Polynomial rings of the chiral $SU(N)_2$ models

A Lima-Santos
Departamento de Física, Universidade Federal de São Carlos, Caixa Postal 676, 13569-905 São Carlos, Brazil

Abstract. Via explicit diagonalization of the chiral $SU(N)_2$ fusion matrices, we discuss the possibility of representing the fusion ring of the chiral $SU(N)$ models, at level $K = 2$, by a polynomial ring in a single variable when $N$ is odd and by a polynomial ring in two-variable when $N$ is even.

1. Introduction

Six years ago, Gepner conjectured that the fusion ring of theories with $SU(N)$ current algebra is isomorphic to a ring in $N - 1$ variables associated to the fundamental representations, quotiented by an ideal of constraints that derive with a potential [1].

Four years ago, Di Francesco and Zuber postulated a necessary and sufficient condition for a one-variable polynomial ring [2]: Assume that among the matrices $N_i, i = 1, ..., n$, there exists at least one, call it $N_f$, with non degenerated eigenvalues. Thus, any other $N_i$ may be diagonalized in the same basis as $N_f$ and there exists a unique polynomial $P_i(x)$ of degree at most $n - 1$ such that its eigenvalues $\gamma_i^{(l)}$ satisfy

$$\gamma_i^{(l)} = P_i(\gamma_f^{(l)})$$

$P_i$ being given by the Lagrange interpolation formula. Therefore, any $N_i$ may be written as

$$N_i = P_i(N_f)$$

with a polynomial $P_i$ ; as both $N_i$ and $N_f$ have integral entries, $P_i(x)$ must have rational coefficients.

The $n \times n$ matrix $N_f$, on the other hand, satisfies its characteristic equation $\mathcal{P}(x) = 0$, that is also its minimal equation, as $N_f$ has no degenerate eigenvalues. The constraint on $N_f$ is thus

$$\mathcal{P}(N_f) = 0$$

that may of course be integrated to yield a ”potential” $\mathcal{W}(x)$, which is a polynomial of degree $n + 1$. In this way, Di Francesco and Zuber have characterized the rational
conformal field theories (RCFTs) which have a description in terms of a fusion potential in one variable. Moreover, they have also proposed a generalized potential to describe other theories. In reference\cite{3} Aharony have determined a simple criterion to a generalized description of RCFTs by fusion potentials in more than one variable.

In this note we tackle this problem and discuss the possibility of representing the fusion rings of the chiral $SU(N)_2$ models, by polynomial rings in two variables. Exploiting the Di Francesco and Zuber condition we show that these polynomial rings in two variables are reduced to polynomial rings in a single variable in the cases for which $N$ is odd (or $N = 2$).

In section 2 we discuss some algebraic setting of the chiral RCFTs. Section 3 describes the primary fields of the chiral $SU(N)$ models, at level $K = 2$, in *cominimal equivalence classes*. In the last section we report a computer study which diagonalizes the fusion matrices of the chiral $SU(N)_2$ models and gives their polynomial rings in one and two variables.

2. Fusion algebras

Fusion algebras are found to play an important role in the study of RCFTs. Beside the fact that the fusion rules can be expressed in terms of the unitary matrix $S$ that encodes the modular transformations of the characters of the RCFT

$$N^k_{ij} = \sum_l \frac{S^*_{il}}{S_{0l}} S_{jl} S^*_{kl}. \quad (4)$$

Here "0 " refers to the identity operator, and the labels $i, ..., l$ run over $n$ values corresponding to the primary fields of the chiral algebra of the RCFT. There is a more fundamental reason to look for representations of the fusion algebra, based on the concept of operator products\cite{6}. When one tries to compute the operator product coefficients, one is almost inevitably led to the concept of fusion rules, i.e. formal products

$$A_i A_j = \sum_k N^k_{ij} A_k. \quad (5)$$

of primary fields describing the basis-independent content of the operator product algebra.

By definition, the fusion rule coefficients possess the property of integrality $N^k_{ij} \in \mathbb{Z}_{\geq 0}$. In addition, they inherit several simple properties:

- **symmetry**: $N^k_{ij} = N^k_{ji}$.
- **associativity**: $\sum_k N^k_{ij} N^m_{kl} = \sum_k N^k_{jl} N^m_{ik}$.
- **existence of unit**: there is an index "0 " (identity operator) such that $N^i_{0i} = \delta^i_i$ and
- **charge conjugation**: $N_{ijl} = \sum_k N^k_{ij} C_{kl} = (N^l_{ij})^\dagger$ is completely symmetric in the indices $i, j, l$. 


Because of these properties, one can interpret the fusion rule coefficients as the structure constants of a commutative associative ring with basis given by the primary fields.

The matrix $S$ implements the modular transformation $\tau \to -1/\tau$ and obeys $S^2 = C$. In addition, the diagonal matrix $T_{ii} = \exp(2i\pi(\Delta_i - c/24))$, where $\Delta_i$ is the conformal dimension of the primary field $i$ and $c$ is the central charge, implements the modular transformation $\tau \to \tau + 1$ and obeys $(ST)^3 = C$, which implies a relation between the structure constants $N^k_{ij}$ and the conformal dimensions $\Delta_i$ [7]:

$$N_{ijkl}(\Delta_i + \Delta_j + \Delta_k + \Delta_l) = \sum_r N_{ijklr}\Delta_r$$

where

$$N_{ijkl} = N^r_{ij}N^n_{kl} \quad \text{and} \quad N_{ijklr} = N^r_{ij}N_{klr} + N^r_{jk}N_{ilr} + N^r_{ik}N_{jlr}$$

It was suggested in [8] that these proprieties fully characterize a RCFT, and that any commutative ring satisfying these properties is the fusion ring of some RCFT.

The matrices $N_i$ defined by $(N_i)_{jk} = N^k_{ij}$ form themselves a trivial representation of the fusion algebra

$$N_i N_j = \sum_k N^k_{ij} N_k$$

as follows from unitarity of the matrix $S$; this expresses the associativity property of the algebra [9]. The relation [9] implies that the matrix $S$ diagonalizes the matrices $N_i$ and their eigenvalues are of the form

$$\gamma^{(l)}_i = \frac{S^i_l}{S^0_l}$$

and obey the sum rules

$$\gamma^{(l)}_i \gamma^{(l)}_j = \sum_k N^k_{ij} \gamma^{(l)}_k$$

The general study of these fusion algebras and their classification have been the object of much work [8]-[11].

The numbers

$$d_i \doteq \gamma^{(0)}_i = \frac{S^i_0}{S^0_0}$$

appear as statistical dimensions of superselection sectors [12],[13] in algebraic quantum field theory; as square roots of indices for inclusions of von Neumann algebras [14]; as relative sizes of highest weight modules of chiral symmetry algebras in conformal field theory [1]; and in connection with truncated tensor products of quantum groups (see [15] for an accomplished review). According to [14], these numbers obey the statistical dimension sum rules

$$d_i d_j = \sum_k N^k_{ij} d_k.$$
which shows that \( d_i \) is a Frobenius eigenvalue of \( N_i \).

3. \( SU(N)_2 \) cominimal equivalence classes

At the level \( K = 2 \) the central charge of the chiral \( SU(N) \) models is given by

\[
c = \frac{2(N-1)}{N+2}
\]

and their primary fields are identified with the order fields \( \sigma_k \), \( k = 0, 1, ..., N-1 \); \( Z_N \)-neutral fields \( \epsilon^{(j)}, j = 1, 2, ... \leq N/2 \) and the parafermionic currents \( \Psi_k, k = 1, ..., N-1 \), in Zamolodchikov-Fateev’s parafermionic theories\[16\]. For each primary field we define a “charge” \( \nu = 0, 1, ..., 2(N-1) \mod 2N \) and we collect the \( N(N+1)/2 \) primary fields in \( N \) cominimal equivalence classes \[17\], \([\phi^k_\nu]\), \( k = 0, 1, ..., N-1 \), according to their statistical dimensions:

\[
d_k = \frac{k-1}{\prod_{i=0}^{k-1} s(N-i)}; \quad s(x) = \sin \left( \frac{x\pi}{N+2} \right) \\
d_0 = 1, \quad d_{N-k} = d_k, \quad k = 1, 2, ..., N-1
\]

\( SU(N)_2 \) representations of the order fields \( \phi^k_\nu \), \( k = 1, ..., N-1 \) are the fully antisymmetric Young tableaux with \( k \) boxes (i.e. the reduced tableau which is a column with \( k \) boxes). Tableaux of fields comprising a cominimal equivalence class \( \phi^k_\nu \) in which the representation \( \phi^k_\nu \) appears, \( (\nu = k \mod 2, \text{ i.e.}, \nu = k, k+2, \cdots, 2N-2-k) \), are obtained by adding \( (\nu-k)/2 \) rows of width 2 to the top of the reduced tableau of \( \phi^k_\nu \). Therefore \( \phi^k_\nu \) is a Young tableau of two columns with \( \nu \) boxes, since \( (\nu+k)/2 \) boxes in the first column and \( (\nu-k)/2 \) in the second column.

The conformal weights of the fields comprising a cominimal equivalence class in which the representation \( \phi^k_\nu \) appears are simply related to the conformal weight of \( \phi^k_\nu \) by

\[
\Delta^k_\nu = \Delta^k_\nu + \frac{\nu - k}{4N}(2N - \nu - k)
\]

and the conformal dimensions of the order fields \[16\] are given by

\[
\Delta^k_\nu = \frac{k(N-k)}{2N(N+2)}
\]

These equivalence classes are generated by \( Z_N \) symmetry which connect the representations belonging to each class through of the fusion rules\[18\]

\[
\phi^{k_1}_{\nu_1} \times \phi^{k_2}_{\nu_2} = \sum_{k = |k_1-k_2| \mod 2}^{\min(k_1+k_2,2N-k_1-k_2)} \phi^{k}_{\nu_1+\nu_2}
\]

In particular, the elementary field \( \phi^1_1 \), \( (\phi^1_1 \times \phi^k_\nu = \phi^{k-1}_{\nu+1} + \phi^{k+1}_{\nu+1}) \) connects the equivalence class of \( \phi^k_\nu \) with adjacent classes, while the field \( \phi^0_2 \), \( (\phi^0_2 \times \phi^k_\nu = \phi^{k}_{\nu+2}) \), connects the
fields in the same cominimal equivalence class. Thus, the $SU(N)_2$ fusion ring can be generated by these two fields. For example, the 10 primary fields of $SU(4)_2$ can be collected in 4 cominimal equivalence classes as

$$
\begin{align*}
\phi_0^1 & \rightarrow d_1 = \frac{s(4)}{s(1)} \\
\phi_0^2 & \rightarrow d_2 = \frac{s(3)}{s(1)} \\
\phi_0^3 & \rightarrow d_3 = \frac{s(2)}{s(1)} \\
\phi_0^4 & \rightarrow d_4 = \frac{s(1)}{s(1)}
\end{align*}
$$

These cominimal equivalence classes provide a representation of the $Z_4$ symmetry and the primary fields corresponding to representations in the same class differ only by free fields.

4. $SU(N)_2$ polynomial rings

Let us start by considering the case $SU(4)_2$ (the $SU(2)_2$ and $SU(3)_2$ cases were considered in [2]):

The variables $x$ and $y$ are associated to the fields $\phi_1^1$ and $\phi_2^0$, respectively. Using $\phi_0^0 = 1$, the fusion rules (17) (see table (18)) give the expressions of the other fields

$$
\begin{align*}
\phi_0^1 & = 1 \\
\phi_1^0 & = x \\
\phi_2^0 & = y \\
\phi_3^0 & = x^2 - y \\
\phi_4^0 & = xy \\
\phi_5^0 & = y^2 \\
\phi_6^0 & = y^3
\end{align*}
$$

and from the identification $\phi_\nu^k = \phi_\nu^{4-k} \mod 8$ we get the following constraints

$$
\begin{align*}
x^4 - 3x^2y + y^2 &= 1 \\
x^3y - 2xy^2 &= x \\
x^2y^2 - y^3 &= x^2 - y \\
x^3 - 2xy &= xy^3 \\
y^4 &= 1
\end{align*}
$$

These constraints can be combined and reduced to a one-variable constraint

$$
x^{10} - 8x^6 - 9x^2 = 0,
$$

which is equal to the characteristic equation of the fusion matrix $N_{\phi_1^1}$, and its eigenvalue 0 is doubly degenerate implying that $x$ may not be inverted on the ring. Similarly, one can eliminate $x$ from (20) and get a one-variable constraint in $y$

$$
y^{10} - y^8 - 2y^6 + 2y^4 + y^2 - 1 = 0
$$
which is equal to the characteristic equation of the fusion matrix $N_{\phi_2}$, whose eigenvalues are degenerate. Thus, the fusion ring of the $SU(4)_2$ model can be expressed in terms of two variables associated with the representations $\phi_1^1$ and $\phi_2^0$ which satisfy independent constraint equations.

Next, let us consider the 15 primary fields of the chiral $SU(5)_2$ model which can be collected in 5 cominimal equivalence classes as:

$$
\begin{align*}
\phi_0^0 & \rightarrow d_0 = \frac{s(6)}{s(1)} \\
\phi_0^1 & \rightarrow d_1 = \frac{s(5)}{s(1)} \\
\phi_0^2 & \rightarrow d_2 = \frac{s(4)}{s(1)} \\
\phi_0^3 & \rightarrow d_3 = \frac{s(3)}{s(1)} \\
\phi_0^4 & \rightarrow d_4 = \frac{s(2)}{s(1)}
\end{align*}
$$

(23)

The variables $x$ and $y$ are associated to the fields $\phi_1^1$ and $\phi_2^0$, respectively. Using $\phi_0^1 = 1$, the fusion rules (17) give the expressions of the other fields

$$
\begin{align*}
\phi_0^0 &= 1 \\
\phi_0^1 &= x \\
\phi_0^2 &= y \\
\phi_0^3 &= y^2 \\
\phi_0^4 &= y^3 \\
\phi_0^5 &= y^4 \\
\phi_0^6 &= x \\
\phi_0^7 &= x^2 \\
\phi_0^8 &= x^3 \\
\phi_1^1 &= x^2 y - y^2 \\
\phi_1^2 &= x^2 y^2 - y^3 \\
\phi_1^3 &= x^3 - 2xy \\
\phi_1^4 &= x^3 y - 2x^2 y^2 \\
\phi_1^5 &= x^4 - 3x^2 y + y^2 \\
\phi_0^4 &= x^4 - 3x^2 y + y^2
\end{align*}
$$

(24)

and the identification $\phi_\nu^k = \phi_{5+k}^\nu \mod 10$ gives us the following constraint equations

$$
\begin{align*}
x^5 - 4x^3 y + 3xy^2 &= 1 \\
x^4 y - 3x^2 y^2 + y^3 &= x \\
x^3 y^2 - 2xy^3 &= x^2 - y \\
x^5 &= 1
\end{align*}
$$

(25)

These constraints can be combined and reduced to a one-variable constraint equation

$$
x^{15} - 16x^{10} - 57x^5 + 1 = 0
$$

(26)

which is equal to the characteristic equation of the fusion matrix $N_{\phi_1^1}$, whose eigenvalues are non-degenerate. It means that $x$ may be inverted on the ring: we can eliminate $y$ from the constraint equations (25) as:

$$
y = \frac{1}{181} (-14x^{12} + 221x^7 + 910x^2).
$$

(27)
Substituting this value of $y$ into (24) we will get a polynomial ring in a single variable:

$$ P_0^0(x) = 1 \quad P_1^0(x) = x $$

$$ P_2^0(x) = \frac{1}{15}(910x^2 + 221x^7 - 14x^{12}) \quad P_3^0(x) = \frac{1}{15}(910x^3 + 221x^8 - 14x^{13}) $$

$$ P_4^0(x) = \frac{1}{15}(4592x^4 + 1260x^9 - 79x^{14}) \quad P_5^0(x) = \frac{1}{15}(79 + 89x^5 - 4x^{10}) $$

$$ P_6^0(x) = \frac{1}{15}(404x + 155x^6 - 9x^{11}) \quad P_7^0(x) = \frac{1}{15}(404x^2 + 155x^7 - 9x^{12}) $$

$$ P_8^0(x) = \frac{1}{15}(2043x^3 + 597x^8 - 37x^{13}) \quad P_3^1(x) = -\frac{1}{131}(1639x^3 + 442x^8 - 28x^{13}) $$

$$ P_2^1(x) = \frac{1}{131}(729x^2 + 221x^7 - 14x^{12}) \quad P_5^1(x) = -\frac{1}{131}(144 + 66x^5 - 5x^{10}) $$

$$ P_4^1(x) = \frac{1}{131}(3682x^4 + 1039x^9 - 65x^{14}) \quad P_4^1(x) = \frac{1}{131}(2043x^4 + 597x^9 - 37x^{14}) $$

$$ P_6^1(x) = \frac{1}{131}(325x + 66x^6 - 5x^{11}) $$

These $P_{\nu}^k(x)$ polynomials define (modulo $x^{15} - 16x^{10} - 57x^5 + 1$) one-variable SU(5)$_2$ polynomial ring.

Similarly, one can eliminate $x$ from (25) and get a one-variable constraint in $y$

$$ y^{15} - 3y^{10} + 3y^5 - 1 = 0 \quad (28) $$

which is equal to the characteristic equation of the fusion matrix $N_{\phi_2^0}$, but their eigenvalues are degenerate.

We now extend this construction to the whole set of SU($N$)$_2$ models. For each irreducible representation $\phi_{\nu}^k$ we associate the following polynomials

$$ P_{\nu}^k(x, y) = \sum_{n=0}^{[\frac{k}{2}]} (-1)^n \frac{(k - n)!}{n!(k - 2n)!} x^{k-2n} y^{n+\frac{\nu-k}{2}} \quad (29) $$

where $k = 0, 1, \ldots, N-1$, $\nu = k$ mod 2, i.e. $\nu = k, k+2, \ldots, 2(N-1)-k$ and $[\frac{k}{2}]$ means the largest integer less than or equal to $k/2$.

The identification $\phi_{\nu}^k = \phi_{N+\nu}^{N-k}$ mod $2N$ gives the corresponding one-variable constraint equations:

$$ x^{\frac{N}{2}} \prod_{n=1}^{\frac{N}{2}} (x^N + (-1)^n d^N(n)) = 0, \quad (y^{\frac{N}{2}} - 1)^{\frac{N+2}{2}} (y^{\frac{N}{2}} + 1)^{\frac{N}{2}} = 0 \quad (30) $$

for the cases when $N$ is even, and

$$ \prod_{n=1}^{\frac{N+1}{2}} (x^N - d^N(n)) = 0, \quad (y^N - 1)^{\frac{N+1}{2}} = 0 \quad (31) $$
for the cases when $N$ is odd. In these expressions have introduced the numbers
\[
d(n) = \frac{\sin(n\pi \frac{N}{N+2})}{\sin(n\pi \frac{N}{N+2})}, \quad n = 1, 2, \ldots, \leq \frac{N+2}{2}.
\] (32)

Inspecting the constraint equations in the variable $y$ we can see that the fusion matrices $N_{\phi_2}$ are degenerate for all $SU(N)_2$ models. It means that we can not eliminate the variable $x$ from the polynomials (29). If $N$ is even and $N > 2$, we can see from (30) that among the eigenvalues of fusion matrices $N_{\phi_1}$ only zero is degenerate ($N/2$ times), following that $x$ also can not be inverted on these rings. It means that we also can not eliminate the variable $y$ from (29) and the corresponding fusion ring is represented by a polynomial ring in two variables.

On the other hand, if $N$ is odd or $N = 2$, the eigenvalues of fusion matrices $N_{\phi_1}$ are not degenerate and $x$ may be inverted on the ring. We can therefore solve for $y$ as function of $x$ using the corresponding constraint equations which were reduced to (31) and the fusion ring is faithfully represented by one variable polynomials. For instance, the next $N$-odd models is $SU(7)_2$ for which the constraint equation is $x^{28} - 64x^{21} - 157x^{14} + 1640x^7 + 1 = 0$ and it is possible eliminate $y$ from (29) using:
\[
y = \frac{1}{664276}(2958x^{23} - 189549x^{16} - 4653716x^9 + 5504583x^2).
\] (33)

and we get the resulting fusion ring as a polynomial ring in one variable.

At this point we can proceed to the generalization of these results by explicit diagonalization of fusion matrices of the chiral $SU(N)_2$ models. To each irreducible representation $\phi_\nu^k$ we associate a factored characteristic equation $\det(x1 - N_{\phi_\nu^k}) = 0$ which depend on the parafermionic charge $\nu$ according to $N = \frac{p}{q} \nu$, where $p$ and $q$ are positive integers mutually coprime:
\[
\prod_{n=1}^{N+1} (x^p - d^p_k(n))^\frac{\nu}{q} = 0, \quad \text{if } p.q \text{-odd}
\] (34)
\[
\prod_{n=1}^{N+1} (x^p + (-1)^n d^p_k(n))^\frac{\nu}{q} = 0, \quad \text{if } p.q \text{-even}
\] (35)

for $N$-odd, and
\[
(x^p - d^p_k(l))^\frac{\nu}{q} \prod_{n=1}^{N+1} (x^p - d^p_k(n))^\frac{\nu}{q} = 0, \quad \text{if } p.q \text{-odd}
\] (36)
\[
(x^p + (-1)^l d^p_k(l))^\frac{\nu}{q} \prod_{n=1}^{N+1} (x^p + (-1)^n d^p_k(n))^\frac{\nu}{q} = 0, \quad \text{if } p.q \text{-even}
\] (37)
where \( l = (N + 2)/2 \), for \( N \)-even.

Here we have introduced a generalization of the numbers \( d(n) \) of eq.(32):

\[
d_k(n) = \frac{\sin\left(\frac{n(N + 1 - k)\pi}{N + 2}\right)}{\sin\left(\frac{n\pi}{N + 2}\right)}, \quad k = 0, 1, 2, \ldots, N - 1,
\]

\[
n = 1, 2, \ldots, \frac{N + 2}{2}
\]

which satisfy the following sum rules

\[
d_i(n)d_j(n) = \sum_k (N_i)_j^k d_k(n).
\]

From these numbers we observe that the characteristic polynomials of the fusion matrices of the fields comparing the same comimimal equivalence classes have equivalent spectra of zeros, i.e., they differ only in the \( Z_N \)-degeneracy of their eigenvalues which depend on of the parafermionic charge through the relation \( N = pv/q \).

Therefore there are many alternative ways of constructing the \( SU(N)_2 \) polynomial rings in two-variables: Take for \( y \) any field belonging to any equivalence class, \([\phi_k^l]\). The fusion rules (17) gives us four possibilities (at most) to choose the field associated with the variable \( x \). The corresponding constraint equations are given by (34-37). If at least one of the fusion matrices associated with \( x \) and \( y \) is non degenerate, it is possible to eliminate one of variables resulting in a polynomial ring in a single variable.

These results tell us that \( SU(N)_K \), for \( N \)-old possess a single variable polynomial ring at level \( K = 2 \). For other values of \( K \), as observed by Gannon [19], \( SU(2) \) and \( SU(3) \) are the only \( SU(N) \) whose fusion ring at all level \( K \) can be represented by polynomials in only one variable. For each \( N > 3 \), there will be infinitely many \( K \) for which the fusion ring \( SU(N)_K \) requires more than one variable, and infinitely many other \( K \) for which one variable will suffice.

Acknowledgments

I would like to thank Profs. Roland Köberle and Angela Forester for useful discussions.

References

[1] Gepner D, 1991 *Commun. Math. Phys.* 141 381.
[2] Di Francesco P and Zuber J.-B, 1993 *J. Phys. A: Math. Gen.* 26 1441.
[3] Ararony O1993 *Phys. Lett. B* 306 276.
[4] Verlinde E 1988 *Nucl. Phys. B* 300 [FS22] 360.
[5] Dijkgraaf R and Verlinde E 1988 *Nucl. Phys. B* (Proc. Suppl.) 5 87.
[6] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys. B* 241 333.
[7] Vafa C 1988 *Phys. Lett. B* 300 360.
[8] Gepner D and Kapustin A 1995, Phys. Lett. B 349 71.
[9] Kawai T 1989 Phys. Lett. B 217 247.
[10] Caselle M, Ponzano G and Ravanini F 1990 Phys. Lett. B 251 260.
[11] Caselle M, Ponzano G and Ravanini F 1992 Int. J. Mod. Phys. B 6 2075.
[12] Kastler D, Mebkhout M and Rehren K.-H, in: The Algebraic Theory of Superselection Sectors. Introduction and Recent Results, Kastler D ed. (World Scientific, Singapore, 1990).
[13] Rehren K.-H 1990 Lecture Notes in Physics 370 139. Springer.
[14] Fredenhagen K, Rehren K.-H and Schroer B 1989 Commun. Math. Phys. 125 201.
[15] Fuchs J, Quantum Dimensions, CERN-TH.6156/91- Communications in Theoretical Physics (published by the Allahabad Mathematical Society).
[16] Zamolodchikov A B and Fateev V A 1985 Sov. Phys. JETP 62 215.
[17] Naculich S G, Riggs H A and Schnitzer H 1990 Phys. Lett. B 246 417.
[18] Gepner D and Qiu Z 1987 Nucl. Phys. B 285 423.
[19] Gannon T 1994, "The classification of $SU(n)_k$ Automorphism Invariants" hep-th/9408119.