One-dimensional symmetry of positive bounded solutions to
the subcubic and cubic nonlinear Schrödinger equation in
the half-space in dimensions $N = 4, 5$

Christos Sourdis

Received: 2 May 2022 / Accepted: 7 June 2022 / Published online: 2 July 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We are concerned with the half-space Dirichlet problem

$$\begin{cases}
-\Delta v + v = |v|^{p-1}v & \text{in } \mathbb{R}^N_+,

v = c & \text{on } \partial \mathbb{R}^N_+,

\lim_{x_N \to \infty} v(x', x_N) = 0 & \text{uniformly in } x' \in \mathbb{R}^{N-1},
\end{cases}$$

where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$ for some $N \geq 2$, and $p > 1$, $c > 0$ are constants. It was shown recently by Fernandez and Weth [Math. Ann. (2021)] that there exists an explicit number $c_p \in (1, \sqrt{e})$, depending only on $p$, such that for $0 < c < c_p$ there are infinitely many bounded positive solutions, whereas, for $c > c_p$ there are no bounded positive solutions. They also posed as an interesting open question whether the one-dimensional solution is the unique bounded positive solution in the case where $c = c_p$. If $N = 2, 3$, we recently showed this one-dimensional symmetry property in [Partial Differ. Equ. Appl. (2021)] by adapting some ideas from the proof of De Giorgi’s conjecture in low dimensions. Here, we first focus on the case $1 < p < 3$ and prove this uniqueness property in dimensions $2 \leq N \leq 5$. Then, for the cubic NLS, where $p = 3$, we establish this for $2 \leq N \leq 4$. Our approach is completely different and relies on showing that a suitable auxiliary function, inspired by a Lyapunov-Schmidt type decomposition of the solution, is a nonnegative super-solution to a Lane-Emden-Fowler equation in $\mathbb{R}^{N-1}$, for which an optimal Liouville type result is available.

Mathematics Subject Classification 35B09

1 Introduction

Recently in [11], the authors studied the half-space Dirichlet problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
-\Delta v + v = |v|^{p-1}v \text{ in } \mathbb{R}^N_+ , \\
v = c \text{ on } \partial \mathbb{R}^N_+ , \\
\lim_{x_N \to \infty} v(x', x_N) = 0 \text{ uniformly in } x' \in \mathbb{R}^{N-1} ,
\end{array} \right.
\end{align*}
\] (1.1)

where \( \mathbb{R}^N_+ = \{ x \in \mathbb{R}^N : x_N > 0 \} \) for some \( N \geq 1 \), and \( p > 1, c > 0 \) are constants. We note that \( V(x, t) = e^{it}v(x) \) is a standing wave solution to the focusing nonlinear Schrödinger equation with the pure odd power nonlinearity of exponent \( p \). Let us clarify here that throughout this paper solutions will be understood in the classical sense (i.e. at least of class \( C^2 \), and continuous up to the boundary).

If \( N = 1 \), then the corresponding ODE has a unique positive even solution that decays to zero at infinity, it is given explicitly by the following formula

\[
t \to w_0(t) = c_p \left[ \cosh \left( \frac{p - 1}{2} t \right) \right]^{\frac{1}{p-1}} \text{ with } c_p = \left( \frac{p + 1}{2} \right)^{\frac{1}{p-1}} = w_0(0) = \sup_{t \in \mathbb{R}} w_0(t).
\]

(1.2)

Still for \( N = 1 \), it was shown in the aforementioned reference that if \( 0 < c < c_p \) then (1.1) possesses exactly two positive solutions given by

\[
t \to w_0(t + t_{c,p}) \text{ and } t \to w_0(t - t_{c,p})
\]

with

\[
t_{c,p} = \frac{2}{p-1} \ln \left( \sqrt{\frac{p+1}{2c(p-1)}} + \sqrt{\frac{p+1}{2c(p-1)} - 1} \right);
\]

if \( c = c_p \) then \( w_0 \) is the unique positive solution; if \( c > c_p \) then there are no positive solutions.

We note in passing that the above solutions play an important role in a class of boundary layer problems (see [15]).

If \( N \geq 2, p > 1, \) and \( 0 < c < c_p \), using variational methods, it was shown in the same reference [11] that (1.1) admits at least three positive bounded solutions that are geometrically distinct in the sense that they are not translates of each other in the \( x' \) direction. In particular, under the further restriction that \( p + 1 \) is smaller than the critical Sobolev exponent in \( \mathbb{R}^N, N \geq 2 \), then (1.1) admits a positive bounded solution of the form

\[
x \to w_0(x_N + t_{c,p}) + \tilde{v}(x) \text{ with } \tilde{v} \in H^1_0(\mathbb{R}^N_+) \setminus \{0\} \text{ nonnegative},
\]

(\( H^1_0(\mathbb{R}^N_+) \) denotes the usual Sobolev function space with zero trace on \( \partial \mathbb{R}^N_+ \)).

On the other hand, if \( c > c_p, p > 1 \), it was shown therein that (1.1) has no bounded positive solutions. This was accomplished by means of the famous sliding method [2].

Still in the same reference [11], it was posed as an interesting open question whether the function \( x \to w_0(x_N) \) is the unique bounded positive solution to (1.1) in the case \( c = c_p, p > 1 \). In this regard, as we explained in [19], we note that the aforementioned sliding argument can be applied even in this case to establish that

\[
w_0(x_N) < v(x), x \in \mathbb{R}^N_+, \text{ or } w_0 \equiv v.
\]

(1.3)

If \( N = 2, 3 \), in the aforementioned reference, we were able to exclude the first scenario in (1.3) by adapting some ideas from the proof of the famous De Giorgi conjecture in the plane (see [3, 12]). More precisely, a key observation in the proof was that the convexity of the nonlinearity implies that, in the first case of (1.3), the difference \( v - w_0 \) would be a
positive super-solution to the linearized problem on \( w_0 \) (see also [6, Ch. 1]). Actually, this property brought the problem closer in spirit to that of the one-dimensional symmetry of bounded, stable solutions to semilinear elliptic equations in the plane (a stronger version of De Giorgi’s conjecture, see [4, 9]). The fact that we were dealing with the half-space and not the full space also created some technical difficulties in applying the approach of the aforementioned references. We point out that we were able to gain one more dimension, compared to the aforementioned works, owing to the uniform exponential decay of solutions as \( x_N \rightarrow \infty \).

In the current work, we will first restrict ourselves to the subcubic regime \( 1 < p < 3 \) and prove the following.

**Theorem 1.1** If \( 1 < p < 3 \) and \( 2 \leq N \leq 5 \), then the only positive bounded solution of (1.1) with \( c = c_p \) is \( v(x) = w_0(x_N) \), where \( c_p \) and \( w_0 \) are as in (1.2).

Then, for the cubic case we can show the following.

**Theorem 1.2** If \( p = 3 \) and \( 2 \leq N \leq 4 \), then the same conclusion of Theorem 1.1 holds.

We would like to highlight that byproducts of our proofs are an astonishingly simple proof of our aforementioned result in [19] in two and three dimensions (see Remark 2.1 below) and a nontrivial integral estimate that holds in all dimensions (see Remark 2.2 below).

Our main observation behind the proofs is that the auxiliary function

\[
u(x') = \int_0^{\infty} \left( v(x', x_N) - w_0(x_N) \right) \left( -w'_0(x_N) \right) \, dx_N, \quad x' \in \mathbb{R}^{N-1},
\]

is a nonnegative super-solution to the quadratic Lane-Emden-Fowler equation \( \Delta u + u^2 = 0 \) if \( 1 < p < 3 \), after a suitable re-scaling, or the cubic one if \( p = 3 \). We point out that this property is valid for all \( N \geq 2 \). Then, restricting ourselves to sufficiently low dimensions, we can apply a well known but seldom utilized Liouville type result to conclude (see Appendix A below). We note in passing that the above auxiliary function \( u \) corresponds to the projection of the difference \( v - w_0 \) on \( w'_0 \) which is in the kernel of the linearized problem on \( w_0 \). Lastly, let us remark that an approach of this nature has been frequently applied in various parabolic problems for obtaining Liouville type results for ancient or eternal solutions (see for instance [14]).

Unfortunately, it seems that our method of proof does not provide useful information if \( p > 3 \) and \( N \geq 4 \).

For completeness, let us mention that (1.1) with \( c = 0 \) is fairly well understood. Indeed, if \( p + 1 > 2 \) is subcritical in the sense of the Sobolev imbedding in \( \mathbb{R}^N \), \( N \geq 2 \), it was shown in [7] that there are no nontrivial solutions in \( H_0^1(\mathbb{R}^N) \). Furthermore, it is known that there are no positive bounded solutions (we point out that this holds without any restriction on \( p > 1 \), see [10]). Lastly, we refer to [17] for nonexistence results via the Morse index.

The proofs of Theorems 1.1 and 1.2 will be given in the following section together with some related remarks. Finally, in Appendix A we recall a well known Liouville type theorem concerning the nonexistence of positive super-solutions to the Lane-Emden-Fowler equation in the whole space.

### 2 Proofs of the main results

In this section, we will give the Proofs of Theorems 1.1 and 1.2 along with some related remarks.
2.1 Proof of Theorem 1.1

Proof As we have already mentioned, relation (1.3) is valid. We wish to show that the second alternative is the one which holds. To this end, let us argue by contradiction and suppose that

\[ w_0(x_N) < v(x', x_N), \quad (x', x_N) \in \mathbb{R}_+^N. \tag{2.1} \]

Then, as we know from [19], the difference

\[ \varphi(x', x_N) = v(x', x_N) - w_0(x_N), \quad (x', x_N) \in \mathbb{R}_+^N, \tag{2.2} \]

furnishes a positive super-solution to the linearization of (1.1) on \( w_0 \). Here, we will need to be more precise and make full use of the PDE that \( \varphi \) satisfies, which is nonlinear and we shall now recall. Since both \( v \) and \( w_0 \) satisfy the same PDE in (1.1) and are positive, we find that

\[
\begin{align*}
\Delta (v - w_0) &= v - v^p - w_0 + w_0^p \\
&= (1 - pw_0^{p-1})(v - w_0) - (vw_0^p - pw_0^{p-1}(v - w_0)).
\end{align*}
\]

Hence, recalling the definition of \( \varphi \) from (2.2), and applying the mean value theorem, we arrive at

\[
\Delta \varphi = (1 - pw_0^{p-1})\varphi - \frac{p(p - 1)}{2}x^{p-2}\varphi^2, \quad \varphi > 0 \text{ in } \mathbb{R}_+^N; \quad \varphi = 0 \text{ on } \partial \mathbb{R}_+^N, \tag{2.3}
\]

where

\[ w_0(x_N) < \xi(x) < v(x), \quad x = (x', x_N) \in \mathbb{R}_+^N. \tag{2.4} \]

Let us now consider the positive auxiliary function

\[ u(x') = \int_0^\infty \varphi(x', x_N)Z(x_N)dx_N, \quad x' \in \mathbb{R}_+^{N-1}, \tag{2.5} \]

where \( \varphi \) is as in (2.2), and

\[ Z(x_N) \equiv -w'_0(x_N). \tag{2.6} \]

For future reference, we observe that differentiation of the ODE satisfied by \( w_0 \) yields

\[
-Z'' + \left(1 - pw_0^{p-1}(x_N)\right)Z = 0, \quad x_N > 0. \tag{2.7}
\]

Furthermore, using (1.2), we get

\[ Z(0) = 0, \quad Z > 0 \text{ in } (0, \infty) \text{ and } Z/w_0 \to 1 \text{ as } x_N \to \infty. \tag{2.8} \]

By standard elliptic estimates (see for instance [8]), we deduce that the second derivatives of \( \varphi \) remain uniformly bounded as \( x_N \to \infty \). Hence, thanks to the exponential decay of \( Z \), we can compute that
\[ \Delta_{x'} u(x') = \int_{0}^{\infty} \Delta_{x'} \varphi(x', x_N) Z(x_N) dx_N \]

\[ = \int_{0}^{\infty} (-\varphi_{x_N} + \Delta \varphi) Z dx_N \]

using (2.3) : \[ \int_{0}^{\infty} \left[ -\varphi_{x_N} + (1 - pw_0^{p-1}) \varphi - \frac{p(p-1)}{2} \xi^{p-2} \varphi^2 \right] Z dx_N \]

integrating by parts : \[ \int_{0}^{\infty} \varphi \left[ -Z'' + (1 - pw_0^{p-1}) Z \right] dx_N - \frac{p(p-1)}{2} \int_{0}^{\infty} \xi^{p-2} \varphi^2 Z dx_N \]

\[ + \varphi_{x_N} (x', 0) Z(0) - \varphi(x', 0) Z'(0) \]

via (2.3), (2.7), (2.8) : \[ - \frac{p(p-1)}{2} \int_{0}^{\infty} \xi^{p-2} \varphi^2 Z dx_N. \quad (2.9) \]

We first consider the case where \( 1 < p \leq 2 \). Then, by virtue of (2.4), we obtain

\[ \xi^{p-2} \geq v^{p-2} \geq \|v\|_{L^\infty(\mathbb{R}^N_+)}^{p-2} \quad \text{in} \quad \mathbb{R}^N_+. \quad (2.10) \]

On the other hand, thanks to the Cauchy-Schwarz inequality and recalling the definition of \( Z \) from (2.6) (keeping in mind also (2.8)), we have

\[ u^2(x') = \left( \int_{0}^{\infty} \varphi Z dx_N \right)^2 \leq \int_{0}^{\infty} \varphi^2 Z dx_N \int_{0}^{\infty} Z dx_N = c_p \int_{0}^{\infty} \varphi^2 Z dx_N. \quad (2.11) \]

Hence, by combining (2.9), (2.10) and (2.11), we infer that

\[ - \Delta_{x'} u \geq C u^2, \quad u > 0 \quad \text{in} \quad \mathbb{R}^{N-1}. \quad (2.12) \]

for some constant \( C > 0 \) that depends only on \( p \) and the supremum of \( v \). However, in view of Theorem A.1 in Appendix A, after a simple rescaling in the above inequality, we can arrive at a contradiction, provided that we restrict \( N \geq 2 \) so that

\[ 2 \leq \rho_{sg}(N-1) \Leftrightarrow N = 2 \text{ or } N = 3 \text{ or } 2 \leq \frac{N-1}{N-3} \Leftrightarrow 2 \leq N \leq 5. \]

To complete the proof, it remains to consider the case \( 2 < p < 3 \). Then, the ordering (2.4) gives

\[ \xi^{p-2} > w_0^{p-2} \quad \text{in} \quad \mathbb{R}^N_+. \quad (2.13) \]

Now, we observe that

\[ u^2(x') = \left( \int_{0}^{\infty} \varphi Z dx_N \right)^2 \]

by the Cauchy-Schwarz inequality : \[ \leq \int_{0}^{\infty} \frac{Z}{\xi^{p-2}} dx_N \int_{0}^{\infty} \xi^{p-2} \varphi^2 Z dx_N \]

using (2.13) : \[ \leq \int_{0}^{\infty} \frac{Z}{w_0^{p-2}} dx_N \int_{0}^{\infty} \xi^{p-2} \varphi^2 Z dx_N \]

via (2.8) : \[ \leq \| \frac{Z}{w_0} \|_{L^\infty(0, \infty)} \int_{0}^{\infty} \frac{Z}{w_0}^{3-p} dx_N \int_{0}^{\infty} \xi^{p-2} \varphi^2 Z dx_N \]

since \( p < 3 \) and \( w_0 \) has exponential decay : \[ C' \int_{0}^{\infty} \xi^{p-2} \varphi^2 Z dx_N, \]

for some positive constant \( C' \) that depends only on \( p \). Thus, by combining (2.9) and the above relation, we see that \( u \) satisfies (2.12), for possibly a different positive constant, and can conclude as before. \[ \square \]
Remark 2.1 The approach of the current paper can be used to give a streamlined proof of our previously mentioned result in [19]. Indeed, it follows from (2.9) that \(u\) is a nonnegative super-harmonic function in \(\mathbb{R}^{N-1}, N \geq 2\), for any \(p > 1\). Therefore, if \(N = 2, 3\), by the famous theorem of Hadamard and Liouville (see for instance [9, Thm. 3.1]), we deduce that \(u\) is a constant function. In turn, using (2.9) once more, we conclude that \(\varphi \equiv 0\), as desired. It is worth mentioning that an argument of this nature can be found in [5, Lem. 6.1] for the study of the linearized Allen-Cahn equation.

Remark 2.2 We recall from the Proof of Theorem 1.1 that the auxiliary function \(u\), as defined in (2.5), satisfies (2.12) for all \(N \geq 2\), provided that \(1 < p < 3\). Hence, by applying [18, Lem. 2.4], for any \(\gamma \in (0, 2)\), we infer that there exists a positive constant \(M > 0\) such that

\[
\int_{|x'| < R} u^\gamma d x' \leq M R^{N-1-2\gamma}, \quad R > 0.
\]

2.2 Proof of Theorem 1.2

Proof The proof proceeds as that of Theorem 1.1, invoking a contradiction argument. However, instead of using the mean value theorem, we can compute explicitly that the positive auxiliary function \(u\), as defined in (2.5), satisfies

\[
\Delta_{x'} u = -\int_0^\infty (3w_0 \varphi^2 + \varphi^3) Z dx_N \leq -\int_0^\infty \varphi^3 Z dx_N, \ x' \in \mathbb{R}^{N-1}.
\]

Next, by Hölder’s inequality we get

\[
u^3(x') = \left( \int_0^\infty \varphi Z dx_N \right)^3 \leq \left( \int_0^\infty Z dx_N \right)^2 \int_0^\infty \varphi^3 Z dx_N = c_p^2 \int_0^\infty \varphi^3 Z dx_N,
\]

\(x' \in \mathbb{R}^{N-1}\) (recall the definition of \(Z\) from (2.6), and the definition of \(c_p\) from (1.2)). Hence, we deduce that

\[-\Delta_{x'} u \geq \frac{1}{c_p^3} u^3, \ u > 0, \ x' \in \mathbb{R}^{N-1}.
\]

On the other hand, this contradicts the Liouville type theorem in Appendix A since it is easy to check that

\[3 \leq p_{sg}(N - 1) \iff N = 2 \text{ or } N = 3 \text{ or } 3 \leq \frac{N - 1}{N - 3} \iff 2 \leq N \leq 4,
\]

and the proof is completed. \(\square\)

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A. A Liouville type theorem

For the reader’s convenience, we state below a well known result due to [13] (see also [1] and [16, Ch. I] for simpler proofs and extensions), which we used in the proofs of Theorems 1.1 and 1.2.
Theorem A.1 Let $1 < p \leq p_{sg}(n)$, where
\[
p_{sg}(n) = \begin{cases} 
\infty & \text{if } n \leq 2, \\
\frac{n}{n-2} & \text{if } n > 2.
\end{cases}
\]
Then, the inequality
\[-\Delta u \geq u^p, \quad x \in \mathbb{R}^n,
\]
does not possess any positive classical solution.

Remark A.2 As was remarked in [16], the condition $p \leq p_{sg}$ in Theorem A.1 is optimal, as shown by the explicit example $u(x) = k \left(1 + |x|^2\right)^{-1/(p-1)}$ with $n \geq 3$, $p > p_{sg}$ and $k > 0$ small enough.

References

1. Armstrong, S.N., Sirakov, B.: Nonexistence of positive supersolutions of elliptic equations via the maximum principle. Comm. Partial Differ. Equ. 36, 2011–2047 (2011)
2. Berestycki, H., Nirenberg, L.: On the method of moving planes and the sliding method. Bol. Soc. Brasil. Mat. (N.S.) 22, 1–37 (1991)
3. Berestycki, H., Caffarelli, L., Nirenberg, L.: Further qualitative properties for elliptic equations in unbounded domains. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25, 69–94 (1997)
4. Dancer, E.N.: Stable and finite Morse index solutions on $\mathbb{R}^n$ or on bounded domains with small diffusion. Trans. Amer. Math. Soc. 357, 1225–1243 (2005)
5. del Pino, M., Kowalczyk, M., Wei, J.: On De Giorgi’s conjecture in dimension $N \geq 9$. Ann. Math. 174, 1485–1569 (2011)
6. Dupaigne, L.: Stable Solutions Of Elliptic Partial Differential Equations. Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math. 143, CRC Press, Boca Raton (2011)
7. Esteban, M.J., Lions, P.-L.: Existence and nonexistence results for semilinear elliptic problems in unbounded domains. Proc. Roy. Soc. Edinburgh Sect. A 93, 1–14 (1982)
8. Evans, L.C.: Partial Differential Equations. In: Graduate Series in Mathematics 19, 2nd edn. AMS, Amsterdam (2010)
9. Farina, A.: Liouville-Type Theorems for Elliptic Problems, In: M. Chipot (Ed.), Handbook of Differential Equations IV: Stationary Partial Differential Equations, pp. 61–116. Elsevier B.V. (2007)
10. Farina, A.: Some results about semilinear elliptic problems on half-spaces. Math. Eng. 2, 709–721 (2020)
11. Fernandez, A.J., Weth, T.: The nonlinear Schrödinger equation in the half-space. Math. Ann. (2021). https://doi.org/10.1007/s00208-020-02129-8
12. Ghoussoub, N., Guí, C.: On a conjecture of De Giorgi and some related problems. Math. Ann. 311, 481–491 (1998)
13. Gidas, B.: Symmetry properties and isolated singularities of positive solutions of nonlinear elliptic equations. Nonlinear partial differential equations in engineering and applied science, Lecture Notes in Pure and Appl. Math. 54, pp. 255–273. Dekker, New York (1980)
14. Merle, F., Zaag, H.: Optimal estimates for blow-up rate and behavior for nonlinear heat equations. Comm. Pure Appl. Math. 51, 139–196 (1998)
15. Omel’chenko, O.E., Recke, L., Butuzov, V.F., Nefedov, N.N.: Time-periodic boundary layer solutions to singularly perturbed parabolic problems. J. Differential Equations 262, 4823–4862 (2017)
16. Quittner, P., Souplet, Ph.: Superlinear Parabolic Problems. In: Blow-up, global existence and steady states, 2nd edn. Birkhäuser Advanced Texts, Birkhäuser, Basel (2019)
17. Selmi, A., Harrabi, A., Zaidi, C.: Nonexistence results on the space or the half space of $-\Delta u + \lambda u = |u|^{p-1}u$ via the Morse index, Commun. Pure. Appl. Anal. 19, 2839–2852 (2020)
18. Serrin, J., Zou, H.: Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities. Acta Math. 189, 79–142 (2002)
19. Sourdis, C.: One-dimensional symmetry of positive bounded solutions to the nonlinear Schrödinger equation in the half-space in two and three dimensions. Partial Differ. Equ. Appl. 2, 79 (2021). https://doi.org/10.1007/s42985-021-00125-4

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.