

On Douglas-Shapiro-Shields Factorizations

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Abstract. In this note we consider the kernels of vectorial Hankel operators and examine a question which functions are admitted to canonical ‘pseudo’-Douglas-Shapiro-Shields factorizations.

1. Introduction

Let \( \mathbb{T} \) be the unit circle in the complex plane \( \mathbb{C} \). For a separable complex Hilbert space \( E \), let \( L^2_E \) be the set of all strongly measurable functions \( f : \mathbb{T} \rightarrow E \) such that

\[
\|f\|_2 := \left( \int_{\mathbb{T}} \|f(z)\|^2_E dm(z) \right)^{1/2} < \infty.
\]

Then \( H^2_E \) denotes the corresponding \( E \)-valued Hardy space, i.e., the set of all \( f \in L^2_E \) with \( \widehat{f}(n) = 0 \) for \( n < 0 \). Let \( \mathcal{B}(D, E) \) denote the set of all bounded linear operators between separable complex Hilbert spaces \( D \) and \( E \), and abbreviate \( \mathcal{B}(E, E) \) to \( \mathcal{B}(E) \). A function \( \Phi : \mathbb{T} \rightarrow \mathcal{B}(D, E) \) is called WOT measurable if \( z \mapsto \Phi(z)x \) is weakly measurable for every \( x \in D \). Let \( L^\infty(\mathcal{B}(D, E)) \) denote the set of all bounded WOT measurable \( \mathcal{B}(D, E) \)-valued functions on \( \mathbb{T} \). Define \( H^\infty(\mathcal{B}(D, E)) \) by the set of functions \( \Phi \in L^\infty(\mathcal{B}(D, E)) \) whose Fourier coefficients \( \widehat{\Phi}(n) = 0 \) for \( n < 0 \). A function \( \Delta \in H^\infty(\mathcal{B}(D, E)) \) is called an inner function if \( \Delta^* \Delta = I_D \) a.e. on \( \mathbb{T} \) and is called two-sided inner function if \( \Delta \) is inner and \( \Delta \Delta^* = I_E \) a.e. on \( \mathbb{T} \). For a function \( \Phi \in H^\infty(\mathcal{B}(D, E)) \), an inner function \( \Delta \) with values in \( \mathcal{B}(D', E) \) is called a left inner divisor of \( \Phi \) if \( \Phi = \Delta A \) for \( A \in H^\infty(\mathcal{B}(D, D')) \).

For \( \Phi \in H^\infty(\mathcal{B}(D_1, E)) \) and \( \Psi \in H^\infty(\mathcal{B}(D_2, E)) \), we say that \( \Phi \) and \( \Psi \) are left coprime if the only common left

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inner divisor of both $\Phi$ and $\Psi$ is a unitary operator. Also, for $\Phi \in H^\infty(\mathcal{B}(E, D_1))$ and $\Psi \in H^\infty(\mathcal{B}(E, D_2))$, we say that $\Phi$ and $\Psi$ are right coprime if $\Phi$ and $\Psi$ are left coprime, where $\Phi(z) := \Phi(\overline{z})$.

A Hankel operator with symbol $\Phi \in L^\infty(\mathcal{B}(D, E))$ is an operator $H_\Phi : H^2_E \to H^2_E$ defined by

$$H_\Phi f := J P^J(\Phi f) \quad \text{for } f \in H^2_E,$$

where $P^J$ is the orthogonal projection of $L^2_E$ onto $(H^2_E)^\perp$ and $J$ denotes the unitary operator from $L^2_E$ onto $L^2_E$ given by $J(f)(z) := \overline{f}(\overline{z})$ for $f \in L^2_E$. A shift operator $S_E$ on $H^2_E$ is defined by

$$(S_E f)(z) := z f(z) \quad \text{for each } f \in H^2_E.$$ We can see that the kernel of a Hankel operator $H_\Phi$ is an invariant subspace of the shift operator on $H^2_E$. Thus by the Beurling-Lax-Halmos Theorem (cf. [2], [15], [14], [17]),

$$\ker H_\Phi = \Delta H^2_E,$$

for some inner function $\Delta \in H^\infty(\mathcal{B}(E', E))$. Some kernels of products of Hankel operators with scalar symbols are also invariant subspaces of the shift operator on $H^2$ (cf. [11] [8], [9]).

Related to this is the notion of Douglas-Shapiro-Shields (DSS) factorization. For a function $\Phi \in L^\infty(\mathcal{B}(E', E))$, the Douglas-Shapiro-Shields (briefly, DSS) factorization of $\Phi$ is (cf. [4], [6], [7], [12]):

$$\Phi = \Delta A^*,$$

where $\Delta \in H^\infty(\mathcal{B}(E))$ is two-sided inner and $A \in H^\infty(\mathcal{B}(E, E'))$. It is known (cf. [4], [7], [12]) that if $\Phi \in L^\infty(\mathcal{B}(E', E))$ admits a DSS factorization of the form (2), then $\Delta$ can be obtained from the equation

$$\ker H_\Phi = \Delta H^2_E,$$

in this case, $\Delta$ and $A$ are right coprime. The DSS factorization satisfying (3) is called canonical. Consequently, each function that admits a DSS factorization can be arranged in a canonical form.

We recall (cf. [1], [16]) that for a scalar function $\varphi$ defined on $\mathbb{T}$, $\varphi$ is said to be of bounded type if

$$\varphi = h_1/h_2 \quad \text{a.e. on } \mathbb{T}$$

for some $h_1, h_2 \in H^\infty$. If $\Phi$ is a matrix-valued $L^\infty$-function then $\Phi$ is said to be of bounded type if each entry of $\Phi$ is of bounded type. It is also known that if $\Phi$ is a matrix-valued function then (cf. [3], [12])

$$\Phi^* \text{ is of bounded type } \iff \Phi \text{ admits a (canonical) DSS factorization.}$$

If the condition “$\Delta$ is two-sided” is dropped in (2), what can we say about a DSS factorization? More concretely, we would like to ask:

**Question 1.1.** If $\Phi \in L^\infty(\mathcal{B}(E', E))$ is expressed as

$$\Phi = \Delta A^*,$$

where $\Delta \in H^\infty(\mathcal{B}(D, E))$ is inner and $A \in H^\infty(\mathcal{B}(D', E'))$, does it follow that $\Delta$ can be obtained from the equation

$$\ker H_\Phi = \Delta H^2_E?$$

In this note we consider Question 1.1.

We remark that the kernels of Hankel operators with operator-valued symbols are studied recently in [4] where the degree of cyclicity of the set obtained by the analytic part of the symbol is shown to be connected with the size of the inner matrix $\Delta$ as in (5) (the case of matrix-valued symbol is studied in [13] where an index of the adjoint of the symbol is also connected with the same thing). We will use the degree of cyclicity to give a more explicit answer to Question 1.1 for matrix-valued symbols. The following inverse question is investigated in [10]: Given an (nonsquare) inner matrix $\Delta$, find all matrix-valued $\Phi$ in $L^\infty(\mathcal{B}(D, E))$ such that $\ker H_\Phi = \Delta H^2_E$. A complete answer to this inverse question is given in the case $\Delta$ is a $2 \times 1$ inner matrix or $\Delta$ is an inner matrix such that $\Delta^*$ is of bounded type.
2. The main results

For an inner function \( \Delta \in H^\infty(\mathcal{B}(D, E)) \), \( \mathcal{H}(\Delta) \) denotes the orthogonal complement of the subspace \( \Delta H^2_D \) in \( H^2_E \), i.e.,
\[
\mathcal{H}(\Delta) := H^2_E \ominus \Delta H^2_D.
\]

For a function \( \Phi : T \to \mathcal{B}(D, E) \), write \( \Phi(z) := \Phi(\Xi) \).

We now answer Question 1.1 affirmatively.

Theorem 2.1. If \( \Phi \in L^\infty(\mathcal{B}(E', E)) \) is expressed as
\[
\Phi = \Delta A',
\]
where \( \Delta \in H^\infty(\mathcal{B}(D, E)) \) is inner and \( A \in H^\infty(\mathcal{B}(D, E')) \), then we can write
\[
\Phi = \Delta \Lambda B_0^* ,
\]
where \( B_0 \in H^\infty(\mathcal{B}(E_0, E')) \) and \( \Lambda \in H^\infty(\mathcal{B}(E_0, E)) \) is an inner function which comes from the equation
\[
\ker H_{\Phi^*} = \Lambda H^2_{E_0^*}
\]
for some Hilbert space \( E_0 \). Moreover, in the factorization (7), \( \Lambda \) and \( B_0 \) are right coprime.

Proof. Suppose that \( \Phi \in L^\infty(\mathcal{B}(E', E)) \) can be written as
\[
\Phi = \Delta A',
\]
where \( \Delta \in H^\infty(\mathcal{B}(D, E)) \) is inner and \( A \in H^\infty(\mathcal{B}(D, E')) \). Define
\[
\Lambda := \text{left-g.c.d.}\{ \Theta : \Theta = \Theta B^* \text{ with } \Theta \in H^\infty(\mathcal{B}(D, E)) \text{ inner and } B \in H^\infty(D, E') \},
\]
where left-g.c.d. means the greatest common left inner divisor. If \( \Phi = \Theta B^* \) for some inner function \( \Theta \in H^\infty(D, E) \) and \( B \in H^\infty(D, E') \). Then \( \Theta H^2_{D^*} \subseteq \ker H_{\Phi^*} \). We thus have
\[
\Lambda H^2_{E_0^*} \subseteq \ker H_{\Phi^*} \quad \text{for some Hilbert space } E_0.
\]

For the reverse inclusion, suppose \( \ker H_{\Phi^*} \neq \{0\} \). Then in view of the Beurling-Lax-Halmos Theorem that \( \ker H_{\Phi^*} = \Delta_1 H^2_{E_1^*} \) for some nonzero inner function \( \Delta_1 \) with values in \( \mathcal{B}(E_1, E) \). Thus we have \( \Delta H^2_D \subseteq \Delta_1 H^2_{E_1^*} \), which implies that \( \Delta_1 \) is a left inner divisor of \( \Delta \). Write
\[
\Delta = \Delta_1 \Omega,
\]
where \( \Omega \) is inner function with values in \( \mathcal{B}(D, E_1) \). Since \( \ker H_{\Phi^*} = \Delta_1 H^2_{E_1^*} \), it follows that for all \( f \in H^2_{E_1^*} \),
\[
A \Omega^* f = \Phi^* \Lambda_1 f \in H^2_{E_1^*}.
\]

Put \( B := A \Omega^* \). Then \( B \in L^\infty(\mathcal{B}(E_1, E')) \). It thus follows from (12) that for all \( x \in E_1 \) and \( n = 1, 2, 3, \ldots \),
\[
\hat{B}(-n)x = \int_{E_1} z^n B(z) x dm(z) = 0.
\]

Thus \( B \) belongs to \( H^\infty(\mathcal{B}(E_1, E')) \). Since \( \Delta_1 B^* = \Phi \), it follows that \( \Delta_1 H^2_{E_1^*} \subseteq \Delta H^2_{E_1^*} \), which together with (11) gives \( \Delta_1 H^2_{E_1^*} = \Delta H^2_{E_1^*} \). Thus \( \Delta_1 = \Delta A \) for some unitary operator \( U \in \mathcal{B}(E_1, E_0) \). Put \( B_0 := BU^* \in H^\infty(\mathcal{B}(E_0, E')) \).

Then
\[
\Phi = \Delta \Lambda B_0^* \quad \text{and} \quad \ker H_{\Phi^*} = \Lambda H^2_{E_0^*}.
\]
We now claim that $\Lambda_A$ and $B_0$ are right coprime. To see this we assume that $\Omega$ is a common left inner divisor of $\Lambda_A$ and $B_0$. Then we can write

$$\overline{\Lambda}_A = \Omega \Lambda_2 \quad \text{and} \quad \overline{B}_0 = \Omega B_2,$$

where $\Lambda_2 \in H^\infty(\mathcal{B}(E, E_1))$ and $B_2 \in H^\infty((E', E_1))$. Then $\overline{\Lambda}_2$ is a left inner divisor of $\Lambda_A$, and we have that

$$\Phi = \Lambda_A B_0 = \overline{\Lambda}_2 \overline{\Omega} \overline{\Phi}_2 = \overline{\Delta}_2 \overline{\Phi}_2.$$

Thus

$$\overline{\Delta}_2 H^2_{E_1} \subseteq \ker \Phi' = \Lambda_A H^2_{D_0},$$

which implies that $\Lambda_A$ is a left inner divisor of $\overline{\Delta}_2$. It thus follows that $\overline{\Omega}$ is a unitary operator and so is $\Omega$. Therefore $\Lambda_A$ and $B_0$ are right coprime. This completes the proof. \hfill $\square$

**Remark 2.2.** The expression (6) will be called a pseudo-DSS factorization and the expression (7) will be called a canonical pseudo-DSS factorization. Thus Theorem 2.1 says that if a function $\Phi \in L^\infty(\mathcal{B}(E', E))$ admits a pseudo-DSS factorization then we can always arrange the pseudo-DSS factorization of $\Phi$ in a canonical form.

For an inner function $\Delta \in H^\infty(\mathcal{B}(D, E)), \Omega \in H^\infty(D')$, define the kernel of $\Delta^*$ by

$$\ker \Delta^* := \{ f \in H^2_D : \Delta^*(z)f(z) = 0 \ \text{for almost all } z \in \mathbb{T} \}.$$

Since $\ker \Delta^*$ is an invariant subspace for the shift operator $S_D$, it follows from the Beurling-Lax-Halmos Theorem that $\ker \Delta^* = \Omega H^2_D$ for some inner function $\Omega \in H^\infty(D', E)$.

The following lemma gives a concrete description for the kernel of $\Delta^*$.

**Lemma 2.3.** [4] [10] Let $\Delta$ be an inner function with values in $\mathcal{B}(D, E)$. Then we may write $\ker \Delta^* = \Omega H^2_D$ for some inner function $\Omega \in H^\infty(D', E)$. Put

$$\Delta_c := \text{left-g.c.d.}\{ [g]^1 : g \in \ker \Delta^* \},$$

where $[g] : \mathbb{T} \to \mathcal{B}(C, E)$ is defined by $[g](z)\alpha := \alpha g(z)$ ($\alpha \in C$) and $[g]^1$ denotes the inner part of $[g]$. Then,

(a) $\Omega = \Delta_c$;

(b) $[\Delta, \Delta_c]$ is an inner function with values in $\mathcal{B}(D \oplus D', E)$;

(c) $\ker H_{\Delta_c} = [\Delta, \Delta_c] H^2_{D \oplus D'} \equiv \Delta H^2_D \oplus \Delta_c H^2_{D'}$.

**Definition 2.4.** $\Delta_c$ is called the complementary factor of an inner function $\Delta$.

We then have:

**Corollary 2.5.** Suppose $\Delta$ is an inner function with values in $H^\infty(\mathcal{B}(D, E))$ and $A \in H^\infty(\mathcal{B}(D, E'))$. If $\Delta$ admits a DSS factorization then

$$\ker H_{\Delta^*} = \Theta H^2_D,$$

where $\Theta = [\Delta, \Delta_c] \Omega$ is two-sided inner with

$$\Omega := \text{left-g.c.d.}\{ [\Delta, \Delta_c], [A, 0] \} \quad \text{(where } [A, 0] \in H^\infty(\mathcal{B}(D \oplus D', E'))) \text{.}$$
Proof. Let 
\[ \Omega := \text{left-g.c.d.}([\Lambda, \tilde{\Lambda}], [A, 0]). \]

Since \( \Lambda \) admits a DSS factorization, it follows from Lemma 2.3 that \([\Lambda, \Lambda_c]\) is two-sided inner, and so is \([\Lambda, \tilde{\Lambda}]. \) Thus \( \Omega \) is two-sided inner, and hence we may write 
\[ [\Lambda, \Lambda_c] = \Theta \tilde{\Omega} \quad \text{and} \quad [A, 0] = B\tilde{\Omega} \quad (\Theta \in H^\infty(B(E)), \ B \in H^\infty(B(E, E'))), \]
where \( \Theta \) and \( B \) are right coprime. Thus we have that 
\[ \Delta A^* = [\Lambda, \Lambda_c][A, 0]^* = \Theta B^*. \]

But since \( \tilde{\Omega} \) is two-sided inner, so is \( \Theta, \) and hence \( \ker H_{\Delta^*} = \Theta^2. \) This completes the proof. \( \square \)

The following example shows that Corollary 2.5 may fail if the condition “\( \Lambda \) admits a DSS factorization” is dropped.

Example 2.6. Let \( h(z) := e^z \in H^\infty. \) Put 
\[ f(z) := \frac{h(z)}{\sqrt{2}\|h\|_\infty}. \]

Clearly, \( f \) is not of bounded type. Let \( h_1(z) := \sqrt{1 - |f(z)|^2}. \) Then \( h_1 \in L^\infty \) and \( |h_1| \geq \frac{1}{\sqrt{2}}. \) Thus there exists an outer function \( g \) such that \( |h_1(z)| = |g| \) a.e. on \( T, \) (cf. [5, Corollary 6.25], [4]). Let 
\[ \Lambda := \begin{bmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A := \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Then \( \Lambda \) is inner and \( \Lambda^* \) is not of bounded type, so that by (4), \( \Lambda \) does not admit a DSS factorization. Write 
\[ \Delta_1 := \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{and} \quad \Delta_2 := \begin{bmatrix} 1 \\ \sqrt{2}z \end{bmatrix}. \]

Then it follows from Lemma 2.3 that 
\[ \ker H_{\Delta_1} = \ker H_{\Delta_1} \oplus \ker H_{\Delta_2} = \Delta_1 H^2 \oplus \begin{bmatrix} \frac{z}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -1 \end{bmatrix} H^2_{\mathbb{C}^2}, \]
and hence, 
\[ [\Lambda, \Lambda_c] = \begin{bmatrix} f & 0 & 0 \\ g & 0 & 0 \\ 0 & \frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \]

Since \( \ker H_{\Delta_1} = H^2 \oplus H^2 \oplus \ker H_{\Delta_2}, \) it follows that 
\[ \ker H_{\Delta^*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{z}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} H^2_{\mathbb{C}^4} \equiv \Theta H^2_{\mathbb{C}^4}. \]
Since $\tilde{f}$ is invertible in $H^\infty$, it follows that $A$ and $\Delta$ are right coprime. On the other hand, since
\[
\begin{pmatrix} A & 0 \end{pmatrix} H^2 C_4 \uplus \begin{pmatrix} \Delta & \Delta_c \end{pmatrix} H^2 C_4 = \begin{pmatrix} \tilde{f} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} H^2 C_4 \uplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} H^2 = H^2 C_3,
\]
it follows that
\[
\Omega \equiv \text{left-g.c.d.}(\begin{pmatrix} \Delta & \Delta_c \end{pmatrix}, \begin{pmatrix} A & 0 \end{pmatrix}) = I_3.
\]
We thus have $\Theta \equiv [\Delta, \Delta_c] \tilde{\Omega}$.

Let $M_{n \times m}$ denote the set of all $n \times m$ complex matrices and write $M_n \equiv M_{n \times n}$.

**Remark 2.7.** It is clear that if $\Phi \in L^\infty(B(E, E))$ is such that $\ker H_\Phi = \{0\}$ then, by Theorem 2.1, $\Phi$ does not admit a pseudo-DSS factorization. We next give a less trivial example in the sense that $\ker H_\Phi \neq \{0\}$, but $\Phi$ still does not admit a pseudo-DSS factorization. Suppose that $\theta_1$ and $\theta_2$ are coprime inner functions. Consider
\[
\Phi = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & a \end{pmatrix} \in H^\infty M_{3 \times 3},
\]
where $a \in H^\infty$ is such that $a$ is not of bounded type. Then a direct calculation shows that
\[
\ker H_\Phi = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{pmatrix} H^2 C_2 \equiv \Theta H^2 C_2.
\]
Assume that $\Phi$ admits a pseudo-DSS factorization. Then, by Theorem 2.1, we may write
\[
\Phi = \Theta B^\ast
\]
for some $B \in H^\infty_{M_{3 \times 2}}$. However, for any $B \in H^\infty_{M_{3 \times 2}}$
\[
\Phi = \Theta B^\ast = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \\ 0 & 0 \end{pmatrix} B^\ast = \begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix},
\]
a contradiction.

3. When $\Phi$ is a matrix-valued symbol

Theorem 2.1 gives a satisfactory answer to Question 1.1, however as we have seen in a previous example, it is not a simple matter to find the canonical pseudo-DSS factorization of $\Phi$. Equivalently, we need to find $\ker H_\Phi$. In the case when $\Delta$ admits a DSS factorization, Corollary 2.5 gives a practical way of finding $\ker H_\Phi$. Here we extend Corollary 2.5 to more general situations when $\Phi$ is a matrix-valued symbol. Since Corollary 2.5 covers the case when $\Phi \equiv \Delta A^\ast$ admits a DSS factorization, here we will assume $\Phi$ does not admit a DSS factorization.

**Proposition 3.1.** Suppose $\Phi \in L^\infty_{M_{2 \times m}}$ does not admit a DSS factorization. Let
\[
\Phi = \Delta A^\ast \quad \text{(pseudo-DSS factorization)}.
\]
If $[\Delta, \Delta_c]$ is in $H^\infty_{M_{2 \times (n-1)}}$ and $\Omega \equiv \text{left-g.c.d.}(\begin{pmatrix} \Delta & \Delta_c \end{pmatrix}, \begin{pmatrix} A & 0 \end{pmatrix})$ is two-sided inner, then $\Phi = \Theta B^\ast$ is a canonical pseudo-DSS factorization for some $B \in H^\infty_{M_{2 \times (n-1)}}$, where
\[
\Theta \equiv [\Delta, \Delta_c] \tilde{\Omega}.
\]
Proof. By Theorem 2.1, we need to show $\Delta_A$ given by (8) is the same as the $\Theta$ given by (15). By the definition of $\Omega$,

$$[\Delta, \Delta_c] = \Theta \bar{\Omega} \quad \text{and} \quad [A, 0] = B \bar{\Omega}$$

for some $B \in H^0_{M_{\text{Max}(s-1)}}$ and $\Theta$ and $B$ are right coprime. Since

$$\Phi = \Delta A^* = [\Delta, \Delta_c] [A, 0]^* = \Theta \bar{\Omega} B^* = \Theta B',$$

it follows that $\Theta H^2_{C_{r-1}} \subseteq \ker H_{c_F^s} \equiv \Delta_A H^2_{C_r}$. Thus $\Theta = \Delta_A \Gamma$ for some inner function $\Gamma \in H^0_{M_{\text{Max}(s-1)}}$. Since $\Phi \in L^0_{M_{\text{Max}}}$ does not admit a DSS factorization, it follows that $r < n$, and hence $r = n - 1$. It thus follows from Theorem 2.1 that

$$\Delta_A \Gamma B^* = \Theta B^* = \Phi = \Delta_A B_0^* \quad \text{for some } B_0 \in H^0_{M_{\text{Max}(s-1)}}.$$

Thus $\Gamma B^* = B_0^*$, and hence $B = B_0 \Gamma$. Hence, the fact that $\Theta$ and $B$ are right coprime implies that $\Gamma$ is a unitary constant, and therefore $\Theta = \Delta_A$. \hfill $\square$

We give an example to illustrate the above proposition.

**Example 3.2.** We use the same notation as in Example 2.6. Let

$$A := [1, 1].$$

Then $\Phi = \Delta A^* = [f \ g \ \bar{\gamma}_2 \ \bar{\gamma}_3]^*$ does not admit a DSS factorization. Note that

$$\Omega = \text{left-g.c.d} \left( \left[ \tilde{A}_0, \tilde{A}_c \right] \right) = I_3.$$

It follows from the above proposition that $\ker H_{c_F} = [\Delta, \Delta_c] H^2_{C_r}$.

Next we extend the above proposition by using the notion of degree of cyclicity due to V.I. Vasyunin and N.K. Nikolskii [18] (or [16]): If $F \subseteq H^2_{C_r}$, then the degree of cyclicity, denoted by $\text{dc}(F)$, of $F$ is defined by the number

$$\text{dc}(F) := n - \max_{\zeta \in \mathbb{D}} \text{dim} \left( \left[ \tilde{A}_i \tilde{A}_c \right] \right) = I_3.$$

where $E^*_F$ denotes the smallest $S^*_F$-invariant subspace containing $F$, i.e., $E^*_F = \sqrt{\{S^*_F F : n \geq 0\}}$. It is known from [4, Lemma 2.13] that if $\Phi \equiv [\Phi_1, \cdots, \Phi_n] \in L^0_{C_{r-1}}$ is an $m \times n$ matrix-valued function then

$$\ker H_{c_F} = \Theta H^2_{C_r} \iff \text{dc}(\Phi_+) = n - r, \quad (16)$$

where $\Theta$ is an $m \times r$ inner matrix function and $\Phi_+ := [(\Phi_1)_+, \cdots, (\Phi_n)_+] \subseteq H^0_{C_{r-1}}$ (where $(\Phi_i)_+$ denotes the analytic part of $\Phi_i$).

**Remark 3.3.** Suppose $\Phi \in L^0_{M_{\text{Max}}}$ does not admit a DSS factorization. Let

$$\Phi = \Delta A^* \quad \text{(pseudo-DSS factorization)}.$$

Suppose that $[\Delta, \Delta_c] \in H^0_{M_{\text{Max}}}$, $\Omega \equiv \text{left-g.c.d} \left( \left[ \tilde{A}_0, \tilde{A}_c \right] \right)$ is two-sided inner and $\text{dc}(\Phi_+) = n - s$. Then by the same argument as the proof of Proposition 3.1, we have that $\Theta = \Delta_A \Gamma$ for some inner function $\Gamma$. By Theorem 2.1, $\ker H_{c_F} = \Delta_A H^2_{C_r}$, for some $r \leq n$. By the assumption, $\text{dc}(\Phi_+) = n - s$ and by (16), $r = s$. Therefore $\Theta = \Delta_A \Gamma$ implies that $\Gamma$ is a $s \times s$ two-sided inner matrix. Thus by the same argument as the proof of Proposition 3.1, we have that

$$\Phi = \Theta B^* \quad \text{(canonical pseudo-DSS factorization)}.$$

where $\Theta = [\Delta, \Delta_c] \bar{\Omega}$. 

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