BLOW-UP RATES OF LARGE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. In this paper we analyze the blow-up rates of large solutions to the semilinear elliptic problem
\[ \Delta u = b(x)f(u), \quad x \in \Omega, \quad u|_{\partial \Omega} = +\infty, \]
where \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N \), \( f \) is rapidly varying or normalised regularly varying with index \( p > 1 \) at infinity, and \( b \in C^\alpha(\bar{\Omega}) \) which is non-negative in \( \Omega \) and positive near the boundary and may be vanishing on the boundary.

1. Introduction and the main results. In this paper we analyze the blow-up rates of large solutions to the following semilinear elliptic problem
\[ \Delta u = b(x)f(u), \quad x \in \Omega, \quad u|_{\partial \Omega} = +\infty, \quad (1.1) \]
where the last condition means that \( u(x) \to +\infty \) as \( d(x) = \text{dist}(x, \partial \Omega) \to 0 \), \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^N \), \( b \) satisfies
\( (b_1) \) \( b \in C^\alpha(\bar{\Omega}) \) for some \( \alpha \in (0, 1) \), is non-negative in \( \Omega \);
\( (b_2) \) there exists \( k \in \Lambda \) such that
\[ 0 < b_1 =: \lim_{d(x) \to 0} \inf b(x) \frac{k(x)}{k^2(d(x))} \leq b_2 =: \lim_{d(x) \to 0} \sup b(x) \frac{k(x)}{k^2(d(x))} < \infty, \]
where \( \Lambda \) is the set of all positive non-decreasing functions in \( C^1(0, \delta_0) \) \( (\delta_0 > 0) \) which satisfy
\[ \lim_{t \to 0^+} \frac{1}{d} \frac{d}{dt} \left( \frac{K(t)}{k(t)} \right) =: C_k \in \mathbb{R}, \quad K(t) = \int_0^t k(s)ds, \quad (1.2) \]
and \( f \) satisfies
\( (f_1) \): \( f \in C^1(\mathbb{R}), \quad f(s) > 0, \quad \forall s \in \mathbb{R}, \quad f \) is increasing on \( \mathbb{R} \) (or \( f_0 \) \( f \in C^1[0, \infty) \), \( f(0) = 0, \quad f \) is increasing on \( [0, \infty) \));
\( (f_2) \): \( \int_0^\infty \frac{dv}{f(v)} < \infty; \)

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(f₃): there exists $C_f > 0$ such that
\[
\lim_{s \to +\infty} f'(s) \int_s^{\infty} \frac{d\nu}{f(\nu)} = C_f.
\]

We note that the set $\Lambda$ was first introduced by Cîrstea and Rădulescu [6] for studying the boundary behaviour and uniqueness of solutions to problem (1.1) with the weight $b$ satisfying $(b_1)$ and the assumption that

(b₃) there exist $k \in \Lambda$ and $b_0 > 0$ such that
\[
\lim_{d(x) \to 0} \frac{b(x)}{k^2(d(x))} = b_0.
\]

The problem (1.1) arises from many branches of mathematics and has been discussed and extended by many authors in many contexts, for instance, the existence, boundary behavior and uniqueness of solutions, see, [1]-[4], [6]-[15], [17]-[26], [28], [29], [32]-[37] and the references therein.

For $b \equiv 1$ on $\Omega$ and $f$ satisfying $(f_1)$ (or $(f_{01})$), Keller-Osserman ([19, 29]) first supplied a necessary and sufficient condition
\[
\int_1^{\infty} \frac{ds}{\sqrt{2F(s)}} < \infty, \quad F(s) = \int_0^s f(\nu) d\nu, \tag{1.3}
\]
for the existence of solutions to problem (1.1).

Loewner and Nirenberg [22] showed that if $f(u) = u^{p_0}$ with $p_0 = (N+2)/(N-2)$, $N > 2$, then problem (1.1) has a unique positive solution $u$ which satisfies
\[
\lim_{d(x) \to 0} u(x) \left(\frac{d(x)}{d(x)}\right)^{(N-2)/2} = \left(\frac{N(N-2)}{4}\right)^{(N-2)/4}.
\]

Bandle and Marcus [3] established the following results.

If $f$ satisfies $(f_{01})$ and the condition that

(f₁) there exist $p > 1$ and $S_0 \geq 1$ such that $f(\xi s) \leq \xi^p f(s)$ for all $\xi \in (0, 1)$ and $s \geq S_0/\xi$,

then for any solution $u$ of problem (1.1)
\[
\frac{u(x)}{\phi(d(x))} \to 1 \quad \text{as} \quad d(x) \to 0, \tag{1.4}
\]
where $\phi$ satisfies
\[
\int_{\phi(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t, \quad \forall \ t > 0. \tag{1.5}
\]

If $f$ further satisfies

(f₅) $f(s)/s$ is increasing on $[0, \infty)$, then problem (1.1) has a unique solution.

Lazer-McKenna [21] showed that if $f$ satisfies (f₁) (or $(f_{01})$) and

(f₆) there exists $S_0 > 0$ such that $f'$ is non-decreasing on $[S_0, \infty)$, and

\[
\lim_{s \to \infty} \frac{f'(s)}{\sqrt{F(s)}} = \infty,
\]

then for any solution $u$ of problem (1.1)
\[
u(x) - \phi(d(x)) \to 0 \quad \text{as} \quad d(x) \to 0. \tag{1.6}
\]

Recently, when $f$ satisfies $(f_{01})$, (f₂) and (f₃) with $C_f > 1$, and $b \in C(\overline{\Omega})$ which is non-negative in $\Omega$ and satisfies (b₂) with $k(t) = t^\sigma$, $\sigma \geq 0$, García - Melián [15] showed the boundary behavior and uniqueness of solutions to problem (1.1).
In [37], the authors related the constants $C_f$ and $C_k$ and got the boundary behavior and uniqueness of solutions to problem (1.1) under $(b_3)$.

Cîrstea and Rădulescu [6], Cîrstea and Du [7], Cîrstea [8] introduced the Karamata regular variation theory to study the boundary behavior and uniqueness of solutions to problem (1.1) and obtained a series of very rich significant information about the qualitative behavior of the boundary blow-up solutions in a general framework that removes previous restrictions in the literature.

For other boundary behavior and uniqueness results, see, for instance, [1]-[2], [4], [9]-[14], [17, 18], [23]-[26], [28, 33, 35, 36] and the references therein.

Inspired by the above works, in this paper we obtained the boundary behavior and uniqueness of solutions to problem (1.1) under $(b_2)$.

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic process (see Seneta [31], Bingham et al [5], Resnick [30] and Maric [27].), and has been applied to study the asymptotic behavior of solutions to differential equations and problem (1.1) (see Maric [27], Cîrstea and Rădulescu [6], Cîrstea and Du [7], Cîrstea [8], the authors [37] and the references therein.).

Our main results are summarized as the following.

**Theorem 1.1.** Let $f$ satisfy $(f_1)$ or $(f_{11})$, $(f_2)$, $(f_3)$ and let $b$ satisfy $(b_1)$ and $(b_2)$.

(I): If

\[ C_k > 0 \quad \text{or} \quad C_f > 1, \]

then for any solution $u$ of problem (1.1)

\[ \xi_2^{- (C_f - 1)} \leq \liminf_{d(x) \to 0} \psi(K^2(d(x))) \leq \limsup_{d(x) \to 0} \psi(K^2(d(x))) \leq \xi_1^{- (C_f - 1)}, \]

where $\psi$ satisfies

\[ \int_0^{\infty} \frac{ds}{f(s)} = t, \quad \forall \ t > 0, \]

and

\[ \xi_1 = \frac{b_1}{2(C_k + 2C_f - 2)}, \quad \xi_2 = \frac{b_2}{2(C_k + 2C_f - 2)}. \]

In particular, when $C_f = 1$,

\[ \lim_{d(x) \to 0} \frac{u(x)}{\psi(K^2(d(x)))} = 1. \]

(II): If

\[ C_k > 0, \]

then problem (1.1) admits a unique solution.

**Remark 1.2.** For the existence of the minimal solution to problem (1.1), see [20] [32] and [34].

**Remark 1.3.** By the following Lemmas 2.9 and 2.10, one can see that (1.7) is equivalent to the condition that

\[ C_k + 2C_f > 2. \]

The outline of this paper is as follows. In Section 2, we need preliminary considerations. In Section 3 we prove Theorem 1.1 (I). The proof of Theorem 1.1 (II) is given in Section 4.
2. Preliminaries. In this section, we present some bases of Karamata regular variation theory which come from Seneta [31], Preliminaries in Resnick [30], Introductions and the Appendix in Maric [27].

**Definition 2.1.** A positive measurable function $f$ defined on $[a, \infty)$, for some $a > 0$, is called **regularly varying at infinity** with index $p$, written as $f \in RV_p$, if for each $\xi > 0$ and some $p \in \mathbb{R}$,

$$\lim_{s \to \infty} \frac{f(\xi s)}{f(s)} = \xi^p$$

(2.1)

In particular, when $p = 0$, $f$ is called **slowly varying at infinity**.

**Definition 2.2.** A positive measurable function $f$ defined on $[a, \infty)$, for some $a > 0$, is called **rapidly varying at infinity** if for each $p > 1$

$$\lim_{s \to \infty} \frac{f(s)}{s^p} = \infty.$$  

(2.2)

Clearly, if $f \in RV_p$, then $L(s) := f(s)/s^p$ is slowly varying at infinity. Some basic examples of slowly varying functions at infinity are

(i): every measurable function on $[a, \infty)$ which has a positive limit at infinity;

(ii): $(\ln s)^q$ and $(\ln(\ln s))^q$, $q \in \mathbb{R}$;

(iii): $e^{(\ln s)^q}$, $0 < q < 1$.

We also say that a positive measurable function $g$ defined on $(0, a)$ for some $a > 0$, is **regularly varying at zero** with index $p$ (written as $g \in RVZ_p$) if $t \to g(1/t)$ belongs to $RV_{-p}$. Similarly, $g$ is called **rapidly varying at zero** if $t \to g(1/t)$ is rapidly varying at infinity.

**Proposition 2.3** (Uniform convergence theorem). If $f \in RV_p$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $p < 0$, then uniform convergence holds on intervals of the form $(a_1, \infty)$ with $a_1 > 0$; if $p > 0$, then uniform convergence holds on intervals $(0, a_1)$ provided $f$ is bounded on $(0, a_1]$ for all $a_1 > 0$.

**Proposition 2.4** (Representation theorem). A function $L$ is slowly varying at infinity if and only if it may be written in the form

$$L(s) = \varphi(s)\exp\left(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$

(2.3)

for some $a_1 \geq a$, where the functions $\varphi$ and $y$ are measurable and for $s \to \infty$, $y(s) \to 0$ and $\varphi(s) \to \varphi_0$, with $\varphi_0 > 0$.

We say that

$$\hat{L}(s) = c_0 \exp\left(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau\right), \quad s \geq a_1,$$

(2.4)

is **normalised** slowly varying at infinity and

$$f(s) = s^p \hat{L}(s), \quad s \geq a_1,$$

(2.5)

is **normalised** regularly varying at infinity with index $p$ (and written as $f \in NRV_p$).

Similarly, $g$ is said **normalised** regularly varying at zero with index $p$, written as $g \in NRVZ_p$ if $t \to g(1/t)$ belongs to $NRV_{-p}$. 
A function $f \in RV_p$ belongs to $NRV_p$ if and only if
\[ f \in C^1[a_1, \infty) \text{ for some } a_1 > 0 \text{ and } \lim_{s \to \infty} \frac{sf'(s)}{f(s)} = p. \tag{2.6} \]

**Proposition 2.5.** If functions $L, L_1$ are slowly varying at infinity, then
\begin{itemize}
  \item[(i)] $L^q$ for every $q \in \mathbb{R}$, $c_1 L + c_2 L_1$ ($c_1 \geq 0$, $c_2 \geq 0$ with $c_1 + c_2 > 0$), $L \circ L_1$ (if $L_1(t) \to \infty$ as $t \to \infty$), are also slowly varying at infinity.
  \item[(ii)] For every $q > 0$ and $t \to \infty$, $t^q L(t) \to \infty$ and $t^{-q} L(t) \to 0$.
  \item[(iii)] For $q \in \mathbb{R}$ and $t \to \infty$, \( \frac{\ln(L(t))}{\ln t} \to 0 \) and \( \frac{\ln^r(L(t))}{\ln t} \to q. \)
\end{itemize}

**Proposition 2.6.** If $f_1 \in RV_{q_1}, f_2 \in RV_{q_2}$ with $\lim_{t \to \infty} f_2(t) = \infty$, then $f_1 \circ f_2 \in RV_{q_1q_2}$.

**Proposition 2.7** (Asymptotic behaviour). If a function $L$ is slowly varying at infinity, then for $a \geq 0$ and $t \to \infty$,
\begin{itemize}
  \item [(i)] $\int_0^1 s^q L(s)ds \equiv (q + 1)^{-1}t^{1+q}L(t)$, for $q > -1$;
  \item [(ii)] $\int_1^\infty s^q L(s)ds \equiv (-q - 1)^{-1}t^{-1+q}L(t)$, for $q < -1$.
\end{itemize}

**Proposition 2.8** (Asymptotic behaviour). If a function $H$ is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,
\begin{itemize}
  \item [(i)] $\int_0^a s^q H(s)ds \equiv (q + 1)^{-1}t^{1+q}H(t)$, for $q > -1$;
  \item [(ii)] $\int_a^\infty s^q H(s)ds \equiv (-q - 1)^{-1}t^{-1+q}H(t)$, for $q < -1$.
\end{itemize}

Our results in this section are summarized as the following.

**Lemma 2.9.** Let $k \in \Lambda$. Then
\begin{itemize}
  \item [(i)] $C_k \in [0, 1]$;
  \item [(ii)] when $C_k \in (0, 1)$, $k \in NRVZ(1-C_k)/C_k$;
  \item [(iii)] when $C_k = 1$, $k$ is normalised slowly varying at zero;
  \item [(iv)] when $C_k = 0$, $k$ is rapidly varying at zero.
\end{itemize}

**Proof.** By the l’Hospital’s rule and (1.2), we have
\[ \lim_{t \to 0} \frac{K(t)}{tk(t)} = \lim_{t \to 0} \frac{K(t)}{tk(t)} \frac{K(t)}{\ln t} = \lim_{t \to 0} \frac{K(t)}{k(t)} = 1 - \lim_{t \to 0} \frac{K(t)}{k(t)} = C_k. \tag{2.7} \]

(i) Since $k'(t) \geq 0$, $t \in (0, \delta_0)$, we see that $C_k \in [0, 1]$.

(ii) and (iii) It follows by (2.7) that
\[ \lim_{t \to 0} \frac{tk'(t)}{k(t)} = \lim_{t \to 0} \frac{K(t)}{k^2(t)} = \lim_{t \to 0} \frac{tk(t)}{K(t)} = \frac{1 - C_k}{C_k}, \tag{2.8} \]
\[ \text{i.e., when } C_k \in (0, 1), k \in NRVZ(1-C_k)/C_k, \text{ when } C_k = 1, k \text{ is normalised slowly varying at zero.} \]

(iv) When $C_k = 0$, for arbitrary $\gamma > 0$, it follows by (2.8) that $\lim_{t \to 0} \frac{tk'(t)}{k(t)} = +\infty$ and there exists $t_{0\gamma} > 0$ such that
\[ \frac{k'(t)}{k(t)} > (\gamma + 1)t^{-1}, \forall t \in (0, t_{0\gamma}]. \tag{2.9} \]

Integrate (2.9) from $t$ to $t_{0\gamma}$, we obtain
\[ \ln(k(t)) - \ln(k(t)) > (\gamma + 1)(\ln t_{0\gamma} - \ln t), \forall t \in (0, t_{0\gamma}], \]
Lemma 2.10 (Lemma 2.1 in [37]). If $f$ satisfies (f₁) (or (f₀₁)), (f₂) and (f₃), then

(i): $C_f \in [1, \infty)$;
(ii): there exists $S_0 > 0$ such that $f(s)/s^p$ is increasing in $[S_0, \infty)$, where $p \in \left(1, \frac{C_f}{C_f - 1}\right)$ for $C_f > 1$ and $p \in (1, \infty)$ for $C_f = 1$;
(iii): $f$ satisfies the Keller-Osserman condition (1.3);
(iv): (f₃) holds for $C_f > 1$ if and only if $f \in N R V_{C_f/(C_f - 1)}$;
(v): when $C_f = 1$, $f$ is rapidly varying at infinity.

Lemma 2.11 (Lemma 2.2 in [37]). Let $f$ satisfy (f₁) (or (f₀₁))- (f₃) and let $\psi$ be the solution to the problem

$$\int_{\psi(t)}^\infty \frac{ds}{f(s)} = t, \forall \ t > 0.$$ 

Then

(i): $-\psi'(t) = f(\psi(t))$, $\psi(t) > 0$, $t > 0$, $\psi(0) =: \lim_{t \to 0^+} \psi(t) = +\infty$ and $\psi''(t) = f(\psi(t))\psi'(t)$, $t > 0$;
(ii): $\psi \in N R V_{Z_{-(C_f - 1)}}$;
(iii): $-\psi' = f \circ \psi \in N R V_{Z_{C_f}}$;
(iv): $\lim_{t \to 0^+} \frac{\ln(\psi(t))}{\ln t} = C_f - 1$ and $\lim_{t \to 0^+} \frac{\ln f(\psi(t))}{\ln t} = C_f$.

By the definition of the set $\Lambda$, Definition 2.1, (2.6), (2.7), Proposition 2.6 and Lemmas 2.9-2.11, we have the following results.

Lemma 2.12. Let $k \in \Lambda$, $f$ satisfy (f₁) (or (f₀₁))- (f₃) and $\psi$ be given in Lemma 2.11. Then

(i): $\lim_{t \to 0^+} \frac{K(t)}{K(t)} = 0$;
(ii): $C_k \in [0, 1]$ and $\lim_{t \to 0^+} \frac{K(t)K'(t)}{K^2(t)} = 1 - C_k$;
(iii): when $C_k > 0$, $\lim_{t \to 0^+} \frac{\psi(K(t))K'(t)}{K(t)K''(t)} = 1/C_k$ and $\lim_{t \to 0^+} \frac{K(3t)/K(t)}{2^1/C_k} = 3^{-1/C_k}$;
(iv): when $C_k > 0$, $\lim_{t \to 0^+} \frac{\psi(K(3t))/\psi(K(t))}{\psi(K(3t))/\psi(K(t))} = 3^{-2(C_f - 1)/C_k}$;
(v): when $C_k > 0$ and $\xi > 0$, $\lim_{t \to 0^+} \frac{\psi(K(t))/\psi(t)}{\psi(K(t))/\psi(t)} = (C_f - 1)\xi^{C_f/((C_f - 1)}$.

3. Boundary behaviour. In this section we give the boundary behaviour of solutions to problem (1.1).

First, in the same proof of Lemma 2.4 in [9], we have the following result.

Lemma 3.1 (the comparison principle). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $f$ be an increasing function and let $b$ satisfy (b₁) and be positive near the boundary $\partial \Omega$. Assume that $u_1, u_2 \in C^2(\Omega)$ satisfy $\Delta u_1 \geq b(x)f(u_1)$ and $\Delta u_2 \leq b(x)f(u_2)$ in $\Omega$. If $\liminf_{x \to \partial \Omega}(u_2 - u_1)(x) \geq 0$, then $u_2 \geq u_1$ in $\Omega$.

Now let $v_0 \in C^{2+\alpha}(\Omega) \cap C^1(\Omega)$ be the unique solution to the problem

$$-\Delta v_0 = 1, \quad v_0 > 0, \quad x \in \Omega, \quad v_0|_{\partial \Omega} = 0.$$ 

By the Höpf maximum principle in [16], we see that

$$\nabla v_0(x) \neq 0, \quad \forall \ x \in \partial \Omega \quad \text{and} \quad c_1d(x) \leq v_0(x) \leq c_2d(x), \quad \forall \ x \in \Omega.$$ 

(3.2)
where $c_1$, $c_2$ are positive constants.

For any $\delta > 0$, we define

$$\Omega_{\delta} = \{x \in \Omega : d(x) < \delta\}.$$ 

Since $\Omega$ is smooth, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$ and

$$|\nabla d(x)| = 1 \text{ and } \Delta d(x) = -(N - 1)H(\bar{x}) + o(1), \forall x \in \Omega_{\delta_0},$$

where $\bar{x}$ is the nearest point to $x$ on $\partial \Omega$, and $H(\bar{x})$ denotes the mean curvature of $\partial \Omega$ at $\bar{x}$.

**Lemma 3.2.** Under the hypotheses in Theorem 1.1 (I), any solution $u$ of problem (1.1) satisfies (1.8).

**Proof.** Let $\varepsilon \in (0, b_1/4)$ and

$$\tau_1 = \xi_1 - 2\varepsilon \xi_1 / b_1, \quad \tau_2 = \xi_2 + 2\varepsilon \xi_2 / b_2.$$ 

It follows that

$$\xi_1 / 2 < \tau_1 < \tau_2 < 2 \xi_2,$$

where $\xi_1$ and $\xi_2$ are given in (1.10).

By (b1), (b2) and Lemmas 2.9-2.12, we see that there is $\delta_x \in (0, \delta_0/2)$ (which is corresponding to $\varepsilon$) sufficiently small such that

**$r_1$:** $(b_1 - \varepsilon)k^2(d(x) - \rho) \leq b(x), \ x \in D^-_{\rho} = \Omega_{2\delta_x} \setminus \Omega_{\rho}$ and $b(x) \leq (b_2 + \varepsilon)k^2(d(x) + \rho), \ x \in D^+_{\rho} = \Omega_{2\delta_x} - \rho$, where $\rho \in (0, \delta_x)$.

**$r_2$:** For $i = 1, 2$,

$$8\xi_2 \left| \tau_i K^2(t)f'(\psi(\tau_i K^2(t))) - C_f \right| + 4\xi_2 \left| \frac{k'(t)K(t)}{k^2(t)} - (1 - C_k) \right| + 4\xi_2 \left| K(t) \right| \Delta d(x) < \varepsilon, \ \forall (x, t) \in \Omega_{2\delta_x} \times (0, 2\delta_x).$$

Let

$$d_1(x) = d(x) - \rho, \ d_2(x) = d(x) + \rho, \quad (3.4)$$

$$\bar{u}_x = \psi(\tau_1 K^2(d_1(x))), \quad x \in D^-_{\rho} \text{ and } \underline{u}_x = \psi(\tau_2 K^2(d_2(x))), \ x \in D^+_{\rho}. \quad (3.5)$$

It follows that, for $x \in D^-_{\rho}$

$$\Delta \bar{u}_x(x) - b(x)f(\bar{u}_x(x)) = \psi''(\tau_1 K^2(d_1(x))) \left(2\tau_1 K(d_1(x))k(d_1(x))\right)^2 + 2\tau_1 \psi'(\tau_1 K^2(d_1(x)))$$

$$\left(k^2(d_1(x)) + K(d_1(x))k'(d_1(x)) + K(d_1(x))k(d_1(x))\right)\Delta d(x)$$

$$- b(x)f(\psi(\tau_1 K^2(d_1(x)))) \leq f(\psi(\tau_1 K^2(d_1(x))))k^2(d_1(x)) \left(4\tau_1 \psi'(\tau_1 K^2(d_1(x)))

f'(\psi(\tau_1 K^2(d_1(x)))) - C_f \right) + 4\tau_1 C_f - 2\tau_1$$

$$- 2\tau_1 \left\{ \frac{k'(d_1(x))K(d_1(x))}{k^2(d_1(x))} - (1 - C_k) \right\} - 2\tau_1(1 - C_k)$$

$$- 2\tau_1 \frac{K(d_1(x))}{k(d_1(x))} \Delta d(x) - (b_1 - \varepsilon) \leq 0,$$

i.e., $\bar{u}_x$ is a supersolution to equation (1.1) in $D^-_{\rho}$.

In a similar way, we can show that $\underline{u}_x$ is a subsolution to equation (1.1) in $D^+_{\rho}$.
Now let \( u \) be an arbitrary solution to problem (1.1). We assert that there exists a positive constant \( M \) such that
\[
 u \leq Mv_0(x) + \bar{u}_\varepsilon, \quad x \in D_\rho^- ,
\]
\[
 \underline{u}_\varepsilon \leq u + Mv_0(x), \quad x \in D_\rho^+ ,
\]
where \( v_0 \) is the solution to problem (3.1).

In fact, we may choose a large \( M \) such that
\[
 u \leq Mv_0(x) + \bar{u}_\varepsilon \text{ on } \Gamma_{2\delta_x} := \{ x \in \Omega : d(x) = 2\delta_x \}.
\]

By \((f_1)\) or \((f_0)\), we see that \( \bar{u}_\varepsilon + Mv_0 \) is also a supersolution to equation (1.1) in \( D_\rho^- \). Since \( u < \bar{u}_\varepsilon \) on \( \Gamma_\rho := \{ x \in \Omega : d(x) = \rho \} \), (3.6) follows by Lemma 3.1.

In a similar way, we can show (3.7).

Hence, \( x \in D_\rho^- \cap D_\rho^+ \), by letting \( \rho \to 0 \), we have
\[
 1 - \frac{Mv_0(x)}{\psi(\tau_2K^2(d(x)))} \leq \frac{u(x)}{\psi(\tau_2K^2(d(x)))} \quad \text{and} \quad \frac{u(x)}{\psi(\tau_1K^2(d(x)))} \leq 1 + \frac{Mv_0(x)}{\psi(\tau_1K^2(d(x)))}.
\]

Consequently,
\[
 1 \leq \lim_{d(x) \to 0} \inf \frac{u(x)}{\psi(\tau_2K^2(d(x)))} \quad \text{and} \quad \lim_{d(x) \to 0} \sup \frac{u(x)}{\psi(\tau_1K^2(d(x)))} \leq 1. \quad (3.8)
\]

Thus by letting \( \varepsilon \to 0 \), we have
\[
 1 \leq \lim_{d(x) \to 0} \inf \frac{u(x)}{\psi(\xi_2K^2(d(x)))} \quad \text{and} \quad \lim_{d(x) \to 0} \sup \frac{u(x)}{\psi(\xi_1K^2(d(x)))} \leq 1. \quad (3.9)
\]

By Lemma 2.11 (ii) and Proposition 2.3, we have
\[
 \lim_{d(x) \to 0} \frac{\psi(\xi_2K^2(d(x)))}{\psi(\xi_1K^2(d(x)))} = \xi_2^{-(C_f-1)} \quad \text{and} \quad \lim_{d(x) \to 0} \frac{\psi(\xi_1K^2(d(x)))}{\psi(\xi_1K^2(d(x)))} = \xi_1^{-(C_f-1)}.
\]

Thus
\[
 \xi_2^{-(C_f-1)} \leq \lim_{d(x) \to 0} \inf \frac{u(x)}{\psi(K^2(d(x)))} \leq \lim_{d(x) \to 0} \sup \frac{u(x)}{\psi(K^2(d(x)))} \leq \xi_1^{-(C_f-1)} \quad (3.10)
\]

In particular, when \( C_f = 1 \), we have
\[
 \lim_{d(x) \to 0} \frac{u(x)}{\psi(K^2(d(x)))} = 1. \quad (3.11)
\]

The proof is finished. \( \square \)

4. **Uniqueness.** In this section we prove the uniqueness of solutions to problem (1.1).

First we need the following lemma.

**Lemma 4.1.** If \( C_f > 1 \) and \( C_k > 0 \), then for arbitrary solutions \( u_1 \) and \( u_2 \) of problem (1.1),
\[
 \lim_{d(x) \to 0} \frac{u_2(x)}{u_1(x)} = 1. \quad (4.1)
\]
Proof. The method is similar to the proof of Theorem 2 in [15]. First we see by Lemma 2.10 (ii) that
\[ \frac{f(s)}{s^p} \text{ is increasing in } [S_0, \infty) \] (4.2)
for some \( p > 1 \) and \( S_0 \) large enough.

Consequently,
\[ \frac{f(s)}{s} \text{ is also increasing in } [S_0, \infty). \] (4.3)

By \( C_f > 1 \) and (1.8), we have
\[ \lim_{d(x) \to 0} \sup_{u_1(x)} \frac{u_2(x)}{u_1(x)} \leq \left( \frac{x_2}{x_1} \right)^{C_f-1}. \] (4.4)

Let
\[ M_0 := \lim_{d(x) \to 0} \sup_{u_1(x)} \frac{u_2(x)}{u_1(x)}. \] (4.5)

To prove the theorem it suffices to show that \( M_0 \leq 1 \). Assume the contrary, \( M_0 > 1 \). Let \( \varepsilon \in (0, \min \{ (M_0 - 1)/2, M_0 - (1 + m_0)^{1/(q - 1)} \}) \) small enough, where \( m_0 \in (0, M_0^{-1} - 1) \). Then we have
\[ M_0 - \varepsilon > 1 \text{ and } (M_0 - \varepsilon)^q \geq (1 + m_0)(M_0 - \varepsilon). \] (4.6)

By (4.5), there exist \( \delta_\varepsilon > 0 \) (which is corresponding to \( \varepsilon \)) sufficiently small and \( x_0 \) such that
\[ \frac{u_2(x)}{u_1(x)} < M_0 + \varepsilon, \quad x \in \Omega_{\delta_\varepsilon}, \] (4.7)
and
\[ \frac{u_2(x_0)}{u_1(x_0)} > M_0 - \varepsilon, \quad x_0 \in \Omega_{2\delta_\varepsilon/3}. \] (4.8)

Moreover, by (1.8) and Lemma 2.12 \((C_k > 0)\), we may assume that
\begin{enumerate}
  \item [(I_1):] \( u_1, u_2 \geq S_0 \text{ in } \Omega_{\delta_\varepsilon}; \)
  \item [(I_2):] \( u_1(x) \geq (2\xi_2)^{-(C_f-1)} \psi(K^2(d(x))) \text{ in } \Omega_{\delta_\varepsilon}; \)
  \item [(I_3):] \( u_1(x) \leq 2\xi_1^{-(C_f-1)} \psi(K^2(d(x))) \text{ in } \Omega_{\delta_\varepsilon}; \)
  \item [(I_4):] \( f((2\xi_2)^{-(C_f-1)} \psi(K^2(d(x)))) \geq c_0 \psi(K^2(d(x))) \geq c_0 \psi(K^2(d(x))) \text{ in } \Omega_{\delta_\varepsilon}, \)
  where \( c_0 = \frac{C_f-1}{4(2\xi_2)^{C_f-1}} \)
  \item [(I_5):] \( \frac{d(x)k(d(x))}{K(3d(x))} \geq \frac{6^{-1/C_k}}{C_k} \text{ in } \Omega_{\delta_\varepsilon}; \)
  \item [(I_6):] \( \frac{\psi(K^2(3d(x)))}{\psi(K^2(d(x)))} \geq 6^{-2(C_f-1)/C_k} \text{ in } \Omega_{\delta_\varepsilon}. \)
\end{enumerate}

Define
\[ D = \{ x \in \Omega_{\delta_\varepsilon} : u_2(x) > (M_0 - \varepsilon)u_1(x) \} \cap B_\rho(x_0), \] (4.9)
where \( \rho = d(x_0)/2 \).

Now let \( \Omega = B_\rho(x_0) \) in problem (3.1). Then
\[ v_0(x) = (2N)^{-1} (\rho^2 - |x - x_0|^2). \]
Using (f1) (or (f01)), (b2), (4.2), (4.6), (I1)-(I2), (I4)-(I6) and \( \rho \leq d \leq 3\rho < \delta \) in \( D \), we have in the set \( D \)

\[
\Delta (u_2 - (M_0 - \varepsilon)u_1) = b(x) (f(u_2) - (M_0 - \varepsilon)f(u_1)) 
\]

\[
\geq b(x) (f((M_0 - \varepsilon)u_1) - (M_0 - \varepsilon)f(u_1)) \geq b(x) ((M_0 - \varepsilon)^9 - (M_0 - \varepsilon))f(u_1) 
\]

\[
\geq b_1m_0 (M_0 - \varepsilon)k^2(d(x))f(u_1) \geq b_1m_0c_0(M_0 - \varepsilon)k^2(\rho)(K(3\rho))^{-2}\psi(K^2(3\rho)) 
\]

\[
\geq C, 
\]

where

\[
C = b_1m_0c_0(M_0 - \varepsilon)6^{-2/c_0}6^{-2(C_1^{-1})/c_k}c_k^{-2}\rho^{-2}\psi(K^2(\rho)). \tag{4.10}
\]

It follows that

\[
\Delta (u_2 - (M_0 - \varepsilon)u_1 + Cv_0) \geq 0 \quad \text{in} \quad D. \tag{4.11}
\]

Then, the maximum principle implies the existence of \( y_0 \in \partial D \) such that

\[
u_2(x_0) - (M_0 - \varepsilon)u_1(x_0) + Cv_0(x_0) < u_2(y_0) - (M_0 - \varepsilon)u_1(y_0) + Cv_0(y_0). \tag{4.12}
\]

Note that \( y_0 \in \partial D \), thus if \( u_2(y_0) = (M_0 - \varepsilon)u_1(y_0) \), (4.12) implies that \( y_0 \in \partial B_\rho(x_0) \), and in particular

\[
C(2N)^{-1} \rho^2 \leq u_2(y_0) - (M_0 - \varepsilon)u_1(y_0). \tag{4.13}
\]

Moreover, by \( \rho \leq d(y_0) \leq 3d(x_0)/2 < \delta \), (4.10) and (I3), we have

\[
C(2N)^{-1} \rho^2 \geq (M_0 - \varepsilon)C_0\psi(K^2(d(y_0))) \geq (M_0 - \varepsilon)C_02^{-1}\xi_1^{C_j^{-1}}u_1(y_0),
\]

where

\[
C_0 = b_1m_0c_0(2N)^{-1}6^{-2/c_0}6^{-2(C_1^{-1})/c_k}c_k^{-2}. \tag{4.14}
\]

It follows by (4.7) that

\[
(M_0 + \varepsilon)u_1(y_0) > u_2(y_0) \geq (M_0 - \varepsilon)u_1(y_0) + 2^{-1}C_0(M_0 - \varepsilon)\xi_1^{C_j^{-1}}u_1(y_0),
\]

i.e.,

\[
M_0 + \varepsilon \geq \left( 1 + 2^{-1}C_0\xi_1^{C_j^{-1}} \right) (M_0 - \varepsilon). \tag{4.15}
\]

After letting \( \varepsilon \to 0 \), we arrive at a clear contradiction. Thus, our initial assumption \( M_0 > 1 \) is incorrect, and we have \( M_0 \leq 1 \). This concludes the proof. \( \square \)

Finally, we prove the uniqueness of solutions to problem (1.1).

Let \( u_0 \) be the minimal solution of problem (1.1), and let \( u \) be any other solution to problem (1.1). We prove \( u = u_0 \) in \( \Omega \). In fact, by the definition of the minimal solution, we have

\[
u_0 \leq u \quad \text{in} \quad \Omega. \tag{4.16}
\]

Moreover, by the asymptotic behavior (1.11) and (4.1) we have

\[
\lim_{d(x) \to 0} \frac{u(x)}{u_0(x)} = 1. \tag{4.17}
\]

For \( \varepsilon > 0 \) arbitrary, setting \( w := (1 + \varepsilon)u_0 \), we have

\[
\lim_{d(x) \to 0} (w(x) - u(x)) = \lim_{d(x) \to 0} u(x) \left( \frac{(1 + \varepsilon)u_0(x)}{u(x)} - 1 \right) = +\infty. \tag{4.18}
\]

Now, for small \( \varepsilon > 0 \), we define the (open) set

\[
D_\varepsilon := \{ x \in \Omega : w(x) < u(x) \}. \tag{4.19}
\]

We may assume that \( D_\varepsilon \) is nonempty for \( \varepsilon \) small enough, for otherwise there is nothing to prove. Indeed, notice that \( D_\varepsilon \) monotonically increases as \( \varepsilon \downarrow 0 \). Moreover, we may also assume that \( D_\varepsilon \to \Omega \) as \( \varepsilon \to 0 \), for if there exists \( x_0 \in \Omega \) and a sequence
that we have \( u(0) \geq u(x_0) \). The strong maximum principle then yields \( u \equiv u_0 \) in \( \Omega \). Finally, we have \( D_\varepsilon \subset \subset \Omega \) by (4.18).

Next we choose \( \eta > 0 \) so that \( u_0 > S_0 \) in \( \Omega_\eta \) and define \( D_{\varepsilon, \eta} = D_\varepsilon \cap D_\eta \). Notice that \( D_{\varepsilon, \eta} \) is a non-empty open set for small \( \varepsilon \). Moreover, we have by (4.3) that

\[
\Delta w = (1 + \varepsilon) b(x) f(u_0) \leq b(x) f(w), \quad x \in D_{\varepsilon, \eta}. \tag{4.20}
\]

It follows by \((f_1)\) or \((f_{01})\) that

\[
\Delta (u - w) \geq b(x) (f(u) - f(w)) \geq 0, \quad x \in D_{\varepsilon, \eta}. \tag{4.21}
\]

Thus, there is

\[
u(x) - w(x) \leq \max_{\partial D_{\varepsilon, \eta}} (u - w), \quad x \in D_{\varepsilon, \eta}, \tag{4.22}
\]

by view of the maximum principle.

Since \( \partial D_{\varepsilon, \eta} = (\partial D_\varepsilon \cap \partial D_\eta) \cup (D_\varepsilon \cap \partial D_\eta) \), \( D_\varepsilon \cap \partial \Omega = \emptyset \) and \( (u - w)|_{\partial D_\varepsilon} = 0 \), we see that the maximum of \( u - w \) is achieved on \( D_\varepsilon \cap \partial D_\eta = D_\varepsilon \cap \{ x : d(x) = \eta \} \).

Hence

\[
u(x) - w(x) \leq \max_{D_\varepsilon \cap \partial D_\eta} (u(x) - w(x)), \quad x \in D_{\varepsilon, \eta}. \tag{4.23}
\]

Letting \( \varepsilon \to 0 \) in (4.23) we obtain

\[
u(x) - u_0 \leq \max_{D_\varepsilon \cap \partial D_\eta} (u - u_0) =: \theta \quad \text{in} \quad \Omega_\eta. \tag{4.24}
\]

On the other hand, by (4.16) and \((f_1)\) or \((f_{01})\), we have

\[
\Delta (u - u_0) = b(x) (f(u) - f(u_0)) \geq 0, \quad x \in \Omega^\theta : = \{ x \in \Omega : d(x) > \eta \}. \tag{4.25}
\]

The maximum principle implies that \( u - u_0 \leq \theta \) in \( \Omega^\theta \), and hence \( u - u_0 \leq \theta \) in the whole \( \Omega \). Then the strong maximum principle gives \( u - u_0 \equiv \theta \). We obtain that \( f(u) = f(u + \theta) \) in \( \Omega \), which can only hold if \( \theta = 0 \). Thus \( u \equiv u_0 \), and this shows the uniqueness.

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