Stationary Solutions of the Klein-Gordon Equation in a Potential Field

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Abstract
We seek to introduce a mathematical method to derive the Klein-Gordon equation and a set of relevant laws strictly, which combines the relativistic wave functions in two inertial frames of reference. If we define the stationary state wave functions as special solutions like $\Psi(r,t) = \psi(r)e^{-iEt/\hbar}$, and define $m = E/c^2$, which is called the mass of the system, then the Klein-Gordon equation can clearly be expressed in a better form when compared with the non-relativistic limit, which not only allows us to transplant the solving approach of the Schrödinger equation into the relativistic wave equations, but also proves that the stationary solutions of the Klein-Gordon equation in a potential field have the probability significance. For comparison, we have also discussed the Dirac equation. By introducing the concept of system mass into the Klein-Gordon equation with the scalar and vector potentials, we prove that if the Schrödinger equation in a certain potential field can be solved exactly, then under the condition that the scalar and vector potentials are equal, the Klein-Gordon equation in the same potential field can also be solved exactly by using the same method.

Keywords: Relativistic wave equations; Klein-Gordon equation; Bound state
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1 Introduction
Both special relativity and experiments indicate that, the mass of a many-particle system in a bound state is less than the sum of the rest mass of every particle forming the system, and the difference gives the mass defect of the system, while the product of the mass defect and the square of the speed of light gives the binding energy of the system. As the binding energy is quantized, the sum of it and the rest mass of
every particle forming the system is the energy level of the system. For instance, the mass of an atomic nucleus is obviously less than the sum of the rest mass of every nucleon forming the atomic nucleus. Therefore, in order to express the mass defect explicitly, there is a necessity to introduce the concept of system mass, which differs from the sum of the rest mass of every particle forming the system. By introducing this concept, we can express relativistic wave equations in a better form when compared with the non-relativistic limit. By means of this method, we are able to solve the Klein-Gordon equation just like solving the Schrödinger equation, and the solutions can also have the probability significance just like the solutions of the Schrödinger equation. As the Schrödinger equation has been thoroughly studied, it surely creates conditions for studying the Klein-Gordon equation thoroughly.

2 Relativistic Wave Equations and the Probability Significance of Their Stationary Solutions

As we know, for a free particle, its energy and momentum are both constant, therefore, it is quite natural to assume that its matter wave is a plane wave. According to the de Broglie relation

\[ \hbar k = p, \quad E = \hbar \omega. \]

We have the wave function of a free particle:

\[ \Psi_p(r,t) = Ae^{-\frac{\hbar}{\pi}(Et - p \cdot r)}. \]  

(1)

Where \( E \) is the total energy of the particle containing the intrinsic energy \( m_0c^2 \), and \( p \) is the momentum of the particle.

The quantization method of quantum mechanics is assuming \( E \) and \( p \) are correspondingly equivalent to the following two differential operators:

\[ E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad p \rightarrow -i\hbar \nabla. \]

This relation is obviously tenable for the wave function of a free particle, while arbitrary wave function is equal to the linear superposition of the plane waves of free particles with all possible momentum, i.e.

\[ \Psi(r,t) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(p,t)e^{-\frac{i\hbar}{\pi}(Et - p \cdot r)} dp_x dp_y dp_z, \]  

(2)

\[ c(p,t) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(r,t)e^{\frac{i\hbar}{\pi}(Et - p \cdot r)} dx dy dz. \]  

(3)

Therefore, this quantization method is also tenable for arbitrary wave functions. Later, we will prove the right-hand side of (1) is relativistic, thus this quantization method itself is relativistic. Then why the Schrödinger equation derived from
this method is non-relativistic? That is because the relation between \( E \) and \( p \) is non-relativistic, which is due to the fact that the kinetic energy is non-relativistic. Therefore, if we want to establish the relativistic wave equations, we need to introduce the relativistic kinetic expression.

Assuming that any particle with the rest mass \( m_0 \), no matter how high the speed is, no matter it is in a potential field or in free space, and no matter how it interacts with other particles, its kinetic energy is:

\[
E_k = (c^2 p^2 + m_0^2 c^4)^{1/2} - m_0 c^2.
\]  

(4)

If \( E_k + m_0 c^2 \) is denoted by \( E \), then (4) can be expressed as

\[
E^2 = c^2 p^2 + m_0^2 c^4.
\]

Thus, it leaves us a matter of mathematical techniques. Therefore, we introduce the mathematical method in reference [1]-[4]. Power functions and exponential functions play special roles in this method, which are called the base functions as we can establish mapping relations between them and arbitrary functions in a certain range. For instance, in quantum mechanics, (2) determines the mapping relations between wave functions of free particles and arbitrary wave functions. Similarly to reference [1]-[4], we can introduce the base function and relevant concepts in quantum mechanics:

**Definition 1** The right-hand side of (1) is defined as the base function of quantum mechanics, where \( E \) and \( p \) are called the characters of base functions, while \( E \) and \( p \) are not only suitable for free particles, but also suitable for any system, and the relation between \( E \) and \( p \) is called the characteristic equation of wave equations. Different system has different characteristic equations. For instance, \( E^2 = c^2 p^2 + m_0^2 c^4 \) is the characteristic equation for free-particle system.

According to differential laws, we have

\[
i \hbar \frac{\partial}{\partial t} \Psi_p = E \Psi_p, \quad -i \hbar \nabla \Psi_p = p \Psi_p.
\]

(5)

**Definition 2** Let \( m_0 = m_{01} + m_{02} + \cdots + m_{0N} \) be the total rest mass of an \( N \)-particle system, \( E' \) be the sum of the kinetic energy and potential energy of all the \( N \) particles, then the actual mass of the system, which is called the system mass, is defined as

\[
m = m_0 + \frac{1}{c^2} E'.
\]

(6)

**Definition 3** If the system is in a bound state (\( E' < 0 \)), then the absolute value of \( E' \) is

\[
|E'| = m_0 c^2 - mc^2 = \Delta mc^2,
\]

which is called the binding energy of the system, where \( \Delta m = m_0 - m \) is the mass defect of the system.

**Definition 4** The total energy of the system \( E \) is defined as the sum of the rest energy, kinetic energy and potential energy of all the particles forming the system, i.e. \( E = m_0 c^2 + E' \).
According to Definition 2 and 4, the total energy of the system is equal to the product of the system mass and the square of the speed of light, i.e. \( E = mc^2 \), thus the system mass is uniquely determined by the energy level of the system.

**Definition 5** In relativistic quantum mechanics, the stationary state wave function is defined as the following special solution, i.e.

\[
\Psi(r, t) = \psi(r) e^{-iEt/\hbar}.
\]

Let \( U(r) \) be the potential energy of a particle moving in a potential field, then according to Definition 2, we have \( E_k = E' - U \). Combining it with (4), we have

\[
E' - U = (c^2p^2 + m_0^2c^4)^{1/2} - m_0c^2.
\]

This is the characteristic equation of relativistic wave equations for a single-particle system. In order to make it easier to solve the corresponding relativistic wave equation, the characteristic equation (5) should be transformed to remove the fractional power, then we have

\[
(E' - U + m_0c^2)^2 = c^2p^2 + m_0^2c^4.
\]

Expanding the left-hand side of (11) and by using (6), we have

\[
\begin{align*}
E' &= \frac{p^2}{m_0 + m} + \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2}, \\
E &= \frac{p^2}{m_0 + m} + \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} + m_0c^2.
\end{align*}
\]

The essence of this expression is the relativistic Hamiltonian. Therefore, taking (11) as the characteristic equation, multiplying both sides of the equation by the base function \( \Psi_p(r, t) \), and by using (5), we have

\[
\begin{align*}
\hbar \frac{\partial \Psi_p}{\partial t} &= -\frac{\hbar^2}{m_0 + m} \nabla^2 \Psi_p + \frac{2m}{m_0 + m}U \Psi_p - \frac{U^2}{(m_0 + m)c^2} \Psi_p + m_0c^2 \Psi_p.
\end{align*}
\]

According to (2), in the operator equation which is tenable for the base function \( \Psi_p \), as long as each operator in the operator equation is a linear operator and each linear operator does not explicitly contain the characters \( E \) and \( p \) of \( \Psi_p \), then this operator equation is also tenable for an arbitrary wave function \( \Psi(r, t) \). Whereas, considering that the system mass \( m \) is equivalent to the character \( E \) of \( \Psi_p \), this operator equation is not tenable for arbitrary wave functions, but tenable for an stationary state wave function like (7). Thus we have:

A particle with the mass \( m_0 \) moving in the potential field \( U(r) \) can be described by the wave function \( \Psi(r, t) \), an arbitrary stationary state wave function \( \Psi \) satisfies the following relativistic wave equation

\[
\begin{align*}
\hbar \frac{\partial \Psi}{\partial t} &= -\frac{\hbar^2}{m_0 + m} \nabla^2 \Psi + \frac{2m}{m_0 + m}U \Psi - \frac{U^2}{(m_0 + m)c^2} \Psi + m_0c^2 \Psi.
\end{align*}
\]
The corresponding $\psi(r)$ satisfies the following relativistic wave equation

$$ E\psi = -\frac{\hbar^2}{m_0 + m}\nabla^2 \psi + \frac{2m}{m_0 + m}U\psi - \frac{U^2}{(m_0 + m)c^2}\psi + m_0c^2\psi. \quad (12) $$

Or, if the energy corresponding to the energy operator is interpreted as $E'$ in Definition 2, then the relativistic wave equation is

$$ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{m_0 + m}\nabla^2 \Psi + \frac{2m}{m_0 + m}U\Psi - \frac{U^2}{(m_0 + m)c^2}\Psi. \quad (13) $$

The corresponding stationary form is

$$ E'\psi = -\frac{\hbar^2}{m_0 + m}\nabla^2 \psi + \frac{2m}{m_0 + m}U\psi - \frac{U^2}{(m_0 + m)c^2}\psi. \quad (14) $$

As $E - m_0c^2 = E'$, (11) and (13) are equivalent. In the non-relativistic limit, we have

$$ m \to m_0, \quad \frac{U^2}{(m_0 + m)c^2} \to 0, $$

(13) approaches the Schrödinger equation.

Let $w(r, t) = \Psi^*(r, t)\Psi(r, t)$, differentiating both sides of the equation with respect to time $t$, and considering

$$ \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{m_0 + m}\nabla^2 \Psi + \frac{1}{i\hbar} \frac{2m}{m_0 + m}U\Psi - \frac{1}{i\hbar} \frac{U^2}{(m_0 + m)c^2}\Psi + \frac{1}{i\hbar} m_0c^2\Psi, $$

$$ \frac{\partial \Psi^*}{\partial t} = -\frac{i\hbar}{m_0 + m}\nabla^2 \Psi^* - \frac{1}{i\hbar} \frac{2m}{m_0 + m}U\Psi^* + \frac{1}{i\hbar} \frac{U^2}{(m_0 + m)c^2}\Psi^* - \frac{1}{i\hbar} m_0c^2\Psi^*. $$

We have the probability conservation equation, i.e.

$$ \frac{\partial w}{\partial t} = \frac{i\hbar}{m_0 + m}(\Psi^*\nabla^2 \Psi - \Psi \nabla^2 \Psi^*) = \frac{i\hbar}{m_0 + m} \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) = -\nabla \cdot J. $$

In other words, $w(r, t) = \Psi^*(r, t)\Psi(r, t)$ can be interpreted as the probability density. Thus we have

Let $w(r, t) = \Psi^*(r, t)\Psi(r, t)$ be the probability density, then the corresponding probability current density vector is

$$ J = \frac{i\hbar}{m_0 + m}(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi), \quad (15) $$

making the following equation tenable, i.e.

$$ \frac{\partial w}{\partial t} + \nabla \cdot J = 0. \quad (16) $$

Here $\Psi(r, t)$ is a stationary state wave function like (11), determined by wave function (11) or (13) and natural boundary conditions.
Now, let us introduce the concept of system mass into the Dirac equation, so as to compare it with (14). According to reference [5], we can express the Dirac equation as the following two equations:

\[ c \hat{\sigma} \cdot \mathbf{p} \psi_2 + \mu_0 c^2 \psi_1 + U \psi_1 = E \psi_1, \]

\[ c \hat{\sigma} \cdot \mathbf{p} \psi_1 - \mu_0 c^2 \psi_2 + U \psi_2 = E \psi_2. \]

Considering \( E = \mu c^2 \), solve the second equation for \( \psi_2 \), then substitute the result into the first equation, thus we have

\[ \frac{c^2}{(\mu_0 + \mu)c^2 - U} \left( \begin{array}{c} c^2 \hat{\sigma} \cdot \mathbf{p} \\ \mu c \end{array} \right) \psi_1 + \mu_0 c^2 \psi_1 + U \psi_1 = E \psi_1. \]

Where

\[ \left( \begin{array}{c} c^2 \hat{\sigma} \cdot \mathbf{p} \\ \mu c \end{array} \right) = \frac{c^2}{(\mu_0 + \mu)c^2 - U} \left( \begin{array}{c} c^2 \hat{\sigma} \cdot \mathbf{p} \\ \mu c \end{array} \right) \psi_1 + \left( \begin{array}{c} c^2 \hat{\sigma} \cdot \mathbf{p} \\ \mu c \end{array} \right). \]

As \( \hat{\sigma} \cdot \mathbf{p} \hat{\sigma} \cdot \mathbf{p} = p^2 \), \( E = E' + \mu_0 c^2 \), the equation can be transformed into the following form:

\[ E' \psi_1 = \frac{c^2p^2}{(\mu_0 + \mu)c^2 - U} \psi_1 + U \psi_1 + \left( \begin{array}{c} c^2 \hat{\sigma} \cdot \mathbf{p} \\ \mu c \end{array} \right). \]

As it is known, for any scalar function \( U \), we have

\[ \hat{\sigma} \cdot \mathbf{p} U \hat{\sigma} \cdot \mathbf{p} \psi_1 = \{-ih(\nabla U \cdot \mathbf{p}) + \hbar \hat{\sigma} [(\nabla U) \times \mathbf{p}] \} \psi_1. \]

If \( U \) is simply a function of \( r \), then

\[ \nabla U = \frac{1}{r} \frac{dU}{dr} \mathbf{r}, \]

\[ -ih(\nabla U \cdot \mathbf{p}) \psi_1 = -\hbar^2 \nabla U \cdot \nabla \psi_1 = -\hbar^2 \frac{dU}{dr} \frac{\partial \psi_1}{\partial r}. \]

\[ \langle \nabla U \rangle \times \mathbf{p} = \frac{1}{r} \frac{dU}{dr} (r \times \mathbf{p}) = \frac{1}{r} \frac{dU}{dr} \mathbf{L}. \]

Thus the equation can be expressed as

\[ E' \psi_1 = \frac{c^2p^2}{(\mu_0 + \mu)c^2 - U} \psi_1 + U \psi_1 + \frac{c^2}{(\mu_0 + \mu)c^2 - U} \frac{dU}{dr} \left( \frac{2}{r} \mathbf{S} \cdot \mathbf{L} \psi_1 - \hbar^2 \frac{\partial \psi_1}{\partial r} \right). \]

Where \( \mathbf{S} = (1/2)\hbar \hat{\sigma} \) is the spin angular momentum operator. Multiplying both sides of the equation by \( (\mu_0 + \mu)c^2 - U \), noting that \( E' = E - \mu_0 c^2 = \mu c^2 - \mu_0 c^2 \), we have

\[ E'(\mu_0 + \mu)c^2 \psi_1 = \frac{c^2p^2 \psi_1 + 2\mu c^2 U \psi_1 - U^2 \psi_1}{(\mu_0 + \mu)c^2 - U} \frac{dU}{dr} \left( \frac{2}{r} \mathbf{S} \cdot \mathbf{L} \psi_1 - \hbar^2 \frac{\partial \psi_1}{\partial r} \right). \]
Therefore, the Dirac equation can be expressed as the following form:

\[
E'\psi = -\frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{\mu_0 + \mu} U\psi - \frac{U^2}{(\mu_0 + \mu)c^2} \psi + \frac{1}{(\mu_0 + \mu)(\mu_0 + \mu)c^2 - U} \frac{dU}{dr} \left( \frac{2}{r} \mathbf{S} \cdot \mathbf{L} \psi - \hbar^2 \frac{\partial \psi}{\partial r} \right).
\]

(17)

Where \( \mu \) is the system mass corresponding to \( \mu_0 \), \( \mathbf{S} \) is the spin angular momentum operator, \( \mathbf{L} \) is the orbital angular momentum operator, and \( U \) is simply a function of \( r \).

3 The System Mass of Pionic Hydrogen Atoms and Relativistic Wave Functions

A pionic hydrogen atom is a system formed by a negative pion and an atomic nucleus. Reference [6] expounded the relativistic energy levels of such a system. Let us take the pionic hydrogen atom for example. We can transplant the mathematical methods of the non-relativistic quantum mechanics in reference [7] into the relativistic quantum mechanics, thus determine the system mass and relativistic wave functions of such a system.

Considering a negative pion \( \pi^- \) with the rest mass \( m_0 \) moving in an nuclear electric field, taking the atomic nucleus as the origin of coordinates, then the potential energy of the system is

\[
U = -\frac{Ze_s^2}{r}, \quad e_s = e(4\pi\varepsilon_0)^{-1/2},
\]

according to (14), the relativistic stationary state wave equation of the system is

\[
E'\psi = -\frac{\hbar^2}{m_0 + m} \nabla^2 \psi - \frac{2mZe_s^2}{m_0 + m} \psi - \frac{1}{(m_0 + m)c^2} \frac{Z^2e_s^4}{r^2} \psi.
\]

The equation expressed in spherical polar coordinates is

\[
E'\psi = -\frac{\hbar^2}{m_0 + m} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} \right] \psi
- \frac{2mZe_s^2}{m_0 + m} \frac{1}{r} \frac{Z^2e_s^4}{r^2} \psi.
\]

By following the solving procedure of the Schrödinger equation in reference [7], we use the method of separation of variables to solve this equation. Let \( \psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi) \), by substituting it into the equation, we have

\[
\frac{(m_0 + m)E'r^2}{\hbar^2} + \frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) + 2m \frac{Ze_s^2}{\hbar^2} r + \frac{Z^2e_s^4}{\hbar^2 c^2} \frac{1}{r^2} R = \lambda.
\]

(18)
\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} + \lambda Y = 0. \tag{19}
\]

According to (19), denoting \( \lambda = l(l+1) \), \( l = 0, 1, 2, \ldots \), obviously the solution of the equation is the spherical harmonics \( Y_{lm}(\theta, \varphi) \).

Now let us solve the radial equation (18), discussing the situation of the bound state \( (E' < 0) \). Let

\[
\alpha' = \left[ \frac{4(m_0 + m)|E'|}{\hbar^2} \right]^{1/2}, \quad \beta = \frac{2mZe_s^2}{\alpha' \hbar^2}. \tag{20}
\]

Let \( R(r) = u(r)/r \), considering

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{1}{r} \frac{d^2}{dr^2}(rR),
\]

and by using the variable substitution \( \rho = \alpha' r \), then the equation (18) can be expressed as (\( \alpha \) is the fine structure constant, which is absolutely different from \( \alpha' \)):

\[
d^2u \frac{d}{d\rho^2} + \left[ \frac{\beta}{\rho} - \frac{l(l+1)}{\rho^2} - Z^2 \alpha^2 \right] u = 0. \tag{21}
\]

Firstly, let us study the asymptotic behaviour of this equation, when \( \rho \to \infty \), the equation can be transformed into the following form:

\[
d^2u \frac{d}{d\rho^2} - \frac{1}{4} u = 0, \quad u(\rho) = e^{\pm \rho/2}.
\]

As \( e^{\rho/2} \) is in conflict with the finite conditions of wave functions, we substitute \( u(\rho) = e^{-\rho/2} f(\rho) \) into the equation, then we have the equation satisfied by \( f(\rho) \):

\[
d^2f \frac{d}{d\rho^2} + \left[ \frac{\beta}{\rho} - \frac{l(l+1)}{\rho^2} - Z^2 \alpha^2 \right] f = 0. \tag{22}
\]

Now let us solve this equation for series solutions. Let

\[
f(\rho) = \sum_{\nu=0}^{\infty} b_\nu \rho^{s+\nu}, \quad b_0 \neq 0. \tag{23}
\]

In order to guarantee the finiteness of \( R = u/r \) at \( r = 0 \), \( s \) should be no less than 1. By substituting (23) into (22), as the coefficient of \( \rho^{s+\nu-1} \) is equal to zero, we have the relation satisfied by \( b_\nu \):

\[
b_{\nu+1} = \frac{s+\nu-\beta}{(s+\nu)(s+\nu+1)-l(l+1)+Z^2 \alpha^2} b_\nu. \tag{24}
\]

If the series are infinite series, then when \( \nu \to \infty \) we have \( b_{\nu+1}/b_\nu \to 1/\nu \). Therefore, when \( \rho \to \infty \), the behaviour of the series is the same as that of \( e^\rho \), then we have

\[
R = \frac{\alpha'}{\rho} u(\rho) = \frac{\alpha'}{\rho} e^{-\rho/2} f(\rho).
\]
Where $f(\rho)$ tends to infinity when $\rho \to \infty$, which is in conflict with the finite conditions of wave functions. Therefore, the series should only have finite terms. Let $b_{n_r} \rho^{n_r}$ be the highest-order term, then $b_{n_r+1} = 0$, by substituting $\nu = n_r$ into (24) we have $\beta = n_r + s$. On the other hand, the series starts from $\nu = 0$, and doesn’t have the term $\nu = -1$, therefore, $b_{-1} = 0$. Substituting $\nu = -1$ into (24), considering $b_0 \neq 0$, we have $s(s-1) = l(l+1) - Z^2 \alpha^2$. Denoting $n = n_r + l + 1$, then the following set of equations can be solved for $s$ and $\beta$:

$$
\begin{cases}
  s(s-1) = l(l+1) - Z^2 \alpha^2 \\
  \beta = n_r + s \\
  n = n_r + l + 1.
\end{cases}
$$

(25)

We have $s = 1/2 \pm \sqrt{(l+1/2)^2 - Z^2 \alpha^2}$, taking $s = 1/2 + \sqrt{(l+1/2)^2 - Z^2 \alpha^2}$, then

$$
\beta = n_r + s = n - l - 1/2 + \sqrt{(l+1/2)^2 - Z^2 \alpha^2} = n - \sigma_t.
$$

Consider $m = m_0 - |E'|/c^2$, then $|E'|$ satisfies the following second-order algebraic equation:

$$
Z^2 \alpha^2 + (n - \sigma_t)^2 |E'|^2 - 2m_0|Z^2 \alpha^2 + (n - \sigma_t)^2||E'| + Z^2 \alpha^2 m_0^2 c^2 = 0.
$$

Obviously, the expression of $|E'|$ obtained by solving the equation is related to both $n = 1, 2, \ldots$ and $l = 0, 1, \ldots, n-1$, thus $E = m_0 c^2 - |E'|$ can be denoted by $E_n$, then we have

$$
E_n = m_0 c^2 - |E'|
$$

$$
= m_0 c^2 - m_0 c^2 \left( 1 \pm \frac{n - \sigma_t}{\sqrt{Z^2 \alpha^2 + (n - \sigma_t)^2}} \right) = \mp \frac{(n - \sigma_t)m_0 c^2}{\sqrt{Z^2 \alpha^2 + (n - \sigma_t)^2}}.
$$

If simply taking the positive solution, then we have

$$
E_n = \frac{m_0 c^2}{\sqrt{1 + Z^2 \alpha^2 / (n - \sigma_t)^2}}.
$$

Therefore, for a hydrogen-like atom formed by pion capture in an atomic nucleus, if we ignore the effects of nuclear motion, then its relativistic energy level is

$$
E_n = m_0 c^2 \left[ 1 + \frac{Z^2 \alpha^2}{(n - (l + 1/2) + \sqrt{(l+1/2)^2 - Z^2 \alpha^2})^2} \right]^{-1/2}
$$

$$
= m_0 c^2 \left[ 1 - \frac{Z^2 \alpha^2}{2n^2} - \frac{Z^4 \alpha^4}{2n^4} \left( \frac{n}{l+1/2} - \frac{3}{4} \right) + \cdots \right]. 
$$

(26)
Where \( n = 1, 2, \ldots \) is the principal quantum number, \( l = 0, 1, \ldots, n-1 \) is the angular quantum number, \( m_0 \) is the rest mass of \( \pi^- \), \( Z \) is the atomic number, \( \alpha \) is the fine structure constant, \( c \) is the speed of light in vacuum. The result is completely the same as that of reference [6].

As \( E_n = mc^2 \), the system mass \( m \) corresponding to the positive solution is

\[
m = \frac{m_0}{\sqrt{1 + Z^2 \alpha^2/(n-\sigma_1)^2}}. \tag{27}
\]

Therefore, when the system is in a bound state, the system mass takes discrete values.

According to (15) and (16), for a pionic hydrogen atom, the solution of the Klein-Gordon equation has the probability significance, further, we ought to obtain the relativistic wave functions of such a system. Solving the radial equation (18) can come down to solving the equation (22), therefore, by substituting the solutions \( b \) of the Klein-Gordon equation into (23), then

\[
\sigma(r) = (\nu + l + 1 - n)(\nu + l + 2 + \nu - \sigma_1) + Z^2 \alpha^2 b_\nu.
\]

Substituting it into (23), then

\[
\sigma(r) = \frac{A}{r} \quad \text{and} \quad T(r) = \frac{B}{r^2},
\]

where \( A \) and \( B \) are defined by (20), thus we have

\[
\sigma(r) = \frac{A}{r} \quad \text{and} \quad T(r) = \frac{B}{r^2}.
\]

According to (15) and (16), for a pionic hydrogen atom, the solution of the Klein-Gordon equation has the probability significance, further, we ought to obtain the relativistic wave functions of such a system. Solving the radial equation (18) can come down to solving the equation (22), therefore, by substituting the solutions \( s = l + 1 - \sigma_1, \beta = n - \sigma_1 \) of the equation (25) into (24), we have

\[
\rho \equiv \frac{m Ze^2}{\hbar^2} = \frac{m Ze^2}{(n-\sigma_1)\hbar^2} = \frac{2Z}{(n-\sigma_1)a_0} r, \quad a_0 = \frac{\hbar^2}{me_s^2}.
\]

According to (24), we have

\[
\rho = \alpha' r = \frac{2m Ze^2}{\hbar^2 \beta} r = \frac{2m Ze^2}{(n-\sigma_1)\hbar^2} r = \frac{2Z}{(n-\sigma_1)a_0} r.
\]
Therefore, for a hydrogen-like atom formed by pion capture in an atomic nucleus, if we ignore the effects of nuclear motion, then its relativistic stationary state wave function is

\[ \Psi(r, t) = \psi_{nlm}(r, \theta, \varphi)e^{-iE_nt/h} = R_{nl}(r)Y_{lm}(\theta, \varphi)e^{-iE_nt/h}. \] (28)

Where \( Y_{lm}(\theta, \varphi) \) is the normalized spherical harmonics. Relativistic effects do not cause the change of the angular wave function \( Y_{lm}(\theta, \varphi) \), but the change of the radial wave function \( R_{nl}(r) \), i.e.

\[ R_{nl}(r) = N_{nl}e^{-\frac{\hbar^2}{2}\cdot\frac{n\cdot l\cdot \sigma_l}{a_0}r} \left( \frac{2Z}{n-l-\sigma_l} \right)^{l-\sigma_l} L_{n+l+1-\sigma_l}^\sigma_0 \] (29)

\[ \sigma_l = l + \frac{1}{2} - \left[ \left( l + \frac{1}{2} \right)^2 - Z^2\alpha^2 \right]^{1/2} \times \frac{\sum_{k=1}^\infty 2^{k-1}(2k-3)!!}{k!(2l+1)^{2k-1}}(Z\alpha)^2. \]

Where \( n \) is the principal quantum number, \( l \) is the angular quantum number, \( m \) is the magnetic quantum number (do not confuse it with the system mass \( m \)), \( Z \) is the atomic number, \( \alpha = e_s^2/\hbar c \) is the fine structure constant, and

\[ a_0 = \frac{\hbar^2}{m\epsilon_s^2}, \quad m = m_0 \left[ 1 + \frac{Z^2\alpha^2}{(n-\sigma_l)^2} \right]^{-1/2}. \]

Where \( m \) is the system mass, and \( m_0 \) is the rest mass of a particle. The expression of \( L_{n+l+1-\sigma_l}(\rho) \) is

\[ L_{n+l+1-\sigma_l}^\sigma_0(\rho) = \sum_{\nu=0}^{n-1-l} \frac{(-1)^{\nu+1}(n+l)!^2\rho^\nu}{(n-l-1-\nu)!\Gamma(2l+\nu+2-\sigma_l)\Gamma(\nu+1-\sigma_l)\eta(l, \nu)}, \] (30)

which is called the relativistic associated Laguerre polynomials, where

\[ \eta(l, \nu) = \prod_{k=1}^\nu \left( 1 + \frac{Z^2\alpha^2}{(k-\sigma_l)(2l+1+k-\sigma_l)} \right). \]

\( R_{nl}(r) \) is also normalized, and \( N_{nl} \) is the normalized constant.

According to (15) and (16), by using the Gauss's theorem in vector analysis, the relativistic stationary state wave function \( \Psi(r, t) \) satisfies the normalization condition:

\[ \int_\infty^\infty w(r, t)\, dr = \int_\infty^\infty \Psi^* \Psi(r, t)\, dr = 1, \]

which can also be expressed as

\[ \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} Y_{nm}^* r^2 J_{nm}(\theta, \phi) Y_{lm}(\theta, \phi) \sin \theta\, d\theta\, d\phi = 1. \]

Thus the normalized constant \( N_{nl} \) can be determined.

The value of the fine structure constant is very small (\( \alpha \approx 1/137 \)), for \( Z = 1 \), in the non-relativistic limit, \( \eta(l, \nu) \to 1 \), \( \sigma_l \to 0 \), Thus \( L_{n+l+1-\sigma_l}(\rho) \to L_{n+l+1}^\sigma(\rho) \). \( L_{n+l+1}^\sigma(\rho) \) is the associated Laguerre polynomials.
4 The Klein-Gordon Equation

Now, let us prove that the wave equations (11) and (13) are relativistic. Assuming there are two equations, one of which is relativistic, while it is unknown whether or not the other is relativistic, if we can prove these two equations have the same solution, then the other is also relativistic. According to this thinking, if we directly quantize (9), then \( \psi(r) \), which is the main part of the stationary state wave function, satisfies the following wave equation:

\[
(E' + m_0c^2 - U(r))^2 \psi = m_0^2 c^4 \psi - c^2 \hbar^2 \nabla^2 \psi,
\]

it has the same solution as (14). We can say they are equivalent, and it can be equivalently expressed as:

\[
(E - U(r))^2 \psi = m_0^2 c^4 \psi - c^2 \hbar^2 \nabla^2 \psi.
\] (31)

Further, we have the more general form of this equation

\[
\left( i\hbar \frac{\partial}{\partial t} - U(r) \right)^2 \Psi = m_0^2 c^4 \Psi - c^2 \hbar^2 \nabla^2 \Psi.
\] (32)

Under strong coupling conditions, the relativistic effects of moving particles in a potential field become significant, therefore, seeking for the exact solution of the Klein-Gordon equation or the Dirac equation in a typical potential field has drawn increasing attention in the last few years. Let \( m_0 \) be the rest mass, and \( E \) be the total energy containing the rest energy, according to reference [8]-[13], the Klein-Gordon equation with the vector potential \( U(r) \) and the scalar potential \( S(r) \) in spherical coordinates is:

\[
[-\hbar^2 c^2 \nabla^2 + (m_0c^2 + S(r))^2] \psi(r) = [E - U(r)]^2 \psi(r).
\] (33)

When \( S(r) = 0 \), we have (31). In other words, by introducing the concept of system mass, the Klein-Gordon equation can be expressed in a better form when compared with the non-relativistic limit, making it easier to transplant the concepts and methods of the non-relativistic quantum mechanics into the relativistic quantum mechanics.

If we introduce the concept of system mass, (33) can be expressed as

\[
E' \psi = - \frac{\hbar^2}{m_0 + m} \nabla^2 \psi + \frac{2m}{m_0 + m} U \psi - \frac{U^2}{(m_0 + m)c^2} \psi + \frac{2m_0}{m_0 + m} S \psi + \frac{S^2}{(m_0 + m)c^2} \psi.
\] (34)

Where \( m \) is the system mass, and \( E' = E - m_0c^2, E = mc^2 \). (34) formally remains similar with the Schrödinger equation, and the solution of the equation also has the probability significance. Now, let us seek for the relation between the solutions of
in two different inertial frames of reference. Let \( K \) and \( K' \) be two inertial frames of reference, and their coordinate axes be correspondingly parallel, \( K' \) be moving at a speed of \( v \) along the positive x-axis relative to \( K \), and the coincident moment of \( O \) and \( O' \) be the starting time point. In the inertial frame of reference \( K \), particles are moving in the potential field \( U \), while in the inertial frame of reference \( K' \), particles are moving in the potential field \( U' \). We use the following transformation relation as the characteristic equation for a single-particle system, i.e. the Lorentz transformation from \( K \) to \( K' \) is

\[
\begin{align*}
p_x' &= \frac{1}{\sqrt{1 - \beta^2}} p_x - \frac{v/c^2}{\sqrt{1 - \beta^2}} (E - U), \\
p_y' &= p_y, \\
p_z' &= p_z,
\end{align*}
\]

(35)

The Lorentz transformation from \( K' \) to \( K \) is

\[
\begin{align*}
p_x &= \frac{1}{\sqrt{1 - \beta^2}} p_x' + \frac{v/c^2}{\sqrt{1 - \beta^2}} (E' - U') \\
p_y' &= p_y', \\
p_z &= p_z',
\end{align*}
\]

(36)

\[
E' - U' = \frac{1}{\sqrt{1 - \beta^2}} (E - U) - \frac{v}{\sqrt{1 - \beta^2}} p_x.
\]

Here \( E' \) is the total energy of the particle moving in the potential field \( U' \), which is different from the \( E' \) in Definition 2.

In the base function \( \Psi_p(r, t) = A \exp[-i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar] \), \( Et - \mathbf{p} \cdot \mathbf{r} \) is the phase of a plane wave, when transformed from \( K \) to \( K' \), \( E' t' - \mathbf{p}' \cdot \mathbf{r}' \) and \( Et - \mathbf{p} \cdot \mathbf{r} \) differ by at most an integral multiple of \( 2\pi \). By the periodicity of phase, we have \( Et - \mathbf{p} \cdot \mathbf{r} \) is a relativistic invariant. Therefore, by choosing \( A' \) properly, we have

\[
A \exp[-i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar] = A' \exp[-i(E' t' - \mathbf{p}' \cdot \mathbf{r}')/\hbar] \quad \text{or} \quad \Psi_p(r, t) = \Psi'_p(r', t').
\]

Thus if we multiply the left-hand side of the characteristic equation (35) by \( \Psi'_p(r', t') \), and multiply the right-hand side of the equation by \( \Psi_p(r, t) \), the equation is still tenable. By using (35) and similar differential relations, we have the following set of relations:

\[
\begin{align*}
-i \hbar \frac{\partial \Psi_p'}{\partial x'} &= - \frac{i \hbar}{\sqrt{1 - \beta^2}} \left( \frac{\partial \Psi_p}{\partial x} + \frac{v}{c^2} \frac{\partial \Psi_p}{\partial t} \right) + \frac{v/c^2}{\sqrt{1 - \beta^2}} U \Psi_p, \\
-i \hbar \frac{\partial \Psi_p'}{\partial y'} &= - i \hbar \frac{\partial \Psi_p}{\partial y}, \\
-i \hbar \frac{\partial \Psi_p'}{\partial z'} &= - i \hbar \frac{\partial \Psi_p}{\partial z}.
\end{align*}
\]
\[-U'\Psi' + i\hbar \frac{\partial \Psi'}{\partial t'} = -\frac{1}{\sqrt{1-\beta^2}} U \Psi + \frac{i\hbar}{\sqrt{1-\beta^2}} \left( \frac{\partial \Psi_p}{\partial t} + v \frac{\partial \Psi_p}{\partial x} \right).\]

By using (2), we have:

Let \(K\) and \(K'\) be two inertial frames of reference, and their coordinate axes be correspondingly parallel, \(K'\) be moving at a speed of \(v\) along the positive x-axis relative to \(K\), and the coincident moment of \(O\) and \(O'\) be the starting time point, then for a single-particle system, the Lorentz transformation of wave functions from \(K\) to \(K'\) is

\[-i\hbar \frac{\partial \Psi'}{\partial x'} = -\frac{i\hbar}{\sqrt{1-\beta^2}} \left( \frac{\partial \Psi'}{\partial x'} + v \frac{\partial \Psi'}{\partial t'} \right) - \frac{v/c^2}{\sqrt{1-\beta^2}} U' \Psi'\]

\[-i\hbar \frac{\partial \Psi'}{\partial y'} = -i\hbar \frac{\partial \Psi'}{\partial y'}\]

\[-i\hbar \frac{\partial \Psi'}{\partial z'} = -i\hbar \frac{\partial \Psi'}{\partial z'}\]

\[-U' \Psi' + i\hbar \frac{\partial \Psi'}{\partial t'} = -\frac{1}{\sqrt{1-\beta^2}} U' \Psi + \frac{i\hbar}{\sqrt{1-\beta^2}} \left( \frac{\partial \Psi'_p}{\partial t'} + v \frac{\partial \Psi'_p}{\partial x'} \right).\]

Similarly, the Lorentz transformation of wave functions from \(K'\) to \(K\) is

\[-i\hbar \frac{\partial \Psi}{\partial x} = -\frac{i\hbar}{\sqrt{1-\beta^2}} \left( \frac{\partial \Psi'}{\partial x'} - \frac{v}{c^2} \frac{\partial \Psi'}{\partial t'} \right) - \frac{v/c^2}{\sqrt{1-\beta^2}} U \Psi\]

\[-i\hbar \frac{\partial \Psi}{\partial y} = -i\hbar \frac{\partial \Psi'}{\partial y'}\]

\[-i\hbar \frac{\partial \Psi}{\partial z} = -i\hbar \frac{\partial \Psi'}{\partial z'}\]

\[-U \Psi + i\hbar \frac{\partial \Psi}{\partial t} = -\frac{1}{\sqrt{1-\beta^2}} U' \Psi' + \frac{i\hbar}{\sqrt{1-\beta^2}} \left( \frac{\partial \Psi'_p}{\partial t'} - v \frac{\partial \Psi'_p}{\partial x'} \right).\]

Where \(\beta = v/c\), \(U\) is the potential filed in \(K\), and \(U'\) is the potential filed in \(K'\).

5 Conclusion

In summary, by introducing Definition 1-5, using (2) and assuming that the relativistic kinetic expression is tenable in a larger sense, the Klein-Gordon equation in a potential field (32) is derived more strictly, and we also prove that its stationary solutions \((\psi(r)e^{-E(t)/h})\) satisfy the equation (11), thus they have the probability significance.

Under the condition that the scalar and vector potentials are equal, \(S(r) = U(r)\), (34) degenerates into

\[E'\psi = -\frac{\hbar^2}{m_0 + m} \nabla^2 \psi + 2U \psi,\]

which is formally similar to the Schrödinger equation. Therefore, the following conjecture brought out in reference [13] is correct, i.e. if the bound states of the Schrödinger equation in a certain potential field can be solved exactly, then under the condition...
that the scalar and vector potentials are equal, the bound states of the Klein-Gordon equation in the same potential field can also be solved exactly by using the same method. Reference [10]-[13] have given several exact bound state solutions of the Klein-Gordon equation and Dirac equation in a non-central potential field with equal scalar and vector potentials. And under the condition that the scalar and vector Manning-Rosen potentials are equal, the bound state solutions of the s-wave Klein-Gordon equation and Dirac equation are respectively derived in reference [9]. By using (39), these results can be derived much easier.

According to reference [14], the radial Schrödinger equation with the Hulthén potential is

\[
\frac{d^2u(r)}{dr^2} + \frac{2m_0}{h^2} \left( E' + Ze^2 \lambda \frac{e^{-\lambda r}}{1 - e^{-\lambda r}} - \frac{l(l+1)h^2}{2m_0r^2} \right) u(r) = 0. \tag{40}
\]

According to (39), under the condition that the scalar and vector potentials are equal, the radial Klein-Gordon equation with the Hulthén potential is

\[
\frac{d^2u(r)}{dr^2} + m_0 + m \frac{2m_0}{h^2} \left( E' + 2Ze^2 \lambda \frac{e^{-\lambda r}}{1 - e^{-\lambda r}} - \frac{l(l+1)h^2}{(m_0 + m)r^2} \right) u(r) = 0. \tag{41}
\]

Obviously, solving this equation does not increase any difficulty, thus the results of the scattering states of the Hulthén potential in reference [14] can be extended from the Schrödinger equation to the Klein-Gordon equation with equal scalar and vector potentials. Therefore, we have also come to the conclusion that, if the scattering states of the Schrödinger equation in a certain potential field can be solved exactly, then under the condition that the scalar and vector potentials are equal, the scattering states of the Klein-Gordon equation in the same potential field can also be solved exactly by using the same method.

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