On metric geometry of conformal moduli spaces of four-dimensional superconformal theories

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Abstract: Conformal moduli spaces of four-dimensional superconformal theories obtained by deformations of a superpotential are considered. These spaces possess a natural metric (a Zamolodchikov metric). This metric is shown to be Kahler. The proof is based on superconformal Ward identities.
1. Introduction

Since the seminal work of Leigh and Srassler [1] who showed that four-dimensional superconformal theories often come in continuous families called conformal moduli spaces there has been a considerable progress in a study of these spaces.

An important progress has been made in connection with the AdS/CFT correspondence. Supergravity duals of superconformal deformations were studied perturbatively in [2] and more generally in [3], where it was shown that a dimension of the conformal moduli space in the supergravity case is equal to a certain well-defined index. Some examples of exact supergravity duals of superconformal theories became accessible with a specific construction of families of Sasaki-Einstein spaces. In [4] there was found an exact dual of a so called $\beta$-deformation of the $\mathcal{N} = 4$ theory, which is a particular kind of a general Leigh-Strassler deformation. It was shown also that all toric quiver gauge theories admit $\beta$-deformations [5] which have supergravity duals [6].

Any conformal moduli space has a natural metric defined on it, namely, the Zamolodchikov metric [7]. It is defined in terms of exactly marginal operators $O_i(x)$. This metric turns out to be Kahler for the exact supergravity dual of a $\beta$-deformation [4] and was shown to be Kahler in general in the supergravity [8]. All these results concerned $SU(N)$ field theories with infinite $N$ and it was not clear whether they hold for $1/N$ corrections as well.

On the other side, there are well established results about two-dimensional conformal theories with $(2,2)$ supersymmetry, which are analogs of four-dimensional $\mathcal{N} = 1$ superconformal theories. The conformal moduli spaces of these theories are coset spaces and the metric on them is Kahler [9].

In this paper we show that the Zamolodchikov metric on conformal moduli spaces of four-dimensional superconformal theories with 8 supercharges is Kahler (as it was initially conjectured in [10]). We use purely field theoretic methods, so the results hold for any $N$. The main tool is the superconformal Ward identities which constrain a possible form of various correlators of exactly marginal operators. As an example we work with a space of deformations of $\mathcal{N} = 4$ theory, but our results are general.

The proof of a Kahlerity is given in section 2. Section 3 is a brief summary and a discussion of open questions.
2. Proof of the Kahleriity of the Zamolodchikov metric

We consider a 4D $\mathcal{N} = 1$ superconformal theory obtained by a Leigh-Strassler \cite{LeighStrassler} deformation of the superpotential of the $\mathcal{N} = 4$ theory, although our treatment is applicable to other theories as well. The Lagrangian of the theory is

$$L = \sum_{i=1}^{3} \text{Tr} \int d^4 \theta e^{-V} \Phi_i \Phi_i + \frac{\tau}{32 \pi i} \text{Tr} \int d^2 \theta W^2 +$$

$$+ \text{Tr} \int d^2 \theta (h_0 \epsilon^{ijk} \Phi_i \Phi_j \Phi_k + h_1 f^{ijk} \Phi_i \Phi_j \Phi_k + h_2 (\Phi^3_1 + \Phi^3_2 + \Phi^3_3)) + c.c., \quad (2.1)$$

where $\tau$ is a usual combination of a coupling constant and a $\theta$ angle: $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$ and $\epsilon^{ijk}$ and $f^{ijk}$ are antisymmetric and symmetric $SU(3)$ invariants respectively. The coupling constants $h_{0,1,2}$ are complex-valued, however their common phase is of no importance since it can be compensated by a redefinition of superfields.

The argument of Leigh and Strassler shows that this theory is conformal as long as the coupling constants satisfy the equation

$$\gamma(\tau, h_0, h_1, h_2, \bar{\tau}, \bar{h}_0, \bar{h}_1, \bar{h}_2) = 0, \quad (2.2)$$

where $\gamma$ is an anomalous dimension of chiral superfields (the form of the Lagrangian guarantees that all three chiral superfields have the same dimension because of various discrete symmetries between the three superfields). Solutions of this equation which differ by a common phase of $h_{0,1,2}$ should be identified. The $U(1)$ quotient space obtained this way is a conformal moduli space $\mathcal{M}_c$.\footnote{There are additional discrete identifications of points on $\mathcal{M}_c$ due to the $SL(2,\mathbb{Z})$ duality.} It has a real dimension 6. In what follows we take the gauge coupling constant $\tau$ to be fixed and allow the superpotential couplings $h_k$ to change so that the eq. 2.2 would be satisfied, so we concentrate on four-dimensional slices of $\mathcal{M}_c$, which we will denote by $\mathcal{M}_c(\tau)$.

The first point that we need to establish is a fact that there are (local) complex coordinates on $\mathcal{M}_c(\tau)$ such that the coupling constants $h_{0,1,2}$ are their holomorphic functions. To see this we notice that the definition of $\mathcal{M}_c(\tau)$ resembles a definition of a complex projective space $\mathbb{CP}_n$ in terms of an affine space $\mathbb{C}^{n+1}$. Indeed, in both cases the definition includes a single real equation (2.2 for $\mathcal{M}_c(\tau)$ and an equation of a sphere for $\mathbb{CP}_n$) and a $U(1)$ which rotates phases of coordinates of the ambient affine complex space. We need to choose coordinates on $\mathcal{M}_c(\tau)$ which are invariant under this $U(1)$. We know that the point $h_0 = g, h_1 = h_2 = 0$ belongs to $\mathcal{M}_c(\tau)$. We choose some neighborhood of this point and analogously to $\mathbb{CP}_n$ define local complex coordinates $z_n, n = 1, 2$ on $\mathcal{M}_c(\tau)$ to be $z_{1,2} = h_{1,2}/h_0$. In order to see that these are good coordinates we need to show that having chosen $z$'s, we can reconstruct $h$'s without any need in $\bar{h}$'s. Having chosen $z$'s, we get $h_{1,2} = z_{1,2} h_0$, and the eq. 2.2 becomes

$$\gamma(\tau, h_0, z_1 h_0, z_2 h_0, \bar{\tau}, \bar{h}_0, z_1 \bar{h}_0, z_2 \bar{h}_0) = 0. \quad (2.3)$$
In this eq. a phase of $h_0$ is not relevant because it can be changed by a $U(1)$ rotation. We can choose $h_0$ to be real, for example, and then the last eq. becomes an equation for $|h_0|$. The eq. is satisfied for $z_{1,2} = 0$ (the solution is $h_0 = g$) and by the implicit function theorem it has solutions for $z_{1,2}$ being sufficiently close to 0 (the conditions of the theorem are satisfied since for small coupling constants the expansion of $\gamma$ is $\gamma \sim |h_0|^2 + |h_1|^2 + |h_2|^2 - g^2$ with some coefficients which are not relevant for the argument). So we conclude that the required choice of coordinates is possible.

The space of conformal deformations of the superpotential of $\mathcal{N} = 4$ theory was studied more generally in [2, 3], without imposing any discrete symmetries on the Lagrangian. In this case the superpotential is of a general form $h_{ijk} \text{Tr}(\Phi_i \Phi_j \Phi_k)$ with symmetric complex-valued coefficients $h_{ijk}$. Instead of a single anomalous dimension $\gamma$ in this formulation there appears a Hermitian matrix $\gamma_{ij}$ of anomalous dimensions, and instead of a single Leigh-Strassler equation 2.2 there are 8 equations corresponding to the traceless part of $\gamma_{ij}$. The number of independent coupling constants is 10 and there is a global $SU(3)$ which identifies different solutions, so the real dimension of the space of superpotential deformations is again 4. As discussed in [3], close to the origin $M_c(\tau) \approx 10\mathbb{C}/SL(3, \mathbb{C})$. This manifold is a solution of the $D$-term constraint of the global $SU(3)$. A solution of a $D$-term constraint is a result of a division of the ambient space $10\mathbb{C}$ by a complexified group, which is $SL(3, \mathbb{C})$. It is also argued in [3] that $M_c(\tau)$ is a complex manifold, in an agreement with the argument above.

As discussed in the introduction, any conformal moduli space is endowed with a natural metric, the Zamolodchikov metric [6], defined in terms of correlators of exactly marginal operators $O_i(x)$. In a $d$-dimensional theory a mass dimension of exactly marginal operators is $d$ and a conformal invariance fixes a form of a correlator of two such operators to be $<O_i(x)O_i(y)> \sim |x - y|^{-2d}$. The correlator is similar to an inner product of two tangent vectors, and the Zamolodchikov metric is defined as a coefficient in this identity:

$$<O_i(x)O_i(y)> = g_{ij}|x - y|^{-2d}. \quad (2.4)$$

We study the Zamolodchikov metric on the conformal moduli space of 4-dimensional superconformal theories $\mathcal{M}_c(\tau)$. In this case the supersymmetry imposes additional constraints on the metric. We will show that these additional constraints make the metric Kahler, as was proposed in [10].

As a starting point we use the fact that according to the representation theory of superconformal groups [11] there is a relation between a mass dimension and an $R$-charge of a chiral primary field, namely $d = \frac{3}{2}r$ (a corresponding relation for an antichiral primary field is $d = -\frac{3}{2}r$). A chiral field is a lowest component of a chiral superfield, and corresponding relations for highest ($F$-) components of chiral and antichiral superfields are $d - 1 = \frac{3}{2}(r + 2)$ and $d - 1 = -\frac{3}{2}(r - 2)$ correspondingly. An exactly marginal operator is of dimension 4 and has a vanishing $R$-charge, and these relations are satisfied. Therefore any exactly marginal operator must be a linear combination of $F$-components of chiral and antichiral superfields.

Next we notice that an exactly marginal operator is a derivative of the Lagrangian w.r.t. a coordinate on $\mathcal{M}_c$. We have shown that there are complex coordinates $z_n$ on
\( \mathcal{M}_c(\tau) \) such that the coupling constants \( h_k \) are holomorphic functions of \( z_n \). In these coordinates the derivatives \( \partial L/\partial z_n \) are proportional to linear combinations of \( \partial L/\partial h_k \). These derivatives are \( F \)-components of chiral superfields (with no contribution of antichiral ones). Correspondingly, \( \partial L/\partial \bar{z}_n \) are \( F \)-components of antichiral superfields.

We use now superconformal Ward identities to study restrictions that supersymmetry imposes on the form of the Zamolodchikov metric. In our coordinate system there are four different kinds of components of the metric: \( g_{mn}, g_{\bar{m}n}, g_{mn}, g_{\bar{m}\bar{n}} \). First of all we show that \( g_{mn} = g_{\bar{m}n} = 0 \). In order to see this consider a correlator of two chiral superfields on \( \mathcal{N} = 1 \) superspace. These superfields are functions of \( \theta \) and a chiral coordinate \( x_\mu^+ = x_\mu + \frac{i}{2} \theta \sigma \). Therefore the correlator is a function of these coordinates:

\[
< \Phi(x_1^+, \theta_1)\Phi(x_2^+, \theta_2) >= f(x_1^+, x_2^+, \theta_1, \theta_2).
\]

We are interested in the correlator of \( F \)-components, which is a coefficient of \( \theta^2 \bar{\theta}^2 \) on the RHS. In order to see that this coefficient vanishes apply to both sides the superconformal generator \( S^\alpha \) which in the chiral representation is (see \([12]\))

\[
S^\alpha \sim \bar{\sigma}^\alpha \mu x^\mu \partial_\alpha.
\]

LHS of \((2.5)\) vanishes under the action of \( S^\alpha \), whereas the coefficient of \( \theta^2 \bar{\theta}^2 \) of RHS gets multiplied by \( \bar{\sigma}^\alpha \mu (x^\mu_1 + x^\mu_2) \). The coefficient itself depends on \( x_1 \) and \( x_2 \) as \( |x_1 - x_2|^{-8} \) and therefore can vanish only if it is absent. So we see that the correlator of the \( F \)-terms vanishes (actually, the only possible term in a correlator of two chiral superfields is a contact term \([4, 5, 13]\), we will discuss this fact and use it later). This in turn means the term \( g_{mn} \) of the metric vanishes. Similar computation for antichiral superfields shows that \( g_{m\bar{n}} \) vanishes as well. We see that the metric in our coordinate system is Hermitian.

Next we show that the metric on \( \mathcal{M}_c(\tau) \) is Kahler. A definition of the Kahler metric \( g_{mn} = \partial_m \partial_n K \), where \( K \) is a Kahler potential, is equivalent to the integrability condition \( \partial_m g_{nl} = \partial_n g_{ml} \) (and its complex conjugate). This is the condition that we are going to check.

The definition of the Zamolodchikov metric implies that

\[
\partial_m g_{nl} = \partial_m < \frac{\partial L}{\partial z_n}(x) \frac{\partial L}{\partial \bar{z}_l}(y) > |x - y|^8 = \left( < \frac{\partial L}{\partial z_n}(x) \frac{\partial L}{\partial \bar{z}_l}(y) \int d^4 u \frac{\partial L}{\partial z_m}(u) > 
+ < \frac{\partial^2 L}{\partial z_m \partial z_n}(x) \frac{\partial L}{\partial \bar{z}_l}(y) > + < \frac{\partial L}{\partial \bar{z}_m}(x) \frac{\partial^2 L}{\partial z_n \partial \bar{z}_l}(y) > \right) |x - y|^8
\]

A few remarks about this equality are in order. First of all, the second term on the RHS is symmetric under the interchange of \( m \) and \( n \) and therefore satisfies the integrability condition. Next, as we have shown, \( \partial L/\partial z_n \) depends only on \( z \)'s and not on \( \bar{z} \)'s, and therefore the last term on the RHS vanishes. Finally, consider the first term which involves a three-point function of exactly marginal operators. The conformal invariance in principle allows an appearance of a term \( |x - y|^{-4} |x - u|^{-4} |y - u|^{-4} \) in the three-point function of operators of the mass dimension 4. However, such a behavior is forbidden for exactly
marginal operators since it would lead to a non-trivial \( \beta \)-function of a deformation operator in a perturbed theory. But there can appear contact terms in the three-point function. Such contact terms are essential in a geometry of \( \mathcal{M}_c \), as pointed out in [16] for a two-dimensional case. We show now that the superconformal invariance allows only for contact terms between operators of the same chirality. Indeed, if there is a contact term in the OPE of two \( F \)-terms of the same chirality:

\[
F_m(x_1)F_n(x_2) \sim A^k_{mn} \delta^4(x_1 - x_2)F_k(x_1)
\]

then this contact term can be promoted to a contact term on the superspace:

\[
\Phi_m(x_1^+, \theta_1)\Phi_n(x_2^+, \theta_2) \sim A^k_{mn} \delta^4(x_1^+ - x_2^+)\delta^2(\theta_1 - \theta_2)\Phi(x_1^+, \theta_1),
\]

with \( \delta^2(\theta_1 - \theta_2) \equiv (\theta_1 - \theta_2)^2 \). This is precisely the contact term mentioned in our discussion of the two-point functions. However, there is no supersymmetry covariant generalization of a contact term between chiral and antichiral superfields, and therefore a contact term between \( F \)-terms of superfields of opposite chiralities is not consistent with the supersymmetry. So indeed the only contact terms that are allowed are those between fields of the same chirality.

This last point leads to the following observation about eq. 2.7. If the OPE between the exactly marginal operators is \( \partial_m L(x)\partial_n L(y) \sim A^k_{mn} \partial_k L(x) \) then the first term on the RHS is \( A^k_{mn} \partial_k L(x) \). Since \( A^k_{mn} = A^k_{nm} \) this term is symmetric under \( m \leftrightarrow n \).

We see that the whole RHS of eq. 2.7 is symmetric under \( m \leftrightarrow n \), and the integrability condition holds. We conclude therefore that the Zamolodchikov metric on \( \mathcal{M}_c(\tau) \) is Kahler.

3. Discussion

In this paper we considered a metric geometry of a conformal moduli space of four-dimensional \( \mathcal{N} = 4 \) theory which involves deformations of a superpotential. We have shown that a natural Zamolodchikov metric on this space is Kahler. The proof is based on superconformal Ward identities for correlators of superfields on the \( \mathcal{N} = 1 \) superspace. It involves three logical steps:

- Show that there are complex coordinates on the conformal moduli space such that (complex) coupling constants of the theory depend on them holomorphically. This is not trivial because \( \mathcal{M}_c(\tau) \) is a \( U(1) \) quotient. But a similarity with a complex projective space allows one to define holomorphic coordinates.

- Show that the metric in these coordinates is Hermitian. The choice of coordinates made in the previous step is crucial here and in the next step because it guarantees that the exactly marginal operators which span a tangent space to \( \mathcal{M}_c(\tau) \) are \( F \)-terms of superfields of a definite chirality.

- Show that the integrability condition which guarantees the existence of a Kahler potential is satisfied. The main point here is a consideration of a three-point function of exactly marginal operators. It is a combination of contact terms and those of them that violate the integrability condition are forbidden by the supersymmetry.
There are a few remaining questions, though. One of them is to derive explicitly the Kahler potential for the metric. Since the metric is a two-point function of exactly marginal operators the potential should be connected somehow to a partition function of the theory, whose second derivatives w.r.t. coordinates of $M_c(\tau)$ give correlators of exactly marginal operators integrated over the whole space-time. However, the partition function of any supersymmetric theory vanishes. The integrals of correlators diverge, but their regulated versions vanish (for example, in a dimensional regularization any power-like divergence is set to zero). Therefore a correct version of the Kahler potential is a partition function regularized in some way which would break the supersymmetry. These observations were made in [10]. To derive the Kahler potential is a subject of a future work.

Another issue to be studied is a generalization of our result to the gauge coupling as well. The corresponding term in the Lagrangian is again a chiral superfield and should be amenable to similar treatment. The only possible problem is whether there is a choice of “good” complex coordinate on $M_c$ such that the gauge coupling $\tau$ would be their holomorphic function. An intuition based on the supergravity supports an affirmative answer.

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