Nullification of the Nullification

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A recent claim by Meyer, Kent and Clifton (MKC), that their models “nullify” the Kochen-Specker theorem, has attracted much comment. In this paper we present a new counter-argument, based on the fact that a classical measurement reveals, not simply a pre-existing value, but pre-existing classical information. In the MKC models measurements do not generally reveal pre-existing classical information. Consequently, the Kochen-Specker theorem is not nullified. We go on to prove a generalized version of the Kochen-Specker theorem, applying to non-ideal quantum measurements. The theorem was inspired by the work of Simon et al and Larsson (SBZL). However, there is a loophole in SBZL’s argument, which means that their result is invalid (operational non-contextuality is not inconsistent with the empirical predictions of quantum mechanics). Our treatment resolves this difficulty. We conclude by discussing the question, whether the MKC models can reproduce the empirical predictions of quantum mechanics.

I. INTRODUCTION

Prior to the work of Meyer [1], the Kochen-Specker theorem [2–6] (KS theorem) was generally regarded as one of the key foundational results of quantum mechanics, establishing one of the most important ways in which quantum mechanics enforces a radical departure from the assumptions of classical physics. Classically, a measurement is a process which reveals the pre-existing value of the observable measured. Until recently it was generally accepted that the KS theorem shows that quantum mechanics cannot be interpreted in such a way as to preserve this feature of classical physics.

Unfortunately, the KS theorem, as originally formulated, only applies to ideal, or von Neumann measurement processes. In practice, strict ideality is seldom, and perhaps never actually attainable. It follows that the KS theorem, in its original form, is not sufficient to demonstrate a contradiction between classical assumptions and the empirically verifiable predictions of quantum mechanics, regarding the outcome of real, laboratory measurements.

At first sight, this may not seem a serious problem. The natural assumption would be that the original KS theorem is a limiting case of a more general theorem, which does apply to non-ideal measurements. We will eventually argue that this is, in fact, the correct assumption. However, the question is greatly complicated by the work of Meyer [1], Kent [7] and Clifton and Kent [8] (MKC in the sequel), who have argued that “finite precision measurement nullifies the Kochen-Specker theorem”.

MKC’s claimed “nullification” of the KS theorem has given rise to some controversy. It has been discussed by (in chronological order) Cabello [12], Havlicek et al [13], Mermin [14], Appleby [15,16], Simon et al [17], Larsson [18], Simon [19], Cabello [20] and Boyle and Schafir [21]. For further discussion of the problem of demonstrating an empirical contradiction with the predictions of non-contextual hidden variables theories see Cabello and García-Alcaine [22], Basu et al [23], Cabello [24], Michler et al [25], Simon et al [26] and Cereceda [27]. Finally, it should be noted that MKC’s argument was inspired by the previous work of Pitowsky [9], Hales and Straus [10] and Godsil and Zaks [11] (we briefly comment on Pitowsky’s work in the conclusion).

The arguments in this paper are largely new. They go significantly beyond any that have previously been given. We begin by nullifying MKC’s claimed nullification. That is, we show that MKC’s ingenious mathematical constructions, though deeply interesting in their own right, do not invalidate the essential physical point of the KS theorem. Having cleared the conceptual ground, we then go on to establish a generalized version of the KS theorem, which does apply to non-ideal measurements. In order to make the treatment comprehensive we conclude by discussing a number of related questions. In particular, we discuss a recent claim by Cabello [20], that the MKC models do not reproduce the empirically verifiable predictions of quantum mechanics.

Our main criticism of MKC’s argument is contained in Sections II–IV.

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Section II contains some preliminary considerations. MKC assume that an experimenter can never precisely know what has “actually” been measured. This assumption plays an important role in their argument because it enables them to postulate that it is physically impossible to measure each of the observables in a KS-uncolourable set. We show that the assumption depends on some misconceptions regarding non-ideal quantum measurements.

Section III contains the crux of the critical part of our argument. The physical point of the KS theorem is to show that quantum measurements cannot generally be interpreted as measurements in the classical sense. A classical measurement is not simply a process which reveals a pre-existing value. Rather, it is a process which reveals a pre-existing piece of classical information, represented by a proposition of the form “observable A took value x”. For this condition to be satisfied it is essential that the value x and observable A both be adequately specified. It appears to MKC that their models nullify the KS theorem because, in their models, a measurement does always reveal the value of something. However, it does not reveal the precise identity of that something. Since the MKC valuations are radically discontinuous this means that measurements do not generally reveal any classical information. Consequently, they cannot generally be regarded as measurements in the classical sense. It follows that the KS theorem is not nullified.

Section IV contains some supplementary considerations, which reinforce the conclusion reached in Section III.

The argument just outlined goes beyond the arguments given in our previous papers because it shows that MKC’s claim, to have nullified the KS theorem, is simply false. In Appleby and Larsson we showed that the MKC models are highly non-classical. In fact, we showed in ref. that the MKC models exhibit a novel kind of contextuality, which is even more strikingly at variance with classical assumptions than the usual kind of contextuality, featuring in the KS theorem. Consequently, the argument in ref. is, by itself, sufficient to refute MKC’s suggestion, that their models provide a classical explanation of non-relativistic quantum mechanics. The argument in ref. though less clear-cut, lends additional support to this conclusion. However, neither argument directly bears on MKC’s claim, to have nullified the KS theorem as such. Indeed, at the time we wrote these papers it appeared to us that MKC actually had “nullified the KS theorem strictly so-called” (as we put it).

It is important to establish that finite precision does unequivocally not nullify the KS theorem because, in the absence of such a demonstration, it is impossible to achieve an unobstructed understanding of contextuality, as it applies in a real, laboratory setting. The question has immediate, practical relevance, in view of the current interest in experimental investigations of contextuality.

Having cleared the conceptual ground in Sections II–IV we go on, in Section V, to formulate and prove a generalized KS theorem, applying to non-ideal measurements. The argument in this section is inspired by the argument of Simon et al. and Larsson (SBZL in the sequel). However, there are some important differences. SBZL work with an “operational” concept of contextuality. Their motive for introducing this concept is to circumvent MKC’s argument. Clearly, the motive no longer applies, once it is established that MKC’s argument is invalid. We are consequently able to formulate a generalized KS theorem in terms of the ordinary concept of contextuality. This has several advantages. In the first place it means that the result we prove is a straightforward generalization of the ordinary KS theorem. In the second place, there is a loophole in SBZL’s argument, which means that their result is actually incorrect. As we show in Appendix C, by means of a counter-example, operational non-contextuality is not inconsistent with the empirical predictions of quantum mechanics. The modified argument we give in Section V resolves this difficulty. Lastly, the result we prove implies a significantly different conclusion, as to the conditions which must be satisfied in order refute non-contextual theories experimentally.

Sections II–V contain the main part of our argument. In Sections VI–VIII we address a number of related questions. In Section VI we briefly discuss the POVM arguments given by Kent and Clifton and Kent. In Section VII we give an improved version of the argument in Appleby. In the first place we have improved the argument so as to take account of the points made in Section II of this paper. In the second place, the version we give now does not involve joint measurements of non-commuting observables. In the third place, we have strengthened the argument, using ideas derived from the subsequent work of SBZL.

In Section VIII we discuss the question, whether there actually exists a complete theory of MKC type which is empirically equivalent to quantum mechanics. In particular, we discuss a recent claim by Cabello, that theories of this type make experimentally testable predictions which conflict with those of quantum mechanics. We argue that, although Cabello makes some very pertinent points, the question remains open.

II. WHAT IS “ACTUALLY” MEASURED?

MKC’s argument partly depends on a misconception, regarding non-ideal quantum measurements. We begin our critical discussion by clarifying this point. The criticisms which follow are preliminary to our main critical argument, contained in the next section.
MKC postulate that there are many observables which it is physically impossible to measure. If an experimenter attempts to measure an observable $\hat{A}$ in this forbidden set, then what is actually measured is a slightly different observable $\hat{B}$. The measurement reveals the pre-existing value of the observable $\hat{B}$ which is actually measured.

It can be seen from this that MKC tacitly assume:

**Property 1** For a given finite precision measuring apparatus there is a single, uniquely defined observable, which is the only observable that is “actually” measured.

**Property 2** The nominal observable, which is recorded in the experimenter’s notebook as having been measured, is typically not the observable which is “actually” measured.

We will show that these assumptions are unjustified.

Before proceeding further, let us clarify the meaning of the term “nominal observable”, as it appears in the above statement. The term essentially corresponds to Simon *et al’s* [17] “switch position”. However, unlike Simon *et al* we are not appealing to the concept of an “operational observable” (i.e. the concept that different instruments define different observables—see Appendix C). We are simply recognizing the fact that the complete record of a measurement (quantum or classical) must include, not only a specification of the value obtained, but also a specification of the observable measured. Suppose, for instance, that an experimenter makes 1,000 different measurements of 1,000 different observables. If the experimenter simply writes down a list of 1,000 numbers, without any indication as to how those numbers were obtained, then the record will obviously be useless (this point, that the outcome of a measurement [quantum or classical] is, not simply a value, but a determinate proposition, will play a central role in the argument of the next section).

It should be stressed that, although we refer, for convenience, to the actions of a human experimenter, the same point would apply if the measurements were performed by a completely arbitrary “information gathering and utilizing system” [28]. If the system is to utilize the information it acquires then it must record, in its memory area, a specification of the observables measured, as well as the values obtained.

The nominal observable need not be specified precisely. Rather than recording the information “observable $\hat{S}_z$ was measured and value +1/2 obtained”, one might instead record the information “observable $n \cdot \hat{S}$ was measured, for $n \in U$, and value +1/2 obtained” (where $U$ is some subset of the unit 2-sphere). In the following we will assume that the nominal observable is specified precisely, because this is legitimate, and because it happens to be the usual practice. However, the argument could easily be modified so as to allow for the possibility that the nominal observable is only specified partially.

Let us now consider MKC’s assumption that, for each quantum measurement, there is one, and only one observable which is “actually” measured. The fact that this assumption is unjustified becomes apparent when one takes into view (as MKC do not) the detailed physical implementation of an abstract quantum measurement scheme.

![Schematic illustration of the Stern-Gerlach arrangement.](image)

**FIG. 1.** Schematic illustration of the Stern-Gerlach arrangement. The surface irregularities of the pole pieces, and asymmetries in the environment, mean that the field is not perfectly symmetric. There is no single direction which represents the alignment of the field more truly than any other. It follows that there is no single spin component which is measured more truly than any other.

Consider, for example, a non-ideal measurement of the spin component $n \cdot \hat{S}$ using a Stern-Gerlach apparatus. The vector $n$ is determined by the axis of symmetry of the arrangement. MKC take it that, in such a case, although the axis
is not precisely known to the experimenter, there does actually exist a single, sharply defined axis of symmetry; and that there correspondingly exists a single, sharply defined spin component which is the only spin component that is actually measured. However, this is to overlook the fact that the magnetic field will not, in practice, be perfectly symmetric (due to irregularities in the shape of the pole-pieces, asymmetries in the placement of surrounding objects, etc.). As illustrated in Fig. 1, the numerous slight departures from perfect symmetry introduce some unavoidable “fuzziness” into the concept “axis of symmetry of the field”. Under these conditions there is no single vector \( \mathbf{n} \) which represents the axis of symmetry more truly than any other. This being so, there are no obvious grounds for picking out any single spin component as the only component which is “actually” measured.

A measurement \( \hat{\mathbf{n}} \cdot \hat{\mathbf{S}} \) of \( \mathbf{n} \cdot \mathbf{S} \) is a process which discriminates the eigenstates of \( \mathbf{n} \cdot \mathbf{S} \). Given an unknown eigenstate of \( \mathbf{n} \cdot \mathbf{S} \), an ideal measuring instrument will inform the experimenter, with certainty, which particular eigenstate it was. A non-ideal measuring instrument will inform the experimenter, with high probability, which particular eigenstate it was. This is the definition of a quantum measurement process \( \mathbb{R} \): if a process performs the function, of discriminating the eigenstates of an operator, then it is a measurement of that operator, ideal or non-ideal as the case may be.

Of course, it is seldom, if ever the case that a real laboratory instrument is strictly ideal, and the Stern-Gerlach arrangement is no exception to this rule. It is important, furthermore, to note that, not only does a Stern-Gerlach arrangement fail to perform an ideal measurement of the nominal spin component, which is recorded in the experimenter’s notebook as having been measured, it fails to perform an ideal measurement of any other spin component either (for a discussion of the unavoidable sources of non-ideality in Stern-Gerlach measurements see Busch et al. [31]). The arrangement may, however, be used to perform non-ideal measurements (which is the most that can reasonably be demanded of a real, laboratory instrument).

We can characterize the degree of non-ideality in quantitative terms by considering the probability that the process will fail to correctly identify a given eigenstate. Suppose that the arrangement is used to measure the component \( \mathbf{n} \cdot \mathbf{S} \) for a spin-1/2 particle. Let \( p_{\pm\pm}(\mathbf{n}) \) (respectively \( p_{\mp\mp}(\mathbf{n}) \)) be the probability that, when the system particle is initially in the spin up (respectively spin-down) eigenstate of \( \mathbf{n} \cdot \mathbf{S} \) the measurement outcome is \(-1/2\) (respectively \(+1/2\)). Then it may be said that, the smaller these probabilities are, the more nearly ideal the measurement.

There is no vector \( \mathbf{n} \) for which \( p_{\pm\pm}(\mathbf{n}) = p_{\mp\mp}(\mathbf{n}) = 0 \) (see Busch et al. [31]). There are, however, vectors \( \mathbf{n} \) for which the failure probabilities \( p_{\pm\pm}(\mathbf{n}), p_{\mp\mp}(\mathbf{n}) \) are both small. If \( \mathbf{n} \) satisfies this condition the arrangement can be used to discriminate the eigenstates of \( \mathbf{n} \cdot \mathbf{S} \) with a high degree of reliability. In other words, it can be used to perform non-ideal measurements of \( \mathbf{n} \cdot \mathbf{S} \).

The failure probabilities \( p_{\pm\pm}(\mathbf{n}), p_{\mp\mp}(\mathbf{n}) \) vary continuously with \( \mathbf{n} \). This means that, if the probabilities are small for one vector \( \mathbf{n} \), then they will also be small for every other vector which is close to \( \mathbf{n} \). It follows that, for a given arrangement, there are infinitely many different spin components which are all non-ideally measured. Furthermore the class of observables which are non-ideally measured includes the nominal observable (provided it is correctly recorded, and provided the instrument does not malfunction).

We began our discussion of the Stern-Gerlach arrangement by noting that, in practice, the arrangement will not have a sharply defined axis of symmetry. It can now be seen that, even if the symmetry were exact, MKC’s assumption, that there is only one observable that is “actually” measured, would still not be justified. It is true that there might (perhaps) then be a well-defined, natural sense in which the measurement of \( \mathbf{n} \cdot \mathbf{S} \) was “most nearly ideal” when \( \mathbf{n} \) was parallel to the axis of symmetry. But it would not follow that a neighbouring component \( \mathbf{n}' \cdot \mathbf{S} \), defined by a vector \( \mathbf{n}' \) which was not exactly parallel to the axis of symmetry, was not “actually” measured at all.

A process does not have to be precisely optimal in order to count as a measurement. If the probabilities \( p_{\pm\pm}(\mathbf{n}) \) and \( p_{\mp\mp}(\mathbf{n}) \) become slightly larger, then that simply means that the process does not discriminate eigenstates quite so efficiently as before. It does not mean that the process thereby ceases to be a measurement.

The real photon detectors which have been constructed to date are imperfect, in that there is no non-zero probability that the detector will fail to register the presence of a photon. This does not mean that a real photon detector does not “actually” detect photons at all. Furthermore, a photon detector whose performance becomes slightly degraded, so that it is no longer quite so efficient as before, does not, on that account, cease to be a photon detector.

In general, however, there is no unambiguous, non-arbitrary, physically well-motivated way to define the concept “unique spin component for which the measurement is most nearly ideal”. For instance, minimizing the function \( (1/2) (p_{\pm\pm}(\mathbf{n}) + p_{\mp\mp}(\mathbf{n})) \), and minimizing the function \( ((1/2) (p_{\pm\pm}(\mathbf{n})^2 + p_{\mp\mp}(\mathbf{n})^2))^{1/2} \) leads to definitions of this concept which are, in general, incompatible (quite apart from the fact that the minima may not be unique). In the general case there are many spin components which all have an equally valid claim to the status “spin component for which the measurement is most nearly ideal”.

Thus far we have been considering the case of the Stern-Gerlach apparatus. However, it is shown in Appendix A that the above criticisms apply much more generally, to the case of any approximate von Neumann measurement (i.e. any non-ideal measurement described by a unitary operator which is close to a unitary operator describing a von Neumann measurement). This provides strong grounds for the assertion that Properties 1 and 2 assumed by MKC...
should be replaced by

**Property 1’** For a given finite precision measuring apparatus there are infinitely many different observables which are non-ideally measured.

**Property 2’** The nominal observable is among the observables which are non-ideally measured (provided the nominal observable is recorded correctly, and provided there is no malfunction in the measuring apparatus).

It may be worth remarking that these statements are also valid classically. Consider, for example a classical measurement of the velocity component $v_z$ of some macroscopic object, using the Doppler shift of the radiation it emits. In order to perform this measurement it is necessary that the line running from source to detector should be parallel to the $z$-axis. The fact that source and detector are both extended objects, which are not perfectly symmetric, means that, in practice, this line is not even precisely defined, let alone precisely known. However, this does not imply that $v_z$ may not be the component which is “actually” measured. Rather, it is to be regarded as one of the sources of error in the measurement of $v_z$. A classical measurement of a classical observable $A$ is a process which reveals, with a high degree of reliability, and to a high degree of accuracy, the pre-existing value of $A$. On this definition the procedure described effects a non-ideal classical measurement of $n \cdot v$ for every $n$ which is nearly (but perhaps not precisely) parallel to the $z$-axis.

The effect of the argument in this section is to restore the natural assumption of most physicists, that the observable which an experimenter records as having been non-ideally measured is also measured in fact (provided the experimenter does not make a mistake, and provided there is no malfunction in the measuring apparatus).

Let us now consider the bearing which this has on MKC’s argument. Our criticisms do not invalidate MKC’s postulate, that a measurement of $\hat{A}$ may reveal the value of some other, neighbouring observable $\hat{B}$. They do, however, show that MKC cannot justifiably claim that $\hat{B}$ is the only observable which is “actually” measured, and that $\hat{A}$ is not really measured at all. MKC cannot even claim that $\hat{B}$ is the uniquely defined observable for which the measurement is most nearly ideal. In general, the observable $\hat{B}$ has no preferred status. Properly understood, MKC’s postulate comes to this: that the hidden dynamics somehow selects an essentially arbitrary observable $\hat{B} \approx \hat{A}$, whose value it then reveals.

The point is important because, once MKC’s proposal is formulated correctly, it becomes easier to see the fallacy in their argument.

### III. NULLIFICATION OF THE NULLIFICATION

This section contains the main part of our critical argument.

The KS theorem shows

**Proposition 1** No hidden variables interpretation can have the property that, for every observable $\hat{A}$, an ideal measurement of $\hat{A}$ always reveals the pre-existing value of $\hat{A}$.

MKC’s claim, when appropriately re-formulated, so as to remove any unjustified appeal to the concept of “the unique observable which is actually measured”, is that this proposition is “nullified” by

**Proposition 2** There do exist hidden variables interpretations with the property that, for every observable $\hat{A}$, a finite precision measurement of $\hat{A}$ always reveals the pre-existing value of some unknown observable $\hat{B}$ in a small neighbourhood of $\hat{A}$.

In considering this claim the first problem we face is that the term “nullified” is not part of the standard lexicon of science and mathematics, and so its meaning is potentially ambiguous. Propositions 1 and 2 are logically consistent, so there can be no question of the KS theorem actually being refuted. It is not immediately apparent what, exactly, is meant by the claim that Proposition 2 “nullifies” Proposition 1, without contradicting it.

In this connection it should be noted that the reference to finite precision measurements, on which MKC themselves place much emphasis, is actually irrelevant. MKC postulate that, for each non-ideal measuring apparatus, the hidden dynamics somehow arbitrarily selects an observable $\hat{B}$, close to the nominal observable, whose value the measurement then reveals. One could, with equal justification, postulate that the same is true for each ideal measuring apparatus: implying

**Proposition 2’** There do exist hidden variables interpretations with the property that, for every observable $\hat{A}$, an ideal measurement of $\hat{A}$ always reveals the pre-existing value of some unknown observable $\hat{B}$ in a small neighbourhood of $\hat{A}$.
If the KS theorem is nullified by Proposition 2, then it must, presumably, also be nullified by Proposition 2’. It would seem to follow that such nullification as there might be cannot be attributed specifically to the finite precision.

In the following we will take the claim to be that the MKC models invalidate what had previously been regarded as the key physical implication of the KS theorem: namely, the implication that quantum measurements cannot systematically be interpreted as measurements in the classical sense. In the MKC models a measurement does not always reveal the value of the nominal observable. It does, however, always reveal the value of an observable which is extremely close to the nominal observable. MKC assume that is enough to recover the classical concept of measurement.

The question we now have to decide is whether MKC are correct to assume that a process which does not reveal the pre-existing value of the nominal observable may still count as a measurement in the classical sense. We will argue that they are not.

The crucial point to realize is that a classical measurement is not simply a process which reveals a pre-existing value. Rather, it is a process which reveals a pre-existing piece of classical information, represented by a proposition of the form “observable A took value x”. For this to be true it is essential that the observable A and the value x both be adequately specified. A process which reveals the value of some completely unknown observable does not reveal any classical information, and is not a classical measurement.

The following example may serve to illustrate this point. Suppose that an experimenter is given a sealed box, containing 100 individually labelled rods. Suppose that, for each integer \( n = 1, \ldots, 100 \) the box contains exactly one rod of length \( n \) cm. Suppose that the experimenter, without looking inside the box, uses a random number generator to select a particular integer \( 1 \leq n_0 \leq 100 \). It could be said that this procedure reveals the length of one of the rods in the box. However, it does not reveal the identity of that rod. Consequently, the process does not reveal any classical information, and so it cannot be considered a classical measurement.

It may, at first sight, seem that these remarks do not apply to the situation envisaged by MKC since, in their models, the observable \( B \), whose value is revealed by the measurement, is not completely unknown. In fact, it is assumed that \( B \) is extremely close to the nominal observable \( A \): so there is a sense in which \( B \) is specified very precisely. However, it is not specified precisely enough.

The point here is that the MKC valuations are radically discontinuous. If the valuation could be assumed continuous in the vicinity of \( A \), then a process which revealed the value of some neighbouring observable \( B \) could be regarded as a measurement in the classical sense. However, there is in fact always a region on which the valuation is highly discontinuous, and so this assumption would not generally be justified.

Consider, for example, the colourings of the unit 2-sphere \( S^2 \) discussed by Meyer (and described in more detail by Havlicek et al [13]). These colourings have the property that, given any \( n \in S^2 \), every neighbourhood of \( n \) contains infinitely many points evaluating to 0, and infinitely many points evaluating to 1. Suppose, now, that one performs a finite precision measurement of the nominal observable \((n \cdot \hat{S})^2\) and obtains (say) the value 1. Then one knows that, there is a point \( n' \) in some small neighbourhood \( U \) of \( n \) which evaluates to 1. However, \( U \) contains infinitely many points evaluating to 1, and infinitely many points evaluating to 0. The process does not reveal any more information, as to which particular point takes which particular value, than could be obtained by tossing a coin. Consequently, it cannot be considered a classical measurement. It is not a classical measurement for the same reason that random number generators cannot be used to make classical measurements of length.

Essentially the same criticism applies to an arbitrary model of MKC type. In the general case the highly discontinuous behaviour just discussed may not occur everywhere, on the whole of \( S^2 \). However, there will always be a non-empty open subset of \( S^2 \) on which such behaviour occurs.

The detailed proof of this statement is given in Appendix 3. We will confine ourselves here to summarising the main points.

Consider an arbitrary MKC colouring \( f:S^2_0 \to \{0, 1\} \). Here \( S^2_0 \) is any dense, KS-colourable subset of the unit 2-sphere \( S^2 \) having the property that the set of triads contained in \( S^2_0 \) is dense in the space of all triads (MKC assume that \( S^2_0 \) is also countable; however the argument which follows does not depend on this assumption).

We define the discontinuity region \( D \subseteq S^2 \) to consist of those points \( n \in S^2 \) with the property that each neighbourhood of \( n \) contains infinitely many points evaluating to 0, and infinitely many points evaluating to 1 (it is not assumed that \( n \) itself \( \in S^2_0 \)). We define the continuity region \( C \subseteq S^2 \) to consist of those points \( n \in S^2 \) with the property that \( f \) is continuous on \( U \cap S^2_0 \), for some neighbourhood \( U \) of \( n \). In the case of the Meyer colourings \( C = \emptyset \) and \( D = S^2 \). However, this is not true generally.

It is readily verified that these regions partition \( S^2 \) (the complete unit 2-sphere, not just \( S^2_0 \)) into two disjoint subsets: \( C \cup D = S^2 \) and \( C \cap D = \emptyset \). It is also readily verified that \( C \) is open and \( D \) is closed.

The key result, proved in Appendix 3, is that \( C \) is not only open, but also KS-colourable. Furthermore, the KS-colouring \( f \) defined on \( C \cap S^2_0 \) uniquely extends to a continuous, induced KS-colouring \( f:C \to \{0, 1\} \).

We show in Appendix 3 that this result places significant constraints on the minimum size of the discontinuity region. In the first place, \( D \) must have non-empty interior (i.e. it must contain a non-empty open subset—implying that
it cannot simply consist of a set of isolated points or lines. In the second place, there is a fixed, model-independent, non-zero lower bound on the solid angle subtended by $D$.

Let us now consider the bearing of these results on MKC’s claim, to have “nullified” the KS-theorem. The argument we gave above (in connection with the Meyer colourings) shows that the $f$-value assigned to a specific point $\in D \cap S_0^2$ cannot be ascertained by any quantum measurement and is, in this sense, inaccessible. On the other hand the $f$-value assigned to a specific point $\in C \cap S_0^2$ can be ascertained by a quantum measurement, and so it is accessible (provided the observable whose value is revealed is sufficiently close to the nominal observable). It is easy to see that the same is also true of the $f$-value assigned to a specific point in $C$ (where $f$ is the induced KS-colouring defined above).

From an empirical point of view it is not the $f$-values defined on $S_0$ which are significant, but the $f$-values defined on $C$. It is only when $n \in C$ that a quantum measurement of $(n \cdot \hat{S})^2$ can be interpreted as a measurement in the classical sense, which reveals the corresponding $f$-value (provided the observable whose value is revealed is sufficiently close to the nominal observable).

The fact that there exist non-empty, KS-colourable open subsets of $S^2$ is trivial (consider, for example, the cone $C = \{n \in S^2: |n_z| > 1/\sqrt{2}\}$). There is no sense in which this fact can validly be said to nullify the KS theorem.

In conclusion, it can be seen that the finely calculated approach of MKC, where one marks in individual points using a pencil, gets no nearer to “nullifying” the KS theorem than does a cruder approach, where one paints in whole open regions using a brush. It can also be seen that a would-be nullifier gets no nearer to his/her goal by postulating that a measurement reveals the value of an observable which is typically not the nominal observable.

IV. NULLIFYING MADE EASY

This section contains some supplementary considerations. Its purpose is to reinforce the point made in the last section that, for a model to nullify the KS theorem, rather more is required than the bare fact, that a measurement always reveals the pre-existing value of something.

If the KS theorem could be nullified in the manner that MKC propose, then one could achieve that end more straightforwardly, without appealing to MKC’s sophisticated set-theoretic constructions, by means of one of the following, much less sophisticated models.

Since it is a hidden variables theory that is in question, Ockham’s razor does not apply, and so we are free to multiply hypothetical entities at will. Let us accordingly postulate that, corresponding to each observable $(n \cdot \hat{S})^2$ of the conventional theory, there exist two beables $s_A(n)$ and $s_B(n)$. Each beable is assigned the value 0 or 1. We postulate that these values fluctuate randomly, and independently, but in such a way that, for each $n$, every time, exactly one of the beables $s_A(n)$, $s_B(n)$ is assigned the value 0, and exactly one is assigned the value 1. In every other respect the model coincides with conventional quantum mechanics. We are thus assuming that the values taken by the beables have no causal influence on the outcome of a measurement—or, for that matter, on anything else. Their function is simply to exist. We will refer to this as model 1.

It is trivially true that, in model 1, the effect of measuring $(n \cdot \hat{S})^2$ is to reveal the pre-existing value of one of the two quantities $s_A(n)$ or $s_B(n)$. This is to be compared with the situation in the MKC models, where the effect is to reveal the value of one out of an infinite set of vectors contained in some small neighbourhood of $n$. It would consequently seem that, if one is to accept that the MKC models nullify the KS theorem, then one must accept that model 1 nullifies it equally well.

Of course, it is intuitively evident that the KS theorem is not really nullified by model 1. However, it is worth examining the reason for this.

Model 1 does not nullify the KS theorem because, even though the measurement reveals a pre-existing value, the experimenter does not learn anything thereby. The experimenter already knows, before the measurement, just from the way the model is defined, that exactly one of the two quantities $s_A(n)$ or $s_B(n)$ takes the value 0. The experimenter does not know more after the measurement. The procedure cannot be used to acquire additional knowledge concerning $s_A(n)$, $s_B(n)$ and so it is not, in the classical sense, a measurement of these quantities. In short, model 1 does not nullify the KS theorem for essentially the same reason that the MKC models do not nullify it.

It is also illuminating to consider the following, slightly different model, which we will refer to as model 2. Model 2 is the same as model 1, except that the quantities $s_A(n)$, $s_B(n)$ are not assumed to fluctuate. Instead, it is postulated that, for each $n$, $s_A(n)$ always takes the value 0, and $s_B(n)$ always takes the value 1.

In model 2 a measurement of $(n \cdot \hat{S})^2$ not only reveals the pre-existing value of $s_A(n)$ or $s_B(n)$: the experimenter even knows which of the two quantities took this value. Nevertheless, there is still no nullification. Suppose that an experimenter measures $(n \cdot \hat{S})^2$ and obtains the value 0. Then, on the assumptions of model 2, the experimenter knows that $s_A(n)$ took the value 0. However, the experimenter knew this already, before carrying out the measurement.
The procedure has not conveyed any additional information, regarding $s_A(n)$, and so it is not, in the classical sense, a measurement of $s_A(n)$.

The triviality of the constructions makes it evident that models 1 and 2 do not really nullify the KS theorem. However, the underlying fallacy is essentially the same as the fallacy underlying the much more sophisticated arguments of MKC. The only difference is that, in the case of the MKC models, partly because the constructions are so much more sophisticated, the fallacy is less obvious.

V. KS THEOREM FOR NON-IDEAL MEASUREMENTS

In Sections II–IV we established the negative result: that finite precision does not “nullify” the KS theorem. The purpose of this section is to establish the positive result: that the KS theorem has a natural generalization, applying to the non-ideal measurements which might actually be performed, in the laboratory.

The original version of the KS theorem shows

**Proposition 1** No hidden variables interpretation can have the property that, for every observable $\hat{A}$, an ideal measurement of $\hat{A}$ always reveals the pre-existing value of $\hat{A}$.

—meaning that ideal quantum measurements cannot systematically be interpreted as ideal classical measurements, of the same observables.

The generalized version of the KS theorem shows

**Proposition 1′** No hidden variables interpretation can have the property that, for every observable $\hat{A}$, a near-ideal measurement of $\hat{A}$ always has a high probability of revealing a good approximation to the pre-existing value of $\hat{A}$.

—meaning that near-ideal quantum measurements cannot systematically be interpreted as near-ideal classical measurements, of the same observables.

Proposition 1′ is a qualitative statement, since the terms “near-ideal”, “high probability” and “good approximation” are not sharply defined. In the following we will prove an inequality which gives unambiguous, quantitative expression to Proposition 1′, as it applies to the standard example of a spin 1 system (see Inequality (3) below).

Proposition 1′ does not exclude the possibility that a near-ideal quantum measurement may reveal the pre-existing value of something other than the nominal observable. As we saw in Sections II and IV this possibility has no bearing on the essential point of the KS theorem, which is to show that there is a radical difference between the quantum mechanical and classical concepts of measurement.

Proposition 1′ does not only exclude the possibility of interpreting quantum measurements as ideal classical measurements. This is important, since there would otherwise be no conflict with classical assumptions. One does not classically expect a real, laboratory measurement to be strictly ideal (in the classical sense).

The argument which follows was inspired by the work of Simon et al. and Larsson (SBZL). However, we do not use their “operational” definition of contextuality. This makes the relationship with the original KS theorem much clearer. Furthermore, there is a loophole in SBZL’s argument, which means that the result stated by them is actually invalid (operational non-contextuality is not inconsistent with the empirical predictions of quantum mechanics). Lastly, the result we prove implies a significantly different conclusion, as to the conditions which must be satisfied in order to refute non-contextual theories experimentally. For further discussion of SBZL’s approach see the end of this section, and Appendix 3.

Consider the standard case of a spin-1 system, with spin vector $\hat{S}$. Consider an apparatus which can be used to perform a joint, possibly non-ideal measurement of the observables $(e_1 \cdot S)^2$, $(e_2 \cdot S)^2$, $(e_3 \cdot S)^2$ for any orthonormal triad $\tau = \{e_1, e_2, e_3\}$.

We are here assuming that it is physically possible to (non-ideally) measure every triad $\tau$—contrary to what is postulated by MKC. The justification for this assumption is that, if it is physically possible to (non-ideally) measure one triad, then it follows from rotational invariance, together with the argument in Section II and Appendix 3 of this paper, that it is physically possible to (non-ideally) measure every other triad. Suppose, for instance, that one is given an apparatus which (non-ideally) measures the triad $\tau$, and one wants to measure the triad $\tau'$ which is obtained by rotating $\tau$ through the Euler angles $\theta, \phi, \psi$. Then one simply rotates the apparatus through these angles. The fact that the rotation cannot be performed with infinite precision does not mean that the apparatus may not measure $\tau'$ at all. It only means that the measurement of $\tau'$ may be less nearly ideal than would otherwise be the case (i.e. it may discriminate the eigenstates of $\tau'$ less reliably than would otherwise be the case). For further discussion of the practical realization of such measurements see Swift and Wright [33].
Let \( p_q(\tau, \psi) \) be the probability that a measurement of \( \tau \) results in one of the "illegal" combinations 000, 001, 010, 100, 111, given that the system was prepared in the state \( \psi \). Define

\[
\epsilon_q = \sup_{\tau, \psi} \{ p_q(\tau, \psi) \}
\]  

(1)

The quantity \( \epsilon_q \) partially characterizes the degree of non-ideality of the quantum measuring device (the characterization is only partial because the condition \( \epsilon_q = 0 \), though necessary, is not sufficient for strict ideality). It corresponds to SBZL’s quantity \( \epsilon \) (mutatis mutandi).

Let \( f_\lambda(n) \) be the pre-existing value of \((n \cdot \hat{S})^2\) when the hidden state is \( \lambda \) (it should be stressed that \( f_\lambda \) is defined for every \( n \), and so it is not a KS-colouring). Let \( p_c(\tau, \lambda) \) be the probability that, with the system initially in the hidden state \( \lambda \), a measurement of \( \tau \) fails to reveal the \( f_\lambda \)-values of \( \tau \). Define

\[
\epsilon_c = \sup_{\tau, \lambda} \{ p_c(\tau, \lambda) \}
\]  

(2)

The quantity \( \epsilon_c \) characterizes the degree of non-ideality of the apparatus regarded as a classical measuring device, whose function is to reveal pre-existing values. It should be noted that \( \epsilon_c \) depends on the particular hidden variables theory considered. Unlike \( \epsilon_q \) it has no empirical significance.

We will now prove

\[
N\epsilon_q + \epsilon_c \geq 1
\]  

(3)

where \( N \) is the smallest number of triads which do not admit a consistent colouring.

This inequality shows that \( \epsilon_q \) and \( \epsilon_c \) cannot both be made arbitrarily small. It thus gives precise, quantitative expression to the qualitative statement of Proposition 1’.

Proposition 1’ is a general statement, so it allows for the possibility that a measurement may reveal a good approximation to the pre-existing value, without revealing it exactly. In the case considered here there are only two possible values, and so this contingency cannot arise.

To prove this inequality, consider the probability \( p_i(\tau, \lambda) \) that, when the hidden state is \( \lambda \), a measurement of \( \tau \) gives one of the “illegal” results 000, 001, 010, 100, 111. We have, for all \( \psi \),

\[
p_q(\tau, \psi) = \int_\Lambda p_i(\tau, \lambda) \, d\nu_\psi
\]  

(4)

where \( \Lambda \) is the hidden state space (of the system by itself), and \( \nu_\psi \) is the probability measure on \( \Lambda \) which corresponds to the quantum state \( \psi \).

Now choose a KS-uncolourable set which contains the smallest possible number of distinct orthornormal triads. Let \( \tau_1, \ldots, \tau_N \) be an enumeration of these triads. By construction there exists, for each \( \lambda \), an index \( r_\lambda \) such that the \( f_\lambda \)-values of \( \tau_{r_\lambda} \) are “illegal”. It is then a straightforward consequence of the definitions that

\[
1 - p_i(\tau_{r_\lambda}, \lambda) \leq p_c(\tau_{r_\lambda}, \lambda)
\]  

(5)

implying

\[
p_i(\tau_{r_\lambda}, \lambda) \geq 1 - \epsilon_c
\]  

(6)

for all \( \lambda \). Taking this result in conjunction with Eqs. (1) and (4) we deduce

\[
N\epsilon_q \geq \sum_{r=1}^N p_q(\tau_r, \psi) \geq \int \left( \sum_{r=1}^N p_i(\tau_r, \lambda) \right) \, d\nu_\psi \geq \int p_i(\tau_{r_\lambda}, \lambda) \, d\nu_\psi \geq 1 - \epsilon_c
\]  

(7)

— which is the result claimed.

It is worth remarking that this argument actually establishes the stronger, state-dependent inequality

\[
N\epsilon_q(\psi) + \epsilon_c \geq 1
\]  

(8)

where \( \epsilon_q(\psi) = \sup_\tau \{ p_q(\tau, \psi) \} \).

If the measurement faithfully reveals pre-existing values then Inequality (8) implies \( N\epsilon_q \geq 1 \). This is similar to the result claimed by SBZL [17] [19], that \( N\epsilon_q \geq 1 \) in any theory which is operationally non-contextual (substituting our \( \epsilon_q \) for their \( \epsilon \)). However, it is not the same because a theory which does not faithfully reveal pre-existing values may
still be non-contextual in the sense of SBZL’s operational definition. SBZL’s claim is actually incorrect, since there exist operationally non-contextual theories with the property that \( N \epsilon_q = 0 \) (see Appendix C).

Simon et al’s \[17\] statement, that there is a conflict with classical assumptions as soon as \( \epsilon_q < 1/N \), also requires modification. Suppose that \( \epsilon_q \) is only slightly less than \( 1/N \). Then it is true that the classical failure probability cannot be strictly zero. However, it may still be very small—which would be entirely consistent with classical assumptions regarding a real laboratory instrument. Indeed, the possibility is not excluded that \( 0 < \epsilon_c \ll \epsilon_q < 1/N \)—implying that the apparatus actually functions much better when regarded from the classical point of view, as an instrument for revealing pre-existing values, than it does when regarded from the quantum point of view, as an instrument for discriminating eigenstates. It follows that, in order to demonstrate an experimental conflict with the classical concept of measurement, it is not enough simply to construct an apparatus for which \( \epsilon_q < 1/N \). Instead, one must reduce \( \epsilon_q \) to the point where the lower bound on \( \epsilon_c \) becomes \( \sim 1 \).

The purpose of this section was to establish the basic point of principle, that the KS theorem does have a valid generalization applying to non-ideal measurements. For this purpose it was sufficient to confine ourselves to the single example of a spin 1 system. However, it would clearly be interesting to examine other examples. Indeed, we have not shown that Inequality \( \text{(3)} \) is the only, or even the most useful inequality applying a spin 1 system. One would expect there to be many different ways to give quantitative expression to the qualitative principle represented by Proposition 1’. This is a question which requires further investigation.

VI. KENT AND CLIFTON’S POVM ARGUMENT

In this section we briefly consider the argument of Kent \[7\] and Clifton and Kent \[8\] to (as they put it) “rule out falsifications of non-contextual models based on generalized observables represented by POVM measures”. We include this section for the sake of completeness. A reader who is so inclined may skip to the next section without loss of continuity.

A POVM (or positive operator valued measure \[11,12,13\]) is an indexed set of positive operators \( \hat{F}_i \) with the property \( \sum_i \hat{F}_i = 1 \) (following Kent and Clifton we take the set to be finite). Kent and Clifton take the view that to each distinct POVM there corresponds a distinct generalized observable. A generalized observable, in effect, simply is a POVM.

Kent and Clifton consider that a non-contextual theory of generalized observables would be one in which generalized measurements reveal the pre-existing truth values of the corresponding POVMs. Their argument to show that such theories exist is similar to the argument which, they claim, nullifies the ordinary KS theorem. The objection to it is also similar.

Let \( \mathcal{A} \) be the set of all positive operators. Kent and Clifton postulate that the POVM which is actually measured is always contained in a certain dense subset \( \mathcal{A}_d \subset \mathcal{A} \), which is colourable in a generalized sense. If \( \hat{F}_i \) is the POVM that is actually measured, then the outcome will be the unique index \( i_0 \) for which \( \hat{F}_{i_0} \) evaluates to “true”.

The objection to this argument is essentially the point we have already made, in Sections III and IV. The problem is that the experimenter does not know the identity of the positive operator whose truth value is revealed. Suppose that an experimenter measures the nominal POVM \( \hat{F}_i \), and obtains the result \( i_0 \). Then s/he knows that there is another, typically different POVM \( \hat{F}_{i_0}' \) for which \( \hat{F}_{i_0}' \) is “true”. The experimenter knows that something is “true”. However, the experimenter does not know what that something is. Consequently, the procedure is not genuinely informative.

If one is told “proposition \( p \) is true”, but is not given any indication what the symbol \( p \) denotes, then one has not really been told anything.

Kent and Clifton consider that their POVM argument includes, as a special case, an alternative proof that non-ideal measurements of conventional observables can be explained non-contextually \[17\]. Let us consider how this might work.

POVMs play an important role in the mathematical description of non-ideal measurements of conventional observables. Consider, for example, the non-ideal measurement of the conventional observable \( \hat{A} = \sum_a a |a\rangle \langle a| \) described by Eq. (A2) in Appendix A. Define

\[
\hat{E}_a = |a\rangle \langle a| + \sum_b \epsilon_{b,aa} |a\rangle \langle b| + \sum_b \epsilon^*_{b,aa} |b\rangle \langle a| + \sum_{b_1,b_2,c} \epsilon^*_{b_1,ca} \epsilon_{b_2,ca} |b_1\rangle \langle b_2|
\]  

It is readily verified that the operators \( \hat{E}_a \) constitute a POVM. Their physical significance is that, if the system is initially in the state \( |\psi\rangle \), then \( \langle \psi| \hat{E}_a |\psi\rangle \) is the probability that the result of the measurement will be \( a \).

Let us note, parenthetically, that it may be questioned whether the concept of a generalized observable is appropriate in the case of POVMs which arise in this way, as part of the mathematical description of non-ideal measurements of conventional observables \[11,12,13\].
Clifton and Kent consider that, in the case of the measurement process described by Eq. (A2), what is actually measured is, not really A at all, but rather the POVM \( \hat{E}_a \). Of course, the unitary operator \( U \) in Eq. (A2), and the POVM \( \hat{E}_a \) defined in terms of it, could not, in practice, be known, with infinite precision. Consequently, Clifton and Kent’s POVM argument, when specialized to non-ideal measurements of conventional observables, establishes the following: there exist models in which the effect of non-ideally measuring an observable \( A = \sum a |a\rangle\langle a| \) is to reveal the pre-existing truth values of some unknown POVM \( \hat{E}_a \approx |a\rangle\langle a| \). Conceived as a way of nullifying the KS theorem this result is no more effectual than Proposition 2, which we criticized in Sections III and IV.

VII. CONTEXTUALITY OF SEQUENTIAL ANALYZERS

In this section we discuss the predictions of the MKC models regarding sequential measurements, using three separate analyzers. If the arrangement is near-ideal, then the observable whose value is revealed by one analyzer must depend on the settings of the other two analyzers. In other words, it must depend on the overall experimental context.

This result serves to reinforce the conclusion reached earlier, that the MKC models are highly non-classical. It also raises the question, whether the MKC models are capable of reproducing the empirical predictions of quantum mechanics—as discussed in the next section.

The argument which follows is an improved version of an argument given in Appleby [15]. In the first place we have improved the argument so as to take account of the points made in Section II of this paper. In the second place, the version we give now does not involve joint measurements of non-commuting observables. In the third place, we have strengthened the argument, using ideas derived from the subsequent work of SBZL.

Consider sequential measurements of the observables \((e_1 \cdot \hat{S})^2, (e_2 \cdot \hat{S})^2, (e_3 \cdot \hat{S})^2\). The system passes through a succession of analyzers, which measure each observable in turn.

FIG. 2. Sequential measurement of \((e_1 \cdot \hat{S})^2, (e_2 \cdot \hat{S})^2, (e_3 \cdot \hat{S})^2\). The system passes through a succession of analyzers, which measure each observable in turn.

Consider sequential measurements of the observables \((e_1 \cdot \hat{S})^2, (e_2 \cdot \hat{S})^2, (e_3 \cdot \hat{S})^2\), using the apparatus illustrated in Fig. 2. The apparatus consists of three identical, non-ideal analyzers chained together. \( \hat{S} \) denotes the spin vector of a spin 1 system, and \( \{e_1, e_2, e_3\} \) is an arbitrary orthonormal triad. The assumption that \( \{e_1, e_2, e_3\} \) can be chosen arbitrarily is justified in Section V.

The fact that there may be errors in the alignment of the analyzers is already included in the fact that the analyzers are not assumed ideal (see Section II). It may therefore be assumed that the nominal triad \( \{e_1, e_2, e_3\} \) is strictly orthogonal.

Suppose that one of these analyzers is used on its own, to measure the single observable \((n \cdot \hat{S})^2\). MKC postulate that the measurement will reveal the value of some neighbouring vector \( n' \in S_0^3 \) (where, as usual, \( S_0^3 \) denotes the dense, KS-colourable set on which the MKC valuations are defined). For each \( n \) there is a probability measure \( \mu_n \) such that \( \mu_n(A) \) is the probability that \( n' \in A \subseteq S_0^3 \).

Now consider the situation when the analyzers are used jointly, to measure \((e_1 \cdot \hat{S})^2, (e_2 \cdot \hat{S})^2, (e_3 \cdot \hat{S})^2\). Let \( \{e_1', e_2', e_3'\} \) be the triad whose values are revealed by the measurement. If the behaviour of each analyzer is independent of the context in which it is used, then the probability that \( \{e_1', e_2', e_3'\} \in A_1 \times A_2 \times A_3 \) must be \( \mu_{e_1'}(A_1)\mu_{e_2'}(A_2)\mu_{e_3'}(A_3) \).

It can be seen that, if the behaviour of the analyzers is context-independent, then the triad \( \{e_1', e_2', e_3'\} \) is not guaranteed to be strictly orthogonal (unlike the triad \( \{e_1, e_2, e_3\} \), which is orthogonal by assumption). Consequently, the measurement outcome is not guaranteed to be one of the “legal” combinations 110, 101, 011. In Appendix B we use this to show that, if the analyzers are sufficiently near-ideal, then the assumption of context-independence conflicts with the empirical predictions of quantum mechanics. Specifically, we show that context-independence implies

\[
N \epsilon_q \geq \frac{1}{2}
\]

where \( \epsilon_q \) is the quantum non-ideality index defined by Eq. (1), and \( N \) is the smallest number of distinct orthonormal triads which cannot be consistently coloured.
Inequality (10) shows that, if the analyzers are sufficiently near-ideal, so that $\epsilon_q < 1/(2N)$, then the observable whose value is revealed by one analyzer must depend on the settings of the other two analyzers. This is a form of contextuality.

Inequality (10) was derived on the assumption that a measurement of $(\mathbf{n} \cdot \hat{\mathbf{S}})^2$ always reveals the pre-existing value of $\mathbf{n}' \in S^6_0$. Suppose we make the weaker assumption, that the value of $\mathbf{n}'$ is only revealed with probability $\geq 1 - \delta_c$. Then a similar argument to that given in Appendix B shows

$$N\epsilon_q + \delta_c \geq \frac{1}{2}$$

(11)

if the behaviour of the analyzers is context-independent. This inequality should be compared with Inequality (8).

It should be noted that essentially the same argument goes through in the case where the measurements are made non-sequentially. We have chosen to focus on the sequential case because that is where the conflict with classical assumptions emerges most clearly.

Kent [13] has argued that sequential measurements are not joint measurements. According to him a process does not count as a joint measurement unless the different observables are measured “all at once”, in parallel so to speak. We consider this objection to be unjustified, for the reasons explained in Appendix E.

VIII. ARE THE MKC MODELS EMPIRICALLY EQUIVALENT TO QUANTUM MECHANICS?

The purpose of this section is to discuss the question, whether there actually exists a complete theory of MKC type which is empirically equivalent to quantum mechanics. In particular, we will discuss a recent claim by Cabello [20], that theories of MKC type make empirically testable predictions which definitely conflict with those of quantum mechanics. Our conclusion will be that, although Cabello makes some very pertinent points, the question remains open.

To begin with it should be noted that, as Clifton and Kent have themselves emphasize, the MKC models are, as they stand, incomplete, since they do not include any dynamical postulate. The result proved in the last section illustrates this point. We showed that, if the analyzers in Fig. 2 are sufficiently near-ideal, MKC’s assumptions require there to be a delicately adjusted “conspiracy”, whereby the behaviour of one analyzer is influenced by the settings of the other two analyzers in just that particular manner which is required to secure agreement with the empirical predictions of quantum mechanics. In a complete hidden variables theory this conspiracy would be explained in terms of the dynamics of the hidden variables describing the system+apparatus+environment composite. However, MKC do not consider the dynamics. Consequently, their account does not show that it is actually possible to construct a complete, dynamical theory having all the required properties.

Let $\mathcal{P}_d$ be the set of projections on which the MKC valuations are defined, and let $\Lambda$ be the hidden state space. Clifton and Kent show that one can associate to each $\hat{P} \in \mathcal{P}_d$ a function $P: \Lambda \to \{0, 1\}$, and to each quantum state $\psi$ a probability measure $\nu_\psi$ such that

$$\langle \psi | \hat{P} | \psi \rangle = \int_\Lambda P(\lambda) \, d\nu_\psi$$

(12)

Let us call this their statistical theorem.

In order to obtain a complete hidden variables theory, they would need to supplement this result with a dynamical theorem. They would need to show that there exists a one-parameter family of mappings $F_t: \Lambda \to \Lambda$ such that $F_{t_1} \circ F_{t_2} = F_{t_1 + t_2}$ and

$$\nu_{\psi_t} = \nu_\psi \circ F_t^{-1}$$

(13)

for all $t$, $\psi$ (where $\psi_t$ denotes the Schrödinger picture state into which $\psi$ evolves after a time $t$). It is a non-trivial question, whether such mappings actually exist.

There are three weaknesses in MKC’s argument. MKC’s mathematical results—the existence of dense KS-colourable sets, and Clifton and Kent’s statistical theorem—are clearly formulated, and rigorously proved. However, the physical part of their argument, to show that these results have the effect of “nullifying” the KS theorem, is much less clear, as Cabello and Mermin have stressed. The counter-argument we presented in Sections II–IV was directed at this point of weakness. We showed that the physical part of MKC’s argument disintegrates when subjected to careful analysis.

The second point of weakness is the lack of any dynamical theorem. If it could be shown that it is definitely not possible to define a dynamical evolution satisfying Eq. (13), then it would follow that there does not exist a complete theory of MKC type which is empirically equivalent to quantum mechanics.
The third weakness is that MKC only consider joint measurements of commuting observables. In a genuinely non-contextual theory there should be a principle of *approximate* non-contextuality, applying to joint measurements of non-commuting observables.

The argument in Cabello\(^{20}\), however, is not directed at any of these points of weakness. Instead, Cabello claims to have derived a result (his Lemmas 3 and 4) which directly contradicts Clifton and Kent’s statistical theorem. There are only two possibilities here: either there must be a technical mistake in the proof of Clifton and Kent’s statistical theorem, or else there must be a technical mistake in the proof of Cabello’s Lemmas 3 and 4. It appears to us that the mistake is in fact on Cabello’s side.

MKC postulate that the effect of measuring an observable \( \hat{P} \) is to reveal the pre-existing value of some other observable \( \hat{P}' \) which is very close to \( \hat{P} \). Cabello (see footnote 27 in his paper) takes this to imply that a sufficiently high precision measurement of \( \hat{P} \) will, with high probability, reveal the pre-existing value of \( \hat{P} \) itself (if \( \hat{P} \) is in the set \( \mathcal{P}_d \) on which the valuation is defined). In other words he assumes (in the notation of Eq. (12) above) that, if \( \hat{P}' \approx \hat{P} \), then there is a high probability that \( P'(\lambda) = P(\lambda) \). He fails to allow for the fact, which we proved in Section \( \ref{sec:3} \) and Appendix \( \ref{app:a} \) that the MKC valuations are radically discontinuous. This means that a knowledge of the value of \( P' \) does not, in general, convey any information at all regarding the value of \( \hat{P} \), no matter how small the difference \( \hat{P}' - \hat{P} \).

If \( \hat{P}' \approx \hat{P} \) then it follows from Eq. (12) that \( \int_{\Lambda} (P'(\lambda) - P(\lambda)) \, d\nu_\psi \approx 0 \) for all \( \psi \). However, this does not imply \( \int_{\Lambda} |P'(\lambda) - P(\lambda)| \, d\nu_\psi \approx 0 \), as assumed by Cabello.

Cabello’s argument can, in fact, be turned around, to disprove the assumption on which it depends. It follows from Clifton and Kent’s statistical theorem that the conclusion to Cabello’s Lemma 3 is false. So what Cabello has really done is to prove, by *reductio ad absurdum*, that there are cases where \( \int_{\Lambda} |P'(\lambda) - P(\lambda)| \, d\nu_\psi \) is not negligible, even though \( \hat{P}' \approx \hat{P} \). He has similarly proved that there are cases for which there is a non-negligible probability of two nearly orthogonal projectors both evaluating to 1.

Of course, in a genuinely non-contextual theory, there *would* be a high probability of \( \hat{P}' \) having the same value as \( \hat{P} \), if \( \hat{P}' - \hat{P} \) is sufficiently small. The fact that the MKC models do not have this property means that they are not non-contextual—as we showed in Section \( \ref{sec:3} \). However, Cabello’s argument is not directed at MKC’s claim, that their models are non-contextual. It is directed, instead, at MKC’s claim, that their models reproduce the empirical predictions of quantum mechanics. There is no evident contradiction between this latter claim and the fact that the valuations in their models are highly discontinuous.

It should, however, be stressed that, although it has not been proved that there does not exist a complete theory of MKC type which is empirically equivalent to quantum mechanics, the contrary proposition has not been proved either. The question, in short, remains open. Furthermore, although Cabello fails to establish his main contention, some of his intermediate results are of considerable interest. It is possible that further investigation along these lines would show that Cabello’s contention is, after all, correct. However, it appears to us that such a programme is unlikely to succeed unless it is directed at one of the points of weakness indicated above, rather than at Clifton and Kent’s statistical theorem (which seems to have been rigorously proved).

**IX. CONCLUSION**

A substantial part of this paper has been devoted to a criticism of MKC. We ought to stress, by way of conclusion, that we actually consider that they have made a very important contribution to this subject. We have argued that MKC’s claim, that their models “nullify” the KS theorem, was premature; in fact, fallacious. However, the models themselves are deeply interesting. They confirm the fact that there are depths to the KS theorem which, in the past, have not been sufficiently appreciated (“confirm” because this point had, in fact, already been established by the constructions of Pitowsky\(^{14}\)).

MKC take it to be almost self-evident that the existence of dense KS-colourable subsets of \( S^2 \) “nullifies” the KS theorem. We have argued that they are mistaken, and that there is actually no nullification. However, the mistake is far from obvious. MKC’s contention seems, at first sight, highly plausible. At least, it seemed highly plausible to the present author, before he began to seriously engage with the problem.

In order to see that the MKC models do not “nullify” the KS theorem it is necessary to appreciate the point made in Section \( \ref{sec:3} \) that a process does not count as a classical measurement unless it reveals, not simply a pre-existing value, but a pre-existing piece of classical information. To put it another way, the outcome of a classical measurement is, not simply a bare number, but a determinate proposition, which expresses some pre-existing classical fact. It is also necessary to appreciate the points made in Section \( \ref{sec:3} \) and Appendix \( \ref{app:a} \) regarding non-ideal quantum measurements. In previous accounts these points have not been explicitly recognized. They are, however, crucial, as the MKC models demonstrate.
Although the MKC models do not “nullify” the KS theorem, they do cast important new light on the physical implications of the KS theorem. They show that there are subtleties to the physical interpretation of the KS theorem which have not, in the past, been sufficiently emphasized. For this reason the work of MKC is valuable.

Although we have presented our discussion in the form of a criticism it might, in some ways, be more appropriate if it was seen, in terms which are less prejudicial, as one more contribution to a line of investigation which Pitowsky \[9\] initiated, and which MKC have continued. It is a line of investigation which might be worth pursuing further.

Pitowsky’s constructions are particularly interesting in this respect, because he challenges traditional assumptions at a much more fundamental level. For instance, he challenges the assumption that sets which are not Lebesgue-measurable have no physical significance. Pitowsky’s idea, that “one can, as it were, reduce physical paradox to a mathematical ‘pathology’ ”, is unlikely to lead to a classical explanation of quantum phenomena (a “pathological” theory—which is to say, a theory that seems strange as judged by the standards of ordinary intuition—is a non-classical theory, almost by definition). However, that does not seem to be Pitowsky’s intention. Rather, he seeks to gain additional insight, into what quantum mechanics is actually telling us, by considering the problem from a new and unusual perspective. It is a deeply interesting idea, which deserves to be more widely known.

**Acknowledgments**

It is a pleasure to thank Prof. A. Zeilinger for his hospitality during the programme “Quantum Measurement and Information” held at ESI in Vienna. It is also a pleasure to thank N.D. Mermin, A. Peres, A. Kent, P. Busch, D. Home, K. Svozil, I. Pitowsky, J. Larsson, G. Mahler and two anonymous referees for their stimulating and helpful comments.

**APPENDIX A: APPROXIMATE VON NEUMANN MEASURMENTS**

In Section \[4\] we criticized MKC’s concept, of the single observable that is *actually* measured, for the special case of the Stern-Gerlach apparatus. The purpose of this Appendix is to show that our argument generalizes, to the case of any approximate von Neumann measurement (i.e. any measurement described by a unitary operator which is close to the unitary operator describing a von Neumann measurement).

Consider, to begin with, a strict von Neumann measurement of the observable $\hat{A} = \sum_{a \in \mathcal{E}} a |a\rangle \langle a|$ (where $\mathcal{E}$ is the set of eigenvalues) acting on an $n$-dimensional Hilbert space $\mathcal{H}$. For the sake of simplicity we assume that $\hat{A}$ is non-degenerate. In order to perform the measurement $\hat{A}$ is coupled to a pointer observable $\hat{\mu} = \sum_{a \in \mathcal{E}} a |a\rangle_{pt} \langle a|$ acting on a different $n$-dimensional Hilbert space $\mathcal{H}_{pt}$ (the state space of the measuring instrument). The measurement interaction causes the evolution

$$\hat{U}|a\rangle \otimes |\phi_0\rangle_{pt} = |a\rangle \otimes |a\rangle_{pt} \quad (A1)$$

for all $a$, where $\hat{U}$ is the unitary evolution operator describing the measurement interaction, and $|\phi_0\rangle_{pt}$ is the (fixed) initial state of the measuring instrument.

The measurement described by Eq. \[(A1)\] is ideal because \[[3]\]

1. It discriminates eigenstates with perfect reliability (if the system is initially in the eigenstate $|a\rangle$, then the final pointer reading will certainly be $a$).

2. The final state correlations are perfect (if the final pointer reading is found to be $a$, then the final state of the system is $|a\rangle$).

In practice it would be difficult, if not impossible to construct an apparatus such that $\hat{U}$ satisfied Eq. \[(A1)\] *precisely*. A more realistic model of the kind of measurement which it might actually be possible to perform, in the laboratory, would be

$$\hat{U}|a\rangle \otimes |\phi_0\rangle_{pt} = |a\rangle \otimes |a\rangle_{pt} + \sum_{b,c \in \mathcal{E}} \epsilon_{a,bc} |b\rangle \otimes |c\rangle_{pt} \quad (A2)$$

where the coefficients $\epsilon_{a,bc}$ are small.

Of course, the model measurement process described by Eq. \[(A2)\], though more realistic than the one described by Eq. \[(A1)\], is still highly idealized. A real laboratory instrument is a complex macroscopic system having many other degrees of freedom apart from the pointer observable. It is also an open system, undergoing non-negligible interactions
with its environment (these interactions play an important role in accounting for the fact that system+instrument are in a mixed state at the end of the measurement process [44]). Lastly, it should be noted that in a real laboratory measurement the final state correlations are often (though not always) very poor (for instance, in the traditional Stern-Gerlach arrangement, where the particle hits a photographic plate, the result of the measurement does not contain useful information regarding the final spin state of the particle). However, the model is sufficiently realistic for present purposes.

The unitary operator $\hat{U}$ in Eq. (A2) is close to the unitary operator describing a von Neumann measurement. In other words, it describes an approximate von Neumann measurement. The process is not an ideal measurement of $\hat{A}$. It is, however, a non-ideal measurement (which is the most that can reasonably be demanded of a real laboratory procedure).

Specifically:

1′. It discriminates eigenstates with a high degree of reliability (if the system is initially in the eigenstate $|a\rangle$, then there is a high probability that the final pointer reading will approximately be $a$).

2′. The final state correlations are good (if the final pointer reading is found to be $a$ then $\text{Tr}((\hat{A} - a)^2 \hat{\rho}) \approx 0$, where $\hat{\rho}$ is the reduced density matrix describing the final state of the system, after the pointer has been read).

These properties should be compared with the defining characteristics of an ideal measurement (items 1 and 2 above).

We now show that the process also effects a non-ideal measurement of every other observable which is sufficiently close to $\hat{A}$.

Choose some continuous parameterization for the ONBs (orthonormal bases) in $\mathcal{H}$, so that $|a, \nu\rangle$ is the ONB with parameter value $\nu$. Let $\nu_0$ be parameter value corresponding to the original ONB: $|a, \nu_0\rangle = |a\rangle$. Then an arbitrary observable $\hat{B}$ can be expressed in the form

$$\hat{B} = \hat{A}(\nu, \delta) = \sum_{a \in \mathcal{E}} (a + \delta_a)|a, \nu\rangle \langle a, \nu|$$

(A3)

for suitable $\nu, \delta$. If $\hat{B}$ is close to $\hat{A}$ then we can choose $\nu \approx \nu_0$ and $\delta \approx 0$.

Eq. (A2) can be re-written in terms of the ONB $|a, \nu\rangle$:

$$\hat{U}|a, \nu\rangle \otimes |\phi_0\rangle_{pt} = |a, \nu\rangle \otimes |a\rangle_{pt} + \sum_{b,c \in \mathcal{E}} \epsilon_{a,bc}(\nu) |b, \nu\rangle \otimes |c\rangle_{pt}$$

(A4)

where the coefficients $\epsilon_{a,bc}(\nu)$ vary continuously with $\nu$. It follows that, if $\nu \approx \nu_0$ and $\delta \approx 0$, then the coefficients $\epsilon_{a,bc}(\nu)$ will be small, and so the process will be a non-ideal measurement of $\hat{A}(\nu, \delta)$ (in the sense of the definition expressed by items 1′ and 2′ above).

We conclude that $\hat{U}$ describes a non-ideal measurement of $\hat{B}$ for every $\hat{B}$ close to $\hat{A}$—as claimed.

MKC assume that, given any non-ideal measurement of $\hat{A}$, there will be a neighbouring observable $\hat{B}$ which is measured ideally, and which they take to be the only observable that is "actually" measured. We noted in Section II that there exists at least one class of non-ideal measurements for which this assumption is invalid: namely, Stern-Gerlach measurements. A Stern-Gerlach measurement is not an ideal measurement of anything; not an ideal measurement of the nominal observable, which the experimeter records as having been measured, and not an ideal measurement of any other observable either [31]. Measurements of this kind might be described as completely non-ideal. It is easily seen that they are, in a certain well-defined sense, typical, or generic.

If the measurement described by Eq. (A2) is of the special kind assumed by MKC then

$$\hat{U}|a, \nu\rangle \otimes |\phi_0\rangle_{pt} = e^{i\theta_a}|a, \nu\rangle \otimes |a\rangle_{pt}$$

(A5)

for some $\nu \approx \nu_0$ and vector $\theta$. We may thus identify the set of all such measurements with a neighbourhood of the identity in the group $U(n)$, having dimension $n^2$. On the other hand, the space of coefficients $\epsilon_{a,bc}$ may be identified with a neighbourhood in the coset space $U(n^2)/U(n^2 - n)$, having dimension $n^2(2n - 1)$. It follows that, if $n > 1$, the set of measurements assumed by MKC constitute a proper, lower-dimensional submanifold of the manifold of all non-ideal measurements. This means that, if one were to select a set of coefficients $\epsilon_{a,bc}$ satisfying the unitarity condition at random, then, for many reasonable choices of probability measure, there would be probability zero that the measurement which resulted was of the special kind assumed by MKC. In any case, the contrary hypothesis, that a randomly chosen laboratory measurement is likely to be of the kind assumed by MKC, cannot be considered plausible.

Suppose, however, that a laboratory measurement of $\hat{A}$ did happen to be of the very special kind assumed by MKC. Then there would be some observable $\hat{B} \approx \hat{A}$ which was measured ideally. However, MKC would still not be justified
in arguing that \( \hat{B} \) is the only observable that is “actually” measured. The fact that the measurement of \( \hat{B} \) is slightly better than the measurement of \( \hat{A} \) does not imply that \( \hat{A} \) is not measured at all. A measurement does not have to be strictly ideal in order to count as a measurement (otherwise one would have to say that a Stern-Gerlach apparatus does not measure spin, and that real photon detectors do not detect photons).

APPENDIX B: LEMMAS CONCERNING THE REGIONS \( C \) AND \( D \).

The purpose of this appendix is to prove those results to which we appealed, without proof, in Section \[\square\]. For ease of reference we state the results as lemmas. The reader should refer to Section \[\square\] for definitions of notation and terminology.

Lemma 1  The continuity region \( C \) is KS-colourable. Furthermore \( f \) extends by continuity to a uniquely defined continuous KS-colouring \( \hat{f} : C \to \{0,1\} \).

Let \( C_0 \) (respectively \( C_1 \)) consist of those vectors \( \mathbf{n} \in C \) having a neighbourhood \( U \) such that \( f(\mathbf{n}) = 0 \) (respectively \( f(\mathbf{n}) = 1 \)) for all \( \mathbf{n}' \in U \cap S_0^2 \).

\( C_0 \) and \( C_1 \) partition \( C \) into two disjoint (open) subsets. We may therefore define a function \( \hat{f} : C \to \{0,1\} \) by setting \( \hat{f}(\mathbf{n}) = 0 \) if \( \mathbf{n} \in C_0 \), and \( \hat{f}(\mathbf{n}) = 1 \) if \( \mathbf{n} \in C_1 \).

Now let \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) be any orthonormal triad \( \subseteq C \). By construction, there exists a triad \( \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} \subseteq C \cap S_0^2 \) such that \( \hat{f}(\mathbf{e}_r) = f(\mathbf{e}'_r) \) for \( r = 1,2,3 \). It follows that \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) must \( \hat{f} \)-evaluate to one of the “legal” combinations 011, 101, 110. A similar argument shows that two orthogonal vectors \( \in C \) cannot both \( \hat{f} \)-evaluate to 0, even if their vector product \( \notin C \).

It follows that \( \hat{f} \) is a KS-colouring of the set \( C \). The claim is now immediate.

Lemma 2  The discontinuity region \( D \) has non-empty interior.

Let \( \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_R\} \) be any finite KS-uncolourable set. The fact that \( C \) (the complement of \( D \)) is KS-colourable means that at least one of these vectors must \( \notin D \). Suppose that the labelling is such that, for some \( r \geq 1 \), \( \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_r\} \subseteq D \) and \( \{\mathbf{n}_{r+1}, \mathbf{n}_{r+2}, \ldots, \mathbf{n}_R\} \subseteq C \).

For each \( s = 1, \ldots, r \) let \( \epsilon_s \) be the angular distance from \( \mathbf{n}_s \) to \( C \), and for each \( t = r + 1, \ldots, R \) let \( \delta_t \) be the angular distance from \( \mathbf{n}_t \) to \( D \). Define \( \delta = \min_{r+1 \leq t \leq R} (\delta_t) \). The fact that \( C \) is open means that \( \delta > 0 \) (unless \( C \) is empty—in which case the claim is immediate).

If it should happen that \( \epsilon_s > 0 \) for some \( s \), then \( \mathbf{n}_s \) is in the interior of \( D \)—which means that the interior of \( D \) is non-empty. Otherwise we can find a suitable rotation, through an angle \( < \delta \), which will take one or more of the vectors \( \mathbf{n}_1, \ldots, \mathbf{n}_r \) into \( C \), while not taking any of the vectors \( \mathbf{n}_{r+1}, \ldots, \mathbf{n}_R \) out of \( C \). After re-labelling this gives us a new KS-uncolourable set \( \{\mathbf{n}'_1, \mathbf{n}'_2, \ldots, \mathbf{n}'_r\} \subseteq D \) and \( \{\mathbf{n}'_{r+1}, \mathbf{n}'_{r+2}, \ldots, \mathbf{n}'_R\} \subseteq C \) for some \( r' < r \).

If it should still happen that none of the vectors \( \mathbf{n}'_1, \ldots, \mathbf{n}'_r \) is in the interior of \( D \) we can repeat the procedure.

It is impossible to rotate all the vectors into \( C \), so after sufficiently many iterations of the procedure at least one of the vectors must be in the interior of \( D \). It follows that the interior of \( D \) cannot be empty.

We conclude by noting that this argument actually establishes the more general statement, that the complement of any open KS-colourable set must have non-empty interior.

Lemma 3  There is a fixed, model-independent, non-zero lower bound on the solid angle subtended by \( D \).

Let us note that it already follows from Lemma 2 that the solid angle subtended by \( D \) must be \( > 0 \), for any given colouring. Lemma 3 implies a stronger statement: that one cannot, by choosing the colouring appropriately, make the solid angle arbitrarily small.

We begin by establishing some notation. Let \( \mu \) be the invariant measure on \( S^2 \), normalized so that \( \mu(S^2) = 1 \) (i.e. the solid angle scaled by \( 1/(4\pi) \)). Let \( \mathcal{O} \) (respectively \( \mathcal{B} \)) be the set of all open (respectively Borel, or \( \mu \)-measurable) subsets of \( S^2 \) which are KS-colourable. Define

\[
\begin{align*}
\mu_{\mathcal{O}} &= 1 - \sup_{U \in \mathcal{O}} (\mu(U)) \\
\mu_{\mathcal{B}} &= 1 - \sup_{B \in \mathcal{B}} (\mu(B))
\end{align*}
\]

(B1) \hspace{1cm} (B2)

Clearly, \( \mu_{\mathcal{O}} \geq \mu_{\mathcal{B}} \) (it is a non-trivial question as to whether \( \mu_{\mathcal{O}} \) actually \( = \mu_{\mathcal{B}} \)).

We will show that \( \mu_{\mathcal{B}} > 0 \). In view of the fact that \( C \in \mathcal{O} \) and \( D = S^2 - C \) this implies
\[ \mu(D) \geq d_\mathcal{O} \geq d_B > 0 \] (B3)

which establishes the claim.

Before proceeding to the proof, let us note that it follows from the last paragraph in the proof of Lemma 2 that \( \mu(U) > 0 \) for all \( U \in \mathcal{O} \). The proof which follows is needed to show that there is not a sequence of sets \( U_n \in \mathcal{O} \) such that \( \mu(U_n) \to 0 \). This may appear intuitively obvious. However, the fact that there exist dense KS-colourable subsets of \( S^2 \) was intuitively surprising (at least to the present writer), and so it behoves us to be careful.

We now proceed to the proof that \( d_B > 0 \).

Choose some fixed, finite, KS-uncolourable set \( \{n_1, n_2, \ldots, n_{2M}\} \subset S^2 \), with the property that \( n_{M+r} = -n_r \) for \( r = 1, \ldots, M \).

Let \( \theta_0 \) be the minimum angular separation of the vectors belonging to this set:

\[ \theta_0 = \min_{1 \leq r,s \leq M} \left( \cos^{-1}(n_r \cdot n_s) \right) \quad (B4) \]

Surround each vector \( n_r \) with a circular patch \( E_r \) of radius \( \theta_0/2 \):

\[ E_r = \{ m \in S^2 : \cos^{-1}(m \cdot n_r) \leq \theta_0/2 \} \quad (B5) \]

Let \( B \) be any set \( \in \mathcal{B} \), and let \( B^c = S^2 - B \) be its complement. We may assume, without loss of generality, that \( B \) is invariant under the parity operation (implying that \( B^c \) is also invariant). By construction, the sets \( E_r \) are non-intersecting, with the possible exception of a set of measure zero on their boundaries. Consequently

\[ \mu(B^c) \geq \sum_{r=1}^{2M} \mu(E_r \cap B^c) \quad (B6) \]

We next define, for each \( r \), a function \( g_r : S^2 \to E_r \) by

\[ g_r(m) = e^{(\theta_0/2) m \cdot L_r} n_r \quad (B7) \]

where \( L_1, L_2, L_3 \) are the generators of \( SO(3) \). Thus, \( g_r(m) \) is the vector obtained by rotating \( n_r \) through a (fixed) angle \( \theta_0/2 \) about the (variable) axis \( m \). It is easy to see that, as \( m \) ranges over the whole of \( S^2 \), the interior of \( E_r \) is covered twice, and the boundary once. Consequently

\[ \mu(E_r \cap B^c) = \frac{1}{2} \int_{B_r^c} J_r(m) \, d\mu \quad (B8) \]

where \( J_r \) is the Jacobian of \( g_r \), and \( B_r^c = g_r^{-1}(E_r \cap B^c) \). Define, for each \( m \in S^2 \),

\[ J(m) = \min_{1 \leq r \leq 2M} (J_r(m)) \quad (B9) \]

Eqs. (B4) and (B8) then imply

\[ \mu(B^c) \geq \frac{1}{2} \left( \int_{\bigcup_{r=1}^{M} B_r^c} J(m) \, d\mu + \int_{\bigcup_{r=M+1}^{2M} B_r^c} J(m) \, d\mu \right) \quad (B10) \]

We now observe that, for each fixed \( m \in S^2 \), the set \( \{g_1(m), g_2(m), \ldots, g_{2M}(m)\} \), being obtained by rotating the KS-uncolourable set \( \{n_1, n_2, \ldots, n_{2M}\} \), must itself be KS-uncolourable. On the other hand, \( B \) is KS-colourable. It follows that, for each \( m \in S^2 \), we must have \( g_r(m) \in B^c \) and \( g_{M+r}(m) \in B^c \) for some \( r \leq M \). Consequently

\[ \bigcup_{r=1}^{M} B_r^c = \bigcup_{r=M+1}^{2M} B_r^c = S^2 \]

and

\[ \mu(B^c) \geq \int J(m) \, d\mu \quad (B11) \]

Since the right-hand side of this inequality is independent of \( B \) we may deduce

\[ \sup_{B \in \mathcal{B}} (\mu(B)) \leq 1 - \int J(m) \, d\mu \quad (B12) \]

To complete the proof we note that \( J(m) \) is a continuous, non-negative function having finitely many (in fact \( 2M \)) zeros. It follows that

\[ d_B \geq \int J(m) \, d\mu > 0 \quad (B13) \]
Concluding Remarks

It is natural to consider, in addition to the sets \( O \) and \( B \) defined above, the set \( C \), consisting of all closed, KS-colourable subsets of \( S^2 \). Define \( d_C = 1 - \sup_{K \in C} (\mu(K)) \). Clearly, \( d_C \geq d_B \). It is a standard result (see, for example, Halmos \[45\], Chap. 10) that, for each Borel set \( B \), there exists a sequence of closed sets \( K_n \subseteq B \) such that \( \mu(B) = \lim_{n \to \infty} (\mu(K_n)) \). It follows that we also have \( d_C \geq d_B \). Combining these inequalities we deduce

\[
d_O \geq d_C = d_B
\]

The numbers \( d_O \) and \( d_C = d_B \) could be regarded as non-classicality indices (the larger they are, the more radically quantum mechanics departs from classical assumptions). From this point of view it would be interesting to know their actual values.

The evaluation of the integral on the right-hand side of Inequality (B13), though straightforward, would be somewhat tedious. We will therefore content ourselves with noting that it follows from Eq. (B5) that

\[
\int J(m) \, d\mu < \int J_r(m) \, d\mu = 2 \int_{E_r} d\mu = 2 \sin^2 \left( \frac{\theta_0}{4} \right)
\]

where \( r \) is any integer in the range 1, \ldots, 2M, and where \( \theta_0 \) is the minimum angular separation of the vectors in the KS-uncolourable set considered. In the case of the Conway-Kochen set \( \mathbb{E} \), one has \( \theta_0 = 18.4^\circ \) (the angle between the vectors \((0,1,2)\) and \((0,2,2)\)), implying that the right-hand side of Inequality (B13) is \( < 0.013 \).

One would like to know if \( d_O \) is of the same order as the lower bound set by Inequality (B13)—implying that \( \geq 99\% \) of the area of \( S^2 \) can be covered with an open KS-colourable subset—or whether the actual value is significantly larger. This is a question which it might be interesting to investigate further. It might also be interesting to investigate the analogous quantities defined on Hilbert spaces of dimension \( > 3 \). In particular, it might be interesting to investigate the way in which these quantities vary with the dimension of the Hilbert space.

APPENDIX C: THE “OPERATIONAL” APPROACH OF SIMON ET AL AND LARSSON

Simon et al \[17,19\] and Larsson \[18\] (SBZL) claim to have proved an operational generalization of the KS theorem, applying to real experiments (Larsson, however, does not use the word “operational”). Their result inspired the generalized KS theorem proved in Section IV of this paper. It also inspired part of the argument in Section VII. However, although we are very significantly indebted to SBZL, we also have some significant criticisms. We will argue, in fact, that their operational version of the theorem is invalid.

In the following we begin by criticizing SBZL’s operational definition of contextuality. We go on to identify a loophole in SBZL’s argument. We conclude by exhibiting a model which gets through the loophole, and which provides a concrete counter-example to their operational version of the generalized KS theorem.

SBZL consider an instrument which performs measurements of each of the \( N \) triads contained in some fixed KS uncolourable set. The triad to be measured is determined by three switches. They assume that the system+instrument combination is fully specified by: (1) the three switch positions \( e_1, e_2, e_3 \), (2) the system hidden variables \( \lambda_S \), (3) the instrument hidden variables \( \lambda_I \) (Larsson, however, denotes the pair \((\lambda_S, \lambda_I)\) by the single symbol \( \lambda \)). In general, the \( r \)th pointer reading is given by a function \( X_r(e_1, e_2, e_3, \lambda_S, \lambda_I) \) which depends on all three switch positions, as well as the hidden variables. SBZL take it that a theory is non-contextual in an operational sense if and only if there exists, for each measuring instrument, a single function \( X \) such that

\[
X_r(e_1, e_2, e_3, \lambda_S, \lambda_I) = X(e_r, \lambda_S, \lambda_I)
\]

for \( r = 1, 2, 3 \).

This definition has the following features: (1) if the instrument is different then the function \( X \) may be different and (2) \( X \) depends on the instrument hidden variables \( \lambda_I \), as well as the system hidden variables \( \lambda_S \). For these reasons it is not a definition of non-contextuality in the ordinary sense.

A non-contextual theory (in the ordinary sense) is a theory in which measurements reveal the pre-existing values of the observables measured. It is, in other words, a classical theory, at least so far as the concept of measurement is concerned. Non-contextuality in SBZL’s operational sense, by contrast, does not have the implication that measurements reveal pre-existing values (of the observables measured). It is not a signature of classicality. In fact, we will show that it is not even inconsistent with the empirical predictions of quantum mechanics. It is therefore inappropriate. At least, it is inappropriate if the aim is to identify hypotheses which the empirically verifiable predictions of quantum mechanics exclude.

\[18\]
Before proceeding further it will be convenient to digress, and consider the ordinary concept of non-contextuality in a little more detail.

The context of a measurement is the particular procedure used to carry out the measurement. Alternatively, it is what Bohr [10] describes as the “whole experimental arrangement”, and what Bell [2] describes as the “complete disposition of the apparatus”. The term “contextuality” or “context-dependence” refers to the phenomenon in which different procedures for measuring the same observable have different outcomes, even though the hidden state of the system is initially the same.

A theory in which measurements reveal pre-existing values (of the observables measured) is obviously guaranteed to be non-contextual. The converse is also true: if a theory is non-contextual, then measurements reveal pre-existing values (of the observables measured). It may perhaps happen that pre-existing values are not explicitly mentioned, in the formal description of a non-contextual theory. Nevertheless, they must be present implicitly (since, if the theory predicts that a measurement of $A$ will always produce the value $a$, then that effectively means the theory assigns to $A$ the value $a$). To say that a theory is non-contextual is logically equivalent to saying that it is a theory in which measurements reveal pre-existing values (of the observables measured).

There is a tendency, in formal discussions, to focus on one particular kind of contextuality: namely, the kind where the result of measuring $A$ depends on which other commuting observables $B, C, \ldots$ are jointly measured along with $A$. It is an important kind of contextuality because it lends itself to the proof of general theorems. However, there are many other kinds.

A difference in measurement context is any difference in the detailed specification of the apparatus used for performing the measurement. It need not be a major difference. Suppose, for example, that the shape of the pole pieces in a Stern-Gerlach arrangement was slightly changed. This would count as a change of context. A real laboratory instrument admits numerous minor variations on the basic design. A theory is not genuinely non-contextual unless measurement outcomes are insensitive to every such variation.

In Eq. (C1) the function $X$ depends on the instrument, and on $\lambda_1$. It follows that SBZL’s operational criterion does not imply that measurement outcomes are genuinely independent of context. SBZL tend to focus on the switch positions and pointer readings as the features of the measuring apparatus of which the experimenter has knowledge (“the switch position is all he knows about”, as it is expressed in ref. [17]). However a real measuring apparatus has many other controllable parameters: for example, the number of turns in each coil, the exact position of each screw, the pressure in each evacuated chamber, etc. etc. A small change in one of these parameters will be reflected by a change in the probability distribution on the variables $\lambda_1$, if not by a change in the function $X$ itself. Consequently, it may have a large effect on the measurement outcome, even in a theory satisfying SBZL’s operational criterion.

SBZL’s operational criterion is, in one way, too weak. Not only does their criterion fail to exclude models which are highly non-classical. It is not even inconsistent with the empirical predictions of quantum mechanics (see the example at the end of this appendix).

Their criterion is, in another way, too strong. We stated earlier that a non-contextual theory is one in which measurements reveal pre-existing values (of the observables measured). However, this only applies to ideal measurements. Non-ideal measurements do not certainly reveal exact pre-existing values, and so they may only be approximately independent of context. SBZL’s criterion is meant to apply to real laboratory measurements. It is therefore unduly restrictive in requiring that, for given $\lambda_S, \lambda_1$, measurements of $e_1$ in the contexts $(e_1, e_2, e_3)$ and $(e_1', e_2', e_3')$ must, with certainty, have the same outcome (this point is closely related to the point we made at the end of Section V, when we argued that the condition $N \epsilon_1 < 1$ is not sufficient to ensure a contradiction with classical assumptions).

In short, the criterion for a theory to be genuinely non-contextual is, not Eq. (C1), but

$$X_r(e_1, e_2, e_3, \lambda_S, \lambda_1) = f(e_r, \lambda_S)$$

(C2)

for all $\lambda_S, \lambda_1$ in a set of probability measure $\approx 1$, and for some function $f$ which (unlike $X$ and $X_r$) is the same for all measuring instruments. It will be seen that this is, in effect, the criterion we used in Section V.

Let us now consider SBZL’s argument.

SBZL consider a set of $N$ switch settings $(e_1^1, e_2^1, e_3^1), \ldots, (e_1^N, e_2^N, e_3^N)$ which cannot consistently be coloured. For each $r$, let $A_r$ be the set of hidden variables $(\lambda_S, \lambda_1)$ such that

$$X(e_1^r, \lambda_S, \lambda_1) + X(e_2^r, \lambda_S, \lambda_1) + X(e_3^r, \lambda_S, \lambda_1) \neq 2$$

(C3)

where $X$ is the function defined by Eq. (C1) above.

Let $A_S$ and $A_1$ be the hidden state spaces of system and instrument respectively, and let $\mu$ be the probability measure on $A_S \times A_1$ describing the system+instrument combination. SBZL make the crucial assumption, that $\mu$ is the same for all switch settings.

It follows from the definitions that $\mu(A_r) \leq \epsilon_1$ for all $r$ (where $\epsilon_1$ is the quantity which SBZL denote $\epsilon$, and which is defined by Eq. (I) of this paper). In view of the fact that $\cup_{r=1}^N A_r = A_S \times A_1$ this implies
SBZL conclude that theories which are non-contextual in their operational sense cannot reproduce the quantum mechanical predictions for an apparatus with \( \epsilon_q < 1/N \).

The problem with this argument is the fact that it rests on the assumption that \( \mu \) is the same for all switch settings. The assumption is not justified. Changing the switch settings changes the quantum state of the instrument. In a hidden variables theory the quantum state of the instrument is represented by a probability measure on \( \Sigma \times \Lambda_1 \) (where \( \Sigma \) is the space of switch settings). It is reasonable to assume that this probability measure will be of the form \( \delta \times \nu \), where \( \delta \) is a \( \delta \)-function measure on \( \Sigma \). However, there is no reason to assume that \( \nu \) is the same for all switch settings—which is what would be needed if SBZL’s assumption was to be justified.

To find an example of a hidden variables theory to which SBZL’s assumption is inapplicable, one does not need to look further than the Bohm theory. In the Bohm theory the probability distribution is given by the square of the modulus of the wave function, in the \( x \)-space representation. Changing the switch settings changes the wave function, and so it will, in general, change the probability distribution.

It is not simply that SBZL’s argument is not sufficient to establish their conclusion. Their conclusion is not true—as we now show, by means of a counter-example. The example we choose is a model of MKC type. It has the property \( \epsilon_q = 0 \). The model is not non-contextual in the ordinary sense, defined by Eq. (C2). Consequently, there is no contradiction with the result proved in Section V: that theories which are non-contextual in the ordinary sense cannot reproduce the quantum mechanical predictions for an apparatus with \( \epsilon_q = 0 \). The model actually shows that non-contextuality in SBZL’s operational sense is not inconsistent with the empirical predictions of quantum mechanics, even as regards the outcome of measurements for which \( \epsilon_q = 0 \).

Comparing this equation with Eq. (C1) it can be seen that the model is non-contextual in SBZL’s operational sense (with \( X(n, \lambda_S, \lambda_I) = f_{\lambda_0}(e_{I_{18}(\lambda_I)}(\lambda_I)) \)). On the other hand the fact that \( f_{\lambda_0} \) is a KS-colouring implies \( \epsilon_q = 0 \).

The model thus shows that SBZL’s operational version of the generalized KS theorem is invalid.

Of course, the model is non-contextual in the ordinary sense, as defined by Eq. (C2). Consequently, there is no contradiction with the result proved in Section V: that theories which are non-contextual in the ordinary sense cannot reproduce the quantum mechanical predictions for an apparatus with \( \epsilon_q < 1/N \).

The model actually shows that non-contextuality in SBZL’s operational sense is not inconsistent with the empirical predictions of quantum mechanics, even as regards the outcome of measurements for which \( \epsilon_q = 0 \). Consequently, it also provides some additional grounds for considering their operational criterion to be inappropriate.

**APPENDIX D: CONTEXTUALITY OF SEQUENTIAL ANALYZERS**

The purpose of this appendix is to prove Inequality (11), establishing that the observable whose value is revealed by a near-ideal analyzer must depend on context.

Let \( f_\lambda \) be the KS-colouring of \( S_0^2 \) defined by the hidden state \( \lambda \). Define an induced valuation \( f_\lambda \) of the full unit 2-sphere \( S^2 \) by

\[
0 \quad \text{if } \mu_n(\{n' \in S_0^2 : f_\lambda(n') = 1\}) < 0.5
\]
\[
1 \quad \text{if } \mu_n(\{n' \in S_0^2 : f_\lambda(n') = 1\}) \geq 0.5
\]

for all \( n \). The significance of this construction is that, if the hidden state is \( \lambda \), then there is probability \( \geq 0.5 \) that a measurement of \( n \) will have outcome \( f_\lambda(n) \).

We now make the assumption, that the behaviour of the analyzers is independent of context.

Let \( \tau \) be any orthonormal triad which \( f_\lambda \)-evaluates to one of the “illegal” combinations \( 000, 011, 101, 110, 111 \). We will show that, if \( \tau \) is measured in the hidden state \( \lambda \), there is probability \( \geq 0.5 \) that the result will be “illegal”.

In fact, suppose that the \( f_\lambda \)-values of \( \tau \) are 000. Let \( p_r \) be the probability that the result of measuring \( e_r \) is 0. The definition of \( f_\lambda \) implies that \( p_r \geq 0.5 \) for each \( r \). The assumption of context-independence implies that the probabilities

\[
1 = \mu \left( \bigcup_{r=1}^N A_r \right) \leq \sum_{r=1}^N \mu(A_r) \leq N \epsilon_q
\]

\[\text{(C4)}\]
are independent. Consequently, the probability that the measurement outcome will be one of the “legal” combinations \(001, 010, 100\) is

\[
p_{\text{legal}} = p_1 p_2 (1 - p_3) + p_1 p_3 (1 - p_2) + p_2 p_3 (1 - p_1)
\]

It is straightforward to verify that the expression on the right hand side is always \(\leq 0.5\). A similar argument shows that \(p_{\text{legal}}\) is also \(\leq 0.5\) if the \(f_\lambda\)-values of \(\tau\) are one of the other four “illegal” combinations. We conclude that, in every case, there is probability \(\geq 0.5\) that the measurement outcome will be “illegal”—as claimed.

Now let \(\tau_1, \ldots, \tau_N\) be a set of triads which cannot consistently be coloured. For each \(\lambda\) choose an index \(r_\lambda\) such that the \(f_\lambda\)-values of \(\tau_{r_\lambda}\) are “illegal”. Let \(\Lambda\) be the set of all hidden states \(\lambda\). For each index \(r\) define \(\Lambda_r = \{ \lambda \in \Lambda; r_\lambda = r \}\). On the one hand, it follows from the result proved in the last paragraph that, if \(\lambda \in \Lambda_r\), then there is probability \(\geq 0.5\) that a measurement of \(\tau\) will have an “illegal” outcome. On the other hand the sets \(\Lambda_1, \ldots, \Lambda_N\) partition \(\Lambda\) into \(N\) disjoint subsets, so there must be an index \(r_0\) such that there is probability \(\geq 1/N\) that \(\lambda \in \Lambda_{r_0}\). Combining these statements we deduce that there is probability \(\geq 1/(2N)\) that a measurement of \(\tau_{r_0}\) will have an “illegal” outcome. However, it follows from the definition that this probability must be \(\leq \epsilon_\lambda\).

We deduce that, if the behaviour of the analyzers is context-independent, then \(N \epsilon_\lambda \geq 1/2\).

**APPENDIX E: SEQUENTIAL MEASUREMENTS ARE JOINT MEASUREMENTS**

Kent [38] has argued that a sequential measurement, such as the one illustrated in Fig. 2 is not a joint measurement. According to him, a process does not count as a joint measurement unless the different observables are measured all together, in parallel so to speak. The purpose of this appendix is to explain why we consider this position to be unjustified.

Measuring instruments are defined functionally, in terms of what they do. This is true both classically and quantum mechanically. If a system performs the function, of an instrument which measures \(A\), then it is an instrument which measures \(A\).

The function of a classical measuring instrument is to reveal pre-existing values. Suppose that one has a sealed box with a dial and two terminals on the front, and suppose that one wishes to confirm that it is a classical ammeter. This question can be decided without looking inside the box. It is sufficient to test the instrument with known currents, and check that it reads correctly. Anything which does the job of a classical ammeter is a classical ammeter.

A micrometer screw guage and an interferometer look very different. However, they both count as length measuring instruments. This is because they both do the job of a length measuring instrument.

Similar considerations apply in quantum mechanics. The function of a quantum measuring instrument is to discriminate eigenstates. Suppose one has a sealed box, with a hole into which spin 1 particles can be injected, and a digital readout. Suppose it is found, after testing, that the box correctly identifies the joint eigenstates of \((e_1 \cdot S)^2\), \((e_2 \cdot S)^2\), \((e_3 \cdot S)^2\). Then that is enough to establish that it jointly measures these observables.

Anything which discriminates the joint eigenstates of \((e_1 \cdot S)^2\), \((e_2 \cdot S)^2\), \((e_3 \cdot S)^2\), jointly measures these observables. In particular, the apparatus illustrated in Fig. 2 jointly measures them.

A quantum measurement can always be conceived in “black-box” terms. In order to decide whether a process is a quantum measurement of a given set of commuting observables it is only necessary to consider the state of system-apparatus at the time when the measurement interaction commences and again, at the time when the interaction is concluded (the difference between these times necessarily being \(> 0\)). If the initial and final states are related in the required manner, then the process is a measurement of the observables concerned. Questions as to the detailed behaviour of the state between these times, during the course of the interaction, are irrelevant.

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