Construction of negatively curved complete intersections

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Abstract

Using the Donaldson-Auroux theory, we construct complete intersections in complex projective manifolds, which are negatively curved in various ways. In particular, we prove the existence of compact simply connected Kähler manifolds with negative holomorphic bisectional curvature. We also construct hyperbolic hypersurfaces and we obtain bounds for their Kobayashi hyperbolic metric.

In this article, we give new applications of the asymptotic methods invented by Donaldson in [6]. Although the theory was primarily designed to prove structure theorems in symplectic geometry, applications to complex geometry already appear in [6, Corollary 33]. Nonetheless, it seems that for a number of years complex applications haven’t been continued. Recently, Giroux and Pardon have used Donaldson’s techniques in their study of Stein manifolds [7]. In this paper we give other applications in the context of complex projective geometry.

Let $X$ be a complex projective manifold endowed with an ample line bundle $L \to X$. A hypersurface of degree $k$ in $X$ is, by definition, the zero set $Y = \{s_k = 0\}$ of a holomorphic section $s_k$ of the line bundle $L^\otimes k$ (the $k$th tensor power of $L$). More generally, a complete intersection $Y \subset X$ of dimension $d$ and degree $k$ is the zero set $Y = \{s_k = 0\}$ of a holomorphic section $s_k$ of $\mathbb{C}^{n-d} \otimes L^\otimes k = L^\otimes k \oplus \cdots \oplus L^\otimes k$ where $n = \dim X$. In this paper, we will always assume that $Y$ is smooth (with $s_k$ transverse to 0).

Fix a Hermitian metric $\mu$ on $X$. Our main results are existence results for complete intersections in $X$ whose curvature for the induced metric has various negativity properties.

**Theorem 1.** Let $X$ be a complex projective manifold of dimension $n$ equipped with a Hermitian metric $\mu$ and an ample line bundle $L \to X$.

(a) If $n \geq 3$ then, for every sufficiently large $k$, there exists a curve $Y \subset X$ which is a complete intersection of degree $k$ with negative curvature (for the induced metric).

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(b) If \( d \leq n - 2 \) then, for every sufficiently large \( k \), there exists a complete intersection \( Y \) of dimension \( d \) and degree \( k \) with negative Ricci curvature in \( X \).

(c) If \( d \leq n - 1 \) and \( n \geq 3 \) then, for every sufficiently large \( k \), there exists a complete intersection \( Y \) of dimension \( d \) and degree \( k \) with negative scalar curvature in \( X \).

(d) If \( n \geq 3d \) then, for every sufficiently large \( k \), there exists a complete intersection \( Y \) of dimension \( d \) and degree \( k \) with negative holomorphic sectional curvature in \( X \).

(e) If \( n \geq 4d - 1 \) then, for every sufficiently large \( k \), there exists a complete intersection \( Y \) of dimension \( d \) and degree \( k \) with negative holomorphic bisectional curvature in \( X \).

We could find no trace of such results in the literature, so we presume they are new.

Remark. In fact, we prove a stronger result. In each case, we construct a sequence of submanifolds \( (Y_k)_{k \geq 1} \) independent of the metric \( \mu \), such that, for every \( k \), the submanifold \( Y_k \subset X \) is a complete intersection of degree \( k \) and dimension \( d \) and, for every metric \( \mu \), if \( k \) is sufficiently large (in a sense which depends on \( \mu \)), then the corresponding curvature of \( Y_k \) is negative. In particular, if \( \mu_1, \mu_2, ... \) are a finite family of Hermitian metrics then there exists a complete intersection whose curvature is negative for every metric in the family. The same holds for a compact family of Hermitian metrics.

A simple counter-example shows that Theorem 1(a) is sharp: no curve in the projective plane \( \mathbb{CP}^2 \) can have nonpositive curvature everywhere for the metric induced by the Fubini-Study metric. Similarly, if \( X \) is an abelian variety with a constant Kähler metric then no complex hypersurface \( Y \subset X \) can have negative Ricci curvature everywhere and therefore Theorem 1(b) is sharp. For \( n \geq 2d \), Brotbek, Darondeau and Xie (\[5\], \[15\]) have constructed complete intersections with ample cotangent bundles. In a way, Theorem 1(e) is a metric version of this algebraic result, under the stronger hypothesis \( n \geq 4d - 1 \).

Theorem 1(e) answers a classical question concerning the bisectional curvature (see \[16\] and \[17\], Question 35). If \( X \) is simply connected then the Lefschetz theorem implies that complete intersections of dimension \( d \geq 2 \) are simply connected. Hence, Theorem 1(e) implies the following result:

**Corollary 2.** There exist compact simply connected Kähler manifolds with negative holomorphic bisectional curvature.

In particular, this corollary shows that a simply connected complete Kähler manifold with negative holomorphic bisectional curvature need not be Stein, which was not known (\[17\], Question 35).
If $Y \subset X$ is a complex submanifold then the metric $\mu$ induces Hermitian metrics on the (complex) cotangent bundle $T^* Y$, on the normal bundle $NY$ and, more generally, on their exterior powers $\bigwedge^l(T^* Y)$ and $\bigwedge^l(NY)$. Our next result constructs submanifolds for which the Hermitian bundle $\bigwedge^l(T^* Y)$ (or $\bigwedge^l(NY)$) is Griffiths-positive.

**Theorem 3.** Let $X$ be a complex projective manifold of dimension $n$ equipped with a Hermitian metric $\mu$ and an ample line bundle $L \to X$.

(a) If $2d(1+l) \leq l(n+l)$ then, for every sufficiently large $k$, there exists a complete intersection $Y$ of dimension $d$ and degree $k$ such that the Hermitian bundle $\bigwedge^l(T^* Y)$ is Griffiths-positive (for the induced metric).

(b) Similarly, if $l(n-l) \leq 2d(l-1)$ then there exists $Y$ such that $\bigwedge^l(NY)$ is Griffiths-positive.

Remark. If $l = 1$ then Theorem 3(a) is just Theorem 1(e). Indeed, the cotangent bundle is Griffiths-positive if and only if the tangent bundle is Griffiths-negative, in other words if and only if the holomorphic bisectional curvature is negative (see definitions in Sections 2.5 and 3 below). Clearly, if $l = 1$ then Theorem 3(b) is empty. Furthermore, Theorem 3(a) with $l = d$ is a reformulation of Theorem 1(b) if the metric $\mu$ is Kähler.

Masuda and Noguchi [11] have proved that every projective manifold contains hyperbolic hypersurfaces (in the sense of Brody or Kobayashi). Theorem 4 below gives a new construction of such hyperbolic hypersurfaces whose Kobayashi hyperbolic metric actually satisfies quantitative bounds. (Recently, it has been shown that almost every hypersurface of sufficiently large degree is hyperbolic, see [4] and [14].)

**Theorem 4.** Let $X$ be a complex projective manifold equipped with a Hermitian metric $\mu$ and an ample line bundle $L \to X$ and let $D \subset \mathbb{C}$ denote the unit disk.

Then, for every sufficiently large $k$, there exists a hyperbolic hypersurface $Y$ of degree $k$ such that every holomorphic map $f : D \to Y$ satisfies

$$\|f'(0)\| \leq \frac{C}{\sqrt{k}}$$

for some constant $C$ which only depends on $X$, $L$ and $\mu$.

To complete this introduction, let’s state without proof two similar results. The first one is a version of Theorem 4 for the complements $X \setminus Y$:

**Theorem 5.** Let $X$ be a complex projective manifold equipped with a Hermitian metric $\mu$ and an ample line bundle $L$.

Then, for every sufficiently large $k$, there exists a hypersurface $Y$ of degree $k$ such that every holomorphic map $f : D \to X \setminus Y$ satisfies

$$\|f'(0)\| \leq \frac{C}{\sqrt{k}}$$

for some constant $C$ which only depends on $X$, $L$ and $\mu$. 

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The second one concerns holomorphic 1–forms:

**Theorem 6.** Let $X$ be a complex projective manifold equipped with a Hermitian metric $\mu$ and an ample line bundle $L$.

Then, for every sufficiently large $k$, there exists a holomorphic 1–form $\alpha$ with values in $L^\otimes k$ such that every holomorphic map $f : \mathbb{D} \to X$ satisfies

$$\left( \text{the pull-back form } f^*\alpha \text{ is identically 0 on } \mathbb{D} \right) \Rightarrow \left( \|f'(0)\| \leq \frac{C}{\sqrt{k}} \right)$$

for some constant $C$ which only depends on $X$, $L$ and $\mu$.

In particular, in dimension 2, the form $\alpha$ defines a (singular) holomorphic foliation whose leaves do not contain large disks.

Though they are not included here, the proofs of Theorems 5 and 6 are very similar to those of Theorems 1, 3 and 4. They just require extensions of the formalism presented in Sections 1 and 5 which will be developed elsewhere.

The organization of the paper is as follows. The Donaldson-Auroux theory is discussed in Section 1. Results are stated in the language of [12]. In Sections 2, 3 and 4 we use the theory of Donaldson and Auroux in order to prove Theorem 1, Theorem 3 and Theorem 4. In Section 5 we prove the main result of Section 1.

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1 **Asymptotic transversality theory**

1.1 **Limit submanifolds**

Let $X$ be a complex projective manifold of dimension $n$ equipped with an ample line bundle $L \to X$. Recall that a submanifold $Y$ is a complete intersection of dimension $d$ and degree $k$ if it is the common zero set of $n - d$ holomorphic sections of $L^\otimes k$.

In [6] and [1], Donaldson and Auroux construct complete intersections $Y_k$ of fixed dimension $d$ and of degree $k$ going to $\infty$ which satisfy some remarkable compactness properties. More specifically, the geometry of $Y_k$, when regarded at scale $\frac{1}{\sqrt{k}}$, remains bounded (see Definition 7 below), in particular the submanifolds $(Y_k)_{k \geq 1}$ admit limits which are submanifolds in $\mathbb{C}^n$ just because a complex manifold viewed at smaller and smaller scale looks like $\mathbb{C}^n$. We will
give below some precise definitions and results in which we try to emphasize the role of this limit submanifolds.

Let’s give formal definitions. For every sufficiently large $k$, let $Y_k \subset X$ be a smooth complete intersection of dimension $d$ and degree $k$, let $B = \mathbb{B}(1) \subset \mathbb{C}^n$ be the unit ball and, more generally, let $B(r)$ denote the ball of radius $r$. Given holomorphic charts $\varphi_k : \mathbb{B} \to X$, for $k$ in some infinite set $I$ of integers, we define the corresponding rescaled charts $R\varphi_k : \mathbb{B}(\sqrt{k}) \to X$ by the following formula:

$$R\varphi_k(p) = \varphi_k \left( \frac{p}{\sqrt{k}} \right)$$

where the point $p$ belongs to $\mathbb{B}(\sqrt{k})$. For every sufficiently large $k \in I$, we define the renormalized submanifold $R Y_k = (R \varphi_k)^{-1}(Y_k) \subset \mathbb{C}^n$.

This is a rather abbreviated notation. It is important to remember that the submanifold $R Y_k$ depends on the chart $\varphi_k$, even though the chart doesn’t appear in the notation.

**Definition 7.** The sequence $(Y_k)_{k \gg 1}$ is renormalizable if for every sequence $(\varphi_k : \mathbb{B} \to X)_{k \in I}$ of holomorphic charts (indexed by some infinite set $I$ of sufficiently large integers) which satisfies properties 1 and 2 below, the corresponding family $(R Y_k)_{k \in I}$ of renormalized submanifolds is relatively compact for the smooth compact-open topology. The charts $(\varphi_k : \mathbb{B} \to X)_{k \in I}$ are supposed to satisfy the following properties:

1. Normality: The family $(\varphi_k)_{k \in I}$ is normal (i.e. relatively compact for the smooth compact-open topology on holomorphic maps).

2. Non degeneracy: If $\varphi_\infty : \mathbb{B} \to X$ is a limit map of the normal sequence $(\varphi_k)_{k \in I}$ then the tangent map $d\varphi_\infty(0)$ is an isomorphism.

Note that we don’t assume that the points $(\varphi_k(0))_{k \in I}$ form a constant sequence in $X$.

Let $(R Y_k)_{k \in J}$ be a subsequence which converges to some smooth holomorphic submanifold $Y_\infty \subset \mathbb{C}^n$ of dimension $d$. Such a $Y_\infty$ is called a limit submanifold of the renormalizable sequence $(Y_k)_{k \gg 1}$.

The following theorem is a reformulation in the above terminology of results due to Donaldson [6] and Auroux [1].

**Theorem 8.** Let $X$ be a complex projective manifold of dimension $n$ equipped with an ample line bundle $L$ and let $d$ be an integer with $1 \leq d < n$.

Then, for every sufficiently large $k$, there exists a complete intersection $Y_k \subset X$ of dimension $d$ and degree $k$ such that submanifolds $(Y_k)_{k \gg 1}$ form a renormalizable sequence.

This Theorem is a special case of Theorem 9 below.
1.2 Jets of submanifolds

Let \( \text{Jet}_{d,n}^l \) denote the space of \( l \)-jets of complex submanifolds in \( \mathbb{C}^n \) of dimension \( d \). In order to describe \( \text{Jet}_{d,n}^l \), consider the set \( \widetilde{\text{Jet}}_{d,n}^l \) of \( l \)-jets of regular parametrizations of submanifolds. Hence, the space \( \widetilde{\text{Jet}}_{d,n}^l \) is a (smooth, quasi-projective) algebraic manifold whose elements are \( l \)-jets of (germs of) holomorphic maps \((\mathbb{C}^d, 0) \rightarrow \mathbb{C}^n \) whose differential at 0 is injective. In particular, it is endowed with the Zariski topology. Note that: \( \text{Jet}_{d,n}^l = \widetilde{\text{Jet}}_{d,n}^l / G_d^l \) where \( G_d^l \) is the group of \( l \)-jets of (germs of) biholomorphisms \((\mathbb{C}^d, 0) \rightarrow (\mathbb{C}^d, 0) \). In this paper \( \text{Jet}_{d,n}^l \) is endowed with the quotient topology and closed subsets in \( \text{Jet}_{d,n}^l \) are called closed complex algebraic subsets. This terminology comes from the fact that one can endow \( \text{Jet}_{d,n}^l \) with a natural structure of algebraic manifold and the quotient topology above is just the corresponding Zariski topology.

Notice \( \text{Jet}_{d,n}^{l+1} \rightarrow \text{Jet}_{d,n}^l \) is a fiber bundle. Let \( Y \subset \mathbb{C}^n \) be a complex submanifold of dimension \( d \) and let \( p \in Y \) be a point. Of course, \( \text{Jet}_{d,n}^0 = \mathbb{C}^n \) since the 0-jet of \( Y \) at \( p \) is just the point \( p \). The 1-jet of \( Y \) at \( p \) is a pair \((p, T)\) where \( T \subset \mathbb{C}^n \) is a linear subspace (tangent space). Denote by \( \text{Grass}(d, \mathbb{C}^n) \) the grassmannian of linear subspaces of dimension \( d \) in \( \mathbb{C}^n \). Then \( \text{Jet}_{d,n}^1 = \mathbb{C}^n \times \text{Grass}(d, \mathbb{C}^n) \).

The 2-jet of \( Y \) at \( p \) is a triple \((p, T, \Pi)\) where \( T \) is a linear subspace and \( \Pi \) is a symmetric bilinear map \( T \times T \rightarrow \mathbb{C}^n / T \) (the second fundamental form). Hence, \( \text{Jet}_{d,n}^2 \rightarrow \text{Jet}_{d,n}^1 \) is a vector bundle having rank \( \frac{2(d+1)(n-d)}{2} \). The fiber over \((p, T) \in \text{Jet}_{d,n}^1 \) equals the space \( \text{Sym}^2(T, \mathbb{C}^n / T) \) of symmetric bilinear maps from \( T \times T \) to \( \mathbb{C}^n / T \). If \( a \in \text{Jet}_{d,n}^2 \) then we will use the notation \( a = (p_a, T_a, \Pi_a) \).

1.3 Avoidance theorem

Consider a subset \( A \) in \( \text{Jet}_{d,n}^l \) satisfying the following two conditions:

1. \( A \) is a closed complex algebraic subset
2. \( A \) is invariant under the natural action of affine transformations of \( \mathbb{C}^n \) upon \( \text{Jet}_{d,n}^l \).

Our main tool is a generalization of Theorem 8

**Theorem 9.** Let \( X \) be a complex projective manifold of dimension \( n \) equipped with an ample line bundle \( L \) and let \( A \subset \text{Jet}_{d,n}^l \) be a subset satisfying the above conditions 1 and 2.

If \( \text{codim}(A) > d \) then, for every sufficiently large \( k \), there exists a complete intersection \( Y_k \subset X \) of dimension \( d \) and degree \( k \) such that submanifolds \((Y_k)_{k \geq 1}\) form a renormalizable sequence which avoids the subset \( A \) asymptotically in the sense that the \( l \)-jets of its limit submanifolds lie in the complement \( \text{Jet}_{d,n}^l \setminus A \).

The proof is an application of Donaldson and Auroux’s techniques. We will give a detailed proof of Theorem 9 in Section 5.
In order to apply the avoidance theorem, we need to produce large codimension subsets. That’s the use of the main theorem of elimination theory.

**Theorem 10.** Let $X_1, X_2$ be complex algebraic manifolds. Assume $X_1$ is quasi-projective and $X_2$ is projective. Let $A \subset X_1 \times X_2$ be a closed algebraic subset and denote by $\pi_1$ the first projection.

Then $\pi_1(A)$ is a closed algebraic subset in $X_1$ with $\dim(\pi_1(A)) \leq \dim(A)$ (hence: $\text{codim}(\pi_1(A)) \geq \text{codim}(A) - \dim(X_2)$).

This classical result is proved in [13], p 104-107.

2 Proof of Theorem 1

As we shall see, all cases in Theorem 1 are proved in the same way.

Let $X$ be a complex projective manifold of dimension $n$ equipped with a Hermitian metric $\mu$ and an ample line bundle $L \to X$. All types of curvature in Theorem 1 are non-positive for a complex submanifold $Y \subset \mathbb{C}^n$. (Here, $\mathbb{C}^n$ is endowed with a constant Kähler metric and $Y$ is endowed with the induced metric.) For each type of curvature, we will construct a closed algebraic subset $A \subset \text{Jet}^2_{d,n}$ of codimension $> d$ which satisfies the invariance assumption of Theorem 9 and the following assumption:

If $Y \subset \mathbb{C}^n$ is a complex submanifold of dimension $d$ and if $p \in Y$ is a point such that the 2−jet of $Y$ at $p$ belongs to the complement $\text{Jet}^2_{d,n} \setminus A$ then the curvature of $Y$ at $p$ is negative.

If such an $A$ exists then apply Theorem 9. This theorem provides a renormalizable sequence of complete intersections $(Y_k)_{k \gg 1}$ in $X$ of dimension $d$ which avoids the subset $A$ asymptotically. Then the limit submanifolds $Y_\infty$ have negative curvature. Let’s prove that, for every sufficiently large $k$, the submanifold $Y_k$ has negative curvature.

Our proof goes by contradiction. Assume that, for infinitely many integers $k$, there exists some point $p_k \in Y_k$ where the curvature of $Y_k$ isn’t negative. This holds for both metrics $\mu$ and $\mu_k = k\mu$. Pick charts $\varphi_k : B \to X$ with $\varphi_k(0) = p_k$. Then the curvature of the rescaled submanifold $RY_k \subset \mathbb{C}^n$ isn’t negative with respect to the renormalized metric $(R\varphi_k)^*\mu_k = k(R\varphi_k)^*\mu$. In order to get limit submanifolds, WLOG, assume that the charts $(\varphi_k)_k$ are normal and non-degenerate (cf. Definition 7).

Up to subsequences, the rescaled submanifolds $(RY_k)_k$ converge to some limit submanifold $Y_\infty$ and the renormalized metrics converge to some constant Kähler metric on $\mathbb{C}^n$. Conclude that the curvature of $Y_\infty$ at $0 \in \mathbb{C}^n$ is non-negative (since $(RY_k)$ converge to $Y_\infty$). This is a contradiction.

Hence, in order to prove Theorem 1, one only needs to produce a suitable subset $A \subset \text{Jet}^2_{d,n}$ for each type of curvature.
2.1 Curves with negative curvature

Here, we prove Theorem 1(a).

**Proposition 11.** A complex curve $Y \subset \mathbb{C}^n$ has non-positive Gauss curvature for the metric induced on $Y$ by any constant Kähler metric on $\mathbb{C}^n$. Moreover the points where the curvature vanishes are the inflection points.

**Proof.** Consider the Gauss map $G$ from $Y$ to $\mathbb{C}P^{n-1}$. For every $y \in Y$, the value $G(y)$ is just the tangent line $T_y Y$. Let $O(-1)$ be the usual tautological line bundle over $\mathbb{C}P^{n-1}$, endowed with the Hermitian metric induced by the constant Kähler metric over $\mathbb{C}^n$. The tangent bundle is the pull-back $G^* (O(-1))$ of this line bundle by the Gauss map. Moreover, the metric of $TY$ equals the pull-back metric.

It is well known that the curvature of $O(-1)$ is negative. Hence, since the map $G$ is holomorphic, the pull-back $G^* (O(-1))$ has non-positive curvature. Moreover, the curvature of $G^* (O(-1))$ vanishes at critical points of $G$ (i.e. inflection points).

In the set $\text{Jet}_{1,n}^2$ of $2-$jets of curves, let $A$ be the subset of $2-$jets of curves at inflection points. Notice that the affine transformations preserve $A$. The subset $A$ is the zero section of the vector bundle $\text{Jet}_{1,n}^2 \rightarrow \text{Jet}_{1,n}^1$. Hence: $\text{codim}(A) = n - 1$. Here $d = 1$ and $n \geq 3$. Therefore, condition $\text{codim}(A) > d$ is satisfied and the proof of Theorem 1(a) is complete.

2.2 Ricci curvature

Here, we prove Theorem 1(b).

Let $Y \subset \mathbb{C}^n$ be a complex submanifold and denote by $\Pi$ the second fundamental form of $Y$, which depends only on the $2-$jet of $Y$. By Proposition 9.5 in Chapter IX of [9], the Ricci curvature of $Y$ at a point $p$ satisfies:

$$\text{Ric}(v, v) = -2 \sum_{i=1}^{d} ||\Pi(e_i, v)||^2$$

for every $v \in T_p Y$, where $(e_i)_{1 \leq i \leq d}$ is a unitary basis in $T_p Y$. Hence, if there exists $v' \in T_p Y$ such that $\Pi(v', v) \neq 0$, then $\text{Ric}(v, v)$ is negative. This leads us to consider the set $A \subset \text{Jet}_{d,n}^2$ of $2-$jets $a = (p_a, T_a, \Pi_a)$ such that:

$$\exists u \neq 0 \in T_a, \quad \forall u' \in T_a, \quad \Pi_a(u, u') = 0$$

with the notations of Section 1.2. Obviously, $A$ is invariant under the affine transformations. We just need to show that $A$ is a closed algebraic subset of codimension $> d$.

For this purpose, we regard the set $A$ as the image of some subset $A_1$ of $\text{Jet}_{d,n}^2 \times \mathbb{C}P^{n-1}$ which we will define below. A point in $\text{Jet}_{d,n}^2 \times \mathbb{C}P^{n-1}$ is a pair $(a, b)$ where $a$ is the $2-$jet of a submanifold $Y$ and $b$ is a line in $\mathbb{C}^n$. 
The condition \( b \subset T_a \) defines a closed algebraic subset \( B \subset \text{Jet}^2_{d,n} \times \mathbb{CP}^{n-1} \) of codimension \( n - d \). Define another closed algebraic subset \( A_1 \subset B \) by using the following condition:

\[
\forall u \in b, \forall u' \in T_a \quad \Pi_a(u, u') = 0.
\]

In short, our condition writes: \( \Pi_a(u, T_a) = 0 \) where \( u \) is a generator of the line \( b \).

The map \( \Pi_a(u, ) \) is any linear map from \( T_a \) to \( \mathbb{C}^n/T_a \). Of course, the space of linear maps from \( T_a \) to \( \mathbb{C}^n/T_a \) has dimension \( d(n - d) \). Hence, since the subset \( A_1 \) is defined by \( \Pi_a(u, ) = 0 \), the codimension of \( A_1 \) in \( B \) is \( d(n - d) \) and the codimension of \( A_1 \) in \( \text{Jet}^2_{d,n} \times \mathbb{CP}^{n-1} \) equals:

\[
\text{codim}(A_1) = d(n - d) + \text{codim}(B)
\]

Apply Theorem 10. Since \( A \) is the image of \( A_1 \) via the projection \( \text{Jet}^2_{d,n} \times \mathbb{CP}^{n-1} \rightarrow \text{Jet}^2_{d,n} \), the subset \( A \) is closed and:

\[
\text{codim}(A) \geq \text{codim}(A_1) - \text{dim}(\mathbb{CP}^{n-1})
\]

\[
= (d + 1)(n - d) - (n - 1)
\]

\[
= d(n - d - 1) + 1.
\]

Under the assumption \( d \leq n - 2 \), one obtains: \( \text{codim}(A) > d \).

### 2.3 Scalar curvature

Here, we prove Theorem 1(c).

Consider the scalar curvature.

\[
\text{Scal} = 2 \sum_{i=1}^{d} \text{Ric}(e_i, e_i).
\]

Hence, the complex submanifold \( Y \subset \mathbb{C}^n \) satisfies:

\[
\text{Scal} = -4 \sum_{i,j=1}^{d} \|\Pi(e_i, e_j)\|^2.
\]

Therefore, the scalar curvature vanishes if and only if \( \Pi \) does. The condition \( \Pi_a = 0 \) defines a codimension \( \frac{d(d+1)(n-d)}{2} \) closed algebraic subset \( A \subset \text{Jet}^2_{d,n} \).

The assumptions \( n \geq 3 \) and \( d \leq n - 1 \) ensure: \( \text{codim}(A) > d \).

### 2.4 Holomorphic sectional curvature

Here, we prove Theorem 1(d).

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The holomorphic sectional curvature HolSec(v) equals \( R(v, Jv, v, Jv) \) where \( R \) is the usual Riemann tensor and \( J \) is the multiplication by \( i \) on the tangent space. If \( Y \subset \mathbb{C}^n \) is a complex submanifold then, by Proposition 9.2 in Chapter IX of [9]:

\[
\text{HolSec}(v) = -2\|\Pi(v, v)\|^2.
\]

The strategy is the same as in Section 2.2. We use the same subset \( B \subset \text{Jet}^2_{d,n} \times \mathbb{C}P^{n-1} \) and some slightly different subset \( A_1 \subset B \).

As before, we use the notation \((a, b)\) for a point in \( \text{Jet}^2_{d,n} \times \mathbb{C}P^{n-1} \). The subset \( A_1 \) is defined by the condition \( \Pi_a(b, b) = 0 \) so the codimension of \( A_1 \) in \( B \) equals \( n - d \). Hence, by the same calculation as in Section 2.2 we get \( \text{codim}(A) \geq n - 2d + 1 \) where \( A \) is the image of \( A_1 \) in \( \text{Jet}^2_{d,n} \). We obtain \( \text{codim}(A) > d \) because \( n \geq 3d \).

### 2.5 Holomorphic bisectional curvature

Here, we prove Theorem 1(e).

The holomorphic bisectional curvature is defined in [8] by the formula:

\[
\text{HolBisec}(v, v') = R(v, Jv, v', Jv').
\]

For a submanifold \( Y \subset \mathbb{C}^n \), formula (9) in [8] writes:

\[
\begin{align*}
\text{HolBisec}(v, v') &= -\|\Pi(v, v')\|^2 - \|\Pi(v, Jv')\|^2 \\
&= -2\|\Pi(v, v')\|^2.
\end{align*}
\]

We define a subset \( B \subset \text{Jet}^2_{d,n} \times \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \) by the following characterization. The point \((a, b, b')\) belongs to \( B \) if and only if \( b, b' \subset T_a \).

\[
\text{codim}(B) = 2(n - d)
\]

Then the subset \( A_1 \subset B \) is defined by the condition \( \Pi_a(b, b') = 0 \), so the codimension of \( A_1 \) in \( B \) is \( n - d \) and the codimension of \( A_1 \) in \( \text{Jet}^2_{d,n} \times \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} \) equals \( 3(n - d) \). As usual, \( A \) is the image of \( A_1 \) in \( \text{Jet}^2_{d,n} \). We obtain:

\[
\begin{align*}
\text{codim}(A) &\geq 3(n - d) - \dim(\mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}) \\
&= n - 3d + 2
\end{align*}
\]

Under the assumption \( n \geq 4d - 1 \), we obtain: \( \text{codim}(A) > d \). This completes the proof of Theorem 1.

### 3 Proof of Theorem 3

**Definition 12.** Let \( Y \) be a complex manifold equipped with a Hermitian vector bundle \( E \rightarrow Y \). The Hermitian bundle \( E \) is Griffiths-positive if the curvature \( F \) of \( E \) satisfies

\[
\langle F(Jv, v)v', Jv' \rangle > 0
\]
for all vectors \( v \in T_pY \) and \( v' \in E_p \), where \( p \) is a point in \( Y \), where \( J \) is the multiplication by \( i \) on the tangent space \( T_pY \) and \( J' \) is the multiplication by \( i \) on the fiber \( E_p \) over \( p \).

The proof of Theorem 3 proceeds in two steps: (1) We study the corresponding problem in \( \mathbb{C}^n \). (2) We apply the Donaldson-Auroux machinery.

Step 1. Let \( Y \subset \mathbb{C}^n \) be a complex submanifold of dimension \( d \). Consider the Gauss map \( G \) defined by \( G(p,[v]) = [v] \) from the total space of the projective bundle \( P(\bigwedge^l(TY)) \) to the projective space \( P(\bigwedge^l(\mathbb{C}^n)) \). Similarly, define \( G' \) from the total space of \( P(\bigwedge^l(NY)^*) \) to \( P(\bigwedge^l(\mathbb{C}^n)^*) \).

Endow \( \mathbb{C}^n \) with a constant Kähler metric. This induces Hermitian metrics on both bundles \( \bigwedge^l(T^*Y) \) and \( \bigwedge^l(NY) \).

**Lemma 13.** The Hermitian bundle \( \bigwedge^l(T^*Y) \) is Griffiths-positive if and only if the Gauss map \( G \) is an immersion.

Of course, the same holds for \( \bigwedge^l(NY) \) and \( G' \).

**Proof.** Denote by \( \mathcal{O}(1) \) the usual hyperplane line bundle over \( P(\bigwedge^l(\mathbb{C}^n)) \). Endow \( \mathcal{O}(1) \) with the Hermitian metric induced by some constant Kähler metric over \( \mathbb{C}^n \). Denote by \( G^*(\mathcal{O}(1)) \) the pull-back Hermitian line bundle. The important fact is that, by a reformulation of the definition of the Griffiths positivity (see for example [10, p. 19-20]), \( \bigwedge^l(T^*Y) \) is Griffiths-positive if and only if the Hermitian line bundle \( G^*(\mathcal{O}(1)) \) (also known as the Serre line bundle of \( \bigwedge^l(T^*Y) \)) has positive curvature.

Use the fact that the curvature of \( \mathcal{O}(1) \) is positive, as in the proof of Proposition [11] since the map \( G \) is holomorphic, the pull-back \( G^*(\mathcal{O}(1)) \) has semi-positive curvature and furthermore, the curvature of \( G^*(\mathcal{O}(1)) \) is positive if and only if \( G \) is an immersion. 

\]

Step 2. In order to apply Theorem 9, we consider some appropriate subset \( A \subset Jet^2_{d,n} \). Note that the 1–jet of the Gauss map of a submanifold at a point depends only on the 2–jet of the submanifold at that point. Let \( A \) be the set of 2–jets of submanifolds \( Y \subset \mathbb{C}^n \) whose Gauss map \( G \) is not an immersion (over some neighborhood of the corresponding point \( p \in \mathbb{C}^n \)).

**Lemma 14.** The set \( A \) is a closed algebraic subset. Moreover, if \( 2d(1 + l) \leq l(n + l) \) then: \( \text{codim}(A) > d \).

Similarly, if \( l(n - l) \leq 2d(1 - l) \) then the codimension of the set of 2–jets of submanifolds \( Y \) with non-immersive \( G' \) is \( > d \).

**Proof.** First, pick a complex submanifold \( Y \subset \mathbb{C}^n \) of dimension \( d \) and some point \( p \in Y \). The domain of \( G \) is the set \( P(\bigwedge^l(TY)) \) of classes of \( l \)–vectors in \( TY \). Denote by \( G_{\text{dec}} \) the restriction of \( G \) to the set \( P_{\text{dec}}(\bigwedge^l(TY)) \) of classes of decomposable \( l \)–vectors. The following four conditions are equivalent:
(i) The map $G$ is an immersion in some neighborhood of the fiber $P(\Lambda^l(T_pY))$ in the total space $P(\Lambda^l(TY))$.

(ii) The restriction map $G_{\text{dec}}$ is an immersion in a neighborhood of the fiber $P_{\text{dec}}(\Lambda^l(T_pY))$ in the total space $P_{\text{dec}}(\Lambda^l(TY))$.

(iii) For every non-zero vector $u \in T_pY$, the subspace:
$$K_u = \{ u' \in T_pY, \Pi(u, u') = 0 \}$$

(where $\Pi$ is the second fundamental form of the submanifold $Y$ at $p$), satisfies: $\dim(K_u) < l$.

(iv) For every non-zero vector $u \in T_pY$ and every linear subspace $V \subset T_pY$ of dimension $l$, there exists $u' \in V$ such that $\Pi(u, u') \neq 0$.

(i) $\Rightarrow$ (ii) and equivalence (iii)$\Leftrightarrow$(iv) are clear.

Let’s prove the contrapositive of (ii)$\Rightarrow$(iv). Assume there exists a non-zero $u \in T_pY$ and a linear subspace $V$ of dimension $l$, such that $\Pi(u, V) = 0$. Let $u_1, ..., u_l$ be a basis in $V$. For $1 \leq \alpha \leq l$, extend $u_\alpha$ to a local section of $TY$. The exterior product $u_1 \wedge \cdots \wedge u_l$ is a non-zero decomposable local section of $\Lambda^l(TY)$. Denote by $\lambda$ the induced section of $P_{\text{dec}}(\Lambda^l(TY))$.

Consider $u_\alpha$ as a map from a neighborhood of $p$ in $Y$ to $\mathbb{C}^n$. The differential of $u_\alpha$ at point $p$ is a linear map $(du_\alpha)_p$ from $T_pY$ to $\mathbb{C}^n$.

Since $\Pi(u, u_\alpha) = 0$, (WLOG) we can assume that we have constructed each extension $u_\alpha$ in such a way that $(du_\alpha)_p(u) = 0$. Then $d(u_1 \wedge \cdots \wedge u_l)_p(u)$ is zero and $(d\lambda)_p(u)$ belongs to the kernel of the differential $(dG_{\text{dec}})_p$. Therefore the map $G_{\text{dec}}$ isn’t an immersion at point $\lambda(p)$. This completes the proof of (the contrapositive of) (ii)$\Rightarrow$(iv).

In order to prove (iii)$\Rightarrow$(i), we need to look at the relations between the second fundamental form $\Pi$ and the sections of $\Lambda^l(TY)$.

Since $Y$ is a submanifold in $\mathbb{C}^n$, a local section of $TY$ provides a map $f$ from a neighborhood of $p$ in $Y$ to $\mathbb{C}^n$. Its differential $(df)_p$ is a linear map from $T_pY$ to $\mathbb{C}^n$. If $u \in T_pY$ is a vector then $(df)_p(u) \in \Lambda^l(\mathbb{C}^n)$ and, by definition of the second fundamental form, the class of $(df)_p(u) \mod T_pY$ equals $\Pi(f(u), f(p))$.

More generally, a local section of $\Lambda^l(TY)$ provides a local map $f$ from $Y$ to $\Lambda^l(\mathbb{C}^n)$. A priori, if $u \in T_pY$ then $(df)_p(u) \in \Lambda^l(\mathbb{C}^n)$. Actually, $(df)_p(u)$ belongs to some linear subspace $\Gamma \subset \Lambda^l(\mathbb{C}^n)$. Here, $\Gamma$ is the image of $\mathbb{C}^n \otimes (T_pY)^{\otimes l-1} \subset (\mathbb{C}^n)^{\otimes l}$ under the usual projection $(\mathbb{C}^n)^{\otimes l} \rightarrow \Lambda^l(\mathbb{C}^n)$.

Let $\Pi\Lambda_u^l$ be the unique linear map from $\Lambda^l(T_pY)$ to $(\mathbb{C}^n/T_pY) \otimes \Lambda^{l-1}(T_pY)$ such that:
$$\Pi\Lambda_u^l(u_1 \wedge \cdots \wedge u_l) = \sum_{\alpha=1}^l (-1)^{\alpha+1} \Pi(u, u_\alpha) \otimes (u_1 \wedge \cdots \wedge \hat{u}_\alpha \wedge \cdots \wedge u_l)$$

for all $u_1, ..., u_l \in T_pY$. By the Leibniz rule, the class of $(df)_p(u) \mod \Lambda^l(T_pY)$ equals $\Pi\Lambda_u^l(f(p))$, where we use the following identification:
$$\Gamma/\Lambda^l(T_pY) = (\mathbb{C}^n/T_pY) \otimes \Lambda^{l-1}(T_pY).$$
The kernel of the map $\Pi A^l_u$ equals $\bigwedge^l(K_u)$.

Let’s prove (iii)$\Rightarrow$(i). Assume $(dG)_q(v) = 0$ where $q$ is a point in $P(\bigwedge^l(T_pY))$ and the vector $v$ is tangent to the total space $P(\bigwedge^l(TY))$ at $q$. In order to show that $G$ is an immersion, we shall prove $v = 0$.

Denote by $u \in T_pY$ the projection of $v$. The total space $P(\bigwedge^l(TY))$ is a subset of the product $Y \times \bigwedge^l(C^n)$. The components of $v$ are $u$ and $(dG)_q(v)$. We know the second one is zero. Hence, we just need to prove $u = 0$.

Since $(dG)_q(v)$ is zero, there exists a map $f$ from a neighborhood of $p$ to $\bigwedge^l(C^n)$ such that:

- $f$ corresponds to a local section of $\bigwedge^l(TY)$
- the vector $f(p) \in \bigwedge^l(T_pY)$ is a generator of the line $q \in P(\bigwedge^l(T_pY))$
- $(df)_p(u) = 0$ and therefore $\Pi A^l_u(f(p)) = 0$.

Assume $u \neq 0$. The kernel of $\Pi A^l_u$ equals $\bigwedge^l(K_u)$ and assumption (iii) states that $\dim(K_u) < l$. Now, $f(p) \in \bigwedge^l(K_u)$ is a non-zero $l$–vector over a vector space of dimension $< l$. This is a contradiction. Hence, $G$ is an immersion. This completes equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

The equivalence (i)$\Leftrightarrow$(iv) provides another definition of the set $A$:

$$A = \{a = (p_a, T_a, \Pi a) \in \text{Jet}_{d,n}^2, \exists b \in CP^{n-1}, \exists b' \in \text{Grass}(l, C^n) \text{ such that } b, b' \subset T_a \text{ and } \Pi a(b, b') = 0 \}.$$ 

In order to study $A$, first consider some simpler sets. The incidence subset $A_1$:

$$A_1 = \{(a, b) \in \text{Jet}_{d,n}^1 \times CP^{n-1}, \ b \subset T_a \}$$

has codimension $n - d$ in $\text{Jet}_{d,n}^1 \times CP^{n-1}$. Similarly:

$$A_2 = \{(a, b') \in \text{Jet}_{d,n}^1 \times \text{Grass}(l, C^n), \ b' \subset T_a \}$$

has codimension $l(n - d)$ in $\text{Jet}_{d,n}^1 \times \text{Grass}(l, C^n)$ and:

$$A_3 = \{(a, b, b') \in \text{Jet}_{d,n}^1 \times CP^{n-1} \times \text{Grass}(l, C^n), \ b, b' \subset T_a \}$$

has codimension $(1 + l)(n - d)$ in $\text{Jet}_{d,n}^1 \times CP^{n-1} \times \text{Grass}(l, C^n)$.

Denote by $\pi$ the natural projection:

$$\pi : \text{Jet}_{d,n}^2 \times CP^{n-1} \times \text{Grass}(l, C^n) \to \text{Jet}_{d,n}^1 \times CP^{n-1} \times \text{Grass}(l, C^n).$$

Consider:

$$A_4 = \{(a, b, b') \in \text{Jet}_{d,n}^2 \times CP^{n-1} \times \text{Grass}(l, C^n), \text{ such that } \pi(a, b, b') \in A_3 \text{ and } \Pi a(b, b') = 0 \}.$$
The set $A_4$ is a subbundle of the bundle $\pi: \pi^{-1}(A_3) \to A_3$ with:

$$\dim(A_4) = \dim(\pi^{-1}(A_3)) - l(n - d)$$

Indeed, consider a point $(a, b, b') \in A_3 \subset \text{Jet}^{d,n}_{d,n} \times \mathbb{C}P^{n-1} \times \text{Grass}(l, \mathbb{C}^n)$. The fiber of $\pi$ over $(a, b, b')$ equals $\text{Sym}^2(T_a, \mathbb{C}^n/T_a)$ and the fiber of the subbundle $A_4$ is the subspace of the symmetric maps from $T_a \times T_a$ to $\mathbb{C}^n/T_a$ whose restriction to the subspace $b \times b' \subset T_a \times T_a$ is $0$. This fiber has codimension $l(n - d)$ in $\text{Sym}^2(T_a, \mathbb{C}^n/T_a)$.

Hence:

$$\text{codim}(A_4) = \text{codim}(A_3) + l(n - d) = (1 + 2l)(n - d).$$

The set $A$ is the image of $A_4$ by the first projection $\text{Jet}^2_{d,n} \times \mathbb{C}P^{n-1} \times \text{Grass}(l, \mathbb{C}^n) \to \text{Jet}^2_{d,n}$. By Theorem 10, the subset $A$ is closed and satisfies:

$$\text{codim}(A) \geq \text{codim}(A_4) - \dim(\mathbb{C}P^{n-1} \times \text{Grass}(l, \mathbb{C}^n)) = (1 + 2l)(n - d) - n + 1 - l(n - l) = 1 - d + l(n + l) - 2dl.$$

Under the assumption $2d(1 + l) \leq l(n + l)$, we obtain: $\text{codim}(A) > d$.

Now we are able to complete the proof of Theorem 3. Our argument is almost the same as in Section 2. Apply Theorem 9, Lemma 13 and Lemma 14 since $\text{codim}(A) > d$, there exists a renormalizable sequence $(Y_k)_{k \gg 1}$ such that, for every limit submanifold $Y_\infty$, the Hermitian bundle $\bigwedge^l(T^*Y_\infty)$ is Griffiths-positive.

Since the geometry of $Y_k$ tends to the geometry of $Y_\infty$, it seems quite likely that, for every sufficiently large $k$, the Hermitian bundle $\bigwedge^l(T^*Y_k)$ is Griffiths-positive. Actually, a detailed proof of this fact can be achieved by the same argument as in Section 2: proof by contradiction and extraction of subsequences.

This completes the proof of Theorem 3 (of course, case (b) is similar).

4 Proof of Theorem 4

**Definition 15.** Let $(Y_k)_{k \gg 1}$ be a renormalizable sequence of hypersurfaces. We say the hypersurfaces $(Y_k)_{k \gg 1}$ contain lines asymptotically if there is a limit hypersurface $Y_\infty$ which contains some (affine, complex) line in $\mathbb{C}^n$.

The proof of Theorem 4 proceeds in two steps. First, by using Theorem 9, we will construct a renormalizable sequence of hypersurfaces $(Y_k)_{k \gg 1}$ containing no line asymptotically. Then, we will prove that such hypersurfaces are hyperbolic and satisfy the required estimates.

In order to apply Theorem 9, we define some relevant closed subset in $\text{Jet}^l_{d,n-1,n}$. Consider the set $A$ of $l$-jets of the hypersurfaces in $\mathbb{C}^n$ which are tangent to order $l$ (at least) with some affine line.
Lemma 16. A is a closed algebraic subset in $\text{Jet}^{l}_{n-1,n}$ of codimension $\geq l+1-n$.

Proof. We will express $A$ as the projection of a subset $A_1 \subset \text{Jet}^{l}_{n-1,n} \times \mathbb{C}P^{n-1}$. Pick $(a,b) \in \text{Jet}^{l}_{n-1,n} \times \mathbb{C}P^{n-1}$. Let $Y$ be a germ of hypersurface at a point $p$ with $l$-jet $a$ and let $L$ be the affine line passing through $p$ with direction $b$. The subset $A_1$ is defined in the following way: $(a,b) \in A_1$ if and only if the line $L$ is tangent to order $l$ (or more) with $Y$ at $p$. Clearly, $A$ is the projection of $A_1$ in $\text{Jet}^{l}_{n-1,n}$. By Theorem 10, $A$ is a closed algebraic subset with:

$$\text{codim}(A) \geq l - \dim(\mathbb{C}P^{n-1}) = l + 1 - n.$$ 

Proposition 17. Let $X$ be a projective manifold and let $L$ be an ample line bundle over $X$.

Then, for every sufficiently large $k$, there exists a hypersurface $Y_k$ of degree $k$ such that the hypersurfaces $(Y_k)_{k \gg 1}$ form a renormalizable sequence and contain no line asymptotically.

Proof. Notice that $A$ is invariant under the affine transformations. Pick some large $l$, so that $\text{codim}(A) > d = n-1$. By Theorem 10 there exist renormalizable hypersurfaces $(Y_k)_{k \gg 1}$ whose limit hypersurfaces have no contact of order $l$ (or more) with any line. In particular, they contain no line. \qed

Brody has proved a clever reparametrization lemma:

Lemma 18. For every holomorphic map $f : \mathbb{D} \to X$, there exists a holomorphic map $g : \mathbb{D} \to X$ such that:

$$g(\mathbb{D}) \subset f(\mathbb{D})$$

$$\sup_{\mathbb{D}} \|g'(z)\| \leq C\|g'(0)\|$$

$$\|f'(0)\| \leq C\|g'(0)\|$$

where $C$ is a universal bound (in particular, independent of $f$).

Proof. The idea of the proof is to go back and forth between the Euclidian metric and the Poincaré metric. First, remark that the Jacobian of the map $p \mapsto \frac{p}{2}$ from $\mathbb{D}$ endowed with the Poincaré metric to $\mathbb{D}$ endowed with the Euclidian metric is a positive function on $\mathbb{D}$ which tends to zero on the boundary $\partial \mathbb{D}$. Therefore the jacobian $j_1$ of the map $f_1(p) = f\left(\frac{p}{2}\right)$ from $\mathbb{D}$ endowed with the Poincaré metric to $X$ tends to zero and, hence, the supremum of $j_1$ is achieved at some point $p_0$ in $\mathbb{D}$. Moreover, $\|f'(0)\| = C_1j_1(0)$ where $C_1$ is some universal constant.
Pick a conformal transformation \( h : \mathbb{D} \to \mathbb{D} \) with \( h(0) = p_0 \). We set \( f_2 = f_1 \circ h \). Since conformal mappings preserve the Poincaré metric, the corresponding Jacobian \( j_2 \) satisfies \( j_2 = j_1 \circ h \), so that:

\[
\sup_{\mathbb{D}} j_2 = \sup_{\mathbb{D}} j_1 = j_1(p_0) = j_2(0).
\]

To complete the proof, we go back to the Euclidian metric. The Jacobian of the map \( p \mapsto p_2 \) from \( \mathbb{D} \) endowed with the Euclidian metric to \( \mathbb{D} \) endowed with the Poincaré metric is bounded by a constant \( C_2 \). Hence the map \( f_3(p) = f_2 \left( \frac{p}{2} \right) \) satisfies:

\[
\sup_{\mathbb{D}} \| f'_3 \| \leq C_2 \sup_{\mathbb{D}} j_2 = C_2 C_3 \| f'_3(0) \|
\]

and:

\[
\| f'(0) \| = C_1 j_1(0) \leq C_1 j_1(p_0) = C_1 j_2(0) = C_1 C_3 \| f'_3(0) \|
\]

where \( C_3 \) is another constant. The proof is completed (set \( g = f_3 \)).

Theorem 4 instantly follows from Proposition 17 and from the following renormalized version of Brody’s techniques:

**Proposition 19.** Let \((Y_k)_{k \gg 1}\) be renormalizable hypersurfaces. Assume that for infinitely many \( k \), there exists a holomorphic map \( f_k : \mathbb{D} \to Y_k \) such that the numbers \( \sqrt{k} \| f'_k(0) \| \) tend to \( \infty \).

Then the hypersurfaces \((Y_k)_{k \gg 1}\) contain lines asymptotically.

**Proof.** Using Lemma 18 we can assume (WLOG) that:

\[
\sup_{\mathbb{D}} \| f'_k \| \leq C \| f'_k(0) \|
\]

for a constant \( C \) which doesn’t depend on \( k \). By the assumption on \( \sqrt{k} \| f'_k(0) \| \), the vector \( f'_k(0) \) is large with respect to the rescaled metric \( \mu_k = k \mu \). We can also assume that \( f'_k(0) \) is small with respect to the metric \( \mu \). Indeed, if \( \| f'_k(0) \| \) doesn’t tend to 0, and hence \( \| f'_k(0) \| \geq k^{-\frac{1}{2}} \) for infinitely many \( k \), then replace \( f_k \) with the rescaled map \( z \mapsto f_k \left( \frac{k^{-\frac{1}{2}} \| f'_k(0) \|-1}{2} \right) \). The diameter of \( f_k(\mathbb{D}) \) (with respect to \( \mu \)) tends to 0.

Pick some holomorphic chart \( \varphi_k : \mathbb{B} \to X \) centered at \( f_k(0) \). We can assume that the family \((\varphi_k)\) is normal and non-degenerate. Moreover, we can assume (WLOG) that every limit map \( \varphi_\infty : \mathbb{B} \to X \) of the normal family is a chart. Then, the restrictions \( \varphi_k |_{\mathbb{B} \left( \frac{1}{2} \right)} \) from the ball \( \mathbb{B} \left( \frac{1}{2} \right) \) to \( \varphi_k \left( \mathbb{B} \left( \frac{1}{2} \right) \right) \) have bounded distortion. The non-degeneracy of the family \((\varphi_k)\) implies that \( \varphi_k \left( \mathbb{B} \left( \frac{1}{2} \right) \right) \) contains a ball of radius independent of \( k \) centered at \( f_k(0) \) in \( X \). Hence, for \( k \) sufficiently large:

\[
f_k(\mathbb{D}) \subset \varphi_k \left( \mathbb{B} \left( \frac{1}{2} \right) \right)
\]
Define $g_k : \mathbb{D} \to \varphi_k^{-1}(Y_k) \subset \mathbb{C}^n$ by the formula:

$$g_k = \varphi_k^{-1} \circ f_k$$

Since $\varphi_k$ has bounded distortion on $B\left(\frac{1}{2}\right)$, the maps $f_k$ and $g_k$ behave similarly, in particular:

1. $\sqrt{k}\|g_k'(0)\|$ tends to $\infty$
2. $\sup_{D} \|g_k\| \leq C\|g_k'(0)\|$ for some $C$.

We now define a rescaled version $h_k$ of $g_k$. We set $r_k = \sqrt{k}\|g_k'(0)\|$ and we define $h_k$ on a large disk $D\left(r_k\right)$ by the following formula:

$$h_k(z) = \sqrt{k}g_k\left(\frac{z}{r_k}\right) \in \mathbb{C}^n$$

Now:

$$\|h_k'(0)\| = \frac{\sqrt{k}}{r_k}\|g_k'(0)\| = 1$$

$$\sup_{D\left(r_k\right)} \|h_k'\| = \frac{\sqrt{k}}{r_k}\sup_{D} \|g_k\|$$

$$\leq C\frac{\sqrt{k}}{r_k}\|g_k'(0)\| = C$$

We conclude that the maps $(h_k)$ form a normal family because $\|h_k'\|$ is uniformly bounded and $h_k(0) = 0$. Therefore some subsequence converges to a map $h_\infty$ with bounded derivative and $\|h_\infty'(0)\| = 1$. Since $r_k$ tends to $\infty$, $h_\infty$ is an entire map.

Recall a classical fact. The image of a non-constant entire map $h_\infty : \mathbb{C} \to \mathbb{C}^n$ with bounded derivative is a line (since the derivative is bounded, Liouville’s theorem implies it is constant).

The image $g_k(\mathbb{D})$ is contained in $\varphi_k^{-1}(Y_k)$, so $h_k(D(r_k))$ is contained in $RY_k$. Passing to a subsequence if necessary, the rescaled hypersurfaces $(RY_k)$ converge to some limit hypersurface $Y_\infty$ and the line $h_\infty(\mathbb{C})$ is contained in $Y_\infty$. This completes the proof of Proposition 19 (and of Theorem 4).

\[\square\]

5 Proof of Theorem 9

We will give the proof of Theorem 9. The steps of our proof are the same as in [6]. We will also use the argument proposed in [3] to simplify a difficult step of the proof.
5.1 Framed charts

Let $X$ be a projective manifold of dimension $n$ and let $L \to X$ be an ample line bundle endowed with a Hermitian metric of curvature $-i2\pi\omega$ where $\omega$ is a Kähler form. We define a Riemannian metric $g$ by the usual formula:

$$g(v, w) = \omega(v, Jw).$$

Hence, the tangent space $T_pX$ is a Hermitian vector space.

Let $\mathbb{B}(r) \subset \mathbb{C}^n$ denote the ball of radius $r$.

**Definition 20.** A framed chart is a pair $(\varphi, \tau)$ where $\varphi : \mathbb{B}(r) \to X$ is a holomorphic chart and $\tau : \mathbb{C} \times \mathbb{B}(r) \to \varphi^*L$ is a holomorphic trivialization of the pull-back line bundle $\varphi^*L$.

We denote by $\tau^k$ and $m\tau^k$ the corresponding trivializations of the line bundle $\varphi^*L^\otimes k$ and of the vector bundle $\varphi^*(\mathbb{C}^m \otimes L^\otimes k)$.

Let $s_k$ be a holomorphic (or smooth) section of $\mathbb{C}^m \otimes L^\otimes k$. Via the trivialization $m\tau^k$, the pull-back section $\varphi^*s_k$ defines a holomorphic map $M(s_k, \tau) : \mathbb{B}(r) \to \mathbb{C}^m$. Then, the renormalized map $R(s_k, \tau) : \mathbb{B}(\sqrt{kr}) \to \mathbb{C}^m$ is defined as the composite map of $M(s_k, \tau)$ with the homothety $\mathbb{B}(\sqrt{kr}) \to \mathbb{B}(r)$ of ratio $\frac{1}{\sqrt{k}}$. We use shorter notations $M_{sk}$ and $R_{sk}$ instead of $M(s_k, \tau)$ and $R(s_k, \tau)$ whenever $\tau$ is clear from the context. Hence:

$$R_{sk}(v) = M_{sk}\left(\frac{v}{\sqrt{k}}\right)$$

for all $v \in \mathbb{B}(\sqrt{kr})$.

Recall that $L$ is endowed with a Hermitian metric. Denote by $e^{-h}$ the metric on the trivial bundle $\mathbb{C} \times \mathbb{B}(r) \to \mathbb{B}(r)$ pull-back of the metric of $L$ via the trivialization $\tau$. This defines a smooth map $h : \mathbb{B}(r) \to \mathbb{R}$.

**Definition 21.** The framed chart $(\varphi, \tau)$ is standard at the origin if the following two conditions hold:

(a) The tangent map $d\varphi(0) : \mathbb{C}^n \to T_{\varphi(0)}X$ is an isometry of Hermitian vector spaces, where $T_{\varphi(0)}X$ is endowed with the metric $g_{\varphi(0)}$ and $\mathbb{C}^n$ with the standard metric.

(b) The function $h$ has a non-degenerate maximum at 0 with value $h(0) = 0$ and the second derivative $d^2\overline{h}(0)$ satisfies: $d^2\overline{h}(0)(v, w) = \pi\langle v, w \rangle$ for all vectors $v, w \in \mathbb{C}^n$ where $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ denotes the usual inner product.

Notice that condition (b) implies that: $h(v) = \frac{\pi}{4}\|v\|^2 + o(\|v\|^2)$ and hence:

$$\frac{\pi}{4}\|v\|^2 \leq h(v) \leq \frac{3\pi}{4}\|v\|^2$$
on a neighborhood of the origin.

For every \( p \in X \), there exists a framed chart \((\varphi_p, \tau_p)\) centered at \( p \), standard at the origin, because the curvature of \( L \) equals \(-i2\pi\omega\).

**Definition 22.** For every index \( i \) in a set \( I \), let \( \varphi_i : \mathbb{B}(r) \to X \) be a holomorphic charts defined on a ball \( \mathbb{B}(r) \) independent of \( i \). The family \((\varphi_i)_{i \in I}\) is normal if the charts \((\varphi_i)_{i \in I}\) form a relatively compact subset in the set of maps \( \mathbb{B}(r) \to X \) for the smooth compact-open topology.

Similarly, let \((\varphi_i, \tau_i)\) be framed charts defined on a ball \( \mathbb{B}(r) \). For every \( i \in I \), the map \( \tau_i : \mathbb{C} \times \mathbb{B}(r) \to \varphi_i^*L \) induces a map \( \tilde{\tau}_i : \mathbb{C} \times \mathbb{B}(r) \to L \) where we denote by \( L \) the total space of the line bundle. The family of framed charts \((\varphi_i, \tau_i)_{i \in I}\) is normal if the maps \((\tilde{\tau}_i)_{i \in I}\) form a relatively compact family in the set of maps \( \mathbb{C} \times \mathbb{B}(r) \to L \) for the smooth compact-open topology.

### 5.2 Renormalizable sequences of sections

In Section 1.1, we have defined renormalizable sequences of submanifolds. Similarly, we will define renormalizable sequences of sections.

**Definition 23.** For every sufficiently large \( k \), let \( s_k \) be a holomorphic section of the vector bundle \( \mathbb{C}^m \otimes L^{\otimes k} \) where the rank \( m \) is independent of \( k \).

The sequence \((s_k)_{k \geq 1}\) is renormalizable if for all normal sequence of framed charts \((\varphi_k, \tau_k)_{k \in I}\) standard at the origin (defined on a ball \( \mathbb{B}(r) \) and indexed by some infinite set \( I \) of sufficiently large integers), the corresponding family \((R_{s_k} : \mathbb{B}(\sqrt{k}r) \to \mathbb{C}^m)_{k \in I}\) of rescaled maps is normal.

The limits of (subsequences of) the normal family \((R_{s_k} : \mathbb{B}(\sqrt{k}r) \to \mathbb{C}^m)_{k \in I}\) are holomorphic maps \( s_\infty : \mathbb{C}^n \to \mathbb{C}^m \). Such a \( s_\infty \) is called a limit section.

**Remark.** The map \( s_\infty : \mathbb{C}^n \to \mathbb{C}^m \) is a section of the trivial bundle \( \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n \). This trivial bundle should be viewed as an "infinitesimal bundle". Though this bundle is trivial, it is naturally endowed with the non-trivial Hermitian metric \( e^{-h_\infty} \) where the map \( h_\infty : \mathbb{C}^n \to \mathbb{R} \) is defined by the formula \( h_\infty(v) = \frac{\|v\|^2}{2} \). Hence, the norm of a constant section \( s \) of \( \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n \) is a Gaussian function \( \|s(v)\| = \|s(0)\| \exp(-\frac{\|v\|^2}{2}) \).

A sequence \((s_k)_{k \geq 1}\) of sections is renormalizable if and only if, for every \( l \), the corresponding sequence \((\|s_k\| \|v\|)_{k \geq 1}\) of norms \( \mathcal{C}^l \) measured with the rescaled metric \( g_k = k \cdot g \) is bounded. Of course, the definition of the norm \( \mathcal{C}^l \) for sections uses connections. We use the Levi-Civita connection over \( X \) and the Chern connection on \( \mathbb{C}^m \otimes L^{\otimes k} \). Notice that both metrics \( g \) and \( g_k \) induce the same Levi-Civita connection.

### 5.3 Avoidance theorem for sections

The jet of order \( l \) at the origin of a (germ of) holomorphic map \( \mathbb{C}^n \to \mathbb{C}^m \) is a polynomial map \( \mathbb{C}^n \to \mathbb{C}^m \) of degree \( \leq l \). Hence, the space \( \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m) \) of
Theorem 24. Let $X$ be a projective manifold of dimension $n$ and let $L \to X$ be an ample line bundle endowed with a Hermitian metric of positive curvature. Let $A \subset \mathbb{C}^n \times \operatorname{Pol}_l(\mathbb{C}^n,\mathbb{C}^m)$ be a closed complex algebraic subset of codimension $\geq n + 1$ satisfying the following invariance conditions:

1. $A$ is invariant under the natural action of affine unitary transformations of $\mathbb{C}^n$ upon $\mathbb{C}^n \times \operatorname{Pol}_l(\mathbb{C}^n,\mathbb{C}^m)$.
2. Let $f : (\mathbb{C}^n, p) \to \mathbb{C}^m$ be the $l$--jet of a holomorphic map and let $u : (\mathbb{C}^n, p) \to \mathbb{C}$ be the $l$--jet of a holomorphic function. If $f$ belongs to $A$ then $uf$ belongs to $A$.

Then for every sufficiently large $k$, there exists a holomorphic section $s_k$ of $\mathbb{C}^m \otimes L^\otimes k$ such that the sections $(s_k)_{k \gg k}$ form a renormalizable sequence which avoids $A$ asymptotically in the sense that the jets of order $l$ of its limit sections lie in the complement $(\mathbb{C}^n \times \operatorname{Pol}_l(\mathbb{C}^n,\mathbb{C}^m)) \setminus A$.

This result has similarities with the main theorem in [2]. We will give the proof of Theorem 24 in section 5.4. Then, in section 5.5 we will prove that this theorem implies Theorem 3

5.4 Donaldson’s construction

Here, we prove Theorem 24

First, we will define tools that we will use in order to prove Theorem 24. For every $p \in X$, there exist a framed chart $(\varphi_p, \tau_p)$ centered at $p$ and standard at the origin. Moreover, if we denote by $e^{-h}$ the pull-back Hermitian metric on the trivial bundle $\mathbb{C} \times B(r) \to B(r)$, we can assume that the map $h : B(r) \to \mathbb{R}$ satisfies the condition $\frac{n}{4k}\|v\|^2 \leq h(v) \leq \frac{2n}{k}\|v\|^2$ on $B(r)$.

Since $X$ is compact, we can assume that the charts $(\varphi_p : B(r) \to X)_{p \in X}$ are defined on a ball $B(r)$ whose radius does not depend on $p$ and we can assume that the framed charts $(\varphi_p, \tau_p)_{p \in X}$ form a normal family.

If $s_k$ is a holomorphic section of $\mathbb{C}^m \otimes L^\otimes k$, we denote by $R_p s_k$ the corresponding renormalized holomorphic map $B(\sqrt{k}r) \to \mathbb{C}^m$ with respect to the chart $\varphi_p$ and the trivialization $\tau_p$. The jet of order $l$ at the origin of $R_p s_k$ is a polynomial map $H : \mathbb{C}^n \to \mathbb{C}^m$ of degree $\leq l$.

We fix a norm $\|\cdot\|$ on $\operatorname{Pol}_l(\mathbb{C}^n,\mathbb{C}^m)$ (they are all equivalent). The Riemannian metric $g$ induces a renormalized metric $g_k = kg$ and we denote by $d$ and $d_k = \sqrt{k}d$ the corresponding distance functions.

Lemma 25. For every sufficiently large $k$ and for every $H \in \operatorname{Pol}_l(\mathbb{C}^n,\mathbb{C}^m)$, there exists a global holomorphic section $\sigma_k = \sigma_k(H, p)$ satisfying the following conditions:
1. The jet of order \( l \) of \( R_p \sigma_k \) at the origin equals \( H \).

2. The mass of \( \sigma_k \) is concentrated around \( p \), in the sense that, for every \( l' \), the norm of the jet of order \( l' \) of \( \sigma_k \) at \( q \) is bounded by:

\[
C \| H \| \exp \left( -\frac{kd(p, q)^2}{C} \right)
\]

where the constant \( C \) only depends on \( l \) and \( l' \) (in particular, \( C \) is independent of \( k \) and \( H \)).

Proof. Let \( \tilde{\sigma}_k = \tilde{\sigma}_k(H, p) \) be the smooth section of \( \mathbb{C}^m \times L^{\otimes k} \) defined by the following two conditions:

1. The rescaled smooth map \( R_p \tilde{\sigma}_k : \mathbb{B}(\sqrt{k}r) \to \mathbb{C}^m \) satisfies:

\[
R_p \tilde{\sigma}_k(v) = \rho \left( \frac{v}{\sqrt{k}} \right) H(v)
\]

where \( \rho : \mathbb{B}(r) \to \mathbb{R}_{\geq 0} \) is a cut-off function.

2. We extend \( \tilde{\sigma}_k \) by 0 on the complement \( X \setminus \varphi(\mathbb{B}(r)) \).

Recall that the pull-back metric \( e^{-h} \) of the metric of \( L \) satisfies \( \frac{4}{r^2} \| v \|^2 \leq h(v) \leq \frac{4}{r^2} \| v \|^2 \) on \( \mathbb{B}(r) \). The pull-back of the metric of \( L^{\otimes k} \) is \( e^{-kh} \) and the pull-back of the metric of \( \mathbb{C}^m \otimes L^{\otimes k} \) equals \( e^{-kh} \langle \ldots \rangle \) where \( \langle \ldots \rangle \) is the usual inner product of \( \mathbb{C}^m \). Hence, by the Leibniz rule, the norm of the jet of order \( l' \) of \( \tilde{\sigma}_k \) at \( q \) is bounded by \( C \| H \| \exp \left( -\frac{kd(p, q)^2}{C} \right) \). Here, we measure the norms of jets with the metric \( g_k \).

The support \( \text{supp} (\overline{\partial} \tilde{\sigma}_k) \) is contained in the image \( \varphi(\text{supp}(d\rho)) \). Hence, every \( q \in \text{supp} (\overline{\partial} \tilde{\sigma}_k) \) satisfies \( d(p, q) > \frac{1}{C} \) for a constant \( C > 0 \) and the norm of the jet of order \( l' \) of \( \overline{\partial} \tilde{\sigma}_k \) at \( q \) is bounded by \( C \| H \| \exp \left( -\frac{k}{C} \right) \) for a (new) constant \( C \).

We solve the Cauchy-Riemann equation \( \overline{\partial} \xi_k = -\overline{\partial} \tilde{\sigma}_k \). There exists a section \( \xi_k = \xi_k(H, p) \) such that \( \tilde{\sigma}_k + \xi_k \) is holomorphic and:

\[
\| \xi_k \|_{L^2(X, g_k)} \leq C \| \overline{\partial} \tilde{\sigma}_k \|_{L^2(X, g_k)} \leq C \| H \| \exp \left( -\frac{k}{C} \right)
\]

(estimate (35) in [6]). A priori, the result from [6] holds for sections of the line bundle \( L^{\otimes k} \) but, clearly, we can extend this result to sections of the direct sum \( \mathbb{C}^m \otimes L^{\otimes k} \). We can assume that \( \xi_k(H, p) \) depends linearly on \( H \) (choose values on a basis of \( \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \) and take linear combinations). We set \( \overline{\sigma}_k = \tilde{\sigma}_k + \xi_k \). When \( k \) tends to \( \infty \), the norm \( L^2 \) of \( \xi_k \) and the norms \( C' \) of \( \overline{\partial} \xi_k \) decrease exponentially. Recall the following elliptic estimate:

\[
\| \xi_k \|_{C'(X, g_k)} \leq C \left( \| \xi_k \|_{L^2(X, g_k)} + \| \overline{\partial} \xi_k \|_{C'(X, g_k)} \right).
\]
Hence, the norm $C'$ of $\xi_k$ satisfies a similar estimate:

$$\|\xi_k\|_{C'(X,g_k)} \leq C\|H\| \exp\left(-\frac{k}{C}\right).$$

Collecting the estimates of $\tilde{\sigma}_k$ and $\xi_k$, we obtain the following bound for the norm of the jet of $\sigma_k = \tilde{\sigma}_k + \xi_k$ at $q$:

$$C\|H\| \exp\left(-\frac{kd(p,q)^2}{C}\right)$$

where $C$ is another constant.

Because the $C'$-norm of $\xi_k$ is bounded by $C\|H\| \exp\left(-\frac{k}{C}\right)$, the jet of the renormalized holomorphic map $R_p\sigma_k(H,p)$ at the origin is closed but not equal to $H$. Hence, if we denote by $I_k(H)$ the jet of $R_p\sigma_k(H,p)$ at the origin, the linear map $I_k : \mathcal{P}ol_l(\mathbb{C}^n,\mathbb{C}^m) \to \mathcal{P}ol_l(\mathbb{C}^n,\mathbb{C}^m)$ lies in a small neighborhood (of radius $C\exp\left(-\frac{k}{C}\right)$) of the identity map if $k$ is sufficiently large. In particular $I_k$ is an isomorphism of bounded distortion and we can set:

$$\sigma_k(H,p) = \sigma_k(I_k^{-1}(H),p).$$

The mass of the section $\sigma_k(H,p)$ is concentrated around the center of the chart $\varphi_p$. The main tools in Donaldson’s construction are suitable combinations of such concentrated sections.

For every $k$, let $X_k \subset X$ be a $g_k$-discretization of $X$, that is, a finite subset satisfying the following two conditions:

1. For every $p \in X$, there exists $q \in X_k$ such that $d_k(p,q) < 1$
2. Every $p$ and $q \in X_k$ satisfy $d_k(p,q) \geq 1$

where $d_k = \sqrt{k}d$ is the distance function of the rescaled metric $g_k$.

**Lemma 26.** For every sufficiently large $k$ and for every $p \in X_k$, let $H_{k,p} \in \mathcal{P}ol_l(\mathbb{C}^n,\mathbb{C}^m)$ be a polynomial map such that $\|H_{k,p}\| \leq 1$. We define a section $s_k$ of $\mathbb{C}^m \otimes L^\otimes k$ by the following formula:

$$s_k = \sum_{p \in X_k} \sigma_k(H_{k,p},p).$$

Then the sections $(s_k)_{k \geq 1}$ form a renormalizable sequence. In particular, if $K \subset \mathbb{C}^n$ is a compact subset then, for every $q \in X$ and $z \in K$, the norm of the jet of order $l$ of $R_q s_k$ at $z$ is bounded by a constant independent of $k$, $q$ and $z$.

**Proof.** The norm of the jet of order $l'$ of $\sigma_k(H_{k,p},p)$ at $q$ is bounded by $C \exp\left(-\frac{d_k(p,q)^2}{C}\right)$. Hence the norm of the jet of $s_k$ is bounded by:

$$C \sum_{p \in X_k} \exp\left(-\frac{d_k(p,q)^2}{C}\right).$$
We fix \( q \in X \) and we define a partition:

\[
X_k = \bigcup_{a \geq 0} X_{k,a}
\]

where, for every integer \( a \), the subset \( X_{k,a} \) is the set of points \( p \in X_k \) satisfying \( a \leq d_k(p, q) < a + 1 \). By a volume argument, the cardinality of \( X_{k,a} \) is bounded by a polynomial \( P(a) \) where \( P \) is independent of \( k \) and \( q \). The norm of the jet of \( s_k \) is bounded by:

\[
C \sum_{a \geq 0} \sum_{p \in X_{k,a}} \exp \left( -\frac{d_k(p, q)^2}{C} \right) \leq C \sum_{a \geq 0} \exp \left( -\frac{a^2}{C} \right) P(a).
\]

This bound is finite and independent of \( k \).

Remark. If \( H_{k,p} = 0 \) for every \( p \) such that \( d_k(p, q) \leq D \), then the norm of the \( l \)-jet of \( R_q s_k \) at \( z \) is bounded by:

\[
C \sum_{a \geq D-1} \exp \left( -\frac{a^2}{C} \right) P(a)
\]

and hence by:

\[
C \exp \left( -\frac{D^2}{C} \right)
\]

where \( C \) is another constant.

As we have seen, the space of jets of order \( l \) of holomorphic maps \( \mathbb{C}^n \to \mathbb{C}^m \) equals \( \mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \). We define the (fibered) distance between two subsets \( A \) and \( B \) of \( \mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \). By definition, the distance \( \text{dist}(A, B) \) is the \emph{infimum} of the distances between pairs of points \( (z, H_1) \in A \) and \( (z, H_2) \in B \) having the same projection \( z \in \mathbb{C}^n \).

**Lemma 27.** Let \( A \subset \mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \) be a closed complex algebraic subset of codimension \( \geq n+1 \) and let \( K \subset \mathbb{C}^n \) be a compact subset. For every sufficiently large \( k \) and for every \( p \in X_k \), let \( H_{k,p} \in \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \) be a polynomial map such that \( \|H_{k,p}\| \leq 1 \). We set:

\[
s_k = \sum_{p \in X_k} \sigma_k(H_{k,p}, p).
\]

Then, for every \( q \in X \) and \( 0 < \varepsilon < \frac{1}{4} \), there exists a polynomial map \( H \in \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \) satisfying \( \|H\| \leq \varepsilon \) such that, setting

\[
\tilde{s}_k = s_k + \sigma_k(H, q),
\]

the distance between \( A \) and the set \( B \) of jets of order \( l \) of \( R_q \tilde{s}_k \) at points in \( K \) satisfies the following estimate:

\[
\text{dist}(A, B) \geq \varepsilon (-\log \varepsilon)^{-N}
\]

where \( N \) is independent of \( k \) and \( q \) and of the polynomial maps \( H_{k,p} \).
Proof. We can assume (WLOG) that $K$ is a polydisk. We approximate the holomorphic map $f = R_q s_k : B(\sqrt{K_F}) \to \mathbb{C}^m$ by its polynomial expansion $\overline{f}$. More precisely, for every $0 < \eta < \frac{1}{4}$, there exists a polynomial map $\overline{f} : \mathbb{C}^n \to \mathbb{C}^m$ satisfying the following two conditions:

1. For every $z \in K$, the Taylor polynomial $T_l(f - \overline{f}, z)$ of order $l$ of the map $f - \overline{f}$ at $z$ satisfies the following estimate:
   \[ \|T_l(f - \overline{f}, z)\| \leq \eta. \]

2. The degree of $\overline{f}$ is $\leq (-\log \eta)^N$ where $N$ is independent of $k, q$ and of the maps $H_{k,p}$.

For every $z \in \mathbb{C}^n$ and every $H \in \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)$, we set:
\[ \psi(z, H) = H - T_l(\overline{f}, z). \]

The map $\psi : \mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m) \to \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)$ is a polynomial map of degree $\leq (-\log \eta)^N$.

Notice that, since:
\[ \dim(A) \leq \dim(\mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)) - (n + 1) < \dim(\text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)), \]

the restriction map $\psi_A$ isn’t surjective and $\psi(A)$ is contained in a hypersurface $\mathcal{H} \subset \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)$ whose degree is a polynomial function of the degree of $\psi$, say $(-\log \eta)^N$ where $N$ is another constant.

We denote by $B(\varepsilon) \subset \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)$ the ball of radius $\varepsilon$. The $2\eta$–neighborhood $\mathcal{H}_{2\eta}$ of $\mathcal{H}$ satisfies the following estimate:
\[ \text{vol}(\mathcal{H}_{2\eta} \cap B(\varepsilon)) \leq C \eta \varepsilon^{2\delta - 1} (\deg \mathcal{H})^C \leq C \eta \varepsilon^{2\delta - 1} (-\log \eta)^CN \]

where $C$ is a constant and $\delta$ is the complex dimension of $\text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)$.

By a simple volume argument, if we set $\eta = \varepsilon(-\log \varepsilon)^{-N}$ where $N$ is sufficiently large then there exists $H \in \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)$ such that $\|H\| < \varepsilon$ and $H \not\in \mathcal{H}_{2\eta}$.

We set $\overline{s}_k = s_k + \sigma_k(H, q)$. For every $z$:
\[ T_l(R_q \overline{s}_k, z) = H + T_l(R_q s_k, z) = H + T_l(f, z). \]

Then, for every $(z, H_1) \in A$:
\[ \|T_l(R_q \overline{s}_k, z) - H_1\| = \|H + T_l(f, z) - H_1\| \geq \|H - \psi(z, H_1)\| - \|T_l(f - \overline{f}, z)\| \geq 2\eta - \eta = \eta = \varepsilon(-\log \varepsilon)^{-N}. \]

Moreover, we can assume that $N$ is independent of the polynomial maps $H_{k,p}$. Indeed, if we consider all maps $s_k$, for all sufficiently large $k$ and for
all polynomial maps \((H_{k,p})_{p \in X_k}\) such that \(\|H_{k,p}\| \leq 1\), then the corresponding maps \((R_q s_k)\) form a normal family and so we can assume that every constant in this proof is independent of the maps \(H_{k,p}\).

\[\text{Lemma 28.}\]

Let \(A \subset \mathbb{C}^n \times \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m)\) be a closed complex algebraic subset of codimension \(\geq n + 1\) and let \(K \subset \mathbb{C}^n\) be a compact subset.

Then for every sufficiently large \(k\), there exists a holomorphic section \(s_k\) of \(\mathbb{C}^m \otimes L \otimes k\) such that the sections \((s_k)_{k \geq h}\) form a renormalizable sequence and, for every \(q \in X_k\), the distance between \(A\) and the set \(B\) of jets of order \(l\) of \(R_q s_k\) at points in \(K\) satisfies \(\text{dist}(A,B) \geq \varepsilon\) where \(\varepsilon > 0\) is independent of \(k\) and \(q\).

\[\text{Proof.}\]

We will construct a section:

\[s_k = \sum_{p \in X_k} \sigma_k(H_{k,p}, p)\]

where the maps \(H_{k,p}\) are suitable elements in \(\mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m)\). The construction follows the lines of [6] and we will skip some calculations.

For every \(D > 0\), there exists a partition:

\[X_k = \bigcup_{1 \leq i \leq i_D} X_{k,i}\]

such that every pair of points \(p, q\) in the same subset \(X_{k,i}\) satisfies \(d_k(p, q) \geq D\) and the number \(i_D\) of subsets in the partition is a polynomial function of \(D\) independent of \(k\).

If \(p\) lies in \(X_{k,i}\), we consider the following partition:

\[X_k = \{p\} \cup S_1 \cup S_2 \cup S_3\]

where:

\[S_1 = \bigcup_{j < i} X_{k,j}\]

\[S_2 = X_{k,i} \setminus \{p\}\]

\[S_3 = \bigcup_{j > i} X_{k,j}\]

Hence:

\[s_k = \sigma_k(H_{k,p}, p) + s_k^1 + s_k^2 + s_k^3\]

where \(s_k^l = \sum_{q \in S_l} \sigma_k(H_{k,q}, q)\).

Let \(\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{i_D}\) be a sequence of sufficiently small positive numbers. By Lemma 27 for every sufficiently large \(k\), there exists a family \((H_{k,p})_{p \in X_k}\) of polynomial maps in \(\mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m)\) satisfying the following two conditions, for every \(p \in X_{k,i}\):

1. \(\|H_{k,p}\| \leq \varepsilon_i\).
2. For every $z \in K$:

$$\text{dist}(A_z, T_l(R_p(s_k^1 + \sigma_k(H_{k,p}, p))), z)) \geq \varepsilon_i(- \log \varepsilon_i)^{-N}$$

where $A_z = \{ H \in \mathcal{P}(\mathbb{C}^n, \mathbb{C}^m), (z, H) \in A \}$ and, as usual, the polynomial map $T_l(R_p(s_k^1 + \sigma_k(H_{k,p}, p)), z) \in \mathcal{P}(\mathbb{C}^n, \mathbb{C}^m)$ is the $l$-jet of the map $R_p(s_k^1 + \sigma_k(H_{k,p}, p))$ at $z$.

By Lemma 26

$$\|T_l(R_p s_{k^3}, z)\| \leq C \varepsilon_{i+1}$$

Similarly, by the remark following Lemma 26

$$\|T_l(R_p s_{k^2}, z)\| \leq C \varepsilon_i \exp\left(- \frac{D^2}{C}\right)$$

because, for every $q \in X_{k,i} \setminus \{p\}$:

$$d_k\left(\varphi_p\left(\frac{z}{\sqrt{k}}\right), q\right) \geq d_k(p, q) - d_k\left(\varphi_p\left(\frac{z}{\sqrt{k}}\right), p\right) \geq D - C$$

where the constant $C$ is independent of $D$.

Since $i_D$ is a polynomial function of $D$, a calculation shows that for every $C$ and $N$, if $D$ is sufficiently large then there exists a sequence $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{i_D}$ satisfying the following two estimates:

$$C \varepsilon_{i+1} \leq \frac{1}{4} \varepsilon_i(- \log \varepsilon_i)^{-N}$$

$$C \exp\left(- \frac{D^2}{C}\right) \leq \frac{1}{4}(- \log \varepsilon_i)^{-N}.$$ 

Hence, if $p \in X_{k,i}$:

$$\text{dist}(A_z, T_l(R_p s_k, z)) \geq \text{dist}(A_z, T_l(R_p(s_k^1 + \sigma_k(H_{k,p}, p)), z))$$

$$- \|T_l(R_p s_{k^2}, z)\| - \|T_l(R_p s_{k^3}, z)\|$$

$$\geq \frac{1}{2} \varepsilon_i(- \log \varepsilon_i)^{-N}.$$ 

The infimum of the right-hand side for $1 \leq i \leq i_D$ is positive and independent of $k \gg 1$, $p \in X_k$ and $z \in K$.

Now, we will complete the proof of Theorem 24 The trivial holomorphic line bundle $\mathcal{L} : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ is endowed with the non-trivial Hermitian metric $\exp\left(- \frac{\pi}{4} \|z\|^2\right)$. Let $u : \mathbb{C}^n \to \mathbb{C}^n$ be an affine unitary transformation:

$$u(z) = \alpha(z) + \beta.$$ 

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A lift of \( u \) is an automorphism \( \bar{u} \) of the Hermitian bundle \( \mathcal{L} \):

\[
\bar{u}(w, z) = \left( \lambda w \exp \left( \frac{\pi}{2} \| \alpha^{-1} (\beta) \|^2 + \pi (\alpha^{-1} (\beta), z) \right), u(z) \right)
\]

where \( w \in \mathbb{C}, \ z \in \mathbb{C}^n \) and \( \lambda \) is a unit complex number. Since the Hermitian inner product \( \langle \cdot, \cdot \rangle \) is linear in the second argument, the map:

\[
z \mapsto \lambda \exp \left( \frac{\pi}{2} \| \alpha^{-1} (\beta) \|^2 + \pi (\alpha^{-1} (\beta), z) \right)
\]

is holomorphic. Hence, if a subset \( A \subset \mathbb{C}^n \times \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m) \) satisfies the invariance hypotheses of Theorem 24 then, in particular, \( A \) is invariant under the action of \( \bar{u} \) upon \( \mathbb{C}^n \times \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m) \).

Let \( A \) be a subset of \( \mathbb{C}^n \times \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m) \) satisfying the hypotheses of Theorem 24 and let \( \mathbb{B} \subset \mathbb{C}^n \) be the closed unit ball. We apply Lemma 28 to \( A \) and \( K = \mathbb{B} \). There exists a renormalizable sequence of holomorphic sections \( (s_k)_{k \geq 1} \) such that every limit sections \( s_\infty \) of the sequence of renormalized maps \( (R(s_k, \tau_{p_k}))_{k \geq 1} \) avoids \( A \cap (\mathbb{B} \times \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m)) \), if \( p_k \) lies in \( X_k \). Hence, if \( z \in \mathbb{B} \) then \( T_l(s_\infty, z) \notin A \).

We will prove that a similar result holds if we replace the subset \( A \cap (\mathbb{B} \times \mathcal{P}ol_l(\mathbb{C}^n, \mathbb{C}^m)) \) with \( A \) and if we consider any other normal sequence of framed charts \( (\hat{\varphi}_k, \hat{\tau}_k)_{k \geq 1} \) standard at the origin.

Let \( \hat{s}_\infty \) be a limit section of the normal sequence \( (R(s_k, \hat{\tau}_k))_{k \geq 1} \) and let \( \hat{z}_0 \in \mathbb{C}^n \) be a point. We set \( \hat{\rho}_k = \hat{\varphi}^{-1}_k \left( \frac{\hat{z}}{\sqrt{k}} \right) \). Since \( X_k \) is a \( g_k \)-discretization, there exists a point \( p_k \in X_k \) satisfying \( d_k(p_k, \hat{\rho}_k) \leq 1 \). Let \( u_k \) be the renormalized change of coordinates:

\[
u_k(z) = \sqrt{k} \hat{\varphi}^{-1}_k \left( \frac{z}{\sqrt{k}} \right).
\]

The local biholomorphisms \( (u_k)_{k \geq 1} \) form a normal family and every limit \( u_\infty \) is an affine unitary transformation \( \mathbb{C}^n \to \mathbb{C}^n \).

Let \( \bar{u}_k \) denote the transition map between the trivializations

\[
z \mapsto \hat{\tau}_k \left( \frac{z}{\sqrt{k}} \right) \quad \text{and} \quad z \mapsto \tau_{p_k} \left( \frac{z}{\sqrt{k}} \right).
\]

Similarly, the maps \( (\bar{u}_k)_{k \geq 1} \) form a normal family and every limit \( \bar{u}_\infty \) is the lift of a limit \( u_\infty \) of \( (u_k)_{k \geq 1} \).

The map \( \hat{s}_\infty \) is the limit of a convergent subsequence \( (R(s_k, \hat{\tau}_k))_{k \in I} \). We can assume that the corresponding sequences \( (u_k)_{k \in I} \) and \( (\bar{u}_k)_{k \in I} \) converge to \( u_\infty \) and \( \bar{u}_\infty \). Moreover, since \( d_k(p_k, \bar{\rho}_k) \leq 1 \) for all \( k \in I \), we can assume that the points \( (u_k(\hat{z}_0))_{k \in I} \) converge to some point \( z_0 \in \mathbb{B} \subset \mathbb{C}^n \).

Then the image of \( T_l(\hat{s}_\infty, \hat{z}_0) \) under the action of \( \bar{u}_\infty \) equals \( T_l(s_\infty, z_0) \) where \( s_\infty \) is a limit map of the sequence \( (R(s_k, \tau_{p_k}))_{k \geq 1} \). By definition of the sequence \( (s_k)_{k \geq 1} \), the jet \( T_l(s_\infty, z_0) \) does not lie in \( A \). Since \( A \) is invariant under the action of \( \bar{u}_\infty \), we conclude that \( T_l(s_\infty, z_0) \notin A \). The sequence \( (s_k)_{k \geq 1} \) avoids \( A \) asymptotically and the proof of Theorem 24 is completed.
5.5 Proof of the avoidance theorem for submanifolds

Here, we complete the proof of Theorem 9.

**Definition 29.** For every sufficiently large $k$, let $s_k$ be a section of $\mathbb{C}^m \otimes L^\otimes k$ such that the sections $(s_k)_{k \geq 1}$ form a renormalizable sequence. The sequence $(s_k)_{k \geq 1}$ is transverse to 0 asymptotically if every limit section $s_\infty : \mathbb{C}^n \to \mathbb{C}^m$ is transverse to 0.

In order to construct submanifolds of dimension $d$, we set $m = n - d$.

**Lemma 30.** For every sufficiently large $k$, let $s_k$ be a holomorphic section of $\mathbb{C}^{n-d} \otimes L^\otimes k$ such that the sections $(s_k)_{k \geq 1}$ form a renormalizable sequence transverse to 0 asymptotically.

Then, for every sufficiently large $k$, the section $s_k$ is transverse to 0 and the zero sets $(Y_k = \{s_k = 0\})_{k \geq 1}$ form a renormalizable sequence of complete intersections of dimension $d$. Moreover, for every limit submanifold $Y_\infty$ of $(Y_k)_{k \geq 1}$, there exists a limit section $s_\infty$ of $(s_k)_{k \geq 1}$ such that $Y_\infty$ equals $\{s_\infty = 0\}$, up to a linear transformation of $\mathbb{C}^n$.

**Proof.** First, we will prove by contradiction that, for every sufficiently large $k$, the section $s_k$ is transverse to 0. Assume that, for infinitely many $k$, there exists a point $p_k \in \{s_k = 0\}$ such that $s_k$ is not transverse to 0 at $p_k$. For every sufficiently large $k$, let $(\varphi_k, \tau_k)$ be a framed chart centered at $p_k$ standard at the origin. Assume that the framed charts $(\varphi_k, \tau_k)_{k \geq 1}$ form a normal family. We denote by $R s_k$ the corresponding renormalized map. A subsequence $(R s_k)_{k \in I}$ tends to a map $s_\infty$ transverse to 0. Hence, for every sufficiently large $k \in I$, the map $R s_k$ is transverse to 0, say at the origin. On the other hand, $s_k$ is not transverse to 0 at $p_k = \varphi_k(0)$. This is a contradiction.

Now, since $s_k$ is transverse to 0, the complete intersection $Y_k = \{s_k = 0\}$ is smooth. We will prove that the submanifolds $(Y_k)_{k \geq 1}$ form a renormalizable sequence.

For every large $k$, let $\hat{\varphi}_k : \mathbb{B} \to X$ be a holomorphic chart such that the charts $(\hat{\varphi}_k)_{k \geq 1}$ form a normal family and satisfy the non-degeneracy condition of Definition 7.

We didn’t assume that $d\hat{\varphi}_k(0)$ is unitary. Nonetheless, we write $\hat{\varphi}_k = \varphi_k \circ u_k$ were $u_k$ is a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$ and $\varphi_k$ is a holomorphic map such that $d\varphi_k(0)$ is unitary. The non-degeneracy condition implies that we can assume that the sequence $(u_k)_{k \geq 1}$ is relatively compact in $GL(n, \mathbb{C})$. Hence, the chart $\varphi_k$ is defined on an ellipsoid containing a ball $\mathbb{B}(r)$ whose radius $r$ is independent of $k$ and the charts $(\varphi_k : \mathbb{B}(r) \to X)_{k \geq 1}$ form a normal family. Notice that every limit submanifold $\hat{Y}_\infty$ defined by using the charts $(\hat{\varphi}_k)_{k \geq 1}$ is the image $u_\infty(Y_\infty)$ of a limit submanifold $Y_\infty$ defined by using the charts $(\varphi_k)_{k \geq 1}$, where $u_\infty$ is a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$.

Now we consider the charts $(\varphi_k)_{k \geq 1}$ and we will show that the corresponding renormalized submanifolds $(R Y_k)_{k \geq 1}$ form a normal sequence.

For every $k$, there exist a holomorphic trivialization $(\tau_k : \mathbb{C} \times \mathbb{B}(r) \to \varphi_k^* L)_{k \geq 1}$ such that the framed chart $(\hat{\varphi}_k, \tau_k)$ is standard at the origin and the
family \((\varphi_k, r_k)_{k \geq 1}\) is normal. As usual, we denote by \(R_{s_k} : B(\sqrt{k}r) \to \mathbb{C}^m\) the corresponding renormalized map.

Since the sequence \((s_k)_{k \geq 1}\) is renormalizable, every subsequence of \((R_{s_k})_{k \geq 1}\) admits a subsubsequence which converges to a limit map \(s_\infty : \mathbb{C}^n \to \mathbb{C}^m\). By assumption, \(s_\infty\) is transverse to \(0\). Hence the submanifolds \(R_{s_k} = \{R_{s_k} = 0\}\) converge to the submanifold \(\{s_\infty = 0\} \subset \mathbb{C}^n\) in the smooth compact-open topology.

**Lemma 31.** Let \(A \subset \text{Jet}^l_{d, n}\) be a set of \(l\)-jets of complex submanifolds, which is a closed complex algebraic subset of \(\text{Jet}^l_{d, n}\). We set \(m = n - d\). Then there exists a set \(B\) of \(l\)-jets of holomorphic maps \(\mathbb{C}^n \to \mathbb{C}^m\), which is a closed complex algebraic subset of \(\mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)\) of codimension \(\geq \min\{n + 1, n - d + \text{codim}(A)\}\) and satisfies the following property:

If \(s : (\mathbb{C}^n, p) \to \mathbb{C}^m\) is a germ of holomorphic map whose \(l\)-jet doesn’t lie in \(B\) then \(s\) is transverse to \(0\) and, setting \(Y = \{s = 0\}\), the \(l\)-jet of the submanifold \(Y\) at \(p\) doesn’t lie in \(A\).

Moreover, we can assume that:

1. If \(A\) is invariant under the natural action of affine transformations of \(\mathbb{C}^n\) upon \(\text{Jet}^l_{d, n}\) then \(B\) is invariant under their natural action upon \(\mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)\).

2. The subset \(B\) is invariant under the action (by multiplication) of the germs of holomorphic functions.

**Proof.** Let \(E \subset \mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)\) be the set of \(l\)-jets \((z, H)\) whose values \(H(z)\) equal \(0\). Hence \(E\) is a linear subspace of codimension \(m\). Let \(B_0 \subset E\) be the closed algebraic subset of jets such that the tangent map \(dH(z)\) is not surjective. The codimension of \(B_0\) in \(E\) is \(\geq n - m + 1\).

An element \([s] \in E \setminus B_0\) is the jet of a regular equation and hence, \([s]\) defines a jet of submanifold \(\mathcal{Y}(s) \in \text{Jet}^l_{d, n}\). This map \(\mathcal{Y} : E \setminus B_0 \to \text{Jet}^l_{d, n}\) is a regular algebraic submersion. Therefore, \(\mathcal{Y}^{-1}(A) \subset E \setminus B_0\) is a closed algebraic subset whose codimension in \(E \setminus B_0\) equals \(\text{codim}(A)\).

We set \(B = B_0 \cup \mathcal{Y}^{-1}(A)\). Then \(B\) is a closed algebraic subset whose codimension in \(E\) is \(\geq \min\{n - m + 1, \text{codim}(A)\}\). Hence \(B\) has the required codimension in \(\mathbb{C}^n \times \text{Pol}_l(\mathbb{C}^n, \mathbb{C}^m)\).

Clearly, the subset \(B\) satisfies the second invariance condition and if \(A\) is invariant then \(B\) also satisfies the first one.

**Now, we are able to complete the proof of Theorem 9.** Let \(A \subset \text{Jet}^l_{d, n}\) be a closed algebraic subset satisfying the hypotheses of Theorem 9. In particular, \(\text{codim}(A) > d\).
By Lemma 31, the corresponding subset $B \subset \mathbb{C}^n \times \mathcal{P}ol(C^n, \mathbb{C}^m)$ satisfies the following condition:

$$\text{codim}(B) \geq \min\{n + 1, n - d + \text{codim}(A)\} = n + 1$$

as well as the other hypotheses of Theorem 24. Hence, there exists a renormalizable sequence of sections $(s_k)_{k \gg 1}$ which avoids $B$ asymptotically. By definition of $B$, the sequence $(s_k)_{k \gg 1}$ is transverse to 0 asymptotically and, for every limit section $s_\infty$, the jets of the submanifold $\{s_\infty = 0\}$ do no lie in $A$. By Lemma 30 if we set $Y_k = \{s_k = 0\}$ then the complete intersections $(Y_k)_{k \gg 1}$ form a renormalizable sequence satisfying the required conditions and the proof of Theorem 9 is completed.

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