An introduction to the covariant quantization of superstrings

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Abstract

We give an introduction to a new approach to the covariant quantization of superstrings. After a brief review of the classical Green–Schwarz superstring and Berkovits’ approach to its quantization based on pure spinors, we discuss our covariant formulation without pure spinor constraints. We discuss the relation between the concept of grading, which we introduced to define vertex operators, and homological perturbation theory, and we compare our work with recent work by others. In the appendices, we include some background material for the Green–Schwarz and Berkovits formulations, in order that this presentation be self-contained.

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1. Introduction

String theory is mostly based on the Ramond–Neveu–Schwarz (RNS) formulation, with worldsheet fermions \(\psi^m\) in the vector representation of the spacetime Lorentz group \(SO(9, 1)\). This formulation exhibits classically an \(N = 1\) local supersymmetry of the worldsheet. The BRST symmetry of the RNS formulation is based on the super-reparametrization invariance of the worldsheet. The fundamental fields are the bosons \(x^m\), the fermions \(\psi^m\), the reparametrization ghosts \(b_\tau, c^\tau\) and the superghosts \(\beta_\chi, \gamma\). Physical states correspond to vertex operators which (i) belong to the BRST cohomology and (ii) are annihilated by the zero mode \(b_0\) of the antighost for the open string, or by \(b_0\) and \(\tilde{b}_0\) for the closed string. To obtain a set of physical states which form a representation of spacetime supersymmetry, the GSO projection is applied to remove half of the physical states. Spacetime supersymmetry is thus not manifest, and the study of Ramond–Ramond backgrounds is not feasible. Therefore, one would prefer a formulation with spacetime fermions \(\theta^\mu\) belonging to a representation of \(Spin(9, 1)\) because it would keep spacetime supersymmetry (susy) manifest.
At the classical level, such a formulation was constructed by Green and Schwarz in 1984 [1]. Their classical action contains two fermions $\theta^\alpha (i = 1, 2)$ and the bosonic coordinates $x^m$. Each of the $\theta$ is real and can be chiral or anti-chiral (type IIA/B superstrings): they are 16-component Majorana–Weyl spinors which are spacetime spinors and worldsheet scalars. We denote chiral spinors by contravariant indices $\theta^\alpha$ with $\alpha = 1, \ldots, 16$; antichiral spinors are denoted by $b_\alpha$, also with $\alpha = 1, \ldots, 16$. We shall only consider chiral $\theta$ below.

The rigid spacetime supersymmetry is given by the usual nonlinear coordinate representation

$$\delta_{\epsilon_i} \theta^\alpha = \epsilon^\alpha, \quad \delta_{\epsilon_i} x^m = \epsilon^m \Gamma^m \theta^\alpha = i\epsilon^\alpha \theta^\alpha \gamma^m, \quad (1)$$

where $\gamma^m$ are ten real symmetric $16 \times 16$ matrices and the flavour indices $i = 1, 2$ are summed over. (In appendix A, Dirac matrices and Majorana–Weyl spinors are reviewed.)

Susy-invariant building blocks are

$$\Pi^\mu_1 \equiv \partial_\mu x^m - i\theta^i \gamma^m \partial_\mu \theta^i, \quad \partial_\mu \theta^\alpha \quad (2)$$

where $\mu = 0, 1$ and $\partial_0 = \partial_t$ and $\partial_1 = \partial_\sigma$. A natural choice for the action on a flat background spacetime and curved worldsheet would seem to be

$$L_1 = -\frac{1}{2\pi} \sqrt{-h^{\mu\nu}} \eta_{\mu\nu} \Pi^\mu_1 \Pi^n_1, \quad (3)$$

with $h^{\mu\nu}$ the worldsheet metric, because it is the susy-invariant line element (a natural generalization of the action for the bosonic string). However, it yields no kinetic term for the fermions. Even if one could produce a kinetic term, there would still be the problem that one would have $\frac{1}{2}(16 + 16) = 16$ fermionic propagating modes and eight bosonic propagating modes. Such a theory could not yield a linear representation of supersymmetry.

A resolution of this problem became possible when Siegel found a new local fermionic symmetry ($\kappa$-symmetry) for the point particle [2]. Green and Schwarz tried to find this symmetry in their string, and they discovered that it is present, but only after adding a Wess–Zumino–Novikov–Witten term to the action. Using this symmetry one could impose the gauge $\Gamma^\mu \theta^1 = \Gamma^\mu \theta^2 = 0$ (where $\Gamma^\mu = \Gamma^0 \pm \Gamma^5$), and if one then also fixed the local scale and general coordinate symmetry by $h^{\mu\nu} = \eta^{\mu\nu}$, and the remaining conformal symmetry by $x^\tau(\sigma, t) = x_0^\tau + p^\tau t$, the action became a free string theory with eight fermionic degrees of freedom and eight bosonic degrees of freedom. Susy was linearly realized and quantization posed no problem.

However, in this combined $\kappa$-light cone gauge, manifest $SO(9, 1)$ Lorentz invariance is lost, and with it all the reasons for studying the superstring in the first place. (We shall call the string of Green and Schwarz the superstring, to distinguish it from the RNS string which we call the spinning string.)

Going back to the original classical action, it was soon realized that second-class constraints were present, due to the definition of the conjugate momenta of the $\theta$. These second-class constraints could be handled by decomposing them w.r.t. a non-compact $SU(5)$ subgroup of $SO(9, 1)$ (see appendix D), but then again manifest Lorentz invariance was lost. An approach to quantization which could deal with second-class constraints and keep covariance was needed. By using a proposal of Faddeev and Fradkin, one could turn second-class constraints into first-class constraints by adding further fields, but upon quantization one now obtained an infinite set of ghosts-for-ghosts, and problems with the calculation of anomalies were encountered. At the end of the 1980s, several authors tried different approaches, but they always encountered infinite sets of ghosts-for-ghosts, and 15 years of pain followed [3].

A few years ago Berkovits developed a new line of thought [4]. Taking a flat background and a flat worldsheet metric, the central charge $c$ in one sector of ten free bosons $x^m$ and one

3 One $\theta$ is always left-moving and the other $\theta$ is always right-moving, whether or not they have the same chirality.
\( \theta \) is \( c = 10 - 2 \times 16 = -22 \) (there is a conjugate momentum \( p_{\theta} \) for \( \theta^\alpha \)). He noted that if one decomposes a chiral spinor \( \lambda^\alpha \) under the non-compact \( SU(5) \) subgroup of \( SO(9,1) \), it decomposes as \( 16 \to 10 + \overline{5} + 1 \) (see appendix D). Imposing the constraint

\[
\lambda^T \gamma^m \lambda = 0,
\]

also known as pure spinor constraint, one can express the \( 5^* \) in terms of the \( 10 \) and \( 1 \) and hence it seemed that by adding a commuting pure spinor (with conjugate momenta for the \( 10 \) and \( 1 \)), one could obtain vanishing central charge: \( c = 10, - 2 \times 16, p_\theta + 2 \times (10 + 1) \lambda, p_\lambda = 0 \). In the past few years, he has developed this approach further.

Having a constraint such as (4) in a theory leads to problems at the quantum level in the computation of loop corrections and in the definition of the path integral. A similar situation occurred in superspace formulations of supergravity, where one must impose constraints on the supertorsions; in that case the constraints were solved and the covariance was sacrificed. One could work only with \( 10 \) and \( 1 \), but then one would again violate manifest Lorentz invariance.

We have developed an approach [5–9] which starts with the same \( \theta^\alpha, p_{\theta} \), and \( \lambda^\alpha \) as used by Berkovits, but we relax the constraint (4) by adding new ghosts. In Berkovits’ and our approach one has the BRST law \( s \theta^\alpha = i \lambda^\alpha \), with real \( \theta^\alpha \), but in Berkovits’ approach \( \lambda^\alpha \) must be complex in order that (4) have a solution at all, whereas in our approach \( \lambda^\alpha \) is real. The law \( s \theta^\alpha = i \lambda^\alpha \) is an enormous simplification over the law one would obtain from the \( \kappa \)-symmetry law \( \delta \theta^\alpha = \Pi^\alpha_\mu (\gamma_\mu \kappa^\mu)^\alpha \) with selfdual \( \kappa^\mu_\nu \). It is this simpler starting point that avoids the infinite set of ghosts-for-ghosts. Perhaps our finite set of fields corresponds to resummations of the infinite set of fields encountered in previous approaches. First, we give a brief review of the classical superstring action from which we shall only extract a set of first-class constraints \( dz^\alpha \). These first-class constraints are removed from the action and used to construct a BRST charge.

We deduce the full theory by requiring nilpotency of the BRST charge: each time nilpotency on a given field does not hold we add a new field (ghost) and define its BRST transformation rule such that nilpotency holds. \textit{A priori}, one might expect that one would end up again with an infinite set of ghosts-for-ghosts, but to our happy surprise the iteration procedure stops after a finite number of steps.

In some modern approaches the difference between the action and the BRST charge becomes less clear (in the BV formalism the action is even equal to the BRST charge, and in string field theory the action is expressed in terms of the BRST charge). So the transplantation of the first-class constraints from the action to the BRST charge may not be as drastic as it may sound at first. We may in this way create a different off-shell formulation of the same physical theory. The great advantage of this procedure is that one is left with a free action, so that propagators become very easy to write down, and OPEs among vertex operators become as easy as in the RNS approach.

We shall now present our approach. We have a new definition of physical states, and we obtain the correct spectrum for the open string as well as for the closed superstring, both at the massless level and at the massive levels. Since these notes are intended as introduction to our work, we give much background material in the appendices. Such material is not present in our papers, but may help to understand the reasons and the technical aspects of our approach. Our approach differs from the conventional approaches to the BRST quantization of strings. One would therefore like to see it work in a simpler example. For that reason we have applied our ideas to pure Yang–Mills field theory [9]. One gets also in that case the correct cohomology.
We have found since the conference some deep geometrical meanings of the new ghosts, but we have not yet found the underlying classical action to which our quantum theory corresponds. Sorokin, Tonin and collaborators have recently shown [10] how one can obtain Berkovits’ theory from an \( N = (2, 0) \) worldsheet action with superdiffeomorphism embeddings, and it is possible that a similar approach yields our theory.

2. The classical Green–Schwarz action

As we already mentioned, a natural generalization of the bosonic string with \( \mathcal{L} \sim (\partial_{\alpha}x^{m})^{2} \) with spacetime supersymmetry is the supersymmetric line element given in (2) and (3). If one considers the interaction term \( \partial_{\alpha}x^{m}(\theta_{\gamma}x^{n}\partial^{n}\theta) \) and if one chooses the light-cone gauge \( x^{+} = x^{0} + p^{+}t \) one obtains a term \( p^{+}\theta_{\gamma}x^{n}\partial_{\alpha}\theta = (\sqrt{p^{+}\theta})\gamma_{n}\partial_{\alpha}(\sqrt{p^{+}\theta}) \). This is not a satisfactory kinetic term because we also would need a term with \( p^{+}\theta_{\gamma}x^{n}\partial_{\alpha}\theta \). Such a term would be obtained if the action contains a term of the form \( (\partial_{\sigma}x^{+})\theta_{\gamma}x^{n}\partial^{n}\theta \). The extra kinetic term \( \epsilon^{\mu\nu}(\partial_{\mu}x^{m})\theta_{\gamma}x^{n}\partial^{n}\theta \) is part of a Wess–Zumino term (see appendix B).

Rigid susy (1) and \( \delta_{\epsilon}(\partial_{\sigma}x^{+}) = 0 \) would lead to \( \epsilon p^{+}\partial_{\alpha}\theta = 0 \). This suggests that the light-cone gauge for \( \theta \) should read \( \gamma^{+}\theta = 0 \), or, in terms of \( 32 \times 32 \) matrices, \( \Gamma^{+}\theta = 0 \). If \( \Gamma^{+}\theta = 0 \), also \( \theta^{I}C^{+I} = 0 \), and inserting \( [\Gamma^{+}, \Gamma^{-}] = 1 \), one would also find that \( \theta^{I}C^{+I}\partial_{\alpha}\theta = 0 \) for \( I = 1, \ldots, 8 \). So, then we would find in the light-cone gauge that the action for \( \theta \) becomes a free action, a good starting point for string theory at the quantum level.

In order that these steps are correct, we would need a local fermionic symmetry which would justify the gauge \( \Gamma^{+}\theta = 0 \). Pursuing this line of thought, one then arrives at the crucial question: does the sum of the supersymmetric line element and the WZNW term contain a new fermionic symmetry with half as many parameters as there are \( \theta \) components? The answer is affirmative, and the \( \kappa \)-symmetry is briefly discussed at the end of appendix B, but since we shall not need the explicit form of the \( \kappa \)-symmetry transformation laws, we do not give them.

The superstring action is very complicated already in a flat background. We extract from it a set of first-class constraints \( d_{\alpha} = 0 \), from which we build the BRST charge, and at all stages we work with a free action. The precise way to obtain \( d_{\alpha} \) from the classical superstring action is discussed in appendix C.

3. Determining the theory from the nilpotency of the BRST charge

We now start our programme of determining the theory (the BRST charge and the ghost content) by requiring nilpotency of the BRST transformations. We consider one \( \theta \) for simplicity (we have also extended our work to two \( \theta \) [8]). We shall be careful (for once) with aspects such as reality and normalizations. The BRST transformations preserve reality and are generated by \( \Lambda_{Q} \) where \( \Lambda \) is imaginary and anticommuting. It then follows that \( Q \) should also be anti-Hermitian in order that \( \Lambda Q \) be anti-Hermitian. For any field, we define the \( s \) transformations as BRST transformations without \( \Lambda \), so \( \delta_{\Phi} = [\Lambda Q, \Phi] \) and \( s\Phi = [Q, \Phi]_{\Lambda} \). The \( s \)-transformations have reality properties which follow from the BRST transformations (which preserve reality).

We begin with

\[
Q = \int i\lambda^{\alpha}d_{\alpha},
\]

where \( d_{\alpha} \) is given in appendix C and \( f = \frac{1}{2\pi i f} d\bar{z} \). This \( Q \) is indeed anti-Hermitian because \( d_{\alpha} \) is anti-Hermitian. (We have performed a Wick rotation in appendix C, in order to use
establish that the anomaly with a Hermitian cz

Since Spacetime susy requires that the OPE of Q is evaluated using \( \delta \lambda^a(z) \delta x^a(w) \sim -\eta^a(z-w)\), and since we work with a free action, \( \bar{\partial}d_{ca} = 0 \) so that in flat space \( \lambda^a d_{ca} \) is a holomorphic current, namely \( \bar{\partial}(\lambda^a d_{ca}) = 0 \).

The field \( d_{ca} \) contains a term \( p_{ca} \), where \( p_{ca} \) is the momentum conjugate to \( \partial w \) and it is anti-Hermitian since \( p_{ca} \) is anti-Hermitian as can be seen from the action \( \int d^2z p_{ca} \bar{\partial} \theta \). The factor \( \frac{1}{2} \) in \( d_{ca} \) in equation (39) can be checked by noting that the OPE\(^4 \) of \( d_{ca} \) with \( d_{\beta} \) is proportional to \( \Pi^m \). The expression for \( \Pi^m \) is real and fixed by spacetime susy.

The operators \( d_{ca} \) generate a closed algebra of currents with a central charge

\[
d_{\alpha}(z) d_{\beta}(w) \sim \frac{2i}{z-w} \gamma^{a}_{\alpha\beta} \Pi_m(w), \quad d_{\alpha}(z) \Pi^m(w) \sim -\frac{2i}{z-w} \gamma^{m}_{\alpha\beta} d_{\beta}(w),
\]

\[
\Pi^m(z) \Pi^m(w) \sim \frac{1}{(z-w)^2} \eta^{mm}, \quad d_{\alpha}(z) \bar{\theta}(w) \sim \frac{1}{z-w} \delta^\alpha_\mu.
\]

Acting with (5) on \( \theta^a \), one obtains \( s \theta^a = i\lambda^a \), and acting on \( \lambda^a \) yields \( s \lambda^a = 0 \). Nilpotency on \( \theta^a \) and \( \lambda^a \) is achieved. Repeating this procedure on \( x^m \) gives \( s x^m = \lambda \gamma^m \), but since \( x^m = i\gamma^m \lambda \) does not vanish, we introduce a new real anticommuting ghost \( \xi^m \) by setting \( s x^m = \lambda \gamma^m + \xi^m \) and choosing the BRST transformation law of \( \xi^m \) such that the nilpotency on \( x^m \) is obtained. This leads to \( i\xi^m = -i\lambda \gamma^m \). Nilpotency on \( x^m \) is now achieved, but \( s \) has acquired an extra term \( Q' = -\frac{i}{2} \lambda^m \Pi_m \) where we recall \( \Pi_m = \partial_z x^m - i\theta \gamma^m \partial_z \theta \). Nilpotency on \( p_{ca} \), or equivalently on \( d_{ca} \), is obtained by further modifying the sum of \( Qd_{ca} = -2i\Pi^m(z\lambda) \) and \( Q' d_{ca} = -2i\xi^m(z\gamma_m \partial_z \theta) \) by adding \( Q'' d_{ca} = (\partial_z \chi) \) and fixing the BRST law of \( \chi \) such that the nilpotency on \( d_{ca} \) is achieved.\(^6 \) This yields \( Q\chi = 2\xi^m(z\gamma_m \lambda) \) and \( Q'\chi = 0 \) due to a Fierz rearrangement involving three chiral spinors (see equation (26)). At this point we have achieved nilpotency on \( \theta^a, x^m, d_{ca} \) and \( \lambda^a, \xi^m, \chi \) and find that \( s\Phi = [Q, \Phi] \) with

\[
Q = \int \left[ i\lambda^a d_{ca} - \xi^m \Pi_m - \chi_i \partial_i \theta^a - 2\xi^m (\kappa^i \gamma_m \lambda) - i\beta_i \gamma^m \lambda \right]
\]

reproduces all BRST laws for all fields introduced so far except for the three antighosts.

Unfortunately, the BRST charge (7) fails to be nilpotent and therefore the concept of BRST cohomology is at this point meaningless. In order to repair this problem, we could proceed in two different ways: (i) either continuing with our programme of requiring nilpotency on each field separately (continuing with the antighosts \( p_{ca}, \kappa^a, \omega_{ca} \)); or (ii) terminating this process by hand in one stroke by adding a ghost pair \( (b, c_i) \) as we now explain. We begin with

\[
\{Q, Q\} = \int A_z, \quad A_z = \xi m \partial_z \xi^m + i\lambda^a \partial_z \chi_a - i\chi_a \partial_z \lambda^a.
\]

The non-closure term \( A_z \) is due to the double poles in (6). By direct computation we establish that the anomaly \( \int A_z \) is BRST invariant, as it should be according to consistency, \( \{Q, A_z\} = \partial_z Y \) where \( Y = i\xi^m \lambda \gamma^m \lambda \). If we define

\[
Q' = Q + \int \left( \frac{1}{2} c_i - b B_i \right),
\]

with a Hermitian \( c_i \) and an anti-Hermitian \( b \), we find that

\[
\{Q', Q'\} = \int \left( (A_z - B_i) + b [Q, B_i] \right).
\]

\(^4 \) The OPE of \( d_{ca} \) with \( d_{\beta} \) is evaluated using \( \delta \lambda^a(z) \delta x^a(w) \sim -\eta^a(z-w) \), and \( p_{ca}(z) \theta(w) \sim \delta^a(z-w) \).

\(^5 \) Spacetime susy requires that \( Q' \) depends on \( \Pi^m \) instead of, for example, \( \partial w \).

\(^6 \) Since \( Q + Q'^2 \) is \( d_{ca} = \delta_i (-2\xi^m \gamma_m \lambda) \), we add a term \( \partial_z \chi_a \) instead of a field \( X_{ca} \).
and, requiring that $Q'$ be nilpotent, a solution for $B_z$ is obtained by imposing\(^7\)
\[
[Q, B_z] = 0, \quad B_z = A_z + \partial_z X, \quad [Q, X] = -Y
\]
which is satisfied by $X = -\frac{i}{2} \chi_{\alpha} \lambda^\alpha$. Then one gets\(^8\)
\[
B_z = \xi_m \partial_z \xi^m + \frac{i}{2} \lambda^\alpha \partial_z \chi_{\alpha} - \frac{3i}{2} \chi_{\alpha} \partial_z \lambda^\alpha. \tag{12}
\]

However, any $Q'$ of the form $\int c_z + 'more'$ can be always brought in the form $\int c_z$ by a similarity transformation, namely as follows
\[
Q' = \left[ e^{i(R_z - b S_z - b \partial_z b T)} \int c_z e^{i(R_z + b S_z + b \partial_z b T)} \right]
= \int (c_z + S_z - b \partial_z T) + \frac{1}{2} \left[ \int (S_z - b \partial_z T), U \right] + \frac{1}{6} \left[ \int (S_z - b \partial_z T), U \right], \tag{13}
\]
where $U = \int (R_z + b S_z + b \partial_z b T)$. The $R_z$, $S_z$ and $T$ are Hermitian polynomials in all fields except $c_z$, $b$ with ghost numbers 0, 1, 2, respectively. The solution in (9) and (12) corresponds to a particular choice of $R_z$, $S_z$ and $T$,\(^9\) but any other choice also yields a nilpotent BRST charge.

There is now a problem: the operator $Q' = e^{-U} \int c_z e^U$ has trivial cohomology in the space of local vertex operators, because any $O(w)$ satisfying $\int c_z O(w) = 0$ can always be written as $O(w) = \int c_z G(w)$ where $G(w) = b_0 O(w)$. (Note that $O(w)$ cannot depend on $b_0$ because $\int c_z O(w) = 0$, and $c_0 = \int c_z$.)

We shall restrict the space of vertex operators in which $Q$ acts, in order to obtain non-trivial cohomology. We achieve this by introducing a new quantum number, called grading, and requiring that vertex operators have non-negative grading. In the smaller space of non-negative grading (see the following section) the similarity transformation cannot transform each $Q$ into the form $\int c_z$, and we shall indeed obtain non-trivial cohomology, namely the correct cohomology.

We have at this point obtained a new nilpotent BRST charge, and a set of ghost (and antighost) fields (whose geometrical meaning at this point is becoming clear). It is time to revert to the issue of the central charge. Since all fields are free fields, one simply needs to add the central charge of each canonical pair: this yields $c = 20$. So the central charge does not vanish, and to remedy this obstruction, we add by hand a real anticommuting vector pair $(\omega^m, \eta^m)$ which contributes $-2 \times 10$ to $c$. The BRST charge does not contain $\omega^m$ and $\eta^m$, hence $\omega^m$ and $\eta^m$ are BRST inert.

The reader (and the authors) may feel uncomfortable with these rescue missions by hand; a good theory should produce all fields automatically without outside help. Fortunately, we can announce that a more fundamental way of proceeding, by continuing to require nilpotency on the antighosts and then on the new fields which are introduced in this process, produces the pair $(\omega^m, \eta^m)$. We are in the process of writing these considerations, and hopefully also the pair $(b, c_z)$ will be automatically produced in this way.

Our results obtained by elementary methods and ad hoc additions, display nevertheless a few striking regularities, which confirm us in our belief that we are on the right track. For example, the grading which we discuss in the following section is generated by a current
\(^7\) The relation $[Q, X] = -Y$ follows from acting with $Q$ on $B_z = A_z + \partial_z X$.
\(^8\) One can even obtain a nilpotent current: $j' = j + c - b B + \delta(b X)$. Note that $Q$, but not $j$, commutes with $\int b B$.
\(^9\) Namely, $T = \frac{1}{2} X$, $\int S_z = Q$, $R_z = 0$. All the terms displayed in (13) contribute.
An introduction to the covariant quantization of superstrings whose anomaly vanishes. This need not have happened, and provides welcome support for the various steps we have taken, but it hints of course at something more fundamental.

4. The notion of the grading

In our work we define physical states by means of vertex operators which satisfy two conditions

(i) They are in the BRST cohomology
(ii) They should have non-negative grading [6].

The grading is a quantum number which was initially obtained from the algebra of the abstract currents \( d_{\alpha}, \Pi^\alpha \) and \( \partial_\theta \). Assigning grading \(-1\) to \( d_{\alpha} \), we assign grading \(+1\) to the corresponding ghost \( \lambda^\alpha \). We then require that the grading be preserved in the operator product expansion. From \( dd \sim \Pi \) we deduce that \( \Pi^\alpha \) has grading \(-2\), so \( \xi^m \) has grading \(2\). Then \( d\Pi \sim \partial_\theta \) assigns grading \(-3\) to \( \partial_\theta \), and thus grading \(+3\) to \( \chi \). (To avoid confusion note that in some of our published work we use half these gradings.) The grading of the ghosts \( b \) and \( c \) is more subtle, but it can be obtained in the same spirit. From \( d\partial_\theta \sim (z-w)^{-2} \) and \( \Pi \Pi \sim (z-w)^{-2} \) we introduce a central charge generator \( I \) which has grading \(-4\). The corresponding ghost \( \epsilon \) has grading \(4\). All antighosts have opposite grading from the ghosts. The trivial ghost pair \( \omega^m, \eta^m \) has grading \((4, -4)\) because it is part of a quartet of which the grading of the other members is already known [6]. With these grading assignments to the ghost fields, the BRST charge can be decomposed into terms with definite but different gradings. It turns out that all the terms have non-negative grading: \( Q = \sum_{n=0}^{4} Q_n \). This \( Q \) maps the subspace of the Hilbert space with non-negative grading into itself. In [7], the equivalence with Berkovits’ pure spinor formulation has been proved.

According to the grading condition (ii), the most general expression for the massless vertex in the case of open superstring is given by

\[
O = \lambda^\alpha A_\alpha + \xi^m A_m + \chi_\alpha W^\alpha + \omega_\alpha B^m + b\text{-terms},
\]  

where \( A_\alpha, A_m, W^\alpha \) and \( B^m \) are arbitrary superfields, so \( A_\alpha = A_\alpha(x, \theta) \), etc. Requiring non-negative grading, the following combinations:

\[
b\lambda^\alpha \lambda^\beta, \quad b\lambda^\alpha \xi^m,
\]  

are not allowed. Note that the vertex operator does not have a specific grading but contains terms with several (non-negative) gradings.

Finally, requiring BRST invariance of \( O \), one easily derives the equations of motion for \( N = 1 \) SYM in \( D = (9, 1) \). From the \( b\)-terms in \( O \) one only finds that the superfields in these terms are expressed in terms of \( A_\alpha, A_m, W^\alpha \) and \( B^m \). However, in the sectors with \( \lambda^\alpha \lambda^\beta \) and \( \lambda^\alpha \xi^m \) one learns that all remaining superfields appearing in this vertex operator can be expressed in terms of \( A_\alpha(x, \theta) \), for example

\[
A_m = \frac{1}{8} \gamma^m_{\alpha\beta} D_\alpha A_\beta, \quad W^\alpha = \frac{1}{10} \gamma^m_{\alpha\beta} (D_\beta A_m - \partial_m A_\beta).
\]  

The superfield \( A_\alpha \) itself satisfies

\[
\gamma^m_{\alpha\beta} D_\alpha A_\beta = 0,
\]

which contains the linearized Dirac and Yang–Mills equations upon expanding in terms of \( \theta \).

Along the same lines, one can study the closed string or massive vertex operators and one finds the complete correct spectrum of the open or closed superstring. Other interesting cases one might study are the superstring in lower dimensions, or a finitely reducible gauge theory.

The notion that one must restrict the space of the vertex operators is not new by itself: in the spinning (RNS) string, one should restrict the commuting susy ghosts to non-negative
are annihilated by which acts on the fields $\delta$ ghost number; it turns out that the pure ghost number can define a further quantum number by a linear combination of the antifield number and the

10 This operator is known in the literature as the Koszul–Tate resolutor.

5. Grading, reducibility, homological perturbation theory and BRST nilpotency

In the previous section we have introduced a new quantum number for fields, the grading, and a new definition of physical states which required that vertex operators have non-negative grading. The results (the correct physical spectrum) justify to some extent this notion of a grading. We now present a new understanding: the grading number is the pure ghost number (resolution degree) of homological perturbation theory.

According to homological perturbation theory (HPT) [12], once one has an initial BRST-like symmetry so which is nilpotent modulo constraints $G_a$ and gauge transformations (with possibly field-dependent parameter $\epsilon^a$ where the gauge transformations are due to OPEs of the fields with the constraints), we may introduce new fields $P_a$ and a new nilpotent operator $\delta_{-1}$ such that

$$\delta_{-1} P_a = -G_a, \quad \delta_{-1} (\text{other fields}) = 0.$$ (18)

The new fields $P_a$ carry a new quantum number usually called antifield number and the operator $\delta_{-1}$ lowers this number. The solutions of $\delta_{-1} X = 0$, but $X \neq \delta_{-1} Y$ are called homology instead cohomology classes because of this lowering. Next one relaxes the constraints $G_a = 0$ and if there is nontrivial homology, one introduces a new ghost which removes this spurious homology. In our case $G_a = \lambda \gamma^m \lambda$, and there is a new homology, namely $\xi^m \gamma^a (\gamma_0 \lambda)_a$. Indeed $\delta_{-1} (\xi^m \gamma^a (\gamma_0 \lambda)_a) = 0$, but $\xi^m \gamma^a (\gamma_0 \lambda)_a \neq \delta_{-1} X$ because only $\xi^m$ transforms under $\delta_{-1}$. Thus we add a ghost $\chi_a$ and define $\delta_{-1} \chi_a = \xi^m \gamma^a (\gamma_0 \lambda)_a$. Then $\xi^m \gamma^a (\gamma_0 \lambda)_a$ becomes trivial homology. Now we repeat the argument. There is again a new homological class; it is given by $\xi^m \xi^n + \lambda \gamma^{mn} \chi$. Indeed, $\delta_{-1} (\xi^m \xi^n + \lambda \gamma^{mn} \chi) = \lambda \gamma^{m \lambda} \xi^n + \lambda \gamma^{mn} (\xi^m \gamma_0 \lambda)$ and $\lambda \gamma^{mn} \gamma_0 = \delta_{-1} (\lambda \gamma^{mn} \lambda)$. Again, HPT would instruct us to introduce new ghosts $B^{mn}$ and define $\delta_{-1} B^{mn} = \xi^m \xi^n + \lambda \gamma^{mn} \chi$. This is the conventional path. However, we followed another path. Namely, we introduced an antighost $b$ with antifield number $-3$ and this removes the extra homology class $\xi^m \xi^n + \lambda \gamma^{mn} \chi$ because it is now equal to $\delta_{-1} b (\xi^m \xi^n + \lambda \gamma^{mn} \chi)$ if at the same time we add a term $\int c_z$ to $\delta_{-1}$. At this point we have a nilpotent $\delta_{-1}$ without any non-trivial homology classes.

Note that we only introduced $b$ at the level of $\xi^m \xi^n$. We could have introduced $b$ one step earlier, namely when we removed $\xi^m (\gamma_0 \lambda)_a$; in that case we would not have needed a ghost $\chi_a$ (which is however useful for the central charge) and still we would have obtained a nilpotent $\delta_{-1}$ without any non-trivial homology classes. However, we could not have introduced $b$ at the very beginning when we had the non-trivial homology $\lambda \gamma^m \lambda$, because we would have obtained a trivial spectrum of the BRST charge.

It has been proved that one can always add further operators $s_1, s_2, \ldots$ to $\delta_{-1} + s_0$ such that $s \equiv \delta_{-1} + s_0 + s_1 + s_2 + \cdots$ is nilpotent. The form of $s_0$ with $n > 1$ follows from the requirement that $s^n = 0$. The $s_a$ have definite antifield number equal to $n$. In addition, one can define a further quantum number by a linear combination of the antifield number and the ghost number; it turns out that the pure ghost number $n_{\text{pg}}$, defined as the sum of the ghost number $n_\chi$ plus the antifield number $n_{\text{af}}$, coincides with our grading number.

In the superstring case, $s_0$ should be identified with Berkovits’ BRST-like symmetry in (5) which acts on the fields $\Phi = (\theta^a, x^m, d_a, \lambda^a, u_a)$, where $u_a$ is the conjugate momentum.
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of $\lambda^\alpha$. This $s_0$ should be nilpotent up to the pure spinor constraint $\lambda^m\gamma_\lambda = 0$ and up to the gauge transformations $\Delta_\xi \Phi(w) = \oint dz (\epsilon_\alpha \lambda^m \gamma^\lambda_\lambda(z)) \Phi(w)$ (the bracket $[\Phi, G_\alpha]$ in our case is written in terms of the operator product). Indeed,

\begin{align*}
Q^2_B \lambda^m &= -\frac{1}{2} \lambda^m \gamma^\lambda_\lambda, \\
Q^2_B \theta^\alpha &= 0, \\
Q^2_B d_\alpha &= -\frac{1}{2} \theta^2(\lambda^m \gamma^\lambda_\lambda), \\
Q^2_B w_\alpha &= \Pi_m (\gamma^m \gamma^\lambda_\lambda) \omega = \Delta_\Pi w_\alpha. 
\end{align*}

The field $\Phi$ corresponds in our case to $\xi_m$ and $\delta_\omega \Phi = -G_\alpha$ corresponds to $\{Q_0, \xi^m\} = -\frac{1}{2} \lambda^m \gamma^\lambda_\lambda$ where $Q_0 = \oint dz \xi_c^m (z)$ is the grading zero part of the BRST charge. Further, we identify $Q_2 \equiv s_1$, $Q_3 \equiv s_2$ and $Q_4 \equiv s_3$ where $Q_2$, $Q_3$ and $Q_4$ are given in [6].

The fields $\Phi$ have by definition vanishing antifield number. Hence $n_{pg}(\lambda) = n_g(\lambda) + n_{af}(\lambda) = 1 + 0 = 1$ which agrees (up to a factor 2) with our grading. Similarly also for all other fields $n_{pg}$ is equal to twice our grading. Hence, our notion of grading is closely related to the notion of antifield number in homological perturbation theory.

There is however, a difference between our approach and standard homological perturbation theory. In the latter case one has by definition only fields with positive antifield number (contributing to $s_n$ with $n \geq 0$), but in our case we have antighosts in the theory, and if the ghosts have positive antifield number, it is reasonable to assign the opposite (negative) antifield number to the antighosts (in this way the action is neutral). One must introduce a floor from which to work upwards, in a similar way as Dirac introduced the concept of a sea to excluded unbounded negative energy. We have constructed such a floor by hand, by requiring that the vertex operators have a lower bound on their grading; from the previous correspondence it even follows that this lower bound is zero.

We end with some comments on the previous discussion. In any application of HPT one can distinguish the following aspects:

1. the constraints one starts with may be reducible or irreducible. As constraints, following Berkovits, we choose $\lambda^m \gamma^\lambda_\lambda = 0$ because we decompose $(\lambda^m d_\alpha)(z)(\lambda^b d_\beta)(w) \sim (\lambda^m \gamma^\lambda_\lambda \Pi_m)(z-w)$ into constraints $\lambda^m \gamma^\lambda_\lambda$ and generators $\Pi_m$. Our set of constraints is reducible because there exists (field dependent and in general composite) parameters $c_m$ such that $c_m \gamma^m \gamma^\lambda_\lambda = 0$, namely $c_m = (\gamma^m \lambda)$.

2. One either works at the classical level or at the quantum level. We have been working at the quantum level.

3. The algebra of first-class constraints may contain only first-order poles, or also second-order poles in $z-w$. We did encounter second-order poles, but note that they were not due to double contractions, but rather to derivatives of first-order poles.

4. We deviated from the conventional HPT by introducing the antighost $b$. It may be that our pair $b, c_z$ has some relation to Jacobians which arise in the path integral treatment of WZWN models.

A more detailed discussion of the relation between Berkovits’ formalism, our formalism, HPT and equivariant cohomology is in preparation [16].

Acknowledgments

At the July 2002 string workshop in Amsterdam, P Townsend suggested to apply our ideas to a simpler model, and [9] contains the result. E Verlinde suggested not to short-circuit our derivation of the BRST charge by introducing the ghost pair $(c_z, b)$ by hand, but to go on applying our method. This indeed works and the result will be published elsewhere. This work was done in part at the Ecole Normale Superieure at Paris whose support we gratefully acknowledge. In addition, we were partly funded by NSF Grant PHY-0098527.
Appendix A. Majorana and Weyl spinors in $D = (9, 1)$

In $D = (9, 1)$ dimensions, we use ten real $D = (9, 1)$ Dirac matrices $\Gamma^m = \{ I \otimes (i \tau_2), \sigma^\mu \otimes \tau_\mu, \chi \otimes \tau_1 \}$ where $m = 0, \ldots, 9$ and $\mu = 1, \ldots, 8$. The $\sigma^\mu$ are eight real symmetric $16 \times 16$ off-diagonal Dirac matrices for $D = (8, 0)$, while $\chi$ is the real $16 \times 16$ diagonal chirality matrix in $D = 8$.\(^{11}\) So $\chi = \sigma_1, \ldots, \sigma_8, \chi^2 = \chi$ and $\chi^2 = 1$. The chirality matrix in $D = (9, 1)$ is then $I \otimes \tau_1$ and the $D = (9, 1)$ charge conjugation matrix $C$, satisfying $C \Gamma^m = -\Gamma^{m,T} C$, is given by $C = \Gamma^0$. If one uses spinors $\Psi^\tau = (\lambda_L, \zeta_R)$ with spinor indices $\lambda^I_\mu$ and $\zeta_{\mu \beta}$, the index structure of the Dirac matrices, the charge conjugation matrix $C$, and the chirality matrix $\Gamma_\mu \equiv \Gamma^0 \Gamma^1 \ldots \Gamma^9 = I_{16 \times 16} \otimes \tau_3$ is as follows:

$$
\Gamma^m = \begin{pmatrix}
0 & (\sigma^m)^{\alpha \beta} \\
(\bar{\sigma}^m)^{\beta \alpha} & 0
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & c^\beta_\gamma \\
\bar{c}^\beta_\gamma & 0
\end{pmatrix},
$$

where $\sigma^m = \{ I, \sigma^\mu, \chi \}$ and $\bar{\sigma}^m = \{ -I, \sigma^\mu, \chi \}$. The matrices $c^\beta_\gamma$ and $\bar{c}^\beta_\gamma$ are numerically equal to $I_{16 \times 16}$ and $-I_{16 \times 16}$, respectively. Thus the $\lambda^I_\mu$ are chiral and the $\zeta_{\mu \beta}$ are antichiral. This explains the spinorial index structure of the $\Gamma^m$.

In applications we need the matrices $C \Gamma^m$ (for example in (4)). Direct matrix multiplication shows that $C \Gamma^m$ is given by

$$
C \Gamma^m = \begin{pmatrix}
(\bar{\sigma}^m)^{\alpha \beta} & 0 \\
0 & (\sigma^m)^{\alpha \beta}
\end{pmatrix} \equiv \begin{pmatrix}
\gamma^m_{\alpha \beta} & 0 \\
0 & (\gamma^m)^{\alpha \beta}
\end{pmatrix},
$$

using

$$
\Gamma^m,\gamma = \begin{pmatrix}
0 & (\sigma^m,\gamma)^{\alpha \beta} \\
(\bar{\sigma}^m,\gamma)^{\beta \alpha} & 0
\end{pmatrix}. \quad (23)
$$

We only use the real $16 \times 16$ symmetric matrices $\gamma^m_{\alpha \beta} = \bar{\sigma}^m_{\alpha \beta}$ and $\gamma^{\alpha \beta}_m = -\sigma^m_{\alpha \beta}$ in the text, and we omit the dots for reasons we now explain.

The Lorentz generators are given by

$$
L^{mn} = \frac{1}{2} (\Gamma^m \Gamma^n - \Gamma^n \Gamma^m) = \begin{pmatrix}
\frac{1}{2} \sigma^m_{\alpha \beta} \bar{\sigma}^n_{\gamma \delta} - m \leftrightarrow n & 0 \\
0 & \frac{1}{2} \bar{\sigma}^m_{\alpha \beta} \sigma^n_{\gamma \delta} - m \leftrightarrow n
\end{pmatrix}. \quad (24)
$$

Hence the chiral spinors $\lambda^I_\mu$ and the antichiral $\zeta_{\mu \beta}$ form separate representations of $SO(9, 1)$. These representations are inequivalent because $\sigma^m$ and $\bar{\sigma}^m$ are equal except for $m = 0$ where $\sigma^0 = I$ but $\bar{\sigma}^0 = -I$, and there is no matrix $S$ satisfying $S \sigma^\mu = -\sigma^\mu S$ and $S \chi = -\chi S$. (From $S \sigma^\mu = -\sigma^\mu S$ it follows that $S \chi = +\chi S$.) We denote these real inequivalent representations by $16$ and $\overline{16}$, respectively.

In $D = (9, 1)$ dimensions one cannot raise or lower spinor indices with the charge conjugation matrix, because $C$ is off-diagonal. In $D = (3, 1)$, on the other hand, $C$ is diagonal and is given by $C = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$, and therefore one can raise and lower the indices with the

\(^{11}\) The eight real $16 \times 16$ matrices of $D = (8, 0)$ can be obtained from a set of seven purely imaginary $8 \times 8$ matrices $\lambda^I_\mu$ for $D = (7, 0)$ as follows $\sigma^\mu = \{ \lambda^1 \otimes \sigma_2, I_{8 \times 8} \otimes \sigma_1 \}$. The seven $8 \times 8$ matrices $\lambda^I_\mu$ themselves can be obtained from the representation $\gamma^4 = \sigma^8 \otimes t^2$, $\gamma^4 = 1 \otimes t^1$, and $\gamma^4 = 1 \otimes t^4$ for, $D = (3, 1)$ with real symmetric matrices $\gamma^4, \gamma^5, \gamma^6$ and imaginary antisymmetric $\gamma^1, \gamma^2$ as follows:

$$
\lambda^1 = \{ \gamma^2 \otimes \sigma_2, \gamma^5 \otimes \sigma_2, \gamma^5 \otimes \sigma_2, \gamma^1 \otimes 1, \gamma^3 \otimes 1, i \gamma^2 \gamma^5 \otimes \sigma_1, i \gamma^2 \gamma^5 \otimes \sigma_3 \}. \quad (22)
$$
charge conjugation matrices $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$. For that reason one has in $D = (3, 1)$ two independent representations: $\lambda^\alpha \sim \lambda_\alpha$ and $\chi^\dot{\alpha} \sim \chi_{\dot{\alpha}}$.

In $D = (9, 1)$ dimensions, one can also define spinors $\kappa_\alpha$ and $\eta^\alpha$ which transform under Lorentz transformations such that $\kappa_\alpha \lambda^\alpha$ and $\eta^\alpha \chi^\dot{\alpha}$ are invariant. If we denote the generators of $\lambda^\alpha$ by $(\gamma^k, \gamma^k)$ with $k, l = 1, \ldots, 9$, those for $\kappa_\alpha$ are given by $(-\gamma^k, \gamma^k)$). Of course these matrices form also a representation of the Lorentz group, but they are not inequivalent representations. It is easy to check that in the representation given above, the Lorentz generators for the spinors $\lambda^\alpha$, $\chi_\alpha$, $\kappa_\alpha$, and $\eta^\alpha$ are given, respectively, by

$$
\begin{align*}
(\gamma^k, \gamma^k), & \quad (\gamma^k, -\gamma^k), & \quad (\gamma^k, -\gamma^k), & \quad (\gamma^k, \gamma^k).
\end{align*}
$$

Thus in $D = (9, 1)$ dimensions $\kappa_\alpha$ transforms like $\chi_\alpha$, and $\eta^\alpha$ like $\lambda^\alpha$. Hence, one may omit the dots without causing confusion, but it matters whether one has upper or lower indices. For $D = (3, 1)$ dimensions one has just the opposite situation: the representation to which $\lambda^\alpha$ and $\kappa_\alpha$ belong is inequivalent to the representation to which $\chi_\alpha$ and $\eta^\alpha$ belong.

We conclude that chiral spinors are given by $\lambda^\alpha$, antichiral spinors by $\chi_\alpha$, and in the text we use the 20 real symmetric $16 \times 16$ matrices $\gamma_\mu^{\alpha\beta}$ and $\gamma^{\alpha\beta}$ (omitting again the dots in the latter). The matrices $\gamma_\mu^\alpha$ satisfy $\gamma_\mu^\alpha \gamma_\nu^{\alpha\beta} + \gamma_\nu^{\alpha\beta} \gamma_\mu^{\beta\alpha} = 2\eta^{\mu\nu} \delta^\alpha_\beta$ and $\gamma_\mu^{\alpha\beta} \gamma_\nu^{\beta\alpha} = 0$. The latter relation makes Fierz rearrangements very easy. The usual Fierz rearrangement for three chiral spinors becomes then simply the statement that $\gamma_\mu^{\alpha\beta} \gamma_\nu^{\beta\alpha}$ vanishes when totally symmetrized in the indices $\alpha, \beta$, and $\gamma$. In particular,

$$
(\lambda^\gamma \lambda^\alpha) \gamma^\mu \lambda = 0.
$$

### Appendix B. The WZNW term

We follow [15]. The WZNW term $\mathcal{L}_{WZ}$ is proportional to $\epsilon^{\mu\nu}$ (with $\mu, \nu = 0, 1$) hence $\mathcal{L}_{WZ} d^2x$ can be written as a 2-form

$$
\omega_2 \equiv \mathcal{L}_{WZ} d^2x.
$$

Since $\omega_2$ is susy invariant up to a total derivative, we have

$$
\delta_\chi \omega_2 = dX.
$$

Define now a 3-form $\omega_3$ as follows: $\omega_3 = d\omega_2$. Then clearly,

$$
\delta_\chi \omega_3 = 0, \quad \text{and} \quad d\omega_3 = 0.
$$

From $\delta_\chi \omega_3 = 0$ it is natural to try to construct $\omega_3$ from the susy-invariant 1-forms $\Pi^m = dx^m - i \sum_j \theta^j \gamma^m \theta^j$ and $d\theta^j$. Lorentz invariance then yields only one possibility

$$
\omega_3 = a_{ij} \Pi^m d\theta^i \gamma_m d\theta^j
$$

where $a_{ij}$ is a real symmetric $N \times N$ matrix. We diagonalize $a_{ij}$ by a real orthogonal transformation (which leaves $\Pi^m$, and thus $\mathcal{L}_1$ in (3) invariant). Then $d\omega_3 = i(\sum_i d\theta^i \gamma^m d\theta^i)(\sum_j a_{ij} d\theta^k \gamma_m d\theta^j)$. In $d\omega_3$ the direct terms cancel due to the standard identity $\gamma^m d\theta^i \gamma^1 d\theta^1 \gamma^m \gamma^1 = 0$, while the cross-terms cancel only if $N = 2$ and if the diagonal matrix $a_{ij}$ has entries $(+1, -1)$. Hence

$$
\omega_3 = i\Pi^m (d\theta^1 \gamma_m d\theta^1 - d\theta^2 \gamma^m d\theta^2).
$$

Using that $\omega_1 = d\omega_2$, we find the WZNW term up to an overall constant

$$
\mathcal{L}_{WZ} = \frac{1}{\pi} \epsilon^{\mu\nu}[-i d_{\mu} x^m (\theta^1 \gamma_m \partial^1 \theta^1 - \theta^2 \gamma_m \partial^2 \theta^2) + \theta^1 \gamma_m \partial^1 \theta^1 \gamma^m \partial^2 \theta^2].
$$
Indeed, with $\epsilon_{\mu\nu} d^2 x = dx^\mu dx^\nu$ one gets
\[d(\mathcal{L}_{WZ} d^2 x) \sim -i dx^m (\theta^1 \gamma_m d\theta^1 - d\theta^2 \gamma_m d\theta^2)
+ (\theta^1 \gamma_m d\theta^1 d\theta^2 \gamma_m d\theta^2 - d\theta^1 \gamma_m d\theta^1 \theta^2 \gamma_m d\theta^2)\]
which is equal to
\[\omega_1 = -i(dx^m - i\theta^1 \gamma_m d\theta^1 - i\theta^2 \gamma_m d\theta^2)(d\theta^1 \gamma_m d\theta^1 - d\theta^2 \gamma_m d\theta^2).\]

Note that the WZNW term is antisymmetric in $\theta^1$ and $\theta^2$ while $\mathcal{L}_1$ is symmetric. Only the sum of $\mathcal{L}_1$ and $\mathcal{L}_{WZ}$ is $\kappa$-invariant, up to a total derivative. The $\kappa$-transformation rule for $x^m$ is $\delta_\kappa x^m = -i\sum \delta_\kappa \theta^1 \gamma^m \delta_\kappa \theta^1$ with the opposite sign to the susy rule. The expressions for $\delta_\kappa \theta^1$ and $\delta_\kappa \sqrt{-h} h^{\mu\nu}$ are complicated, involving self-dual and antiselfdual anticommuting gauge parameters with three indices, but we do not need them. We begin with the BRST law $s\theta^a = i\lambda^a$ where $\lambda^a$ is an unconstrained ghost field, but the precise classical action to which this corresponds is not known at present. That does not matter as long as we can construct the complete quantum theory, although knowledge of the classical action might clarify the results obtained at the quantum level.

For the open string one has the following boundary conditions at $\sigma = 0, \pi$ [5]:
\[
\theta^{1i} = \theta^{2i}, \quad \epsilon^{1i} = \epsilon^{2i}, \quad h^{\theta \phi} \partial x^m = 0, \quad \kappa_i^1 = \kappa_i^2 = \sqrt{-h} \kappa_i^{1\sigma} = \sqrt{-h} \kappa_i^{2\sigma}.
\]

**Appendix C. A useful identity for the superstring**

The superstring action in the gauge $h^{\mu\nu} = \gamma^{\mu\nu}$ is given by
\[\mathcal{L} = -\frac{1}{2\pi} \eta_{mn} \Pi^m_{\mu} \Pi^{\mu n} + \mathcal{L}_{WZ},\]
\[\mathcal{L}_{WZ} = \frac{1}{2\pi} \epsilon^{\mu\nu} \left[ -i\partial_\mu x^a \partial_\nu \theta^a + \theta^1 \gamma^a \partial_\mu \theta^1 + \theta^2 \gamma^a \partial_\mu \theta^2 \right],\]
where $\Pi^m_{\mu}$ is given in (2). For definiteness we choose $\epsilon^{01} = 1$ and $\eta^{\mu\nu}$ as well as $\eta^{mn}$ have $\eta^{00} = -1$. This action is real.

By just writing out all the terms, the action can be rewritten with chiral derivatives as
\[\mathcal{L} = \frac{1}{2} \eta_{mn} \partial x^m \partial x^n - i\bar{a} \gamma^m \partial_\nu \partial_\mu \theta^a - i\bar{a} \partial x^m \gamma^a \partial_\mu \theta^a - \frac{1}{2} (\theta^1 \gamma^m \partial_\mu \theta^1 + \theta^2 \gamma^m \partial_\mu \theta^2)\]
\[- \frac{1}{2} (\theta^1 \gamma^m \partial_\nu \theta^1 + \theta^2 \gamma^m \partial_\nu \theta^2),\]
with $\partial = \partial_\sigma - \partial_\sigma$ and $\bar{\partial} = \bar{\partial}_\sigma + \partial_\sigma$.

Except for the purely bosonic terms, all terms involve either $\bar{\partial} \theta^1$ or $\bar{\partial} \theta^2$. Hence we can write the action as
\[-\mathcal{L} = \frac{1}{2} \eta_{mn} \partial x^m \partial x^n + (p_{1a})_{\text{sol}} \bar{\partial} \theta^1 + (p_{2a})_{\text{sol}} \bar{\partial} \theta^2\]
where $(p_{\alpha a})_{\text{sol}}$ are complicated composite expressions.

We can then also write the action with independent $p_{ia}$ if we impose the constraint that
\[d_{ia} = p_{ia} - (p_{ia})_{\text{sol}}\] vanishes. The complete expressions for $d_{ia}$ are given by
\[d_{1a} = p_{1a} + i\bar{a} x^m \gamma_m \theta^1 + \frac{1}{2} (\gamma^m \partial_\mu \theta^1)(\theta^1 \gamma_m \partial_\mu \theta^1 + \theta^2 \gamma_m \partial_\mu \theta^2),\]
\[d_{2a} = p_{2a} + i\bar{a} x^m \gamma_m \theta^2 + \frac{1}{2} (\gamma^m \partial_\mu \theta^2)(\theta^1 \gamma_m \partial_\mu \theta^1 + \theta^2 \gamma_m \partial_\mu \theta^2).\]
In the text we work with the free action with independent fields $p_{ia}$. The $d_{ia}$ are transferred to the BRST charge where they are multiplied by the independent unconstrained real chiral commuting spinors $\lambda^a$. To make use of the calculation technique of conformal field theory, we make a Wick rotation $t \rightarrow -i\tau$, $\partial_\sigma \rightarrow +i\partial_\tau$ and $\partial = \partial_\sigma - \partial_\tau \rightarrow \partial_\tau = \partial_\sigma - i\partial_\tau$ and analogously for $\bar{\partial}$. We also restrict ourselves to only one sector with $\theta = \theta^1$ and $d_{ia} = d_{1a}$, by setting $\theta^2 = 0$. For a treatment which describes both sectors, we refer to [8].
Appendix D. Solution of the pure spinor constraints

In this appendix we discuss a solution of the constraint that the chiral spinors $\lambda$ are pure spinors. The equation to be solved reads

$$\lambda^a \gamma^{\alpha \beta} \lambda_b = 0,$$

where $\lambda^a$ are complex chiral (16-component) spinors. We shall decompose $\lambda$ w.r.t. a non-compact version of an SU(5) subgroup of SO(9, 1) as $|\tilde{\lambda}\rangle = \lambda_+ |0\rangle + \frac{1}{5!} \lambda_{\mu \nu \rho \delta}^a a_i^\mu a_j^\nu a_k^\rho a_l^\delta |0\rangle$. This decomposition corresponds to $\mathbf{16} = \mathbf{1} + \mathbf{10} + \mathbf{5}$. Then we shall show that the constraints express the 5 in terms of the $\mathbf{1}$ and $\mathbf{10}$. Hence there are 11 independent complex components in $\lambda$. We shall prove that $\lambda$ is complex and not a Majorana spinor, so $\lambda_D \equiv \lambda^\dagger i \gamma^0$ differs from $\lambda_M \equiv \lambda^T C$. (Recall that a Majorana spinor is defined by the condition $\lambda_D = \lambda_M$.)

The Dirac matrices in $D = (9, 1)$ dimensions satisfy $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$, where $\eta^{mn}$ is diagonal with entries $(-1, +1, \ldots, 1)$ for $m, n = 0, \ldots, 9$. We combine them into five annihilation operators $a_j$ and five creation operators $a^j$ as follows:

$$a_1 = \frac{1}{2} (\Gamma^1 - i \Gamma^2), \quad a_2 = \frac{1}{2} (\Gamma^3 + i \Gamma^4), \quad \ldots, \quad a_5 = \frac{1}{2} (\Gamma^9 - \Gamma^{10}).$$

$$a^1 = \frac{1}{2} (\Gamma^1 + i \Gamma^2), \quad a^2 = \frac{1}{2} (\Gamma^3 - i \Gamma^4), \quad \ldots, \quad a^5 = \frac{1}{2} (\Gamma^9 + \Gamma^{10}).$$

Clearly $\{a_i, a^j\} = \delta^i_j$ for $i, j = 1, \ldots, 5$. We introduce a vacuum $|0\rangle$ with $a_i |0\rangle = 0$. By acting with one or more $a^j$ on $|0\rangle$, we obtain 32 states $|A\rangle$ with $A = 1, \ldots, 32$. Similarly, we introduce a state $|0\rangle$ which satisfies $0a^j = 0$ and we create 32 states $|B\rangle$ by acting with one or more $a_j$ on $|0\rangle$. We choose the states $|B\rangle$ as $|A\rangle$'. For example, if $|A\rangle = a^1 a^2 \ldots a^7 |0\rangle$ then $|A\rangle = |0\rangle a_i \ldots a_5$. Then $|A\rangle |B\rangle = \delta^i_j$.

**Lemma 1.** The matrix elements $\langle B|a^j|A\rangle \equiv (\Gamma^j)^B_A$ and $\langle B|a_i|C\rangle \equiv (\Gamma^i)^B_C$ form a representation of the Clifford algebra.

**Proof.** This follows from $\sum_i |C\rangle \langle C| = I$. Namely, $\sum_i |C\rangle \langle C| = |0\rangle \langle 0| + \sum a^i |0\rangle \langle 0| a_i + \cdots + a^1 a^2 \ldots a^7 |0\rangle \langle 0| a_5 \ldots a_1$, where the sum over $C$ runs over the 32 states shown. For any state $|A\rangle$ one has $|A\rangle = \sum_i |C\rangle \langle C|A\rangle$, because $\langle C|A\rangle = \delta^i_A$ by construction.

**Lemma 2.** The chirality matrix $\Gamma_\sigma = \Gamma^1 \Gamma^2 \ldots \Gamma^9 \Gamma^{10}$ satisfies $\Gamma_\sigma^2 = 1$, and $\Gamma_\sigma^1 = \Gamma_\sigma$. It is given by

$$\Gamma_\sigma = (2a_i a^1 - 1) \ldots (2a_5 a^5 - 1).$$

**Proof.** $\Gamma_\sigma = (a^1 + a_1)(a^1 - a_1) \ldots (a^5 + a_5)(a^5 - a_5)$ and $(a^1 + a_1)(a^1 - a_1) = 2a_1 a^1 - 1$. As a check note that $2a_1 a^1 - 1 = 1$, and that $\{\Gamma_\sigma, a^1\} = 0$ because $\{2a_1 a^1 - 1, a^1\} = 0$. Similarly $\Gamma_\sigma a_1 = 0$. Further, $\Gamma_\sigma |0\rangle = |0\rangle$.

**Lemma 3.** $\langle B|a_i|C\rangle = \langle C|a_j|B\rangle$ is real.

**Proof.** This follows from the fact that one obtains the second matrix element from the first by left–right reflection, and from the fact that the anticommutation relations have the same symmetry and are real: $[a^j, a_i] = [a_i, a^j] = \delta^j_i$.

**Lemma 4.** The matrix representation of $\Gamma^1, \Gamma^3, \Gamma^5, \Gamma^7, \Gamma^9$ is real and symmetric while that of $\Gamma^2, \Gamma^4, \Gamma^6, \Gamma^8$ is purely imaginary and antisymmetric, and that of $\Gamma^0$ is real and antisymmetric.

**Proof.** $\langle A|a^j \pm a_j|B\rangle = \langle B| \pm a^j + a_j|A\rangle$. 


Lemma 5. The charge conjugation matrix $C$, defined by $C \Gamma^m = -\Gamma^{m,\dagger} C$ is given by

$$C = -\Gamma^2 \Gamma^4 \Gamma^6 \Gamma^8 \Gamma^0 = (a_1 - a^7)(a_2 - a^3) \cdots (a_5 - a^5).$$

The minus sign is added for later convenience.

Proof. $\Gamma^1, \Gamma^3, \Gamma^5, \Gamma^7, \Gamma^9$ anticommute with $C$, while $\Gamma^2, \Gamma^4, \Gamma^6, \Gamma^8, \Gamma^0$ commute with $C$, the former are symmetric while the latter are antisymmetric.

\[\square\]

Theorem 1. A chiral spinor $\lambda$ can be expanded as follows:

$$|\lambda\rangle = \lambda_+|0\rangle + \frac{1}{2!} \lambda_{ij} a^i a^j|0\rangle + \frac{1}{4!} \lambda^i \epsilon_{ijklm} a^j a^k a^l a^m|0\rangle.$$  \hspace{1cm} \text{(43)}

Proof. $\Gamma_\mu|0\rangle = |0\rangle$; hence $\Gamma_\mu|\lambda\rangle = |\lambda\rangle$. The 16 non-vanishing components of $|\lambda\rangle$ are the projections of the ket $|\lambda\rangle$ onto the corresponding 16 bras: in particular

$$\lambda_+ = \langle 0|\lambda \rangle = \langle \lambda|0 \rangle,$$

$$\lambda_{ij} = \langle 0|a_ia_j|\lambda \rangle = \langle \lambda|a^i a^j|0 \rangle,$$

$$\lambda^i = \frac{1}{4!} \epsilon^{ijklm} \langle 0|a_ia_k a_l a_m|\lambda \rangle = \frac{1}{4!} \epsilon^{ijklm} \langle \lambda|a^i a^k a^l a^m|0 \rangle.$$  \hspace{1cm} \text{(44)}

We are now ready to solve the ten constraints $\tilde{\lambda}^I \Gamma^{m}_{\alpha \beta} \lambda^\beta = 0$. These relations are equivalent to the five constraints $\tilde{\lambda}^I C a^I \lambda = 0$ and the five other constraints $\tilde{\lambda}^I C a^I \lambda = 0$. They can be rewritten as follows:

$$\langle \lambda|C a^I \lambda \rangle = 0, \hspace{1cm} \langle \lambda|C a_j \lambda \rangle = 0.$$  \hspace{1cm} \text{(45)}

Theorem 2. $\langle A|C|B \rangle \neq 0$ iff $A^I B$ is proportional to precisely $a^I a^2 a^3 a^4 a^5$.

Proof. $a_j C = -C a^j$ and $a^I C = -C a^I$. Further $C|0\rangle = -a^1 a^2 a^3 a^4 a^5|0\rangle$ and $|0\rangle C = |0\rangle a_1 a_2 a_3 a_4 a_5$. Pulling all $a_j$'s in $\langle A|$ to the right of $C$, we obtain, up to an overall sign, $|0\rangle C A^I |0\rangle$ and this is only non-vanishing if all $a^k$ in $A^I B$ match the $a_k$ in $|0\rangle C$. It follows that $|0\rangle C a^1 a^2 a^3 a^4 a^5|0\rangle = 1$.

First set of constraints

$$\langle \lambda|C a^I |\lambda\rangle = \langle 0|C \left( \lambda_+ + \frac{1}{2!} \lambda_{ij} a^i a^j + \frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm} \right) a^I |\lambda\rangle$$

$$= 2 \left( \lambda_+ \lambda^I + \frac{1}{8}\epsilon^{ijklm} \lambda_{jk} \lambda_{lm} \right).$$  \hspace{1cm} \text{(46)}

Second set of constraints

$$\langle \lambda|C a_i |\lambda\rangle = \langle 0|C \left( \lambda_+ + \frac{1}{2!} \lambda_{ij} a^i a^j + \frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm} \right) a_i |\lambda\rangle$$

$$= -2\lambda_{ij} \lambda^i.$$  \hspace{1cm} \text{(47)}

Main result. The solution of the first set of constraints $\lambda_+ \lambda^I + \frac{1}{8}\epsilon^{ijklm} \lambda_{jk} \lambda_{lm} = 0$ is given by

$$\lambda^I = -\frac{1}{8\lambda_+} \epsilon^{ijklm} \lambda_{jk} \lambda_{lm}.$$  \hspace{1cm} \text{(48)}

The solution automatically satisfies the second set of constraints because

$$\lambda^I \lambda_{in} = \epsilon^{ijklm} \lambda_{jk} \lambda_{lm} \lambda_{in} = 0.$$  \hspace{1cm} \text{(49)}

Proof. A totally antisymmetric tensor with six indices in five dimensions vanishes. Hence $\lambda^I \lambda_{in}$ is equal to a sum of five terms, due to exchanging $n$ with $j, k, l, m$ and $i$, respectively.
Interchanging $n$ with $i$ yields minus the original tensor, but also interchanging $n$ with $j, k, l$ and $m$ yields each time minus the original expression. Hence the expression vanishes.

Comment 1. The fact that a pure chiral spinor contains 11 independent complex components leads to a vanishing central charge in Berkovits’ approach with variables $x^n, \theta^a$ and the conjugate momentum $p_n$, and $\lambda^a$ with conjugate momentum $p_{\lambda^a}$: $c = +10 + 2 \times 16 + 2 \times 1 = 0$. In our approach we have 16 independent real components in $\lambda^a$ and 16 conjugate momenta $p_{\lambda^a}$ with $a = 1, \ldots, 16$. Also in our case $c = 0$, but there are more ghosts, and there is nowhere a decomposition w.r.t. a subgroup of $SO(9, 1)$.

Comment 2. In the decomposition in theorem 1, one can choose all $\lambda$ to be real, and $\lambda^i$ to be expressed in terms of $\lambda_+ \lambda_ij$ as in (48). Then $\lambda$ is a real chiral spinor. However, the Dirac matrices are complex, so under a Lorentz transformation $\lambda$ becomes complex in a general Lorentz frame.

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