Spectrum, Algebraicity and Normalization in Alternate Bases

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Cantor real bases and alternate bases

Let $\beta = (\beta_n)_{n \geq 0}$ be a sequence of real numbers greater than 1 and such that $\prod_{n=0}^{\infty} \beta_n$ is infinite.

A $\beta$-representation of a real number $x$ is an infinite sequence $a = (a_n)_{n \geq 0}$ of integers such that

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0 \beta_1} + \frac{a_2}{\beta_0 \beta_1 \beta_2} + \cdots$$

An alternate base is a periodic Cantor base. In this case, we simply write $\beta = (\beta_0, \ldots, \beta_{p-1})$ and we use the convention that

- $\beta_n = \beta_n \mod p$
- $\beta^{(n)} = (\beta_n, \ldots, \beta_{n+p-1})$

for all $n \geq 0$. We call the number $p$ the length of the alternate base $\beta$. 
Greedy algorithm

For $x \in [0, 1]$, a distinguished $\beta$-representation

$$d_\beta(x) = (\varepsilon_n)_{n \geq 0},$$
called the $\beta$-expansion of $x$, is obtained from the greedy algorithm:

- $r_0 = x$
- $\varepsilon_n = \lfloor \beta_n r_n \rfloor$ and $r_{n+1} = \beta_n r_n - \varepsilon_n$ for $n \in \mathbb{N}$.

For each $n$, we have $\varepsilon_n \in \{0, 1, \ldots, \lfloor \beta_n \rfloor \}$.

Thus, the $\beta$-expansions are written over the alphabet $\{0, 1, \ldots, \max_{0 \leq i < p} \lfloor \beta_i \rfloor \}$. 
Parry’s theorem for alternate bases and alternate $\beta$-shift

The quasi-greedy $\beta$-expansion of 1 is $d^*_\beta(1) = \lim_{x \to 1^-} d\beta(x)$.

**Theorem (Charlier & Cisternino 2021)**

An infinite sequence $a_0a_1a_2\cdots$ of non-negative integers belongs to the set $\{d\beta(x): x \in [0,1]\}$ if and only if $a_n a_{n+1} a_{n+2} \cdots <_{\text{lex}} d^*_\beta(n)(1)$ for all $n \in \mathbb{N}$.

For an alternate base $\beta$, the set $\{d\beta(x): x \in [0,1]\}$ is not shift-invariant in general.

The $\beta$-shift is defined as the topological closure of the set

$$\bigcup_{i=0}^{p-1} \{d_{\beta(i)}(x): x \in [0,1]\}.$$ 

**Theorem (Charlier & Cisternino 2021)**

The $\beta$-shift is sofic if and only if $d^*_{\beta(i)}(1)$ is eventually periodic for all $i \in \{0, \ldots, p-1\}$.

In view of this result, we refer to such alternate bases as the Parry alternate bases.
Example

Let $\beta = \left( \frac{1 + \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{6} \right)$. We can compute $d^*_{\beta(0)}(1) = 200(10)^\omega$ and $d^*_{\beta(1)}(1) = (10)^\omega$.

The following finite automaton accepts the set of factors of elements in the $\beta$-shift.
Aims of this work

- **Algebraic properties of Parry alternate bases.**
  1. A necessary condition for being a Parry alternate base is that the product \( \delta = \prod_{i=0}^{p-1} \beta_i \) is an algebraic integer and \( \beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta) \).
  2. A sufficient condition for being a Parry alternate base if that \( \delta \) is a Pisot number and \( \beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta) \).

- **Normalization of alternate base representations.**

  If \( \delta \) is a Pisot number and \( \beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta) \), then the normalization function is computable by a finite Büchi automaton. Such an automaton is effectively given.
Spectrum as a tool

The notion of spectrum associated with a real base $\beta > 1$ and an alphabet of the form $A_d = \{0, 1, \ldots, d\}$ with $d \in \mathbb{N}$ was introduced by Erdős, Joó and Komornik in 1990.

For our purposes, we use a generalized concept of complex spectrum, and study its topological properties.

Let $\delta \in \mathbb{C}$ such that $|\delta| > 1$ with an alphabet $A \subset \mathbb{C}$.

The spectrum associated with $\delta$ and $A$ is the set

$$X^A(\delta) = \left\{ \sum_{i=0}^{\ell-1} a_i \delta^{\ell-1-i} : n \in \mathbb{N}, \ a_i \in A \right\}.$$

We say that a word $a_0 \cdots a_{\ell-1}$ over $A$ corresponds to the element $\sum_{i=0}^{\ell-1} a_i \delta^{\ell-1-i}$ in the spectrum $X^A(\delta)$. 
The following result shows that topological properties of the spectrum are linked with arithmetical aspects of the numeration system.

**Theorem (Frougny & Pelantová 2018)**

Let $\beta > 1$ and $d \in \mathbb{N}$. Then $Z(\beta, d)$ is accepted by a finite Büchi automaton if and only if the spectrum $X^d(\beta)$ has no accumulation point in $\mathbb{R}$.

For the case of real bases and symmetric integer alphabets, there is a complete characterization of the bases which give spectra without accumulation points in dependence on the alphabet.

**Theorem (Akiyama & Komornik 2013, Feng 2016)**

Let $\beta > 1$ and $d \in \mathbb{N}$. The spectrum $X^d(\beta)$ has no accumulation point in $\mathbb{R}$ if and only if either $\beta - 1 \geq d$ or $\beta$ is a Pisot number.
Set of $\delta$-representations of zero and complex zero automaton

For a complex base $\delta$ and an alphabet $A$ of complex numbers, we define

$$Z(\delta, A) = \{ a \in A^\mathbb{N} : \sum_{n=0}^{+\infty} \frac{a_n}{\delta^{n+1}} = 0 \}.$$ 

Generalizing ideas from Frougny, we define a Büchi automaton

$$Z(\delta, A) = (Q, 0, Q, A, E).$$

- **States:** $Q = X^A(\delta) \cap \{ z \in \mathbb{C} : |z| \leq \frac{M}{|\delta|-1} \}$ where $M = \max\{|a| : a \in A\}$.
- **Transitions:** $E = \{(z, a, z\delta + a) : z \in Q, a \in A\}$.

**Proposition**

*The Büchi automaton $Z(\delta, A)$ accepts the set $Z(\delta, A)$.***
Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

Let $\delta$ be a complex number such that $|\delta| > 1$ and let $A$ be an alphabet of complex numbers. Then the following assertions are equivalent.

1. The set $Z(\delta, A)$ is accepted by a finite Büchi automaton.
2. The zero automaton $\mathcal{Z}(\delta, A)$ is finite.
3. The spectrum $X^A(\delta)$ has no accumulation point in $\mathbb{C}$. 
Towards an analogous result for alternate bases

- We consider a fixed alternate base $\beta = (\beta_0, \ldots, \beta_{p-1})$.
- We set $\delta = \prod_{i=0}^{p-1} \beta_i$.
- We consider a $p$-tuple $D = (D_0, \ldots, D_{p-1})$ where, for all $i \in \{0, \ldots, p - 1\}$, $D_i$ is an alphabet of integers containing 0.
- We use the convention that for all $n \in \mathbb{Z}$, $D_n = D_{n \mod p}$ and $D^{(n)} = (D_n, \ldots, D_{n+p-1})$.

Grouping terms $p$ by $p$, the equality

$$x = \frac{a_0}{\beta_0} + \frac{a_1}{\beta_0\beta_1} + \cdots + \frac{a_{p-1}}{\beta_0\beta_1\cdots\beta_{p-1}} + \cdots$$

can be written as

$$x = \sum_{i=0}^{p-1} \frac{a_i\beta_{i+1}\cdots\beta_{p-1}}{\delta} + \sum_{i=0}^{p-1} \frac{a_{p+i}\beta_{i+1}\cdots\beta_{p-1}}{\delta^2} + \cdots$$

If we add the constraint that each letter $a_n$ belongs to $D_n$, then we obtain a $\delta$-representation of $x$ over the alphabet

$$D = \left\{ \sum_{i=0}^{p-1} a_i\beta_{i+1}\cdots\beta_{p-1} : \forall i \in \{0, \ldots, p - 1\}, \ a_i \in D_i \right\}.$$
Alternate spectrum

For $\delta = \prod_{i=0}^{p-1} \beta_i$ and the alphabet $\mathcal{D}$, we consider the spectrum $X^\mathcal{D}(\delta)$.

For each $i \in \{0, \ldots, p - 1\}$, we let $X(i)$ denote the spectrum built from the shifted base $\beta^{(i)}$ and the shifted $p$-tuple of alphabets $\mathcal{D}^{(i)}$.

In particular, we have $X(0) = X^\mathcal{D}(\delta)$.

**Lemma**

*For each $i \in \{0, \ldots, p - 1\}$, we have $X(i) \cdot \beta_i + D_i = X(i + 1)$ where $X(p) = X(0)$.***
Alternate zero automaton

For each $i \in \{0, \ldots, p-1\}$, we define

$$M^{(i)} = \sum_{n=i}^{+\infty} \frac{\max(D_n)}{\prod_{k=i}^{n} \beta_k} \quad \text{and} \quad m^{(i)} = \sum_{n=i}^{+\infty} \frac{\min(D_n)}{\prod_{k=i}^{n} \beta_k}.$$  

We define a Büchi automaton associated with an alternate base $\beta$ and a $p$-tuple of alphabets $D$ as

$$Z(\beta, D) = (Q_{\beta,D}, (0,0), Q_{\beta,D}, \cup_{i=0}^{p-1} D_i, E)$$

where

- $Q_{\beta,D} = \bigcup_{i=0}^{p-1} (\{i\} \times (X(i) \cap [-M^{(i)}, -m^{(i)}]))$

- $E$ is the set of transitions defined as follows: for $(i, s), (j, t) \in Q_{\beta,D}$ and $a \in \cup_{i=0}^{p-1} D_i$, there is a transition $(i, s) \xrightarrow{a} (j, t)$ if and only if $j \equiv i + 1 \pmod{p}$, $a \in D_i$ and $t = \beta_i s + a$.

Proposition

The Büchi automaton $Z(\beta, D)$ accepts the set

$$Z(\beta, D) = \{a \in \prod_{n=0}^{+\infty} D_n : \sum_{n=0}^{+\infty} \frac{a_n}{\prod_{k=0}^{n} \beta_k} = 0\}.$$
An example

Consider the alternate base $\beta = \left(\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6}\right)$ and $D = \{-2, -1, 0, 1, 2\}, \{-1, 0, 1\}$. Then $M^{(0)} = \text{val}_\beta((21)^\omega) \simeq 1.67994$ and $M^{(1)} = \text{val}_\beta((12)^\omega) \simeq 1.86852$.

Zero automaton $\mathcal{E}(\beta, D)$

For instance, the infinite words $1(\overline{10})^\omega$ and $(0\overline{12121})^\omega$ have value 0 in base $\beta$ (where $\overline{1}$ and $\overline{2}$ designate the digits $-1$ and $-2$ respectively).
Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

Let $\beta$ be an alternate base of length $p$ and let $D$ be a $p$-tuple of alphabets of integers containing 0. Then the following assertions are equivalent.

1. The set $Z(\beta, D)$ is accepted by a finite Büchi automaton.
2. The zero automaton $Z(\beta, D)$ is finite.
3. The spectrum $X^D(\delta)$ has no accumulation point in $\mathbb{R}$. 
Necessary conditions on $\beta$ to be a Parry alternate base

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

If $\beta$ is a Parry alternate base, then

- $\delta$ is an algebraic integer
- $\beta_i \in \mathbb{Q}(\delta)$ for all $i \in \{0, \ldots, p - 1\}$.

Let me give some intuition on an example.

Let $\beta = (\beta_0, \beta_1, \beta_2)$ be a base such that the expansions of 1 are given by

$$d_\beta(1) = 30^\omega, \quad d_{\beta(1)}(1) = 110^\omega, \quad d_{\beta(2)}(1) = 1(110)^\omega.$$ 

We derive that $\beta_0, \beta_1, \beta_2$ satisfy the following set of equations

$$\frac{3}{\beta_0} = 1, \quad \frac{1}{\beta_1} + \frac{1}{\beta_1 \beta_2} = 1, \quad \frac{1}{\beta_2} + \left( \frac{1}{\beta_2 \beta_0} + \frac{1}{\delta} \right) \frac{\delta}{\delta - 1} = 1,$$

where $\delta = \beta_0 \beta_1 \beta_2$.

Multiplying the first equation by $\delta$, the second one by $\beta_1 \beta_2$ and the third one by $(\delta - 1)\beta_2$, we obtain the identities

$$3\beta_1 \beta_2 - \delta = 0, \quad -\beta_1 \beta_2 + \beta_2 + 1 = 0, \quad \beta_1 \beta_2 + (2 - \delta)\beta_2 + \delta - 1 = 0.$$

In a matrix formalism, we have

$$\begin{pmatrix} 3 & 0 & -\delta \\ -1 & 1 & 1 \\ 1 & 2 - \delta & \delta - 1 \end{pmatrix} \begin{pmatrix} \beta_1 \beta_2 \\ \beta_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
The existence of a non-zero vector \((\beta_1, \beta_2, 1)^T\) as a solution of this equation forces that the determinant of the coefficient matrix is zero:

\[
\delta^2 - 9\delta + 9 = 0.
\]

Hence we must have \(\delta = \frac{9 + 3\sqrt{5}}{2} = 3\varphi^2\) where \(\varphi = \frac{1 + \sqrt{5}}{2}\) is the golden ratio.

We then obtain

\[
\beta_1\beta_2 = \frac{\delta}{3} = \varphi^2 \text{ and } \beta_2 = \beta_1\beta_2 - 1 = \varphi^2 - 1 = \varphi.
\]

Consequently,

\[
\beta_1 = \frac{\beta_1\beta_2}{\beta_2} = \frac{\varphi^2}{\varphi} = \varphi \text{ and } \beta_0 = \frac{\delta}{\beta_1\beta_2} = \frac{3\varphi^2}{\varphi^2} = 3.
\]

Indeed, the triple \(\beta = (3, \varphi, \varphi)\) is an alternate base giving precisely the given expansions of 1.
For obtaining the values $\beta_0, \beta_1, \beta_2$ from the known eventually periodic expansions we have used the fact that $\beta_0, \beta_1, \beta_2$ and $\delta = \beta_0\beta_1\beta_2$ are solutions of a system of polynomial equations in four unknowns $x_0, x_1, x_2, y$, in our case

\[
\begin{align*}
3x_1x_2 - y &= 0 \\
-x_1x_2 + x_2 + 1 &= 0 \\
x_1x_2 + (2 - y)x_2 + y - 1 &= 0 \\
x_1x_2x_3 &= y.
\end{align*}
\]

The solution of the system yielded that $\delta$ is a root of a monic polynomial with integer coefficients, i.e., is an algebraic integer.

The same strategy can be applied to any Parry alternate base.
A sufficient condition on $\beta$ to be a Parry alternate base

As previously:

- $\delta = \prod_{i=0}^{p-1} \beta_i$
- $D = (D_0, \ldots, D_{p-1})$ is a $p$-tuple of alphabets of integers containing 0
- $D$ is the corresponding alphabet of real numbers.

**Proposition**

If $D_i \supseteq \{-\lfloor \beta_i \rfloor, \ldots, \lfloor \beta_i \rfloor\}$ for all $i \in \{0, \ldots, p-1\}$ and if the spectrum $X^D(\delta)$ has no accumulation point in $\mathbb{R}$, then $\beta$ is a Parry alternate base.

**Proposition**

If $\delta$ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then the spectrum $X^D(\delta)$ has no accumulation point in $\mathbb{R}$.

As a consequence, we get

**Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)**

If $\delta$ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\beta$ is a Parry alternate base.
Some remarks

▶ The condition of \( \delta \) being a Pisot number is neither sufficient nor necessary for \( \beta \) to be a Parry alternate base.

1. Even for \( p = 1 \), there exist Parry numbers which are not Pisot.
2. To see that it is not sufficient for \( p \geq 2 \), consider the alternate base \( \beta = (\sqrt{\beta}, \sqrt{\beta}) \) where \( \beta \) is the smallest Pisot number. The product \( \delta \) is the Pisot number \( \beta \). However, the \( \beta \)-expansion of 1 is equal to \( d_{\sqrt{\beta}}(1) \), which is aperiodic.

▶ The bases \( \beta_0, \ldots, \beta_{p-1} \) need not be algebraic integers in order to have a Parry alternate base.

To see this, consider \( \beta = \left( \frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6} \right) \). For this base, we have \( d_{\beta_0}(1) = 2010^\omega \) and \( d_{\beta_1}(1) = 110^\omega \). However, \( \frac{5+\sqrt{13}}{6} \) is not an algebraic integer.

▶ For the same non Pisot algebraic integer \( \delta \), there may exist a Parry alternate base \( \alpha = (\alpha_0, \ldots, \alpha_{p-1}) \) and a non-Parry alternate base \( \beta = (\beta_0 \cdots \beta_{p-1}) \) such that \( \prod_{i=0}^{p-1} \alpha_i = \prod_{i=0}^{p-1} \beta_i = \delta \).
Generalization of Schmidt’s results

Define $\text{Per}(\beta) = \{x \in [0, 1) : d_\beta(x) \text{ is ultimately periodic}\}$.

**Theorem (Charlier, Cisternino & Kreczman 2022)**

1. If $Q \cap [0, 1) \subseteq \bigcap_{i=0}^{p-1} \text{Per}(\beta^{(i)})$ then $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ and $\delta$ is either a Pisot number or a Salem number.

2. If $\delta$ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$ then $\text{Per}(\beta) = \mathbb{Q}(\delta) \cap [0, 1)$.

**Theorem (Charlier, Cisternino & Kreczman 2022)**

If $\delta$ is an algebraic integer that is neither a Pisot number nor a Salem number then $\text{Per}(\beta) \cap \mathbb{Q}$ is nowhere dense in $[0, 1)$. 
Alternate bases whose set of zero representations is accepted by a finite Büchi automaton

Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)

The following assertions are equivalent.

1. The set $Z(β, D)$ is accepted by a finite Büchi automaton for all $p$-tuple of alphabets of integers $D = (D_0, \ldots, D_{p−1})$.

2. The set $Z(β, D)$ is accepted by a finite Büchi automaton for one $p$-tuple of alphabets of integers $D = (D_0, \ldots, D_{p−1})$ such that $D_i \supseteq \{-\lfloor β_i \rfloor, \ldots, \lfloor β_i \rfloor \}$ for all $i \in \{0, \ldots, p−1\}$ and $\lfloor β_j \rfloor \geq \lceil δ \rceil − 1$ for some $j \in \{0, \ldots, p−1\}$.

3. $δ$ is a Pisot number and $β_0, \ldots, β_{p−1} \in \mathbb{Q}(δ)$. 
Normalization in alternate base

The normalization function is the partial function $\nu_{\beta,D}$ mapping any $\beta$-representation $a \in \prod_{n \in \mathbb{N}} D_n$ of a real number $x \in [0,1)$ to the $\beta$-expansion of $x$.

We say that $\nu_{\beta,D}$ is computable by a finite Büchi automaton if there exists a finite Büchi automaton accepting the set

$$\left\{ (u,v) \in \prod_{n \in \mathbb{N}} (D_n \times \{0,\ldots,[\beta_n]-1\}) : \text{val}_\beta(u) = \text{val}_\beta(v) \text{ and } \exists x \in [0,1), v = d_\beta(x) \right\}.$$

First ingredient.
Consider two $p$-tuples of alphabets $D = (D_0,\ldots,D_{p-1})$ and $D' = (D'_0,\ldots,D'_{p-1})$.
We set $D - D' = (D_0 - D'_0,\ldots,D_{p-1} - D'_{p-1})$.

From the zero automaton $\mathcal{E}(\beta,D - D')$, we define a converter $C_{\beta,D,D'}$ from $D$ to $D'$, that is, a Büchi automaton accepting the set

$$\left\{ (u,v) \in \prod_{n \in \mathbb{N}} (D_n \times D'_n) : \text{val}_\beta(u) = \text{val}_\beta(v) \right\}.$$

Proposition
If $\delta$ is a Pisot number and $\beta_0,\ldots,\beta_{p-1} \in \mathbb{Q}(\delta)$, then the converter $C_{\beta,D,D'}$ is finite.
Second ingredient.

In the case where $\beta$ is a Parry alternate base, we can define a Büchi automaton accepting the set $\{d_\beta(x) : x \in [0, 1)\}$.

For $\beta = (\frac{1+\sqrt{13}}{2}, \frac{5+\sqrt{13}}{6})$, we have seen that $d_{\beta(0)}^*(1) = 200(10)^\omega$ and $d_{\beta(1)}^*(1) = (10)^\omega$.

Combining these two automata, we obtain the following result.

**Theorem (Charlier, Cisternino, Masáková & Pelantová 2022)**

*If $\delta$ is a Pisot number and $\beta_0, \ldots, \beta_{p-1} \in \mathbb{Q}(\delta)$, then the normalization function $\nu_{\beta,D}$ is computable by a finite Büchi automaton.*
Thank you!