ARITHMETIC LEVEL RAISING ON TRIPLE PRODUCT OF SHIMURA CURVES AND GROSS–KUDLA–SCHOEN DIAGONAL CYCLES II: BIPARTITE EULER SYSTEM

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Abstract. In this article, we study the Gross–Kudla–Schoen diagonal cycle on the triple product of Shimura curves at a place of good reduction and prove an unramified arithmetic level raising theorem for the cohomology of this triple product. We deduce from it a reciprocity law which relates the image of the diagonal cycle under the Abel–Jacobi map to certain period integral of Gross–Kudla type. Combing this with the first reciprocity law we proved in a previous work, we show that the Gross–Kudla–Schoen diagonal cycles along with the Gross–Kudla periods form a bipartite Euler system for the symmetric cube motive of a modular form. As an application we provide some evidence for the rank one case of the Bloch–Kato conjecture for the symmetric cube motive of a modular form.

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1. Introduction

In a seminal work of Bertolini–Darmon [BD3], the authors constructed a Kolyvagin type Euler system using Heegner points on various Shimura curves. The cohomology classes in this system satisfy beautiful reciprocity laws that resemble the so-called Jochnowitz’s congruence. More precisely, these reciprocity laws relate the Kummer images of the Heegner points to certain toric period integrals via level-raising congruences. One can consider these Heegner point classes as the incarnations of the first derivatives of certain Rankin–Selberg $L$-functions at the central critical points in light of the Gross–Zagier type formula while the toric period integrals are the algebraic parts of the central critical values of the Rankin–Selberg type $L$-functions as shown by the Gross’ formula. The reciprocity laws alluded above are reflections of the congruences between the algebraic parts of the first derivatives of the Rankin–Selberg $L$-functions and the algebraic parts of the central critical values of the Rankin–Selberg type $L$-functions observed by Jochnowitz using computational method. Using these reciprocity laws, Bertolini–Darmon are able to verify the one-sided divisibility of the Iwasawa main conjecture for an elliptic curve over the anticyclotomic extension of an imaginary quadratic field. These reciprocity laws are also the backbone of Wei Zhang’s work [Zhang] on verifying the Kolyvagin’s conjecture. The method of Bertolini–Darmon is axiomatized by [How] and is given the name of a bipartite Euler system. In [How], it was also mentioned that the cycle on the triple product of Shimura curves studied in [CK] and [GS] known as the Gross–Kudla–Schoen diagonal cycle might form another example of such a bipartite Euler system. The main theme of this article is to investigate Howard’s conjecture for the Gross–Kudla–Schoen diagonal cycle.

The present article completes what was started in [Wang] where we verified one of the reciprocity laws (the first reciprocity law). We will show below that the Gross–Kudla–Schoen diagonal cycles and the Gross–Kudla periods on the triple product of Shimura curves satisfy all the reciprocity laws needed to form a bipartite Euler system for the triple product motive of modular forms. However these cycles do not give a bipartite Euler system in the strict sense of [How] for obvious rank reasons. On the other hand, when we consider the symmetric cube component of the triple tensor product motive, then the image of these Gross–Kudla–Schoen diagonal cycles in the symmetric cube component do form a bipartite Euler system. Using this bipartite Euler system, we provide some evidence towards the rank one case of the Bloch–Kato conjecture for the symmetric cube motive of a modular form. A refinement of the method in this article should also be able to handle the more general case of the triple product motive of modular forms. We decide to treat this in another occasion where we also plan to extend the results here to an Iwasawa theoretic setting.

Note that the Gross–Kudla–Schoen diagonal cycle is also studied extensively using the $p$-adic method. This was initiated in [DR1] and [DR2] where certain $p$-adic reciprocity laws in the style of Perrin-Riou are proved. The reciprocity laws in this article can be viewed as the counter part of those in [DR1] and [DR2] in the level raising situation. We will refer to [BDRSV] and [CH] for further development and arithmetic applications in this direction. It also should be possible to combine the results in this article with the $p$-adic methods to prove interesting Iwasawa theoretic results as was done in [BD3]. In a parallel direction, Liu and Tian [Liu1], [Liu2] and [LT] have constructed the desired bipartite Euler system for the triple product motive realized on certain Hilbert modular threefold.
1.1. Main results. In order to state our main results, we introduce some notations. Let \( l \geq 5 \) be a prime. Let \( \mathbf{f} = (f_1, f_2, f_3) \) be a triple of normalized newforms in \( \mathcal{S}^n_{\text{new}}(\Gamma_0(N))^3 \). Throughout this article, we assume that we have a factorization \( N = N^+N^- \) such that \( (N^+, N^-) = 1 \) and \( N^- \) is square-free with \( \text{even} \) number of prime factors. For \( i \in \{1, 2, 3\} \), let \( E_i = \mathbb{Q}(f_i) \) be the Hecke field of \( f_i \) and \( \lambda_i \) be a place of \( E_i \) above the prime \( l \). We denote by \( \mathcal{O}_{\lambda_i} \) the valuation ring of the completion \( E_{\lambda_i} \) of \( E \) at \( \lambda_i \). Let \( \varpi_i \) be a uniformizer of \( \mathcal{O}_{\lambda_i} \) and \( \lambda_i = (\varpi_i) \) be the maximal ideal. Let \( \mathcal{O}_{\lambda_i,n} = \mathcal{O}_{\lambda_i}/\lambda_i^n \) for any integer \( n \geq 1 \). Let \( k_i \) be the residue field of \( E_{\lambda_i} \). For \( i \in \{1, 2, 3\} \), we denote by
\[
\rho_{f_i, \lambda_i} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(E_{\lambda_i}) = \text{GL}(V_{f_i, \lambda_i})
\]
the \( \lambda_i \)-adic representation attached to \( f_i \) and whose residual representation will be denoted by \( \bar{\rho}_{f_i, \lambda_i} \). We introduce the following subsets of the set of prime divisors of \( N; \Sigma^+ \) is the set of prime divisors of \( N^+; \Sigma_{\text{ram}} \) is the set of prime divisor \( r \) of \( N^- \) such that \( l \mid r^2 - 1 \).

**Assumption 1.** We make the following assumptions on \( \bar{\rho}_{f_i, \lambda_i} \) and thus on \( \mathbf{m}_i \):

1. \( \bar{\rho}_{f_i, \lambda_i}|_{G_{\mathbb{Q}(\zeta)}} \) is absolutely irreducible;
2. \( \bar{\rho}_{f_i, \lambda_i} \) is minimally ramified at primes in \( \Sigma^+ \cup \Sigma_{\text{ram}} \) and is ramified at primes in \( \Sigma_{\text{ram}} \);
3. The image of \( \bar{\rho}_{f_i, \lambda_i} \) contains \( \text{GL}_2(F_l) \).

We introduce an auxiliary prime \( d \) and assume it is clean with respect to \( (\mathcal{O}_{f_1, \lambda_1}, \mathcal{O}_{f_2, \lambda_2}, \mathcal{O}_{f_3, \lambda_3}) \) in the sense of Definition 2.4. This auxiliary prime is used to make the moduli problems associated to the Shimura curves in this article representable. Let \( B = B_{N^-} \) be the indefinite quaternion algebra of discriminant \( N^- \), let \( X_d \) be the Shimura curve over \( B \) with a level structure given by an Eichler order of level \( N^+ \) and an auxiliary \( d \)-level structure. Let \( p \) be a prime away from \( Nd, B = B_{pN^-} \) be the definite quaternion algebra over \( \mathbb{Q} \) and \( \mathbb{Z}_d(B) \) be the Shimura set associated to \( B \) with a level structure given by an Eichler order of level \( N^+ \) and an auxiliary \( d \)-level structure. For a finite set of primes \( S \) away from \( Nd \), let \( T[S] \) be the Hecke algebra containing all the Hecke operators away from \( Nd \) and all the primes in \( S \). If \( S \) is empty, then we omit it from the notation. The Hecke eigensystem of \( f_i \) gives rise to a morphism \( \phi_i : T \rightarrow \mathcal{O}_{\lambda_i} \) whose reduction modulo \( \lambda_i^n \) will be denoted by \( \phi_{i,n} \). The kernel of this morphism will be denoted by \( \mathfrak{p}_{i,n} \) and the unique maximal ideal in \( T \) containing \( \mathfrak{p}_{i,n} \) will be denoted by \( \mathfrak{m}_i \). We will denote by \( \mathfrak{m}_f \) the triple of maximal ideals \( (\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3) \). The triple product of Hecke algebras \( T \otimes T \otimes T \) acts naturally on the étale cohomology of the triple product of Shimura curves \( X_d^3 \) by the Künneth formula, it makes sense to localize the cohomology of \( X_d^3 \) at \( \mathfrak{m}_f \) (\( (\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3) \)). We set \( T_{\mathfrak{m}_f}^\otimes = T_{\mathfrak{m}_1} \otimes T_{\mathfrak{m}_2} \otimes T_{\mathfrak{m}_3} \). The maps \( (\phi_1, \phi_2, \phi_3) \) induce a map \( T_{\mathfrak{m}_f}^\otimes \rightarrow \mathcal{O}_{\Delta} \) where \( \mathcal{O}_{\Delta} = \mathcal{O}_{E_{\lambda_1}} \otimes \mathcal{O}_{E_{\lambda_2}} \otimes \mathcal{O}_{E_{\lambda_3}} \). It makes sense to define the \( \mathcal{O}_{\Delta}[G_{\mathbb{Q}}] \)-module
\[
M(\mathbf{f}, d)(-1) = H^3(X_d^3_{\mathfrak{m}_f}, \mathbb{Z}_l(2))_{\mathfrak{m}_f} \otimes T_{\mathfrak{m}_f}^\otimes \mathcal{O}_{\Delta}.
\]
Similarly, we define the \( \mathcal{O}_{\Delta,n}[G_{\mathbb{Q}}] \)-module
\[
M_n(\mathbf{f}, d)(-1) = H^3(X_d^3_{\mathfrak{m}_f}, \mathbb{Z}_l(2))_{\mathfrak{m}_f} \otimes T_{\mathfrak{m}_f}^\otimes \mathcal{O}_{\Delta,n}
\]
with \( \mathcal{O}_{\Delta,n} = \mathcal{O}_{\lambda_1,n} \otimes \mathcal{O}_{\lambda_2,n} \otimes \mathcal{O}_{\lambda_3,n} \) using the map \( T_{\mathfrak{m}_f}^\otimes \rightarrow \mathcal{O}_{\Delta,n} \) induced by \( (\phi_{1,n}, \phi_{2,n}, \phi_{3,n}) \).

We introduce the following notion of an \( n \)-admissible prime for the triple \( \mathbf{f} \). These primes are the level raising primes for the triple \( \mathbf{f} \).

**Definition 1.1.** A prime \( p \) is \( n \)-admissible for the triple \( \mathbf{f} = (f_1, f_2, f_3) \) if
(1) \( p \nmid Nl \);
(2) \( l \nmid p^2 - 1 \);
(3) \( \overline{p}_i \mid p + 1 - \epsilon_{p,ia_p}(f_i) \) with \( \epsilon_{p,i} \in \{\pm 1\} \) for \( i = 1, 2, 3 \);
(4) \( \epsilon_{p,1}\epsilon_{p,2}\epsilon_{p,3} = 1 \).

Let \( p \) be an \( n \)-admissible prime for \( \mathfrak{f}_n \). For an integer \( n \geq 1 \) and \( i \in \{1, 2, 3\} \), we let \( \phi^{[p]}_{i,n} : T[p] \to \mathcal{O}_{\lambda_i,n} \) be the map that agrees with \( \phi_{i,n} \) at all the Hecke operators away from \( p \). We denote by \( \mathfrak{p}^{[p]}_{i,n} \) the kernel of \( \phi^{[p]}_{i,n} \). Then we can prove the following arithmetic level raising theorem for the triple product of Shimura curves.

**Theorem 2.** Let \( p \) be an \( n \)-admissible prime for \( \mathfrak{f}_n \). We assume that each maximal ideal in the triple \( \mathfrak{m}_\mathfrak{f} = (m_1, m_2, m_3) \) satisfies Assumption\[.] Then we have the following isomorphism

\[
\Phi_{\mathfrak{f}_n} : \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \prod_{\mathfrak{p}^{[p]}_{i,n}} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}) / \mathfrak{p}^{[p]}_{i,n} \cong H^1(F_p, \mathcal{M}_n(\mathfrak{f}, d)(-1))
\]

between \( \mathcal{O}_{\Delta,n} \)-modules.

Let \( p \) be an \( n \)-admissible prime for \( \mathfrak{f}_n \). We consider the diagonal embedding \( \theta : X_d \to X_d^3 \) of \( X_d \) into its triple fiber product \( X_d^3 \). This gives a class \( \Delta_d = \theta_*[X_d] \in \text{CH}^2(X_d^3) \) which will be referred to as the *Gross–Kudla–Schoen diagonal cycle* as in the title of this article. In §3.3, we will introduce the Abel–Jacobi map

\[
\text{AJ}_{\mathfrak{f}_n} : \text{CH}^2(X_d^3) \to H^1(Q, \mathcal{M}(\mathfrak{f}_n)(-1))
\]

for \( \mathcal{M}(\mathfrak{f}, d)(-1) \) and the Abel–Jacobi map

\[
\text{AJ}_{\mathfrak{f}_n} : \text{CH}^2(X_d^3) \to H^1(Q, \mathcal{M}_n(\mathfrak{f}, d)(-1)).
\]

for \( \mathcal{M}_n(\mathfrak{f})(-1) \). We denote by \( \Theta(\mathfrak{f}, d) \in H^1(Q, \mathcal{M}(\mathfrak{f}, d)(-1)) \) the image of \( \Delta_d \) under \( \text{AJ}_{\mathfrak{f}_n} \) and by \( \Theta_n(\mathfrak{f}, d) \in H^1(Q, \mathcal{M}_n(\mathfrak{f}, d)(-1)) \) the image of \( \Delta_d \) under \( \text{AJ}_{\mathfrak{f}_n} \). We will define a natural bilinear pairing

\[
(\ , \ ) : \bigoplus_{i=1}^{3} \prod_{\mathfrak{p}^{[p]}_{i,n}} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}) / \mathfrak{p}^{[p]}_{i,n} \times \bigoplus_{j=1}^{3} \prod_{\mathfrak{p}^{[p]}_{j,n}} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_j}) / \mathfrak{p}^{[p]}_{j,n} \to \mathcal{O}_{\Delta,n}
\]

in §3.3 which come from Poincaré duality on \( Z_d(\mathcal{B}) \). The localization \( \text{loc}_p(\Theta_n(\mathfrak{f}, d)) \) of \( \Theta_n(\mathfrak{f}, d) \) can be viewed as an element in \( H^1(F_p, \mathcal{M}_n(\mathfrak{f}, d)(-1)) \) which in turn can be viewed as an element in \( \bigoplus_{j=1}^{3} \prod_{\mathfrak{p}^{[p]}_{j,n}} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_j}) / \mathfrak{p}^{[p]}_{j,n} \) via the isomorphism \( \Phi_{\mathfrak{f}_n} \) in the above Theorem. For \( j \in \{1, 2, 3\} \), we will denote by \( \text{loc}^{(j)}(\Theta_n(\mathfrak{f}, d)) \) the component of \( \text{loc}_p(\Theta_n(\mathfrak{f}, d)) \) in the \( j \)-th copy of the space \( \bigoplus_{j=1}^{3} \prod_{\mathfrak{p}^{[p]}_{j,n}} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_j}) / \mathfrak{p}^{[p]}_{j,n} \). The following reciprocity formula relating the Gross–Kudla–Schoen diagonal cycle class to certain Gross–Kudla type period integral. This is usually referred to as the *second reciprocity law* for the diagonal cycle \( \Delta_d \). Let \( \Delta_d(\mathcal{B}) = \vartheta_* Z_d(\mathcal{B}) \) be the diagonal cycle for the diagonal embedding \( \vartheta : Z_d(\mathcal{B}) \to Z_d^3(\mathcal{B}) \) of \( Z_d(\mathcal{B}) \) into its triple product \( Z_d^3(\mathcal{B}) \).

**Theorem 3.** Let \( p \) be an \( n \)-admissible prime for \( \mathfrak{f}_n \). Suppose each \( \mathfrak{m}_i \) satisfies Assumption\[.] Then the following formula

\[
(\text{loc}^{(j)}_p(\Theta_n(\mathfrak{f}, d)), \phi_1 \otimes \phi_2 \otimes \phi_3) = \sum_{z \in \Delta_d(\mathcal{B})} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z)
\]
holds for every \( \phi_1 \otimes \phi_2 \otimes \phi_3 \in \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), E_{\lambda_j}/\mathcal{O}_{\lambda_j})[p_{i,n}^{[p]}) \) and \( j \in \{1, 2, 3\} \).

Next we consider a pair of \( n \)-admissible primes \( (p, q) \) for \( \mathcal{f} \). Consider the indefinite quaternion algebra \( B^\perp \) of discriminant \( N - pq \). Then we can associate to it a Shimura curve over \( \mathbb{Q} \) denoted by \( X_d^2 \). We have a triple of morphisms \( \phi_{i,n}^{[pq]} : T_{m_i}^{[pq]} \to \mathcal{O}_{\lambda_i,n} \), with kernel \( p_{i,n}^{[pq]} \) that agree with the morphisms \( \phi_{i,n} \) away from \( pq \). We denote by \( m_i^{[pq]} = (m_1^{[pq]}, m_2^{[pq]}, m_3^{[pq]}) \), the triple of maximal ideals of \( T_{m_i}^{[pq]} \) given by \( m_i^{[pq]} = T_{[pq]} \cap m_i \). We consider the natural diagonal morphism \( \theta_q^2 : X_d^3 \to X_d^{3'} \) of \( X_d^3 \) into the triple fiber product \( X_d^{3'} \). We define the \( \mathcal{O}_{\lambda,n}[G_{\mathbb{Q}}] \)-module \( M_{n}^{[pq]}(\mathcal{f}, d) \) by

\[
M_{n}^{[pq]}(\mathcal{f}, d) = H^3(X_d^3, \mathbb{Q}_{ac}, Z_d(1))_{m_i^{[pq]}} \otimes_{T_{m_i}^{[pq]}} \mathcal{O}_{\lambda,n}
\]

where \( T_{m_i}^{[pq]} \otimes T_{m_i}^{[pq]} \otimes T_{m_i}^{[pq]} \). There is an Abel-Jacobi map

\[
AJ_{\mathcal{f}, n}^{[pq]} : CH^2(X_d^3) \to H^1(\mathbb{Q}, M_{n}^{[pq]}(\mathcal{f}, d)(-1))
\]

for \( M_{n}^{[pq]}(\mathcal{f}, d)(-1) \) constructed similarly for that of \( M_n(\mathcal{f}, d) \). We denote by \( \Theta_{n}^{[pq]}(\mathcal{f}, d) \in H^1(\mathbb{Q}, M_{n}^{[pq]}(\mathcal{f}, d)(-1)) \) the image of the diagonal cycle \( \Delta_d^2 = \theta^2_d[X_d^3] \) under \( AJ_{\mathcal{f}, n}^{[pq]} \). The main results of our previous work \([8]\) provide an isomorphism

\[
\bigoplus_{j=1}^{3} \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})[p_{i,n}^{[pq]}]) \cong H^1_{\text{sing}}(\mathbb{Q}_{\ell}, M_{n}^{[pq]}(\mathcal{f}, d)(-1))
\]

of modules of rank 3 over \( \mathcal{O}_{\lambda,n} \) under which

\[
(\partial_q^j)\Theta_{n}^{[pq]}(\mathcal{f}, d, \phi_1 \otimes \phi_2 \otimes \phi_3) = (q + 1)^3 I(\phi_1, \phi_2, \phi_3)
\]

for any \( \phi_1 \otimes \phi_2 \otimes \phi_3 \in \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})[p_{i,n}^{[pq]}] \) and \( j \in \{1, 2, 3\} \). Here \( \partial_q\Theta_{n}^{[pq]}(\mathcal{f}, d) \) is the singular residue of \( \Theta_{n}^{[pq]}(\mathcal{f}, d) \) at \( q \) and \( \partial_q(\phi_1 \otimes \phi_2 \otimes \phi_3) \) is the \( j \)-th component of \( \partial_q\Theta_{n}^{[pq]}(\mathcal{f}, d) \). And we define the Gross–Kudla period \( I(\phi_1, \phi_2, \phi_3) \) by

\[
I(\phi_1, \phi_2, \phi_3) = \sum_{z \in \Delta_d(\overline{B})} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z).
\]

Combing Theorem \( 3 \) and \((1.1)\), we arrive at the following equation

\[
(\partial_q(\phi_1 \otimes \phi_2 \otimes \phi_3)) = I(\phi_1, \phi_2, \phi_3) = (1 + q)^3 I(\phi_1, \phi_2, \phi_3).
\]

This shows that \( (\Theta_{n}(\mathcal{f}, d), \Theta_{n}^{[pq]}(\mathcal{f}, d)) \) along with the Gross–Kudla periods \( I(\phi_1, \phi_2, \phi_3) \) form a bipartite Euler system in a weaker sense: these classes do satisfy all the required reciprocity laws, however the singular and the finite part are both of rank 3 as opposed to of rank 1 as required by the definition in \( [8] \).
1.2. The symmetric cube motive. Consider the case when \( f = (f, f, f) \) for a single newform \( f \in S^2_{\text{new}}(T_0(N)) \) with \( N = N^+N^- \) such that \( (N^+, N^-) = 1 \) and \( N^- \) is square-free with even number of prime factors. Let \( V_{f,\lambda} \) be the representation space of the Galois representation \( \rho_{f,\lambda} \) attached to \( f \) and \( V(\mathfrak{f}) = V^{\otimes 3}_{f,\lambda} \) be the triple tensor product representation. Then we have the following factorization

\[
V(\mathfrak{f})(-1) = \text{Sym}^3 V_{f,\lambda}(-1) \oplus V_{f,\lambda} \oplus V_{f,\lambda}
\]

and we refer to \( V^\circ(\mathfrak{f})(-1) := \text{Sym}^3 V_{f,\lambda}(-1) \) as the symmetric cube component of \( V(\mathfrak{f})(-1) \). The triple product \( L \)-function \( L(f \otimes f \otimes f, s) \) factors accordingly as

\[
L(f \otimes f \otimes f, s) = L(\text{Sym}^3 f, s)L(f, s - 1)^2.
\]

We project the class \( \Theta(\mathfrak{f}, d) \in H^1(Q, V(\mathfrak{f})(-1)) \) which is the image of the diagonal cycle \( \Delta_d = \theta_s [X_d] \) under the Abel-Jacobi map

\[
\text{AJ}_L : \text{CH}^2(X^3_d) \to H^1(Q, V(\mathfrak{f})(-1))
\]

to its symmetric cube component and we obtain thus a class \( \Theta^\circ(\mathfrak{f}, d) \in H^1(Q, V^\circ(\mathfrak{f})(-1)) \). In light of the conjectural Gross–Zagier formula for the triple product \( L \)-function and the conjectural injectivity of the Abel-Jacobi map in this setting, the class \( \Theta^\circ(\mathfrak{f}, d) \) should be considered as an algebraic incarnation of the first derivative \( L'(\text{Sym}^3 f, s) \) at \( s = 2 \). Using the reciprocity laws proved in this article and in [Wang], we prove the following theorem towards the rank 1 case of the Bloch–Kato conjecture for the symmetric cube motive of the modular form \( f \) at the end of this article.

**Theorem 4.** Suppose that the modular form \( f \) satisfies the following assumptions.

1. The residual Galois representation \( \bar{\rho}_{f,\lambda}|_{G_\mathbb{Q}(p)} \) is absolutely irreducible;
2. The residual Galois representation \( \bar{\rho}_{f,\lambda} \) is minimally ramified at primes in \( \Sigma^+ \cup \Sigma^- \).
   Moreover \( \bar{\rho}_{f,\lambda} \) is ramified at primes in \( \Sigma^- \);
3. The image of \( \bar{\rho}_{f,\lambda} \) contains \( \text{GL}_2(F_1) \).

If the class \( \Theta^\circ(\mathfrak{f}, d) \in H^1(Q, V^\circ(\mathfrak{f})(-1)) \) is non-zero, then the symmetric cube Bloch–Kato Selmer group

\[
H^1_f(Q, V^\circ(\mathfrak{f})(-1))
\]

is of dimension 1 over \( E_\lambda \).

1.3. Notations and conventions. We will use common notations and conventions in algebraic number theory and algebraic geometry. The cohomologies appeared in this article will be understood as the étale cohomologies. For a field \( K \), we denote by \( K^{ac} \) the separable closure of \( K \) and put \( G_K := \text{Gal}(K^{ac}/K) \) the absolute Galois group of \( K \). We let \( \mathbf{A} \) be the ring of adèles over \( \mathbb{Q} \) and \( \mathbf{A}^\infty \) be the subring of finite adèles. For a prime \( p \), \( \mathbf{A}^\infty,(p) \) is the prime-to-\( p \) part of \( \mathbf{A}^\infty \).

When \( K \) is a local field, we denote by \( \mathcal{O}_K \) its valuation ring and by \( k \) its residue field. We let \( I_K \) be the inertia subgroup of \( G_K \). For a \( G_K \)-module \( M \), we have the following exact sequence of Galois cohomology groups

\[
0 \to H^1_{\text{un}}(K, M) \to H^1(K, M) \xrightarrow{\partial_\text{un}} H^1_{\text{fin}}(K, M) \to 0
\]

(1.2) where \( H^1_{\text{un}}(K, M) = H^1(k, M^{\text{tr}}) \) is called the unramified or the finite part of the cohomology group \( H^1(K, M) \) and \( H^1_{\text{fin}}(K, M) \) is defined as the quotient of \( H^1(K, M) \) by its finite part is called the singular quotient of \( H^1(K, M) \). The natural quotient map \( H^1(K, M) \xrightarrow{\partial_\text{un}} H^1_{\text{fin}}(K, M) \)
will be referred to as the singular quotient map. The element $\partial_p x$ will be referred to as the singular residue of $x$ for $x \in H^1(K,M)$.

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2. Arithmetic level raising for Shimura curves

2.1. Shimura curves and Shimura sets. Let $N = N^+ N^-$ be a factorization of $N$ such that $N^-$ is square-free and has even number of prime divisors. Let $p$ be a prime away from $N$. Let $B = B_{N^-}$ be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $N^-$. Let $\mathcal{O}_B$ be a maximal order of $B$. For a square-free integer $M$ divisible by $N^+$ and relatively prime to $N^-$, let $\mathcal{O}_{B,M}$ be an Eichler order of level $M$. Let $G$ be the algebraic group over $\mathbb{Q}$ defined by $B^\times$. The Eichler order $\mathcal{O}_{B,M}$ defines an open compact subgroup $K_M$ of $G(\mathbb{A}_f)$.

Let $m$ be a square free integer such that $(m, p N^-) = 1$, we define the open compact subgroup $K_{M,m} = \{g = (g_v) \in \hat{\mathcal{O}}_{B,M}^\times : g_v \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod v \text{ for } v \mid m\}$. When $m = 1$, we simply write $K_{M,m}$ as $K_M$. We define a Shimura curve $X_m$ over $\mathbb{Q}$ whose complex points are are uniformized by the double coset space $X_m(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{H}^\pm \times G(\mathbb{A}_f)/K_{N^+,m}$.

We define an integral model $\mathfrak{X}_m$ over $\mathbb{Z}[1/N \!m]$ which represents the following functor. Let $S$ be a locally Noetherian test scheme over $\mathbb{Z}[1/N \!m]$. Then $\mathfrak{X}_m(S)$ classifies the tuples $(A, \iota, C_{N^+}, \alpha_m)$ up to isomorphism where

1. $A$ is an $S$-abelian scheme of relative dimension 2;
2. $\iota : \mathcal{O}_B \hookrightarrow \text{End}_S(A)$ is an action of $\mathcal{O}_B$ on $A$;
3. $C_{N^+}$ is a finite flat subgroup scheme of $A[N^+]$ of order $(N^+)^2$ which is stable and locally cyclic under the action of $\mathcal{O}_B$;
4. $\alpha_m : (\mathbb{Z}/m)^2 \rightarrow A[m]$ is an $\mathcal{O}_B$-equivariant injection of finite flat group schemes over $S$.

This functor is representable by a smooth projective scheme denoted also by $\mathfrak{X}_m$ over $\mathbb{Z}[1/N \!m]$ if $m \geq 5$. If we omit the data $\alpha_m$ from the above moduli problem, we will obtain a coarse moduli space $\mathfrak{X}$ over $\mathbb{Z}[1/N]$ which defines an integral model of the Shimura curve $X$ over $\mathbb{Q}$ whose complex points are uniformized by $X(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{H}^\pm \times G(\mathbb{A}_f)/K_{N^+}$.

In this article, we will sometimes consider the base-change of $\mathfrak{X}_m$ to $\mathbb{Z}_p^2$ for some prime $p \nmid N \!m$ and we will denote it by the same notation $\overline{\mathfrak{X}}_m$. It is well known that $\mathfrak{X}_m$ is smooth and projective of relative dimension 1 over $\mathbb{Z}_p^2$. The generic fiber of $\mathfrak{X}_m$ will simply be denoted by $X_m$ and its special fiber will be denoted by $\overline{\mathfrak{X}}_m$. Suppose $m = d$ is a prime, we have two degeneracy maps

$$X \xleftarrow{\pi_{1,d}} X_d \xrightarrow{\pi_{0,d}} X$$

where $\pi_{0,d}$ is given by forgetting the level structure $\alpha_d$ at $d$ and $\pi_{1,d}$ is given by sending an object $(A, \iota, C_{N^+}, \alpha_d)$ to $(A', \iota', C_{N^+}')$ where $A'$ is the quotient of $A$ by the subgroup scheme...
\[ C_d = \alpha_d((\mathbb{Z}/d)^2), \] \( l' \) is the induced action of \( \mathcal{O}_B \) on \( A' \) and \( C'_N \) is the quotient of \( C_N \) by \( C_d \).

Let \( x = (A, \iota, C_N, \alpha_m) \in \mathfrak{X}_m \) be a \( \mathbb{F}_p \)-point. Then the \( p \)-divisible group \( A[p^\infty] \) of \( A \) can be written as \( A[p^\infty] = E[p^\infty] \times E[p^\infty] \) for a \( p \)-divisible group \( E[p^\infty] \) associated to an elliptic curve \( E \) and \( \mathcal{O}_B \) acts naturally via \( \mathcal{O}_B \otimes \mathbb{Z}_p = M_2(\mathbb{Z}_p) \). Depending on \( E[p^\infty] \) is ordinary or supersingular, we will accordingly call \( x \) an ordinary or a supersingular point. Let \( \mathfrak{X}_m \) be the closed sub-scheme given by those points that are supersingular and let \( \mathfrak{X}_m^{\text{ord}} = \mathfrak{X}_m - \mathfrak{X}_m^{\text{ss}} \) be its complement. We will refer to \( \mathfrak{X}_m \) as the supersingular locus and to \( \mathfrak{X}_m^{\text{ord}} \) as the ordinary locus. Let \( \mathcal{B} = B_{pN^\circ} \) be the definite quaternion algebra with discriminant \( pN^\circ \) and \( \mathcal{O}_B \) be a maximal order. We will write \( \mathcal{O}_{\mathcal{B}, p} = \mathcal{O}_B \otimes \mathbb{Z}_p \) and define \( K_p = \mathcal{O}_{\mathcal{B}, p}^\times \). Note that we can naturally view the prime-to-\( p \) part \( K_{N, m}^{(p)} \) of \( K_{N, m} \) as an open compact subgroup of \( \mathcal{B}^\times(\mathbb{A}^\infty(m)) \). The scheme \( \mathfrak{X}_m^{\text{ss}} \) is given by a finite set of points and we have the following parametrization of \( \mathfrak{X}_m^{\text{ss}} \).

**Lemma 2.1.** Let \( K_{N, m}^{(p)} = K_{N, m}^{(p)}K_p \). We have an isomorphism
\[ \mathfrak{X}_m^{\text{ss}} \cong \mathcal{B}^\times(\mathbb{Q})/\mathcal{B}^\times(\mathbb{A}^\infty)/K_{N, m}^{(p)}, \]

**Proof.** The lemma is well known and can be proved using essentially the same method of the classical work Deuring and Serre. See [DT, Lemma 9] for example. \( \square \)

We will consider the Shimura set associated to the definite quaternion algebra \( \mathcal{B} \) with level \( K_{N, m} \) given by
\[ Z_m(\mathcal{B}) = \mathcal{B}^\times(\mathbb{Q})/\mathcal{B}^\times(\mathbb{A}^\infty)/K_{N, m} \]
The above lemma gives rise to an isomorphism \( \mathfrak{X}_m^{\text{ss}} \cong Z_m(\mathcal{B}) \). When \( m = 1 \), then by convention \( K_{N, m} \) agrees with \( K_N \) and we write \( Z_1(\mathcal{B}) \) simply by \( Z(\mathcal{B}) \). Suppose \( m = 1 \) is a prime, we have two natural degeneracy maps at \( d \) on these Shimura sets
\[ Z(\mathcal{B}) \xrightarrow{\pi_{1,d}} Z_d(\mathcal{B}) \xrightarrow{\pi_{0,d}} Z(\mathcal{B}) \]
where \( \pi_{0,d} \) is the natural one and \( \pi_{1,d} \) is the one defined as in [Lin2, A.1.2]. Let \( R \) be a ring, we write \( \Gamma(Z_m(\mathcal{B}), R) = H^0(Z_m(\mathcal{B}), R) \) for the space of \( R \)-valued functions on \( Z_m(\mathcal{B}) \).

### 2.2. Shimura curves with Iwahori level
We will consider now the curve \( X_m(p) \) over \( \mathbb{Q} \) whose complex points are given by
\[ X_m(p)(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{H}^\pm \times G(\mathbb{A}^\infty)/K_{pN, m} \]
We define an integral model \( \mathfrak{X}_m(p) \) over \( \mathbb{Z}[1/Nm] \) which represents the following functor. Let \( S \) be a locally Noetherian test scheme over \( \mathbb{Z}[1/Nm] \). Then \( \mathfrak{X}_m(p)(S) \) classifies the tuples
\[ (A_1, A_2, i_1, i_2, \pi, C_{N, m}, \alpha_m) \]
up to isomorphism where
1. \( A_i \) for \( i = 1, 2 \) is an \( S \)-abelian scheme of relative dimension 2;
2. \( i_2 : \mathcal{O}_B \hookrightarrow \text{End}_S(A_i) \) is an action of \( \mathcal{O}_B \) on \( A_i \) for \( i = 1, 2 \);
3. \( \pi : A_1 \rightarrow A_2 \) is an isogeny of degree \( p \) that commutes with the action of \( \mathcal{O}_B \);
4. \( C_{N, m} \) is a finite flat subgroup scheme of \( A_1[N, m] \cong A_2[N, m] \) of order \( (N^+)^2 \) which is stable and locally cyclic under the action of \( \mathcal{O}_B \);
(5) \( \alpha_m : (\mathbb{Z}/m)^2 \to \mathcal{A}_1[m] \cong \mathcal{A}_2[m] \) is an \( \mathcal{O}_B \)-equivariant injection of finite flat group schemes over \( S \).

By \[ \text{Buz} \] Theorem 4.7, \( \mathfrak{x}_m(p) \) is regular and proper over \( \mathbb{Z}[1/Nm] \). Again when we consider the base-change of \( \mathfrak{x}_m(p) \) to \( \mathbb{Z}_p^2 \), we use the same symbol for this base-change. We denote by \( X_m(p) \) the generic fiber of \( \mathfrak{x}_m(p) \) and \( \mathfrak{x}_m(p) \) the special fiber of \( \mathfrak{x}_m(p) \). We have the following descriptions of \( \mathfrak{x}_m(p) \) over \( \mathbb{F}_p^2 \).

**Lemma 2.2.** The scheme \( \mathfrak{x}_m(p) \) consists of two irreducible components both isomorphic to \( \mathfrak{x}_m \), crossing transversely at the supersingular locus of \( \mathfrak{x}_m(p) \) which can be identified with the supersingular locus of \( \mathfrak{x}_m \).

**Proof.** This is well-known and see \[ \text{Buz} \] Theorem 4.7(v) for a proof of this. \( \square \)

Let \( \pi_1 : \mathfrak{x}_m(p) \to \mathfrak{x}_m \) be the morphism given by
\[
(\mathfrak{A}_1, \mathfrak{A}_2, \iota_1, \iota_2, \pi, C_{N+}, \alpha_m) \mapsto (\mathfrak{A}_1, \iota_1, C_{N+}, \alpha_m)
\]
and \( \pi_2 : \mathfrak{x}_m(p) \to \mathfrak{x}_m \) be the morphism given by
\[
(\mathfrak{A}_1, \mathfrak{A}_2, \iota_1, \iota_2, \pi, C_{N+}, \alpha_m) \mapsto (\mathfrak{A}_2, \iota_2, C_{N+}, \alpha_m)
\]
We can define two closed immersions \( i_1 : \mathfrak{x}_m \to \mathfrak{x}_m(p) \) and \( i_2 : \mathfrak{x}_m \to \mathfrak{x}_m(p) \) as in the proof of \[ \text{Buz} \] Theorem 4.7(v) such that
\[
(\pi_1 \circ i_1, \pi_2 \circ i_1) = \left( \begin{array}{cc} \text{id} & \text{Frob} \end{array} \right)
\]
\[
(\pi_2 \circ i_2) = \left( \begin{array}{cc} S_p^{-1} \text{Frob}_p & \text{id} \end{array} \right)
\]
where \( S_p \) corresponds to the central element in the spherical Hecke algebra of \( GL_2(\mathbb{Q}_p) \).

We will need the following result known as the Ihara’s lemma. This is proved for the case of classical modular curves by Ribet \[ \text{Rib1} \] and Diamond–Taylor \[ \text{DT} \] for Shimura curves. See also \[ \text{MS21} \] for an interesting approach using the Taylor–Wiles patching method to prove an analogue result for a Shimura curve over a totally real field. Recall that a Hecke module is **Eisenstein** if its support consists of Eisenstein maximal ideals. A maximal ideal is Eisenstein if its associated Galois representation is reducible.

**Theorem 2.3** (Ihara’s lemma). We have the following statements.

1. The kernel of the pull-back map
   \[
   (\pi_1^* + \pi_2^*) : H^1(X_m, \mathbb{Q}^{ac}, k_\lambda)^{\oplus 2} \to H^1(X_m(p), \mathbb{Q}^{ac}, k_\lambda)
   \]
is **Eisenstein**.

2. The cokernel of the push-forward map
   \[
   (\pi_1^*, \pi_2^*) : H^1(X_m(p), \mathbb{Q}^{ac}, k_\lambda) \to H^1(X_m, \mathbb{Q}^{ac}, k_\lambda)^{\oplus 2}
   \]
is **Eisenstein**.

### 2.3. Unramified level raising for Shimura curves

Let \( f \) be a normalized newform in \( S^\text{new}_2(\Gamma_0(N)) \). Let \( E = \mathbb{Q}(f) \) be the Hecke field of \( f \) and let \( \lambda \) be a place above \( l \) in \( E \). We denote by \( \mathcal{O}_\lambda \) the valuation ring of \( E_\lambda \) and by \( k_\lambda \) its residue field. Let \( \varpi \) be a uniformizer of \( \mathcal{O}_\lambda \) and \( \lambda = (\varpi) \) be the maximal ideal. Then by the construction of Eichler–Shimura, we can attach a Galois representation
\[
\rho_{f, \lambda} : G_\mathbb{Q} \to GL_2(E_\lambda) = GL(V_{f, \lambda})
\]
to \( f \) such that \( \text{tr}(\rho_{f,\lambda}(\text{Frob}_p)) = a_p(f) \) where \( a_p(f) \) is the Fourier coefficient of \( f \) at \( p \) for every \( p \nmid N \) and such that the determinant \( \det(\rho_{f,\lambda}) = \epsilon_f \) is the \( l \)-adic cyclotomic character. We denote by \( \overline{\rho}_{f,\lambda} : G_\mathbb{Q} \rightarrow \text{GL}_2(k_\lambda) \) the residual representation of \( \rho_{f,\lambda} \).

We fix an auxiliary prime \( m = d \geq 5 \) considered in the last subsection. Let \( T \) be the abstract Hecke algebra unramified away from \( Nd \). This means \( T \) is the restricted tensor product of the spherical Hecke algebra \( T_v \) for \( \text{GL}_2(\mathbb{Q}_v) \) with \( v \nmid Nd \). The spherical Hecke algebra \( T_v \) is generated by the operators \( T_v, S_v \). The Hecke eigensystem of \( f \) gives rise to a map \( \phi_f : T \rightarrow O_\lambda \) sending \( T_v \) to \( a_v(f) \) and sending \( S_v \) to 1 for \( v \nmid Nd \).

For a finite set of primes \( S \) away from \( Nd \), let \( T[S] \) be the Hecke algebra containing all the Hecke operators away from \( Nd \) and away from the primes in \( S \). We introduce the following ideals of \( T[S] \)

\[
\mathfrak{m}^{[S]} = \ker[T \overset{\phi_f}{\rightarrow} O_\lambda \rightarrow O_\lambda / \lambda] \cap T^{[S]}
\]

and

\[
\mathfrak{p}_n^{[S]} = \ker[T \overset{\phi_f}{\rightarrow} O_\lambda \rightarrow O_\lambda / \lambda^n] \cap T^{[S]}.
\]

If \( S \) is empty, then we omit it from all the notations. We say \( \mathfrak{m} \) is residually irreducible if \( \overline{\rho}_{f,\lambda} \) is absolutely irreducible.

**Definition 2.4.** We say the auxiliary prime \( d \) is clean for \( \overline{\rho}_{f,\lambda} \) if the degeneracy maps \( (\pi_{1,d,*}, \pi_{2,d,*}) \) on \( X_d \) induces an isomorphism

\[
(\pi_{1,d,*}, \pi_{2,d,*}) : H^1(X_d, O_\lambda, \mathcal{O}_\lambda) / \mathfrak{m} \overset{\sim}{\rightarrow} H^1(X_d, O_\lambda, \mathcal{O}_\lambda) / \mathfrak{m}^2
\]

and the degeneracy maps \( (\pi_{1,d,*}, \pi_{2,d,*}) \) on \( Z_d(\mathcal{B}) \) induces an isomorphism

\[
(\pi_{1,d,*}, \pi_{2,d,*}) : \Gamma(Z_d(\mathcal{B}), \mathcal{O}_\lambda) / \mathfrak{m}^{[n]} \overset{\sim}{\rightarrow} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_\lambda) / \mathfrak{m}^{[n]}.
\]

From here on, we will always fix a clean \( d \) for \( \overline{\rho}_{f,\lambda} \). Furthermore, we will impose the following assumptions on \( \overline{\rho}_{f,\lambda} \):

**Assumption 2.5.** We make the following assumptions on \( \overline{\rho}_{f,\lambda} \) and thus on \( \mathfrak{m} \):

1. \( \overline{\rho}_{f,\lambda} | G_{\mathbb{Q}(\zeta_d)} \) is absolutely irreducible;
2. \( \overline{\rho}_{f,\lambda} \) is minimally ramified at primes in \( \Sigma^+ \cup \Sigma_{\text{ram}}^- \) and is ramified at primes in \( \Sigma_{\text{ram}}^- \);
3. The image of \( \overline{\rho}_{f,\lambda} \) contains \( \text{GL}_2(\mathbb{F}_l) \).

**Proposition 2.6.** Under Assumption 2.5, we have an isomorphism

\[
H^1(X_{\mathbb{Q}^c}, O_\lambda(1))/\mathfrak{m} \cong \overline{\rho}_{f,\lambda}
\]

which induces an isomorphism

\[
H^1(X_{d,\mathbb{Q}^c}, O_\lambda(1))/\mathfrak{m} \cong \overline{\rho}_{f,\lambda}^{\otimes 2}.
\]

**Proof.** By [BLR2], it follows that \( H^1(X_{\mathbb{Q}^c}, O_\lambda(1))/\mathfrak{m} \) is isomorphic to \( \overline{\rho}_{f,\lambda}^{\otimes d_m} \) for some \( d_m \geq 1 \). Therefore \( H^1(X_{d,\mathbb{Q}^c}, O_\lambda(1))/\mathfrak{m} \) is isomorphic to \( 2d_m \) copies of \( \overline{\rho}_{f,\lambda} \), by the cleanness of \( d \). On the other hand, we know that \( \Gamma(Z(\mathcal{B}), O_\lambda) / \mathfrak{m} \) is one dimensional over \( k_\lambda \) under the Assumption 2.5 by [CH1 Proposition 6.8] (Note that our assumption is stronger than the (CR+) assumption in [CH1 Proposition 6.8]). By the same argument of [LTXZZ Lemma 6.4.2], \( d_m \) is equal to the dimension of \( \Gamma(Z(\mathcal{B}), O_\lambda) / \mathfrak{m} \) over \( k_\lambda \) and hence is equal to 1.

**Remark 2.7.** Note that \( H^1(X_{\mathbb{Q}^c}, O_\lambda(1))/\mathfrak{m} \) provide a \( G_\mathbb{Q} \)-stable \( O_\lambda \)-lattice \( \rho_{O_\lambda} \) of \( \rho_{f,\lambda} \). Then we have \( H^1(X_{d,\mathbb{Q}^c}, O_\lambda(1))/\mathfrak{m} \cong \rho_{O_\lambda}^{\otimes 2} \) by the above proposition. We denote by \( \rho_{\lambda,n} \) the natural reduction of \( \rho_{O_\lambda} \) by \( \lambda^n \). Then we have \( H^1(X_{d,\mathbb{Q}^c}, O_\lambda)/\mathfrak{m} \cong \rho_{\lambda,n}^{\otimes 2} \).
Let \( \text{cl} : \text{CH}^1(X_d) \to H^2(X_d, O_X(1)) \) be the cycle class map for \( X_d \). Suppose that \( m \) satisfies Assumption 2.5. Using the Hochschild–Serre spectral sequence and the fact that \( H^0 \) and \( H^2 \) of \( X_d \) are both Eisenstein [DT], we have an isomorphism
\[
H^2(X_d, O_X(1)) \cong H^1(F_{p^2}, H^1(X_d, O_X(1))_{m[n]}).
\]
Then the cycle class map \( \text{cl} \) induces the Abel–Jacobi map
\[
\text{AJ}_f : \text{CH}^1(X_d) \to H^1(F_{p^2}, H^1(X_d, O_X(1))_{m[n]}).
\]
The Abel–Jacobi image \( \text{AJ}_f([X_d^{ss}]) \) of the class of the supersingular locus \( [X_d^{ss}] \in \text{CH}^1(X_d) \) can be calculated by the following exact sequence
\[
0 \to H^1(X_d, O_X(1))_{m[n]} \to H^1(X_d, O_X(1))_{m[n]} \to H^0(X_d, O_X(1))_{m[n]} \to 0.
\]
More precisely, the above exact sequence induces the following coboundary map
\[
\Phi : H^0(X_d, O_X(1)) \xrightarrow{G_{p^2}} H^1(F_{p^2}, H^1(X_d, O_X(1))_{m[n]} = H^1(F_{p^2}, H^1(X_d, O_X(1))_{m[n]}).
\]
Note that we have identified \( H^1(X_d, O_X(1))_m \) with \( H^1(X_d, O_X(1))_{m[n]} \) and will identify \( H^1(X_d, O_X(1))_{/p} \), with \( H^1(X_d, O_X(1))_{/p[n]} \); these identifications follow from Proposition 2.6 and will be used elsewhere in this article without further remark. The Abel–Jacobi image \( \text{AJ}_f([X_d^{ss}]) \) of \( [X_d^{ss}] \) is represented by the image of the characteristic function \( 1_{X_d^{ss}} \) of the supersingular locus \( X_d^{ss} \) under \( \Phi \). On the other hand, we can consider the excision exact sequence
\[
0 \to H^0(X_d, O_X(1))_{m[n]} \to H^1(X_d, O_X(1))_{m[n]} \to H^1(X_d, O_X(1))_{m[n]} \to 0
\]
whose connecting homomorphism gives the map
\[
\Phi^* : H^1(X_d, O_X(1))_{m[n]} \to H^1(F_{p^2}, H^0(X_d, O_X(1))_{m[n]}).
\]
These maps induce
\[
\Phi_n : H^0(X_d, O_X(1))_{/p[n]} \xrightarrow{G_{p^n}} H^1(F_{p^n}, H^1(X_d, O_X(1))_{/p[n]}),
\]
\[
\Phi^*_n : H^1(X_d, O_X(1))_{/p[n]} \xrightarrow{G_{p^n}} H^1(F_{p^n}, H^0(X_d, O_X(1))_{/p[n]}),
\]
for each \( n \geq 1 \). Since \( X_d^{ss} \) is naturally defined over \( F_{p^2} \), we can naturally identify the source of \( \Phi_n \) and the target of \( \Phi^*_n \) with \( \Gamma(Z_d(B), O_X(1))_{/p[n]} \). It is clear that the two maps \( \Phi_n \) and \( \Phi^*_n \) are dual to each other under the Tate local duality for \( F_{p^2} \) and the Poincaré duality for \( X_d \). The theorem below is usually referred to as the arithmetic level raising theorem for Shimura curves. It is proved in [LT, Proposition 4.8] and [Xiao]. We will give a slightly different proof of this theorem following the strategy of [Xiao].

**Proposition 2.8.** The map \( \Phi_n \) is surjective and \( \Phi^*_n \) is injective.

**Proof.** We prove that \( \Phi_n \) is surjective and the injectivity of \( \Phi^*_n \) follows by duality. We consider the localized weight spectral sequence [RZ, Sam] for \( H^1(X_d, O_X(1))_{m[n]} \) and its induced
monodromy filtration:
\[
0 \subset E_{1,0}^{1} M_{1} H^{1}(X_{d}(p) Q_{p}, O_{\lambda}(1))_{m[p]} \subset E_{1,1}^{0} M_{0} H^{1}(X_{d}(p) Q_{ac}, O_{\lambda}(1))_{m[p]} \subset E_{2,0}^{1} M_{-1} H^{1}(X_{d}(p) Q_{ac}, O_{\lambda}(1))_{m[p]}.
\]

By [Ill1, example 2.4.6] and the discussions in Lemma 2.2, we have
\[
\begin{align*}
E_{2,0}^{1} &= H^{0}(\mathcal{X}_{d,c}^{ss}, O_{\lambda}(1))_{m[p]}; \\
E_{2,1}^{1} &= H^{1}(\mathcal{X}_{d,c}^{ss}, O_{\lambda}(1))_{m[p]}; \\
E_{2,2}^{1} &= H^{0}(\mathcal{X}_{d,s}^{ss}, O_{\lambda}(1))_{m[p]}.
\end{align*}
\]

Next consider the pushforward map
\[
(\pi_{1*}, \pi_{2*}): H^{1}(X_{d}(p) Q_{p}, O_{\lambda}(1))_{m[p]} \to H^{1}(X_{d}, Q_{ac}, O_{\lambda}(1))_{m[p]}.
\]
This is surjective by Ihara’s lemma Theorem 2.3 and Nakayama’s lemma. It is well-known that the composite
\[
E_{2,0}^{1} \to H^{1}(X_{d}(p) Q_{ac}, O_{\lambda}(1))_{m[p]} \xrightarrow{(\pi_{1*}, \pi_{2*})} H^{1}(X_{d}, Q_{ac}, O_{\lambda}(1))_{m[p]} \to \text{coker}(\nabla)
\]
is zero. Indeed, the term \(E_{2,1}^{1}\) corresponds to the toric part of the Neron model of the Jacobian of \(X_{d}(p)\) over \(Q_{p}\) and projects to zero to the \(p\)-old part of \(X_{d}(p)\), see [Ri1, Theorem 3.10]. Therefore we obtain the following commutative diagram
\[
\begin{array}{ccc}
H^{1}(X_{d}, Q_{ac}, O_{\lambda}(1))_{m[p]}^{\otimes 2} & \xrightarrow{(i_{1*}, i_{2*})} & H^{1}(X_{d}(p) Q_{ac}, O_{\lambda}(1))_{m[p]} \\
\downarrow & & \downarrow \phi' \\
H^{1}(X_{d}, Q_{ac}, O_{\lambda}(1))_{m[p]}^{\otimes 2} & \xrightarrow{\nabla} & H^{1}(X_{d}, Q_{ac}, O_{\lambda}(1))_{m[p]}^{\otimes 2} \\
& & \downarrow \text{coker}(\nabla)
\end{array}
\]
where the top row of the diagram is the monodromy filtration of \(H^{1}(X_{d} \otimes Q_{p}, O_{\lambda}(1))_{m[p]}\) which is exact on the right. The map \(\phi'\) is the one naturally induced by \((\pi_{1*}, \pi_{2*})\). The map \(\nabla\) is by definition given by the composite of \((\pi_{1*}, \pi_{2*})\) and \((i_{1*}, i_{2*})\). By (2.1), the map \(\nabla\) is given by the matrix
\[
\begin{pmatrix}
id & \text{Frob}_{p} \\
\text{Frob}_{p} & id
\end{pmatrix}
\]
since the central element \(S_{p}\) has trivial action. It follows then that we have an isomorphism
\[
\text{coker}(\nabla) = H^{1}(\mathcal{F}_{p}, H^{1}(X_{d}, Q_{ac}, O_{\lambda}(1))_{m[p]}).
\]

Since \((\pi_{1*}, \pi_{2*})\) is surjective, the map \(\Phi'\) is surjective as well. Let \(\Phi'_{n}\) be the reduction of \(\Phi'\) modulo \(p_{n}^{[p]}\). Therefore we are left to show that \(\Phi'_{n}\) agree with the map \(\Phi_{n}\). To show this, we rely on some results proved in [Ill2]. More precisely, the natural quotient map
\[
H^{1}(X_{d}(p) Q_{ac}, O_{\lambda}(1))_{m[p]} \to H^{0}(\mathcal{X}_{d,c}^{ss}, O_{\lambda})_{m[p]}
\]
in the monodromy filtration factors through \(H^{1}(\mathcal{X}_{d,c}^{ord}, O_{\lambda})\):
\[
H^{1}(X_{d}(p) Q_{ac}, O_{\lambda}(1))_{m[p]} \xrightarrow{i_{1}} H^{1}(\mathcal{X}_{d,c}^{ord}, O_{\lambda}(1))_{m[p]} \to H^{0}(\mathcal{X}_{d,c}^{ss}, O_{\lambda})_{m[p]} \to 0
\]
where $H^1(\mathfrak{X}_{d,F_p^c}^{\text{ord}}, \mathcal{O}_\lambda(1))_{\mathfrak{m}[\nu]} \to H^0(\mathbb{Z}_{d}(\mathcal{B}), \mathcal{O}_\lambda)_{\mathfrak{m}[\nu]}$ comes from the natural excision exact sequence for $H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))_m$ and the $i_1^*$ is the pullback of the cohomology of nearby cycles

$$H^1(\mathfrak{X}_{d}(p)_{\mathfrak{F}_p^c}, R\Psi(\mathcal{O}_\lambda)(1))_{\mathfrak{m}[\nu]} \xrightarrow{i_1^*} H^1(\mathfrak{X}_{d,F_p^c}^{\text{ord}}, R\Psi(\mathcal{O}_\lambda)(1))_{\mathfrak{m}[\nu]}.$$ 

For the proof of these facts, see [Ill2 Proposition 1.5]. Let $x \in H^0(\mathbb{Z}_{d}(\mathcal{B}), \mathcal{O}_\lambda)_{\mathfrak{m}[\nu]}$ and $\tilde{x}$ be a preimage of $x$ in $H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))_{\mathfrak{m}[\nu]}$. Since $\phi_{1*}$ is the identity map, we can take $\phi_{1*}(\tilde{x})$ as a preimage of $\tilde{x}$ in $H^1(\mathfrak{X}_{d}(p)_{\mathfrak{F}_p^c}, R\Psi(\mathcal{O}_\lambda)(1))_{\mathfrak{m}[\nu]}$. Therefore for $x \in H^0(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))_{\mathfrak{m}[\nu]}$, we have $\Phi'(x) = (\pi_{1*}\phi_{1*}(\tilde{x}), \pi_{2*}\phi_{2*}(\tilde{x})) = (\tilde{x}, \text{Frob}_p(\tilde{x}))$. Since the natural quotient map $H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))_{\mathfrak{m}[\nu]} \to \text{coker}(\nabla)$ is given by sending $(x, y) \in H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))_{\mathfrak{m}[\nu]}$ to $(x - \text{Frob}_p(y))$ in light of the form of $\nabla$, we have $\Phi'(x) = (1 - \text{Frob}_p^2)\tilde{x}$. But this is precisely the definition of $\Phi(x)$. This finishes the proof that the map $\Phi$ is surjective and thus $\Phi_n$ is also surjective.

\[\Phi_n : H^0(\mathfrak{X}_{d,F_p^c}^{\text{sc}}, \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}^{G_F} \xrightarrow{\sim} H^1(F_p, H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))/\mathfrak{m}[\nu]),\]

which can be identified with an isomorphism

$$\Phi_n : \Gamma(Z_{d}(\mathcal{B}), \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}^{G_F} \xrightarrow{\sim} H^1(F_p, H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda,n(1))/\mathfrak{m}[\nu]).$$

(2) Similarly, we have a canonical isomorphism

$$\Phi_n^* : H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}^{G_F} \xrightarrow{\sim} H^1(F_p, H^0(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}^{G_F}),$$

which can be identified with an isomorphism

$$\Phi_n^* : H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}^{G_F} \xrightarrow{\sim} \Gamma(Z_{d}(\mathcal{B}), \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}^{G_F}.$$

\textbf{Proof.} We only need to show that the two sides of $\Phi_n$ have the same cardinality. But by Proposition 2.9, we know that

$$H^1(F_p, H^1(\mathfrak{X}_{d,F_p^c}, \mathcal{O}_\lambda(1))/\mathfrak{m}[\nu]) = H^1(F_p, \mathfrak{m}[\nu])$$

is free of rank two over $\mathcal{O}_{\lambda,n}$. Note Assumption 2.5 implies that $\Gamma(Z_{d}(\mathcal{B}), k_{\lambda})_{\mathfrak{m}}$ is one-dimensional by [CH1 Proposition 6.8]. By the cleanness of $d$, it follows that $\Gamma(Z_{d}(\mathcal{B}), k_{\lambda})_{\mathfrak{m}[\nu]}$ is two dimensional and hence $\Gamma(Z_{d}(\mathcal{B}), \mathcal{O}_\lambda)_{/\mathfrak{m}[\nu]}$ is free of rank two over $\mathcal{O}_{\lambda,n}$. The theorem follows.

\[\square\]
Proposition 2.11. Let $p$ be an $n$-admissible prime for $f$. We assume that Assumption 2.5 holds, then we have the following isomorphism

$$H^1(X_d, F^{ac}_p, \mathcal{O}_\lambda(1))_{/p_n} \cong (\mathbb{Z}_d(B), \mathcal{O}_{\lambda,n}(1))_{/F[p]} \oplus (\mathbb{Z}_d(B), \mathcal{O}_{\lambda,n})_{/F[p]}$$

as $\mathcal{O}_{\lambda,n}[G_{F[p]}]$-modules.

Proof. Since $p$ is $n$-admissible and Proposition 2.6 holds, we have an exact sequence of $\mathcal{O}_{\lambda,n}[G_{F[p]}]$-modules

$$0 \to H^1(X_d, F^{ac}_p, \mathcal{O}_\lambda(1))_{/p_n} \to H^1(F^{ac}_p, X_d, \mathcal{O}_\lambda(1))_{/p_n} \to H^1(F^{ac}_p, H^1(X_d, F^{ac}_p, \mathcal{O}_\lambda(1))_{/p_n}) \to 0.$$

This exact sequence clearly splits and we obtain an identification

$$H^1(X_d, F^{ac}_p, \mathcal{O}_\lambda(1))_{/p_n} = (\mathbb{Z}_d(B), \mathcal{O}_{\lambda,n}(1))_{/F[p]} \oplus (\mathbb{Z}_d(B), \mathcal{O}_{\lambda,n})_{/F[p]}$$

using $\Phi_n$ and $\Phi_n^*$. □

We derive from this proposition the following corollary. This corollary is not strictly needed for later applications but we include it as we think it is of independent interest.

We define the $p$-new part $H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\dagger$ of $H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/F[p]}$ by

$$H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\dagger = \ker[H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/F[p]} \to H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/F[p]}]$$

and the $p$-new quotient $H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\ddagger$ of $H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/F[p]}$ by

$$H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\ddagger = \ker[H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/F[p]} \to H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/F[p]}].$$

Corollary 2.12. Let $p$ be an $n$-admissible prime for $f$. We assume that Assumption 2.5 holds. Then we have the following isomorphisms of $\mathcal{O}_{\lambda,n}[G_{\mathbb{Q}}]$-modules

$$H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\dagger \cong H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))_{/p_n} \cong H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\ddagger.$$

Proof. We have a commutative diagram

$$\begin{array}{cccccc}
\ker(\overline{\tau}) & \longrightarrow & \ker(\alpha) & \longrightarrow & H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda}(1))_{/p_n} & \\
\downarrow & & \downarrow & & \downarrow & \\
H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda}(1))_{/p_n} & \longrightarrow & H^1(X_d, F^{ac}_p, R\Phi(\mathcal{O}_{\lambda}(1))_{/p_n} & \longrightarrow & H^1(X_d, F^{ac}_p, R\Phi(\mathcal{O}_{\lambda}(1))_{/p_n} & \\
\downarrow & & \downarrow & & \downarrow & \\
H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda}(1))_{/p_n} & \longrightarrow & H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda}(1))_{/p_n} & \longrightarrow & 0 & \\
\downarrow & & \downarrow & & & \\
\ker(\overline{\alpha}) & \longrightarrow & \ker(\alpha) & \longrightarrow & H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda}(1))_{/p_n} & \\
\end{array}$$

where $\overline{\tau}$ and $\alpha$ are both induced by the degeneracy maps $(\pi_{1*}, \pi_{2*})$. Here the second row is the specialization exact sequence. Note that $\ker(\alpha) = H^1(X_d, F^{ac}_p, \mathcal{O}_{\lambda,n}(1))^\dagger$ and
H^1(X_d(p)_{\mathbb{F}_p}, R\Phi(O_\lambda)(1))_{/p^n} \cong \Gamma(Z_d(\mathcal{B}), O_\lambda)_{/p^n} \text{ canonically. To understand } \ker(\pi), \text{ we consider the following commutative diagram}

\[
\begin{array}{ccl}
\ker(\pi) & \longrightarrow & \ker(\nabla) \\
\downarrow & & \downarrow \\
H^0(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p^n} & \longrightarrow & H^1(X_d(p)_{\mathbb{F}_p}, O_\lambda(1))_{/p^n} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^1(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p^n} \\
\downarrow & & \downarrow \\
\text{coker}(\pi) & \longrightarrow & \text{coker}(\nabla)
\end{array}
\]

which implies (by the snake lemma) that \( \ker(\pi) \) sits in the exact sequence

\[0 \to H^0(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p^n} \to \ker(\pi) \to \ker(\nabla) \to 0\]

and that \( \text{coker}(\pi) = \text{coker}(\nabla) \). But we have seen in the proof of Proposition 2.8 that \( \ker(\nabla) \cong H^0(\mathbb{F}_p, H^1(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p^n}) \) and that \( \text{coker}(\nabla) \cong H^1(\mathbb{F}_p, H^1(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p^n}) \). By Proposition 2.11, they are both isomorphic to \( \Gamma(Z_d(\mathcal{B}), O_\lambda)_{/p^n} \). Hence by the snake lemma again, we have \( \ker(\pi) = \ker(\alpha) \) and thus

\[0 \to \Gamma(Z_d(\mathcal{B}), O_\lambda(1))_{/p^n} \to \ker(\alpha) = H^1(X_d(p)_{\mathbb{Q}_c}, O_{\lambda,n}(1))_{/p} \to \Gamma(Z_d(\mathcal{B}), O_\lambda)_{/p^n} \to 0\]

which splits as \( \ker(\pi) \) is unramified. Comparing this exact sequence with Proposition 2.11 we see that

\[H^1(X_d(p)_{\mathbb{Q}_c}, O_{\lambda,n}(1))_{/p} \cong H^1(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p} \]

as desired. The other isomorphism \( H^1(X_d(p)_{\mathbb{Q}_c}, O_{\lambda,n}(1))_{/p} \cong H^1(X_d, \mathbb{Q}_c, O_\lambda(1))_{/p} \) follows from a similar argument.

3. Arithmetic level raising for triple product of Shimura curves

3.1. The triple tensor product Galois representation. Let \( f = (f_1, f_2, f_3) \in S_2^{\text{new}}(\Gamma_0(N))^3 \) be a triple of normalized newforms of level \( \Gamma_0(N) \) with \( q \)-expansions:

\[f_1 = \sum_{n \geq 1} a_n(f_1)q^n,\]
\[f_2 = \sum_{n \geq 1} a_n(f_2)q^n,\]
\[f_3 = \sum_{n \geq 1} a_n(f_3)q^n.\]

We assume \( N = N^+N^- \) such that \( (N^+, N^-) = 1 \) and \( N^- \) is square-free with even number of prime divisors.

For \( i \in \{1, 2, 3\} \), let \( E_i = \mathbb{Q}(f_i) \) be the Hecke field of \( f_i \). Let \( \lambda_i \) be a place in \( E_i \) above \( l \) and write \( E_{\lambda_i} \) for the completion of \( E_i \) at \( \lambda_i \). Let \( O_{\lambda_i} \) be the valuation ring of \( E_{\lambda_i} \). We fix a maximal ideal \( \lambda_i \) of \( O_{\lambda_i} \) and denote by \( k_{\lambda_i} \) its residue field. Let \( \rho_{f_i, \lambda_i} : G_{\mathbb{Q}} \to \text{GL}_2(E_{\lambda_i}) \)
be the Galois representation attached to $f_i$ whose residue representation is $\mathfrak{f}_{f_i,\lambda_i}$. Let $d$ be a clean prime for each $\mathfrak{p}_{f_i,\lambda_i}$. We have the morphism $\phi_i : \mathfrak{T} \to O_{\lambda_i}$ corresponding to the Hecke eigensystem of $f_i$ for each $i$.

For a finite set of primes $S$ away from the set of prime divisors of $Nd$, we introduce some ideals of $T^{[S]}$ associated to the triple $\mathfrak{f}$:

$$m_i^{[S]} = \ker[T \xrightarrow{\phi_i} O_{\lambda_i} \to O_{\lambda_i}/\lambda] \cap T^{[S]}$$

$$p_{i,n}^{[S]} = \ker[T \xrightarrow{\phi_i} O_{\lambda_i} \to O_{\lambda_i}/\lambda^n] \cap T^{[S]}$$

for each $n \geq 1$ and $i \in \{1, 2, 3\}$. If $S$ is empty, then we omit it from all the notations. We have the triple of ideals $m_{i,n}^{[S]} = (m_1^{[S]}, m_2^{[S]}, m_3^{[S]})$ and $p_{i,n}^{[S]} = (p_1^{[S]}, p_2^{[S]}, p_3^{[S]})$. It will be assumed that $m_i$ satisfies Assumption 2.5 By Proposition 2.6, $H^1(X_d, Q_{ac}, O_{\lambda_i}(1))_{m_i}$ defines a $G_{\mathbb{Q}}$-stable $O_{\lambda_i}$-lattice $\rho_{\mathfrak{O}_{\lambda_i}}$ in $\rho_{\mathfrak{f}_{i,n}}$. We denote by $\rho_{i,n}$ the reduction of $\rho_{\mathfrak{O}_{\lambda_i}}$ modulo $\lambda^n$. Then $H^1(X_d, Q_{ac}, O_{\lambda_i}(1)_{p_{i,n}} \cong \rho_{i,n}^{[S]}$ for every $n \geq 1$.

We will be concerned with the following twist of the 8-dimensional triple tensor product Galois representation

$$V(\mathfrak{f})(-1) = \rho_{f_1,\lambda_1} \otimes \rho_{f_2,\lambda_2} \otimes \rho_{f_3,\lambda_3}(-1)$$

associated to the triple $\mathfrak{f}$ over $\mathfrak{E}_{\lambda} = E_{\lambda_1} \times E_{\lambda_2} \times E_{\lambda_3}$. There is a natural $G_{\mathbb{Q}}$-stable lattice given by

$$M(\mathfrak{f})(-1) = \rho_{\mathfrak{O}_{\lambda_1}} \otimes \rho_{\mathfrak{O}_{\lambda_2}} \otimes \rho_{\mathfrak{O}_{\lambda_3}}(-1)$$

over $\mathfrak{O}_{\lambda} = \mathfrak{O}_{\lambda_1} \otimes \mathfrak{O}_{\lambda_2} \otimes \mathfrak{O}_{\lambda_3}$. We define the $\mathfrak{O}_{\lambda}[G_{\mathbb{Q}}]$-module $M(\mathfrak{f}, d)$ by

$$M(\mathfrak{f}, d) = \otimes_{i=1}^3 H^1(X_d, Q_{ac}, O_{\lambda_i}(1))_{m_i} \cong \rho_{\mathfrak{O}_{\lambda_1}}^{[S]} \otimes \rho_{\mathfrak{O}_{\lambda_2}}^{[S]} \otimes \rho_{\mathfrak{O}_{\lambda_3}}^{[S]}(-1).$$

And for each $n \geq 1$, we define the $\mathfrak{O}_{\lambda,n}[G_{\mathbb{Q}}]$-module $M_n(\mathfrak{f}, d)$ by

$$M_n(\mathfrak{f}, d) = \otimes_{i=1}^3 H^1(X_d, Q_{ac}, O_{\lambda_i}(1))_{p_{i,n}} \cong \rho_{\mathfrak{O}_{\lambda_1}}^{[S]} \otimes \rho_{\mathfrak{O}_{\lambda_2}}^{[S]} \otimes \rho_{\mathfrak{O}_{\lambda_3}}^{[S]}(-1)$$

where we put $\mathfrak{O}_{\lambda,n} = \mathfrak{O}_{\lambda_1,n} \otimes \mathfrak{O}_{\lambda_2,n} \otimes \mathfrak{O}_{\lambda_3,n}$.

These Galois modules appears naturally in the middle degree cohomology $H^3(X_d, Q_{ac}, Z_l(2))$ of the triple fiber product of Shimura curves $X_d^3$. By the Künneth formula, the triple tensor product Hecke algebra $\mathfrak{T} \otimes \mathfrak{T} \otimes \mathfrak{T}$ acts on $H^3(X_d, Q_{ac}, Z_l(2))$. Therefore we can localize $H^3(X_d, Q_{ac}, Z_l(2))$ at $\mathfrak{m}_l$. Let $T^{[S]}_{\mathfrak{m}_l} = T_{\mathfrak{m}_1} \otimes T_{\mathfrak{m}_2} \otimes T_{\mathfrak{m}_3}$. The maps $\phi_i$ induce maps $\phi_i : T_{\mathfrak{m}_l}^{[S]} \to \mathfrak{O}_{\lambda}$ and $\phi_{\mathfrak{f}_{n,l}} : T_{\mathfrak{m}_l}^{[S]} \to \mathfrak{O}_{\lambda,n}$. We have the following lemma.

**Lemma 3.1.** There is an isomorphism of $\mathfrak{O}_{\lambda}[G_{\mathbb{Q}}]$-modules

$$M(\mathfrak{f}, d)(-1) \cong H^3(X_d, Q_{ac}, Z_l(2))_{m_l} \otimes T^{[S]}_{\mathfrak{m}_l} \mathfrak{O}_{\lambda}$$

and an isomorphism of $\mathfrak{O}_{\lambda,n}[G_{\mathbb{Q}}]$-modules

$$M_n(\mathfrak{f}, d)(-1) \cong H^3(X_d, Q_{ac}, Z_l(2))_{m_l} \otimes T^{[S]}_{\mathfrak{m}_l} \mathfrak{O}_{\lambda,n}$$

for each $n \geq 1$.

**Proof.** This follows from an easy application of the Künneth formula using the fact that $H^0$ and $H^2$ of $X_d, Q_{ac}$ are both Eisenstein as Hecke modules, see [DT, Lemma 3]. \[\square\]
3.2. Unramified level raising for triple product of Shimura curves. We recall next the definition of an \( n \)-admissible prime for \( \mathbf{f} \) introduced in [Wang, Definition 4.3].

**Definition 3.2.** Let \( n \geq 1 \) be an integer. We say that a prime \( p \) is \( n \)-admissible for \( \mathbf{f} = (f_1, f_2, f_3) \) if

1. \( p \nmid N_l; \)
2. \( l \nmid p^2 - 1; \)
3. \( \varpi^i_n \mid p + 1 - \epsilon_{p,i} \alpha_p(f_i) \) with \( \epsilon_{p,i} = \pm 1 \) for \( i \in \{1, 2, 3\}; \)
4. \( \epsilon_{p,1} \epsilon_{p,2} \epsilon_{p,3} = 1. \)

Let \( p \) be an \( n \)-admissible prime for \( \mathbf{f} \). Since the integral model \( \overline{X}_d^3 \) of \( X_d^3 \) acquires good reduction at \( p \), \( M_n(\mathbf{f}, d) \) is unramified at \( p \) as an \( \mathcal{O}_{\Delta,n}[G_{\mathbb{Q}}] \)-module and therefore it makes sense to consider \( H^1(\mathbb{F}_p^2, M_n(\mathbf{f}, d)(-1)) \). We define the \( \mathcal{O}_{\Delta,n} \)-module \( Z_n(\mathbf{f}, d) \) by

\[
Z_n(\mathbf{f}, d) = \bigotimes_{i=1}^3 \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i})/p_{i,n}^{[p\mid]}. 
\]

Let \( \overline{X}_i \) be the \( i \)-th copy of \( \overline{X}_d \) in the triple product \( \overline{X}_d^3 \) for \( i \in \{1, 2, 3\} \). Let \( T_{i,n} = H^1(\overline{X}_i, \mathcal{O}_{\lambda_i}(1))/p_{i,n}. \) By Proposition 2.11 we have a decomposition

\[
T_{i,n} = T_{i,n}^+ \oplus T_{i,n}^- 
\]

where

- \( T_{i,n}^+ \) is given by \( H^1(\overline{X}_i, 1) \) which is identified with \( \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}(1))/p_{i,n}^{[p\mid]} \) via the level raising map \( \Phi_{i,n}^*; \)
- \( T_{i,n}^- \) is given by \( H^1(\mathcal{F}_p^2, H^1(\overline{X}_i, \mathcal{O}_{\lambda_i}(1))/p_{i,n}) \) which is identified with \( \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i})/p_{i,n}^{[p\mid]} \) via the level raising map \( \Phi_{i,n}. \)

**Lemma 3.3.** Under the Poincaré duality between \( T_{i,n} \) and \( T_{i,n}(-1) \), \( T_{i,n}^- \) is dual to \( T_{i,n}^+(-1) \) and \( T_{i,n}^+ \) is dual to \( T_{i,n}^+(-1) \).

*Proof.* This follows immediately from the constructions in Proposition 2.11 \( \square \)

**Lemma 3.4.** Let \( p \) be an \( n \)-admissible prime for \( \mathbf{f} \). Then \( M_n(\mathbf{f}, d) \) is unramified at \( p \) and we have a natural isomorphism

\[
M_n(\mathbf{f}, d) \cong Z_n(\mathbf{f}, d) \oplus Z_n^{\otimes 3}(\mathbf{f}, d)(1) \oplus Z_n^{\otimes 3}(\mathbf{f}, d)(2) \oplus Z_n(\mathbf{f}, d)(3)
\]

between \( \mathcal{O}_{\Delta,n}[G_{\mathbb{F}_p^2}] \)-modules.

*Proof.* This follows from the definition of an \( n \)-admissible prime for \( \mathbf{f} \) and the split exact sequence in Proposition 2.11. Indeed, we have

\[
M_n(\mathbf{f}, d) = T_{1,n} \otimes T_{2,n} \otimes T_{3,n} \\
= \bigoplus_{?, \in \{\pm\}} T_{1,n}^{?,1} \otimes T_{2,n}^{?,2} \otimes T_{3,n}^{?,3} \\
= Z_n(\mathbf{f}, d) \oplus Z_n^{\otimes 3}(\mathbf{f}, d)(1) \oplus Z_n^{\otimes 3}(\mathbf{f}, d)(2) \oplus Z_n(\mathbf{f}, d)(3).
\]

\( \square \)
The following theorem will be referred to as the unramified arithmetic level raising theorem for the triple product of Shimura curves. We call it the unramified arithmetic level raising theorem as we are interested in the cohomology of the triple product of Shimura curves at a place of good reduction.

**Theorem 3.5 (Unramified level raising).** Let $p$ be an $n$-admissible prime for $f$. We assume that each maximal ideal in the triple $m_f = (m_1, m_2, m_3)$ satisfies Assumption [2.5]. Then we have the following isomorphism

\[(3.1) \quad \Phi_{f,n} : H^1(F_p^2, M_n(f, d)(-1)) \cong \bigoplus_{j=1}^{3} \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_i}^{[p]}.\]

**Proof.** By Lemma 3.3 we have

\[
H^1(F_p^2, M_n(f, d)(-1)) \cong H^1(F_p^2, Z_n(f, d)(-1) \oplus Z_n^3(f, d)(1) \oplus Z_n(f, d)(2))
\]

and

\[
\cong \bigoplus_{j=1}^{3} \left( \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_i}^{[p]} \right).
\]

\[\square\]

**Corollary 3.6.** Let $p$ be an $n$-admissible prime for $f$. We assume that each maximal ideal in the triple $m_f = (m_1, m_2, m_3)$ satisfies Assumption [2.5]. Then we have the following isomorphism

\[
\Phi_{f,n} : H^1_{\text{lin}}(Q_p, M_n(f, d)(-1)) \cong \bigoplus_{j=1}^{3} \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_i}^{[p]}.\]

**Proof.** Since $M_n(f, d)$ is unramified, we have

\[
H^1_{\text{lin}}(Q_p, M_n(f, d)(-1)) \cong H^1(F_p, M_n(f, d)(-1)).
\]

By [Rib2] Proposition 3.8, the non-trivial element in Gal($F_p^2/F_p$) acts on $\Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_i}$ by $\epsilon_i$. Therefore the non-trivial element in Gal($F_p^2/F_p$) acts on $\bigoplus_{j=1}^{3} \left( \bigoplus_{i=1}^{3} \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_i}^{[p]} \right)$ by the product of the sign $(\epsilon_1, \epsilon_2, \epsilon_3)$ which is 1 by the definition of an $n$-admissible prime for $f$. Since the map $\Phi_{f,n}$ commutes with the action of Gal($F_p^2/F_p$) by construction, the conclusion immediately follows. \[\square\]

### 3.3. Reciprocity laws for Gross–Kudla–Schoen diagonal cycles

In this subsection, we will always fix an $n$-admissible prime $p$ for $f$. Let $m_f = (m_1, m_2, m_3)$ be the triple of maximal ideals that all satisfy Assumption [2.5]. Recall we have the Shimura curve $X_d$ over $\mathbb{Q}$ and its integral model $X_d$ over $\mathbb{Z}[1/Nd]$. We consider the diagonal embedding $\theta : X_d \to X_d^3$ of $X_d$ into its triple fiber product. This gives a class $\Delta_d = \theta_*[X_d] \in \text{CH}^2(X_d^3)$ which will be referred to as the *Gross–Kudla–Schoen diagonal cycle*. We consider the composite map induced by the étale cycle class map localized at the triple of maximal ideals in $m_f$

\[
\text{cl} : \text{CH}^2(X_d^3) \to H^4(X_d^3, \mathbb{Z}/(2))_{m_f} \otimes_{T_{m_f}^{[3]}} \mathcal{O}_\Delta.
\]

Since $H^4(X_d^3, \mathbb{Q}_{\text{prim}}, \mathbb{Z}/(2))_{m_f} \otimes_{T_{m_f}^{[3]}} \mathcal{O}_\Delta$ is zero by the Künneth formula and our assumption that $m_f$ is absolutely irreducible, the Hochschild–Serre spectral sequence induces the following Abel–Jacobi map

\[
\text{AJ}_f : \text{CH}^2(X_d^3) \to H^1(Q, H^3(X_d^3, \mathbb{Z}/(2))_{m_f} \otimes_{T_{m_f}^{[3]}} \mathcal{O}_\Delta).
\]
We can compose it further with the natural map $M(\mathbf{f}, d)(-1) \rightarrow M_n(\mathbf{f}, d)(-1)$. We will denote the composite map by

$$AJ_{\mathbf{f}, n} : \text{CH}^2(X_d^3) \rightarrow \text{H}^1(Q, M_n(\mathbf{f}, d)(-1)).$$

Let $\Theta_n(\mathbf{f}, d) \in \text{H}^1(Q, M_n(\mathbf{f}, d)(-1))$ be the image of $\Delta_d = \theta_1[X_d]$ under $AJ_{\mathbf{f}, n}$. Recall that $Z_d(\mathcal{B})$ is the Shimura set associated to the definite quaternion algebra $\mathcal{B}$. It can also be identified as the supersingular locus $X_d^3$ of the special fiber $X_d$. We can restrict the map $\theta$ to the supersingular locus $X_d^3 \cong Z_d(\mathcal{B})$. We will write the resulting map as $\theta : Z_d(\mathcal{B}) \rightarrow Z_d(\mathcal{B})^3$ and obtain the diagonal cycle $\Delta_d(\mathcal{B}) = \theta_1[Z_d(\mathcal{B})]$.

We have a bilinear pairing

$$(, ) : \bigoplus_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}) \times \bigoplus_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), E_{\lambda_i}/\mathcal{O}_{\lambda_i}) \rightarrow \mathcal{O}_\lambda$$

given by the formula

$$\left( \bigotimes_{i=1}^{3} \phi_i \right) = \sum_{(z_1, z_2, z_3) \in Z_d(\mathcal{B})^3} \phi_1(z_1) \otimes z_2 \otimes \phi_3(z_3)$$

for $\bigotimes_{i=1}^{3} \phi_i \in \bigotimes_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i})$ and $\bigotimes_{i=1}^{3} \phi_i \in \bigotimes_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), E_{\lambda_i}/\mathcal{O}_{\lambda_i})$ induced by the Poincaré duality on $Z_d(\mathcal{B})^3$. This pairing gives rise naturally to a pairing

$$(, ) : \bigoplus_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}) / \mathfrak{p}_{i,n}^{[p^j]} \times \bigoplus_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), E_{\lambda_i}/\mathcal{O}_{\lambda_i}) / \mathfrak{p}_{i,n}^{[p^j]} \rightarrow \mathcal{O}_{\Delta,n}.$$ 

**Lemma 3.7.** Let $p$ be an $n$-admissible prime for $\mathbf{f}$. The class $\text{loc}_p(\Theta_n(\mathbf{f}, d)) \in \text{H}^1(Q, M_n(\mathbf{f}, d)(-1))$ lies in the finite part $H^{1}_{\text{fin}}(Q, M_n(\mathbf{f}, d)(-1))$ of $H^1(Q, M_n(\mathbf{f}, d)(-1))$.

**Proof.** This follows immediately from the fact that the threefold $X^3_d$ admits good reduction at the prime $p$. See [Lin1], Lemma 3.4. \hfill \square

The above lemma allows us to consider the element

$$\text{loc}_p(\Theta_n(\mathbf{f}, d)) \in \bigoplus_{j=1}^{3} \bigotimes_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}) / \mathfrak{p}_{i,n}^{[p^j]}$$

via the isomorphism $\Phi_{\mathbf{f}, n}$ in Corollary 3.6. We will denote by $\text{loc}_p^{(j)}(\Theta_n(\mathbf{f}, d))$ the component of $\text{loc}_p(\Theta_n(\mathbf{f}, d))$ in the $j$-th copy of $\bigotimes_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda_i}) / \mathfrak{p}_{i,n}^{[p^j]}$. Following the terminology in [BD3], we will refer to the formula in the theorem below as the second explicit reciprocity law for the Gross–Kudla–Schoen diagonal cycle class.

**Theorem 3.8** (The second reciprocity law). Let $p$ be an $n$-admissible prime for $\mathbf{f}$. Suppose each maximal ideal in $\mathfrak{m}_\mathbf{f}$ satisfies Assumption 2.6. Then the following formula

$$\left( \text{loc}_p^{(j)}(\Theta_n(\mathbf{f}, d)), \phi_1 \otimes \phi_2 \otimes \phi_3 \right) = \sum_{z \in \Delta_d(\mathcal{B})} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z)$$

holds for any $\phi_1 \otimes \phi_2 \otimes \phi_3 \in \bigotimes_{i=1}^{3} \Gamma(Z_d(\mathcal{B}), E_{\lambda_i}/\mathcal{O}_{\lambda_i}) / \mathfrak{p}_{i,n}^{[p^j]}$ and $j = 1, 2, 3$. 


Proof. Let $\theta : X_d \to X_d^3$ be the map induced by $\theta : \mathcal{X}_d \to \mathcal{X}_d^3$ on the special fiber. Consider the Abel–Jacobi map $AJ_{\mathcal{L}n} : \text{CH}^2(\mathcal{X}_d^3) \to H^1(\mathbb{F}_{p^2}, M_n(\mathcal{f}, d)(-1))$ constructed similarly as the one on $\text{CH}^2(X_d^3)$. Let $\Delta_d = \{ \theta \mathcal{X}_d \in \text{CH}^2(\mathcal{X}_d^3) \}$ be the diagonal element, we denote by $\Theta_n(\mathcal{f}, d)$ the class given by $AJ_{\mathcal{L}n}(\Delta_d) \in H^1(\mathbb{F}_{p^2}, M_n(\mathcal{f}, d)(-1))$ and define $\Theta_n^{(j)}(\mathcal{f}, d)$ to be the projection to the $j$-th component of $\bigoplus_{j=1}^{3} \otimes_{i=1}^{3} \Gamma(Z_d(B), \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n}$ using the isomorphism $\Phi_{\mathcal{L}n}$ for $j \in \{1, 2, 3\}$. Since $X_d^3$ has good reduction at $p$ and thus $M_n(\mathcal{f}, d)(-1)$ is unramified, the class $\Theta_n(\mathcal{f}, d)$ agrees with $\text{loc}_p(\Theta_n(\mathcal{f}, d))$. Similarly, we can identify $\Theta_n^{(j)}(\mathcal{f}, d)$ with $\text{loc}_p^{(j)}(\Theta_n(\mathcal{f}, d))$ for every $j \in \{1, 2, 3\}$. To finish the proof, we will show without loss of generality

$$\Theta_n^{(1)}(\mathcal{f}, d), \phi_1 \otimes \phi_2 \otimes \phi_3) = \sum_{z \in \Delta_d(B)} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z).$$

To prove this claim, we are free to take a base change of coefficients for the cohomologies involved. Therefore we can assume that $\mathcal{O}_\lambda = \mathcal{O}_\lambda$ for a discrete valuation ring $\mathcal{O}_\lambda$ with maximal ideal $\lambda$. We write $\mathcal{O}_{\lambda,n}$ for $\mathcal{O}_\lambda/\mathcal{P}_n$ with any $n \geq 1$. Recall that the class $\Theta_n^{(1)}(\mathcal{f}, d)$ is the image of $\Theta_n(\mathcal{f}, d)$ under the following maps

$$H^1(\mathbb{F}_{p^2}, M_n(\mathcal{f}, d)(-1)) = H^1(\mathbb{F}_{p^2}, T_{1,n} \otimes T_{2,n} \otimes T_{3,n}(-1)) \to H^1(\mathbb{F}_{p^2}, T_{1,n}^+(−1) \otimes T_{2,n}^− \otimes T_{3,n}^−) \sim T_{1,n}^+(−1) \otimes T_{2,n}^− \otimes T_{3,n}^− \sim \Gamma(Z_d(B), \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n} \otimes \Gamma(Z_d(B), \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n} \otimes \Gamma(Z_d(B), \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n} \sim \Gamma(Z_d(B)^3, \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n}.$$  

We need to prove that the element $\Theta_n^{(1)}(\mathcal{f}, d) \in \Gamma(Z_d(B)^3, \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n}$ agrees with the characteristic function $1_{\overline{B}} \in \Gamma(Z_d(B)^3, \mathcal{O}_\lambda)/\mathcal{P}_{\mathcal{L}3,n}$ of the diagonal cycle $\Delta_d(B) = \partial_\nu Z_d(B)$ of the embedding $\partial : Z_d(B) \to Z_d(B)^3$. Once this is proved, we have

$$\Theta_n^{(1)}(\mathcal{f}, d), \phi_1 \otimes \phi_2 \otimes \phi_3) = (1_{\overline{B}}, \phi_1 \otimes \phi_2 \otimes \phi_3) = \sum_{z \in \Delta_d(B)} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z).$$

To prove this, we introduce some auxiliary notations. Let $ij \in \{12, 23, 31\}$, then we have projections

$$p_{ij} : \overline{X}_d^3 \to \overline{X}_i \times \overline{X}_j.$$  

Notice that the projection map $p_{ij} : H^4(\overline{X}_d, \mathcal{O}_\lambda(2)) \to H^4(\overline{X}_i^2, \mathcal{O}_\lambda(1))$ sends the diagonal class $[\Delta_d]$ to $[\Delta_{ij}]$, where $\Delta_{ij}$ denotes the diagonal of $\overline{X}_d$. Moreover the image of $[\Delta_{ij}]$ in $H^2(\overline{X}_{ij}, \mathcal{O}_\lambda(1))/\mathcal{P}_{\mathcal{L}n}$ is $\text{Hom}_{\mathcal{O}_{\lambda,n}}(\mathcal{G}_{\mathcal{P}^2}, H^1(\overline{X}_i^2, \mathcal{O}_\lambda(1))/\mathcal{P}_{\mathcal{L}n}, H^1(\overline{X}_i^2, \mathcal{O}_\lambda(1))/\mathcal{P}_{\mathcal{L}n})$.
is given by the identity map \(\text{id} \in \text{Hom}_{\mathcal{O}_{\lambda,n}[G_{p^2}]}(T_{i,n}, T_{j,n})\). It follows that the element \(\overline{\Theta}_n^{(1)}(\mathbf{f}, d)\) in
\[
T_{1,n}^+(-1) \otimes T_{2,n}^- \otimes T_{3,n}^- \cong \text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{2,n}^- \otimes T_{3,n}^-)
\]
can be characterized by the property that its image under the natural map
\[
\text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{2,n}^- \otimes T_{3,n}^-) \to \text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{2,n}^-) \times \text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{3,n}^-)
\]
is given by \((\text{id}, \text{id}) \in \text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{2,n}^-) \times \text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{3,n}^-)\). To see this, notice that the composite maps given by
\[
H^1(F_{p^2}, T_{2,n}) \otimes (T_{3,n} \otimes T_{1,n}(-1))_{G_{p^2}} \to H^1(F_{p^2}, T_{1,n} \otimes T_{2,n} \otimes T_{3,n}(-1)) = (T_{3,n} \otimes T_{1,n}(-1))_{G_{p^2}}
\]
are the natural projection maps to the second factors. Also notice that
\[
T_{1,n}^+(-1) \otimes T_{2,n}^- \otimes T_{3,n}^- = \text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{2,n}^- \otimes T_{3,n}^-)
\]
is the intersection of the \(\mathcal{O}_{\lambda,n}\)-modules
\[
\begin{align*}
H^1(F_{p^2}, T_{2,n}) \otimes (T_{3,n} \otimes T_{1,n}(-1))_{G_{p^2}} &= (T_{1,n}^+(-1) \otimes T_{2,n}^- \otimes T_{3,n}^-) \oplus (T_{3,n}^+(-1) \otimes T_{2,n}^+ \otimes T_{1,n}^-), \\
H^1(F_{p^2}, T_{3,n}) \otimes (T_{2,n} \otimes T_{1,n}(-1))_{G_{p^2}} &= (T_{1,n}^+(-1) \otimes T_{2,n}^- \otimes T_{3,n}^-) \oplus (T_{3,n}^+(-1) \otimes T_{2,n}^- \otimes T_{1,n}^-).
\end{align*}
\]
Therefore \(\overline{\Theta}_n^{(1)}(\mathbf{f}, d) \in T_{1,n}^+(-1) \otimes T_{2,n}^- \otimes T_{3,n}^-\) maps to
\[
(\text{id}, \text{id}) \in (T_{3,n} \otimes T_{1,n}(-1))_{G_{p^2}} \times (T_{3,n} \otimes T_{1,n}(-1))_{G_{p^2}}.
\]
This verifies the claimed property of \(\overline{\Theta}_n^{(1)}(\mathbf{f}, d)\) in \(T_{1,n}^+(-1) \otimes T_{2,n}^- \otimes T_{3,n}^-\).

This property shows that under the identification
\[
\text{Hom}_{\mathcal{O}_{\lambda,n}}(T_{1,n}^-, T_{2,n}^- \otimes T_{3,n}^-) \cong \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda}/p_{1,n}^{[p]} \otimes \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda}/p_{2,n}^{[p]} \otimes \Gamma(Z_d(\mathcal{B}), \mathcal{O}_{\lambda}/p_{3,n}^{[p]})
\]
the image of the element \(\overline{\Theta}_n^{(1)}(\mathbf{f}, d)\) is the one given by the characteristic function \(1_{\overline{\mathbb{T}}} \) of the diagonal of \(Z_d(\mathcal{B})\) as this element is the unique one having the above property.

### 3.4. 34. 34. 34. 34. 34. 34. 34. 34. Ramified arithmetic level raising for Shimura curves. We consider the setting in §2.2, let \((p, q)\) be a pair of \(n\)-admissible primes for the modular form \(f\). We have the indefinite quaternion algebra \(B^\sharp\) of discriminant \(Npq\). Then we can associate to it a Shimura curve over \(\mathbb{Q}\) denoted by \(X_d^\sharp\) and its integral model \(X_d^\sharp\) over \(\mathbb{Z}[1/Npq]\) as in [Wang 3A]. The curve \(X_d^\sharp\) admits the Cerednik–Drinfeld uniformization and its special fiber over \(F_{p^2}\) can be explicitly described as in [Wang Proposition 2.2]. In particular, the special fiber \(X_d^\sharp\) is a union \(\mathbf{P}^1(Z_d^\sharp(\mathcal{B})) \cup \mathbf{P}^1(Z_d^\sharp(\mathcal{B}))\) with \(Z_d^\sharp(\mathcal{B}) = Z_d^\sharp(\mathcal{B}) = Z_d(\mathcal{B})\) and the intersection \(\mathbf{P}^1(Z_d^\sharp(\mathcal{B})) \cap \mathbf{P}^1(Z_d^\sharp(\mathcal{B}))\) is the Shimura set \(Z_d(\mathcal{B})\) obtained by adding an Iwahori level structure at \(q\) to the Shimura set \(Z_d(\mathcal{B})\). We prove some parallel results in this setting to Proposition 2.11 and Proposition 2.12.

**Lemma 3.9.** Under Assumption 2.5, we have an isomorphism
\[
H^1(X_d^\sharp, \mathcal{O}_{\lambda}(1))_{/p^\infty} \cong \rho_{\lambda,n}^{\oplus 2}.
\]
Proof. This follows from the same argument as in [BD3 Theorem 5.17] combined with the argument as in Lemma 2.6.

Proposition 3.10. Let \((p, q)\) be a pair of \(n\)-admissible primes for \(f\). We assume that Assumption 2.7 holds, then there is an isomorphism

\[
H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}} \cong \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}} \oplus \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}
\]

of \(\mathcal{O}_{\lambda,n}[G_{Q_q}]\)-modules.

Proof. By the ramified level raising theorem for \(X^2_{d, Q_q^{ac}}\) given by [Wang, Theorem 3.4], we have

\[
H^1_{\sin}(Q_q, H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}}) \cong \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}.
\]

But then

\[
H^1_{\sin}(Q_q, H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}}) = H^1(I_q, H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}})^{G_{F_q}}
\]

\[
\cong \text{Hom}(Z_I(1), H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}})^{G_{F_q}}
\]

\[
\cong H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda)^{G_{F_q}}/_{/p_n^{[\nu]}}
\]

which follows from the fact that \(H^1_{\sin}(Q_q, H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}})\) is unramified as an \(\mathcal{O}_{\lambda,n}[G_{Q_q}]\)-module. By the Tate local duality and Poincaré duality on \(X^2_{d, Q_q^{ac}}\), we have

\[
H^1(F_q, H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}}) \cong \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}.
\]

By the \(n\)-admissibility of \(q\), we have the following exact sequence

\[
0 \to H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda)^{G_{F_q}}/_{/p_n^{[\nu]}}(1) \to H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}} \to H^1(F_q, H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}}) \to 0
\]

which can be identified with

\[
0 \to \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}(1) \to H^1(X^2_{d, Q_q^{ac}}, \mathcal{O}_\lambda(1))/_{/p_n^{[\nu]}} \to \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}} \to 0.
\]

This exact sequence clearly splits and the proposition is proved.

We define the \(q\)-new part \(\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_{\lambda,n})\uparrow\) of the space \(\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}\) by

\[
\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_\lambda)\uparrow = \ker[\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}} \to \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)^{G_{F_q}}/_{/p_n^{[\nu]}}]
\]

and the \(q\)-new quotient \(\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_{\lambda,n})\downarrow\) of the space \(\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}\) by

\[
\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_\lambda)\downarrow = \text{coker}[\Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)^{G_{F_q}}/_{/p_n^{[\nu]}} \to \Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}}]
\]

Corollary 3.11. Under Assumption 2.7, we have the following isomorphisms of \(\mathcal{O}_{\lambda,n}\)-modules

\[
\Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_{\lambda,n})\uparrow \cong \Gamma(Z_d(\overline{B}), \mathcal{O}_\lambda)/_{/p_n^{[\nu]}} \cong \Gamma(Z_{d,1w_q}(\overline{B}), \mathcal{O}_{\lambda,n})\downarrow.
\]
Proof. We consider the localized weight spectral sequence \( \mathbf{RZ} \) for \( H^1(X_{d,q}^{ac}, O_\lambda(1))_{m^{[pq]}} \) and its induced monodromy filtration:

\[
0 \subset E^{1,0}_{2,m^{[pq]}} M_1 H^1(X_{d,q}^{ac}, O_\lambda(1))_{m^{[pq]}} \subset E^{1,1}_{2,m^{[pq]}} M_0 H^1(X_{d,q}^{ac}, O_\lambda(1))_{m^{[pq]}}
\]

(3.4)

By [III, example 2.4.6] and the discussions in Lemma 2.2, we have

\[
E^{1,0}_{2,m^{[pq]}} = \text{kerr}[\Gamma(Z_d(B), O_\lambda(1))]^{\otimes 2}_{m^{[pq]}} \to \Gamma(Z_{d,Iw}(B), O_\lambda(1))_{m^{[pq]}}]
\]

\[
E^{1,1}_{2,m^{[pq]}} = 0;
\]

\[
E^{1,2}_{2,m^{[pq]}} = \text{ker}[\Gamma(Z_{d,Iw}(B), O_\lambda(1))]^{\otimes 2}_{m^{[pq]}} \to \Gamma(Z_d(B), O_\lambda(1))_{m^{[pq]}}]
\]

By [III, example 2.4.6] and the discussions in Lemma 2.2, we have

Then we have an exact sequence

\[
0 \to \Gamma(Z_{d,Iw}(B), O_{\lambda,n}(1)) \to H^1(X_{d,q}^{ac}, O_\lambda(1))_{/\mathfrak{p}_n^{[pq]}} \to \Gamma(Z_{d,Iw}(B), O_{\lambda,n}) \to 0
\]

of \( O_{\lambda,n}[G_{Q_q}] \)-modules which clearly splits as \( H^1(X_{d,q}^{ac}, O_\lambda(1))_{/\mathfrak{p}_n^{[pq]}} \) is unramified by Lemma 3.9. Then the claimed isomorphisms follow from Proposition 3.10. \( \square \)

3.5. Ramified arithmetic level raising for triple product of Shimura curves. Now we shift to the triple product setting. We consider a pair of \( n \)-admissible primes \((p,q)\) for \( \mathfrak{f} \). In [Wang, Theorem 2], we proved the ramified arithmetic level raising theorem for the triple product of Shimura curves \( X_{d}^{23} \) under slightly different assumptions, now we review this result and indicate necessary modifications to incorporate our assumptions. We define the \( O_{\lambda,n}[G_{Q_q}] \)-module \( M_{n}^{[pq]}(\mathfrak{f}) \) over \( O_{\lambda,n} \) by

\[
M_{n}^{[pq]}(\mathfrak{f}, d) = \bigoplus_{i=1}^{3} H^1(X_{d,q}^{ac}, O_\lambda(1))_{/\mathfrak{p}_n^{[pq]}}.
\]

There is an isomorphism

\[
H^1(X_{d,q}^{ac}, O_\lambda(1))_{/\mathfrak{p}_n^{[pq]}} \cong \rho_{i,n}^{\otimes 2} \text{ by Lemma 3.9.}
\]

Hence we have an isomorphism

\[
M_{n}^{[pq]}(\mathfrak{f}, d) \cong M_{n}(\mathfrak{f}, d) \text{ as } O_{\lambda,n}[G_{Q_q}] \text{-modules.}
\]

Recall we have defined

\[
Z_n(\mathfrak{f}, d) = \bigotimes_{i=1}^{3} \Gamma(Z_d(B), O_\lambda)_{/\mathfrak{p}_n^{[pq]}}
\]

in 3.2. The results in the last subsection imply immediately the following lemma.

**Proposition 3.12** (Ramified level raising). Let \((p,q)\) be a pair of \( n \)-admissible primes for \( \mathfrak{f} \). Suppose each maximal ideal in \( m_{\mathfrak{f}} \) satisfies Assumption 2.5.

1. Then \( M_{n}^{[pq]}(\mathfrak{f}, d) \) is unramified at \( q \) and we have a natural isomorphism

\[
M_{n}^{[pq]}(\mathfrak{f}, d) \cong Z_n(\mathfrak{f}, d) \oplus Z_n(\mathfrak{f}, d)(1) \oplus Z_n^{\otimes 3}(\mathfrak{f}, d)(2) \oplus Z_n(\mathfrak{f}, d)(3)
\]

as \( O_{\lambda,n}[G_{Q_q}^{23}] \)-modules.

2. There is an isomorphism

\[
\bigotimes_{j=1}^{3} \bigotimes_{i=1}^{3} \Gamma(Z_d(B), O_\lambda)_{/\mathfrak{p}_n^{[pq]}} \cong H^3_{\text{mor}}(Q_{q}, M_{n}^{[pq]}(\mathfrak{f}, d)(-1))
\]

of \( O_{\lambda,n} \)-modules.
Proof. The first statement follows from immediately from Proposition 3.10. For the second statement, we have

\[ H^1_{sin}(Q_{q^2}, M^{[pq]}(f, d)(-1)) = \text{Hom}(Z_1(1), M^{[pq]}(f, d)(-1))^{G_{F_q}} \]
\[ \cong M^{[pq]}(f, d)(-2)^{G_{F_q}} \]
\[ \cong (Z_n(f, d)(-2) \oplus Z^{G_3}_n(f, d)(-1) \oplus Z^{G_3}_n(f, d) \oplus Z_n(f, d)(1))^{G_{F_q}} \]
\[ \cong Z^{G_3}_n(f, d). \]

Then one proceeds as in [Wang, Corollary 4.11] to descend the result to \( Q_q \).

Let \( \Delta^3_d = [\theta^*_X X^3_d] \in CH^2(X^3_d) \) be the diagonal cycle for the natural diagonal morphism \( \theta^2 : X^2_d \to X^3_d \). Consider the Abel–Jacobi map

\[ \text{AJ}^{[pq]}_{f,n} : CH^2(X^3_d) \to H^1(Q, M^{[pq]}(f, d)(-1)) \]

for \( M^{[pq]}(f, d)(-1) \) defined similarly as in [22]. We denote by \( \Theta^{[pq]}_{n}(f, d) \) the image of \( \Delta^3_d \) in \( H^1(Q, M^{[pq]}(f, d)(-1)) \) under the map \( \text{AJ}^{[pq]}_{f,n} \). We have the localization map

\[ \text{loc}_q : H^1(Q, M^{[pq]}(f, d)(-1)) \to H^1(Q, M^{[pq]}(f, d)(-1)) \]

and the singular quotient map

\[ \partial_q : H^1(Q, M^{[pq]}(f, d)(-1)) \to H^1_{sin}(Q, M^{[pq]}(f, d)(-1)). \]

We therefore obtain an element \( \partial_q \text{loc}_q(\Theta^{[pq]}_{n}(f, d)) \in H^1_{sin}(Q, M^{[pq]}(f, d)(-1)) \). To simplify the notation, we denote this element by \( \partial_q \Theta^{[pq]}_{n}(f, d) \). Proposition 3.12 allows us to view the element \( \partial_q \Theta^{[pq]}_{n}(f, d) \) as an element in \( \bigoplus_{j=1}^3 (\bigotimes_{i=1}^3 \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_{i,n}^{[pq]}} \). We denote by \( \partial_q^{(j)} \Theta^{[pq]}_{n}(f, d) \) the component of \( \partial_q \Theta^{[pq]}_{n}(f, d) \) in the \( j \)-th copy of \( \bigoplus_{j=1}^3 (\bigotimes_{i=1}^3 \Gamma(Z_d(\overline{B}), \mathcal{O}_{\lambda_i})_{p_{i,n}^{[pq]}} \).

Theorem 3.13 (First reciprocity law). Let \((p, q)\) be a pair of \( n \)-admissible primes for \( f \). Suppose each maximal ideal in \( m_f \) satisfies Assumption 2.5. Then the formula

\[ (\partial_q^{(j)} \Theta^{[pq]}_{n}(f, d), \phi_1 \otimes \phi_2 \otimes \phi_3) = (q + 1)^3 \sum_{z \in \Delta_d(\overline{B})} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z) \]

holds for any \( \phi_1 \otimes \phi_2 \otimes \phi_3 \in \bigotimes_{i=1}^3 \Gamma(Z_d(\overline{B}), E_{\lambda_i}/\mathcal{O}_{\lambda_i})_{p_{i,n}^{[pq]}} \) and any \( j \in \{1, 2, 3\} \).

Proof. This follows from [Wang, Theorem 4.12] by using Proposition 3.12 instead of [Wang, Theorem 4.7, Corollary 4.11].

Remark 3.14. Theorem 3.13 and Theorem 3.8 imply the following relation

\[ (\partial_q^{(j)} \Theta^{[pq]}_{n}(f, d), \phi_1 \otimes \phi_2 \otimes \phi_3) = (q + 1)^3 (\text{loc}_q^{(j)}(\Theta_{n}(f, d)), \phi_1 \otimes \phi_2 \otimes \phi_3) \]
\[ = (q + 1)^3 I(\phi_1, \phi_2, \phi_3) \]

where

\[ I(\phi_1, \phi_2, \phi_3) = \sum_{z \in \Delta_d(\overline{B})} \phi_1(z) \otimes \phi_2(z) \otimes \phi_3(z) \]
for any $\phi_1 \otimes \phi_2 \otimes \phi_3 \in \otimes_{i=1}^3 \Gamma(Z(\overline{B}), E_{\lambda_i}/\mathcal{O}_{\lambda_i})[p^i]$ is the Gross–Kudla period. This shows that $(\Theta_n(f, d), \Theta_n^{[p^i]}(f, d))$ along with $I(\phi_1, \phi_2, \phi_3)$ form a bipartite Euler system in a weaker sense: these classes do satisfy all the required reciprocity laws, however the Frobenius eigenvalues on $M_4(f, d)$ is not just $p$ and 1 which is required by the definition in [How] for a bipartite Euler system. This also means the full part and singular part of the local Galois cohomology group of the triple product Galois representation at a level raising prime have rank bigger than one.

4. Applications to the Bloch–Kato conjectures

4.1. The rank 1 case of the Bloch–Kato conjecture. We consider the triple tensor product Galois representation $\rho_{f_1, \lambda_1} \otimes \rho_{f_2, \lambda_2} \otimes \rho_{f_3, \lambda_3}$ over $E_{\Delta} := E_{\lambda_1} \otimes E_{\lambda_2} \otimes E_{\lambda_3}$ attached to $\mathfrak{f} = (f_1, f_2, f_3)$. Recall that

$$V(\mathfrak{f}) = \rho_{f_1, \lambda_1} \otimes \rho_{f_2, \lambda_2} \otimes \rho_{f_3, \lambda_3}.$$ 

By [Wang] Lemma 5.1, we have $H^1(Q_v, V(\mathfrak{f})(-1)) = 0$ for all $v \mid l$. Therefore it makes sense to consider the following Selmer group.

**Definition 4.1.** The Bloch–Kato Selmer group $H^1_f(Q_v, V(\mathfrak{f})(-1))$ of the representation $V(\mathfrak{f})(-1)$ is the subspace of $H^1(Q_v, V(\mathfrak{f})(-1))$ consisting of those classes $s$ such that $\text{loc}_l(s) \in H^1_f(Q_v, V(\mathfrak{f})(-1))$ where

$$H^1_f(Q_v, V(\mathfrak{f})(-1)) = \ker[H^1_f(Q_v, V(\mathfrak{f})(-1)) \rightarrow H^1_f(Q_v, V(\mathfrak{f})(-1))].$$

Let $(\pi_1, \pi_2, \pi_3)$ be the triple of irreducible cuspidal automorphic representation of $GL_2(A)$ associated to the triple $\mathfrak{f} = (f_1, f_2, f_3)$. We have the Garrett–Rankin triple product $L$-function

$$L(f_1 \otimes f_2 \otimes f_3, s) = L(s - \frac{3}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3, r)$$

where $r$ is the natural 8-dimensional representation of the $L$-group of $GL_2 \times GL_2 \times GL_2$ and $L(s - \frac{3}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3, r)$ is the Langlands $L$-function for $r$. The parity of the order of vanishing of $L(f_1 \otimes f_2 \otimes f_3, s)$ at the central critical point $s = 2$ is controlled by the global root number $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3, r) \in \{\pm 1\}$. In this article we will focus on the case when $\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3, r) = -1$. Consider the following Abigail-Jacobi map

$$AJ_{L,Q} : \text{CH}^2(X_d^3) \rightarrow H^1(Q, H^3(X_d^3, \mathbb{Q}^*_l, Z_l(2)) \otimes T_{\text{cris}} E_{\Delta}).$$

We denote by $\Theta(\mathfrak{f}, d) \in H^1(Q, V(\mathfrak{f})(-1))$ the image of $\Delta_d = 0_x[X_d]$ under the composite of $AJ_{L,Q}$ and the natural projection $H^1(Q, H^3(X_d^3, \mathbb{Q}_l, Z_l(2)) \otimes T_{\text{cris}} E_{\Delta}) \rightarrow H^1(Q, V(\mathfrak{f})(-1)).$ Since $X_d^3$ has good reduction at $l$, $\Theta(\mathfrak{f}, d)$ in fact lies in the Bloch–Kato Selmer group $H^1_f(Q_v, V(\mathfrak{f})(-1))$ by [Nek] Theorem 3.1.

In light of the Gross–Zagier formula for the Gross–Kudla–Schoen diagonal cycles [YZZ] and assuming the conjectural injectivity of the Abel–Jacobi map and the non-degeneracy of the height-pairing, there are infinitely many $l$ such that $\Theta(\mathfrak{f}, d)$ is non-zero as long as the first derivative $L'(f_1 \otimes f_2 \otimes f_3, 2)$ is non-vanishing. Therefore we can view $\Theta(\mathfrak{f}, d)$ as an algebraic incarnation of the first derivative $L'(f_1 \otimes f_2 \otimes f_3, 2)$ and it makes sense to formulate the following conjecture towards the rank one case of the Bloch–Kato conjecture for the triple product motive attached to $\mathfrak{f}$.
Conjecture 4.2. Suppose the class $\Theta(\mathbf{f}, d) \in H^1(Q, V(\mathbf{f})(-1))$ is non-zero. Then the Bloch–Kato Selmer group $H^1_f(Q, V(\mathbf{f})(-1))$ is of rank 1 over $E_\Lambda$.

4.2. Bipartite Euler system for the symmetric cube motive. In this subsection, we apply the results in this article to the case when the triple product motive degenerates. Let $\mathbf{f} = (f, f, f)$ for a single modular form $f \in S^new_2(T_0(N))$ such that $N = N^+N^-$ with $(N^+, N^-) = 1$ and $N^-$ is square-free with even number of prime factors. We denote by $\pi$ the automorphic representation of $GL_2(\mathbb{A})$ associated to $f$. Recall that we have the Galois representation $\rho_{f, \lambda}$ and its residual representation $\overline{\rho}_{f, \lambda}$ attached to $f$. The triple tensor product representation $\rho_{f, \lambda}^{\otimes 3}$ admits the following factorization

$$\rho_{f, \lambda}^{\otimes 3}(-1) = \text{Sym}^3 \rho_{f, \lambda}(-1) \oplus \rho_{f, \lambda} \oplus \rho_{f, \lambda}$$

by the Schur functor construction. In the notations of previous sections, this factorization is given by $V(\mathbf{f})(-1) = \text{Sym}^3 V_{f, \lambda}(-1) \oplus V_{f, \lambda} \oplus V_{f, \lambda}$. The triple tensor product $L$-function $L(f \otimes f \otimes f, s)$ factors accordingly as

$$L(f \otimes f \otimes f, s) = L(\text{Sym}^3 f, s)L(f, s - 1)^2.$$

In fact by a result of Kim and Shahadi [KS], the $L(\text{Sym}^3 f, s)$ is entire and $L(f \otimes f \otimes f, s)$ is divisible by $L(f, s - 1)$. In the case where the global root number $\epsilon(\pi \otimes \pi \otimes \pi, \tau)$ is $-1$ for $L(f \otimes f \otimes f, s)$, the $L$-function $L(\text{Sym}^3 f, s)$ has also global root number $-1$ at $s = 2$.

We project the class $\Theta(\mathbf{f}, d) \in H^1(Q, V(\mathbf{f})(-1))$ to the symmetric cube component according to the factorization

$$V(\mathbf{f})(-1) = \text{Sym}^3 V_{f, \lambda}(-1) \oplus V_{f, \lambda} \oplus V_{f, \lambda}.$$

The resulting class will be denoted by $\Theta^\circ(\mathbf{f}, d)$. The symmetric cube component $\text{Sym}^3 V_{f, \lambda}(-1)$ of $V(\mathbf{f})(-1)$ will be denoted by $V^\circ(\mathbf{f})(-1)$ and thus $\Theta^\circ(\mathbf{f}, d) \in H^1(Q, V^\circ(\mathbf{f})(-1))$. The class $\Theta^\circ(\mathbf{f}, d)$ can be considered as an algebraic incarnation of the first derivative $L'(\text{Sym}^3 f, s)$ at $s = 2$. We define the symmetric cube Bloch–Kato Selmer group $H^1_f(Q, V^\circ(\mathbf{f})(-1))$ the same way as in Definition 4.3. Using the reciprocity laws proved in this article, we will prove the following theorem towards the rank 1 case of the Bloch–Kato conjecture for the symmetric cube motive of the modular form $f$.

Theorem 4.3. Suppose that the modular form $f$ satisfies the following assumptions:

1. The residual Galois representation $\overline{\rho}_{f, \lambda}|_{G_{Q_\lambda}}$ is absolutely irreducible;
2. The residual Galois representation $\overline{\rho}_{f, \lambda}$ is minimally ramified at primes in $\Sigma^+ \cup \Sigma^\text{ram}$;
3. The image of $\overline{\rho}_{f, \lambda}$ contains $GL_2(F_\ell)$.

If the class $\Theta^\circ(\mathbf{f}, d) \in H^1(Q, V^\circ(\mathbf{f})(-1))$ is non-zero, then the symmetric cube Bloch–Kato Selmer group

$$H^1_f(Q, V^\circ(\mathbf{f})(-1))$$

is of dimension 1 over $E_\Lambda$.

To prove this theorem, we review a few results on local Tate dualities. Let $T_{f, \lambda}$ be the $\mathcal{O}_{\mathbb{A}}$-lattice in $\rho_{f, \lambda}$ determined by the isomorphism $H^1(X_{d, Q\text{-aut}}, \mathcal{O}_\lambda(1)) \cong T_{f, \lambda}^{\otimes 2}$ in Proposition 2.6. Let $A_{f, \lambda} = V_{f, \lambda}/T_{f, \lambda}$ be the divisible $G_{Q\text{-mod}}$-module associated to $f$. Let $n \geq 1$ be an integer. We denote the $\lambda^n$ torsion subgroup of $A_{f, \lambda}$ by $A_{f, n}$ and set $T_{f, n} = T_{f, \lambda}/\lambda^n$. We put $N^\circ_n(\mathbf{f})(-1) = \text{Sym}^3 A_{f, \lambda}(-1)$ and $N^\circ_n(\mathbf{f})(-1) = \text{Sym}^3 A_{f, n}(-1)$. Similarly, we let $M^\circ_n(\mathbf{f})(-1) = \text{Sym}^3 T_{f, \lambda}(-1)$ and $M^\circ_n(\mathbf{f})(-1) = \text{Sym}^3 T_{f, n}(-1)$. Note that $N^\circ_n(\mathbf{f})(-1)$ and
LEMMA 4.5. We have the following statements.

1. The sum \( \sum_v ( \ , \ )_v \) restricted to \( H^1(Q_v, N_n^0(f)(-1)) \times H^1(Q_v, M_n^0(f)(-1)) \) is trivial. Here \( v \) runs through all places in \( Q \).

2. For every \( v \neq l \), there exists an integer \( n_v \geq 1 \) independent of \( n \) such that the image of the pairing

\[
(\ , \ )_v : H^1(Q_v, N_n^0(f)(-1)) \times H^1(Q_v, M_n^0(f)(-1)) \to H^1(Q_v, O_n(1)) \cong O_n
\]

is annihilated by \( \varpi^{n_v} \).

3. For every \( v \neq l \), the finite part \( H^1_{\text{fin}}(Q_v, N_n^0(f)(-1)) \) is orthogonal to \( H^1_{\text{fin}}(Q_v, M_n^0(f)(-1)) \) under the pairing \( (\ , \ )_v \). Similarly, \( H^1_{\text{fin}}(Q_l, N_n^0(f)(-1)) \) is orthogonal to \( H^1_{\text{fin}}(Q_l, M_n^0(f)(-1)) \).

4. Let \( p \) be a \((n, 1)\)-admissible prime for \( f \), then we have a perfect pairing

\[
H^1_{\text{fin}}(Q_p, N_n^0(f)(-1)) \times H^1_{\text{fin}}(Q_p, M_n^0(f)(-1)) \to O_n
\]

of free \( O_n \)-modules of rank 1.

Proof. The statement (1) follows from global class field theory. Part (2) follows from the fact that \( H^1(Q_v, V^0(f)) = 0 \) for all \( v \neq l \) and thus \( H^1(Q_v, V^0(f)) \) is torsion for all \( v \neq l \), see [Liv91] Lemma 4.3]. Part (3) is well known, see [DDT98] Theorem 2.17(e)] for the first statement and [Liv91] Lemma 4.8] for the second statement.

For (4), it follows from the definition of an \((n, 1)\)-admissible prime for \( f \) that \( M_n(f) \) is unramified at \( p \) and \( M_n(f) \cong O_n \oplus O_n^{\oplus 3} \oplus O_n^{\oplus 3} \oplus O_n(3) \) as a Galois module for \( G_{Q_p} \). Then it follows from a simple computation that \( M_n(f) \cong O_n \oplus O_n(1) \oplus O_n(2) \oplus O_n(3) \). From this, it follows immediately that both \( H^1_{\text{fin}}(Q_p, M_n(f)(-1)) \) and \( H^1_{\text{fin}}(Q_p, M_n(f)(-1)) \) are free of rank 1 over \( O_n \). The last claim also follows form this. □
We have the Abel–Jacobi map
\[ \text{AJ}_{\mathbb{L}_n}^\circ : \text{CH}^2(X_{3d}^3) \to H^1(\mathbb{Q}, M_n^\circ(\mathfrak{f})(-1)) \]
for \( M_n^\circ(\mathfrak{f})(-1) \) constructed by composing the Abel–Jacobi map \( \text{AJ}_{\mathbb{L}_n}^\circ \) with the natural projection map from \( M_n(\mathfrak{f}, d)(-1) \) to its symmetric cube component \( M_n^\circ(\mathfrak{f})(-1) \). We will denote by \( \Theta_n^\circ(\mathfrak{f}, d) \in H^1(\mathbb{Q}, M_n^\circ(\mathfrak{f})(-1)) \) the image of the Gross–Kudla–Schoen diagonal cycle \( \Delta_d = \theta_n^\circ[X_d] \) under the map \( \text{AJ}_{\mathbb{L}_n}^\circ \).

Let \( (p, q) \) be a pair of \((n, 1)\)-admissible primes for \( f \). Then we have the Shimura curve \( X_d^3 \) and its integral model \( X_d^3 \) over \( \mathbb{Z}[1/Nd] \) considered in the setting of ramified arithmetic level-raising. We have another Abel–Jacobi map
\[ \text{AJ}_{\mathbb{L}_n}^{[pq]} : \text{CH}^2(X_{3d}^{pq}) \to H^1(\mathbb{Q}, M_n^\circ(\mathfrak{f})(-1)) \]
for \( M_n^\circ(\mathfrak{f})(-1) \) constructed by composing the Abel–Jacobi map \( \text{AJ}_{\mathbb{L}_n}^{[pq]} \) with the projection from \( M_n^{[pq]}(\mathfrak{f}, d)(-1) \equiv M_n(\mathfrak{f}, d)(-1) \) to its symmetric cube component \( M_n^\circ(\mathfrak{f})(-1) \). We denote by \( \Theta_n^{[pq]}(\mathfrak{f}, d) \in H^1(\mathbb{Q}, M_n^\circ(\mathfrak{f})(-1)) \) the image of \( \Delta_d^{[pq]} = \theta_n^{[pq]}[X_d^{pq}] \) under the map \( \text{AJ}_{\mathbb{L}_n}^{[pq]} \). Since the pair \( (\Theta_n^\circ(\mathfrak{f}, d), \Theta_n^{[pq]}(\mathfrak{f}, d)) \) satisfy reciprocity laws as in Theorem 3.13 and Theorem 3.8, the collection of \( (\Theta_n^\circ(\mathfrak{f}, d), \Theta_n^{[pq]}(\mathfrak{f}, d)) \) along with the Gross–Kudla periods form a bipartite Euler system in a slightly broader sense of \( \text{How} \): the Galois representations are assumed to be two-dimensional in \( \text{How} \).

**Proof of Theorem 4.3**. We proceed by assuming that the dimension of \( H^1_f(\mathbb{Q}, V^\circ(\mathfrak{f})(-1)) \) is greater than 1 and we will derive a contradiction from this. Consider the element \( \Theta^\circ(\mathfrak{f}, d) \) in the statement of the theorem. Assume it is non-zero, then we can always find an integer \( n \geq 1 \) large enough such that the element \( \Theta_n^\circ(\mathfrak{f}, d) \) is non-zero in \( H^1(\mathbb{Q}, M_n^\circ(\mathfrak{f})(-1)) \). By [Lin1] Lemma 5.9, we can find a free \( O_n \)-module \( S_n \) of rank 2 in \( H^1_f(\mathbb{Q}, N_n^\circ(\mathfrak{f})(-1)) \) with a basis \( \{s, s'\} \) such that \( \Theta_n^\circ(\mathfrak{f}, d) = \varpi^{n_0}s \) for some \( n_0 < n \).

Under the assumptions in the theorem, there are infinitely many \((n, 1)\)-admissible primes for \( f \) by the proof of [BD3] Theorem 3.2. By the same argument as in [LT] Theorem 5.7 which relies on [Lin2] Lemma 4.16, Lemma 4.11, we can choose a pair of admissible primes \((p, q)\) such that
- the image of \( \text{loc}_p(s') \) in \( H_{\text{fin}}^1(\mathbb{Q}_p, N_1^1(\mathfrak{f})(-1)) \) is 0;
- the image of \( \text{loc}_p(s) \) in \( H_{\text{fin}}^1(\mathbb{Q}_p, N_1^1(\mathfrak{f})(-1)) \) is non-zero;
- the image of \( \text{loc}_q(s') \) in \( H_{\text{fin}}^1(\mathbb{Q}_q, N_1^1(\mathfrak{f})(-1)) \) is non-zero.

By [Lin1] Lemma 3.4, we know
- \( \text{loc}_p(s') \in H_{\text{fin}}^1(\mathbb{Q}_p, N_p^0(\mathfrak{f})(-1)) \) for all \( v \nmid lN \);
- \( \text{loc}_p(s') \in H_f^1(\mathbb{Q}_q, N_p^0(\mathfrak{f})(-1)) \).

We consider the element \( \Theta_n^{[pq]}(\mathfrak{f}, d) \in H^1(\mathbb{Q}, M_n^\circ(\mathfrak{f})(-1)) \). This element has the following properties
- \( \text{loc}_v(\Theta_n^{[pq]}(\mathfrak{f}, d)) \in H_{\text{fin}}^1(\mathbb{Q}_v, M_v^0(\mathfrak{f})(-1)) \) for all \( v \nmid lpqNd \);
- \( \text{loc}_v(\Theta_n^{[pq]}(\mathfrak{f}, d)) \in H_f^1(\mathbb{Q}_v, M_v^0(\mathfrak{f})(-1)) \).

These properties follow from the fact that \( X_d^{pq} \) has good reduction away from \( pqNd \). By Lemma 4.5 (2) and (3), there is an integer \( n_N \geq 1 \) such that
\begin{equation}
\varpi^n \mid \sum_{v \notin \{p, q\}} \varpi^{n_N}(s', \Theta_n^{[pq]}(\mathfrak{f}, d))_v.
\end{equation}
Since \( \text{loc}_p(s') = 0 \) in \( H^1_{\text{fug}}(\mathbb{Q}_p, N_I f)(-1) \), we have \( (s', \Theta_n^{[pq]}(f, d))_p = 0 \). Let \( \phi \in \Gamma(\mathcal{O}_\Lambda, E_\Lambda/\mathcal{O}_\Lambda)[p_n] \) be a generator for this rank one module. By the choice of \( s \), we have

\[
\varpi^{n_0} \mid (\text{loc}_p(\Theta_n^{[pq]}), \phi \otimes \phi \otimes \phi) \quad \text{but} \quad \varpi^{n_0+1} \nmid (\text{loc}_p(\Theta_n^{[pq]}(f, d)), \phi \otimes \phi \otimes \phi).
\]

Since we have

\[
(\partial_q \Theta_n^{[pq]}(f, d), \phi \otimes \phi \otimes \phi) = (1 + q^2)(\text{loc}_p(\Theta_n^{[pq]}(f, d)), \phi \otimes \phi \otimes \phi)
\]

by Theorem 3.13 and \( l \nmid q + 1 \),

\[
(4.3) \quad \varpi^{n_0} \mid (s', \Theta_n^{[pq]}(f, d))_q \quad \text{but} \quad \varpi^{n_0+1} \nmid (s', \Theta_n^{[pq]}(f, d))_q.
\]

We can choose \( n, n_0, n_N \) such that \( n > n_0 + n_N \). By Lemma 4.5 (1),

\[
\varpi^n \mid \sum_v (s', \Theta_n^{[pq]}(f, d))_v
\]

where the sum runs through all the places \( v \) of \( \mathbb{Q} \). This implies

\[
\varpi^{n-n_N} \mid (s', \Theta_n^{[pq]}(f, d))_q
\]

by (4.2). This is a contradiction to (4.3). \( \square \)

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