Multi-valued Connectives for Fuzzy Sets

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Abstract

We present a procedure for the construction of multi-valued t-norms and t-conorms. Our procedure makes use of a pair of single-valued t-norms and the respective dual t-conorms and produces interval-valued t-norms \( \cap \) and t-conorms \( \cup \). In this manner we combine desirable characteristics of different t-norms and t-conorms; if we use the t-norm \( \land \) and t-conorm \( \lor \), then \( (X, \cap, \cup) \) is a superlattice, i.e. the multivalued analog of a lattice.

1 Introduction

The fuzzy literature contains many examples of t-norms, which are a generalization of (classical) set intersection. All of these t-norms are (as far as we know) single-valued. To be precise: given a set (of membership values, truth values etc.) \( X \), a t-norm is a binary function \( T : X \times X \to X \) satisfying certain properties. Hence, given two elements of \( X \), call them \( x, y \), then \( T(x, y) \) is also an element of \( X \). Note that this is also true in the context of interval-valued fuzzy sets, fuzzy sets of type 2 and other variants. For example, a t-norm which operates on interval-valued fuzzy sets combines two intervals to produce one interval. Similar remarks can be made about t-conorms, which are a generalization of (classical) set union. We will refer to both t-norms and t-conorms as connectives.

In this paper we introduce multi-valued connectives. In other words, we are interested in binary functions which map elements of \( X \) to subsets of \( X \). Before formally presenting our results let us briefly discuss the reasons for introducing multi-valued connectives.

Fuzzy theorists have often argued that a major motive behind the theory of fuzzy sets has been the treatment of uncertainty. Many examples appear in the literature; for instance Nguyen [25] mentions classes with vaguely defined boundaries and numbers which are only known to lie within an interval as two examples where fuzzy sets can be fruitfully applied.

The above examples (and many similar ones appearing throughout the literature) involve uncertainty about the degree to which objects belong to sets; on the other hand the manner in which fuzzy sets are combined (e.g. by unions, intersections etc.) does not involve any uncertainty. For example, given two fuzzy sets \( A \) and \( B \) and an element \( x \), the degree to which \( x \) belongs to both \( A \) and \( B \) is given by \( A(x) \land B(x) \); no uncertainty is involved in the application of the \( \land \) connective. A natural extension of the principle of fuzziness is to consider uncertain connectives; the use of multi-valued t-norms and t-conorms is a simple step in this direction.

Hence the plan of this paper is as follows. We work in the context of a deMorgan lattice \( (X, \land, \lor, \neg) \) (where \( \neg \) is negation), hence our results will hold equally for fuzzy and L-fuzzy sets. We introduce multi-valued operations \( \cap : X \times X \to \mathcal{P}(X) \) and \( \cup : X \times X \to \mathcal{P}(X) \) (where \( \mathcal{P}(X) \) is the power set of \( X \)). Then we show that \( \cap \) has properties which are analogous (in the multi-valued context) of the properties usually required of t-norms; similarly \( \cup \) has properties analogous to those usually required of t-conorms. Finally, we show that the structure \( (X, \cap, \cup) \) is the analog (in the multi-valued context) of a lattice.
The last remark requires some additional explanation. Let us first remark that there is an extensive literature in the study of multi-valued algebraic operations (called *hyperoperations*) and the corresponding algebraic structures (*hyperalgebras*). The books [3, 5] present an extensive study of hyperalgebras such as *hypergroups* (the multi-valued analog of group, see also [2, 30] and for fuzzy *hypergroups* [8, 14, 16, 11, 31, 32] and other relations between hypergroups and fuzzy sets [10, 7, 21, 28]), hyperrings (the multi-valued analog of ring, see also [29, 10]), hyperlattices and *superlattices* (the multi-valued analog of lattice, see also [1, 9, 18, 19, 20, 22, 24, 26, 27, 23]) etc. As will be seen in Section 3, our \((X, \sqcap, \sqcup)\) is a *superlattice* [23].

While multi-valued operations have been studied extensively in the hyperalgebraic literature, we believe (as already mentioned) that they have not been previously discussed in the fuzzy literature. However, our approach is quite similar to the one used by Jenei in [13]. Jenei introduces *t-norms* and *t-conorms* for intervals, i.e. his connectives are *single-valued* functions which map pairs of intervals to intervals. Hence these are essentially connectives for interval-valued fuzzy sets; the same idea is discussed in [25] and several other places. However, the actual construction of the interval-valued t-norms and t-conorms is the same as the one used by us (indeed Jenei’s paper has been a major inspiration to us). Jenei argues that his connectives are preferable to classical ones because they combine a large number of desirable properties; this remark also holds for our \(\sqcap\) and \(\sqcup\) and can be considered as an additional reason for their introduction.

2 Preliminaries

We will present our results in the context of L-fuzzy sets, i.e. all the results presented below hold when membership takes values in a lattice (rather than in the unit interval of real numbers). This generality can be obtained at no additional cost, i.e. the proofs of our results are essentially the same for the cases of real numbers and general lattice.

Hence, in what follows we assume the existence of a deMorgan lattice \((X, \wedge, \vee, ')\) (where ‘ denotes *negation*) with a minimum element 0 and a maximum element 1. The order compatible with \(\wedge, \vee\) will be denoted by \(\leq\). Lattice *intervals* are defined in the standard manner: for every \(x, y \in X\) with \(x \leq y\) we define \([x, y] = \{z : x \leq z \leq y\}\). The empty interval is the empty set \(\emptyset\) and can be symbolized as \([x, y]\) for any pair \(x, y\) such that \(x \not\leq y\). The collection of all intervals of \(X\), including the empty interval, will be symbolized by \(I(X)\). We define, in standard manner, an order on \(I(X)\).

**Definition 2.1** For every \([x, y], [u, v] \in I(X)\) we write \([x, y] \leq [u, v]\) iff \(x \leq u\) and \(y \leq v\).

**Proposition 2.2** \(\leq\) is an order on \(I(X)\) and \((I(X), \leq)\) is a lattice where

\[
\inf ([x, y], [u, v]) = [x \wedge y, u \wedge v], \quad \sup ([x, y], [u, v]) = [x \vee y, u \vee v]
\]

for every \([x, y], [u, v] \in I(X)\).

In the lattice context we can define a t-norm \(T\) to be any function \(T : X \times X \to X\) which satisfies the following properties.

**Definition 2.3** A function \(T : X \times X \to X\) is a t-norm if it satisfies the following for every \(x, y, z \in X\).

1. \(T(1, x) = x\).
2. \(T(x, y) = T(y, x)\).
3. \(T(x, T(y, z)) = T(T(x, y), z)\).
4. \( x \leq y \Rightarrow T(x,z) \leq T(y,z) \).

Similarly, a t-conorm \( S \) is any function \( S : X \times X \rightarrow X \) which satisfies the following properties.

**Definition 2.4** A function \( S : X \times X \rightarrow X \) is a t-conorm if it satisfies the following for every \( x, y, z \in X \):

1. \( S(0,x) = x \).
2. \( S(x,y) = S(y,x) \).
3. \( S(x,S(y,z)) = S(S(x,y),z) \).
4. \( x \leq y \Rightarrow S(x,z) \leq S(y,z) \).

**Notation 2.5** We will write \( T(x,y,z) \) for \( T(T(x,y),z) = T(x,T(y,z)) \) and \( S(x,y,z) \) for \( S(S(x,y),z) = S(x,S(y,z)) \) (by associativity).

**Definition 2.6** Given a t-norm \( T \) and a t-conorm \( S \), we say that \( T \) and \( S \) are dual (with respect to the negation \( ' \)) iff \( (T(x,y))' = S(x',y') \).

**Definition 2.7** For every \( [x,y] \in I(X) \), we define \([x,y]' = \{z'\}_{z \in [x,y]} \).

**Remark.** In the sequel we will occasionally make use of certain well-known properties of t-norms and t-conorms which follow from Definitions 2.3 and 2.4. For example, \( T(0,x) = 0, S(1,x) = 1, x \leq y \Rightarrow T(z,x) \leq T(z,y) \), \( x \leq y \Rightarrow S(z,x) \leq S(z,y) \) etc. Also, using Definition 2.7 it is straightforward that \([x,y]' = [y',x'] \). Finally, proofs of the following propositions can be found in [25].

**Proposition 2.8** \( \land \) is a t-norm and \( \lor \) is its dual t-conorm.

**Proposition 2.9** Given a t-norm \( T \) and a t-conorm \( S \), for every \( x, y \in X \) we have: \( T(x,y) \leq x \land y \) and \( x \lor y \leq S(x,y) \).

**Proposition 2.10** For all \( x, y \in X \) we have: \( T(x,y) \leq x, x \leq S(x,y) \).

We now present some material relating to hyperoperations. For more details see [3].

**Definition 2.11** A hyperoperation is a mapping \( * : X \times X \rightarrow P(X) \), where \( P(X) \) is the power-set of \( X \).

**Remark.** In other words, while an operation maps every pair of elements to an element, a hyperoperation maps every pair of elements to a set. The following is a standard notation used in the hyperoperations literature.

**Notation 2.12** If \( * \) is a hyperoperation on \( X \), then for every \( x, y, z \in X \) we define

\[
x * (y * z) = \bigcup_{u \in y * z} x * u, \quad (x * y) * z = \bigcup_{u \in x * y} u * z.
\]

A particular hyperstructure of interest in this paper is the superlattice [12, 23].

**Definition 2.13** Given hyperoperations \( \lor, \land \) on \( (X, \land, \lor) \), we say that \( (X, \lor, \land) \) is a superlattice iff the following properties hold for all \( x, y, z \in X \).
Proposition 3.2
Condition 3.1 is automatically satisfied for every $X$.

In the following $T(x,y)$ will denote an arbitrary t-norm and $S(x,y)$ its dual t-conorm (with respect to some arbitrary negation $x'$). The only condition we impose on $T(x,y)$ and $S(x,y)$ is the following.

Condition 3.1 For all $x,y,z \in X$ we have:

1. $T(x \vee y, z) = T(x,z) \vee T(y,z)$.
2. $T(x \wedge y, z) = T(x,z) \wedge T(y,z)$.
3. $S(x \vee y, z) = S(x,z) \vee S(y,z)$.
4. $S(x \wedge y, z) = S(x,z) \wedge S(y,z)$.

Proposition 3.2 Condition 3.1 is automatically satisfied for every $T,S$ pair if $X$ is the interval $[0,1]$ of real numbers.

Proof. We only prove the first part of Condition 3.1 (the remaining parts are proved similarly). Without loss of generality suppose that $x \leq y$. Then $x \vee y = y$ and so $T(x \vee y, z) = T(y,z)$. But also $x \leq y \Rightarrow T(x,z) \leq T(y,z) \Rightarrow T(x,z) \vee T(y,z) = T(y,z)$. ■

We now define the interval-valued fuzzy connectives $\cap, \cup$.

Definition 3.3 For all $x,y \in X$ we define $x \cap y = [T(x,y), x \wedge y]$, $x \cup y = [x \vee y, S(x,y)]$.

Proposition 3.4 For all $x,y,z \in X$ such that $y \leq z$, we have: $x \cap [y,z] = [T(x,y), x \wedge z]$ and $x \cup [y,z] = [x \vee y, S(x,z)]$.

Proof. Choose any $w \in x \cap [y,z] = \cup_{u \in [y,z]} x \cap u = \cup_{y \leq u \leq z} [T(x,u), x \wedge u]$. Then there exists some $u$ such that: $y \leq u \leq z$ and $T(x,u) \leq w \leq x \wedge u$. It follows that $w \leq x \wedge u \leq x \wedge z$ and and $T(x,y) \leq T(x,u) \leq w$. Hence $w \in [T(x,y), x \wedge z]$ and so

$$x \cap [y,z] \subseteq [T(x,y), x \wedge z]. \quad (1)$$

On the other hand, choose any $w \in [T(x,y), x \wedge z]$ and define

$$u = (y \wedge w) \wedge z = y \wedge (w \wedge z) \quad (2)$$
Now (the second equality in (2) follows from distributivity).

\[
\begin{align*}
  u &= (y \lor w) \land z \leq z \\
  u &= y \lor (w \land z) \geq y.
\end{align*}
\]

Hence

\[u \in [y, z]\] (5)

On the other hand, \(u \land x = (y \lor w) \land z \land x\). But \(w \leq y \lor w\) and \(w \leq z \land x\), hence \(w \leq u \land x\). Also \(T(u, x) = T(y \lor (w \land z), x) = T(y, x) \lor T(w \land z, x)\). But \(T(y, x) \leq w\) and \(T(w \land z, x) \leq T(w, x) \leq w\). Hence \(T(u, x) = T(y \lor (w \land z), x) \leq w\). Hence

\[w \in [T(u, x), u \land x].\] (6)

(5) and (6) imply that \(w \in x \cap [y, z]\) and so

\[T(x, y), x \land z] \subseteq x \cap [y, z];\] (7)

(11) and (7) imply that \([T(x, y), x \land z] = x \cap [y, z]\) and we have proved the first part of the theorem; the second part is proved dually. ■

The following proposition shows that \(\cap, \cup\) have the analogs of t-norm, t-conorm properties (in the context of hyperoperations).

**Proposition 3.5** For all \(x, y, z \in X\) we have:

1. \(x \in 1 \cap x, 0 \in 0 \cap x, x \in 0 \cup x, 1 \in 1 \cup x\).
2. \(x \cap y = y \cap x, x \cup y = y \cup x\).
3. If \(x \leq y\), then \(x \cap z \leq y \cap z\) and \(x \cup z \leq y \cup z\).
4. \((x \cap y) \cap z = x \cap (y \cap z) = [T(x, y, z), x \land y \land z]\) and \((x \cup y) \cup z = x \cup (y \cup z) = [x \lor y \lor z, S(x, y, z)]\).

**Proof.** The first part of 1 is proved as follows: \(1 \cap x = [T(1, x), 1 \land x] = [x, x] \ni 1\). Similarly, for the second part: \(0 \cap x = [T(0, x), 0 \land x] = [0, 0] \ni 0\). The remaining two parts are proved similarly. 2 is immediate. Regarding 3 we have: \(x \cap z = [T(x, z), x \land z], y \cap z = [T(y, z), y \land z]\); now, if \(x \leq y\) then \(T(x, z) \leq T(y, z)\) and \(x \land z \leq y \land z\) which shows that \(x \cap z \leq y \cap z\); \(x \cup z \leq y \cup z\) is proved dually. Let us now turn to 4.

First, take any \(w \in (x \cap y) \cap z = \cup_{u \in x \cap y} u \cap z = \cup_{T(x, y) \leq u \leq x \land y} [T(u, z), u \land z].\) Hence there exists some \(u\) such that \(T(x, y) \leq u \leq x \land y\) and \(T(u, z) \leq w \leq u \land z\). Hence \(w \leq u \land z \leq x \land y \land z\) and \(w \geq T(u, z) \geq T(T(x, y), z) = T(x, y, z).\) It follows that \(w \in [T(x, y, z), x \land y \land z]\) and so

\[(x \cap y) \cap z \subseteq [T(x, y, z), x \land y \land z].\]

(8)

Second, take any \(w \in [T(x, y, z), x \land y \land z]\) and define

\[u = (T(x, y) \lor w) \land (x \land y) = T(x, y) \lor (w \land x \land y)\] (9)

(the second equality in (9) follows from distributivity). Now

\[u = (T(x, y) \lor w) \land (x \land y) \Rightarrow u \leq x \land y\] (10)

\[u = T(x, y) \lor (w \land x \land y) \Rightarrow u \geq T(x, y)\] (11)

5
Furthermore \( u \land z = (T(x, y) \lor w) \land (x \land y) \land z \). But \( w \leq T(x, y) \lor w \) and \( w \leq (x \land y) \land z \). Hence \( w \leq u \land z \). Also, \( T(u, z) = T(T(x, y) \lor (w \land x \land y), z) = T(T(x, y), z) \lor T(w \land x \land y, z) \). Now \( T(T(x, y), z) = T(x, y, z) \leq w \) and \( T(w \land x \land y, z) \leq T(w, z) \leq w \). Hence \( T(u, z) \leq w \). In short
\[
w \in [T(u, z), u \land z].
\] (13)

From (12) and (13) we conclude \( w \in (x \cap y) \cap z \) and so
\[
[T(x, y, z), x \land y \land z] \subseteq (x \cap y) \cap z.
\] (14)

From (5) and (14) we conclude \([T(x, y, z), x \land y \land z] = (x \cap y) \cap z\) and we have established the first part of 4; the second part is proved dually.

**Proposition 3.6** For all \( x, y \in X \) we have:

1. \( x \in x \cap x, x \in x \cup x \).
2. \( x \in x \cap (x \cup y), x \in x \cup (x \cap y) \).
3. \( x \leq y \Leftrightarrow y \in x \cup y \Leftrightarrow x \in x \cap y \).

**Proof.** For 1: \( x \cap x = [T(x, x), x \land x] \), but \( T(x, x) \leq x \) and \( x \land x = x \), hence \( x \in x \cap x \); the second part is proved dually.

For 2: \( x \cap (x \cup y) = x \cap [x \cup y, S(x, y)] = [T(x, x \cup y), x \land S(x, y)] \). But \( T(x, x \cup y) = T(x, x) \lor T(x, y) \) and we have \( T(x, x) \leq x, T(x, y) \leq x \); hence \( T(x, x \cup y) \leq x \). Also, \( x \leq S(x, y) \) and so \( x \land S(x, y) = x \). Hence \( x \in [T(x, x \cup y), x \land S(x, y)] = x \cap (x \cup y) \). The second part of 2 is proved dually.

Finally, for 3, \( x \cup y = [x \lor y, S(x, y)] \), but \( x \leq y \Rightarrow x \lor y = y \) and \( y \leq S(x, y) \); hence \( y \in [y, S(x, y)] = x \cup y \). Conversely, \( y \in x \cup y = [x \lor y, S(x, y)] \Rightarrow x \lor y \leq y \Rightarrow x \lor y = y \Rightarrow x \leq y \). The second part of 3 is proved dually.

**Proposition 3.7** For all \( x, y \in X \) we have: \((x \cup y)' = x' \cap y' \) and \((x \cap y)' = x' \cup y' \).

**Proof.** We prove only the first part (the second part is proved dually). We have
\[
(x \cup y)' = [x \lor y, S(x, y)]' \\
= \{z': x \lor y \leq z \leq S(x, y)\} \\
= \{z': x' \land y' \geq z' \geq (S(x, y))'\} \\
= x' \cap y'.
\]

**Proposition 3.8** For all \( x, y, z \in X \) we have:

1. \([T(x, y \lor z), x \land (y \lor z)] \subseteq (x \cap (y \lor z)) \cap ((x \cap y) \cup (x \cap z)) \).
2. \([x \lor (y \land z), S(x, y \land z)] \subseteq (x \lor (y \lor z)) \cap ((x \lor y) \cap (x \lor z)) \).
Proposition 4.1 Suppose that for all $x, y \in X$ we have $T_1 (x, y) \leq T_2 (x, y)$ and $S_2 (x, y) \leq S_1 (x, y)$. For all $x, y \in X$ define $x \sqcap y$ and $x \sqcup y$ as in (14). Then

\begin{align}
(\forall x, y \in X : T_2 (x, y) = x \land y) & \iff (\forall x, y \in X : x \leq y \iff x \in x \sqcap y) \\
(\forall x, y \in X : S_2 (x, y) = x \lor y) & \iff (\forall x, y \in X : x \leq y \iff y \in x \sqcup y)
\end{align}

4 Generalizations

We can generalize the construction of the multi-valued connectives (Definition 3.3) in the following manner. Suppose that $T_1, T_2$ are t-norms and $S_1, S_2$ their dual t-conorms. Furthermore, suppose that for all $x, y \in X$ we have $T_1 (x, y) \leq T_2 (x, y)$ and $S_2 (x, y) \leq S_1 (x, y)$. For all $x, y \in X$ define

\[
x \sqcap y = [T_1 (x, y), T_2 (x, y)], x \sqcup y = [S_2 (x, y), S_1 (x, y)].
\]

Then it is still possible that $\sqcap, \sqcup$ have the t-norm, t-conorm properties of Proposition 3.5. As an example take

\[
T_1 (x, y) = \max (0, x + y - 1), \quad T_2 (x, y) = xy, \quad S_1 (x, y) = \min (1, x + y), \quad S_2 (x, y) = x + y - xy.
\]

It is easy to check that all the properties of Proposition 3.5 still hold.

However, an additional attractive point of our construction is that $(X, \sqcap, \sqcup)$ behaves similarly to a lattice (i.e. it is a superlattice). Can we obtain this behavior for $T_2$ different from $\land$ and $S_2$ different from $\lor$? A first answer turns out to be negative.

Proposition 3.9 The hyperalgebra $(X, \sqcap, \sqcup)$ is a superlattice.

Proof. The proof consists in checking that all the properties listed in Definition 2.13 are satisfied when we use $\sqcap$ in place of $\triangle$ and $\sqcup$ in place of $\lor$. Indeed, A1, A4 and A5 are parts 1, 2 and 3 of Proposition 3.6 and A2, A3 are parts 2 and 4 of Proposition 3.5. ■
be defined in a general set different from $\leq$. To go the other way, suppose that $(\forall x, y \in X : x \leq y \iff x \in x \sqcap y)$ holds. Choose any $x, y \in X$. Since $x \land y \leq y$, we must have

$$x \land y \in x \sqcap y = [T_1(x \land y, y), T_2(x \land y, y)] \subseteq [T_1(x \land y, y), T_2(x, y)].$$

Hence $x \land y \leq T_2(x, y)$; but also $T_2(x, y) \leq x \land y$. It follows that $T_2(x, y) = x \land y$ and this holds for every $x, y \in X$. Hence (20) has been proved. \(\blacksquare\)

From the above proposition we see that $(X, \sqcap, \sqcup)$ is a superlattice compatible with the original order $\leq$ iff $x \sqcap y$ and $x \sqcup y$ are defined according to Definition 3.3.

However it may still be possible to define $x \sqcap y$ and $x \sqcup y$ in such a manner that $(X, \sqcap, \sqcup)$ is a superlattice in a more general sense. Namely, suppose that A1-A4 are satisfied and A5 is replaced by the following conditions.

A6 $y \in x \triangledown y \iff x \in x \bigtriangleup y$.

A7 $(x \in x \triangledown y$ and $y \in x \triangledown y) \Rightarrow x = y$.

A8 $(x \in x \triangledown y$ and $y \in y \triangledown z) \Rightarrow x \in x \triangledown z$.

If A1-A4 and A6-A8 hold, then we can define a relation $\leq$ on $X$ as follows: “$x \leq y$ iff $y \in x \triangledown y$”. It turns out that using A6-A8 it can be shown that $\leq$ is an order on $X$, which will, in general, be different from $\leq$; in fact A1-A4 and A6-A8 do not use $\leq$ at all, hence the hyperoperations $\triangledown, \bigtriangleup$ can be defined in a general set $X$ (not necessarily a lattice).

In this light, it may be possible for some pairs $T_1$, $T_2$ and $S_1$, $S_2$ to define $\sqcup$ and $\sqcap$ as in (19) and then show that A1-A4 and A6-A8 hold; in such a case $\sqcap$ and $\sqcup$ will define an order $x \leq y$ on $X$ as follows: “$x \leq y$ iff $y \in x \sqcup y$” and $\sqcup, \sqcap$ are reasonable candidates for multi-valued t-norm and t-conorm on $X$. However, we emphasize again that $\sqcup, \sqcap$ will not fully respect the “intrinsic” order $\leq$.

5 Conclusion

We have presented a procedure for constructing multi-valued t-norms and t-conorms. Our construction uses a pair of single-valued t-norms and the pair of dual t-conorms and constructs interval-valued t-norms $\sqcap$ and t-conorms $\sqcup$. In this manner we can combine desirable characteristics of different t-norms and t-conorms; furthermore if we use the t-norm $\land$ and t-conorm $\lor$, then $(X, \sqcap, \sqcup)$ is a superlattice, i.e. the multivalued analog of a lattice.

Let us close with some issues which require further research. First, it will be interesting to obtain further “deMorgan-like” properties of $(X, \sqcap, \sqcup, \lor)$ and develop a logic based on multi-valued connectives. Of particular interest is the study of the resulting implication operator, the law of excluded middle and the law of contradiction. Second, note that the fuzzy implication operator is closely connected to the fuzzy inclusion measure, so it would be interesting to consider interval-valued inclusion measures. Third, we are interested in analyzing $(X, \sqcap, \sqcup)$ from a geometric point of view, paying special attention to issues such as metric properties, continuity, convexity and betweenness. Finally, it will be interesting to develop a procedure for developing a family of interval-valued t-norms $\{\sqcap_a\}_{a \in [0,1]}$ which have the a-cut properties, because the $\sqcap_a$’s can then be used to construct a fuzzy-valued t-norm $\sqcap$. Similarly, one could use a family $\{\lor_a\}_{a \in [0,1]}$ to construct a fuzzy-valued t-conorm $\lor$.
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