedge \ldots \wedge dx_n) \).

This space is canonically isomorphic to the Hochschild homology of the category of matrix factorizations of \( w \). Let \( G_w \subset (\mathbb{C}^*)^n \) be the group of all diagonal symmetries of \( w \). For each \( \gamma \in G_w \) consider the subspace of invariants \((\mathbb{A}^n)^{\gamma} \subset \mathbb{A}^n\) and set \( w_\gamma = w|_{(\mathbb{A}^n)^{\gamma}} \). Then \( w_\gamma \) still has an isolated singularity at zero. Let \( G \subset G_w \) be a finite subgroup containing the exponential grading element

\[
J = (\exp(2\pi i q_1), \ldots, \exp(2\pi i q_n)) \in G_w,
\]

and let \( R = \mathbb{C}[\hat{G}] \) be the representation ring of \( G \). Our construction associates with such \( G \) a CohFT with coefficients in \( R \) on the state space

\[
\mathcal{H}(w, G) = \bigoplus_{\gamma, \gamma' \in G} H(w_{\gamma, \gamma'})^G,
\]

where \( w_{\gamma, \gamma'} \) is the restriction of \( w \) to the subspace of invariants \((\mathbb{A}^n)^{\{\gamma, \gamma'\}} \subset \mathbb{A}^n\). We view this space as an \( R \)-module via the \( G \)-grading given by \( \gamma' \). The component corresponding to \( \gamma' = 1 \) admits a certain twist by a Todd class, and conjecturally this reduced CohFT is isomorphic to the FJR-theory. In [8] the FJR-theory for the quintic threefold is related to a certain specialization of the corresponding Gromov-Witten theory. One can speculate that our CohFTs associate with other elements \( \gamma' \in G \) are also related to some specializations of the Gromov-Witten invariants in the Calabi-Yau case.

As in [14], the main geometric ingredient of our construction is the moduli stacks of the so-called \( w \)-curves which are orbicurves \( \mathcal{C} \) with marked (orbi-)points together with a collection of line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_n \) satisfying certain constraints coming from the monomials of \( w \). However, we observe that these coverings should really be viewed as an attribute of the group of symmetries \( G \) rather than of the potential \( w \). More precisely, we reformulate the notion of a \( w \)-structure using principal \( \Gamma \)-bundles, where \( \Gamma \) is an extension of \( \mathbb{G}_m \) by \( G \). This leads to the moduli stacks of \( \Gamma \)-spin curves that replace \( w \)-curves considered in [14]. This technical device allows us to remove the assumption that \( G_w \) is finite imposed in [14] and to show that every finite subgroup of \( G_w \) containing \( J \) is admissible in the sense of [14].

Our results give also a categorification of these CohFTs in a certain weak sense. Namely, we construct a collection of functors inducing the CohFT maps after passing to Hochschild homology (up to rescaling). These functors are given by kernels which are certain matrix
factorizations, called fundamental matrix factorizations, on the product of the moduli spaces of $\Gamma$-spin curves with affine spaces. In some sense these fundamental matrix factorizations play a role similar to that of the virtual fundamental class in the GW-theory. The factorization axiom holds on the categorified level after passing to appropriate finite covers of the relevant moduli spaces. It seems plausible that there should also be a version of quantum K-theory in our setup (see [38]).

Note that representations of functors between categories of matrix factorizations by kernels are discussed extensively in the work of Ballard, Favero and Katzarkov [4] (in the context of graded algebras), where the authors give interesting applications to Homological Mirror Symmetry and Hodge theory.

One of the obstacles that prevents us from comparing the specialization of our CohFT at $\gamma' = 1$ with the FJR-theory is that we were able to prove the analog of the Dimension Axiom of [14, Sec. 4] only in some special cases. In particular, we verified it for all simple singularities, which together with the computation of the corresponding Frobenius algebras enables us to show that in this case our reduced CohFT is isomorphic to that of Fan-Jarvis-Ruan (see Section 7).

The paper is organized as follows. In Section 1 we review basics of the theory of matrix factorizations. In particular, we discuss the relation between the derived category of matrix factorizations and the singularity category, prove auxiliary results about push-forwards of matrix factorizations and establish some properties of Koszul matrix factorizations, crucial for our main construction. In Section 2 we specialize to the case of a quasihomogeneous isolated singularity. We consider the dg-category of equivariant matrix factorizations and compute its Hochschild homology space with the canonical pairing. We also discuss functors between categories of matrix factorizations given by kernels. In Section 3 we recast the notion of a $w$-structure from [14] in terms of torsors over some commutative algebraic groups and consider the corresponding moduli stacks of $\Gamma$-spin curves. Section 4 is the technical core of the paper. Here we construct the fundamental matrix factorization over the product of the moduli space of $\Gamma$-spin curves with an affine space. The construction shares some of the features with the construction of the Witten’s top Chern class in [48], however, it uses also some new ingredients, notably the push-forward of matrix factorizations. Also, the proof of independence of the fundamental matrix factorization on choices of resolutions is based on a different idea (we use properties of regular Koszul matrix factorizations). In Section 5 we define the CohFTs associated with a pair $(w, G)$ using the functor associated with the fundamental matrix factorization and passing to the induced map on Hochschild homology. We also prove for our theory analogs of all properties established in [14, Sec. 4] for the FJR-theory except the Dimension Axiom (see 5.6). In Section 6 we give a recipe for calculating genus zero three-point correlators for our theory which are responsible for the Frobenius algebra structure on the state space. In Section 7 we compute all such correlators in the case when $w$ is a simple singularity of type $A$, $D$, $E_6$, $E_7$ or $E_8$. We also prove in Section 7.6 that our reduced CohFT is isomorphic to the FJR-theory in this case. In Appendix we discuss the constructions of functoriality for Hochschild homology. In particular, we prove that the construction used in [49] is compatible with the standard one (see Theorem 8.0.3).

Notation and conventions. We work with schemes and stacks over $\mathbb{C}$ and all our (dg-)}
categories are $\mathbb{C}$-linear. For a finite abelian group $G$ we denote by $\hat{G}$ the dual abelian group. For a commutative algebraic group $\Gamma$ we denote by $X(\Gamma)$ the group of characters of $\Gamma$. We say that a triangulated category $\mathcal{D}$ is generated by a set of objects $(E_i)$ if the minimal full triangulated subcategory containing $E_i$ and closed under taking direct summands is the entire $\mathcal{D}$. For an additive category $\mathcal{C}$ we denote by $\text{Com}(\mathcal{C})$ the category of complexes over $\mathcal{C}$.

We always assume that our algebraic stacks are Noetherian and semi-separated. For such an algebraic stack $X$ we denote by $\text{Coh}(X)$ (resp., $\text{Qcoh}(X)$; resp., $D^b(X)$) the category of coherent sheaves (resp., quasicoherent sheaves; resp., bounded derived category of coherent sheaves) on $X$. By [2, Cor. 2.11], $D^b(X)$ is equivalent to the full subcategory of the bounded derived category of $\mathcal{O}_X$-modules consisting of complexes with coherent cohomology. By a vector bundle on a stack we mean a locally free sheaf of $\mathcal{O}$-modules of finite rank.

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1 Matrix factorizations on stacks

In this section we review the theory of matrix factorizations on stacks from [50]. We also establish some technical results on push-forwards of matrix factorizations and on Koszul matrix factorizations.

1.1 Categories of matrix factorizations

Let us recall some basic definitions from [50].

**Definition 1.1.1.** Let $X$ be an algebraic stack, $L$ a line bundle on $X$, and $W \in H^0(X, L)$ a section (called a potential). A matrix factorization $\bar{E} = (E_\bullet, \delta_\bullet)$ of $W$ on $X$ consists of a pair of vector bundles (i.e., locally free sheaves of finite rank) $E_0, E_1$ on $X$ together with homomorphisms

$$\delta_1 : E_1 \rightarrow E_0 \quad \text{and} \quad \delta_0 : E_0 \rightarrow E_1 \otimes L,$$

such that $\delta_0 \delta_1 = W \cdot \text{id}$ and $\delta_1 \delta_0 = W \cdot \text{id}$.

Sometimes we will assume that the potential $W$ is not a zero divisor, i.e., the morphism $W : \mathcal{O}_X \rightarrow L$ is injective.

In the case $W = 0$ we have $\delta_0 \delta_1 = \delta_1 \delta_0 = 0$, so we can define cohomology of a matrix factorization $\bar{E}$ by

$$H^0(\bar{E}) = \ker(\delta_0) / \delta_1(E_1), \quad H^1(\bar{E}) = \ker(\delta_1) / \delta_0(E_0 \otimes L^{-1}). \quad (1.1)$$
Definition 1.1.2. We define the dg-category $\text{MF}(X,W)$ of matrix factorizations of $W$ as follows. For a pair of matrix factorizations $\bar{E}$ and $\bar{F}$ we define a $\mathbb{Z}$-graded complex of morphisms $\text{Hom}_{\text{MF}}(\bar{E}, \bar{F})$ by setting

$$\text{Hom}_{\text{MF}}(\bar{E}, \bar{F})^{2n} = \text{Hom}(E_0, F_0 \otimes L^n) \oplus \text{Hom}(E_1, F_1 \otimes L^n),$$

$$\text{Hom}_{\text{MF}}(\bar{E}, \bar{F})^{2n+1} = \text{Hom}(E_0, F_1 \otimes L^{n+1}) \oplus \text{Hom}(E_1, F_0 \otimes L^n).$$

The differential on $\text{Hom}_{\text{MF}}(\bar{E}, \bar{F})$ is given by

$$df = \delta_E \circ f - (-1)^{|f|} f \circ \delta_E.$$  \quad (1.2)

We denote by $\text{HMF}(X,W) = H^0\text{MF}(X,W)$ the corresponding homotopy category. We will usually omit $X$ from the notation. As in the standard case considered in [43] the category $\text{HMF}(W)$ has a triangulated structure (see [50, Def. 1.3]).

One can conveniently write the complexes $\text{Hom}_{\text{MF}}(\bar{E}, \bar{F})$ using the half-twist notation (see [50, Def. 1.2]):

$$\text{Hom}_{\text{MF}}(\bar{E}, \bar{F})^i = \text{Hom}_{i \mod 2}(E(L^{1/2}), F(L^{1/2}) \otimes L^{1/2}),$$

where $E(L^{1/2}) = E_0 \oplus (E_1 \otimes L^{1/2})$ and $\text{Hom}_{i \mod 2}$ denotes morphisms of $\mathbb{Z}/2$-graded bundles, homogeneous of degree $i \mod 2$.

We also consider the dg-category $\text{MF}^\infty(X,W)$ and the corresponding homotopy category $\text{HMF}^\infty(X,W)$ of \textit{quasi-matrix factorizations} defined using locally free sheaves of not necessarily finite rank (see [50, Def. 1.4]). An even larger dg-category $\text{QMF}(X,W)$ of \textit{quasicoherent matrix factorizations} is obtained if we allow $E_0$ and $E_1$ to be arbitrary quasicoherent sheaves. This category is featured prominently in more recent treatments of matrix factorizations in the non-affine case (see [54] and [40]). We will mostly use it in the case when $W = 0$. Note that in this case we have a natural class of quasi-isomorphisms in the corresponding homotopy category (defined using the cohomology (1.1)). Localizing with respect to quasi-isomorphisms we get the derived category $\text{DQMF}(X,0)$. We denote by $\text{DQMF}_c(X,0) \subset \text{DQMF}(X,0)$ the full subcategory of quasicoherent matrix factorizations of 0 with coherent cohomology. Similarly, replacing quasicoherent sheaves with coherent sheaves one can talk about \textit{coherent matrix factorizations} of 0 and define the derived category of coherent matrix factorizations $\text{DCMF}(X,0)$.

Let us introduce some natural operations on matrix factorizations.

For a bounded complex of vector bundles on $X$, $(C^\bullet, \delta_C)$, and a quasicoherent matrix factorization $\bar{E} = (E_\bullet, \delta_\bullet)$ of $W \in H^0(X,L)$ we define the matrix factorization $C^\bullet \otimes \bar{E}$ of $W$ by setting

$$(C^\bullet \otimes E)(L^{1/2}) = C(L^{-1/2}) \otimes E(L^{1/2}),$$  \quad (1.3)

with the differential $\delta_C \otimes \text{id} + \text{id} \otimes \delta$, where $C(L^{-1/2}) = \oplus_{n \in \mathbb{Z}} C^n \otimes L^{-n/2}$ with the $\mathbb{Z}/2$-grading induced by the $\mathbb{Z}$-grading and the induced differential $\delta_C : C(L^{-1/2}) \to C(L^{-1/2}) \otimes L^{1/2}$. Explicitly,

$$(C^\bullet \otimes E)_i = \oplus_{n \in \mathbb{Z}} C^n \otimes E(L^{1/2})_{n+i} \otimes L^{-(n+i)/2}, \quad \text{for } i = 0, 1.$$
Note that we have natural isomorphisms

\[(C^*[1]) \otimes \bar{E} \simeq C^* \otimes (\bar{E}[1]) \simeq (C^* \otimes \bar{E})[1].\]

With a quasicoherent matrix factorization \(\bar{E} = (E_\bullet, \delta_\bullet)\) we associate a \(\mathbb{Z}\)-graded complex of vector bundles on \(X_0\)

\[
\text{com}(\bar{E}) : \ldots \to (E_0 \otimes L^{-1})|_{X_0} \xrightarrow{\delta_0} E_1|_{X_0} \xrightarrow{\delta_1} E_0|_{X_0} \xrightarrow{\delta_0} (E_1 \otimes L)|_{X_0} \to \ldots \quad (1.4)
\]

where \(\delta_i\) is induced by \(\delta_i\), and \(E_0|_{X_0}\) is placed in degree 0 (for quasi-matrix factorizations this construction was considered in [50, Sec. 1]). By [50, Lem. 1.5], this complex is exact provided \(W\) is not a zero divisor and \(\bar{E}\) is a quasi-matrix factorization. This construction extends to a dg-functor

\[
\text{com} : \text{QMF}(W) \to \text{Com(Qcoh}(X_0))
\]

that induces an exact functor between the corresponding homotopy categories. It is easy to see that for a bounded complex \(C^*\) of vector bundles on \(X\) one has a natural isomorphism

\[
\text{com}(C^* \otimes \bar{E}) \simeq C^*|_{X_0} \otimes \text{com}(\bar{E}). \quad (1.5)
\]

In the case \(W = 0\) the complex \(\text{com}(\bar{E})\) satisfies

\[
H^{2n} \text{com}(\bar{E}) = H^0(\bar{E}) \otimes L^n, \quad H^{2n-1} \text{com}(\bar{E}) = H^1(\bar{E}) \otimes L^n.
\]

In particular, a closed morphism \(q\) of quasicoherent matrix factorizations of 0 is a quasi-isomorphism if and only if \(\text{com}(q)\) is a quasi-isomorphism.

**Definition 1.1.3.** For a pair of potentials \(W, W' \in H^0(X, L)\) the tensor product dg-functor

\[
\text{MF}(W) \otimes \text{MF}(W') \to \text{MF}(W + W')
\]

is defined as follows. For \(\bar{E} = (E, \delta_E)\) and \(\bar{F} = (F, \delta_F)\) we set

\[
(\bar{E} \otimes \bar{F})_0 = E_0 \oplus F_0 \oplus E_1 \otimes F_1 \otimes L \text{ and } E_0 \otimes F_1 \oplus E_1 \otimes F_0
\]

with the differential is induced by \(\delta_E\) and \(\delta_F\). Note that

\[
(E \otimes F)(L^{1/2}) = E(L^{1/2}) \otimes F(L^{1/2}).
\]

**Definition 1.1.4.** For a bounded complex \((C^*, \delta_C)\) of vector bundles and a line bundle \(L\) on \(X\) let us define \((\text{mf}(C^*), \delta)\), a matrix factorization of 0 in \(H^0(X, L)\), by setting

\[
\text{mf}(C^*)_0 = \bigoplus_n C^{2n} \otimes L^{-n}, \quad \text{mf}(C^*)_1 = \bigoplus_n C^{2n-1} \otimes L^{-n}
\]

with the differential \(\delta\) induced by \(\delta_C\).
A straightforward check shows that the tensor product operations (1.3) and (1.6) are consistent.

**Lemma 1.1.5.** One has a natural isomorphism

\[
C^\bullet \otimes \bar{E} \simeq \text{mf}(C^\bullet) \otimes \bar{E}
\]

in \(\text{MF}(W)\), on the left we use the operation (1.3). Hence, by (1.5) we have

\[
\text{com}(\text{mf}(C^\bullet) \otimes \bar{E}) \simeq C^\bullet|_{X_0} \otimes \text{com}(\bar{E}).
\]

We have the duality dg-functor

\[
\text{MF}(W) \to \text{MF}(-W)
\]

(1.8)

sending \(E = (E, \delta_E)\) to \(E^* = (E(L^{1/2})^\vee(L^{-1/2}), \delta^*)\). In other words, the even part of \(E^*\) is \(E_0^\vee\) and its odd part is \(E_1^\vee \otimes L^{-1}\).

**Lemma 1.1.6.** For a pair of matrix factorizations \(\bar{E}\) and \(\bar{F}\) in \(\text{MF}(X, W)\) we have an isomorphism of complexes

\[
\mathcal{H}\text{om}_{\text{MF}}(\bar{E}, \bar{F}) \simeq H^0(X, \text{com}(\bar{E}^* \otimes \bar{F})),
\]

where \(\bar{E}^* \otimes \bar{F} \in \text{MF}(X, 0)\).

For a morphism of stacks \(f : X' \to X\), a line bundle \(L\) over \(X\) and a section \(W \in H^0(X, L)\) we have natural pull-back functors on matrix factorizations: a dg-functor

\[
f^* : \text{MF}(X, W) \to \text{MF}(X', f^*W),
\]

where \(f^*W\) is the induced section of \(f^*L\) on \(X'\), and the induced exact functor

\[
f^* : \text{HMF}(X, W) \to \text{HMF}(X, f^*W).
\]

**Definition 1.1.7.** The external product of \((X, W)\) and \((X', W')\) (where \(W \in H^0(X, L)\) and \(W' \in H^0(X', L')\)) is defined as a pair \((U, W \otimes W')\), where \(U \to X \times X'\) is the \(\mathbb{G}_m\)-torsor associated with the line bundle \(L \otimes L^{-1}\). Let \(p_1 : U \to X\) and \(p_2 : U \to X'\) be the natural projections. Then we have an isomorphism \(L_U = p_1^*L \simeq p_2^*L'\), and we define

\[
W \otimes W' = p_1^*W + p_2^*W' \in H^0(U, L_U).
\]

Combining the pull-back and tensor product functors defined above we obtain the external tensor product functor

\[
\text{MF}(X, W) \otimes \text{MF}(X', W') \to \text{MF}(U, p_1^*W + p_2^*W).
\]

(1.9)

We can define Koszul matrix factorizations \(\{\alpha, \beta\}\) in our setting (see [33], [49]).
Definition 1.1.8. Let $X$, $L$ and $W$ be as above, and let $V$ be a vector bundle $V$ on $X$. For global sections

$$\alpha \in H^0(X, V \otimes L), \ \beta \in H^0(X, V^\vee)$$
such that $\langle \alpha, \beta \rangle = W$

we define the Koszul matrix factorization $\{\alpha, \beta\}$ of $W$ by

$$\{\alpha, \beta\} = \left( \bigwedge^\bullet (V \otimes L^{1/2})(L^{-1/2}), \delta_{\alpha, \beta} \right), \quad (1.10)$$

with the $\mathbb{Z}/2$-grading on $\bigwedge^\bullet (V \otimes L^{1/2})$ induced by the $\mathbb{Z}$-grading. The differential is given by

$$\delta_{\alpha, \beta} = \alpha \wedge + \iota(\beta),$$

where $\iota(\beta)$ is the contraction by $\beta$.

Explicitly,

$$\{\alpha, \beta\}_0 = O_X \oplus (\wedge^2 V \otimes L) \oplus (\wedge^4 V \otimes L^2) \oplus \ldots,$$

$$\{\alpha, \beta\}_1 = V \oplus (\wedge^3 V \otimes L) \oplus (\wedge^5 V \otimes L^2) \oplus \ldots.$$

Lemma 1.1.9. For $\alpha \in H^0(X, V \otimes L)$ let

$$K^\bullet(\alpha) = \left( \bigwedge^\bullet (V \otimes L), \alpha \wedge ? \right) \quad (1.11)$$

be the Koszul complex. Then one has an isomorphism of matrix factorizations in $\text{MF}(0)$

$$\{\alpha, 0\} \simeq \text{mf}(K^\bullet(\alpha)).$$

The proof is straightforward.

1.2 Equivariant matrix factorizations

Let $X$ be a stack and $\Gamma$ an affine algebraic group acting on $X$. Let also $W$ be a regular function on $X$, semi-invariant with respect to $\Gamma$. Thus, we have a character $\chi : \Gamma \to \mathbb{G}_m$ such that

$$W(\gamma \cdot x) = \chi(\gamma)W(x)$$

for $\gamma \in \Gamma$, $x \in X$. Recall that $\Gamma$-equivariant matrix factorizations of $W$ with respect to the character $\chi$ are defined as pairs of $\Gamma$-equivariant vector bundles $E_0, E_1$ on $X$ together with $\Gamma$-invariant homomorphisms

$$\delta_1 : E_1 \to E_0 \quad \text{and} \quad \delta_0 : E_0 \to E_1 \otimes \chi,$$

such that $\delta_0 \delta_1 = W \cdot \text{id}$ and $\delta_1 \delta_0 = W \cdot \text{id}$.

The dg-category $\text{MF}_{\Gamma}(X, W)$ of $\Gamma$-equivariant matrix factorizations is naturally equivalent to the category $\text{MF}(X/\Gamma, \underline{W})$, where $\underline{W}$ is the section induced by $W$ of the line bundle over $X/\Gamma$ associated with $\chi$ (see [50, Prop. 2.2]).
Example 1.2.1. Let $V$ be a $\Gamma$-equivariant vector bundle on $X$. For $\Gamma$-invariant sections $\alpha \in H^0(X, V \otimes \chi)^G$ and $\beta \in H^0(U, V^\vee)^\Gamma$ the Koszul matrix factorization $\{\alpha, \beta\}$ (see (1.10)) is $\Gamma$-equivariant.

We will often consider the following special situation. Let $\Gamma$ be a commutative algebraic group with a surjective homomorphism $\chi : \Gamma \to \mathbb{G}_m$ such that $G := \ker(\Gamma)$ is finite, and let $X$ be a stack with the trivial action of $\Gamma$.

Let $\chi_1, \ldots, \chi_d \in \widehat{\Gamma}$ be a system of representatives for the cosets of the subgroup $\langle \chi \rangle \subset \widehat{\Gamma}$ of characters of $\Gamma$. A $\Gamma$-equivariant quasicoherent matrix factorization $\bar{E}$ consists of a pair of $\hat{\Gamma}$-graded quasicoherent sheaves $E_0 = \bigoplus_{\xi \in \hat{\Gamma}} E_{0, \xi}$, $E_1 = \bigoplus_{\xi \in \hat{\Gamma}} E_{1, \xi}$ and a differential $\delta$ on $E_0 \oplus E_1$, such that

$$\delta(E_{1, \xi}) \subset E_{0, \xi} \quad \text{and} \quad \delta(E_{0, \xi}) \subset E_{1, \xi^{-1}}.$$  

Now we associate with $\bar{E}$ the complex of $G$-equivariant quasicoherent sheaves on $X$

$$\com_G(\bar{E}) := \bigoplus_{i=1}^d \com(E)_{\chi_i}, \text{ where }$$

$\com(E)_{\chi_i} : \ldots E_{0, \chi_i} \rightarrow E_{1, \chi_i} \rightarrow E_{0, \chi_i} \rightarrow E_{1, \chi_i^{-1}} \rightarrow E_{0, \chi_i^{-1}} \rightarrow \ldots$

where the action of $G$ is given by restricting the action of $\Gamma$. Note that we have an isomorphism of $\mathbb{Z}/2$-graded complexes

$$\com_G(\bar{E}) \simeq E_*.$$  

This implies the following result.

Proposition 1.2.2. In the above situation the functor

$$\com_G : \text{QMF}_{\Gamma, \chi}(X, 0) \rightarrow \text{Com}(\text{Qcoh}_G(X))$$

is an equivalence of dg-categories, which restricts to an equivalence

$$\text{MF}_{\Gamma, \chi}(X, 0) \rightarrow \text{Com}^b(\text{Bun}_G(X)),$$

where $\text{Bun}_G(X)$ is the category of $G$-equivariant vector bundles on $X$ and $\text{Com}^b(\cdot)$ denotes the category of bounded complexes. The functor $\com_G$ also induces an equivalence between the derived category of quasicoherent matrix factorizations with coherent cohomology $D^\text{MF}_\ell(X/\Gamma, 0)$ and the derived category $D^b_{\ell}(\text{Qcoh}_G(X))$ of complexes with bounded coherent cohomology.

Remark 1.2.3. If we choose $\chi_1$ to be the trivial character then the $G$-invariant part of $\com_G(\bar{E})$ is isomorphic to $\Gamma$-invariants of the complex $\com(\bar{E})$ (see (1.4)).

Remark 1.2.4. In the above situation let $C^\bullet$ be a bounded complex of $\Gamma$-equivariant vector bundles on $X$. Recall that we can associate with it a matrix factorization $\text{mf}(C^\bullet) \in \text{MF}_{\Gamma, \chi}(X, 0)$ (see (1.7)). Then we have natural isomorphisms of $G$-equivariant sheaves

$$H^{\text{even}} \com_G(\text{mf}(C^\bullet)) \simeq H^0(\text{mf}(C^\bullet)) \simeq H^{\text{even}}(C^\bullet),$$

$$H^{\text{odd}} \com_G(\text{mf}(C^\bullet)) \simeq H^1(\text{mf}(C^\bullet)) \simeq H^{\text{odd}}(C^\bullet).$$

(1.13)
1.3 Connection with categories of singularities

As was proved by Orlov [43], the homotopy category of matrix factorizations of a nonzero function $W$ on a smooth affine variety $X$ is equivalent to a certain triangulated category $D_{Sg}(X_0)$ that “measures the singularity” of the hypersurface $X_0 = (W = 0)$. The singularity category $D_{Sg}(X_0)$ is the quotient of the derived category $D^b(X_0)$ of coherent sheaves on $X_0$ by the subcategory of perfect complexes. This definition also makes sense for stacks. In [50] we proved an extension of Orlov’s result to smooth stacks (satisfying certain technical assumptions) by replacing the homotopy category of matrix factorizations with the appropriate derived category (see below). The stacks that we allow are called FC DRP-stacks (see [50, Def. 3.1]), where FCD stands for “finite cohomological dimension” and RP for “resolution property”.

Let $X$ be a stack, and let $W \in H^0(X, L)$ be a potential, where $L$ is a line bundle on $X$. Assume that $W$ is not a zero divisor, and let $X_0 = W^{-1}(0)$ be the zero locus of $W$. As in [43], we consider the natural functor

$$\mathcal{C} : \text{HMF}(X, W) \to D_{Sg}(X_0)$$

that associates with a matrix factorization $(E_\bullet, \delta)$ the cokernel of $\delta_1 : E_1 \to E_0$. This functor is exact (see Lemma 3.12 of [50]).

In the case when $X$ is a smooth affine scheme and $L$ is trivial, the functor $\mathcal{C}$ is an equivalence by [43, Thm. 3.9]. In the non-affine case we need to localize the category $\text{HMF}(X, W)$. Namely, we consider the full subcategory

$$\text{LHZ}(X, W) \subset \text{HMF}(X, W)$$

consisting of matrix factorizations $\tilde{E}$ that are locally contractible (i.e., there exists an open covering $U_i$ of $X$ in smooth topology such that $\tilde{E}|_{U_i} = 0$ in $\text{HMF}(U_i, W|_{U_i})$). Then we define the derived category of matrix factorizations as the quotient

$$\text{DMF}(X, W) = \text{HMF}(X, W)/\text{LHZ}(X, W).$$

We proved in [50, Thm. 3.14] that in the case when $X$ is a smooth FCDRP-stack the functor $\mathcal{C}$ induces an exact equivalence

$$\overline{\mathcal{C}} : \text{DMF}(W) \to D_{Sg}(X_0).$$

We will need the following property of the functor $\mathcal{C}$.

**Proposition 1.3.1.** Assume that $W$ is not a zero divisor. For any $\tilde{E} = (E, \delta) \in \text{HMF}(X, W)$ and a bounded complex $C^\bullet$ of vector bundles on $X$ there is an isomorphism

$$\mathcal{C}(C^\bullet \otimes \tilde{E}) \simeq C^\bullet|_{X_0} \otimes \mathcal{C}(\tilde{E})$$

in $D_{Sg}(X_0)$, where we use the operation of tensor product of a matrix factorization with a complex of vector bundles on $X$ (see (1.3)).
Proof. Since $\mathcal{C}$ commutes with the translation functors, it is enough to consider the case when $C^\bullet$ is concentrated in non-positive degrees. Recall that

$$(C^\bullet \otimes E)_i = \bigoplus_{n \geq 0} C^{-n} \otimes E(L^{1/2})_{n-i} \otimes L^{(n-i)/2}, \text{ for } i = 0, 1,$$

and the (injective) differential $\delta_1 : (C^\bullet \otimes E)_1 \to (C^\bullet \otimes E)_0$ is induced by $\delta$ and the differential $d_C$ on $C^\bullet$. Let us consider the sheaves

$$\mathcal{F} := \mathcal{C}(\bar{E}) = \ker(\delta_1 : E_1 \to E_0) \text{ and } \widetilde{\mathcal{F}} = \mathcal{C}(C^\bullet \otimes \bar{E}) = \ker(\delta_1).$$

We are going to construct an exact triple of bounded complexes of coherent sheaves on $X_0$, concentrated in non-negative degrees, of the form

$$0 \to S^\bullet \to G^\bullet \xrightarrow{p} C^\bullet|_{X_0} \otimes \mathcal{F} \to 0,$$

such that the terms of $S^\bullet$ are locally free and $G^\bullet$ is a resolution for $\widetilde{\mathcal{F}}$. This will imply that $G^\bullet$ is isomorphic to $C^\bullet|_{X_0} \otimes \mathcal{F}$ in $D_{sg}(X_0)$, and the assertion will follow. We define the complex $G^\bullet$ by

$$G^{-i} = C^{-i} \otimes \mathcal{F} \oplus \bigoplus_{n \geq i} C^{-n} \otimes E(L^{1/2})_{n-i} \otimes L^{(n-i)/2}|_{X_0} \text{ for } i \geq 0,$$

with the differential induced by $\delta$, $d_C$, and the embedding $\mathcal{F} \to E_1 \otimes L|_{X_0}$. The map

$$p : G^\bullet \to C^\bullet \otimes \mathcal{F}$$

is defined as the natural projection. It remains to show that $G^\bullet$ is a resolution of $\widetilde{\mathcal{F}}$. It is easy to check that the canonical map $(C^\bullet \otimes E)_0 \to \widetilde{\mathcal{F}}$ factors through a map $G^0 \to \widetilde{\mathcal{F}}$ that extends to a morphism of complexes $G^\bullet \to \widetilde{\mathcal{F}}$. To see that it is a quasi-isomorphism, we use the increasing filtrations on both sides induced by the stupid filtration $[C^\bullet]_{\geq-n}$ on $C^\bullet$. The associated quotients of the induced filtration on $\mathcal{F}$ are

$$C^{-n} \otimes \ker((E(L^{1/2})_{n-1} \otimes L^{(n-1)/2} \to E(L^{1/2})_n \otimes L^n/2), \quad (1.16)$$

where $n \geq 0$. On the other hand, the associated quotients of the induced filtration of $G^\bullet$ are the complexes

$$C^{-n} \otimes G_0 \to C^{-n} \otimes E_1 \otimes L|_{X_0} \to \ldots \to C^{-n} \otimes E(L^{1/2})_n \otimes L^n/2 \quad (1.17)$$

concentrated in degrees $[-n, 0]$ with $n \geq 0$. It remains to observe that by [50, Lem. 1.5], the complex (1.17) is a resolution of the sheaf (1.16).

Remark 1.3.2. In the case $W = 0$ the definition of the derived category of matrix factorizations still makes sense. For instance, as in Proposition 1.2.2, let us consider the case $X = Y/\Gamma$ with $\Gamma$ acting trivially on $Y$ and equipped with a surjective character $\chi : \Gamma \to \mathbb{G}_m$ (which defines the line bundle $L$). Assume also that $G = \ker(\chi)$ is finite. Then the category $\text{DMF}_{\Gamma, \chi}(Y, 0) = \text{DMF}(X, 0)$ is equivalent to the category of $\bar{\mathbb{G}}$-graded objects in the usual derived category of bounded ($\mathbb{Z}$-graded) complexes of vector bundles on $Y$ (since a bounded acyclic complex of projective modules is contractible).
As we have shown in [50, Sec. 4], the functor $\mathcal{C}$ extends naturally to quasi-matrix factorizations. More precisely, we have a functor
\[ \mathcal{C}^\infty : \text{DMF}^\infty(X, W) \to D^\prime_{\text{Sg}}(X_0), \]
where $\text{DMF}^\infty(X, W)$ is the derived category of quasi-matrix factorizations defined as the quotient of the homotopy category $\text{HMF}^\infty(X, W)$ by the subcategory of quasi-matrix factorizations that are locally homotopic to zero, and $D^\prime_{\text{Sg}}(X_0)$ is the quotient of $D^b(\text{Qcoh}(X_0))$ by the subcategory of bounded complexes of locally free sheaves.

If $f : X \to Y$ is a smooth affine morphism with integral fibers, where $X$ and $Y$ are stacks, and $W \in H^0(Y, L)$ is a potential, then we have a natural push-forward functor that takes a quasi-matrix factorization of $f^*W$ on $X$ to a quasi-matrix factorization of $W$ on $Y$ (see [50, Def. 4.8]). Furthermore, if $W$ is not a zero divisor and $Y$ is smooth with the resolution property, then there is an induced functor of derived categories $f_* : \text{DMF}^\infty(X, f^*W) \to \text{DMF}^\infty(Y, W)$, so that we have a commutative diagram
\[
\begin{array}{ccc}
\text{DMF}^\infty(X, f^*W) & \xrightarrow{\mathcal{C}^\infty} & D^\prime_{\text{Sg}}(X_0) \\
f_* & & g_* \\
\downarrow & & \downarrow \\
\text{DMF}^\infty(Y, W) & \xrightarrow{\mathcal{C}^\infty} & D^\prime_{\text{Sg}}(Y_0)
\end{array}
\]
where $Y_0$ (resp., $X_0$) be the zero locus of $W$ (resp., $f^*W$), $g : X_0 \to Y_0$ is the morphism induced by $f$, and the right vertical arrow is induced by the push-forward functor $g_* : D^b(\text{Qcoh}(X_0)) \to D^b(\text{Qcoh}(Y_0))$.

### 1.4 Supports

Let $X$ be a stack and let $W \in H^0(X, L)$ be a section. We denote by $\overline{\text{DMF}}(X, W)$ the idempotent closure of $\text{DMF}(X, W)$. For a closed substack $Z \subset X$ the full subcategory $\overline{\text{DMF}}(X, Z; W) \subset \overline{\text{DMF}}(X, W)$ of matrix factorizations with support on $Z$ is defined as the common kernel of the restriction functors
\[ E \mapsto i^*_x \text{com}(E) \]
for closed points $x \in X_0 \setminus Z$, where $X_0 = W^{-1}(0) \subset X$ (see [50, (5.1)]). In the case when $W$ is a non-zero divisor and $X$ is a smooth FCDRP-stack we proved that a matrix factorization $\overline{E} = (E, \delta)$ belongs to $\overline{\text{DMF}}(X, Z; W)$ if and only if it restricts to zero in $\overline{\text{DMF}}(X \setminus Z, W|_{X \setminus Z})$ (see the proof of [50, Prop. 5.6]).

**Lemma 1.4.1.** Let $\{\alpha, \beta\}$ be the Koszul matrix factorization of $W \in H^0(X, L)$ associated with sections $\alpha \in V \otimes L$ and $\beta \in V^\vee$ such that $\langle \alpha, \beta \rangle = W$ (see (1.10)). Then $\{\alpha, \beta\}$ is supported on the zero locus of the section $\langle \alpha, \beta \rangle \in (V \otimes L) \oplus V^\vee$.  

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Proof. By definition of support of a matrix factorization, we have to check that if \( V \) is a vector space and \( \alpha \in V \) and \( \beta \in V^* \) are such that \( \langle \alpha, \beta \rangle = 0 \) and \( (\alpha, \beta) \neq (0, 0) \), then the complex
\[
\{\alpha, \beta\} = (\bigwedge^\bullet V, \alpha \wedge + \iota(\beta))
\]
is acyclic. Since \( \langle \alpha, \beta \rangle = 0 \), we can find a direct sum decomposition \( V = V_1 \oplus V_2 \) such that \( \alpha \in V_1 \) and \( \beta \in V_2^* \). Then \( \{\alpha, \beta\} \) becomes the total complex of the tensor product of the complexes \( (\bigwedge^\bullet V_1, \alpha \wedge) \) and \( (\bigwedge^\bullet V_2, \iota(\beta)) \), at least one of which is acyclic because \( (\alpha, \beta) \neq (0, 0) \).

Recall that for a pair of potentials \( W, W' \in H^0(X, L) \) we have the tensor product bifunctor (see (1.6))
\[
HMF(X, W) \times HMF(X, W') \to HMF(X, W + W').
\] (1.18)

Proposition 1.4.2. Let \( X \) be a stack and \( W, W' \in H^0(X, L) \) two sections.
(i) For a pair of closed substacks \( Z \subset X \) and \( Z' \subset X \) the bifunctor (1.18) induces an exact bifunctor
\[
\overline{DMF}(X, Z; W) \times \overline{DMF}(X, Z'; W') \to \overline{DMF}(X, Z \cap Z'; W + W').
\]
(ii) Assume in addition that \( X \) is a smooth FCDRP stack and \( W \) and \( W' \) are not zero divisors. The the tensor product induces a bifunctor
\[
\overline{DMF}(X, W) \times \overline{DMF}(X, W') \to \overline{DMF}(X, \Sigma \cap \Sigma'; W + W').
\]
where \( \Sigma \) (resp., \( \Sigma' \)) is the singularity locus of the hypersurface \( W = 0 \) (resp., \( W' = 0 \)).

Proof. (i) The tensor product functor (1.18) is compatible with pull-backs, hence, if at least one of the matrix factorizations \( \overline{E} \in HMF(X, W) \) or \( \overline{F} \in HMF(X, W') \) is locally contractible then so is \( \overline{E} \otimes \overline{F} \). Therefore, this functor descends to derived categories. Similarly, for \( \overline{E} \in \overline{DMF}(X, Z; W) \) and \( \overline{F} \in \overline{DMF}(X, Z'; W') \) the tensor product \( \overline{E} \otimes \overline{F} \) is contractible in a neighborhood of \( x \notin Z \cap Z' \) (since either \( \overline{E} \) of \( \overline{F} \) is contractible near \( x \)).

(ii) This follows from part (i) and [50, Cor. 5.3].

1.5 Push-forwards

In [50, Sec. 6] we defined the push-forwards for matrix factorizations with relatively proper support. Let \( f : X \to Y \) be a representable morphism of smooth FCDRP-stacks and \( W \in H^0(Y, L) \) a potential such that \( W \) and \( f^*W \) are not zero divisors. Let \( Z \subset X_0 \) be a closed substack of the zero locus of \( f^*W \), such that the induced morphism \( f : Z \to Y \) is proper. Let \( Y_0 \subset Y \) denote the zero locus of \( W \), and let \( f_0 : X_0 \to Y_0 \) be the map induced by \( f \). The derived push-forward functor
\[
R_{f_0*} : D^b(X_0, Z) \to D^b(Y_0, f(Z))
\]
induces a functor
\[
\overline{DS}_g(X_0, Z) \to \overline{DS}_g(Y_0, f(Z)),
\]

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and hence, by [50, Prop. 5.6], a functor
\[
Rf_* : \underline{\text{DMF}}(X, Z; f^* W) \to \underline{\text{DMF}}(Y, f(Z); W).
\]

In the case when \( f \) is proper this functor preserves the usual derived categories inside their idempotent completions (see [50, Rem. 6.2]). We proved in [50, Prop. 6.3] that this functor is compatible with the natural push-forwards for quasi-matrix factorizations with respect to smooth affine morphisms with integral fibers (see [50, Def. 4.8]). More generally, if \( f : X \to Y \) is an arbitrary affine morphism, then for any \( W \in H^0(Y, L) \) we have the naive push-forward functor
\[
f_* : \text{QMF}(X, f^* W) \to \text{QMF}(Y, W) : \bar{E} = (E, \delta) \mapsto (f_* E, f_* \delta)
\]
for quasicoherent matrix factorizations (see Section 1.1). Note that in the case when \( W = 0 \) this functor respects quasi-isomorphisms.

**Remark 1.5.1.** In the case when \( f \) is a finite morphism, the functor (1.19) is compatible with the naive push-forward (1.20) in the following sense. For a matrix factorization \( \bar{E} = (E, \delta) \) the naive push-forward \( f_* \bar{E} \) is a coherent matrix factorization with an additional property that the multiplication by \( W \) is an injective endomorphism. The cokernel functor \( \mathfrak{C} : \text{MF}(Y, W) \to D_{\text{sg}}(Y_0) \) extends naturally to the category of such coherent matrix factorizations, so we can view \( f_* \bar{E} \) as an object of the derived category \( \text{DMF}(Y, W) \) (cf. [50, Def. 3.21, Rem. 6.2]). Since \( \mathfrak{C}(f_* \bar{E}) \simeq f_{0*}(\bar{E}) \), we obtain an isomorphism in \( \text{DMF}(Y, W) \) of \( f_* \bar{E} \) with the push-forward of \( \bar{E} \) given by 1.19.

**Remark 1.5.2.** As it was pointed out to us by Leonid Positselski, an alternative construction of push-forwards can be given using exotic derived categories of [53] and the results of [54]. One can start with the natural push-forward functors between the coderived categories of quasicoherent matrix factorizations which are defined using injective resolutions (see [53, Sec. 3.7]). Then one can use the fact that for a regular scheme of finite Krull dimension the coderived category of quasi-matrix factorizations is equivalent to the coderived category of quasicoherent matrix factorizations (see [54, Thm. 1(a)]). Note that the absolute derived category of matrix factorizations, which is a full subcategory in the above coderived category, is equivalent to the corresponding hypersurface singularity category (see [53, Thm. 2], [40, Prop. 2.13] or [46]) and hence to our category \( \text{DMF}(X, W) \).

The following lemma gives an important relation between the push-forward with respect to the embedding of the zero locus of \( W \) and the tensor product of matrix factorizations of \( W \) and of \(-W\).

**Lemma 1.5.3.** Let \( X \) be a smooth FCDRP-stack, \( W \in H^0(X, L) \) a potential and \( \bar{E} \) a matrix factorization of \( W \).

(i) Let us define the matrix factorization \( C_\bullet(\bar{E}) \) of 0 on \( X \) as follows:

\[
C_0(\bar{E}) = E_0 \oplus E_1 \otimes L, \quad C_1(\bar{E}) = E_1 \oplus E_0,
\]

with the differential \( \delta(x, y) = (\delta_E(x) + y, -W(x) - \delta_E(y)) \). Then \( C_\bullet(\bar{E}) \) is contractible (i.e., homotopy equivalent to 0).
(ii) Let $\bar{F}$ be a matrix factorization of $-W$, and let $i : X_0 \hookrightarrow X$ be the inclusion of the zero locus of $W$. Assume that $W$ is not a zero divisor. Then the map

$$\mathcal{F} \mapsto i_* (\mathcal{F} \otimes_{\mathcal{O}_{X_0}} i^* \bar{F})$$

(1.21)
gives a well-defined functor $D_{Sg}(X_0) \to D\text{CMF}(X, 0)$, where $D\text{CMF}(X, 0)$ is the derived category of coherent matrix factorizations (see Section 1.1). Also, the natural map of coherent matrix factorization of $0$ on $X$

$$q : \bar{E} \otimes \bar{F} \to i_* (\mathfrak{C}(\bar{E}) \otimes_{\mathcal{O}_{X_0}} i^* \bar{F})$$,

induced by the projection $E_0 \to \mathfrak{C}(\bar{E}) = \text{coker}(E_1 \to E_0)$, is a quasi-isomorphism.

**Proof.** (i) The contracting homotopy sends $(x, y)$ to $(0, y)$.

(ii) By [50, Lem. 1.5], the complex $\text{com}(i^* \bar{F}) = \text{com}(\bar{F})$ of bundles on $X_0$ is exact. Therefore, using isomorphism (1.5) we see that for a perfect complex $C^\bullet$ on $X_0$ the coherent matrix factorization of zero $C^\bullet \otimes i^* \bar{F}$ is acyclic. Hence, (1.21) gives a well-defined functor. We have to check that the map $\text{com}(q)$ is a quasi-isomorphism (see (1.4)). But this map fits into an exact sequence of complexes of sheaves on $X_0$

$$0 \to K^\bullet \to \text{com}(\bar{E} \otimes \bar{F}) \xrightarrow{\text{com}(q)} i_* (\mathfrak{C}(\bar{E}) \otimes \text{com}(\bar{F})) \to 0,$$

where $K^\bullet = E_1 \otimes \text{com}(C_\bullet(\bar{F}))$. Hence, by part (i), $K^\bullet$ is acyclic.

We will need the push-forward functors in the following situation.

**Example 1.5.4.** Let $\pi : E \to X$ be a smooth affine morphism with integral fibers, where $X$ is a smooth FCDRP-stack, and let $W \in H^0(X, \mathcal{O}_X)$ be a potential. Suppose that we have a commutative algebraic group $\Gamma$ acting on both $E$ and $X$ compatibly, so that $W$ is semi-invariant with respect to $\Gamma$ and a character $\chi : \Gamma \to \mathbb{G}_m$, where $\chi$ is surjective with finite kernel $G$. Let $Z \subset E$ be a $\Gamma$-invariant closed substack such that $\pi^* W|_Z = 0$ and $Z$ is proper over $X$. We will define the push-forward functor in the following two cases.

**Case 1.** Assume that $W$ is not a zero divisor. Then by [50, Prop. 6.1] (applied to the morphism $E/\Gamma \to X/\Gamma$), we have the push-forward functor

$$\pi_* : \text{DMF}_\Gamma(E, Z; \pi^* W) \to \text{DMF}_\Gamma(X, W).$$

If in addition, a subgroup $I \subset G$ acts trivially on $X$ then we can combine the above functor with taking $I$-invariants to get a functor

$$\pi^I_* : \text{DMF}_\Gamma(E, Z; \pi^* W) \to \text{DMF}_{\Gamma/I}(X, W).$$

(1.22)

**Case 2.** Assume that $W = 0$ and the action of $\Gamma$ on $X$ (but not necessarily on $E$) is trivial. Then we can still define the push-forward functor as follows. Given a $\Gamma$-equivariant matrix factorization $\bar{P}$ of $0$ on $E$, supported on $Z$, we can consider the push-forward $\pi_* \bar{P}$ (see [50, Def. 4.8]) which will be a $\Gamma$-equivariant quasi-matrix factorization of $0$ on $X$. Let us consider the corresponding complex $\text{com}_G(\pi_* \bar{P})$ of $G$-equivariant quasicoherent sheaves defined by
(1.12) (using representatives for the \(\langle \chi \rangle\)-cosets in the character group of \(\Gamma\)). The assumption on the support of \(\bar{P}\) implies that this complex has bounded coherent cohomology. Since the category of such complexes is equivalent to the bounded derived category of coherent sheaves, we obtain a functor

\[
\pi_* : DMF^\Gamma(E, Z; 0) \to D^b_G(X) \simeq DMF^\Gamma(X, 0). \tag{1.23}
\]

The map on the Grothendieck groups induced by this functor sends the class of a matrix factorization \(\bar{P}\) to \([\pi_0H^0(P, \delta)] - [\pi_1H^1(P, \delta)]\).

In both cases the diagram of functors

\[
\begin{array}{ccc}
DMF^\Gamma(E, Z; \pi^*W) & \xrightarrow{\pi_*} & DMF^\Gamma(X, W) \\
\downarrow & & \downarrow \\
DMF^\infty_\Gamma(E, f^*W) & \xrightarrow{\pi_*} & DMF^\infty_\Gamma(X, W)
\end{array}
\tag{1.24}
\]

is commutative. Indeed, for \(W = 0\) (and \(\Gamma\) acting trivially on \(X\)) this is clear from the definition, and for \(W \neq 0\) this follows from [50, Prop. 6.3].

Note that if we have a \(\Gamma\)-equivariant closed substack \(i : E' \to E\) such that \(Z \subset E'\), such that \(\pi' = \pi|_{E'}\) is still smooth with integral fibers, then we can consider the push-forward functors (1.19) associated with the projections \(\pi : E \to X\) and \(\pi' : E' \to X\) and also the functor \(i_* : DMF(E', f^*W|_{E'}) \to DMF(E, f^*W)\). In this situation one has an isomorphism of functors

\[
\pi_* \circ i_* \simeq \pi'_*
\]

from \(DMF(E', Z; (\pi'|_{E'})^*W|_{E'})\) to \(DMF(X, W)\).

We have the following analog of the projection formula.

**Proposition 1.5.5.** Let \(\pi : E \to X\), \(Z \subset E\), \(\Gamma\) and \(W\) be as in one of the two cases of Example 1.5.4, where \(\Gamma\) acts trivially on \(X\). Then for \(\bar{P} \in DMF^\Gamma(E, Z; \pi^*W)\) and \(\bar{Q} \in DMF^\infty_\Gamma(X, -W)\) one has a functorial isomorphism

\[
\pi_*(\bar{P} \otimes \pi^*\bar{Q}) \simeq \pi_*(\bar{P}) \otimes \bar{Q}.
\]

**Proof.** The case \(W = 0\) immediately reduces to the usual projection formula, so we will assume that \(W\) is not a zero divisor. By Lemma 1.5.3(ii), we have natural quasi-isomorphisms

\[
\pi_*(\bar{P} \otimes \pi^*\bar{Q}) \simeq i_*\pi_{0*}(\mathcal{C}(\bar{P}) \otimes \pi_0^*\pi_0^\dagger\pi^*_0\bar{Q})
\]

and

\[
\pi_*(\bar{P}) \otimes \bar{Q} \simeq i_* (\mathcal{C}(\pi_*\bar{P}) \otimes \pi^*_0i^*\bar{Q}),
\]

where \(E_0 = \pi^{-1}(X_0)\), \(\pi_0 : E_0 \to X_0\) is the restriction of \(\pi\), and \(i : X_0 \to X\) is the natural embedding. It remains to use the isomorphism

\[
\pi_{0*}\mathcal{C}(\bar{P}) \simeq \mathcal{C}(\pi_*\bar{P})
\]

and the usual projection formula for \(\pi_0\). \(\square\)

Our push-forward functors also have the following base change property.
Proposition 1.5.6. (i) Suppose we have a cartesian diagram of smooth FCDRP-stacks

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

where \( f \) is a flat representable morphism, and let \( W \in H^0(Y, L) \) be a potential such that its pull-backs \( W_X, W_{X'}, \) and \( W_{Y'} \) to \( X, X' \) and \( Y' \), respectively, are not zero-divisors. Let \( X_0, Y_0, X'_0 \) and \( Y'_0 \) be the zero loci of these potentials, \( Z \subset X_0 \) a closed substack, proper over \( Y \), and \( Z' \subset X'_0 \) the induced closed substack. Then the diagram of functors

\[
\begin{array}{ccc}
\text{DMF}(X, Z, W_X) & \xrightarrow{v^*} & \text{DMF}(X', Z', W_{X'}) \\
\downarrow{Rf_*} & & \downarrow{Rf'_*} \\
\text{DMF}(Y, W) & \xrightarrow{u^*} & \text{DMF}(Y', W_{Y'})
\end{array}
\]

is commutative.

(ii) Let \( (\pi : E \to X, W, \Gamma, \chi, Z) \) be as in Example 1.5.4, where either \( W \) is not a zero-divisor or \( W = 0 \) and \( \Gamma \) acts trivially on \( X \). Suppose we have a cartesian diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{v} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
X' & \xrightarrow{u} & X
\end{array}
\]

of stacks with \( \Gamma \)-action, where all the maps are \( \Gamma \)-equivariant. Assume that the action of \( \Gamma \) on \( X' \) is trivial and \( i^*W = 0 \). Then for any matrix factorization \( \bar{P} \) of \( W \) on \( E \), supported on \( v^{-1}(Z) \), there is an isomorphism

\[ u^*\pi_*P \simeq \pi'_*v^*P \]

in \( D^b_\Gamma(X') \cong \text{DMF}_\Gamma(X', 0) \).

Proof. (i) This follows easily from the usual base change formula.

(ii) It is enough to check the corresponding isomorphism in \( \text{DMF}^\infty_\Gamma(X', 0) \). Therefore, the statement follows from the commutativity of the diagrams (1.24) for \( \pi \) and \( \pi' \) together with the usual base change formula. \( \square \)

As in the case of usual sheaves, the base change formula leads to a relative K"unneth isomorphism for push-forwards of matrix factorizations.
**Proposition 1.5.7.** Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be smooth morphisms with connected fibers of FCDRP-stacks over a smooth FCDRP-stack $S$. Let $W_1 \in H^0(Y_1, L_1)$ and $W_2 \in H^0(Y_2, L_2)$ be potentials, and let $W = p_{Y_1}^*W_1 + p_{Y_2}^*W_2$ be the corresponding potential on $Y = Y_1 \times_{S} Y_2$, where $p_{Y_i} : Y \rightarrow Y_i$ are the projections. Let $Z_1 \subset X_1$ (resp., $Z_2 \subset X_2$) be a closed substack, proper over $Y_1$ (resp., $Y_2$). Consider the relative product map

$$f = f_1 \times_S f_2 : X_1 \times_S X_2 \rightarrow Y = Y_1 \times_S Y_2.$$ 

Assume that $W_1$, $W_2$ and $W$ are non-zero-divisors. Then for $\bar{E}_1 \in \text{DMF}(X_1, Z_1, f_1^*W_1)$ and $\bar{E}_2 \in \text{DMF}(X_2, Z_2, f_2^*W_2)$ there is a functorial isomorphism

$$Rf_*(p_{X_1}^*(\bar{E}_1) \otimes p_{X_2}^*(\bar{E}_2)) \simeq p_{Y_1}^*(Rf_1^*(\bar{E}_1)) \otimes p_{Y_2}^*(Rf_2^*(\bar{E}_2))$$

in $\text{DMF}(Y, W)$, where $p_{X_i} : X_1 \times_S X_2 \rightarrow X_i$ are the projections.

The same assertion holds if one (or both) of the morphisms $f_i$ is of the form $E/\Gamma \rightarrow X/\Gamma$, where $E \rightarrow$ is a smooth affine morphism, $\Gamma$ is a commutative group acting trivially on $Z$ and $W_i = 0$ (see Case 2 of Example 1.5.4).

**Proof.** Let us denote $P = Rf_*(p_{X_1}^*(\bar{E}_1) \otimes p_{X_2}^*(\bar{E}_2))$ and $P_i = Rf_i^*(\bar{E}_i)$ for $i = 1, 2$. Consider the commutative diagram with a cartesian square

$$
\begin{array}{ccc}
X & \xrightarrow{id \times f_2} & X_1 \times_S Y_2 \\
\downarrow{p_{X_1}} & & \downarrow{p_1} \\
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow{f} & & \downarrow{p_{Y_1}} \\
Y_1 \times S Y_2 & \xrightarrow{1 \times id} & Y_1 \\
\end{array}
$$

(1.25)

Note that the composition of the arrows in the first row is equal to $f$. Thus, we have

$$P \simeq R(f_1 \times id)_*(R(id \times f_2)_*(p_{X_1}^*(\bar{E}_1) \otimes p_{X_2}^*(\bar{E}_2))) \simeq R(f_1 \times id)_*(p_{Y_1}^*(\bar{E}_1) \otimes R(id \times f_2)_*p_{X_1}^*(\bar{E}_2)).$$

Applying the base change formula in the cartesian square

$$
\begin{array}{ccc}
X & \xrightarrow{p_{X_2}} & X_2 \\
\downarrow{id \times f_2} & & \downarrow{f_2} \\
X_1 \times_S Y_2 & \xrightarrow{p_2} & Y_2 \\
\end{array}
$$

we get

$$R(id \times f_2)_*p_{X_2}^*(\bar{E}_2) \simeq p_2^*P_2.$$ 

Hence,

$$P \simeq R(f_1 \times id)_*(p_{Y_1}^*(\bar{E}_1)) \otimes p_{X_2}^*(P_2).$$

Finally, the base change formula in the cartesian square of (1.25) shows that

$$R(f_1 \times id)_*(p_{Y_1}^*(\bar{E}_1)) \simeq p_{Y_1}^*(P_1).$$

$\Box$
1.6 Regular Koszul matrix factorizations

Here we study Koszul matrix factorizations \{\alpha, \beta\} in the case when \( \beta \) is a regular section. Such matrix factorizations should be viewed as deformations of the Koszul complex \{0, \beta\}.

In this section we assume that \( X \) is a smooth FCDRP-stack. We fix a potential \( W \in H^0(X, L) \), a vector bundle \( V \) on \( X \), and sections \( \alpha \in H^0(X, V \otimes L) \) and \( \beta \in H^0(X, V^\vee) \), such that \( \langle \alpha, \beta \rangle = W \).

**Definition 1.6.1.** The Koszul matrix factorization (see (1.10))

\[
\{\alpha, \beta\} = \left( \bigwedge^\bullet (V \otimes L^{1/2})(L^{-1/2}), \delta_{\alpha, \beta} \right),
\]

is called regular if \( \beta \) is a regular section of \( V^\vee \).

In this section we always assume that \( \{\alpha, \beta\} \) is regular (with the exception of Proposition 1.6.4) and denote by \( i : X' \hookrightarrow X \) the embedding of the zero locus of \( \beta \). Note that \( X' \) is contained in \( X_0 \), the zero locus of \( W \).

**Lemma 1.6.2.** (i) If \( W \) is not a zero divisor then

\[ C(\{\alpha, \beta\}) \simeq O_{X'} \text{ in } D_{\text{sg}}(X_0), \]

where \( C \) is the cokernel functor (1.14), and \( O_{X'} \) is viewed as a coherent sheaf on \( X_0 \).

(ii) If \( W = 0 \) then

\[ H^0(\{\alpha, \beta\}) \simeq i_* O_{X'} \text{ and } H^1(\{\alpha, \beta\}) = 0, \]

where \( H^i(\hat{P}) := H^i(\text{com}(\hat{P})) \) (see (1.4)). In other words, the natural morphism of quasicoherent matrix factorizations

\[ \{\alpha, \beta\} \to i_* \text{mf}(O_{X'}) \]

is a quasi-isomorphism (where \( \text{mf}(O_{X'})_0 = O_{X'}, \text{mf}(O_{X'})_1 = 0 \)).

**Proof.** (i) We have a natural map \( C(\{\alpha, \beta\}) \to i_* O_{X'} \) induced by the projection \( \bigwedge^\bullet \to \bigwedge^0 \) and by the map \( O_X \to O_{X'} = \text{coker}(\beta^\vee : V \to O_X) \). It is enough to prove that this map is an isomorphism locally, so the statement reduces to the affine case proved in [49, Prop. 2.3.1].

(ii) Since \( \langle \alpha, \beta \rangle = W = 0 \), the complex \( \text{com}(\{\alpha, \beta\}) \) can be identified with the total complex of the bicomplex

\[ K = \bigoplus_{i, j} K^{i,j}, \text{ where } K^{i,j} = \bigwedge^{i-j} V \otimes L^i, \]

with differentials given by \( \iota(\beta) \) and \( \alpha \wedge ? \). The regularity of \( \beta \) implies that the cohomology of the differential \( \iota(\beta) \) is concentrated along the diagonal \( \bigoplus_i K^{i,i} \). From the spectral sequence we immediately see that

\[ H^{2n+1}(\text{com}(\{\alpha, \beta\})) = 0 \text{ and } H^{2n}(\text{com}(\{\alpha, \beta\})) \simeq i_* O_{X'} \otimes L^n. \]

\[ \square \]
**Proposition 1.6.3.** (i) Let $W_1 \in H^0(X, L)$ be another potential, such that $i : X' \hookrightarrow X$, the zero locus of $W$, is smooth and the restriction of $W_1$ to $X'$ is not a zero divisor. Then for every matrix factorization $\bar{P} = (P, \delta) \in \text{MF}(X, W_1)$ we have a functorial isomorphism in $\text{DMF}(X, W + W_1)$

$$q : \bar{P} \otimes \{\alpha, \beta\} \xrightarrow{\sim} i_* i^* \bar{P},$$

where on the right-hand side $i_*$ is the push-forward functor

$$i_* : \text{DMF}(X', W_1|_{X'}) \to \text{DMF}(X, W + W_1).$$

(ii) Assume that $W$ is not a zero divisor. Then for every matrix factorization $\bar{P} \in \text{MF}(X, -W)$ we have a quasi-isomorphism in $\text{QMF}(X, 0)$

$$q : \bar{P} \otimes \{\alpha, \beta\} \to i_* i^* \bar{P}.$$

**Proof.** (i) The projection $\bigwedge^\bullet (V \otimes L^{1/2})(L^{-1/2}) \to O_X$ induces a natural morphism

$$q : P \otimes \bigwedge^\bullet (V \otimes L^{1/2})(L^{-1/2}) \to i_* i^* (P \otimes \bigwedge^\bullet (V \otimes L^{1/2})(L^{-1/2})) \to i_* i^* P$$

of coherent matrix factorizations of $W + W_1$, where we use the fact that the naïve push-forward $i_* i^* P$ is compatible with the push-forward functor (1.19) (see Remark 1.5.1). To show that $q$ induces an isomorphism in $\text{DMF}(X, W + W_1)$, we can argue locally. Thus, we can assume that $L$ and $V$ are trivial bundles. We will use induction in the rank $r$ of $V$. In the case when $r = 1$, i.e., $V = O$, we have $\beta = f, \alpha = g$, where $f$ and $g$ are functions on $X$ such that $W = fg$ and $i : X' = Z(f) \hookrightarrow X$ is a divisor. By definition,

$$\mathcal{E}(\bar{P} \otimes \{g, f\}) = \text{coker}(D : P_1 \oplus P_0 \to P_0 \oplus P_1),$$

where

$$D(p_1, p_0) = (\delta(p_1) + f \cdot p_0, \delta(p_0) - g \cdot p_1).$$

Let us consider the exact triple of two-term complexes

$$0 \to [0 \to P_1] \to [P_1 \oplus P_0 \xrightarrow{D} P_0 \oplus P_1] \to [P_1 \oplus P_0 \xrightarrow{\partial} P_0] \to 0,$$

where $\partial(p_1, p_0) = \delta(p_1) + f \cdot p_0$. Since $D$ is a part of the differential of the matrix factorization $\bar{P} \otimes \{g, f\}$ of $W + W_1$, it is injective. Thus, from the above exact triple we get an exact sequence of sheaves

$$0 \to \ker(\partial) \xrightarrow{\gamma} P_1 \to \text{coker}(D) \to \text{coker}(\partial) \to 0,$$

where $\gamma(p_1, p_0) = \delta(p_0) - g \cdot p_1$. Now

$$\text{coker}(\partial) \simeq P_0/(\delta(P_1) + fP_0) \simeq i_* \mathcal{E}(i^* \bar{P}).$$

Since $W_1|_{X'} \neq 0$, the morphism $\delta|_{X'} : P_1/fP_1 \to P_0/fP_0$ is injective. In other words, $\delta^{-1}(fP_0) \cap P_1 = fP_1$. This implies that the map $p_1 \mapsto (fp_1, -\delta(p_1))$ gives an isomorphism

$$\lambda : P_1 \cong \ker(\partial).$$
The composition of $\gamma$ with $\lambda$ sends $p_1$ to $-\delta^2(p_1) - fg \cdot p_1 = -(W + W_1) \cdot p_1$. Thus, the exact sequence (1.27) is isomorphic to

$$0 \to P_1 \xrightarrow{-W_1} P_1 \to \mathcal{E}(\bar{P} \otimes \{g, f\}) \to i_* \mathcal{E}(i^* \bar{P}) \to 0.$$ 

Since $\mathcal{E}(i_* i^* P) \simeq i_* \mathcal{E}(i^* \bar{P})$, we get an exact sequence

$$0 \to P_1/(W + W_1)P_1 \to \mathcal{E}(\bar{P} \otimes \{g, f\}) \xrightarrow{\mathcal{E}(q)} \mathcal{E}(i_* i^* \bar{P}) \to 0$$

that implies that the map $\mathcal{E}(q)$ is an isomorphism in $D_{Sg}(X_0)$, and so by [50, Thm. 3.14], $q$ is an isomorphism. This gives the base of induction.

When $r = \text{rk} V > 1$, decompose $V$ as $V = O_X \oplus V'$, and let $\{\alpha, \beta\} = \{g, f\} \otimes \{\alpha', \beta'\}$ be the corresponding decomposition, where $\beta' \in (V')^\vee$, $\alpha' \in V'$, $f, g \in H^0(O_X)$. Let $j : X'' \hookrightarrow X$ be the zero locus of $\beta'$. Note that $X' \subset X''$ is the zero locus of the function $j^* f$. Since $\beta'$ is a regular section of $V'$, we can apply the induction hypothesis to the matrix factorization $\{\alpha', \beta'\}$ on $X$ and conclude that the natural map

$$\bar{P} \otimes \{g, f\} \otimes \{\alpha', \beta'\} \to j_* (j^* \bar{P} \otimes \{j^* g, j^* f\})$$

is an isomorphism. Applying the case $r = 1$ to the matrix factorization $\{j^* g, j^* f\}$, we see that

$$j^* \bar{P} \otimes \{j^* g, j^* f\} \simeq k_* k^* j^* \bar{P} \simeq k_* i^* \bar{P},$$

where $k : X' \hookrightarrow X''$ is the natural embedding. This establishes the induction step. Now the assertion follows.

(ii) Since $\mathcal{E}(\{\alpha, \beta\}) \simeq i_* O_{X'}$ in $D_{Sg}(X_0)$ (see Lemma 1.6.2(i)), this follows from Lemma 1.5.3(ii). \qed

The following result deals with the situation when the section $\beta \in H^0(X, V^\vee)$ is not regular but is the image of a regular section of a subbundle of $V^\vee$.

**Proposition 1.6.4.** Let $U \subset V$ be a subbundle. Assume that we have a regular section $\beta' \in H^0(X, (V/U)^\vee)$ such that $\beta = \iota(\beta')$, where $\iota : (V/U)^\vee \to V^\vee$ is the natural inclusion.

(i) Assume that $W$ is not a zero divisor. Then we have the following equality in the Grothendieck group of $D_{Sg}(X_0)$:

$$[\mathcal{E}(\{\alpha, \beta\})] = [i'_* i^* \bigwedge^\bullet (U \otimes L^{1/2})(L^{-1/2})],$$

where $i' : X' \hookrightarrow X_0$ is the inclusion.

(ii) Assume that $W = 0$. Then

$$[\{\alpha, \beta\}] = [i_* \text{mf } i^* \bigwedge^\bullet (U \otimes L^{1/2})(L^{-1/2})],$$

in the Grothendieck group of the derived category of coherent matrix factorizations (see Section 1.1).
Proof. Note that the differential $\delta_{\alpha,\beta}$ is compatible with the filtration of the exterior algebra $\Lambda^\bullet (V \otimes L^{1/2})(L^{-1/2})$ by powers of the ideal generated by $U$. Hence, in the Grothendieck group we can replace $\{\alpha, \beta\}$ by the associated quotient, which is isomorphic to the tensor product $C^\bullet \otimes \{\overline{\alpha}, \beta'\}$, where

$$C^\bullet = \Lambda^\bullet (U \otimes L^{1/2})(L^{-1/2})$$

and $\overline{\alpha}$ is the section of $(V/U) \otimes L$ induced by $\alpha$. Since $\beta'$ is regular, in case (i) we have $\mathcal{E}(\{\overline{\alpha}, \beta'\}) \cong i'_* \mathcal{O}_{X'}$ by Lemma 1.6.2(i). By Proposition 1.3.1, this implies that

$$\mathcal{E}(C^\bullet \otimes \{\overline{\alpha}, \beta'\}) \cong i'_* i^* C^\bullet.$$ 

In case (ii) we deduce from Lemma 1.6.2(ii) combined with Lemma 1.1.5 the following quasi-isomorphisms of coherent matrix factorizations:

$$C^\bullet \otimes \{\overline{\alpha}, \beta'\} \cong C^\bullet \otimes i_* \text{mf}(\mathcal{O}_{X'}) \cong i'_* (i^* C^\bullet \otimes \text{mf}(\mathcal{O}_{X'})) \cong i'_* \text{mf}(i^* C^\bullet).$$

$\square$

2 Matrix factorizations of a quasihomogeneous isolated singularity

Throughout this section we fix a quasihomogeneous potential $w$ on $\mathbb{A}^n$ with an isolated singularity at 0. Recall that the latter condition means the quotient $\mathbb{C}[x_1, \ldots, x_n]/(\partial_1 w, \ldots, \partial_n w)$ is finite-dimensional (where $\partial_i$ denotes the partial derivative with respect to $x_i$). We fix the set of coprime positive degrees $(d_1, \ldots, d_n)$, such that $w$ is homogeneous of degree $d$ with respect to the grading $\text{deg}(x_i) = d_i$.

From now on we denote by $w$ such a potential.

In this section we calculate the Hochschild homology of the dg-category of $\Gamma$-equivariant matrix factorizations of $w$, where $\Gamma$ is a certain one-dimensional subgroup of $\mathbb{G}_m^n$, and the canonical bilinear form on this Hochschild homology. This is a $\mathbb{Z}$-graded analog of the computations of the Hochschild homology of the $\mathbb{Z}/2$-dg-category of matrix factorizations of $w$ and of the canonical bilinear form on it performed in [10] and [49]. We also discuss functors between categories of equivariant matrix factorizations given by kernels.

2.1 Symmetry groups

With $w$ we associate certain natural groups as follows. Let us write

$$w = \sum_{k=1}^{N} c_k M_k,$$

where $M_k(x_1, \ldots, x_k)$ are monomials and $c_k \in \mathbb{C}^\times$. We have a homomorphism

$$\rho : \mathbb{G}_m^n \to \mathbb{G}_m^N : (\lambda_\bullet) \mapsto (M_k(\lambda_\bullet)).$$
Let $\Gamma_w \subset G^n_m$ be the preimage of the diagonal $G_m \subset G^n_m$ under $\rho$. In other words, $\Gamma_w$ is the maximal subgroup of diagonal transformations of $\mathbb{A}^n$ under which $w$ is semi-invariant. Let $\chi_w : \Gamma_w \to G_m$ be the natural character. It is easy to see that $G_w = \ker(\rho) = \ker(\chi_w)$, so we have a canonical extension of commutative algebraic groups

$$1 \to G_w \to \Gamma_w \xrightarrow{\chi_w} G_m \to 1.$$  

The choice of coprime degrees $d = (d_1, \ldots, d_n)$ defines an injective homomorphism

$$i_d : G_m \to \Gamma_w : \lambda \to (\lambda^{d_1}, \ldots, \lambda^{d_n})$$

(2.1)

such that $\chi_w \circ i_d(\lambda) = \lambda^d$. Thus, the intersection of $G_w$ with $i_d(G_m)$ is the cyclic subgroup of order $d$ generating by the exponential grading element

$$J = (\exp(2\pi i q_1), \ldots, \exp(2\pi i q_n)) \in G_w,$$

where $q_j = d_j/d$. (2.2)

Note that we have a short exact sequence

$$1 \to \mathbb{Z}/d \xrightarrow{i} G_w \times G_m \xrightarrow{(i,i_d)} \Gamma_w \to 1$$

where $i : G_w \to \Gamma_w$ is the natural embedding and $i(1) = (J, \exp(-2\pi i/d))$.

We will often use the following correspondence between certain subgroups of $\Gamma_w$ and subgroups of $G_w$.

**Lemma 2.1.1.** There is a natural bijection between the set of algebraic subgroups $\Gamma \subset \Gamma_w$ containing $i_d(G_m)$ and the set of algebraic subgroups $G \subset G_w$ containing the element $J$ that associates with $\Gamma$ the intersection $G = \Gamma \cap G_w$ and with $G$ the image $\Gamma = (\iota, i_d)(G \times G_m)$.

**Proof.** Let $G \subset G_w$ be a subgroup containing $J$. Then intersection of $\Gamma = (\iota, i_d)(G \times G_m)$ with $G_w = \ker(\chi_w)$ consists of the elements $i(g)i_d(\lambda)$, where $g \in G$, $\lambda \in G_m$, such that $\chi_w(i(g)i_d(\lambda)) = \lambda^d = 1$. Since in this case $i_d(\lambda)$ belongs to the subgroup generated by $J$, it follows that $\Gamma \cap G_w = G$. Now let $\Gamma \subset \Gamma_w$ be any subgroup containing $i_d(G_m)$. Consider the subgroup

$$\tilde{\Gamma} = (\iota, i_d)^{-1}(\Gamma) \subset G_w \times G_m.$$  

Then $\tilde{\Gamma}$ contains $1 \times G_m$, so we have $\tilde{\Gamma} = G \times G_m$. Furthermore, $\tilde{\Gamma}$ contains the element $i(1) \in \ker(\iota, i_d)$, hence $G$ contains $J$. \qed

Let us fix a commutative algebraic group $\Gamma$ equipped with a homomorphism $\xi : \Gamma \to \Gamma_w$ such that the composition

$$\chi = \chi_w \circ \xi : \Gamma \to G_m$$

is surjective and and the kernel of $\chi$ is finite. We set $G = \ker(\chi) \subset \Gamma$, so that we have an exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} G_m \to 1.$$  

(2.3)

We will work with the dg-category of $\Gamma$-equivariant matrix factorizations of $w$

$$\text{MF}_\Gamma(w) = \text{MF}_{\Gamma,\chi}(\mathbb{A}^n, w).$$

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The corresponding homotopy category \( \text{HMF}_\Gamma(w) = \text{HMF}(_A^n/\Gamma, w) \) is equivalent to the equivariant category of singularity of the hypersurface \( w = 0 \) (see [50, Prop. 3.19]), i.e., in this case we have \( \text{DMF}(_A^n/\Gamma, w) = \text{HMF}(_A^n/\Gamma, w) \).

Note that if we view \( \text{MF}_\Gamma(_A^n, w) \) as a \( \mathbb{Z}/2 \)-graded dg-category then it can be identified with the full subcategory in \( \text{MF}_G(_A^n, w) \).

### 2.2 Functors defined by kernels

By analogy with Fourier-Mukai transforms on derived categories of coherent sheaves we can use matrix factorizations of the external product of potentials as kernels representing functors between the categories of matrix factorizations.

Let \( W \) be a function on a smooth FCDRP-stack \( X \), which is not a zero divisor and is semi-invariant with respect to a group \( H \) acting on \( X \) and a surjective character \( \chi_W : H \to \mathbb{G}_m \) with finite kernel. Also let \( K \) be a subgroup of \( H \times \Gamma \) such that the restrictions of the characters \( \chi_W \times 1 \) and \( 1 \times \chi_w \) to \( K \) are equal. Consider the potential

\[
(W \oplus w) := p_1^* W + p_2^* w
\]
on \( X \times A^n \), where \( p_1 : X \times A^n \to X \) and \( p_2 : X \times A^n \to A^n \) are the projections. Then \( W \oplus w \) is semi-invariant with respect to \( K \). Note that if we take \( K \) to be maximal then \((X \times A^n)/K, W \oplus w)\) can be identified the external product of \((X/H, W)\) with \((A^n/\Gamma, w)\) (see Definition 1.1.7). Since the singularity locus of \( W \oplus w \) is a subset of \( X_0 \times \{0\} \), Proposition 1.4.2 implies that for any \( P \in \text{DMF}_K(W \oplus w) \) and \( Q \in \text{DMF}_\Gamma(A^n, -w) \) the tensor product \( P \otimes Q \in \text{DMF}_{K}(p_1^* W) \) belongs to the subcategory \( \text{DMF}_{X_0 \times \{0\}}(X \times A^n/K, p_1^* W) \subset \text{DMF}(X \times A^n/K, p_1^* W) \).

Assume moreover that the projection \( K \to H \) is surjective with the kernel \( K_0 \). Then we can apply the functor \( p_1^{K_0} \) to \( P \otimes K \) (see (1.22)), so we obtain a functor

\[
\Phi_P : \text{DMF}_\Gamma(A^n, -w) \to \text{DMF}_H(X, W) : Q \mapsto p_1^*(P \otimes Q)^{K_0}.
\]

The same construction works in the case \( W = 0 \) if we assume that \( H \) acts trivially on \( X \) (see Example 1.5.4).

**Remark 2.2.1.** 1. The functor \( \Phi_P \) has a natural dg-version. Namely, the category \( \text{DMF}^\infty_H(X, W) \) can be identified with the full subcategory of the derived category of quasi-matrix factorizations \( \text{DMF}^\infty_H(X, W) \) (see Cor. 4.5 and Lem. 4.6 of [50]). The latter category has a natural dg-version \( \text{DMF}^\infty_{H}^{dg}(X, W) \) obtained using the construction of dg-quotient (see [32, Thm. 4.8]). Thus, we get a dg-version of \( \text{DMF}_H(X, W) \) by taking the corresponding full dg-subcategory of \( \text{DMF}^\infty_{H}^{dg}(X, W) \). Now to obtain a dg-functor \( \Phi_P^{dg} \) inducing \( \Phi_P \) we can use the natural push-forward dg-functor for quasi-matrix factorizations (see [50, Def. 4.8]). Furthermore, all the isomorphisms of functors of the form \( \Phi_P \) discussed below will be induced by morphisms between dg-functors.

2. By Corollary 8.0.4, the map on Hochschild homology induced by the dg-functor \( \Phi_P^{dg} \) depends only on the class of \( P \) in the Grothendieck group. In particular, since a different
the other hand, since the $\Gamma$-equivariant map

$$\phi$$

is surjective, we can find a $\Gamma$-equivariant element $\phi$ of

$$\mathbb{C}$$

where $e_1, \ldots, e_n$ is a basis of $T^*$, then $\psi = \sum_i e_i^* \otimes x_i$, where $(e_i^*)$ is the dual basis of $T$. On the other hand, since the $\Gamma$-equivariant map

$$\langle \phi^*, \psi \rangle : T^* \otimes \mathbb{C}[x] \to \mathfrak{m}$$

is surjective, we can find a $\Gamma$-equivariant element $\phi \in T^* \otimes \mathbb{C}[x] \otimes \chi$, such that $\langle \phi, \psi \rangle = w$. The pair $\phi, \psi$ defines a $\Gamma$-equivariant Koszul matrix factorization

$$\mathbb{C}^* = \{ \phi, \psi \}$$

of $w$ on $\mathbb{A}^n$ with the respect to the character $\chi$.

Let $\langle \chi \rangle \subset \hat{\Gamma}$ denote the subgroup (isomorphic to $\mathbb{Z}$) generated by $\chi$ in the group of characters of $\Gamma$. Let $\chi_1, \ldots, \chi_r$ be a set of representatives for $\langle \chi \rangle$-cosets in $\hat{\Gamma}$.

2.3 Generators of categories of matrix factorizations

Let us define an object $\mathbb{C}^* \in \text{MF}_\Gamma(\mathbb{A}^n, w)$, the stabilized residue field, similarly to the $\mathbb{Z}/2$-graded considered in [49, sec. 2.5]. Let $T = (\mathfrak{m}/\mathfrak{m}^2)^*$ be the tangent space to $\mathbb{A}^n$ at the origin. The projection $\mathfrak{m} \to T^*$ admits a $\Gamma$-equivariant splitting $s : T^* \to \mathfrak{m}$, which defines an element $\psi \in (T \otimes \mathbb{C}[x])^\Gamma$. If we choose generators $x_1, \ldots, x_n$ of $\mathfrak{m} \subset \mathbb{C}[x]$ as $x_i = s(e_i)$, where $e_1, \ldots, e_n$ is a basis of $T^*$, then $\psi = \sum_i e_i^* \otimes x_i$, where $(e_i^*)$ is the dual basis of $T$. On the other hand, since the $\Gamma$-equivariant map

$$\langle \phi, \psi \rangle : T^* \otimes \mathbb{C}[x] \to \mathfrak{m}$$

is surjective, we can find a $\Gamma$-equivariant element $\phi \in T^* \otimes \mathbb{C}[x] \otimes \chi$, such that $\langle \phi, \psi \rangle = w$. The pair $\phi, \psi$ defines a $\Gamma$-equivariant Koszul matrix factorization

$$\mathbb{C}^* = \{ \phi, \psi \}$$

of $w$ on $\mathbb{A}^n$ with the respect to the character $\chi$.

Let $\langle \chi \rangle \subset \hat{\Gamma}$ denote the subgroup (isomorphic to $\mathbb{Z}$) generated by $\chi$ in the group of characters of $\Gamma$. Let $\chi_1, \ldots, \chi_r$ be a set of representatives for $\langle \chi \rangle$-cosets in $\hat{\Gamma}$.
Proposition 2.3.1. The matrix factorization \( \bigoplus_{i=1}^{r} \mathbb{C}^* \otimes \chi_i \) is a generator of the triangulated category \( \text{DMF}_1(\mathbb{A}^n, w) = \text{HMF}_1(\mathbb{A}^n, w) \).

Proof. Under the equivalence of \( \text{DMF}_1(\mathbb{A}^n, w) \) with the equivariant singularity category of the hypersurface \( S = (w = 0) \) (see [50, Thm. 3.14]) the stabilized residue field \( \mathbb{C}^* \) corresponds to the skyscraper sheaf at the origin. Since the singularity locus of \( S \) is the origin, by [50, Cor. 5.3], the category \( D_{SG}(X/\Gamma) \) is equivalent to \( D_{SG}(X/\Gamma, 0/\Gamma) \). The latter category is generated by the skyscraper sheaf at the origin twisted by characters of \( \Gamma \). Since the twisting by \( \chi \) is isomorphic to the square of the translation functor, it is enough to consider representatives of \( \hat{\Gamma}/\langle \chi \rangle \) (cf. [52, sec. 12] for a similar reasoning). \( \square \)

Corollary 2.3.2. Let \( w' \) be another quasi-homogeneous potential on \( \mathbb{A}^m \) with an isolated singularity at \( 0 \), semi-invariant with respect to \((\Gamma', \chi')\). Let \( \Pi \subset \Gamma' \times \Gamma \) denote the preimage of the diagonal under the homomorphism \( \chi' \times \chi : \Gamma' \times \Gamma \to G_m \times G_m \). Then the external tensor product dg-functor

\[
\text{MF}_{\Gamma'}(\mathbb{A}^m, w') \otimes \text{MF}_{\Gamma}(\mathbb{A}^n, w) \to \text{MF}_{\Pi}(\mathbb{A}^m \times \mathbb{A}^n, w' \oplus w)
\]

induces an equivalence of perfect derived categories.

Proof. Use Proposition 2.3.1 and argue as in [10, Sec. 6.1]. \( \square \)

Remark 2.3.3. A more general version of the above Corollary is proved independently in the work of Ballard, Favero and Katzarkov [4, Prop. 6.7].

2.4 The diagonal matrix factorization

Here we construct a \( \Gamma_w \)-equivariant version of the diagonal matrix factorization representing the identity functor (see [10], [49]). We keep the notation of the previous section. Consider the \( \Gamma_w \)-invariant elements \( \alpha \in V \otimes k[x, y] \otimes \chi \) and \( \beta \in V^* \otimes k[x, y] \), given by

\[
\alpha = \sum_j e_j \otimes w_j, \quad \beta = \sum_j e_j^* \otimes (y_j - x_j),
\]

where \( w_j(x, y) \) are polynomials such that

\[
w(y) - w(x) = \sum_{j=1}^{n} (y_j - x_j)w_j(x, y)
\]

(such polynomials exist because \( \Gamma_w \) is reductive). We define a \( \Gamma_w \)-equivariant matrix factorization of \( \tilde{w} = -w(x) + w(y) \) on \( \mathbb{A}^n \times \mathbb{A}^n \) (with respect to the diagonal action of \( \Gamma_w \) on \( \mathbb{A}^n \times \mathbb{A}^n \)) by

\[
\Delta_w^\text{st} = \{ \alpha, \beta \}.
\]

Now let \( X \) be a smooth FCDRP-stack, and let \( W \) be a function on \( X \), as in section 2.2, i.e., \( W \) is semi-invariant with respect to \((H, \chi_W)\) and we have a subgroup \( K \subset H \times \Gamma \) such that \( \chi_W \times \text{id}|_K = \text{id} \times \chi|_K \). We also assume that either \( W \) is not a zero divisor, or \( W = 0 \) and \( H \) acts trivially on \( X \).
Proposition 2.4.1. (i) The functor
\[ \Phi_{\Delta_{st}^w}: \text{DMF}_\Gamma(\mathbb{A}^n, w) \to \text{DMF}_\Gamma(\mathbb{A}^n, w) \]
associated with the kernel \( \Delta_{st}^w \in \text{DMF}_\Gamma(\mathbb{A}^n \times \mathbb{A}^n, \overline{w}) \) by (2.4), is isomorphic to the identity functor.

(ii) Let \( \pi: \mathbb{A}^n \to \text{pt} \) be the projection and \( \tilde{\Delta}_{st} = p_{23}^{*} \Delta_{st}^w \in \text{DMF}_K(X \times \mathbb{A}^n \times \mathbb{A}^n, 0 \oplus (-w) \oplus w) \) be the pull-back of \( \Delta_{st} \) under the projection \( p_{23}: X \times \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n \). Then the following diagram of functors is commutative up to an isomorphism:

\[
\begin{array}{ccc}
\text{DMF}_K(X \times \mathbb{A}^n \times \mathbb{A}^n, W \oplus w \oplus (-w)) & \xrightarrow{\otimes \tilde{\Delta}_{st}^w} & \text{DMF}_K(X \times \{(0,0)\}, X \times \mathbb{A}^n, W \oplus 0 \oplus 0) \\
(id_X \times \Delta)^* \downarrow & & (id_X \times \pi \times \pi)^{K_0} \downarrow \\
\text{DMF}_{K,X \times \{0\}}(X \times \mathbb{A}^n, W \oplus 0) & \xrightarrow{(id_X \times \pi)^{K_0}} & \text{DMF}_H(X, W)
\end{array}
\]

where \( \Delta: \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n \) is the diagonal embedding, \( K_0 = \ker(K \to H) \), and we use the push-forward functors combined with taking \( K_0 \)-invariants as in (1.22).

Proof. (i) This can be derived from Proposition 2.3.1 similarly to the non-equivariant case considered in [10] (see also [49]).

(ii) The proof is based on the fact that \( \tilde{\Delta}_{st} \) is a regular Koszul matrix factorization with the zero locus \( X \times \Delta(\mathbb{A}^n) \subset X \times \mathbb{A}^n \times \mathbb{A}^n \). Assume first that \( W \) is not a zero divisor. Then by Proposition 1.6.3(i), we have
\[ P \otimes \tilde{\Delta}_{st} \simeq (id_X \times \Delta)_*(id_X \times \Delta)^* P, \]
which implies the result since
\[ (id_X \times \pi \times \pi)_* \circ (id_X \times \Delta)_* = (id_X \times \pi)_*. \]
In the case \( W = 0 \) the proof is similar, but we use Proposition 1.6.3(ii) instead. \( \square \)

Corollary 2.4.2. The dg-category \( \text{MF}_\Gamma(\mathbb{A}^n, w) \) is dg-Morita equivalent to a smooth proper dg-algebra.

Proof. This follows from the existence of a compact generator (see Proposition 2.3.1), from Corollary 2.3.2 and from the fact that the diagonal bimodule is represented by a matrix factorization (cf. [10, Sec. 7]).
2.5 Hochschild homology and the Chern character for dg-categories

Below we will use the formalism of [49, Sec. 1] (see also [30] and [57]).

Let $\mathcal{C}$ be a dg-category over a field $k$. We denote by $D(\mathcal{C})$ the derived category of right $\mathcal{C}$-modules, by $\text{Per}(\mathcal{C}) \subset D(\mathcal{C})$ the perfect derived category, and by $\text{Per}_{dg}(\mathcal{C})$ the dg-category of homotopically finitely presented right $\mathcal{C}$-modules (see [57, Sec. 7]). The Hochschild homology of $\mathcal{C}$ is given by

$$HH_*(\mathcal{C}) = \text{Tr}_{\mathcal{C}}(\Delta_{\mathcal{C}}),$$

where $\Delta_{\mathcal{C}}$ is the diagonal $\mathcal{C} - \mathcal{C}$-bimodule $E \otimes F^\vee \mapsto \text{Hom}_\mathcal{C}(F, E)$ and

$$\text{Tr}_{\mathcal{C}} : D(\mathcal{C}^{op} \otimes \mathcal{C}) \to D(k) \quad (2.6)$$

is the trace functor given by the derived tensor product with $\Delta_{\mathcal{C}}$.

As in [49, Sec. 1.2] we consider only dg-categories $\mathcal{C}$ such that the $\mathcal{C} - \mathcal{C}$-bimodule $\Delta_{\mathcal{C}}$ is perfect, the complexes $\text{Hom}_\mathcal{C}(A, B)$ for $A, B \in \mathcal{C}$ have finite dimensional cohomology, and the derived category $D(\mathcal{C})$ has a compact generator. Such dg-categories are dg Morita equivalent to homologically smooth and proper dg-algebras and can be characterized by the condition that $\text{Per}_{dg}(\mathcal{C})$ is saturated, i.e., proper, smooth and triangulated (see [58, Sec. 2.2]).

Any dg-functor $F : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{D})$ between dg-categories of the above type induces a map $F_* : HH_*(\mathcal{C}) \to HH_*(\mathcal{D})$. This map can be defined in several equivalent ways. We will use a slight modification of the construction given in [49, Sec. 1.2] (see Appendix for the details).

First, consider the induced functor

$$F^{(2)} : \text{Per}(\mathcal{C}^{op} \otimes \mathcal{C}) \to \text{Per}(\mathcal{D}^{op} \otimes \mathcal{D})$$

that sends the representable module $h_{C_1^{op} \otimes C_2}$ to $h_{F(C_1)^{op} \otimes F(C_2)}$.

There is a canonical morphism of functors

$$\text{Tr}_{\mathcal{C}} \to \text{Tr}_{\mathcal{D}} \circ F^{(2)} \quad (2.7)$$

and a canonical morphism

$$F^{(2)}(\Delta_{\mathcal{C}}) \to \Delta_{\mathcal{D}} \quad (2.8)$$

in $\text{Per}(\mathcal{D}^{op} \otimes \mathcal{D})$.

Now the map $F_*$ is defined as the composition

$$\text{Tr}_{\mathcal{C}}(\Delta_{\mathcal{C}}) \to \text{Tr}_{\mathcal{D}} F^{(2)}(\Delta_{\mathcal{C}}) \to \text{Tr}_{\mathcal{D}}(\Delta_{\mathcal{D}}),$$

where the first arrow is induced by (2.7) and the second is induced by (2.8).

For an object $E \in \text{Per}_{dg}(\mathcal{C})$ we define its Chern character using the functor $1_E : \text{Per}_{dg}(k) \to \text{Per}_{dg}(CC)$ sending $k$ to $E$ as follows

$$\text{ch}(E) = (1_E)_*(1) \in H_0(\mathcal{C}).$$

The maps $F_*$ are compatible with the composition (see [49, Lem. 1.2.1]), which implies the functoriality of the Chern character

$$\text{ch}(F(E)) = F_*(\text{ch}(E))$$

for $E \in \text{Per}(\mathcal{C})$.  

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2.6 Hochschild homology of $\text{MF}_\Gamma(w)$

Let $A_w = \mathbb{C}[x]/(\partial_1 w, \ldots, \partial_n w)$ be the Milnor ring of the isolated singularity $w$, and consider the space $H(w) = A_w \otimes dx$, where $dx = dx_1 \wedge \ldots \wedge dx_n$, equipped with the action of $\Gamma$, induced by its action on $\mathbb{A}^n$.

Recall that the Hochschild homology of the $\mathbb{Z}/2$-dg-category $\text{MF}(w)$ is isomorphic to the space $H(w)$ in degree $n \mod(2)$ (see [10]). The Hochschild homology of $\mathbb{Z}/2$-dg-category $\text{MF}_G(w)$ is given by

$$HH_*(\text{MF}_G(w)) \simeq \bigoplus_{\gamma \in G} H(w_\gamma)^G$$

where $w_\gamma$ is the restriction of $w$ to the subspace $(\mathbb{A}^n)^\gamma \subset \mathbb{A}^n$ (see [49]). Here we establish a version of this isomorphism for the $\mathbb{Z}$-graded dg-category $\text{MF}_\Gamma(w)$.

Let $\Gamma^{(2)} \subset \Gamma \times \Gamma$ denote the preimage of the diagonal under the homomorphism $\chi \times \chi : \Gamma \times \Gamma \to \mathbb{G}_m \times \mathbb{G}_m$. Let $\chi^{(2)} : \Gamma^{(2)} \to \mathbb{G}_m$ be the character induced by $\chi$ and by one of the projections $\Gamma^{(2)} \to \Gamma$. Let us consider the dg-category $\text{MF}_\Gamma^{(2)}(\mathbb{A}^n \times \mathbb{A}^n, \tilde{w})$, where $\tilde{w}(x, y) = -w(x) + w(y)$. Similar to the $\mathbb{Z}/2$-graded case (see [10, sec. 6.1]) we can interpret the corresponding perfect derived category as the category of dg-functors $\text{MF}_\Gamma(\mathbb{A}^n) \to \text{MF}_\Gamma(\mathbb{A}^n)$. Namely, to a kernel $K \in \text{MF}_\Gamma^{(2)}(\mathbb{A}^n \times \mathbb{A}^n, \tilde{w})$ we associate the dg-functor (2.4)

$$\Phi_K : \text{MF}_\Gamma^{\infty}(\mathbb{A}^n, w) \to \text{MF}_\Gamma^{\infty}(\mathbb{A}^n, \tilde{w}) : \tilde{E} \mapsto p_2_*(p_1^*\tilde{E} \otimes K)^{G \times \{1\}}.$$ 

Note that here the invariants are taken with respect to the action of the group $G$ on the first factor of the product $\mathbb{A}^n \times \mathbb{A}^n$. Since $\text{Per}_{dg}(\text{MF}_\Gamma(\mathbb{A}^n, w))$ is saturated, Corollary 2.3.2 implies that every dg-functor from this category to itself is represented by a matrix factorization of $\tilde{w}$.

Now we are ready to compute the Hochschild homology of $\text{MF}_\Gamma(w)$. Let $\hat{G}$ be the dual group to $G$. The exact sequence (2.3) induces an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{n \mapsto \chi^n} \hat{\Gamma} \to \hat{G} \to 0.$$ 

Note that we have the natural action of $\hat{\Gamma}$ on the category $\text{DMF}_\Gamma(w)$ given by tensor multiplication with 1-dimensional representations of $\Gamma$. Furthermore, by definition of the triangulated structure on $\text{DMF}_\Gamma(w)$ we have

$$\tilde{E} \otimes \chi \simeq \tilde{E}[2].$$

Hence, the induced action of $\hat{\Gamma}$ on $HH_*(\text{MF}_\Gamma(w))$ factors through an action of $\hat{G}$. In other words, $HH_*(\text{MF}_\Gamma(w))$ has a natural structure of $R$-module for $R = \mathbb{C}[\hat{G}]$.

**Theorem 2.6.1.** (i) The Hochschild homology of the dg-category $\text{MF}_\Gamma(w)$ is given by

$$HH_*(\text{MF}_\Gamma(w)) \simeq \bigoplus_{\gamma \in G} H(w_\gamma)^G,$$ 

(ii)
where $w_\gamma$ is the restriction of $w$ to the subspace of $\gamma$-invariants $(\mathbb{A}^n)^\gamma$, with the $\mathbb{Z}$-grading is given by

$$H(w_\gamma)^G = \bigoplus_{i \in \mathbb{Z}} H(w_\gamma)^G_{\chi^{-i}}[n_\gamma - 2i],$$

where $n_\gamma = \dim(\mathbb{A}^n)^\gamma$. We have an isomorphism of $\mathbb{Z}/2$-graded spaces

$$HH_*(\text{MF}_G(w)) \simeq HH_*(\text{MF}_G(w))$$

identifying the decompositions (2.10) and (2.9).

(ii) The decomposition of $HH_*(\text{MF}_G(w))$ into $\gamma$-isotypical subspaces (where $\gamma \in G$ is viewed as a character of $\hat{G}$) coincides with the decompositions (2.10) and (2.9).

(iii) Let $\Gamma' \subset \Gamma$ be a subgroup such that the restriction of $\chi$ to $\Gamma'$ is surjective, and let $G' = \Gamma' \cap G_w$ be the corresponding subgroup of $G$. Let

$$\text{Res}^G_{G'} : HH_*(\text{MF}_G(w)) \to HH_*(\text{MF}_{\Gamma'}(w))$$

be the map induced by the forgetful functor

$$\Phi : \text{MF}_G(w) \to \text{MF}_{\Gamma'}(w).$$

Then the restriction of $\text{Res}^G_{G'}$ to the component of the decomposition (2.10) corresponding to an element $\gamma \in G$ is equal to zero if $\gamma \notin G'$ and to the canonical embedding $H(w_\gamma)^G \to H(w_\gamma)^{G'}$ if $\gamma \in G'$.

**Proof.** (i) First, let us check that the trace functor (see Section 2.5)

$$\text{Tr} : \text{MF}_{\Gamma}(\mathbb{A}^n \times \mathbb{A}^n, \tilde{w}) \to \text{Com}_{\Gamma}(\mathbb{C} - \text{mod})$$

associates with a matrix factorization $\tilde{E}$ of $\tilde{w}$ the $\Gamma$-invariants in the global sections of the restriction of the complex $\text{com}(\tilde{E})$ (see (1.4)) to the diagonal ($y = x$) in $\mathbb{A}^n \times \mathbb{A}^n$. Indeed, this follows from the isomorphism

$$H^0(\mathbb{A}^n, \text{com}(\tilde{E}^\vee \otimes \tilde{E}')|_{y=x})^\Gamma \simeq H^0(\mathbb{A}^n, \text{com}(\tilde{E}^\vee \otimes \tilde{E}')^\Gamma) \simeq \text{Hom}_{\text{MF}_G}(\tilde{E}, \tilde{E}')$$

for $\tilde{E}, \tilde{E}' \in \text{MF}_G(\mathbb{A}^n, w)$ (see Lemma 1.1.6).

Next, we observe that the identity functor on $\text{MF}_G(\mathbb{A}^n, w)$ is represented by

$$\Delta^\text{st}_G := \bigoplus_{\gamma \in G}(\text{id} \times \gamma)^* \Delta^\text{st} \in \text{MF}_{\Gamma}(\tilde{w}),$$

where $\Delta^\text{st} = \Delta^\text{st}_w$, with the $\Gamma^{(2)}$-equivariant structure induced by the $\Gamma$-equivariant structure on $\Delta^\text{st}$ via the diagonal embedding of $\Gamma$ into $\Gamma^{(2)}$. This follows immediately from Proposition 2.4.1(i) because of the exact sequence $1 \to G \to \Gamma^{(2)} \to \Gamma \to 1$.

Finally, to compute the Hochschild homology we have to apply the functor $\text{Tr}$ to $\Delta^\text{st}_G$. Let us show that

$$\text{com}(\Delta^\text{st}_G)|_{y=x} \simeq \bigoplus_{\gamma \in G} \bigoplus_{i \in \mathbb{Z}} (\mathcal{K}_{w, \chi^{-i}})[2i],$$

(2.13)
where $\mathcal{K}_w$ is the complex
\begin{equation}
\mathcal{K}_w = [\Omega^0_{\mathbb{A}^n} \to \Omega^1_{\mathbb{A}^n} \otimes \chi \to \ldots \to \Omega^n_{\mathbb{A}^n} \otimes \chi^n]
\end{equation}
(2.14)
placed in degrees $[0, n]$ with the differential given by $dw\wedge \cdot$. The summand of (2.13) corresponding to $\gamma = 1$ is
\begin{equation}
\text{com}(\Delta^{\text{st}})|_{y=x} \simeq \bigoplus_{i \in \mathbb{Z}} (\mathcal{K}_w \otimes \chi^{-i})[2i].
\end{equation}
(2.15)
Using the identifications
\begin{equation}
(\Delta^{\text{st}})_0|_{y=x} \simeq \bigoplus_{i \geq 0} \Omega^{2i}_{\mathbb{A}^n} \cdot \chi^i, \quad \text{and} \quad (\Delta^{\text{st}})_1|_{y=x} \simeq \bigoplus_{i \geq 0} \Omega^{2i+1}_{\mathbb{A}^n} \cdot \chi^i,
\end{equation}
the complex $\text{com}(\Delta^{\text{st}})|_{y=x}$ can be presented as
\begin{align*}
\Omega^0 \chi^{-1} & \quad \Omega^0 \quad \Omega^0 \chi \\
\Omega^1 & \quad \Omega^1 \chi \\
\Omega^2 & \quad \Omega^2 \chi \\
\ldots & \quad \ldots \quad \ldots
\end{align*}
where all the arrows are given by $dw\wedge \cdot$. This immediately gives the isomorphism (2.15). The equality of the summands in (2.13) corresponding to $\gamma \neq 1$ is verified similarly by explicitly calculating $(\text{id} \times \gamma)^* \Delta^{\text{st}}|_{y=x} \simeq \Delta^{\text{st}}|_{y=\gamma x}$ as in [49, sec. 2.5].

Since each $w_\gamma$ is an isolated singularity (see [49, Lem. 2.5.3(i)]), the cohomology of the complex $\mathcal{K}_{w_\gamma}$ (2.14) is isomorphic to $H(w_\gamma) \otimes \chi^{n_\gamma}$ concentrated in (cohomological) degree $n_\gamma$. Hence, using (2.13) we see that the cohomology of $\text{com}(\Delta_{G}^{\text{st}})$ is isomorphic to
\begin{equation}
\bigoplus_{\gamma \in G} \bigoplus_{i \in \mathbb{Z}} (H(w_\gamma) \otimes \chi^{n_\gamma-i})[2i - n_\gamma].
\end{equation}
Passing to $\Gamma$-invariants and substituting $i \mapsto n_\gamma - i$ we get the result. The last assertion follows from the computation in [49, Sec. 2.5].

(ii) In general, if $\alpha : \mathcal{C} \to \mathcal{C}$ is an autoequivalence of a dg-category, the induced automorphism $\alpha_*$ of $HH_* (\mathcal{C})$ is defined as follows (see section 2.5). We have an induced equivalence of $\alpha^{(2)}$ of $\mathcal{C}^{op} \otimes \mathcal{C}$ and natural isomorphisms
\begin{equation}
\psi : \text{Tr}_\mathcal{C} \circ (\alpha^{(2)}) \simeq \text{Tr}_\mathcal{C},
\end{equation}
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that induce an automorphism

\[ \alpha_* : HH_*(\mathcal{C}) = \text{Tr}_\mathcal{C}(\Delta_e) \xrightarrow{\phi} \text{Tr}_\mathcal{C}(\alpha^{(2)}(\Delta_e)) \xrightarrow{\psi} \text{Tr}_\mathcal{C}(\Delta_e) = HH_*(\mathcal{C}). \]

Now let us specialize to the case of \( \mathcal{C} = \text{MF}_\Gamma(w) \) and the autoequivalence \( \alpha \) given by the tensoring with a character \( \eta \) of \( \Gamma \). Under the identification of the perfect derived category \( \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}) \) with \( \text{HMF}_{(2)\Gamma}(\mathbb{A}^n \times \mathbb{A}^n, \mathbb{w}) \) (see Corollary 2.3.2) the functor \( \alpha^{(2)} \) corresponds to tensoring with the character \( \eta^{-1} \times \eta|_{\Gamma^{(2)}} \). Recall that by Proposition 2.4.1, the kernel \( \Delta_e \) representing the identity functor in this case is \( \Delta_e^{\text{st}} = \bigoplus_{\gamma \in G}(\text{id} \times \gamma)^*\Delta^{\text{st}} \). It is easy to check that the isomorphism \( \phi \) in this case is given by the multiplication by \( \eta(\gamma)^{-1} \) on the component \( (\text{id} \times \gamma)^*\Delta^{\text{st}} \).

Since, the automorphism \( \alpha_* \) is obtained by the restriction of \( \phi \) to the diagonal in \( \mathbb{A}^n \times \mathbb{A}^n \), we obtain that \( \alpha_* \) acts as \( \eta(\gamma)^{-1} \) on the component of \( HH_*(\text{MF}_\Gamma(w)) \) coming from the term

\[ (\text{id} \times \gamma)^*\Delta^{\text{st}}|_{y=x} \simeq \Delta^{\text{st}}|_{y=\gamma x} \]

of \( \Delta_e^{\text{st}} \), which is exactly the term corresponding to \( \gamma \) in the decomposition (2.10).

(iii) Let us apply the general construction of Section 2.5 (see also Appendix) to the forgetful functor \( \Phi : \mathcal{C} \rightarrow \mathcal{D} \), where \( \mathcal{C} = \text{MF}_\Gamma(w) \) and \( \mathcal{D} = \text{MF}_{\Gamma'}(w) \). By Corollary 2.3.2, we have natural equivalences

\[ \text{Per}(\mathcal{C} \otimes \mathcal{C}^{\text{op}}) \simeq \text{HMF}_{(2)\Gamma}(w \oplus (-w)), \quad \text{Per}(\mathcal{D} \otimes \mathcal{D}^{\text{op}}) \simeq \text{HMF}_{(2)\Gamma'}(w \oplus (-w)). \]

Under the equivalences (2.17) the induced functor \( \Phi^{(2)} : \text{Per}(\mathcal{C} \otimes \mathcal{C}^{\text{op}}) \rightarrow \text{Per}(\mathcal{D} \otimes \mathcal{D}^{\text{op}}) \) is identified with the forgetful functor corresponding to the restriction from \( \Gamma^{(2)} \) to \( (\Gamma')^{(2)} \). The map on Hochschild homology \( \Phi_* \) is given by the composition

\[ \text{Tr}_\mathcal{C}(\Delta_e) \rightarrow \text{Tr}_\mathcal{D} \Phi^{(2)}(\Delta_e) \rightarrow \text{Tr}_\mathcal{D}(\Delta_\mathcal{D}) \]

via the natural transformation

\[ \text{Tr}_\mathcal{C} \rightarrow \text{Tr}_\mathcal{D} \circ \Phi^{(2)} \]

and the natural morphism

\[ \Phi^{(2)}(\Delta_e) \rightarrow \Delta_\mathcal{D}. \]

As we have seen in the beginning of the proof of (i), the functor \( \text{Tr}_\mathcal{C} \) (resp., \( \text{Tr}_\mathcal{D} \)) can be identified under the equivalences (2.17) with the restriction to the diagonal followed by taking \( \Gamma \)-invariants (resp., \( \Gamma' \)-invariants). Since the morphism (2.18) on objects of the form \( E_1 \otimes E_2' \in \mathcal{C} \otimes \mathcal{C}^{\text{op}} \) corresponds to the natural embedding \( \text{Hom}_\mathcal{C}(E_2, E_1) \rightarrow \text{Hom}_\mathcal{D}(\Phi(E_2), \Phi(E_1)) \), we see that the morphism (2.18) corresponds to the natural embedding of \( \Gamma \)-invariants into \( \Gamma' \)-invariants. On the other hand, using equivalences (2.17) and (2.12) we have

\[ \Delta_e \simeq \bigoplus_{\gamma \in G}(\text{id} \times \gamma)^*\Delta^{\text{st}}_w, \quad \Delta_\mathcal{D} \simeq \bigoplus_{\gamma' \in G'}(\text{id} \times \gamma')^*\Delta^{\text{st}}_w. \]

We claim that the morphism (2.19) corresponds under these identifications to the natural projection (identity on summands corresponding to elements \( \gamma \in G' \), and zero on all the
other summands). This will immediately imply the desired statement. By choosing a \( \Gamma \)-equivariant generator of the category of (non-equivariant) matrix factorizations of \( w \), we can reduce the task to a similar question for a dg-algebra \( A \) with an action of the group \( G \). The analog of a representation of the identity functor via the kernel (2.12) is the functorial isomorphism of \( A[G] \)-modules

\[
M \simeq (A[G] \otimes_A M)^G : m \mapsto \sum_{\gamma \in G} \gamma^{-1} \otimes \gamma m,
\]

(2.20)

where \( M \) is any module over \( A[G] \), the twisted group algebra of \( G \). The \( G \)-invariants on the right-hand side of (2.20) are taken with respect to the action of \( G \) on \( A[G] \otimes_A M \) given by

\[
\gamma \cdot (x \otimes m) = x \gamma^{-1} \otimes \gamma m,
\]

while the \( A[G] \)-structure is induced by the the left action of \( A[G] \) on itself. The morphism (2.19) is obtained via the natural isomorphism

\[
\Phi^{(2)}(\Delta \mathcal{C}) \simeq \Phi \circ \Psi,
\]

where \( \Psi : A[G'] - \text{mod} \to A[G] - \text{mod} \) is the right adjoint functor to the restriction functor \( \Phi : A[G] - \text{mod} \to A[G'] - \text{mod} \). For an \( A[G] \)-module \( N \) we have a functorial isomorphism of \( A[G]-\text{modules} \)

\[
\Psi(N) \simeq (A[G] \otimes_A N)^{G'}
\]

(with the same conventions as in (2.20)). Namely, for any \( A[G] \)-module \( M \) the isomorphism

\[
\text{Hom}_{A[G']}(M, N) \to \text{Hom}_{A[G]}(M, (A[G] \otimes_A N)^{G'})
\]

sends \( f : M \to N \) to the homomorphism

\[
m \mapsto \sum_{\gamma \in G} \gamma^{-1} \otimes f(\gamma m).
\]

Thus, the adjunction map \( (\Phi \circ \Psi)(N) \to N \) is given by the map of \( G' \)-modules

\[
(A[G] \otimes_A (A[G] \otimes_A N)^{G'})^G \simeq (P \otimes_A N)^{G'} \xrightarrow{\pi} (A[G'] \otimes N)^{G'} \simeq N
\]

where \( P = (A[G] \otimes_A A[G])^G \simeq A[G] \) as \( A[G'] - A[G'] \)-bimodule, and the map \( \pi \) is induced by the projection \( A[G] \to A[G'] \) (sending \( [\gamma] \) to zero for all \( \gamma \in G \setminus G' \)). This implies our claim.

**Corollary 2.6.2.** Let \( w' \) be a quasi-homogeneous potential on \( \mathbb{A}^m \) with an isolated singularity at 0, semi-invariant with respect to the same group \( \Gamma \) and the same character \( \chi : \Gamma \to \mathbb{G}_m \). Then the tensor product functor induces an isomorphism

\[
\mathcal{H}(w') \otimes_R \mathcal{H}(w) \to \text{HH}_a(MF_\Gamma(\mathbb{A}^m \times \mathbb{A}^n, w' \oplus w)^{G \times G},
\]

(2.21)

where \( R = \mathbb{C}[\hat{G}] \) and the \( G \times G \)-action on the Hochschild homology is induced by the action of \( G \times G \) on \( \mathbb{A}^m \times \mathbb{A}^n \).
Proof. By Corollary 2.3.2 together with the Künneth formula for Hochschild homology (see [55, Sec. 2.4], [49, 1.1.4]), the tensor product induces an isomorphism

\[ \mathcal{H}(w') \otimes_{\mathbb{C}} \mathcal{H}(w) \to HH_{*}(\text{MF}_{\Gamma}(\mathbb{A}^{m} \times \mathbb{A}^{n}, w' \oplus w)). \]

Now by Theorem 2.6.1(iii), the map

\[ HH_{*}(\text{MF}_{\Gamma}(\mathbb{A}^{m} \times \mathbb{A}^{n}, w' \oplus w)) \to HH_{*}(\text{MF}_{\Gamma}(\mathbb{A}^{m} \times \mathbb{A}^{n}, w' \oplus w))^{G \times G} \]

coincides with the projection to the components associated with the image of the diagonal embedding \( G \to G \times G \) (where we use the natural \( G \times G \)-grading on the \( \Gamma^{(2)} \)-equivariant Hochschild homology). Similarly, the map

\[ \mathcal{H}(w') \otimes_{\mathbb{C}} \mathcal{H}(w) \to \mathcal{H}(w') \otimes_{\mathbb{R}} \mathcal{H}(w) \]

can be identified with the projection to \( \bigoplus_{\gamma \in G} e_{\gamma} \mathcal{H}(w') \otimes e_{\gamma} \mathcal{H}(w) \), so the assertion follows. \( \Box \)

Example 2.6.3. Consider \( \Gamma = \mathbb{G}_{m} \) embedded naturally into \( \Gamma_{w} \) via \( \lambda \mapsto (\lambda^{d_{1}}, \ldots, \lambda^{d_{n}}) \), so that the induced character \( \chi : \mathbb{G}_{m} \to \mathbb{G}_{m} \) is \( \lambda \mapsto \lambda^{d} \), and \( G = \mathbb{Z}/d \). Then

\[ HH_{*}(\text{MF}_{\mathbb{G}_{m}}(w)) \simeq \bigoplus_{j \in \mathbb{Z}/d, i \in \mathbb{Z}} H(w_{j})_{di}[n_{j} - 2i], \]

where \( n_{j} \) is the number of \( s \in [1, n] \) such that \( d | jd_{s} \) (we use the \( \mathbb{Z} \)-grading of \( H(w_{j}) \) induced by the \( \mathbb{Z} \)-grading of the variables \( x_{i} \)).

Remark 2.6.4. In the case \( n = 0 \) and \( w = 0 \) the category \( \text{DMF}_{\Gamma}(0) \) is (noncanonically) equivalent to \( D(G - \text{mod}) \). Similarly, \( \text{DMF}_{\Gamma^{(2)}}(0) \) is equivalent to \( D(G^{2} - \text{mod}) \). The diagonal object \( \Delta_{G}^{st} \) in this case can be identified with

\[ \Delta_{G}^{st} \simeq \text{Ind}_{G}^{G^{2}} 1_{G} \simeq \bigoplus_{\eta \in \hat{G}} \eta \otimes \eta^{-1}, \]

where \( 1_{G} \) is the trivial representation of \( G \).

2.7 The Chern character and the canonical bilinear pairing

Recall that there is a canonical pairing on Hochschild homology

\[ (\cdot, \cdot) : HH_{*}(\text{MF}_{\Gamma}(w)^{op}) \otimes HH_{*}(\text{MF}_{\Gamma}(w)) \to \mathbb{C} \quad (2.22) \]

induced by the dg-version of the trace functor (2.6) restricted to perfect bimodules (see [49, Sec. 1.2]). Under the identification of \( \mathbb{Z}/2 \)-graded spaces \( HH_{*}(\text{MF}_{\Gamma}(w)) \simeq HH_{*}(\text{MF}_{G}(w)) \) this pairing coincides with the nondegenerate bilinear pairing on Hochschild homology of \( \text{MF}_{G}(w) \) calculated in [49] (see Theorem 2.6.1). Note that the duality (3.10) gives a natural equivalence \( \text{MF}_{\Gamma}(w)^{op} \simeq \text{MF}_{\Gamma}(-w) \), so that the canonical pairing is induced by the dg-functor

\[ \text{MF}_{\Gamma}(-w) \otimes \text{MF}_{\Gamma}(w) \to \text{Com}_{\Gamma}(\text{C - mod}) : (E, F) \mapsto \text{com}(E \otimes F)^{\Gamma}, \]

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where \( \text{Com}_f(C\text{-mod}) \) is the category of complexes of \( C \)-vector spaces with finite-dimensional total cohomology.

Let us denote by

\[
\mathcal{H}(w) = HH_*(MF_\Gamma(w)) = \bigoplus_{\gamma \in G} H(w^G_\gamma)
\]

(2.23)

the Hochschild homology space computed in Theorem 2.6.1(i).

**Definition 2.7.1.** Let \( R = C[\hat{G}] \) be the group algebra of the dual group \( \hat{G} \). We define an \( R \)-bilinear version of the canonical pairing (2.22)

\[
(\cdot, \cdot)^R : \mathcal{H}(-w) \otimes \mathcal{H}(w) \to R
\]

(2.24)

as the map on Hochschild homology induced by the tensor product functor

\[
MF_\Gamma(-w) \otimes MF_\Gamma(w) \to \text{Com}_f(G\text{-mod}) : (E, F) \mapsto \text{com}_G(E \otimes F)
\]

where \( \text{com}_G \) is given by (1.12), \( \text{Com}_f(G\text{-mod}) \) is the category of complexes of \( G \)-modules with finite-dimensional total cohomology.

We have an isomorphism

\[
\text{com}(E \otimes F)^G \simeq \text{com}_G(E \otimes F)^G,
\]

which implies that

\[
(\cdot, \cdot) = \text{tr} \circ (\cdot, \cdot)^R,
\]

(2.25)

where \( \text{tr} : R \to C \) is given by

\[
\text{tr}(\sum_{\eta \in \hat{G}} c_{\eta} \cdot [\eta]) = c_1.
\]

(2.26)

**Proposition 2.7.2.** The pairing \( (\cdot, \cdot)^R \) is perfect and the corresponding Casimir element \( T_w \) is given by

\[
T_w = \frac{1}{|G|} \sum_{g \in G} (\text{id} \times g)^* \text{ch}(\Delta_{w}^{st}) \in HH_*(MF_\Gamma(\mathbb{A}^n \times \mathbb{A}^n, \tilde{w}))^{G \times G} \simeq \mathcal{H}(-w) \otimes_R \mathcal{H}(w),
\]

where the last isomorphism comes from Corollary 2.6.2.

**Proof.** Since the functor \( \Phi_{\Delta_{w}^{st}} \) given by the kernel \( \Delta_{w}^{st} \) is isomorphic to the identity functor (see Proposition 2.4.1(i)), the composition

\[
MF_\Gamma(w) \xrightarrow{(\otimes p_{23}^{\Delta_{w}^{st}})_{\text{proj}}} MF_\Gamma(w \oplus (-w) \oplus w) \xrightarrow{\Delta_{12}^{\Delta_{w}^{st}}} MF_\Gamma(w)
\]

is isomorphic to the identity. Hence, the composition of the induced maps on Hochschild homology

\[
\mathcal{H}(w) \xrightarrow{\alpha} HH_*(MF_\Gamma(w \oplus (-w) \oplus w)) \xrightarrow{\beta} \mathcal{H}(w)
\]

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is equal to the identity. Since $\beta$ is equivariant with respect to the $G$-action on $HH_*(MF_\Gamma(w \oplus (-w) \oplus w))$ given by the embedding of $1 \times 1 \times G \subset G \times G \times G$ and the trivial $G$-action on $\mathcal{H}(w)$, we have $\beta \circ (|G|^{-1} \cdot \sum_{g \in G}(id \times id \times g)^* \circ \alpha) = id$. But the element
\[ T_w = \frac{1}{|G|} \cdot \sum_{g \in G}(id \times g)^* \text{ch}(\Delta_{w}^{st}) \in HH_*(MF_\Gamma((-w) \oplus w)) \]
is invariant under $G \times G$ (since $\Delta_{w}^{st}$ is equivariant with respect to the diagonal action of $G$). Therefore, we have
\[ \beta(x \otimes T_w) = x \]
for any $x \in \mathcal{H}(w)$. It remains to observe that $x \otimes T_w$ belongs to the subspace
\[ HH_*(MF_\Gamma(w \oplus (-w) \oplus w))^{G \times G \times G} \simeq \mathcal{H}(w) \otimes_R \mathcal{H}(-w) \otimes_R \mathcal{H}(w), \]
and the restriction of $\beta$ to this subspace is equal to $(\cdot, \cdot)^R \otimes id$.

By definition, the pairing $(\cdot, \cdot)^R$ is obtained as the restriction to the space of $G \times G$-invariants of the map
\[ HH_*(MF_\Gamma((-w) \oplus w)) \rightarrow R \]
induced by the functor of restricting to the diagonal.

\[ \square \]

Example 2.7.3. Let $i: \{0\} \hookrightarrow \mathbb{A}^n$ denote the natural embedding, and let
\[ \kappa : HH_*(MF_\Gamma(w)) \rightarrow R \]
be the $R$-linear functional induced by the restriction functor
\[ MF_\Gamma(w) \rightarrow \text{Com}_f(G - \text{mod}) : E \mapsto \text{com}_G(i^*E). \]
Then
\[ \kappa(x) = (x, \text{ch}(\mathbb{C}^{\text{st}}))^R. \]
Indeed, this follows from Lemma 2.2.2 by spelling out the definitions.

Using (2.25) and the formula for the canonical pairing $(\cdot, \cdot)$ from [49, Thm. 4.2.1], we deduce the following explicit formula for the $R$-bilinear pairing $(\cdot, \cdot)^R$.

Lemma 2.7.4. Let us consider the standard residue pairing
\[ \langle f \otimes dx, g \otimes dx \rangle_w = (-1)^{(2)} \text{Res}_0(f \cdot g) \]
on the twisted Milnor ring $\mathcal{A}_w \otimes dx = H(w)$ (where $\text{Res}_0$ is the Grothendieck residue on $\mathcal{A}_w$). For $h \in \mathcal{H}(w)$ let $h_\gamma \in H(w_\gamma)$ be the component of $h$ with respect to the decomposition (2.23). Then the $R$-bilinear canonical pairing (2.24) is given by
\[ (h, h')^R = \sum_{\gamma \in G} c_\gamma \cdot (h_\gamma, h_\gamma')_{w_\gamma} \cdot e_\gamma, \quad (2.27) \]
where
\[
\begin{align*}
\eta & := \frac{1}{|G|} \cdot \sum_{\eta \in \hat{G}} \eta^{-1}(\gamma)[\eta]. \\
c_{\gamma} & := \det(\gamma, T/(T^*)^{-1}),
\end{align*}
\] (2.28)

and \(T^* = m/m^2\).

**Proof.** By Theorem 2.6.1, the decomposition (2.23) coincides with the decomposition
\[
\mathcal{H}(w) = \bigoplus_{\gamma \in G} c_{\gamma} \mathcal{H}(w)
\]
induced by the \(R\)-module structure on \(\mathcal{H}(w)\). Since both sides of (2.27) are \(R\)-bilinear, it is enough to check the equality after applying \(\text{tr}\) (see (2.26)) which holds by [49, Thm. 4.2.1]. \(\square\)

## 3 \(\Gamma\)-spin curves and \(w\)-structures

### 3.1 Abstract \(w\)-structures

Let
\[
w(x_1, \ldots, x_n) = \sum_{k=1}^{N} c_k M_k
\]
be a Laurent polynomial in \(x_1, \ldots, x_n\), where \(c_k \in \mathbb{C}^*\) and
\[
M_k = \prod_{i=1}^{n} x_i^{m_{ki}}
\]
are monomials. We denote by \(m_w : \mathbb{Z}^n \to \mathbb{Z}^N\) the map given by the matrix \((m_{ki})\) of exponents.

**Definition 3.1.1.** Let \(d = (d_1, \ldots, d_n) \in \mathbb{Z}^n\) be a primitive vector and let \(d\) be an integer. A Laurent polynomial is *quasihomogeneous* of degree \(d\) with respect to \(d\) if
\[
w(\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n) = \lambda^d w(x_1, \ldots, x_n).
\]

The above equation is equivalent to
\[
m_w(d) = d \cdot e,
\] (3.1)
where \(e \in \mathbb{Z}^N\) the vector with all components equal to 1. Let us consider the dual homomorphisms \(m^*_w : \mathbb{Z}^N \to \mathbb{Z}^n\), \(e^* : \mathbb{Z}^N \to \mathbb{Z}\) and \(d^* : \mathbb{Z}^n \to \mathbb{Z}\). Then (3.1) implies that
\[
d^* \circ m^*_w = d \cdot e^*.
\]
Let us consider the subgroup \( P_w = \text{im}(m_w^*) \subset \mathbb{Z}^n \). Then we obtain that the restriction \( d^*|_{P_w} \) is divisible by \( d \) and we can define the homomorphism

\[
\deg = \frac{1}{d} d^* : P_w \rightarrow \mathbb{Z},
\]

so that \( \deg \circ m_w^* = e^* \).

**Definition 3.1.2.** Let \( \mathcal{C} \) be a symmetric monoidal category with a unit object \( 1 \), and let \( E \in \mathcal{C} \) be an invertible object. Let also \( w \) be a Laurent polynomial in \( x_1, \ldots, x_n \), quasihomogeneous of degree \( d \) with respect to \( d \).

(i) A \( (w, d) \)-structure in \( \mathcal{C} \) with respect to \( E \) is a monoidal functor

\[
\Phi : \mathbb{Z}^n \rightarrow \mathcal{C}
\]

together with an isomorphism of monoidal functors

\[
\Phi|_{P_w} \sim \Lambda \circ \deg,
\]

where \( \Lambda : \mathbb{Z} \rightarrow \mathcal{C} \) is the monoidal functor \( i \mapsto E^i = E^{\otimes i} \), and \( \deg \) is the homomorphism (3.2).

(ii) Assume in addition that \( w \) is a polynomial. A weak \( (w, d) \)-structure in \( \mathcal{C} \) with respect to \( E \) is a monoidal functor

\[
\Phi : \mathbb{Z}^n \rightarrow \mathcal{C}
\]

together with a morphism of monoidal functors

\[
\phi : \Phi|_{P_w^+} \rightarrow \Lambda \circ \deg|_{P_w^+},
\]

where \( P_w^+ = P_w \cap \mathbb{Z}^n_{\geq 0} \).

Here we view the abelian monoids \( \mathbb{Z}^n \) and \( P_w^+ \) as symmetric monoidal categories where the only morphisms are the identity morphisms.

The following proposition gives a more down-to-earth interpretation of \( w \)-structures.

**Proposition 3.1.3.** Let \( w \) be a Laurent polynomial, quasihomogeneous of degree \( d \) with respect to \( d \), and let \( (\mathcal{C}, E) \) be as above.

(i) Let \( v_1, \ldots, v_n \) be a basis of \( \mathbb{Z}^n \) such that \( k_1v_1, \ldots, k_rv_r \) is a basis of \( P_w \) for positive integers \( k_1, \ldots, k_r \), where \( r = \text{rk} P_w \leq n \). Then isomorphism classes of \( w \)-structures with respect to \( E \) correspond to isomorphism classes of collections of invertible objects \( \Phi(v_1), \ldots, \Phi(v_n) \) in \( \mathcal{C} \) together with isomorphisms

\[
\phi_i : \Phi(v_i)^{\otimes k_i} \rightarrow E^{\deg(k_i v_i)}, \quad i = 1, \ldots, r.
\]

(ii) Assume in addition that \( w \) is a polynomial. For every weak \( (w, d) \)-structure \( (\Phi, \phi) \) in \( \mathcal{C} \) with respect to \( E \in \mathcal{C} \), such that \( \phi \) is an isomorphism, there exists a unique extension of \( \phi \) to an isomorphism of monoidal functors

\[
\tilde{\phi} : \Phi|_{P_w} \rightarrow \Lambda \circ \deg,
\]

i.e., to a \( (w, d) \)-structure.
Proof. (i) Any monoidal functor $\Phi : \mathbb{Z}^n \to \mathcal{C}$ is determined up to an isomorphism by the collection of invertible objects $\Phi(v_1), \ldots, \Phi(v_n)$. The same is true for the group $P_w \simeq \mathbb{Z}^r$ with the basis $k_1v_1, \ldots, k_rv_r$. Now the result follows from (i).

(ii) For $p \in P_w$ we have an isomorphism

$$\Phi(-p) \simeq \Phi(p)^{-1} \simeq E^{-\deg(p)} = E^{\deg(-p)},$$

where the first isomorphism comes from the monoidal structure on $\Phi$ and the second is induced by $\phi$. Since $P_w$ is generated by the vectors $d^w_s(e_s) \in \mathbb{Z}_{\geq 0}^r$, $s = 1, \ldots, N$, every element of $P_w$ can be represented as $p - p'$ with $p, p' \in P_w$. The monoidal structure on $\Phi$ gives an isomorphism $\Phi(p - p') \simeq \Phi(p) \otimes \Phi(p')^{-1}$. Now $\phi(p)$ and $\phi(p')$ induce an isomorphism

$$\tilde{\phi}(p - p') : \Phi(p - p') \to E^{\deg(p)} \otimes E^{-\deg(p')} \simeq E^{\deg(p - p')},$$

which is the unique extension of $\phi$ to $P_w$. \hfill \square

Definition 3.1.4. The $(w, d)$-structure in $\mathcal{C}$ with respect to $E = 1$ given by $\Phi(?) = 1$ with the identity isomorphisms $\phi$ is called the trivial $(w, d)$-structure.

Proposition 3.1.5. Suppose that $\text{End}(1)$ is an algebraically closed field, $E \simeq 1$, and a $(w, d)$-structure $(\Phi, \phi)$ satisfies $\Phi(e_j) \simeq 1$ for every $j$. Then this $(w, d)$-structure is isomorphic to the trivial $(w, d)$-structure.

Proof. By Proposition 3.1.3, such a $(w, d)$-structure corresponds to a collection of isomorphisms $\phi_i : \Phi(v_i)^{\otimes k_i} \to 1$, $i = 1, \ldots, r$. But $\Phi(v_i) \simeq 1$, so $\phi_i$ can be viewed as an element of $\text{End}(1)$. Choose $\xi_i \in \text{End}(1)$ such that $\xi_i^{k_i} = \phi_i$. Then the morphisms

$$\Phi(v_i) \simeq 1 \xrightarrow{\xi_i} 1$$

induce an isomorphism with the trivial $(w, d)$-structure. \hfill \square

Remark 3.1.6. We will sometimes omit the vector of degrees $d$ from notation and talk simply of $w$-structures, when this vector is fixed. In the case when $m_w$ is injective the vector $d$ is uniquely determined by $w$ up to a sign.

3.2 $\Gamma$-spin curves and their moduli

Let us fix an algebraic subgroup $\Gamma \subset \mathcal{G}_m^n$ with a surjective character $\chi : \Gamma \to \mathcal{G}_m$ such that $G = \ker(\chi)$ is finite. Thus, we have an exact sequence of commutative algebraic groups

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathcal{G}_m \to 1.$$

With these data we will associate a finite covering

$$S_{g,r,\Gamma,\chi} \to \overline{M}_{g,r}$$
of the Deligne-Mumford moduli stacks of stable curves. The stacks $S_{g,r,\Gamma,\chi}$ are slight generalizations of the moduli spaces of $u$-curves considered in [14].

Recall (see [3, Sec. 4], [14, Sec. 2.1]) that an orbicurve with marked points $(C, p_1, \ldots, p_r)$ is a proper Deligne-Mumford stack $C$ whose coarse moduli space is a (connected) nodal curve $C$, equipped with marked orbipoints $p_1, \ldots, p_r \subset C$, such that the projection $\rho : C \to C$ is an isomorphism away from the marked points and from the nodes. It is also required that each node is locally modeled by a quotient stack of the form $\{xy = 0\}/(\mathbb{Z}/n)$, where the action of $\mathbb{Z}/n$ is given by $(x, y) \mapsto (\exp(2\pi i/n)x, \exp(-2\pi i/n)y)$. We say that an orbicurve $C$ is smooth if the curve $C$ is smooth. We denote by

$$\omega_{C}^{\log} = \rho^*(\omega_C(p_1 + \ldots + p_r))$$

the log-canonical line bundle with respect to $p_1, \ldots, p_r$ (see [14, Def. 2.1.2])

**Definition 3.2.1.** (i) A $(\Gamma, \chi)$-spin curve (a $\Gamma$-spin curve for short) is an orbicurve with marked points $(C, p_1, \ldots, p_r)$ together with a principal $\Gamma$-bundle $P$ over $C$ and an isomorphism of $\mathbb{G}_m$-bundles

$$\varepsilon : \chi_* P \to P(\omega_{C}^{\log}),$$

(3.4)

where $P(\omega_{C}^{\log})$ is the principal $\mathbb{G}_m$-bundle associated with the line bundle $\omega_{C}^{\log}$. An isomorphism between two $\Gamma$-spin curves is an isomorphism of curves with marked points $f : (C, p_1, \ldots, p_r) \to (C', p'_1, \ldots, p'_r)$ and an isomorphism of $\Gamma$-bundles $t : P \to f^* P'$ compatible with isomorphisms (3.4) for $P$ and $P'$.

(ii) Let $(P, \varepsilon)$ be a $\Gamma$-spin structure on an orbicurve $(C, p_1, \ldots, p_r)$. For each marked point $p_i$ let us consider the homomorphism

$$G(p_i) \to \Gamma \subset (\mathbb{C}^*)^n$$

(3.5)

from the local automorphism group of $C$ at $p_i$ associated with its action on the fiber of $P$ at $p_i$. We denote by

$$\gamma_i = \gamma_i(P) = (\gamma_{i1}, \ldots, \gamma_in) \in (\mathbb{C}^*)^n$$

(3.6)

the image of the canonical generator of the cyclic group $G(p_i)$ under (3.5) The collection $\gamma = (\gamma_1, \ldots, \gamma_r)$ is called the type of the $\Gamma$-spin curve.

(iii) We say that a $\Gamma$-spin curve is stable if $(C, p_1, \ldots, p_r)$ is stable and all the homomorphisms (3.5) are injective.

Since the restriction of $\omega_{C}^{\log}$ to each marked point is trivial, the isomorphisms (3.4) imply that the elements $\gamma_i \in \Gamma$ belong to the subgroup $\ker(\chi) = G$. For a stable $\Gamma$-spin curve we will identify $G(p_i)$ with the subgroup $\langle \gamma_i \rangle \subset G$ generated by $\gamma_i$.

It is useful to rewrite the definition of a $\Gamma$-spin structure in terms of principal bundles that have only algebraic tori as structure groups. Namely, consider the algebraic torus $T = \mathbb{G}_m^n/G$. The embedding $\Gamma \hookrightarrow \mathbb{G}_m^n$ induces an embedding $\varphi : \mathbb{G}_m = \Gamma/G \hookrightarrow T$. Thus, we
have a commutative diagram with exact rows

\[
\begin{array}{cccc}
1 & \rightarrow & G & \rightarrow \Gamma \\
\downarrow{id} & & \downarrow{\chi} & \rightarrow \mathbb{G}_m & \rightarrow 1 \\
1 & \rightarrow & G & \rightarrow \mathbb{G}_m^n & \rightarrow T & \rightarrow 1 \\
\end{array}
\]

(3.7)

such that \( \pi \) induces an isomorphism \( \mathbb{G}_m^n/\Gamma \rightarrow T/\varphi(\mathbb{G}_m) \).

The principal \( \Gamma \)-bundle \( P \) in Definition 3.2.1 gives rise to a \( \mathbb{G}_m^n \)-bundle \( P' \) via the embedding \( \Gamma \rightarrow \mathbb{G}_m^n \). We are going to rewrite the definition of a \( \Gamma \)-spin structure in terms of \( P' \) and the homomorphisms \( \pi \) and \( \varphi \) from diagram (3.7) (see Proposition 3.2.2(i) below). On the other hand, we can view \( P' \) a collection of line bundles \( (\mathcal{L}_1, \ldots, \mathcal{L}_n) \) on \( \mathcal{C} \). We will show that the isomorphism (3.4) can be interpreted as a \((w, d)\)-structure in the category of line bundles on \( \mathcal{C} \) with respect to \( \omega_{e}^{\log} \) for some quasihomogeneous Laurent polynomial \( w \) (see Section 3.1)

Let \( \Gamma_0 \) be the connected component of 1 in \( \Gamma \). \( \Gamma_0 \) is a one-dimensional torus, so we can choose an identification \( \Gamma_0 = \mathbb{G}_m \). The embedding \( \Gamma_0 = \mathbb{G}_m \rightarrow \mathbb{G}_m^n \) takes form \( \lambda \mapsto (\lambda^{d_1}, \ldots, \lambda^{d_n}) \) for some primitive vector \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \). Furthermore, the restriction of the character \( \chi \) to \( \Gamma_0 = \mathbb{G}_m \) is given by \( \lambda \mapsto \lambda^d \) for some \( d \). Note that the degrees \( (d, d) \in \mathbb{Z}^{n+1} \) are determined by \( \Gamma \) uniquely up to a sign.

**Proposition 3.2.2.** (i) The category of \( \Gamma \)-spin structures on \( (\mathcal{C}, p_1, \ldots, p_r) \) is equivalent to the category of pairs \( (P', \varepsilon') \), where \( P' \) is a principal \( \mathbb{G}_m^n \)-bundle and \( \varepsilon' \) is an isomorphism

\[
\varepsilon' : \pi_* P' \rightarrow \varphi_* P(\omega_{e}^{\log}).
\]

(3.8)

(ii) There exists a Laurent polynomial \( w(x_1, \ldots, x_n) \), quasihomogeneous of degree \( d \) with respect to the grading of the variables given by \( d \), such that \( G \) is equal to the subgroup of diagonal symmetries of \( w \).

(iii) For any \( w \) as in (ii) the category of \((w, d)\)-structures in the category of line bundles on \( \mathcal{C} \) with respect to \( \omega_{e}^{\log} \) is equivalent to the category of \((\Gamma, \chi)\)-spin structures on \( (\mathcal{C}, p_1, \ldots, p_r) \).

(iv) Let \( w \) be a quasihomogeneous polynomial of degree \( d > 0 \) with respect to the grading given by \( d \), and let \( G \) be a finite subgroup of the group \( G_w \) of diagonal symmetries of \( w \), such that \( G \) contains the exponential grading element \( J \) (see (2.2)). Let also \( \Gamma \subset \mathbb{G}_m^n \) be the subgroup associated with \( G \subset G_w \) by Lemma 2.1.1. Then to any \( \Gamma \)-spin structure on \( (\mathcal{C}, p_1, \ldots, p_r) \) there corresponds a natural \((w, d)\)-structure in the category of line bundles on \( \mathcal{C} \) with respect to \( \omega_{e}^{\log} \).

**Proof.** (i) A \( \Gamma \)-bundle \( P \) can be viewed as a \( \mathbb{G}_m^n \)-bundle \( P' \) together with a trivialization of the induced \( \mathbb{G}_m^n/\Gamma \)-bundle. Commutativity of the right square in (3.7) shows that

\[
\pi_* P' \simeq \varphi_* \chi_* P.
\]
Hence, the isomorphism (3.4) gives rise to an isomorphism (3.8). Conversely, starting with a pair \((P', \varepsilon')\) we observe that the isomorphism (3.8) induces a trivialization of the \(T/\varphi(\mathbb{G}_m)\)-bundle obtained by the push-out from \(\pi_* P'\), or equivalently a trivialization of the \(\mathbb{G}_m^n/\Gamma\)-bundle obtained by push-out from \(P'\). Thus, we can reduce the structure to \(\Gamma\) and obtain a \(\Gamma\)-bundle \(P\). Now both parts of (3.8) become push-outs of the corresponding parts of (3.4) with respect to \(\varphi\). Since \(\varepsilon'\) is compatible with the trivializations of the push-outs with respect to the projection \(T \to T/\varphi(\mathbb{G}_m)\), it induces an isomorphism (3.4).

(ii) Consider the exact sequence of algebraic tori

\[ 1 \to \mathbb{G}_m \xrightarrow{\varphi} T \to T' \to 1. \]

Since we work over \(\mathbb{C}\), we can find a splitting \(T \simeq \mathbb{G}_m \times T'\). Consider the collection of characters of \(T\)

\[ \eta_0 = (\text{id}, 1), \eta_1 = (\text{id}, \epsilon_1), \ldots, \eta_{n-1} = (\text{id}, \epsilon_{n-1}), \quad (3.9) \]

where \(\text{id}\) is the identity character of \(\mathbb{G}_m\) and \((\epsilon_1, \ldots, \epsilon_{n-1})\) is a basis of the group of characters of the torus \(T'\). We have

\[ \bigcap_{i=0}^{n-1} \ker(\eta_i) = 1, \text{ hence, } \bigcap_{i=0}^{n-1} \ker(\eta_i \circ \pi) = G. \]

Furthermore, each of the characters

\[ M_i = \eta_i \circ \pi : \mathbb{G}_m^n \to \mathbb{G}_m \text{ for } i = 0, \ldots, n-1 \quad (3.10) \]

has the property

\[ M_i|_{\Gamma} = \eta_i \circ \varphi \circ \chi = \chi. \]

Recall that the restriction of \(\chi\) to \(\Gamma_0 = \mathbb{G}_m\) sends \(\lambda\) to \(\lambda^d\). Therefore, each \(M_i\) can be viewed as a Laurent monomial in \(x_1, \ldots, x_n\) of degree \(d\) with respect to \(d\). Thus, we can take

\[ w = \sum_{i=0}^{n-1} M_i. \]

(iii) First, we observe that for an algebraic torus \(T\) the category of principal \(T\)-bundles on \(\mathcal{C}\) is equivalent to the category of monoidal functors from with the character group \(X(T)\) to the monoidal category \(\mathcal{Pic}(\mathcal{C})\) of line bundles on \(\mathcal{C}\). Namely, with a \(T\)-bundle \(P\) we associate the monoidal functor \(\Phi_P\) sending a character \(\eta : T \to \mathbb{G}_m\) to the line bundle corresponding to the induced \(\mathbb{G}_m\)-torsor \(\eta_* P\). Indeed, a choice of a basis of \(X(T)\) shows that both structures are equivalent to collections of line bundles on \(\mathcal{C}\). If \(f : T_1 \to T_2\) is a homomorphism of tori, then the monoidal functor \(\Phi_{f*,P_1} : X(T_2) \to \mathcal{Pic}(\mathcal{C})\) associated with the push-out \(f_* P_1\) of a principal \(T_1\)-bundle is isomorphic to the composition \(\Phi_{P_1} \circ f^*\), where \(f^* : X(T_2) \to X(T_1)\) is the induced homomorphism of the character groups.

Given a Laurent polynomial \(w = \sum_{s=1}^{N} M_s\) as in (ii), we have

\[ G = \ker((m_w)_* : \mathbb{G}_m^n \to \mathbb{G}_m^N), \]

where \(m_w : \mathbb{Z}^n \to \mathbb{Z}^m\) is the linear map defined by the exponents of the monomials \(M_s\) (see Section 3.1). Hence, the map \((m_w)_*\) factors through the projection \(\pi : \mathbb{G}_m^n \to T\)
followed by an embedding of tori $T \hookrightarrow \mathbb{G}_m^N$. The induced homomorphisms of character groups $\mathbb{Z}^N \to X(T)$ and $\pi^* : X(T) \to \mathbb{Z}^n$ are surjective and injective, respectively. Therefore, $\pi^*$ induces an isomorphism of $X(T)$ with $P_w = \text{im}(m_w) \subset \mathbb{Z}^n$ such that the homomorphism $\deg : P_w \to \mathbb{Z}$ gets identified with the homomorphism on the character groups induced by the embedding of tori $\varphi : \mathbb{G}_m \to T$. Now the assertion follows from (i).

(iv) Let us choose a Laurent polynomial $\overline{w}(x_1, \ldots, x_n)$ as in (ii). By adding extra monomials to $\overline{w}$ we can assume that $\overline{w}$ contains all the monomials in $w$. By (iii), a $\Gamma$-spin structure induces a $(\overline{w}, d)$-structure, which in turn gives a $(w, d)$-structure.

**Remarks 3.2.3.** 1. Proposition 3.2.2(ii) implies that for a quasihomogeneous polynomial $w$, every finite subgroup $G$ of the group of diagonal symmetries $G_w$ such that $G$ contains the exponential grading element $J$, is admissible in the sense of [14, Def. 2.3.2].

2. Let $(P, \varepsilon)$ be a $\Gamma$-spin structure on $(\mathbb{C}, p_1, \ldots, p_r)$. Any Laurent monomial $M : \mathbb{G}_m^n \to \mathbb{G}_m$ of degree $d$ with respect to $d$, such that $M|_G = 1$, restricts to the character $\chi$ on $\Gamma$. Indeed, the condition that $M|_G = 1$ implies that $M|_\Gamma = \chi^a$ for some $a \in \mathbb{Z}$. The fact that $a = 1$ follows from the equality $M|_{\Gamma_0} = \chi|_{\Gamma_0}$ which in turn follows from the condition that $M$ has the degree $d$ with respect to $d$. Thus, the isomorphism (3.4) gives rise to isomorphisms

$$M(\mathcal{L}_1, \ldots, \mathcal{L}_n) \sim \omega^\log_{\mathbb{C}},$$

(3.11)

for any Laurent monomial $M$ of degree $d$ with respect to $d$ such that $M|_G = 1$. These are the kind of isomorphisms that appear in the original definition of a $w$-structure in [14]. More precisely, if $w(x_1, \ldots, x_n)$ is a Laurent polynomial, quasihomogeneous of degree $d$ with respect to $d$, such that $G$ is equal to the group $G_w$ of diagonal symmetries of $w$, then by Proposition 3.2.2(iii), the notion of a $\Gamma$-spin curve is equivalent to the notion of a $w$-curve defined in [14, Sec. 2.1] (generalized to the case when $w$ is a Laurent polynomial). Indeed, the Smith normal forms appearing in [14, Def. 2.1.10] is just a way of recording an isomorphism of monoidal functors from a free abelian group in coordinates (see Proposition 3.1.3). More generally, if $G$ is just a subgroup of $G_w$ containing $J$, then by Proposition 3.2.2(iv), any $\Gamma$-spin curve has a natural structure of a $w$-curve.

We will give now yet another way to describe $\Gamma$-spin structures.

**Corollary 3.2.4.** The category of $\Gamma$-spin structures on $(\mathbb{C}, p_1, \ldots, p_r)$ is equivalent to the category of collections of $n$ line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on $\mathbb{C}$ together with isomorphisms (3.11) for the Laurent monomials $M = M_0, \ldots, M_{n-1}$ given by (3.10).

**Proof.** The line bundle $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ are obtained from a $\Gamma$-spin structure as discussed before Proposition 3.2.2. The isomorphisms (3.11) for the monomials $M_0, \ldots, M_{n-1}$ correspond to an isomorphism of $T$-bundles (3.8) under the identification

$$(M_0, M_1, \ldots, M_{n-1}) : T \to \mathbb{G}_m^n.$$

We are going to work with the moduli space of $\Gamma$-spin curves.
Definition 3.2.5. A genus-$g$, stable $\Gamma$-spin curve with $k$ marked points over a base $T$ is a flat family $\mathcal{C} \to T$ of genus-$g$ orbicurves with gerbe markings $p_1, \ldots, p_k \subset \mathcal{C}$ and sections $\sigma_i : T \to p_i$ inducing isomorphism of $T$ with the coarse moduli of $p_i$, together with a relative $\Gamma$-spin structure $(P, \varepsilon)$ on $\mathcal{C}$, such that all the fibers over closed points of $T$ are stable $\Gamma$-spin curves. Here $P$ is a $\Gamma$-bundle on $\mathcal{C}$, and

$$\varepsilon : \chi_* P \simeq P(\omega_{\mathcal{C}/T}^{\log})$$

is an isomorphism of $\mathbb{G}_m$-bundles on $\mathcal{C}$, where $\omega_{\mathcal{C}/T}^{\log}$ is the relative log-canonical line bundle. These structures naturally form a stack $S_{g,r} = S_{g,r,\Gamma,\chi}$, which is the disjoint union of the open and closed substacks $S_g(\gamma)$ for $\gamma \in G^r$, parametrizing $\Gamma$-spin curves of type $\gamma$.

Proposition 3.2.6. The stack $S_{g,r}$ is a smooth proper DM-stack over $\mathbb{C}$ with projective coarse moduli, and the natural forgetful morphism $S_{g,r} \to M_{g,r}$ is quasi-finite. If $G$ is equal to the group $G_w$ of diagonal symmetries of a Laurent polynomial $w$, quasihomogeneous of degree $d$ with respect to $d$, then $S_{g,r}$ is naturally isomorphic to the stack $W_{g,r}(w)$ of $w$-curves constructed in [14, Sec. 2.2].

Proof. First, we observe that although Fan-Jarvis-Ruan assume that $w$ is a polynomial, the definition of a $w$-structure and the results of [14, Sec. 2.2] are valid also in the case of a quasihomogeneous Laurent polynomial with finite group of diagonal symmetries (in fact, this extension is used in [14, Sec. 2.3] to define the moduli spaces associated with admissible subgroups of $G_w$).

Applying Proposition 3.2.2(ii,iii), we see that our moduli stack $S_{g,r}$ is naturally isomorphic to $W_{g,r}(w)$ for some quasihomogeneous Laurent polynomial $w$. Now we can use [14, Thm. 2.2.6] to derive the required properties of the stack $S_{g,r}$. (Alternatively, by modifying the arguments of [14, Thm. 2.2.6] one can work directly with $\Gamma$-spin structures.)

Similarly to [14, Sec. 2.2.3] we consider rigidifications of $\Gamma$-spin curves.

Definition 3.2.7. A rigidification of a $\Gamma$-spin structure $(P, \varepsilon)$ on an orbicurve $\mathcal{C}$ at a marked point $p_i$ is a trivialization of $P|_{p_i}$, i.e., an isomorphism

$$P|_{p_i} \simeq \Gamma/\langle \gamma_i \rangle$$

compatible with the canonical trivialization of $\omega_{\mathcal{C}/p_i}^{\log}$ via the isomorphism (3.4). A rigidification of a $\Gamma$-spin curve $(\mathcal{C}, p_1, \ldots, p_r; P, \varepsilon)$ consists of a collection of rigidifications of $(P, \varepsilon)$ at every marked point $p_i$.

The group $\prod_{i=1}^r G/\langle \gamma_i \rangle$ acts simply transitively on the set of rigidifications of a given $\Gamma$-spin curve. Thus, the moduli stack of rigidified $\Gamma$-spin curves is a $\prod_{i=1}^r G/\langle \gamma_i \rangle$-torsor over $S_g(\gamma)$ that we denote by

$$S_{g,\text{rig}} \to S_g(\gamma) \quad (3.12)$$

Let $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ be the line bundles associated with a $\Gamma$-spin structure $P$. For each $(p_i, j)$ such that the $j$th component of $\gamma_i$ is trivial, a rigidification structure induces a well-defined
trivialization of \( \mathcal{L}_j|_{p_i} \). Below we will define a different version of a rigidification structure that keeps track only of these trivializations.

First, we define restrictions of \( \Gamma \)-spin structures associated with coordinate projections \( G_m^n \to G_m^k \).

**Definition 3.2.8.** Let \( I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \) be a subset such that the vector \( d_I = (d_{i_1}, \ldots, d_{i_k}) \) is not zero, and let 

\[
p_I : G_m^n \to G_m^k
\]

be the corresponding coordinate projection. We say that \( I \) is \((\Gamma, \chi)\)-admissible if the character \( \chi \) factors through the projection \( \Gamma \to \Gamma_I := p_I(\Gamma) \).

The following statement is an immediate corollary of the definitions.

**Lemma 3.2.9.** Assume that \( I \) is \((\Gamma, \chi)\)-admissible. Let \( \chi_I : \Gamma_I \to G_m^k \) be the character of \( \Gamma_I \) induced by \( \chi \). Then any \((\Gamma, \chi)\)-spin structure \( (P, \varepsilon) \) on \( (\mathbb{C}, p_1, \ldots, p_r) \) naturally induces a \((\Gamma_I, \chi_I)\)-spin structure \( (P_I, \varepsilon_I) \) on \( (\mathbb{C}, p_1, \ldots, p_r) \) with \( P_I = (p_I)_*P \). Also, a rigidification of \( (P, \varepsilon) \) induces a rigidification of \( (P_I, \varepsilon_I) \).

\[ \blacksquare \]

For every \( \gamma \in G \subset G_m^n \) let us denote by \( I(\gamma) \subset \{1, \ldots, n\} \) the set of all \( j \) such that the \( j \)th component of \( \gamma \) is trivial.

**Definition 3.2.10.** Assume that the degree \( d_i \) is nonzero for every \( i = 1, \ldots, n \).

(i) A collection \( \gamma = (\gamma_1, \ldots, \gamma_r) \in G^r \) is called \((\Gamma, \chi)\)-admissible if \( I(\gamma_i) \) is \((\Gamma, \chi)\)-admissible for every \( i = 1, \ldots, r \).

(ii) If \( \gamma \) is \((\Gamma, \chi)\)-admissible, we define a restricted rigidification of a \( \Gamma \)-spin structure \( (P, \varepsilon) \) of type \( \gamma \) as a collection of rigidifications of the induced \( \Gamma_I(\gamma_i) \)-structure \( (P_I(\gamma_i), \varepsilon_I(\gamma_i)) \) at \( p_i \) for \( i = 1, \ldots, r \).

Note that by definition, a restricted rigidification of a \( \Gamma \)-spin structure consists of trivializations of \( \mathcal{L}_j|_{p_i} \) for \( j \in I(\gamma_i) \) and \( i = 1, \ldots, r \), satisfying certain compatibilities.

For a \((\Gamma, \chi)\)-admissible collection \( \gamma = (\gamma_1, \ldots, \gamma_r) \in G^r \) let us set

\[
G(\gamma) = \prod_{i=1}^r G_{I(\gamma_i)}, \quad \text{where} \quad G_I = p_I(G).
\]

Then the group \( G(\gamma) \) acts simply transitively on the set of restricted rigidifications of a given \( \Gamma \)-spin curve of type \( \gamma \). We denote by \( S_{g}^{\text{rig},0}(\gamma) \to S_g(\gamma) \) the \( G(\gamma) \)-torsor of restricted rigidifications. We have a natural surjective morphism

\[
S_g^{\text{rig}}(\gamma) \to S_{g}^{\text{rig},0}(\gamma) \quad \text{(3.13)}
\]

compatible with the homomorphism

\[
r \gamma : \prod_{i=1}^r G/\langle \gamma_i \rangle \to G(\gamma).
\]

Therefore, that the \( G(\gamma) \)-torsor \( S_{g}^{\text{rig},0}(\gamma) \) is isomorphic to the push-out of (3.12) with respect to \( r \gamma \).
Lemma 3.2.11. Let \( w(x_1, \ldots, x_n) \) be a polynomial with an isolated singularity at the origin, quasihomogeneous of degree \( d \) with respect to \( d = (d_1, \ldots, d_n) \), where \( d_i \neq 0 \) for \( i = 1, \ldots, n \). Let also \( G \subset G_w \) be a finite subgroup containing the exponential grading element \( J \), and let \( \Gamma \subset G_m \) be the corresponding extension of \( G_m \) by \( G \) (see Lemma 2.2). Then every \( \gamma = (\gamma_1, \ldots, \gamma_r) \in G_r \) is \((\Gamma, \chi)\)-admissible.

Proof. It is enough to check that for every \( \gamma \in G_w \) the set \( I(\gamma) \subset \{1, \ldots, n\} \) is \((\Gamma, \chi)\)-admissible. To do this we use the fact that the restriction of \( w \) to the subspace of \( \gamma \)-invariants still has an isolated singularity at the origin (see [49, Lem. 2.5.3(i)]). In the case \( I(\gamma) = \emptyset \), the assertion is trivial, so we can assume that \( I(\gamma) = \{i_1, \ldots, i_k\} \neq \emptyset \). Let us take any monomial \( M(x_{i_1}, \ldots, x_{i_k}) \) occurring in the restriction of \( w \) to the subspace of \( \gamma \)-invariants. Since \( M \) also occurs in \( w \), the action of \( \Gamma \) on \( M \) rescales it by the character \( \chi \). But \( M \) factors through \( p_{I(\gamma)} : G_m \to G^k_m \), hence, \( \chi \) also factors through \( p_{I(\gamma)} \).

3.3 Invariants of smooth \( \Gamma \)-spin curves

We keep the assumptions and notation of the beginning of Section 3.2. Let \( (C, p_1, \ldots, p_r; P, \varepsilon) \) be a \( \Gamma \)-spin curve of type \( \pi = (\gamma_1, \ldots, \gamma_r) \in G^r \) with a smooth orbicurve \( C \), and let \( L_1, \ldots, L_n \) be the line bundles associated with the \( G_m \)-bundle \( P \). As in Section 3.2, consider the map \( \rho : C \to C \), where \( C \) is the smooth curve obtained by forgetting the orbi-structure at the marked points, and the line bundles \( L_j = \rho_*(L_j) \) on \( C \).

For \( \gamma \in G \) we define

\[
\theta_\gamma = (\theta_1, \ldots, \theta_n) \in \mathbb{Q}^n
\]
as the unique vector with \( 0 \leq \theta_j < 1 \) for \( j = 1, \ldots, n \) such that

\[
\gamma = \exp(2\pi i \theta_\gamma) = (\exp(2\pi i \theta_1), \ldots, \exp(2\pi i \theta_n)) \in (\mathbb{C}^*)^n.
\]

Let \( J = \exp(2\pi i q) \in (\mathbb{C}^*)^n \) be the exponential grading element (2.2), where \( q = (q_1, \ldots, q_n) \in \mathbb{Q}^n \) with \( q_j = d_j/d \). Note that by definition, \( J \in G_m = \Gamma_0 \subset \Gamma \). Furthermore, \( \chi(J) = 1 \), so \( J \) belongs to \( G = \ker(\chi) \subset \Gamma \). The following result is essentially contained in [23, Prop. 2.1.23, 2.2.8].

Proposition 3.3.1. Let \( g \) be the genus of \( C \).

(i) One has the following identity in \( G \):

\[
\gamma_1 \cdot \ldots \cdot \gamma_r = J^{2g-2+r}.
\]

Furthermore, consider the vector

\[
\deg = (\deg L_1, \ldots, \deg L_n) \in \mathbb{Z}^n
\]

Then

\[
\deg = (2g-2+r)q - \theta_1 - \ldots - \theta_r,
\]

where \( \theta_s = \theta_{\gamma_s} \).
(ii) There exists a $\Gamma$-spin structure on $(\mathcal{C}, p_1, \ldots, p_r)$ of type $\overline{\tau}$ if and only if (3.14) is satisfied.

(iii) There exists a simple transitive action of the group $H^1(C,G)$ on the set of isomorphism classes of $\Gamma$-spin structures on $(\mathcal{C}, p_1, \ldots, p_r)$ of type $\overline{\tau}$. In particular, if this set is nonempty, then it has $|G|^{2g}$ elements.

Proof. (i) Let $M = x_1^{k_1} \ldots x_n^{k_n}$ be a Laurent monomial of degree $d$ with respect to $d$, such that $M|_G = 1$, and let $l_M : \mathbb{Z}^n \to \mathbb{Z}$ denote the corresponding linear form $\sum_j k_j e_j^*$. Then it follows immediately from the definition (2.2) that

$$l_M(q) = 1. \tag{3.16}$$

Recall that by (3.11), the line bundle $M(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ is isomorphic to $\omega_C^{\log}$. Considering this isomorphism near each marked point we obtain that

$$l_M(\theta_s) \in \mathbb{Z} \text{ for } s = 1, \ldots, r.$$ 

We claim that the isomorphism (3.11) induces an isomorphism

$$M(L_1, \ldots, L_n) \simeq \omega_C^{\log}(-l_M(\theta_1)p_1 - \ldots - l_M(\theta_r)p_r). \tag{3.17}$$

Indeed, let $p = p_s$ be one of the marked points, and let $z$ be a local coordinate near $p$ on $\mathbb{C}$, so that the generator $g_p$ of the local group $G(p)$ acts on $z$ by the multiplication with $\exp(-2\pi i/m)$. Then we can view $z^m$ as a local coordinate near $p = p_s$ on $C$. For each $j$ let $e_j(p)$ denote a generator of $\mathcal{L}_j$ as an $\mathcal{O}_C$-module near $p$. For every $j = 1, \ldots, n$, we have

$$g_p \cdot e_j(p) = \exp(2\pi i \theta_s) \cdot e_j(p),$$

where $\theta_s = (\theta_{s1}, \ldots, \theta_{sn})$. The line bundle $L_j = \rho_*L_j$ is generated near $p$ by $z^{m\theta_{sj}} \cdot e_j(p)$. The isomorphism (3.11) implies that the action of $G(p)$ on $M(e_\bullet(p))$ is trivial. Hence, the line bundle $\rho_*(M(\mathcal{L}_\bullet))$ is generated near $p$ by $M(e_\bullet(p))$. On the other hand, $L_j^{\otimes k_j}$ is generated by $z^{mk_j\theta_{sj}} \cdot e_j(p)^{\otimes k_j}$ near this point. Thus, we have an isomorphism

$$M(L_\bullet) \xrightarrow{\sim} \rho_*(M(\mathcal{L}_\bullet))(-\sum_{s=1}^{r} a_s p_s), \text{ where}$$

$$a_s = \sum_{j=1}^{n} k_j \theta_{sj}.$$ 

Since $\rho_*(M(\mathcal{L}_\bullet)) \simeq \rho_*(\omega_C^{\log}) \simeq \omega_C^{\log}$, this gives (3.17).

Comparing the degrees in (3.17), we get a system of equations

$$l_M(\text{deg}) = 2g - 2 + r - l_M(\theta_1 + \ldots + \theta_r),$$

where $M$ runs over Laurent monomials of degree $d$ with respect to $d$ such that $M|_G = 1$. Using (3.16) we can rewrite this system as

$$l_M(\text{deg}) = l_M((2g - 2 + r)q - \theta_1 - \ldots - \theta_r)).$$
Now (3.15) follows from the fact that among \( l_M \) there exist \( n \) linearly independent forms: it is enough to take the monomials \( M \) corresponding to the characters (3.10). Since \( \deg \in \mathbb{Z}^n \), this also implies (3.14).

(ii) Given \((\gamma_1, \ldots, \gamma_r) \in G^r \) satisfying (3.14), by Corollary 3.2.4, we have to construct a collection of line bundles \((\mathcal{L}_1, \ldots, \mathcal{L}_n)\) on \( \mathcal{C} \), such that the action of the local group at every marked point \( p_s \) on the fiber of \( \bigoplus_j \mathcal{L}_j \) at \( p_s \) is given by \( \gamma_s \), and isomorphisms

\[
\mathcal{M}_i(\mathcal{L}_1, \ldots, \mathcal{L}_n) \simeq \omega_C^{\log} \quad \text{for} \quad i = 0, \ldots, n - 1,
\]

where \( M_0, \ldots, M_{n-1} \) are the Laurent monomials corresponding to the characters (3.10). First, we claim that there exists a collection of line bundles \((L_1, \ldots, L_n)\) on \( C \) together with isomorphisms (3.17) for \( M = M_0, \ldots, M_{n-1} \). Indeed, since the characters (3.9) form a basis in the character lattice of \( T \), the matrix of exponents of the monomials \( M_0, \ldots, M_{n-1} \) is nondegenerate. Therefore, by the divisibility of the group \( \text{Pic}^0(C) \), it is enough to check that the system of equations on degrees \( \deg = (\deg(L_1), \ldots, \deg(L_n)) \) imposed by (3.18) has a solution. As we have seen in the proof of (i), if the condition (3.14) is satisfied, then \( \deg \) defined by (3.15) gives a solution of this system. Finally, for \( j = 1, \ldots, n \), we define \( \mathcal{L}_j \) as the unique line bundle on \( \mathcal{C} \) with \( \rho_s \mathcal{L}_j \simeq L_j \) such that for every \( s = 1, \ldots, r \) the generator of the local group at \( p_s \) acts on the fiber of \( \mathcal{L}_j \) at \( p_s \) by the \( j \)th component of \( \gamma_s \). As in part (i), this implies that

\[
\rho_s \mathcal{M}_i(\mathcal{L}_1, \ldots, \mathcal{L}_n) \simeq \omega_C^{\log} \quad \text{for} \quad i = 0, \ldots, n - 1,
\]

and that the action of the local groups at the marked points on \( \mathcal{M}_i(\mathcal{L}_1, \ldots, \mathcal{L}_n) \) is trivial. But this is equivalent to (3.18).

(iii) Let \((\mathcal{L}_1, \ldots, \mathcal{L}_n)\) be a collection of line bundles on \( \mathcal{C} \) satisfying (3.18), such that the action of the local group on the fiber of \( \bigoplus_j \mathcal{L}_j \) at each marked point \( p_s \) is given by \( \gamma_s \). Any other such collection has the form \((\mathcal{L}_1 \otimes \mathcal{K}_1, \ldots, \mathcal{L}_n \otimes \mathcal{K}_n)\), where for each \( j \) the local groups of the marked points act trivially on the fibers of \( \mathcal{K}_j \) and

\[
\mathcal{M}_i(\mathcal{K}_1, \ldots, \mathcal{K}_n) \simeq \mathcal{O}_C \quad \text{for} \quad i = 0, \ldots, n - 1.
\]

Thus, different choices of \( \mathcal{L}_\bullet \) correspond to collections of line bundles \( \mathcal{K}_\bullet \) on \( \mathcal{C} \) such that \( \mathcal{K}_j = \rho^* \mathcal{K}_j \) for \( j = 1, \ldots, n \) and the line bundles \( K_j = \mathcal{K}_j \) on \( C \) satisfy

\[
\mathcal{M}_i(K_1, \ldots, K_n) \simeq \mathcal{O}_C \quad \text{for} \quad i = 0, \ldots, n - 1.
\]

Since \( G \) is the kernel of the homomorphism \( \mathbb{G}_m^r \to \mathbb{G}_m^n \) given by \((M_0, \ldots, M_{n-1})\), this is equivalent to a choice of a \( G \)-bundle. Now the assertion follows from the fact that the group of isomorphism classes of \( G \)-bundles on \( C \) is isomorphic \( H^1(C, G) \).

As a consequence of the relation (3.15), following [14] we deduce the formula for the Euler characteristic of the bundle \( \bigoplus_j \mathcal{L}_j \). For \( \gamma \in G \) define the degree shifting number \( \iota_\gamma \) as the sum of coordinates of the vector \( \theta_\gamma - q \).

50
Corollary 3.3.2. One has

\[-\sum_{j=1}^{n} \chi(C, L_j) = D_g(\gamma_1, \ldots, \gamma_r) = (g - 1)\hat{c} + \epsilon_{\gamma_1} + \ldots + \epsilon_{\gamma_r}, \tag{3.19}\]

where

\[\hat{c} = \sum_{j=1}^{n} (1 - 2q_j).\]

We will also use the modified quantities

\[\tilde{D}_g(\gamma_1, \ldots, \gamma_r) = D_g(\gamma_1, \ldots, \gamma_r) + \frac{1}{2} \cdot \sum_{i=1}^{r} N_{\gamma_i}, \tag{3.20}\]

with

\[N_{\gamma} = \dim(\mathbb{A}^{\gamma}).\]

They satisfy the following factorization properties.

Lemma 3.3.3. For any \(\vec{\gamma} = (\gamma_1, \ldots, \gamma_r) \in G^r, \vec{\gamma'} = (\gamma'_1, \ldots, \gamma'_r) \in G^{r'}\) and \(\gamma \in G\) one has

\[\tilde{D}_{g_1}(\vec{\gamma}, \gamma) + \tilde{D}_{g_2}(\vec{\gamma'}, \gamma^{-1}) = \tilde{D}_{g_1+g_2}(\vec{\gamma}, \vec{\gamma'}), \tag{3.21}\]

\[\tilde{D}_g(\vec{\gamma}, \gamma^{-1}) = \tilde{D}_{g+1}(\vec{\gamma}). \tag{3.22}\]

Proof. This follows immediately from the simple relation

\[\epsilon_{\gamma} + \epsilon_{\gamma^{-1}} = \hat{c} - N_{\gamma}\]

established in [14, Prop. 3.2.4]. \(\square\)

4 Matrix factorizations from \(w\)-structures

4.1 \(w\)-structures with respect to the canonical bundle

Let \(\mathcal{T}\) be a symmetric monoidal category with split projectors over a field of characteristic zero. To a monomial \(M(x_1, \ldots, x_n) = x_1^{m_1} \ldots x_n^{m_n}\) we associate a polyfunctor on \(\mathcal{T}^n\)

\[M_{\mathcal{T}} : \mathcal{T}^n \to \mathcal{T} : (A_1, \ldots, A_n) \mapsto M(A_{\bullet}) := S^{m_1} A_1 \otimes \ldots \otimes S^{m_n} A_n,\]

where \(S^{m}(?)\) denote the symmetric powers in \(\mathcal{T}\). In particular, we are going to use this operation in the case when \(\mathcal{T}\) is the derived category of coherent sheaves with the monoidal structure given by the derived tensor product.

Fix a quasihomogeneous polynomial

\[w(x_1, \ldots, x_n) = \sum_{k=1}^{N} c_k M_k. \tag{4.1}\]
Let \( \pi : C \to S \) be a family of nodal curves with a weak \( w \)-structure \((\Phi, \phi)\) in the category of line bundles over \( C \) with respect to the relative canonical bundle \( \omega_{C/S} \). This structure can be specified by a collection of line bundles \( L_j = \Phi(e_j) \), \( j = 1, \ldots, n \) and a collection of morphisms

\[
\phi_k : M_k(L_\bullet) \to \omega_{C/S}
\]

for every monomial \( M_k \) appearing in \( w \). Assume that the restriction of each \( \phi_k \) to any fiber of \( \pi \) is an isomorphism outside a finite number of points.

For each \( k = 1, \ldots, N \), the morphism \( \phi_k \) induces a morphism in the derived category \( D(S) \)

\[
M_k(R\pi_*(L_1), \ldots, R\pi_*(L_n)) \to R\pi_*(\omega_{C/S}) \to \mathcal{O}_S[-1]. \tag{4.2}
\]

Assume that each \( R\pi_*(L_j) \) is represented by a complex \( A_j \beta_j \to B_j \) of vector bundles on \( S \), in such a way that the morphism (4.2) is realized on the level of complexes. Recall that the \( m \)-th symmetric power of (4.3) is the complex

\[
S^m A_j \to S^{m-1} A_j \otimes B_j \to \ldots
\]

concentrated in the degrees 0, 1, \ldots, \( m \). Therefore, the source of the map (4.2) is represented by the complex

\[
M_k(A_\bullet) \to \bigoplus_{j=1}^n B_j \otimes \partial_j M_k(A_\bullet) \to \ldots,
\]

where \( \partial_j M_s \) are the “partial derivatives” of the monomial \( M_k = x_1^{m_{k1}} \cdots x_n^{m_{kn}} \):

\[
\partial_1 M_k = x_1^{m_{k1}-1} x_2^{m_{k2}} \cdots x_n^{m_{kn}} \ldots; \quad \partial_n M_k = x_1^{m_{k1}} x_2^{m_{k2}} \cdots x_n^{m_{kn}-1}.
\]

The differential \( \delta \) is given by

\[
\delta(f_1 \otimes \ldots \otimes f_n) = (\delta_1(f_1) \otimes f_2 \otimes \ldots \otimes f_n, \ldots; f_1 \otimes \ldots \otimes \delta_n(f_n)),
\]

where \( f_j \in S^{m_{kj}} A_j \), and \( \delta_j : S^{m_{kj}} A_j \to B_j \otimes S^{m_{kj}-1} A_j \) is induced by the Koszul differential \( S^{m_{kj}} A_j \to A_j \otimes S^{m_{kj}-1} A_j \) and by the map \( \beta_j : A_j \to B_j \), i.e.,

\[
\delta_j(a_j^{m_{kj}}) = m_{kj} \beta_j(a_j) \otimes a_j^{m_{kj}-1}.
\]

By our assumption, the map (4.2) is realized as a chain map of complexes, so we have a map

\[
\alpha_k : \bigoplus_j B_j \otimes \partial_j M_k(A_\bullet) \to \mathcal{O}_S
\]

such that \( \alpha_k \circ \delta = 0 \). The components of \( \alpha_k \) can be viewed as morphisms

\[
\alpha_{kj} : \partial_j M_k(A_\bullet) \to B_j^\vee,
\]
and the condition $\alpha_k \circ \delta = 0$ can be expressed as follows:

$$\sum_{j=1}^{n} m_{kj} \langle \alpha_{kj}(a_{1}^{mk_1} \otimes \ldots \otimes a_{j}^{mk_j} \otimes \ldots \otimes a_{n}^{mkn}), \beta_j(a_j) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the evaluation pairing. Let $p : X \to S$ be the total space of the vector bundle $A_1 \oplus \ldots \oplus A_n$ over $S$. Then we can view the maps $\alpha_{kj}$ as sections of the induced bundles $p^*B_j^\vee$ on $X$ and the maps $\beta_j$ as sections of $p^*B_j$ on $X$. The above equation can be viewed as the following identity of functions on $X$:

$$\sum_{j=1}^{n} m_{kj} \langle \alpha_{kj}, \beta_j \rangle = 0. \quad (4.4)$$

Now let us define sections $\alpha_j(w) \in \Gamma(X, p^*B_j^\vee)$ by setting

$$\alpha_j^w = \sum_{k=1}^{N} c_k m_{kj} \alpha_{kj}.$$

Then from (4.4) we obtain that

$$\sum_{j=1}^{n} \langle \alpha_j^w, \beta_j \rangle = 0,$$

so the sections

$$\alpha = (\alpha_1^w, \ldots, \alpha_n^w) \in \bigoplus_{j=1}^{n} p^*B_j^\vee \text{ and } \beta = (\beta_1, \ldots, \beta_n) \in \bigoplus_{j=1}^{n} p^*B_j$$

satisfy $\langle \alpha, \beta \rangle = 0$. Recall that $X$ is the total space of the vector bundle $\oplus_i A_i$ over $S$, so it contains $S$ as the zero section.

**Proposition 4.1.1.** Suppose that $w$ has an isolated singularity at the origin. Then the common vanishing locus of $\alpha$ and $\beta$, $Z(\alpha, \beta)$, coincides with the zero section in $X$.

**Proof.** It is obvious that $Z(\alpha, \beta)$ contains the zero section $S \subset X$ It is enough to check the opposite inclusion in the case when $S$ is a point, i.e., $C = C$ is a single orbicurve $C$ and $X$ is a vector space. To compute $Z(\alpha, \beta)$, first observe that $\ker(\beta_j)$ is isomorphic to $H^0(C, L_j)$ (resp., $\coker(\beta_j) \simeq H^1(C, L_j)$). For each monomial $M_k = x_1^{mk_1} \ldots x_n^{mkn}$ the map (4.2) induces morphisms

$$R\Gamma(C, L_j) \otimes \partial_j M_k(R\Gamma(C, L_\bullet)) \to \mathbb{C}[-1]$$

and in particular, well-defined morphisms

$$H^1(C, L_j) \otimes \partial_j M_k(H^0(C, L_\bullet)) \to \mathbb{C}.$$
These maps are also induced by the maps $\alpha_{kj}$, so we deduce that the restriction of $\alpha_{kj}$ to $\partial_j M_k(H^0(L_\bullet))$ factors through the embedding $\kappa_{kj} : H^1(L_j)^* \to B_j^\vee$. Note that by Serre duality $H^1(L_j)^* \simeq \text{Hom}(L_j, \omega_C)$, and the map 

$$\partial_j M_k(H^0(L_\bullet)) \to \text{Hom}(L_j, \omega_C)$$

corresponding to $\alpha_{kj}$ is the composition of the product map

$$\partial_j M_k(H^0(L_\bullet)) \to H^0(\partial_j M_k(L_\bullet))$$

with the natural map

$$\phi_{kj} : H^0(\partial_j M_k(L_\bullet)) \to \text{Hom}(L_j, \omega_C)$$

induced by $\phi_k$. Since $\kappa_{kj}$ is an embedding, the condition that $\alpha_{wj}(a) = 0$ where $a = (a_1, \ldots, a_n)$ and $a_j \in H^0(L_j)$, can be rewritten as

$$\sum_{k=1}^N c_k \phi_{kj}(\partial_j M_k(a)) = 0, \tag{4.5}$$

where the expression

$$M_k(a) = a_1^{m_k} \otimes \cdots \otimes a_n^{m_k}$$

is a section of $M_k(L_\bullet) = L_1^{m_k} \cdots L_n^{m_k}$.

For every generic point $x \in C$, where all $\phi_k$ are isomorphisms, there exist a collection of trivializations of $L_j$'s at $x$ and a trivialization of $\omega_x$ such that each map $\phi_k$ becomes equal to the identity under these trivializations. This follows easily from Proposition 3.1.5, since the restriction of our weak $w$-structure to $x$ becomes a $w$-structure with respect to the unit object. Using these trivializations we can view $a(x) = (a_1(x), \ldots, a_n(x))$ as a point in $\mathbb{C}^n$. Now the left-hand side of (4.5) evaluated at $x$ becomes the $j$th partial derivative of $w$ at $a(x)$ (see (4.1)). Thus, $\partial_j w(a(x)) = 0$ for all $k = 1, \ldots, n$. This implies that $a(x) = 0$, since $w$ has an isolated singularity at 0. Therefore, $a = 0$ on a dense subset of $C$ and since $L_i$ have no torsion, this implies that $a = 0$. So $Z(\alpha, \beta)$ is contained in $S$.

### 4.2 Fundamental matrix factorizations

Let $w$ be a quasihomogeneous polynomial with an isolated singularity, $G \subset G_w$ a subgroup containing $J$ and $\Gamma \subset \Gamma_w$ the corresponding extension of $G_m$ by $G$ (see Lemma 2.1.1). A key role in our construction of the CohFT associated with $w$ and $G$ will be played by certain matrix factorizations of $-p^*w_\gamma$ on $S_{g,1}(\gamma) \times \mathbb{A}^\gamma$, where $\mathbb{A}^\gamma = \prod_{i=1}^r (\mathbb{A}^n)^{\gamma_i}$ and $w_\gamma = \sum_{i=1}^r \text{pr}_i^* w_{\gamma_i}$, where $p : S_{g,1}(\gamma_1, \ldots, \gamma_r) \times \mathbb{A}^\gamma \to \mathbb{A}^\gamma$ and $\text{pr}_i : \mathbb{A}^\gamma \to (\mathbb{A}^n)^{\gamma_i}$ are the projections. This collection of objects can be viewed as a categorified analog of the fundamental class in Gromov-Witten theory.\(^1\)

\(^1\)Such a categorified version of the fundamental class exists in GW-theory as well and can be constructed as in [38]
The construction roughly goes as follows. First, using the universal \( w \)-structure over \( \mathcal{S}^{\text{rig}}_g(\tau) = \mathcal{S}^{\text{rig}}_{g,\Gamma}(\tau) \) provided by Proposition 3.2.2(iv) (together with the rigidity structure), we construct a Koszul matrix factorization \( \{ \alpha, \beta \} \) of the pull-back of \( -p^*w_{\tau} \) on a certain affine bundle \( X \rightarrow \mathcal{S}^{\text{rig}}_g(\tau) \times \mathbb{A}^\tau \). We prove that this matrix factorization is supported on a section of this affine bundle over \( \mathcal{S}^{\text{rig}}_g(\tau) \times \{ 0 \} \). This allows us to apply the construction of [50, Sec. 6] to define the fundamental matrix factorization \( \mathcal{P}^{\text{rig}}_g(\tau) \in \mathcal{D}MFr_w(\mathcal{S}^{\text{rig}}_g(\tau) \times \mathbb{A}^\tau, -p^*w_{\tau}) \) as the push-forward of \( \{ \alpha, \beta \} \). The space \( X \) is the total space of the vector bundle \( \bigoplus_{j=1}^n A_j \) over \( \mathcal{S}^{\text{rig}}_g(\tau) \) for an appropriate choice of resolutions \([d_j : A_j \rightarrow B_j] \) of the derived push-forwards of the line bundles associated with the \( \Gamma \)-spin structure. The element \( \beta \) is the section of the pull-back of \( \bigoplus B_j \) to \( X \) induced by the differentials \( d_j \), while the map \( X \rightarrow \mathbb{A}^\tau \) is induced by the rigidification structure. The construction of the element \( \alpha \), which is a section of \( \bigoplus B'_j \), is more involved. First, we use the isomorphisms (3.11) of the universal \( w \)-structure to construct the data needed for \( \alpha \) on the level of derived categories. Then we use quasi-projectivity of the coarse moduli space to lift these data to the level of complexes.

We start with a family of \( \Gamma \)-spin curves over a base \( S \) of type \( \tau = (\gamma_1, \ldots, \gamma_r) \in G^r \). Recall (see Definition 3.2.5) that this is a family \( \pi : C \rightarrow S \) of nodal orbicurves with marked orbipoints \( p_1 \rightarrow C, \ldots, p_r \rightarrow C \) and a principal \( \Gamma \)-bundle \( P \) together with an isomorphism \( \varepsilon : \pi_*P \simeq \varphi_*P(\omega_{C/S}^{\log}) \). Let \( (\mathcal{L}_1, \ldots, \mathcal{L}_n) \) be a collection of line bundles on \( C \) associated with \( P \). Then \( \varepsilon \) induces isomorphisms

\[
\phi_M : M(\mathcal{L}_\bullet) = M(\mathcal{L}_1, \ldots, \mathcal{L}_n) \rightarrow \omega_{C/S}^{\log}
\]

for every monomial \( M \) occurring in \( w \) (see Remark 3.2.3.2). We also assume that our \( \Gamma \)-spin structure \( (P, \varepsilon) \) is equipped with a restricted rigidification (see Definition 3.2.10). Note that this notion is well defined in our situation by Lemma 3.2.11.

Let \( (C \rightarrow S, p_1, \ldots, p_r) \) be the family of orbicurves with marked points, obtained by forgetting the orbistructure at \( p_i \)'s (and keeping the orbistructure at the nodes), and let \( \rho : C \rightarrow C \) be the natural projection. The family \( (C, p_1, \ldots, p_r) \) can be constructed as follows. Present \( \tilde{C} \) as the union of two open substacks \( \tilde{C}^{\text{reg}} \) and \( \tilde{C}^0 \), obtained by taking the complements of the nodes and of the marked points, respectively. Then take the relative coarse moduli \( C^{\text{reg}} \) of \( \tilde{C}^{\text{reg}} \) with the induced marked points (see [29] or [?]). The family \( C \) is obtained by gluing \( C^{\text{reg}} \) with \( \tilde{C}^0 \).

For each \( j = 1, \ldots, n \) the push-forward \( L_j = \rho_*\mathcal{L}_j \) is a line bundle on \( C \). By abuse of notation we will denote the projection \( C \rightarrow S \) also by \( \pi \). Note that we have natural isomorphisms \( R\pi_*(\mathcal{L}_j) \simeq R\pi_*(\mathcal{L}_j) \).

Let us introduce some more notation. The \( r \)-tuple \( \tau = (\gamma_1, \ldots, \gamma_r) \in G^r \) determines a relation

\[
\Sigma = \Sigma(\tau) = \{(p_i, j) \mid \gamma_{ij} = 1\} \subset \{p_1, \ldots, p_r\} \times \{1, \ldots, n\}.
\]

For a \( \Gamma \)-spin structure of type \( \tau \) one has \( (p_i, j) \in \Sigma \) if and only if the action of \( G(p_i) \) on \( \mathcal{L}_{ij}|_{p_i} \) is trivial.

Note that the cross-section of \( \Sigma \) corresponding to a point \( p_i \) is the subset \( I(\gamma_i) \subset \{1, \ldots, n\} \) listing the coordinates of the subspace of \( \gamma_i \)-invariants \( (\mathbb{A}^n)^{\gamma_i} \subset \mathbb{A}^n \). Hence,
the affine space
\[ \mathbb{A}^r = \prod_{i=1}^{r} (\mathbb{A}^n)^{\gamma_i} \]
has coordinates \( x_j(i) \) labeled by \((p_i, j) \in \Sigma\).

Let us equip the space \( \mathbb{A}^r \) with the potential
\[ w_{\gamma_i} = \sum_{i=1}^{r} \text{pr}_i^* w_{\gamma_i}, \]
where \( w_{\gamma_i} \) is the restriction of \( w \) to \((\mathbb{A}^n)^{\gamma_i}\) and \( \text{pr}_i : \mathbb{A}^r \to (\mathbb{A}^n)^{\gamma_i} \) is the projection.

Let \( \Sigma_j = \{ p_i | (p_i, j) \in \Sigma \} \subset \{ p_1, \ldots, p_r \} \)
be the \( j \)-th cross-section of \( \Sigma \), and for each monomial \( M \) occurring in \( w \) let us set
\[ \Sigma_M = \bigcap_{x_j|M} \Sigma_j. \]

**Lemma 4.2.1.** We have
\[ w_{\gamma_i} = \sum_{k : i \in \Sigma_M} c_k \cdot M_k \]
and
\[ w_{\gamma} = \sum_{k=1}^{N} c_k \cdot M_k^{\oplus \Sigma_M}, \]
where \( M^{\oplus \Sigma_M} = \sum_{i \in \Sigma_M} M(x_1(i), \ldots, x_n(i)). \)

**Proof.** By definition, a monomial \( M \) of \( w \) occurs in \( w_{\gamma_i} \) if and only if \( \gamma_i \) acts trivially on all variables \( x_j \) such that \( x_j|M \), i.e., if and only if \( i \in \Sigma_M. \)

The following important observation will eventually lead to a connection with the function \( w_{\gamma}. \)

**Lemma 4.2.2.** For each monomial \( M \) occurring in \( w \) the isomorphism (4.6) induces an injective morphism
\[ \overline{\phi}_M : M(L_1, \ldots, L_n) \to \omega_C(\Sigma_M) \]
on \( C \), which is an isomorphism near \( \Sigma_M. \)

**Proof.** To simplify notation we consider the case of a single curve (i.e., \( S \) is a point). As we have seen in the proof of Proposition 3.3.1(i), the map
\[ \phi'_M : M(L_*) \to \rho_*(M(L_*) \simeq \omega_C(p_1 + \ldots + p_r), \]
induced by \( \phi_M \), vanishes at \( p_i \) to the order
\[ \frac{1}{m} \sum_{j=1}^{n} k_j \theta_{ij} \geq 0, \]
where we use the notation of Proposition 3.3.1(i). Furthermore, \( \phi_M' \) is an isomorphism near \( p_i \) if and only if \( \theta_{ij} = 0 \) for all \( j \) such that \( x_j \mid M \). This immediately implies the assertion since \((p_i,j) \in \Sigma \) if and only if \( \theta_{ij} = 0 \).

The restricted rigidification structure on \((P, \epsilon)\) induces trivializations

\[
e_j(i) : \mathcal{O}_{p_i} \to \mathcal{L}_j|_{p_i} \quad \text{for} \quad (p_i,j) \in \Sigma,
\]
such that for every monomial \( M \) occurring in \( w \) and every \( p_i \in \Sigma_M \) the induced morphism

\[
\mathcal{O}_{p_i} \xrightarrow{M(e_j(i))} M(L_\bullet) \xrightarrow{\varphi_M} \omega_C(\Sigma_M)|_{p_i} \cong \mathcal{O}_{p_i}
\]
is the identity. For \( j = 1, \ldots, n \) let

\[
e_j : \mathcal{O}_S^{\Sigma_j} \to \pi_*(L_j|_{\Sigma_j}) \quad (4.10)
\]
denote the isomorphism induced by \((e_j(i))\). From now on we will work exclusively with the data \((L_1, \ldots, L_n; e_1, \ldots, e_n)\) on the family of orbicurves \( C/S \) that has trivial orbi-structure at the marked points \( p_1, \ldots, p_r \), together with the morphisms (4.9).

Note that since \( \Sigma_M \subset \Sigma_j \) for every \( j = 1, \ldots, n \) the map \( \varphi_M \) induces a morphism

\[
M(L_\bullet(-\Sigma_\bullet)) = M(L_1(-\Sigma_1), \ldots, L_n(-\Sigma_n)) \to \omega_{C/S}, \quad (4.11)
\]
where we view \( \Sigma_j \) as a subdivisor of \( p_1 + \ldots + p_r \). The collection \((L_j(-\Sigma_j))\) equipped with morphisms (4.11) is a weak \( w \)-structure with respect to \( \omega_{C/S} \). Therefore, the construction of Section 4.1 gives a natural morphism

\[
t_M : M(R\pi_*(L_\bullet(-\Sigma_\bullet))) \to \mathcal{O}_S[-1] \quad (4.12)
\]
induced by (4.11) (see (4.2)).

For each \( j \) we have an exact triangle in \( D^b(S) \)

\[
R\pi_*(L_j(-\Sigma_j)) \to R\pi_*(L_j) \xrightarrow{r_j} \pi_*(L_j|_{\Sigma_j}) \to \ldots
\]
The isomorphism (4.10) gives a map

\[
Z_j : R\pi_*(L_j) \to \mathcal{O}_S^{\Sigma_j}, \quad (4.13)
\]
such that

\[
r_j = e_j \circ Z_j.
\]
Let us denote

\[
Z_M : M(R\pi_*(L_\bullet)) \xrightarrow{M(Z_\bullet)} M(\mathcal{O}_S^{\Sigma_\bullet}) \to M(\mathcal{O}_S^{\Sigma_M}, \ldots, \mathcal{O}_S^{\Sigma_M}) \to \mathcal{O}_S^{\Sigma_M}
\]
the map induced by the composition \( R\pi_*(L_j) \xrightarrow{Z_j} \mathcal{O}_S^{\Sigma_j} \to \mathcal{O}_S^{\Sigma_M} \) and by the algebra structure on \( \mathcal{O}_S^{\Sigma_M} \).
We split the construction of the fundamental matrix factorization into four steps. First, we extend the morphisms (4.12) to morphisms \( \tau_M : E_M \to \mathcal{O}_S[-1] \) where \( E_M \in D^b(S) \) are certain modifications of \( M(R\pi_*(L_\bullet)) \). In Step 2 we realize these morphisms on the level of complexes using appropriate resolutions \([A_j \to B_j]\) of \( R\pi_*(L_j) \). At the same time we realize morphisms \( Z_j \) by maps of vector bundles \( A_j \to \mathcal{O}_{\Sigma_j}S \). These maps combine into a morphism \( Z : X \to A^\gamma \), where \( X \) is the total space of the bundle \( \bigoplus A_j \). In Step 3 we construct a Koszul matrix factorization \{\alpha, \beta\} of \(-Z^*w_\gamma\) and in Step 4 we show that it can be pushed forward to a matrix factorization of \(-p^*w_\gamma\) on \( S \times A^\gamma \), where \( p \) is the projection \( S \times A^\gamma \to A^\gamma \).

**Step 1.** For each monomial \( M \) appearing in \( w \) we construct a canonical commutative diagram in \( D^b(S) \) with an exact triangle as a middle row

\[
\begin{array}{cccccc}
M(R\pi_*(L_\bullet(-\Sigma_\bullet))) & \xrightarrow{\text{nat}} & M(R\pi_*(L_\bullet)) & \xrightarrow{Z_M} & \mathcal{O}_{\Sigma M}^S & \xrightarrow{\text{Tr}} & E_M[1] \\
E_M & \xrightarrow{\epsilon} & M(R\pi_*(L_\bullet)) & \xrightarrow{Z_M} & \mathcal{O}_{\Sigma M}^S & \xrightarrow{\text{Tr}} & E_M[1] \\
\mathcal{O}_S[-1] & \xrightarrow{t_M} & M(R\pi_*(L_\bullet)) & \xrightarrow{Z_M} & \mathcal{O}_{\Sigma M}^S & \xrightarrow{\text{Tr}} & E_M[1] \\
\end{array}
\]

(4.14)

where

\[
\text{nat} : M(R\pi_*(L_\bullet(-\Sigma_\bullet))) \to M(R\pi_*(L_\bullet))
\]

is the natural map and the map \( \text{Tr} : \mathcal{O}_{\Sigma M}^S \to \mathcal{O}_S \) is given by the sum of components, and \( t_M \) is the morphism (4.12).

Let \( M = x_1^{k_1} \ldots x_n^{k_n} \) and set \( |M| = \deg M = k_1 + \ldots + k_n \). Consider the relative \(|M|\)th power \( \pi^M : C^M := C^{|M|} \to S \) of \( C \) over \( S \) and define the line bundle \( L^M_\bullet \) on \( C^M \) by

\[
L^M_\bullet := L_1^{\oplus k_1} \boxtimes \ldots \boxtimes L_n^{\oplus k_n}.
\]

Also, consider the product of symmetric groups

\[
\text{Sym}(M) = S_{k_1} \times \ldots \times S_{k_n}.
\]

(4.15)

From the K"unneth isomorphism we get an identification

\[
M(R\pi_*(L_\bullet)) \simeq (R\pi_*(L^M_\bullet))^{\text{Sym}(M)},
\]

where we take invariants with respect to the natural \( \text{Sym}(M) \)-action. Let

\[
\sigma_M : \Sigma_M \to C \xrightarrow{\Delta_M} C^M
\]
be the closed embedding, where $\Delta_M$ is the diagonal map, and let $\mathcal{J}_M \subset \mathcal{O}_{C^M}$ denote the ideal sheaf of $\sigma_M(\Sigma_M)$. Define a $\text{Sym}(M)$-equivariant coherent sheaf $\mathcal{F}_M$ on $C^M$ by

$$\mathcal{F}_M = \mathcal{J}_M L^M_\bullet$$

and set

$$E_M = \left( R\pi^*_M(\mathcal{F}_M) \right)^{\text{Sym}(M)}.$$ 

Note that we have an exact sequence

$$0 \to \mathcal{F}_M \to L^M_\bullet \to (\sigma_M)_*\sigma^*_M L^M_\bullet \to 0. \quad (4.17)$$

The map $\overline{\phi}_M$ induces an isomorphism

$$\sigma^*_M L^M_\bullet \simeq M(L_\bullet)|_{\Sigma_M} \simeq \mathcal{O}_{\Sigma_M}$$

on $\Sigma_M$, which coincides with the isomorphism obtained from the trivializations of $L_j|_{\Sigma_j}$. Hence, taking the push-forward of the sequence (4.17) to $S$, and considering $\text{Sym}(M)$-invariants we get the horizontal exact triangle of the diagram (4.14).

We have an embedding of sheaves on $C^M$

$$\iota^C_M : L_\bullet(-\Sigma_\bullet)^M \hookrightarrow \mathcal{F}_M. \quad (4.18)$$

After taking the push-forward to $S$ and passing to the $\text{Sym}(M)$-invariants it induces the map $\iota_M : M(R\pi_*(L_\bullet)) \to E_M$.

The map $\tau_M$ is constructed similarly using the morphism of sheaves

$$\kappa_M : \mathcal{F}_M \to (\Delta_M)_*\omega_{C/S} \quad (4.19)$$

defined as follows. Let $\mathcal{J}_\Delta \subset \mathcal{O}_{C^M}$ be the ideal sheaf of the diagonal $\Delta_M(C) \subset C^M$. Let $\psi_M : \Delta^*_M \mathcal{F}_M \to \omega_{C/S}$ be the composition of the isomorphism

$$\Delta^*_M \mathcal{F}_M \simeq \mathcal{J}_M L^M_\bullet / \mathcal{J}_\Delta L_\bullet \simeq M(L_\bullet)(-\Sigma_M). \quad (4.20)$$

with the map $\overline{\phi}_M : M(L_\bullet)(-\Sigma_M) \to \omega_{C/S}$. Now we define $\kappa_M$ as the morphism $\mathcal{F}_M \to (\Delta_M)_*\omega_{C/S}$ corresponding to $\psi_M$ by adjunction.

**Lemma 4.2.3.** The composition $\kappa_M \circ \iota^C_M$ of morphisms (4.18) and (4.19) coincides with the natural map $L_\bullet(-\Sigma_\bullet)^M \to (\Delta_M)_*\omega_{C/S}$ corresponding by adjunction to the map

$$\Delta^*_M L_\bullet(-\Sigma_\bullet)^M \simeq M(L_\bullet(-\Sigma_\bullet)) \to \omega_{C/S}. \quad (4.21)$$

given by (4.11)

**Proof.** This follows by adjunction from the fact that the map (4.21) coincides with the composition of $\Delta^*_M \iota^C_M$ with the map

$$\psi_M : \Delta^*_M \mathcal{F}_M \simeq M(L_\bullet)(-\Sigma_M) \xrightarrow{\overline{\phi}_M} \omega_{C/S}. \quad \square$$
We have a morphism of exact sequences

\[
\begin{array}{cccccc}
0 & \to & F^M & \to & L^M & \to & (\sigma_M)_* \mathcal{O}_{\Sigma M} & \to & 0 \\
\kappa_M & & & & \text{id} & & & & \\
0 & \to & (\Delta_M)_* \omega_{C/S} & \to & (\Delta_M)_* \omega_{C/S}(\Sigma_M) & \to & (\sigma_M)_* \mathcal{O}_{\Sigma M} & \to & 0
\end{array}
\]

where the middle vertical arrow corresponds to \( \overline{\phi}_M \) by adjunction. It induces a morphism of exact triangles

\[
\begin{array}{cccccc}
E_M & \xrightarrow{\epsilon} & M( R\pi_*(L_\bullet) ) & \xrightarrow{Z_M} & \mathcal{O}_S^{\Sigma M} & \xrightarrow{i} & E_M[1] \\
\tau'_M & & \downarrow \text{id} & & \downarrow \tau'_M & & \\
R\pi_*(\omega_{C/S}) & \to & R\pi_*(\omega_{C/S}(\Sigma_M)) & \to & \mathcal{O}_S^{\Sigma M} & \to & R\pi_*(\omega_{C/S})[1]
\end{array}
\]

On the other hand, we have a morphism of exact triangles

\[
\begin{array}{cccccc}
R\pi_*(\omega_{C/S}) & \to & R\pi_*(\omega_{C/S}(\Sigma_M)) & \to & \mathcal{O}_S^{\Sigma M} & \to & R\pi_*(\omega_{C/S})[1] \\
\text{Tr}_{C/S} & & \downarrow \text{id} & & \downarrow \text{Tr}_{C/S}[1] & & \\
\mathcal{O}_S[-1] & \to & [\mathcal{O}_S^{\Sigma M} \to \mathcal{O}_S] & \to & \mathcal{O}_S^{\Sigma M} & \to & \mathcal{O}_S
\end{array}
\]

where \( \text{Tr}_{C/S} : R\pi_*(\omega_{C/S}) \to \mathcal{O}_S[-1] \) is the Grothendieck trace map (see [22]). Composing these two morphisms of exact triangles we get a diagram

\[
\begin{array}{cccccc}
E_M & \xrightarrow{\epsilon} & M( R\pi_*(L_\bullet) ) & \xrightarrow{Z_M} & \mathcal{O}_S^{\Sigma M} & \xrightarrow{i} & E_M[1] \\
\tau_M & & \downarrow \text{id} & & \downarrow \tau_M[1] & & \\
\mathcal{O}_S[-1] & \to & [\mathcal{O}_S^{\Sigma M} \to \mathcal{O}_S] & \to & \mathcal{O}_S^{\Sigma M} & \to & \mathcal{O}_S
\end{array}
\]

and in particular, the canonical map \( \tau_M : E_M \to \mathcal{O}_S[-1] \). This finishes the construction of the diagram (4.14). The equality

\[
t_M = \tau_M \circ \iota_M
\]

(4.23)

(the commutativity of the leftmost triangle in the diagram (4.14)) follows from Lemma 4.2.3.
Note that when $\Sigma_M = \emptyset$ the map $\tau_M$ coincides with the map (4.2) constructed from the data $(L_1, \ldots, L_n; \overline{\phi}_M)$.

**Step 2.** Next, we are going to realize the diagram (4.14) on the level of complexes. More precisely, we will represent each $R\pi_*(L_j)$ by a complex $K_j = [A_j \to B_j]$ of vector bundles on $S$, concentrated in degrees $[0, 1]$, in such a way that the map $Z_j : R\pi_*(L_j) \to \mathcal{O}_S^{\Sigma_j}$ is realized by a surjective chain map of complexes $Z_j : K_j \to \mathcal{O}_S^{\Sigma_j}$. Then the subcomplex $K_j' = \ker(Z_j) = [A_j' \to B_j]$ will represent $R\pi_*(L_j(-\Sigma_j))$ and the map $\eta : (K_j') \hookrightarrow (K_j)$ will be the natural inclusion. For each monomial $M$ appearing in $w$ the map $Z_M : M(K_\bullet) \to \mathcal{O}_S^{\Sigma M}$ will be realized by the composition of

$$M(Z_\bullet) : M(K_\bullet) \to M(\mathcal{O}_S^{\Sigma_1}, \ldots, \mathcal{O}_S^{\Sigma_n})$$

with the natural epimorphism $M(\mathcal{O}_S^{\Sigma_1}, \ldots, \mathcal{O}_S^{\Sigma_n}) \to \mathcal{O}_S^{\Sigma M}$. Also, the complex $K_M = \text{Cone}(Z_M)[-1]$ will represent the object $E_M \in D^b(S)$ and the maps $\tau_M$ and $\iota_M$ will be realized by chain map of complexes, so that the diagram (4.14) will be commutative in the category of complexes.

To construct appropriate complexes representing $R\pi_*(L_j(-\Sigma_j))$ and $R\pi_*(L_j)$ we will need the following result.

**Lemma 4.2.4.** Let $T$ be a proper smooth DM-stack over $\mathbb{C}$ with projective coarse moduli $\mathcal{T}$. Let $p : T \to \mathcal{T}$ be the projection, $\mathcal{O}(1)$ an ample line bundle on $\mathcal{T}$ and $\mathcal{O}_T(1) := p^*\mathcal{O}(1)$.

(i) There exists a vector bundle $\mathcal{V}$ on $T$ such that for any coherent sheaf $\mathcal{F}$ on $T$ the natural map

$$H^0(T, \mathcal{V}^r \otimes \mathcal{F}(n)) \otimes \mathcal{V}(-n) \to \mathcal{F}$$

(4.24)

is surjective for $n \gg 0$. Also,

$$H^{>0}(T, \mathcal{F}(n)) = 0$$

for $n \gg 0$.

(ii) Let $\pi : C \to T$, $p_1, \ldots, p_r : T \to C$ be a family of stable curves with marked points over $T$. Then for every vector bundle $E$ on $C$ there exists an embedding $E \to F$ of vector bundles on $C$ such that $R^i\pi_*(F) = 0$.

**Proof.** (i) It is well known that $T$ is a quotient stack (see [36, Thm. 4.4]). Hence, by [37, Thm. 1], there exists a scheme $Z$ and a finite flat surjective morphism $q : Z \to T$. Since the projection $p : T \to \mathcal{T}$ is proper and quasi-finite, it follows that the map $p \circ q : Z \to \mathcal{T}$ is finite, so $Z$ is projective and $\mathcal{O}_Z(1) = (pq)^*\mathcal{O}(1)$ is ample. Therefore, for any coherent sheaf $\mathcal{F}$ on $T$ the natural morphism of sheaves on $Z$

$$H^0(Z, q^*\mathcal{F}(n)) \otimes \mathcal{O}_Z(-n) \to q^*\mathcal{F}$$

(4.25)

is surjective and $H^{>0}(Z, q^*\mathcal{F}(n)) = 0$ for $n \gg 0$. Note that by projection formula

$$H^i(Z, q^*\mathcal{F}(n)) \simeq H^i(T, q_*\mathcal{O}_Z \otimes \mathcal{F}(n)).$$

Since $\mathcal{O}_T$ is a direct summand of $q_*(\mathcal{O}_Z)$, we deduce the vanishing of $H^{>0}(T, \mathcal{F}(n))$ for $n \gg 0$. On the other hand, applying the push-forward by $q$ to (4.25) we obtain a surjective morphism

$$H^0(Z, q^*\mathcal{F}(n)) \otimes q_*(\mathcal{O}_Z)(-n) \to q_*(\mathcal{O}_Z) \otimes \mathcal{F},$$
which implies the surjectivity of (4.24) for \( n \gg 0 \) with \( \mathcal{V} = q_*(\mathcal{O}_Z) \).

(ii) Let \( L = \omega_{C/T}(p_1 + \ldots + p_r) \). Since the line bundle \( L \) is ample on fibers of \( \pi \), for sufficiently large \( n \) we have \( R^1\pi_*(E^\vee \otimes L^n) = R^1\pi_*(L^n) = 0 \) and the morphism

\[
\pi^*\pi_*(E^\vee \otimes L^n) \otimes L^{-n} \to E^\vee
\]

is surjective. Thus, setting \( F = \pi^*(\pi_*(E^\vee \otimes L^n))^\vee \otimes L^n \) we obtain an embedding of vector bundles \( E \to F \) and

\[
R^1\pi_*(F) \simeq (\pi_*(E^\vee \otimes L^n))^\vee \otimes R^1\pi_*(L^n) = 0.
\]

\( \square \)

Without loss of generality we can assume that \( S \) is connected. Our family of \( \Gamma \)-spin curves \((C/S, p_1, \ldots, p_r; P, \varepsilon)\) induces a map \( S \to S \) to a connected component of the moduli stack of \( \Gamma \)-spin structures. Note that the data \((C/S, p_1, \ldots, p_r; L_1, \ldots, L_n, \phi_M)\) is obtained by the base change from the corresponding universal data over \( S \). By Proposition 3.2.6, \( S \) is a proper smooth DM-stack with projective coarse moduli, so Lemma 4.2.4 can be applied over \( S \). Hence, similar assertions hold for sheaves over \( S \) that are obtained by the pull-back from sheaves over \( S \).

By Lemma 4.2.4(ii) for each \( j \) we can choose an embedding of vector bundles \( L_j \to P_j \), where \( R^1\pi_*(P_j) = 0 \). Hence, we get \( \pi \)-acyclic resolutions

\[
L_j(-\Sigma_j) \to [P_j \to Q_j] \quad \text{and} \quad L_j \to [P_j \oplus L_j|_{\Sigma_j} \to Q_j],
\]

where \( Q_j = P_j/L_j(-\Sigma_j) \simeq \coker(L_j \to P_j \oplus L_j|_{\Sigma_j}) \). Note that the exact sequence

\[
0 \to L(-\Sigma_j) \to L_j \to L_j|_{\Sigma_j} \to 0
\]

is realized by the exact sequence of \( \pi \)-acyclic resolutions

\[
0 \to [P_j \to Q_j] \to [P_j \oplus L_j|_{\Sigma_j} \to Q_j] \to L_j|_{\Sigma_j} \to 0.
\]

Now consider the sheaves \( A_j = \pi_*(P_j \oplus L_j|_{\Sigma_j}) \), \( B_j = \pi_*(Q_j) \) and \( A_j' = \pi_*(P_j) \) on \( S \). Since \( R^{>0}\pi_*(P_j) = 0 \), it follows that \( A_j \) and \( A_j' \) are vector bundles. We have an exact sequence of sheaves on \( C \)

\[
0 \to L_j|_{\Sigma_j} \to Q_j \to Q_j' \to 0,
\]

where \( Q_j' = P_j/L_j \). Since \( Q_j' \) is a vector bundle on \( C \) with \( R^{>0}\pi_*(Q_j') = 0 \) it follows that \( \pi_*(Q_j') \) is a vector bundle, hence \( B_j = \pi_*(Q_j) \) is also a vector bundle.

Thus, we get the complexes \( K_j'' = [A_j' \to B_j] \) and \( K_j = [A_j \to B_j] \) representing \( R\pi_*(L_j(-\Sigma_j)) \) and \( R\pi_*(L_j) \), and the map \( Z_j : A_j = \pi_*(P_j \oplus L_j|_{\Sigma_j}) \to \pi_*(L_j|_{\Sigma_j}) \simeq \mathcal{O}_S^{\Sigma_j} \) induced by the projection, such that \( K_j' = \ker(Z_j) \).

At this point we can realize on the level of complexes the part of the diagram (4.14) not involving the maps \( t_M \) and \( \tau_M \). First, by taking the external tensor products we get for each monomial \( M \) a \( \pi^M \)-acyclic resolution

\[
L^M_\bullet \to R^M_\bullet \quad \text{and} \quad L_\bullet(-\Sigma_\bullet)^M \to R^M,
\]

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where $\hat{R}^M$ is a subcomplex in $R^M$. Recall that the object $E_M \in D^b(S)$ is given by $\text{Sym}(M)$-invariants of the push-forward $R\pi^*_M(\cF_M)$, so we also need a $\pi^M$-acyclic resolution of $\cF_M$. Note that the map $L^M_\bullet \to \Delta_!\cO_{\Sigma_M}$ is realized by a surjective chain map of complexes $r(M) : R^M \to \Delta_!\cO_{\Sigma_M}$ vanishing on $\hat{R}^M$. This implies that the subcomplex $\ker(r(M)) \subset R^M$ is a $\pi^M$-acyclic resolution of $\cF_M$, and the embedding $L_\bullet(-\Sigma^M_\bullet) \hookrightarrow \cF_M$ is realized by an embedding of resolutions $\hat{R}^M \hookrightarrow \ker(r(M))$.

Now the $\text{Sym}(M)$-invariants of the push-forwards with respect to $\pi^M$ of $R^M$, $\hat{R}^M$ and $\ker(r(M))$ will represent $M(R\pi^*_M(L^\bullet_\bullet))$, $M(R\pi^*_M(L^\bullet_\bullet(-\Sigma^\bullet_\bullet)))$ and $E_M$, respectively. Now all the maps in the diagram (4.14) except for $t_M$ and $\tau_M$, are realized on the level of complexes. Furthermore, we have natural isomorphisms

$$\pi^*_M(R^M)^{\text{Sym}(M)} \simeq M(K^\bullet_\bullet), \quad \pi^*_M(\hat{R}^M)^{\text{Sym}(M)} \simeq M(K'_\bullet_\bullet)$$

and an exact sequence

$$0 \to \pi^*_M(\ker(r(M))^{\text{Sym}(M)}) \to M(K^\bullet_\bullet) \xrightarrow{Z_M} \cO_{\Sigma^M} \to 0.$$  

It follows that if instead of $\pi^*_M(\ker(r(M))^{\text{Sym}(M)})$ we use $K_M = \text{Cone}(Z_M)[-1]$, we will still have a representation of the part of the diagram (4.14) not involving the maps $t_M$ and $\tau_M$, on the level of complexes (note that $t_M$ becomes represented by the natural inclusion $M(K'_\bullet_\bullet) \to K_M$).

Note that the above realization depends only on complexes $[A_j \to B_j]$ and surjective maps $Z_j : A_j \to \Delta^S_{\Sigma} j$ (recall that $A'_j = \ker(Z_j)$). To realize the maps $\tau_M$ by chain maps we will modify this realization by replacing the complexes $[A_j \to B_j]$ with new complexes $[\tilde{A}_j \to \tilde{B}_j]$ equipped with surjective quasi-isomorphisms $[\tilde{A}_j \to \tilde{B}_j] \to [A_j \to B_j]$. To do this we will need the following technical assertion similar to Proposition 4.7 of [48].

**Lemma 4.2.5.** Let a stack $T$ and a vector bundle $\mathcal{V}$ be as in Lemma 4.2.4(i). Let $[C_0 \to C_1]$ be a complex of vector bundles on $T$. For each integer $d > 0$ there exists $m_0 > 0$ such that for any $m_1 \geq m_0$ and any surjection

$$C_1 = \mathcal{V}^\vee(-m_1)^{\oplus N} \xrightarrow{\sigma} C_1$$

one has

$$H^{>0}(T, (\hat{C}_0)^{\oplus q_1} \otimes \mathcal{V}^{\otimes q_2}(m)) = 0 \text{ for } m \geq m_0 \text{ and } q_1 + q_2 \leq d, \quad (4.26)$$

where the bundle $\hat{C}_0$ is the fiber product of $C_0$ and $\hat{C}_1$ over $C_1$, so that we have a quasi-isomorphism of complexes

$$[\hat{C}_0 \to \hat{C}_1] \to [C_0 \to C_1].$$

**Proof.** Set $K = \ker(\sigma)$, so that we have exact sequences of vector bundles

$$0 \to K \to \hat{C}_0 \to C_0 \to 0, \quad (4.27)$$

$$0 \to K \to \hat{C}_1 \to C_1 \to 0. \quad (4.28)$$
From the sequence (4.27) we see that (4.26) would follow from the vanishing of

$$H^{>0}(T, (C_0^{q_1})^\vee \otimes (K^\otimes q_2)^\vee \otimes V^\otimes q_3(m))$$

for $$m \geq m_0$$ and $$q_1 + q_2 + q_3 \leq d$$. Taking tensor powers of the sequence dual to (4.28) we get a resolution of the bundle $$(K^\otimes q_2)^\vee$$ with terms that are direct sums of vector bundles of the form $$(C_1^{q_3(q_2-s)})^\vee \otimes V^\otimes s(m_1 s)$$. Thus, it would be enough to find $$m_0$$ such that

$$H^{>0}(T, (C_0^{q_1})^\vee \otimes (C_1^{q_2})^\vee \otimes V^\otimes q_3(m)) = 0$$

for $$m \geq m_0$$ and $$q_1 + q_2 + q_3 \leq d$$. But this is possible by Lemma 4.2.4(i).

Let us apply Lemma 4.2.5 to the complex $$[\oplus_j A_j \rightarrow \oplus_j B_j]$$ and $$d$$ equal to the maximum of the degrees $$|M|$$ of all the monomials $$M$$ occurring in $$w$$. Then we can choose large enough $$m_0$$ and surjections $$B_j = V^\vee(-m_0)^\otimes \Sigma_j \rightarrow B_j$$ and replace each $$[A_j \rightarrow B_j]$$ by a quasi-isomorphic complex $$[\overline{A}_j \rightarrow \overline{B}_j]$$ such that

$$\text{Ext}_S^{>0}((\oplus_j \overline{A}_j)^{\otimes q_1} \otimes (V^\otimes q_2)^\vee(-m), \mathcal{O}_S) = 0$$

for $$m \geq m_0$$ and $$q_1 + q_2 \leq d$$. This implies that

$$\text{Ext}_S^{>0}((\oplus_j \overline{A}_j)^{\otimes q_1} \otimes (\oplus_j \overline{B}_j)^{\otimes q_2}, \mathcal{O}_S) = 0$$

for $$q_1 + q_2 \leq d$$ and $$q_2 \geq 1$$. Hence, for every monomial $$M$$ appearing in $$w$$ the terms of the complex

$$\overline{K}_M = \text{Cone}(M([\overline{A}_* \rightarrow \overline{B}_*]) \xrightarrow{Z_M} \mathcal{O}_S^{\Sigma M}[-1])$$

representing $$E_M$$, satisfy $$\text{Ext}_S^{>0}(\overline{K}_M, \mathcal{O}_S) = 0$$ for $$i \geq 2$$. This easily implies (using the standard spectral sequence) that the space of morphisms

$$\text{Hom}_{D(S)}(\overline{K}_M, \mathcal{O}_S[-1])$$

in the derived category is the same as in the homotopy category of complexes. Thus, replacing $$[A_j \rightarrow B_j]$$ with $$[\overline{A}_j \rightarrow \overline{B}_j]$$ we can realize the map $$\tau_M$$ by a chain map $$K_M \rightarrow \mathcal{O}_S[-1]$$.

**Step 3.** Let $$X$$ be the total space of the bundle $$A_1 \oplus \ldots \oplus A_n$$ over $$S$$ and let $$p : X \rightarrow S$$ be the projection. Note that there is a natural map $$Z : X \rightarrow A^\vee$$ induced by the morphisms $$Z_j : A_j \rightarrow \mathcal{O}_S^{\Sigma_j}$$ constructed in Step 2. Recall (see Section 3.2) that the group $$G(\Sigma) = \prod_{i=1}^r G_i(\gamma_i)$$ acts on the set of restricted rigidifications of a given $$\Gamma$$-spin structure in such a way that an element $$\lambda = (\lambda(i)) \in G(\Sigma)$$ changes the trivializations $$(e_j(i))$$ to $$(\lambda_j(i) \cdot e_j(i))$$, where $$\lambda_j(i)$$, for $$(p_i, j) \in \Sigma$$, are the components of $$\lambda(i) \in G_i(\gamma_i)$$. Suppose that a subgroup $$G_S \subset G(\Sigma)$$ acts on $$S$$, so that the map $$S \rightarrow S$$ is $$G_S$$-invariant and the action on fibers is induced by the above rescaling action of $$G(\Sigma)$$. Then all complexes $$[A_j \rightarrow B_j]$$ are $$G_S$$-equivariant, so we have an action of $$G_S$$ on $$X$$. Since the maps $$Z_j : R\pi_*(L_j) \rightarrow \mathcal{O}_S^{\Sigma_j}$$ were defined using the trivializations of $$L_j|_{\Sigma_j}$$ given by $$(e_j(i))$$, we have

$$\lambda^* Z_j = \lambda_j^{-1} \cdot Z_j$$
for $\lambda \in G_S$, where $\lambda_j = (\lambda_j(i))_{i \in \Sigma_j}$ acts diagonally on $O_{S_j}^{\Sigma_j}$. Hence, the map $Z : X \to A^\gamma$ satisfies
\[ Z \circ \lambda = \lambda^{-1} \cdot Z. \]
Thus, the map $(p, Z) : X \to S \times A^\gamma$ is $G_S$-equivariant, where $\lambda \in G_S \subset G(\gamma)$ acts on $S \times A^\gamma$ by
\[ \lambda \cdot (s, z) = (\lambda \cdot s, \lambda^{-1} \cdot z). \]
The natural action of $G_n^m$ on the fibers of $p : X \to S$ induces an action of the group $\Gamma_w$ (see Section 2.1) on $X$ such that the map $Z$ is $\Gamma_w$-equivariant. Now we will construct a $G_S \times \Gamma_w$-equivariant Koszul matrix factorization of the potential $-Z^*w_\gamma$ on $X$.

The complex $K_M = \text{Cone}(M([A_\bullet \to B_\bullet]) \xrightarrow{Z_M} O_S^{\Sigma M})[-1]$ has the following form
\[
M(A_\bullet) \xrightarrow{(Z_M, -\delta)} O_S^{\Sigma M} \oplus \bigoplus_{j=1}^n \partial_j M(A_\bullet) \otimes B_j \to \\
\bigoplus_{j<j'} (\partial_j \partial_j' M(A_\bullet) \otimes B_j \otimes B_{j'}) \oplus \bigoplus_j (\partial_j^2 M(A_\bullet) \otimes \bigwedge^2 B_j) \to \ldots
\]
where the first term is in degree 0 (here $\delta$ is the differential on the complex $M([A_\bullet \to B_\bullet])$).

Since the chain map $\tau_M$ is equal to $\text{Tr}$ on $O_S^{\Sigma M}[-1] \subset K_M$ (see diagram (4.14)), it corresponds to a map
\[ \alpha_M = (\alpha_{M,j}) : \bigoplus_{j=1}^n B_j \otimes \partial_j M(A_\bullet) \to O_X \]
such that the following diagram is commutative
\[
\begin{array}{ccc}
M(A_\bullet) & \xrightarrow{\delta} & \bigoplus_{j=1}^n \partial_j M(A_\bullet) \otimes B_j \to \\
\downarrow Z_M & & \downarrow \alpha_M \\
O_S^{\Sigma M} & \xrightarrow{\text{Tr}} & O_S
\end{array}
\]
(4.29)

Set
\[ \alpha'_M := (k_j \cdot \alpha_{M,j})_{j=1,\ldots,n}, \]
where $M = x_1^{k_1} \cdots x_n^{k_n}$. Let us view the differential
\[ \beta = \bigoplus_j \beta_j : \bigoplus_j A_j \to \bigoplus_j B_j \]
as a section of the bundle $p^*(\bigoplus_j B_j)$ on $X$ (linear along fibers). Similarly, we can view $\alpha'_M$ as a section of the bundle $p^*(\bigoplus_j B'_j)$ on $X$ (polynomial along fibers). Let $\langle \cdot, \cdot \rangle$ denote the natural pairing between $p^*(\bigoplus_j B_j)$ and $p^*(\bigoplus_j B_j)$.

**Lemma 4.2.6.** One has $\langle \alpha'_M, \beta \rangle = Z^*M^{\Sigma M}$. 

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Proof. The components of the differential $\delta$ have form

$$\delta(M(a_*))_j = \beta_j(a_j) \otimes \partial_j M(a_1, \ldots, a_n) \in B_j \otimes \partial_j M(A_*),$$

where $a_j \in A_j$, $j = 1, \ldots, n$. Hence, the composition $\alpha_M \circ \delta(M(a_*)) : M(A_*) \to O_S$ corresponds to the function $\langle \alpha'_M, \beta \rangle$ on $X$. On the other hand, the components of the map $Z_M$ correspond to the functions $Z^* M(x_1(i), \ldots, M(x_n(i)))$ (where $i \in \Sigma_M$). Thus, the assertion follows from the commutativity of the diagram (4.29).

Now we set

$$\alpha_w = \sum_M c_M \alpha'_M,$$

so that

$$\langle \alpha_w, \beta \rangle = \sum_M c_M Z^* M^\otimes \Sigma_M = Z^* w_\tau.$$

Hence, we have a Koszul matrix factorization $\{-\alpha_w, \beta\}$ of $-Z^* w_\tau$. We claim that it can be equipped with a $G_S \times \Gamma_w$-equivariant structure with respect to the character $\chi_w : \Gamma_w \to \mathbb{G}_m$ (and trivial on $G_S$). The bundles $\bigoplus A_j$ and $\bigoplus B_j$ are equipped with a $G_S \times \Gamma_w$-equivariant structure, in such a way that the action of $\Gamma_w$ is induced by the embedding $\Gamma_w \subset \mathbb{G}_m$. Then $\beta$ can be viewed as a $G_S \times \Gamma_w$-invariant section of $\bigoplus_j B_j$. On the other hand, $\alpha_w$ gives a $G_S \times \Gamma_w$-invariant section of $\chi_w \otimes \bigoplus_j B_j^\vee$. Thus, we obtain a $G_S \times \Gamma_w$-equivariant structure on the matrix factorization $\{-\alpha_w, \beta\}$. More explicitly, we have

$$\{-\alpha_w, \beta\}_0 = \bigoplus_i \bigwedge^{2i} (p^* \bigoplus_j B_j^\vee) \otimes \chi_w,$$

$$\{-\alpha_w, \beta\}_1 = \bigoplus_i \bigwedge^{2i+1} (p^* \bigoplus_j B_j^\vee) \otimes \chi_w,$$

where the differential is given by

$$\delta = \iota(\beta) - \alpha_w \wedge.$$

Step 4. Now we will show that the matrix factorization $\{-\alpha_w, \beta\}$ constructed in Step 3 is supported on the zero section in $X$. This will allow us to apply the push-forward functor (see Example 1.5.4) for the projection $(p, Z) : X \to S \times \mathbb{A}^\tau$. For the universal family of $\Gamma$-spin curves over the moduli space $S_{g,0}^{\text{rig}}$ we get an object

$$P_{g,0}^{\text{rig}}(\tau) := (p, Z)_* \{-\alpha_w, \beta\} \in \overline{DMF}_{G(\tau) \times \Gamma_w}(S_{g,0}^{\text{rig}}(\tau) \times \mathbb{A}^\tau, -w_\tau).$$

Recall that by Step 2 we have $A'_j = \ker(A_j \to O_{\Sigma_j}^\tau)$ and the complex $[A'_j \to B_j]$ represents $R\pi_* (L_j(-\Sigma_j))$, where $\beta'_j = \beta_j|_{A'_j}$. Thus, the total space $X_0$ of the vector bundle $A'_1 \oplus \ldots \oplus A'_n$ over $S$ coincides with the preimage of the origin $Z^{-1}(0) \subset X$. Since the critical locus of $w_\tau$ on $\mathbb{A}^\tau$ is the origin $0 \in \mathbb{A}^\tau$, it is enough to consider the zero locus of the restriction of $\{-\alpha_w, \beta\}$ to $X_0 = Z^{-1}(0)$ (by [50, Cor. 5.3]). Let $\alpha_0$ be the restriction of $\alpha_w$ to $X_0$. The equation (4.23) implies that $\alpha_0$ and $\beta' = (\beta'_j)$ are exactly the sections of the pull-backs of $\bigoplus_j B_j^\vee$ and $\bigoplus_j B_j$ obtained by the construction of section 3 applied to the collection of morphisms (4.11). It follows from Proposition 4.1.1 that the zero locus $Z(s_0)$ is exactly the
zero section $S \subset X_0 \subset X$, as claimed. Hence, by Lemma 1.4.1, \(\{-\alpha_w, \beta\}|_{X_0}\) is supported on the zero section.

Recall that we have a natural morphism

\[
S_g^{\text{rig}}(\gamma) \rightarrow S_g^{\text{rig},0}(\gamma)
\]

compatible with the homomorphism \(\prod_{i=1}^r G/\langle \gamma_i \rangle \rightarrow G(\gamma)\). Hence, by taking the pull-back of \(P_g^{\text{rig},0}(\gamma)\) we obtain a matrix factorization

\[
P_g^{\text{rig}}(\gamma) \in \text{DMF}_{G_w}(S_g^{\text{rig}}(\gamma) \times \mathbb{A}^r, -w\gamma)
\]

which is equivariant with respect to the action of \(\prod_{i=1}^r G/\langle \gamma_i \rangle\).

This finishes the construction of the fundamental matrix factorizations. In Section 4.3 we will show that it does not depend on the choices made (up to an isomorphism).

**Example 4.2.7.** In the case when \(\mathbb{A}^r = 0\) (and \(w\gamma = 0\)) we have \(S_g^{\text{rig},0}(\gamma) = S_g(\gamma)\) and the category \(\text{DMF}_{G_w}(S_g(\gamma))\) is (non-canonically) equivalent to the bounded derived category of \(G_w\)-equivariant coherent sheaves on \(S_g(\gamma)\). For example, for \(w = x^n\), \(G_w = \mathbb{Z}/n\) (and \(\Gamma_w = \mathbb{G}_m\)) this will be the case whenever all \(\gamma_i \in \mathbb{Z}/n\) are nontrivial. In this case the Chern class of \(P_g(\gamma)\) is closely related to the Witten’s virtual top Chern class on the moduli spaces of higher spin curves (see [23], [48] and [7]). To get the Witten’s virtual top Chern class one has to twist it with a certain Todd class (see (5.16) below).

### 4.3 Independence on choices

Here we will show that the isomorphism class of the fundamental matrix factorization \(P_g^{\text{rig},0}(\gamma)\) does not depend on the choices made in Step 2 when realizing the diagram (4.14) on the level of complexes.

First, since \(\text{Hom}_{D^b(S)}(K_M, \mathcal{O}_S[-1])\) can be computed in the homotopy category, all chain map \(K_M \rightarrow \mathcal{O}_S[-1]\) representing \(\tau_M\) are homotopic. A homotopy between two such maps is given by a \(G(\gamma) \times G_w\)-equivariant map

\[
h : \bigoplus_{j<j'} (\partial_j \partial_{j'} M(A_\bullet) \otimes B_j \otimes B_{j'}) \oplus \bigoplus_j (\partial_j^2 M(A_\bullet) \otimes \bigwedge^2 B_j) \rightarrow \mathcal{O}_S \otimes \chi_w.
\]

After dualization \(h\) can be viewed as a section of \(p^* \bigwedge^2 (\bigoplus_j B_j^\vee) \otimes \chi_w\). Now the operator \(\exp(-h)\wedge?\) induces a \(G(\gamma) \times G_w\)-equivariant isomorphism between the matrix factorizations associated with two homotopic choices of \(\tau_M\) (cf. [48, Prop. 4.2] or the proof of [49, Lem.2.5.5]).

To prove independence on the choice of presentations \(R\pi_*(L_j) = [A_j \rightarrow B_j]\), we will use the following property of Koszul matrix factorizations (which is analogous to the results of [48, sec. 3.2]).

**Proposition 4.3.1.** Let \(V\) be a vector bundle on a smooth FCDRP-stack \(X\), \(W \in H^0(X, L)\) a potential, and let \(\{\alpha, \beta\}\) be the Koszul matrix factorization associated with sections \(\alpha \in
$H^0(X, V^\vee \otimes L)$ and $\beta \in H^0(X, V)$ such that $(\alpha, \beta) = W$. Let $V_1 \subset V$ be a subbundle such that $\beta \mod V_1$ is a regular section of $V/V_1$. Assume that the zero locus $X' = Z(\beta \mod V_1)$ is smooth and consider the induced sections

$$\beta' = \beta|_{X'} \in H^0(X', V')$$

of the bundle $V' = V_1|_{X'}$ (note that the restriction $\beta|_{X'}$ belongs to $V_1|_{X'} \subset V|_{X'}$) and

$$\alpha' = \alpha \mod (V_1^\perp|_{X'})$$

of the bundle $L \otimes (V^\vee/V_1^\perp)|_{X'} \cong (V')^\vee$. Assume also that either $W|_{X'}$ is a non-zero-divisor or $W = 0$ and the zero loci $Z(\alpha, \beta)$ and $Z(\alpha', \beta')$ are proper. Then one has an isomorphism

$$\{\alpha, \beta\} \simeq i_*\{\alpha', \beta'\}
\tag{4.32}$$

in $\text{DMF}(X, W)$, where $i : X' \hookrightarrow X$ is the natural embedding.

**Proof.** We have a natural morphism

$$\bigwedge^i (V^\vee \otimes L^{1/2})(L^{-1/2}) \to i_*i^*\bigwedge^i (V^\vee \otimes L^{1/2})(L^{-1/2}) \to i_*\bigwedge^i ((V')^\vee \otimes L^{1/2})(L^{-1/2}),
\tag{4.33}$$

compatible with the differentials in $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$.

Assume first that $W|_{X'}$ is a non-zero-divisor. To show that the map (4.33) is an isomorphism in $\text{DMF}(X, W)$ we can argue locally. Thus, we can assume that $V = V_1 \oplus V_2$, so we can write $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ with $\alpha_i \in H^0(X, V_i^\vee \otimes L)$ and $\beta_i \in H^0(X, V_i)$, $i = 1, 2$. Our assumptions mean that $\beta_2$ is a regular section of $V_2$ and $X'$ is its zero locus. We have a natural isomorphism of matrix factorizations

$$\{\alpha, \beta\} \simeq \{\alpha_1, \beta_1\} \otimes \{\alpha_2, \beta_2\},$$

where $\{\alpha_i, \beta_i\}$ is a matrix factorization of $W_i := \langle \alpha_i, \beta_i \rangle$. Note that $W = W_1 + W_2$ and $W_2|_{X'} = 0$. Thus, we only need to show that the natural morphism

$$\{\alpha_1, \beta_1\} \otimes \{\alpha_2, \beta_2\} \to i_*\{i^*\alpha_1, i^*\beta_1\}$$

is an isomorphism in $\text{DMF}(X, W)$. But this follows from Proposition 1.6.3(i).

Next, consider the case $W = 0$. We use a deformation argument. Namely, let us consider a family of chain maps

$$f_t : \{t\alpha, \beta\} \to i_*\{t\alpha', \beta'\},$$

where $t \in \mathbb{A}^1$. We can view the complex $\mathcal{K} = \text{Cone}(f_t)$ as a complex of sheaves on $X \times \mathbb{A}^1$, flat over $\mathbb{A}^1$. Note that $f_0$ is quasi-isomorphism, since $\{0, \beta\}$ is the usual Koszul complex and $\beta$ is regular, hence $\mathcal{K}|_{X \times \{0\}}$ is acyclic. On the other hand, by Lemma 1.4.1, the cohomology of $\mathcal{K}|_{X \times (\mathbb{A}^1 \setminus \{0\})}$ is supported on $Z \times (\mathbb{A}^1 \setminus \{0\})$, where $Z = Z(\alpha, \beta) \cup Z(\alpha', \beta')$. By our assumption, $Z$ is proper, hence there exists an open neighborhood $U$ of $0 \in \mathbb{A}^1$ such that $\mathcal{K}|_{X \times U}$ is acyclic. In particular, there exists $t \neq 0$ such that $f_t$ is quasi-isomorphism. Using
an isomorphism between \( f_1 \) and \( f_t \) for \( t \neq 0 \) (given by multiplying by \( t^i \) on \( \wedge^i \)), we derive that \( f_1 \) is also a quasi-isomorphism.

Now we are ready to show that different choices of complexes of bundles \( K_j = [A_j \to B_j] \) realizing \( R\pi_*(L_j) \) and surjective maps \( K_j \to O_S^{[j]} \) realizing \( Z_j \), which we made in Step 2 realizing the diagram (4.14) at the level of complexes, lead to isomorphic matrix factorizations. It is enough to check this in the case when one of the complexes \( K_j \) is replaced by a complex \( \tilde{K}_j \) in degrees \([0,1]\) such that we have a quasi-isomorphism \( \tilde{K}_j \to K_j \) for which the composition \( \tilde{K}_j \to K_j \to O_S^{[j]} \) is still surjective. By Lemma 4.4 of [48], we can represent the map \( \tilde{K}_j \to K_j \) in the homotopy category as a composition of an embedding of complexes \( \tilde{K}_j \to K_j + U \) followed by the projection \( K_j + U \to K_j \), where \( U \) is of the form \( U = [F \to \text{id} \to F] \). Thus, it suffices to consider separately two cases: the case when \( \tilde{K}_j \to K_j \) is an embedding and the case when it is a projection onto a direct summand.

First, suppose we have a quasi-isomorphism \( [\tilde{A}_1 \to \tilde{B}_1] \to [A_1 \to B_1] \), such that \( \tilde{A}_1 \to A_1 \) and \( \tilde{B}_1 \to B_1 \) are embeddings of bundles. Let us denote \( C = A_1/A_1 = B_1/B_1 \) and let \( \tilde{Z}_1 : \tilde{A}_1 \to O_S^{[1]} \) be the restriction of \( Z_1 \) (recall that \( \tilde{Z}_1 \) is still surjective). Then for each monomial \( M \), the components

\[
\tilde{\alpha}_j : \partial_j M(\tilde{A}_1, A_2, \ldots, A_n) \to B_j^\vee,
\]

for \( j \neq 1 \), are obtained from \( \alpha_j \) by restriction to \( \tilde{A}_1 \), while to get \( \tilde{\alpha}_1 \) we also have to use the map \( B_1^\vee \to (\tilde{B}_1)^\vee \). In the new realization the space \( X \) is replaced by its subspace \( \tilde{X} \subset X \) which is the total space of \( \tilde{A}_1 + \bigoplus_{j>1} A_j \). Note that \( \tilde{X} \) is the zero locus of the regular section of \( p^* B_1/p^* B_1 = p^* C \) on \( X \), induced by \( \beta_1 \). Thus, we can apply Proposition 4.3.1 to the subbundle

\[
p^* \tilde{B}_1 + \bigoplus_{j>1} B_j \subset \bigoplus_j B_j
\]

to obtain an isomorphism

\[
\{\alpha, \beta\} \simeq i_* \{\tilde{\alpha}, \tilde{\beta}\},
\]

where \( i : \tilde{X} \hookrightarrow X \) is the embedding. Hence, the push-forwards of \( \{\alpha, \beta\} \) and \( \{\tilde{\alpha}, \tilde{\beta}\} \) to \( S \times \mathbb{A}^\mathbb{R} \) are isomorphic.

The case of the quasi-isomorphism of the type \( K_j + U \to K_j \) is similar because we can apply the above argument to the embedding \( K_j \to K_j + U \) which is also a quasi-isomorphism.

This finishes the proof of independence of the isomorphism class of the fundamental matrix factorization in the corresponding derived category on the choices made in Step 2 of the construction.
5 Cohomological field theories associated with a quasi-homogeneous isolated singularity

5.1 Construction of CohFTs

Let $R$ be a commutative $\mathbb{C}$-algebra and $\mathcal{H}$ a finitely generated $\mathbb{Z}/2$-graded projective $R$-module equipped with a perfect symmetric $R$-bilinear pairing $b : \mathcal{H} \otimes_R \mathcal{H} \to R$ (i.e., $b$ induces an isomorphism $\mathcal{H} \cong \text{Hom}_R(\mathcal{H}, R)$). Let $\Delta^R \in \mathcal{H} \otimes_R \mathcal{H}$ be the Casimir element corresponding to $b$. Recall (see [41, III.4]) that a complete Cohomological Field Theory (CohFT) on the state space $(\mathcal{H}, b)$ with coefficients in $R$ is a collection of even $R$-linear maps $\Lambda_{g,r} : \mathcal{H}^{\otimes_R n} \to H^*(\overline{M}_{g,r}) \otimes R$ (5.1) and a fixed element $1 \in \mathcal{H}$ (called flat unit) satisfying certain properties. Here the Casimir $\Delta^R$ is used to formulate the factorization properties and the insertion of the identity $1$ corresponds to forgetting a marked point.

An example of CohFT with coefficients is provided by the $G$-equivariant Gromov-Witten theory considered in [18], where $R = H^*_G(pt)$. Note that if we have a homomorphism $R \to R'$ and a CohFT with coefficients in $R$ then by extending scalars we can obtain a CohFT with coefficients in $R'$ and the state space $\mathcal{H} \otimes_R R'$.

Let us fix a quasihomogeneous polynomial $w(x_1, \ldots, x_n)$ with an isolated singularity. We assume that the degrees $d_j = \deg(x_j)$ are positive. Let $G \subset G_w$ be a finite subgroup in the group of diagonal symmetries of $w$, such that $G$ contains the exponential grading element $J$ (defined by (2.2)). Let $\Gamma \subset \Gamma_w$ be the subgroup associated with $G$ by Lemma 2.1.1, so that there is an exact sequence

$$1 \to G \to \Gamma \xrightarrow{\chi} \mathbb{G}_m \to 1.$$  (5.2)

For each $\gamma \in G$ let us set

$$\mathcal{H}_\gamma := HH_*(\text{MF}_\Gamma(w_\gamma)),$$

where $w_\gamma$ is the restriction of $w$ to the fixed point locus $(\mathbb{A}^n)^\gamma \subset \mathbb{A}^n$ (we consider the induced action of $\Gamma$ on $(\mathbb{A}^n)^\gamma$). Recall from Section 2.6 that $\mathcal{H}_\gamma$ has a natural $\widehat{G}$-action, so we can view it as a module over the ring $R = \mathbb{C}[\widehat{G}]$.

We will construct a CohFT with coefficients in $R$ and the state space

$$\mathcal{H} = \mathcal{H}(w, G) := \bigoplus_{\gamma \in G} \mathcal{H}_\gamma.$$  (5.3)

To define the corresponding $R$-linear maps

$$\Lambda_{g,\Gamma,\chi} = \Lambda_{g}^R(\overline{\gamma}) : \mathcal{H}_{\gamma_1} \otimes_R \cdots \otimes_R \mathcal{H}_{\gamma_n} \to H^*(\overline{M}_{g,r}, R) = H^*(\overline{M}_{g,r}) \otimes R$$  (5.4)

for $\overline{\gamma} = (\gamma_1, \ldots, \gamma_r) \in G^r$ (the components of the CohFT maps (5.1)) we will use the moduli spaces

$$S_{g}^{\text{rig}}(\overline{\gamma}) = S_{g,r,\Gamma,\chi}^{\text{rig}}(\overline{\gamma})$$
introduced in Section 3.2 and the fundamental matrix factorizations $P_g^{\text{rig}}(\gamma)$ (see (4.31)).

Retaining only the action of the subgroup $\Gamma \subset \Gamma_w$ on $P_g^{\text{rig}}(\gamma)$ we obtain an object

$$P_g^{\text{rig}}(\gamma) = DMF(\mathbb{A}^\gamma, -w),$$

which is also invariant under the action of $G^r$ (see Step 4 of the construction of Section 4.2). This object gives a functor

$$\Phi_g(\gamma) : DMF(\mathbb{A}^\gamma, w) \to D_G(S_g^{\text{rig}}(\gamma)) : E \mapsto (p_1)_*(p_2^*E \otimes P_g^{\text{rig}}(\gamma)), \quad (5.5)$$

where $p_1$ and $p_2$ are the projections from the product $S_g^{\text{rig}}(\gamma) \times \mathbb{A}^\gamma$ onto its factors. Here we use the push-forward functor (1.23). Note that the functor $\Phi_g(\gamma)$ is compatible with the action of $G^r$ on both categories.

For a stack $\mathcal{X}$ let us denote by $HH_*(\mathcal{X})$ the Hochschild homology of the dg-version of the derived category of coherent sheaves on $\mathcal{X}$. Note that for a morphism $f : \mathcal{X} \to \mathcal{X}'$ we have pull-back maps $f^* : HH_*(\mathcal{X}') \to HH_*(\mathcal{X})$ induced by the pull-back functor between the derived categories of coherent sheaves.

Recall that for a smooth projective variety $X$ one has the Hochschild-Konstant-Rosenberg isomorphism

$$I_{\text{H KR}} : HH_*(D(X)) \simeq H^*(X, \mathbb{C}),$$

sending $HH_*(D(X))$ to $\bigoplus_{q-p=i} H^{p,q}(X)$. Now let $\mathcal{X}$ be a DM-stack such that there exists a finite flat surjective morphism $\pi : X \to \mathcal{X}$ with $X$ smooth and projective variety. For example, this is the case for $\mathcal{M}_{g,n}$ (see [1]). For such $\mathcal{X}$ using the map on Hochschild homology induced by the pull-back functor $\pi^* : D(\mathcal{X}) \to D(X)$ we obtain a map

$$\alpha_\mathcal{X} : HH_*(D(\mathcal{X})) \to HH_*(D(X)) \xrightarrow{I_{\text{H KR}}} H^*(X, \mathbb{C}) \xrightarrow{\deg \pi} H^*(\mathcal{X}, \mathbb{C})$$

(5.6)

that we will call the HKR map. This map does not depend on a choice of a morphism $X \to \mathcal{X}$. Furthermore, these maps are compatible with pull-backs and push-forwards with respect to finite étale morphisms.

**Lemma 5.1.1.** Let $\mathcal{X}$ be a DM-stack such that there exists a finite flat surjective morphism $\pi : X \to \mathcal{X}$ with $X$ smooth and projective. For an object $F \in D(\mathcal{X})$ let $\text{ch}^{HH}(F) \in HH_*(D(\mathcal{X}))$ be the categorical Chern character (see Section 2.5) and let $\text{ch}^{\text{top}}(F) \in H^*(\mathcal{X}, \mathbb{C})$ be the usual topological Chern character. Then $\alpha_\mathcal{X}(\text{ch}^{HH}(F)) = \text{ch}^{\text{top}}(F)$.

**Proof.** First, note that by [6, Thm. 4.5], for any $G \in D(X)$ we have

$$I_{\text{H KR}}(\text{ch}^{HH}(G)) = \text{ch}^{\text{top}}(G) \in H^*(X, \mathbb{C}).$$

Applying this to $G = \pi^*F$ we obtain

$$I_{\text{H KR}}(\pi^* \text{ch}^{HH}(F)) = \text{ch}^{\text{top}}(\pi^*G) = \pi^* \text{ch}^{\text{top}}(F).$$

Hence,

$$\alpha_\mathcal{X}(\text{ch}^{HH}(F)) = \frac{1}{\deg \pi} \pi_* \pi^* \text{ch}^{\text{top}}(F) = \text{ch}^{\text{top}}(F).$$

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Recall that by Corollary 2.6.2, we have an isomorphism
\[ \iota(\gamma) : \mathcal{H}_\gamma \otimes_R \cdots \otimes_R \mathcal{H}_\gamma \cong \text{HH}_*(\text{MF}_\Gamma(A^\gamma, w^\gamma))^{G^\gamma} \subset \text{HH}_*(\text{MF}_\Gamma(A^\gamma, w^\gamma)). \]  
(5.7)

Let \[ \phi_g(\gamma) : \mathcal{H}_\gamma \otimes_R \cdots \otimes_R \mathcal{H}_\gamma \xrightarrow{\iota(\gamma)} \text{HH}_*(\text{MF}_\Gamma(A^\gamma, w^\gamma)) \xrightarrow{\Phi_g(\gamma) \ast} \text{HH}_*(\text{S}^{\text{rig}}_g(\gamma)) \otimes_R \to \]  
(5.8)
be the composition of this embedding, the map \(\Phi_g(\gamma)\) induced on the Hochschild homology spaces by the functor \(\Phi_g(\gamma)\) and the HKR map (5.6) for \(X = \text{S}^{\text{rig}}_g(\gamma)\).

Let \(\text{st} : \text{S}^{\text{rig}}_g(\gamma) \to \overline{M}_{g,r}\) be the projection. Let us consider the corresponding push-forward map
\[ \text{st}_* : H^*(\text{S}^{\text{rig}}_g(\gamma), \mathbb{C}) \otimes_R \to H^*(\overline{M}_{g,r}, \mathbb{C}) \otimes_R. \]

Now we are ready to define the CohFT on the state space \(\mathcal{H} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma\). The maps (5.4) are given by
\[ \Lambda_g^R(\gamma) = \frac{1}{\deg(\text{st}_g)} \cdot \text{st}_* \circ \phi_g(\gamma). \]  
(5.9)

Note that these maps are even with respect to the natural \(\mathbb{Z}_2\)-gradings on (5.3). For the exponential grading element \(J \in G\) we have \(\mathcal{H}_J = R\) since \(J\) acts on \(A^n\) without fixed points.

We take the element \(1 \in \mathcal{H}_J \subset \mathcal{H}\) to be the flat unit \(1\) of our CohFT. To define a metric on \(\mathcal{H}\) we consider the element \(\zeta_\bullet \in (\mathbb{C}^*)^n\) with components
\[ \zeta_j = \exp(\pi iq_j) \quad \text{for} \quad j = 1, \ldots, n, \]  
(5.10)
where \(q_j = d_j/d\) (see Section 2). Note that \((\zeta_\bullet)^2 = J\) and \(\chi(\zeta_\bullet) = -1\), so
\[ w(\zeta_\bullet x) = -w(x). \]  
(5.11)

We set
\[ (x, y) = \sum_{\gamma} ((\zeta_\bullet)_\ast x_\gamma, y_{\gamma^{-1}})^R_{w_\gamma}, \]  
(5.12)
where \((?, ?)^R_{w_\gamma}\) is the canonical \(R\)-valued bilinear form on \(\mathcal{H}_\gamma\) (see Section 2.7).

**Theorem 5.1.2.** Let \(w(x_1, \ldots, x_n)\) be a quasihomogeneous polynomial with isolated singularity and \(G \subset G_w\) a finite subgroup containing \(J\). The state space \(\mathcal{H} = \mathcal{H}(w, G)\), the metric (5.12), the flat unit \(1 \in \mathcal{H}_J \subset \mathcal{H}\) and the collection of maps \(\Lambda_g^R(\gamma)\) define the CohFT with coefficients in \(R = \mathbb{C}[[\hat{G}]]\).

In Sections 5.2, 5.3 and 5.4 we will check the factorization and the flat unit axioms for this CohFT. In Section 6.1 we will finish the proof of the theorem by verifying the remaining axiom relating the metric on \(\mathcal{H}\) and the maps \(\Lambda_g^R(\gamma, \gamma^{-1}, J)\) (for \(\gamma_1 \neq \gamma_2^{-1}\) the moduli space \(S_0(\gamma_1, \gamma_2, J)\) is empty). To do this we will show that the fundamental matrix factorization on
each component of $S_0^{\text{rig}}(\gamma, \gamma^{-1}, J)$ is obtained from the diagonal matrix factorization $\Delta_{\omega, \zeta}^{\text{st}}$ by the action of an element in $G \times G$.

Since the algebra $R$ is isomorphic to the direct sum of algebras $R \cong \bigoplus_{\gamma' \in G} \mathbb{C} \cdot e_{\gamma'}$, where $e_{\gamma'}$ are idempotents (2.28), our CohFT decomposes into a direct sum of CohFTs indexed by elements of $G$. By Theorem 2.6.1(ii),

$$e_{\gamma'} \cdot \mathcal{H}_{\gamma} = H(\mathbf{w}_{\gamma, \gamma'})^G,$$

where $\mathbf{w}_{\gamma, \gamma'}$ is the restriction of $\mathbf{w}$ to the subspace of $\{\gamma, \gamma'\}$-invariants. Thus, the sub-CohFT corresponding to an element $\gamma' \in G$ has the state space

$$\mathcal{H}(\mathbf{w}, G, \gamma') := \bigoplus_{\gamma \in G} H(\mathbf{w}_{\gamma, \gamma'})^G.$$  

(5.13)

and the components of the maps (5.1) given by

$$\Lambda_{g}^\gamma(\mathcal{V}) = \frac{1}{\deg(st_g)} \cdot (st_g)_* \pi_{\gamma'} \phi_g(\mathcal{V})|_{e_{\gamma'} \mathcal{H}_{\gamma_1} \otimes \cdots \otimes e_{\gamma'} \mathcal{H}_{\gamma_r}},$$  

(5.14)

where $\pi_{\gamma'} : R \to \mathbb{C}$ is the specialization homomorphism.

The CohFT corresponding to $\gamma' = 1$ can be twisted to produce a theory satisfying an analog of the concavity axiom from [14] (see section 5.5). Namely, we define twisted maps

$$\phi_{g}^{tw}(\mathcal{V}) : e_1 \mathcal{H}_{\gamma_1} \otimes \cdots \otimes e_1 \mathcal{H}_{\gamma_r} \to H^*(S_{g}^{\text{rig}}(\mathcal{V}))$$

for $\mathcal{V} \in G^r$ by

$$\phi_{g}^{tw}(\mathcal{V}) = \text{Td}(R \pi_*(\bigoplus_{j=1}^n L_{j}))^{-1} \cdot \pi_1 \phi_g(\mathcal{V})|_{e_1 \mathcal{H}_{\gamma_1} \otimes \cdots \otimes e_1 \mathcal{H}_{\gamma_r}},$$  

(5.15)

where $(L_{\bullet})$ is the universal $\mathbf{w}$-structure and Td is the Todd class, and we set

$$\lambda_g(\mathcal{V}) = \exp(\pi i \tilde{D}_g(\mathcal{V})) \cdot \frac{1}{\deg(st_g)} \cdot (st_g)_* \phi_{g}^{tw}(\mathcal{V}),$$  

(5.16)

where $\tilde{D}_g(\mathcal{V})$ is given by (3.20).

**Theorem 5.1.3.** Let $\mathbf{w}$ and $G$ be as in Theorem 5.1.2. The collection of maps $\lambda_g(\mathcal{V})$ defines a CohFT on the state space $\mathcal{H}(\mathbf{w}, G, 1)$ with the metric obtained by restricting the metric (5.12) and the flat unit element $1 \in H(\mathbf{w}_{1,1})^G$.

We will call the CohFT of this Theorem the *reduced CohFT* associated with $(\mathbf{w}, G)$.

The proof of this theorem will be given in Sections 5.2, 5.3, 5.4 and 6.1 simultaneously with the proof of Theorem 5.1.2.

Note that the state space of the reduced theory

$$\mathcal{H}(\mathbf{w}, G, 1) = \bigoplus_{\gamma \in G} H(\mathbf{w}_\gamma)^G$$

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can be identified with the state space of the CohFT constructed in [14, sec. 5.1]. Hypotheti-
cally, the theories themselves also match.

**Conjecture.** The reduced CohFT associated with \((w, G)\) is isomorphic to the FJR-theory for the same pair constructed in [14].

One of the obstacles on the way to proving this conjecture is that it is not clear how to verify the homogeneity property of the maps (5.16) in our setup in general. For some particular cases this homogeneity is proved in Section 5.6. In particular, it holds for all simple singularities. Combining this with the reconstruction theorem [14, Thm. 6.2.10] we will show in Section 7.6 that the above conjecture holds for them. In the case of the simple singularity of type \(A, w = x^n\), the conjecture also follows from the results of Chiodo [7, sec. 5] (which imply that in this case our definition of the reduced CohFT is compatible with the construction of [48]) together with the reconstruction results of [23] and [12] (see also [16]).

**Remark 5.1.4.** It would be interesting to find quantum K-theory versions of CohFT from Theorems 5.1.2 and 5.1.3 (see [19], [38]).

### 5.2 Behavior of fundamental matrix factorizations under gluing

Let \((\tilde{\pi}: \tilde{C} \to S, p_1, \ldots, p_r, p, q)\) be a family of stable orbicurves with \(r + 2\) marked orbipoints over a connected base \(S\). Also, let \(f: \tilde{C} \to C\) be a morphism of families of stable orbicurves over \(S\), where the family \(C\) has a node \(\sigma\) such that \(f\) is an isomorphism over \(C\setminus \sigma\) and \(f^{-1}(\sigma)\) is the union of \(p\) and \(q\). We will say that \(C\) is obtained from \(\tilde{C}\) by gluing points \(p\) and \(q\). The remaining marked points \(p_1, \ldots, p_r\) can be viewed as marked points on \(C\). Let \((\tilde{P}, \tilde{\varepsilon})\) be a \(\Gamma\)-spin structure of type \(\tilde{\gamma} = (\gamma_1, \ldots, \gamma_r, \gamma_p, \gamma_q)\) on \((\tilde{C}, p_1, \ldots, p_r, p, q)\) equipped with a rigidification \((e^{(1)}, \ldots, e^{(r)}, e^{(p)}, e^{(q)})\), where \(e^{(i)} \in \tilde{P}|_{p_i}, i = 1, \ldots, r, e^{(p)} \in \tilde{P}|_{p}\) and \(e^{(q)} \in \tilde{P}|_{q}\).

Note that \(\gamma_q = \gamma_p^{-1}\) since the orbicurve \(C\) is balanced at the node \(\sigma\).

Now we will define a glued \(\Gamma\)-spin structure \((P, \varepsilon)\) on \((C, p_1, \ldots, p_r)\) such that \(\tilde{P} \simeq f^*P\). Recall that there is a canonical isomorphism \(f^*\omega_{C/S} \simeq \omega_{\tilde{C}/S}(p + q)\), such that the following diagram is commutative

\[
\begin{array}{ccc}
(f^*\omega_{C/S})|_p & \rightarrow & \omega_{\tilde{C}/S}(p + q)|_p \\
\downarrow & & \downarrow \\
(f^*\omega_{C/S})|_q & \rightarrow & \omega_{\tilde{C}/S}(p + q)|_q
\end{array}
\]

Hence, we obtain an isomorphism \(f^*\omega_{C/S}^{\log} \simeq \omega_{\tilde{C}/S}^{\log}\) with a similar property (the residues are opposite). To get a \(\Gamma\)-spin structure on \(\tilde{C}\) we will use the element \(\zeta \in \Gamma \subset (\mathbb{C}^*)^n\) defined by (5.10) which satisfies

\[
\chi(\zeta) = -1.
\]
Consider the isomorphism
\[ u : \widetilde{P} |_p \overset{\sim}{\longrightarrow} \widetilde{P} |_q \] (5.18)
defined by
\[ e(q) = \zeta \cdot u(e(p)), \] (5.19)
and let \( P \) be the \( \Gamma \)-bundle on \( \mathcal{C} \) obtained from \( \widetilde{P} \) by gluing with respect to \( u \). Then the induced trivializations \( \chi_*(e(p)) \) and \( \chi_*(e(q)) \) of \( \mathbb{G}_m \)-torsors \( \chi_*(\widetilde{P}) |_p \) and \( \chi_*(\widetilde{P}) |_q \), respectively, satisfy
\[ \chi_*(e(q)) = -\chi_*(u)(\chi_*(e(p))). \]

Hence, by commutativity of the diagram (5.17), the isomorphism \( \tilde{\varepsilon} : \chi_* (\tilde{P}) \rightarrow P(\omega_{\mathcal{C}/S}^{\log}) \) descends to an isomorphism \( \varepsilon : \chi_* (P) \rightarrow P(\omega_{\mathcal{C}/S}^{\log}) \). Thus, we obtain a \( \Gamma \)-spin structure on \( (\mathcal{C}, p_1, \ldots, p_r) \). Furthermore, the trivializations \( e(i), i = 1, \ldots, r \) define a rigidification structure of \( (P, \varepsilon) \).

Now we are going to compare the matrix factorizations
\[ P \in \overline{\text{DMF}}_{\Gamma_w}(S \times (\mathbb{A}^n)^\top, -\mathbf{w}_\top) \quad \text{and} \quad \tilde{P} \in \overline{\text{DMF}}_{\Gamma_w}(S \times (\mathbb{A}^n)^\top, -\mathbf{w}_\top), \]
associated with the \( \Gamma \)-spin structures \( (P, \varepsilon) \) and \( (\tilde{P}, \tilde{\varepsilon}) \), respectively, by the construction of Section 4.2.

**Theorem 5.2.1.** One has isomorphisms
\[ P \simeq (\text{id}_S \times \text{pr}_{1,\ldots,r})_* (\text{id}_S \times \Delta^\xi \ast \tilde{P}) \] (5.20)
and
\[ P \simeq (\text{id}_S \times \text{pr}_{1,\ldots,r})_* (\tilde{P} \otimes (\text{pr}_p, \text{pr}_q)^* \Delta_{w,\top}^{st}) \] (5.21)
in \( \overline{\text{DMF}}_{\Gamma_w}(S \times (\mathbb{A}^n)^\top, -\mathbf{w}_\top) \), where \( \text{pr}_p^{1,\ldots,r} : (\mathbb{A}^n)^\top \times (\mathbb{A}^n)^\top \rightarrow (\mathbb{A}^n)^\top \) and \( \text{pr}_{1,\ldots,r} : (\mathbb{A}^n)^\top \rightarrow (\mathbb{A}^n)^\top \) are the coordinate projections,
\[ \Delta^\xi : (\mathbb{A}^n)^\top \rightarrow (\mathbb{A}^n)^\top \times (\mathbb{A}^n)^\top : x \mapsto (\zeta \cdot x, x) \]
is the shifted diagonal, and
\[ \Delta_{w,\top}^{st} := (\zeta \cdot \text{id})^* \Delta_{w,\top}^{st} \]
is the shifted diagonal matrix factorization.

We are going to prove Theorem 5.2.1 by going through the steps of the construction of Section 4.2 for both \( \Gamma \)-spin structures \( (P, \varepsilon) \) and \( (\tilde{P}, \tilde{\varepsilon}) \).

For \( j = 1, \ldots, n \), let
\[ \Sigma_j \subset p_1 + \ldots + p_r \subset \mathcal{C} \quad \text{and} \quad \widetilde{\Sigma}_j \subset p_1 + \ldots + p_r + p + q \subset \widetilde{\mathcal{C}} \]
be the subdivisors defined by (4.8) for the collections \( \top \) and \( \top = (\top, \gamma_p, \gamma_q) \), respectively.

For each monomial \( M \) in \( \mathbf{w} \) we set \( \Sigma_M = \cap_{x_j \mid M} \Sigma_j \) and \( \widetilde{\Sigma}_M = \cap_{x_j \mid M} \widetilde{\Sigma}_j \). Note that
\[ \Sigma_j = \widetilde{\Sigma}_j \cap \{p_1 + \ldots + p_r\} \quad \text{and} \quad \Sigma_M = \widetilde{\Sigma}_M \cap \{p_1 + \ldots + p_r\}. \]
Let $C$ and $\tilde{C}$ be the curves obtained from $\mathcal{C}$ and $\tilde{\mathcal{C}}$ by forgetting the orbifold structure at $p_1, \ldots, p_r$, let $\rho : \mathcal{C} \to C$, $\tilde{\rho} : \tilde{\mathcal{C}} \to \tilde{C}$ be the natural projections. We still denote by $f : \tilde{C} \to C$ the morphism induced by $f : \tilde{\mathcal{C}} \to \mathcal{C}$. Also, let $(\mathcal{L}_\bullet)$ (resp., $(\tilde{\mathcal{L}}_\bullet)$) be the line bundles associated with the $\Gamma$-bundle $P$ (resp., $\tilde{P}$), and let $L_j = \rho_* \mathcal{L}_j$ and $\tilde{L}_j = \tilde{\rho}_* \tilde{\mathcal{L}}_j$.

We have the natural exact sequence

$$0 \to \mathcal{O}_C \to f_* \mathcal{O}_{\tilde{C}} \xrightarrow{\delta} \sigma_* \mathcal{O}_S \to 0.$$  \hfill (5.22)

where the map $\delta$ is the difference of evaluations along $p$ and $q$. Tensoring this sequence with $L_j$ and using the projection formula we get an exact triangle

$$R\pi_*(L_j) \to R\tilde{\pi}_*(\tilde{L}_j) \to \pi_*(\sigma^* L_j) \to \ldots$$  \hfill (5.23)

The third term depends on the action of the local group at $\sigma$ on $\sigma^* L_j$. We have two cases:

**Case 1.** $\tilde{\Sigma}_j = \Sigma_j$ (i.e., $\gamma_p$ has a nontrivial $j$th component). Then the third term of the triangle (5.23) vanishes, so we have an isomorphism

$$R\pi_*(L_j) \simeq R\tilde{\pi}_*(\tilde{L}_j).$$  \hfill (5.24)

**Case 2.** $\tilde{\Sigma}_j = \Sigma_j + p + q$ (i.e., $\gamma_p$ has trivial $j$th component). Then we get an exact triangle

$$R\pi_*(L_j) \to R\tilde{\pi}_*(\tilde{L}_j) \xrightarrow{r_p - u_j \cdot r_q} (\tilde{L}_j)|_p \to \ldots,$$  \hfill (5.25)

where $r_p : R\tilde{\pi}_*(\tilde{L}_j) \to (\tilde{L}_j)|_p$ and $r_q : R\tilde{\pi}_*(\tilde{L}_j) \to (\tilde{L}_j)|_q$ are the restriction maps and $u_j : (\tilde{L}_j)|_p \to (\tilde{L}_j)|_q$ is the isomorphism induced by (5.18).

Let $Z_j : R\pi_*(L_j) \to \mathcal{O}_S^{\Sigma_j}$ and $\tilde{Z}_j : R\tilde{\pi}_* \tilde{L}_j \to \mathcal{O}_S^{\tilde{\Sigma}_j}$ be the maps (4.13) induced by the rigidifications of the $\Gamma$-spin structures $(P, \varepsilon)$ and $(\tilde{P}, \tilde{\varepsilon})$. In the second case we will also have to consider the maps

$$Z'_j : R\pi_*(L_j) \to R\tilde{\pi}_* \tilde{L}_j \xrightarrow{\tilde{Z}_j} \mathcal{O}_S^{\tilde{\Sigma}_j}.$$  \hfill (5.26)

Note that the components of this map corresponding to the points $p$ and $q$ satisfy

$$Z'_j(p) = \zeta_j \cdot Z'_j(q)$$  \hfill (5.27)

(this follows from (5.19)).

Now let $\mathcal{F}_M$ (resp., $\tilde{\mathcal{F}}_M$) be the coherent sheaf on $C_M$ (resp., $\tilde{C}_M$) defined by (4.16) for our $\Gamma$-spin structures over $\mathcal{C}$ and $\tilde{\mathcal{C}}$. By Step 1 of the construction of 4.2, we have canonical maps

$$\tau_M : E_M = R\pi_*(\mathcal{F}_M)^{\text{Sym}(M)} \to \mathcal{O}_S[-1] \quad \text{and} \quad \tilde{\tau}_M : \tilde{E}_M = R\tilde{\pi}_*(\tilde{\mathcal{F}}_M)^{\text{Sym}(M)} \to \mathcal{O}_S[-1].$$

We are going to establish a certain compatibility between these maps (see Lemma 5.2.2 below).
First, we need some preparations. Define a subbundle $\mathcal{O}^\tilde{\Sigma}_M(p,q) \subset \mathcal{O}^\tilde{\Sigma}_M$ by

$$
\mathcal{O}^\tilde{\Sigma}_M(p,q) = \begin{cases} 
\mathcal{O}^\tilde{\Sigma}_M, & \text{if } \tilde{\Sigma}_M = \Sigma_M, \\
\ker(\pi_p + \pi_q : \mathcal{O}^\tilde{\Sigma}_M \to \mathcal{O}_S), & \text{if } \tilde{\Sigma}_M = \Sigma_M + p + q,
\end{cases}
$$

where $\pi_p, \pi_q : \mathcal{O}^\tilde{\Sigma}_M \to \mathcal{O}_S$ are the projections corresponding to the marked points $p$ and $q$, respectively.

Suppose first that $\tilde{\Sigma}_M = \Sigma_M + p + q$. This happens exactly when for each $x_j|M$ the element $\gamma_p$ has trivial $j$th component. In this case the action of the local group at the node $\sigma$ on $\sigma^*M(L_1, \ldots, L_n) \simeq \mathcal{O}_\sigma$ is trivial. By (5.27), for every monomial $M$ in $\mathfrak{w}$ we have

$$M(Z'_\bullet)_q = -M(Z'_\bullet)_p,$$

so $M(Z'_\bullet)$ induces a morphism

$$Z'_M : M(R\pi_*(L_*)) \to \mathcal{O}^\tilde{\Sigma}_M(p,q).$$

Let us define a coherent sheaf $\mathcal{F}'_M$ on $\mathcal{C}^M$ from the exact sequence

$$0 \to \mathcal{F}'_M \to L'_M \to (\Delta_M)_*(\mathcal{O}_{\Sigma_M} \oplus \mathcal{O}_\sigma) \to 0$$

and set

$$E'_M = R\pi_*(\mathcal{F}'_M)^{\text{Sym}(M)},$$

where $\text{Sym}(M)$ is the product of symmetric groups (4.15). Then the above exact sequence gives rise to an exact triangle

$$E'_M \to M(R\pi_*(L_*)) \overset{Z'_M}{\to} \mathcal{O}^\tilde{\Sigma}_M(p,q) \to E'_M[1].$$

Now let $f^M : \tilde{\mathcal{C}}^M \to \mathcal{C}^M$ be the map induced by the morphism $f : \tilde{\mathcal{C}} \to \mathcal{C}$. We have a commutative diagram with exact rows

$$
\begin{array}{cccccccc}
0 & \to & \mathcal{F}_M & \overset{\epsilon}{\to} & L^M & \to & (\Delta_M)_*\mathcal{O}_{\Sigma_M} & \to & 0 \\
& & & \downarrow{id} & & & & \\
0 & \to & \mathcal{F}'_M & \to & L^M & \to & (\Delta_M)_*(\mathcal{O}_{\Sigma_M} \oplus \mathcal{O}_\sigma) & \to & 0 \\
& & & (id, k) & & & & \\
0 & \to & f^M_*\mathcal{F}_M & \to & \Delta_*\mathcal{O}_{\Sigma_M}^\Sigma_2 & \to & 0
\end{array}
$$

(5.28)
where \( k : \mathcal{O}_\sigma \to \mathcal{O}_S^\sigma : x \mapsto (x, -x) \). It induces a commutative diagram in \( D^b(S) \) whose rows are exact triangles

where the map \( \text{pr}_{\Sigma_M} \) is induced by the natural projection \( \mathcal{O}_S^{\Sigma_M} \to \mathcal{O}_S^{\Sigma_M} \) and \( Z_M \) is defined similarly to \( Z_M \) for the data associated with \( \tilde{C} \).

In the case \( \Sigma_M = \Sigma_M \) we still have the commutative diagram (5.29) with \( E'_M = E_M \) and the morphism from the second row to the first being identity.

**Lemma 5.2.2.** The diagram

\[
\begin{array}{ccc}
E'_M & \xrightarrow{f} & E_M \\
\downarrow g & & \downarrow \tau_M \\
\tilde{E}_M & \xrightarrow{\tilde{\tau}_M} & \mathcal{O}_S[-1]
\end{array}
\]

is commutative.

**Proof.** Recall (see [22]) that there is a natural map

\[ \text{Tr}_f : f_* \omega_{\tilde{C}/S} \to \omega_{C/S} \]

such that the composition

\[ f_* \omega_{\tilde{C}/S} \xrightarrow{\text{Tr}_f} \omega_{C/S} \to f_* f^* \omega_{C/S} \approx f_*(\omega_{\tilde{C}/S}(p + q)) \]

is the natural map induced by the embedding \( \omega_{\tilde{C}/S} \to \omega_{\tilde{C}/S}(p + q) \), while the composition

\[ R\pi_*(f_* \omega_{\tilde{C}/S}) \xrightarrow{\text{Tr}_f} R\pi_*(\omega_{C/S}) \xrightarrow{\text{Tr}_{C/S}} \mathcal{O}_S[-1] \]

can be identified with \( \text{Tr}_{\tilde{C}/S} \). By definition of the maps \( \tau_M \) and \( \tilde{\tau}_M \), to show that diagram \( 5.30 \) is commutative.
(5.30) is commutative it is sufficient to check commutativity of the diagram

This follows from the commutativity of the diagram

(see (5.28)) since the natural morphism of sheaves

is injective.

As in Step 2 of the construction of the fundamental matrix factorization in Section 4.2, we choose for each $j$ a $\pi$-acyclic resolution

However, we will modify these resolutions using the map $f : \tilde{C} \to C$.

In Case 1, when $\tilde{\Sigma}_j = \Sigma_j$, the pull-back of (5.31) by $f$ gives a $\tilde{\pi}$-acyclic resolution

and a compatible resolution of $\tilde{L}_j(-\tilde{\Sigma}_j)$. In Case 2, when $\tilde{\Sigma}_j = \Sigma_j + p + q$, the pull-back of (5.31) gives a $\tilde{\pi}$-acyclic resolution

and a compatible resolution of $\tilde{L}_j(-\tilde{\Sigma}_j)$. In both cases pushing forward by $f$ we get a two-term resolution for $f_*\tilde{L}_j$ such that both the difference map $f_*\tilde{L}_j \to \tilde{L}_j|_{\Sigma}$ and the map $f_*\tilde{L}_j \to L_j|_{\Sigma}$ factor through this resolution. This leads to a new $\pi$-acyclic resolution for $L_j$ of the form

(5.32)
and a compatible resolution for \( L_j(-\Sigma_j) \), such that the map \( L_j \rightarrow L_j|_{\Sigma_j} \) is induced by a surjective map from the resolution in \((5.32)\). Pushing forward these resolutions to \( S \) we obtain complexes

\[
[A_j' \rightarrow B_j] \subset [A_j' \rightarrow B_j] \subset [A_j \rightarrow B_j] \subset [\tilde{A}_j \rightarrow B_j] \tag{5.33}
\]

representing \( R\pi^*_s(L_j(-\Sigma_j)) \), \( R\pi^*_s(L_j(-\Sigma_j)) \), \( R\pi^*_s(L_j) \) and \( R\pi^*_s\tilde{L}_j \), respectively. Note that we have an exact triple

\[
0 \rightarrow \tilde{A}_j \rightarrow \tilde{A}_j \xrightarrow{\tilde{Z}_j} \mathcal{O}^\Sigma_{S'} \rightarrow 0
\]

and a similar exact triple relating \( A_j' \) and \( A_j \). In the case \( \tilde{\Sigma}_j = \Sigma_j \) we have an equality \( A_j = \tilde{A}_j \), while in the case \( \Sigma_j = \Sigma_j + p + q \) we have an exact triple of complexes

\[
0 \rightarrow [A_j \rightarrow B_j] \xrightarrow{\delta_j(p,q)} [\tilde{A}_j \rightarrow B_j] \xrightarrow{\mathcal{O}_S} 0
\]

representing the exact triangle \((5.25)\), where the morphism \( \delta_j(p,q) : \tilde{A}_j \rightarrow \mathcal{O}_S \) is given by

\[
\delta_j(p,q) = \tilde{Z}_j(p) - \zeta_j \cdot \tilde{Z}_j(q).
\]

(here \( \tilde{Z}_j(p) \) and \( \tilde{Z}_j(q) \) are the components of \( \tilde{Z}_j \)). The maps \( Z_j' : A_j \rightarrow \mathcal{O}^\Sigma_{S'} \) realizing \((5.26)\) are defined as compositions

\[
Z_j' : A_j \rightarrow \tilde{A}_j \xrightarrow{\tilde{Z}_j} \mathcal{O}^\Sigma_{S'}
\]

while \( Z_j : A_j \rightarrow \mathcal{O}^\Sigma_{S'} \) are obtained from them by projecting to \( \mathcal{O}^\Sigma_{S'} \).

Note that all four resolutions \((5.33)\) have the same second term \( B_j \). Thus, as in Step 2 we can assume that \( B_j = V^\vee(-m_0)^{\otimes N_j} \) for large enough \( m_0 \), so that

\[
\operatorname{Ext}^0_S((\oplus_j \tilde{A}_j)^{\otimes q_1} \otimes (V^\vee)^{\otimes q_2}(-m), \mathcal{O}_S) = \operatorname{Ext}^0_S((\oplus_j A_j)^{\otimes q_1} \otimes (V^\vee)^{\otimes q_2}(-m), \mathcal{O}_S) = 0
\]

for \( m \geq m_0 \) and \( q_1 + q_2 \leq d \). Hence, we can assume that

\[
\operatorname{Ext}^0_S((\oplus_j \tilde{A}_j)^{\otimes q_1} \otimes (\oplus_j B_j)^{\otimes q_2}, \mathcal{O}_S) = \operatorname{Ext}^0_S((\oplus_j A_j)^{\otimes q_1} \otimes (\oplus_j B_j)^{\otimes q_2}, \mathcal{O}_S) = 0
\]

for \( q_1 + q_2 \leq d \) and \( q_2 \geq 1 \).

As in Step 2 we realize \( E_M, \tilde{E}_M \) and \( E'_M \) by the complexes

\[
K_M = \operatorname{Cone}(M([A_\bullet \rightarrow B_\bullet]) \xrightarrow{Z_M} \mathcal{O}^\Sigma_{S'})[-1], \quad \tilde{K}_M = \operatorname{Cone}(M([\tilde{A}_\bullet \rightarrow B_\bullet]) \xrightarrow{\tilde{Z}_M} \mathcal{O}^\Sigma_{S'})[-1], \quad \text{and}
\]

\[
K'_M = \operatorname{Cone}(M([A_\bullet \rightarrow B_\bullet]) \xrightarrow{Z'_M} \mathcal{O}^\Sigma_{S'}(p,q))[1], \quad \text{respectively. Then for all } M \text{ appearing in } \mathbf{w} \text{ we will have}
\]

\[
\operatorname{Ext}^i_S(\tilde{K}_M, \mathcal{O}_S) = \operatorname{Ext}^i_S(K'_M, \mathcal{O}_S) = \operatorname{Ext}^i_S(K_M, \mathcal{O}_S) = 0 \quad \text{for } i \geq 2. \tag{5.34}
\]

As before we consider the spaces

\[
X := \operatorname{tot}(A_1 \oplus \ldots \oplus A_n) \subset \tilde{X} := \operatorname{tot}(<\tilde{A}_1 \oplus \ldots \oplus \tilde{A}_n>
\]

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over $S$. We have the following diagram with a cartesian square

$$
\begin{array}{ccc}
X & \overset{q}{\longrightarrow} & \prod_{i=1}^{r} (A^n_i)^{\gamma_i} \times (A^n_p)^{\gamma_p} \\
\downarrow & & \downarrow \text{id} \times \Delta \zeta \\
\tilde{X} & \overset{\tilde{Z}}{\longrightarrow} & \prod_{i=1}^{r} (A^n_i)^{\gamma_i} \times (A^n_p)^{\gamma_p} \times (A^n_q)^{\gamma_q}
\end{array}
$$

(5.35)

where the maps $\tilde{Z}$ and $q$ are given by $\tilde{Z}_1, \ldots, \tilde{Z}_n$ and $Z'_1, \ldots, Z'_n$, respectively, and the composition of two horizontal arrows in the first row is equal to $Z$.

**Proof of Theorem 5.2.1.** Isomorphism (5.21) follows from (5.20) by Proposition 2.4.1(ii), so we only need to prove (5.20).

As in Step 2 of the construction, the vanishing (5.34) can be used to choose for each monomial $M$ chain maps representing $\tau_M$ and $\tilde{\tau}_M$. Also, the morphisms in derived category from $K'_M$ to $\mathcal{O}_S[-1]$ can be calculated in the homotopy category. Hence, we obtain a realization of the commutative diagram (5.30) in the homotopy category. Since the complex $K'_M$ differs from $K_M$ only in terms of degree 0 and 1, we can replace $\tau_M$ by a homotopic chain map, so that the diagram (5.30) will be commutative in the category of complexes.

Let $\tilde{p} : \tilde{X} \rightarrow S$ and $p : X \rightarrow S$ be the projections. Recall that we have a section $\tilde{\beta}$ (resp., $\beta$) of $\tilde{p}^*(\bigoplus_j B_j)$ (resp., $p^*(\bigoplus_j B_j)$), corresponding to the differential $\bigoplus_j \tilde{A}_j \rightarrow \bigoplus_j B_j$ (resp., $\bigoplus_j A_j \rightarrow \bigoplus_j B_j$), and $\beta$ is equal to the restriction of $\tilde{\beta}$ to $X$ by construction. As in Step 3 of the construction, from the chain map $\tilde{\tau}_M$ (resp., $\tau_M$) we get a section $\tilde{\alpha}_M$ (resp., $\alpha_M$) of the bundle $\tilde{p}^*(\bigoplus_j B'_j)$ (resp., $p^*(\bigoplus_j B'_j)$) satisfying

$$\langle \tilde{\alpha}_M, \tilde{\beta} \rangle = \tilde{Z}^* M^{\oplus \Sigma_M}$$

(resp., $\langle \alpha_M, \beta \rangle = Z^* M^{\oplus \Sigma_M}$).

The components of degree 1 of the diagram (5.30) form the following commutative square

$$
\begin{array}{ccc}
\mathcal{O}_S^\Sigma (p, q) \oplus \bigoplus_j \partial_j M(A_\bullet) \otimes B_j & \longrightarrow & \mathcal{O}_S^\Sigma \oplus \bigoplus_j \partial_j M(A_\bullet) \otimes B_j \\
\downarrow & & \downarrow (\text{Tr}, \alpha_M) \\
\mathcal{O}_S^{\tilde{\Sigma}} \oplus \bigoplus_j \partial_j M(\tilde{A}_\bullet) \otimes B_j & \longrightarrow & \mathcal{O}_S^\Sigma \oplus \bigoplus_j \partial_j M(A_\bullet) \otimes B_j \\
\downarrow (\text{Tr}, \tilde{\alpha}_M) & & \\
\mathcal{O}_S & & \mathcal{O}_S
\end{array}
$$

This implies that the restriction of $\tilde{\alpha}_M$ to $X$ is equal to $\alpha_M$. Therefore, the matrix factorization $\{-\alpha_w, \beta\}$ on $X$ is isomorphic to the restriction of $\{-\tilde{\alpha}_w, \tilde{\beta}\}$. As in Step 4, we have

$$\tilde{P} = (\tilde{p}, \tilde{Z})_* \{-\tilde{\alpha}_w, \tilde{\beta}\} \quad \text{and} \quad P = (p, Z)_* \{-\alpha_w, \beta\}.$$
Since $Z$ is the composition of arrows in the first row of the diagram (5.35), we have
\[
P \simeq (\text{id}_S \times \text{pr}_1, \ldots, r_1 \times \{p, q\}_* \{\alpha, \beta\}).
\]

On the other hand, by the base change formula in the cartesian square
\[
\begin{array}{ccc}
X & \xrightarrow{(p, q)} & S \times \prod_{i=1}^r (\mathbb{A}^n)^{\gamma_i} \times (\mathbb{A}^n)^{\gamma_p} \\
\downarrow & & \downarrow \text{id}_S \times \text{id} \times \Delta^\gamma \\
\tilde{X} & \xrightarrow{(\tilde{p}, \tilde{Z})} & S \times \prod_{i=1}^r (\mathbb{A}^n)^{\gamma_i} \times (\mathbb{A}^n)^{\gamma_p} \times (\mathbb{A}^n)^{\gamma_q}
\end{array}
\]
we have
\[
(p, q)_* \{\alpha, \beta\} \simeq (\text{id}_S \times \text{id} \times \Delta^\gamma)^* \tilde{P},
\]
which implies the required isomorphism (5.20).

\section{5.3 Verification of the factorization axiom of CohFT}

The main axiom of CohFT describes the factorization property of the maps $\Lambda_g(\gamma)$ under the gluing morphisms
\[
\rho_{\text{tree}} : \overline{M}_{g_1, r_1+1} \times \overline{M}_{g_2, r_2+1} \to \overline{M}_{g, r} \quad \text{and} \quad \rho_{\text{loop}} : \overline{M}_{g-1, r+2} \to \overline{M}_{g, r},
\]
where $g_1 + g_2 = g$ and $r_1 + r_2 = r$ (see [34, 2.2.6, 2.2.7]). Here we will verify this axiom for the maps (5.9) of our CohFT on the state space $\mathcal{H} = \mathcal{H}(w, G)$ (see (5.3)) and also for the twisted maps (5.16) on the state space $\mathcal{H}(w, G, 1)$. Specifically, in this section we prove the equalities
\[
(\rho_{\text{tree}})^* \circ \Lambda_g(\gamma_1, \ldots, \gamma_r) = \sum_{\gamma \in G} (\Lambda_{g_1}(\gamma_1, \ldots, \gamma_r, \gamma) \otimes \Lambda_{g_2}(\gamma_1', \ldots, \gamma_r', \gamma^{-1})) \circ (\text{id}^\otimes \otimes T_{w, \zeta}). \quad (5.36)
\]
\[
(\rho_{\text{loop}})^* \circ \Lambda_g(\gamma_1, \ldots, \gamma_r) = \sum_{\gamma \in G} (\Lambda_{g-1}(\gamma_1, \ldots, \gamma_r, \gamma, \gamma^{-1})) \circ (\text{id}^\otimes \otimes T_{w, \zeta}), \quad (5.37)
\]
where $\gamma_i$ are elements of $G$ and
\[
T_{w, \zeta} = \frac{1}{|G|} \sum_{h \in G} (\text{id} \times h)^* \text{ch}(\Delta_{w, \zeta}^\text{st}) \in HH_* (\text{MF}_G(A^n \times A^n, w \oplus w))^{G \times G} \simeq HH_* (\text{MF}_G(w)) \otimes_R HH_* (\text{MF}_G(w)) \quad (5.38)
\]
(the last identification follows from Corollary 2.6.2). The proof of the axiom will be finished once we check that $(T_{w, \zeta})_{\gamma \in G}$ are exactly the components of the Casimir element for the metric on $\mathcal{H}$. This will be done in Lemma 6.1.1 below.
To verify (5.36) we first observe that due to condition (3.14) the only potentially nontrivial summand in the right-hand side of (5.36) corresponds to
\[ \gamma = (\gamma_1 \cdot \ldots \cdot \gamma_{r_1})^{-1} J^{2q_1 - 1 + r_1} = \gamma'_1 \cdot \ldots \cdot \gamma'_{r_2} J^{-2q_2 + 1 - r_2}. \]
We have the following commutative diagram involving the moduli spaces of \( \overline{w} \)-structures with rigidifications:
\[
\begin{array}{ccc}
S^\text{rig}_{g_1}(\gamma_1, \ldots, \gamma_{r_1}, \gamma) \times S^\text{rig}_{g_2}(\gamma'_1, \ldots, \gamma'_{r_2}, \gamma^{-1}) & \xrightarrow{\rho^\text{rig}_{\text{tree}, \gamma}} & S^\text{rig}_g(\gamma_1, \ldots, \gamma_{r_1}; \gamma'_1, \ldots, \gamma'_{r_2}) \\
\text{st}_{g_1} \times \text{st}_{g_2} & \downarrow & \text{st}_g \\
\overline{\mathcal{M}}_{g_1, r_1 + 1} \times \overline{\mathcal{M}}_{g_2, r_2 + 1} & \xrightarrow{\rho_{\text{tree}}} & \overline{\mathcal{M}}_{g, r} \\
\end{array}
\]  
(5.39)
where the maps \( \text{st}_g \), \( \text{st}_{g_1} \), and \( \text{st}_{g_2} \) are the natural projections. The map \( \rho^\text{rig}_{\text{tree}, \gamma} \) is given by the gluing construction described in the beginning of Section 5.2. Theorem 5.2.1 applied to
\[ S = S^\text{rig}_{g_1}(\gamma_1, \ldots, \gamma_{r_1}, \gamma) \times \delta^\text{rig}_{g_2}(\gamma'_1, \ldots, \gamma'_{r_2}, \gamma^{-1}) \]
gives the following relation between the fundamental matrix factorizations:
\[
(\rho^\text{rig}_{\text{tree}, \gamma} \times \text{id})^* P^\text{rig}_{g_1, g_2, \Gamma}(\gamma_1, \ldots, \gamma_{r_1}; \gamma'_1, \ldots, \gamma'_{r_2}) \simeq
(\text{id}_S \times \text{pr}_{1, \ldots, r})_*(\pi_1^* P^\text{rig}_{g_1, g_2, \Gamma}(\gamma_1, \ldots, \gamma_{r_1}, \gamma) \otimes \pi_2^* P^\text{rig}_{g_2, \Gamma}(\gamma'_1, \ldots, \gamma'_{r_2}, \gamma^{-1}) \otimes \text{pr}_{p,q}^* \Delta^\text{rig}_{\text{w}, \zeta}),
\]
(5.40)
where \( \pi_1 \) and \( \pi_2 \) are the projections of \( S \times \prod_{i=1}^{r_1} (A^n)_{\gamma_i^+} \times \prod_{i=1}^{r_2} (A^n)_{\gamma_i^+} \times (A^n)_\gamma \times (A^n)^{-1} \) to \( S \times \prod_{i=1}^{r_1} (A^n)_{\gamma_i^+} \times (A^n)_\gamma \times (A^n)^{-1} \) and \( S \times \prod_{i=1}^{r_2} (A^n)_{\gamma_i^+} \times (A^n)^{-1} \), respectively, and \( \text{pr}_{p,q} \) is the projection to \( (A^n)_\gamma \times (A^n)^{-1} \).

The functor
\[
\Phi : \text{DMF}_G(A^n, \overline{\mathcal{M}}_{g, r}, w_{\gamma} \oplus w_{\overline{\gamma}}) \to D_G(S)
\]
given by the kernel in the left-hand side of (5.40), where \( \gamma = (\gamma_1, \ldots, \gamma_{r_1}) \) and \( \overline{\gamma} = (\gamma'_1, \ldots, \gamma'_{r_2}) \), is isomorphic to the composition
\[
\begin{array}{c}
\text{DMF}_G(A^n, \overline{\mathcal{M}}_{g, r}, w_{\gamma} \oplus w_{\overline{\gamma}}) \xrightarrow{\Phi_g(\gamma, \overline{\gamma})} D_G(\delta^\text{rig}_g(\gamma, \overline{\gamma})) \xrightarrow{(\rho^\text{rig}_{\text{tree}, \gamma})^*} D_G(S).
\end{array}
\]
After passing to Hochschild homology and using the HKR map (5.6) we obtain that the map induced by \( \Phi \)
\[
\phi : \bigotimes_{k=1}^{r_1} \mathcal{H}_{n_k} \otimes \bigotimes_{l=1}^{r_2} \mathcal{H}_{n_l} \xrightarrow{\iota} HH_*(\text{MF}_G(A^n, \overline{\mathcal{M}}_{g, r}, w_{\gamma} \oplus w_{\overline{\gamma}})) \xrightarrow{\Phi_*} H^*(S, \mathbb{C}) \otimes R
\]
(where all tensor products are taken over \( R \) and \( \iota = \iota(\gamma, \overline{\gamma}) \) is the map (5.7)) is equal to the composition \( (\rho^\text{rig}_{\text{tree}, \gamma})^* \circ \phi_g(\gamma, \overline{\gamma}) \), where \( \phi_g(\gamma, \overline{\gamma}) \) is the map (5.8). On the other hand, let
\[
\phi' : \bigotimes_{k=1}^{r_1} \mathcal{H}_{n_k} \otimes \bigotimes_{l=1}^{r_2} \mathcal{H}_{n_l} \xrightarrow{\iota} HH_*(\text{MF}_G(A^n, \overline{\mathcal{M}}_{g, r}, w_{\gamma} \oplus w_{\overline{\gamma}})) \xrightarrow{\Phi'_*} H^*(S, \mathbb{C}) \otimes R
\]
be the composition of the HKR map with the map induced on Hochschild homology by the functor $\Phi'$ given by the kernel in the right-hand side of (5.40). Using the projection formula for $\text{id}_S \times pr_1, \ldots, r$ (see Proposition 1.5.5) we see that $\Phi'$ is equal to the composition

$$HH_*(MF_!(\mathbb{A}^T \times \mathbb{A}^T, w_\gamma \oplus w_\gamma)) \otimes pr^*_{\gamma_1, \ldots, \gamma_r} ch(\Delta_{w_\gamma, \zeta})$$

$$HH_*(MF_!(\mathbb{A}^T \times \mathbb{A}^T \times (\mathbb{A}^n)^\gamma \times (\mathbb{A}^n)^{\gamma^{-1}}, w_\gamma \oplus w_\gamma \oplus w_\gamma \oplus w_{\gamma^{-1}})) \otimes \tilde{\Phi}_* H^*(S, \mathbb{C}) \otimes \mathbb{R},$$

where $\tilde{\Phi}_*$ is induced by the functor $\tilde{\Phi}$ associated with the kernel $\pi_{1,1}^* P_{g_1, r}^\text{rig}((\gamma_1, \ldots, \gamma_{r_1}, \gamma) \otimes \pi_2^* P_{g_2, r}^\text{rig}(\gamma_1, \ldots, \gamma_{r_2}, \gamma^{-1})$. Since $\tilde{\Phi}_*$ is invariant with respect to the action of $G \times G$ on the factors $(\mathbb{A}^n)^\gamma \times (\mathbb{A}^n)^{\gamma^{-1}}$, we can replace $\text{ch}(\Delta_{w_\gamma, \zeta})$ by its $G \times G$-averaging $T_{w_\gamma, \zeta}$ in the above formula for $\Phi'$. Since $\iota$ is exactly the embedding of $G^{r_1 + r_2}$-invariants (see Corollary 2.6.2), we obtain that $\phi$ is equal to the composition

$$\bigotimes_{l=1}^{r_1} \mathcal{H}_{\gamma_l} \bigotimes_{l=1}^{r_2} \mathcal{H}_{\gamma^l} \otimes \text{id}_{\mathbb{R}^T \otimes T_{w_\gamma, \zeta}} \bigotimes_{l=1}^{r_1} \mathcal{H}_{\gamma_l} \bigotimes_{l=1}^{r_2} \mathcal{H}_{\gamma^l} \otimes \mathcal{R} \mathcal{H}_{\gamma^{-1}} \otimes \phi_{g_1} \otimes \phi_{g_2} H^*(S, \mathbb{C}) \otimes \mathbb{R},$$

where $\phi_{g_1} = \phi_{g_1}(\gamma_1, \ldots, \gamma_{r_1}, \gamma)$ and $\phi_{g_2} = \phi_{g_2}(\gamma_1', \ldots, \gamma_{r_2}', \gamma^{-1})$. Since the functors $\Phi$ and $\Phi'$ are isomorphic, we have

$$\phi = \phi'.$$

(5.41)

The desired formula (5.36) is obtained from this by applying $(\text{st}_{g_1} \times \text{st}_{g_2})_*$ taking into account the relation

$$\frac{1}{\text{deg}(\text{st}_g)} \rho^*_{\text{tree}}(\text{st}_g)_* = \sum_{\gamma \in G} \frac{1}{\text{deg}(\text{st}_{g_1}) \text{deg}(\text{st}_{g_2})} (\text{st}_{g_1} \times \text{st}_{g_2})_*(\rho^\text{rig}_{\text{tree}, \gamma}),$$

(5.42)

which holds because the space in the left upper corner of diagram (5.39) is an étale covering of the fibered product of $\rho_{\text{tree}}$ and $\text{st}_g$.

The proof of (5.37) is analogous: one has to apply Theorem 5.2.1 to compare matrix factorizations over

$$S = \mathcal{S}^\text{rig}_{g^{-1}}(\gamma_1, \ldots, \gamma_{r}, \gamma, \gamma^{-1}).$$

Also, one has to replace (5.39) with the commutative diagram

$$\begin{array}{c}
\mathbb{M}_{g-1,r+2} \rightarrow \mathbb{M}_{g,r} \\
\text{st}_{g-1,\gamma} \downarrow \quad \rho_{\text{loop}} \downarrow \\
\mathbb{S}^\text{rig}_{g^{-1}}(\gamma_1, \ldots, \gamma_{r}, \gamma, \gamma^{-1}) \rightarrow \mathcal{S}^\text{rig}_{g}(\gamma_1, \ldots, \gamma_{r})
\end{array}$$

and use the corresponding equation

$$\frac{1}{\text{deg}(\text{st}_g)} \rho^*_{\text{loop}}(\text{st}_g)_* = \sum_{\gamma \in G} \frac{1}{\text{deg}(\text{st}_{g^{-1},1,\gamma})} (\text{st}_{g^{-1},1,\gamma})_*(\rho^\text{rig}_{\text{loop}, \gamma}),$$

(5.43)
To check the factorization axiom for the twisted maps (5.16) we note that in the situation of Section 5.2 we have
\[ \text{Td}(R\pi_* (\mathcal{L}_j)) = \text{Td}(R\tilde{\pi}_* (\tilde{\mathcal{L}}_j)). \]
Indeed, in Case 1 this follows from the isomorphism (5.24), and in Case 2 — from the exact triangle (5.25) using the fact that \( \tilde{\mathcal{L}}_j|_p \) is trivial and so has the trivial Todd class. Now it remains to apply the equalities (5.41) and (5.42) (resp., (5.43)), taking into account Lemma 3.3.3.

5.4 Forgetting tails

Here we will check the forgetting tails axiom of CohFT (see [14, 4.2]) which corresponds to the projection
\[ \theta : \overline{\mathcal{M}}_{g, r} \to \overline{\mathcal{M}}_{g, r-1}. \]
To do this we have to compare the fundamental matrix factorizations in the following situation. Let \( S = S^0_{g, \overline{\gamma}}(\overline{\gamma}) \) be the moduli space of \( \Gamma \)-spin curves with a restricted rigidification structure \( \psi \) (see Section 3.2) of type \( (\overline{\gamma}, J) \), where \( \overline{\gamma} = (\gamma_1, \ldots, \gamma_{r-1}) \) and \( J \) is the exponential grading element (see (2.2)). Let \( (\mathcal{C} \to S, p_1, \ldots, p_r; \mathcal{E}) \) be the universal \( \Gamma \)-spin curve. Let \( \mathcal{C}_r \to S \) be the corresponding family obtained by forgetting the orbifold structure at \( p_r \). Let \( \rho_r : \mathcal{C} \to \mathcal{C}_r \) and \( \rho' : \mathcal{C}_r \to C \) be the corresponding projections, so that \( \rho' \circ \rho_r = \rho : \mathcal{C} \to C \) is the morphism of forgetting the orbifold structure at all the marked points (see Section 3.2).

Let \( (\mathcal{L}_1, \ldots, \mathcal{L}_n) \) be the collection of line bundles on \( \mathcal{C} \) associated with the \( \Gamma \)-bundle \( P \).

**Lemma 5.4.1.** On the family \( (\mathcal{C}_r \to S, p_1, \ldots, p_{r-1}) \) there is a natural \( \Gamma \)-spin structure \( (\mathcal{P}, \overline{\epsilon}) \) with a restricted rigidification \( \overline{\psi} \) such that the collection line bundles on \( \mathcal{C}_r \) associated with \( \mathcal{P} \) is isomorphic to \( (\rho_r^* \mathcal{L}_1, \ldots, \rho_r^* \mathcal{L}_n) \). Hence, we obtain a morphism
\[ \overline{\theta} : S = S^0_{g, \overline{\gamma}}(\overline{\gamma}, J) \to S^0_{g, \overline{\gamma}}(\overline{\gamma}) \]
covering the forgetting tail map \( \theta \).

**Proof.** By Corollary 3.2.4, a \( \Gamma \)-spin structure \( (P, \epsilon) \) can be described as an additional structure for the line bundles \( (\mathcal{L}_1, \ldots, \mathcal{L}_n) \) given by isomorphisms
\[ M_i(\mathcal{L}_1, \ldots, \mathcal{L}_n) \simeq \omega^{\log}_{\mathcal{C}/S} \text{ for } i = 0, \ldots, n - 1, \]
where \( M_0, \ldots, M_{n-1} \) are the Laurent monomials (3.10). In the notation of Proposition 3.3.1(i), we have \( l_{M_i}(\mathcal{q}) = 1 \) for \( i = 0, \ldots, n - 1 \). Therefore, the isomorphisms (5.44) induce isomorphisms
\[ M_i(\rho_r^* \mathcal{L}_1, \ldots, \rho_r^* \mathcal{L}_n) \simeq \rho_r^* (\omega^{\log}_{\mathcal{C}/S}(-p_r)) \simeq \omega^{\log}_{\mathcal{C}_{r}/S} \text{ for } i = 0, \ldots, n - 1, \]
where the log-structure on \( \mathcal{C}_r \) is given by the markings \( p_1, \ldots, p_{r-1} \). This is established by the argument similar to the proof of (3.17) applied only at the marked point \( p_r \). Thus, we get a \( \Gamma \)-spin structure \( (\mathcal{P}, \epsilon) \) on \( (\mathcal{C}_r \to S, p_1, \ldots, p_{r-1}) \), which inherits the rigidification at the marked points \( p_1, \ldots, p_{r-1} \). \( \Box \)
Since \((\mathbb{A}^n)^I = 0\) and \(w_J = 0\), the fundamental matrix factorizations associated with the data \((C \to S, p_1, \ldots, p_r; P, \varepsilon, \psi)\) and \((C_r \to S, p_1, \ldots, p_r; \mathcal{P}, \mathcal{E}, \mathcal{P})\) (see Section 4.2) belong to the same category, so we can compare them.

**Proposition 5.4.2.** One has an isomorphism
\[
P_{\text{rig}g}^{0}(\gamma, J) \simeq (\tilde{\theta} \times \text{id})^* P_{\text{rig}g}^{0}(\gamma).
\]

**Proof.** The natural isomorphisms \(\rho^* (\rho_r)^* L_j \simeq \rho_* L_j\) are compatible with the maps (4.9) constructed using the \(w\)-structures on \(C\) and on \(C_r\) and with trivializations (4.10). Since the rest of the construction in Section 4.2 depends only on these data, the assertion follows. \(\square\)

The above proposition immediately implies that
\[
\Lambda^R _g(\gamma) = \Lambda^R _g(\gamma, J) \circ (\text{id} \otimes 1^{\otimes r-1} \otimes 1) \quad \text{and} \quad \lambda _g(\gamma) = \lambda _g(\gamma, J) \circ (\text{id} \otimes 1^{\otimes r-1} \otimes 1)
\]
which is the forgetting tails axiom for the CohFTs of Theorems 5.1.2 and 5.1.3.

### 5.5 Concavity

Here we consider a special class of families of \(\Gamma\)-spin structures, called *concave*, and derive a formula connecting the class of the fundamental matrix factorization in cohomology of the base with the Chern character of a certain vector bundle. This is a generalization of the concavity property of [14, Thm. 4.1.5].

Let \(S\) be a DM-stack admitting a finite flat covering by a smooth projective scheme, and let \(G\) be a finite group. For a \(G\)-equivariant vector bundle \(V\) on \(S\) (where the action of \(G\) on \(S\) is trivial) we have a canonical decomposition
\[
V = \bigoplus_{\eta \in \text{Irr}(G)} V_{\eta} \otimes \eta
\]
compatible with the action of \(G\) (where \(\text{Irr}(G)\) is the set of isomorphism classes of irreducible representations of \(G\), and the bundles \(V_{\eta}\) have the trivial \(G\)-action). Hence, the abstract Chern character (see Section 2.5) \(\text{ch}^{HH}_G(V) \in HH_*(D_G(S))\) decomposes as
\[
\text{ch}^{HH}_G(V) = \sum_{\eta \in \text{Irr}(G)} \text{ch}^{HH}(V_{\eta})[\eta].
\]

Let \(R(G)\) be the representation ring of \(G\) over \(\mathbb{C}\). Applying the map \(\alpha_S \otimes \text{id} : HH_*(S) \otimes R(G) \to H^*(S, \mathbb{C}) \otimes R(G)\) (see (5.6)) we obtain an element with values in \(H^*(S, \mathbb{C}) \otimes R(G)\):
\[
\text{ch}_G(V) = (\alpha_S \otimes \text{id}) \text{ch}^{HH}_G(V) = \sum_{\eta \in \text{Irr}(G)} \chi(V_{\eta})[\eta] \in H^*(S, \mathbb{C}) \otimes R(G) \quad (5.45)
\]
Remark 5.5.1. The above notion is different from the usual $G$-equivariant Chern character of $V$ with values in $H^*_G(S, \mathbb{C})$, which for a finite group acting trivially on $S$ is equal to the non-equivariant Chern character, because in this case $H^*_G(S, \mathbb{C}) = H^*(S, \mathbb{C}) \otimes H^*(BG, \mathbb{C}) = H^*(S, \mathbb{C})$.

If $G$ is commutative then $R(G) = R = \mathbb{C}[\hat{G}]$ and for every $\gamma \in G$ we have the evaluation homomorphism $\pi_\gamma : R \to \mathbb{C}$. Thus, we can consider the components

$$ch_G(V)_\gamma := \pi_\gamma(ch_G(V)) = \sum_{\eta \in \hat{G}} \eta(\gamma) ch(V_\eta) \in H^*(S, \mathbb{C}).$$

Note that for $\gamma = 1$ the component $ch_G(V)_1$ is the usual (non-equivariant) Chern character.

Now assume that we have a family $(\pi : C \to S, p_1, \ldots, p_r)$ of orbicurves of genus $g$ with $r$ marked points and a rigidified $\Gamma$-spin structure $(P, \varepsilon)$, induced by a map $f : S \to S_{\text{rig}}^g(\gamma)$. Such a family is called concave if $\pi_*(\bigoplus_{j=1}^n L_j) = 0$ for $j = 1, \ldots, n$, where $(L_1, \ldots, L_n)$ are the line bundles on $C$ associated with the $\Gamma$-bundle $P$. In this case

$$\mathcal{V} = R^1\pi_*(\bigoplus_{j=1}^n L_j)$$

is a vector bundle on $S$, equipped with a $G$-equivariant structure via the embedding $G \subset G_n^m$. Denote by $P$ the pull-back of the fundamental matrix factorization (4.31) to $S \times \mathbb{A}^\gamma$. For each $\gamma \in G$ let us denote by

$$\kappa_\gamma : H_{\gamma} = HH_*(\Gamma((\mathbb{A}^n)^\gamma, \omega_\gamma)) \to R$$

the map induced by the restriction to the origin functor. Note that $\kappa_\gamma$ is given by the canonical pairing with the Chern character of the stabilization of the residue field $\mathbb{C}^*$ (see Example 2.7.3).

**Proposition 5.5.2.** For a concave family of $\Gamma$-spin curves over $S$ induced by the morphism $f : S \to S_{\text{rig}}^g(\gamma)$, the map

$$f^*\phi_g(\gamma) : \mathcal{H}_{\gamma_1} \otimes_R \cdots \otimes_R \mathcal{H}_{\gamma_r} \to H^*(S, \mathbb{C}) \otimes R$$

is given by

$$f^*\phi_g(\gamma_1, \ldots, \gamma_r) = ch_G(\bigwedge^* \mathcal{V}) \otimes \kappa_{\gamma_1} \otimes \cdots \otimes \kappa_{\gamma_r},$$

where $ch_G \in H^*(S, \mathbb{C}) \otimes R$ is given by (5.45).

**Proof.** By the base change formula, the matrix factorization $P$ is obtained by applying the construction of Section 4.2 directly to the family of $\Gamma$-spin curves over $S$. Note that in our case for each $j = 1, \ldots, n$, the complex $[A_j \to B_j]$ representing $R\pi_*(L_j)$ has the property that $\beta_j : A_j \to B_j$ is the embedding of a subbundle. Hence, the map $\beta = \oplus \beta_j$
viewed as a section of $p^*(\bigoplus_j B_j)$ on $X = \operatorname{tot}(\bigoplus_j A_j)$, is a regular section of the subbundle $p^*(\bigoplus_j A_j) \subset p^*(\bigoplus_j B_j)$. Let $i : S \to X$ be the zero section. Assume first that $A^\gamma \neq 0$ and let $H \subset \mathbb{A}^\gamma$ be the hypersurface $w^\gamma = 0$. Let $i' : S \to Z^{-1}(H)$ denote the natural embedding. Since $B_j/A_j \simeq R^1 \pi_*(\mathcal{L}_j)$, Proposition 1.6.4(i) implies that

$$[\mathcal{C}(-\alpha_w, \beta)] = [\mathcal{I}'_* \bigwedge^\bullet (\mathcal{V}^\vee)]$$

in the Grothendieck group of $D_{Sg}(Z^{-1}(H)/\Gamma)$. Hence, the class of $\mathcal{C}(P)$ in the Grothendieck group of $D_{Sg}(S \times H/\Gamma)$ is given by

$$[\mathcal{C}(P)] = [(p, Z), \mathcal{C}(-\alpha_w, \beta)] = [(\operatorname{id}_S \times k)_* \bigwedge^\bullet (\mathcal{V}^\vee)],$$

where $k : \{0\} \to H$ is the embedding. This implies the result using Lemma 2.2.2 and Remark 2.2.1.2. In the case $A^\gamma = 0$, by Proposition 1.6.4(ii), we have

$$[\{-\alpha_w, \beta\}] = [\mathcal{I}_* \operatorname{mf} \bigwedge^\bullet (\mathcal{V}^\vee)]$$

and so by Remark 1.2.4,

$$[\operatorname{com}_G(P)] = [\bigwedge^\bullet (\mathcal{V}^\vee)].$$

\[ \square \]

**Corollary 5.5.3.** In the situation of Proposition 5.5.2 assume in addition that $(\mathbb{A}^n)^\gamma_i = 0$ for all $i = 1, \ldots, r$. In this case $\mathcal{H}_{\gamma_i} = R$ for every $i$, so we can view the map (5.46) as an element of $H^*(S, \mathbb{C}) \otimes R$. We have

$$f^* \phi_g(\gamma_1, \ldots, \gamma_r) = \operatorname{ch}_G(\bigwedge^\bullet \mathcal{V}^\vee),$$

where $\operatorname{ch}_G$ is given by (5.45). The twisted element $\phi_g^{tw}(\gamma) \in H^*(S, \mathbb{C})$ (see (5.15)) is equal to the top Chern class of $\mathcal{V}^\vee$:

$$\phi_g^{tw}(\gamma) = (-1)^D c_D(\mathcal{V}) = c_D(\mathcal{V}^\vee).$$

where $D$ is the rank of $\mathcal{V}$.

**Proof.** The formula (5.48) is a direct consequence (5.47). The formula (5.49) follows from the fact that the component $\operatorname{ch}_G(\bigwedge^\bullet \mathcal{V}^*)_1$ is the usual Chern character and from the standard relation

$$c_{\operatorname{top}}(\mathcal{V}) = \operatorname{Td}(\mathcal{V}) \cdot \operatorname{ch}(\bigwedge^\bullet \mathcal{V}^\vee).$$

\[ \square \]
5.6 Homogeneity conjecture

Our conjecture that the reduced CohFT of Theorem 5.1.3 is isomorphic to the one constructed in [14] implies a certain homogeneity property of the maps $\lambda_g(\overline{\gamma})$. This suggests the following analog of the Dimension axiom of [14, Thm. 4.1.5] for the maps

$$
\phi_{g}^{tw}(\overline{\gamma}) : e_1\mathcal{H}_{\gamma_1} \otimes \ldots \otimes e_1\mathcal{H}_{\gamma_r} \to H^*(S_{g}^{\text{rig}}(\overline{\gamma})), \mathbb{C})
$$

(see (5.15)).

**Homogeneity Conjecture.** The image of the map $\phi_{g}^{tw}(\overline{\gamma})$ is contained in $H^{2\tilde{D}_g(\overline{\gamma})}(S_{g}^{\text{rig}}(\overline{\gamma})), \mathbb{C})$, where

$$
\tilde{D}_g(\overline{\gamma}) = D_g(\overline{\gamma}) + \frac{1}{2} \sum_{i=1}^{r} N_{\gamma_i} \text{ and }
$$

$$
D_g(\gamma_1, \ldots, \gamma_r) = (g-1)c_w + \iota_{\gamma_1} + \ldots + \iota_{\gamma_r}
$$

(see Section 3.3).

Note that for $\overline{\gamma}$ such that $S_g(\overline{\gamma})$ is nonempty, we have $2\tilde{D}_g(\overline{\gamma}) \in \mathbb{Z}$ and

$$
2\tilde{D}_g(\overline{\gamma}) \equiv N_{\gamma_1} + \ldots + N_{\gamma_r} \mod 2,
$$

where $N_{\gamma} = \dim(\mathbb{A}_n)^{\gamma}$. Since the map $\phi_{g}^{tw}(\overline{\gamma})$ is even with respect to the natural $\mathbb{Z}/2\mathbb{Z}$-grading, we know that the above conjecture holds modulo 2, i.e., the image of $\phi_{g}^{tw}(\overline{\gamma})$ is contained in

$$
\bigoplus_{m \in \mathbb{Z}} H^{2m+N_{\gamma_1}+\ldots+N_{\gamma_r}}(S_{g}^{\text{rig}}(\overline{\gamma})), \mathbb{C}).
$$

Now we are going to prove a certain homogeneity property related to the above conjecture. Let us say that a Koszul matrix factorization $\{\alpha, \beta\}$ of $\mathbb{w}$ has rank $k$ if $\alpha$ and $\beta$ are sections of dual vector bundles of rank $k$.

**Proposition 5.6.1.** Let $\bar{E}_i$, for $i = 1, \ldots, r$, be a Koszul matrix factorization of the potential $w_{\gamma_i}$ of rank $k_i$. Then

$$
\phi_{g}^{tw}(\overline{\gamma})(\text{ch}(\bar{E}_1), \ldots, \text{ch}(\bar{E}_r)) \in H^{2(D_g(\overline{\gamma})+k_1+\ldots+k_r)}(S_{g}^{\text{rig}}(\overline{\gamma})), \mathbb{C}).
$$

**Proof.** Let us write for brevity $S = S_{g}^{\text{rig}}(\overline{\gamma})$. Recall that the fundamental matrix factorization (4.31) is a $\Gamma$-equivariant matrix factorization of $-\mathbb{w}_\gamma$ on $S \times \prod_{i=1}^{r}(\mathbb{A}_n)^{\gamma_i}$ of the form

$$
P = (p, Z) \ast \{\alpha, \beta\},
$$

where $p : X = \text{tot}(A) \to S$ is the projection from the total space of a vector bundle $A = \bigoplus_j A_j$, and $\{\alpha, \beta\}$ is the Koszul matrix factorization of $0$, $\{\alpha, \beta\} = \left( \bigoplus_m p^* B^\vee \right) \otimes \chi^{[m/2]}$, $\delta$, $\delta$.
where \( B = \bigoplus_j B_j \). Here the complexes \([A_j \to B_j]\) represent the objects \( R\pi_*(L_j) \in D^b(S) \).

Since \( \phi_{tw}^\ell \) is defined using the specialization by \( \pi_1 : R \to \mathbb{C} \) corresponding to the element \( 1 \in G \), we can disregard the \( G \)-equivariant structure on \( P \). Therefore, we have

\[
\phi_{tw}^\ell (\gamma)(\text{ch}(\bar{E}_1), \ldots, \text{ch}(\bar{E}_r)) = \exp(\pi i \bar{D}_g(\gamma)) \cdot \frac{1}{\text{deg(st)}} \text{st}_*(Td(R\pi_*(\bigoplus_{j=1}^n L_j))^{-1} \cdot \text{ch}(K)),
\]

where \( K = (p_8)_*(\mathbb{P} \otimes \bar{E}_1 \otimes \cdots \otimes \bar{E}_r) \). Here \( p_8 : S \times A^\top \to S \) is the projection. We view \( K \) as a \( \mathbb{Z}/2 \)-graded complex of quasicoherent sheaves on \( S \) with coherent cohomology. The Chern character of such a complex \( K \) can be calculated as \( \text{ch}(K) = \text{ch}(H_{\text{even}}K) - \text{ch}(H_{\text{odd}}K) \). Thus, we can replace the expression in the right-hand side of (5.50) with a similar expression of characteristic classes in the Chow group \( A^*(S) \otimes \mathbb{Q} \).

It is enough to show that

\[
\text{Td}(R\pi_*(\bigoplus_{j=1}^n L_j))^{-1} \cdot \text{ch}(K) = \text{Td}(B) \cdot \text{Td}(A)^{-1} \cdot \text{ch}(K) \in A^{D_g(\gamma) + k_1 + \ldots + k_r}(S) \otimes \mathbb{Q},
\]

where \( D_g(\gamma) = \text{rk } B - \text{rk } A \) by (3.19). Note that

\[
K = (p_8)_*(p, Z)_*(\{\alpha, \beta\} \otimes \bar{E}_1 \otimes \cdots \otimes \bar{E}_r) \simeq p_*\{\alpha', \beta'\},
\]

where \( \{\alpha', \beta'\} \) is a Koszul matrix factorization of zero of rank \( \text{rk } B + k_1 + \ldots + k_r \) on \( X \) (supported at the zero section \( S \subset X \)). By [7, Lemma 5.3.8], we have

\[
\text{Td}(A)^{-1} \cdot \text{ch}(K) = \text{ch}^S_X(\{\alpha', \beta'\}) \cdot [p],
\]

where \( \text{ch}^S_X(\{\alpha', \beta'\}) \in A^*(S \to X) \) is the localized Chern character of the \( \mathbb{Z}/2 \)-graded complex \( \{\alpha', \beta'\} \) (see [48, sec. 2.2]), and \([p] \in A^{-\text{rk } A}(X \to S)\) is the orientation class of \( p \). Now by [48, Thm. 3.2], the class

\[
\text{Td}(B) \cdot \text{ch}^S_X(\{\alpha', \beta'\}) \in A^*(S \to X)
\]

is concentrated in degree \( \text{rk } B + k_1 + \ldots + k_r \). Hence, the class

\[
\text{Td}(R\pi_*(\bigoplus_{j=1}^n L_j))^{-1} \cdot \text{ch}(P) = \text{Td}(B) \cdot \text{ch}^S_X(\{\alpha', \beta'\}) \cdot [p] \in A^*(S)
\]

lives in degree \( \text{rk } B - \text{rk } A + k_1 + \ldots + k_r \) as claimed. \( \square \)

**Corollary 5.6.2.** The Homogeneity Conjecture holds in the case when \( (A^n)^{\gamma_i} = 0 \) for every \( i = 1, \ldots, r \).

We expect that Proposition 5.6.1 is compatible with the above Homogeneity Conjecture due to the following property of Koszul matrix factorizations.

**Vanishing conjecture for Koszul matrix factorizations.** Let \( W \in \mathbb{C}[[x_1, \ldots, x_n]] \) be an isolated singularity and let \( \{\alpha, \beta\} \) be a Koszul matrix factorization of \( W \) of rank \( r \neq n/2 \). Then \( \text{ch}(\{\alpha, \beta\}) = 0 \).

Note that here we do not assume \( W \) to be quasi-homogeneous.

We can verify the above conjecture in the case when \( \beta \) is linear.
Proposition 5.6.3. The vanishing conjecture holds for a Koszul matrix factorization \( \{\alpha, \beta\} = \{a_1, \ldots, a_r; b_1, \ldots, b_r\} \), where \( b_1, \ldots, b_r \) are linear forms in \( x_1, \ldots, x_n \).

Proof. Note that if one of the elements \( a_i \) does not lie in the maximal ideal \( \mathfrak{m} \subset \mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_n]] \) then \( \{\alpha, \beta\} \) is contractible and so has trivial Chern character. Thus, we can assume that \( a_i \in \mathfrak{m} \) for all \( i \). The inclusion of ideals \( (\partial_1 W, \ldots, \partial_n W) \subset (a_1, \ldots, a_r, b_1, \ldots, b_r) \) implies that \( 2r \geq n \) (since \( W \) has an isolated singularity). It remains to prove that \( ch(\{\alpha, \beta\}) = 0 \) in the case \( 2r > n \). We have \( \alpha \in V \otimes \mathbb{C}[[x]], \beta \in V^* \otimes \mathbb{C}[[x]] \), where \( V \) is a \( k \)-vector space of dimension \( r \), and

\[
\{\alpha, \beta\} = (\wedge^* (V) \otimes \mathbb{C}[[x]], \delta = \alpha \wedge + \iota(\beta)).
\]

By the formula for the Chern character for matrix factorizations \cite[Thm. 3.2.3]{49}, it is enough to check that

\[
str_{\mathbb{C}[[x]]}(\partial_0 \delta \circ \cdots \circ \partial_1 \delta) = str_{\mathbb{C}[[x]]}((\iota(\partial_0 \beta) + \partial_0 \alpha \wedge?) \circ \cdots \circ (\iota(\partial_1 \beta) + \partial_1 \alpha \wedge?)) = 0. \tag{5.51}
\]

By assumption, \( \beta \) is linear in \( x_1, \ldots, x_n \), so \( \partial_i \beta \) are elements of \( V^* \subset V^* \otimes \mathbb{C}[[x]] \). After expanding the product in (5.51) only the terms which contain equal amounts of \( \iota(\partial_i \beta) \) and \( \partial_j \alpha \wedge? \) factors will contribute to the supertrace. Thus, it is enough to check that for every subset \( I \subset \{1, \ldots, n\} \) of cardinality \( n/2 \) one has

\[
str_{\mathbb{C}[[x]]}(A_1 \circ \cdots \circ A_n) = 0, \tag{5.52}
\]

where

\[
A_i = \begin{cases} \iota(\partial_i \beta), & i \in I \\ \partial_i \alpha \wedge?, & i \notin I. \end{cases}
\]

Now we can argue as in \cite[Lem. 4.3.5]{49}. Let \( V_1 \subset V \) be the orthogonal complement to the collection of elements \( (\partial_i \beta)_{i \in I} \). Since \( r > n/2 \), we have \( V_1 \neq 0 \). Let \( V_2 \subset V \) be some complementary subspace to \( V_1 \), so that \( V = V_1 \oplus V_2 \). Then each operator \( \overline{A}_i \) preserves the filtration

\[
\bigwedge^{\geq t} (V_1) \otimes \bigwedge^* (V_2) \otimes \mathbb{C}[[x]]
\]

on \( \bigwedge^* (V) \otimes \mathbb{C}[[x]] \). The induced operator on the associated graded space is of the form \( id \otimes \overline{A}_i \), where \( \overline{A}_i \) acts on \( \bigwedge^* (V_2) \otimes \mathbb{C}[[x]] \). Thus, the supertrace in (5.52) is equal to the product

\[
str_C(id; \bigwedge^* (V_1)) \cdot str_{\mathbb{C}[[x]]}(\overline{A}_1 \circ \cdots \circ \overline{A}_n)
\]

which is equal to zero because \( str_C(id; \bigwedge^* (V_1)) = 0 \).

Corollary 5.6.4. Assume that for each \( \gamma \in G \), the space \( HH_\gamma(MF(w)) \) is generated by the Chern characters of Koszul matrix factorizations \( \{a, b\} \) of rank 1 with \( b \) linear (possibly after a change of variables \( (x_1, \ldots, x_n) \)). Then the Homogeneity Conjecture holds for the CohFT associated with \( w \) and \( G \).

In particular, it holds for all simple singularities.

Proof. The first assertion follows from Propositions 5.6.1 and 5.6.3. In Section 7 we will verify that this criterion can be applied to all simple singularities. \( \square \)
5.7 Index zero

In the case when \( D^g(\gamma) = 0 \) and \( (A^n)^{\gamma_i} = 0 \) for every \( i = 1, \ldots, r \), the Homogeneity Conjecture predicts that \( \phi^{tw}(\gamma) \) belongs to \( \mathcal{H}_0(S_g(\gamma), \mathbb{C}) \), so it should be a multiple of the fundamental class on each connected component. Following [14], we will identify this multiple in terms of the degree of a certain map between affine spaces.

Let \( (\mathcal{C}, p_1, \ldots, p_r; \mathcal{P}, \varepsilon) \) be a \( \Gamma \)-spin curve with smooth orbicurve \( \mathcal{C} \), corresponding to a point in the moduli space \( S = S_g(\gamma) \). Assume that \( D^g(\gamma) = 0 \) and \( (A^n)^{\gamma_i} = 0 \) for every \( i = 1, \ldots, r \). As in Section 4.2, consider \( \rho : \mathcal{C} \to C \), where \( C \) is the smooth curve obtained by forgetting the orbistructure at marked points, and the line bundles \( L_j = \rho_*(\mathcal{L}_j) \) on \( C \), where \( (\mathcal{L}_1, \ldots, \mathcal{L}_n) \) are the line bundles on \( \mathcal{C} \) associated with the \( \Gamma \)-bundle \( \mathcal{P} \). Then for every monomial \( M \) in \( w \) and every \( j = 1, \ldots, n \) we have a morphism

\[
\alpha_j(M) : \partial_j M(H^0(L_j)) \to H^0(\partial_j M(L_j)) \to H^0(\omega_C \otimes L_j^{-1}) \simeq H^1(L_j)^*,
\]

which can be viewed as a section of the vector bundle \( H^1(L_j)^* \otimes \mathcal{O}_X \) on the affine space \( X = \bigoplus_{j=1}^n H^0(L_j) \) (see Section 4.1). As in Section 4.1 we take linear combinations of these sections

\[
\alpha^w_j = \sum_k c_k m_{kj} \alpha_j(M_k),
\]

where \( w = \sum_{k=1}^N c_k M_k \) and \( M_k = x_1^{m_{k1}} \cdots x_n^{m_{kn}} \). Then by Proposition 4.1.1, the section \( \alpha = (\alpha^w_1, \ldots, \alpha^w_n) \) of \( \bigoplus_{j=1}^n H^1(L_j)^* \otimes \mathcal{O}_X \) has zero locus supported only at the origin. We can view \( \alpha \) as a \( \Gamma \)-equivariant morphism between affine spaces

\[
\alpha : X = \bigoplus_{j=1}^n H^0(L_j) \to Y = \bigoplus_{j=1}^n H^1(L_j)^* \otimes \chi. \tag{5.53}
\]

This morphism has the property that the subscheme \( \alpha^{-1}(0) \) is concentrated at the origin in \( X \). Note that our assumption \( D^g(\gamma) = 0 \) implies that \( X \) and \( Y \) are affine spaces of the same dimension.

Let \( X(\Gamma) \) denote the group of algebraic characters of \( \Gamma \). The natural embedding \( \iota : G \to \Gamma \) induces a surjective homomorphism of the group rings

\[
\iota^* : \mathbb{C}[X(\Gamma)] \to \mathbb{C}[\hat{G}] = R.
\]

Also, consider the homomorphism

\[
\varphi : G_m \to \Gamma : \lambda \mapsto (\lambda^{d_1}, \ldots, \lambda^{d_n})
\]

and the induced homomorphism

\[
\varphi^* : \mathbb{C}[X(\Gamma)] \to \mathbb{C}[X(G_m)] = \mathbb{C}[t, t^{-1}],
\]

where we choose a generating character \( t \in X(G_m) \) to be \( t(\lambda) = \lambda^{-1} \). Note that \( \varphi^*(\chi) = t^{-d} \).

Define the \( \mathbb{Z} \)-grading on \( \mathbb{C}[X(\Gamma)] \) by

\[
\varphi^*(\xi) = t^{\deg(\xi)} \text{ for } \xi \in X(\Gamma),
\]

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so that \( \varphi^* \) becomes a homomorphism of \( \mathbb{Z} \)-graded algebras. Let \( \mathbb{C}[X(\Gamma)]^\sim \) be the completion of \( \mathbb{C}[X(\Gamma)] \) with respect to the degree filtration \( (\mathbb{C}[X(\Gamma)])_{\geq q} \). The homomorphism \( \varphi^* \) extends to a homomorphism from \( \mathbb{C}[X(\Gamma)]^\sim \) to the ring of Laurent series in \( t \).

We have the following analog of the Index zero Axiom of [14, Thm. 4.1.5].

**Proposition 5.7.1.** Assume that \( D_g(\overline{\gamma}) = 0 \) and \( (\mathbb{A}^n)^{\alpha_t} = 0 \) for every \( i = 1, \ldots, r \). Let \( x = (\mathcal{C}, \mathcal{L}_\bullet) \in S \) be a point with smooth \( \mathcal{C} \). Let

\[
a_j = h^0(C, L_j), \quad b_j = h^1(C, L_j) \quad \text{and} \quad h = \sum_{j=1}^n a_j + \sum_{j=1}^n b_j = \dim X,
\]

where we use the notation above.

(i) The restriction of the class \( \phi_g(\overline{\gamma}) \in H^*(S, \mathbb{C}) \otimes R \) to the point \( x \in S \) is given by

\[
\phi_g(\overline{\gamma})|_x = (-1)^h \cdot [H^0(\alpha^{-1}(0), \mathcal{O})]_G \in R,
\]

where \([V]_G\) denotes the class of a \( G \)-module in the representation ring \( R \) of \( G \). In particular, the restriction of the specialized class \( \phi_g^{bw}(\overline{\gamma}) \in H^*(S, \mathbb{C}) \) to \( x \in S \) is equal to

\[
\phi_g^{bw}(\overline{\gamma})|_x = (-1)^h \cdot \ell(\alpha^{-1}(0)) \in \mathbb{C}, \tag{5.54}
\]

where \( \ell(\alpha^{-1}(0)) \) is the length of the zero-dimensional scheme \( \alpha^{-1}(0) \).

(ii) Let \( t_j \in X(\Gamma) \) denote the inverse of the character of \( \Gamma \) induced by the \( j \)th projection \( \mathbb{G}_m^n \to \mathbb{G}_m \). Consider the element

\[
P = \prod_{j=1}^n \frac{(1 - \chi t_j)^{b_j}}{(1 - t_j)^{a_j}} \in \mathbb{C}[X(\Gamma)]^\sim.
\]

Then \( P \) belongs to \( \mathbb{C}[X(\Gamma)] \) and

\[
\phi_g(\overline{\gamma})|_x = (-1)^h \cdot [H^0(\alpha^{-1}(0), \mathcal{O})]_G = \iota^* P.
\]

(iii) Consider the subgroup \( \langle J \rangle = G \cap \varphi(\mathbb{G}_m) \subset \Gamma \). Since \( J \) has order \( d \) we can identify the representation ring of \( \langle J \rangle \) with \( \mathbb{C}[u]/(u^d - 1) \) using the character \( u = t|_{\langle J \rangle} \). We have the specialization homomorphism \( \mathbb{C}[t, t^{-1}] \to \mathbb{C}[u]/(u^d - 1) \) sending \( t \) to \( u \). Consider the Laurent series

\[
Q(t) = \prod_{j=1}^n \frac{(1 - t^{-d + d_j})^{b_j}}{(1 - t^{d_j})^{a_j}}.
\]

Then \( Q(t) \) is a Laurent polynomial and viewing \( H^0(\alpha^{-1}(0), \mathcal{O}) \) as a representation of \( \langle J \rangle \) we obtain

\[
(-1)^h \cdot [H^0(\alpha^{-1}(0), \mathcal{O})]_{\langle J \rangle} = Q|_{t = u}.
\]

(iv) One has

\[
\ell(\alpha^{-1}(0)) = \prod_{j=1}^n \frac{(1 - q_j)^{b_j}}{q_j^{a_j}}.
\]
Proof. (i) Let $\mathbf{P}$ be the restriction of the fundamental matrix factorization over $S$ to $x$. By our construction, $\mathbf{P}$ is the $\Gamma$-equivariant matrix factorization of 0 over a point given by
\[ \mathbf{P} = p_* \{-\alpha, 0\} \simeq p_* \text{mf}(K^\bullet(-\alpha)), \quad (5.55) \]
where $p : X \to pt$ is the projection (see Lemma 1.1.9). Since $H^0(\text{mf}(K^\bullet(-\alpha))) \simeq H^{\text{even}}(K^\bullet(-\alpha))$ and $H^1(\text{mf}(K^\bullet(-\alpha))) \simeq H^{\text{odd}}(K^\bullet(-\alpha))$
as $G$-equivariant sheaves (see (1.13)), the assertion follows from the fact that the only nonzero cohomology of the Koszul complex $K^\bullet(-\alpha)$ is the sheaf $\mathcal{O}_{\alpha-1}(0)$ in degree $h(s) = \dim X$.

(ii) Let $C(\Gamma)$ be the category of representations $V$ of $\Gamma$ of the form
\[ V = \bigoplus_{\xi \in X(\Gamma)} V_{\xi} \otimes \xi, \]
where all multiplicities $V_{\xi}$ are finite-dimensional. The assignment
\[ V \mapsto [V] = \sum_{\xi \in X(\Gamma)} \dim V_{\xi} \cdot \xi \]
gives an additive function $K_0(C(\Gamma)) \to \mathcal{C}[X(\Gamma)]^\wedge$, compatible with tensor products. It is easy to see that
\[ P = [p_* K(-\alpha)]_\Gamma. \]
The isomorphism (5.55) implies that the class of $\mathbf{P}$ in $R$ is equal to $\iota^* P$. Also, the cohomology of $p_* K(-\alpha)$ is finite-dimensional, so $P \in \mathcal{C}[X(\Gamma)]$.

(iii) This follows from (ii) applying the specialization with respect to the homomorphism $\varphi^* : X(\Gamma) \to X(\mathbb{G}_m)$ because $\varphi^*(\chi) = t^{-d}$ and $\varphi^*(t) = t^{d_j}$.

(iv) This follows from (iii) by specializing to $t = 1$. \qed

Remark 5.7.2. The Witten map $\mathcal{D}$ considered in [14, Thm. 4.1.5] (restricted to a point) is equal to the complex conjugate of our map $\alpha$ (see (5.53)). Thus, the degree of $\mathcal{D}$ differs from the algebraic degree of $\alpha$ by the factor $(-1)^h$, so our formula (5.54) agrees with Index Zero Axiom of [14, Thm. 4.1.5].

Example 5.7.3. Consider the case when $G = \langle J \rangle$ and $w(x_1, \ldots, x_n)$ is homogeneous of degree $d$, so that $d_j = 1$ and $q_j = 1/d$. Then the degrees of all the line bundles $L_j$ are the same, so the index zero condition means that $\deg(L_j) = g - 1$ for every $j$. In this case the formula of Proposition 5.7.1(iii) gives
\[ \phi_g(\overline{\gamma})|_x = \left( \frac{1 - t^{d+1}}{1 - t} \right) \sum_j a_j |_{t=u} = -u(1 + u + \ldots + u^{d-2}) \sum_j a_j. \]
Specialization at $u = 1$ gives in this case
\[ \phi_g^{tw}(\overline{\gamma}) = (-d + 1) \sum_j a_j. \]
5.8 Sums of singularities

Assume that we have a decomposition \( w = w'(x_1, \ldots, x_{n'}) \oplus w''(y_1, \ldots, y_{n''}) \) with \( n' > 0 \) and \( n'' > 0 \). Then both polynomials \( w' \) and \( w'' \) are also quasi-homogeneous with respect to the restrictions of the degree vector. Assume also that \( G = G' \times G'' \), where \( G' \subset G_{w'} \) (resp., \( G'' \subset G_{w''} \)) is a finite subgroup containing the exponential grading element. Let \( \Gamma' \subset \Gamma_{w'} \) (resp., \( \Gamma'' \subset \Gamma_{w''} \)) be the subgroup associated with \( G' \) (resp., \( G'' \)). Then the character \( \chi : \Gamma \to \mathbb{G}_m \) factors through each of the natural projections \( \Gamma \to \Gamma' \) and \( \Gamma \to \Gamma'' \) and we have a cartesian square of commutative algebraic groups

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\chi'} & \Gamma' \\
\downarrow & & \downarrow \\
\Gamma'' & \xrightarrow{\chi''} & \mathbb{G}_m
\end{array}
\]

Hence, for \( \gamma' \in (G')^\ast \) and \( \gamma'' \in (G'')^\ast \) we have natural isomorphisms of the moduli spaces

\[
S_{g,\Gamma}^{\text{rig}}(\gamma) \simeq S_{g,\Gamma'}^{\text{rig}}(\gamma') \times_{\mathbb{M}_{g,r}} S_{g,\Gamma''}^{\text{rig}}(\gamma''),
\]

where \( \gamma = (\gamma', \gamma'') \in G^\ast \). Also, we have a natural decomposition of the state space \( \mathcal{H}(w, G) \) into a tensor product:

\[
\mathcal{H}(w, G) \simeq \mathcal{H}(w', G') \otimes_{\mathbb{C}} \mathcal{H}(w'', G''),
\]

compatible with the decomposition \( R = R' \otimes R'' \), where \( R' = \mathbb{C}[G'] \), \( R'' = \mathbb{C}[G''] \).

The following result is an analog of [14, Thm. 4.1.5(8)] and [14, Thm. 4.2.2].

**Theorem 5.8.1.** (i) For \( \gamma = (\gamma', \gamma'') \in G^\ast \) one has an isomorphism

\[
P_{g,\Gamma}^{\text{rig}}(\gamma) \simeq p_{g,\Gamma'}^{\text{rig}}(\gamma') \otimes p_{g,\Gamma''}^{\text{rig}}(\gamma''),
\]

in \( \text{DMF}_{\Gamma}(S_{g,\Gamma}^{\text{rig}}(\gamma') \times A^{\gamma'} \times A^{\gamma''}, -w_{\gamma'}) \), where \( p_{\Gamma'} \) and \( p_{\Gamma''} \) are the projections to \( S_{g,\Gamma'}^{\text{rig}}(\gamma') \times A^{\gamma'} \) and \( S_{g,\Gamma''}^{\text{rig}}(\gamma'') \times A^{\gamma''} \), respectively.

(ii) Under the decomposition (5.57), the map \( \Lambda_{g,\Gamma}^{R}(\gamma) \) becomes the tensor product (over \( \mathbb{C} \)) of the maps \( \Lambda_{g,\Gamma'}^{R'}(\gamma') \) and \( \Lambda_{g,\Gamma''}^{R''}(\gamma'') \). The similar result holds for the twisted map \( \lambda_{g}(\gamma) \) (see (5.16)).

**Proof.** (i) Let \( S = S_{g,\Gamma}^{\text{rig}}(\gamma) \) be the moduli space of \( \Gamma \)-spin curves of genus \( g \) and type \( \gamma \). Recall (see Section 4.2) that

\[
P = (p, Z)_{\ast}E,
\]

where \( E \) is a Koszul matrix factorization on the total space of the vector bundle \( p : X \to S \) and \( Z : X \to A_{\gamma} \) is a linear map. We denote by \( X', p', E' \), etc. (resp., \( X'', p'', E'' \), etc.) the similar data constructed for \( (w', \Gamma') \) (resp., \( (w'', \Gamma'') \)) using the induced \( \Gamma' \)-spin structure.
the push-forward with respect to the projection $s^\prime$ of the Todd class and the additivity of the numbers $\gamma_1, \gamma_2, \gamma_3$. In particular, we will verify the metric axiom of the CohFT. The results of Section 6.2 will be used in Section 7 to determine the Frobenius algebras associated with our CohFT for all simple singularities.
6.1 Metric axiom

In this section we will check that the components of the metric on the state space of the CohFT of Theorem 5.1.2 are equal to the maps $\Lambda_0^\delta(\gamma, \gamma^{-1}, J)$, as required by the metric axiom of CohFT.

Since $\mathcal{M}_{0,3}$ is a point, the moduli space $\mathcal{S}_{0,3}$ a finite stack, where a point of $\mathcal{S}_{0,3}$ corresponds to a $\Gamma$-spin curve $(\mathcal{C}, p_1, p_2, p_3; P, \varepsilon)$ with the projection $\rho: \mathcal{C} \to C = \mathbb{P}^1$ obtained by forgetting the orb-structure at $p_1, p_2$ and $p_3$. By Proposition 3.3.1, the type $(\gamma_1, \gamma_2, \gamma_3) \in G^3$ of such a $\Gamma$-spin curve should satisfy $\gamma_1 \gamma_2 \gamma_3 = J$, and there exists a $\Gamma$-spin curve of every such type, unique up to an isomorphism.

First, let us consider the situation where $\gamma_3 = J$ and therefore $\gamma_1 \gamma_2 = 1$. Thus, we will denote $\gamma_1 = \gamma$ and $\gamma_2 = \gamma^{-1}$. As we have seen in Section 5.4, in this case we can drop the point $p_3$ when calculating the fundamental matrix factorization.

Let $L_j = \rho_*(\mathcal{L}_j)$ be the corresponding line bundles on $\mathbb{P}^1$ (where $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ are the line bundles on $\mathcal{C}$ associated with $P$), and let $S \subset \{1, \ldots, n\}$ be the set of $j$ such that the $j$th component of $\gamma$ is trivial. Formula (3.15) for $g = 0$ and $r = 2$ implies that

$$\deg L_j = \begin{cases} 0, & j \in S, \\ -1, & j \notin S. \end{cases}$$

Note that in the notation of Section 4.2

$$\Sigma_j = \begin{cases} \{p_1, p_2\}, & j \in S \\ \emptyset, & j \notin S. \end{cases}$$

Hence, $\Sigma_M = \{p_1, p_2\}$ when $j \in S$ for all $x_j|_M$, and $\Sigma_M$ is empty otherwise. Thus, for each $M$ we have one of the two cases depending on whether $l_M(\deg)$ is zero or not:

(i) $\Sigma_M = \{p_1, p_2\}$ and $l_M(w_1) = l_M(w_2) = 0$;

(ii) $\Sigma_M = \emptyset$ and both $l_M(w_1)$ and $l_M(w_2)$ are positive.

Without loss of generality we can assume that $L_j = \mathcal{O}_{\mathbb{P}^1}$ for $j = 1, \ldots, k$ and $L_j = \mathcal{O}_{\mathbb{P}^1}(-1)$ for $j = k + 1, \ldots, n$ (so $S = \{1, \ldots, k\}$).

The moduli space $\mathcal{S}_0^{\text{rig}}(\gamma, \gamma^{-1}, J)$ is a collection of points $\psi$ corresponding to different choices of rigidification. We are going to compute the restriction $P(\psi)$ of the fundamental matrix factorization $P_0^{\text{rig}}(\gamma, \gamma^{-1}, J)$ to $\{\psi\} \times \mathbb{A}^7$. We claim that there exists a rigidification $\psi_0$ of $(P, \varepsilon)$ such that for $j = 1, \ldots, k$ the induced trivialization of $L_j|_{p_1}$ comes from a global trivialization of $L_j = \mathcal{O}$ and the induced trivialization of $L_j|_{p_2}$ differs from the restriction of a global trivialization by the factor $\zeta_j = \exp(\pi i q_j)$. Indeed, by Corollary 3.2.4, a choice of a restricted rigidification with this property is equivalent to choosing trivializations of the line bundles $L_1, \ldots, L_k$ at $p_1$ and $p_2$, such that the induced trivializations of $M_i(L_1, \ldots, L_k)$ are Laurent monomials in the subset of variables $x_1, \ldots, x_k$, chosen as in (3.10). If we choose arbitrary global trivializations $e_j: \mathcal{O} \to L_j$ for $j = 1, \ldots, k$, then we will get a collection of nonzero residues

$$r_i = \text{Res}_{p_1}(M_i(e_1, \ldots, e_k)), \quad i = 0, \ldots, k - 1.$$
Since the map $(M_0, \ldots, M_{k-1}) : (\mathbb{C}^*)^k \to (\mathbb{C}^*)^k$ is surjective, we can rescale $e_j$’s, so that $r_i = 1$ for each $i = 0, \ldots, k - 1$. Since the residues of global sections of $\omega(p_1 + p_2)$ at $p_1$ and $p_2$ are opposite, the trivializations $e_{j_1} \in L_{j_1}$, and $\zeta_{j_2} e_{j_2} \in L_{j_2}$ define a restricted rigidification of our $\Gamma$-spin structure. Using surjectivity of the map (3.13) we extend it to a rigidification $\psi_0$. We are going to show that $P(\psi_0)$ is essentially the diagonal matrix factorization on $(\mathbb{A}^n)^\gamma \times (\mathbb{A}^n)^\gamma = \mathbb{A}^k \times \mathbb{A}^k$.

To apply the construction of Section 4.2 we need to choose resolutions $[A_j \to B_j]$ for $\Gamma(L_j)$ such that the restriction maps $Z_j : \Gamma(L_j) \to L_j|_{\Sigma_j}$ are realized by surjective maps $A_j \to L_j|_{\Sigma_j}$. When $j = 1, \ldots, k$, we have $L_j \simeq \mathcal{O}$ and we take the resolution

$$\Gamma(L_j) \to [L_j|_{p_1} \oplus L_j|_{p_2} \xrightarrow{\delta} L|_{p_1}],$$

(6.1)

where $\delta$ is the difference map that uses the natural identification $L|_{p_1} \simeq L|_{p_2}$. Using the rigidification $\psi_0$ we can identify $L_j|_{p_1} \oplus L_j|_{p_2}$ with $\mathbb{C}^2$. Then the above resolution becomes $[\mathbb{C}^2 \xrightarrow{\beta} \mathbb{C}]$, where $\beta_j(x, y) = y - \zeta_j x$ by the choice of $\psi_0$. Note that that the restriction map $Z_j : \Gamma(L_j) \to L_j|_{p_1+p_2} \simeq \mathbb{C}^2$ is realized by the identity map $\mathbb{C}^2 \to \mathbb{C}^2$.

When $j = k + 1, \ldots, n$, there exists an isomorphism $L_j = 0(-1)$. Since for such $j$ we have $\Sigma_j = \emptyset$, we can simply set $A_j = B_j = 0$ in this case.

With these choices of resolutions the space $X$ can be identified with $\mathbb{A}^k \times \mathbb{A}^k$, so that the map $Z$ becomes the identity. The bundle $p^*(\bigoplus B_j)$ on $\mathbb{A}^k \times \mathbb{A}^k$ is the trivial bundle of rank $k$ with the basis $e_1, \ldots, e_k$, and that the differential $\beta : \bigoplus A_j \to \bigoplus B_j$ corresponds to the section

$$\beta = \sum_{j=1}^k (y_j - \zeta_j x_j) e_j \in H^0(X, p^*(\bigoplus B_j)).$$

The $w$-structure also induces a section $-\alpha_w \in H^0(X, p^*(\bigoplus B_j))$ such that $\{-\alpha_w, \beta\}$ is a matrix factorization of $-w_{\gamma}(x) - w_{\gamma^{-1}}(y)$ on $\mathbb{A}^k \times \mathbb{A}^k$ (see Section 4.2). Since $\beta$ is regular, the Koszul matrix factorization $\{-\alpha_w, \beta\}$ is a stabilization of the structure sheaf of the shifted diagonal $y = \zeta x$ in $\mathbb{A}^k \times \mathbb{A}^k$ (see Theorem 5.2.1). Therefore,

$$P(\psi_0) = \{-\alpha_w, \beta\} \simeq \Delta_{\alpha_w, \beta},$$

where $\Delta_{\alpha_w, \beta} \simeq (\zeta, \text{id})^* \Delta_{\alpha_w, \beta}$.}

Any other rigidification $\psi \in H^0(\gamma, \gamma^{-1}, J)$ is obtained from $\psi_0$ by the action of an element $(g_1, g_2) \in G \times G$. Hence,

$$P(\psi) \simeq (g_1 \times g_2)^* \Delta_{\alpha_w, \beta} \simeq (\text{id} \times g)^* \Delta_{\alpha_w, \beta},$$

where $g = g_2 g_1^{-1} \in G$. Now we are ready to calculate the maps $\Lambda^R_0(\gamma, \gamma^{-1}, J)(\cdot, \cdot, 1)$.

**Lemma 6.1.1.** For $h \in \mathcal{H}(w_\gamma)$ and $h' \in \mathcal{H}(w_{\gamma^{-1}})$ one has

$$\Lambda^R_0(\gamma, \gamma^{-1}, J)(h, h') = ((\zeta\cdot) h, h')^{R}_{w_\gamma},$$

(6.2)

where

$$(\cdot, \cdot)^{R}_{w_\gamma} : \mathcal{H}(-w_\gamma) \otimes \mathcal{H}(w_\gamma) \to R$$

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is the pairing (2.27), and
\[(\zeta)_* : \mathcal{H}(w_{\gamma \gamma}) = \mathcal{H}(w_{\gamma}) \rightarrow \mathcal{H}(-w_{\gamma}) = \mathcal{H}(-w_{\gamma \gamma})\]
is the isomorphism induced by the automorphism \(x_j \mapsto \zeta_j x_j\) of \(\mathbb{C}[x_1, \ldots, x_n]\).

The Casimir element corresponding to this metric on the state space \(\mathcal{H} = \bigoplus \mathcal{H}(w_{\gamma})\) has components \(T_{w,\zeta} \in \mathcal{H}(w_{\gamma}) \otimes \mathcal{H}(w_{\gamma \gamma})\) given by (5.38).

**Proof.** Recall that \(\Lambda^R_{\gamma, \gamma \gamma}(\gamma, \gamma \gamma, J)\) is defined using the functor \(\Phi^R_0(\gamma, \gamma \gamma, J) : DMF(\mathbb{A}^k \times \mathbb{A}^k, w_{\gamma \gamma} \oplus w_{\gamma \gamma \gamma}) \rightarrow D_G(S^R_0(\gamma, \gamma \gamma, J))\) associated with the kernel \(P_{\text{rig}}(\gamma, \gamma \gamma, J)\) (see (5.5)). Let us consider the component of this functor \(\Phi(\psi) : MF(\mathbb{A}^k \times \mathbb{A}^k, w_{\gamma \gamma} \oplus w_{\gamma \gamma \gamma}) \rightarrow \text{Com}(G - \text{mod})\) corresponding to the point \(\psi \in S^R_0(\gamma, \gamma \gamma, J)\). As we saw above, \(\Phi(\psi)\) is given by tensoring with \(P(\psi) \simeq (\text{id} \times g)^* \Delta^*_{-w_{\gamma \gamma}, \zeta}\) for some \(g \in G\). Hence, by Proposition 2.4.1(ii), we have
\[\Phi(\psi) = \pi_* \circ \Delta^* \circ (\zeta_\bullet \times g^{-1})^*,\]
where \(\pi : \mathbb{A}^k \rightarrow pt\) is the projection. Recall that the canonical pairing \((\cdot, \cdot)_w^R\) is induced by the functor \(\pi_* \circ \Delta^*\), whereas the functor \((g^{-1})^*\) induces the identity map on the Hochschild homology \(HH_*(MF(\gamma_{\gamma \gamma}))\). Formula (6.2) follows from this and from the fact that under the isomorphisms (2.10) the map
\[(\zeta_\bullet)_* : HH_*(MF(\gamma_{\gamma \gamma})) \rightarrow HH_*(MF(-\gamma_{\gamma \gamma}))\]
decomposes into the direct sum of morphisms
\[(\zeta_\bullet)_* : H(\gamma_{\gamma \gamma}) \rightarrow H(-\gamma_{\gamma \gamma}) = H(\gamma_{\gamma \gamma})\]
induced by the automorphism of \(\mathbb{C}[x]\) sending \(x_i\) to \(\zeta_i x_i\), where \(\gamma_{\gamma \gamma} = w|_{(\mathbb{A}^n)(\gamma_{\gamma \gamma})}\).

The second assertion follows from Proposition 2.7.2. \(\square\)

### 6.2 Three-point correlators

Now we are going to consider the maps \(\Lambda^R(\overline{\gamma})\) corresponding to arbitrary markings \(\overline{\gamma} = (\gamma_1, \gamma_2, \gamma_3)\). As in the beginning of the section, we work with a \(w\)-curve \(\mathcal{C}\) with three marked points, where we no longer assume that \(\gamma_3 = J\). Note that in this case (3.15) takes form
\[\deg = q - \theta_1 - \theta_2 - \theta_3.\] (6.3)
Recall that for every \(j = 1, \ldots, n\), the subset \(\Sigma_j \subset \{p_1, p_2, p_3\}\) contains the point \(p_i\) if and only if \((\theta_i)_j = 0\). Using the fact that the coordinates of \(\theta_i\) belong the interval \([0, 1]\) we obtain the following list of possible cases.
Lemma 6.2.1. For every \( j = 1, \ldots, n \), one of the following possibilities is realized:

(i) \( |\Sigma_j| = 2 \) and \( L_j = \emptyset \);
(ii) \( |\Sigma_j| = 1 \) and \( L_j \) is either \( \emptyset \) or \( \emptyset(-1) \);
(iii) \( \Sigma_j = \emptyset \) and \( L_j \) is either \( \emptyset \), or \( \emptyset(-1) \), or \( \emptyset(-2) \).

As before, to compute \( \Lambda_0^R(\gamma) \) we need to choose resolutions \([A_j \to B_j]\) for \( R\Gamma(L_j) \).

In case (i) we will use the resolution (6.1). In case (ii) when \( L_j = \emptyset \) we can use the resolution

\[
A_j = H^0(\emptyset) \to B_j = 0
\]

with \( Z_j : A_j \to \mathbb{C} \) equal to the identity map. In case (ii) when \( L_j = \emptyset(-1) \) we will use the resolution

\[
A_j = H^0(\emptyset) \overset{\text{ev}_\infty}{\longrightarrow} B_j = \mathbb{C},
\]

where \( \text{ev}_\infty \) is the evaluation at the point \( \infty \in \mathbb{P}^1 \) (that we assume to be distinct from \( p_1, p_2, p_3 \)) with \( Z_j \) still equal to the identity. Finally, in case (iii) we will take the resolution with \( A_j = H^0(L_j) \), \( B_j = H^1(L_j) \) and zero differential.

Assume that for any \( j \) such that \( L_j = \emptyset \) one has \( |\Sigma_j| \geq 1 \). Then the corresponding space \( X = \bigoplus_j A_j \) gets identified with \( \mathbb{A}^\mathbb{V} = \prod_j A^{\Sigma_j} \). We will use the coordinates \( x_j(i) \) on this affine space indexed by \((i, j)\) such that \( p_i \in \Sigma_j \). For \( i = 0, 1, 2 \) let us set

\[
S_i = \{ j \mid |\Sigma_j| = i \text{ and } L_j \simeq \emptyset(i - 2) \} \subset \{1, \ldots, n\}
\]

The bundle \( \bigoplus_j B_j \) has generators \( e_j \) labeled by \( j \in S_0 \sqcup S_1 \sqcup S_2 \). For \( j \in S_2 \) the coefficient of \( e_j \) in \( \beta \) has the form \( a_jx_j(i_1) + b_jx_j(i_2) \), where \( \Sigma_j = \{p_{i_1}, p_{i_2}\} \), \( i_1 < i_2 \), and \( a_j \) and \( b_j \) are nonzero constants (depending on a rigidification \( \psi \)). For \( j \in S_1 \) the coefficient of \( e_j \) in \( \beta \) is \( x_j(i) \), where \( \Sigma_j = \{p_i\} \). Finally, for \( j \in S_0 \) the coefficient of \( e_j \) in \( \beta \) is zero.

Proposition 6.2.2. (i) Assume that one has \( |\Sigma_j| \geq 1 \) for each \( j \) such that \( L_j = \emptyset \). Let \( P(\psi) \) be the restriction of the fundamental matrix factorization to a point \( \psi \in S_0^{\rig}(\gamma) \). Then \( P \) is isomorphic to a Koszul matrix factorization

\[
P(\psi) = \{\alpha, \beta\}
\]

of the potential \( w_{\gamma_1}(x_1) + w_{\gamma_2}(x_2) + w_{\gamma_3}(x_3) \), where \( \beta \) is a section of the trivial bundle with generators \( e_j \) numbered by \( j \in S_0 \sqcup S_1 \sqcup S_2 \), of the form

\[
\beta = \sum_{j \in S_1} x_j(i) e_j + \sum_{j \in S_2} (a_jx_j(i_1) + b_jx_j(i_2)) e_j
\]

for some \( a_j, b_j \in \mathbb{C}^* \), where in the first (resp., second) sum we assume that \( \Sigma_j = \{p_i\} \) (resp., \( \Sigma_j = \{p_{i_1}, p_{i_2}\} \)).

Furthermore, the map

\[
\Lambda_0^R(\gamma) : \mathcal{H}_{\gamma_1} \otimes_R \mathcal{H}_{\gamma_2} \otimes_R \mathcal{H}_{\gamma_3} \to R
\]

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is equal to $i^*_\psi \phi_0(\gamma)$ (see (5.8)) for any choice of a rigidification $\psi$, where $i_\psi : \{\psi\} \hookrightarrow S^\rig_0(\gamma)$ is the embedding.

(ii) Now assume that for any $j$ such that $L_j = \emptyset$ one has $|\Sigma_j| = 2$. Consider the composition

$$\phi(S_1, S_2) : H_{\gamma_1} \otimes_R H_{\gamma_2} \otimes_R H_{\gamma_3} \xrightarrow{(\gamma)} HH_*(MF_\Gamma(A^\gamma, w_\gamma)) \rightarrow R,$$

where $(\gamma)$ is the map (5.7) and the second arrow is induced by the functor

$$MF_\Gamma(A^\gamma, w_\gamma) \xrightarrow{\text{com}\circ \pi \circ i^*} \text{Com}_f(G - \text{mod}),$$

where $i : A^{S_2} \rightarrow A^\gamma$ is the embedding of the subspace cut by linear equations

$$a_j x_j(i_1) + b_j x_j(i_2) = 0 \quad \text{for } j \in S_2, \Sigma_j = \{p_i, p_{i_2}\},$$

$$x_j(i) = 0 \quad \text{for } j \in S_1, \Sigma_j = \{p_i\}$$

and $\pi : A^{S_2} \rightarrow \text{pt}$ is the projection. Then one has

$$\Lambda^R_0(\gamma) = \prod_{j \in S_0} (1 - t_j) \cdot \phi(S_1, S_2), \quad (6.4)$$

where $(t_1, \ldots, t_n)$ are the inverses of the characters of $G$ corresponding to the coordinates of the natural map $G \rightarrow \mathbb{G}_m^n$.

Proof. (i) The first assertion follows immediately from the discussion preceding the proposition. For the second, we observe that the group $G_3$ acts transitively on $S^\rig_0(\gamma)$ and for $g \in G_3$ one has

$$P(g\psi) = g^* P(\psi).$$

This implies the statement since the functors $g^*$ on $MF_\Gamma((A^n)^g, w_n)$ for $g \in G$ induce the identity maps on the Hochschild homology and $\Lambda^R_0(\gamma)$ is equal the average of the maps $i^*_\psi \phi_0(\gamma)$ over $\psi \in S^\rig_0(\gamma)$.

(ii) Under our assumptions we have an identification

$$A^{S_1} \times A^{S_2} \times A^{S_2} \simeq A^\gamma$$

so that the section

$$\beta \in \bigoplus_{j} B_j = \bigoplus_{j \in S_0 \cup S_1 \cup S_2} \emptyset \cdot e_j$$

is of the form

$$\sum_{j \in S_1} x_j e_j + \sum_{j \in S_2} (a_j x_j + b_j y_j) e_j,$$

where $(x_j)_{j \in S_1}$ are coordinates on $A^{S_1}$, $(x_j, y_j)_{j \in S_2}$ are coordinates on $A^{S_2} \times A^{S_2}$, and $e_j$ is the generator of the one-dimensional representation $\eta_j$ of $G$. By the second assertion in part (i), it is enough to make calculations for one rigidification $\psi$. By part (i), we have

$$P(\psi) \simeq \{\alpha, \beta\} = \{\alpha_0, 0\} \otimes \{\alpha_{12}, \beta_{12}\} \simeq K^\bullet(\alpha_0) \otimes \{a_{12}, \beta_{12}\},$$

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where we use the decomposition
\[ \bigoplus_j B_j = \left( \bigoplus_{j \in S_0} O_X \cdot e_j \right) \oplus \left( \bigoplus_{j \in S_1 \cup S_2} O_X \cdot e_j \right) \]
and write in components \( \alpha = (\alpha_0, \alpha_{12}), \beta = (0, \beta_{12}) \). Applying Lemma 2.2.2 we obtain an isomorphism
\[ \Phi_{P(\psi)}(\bar{E}_1, \bar{E}_2, \bar{E}_3) \simeq \pi_*(i^*(\bar{E}_1 \otimes \bar{E}_2 \otimes \bar{E}_3) \otimes K^*(\alpha_0)) \]
for the functor associated with \( P(\psi) \). Since \( i^*(\bar{E}_1 \otimes \bar{E}_2 \otimes \bar{E}_3) \) is supported at the origin, on the level of Hochschild homology we can replace the complex \( K^*(\alpha) \) by the alternating sum of its terms. This gives rise to the factor \( \prod_{j \in S_0} (1 - t_j) \) in the formula (6.4).

Note that we have a natural identification
\[ \mathbb{A}^{S_2} = (\mathbb{A}^n)^{\gamma_1 \cdot \gamma_2} \times (\mathbb{A}^n)^{\gamma_1 \cdot \gamma_3} \times (\mathbb{A}^n)^{\gamma_2 \cdot \gamma_3}, \]
so that the embedding \( i \) is the product of three diagonal embeddings complemented by zero in the remaining coordinates.

**Remark 6.2.3.** The assumptions of Proposition 6.2.2(ii) are automatically satisfied if \( d_j = \deg(x_j) = 1 \) for \( j = 1, \ldots, n \) and \( \gamma_i^d = 1 \) for \( i = 1, 2, 3 \), where \( d = \deg(w) \).

### 6.3 Case of homogeneous polynomials and the scalar group action

Let us consider the case when \( w(x_1, \ldots, x_n) \) is homogeneous (i.e., \( \deg(x_j) = 1 \)) and \( G = \mathbb{Z}/d \), where \( d = \deg(w) \), such that \( m \in \mathbb{Z}/d \) acts by the scalar multiplication with \( \exp(2\pi im/d) \). The group \( \Gamma \) in this case is \( \mathbb{G}_m \) acting on \( \mathbb{A}^n \) via scalar multiplications, the character \( \chi : \mathbb{G}_m \to \mathbb{G}_m \) sends \( \lambda \) to \( \lambda^d \). The element \( \zeta \in (\mathbb{C}^*)^n \) has all the components equal to \( \exp(\pi i/d) \).

Our CohFT has the state space
\[ \mathcal{H} = \bigoplus_{m \in \mathbb{Z}/d} \mathcal{H}_m, \]
where \( \mathcal{H}_m = R \) for \( m \in \mathbb{Z}/d \setminus 0 \) and
\[ \mathcal{H}_0 = HH_*(\text{MF}_{\mathbb{G}_m}(\mathbb{A}^n, w)). \]

Let us set for \( m \in \mathbb{Z}/d \setminus 0 \)
\[ e(m) := 1 \in \mathcal{H}_m. \]
Recall that \( \mathcal{H}_0 \) is an \( R \)-module equipped with the canonical \( R \)-valued metric \( (\cdot, \cdot)^R \). We also have a special element
\[ e(0) := ch(C^d) \in \mathcal{H}_0. \]
Using Proposition 6.2.2(ii) we can calculate all the maps \( \Lambda^R_0(m_1, m_2, m_3) \), or equivalently the \( R \)-algebra structure on \( \mathcal{H} \) in terms of these data. Recall that the \( R \)-valued metric on \( \mathcal{H} \) restricts to the canonical metric on \( \mathcal{H}_0 \times \mathcal{H}_0 \) (twisted by \( (\zeta\cdot)_* \) in the first factor) and also restricts to the natural pairing between \( \mathcal{H}_m \) and \( \mathcal{H}_{-m} \), so that
\[ (e(m), e(-m))^R = 1 \text{ for } m \in \mathbb{Z}/d \setminus 0. \]
The flat unit element in this case is $e(1) \in \mathcal{H}_1$. Let us pick as a generator $t$ of the group $\hat{G}$ the character

$$t(m) := \exp(-2\pi im/d)$$

(6.5)

and use the corresponding identification $R = \mathbb{C}[t]/(t^d - 1)$. Below we will often omit $(m_1, m_2, m_3)$ from the notation $\Lambda_0^R(m_1, m_2, m_3)(x, y, z)$, where $x \in \mathcal{H}_{m_1}$, $y \in \mathcal{H}_{m_2}$ and $z \in \mathcal{H}_{m_3}$.

**Theorem 6.3.1.** (a) All the nonzero maps $\Lambda_0^R(\cdot, \cdot, \cdot)$ are given by

$$\Lambda_0^R(e(m_1), e(m_2), e(m_3)) = \begin{cases} 1, & m_1 + m_2 + m_3 = d + 1, \\ (1-t)^n, & m_1 + m_2 + m_3 = 2d + 1, \end{cases}$$

(6.6)

$$\Lambda_0^R(x, e(m), e(d + 1 - m)) = (x, e(0))^R,$$

$$\Lambda_0^R(x, y, e(1)) = ((\zeta_x)_* x, y)^R,$$

(6.7) (6.8)

where $m_1, m_2, m_3 \in [1, d - 1]$, $m \in [2, d - 1]$ and $x, y \in \mathcal{H}_0$.

(b) The maps $1 \mapsto e(1), u \mapsto e(2)$ induce an isomorphism of $R$-algebras

$$R[\mathcal{H}_0, u]/I \simeq \mathcal{H},$$

where $I$ is the ideal generated by the relations

$$u^{d-1} = e(0), \quad xu = (x, e(0))^R, \quad xy = ((\zeta_x)_* x, y)^R u^{d-2}, \text{ where } x, y \in \mathcal{H}_0.$$

**Proof.** (a) Recall that the map $\Lambda_0^R(m_1, m_2, m_3)$ is zero unless

$$m_1 + m_2 + m_3 \equiv 1 \text{ mod}(d).$$

Assume first that $m_1 = m_2 = 0$. Then $m_3 = 1$, and the corresponding equality (6.8) follows from Lemma 6.1.1.

Next, assume that $m_1 = 0$ while $m_2, m_3 \in [1, d - 1]$ are such that $m_2 + m_3 = d + 1$. Then (3.15) gives $L_j = \mathcal{O}(-1)$ for all $j$. Hence, Proposition 6.2.2(ii) applies with $S_0 = S_2 = \emptyset$ and $S_1 = \{1, \ldots, n\}$. Thus, $\Lambda_0^R(0, m_2, m_3)(\cdot, 1, 1) : \mathcal{H}_0 \to R$ is the map induced on Hochschild homology by the functor of restriction to $0 \in \mathbb{A}^n$. Thus, (6.7) follows from Example 2.7.3.

Finally, in the case when $m_1, m_2, m_3 \in [1, d - 1]$ we have either $L_j = \mathcal{O}(-1)$ for all $j$ (when $m_1 + m_2 + m_3 = d + 1$) or $L_j = \mathcal{O}(-2)$ for all $j$ (when $m_1 + m_2 + m_3 = 2d + 1$). In the former case $S_0 = S_1 = S_2 = \emptyset$, while in the latter case $S_1 = S_2 = \emptyset$ and $S_0 = \{1, \ldots, n\}$. Now (6.6) follows from Proposition 6.2.2(ii).

(b) From (a) and the definition of the metric on $\mathcal{H}$ we obtain the following multiplication rules in $\mathcal{H}$. First, we get that $e(1)$ is a unit. For $i, j \in [1, d - 1]$ we have

$$e(i) \cdot e(j) = \begin{cases} e(i + j - 1), & i + j \leq d + 1, \\ (1-t)^n e(i + j - 1), & i + j > d + 1 \end{cases}$$

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(for \(i + j = d + 1\) we use the fact that \((\zeta_\bullet)^* e(0) = e(0)\)). Also, for \(i \in [2, d - 1]\) and \(x \in \mathcal{H}_0\) one has
\[
x \cdot e(i) = (x, e(0))^R \cdot e(i - 1).
\]
Finally, for \(x, y \in \mathcal{H}_0\) one has
\[
x \cdot y = ((\zeta_\bullet)_x, y)^R \cdot e(d - 1).
\]
This is equivalent to our assertion.

Note that the associativity of the product rules obtained in the proof of Theorem 6.3.1(b) amounts to the identity
\[
(x, e(0))^R \cdot (y, e(0))^R = ((\zeta_\bullet)_x, y)^R \cdot (1 - t)^n = (x, y)^R \cdot (1 - t)^n.
\] (6.9)
This can be checked independently using Example 2.7.3. Namely, the left-hand side corresponds to the functor of restriction to the origin in \(\mathbb{A}^n \times \mathbb{A}^n\), and the right-hand-side is obtained by first restricting to the shifted diagonal and then to the origin.

**Remark 6.3.2.** The specialization of \(\mathcal{H}\) at \(t = 1\) is the algebra
\[
\mathbb{C}[\mathcal{H}_{0,0}, u]/(u^{d-1}, xu, xy - ((\zeta_\bullet)_x, y)u^{d-2}),
\]
where \(x, y \in \mathcal{H}_{0,0} = H(w)^{\mathbb{Z}/d}\). Indeed, this follows easily from the fact that \(e(0)\) has zero component in \(\mathcal{H}_{0,0}\) (see [49]). In the case when \(d = n\) the category of \(\mathbb{G}_m\)-equivariant matrix factorizations of \(w\) is equivalent to the derived category of coherent sheaves on the corresponding Calabi-Yau hypersurface \(X \subset \mathbb{P}^n\) (see [44]) and the above algebra is isomorphic to \(H^*(X, \mathbb{C})\).

7 Simple singularities

In this section we will calculate the Frobenius algebra structure on the \(R\)-module \(\mathcal{H} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma\) given by the CohFT of Theorem 5.1.2 for all simple singularities, i.e., for singularities of type \(A_n, D_n, E_6, E_7\) and \(E_8\). We will use the notation and the results of Section 6.

7.1 Singularity of type \(A\)

Consider the \(A_{d-1}\) singularity \(w = x^d\) with the symmetry group \(G = G_w = \langle J \rangle\), where \(J = \exp(2\pi im/d)\) (this is the only possible symmetry group allowed by our construction). Since \(w\) a homogeneous polynomial, we can apply Theorem 6.3.1. We keep the notation of Section 6.3. Thus, for \(m \in \mathbb{Z}/d\) we have
\[
\mathcal{H}_m = \begin{cases} 
R, & m \not= 0, \\
HH_* (\text{MF}_{\mathbb{G}_m} (\mathbb{A}^1, x^d)), & m \equiv 0.
\end{cases}
\]
From the description of the Hochschild homology of \( \text{MF}_{\mathbb{G}_m}(\mathbb{A}^1, x^d) \) (see Theorem 2.6.1) and the calculation of the Chern character \( \text{ch}(\mathcal{C}^*t) \) in [49, Ex. 4.2.2] we have an identification of \( R \)-modules

\[
R/(1 + t + \ldots + t^{d-1}) \xrightarrow{\sim} \mathcal{H}_0 : 1 \mapsto e(0) = \text{ch}(\mathcal{C}^*t).
\]

The metric on \( \mathcal{H}_0 \) is given by

\[
(e(0), e(0)) = (e(0), \zeta^* e(0))^R = (e(0), e(0))^R = 1 - t,
\]

where \( \zeta = \exp(\pi i/d) \) (see (5.12)). Indeed, since \( e(0) = \text{ch}(\mathcal{C}^*t) \), by Lemma 2.2.2, \( (e(0), e(0))^R \) is equal to the class of the \( \mathbb{Z}/d \)-representation \( (\mathcal{C}^*t)|_0 \) in \( R \). Also, \( \zeta^* \) acts trivially on the regular Koszul matrix factorization \( \mathcal{C}^*t \).

Thus, we have an isomorphism of \( R \)-modules

\[
\mathcal{H} = R/(1 + t + \ldots + t^{d-1}) \cdot e(0) \oplus \bigoplus_{m \in \mathbb{Z}/d \setminus 0} R \cdot e(m).
\]

The metric on \( \mathcal{H} \) is given by

\[
(e(m), e(-m)) = \begin{cases} 
1, & m \neq 0, \\
1 - t, & m \equiv 0.
\end{cases}
\]

By Theorem 6.3.1, setting \( u = e(2) \) we obtain an isomorphism of \( R \)-algebras

\[
\mathcal{H} \simeq R[u]/(u^d - 1 + t, (1 + t + \ldots + t^{d-1})u^{d-1}).
\]

If we specialize with respect to \( \pi_1 : t \mapsto 1 \) we get the Frobenius ring \( \mathbb{C}[u]/(u^d-1) \), i.e., the Milnor ring of the same singularity. On the other hand, the specialization with respect to \( \pi_\omega : t \mapsto \omega \), where \( \omega \) is a nontrivial \( d \)th root of unity, gives the semisimple ring \( \mathbb{C}[u]/(u^d - 1 + \omega) \).

### 7.2 Singularity of type \( D \) with the maximal group of diagonal symmetries

Consider the \( D_{d+1} \) singularity \( w = x^d + xy^2 \). The group \( G = G_w \) of diagonal symmetries is isomorphic to \( \mathbb{Z}/2d \), where \( m \in \mathbb{Z}/2d \) acts on \( \mathbb{A}^2 \) by \( \exp(-2\pi im/d), \exp(\pi im/d) \). In this case we denote the line bundles on \( \mathbb{P}^1 \) associated with a \( \Gamma \)-spin structure as \( L_x \) and \( L_y \). We will use the identification \( R = \mathbb{C}[t]/(t^{2d} - 1), \) where \( t : \mathbb{Z}/2d \to \mathbb{C}^* \) is given by (6.5) with \( d \) replaced by \( 2d \).

Let us calculate the \( R \)-modules \( \mathcal{H}_m \) in the decomposition of the state space \( \mathcal{H} = \bigoplus_{m \in \mathbb{Z}/2d} \mathcal{H}_m \).

For \( m = 0 \) we claim that there is an isomorphism of \( R \)-modules

\[
R/(1 - t + t^2 - t^3 + \ldots - t^{2d-1}) \to \mathcal{H}_0 : 1 \mapsto e(0),
\]

where \( e(0) = \text{ch}(E) \) with \( E = \{ x^{d-1} + y^2, x \} \). Indeed, consider the decomposition

\[
\mathcal{H}_0 = \bigoplus_{m \in \mathbb{Z}/2d} H(w_m)_{\mathbb{Z}/2d},
\]

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where \( w_m \) is the restriction of \( w \) to the subspace of \( m \)-invariants. We have
\[
H(w_0)^{\mathbb{Z}/2d} = \mathbb{C} \cdot y \, dx \wedge dy, \quad H(w_d)^{\mathbb{Z}/2d} = 0, \quad \text{and} \quad H(w_m)^{\mathbb{Z}/2d} = \mathbb{C} \quad \text{for} \quad m \neq 0, d.
\]
Now our claim follows from the fact that \( e(0) \) has nonzero components in \( H(w_m)^{\mathbb{Z}/2d} \) for all \( m \neq d \) (this computation is analogous to the case \( m = 0 \) and \( d = 3 \) considered in [49, Ex. 4.1.8]).

For \( m \neq 0, d \) the components \( \mathcal{H}_m \) can be identified with \( R \cdot e(m) \), and for \( m \equiv d \) we have an isomorphism
\[
\mathcal{H}_d = HH_*(MF_{\mathbb{Z}/2d}(\mathbb{A}^1, x^d)) \simeq R/(1 + t^2 + t^4 + \ldots + t^{2d-2}) \cdot e(d),
\]
where \( e(d) = \text{ch}(\mathbb{C}^*) \) (here \( m \in \mathbb{Z}/2d \) acts on \( \mathbb{A}^1 \) by \( \exp(2\pi i m/d) \)). Note that \((\zeta_\bullet)^* e(m) = e(m) \) since \( e(m) \) is the Chern class of an object invariant under \((\zeta_\bullet)^* \).

The metric on \( \mathcal{H} \) is given by
\[
(e(m), e(-m)) = 1 \quad \text{for} \quad m \neq 0, d,
\]
\[
(e(d), e(d)) = 1 - t^{-2},
\]
\[
(e(0), e(0)) = -(1 + t)t^{-2}.
\]
The second equality follows from the case of singularity \( A_{d-1} \). To get the last equality we observe that by Lemma 2.2.2, \((e(0), e(0))^R\) is equal to the class of the \( \mathbb{Z}/2 \)-graded complex of \( G \)-modules (with finite-dimensional cohomology)
\[
\pi_*(\mathcal{E}|_{x=0}) = \pi_*(\{y^2, 0\}) \simeq \mathbb{C}[y]/(y^2) \otimes t^{-2}[1], \quad (7.2)
\]
and the class of \( \mathbb{C}[y]/(y^2) \) is equal to \( 1 + t \).

To calculate \( \Lambda_0^R(e(m_1), e(m_2), e(m_3)) \in R \) we can apply Proposition 6.2.2(ii) in most cases. From now on we assume that \( 0 \leq m_i < 2d \) for \( i = 1, 2, 3 \). Note that \( q = (\frac{1}{d}, \frac{d-1}{2d}) \) while
\[
\boldsymbol{\theta}_i = \theta_{m_i} = \begin{cases} (0, 0), & m_i = 0, \\ (1 - \frac{m_i}{d} \cdot \frac{m_i}{2d}), & 0 < m_i \leq d, \\ (2 - \frac{m_i}{d} \cdot \frac{m_i}{2d}), & d < m_i < 2d. \end{cases}
\]
Thus, the condition \((6.3) \) implies that \( L_x = 0 \) only when two of the \( m_i \)'s are zero. On the other hand, \( L_y = 0 \) if and only if \( m_1 + m_2 + m_3 = d - 1 \), in which case it is possible that only one or none of \( m_i \)'s is trivial. Thus, we have two cases not covered by Proposition 6.2.2(ii):

**Case 1a.** \( m_1 + m_2 + m_3 = d - 1 \) with \( m_i > 0 \) for \( i = 1, 2, 3 \);

**Case 1b.** \( m_1 = 0 \) and \( m_2 + m_3 = d - 1 \) with \( m_3 > 0 \) and \( m_3 > 0 \).

Consider first Case 1a (which can occur only for \( d \geq 4 \)). We have \( \theta_1 = (1 - m_i/d, m_i/(2d)) \), so \( \theta_1 + \theta_2 + \theta_3 = (2 + 1/d, (d - 1)/(2d)) \). Thus, we have \( L_x = 0 (-2) \) and \( L_y = 0 \). Hence, this is the case of index zero considered in Section 5.7 (since \( \chi(0(-2)) + \chi(0) = 0 \)). Also, since \( 0 < m_i < d - 1 \), all these group elements have trivial invariants in \( \mathbb{A}^2 \). Thus, applying Proposition 5.7.1(ii), we obtain
\[
\Lambda_0^R(e(m_1), e(m_2), e(m_3)) = \frac{1 - t_1^{-d} \cdot t_1}{1 - t_2}|_{t_1=t^{-2},t_2=t}.
\]

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where \( t_1 \) and \( t_2 \) are characters of the group \( \Gamma \) satisfying \( t_1^{d-1} = t_2^d \). Thus, we have

\[
\Lambda_0^R(e(m_1), e(m_2), e(m_3)) = \frac{1 - t_2^{-2}}{1 - t_2} |t_2| = -(1 + t)t^{-2}.
\]

In Case 1b we have \( \theta_1 + \theta_2 + \theta_3 = (1 + 1/d, (d - 1)/2d) \), so \( L_x = \emptyset(-1) \) and \( L_y = \emptyset \). Also, we have \( \Sigma_x = \Sigma_y = \{p_1\} \). By Proposition 6.2.2(i), in this case the fundamental matrix factorization \( \mathbf{P}(\psi) \) is the Koszul matrix factorization \( \{ -x^{d-1} - y^2, x \} \) on \( \mathbb{A}^2 \). Hence, by Lemma 2.2.2, the corresponding functor \( \text{MF}_G(w) \to \text{Com}_f(G - \text{mod}) \) associates with \( \bar{E} \) the restriction \( \text{com}_G(E|_{x=0}) \), which we computed in (7.2). This gives

\[
\Lambda_0^R(e(0), e(m), e(d - 1 - m)) = -(1 + t)t^{-2} \text{ for } 0 < m < d - 1.
\]

In all other cases we can apply Proposition 6.2.2(ii). Note that \( \Lambda_0^R(e(m_1), e(m_2), e(m_3)) \) is only nonzero when \( m_1 + m_2 + m_3 \equiv d - 1(2d) \) (since \( J \) corresponds to \( d - 1 \in \mathbb{Z}/2d \)). Note also that by (6.3),

\[
L_y = \emptyset(-a),
\]

where \( m_1 + m_2 + m_3 = d - 1 + 2da \).

**Case 2a.** \( m_1 + m_2 + m_3 = 3d - 1 \) with \( 0 < m_1 < d, 0 < m_2 < d \) and \( d < m_3 < 2d \).

In this case \( L_x = L_y = \emptyset(-1) \), so \( S_0 = S_1 = S_2 = \emptyset \) and we get

\[
\Lambda_0^R(e(m_1), e(m_2), e(m_3)) = 1.
\]

**Case 2b** (occurs only for \( d \geq 4 \)). \( m_1 + m_2 + m_3 = 3d - 1 \) with \( 0 < m_1 < d, d < m_2 < 2d \) and \( d < m_3 < 2d \).

In this case \( L_x = \emptyset(-2) \) and \( L_y = \emptyset(-1) \), so \( S_1 = S_2 = \emptyset \) and \( S_0 = \{x\} \). Thus, we get

\[
\Lambda_0^R(e(m_1), e(m_2), e(m_3)) = 1 - t^{-2}.
\]

**Case 2c.** \( m_1 + m_2 + m_3 = 5d - 1 \) with \( d < m_i < 2d \) for \( i = 1, 2, 3 \).

In this case \( L_x = \emptyset(-1) \) and \( L_y = \emptyset(-2) \), so \( S_1 = S_2 = \emptyset \) and \( S_0 = \{y\} \). Thus,

\[
\Lambda_0^R(e(m_1), e(m_2), e(m_3)) = 1 - t.
\]

**Case 3.** \( m_1 = d, 0 < m_2 < d, d < m_3 < 2d \) with \( m_1 + m_2 = 2d - 1 \). In this case \( L_x = L_y = \emptyset(-1) \), \( S_1 = \{x\} \) and \( S_0 = S_2 = \emptyset \), so we have to calculate the class of the restriction of \( \mathbb{C}^{st} \in \text{MF}_{G_m}(\mathbb{A}^1, x^d) \) to \( 0 \in \mathbb{A}^1 \). Thus,

\[
\Lambda_0^R(e(d), e(m), e(2d - 1 - m)) = 1 - t^{-2} \text{ for } 0 < m < d - 1.
\]

**Case 4.** \( m_1 = 0, m_2 + m_3 = 3d - 1 \), where \( d < m_2 < 2d \) and \( d < m_3 < 2d \). In this case \( L_x = L_y = \emptyset(-1) \), so \( S_1 = \{x, y\} \) and \( S_0 = S_2 = \emptyset \). Hence, we are reduced to computing the class of the restriction of \( \bar{E} \) to \( x = y = 0 \). Therefore, we get

\[
\Lambda_0^R(e(0), e(m), e(3d - 1 - m)) = 1 - t^{-2} \text{ for } d < m < 2d - 1.
\]
Case 5. \( m_1 = 0, m_2 = d, m_3 = 2d - 1 \). In this case \( L_x = 0, \ L_y = \mathcal{O}(-1) \), so we have \( S_1 = \{ y \}, S_2 = \{ x \} \) and \( S_0 = \emptyset \). Thus, we have to calculate the class of the restriction
\[
\bar{E} \boxtimes \mathbb{C}^d \in \text{MF}(\mathbb{A}^2 \times \mathbb{A}^1, w(x_1, y) \oplus x_2^d)
\]
to the linear subspace \( x_1 = x_2, \ y = 0 \). Since tensoring with \( \mathbb{C}^d \) has the same effect as the restriction to the origin, this class is equal to the class of the restriction of \( \bar{E} \) to \( x = y = 0 \), so we obtain as in the previous case
\[
\Lambda_{R}^0(e(0), e(d), e(2d-1)) = 1 - t^{-2}.
\]

The remaining two cases \((m_1, m_2, m_3) = (0, 0, d - 1) \) or \((d, d, d - 1)\) can be computed using (6.2), since \( e(d - 1) \) is the flat unit \( 1 \) for our theory.

Now we can determine the ring structure on \( \mathcal{H} \). The element \( e(d - 1) \) is a unit. We always have
\[
e(m_1)e(m_2) = r(m_1, m_2)e(m_1 + m_2 - d + 1)
\]
with some \( r(m_1, m_2) \in R \). From the formulas for \( \Lambda_{R}^0(?, ?, ?) \) we get the relations
\[
e(d - 2)e(m) = e(m - 1) \text{ for } 1 \leq m \leq d - 2,
\]
\[
e(d - 2)^d = e(d - 2)e(0) = -(1 + t)t^{-2}e(-1),
\]
\[
e(d - 2)e(-m) = e(-m - 1) \text{ for } 1 \leq m \leq d - 1,
\]
\[
e(-1)^2 = (1 - t)e(d - 1).
\]
Thus, \( \mathcal{H} \) is generated as an \( R \)-algebra by elements \( u = e(d - 2) \) and \( v = e(-1) \) subject to the relations
\[
v^2 = 1 - t, \quad u^d = -(1 + t)t^{-2}v, \quad (1 - t)(1 + t^2 + \ldots + t^{2d - 2})u^{d-1} = (1 + t^2 + \ldots + t^{2d - 2})u^{d-1}v = 0.
\]

The specialization \( t = 1 \) gives the Frobenius algebra \( \mathbb{C}[u]/(u^{2d-1}) \) with the pairing given by
\[
(u^{2d-2}, 1) = -2(u^{d-2}e(-1), e(d - 1)) = -2(e(-d + 1), e(d - 1)) = -2.
\]

7.3 Singularity of type \( D \) with the non-maximal symmetry group

The group of diagonal symmetries \( G_w \) of the \( D_{d+1} \) singularity \( w = x^d + xy^2 \) is not generated by the exponential grading element \( J = (\exp(2\pi i/d), \exp(2\pi i k/d)) \) precisely when \( d = 2k + 1 \) is odd. In this case \( J \) has order \( d \), so the subgroup \( G = \langle J \rangle \) has index two in \( G_w \). In this section we will calculate the Frobenius algebra corresponding to this subgroup. We will use the identification \( R = \mathbb{C}[t]/(t^d - 1) \), where the generating character \( t \) is defined by \( t(J) = \exp(-2\pi i / d) \). As before we denote the line bundles of a \( \Gamma \)-spin structure by \( L_x \) and \( L_y \). For \( m \in \mathbb{Z} / d \) we set \( \mathcal{H}_m = \mathcal{H}_{J^m} \). Since all nontrivial powers of \( J \) have no invariants on \( \mathbb{A}^2 \), we have \( \mathcal{H}_m = R \) for \( m \neq 0 \). For such \( m \) we denote by \( e(m) \) the element \( 1 \in \mathcal{H}_m \). To compute \( \mathcal{H}_0 \) as an \( R \)-module consider \( \langle J \rangle \)-equivariant matrix factorizations
\[
\bar{E}_\pm = \{ x^{k+1} \pm ixy, x^k \mp iy \}.
\]
and denote their Chern characters by $e_\pm(0) = ch(\bar{E}_\pm)$. We claim that the map

$$R \oplus R \to \mathcal{H}_0: (r_1, r_2) \mapsto r_1 e_+(0) + r_2 (e_+(0) - e_-(0))$$

induces an isomorphism of $R$-modules

$$R \oplus R / (t - 1) \to \mathcal{H}_0.$$

Indeed, the components of the decomposition

$$\mathcal{H}_0 = \bigoplus_{m \in \mathbb{Z}/d} H(w_j^m) \cdot J$$

are

$$H(w)^J = (\mathbb{C} \cdot x^k + \mathbb{C} \cdot y) \cdot dx \wedge dy, \quad H(w_j^m)^J = \mathbb{C} \text{ for } m \neq 0.$$

Now using [49, Thm. 3.3.3] we obtain that $ch(\bar{E}_\pm)$ have nonzero $m$-components for $m \neq 0$, and their 0-component are given by

$$ch(\bar{E}_\pm)_0 = (\mp i dx^k + y) \cdot dx \wedge dy.$$

This immediately implies our claim.

The metric on $\mathcal{H}$ is given by

$$(e(m), e(-m)) = 1 \text{ for } m \neq 0,$$

$$e_\pm(0), e_\pm(0)) = -(1 + t + \ldots + t^k) t^k$$

$$e_+(0), e_-(0)) = 1 + t + \ldots + t^{k-1}.$$

Indeed, the last two equalities follow from the quasi-isomorphisms

$$\pi_* \bar{E}_\pm |_{x^k + iy = 0} = \pi_* \{2x^{k+1}, 0\} \simeq \mathbb{C}[x]/(x^{k+1}) \otimes t^{-k-1}[1],$$

$$\pi_* \bar{E}_+ |_{x^k + iy = 0} = \pi_* \{0, 2x^k\} \simeq \mathbb{C}[x]/(x^k).$$

The calculation of the three-point correlators $\Lambda^R_0(m_1, m_2, m_3)$ is done similarly to the case of the maximal symmetry group. We have

$$q = (\frac{1}{d}, \frac{k}{d}) \text{ and } \theta_{m_i} = \begin{cases} (0, 0), & m_i = 0, \\ (\frac{2l}{d}, 1 - \frac{l}{d}), & m_i = 2l, 0 < l \leq k, \\ (1 - \frac{2l}{d}, \frac{l}{d}), & m_i = -2l, 0 < l \leq k. \end{cases}$$

The condition (6.3) implies that $L_x = \emptyset$ only when two of the $m_i$’s are trivial.

We have the following cases with $L_y = \emptyset$ and $|\Sigma_y| \leq 1$.

**Case 1a.** $m_i = -2l_i, i = 1, 2, 3$, where $l_1 + l_2 + l_3 = k$, $l_i > 0$. In this case $L_x = \emptyset(-2)$ and $\Sigma_x = \Sigma_y = \emptyset$. Hence, by Proposition 5.7.1(ii), we obtain

$$\Lambda^R_0(e(-2l_1), e(-2l_2), e(-2l_3)) = \frac{1 - t^{-d} \cdot t}{1 - t^k} |_{a=1} = -(1 + t^k)t.$$
Case 1b. \( m_1 = 0, m_2 = -2l_2, m_3 = -2l_3, \) where \( l_2 + l_3 = k, l_i > 0. \) In this case \( L_x = \mathcal{O}(-1) \) and \( \Sigma_x = \Sigma_y = \{p_1\}. \) By Proposition 6.2.2(i), we have
\[
P(\psi) = \{-x^{d-1} - y^2, x\}.
\]
Therefore, by Lemma 2.2.2, the corresponding functor \( MF_\Gamma(w) \to \text{Com}_f(G - \text{mod}) \) is given by the restriction to \( x = 0. \) We have
\[
\pi_* \bar{E}_|_{x=0} = \pi_* \{0, \mp iy\} \cong \mathbb{C}.
\]
Hence,
\[
\Lambda_0^R(e_\pm(0), e(-2l_2), e(-2l_3)) = 1.
\]

In the cases when \( L_y \neq \mathcal{O} \) or \( |\Sigma_y| \geq 2 \) we can apply Proposition 6.2.2(ii). Note that \( \Lambda_0^R(m_1, m_2, m_3) \) is nonzero only when \( m_1 + m_2 + m_3 \equiv 1 \mod d. \)

Case 2a. \( m_1 = 2l_1, m_2 = -2l_2, m_3 = -2l_3, \) where \( 1 \leq l_i \leq k \) and \( l_2 + l_3 - l_1 = k. \) In this case \( L_x = L_y = \mathcal{O}(-1), \) so \( S_0 = S_1 = S_2 = \emptyset, \) and we get
\[
\Lambda_0^R(e(2l_1), e(-2l_2), e(-2l_3)) = 1.
\]

Case 2b. \( m_1 = 2l_1, m_2 = 2l_2, m_3 = -2l_3, \) where \( 1 \leq l_i \leq k \) and \( -l_3 + l_1 + l_2 = k + 1. \) In this case \( L_x = \mathcal{O}(-2) \) and \( L_y = \mathcal{O}(-1), \) so \( S_1 = S_2 = \emptyset \) and \( S_0 = \{x\}. \) Hence,
\[
\Lambda_0^R(e(2l_1), e(2l_2), e(-2l_3)) = 1 - t.
\]

Case 2c. \( m_i = 2l_i, \) where \( 1 \leq l_i \leq k \) and \( l_1 + l_2 + l_3 = k + 1. \) In this case \( L_x = \mathcal{O}(-1) \) and \( L_y = \mathcal{O}(-2), \) so \( S_1 = S_2 = \emptyset \) and \( S_0 = \{y\}. \) Hence,
\[
\Lambda_0^R(e(2l_1), e(2l_2), e(2l_3)) = 1 - t^k.
\]

Case 3. \( m_1 = 0, m_2 = 2l, m_3 = 2(k+1-l), \) where \( 1 \leq l \leq k. \) In this case \( L_x = L_y = \mathcal{O}(-1), \) so \( S_0 = S_2 = \emptyset \) and \( S_1 = \{x, y\}. \) Therefore, \( \Lambda_0^R(0, 2l, 2(k+1-l)) \) sends the class of a matrix factorization of \( w \) on \( \mathbb{A}^2 \) to the class of its restriction to the origin. Thus,
\[
\Lambda_0^R(e_\pm(0), e(2l), e(2(k+1-l))) = 1 - t^k.
\]

The remaining case \( m_1 = m_2 = 0, m_3 = 1 \) follows from Lemma 6.1.1 since \( e(1) \) is the flat unit.

Now let us determine the ring structure on \( \mathcal{H}. \) The element \( e(1) \) is a unit. Note that the product of elements in \( \mathcal{H}_{m_1} \) and \( \mathcal{H}_{m_2} \) lies in \( \mathcal{H}_{m_1+m_2-1}. \) Using the above calculations we obtain
\[
e_\pm(0)^2 = -(1 + t + \ldots + t^k)t^k e(2k), \quad e_+(0)e_-(0) = (1 + t + \ldots + t^{k-1})e(2k),
\]
\[
e_\pm(0)e(2l) = (1 - t^k)e(-2(k + 1 - l)),
\]
\[
e_\pm(0)e(-2l) = e(2(k - l)) \text{ for } l < k,
\]
\[
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\]
This Frobenius algebra is isomorphic to the Milnor ring of the singularity $D$.

Thus, $H_u$ is generated as an $R$-algebra by the elements $u = e(3) = e(2) = e_+(0) u^l$ for $1 \leq l \leq k$, $u^k = -t(1 + t^k) \cdot (e_+(0) + e_-(0))$,

Thus, $H$ is generated as an $R$-algebra by the elements $u$ and $v = e_+(0) - e_-(0)$, subject to the relations

The specialization $t = 1$ gives the Frobenius algebra $\mathbb{C}[u, v]/(uv, v^2 - du^{d-1})$ with the pairing

Note that this Frobenius algebra is isomorphic to the Milnor ring of the singularity $D_{d+1}$.

In the $D_4$ case, i.e., when $k = 1$, we obtain that $H$ is generated as an $R$-algebra by the elements $e_\pm(0)$, subject to the relations

If we take generators $u = (e_+(0) + e_-(0))/2$ and $v = (e_+(0) - e_-(0))/2$, the relations become

The specialization $t = 1$ gives the Frobenius algebra $\mathbb{C}[u, v]/(uv, 3u^2 - v^2)$ with the pairing

This Frobenius algebra is isomorphic to the Milnor ring of the $D_4$ singularity.
7.4 \( E_7 \) singularity

In this case of the \( E_7 \) singularity \( w = x^3 + xy^3 \) the maximal symmetry group \( G = G_w \) is generated by \( J = (\exp(2\pi i/3), \exp(4\pi i/9)) \). We will use the identification of \( G \) with \( \mathbb{Z}/9 \) where \( m \in \mathbb{Z}/9 \) acts on \( \mathbb{A}^2 \) by \( (\exp(-2\pi im/3), \exp(2\pi im/9)) \). We will denote by \( t \) the character of \( \mathbb{Z}/9 \) given by (6.5) (with \( d = 9 \)) and use the identification \( R = \mathbb{C}[t]/(t^9 - 1) \).

First, let us determine the \( R \)-module \( \mathcal{H} = \bigoplus_{m \in \mathbb{Z}/9} \mathcal{H}_m \). We claim that there is an isomorphism of \( R \)-modules

\[
R/((t - 1)(1 + t^3 + t^6)) \to \mathcal{H}_0 : 1 \mapsto e(0),
\]

where \( e(0) = \text{ch}(\bar{E}) \) with \( \bar{E} = \{x^2 + y^3, x\} \). Indeed, the summands of the decomposition (2.10) of \( \mathcal{H}_0 \) are

\[
H(w_0)^G = \mathbb{C} \cdot y^2 \cdot dx \wedge dy, \quad H(w_{\pm 3})^G = H(w_m)^G = \mathbb{C} \quad \text{for} \quad m \neq 0, \pm 3,
\]

where \( w_m \) is the restriction of \( w \) to the subspace of \( m \)-invariants in \( \mathbb{A}^2 \). Our claim follows from the fact that \( e(0) \) has nonzero components in \( H(w_m) \) for all \( m \neq \pm 3 \).

The components \( \mathcal{H}_m \) for \( m \) not divisible by 3 can be identified with \( R \cdot e(m) \), while

\[
\mathcal{H}_{\pm 3} = H H_s(MF_{\mathbb{Z}/9}(\mathbb{A}^1, x^3)) \simeq R/(1 + t^3 + t^6) \cdot e(\pm 3),
\]

where \( e(\pm 3) = \text{ch}(\mathbb{C}^n) \).

The metric on \( \mathcal{H} \) is given by

\[
(e(m), e(-m)) = 1 \quad \text{for} \quad m \neq 0, \pm 3,
\]

\[
(e(3), e(-3)) = 1 - t^{-3},
\]

\[
(e(0), e(0)) = -(1 + t + t^2) t^{-3},
\]

where the last equality follows from the quasi-isomorphism

\[
\pi_* \bar{E}|_{x=0} = \pi_* \{y^3, 0\} \simeq \mathbb{C}[y]/(y^3) \otimes t^{-3}[1].
\]

Now let us compute the three-point correlators \( \lambda^R_0(e(m_1), e(m_2), e(m_3)) \). We have \( q = \left(\frac{1}{3}, \frac{2}{3}\right) \) and

\[
\theta_{m_i} = \begin{cases} 
(0, \frac{1}{3}), & m_i = 3l, 0 \leq l \leq 2, \\
(\frac{2}{3}, \frac{3l+1}{9}), & m_i = 3l + 1, 0 \leq l \leq 2, \\
(\frac{1}{3}, \frac{3l+2}{9}), & m_i = 3l + 2, 0 \leq l \leq 2.
\end{cases}
\]

Note that \( L_x = \emptyset \) implies that \( |\Sigma_x| = 2 \). Hence, we have only one case not covered by Proposition 6.2.2(ii).

Case 1. \( m_1 = 0, m_2 = 1, m_3 = 1 \). In this case \( L_x = \emptyset(-1), L_y = \emptyset, \Sigma_x = \Sigma_y = \{p_1\} \).

Thus, we have \( S_0 = S_2 = \emptyset \) and \( S_1 = \{x\} \). Then by Proposition 6.2.2(i), \( P(\psi) \) is the Koszul matrix factorization \( \{ -x^2 - y^3, x \} \). Thus, \( \lambda^R_0(e(0), e(1), e(1)) \) is given by the class of \( \pi_* (\bar{E}|_{x=0}) \) computed above. Hence,

\[
\Lambda^R_0(e(0), e(1), e(1)) = -(1 + t + t^2) t^{-3}.
\]

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In all remaining cases we can use Proposition 6.2.2(ii). Note that $\Lambda^R_0(e(m_1), e(m_2), e(m_3))$ is nonzero only when $m_1 + m_2 + m_3 \equiv 2 \mod 9$.

**Case 2a.** $m_1 = 3l_1 + 1, m_2 = 3l_2 + 2, m_3 = 3l_3 + 2$, where $0 \leq l_i \leq 2, l_1 + l_2 + l_3 = 2$. We have $L_x = L_y = \emptyset(-1), S_0 = S_1 = S_2 = \emptyset$. Hence, in this case

$$\Lambda^R_0(e(3l_1 + 1), e(3l_2 + 2), e(3l_3 + 2)) = 1.$$  

**Case 2b.** $m_1 = 3l_1 + 1, m_2 = 3l_2 + 2, m_3 = 3l_3 + 2$, where $0 \leq l_i \leq 2, l_1 + l_2 + l_3 = 5$. We have $L_x = L_y = \emptyset(-1), L_y = \emptyset(-2), S_1 = S_2 = \emptyset, S_0 = \{y\}$. Hence, in this case

$$\Lambda^R_0(e(3l_1 + 1), e(3l_2 + 2), e(3l_3 + 2)) = 1 - t.$$  

**Case 3a.** $m_1 = 3l_1, m_2 = 3l_2 + 1, m_3 = 3l_3 + 1$, where $0 \leq l_i \leq 2, l_1 > 0, l_1 + l_2 + l_3 = 3$. We have $L_x = L_y = \emptyset(-1), S_0 = S_2 = \emptyset, S_1 = \{x\}$. Hence, in this case we have to compute the class of the restriction of $e(3l_1)$ to the origin, which gives

$$\Lambda^R_0(e(3l_1), e(3l_2 + 1), e(3l_3 + 1)) = 1 - t^{-3}.$$  

**Case 3b.** $m_1 = 6, m_2 = m_3 = 7$. We have $L_x = \emptyset(-1), L_y = \emptyset(-2), S_2 = \emptyset, S_0 = \{y\}, S_1 = \{x\}$. Hence,

$$\Lambda^R_0(e(6), e(7), e(7)) = (1 - t)(1 - t^{-3}).$$  

**Case 3c.** $m_1 = 0, m_2 = 3l_2 + 1, m_3 = 3l_3 + 1$, where $0 \leq l_i \leq 2, l_2 + l_3 = 3$. We have $L_x = L_y = \emptyset(-1), S_0 = S_2 = \emptyset, S_1 = \{x, y\}$. Hence, we have to compute the restriction of $E$ to the origin, which gives

$$\Lambda^R_0(e(0), e(3l_2 + 1), e(3l_3 + 1)) = 1 - t^{-3}.$$  

**Case 4a.** $m_1 = 3l_1, m_2 = 3l_2, m_3 = 3l_3 + 2$, where $0 \leq l_i \leq 2, l_1 > 0, l_2 > 0, l_1 + l_2 + l_3 = 3$. We have $L_x = \emptyset, L_y = \emptyset(-1), S_0 = S_1 = \emptyset, S_2 = \emptyset$. Hence, in this case we have to compute the class of $\pi_*(\mathbb{C}^* \otimes \mathbb{C}^*)$, which is equivalent to computing the pairing ($(e(3), e(-3))$. Therefore, we get

$$\Lambda^R_0(e(3l_1), e(3l_2), e(3l_3 + 2)) = 1 - t^{-3}.$$  

**Case 4b.** $m_1 = 6, m_2 = 6, m_3 = 8$. We have $L_x = \emptyset, L_y = \emptyset(-2), S_1 = \emptyset, S_0 = \{y\}, S_2 = \{x\}$. Hence,

$$\Lambda^R_0(e(6), e(6), e(8)) = (1 - t)(1 - t^{-3}).$$  

**Case 4c.** $m_1 = 0, m_2 = 3l_2, m_3 = 3l_3 + 2$, where $l_2 > 0, l_3 > 0, l_2 + l_3 = 3$. We have $L_x = \emptyset, L_y = \emptyset(-1), S_0 = S_1 = \{y\}, S_2 = \{x\}$. Hence, we have to compute the restriction of the matrix factorization $E \otimes \mathbb{C}^*$ on the space $\mathbb{A}^2 \times \mathbb{A}^1$ with coordinates $(x, y, x')$ to the subspace $ax + bx' = y = 0$.restricting to $y = 0$ we get $\mathbb{C}^* \otimes \mathbb{C}^*$, so the answer is the same as in Case 4a:

$$\Lambda^R_0(e(0), e(3l_2), e(3l_3 + 2)) = 1 - t^{-3}.$$  

The remaining case $m_1 = m_2 = 0, m_3 = 2$ follows from the metric axiom.

Now we can determine the ring structure on $\mathcal{H}$. Note that $e(m_1) \cdot e(m_2)$ is always proportional to $e(m_1 + m_2 - 2)$ and $e(2)$ is a unit of $\mathcal{H}$. From the above computation of
\[ \Lambda^R(e(m_1), e(m_2), e(m_3)) \] and the formulas for the metric we obtain the following multiplication table:

\[
\begin{align*}
e(1)e(5) &= e(4), & e(1)e(8) &= e(4)e(5) = e(7), \\
e(4)e(8) &= e(7)e(5) = (1-t)e(1), & e(7)e(8) &= (1-t)e(4), \\
e(5)^2 &= e(8), & e(5)e(8) &= (1-t), & e(8)^2 &= (1-t)e(5), \\
e(1)^2 &= e(0), & e(1)e(5) &= e(3), & e(1)e(7) &= e(4)^2 = e(6), & e(4)e(7) &= (1-t)e(0), \\
e(0)e(1) &= -(1+t+t^2)t^{-3}e(-1), & e(6)e(7) &= (1-t)(1-t^{-3}), \\
e(0)e(4) &= e(3)e(1) = (1-t^{-3}), & e(0)e(7) &= e(4)e(5) = e(6)e(1) = (1-t^{-3})e(5), \\
e(3)e(7) &= e(6)e(4) = (1-t^{-3})e(8), & e(3)e(5) &= e(6), & e(3)e(8) &= e(6)e(5) = (1-t)e(0), & e(6)e(8) &= (1-t)e(3), \\
e(3)^2 &= e(0)e(6) = (1-t^{-3})e(4), & e(3)e(6) &= (1-t^{-3})e(7), & e(6)^2 &= (1-t)(1-t^{-3})e(1), \\
e(0)e(3) &= (1-t^{-3})e(1), & e(0)^2 &= -(1+t+t^2)t^{-3}e(7).
\]

It follows that \( \mathcal{H} \) is generated as an \( R \)-algebra by the elements \( u = e(5) \) and \( v = e(1) \) subject to the relations

\[
u^3 = 1 - t, \quad v^3 = -(1+t+t^2)t^{-3}u^2, \quad (1 + t^3 + t^6)uv^2 = 0.
\]

The specialization \( t = 1 \) gives the Frobenius algebra \( \mathbb{C}[u, v]/(v^3 + 3u^2, uv^2) \) with the pairing

\[
(u^2v, 1) = (e(7), e(2)) = 1.
\]

Thus, we again obtain the Milnor ring of the same singularity.

### 7.5 \( E_6 \) and \( E_8 \) singularities

In the case of the \( E_6 \) and \( E_8 \) singularities \( w = x^3 + y^4 \) and \( w = x^3 + y^5 \) the maximal group \( G = G_w \) of diagonal symmetries coincides with \( \langle J \rangle \). Thus, in both cases we have \( w = w_1 \oplus w_2 \) and \( G = G_1 \times G_2 \), where \( (w_1, G_1) \) is an \( A_n \)-singularity for some \( n \). By Theorem 5.8.1, the corresponding Frobenius algebras over \( R \) are tensor products (over \( \mathbb{C} \)) of the Frobenius algebras corresponding to \( (w_1, G_1) \) and \( (w_2, G_2) \). Thus, for \( E_6 \) we have \( R = \mathbb{C}[t]/(t^{12} - 1) \), and \( \mathcal{H}_{E_6} \) is the \( R \)-algebra generated by \( u \) and \( v \) subject to the relations

\[
u^3 = 1 - t^4, \quad (1 + t^4 + t^8)u^2 = 0, \quad v^4 = 1 - t^3, \quad (1 + t^3 + t^6 + t^9)v^3 = 0.
\]

For \( E_8 \) we have \( R = \mathbb{C}[t]/(t^{15} - 1) \), and \( \mathcal{H}_{E_8} \) is generated over \( R \) by \( u \) and \( v \) with the relations

\[
u^3 = 1 - t^5, \quad (1 + t^5 + t^{10})u^2 = 0, \quad v^5 = 1 - t^3, \quad (1 + t^3 + t^6 + t^9 + t^{12})v^4 = 0.
\]

In both cases the specialization \( t = 1 \) gives the Milnor ring of the corresponding singularity.
7.6 Comparison with the Fan-Jarvis-Ruan theory

Let $w$ be a simple singularity, and let $G \subset G_w$ be a subgroup containing $J$. Here we will show that our reduced CohFT for the pair $(w, G)$ is isomorphic to the FJR-theory for the same data constructed in [14]. Recall that the state space

\[ H_{FJR} = \bigoplus_{\gamma \in G} H_{\gamma} \]

of the FJR theory coincides with our state space

\[ H_{\text{red}} = H(w, G, 1) = \bigoplus_{\gamma \in \Gamma} H_{\gamma} \]

(see (5.13)). However, the obvious identification of the state spaces is not compatible with the operations of CohFT.

**Theorem 7.6.1.** The reduced CohFT associated with the pair $(w, G)$ is isomorphic to the FJR-theory for the same pair.

**Proof.** In Sections 7.1–7.5 we showed that for simple singularities all the components $H_{\gamma}$ are generated by the Chern characters of Koszul matrix factorizations of rank 1. Therefore, by Corollary 5.6.4, the Homogeneity Conjecture holds for $(w, G)$. Together with the results of Sections 5.5–5.8 this implies that our reduced CohFT has all the properties established in [14, Sec. 4] for the FJR-theory. Therefore, the Reconstruction Theorem [14, Thm. 6.2.10], proved for the FJR-theory, is valid for our theory as well.

We claim that in order to construct an isomorphism of the theories it is enough to find an isomorphism $\psi : H_{\text{red}} \sim H_{FJR}$ of Frobenius algebras respecting metrics such that for every $\gamma \in G$ with $(A^n)_\gamma = 0$, the restriction of $\psi$ to $H_{\gamma}$ is the identity map $\mathbb{C} = H_{\gamma} \to H_{\gamma} = \mathbb{C}$. Indeed, the latter condition guarantees that $\psi$ respects the CohFT maps in the concave case (see Corollary 5.5.3). For all $(w, G)$ except the $D_4$ singularity with the group $G = \langle J \rangle$, the Reconstruction Theorem implies that both theories are determined by the Frobenius algebra structure on the state space along with a certain four-point correlator which can computed using the Concavity property. In the remaining case $(D_4, \langle J \rangle)$ a similar statement is true as follows from [13, Thm. 4.5]. (The analog of this theorem holds also for our reduced CohFT.) Now let us construct the required isomorphism $H_{\text{red}} \simeq H_{FJR}$ for singularities of each type.

1. **Types $A$, $E_6$ and $E_8$.** In these cases for every $\gamma \in G$ we have either $(A^n)_\gamma = 0$ or $H_{\gamma} = H_{FJR} = 0$. Thus, we should take $\psi$ to be the natural identification $H_{\text{red}} \rightarrow H_{FJR}$. All genus zero correlators in both theories are computed from the Concavity property (see [23], [16]), so they coincide under this identification of the state spaces.

2. **Type $D_{d+1}$, $G = G_w$.** In this case we still have $(A^2)_\gamma = 0$ for all $\gamma \neq 1$, but the component corresponding to $\gamma = 1$ is nonzero. As shown in Section 7.2, the algebra $H_{\text{red}}$ is generated by the element $e(d-2)$ and we have

\[
eq (d-2)^i = (d-1-i) \quad \text{for } 1 \leq i \leq d-1 \text{ and} \\
eq (d-2)^{d-1+i} = -2e(-i) \quad \text{for } 1 \leq i \leq d-1.
\]
Note that these relations follow formally from the formulas for the metric, for the correlators of Case 2a in Section 7.2 and for one of the correlators of Case 1b,
\[
\lambda(e(0), e(d - 2), e(1)) = -2.
\] (7.3)

Now we define the map \( \psi : \mathcal{H}^{red} \to \mathcal{H}^{FJR} \) by
\[
\psi(e(i)) = \begin{cases} 
  e_{-i}, & i \neq 0, \\
  \epsilon \cdot 2ye_0, & i \equiv 0
\end{cases}
\]
with \( \epsilon = \pm 1 \), where in the right-hand side we use the notation of [14, Sec. 5.3.1] with \( n = d \).

Our calculations in Section 7.2 together with calculations of [14, Sec. 5.3.1] imply that this map is compatible with the metrics and with the correlators of Case 2a. Furthermore, as shown in [14, Sec. 5.3.1],
\[
\langle ye_0, e_{d+2}, e_{-1} \rangle = \pm 1.
\] (7.4)

Therefore, we can choose \( \epsilon \) such that \( \psi \) is compatible with the correlators (7.3) and (7.4).

Such \( \psi \) sends powers of \( e(d - 2) \) to the corresponding powers of \( e_{d+2} \), and so it is a ring isomorphism satisfying our requirements.

3. Type \( D_{d+1} \), \( G = \langle J \rangle \), \( d = 2k + 1 \). Assume first that \( k > 1 \). Then the algebra \( \mathcal{H}^{red} \) is generated by the elements \( u = e(3) \) and \( v = e_+(0) - e_-(0) \) subject to the relations \( uv = 0, v^2 = du^{d-1} \) (see Section 7.3). Therefore, using the notation and the calculations of [14, Sec. 5.2.4] we see that there is an algebra isomorphism
\[
\psi : \mathcal{H}^{red} \to \mathcal{H}^{FJR} : u \mapsto e_3, v \mapsto i\alpha x^k e_0 + i\beta ye_0.
\]

Note that we have
\[
u^l = \begin{cases} 
  e(2l + 1), & 0 \leq l \leq k - 1, \\
  -2e(2l - 2k), & k + 1 \leq l \leq 2k,
\end{cases}
\]
and similar relations, expressing \( e_j \) in terms of \( e_3 \), hold in \( \mathcal{H}^{FJR} \). Therefore, \( \psi(e(j)) = e_j \) for \( j \neq 0 \). Finally, the metric on \( \mathcal{H}^{red} \) is determined by \((u^{2k}, 1) = -2, \) and we have the similar relation for the metric on \( \mathcal{H}^{FJR} \). Hence, \( \psi \) respects the metrics.

In the case \( k = 1 \) the algebra \( \mathcal{H}^{red} \) is generated by the elements \( u \) and \( v \) such that \( uv = 0 \) and \( u^2 = v^2/3 = -e(2)/2 \). Calculations of [14, Sec. 5.2.4] show that \( \mathcal{H}^{FJR} \) is generated by the elements \( X = xe_3 \) and \( Y = ye_3 \) with the relations
\[
XY = 0, \quad X^2 = \frac{1}{6}e_2, \quad \text{and} \quad Y^2 = -\frac{1}{2}e_2.
\]

Therefore, the map \( \psi \) defined by
\[
\psi(u) = i\sqrt{3} \cdot X, \quad \psi(v) = \sqrt{3} \cdot Y
\]
gives an algebra isomorphism sending \( e(2) = e_2 \). This \( \psi \) also respects the metrics.

4. Type \( E_7 \). In the notation of [14, Sec. 5.2.2], let us define the algebra isomorphism \( \psi : \mathcal{H}^{red} \to \mathcal{H}^{FJR} \) by sending \( u = e(5) \) to \( e_7 \) and \( v = e(1) \) to \( e_5 \). Using the relations of Section 7.4 and of [14, Sec. 5.2.2] one immediately checks that \( \psi \) sends \( e(2j) \) to \( e_j \) for \( j \neq 0 \) mod 3. Since the metrics are determined by the relations \((e(-2), 1) = (e_8, 1) = 1, \) the map \( \psi \) also preserves the metrics.
APPENDIX. Functoriality of Hochschild homology

We will use the notation of Section 2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be small dg-categories which are dg Morita equivalent to smooth and proper dg-algebras, and let $F : \text{Per}_{dg}(\mathcal{C}) \to \text{Per}_{dg}(\mathcal{D})$ be a dg-functor. In this appendix we recall the construction of the map on Hochschild homology $F_* : HH_*(\mathcal{C}) \to HH_*(\mathcal{D})$

given in [49, Sec. 1.2] and will show that it agrees with the similar map constructed using the standard Hochschild complexes (see [55, Sec. 2.3]).

Recall that our construction in [49, Sec. 1.2] uses the fact that every dg-functor $F$ can be realized as the tensor product functor with a perfect $\mathcal{C} - \mathcal{D}$-bimodule $X$:

$$F(M) = M \otimes_{\mathcal{C}} X.$$ 

Let us consider the $\mathcal{D} - \mathcal{C}$-bimodule $X^T$ given by

$$X^T(D, C^\vee) = \text{Hom}_{\mathcal{D}^{\text{op}} - \text{mod}}(X(C, ?), h_D),$$

where $h_D$ is the representable right $\mathcal{D}$-module associated with $D \in \mathcal{D}$. In [49, Sec. 1.2] we constructed canonical morphisms

$$u : \Delta_\mathcal{C} \to X \otimes_{\mathcal{D}} X^T$$

and

$$c : X^T \otimes_{\mathcal{C}} X \to \Delta_\mathcal{D}$$

in the derived categories of $\mathcal{C} - \mathcal{C}$ and $\mathcal{D} - \mathcal{D}$-bimodules. The map $F_*$ is defined as the composition

$$\text{Tr}_{\mathcal{C}}(\Delta_\mathcal{C}) \xrightarrow{\text{Tr}_{\mathcal{C}}(u)} \text{Tr}_{\mathcal{C}}(X \otimes_{\mathcal{D}} X^T) \simeq \text{Tr}_{\mathcal{D}}(X^T \otimes_{\mathcal{C}} X) \xrightarrow{\text{Tr}_{\mathcal{D}}(c)} \text{Tr}_{\mathcal{D}}(\Delta_\mathcal{D}),$$

where the isomorphism in the middle is the canonical isomorphism constructed in [49, Lem. 1.1.3].

Let us describe a modification of this construction, convenient for our purposes. Consider the functor

$$F^{(2)} : \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C}) \to \text{Per}(\mathcal{D}^{\text{op}} \otimes \mathcal{D})$$

defined using the tensor products with $X$ and $X^T$:

$$F^{(2)}(M) = X^T \otimes_{\mathcal{C}}^{\text{L}} M \otimes_{\mathcal{C}}^{\text{L}} (X \otimes X^T) \simeq M \otimes_{\mathcal{C}^{\text{op}} \otimes \mathcal{C}} (X \otimes X^T)$$

for $M \in \text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$. Note that $F^{(2)}$ sends a representable $\mathcal{C} - \mathcal{C}$-bimodule $h_{C_1^\vee \otimes C_2}$ to $h_{F(C_1)^\vee \otimes F(C_2)}$.

Consider the canonical morphism of functors from $\text{Per}(\mathcal{C}^{\text{op}} \otimes \mathcal{C})$ to $\text{Per}(k)$

$$t_F : \text{Tr}_{\mathcal{C}} \to \text{Tr}_{\mathcal{D}} \circ F^{(2)}$$

induced by the map $u$ and the isomorphism

$$X \otimes_{\mathcal{D}} X^T \simeq (X \otimes X^T) \otimes_{\mathcal{D}^{\text{op}}} \Delta_\mathcal{D}$$
(see [49, eq. (1.8)]). Note that on a representable bimodule $h_{C_1 \otimes C_2}$ the morphism $t_F$ is given by the map
\[ \text{Hom}_C(C_1, C_2) \xrightarrow{F} \text{Hom}_D(F(C_1), F(C_2)). \]
Also consider the canonical morphism in $\text{Per}(D^{op} \otimes D)$
\[ c_F : F^{(2)}(\Delta_e) \to \Delta_D. \]
given by the composition
\[ F^{(2)}(\Delta_e) \simeq \Delta_e \otimes L_{\text{cop}_C} X \otimes X^T \simeq X^T \otimes_X X \xrightarrow{e} \Delta_D. \]

**Proposition 8.0.2.** The map $F_*$ is equal to the composition
\[ \Delta_\text{e} \xrightarrow{t_F(\Delta_\text{e})} \Delta_\text{D} \xrightarrow{\text{Tr}_D(F^{(2)}(\Delta_\text{e}))} \text{Tr}_D(\Delta_\text{D}). \]

**Proof.** By definition, $F_*$ is the composition of the following four morphisms
\[ \Delta_\text{e} \xrightarrow{\text{Tr}_C(\Delta_\text{e})} (X \otimes D X^T) \otimes L_{\text{cop}_C} \Delta_\text{e} \xrightarrow{\sim} (X \otimes X^T) \otimes L_{D^{op} \otimes \text{cop}_C \otimes \text{cop}_D} (\Delta_\text{D} \otimes \Delta_\text{e}) \xrightarrow{\text{Tr}_D(c_F)} \Delta_\text{D}. \]

It remains to notice that the composition of the first two arrows is $t_F(\Delta_\text{e})$ while the composition of the last two arrows is $\text{Tr}_D(c_F)$. \qed

Now we will compare our map $F_*$ with the map on Hochschild homology constructed using the standard complexes. Recall that the diagonal bimodule $\Delta_\text{e}$ has the bar-resolution by representable $\mathcal{C} - \mathcal{C}$-bimodules (see [30, Sec. 6.6]):
\[ \ldots \to \text{Bar}_1(\mathcal{C})(P^\vee, Q) \to \text{Bar}_0(\mathcal{C})(P^\vee, Q) \to \Delta_\text{e}(P^\vee, Q) = \mathcal{C}(P, Q), \quad (8.1) \]
where we use the notation $\mathcal{C}(?, ?) = \text{Hom}_\mathcal{C}(?, ?)$, and for $P, Q \in \mathcal{C}$,
\[ \text{Bar}_n(\mathcal{C})(P^\vee, Q) = \bigoplus_{C_0, \ldots, C_n \in \mathcal{C}} \mathcal{C}(C_n, Q) \otimes \mathcal{C}(C_{n-1}, C_n) \otimes \cdots \otimes \mathcal{C}(C_0, C_1) \otimes \mathcal{C}(P, C_0). \]
Computing $\text{Tr}_\mathcal{C}(\Delta_\text{e})$ with the help of this resolution and taking into account the identification
\[ \text{Bar}_n(\mathcal{C}) \otimes L_{\text{cop}_C} \Delta_\text{e} = \bigoplus_{C_0, \ldots, C_n \in \mathcal{C}} \mathcal{C}(C_n, C_0) \otimes \mathcal{C}(C_{n-1}, C_n) \otimes \cdots \otimes \mathcal{C}(C_0, C_1) \]
leads to the standard *Hochschild complex* $\mathbf{C}^H(\mathcal{C})$ (see e.g. [55, Sec. 2.3]). Thus, we obtain a canonical isomorphism in $D(k)$
\[ \mathbf{C}^H(\mathcal{C}) \simeq \text{Tr}_\mathcal{C}(\Delta_\text{e}). \quad (8.2) \]
Note that if $\mathcal{C}$ has a compact generator $G$, then the restriction functor induces an equivalence of the derived category of $\mathcal{C} - \mathcal{C}$-bimodules with the derived category of bimodules over the dg-algebra $A = \mathcal{C}(G, G)$ that sends $\Delta_\text{e}$ to the diagonal bimodule $A$. Hence we obtain an
isomorphism in \( D(k) \) between \( \text{Tr}_\mathcal{C}(\Delta_\mathcal{C}) \) and the Hochschild homology of \( A \). To realize this isomorphism on the chain level we can use the subcomplex \( \text{Bar}_n(\mathcal{C}, G) \) in the bar-resolution with

\[
\text{Bar}_n(\mathcal{C}, G)(P^\vee, Q) = \mathcal{C}(G, Q) \otimes \mathcal{C}(G, G)^{\otimes n} \otimes \mathcal{C}(P, G).
\]

The corresponding subcomplex \( \text{Tr}_\mathcal{C}(\text{Bar}_n(\mathcal{C}, G)) \) in \( \mathbf{C}^H(\mathcal{C}) \) computes the Hochschild homology of \( A \). Similar subcomplexes can be defined in the situation when \( \mathcal{C} \) is generated by a finite set of compact objects \( G_1, \ldots, G_m \). The isomorphisms (8.2) are compatible with the inclusions \( \{G_1, \ldots, G_m\} \subset \mathcal{C} \) of the full dg-subcategories with objects \( G_1, \ldots, G_m \) (inducing equivalences of derived categories).

The standard complex \( \mathbf{C}^H(\mathcal{C}) \) is functorial with respect to dg-functors between dg-categories. Let us show that the induced maps on Hochschild homology coincide with the maps \( F_* \) defined above.

**Theorem 8.0.3.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a dg-functor. Then the map \( F_* \) coincides with the map on Hochschild homology induced by the chain map of Hochschild complexes

\[
\mathbf{C}^H(F) : \mathbf{C}^H(\mathcal{C}) \to \mathbf{C}^H(\mathcal{D})
\]
given by \( F \).

**Proof.** Assume first that both \( \mathcal{C} \) and \( \mathcal{D} \) have finite number of objects. Then we have the dg-algebras

\[
A = \bigoplus_{C_1, C_2 \in \mathcal{C}} \mathcal{C}(C_1, C_2) \quad \text{and} \quad B = \bigoplus_{D_1, D_2 \in \mathcal{D}} \mathcal{D}(D_1, D_2),
\]

so that the categories of modules over \( \mathcal{C} \) and \( A \) (resp., over \( \mathcal{D} \) and \( B \)) are equivalent. We can view \( F \) as a non-unital homomorphism of dg-algebras \( f : A \to B \) and extend it to a functor between the categories of perfect modules

\[
F : \text{Per}(\mathcal{C}) = \text{Per}(A) \to \text{Per}(B) = \text{Per}(\mathcal{D})
\]

by \( F(M) = M \otimes_A B \) for \( M \in \text{Per}(A) \). We are going to compute the map \( F_* \) in this case using Proposition 8.0.2. In our case the diagonal \( \Delta_\mathcal{C} \) (resp., \( \Delta_\mathcal{D} \)) corresponds to the \( A - A \)-bimodule \( A \) (resp., \( B - B \)-bimodule \( B \)), and \( \text{Tr}_\mathcal{C} \) (resp., \( \text{Tr}_\mathcal{D} \)) is given by the functor \( ? \otimes_{A^{\text{op}}} A \) (resp., \( ? \otimes_{B^{\text{op}}} B \)), where \( A^e = A^{\text{op}} \otimes A \). The functor \( F^{(2)} : \text{Per}(A^e) \to \text{Per}(B^e) \) sends \( M \in \text{Per}(A^e) \) to \( M \otimes_{A^e} B^e \). The natural transformation \( t_F : \text{Tr}_\mathcal{C} \to \text{Tr}_\mathcal{D} \circ F \) is given by the morphisms

\[
t_F(M) : M \otimes_{A^e} A \xrightarrow{id \otimes f} M \otimes_{A^e} B \simeq (M \otimes_{A^e} B^e) \otimes_{B^e} B.
\]

induced by \( f \). Finally, the map \( c_F : F^{(2)}(A) \to B \) in \( D(B^e) \) is the natural map \( A \otimes_{A^e} B^e \to B \) induced by \( f \). Let \( \text{Bar}_\bullet(A) \) (resp., \( \text{Bar}_\bullet(B) \)) be the bar-resolution of the bimodule \( A \) (resp., \( B \)). Then \( c_F \) is realized by the natural morphism of complexes

\[
\text{Bar}_\bullet(A) \otimes_{A^e} B^e \to \text{Bar}_\bullet(B),
\]

given by

\[
B \otimes (A^{\otimes n}) \otimes B \xrightarrow{id \otimes (f^{\otimes n}) \otimes id} B \otimes (B \otimes \ldots \otimes B) \otimes B.
\]
Hence, the morphism $\text{Tr}_D(c_F)$ is realized by the map

$$\text{Bar}_\bullet(A) \otimes_{A^e} B \simeq (\text{Bar}_\bullet(A) \otimes_{A^e} B^e) \otimes_{B^e} B \to \text{Bar}_\bullet(B) \otimes_{B^e} B$$
given by

$$A^\otimes n \otimes B \xrightarrow{f^\otimes n \otimes \text{id}} B^\otimes n \otimes B.$$

Since the map $t_F(\text{Bar}_\bullet(A))$ is given by

$$A^\otimes n \otimes A \xrightarrow{\text{id} \otimes f} A^\otimes n \otimes B,$$

we see that its composition with $\text{Tr}_D(c_F)$ is equal to the natural map

$$\mathbf{C}^H(f) : \mathbf{C}^H(A) \to \mathbf{C}^H(B)$$
of the Hochschild complexes induced by $f$.

Now let us consider the general case. Let $X$ (resp., $Y$) be a compact generator of $\mathcal{C}$ (resp., $\mathcal{D}$). We have the following commutative diagram of dg-functors

$$
\begin{array}{ccc}
\{X\}_e & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\{F(X),Y\}_e & \longrightarrow & \mathcal{D}
\end{array}
$$

where horizontal arrows are inclusions inducing equivalences of derived categories, and $F'$ is the restriction of $F$. By the first part of the proof, the map $(F')_*$ is represented by the chain map $\mathbf{C}^H(F')$ of the Hochschild complexes. The same is true for both horizontal arrows by the discussion preceding the formulation of the theorem. Since the inclusion $\{X\}_e \to \mathcal{C}$ induces an isomorphism on Hochschild homology, our assertion follows.

**Corollary 8.0.4.** The map $F_* : HH_*(\mathcal{C}) \to HH_*(\mathcal{D})$ depends only on the class of $F$ in the Grothendieck group of $\text{Per}(\mathcal{C}^{\text{op}} - \mathcal{D})$.

**Proof.** This follows from [31, Thm. 2.4].

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