Insights into the Core of the Assignment Game via Complementarity

Vijay V. Vazirani*¹

¹University of California, Irvine

Abstract

The assignment game forms a paradigmatic setting for studying the core – its pristine structural properties yield an in-depth understanding of this quintessential solution concept within cooperative game theory. In turn, insights gained provide valuable guidance on profit-sharing in real-life situations. In this vein, we raise three basic questions and address them using the following broad idea.

Consider the LP-relaxation of the problem of computing an optimal assignment. On the one hand, the worth of the assignment game is given by the optimal objective function value of this LP, and on the other, the classic Shapley-Shubik Theorem [SS71] tells us that its core imputations are precisely optimal solutions to the dual of this LP. These two facts naturally raise the question of viewing core imputations through the lens of complementarity. In turn, this leads to a resolution of all our questions.

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1 Introduction

The matching game forms one of the cornerstones of cooperative game theory; in particular, when restricted to bipartite graphs, it is called the assignment game, which has a wide range of applications. A quintessential solution concept in this theory is the core, which captures all possible ways of distributing the total worth of a game among individual agents in such a way that the grand coalition remains intact, i.e., a sub-coalition will not be able to generate more profits by itself and therefore has no incentive to secede from the grand coalition. For an extensive coverage of these notions, see the book by Moulin [Mou14].

The classic paper of Shapley and Shubik [SS71] gave a characterization of the profit-sharing methods that lie in the core of the assignment game. They also provided other insights into the structure of the core of this game, e.g., they characterized “antipodal” points in the core, i.e., imputations which are maximally distant. This in-depth understanding, which follows from the pristine structural properties of the assignment game, make it a paradigmatic setting for studying the core. In turn, insights gained provide valuable guidance on profit-sharing in real-life situations.

In this vein, we raise the following basic questions and resolve them in this paper:

1. Do core imputations spread the profit more-or-less evenly or do they restrict them to certain well-chosen agents? If the latter, what characterizes these “chosen” agents?

2. Let \( i \) and \( j \) be two agents who can be matched and let \( w_{ij} \) be the worth generated if they are matched. By definition, under any core imputation, the sum of the profits of \( i \) and \( j \) is at least \( w_{ij} \), independent of whether \( i \) and \( j \) are matched to each other or matched at all. For which pairs of agents \( (i, j) \) can this sum strictly exceed \( w_{ij} \) and for what reason?

3. An assignment game is said to be degenerate if the optimal assignment is not unique. Although Shapley and Shubik had mentioned this phenomenon, they brushed it away, claiming that “in the most common case” the optimal assignment will be unique, and if not, their suggestion was to perturb the edge weights to make the optimal assignment unique. However, this is far from satisfactory, since perturbing the weights destroys crucial information contained in the original instance and the outcome becomes a function of the vagaries of the randomness imposed on the instance.

Our paper addresses these issues using the following broad idea. First consider the LP-relaxation of the problem of computing an optimal assignment. A well-known theorem in matching theory says that this LP always has an integral optimal solution, i.e., an optimal assignment [LP86]. Therefore, the worth of the assignment game is given by the optimal objective function value of this LP. Next consider the dual of this LP. The Shapley-Shubik Theorem says that the set of core imputations of this game are precisely optimal solutions to this dual LP.

These two facts naturally raise the question of viewing core imputations through the lens of complementarity; in turn, this leads to a resolution of all three issues. Recall that the complementary slackness conditions for a primal-dual pair of LPs relate primal variables with dual constraints and dual variables with primal constraints, see [Sch86].

Under core imputations, the profit allocated to an agent is a function of the value he/she brings to
the various sub-coalitions he/she belongs to, i.e., it is consistent with his/her negotiating power. Indeed, it is well known that core imputations provide profound insights into the negotiating power of individuals and sub-coalitions; Section 5 illustrates this via some well-chosen examples. The first question provides further insights into this issue. Our answer to this question is that the core rewards only essential agents, namely those who are matched by every maximum weight matching.

Our answer to the second issue is most counter-intuitive: we show that a pair of players \((i, j)\) get overpaid by core allocations if and only if they are so incompetent, as a team, that they don’t participate in any maximum weight matching! Since \(i\) and \(j\) are incompetent as a team, \(w_{ij}\) is small. On the other hand, \(i\) and \(j\) do team up with other agents in maximum weight matchings – otherwise, such a matching would have chosen the team \((u, v)\) despite its low weight. This explains the fact that the sum of the profits \(i\) and \(j\) accrue exceeds \(w_{ij}\). Our insight into degeneracy is that it treats teams and agents in markedly different ways, see Section 4.3. Section 3 discusses past approaches to degeneracy.

The following setting, taken from [EK01] and [BKP12], vividly captures the issues underlying profit-sharing in an assignment game. Suppose a coed tennis club has sets \(U\) and \(V\) of women and men players, respectively, who can participate in an upcoming mixed doubles tournament. Assume \(|U| = m\) and \(|V| = n\), where \(m, n\) are arbitrary. Let \(G = (U, V, E)\) be a bipartite graph whose vertices are the women and men players and an edge \((i, j)\) represents the fact that agents \(i \in U\) and \(j \in V\) are eligible to participate as a mixed doubles team in the tournament. Let \(w\) be an edge-weight function for \(G\), where \(w_{ij} > 0\) represents the expected earnings if \(i\) and \(j\) do participate as a team in the tournament. The total worth of the game is the weight of a maximum weight matching in \(G\).

Assume that the club picks such a matching for the tournament. The question is how to distribute the total profit among the agents — strong players, weak players and unmatched players — so that no subset of players feel they will be better off seceding and forming their own tennis club.

2 Definitions and Preliminary Facts

The assignment game, \(G = (U, V, E), \; w : E \to \mathbb{R}_+\), has been defined in the Introduction. We start by giving definitions needed to state the Shapley-Shubik Theorem.

**Definition 1.** The set of all players, \(U \cup V\), is called the grand coalition. A subset of the players, \((S_u \cup S_v)\), with \(S_u \subseteq U\) and \(S_v \subseteq V\), is called a coalition or a sub-coalition.

**Definition 2.** The worth of a coalition \((S_u \cup S_v)\) is defined to be the maximum profit that can be generated by teams within \((S_u \cup S_v)\) and is denoted by \(p(S_u \cup S_v)\). Formally, \(p(S_u \cup S_v)\) is the weight of a maximum weight matching in the graph \(G\) restricted to vertices in \((S_u \cup S_v)\) only. \(p(U \cup V)\) is called the worth of the game. The characteristic function of the game is defined to be \(p : 2^{U \cup V} \to \mathbb{R}_+\).
Definition 3. An imputation\(^1\) gives a way of dividing the worth of the game, \(p(U \cup V)\), among the agents. It consists of two functions \(u : U \rightarrow \mathbb{R}_+\) and \(v : V \rightarrow \mathbb{R}_+\) such that \(\sum_{i \in U} u(i) + \sum_{j \in V} v(j) = p(U \cup V)\).

Definition 4. An imputation \((u, v)\) is said to be in the core of the assignment game if for any coalition \((S_u \cup S_v)\), the total worth allocated to agents in the coalition is at least as large as the worth that they can generate by themselves, i.e., \(\sum_{i \in S_u} u(i) + \sum_{j \in S_v} v(j) \geq p(S)\).

We next describe the characterization of the core of the assignment game given by Shapley and Shubik \([SS71]\)^2.

As stated in Definition 2, the worth of the game, \(G = (U, V, E)\), \(w : E \to \mathbb{R}_+\), is the weight of a maximum weight matching in \(G\). Linear program (1) gives the LP-relaxation of the problem of finding such a matching. In this program, variable \(x_{ij}\) indicates the extent to which edge \((i, j)\) is picked in the solution. Matching theory tells us that this LP always has an integral optimal solution \([LP86]\); the latter is a maximum weight matching in \(G\).

\[
\begin{align*}
\text{max} & \quad \sum_{(i,j) \in E} w_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall i \in U, \\
& \quad \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall j \in V, \\
& \quad x_{ij} \geq 0 \quad \forall (i, j) \in E
\end{align*}
\]

Taking \(u_i\) and \(v_j\) to be the dual variables for the first and second constraints of (1), we obtain the dual LP:

\[
\begin{align*}
\text{min} & \quad \sum_{i \in U} u_i + \sum_{j \in V} v_j \\
\text{s.t.} & \quad u_i + v_j \geq w_{ij} \quad \forall (i, j) \in E, \\
& \quad u_i \geq 0 \quad \forall i \in U, \\
& \quad v_j \geq 0 \quad \forall j \in V
\end{align*}
\]

Theorem 1. (Shapley and Shubik \([SS71]\)) The imputation \((u, v)\) is in the core of the assignment game if and only if it is an optimal solution to the dual LP, (2).

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\(^1\)Some authors prefer to call this a pre-imputation, while using the term imputation when individual rationality is also satisfied.

\(^2\)Shapley and Shubik had described this game in the context of the housing market in which agents are of two types, buyers and sellers. They had shown that each imputation in the core of this game gives rise to unique prices for all the houses. In this paper we will present the assignment game in a variant of the tennis setting given in the Introduction; this will obviate the need to define “prices”, hence leading to simplicity.
3 Related Works

The core is a key solution concept in cooperative game theory for several reasons. First, imputations in the core give a way of distributing the worth of the game among players so the grand coalition remains intact, i.e., no sub-coalition can do better by itself and hence has any incentive to secede. Second, this way of distributing the worth can be considered to be fair, since each agent gets as much as her worth to the grand coalition and all sub-coalitions she is in.

An imputation in the core has to ensure that each of the exponentially many sub-coalitions is “happy” — clearly, that is a lot of constraints. As a result, the core is non-empty only for a handful of games, those with very good structural properties. Besides the assignment game, which is described in Section 2, this holds for the stable matching solution concept, given by Gale and Shapley [GS62], which lies in the core of their game. The only coalitions that matter in this game are ones formed by one agent from each side of the bipartition; a stable matching ensures that no such coalition has the incentive to secede.

Next, let us consider a generalization of the assignment game to the general graph matching game. It consists of an undirected graph $G = (V, E)$ and an edge-weight function $w$. The vertices $i \in V$ are the agents and an edge $(i, j)$ represents the fact that agents $i$ and $j$ are eligible for an activity or a trade. Continuing with the analogy of tennis, given for the assignment game in the Introduction, for this generalization, we will assume that any two agents can form a doubles team. If $(i, j) \in E$, $w_{ij}$ represents the profit generated if $i$ and $j$ play in the tournament. The worth of a coalition $S \subseteq V$, denoted $p(S)$, is given by the weight of a maximum matching in $G$ restricted to $S$. This game may have an empty core.

To deal with the question of emptiness of core, the following two notions have been given in the past. The first is that of least core, defined by Mascher et al. [MPS79]. If the core is empty, there will necessarily be sets $S \subseteq V$ such that $v(S) < p(S)$ for any imputation $v$. The least core maximizes the minimum of $v(S) - p(S)$ over all sets $S \subseteq V$, subject to $v(\emptyset) = 0$ and $v(V) = p(V)$. This involves solving an LP with exponentially many constraints, though, if a separation oracle can be implemented in polynomial time, then the ellipsoid algorithm will accomplish this in polynomial time [GLS88]; see below for a resolution for the case of the matching game.

A more well known notion is that of nucleolus which is contained in the least core. After maximizing the minimum of $v(S) - p(S)$ over all sets $S \subseteq V$, it does the same for all remaining sets and so on. A formal definition is given below.

**Definition 5.** For an imputation $v : V \to \mathcal{R}_+$, let $\theta(v)$ be the vector obtained by sorting the $2^{|V|} - 2$ values $v(S) - p(S)$ for each $\emptyset \subset S \subset V$ in non-decreasing order. Then the unique imputation, $v$, that lexicographically maximizes $\theta(v)$ is called the nucleolus and is denoted $\nu(G)$.

The nucleolus was defined in 1969 by Schmeidler [Sch69], though its history can be traced back to the Babylonian Talmud [AM85]. It has several modern-day applications, e.g., [BST05]. In 1998, [FKFH98] stated the problem of computing the nucleolus of the matching game in polynomial time. For the assignment game with unit weight edges, this was done in [SR94]; however, since the assignment game has a non-empty core, this result was of little value. For the general graph matching game with unit weight edges, this was done by Kern and Paulusma [KP03]. Finally,
the general problem was resolved by Konemann et al. [KPT20]. However, their algorithm makes extensive use of the ellipsoid algorithm and is therefore neither efficient nor does it give deep insights into the underlying combinatorial structure. They leave the open problem of finding a combinatorial polynomial time algorithm. We note that the difference $v(S) - p(S)$ appearing in the least core and nucleolus has not been upper-bounded for any standard family of games, including the general graph matching game.

A different notion was recently proposed in [Vaz22], namely approximate core. That paper gives an imputation in the $2/3$-approximate core for the general graph matching game, i.e., the total profit allocated to a sub-coalition is at least $2/3$ factor of the profit which it can generate by seceding. Moreover, this imputation can be computed in polynomial time and the bound is best possible, since it is the integrality gap of the natural underlying LP. This method used methodology developed in field of approximation algorithms, e.g., see [Vaz01], which uses multiplicative approximation as a norm, and yields polynomial time algorithms.

Whereas the previous two notions involve additive approximation, [Vaz22] uses multiplicative approximation. As argued in [Vaz22], this is more fair: the worst set for least core, say $S$, may have small $p(S)$. If so, the members of this set are unfairly treated, since they may end up getting almost no part of the worth of the game. On the other hand, under multiplicative approximation, the profit of every coalition is guaranteed to be at least $2/3$ fraction of its worth. It therefore appears to be a simpler, more direct and more effective way of arriving at a “fair” division of the worth of a game in the face of an empty core.

Over the years, researchers have approached the phenomenon of degeneracy in the assignment game from directions that are different from ours. Nunez and Rafels [NR08], studied relationships between degeneracy and the dimension of the core. They defined an agent to be active if her profit is not constant across the various imputations in the core, and non-active otherwise. Clearly, this notion has much to do with the dimension of the core, e.g., it is easy to see that if all agents are non-active, the core must be zero-dimensional. They prove that if all agents are active, then the core is full dimensional if and only if the game is non-degenerate. Furthermore, if there are exactly two optimal matchings, then the core can have any dimension between 1 and $m-1$, where $m$ is the smaller of $|U|$ and $|V|$; clearly, $m$ is an upper bound on the dimension.

In another work, Chambers and Echenique [CE15] study the following question: Given the entire set of optimal matchings of a game on $m = |U|$, $n = |V|$ agents, is there an $m \times n$ surplus matrix which has this set of optimal matchings. They give necessary and sufficient conditions for the existence of such a matrix.

4 The Core via the Lens of Complementarity

In this section, we provide answers to the three issues raised in the Introduction.

4.1 The first issue: Allocations made to agents by core imputations

Definition 6. A generic player in $U \cup V$ will be denoted by $q$. We will say that $q$ is:

1. essential if $q$ is matched in every maximum weight matching in $G$. 

2. **viable** if there is a maximum weight matching $M$ such that $q$ is matched in $M$ and another, $M'$ such that $q$ is not matched in $M'$.

3. **subpar** if for every maximum weight matching $M$ in $G$, $q$ is not matched in $M$.

**Definition 7.** Let $y$ be an imputation in the core. We will say that $q$ gets paid in $y$ if $y_q > 0$ and does not get paid otherwise. Furthermore, $q$ is paid sometimes if there is at least one imputation in the core under which $q$ gets paid, and it is never paid if it is not paid under every imputation.

**Theorem 2.** For every player $q \in (U \cup V)$:

$$q \text{ is paid sometimes } \iff q \text{ is essential}$$

**Proof.** The proof follows by applying complementary slackness conditions and strict complementarity to the primal LP (1) and dual LP (2); see [Sch86] for formal statements of these facts. By Theorem 1, talking about imputations in the core of the assignment game is equivalent to talking about optimal solutions to the dual LP.

Let $x$ and $y$ be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each $q \in (U \cup V)$:

$$y_q \cdot (x(\delta(q)) - 1) = 0.$$ 

Suppose $q$ is paid sometimes. Then, there is an optimal solution to the dual LP, say $y$, such that $y_q > 0$. By the Complementary Slackness Theorem, for any optimal solution, $x$, to LP (1), $x(\delta(q)) = 1$, i.e., $q$ is matched in $x$. Varying $x$ over all optimal primal solutions, we get that $q$ is always matched. In particular, $q$ is matched in all optimal assignments, i.e., integral optimal primal solutions, and is therefore essential. This proves the forward direction.

Strict complementarity implies that corresponding to each player $q$, there is a pair of optimal primal and dual solutions, say $x$ and $y$, such that either $y_q = 0$ or $x(\delta(q)) = 1$ but not both. Assume that $q$ is essential, i.e., it is matched in every integral optimal primal solution. Since every fractional optimal primal solution is a convex combination of integral optimal primal solutions, $q$ is fully matched in every optimal solution, $x$, to LP (1), i.e., $x(\delta(q)) = 1$. Therefore there must be an optimal dual solution $y$ such that $y_q > 0$. Hence $q$ is paid sometimes, proving the reverse direction.

Theorem 2 is equivalent to the following. For every player $q \in (U \cup V)$:

$$q \text{ is never paid } \iff q \text{ is not essential}$$

Thus core imputations pay only essential players. Since we have assumed that the weight of each edge is positive, so is the worth of the game, and all of it goes to essential players. Hence we get:

**Corollary 1.** In the assignment game, the set of essential players is non-empty and in every core imputation, the entire worth of the game is distributed among essential players.

Corollary 1 is of much consequence: It tells us that no matter which teams the tennis club picks for the tournament, the players who are allocated profits are precisely the ones who play
in all possible choices of the tennis club. Furthermore, the identification of these players, and
the exact manner in which the total profit is divided up among them, follows the negotiating
process described on Section 5, in which each player ascertains his/her negotiating power based
on all possible sub-coalitions he/she participates in. Perhaps the most remarkable aspect of the
Shapley-Shubik result is that each possible outcome of this very real process is captured by an
inanimate object, namely an optimal solution to the dual, LP (2).

By Corollary 1, core imputations reward only essential players. This raises the following question:
Can’t a non-essential player, say \( q \), team up with another player, say \( p \), and secede, by promising
\( p \) almost all of the resulting profit? The answer is “No”, because the dual (2) has the constraint
\( y_q + y_p \geq w_{qp} \). Therefore, if \( y_q = 0 \), \( y_p \geq w_{qp} \), i.e., \( p \) will not gain by seceding together with \( q \).

4.2 The second issue: Allocations made to teams by core imputations

Definition 8. By a mixed doubles team we mean an edge in \( G \); a generic one will be denoted as
\( e = (u, v) \). We will say that \( e \) is:

1. essential if \( e \) is matched in every maximum weight matching in \( G \).
2. viable if there is a maximum weight matching \( M \) such that \( e \in M \), and another, \( M' \) such
   that \( e \notin M' \).
3. subpar if for every maximum weight matching \( M \) in \( G \), \( e \notin M \).

Definition 9. Let \( y \) be an imputation in the core of the game. We will say that \( e \) is fairly paid in \( y \)
if \( y_u + y_v = w_e \) and it is overpaid if \( y_u + y_v > w_e \). Finally, we will say that \( e \) is always paid fairly
if it is fairly paid in every imputation in the core.

Theorem 3. For every team \( e \in E \):

\[
e \text{ is always paid fairly} \iff e \text{ is viable or essential}
\]

Proof. The proof is similar to that of Theorem 2. Let \( x \) and \( y \) be optimal solutions to LP (1)
and LP (2), respectively. By the Complementary Slackness Theorem, for each \( e = (u, v) \in E : 
\]
\[
x_e \cdot (y_u + y_v - w_e) = 0.
\]

We will prove the reverse direction of the implication first. Suppose \( e \) is viable or essential. Then
there is an optimal solution to the primal, say \( x \), under which it is matched. Therefore, \( x_e > 0 \).
Let \( y \) be an arbitrary optimal dual solution. Then, by the Complementary Slackness Theorem,
\( y_u + y_v = w_e \), i.e., \( e \) is fairly paid in \( y \). Varying \( y \) over all optimal dual solutions, we get that \( e \) is
always paid fairly.

To prove the forward direction, we will use strict complementarity. It implies that corresponding
to each team \( e \), there is a pair of optimal primal and dual solutions \( x \) and \( y \) such that either \( x_e = 0 \)
or \( y_u + y_v = w_e \) but not both.

Assume that team \( e \) is always fairly paid, i.e., under every optimal dual solution \( y \), \( y_u + y_v = w_e \).
By strict complementarity, there must be an optimal primal solution \( x \) for which \( x_e > 0 \). Since

\[\text{Observe that by the first constraint of the dual LP (2), these are the only possibilities.}\]
the polytope defined by the constraints of the primal LP (1) has integral optimal vertices, \( x \) is a convex combination of optimal assignments. Therefore, there must be an optimal assignment in which \( e \) is matched. Therefore \( e \) is viable or essential and the forward direction also holds. \( \square \)

Negating both sides of the implication proved in Theorem 3, we get the following implication. For every team \( e \in E \):

\[
e \text{ is subpar } \iff e \text{ is sometimes overpaid}
\]

Clearly, this statement is equivalent to the statement proved Theorem 3 and hence contains no new information. However, it provides a new viewpoint. These two equivalent statements yield the following assertion, which at first sight seems incongruous with what we desire from the notion of the core and the just manner in which it allocates profits:

*Whereas viable and essential teams are always paid fairly, subpar teams are sometimes overpaid.*

How can the core favor subpar teams over viable and essential teams? An explanation is provided in the Introduction, namely a subpar team \((i, j)\) gets overpaid because \(i\) and \(j\) create worth by playing in competent teams with other players.

Finally, we observe that contrary to Corollary 1, which says that the set of essential players is non-empty, the set of essential teams may be empty, as is the case in Examples 1 and 2 in Section 5.

### 4.3 The third issue: Degeneracy

Next we use Theorems 2 and 3 to get insights into degeneracy. Clearly, if an assignment game is non-degenerate, then every team and every player is either always matched or always unmatched in the set of maximum weight matchings in \( G \), i.e., there are no viable teams or players. Since viable teams and players arise due to degeneracy, in order to understand the phenomenon of degeneracy, we need to understand how viable teams and players behave with respect to core imputations; this is done in the next corollary.

**Corollary 2.** In the presence of degeneracy, imputations in the core of an assignment game treat:

- viable players in the same way as subpar players, namely they are never paid.
- viable teams in the same way as essential teams, namely they are always fairly paid.

### 5 Insights Provided by the Core into the Negotiating Power of Agents

**Example 1.** Consider an assignment game whose bipartite graph, on the three agents \( u, v_1, v_2 \), and two edges, is given in Figure 1. Clearly, one of \( v_1 \) and \( v_2 \) will be left out in any matching. First assume that the weight of both edges is 1. If so, the unique imputation in the core gives zero to \( v_1 \) and \( v_2 \), and 1 to \( u \). Next assume that the weights of the edges \((u, v_1)\) and \((u, v_2)\) are 1 and \(1 + \epsilon\) respectively, for a small \( \epsilon > 0 \). If so, the unique imputation in the core gives 0, \( \epsilon \) and 1 to \( v_1 \), \( v_2 \) and \( u \), respectively.
How fair are the imputations given in Example 1? As stated in the Introduction, imputations in the core have a lot to do with the negotiating power of individuals and sub-coalitions. Let us argue that when the imputations given above are viewed from this angle, they are fair in that the profit allocated to an agent is consistent with their negotiating power, i.e., their worth. In the first case, whereas $u$ has alternatives, $v_1$ and $v_2$ don’t. As a result, $u$ will squeeze out all profits from whoever she plays with, by threatening to partner with the other player. Therefore $v_1$ and $v_2$ have to be content with no rewards! In the second case, $u$ can always threaten to match up with $v_2$. Therefore $v_1$ has to be content with a profit of $\epsilon$ only.

**Example 2.** Consider an assignment game whose bipartite graph, shown in Figure 2, has four edges, $(u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3)$ on the five agents $u_1, u_2, v_1, v_2, v_3$. Let the weights of these four edges be 1, 1.1, 1.1 and 1, respectively. The worth of this game is clearly 2.1.

In Example 2, at first sight, $v_2$ looks like the dominant player, since he has two choices of partners, namely $u_1$ and $u_2$, and because teams involving him have the biggest earnings, namely 1.1 as opposed to 1. Yet, the unique core imputation in the core awards $1,1,0,0.1,0$ to agents $u_1, u_2, v_1, v_2, v_3$. 
$u_1, u_2, v_1, v_2, v_3$, respectively.

The question arises: “Why is $v_2$ allocated only 0.1? Can’t he negotiate a higher profit, given his favorable circumstance?” The answer is “No.” The reason is that $u_1$ and $u_2$ are in an even stronger position than $v_2$, since both of them have a ready partner available, namely $v_1$ and $v_3$, respectively, with whom each can earn 1. Therefore, the core imputation awards 1 to each of them, giving the leftover profit of 0.1 to $v_2$. Hence the core imputation has indeed allocated profits according to the negotiating power of each agent.

**Example 3.** Consider the assignment game whose bipartite graph, in eight vertices, is shown in Figure 3. Assume that the weights of $(u_1, v_1)$ and $(u_2, v_2)$ are 100 each, the weights of $(u_1, v_3)$ and $(u_2, v_4)$ are 51 each, and the weights of $(u_3, v_2)$ and $(u_4, v_1)$ are 50 each. Clearly, the worth of the game is 202 and is given by matching the last four edges. There are two antipodal imputations in the core. The woman-optimal imputation gives 51 to each of $u_1$ and $u_2$, 50 to each of $v_1$ and $v_2$, and zero to the rest. The man-optimal imputation gives 50 to each of $u_1, u_2, v_1$ and $v_2$, 1 to each of $v_3$ and $v_4$, and zero to $u_3$ and $u_4$.

![Figure 3: The graph for Example 3.](image)

Let us partition the players into two sets: the *bottom players* consisting of $u_3, u_4, v_3$ and $v_4$, and the *top players* consisting of the rest. The question is why do the bottom players get so little profit as compared to the top players? In particular, the bottom players get zero profit in the woman-optimal imputation and a total of only 2 in the man-optimal imputation.
The reason is that the top players can generate a worth of 200 on their own, via the teams \((u_1, v_1)\) and \((u_2, v_2)\). This gives them the power to negotiate a total profit of at least 200, leaving a profit of at most 2 for the first four players. Indeed, the core imputations must respect this, to prevent the top players from seceding.

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