A SIMPLE PROOF OF CURTIS’ CONNECTIVITY THEOREM FOR LIE POWERS

SERGEI O. IVANOV, VLADISLAV ROMANOVSKII, AND ANDREI SEMENOV

ABSTRACT. We give a simple proof of the Curtis’ theorem: if $A_\bullet$ is $k$-connected free simplicial abelian group, then $L^n(A_\bullet)$ is an $k + \lceil \log_2 n \rceil$-connected simplicial abelian group, where $L^n$ is the functor of $n$-th Lie power. In the proof we do not use Curtis’ decomposition of Lie powers. Instead of this we use the Chevalley-Eilenberg complex for the free Lie algebra.

1. Introduction

In [5] Curtis constructed a spectral sequence that converges to the homotopy groups $\pi_\ast(X)$ of a simply connected space $X$. It was described in the language of simplicial groups. This spectral sequence was an early version of the unstable Adams spectral sequence (see [6] §9, [2]). Recall that a simplicial group $G_\bullet$ is called $n$-connected if $\pi_i(G_\bullet) = 0$ for $i \leq n$. For a group $G$ we denote by $\gamma_n(G)$ the $n$-th term of its lower central series. In order to prove the convergence of this spectral sequence, Curtis proved a theorem, that we call “Curtis’ connectivity theorem for lower central series”. It can be formulated as follows.

Theorem ([5]). If $G_\bullet$ is a $k$-connected free simplicial group for $k \geq 0$, then the simplicial group $\gamma_n(G_\bullet)$ is $k+ \lceil \log_2 n \rceil$-connected.

Curtis gave a tricky proof of this theorem using some delicate calculations with generators in free groups. Later Rector [11] described a mod-$p$ analogue of this spectral sequence where the lower central series is replaced by the mod-$p$ lower central series. Then Quillen [10] found a more conceptual way to prove the connectivity theorem for the mod-$p$ lower central series using simplicial profinite groups. This result was enough to prove the convergence of the mod-$p$ version of the spectral sequence. Quillen reduced this connectivity theorem to an earlier result of Curtis, which we call “Curtis’ connectivity theorem for Lie powers”. Denote by $L^n : \text{Ab} \to \text{Ab}$ the functor of $n$-th Lie power. Then the theorem can be formulated as follows.

Theorem ([4]). If $A_\bullet$ is a $k$-connected free simplicial abelian group, then the simplicial abelian group $L^n(A_\bullet)$ is $k+ \lceil \log_2 n \rceil$-connected.

The Curtis’ proof of this theorem is quite complicated and takes up most of the paper (see [4] §4 – §7). He used so-called “decomposition of Lie powers” into smaller functors. The decomposition is a kind of filtration on the functor $L^n$ (see [4] §4). The goal of this paper is to give a simpler proof of this theorem without the decomposition. Instead of this we use the Chevalley-Eilenberg complex for the free Lie algebra. We also generalize the statement to the case of modules over arbitrary commutative ring.

Let $R$ be a commutative ring. We say that a functor $\mathcal{F} : \text{Mod}(R) \to \text{Mod}(R)$ is $n$-connected if for any $k \geq 0$ and any $k$-connected free simplicial module $A_\bullet$ the simplicial module $\mathcal{F}(A_\bullet)$ is $k + n$-connected. In these terms we prove the following.

Theorem. The functor of Lie power $L^n : \text{Mod}(R) \to \text{Mod}(R)$ is $\lceil \log_2 n \rceil$-connected.

The work is supported by a grant of the Government of the Russian Federation for the state support of scientific research, agreement 14.W03.31.0030 dated 15.02.2018. The third author was also supported by “Native Towns”, a social investment program of PJSC “Gazprom Neft”.

1
We also note that this estimation of connectivity of $L^n$ is the best possible for $n = 2^m$. This is an easy corollary of the description of homotopy groups of 2-restricted Lie powers on the language of lambda-algebra given in \cite{6} and \cite{2} (see also \cite{9}, \cite{8}).

**Proposition.** If $R = \mathbb{Z}$ or $R = \mathbb{Z}/2$ the functor $L^n : \text{Mod}(R) \to \text{Mod}(R)$ is not $n + 1$-connected.

Note that our proof of the main theorem is quite elementary. However, the proposition is a corollary of some non-elementary results about the lambda-algebra.

Assume that $\mathfrak{g}$ is a Lie algebra which is free as a module over the ground commutative ring $R$. By the Chevalley-Eilenberg complex of $\mathfrak{g}$ we mean the chain complex whose components are exterior powers $\Lambda^i \mathfrak{g}$ and whose homology is homology of the Lie algebra with trivial coefficients $H_*(\mathfrak{g})$. We consider the free Lie algebra as a functor from the category of free modules to the category of Lie algebras. The free Lie algebra has a natural grading whose components are Lie powers $L^*(A) = \bigoplus_{n\geq 1} L^n(A)$. Here we treat Lie powers as functors from the category of free modules $L^n : \text{FMod}(R) \to \text{Mod}(R)$. The grading on the free Lie algebra induces a grading on the Chevalley-Eilenberg complex whose components give exact sequences of functors on the category of free modules:

$$0 \to \Lambda^2 \to L^2 \to 0,$$

$$0 \to \Lambda^3 \to \text{Id} \otimes L^2 \to L^3 \to 0,$$

$$0 \to \Lambda^4 \to \Lambda^2 \otimes L^2 \to (\text{Id} \otimes L^3) \otimes \Lambda^2 L^2 \to L^4 \to 0,$$

$$\cdots$$

$$0 \to \Lambda^n \to \cdots \to \bigoplus_{k_1 + \cdots + k_n = i} \Lambda^{k_1} L^1 \otimes \Lambda^{k_2} L^2 \otimes \cdots \otimes \Lambda^{k_n} L^n \to \cdots \to L^n \to 0$$

(see Corollary \cite{2.2}), where $\Lambda^{k_s} L^s$ denotes the composition of the functor of Lie power and the functor of exterior power. We use these complexes for induction in the proof of the main result.

## 2. Graded Chevalley-Eilenberg complex

Throughout the paper $R$ denotes a commutative ring. All algebras, modules, simplicial modules, tensor products and exterior powers are assumed to be over $R$.

Let $\mathfrak{g}$ be a Lie algebra which is free as a module. If we tensor the Chevalley-Eilenberg resolution $V_\bullet(\mathfrak{g})$ (see \cite{3} XIII §7-8) on the trivial module $R$, we obtain a complex $C_\bullet(\mathfrak{g}) \cong R \otimes_{\mathfrak{g}} V_\bullet(\mathfrak{g})$ that we call the Chevalley-Eilenberg complex. Its components are exterior powers of the Lie algebra $C_i(\mathfrak{g}) = \Lambda^i \mathfrak{g}$ and the differential is given by the formula

$$d(x_1 \wedge \cdots \wedge x_i) = \sum_{s,t} (-1)^{s+t} [x_s, x_t] \wedge x_1 \wedge \cdots \wedge \hat{x}_s \wedge \cdots \wedge \hat{x}_t \wedge \cdots \wedge x_i.$$

The homology of this complex is isomorphic to the homology of the Lie algebra $\mathfrak{g}$ with trivial coefficients

$$H_i(\mathfrak{g}, R) = H_i(C_\bullet(\mathfrak{g})).$$

Let $\mathfrak{g}$ be a graded Lie algebra $\mathfrak{g} = \bigoplus_{n \geq 1} \mathfrak{g}_n$. By a graded Lie algebra we mean a usual Lie algebra (not a Lie superalgebra) $\mathfrak{g}$ together with a decomposition into direct sum of modules $\mathfrak{g} = \bigoplus_{n \geq 1} \mathfrak{g}_n$ such that $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ for all $n, m \geq 1$. The degree of a homogeneous element $x \in \mathfrak{g}_n$ is denoted by $|x| = n$.

For $n \geq 1$ we consider a submodule $C^{(n)}_i(\mathfrak{g})$ of $C_i(\mathfrak{g})$ spanned by elements $x_1 \wedge \cdots \wedge x_i$, where $x_1, \ldots, x_i$ are homogeneous and $|x_1| + \cdots + |x_i| = n$.

$$C^{(n)}_i(\mathfrak{g}) = \text{span}\{x_1 \wedge \cdots \wedge x_i \in \Lambda^i \mathfrak{g} \mid |x_1| + \cdots + |x_i| = n\}.$$ 

It is easy to see that $d(C^{(n)}_i(\mathfrak{g})) \subseteq C^{(n)}_{i-1}(\mathfrak{g})$, and hence we obtain a subcomplex $C^{(n)}_\bullet(\mathfrak{g})$ of $C_\bullet(\mathfrak{g})$. 
Proposition 2.1. Let $g = \bigoplus_{n \geq 1} g_n$ be a graded Lie algebra, where $g_n$ is free as module for each $n$. Then the Chevalley-Eilenberg complex $C_\bullet(g)$ has a natural grading

$$C_\bullet(g) = \bigoplus_{n \geq 1} C_\bullet^{(n)}(g),$$

and there is a natural isomorphism

$$C_\bullet^{(n)}(g) \cong \bigoplus_{k_1 + \cdots + k_n = i} \Lambda^{k_1} g_1 \otimes \Lambda^{k_2} g_2 \otimes \cdots \otimes \Lambda^{k_n} g_n.$$

Here the sum runs over the set of ordered n-tuples of non-negative integers $(k_1, \ldots, k_n)$ such that $k_1 + \cdots + k_n = i$ and $k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n = n$.

Proof. For any modules $A, B$ there is an isomorphism $\Lambda^i(A \otimes B) \cong \bigoplus_{k+\ell=i} \Lambda^k(A) \otimes \Lambda^\ell(B)$. By induction we obtain the isomorphism

$$\Lambda^i\left(\bigoplus_{s=1}^N A_s\right) \cong \bigoplus_{k_1 + \cdots + k_N = i} \Lambda^{k_1} A_1 \otimes \cdots \otimes \Lambda^{k_N} A_N.$$

Using the fact that the exterior power commutes with direct limits, we obtain the isomorphism for any infinite sequence of modules $A_1, A_2, \ldots$

$$\Lambda^i\left(\bigoplus_{s=1}^\infty A_s\right) \cong \bigoplus_{k_1 + k_2 + \cdots = i} \Lambda^{k_1} A_1 \otimes \Lambda^{k_2} A_2 \otimes \cdots.$$

Here we consider only sequences of non-negative integers $k_1, k_2, \ldots$ where there only finitely many non-zero elements, and hence, each summand in the sum is a finite tensor product.

Take $A_n = g_n$. If we have an element $x_1 \wedge \cdots \wedge x_i$ with homogeneous $x_s \in g$ from the $R$-submodule corresponding to a summand $\Lambda^{k_1} g_1 \otimes \Lambda^{k_2} g_2 \otimes \cdots$, then $|x_1| + \cdots + |x_n| = k_1 \cdot 1 + k_2 \cdot 2 + \cdots$. The assertion follows. \qed

Let $A$ be a free module. We denote by $L^\bullet(A)$ the free Lie algebra generated by $A$. For any basis $(a_s)$ of $A, L^\bullet(A)$ is isomorphic to the free Lie algebra generated by the family $(a_s)$. The Lie algebra $L^\bullet(A)$ is free as a module (see [13], [12 Cor. 0.10]). Its enveloping algebra is the tensor algebra $T^\bullet(A)$. The map $L^\bullet(A) \rightarrow T^\bullet(A)$ is injective [12 Cor. 0.3]. Hence, $L^\bullet(A)$ can be described in terms of tensor algebra. Consider the tensor algebra $T^\bullet(A)$ as a Lie algebra with respect to the commutator. Then $L^\bullet(A)$ can be described as the Lie subalgebra of $T^\bullet(A)$ generated by $A$ (see also [6] §7.4).

The Lie algebra $L^\bullet(A)$ has a natural grading

$$L^\bullet(A) = \bigoplus_{n=1}^\infty L^n(A),$$

where $L^n(A)$ is generated by $n$-fold commutators. Equivalently $L^n(A)$ can be described using the embedding into the tensor algebra as $L^n(A) = L(A) \cap T^n(A)$. The homology of $L^\bullet(A)$ the free Lie algebra can be described as follows $H_i(L^\bullet(A)) = 0$ for $i > 1$ and $H_1(L^\bullet(A)) = A$. For simplicity we set

$$C_\bullet^{(n)}(A) := C_\bullet^{(n)}(L^\bullet(A)).$$

All these constructions are natural by $A$. Denote by $L^n$ the functor of $n$-th Lie power from the category of free modules to the category of modules

$$L^n : \text{FMod}(R) \rightarrow \text{Mod}(R).$$

Moreover, we treat $C_\bullet^{(n)}$ as a complex in the category of functors $\text{FMod}(R) \rightarrow \text{Mod}(R)$. Then Proposition 2.1 implies the following corollary.
Corollary 2.2. For \( n \geq 2 \) the complex \( C_\bullet^{(n)} \) of functors \( FMod(R) \to Mod(R) \) is acyclic and has the following components

\[
C_\bullet^{(n)} = \bigoplus_{k_1 + \cdots + k_n = i} \Lambda^{k_1} L^1 \otimes \Lambda^{k_2} L^2 \otimes \cdots \otimes \Lambda^{k_n} L^n,
\]

where \( \Lambda^k L^s \) denotes the composition of the functor of Lie power and the functor of exterior power. Here the sum runs over the set of ordered \( n \)-tuples of non-negative integers \( (k_1, \ldots, k_n) \) such that \( k_1 + \cdots + k_n = i \) and \( k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n = n \).

Remark 2.3. Note that \( C_\bullet^{(n)} = 0 \) for \( i \notin \{1, \ldots, n\} \), and that there are isomorphisms \( C_\bullet^{(n)} = \Lambda^n \) and \( C_1^{(n)} = L^n \). In other words, \( C_\bullet^{(n)} \) is an exact sequence that connects \( \Lambda^n \) and \( L^n \).

3. Connectivity of functors

For \( n \geq 0 \) we say that a simplicial module \( A_\bullet \) is \( n \)-connected, if \( \pi_i(A_\bullet) = 0 \) for \( i \leq n \).

Lemma 3.1. Let \( A_\bullet \) be an \( n \)-connected simplicial module and \( B_\bullet \) be an \( m \)-connected free simplicial module. Then \( A_\bullet \otimes B_\bullet \) is \( n + m + 1 \)-connected.

Proof. Consider their component-wise tensor product \( A_\bullet \otimes B_\bullet \). The Eilenberg-Zilber theorem implies that \( \pi_s(A_\bullet \otimes B_\bullet) \cong H_s(NA_\bullet \otimes NB_\bullet) \), where \( NA_\bullet \) denotes the Moore complex of \( C_\bullet \). Since \( NB_\bullet \) is a direct summand of \( B_\bullet \), it is projective module. This gives the following variant of the Künneth spectral sequence:

\[
E^2_{pq} = \bigoplus_{s+t=q} \text{Tor}_p^{R}(\pi_s(A_\bullet), \pi_t(B_\bullet)) \Rightarrow \pi_{p+q}(A_\bullet \otimes B_\bullet).
\]

If \( s + t \leq n + m + 1 \), then either \( s < n + 1 \) or \( t < m + 1 \). Hence \( E^2_{pq} = 0 \) for \( p + q \leq n + m + 1 \). Therefore, \( A_\bullet \otimes B_\bullet \) is \( n + m + 1 \)-connected.

A functor from the category of modules to itself

\[
\mathcal{F} : Mod(R) \to Mod(R)
\]

is said to be \( n \)-connected if for any \( k \geq 0 \) and any \( k \)-connected free simplicial module \( A_\bullet \) the simplicial module \( \mathcal{F}(A_\bullet) \) is \( k + n \)-connected.

Lemma 3.2. Let \( \mathcal{F} : Mod(R) \to Mod(R) \) be an \( n \)-connected functor and \( \mathcal{G} : Mod(R) \to Mod(R) \) be \( m \)-connected functor. Assume that \( \mathcal{G} \) sends free modules to free modules. Then the composition \( \mathcal{F} \mathcal{G} \) is \( n + m \)-connected and the tensor product \( \mathcal{F} \otimes \mathcal{G} \) is \( n + m + 1 \)-connected.

Proof. The fact about the composition is obvious. The fact about the tensor product follows from Lemma 3.1.

Lemma 3.3. Let

\[
0 \to \mathcal{F}_n \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{G} \to 0
\]

be an exact sequence of functors such that \( \mathcal{F}_i \) is \( n - i \)-connected. Then \( \mathcal{G} \) is \( n \)-connected.

Proof. The proof is by induction. For \( n = 0 \) this is obvious. Assume that \( n \geq 1 \) and that the statement holds for smaller numbers. Set \( \mathcal{H} := \text{Ker}(\mathcal{F}_0 \to \mathcal{G}) \). Then by the induction hypothesis \( \mathcal{H} \) is \( n - 1 \)-connected. The long exact sequence

\[
\cdots \to \pi_i(\mathcal{H}(A_\bullet)) \to \pi_i(\mathcal{F}_0(A_\bullet)) \to \pi_i(\mathcal{G}(A_\bullet)) \to \pi_{i-1}(\mathcal{H}(A_\bullet)) \to \cdots
\]

implies that \( \mathcal{G} \) is \( n \)-connected.

Proposition 3.4. The functor of exterior power \( \Lambda^n \) is \( n - 1 \)-connected.
Hence graded chain complexes). Moreover, if Lemma 3.5.

For any two sequences of positive integer numbers

\[ \Lambda^n(B_\bullet[1]) \sim \Gamma^n(B_\bullet)[n]. \]

Any 0-connected free simplicial module \( A_\bullet \) is homotopy equivalent to a simplicial module of the form \( B_\bullet[1] \), where \( B_\bullet \) is also a free simplicial module (it follows from the same fact for non-negatively graded chain complexes). Moreover, if \( A_\bullet \) is \( k \)-connected, we can chose \( B_\bullet \) so that \( B_i = 0 \) for \( i \leq k - 1 \). Hence \( \pi_i(\Lambda^n(A_\bullet)) = \pi_i(\Lambda^n(B_\bullet[1])) = \pi_{i-n}(\Gamma^n(B_\bullet)) = 0 \) for \( i \leq k + n - 1 \).


Lemma 3.5. For any two sequences of positive integer numbers \( u_1, \ldots, u_m \) and \( v_1, \ldots, v_m \) the following inequality holds

\[ \sum_{s=1}^{m} (u_s + \log_2 v_s) \geq 1 + \log_2 \left( \sum_{s=1}^{m} u_s v_s \right). \]

**Proof.** It is easy to prove by induction that \( \prod_{s=1}^{m} 2^{u_s} v_s \geq 2 \sum_{s=1}^{m} u_s v_s. \) If we apply logarithms, we obtain the required statement.


Theorem 3.6. The functor of Lie power \( L^n \) is \( \lceil \log_2 n \rceil \)-connected.

**Proof.** The proof is by induction. For \( n = 1 \) we have \( L^1 = \text{Id} \) and this is obvious. Assume that \( n \geq 2 \) and that the statement holds for all smaller numbers. Consider the acyclic chain complex \( C_i^{(n)} \) (Corollary 2.2). Using Lemma 3.3 we obtain that it is enough to check that the functor

\[ C_i^{(n)} = \bigoplus_{k_1, \ldots, k_n n = n} \Lambda^{k_1} L^1 \otimes \Lambda^{k_2} L^2 \otimes \cdots \otimes \Lambda^{k_n} L^n \]

is \( \lceil \log_2 n \rceil \)-connected for \( i \geq 2 \). It is enough to prove this for each summand.

Fix an \( n \)-tuple of \( (k_1, \ldots, k_n) \) such that \( k_1 + \cdots + k_n = i \geq 2 \) and \( k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n = n. \) Note that \( i \geq 2 \) implies \( k_n = 0. \) Some of the numbers \( k_j \) equal to zero. Denote by \( j_1, \ldots, j_m \) the indexes corresponding to non-zero numbers \( k_{j_s} \neq 0. \) By Lemma 3.2 the functor \( \Lambda^{k_j} L^{j_s} \) is \( k_j - 1 + \lceil \log_2 j_s \rceil \)-connected for \( j < n. \) Then again by Lemma 3.2 the tensor product \( \Lambda^{k_1} L^1 \otimes \Lambda^{k_2} L^2 \otimes \cdots \otimes \Lambda^{k_n} L^n \) is \( \sum_{s=1}^{m} (k_{j_s} - 1 + \lceil \log_2 j_s \rceil) + m - 1 \)-connected. Using Lemma 3.5 we obtain

\[ \sum_{s=1}^{m} (k_{j_s} - 1 + \log_2 j_s) + m - 1 \geq \sum_{s=1}^{m} (k_{j_s} + \log_2 j_s) - 1 \geq \log_2 n. \]

The assertion follows.

4. Connectivity of \( L^{2^n} \)

For \( k \geq 0 \) we denote by \( R[k + 1] \) the chain complex concentrated in \( k + 1 \)-st degree whose \( k + 1 \)-st component is equal to \( R. \) The Dold-Kan corresponding simplicial module is denoted by \( K(R, k + 1). \) Note that \( K(R, k + 1) \) is a \( k \)-connected free simplicial module.

Proposition 4.1. Let \( R = \mathbb{Z} \) or \( R = \mathbb{Z}/2 \) and \( n \geq 0. \) Then \( L^{2^n} \) is not \( n + 1 \)-connected. Moreover, for any \( k \geq 0 \)

\[ \pi_{n+1+k}(L^{2^n}(K(R, k + 1))) \neq 0. \]

**Proof.** (1) Let \( R = \mathbb{Z}/2. \) We fix \( k \) and set \( V_\bullet = K(\mathbb{Z}/2, k + 1). \) Denote by \( L^n_{\text{res}} : \text{Vect}(\mathbb{Z}/2) \rightarrow \text{Vect}(\mathbb{Z}/2) \) the functor of 2-restricted Lie power (see \([11 \ S.7], \ [16 \ S.7] \)). The homotopy groups \( \pi_* \left( L^n_{\text{res}}(V_\bullet) \right) \) are described in terms of the lambda-algebra in \([61 \ Th. 8.8] \) (see also \([2] \) and discussion after Theorem 7.11 in \([3] \)):

\[ \pi_{i+k+1}(L^{2^n}_{\text{res}}(V_\bullet)) \cong \Lambda^{i,n}(k + 2), \]
where $A^{i,n}(k + 2)$ denotes the vector sub-space of the lambda algebra $A$ with the basis given by compositions $\lambda_i \ldots \lambda_{i_n}$, where $i_{s+1} \leq 2i_s$, $i_1 + \cdots + i_n = i$ and $i_1 \leq k + 2$. In particular, $\lambda_1^n \in A^{n,n}(k + 2) \neq 0$. Hence
\[
\pi_{n+1+k}(L^n_{\text{res}}(V_*)) \neq 0.
\]

For arbitrary simplicial Lie algebra $g_\ast$ and $t \geq 1$ we define the map $\tilde{\lambda}_1 : g_t \to g_{t+1}$ by the formula $\tilde{\lambda}_1(x) = [s_0x, s_1x]$, where $s_0, s_1$ are degeneracy maps. Denote by $i_{k+1}$ the unit of $(V_*)_{k+1} = R$. Then $\lambda^n_1(i_{k+1}) \in (L^n(V_*))_{n+k+1}$ is the element representing $\lambda^n_1 \in A^{n,n}(k + 2) \cong \pi_{n+1+k}(L^n_{\text{res}}(V_*))$ (see [6] Prop. 8.6]. By the definition of $\tilde{\lambda}_1$, the element $\tilde{\lambda}_1^n(i_{k+1})$ lies in the unrestricted part $L^n_{\text{res}}(V_*)$ of $L^n_{\text{res}}(V_*)$. Therefore $\tilde{\lambda}_1^n(i_{k+1})$ represents a nontrivial element of $\pi_{n+1+k}(L^n(V_*))$ and hence
\[
\pi_{n+1+k}(L^n_{\text{res}}(V_*)) \neq 0.
\]

(2) Now assume that $R = \mathbb{Z}$ and set $A_* = K(\mathbb{Z}, k + 1)$. We denote by $L^*_\mathbb{Z}/2$ the functor of Lie power over $\mathbb{Z}/2$, which we already discussed, and by $L^*_{\mathbb{Z}/2}$ the functor of Lie power over $\mathbb{Z}$. Then for any free abelian group $A$ we have $L^*_\mathbb{Z}(A) \otimes \mathbb{Z}/2 \cong L^*_{\mathbb{Z}/2}(A \otimes \mathbb{Z}/2)$. The universal coefficient theorem gives the following short exact sequence
\[
0 \longrightarrow \pi_i(L^n_{\mathbb{Z}}(A_*)) \otimes \mathbb{Z}/2 \longrightarrow \pi_i(L^n_{\mathbb{Z}/2}(V_*)) \longrightarrow \text{Tor}_i(\pi_{i-1}(L^n_{\mathbb{Z}}(A_*)), \mathbb{Z}/2) \longrightarrow 0.
\]

Since the functor $L^n_{\mathbb{Z}}$ is $n$-connected, $\pi_{n+k}(L^n_{\mathbb{Z}}(A_*)) = 0$. Therefore
\[
\pi_{n+1+k}(L^n_{\mathbb{Z}}(A_*)) \otimes \mathbb{Z}/2 \cong \pi_{n+1+k}(L^n_{\mathbb{Z}/2}(V_*)).
\]

We already proved that $\pi_{n+1+k}(L^n_{\mathbb{Z}/2}(V_*)) \neq 0$. Hence $\pi_{n+1+k}(L^n_{\mathbb{Z}}(A_*)) \neq 0$. \hfill \Box

**Remark 4.2.** The Proposition [4.1] can be also deduced from results of D. Leibowitz [8] or from unpublished results of R. Mikhailov [9], where he describes all derived functors in the sense of Dold-Puppe of Lie powers in the case $R = \mathbb{Z}$.

**References**

[1] Yu. A. Bahturin. Identical relations in Lie algebras. VNU Science Press, b.v., Utrecht, 1987.
[2] A. K. Bousfield, E. B. Curtis, D. M. Kan, D. G. Quillen, D. L. Rector, J. W. Schlesinger. The mod-p lower central series and the Adams spectral sequence, Topology 5 (1966), 331-342.
[3] H. Cartan, S. Eilenberg. Homological algebra. Princeton University Press, Princeton, N. J., 1956.
[4] E. B. Curtis. Lower central series of semi-simplicial complexes, Topology 2 (1963), 159-171.
[5] E. B. Curtis. Some relations between homotopy and homology. Ann. Math. 82 (1965), 386-413.
[6] E. B. Curtis, Simplicial Homotopy Theory, Advances in Math. 6 (1971), 107–209.
[7] L. Illusie, Complexes cotangent et déformations. I, Lecture Notes in Mathematics, Vol. 239, Springer-Verlag, Berlin-New York, 1971.
[8] D. Leibowitz. The $E^1$ term of the lower central series spectral sequence for the homotopy of spaces, Brandeis University Ph.D. thesis (1972).
[9] R. Mikhailov. Homotopy theory of Lie functors. \url{arXiv:1808.00681}
[10] D. G. Quillen. An application of simplicial profinite groups. Comment. Math. Helv. 44 (1969), 45–60.
[11] D. L. Rector. An unstable Adams spectral sequence, Topology 5 (1966), 343-346.
[12] C. Reutenauer, Free Lie algebras, Oxford University Press, 1993.
[13] A. I. Shirshov. Subalgebras of free Lie algebras, Mat. Sbornik N.S. 33(75), (1953), 441–452.
