Domino tilings with diagonal impurities

Fumihiko Nakano and Taizo Sadahiro

Abstract

This paper studies the dimer model on the dual graph of the square-octagon lattice, which can be viewed as the domino tilings with impurities in some sense. In particular, under a certain boundary condition, we give an exact formula representing the probability of finding an impurity at a given site in a uniformly random dimer configuration in terms of simple random walks on the square lattice.

1 Introduction

Although the dimer models on planar bipartite lattice graphs have been greatly advanced over the last decade (see e.g., [5], [3]), much less is known about non-bipartite cases. This paper deals with a non-bipartite lattice Γ, the dual of the square-octagon lattice. As will be clear later, the dimer model on Γ can be viewed as the domino tiling model containing certain impurities. Our main aim in this paper is to study the behavior of these impurities. In particular, under a certain boundary condition, we give an exact formula representing the probability of finding an impurity at a given site in a uniformly random dimer configuration in terms of the simple random walks on the square lattice.

Figure 1: Square-Octagon graph (dashed) and its dual

We define the dual-square-octagon graph Γ as follows: the vertices \( V(\Gamma) \) is \( \mathbb{Z}^2 \) which is divided into two subsets \( W = \{(x, y) \in V(\Gamma) \mid x + y \text{ is even} \} \) and \( B = V(\Gamma) \setminus W \), and there is an edge between \( v \) and \( v' \in V(\Gamma) \) if and only if

\[
v - v' \in \{\pm(1, 0), \pm(0, 1)\}
\]

1Faculty of Science, Department of Mathematics and Information Science, Kochi University, 2-5-1, Akebonomachi, Kochi, 780-8520, Japan. e-mail : nakano@math.kochi-u.ac.jp
2Faculty of Administration, Prefectural University of Kumamoto, Tsukide 3-1-100, Kumamoto, 862-8502, Japan. e-mail : sadahiro@pu-kumamoto.ac.jp
or

\[ v, v' \in W \text{ and } v - v' \in \{ \pm(1, 1), \pm(1, -1) \}. \]

Thus, \( \Gamma \) is the dual graph of the \textit{square-octagon} lattice graph (see Figure 1). We say a vertex is white (resp. black) if it is in \( W \) (resp. \( B \)). We call an edge connecting two white vertices a \textit{diagonal} edge. The edge set \( E(\Gamma) \) of \( \Gamma \) is divided into two disjoint subsets \( E_1 \) and \( E_2 \), where \( E_2 \) is the set of diagonal edges and \( E_1 = E(\Gamma) \setminus E_1 \). Therefore, the graph \( \Gamma \) is obtained from the ordinary square lattice graph by adding the edges \( E_2 \). We denote by \( \{v, v'\} \) the unoriented edge between two vertices \( v \) and \( v' \). In the following we sometimes need to orient the edges, and we denote by \((v, v')\) the oriented edge \textit{from} \( v \) \textit{to} \( v' \).

A dimer covering (or perfect matching) \( M \) of a graph \( G = (V(G), E(G)) \) is a subset of the edge set \( E(G) \) such that each element of the vertex set \( V(G) \) is incident to exactly one element of \( M \). We call an edge \( e \) in a dimer covering a \textit{dimer}. We say a subgraph \( G \) of \( \Gamma \) is \textit{simply connected}, if \( G \) and \( \Gamma \setminus G \) are both connected. We say a subgraph of \( \Gamma \) is \textit{normal}, if it is simply connected and induced by a finite subset of \( V(\Gamma) \). This paper deals with the dimer coverings of normal subgraphs of \( \Gamma \). A dimer covering of a normal graph is equivalent to a tilings of the corresponding region by square-octagon and octagon-octagon tiles (see Figure 2).

![Figure 2: Tiling and dimer covering](image)

For a normal subgraph \( G \) of \( \Gamma \), we denote

\[ W_G = V(G) \cap W, \quad B_G = V(G) \cap B. \]

Let \( M \) be a dimer covering of a normal graph \( G \) and let \( k \) be the number of diagonal edges in \( M \). Then

\[ k = \frac{|W_G| - |B_G|}{2}. \]

(1)

Hence, the number of diagonal edges in a dimer covering \( M \) of \( G \) is an invariant of \( G \), not depending on the choice of \( M \). If a dimer covering \( M \) of \( G \) does not contain diagonal edge, then it is a dimer covering of the ordinary square lattice graph, also known as the \textit{domino tiling}, which has been extensively studied. In this respect, it may be natural to call a dimer \( e \in E_2 \) of \( G \) \textit{impurity}. Our main aim in this paper is to study the behavior of these impurities. In our forthcoming paper it will be shown that the local transformations which will be introduced as the \( t \)-moves and the \( s \)-moves in the next section connects all dimer coverings, that is, any dimer covering of a normal graph \( G \) can be transformed into any other dimer covering of \( G \) by applying some sequence of the local transformations. This property enables one to construct an ergodic Markov chain whose
state space is the dimer coverings. Figure 3 shows the result of a simulation of the Markov chain whose stationary distribution is uniform, where we can see that the impurities tend to be located near the diagonal edges on the boundary of the graph.

Figure 3: Markov chain simulation: the initial configuration (left) and the configuration after $10^6$ steps (right)

The rest of this paper is organized as follows. Section 2 provides some basic properties of the dimer model on $\Gamma$. In Section 3 we show an exact result on the easiest case where dimer coverings contains exactly one impurity.

## 2 Local moves and impurities’ orbits

Let $\{a, b\}$ and $\{c, d\}$ be dimers contained in a dimer covering $M$ of a normal graph $G$, which satisfy one of the followings:

- **S**: $a, b, c, d$ are the four vertices of a unit square.
- **T**: $\{a, b\}, \{b, c\} \in E_2$ and $\{c, d\}, \{d, a\} \in E_1$.

In case of **S** (resp. **T**), we call the transformation which transforms $M$ into another by replacing $\{\{a, b\}, \{c, d\}\}$ with $\{\{b, c\}, \{d, a\}\}$ an *s-move* (resp. *t-move*), which is shown in Figure 4.

![Figure 4: local moves](image-url)
We divide the white vertices $W$ into two parts $W_0 = 2\mathbb{Z} \times 2\mathbb{Z}$ and $W_1 = W_0 + (1,1)$ and define two graphs $\Lambda$ and $\Lambda^\perp$ as follows: $\Lambda$ has vertices $W_0$ and it has an edge between $v$ and $v' \in W_0$ if and only if $v - v' \in \{\pm(2,0), \pm(0,2)\}$, $\Lambda^\perp$ is the dual graph of $\Lambda$ having vertices $W_1$. Let $w_1, w_2, w_3, w_4$ be four white vertices which are adjacent to a black vertex $b$ listed in counter-clockwise order as shown in Figure 5. Then one of the two sets $\{w_1, w_3\}$ and $\{w_2, w_4\}$ is contained in $W_0$ and the other is in $W_1$. Let us assume that $w_1, w_2$ and $b$ are contained in a normal graph $G$ and $w_3$ and $w_4$ are not necessarily contained in $G$. For a dimer covering $M$ of $G$ we draw an arc centered at $w_1$ (resp. $w_2$) which starts at the middle point of the edge $\{w_1, w_2\}$ and ends at a point on the edge $\{w_1, b\}$ (resp. $\{w_2, b\}$) if $\{w_2, b\}$ or $\{w_4, b\}$ (resp. $\{w_1, b\}$ or $\{w_3, b\}$) is contained in $M$. Then a dimer covering $M$ of $G$ defines curves on the plane composed of these arcs, which we call the slit-curves. Figure 6 shows an example of slit-curves.

Figure 5: Arcs and dimers

Figure 6: Left: Slit-curves generated by a dimer covering. Right: Primary forest (solid) and dual forest (dashed)
When a $t$-move $\tau$ transforms a dimer covering $M$ into another dimer covering $M'$ by replacing an impurity $e$ with another impurity $e'$, we simply say $\tau$ transforms $e$ to $e'$ and simply denote as

$$e' = \tau(e).$$

**Proposition 1.** A $t$-move keeps all slit-curves unchanged. There exists a sequence of $t$-moves which transforms an impurity $e$ to another impurity $e' \in E_2$ if and only if $e$ and $e'$ intersects with a common slit-curve.

**Proof.** From the definition, a $t$-move clearly keeps slit-curves unchanged. It is clear that an impurity $e$ can be transformed to $e'$ by one $t$-move if and only if $e$ and $e'$ have a common terminal vertex and there exists a slit-curve which intersects with both of $e$ and $e'$. Now the last statement can be easily proved by the induction on the length of the portion of the slit-curve from $e$ to $e'$. □

**Corollary 1.** If a slit-curve does not terminate on a diagonal edge, it does not intersects with impurities. For each slit-curve $C$, there is at most one impurity intersecting with $C$.

**Proof.** Assume that a slit-curve terminates on an edge $\{v_1, v_2\} \in E_1$ as shown in Figure 7. Then the dimer $\{v_2, v_3\}$ must be contained in the dimer covering which generate the slit-curves. For the sake of contradiction, assume that an impurity intersects with this slit-curve. The impurity can not be transformed to $\{v_1, v_3\}$ in the figure since the vertex $v_3$ must be incident to a dimer $\{v_2, v_3\}$, which contradicts Proposition 1. The last statement follows from Proposition 1 and the fact that a $t$-move can transform only one impurity.

![Figure 7: A slit-curve terminating on an edge in $E_1$](image)

**Proposition 2.**

1. A slit-curve does not intersects with another slit-curve.

2. A slit-curve does not form a loop.

**Proof.** From the definition of the slit-curves, 1 is obvious. For the sake of the contradiction, assume that a slit-curve $C$ form a loop. Without loss of generality we may assume $C$ is the innermost loop. Then the subgraph of $G$ induced by the vertices contained inside $C$ is a tree, since otherwise a slit-curve exists inside $C$ and it must form a loop. Since the tree $T$ inside $C$ can not have black leaves, i.e., black vertices which is incident to only one edge of $T$ and hence $T$ has odd number of vertices, $n$ whites and $n - 1$ blacks. Thus $C$ must intersect with an impurity. Let us denote by $B_C$ (resp. $W_C$) the black (resp. white) vertices which are outside of $C$ and adjacent to vertices inside $C$. Then by the induction on the number of white vertices inside $C$, we have $|B_C| = |W_C|$. (See Figure 8) Let $b \in B_C$. Then there exists exactly one vertex $w \in W_C$ such that $\{b, w\} \in M$. Therefore every element of $W_C$ is incident to a dimer outside of $C$, which is a contradiction.
Figure 8: If a slit-curve $C$ formed a loop, there would be an odd number of vertices inside $C$ and no vacant vertex around $C$.

**Proposition 3.** Remove the edges of $G$ which intersect with slit-curves. Then, each of the connected components of the resulting graph is a tree, that is, the slit-curves determine a spanning forest of $G$.

**Proof.** If a connected component of the graph obtained by removing edges intersecting with slit-curves contains a loop, then a slit-curve must form a loop, which contradicts Proposition 2.

Each tree in the forest obtained by removing edges intersecting with slit-curves does not contain a path of the form like $\longrightarrow$, i.e., a path bended at a black vertex, hence it can be viewed as a tree in $\Lambda$ or $\Lambda^\perp$, by removing the edges incident to black leaves, i.e., the black vertices each of which is incident to exactly one edge in the tree. We call the set of these trees in $\Lambda$ (resp. $\Lambda^\perp$) the primary forest (resp. dual forest) obtained from $M$ and denote it by $F(M)$ (resp. $F^\perp(M)$).

Let $\mathcal{M}(G)$ denote the set of dimer coverings of a normal graph $G$. We introduce a relation $\sim$ on $\mathcal{M}(G)$ as follows: For two dimer coverings $M_1, M_2 \in \mathcal{M}(G)$, we define

$$M_1 \sim M_2$$

if $M_1$ can be transformed into $M_2$ by applying a sequence of $t$-moves. Then the relation $\sim$ is clearly a equivalence relation and the set of equivalence classes are denoted by $\mathcal{M}(G)/\sim$.

## 3 Configurations with only one impurity

In this section, we show an exact enumerative formula of dimer coverings of graphs of special shapes, each of which has dimer coverings with only one impurity.

### 3.1 Temperley bijection

Let $H$ be a simply connected subgraph of $\Lambda$. We define the dual graph $H^\perp$ of $H$ in the following way. $H^\perp$ has the vertices $V(H^\perp)$ consisting of vertices corresponding to the faces of $H$, more specifically,

$$V(H^\perp) = \{(x, y) \in W_1 = (1, 1) + 2\mathbb{Z} \times 2\mathbb{Z} \mid (x, y) \text{ is in a face of } H\} \cup \{f^*\},$$
where $f^*$ is the vertex taken from $W_1$ so that the graph induced by $V(H^\perp)$ in $\Lambda^\perp$ is simply connected in $\Lambda$. Let $d^*$ be the number of edges $l_i$ of $\Lambda$ which connect $f^*$ and other vertices in $V(H^\perp)$. Then, since $H$ is simply connected, $d^*$ is at most 3. $H^\perp$ has edges $E(H^\perp)$ each of which corresponds to an edge bounding a face of $H$. Then, by embedding $H$ and $H^\perp$ simultaneously into the plane so that each edge $e^\perp$ of $H^\perp$ crosses the corresponding edge $e$ of $H$ only once at the middle point of $e$, where we add a new black vertex, and we obtain a bipartite graph $N'$.

Figure 9: By superimposing $H$ and its dual $H^\perp$ and removing $f^*$ and $v^*$, we obtain a balanced bipartite graph $N$. $G$ is the normal graph induced by $V(N) \cup \{f^*, v^*\}$.

Remove from $N'$ the vertex $f^*$ and a vertex $v^* \in V(H)$ which is adjacent to $f^*$ in $\Gamma$ and incident to the outer face of $H$. Then we obtain a balanced bipartite graph $N$, which contains the same number of black and white vertices. Burton and Pemantle [2] (see also [6]) showed that there is a bijection between the set of dimer coverings of $N$ and the set $T$ of spanning trees of $H^\perp$. Here, we review this bijection. Let $T$ be a spanning tree of $H$. Then the edges of $H^\perp$ that do not cross the edges of $T$ form a spanning tree of $H^\perp$, called the dual tree and denoted by $T^\perp$. This correspondence makes a bijection between the set of spanning trees of $H$ and that of $H^\perp$. We define the root of $T$ (resp. $T^\perp$) to be $f^*$ (resp. $v^*$), and orient $T$ and $T^\perp$ so that they point toward the roots. Then the subset $M = \{(x, \frac{x+y}{2}) \mid (x, y) \in T \text{ or } T^\perp\}$ of edges of $N$ is a dimer covering of $N$, where $(x, y)$ denotes the oriented edge from $x$ to $y$. This map $T \mapsto M$ is the bijection called the Temperley bijection [2, 6]. Conversely, let $M$ be a dimer covering of $N$. Then the map

$$\varphi : M \mapsto T = \left\{ (x, y) \mid \{x, \frac{x+y}{2}\} \in M, x \in V(H) \right\}.$$  

is the inverse of this bijection.

Let $G$ be the normal subgraph of $\Gamma$ which is induced by the vertices $V(G) = V(N) \cup \{v^*, f^*\}$. (see Figure 9) Then a dimer covering $M$ of $G$ contains exactly one impurity. By Corollary [1] the slit-curve which intersects with the impurity terminates at the middle points of the two diagonal edges $e_1^*$ and $e_2^*$ on the boundary of $G$, each of which are adjacent to $f^*$. Therefore, there exists a unique dimer covering $M'$ such that $M \sim M'$ and $e_1^* \in M'$. Since $M'$ can be regarded as the
Lemma 1. Let $G$ be the graph described as above and let $M$ be a dimer covering of $G$. Then
\[ F(M) = \{ \varphi \circ \pi(M) \}. \]
Let $T^\perp$ be the dual tree of $\varphi \circ \pi(M)$. Then $F^\perp(M)$ can be obtained from $T^\perp$ by removing all edges incident to $f^*$ from $T^\perp$ except for edges in $\{l_1, \ldots, l_d^*\}$.

Proof. After removing all edges intersecting with slit-curves from $N$, the resulting trees whose white vertices are in $V(\Lambda)$ have no black leaves. Since $t$-moves keep the slit-curves unchanged, the slit-curves can not intersects with $T = \varphi(\pi(M))$ and $T^\perp$, and $T$ is connected, hence $F(M) = \{ \varphi \circ \pi(M) \}$. \hfill \Box

Theorem 1. Let $G$ and $H^\perp$ be graphs as described above. Let $\mathcal{M}$ be the set of the dimer coverings of $G$ and let $\mathcal{T}$ be the set of the spanning trees of $H$. Then the map
\[ \bar{\varphi} : (\mathcal{M}/ \sim) \ni [M] \mapsto \varphi \circ \pi(M) \in \mathcal{T} \]
is a bijection.

Proof. If $M_1 \sim M_2$ then $\pi(M_1) = \pi(M_2)$. Thus $\bar{\varphi}$ is well-defined. Since $\pi$ is surjective and $\varphi$ is a bijection, $\bar{\varphi}$ is also surjective. If $\varphi([M_1]) = \varphi([M_2])$, then $\pi(M_1) = \pi(M_2)$ and hence $M_1 \sim M_2$, that is, $\bar{\varphi}$ is injective. \hfill \Box

3.2 Probability of finding the impurity at a given site

Let $e = \{x, y\} \in E_2$ be a diagonal edge in a normal graph $G$. Then each class $[M] \in \mathcal{M}/ \sim$ contains at most one dimer covering with impurity $e$. Thus, to count the dimer coverings with a fixed impurity, we can instead count the trees of $H$ corresponding to such dimer coverings. Let $M \in \mathcal{M}$ be a dimer covering such that $T = \varphi([M])$. Then $M$ determines the slit-curves, among which the one $C^*$ intersecting with the impurity terminates at the middle points of $e_1^* = \{f^*, v^*\}$ and $e_2^*$. Thus the slit-curve $C^*$ surrounds a tree $T^* \in F^\perp(M)$. (See Figure 10) Therefore, we have,
Lemma 2. A spanning tree $T$ of $H$ can be represented as $T = \varphi \circ \pi(M)$ for some dimer covering $M$ containing impurity $e = \{x, y\}$ if and only if $x \in T^*$.

We choose a spanning tree $T$ of $H$ uniformly at random and define $p_v$ by

$$p_v = \Pr(v \in T^*) \quad (2)$$

for each $v \in V(H)$. To obtain a uniformly random spanning tree of $H$, we can instead choose a uniformly random spanning tree of $H^\perp$. By the last half of Lemma \[1\] and the results of Pemantle \[4\] on the random spanning trees and the loop erased random walks, $p_v$’s can be computed via the negative Laplacian which is defined as follows. The negative Laplacian $A' = (a_{i,j})$ of $H^\perp$ is the $|V(H^\perp)|$-dimensional square matrix defined as follows: Each of rows and columns of $A'$ corresponds to a vertex of $H^\perp$ and

$$a_{i,j} = \begin{cases} 
-1 & i \text{ and } j \text{ are adjacent,} \\
0 & i \text{ and } j \text{ are not adjacent,} \\
4 & i = j, i \neq f^*, \\
\text{the number of edges incident to } f^* & i = j = f^*. 
\end{cases}$$

Let $A$ be the matrix obtained by removing the row and column corresponding to $f^*$. Then the vector $p = (p_w)_{w \in V \setminus \{f^*\}}$ satisfies

$$Ap = b,$$

where $b = (b_w)_{w \in V \setminus \{f^*\}}$ is defined by

$$b_w = \begin{cases} 
1 & w \text{ is adjacent to } f^* \text{ in } \Lambda^\perp, \\
0 & \text{otherwise}.
\end{cases}$$

Theorem 2. Let $H$, $H^\perp$, $A$ and $p_v$ be as described above. Let $e = \{v, w\}$ be a diagonal edge of $G_{m,n}$, where $v \in H^\perp_{m,n}$. Then the number of dimer coverings containing the impurity $e$ is

$$|\det A|p_v.$$

Therefore the total number of dimer coverings of $G$ is

$$|\mathcal{M}| = |\det A| \left( 4 \sum_{v \in V(H^\perp)} p_v + d^* - 3 \right) = |\det A| \left( 4(1, A^{-1}b) + d^* + 1 \right),$$

where $1 = (1, 1, \ldots, 1)^t$ and $d^*$ is the number of edges in $\Lambda^\perp$ connecting $f^*$ and other vertices in $V(H^\perp)$.

Proof. By Kirchhoff’s Matrix-Tree Theorem (see Chapter 6 of \[1\]), the number of spanning trees of $H^\perp$ is $|\det A|$. Each $v \neq f^* \in V(H^\perp)$ is incident to exactly four diagonal edges of $G$ and $f^*$ is incident to $d^* - 1$ diagonal edges. \[\square\]
As an immediate corollary of Theorem 2 we have the following:

**Theorem 3.** If we choose a dimer covering of $G$ uniformly at random, the probability of finding a given impurity $e = \{v, w\}$ with $v \in V(H^\perp)$ is

$$p_v \frac{4}{4 \sum_{w \in V(H^\perp)} P_w + d^* - 3}.$$

**Example 1.** Let $H$ be the graph consisting of $n$ squares in $\Lambda$ as shown in the left side of Figure 11, and let $H^\perp$ have the vertex $f^* = (2n + 1, 1)$ and vertices indexed as shown in the middle of Figure 11. Then we obtain the graph $G$ as shown in the right side of Figure 11. Then we have

$$A = \begin{pmatrix} 4 & -1 \\ -1 & 4 & -1 \\ 0 & -1 & 4 & -1 \\ \vdots \\ 4 & -1 \\ 4 \end{pmatrix}.$$

The determinant of $A$ and the probability $p_j$ can be evaluated explicitly:

$$\det A = \frac{1}{2\sqrt{3}} \left(\lambda_+^{n+1} - \lambda_-^{n+1}\right),$$

where $\lambda_\pm = 2 \pm \sqrt{3}$.

$$p_j = \frac{p_1}{2\sqrt{3}} \left(1 - \left(\frac{\lambda_-}{\lambda_+}\right)^j\right),$$

where $p_1 = 2\sqrt{3} \left(\lambda_+^{n+1} - \lambda_-^{n+1}\right)^{-1}$. Thus we have

$$\sum_j p_j = \frac{p_1}{2\sqrt{3}} \left(\frac{\lambda_+^{n+1} - \lambda_+}{\lambda_+ - 1} - \frac{\lambda_-^{n+1} - \lambda_-}{\lambda_- - 1}\right) + 2 = \frac{p_1}{2\sqrt{3}} \lambda_+^n (1 + o(1)).$$

Thus the probability of finding an impurity $e = \{j, v\}$ in a random dimer covering is

$$\frac{1}{4} \lambda_+^{-(n-j)} (1 + o(1)).$$

![Figure 11: $H$ (left), $H^\perp$ (center), and $G$ (right)](image-url)
Example 2. Let $H$ be the graph as shown in the left side of Figure 12 and let $H^\perp$ be as shown in the middle of Figure 12. Then we obtain $G$ shown in the right side of Figure 12 and we have

$$A = \begin{pmatrix}
4 & -1 & -1 \\
-1 & 4 & 0 \\
-1 & 0 & 4
\end{pmatrix}, \quad b = (0, 1, 1)^t.$$

Thus, $\det A = 56$, $p = A^{-1}b = (\frac{1}{7}, \frac{2}{7}, \frac{2}{7})^t$ and the number of dimer coverings is 328. We have $56 \times \frac{2}{7} = 16$ dimer coverings with a fixed impurity incident to the vertex 3, which are shown in Figure 13.

![Figure 12: $H$, $H^\perp$ and $G$](image1)

![Figure 13: Dimer coverings with a fixed impurity](image2)

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