Invited Paper

Inclusion of periodic solutions for forced delay differential equation modeling El Niño

Shin’ichi Oishi\textsuperscript{1a)} and Kouta Sekine\textsuperscript{2}

\textsuperscript{1} Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University, Tokyo 169-8555 and JST CREST, Japan

\textsuperscript{2} Faculty of Information Networking for Innovation and Design, Toyo University, 1-7-11 Akabanedai, Kita-ku, Tokyo, 115-0053, Japan

\textsuperscript{a)} oishi@waseda.jp

Received July 13, 2020; Revised January 7, 2021; Published July 1, 2021

Abstract: A computer assisted proof is presented for the existence of various periodic solutions for forced Suarez-Schopf’s equation, which are delay differential equations modeling El Niño. Tight inclusions of periodic solutions are calculated through numerical verification method by utilizing a structure of Galerkin’s equation for forced Suarez-Schopf’s equation effectively. The existence of various periodic solutions has been proved via computer assisted proofs including various subharmonics. Especially, coexistence of several subharmonics are proved and numerical simulations are presented suggesting an appearance of chaos.

Key Words: asymptotic diagonally dominant matrix, El Niño phenomena, periodic solution, subharmonic windows, chaos, computer assisted proof

1. Introduction

In 1988, Suarez and Schopf [1] have introduced a \textit{delayed action oscillator} equation

\[
\frac{dx(t)}{dt} - x(t) + x^3(t) + \alpha x(t - \tau) = 0
\]  

as a simple model of El Niño. We will call Eq. (1) as Suarez and Schopf’s equation, or SS equation in short. The variable \(x\) represents a deviation of a sea surface temperature near Peru from average. The term \(f(x) = -x(t) + x^3(t)\) represents an effect of energy exchange between sea surface and atmosphere. The term \(\alpha x(t - \tau)\) represents an effect of delay by wave propagation on the equator from this area to the east end of Asia (the eastward Kelvin wave), reflected at there, and reflecting back to near Peru (the westward Rossby wave). The parameter \(\tau (> 0)\) expresses a turn around time of these Kelvin and Rossby waves propagation. The parameter \(\alpha (> 0)\) expresses a strength of the effect of this delayed wave. Suarez and Schopf [1] have shown that, by the introduction of the delay term, periodic
solutions with the long periods exist comparable to those of the real El Niño phenomena despite SS equation is one of the simplest equation describing balance between the active nonlinear effect and the delay effect. However, it is pointed out that SS equation cannot describe a synchronization to seasonal change and many of complex behavior of El Niño phenomena. To overcome this, several authors consider to add an effect of seasonal forcing. Among them, Ghil, Zaliapin and Thompson [11] have introduced another model of El Niño Southern Oscillation, or ENSO in short, as

\[ \frac{dx(t)}{dt} + b \tanh(\kappa x(t - \tau)) + c \cos 2\pi t = 0. \]  

(2)

Here, the parameters \( b, c, \) and \( \kappa \) express strengths of the effects of nonlinear interaction, the seasonal force, and the delayed wave, respectively. In [11] and [12] it has been shown that a complex dynamics can be described by Eq. (2). Keane, Krauskopf, and Postlethwaite [13] have investigate Eq. (2) and shown the coexistence of stable tori, how they relate to each other, and bifurcate, which involves bifurcations of invariant tori.

This paper proposes to add a seasonal forcing term to SS equation and consider the following forced delay action oscillator equation

\[ \frac{dx(t)}{dt} - x(t) + x^3(t) + \alpha x(t - \tau) - \beta \cos \omega t = 0, \]  

(3)

which we will call forced Suarez and Schopf’s equation, or fSS equation in short. Here, the parameters \( \beta (\geq 0) \) and \( \omega (> 0) \) express the strength of the effect of the seasonal force, and the angular frequency of the seasonal force, respectively. Compared with Ghil, Zaliapin and Thompson’s equation, Eq. (3) has a cubic nonlinearity, and additive linear delay and sinusoidal forcing terms. Thus, it is a very simple model describing a balance among effects of nonlinearity, delay and seasonal forcing. In this paper, we will prove various periodic solutions exist and various bifurcation phenomena among them are happen including various subharmonic solutions, symmetry breaking bifurcations, period doubling bifurcations. The analysis of fSS equation is still not so simple because fSS equation is a delay differential equation so that it is an infinite dimensional nonlinear equation. We have analyzed fSS equation using a computer assisted proof based on verified numerical computations. One of the main results is to prove the following theorem by this approach:

**Theorem 1.1.** Let \( \alpha = 0.67, \beta = 0.85, \tau = 1.85 \) and \( \omega = 1.3 \). Then, at this parameter set, Eq. (3) has \( 1/2, 1/3 \) symmetric and asymmetric, \( 1/4, 1/5 \) symmetric and asymmetric, and \( 1/7 \) symmetric subharmonic solutions.

This coexistence of several subharmonics in the same parameter region suggests the existence of chaos. We will report various chaotic phenomena can be seen in various parameter sets.

In 1965, Urabe [2] has initiated a study of including a periodic solution for the Duffing equation. In 1972, Bouc [3] has given a functional analytic estimate of a key constant in Urabe’s theory. Based on these result [2, 3], in [4], the first author of the present paper has presented a method of computer assisted proof of the existence of exact solutions for general nonlinear operator equations. Recently, extensive studies have conducted on computer assisted proofs of the existence of periodic solutions for delay differential equations using numerical verification method [5–8]. In [9], the first author of the present paper has presented a method for including periodic solutions to the delay Duffing equation by extending the method presented in [4]. In the present paper, using the method in [9], we will include exact periodic solutions of fSS equation. From the point of view of the verified numerical computations, by introducing a concept of asymptotic diagonal dominant matrix, we will show that an acceleration is possible for a convergence of verification algorithm.

The fSS equation has a symmetry, i.e., if \( x(t) \) is a solution, \( \tilde{x}(t) = -x(t + \pi) := Sx(t) \) becomes its solution, too. A periodic solution \( x(t) \) satisfying \( Sx = x \) is called odd symmetric. In Section 2, we show that for any given set of \( \alpha, \beta, \omega \) and \( \tau \), fSS equation has at least one odd symmetric periodic solution with the same period as the external force, which we will call a fundamental periodic solution. A computer assisted method is given for including exact periodic solutions through numerical verification.
method. We will show that under a suitably chosen framework the Jacobian of Gerlerkin’s equation for forced Suarez-Schopf’s equation becomes an asymptotic block diagonally dominant matrix. We have developed a method utilizing this structure. Appendix A discusses this new concept of an asymptotic block diagonally dominant matrix in detail. In Section 3, we will show in certain parameter ranges of $\alpha, \beta, \omega$, a fundamental periodic solution curve obtained by changing the parameter $\tau$ has saddle node bifurcation points. By solving an extended system which is satisfied by a saddle-node points, we have numerically identifies a saddle-node bifurcation curve in $\tau, \beta$ plane. We will report a kind of global bifurcation named island bifurcation is observed and a loop of odd periodic solutions is generated.

Fixing $\alpha$ and $\omega$, in $\tau, \beta$ parameter space, we will show that a symmetry breaking bifurcation curve can be parameterized as $\{\beta(\tau)|\tau \in [0, 2\pi/\omega]\}$. We observe that $(\tau, \omega)$ is below this curve may become a necessary and sufficient condition for the existence of an asymmetric periodic solutions with the angular frequency $\omega$. In Section 4, it is shown that a variety of subharmonic solutions exist. In fact, we will prove the existence and have tight inclusions of 1/2, 1/3 symmetric, 1/3 asymmetric, 1/4, 1/5 symmetric, 1/5 asymmetric, 1/6, 1/7 symmetric and asymmetric and several further higher subharmonics including 1/9, 1/11, 1/13, 1/21 subharmonics. For 1/2, 1/3 symmetric, 1/4, 1/5 symmetric, 1/7 symmetric subharmonics, if we fix $\alpha, \beta$ and $\omega$, then in $x, \tau$ space continuous component of such subharmonic solutions forms a loop with two saddle-node bifurcation points. From this fact, we can determine an existence region of such subharmonic solutions by solving the extended system satisfied by saddle-node bifurcation points. This is also true for 1/3 asymmetric and 1/5 asymmetric solution branch except they are generated by symmetry breaking bifurcations. We will show there exist cases in which a period doubling bifurcation of 1/2 subharmonics generate 1/4 subharmonics and a symmetry breaking bifurcations generate solution curves of 1/3 asymmetric and 1/5 asymmetric harmonics from solution curves of 1/3 symmetric and 1/5 symmetric harmonics, respectively. At the parameter value $\alpha = 0.67, \omega = 1.3$, and $\beta = 0.85$, coexistence of various subharmonics is shown. Furthermore, for $\tau \in [1.703, 1.860]$ and $\tau \in [2.031, 2.164]$, we can observe mainly chaotic behavior, except several windows of attractive periodic solutions. Here, we should remark that Tsuda, Tamura, Sueoka and Fujii [14] have treated

$$
\frac{d^2x(t)}{dt^2} - (h - \gamma x^2(t)) \frac{dx(t)}{dt} + \omega_0^2 x(t) + qx^3(t) + bx(t - \tau) - f \cos \omega t + G
$$

and have observed the coexistence of several subharmonics and chaotic phenomena. Here, $h, \gamma, \omega_0, \omega, q, b, \tau, f$ and $G$ are real parameters. In Section 5, concluding remarks are presented.

In this paper, MATLAB is used for numerical integration and for calculating Galerkin’s approximate periodic solutions. For computer assisted proof we use VCP library developed by one of the authors, Kouta Sekine1. To handle smoothly various data types in verified computations, Masahide Kashiwagi has developed C++ class library named kv library2. This library is written on the philosophy of policy-based programming. VCP library is written on kv library to enjoy high performance computing technology based on the optimized BLAS such as MKL. Thus, using VCP one can write a high performance verification program with a policy-based programming philosophy. Details are referred to the cited home pages.

Stability is not clear for an approximate solution obtained by Galerkin’s method. To verify the stability rigorously is not scope of this paper because it needs a lot of space. In stead, we will say such a solution is stable if it is also obtained as a stationary state of solving the initial value problem of fSS via Euler’s method.3

2. Verification method for fSS equation

In the first place, we note that if $\sigma(t) \in C([\tau, 0])$ is given, then Eq. (3) with the initial history $x(t) = \sigma(t), t \in [\tau, 0]$ has a global solution $x \in C^1([0, T])$ with any $T > 0$. Here, for a closed

1VCP library can be downloaded from https://verified.computation.jp/VCP_Lib/vcp_latest.zip
2C++ numerical verification library with guaranteed accuracy. See details of kv library, http://verifiedly.me/kv/.
3In the rest of paper, to save space, we use the terminology ‘the result of numerical integration’ for pointing ‘the result of solving initial value problems of fSS via Euler’s method’.
finite interval \( I, C(I) \) and \( C^1(I) \) are sets of continuous functions and of continuous functions on \( I \), with norms \( \|x\|_\infty = \max_{t \in I} |x(t)| \) and \( \|x\|_{C^1} = \|x\|_\infty + \|x'\|_\infty \), respectively. Moreover, \( x' \) is the derivative of \( x(t) \) with respect to \( t \). This existence of a global solution can be seen as follows. Let \( 0 \leq t \leq \tau \). Since \( x(t - \tau) = \sigma(t - \tau) \) is a given function, by the Picard-Lindelöf theorem (see, for instance, [15] Theorem 1.3.1), it is seen that there exists a positive \( \epsilon \leq \tau \), a solution \( x^* \in C^1[0, \epsilon] \) of the problem exists and unique. Let \( \sigma_0 = \max_{t \leq \tau} |\sigma(t)| \). Then, we will show that for any \( \delta > 0 \), \( \|x^*\|_{C^1[0,\epsilon]} < \max\{\sqrt{\alpha}, \beta, \sigma_0\} + \delta := c \). Let assume the contrary, i.e., assume that \( x(\tau) = c \) is attained at certain \( \tau \in (0, \epsilon] \). Since \( c > \beta \), \( x(\tau) - \beta \cos \omega \tau > 0 \). On the other hand, as \( c > \max\{\sqrt{\alpha}, \sigma_0\} \),

\[
[x(\tau)]^3 + \alpha x(\tau - \tau) \geq c^3 - \alpha c > 0.
\]

Thus,

\[
\frac{dx(\tau)}{dt} = -\left\{x(\tau) - [x(\tau)]^3 + \alpha x(\tau - \tau) - \beta \cos \omega \tau \right\} < 0
\]

which contradicts to the expectation \( dx(\tau)/dt \geq 0 \) because we have assumed \( \|x\|_{C^1[0,\epsilon]} = c \). Similarly, if we assume that if \( x(t) = -c \) is attained at some \( \tau \in [0, \epsilon] \), we have a contradiction. Thus, we have \( \|x^*\|_{C^1[0,\epsilon]} \leq c \). This shows that \( x^* \) can be extended to a whole interval \([0, \tau]\) with the bound \( \|x^*\|_{C^1[0,\tau]} \leq \max\{\sqrt{\alpha}, \beta, \sigma_0\} \). Using the idea of the step method, we can continue \( x^* \) continuously to a solution of the problem, say also \( x^* \), for \( t \in [0, \tau) \). This extended \( x^* \) is \( C^1 \) and have a bound \( \|x^*\|_{C^1[0,T]} \leq \max\{\sqrt{\alpha}, \beta, \sigma_0\} \) for any \( T > 0 \).

In this paper, we will prove that the fSS equation has a various kind of periodic solutions including various subharmonic solutions. To prove the existence of such periodic solutions, we use a computer assisted proof consisting of

**Step 1** First, using Fourier-Galerkin method, we obtain a very good approximation.

**Step 2** Then, consider the infinite dimensional Newton method starting with this approximate solution. Using the convergence theorem of Newton’s method, we will prove the existence of exact periodic solution in a neighborhood of this approximate solution. To show that the conditions of the convergence theorem of Newton’s method are held, we will use the numerical computations with result verifications based on the interval arithmetic.

By a variable transformation \( s = \omega t \), we rewrite Eq. (3). If we further write \( s \) by \( t \) and \( \tilde{x}(s) = x(s/\omega) \) by \( x(t) \), we have

\[
\frac{dx(t)}{dt} + \frac{1}{\omega} (-x(t) + x^3(t) + \alpha x(t - \omega t) - \beta \cos t) = 0 \tag{5}
\]

for \( t \in [0, 2\pi] \). We now show a functional analytic framework for inclusion of periodic solutions, which becomes a base of Step 2. Assume that positive real parameters \( \alpha \) and \( \tau \) are fixed. In the first place, we note that Eq. (5) has a symmetry, i.e., if \( x(t) \) is a solution, then \( -x(t + \pi) = Sx(t) \) becomes its solution, too. A periodic solution \( x(t) \) satisfying \( Sx = x \) is called odd symmetric. In the following, for the moment, we are concerned with an odd symmetric periodic solution. Let \( L^2(0, 2\pi) \) be a space of all square integral functions on the interval \([0, 2\pi]\). \( L^2(0, 2\pi) \) is Hilbert space with the inner product

\[
(x, y) = \int_0^{2\pi} x(t)y(t)dt
\]

for \( x, y \in L^2(0, 2\pi) \). We denote

\[
\|x\|_2 = \sqrt{\int_0^{2\pi} |x(t)|^2dt}.
\]

Let \( X = Y = L^2(0, 2\pi) \). Let

\[
H^1_S = \{x, \dot{x} \in L^2(0, 2\pi) | Sx = x \}.
\]

Here, \( \dot{x} \) is a derivative of \( x \) with respect to \( t \) in the distribution sense. Then, \( H^1_S \) becomes Sobolev space, with the inner product
\((x, y)_{H^1} = (x, y) + (\dot{x}, \dot{y}).\)

We denote
\[\|x\|_{H^1} = \sqrt{\|x\|_2^2 + \|\dot{x}\|_2^2}.\]

Let \(D = H^1_\lambda\) and \(L: D \to Y\) be defined for \(x \in D\) by
\[Lx = \frac{dx}{dt}.\]

Assume \(\omega > 0\) is fixed. Let \(N: D \to Y\) and \(F: D \to Y\) be defined by
\[Nx = \frac{1}{\omega}(-x(t) + x^3(t) + \alpha x(t - \omega \tau) - \beta \cos t).\]

and
\[Fx = Lx + Nx,\]

respectively. If \(x \in D\) satisfies \(Fx = 0\), \(x\) becomes a periodic solution of forced Suarez and Schopf’s equation. It is easy to see that \(F: D \to Y\) is Fréchet differentiable with respect to \(x\) and for \(p \in D\) the Fréchet derivative \(D_x F(x)p\) becomes
\[D_x F(x)p = \frac{dp(t)}{dt} + \frac{1}{\omega}[-p(t) + 3x^2(t)p(t) + \alpha p(t - \omega \tau)].\]

It is also easy to see that an operator \(L: D \to Y\) is the Fredholm operator with index 0. In fact, the null space of \(L: D \to Y\) is given by \(N(L) = \{x = \text{const} \in D\}\). The range of \(L\) is given by
\[R(L) = \left\{x \in L^2(0, 2\pi) \mid \int_0^{2\pi} x(t) dt = 0 \right\}.\]

Thus, \(\text{dim} \ N(L) = 1\), \(R(L)\) is closed in \(Y\), and \(\text{codim} \ R(L) = 1\). Thus, \(L: D \to Y\) is Fredholm with index 0. Here, the index of Fredholm operator \(L: D \to Y\) is defined by \(\text{ind}(L) = \dim N(L) - \text{codim} R(L)\). Moreover, the operator \(N: D \to Y\) is compact. Since the compact perturbation of a Fredholm operator keeps the Fredholm property and its index, \(F = L + N: D \to Y\) is Fredholm with index 0. Sard-Smale’s lemma [17] asserts that without loss of generality, we can assume that 0 is a regular value of \(F\) and in this case, a solution manifold \(\{x \in D | F(x) = 0\}\) becomes 0 dimensional, i.e., it is a union of isolated points.

Furthermore, it is seen that \(F = L + N: D \subset X \to Y\) is proper, i.e., if \(M\) is a compact set in \(X\), then \(F^{-1}(M)\) is compact in \(D\). This property follows from the fact that a solution of \(F(x) = y\) becomes analytically [18] and from the fact that a bounded set of analytic functions is compact in \(D\). Based on the above mentioned observations, we can now prove the following theorem:

**Theorem 2.1.** Let \(\omega\) be a positive real number. Then, forced Suarez and Schopf’s equation (5) has at least a solution \(x^* \in H^1_\lambda\). In other word, \(F: D \subset X \to Y\) has at least a zero in \(H^1_\lambda\). This fundamental periodic solution \(x^*\) enjoys a priori bounds
\[\|x^*\|_2 \leq \sqrt{2\pi} \max \{\sqrt{2 + \alpha, \beta}\}, \quad \|\dot{x}^*\|_2 \leq \frac{\sqrt{2\pi}}{\omega} (\alpha \max \{\sqrt{2 + \alpha, \beta}\} + \beta).\]

**Proof** Let \(\lambda \in [0, 1]\). Put
\[H(x, \lambda) = \frac{dx}{dt} + \frac{1}{\omega} \left[ x(t) + \lambda [-2x(t) + x^3(t) + \alpha x(t - \omega \tau)] - \beta \cos t \right].\]

Then, \(H: D \times [0, 1] \subset X \to Y\) is proper Fredholm with index one. Furthermore, it is obvious that \(H: D \times [0, 1] \subset X \to Y\) is \(C^2\). Thus, Sard-Smale’s lemma [17] says that we can assume without loss of generality that 0 is a regular value of \(H\) and in this case, a solution manifold \(\{(x, \lambda) \in D \times [0, 1] | H(x, \lambda) = 0\}\) becomes one dimensional, i.e., it is a union of one dimensional manifolds.

Since \(H(x, 0)\) becomes linear, it is easy to see that \(H(x, 0) = 0\) has a unique solution, say \(\tilde{x}\), in \(D\).

In fact, we have
\[ \ddot{x}(t) = \frac{\beta}{1 + \omega^2}(\cos t + \omega \sin t). \]

Thus, \[ \|\ddot{x}\|_\infty := \max_{0 \leq t \leq 2\pi} |\ddot{x}(t)| = \beta. \] Let us now consider the case of \( \lambda \in (0, 1] \). We note that it is known that the solution of \( H(x, \lambda) = 0 \) becomes analytic \([18]\). Based on this, we now show that the solution curve of \( H(x, \lambda) = 0 \) connected with \( (\tilde{x}, 0) \) is located in the bounded region \( \{ x \in H_\beta : \|x - \tilde{x}\|_\infty < c \} \times (0, 1] \) with \( c = \max \{ \sqrt{2} + \alpha, \beta \} + \epsilon \). Here, \( \epsilon \) is an arbitrary positive real number.

Let assume contrary that \( \|x^\lambda\|_\infty = c \) on the solution manifold of \( H(x^\lambda, \lambda) = 0 \) with certain \( \lambda \in (0, 1] \). If \( x^\lambda(t) = c \) is attained at some \( \bar{t} \in [0, 2\pi] \). Since \( c > \beta \),

\[ x^\lambda(\bar{t}) - \beta \cos \bar{t} > 0. \]

On the other hand, as \( c > \sqrt{2 + \alpha} \),

\[ -2x^\lambda(\bar{t}) + [x^\lambda(\bar{t})]^3 + \alpha x^\lambda(\bar{t} - \omega \tau) \geq c^3 - (2 + \alpha)c > 0. \]

Thus,

\[ \frac{dx^\lambda(\bar{t})}{dt} = -\frac{1}{\omega} \{ x^\lambda(\bar{t}) + \lambda [-2x^\lambda(\bar{t}) + [x^\lambda(\bar{t})]^3 + \alpha x^\lambda(\bar{t} - \omega \tau)] - \beta \cos \bar{t} \} < 0, \]

which contradicts to the expectation

\[ \frac{dx^\lambda(\bar{t})}{dt} \geq 0, \]

because \( \|x^\lambda\|_\infty = c \). Similarly, if we assume that if \( x^\lambda(t) = -c \) is attained at some \( \bar{t} \in [0, 2\pi] \). Since \( c > \beta \),

\[ x^\lambda(\bar{t}) - \beta \cos \bar{t} < 0. \]

On the other hand, as \( c > \sqrt{2 + \alpha} \),

\[ -2x^\lambda(\bar{t}) + [x^\lambda(\bar{t})]^3 + \alpha x^\lambda(\bar{t} - \omega \tau) \leq -c^3 + (2 + \alpha)c < 0. \]

Thus,

\[ \frac{dx^\lambda(\bar{t})}{dt} = -\frac{1}{\omega} \{ x^\lambda(\bar{t}) + \lambda [-2x^\lambda(\bar{t}) + [x^\lambda(\bar{t})]^3 + \alpha x^\lambda(\bar{t} - \omega \tau)] - \beta \cos \bar{t} \} > 0, \]

which contradicts to the expectation

\[ \frac{dx^\lambda(\bar{t})}{dt} \leq 0, \]

because \( \|x^\lambda\|_\infty = c \). Thus, we have shown that the solution curve of \( H(x, \lambda) = 0 \) connected with \( (\tilde{x}, 0) \) is located in the region \( \{ x \in H_\beta : \|x - \tilde{x}\|_\infty < c \} \times (0, 1] \).

On a solution manifold \( \{(x, \lambda) \in D \times [0, 1] | H(x, \lambda) = 0 \} \), we have

\[ (H(x, \lambda), \dot{x}) = 0. \quad (6) \]

We note that for \( x \in D, (x, \dot{x}) = 0 \) and \( (x^3, \dot{x}) = 0 \). Moreover, from the Schwarz inequality, we have \( (x(t - \omega \tau), \dot{x}(t)) \leq \|x\|_2 \|\dot{x}\|_2 \) and \( (\cos t, \dot{x}) \leq \sqrt{2\pi} \|\dot{x}\|_2 \). Then, from Eq. (6), we have

\[ \|\dot{x}\|_2 \leq \frac{\sqrt{2\pi}}{\omega} (\alpha c + \beta). \]

Thus, it turns out that the solution manifold \( H(x, \lambda) = 0 \) in \( \{(x, \lambda) \in D \times [0, 1] \} \) is located in the bounded region

\[ \left\{ (x, \lambda) \in D \times [0, 1] \mid \|x\|_2 \leq \sqrt{2\pi} c, \|\dot{x}\|_2 \leq \frac{\sqrt{2\pi}}{\omega} (\alpha c + \beta) \right\} \]

and is compact. Especially, the solution curve connected with \( (\tilde{x}, 0) \) should finally reaches to \( \lambda = 1 \) plane at \( (x^*, 1) \in \Omega \times \{1\} \). From the definition of \( H \), \( x^* \) is a solution of \( F(x) = 0 \). \( \square \)

Let \( x \in H_\beta^1 \). Then, \( x \) has the following Fourier series expansion:

580
\[ x = \sum_{i=1}^{\infty} [a_{2i-1} \cos (2i-1)t + b_{2i-1} \sin (2i-1)t]. \]  

(7)

Namely, the set \{cos (2i-1)t, sin (2i-1)t\}_{i=1}^{\infty} forms a complete orthonormal system in \(H^1_S\). Based on this, we define a projection operator \(P_n : H^1_S \rightarrow H^1_S\) by

\[
(P_n x)(t) = \sum_{i=1}^{n} (a_{2i-1} \cos (2i-1)t + b_{2i-1} \sin (2i-1)t)
\]

with

\[
a_{2i-1} = \frac{1}{\pi} \int_{0}^{2\pi} x(t) \cos (2i-1)t \, dt, \quad \text{and} \quad b_{2i-1} = \frac{1}{\pi} \int_{0}^{2\pi} x(t) \sin (2i-1)t \, dt.
\]

The following is a direct consequence of the Perseval equality. Cesari has shown the following theorem [16]:

**Theorem 2.2.** Let \(n\) be a positive integer. For \(x \in H^1_S\), it holds

\[
\|x - P_n x\|_2 \leq \sigma_n \left\| \frac{dx}{dt} \right\|_2, \quad \sigma_n = \frac{1}{2n+1}. \quad \Box
\]

We assume that an approximate periodic solution of Eq. (3), \(x_0(t) \in \mathcal{D}\), is obtained as

\[
x_0(t) = \sum_{i=1}^{m} [a_{2i-1} \cos (2i-1)t + b_{2i-1} \sin (2i-1)t]
\]

with \(m\) being a positive integer. Let \(x \in \mathcal{D}\) be

\[
x(t) = \sum_{i=1}^{\infty} [c_{2i-1} \cos (2i-1)t + d_{2i-1} \sin (2i-1)t].
\]

Let \(n\) be a positive integer. Let us now consider Galerkin’s approximation \(P_n D_x F(x_0) P_n x\). We express \(P_n x\) by its Fourier coefficient as

\[
P_n x \Leftrightarrow \begin{pmatrix} c_1 & d_1 & c_3 & d_3 & c_5 & d_5 & \cdots & c_{2n-1} & d_{2n-1} \end{pmatrix}^t.
\]

(10)

Let \(M_n(\mathbb{R})\) be the set of all real matrices of order \(n\). Let \(G_{(m,n)} \in M_n(\mathbb{R})\) be defined by

\[
P_n D_x F(x_0) P_n x \Leftrightarrow G_{(m,n)} \begin{pmatrix} c_1 & d_1 & c_3 & d_3 & c_5 & d_5 & \cdots & c_{2n-1} & d_{2n-1} \end{pmatrix}^t.
\]

(11)

Then, \(G_{(m,n)}\) is an expression of a Jacobian of Galerkin’s approximation through Fourier’s basis.

The following theorem can be proved based on the arguments in [4]:

**Theorem 2.3.** Let \(X = Y = L^2(0,2\pi)\), and \(\mathcal{D} = H^1_S\). Let \(\sigma_n = 1/(2n-1)\). Let \(U_n = P_n X\) and \(V_n = P_n Y\). Let \(x_0 \in \mathcal{D}\). Then, the Fréchet derivative \(D_x F(x_0) : \mathcal{D} \rightarrow Y\),

\[
D_x F(x_0)p = \frac{1}{\omega} \left[ -p(t) + 3x_0^2(t)p(t) + \alpha p(t - \omega t) \right], \quad (p \in \mathcal{D})
\]

can be extended to a bounded operator from \(X \rightarrow Y\) such that

\[
\|D_x F(x_0)\|_{L(X,Y)} \leq \frac{1}{\omega} \left( 1 + \alpha + 3 \|x_0(t)\|_{L^2}^2 \right) =: K_0.
\]

(12)

Let \(G_{(m,n)} : U_n \rightarrow V_n\) be defined by Eq. (11). Assume

\[
\|G_{(m,n)}^{-1}\|_{L(U_n,V_n)} \leq M_n.
\]

(13)

If \(1 - c_n K_0(1 + M_n K_0) > 0\), \(D_x F(x_0)^{-1} : Y \rightarrow \mathcal{D}\) exists and satisfies

\[
\|D_x F(x_0)^{-1}\|_{L(Y,\mathcal{D})} \leq \frac{\sqrt{(1 + M_n K_0)^2 + (\sigma_n(1 + M_n K_0) + M_n)^2}}{1 - \sigma_n K_0(1 + M_n K_0)} := M.
\]

(14)
Proof} Let \( x \in D \). It follows
\[
\|x\|_2 \leq \|x - P_n x\|_2 + \|P_n x\|_2 \leq \sigma_n \|Lx\|_2 + \|P_n x\|_2.
\]
Then,
\[
\|Lx\|_2 \leq \|D_x F(x_0) x\|_2 + \|D_x N(x_0) x\|_2 \\
\leq \|D_x F(x_0) x\|_2 + \|D_x N(x_0) x\|_{\|L(X,Y)\|_2} \\
\leq \|D_x F(x_0) x\|_2 + K_0 \|\sigma_n \|Lx\|_2 + \|P_n x\|_2\|
\]
Now, consider linearized Galerkin’s equation
\[
s = LP_n x + P_n[D_x N(x_0) P_n x].
\]
Then, noticing \(LP_n x = P_n Lx\) and \(\|P_n x\|_2 \leq \|x\|_2\),
\[
\|s\|_2 \leq \|P_n D_x F(x_0) x\|_2 + \|P_n D_x N(x_0) (x - P_n x)\|_2 \\
\leq \|D_x F(x_0) x\|_2 + \|D_x N(x_0) (x - P_n x)\|_2 \\
\leq \|D_x F(x_0) x\|_2 + \|\sigma_n K_0 \|Lx\|_2.
\]
From the assumption for \(G_{(m,n)} : U_n \rightarrow V_n\)
\[
\|G_{(m,n)}^{-1} \|_{L(V_n,F_{(m,n)})} \leq M_n,
\]
we have
\[
\|P_n x\|_2 \leq M_n \|s\|_2.
\]
Thus,
\[
\|Lx\|_2 \leq \|D_x F(x_0) x\|_2 + K_0 \|\sigma_n \|Lx\|_2 + M_n (\|D_x F(x_0) x\|_2 + \|\sigma_n K_0 \|Lx\|_2).\]
From this, it follows that
\[
\|Lx\|_2 \leq \frac{1 + M_n K_0}{1 - \sigma_n K_0(1 + M_n K_0)} \|D_x F(x_0) x\|_2.
\]
Similarly we have
\[
\|x\|_2 \leq \sigma_n \|Lx\|_2 + M_n (\|D_x F(x_0) x\|_2 + \|\sigma_n K_0 \|Lx\|_2) \\
= \sigma_n (1 + \sigma_n M_n K_0) \|Lx\|_2 + M_n \|D_x F(x_0) x\|_2 \\
\leq \frac{\sigma_n (1 + \sigma_n M_n K_0) + M_n}{1 - \sigma_n K_0(1 + M_n K_0)} \|D_x F(x_0) x\|_2.
\]
Thus, we have
\[
\|x\|_{W^1} = \sqrt{\|x\|_2^2 + \|Lx\|_2^2} \leq \frac{\sqrt{(1 + M_n K_0)^2 + (\sigma_n (1 + \sigma_n M_n K_0) + M_n)^2}}{1 - \sigma_n K_0(1 + M_n K_0)} \|D_x F(x_0) x\|_2.
\]
This shows \(D_x F(x_0) : D \rightarrow Y\) is injective. Since \(D_x F(x_0) : D \rightarrow Y\) is the Fredholm operator with index 0, which is a definition of \(F : D \rightarrow Y\) is the Fredholm operator with index 0, \(D_x F(x_0) : D \rightarrow Y\) is also onto. Thus, it turns out that \(D_x F(x_0) : D \rightarrow Y\) is invertible and theorem follows. \(\square\)

**Remark 1.** To calculate \(K_0\), we need a bound for \(\|x_0\|_\infty\). We here show a method using the Lagrange interpolation. In case of having trigonometric series expression of \(x_0\), this method gives a tight upper bound with a cost of \(2\pi/h\) function evaluations, where \(h\) is a sampling interval. For \(x_0 \in C^2 [0, 2\pi]\) and a nonempty interval \([a, b] \subset [0, 2\pi]\), let \(\tilde{x}(t)\) be a linear interpolation of \(x_0(a)\) and \(x_0(b)\) on \([a, b]\). Then, the error estimate of the Lagrange interpolation gives

582
This shows a tight upper bound for \( \|x_0\|_{C[a,b]} \) presented as a generalization of the convergence theorem for a simplified Newton method to \( \mathbb{R}^n \). Let there exists a real valued function ˜\( \phi \) satisfying the property that ˜\( \phi \) being not Lipschitz continuous:

\[
\frac{d}{dt} (\int_0^t F(s) ds) = F(t)
\]

where \( e = 10 \). We take \( h = 2\pi/1000 \) and put

\[
h_0 = \max_{0 \leq i \leq 1000} \{ |x_0(ih)| \}.
\]

Then, \( c_0 \geq \|\tilde{x}_0\|_\infty \). By estimating \( h_0 \) and \( c_0 \) using interval arithmetic, we have \( h_0 \leq 0.9223 \) and \( c_0 \leq 7.1576 \). From Eq. (16) it follows

\[
\|x_0\|_\infty \leq h_0 + \frac{c_0}{2} h^2 \leq 0.9223 + \frac{7.1576}{2} h^2 \leq 0.9224.
\]

This shows a tight upper bound for \( \|x_0\|_\infty \) is obtained by this approach. \( \square \)

On the other hand, To calculate \( M \), we need further an estimate for \( M_n \). In the next subsection, we will discuss how to calculate \( M_n \), which is one of main topics of this paper. In [9], the following is presented as a generalization of the convergence theorem for a simplified Newton method to \( F(x) = 0 \) in case of \( D_x F \) being not Lipschitz continuous:

**Theorem 2.4.** [9] Let \( X = Y = L^2(0, 2\pi) \) and \( D = H^1_0 \). Let \( F : D \to Y \), and assume that \( F \) is continuously Fréchet differentiable with respect to \((x, \omega)\). Assume that \( D_x F(x_0)^{-1} : Y \to D \) exists and define \( g : D \to D \) by

\[
g(x) = x - D_x F(x_0)^{-1} F(x).
\]

Assume that

\[
\|D_x F(x_0)^{-1}\|_{L(Y,D)} \leq M
\]

and \( M\|F(x_0)\|_Y \leq \eta \). Let \( r_0 > 0 \) and \( B(x_0, r_0) = \{ x \mid \|x - x_0\|_D \leq r_0 \} \). Moreover, we assume also that there exists \( R > 0 \) such that for \( 0 < r < R \) if \( x \in B(x_0, r) \), then there exists a real valued continuous function \( \tilde{b} \) of \( r \) satisfying

\[
M\|D_x F(x) - D_x F(x_0)\|_{L(D,Y)} \leq \tilde{b}(r) = b(r)
\]

with the property that \( \tilde{b}(r) \) is positive for any \( r > 0 \) and monotonically decreasing to zero as \( r \to 0 \). If \( \eta + r_0 b(r_0) \leq r_0 \), then \( g \) becomes a contraction mapping on \( B(x_0, r_0) \) so that it has a unique fixed point \( x^* \) on \( B(x_0, r_0) \). Since \( DF(x^*) \) becomes invertible, the point \( x^* \in D \) becomes a zero of \( F \). \( \square \)

**Proof** For \( x_1, x_2 \in B(x_0, r_0) \), put \( x_t = x_1 + t(x_2 - x_1) \). Then, \( x_t \in B(x_0, r_0) \) for \( 0 \leq t \leq 1 \). The Fréchet derivative of \( g : D \to D \) at \( x_t \) is given by \( D_z g(x_t) = I - D_x F(x_0)^{-1} D_x F(x_t) \). Thus, we have

\[
\|g(x_1) - g(x_2)\|_D = \left\| \int_0^1 D_z g(x_t)(x_1 - x_2) dt \right\|_D
\]

\[
\leq \|D_x F(x_0)^{-1}\|_{L(Y,D)} \left\| \int_0^1 (D_x F(x_t) - D_x F(x_0))(x_1 - x_2) dt \right\|_Y
\]

\[
\leq b(r_0)\|x_1 - x_2\|_D.
\]
From this if $b(r_0) < 1$, $g$ is contractive on the closed ball $B(x_0, r_0)$. Similarly if $x_1 \in B(x_0, r_0)$,

$$
\|g(x_1) - x_0\|_D \leq \|g(x_0) - x_0\|_D + \|g(x_1) - g(x_0)\|_D \\
\leq \eta + b(r_0)\|x_1 - x_0\|_D,
$$

which shows if $\eta + r_0b(r_0) \leq r_0$, $g(B(x_0, r_0)) \subset B(x_0, r_0)$. Thus, the assertion follows from the contraction mapping principle. \(\square\)

Let $x \in B(x_0, r)$. We now estimate a function $\tilde{b}(r)$. Let $p \in D$, $x \in B(x_0, r)$. From

$$
D_x F(x)p - D_x F(x_0)p = \frac{3}{\omega_0} (x^2(t) - \frac{3}{2} x_0^2(t)) p(t)
$$

we have

$$
\|D_x Fx - D_x Fx_0\|_{L(D, Y)} = \frac{6\sqrt{2\pi}\|x_0\|_\infty r}{\omega_0} \equiv \tilde{b}(r).
$$

2.1 Inclusion of periodic solutions

Let us assume that an approximate periodic solution be calculated by Fourier-Galerkin’s method explained in the above. We now show how to check the conditions of Theorem 2.4 to prove the

Let us assume that an approximate periodic solution be calculated by Fourier-Galerkin’s method

and

Thus, the assertion follows from the
difficulty can be resolved by Theorem 2.5, which tunes Theorem 2.3 to FSS equation. In proof of this theorem, a concept of

order delay differential equations. In the following, we will show that this difficulty can be resolved by Theorem 2.5, which tunes Theorem 2.3 to FSS equation. In proof of this theorem, a concept of asymptotic block diagonally dominant matrices plays important role. Details of this new concept are discussed in Appendix A.

Theorem 2.5. Let $m$ be a positive integer. Let $X = Y = L^2(0, 2\pi)$, and $D = H^1_0$. Let $U_m = P_m X$ and $V_m = P_m Y$. Let an approximate periodic solution $x_0$ be obtained as

$$
x_0(t) = \sum_{i=1}^m (a_{2i-1} \cos (2i-1)t + b_{2i-1} \sin (2i-1)t). \quad (20)
$$

The Fréchet derivative $D_x N(x_0) : D \rightarrow Y$ can be extended to a bounded operator from $X \rightarrow Y$ such that

$$
\|D_x N(x_0)\|_{L(X, Y)} \leq \frac{1}{\omega_0} (1 + \alpha + 3\|x_0(t)\|_\infty^2) := K_0. \quad (21)
$$

Put further that

$$
p(t) = \sum_{i=1}^n (c_{2i-1} \cos (2i-1)t + d_{2i-1} \sin (2i-1)t). \quad (21)
$$

Now consider Jacobian of Galerkin’s equation, $G_{(m,n)}$, defined by

$$
P_n D_x F(x_0)P_n(p) \Leftrightarrow G_{(m,n)} \left( \begin{array}{cccccccc}
    c_1 & d_1 & c_3 & d_3 & c_5 & d_5 & \cdots & c_{2n-1} & d_{2n-1} \end{array} \right)^T. \quad (22)
$$

Choose that $n \geq 4m$. For a given $(m, n)$, let $A_m, B_{(m,n)}, C_{(m,n)}, D_{(m,n)}$ be defined by $A, B, C, D$ in Eq. (A-12) with $b = 4m$, respectively. Let

$$
d_m = \frac{(2m-1) + \frac{1}{2} |\alpha_0 - 1| + \sqrt{2} \alpha}{(2m-1)^2 + \frac{1}{2} \alpha^2 + (\alpha_0 - 1)^2 - 2 \sqrt{\alpha (\alpha_0 - 1)^2 + (2m - 1)^2}},
$$

$$
k_1 = \|A^{-1}_m B_{(m,4m)}\|_\infty, \quad k_2 = d_m \|C_{(m,n)} D_{(m,n)}\|_\infty,
$$

$$
k_1^1 = \|A^{-1}_m C_{(m,4m)}\|_1, \quad k_2^1 = d_m \left\| \frac{B_{(m,4m)}}{D_{(m,4m)} f} \right\|_1 = k_2.
$$

584
If \( \max \{k_1, k_2, k_1', k_2' \} < 1 \), then \( M_m = M^{\text{block}}_\infty M^{\text{block}}_1 \) becomes an upper bound of \( \|G^{-1}_{(m,n)}\|_2 \). Here,

\[
M^{\text{block}}_\infty = \max \left\{ \frac{\|A_m^{-1}\|_\infty \cdot d_m}{1 - \max \{k_1, k_2\}} \right\}, \quad M^{\text{block}}_1 = \max \left\{ \frac{\|A_m^{-1}\|_1 \cdot d_m}{1 - \max \{k_1', k_2'\}} \right\}.
\]

Since \( n \) can be taken arbitrary large, it follows that \( D_x F(x_0)^{-1} : Y \rightarrow D \) exists and satisfies

\[
\|D_x F(x_0)^{-1}\|_{L(Y,D)} \leq \sqrt{(1 + M_m K_0)^2 + M_m^2} := M. \quad \Box
\]

**Proof of Theorem 2.5** Let \( n, m \) be positive integers. Let us start discussions with an assumption that an approximate periodic solution is obtained as

\[
x_0(t) = \sum_{i=1}^{m} (a_{2i-1} \cos (2i-1)t + b_{2i-1} \sin (2i-1)t).
\]

Then, we have

\[
x^2_0(t) = \sum_{i=1}^{2m-1} (\alpha_{2i} \cos 2it + \beta_{2i} \sin 2it)
\]

with

\[
\begin{align*}
\alpha_0 & = \frac{1}{2} (a_1^2 + a_3^2 + \cdots + a_{2m-1}^2) + \frac{1}{2} (b_1^2 + b_3^2 + \cdots + b_{2m-1}^2), \\
\alpha_2 & = \left( \frac{1}{2} a_1^2 + a_1 a_3 + a_3 a_5 + a_5 a_7 + \cdots + a_{2m-3} a_{2m-1} \right) \\
& \quad + \left( -\frac{1}{2} b_1^2 + b_1 b_3 + b_3 b_5 + b_5 b_7 + \cdots + b_{2m-3} b_{2m-1} \right), \\
\beta_2 & = (a_1 b_3 + a_3 b_5 + a_5 b_7 + \cdots + a_{2m-3} b_{2m-1}) - (b_1 a_3 + b_3 a_5 + b_5 a_7 + \cdots + b_{2m-3} a_{2m-1}), \\
\alpha_{4m-2} & = \frac{1}{2} (a_{2m-1}^2) - \frac{1}{2} b_{2m-1}^2, \\
\beta_{4m-2} & = \frac{1}{2} a_{2m-1} b_{2m-1}.
\end{align*}
\]

Put further that

\[
p(t) = \sum_{i=1}^{n} (c_{2i-1} \cos (2i-1)t + d_{2i-1} \sin (2i-1)t).
\]

Now consider Jacobian of Galerkin’s equation, \( G_{(m,n)} \), defined by

\[
P_n D_x F(x_0) Q_n(p) \Leftrightarrow G_{(m,n)} \left( \begin{array}{ccccccc} c_2 & d_1 & c_3 & d_3 & c_5 & d_5 & \cdots & c_{2n-1} & d_{2n-1} \end{array} \right)^t.
\]

Choose that \( n \geq 4m \). Then, it is seen that \( G_{(m,n)} \) becomes a band matrix with the band width \( 4m - 1 \). For example, Fig. 1 shows the sparse patterns of \( G_{(m,n)} \) for \((m,n) = (15,70)\), \((m,n) = (20,90)\) and \((m,n) = (50,210)\). For a given \((m,n)\), let \( A_{(m,n)} \), \( B_{(m,n)} \), \( C_{(m,n)} \), \( D_{(m,n)} \) be defined by \( A, B, C, D \) in Eq. (A-12) with \( b = 4m \), respectively. Then, it is easy to see that \( A_{(m,n)} = A_{(m,n')} \) for \( n, n' \geq 0 \). Thus, we denote \( A_{(m,n)} = A_m \). It is also seen that \( \|A^{-1}_{(m,n)} B_{(m,n)}\|_\infty = \|A^{-1}_{(m,n')} B_{(m,n')}\|_\infty \) for \( n, n' \geq 4m \). To see why \( G_{(m,n)} \) becomes a band matrix, let us calculate \( x^2_0(t)p(t) \), because the band structure is coming from this term:

![Fig. 1. Sparse diagram for \( G_{(m,n)} \).](image-url)
\[ x_0^2(t)p(t) = \left( \alpha_0 + \sum_{i=1}^{2m-1} (\alpha_{2i} \cos 2it + \beta_{2i} \sin 2it) \right) \left( \sum_{i=1}^{n} (\epsilon_{2i-1} \cos (2i-1)t + d_{2i-1} \sin (2i-1)t) \right) \\
= \sum_{i=1}^{2m+n-1} (\gamma_{2i-1} \cos (2i-1)t + \delta_{2i-1} \sin (2i-1)t). \]

Here,
\[
\gamma_{4m+1} = 0.5 \left( \alpha_{2(2m+1)} c_3 + \alpha_{2(2m+2)} c_5 + \cdots + \alpha_{2(4m-1)} c_{20} + 2\alpha_0 c_{4m+1} + \alpha_{24m+3} + \cdots + \alpha_{2(2m+1)} c_{8m-1} \right)
- \left( \beta_{2(2m+1)} d_3 + \beta_{2(2m+2)} d_5 + \cdots + \beta_2 d_{4m-1} - \beta_2 d_{4m+3} - \cdots - \beta_{2(2m+1)} d_{8m-1} \right),
\]
\[
\delta_{4m+1} = 0.5 \left( \beta_{2(2m+1)} c_3 + \beta_{2(2m+2)} c_5 + \cdots + \beta_{24m-1} - \beta_2 c_{4m+3} - \cdots - \beta_{2(2m+1)} c_{8m-1} \right)
+ \left( \alpha_{2(2m+1)} d_3 + \alpha_{2(2m+2)} d_5 + \cdots + \alpha_2 d_{4m-1} + 2\alpha_0 d_{4m+1} + \alpha_2 d_{4m+3} + \cdots + \alpha_{2(2m+1)} d_{8m-1} \right),
\]

From this, it is seen that \( 2(C_{(m,n)} D_{(m,n)}) \) is given by

\[
\begin{pmatrix}
0 & 0 & \alpha_{2(2m-1)} & -\beta_{2(2m-1)} & \alpha_{2(2m-2)} & -\beta_{2(2m-2)} & \cdots & \alpha_{2} & -\beta_{2} & 0 & 0 & \alpha_{2} & \beta_{2} \\
0 & 0 & \beta_{2(2m-1)} & \alpha_{2(2m-1)} & \alpha_{2(2m-2)} & \beta_{2(2m-2)} & \cdots & \beta_{2} & \alpha_{2} & 0 & 0 & -\beta_{2} & \alpha_{2} \\
0 & 0 & 0 & 0 & \alpha_{2(2m-1)} & -\beta_{2(2m-1)} & \cdots & \alpha_{4} & -\beta_{4} & \alpha_{2} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{4} & \beta_{4} & \cdots & \alpha_{2(2m-1)} & \beta_{2(2m-1)} & 0 & 0 & 0 & 0 & 0 & \cdots \\
-\beta_{4} & \alpha_{4} & \cdots & -\beta_{2(2m-1)} & \alpha_{2(2m-1)} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\alpha_{2} & \beta_{2} & \cdots & \alpha_{2(2m-1)} & \beta_{2(2m-1)} & \alpha_{2(2m-2)} & \beta_{2(2m-2)} & \cdots & \alpha_{2} & \beta_{2} & 0 & 0 & 0 \\
-\beta_{2} & \alpha_{2} & \cdots & -\beta_{2(2m-1)} & \alpha_{2(2m-1)} & -\beta_{2(2m-2)} & \alpha_{2(2m-2)} & -\beta_{2(2m-2)} & \cdots & \beta_{2} & \alpha_{2} & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

This shows that \( G_{(m,n)} \) becomes a band matrix. We note that \( 2 \times 2 \) diagonal block of \( D_{(m,n)} \) is given by

\[
\begin{pmatrix}
\frac{1}{\omega}(-1 + \alpha_0 + \alpha \cos (2i - 1)\omega \tau) & 2i - 1 - \frac{\alpha}{\omega} \sin (2i - 1)\omega \tau \\
-(2i - 1) - \frac{\alpha}{\omega} \sin (2i - 1)\omega \tau & +\frac{1}{\omega}(-1 + \alpha_0 + \alpha \cos (2i - 1)\omega \tau)
\end{pmatrix}
:= D(2i)
\]

with \( i > m \). We note that for \( n, n' \geq 4m \),

\[
\| (C_{(m,n)} D_{(m,n)}) \|_{\infty} = \| (C_{(m,n') D_{(m,n')}}) \|_{\infty} = \sum_{i=1}^{2m-1} (|\alpha_{2i}| + |\beta_{2i}|) := g_{m}. \]

Namely, if \( n \geq 4m \), \( \| C_{(m,n)} D_{(m,n)} \|_{\infty} \) becomes independent of \( n \). Furthermore, \( D(2i) \) is independent of \( n \) and

\[
\| D(2i)^{-1} \|_{\infty} \leq \frac{(2i - 1) + \frac{1}{\omega}(|\alpha_0 - 1| + \sqrt{2}\alpha)}{(2i - 1)^2 + \frac{1}{\omega^2}[\alpha^2 + (\alpha_0 - 1)|^2 - 2\frac{\alpha}{\omega} \sqrt{(\alpha_0 - 1)^2 + (2i - 1)^2}] := d_i.
\]

Thus, if \( n \geq 4m \)

\[
\| D_{(m,n)}^{-1} C_{(m,n)} D_{(m,n)} \|_{\infty} \leq \| D_{(m,n)}^{-1} \|_{\infty} \| C_{(m,n)} D_{(m,n)} \|_{\infty} \leq d_m g_m.
\]

Thus, from Theorem A.1 if \( \max \{k_1, k_2\} < 1 \), for any \( n \) satisfying \( n \geq 4m \) we have

\[
\| G_{(m,n)}^{-1} \|_{\infty} \leq \max \{|A_{m,n}^{-1}|_{\infty} ; d_m\} := M_{\infty}^{\text{block}}
\]

with \( k_1 = |A_{m,n}^{-1} B_{(m,4m)}|_{\infty} \) and \( k_2 = d_m g_m \).
A bound for \( \|G_{(m,n)}^{-1}\|_1 \) can be obtained similarly. It is seen that for \( n \geq 4m \)
\[
\left( \begin{array}{c}
B_{(m,n)} \\
D_{(m,n)}
\end{array} \right)^t = (C_{(m,n)} D_{(m,n)})_f.
\]
Thus, if we put
\[
k_1^1 = \|A_m^{-1}C_{(m,4m)}\|_1 \quad \text{and} \quad k_2^1 = d_m \left\| \left( \begin{array}{c} B_{(m,4m)} \\
D_{(m,4m)}_f \end{array} \right) \right\|_1,
\]
then
\[
k_2^1 = d_m \left\| \left( \begin{array}{c} B_{(m,4m)} \\
D_{(m,4m)}_f \end{array} \right) \right\|_\infty = d_m \left\| (C_{(m,4m)} D_{(m,4m)})_f \right\|_\infty = d_m g_m = k_2.
\]
From Theorem A.4, if \( \max \{k_1^1, k_2^1\} < 1 \), for any \( n \) satisfying \( n \geq 4m \) we have
\[
\|G_{(m,n)}^{-1}\|_1 \leq \frac{\max \{\|A_m^{-1}\|_\infty, d_m\}}{1 - \max \{k_1^1, k_2^1\}} := M_1^{\text{block}}
\]
with \( k_1^1 = \|A_m^{-1}C_{(m,4m)}\|_1 \) and \( k_2^1 = d_m g_m \).

**Remark 2.** The above mentioned discussion gives a lower bound of minimum singular value of \( G_{(m,n)} \) independent of \( n \) provided that \( n \geq 4m \). Our computations so far indicate that the minimum singular value of \( G_{(m,n)} \) itself is independent of \( n \) provided that \( m, n \) are not too small with \( n \geq 4m \).

**Remark 3.** If a targeted solution is regular, then usually the norm of the inverse of Jacobian matrix of Galerkin's equation is bounded by a constant multiple of norm of inverse of the Fréchet derivative of the operator \( F \). Our computations so far indicate that \( k_1 < k_2 \). Furthermore, \( d_m \to 0 \) as \( m \to \infty \) so the \( k_2 \to 0 \) as \( n \to \infty \). This is same for \( k_1^1 \) and \( k_2^1 \). Thus, if we take sufficiently large \( m \), then we can expect that the verification of the targeting periodic solution successes.

**Fig. 2.** Approximate fundamental periodic solution used in the computer assisted proof. Including this figure, in the following, time wave forms of periodic solutions are showing on time intervals having lengths of several times of periods.

Based on such calculations, we first show an inclusion of the fundamental periodic solution for Eq. (5) at \( \alpha = 0.67, \beta = 0.85, \omega = 1.3 \) with \( \tau = 1.5 \) and \( \tau = 1.85 \). We note that Newton’s method starting with approximate solutions
\[
\begin{align*}
x_1 &= -0.6057 \cos t + 0.6140 \sin t + 0.0209 \cos 3t - 0.031 \sin 3t \\
x_2 &= -0.4614 \cos t + 0.3125 \sin t + 0.0126 \cos 3t - 0.0022 \sin 3t
\end{align*}
\]
are convergent for an accurate approximate fundamental solution of Galerkin’s approximates equation of Eq. (5) with \( m = 41 \). At this accurate approximate fundamental solution, we can succeed to prove the existence of an exact fundamental periodic solution using a computer assisted method presented in this section as shown in Table I. Here, \( c = 10 \). Note again this table holds for any \( n \) bigger than \( 4m \). Thus, we have calculated this table at \( n = 4m + 10 \). Numerical values between the third column to the seventh column give upper bounds for corresponding variables. We have calculated \( M \) assuming \( n = 10^9 \). The residual \( r = \|F(z_0)\|_\infty \) and \( K_0 \) depend only on an approximate solution \( z_0 \). If we choose \( \eta = 2Mr \) and if \( n \geq 4(m + 1) \), then \( \eta \) is also independent of \( n \). In this table, the inequality \( b(r_0) < 1 \)
is a sufficient condition for existence of an exact periodic solution \( x^* \) in the \( r_0 \)-neighborhood of the approximate solution \( x_0 \) shown in Fig. 2, i.e., \( x^* \in B(x_0, r) \). Here, we note in Fig. 2, the graph of the approximate solution is shown for the original fSS equation (3). This is because we will compare graphs of Galerkin’s approximations with those of approximations obtained by numerical integration for the original fSS equation (3) including chaotic cases. Throughout this paper, the graph of the approximate solution is shown for the original fSS equation (3).

### 3. Various bifurcations of fundamental periodic solution curve

In this section, we will consider fSS equation (3) fixing \( \omega > 0, 0 < \alpha < 1, \) and \( \beta \geq 0 \) taking \( \tau \geq 0 \) as a parameter. We will report various bifurcations of fundamental periodic solution curve.

#### 3.1 Saddle node bifurcation generates island bifurcation, double islands bifurcation, and chaos

We first consider a solution curve of odd symmetric periodic solutions for fSS equation obtained by changing \( \tau \geq 0 \) for various fixed \( \omega > 0, 0 < \alpha < 1, \) and \( \beta \geq 0 \). We show that saddle node bifurcations occur for certain choice of parameters. It is noted here that fundamental periodic solution curve itself becomes periodic with respect to \( \tau \) with the period \( 2\pi/\omega \), which is come from the periodicity of Eq. (5) itself.

![Fig. 3. Saddle node bifurcation.](image)

It is known that the saddle-node bifurcation point of a 2\( \pi \) periodic solution curve of Eq. (5) is a regular solution of the following extended system

\[
F(x, \tau) = 0, \quad D_x F(x, \tau) \phi = 0, \quad \frac{1}{\pi} \int_0^{2\pi} \phi(t) \cos \tau t \, dt = 0.1. \tag{26}
\]

Here, we add \( \tau \) as unknown variable and add a phase condition \( \frac{1}{\pi} \int_0^{2\pi} \phi(t) \cos \tau t \, dt = 0.1 \) as an additional constraint. Thus, the operator from \( D^2 \times \mathbb{R} \) to \( Y^2 \times \mathbb{R} \) defined by \( (x, \phi, \tau)^t \mapsto (F(x, \tau), D_x F(x, \tau) \phi, \frac{1}{\pi} \int_0^{2\pi} \phi(t) \cos \tau t \, dt)^t \) becomes a Fredholm operator with index zero. Based on this, in the first place, let us consider the case of \( \omega = 1 \) and \( \alpha = 0.75 \). In the following, we observe the change of saddle node bifurcation points by taking \( \beta \) as a parameter. By solving Eq. (26) using Fourier-Galerkin’s method, we have calculated an approximation of saddle node bifurcation curve in \( \tau-\beta \) space as shown in Fig. 3, from which it is seen that there occurs super and sub cusp bifurcations of the saddle node bifurcation curve. From around \( \beta = 1.1 \), if we decrease \( \beta \) then a cusp bifurcation occurs and two saddle node bifurcation points appear in fundamental periodic solution curve. This situation is continued until \( \beta = 0.33 \). Then, at around \( \beta = 0.33 \), the saddle node bifurcation curve is annihilated by another cusp bifurcation. As a result, saddle node bifurcation curve becomes closed curve in \( \tau-\beta \) parameter space.

Similar but different type of saddle node bifurcation curve is observed at \( \omega = 0.6875 \) and \( \alpha = 0.75 \). By solving Eq. (26), we obtain an approximation of saddle node bifurcation curve as shown in Fig. 4. In this case, as before, saddle node bifurcation curve becomes closed curve in \( \tau-\beta \) parameter space.
From around $\beta = 1.1$, as $\beta$ is decreasing, a cusp bifurcation occurs and two saddle node bifurcation points appear in fundamental periodic solution curve. This situation is continued until $\beta = 0.16$. Then, a different situation happens at around $\beta = 1.6$, i.e., a loop of odd symmetric periodic solution curve appear. If further $\beta$ is decreasing, at around $\beta = 0.04$, the saddle node bifurcation points disappear by the annihilation of a loop of odd symmetric periodic solution curve. In the previous paper [9], we have named such a global bifurcation as an island bifurcation.

Generation of a loop of odd symmetric solutions in $x$-$\tau$ space is also seen at $\omega = 0.5$ and $\alpha = 0.75$ as shown in Fig. 5. In this case, further we have observed a generation of the second loop of odd symmetric solutions in $x$-$\tau$ space. This is also a global bifurcation generated via hitting of the bottom of the saddle-node bifurcation curve at $\beta = 0$ line and its part below $\beta = 0$ line is mirror reflected to $\beta > 0$ region. This kind of the mirror reflection is possible because if $x$ is a solution of fSS equation (3) for $\beta$, $-x$ is also a solution of fSS for $-\beta$. This double islands bifurcation, which is our naming of this global bifurcation, happens at $\beta$ same as that of the cusp point of the left end of the saddle-node bifurcation curve. In Fig. 5 saddle-node bifurcation curves are also shown for $\omega = 0.35, 0.3, 0.25$ and $\omega = 0.2$. Especially, for $\omega = 0.2$ during $\tau \in [2.8, 4.5]$, periodic solutions become near singular. The following are initial functions for Newton’s method to Galerkin’s approximation of Eq. (5):

$$
\begin{align*}
    x_{sil} &= 1.3651 \cos t - 0.3521 \sin t - 0.2015 \cos 3t + 0.0124 \sin 3t + 0.0619 \cos 5t + 0.0421 \sin 5t \\
    x_{silm} &= -1.1717 \cos t - 0.6258 \sin t - 0.1094 \cos 3t + 0.1478 \sin 3t + 0.0180 \cos 5t + 0.0607 \sin 5t \\
    x_{sim1} &= -0.0826 \cos t - 0.0115 \sin t \\
    x_{sim1} &= 0.8963 \cos t - 0.0025 \sin t + 0.0412 \cos 3t - 0.1229 \sin 3t - 0.0236 \cos 5t - 0.0040 \sin 5t \\
    x_{sit2} &= -0.6401 \cos t - 0.2838 \sin t - 0.0527 \cos 3t - 0.0181 \sin 3t - 0.0061 \cos 5t + 0.0022 \sin 5t
\end{align*}
$$

The verification result is shown in Table II.
Table II. Verifications for symmetric periodic solutions in Fig. 5 ($\alpha = 0.75, \beta = 0.11, \omega = 0.5, \kappa = K_0(1 + M_2^{block}K_0)$ and $k = \max\{k_1, k_2, k_1^1, k_1^2\}$).

| name | $\tau$ | $\beta$ | $m$ | $k$ | $M_2^{block}$ $M$ | $r_0$ | $b(r_0)$ | $||x||_2$ |
|------|--------|--------|----|----|-----------------|------|----------|--------|
| $x_{sil}$ | 4 | 0.11 | 101 | $2e^{-1}$ | $5e^4$ | 16.7 | $3e^2$ | $2e^{-10}$ | $2e^{-6}$ | 1.4266 |
| $x_{sil}$ | 4 | 0.11 | 101 | $e^{-1}$ | $4e^4$ | 15.5 | $2e^2$ | $8e^{-11}$ | $8e^{-7}$ | 1.3428 |
| $x_{sym}$ | 4 | 0.11 | 41 | $7e^{-4}$ | $3e^2$ | 4.2 | $2e^1$ | $8e^{-15}$ | $3e^{-14}$ | 0.083 |
| $x_{il1}$ | 1.8 | 0.11 | 61 | $7e^{-2}$ | $7e^3$ | 8.4 | $8e^1$ | $7e^{-12}$ | $2e^{-8}$ | 0.9060 |
| $x_{il2}$ | 1.8 | 0.11 | 61 | $5e^{-2}$ | $2e^3$ | 5.7 | $4e^1$ | $2e^{-12}$ | $2e^{-9}$ | 0.7024 |

Fig. 6. Change of saddle-node bifurcation curves.

In order to observe further changes, we are calculating saddle-node bifurcation curves by decreasing $\omega$. The result is shown in Fig. 6. This figure shows that as decreasing $\omega$, at around $\tau = 3$, the two branches of saddle node curves are approaching. To see a behavior of periodic solutions in this ill-conditioned area, we have simulated FSS equation via numerical integration. Figure 7 shows results obtained numerical integration. The first figure in Fig. 7 is a stroboscopic bifurcation diagram. This figure shows that there are two stable asymmetric $2\pi$ periodic solutions at $\tau = 2$. Increasing $\tau$, experienced a subcritical symmetric breaking bifurcation at $\tau = 2.14$ this is deformed into a stable symmetric $2\pi$ periodic solution. At $\tau = 2.14$, only this is stable. However, at $\tau = 2.266$, there are two stable symmetric $2\pi$ periodic solutions. At $\tau = 2.375$, only one stable symmetric $2\pi$ periodic solution is observed. It is noted that at $\tau = 2.14$ and at $\tau = 2.375$, branches of stable one is exchanged. Then, further increase $\tau$ to $\tau = 3$ then, we can observe chaotic phenomena happens. This chaos is generated by the instability caused through an approaching of two branches of the saddle node bifurcation curve. We have checked the existence of $2\pi$ periodic solutions suggested by the numerical simulation. For the purpose, an approximate solution is calculated by Galerkin’s method. Newton’s iterations for Galerkin’s equation at $\tau = 2$ is convergent starting from the following initial functions:

$$x_{\tau=2} = 0.0503 + 0.8065 \cos t + 0.1832 \sin t + 0.0726 \cos 2t + 0.0868 \sin 2t - 0.14 \cos 3t - 0.1545 \sin 3t + 0.05 \cos 4t - 0.208 \sin 4t - 0.114 \cos 5t + 0.0245 \sin 5t$$

\(^4\)Details of symmetric breaking bifurcation is discussed in the next subsection.
A verification result is shown in Table III:

| name | τ | m | k | \(M^\text{block}\) | M | \(r_\theta\) | \(b(r_\theta)\) | \(|x|_2\) |
|------|---|---|---|---------------|---|---------|---------|------|
| \(x_{\tau=2}\) | 2 | 151 | \(2e^{-1}\) | \(5e^6\) | 30 | \(7e^2\) | \(2e^{-9}\) | \(3e^{-6}\) | 0.9020 |

### 3.2 Autonomous case

For considering \(\beta(\geq 0)\) is small, we need to know properties of periodic solutions of autonomous Suarez and Schopf’s Equation

\[
\frac{dx(t)}{dt} - x(t) + x^3(t) + \alpha x(t - \tau) = 0. \tag{27}
\]

Let the parameter range be \(0 < \alpha < 1\) and \(\tau > 0\). The following is shown in [1]. Equation (27) has three equilibrium points \(x = 0, x_\pm = \pm \sqrt{1 - \alpha}\). The state \(x = 0\) is an unstable fixed point and a solution starting near \(x = 0\) grows without oscillating. In [1], a perturbation \(\delta x\) at \(x_\pm\) is considered. Put \(x(t) = \delta x(t) + x_\pm(t)\), then the first order approximation of Eq. (27) becomes

\[
\frac{d\delta x(t)}{dt} = \frac{\delta x(t)}{\tau} + 3x_\pm^2\delta x(t) + \alpha \delta x(t - \tau).
\]

If we put \(\delta x(t) = c_\delta e^{\lambda t}\) with \(c_\delta\) and \(\lambda\) being real and complex constants, respectively, then we have a condition on \(\lambda\) as

\[
\lambda - (3\alpha - 2) + \alpha e^{-\lambda \tau} = 0,
\]

which is called a characteristic equation of Eq. (27) at \(x = x_\pm\). One of the conditions of an occurrence of the Hopf bifurcation is \(\text{Re}[\lambda]\) becomes zero, which gives a condition on \(\alpha\) and \(\tau\) as

\[
\tau = \left(\frac{2n\pi + \arccos\left(\frac{3\alpha - 2}{\alpha}\right)}{\alpha}\right)\left(\alpha^2 - (2 - 3\alpha)^2\right)^{-1/2}, \quad (n = 0, 1, 2, \ldots).
\]

Figure 8 shows the Hopf bifurcation curves. If the parameter \((\alpha, \tau)\) is chosen in the region A, then a solution approaches to one of the stable equilibrium points \(x_\pm\). If the parameter is set in the region B, a solution approaches to a limit cycle generated by the Hopf bifurcation at \(n = 0\). In the region C, D, E, F, and G there exist, al least, 2, 3, 4, 5 and 6 periodic solutions, respectively, generated by successive Hopf bifurcations.

![Fig. 8. Hopf bifurcation curve.](image)

Based on this known result, we are now consider to calculate periodic solutions. Let \(T > 0\) be an unknown period of the periodic wave. Define the angular frequency by \(\omega = 2\pi/T\). By a variable transformation \(s = \omega t\), we rewrite Eq. (27). If we further write \(s\) by \(t\) and \(\dot{x}(s) = x(s/\omega)\) by \(x(t)\), we have

\[
\frac{dx(t)}{dt} + \frac{1}{\omega}(\dot{x}(t) + x^3(t) + \alpha x(t - \omega T)) = 0 \tag{28}
\]

for \(t \in [0, 2\pi]\) with \(\omega\) being an unknown variable. Since a periodic solution of Eq. (28) has a shift invariance, to fix it we pose a phase condition as \(\int_0^{2\pi} x(t) \cos t dt = 0\). Solving Eq. (28) by Galerkin’s method, we found an approximate periodic solution branch generated by the first Hopf bifurcation shown in Fig. 9. We have also verified the existence of exact periodic solutions via verified computations around these approximate solutions. We will present such a verification procedure in a separate paper.

Results of numerical integration implies autonomous oscillations generated by the first Hopf bifurcation are stable and others are unstable.
3.3 Symmetry breaking bifurcation curve as death parasol of asymmetric solutions, almost periodic solutions, and chaos

From this subsection, we are concerned with also asymmetric $2\pi$ periodic solutions. For the purpose, instead of $H^{1}_{S}$, we consider $H^{1}(0, 2\pi) = \{x, \dot{x} \in L^{2}(0, 2\pi)\}$ and $D = H^{1}(0, 2\pi)$. In this case, projection operator $P_{n}$ should be replaced as follows: Let $x \in H^{1}(0, 2\pi)$. We define a projection operator $P_{n}: H^{1}(0, 2\pi) \rightarrow H^{1}(0, 2\pi)$ by

$$(P_{n}x)(t) = \frac{a_0}{2} + \sum_{i=1}^{n} [a_i \cos it + b_i \sin it]$$

with

$$a_i = \frac{1}{\pi} \int_{0}^{2\pi} x(t) \cos it \, dt, \quad b_i = \frac{1}{\pi} \int_{0}^{2\pi} x(t) \sin it \, dt.$$ 

Then, remaining becomes same as those for the symmetric case. Especially, Theorem 2.5 holds also for asymmetric cases by replacing $H^{1}_{S}$ by $H^{1}(0, 2\pi)$, by changing definition of $D, P_{n}$ and Galerkin’s approximation as

$$x_0 = \frac{a_0}{2} + \sum_{i=1}^{m} [a_i \cos it + b_i \sin it].$$

Based on this, we first prove Theorem 3.1. Namely, when $\beta = 0$, fSS equation has three constant solutions, $x(t) \equiv 0$ and $x(t) \equiv \pm \sqrt{1 - \alpha}$. For sufficiently small $\beta > 0$, fSS equation has three different analytic $2\pi$ periodic solutions. Two of them are asymmetric periodic solutions and have a bound $\|x\|_{\infty} \leq 2.5$.

**Proof** Let $\omega > 0$ and $0 < \alpha < 1$ be fixed. Taking $\beta > 0$ as a parameter. Put

$$F(x, \beta) = \frac{dx}{dt} + \frac{1}{\omega} (-x(t) + x^3(t) + \alpha x(t - \omega \tau) - \beta \cos t).$$

Then, $F: D \times [0, 1] \subset X \rightarrow Y$ is proper Fredholm with index one. Furthermore, it is obvious that $F: D \times [0, 1] \subset X \rightarrow Y$ is $C^2$. Thus, Sard-Smale’s lemma [17] says that we can assume without loss of generality that 0 is a regular value of $F$ and in this case, a solution manifold $\{(x, \beta) \in D \times [0, 1] | F(x, \beta) = 0\}$ becomes one dimensional, i.e., it is a union of one dimensional manifolds.

In the first place, we are concerned with the case that the solution curve $(x, \beta)$ of $F(x, \beta) = 0$ is connected to $(\sqrt{1 - \alpha}, 0)$. It is easy to see that $F(x, 0) = 0$ has a unique solution $(\sqrt{1 - \alpha}, 0)$ in the vicinity of $(\sqrt{1 - \alpha}, 0)$. Let us now consider the case of $\alpha \in [0, 1]$ and $\beta \in (0, 10]$. We note that it is know that the solution of $F(x^\beta, \beta) = 0$ becomes analytic [18]. Based on this, we now show that the solution curve of $F(x, \beta) = 0$ connected with $(\sqrt{1 - \alpha}, 0)$ is located in the bounded region.

**Fig. 9.** The odd symmetric periodic solution curve generated by the first Hopf bifurcation ($\alpha = 0.75$).
\{ x \in H^1_S \| x - \sqrt{1-\alpha} \|_\infty < c \} \times (0,10) \text{ with } c = 2.5. \text{ Put } x^\beta = \sqrt{1-\alpha} + \tilde{x}^\beta. \text{ Let assume contrary that } \| \tilde{x}^\beta \|_\infty = c \text{ on the solution manifold of } F(x^\beta, \beta) = 0 \text{ with certain } \beta \in (0,10). \text{ If } \tilde{x}^\beta(\tau) = c > 0 \text{ is attained at some } \tau \in [0,2\pi]. \text{ Figure 10 shows if } c = 2.5.

\begin{align*}
-x^\beta(\tau) + [x^\beta(\tau)]^3 + \alpha x^\beta(\tau - \omega \tau) + \beta \cos \tau \\
\geq -c - \sqrt{1-\alpha} + c^3 + 3\sqrt{1-\alpha}c^2 + 3(1-\alpha)c + (1-\alpha)^3/2 + \alpha(\sqrt{1-\alpha} - c) + \beta \cos \tau \\
= c^3 + 3\sqrt{1-\alpha}c^2 + (2-4\alpha)c - \beta > 0.
\end{align*}

Thus, \[ \frac{dx^\beta(\tau)}{d\tau} = -\frac{1}{\omega} (-x^\lambda(\tau) + x^\beta(\tau) + \alpha x^\beta(\tau - \omega \tau) - \beta \cos \tau) < 0. \]

which contradicts to the expectation \[ \frac{dx^\beta(\tau)}{d\tau} = 0 \]

because \( \| x^\beta - \sqrt{1-\alpha} \|_\infty = c. \) It is obvious, \( \beta > 0 \) is sufficiently small, this solution \( x^\beta \) is an asymmetric \( 2\pi \) periodic solution enjoying \( \| x^\beta \|_\infty < 2.5. \)

For sufficiently small \( \beta > 0, Sx^\beta \) becomes a different \( 2\pi \) asymmetric solution of \( F(x, \beta) = 0 \) connected to \( (-\sqrt{1-\alpha},0). \) Moreover, it is easy to see that for sufficiently small \( \beta > 0 \) there exists a solution of \( F(x, \beta) = 0 \) connected to \( x \equiv 0. \) Thus, if \( \beta > 0 \) is small enough, we have three different periodic solutions. Two of them are asymmetric periodic solutions.

It is known that the symmetry breaking bifurcation point is a regular solution of the following extended system

\begin{equation}
F(x, \tau) = 0, \quad D_x F(x, \tau) \phi = 0, \quad \frac{1}{\pi} \int_0^{2\pi} \phi(t) dt = 0.1, \quad (30)
\end{equation}

where, \( x \in D_s = \{ y \in D | Sy = y \} \) and \( \phi \in D_a = \{ \psi \in D | S \psi = -\psi \}. \) Fixing \( \alpha \) and \( \omega, \) in \( \tau-\beta \) space, Fig. 11 shows curves consisting of the symmetry breaking bifurcation points. In \( \tau-\beta \) space, symmetry breaking bifurcation curves can be parameterized by \( \tau \) as \( (\beta(\tau), \tau). \) It’s shape resembles us a parasol. Every examinations doing so far by us indicate that if and only if a point \( (\beta, \tau) \) in \( \tau-\beta \) space is under this parasol, there exist asymmetric \( 2\pi \) periodic solutions. Thus, we would like to call this parasol

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11.png}
\caption{Symmetry breaking bifurcation curve as death parasol.}
\end{figure}
as death parasol since asymmetric periodic solutions can be observed experimentally only under this curve.

The first three figures in Fig. 12 shows also the case of $\alpha = 0.75, \tau = 1.85,$ and $\omega = 0.6875$. In this figure, while $s$ is a label for a symmetric $2\pi$ periodic solution, $a$ is that for an asymmetric $2\pi$ periodic solution. The first figure from left in Fig. 12 shows the asymmetric periodic solution curve passes the strait between the island and the fundamental periodic solution curve. The next two figures show that the asymmetric periodic solution curve departs from the symmetric breaking bifurcation point on the fundamental periodic solution curve and returns to another symmetric breaking bifurcation point. The fourth figure shows the passing a strait is seen also for the double islands case. Here, we note again the fundamental periodic solution curve itself is periodic with the period $2\pi/\omega$ so that $\tau-x$ space can be considered as a cylinder.

![Fig. 12. Asymmetric 2\pi periodic solution curves.](image)

Since the death parasol is a phenomenon occurs when $\beta$ is small. Thus, we would like to mention here other interesting phenomena occur when $\beta$ is small. Figure 13 shows that when $0 < \beta \leq 0.034$, almost periodic waves are observed by numerical simulation. This kind of phenomena is known to happen when an autonomous oscillation is not completely synchronized to the external force because $\beta$ is too small to synchronize completely.

![Fig. 13. When $\beta \leq 0.034$ almost periodic waves are observed by numerical simulation ($\alpha = 0.75, \tau = 1.85, \omega = 0.6875$).](image)

It is pointed out here that if the autonomous system obtained by putting $\beta = 0$ has no self oscillating mode, then a different behavior is seen as shown in Fig. 14. At $\beta = 0$, there is no autonomous oscillations. At $\beta = 0.07$ we can see two stable asymmetric $2\pi$ periodic solutions. At $\beta = 0.76$ it is seen that two stable $1/2$ subharmonics are generated via a period doubling bifurcation. It means that two asymmetric $2\pi$ periodic solutions becomes unstable. At $\beta = 0.08$ chaotic state is observed. This chaos consists of orbits wandering around two unstable asymmetric $2\pi$ periodic solutions. At $\beta = 0.08$, at $\beta = 0.11$ and at $\beta = 0.13$, it is seen that there exist $1/5$ symmetric, $1/3$ symmetric and $1/7$ asymmetric subharmonics, respectively. At $\beta = 0.2$, it might be seen as an almost periodic solution. This means that the amplitude of the external force $\beta$ becomes large enough to weekly synchronize waves. Then after becoming $\beta$ sufficiently large, synchronized $2\pi$ periodic solutions are observed. Based on this observation, we have calculated approximate subharmonics shown in Fig. 14 by Newton-Galerkin’s method starting with
Fig. 14. Case of no self oscillating mode exist ($\alpha = 0.6, \tau = 1.75, \omega = 0.7$) at $\beta = 0$. Three rows from top are obtained by numerical integration and others are obtained by Galerkin’s method.

Table IV. Verification for 1/5 and 1/7 symmetric subharmonic in Figs. 14 ($\alpha = 0.6, \tau = 1.75, \omega = 0.7$).

| name | $\beta$ | $m$ | $k_1$ | $k_2$ | $\kappa$ | $M_{stock}$ | $M$ | $r_0$ | $b(r_0)$ | $\|x\|_2$ |
|------|--------|-----|-------|-------|---------|-------------|-----|------|---------|---------|
| $x_5$ | 0.09   | 201 | $4.3e^{-2}$ | $8.5e^{-2}$ | $5.2e^5$ | 26 | 693 | $9e^{-10}$ | $5e^{-5}$ | 0.8097 |
| $x_{7a}$ | 0.129 | 351 | $4.5e^{-2}$ | $8.8e^{-2}$ | $1.2e^6$ | 18 | 715 | $3e^{-9}$ | $3e^{-4}$ | 0.83419 |

$x_5 = -0.3206 \cos t - 0.6369 \sin t - 0.2586 \cos 3t - 0.2015 \sin 3t$
$-0.0412 \cos 5t + 0.1674 \sin 5t - 0.0525 \cos 7t - 0.0515 \sin 7t$
$-0.0644 \cos 9t + 0.0145 \sin 9t$

$x_{7a} = 0.2117 \cos t + 0.2300 \sin t + 0.4846 \cos 3t - 0.4552 \sin 3t$
$+0.0384 \cos 5t + 0.2050 \sin 5t - 0.1896 \cos 7t + 0.2549 \sin 7t$
$-0.0664 \cos 9t + 0.0542 \sin 9t$

$x_{7b} = -0.1419 \cos t - 0.1447 \sin t + 0.1161 \cos 3t - 0.1175 \sin 3t$
$0.0911 \cos 5t + 0.0770 \sin 5t - 0.2074 \cos 7t + 0.0968 \sin 7t$
$-0.0500 \cos 9t - 0.0232 \sin 9t + 0.0100 \cos 11t - 0.0356 \sin 11t$

It should be noted here that numerical simulation implies that the first three are stable and the last

595
one unstable. We have included exact 1/5 and 1/7 symmetric subharmonics as shown in Table IV.

4. Subharmonic solutions, bifurcations among them, and chaos

In this section, in the first place fixing $\alpha, \omega$ and $\beta$ and taking $\tau$ as a parameter, we show various properties of subharmonics and bifurcations among them for fSS equation.

Let $n = 2, 3, \ldots$. We consider

$$\frac{d\tilde{x}(s)}{ds} + \frac{1}{\tilde{\omega}}(-\tilde{x}(s) + \tilde{x}^3(s) + \alpha \tilde{x}(s - \omega \tau) - \beta \cos ns) = 0 \tag{31}$$

and calculate its $2\pi$ periodic solution. If $\tilde{x}(s)$ is a $2\pi$-periodic solution of Eq. (31), then $x(t) = \tilde{x}(s = t/n)$ becomes a $2n\pi$ periodic solution, i.e. a $1/n$ subharmonic solution, of fSS:

$$\frac{dx(t)}{dt} + \frac{1}{\omega}(-x(t) + x^3(t) + \alpha x(t - \omega \tau) - \beta \cos t) = 0 \tag{32}$$

with $\omega = n\tilde{\omega}$. It is easily seen that there exist symmetries satisfied by subharmonic solutions. Let $n$ be a positive integer, then it is seen that Eq. (32) has $n$ symmetries for $2n\pi$ periodic solutions:

$$S_1 x(t) = -x(t + \pi), \quad S_2 (t) = x(t + 2\pi), \quad S_3 (t) = x(t + 4\pi), \quad \ldots, \quad S_n (t) = x(t + (n - 1)\pi).$$

Namely, if $x(s)$ is a $2n\pi$ periodic solution of Eq. (32), then $S_i x(t)$, $(i = 1, 2, \ldots, n)$ are also its $2n\pi$ periodic solutions. A solution satisfying $S_1 x = x$ is called an odd symmetric solution. If $x(t)$ is a $2\pi$ periodic solution of Eq. (32) for $\tau$, then it also a $2\pi$ periodic solution of Eq. (32) for $\tau + 2\pi/\omega$.

Fixing $\alpha, \beta, \tilde{\omega}$, define the operator $G : H^1(0, 2\pi) \times \mathbb{R} \rightarrow L^2(0, 2\pi)$ as

$$G(\tilde{x}, \tau) = \frac{d\tilde{x}(s)}{ds} + \frac{1}{\omega}(-\tilde{x}(s) + \tilde{x}^3(s) + \alpha \tilde{x}(s - \omega \tau) - \beta \cos ns).$$

Then, $G : H^1(0, 2\pi) \times \mathbb{R} \rightarrow L^2(0, 2\pi)$ becomes the Fredholm operator with index one. From Sard and Smale’s Lemma, without loss of generality, we can assume $0$ is a regular value of $G$. In this case, a solution set of $G(\tilde{x}, \tau) = 0$ in $\tilde{x}$-$\tau$ space consists of a set of one dimensional curves. Since $G$ is $2\pi/\tilde{\omega}$ periodic with respect to $\tau$, such solution curves in $\tilde{x}$-$\tau$ space can be classified in the following ones:

1. Loops,

2. One dimensional curve starting with a bifurcation point and end with a bifurcation point. For such a bifurcation points, period doubling bifurcation points and symmetry breaking bifurcation points are possible.

Now, taking $\tau$ and $\beta$ are parameters. From this observation, an existence area of subharmonics in $\tau$-$\beta$ parameter space can be determine by the following ways:

1. As mentioned above, a solution set of $G(\tilde{x}, \tau) = 0$ in $\tilde{x}$-$\tau$ space becomes an one dimensional curve. In general, for $1/2n$ subharmonic solution curves and $1/(2n + 1)$ symmetric subharmonic solution curves, $(n = 1, 2, \ldots)$, this curve becomes a loop. Since both ends of this loop are saddle-node bifurcation points, boundaries of an existence region for such a subharmonics in $\tau$-$\beta$ space consist of saddle-node bifurcation curves. Thus, such an area can be determined by solving a system of equations satisfied by saddle-node bifurcation points by changing $\beta$.

2. For the case of $1/(2n + 1)$, $(n = 1, 2, \ldots)$ symmetric subharmonic solution curves, since they are generated from symmetry breaking bifurcations, boundaries of an existence region for such a subharmonics in $\tau$-$\beta$ space consist of symmetry breaking bifurcation curves. Thus, such an area can be determined by solving a system of equations satisfied by symmetry bifurcation points by changing $\beta$.

In fact, we have determined the boundaries of the existence of subharmonics in $\tau$-$\beta$ space by this method. In the following, we will report such a results. Throughout this paper, while Galerkin’s approximations are shown for Eq. (31), to compare results of numerical integration and those of Galerkin’s approximation, graphs of the approximate solutions are shown for the original fSS equation (3).
4.1 Stable and unstable 1/2 subharmonics: Window, coexistence with higher subharmonics, and chaos

By seeking $2\pi$ periodic solution of Eq. (31) with $n = 2$, we have identified several area where approximate 1/2 subharmonic solutions for Eq. (32) exist. The first figure in Fig. 15 shows an example of the area where approximate 1/2 subharmonic solutions exist. The solid line in the second figure of Fig. 15 shows a slice of this area by the line $\beta = 1.76$. On the other hand, the dashed line shows a fundamental periodic solutions’ curve. Galerkin’s approximate solution of Eq. (31) for $n = 1, 2$ can be obtained by Newton’s method starting with the following initial approximation:

$$
\begin{align*}
    x_1 &= -0.5003 \cos t + 0.6636 \sin t + 0.0137 \cos 3t - 0.0245 \sin 3t - 0.0005 \cos 5t + 0.0017 \sin 5t \\
    x_2 &= -0.0001 - 0.0039 \cos t - 0.0147 \sin t - 0.5002 \cos 2t + 0.6637 \sin 2t \\
    &\quad -3.1951e^{-4} \cos 3t - 2.4467e^{-3} \sin 3t + 8.0741e^{-6} \cos 4t + 4.5587e^{-7} \sin 4t \\
    &\quad +1.3006e^{-3} \cos 5t + 2.0739e^{-3} \sin 5t
\end{align*}
$$

![Fig. 15](image)

**Fig. 15.** 1/2 subharmonic solutions ($\alpha = 0.8, \omega = 1.6, \beta = 1.76$).

We have verified the existence of exact $2\pi$ periodic solutions for Eq. (31) with $n = 1, 2$ around the approximate solution shown in Fig. 15. Here, we note again throughout this paper, to compare wave forms, figures of approximate solutions are drawn for the original fSS equation (3). Thus, in Fourier’s expression, coefficients of $\cos 2t$ and $\sin 2t$ for $x_2$ corresponds to those of $\cos t$ and $\sin t$ for $x_1$, respectively. The fact that they are almost same suggests $x_2$ is generated from $x_1$ by period doubling bifurcation. The verification results of existence of exact periodic solutions are shown in Table V.

| name | $\tau$ | $m$ | $k$ | $\kappa$ | $M_2^\text{block}$ | $M$ | $r_0$ | $b(r_0)$ | $\|x\|_2$ |
|------|-------|-----|-----|---------|-----------------|-----|------|----------|------|
| $x_1$ | 2.0212 | 61 | $2e^{-2}$ | $9e^1$ | 5.1 | 14.2 | $3e^{-13}$ | $4e^{-11}$ | 0.8315 |
| $x_2$ | 2.0212 | 61 | $4e^{-2}$ | $9e^5$ | $6.4e^3$ | $3.3e^4$ | $3e^{-9}$ | $e^{-4}$ | 0.8317 |

4.2 1/3 subharmonic solutions: Chaos and window

By solving the extended system satisfied by the saddle-node bifurcation points, we have determined regions where 1/3 symmetric subharmonics exist.

We have found in $\tau$-$\beta$ parameter space, an existence areas of 1/3 asymmetric subharmonic solutions as shown in the first figure of Fig. 16. The second figure is an example of an asymmetric 1/3 subharmonic in this area. Via continuation method, from this solution, we follows a solution curve of asymmetric 1/3 subharmonics at $\beta = 1.2$. Then, we find that a loop of 1/3 asymmetric subharmonic exists during $2.164 < \tau < 2.376$. The fourth figure shows a bifurcation diagram obtained by numerical integration. It implies that during $2.167 < \tau < 2.319$ the asymmetric 1/3 subharmonic is unstable and coexistence with a stable 1/5 symmetric, a stable 1/6 subharmonics and chaos is observed. Then, we can observe a subcritical period doubling bifurcation. In fact, at $\tau = 2.3125$ a stable 1/6 subharmonic window exits. After a subcritical period doubling bifurcation, a stable 1/3 asymmetric subharmonic is observed at $\tau = 2.3447$. Figure 17 shows an approximate stable 1/3 asymmetric subharmonic at $\tau = 2.3447$. Its Galerkin’s approximation can be obtained by Newton’s method starting with the following initial function:
During $2.322 < \tau < 2.373$ the 1/3 asymmetric subharmonics become stable and form a window. This change of stability is different from the first example, in which almost whole loops of 1/3 symmetric subharmonics are stable. At $\tau = 2.4375$, a chaotic motion is observed, again. Then, at $\tau = 2.457$ a stable 1/10 subharmonic window appears. Through a subcritical period doubling bifurcation, a stable 1/5 asymmetric subharmonic is generated at $\tau = 2.4688$.

Its Galerkin’s approximation can be obtained by Newton’s method starting with the following initial function:

$$x_5 = -0.0033 + 0.1880 \cos t + 0.3035 \sin t - 0.0308 \cos 2t + 0.0032 \sin 2t$$
$$-0.8559 \cos 3t + 0.0038 \sin 3t - 0.0408 \cos 4t - 0.2112 \sin 4t - 0.1907 \cos 5t + 0.7641 \sin 5t$$

Using verified numerical computation, we have proved that there exits an exact 1/5 asymmetric subharmonic nearby the approximate solution as shown in Table VI.

| name | $\tau$ | $m$ | $k$ | $\kappa$ | $M_{block}^2$ | $M$ | $r_0$ | $b(r_0)$ | $||x||_2$ |
|------|------|-----|-----|-----------|--------------|----|------|---------|----------|
| $x_3$ | 2.3447 | 201 | $8e^{-2}$ | $3.1e^4$ | 4.2 | 81 | $3e^{-10}$ | $2e^{-6}$ | 1.2476 |
| $x_5$ | 2.4688 | 251 | $1.4e^{-1}$ | $8.1e^3$ | 23 | $7.4e^3$ | $8e^{-9}$ | $8e^{-4}$ | 1.2473 |

### 4.3 1/4 subharmonic solutions

We have identified area where approximate 1/4 subharmonic solutions exist for Eq. (31) with $n = 4$ by solving an extended system of equations satisfied by saddle node bifurcation points of 1/4 subharmonic solution curves. In fact, for instance, at $\alpha = 0.75$, $\omega = 2.72$ and $\beta = 1.2$, if we solve Eq. (31) with $n = 4$ by changing $\tau$, we have a loop as shown in the second figure in Fig. 18. Thus, in $\tau$-$\beta$ space, we...
can determine the existence area of 1/4 subharmonic as Fig. 18. Gakerkin’s approximations shown in Fig. 18 can be obtained by Newton’s method starting with the following initial approximations:

\[
\begin{align*}
    x_{41} &= 0.0568 + 0.3668 \cos t + 0.6880 \sin t + 0.0214 \cos 2t - 0.1161 \sin 2t \\
    &\quad - 0.0145 \cos 3t + 0.0372 \sin 3t - 0.0053 \cos 4t + 0.372 \sin 4t \\
    x_{42} &= 0.0121 + 0.5810 \cos t + 0.4718 \sin t + 0.1278 \cos 2t - 0.0942 \sin 2t \\
    &\quad + 0.0308 \cos 3t + 0.0467 \sin 3t - 0.0097 \cos 4t + 0.32 \sin 4t
\end{align*}
\]

![Fig. 18. Area existing 1/4 subharmonic solutions, loop of 1/4 Subharmonic Solutions and 1/4 subharmonic solutions.](image)

An inclusion result of the exact 1/4 subharmonic solution nearby these approximate is presented in Table VII. To check stability, Fig. 19 shows a stroboscopic bifurcation diagram obtained by numerical integration. From the first figure, it is seen that 1/4 subharmonics form two windows including \(\tau = 1.7812\) and \(\tau = 2.875\), respectively.

![Fig. 19. Stroboscopic bifurcation diagram obtained by numerical integration.](image)

| name  | m    | \(k_1\) | \(k_2\) | \(\kappa\) | \(M_0^{block}\) | \(M\) | \(r_0\) | \(b(r_0)\) | \(\|x\|_2\) |
|-------|------|--------|--------|----------|----------------|------|-------|----------|--------|
| \(x_{41}\) | 151  | 2.3e-2 | 4.5e-2 | 6e3      | 9.6           | 8.1e2 | 5e-11 | 2e-7    | 0.85783 |
| \(x_{42}\) | 151  | 2.4e-2 | 4.7e-2 | 5e3      | 8.7           | 7.2e2 | 5e-11 | e-7     | 0.83273 |

4.4 1/5 subharmonic solution

The first figure in Fig. 20 shows an example of regions where 1/5 symmetric subharmonic solutions exit. Figure 20 also shows solution curves of 1/5 symmetric subharmonics and several 1/5 subharmonics. Galerkin’s approximations for these approximate 1/5 subharmonics can be calculated by Newton’s method starting with the following initial approximations:

\[
\begin{align*}
    x_{5s1} &= -0.1951 \cos t + 1.0336 \sin t - 0.1337 \cos 3t + 0.1278 \sin 3t + 0.0656 \cos 5t + 0.6898 \sin 5t, \\
    x_{5s2} &= -0.1685 \cos t + 1.1428 \sin t - 0.1788 \cos 3t + 0.0833 \sin 3t + 0.0279 \cos 5t + 0.4598 \sin 5t, \\
    x_{5s3} &= 0.5939 \cos 0t - 0.4192 \sin t - 0.0116 \cos 3t + 0.0627 \sin 3t + 0.1045 \cos 5t + 0.4983 \sin 5t, \\
    x_{5s4} &= 0.9365 \cos t + 0.4122 \sin t + 0.1068 \cos 3t - 0.1188 \sin 3t + 0.0652 \cos 5t + 0.4081 \sin 5t,
\end{align*}
\]
An inclusion result of the exact 1/5 subharmonic solution nearby an approximate one shown in Fig. 20 is presented in Table VIII.

### Table VIII. Verifications for 1/5 sub-harmonic periodic solutions in Fig. 20 \((\alpha = 0.83, \omega = 4, \kappa = K_0(1 + M^\text{block}_2 K_0))\).

| name  | \(\tau\) | \(\beta\) | \(m\) | \(k_1\) | \(k_2\) | \(\kappa\) | \(M^\text{block}_2\) | \(M\) | \(r_0\) | \(b(r_0)\) | \(\|x\|_2\) |
|-------|------|------|------|------|------|------|---------------|------|------|--------|--------|
| \(x_{5s1}\) | 2.6  | 3.5  | 201  | 2.7e^{-2} | 5.4e^{-2} | \(e^4\) | 5.7  | 6.9e^{1}  | 7e^{-11} | 3e^{-7} | 1.2764 |
| \(x_{5s2}\) | 2.3  | 2    | 201  | 2.2e^{-2} | 4.2e^{-2} | 6e^{3} | 4.4  | 4.9e^{1}  | 4e^{-11} | 7e^{-8} | 1.2607 |
| \(x_{5s3}\) | 1.65 | 2    | 201  | 1.6e^{-2} | 3.1e^{-2} | 9e^{3} | 17.6 | 1.4e^{2}  | 5e^{-11} | 2e^{-7} | 0.8908 |
| \(x_{5s4}\) | 2    | 1.5  | 201  | 2.0e^{-2} | 3.9e^{-2} | 9e^{3} | 9.5  | 9.4e^{1}  | 6e^{-11} | 2e^{-7} | 1.1161 |

An inclusion result of the exact 1/5 subharmonic solution nearby an approximate one shown in Fig. 20 is presented in Table VIII.

### 4.5 Proof of main Theorem: Coexistence of various subharmonics and generation of chaos

In this subsection, we will present proof of Theorem 1.1. Thus, we are concerned with the parameter value \(\alpha = 0.67, \omega = 1.3\). First, we give an auxiliary information, why we have chosen the parameter set \(\alpha = 0.67, \beta = 0.85, \omega = 1.3\) and \(\tau = 1.85\). We found a region in the \(\beta-\tau\) parameter space where various subharmonic solutions coexist as shown in Fig. 21.

![Fig. 21. Coexistence of various subharmonics, existing areas.](image-url)
are generated, respectively. Figure 23 shows wave forms of subharmonics coexist in the region c. This observation can sharpened as the following theorem:

**Theorem 1.1.** Let $\alpha = 0.67, \beta = 0.85, \tau = 1.85$ and $\omega = 1.3$. Then, at this parameter set,
Eq. (3) has 1/2, 1/3 symmetric and 1/3 asymmetric, 1/4, and 1/5 symmetric, 1/5 asymmetric, and 1/7 symmetric subharmonic solutions.

**Proof** At $\alpha = 0.67$, $\beta = 0.85$, $\tau = 1.85$ and $\omega = 1.3$, we have obtained approximate subharmonics as follows:

\[
\begin{align*}
x_{2a} &= 0.1663 + 0.5047 \cos t - 0.2189 \sin t - 0.4285 \cos 2t + 0.6028 \sin 2t \\
x_{2b} &= 0.0149 + 0.3859 \cos t + 0.1157 \sin t - 0.4638 \cos 2t + 0.4553 \sin 2t \\
x_3 &= -7.0287 e^{-2} \cos t + 3.5873 e^{-1} \sin t - 4.7960 e^{-1} \cos 3t + 4.1698 e^{-1} \sin 3t \\
x_{3s} &= 5.2568 e^{-2} \cos t + 1.1658 e^{-1} \sin t - 4.6460 e^{-1} \cos 3t + 3.2481 e^{-1} \sin 3t \\
x_{3a} &= -1.2354 e^{-2} + 3.1643 e^{-1} \cos t - 1.1550 e^{-1} \sin t - 5.2924 e^{-2} \cos 2t - 4.9482 e^{-2} \sin 2t - 4.7663 e^{-1} \cos 3t + 4.0543 e^{-1} \sin 3t + 1.4402 e^{-2} \cos 4t - 1.1069 e^{-2} \sin 4t \\
x_4 &= 1.0811 e^{-1} + 2.2633 e^{-2} \cos t - 5.7651 e^{-2} \sin t + 4.5184 e^{-1} \cos 3t - 1.0328 e^{-1} \sin 3t - 1.9786 e^{-1} \cos 3t - 5.5603 e^{-2} \cos 3t - 4.3017 e^{-1} \cos 4t + 5.3886 e^{-1} \sin 4t \\
x_5 &= 1.5299 e^{-1} \cos t + 2.9846 e^{-1} \sin t - 6.6655 e^{-1} \cos 3t - 1.0767 e^{-1} \sin 3t - 3.1262 e^{-1} \cos 5t + 6.8066 e^{-1} \sin 5t \\
x_{5s} &= 2.1244 e^{-2} \cos t + 1.8173 e^{-1} \sin t - 5.2577 e^{-1} \cos 3t - 2.8099 e^{-1} \sin 3t - 3.6935 e^{-1} \cos 5t + 6.1490 e^{-1} \sin 5t \\
x_{5a} &= -1.2005 e^{-2} + 5.9277 e^{-2} \cos t - 2.9662 e^{-1} \sin t + 1.4658 e^{-2} \cos 2t - 2.8940 e^{-2} \sin 2t - 3.3399 e^{-1} \cos 3t + 5.6146 e^{-1} \sin 3t + 6.1632 e^{-2} \cos 4t + 1.0855 e^{-1} \sin 4t - 3.1801 e^{-1} \cos 5t + 6.6542 e^{-1} \sin 5t \\
x_{7c2} &= 0.0605 \cos t - 0.0839 \sin t - 0.3729 \cos 3t - 0.0147 \sin 3t - 0.0163 \cos 5t - 0.0467 \sin 5t - 0.4735 \cos 7t + 0.4395 \sin 7t \\
x_{7c} &= 0.0471 \cos t - 0.0123 \sin t - 0.296 \cos 3t + 0.0874 \sin 3t - 0.0537 \cos 5t - 0.0863 \sin 5t - 0.4689 \cos 7t + 0.4019 \sin 7t
\end{align*}
\]

Starting with these approximate solutions, we have confirmed that Newton’s iterates for the corresponding Galerkin’s equation converge and we can get accurate approximate solutions shown in Fig. 23. The verification result for these accurate approximate solutions is shown in Table IX.

| Name | $\mu$ | $\kappa$ | $M_{J}^{lock}$ | $M$ | $b(r_0)$ | $||x||_2$ |
|------|------|------|------|------|------|------|
| $x_{2a}$ | 1/2 | 51 | $1.3 e^{-1}$ | $2.3 e^{3}$ | 3.9 | $3.4 e^{1}$ | $8.9 e^{-8}$ | $8.4 e^{-5}$ | 0.93942 |
| $x_{2b}$ | 1/2 | 51 | $1.1 e^{-1}$ | $1.8 e^{3}$ | 5.5 | $3.9 e^{1}$ | $2.2 e^{-9}$ | $1.8 e^{-6}$ | 0.76757 |
| $x_3$ | 1/3 | 101 | $5.7 e^{-2}$ | $9.3 e^{3}$ | 13.0 | $1.2 e^{1}$ | $1.6 e^{-11}$ | $4.6 e^{-8}$ | 0.73431 |
| $x_{3s}$ | 1/3 | 101 | $3.9 e^{-2}$ | $2.8 e^{3}$ | 7.2 | $5.3 e^{1}$ | $3.6 e^{-12}$ | $3.2 e^{-9}$ | 0.58149 |
| $x_{3a}$ | 1/3 | 101 | $6.6 e^{-2}$ | $8.1 e^{3}$ | 9.8 | $9.3 e^{1}$ | $3.0 e^{-11}$ | $7.5 e^{-8}$ | 0.71576 |
| $x_4$ | 1/4 | 151 | $1.3 e^{-1}$ | $2.3 e^{4}$ | 4.3 | $7.5 e^{1}$ | $1.6 e^{-10}$ | $7.1 e^{-7}$ | 0.86860 |
| $x_5$ | 1/5 | 201 | $1.1 e^{-1}$ | $1.5 e^{5}$ | 11.0 | $2.7 e^{2}$ | $7.2 e^{-10}$ | $1.7 e^{-5}$ | 1.06690 |
| $x_{5a}$ | 1/5 | 201 | $1.1 e^{-1}$ | $3.5 e^{4}$ | 3.1 | $6.9 e^{1}$ | $1.2 e^{-10}$ | $6.3 e^{-7}$ | 0.95050 |
| $x_{5a}$ | 1/5 | 201 | $1.4 e^{-1}$ | $9.5 e^{4}$ | 6.5 | $1.6 e^{2}$ | $9.5 e^{-10}$ | $1.4 e^{-5}$ | 1.04422 |
| $x_{7c2}$ | 1/7 | 301 | $6.8 e^{-2}$ | $4.6 e^{4}$ | 2.5 | $58$ | $9 e^{-11}$ | $4 e^{-7}$ | 0.75675 |
| $x_{7c}$ | 1/7 | 301 | $6.3 e^{-2}$ | $3.6 e^{4}$ | 2.3 | $54$ | $7 e^{-11}$ | $3 e^{-7}$ | 0.70162 |

Using a numerical integration, we have calculated a bifurcation diagram in Fig. 24. In two intervals of $\tau$, in [1.703, 1.860] and in [2.031, 2.164], we can observe mainly chaotic behavior. In Fig. 24, stroboscopic pictures of Poincaré map are also shown, which shows chaotic attractors.
5. Concluding remarks

We have considered the following forced delayed action oscillator equation

$$\frac{dx(t)}{dt} - x(t) + x^3(t) + \alpha x(t - \tau) - \beta \cos \omega t = 0.$$  \hfill (33)

In this paper, we have shown that FSS equation has an odd symmetric periodic solution for every fixed positive real $\omega$. Then, we have show that tight inclusions of exact periodic solutions can be obtained by utilizing a structure of Galerkin’s equation for FSS effectively. Namely, it has been shown that an upper bound $\|G_m^1\|_\infty \|G^{-1}_m\|_1$ of a $\|G_{m,n}\|_2$ with $G_{m,n}$ being the Jacobian of the Galerkin approximation defined by Eq. (22) becomes constant regardless its dimension under suitably chosen conditions. A numerical linear algebraic proof of this observation is presented in Appendix A. Namely, we have introduced a definition of asymptotic block diagonal dominant and using this concept a kind of acceleration of convergence of verification is possible. The author thinks that this method is also useful to include exact solutions nearby Galerkin’s approximate solutions for various nonlinear differential equations, especially, for first order ones.

1. In certain parameter range of $\alpha, \beta, \omega$, a fundamental periodic solution curve obtained by changing the parameter $\tau$ has saddle node bifurcation points. By observing the shape of a saddle-node bifurcation curve a kind of global bifurcation named island bifurcation and double islands bifurcation are observed.
2. Fixing $\alpha$ and $\omega$, in $\tau$-$\beta$ parameter space, we can determine a symmetry breaking bifurcation curves in a form $\{\beta(\tau) | \tau \in [0, 2\pi/\omega]\}$. We have observed that if a parameter $(\tau, \omega)$ is below this curve, there exists an asymmetric periodic solutions having the angular frequency $\omega$.

3. It is shown that variety of subharmonic solutions exist. We have shown $1/2$, $1/3$ symmetric, $1/3$ asymmetric, $1/4$, $1/5$ symmetric $1/5$ asymmetric, $1/7$ symmetric subharmonics exist.

4. At the parameter value $\alpha = 0.67, \omega = 1.3$, and $\beta = 0.85$, coexistence of various subharmonics at the same parameter value is shown. For $\tau \in [1.703, 1.860]$ and $\tau \in [2.031, 2.164]$, we can observe mainly chaotic behavior, except several windows of attractive periodic solutions.

Reflecting these observations, we have postulated Theorem 1.1. In Section 4, we have presented a proof of this theorem using a computer assisted proof based on verified numerical computations.

Appendix

A. Appendix–Asymptotic diagonally dominant matrix

Let $M_n(\mathbb{C})$ be the set of all complex matrices of order $n$. Let $G = (g_{ij}) \in M_n(\mathbb{C})$ and $G^\ast$ be its Hermitian conjugate, $G^* = (\overline{g}_{ij}) \in M_n(\mathbb{C})$. Here, $\overline{g}_{ij}$ is the complex conjugate of $g_{ij}$. The singular values of $G$ are positive roots of eigenvalues of $GG^\ast$ or $G^\ast G$. Denote

$$R_i(G) = \sum_{j=1, j\neq i}^{n} |g_{ij}|, \quad C_i(G) = \sum_{j=1, j\neq i}^{n} |g_{ji}|, \quad \text{and} \quad H(G) = \frac{1}{2}(G + G^\ast).$$

Denote further the smallest and the largest singular values of $G$ by $\sigma_n(G)$ and $\sigma_1(G)$, respectively. A matrix $G$ is called strictly row diagonally dominant or strictly column diagonally dominant iff

$$\min_{1 \leq i \leq n} \{ |g_{ii}| - R_i(G) \} > 0, \quad \text{or} \quad \min_{1 \leq i \leq n} \{ |g_{ii}| - C_i(G) \} > 0,$$

respectively. If $G$ is both strictly row and column diagonally dominant, it is called strictly diagonally dominant. In the following, for simplicity, we say a matrix is diagonally dominant if this matrix is strictly diagonally dominant. In 1949, Taussky [19] has shown if $G$ is row diagonally dominant, $G$ is invertible and

$$\|G^{-1}\|_\infty \leq \frac{1}{\min_{1 \leq i \leq n} \{ |g_{ii}| - R_i(G) \}}. \quad \text{(A-1)}$$

Here, $\|G\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |g_{ij}|$. Since $\|G\|_1 = \|G^\ast\|_\infty$ with $G^\ast$ being the transpose of $G$, it turns out that if $G$ is column diagonally dominant, $G$ is invertible and

$$\|G^{-1}\|_1 \leq \frac{1}{\min_{1 \leq i \leq n} \{ |g_{ii}| - C_i(G) \}}. \quad \text{(A-2)}$$

The matrix norm $\|G\|_2$ is given by $\|G\|_2 = \sigma_1(G)$. If $G$ is invertible, $\|G^{-1}\|_2 = 1/\sigma_n(G)$. From $\|G\|_2 \leq \sqrt{\|G\|_\infty \|G\|_1}$ and from Eqs. (A-1) and (A-2), we have Varah’s bound [20] for a diagonally dominant matrix $G$ as

$$\sigma_n(G) \geq \sqrt{\min_{1 \leq i \leq n} \{ |g_{ii}| - R_i(G) \} \cdot \min_{1 \leq i \leq n} \{ |g_{ii}| - C_i(G) \}} (> 0). \quad \text{(A-3)}$$

In 1955, Fan and Hoffman [21] have shown that $\sigma_n(G) \geq \lambda_n(H(G))$, where $\lambda_n(H(G))$ is the smallest eigenvalue of $H(G)$. If $\lambda_n(H(G)) \geq 0$, then $\lambda_n(H(G)) = \sigma_n(H(G))$. It follows that $\sigma_n(G) \geq \sigma_n(H(G))$ provided that $\lambda_n(H(G)) \geq 0$. Note that

$$R_i(H(G)) = C_i(H(G)) \leq \frac{1}{2}[R_i(G) + C_i(G)], \quad (1 \leq i \leq n).$$

Thus, if $\lambda_n(H(G)) > 0$ and if $|g_{ii}| > [R_i(G) + C_i(G)]/2$, $(1 \leq i \leq n)$, we have
\[ \sigma_n(G) \geq \sigma_n(H(G)) \geq \min_{1 \leq i \leq n} \{|g_{ii}| - R_i(H(G))\} \geq \min_{1 \leq i \leq n} \left\{ |g_{ii}| - \frac{1}{2} [R_i(G) + C_i(G)] \right\} > 0, \]  

(A-4)

which is Johnson’s bound [22].

In this appendix, in the first place, we will define an asymptotic diagonally dominant matrix by row and show an extension of Taussky’s theorem [19] for this class of matrices. Then, we will define an asymptotic diagonally dominant matrix by columns. We will extend Corollary 1 in Varah [20] to this class of matrices. Moreover, we will define asymptotic block diagonally dominant matrices and show that Varah’s results [20] for block diagonally dominant matrices can be extended to this class of matrices. As an illustration, we will show a few examples.

A.1 Asymptotic diagonally dominant matrix

In this section, definition is given for an asymptotic diagonally dominant matrix and a lower bound calculable by verified numerical computations is presented for its smallest singular value. An extension is shown for a block asymptotic diagonally dominant matrix. Several examples are shown for illustration. We define an asymptotic diagonally dominant matrix and show an extension of Taussky’s theorem [19]:

**Theorem A.1.** Let \( b, n \) be integers satisfying \( 0 \leq b < n \). Let \( G \in M_n(\mathbb{C}) \) be defined by

\[ G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]

with an \( b \times b \) sub-matrix \( A \), with an \( (n-b) \times (n-b) \) sub-matrix \( D \), with an \( b \times (n-b) \) matrix \( B \) and with an \( (n-b) \times b \) matrix \( C \). Let \( D_d \) and \( D_f \) be the diagonal part and off-diagonal part of \( D \), respectively. We call \( G \) is asymptotic row diagonally dominant iff

\[ \min_{b < k \leq n} \{|g_{kk}| - \sum_{j=1, j \neq k}^{n} |g_{kj}| \} > 0. \]

Assume that \( G \) is asymptotic row diagonally dominant. Then, \( D_d^{-1} \) exists. If \( A^{-1} \) exists and if \( \max \{\|A^{-1}B\|_\infty, \|D_d^{-1}(C D_f)\|_\infty\} < 1 \), \( G^{-1} \) exists and

\[ \|G^{-1}\|_\infty \leq \frac{\max \{\|A^{-1}\|_\infty, \|D_d^{-1}\|_\infty\}}{1 - \max \{\|A^{-1}B\|_\infty, \|D_d^{-1}(C D_f)\|_\infty\}} := M_\infty. \]

**Proof** Let \( H \) be defined by

\[ H = \begin{pmatrix} A & O \\ O & D_d \end{pmatrix}. \]

(A-5)

Since \( A \) is invertible, \( H^{-1} \) exists. We note

\[ \|H^{-1}\|_\infty = \left\| \begin{pmatrix} A^{-1} & O \\ O & D_d^{-1} \end{pmatrix} \right\|_\infty = \max \{\|A^{-1}\|_\infty, \|D_d^{-1}\|_\infty\} \]

(A-6)

and

\[ \|H^{-1}G - I_n\|_\infty = \left\| \begin{pmatrix} I_b & A^{-1}B \\ D_d^{-1}C & D_d^{-1}(D_d + D_f) \end{pmatrix} - I_n \right\|_\infty = \max \{\|A^{-1}B\|_\infty, \|D_d^{-1}(C D_f)\|_\infty\}. \]

(A-7)

Thus, \( \|H^{-1}G - I_n\|_\infty < 1 \) so that \( G^{-1} \) exists and

\[ \|G^{-1}\|_\infty \leq \frac{\|H^{-1}\|_\infty}{1 - \|H^{-1}G - I_n\|_\infty}. \]

(A-8)
Remark 4. It is noted that \((1 - \|D_d^{-1}(C\ D_f)\|_\infty)^{-1}\) becomes

\[
1 - \max_{b<k \leq n} \frac{1}{|g_{kk}|} \left| \sum_{j=1, j \neq k}^{n} |g_{kj}| \right| = \min_{b<k \leq n} \frac{1}{|g_{kk}|} \left( |g_{kk}| - \sum_{j=1, j \neq k}^{n} |g_{kj}| \right) := K
\]

Thus, if \(G\) is asymptotic diagonal dominant, then \(D_d^{-1}\) exists and \(K\) becomes positive.

Theorem A.2. Let \(b, n\) be integers satisfying \(0 \leq b < n\). Let \(G \in M_n(C)\) be defined by

\[
G = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

with an \(b \times b\) square sub-matrix \(A\) and with an \((n-b) \times (n-b)\) square sub-matrix \(D\), where \(0 \leq b \leq n\).

Assume that \(G\) is asymptotic column diagonally dominant, i.e.

\[
\begin{pmatrix}
I_b & B \\
0 & D
\end{pmatrix}
\]

is column diagonally dominant. If \(\max \{\|CA^{-1}\|_1, \left\| \begin{pmatrix} B \\ D_f \end{pmatrix} D_d^{-1} \right\|_1 \} < 1\), it follows that \(G^{-1}\) exists and

\[
\|G^{-1}\|_1 \leq \frac{\max \{\|CA^{-1}\|_1, \|D_d^{-1}\|_1 \}}{1 - \max \{\|CA^{-1}\|_1, \left\| \begin{pmatrix} B \\ D_f \end{pmatrix} D_d^{-1} \right\|_1 \} } := M_1. \quad \Box
\]

A.2 Extension to block matrix

Varah [20] has considered the case of \(G\) being partitioned into blocks \(G_{ij}\) with the diagonal blocks square and has shown that if one replaces \(|g_{kj}|\) by \(\|G_{kj}\|_\infty\) and \(|g_{kk}|\) by \(\|G_{kk}^{-1}\|_\infty^{-1}\), then the bound given by Eq. (A-3) for a diagonally dominant matrix can be extended to a block diagonally dominant matrix. We can extend Varoh’s theorem [20] to an asymptotic block diagonally dominant matrix.

Theorem A.3. Let \(b, n\) be integers satisfying \(0 \leq b < n\). Let \(G \in M_n(C)\) be defined by

\[
G = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

with an \(b \times b\) sub-matrix \(A\) and with an \((n-b) \times (n-b)\) sub-matrix \(D\), where \(0 \leq b \leq n\). Assume that \(D\) is partitioned into blocks \(D_{ij}\) with the diagonal blocks square. Let \(D_d\) and \(D_f\) be the block diagonal part and block off-diagonal part of \(D\), respectively. Assume that \(G\) is asymptotic block row diagonally dominant, i.e., \(D_d\) is invertible and \(\|D_d^{-1}(C\ D_f)\|_\infty < 1\).

If \(\max \{\|A^{-1}\|_\infty, \|D_d^{-1}\|_\infty \} < 1\), it follows that \(G^{-1}\) exists and

\[
\|G^{-1}\|_\infty \leq \frac{\max \{\|A^{-1}\|_\infty, \|D_d^{-1}\|_\infty \}}{1 - \max \{\|A^{-1}\|_\infty, \|D_d^{-1}(C\ D_f)\|_\infty \} } := M_{block}. \quad \Box
\]

Proof of Theorem A.3 is the same as that for Theorem A.1 except in the definition of \(H, D_d\) and \(D_f\) are the block diagonal part and block off-diagonal part of \(D\), respectively.

Theorem A.4. Let \(b, n\) be integers satisfying \(0 \leq b < n\). Let \(G \in M_n(C)\) be defined by

\[
G = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
with an $b \times b$ sub-matrix $A$ and with an $(n - b) \times (n - b)$ sub-matrix $D$, where $0 \leq b \leq n$. Assume that $D$ is partitioned into blocks $D_{ij}$ with the diagonal blocks square. Let $D_d$ and $D_f$ be the block diagonal part and block off-diagonal part of $D$, respectively. Assume that $G$ is asymptotic block column diagonally dominant, i.e. $D_d$ is invertible and

$$\left\| \left( \begin{array}{c|c} B & \end{array} \right) \right\|_{1} < 1.$$  

If $\max \{\|CA^{-1}\|_1, \left\| \left( \begin{array}{c|c} B & \end{array} \right) \right\|_{1} \} < 1$, it follows that $G^{-1}$ exists and

$$\|G^{-1}\|_1 \leq \frac{\max \{\|A^{-1}\|_1, \|D_d^{-1}\|_1\}}{1 - \max \{\|CA^{-1}\|_1, \left\| \left( \begin{array}{c|c} B & \end{array} \right) \right\|_{1} \}} := M_1^{block}. \quad (A-15)$$

### A.3 Examples

Let $M_\infty$, $M_1$, $M_1^{block}$ and $M_2^{block}$ are bounds of $\|G^{-1}\|$ defined in the previous section and $M_2 = \sqrt{M_\infty M_1}$, $M_2^{block} = \sqrt{M_1^{block} M_1^{block}}$. Let further $H_2$ and $H_2^{block}$ be bounds obtained by applying Theorem A.1 and A.3 to $H(G)$, respectively, provided that $\lambda_n(H(G)) > 0$.

#### A.3 Example 1

Let us consider cases of $n = 10$ and $n = 1000$. Put

$$G = \begin{pmatrix}
1 & 1.25 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 2 & 1.25 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 3 & 1.25 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & n-1 & 1.25 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & n & 0
\end{pmatrix}.$$  

If we take $b = 4$, then

$$A = \begin{pmatrix}
1 & 1.25 & 0 & 0 \\
1 & 2 & 1.25 & 0 \\
0 & 1 & 3 & 1.25 \\
0 & 0 & 1 & 4
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
0 & 0 & \cdots \\
1.25 & 0 & \cdots
\end{pmatrix},$$

$$A^{-1} = \begin{pmatrix}
5.38 & -4.38 & 2.04 & -0.63 \\
-3.51 & 3.51 & -1.63 & 0.51 \\
1.30 & -1.30 & 0.97 & -0.30 \\
-0.32 & 0.32 & -0.24 & 0.32
\end{pmatrix} + [-0.01, 0.01] \cdot \begin{pmatrix}
1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & -1 & 1 & 1
\end{pmatrix}.$$  

$$A^{-1} B = \begin{pmatrix}
-0.63 + [-0.01, 0.01] & 0 & \cdots \\
0.51 + [-0.01, 0.01] & 0 & \cdots \\
-0.30 + [-0.01, 0.01] & 0 & \cdots \\
0.32 + [-0.01, 0.01] & 0 & \cdots
\end{pmatrix}.$$  

Thus, it follows

$$\|A^{-1}\|_\infty = \max \sum_{j=1, j\neq i}^{4} |A_{ij}^{-1}| \leq 12.5,$$

and

$$\|A^{-1} B\|_\infty = 0.64 \cdot 1.25 \leq 0.8002.$$
If we take, $D_d = \text{diag}(5, 6, \cdots, 10^9)$, then $\|D^{-1}_d\|_\infty = \|\text{diag}(5^{-1}, 6^{-1}, \cdots, 10^{-9})\|_\infty = 0.2$.

$$
\| (C D_f) \|_\infty = \left\| \begin{pmatrix} \cdots & 0 & 1 & 0 \cdots \\ \cdots & 0 & 0 & 1 & 0 \cdots \\ \cdots & 0 & 0 & 0 & 1 & 0 \cdots \\ \cdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \right\|_\infty \leq 2.25,
$$

and $\|D^{-1}_d(C D_f)\|_\infty \leq 1.25 \cdot 0.26 \leq 0.59$, respectively. Thus, $M_\infty \leq 12.5/(1 - 0.8002) \leq 62.6$. More precise calculation with the double precision accuracy, we have $M_\infty \leq 61.5$. Based on this kind of calculations, we have estimated upper bounds of $\|G^{-1}\|$, which are shown in Table A-I. It is noted that estimates in Table A-I also hold for any $G_n$, provide that a positive integer $n'$ is greater than or equal to $n$.

### Table A-I. Estimates of upper bound of $\|G_n^{-1}\|$.  

| $n$ | $b$ | $M_\infty$ | $M_1$ | $M^\text{Block}_1$ | $M^\text{Block}_2$ | $M^\text{Block}_3$ | $H_2^\text{Block}$ | $\sigma_n^{-1}$ | $\lambda_n^{-1}$ |
|-----|-----|-----------|-------|-------------------|-------------------|-------------------|------------------|---------------|---------------|
| 10  | 4   | 61.5      | 19.2  | 61.5              | 15.7              | 34.3              | 31.0            | 28.7          | 28.7          |
| $10^3$ | 4 | 61.5      | 19.2  | 61.5              | 15.7              | 34.3              | 31.0            | 28.7          | 28.7          |
| 10  | 6   | 19.1      | 15.9  | 16.5              | 13.5              | 17.4              | 15.0            | 18.9          | 16.0          |
| $10^3$ | 6 | 19.1      | 15.9  | 16.5              | 13.5              | 17.4              | 15.0            | 18.9          | 16.0          |
| 10  | 8   | 17.3      | 14.4  | 14.4              | 12.8              | 15.8              | 14.1            | 17.1          | 15.0          |
| $10^3$ | 8 | 17.3      | 14.4  | 14.4              | 12.8              | 15.8              | 14.1            | 17.1          | 15.0          |

### A.3 Example 2

Let us consider cases of $n = 10$, $n = 100$, and $n = 1000$. Let $G \in M_n(\mathbb{R})$ be defined by

$$
G = \begin{pmatrix} 
1 & 1 & 0.5 & 0.25 & 0.125 & 0.0625 & 0.03125 & 0.015625 \\
0.5 & 2 & 1 & 0.5 & 0.25 & 0.125 & 0.0625 & 0.03125 \\
0.25 & 0.5 & 3 & 1 & 0.5 & 0.25 & 0.125 & 0.0625 \\
0.125 & 0.25 & 0.5 & 4 & 1 & 0.5 & 0.25 & 0.125 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots 
\end{pmatrix}.
$$

If we take $b = 4$, then

$$
A = \begin{pmatrix} 
1 & 1 & 0.5 & 0.25 \\
0.5 & 2 & 1 & 0.5 \\
0.25 & 0.5 & 3 & 1 \\
0.125 & 0.25 & 0.5 & 4 \\
\end{pmatrix},
$$

$$
B = \begin{pmatrix} 
0.125 & 0.0625 & 0.03125 & 0.015625 \\
0.25 & 0.125 & 0.0625 & 0.03125 \\
0.5 & 0.25 & 0.125 & 0.0625 \\
1 & 0.5 & 0.25 & 0.125 \\
\end{pmatrix},
$$

$$
A^{-1} = \begin{pmatrix} 
1 \frac{1}{7} & \frac{2}{7} & 0 & 0 \\
-0.3015 & 0.6985 & -0.1785 & -0.0238 \\
-0.0555 & -0.0555 & 0.3751 & -0.0833 \\
-0.0158 & -0.0158 & -0.0357 & 0.2620 \\
\end{pmatrix},
$$

$$
A^{-1}B = \begin{pmatrix} 
0 & 1 & \frac{1}{42} & \frac{1}{12} & \frac{1}{4.2} \\
0.5 & 0.25 & 0.125 & \vdots & \ddots 
\end{pmatrix}.
$$

608
Thus, it follows

\[ \| A^{-1} \|_\infty = \max_{j=1,j\neq i}^4 |A_{ij}^{-1}| = 2, \]

and

\[ \| A^{-1} B \|_\infty \leq \frac{5}{21} \cdot 2 \leq 0.4762. \]

If we take, \( D_d = \text{diag}(5, 6, \cdots, 10^9) \), then

\[ \| D_d^{-1} \|_\infty = \| \text{diag}(5^{-1}, 6^{-1}, \cdots, 10^{-9}) \|_\infty = 0.2. \]

This together with

\[ \| (C \ D_f) \|_\infty \leq 3 \]

gives \( \| D_d^{-1} (C \ D_f) \|_\infty \leq 0.2 \cdot 3 = 0.6. \) Thus, \( \| G^{-1} \|_\infty := M_\infty \leq 2/(1 - 0.6) \leq 5, \)

which is valid for any positive integer \( n \). Based on this kind of calculations, we have estimated upper bounds of \( \| G^{-1} \| \),

which are shown in Table A-II (\( \sigma_n = \sigma_n(G) \) and \( \lambda_n = \lambda_n(H(G)) \)):

| \( n \) | \( b \) | \( M_\infty \) | \( M_1 \) | \( M_2 \) | \( M_\text{block} \) | \( H_2 \) | \( H_\text{block} \) | \( \sigma_n^{-1} \) | \( \lambda_n^{-1} \) |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 4 | 4.71 | 3.96 | 3.77 | 3.08 | 4.32 | 3.41 | 4.58 | 3.14 | 1.62 | 1.68 |
| 10^3 | 4 | 4.85 | 4.02 | 3.82 | 3.11 | 4.42 | 3.45 | 4.68 | 3.18 | 1.62 | 1.68 |

References

[1] M.J. Suarez and P.S. Schopf, “A delayed action oscillator for ENSO,” Journal of the Atmospheric Sciences, vol. 45, no. 21, pp. 3283–3287, 1988.

[2] M. Urabe, “Galerkin’s procedure for nonlinear periodic systems,” Arch. Rat. Mech. Anal., vol. 20, pp. 120–152, 1965.

[3] R. Bouc, “Sur la methode de Galerkin-Urabe pour les systemes differentielles periodiques,” Internat. J. Non-Linear Mech., vol. 7, pp. 175–188, 1972.

[4] S. Oishi, “Numerical verification of existence and inclusion of solutions for nonlinear operator equations,” J. Computational and Applied Math., vol. 235, pp. 870–878, 2010.

[5] T. Minamoto and M.T. Nakao, “A numerical verification method for a periodic solution of a delay differential equation,” Journal of Computational and Applied Mathematics, vol. 235, pp. 870–878, 2010.

[6] R. Szczelina and P. Zgliczynski, “Algorithm for rigorous integration of delay differential equations and the computer-assisted proof of periodic orbits in the mackey-glass equation,” Found Comput Math, vol. 18, no. 6, pp. 1299–1332, 2018.
[12] I. Zaliapin and M. Ghil, “A delay differential model of ENSO variability—Part 2: Phase locking, multiple solutions and dynamics of extrema,” Nonlin. Processes Geophys., vol. 17, pp. 123–135, 2010.

[13] K. Andrew, K. Bernd, and P. Claire, “Delayed feedback versus seasonal forcing: Resonance phenomena in an El Niño southern oscillation model,” SIAM J. APPLIED DYNAMICAL SYSTEMS, Society for Industrial and Applied Mathematics, vol. 14, no. 3, pp. 1229–1257, 2015.

[14] Y. Tsuda, H. Tamura, A. Sueoka, and T. Fujii, “Chaotic behaviour of a nonlinear vibrating system with a retarded argument,” JSME International Journal, Series III, vol. 35, no. 2, pp. 259–267, 1992.

[15] A.C. Earl and L. Norman, “Theory of ordinary differential equations,” New York, McGraw-Hill, 1955.

[16] L. Cesari, “Functional analysis and periodic solutions of nonlinear equations,” Contrib. Differential Equations, vol. 1, pp. 149–187, 1963.

[17] S. Smale, “An infinite dimensional version of sard’s theorem,” American Journal of Mathematics, vol. 87, pp. 861–866, 1965.

[18] R.N. Nassbaum, “Periodic solutions of analytic functional differential equations are analytic,” Michigan Math. J., vol. 20, pp. 249–255, 1973.

[19] O. Taussky, “A recurring theorem on determinants,” Amer. Math. Monthly, vol. 56, pp. 672–676, 1949.

[20] J.M. Varah, “A Lower bound for the smallest singular value of a matrix,” Linear Algebra Appl., vol. 11, pp. 3–5, 1975.

[21] K. Fan and A.J. Hoffman, “Some metric inequalities in the space of matrices,” Proc. Amer. Math. Soc., vol. 6, pp. 111–116, 1955.

[22] C.R. Johnson, “A Gersgorin-type lower bound for the smallest singular value,” Linear Algebra Appl., vol. 112, pp. 1–7, 1989.

[23] D.G. Feingold and R.S. Varga, “Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem,” Pacific J. Math., vol. 12, pp. 1241–1250, 1962.

[24] R.S. Varga, Gersgorin and his circles, Springer Series in Computational Mathematics, vol. 36, Springer-Verlag Berlin Heidelberg, 2004.

[25] A.M. Ostrowski, “On some metrical properties of operator matrices and matrices partitioned into blocks,” J. Math. Anal. Appl., vol. 2, pp. 161–209, 1961.

[26] M. Fiedler and V. Pták, “Generalized norms of matrices and the location of the spectrum,” Czechoslovak Math. J., vol. 12, no. 87, pp. 568–571, 1962.

[27] S.V. Richard, “On diagonal dominance arguments for bounding $\|A^{-1}\|_∞$,” Linear Algebra Appl., vol. 14, pp. 211–217, 1976.