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Depth-preserving property of the local Langlands correspondence for quasi-split classical groups in large residual characteristic

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Abstract. For a quasi-split classical group over a $p$-adic field with sufficiently large residual characteristic, we prove that the maximum depth of a representation in each $L$-packet equals the depth of the corresponding $L$-parameter. Furthermore, for quasi-split unitary groups, we show that the depth is constant in each $L$-packet. The key is an analysis of the endoscopic character relation via harmonic analysis based on Bruhat–Tits theory. These results are slight generalizations of a result of Ganapathy and Varma.

Notation. Let $p$ be a prime number. In this paper, we always assume that $p$ is not equal to 2. We fix a $p$-adic field $F$. We denote the Weil group of $F$ by $W_F$. For an algebraic variety $J$ over $F$, we denote the set of its $F$-valued points by $J(F)$. When $J$ is a connected reductive group, we write $\hat{J}$ and $L_J = \hat{J} \rtimes W_F$ for its Langlands dual group and $L$-group, respectively. In this paper, the word “quasi-split classical group” means one of the following groups:

- a quasi-split special orthogonal group,
- a symplectic group, or
- a quasi-split unitary group.

1. Introduction

Let $F$ be a $p$-adic field and $G$ either a general linear group or a quasi-split classical group over $F$. We denote the set of equivalence classes of irreducible smooth representations of $G = G(F)$ by $\Pi(G)$, and the set of conjugacy classes of $L$-parameters of $G$ by $\Phi(G)$. Then the local Langlands correspondence for $G$, which has been established by Harris–Taylor (general linear groups, [19]), Arthur (symplectic or orthogonal groups, [7]), and Mok (unitary groups, [26]) gives a natural map from the set $\Pi(G)$ to the set $\Phi(G)$ with finite fibers (called $L$-packets). In other words, the local Langlands correspondence gives a natural partition of the set $\Pi(G)$ into finite sets parametrized by $L$-parameters:

$$\Pi(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi^G_{\phi}.$$
It is known that the local Langlands correspondence satisfies a lot of natural properties beyond its characterization. One example of such a phenomenon is the *depth-preserving* property in the case of general linear groups. To be more precise, we recall that a numerical invariant called *depth* is defined for any \( \pi \in \Pi(G) \) by

\[
\text{depth}(\pi) := \inf \{ r \in \mathbb{R}_{\geq 0} \mid \pi^{G, x, r+} \neq 0 \text{ for some } x \in B(G, F) \} \in \mathbb{R}_{\geq 0}.
\]

Here \( B(G, F) \) denotes the Bruhat–Tits building of \( G \) and \( G_{x, \bullet} \) denotes the Moy–Prasad filtration of \( G \) with respect to the point \( x \). On the other hand, using the upper ramification filtration \( I^+_F \) of the inertia subgroup \( I_F \), we can also define the *depth* for any \( \phi \in \Phi(G) \) by

\[
\text{depth}(\phi) := \inf \{ r \in \mathbb{R}_{\geq 0} \mid \phi(w, 1) = 1 \times w \text{ for every } w \in I^+_F \} \in \mathbb{R}_{\geq 0}.
\]

Then it is known that the local Langlands correspondence for \( GL_N \) preserves depth (see, e.g., [2]). Therefore it is natural to attempt to investigate the relationship between the depth of representations and that of \( L \)-parameters under the local Langlands correspondence for other groups. In a recent paper [14], Ganapathy and Varma give the following partial answer to this problem in the case of quasi-split symplectic or special orthogonal groups:

**Theorem 1.1.** [14, Corollary 10.6.4] Let \( H \) be a quasi-split symplectic or special orthogonal group over \( F \). We assume that the residual characteristic \( p \) is large enough (see Hypothesis 4.7 for the precise specification). Let \( \phi \) be a tempered \( L \)-parameter of \( H \), and \( \Pi^H_\phi \) the \( L \)-packet of \( H \) for \( \phi \). Then

\[
\max \{ \text{depth}(\pi) \mid \pi \in \Pi^H_\phi \} \leq \text{depth}(\phi).
\]

We remark that although only the cases of symplectic and orthogonal groups are treated in [14], the same proof works for the case of unitary groups.

Our main theorem in this paper is the following enhancement of the result of Ganapathy–Varma.

**Theorem 1.2.** (Theorem 4.11) Let \( H \) be a quasi-split classical group over \( F \). We assume that the residual characteristic \( p \) is large enough (see Hypothesis 4.7 for the precise specification). Let \( \phi \) be an \( L \)-parameter of \( H \), and \( \Pi^H_\phi \) the \( L \)-packet of \( H \) for \( \phi \). Then

\[
\max \{ \text{depth}(\pi) \mid \pi \in \Pi^H_\phi \} = \text{depth}(\phi).
\]

We note that we can furthermore show the constancy of the depth in each \( L \)-packet when \( H \) is unitary (we will explain this later in this introduction, see Theorem 1.5). We also note that, in Theorem 1.2, \( \phi \) is not necessarily assumed to be tempered in contrast to Theorem 1.1. However, the non-tempered case can be easily reduced to the tempered case by the Langlands classification and the fact that the parabolic induction preserves the depth (see the end of Sect. 4.3).

From now on, let \( H \) be a quasi-split classical group over \( F \). Before we explain the sketch of our proof, we recall the *endoscopic character relation*, which is necessary to characterize the local Langlands correspondence for \( H \). The point is that \( H \) may
be regarded as an *endoscopic group* of a (twisted) general linear group \( GL_N \) over \( F \) (see Sect. 2.1). In particular, we have an embedding \( \iota \) of the \( L \)-group of \( H \) into that of \( GL_N \). Let \( \phi \) be a tempered \( L \)-parameter of \( H \), which is a homomorphism from \( WF \times SL_2(\mathbb{C}) \) to \( L H \). By composing the \( L \)-embedding \( \iota \) with \( \phi \), we obtain an \( L \)-parameter of \( GL_N \). Let \( \phi \) be a tempered \( L \)-parameter of \( H \), which is a homomorphism from \( WF \times SL_2(\mathbb{C}) \) to \( L H \). By composing the \( L \)-embedding \( \iota \) with \( \phi \), we obtain an \( L \)-parameter of \( GL_N \) (note that each \( L \)-packet of \( GL_N \) is a singleton). The other is an \( L \)-packet \( \Pi^H \), which is a finite set of representations of \( H \), corresponding to \( \phi \) under the local Langlands correspondence for \( H \).

In this situation, we say that \( \pi_{\phi}^{GL_N} \) is the endoscopic lift of \( \Pi^H \) from \( H \) to \( GL_N \). The endoscopic character relation is the following identity between the twisted character \( \Theta_{\phi,\theta}^{GL_N} \) of \( \pi_{\phi}^{GL_N} \) and the sum of characters \( \Theta_\pi \) of representations \( \pi \) belonging to \( \Pi^H \):

\[
\Theta_{\phi,\theta}^{GL_N}(f) = \sum_{\pi \in \Pi^H} \Theta_\pi(f^H).
\]

Here \( f \) is any test function of \( GL_N(F) \) and \( f^H \) is its Langlands–Shelstad–Kottwitz transfer to \( H \) (see Sect. 2.2 for the details). By noting that \( \text{depth}(\phi) = \text{depth}(\iota \circ \phi) \), we can see that the depth-preserving problem of the local Langlands correspondence for \( H \) is equivalent to that of the endoscopic lifting from \( H \) to \( GL_N \).

To explain the strategy of our proof of Theorem 1.2, we recall Ganapathy–Varma’s method in their proof of Theorem 1.1. The key ingredient is the character expansion, which describes the behavior of the character \( \Theta_\pi \) of an irreducible smooth representation \( \pi \) of \( H \) in some small neighborhood of the origin in terms of the Fourier transforms of the nilpotent orbital integrals. More precisely, if we have an appropriate exponential map \( c_H \) from the Lie algebra \( \mathfrak{h} \) to \( H \), then

\[
\Theta_\pi(f \circ c_H^{-1}) = \sum_{O \in \text{Nil}(\mathfrak{h})} c_O \cdot \hat{\mu}_O(f)
\]

for any function \( f \) on \( \mathfrak{h} \) supported on a “small neighborhood” of the origin.

The crucially important fact is that the validity range of the character expansion (the optimal neighborhood where the character expansion is valid) of \( \Theta_\pi \) is described via \( \text{depth}(\pi) \). To be more precise, for any \( r \in \mathbb{R} \geq 0 \), we put \( H_{r+} \) to be the union of the \((r+)\)-th Moy–Prasad filtrations of parahoric subgroups (see Sect. 2.5). In [12], DeBacker proved that the character expansion of \( \Theta_\pi \) is valid on \( H_{\text{depth}(\pi)+} \). Furthermore, Ganapathy and Varma proved that its converse is also true, that is, if \( \Theta_\pi \) has a character expansion on \( H_{s+} \) for some \( s \in \mathbb{R}_{\geq 0} \), then \( \text{depth}(\pi) \leq s \).
proof of this optimality statement is based on another result of DeBacker’s on a parametrization of nilpotent orbits (established in [13]) and is summarized as follows. First, by using DeBacker’s parametrization, they constructed a test function $f$ on $h$ such that

- $f \circ c_H^{-1}$ is bi-invariant under the $(s+)$-th Moy-Prasad filtration $H_{x,s+}$ for some $x \in B(H, F)$ and
- the right-hand side of the expansion ($\star$) does not vanish.

Then the equality ($\star$) implies that $\Theta_\pi(f \circ c_H^{-1}) \neq 0$. The key observation here is that, by the definition of the character $\Theta_\pi$,

$$\Theta_\pi(f \circ c_H^{-1}) = \text{tr}(\pi(f \circ c_H^{-1})|\pi H_{x,s+}).$$

Thus the non-vanishing of $\Theta_\pi(f \circ c_H^{-1})$ in particular implies that $\pi H_{x,s+} \neq 0$. Hence $\text{depth}(\pi) \leq s$ by the definition of the depth.

In fact, the character expansion is available also for twisted characters. For every function $f$ on the Lie algebra of $GL_N$ supported on a small neighborhood of the origin,

$$\Theta_{\phi,\theta}^{GL_N}(f \circ c^{-1}) = \sum_{O \in \text{Nil}(g_{\theta})} c_O \cdot \hat{\mu}_O(f_{\theta}).$$

(\star')

Here $c$ is a kind of exponential map (see Sect. 2.4), $g_{\theta}$ is the Lie algebra of the identity component $G_{\theta}$ of the subgroup of fixed points of an involution $\theta$ of $G := GL_N$ (see Sect. 2.1), and $f_{\theta}$ is a function on $g_{\theta}$ which is a semisimple descent of $f$ (see Sect. 3). Note that, although $G_{\theta}$ is a classical group, it may differ from the endoscopic group $H$. (Only when $H$ is a unitary group, $G_{\theta}$ coincides with $H$; see Sect. 4.1). For this expansion of twisted characters, Adler and Korman established a twisted version of DeBacker’s result in [4], that is, the expansion is valid on a subgroup $U_{\text{depth}(\pi_{\phi}^{GL_N})+}$ whose depth is given by depth($\pi_{\phi}^{GL_N}$) (see Sect. 3).

The idea of the proof of Theorem 1.1 due to Ganapathy and Varma is to compare the validity ranges of the character expansions through the endoscopic character relation as follows (let $\Pi_{\phi}^H$ be a tempered $L$-packet and $\pi_{\phi}^{GL_N}$ its endoscopic lift):

1. The character expansion of $\Theta_{\phi,\theta}^{GL_N}$ is valid on $U_{r+}$, where we put $r := \text{depth}(\pi_{\phi}^{GL_N})$ (Adler–Korman’s result [4]).
2. By the endoscopic character relation, the character $\Theta_\pi$ of any $\pi \in \Pi_{\phi}^H$ has the character expansion on $H_{r+}$.
3. By the optimality result of Ganapathy–Varma, we get the inequality $\text{depth}(\pi) \leq r$ for any $\pi \in \Pi_{\phi}^H$.

Then it is natural to expect that we can get the converse inequality by swapping the roles of $GL_N$ and $H$ in this argument, that is:

1' The character expansion of the sum $\sum_{\pi \in \Pi_{\phi}^H} \Theta_\pi$ is valid on $H_{r+}$, where we put $r := \max\{\text{depth}(\pi) | \pi \in \Pi_{\phi}^H\}$ (DeBacker’s result [12]).
(2)’ By the endoscopic character relation, the twisted character $\Theta^{GL_N}_{\phi, \theta}$ has the character expansion on $U_{r^+}$.

(3)’ By establishing the optimality result for the twisted characters following the method of Ganapathy–Varma, we get the inequality $\text{depth}(\pi^{GL_N}) \leq r$.

However, things do not go on so immediately. The problem is in the step (3)’.

By tracing the argument of Ganapathy–Varma, we can construct a test function $f_{\theta}$ on $g_{\theta}$ whose depth is given by $r^+$ such that the right-hand side of the expansion (⋆)′ does not vanish. However, in the twisted case, we furthermore have to describe what kind of a function $f$ on $G$ realizes $f_{\theta}$ as its semisimple descent.

The following is the key proposition of this paper, which describes the semisimple descent for the characteristic functions of Moy–Prasad filtrations of parahoric subgroups. (Note that the Bruhat–Tits building $B(G_{\theta}, F)$ of $G_{\theta}$ can be identified with the $\theta$-fixed points of $B(G, F)$.)

**Proposition 1.3.** (Corollary 3.7) Assume $p \neq 2$. Let $x \in B(G_{\theta}, F) \cong B(G, F)^{\theta}$ and $r \in \mathbb{R}_{>0}$. Then, for any coset $[h] \in G_{\theta, x, r}/G_{\theta, x, r^+}$, its characteristic function $1_{[h]} \in C^\infty_c(G_{\theta})$ is a semisimple descent of a linear combination of the characteristic functions of certain cosets of $G_{x, r}/G_{x, r^+}$ which can be determined explicitly.

Proposition 1.3 guarantees that we can find a bi-$G_{x, r^+}$-invariant function $f$ with semisimple descent $f_{\theta}$. Then we get the nonvanishing of $\Theta^{GL_N}_{\phi, \theta}(f)$ by (⋆)′. This implies that $(\pi^{GL_N}_{\phi})_{G_{x, r^+}} \neq 0$, in particular, depth$(\pi^{GL_N}_{\phi}) \leq r$. This completes the above argument (3)’ and we get the inequality

$$\max\{\text{depth}(\pi) \mid \pi \in \Pi^H_{\phi}\} \geq \text{depth}(\phi).$$

Combining this with Theorem 1.1, we get Theorem 1.2. We note that Proposition 1.3 in the case where $x$ is a hyperspecial point is proved in [14]. However, in order to make the step (3)’ work, we have to treat all points $x$.

We remark that Theorems 1.1 and 1.2 requires a condition on $p$ since all of the above-mentioned results on the character expansion need certain assumptions on $p$. In DaBacker’s paper [12], it is only stated that the hypotheses are satisfied whenever $p$ is sufficiently large. Thus we will explicate how large $p$ must be for classical groups in Lemma 4.4, which implies that Hypothesis 4.7 is enough for the proof of Theorems 1.1 and 1.2.

Next, we explain that we can furthermore show that the depth is constant in each $L$-packet when $H$ is unitary. In this case, $G_{\theta} = H$ as mentioned before. Moreover, the semisimple descent coincides with the Langlands–Shelstad–Kottwitz transfer. Thus Proposition 1.3 implies the following generalization of the fundamental lemma to a positive depth direction:

**Theorem 1.4.** (Theorem 5.1) Assume that $H$ is a quasi-split unitary group and $p \neq 2$. Let $x \in B(H, F) = B(G_{\theta}, F)$ and $r \in \mathbb{R}_{>0}$. Then $\text{vol}(H_{x, r})^{-1} \mathbb{I}_{H_{x, r}} \in C^\infty_c(H)$ is a transfer of $\text{vol}(G_{x, r})^{-1} \mathbb{I}_{G_{x, r}} \in C^\infty_c(G)$. 
We remark that a similar assertion for \( r = 0 \) (namely, the fundamental lemma for parahoric subgroups) in the case where \( E \) is unramified over \( F \) is proved in [20] (see also [15]).

We can deduce the following theorem from Theorem 1.4:

**Theorem 1.5.** (Theorem 5.6) Assume that \( H \) is a quasi-split unitary group and \( p \neq 2 \). Let \( \phi \) be an \( L \)-parameter of \( H \), and \( \Pi^H_\phi \) the \( L \)-packet of \( H \) for \( \phi \). Then

\[
\min\{\text{depth}(\pi) \mid \pi \in \Pi^H_\phi\} \geq \text{depth}(\phi).
\]

In particular, by combining it with Theorem 1.1,

\[
\text{depth}(\pi) = \text{depth}(\phi)
\]

for every \( \pi \in \Pi^H_\phi \) under the assumption that \( p \geq N + 1 \) (Hypothesis 4.7).

The key principle here is that, for any irreducible representation \( \pi \) of \( H \) and a compact open subgroup \( K \) of \( H \), \( \Theta_{\pi}(\mathbb{1}_K) = \dim(\pi^K) \); in particular, \( \Theta_{\pi}(\mathbb{1}_K) \neq 0 \) if and only if \( \pi^K \neq 0 \). Hence, when \( \pi \in \Pi^H_\phi \) satisfies \( \pi^{H_{x,r+}} \neq 0 \) with some \( x \in \mathcal{B}(H,F) \) and \( r \in \mathbb{R}_{>0} \), we know that \( \Theta^G_{\phi,\theta}(\mathbb{1}_{G_{x,r+}}) \neq 0 \) by applying the endoscopic character relation to the test functions considered in Theorem 1.4. Since \( \Theta^G_{\phi,\theta}(\mathbb{1}_{G_{x,r+}}) \neq 0 \) implies that \( (\pi^G_{\phi})^{G_{x,r}} \neq 0 \), we get the inequality of Theorem 1.5.

Note that we cannot carry out this argument in the reverse direction. This is because the converse implication \( (\pi^G_{\phi})^{G_{x,r}} \neq 0 \Rightarrow \Theta^G_{\phi,\theta}(\mathbb{1}_{G_{x,r+}}) \neq 0 \) may not be true in general (see Remark 5.4). Hence, even if the depth of \( \pi \) is given by \( r \), it is not clear whether we can find a point \( x \in \mathcal{B}(G, F) \) satisfying \( \Theta^G_{\phi,\theta}(\mathbb{1}_{G_{x,r+}}) \neq 0 \).

We finally remark that the inequality of Theorem 1.1 does not necessarily hold in general. For example, in [29, Section 7.4] Reeder and Yu constructed a candidate of the \( L \)-parameters of “simple supercuspidal representations” of \( \text{SU}_p(\mathbb{Q}_p) \) for an odd prime \( p \), by assuming Hiraga–Ichino–Ikeda’s formal degree conjecture. In this example, the depth of simple supercuspidal representations and the depth of their \( L \)-parameters are given by \( \frac{1}{2p} \) and \( \frac{1}{2(p+1)} \), respectively. See also [1] for some examples and counterexamples of the depth-preserving property of the local Langlands correspondence (especially, [1, Section 3.3]).

We explain the organization of this paper. In Sect. 2, we collect some basic preliminaries which will be needed in this paper. In Sect. 3, we compute the semisimple descent for the Moy–Prasad filtrations of parahoric subgroups of general linear groups. In Sect. 4, by using the results in Sect. 3, we evaluate the maximum depth of a representation in each \( L \)-packet of classical groups according to the converse of Ganapathy–Varma’s method. In Sect. 5, by focusing on the unitary case, we evaluate the minimum depth of a representation in each \( L \)-packet.

### 2. Basic preliminaries

#### 2.1. Twisted endoscopy for general linear groups

Let \((G, H)\) be one of the following pairs of connected reductive groups over \( F \):
(1) $G = \text{GL}_{2n+1}$ and $H$ is the symplectic group $\text{Sp}_{2n}$.
(2) $G = \text{GL}_{2n}$ and $H$ is the split special orthogonal group $\text{SO}_{2n+1}$.
(3) $G = \text{GL}_{2n}$ and $H$ is a quasi-split special orthogonal group $\text{SO}_{2n}$.
(4) $G = \text{Res}_{E/F} \text{GL}_{N,E}$ for a quadratic extension $E/F$, and $H$ is the quasi-split unitary group $\text{U}_{E/F}(N)$ with respect to $E/F$ in $N$ variables.

We consider the following automorphism of $G$ defined over $F$:

$$\theta: G \rightarrow G; \quad g \mapsto J_N(c(g))^{-1}J_N^{-1}. $$

Here $c$ is

\[
\begin{cases}
\text{the identity map} & \text{if } G = \text{GL}_N, \\
\text{the Galois conjugation of } E/F \text{ if } G = \text{Res}_{E/F} \text{GL}_N,E, \\
\end{cases}
\]

and $J_N$ is the anti-diagonal matrix whose $(i, N+1-i)$-th entry is given by $(-1)^{i-1}$. Then $H$ is an endoscopic group for $(G, \theta)$.

We define a disconnected reductive group $G \rtimes \langle \theta \rangle$ to be the semi-direct product of $G$ and the group $\langle \theta \rangle$ generated by $\theta$. We write $\bar{G}$ for the connected component $G \rtimes \langle \theta \rangle$ of this group which does not contain the unit element. Note that $\bar{G}$ has left and right actions of $G$ and is a bi-$G$-torsor with respect to these actions. In other words, $(\bar{G}, G)$ is a “twisted space” in the sense of Labesse ([23]).

We let $G^\theta_\varnothing$ denote the identity component $(G^\theta)^\circ_\varnothing$ of the subgroup $G^\theta$ of the $\varnothing$-fixed points of $G$. Note that this is given by

$$G^\theta_\varnothing = \begin{cases} 
\text{SO}_{2n+1} & \text{if } G = \text{GL}_{2n+1}, \\
\text{Sp}_{2n} & \text{if } G = \text{GL}_{2n}, \\
\text{Res}_{E/F}(N) & \text{if } G = \text{Res}_{E/F} \text{GL}_{N,E}.
\end{cases}$$

2.2. Orbital integral and the Langlands–Shelstad–Kottwitz transfer. Let $J$ be a connected reductive group over $F$. For an open subset $\mathcal{V}$ of $J$ which is invariant under $J$-conjugation, we denote the set of conjugacy classes of strongly regular semisimple elements of $J$ belonging to $\mathcal{V}$ by $\Gamma(\mathcal{V})$. For an element $f \in C_c^\infty(\mathcal{V})$ and $\gamma \in \Gamma(\mathcal{V})$, we define the normalized orbital integral of $f$ at $\gamma$ by

$$I_\gamma(f) := |D_J(\gamma)|^{-\frac{1}{2}} \int_{J_\gamma \setminus J} f(g^{-1} \gamma g) \, dg,$$

where $D_J(\gamma)$ is the Weyl discriminant of $\gamma$ in $J$, $J_\gamma$ is the set of $F$-valued points of the centralizer $J_\gamma$ of $\gamma$ in $J$, and $dg$ is a right $J$-invariant measure on $J_\gamma \setminus J$ induced by Haar measures on $J$ and $J_\gamma$. Then we can regard $I(f)$ as a function on $\Gamma(\mathcal{V})$. We let $\mathcal{I}(\mathcal{V})$ denote the vector space of such normalized orbital integrals:

$$\mathcal{I}(\mathcal{V}) := \{ I(f) \mid f \in C_c^\infty(\mathcal{V}) \} \subset \{ \mathbb{C}\text{-valued functions on } \Gamma(\mathcal{V}) \}. $$

We note that, for every irreducible smooth representation $\pi$ of $J$, its character $\Theta_\pi: C_c^\infty(\mathcal{V}) \rightarrow \mathbb{C}$ factors through the map $C_c^\infty(\mathcal{V}) \rightarrow \mathcal{I}(\mathcal{V})$. (This follows from the fact that $\Theta_\pi$ is locally $L^1$ ([17, Theorem 16.3]) by using the Weyl integration formula ([16, Lemma 42]).)
Now we furthermore assume that $\mathcal{V}$ is invariant under stable conjugacy. Then, for $f \in C_c^\infty(\mathcal{V})$ and a strongly regular semisimple element $\gamma \in \Gamma(\mathcal{V})$, we can define the stable orbital integral of $f$ at $\gamma$ by

$$SI_\gamma(f) := \sum_{\gamma' \sim \gamma \sim} I_{\gamma'}(f),$$

where the sum is over the set of $J$-conjugacy classes within the stable conjugacy class of $\gamma$. If we put $SI(\mathcal{V})$ to be the vector space of such stable orbital integrals, then we have a canonical surjection $I(\mathcal{V}) \to SI(\mathcal{V})$; $I(f) \mapsto SI(f)$.

For the twisted space $\tilde{\mathcal{G}}$, we can define similar objects; for an open subset $\tilde{\mathcal{V}}$ of $\tilde{\mathcal{G}}$ which is invariant under $G$-conjugation, we let $\Gamma(\tilde{\mathcal{V}})$ denote the set of conjugacy classes of strongly regular semisimple elements of $\tilde{\mathcal{G}}$ belonging to $\tilde{\mathcal{V}}$. For an element $f \in C_c^\infty(\tilde{\mathcal{G}})$, we define the normalized orbital integral of $f$ at $\tilde{\delta}$ by

$$I_\delta(f) := |D_{\tilde{\mathcal{G}}}(\tilde{\delta})|^\frac{1}{2} \int_{G_{\tilde{\delta}} \backslash G} f(g^{-1}g)\,dg,$$

where $D_{\tilde{\mathcal{G}}}(\tilde{\delta})$ is the Weyl discriminant of $\tilde{\delta}$ in $\tilde{\mathcal{G}}$, $G_{\tilde{\delta}}$ is the set of $F$-valued points of the centralizer $G_{\tilde{\delta}}$ of $\tilde{\delta}$ in $G$, and $dg$ is a right $G$-invariant measure on $G_{\tilde{\delta}} \backslash G$ induced by Haar measures on $G$ and $G_{\tilde{\delta}}$. We let $\mathcal{I}(\tilde{\mathcal{V}})$ denote the vector space of such normalized orbital integrals:

$$\mathcal{I}(\tilde{\mathcal{V}}) := \{I(f) \mid f \in C_c^\infty(\tilde{\mathcal{V}})\} \subset \{\mathbb{C}\text{-valued functions on } \Gamma(\mathcal{V})\}.$$ 

For every $\theta$-stable irreducible smooth representation $\pi$ of $G$, its $\theta$-twisted character $\Theta_{\pi, \theta} : C_c^\infty(\tilde{\mathcal{V}}) \to \mathbb{C}$ factors through the map $C_c^\infty(\tilde{\mathcal{V}}) \to \mathcal{I}(\tilde{\mathcal{V}})$. (Similarly to the standard case, this follows from the fact that $\Theta_{\pi, \theta}$ is locally $L^1$ ([11, Theorem 1]) by using the Weyl integration formula in the twisted case ([25, Partie I, Corollaire 7.3.7]).) 

Now we can define the notion of transfer for test functions. Let $(G, H)$ be a pair as in the previous subsection. We denote by $\Delta^IV$ the Kottwitz–Shelstad transfer factor with respect to $(G, H)$ without the fourth factor $\Delta^IV$ (see [22, Section 5] for the definition). Note that, in order to normalize $\Delta^IV$, we have to choose a $\theta$-stable Whittaker datum of $G$. From now on, we fix a $\theta$-stable Whittaker datum of $G$.

**Definition 2.1. (matching orbital integral, transfer of test functions)** We say $f \in C_c^\infty(\tilde{\mathcal{G}})$ and $f^H \in C_c^\infty(\tilde{\mathcal{H}})$ have matching orbital integrals if, for every strongly $G$-regular semisimple element $\gamma$ of $H$,

$$SI_\gamma(f^H) = \sum_{\tilde{\delta} \sim \gamma \sim} \Delta^IV(\gamma, \tilde{\delta})I_{\tilde{\delta}}(f),$$

where the sum is over the set of $G$-conjugacy classes of strongly regular semisimple elements of $\tilde{\mathcal{G}}$ such that $\gamma$ is a norm of $\tilde{\delta}$. In this situation, we say that $f^H$ is a transfer of $f$. 


Here we note that, we choose measures appearing in the above orbital integrals as in the manner of [22, Section 5.5]. See also [14, Remark 6.6.2].

On the existence of transfer of test functions, we have the following highly non-trivial theorem, which was established by a great deal of efforts of a lot of people represented by Waldspurger, Ngô, and so on (see, e.g., [7, page 54] for the details):

**Theorem 2.2.** (Langlands–Shelstad–Kottwitz’s transfer conjecture) For every \( f \in C_\infty(\wh{G}) \), there exists a transfer \( f^H \in C_\infty(H) \) of \( f \).

By this theorem, we have the transfer map \( \mathcal{I}(\wh{G}) \to S\mathcal{I}(H) \) characterized by the matching orbital integral condition.

### 2.3. Arthur’s theory and the endoscopic character relation.

In this subsection, we recall Arthur’s theory of the endoscopic classification of representations of classical groups over \( F \).

To state Arthur’s theorem, we introduce several notations. Let \( J \) be either a general linear group or a quasi-split classical group over \( F \). When \( J \) is an even special orthogonal group \( SO_{2n} \), we let \( \Out(J) \) denote the group of automorphisms of \( J \) given by the conjugation via \( O_{2n}/SO_{2n} \). When \( J \) is not an even special orthogonal group, we simply put \( \Out(J) \) to be the trivial group. We denote the set of equivalence classes of irreducible smooth (resp. tempered) representations of \( J \) by \( \tilde{\Pi}(J) \) (resp. \( \Pi_{\text{temp}}(J) \)). Then the group \( \Out(J) \) acts on these sets. We write \( \tilde{\Pi}(J) \) (resp. \( \Pi_{\text{temp}}(J) \)) for the set of \( \Out(J) \)-orbits in \( \Pi(J) \) (resp. \( \Pi_{\text{temp}}(J) \)). For an \( \Out(J) \)-orbit \( \tilde{\pi} \) in \( \tilde{\Pi}(J) \), we put

\[
\Theta_{\tilde{\pi}} := \frac{1}{|\tilde{\pi}|} \sum_{\pi \in \tilde{\pi}} \Theta_\pi,
\]

where \( \Theta_\pi \) is the character of \( \pi \).

We denote the set of \( \hat{\tilde{\pi}} \)-conjugacy classes of \( L \)-parameters (resp. tempered \( L \)-parameters) of \( J \) by \( \Phi(J) \) (resp. \( \Phi_{\text{temp}}(J) \)), and their \( \Out(J) \)-orbits by \( \tilde{\Phi}(J) \) (resp. \( \tilde{\Phi}_{\text{temp}}(J) \)).

Now let \( (G, H) \) be one of the pairs considered in Sect. 2.1. Then, since \( H \) is an endoscopic group of \( G \), we can regard an element \( \phi \in \Phi_{\text{temp}}(H) \) as an \( L \)-parameter of \( G \). This operation induces an injection from \( \tilde{\Phi}_{\text{temp}}(H) \) to \( \Phi_{\text{temp}}(G) \). For any \( \phi \in \tilde{\Phi}_{\text{temp}}(H) \), we let \( \pi^{G}_{\phi} \) denote the irreducible tempered representation of \( G \) corresponding to \( \phi \) under the local Langlands correspondence for \( G \). As \( \pi^{G}_{\phi} \) is self-dual, hence \( \theta \)-stable (i.e., \( \pi^{G}_{\phi} \cong (\pi^{G}_{\phi})^{\theta} \)), we can define its twisted character \( \Theta^{G}_{\phi, \theta} \). Note that the twisted character \( \Theta^{G}_{\phi, \theta} \) depends on the choice of an intertwiner \( \pi^{G}_{\phi} \cong (\pi^{G}_{\phi})^{\theta} \). In this paper, we normalize \( \Theta^{G}_{\phi, \theta} \) by choosing a unique intertwiner preserving the fixed \( \theta \)-stable Whittaker datum of \( G \).

The following is the local part of Arthur’s theory (the local Langlands correspondence for \( H \)):
Theorem 2.3. [7, Theorems 1.5.1 and 2.2.1], [26, Theorems 2.5.1 and 3.2.1] We have a partition
\[ \tilde{\Pi}_{\text{temp}}(H) = \bigsqcup_{\phi \in \tilde{\Phi}_{\text{temp}}(H)} \Pi^H_{\phi}, \]
where each \( \Pi^H_{\phi} \) is a finite set (called an L-packet) which is stable, that is, the sum \( \Theta^H_{\phi} \) of the characters of representations belonging to \( \Pi^H_{\phi} \) factors through the map \( C^\infty_c(H) \to \mathcal{SI}(H) \). Furthermore, for every \( f \in C^\infty_c(\tilde{G}) \),
\[ \Theta^G_{\phi, \theta}(f) = \Theta^H_{\phi}(f^H) \left( = \sum_{\pi \in \Pi^H_{\phi}} \Theta_\pi(f^H) \right), \]
where \( f^H \in C^\infty_c(H) \) is a transfer of \( f \) (see Definition 2.1 and Theorem 2.2).

We call the representation \( \pi^G_{\phi} \) the endoscopic lift of the L-packet \( \Pi^H_{\phi} \) from \( H \) to \( G \), and the identity in Theorem 2.3 the endoscopic character relation.

Remark 2.4. In general, it is expected that the local Langlands correspondence gives also a description of the internal structure of each L-packet \( \Pi^H_{\phi} \); conjecturally, there exists a bijection between the L-packet \( \Pi^H_{\phi} \) and a set of irreducible representations of a finite group \( S_{\phi} \) associated with \( \phi \). Then, in general, the \( H \)-side of the endoscopic character relation is expected to be given by the weighted sum
\[ \sum_{\pi \in \Pi^H_{\phi}} \dim(\rho_\pi) \Theta_\pi(f^H) \]
of the characters of \( \pi \in \Pi^H_{\phi} \), where \( \rho_{\pi} \) is the irreducible representation of \( S_{\phi} \) corresponding to \( \pi \). When \( H \) is a quasi-split classical group, the finite group \( S_{\phi} \) associated with \( \phi \) is always abelian. Hence \( \dim(\rho_{\pi}) \) equals 1.

2.4. Cayley transform for classical groups. In this subsection, we recall the definition of the Cayley transform and its fundamental properties proved in [14].

Let \( J \) be either a general linear group or a quasi-split classical group over \( F \). We set \( j := \text{Lie}(J)(F) \), where \( \text{Lie}(J) \) is the Lie algebra of \( J \). For an element \( g \in J \), we say that \( g \) is topologically unipotent if \( \lim_{n \to \infty} g^{p^n} = 1 \). We write \( J_{\text{tu}} \) for the set of topologically unipotent elements of \( J \). On the other hand, if \( J \) is a classical group associated with a \( F \)-vector space \( V \), then we can regard \( j \) as a subspace of \( \text{End}(V) \). Thus, for an element \( X \in j \), we can consider its power as a matrix. We say that \( X \in j \) is topologically nilpotent if \( \lim_{n \to \infty} X^n = 0 \). We write \( j_{\text{tn}} \) for the set of topologically nilpotent elements of \( j \).

Now let \( (G, H) \) be one of the pairs in Sect. 2.1.

Definition 2.5. (Cayley transform for general linear groups) Let \( c \) be the map from \( \mathfrak{gl}_N(F)_{\text{tn}} \) to \( \text{GL}_N(F)_{\text{tu}} \) defined by
\[ c : \mathfrak{gl}_N(F)_{\text{tn}} \to \text{GL}_N(F)_{\text{tu}}; \quad X \mapsto \frac{1 + X}{2}, \]
We call this map \( c \) the Cayley transform for \( \text{GL}_N \).
**Proposition 2.6.** (1) The Cayley transform \( c \) for \( GL_N \) is a homeomorphism, and its inverse is given by
\[
c^{-1} : GL_N(F)_{tu} \to gl_N(F)_{tu}; \quad g \mapsto \frac{2(g - 1)}{g + 1}.
\]
(2) For any \( A \in GL_N(F) \),
\[
c \circ (X \mapsto -tX) = \left( g \mapsto tg - 1 \right) \circ c \quad \text{and} \quad \text{Int}(A) \circ c = c \circ \text{Ad}(A).
\]
In particular, for any quasi-split classical group \( \mathbf{J} \) over \( F \), \( c \) defines a homeomorphism from \( j_{tn} \) to \( J_{tu} \).

**Proof.** The first assertion follows from Lemma 3.2.3 in [14] and an easy computation. The second assertion is cited from Remark 3.2.4 in [14]. \( \square \)

For the classical group \( \mathbf{H} \), we introduce a variant \( c' \) of the Cayley transform as follows:

**Definition 2.7.** Let \( c' \) be the map from \( h_{tn} \) to \( H_{tu} \) defined by
\[
c' : h_{tn} \to H_{tu}; \quad X \mapsto \begin{cases} c(X)^2 & \text{if } \mathbf{H} = \text{Sp}_{2n}, \text{or } \mathbf{H} = \text{SO}_{2n+1}, \\ c(X) & \text{if } \mathbf{H} = \text{SO}_{2n} \text{ or } \mathbf{H} = \text{U}_{E/F}(N). \end{cases}
\]

**Remark 2.8.** Note that the modification of \( c \) as in Definition 2.7 is necessary so that the pair \((c, c')\) preserves the norms in endoscopy in the following sense: for any \( X \in g_{\theta, tn} \) and \( Y \in h_{tn} \), \( Y \) is a norm of \( X \) if and only if \( c'(Y) \) is a norm of \( c(X) \times \theta \). This fact is proved in [14, Lemma 6.3.1] in the case where \( \mathbf{H} = \text{Sp}_{2n} \) or \( \mathbf{H} = \text{SO}_{2n+1} \) and [14, Corollary 7.4.5] in the case where \( \mathbf{H} = \text{SO}_{2n} \). (We remark that the above statement on norms should be modified as in [14, Corollary 7.4.5] in the case where \( \mathbf{H} = \text{Sp}_{2n} \) since \( c' \) is defined to be \( c(X)^2 \) also in this case in [14].)

The case where \( \mathbf{H} \) is unitary will be considered in the proof of Proposition 4.1.

### 2.5. Moy–Prasad filtrations of classical groups.

In this subsection, we collect basic properties of the Moy–Prasad filtrations of parahoric subgroups of classical groups. We use the notations of [14, Section 10.1] as follows. Let \( \mathbf{J} \) be a connected reductive group over \( F \). We denote its Bruhat–Tits building by \( B(\mathbf{J}, F) \). For a point \( x \in B(\mathbf{J}, F) \), we have the corresponding parahoric subgroup \( J_{x, 0} \) of \( \mathbf{J} \) and its Moy–Prasad filtration \( \{J_{x, r}\}_{r \in \mathbb{R}_{\geq 0}} \). Similarly to these filtrations, we also have the Moy–Prasad filtration \( \{j_{x, r}\}_{r \in \mathbb{R}} \) of the Lie algebra \( j = \text{Lie } \mathbf{J}(F) \). Here we note that we use the normalized valuation of \( F \) to define these filtrations and that it differs from the original definition in [27] (in [27], the normalized valuation of the splitting field of \( \mathbf{J} \) is used to define the Moy–Prasad filtrations). In other words, we normalize the indices of the Moy–Prasad filtrations so that, for any uniformizer \( \sigma_F \) of \( F \), the following holds:
\[
j_{x, r+1} = \sigma_F \cdot j_{x, r}.
\]
We write $J_{x,r,+}$ and $j_{x,r,+}$ for the quotients $J_{x,r}/J_{x,r,+}$ and $j_{x,r}/j_{x,r,+}$, respectively (here $r^+$ means $r + \varepsilon$ for a sufficiently small positive number $\varepsilon$). We put

$$J_{r,+} := \bigcup_{x \in B(J,F)} J_{x,r,+} \quad \text{and} \quad j_{r,+} := \bigcup_{x \in B(J,F)} j_{x,r,+}$$

(note that $J_+ = J_{0,+}$).

When $J$ is either a general linear group or a quasi-split classical group over $F$, we have the following property of the Cayley transform:

**Proposition 2.9.** [14, Lemmas 10.2.1 and 10.2.3] Let $J$ be either a general linear group or a quasi-split classical group over $F$. Let $x \in B(J,F)$ and $r \in \mathbb{R}_{>0}$. The Cayley transform $c$ induces a homeomorphism

$$c : j_{x,r} \to J_{x,r}.$$ 

In particular, $c$ induces an isomorphism of abelian groups

$$c : j_{x,r,+} \xrightarrow{\approx} J_{x,r,+}.$$ 

Here we remark that, in [14], they treat only the cases of symplectic groups and orthogonal groups. However, by the exact same arguments, we can show the above proposition for quasi-split unitary groups.

We next recall the compatibility of the Moy–Prasad filtrations of general linear groups with the involution $\theta$ (here we use the notations in Sect. 2.1). First, we note that the Bruhat–Tits building $B(G, F)$ of $G_\theta$ can be $G_\theta$-equivariantly identified with the $\theta$-fixed points of $B(G, F)$ (see [14, Remark 10.2.2]). In the rest of this paper, we always use this identification $B(G, F) \cong B(G, F)^\theta$. Under this identification, we have the following properties:

**Proposition 2.10.** [14, Lemma 10.2.3] For any $x \in B(G_\theta, F)$ and $r \in \mathbb{R}_{>0}$,

$$(G_{x,r})^\theta = G_{\theta,x,r} \quad \text{and} \quad (g_{x,r})^{d\theta} = g_{\theta,x,r}.$$ 

Moreover, we can identify $G_{\theta,x,r,+}$ and $g_{\theta,x,r,+}$ with $(G_{x,r,+})^\theta$ and $(g_{x,r,+})^{d\theta}$, respectively.

**Proof.** The first assertion can be deduced from the comparison theorem of the lattice filtrations and the Moy–Prasad filtrations [24] and Proposition 2.9 (see [14, Remark 10.2.2]).

We check the second assertion. By the first assertion, we can identify $G_{\theta,x,r,+}$ and $g_{\theta,x,r,+}$ as subsets of $(G_{x,r,+})^\theta$ and $(g_{x,r,+})^{d\theta}$, respectively. We show that $G_{\theta,x,r,+} = (G_{x,r,+})^\theta$ and that $g_{\theta,x,r,+} = (g_{x,r,+})^{d\theta}$. By Proposition 2.9 and the commutativity of $c$ and $\theta$ (Proposition 2.6), it suffices to show only the latter equality $g_{\theta,x,r,+} = (g_{x,r,+})^{d\theta}$. Let $X$ be an element of $g_{x,r}$ satisfying $d\theta(X + g_{x,r,+}) = X + g_{x,r,+}$. If we put $Y := d\theta(X) - X$ and $X' := X + \frac{1}{2}Y$, then $Y \in g_{x,r,+}$ and $X' \in g_{\theta,x,r}$. Thus the coset $X' + g_{\theta,x,r,+}$ of $g_{\theta,x,r,+}$ maps to $X + g_{x,r,+}$ under the injection $g_{\theta,x,r,+} \hookrightarrow (g_{x,r,+})^{d\theta}$. This completes the proof. \qed
2.6. Depth of representations. In this subsection, we recall the notion of depth of representations. Let \( J \) be a connected reductive group over \( F \). Moreover, we assume that \( J \) is tamely ramified over \( F \). (Note that this is satisfied for any group treated in this paper by the assumption that \( p \neq 2 \).)

**Definition 2.11.** For an irreducible smooth representation \( \pi \) of \( J \), we define its depth by
\[
\text{depth}(\pi) := \inf\{ r \in \mathbb{R}_{\geq 0} \mid \pi^J_{x,r+} \neq 0 \text{ for some } x \in \mathcal{B}(J, F) \} \in \mathbb{R}_{\geq 0}.
\]

**Proposition 2.12.** ([27, Theorem 3.5]) For every irreducible smooth representation \( \pi \) of \( J \), its depth is attained by a point of \( \mathcal{B}(J, F) \).

**Definition 2.13.** For an \( L \)-parameter \( \phi \) of \( J \), we define its depth by
\[
\text{depth}(\phi) := \inf\{ r \in \mathbb{R}_{\geq 0} \mid \phi(w, 1) = 1 \times w \text{ for every } w \in I_F^{0+} \} \in \mathbb{R}_{\geq 0}.
\]

Here \( I_F^0 \) is the ramification filtration of the inertia subgroup \( I_F \) of \( W_F \).

**Remark 2.14.** By the assumption of the tamely-ramifiedness of \( J \), for any \( L \)-parameter \( \phi \) of \( J \), its depth depends only on the \( \hat{J} \)-conjugacy class of \( \phi \). Therefore we can define the depth also for an element of \( \Phi(J) \). Indeed, as \( J \) is tamely ramified, the wild inertia subgroup \( I_0 \) acts on \( \hat{J} \) trivially. Thus, if we write \( \phi(w, 1) = x \times w \) for \( w \in I_F^{0+} \), then its conjugation via \( y \in \hat{J} \) is given by
\[
(y \times 1) \cdot \phi(w, 1) \cdot (y \times 1)^{-1} = (yxw(y)^{-1}) \times w = yxy^{-1} \times w.
\]

Therefore the \( \hat{J} \)-part of this element is trivial if and only if that of \( \phi(w, 1) \) is trivial. Hence, by the definition of the depth, the depth of the \( y \)-conjugation of \( \phi \) is equal to that of \( \phi \).

In the case of general linear groups, the following depth-preserving property of the local Langlands correspondence is known:

**Theorem 2.15.** ([33, 2.3.6] and [2, Proposition 4.2]) Let \( G \) be either \( \text{GL}_N \) or \( \text{Res}_{E/F} \text{GL}_{N,E} \) with quadratic extension \( E \) of \( F \). Then, for every \( \pi \in \Pi(G) \) and \( \phi \in \Phi(G) \) corresponding under the local Langlands correspondence for \( G \),
\[
\text{depth}(\pi) = \text{depth}(\phi).
\]

**Proof.** The case of \( \text{GL}_N \) is treated in [33, 2.3.6] and [2, Proposition 4.2]. We consider the case of \( \text{Res}_{E/F} \text{GL}_{N,E} \). Let \( \pi \) be an irreducible smooth representation of \( \text{GL}_N(E) \). Then, as
\[
(\text{Res}_{E/F} \text{GL}_{N,E})(F) = \text{GL}_N(E) = \text{GL}_{N,E}(E),
\]
we can define the depth of the representation \( \pi \) in two ways, that is,
- the depth of \( \pi \) as a representation of the group of \( F \)-valued points of \( \text{Res}_{E/F} \text{GL}_{N,E} \), and
- the depth of \( \pi \) as a representation of the group of \( E \)-valued points of \( \text{GL}_{N,E} \).

We distinguish these by writing $\text{depth}_F(\pi)$ and $\text{depth}_E(\pi)$ for the former and latter ones, respectively. Note that, in this usage of notations, our task is to show the equality

$$\text{depth}_F(\pi) = \text{depth}(\phi).$$

We first consider the relation between $\text{depth}_F(\pi)$ and $\text{depth}_E(\pi)$. Since we assume that the residual characteristic $p$ is not equal to 2,

- we have an identification of Bruhat–Tit buildings

$$B(\text{Res}_{E/F} GL_N, E) \cong B(GL_N, E),$$

and

- for any points $x_F \in B(\text{Res}_{E/F} GL_N, E)$ and $x_E \in B(GL_N, E)$ corresponding under this identification,

$$\left(\text{Res}_{E/F} GL_N(E)F\right)_{x_F, r} = GL_N(E)_{x_E, er}$$

for any $r \in \mathbb{R}_{\geq 0}$, where $e$ is the ramification index of $E/F$.

See, for example, [8, Appendix A.7] or [5, Appendix A]. (Note that, if we use the valuation of $E$ normalized so that its valuation group is given by $\mathbb{Z}$ to define the Moy–Prasad filtration for $(\text{Res}_{E/F} J)E$, then the above equality should be $(\text{Res}_{E/F} GL_N(E)F)_{x_F, r} = GL_N(E)_{x_E, r}$. However, as we use the valuation of $E$ extending that of $F$, the index of the Moy–Prasad filtration for $GL_N, E$ is multiplied by $e$.) Thus, by the definition of the depth of representations,

$$\text{depth}_F(\pi) = \frac{1}{e} \text{depth}_E(\pi). \quad (1)$$

We next consider the depth of the $L$-parameter $\phi$ of $\text{Res}_{E/F} GL_N, E$ corresponding to $\pi$. If we put $\phi_E$ to be the $L$-parameter of $GL_N, E$ corresponding to $\pi$, then

$$\text{depth}_E(\pi) = \text{depth}(\phi_E) \quad (2)$$

by the depth-preserving property of the local Langlands correspondence for $GL_N, E$ (the result explained in the first paragraph of this proof). On the other hand, the $L$-parameter $\phi$ of $\text{Res}_{E/F} GL_N, E$ corresponding to $\pi$ is given by

$$\phi |_{W_F} : W_F \to L\text{Res}_{E/F} GL_N, E = (GL_N(\mathbb{C}) \times GL_N(\mathbb{C})) \rtimes W_F$$

$$\sigma \mapsto \left(\phi_E(\sigma), \phi_E(w_c^{-1} \sigma w_c)\right) \rtimes \sigma \quad \text{for } \sigma \in W_E,$$

$$w_c \mapsto \left(\phi_E(w_c^2), I_N\right) \rtimes w_c$$

on $W_F$. Here $w_c$ is any element of $W_F \smallsetminus W_E$ and $I_N$ is the identity matrix of size $N$ (see, for example, [28, Section 4.7] or [26, Section 2.2]). In particular, by noting that $I_F^r = I_E^r$ for any $r \in \mathbb{R}_{>0}$ (this follows from the assumption that $p$ is not equal to 2, hence $E$ is tamely ramified over $F$),

$$\text{depth}(\phi) = \frac{1}{e} \text{depth}(\phi_E).$$

Thus, by combining this equality with (1) and (2), we get the desired equality. ☐
3. Semisimple descent for the Moy–Prasad filtrations of $GL_N$

Let $G$ be either $GL_{N,E}$ or $Res_{E/F} GL_{N,E}$ and $\theta$ the involution of $G$ as in Sect. 2.1. In this section, we investigate the semisimple descent of the characteristic functions of the Moy–Prasad filtrations of parahoric subgroups of $G$.

Now let us recall the semisimple descent of test functions supported on the topologically unipotent elements. We define the map $tc$ as follows:

$$tc: G \times G_{\theta} \rightarrow \tilde{G} ; \quad (g, x) \mapsto g \cdot (x \rtimes \theta) \cdot g^{-1}.$$ 

Let $\mathcal{U}$ (resp. $\mathcal{U}_r$) be the image of $G \times G_{\theta,\tu}$ (resp. $G \times G_{\theta,r}$) under the map $tc$. Then the canonical inclusion

$$G_{\theta,\tu} \hookrightarrow \mathcal{U}; \quad x \mapsto x \rtimes \theta$$

induces a bijection $\Gamma(G_{\theta,\tu}) \cong \Gamma(\mathcal{U})$ (see [14, Corollary 4.0.4]). Similarly, for any $r \in \mathbb{R}_{>0}$, we have a bijection $\Gamma(G_{\theta,r}) \cong \Gamma(\mathcal{U}_r)$.

**Definition 3.1.** (Semisimple descent at $\theta$) For $f \in C^\infty_c(\mathcal{U})$ and $f_\theta \in C^\infty_c(G_{\theta,\tu})$ (resp. $f \in C^\infty_c(\mathcal{U}_r)$ and $f_\theta \in C^\infty_c(G_{\theta,r})$), we say that $f_\theta$ is a semisimple descent of $f$ if $I(f_\theta)$ coincides with $I(f)$ as $\mathbb{C}$-valued functions on $\Gamma(G_{\theta,\tu}) \cong \Gamma(\mathcal{U})$ (resp. $\Gamma(G_{\theta,r}) \cong \Gamma(\mathcal{U}_r)$). In other words, for every strongly regular semisimple element $\gamma \in G_{\theta,\tu}$ (resp. $\gamma \in G_{\theta,r}$),

$$I_\gamma(f_\theta) = I_{\gamma \rtimes \theta}(f).$$

Here we note that, on the left-hand side, we consider the orbital integral of $f_\theta$ at $\gamma$ in the group $G^\theta$. For every $\gamma \in G_{\theta,\tu}$, the centralizer $(G^\theta)_\gamma$ of $\gamma$ in $G^\theta$ coincides with the centralizer $G_{\gamma \rtimes \theta}$ of $\gamma \rtimes \theta$ in $G$ (this is a consequence of [14, Lemma 4.0.1]). We use the same Haar measure on these centralizer groups in the above orbital integrals (see [14, Definition 4.2.2]). Then we have the following:

**Proposition 3.2.** [14, Lemma 10.4.2] As subsets of the set of $\mathbb{C}$-valued functions on $\Gamma(G_{\theta,\tu}) \cong \Gamma(\mathcal{U})$, $\mathcal{I}(G_{\theta,\tu}) = \mathcal{I}(\mathcal{U})$. Similarly, for any $r \in \mathbb{R}_{>0}$, $\mathcal{I}(G_{\theta,r}) = \mathcal{I}(\mathcal{U}_r)$.

$$C^\infty_c(G_{\theta,\tu}) \xrightarrow{\mathcal{I}} \mathcal{I}(G_{\theta,\tu}) \xrightarrow{\mathcal{I}} \{\Gamma(G_{\theta,\tu}) \rightarrow \mathbb{C}\}$$

$$C^\infty_c(\mathcal{U}) \xrightarrow{\mathcal{I}} \mathcal{I}(\mathcal{U}) \xrightarrow{\mathcal{I}} \{\Gamma(\mathcal{U}) \rightarrow \mathbb{C}\}$$

We put $E_0 = F$ if $G = GL_N$, and $E_0 = E$ if $G = Res_{E/F} GL_{N,E}$. Let $e$ be the ramification index of the extension $E_0/F$. We fix a uniformizer $\sigma$ of $E_0$ such that $\sigma^e$ belongs to $F^\times$ (thus, $\sigma^e$ is a uniformizer of $F$). For any $O_{E_0}$-lattice $L$ of $g$ and a positive integer $k \in \mathbb{Z}_{>0}$, we put

$$L^k := \bigoplus_{k} L \cdot \cdots \cdot L := \text{Span}_{O_{E_0}}\{Z_1 \cdots Z_k \mid Z_1, \ldots, Z_k \in L\}.$$
Proposition 3.3. (generalization of [14, Lemma 4.2.4 (i)]) Let $L$ be a $d\theta$-stable lattice of $\mathfrak{g}_{\text{lin}}$ satisfying $L^2 \subset L$ and $L^M \subset \sigma L$ with positive integer $M \in \mathbb{Z}_{>0}$. We put $K_L := c(L)$ and $K_{L,\theta} := K_L \cap G_{\theta}$. Then $tc(K_L, K_{L,\theta}) = K_L \times \theta$.

Proof. We follow the proof of [14, Lemma 4.2.4 (i)]. By Proposition 2.9, $c$ defines a homeomorphism from $L$ to $K_L = c(K_L)$. Since the action $d\theta$ on $L$ is translated to $\theta$ on $K_L$ by Proposition 2.6, $K_L$ is $\theta$-stable. Moreover, $K_L$ is a subgroup of $G$ by the assumption that $L^2 \subset L$ ([14, Lemma 4.2.3 (i)]). Hence the inclusion $tc(K_L, K_{L,\theta}) \subset K_L \times \theta$ is clear. Thus our task is to show the converse inclusion $tc(K_L, K_{L,\theta}) \supset K_L \times \theta (= c(L) \times \theta)$.

As the order of the automorphism $d\theta$ of $L$ is 2 and we assume that $p \neq 2$, we can consider the eigenspace decomposition $L = L_+ \oplus L_-$, where $L_+$ (resp. $L_-$) denotes the eigenspace with eigenvalue 1 (resp. $-1$) with respect to $d\theta$.

By the submersivity of the map $tc$ ([14, Lemma 4.0.6]), $tc(K_L, K_{L,\theta})$ is an open subset of $K_L \times \theta = c(L) \times \theta$. Combining this with the compactness of $L_+$, we can easily check that there exists an integer $m \in \mathbb{Z}_{>0}$ such that $tc(K_L, K_{L,\theta}) \supset c(L_+ \oplus \sigma^m L_-) \times \theta$. If we can show that $tc(K_L, K_{L,\theta}) \supset c(L_+ \oplus \sigma^m L_-) \times \theta$, then we get the inclusion $tc(K_L, K_{L,\theta}) \supset c(L) \times \theta$ by the reverse induction on $m$.

Let us take an element $g \in c(L_+ \oplus \sigma^m L_-)$. In order to show that $g \times \theta \in tc(K_L, K_{L,\theta})$, we first show the following claim:

Claim. For any $k \in \mathbb{Z}_{>0}$, there exists $y \in K_L$ satisfying

$$y^{-1} g\theta(y) \in c(L_+ + \sigma^m L^k).$$

Proof of Claim. We first note that $L_k \subset L$ for any $k$ by the assumption that $L^2 \subset L$. We show the claim by induction on $k$. If $k = 1$, the assertion is obvious since

$$c(L_+ + \sigma^m L^1) = c(L_+ + \sigma^m (L_+ \oplus L_-)) = c(L_+ \oplus \sigma^m L_-)$$

and $g$ already belongs to this set. Next we assume the assertion for $k$, and show the assertion for $k + 1$. By the induction hypothesis (the assertion for $k$), we can take an element $y \in K_L$ satisfying $y^{-1} g\theta(y) \in c(L_+ + \sigma^m L^k)$. We take $X_1 \in L_+$ and $X_2 \in L_- \cap \sigma^m L^k$ satisfying $y^{-1} g\theta(y) = c(X_1 + X_2)$ (note that $L^k$ is $d\theta$-stable, hence we can find such $X_2$). It is enough to find an element $y' \in K_L$ satisfying

$$y'^{-1} y^{-1} g\theta(y) y' \in c(L_+ + \sigma^m L^{k+1}).$$

We put $X := X_1 + X_2$ and $Y := \frac{1}{2} X_2$. We show that $y' := c(Y)$ satisfies the above condition. We first note that, for any $W \in \mathfrak{g}_{\text{lin}}$, the power series expansion of $c(W)$ is given by

$$c(W) = \frac{1 + \frac{W}{2}}{1 - \frac{W}{2}} = 1 + W + 2 \cdot \left(\frac{W}{2}\right)^2 + 2 \cdot \left(\frac{W}{2}\right)^3 + \cdots.$$

Thus, by noting that $\sigma^m L^{k+1}$ is closed in $\mathfrak{g}$, hence complete,

$$c(X) \in c(X_1) + X_2 + \sigma^m L^{k+1} \text{ and } c(-Y) \in 1 - Y + \sigma^m L^{k+1}.$$
On the other hand, since $\theta$ and $c$ commute (Proposition 2.6) and $d\theta$ acts on $Y$ via negation, $\theta(c(Y)) = c(-Y)$. Thus
\[
y'^{-1} \cdot c(X) \cdot \theta(y') = c(-Y) \cdot c(X) \cdot c(-Y)
\]
\[
\in (1 - Y)(c(X_1) + X_2)(1 - Y) + \sigma^m L^{k+1}
\]
\[
= c(X_1) + X_2 - 2Y + \sigma^m L^{k+1}
\]
\[
= c(X_1) + \sigma^m L^{k+1}
\]
(note that $Yc(X_1)$ and $c(X_1)Y$ belong to $Y + \sigma^m L^{k+1}$). Since $X$ and $-Y$ belong to $L$, their images $c(X)$ and $c(-Y)$ belong to $K_L$. In particular, the product $c(-Y) \cdot c(X) \cdot c(-Y)$ lies in $K_L = c(L)$. If we put
\[
c(-Y) \cdot c(X) \cdot c(-Y) = c(X_1) + Z
\]
for $Z \in \sigma^m L^{k+1}$, then, by Proposition 2.6 (1), the inverse image of $c(-Y) \cdot c(X) \cdot c(-Y)$ via $c$ is given by
\[
c^{-1}(c(X_1) + Z) = \frac{2(c(X_1) + Z - 1)}{c(X_1) + Z + 1}
\]
\[
= \frac{c(X_1) + Z - 1}{1 + \frac{(c(X_1) + Z - 1)}{2}}
\]
\[
\in (c(X_1) + Z - 1)\left(1 - \frac{c(X_1) + Z - 1}{2} + \frac{(c(X_1) + Z - 1)^2}{2^2} - \ldots\right)
\]
\[
\in (c(X_1) - 1)\left(1 - \frac{c(X_1) - 1}{2} + \frac{(c(X_1) - 1)^2}{2^2} - \ldots\right) + \sigma^m L^{k+1}
\]
\[
= c^{-1}(c(X_1)) + \sigma^m L^{k+1}
\]
\[
= X_1 + \sigma^m L^{k+1}.
\]
Here we used the fact that $c(X_1)Z$ and $Zc(X_1)$ belong to $\sigma^m L^{k+1}$. Therefore the $\theta$-conjugated element $y'^{-1} \cdot c(X) \cdot \theta(y')$ belongs to $c(L_+ + \sigma^m L^{k+1})$. This completes the proof of the claim. \hfill \Box

Now we come back to the proof of Proposition 3.3. By the above claim for $k = M$, we can find an element $y \in K_L$ satisfying $y^{-1}g\theta(y) \in c(L_+ + \sigma^m L^M)$. Since $L^M \subset \sigma L$ by the assumption, we get
\[
y^{-1}g\theta(y) \in c(L_+ + \sigma^{m+1}L) = c(L_+ + \sigma^{m+1}L_-).
\]
As $c(L_+ + \sigma^{m+1}L_-) \triangleright \theta \subset tc(K_L, K_L, \theta)$, we can conclude that $g \triangleright \theta \in tc(K_L, K_L, \theta)$. This completes the proof. \hfill \Box

**Corollary 3.4.** Let $x \in B(G_\theta, F)$ and $r \in \mathbb{R}_{>0}$. Then $tc(G_{x,r} \cdot G_{\theta,x,r}) = G_{x,r} \cdot \theta$.

**Proof.** We note that $g_{x,r}$ is a $d\theta$-stable $\mathcal{O}_{E_0}$-lattice of $g_{m}$ satisfying $g^2_{m,r} \subset g_{x,r}$ (see [14, Lemma 10.2.3 (d)]). Since $G_{x,r} = c(g_{x,r})$ and $G_{\theta,x,r} = G_{x,r} \cap G_{\theta}$, it suffices to show that the assumption of Proposition 3.3 is satisfied, that is, $g^M_{x,r} \subset \sigma g_{x,r}$ for some positive integer $M \in \mathbb{Z}_{>0}$.
Let $i$ be the Iwahori sublattice of $g = \mathfrak{gl}_N(E_0)$ corresponding to an alcove $C$ of $B(G, F)$ whose closure $\overline{C}$ contains the point $x$. Let $s \in \mathbb{Z}_{>0}$ be the integer satisfying $r \leq s < r + 1$. Then

$$i_{s-1+} \supset g_{x,r} \supset i_s.$$  

Here $i_*$ is the Moy–Prasad filtration of $i$ attached to the barycenter of the alcove $C$. We take an element $\varphi \in \mathfrak{gl}_N(E_0)$ satisfying

- $\varphi^N = \varphi \cdot I_N$, where $I_N$ is the identity matrix of size $N$, and
- $i_{t+} = \varphi \cdot i_t = i_t \cdot \varphi$, for any integer $t \in \mathbb{Z}$.

Note that we can always find such an element $\varphi$. Indeed, every Iwahori sublattice of $g$ is $\text{GL}_N(E_0)$-conjugate to the standard Iwahori sublattice $i_{\text{st}}$ (see, e.g., [18, Remark 2]):

$$i_{\text{st}} = \begin{pmatrix} O_{E_0} & & \\ & \ddots & \\ & & O_{E_0} \end{pmatrix}.$$  

Thus, it is enough to check that we can find an element $\varphi$ satisfying the above conditions for this standard Iwahori sublattice. For example, the following element $\varphi_{\text{st}}$ satisfies the conditions for $i_{\text{st}}$:

$$\varphi_{\text{st}} := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \varphi & 0 & \cdots & 0 \end{pmatrix}.$$  

We also note that

$$i_t = \varphi^e \cdot i_{t-1} = \varphi^e N \cdot i_{t-1}.$$  

Then

$$g_{x,r}^{(e+1)N} \subset i_{s-1+}^{(e+1)N} = \varphi^{(e+1)N} \cdot i_{s-1}^{(e+1)N}.$$  

Since $i_{s-1}^{(e+1)N} \subset i_{t-1}$ by $s - 1 \geq 0$, the right-hand side is contained in

$$\varphi^{(e+1)N} \cdot i_{s-1} = \varphi i_s \subset \varphi g_{x,r}.$$  

This implies that the assumption of Proposition 3.3 is satisfied with $M = (e + 1)N$. □

We next extend this corollary to the cosets of the Moy–Prasad filtrations of parahoric subgroups. First recall that, for $x \in B(G_\theta, F)$ and $r \in \mathbb{R}_{>0}$, we can canonically identify $G_{\theta,x,r,r+}$ with $(G_{x,r,r+})^\theta \subset G_{x,r,r+}$ (Proposition 2.10). Note that $G_{x,r}$ acts on $G_{x,r,r+}$ via $\theta$-conjugation (in other words, $G_{x,r}$ acts on $G_{x,r,r+} \rtimes \theta$ via conjugation). If two elements $[g_1], [g_2] \in G_{x,r,r+}$ are $\theta$-conjugate by $G_{x,r}$, then we write $[g_1] \sim_{\theta} [g_2]$. When we regard an element $[h] \in G_{\theta,x,r,r+}$ (resp. $[g] \in G_{x,r,r+}$) as a subset of $G_{\theta,x,r}$ (resp. $G_{x,r}$), we denote it by $[h]_{G_{\theta}}$ (resp. $[g]_{G}$).
Corollary 3.5. Let \( x \in B(G_{\theta}, F) \) and \( r \in \mathbb{R}_{>0} \). Let \([h] \in G_{\theta,x,r,r+}\). Then
\[
\text{tc}(G_{x,r}, [h]_{G_{\theta}}) = \bigsqcup_{[g] \in G_{x,r,r+}} [g]_{G} \rtimes \theta.
\]

Proof. We have
\[
\text{tc}(G_{x,r}, G_{\theta,x,r}) = \text{tc}(G_{x,r}, \bigsqcup_{[h] \in G_{\theta,x,r,r+}} [h]_{G_{\theta}}) = \bigsqcup_{[h] \in G_{\theta,x,r,r+}} \text{tc}(G_{x,r}, [h]_{G_{\theta}}).
\]

Here note that the union on the right-hand side is in fact disjoint. Indeed, if \([h_1], [h_2] \in G_{\theta,x,r,r+}\) satisfy \(\text{tc}(G_{x,r}, [h_1]_{G_{\theta}}) \cap \text{tc}(G_{x,r}, [h_2]_{G_{\theta}}) \neq \emptyset\), then, for some element \(y\) of \(G_{x,r}\), \([h_2] = y[h_1]\theta(y)^{-1}\) in \(G_{x,r,r+}\). In other words, \([h_2] = [y] + [h_1] - [\theta(y)]\). By this equality,
\[
2[h_2] = [h_2] + \theta([h_2]) = [y] + [h_1] - \theta([y]) + \theta([y] + [h_1] - \theta([y])) = [h_1] + \theta([h_1]) = 2[h_1].
\]

Since we assume that \(p\) is not equal to 2, \([h_1] = [h_2]\).

On the other hand, by Corollary 3.4,
\[
\text{tc}(G_{x,r}, G_{\theta,x,r}) = G_{x,r} \rtimes \theta = \bigsqcup_{[g] \in G_{x,r,r+}} [g]_{G} \rtimes \theta.
\]

Then, by noting that
\[
\text{tc}(G_{x,r}, [h]_{G_{\theta}}) \subset \bigsqcup_{[g] \in G_{x,r,r+}} [g]_{G} \rtimes \theta,
\]
we get the assertion. \(\square\)

Now we consider the semisimple descent of the characteristic functions of the cosets of the Moy–Prasad filtrations. We first recall the following lemma:

Lemma 3.6. [14, Lemma 4.2.4 (ii)] Let \(C_{\theta}\) be a compact open subset of \(G_{\theta,u,u}\), and \(K\) a compact open subgroup of \(G\). We assume that \(C_{\theta}\) is closed under conjugation by \(K \cap G^{\theta}\). Then \(\text{vol}(K \cap G^{\theta})^{-1} \mathbb{1}_{C_{\theta}} \in C_{c}^{\infty}(G_{\theta})\) is a semisimple descent of \(\text{vol}(K)^{-1} \mathbb{1}_{\text{tc}(K,C_{\theta})} \in C_{c}^{\infty}(\widetilde{G})\).

Here we note that, in [14], this lemma is proved for pairs \((G, H)\) only in the cases of (1), (2), and (3). However we can show the same assertion for the case of (4) by the same argument, that is, we can check the matching of Weyl discriminants for topologically unipotent elements ([14, Lemma 4.1.3] for the case of (4)) and use Kottwitz’s descent lemma ([21, Lemma 2.3]). By combining Lemma 3.6 with Corollary 3.5, we get the following consequence:
Corollary 3.7. Let \( x \in \mathcal{B}(G_{\theta}, F) \) and \( r \in \mathbb{R}_{>0} \). Let \([h] \in G_{\theta,x,r,r+}\). Then \( \text{vol}(G_{\theta,x,r})^{-1} \mathbb{1}_{\{\theta\}} \in C_c^\infty(G_{\theta}) \) is a semisimple descent of \( G_{x,r} \):

\[
\text{vol}(G_{x,r})^{-1} \sum_{\{g\} \in G_{x,r,r+}} \mathbb{1}_{\{g\}G \times \theta} \in C_c^\infty(\tilde{G}).
\]

Proof. We take \( C_{\theta} \) and \( K \) in Lemma 3.6 to be \([h] \in G_{\theta}, x \in G_{x,r}\), respectively. Then \( G_{x,r} \cap G^\theta = G_{x,r} \cap G_{\theta} = G_{\theta,x,r} \) (note that \( G^\theta \neq G_{\theta} \) only when \( G = \text{GL}_{2n+1} \), and that, in this case, \( G^\theta = O_{2n+1} = \pm SO_{2n+1} = \pm G_{\theta} \)). In particular, since \( G_{\theta,x,r,r+} \) is abelian, \([h] \in G_{\theta}\) is stable under the conjugation by \( K \cap G^\theta \). Hence the assumption of Lemma 3.6 is satisfied. Therefore \( \text{vol}(G_{\theta,x,r})^{-1} \mathbb{1}_{\{h\}} \in C_c^\infty(G_{\theta}) \) is a semisimple descent of \( \text{vol}(G_{x,r})^{-1} \mathbb{1}_{\{c\}} \in C_c^\infty(\tilde{G}) \). By combining this with Corollary 3.5, we get the assertion. \( \square \)

4. Evaluation of the maximum depth in an \( L \)-packet

4.1. Semisimple descent and the transfer. In this subsection, we recall the compatibility of semisimple descent with endoscopic transfer.

We first note that, for each pair \((G, H)\) in Sect. 2.1, the relationship between \( G_{\theta} \) and \( H \) is described as follows:

1. In this case, \( G_{\theta} = \text{SO}_{2n+1} \) and \( H = \text{Sp}_{2n} \). Thus \((G_{\theta})_{sc}, H_{sc}\) can be extended to a nonstandard endoscopic triplet.
2. In this case, \( G_{\theta} = \text{Sp}_{2n} \) and \( H = \text{SO}_{2n+1} \). Thus \((G_{\theta})_{sc}, H_{sc}\) can be extended to a nonstandard endoscopic triplet.
3. In this case, \( G_{\theta} = \text{Sp}_{2n} \) and \( H = \text{SO}_{2n} \). Thus \( H \) can be regarded as a standard endoscopic group of \( G_{\theta} \).
4. In this case, \( G_{\theta} = H = U_{E/F}(N) \).

In particular, in each case, we can define the notion of matching orbital integral for test functions of \( C_c^\infty(g_{\theta}) \) and \( C_c^\infty(h) \). See [31, Section 1.8] for the cases of (1) and (2), and [22, Section 5.5] for the case of (3). In the case of (4), we say that \( f^{\theta} \in C_c^\infty(g_{\theta}) \) and \( f^{h} \in C_c^\infty(h) \) have matching orbital integrals if \( SI(f^{\theta}) = SI(f^{h}) \) (in other words, coincide as elements of \( SI(g_{\theta}) = SI(h) \)). Here we use the same notations as in the group case such as \( SI \) and \( SI_t \). Furthermore, in every case, similarly to Theorem 2.2, we have a transfer map from \( \mathcal{I}(g_{\theta}) \) to \( \mathcal{I}(h) \) characterized by this matching orbital integral condition. See also [14, Sections 6.6 and 10.5].

Then, the relationship between the three notions of matching orbital integrals, that is, the transfer from \( \mathcal{I}(\tilde{G}) \) to \( SI(H) \), the semisimple descent from \( \mathcal{I}(U) \) to \( \mathcal{I}(G_{\theta, w_0}) \), and the transfer from \( \mathcal{I}(g_{\theta, w_0}) \) to \( \mathcal{I}(h_{w_0}) \), can be stated as follows:

Proposition 4.1. Let \( f \in C_c^\infty(U) \) and \( f_{\theta} \in C_c^\infty(G_{\theta, w_0}) \) such that \( f_{\theta} \) is a semisimple descent of \( f \). Then, for \( f^H \in C_c^\infty(H_{w_0}) \), \( f^H \) is a transfer of \( f \) if and only if \( f_{\theta} \circ c \in C_c^\infty(g_{\theta, w_0}) \) and \( f^H \circ c' \in C_c^\infty(h_{w_0}) \) have matching orbital integrals.
This proposition can be interpreted as the commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathcal{I}(G_{\theta, tu}) & \xrightarrow{\text{transfer}} & \mathcal{I}(U) \\
\cong & \uparrow((c^{-1})^*) & \cong \\
\mathcal{I}(g_{\theta, tn}) & \xrightarrow{\text{transfer}} & \mathcal{I}(H_{tu})
\end{array}
\]

The cases of (1) and (2) of this proposition are proved in [14, Lemma 6.6.4], and the case of (3) is proved in [14, Lemma 7.7.2]. Finally, we can show the assertion for the case of (4) in the same manner as in these three cases. However, for the sake of completeness, we explain the proof. In the rest of this subsection, we focus on the case where \( G = \text{Res}_{E/F} \text{GL}_{N,E} \) and \( G_{\theta} = H = U_{E/F}(N) \).

First, we recall a description of the norm correspondence between \( G \) and \( H \) in terms of the eigenvalues of elements. Let \( \tilde{\delta} \) and \( \gamma \) be semisimple elements of \( \tilde{G} \) and \( H \), respectively. Then, by the semisimplicity, these elements can be diagonalized, that is, we can take \( x \in G(F) \) and \( y \in H(F) \) satisfying

\[
x \tilde{\delta} x^{-1} = \left( \text{diag}(t_1, \ldots, t_N), \text{diag}(s_1, \ldots, s_N) \right) \times \theta
\]

and

\[
y \gamma y^{-1} = \text{diag}(v_1, \ldots, v_N).
\]

Here note that \( G_F = (\text{Res}_{E/F} \text{GL}_{N,E})_F \cong \text{GL}_{N,F} \times \text{GL}_{N,F} \). Then, \( \gamma \) is a norm of \( \tilde{\delta} \) if and only if

\[
\{ v_1, \ldots, v_N \} = \left\{ \frac{t_1}{s_N}, \ldots, \frac{t_N}{s_1} \right\}.
\]

On the other hand, also the topological unipotency is characterized in terms of the eigenvalues. Precisely speaking, for a semisimple element \( g \) of a classical group over \( F \), it is topologically unipotent if and only if \( \text{val}(\alpha(g) - 1) > 0 \) for every eigenvalue \( \alpha(g) \) of \( g \). Hence if \( g \in G_{\theta, tu} \) is a semisimple element, then, for some element \( z \in G_{\theta}(F) \), \( z g z^{-1} = \text{diag}(t_1, \ldots, t_N) \) and \( \text{val}(t_i - 1) > 0 \) for every \( 1 \leq i \leq N \). Thus, if \( \tilde{\delta} \in \tilde{G} \) is a semisimple element belonging to \( U \), by the above interpretation of the norm correspondence via the eigenvalues, every norm of \( \tilde{\delta} \) belongs to \( H_{tu} \).

Finally we recall that, for the pair \( (G = \text{Res}_{E/F} \text{GL}_{N,E}, H = U_{E/F}(N)) \), the Kottwitz–Shelstad transfer factor \( \Delta_{IV} \) is trivial. See, for example, [32, 1.10 Proposition]. Note that we implicitly consider the standard base change \( L \)-embedding from \( L H \) to \( L G \) (in the sense of Rogawski, see [28, Section 4.7] or [26, Section 2.1]), and \( I^- = \emptyset \) (hence \( d^- = 0 \)) in the notation of [32]. In particular, every term in the formula of [32, 1.10 Proposition] is trivial.

**Proof of Proposition 4.1 in the case of (4).** We take \( f \in C_{c}^{\infty}(U) \) and \( f_{\theta} \in C_{c}^{\infty}(G_{\theta, tu}) \) such that \( f_{\theta} \) is a semisimple descent of \( f \). Let \( f^{H} \in C_{c}^{\infty}(H_{tu}) \). Then,
by the definition of the transfer, \( f^H \) is a transfer of \( f \) if and only if, for every strongly regular semisimple element \( \gamma \in H \),

\[
SI_\gamma(f^H) = \sum_{\tilde{\delta} \sim \gamma/\sim} \Delta^{IV}(\gamma, \tilde{\delta})I_\delta(f). \tag{*}
\]

However, by the above observation on the norm correspondence for topologically unipotent elements, if \( \gamma \notin H_{tu} \), then every \( \tilde{\delta} \in \tilde{G} \) corresponding to \( \gamma \) does not belong to \( \mathcal{U} \). Thus, since \( f^H \) and \( f \) are supported in the topologically unipotent elements, the condition \((*)\) is trivial for \( \gamma \) such that \( \gamma \notin H_{tu} \).

Now we consider the condition \((*)\) for \( \gamma \in H_{tu} \). We put \( \gamma = c'(Y) \) (recall that every element of \( H_{tu} \) can be written in this form since \( c(h_{tu}) = H_{tu} \) and the map \( h \mapsto h^2 \) gives a bijection from \( H_{tu} \) to itself, see [14, Lemma 3.2.7]). For such an element \( \gamma \), let us consider the index set of the sum on the right-hand side of \((*)\). First, by the above description of the norm correspondence in terms of the eigenvalues, the element \( c(Y) \rtimes \theta \) is contained in this index set (in other words, \( c'(Y) \) is a norm of \( c(Y) \rtimes \theta \)). Moreover, every other element appearing in the index set is \( G(F)-\)conjugate to this element \( c(Y) \rtimes \theta \). Thus, the index set is given by the \( G \)-conjugacy classes within the stable conjugacy class of \( c(Y) \rtimes \theta \) in \( \mathcal{U} \). Since the inclusion

\[
G_{\theta, tu} \hookrightarrow \mathcal{U}; \ x \mapsto x \rtimes \theta
\]

induces a bijection between the conjugacy classes and a bijection between the stable conjugacy classes ([14, Corollary 4.0.4]), we get the equality

\[
\{\tilde{\delta} \in \mathcal{U} \mid \gamma \text{ is a norm of } \tilde{\delta}/(G\text{-conjugacy})
\]  
\[
= \{\delta \in G_{\theta, tu} \mid \delta \text{ is stably conjugate to } c(Y)/(G_{\theta}\text{-conjugacy})\}.
\]

Thus, by combining this observation with the triviality of \( \Delta^{IV} \), the condition \((*)\) is equivalent to

\[
SI_\gamma(f^H) = \sum_{\delta \sim \gamma \rtimes \theta} I_{\delta \rtimes \theta}(f).
\]

However, since \( f_\theta \) is a semisimple descent of \( f \), \( I_{\delta}(f_\theta) = I_{\delta \rtimes \theta}(f) \). Thus the above equality is furthermore equivalent to

\[
SI_\gamma(f^H) = SI_{c(Y)}(f_\theta).
\]

As \( SI_\gamma(f^H) = SI_Y(f^H \circ c) \) and \( SI_{c(Y)}(f_\theta) = SI_Y(f_\theta \circ c) \), we can rephrase this condition as that \( f_\theta \circ c \) and \( f^H \circ c' \) have matching orbital integrals.

\[\square\]

If we assume that the residual characteristic \( p \) is large enough, then we can extend this proposition to functions supported on \( \mathcal{U}_r, G_{\theta, r}, \) and \( H_r \). More precisely, for a pair \((G, H)\) in Sect. 2.1, if we put the condition that

\[
p > \begin{cases} 
2n & \text{the cases of (1), (2), and (3),} \\
N & \text{the case of (4),}
\end{cases}
\]
then every maximal torus in $G$, $G_{\theta}$, and $H$ splits over a tamely ramified extension. Then, for every $r > 0$, the regions $U_r$, $G_{\theta,r}$ and $H_r$ can be characterized in terms of the eigenvalues. As a consequence, we can get the following diagram (see [14, Remarks 10.1.5 and 10.5.1] for details):

$$
\begin{array}{ccc}
\mathcal{I}(G_{\theta,r}) & \xrightarrow{\text{transfer}} & SI(H_r) \\
\cong & & \cong \\
\mathcal{I}(G_{\theta,r}) & \xrightarrow{\text{transfer}} & SI(H_r)
\end{array}
$$

4.2. **Comparison of the validity ranges of character expansions.** We next recall the character expansion of the characters of representations. Let $J$ be a connected reductive group over $F$. For an irreducible smooth representation $\pi$ of $J$, we denote its character by $\Theta_\pi$. For a nilpotent orbit $O$ of $j$, we write $\hat{\mu}_O$ for the following $J$-invariant distribution on $C^\infty_c(j)$:

$$
\hat{f} \mapsto \hat{\mu}_O(\hat{f}),
$$

where $\mu_O$ is the orbital integral with respect to the nilpotent orbit $O$ and $\hat{f}$ is the Fourier transform of $f$. Here we do not recall the normalizations (i.e., the choices of measures) of these orbital integrals and the Fourier transform. See, for example, Sections 3.1 and 3.4 in [12] for the details.

**Definition 4.2. (character expansion)** Let $r \in \mathbb{R}_{>0}$ and $c_j$ be a $J$-equivariant homeomorphism from $j_r$ to $J_r$. Let $\pi$ be an irreducible smooth representation of $J$. We say that $\Theta_\pi$ has a character expansion on $J_r$ with respect to $c_j$ if there exists a complex number $c_O$ for each nilpotent orbit $O$ of $j$ such that, for every $f \in C^\infty_c(j_r)$, the following equality holds:

$$
\Theta_\pi(f \circ c_j^{-1}) = \sum_{O \in \text{Nil}(j)} c_O \cdot \hat{\mu}_O(f).
$$

In other words, as $J$-invariant distributions on $C^\infty_c(j_r)$,

$$
\Theta_\pi \circ (c_j^{-1})^* = \sum_{O \in \text{Nil}(j)} c_O \cdot \hat{\mu}_O.
$$

**Theorem 4.3.** [12, Theorem 3.5.2] Let $H$ be a quasi-split classical group over $F$ as in Sect. 2.1 and $c'$ the Cayley transform defined in Sect. 2.4. We assume that the residual characteristic is large enough to satisfy hypotheses in [12]. Let $\Theta_\pi$ be the character of an irreducible smooth representation $\pi$ of $H$ of depth $r$. Then $\pi$ has a character expansion on $H_{r+}$ with respect to $c'$.

Let us investigate when the hypotheses in [12] are satisfied.

**Lemma 4.4.** We assume that

$$
p \geq \begin{cases} 
4n + 1 & \text{the cases of (1) and (2)}, \\
4n - 3 & \text{the case of (3)}, \\
2N + 1 & \text{the case of (4)}.
\end{cases}
$$

Then the hypotheses in [12] which are needed for Theorem 4.3 are satisfied.
Proof. We first note that the hypotheses of the above theorem consist of Hypotheses 2.2.2, 2.2.4, 2.2.5, 2.2.6, 2.2.8, 3.2.1, 3.4.1, 3.4.3 in [12]. Among them, 3.2.1, 3.4.1, 3.4.3 are checked in [14, Lemma 10.3.1] for $p > 2$. Thus we consider the other hypotheses.

Hypothesis 2.2.4 If we put $h_H$ to be the Coxeter number of $H$, then $\text{ad}(X)^2 h_H^{-1} = 0$ for every nilpotent element $X$ of $\mathfrak{h}$ (see, for example, [10, Proposition 5.5.2]). Hence this hypothesis is satisfied when $2h_H - 1 \leq p - 2$. As $h_H$ is given by

$$\begin{cases} 2n & \text{the cases of (1) and (2)}, \\ 2n - 2 & \text{the case of (3)}, \\ N & \text{the case of (4)}, \end{cases}$$

this condition is equivalent to the condition in our assertion.

Hypothesis 2.2.5 If we define $\exp_i(X)$ to be $\sum_{i=0}^{m} \frac{X^i}{i!}$, then we can check that the adjoint action of $\exp_i(X)$ on $\mathfrak{h}$ is given by $\sum_{i=0}^{m} \frac{\text{ad}(X)^i}{i!}$ by an easy computation. Furthermore, the uniqueness of such a map $\exp_i$ can be checked as follows. To show the uniqueness of $\exp_i$, it is enough to show that the adjoint map $\text{Ad}: H_u \to \text{GL}(\mathfrak{h})$ is injective, where $H_u$ is the set of unipotent elements of $H$.

We first show that if a unipotent element $h \in H_u$ maps to the identity via $\text{Ad}$, then $h$ is equal to the unit element $1$. We take a unipotent element $h \in H_u$ and assume that $\text{Ad}(h) = \text{id}$, that is, $\text{Ad}(h)(Y) = hYh^{-1}$ equals $Y$ for every $Y \in \mathfrak{h}$. If we write $h = c(2X)$, where $X$ is a nilpotent element of $\mathfrak{h}$, then

$$\frac{1 + X}{1 - X} \cdot Y \cdot \frac{1 - X}{1 + X} = Y$$

for every $Y \in \mathfrak{h}$. However, this equality is equivalent to $[X, Y] = XY - YX = 0$, which implies that $X$ belongs to the center of $\mathfrak{h}$. Since $X$ is nilpotent, this implies that $X = 0$. Hence $h = 1$.

We next show that if two unipotent elements $h_1$ and $h_2$ map to the same element under $\text{Ad}$, then $h_1 = h_2$. We take nilpotent elements $X_1$ and $X_2$ such that $h_1 = c(2X_1)$ and $h_2 = c(2X_2)$. Since $\text{Ad}(h_1) = \text{Ad}(h_2)$, in particular $\text{Ad}(h_1)(X_2) = \text{Ad}(h_2)(X_2)$. Thus

$$\frac{1 + X_1}{1 - X_1} \cdot X_2 \cdot \frac{1 - X_1}{1 + X_1} \cdot \frac{1 + X_2}{1 - X_2} \cdot X_2 \cdot \frac{1 - X_2}{1 + X_2} = X_2,$$

and this equality implies that $X_1$ commutes with $X_2$. Hence $h_1$ and $h_2$ commute and $h_1 h_2^{-1}$ is again unipotent. Then, by applying the result proved in the previous paragraph to $h_1 h_2^{-1}$, $h_1 h_2^{-1} = 1$. This completes the proof.

Hypothesis 2.2.6 We present a proof suggested by the referee here. It is well-known that, for any nilpotent element $X \in \mathfrak{h}$, we can find a Lie algebra homomorphism $\phi: \mathfrak{sl}_2 \to \mathfrak{h}$ ("$\mathfrak{sl}_2$-triple") satisfying $\phi(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}) = X$ since we work over the field $F$ of characteristic zero.

By combining $\phi$ with the adjoint representation of $\mathfrak{h}$, we may regard $\phi$ as an $\mathfrak{sl}_2$-representation on $\mathfrak{h}$; $\mathfrak{sl}_2 \xrightarrow{\phi} \mathfrak{h} \xrightarrow{\text{ad}} \text{End}(\mathfrak{h})$. Hence we can take its lift to an $\mathfrak{sl}_2$-representation $\text{SL}_2 \to \text{GL}(\mathfrak{h})$. The image of this homomorphism is contained
in the image $H_{ad}$ of the adjoint representation $H \xrightarrow{Ad} GL(\mathfrak{h})$ since it is so at the Lie algebra level (see [9, Section 7.1 (2)]). Since $SL_2$ is simply-connected, the homomorphism $SL_2 \to H_{ad}$ lifts to a homomorphism $SL_2 \to H$.

Finally, let us check the uniqueness up to rational conjugacy of an $sl_2$-triple. Suppose that $(h, e, f)$ and $(h', e, f')$ are $F$-rational $sl_2$-triples of $\mathfrak{h}$ sharing the same nilpotent element $e \in \mathfrak{h}$. By [10, Proposition 5.5.10], there exists an element $g \in M := C_H(e)^o$ satisfying $ghg^{-1} = h'$ and $gf^g = f'$. As $(h, e, f)$ and $(h', e, f')$ are $F$-rational, the map $\sigma \mapsto g^{-1}\sigma(g)$ gives a 1-cocycle from $Gal(\overline{F}/F)$ to $M$. If we let $C$ denote the centralizer of the $sl_2$-triple $(h, e, f)$ in $M$, then $M = R_u(M) \rtimes C$, where $R_u(M)$ is the unipotent radical of $M$ (see [10, Proposition 5.5.9]). This implies that the composition of the maps between Galois cohomology groups

$$H^1(F, C) \to H^1(F, M) \to H^1(F, C)$$

is the identity map, where the first map is induced from the inclusion map $C \hookrightarrow M$ and the second one is induced from the quotient map $M \to C$. By noting that the cohomology class of $[\sigma \mapsto g^{-1}\sigma(g)]$ is trivial on the middle term, we know that $[\sigma \mapsto g^{-1}\sigma(g)]$ is also trivial in $H^1(F, C)$. This implies that, by replacing $g$ with $gc$ for some $c \in C$, we may assume that $g$ is $F$-rational.

**Hypothesis 2.2.2** According to [13, Appendix A], Hypothesis 2.2.2 follows from Hypotheses 2.2.4, 2.2.5 and 2.2.6 as long as the Killing form $B$ on the Lie algebra $\mathfrak{h}'$ of the derived group of $H$ induces an identification of $\mathfrak{h}'_{x, s}$ with $\mathfrak{h}'_{x, s}^*$ for any $s \in \mathbb{R}$, where $\mathfrak{h}'_{x, s}^* = \{X \in \mathfrak{h}'^* \mid \langle X, Y \rangle \subset F\}$ for any $Y \in \mathfrak{h}'_{x, s}^*$. Let us investigate how large $p$ must be so that this condition is satisfied. As explained in the third paragraph of [13, Appendix A.2], it is enough to consider this problem for the split form of $\mathfrak{h}'$ by Galois descent. Thus it suffices to treat the cases where $\mathfrak{h}'$ equals $so_{2n+1}$ ($H = SO_{2n+1}$), $sp_{2n}$ ($H = Sp_{2n}$), $so_{2n}$ ($H = SO_{2n}$), or $sl_N$ ($H = U_{E/F}(N)$). In the proof of [6, Proposition 4.1], it is explained that the modified form $B' := \ell^{-1}B$ satisfies this property, where the constant $\ell$ is the minimum value of $B(\alpha^\vee, \alpha^\vee)$ for coroots $\alpha^\vee$ in $\mathfrak{h}'$. Therefore, if $p$ is large so that $\ell$ is invertible in $O_F$, we know that also $B$ identifies $\mathfrak{h}'_{x, s}$ with $\mathfrak{h}'_{x, s}^*$. By an easy calculation, we can check that the constant $\ell$ is given by

$$\begin{align*}
4(n + 1) & \quad \text{if } \mathfrak{h}' = sp_{2n} \text{ (the case of (1))}, \\
4(2n - 1) & \quad \text{if } \mathfrak{h}' = so_{2n+1} \text{ (the case of (2))}, \\
4(2n - 2) & \quad \text{if } \mathfrak{h}' = so_{2n} \text{ (the case of (3))}, \\
4N & \quad \text{if } \mathfrak{h}' = sl_N \text{ (the case of (4))}.
\end{align*}$$

As we assume that $p \neq 2$, $\ell$ belongs to $O_F^\times$ if and only if $p$ does not divide

$$\begin{align*}
n + 1 & \quad \text{if } \mathfrak{h}' = sp_{2n} \text{ (the case of (1))}, \\
2n - 1 & \quad \text{if } \mathfrak{h}' = so_{2n+1} \text{ (the case of (2))}, \\
n - 1 & \quad \text{if } \mathfrak{h}' = so_{2n} \text{ (the case of (3))}, \\
N & \quad \text{if } \mathfrak{h}' = sl_N \text{ (the case of (4))}.
\end{align*}$$

This is satisfied when the inequality of the statement holds.
Hypothesis 2.2.8 This follows from [3, Proposition 1.6.3]. (Note that the map “$\phi_x$” in [12, Hypothesis 2.2.8] need not be the same as the map exp, in Hypothesis 2.2.5 nor the map $c$.) □

From now on, we assume that $p$ satisfies the inequality of Lemma 4.4.

We next consider the twisted version of the character expansion.

Definition 4.5. (twisted version of the character expansion) Let $r \in \mathbb{R}_{>0}$ and $c$ be the Cayley transform of $G$ defined in Sect. 2.4. Let $\pi$ be a $\theta$-stable irreducible smooth representation of $G$. Then we have the $\theta$-twisted character $\Theta_{\pi,\theta}$ of $\pi$ which is normalized by the fixed $\theta$-stable Whittaker datum of $G$. We say that $\Theta_{\pi,\theta}$ has a character expansion on $U_r$ with respect to $c$ if there exists a complex number $c_\mathcal{O}$ for each nilpotent orbit $\mathcal{O}$ of $g_\theta$ such that, for every $f \in C_c^\infty(g_\theta, r)$, the following equality holds:

$$\Theta_{\pi,\theta}(f \circ c^{-1}) = \sum_{\mathcal{O} \in \text{Nil}(g_\theta)} c_\mathcal{O} \cdot \hat{\mu}_\mathcal{O}(f).$$

Here, we regard $f \circ c^{-1}$ as an element of $I(U_r)$ via the identification $I(G_{\theta, r}) = I(U_r)$ (note that $\Theta_{\pi,\theta}$ is $G$-invariant, hence factors through $I(U_r)$). In other words, as elements of $I(g_\theta, r)^*$,

$$\Theta_{\pi,\theta} \circ (c^{-1})^* = \sum_{\mathcal{O} \in \text{Nil}(g_\theta)} c_\mathcal{O} \cdot \hat{\mu}_\mathcal{O}.$$

Theorem 4.6. [4, Corollary 12.9] We assume that the residual characteristic is large enough (the same assumption as that for $G_\theta$ in Theorem 4.3). Let $\pi$ be a $\theta$-stable irreducible smooth representation of $G$ of depth $r$. Then $\Theta_{\pi,\theta}$ has a character expansion on $U_{r+}$ with respect to $c$.

Note that, as we explained in the end of Sect. 2.1, the group $G_\theta$ is given by

$$G_\theta = \begin{cases} \text{SO}_{2n+1} & \text{if } G = \text{GL}_{2n+1}, \\ \text{Sp}_{2n} & \text{if } G = \text{GL}_{2n}, \\ U_{E/F}(N) & \text{if } G = \text{Res}_{E/F} \text{GL}_{N,E}. \end{cases}$$

In particular, by Lemma 4.4, the assumption on the residual characteristic $p$ needed in Theorem 4.6 is that

$$p \geq \begin{cases} 4n + 1 & \text{the cases of (1), (2), and (3),} \\ 2N + 1 & \text{the case of (4).} \end{cases}$$

From now on, we assume that the residual characteristic $p$ is large enough so that Theorems 4.3 and 4.6 hold. In other words, we assume the following:

Hypothesis 4.7. The residual characteristic $p$ is greater than or equal to

$$\begin{cases} 4n + 1 & \text{the cases of (1), (2), and (3),} \\ 2N + 1 & \text{the case of (4).} \end{cases}$$
Now let us compare the validity ranges of local character expansions via the endoscopic character relation. Let \((G, H)\) be one of the pairs defined in Sect. 2.1. Let \(\phi\) be a tempered \(L\)-parameter of \(H\). Then, by Theorem 2.3, we get a tempered \(L\)-packet \(\Pi_{\phi}^H\) of \(H\) and an irreducible \(\theta\)-stable tempered representation \(\pi_{G}^G\) of \(G\) corresponding to the \(L\)-parameter \(\phi\).

**Lemma 4.8.** We put \(r_H := \max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^H\}\). Then the \(\theta\)-twisted character \(\Theta_{\phi, \theta}^G\) of \(\pi_{\phi}^G\) has a character expansion on \(\mathcal{U}_{\eta H+}\) with respect to \(c\).

**Proof.** Before we start to prove this lemma, we recall that the following diagram commutes for every \(r \in \mathbb{R}_{>0}\) (this is obtained by taking the dual of the diagram in the end of Sect. 4.1):

\[
\begin{array}{ccc}
\mathcal{I}(G_{\theta, r})^* & \xrightarrow{(\text{transfer})^*} & \mathcal{I}(U_r)^* \\
\cong & & \cong \mathcal{I}(U_r)^* \\
\mathcal{I}(g_{\theta, r})^* & \xleftarrow{(\text{transfer})^*} & \mathcal{I}(h_r)^* \\
\end{array}
\]

By the definition of \(r_H\) and Theorem 4.3, the sum \(\Theta_{\phi}^H\) of the characters of representations belonging to \(\Pi_{\phi}^H\) has a character expansion on \(H_{\eta H+}\) with respect to \(c\), that is, as elements of \(\mathcal{I}(h_{\eta H+})^*\) (hence of \(\mathcal{ST}(h_{\eta H+})^*\)),

\[
\Theta_{\phi}^H \circ (c^{-1})^* = \sum_{O_H \in \text{Nil}(h)} c_{O_H} \cdot \hat{\mu}_{O_H}.
\]

Similarly, if we put \(r_G\) to be the depth of \(\pi_{G}^G\), then \(\Theta_{\phi, \theta}^G\) has a character expansion on \(U_{\eta G+}\) by Theorem 4.6, that is, as elements of \(\mathcal{I}(U_{\eta G+})^* = \mathcal{I}(g_{\theta, r_G+})^*\),

\[
\Theta_{\phi, \theta}^G \circ (c^{-1})^* = \sum_{O \in \text{Nil}(g_{\theta})} c_{O} \cdot \hat{\mu}_{O}.
\]

On the other hand, by the endoscopic character relation (Theorem 2.3), the pull back of \(\Theta_{\phi}^H\) via the transfer map coincides with \(\Theta_{\phi, \theta}^G\) as elements of \(\mathcal{I}(\hat{G})^*\). Thus, by the commutativity of the above diagram,

\[
\Theta_{\phi, \theta}^G \circ (c^{-1})^* = (\text{transfer})^* \left( \sum_{O_H \in \text{Nil}(h)} c_{O_H} \cdot \hat{\mu}_{O_H} \right)
\]

as an element of \(\mathcal{I}(g_{\theta, r_H+})^*\). However, by the homogeneity argument (see [14, Lemma 10.5.3]),

\[
\sum_{O \in \text{Nil}(g_{\theta})} c_{O} \cdot \hat{\mu}_{O} = (\text{transfer})^* \left( \sum_{O_H \in \text{Nil}(h)} c_{O_H} \cdot \hat{\mu}_{O_H} \right)
\]

in \(\mathcal{I}(g_{\theta, r_H+})^*\). In particular, the equality \((*)\) holds in \(\mathcal{I}(g_{\theta, r_H+})^*\). Thus the \(\theta\)-twisted character \(\Theta_{\phi, \theta}^G\) has a character expansion on \(U_{\eta H+}\) with respect to \(c\). \(\Box\)
4.3. Utilization of DeBacker’s parametrization of nilpotent orbits. For any tempered $L$-parameter $\phi$ of $H$, we have the inequality

$$\max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^H\} \leq \text{depth}(\phi)$$

by Theorem 1.1 ([14, Corollary 10.6.4]). Note that, although only the cases of symplectic and special orthogonal groups are treated in [14], we can carry out the same strategy also for unitary groups. Especially, as explained in the paragraph before the proof of Proposition 4.1, the transfer factor $\Delta^IV$ is always trivial in the unitary group case. Hence, the exact same argument works without any more extra computation.

The following is one of the main results of this paper.

**Theorem 4.9.** Assume Hypothesis 4.7. For any tempered $L$-parameter $\phi$ of $H$,

$$\max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^H\} = \text{depth}(\phi).$$

**Proof.** Thanks to the inequality $\max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^H\} \leq \text{depth}(\phi)$ of Ganapathy–Varma, our task is to show the converse inequality

$$\max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^H\} \geq \text{depth}(\phi).$$

It is enough to show $r \geq \text{depth}(\phi)$ for every $r \in \mathbb{R}_{>0}$ which is strictly greater than $\max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^H\}$. For this, we consider the converse direction of the arguments of Ganapathy–Varma in [14, Corollary 10.6.4].

By Lemma 4.8, the twisted character $\Theta_{\phi,\theta}^G$ of the endoscopic lift of $\Pi_{\phi}^H$ has a local character expansion on $U_r$:

$$\Theta_{\phi,\theta}^G \circ (c^{-1})^* = \sum_{O \in \text{Nil}(g_\theta)} c_O \cdot \hat{\mu}_O. \quad (\ast)$$

We take a maximal (in the sense of the closure relation) nilpotent orbit $O_*$ of $g_\theta$ satisfying $c_{O_*} \neq 0$. Here we note that there exists a nilpotent orbit $O$ whose $c_O$ is not zero (thus, the distribution $\Theta_{\phi,\theta}^G$ is not identically zero on $\mathcal{I}(U_r)$). Indeed, if $\Theta_{\phi,\theta}^G$ is identically zero on $\mathcal{I}(U_r)$, then also $\Theta_{\phi}^H$ is zero on $\mathcal{I}(H_r)$ by the injectivity of the pullback via the transfer, which is proved in the proof of [14, Lemma 10.5.4]. Here we note that the image of the transfer from $\mathcal{I}(U_r)$ to $\mathcal{I}(H_r)$ consists of stable orbital integrals which are invariant under the action of $\widetilde{\text{Out}}(H)$. Thus, strictly speaking, the injectivity of the pullback via the transfer holds only for distributions which are invariant under the action of $\widetilde{\text{Out}}(H)$. However it does not cause any problem here since our $L$-packet $\Pi_{\phi}^H$ consists of $\widetilde{\text{Out}}(H)$-orbits (see Sect. 2.3). Then, for any $x \in B(H, F)$ and $s \in \mathbb{R}_{>0}$,

$$\Theta_{\phi}^H(1_{H_{x,s}}) = \sum_{\pi \in \Pi_{\phi}^H} \text{tr}(\pi(1_{H_{x,s}}) \mid \pi_{H_{x,s}}) = \sum_{\pi \in \Pi_{\phi}^H} \dim(\pi_{H_{x,s}}) \cdot \text{vol}(H_{x,s}).$$

In particular, by the admissibility of representations in $\Pi_{\phi}^H$, the distribution $\Theta_{\phi}^H$ does not vanish at $1_{H_{x,s}}$ for a sufficiently large $s$. This is a contradiction.
By the same argument as in the second paragraph of the proof of [14, Corollary 10.6.4] (a consequence of DeBacker’s parametrizing result of nilpotent orbits via Bruhat–Tits theory, see [13, Theorem 5.6.1] and also [12, Section 2.5]), for this nilpotent orbit \( \mathcal{O}_\ast \), we can find a point \( x \in \mathcal{B}(G_\theta, F) \) and an element \( X \in g_{\theta,x,-r} \) satisfying the following conditions:

- \( X \in \mathcal{O}_\ast \), and
- if a nilpotent orbit \( \mathcal{O} \) meets \( X + g_{\theta,x,-r+r} \), then \( \mathcal{O}_\ast \subset \overline{\mathcal{O}} \).

Here we remark that, in order to use DeBacker’s parametrization, we have to put some assumptions on the residual characteristic. However, these are the same as the assumptions used in Theorem 4.6. Hence we do not have to add any further assumption.

As in the third paragraph of the proof of [14, Corollary 10.6.4], we define a homomorphism \( \chi_X \) from \( g_{\theta,x,r} \) to \( \mathbb{C}^\times \) to be

\[
Y \mapsto \psi_F(\text{tr}(XY)),
\]

where \( \psi_F \) is a nontrivial additive character of \( F \) of level zero. Then, since \( X \) belongs to \( g_{\theta,x,-r} \), this homomorphism \( \chi_X \) is \( g_{\theta,x,r+r} \)-invariant. Hence, by composing the inverse of the Cayley transform isomorphism \( g_{\theta,x,r+r} \cong G_{\theta,x,r+r} \) (Proposition 2.9) and considering the zero extension, we can regard \( \chi_X \circ \phi^{-1} \) as an element of \( C_\mathfrak{c}^\infty(G_{\theta,x,r}) \) which is bi-\( G_{\theta,x,r} \)-invariant.

By Corollary 3.7, there exists a bi-\( G_{x,r} \)-invariant test function \( \tilde{f} \) of \( C_\mathfrak{c}^\infty(G_{x,r} \times \theta) \) such that \( \chi_X \circ \phi^{-1} \) is a semisimple descent of \( \tilde{f} \). Since \( G_{x,r+r} \) is \( \theta \)-stable and \( \tilde{f} \) is bi-\( G_{x,r} \)-invariant,

\[
\Theta_{\phi,\theta}^G(\tilde{f}) = \text{tr}(\pi^G_\phi(f) \circ I_\theta \mid (\pi^G_\phi)^G_{x,r+r})
\]

by the definition of the \( \theta \)-twisted character distribution. Here, \( I_\theta \) is the intertwiner \( I_\theta: \pi^G_\phi \cong (\pi^G_\phi)\theta \) normalized by using the fixed \( \theta \)-stable Whittaker datum of \( G \), and \( f \) is an element of \( C_\mathfrak{c}^\infty(G_{x,r}) \) satisfying \( \tilde{f}(g \times \theta) = f(g) \) for any \( g \in G_{x,r} \). Thus, if we can show the non-vanishing of \( \Theta_{\phi,\theta}^G(\tilde{f}) \), then the depth of \( \pi^G_\phi \) (hence the depth of \( \phi \), by Theorem 2.15) is at most \( r \) and the proof is complete.

By the local character expansion as justified around (*),

\[
\Theta_{\phi,\theta}^G(\tilde{f}) = \sum_{\mathcal{O} \in \text{Nil}(g_{\theta})} c_{\mathcal{O}} \cdot \hat{\mu}_{\mathcal{O}}(\chi_X).
\]

By noting that the Fourier transform of \( \chi_X \) on \( g_\theta \) with respect to the function

\[
\mathfrak{g}_\theta \times \mathfrak{g}_\theta \to \mathbb{C}^\times; \quad (Y_1, Y_2) \mapsto \psi_F(\text{tr}(Y_1Y_2))
\]

is given by \( \text{vol}(\mathfrak{g}_{\theta,x,r}) \cdot \mathbb{1}_{X + \mathfrak{g}_{\theta,x,-r+r}} \),

\[
\sum_{\mathcal{O} \in \text{Nil}(g_{\theta})} c_{\mathcal{O}} \cdot \hat{\mu}_{\mathcal{O}}(\chi_X) = \sum_{\mathcal{O} \in \text{Nil}(g_{\theta})} c_{\mathcal{O}} \cdot \mu_{\mathcal{O}}(\hat{\chi}_X)
\]

\[
= \text{vol}(\mathfrak{g}_{\theta,x,r}) \sum_{\mathcal{O} \in \text{Nil}(g_{\theta})} c_{\mathcal{O}} \cdot \mu_{\mathcal{O}}(\mathbb{1}_{X + \mathfrak{g}_{\theta,x,-r+r}}).
\]
Furthermore, by the properties of $O_*$ and $X$, we can compute the right-hand side as follows:

$$\text{vol}(g_{\theta,x,r}) \sum_{O \in \text{Nil}(g_\theta)} c_O \cdot \mu_O(1_{X + g_{\theta,x,-r+}}) = \text{vol}(g_{\theta,x,r}) \sum_{O \in \text{Nil}(g_\theta)} c_O \cdot \mu_O(1_{X + g_{\theta,x,-r+}})$$

In particular, this is not equal to zero. This completes the proof. □

We finally comment on the non-tempered case. In Theorem 2.3, the local Langlands correspondence is stated only for tempered $L$-packets, and we do not have the endoscopic character relation for non-tempered $L$-packets. However, via the theory of Langlands classification, we can extend the local Langlands correspondence from the tempered case to the non-tempered case. To be more precise, let $J$ be a connected reductive group over $F$. Then, for any triple $(P_J, \sigma, \chi)$ consisting of

- a parabolic subgroup $P_J$ of $J$ with Levi decomposition $P_J = M_J N_J$,
- an irreducible tempered representation $\sigma$ of $M_J$, and
- an unramified character $\chi$ of $M$ satisfying a regularity condition,

the parabolic induction $n\text{-Ind}_{P_J}^J(\sigma \otimes \chi)$ possesses a unique irreducible quotient. Langlands classification asserts that any irreducible smooth representation can be realized as such an irreducible quotient in an essentially unique way. When an irreducible smooth representation $\pi$ of $J$ is obtained from a triple $(P_J, \sigma, \chi)$, we define the $L$-parameter of $\pi$ by twisting the $L$-parameter of $\sigma$ via the unramified character $\chi$. (Here note that the $L$-group of $M_J$ may be regarded as a subgroup of that of $J$.) See, for example, [30] for the details of the Langlands classification and the way to extend the local Langlands correspondence from the tempered case to non-tempered case.

The key fact here is the following:

**Proposition 4.10.** [27, Theorem 5.2] Let $J$ be a connected reductive group over $F$ and $P_J$ a parabolic subgroup of $J$ with Levi decomposition $P_J = M_J N_J$. Let $\rho$ be an irreducible smooth representation of $M_J$ and $\pi$ an irreducible subquotient of $n\text{-Ind}_{P_J}^J \rho$. Then $\text{depth}(\pi) = \text{depth}(\rho)$.

In our situation, any Levi subgroup of the classical group $H$ is given by the product of several smaller general linear groups and a classical group of the same type as $H$. Especially, Proposition 4.10 is applicable to any tempered $L$-packet of any Levi subgroup of $H$. Therefore, by also noting that the unramified twist does not change the depth of any $L$-parameter, Proposition 4.10 implies the following:

**Theorem 4.11.** We assume Hypothesis 4.7. For any $L$-parameter $\phi$ of $H$,

$$\max \{ \text{depth}(\pi) \mid \pi \in \Pi^H_{\phi} \} = \text{depth}(\phi).$$
5. Evaluation of the minimum depth in an \( L \)-packet for unitary groups

In this section, we consider the case where \( G := \text{Res}_{E/F} \text{GL}_{N,E} \) for a quadratic extension \( E/F \), and \( H \) is the quasi-split unitary group \( U_{E/F}(N) \) with respect to \( E/F \) in \( N \) variables. Recall that \( G_0 = H \).

By combining Corollary 3.7 with Proposition 4.1, we get the following:

**Theorem 5.1.** Let \( x \in \mathcal{B}(H, F) = \mathcal{B}(G_0, F) \) and \( r \in \mathbb{R}_{>0} \). Then \( \text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \in C_c^\infty(H) \) is a transfer of \( \text{vol}(G_{x,r})^{-1} \mathbb{1}_{G_{x,r}} \circ \theta \in C_c^\infty(G) \).

**Proof.** By Corollary 3.7, \( \text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \in C_c^\infty(H) \) is a semisimple descent of \( \text{vol}(G_{x,r})^{-1} \mathbb{1}_{G_{x,r}} \circ \theta \in C_c^\infty(G) \). Thus, by Proposition 4.1, in order to show the assertion, it is enough to check that \( \text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \circ c \) and \( \text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \circ c' \) have matching orbital integrals. However, since we assume that the residual characteristic \( p \) is not equal to 2, the map \( x \mapsto x^2 \) from \( H_{0x} \) to itself is bijective and induces a bijection on \( H_{x,r} \) (see, e.g., [14, Lemma 3.2.7] for a proof). Thus

\[
\text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \circ c = \text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \circ c' = \text{vol}(H_{x,r})^{-1} \mathbb{1}_{0_{x,r}}.
\]

In particular, they have matching orbital integrals. \( \Box \)

**Proposition 5.2.** For any tempered \( L \)-parameter \( \phi \) of \( H \),

\[
\min\{\text{depth}(\pi) \mid \pi \in \Pi_\phi^H\} \geq \text{depth}(\phi).
\]

**Proof.** Let \( \pi \) be a member of \( \Pi_\phi^H \) having the minimal depth in \( \Pi_\phi^H \), and we put \( r \) to be the depth of \( \pi \). Then, for a point \( x \in \mathcal{B}(H, F) \), \( \pi \) has a non-zero \( H_{x,r} \)-fixed vector by Proposition 2.12. Thus

\[
\Theta_{\pi}(\mathbb{1}_{H_{x,r}+}) = \text{tr}(\pi(\mathbb{1}_{H_{x,r}+}) \mid \pi^H_{x,r+}) = \dim(\pi^H_{x,r+}) \cdot \text{vol}(H_{x,r+}) > 0.
\]

Moreover, for any other member \( \pi' \) of \( \Pi_\phi^H \),

\[
\Theta_{\pi'}(\mathbb{1}_{H_{x,r}+}) = \dim(\pi'_{x,r+}) \cdot \text{vol}(H_{x,r+}) \geq 0.
\]

Therefore we can conclude that \( \Theta_{\phi}^H(\mathbb{1}_{H_{x,r}+}) \) is not zero.

On the other hand, by Theorem 5.1, \( \text{vol}(H_{x,r})^{-1} \mathbb{1}_{H_{x,r}} \in C_c^\infty(H) \) is a transfer of \( \text{vol}(G_{x,r})^{-1} \mathbb{1}_{G_{x,r}} \circ \theta \in C_c^\infty(G) \). Therefore, by applying the endoscopic character relation (Theorem 2.3) to these functions,

\[
\text{vol}(G_{x,r})^{-1} \cdot \Theta_{\phi,0}^G(\mathbb{1}_{G_{x,r}+} \circ \theta) = \text{vol}(H_{x,r})^{-1} \cdot \Theta_{\phi}^H(\mathbb{1}_{H_{x,r}+}) \neq 0.
\]

Since

\[
\Theta_{\phi,0}^G(\mathbb{1}_{G_{x,r}+} \circ \theta) = \text{tr}(\pi_{G}^\phi(\mathbb{1}_{G_{x,r}+}) \circ I_\theta \mid (\pi_{G}^\phi)_{G_{x,r+}}),
\]

where \( I_\theta \) is the isomorphism \( I_\theta : \pi_{G}^\phi \cong (\pi_{G}^\phi)^\theta \) normalized by the fixed Whittaker datum, in particular \( (\pi_{G}^\phi)_{G_{x,r+}} \) is not zero. Therefore the depth of \( \pi_{G}^\phi \) is not greater than \( r \). As \( \text{depth}(\pi_{G}^\phi) = \text{depth}(\phi) \) by Theorem 2.15, we get the assertion. \( \Box \)
Remark 5.3. In the above proof, we only have to assume that \( p \) is not equal to 2.

Remark 5.4. We cannot deduce the converse inequality simply by swapping the roles of \( G \) and \( H \) because of the existence of the intertwiner \( I_\theta \). As noted in the proof of Proposition 5.2, the twisted character \( \Theta_{G,\theta}^{G} (I_{G_{x,r+}} \times \phi) \) is given by \( \text{tr}(\pi_{\phi}^{G}(I_{G_{x,r+}}) \circ I_\theta | (\pi_{\phi}^{G})^{G_{x,r+}}) = \text{tr}(I_\theta | (\pi_{\phi}^{G})^{G_{x,r+}}) \). Thus, even if \( \pi \) has a nonzero \( G_{x,r+} \)-fixed vector, the trace \( \text{tr}(I_\theta | (\pi_{\phi}^{G})^{G_{x,r+}}) \) of the intertwiner \( I_\theta \) may not be necessarily nonzero. Note that this might happen when the dimension of the space \( (\pi_{\phi}^{G})^{G_{x,r+}} \) is greater than 1.

Theorem 5.5. We assume Hypothesis 4.7. Let \( \phi \) be a tempered \( L \)-parameter of \( H \), and \( \Pi_{\phi}^{H} \) the \( L \)-packet of \( H \) for \( \phi \). Then, for every \( \pi \in \Pi_{\phi}^{H} \),

\[
\text{depth}(\pi) = \text{depth}(\phi).
\]

Proof. By combining Proposition 5.2 with the unitary version of Theorem 1.1,

\[
\min\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^{H}\} \geq \text{depth}(\phi) \geq \max\{\text{depth}(\pi) \mid \pi \in \Pi_{\phi}^{H}\}.
\]

Thus \( \text{depth}(\pi) \) is constant on \( \Pi_{\phi}^{H} \) and equal to \( \text{depth}(\phi) \). \( \square \)

By the same argument as in the proof of Theorem 4.11, we can extend this result to non-tempered \( L \)-parameters:

Theorem 5.6. We assume Hypothesis 4.7. Let \( \phi \) be an \( L \)-parameter of \( H \), and \( \Pi_{\phi}^{H} \) the \( L \)-packet of \( H \) for \( \phi \). Then, for every \( \pi \in \Pi_{\phi}^{H} \),

\[
\text{depth}(\pi) = \text{depth}(\phi).
\]

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Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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