DIFFERENTIAL GEOMETRY OF COMPOSITE MANIFOLDS.

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Abstract

In classical field theory, the composite fibred manifolds $Y \to \Sigma \to X$ provides the adequate mathematical formulation of gauge models with broken symmetries, e.g., the gauge gravitation theory. This work is devoted to connections on composite fibred manifolds. In particular, we get the horizontal splitting of the vertical tangent bundle of a composite fibred manifold, besides the familiar one of its tangent bundle. This splitting defines the modified covariant differential and implies the special fashion of Lagrangian densities of field models on composite manifolds. The spinor composite bundles are examined.

1 Introduction

The geometric description of classical fields by sections of fibred manifolds $Y \to X$ is generally accepted.

Remark. All maps throughout are of class $C^\infty$. Manifolds are real, Hausdorff, finite-dimensional, second-countable and connected. By a fibred manifold is meant a surjective submersion

$$\pi : Y \to X$$

provided with an atlas of fibred coordinates $(x^\lambda, y^i)$. A locally trivial fibred manifold is termed the bundle. We denote by $VY$ and $V^*Y$ vertical tangent and vertical cotangent bundles of $Y$ respectively. For the sake of simplicity, the pullbacks

$$Y \times_TX \quad \text{and} \quad T^*_X$$

are denoted by $TX$ and $T^*_X$ respectively. We specify the following types of differential forms on fibred manifolds:

- the exterior horizontal forms $Y \to \check{T}^*X$,
- the tangent-valued horizontal forms $Y \to \check{T}^*X \otimes TY$, including soldering forms $Y \to T^*X \otimes VY$,
- and the pullback-valued forms $Y \to \check{T}^*Y \otimes TX$ and $Y \to \check{T}^*Y \otimes T^*X$.

The present article is devoted to composite fibred manifolds

$$Y \to \Sigma \to X$$

(1)
where $Y \to \Sigma$ is a bundle denoted by $Y_{\Sigma}$ and $\Sigma \to X$ is a fibred manifold.

In analytical mechanics, composite fibred manifolds

$$Y \to \Sigma \to R$$

characterize systems with variable parameters represented by sections of $\Sigma \to R$ \[7\]. In gauge theory of principal connections on a principal bundle $P$ whose structure group is reducible to its closed subgroup $K$, the composite fibred manifold

$$P \to P/K \to X$$

describes spontaneous symmetry breaking \[5, 8, 11\]. Global sections of $P/K \to X$ are treated the Higgs fields.

Application of composite fibred manifolds (1) to field theory is founded on the following speculations. Given a global section $h$ of the fibred manifold $\Sigma \to X$, the restriction $Y_h$ of the bundle $Y_{\Sigma}$ to $h(X)$ is a fibred submanifold of $Y \to X$. There is the 1:1 correspondence between the global sections $s_h$ of $Y_h$ and the global sections of the composite fibred manifold \[1\] which cover $h$. In physical terms, one says that sections $s_h$ of $Y_h$ describe fields in the presence of a background parameter field $h$, whereas sections of the composite fibred manifold $Y$ describe all the pairs $(s_h, h)$. It is important when the bundles $Y_h$ and $Y_{h\neq h'}$ fail to be equivalent in a sense. The configuration space of these pairs is the first order jet manifold $J^1 Y$ of the composite fibred manifold $Y$ and their phase space is the Legendre bundle $\Pi (10)$ over $Y$.

In particular, the gauge gravitation theory is phrased in terms of composite fibred manifolds including composite spinor bundles whose sections describe spinor fields on a world manifold. \[6, 10\].

Let $LX$ be the principal bundle of linear frames in tangent spaces to a world manifold $X^4$. In gravitation theory, its structure group

$$GL_4 = GL^+(4, R)$$

is reduced to the connected Lorentz group $L = SO(3, 1)$. In accordance with the well-known theorem, there is the 1:1 correspondence between the reduced $L$-principal subbundles $L^b X$ of $LX$ and the global sections $h$ of the quotient bundle

$$\Sigma := LX/L \to X^4.$$  \[2\]

These sections are identified to the tetrad gravitational fields.

Let us consider a bundle of complex Clifford algebras $C_{3,1}$ over $X^4$. Its subbundles are both a spinor bundle $S_M \to X^4$ and the bundle $Y_M \to X^4$ of Minkowski spaces of generating elements of $C_{3,1}$. There is the bundle morphism

$$\gamma : Y_M \otimes S_M \to S_M$$

which determines representation of elements of $Y_M$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_M$. To describe spinor fields on a world manifold, one requires that
the bundle $Y_M$ is isomorphic to the cotangent bundle $T^*X$ of $X^4$. It takes place if $Y_M$ is associated with some reduced $L$-principal subbundle $L^hX$ of the linear frame bundle $LX$. Then, there exists the representation

$$\gamma_h : T^*X \otimes S_h \rightarrow S_h$$

of cotangent vectors to a world manifold $X^4$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$ associated with the lift of $L^hX$ to a $SL(2, \mathbb{C})$-principal bundle. Sections of $S_h$ describe spinor fields in the presence of a tetrad gravitational field $h$.

The crucial point consists in the fact that, for different sections $h$ and $h'$, the representations $\gamma_h$ and $\gamma_{h'}$ fail to be equivalent. It follows that a spinor field must be regarded only in a pair with a certain tetrad field $h$. There is the 1:1 correspondence between these pairs and the sections of the composite bundle

$$S \rightarrow \Sigma \rightarrow X^4$$

(3)

where $S \rightarrow \Sigma$ is a spinor bundle associated with the $SL(2, \mathbb{C})$-lift of the $L$-principal bundle $LX \rightarrow \Sigma$.

Dynamics of fields represented by sections of a fibred manifold $Y \rightarrow X$ is phrased in terms of jet manifolds [1, 2, 4, 8, 12].

Remark. Recall that the $k$-order jet manifold $J^kY$ of a fibred manifold $Y$ comprises the equivalence classes $j^k_xs$, $x \in X$, of sections $s$ of $Y$ identified by the $(k + 1)$ terms of their Taylor series at $x$. The first order jet manifold $J^1Y$ of $Y$ is both the fibred manifold $J^1Y \rightarrow X$ and the affine bundle $J^1Y \rightarrow Y$ modelled on the vector bundle $T^*X \otimes VY$. It is endowed with the adapted coordinates $(x^\lambda, y^i, \dot{y}^i_\lambda)$:

$$y^i_{\lambda} = \left( \frac{\partial y^i}{\partial y^j} \dot{y}^j_\mu + \frac{\partial y^i}{\partial x^\mu} \frac{\partial x^\lambda}{\partial x^\mu} \right).$$

We identify $J^1Y$ to its image under the canonical bundle monomorphism

$$\lambda : J^1Y \rightarrow T^*X \otimes \dot{Y},$$

$$\lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i).$$

(4)

Given a fibred morphism of $\Phi : Y \rightarrow Y'$ over a diffeomorphism of $X$, its jet prolongation $J^1\Phi : J^1Y \rightarrow J^1Y'$ reads

$$y^i_{\mu} \circ J^1\Phi = (\partial_\lambda \Phi^i + \partial_j \Phi^i y^j_\lambda) \frac{\partial x^\lambda}{\partial x^\mu}.$$

A section $\overline{s}$ of the fibred jet manifold $J^1Y \rightarrow X$ is called holonomic if it is the jet prolongation $\overline{s} = J^1s$ of a section $s$ of $Y \rightarrow X$. There is the 1:1 correspondence between the global sections

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

(5)
of the affine jet bundle $J^1Y \to Y$ and the connections on the fibred manifold $Y \to X$. These global sections form the affine space modelled on the linear space of soldering forms on $Y$. Every connection $\Gamma$ on $Y \to X$ yields the first order differential operator

$$D_\Gamma : J^1Y \to T^*X \otimes VY,$$

$$D_\Gamma = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i,$$

on $Y \to X$ which is called the covariant differential relative to the connection $\Gamma$. $\square$

In field theory, we can restrict ourselves to the first order Lagrangian formalism when the configuration space of sections of $Y \to X$ is the first order jet manifold $J^1Y$ of $Y$. A Lagrangian density on $J^1Y$ is defined to be a morphism

$$L : J^1Y \to \wedge^n T^*X,$$

$$L = L\omega, \quad \omega = dx^1 \wedge ... \wedge dx^n.$$

Note that since the jet bundle $J^1Y \to Y$ is affine, every polynomial Lagrangian density of field theory factors

$$L : J^1Y \to T^*X \otimes VY \to \wedge^n T^*X. \quad (6)$$

The feature of the dynamics of field systems on composite fibred manifolds consists in the following.

Let $Y$ be a composite manifold (1) provided with the fibred coordinates $(x^\lambda, \sigma^m, y^i)$ where $(x^\lambda, \sigma^m)$ are fibred coordinates of $\Sigma \to X$. Every connection

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \quad (7)$$

on the bundle $Y \to \Sigma$ yields the splitting

$$VY = VY_\Sigma \oplus (Y \times V\Sigma) \quad (8)$$

and, as a consequence, the first order differential operator

$$\bar{D} : J^1Y \to T^*X \otimes VY_\Sigma,$$

$$\bar{D} = dx^\lambda \otimes (y^i_\lambda - \tilde{A}^i_\lambda - A^i_m \sigma^m_\lambda) \partial_i,$$

on $Y$. Let $h$ be a global section of $\Sigma \to X$ and $Y_h$ the restriction of the bundle $Y_\Sigma$ to $h(X)$. The restriction of $\bar{D}$ to $J^1Y_h \subset J^1Y$ comes to the familiar covariant differential relative to a certain connection $A_h$ on $Y_h$. Thus, it is $\bar{D}$ that we may utilize in order to construct a Lagrangian density

$$L : J^1Y \overset{\bar{D}}{\to} T^*X \otimes VY_\Sigma \to \wedge^n T^*X \quad (9)$$

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for sections of a composite manifold. It should be noted that such a Lagrangian density is never regular because of the constraint conditions

$$A^i_m \partial^\mu_i \mathcal{L} = \partial^\mu_m \mathcal{L}.$$ 

If a Lagrangian density is degenerate, the corresponding Euler-Lagrange equations are underdetermined.

To describe constraint field systems, the multisymplectic generalization of the Hamiltonian formalism in mechanics is utilized [1, 3, 7, 8, 12]. In the framework of this approach, the phase space of sections of $Y \to X$ is the Legendre bundle

$$\Pi = \wedge^n T^* X \otimes TX \otimes V^* Y$$

over $Y$. It is provided with the fibred coordinates $(x^\lambda, y^i, p^\lambda_i)$. Note that every Lagrangian density $L$ on $J^1Y$ determines the Legendre morphism

$$\tilde{L} : J^1Y \to \Pi, \quad (x^\mu, y^i, p^\mu_i) \circ \tilde{L} = (x^\mu, y^i, \partial^\mu_i L).$$

Its image plays the role of the Lagrangian constraint space.

The Legendre bundle (10) carries the multisymplectic form

$$\Omega = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial^\lambda.$$ 

We say that a connection $\gamma$ on the fibred Legendre manifold $\Pi \to X$ is a Hamiltonian connection if the form $\gamma \lrcorner \Omega$ is closed. Then, a multimomentum Hamiltonian form $H$ on $\Pi$ is defined to be an exterior form such that

$$dH = \gamma \lrcorner \Omega$$

for some Hamiltonian connection $\gamma$.

The major feature of Hamiltonian systems on composite fibred manifolds lies in the following [1]. Let $Y$ be a composite fibred manifold (1). The Legendre bundle $\Pi$ over $Y$ is coordinatized by

$$(x^\lambda, \sigma^m, y^i, p^\lambda_m, p^\lambda_i).$$

Let $A_\Sigma$ be a connection (2) on the bundle $Y_\Sigma$. With a connection $A_\Sigma$, the splitting

$$\Pi = \wedge^n T^* X \otimes TX \otimes [V^* Y_\Sigma \oplus (Y \times V^* \Sigma)]$$

of the Legendre bundle $\Pi$ is performed as an immediate consequence of the splitting (3). Given the splitting (11), the Legendre bundle $\Pi$ can be provided with the coordinates

$$\overline{p}^\lambda_i = p^\lambda_i, \quad \overline{p}^\lambda_m = p^\lambda_m + A^i_m \overline{p}^\lambda_i.$$
which are compatible with this splitting.

In particular, let $h$ be a global section of the fibred manifold $\Sigma$. It is readily observed that, given the splitting (11), the submanifold

$$\{\sigma = h(x), \overline{p}_m = 0\}$$

of the Legendre bundle $\Pi$ over $Y$ is isomorphic to the Legendre bundle $\Pi_h$ over the restriction $Y_h$ of $Y_\Sigma$ to $h(X)$.

Let a multimomentum Hamiltonian form be associated with a Lagrangian density (9) which contains the velocities $\sigma^m_{\mu}$ only inside the differential $\tilde{D}$. Then, the Lagrangian constraints read

$$\overline{p}_m = 0.$$

## 2 Composite fibred manifolds

The composite fibred manifold (or simply the composite manifold) is defined to be composition of surjective submersions

$$\pi_{\Sigma X} \circ \pi_{Y \Sigma} : Y \to \Sigma \to X.$$ (12)

Roughly speaking, it is the fibred manifold $Y \to X$ provided with the particular class of coordinate atlases $(x^\lambda, \sigma^m, y^i)$:

$$x'^\lambda = f^\lambda(x^\mu), \quad \sigma'^m = f^m(x^\mu, \sigma^n), \quad y'^i = f^i(x^\mu, \sigma^n, y^j),$$

where $(x^\mu, \sigma^m)$ are fibred coordinates of $\Sigma \to X$ and $y^i$ are bundle coordinates of $Y_\Sigma$. Note that if the fibred manifold $\Sigma$ also is a bundle, the composite manifold $Y$ fails to be a bundle in general. We further propose that $\Sigma$ has a global section.

Recall the following assertions [8] [13].

**Proposition 1.** Let $Y$ be the composite manifold (12). Given a section $h$ of $\Sigma$ and a section $s_\Sigma$ of $Y_\Sigma$, their composition $s_\Sigma \circ h$ obviously is a section of the composite manifold $Y$. Conversely, if the bundle $Y_\Sigma$ has a global section, every global section $s$ of the fibred manifold $Y \to X$ is represented by some composition $s_\Sigma \circ h$ where $h = \pi_{Y \Sigma} \circ s$ and $s_\Sigma$ is an extension of the local section $h(X) \to s(X)$ of the bundle $Y_\Sigma$ over the closed imbedded submanifold $h(X) \subset \Sigma$. $\square$

**Proposition 2.** Given a global section $h$ of $\Sigma$, the restriction $Y_h = h^*Y_\Sigma$ of the bundle $Y_\Sigma$ to $h(X)$ is a fibred imbedded submanifold of $Y$. $\square$

**Corollary 3.** There is the 1:1 correspondence between the sections $s_h$ of $Y_h$ and the sections $s$ of the composite manifold $Y$ which cover $h$. $\square$
Given fibred coordinates \((x^\lambda, \sigma^m, y^i)\) of the composite manifold \(Y\), the jet manifolds 
\(J^1\Sigma, J^1Y_\Sigma\) and \(J^1Y\) are coordinatized respectively by 
\[(x^\lambda, \sigma^m, \sigma^m_\lambda), \quad (x^\lambda, \sigma^m, y^i, \widetilde{y}^i_j, y^i_m), \quad (x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda).\]

**Proposition 4.** There exists the canonical surjection 
\[
\rho : J^1\Sigma \times J^1Y_\Sigma \to J^1Y,
\]
\[
\rho(j^1_x h, j^1_x (s_\Sigma \circ h)) = j^1_x (s_\Sigma \circ h),
\]
\[
y^i_\lambda \circ \rho = y^i_m \sigma^m_\lambda + \widetilde{y}^i_j,
\]  
where \(s_\Sigma\) and \(h\) are sections of \(Y_\Sigma\) and \(\Sigma\) respectively. \(\square\)

**Corollary 5.** Let \(A_\Sigma\) be the connection (7) on the bundle \(Y_\Sigma\) and \(\Gamma\) the connection (3) on the fibred manifold \(\Sigma\). Building on the morphism (13), one can construct the composite connection 
\[
A = dx^\lambda \otimes [\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_m \Gamma^m_\lambda + \widetilde{A}^i_\lambda) \partial_i]
\]
on the composite manifold \(Y\) in accordance with the commutative diagram

\[
\begin{array}{ccc}
J^1\Sigma \times J^1Y_\Sigma & \xrightarrow{\rho} & J^1Y \\
\Gamma \times A_\Sigma
& \downarrow
& \\
\Sigma \times Y_\Sigma & \leftarrow & Y
\end{array}
\]

\(\square\)

Let a global section \(h\) of \(\Sigma\) be an integral section of the connection \(\Gamma\) on \(\Sigma\), that is, 
\(\Gamma \circ h = J^1h\). Then, the composite connection (14) on \(Y\) is reducible to the connection 
\[
A_h = dx^\lambda \otimes [\partial_\lambda + (A^i_m \partial_\lambda h^m + \widetilde{A}^i_\lambda) \partial_i]
\]
on the fibred submanifold \(Y_h\) of \(Y \to X\) in accordance with the commutative diagram

\[
\begin{array}{ccc}
J^1Y_h & \xrightarrow{J^1i_h} & J^1Y \\
A_h
& \downarrow
& \\
Y_h & \leftarrow & Y
\end{array}
\]

In particular, every connection \(A_\Sigma\) (7) on \(Y_\Sigma\), whenever \(h\), is reducible to the connection (15) on \(Y_h\).
3 Connections on composite manifolds

Given a composite manifold \((\Sigma)\), we have the exact sequences

\[
0 \rightarrow V\Sigma \hookrightarrow YY \rightarrow Y \times V\Sigma \rightarrow 0,
0 \rightarrow Y \times V^*\Sigma \hookrightarrow V^*Y \rightarrow V^*Y\Sigma \rightarrow 0
\]

over \(Y\), besides the familiar ones

\[
0 \rightarrow VY \hookrightarrow TY \rightarrow Y \times TX \rightarrow 0,
0 \rightarrow Y \times T^*X \hookrightarrow T^*Y \rightarrow V^*Y \rightarrow 0.
\]

**Proposition 6.** There exist the canonical splittings

\[
J^1Y \Sigma \times YY = V\Sigma \bigoplus (Y \times V\Sigma),
\]

\[
\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - y^i_m \dot{\sigma}_m) \partial_i + \dot{\sigma}^m (\partial_m + y^i_m \partial_i),
\]

of the vertical tangent bundle \(VY\) of \(Y \rightarrow X\) and

\[
J^1Y \times V^*Y = V^*\Sigma \bigoplus (Y \times V^*\Sigma),
\]

\[
\dot{y}_i dy^i + \dot{\sigma}_m d\sigma^m = \dot{y}_i(dy^i - y^i_m d\sigma^m) + (\dot{\sigma}_m + y^i_m \dot{y}_i) d\sigma^m,
\]

of the vertical cotangent bundle \(V^*Y\) of \(Y\). □

The proof is starightforward.

These splittings add the familiar canonical horizontal splittings

\[
J^1Y \times TY = TX \bigoplus VY,
J^1Y \times T^*Y = T^*X \bigoplus V^*Y.
\]

**Corollary 7.** Every connection \((\Sigma)\) on the bundle \(Y\Sigma\) determines:

(i) the horizontal splitting

\[
VY = V\Sigma \bigoplus (Y \times V\Sigma),
\]

\[
\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\dot{y}^i - A^i_m \dot{\sigma}_m) \partial_i + \dot{\sigma}^m (\partial_m + A^i_m \partial_i),
\]

of the vertical tangent bundle \(VY\) of \(Y\).
(ii) the dual horizontal splitting
\[ V^*Y = V^*Y_\Sigma \oplus (Y \times V^*\Sigma), \]
\[ \dot{y}_i dy^i + \dot{\sigma}_m d\sigma^m = \dot{y}_i(dy^i - A^i_m d\sigma^m) + (\dot{\sigma}_m + A^i_m \dot{y}_i) d\sigma^m, \]
of the vertical cotangent bundle \( V^*Y \) of \( Y \).

\[ \blacksquare \]

It is readily observed that the splittings (16) and (17) are uniquely characterized by the form
\[ \omega \wedge A_\Sigma = \omega \wedge d\sigma^m \otimes (\partial^i_m + A^i_m \partial^i), \]
and different connections \( A_\Sigma \) can define the same splittings (16) and (17).

Building on the horizontal splitting (16), one can construct the following first order differential operator on the composite manifold \( Y \):
\[ \tilde{D} = D_A : J^1Y \rightarrow T^*X \otimes VY \rightarrow T^*X \otimes VY_\Sigma, \]
\[ \tilde{D} = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda + A^i_m (\sigma^m_\lambda - \Gamma^m_\lambda)) \partial^i = dx^\lambda \otimes (y^i_\lambda - \tilde{A}^i_\lambda + A^i_m \sigma^m_\lambda) \partial^i, \] (19)
where \( D_A \) is the covariant differential relative to the composite connection \( A_\Sigma \) which is composition of \( A_\Sigma \) and some connection \( \Gamma \) on \( \Sigma \). We shall call \( \tilde{D} \) the vertical covariant differential. This possesses the following property.

Given a global section \( h \) of \( \Sigma \), let \( \Gamma \) be a connection on \( \Sigma \) whose integral section is \( h \). It is readily observed that the vertical covariant differential (19) restricted to \( J^1Y_h \subset J^1Y \) comes to the familiar covariant differential relative to the connection \( A_h \) (15) on \( Y_h \). Thus, it is the vertical covariant differential (19) that we may utilize in order to construct a Lagrangian density (9) for sections of a composite manifold.

Now, we consider connections on a composite manifold \( Y \) when \( Y_\Sigma \) is a vector bundle. Let a connection \( A \) on \( Y \) be projectable to a connection \( \Gamma \) on \( \Sigma \) in accordance with the commutative diagram
\[ J^1Y \xrightarrow{\pi_{Y_\Sigma}} J^1\Sigma \]
\[ A \quad Y \xrightarrow{\pi_{Y_\Sigma}} \Sigma \]

Let
\[ A = dx^\lambda \otimes (\partial^i_\lambda + \Gamma^m_\lambda (\sigma) \partial^i + A^i_j \sigma^m_\lambda y^i j \partial^i), \] (20)
be a linear morphism over \( \Gamma \). The following constructions generalize the familiar notions of a dual linear connection and a tensor product linear connection on vector bundles.

Let \( Y^* \rightarrow \Sigma \rightarrow X \) be a composite manifold where \( Y^* \rightarrow \Sigma \) is the vector bundle dual to \( Y_\Sigma \). Given the projectable connection (20) on \( Y \) over \( \Gamma \), there exists the unique projectable connection
\[ A^* = dx^\lambda \otimes (\partial^i_\lambda + \Gamma^m_\lambda \partial^i - A^i_j \sigma^m_\lambda y^i j \partial^i). \]
on $Y^* \to X$ over $\Gamma$ such that the following diagram commutes:

\[
\begin{array}{ccc}
J^1Y \times \Sigma & \xrightarrow{J^1Y^*} & J^1\Sigma \times (T^*X \times \mathbb{R}) \\
\downarrow \text{A} \times \text{A}^* & & \downarrow \Gamma \times \tilde{0} \times \text{Id} \\
Y \times Y^* & \to & \Sigma \times \mathbb{R}
\end{array}
\]

where $\tilde{0}$ is the zero section of $T^*X$. We term $A^*$ the connection dual to $A$ over $\Gamma$.

Let $Y \to \Sigma \to X$ and $Y' \to \Sigma \to X$ be composite manifolds where $Y \to \Sigma$ and $Y' \to \Sigma$ are vector bundles. Let $A$ and $A'$ be the connections on $Y$ and $Y'$ respectively which are projectable to the same connection $\Gamma$ on $\Sigma$. There is the unique projectable connection

\[
A \otimes A' = dx^\lambda \otimes [\partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_{j\lambda} y^j + A'_{k\lambda} y^k) \partial_{ik}]
\]

on the tensor product $Y \otimes Y' \to X$ such that the following diagram commutes:

\[
\begin{array}{ccc}
J^1Y \times J^1Y' & \xrightarrow{J^1\otimes J^1} & J^1(Y \otimes Y') \\
\downarrow \text{A} \times \text{A}' & & \downarrow \text{A} \otimes \text{A}' \\
Y \times Y' & \to & Y \otimes Y'
\end{array}
\]

It is called the tensor product connection over $\Gamma$.

In particular, let $Y \to X$ be a fibred manifold and $\Gamma$ a connection on $Y$. The vertical tangent morphism $V\Gamma$ to $\Gamma$ determines the connection

\[
VT : VY \to VJ^1Y = J^1VY,
\]

\[
VT = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial y^i + \partial_j \Gamma^i_\lambda \dot{y}^i \partial \dot{y}^j),
\]

on the composite manifold $VY \to Y \to X$. The connection $VT$ is projectable to $\Gamma$, and it is a linear bundle morphism over $\Gamma$. It yields the connection

\[
V^*\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial y^i - \partial_j \Gamma^i_\lambda \dot{y}_j \partial \dot{y}^i)
\]

(22)

on the composite manifold $V^*Y \to Y \to X$ which is the dual connection to $VT$ over $\Gamma$.

For instance, every connection $\Gamma$ on a fibred manifold $Y \to X$ gives rise to the connection

\[
\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^i_\lambda (y) \partial_i + (-\partial_j \Gamma^i_\lambda (y) P_{ij}^\mu - K_{\nu\lambda}(x) P_{ij}^\nu + K_{\alpha\lambda}(x) P_{ij}^\lambda) \partial_{ij}]
\]

(23)

on the fibred Legendre manifold $\Pi \to X$ where $K$ is a symmetric linear connection

\[
K^\alpha_{\lambda} = -K_{\nu\lambda}(x) \dot{x}^\nu,
\]

\[
K^\alpha_{\alpha\lambda} = K_{\nu\alpha\lambda}(x) \dot{x}_\nu,
\]

\[
K_{\mu\nu\lambda} = K_{\mu\lambda}(x),
\]
on the bundles $TX$ and $T^*X$. The connection \((23)\) is the tensor product \((21)\) [over $\Gamma$] of the connection $\Gamma \times K$ on the pullback composite manifold

$$Y \times X^n 1 \ T^*X \to Y \to X$$

and the connection $V^*Y \ (22)$ on the composite manifold $V^*Y \to Y \to X$. The connection \((23)\) covers the connection $\Gamma$ on the fibred manifold $Y \to X$. These connections play the prominent role in the multisymplectic Hamiltonian formalism. Let $\Gamma$ be a connection on the fibred manifold $Y \to X$ and

$$\tilde{\Gamma} := \Gamma \circ \pi_{\Pi Y} : \Pi \to Y \to J^1Y,$$

$$\tilde{\Gamma} = dx^\lambda \otimes (\partial \lambda + \Gamma^i_\lambda \partial^i),$$

its pullback by $\pi_{\Pi Y}$ onto the Legendre bundle $\Pi$ over $Y$. Then, the lift $\tilde{\Gamma} \ (23)$ of $\Gamma$ onto the fibred Legendre manifold $\Pi \to X$ obeys the identity

$$\tilde{\Gamma}|\Omega = d(\tilde{\Gamma}|\theta).$$

A glance at this identity shows that $\tilde{\Gamma}$ is a Hamiltonian connection.

4 Composite fibration of principal bundles

Let $\pi_P : P \to X$ be a principal bundle with a structure Lie group $G$ which acts freely and transitively on $P$ on the right:

$$r_g : p \mapsto pg, \quad p \in P, \quad g \in G.$$

Let $K$ be a closed subgroup of $G$. We have the composite manifold

$$\pi_{\Sigma X} \circ \pi_{P\Sigma} : P \to P/K \to X \tag{24}$$

where

$$P_{\Sigma} := P \to P/K$$

is a principal bundle with the structure group $K$ and

$$\Sigma_K = P/K = (P \times G/K)/G$$

is the $P$-associated bundle with the standard fiber $G/K$ on which the structure group $G$ acts on the left.

Let the structure group $G$ be reducible to its closed subgroup $K$. Recall the 1:1 correspondence

$$\pi_{P\Sigma}(P_h) = (h \circ \pi_P)(P_h)$$
between the global sections $h$ of the bundle $P/K \to X$ and the reduced $K$-principal subbundles $P_h$ of $P$ which consist with restrictions of the principal bundle $P_\Sigma$ to $h(X)$.

Given the composite manifold (24), the canonical morphism (13) results in the surjection

$$J^1P_\Sigma/K \times J^1\Sigma \to J^1P/K$$

over $J^1\Sigma$. Let $A_\Sigma$ be a principal connection on $P_\Sigma$ and $\Gamma$ a connection on $\Sigma$. The corresponding composite connection (14) on the composite manifold (24) is equivariant under the canonical action of $K$ on $P$. If the connection $\Gamma$ has an integral global section $h$ of $P/K \to X$, the composite connection (14) is reducible to the connection (15) on $P_h$ which consists with the principal connection on $P_h$ induced by $A_\Sigma$.

Let us consider the composite manifold

$$Y = (P \times V)/K \to P/K \to X$$

(25)

where the bundle

$$Y_\Sigma := (P \times V)/K \to P/K$$

is associated with the $K$-principal bundle $P_\Sigma$. Given a reduced subbundle $P_h$ of $P$, the associated bundle

$$Y_h = (P_h \times V)/K$$

is isomorphic to the restriction of $Y_\Sigma$ to $h(X) \subset \Sigma_K$.

Note that the manifold $(P \times V)/K$ possesses also the structure of the bundle

$$Y = (P \times (G \times V)/K)/G$$

associated with the principal bundle $P$. Its standard fibre is $(G \times V)/K$ on which the structure group $G$ of $P$ (and its subgroup $K$) acts by the law

$$G \ni g : (G \times V)/K \to (gG \times V)/K.$$  

It differs from action of the structure group $K$ of $P_\Sigma$ on this standard fibre. As a shorthand, we can write the latter in the form

$$K \ni g : (G \times V)/K \to (G \times gV)/K.$$  

However, this action fails to be canonical and depends on existence and specification of a global section of the bundle $G \to G/K$. If the standard fibre $V$ of the bundle $Y_\Sigma$ carries representation of the whole group $G$, these two actions are equivalent, otherwise in general case.
5 Composite spinor bundles

By $X^4$ is further meant an oriented world manifold which satisfies the well-known global topological conditions in order that a spinor structure can exist. To summarize these conditions, we assume that $X^4$ is not compact and the linear frame bundle $LX$ over $X^4$ is trivial.

Given a Minkowski space $M$ with the Minkowski metric $\eta$, let

$$A_M = \oplus_n M^n, \quad M^0 = \mathbb{R}, \quad M^{n>0} = \otimes M,$$

be the tensor algebra modelled on $M$. The complexified quotient of this algebra by the two-sided ideal generated by elements

$$e \otimes e' + e' \otimes e - 2\eta(e, e') \in A_M, \quad e \in M,$$

constitutes the complex Clifford algebra $\mathbb{C}_{1,3}$. A spinor space $V$ is defined to be a linear space of some minimal left ideal of $\mathbb{C}_{1,3}$ on which this algebra acts on the left. We have the representation

$$\gamma : M \otimes V \rightarrow V$$

of elements of the Minkowski space $M \subset \mathbb{C}_{1,3}$ by Dirac’s $\gamma$-matrices on $V$:

$$\gamma(e^a \otimes y^A v_A) = \gamma^{aA} y^B v_A,$$

where $\{e^0...e^3\}$ is a fixed basis for $M$, $v_A$ is a basis for $V$.

Let us consider the transformations preserving the representation (26). These are pairs $(l, l_s)$ of Lorentz transformations $l$ of the Minkowski space $M$ and invertible elements $l_s$ of $\mathbb{C}_{1,3}$ such that

$$lM = l_s M l^{-1}_s,$$

$$\gamma(lM \otimes l_s V) = l_s \gamma(M \otimes V).$$

Elements $l_s$ form the Clifford group whose action on $M$ however is not effective. We restrict ourselves to its spinor subgroup $L_s = SL(2, \mathbb{C})$ whose generators act on $V$ by the representation

$$I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b].$$

Let us consider a bundle of complex Clifford algebras $\mathbb{C}_{3,1}$ over $X^4$. Its subbundles are both a spinor bundle $S_M \rightarrow X^4$ and the bundle $Y_M \rightarrow X^4$ of Minkowski spaces of generating elements of $\mathbb{C}_{3,1}$. To describe spinor fields on a world manifold, one must require $Y_M$ be isomorphic to the cotangent bundle $T^*X$ of a world manifold $X^4$. It takes place if the linear frame bundle $LX$ contains a reduced $L$ subbundle $L^bX$ such that

$$Y_M = (L^bX \times M)/L.$$
In this case, the spinor bundle $S_M$ is associated with the $L_s$-lift $P_h$ of $L^hX$:

$$S_M = S_h = (P_h \times V)/L_s.$$  \hfill (27)

There is the above-mentioned 1:1 correspondence between the reduced subbubdles $L^hX$ of $LX$ and global sections $h$ of the bundle $\Sigma$ \hfill (\ref{2}).

Given a tetrad field $h$, let $\Psi^h$ be an atlas of $LX$ such that the corresponding local sections $z^h_\xi$ of $LX$ take their values into $L^hX$.

$$S_M = S_h = (P_h \times V)/L_s.$$  \hfill (27)

Given a tetrad field $h$, let $\Psi^h$ be an atlas of $LX$ such that the corresponding local sections $z^h_\xi$ of $LX$ take their values into $L^hX$.

$$S_M = S_h = (P_h \times V)/L_s.$$  \hfill (27)

They are $GL_4$-valued functions of atlas transformations

$$dx^\lambda = h^\lambda_a(x)h^a$$  \hfill (28)

between the fibre bases $\{dx^\lambda\}$ and $\{h^a\}$ for $T^*X$ associated with $\Psi^T$ and $\Psi^h$ respectively.

Given a tetrad field $h$, one can define the representation

$$\gamma_h : T^*X \otimes S_h = (P_h \times (M \otimes V))/L_s \rightarrow (P_h \times \gamma(M \otimes V))/L_s = S_h$$  \hfill (29)

of cotangent vectors to a world manifold $X^4$ by Dirac’s $\gamma$-matrices on elements of the spinor bundle $S_h$. With respect to an atlas $\{z^h_\xi\}$ of $P_h$ and the associated atlas $\{z^h_\xi\}$ of $LX$, the morphism (29) reads

$$\gamma_h(h^a \otimes y^A v_A(x)) = \gamma^{aA} B y^B v_A(x)$$

where $\{v_A(x)\}$ are the associated fibre bases for $S_h$. As a shorthand, one can write

$$\tilde{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)h^a.$$  \hfill (28)

On the physical level, one can say that, given the representation (29), sections of the spinor bundle $S_h$ describe spinor fields in the presence of the tetrad gravitational field $h$.

Indeed, let $A_h$ be a principal connection on $S_h$ and

$$D : J^1 S_h \rightarrow T^*X \otimes V S_h,$$

$$D = (y^A - A^{ab}_{\lambda}(x)I_{ab} A^B y^B)dx^\lambda \otimes \partial_A,$$

the corresponding covariant differential. Given the representation (29), one can construct the Dirac operator

$$D_h = \gamma_h \circ D : J^1 S_h \rightarrow T^*X \otimes V S_h \rightarrow V S_h,$$

$$\tilde{y}^A \circ D_h = h^\lambda_a(x)\gamma^{aA} B (y^B - A^{ab}_{\lambda}(x)I_{ab} A^B y^B).$$

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We here use the fact that the vertical tangent bundle $V S_h$ admits the canonical splitting

$$V S_h = S_h \times S_h,$$

and $\gamma_h$ in the expression (30) is the pullback

$$\gamma_h : T^* X \otimes V S_h \rightarrow V S_h,$$

$$\gamma_h(h^a \otimes \dot{y}^A \partial_A) = \gamma^a B \dot{y}^B \partial_B,$$

over $S_h$ of the bundle morphism (29).

In the presence of different tetrad fields $h$ and $h'$, spinor fields are described by sections of spinor bundles $S_h$ and $S_{h'}$ associated with $L_s$-lifts $P_h$ and $P_{h'}$ of different reduced $L$-principal subbundles of $L X$. Therefore, the representations $\gamma_h$ and $\gamma_{h'}$ are not equivalent [6, 10]. It follows that a spinor field must be regarded only in a pair with a certain tetrad field. In accordance with Corollary 3, there is the 1:1 correspondence between these pairs and sections of the composite spinor bundle (3).

We have the composite manifold

$$\pi_{\Sigma X} \circ \pi_{P \Sigma} : L X \rightarrow \Sigma \rightarrow X^4$$

where $\Sigma$ is the quotient bundle (2) and

$$L X_{\Sigma} := L X \rightarrow \Sigma$$

is the $L$-principal bundle.

Building on the double universal covering of the group $GL_4$, one can perform the $L_s$-principal lift $P_{\Sigma}$ of $L X_{\Sigma}$ such that

$$P_{\Sigma}/L_s = \Sigma, \quad L X_{\Sigma} = r(P_{\Sigma}).$$

In particular, there is imbedding of $P_h$ onto the restriction of $P_{\Sigma}$ to $h(X^4)$.

Let us consider the composite spinor bundle (2) where

$$S_{\Sigma} = (P_{\Sigma} \times V)/L_s$$

is associated with the $L_s$-principal bundle $P_{\Sigma}$. It is readily observed that, given a global section $h$ of $\Sigma$, the restriction $S_{\Sigma}$ to $h(X^4)$ is the spinor bundle $S_h$ (27) whose sections describe spinor fields in the presence of the tetrad field $h$.

Let us provide the principal bundle $L X$ with a holonomic atlas $\{ \psi_T^\xi, U_\xi \}$ and the principal bundles $P_{\Sigma}$ and $L X_{\Sigma}$ with associated atlases $\{ z^s, U_\epsilon \}$ and $\{ z^s = \epsilon \circ z^s \}$. With respect to these atlases, the composite spinor bundle is endowed with the fibred coordinates $(x^A, \sigma^a, y^A)$ where $(x^A, \sigma^a)$ are fibred coordinates of the bundle $\Sigma$ such that $\sigma^a$ are the matrix components of the group element

$$GL_4 \ni (\psi_T^\xi \circ z^s)(\sigma) : R^4 \rightarrow R^4, \quad \sigma \in U_\epsilon, \quad \pi_{\Sigma X}(\sigma) \in U_\xi.$$
Given a section \( h \) of \( \Sigma \), we have

\[
\begin{align*}
  z^h(x) &= (z \circ h)(x), \quad h(x) \in U_\varepsilon, \quad x \in U_\xi, \\
  (\sigma^{\mu}_{a} \circ h)(x) &= h^\mu_a(x),
\end{align*}
\]

where \( h^\mu_a(x) \) are tetrad functions (28).

The jet manifolds \( J^1 \Sigma \), \( J^1 S_\Sigma \), and \( J^1 S \) are coordinatized by

\[
(x^\lambda, \sigma^\mu_a, \sigma^\mu_{a\lambda}), \quad (x^\lambda, \sigma^\mu_a, y^A, y_A^A), \quad (x^\lambda, \sigma^\mu_a, y^A, \sigma^\mu_{a\lambda}, y_A^A).
\]

Note that, whenever \( h \), the jet manifold \( J^1 S_h \) is a fibred submanifold of \( J^1 S \to X^4 \) given by the coordinate relations

\[
\sigma^\mu_a = h^\mu_a(x), \quad \sigma^\mu_{a\lambda} = \partial_\lambda h^\mu_a(x).
\]

Let us consider the bundle of Minkowski spaces

\[
(LX \times M)/L \to \Sigma
\]

associated with the \( L \)-principal bundle \( LX_{\Sigma} \). Since \( LX_{\Sigma} \) is trivial, it is isomorphic to the pullback \( \Sigma \times T^*X \) which we denote by the same symbol \( T^*X \). Building on the morphism (26), one can define the bundle morphism

\[
\gamma_\Sigma : T^*X \otimes S_\Sigma = (P_\Sigma \times (M \otimes V))/L_s \to (P_\Sigma \times \gamma(M \otimes V))/L_s = S_\Sigma,
\]

\[
\hat{dx}^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a,
\]

over \( \Sigma \). When restricted to \( h(X^4) \subset \Sigma \), the morphism (32) comes to the morphism \( \gamma_h \) (29). Because of the canonical vertical splitting

\[
VS_\Sigma = S_\Sigma \times S_\Sigma,
\]

the morphism (32) yields the corresponding morphism

\[
\gamma_\Sigma : T^*X \otimes VS_\Sigma \to VS_\Sigma.
\]

We use this morphism in order to construct the total Dirac operator on sections of the composite spinor bundle \( S \) (3). We are based on the following fact.

Let

\[
\tilde{A} = dx^\lambda \otimes (\partial_\lambda + \tilde{A}_B^a \partial_B) + d\sigma^\mu_a \otimes (\partial_\mu + A_B^a \partial_B)
\]

be a connection on the bundle \( S_{\Sigma} \). It determines the horizontal splitting (16) of the vertical tangent bundle \( VS \) and the vertical covariant differential (14). The composition of morphisms (33) and (14) is the first order differential operator

\[
D = \gamma_\Sigma \circ \tilde{D} : J^1 S \to T^*X \otimes VS_\Sigma \to VS_\Sigma,
\]

over \( \Sigma \).
\[
\hat{y}^A \circ D = \sigma_a^\gamma y^a_B (y^B_\lambda - \bar{A}^\lambda_B - A^B_\mu \sigma^\mu_{aA}),
\]
on \mathcal{S}. One can treat it the total Dirac operator since, whenever a tetrad field \(h\), the restriction of \(D\) to \(J^1 S_h \subset J^1 \mathcal{S}\) comes to the Dirac operator \(D_h\) (30) with respect to the connection
\[
A_h = dx^\lambda \otimes [\partial_\lambda + (\bar{A}^\lambda_B + A^B_\mu \partial_\lambda h^\mu) \partial_B]
\]
on \(S_h\).

To construct the connection (34) in explicit form, let us set up the principal connection on the bundle \(LX_\Sigma\) which is given by the local connection form
\[
A_\Sigma = (\bar{A}^{ab}_\mu dx^\mu + A^{abc}_\mu d\sigma^\mu_c) \otimes I_{ab},
\]
(35)
\[
\bar{A}^{ab}_\mu = \frac{1}{2} K^\nu_{\lambda \mu} \sigma^\lambda_c (\eta^{cb} \sigma^a_\nu - \eta^{ca} \sigma^b_\nu),
\]
\[
A^{abc}_\mu = \frac{1}{2} (\eta^{cb} \sigma^a_\mu - \eta^{ca} \sigma^b_\mu),
\]
(36)
where \(K\) is some symmetric connection on \(TX\) and (36) corresponds to the canonical left-invariant free-curvature connection on the bundle
\[
GL_4 \rightarrow GL_4/L.
\]
Given a tetrad field \(h\), the connection (35) is reduced to the Levi-Civita connection
\[
A_h = \frac{1}{2} [K^\nu_{\lambda \mu} \sigma^\lambda_c (\eta^{cb} \sigma^a_\nu - \eta^{ca} \sigma^b_\nu) + \partial_\mu h^\nu_c (\eta^{cb} \sigma^a_\nu - \eta^{ca} \sigma^b_\nu)]
\]
on \(L^h X\).

The connection (34) on the spinor bundle \(S_\Sigma\) which is associated with \(A_\Sigma\) (35) reads
\[
\bar{A}_\Sigma = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} \bar{A}^{ab}_{\lambda \mu} I_{ab}^A y^A \partial_B) + d\sigma^\mu_c \otimes (\partial_\mu + \frac{1}{2} A^{abc}_\mu I_{ab}^A y^A \partial_B).
\]
It determines the canonical horizontal splitting (14) of the vertical tangent bundle \(V S\) given by the form (18)
\[
\omega \wedge \otimes d\sigma^\mu_c [\partial_\mu + \frac{1}{8} \eta^{cb} \sigma^a_\mu [\gamma_a, g_b] B y^A \partial_B].
\]

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