On the existence of maximal semidefinite invariant subspaces for $J$-dissipative operators

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Abstract. For a certain class of operators we present some necessary and sufficient conditions for a $J$-dissipative operator in a Krein space to have maximal semidefinite invariant subspaces. We investigate the semigroup properties of restrictions of the operator to these invariant subspaces. These results are applied to the case when the operator admits a matrix representation with respect to the canonical decomposition of the space. The main conditions are formulated in terms of interpolation theory for Banach spaces.

Bibliography: 25 titles.

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§ 1. Introduction

In this paper, we consider the problem of whether invariant semidefinite subspaces exist for $J$-dissipative operators defined in Krein spaces. We recall that a Krein space (see [1]) is a Hilbert space $H$ in which, along with the ordinary inner product $(\cdot, \cdot)$, an indefinite inner product (indefinite metric) $[x, y] = (Jx, y)$ is also defined, where $J = P^+ - P^-$ and the operators $P^\pm$ are orthogonal projections in $H$, $P^+ + P^- = I$ (see [1]). We write $H^\pm = R(P^\pm)$. The operator $J$ is called the canonical symmetry. A Krein space is called a Pontryagin space if $\dim R(P^+) < \infty$ or $\dim R(P^-) < \infty$, and the quantity $\kappa = \min(\dim R(P^+), \dim R(P^-))$ is called the rank of indefiniteness of the Pontryagin space and we then use the symbol $\Pi_\kappa$ for it. A subspace $M \subset H$ is said to be nonnegative (positive, uniformly positive) if $[x, x] \geq 0$ ($[x, x] > 0$, $[x, x] > \delta \|x\|^2$, $\delta > 0$, respectively) for all $x \in M$. Nonpositive, negative, and uniformly negative subspaces are defined in a similar way. We say that $M$ is a maximal nonpositive (nonnegative, positive, and so on) subspace if it is nonpositive (nonnegative, positive) and admits no nontrivial nonpositive (nonnegative, positive, and so on) extensions. A densely defined operator $A: H \to H$ is said to be dissipative (strictly dissipative, uniformly dissipative) if $-\text{Re}(Au, u) \geq 0$ ($-\text{Re}(Au, u) > 0$, $-\text{Re}(Au, u) \geq \delta \|u\|^2$, $\delta > 0$, respectively) for any $u \in D(A)$ and $J$-dissipative (strictly $J$-dissipative, uniformly $J$-dissipative) if the operator $JA$ is dissipative (strictly dissipative, uniformly dissipative, respectively). An operator $A: H \to H$ is said to be maximal dissipative (maximal $J$-dissipative) if it is dissipative ($J$-dissipative) and admits no nontrivial dissipative ($J$-dissipative) extensions.

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Let $L$ be a $J$-dissipative operator. A subspace $M \subset H$ is said to be $L$-invariant if $M \cap D(L)$ is dense in $M$ and $L(M \cap D(L)) \subset M$.

The main problem considered in the paper concerns the existence of maximal semidefinite (that is, sign-definite) invariant subspaces for a given $J$-dissipative operator. The first results in this direction were published quite some time ago by Pontryagin [2] who proved that every $J$-self-adjoint operator $L$ in a Pontryagin space (assume, for definiteness, that $\dim H^+ = \kappa < \infty$) has a maximal nonnegative invariant subspace $M$, $\dim M = \kappa$, such that the spectrum of the operator $L|_M$ is contained in the closed upper half-plane. Pontryagin’s results were generalized to diverse classes of operators by various authors (see [3]–[11]). See [1] for quite a detailed bibliography and a series of results. Among the most recent works, we note the papers [11]–[14], where the most general results up till now were obtained. In these papers, the whole space $H$ is identified with the Cartesian product $H^+ \times H^-$ and the operator $L$ with a matrix operator $L: H^+ \times H^- \to H^+ \times H^-$ of the form

$$
L = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
$$

(1.1)

$$
A_{11} = P^+LP^+, \quad A_{12} = P^+LP^-, \quad A_{21} = P^-LP^+, \quad A_{22} = P^-LP^-.
$$

In this case, the canonical symmetry is of the form $J = \begin{pmatrix} I & 0 \\
0 & -I \end{pmatrix}$, where $I$ stands for the identity operator. In [14] the main condition for a maximal nonnegative invariant subspace to exist for an operator $L$ is the condition that the operator $A_{12}(A_{22} - \mu)^{-1}$ is compact for some $\mu$ in the left half-plane. For operators of a certain class we present sufficient conditions for the existence of maximal semidefinite subspaces and also necessary conditions. In contrast to the papers mentioned above, our results use interpolation theory for Banach spaces, and these conditions are related to some interpolation equations holding. We also study the properties of the restrictions of the operator to these invariant subspaces. The investigation of this last problem was initiated in [14]. The results thus obtained are then used to study operators which can be represented in the form (1.1). Here the compactness condition for the operator $A_{12}(A_{22} - \mu)^{-1}$ does not arise; instead, conditions for the subordination (in a certain sense) of the operators $A_{12}$ and $A_{21}$ to the operators $A_{11}$ and $A_{22}$ occur. In fact, a subordination condition is a condition of diagonal predominance (in a certain sense) for the operator matrix in (1.1). In the sufficiency part, our results sharpen those of [15] (see also [16] and [17]).

§ 2. Preliminaries and results

For given Banach spaces $X$ and $Y$ we denote by $L(X,Y)$ the space of continuous linear operators defined on $X$ and with range in $Y$. If $X = Y$, then we write $L(X)$ instead of $L(X,X)$. We denote the spectrum and the resolvent set of $L$ by $\sigma(L)$ and $\rho(L)$, respectively, and the domain and the range of $L$: $X \to X$ by $D(L)$ and $R(L)$, respectively. If $M \subset X$ is a subspace, then by the restriction of the operator $L$ to $M$ we mean the operator $L|_M: M \to X$ with domain $D(L|_M) = D(L) \cap M$ which coincides with $L$ on $D(L|_M)$. An operator $A$ such that $-A$ is dissipative (maximal dissipative) is said to be accretive (maximal accretive, respectively). Therefore, up to sign, the statements which hold for accretive operators are valid for dissipative
operators as well. Below we replace the word ‘maximal’ by the symbol $m$, and thus we write $m$-dissipative instead of ‘maximal dissipative’. If $A$ is an operator in a Krein space $H$, then $A^*$ denotes the adjoint operator and $A^c$ the adjoint with respect to the indefinite metric in $H$; the latter has the standard properties of an adjoint operator (see [1]). Let $A_0$ and $A_1$ be two Banach spaces continuously embedded in a Hausdorff topological vector space $E$, that is, $A_0 \subset E$ and $A_1 \subset E$. A pair $\{A_0, A_1\}$ of this kind is called an interpolation pair. Recall the definition of the interpolation space $(A_0, A_1)_{\theta,q}$. We describe the $K$-method of interpolation. For every $t$, $0 < t < \infty$, the functional

$$K(t, a, A_0, A_1) = K(t, a) = \inf_{a = a_0 + a_1} \left( \|a_0\|_{A_0} + t \|a_1\|_{A_1} \right), \quad a_0 \in A_0, \quad a_1 \in A_1,$$

defines a norm in the space $A_0 + A_1$ which is equivalent to the standard norm of the space, that is, to the norm $\inf_{a = a_0 + a_1} \left( \|a_0\|_{A_0} + \|a_1\|_{A_1} \right)$, $a_0 \in A_0$ and $a_1 \in A_1$ (see [18], § 1.2.1). Let $0 < \theta < 1$. For $1 \leq q < \infty$ we have

$$(A_0, A_1)_{\theta,q} = \left\{ a \big| a \in A_0 + A_1, \|a\|_{(A_0, A_1)_{\theta,q}} = \left( \int_0^\infty \left[ t^{-\theta} K(t, a) \right]^q \frac{dt}{t} \right)^{1/q} < \infty \right\}.$$

We first describe some facts which we use in this paper.

**Proposition 1.** Let $H$ be a Hilbert space (a Krein space).

1. Every maximal dissipative ($J$-dissipative) operator $A$ is closed and, if $A$ is $m$-dissipative, then $\mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Re } z > 0 \} \subset \rho(A)$ (see [19], Proposition C.7.2).

2. If $A$ is $m$-dissipative, then the operators $A + i\omega I$, $\omega \in \mathbb{R}$, and $A - \varepsilon I$, $\varepsilon > 0$, are $m$-dissipative (see [19], Proposition C.7.2).

3. If $A$ is $m$-$J$-dissipative, then the operator $A + i\omega I$, $\omega \in \mathbb{R}$, is $m$-$J$-dissipative as well. (This property follows easily from the definition.)

4. If $A$ is $m$-$J$-dissipative and $\ker A = \{0\}$, then $\ker A^c = \{0\}$ (see [1], Ch. 2, § 2, Corollary 2.17; we note that [1] uses a different definition of $J$-dissipative operator, namely, that $\text{Im}[Au, u] \geq 0$ for $u \in D(A)$).

5. If $A$ is a maximal uniformly dissipative ($J$-dissipative) operator, then $i \mathbb{R} \in \rho(A)$ (see [1], Ch. 2, § 2, Assertion 2.32).

6. If $A$ is $m$-dissipative ($m$-$J$-dissipative), then the operator $A$ is injective if and only if $R(A)$ is dense in $H$ (see [19], Proposition 7.0.1).

7. If $A$ is $m$-$J$-dissipative, then the operator $A^*$ ($A^c$) is also $m$-dissipative ($m$-$J$-dissipative) (see [19], Proposition C.7.2 or [1], Ch. 2, § 2, Proposition 2.7).

Let $H$ be a complex Hilbert space with norm $\| \cdot \|$, let $L : H \to H$ be a closed densely defined operator, and let $\rho(L) \neq \emptyset$. We take a $\lambda \in \rho(L)$. We equip the space $H_k = D((L - \lambda I)^k)$, where $k$ is a positive integer, with the norm $\|u\|_{H_k} = \|(L - \lambda I)^k u\|$. If $k < 0$ is an integer, then we let $H_k$ denote the completion of $H$ with respect to the norm $\|u\|_{H_k} = \|(L - \lambda I)^k u\|$. Using Hilbert’s resolvent identity

$$(L - \lambda I)^{-1} - (L - \gamma I)^{-1} = (\lambda - \gamma)(L - \lambda I)^{-1}(L - \gamma I)^{-1}, \quad (2.1)$$

one can show that the norm in $H_k$ does not depend on the parameter $\lambda \in \rho(L)$, and thus $H_k$ is well defined. It appears that the spaces $H_k$ were first introduced

and described in [20], §5. Nowadays, the spaces $H_k$ with $k < 0$ are often referred to as extrapolation spaces, and the entire family of spaces $\{H_k\}$ is referred to as a Sobolev tower (see [21]). Using the definitions, it is easy to show that the norm in the space $H_{-k}$, $k > 0$, coincides with the norm

$$\|u\|_{H_{-k}} = \sup_{v \in D((L^*)^k)} \frac{|(u, v)|}{\|v\|_{D((L^* - \lambda I)^k)}}.$$ 

Under the additional assumption that the operator $L$ is positive, that is, \(\{x \in \mathbb{R} : x \leq 0\} \subset \rho(L)\), the following inequality holds:

$$\|(L - \lambda I)^{-1}\| \leq \frac{c}{1 + |\lambda|} \quad \forall \lambda \leq 0, \tag{2.2}$$

and the interpolation spaces $(H_m, H_n)_{\theta, p}$ were also described in [20]. A description of equivalent norms in these interpolation spaces is presented in [18], §1.14.3, for example. Other classes of spaces in which the sectorial property of the operator is heavily used are described, for example, in [19] (see also the bibliography in [19] and [22]). The interpolation properties of these spaces are also described in [19], and, in particular, another class of equivalent norms. We recall that an operator $L : H \to H$ is said to be sectorial if there is a $\theta \in [0, \pi)$ such that $\sigma(L) \subset \overline{S}_\theta$, $S_\theta = \{z : \arg z < \theta\}$, $\mathbb{C} \setminus S_\theta \subset \rho(L)$, and for any $\omega > \theta$ there is a constant $c(\omega)$ such that

$$\|(L - \lambda I)^{-1}\| \leq \frac{c}{|\lambda|} \quad \forall \lambda \in \mathbb{C} \setminus S_\omega. \tag{2.3}$$

Suppose that $L$ is sectorial and injective (we do not assume that $0 \in \rho(L)$). In this case, an analogue of the space $H_1$ is the space $D_L$ which is the completion of $D(L)$ with respect to the norm $\|Lu\| = \|u\|_{D_L}$, and that of the space $H_{-1}$ is the completion of $R(L)$ with respect to the norm $\|L^{-1}u\| = \|u\|_{R_L}$. If $0 \in \rho(L)$, then $H_1 = D_L$ and $H_{-1} = R_L$; otherwise these equations fail to hold.

Let $L : H \to H$ be an $m$-$J$-dissipative operator in a Krein space $H$ with indefinite metric $[\cdot, \cdot] = (J \cdot, \cdot)$, where $J$ stands for the canonical symmetry and $(\cdot, \cdot)$ for the inner product in $H$. We define the space $F_1$ to be the completion of $D(L)$ with respect to the norm

$$\|u\|_{F_1}^2 = - \text{Re}[Lu, u] + \|u\|^2, \quad \|u\| = \|u\|_H,$$

and the space $F_{-1}$ to be the completion of $H$ with respect to the norm

$$\|u\|_{F_{-1}} = \sup_{v \in F_1} \frac{|(u, v)|}{\|v\|_{F_1}}.$$ 

The space $F_1$ can be identified with a dense subspace of $H$ if

$$\exists c > 0 : \quad |(Lu, v)| \leq c\|u\|_{F_1}\|v\|_{F_1} \quad \forall u, v \in D(L). \tag{2.4}$$

The proof can be found in [16], §4, Ch. 1. This also follows from the arguments and results in [19], §7.3.2 (for example, from Assertion 7.3.4, in which it is sufficient to take $A = -JL + I$).
We note the rather obvious fact that condition (2.4) is equivalent to the condition
\[ \exists c > 0 : \ |\text{Im}[Lu, u]| \leq c \|u\|_{F_1}^2 \quad \forall u \in D(L). \quad (2.5) \]
It suffices to take the sesquilinear form \( a(u, v) = -[Lu, v] + (u, v) \) and to apply Proposition C.1.3 in [19].

**Proposition 2.** Let \( L \) be an \( m \)-dissipative operator. Then \(-L\) is sectorial with \( \theta = \pi/2 \) and
\[ (H_1, H_{-1})_{1/2, 2} = H. \quad (2.6) \]
If, in addition, \( L \) is injective, then
\[ (D_L, R_L)_{1/2, 2} = H. \quad (2.7) \]

The sectorial property with \( \theta = \pi/2 \) is proved in [19], §7.1.1. Equation (2.7) is a consequence of Theorems 2.2 and 4.2 in [22] and of the arguments presented there after Theorem 2.2 (see also §7.3.1, Theorem 7.3.1, and equation (7.18) in [19]). We take the operator \( L - \varepsilon I \), for some \( \varepsilon > 0 \), for the operator \( L \). It is \( m \)-dissipative and \( 0 \in \rho(-L + \varepsilon I) \). Using (2.7), we obtain (2.6).

Suppose that the Hilbert space embedding \( H_1 \subset H \) is dense. We can construct the negative space \( H'_1 \) from the pair \( H_1, H \) as the completion of \( H \) with respect to the norm
\[ \|u\|_{H'_1} = \sup_{v \in H_1} \frac{|(u, v)|}{\|v\|_{H_1}}, \]
where the parentheses \((\cdot, \cdot)\) stand for the inner product in \( H \). In this situation, the following well-known statement holds (see [23], Ch. 1 and [24], Equation (2.2)).

**Proposition 3.** The space of antilinear continuous functionals on \( H_1 \) can be identified with \( H'_1 \), the norm in \( H_1 \) is equivalent to the norm \( \sup_{v \in H_1} |(v, u)|/\|v\|_{H'_1} \), and
\[ (H_1, H'_1)_{1/2, 2} = H. \quad (2.8) \]

**Lemma 2.1.** Let \( L : H \to H \) be an \( m-J \)-dissipative operator and let condition (2.4) hold. In this case,
\[ (H_1, H_{-1})_{1/2, 2} = H \quad (2.9) \]
if and only if
\[ (F_1, F_{-1})_{1/2, 2} = H. \quad (2.10) \]

**Proof.** The \( m-J \)-dissipative property of \( L \) implies the \( m \)-dissipative property of \( JL \) in \( H \). In this case, the operator \( JL - \varepsilon I \) is also \( m \)-dissipative for any \( \varepsilon \geq 0 \) (property 2 in Proposition 1). Thus, the operators \( L - \varepsilon J \) (and, in particular, the operator \( L_0 = L - J \)) are \( m-J \)-dissipative, and \( D(L_0) = D(L) \). Moreover, the operator \( L_0 \) is uniformly \( J \)-dissipative, and
\[ -\text{Re}[L_0 u, u] = \|u\|_{F_1}^2 \quad \forall u \in D(L). \quad (2.11) \]
Then \( i \mathbb{R} \in \rho(L_0) \) by property 5 in Proposition 1. We shall write out the norm in the space \( H_{-1} \) for \( u \in H \). We have
\[ \|u\|_{H_{-1}} = \sup_{v \in D(L^*)} \frac{|(u, v)|}{\|v\|_{D(L^*)}} = \sup_{v \in D(L^*)} \frac{|((L - J)^{-1}u, (L^* - J)v)|}{\|v\|_{D(L^*)}}. \]
Since the norms \(\| (L^* - J)v \| \) and \(\| v \|_{D(L^*)} \) are equivalent, the previous equation implies a bound of the form \(\| u \|_{H^{-1}} \leq c\| (L - J)^{-1}u \| \) for any \( u \in H \) and for some constants \( c \) and \( c_1 \). Thus, we can equip \( H^{-1} \) with the equivalent norm \(\| (L - J)^{-1}u \|\). The operator \( L_0 \) satisfies all the conditions of Lemma 4.2 in [16], Ch. 1 (or of Lemma 4.6 in [17]), which implies the statement of the lemma holds.

**Lemma 2.2.** Let \( L : H \to H \) be an \( m-J \)-dissipative operator satisfying (2.4). Then \( L \) admits an extension to an operator \( \tilde{L} \) of class \( L(F_1, F_{-1}) \), and

\[
D(L) = \{ u \in F_1 : \tilde{L}u \in H \}. \tag{2.12}
\]

If \( L^c \) is the operator adjoint with respect to the indefinite metric, then \( L^c \) is an \( m-J \)-dissipative operator satisfying (2.4), where \( D(L^c) \subset F_1 \) and the embedding is dense.

**Proof.** Condition (2.4) implies the bound

\[
\| Lu \|_{F_{-1}} \leq c\| u \|_{F_1}. \tag{2.13}
\]

As \( D(L) \) is dense in \( F_1 \), this implies that \( L \) admits an extension to an operator \( \tilde{L} \in L(F_1, F_{-1}) \), and the inequality \( -\Re \langle \tilde{L}u, u \rangle \geq 0 \) holds for any \( u \in F_1 \). Since \( L \) is maximal \( J \)-dissipative, (2.12) follows. Indeed, let \( D_0 = \{ u \in F_1 : \tilde{L}u \in H \} \). Obviously, \( D(L) \subset D_0 \). The operator \( \tilde{L} : H \to H \) with domain \( D_0 \) is \( J \)-dissipative and is an extension of \( L \). Since \( L \) is maximal \( J \)-dissipative, we conclude that \( \tilde{L} = L \), and thus (2.12) holds. Consider the operator \( L_0 = L - J \). As we noted in the previous lemma, \( L_0 \) is a maximal uniformly \( J \)-dissipative operator with \( D(L_0) = D(L) \). By property 5 of Proposition 1, \( i\mathbb{R} \in \rho(L_0) \). We claim that \( \tilde{L}_0 = \tilde{L} - J \) is an isomorphism of \( F_1 \) onto \( F_{-1} \). Indeed, if \( f \in F_{-1} \), then the expression \( [f, v] \) is an antilinear continuous functional on \( F_1 \), and it follows from the Lax-Milgram theorem (see, for example, [19], Theorem C.5.3) that there is a \( u \in F_1 \) such that \( \langle \tilde{L}_0 u, v \rangle = [f, v] \) for any \( v \in F_1 \). We denote \( L_0 \) and \( \tilde{L}_0 \) below by the same symbol \( L_0 \). It follows from assertions presented in [1], Ch. 2, § 1 that \( (L_0^{-1})^c = (L_0^c)^{-1} \). We note that the negative space \( F_1^1 \) constructed from \( F_1 \) and \( H \) is the completion of \( H \) with respect to the norm \( \| Ju \|_{F_{-1}} \), and thus \( J \) is an isomorphism taking \( F_{-1} \) onto \( F_1^1 \) with \( J^{-1} = J \). Using Proposition 3, we see that every antilinear continuous functional over \( F_1 \) can be represented in the form \( l(v) = [f, v], f \in F_{-1} \), and every linear continuous functional on \( F_{-1} \) can be represented in the form \( l(f) = [f, v], v \in F_1 \). Moreover, we see that the norm in \( F_1 \) is equivalent to the norm \( \| v \|_{F_1} = \sup_{f \in F_{-1}} \| (f, v) \|/\| f \|_{F_{-1}} \). We have

\[
[L_0^{-1}u, v] = [u, (L_0^c)^{-1}v] \quad \forall u, v \in H. \tag{2.14}
\]

Since \( L_0 \) is an isomorphism between \( F_1 \) and \( F_{-1} \), from the last equation it is easy to see that \( (L_0^c)^{-1}v \in F_1 \). Indeed, the left-hand side in (2.14) is a continuous linear functional \( l(u) \) on \( F_{-1} \). Hence, there is an element \( g \in F_1 \) such that

\[
[L_0^{-1}u, v] = [u, g] \quad \forall u \in F_{-1}. \tag{2.15}
\]

Then it follows from (2.14) that \( [u, g] = [u, (L_0^c)^{-1}v] \) for any \( u \in H \). Hence, \( g = (L_0^c)^{-1}v \in F_1 \). Since every element in \( D(L^c) \) can be represented in the form
Let \((L_0^0)^{-1}v\), where \(v \in H\), it follows that \(D(L_0^0) \subset F_1\), and therefore (2.14) implies that \(D(L_0^0)\) is dense in \(F_1\). If not, there exists \(u \in F_{-1}\) such that the right-hand side of (2.14) vanishes for any \(v \in H\). Then the left-hand side also vanishes, and thus \(L_0^{-1}u = 0\), that is, \(u = 0\).

**Lemma 2.3.** Let \(L: H \rightarrow H\) be an \(m\)-\(J\)-dissipative operator, let (2.4) hold, and let a constant \(m > 0\) exist such that

\[
\|u\|^2 \leq m(-\text{Re}[Lu, u] + \|u\|^2_{F_{-1}}) \quad \forall u \in D(L). \tag{2.16}
\]

Then \(\omega_0 \geq 0\) exists such that

a) \(I_{\omega_0} = \{i\omega : |\omega| \geq \omega_0\} \subset \rho(L)\);

b) if \(i\omega \in \rho(L), \omega \in \mathbb{R}\), then the operator \(L - i\omega I: F_1 \rightarrow F_{-1}\) is continuously invertible;

c) there is a constant \(c > 0\) such that

\[
\|(L - i\omega I)^{-1}u\|_{F_{-1}} \leq \frac{c\|u\|_{F_{-1}}}{1 + |\omega|} \quad \forall \omega \in I_{\omega_0}, \quad \forall u \in F_{-1}; \tag{2.17}
\]

d) if \(i\mathbb{R} \subset \rho(L)\), then there is a constant \(c > 0\) such that the bound c) holds for any \(\omega \in \mathbb{R}\);

e) if one of (2.9) and (2.10) is satisfied, then there is a constant \(c > 0\) such that

\[
\|(L - i\omega I)^{-1}u\| \leq \frac{c\|u\|}{1 + |\omega|} \quad \forall \omega \in I_{\omega_0}, \quad \forall u \in H, \tag{2.18}
\]

and if, moreover, \(i\mathbb{R} \subset \rho(L)\), then

\[
\exists c > 0 : \|(L - i\omega I)^{-1}u\| \leq \frac{c\|u\|}{1 + |\omega|} \quad \forall \omega \in \mathbb{R}, \quad \forall u \in H. \tag{2.19}
\]

**Proof.** Consider the equation

\[
Lu - i\omega u = f, \quad \omega \in \mathbb{R}, \tag{2.20}
\]

where \(u \in D(L)\). This implies that \(-\text{Re}[Lu, u] = -\text{Re}[f, u]\), and by (2.16) this can be represented in the form

\[
-\text{Re}[Lu, u] + \delta\|u\|^2 \leq -\text{Re}[f, u] - \delta m \text{Re}[Lu, u] + \delta m\|u\|^2_{F_{-1}}, \quad \delta > 0.
\]

Choosing \(\delta = 1/(1 + m)\), we obtain

\[
\|u\|^2_{F_1} \leq -(m + 1) \text{Re}[f, u] + m\|u\|^2_{F_{-1}}. \tag{2.21}
\]

Inequality (2.13) and equation (2.20) imply the bound

\[
|\omega|^2\|u\|^2_{F_{-1}} \leq 2c^2\|u\|^2_{F_1} + 2\|f\|^2_{F_{-1}}. \tag{2.22}
\]

Dividing this inequality by \(4c^2\) and adding the result to the previous one, we obtain

\[
\frac{1}{2}\|u\|^2_{F_1} + \delta_0|\omega|^2\|u\|^2_{F_{-1}} \leq c_2\|f\|^2_{F_{-1}} - (m + 1) \text{Re}[f, u] + m\|u\|^2_{F_{-1}}, \quad \delta_0 > 0. \tag{2.23}
\]
The Schwarz inequality $|\langle f, u \rangle| \leq \|f\|_{F^{-1}}\|u\|_{F_1}$ holds for $f \in F^{-1}$ and $u \in F_1$. Using this inequality on the right-hand side of (2.23) and then applying the numerical inequality

$$|ab| \leq \frac{\varepsilon|a|^p}{p} + \frac{|b|^q}{q^{1/p}}; \quad \varepsilon \in (0, \infty), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p \in (1, \infty),$$

(2.24)

where we set $p = q = 2$, we obtain

$$\frac{1}{2}\|u\|^2_{F_1} + \delta_0|\omega|^2\|u\|^2_{F^{-1}} \leq c_3(\varepsilon)\|f\|^2_{F^{-1}} + 2\varepsilon\|u\|^2_{F_1} + m\|u\|^2_{F^{-1}}.$$ 

Choosing $\varepsilon = 1/8$ and transferring the term containing $\|u\|_{F_1}$ to the left-hand side, we arrive at the inequality

$$\|u\|^2_{F_1} + |\omega|^2\|u\|^2_{F^{-1}} \leq c_4\|f\|^2_{F^{-1}} + c_5\|u\|^2_{F^{-1}},$$

(2.26)

where $c_4$ and $c_5$ are some constants independent of $u$. Now choosing $\omega_0 = 2\sqrt{c_5} + 1$, we obtain (2.17), which implies that $\text{ker}(L - i\omega I) = \{0\}$ for $\omega \in I_{\omega_0}$. It follows from property 6 of Proposition 1 that $R(L - i\omega I)$ is dense in $H$, and hence also in $F^{-1}$. By (2.17), the operator $(L - i\omega I)^{-1}$ defined on a dense subset of $F^{-1}$ admits an extension by continuity to the whole of $F^{-1}$. This implies that the operator $\tilde{L} - i\omega I$ is an isomorphism taking $F_1$ onto $F^{-1}$, where $\tilde{L}: F_1 \rightarrow F^{-1}$ is an extension of $L$ to a continuous mapping from $F_1$ to $F^{-1}$. It follows from equation (2.12) that $i\omega \in \rho(L)$ for any $\omega \in I_{\omega_0}$.

Suppose now that $i\omega \in \rho(L)$ and $\omega \in \mathbb{R}$. We also take an arbitrary number $\omega_1 \in I_{\omega_0}$. By (2.1),

$$(L - i\omega I)^{-1} = (L - i\omega_1 I)^{-1} - i(\omega_1 - \omega)(L - i\omega I)^{-1}(L - i\omega_1)^{-1}.$$  

Then for $f \in H$ we have

$$\|(L - i\omega I)^{-1}f\|_{F_1} \leq \|(L - i\omega_1 I)^{-1}f\|_{F_1} + (|\omega| + |\omega_1|)\|(L - i\omega I)^{-1}(L - i\omega_1 I)^{-1}\|_{F_1}.$$  

Since $0 \in \rho(L - i\omega I)$, there is a constant $c > 0$ such that

$$\|u\|_{F_1} \leq c\|(L - i\omega I)u\| \quad \forall u \in D(L).$$  

Using this inequality and the embedding $F_1 \subset H$, we obtain

$$\|(L - i\omega I)^{-1}f\|_{F_1} \leq \|(L - i\omega_1 I)^{-1}f\|_{F_1} + c(|\omega| + |\omega_1|)\|(L - i\omega_1 I)^{-1}\|_{F_1}.$$  

Since $|\omega_1| \geq \omega_0$, it follows from what we have already proved above that the right-hand side can be estimated by $c\|f\|_{F^{-1}}$. Thus, the following bound holds:

$$\|(L - i\omega I)^{-1}f\|_{F_1} \leq c\|f\|_{F^{-1}} \quad \forall f \in R(L - i\omega I).$$  

As above, we can see that $\tilde{L} - i\omega I: F_1 \rightarrow F^{-1}$ is continuously invertible.

We can readily prove statement d) using b) and c). We will now obtain statement e) using interpolation.
In what follows, we identify $L$ and its extension $\tilde{L}$. The operator $(L - i\omega I)^{-1}$, where $\omega \in I_{\omega_0}$, is a continuous mapping from $F_{-1}$ into $F_{-1}$, and the following inequality holds:

$$\|(L - i\omega I)^{-1}u\|_{F_{-1}} \leq \frac{c\|u\|_{F_{-1}}}{1 + |\omega|} \quad \forall \omega \in I_{\omega_0} \quad (2.27)$$

for some constant $c > 0$ and all $u \in F_{-1}$. We take $u \in F_1$. Then

$$g = Lu - i\omega_0 u \in F_{-1}, \quad (L - i\omega I)^{-1} g \in F_1,$$

and thus

$$(L - i\omega_0 I)(L - i\omega I)^{-1} u = (L - i\omega I)^{-1} g \in F_1;$$

moreover,

$$\|(L - i\omega I)^{-1} u\|_{F_1} \leq c \|(L - i\omega I)^{-1} g\|_{F_{-1}} \leq \frac{c_1 \|g\|_{F_{-1}}}{1 + |\omega|} \leq \frac{c_2 \|u\|_{F_1}}{1 + |\omega|} \quad (2.28)$$

for any $\omega \in I_{\omega_0}$ and $u \in F_1$. Thus, $(L - i\omega I)^{-1}|_{F_1} \in L(F_1)$. It follows from the standard properties of interpolation spaces (see, for example, part a) of Theorem 1.3.3 in [18]) that

$$(L - i\omega I)^{-1}|_{(F_1,F_{-1})_{1/2,2}} \in L((F_1,F_{-1})_{1/2,2}),$$

and the bound (2.18) follows from the bounds (2.27) and (2.28) and from the equation (2.10). The last statement is obvious.

**Lemma 2.4.** Let $H$ be a Kreĭn space and let $L : H \to H$ be an $m$-$J$-dissipative operator such that

1) there are maximal non-negative and non-positive subspaces $M^\pm$ which are $L$-invariant and such that $H = M^+ + M^-$, where the sum is direct;

2) the subspace $(M^+ \cap D(L)) + (M^- \cap D(L))$ is dense in $D(L)$.

In this case, if $i\omega \in \rho(L)$ and $\omega \in \mathbb{R}$, then $i\omega \in \rho(L|_{M^\pm})$.

**Proof.** We set $H_1^\pm = M^\pm \cap D(L)$ and $H_1 = D(L)$ and equip $H_1$ with the graph norm. Without loss of generality we may assume that $\omega = 0$; if not we consider the operator $L - i\omega I$. By assumption, $L(H_1^\pm) \subset M^\pm$. It follows from condition 1) that $M^\pm$ are closed subspaces of the space $H$. It readily follows from the definition that $H_1^\pm$ are closed subspaces of the space $H_1$. In this case, since $0 \in \rho(L)$, it follows that $L(H_1^\pm)$ are closed subspaces of the space $H$. By 2), $L(H_1^+ + H_1^-)$ is dense in $H$. Therefore, the subspaces $L(H_1^\pm)$ are dense in $M^\pm$. Since $L(H_1^\pm)$ are closed, it follows that $L(H_1^\pm) = M^\pm$. It follows from the inverse mapping theorem that $0 \in \rho(L|_{M^\pm})$.

**Lemma 2.5.** Let $H$ be a Kreĭn space and let $L : H \to H$ be an $m$-$J$-dissipative operator such that conditions (2.4) and (2.16) are satisfied and there are $L$-invariant subspaces $M^\pm$, one maximal nonnegative and one maximal nonpositive, for which $C^\pm \subset \rho(L|_{M^\pm})$, $H = M^+ + M^-$, where the sum is direct, and

$$C^\pm = \{z \in \mathbb{C} : \pm \text{Re } z > 0\}.$$
Let $F_{-1}^\pm$ denote the completions of $M^\pm$ with respect to the norm of the space $F_{-1}$. Then there is a constant $c > 0$ such that
\[
\|(L - zI)^{-1}u\|_{F_{-1}} \leq \frac{c\|u\|_{F_{-1}}}{1 + |z|}, \quad \forall z \in \overline{C^\pm}, \quad \forall u \in F_{-1}^\pm, \tag{2.29}
\]
where the symbol $L$ stands for the extension of $L$ to an operator of class $L(F_1, F_{-1})$. The operators $\pm L|_{F_{-1}^\pm}$, viewed as operators from $F_{-1}^\pm$ to $F_{-1}^\pm$ with domain of definition $F_{-1}^\pm = F_1 \cap M^\pm$, are infinitesimal generators of analytic semigroups.

Proof. By Lemma 2.3, there exists $\omega_0 > 0$ such that $L_{\omega_0} \in \rho(L)$. Since $i \mathbb{R} \subset \rho(L|_{M^\pm})$, we see that $D(L) = H_1^+ + H_1^-$, where $H_1^\pm = D(L) \cap M^\pm$. It follows from the equation obtained in Lemma 2.3, the bound (2.29) holds for any $z$ on the imaginary axis. To prove (2.29) holds, we can simply repeat the arguments of Lemma 2.3 for the operators $L|_{M^\pm}$. When obtaining the bound, we use the inequalities $\text{Re} z [u, u] \geq 0$ for any $u \in M^\pm$ and $z$ such that $\pm \text{Re} z \geq 0$. The last statement of the lemma follows from the properties of analytic semigroups (see [21], Ch. 2, Theorem 4.6 (e)).

§ 3. Main results

Most of the results in the following theorem were obtained in [15] (see also [16], Ch. 1, Theorem 4.1).

Theorem 1. Let $L: H \to H$ be an $m$-$J$-dissipative operator in a Krein space $H$ such that $i \mathbb{R} \in \rho(L)$ and (2.9) and (2.19) hold. Then there are $L$-invariant subspaces $H^\pm$, one maximal nonnegative and one maximal nonpositive. Moreover, the whole of $H$ can be represented as the direct sum $H = H^+ \oplus H^-$, $\sigma(L|_{H^\pm}) \subset \mathbb{C}^\pm$, and the operators $\pm L|_{H^\pm}$ are infinitesimal generators of analytic semigroups.

The final statement in the theorem, which claims that $L|_{H^\pm}$ are infinitesimal generators of analytic semigroups, in not contained in the corresponding theorem in [15]. However, this is a simple consequence of the cited theorem and was in fact obtained in [24] (see [24], Corollary 3.3).

In the following theorem, we generalize Theorem 4.2 of [16], Ch. 1; in contrast to [16], we do not assume that the operator $L$ is uniformly $J$-dissipative and we also obtain some new statements.

Theorem 2. Let $L: H \to H$ be an $m$-$J$-dissipative operator satisfying the condition $i \mathbb{R} \subset \rho(L)$ and also (2.4) and (2.9). Then there are $L$-invariant subspaces $H^\pm$, one maximal nonnegative and one maximal nonpositive. Moreover, the whole space $H$ can be represented as the direct sum $H = H^+ \oplus H^-$, $\sigma(L|_{H^\pm}) \subset \mathbb{C}^\pm$, and the operators $\pm L|_{H^\pm}$ are infinitesimal generators of analytic semigroups. If, in addition, $L$ is strictly $J$-dissipative or uniformly $J$-dissipative, then the corresponding subspaces $H^+$ and $H^-$ can be chosen to be positive and negative or uniformly positive and uniformly negative, respectively.

Proof. By Lemma 2.3, the bound (2.19) holds (we note that (2.16) holds by equation (2.10) in Lemma 2.1). The operators $P^\pm$ of parallel projection onto $H^\pm$
We will find a sequence \( v \) constructed as follows. As is well known, \( \delta > 0 \) exists such that

\[
S = \mathbb{C} \setminus (S^+ \cup S^-) \subset \rho(L),
\]

\[
S^+ = \left\{ \lambda : |\arg \lambda | < \frac{\pi}{2} - \delta, |z| > \delta \right\}, \quad S^- = \left\{ \lambda : |\arg \lambda | > \frac{\pi}{2} + \delta, |z| > \delta \right\},
\]
and the following bound for the resolvent holds for any ray \( \arg \lambda = \theta < S \):

\[
\|(L - \lambda)^{-1} f\| \leq c_1 \|f\|(1 + |\lambda|)^{-1}.
\]

We have

\[
P^\pm f = - \frac{1}{2\pi i} \int_{\Gamma^\pm} \frac{L(L + \lambda)^{-1}}{\lambda} f \, dz,
\]
where the integration over \( \Gamma^\pm \) is carried out in the positive direction with respect to the domains \( S^\pm \). It can readily be seen that the integrals converge in norm for \( f \in D(L) \), and thus the quantities \( P^\pm f \) are well defined. Using Lemma 2.1, we can show that the operators \( P^\pm \) admit extensions to operators of class \( L(H) \). Indeed, it follows from Theorem 1.15.2 (see also § 1.15.4) in \([18]\) and from the reiteration theorem (see \([18]\)) that

\[
(H_1, H)_{\theta, 2} = (H_1, H_{-1})_{\theta/2, 2}, \quad (H, H_{-1})_{1 - \theta, 2} = (H_1, H_{-1})_{1 - \theta/2, 2}, \quad H = (H_1, H_{-1})_{1/2, 2} = ((H_1, H_{-1})_{\theta/2, 2}, (H_1, H_{-1})_{1 - \theta/2, 2})_{1/2, 2}.
\]

The first half of the last equation follows from (2.9). At the same time, the projections \( P^\pm \) admit extensions to continuous mappings of classes \( L((H_1, H)_{\theta, 2}) \) and \( L((H, H_{-1})_{1 - \theta, 2}) \) (see \([24]\), § 3). In this case, the last equation in (3.2) ensures the result holds. Moreover, by definition, we see that \( (P^+ + P^-)f = f \) for any \( f \in D(L) \), and thus for any \( f \in H \). Thus, the space \( H \) can be represented as the direct sum \( H = H^+ + H^- \), where \( H^+ = \{ u \in H : P^+ u = u \} \).

We claim that \( H^+ \) is nonnegative and \( H^- \) is nonpositive. We prove this by analogy with the corresponding proofs in \([15]\) and \([16]\). For example, consider \( H^+ \). Since \( D(L_k) \) is dense in \( H \) for any \( k > 0 \) (see \([18]\), § 1.14.1) and the operator \( P^+ \) is bounded, it follows that for any \( u \in H^+ \) there is a sequence \( u_n \in D(L^2) \cap H^+ \) such that \( \|u_n - u\| \to 0 \) as \( n \to \infty \). Indeed, the operators \( P^\pm \) and \( L \) commute, which means that, if \( f \in D(L^k) \), \( k \geq 1 \), then \( P^\pm f \in D(L^k) \) and \( L^k P^\pm f = P^\pm L^k f \). We will find a sequence \( v_n \in D(L^2) \) such that \( \|v_n - u\| \to 0 \) as \( n \to \infty \). We write \( u_n = P^+ v_n \in D(L^2) \cap H^+ \). Then \( \|u_n - u\| = \|P^+(v_n - u)\| \to 0 \) as \( n \to \infty \). We define an operator

\[
P u = -\frac{1}{2\pi i} \int_{\Gamma^+} e^{-\lambda t} \frac{L(L + \lambda)^{-1}}{\lambda} u \, d\lambda, \quad t > 0.
\]

Take \( v_n(t) = Pu_n \). Using (3.3), we can readily see that

\[
\|v_n\|_{L_2(0, \infty; H)} \leq \frac{1}{2\pi} \int_{\Gamma^+} \frac{c\|u_n\|}{|\lambda| \sqrt{\text{Re} \lambda}} \, |d\lambda| \leq c_1 \|u_n\|,
\]

(see \([15]-[17]\)) such that \( H = P^+ H + P^- H \) and \( P^+ P^- = P^- P^+ = 0 \) can be constructed as follows. As is well known, \( \delta > 0 \) exists such that

\[
S = \mathbb{C} \setminus (S^+ \cup S^-) \subset \rho(L),
\]

\[
S^+ = \left\{ \lambda : |\arg \lambda | < \frac{\pi}{2} - \delta, |z| > \delta \right\}, \quad S^- = \left\{ \lambda : |\arg \lambda | > \frac{\pi}{2} + \delta, |z| > \delta \right\},
\]
where the constant $c$ is taken from the inequality $\|L(L + \lambda)^{-1}u_n\| \leq c\|u_n\|$ and $\Gamma^\pm = \partial S^\pm$. As the integrals obtained from (3.3) by formal differentiation with respect to $t$ and a formal application of the operator $L$ converge in norm (this convergence can be proved in a similar way), we can say that $v_n(t) \in L_2(0, \infty; F_1)$, and the generalized derivative, $v'(t)$, in the sense of the theory of distributions has the property $v_n'(t) \in L_2(0, \infty; H)$. This implies that $v_n(t) \in C([0, \infty); H)$ (possibly after modifying $v_n(t)$ on a set of measure zero), that is, $v_n(t)$ is a continuous function with values in $H$. It can readily be seen that

\begin{align*}
v_n(0) &= u_n, \quad (3.5) \\
Sv_n &= v_n' - Lv_n = 0. \quad (3.6)
\end{align*}

Using equations (3.5) and (3.6) and integrating by parts, we obtain

$$0 = \text{Re} \int_0^\infty [Sv_n(t), v_n(t)] \, dt = -[v_n(0), v_n(0)]_0 - \int_0^\infty \text{Re} [Lv_n, v_n](\tau) \, d\tau. \quad (3.7)$$

This implies that

$$[u_n, u_n]_0 = -\int_0^\infty \text{Re}[Lv_n, v_n](\tau) \, d\tau, \quad (3.8)$$

that is, $[u_n, u_n] \geq 0$. Passing to the limit with respect to $n$ in this inequality, we obtain $[u, u] \geq 0$, that is, $H^+$ is a nonnegative subspace. We can prove that $H^-$ is nonpositive similarly.

We claim that $H^+$ is maximal. Suppose the contrary, that is, let there be a nonnegative subspace $M$ such that $H^+ \subset M$ and $H^+ \neq M$. We take an element $\varphi \in M \setminus H^+$. Then $\varphi + \psi \in M$ and $[\varphi + \psi, \varphi + \psi]_0 \geq 0$ for any $\psi \in H^+$. We represent $\varphi$ in the form $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in H^+$ and $\varphi_2 \in H^-$. Taking $\psi = -\varphi_1$, we see that $\varphi_2 \in M$ and $[\varphi_2, \varphi_2]_0 \geq 0$. However, $\varphi_2 \in H^-$. Then $[\varphi_2, \varphi_2]_0 = 0$. Since $[\varphi, \varphi] \geq 0$ for any $\varphi \in M$ and $-[\varphi, \varphi] \geq 0$ for any $\varphi \in H^-$, the Cauchy-Bunyakovskii inequality

$$|[\varphi, \psi]| \leq \|[\varphi, \varphi]^{1/2}[\psi, \psi]^{1/2} \quad (3.9)$$

holds for any $\varphi, \psi \in M$ and for any $\varphi, \psi \in H^-$. In turn, it follows from this inequality that $[\varphi_2, \psi]_0 = 0$ for any $\psi \in H^+ \subset M$ and for any $\psi \in H^-$. This implies that $\varphi_2 = 0$. Similarly we can prove that $H^-$ is maximal.

We claim that, if the operator $L$ is strictly (uniformly) $J$-dissipative, then the subspace $H^\pm$ is strictly (uniformly, respectively) definite. Let $L$ be uniformly $J$-dissipative. Since $L$ is uniformly $J$-dissipative, we can use the equivalent norm $\|u\|_{F_1}^2 = -\text{Re}[Lu, u]$ for the norm in $F_1$. Then (3.6) and (3.8) imply that

$$[u_n, u_n] \geq \delta_0 (\|v_n'\|^2_{L_2(0, \infty; F_{-1})} + \|v_n\|^2_{L_2(0, \infty; F_1)}), \quad (3.10)$$

where $\delta_0$ stands for a positive constant independent of $n$. Applying (3.10) to the difference $u_n - u_m$, because $u_n$ is a Cauchy sequence in $H$, it follows that $v_n$ is a Cauchy sequence in $L_2(0, \infty; F_1)$, and hence it converges to some function $v(t) \in L_2(0, \infty; F_1)$, where $v_n'(t) \rightarrow v'(t)$ in $L_2(0, \infty; F_{-1})$. By the trace theorem
(see [18], Theorem 1.8.3) and using (2.10) we have \( v_n(0) \to v(0) \) in \( H \), and there is a constant \( \delta > 0 \) for which
\[
\|v'_n\|^2_{L^2(0,\infty;F^{-1})} + \|v_n\|^2_{L^2(0,\infty;F^{-1})} \geq \delta \|u_n\|^2_H.
\]
This inequality and (3.10) imply that \([u_n, u_n] \geq \delta \|u_n\|^2_H/2\). Now passing to the limit with respect to \( n \), we obtain the inequality \([u, u] \geq \delta \|u\|^2_H/2\), which holds for any \( u \in H^+ \) and which implies that the subspace \( H^+ \) is uniformly positive. Similarly we can prove that the subspace \( H^- \) is uniformly negative.

Let the operator \( L \) be strictly \( J \)-dissipative. We claim that the subspaces \( H^\pm \) thus obtained are definite. For example, consider the subspace \( H^+ \). Suppose the contrary. Let \( H^+ \) be degenerate, that is, \( H^+_0 = H^+ \cap (H^+)_{\perp} \neq \{0\} \), where the symbol \( [\perp] \) stands for the orthogonal complement with respect to the indefinite metric. Then \([\varphi, \varphi] = 0\) for any \( \varphi \in H^+_0 \). It follows from the inequality (3.9) that
\[
[\varphi, \psi] = 0
\]
for any \( \varphi \in H^+_0 \) and \( \psi \in H^+ \). Take \( v \in H^+_0 \). By construction, the subspaces \( H^\pm \) satisfy the conditions of Lemma 2.4. Using Lemma 2.4, we see that \( L^{-1}v \in H^+ \), and thus \([L^{-1}v, v]\) = 0 for any \( v \in H^+_0 \). This, together with the dissipativity condition, implies that \( v \in \ker(L^{-1} + (L^{-1})^c) = \ker(L^{-1} + (L^c)^{-1}) \) for these \( v \) (see [1], Ch. 2, §1). Then for \( u = L^{-1}v \) we have \( u = -(L^c)^{-1}Lu \), and thus \( u \in D(L) \cap D(L^c) \). Moreover, \( Lu + L^cu = v - v = 0 \). Hence, \( \text{Re}[Lu, u] = 0 \), which contradicts the condition that \( L \) is strictly dissipative.

Now we turn to the necessary conditions. In essence, we shall prove that the conditions of Theorem 2 are also necessary for the statement of the theorem to hold and present some other statements.

**Theorem 3.** Let \( L: H \to H \) be an \( m \)-\( J \)-dissipative operator such that
1) there is an \( \omega \in \mathbb{R}: \omega \not\in \rho(L) \);
2) there are \( L \)-invariant subspaces \( M^\pm \), one maximal uniformly positive and one maximal uniformly negative, such that \( H = M^+ + M^- \), where the sum is direct;
3) the subspace \((M^+ \cap D(L)) + (M^- \cap D(L))\) is dense in \( D(L) \).

Then equation (2.9) holds.

**Proof.** Without loss of generality we may assume that \( \omega = 0 \). We introduce the operators \( P^\pm \) of parallel projection onto \( M^\pm \), respectively, that correspond to the decomposition \( H = M^+ + M^- \). Thus, \( P^+ + P^- = I \), \( P^+P^- = P^-P^+ = 0 \), and \( P^\pm u = u \) for any \( u \in M^\pm \). By Lemma 2.4, \( L^{-1} \varphi \in M^\pm \cap H_1 \) for any \( \varphi \in M^\pm \), where \( H_1 = D(L) \). Therefore,
\[
P^\pm L^{-1}u = L^{-1}P^\pm u \tag{3.11}
\]
for any \( u \in H \). This implies that \( P^\pm|_{H_1} \in L(H_1) \), the equations \( P^\pm Lu = LP^\pm u \) hold for any \( u \in H_1 \), and \( H_1 = H^+_1 + H^-_1 \) for \( H^\pm_1 = R(P^\pm) \). It also follows from (3.11) that the operators \( P^\pm \) admit extensions to operators of class \( L(H^-_1) \) and, using the equation \( P^+ + P^- = I \), we see as above that \( H^-_1 = H^+_1 + H^-_1 \) and \( H^\pm_1 = R(P^\pm) \). Then
\[
P^\pm \in L(H_{1-2\theta}) \quad \forall \theta \in (0, 1), \quad H_{1-2\theta} = (H_1, H^-_1)_{\theta, 2},
\]
and it follows from Theorem 1.17.1 in [18] that
\[ H_{1-2\theta}^\pm = (H_1^\pm, H_{-1}^\pm)_{\theta, 2} = \{ u \in H_{1-2\theta} : P^\pm u = u \} = P^\pm H_{1-2\theta}. \]

In particular, we also have
\[ H_{1-2\theta} = P^+ H_{1-2\theta} + P^- H_{1-2\theta} = H_{1-2\theta}^+ + H_{1-2\theta}^- \quad \text{(the sum is direct).} \quad (3.12) \]

Let \([\cdot, \cdot]\) stand for the indefinite metric in \(H\). Since \(M^\pm\) are uniformly definite subspaces, it follows that the expressions \((u, v)_\pm = \pm[u, v]\) are equivalent inner products on \(M^\pm\), respectively. We shall consider the operator \(L|_{M^+} : M^+ \to M^+\). The operator \(L|_{M^+}\) is \(m\)-dissipative with respect to this new inner product. By Proposition 2, \(H_0^+ = (H_1^+, H_{-1}^+)_{1/2, 2} = M^+\). Similarly we can show that \(H_0^- = (H_1^-, H_{-1}^-)_{1/2, 2} = M^-\). The statement now follows from (3.12), taking \(\theta = 1/2\).

**Theorem 4.** Let \(L : H \to H\) be an \(m\)-\(J\)-dissipative operator such that condition (2.4) holds and there is an \(L\)-invariant maximal uniformly positive (or uniformly negative) subspace \(M\) such that \(i\mathbb{R} \cap \rho(L|_M) \cap \rho(L) \neq \emptyset\). Then equation (2.9) holds.

**Proof.** We write \(H_1^+ = H_1 \cap M, H_1 = D(L),\) and \(F_1^+ = F_1 \cap M\) (the spaces \(F_1\) and \(F_{-1}\) were constructed before stating Proposition 2). To be definite, we assume that \(M\) is uniformly positive and \(0 \in \rho(L|_M) \cap \rho(L)\). Since \(M\) is maximal and uniformly positive, it follows that \(M\) is projectionally complete, that is, the whole of \(H\) can be represented in the form of a direct sum \(H = M + N, N = M^{[\perp]}\), where \(N\) is uniformly negative (see [1], Ch. 1, Corollary 7.17), and the corresponding projection \(P\) onto \(M\) in parallel to \(N\) is \(J\)-self-adjoint. We introduce an operator \(L_0 = L - PJ P - (I - P)J(I - P)\) by setting \(D(L_0) = D(L)\). Then
\[ -\text{Re}[L_0 u, u] = -\text{Re}[Lu, u] + \|Pu\|^2 + \|\!\!\|I - P\!\!\|\!\!u\|^2 \geq \|u\|^2_{F_1}/2. \]

Thus, the quantity \((-\text{Re}[L_0 u, u])\) is the square of an equivalent norm in \(F_1\). The operator \(L\) satisfies (2.4), and hence, by Lemma 2.2, it admits the extension \(\tilde{L}\) to an operator of class \(L(F_1, F_{-1})\). We write
\[ \tilde{L}_0 = \tilde{L} - PJ P - (I - P)J(I - P) \in L(F_1, F_{-1}). \]

By construction, the operator \(L_0\) also satisfies (2.4). As in Lemma 2.2, we can prove that \(\tilde{L}_0\) is an isomorphism from \(F_1\) onto \(F_{-1}\). We claim that \(0 \notin \rho(L_0)\). Consider the restriction of the operator \(\tilde{L}_0\) to \(H_0 = \{ u \in F_1 : \tilde{L}_0 u \in H \}\). On the one hand, \(D(L) \subset D_0\). Indeed, if \(u \in D(L)\), then
\[ u \in F_1, \quad Lu \in H, \quad PJ Pu, (I - P)J(I - P)u \in H, \]
and thus \(u \in D_0\). Let \(u \in D_0\). Then \(u \in F_1\), and therefore
\[ PJ Pu, (I - P)J(I - P)u \in H, \]
which implies that \(\tilde{L}u \in H\). Thus,
\[ D_0 \subset \{ u \in F_1 : \tilde{L}u \in H \} = D(L) \]
(see equation (2.12)). Hence, $D_0 = D(L)$. Since $\widetilde{L}_0$ is an isomorphism from $F_1$ onto $F_{-1}$, it follows immediately from the definition of the class $D_0$ that $0 \in \rho(L_0)$.

In what follows we write $L$ and $L_0$ instead of $\widetilde{L}$ and $\widetilde{L}_0$, respectively. We set $F_1^+ = F_1 \cap M$. Since $0 \in \rho(L|_M) \cap \rho(L)$, it follows that $L$ takes $D(L) \cap M$ onto $M$.

For $u \in D(L) \cap M$ we have

$$\|u\|_{F_1} \leq \sqrt{2(\text{Re}[-L_0 u, u])}^{1/2} \leq 2 \sup_{v \in F_1^+} \frac{|[L_0 u, v]|}{\|v\|_{F_1}} \leq 2 \sup_{v \in F_1^+} \frac{|[L_0 u, v]|}{\|v\|_{F_1}} = 2\|L_0 u\|_{F_{-1}} \leq 2c\|u\|_{F_1},$$

where $c$ is a constant independent of $u$. This implies that

$$\sup_{v \in F_1^+} \frac{|[L_0 u, v]|}{\|v\|_{F_1}} \leq \sup_{v \in F_1} \frac{|[L_0 u, v]|}{\|v\|_{F_1}} \leq 2c \sup_{v \in F_1^+} \frac{|[L_0 u, v]|}{\|v\|_{F_1}} \quad (3.13)$$

for any $u \in D(L) \cap M$. In particular, this gives the inequality

$$\sup_{v \in F_1^+} \frac{|[u, v]|}{\|v\|_{F_1}} \leq \sup_{v \in F_1} \frac{|[u, v]|}{\|v\|_{F_1}} \leq 2c \sup_{v \in F_1^+} \frac{|[u, v]|}{\|v\|_{F_1}} \quad \forall u \in M. \quad (3.14)$$

We denote the completion of the subspace $M$ with respect to the norm $\sup_{v \in F_1^+} \|[u, v]/\|v\|_{F_1}$ by $F_{-1}$. By (3.14), this norm is equivalent to the ordinary norm on the space $F_{-1}$, and thus we can identify $F_{-1}$ with a closed subspace of $F_{-1}$. The operator $L_0$ is an isomorphism of $F_1^+$ onto $F_{-1}$. Since $M$ is uniformly positive, the expression $\langle \cdot, \cdot \rangle$ defines an inner product on $M$ which is equivalent to the original one. Therefore, the space $F_{-1}$ coincides with the negative space constructed from the pair $F_1^+$, $M$, and, by Proposition 3,

$$(F_1^+, F_{-1})_{1/2, 2} = M.$$  

We claim that the subspace $N$ is invariant with respect to the operator $L^c$ which is $m$-$J$-dissipative like $L$, and $0 \in \rho(L^c)$ (see [1], Ch. 2, §1, Theorem 1.16). By Lemma 2.2, $D(L^c)$ is densely embedded in $F_1$. Therefore, $L^c$, and $L$ itself, admits an extension to a continuous mapping from $F_1$ into $F_{-1}$. By Proposition 1.11 in [1], Ch. 2, §1, $L^c(D(L^c) \cap N) \subset N$. Since $0 \in \rho(L|_M)$, it follows that $L^{-1}M \subset M$, and again we see that $(L^c)^{-1}N \subset N$, because $(L^{-1}c) = (L^c)^{-1}$ (see [1], Ch. 2, §1, Proposition 1.6). These two embeddings, and the fact that $0 \in \rho(L^c)$, imply that $0 \in \rho(L^c|_N)$. The fact that $D(L^c) \cap N$ is dense in $N$ is obvious. Indeed, suppose that this is not the case. Then there is an element $\varphi \in D(L^c) \cap N$ such that $\langle \varphi, \psi \rangle = 0$ for any $\psi \in N$. Let us take $\psi = (L^c)^{-1}v$ for some $v \in N$. Then $0 = \langle \varphi, (L^c)^{-1}v \rangle = [L^{-1}\varphi, v]$, which implies that $L^{-1}\varphi \in D(L) \cap M$, and thus $\varphi \in M$. In other words, $\varphi \in M \cap N = \{0\}$. We set $F_1^- = F_1 \cap N$ and let $F_{-1}^-$ denote the completion of $N$ with respect to the norm $\sup_{v \in F_1^-} \|[u, v]/\|v\|_{F_1}$. Repeating the above arguments for the subspace $M$, we conclude that this norm is equivalent to the ordinary norm of the space $F_{-1}$, and thus one can identify $F_{-1}^-$ with a closed subspace of $F_{-1}$. Moreover,

$$(F_1^-, F_{-1})_{1/2, 2} = N.$$
We will show that
\[ F_i = F_i^+ + F_i^-, \quad P \in L(F_i), \quad i = 1, -1, \quad (3.15) \]
where the sum is direct. Let \( i = -1 \) and \( u = u^+ + u^- \) for \( u^+ \in M \) and \( u^- \in N \). By (3.14),
\[
\|u^+\|_{F_{-1}} \leq 2c \sup_{v \in F_i^+} \frac{[u^+, v]}{\|v\|_{F_i}} = 2c \sup_{v \in F_i^+} \frac{[u^+ + u^-, v]}{\|v\|_{F_i}}
\leq 2c \sup_{v \in F_i^+} \frac{[u^+ + u^-, v]}{\|v\|_{F_i}} \leq 2c\|u\|_{F_{-1}}.
\]
We can obtain the inequality \( \|u^-\|_{F_{-1}} \leq 2c\|u\|_{F_{-1}} \) similarly; therefore,
\[
\|u\|_{F_{-1}} \geq \delta(\|u^+\|_{F_{-1}} + \|u^-\|_{F_{-1}}).
\]
This inequality implies that the projections \( P \) and \( (I - P) \) admit extensions to operators of class \( L(F_{-1}) \), and thus (3.15) holds for \( i = -1 \). We claim that \( P|_{F_1} \in L(F_1) \).
Indeed, for \( v \in M \) and \( u \in F_1 \) we have
\[
[v, Pu] = [v, u] \leq \|v\|_{F_{-1}} \|u\|_{F_1}.
\]
Hence, the expression \([v, Pu]\) admits an extension to a continuous linear functional on \( F_{-1}^+ \), and therefore there is a \( u^+ \in F_1^+ \) such that \([v, Pu] = [v, u^+]\) for any \( v \in M \).
This implies that \( Pu = u^+ \in F_1^+ \), and
\[
\|Pu\|_{F_1} \leq c_1 \sup_{v \in F_{-1}^+} \frac{[v, Pu]}{\|v\|_{F_{-1}}} \leq c_1 \sup_{v \in F_{-1}^+} \frac{[v, u]}{\|v\|_{F_{-1}}} \leq c_1\|u\|_{F_1}.
\]
Thus, \( P, (I - P) \in L(F_1) \), and therefore the equation (3.15) holds for \( i = 1 \). Theorem 1.17.1 in [18] implies that
\[
(F_1, F_{-1})_{1/2, 2} = (F_1^+, F_{-1}^+)_{1/2, 2} + (F_1^-, F_{-1}^-)_{1/2, 2} = M + N = H.
\]
It follows from Lemma 2.1 that we also have \((H_1, H_{-1})_{1/2, 2} = H\).

**Theorem 5.** Let \( L : H \to H \) be an \( m \)-J-dissipative operator such that \( i \mathbb{R} \subset \rho(L) \), conditions (2.4) and (2.16) hold, and there are \( L \)-invariant subspaces \( M^\pm \) of \( H \), one maximal nonnegative and one maximal nonpositive, for which
\[
C^\pm \subset \rho(L|_{M^\pm}), \quad H = M^+ + M^-,
\]
where the sum is direct. Then (2.9) holds. If \( L \) is strictly \( J \)-dissipative or uniformly \( J \)-dissipative, then the subspaces \( M^+ \) and \( M^- \) are definite or uniformly definite, respectively.

**Proof.** As in Lemma 2.5, we define the subspaces \( F_1^\pm \subset F_1 \) and \( F_{-1}^\pm \subset F_{-1} \). By Lemma 2.5, the operators \( \pm L|_{F_{-1}^\pm} : F_{-1}^\pm \to F_{-1}^\pm \) are infinitesimal generators of analytic semigroups. For example, consider the operator \( L^+ = L|_{F_1^+} \) and take some
$T > 0$. As follows from the trace theorem (see [18], Theorem 1.8.3), for a given element $u_0 \in (F^+_1, F^-_{-1})_{1/2,2}$ there is a function $v(t)$ such that $v_t \in L_2(0,T; F^+_{-1})$, $v \in L_2(0,T; F^+_1)$, and $v(0) = u_0$. By well-known results (see, for example, Corollary 1.7 in [25]), the Cauchy problem

$$u_t - L^+ u = f, \quad u(0) = 0$$  \tag{3.16}

has the maximal regularity property, that is, for any function $f \in L_2(0,T; F^+_{-1})$ there is a unique solution of the problem such that $u_t, L^+ u \in L_2(0,T; F^+_{-1})$. Using the change of variable $u = v(t) + V(t)$, we reduce the problem

$$u_t - L^+ u = 0, \quad u|_{t=0} = u_0$$  \tag{3.17}

to a problem of the form (3.16), and thus problem (3.17) has a unique solution such that $u_t, L^+ u \in L_2(0,T; F^+_{-1})$ for any $u_0 \in (F^+_1, F^-_{-1})_{1/2,2}$. We take $u_0 \in F^+_1 \subset (F^+_1, F^-_{-1})_{1/2,2}$. As is well known, there is a $\delta > 0$ such that

$$S_\delta = \left\{ \lambda : |\arg \lambda| \leq \frac{\pi}{2} + \delta \right\} \subset \rho(L^+)$$

and, on any ray $\arg \lambda = \theta \subset S_\delta$, the resolvent admits the bound

$$\|(L^+ - \lambda I)^{-1} f\|_{F^-_{-1}} \leq c_1 \|f\|_{F^-_{-1}} (1 + |\lambda|)^{-1}. \tag{3.18}$$

Then the solution of problem (3.17) can be represented in the form (see, for example, [25])

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (L - \lambda I)^{-1} u_0 \, dz,$$

where the integration along $\Gamma = \partial S^\delta$, $S^\delta = S_\delta \cup \{z : |z| < \varepsilon\}$, is carried out in the positive direction with respect to the domain $S^\delta$, and the parameter $\varepsilon$ is so small that $\{z : |z| < \varepsilon\} \subset \rho(L^+)$. Using the bound for the resolvent in Lemma 2.5 and estimating the integral as in the proof of Theorem 2, we obtain the bound

$$\|u(t)\|_{L_2(0,T; F^-_{-1})} \leq c \|u_0\|. \tag{3.19}$$

It follows from (3.17) that $[u_t, u](t) - [Lu, u] = 0$. Integrating with respect to $t$ from 0 to $T$ and using the fact that the expression $[u, u]$ is nonnegative, we obtain

$$-2 \int_0^T \text{Re}[Lu, u](t) \, dt \leq [u_0, u_0]. \tag{3.20}$$

Using inequality (2.16) and the bounds (3.19) and (3.20), we obtain

$$\int_0^T \|u(t)\|^2 \, dt \leq c \left( -\int_0^T \text{Re}[Lu, u](t) \, dt + \|u(t)\|^2_{L_2(0,T; F^-_{-1})} \right) \leq c_1 \|u_0\|^2. \tag{3.21}$$

Using (3.20) again, we arrive at the inequality

$$\|u(t)\|^2_{L_2(0,T; F^-_{1})} \leq c_2 \|u_0\|^2. \tag{3.22}$$
Then the definition of the norm in the interpolation space gives the bounds
\[
\|u_t\|_{L^2(0,T;F_{-1})}^2 \leq c_3\|u_0\|^2.
\]
(3.23)
Then (3.22) and (3.23) give
\[
\|u_t\|_{L^2(0,T;F_{-1})}^2 + \|u(t)\|_{L^2(0,T;F_1)}^2 \leq c_4\|u_0\|^2.
\]
(3.24)
However, by the trace theorem (see [18], Theorem 1.8.3), the left-hand side of this inequality is bounded below by \(\delta\|u_0\|_{(F_1^+, F_{-1}^+)_{1/2,2}}^2\), where \(\delta\) is some positive constant independent of \(u_0\) and \(u\). The inequality thus obtained implies an embedding \(M^+ \subset (F_1^+, F_{-1}^+)_{1/2,2}\) and the bound
\[
\|u_0\|_{(F_1^+, F_{-1}^+)_{1/2,2}} \leq c_5\|u_0\|.
\]
(3.25)
Using similar reasoning, we can see that there is an embedding \(M^- \subset (F_1^-, F_{-1}^-)_{1/2,2}\), together with the bound
\[
\|v_0\|_{(F_1^-, F_{-1}^-)_{1/2,2}} \leq c_6\|v_0\| \quad \forall v_0 \in M^-.
\]
(3.26)
In this case, instead of the auxiliary problem (3.17), we use the problem
\[
\dot{u}_t + L^- u = 0, \quad u|_{t=0} = v_0 \in F_1^-; \quad L^- = L|_{M^-}. \tag{3.27}
\]
Let \(u_0 \in F_1^+\) and \(v_0 \in F_1^-\). Since \(F_1^\pm\) and \(F_{-1}^\pm\) are closed subspaces of \(F_1\) and \(F_{-1}\) with the same norms, it follows that
\[
K(t, u_0, F_1, F_{-1}) \leq K(t, u_0, F_1^+, F_{-1}^-), \quad K(t, v_0, F_1, F_{-1}) \leq K(t, u_0, F_1^-, F_{-1}^-).
\]
Then the definition of the norm in the interpolation space gives the bounds
\[
\|u_0\|_{(F_1^-, F_{-1}^-)_{1/2,2}} \leq \|u_0\|_{(F_1^+, F_{-1}^+)_{1/2,2}}, \quad \|v_0\|_{(F_1^-, F_{-1}^-)_{1/2,2}} \leq \|v_0\|_{(F_1^+, F_{-1}^+)_{1/2,2}}. \tag{3.28}
\]
These inequalities, together with the triangle inequality and the bounds (3.25) and (3.26), imply that
\[
\|u_0 + v_0\|_{(F_1^-, F_{-1}^-)_{1/2,2}} \leq \|u_0\|_{(F_1^+, F_{-1}^+)_{1/2,2}} + \|v_0\|_{(F_1^-, F_{-1}^-)_{1/2,2}} \\
\leq c(\|u_0\| + \|v_0\|) \leq c_1\|u_0 + v_0\|_H,
\]
where \(c\) and \(c_1\) are positive constants independent of \(u_0\) and \(v_0\). The last inequality follows from the fact that the direct sum of the spaces \(M^+\) and \(M^-\) is the whole of \(H\). By the last inequality, \(H \subset (F_1, F_{-1})_{1/2,2}\). This inclusion implies the equation \(H = (F_1, F_{-1})_{1/2,2}\) (see [16], Ch. 1, Lemma 3.17), and hence (2.9) holds.

We claim that the subspace \(M^+\) is positive if \(L\) is strictly \(J\)-dissipative and uniformly positive if \(L\) is uniformly \(J\)-dissipative. Analogous statements hold for the subspace \(M^-\).

We first assume that \(L\) is strictly \(J\)-dissipative. Suppose the contrary; let \(M^+\) be degenerate, that is, let there be a nonzero element \(\psi \in M^+ \) for which \([\psi, \psi] = 0\). It follows from the Cauchy-Bunyakovskii inequality for the form \([\cdot, \cdot]\) on \(M^+\) that
ψ ∈ \(M^+ \cap (M^+)^{[1]}\), that is, \([\psi, v] = 0\) for any \(v \in M^+\). Since \(0 \in \rho(L|_{M^+})\), it follows that \([L^{-1}\psi, \psi] = 0\). On the other hand, \(\text{Re}[Lu, u] > 0\) for any \(u \in D(L)\). In particular, this yields \(\text{Re}[L^{-1}u, u] > 0\) for any \(u \in H\), and thus \(\text{Re}[L^{-1}\psi, \psi] > 0\), a contradiction. We assume now that \(L\) is uniformly \(J\)-dissipative. Take \(u_0 \in F_1^+\) and consider the solution \(u(t)\) of problem (3.17). This solution satisfies (3.20). Since \(L\) is uniformly \(J\)-dissipative, (3.20) implies that

\[
\|u\|^2_{L^2(0,T;F_1)} \leq c[u_0,u_0],
\]

where \(c\) is a constant independent of \(u\). Then the equation (3.17) implies that

\[
\|u_t\|^2_{L^2(0,T;F_{-1})} \leq c_1 \|u\|^2_{L^2(0,T;F_1)} \leq c_2[u_0,u_0].
\]

The last two inequalities, together with the trace theorem (see [18], Theorem 1.8.3), yield

\[
\|u_0\|^2_{(F_1,F_{-1})_{1/2,2}} \leq c_3[u_0,u_0].
\]

However, \((F_1,F_{-1})_{1/2,2} = H\), as was proved above. Thus, \(M^+\) is uniformly positive. The proof that \(M^-\) is uniformly negative is similar.

As a consequence of Theorems 2 and 5, we have, for example, the following result.

**Theorem 6.** Let \(L: H \to H\) be a maximal uniformly \(J\)-dissipative operator satisfying condition (2.4). Then, if (2.9) holds, there are \(L\)-invariant subspaces \(M \pm\) of \(H\), one maximal uniformly positive and one maximal uniformly negative, such that

\[
\overline{C} \subset \rho(L|_{M \pm}), \quad H = M^+ + M^-,
\]

where the sum is direct and the operators \(\pm L|_{M \pm}\) are infinitesimal generators of analytic semigroups. If there are \(L\)-invariant subspaces \(M \pm\) of \(H\), one maximal nonnegative and one maximal nonpositive, such that

\[
\overline{C} \subset \rho(L|_{M \pm}), \quad H = M^+ + M^-,
\]

where the sum is direct, then the equation (2.9) holds, \(M^+\) is uniformly positive, and \(M^-\) is uniformly negative.

Thus, in this case equation (2.9) is a necessary and sufficient condition for the existence of the desired subspaces.

**Remark 1.** All the theorems in this section can be sharpened in the following sense. The condition \(i \mathbb{R} \subset \rho(L)\) can be weakened. One may assume that \(i \mathbb{R} \subset \rho(L)\), apart from finitely many isolated eigenvalues of finite multiplicity of the operator \(L\). Using Riesz projections, we can reduce the problem to the case of \(i \mathbb{R} \subset \rho(L)\), which we have already studied above.

Now we shall present some corollaries in the case when the operator \(L: H \to H\) \((H\) stands for a Krein space with canonical symmetry \(J = P^+ - P^-\)) is represented in the form (1.1). We identify the whole of \(H\) equipped with the inner product \((\cdot, \cdot)\).
and the norm $\| \cdot \|$ with the Cartesian product $H^+ \times H^-$ (for $H^\pm = R(P^\pm)$) and the operator $L$ with a matrix operator $L : H^+ \times H^- \to H^+ \times H^-$ of the form

$$L = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

$$A_{11} = P^+LP^+, \quad A_{12} = P^+LP^-, \quad A_{21} = P^-LP^+, \quad A_{22} = P^-LP^-.$$

In this case, the canonical symmetry can be represented in the form $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Let $u = (u^+, u^-) \in H$ and $v = (v^+, v^-) \in H$. We write out the inner product $(u, v)$ in $H$ in the form $(u^+, v^+) + (u^-, v^-)$ for $u^\pm, v^\pm \in H^\pm$; then $(Ju, v) = (u^+, v^+) - (u^-, v^-)$. We will first describe conditions ensuring that the conditions of Theorem 1 hold. We write $L_0 = (\begin{smallmatrix} A_{11} & 0 \\ 0 & A_{22} \end{smallmatrix})$.

**Theorem 7.** Let $L : H \to H$ be an $m$-$J$-dissipative operator on the Krein space $H = H^+ \times H^-$ such that $i \mathbb{R} \subset \rho(L) \cap \rho(L_0)$ and let condition (2.19) hold together with the following condition: $D(L) = D(L_0)$, and there exist $\lambda \in \rho(L)$ and $\mu \in \rho(L_0)$ such that operator $(L - \lambda \mathbb{I})^{-1}(L_0 - \mu \mathbb{I}) : D(L_0) \to D(L)$ admits an extension to an isomorphism taking $H$ onto $H$. Then there are $L$-invariant subspaces $M^\pm$, one maximal nonnegative and one maximal nonpositive. Moreover, the whole of $H$ can be represented in the form of a direct sum $H = M^+ + M^-$, we have $\sigma(\mp L|_{M^\pm}) \subset \mathbb{C}^\pm$, and the operators $\pm L|_{M^\pm}$ are infinitesimal generators of analytic semigroups.

**Proof.** We note that, by hypothesis, the operators

$$A_{11} : H^+ \to H^+ \quad \text{and} \quad -A_{22} : H^- \to H^-$$

are $m$-dissipative. To quote Theorem 1, it suffices to prove that the interpolation equation (2.9) holds. By assumption, we have $D(L) = D(L_0) = H_1 = H_1^+ \times H_1^-$, where $H_1^+ = D(A_{11})$ and $H_1^- = D(A_{22})$. By the last condition in the theorem, there are constants $c_1$ and $c_2 > 0$ such that

$$c_1 \|(L - \lambda \mathbb{I})^{-1}u\| \leq \|(L_0 - \mu \mathbb{I})^{-1}u\| \leq c_2 \|(L - \lambda \mathbb{I})^{-1}u\| \quad \forall u \in H,$$

and the space $H_{-1}$, which is the completion of $H$ with respect to the norm $\|L - \lambda \mathbb{I}\|^{-1}u\|$, coincides with the same space constructed from the operator $L_0$, that is, with the space $H_{-1}^+ \times H_{-1}^-$, where $H_{-1}^\pm$ stand for the completions of $H^\pm$ with respect to the norms $\|(A_{11} - \mu \mathbb{I})^{-1}u^+\|$ and $\|(A_{22} - \mu \mathbb{I})^{-1}u^-\|$, respectively. It follows from Proposition 2 that

$$(H_1^+, H_{-1}^+)_{1/2, 2} = H^+, \quad (H_1^-, H_{-1}^-)_{1/2, 2} = H^-,$$

and Theorem 1.17.1 in [18] implies that

$$(H_1, H_{-1})_{1/2, 2} = (H_1^+, H_{-1}^+)_{1/2, 2} \times (H_1^-, H_{-1}^-)_{1/2, 2},$$

and thus

$$(H_1, H_{-1})_{1/2, 2} = H^+ \times H^- = H.$$

In the next theorem we assume that $A_{11}$ and $-A_{22}$ are $m$-dissipative operators satisfying the condition

$$\exists c > 0 : \|(A_{11}u^+, v^+)\| \leq c \|u^+\|_{F_1^+} \|v^+\|_{F_1^+}, \quad \|(A_{22}u^-, v^-)\| \leq c \|u^-\|_{F_1^-} \|v^-\|_{F_1^-}$$

(3.30)
for any $u^\pm, v^\pm \in H_1^\pm$, where $H_1^+ = D(A_{11})$, $H_1^- = D(A_{22})$, and
\[
\|u^+\|^2_{F_1^+} = \text{Re}(-A_{11}u^+, u^+) + \|u^+\|^2, \quad \|u^-\|^2_{F_1^-} = \text{Re}(A_{22}u^-, u^-) + \|u^-\|^2.
\]

Let $F_1^\pm$ denote the completions of $D(A_{11})$ and $D(A_{22})$ with respect to the norms $\| \cdot \|_{F_1^\pm}$, respectively, and let $F_{-1}^\pm$ denote the negative spaces constructed from the pairs $F_1^\pm$ and $H^\pm$. We also assume that the operators $A_{12}$ and $A_{21}$ are subordinated in a certain sense to the operators $A_{11}$ and $A_{22}$, namely, $D(A_{22}) \subset D(A_{12})$, $D(A_{11}) \subset D(A_{21})$, and
\[
\exists c > 0 : \|A_{12}u^-\|_{F_{-1}^+} \leq c\|u^+\|_{F_{-1}^-}, \quad \|A_{21}u^+\|_{F_{-1}^-} \leq c\|u^+\|_{F_{-1}^+} \quad \forall u^\pm \in H_1^\pm;
\]
\[
\exists c_0 > 0 : \|u^+\|^2_{F_1^+} + \|u^-\|^2_{F_1^-} \leq c_0(\text{Re}[-L\bar{u}, u] + \|\bar{u}\|^2) \quad (3.32)
\]
for any $\bar{u} = (u^+, u^-) \in H_1 = H_1^+ \times H_1^-$. 

**Theorem 8.** Let $L : H \to H$ be an $m$-$J$-dissipative operator such that $i\mathbb{R} \subset \rho(L)$ and let the conditions (3.30)–(3.32) hold. Then there are $L$-invariant subspaces $M^\pm$, one maximal nonnegative and one maximal nonpositive. Here the whole of $H$ can be represented as the direct sum $H = M^+ + M^-$, $\sigma(\mp L|_{M^\pm}) \subset \mathbb{C}^\pm$, and the operators $\pm L|_{M^\pm}$ are infinitesimal generators of analytic semigroups. If, in addition, $L$ is strictly $J$-dissipative or uniformly $J$-dissipative, then the corresponding subspaces $M^+$ and $M^-$ can be chosen to be positive and negative or uniformly positive and uniformly negative, respectively.

**Proof.** Using (3.32) it is easy to see that the norm in the space $F_1$ coincides with the norm $\|u\|^2_{F_1} = \|u^+\|^2_{F_1^+} + \|u^-\|^2_{F_1^-}$ for $u = (u^+, u^-)$. Using the definition of the space $F_{-1}$ and of the indefinite metric in $H$ we see that $F_{-1} = F_{-1}^+ \times F_{-1}^-$. By part 3 of Theorem 1.17.1 in [18],
\[
(F_1, F_{-1})_{1/2, 2} = (F_1^+, F_{-1}^+)_{1/2, 2} \times (F_1^-, F_{-1}^-)_{1/2, 2} = H^+ \times H^- = H.
\]

The statement follows from Theorem 2.

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