Eigenvalues and Homology of Flag Complexes and Vector Representations of Graphs

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Abstract

The flag complex of a graph $G = (V, E)$ is the simplicial complex $X(G)$ on the vertex set $V$ whose simplices are subsets of $V$ which span complete subgraphs of $G$. We study relations between the first eigenvalues of successive higher Laplacians of $X(G)$. One consequence is the following

**Theorem:** Let $\lambda_2(G)$ denote the second smallest eigenvalue of the Laplacian of $G$. If $\lambda_2(G) > \frac{k}{k+1} |V|$ then $\tilde{H}^k(X(G); \mathbb{R}) = 0$.

Applications include a lower bound on the homological connectivity of the independent sets complex $I(G)$, in terms of a new graph domination parameter $\Gamma(G)$ defined via certain vector representations of $G$. This in turns implies a Hall type theorem for systems of disjoint representatives in hypergraphs.

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1 Introduction

Let $G = (V, E)$ be a graph with $|V| = n$ vertices. The Laplacian of $G$ is the $V \times V$ positive semidefinite matrix $L_G$ given by

$$L_G(u, v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise} \end{cases}$$

Let $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ denote the eigenvalues of $L_G$. The second smallest eigenvalue $\lambda_2(G)$, called the spectral gap, is a parameter of central importance in a variety of problems. In particular it controls the expansion properties of $G$ and the convergence rate of a random walk on $G$ (see e.g. [6]). The Flag Complex of $G$ is the simplicial complex $X(G)$ on the vertex set $V$ whose simplices are all subsets $\sigma \subset V$ which form a complete subgraph of $G$. Topological properties of $X(G)$ play key roles in recent results in matching theory (see below).

In this paper we study relations between $\lambda_2(G)$, the cohomology of $X(G)$, and a new graph domination parameter $\Gamma(G)$ which is defined via certain vector representations of $G$. As an application we obtain a Hall type theorem for systems of disjoint representatives in families of hypergraphs.

For $k \geq -1$ let $C^k(X(G))$ denote the space of real valued simplicial $k$-cochains of $X(G)$ and let $d_k : C^k(X(G)) \to C^{k+1}(X(G))$ denote the coboundary operator. For $k \geq 0$ define the reduced $k$-dimensional Laplacian of $X(G)$ by $\Delta_k = d_{k-1}d^*_{k-1} + d^*_kd_k$ (see section 2 for details). Let $\mu_k(G)$ denote the minimal eigenvalue of $\Delta_k$. Note that $\mu_0(G) = \lambda_2(G)$. Our main result is the following

**Theorem 1.1** For $k \geq 1$

$$k\mu_k(G) \geq (k+1)\mu_{k-1}(G) - n.$$  (1)

As a direct consequence of Theorem 1.1 we obtain

**Theorem 1.2** If $\lambda_2(G) > \frac{kn}{k+1}$ then $\tilde{H}^k(X(G), \mathbb{R}) = 0$.

**Remarks:**

1. Theorem 1.2 is related to a well-known result of Garland (Theorem 5.9
in [7]) and its extended version by Ballmann and Świątkowski (Theorem 2.5 in [5]). Roughly speaking, these results (in their simplest untwisted form) guarantee the vanishing of \( \tilde{H}^k(X; \mathbb{R}) \) provided that for each \((k - 1)\)-simplex \( \tau \) in \( X \), the spectral gap of the 1-skeleton of the link of \( \tau \) is sufficiently large. Theorem 1.2 is, in a sense, a global counterpart of this statement for flag complexes.

2. Let \( n = r\ell \) where \( r \geq 1, \ell \geq 2 \), and let \( G \) be the Turán graph \( T_r(n) \), i.e. the complete \( r \)-partite graph on \( n \) vertices with all sides equal to \( \ell \). The flag complex \( X(T_r(n)) \) is homotopy equivalent to the wedge of \((\ell - 1)^r \) \((r - 1)\)-dimensional spheres. It can be checked that \( \mu_k(T_r(n)) = \ell(r - k - 1) \) for all \( 0 \leq k \leq r - 1 \), hence (1) is satisfied with equality. Furthermore, \( \lambda_2(G) = \ell(r - 1) = \frac{r-1}{r} n \) while \( \tilde{H}^{r-1}(X(G)) \neq 0 \). Therefore the assumption in Theorem 1.2 cannot be replaced by \( \lambda_2(G) \geq \frac{kn}{k+1} \).

We next study some graph theoretical consequences of Theorem 1.2. The Independence Complex \( I(G) \) of \( G \) is the simplicial complex on the vertex set \( V \) whose simplices are all independent sets \( \sigma \subset V \). Thus \( I(G) = X(\overline{G}) \) where \( \overline{G} \) denotes the complement of \( G \). Recent work on hypergraph matching, starting in [4] with later developments in [1, 10, 2, 3, 11], has utilized topological properties of \( I(G) \) to derive new Hall type theorems for hypergraphs. The main ingredient in these developments are lower bounds on the homological connectivity of \( I(G) \). For a simplicial complex \( Z \) let \( \eta(Z) = \min \{ i : \tilde{H}^i(Z, \mathbb{R}) \neq 0 \} + 1 \). It turns out that various domination parameters of \( G \) may be used to provide lower bounds on \( \eta(I(G)) \). For a subset of vertices \( S \subset V \) let \( N(S) \) denote all vertices that are adjacent to at least one vertex of \( S \) and let \( N'(S) = S \cup N(S) \). \( S \) is a dominating set if \( N'(S) = V \). \( S \) is a totally dominating set if \( N(S) = V \). Here are a few domination parameters:

- The \textit{domination number} \( \gamma(G) \) is the minimal size of a dominating set.
- The \textit{total domination number} \( \tilde{\gamma}(G) \) is the minimal size of a totally dominating set.
- The \textit{independent domination number} \( i\gamma(G) \) is the maximum, over all independent sets \( I \) in \( G \), of the minimal size of a set \( S \) such that \( N(S) \supseteq I \).
The strong fractional domination number, \( \gamma^*_s(G) \), is the minimum of \( \sum_{v \in V} f(v) \), over all nonnegative functions \( f : V \to \mathbb{R} \) such that \( \sum_{u \in E} f(u) + \deg(v) f(v) \geq 1 \) for every vertex \( v \).

Some known lower bounds on \( \eta \) are: \( \eta(I(G)) \geq \tilde{\gamma}(G)/2 \) [10], \( \eta(I(G)) \geq i\gamma(G) \) [4], \( \eta(I(G)) \geq \gamma^*_s(G) \) [11].

Here we introduce a new domination parameter, defined by vector representations. It is similar in spirit to the \( \Theta \) function defined by Lovász [9]. It uses vectors to mimic domination, in a way similar to that in which the \( \Theta \) function mimics independence of sets of vertices. It is defined as follows. A vector representation of a graph \( G = (V,E) \) is an assignment \( P \) of a vector \( P(v) \in \mathbb{R}^\ell \) for some fixed \( \ell \) to every vertex \( v \) of the graph, such that the inner product \( P(u) \cdot P(v) \geq 1 \) whenever \( u, v \) are adjacent in \( G \) and \( P(u) \cdot P(v) \geq 0 \) if they are not adjacent. We shall identify the representation with the matrix \( P \) whose \( v \)-th row is the vector \( P(v) \).

Let \( 1 \) denote the all 1 vector in \( \mathbb{R}^V \). A non-negative vector \( \alpha \) on \( V \) is said to be dominating for \( P \) if \( \sum_{v \in V} \alpha(v) P(v) \cdot P(u) \geq 1 \) for every vertex \( u \), namely \( \alpha P P^T \geq 1 \). (Note that taking \( \alpha \) to be the characteristic function of some totally dominating set satisfies this condition regardless of the representation.) The value of \( P \) is

\[
|P| = \min \{ \alpha \cdot 1 : \alpha \geq 0, \alpha P P^T \geq 1 \} .
\]

The supremum of \( |P| \) over all vector representations \( P \) of \( G \) is denoted by \( \Gamma(G) \). Our main application of Theorem 1.2 is the following

**Theorem 1.3** \( \eta(I(G)) \geq \Gamma(G) \).

**Remark:** One natural vector representation of \( G \) is obtained by taking \( P(v) \in \mathbb{R}^E \) to be the edge incidence vector of the vertex \( v \). For this representation \( |P| = \gamma^*_s(G) \) hence \( \Gamma(G) \geq \gamma^*_s(G) \). The bound \( \eta(I(G)) \geq \gamma^*_s(G) \) was previously obtained in [11]. Theorem 1.3 is however stronger and often gives much sharper estimates for \( \eta(I(G)) \), see e.g. the case of cycles described in Section 4.

We next use Theorem 1.3 to derive a new Hall type result for hypergraphs. Let \( \mathcal{F} \subset 2^V \) be a hypergraph on a finite ground set \( V \). The width \( w(\mathcal{F}) \) of
\( F \) is the minimal \( t \) for which there exist \( F_1, \ldots, F_t \in \mathcal{F} \) such that for any \( F \in \mathcal{F} \), \( F_i \cap F \neq \emptyset \) for some \( 1 \leq i \leq t \).

The fractional width \( w^*(\mathcal{F}) \) of \( \mathcal{F} \) is the minimum of \( \sum_{E \in \mathcal{F}} f(E) \) over all non-negative functions \( f : \mathcal{F} \to \mathbb{R} \) with the property that for every edge \( E \in \mathcal{F} \) the sum \( \sum_{F \in \mathcal{F}} f(F) |E \cap F| \) is at least 1. A matching in \( \mathcal{F} \) is a subhypergraph \( \mathcal{M} \subset \mathcal{F} \) such that \( F \cap F' = \emptyset \) for all \( F \neq F' \in \mathcal{M} \). Let \( \{\mathcal{F}_i\}_{i=1}^m \) be a family of hypergraphs. A system of disjoint representatives (SDR) of \( \{\mathcal{F}_i\}_{i=1}^m \) is a matching \( F_1, \ldots, F_m \) such that \( F_i \in \mathcal{F}_i \) for \( 1 \leq i \leq m \). Haxell [8] proved the following

**Theorem 1.4** [8] If \( \{\mathcal{F}_i\}_{i=1}^m \) satisfies \( w(\cup_{i \in I} \mathcal{F}_i) \geq |I| - 1 \) for all \( \emptyset \neq I \subset [m] \), then \( \{\mathcal{F}_i\}_{i=1}^m \) has an SDR.

Here we use Theorem 1.3 to show

**Theorem 1.5** If \( \{\mathcal{F}_i\}_{i=1}^m \) satisfies \( w^*(\cup_{i \in I} \mathcal{F}_i) > |I| - 1 \) for all \( \emptyset \neq I \subset [m] \), then \( \{\mathcal{F}_i\}_{i=1}^m \) has an SDR.

The paper is organized as follows. In section 2 we recall some topological terminology and the simplicial Hodge theorem. Theorems 1.1 and 1.2 are proved in section 3. The proofs utilize the approach of Garland [7] and its exposition by Ballmann and Świątkowski [5]. In section 4 we relate the \( \Gamma \) parameter to homological connectivity and prove Theorem 1.3. In section 5 we recall a homological Hall type condition (Proposition 5.1) for the existence of colorful simplices in a colored complex. Combining this condition with Theorem 1.3 then completes the proof of Theorem 1.5.

## 2 Topological Preliminaries

Let \( X \) be a finite simplicial complex on the vertex set \( V \). Let \( X(k) \) denote the set of \( k \)-dimensional simplices in \( X \), each taken with an arbitrary but fixed orientation. A simplicial \( k \)-cochain is a real valued skew-symmetric function on all ordered \( k \)-simplices of \( X \). For \( k \geq 0 \) let \( C^k(X) \) denote the space of \( k \)-cochains on \( X \). The \( i \)-face of an ordered \( (k+1) \)-simplex \( \sigma = [v_0, \ldots, v_{k+1}] \) is the ordered \( k \)-simplex \( \sigma_i = [v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}] \). The coboundary operator \( d_k : C^k(X) \to C^{k+1}(X) \) is given by

\[
  d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i).
\]
It will be convenient to augment the cochain complex \( \{ C^i(X) \}_{i=0}^\infty \) with the \((−1)\)-degree term \( C^{-1}(X) = \mathbb{R} \) with the coboundary map \( d_{−1} : C^{-1}(X) \rightarrow C^0(X) \) given by \( d_{−1}(a)(v) = a \) for \( a \in \mathbb{R} \), \( v \in V \). Let \( Z^k(X) = \ker(d_k) \) denote the space of \( k \)-cocycles and let \( B^k(X) = \text{Im}(d_{k−1}) \) denote the space of \( k \)-coboundaries. For \( k \geq 0 \) let \( \tilde{H}^k(X) = Z^k(X)/B^k(X) \) denote the \( k \)-th reduced cohomology group of \( X \) with real coefficients. For each \( k \geq −1 \) endow \( C^k(X) \) with the standard inner product \( (\phi, \psi) = \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma) \) and the corresponding \( L^2 \) norm \( ||\phi|| = (\sum_{\sigma \in X(k)} \phi(\sigma)^2)^{1/2} \). Let \( d_k^* : C^{k+1}(X) \rightarrow C^k(X) \) denote the adjoint of \( d_k \) with respect to these standard inner products. The reduced \( k \)-Laplacian of \( X \) is the mapping

\[
\Delta_k = d_{k−1}d_k^* + d_k^*d_k : C^k(X) \rightarrow C^k(X).
\]

Note that if \( G \) denotes the 1-skeleton of \( X \) and \( J \) is the \( V \times V \) all ones matrix, then the matrix \( J + L_G \) represents \( \Delta_0 \) with respect to the standard basis. In particular, the minimal eigenvalue of \( \Delta_0 \) equals \( \lambda_2(G) \).

The space of harmonic \( k \)-cochains \( \tilde{H}^k(X) = \ker \Delta_k \) consists of all \( \phi \in C^k(X) \) such that both \( d_k\phi \) and \( d_k^*\phi \) are zero. The simplicial version of Hodge Theorem is the following well-known

**Proposition 2.1** \( \tilde{H}^k(X) \cong \tilde{H}^k(X) \) for \( k \geq 0 \).

In particular, \( \tilde{H}^k(X) = 0 \) iff the minimal eigenvalue of \( \Delta_k \) is positive.

### 3 Eigenvalues of Higher Laplacians

Let \( X = X(G) \) be the flag complex of a graph \( G = (V, E) \) on \( |V| = n \) vertices. For an \( i \)-simplex \( \eta \in X \) let \( \deg(\eta) \) denote the number of \((i + 1)\)-simplices in \( X \) which contain \( \eta \). The link of a simplex \( \sigma \in X \) is the complex

\[
\text{lk}(\sigma) = \{ \tau \in X : \sigma \cup \tau \in X, \sigma \cap \tau = \emptyset \}.
\]

For two ordered simplices \( \sigma \in X \), \( \tau \in \text{lk}(\sigma) \) let \( \sigma \tau \) denote their ordered union.

**Claim 3.1** For \( \phi \in C^k(X) \)

\[
||d_k\phi||^2 = \sum_{\sigma \in X(k)} \deg(\sigma)\phi(\sigma)^2 - 2\sum_{\eta \in X(k−1)} \sum_{vw \in \text{lk}(\eta)} \phi(v\eta)\phi(w\eta).
\]
Proof: Recall that for $\tau \in X(k+1)$ we denoted by $\tau_i$ the ordered $k$-simplex obtained by removing the $i$-th vertex of $\tau$. Thus

$$||d_k \phi||^2 = \sum_{\tau \in X(k+1)} d_k \phi(\tau)^2 = \sum_{\tau \in X(k+1)} \sum_{i=0}^{k+1} (-1)^i \phi(\tau_i) \sum_{j=0}^{k+1} (-1)^j \phi(\tau_j) =$$

$$\sum_{\tau \in X(k+1)} \sum_{i=0}^{k+1} \phi(\tau_i)^2 + \sum_{\tau \in X(k+1)} \sum_{i \neq j} (-1)^{i+j} \phi(\tau_i) \phi(\tau_j) =$$

$$\sum_{\sigma \in X(k)} \deg(\sigma) \phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \text{lk}(\eta)} \phi(v\eta) \phi(w\eta) .$$

Hence

$$\sum_{u \in V} ||d_{k-1} \phi_u||^2 =$$

$$\sum_{\sigma \in X(k)} \deg(\sigma) \phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in \text{lk}(\eta)} \phi(v\eta) \phi(w\eta) .$$

Claim 3.2 For $\phi \in C^k(X)$

$$\sum_{u \in V} ||d_{k-1} \phi_u||^2 =$$

$$\sum_{\tau \in X(k-1)} \deg(\tau) \phi_u(\tau)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(\eta)} \phi_u(v\eta) \phi_u(w\eta) .$$

Proof: Applying Claim 3.1 with $\phi_u \in C^{k-1}(X)$ we obtain

$$||d_{k-1} \phi_u||^2 = \sum_{\tau \in X(k-1)} \deg(\tau) \phi_u(\tau)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(\eta)} \phi_u(v\eta) \phi_u(w\eta) .$$

Hence

$$\sum_{u \in V} ||d_{k-1} \phi_u||^2 =$$

$$\sum_{u \in V} \sum_{\tau \in X(k-1)} \deg(\tau) \phi_u(\tau)^2 - 2 \sum_{u \in V} \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(\eta)} \phi_u(v\eta) \phi_u(w\eta) =$$

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\[
\sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) \phi(\sigma)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in \text{lk}(\eta)} \sum_{u \in \text{lk}(\eta) \cap \text{lk}(w\eta)} \phi(v\eta) \phi(w\eta) \right) \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) \phi(\sigma)^2 - 2k \sum_{\tau \in X(k-1)} \sum_{vw \in \text{lk}(\tau)} \phi(v\tau) \phi(w\tau) \right).
\]

The last equality follows from the fact that since \(X\) is a flag complex, if \(\eta \in X(k-2)\), \(vw \in \text{lk}(\eta)\) and \(u \in \text{lk}(v\eta) \cap \text{lk}(w\eta)\), then \(vw \in \text{lk}(u\eta)\).

\[\square\]

Claims 3.1 and 3.2 imply

\[k(||d_k\phi||^2 - \sum_{\sigma \in X(k)} \deg(\sigma) \phi(\sigma)^2) = \]

\[\sum_{u \in V} ||d_{k-1}^*\phi_u||^2 - \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) \phi(\sigma)^2 \right). \tag{2}\]

**Claim 3.3** For \(\phi \in C^k(X)\)

\[\sum_{u \in V} ||d_{k-2}^*\phi_u||^2 = k||d_{k-1}^*\phi||^2. \tag{3}\]

**Proof:** For \(\tau \in X(k-1)\)

\[d_{k-1}^*\phi(\tau) = \sum_{v \in \text{lk}(\tau)} \phi(v\tau). \]

Therefore

\[||d_{k-1}^*\phi||^2 = \sum_{\tau \in X(k-1)} d_{k-1}^*\phi(\tau)^2 = \]

\[\sum_{\tau \in X(k-1)} \left( \sum_{v \in \text{lk}(\tau)} \phi(v\tau) \right) \left( \sum_{w \in \text{lk}(\tau)} \phi(w\tau) \right) = \sum_{\tau \in X(k-1)} \sum_{(v,w) \in \text{lk}(\tau)^2} \phi(v\tau) \phi(w\tau). \tag{4}\]

Substituting \(\phi_u\) in (4) we obtain

\[\sum_{u \in V} ||d_{k-2}^*\phi_u||^2 = \sum_{u \in V} \sum_{\eta \in X(k-2)} \sum_{(v,w) \in \text{lk}(\eta)^2} \phi_u(v\eta) \phi_u(w\eta) = \]
Let \( \phi \in C^k(X) \). Summing (2) and (3) we obtain the following key identity:

\[
\sum_{\eta \in X(k-2)} \sum_{u \in \text{lk}(\eta)} \sum_{(v,w) \in \text{lk}(\eta)^2} \phi(vu\eta)\phi(wu\eta) = k \sum_{\tau \in X(k-1)} \sum_{(v,w) \in \text{lk}(\tau)^2} \phi(v\tau)\phi(w\tau) = k||d^{*}_{k-1}\phi||^2.
\]

\(\blacksquare\)

To estimate the righthand side of (5) we need the following

Claim 3.4 For \( \sigma \in X(k) \)

\[
\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \leq n .
\] (6)

Proof: Recall that \( N(v) \) is the set of neighbors of \( v \) in \( G \). Let \( \sigma = [v_0, \ldots, v_k] \) then for any \( I \subset \{0, \ldots, k\} \)

\[
\deg([v_i : i \in I]) = |\bigcap_{i \in I} N(v_i)| .
\]

Therefore

\[
\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) = \sum_{i=0}^{k} |\bigcap_{j \neq i} N(v_j)| - k |\bigcap_{j=0}^{k} N(v_j)| .
\] (7)

The Claim now follows since each \( v \in V \) is counted at most once on the righthand side of (7).

\(\blacksquare\)
Proof of Theorem 1.1: Let $0 \neq \phi \in C^k(X)$ be an eigenvector of $\Delta_k$ with eigenvalue $\mu_k(G)$. By double counting

$$\sum_{u \in V} ||\phi_u||^2 = (k + 1)||\phi||^2.$$  \hspace{1cm} (8)

Combining (5),(6) and (8) we obtain

$$k\mu_k(G)||\phi||^2 = k(\Delta_k \phi, \phi) \geq \sum_{u \in V}(\Delta_{k-1} \phi_u, \phi_u) - n \sum_{\sigma \in X(k)} \phi(\sigma)^2 \geq$$

$$\mu_{k-1}(G) \sum_{u \in V} ||\phi_u||^2 - n||\phi||^2 = ((k + 1)\mu_{k-1}(G) - n)||\phi||^2.$$  

\hspace{1cm} \Box

Proof of Theorem 1.2: Inequality (1) implies by induction on $k$ that $\mu_k(G) \geq (k + 1)\mu_0(G) - kn$. Therefore, if $\mu_0(G) = \lambda_2(G) > \frac{kn}{k+1}$ then $\mu_k(G) > 0$ and $\tilde{H}^k(X(G), \mathbb{R}) = 0$ follows from the simplicial Hodge Theorem.

\hspace{1cm} \Box

4 Vector Domination and Homology

Let $G = (V, E)$ be a graph with $|V| = n$. We first reformulate Theorem 1.2 in terms of the independence complex $I(G)$.

Theorem 4.1 $\eta(I(G)) \geq \frac{n}{\lambda_n(G)}$.

Proof: Let $\ell = \left\lceil \frac{n}{\lambda_n(G)} \right\rceil$. Since $\lambda_n(G) = n - \lambda_2(\overline{G})$ it follows that $\lambda_2(\overline{G}) > \frac{\ell - 2}{\ell - 1}n$. Therefore by Theorem 1.2, $\tilde{H}^i(I(G)) = \tilde{H}^i(X(\overline{G})) = 0$ for $i \leq \ell - 2$. Hence $\eta(I(G)) \geq \ell$.

\hspace{1cm} \Box

The proof of Theorem 1.3 depends on Theorem 4.1 and the following
Claim 4.2 Let $P$ be a vector representation of $G = (V, E)$. Then

\[
\lambda_n(G) \leq \max_{u \in V} P(u) \cdot \sum_{v \in V} P(v) .
\]

Proof: Let $x = (x(v) : v \in V)$ be a vector in $\mathbb{R}^V$. Then

\[
x^T L_G x = \sum_{uv \in E} (x(u) - x(v))^2 \leq \frac{1}{2} \sum_{(u,v) \in V \times V} (x(u) - x(v))^2 P(u) \cdot P(v) = \\
\sum_{u \in V} x(u)^2 P(u) \cdot \sum_{v \in V} P(v) - \| \sum_{v \in V} x(v)P(v) \|^2 \leq \\
\| x \|^2 \max_{u \in V} P(u) \cdot \sum_{v \in V} P(v)
\]

The Claim follows since $\lambda_n(G) = \max \left\{ \frac{x^T L_G x}{\| x \|^2} : 0 \neq x \in \mathbb{R}^V \right\}$.

Let $\mathbb{Z}_+$ denote the positive integers and let $\mathbb{Q}_+$ denote the positive rationals. For a vector $a = (a(v) : v \in V) \in \mathbb{Z}_+^V$ let $G_a$ denote the graph obtained by replacing each $v \in V$ by an independent set of size $a(v)$. Formally $V(G_a) = \{(v,i) : v \in V, 1 \leq i \leq a(v)\}$ and $\{(u,i),(v,j)\} \in E(G_a)$ if $\{u,v\} \in E$. The projection $(v,i) \rightarrow v$ induces a homotopy equivalence between $I(G_a)$ and $I(G)$. In particular $\eta(I(G_a)) = \eta(I(G))$.

Proof of Theorem 1.3: Let $P$ be a representation of $G$. By linear programming duality

\[
|P| = \min \{ \alpha \cdot 1 : \alpha \geq 0, \alpha PP^T \geq 1 \} = \\
\max \{ \alpha \cdot 1 : \alpha \geq 0, \alpha PP^T \leq 1 \} = \\
\sup \{ \alpha \cdot 1 : \alpha \in \mathbb{Q}_+^V, \alpha PP^T \leq 1 \}.
\]
Let \( \alpha \in \mathbb{Q}^V \) such that \( \alpha PP^T \leq 1 \). Write \( \alpha = \frac{a}{k} \) where \( k \in \mathbb{Z}_+ \) and \( a = (a(v) : v \in V) \in \mathbb{Z}_+^V \). Let \( N = |V(G_a)| = \sum_{u \in V} a(u) \). Consider the representation \( Q \) of \( G_a \) given by \( Q((u, i)) = P(u) \) for \( (u, i) \in V(G_a) \). By Claim 4.2
\[
\lambda_N(G_a) \leq \max_{(u,i) \in V(G_a)} Q((u,i)) \cdot \sum_{(v,j) \in V(G_a)} Q((v,j)) = \max_{u \in V} P(u) \cdot \sum_{v \in V} a(v)P(v) \leq k.
\]

Hence by Theorem 4.1
\[
\alpha \cdot 1 = \frac{1}{k} \sum_{v \in V} a(v) = \frac{N}{k} \leq \frac{N}{\lambda_N(G_a)} \leq \eta(I(G_a)) = \eta(I(G)).
\]

Remarks:

1. Let \( C_n \) denote the \( n \)-cycle on the vertex set \( V = \{0, \ldots, n-1\} \). For \( n = 3k \) define a representation \( P \) of \( C_{3k} \) by
\[
P(\ell) = \begin{cases} 
e_{2j} & \ell = 3j \\ e_{2j} + e_{2j+1} & \ell = 3j + 1 \\ e_{2j+1} + e_{2j+2} & \ell = 3j + 2 \\ \end{cases}
\]
where \( e_0, \ldots, e_{2k-1} \) are orthogonal unit vectors and the indices are cyclic modulo \( 2k \). Let \( \alpha \in \mathbb{R}^V \) be given by \( \alpha(\ell) = 1 \) if \( 3 \) divides \( \ell \) and zero otherwise. Since \( \alpha PP^T = 1 \), it follows by linear programming duality that \( \Gamma(C_{3k}) \geq \alpha \cdot 1 = k \). On the other hand (see Claim 3.3 in [11]) \( \eta(I(C_n)) = \lfloor \frac{n+1}{3} \rfloor \). Therefore \( \eta(I(C_{3k})) = \Gamma(C_{3k}) = k \). For \( n = 3k + 1 \) it can similarly be shown that \( \eta(I(C_{3k+1})) = \Gamma(C_{3k+1}) = k \). The case \( n = 3k - 1 \) is more involved and we only have the bounds \( k \frac{1}{2} \leq \Gamma(C_{3k-1}) \leq \eta(I(C_{3k-1})) = k \). Note that for cycles the bound \( \eta(I(G)) \geq \gamma^*_s(G) \) is weaker since \( \gamma^*_s(C_n) = \frac{n}{4} \).

2. It can be shown that for any graph \( \Gamma(G) \geq \text{Sup}\{\gamma^*_s(G_a) : a \in \mathbb{Z}_+^V\} \). We do not know of examples with strict inequality.
5 A Hall Type Theorem for Fractional Width

Let $Z$ be a simplicial complex on the vertex set $W$ and let $\bigcup_{i=1}^{m} W_i$ be a partition of $W$. A simplex $\tau \in Z$ is colorful if $|\tau \cap W_i| = 1$ for all $1 \leq i \leq m$. For $W' \subset W$ let $Z[W']$ denote the induced subcomplex on $W'$. The following Hall’s type sufficient condition for the existence of colorful simplices appears in [4] and in [10].

**Proposition 5.1** If for all $\emptyset \neq I \subset [m]$

$$\eta(Z[\bigcup_{i \in I} W_i]) \geq |I|$$

then $Z$ contains a colorful simplex.

Let $G$ be a graph on the vertex set $W$ with a partition $W = \bigcup_{i=1}^{m} W_i$. A set $S \subset W$ is colorful if $S \cap W_i \neq \emptyset$ for all $1 \leq i \leq m$. The induced subgraph on $W' \subset W$ is denoted by $G[W']$. Combining Theorem 1.3 and Proposition 5.1 we obtain the following

**Theorem 5.2** If $\Gamma(G[\bigcup_{i \in I} W_i]) > |I| - 1$ for all $\emptyset \neq I \subset [m]$ then $G$ contains a colorful independent set.

Let $\mathcal{F} \subset 2^V$ be a hypergraph, possibly with multiple edges. The line graph $G_{\mathcal{F}} = (W, E)$ associated with $\mathcal{F}$ has vertex set $W = \mathcal{F}$ and edge set $E$ consisting of all $\{F, F'\} \subset \mathcal{F}$ such that $F \cap F' \neq \emptyset$. A matching in $\mathcal{F}$ corresponds to an independent set in $G_{\mathcal{F}}$. For each $F \in \mathcal{F}$ let $P(F) \in \mathbb{R}^V$ denote the incidence vector of $F$. $P$ is clearly a vector representation of $G_{\mathcal{F}}$ and satisfies $|P| = w^*(\mathcal{F})$. Thus $\Gamma(G_{\mathcal{F}}) \geq w^*(\mathcal{F})$.

**Proof of Theorem 1.5:** Let $\mathcal{F}$ denote the disjoint union of the $\mathcal{F}_i$’s, and consider the graph $G_{\mathcal{F}} = (W, E)$ with the partition $W = \bigcup_{i=1}^{m} W_i$ where $W_i = \mathcal{F}_i$. Then for any $\emptyset \neq I \subset [m]$

$$\Gamma(G_{\mathcal{F}}[\bigcup_{i \in I} W_i]) = \Gamma(G_{\bigcup_{i \in I} \mathcal{F}_i}) \geq w^*(\bigcup_{i \in I} \mathcal{F}_i) > |I| - 1 .$$

Theorem 5.2 implies that $G_{\mathcal{F}}$ contains a colorful independent set, hence $\{\mathcal{F}_i\}_{i=1}^{m}$ contains an SDR.

\[\square\]
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