PURELY INFINITE, SIMPLE C*-ALGEBRAS ARISING FROM FREE PRODUCT CONSTRUCTIONS. III

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Abstract. In the reduced free product of C*-algebras, \((A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)\) with respect to faithful states \(\phi_1\) and \(\phi_2\), \(A\) is purely infinite and simple if \(A_1\) is a reduced crossed product \(B \rtimes_{\alpha, r} G\) for \(G\) an infinite group, if \(\phi_1\) is well behaved with respect to this crossed product decomposition, if \(A_2 \neq \mathbb{C}\) and if \(\phi\) is not a trace.

The reduced free product construction for C*-algebras was invented independently by Voiculescu [11] and, in a more limited sense, Avitzour [1]. (The term “reduced” is to distinguish this construction from the universal or “full” free product of C*-algebras.) It is a natural construction in Voiculescu’s free probability theory (see [12]). Given unital C*-algebras \(A_i\) with states \(\phi_i\) whose GNS representations are faithful \((i \in I)\), the construction yields

\[(A, \phi) = \bigoplus_{i \in I} (A_i, \phi_i),\]

where \(A\) is a unital C*-algebra containing copies \(A_i \hookrightarrow A\) and generated by \(\bigcup_{i \in I} A_i\), and where \(\phi\) is a state on \(A\) with faithful GNS representation that restricts to give \(\phi_i\) on \(A_i\) for every \(i \in I\) and such that \((A_i)_{i \in I}\) is free with respect to \(\phi\). Moreover, \(\phi\) is a trace if and only if every \(\phi_i\) is a trace; by [4], \(\phi\) is faithful on \(A\) if and only if \(\phi_i\) is faithful on \(A_i\) for every \(i \in I\).

It is a very interesting open question whether every simple, unital C*-algebra must either have a trace or be purely infinite. Purely infinite C*-algebras were defined by J. Cuntz [3]. A simple unital C*-algebra \(A\) is purely infinite if and only if for every positive element \(x \in A\) there is \(y \in A\) with \(y^*xy = 1\). An equivalent condition is that every hereditary C*-subalgebra of \(A\) contains an infinite projection.

Let

\[(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)\]

be a reduced free product of C*-algebras. In [8] it was shown that if \(\phi_1\) or \(\phi_2\) is nontracial and if \(A_1\) and \(A_2\) are not too small in a specific sense, then \(A\) is properly infinite. It is a plausible conjecture that whenever \(A\) is simple and at least one of \(\phi_1\) and \(\phi_2\) is not a trace, the C*-algebra \(A\) must be purely infinite. The first results in this direction were [7], where in a certain class of examples when \(\phi_1\) was assumed to be nonfaithful, \(A\) was shown to be purely infinite and simple. In [5], assuming \(\phi_1\)
and \( \phi_2 \) faithful, \( A \) was shown to be purely infinite and simple in the case when the centralizer of \( \phi_1 \) in \( A_1 \) contains a diffuse abelian subalgebra and when \( A_2 \) contains a partial isometry that, loosely speaking, scales \( \phi_2 \) by a constant \( \lambda \neq 1 \). In [9], reduced free products of (countably) infinitely many \( C^* \)-algebras that are not too small in a specific sense were shown to be purely infinite.

In this note, we prove a theorem implying that \( A \) is purely infinite and simple under somewhat different conditions. For example, if \( A_1 = C(\mathbb{T}) \) is the algebra of all continuous functions on the circle and if \( \phi_1 \) is given by integration with respect to Haar measure, then \( A \) is simple and purely infinite provided only that \( A_2 \neq C \) and \( \phi_2 \) is faithful but not a trace.

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**Notation.** We begin with some notation, which has appeared elsewhere. Given an algebra \( \mathfrak{A} \) and subsets \( S_i \subseteq \mathfrak{A} \quad (i \in I) \), let \( \Lambda^\alpha((S_i)_{i \in I}) \) be the set of all words \( w = a_1 a_2 \cdots a_n \) where \( n \geq 1 \), \( a_j \in S_{\iota_j} \), and \( \iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \ldots, \iota_{n-1} \neq \iota_n \). We will refer to the elements \( a_1, \ldots, a_n \) as the letters of the word \( w \); we will sometimes regard the word as a product of specific letters, and sometimes as an actual element of the algebra \( \mathfrak{A} \), as it suits the situation.

Moreover, if a \( C^* \)-algebra \( A \) and a state \( \phi : A \to \mathbb{C} \) are specified, we will denote the kernel of \( \phi \) by \( A^\phi \).

**Theorem.** Let \( A_1 \) be a reduced crossed product \( C^* \)-algebra, \( A_1 = B \rtimes_{\alpha,G} G \), where \( G \) is an infinite discrete group and where \( B \) is a unital \( C^* \)-algebra. Denote by \( u_g \) \((g \in G)\) the unitaries in \( A_1 \) arising from the reduced crossed product construction and implementing the automorphisms \( \alpha_g \) on \( B \). Let \( \phi_1 \) be a faithful state on \( B \) that is preserved by all the automorphisms \( \alpha_g \) and denote also by \( \phi_1 \) its extension to the state on \( A_1 \) that vanishes on the subspace \( B u_g \) for every nontrivial \( g \in G \). Let \( A_2 \) be a unital \( C^* \)-algebra, \( A_2 \neq \mathbb{C} \), with a faithful state \( \phi_2 \); let

\[
(A, \phi) = (A_1, \phi_1) \ast (A_2, \phi_2)
\]

be the reduced free product of \( C^* \)-algebras. Suppose that at least one of \( \phi_1 \) and \( \phi_2 \) is not a trace.

Then \( A \) is purely infinite and simple.

**Proof.** Our strategy will be to show that \( A \) is itself the reduced crossed product of a \( C^* \)-subalgebra \( D \) by the group \( G \), where \( D \) is (isomorphic to) the reduced free product of infinitely many \( C^* \)-algebras; a result from [9] will thereby show that \( D \) is purely infinite and simple. We will then show that the action of \( G \) on \( D \) is properly outer; a result of Kishimoto and Kumjian [10] will thereby imply that \( A \) is purely infinite and simple.

**Claim 1.** The family

\[
(B, (u_g^* A_2 u_g)_{g \in G})
\]

is free with respect to \( \phi \).

**Proof.** We must show that

\[
(1) \quad \Lambda^\alpha(B^\phi, (u_g^* A_2^g u_g)_{g \in G}) \subseteq \ker \phi.
\]
Let $x$ be a word belonging to the left–hand side of $(1)$. Splitting off the unitaries $u_g^*$ and $u_g$ from the letters in $x$, then grouping together any neighbors in the resulting word belonging to $A_1$ and using that $u_g, B_2 u_g^* \subseteq A_0^0$ whenever $g_1, g_2 \in G$ and that $u_{g_1} u_{g_2}^* \in A_0^0$ if $g_1 \neq g_2$, we see that $x$ is equal to a word $x' \in A_0^0(A_1^0, A_2^0)$. Hence $x \in \ker \phi$ by freeness. This finishes the proof of Claim 1.

Let $D$ be the $C^*$--subalgebra of $A$ generated by $B \cup \bigcup_{g \in G^2} u_g^* A_2 u_g$.

Claim 2. $D$ is simple and purely infinite.

Proof. Since $A_2 \neq C$, there is a self–adjoint element $x \in A_2 \setminus C$. Let $\mu$ be the distribution of $x$; namely, $\mu$ is the probability measure whose support is the spectrum of $x$ and such that $\phi_2(x^k) = \int_R t^k d\mu(t)$ for all $k \geq 1$. A consequence of Bercovici and Voiculescu’s result [2 Prop. 8] is that for some $n$ large enough, the measure arising as the $n$–fold additive free convolution

$$\mu_n \defeq \mu \boxplus \mu \boxplus \cdots \boxplus \mu$$

has support equal to an interval $[a, b]$ and is absolutely continuous with respect to Lebesgue measure. If $g_1, g_2, \ldots, g_n$ are distinct elements of $G$, then by Claim 1 the distribution of $y \defeq \sum_{j=1}^n u_{g_j}^* x u_{g_j}$ is $\mu_n$; therefore $y$ generates an abelian subalgebra of

$$D(g_1, \ldots, g_n) \defeq C^*(\bigcup_{j=1}^n u_{g_j}^* A_2 u_{g_j})$$

on which $\phi$ is given by a measure without atoms; it follows from [6 Prop. 4.1] that $D(g_1, \ldots, g_n)$ contains a unitary $v$ satisfying $\phi(v) = 0$ (in fact, this proposition gives $\phi(v^k) = 0$ for all nonzero integers $k$, but we will not need this). Therefore, partitioning the family $(u_{g_j} A_2 u_g)_{g \in G}$ into subcollections of cardinality $n$, and including $B$ in one of these subcollections, we see that $D$ is isomorphic to the free product of infinitely many $C^*$–algebras with respect to faithful states,

$$(D, \phi) \cong \prod_{k=1}^\infty (D_k, \psi_k),$$

where each $D_k$ contains a unitary that evaluates to zero under $\psi_k$. Moreover, since either $\phi_2$ or $\phi_1|_B$ is not a trace, at least one of the $\psi_k$ is not a trace. By [9 Thm. 2.1], $D$ is therefore simple and purely infinite. This finishes the proof of Claim 2.

Claim 3. $D$ has trivial relative commutant in $A$.

Proof. Let

$$D_0 = C^*(\bigcup_{g \in G} u_g^* A_2 u_g) \subseteq D;$$

we will show that $D_0$ has trivial relative commutant in $A$, which will imply the same for $D$. Suppose that $x \in A$ and $x$ commutes with $D_0$; our goal is to show that $x$ must belong to $C$. Let $x_0 = x - \phi(x)1$ and suppose, to obtain a contradiction, that $x_0 \neq 0$. Since $\phi$ is faithful, $\|x_0\|^2 = \phi(x_0^* x_0)^{1/2} > 0$. Choose $\epsilon$ so that $0 < \epsilon < \frac{\|x_0\|^2}{2}$. Since

$$C \setminus \text{span } A_0 \left(B^0 \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_0^2\right)$$

we will show that $D_0$ has trivial relative commutant in $A$, which will imply the same for $D$. Suppose that $x \in A$ and $x$ commutes with $D_0$; our goal is to show that $x$ must belong to $C$. Let $x_0 = x - \phi(x)1$ and suppose, to obtain a contradiction, that $x_0 \neq 0$. Since $\phi$ is faithful, $\|x_0\|^2 = \phi(x_0^* x_0)^{1/2} > 0$. Choose $\epsilon$ so that $0 < \epsilon < \frac{\|x_0\|^2}{2}$. Since

$$C \setminus \text{span } A_0 \left(B^0 \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_0^2\right)$$
is a dense $*$-subalgebra of $A$, and since $\Lambda^o(B^o \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_2^o) \subseteq \ker \phi$, there is a sum of finitely many words, $y = w_1 + w_2 + \cdots + w_m$ with $w_1, w_2, \ldots, w_m \in \Lambda^o(B^o \cup \bigcup_{g \in G \setminus \{e\}} Bu_g, A_2^o)$, such that $\|x_0 - y\| < \varepsilon$. Let $F$ be the finite subset of $G$ whose elements are the identity element and all nontrivial elements $g \in G$ for which some $w_j$ has a letter coming from $Bu_g$. From the proof of Claim 2, there is $n \in \mathbb{N}$ such that for any $n$ distinct elements, $g_1, g_2, \ldots, g_n$ of $G$, there is a unitary

$$v \in D(g_1, g_2, \ldots, g_n) = C^* \left( \bigcup_{j=1}^{n} u_{g_j}^* A_2 u_{g_j} \right)$$

with $\phi(v) = 0$. We take this unitary $v$ having ensured that the $n$ distinct elements satisfy $g_j \notin F$ and $g_j^{-1} \notin F$ for every $j \in \{1, \ldots, n\}$.

Let us show that $vy$ and $yv$ are orthogonal with respect to the inner product on $A$ induced by $\phi$, i.e. that $\langle vy, yv \rangle_A = \phi(v^*y^*vy) = 0$. Since

$$C_1 + \text{span} \Lambda^o(u_{g_1}^* A_2^o u_{g_1}, u_{g_2}^* A_2^o u_{g_2}, \ldots, u_{g_n}^* A_2^o u_{g_n})$$

is a dense $*$-subalgebra of $D(g_1, \ldots, g_n)$ and since (as can be seen using Claim 1)

$$\Lambda^o(u_{g_1}^* A_2^o u_{g_1}, u_{g_2}^* A_2^o u_{g_2}, \ldots, u_{g_n}^* A_2^o u_{g_n}) \subseteq \ker \phi,$$

for every $\eta > 0$ there is a sum of finitely many words $z = w'_1 + w'_2 + \cdots + w'_j$ with

$$w'_1, \ldots, w'_j \in \Lambda^o(u_{g_1}^* A_2^o u_{g_1}, u_{g_2}^* A_2^o u_{g_2}, \ldots, u_{g_n}^* A_2^o u_{g_n}),$$

such that $\|v - z\| < \eta$. But we see that each $w'_j$ is equal to a word

$$w''_j \in \Lambda^o(\{ u_g \mid g \in G \setminus \{e\} \}, A_2^o)$$

where $w''_j$ begins with $u_{g_{j-1}}$ and ends with $u_{g_k}$ for some $j, k \in \{1, \ldots, n\}$, and where $w''_j$ has length at least three. Since

$$w_1, \ldots, w_m \in \Lambda^o(B^o \cup \bigcup_{g \in F \setminus \{e\}} Bu_g, A_2^o),$$

when we consider a product $(w''_i)^* w''_j w''_j w_{j_2}$ for arbitrary $i_1, i_2 \in \{1, \ldots, p\}$ and

$$j_1, j_2 \in \{1, \ldots, m\},$$

the choice of the elements $g_1, \ldots, g_n$ ensures that there is not too much cancellation and we are left with a reduced word

$$(w''_i)^* w''_j w''_j w_{j_2} = w \in \Lambda^o(B^o \cup \bigcup_{g \in F \setminus \{e\}} Bu_g, A_2^o);$$

hence $\phi((w''_i)^* w''_j w''_j w_{j_2}) = 0$. This implies that $\phi(z^* y^* y) = 0$. Since $\eta > 0$ was arbitrary and $|\phi(v^* y^* y) - \phi(z^* y^* y)| \leq \eta(2 + \eta) \|y\|^2$, we have $\phi(v^* y^* y) = 0$, i.e. $vy$ and $yv$ are orthogonal.

We now obtain the contradiction. Since $x_0$ belongs to the commutant of $D_0$, we must have $vx_0 - x_0v = 0$. But by orthogonality of $vy$ and $yv$,

$$\|vy - yv\| \geq \|vy - yv\|_2 > \|vy\| \geq \|vy\|_2$$

and hence

$$\|vx_0 - x_0v\| \geq \|vy - yv\| - 2\varepsilon > \|y\| - 2\varepsilon \geq \|x_0\| - 3\varepsilon > 0,$$

which is a contradiction. This finishes the proof of Claim 3.

Claim 4. For every nontrivial $g \in G$, $\beta_g \overset{\text{def}}{=} A(u_g)$ is an outer automorphism of $D$, $g \mapsto \beta_g$ is a group homomorphism and $A$ is isomorphic to the reduced crossed product $D \rtimes_{\beta, r} G$. 

Proof. Clearly, \( \text{Ad}(u_g) \) is an automorphism of \( D \), for every \( g \in G \), and \( g \mapsto \beta_g \) is a group homomorphism. From the density of (2) in \( A \) and the fact that \( u_g B = B u_g \), we see that \( \text{span} \left( \bigcup_{g \in G} D u_g \right) \) is dense in \( A \). Moreover, whenever \( g' \in G \) is nontrivial, \( D u_{g'} \subseteq \ker \phi \); this can be seen by approximating an arbitrary element of \( D u_{g'} \) by sums of words each belonging to \( \{u_g'\} \cup \Lambda^0(B^g,(u_g^* A_g^2 u_g)_{g \in G}) u_{g'} \). As the GNS representation of \( \phi \) is faithful on \( A \), one sees that \( A \) is isomorphic to the reduced crossed product \( D \rtimes_{\beta,v} G \).

We will now show that \( \beta_g \) is an outer automorphism of \( D \) whenever \( g \neq e \). Indeed, if it were inner then letting \( v_g \in D \) be such that \( \beta_g = \text{Ad}(v_g) \), we would have \( u_g^* v_g \) commuting with \( D \). By Claim 3, this would imply that \( u_g \) is a scalar multiple of \( v_g \), hence belongs to \( D \), which contradicts that \( D u_g \subseteq \ker \phi \). This finishes the proof of Claim 4.

Now that \( A \) is seen to be the crossed product of a simple, purely infinite \( C^* \)-algebra by an infinite discrete group acting by outer automorphisms, Kishimoto and Kumjian’s result [10, Lemma 10] shows that \( A \) is simple and purely infinite. □

References

1. D. Avitzour, Free products of \( C^* \)-algebras, Trans. Amer. Math. Soc. 271 (1982), 423-465. [MR 83h:46070]
2. H. Bercovici, D. Voiculescu, Superconvergence to the free central limit theorem and failure of Cramér theorem for free random variables, Prob. Theory Relat. Fields 102 (1995), 215-222. [MR 96k:60115]
3. J. Cuntz, \( K \)-theory for certain \( C^* \)-algebras, Ann. of Math. 113 (1981), 181-197. [MR 84c:46058]
4. K.J. Dykema, Faithfulness of free product states, J. Funct. Anal. 154 (1998), 223-229. [MR 99e:46066]
5. , Purely infinite simple \( C^* \)-algebras arising from free product constructions, II, Math. Scand. (to appear).
6. K.J. Dykema, U. Haagerup, M. Rørdam, The stable rank of some free product \( C^* \)-algebras, Duke Math. J. 90 (1997), 95-121; correction, vol. 94, 1998, p. 213. [MR 99g:46077a]
7. K.J. Dykema, M. Rørdam, Purely infinite simple \( C^* \)-algebras arising from free product constructions, Can. J. Math. 50 (1998), 323-341. [MR 99d:46074]
8. , Projections in free product \( C^* \)-algebras, Geom. Funct. Anal. 8 (1998), 1-16. [MR 99d:46075]
9. , Projections in free product \( C^* \)-algebras, II, Math. Z. (to appear).
10. A. Kishimoto, A. Kumjian, Crossed products of Cuntz algebras by quasi-free automorphisms, Fields Inst. Commun. 13 (1997), 173-192. [MR 98h:46076]
11. D. Voiculescu, Symmetries of some reduced free product \( C^* \)-algebras, Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Mathematics, vol. 1132, Springer–Verlag, 1985, pp. 556–588. [MR 87d:46079]
12. D. Voiculescu, K.J. Dykema, A. Nica, Free Random Variables, CRM Monograph Series vol. 1, American Mathematical Society, 1992. [MR 94c:46133]

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