ZAGIER DUALITY AND INTEGRALITY OF FOURIER COEFFICIENTS FOR WEAKLY HOLOMORPHIC MODULAR FORMS

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Abstract. In this note, we generalize the isomorphism proved in [27] to the case when the discriminant form is not necessarily induced from real quadratic fields. In particular, this general setting includes all of the subspaces with \( \epsilon \)-conditions of weakly holomorphic modular forms of integral weight, only two special cases of which were treated in [27]. With this established, we shall prove the Zagier duality for the canonical bases. Finally we raise a question on the integrality of the Fourier coefficients of these bases elements, or equivalently we concern the existence of a Miller-like basis for vector valued modular forms.

Introduction

Let \( D \) be a discriminant form of even signature \( r \) and let \( k \in \mathbb{Z} \) such that \( k \equiv \frac{r}{2} \mod 2 \). \( D \) determines the so called Weil representation \( \rho_D \) of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{C}[D] \), the group algebra of \( D \). Let \( N \) be the level of \( D \) and \( \{ e_\gamma : \gamma \in D \} \) be the standard basis for \( \mathbb{C}[D] \). Then it is known that there is a uniquely determined Dirichlet character \( \chi_D \) modulo \( N \) such that \( \rho_D(M)e_0 = \chi_D(M)e_0 \) for each \( M \in \Gamma_0(N) \). This gives a map from vector valued modular forms of type \( \rho_D \) to scalar valued modular forms of level \( N \) and character \( \chi_D \). In the other direction, by averaging over the cosets of \( \Gamma_0(N) \) in \( \text{SL}_2(\mathbb{Z}) \), a scalar valued modular form can be sent to a vector valued modular form.

Some properties of these maps are known. By explicitly working out the formulas for the Weil representation \( \rho_D \), Scheithauer [23] proved the surjectivity of the second map when \( N \) is square-free. When \( D \) has an odd prime discriminant, by introducing the plus and minus subspaces, Bruinier and Bundschuh [4] proved that the above maps are actually inverse isomorphisms. The author [27] generalized their results to the case when the discriminant form is given by the norm form of a real quadratic field, by introducing subspaces for sign vectors \( \epsilon \). In particular, \( N \) is a fundamental discriminant of a real quadratic field. Corresponding results, such as rationality

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of Fourier coefficients and obstruction theorem for weakly holomorphic modular forms with prescribed principal parts, were also derived to this more general case. Those weakly holomorphic modular forms are holomorphic on the upper half plane but may possess poles at the cusps; they play an important role in Borcherds’s theory of automorphic products (See for example [1]). However, with the application of Borcherds’s automorphic products in mind, we did not consider such an isomorphism for a general sign vector $\epsilon$. As a first part of this paper, we shall work out those isomorphisms (Theorem 3.3), the obstruction theorem (Theorem 4.4) and other results. More precisely, we shall treat the case when $|D| = N$, $N = N_2N'_2$ with $N'_2$ odd square-free and $N_2 = 1, 4$ or $8$, and $\chi_D$ is a primitive real character of modulus $N$. As we shall see, by varying $D$, our isomorphisms cover all $\epsilon$-subspaces.

In a paper of Zagier [26], he proved a duality, known as Zagier duality, between the Fourier coefficients of certain weakly holomorphic modular forms of weight $1/2$ and the Fourier coefficients of certain weakly holomorphic modular forms of weight $3/2$.

Zagier duality has been discovered in many cases ever since. In the half integral weight case, Bringmann and Ono [3] proved Zagier duality between Maass-Poincaré series and Poincaré series, Folsom and Ono [11] discovered Zagier duality between weight $1/2$ harmonic weak Maass forms and weight $3/2$ weakly holomorphic modular forms, and Kim [13] found Zagier duality for weakly holomorphic modular forms of level $4p$ for some suitable prime $p$. For a more detailed description of these results, please see the introduction in [5]. In the case of integral weight, Cho and Choie [5] proved such duality between vector valued harmonic weak Maass forms and vector valued weak holomorphic modular forms, generalizing Guerzhoy’s result [12] for the full level where Guerzhoy called these pairs grids. Rouse [21], with the nice isomorphism between vector valued and scalar valued modular forms proved by Bruinier and Bundschuh in [4], found the duality between weakly holomorphic modular forms of level $p = 5, 13, 17$. His argument involves the explicit decomposition into the plus and the minus subspaces and the explicit action of Hecke operators. Later Choi [6] gave a simpler proof of this duality, where everything boils down to the well known residue theorem on compact Riemann surfaces. Interestingly, Duke and Jenkins [10] proved Zagier duality for level 1 weakly holomorphic modular forms by finding a canonical basis, a Miller-like basis, and then a double variable generating series. Choi and Kim [7] generalized their results to the case of prime level $p$ such that the genus of the $\Gamma_0^+(p)$ is 0.

With our isomorphisms and the obstruction theorem established in the first part, we will introduce the notion of reduced modular forms. These modular forms form a basis for the whole space of weakly holomorphic modular forms with some $\epsilon$-condition and we called it the canonical
basis. We then prove the Zagier duality for such canonical bases (Theorem 5.7); namely such canonical bases and their dual bases make up the grids for the Zagier duality.

Finally, we consider the integrality of Fourier coefficients $a(n)$ of our bases of reduced modular forms. This is crucial in [14] and [15], since there the coefficients $s(n)a(n)$ represent the multiplicity of roots in some generalized Kac-Moody superalgebras. Here $s(n)$ appears in our isomorphism and is defined to be $2^{\omega((n,N))}$ (Section 1). It is well-known that there exists the so called Miller basis $\{f_1, f_2, ..., f_d\}$ for the space of level 1 cusp forms. More precisely, all Fourier coefficients are integral and the first $d$ coefficients of the basis give us the identity matrix. Such a basis can be extended to the space of holomorphic modular forms easily, and further to that of weakly holomorphic modular forms ([10]).

In the case of higher level, cusps other than $\infty$ appear. Since we know that the holomorphy of a modular form with some $\epsilon$-condition at $\infty$ dominates that at any other cusp (Proposition 3.4 and Corollary 3.6), it is natural to consider these modular forms. Because of the isomorphism, it is better, actually stronger, to require that $s(n)a(n)\in\mathbb{Z}$ for a modular form $f = \sum_n a(n)q^n$. After refining the definition of a Miller basis, we propose the problem on the existence of the Miller basis. Roughly speaking, this concerns the existence of a Miller-like basis for vector valued modular forms. The existence of the canonical basis shows that if a Miller basis exists it has to be the canonical basis up to scalars, so the existence of a Miller basis is equivalent to our stronger integrality of the canonical basis. We do not know how to solve such a problem systematically for all $N$. However, we can still do something. For example, we can reduce the problem to testing a finite number of reduced modular forms for each fixed $N$, hence to computational verification by Sturm’s theorem.

Here is the layout of this paper according to sections.

1. We provide definitions and fix notations for discriminant forms and modular forms.
2. We fix one discriminant form and the corresponding sign vector $\epsilon$, and we also reproduce some results in [27] to our more general setting.
3. We establish the isomorphism and consider the behavior of a modular form with $\epsilon$-condition at other cusps.
4. We prove the obstruction theorem and consider the rationality of Fourier coefficients.
5. We introduce reduced modular forms and prove the Zagier duality. Some examples are presented.
6. We propose the problem on the integrality of Fourier coefficients of reduced modular forms and provide some ideas and one example on how to solve this computationally.
Throughout this paper, the word “isomorphism” is reserved for the isomorphism in Theorem 3.3 and $p$ will always denote a prime. For most of the time, $k$ is an integer and stands for the weight of some modular forms, and $q = e(\tau)$ with $e(\tau) = e^{2\pi i \tau}$, except that when we treat discriminant forms $k = 0$ or 1 in the $p$-excess and $q$ denotes a prime power.

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1. Preliminaries

We recall some definitions on discriminant forms and modular forms, and fix some notations. For more details on discriminant forms, one may consult [8], [20], or [22].

A discriminant form is a finite abelian group $D$ with a quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, such that the symmetric bilinear form defined by $(\beta, \gamma) = q(\beta + \gamma) - q(\beta) - q(\gamma)$ is nondegenerate, namely, the map $D \to \text{Hom}(D, \mathbb{Q}/\mathbb{Z})$ defined by $\gamma \mapsto (\gamma, \cdot)$ is an isomorphism. We shall also write $q(\gamma) = \gamma^2$. We define the level of a discriminant form $D$ to be the smallest positive integer $N$ such that $Nq(\gamma) = 0$ for each $\gamma \in D$. It is well-known that if $L$ is an even lattice then $L'/L$ is a discriminant form, where $L'$ is the dual lattice of $L$. Conversely, any discriminant form can be obtained this way. With this, we define the signature $\text{sign}(D) \in \mathbb{Z}/8\mathbb{Z}$ to be the signature of $L$ modulo 8 for any even lattice $L$ such that $L'/L = D$.

Every discriminant form can be decomposed into a direct sum of Jordan $p$-components for primes $p$ and each Jordan $p$-component can be written as a direct sum of indecomposible Jordan $q$-components with $q$ powers of $p$. Such decompositions are not unique in general. To fix our notations, we recall the possible indecomposible Jordan $q$-components as follows.

Let $p$ be an odd prime and $q > 1$ be a power of $p$. The indecomposable Jordan components with exponent $q$ are denoted by $q^\delta$ with $\delta = \pm 1$; it is a cyclic group of order $q$ with a generator $\gamma$, such that $q(\gamma) = \alpha_q$ and $\delta = \left(\frac{2\alpha_q}{p}\right)$. These discriminant forms both have level $q$.

If $q > 1$ is a power of 2, there are also precisely two indecomposable even Jordan components of exponent $q$, denoted $q^\delta 2$ with $\delta = \pm 1$; it is a direct sum of two cyclic groups of order $q$, generated by two generators $\gamma, \gamma'$, such that

- $q(\gamma) = q(\gamma') = 0$ and $(\gamma, \gamma') = \frac{1}{q}$, if $\delta = 1$,
- $q(\gamma) = q(\gamma') = \frac{1}{q}$ and $(\gamma, \gamma') = \frac{1}{q}$, if $\delta = -1$. 

Such components have level $q$. There are also odd indecomposable Jordan components in this case, denoted by $q_t^{±1}$ with $±1 = (2/t)\mod 8$. Explicitly, $q_t^{±1}$ is a cyclic group of order $q$ with a generator $γ$ such that $q(γ) = 2t$. Clearly, these discriminant forms have level $2q$.

To give a finite direct sum of indecomposable Jordan components of the same exponent $q$, we multiply the signs, add the ranks, and add all subscripts $t$ ($t = 0$ if there is no subscript). So in general, the $q$-component of a discriminant form is given by $q_t^{δ_qn}$ where $δ_q = −1$ if $q$ is odd or the form is even). Set $k = k(q_t^{δ_qn}) = 1$ if $q$ is not a square and $δ_q = −1$, and 0 otherwise. If $q$ is odd, then define $p$-excess($q_t^{±n}$) = $n(q−1) + 4k \mod 8$, and if $q$ is even, then define oddity($q_t^{±n}$) = $2$-excess($q_t^{±n}$) = $t + 4k \mod 8$.

Let $D$ be a discriminant form and assume that $D$ has a Jordan decomposition $D = \oplus q_t^{δ_qnq}$ where the sum is over distinct prime powers $q$. Then

$$p\text{-excess}(D) = \sum_{q|p/q} p\text{-excess}(q_t^{δ_qnq}).$$

We have the oddity formula:

$$\text{sign}(D) + \sum_{p>2} p\text{-excess}(D) = \text{oddity}(D) \mod 8.$$

Throughout this note, $k$ will be an integer and $\mathbb{H}$ will denote the upper half plane. Let $M ∈ GL_2^+(\mathbb{R})$, a real square matrix of size two and of positive determinant, and $f$ be a function on $\mathbb{H}$. The weight-$k$ slash operator of $M$ is defined as

$$(f|_k M)(τ) = (\det(M))^k/2(cτ + d)^{-k}f(Mτ), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $τ$ is the variable on $\mathbb{H}$ and $Mτ = (aτ + b)(cτ + d)^{-1}$. In $GL_2^+(\mathbb{R})$, we denote

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W(m) = \begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix}, \quad V(m) = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix},$$

for a positive integer $m$. We know that $T, S$ are the standard generators for $SL_2(\mathbb{Z})$. Given any discriminant form $D$, let $r = \text{Sign}(D)$ denote the signature of $D$; we assume throughout this note that $r$ is even and $k \equiv r/2 \mod 2$. Let $\{e_γ : γ ∈ D\}$ be the standard basis of the group algebra $\mathbb{C}[D]$. The Weil representation $ρ_D$ attached to $D$ is a unitary representation of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$ such that

$$ρ_D(T)e_γ = e(q(γ))e_γ,$$
\[ \rho_D(S)e_\gamma = \frac{i^{-\frac{r}{2}}}{\sqrt{|D|}} \sum_{\beta \in D} e(-\langle \beta, \gamma \rangle) e_\beta, \]

where \( e(x) = e^{2\pi i x} \) and \(|D|\) is the order of \( D \). In particular, we have \( \rho_D(-I)e_\gamma = (-1)^r/2 e_{-\gamma} \).

Denote by \( \text{Aut}(D) \) the automorphism group of \( D \), that is, the group of group automorphisms of \( D \) that preserve the norm (or the quadratic form). The action of elements in \( \text{Aut}(D) \) and that of \( \rho_D \) commute on \( \mathbb{C}[D] \). We caution here that our \( \rho_D \) is the same as that in [1] and in [4], but conjugate to the one used in [22] and [23].

We denote by \( \mathcal{A}(k, \rho_D) \) the space of functions \( F = \sum_{\gamma \in D} F_\gamma e_\gamma \) on \( \mathbb{H} \), valued in \( \mathbb{C}[D] \), such that

- \( F|kM := \sum_{\gamma} F_\gamma|kMe_\gamma = \rho_D(M)F \) for all \( M \in \text{SL}_2(\mathbb{Z}) \),
- \( F \) is holomorphic on \( \mathbb{H} \) and meromorphic at \( \infty \); namely, for each \( \gamma \in D \), \( F_\gamma \) is holomorphic on \( \mathbb{H} \) and has Fourier expansion at \( \infty \) with at most finitely many negative power terms.

More explicitly, if \( F = \sum_{\gamma} F_\gamma \in \mathcal{A}(k, \rho_D) \), then

\[ F_\gamma(\tau) = \sum_{n \in q(\gamma) + \mathbb{Z}, n \gg -\infty} a(\gamma, n)q^n. \]

Denote by \( \mathcal{M}(k, \rho_D) \) and \( S(k, \rho_D) \) the subspace of holomorphic modular forms and the subspace of cusp forms, respectively. We define \( \mathcal{A}^{\text{inv}}(k, \rho_D) \) to be the subspace of functions that are invariant under \( \text{Aut}(D) \). The assumption that \( k \equiv \frac{r}{2} \) actually says \( F_\gamma = F_{-\gamma} \) for \( F \in \mathcal{A}(k, \rho_D) \), so it must be imposed if we would like to have \( F \in \mathcal{A}^{\text{inv}}(k, \rho_D) \), since \( \gamma \mapsto -\gamma \) defines an element in \( \text{Aut}(D) \). Similarly, we define \( \mathcal{M}^{\text{inv}}(k, \rho_D) \) and \( S^{\text{inv}}(k, \rho_D) \).

For each positive integer \( N \), let \( \Gamma_0(N) \) denote the congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) whose elements have left lower entry divisible by \( N \). For each Dirichlet character \( \chi \) of modulo \( N \), we denote by \( A(N,k,\chi) \) the space of weakly holomorphic modular forms of level \( N \), weight \( k \) and character \( \chi \); namely the space of holomorphic functions \( f \) on \( \mathbb{H} \) such that \( f|M = \chi(M)f \) for each \( M \in \Gamma_0(N) \) and \( f \) is meromorphic at cusps. The subspace of holomorphic forms and that of cuspforms are denoted by \( M(N,k,\chi) \) and \( S(N,k,\chi) \) respectively. Clearly, for these modular form spaces to be non-zero, we need \( \chi(-1) = (-1)^k \). In the next section, for fixed discriminant forms \( D \), we shall see that \( \rho_D \) determines a Dirichlet character \( \chi_D \) and \( \chi_D(-1) = (-1)^{\frac{r}{2}} \). So the conditions we impose on \( k \) are consistent.

Let \( f = \sum_n a(n)q^n \in A(N,k,\chi) \). Then \( P(q^{-1}) = \sum_{n<0} a(n)q^n \) is a polynomial without constant term in \( q^{-1} \) and we call \( P(q^{-1}) \) the principal part of \( f \) (at \( \infty \)).
For any positive integer $m$, we denote by $\omega(m)$ the number of distinct prime divisors of $n$. For any pair $m, N$ of integers, we denote by $(m, N)$ the greatest common divisor of $m$ and $N$, which should not be confused with the bilinear form. If $N > 0$ and $m > 1$, we denote $N_m$ to be the $m$-part of $N$; that is, $N_m \mid N$ is positive, contains only primes that divide $m$, and $(N/N_m, m) = 1$. If $p$ be a prime and $l$ a non-negative integer, we denote $p^l \mid |N$ if $p^l \mid N$ but $p^{l+1} \nmid N$.

For a Dirichlet character $\chi$ modulo $N$, we shall denote its $p$-component by $\chi_p$, hence $\chi = \prod_{p \mid N} \chi_p$. For each positive divisor $m$ of $N$, we define $\chi_m = \prod_{p \mid m} \chi_p$ and $\chi'_m = \prod_{p \mid m} \chi_p$. Let $W(\chi)$ denote the Gauss sum of $\chi$ and we know that if $p > 2$ and $\chi_p = \left(\frac{\cdot}{p}\right)$, then $W(\chi_p) = \varepsilon_p p^{\frac{1}{2}}$; here $\varepsilon_p = 1$ if $p \equiv 1 \mod 4$, and $i$ if $p \equiv 3 \mod 4$. Similarly,

- if $\chi_2 = \left(\frac{-1}{2}\right)$, $W(\chi_2) = \varepsilon_2 4^{\frac{1}{2}}$ with $\varepsilon_2 = i$;
- if $\chi_2 = \left(\frac{2}{2}\right)$, $W(\chi_2) = \varepsilon_2 8^{\frac{1}{2}}$ with $\varepsilon_2 = 1$;
- if $\chi_2 = \left(\frac{-2}{2}\right)$, $W(\chi_2) = \varepsilon_2 8^{\frac{1}{2}}$ with $\varepsilon_2 = i$.

Here $\varepsilon_p$ is not to be confused with the sign vectors $\epsilon_p$ defined in the following section.

For integers $i, j$, we define $\delta_{i,j} = 1$ if $i = j$, and $0$ otherwise.

## 2. Discriminant Forms and $\epsilon$-Condition

In this section, we first fix a discriminant form and investigate its properties, and then define the $\epsilon$-condition on scalar valued modular forms.

From now on and until the end of Section 3, we fix a discriminant form $D = \oplus_p D_p$ of the following form: if $p > 2$, then $D_p = p^{\delta_p}$ with $\delta_p \in \{\pm 1\}$; $D_2$ is trivial, or $2t_1^2$ with $t \in \{\pm 2\}$, or $2^{t_1} \oplus 4^{t_2}$ with $\delta_2 = \left(\frac{2}{t_2}\right)$ and $t_1 \in \{\pm 1\}$, $t_2 \in \{\pm 1, \pm 3\}$. We denote the level of $D$ by $N$; note that $|D| = N$. Note that $D_2$ means something else in [22].

Now let $D^* = D[-1] = \oplus_p D_p^*$ be the discriminant form with the same group but with quadratic form $q^* = -q$. We call $D^*$ the dual of $D$. It is not hard to see that if $p > 2$ then $D_p^* = p^{\delta_p^*}$ with $\delta_p^* = \left(\frac{-1}{p}\right)\delta_p$. If $D_2 = 2t_1^{t_2}$, then $D_2^* = 2^{\frac{1}{t_2} - t_1}$, and if $D_2 = 2^{t_1} \oplus 4^{t_2}$ then $D_2^* = 2^{\frac{1}{t_1}} \oplus 4^{\frac{1}{t_2} - t_2}$. It is clear that $D^*$ has the same level $N$ as $D$ does.

For a modular form $F \in \mathcal{A}^{\text{inv}}(k, \rho_D)$, define $W$ the span of $F_\gamma$, $\gamma \in D$, and $W'$ the span of $F_0|N$, $M \in SL_2(\mathbb{Z})$. Let $W_0$ be the subspace of $T$-invariant functions in $W$. 
Lemma 2.1. If $\beta, \gamma \in D$ with $q(\beta) = q(\gamma)$, then there exists $\sigma \in \text{Aut}(D)$ such that $\sigma \beta = \gamma$.

Lemma 2.2. Let $S \subset D$. If $\sum_{\gamma \in S} F_\gamma \in W'$, then $F_\gamma \in W'$ for any $\gamma \in S$.

Lemma 2.3. $W_0 = \text{span}_\mathbb{C}\{F_0\}$. Actually, if $f = \sum_{\gamma \in D} a_\gamma F_\gamma \in W_0$, then $f = a_0 F_0$.

Lemma 2.4. $W = W'$. In particular, if $F_0 = 0$, then $F = 0$.

Define the primitive Dirichlet character $\chi = \chi_D = \prod_p \chi_p$ of modulus $N$ as follows: if $p > 2$ and $p \mid N$, then $\chi_p = \left( \frac{p}{N} \right)$; $\chi_2$ is trivial if $D_2$ is trivial, $\chi_2 = (\frac{-1}{-N})$ if $D_2 = 2t^2$, and $\chi_2 = (\frac{-2a}{-N})$ with $a = \left( \frac{t_1}{t_2} \right)$ if $D_2 = 2t_1^2 \oplus 4t_2^2$. Such a character is determined by the Weil representation associated to $D$, justifying the notation.

For each sign vector $\epsilon' = (\epsilon'_p)_{p \mid N} \in \{ \pm 1 \}^{\omega(N)}$, we define the subspace in $A(N,k,\chi_D)$

$$A'(N,k,\chi_D) = \left\{ f = \sum_n a(n)q^n \in A(N,k,\chi_D) \left| a(n) = 0, \text{ if } (n,N) = 1 \text{ and } \chi_p(n) = -\epsilon'_p \text{ for some } p \mid N \right\}. \right.$$ 

and we know that $A(N,k,\chi_D) = \oplus_{\epsilon'} A'(N,k,\chi_D)$ (Proposition 4.9, [27]) where $\epsilon'$ runs through the whole set $\{ \pm 1 \}^{\omega(N)}$. The following lemma is Corollary 4.12 in [27].

Lemma 2.5. Assume $f \in A(N,k,\chi_D)$. Then $f \in A'(N,k,\chi_D)$ if and only if

$$f|_k U(N_p) \eta_p = \epsilon'_p \epsilon_p \chi_p(-1)N_p^{k+1} f, \text{ for each } p \mid N.$$ 

It follows that we may drop the condition $(n,N) = 1$ from the definition of $A'(N,k,\chi_D)$. See next section for the meaning of these operators.

Among these subspaces, we specify one of them, $A'(N,k,\chi)$, with $\epsilon = (\epsilon_p)_{p \mid N}$ defined as follows: if $p > 2$, then $\epsilon_p = \chi_p(2N/p)\delta_p$; if $D_2 = 2t^2$, $\epsilon_2 = \chi_2(Nt/8)$, and if $D_2 = 2t_1^2 \oplus 4t_2^2$, we set $\epsilon_2 = \chi_2(t_2N/8)$. By the definition of $D^*$, it can be seen easily that the sign vector $\epsilon^*$ for $D^*$ is given by $\epsilon^*_p = \chi_p(-1)\epsilon_p$ for each $p \mid N$.

For our choice of $N$, it is well-known that the inequivalent cusps of $\Gamma_0(N)$ are represented precisely by $\frac{1}{m}$ with $m$ running over the positive divisors of $N$. In particular, $\frac{1}{N} \sim \infty$ and $1 \sim 0$ as cusps for $\Gamma_0(N)$. 

We reproduce a few lemmas but omit their proofs since the corresponding proofs in [27] can be carried over. They correspond to Proposition 3.3, Lemma 4.1, Lemma 4.2 and Proposition 4.3 in [27], respectively.
The Isomorphism

In this section, we will generalize the results in [27]. Actually, we show that the isomorphism in [27] and other results also hold for our general sign vector \( \epsilon \). We shall be brief on proofs in this section and for more details please see [27].

Before we establish the isomorphism between \( \mathcal{A}^{\text{inv}}(k, \rho_D) \) and \( \mathcal{A}^{\epsilon}(N,k,\chi_D) \), we first recall some Hecke operators. For \( m \mid N \), the Hecke operator \( U(m) \) on \( \mathcal{A}(N,k,\chi_D) \) is defined as

\[
(f|kU(m))(\tau) = m^{\frac{k}{2}-1} \sum_{j \mod m} f \begin{pmatrix} 1 \\ j \\ m \end{pmatrix}.
\]

If \( f = \sum_{n \in \mathbb{Z}} a(n)q^n \), then \( f|kU(m) = \sum_{n \in \mathbb{Z}} a(mn)q^n \).

We shall need the so-called \( W \)-operators; here we follow Miyake’s notations ([19]) and denote them by \( \eta_m \). For a positive divisor \( m \) of \( N \), choose \( \gamma_m \in \text{SL}_2(\mathbb{Z}) \) such that

\[
\gamma_m \equiv \begin{cases} S & \text{mod } (N_m)^2 \\ I & \text{mod } (N/N_m)^2 \end{cases},
\]

and define \( \eta_m = \gamma_m V(N_m) \) and denote \( \eta'_m = \eta_{N/N_m} \). Recall that \( N_m \) means the \( m \)-part of \( N \). For completeness, we copy the following lemma from Lemma 2.1 in [27]. See Section 1 for the meaning of other notations.

**Lemma 3.1.** Let \( f \in A(N, k, \chi_D) \) and \( m, m_1, m_2 \) be positive divisors of \( N \).

1. The action \( f|k\eta_m \) is independent of the choice of \( \gamma_m \) and it defines an operator on \( A(N, k, \chi_D) \).
2. \( f|k\eta_N = f|kW(N) \).
3. If \( (m_1, m_2) = 1 \), \( f|k\eta_{m_1m_2} = \chi_{m_2}(N_{m_1})f|k\eta_{m_1}\eta_{m_2} \). In particular, \( f|k\eta_m\eta'_m = \chi'_m(N_m)f|kW(N) \).
4. Moreover, if \( m = p_1p_2\cdots p_k \) is square-free, then \( f|k\eta_m = \prod_{i<j} \chi_{p_i}(N_{p_j})f|k\eta_{p_1}\eta_{p_2}\cdots\eta_{p_k} \).
5. \( f|k\eta_{m}^{2} = \chi_{m}(-1)\chi'_{m}(N_{m})f \).
6. If \( (m_1, m_2) = 1 \), \( f|k\eta_{m_1}U(m_2) = \chi_{m_1}(m_2)f|kU(m_2)\eta_{m_1} \).

From now on, we shall drop the weight in the notations of the operators if no confusion is possible. We construct the isomorphisms in the following definition.

**Definition 3.2.** Define a map \( \phi : \mathcal{A}^{\text{inv}}(k, \rho_D) \to \mathcal{A}^{\epsilon}(N,k,\chi_D) \) by

\[
F \mapsto i^{\frac{k}{2}}2^{-\omega(N)}N^{-\frac{k-1}{2}}F_0|W(N).
\]
Conversely, we define \( \psi : A^e(N, k, \chi_D) \to A^{\text{inv}}(k, \rho_D) \) by

\[
f \mapsto i^s N^\frac{k-1}{2} \sum_{M \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} (f | W(N) | M) \rho_D(M^{-1}) e_0.
\]

For each integer \( n \) we define \( s(n) = 2^{\omega((n, N))} \); it depends on \( N \). For example, if \( N = 12 \), then \( s(0) = s(6) = 4 \) and \( s(2) = s(3) = 2 \).

**Theorem 3.3.** The maps \( \phi \) and \( \psi \) are inverse isomorphisms between \( A^{\text{inv}}(k, \rho_D) \) and \( A^e(N, k, \chi_D) \).

Explicitly, if \( f = \sum_n a(n)q^n \in A^e(N, k, \chi_D) \) and \( \psi(f) = F = \sum_{\gamma} F_\gamma e_\gamma \), then

\[
F_\gamma(\tau) = s(Nq(\gamma)) \sum_{n \equiv Nq(\gamma) \mod N\mathbb{Z}} a(n) q^n = \sum_{n \equiv Nq(\gamma) \mod N\mathbb{Z}} s(n) a(n) q^n.
\]

**Proof.** Following the same lines as in [27], we sketch the proof.

That \( \psi \) is well-defined follows easily, of which the invariance follows from the fact that the actions of \( SL_2(\mathbb{Z}) \) and \( \text{Aut}(D) \) on \( \mathbb{C}[D] \) commute. On the other hand, \( \phi(F) \) belongs to \( A(N, k, \chi) \) by Proposition 4.5 in [22]. To see that \( \phi(F) \) satisfies the \( \epsilon \)-condition, we observe that

\[
F_0 | W(N) = i^s N^\frac{k-1}{2} \sum_{\gamma \in D} F_\gamma(N\tau) = i^s N^\frac{k-1}{2} \sum_{n \in \mathbb{Z}} \left( \sum_{\gamma : q(\gamma) \equiv n^\frac{k}{N}} a(\gamma, nN^{-1}) \right) q^n := \sum_{n \in \mathbb{Z}} a(n) q^n.
\]

Now it is easy to see that \( q \) represents \( \frac{n}{N} \) if and only if \( q_p \) represents \( \frac{nN/N_p}{N_p} \) for each prime \( p | N \). We may verify case by case that \( a(n) = 0 \) if \( \chi_p(n) = -\epsilon_p \) for some \( p | N \). So \( \phi \) is well-defined.

The same argument in Proposition 4.5 of [27] can be carried over to prove that \( \psi \circ \phi = id \); note that we need Lemma 2.3. The proof of \( \phi \circ \psi = id \) is similar to that of Proposition 4.15 in [27], where explicit formulas in Theorem 4.7 of [22] and Lemma 3.1 are needed. In order to convince the reader that this is the case, we sketch the proof of \( \phi \circ \psi = id \) when \( D_2 = 2_{l_1}^1 \oplus 4_{l_2}^2 \). We leave other cases to the reader.

For each cusp \( s \), define

\[
F_s = (-1)^s \sum_{M \in \Gamma_0(N) \backslash SL_2(\mathbb{Z})} (f | W(N) | M) \rho_D(M^{-1}) e_0.
\]

It suffices to prove that \( \sum_s (F_s, e_0) | W(N) = 2^{\omega(N)} f \). Denote by \( m_1 \) an odd positive divisor of \( N \).

We first consider a cusp \( s \) of the form \( \frac{1}{N/m_1} \). Since the cosets in

\[\{ M \in \gamma_0(N) \backslash SL_2(\mathbb{Z}) : M \infty ~ s \}\]
can be represented by \( \{ \gamma_{m_1} T^j : j \mod m_1 \} \), we first note that
\[
(F_s, e_0)|W(N) = \left( -1 \right)^k m_1^{1-\frac{1}{2}} f|W(N)\eta_{m_1} U(m_1)W(N) \right),
\]
where \( B = (\rho_D(\gamma_{m_1}^{-1}) e_0, e_0) \) and \( A \) is the product of other factors. By repeatedly use of Lemma 3.1, one can show that
\[
A = m_1^{\frac{1}{2}} \chi_{m_1} (-2) \prod_{p|m_1} \delta_p \varepsilon_p \cdot f.
\]

By Theorem 4.7 in [22] (note that his Weil representations are conjugate to ours), we see that
\[
B = -\chi_2 (-1) \chi_{2m_1} (2) \left( \frac{-1}{t_1 t_2} \right) \prod_{p|m_1} \delta_p \varepsilon_p.
\]

Since \( \prod_{p|m_1} \varepsilon_p^2 = \chi_{m_1} (-1) \) and \( \chi_2 (-1) = \left( \frac{-1}{t_1 t_2} \right) \), we have \( AB = f \). Similar computations show that \((F_s, e_0)|W(N) = f\) if \( s \) is of the form \( \frac{1}{8m_1} \).

For a cusp \( s \) of the form \( \frac{1}{2m_1} \) or \( \frac{1}{4m_1} \). We claim that \((F_s, e_0)|W(N) = 0\). To explain this, we employ the notations in [22]. Let \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) such that \( M \infty \sim s \). Then we have \( 2 || c \) or \( 4 || c \). Therefore, \( x_c \neq 0 \) and \( 0 \not\in D^{c^*} \), and hence \((F_s, e_0)|W(N) = 0\).

Putting everything together, we see that \( \sum_s (F_s, e_0)|W(N) = 2^{\omega(N)} f \). \( \square \)

We now investigate the behavior of a weakly holomorphic form that satisfies the \( \epsilon \)-condition at all cusps.

**Proposition 3.4.** Let \( f = \sum_n a(n) q^n \in \mathcal{A}(N, k, \chi_D) \) and let \( s \) be a cusp and \( q_s \) be the local parameter at \( s \). Fix any positive odd divisor \( m_1 \) of \( N \). Then

1. If \( s \sim \frac{1}{m_1} \) or \( s \sim \frac{1}{2m_1} \), then the Fourier expansion of \( f \) at \( s \) contains precisely powers of the form \( q_{s}^n \) with \( a(nN/m) \neq 0 \).
2. If \( 4 \mid N \) and \( s \sim \frac{1}{2m_1} \), then the Fourier expansion of \( f \) at \( s \) contains at most powers of the form \( q_{s}^{n/2} \) with \( a(nN/2m_1) \neq 0 \).
3. If \( 8 \mid N \) and \( s \sim \frac{1}{4m_1} \), then the Fourier expansion of \( f \) at \( s \) contains at most powers of the form \( q_{s}^{n/2} \) with \( a(nN/4m_1) \neq 0 \).

**Proof.** We denote \( f = (\ast) g \) if \( f = cg \) for some \( c \in \mathbb{C}^\times \).
For (1), let \( m = m_1 \) or \( m = N_2m_1 \) accordingly. It is not hard to see that \( \gamma_{N/m} \sim s \), so it suffices to consider the Fourier expansion of \( f|\gamma_{N/m} \). Since \( f|U(N/m)\eta_{N/m} = (\ast)f \), we have

\[
f|\gamma_{N/m} = f|\eta_{N/m}V(N/m)^{-1} = (\ast)f|U(N/m)V(N/m)^{-1}.
\]

Since the width of the cusp \( s \) is \( N/m \), Part (1) follows.

For (2), let us deal with the case when \( 4 \mid N \). Let \( \beta_{2m_1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) be a matrix that is congruent to \( \left( \begin{array}{cc} 1 & 0 \\ 2 & 1 \end{array} \right) \mod 4^2 \), \( \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \mod (N/4m_1)^2 \), and \( \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \mod m_1^2 \).

Clearly, \( \beta_{2m_1} \sim s \). Let us pass to vector valued modular forms. From the isomorphism \( f \mapsto F \), we see that

\[
f|\beta_{2m_1} = (\ast)F_0|W(N)\beta_{2m_1} = (\ast)F_0 \left| \begin{array}{cc} -c & -d \\ Na & Nb \end{array} \right|.
\]

Since \( (c, N) = 2m_1 \), we choose any integers \( u, v \) such that \( \beta = \left( \begin{array}{cc} -c/2m_1 & u \\ aN/2m_1 & v \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \). We then have

\[
f|\beta_{2m_1} = (\ast)F_0 \left| \beta \left( \begin{array}{cc} 2m_1 & -vd - uNb \\ 0 & N/2m_1 \end{array} \right) \right|.
\]

By Theorem 4.7 in [22], we have

\[
F_0|\beta = (\ast) \sum_{\gamma \in D'_{c'}} e(d'\gamma_2^2/2)F_{\gamma},
\]

where \( d' = aN/2m_1 \), \( d' = v \), \( D'_{c'} = \gamma_2 + \gamma_2' + \oplus_{p|m_1}D_p \) with \( \gamma_2, \gamma_2' \) the generators of \( D_2 \), and for the meaning of \( \gamma_2^2/2 \) see [22]. In particular, if \( F_\gamma \) appears in \( F_0|\beta \), we must have \( q(\gamma) \in \frac{1}{2m_1} \mathbb{Z} \).

Moreover, from the isomorphism, we see that such \( F_\gamma \) contains only \( q^{s} \) with \( a(n) \neq 0 \) and \( \frac{N}{2m_1} \mid n \). Therefore, since the width at \( s \) is \( N/4m_1 \), we have \( f|\beta_{2m_1} \) contains at most terms of the form \( q^n \) with \( a(nN/2m_1) \neq 0 \).

Part (3) and the case when \( 8 \mid N \) for Part (2) follow in the same way, and we omit the details.

\( \square \)

**Remark 3.5.** Part (1) of Proposition 3.4 can be easily made precise using the \( \epsilon \)-condition. We can also made Part (2) precise using the argument in the proof of Corollary 4.13 in [27]. For
example, if $4 \mid N$ and we choose for any odd $m_1 \mid N$

$$\alpha_{2m_1} = \eta_{2m_1}^{-1} \begin{pmatrix} 1 & -1/2 \\ 0 & 1 \end{pmatrix} \eta_{2m_1} \eta_{N/4m_1},$$

then $\alpha_{2m_1} V(N/4m_1)^{-1} \in \text{SL}_2(\mathbb{Z})$ and it sends $\infty$ to the cusp $\frac{1}{2m_1}$. Our $\alpha_{2m_1}$ here differs from the one used in [27] by $V(N/4m_1)$. Messy but elementary computations give us that

$$f|\alpha_{2m_1} = \left( -2^{1-\frac{k}{2}} (N/4m_1)^{\frac{k+1}{2}} \chi_2(2) \chi_{m_1} (N/4m_1) \prod_{p \neq p' \mid \frac{N}{4m_1}} \chi_p(p') \prod_{p \mid \frac{N}{m_1}} \epsilon_p e_p^{-1} \right) \times \left( \sum_n \chi_2(n)a(2n)q^n \right) | U(N/4m_1) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

**Corollary 3.6.** Let $f \in A^c(N,k,\chi_D)$.

1. If $f$ is holomorphic (or vanishes, respectively) at $\infty$, then $f \in M^c(N,k,\chi_D)$ (or $S^c(N,k,\chi_D)$, respectively).

2. If $f = q^{-d} + O(1)$ with $d$ a positive integer coprime to $N$, then $f$ is holomorphic at cusps other than $\infty$.

3. If $k \leq 0$ and $f, g \in A^c(N,k,\chi_D)$ have the same principal part at $\infty$, then $f = g$.

**Proof.** The first part follows trivially from Proposition 3.4. The second part also follows from this proposition; actually, if $s \sim \frac{1}{m_1}$ with $m_1$ odd and $q_n^s$ with $n < 0$ appears, then we must have $N/m \mid d$ which is absurd since $m < N$ and $(d,N) = 1$, and the other cases follows similarly.

For (3), we note that $f - g \in A^c(N,k,\chi)$ is holomorphic at $\infty$, hence holomorphic at all cusps by Part (1). But $k \leq 0$, hence $f - g = 0$. Alternatively, we may derive this directly from the isomorphism. Indeed, if $F = \psi(f - g)$, then $F_0$ is holomorphic from the isomorphism and in particular $F_0$ is holomorphic, hence 0. So $F = 0$ by Lemma 2.4 and hence $f - g = 0$. \qed

### 4. Obstructions and Rationality of Fourier Coefficients

In this section, we translate Borcherds’s theorem of obstructions to scalar valued modular forms using the isomorphism in the previous section. In other words, we investigate the existence of weakly holomorphic modular forms with prescribed principal parts. At the end of this section, we shall also mention the rationality of Fourier coefficients of modular forms that satisfy some
\( \epsilon' \)-condition. The arguments are essentially the same as that in Section 5 of [27], so we will only mention some differences.

We shall vary \( D \) by choosing different data for \( \delta_p \) or \( t, t_1, t_2 \), and hence vary \( \epsilon \) and other data, from now on.

Let \( m \) be a positive integer. Recall that if \( F = \sum \gamma F_\gamma e^\gamma \) and \( G = \sum \gamma G_\gamma e^\gamma \) with \( F_\gamma, G_\gamma \in \mathbb{C}(q_m) \), the field of Laurent series in \( q_m = q^{\frac{1}{m}} \), we have the following pairing:

\[
\langle F, G \rangle = \text{the constant term of } \sum \gamma F_\gamma G_\gamma.
\]

The following proposition holds for each \( D \) and its Weil representation. This is Corollary 4.3 in [27], which follows easily from Borcherds’s Theorem 3.1 in [2].

**Proposition 4.1.** Let \( P \) be a \( \mathbb{C}[D] \)-valued polynomial in \( q_{N}^{-1} \) that is invariant under \( \text{Aut}(D) \). Then there exists \( F \in A_{\text{inv}}^\epsilon(N, k, \chi) \) such that \( F - P \) vanishes at \( q_N = 0 \), if and only if \( \langle P, G \rangle = 0 \) for each \( G \in M_{\text{inv}}^\epsilon(2-k, \rho_D^*) \).

We now fix \( N \) such that \( N_p = p \) if \( p \mid N \) is odd and \( N_2 = 1, 4 \) or 8. Define \( \chi_p = \left( \frac{p}{N} \right) \) if \( p \mid N \) is odd, and define \( \chi_2 \) to be 1 if \( 2 \mid N \), \( \left( \frac{-1}{2} \right) \) if \( 2^2 \mid N \), \( \left( \frac{2}{2} \right) \) if \( N \equiv 8 \mod 32 \), and \( \left( \frac{-2}{2} \right) \) if \( N \equiv 24 \mod 32 \). We define \( \chi = \prod_p \chi_p \). In any case, \( \chi \) is a primitive character modulo \( N \).

**Remark 4.2.** By varying our discriminant form \( D = \oplus_p D_p \), the isomorphism actually covers \( A^\epsilon(N, k, \chi) \) for all \( \epsilon \). More explicitly, for any sign vector \( \epsilon \),

- if \( 2 \mid N \), then we choose \( D_p = p^{\delta_p} \) with \( \delta_p = \chi_p(2N/p)e_p \);
- if \( 2^2 \mid N \), then we choose the same \( D_p \) for odd \( p \) and choose \( D_2 = 2_t^{\pm 2} \) with \( t \in \{ \pm 2 \} \) determined by \( t = 2\chi_2(N/4)e_2 \);
- if \( 2^4 \mid N \), then we choose the same \( D_p \) for odd \( p \) and choose \( D_2 = 2_{t_1}^{+1} + 2_{t_2}^{+2} \) with \( t_2 \in \{ \pm 1, \pm 3 \} \) determined by \( \chi_2(t_2) = \chi_2(N/8)e_2 \) and \( t_1 \in \{ \pm 1 \} \) determined by \( \left( \frac{-1}{t_1t_2} \right) = \chi_2(3) \).

For example, here we cover the cases when \( N = 15 \) and \( N = 20 \) for all possible \( \epsilon \). Note that the case when \( N \) is an odd prime is already contained in [3].

Let us assume that \( k \leq 0 \) and hence \( 2 - k \geq 2 \). Let us denote by \( E(N, 2-k, \chi) \) the space of Eisenstein series of level \( N \), weight \( 2-k \) and character \( \chi \). It is well-known that \( \dim(E(N, 2-k, \chi)) = 2^{\omega(N)} \), with a basis concretely given by \{ \( E_m : m \mid N, m = N_m \) \} where (see Theorem
4.5.2 and Theorem 4.6.2 in [9]:

\[ E_m = \delta_{1,m} L(k-1, \chi) + 2 \sum_{n=1}^{\infty} \left( \sum_{d \mid n} \chi_m(n/d) \chi'_m(d) d^{1-k} \right) q^n. \]

For each \( \epsilon \) and \( m \mid N \), we denote \( \epsilon_m = \prod_{p \mid m} \epsilon_p \). Define

\[ E^\epsilon = \frac{1}{s(0)L(k-1, \chi)} \sum_{m \mid N} \epsilon_m E_m, \]

and assume \( E^\epsilon = \sum_{q} B(n) q^n \). We see that \( B(0) = s(0)^{-1} \) and such normalization is different from [4] or [27].

**Lemma 4.3.** For each sign vector \( \epsilon \), we have \( E^\epsilon(N, 2 - k, \chi) = \text{span}_\mathbb{C} \{ E^\epsilon \} \).

**Proof.** We first note that each \( E^\epsilon(N, 2 - k, \chi) \) has dimension 1 by Lemma 5.4 of [27]. It suffices to show that \( E^\epsilon \) or \( \sum_{m} \epsilon_m E_m \) satisfies the \( \epsilon \)-condition.

Let \( n \) be any positive integer such that for some \( p \mid N \) we have \( \chi_p(n) = - \epsilon_p \). We need to show that the \( q^n \) coefficient of \( \sum_{m} \epsilon_m E_m \) is 0. Indeed, such a coefficient is

\[ 2 \sum_{m \mid N} \frac{\epsilon_m}{d \mid n} \chi_m(n/d) \chi'_m(d) d^{1-k} = 2 \sum_{m \mid N} \epsilon_m \sum_{d \mid n} d^{1-k} (\epsilon_p \chi_{pm}(n/d) \chi'_{pm}(d) + \chi_m(n/d) \chi'_m(d)) = 0, \]

since \( \epsilon_p \chi_{pm}(n/d) \chi'_{pm}(d) = \epsilon_p \chi_p(n) \chi_m(n/d) \chi'_m(d) \).

For an integer \( m \), we say that \( m \) is an \( \epsilon \)-integer if \( \chi_p(m) \neq - \epsilon_p \) for each \( p \mid N \). For any \( P = \sum_{n} a(n) q^n \in \mathbb{C}[q^{-1}] \), we say \( P \) is an \( \epsilon \)-polynomial in \( q^{-1} \) if \( n \) is an \( \epsilon \)-integer whenever \( a(n) \neq 0 \).

We state the obstruction theorem for scalar valued modular forms. (See Theorem 6 in [4] in the case of prime level.) Here we remark that the case \( k \geq 2 \) is trivial, since \( 2 - k \leq 0 \) and the obstruction space is trivial.

**Theorem 4.4.** Let \( k \leq 0 \) and \( \epsilon \) be a sign vector. Let \( P = \sum_{n<0} a(n) q^n \) be an \( \epsilon \)-polynomial in \( q^{-1} \). Then there exists \( f \in A^\epsilon(N, k, \chi) \) with \( f = \sum_{n \in \mathbb{Z}} a(n) q^n \), if and only if

\[ \sum_{n<0} s(n) a(n) b(-n) = 0, \]

for each \( g = \sum_{n \geq 0} b(n) q^n \in S^\epsilon(N, 2 - k, \chi) \). If \( f \) exists, it is unique and its constant term is given by

\[ a(0) = - \sum_{n<0} s(n) a(n) B(-n). \]
The statements follow from Proposition 4.1, Remark 4.2 and Lemma 4.3. For details, please see the proof of Theorem 5.5 in [27]. □

For completeness, we reproduce a couple of results in [27] following the lines in [4]. These results concern the rationality of the Fourier coefficients, which is important in Borcherds’s theory of automorphic products. For \( f = \sum_n a(n)q^n \) and \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \), define \( f^\sigma = \sum_n a(n)^\sigma q^n \). Let \( k \) be an integer, possibly positive.

**Lemma 4.5.** If \( f \in A(N, k, \chi) \), so is \( f^\sigma \).

**Proof.** This is Lemma 5.6 in [27]. □

The following generalizes Proposition 5.7 in [27] to our present more general setting.

**Proposition 4.6.** Let \( k \leq 0 \) be an integer and fix any sign vector \( \epsilon \). Let \( f = \sum_n a(n)q^n \in A^\epsilon(N, k, \chi) \) and suppose that \( a(n) \in \mathbb{Q} \) for \( n < 0 \). Then all coefficients \( a(n) \) are rational with bounded denominator.

**Proof.** Let \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \). We note that \( f^\sigma \in A^\epsilon(N, k, \chi) \). Indeed, from Lemma [1.5] we see that \( f^\sigma \in A(N, k, \chi) \); moreover, the Galois action preserves the \( \epsilon \)-condition.

Now consider \( h = f - f^\sigma \in A^\epsilon(N, k, \chi) \). It is obvious that \( h \) is holomorphic at \( \infty \), hence \( h \in M(N, k, \chi) \) by Corollary [3.6]. But \( k \leq 0 \), so \( M(N, k, \chi) = \{0\} \). It follows that \( f \) has rational coefficients. Since \( f\Delta^{k'} \in M(N, k + 12k', \chi) \) for large \( k' \), it has coefficients with bounded denominator, hence so does \( f \). □

5. **Zagier Duality**

In this section, we prove the Zagier duality for integral weight weakly holomorphic modular forms.

We assume that \( N_2 \), the 2-part of \( N \), is 1 or 4, for simplicity. Actually this condition on \( N \) will not be used until Theorem 5.7 on Zagier duality. We keep other notations in the previous section. To better describe the statements, we introduce a notion on weakly holomorphic modular forms with \( \epsilon \)-condition.

**Definition 5.1.** Let \( k, m \) be integers and \( \epsilon \) any sign vector. We call \( f = \sum_n a(n)q^n \in A^\epsilon(N, k, \chi) \) reduced of order \( m \), if \( f = \frac{1}{s(m)}q^m + O(q^{m+1}) \) and for each \( n > m \) with \( a(n) \neq 0 \) there does not exist \( g \in A^\epsilon(N, k, \chi) \) such that \( g = q^n + O(q^{n+1}) \).
Lemma 5.2. For any integer \( m \), there exists at most one \( f \in A^\epsilon(N, k, \chi) \) such that \( f \) is reduced of order \( m \).

Proof. Suppose there are two such modular forms, say \( f = \sum_n a(n)q^n \) and \( g = \sum_n b(n)q^n \), we prove that they must be equal. Indeed, if \( f \neq g \), then let \( n \) be the smallest such that \( a(n) \neq b(n) \). Now \( f - g \in A^\epsilon(N, k, \chi) \) and its Fourier expansion begins with the \( q^n \)-term. Since at least one of \( a(n) \) and \( b(n) \) is not 0, this contradicts to the assumption that both \( f \) and \( g \) are reduced. \( \square \)

If the data \( N, k, \chi \) and \( \epsilon \) are clear from the context, we shall denote by \( f_m \) the reduced modular form of order \( m \) in \( A^\epsilon(N, k, \chi) \) if it exists. We shall also denote by \( f'_m \) the reduced modular form of order \( m \) in \( A^\epsilon(N, 2 - k, \chi) \) if it exists.

For each integer \( m \), let \( A^\epsilon_m(N, k, \chi) \) denote the subspace of modular forms \( f \in A^\epsilon(N, k, \chi) \) such that \( f(n) = \sum_{n \geq m} a(n)q^n \). For example, \( A^\epsilon_0(N, k, \chi) = M^\epsilon(N, k, \chi) \) and \( A^\epsilon_1(N, k, \chi) = S^\epsilon(N, k, \chi) \) by Corollary 3.6.

Proposition 5.3. For any integer \( m \), the set \( \{ f_n : n \geq m, f_n \text{ exists} \} \) is a basis for \( A^\epsilon_m(N, k, \chi) \). In particular, we have canonical bases for the spaces \( A^\epsilon(N, k, \chi), M^\epsilon(N, k, \chi) \) and \( S^\epsilon(N, k, \chi) \) that consist of reduced modular forms.

Proof. Clearly such a set is linearly independent since they have different lowest power terms. We then only need to show that reduced forms span the whole space. Let \( f \) be any non-zero form in \( A^\epsilon(N, k, \chi) \) and we may assume that \( f = \frac{1}{s(m)} q^m + O(q^{m+1}) \). If \( m > 0 \), then \( f \) is a cusp form. We know that \( m \) cannot be big, since otherwise it forces \( f = 0 \) by Sturm’s Theorem (see [25] or Theorem 6.4 below). Such a bound depends only on \( N \).

If \( f \) is reduced, then we are done. If not, we have a finite set of integers \( n \) such that \( n > m \) and a modular form \( g_n = q^n + O(q^{n+1}) \in A^\epsilon(N, k, \chi) \) exists. By induction on \( m \), we may assume that \( g_n \) is a linear combination of reduced modular forms for each \( n \). It is clear that \( f - \sum_n c_n g_n \) is reduced for some scalars \( c_n \), hence \( f \) itself is a linear combination of reduced modular forms. \( \square \)

For an integer \( m \), we say that \( m \) satisfies the \( \epsilon \)-condition, if \( \chi_p(m) \neq -\epsilon_p \) for each \( p \mid N \).

Lemma 5.4. Assume \( k \geq 2 \) and \( m \leq 0 \). Then there exists a reduced modular form of order \( m \) in \( A^\epsilon(N, k, \chi) \) if and only if \( m \) is an \( \epsilon \)-integer. If it exists, such a form contain precisely one non-positive power term, that is \( \frac{1}{s(m)} q^m \).
**Proof.** By Proposition 4.1 and our isomorphism, we can have the obstruction theorem for scalar valued modular forms in the case when \( k \geq 2 \). Here \( 2 - k \leq 0 \), hence the obstruction space is trivial. Therefore, any \( \epsilon \)-polynomial in \( q^{-1} \) lifts to a modular form in \( A^*(N, k, \chi) \). In particular, all \( \epsilon \)-monomials in \( q^{-1} \) can be lifted. Then for a reduced modular form, there is only one non-positive power term in \( q \).

**Lemma 5.5.** Assume \( k \leq 0 \) and \( m < 0 \). Let \( \{ f'_n : n \in S \} \) be the basis of reduced modular forms for \( S^*(N, 2 - k, \chi) \); here \( S \) is a uniquely determined finite set of positive integers. Then \( f_m \) exists if and only if \( m \) is an \( \epsilon \)-integer and \(-m \not\in S\).

**Proof.** Assume that \( S = \{ n_1, n_2, \cdots, n_k \} \) with \( n_i < n_{i+1} \) for \( 1 \leq i < k \), and \( f'_{n_i} = \sum_n a_i(n)q^n \) for each \( i \).

It is clear that if \(-m = n_i \), then \( f_m \) does not exist because of the obstruction by \( f'_{n_i} \). Conversely, suppose \(-m \not\in S\) and consider the polynomial

\[
P = \frac{1}{s(m)}q^m - \frac{1}{s(m)} \sum_i s(n_i)a_i(m)q^{-n_i}.
\]

Since \( f'_{n_i} \) are reduced, we must have \( s(n_i)a_j(n_i) = \delta_{i,j} \). From this, we see that \( P \) satisfies the obstruction conditions. Moreover, since \( n_i \) satisfy the \( \epsilon^* \)-condition, \(-n_i \) satisfies the \( \epsilon \)-condition and \( P \) is an \( \epsilon \)-polynomial in \( q^{-1} \). By Theorem 4.4, there exist a modular form with \( P \) the principal part, and the existence of \( f_m \) follows.

From now on until the end of this section, we assume that \( k \leq 0 \) and hence \( 2 - k \geq 2 \).

**Lemma 5.6.** Let \( \epsilon = (\epsilon_p) \) be any sign vector and let \( \epsilon^* = (\epsilon^*_p) \) with \( \epsilon^*_p = \chi_p(-1)\epsilon_p \). Assume \( m, d \in \mathbb{Z}, m < 0 \), and that both of the reduced modular forms

\[
f_m = \sum_{n \in \mathbb{Z}} a_m(n)q^n \in A^*(N, k, \chi) \quad \text{and} \quad f'_d = \sum_{n \in \mathbb{Z}} b'_d(n)q^n \in A^*(N, 2 - k, \chi)
\]

exists. Then \( a(-n)b(n) = 0 \) for any \( d < n < -m \).

**Proof.** Assume first that \( d \leq 0 \). Since \( 2 - k \geq 2 \) and \( f'_d \) is reduced, we must have \( b(n) = 0 \) if \(-d < n \leq 0 \) by Lemma 5.4. Therefore, we only need to consider the case when \( 0 < n < -m \).

We fix any \( 0 < n_0 < -m \) such that \( a(-n_0) \neq 0 \) and it suffices to prove that \( b(n_0) = 0 \). We first note that \( f'_{-n_0} \) does not exist and by Lemma 5.5 this means that \( f'_d \) exists in the dual cusp form space. Since \( f'_d \) is also reduced, we must have \( b(n_0) = 0 \).

Similarly, if \( d > 0 \), then for any \( d < n < -m \) with \( a(n_0) \neq 0 \), we must have the existence of \( f'_{n_0} \). That \( f'_d \) is reduced implies \( b(n_0) = 0 \).
Theorem 5.7. Let $\epsilon = (\epsilon_p)$ be any sign vector and let $\epsilon^* = (\epsilon^*_p)$ with $\epsilon^*_p = \chi_p(-1)\epsilon_p$. Assume $m, d \in \mathbb{Z}$ with $m < 0$. Assume that both of the reduced modular forms

$$f_m = \sum_{n \in \mathbb{Z}} a_m(n)q^n \in A'(N,k,\chi)$$

and

$$f'_d = \sum_{n \in \mathbb{Z}} b_d(n)q^n \in A'^*(N,2-k,\chi)$$

exists. Then we have

$$a_m(-d) = -b_d(-m).$$

Proof. We denote $f = f_m$ and $f' = f'_d$. Since $ff' \in A(N,2,1)$, we have a meromorphic 1-form $ff'd\tau$ on the compact Riemann surface $X_0(N)$, so the sum of residues of $ff'd\tau$ must vanish. Since $ff'$ is holomorphic on $\mathbb{H}$, $ff'd\tau$ is holomorphic on $X_0(N)$ except at the cusps. For a cusp $s$ with width $h_s$, let $\alpha \in \text{SL}_2(\mathbb{Z})$ such that $\alpha \infty \sim s$. Then we know that the residue of $ff'd\tau$ at $s$ is given by the constant term in $q_s$ of $h_s^2\pi i (ff')|\alpha$. Here $q_s = q^{1/h_s}$. The reader may see Section 2.3 of Miyake’s book [19] for more details on this. Let $m_1$ be an odd positive divisor of $N$.

We first deal with the case when $N$ is odd. For a cusp $s \sim \frac{1}{N/m_1}$, $\gamma_{m_1} \sim s$ and then it is easy to see that the residue of $ff'd\tau$ at $s$ is given by the constant term of $\frac{1}{2\pi i}(f|\eta_{m_1})(f'|\eta_{m_1})$.

Since $f \in A'(N,k,\chi)$, by Lemma 2.5 and Lemma 3.1 in [27], we obtain

$$f|\eta_{m_1} = \left( \prod_{p|m_1} \epsilon_p \epsilon_p' \chi_p'(p) \right) m_1^{\frac{k-1}{2}} f|U(m_1).$$

Similarly

$$f'|\eta_{m_1} = \left( \prod_{p|m_1} \epsilon_p \epsilon_p' \chi_p'(p) \right) m_1^{\frac{1-k}{2}} f'|U(m_1).$$

Hence $(f|\eta_{m_1})(f'|\eta_{m_1}) = (f|U(m_1))(f'|U(m_1))$.

For ease of notations, for $a, b \in \mathbb{Z}$, we define

$$c(a,b) = \begin{cases} 0 & \text{if } a \nmid b, \\ \frac{1}{\sigma(b)} & \text{if } a \mid b. \end{cases}$$

Then the constant term of $(f|U(m_1))(f'|U(m_1))$ is given by

$$c(m_1, m)b_d(-m) + c(m_1, d)a_m(-d),$$

where other terms vanish by Lemma 5.6. Summing over all $m_1 \mid N$, we have

$$\sum_{m_1 | N} c(m_1, m) = \sum_{m_1 | N} c(m_1, d) = 1.$$
Since the inequivalent cusps are precisely represented by \( \frac{1}{N/m_1} \), the sum of all residues of \( f'f'\d\tau \) is given by \( \frac{1}{2\pi i} (a_m(-d) + b_d(-m)) \). It follows that \( a_m(-d) = -b_d(-m) \) and we are done with this case.

Now let us assume \( 2^2 || N \). The idea is the same, but the computations in this case are more complicated. For each \( m_1 \mid \frac{N}{2} \), we have the same expression as above for \( f|\eta_{m_1} \) and for \( f'|\eta_{m_1} \), and from the same argument we see that at the cusp \( s \sim \frac{1}{N/m_1} \), the residue of \( f'f'\d\tau \) at \( s \) is given by the constant term of \( \frac{1}{2\pi i} (f|\eta_{m_1})(f'|\eta_{m_1}) = \frac{1}{2\pi i} (f|U(m_1))(f'|U(m_1)). \) This is

\[
\frac{1}{2\pi i} (c(m_1, m)b_d(-m) + c(m_1, d)a_m(-d)).
\]

Similarly, at the cusp \( s \sim \frac{1}{N/4m_1} \), the residue of \( f'f'\d\tau \) at \( s \) is given by the constant term of \( \frac{1}{2\pi i} (f|\eta_{2m_1})(f'|\eta_{2m_1}) = \frac{1}{2\pi i} (f|U(4m_1))(f'|U(4m_1)). \) This is

\[
\frac{1}{2\pi i} (c(4m_1, m)b_d(-m) + c(4m_1, d)a_m(-d)).
\]

We are left with the cusps of the form \( s \sim \frac{1}{2m_1} \) with \( m_1 \mid \frac{N}{4} \). This time the matrix \( \alpha_{2m_1} \) in Remark 3.5 will do the job. From the expression there, we see that the residue of \( f'f'\d\tau \) at \( s \) is given by the constant term of \( \frac{1}{2\pi i} (f|\alpha_{2m_1})(f'|\alpha_{2m_1}) \), which is given by

\[
\frac{1}{2\pi i} \left( \chi_2(m/2)^2 c(N/2m_1, m)b_d(-m) + \chi_2(d/2)^2 c(N/2m_1, d)a_m(-d) \right).
\]

Note here that if \( 2 \nmid m \), the value of the first term is understood to be 0 even though \( \chi_2(m/2) \) is not defined; the \( c \)-factor is 0 anyway. The same interpretation applies to the second term.

By elementary computations, we see that

- \( \sum_{m_1 \mid N_1} c(4m_1, m) = 1/2 \) if \( 4 \mid m \) and 0 otherwise,
- \( \sum_{m_1 \mid N_1} c(m_1, m) = 1/2 \) if \( 2 \mid m \) and 1 otherwise,
- \( \sum_{m_1 \mid N_1} \chi_2(m/2)^2 c(N/2m_1, m) = 1/2 \) if \( 2 \parallel m \) and 0 otherwise,

and they also hold with \( m \) replaced by \( d \). Summing over \( m_1 \mid \frac{N}{4} \), we see that in this case we also have \( a_m(-d) = -b_d(-m) \). This finishes the proof. \( \square \)

**Remark 5.8.** When \( N = 5, 13, 17 \), this is due to Rouse [21] and when \( N \) is a general odd prime, this is proved by Choi in [6] (note that in this paper the factor \( \varepsilon_p \) is missing in and after Lemma 1.5). Note that we not only generalize their results to a more general setting, that is \( N \) is odd or \( 4 || N \), but also extend the duality to include holomorphic forms. Therefore, we have complete grids.
We finish this section with a few examples. In the “automorphic correction” paper, for a special \( \epsilon \), cases when the obstruction space is trivial were treated, namely when \( N = 8, 12 \) or 21. To best illustrate the theory, here we consider the case when \( N = 15 \).

We know that \( \chi = \left( \frac{1}{\tau} \right) \), and \( \chi_3 = \left( \frac{3}{\tau} \right) \), \( \chi_5 = \left( \frac{5}{\tau} \right) \). There are four distinct sign vectors \( \epsilon \):

\[
\epsilon_1 = (-1, -1), \quad \epsilon_2 = (1, -1), \quad \epsilon_3 = (-1, 1), \quad \epsilon_4 = (1, 1).
\]

Among them, \( \epsilon_1 \) and \( \epsilon_2 \) are dual to each other, and \( \epsilon_3 \) and \( \epsilon_4 \) are dual to each other.

Since in this case the signature \( r \) of all possible discriminant forms \( 3^{\pm 1} \oplus 5^{\pm 1} \) satisfies \( \frac{r}{2} \equiv \frac{(3-1)+(5-1)}{2} \equiv 1 \) mod 2, we should consider odd weights instead. For simplicity, let us consider the case \( k = -1 \) and \( 2 - k = 3 \). We consider examples for two \( \epsilon \) in a moment, one of which has trivial obstructions while the other does not.

We first look at the data for weight 3 homomorphic modular forms. We know that \( S(15, 3, \chi) = \mathbb{C}g_1 + \mathbb{C}g_2 \), with

\[
g_1 = q - 3q^4 - 3q^6 + 9q^9 + 5q^{10} + O(q^{15}) \in S^{\epsilon_4}(15, 3, \chi),
g_2 = q^2 - 3q^3 + 5q^5 - 7q^8 + 9q^{12} + O(q^{15}) \in S^{\epsilon_1}(15, 3, \chi).
\]

The Eisenstein space \( E(15, 3, \chi) = \sum_i \mathbb{C}E^{\epsilon_i} \) with

\[
E^{\epsilon_1} = \frac{1}{4} - \frac{5}{8} q^2 - \frac{5}{8} q^3 - \frac{13}{8} q^5 - \frac{85}{8} q^8 - \frac{105}{8} q^{12} + O(q^{15}),
E^{\epsilon_2} = \frac{1}{4} + \frac{1}{2} q^3 + 6q^7 + \frac{15}{2} q^{10} + \frac{21}{2} q^{12} + 21q^{13} + O(q^{15}),
E^{\epsilon_3} = \frac{1}{4} + \frac{3}{2} q^5 + \frac{5}{2} q^6 + 5q^9 + 15q^{11} + 30q^{14} + O(q^{15}),
E^{\epsilon_4} = \frac{1}{4} - \frac{1}{8} q - \frac{21}{8} q^4 - \frac{25}{8} q^6 - \frac{41}{8} q^9 - \frac{65}{8} q^{10} + O(q^{15}).
\]

We note that \( E^{\epsilon_1} \) and \( E^{\epsilon_4} \) are not reduced, because of the existence of \( g_1, g_2 \), and this is why their Fourier coefficients have bigger denominators than expected. Such integrality will be considered in Section 6.

**Example 5.9.** Let \( \epsilon = \epsilon_1 \) hence \( \epsilon^* = \epsilon_2 \). From above data, we see that there exist no obstructions for \( A^\epsilon(15, -1, \chi) \). From the \( \epsilon \)-condition, we see that \( f_m \) exists if and only if \( m \equiv 0, 2, 3, 5, 8, 12 \) mod 15. The basis of reduced modular forms for \( A^\epsilon(15, -1, \chi) \) starts with:

\[
f_{-3} = \frac{1}{2} q^3 - \frac{1}{2} + 3q^2 - \frac{1}{2} q^3 - 3q^5 - 3q^8 + 6q^{12} + O(q^{15}),
f_{-7} = q^7 - 6 + 12q^2 + 33q^3 + 39q^5 - 140q^8 - 144q^{12} + O(q^{15}),
\]
On the other hand, the basis of reduced forms for $A^\epsilon(15, 3, \chi)$ begins with $f'_0 = E^{\epsilon^2}$:

\[
\begin{align*}
    f'_0 &= \frac{1}{4} + \frac{1}{2} q^3 + 6q^7 + \frac{15}{2} q^{10} + \frac{21}{2} q^{12} + 21 q^{13} + O(q^{15}) \\
    f'_{-2} &= q^{-2} - 3q^3 - 12q^7 - 45q^{10} + 36q^{12} + 146 q^{13} + O(q^{15}),
\end{align*}
\]

We can easily detect the Zagier duality for these basis elements.

**Example 5.10.** Now let $\epsilon = \epsilon_3$ and $\epsilon^* = \epsilon_4$. Because of the existence of $g_1$, there is a non-trivial obstruction condition for $A^\epsilon(15, -1, \chi)$. The if $f_m$ exists, the $\epsilon$ condition says $m \equiv 0, 5, 6, 9, 11, 14 \pmod{15}$. By Lemma 5.5, we see that $f_m$ exists if and only if $m \neq -1$ and $m \equiv 0, 5, 6, 9, 11, 14 \pmod{15}$.

The basis of reduced modular forms for $A^\epsilon(15, -1, \chi)$ starts with

\[
\begin{align*}
    f_{-4} &= q^{-4} + 3q^{-1} + 3 - 7q^5 + 3q^6 - 21q^9 - 11q^{11} + 44q^{14} + O(q^{15}) \\
    f_{-6} &= \frac{1}{2} q^{-6} + 3q^{-1} + \frac{7}{2} + 21q^5 - \frac{49}{4} q^6 + 19q^9 - 147q^{11} + 99q^{14} + O(q^{15}),
\end{align*}
\]

On the other hand, the basis of reduced modular forms for $A^\epsilon(15, 3, \chi)$ starts with

\[
\begin{align*}
    f'_1 &= q - 3q^4 - 3q^6 + 9q^9 + 5q^{10} + O(q^{15}) \\
    f'_0 &= \frac{1}{4} - 3q^4 - \frac{7}{2} q^6 - 4q^9 - \frac{15}{2} q^{10} + O(q^{15}), \\
    f'_{-5} &= \frac{1}{2} q^{-5} + 7q^4 - 21q^6 - 99q^9 + 67q^{10} + O(q^{15}),
\end{align*}
\]

Here $f'_1 = g_1$ and $f'_0$ can be obtained by $E^{\epsilon^4} + \frac{1}{8} g_1$. The duality is also clear from these data.

**Remark 5.11.** To obtain such explicit Fourier expansions for general $N$, one can use $\eta$-quotients in [16]. Indeed, by multiplying some $\eta$-quotients we send $f_m$ to some space of holomorphic modular forms. Then by choosing a basis the computation of $f_m$ reduces to solving a linear system. All of these can be done efficiently with SAGE. By obtaining the order of $f_m$ at various cusps using Proposition 3.4, we may choose a suitable $\eta$-quotient to save some weight and reduce the dimension of the resulting space of holomorphic modular forms. Similar constructions in the case of $N = 5, 13, 17$ can be found in [21] and in [17].
6. Integrality of Reduced Modular Forms

In this last section, we consider the integrality problem for Fourier coefficients of reduced modular forms. Let \( k \) be an integer such that \( k \neq 1 \).

For your reference, we reproduce the first lemma in Miller’s thesis below. See also for example Lemma 2.20 in [24].

**Lemma 6.1** (V. Miller). The cusp form space for \( \text{SL}_2(\mathbb{Z}) \) has a basis \( g_1, g_2, ..., g_d \) such that if \( a_i(g_j) \) is the \( i \)-th coefficient of \( f_j \) then \( a_i(g_j) \in \mathbb{Z} \) for all \( i, j \) and \( a_i(g_j) = \delta_{i,j} \) for \( 1 \leq i \leq d \). Such a basis is called the Miller basis.

By including the Eisenstein series, such a basis extends to a basis for the space holomorphic modular forms easily. If we further allow poles at the cusp, Duke and Jenkins [10] proved the existence of such a basis where they denoted the basis elements by \( f_{k,m} \). In particular, all Fourier coefficients of \( f_{k,m} \) are integral. We note the difference that their \( f_{k,m} \) begin with \( q^{-m} \) instead of \( q^m \).

These modular forms in such a Miller basis are reduced modular forms in our terminology. We consider the existence of a Miller basis for higher level weakly holomorphic modular forms. More precisely, we consider the case when the level \( N \) satisfies \( N_2 = 1, 4, 8 \) and \( N/N_2 \) is square-free (that is, those considered in Section 4) and \( \chi \) is a primitive real character of modulus \( N \). Because of the existence of cusps other than \( \infty \), it is more natural to consider subspaces with \( \epsilon \)-condition, since we know that the holomorphy of such modular forms at \( \infty \) dominates that at other cusps. Actually, we expect more integrality from a canonical basis like Miller basis than the conditions above, based on the connection with vector valued modular forms.

**Definition 6.2.** We call a subset of modular forms \( \{g_m : m \in S\} \subset A^\epsilon(N, k, \chi) \), with \( S \subset \mathbb{Z} \) and \( g_m = \sum_n a_m(n)q^n \), a Miller basis if it is a basis and

- \( a_m(m) = \frac{1}{s(m)} \) and \( a_m(n) = 0 \) if \( n < m \),
- \( a_m(m') = \delta_{m,m'} \) if \( m, m' \in S \),
- \( a_m(n)s(n) \in \mathbb{Z} \) for all \( m, n \).

From Example 5.9 and Example 5.10, we easily see that the integrality condition \( a_m(n)s(n) \in \mathbb{Z} \) is stronger than the naive condition \( a_m(n) \in \mathbb{Z} \). For example, in Example 5.9,

\[
f_{-3} = \frac{1}{2}q^{-3} - \frac{1}{2} + 3q^2 - \frac{1}{2}q^3 - 3q^5 - 3q^8 + 6q^{12} + O(q^{15}).
\]
We see that \(2f_{-3} = \sum_n a(n)q^n\) satisfies the naive integrality condition, but this way we bury the fact that for all \(n\) with \((n, 15) = 1\) we have \(a(n)\) is even. Taking into account of our isomorphism, such a basis is actually a Miller-like basis for vector valued modular forms. Note that the rationality for vector valued holomorphic modular form spaces is known by McGraw [18], which obviously does not imply the existence of a Miller-like basis even for holomorphic modular forms.

Such a basis, if exists, is clearly unique and it has to coincide with the basis of reduced modular forms. Therefore, the existence of the Miller basis is the same as the stronger integrality for reduced modular forms. Naturally, we ask the following question:

**Question.** Does the Miller basis always exist? Putting it in another way, is it true that for any reduced modular form \(f_m = \sum_n a(n)q^n\) we always have \(s(n)a(n) \in \mathbb{Z}\)?

Such integrality first appeared in [14] and then in [15], where such integrality is crucial, since \(s(n)a(n)\) represents the multiplicity of some root in a generalized Kac-Moody superalgebra. In [15], we applied a trick to the reduced forms where only the constant term is possibly half-integral. Although numerical evidence indicates an affirmative answer to the above question, at present time, we do not know how to treat systematically the existence of a Miller basis for all \(N\). However, there is still something we can say. In the rest of this section, we will illustrate how to computationally prove it for a fixed \(N\).

Recall that we denote by \(f_m\) the reduced modular form of order \(m\), when it exists and the data \(N, k, \epsilon\) are clear. We write \(f_m = \sum_n a_m(n)q^n\). Let \(m_\epsilon\) denote the maximal \(m\) such that \(f_m'\) exists in \(A^\epsilon(N, 2 - k, \chi)\). If we would like to emphasize the sign vector, we shall write \(f_{m, \epsilon}, a_{m, \epsilon}(n),\) and \(f_{m, \epsilon}'\) accordingly.

The following lemma reduces the question to testing a finite number of reduced modular forms.

**Lemma 6.3.** The Miller basis exists for \(A^\epsilon(N, k; \chi)\) if and only if for each \(f_m = \sum_n a(n)q^n\) with \(m \geq -N - m_\epsilon\) we have \(s(n)a(n) \in \mathbb{Z}\).

**Proof.** Consider any reduced modular form \(f_{m'}\) with \(m' < -N - m_\epsilon\). There exists integers \(-N - m_\epsilon \leq m'_0 < m_\epsilon\) and \(l > 1\) such that \(m' = -Nl + m'_0\). The existence of \(f_{m'}\) implies that of \(f_{m'_0}\) by Lemma 5.4 and Lemma 5.5. Consider now
\[
g = j(N\tau)^{m_3} f_{m'_0} = \sum_n b(n)q^n \in A^\epsilon(N, k, \chi).
\]
Here \( j = q^{-1} + O(1) \) is the weight 0 modular form of level 1. Since \( j \) has integral Fourier coefficients, by the assumption on \( f_{m_0} \), we see that \( b(n)s(n) \in \mathbb{Z} \) for each \( n \).

Now \( g \) and \( f_{m'} \) share the same lowest power term, and we must have that
\[
f_{m'} = g - \sum_{m > m'} s(m)b(m)f_m.
\]
Hence \( s(n)a_{m'}(n) = s(n)b(n) - s(m)b(m)s(n)a_m(n) \in \mathbb{Z} \) by the assumption. We are done. \( \square \)

To consider the integrality of a fixed reduced modular form, the following theorem of Sturm will be useful. For a commutative ring \( R \), denote by \( R[q] \) the ring of power series in \( q \) with coefficients in \( R \).

**Theorem 6.4** (Sturm, [25]). Let \( \Gamma \) be any congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \), \( O_F \) be the ring of integers in a number field \( F \), and \( \mathfrak{p} \) be any prime ideal. Assume \( f = \sum_n a_nq^n \in M(\Gamma, k, 1) \cap O_F[q] \). If \( a_n \in \mathfrak{p} \) for \( n \leq \frac{k}{12}[\text{SL}_2(\mathbb{Z}) : \Gamma] \), then \( a_n \in \mathfrak{p} \) for all \( n \).

**Corollary 6.5.** Let \( \Gamma \) be any congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Assume \( f = \sum_n a_nq^n \in M(\Gamma, k, 1) \cap \mathbb{Q}[q] \) with bounded denominator. If \( a_n \in \mathbb{Z} \) for \( n \leq \frac{k}{12}[\text{SL}_2(\mathbb{Z}) : \Gamma] \), then \( a_n \in \mathbb{Z} \) for all \( n \).

**Proof.** Let \( M \) be the smallest positive integer such that \( Mf \in \mathbb{Z}[q] \) and we need to prove that \( M = 1 \). Suppose \( M > 1 \) and let \( p \) be any prime divisor of \( M \).

Now \( Mf \in M(\Gamma, k, 1) \cap \mathbb{Z}[q] \) and \( p \mid Ma_n \) for all \( n \) up to \( \frac{k}{12}[\text{SL}_2(\mathbb{Z}) : \Gamma] \), by Theorem 6.4, we have \( p \mid Ma_n \) for all \( n \). Therefore, \( p^{-1}Mf \in \mathbb{Z}[q] \), contradicting to the minimality of \( M \). \( \square \)

**Example 6.6.** Let us treat the simplest case \( N = 3 \) and \( k = -1 \). Firstly, we have \( \chi = \left( \frac{1}{3} \right) \), \( \epsilon_1 = +1 \) and \( \epsilon_2 = -1 \). Since \( S(3, 3, \chi) = \{0\} \), no obstructions for \( A^\epsilon(3, 0, \chi) \) for each \( \epsilon \). The bound in Lemma 6.3 is now \( -3 \). To establish the existence of the Miller basis, we only have to consider the integrality of
\[
\begin{align*}
f_{-2, \epsilon_1} & \quad \text{and} \quad f_{-3, \epsilon_1} \quad \text{for} \quad A^{\epsilon_1}(3, 0, \chi), \\
f_{-1, \epsilon_2} & \quad \text{and} \quad f_{-3, \epsilon_2} \quad \text{for} \quad A^{\epsilon_2}(3, 0, \chi).
\end{align*}
\]
Explicitly the first few term of these modular forms are
\[
\begin{align*}
f_{-1, \epsilon_2} & = q^{-1} + 9 - 82q^2 + 189q^3 - 892q^5 + 1782q^6 - 6234q^8 + O(q^9) \\
f_{-2, \epsilon_1} & = q^{-2} - 27 + 328q - 7128q^3 + 24854q^4 - 221850q^6 + 591632q^7 + O(q^9), \\
f_{-3, \epsilon_1} & = \frac{1}{2}q^{-3} - 36 - 1701q - 50058q^3 - 499608q^4 - 4023392q^6 - 27788508q^7 + O(q^9),
\end{align*}
\]
\[ f_{-3, \epsilon_2} = \frac{1}{2} q^{-3} + 45 + 16038q^2 + 50058q^3 + 2125035q^5 + 4023310q^6 + 89099838q^8 + O(q^9). \]

The coefficients for \( f_{-3, \epsilon_1} \) and \( f_{-3, \epsilon_2} \) are not integral for some large powers, for example their coefficients for \( q^{45} \) are both half integral. The computation of these Fourier expansions involves the following \( \eta \)-quotients:

\[
H_1 = \eta(\tau)^{-3} \eta(3\tau)^9 \quad \text{and} \quad H_2 = \eta(\tau)^9 \eta(3\tau)^{-3}.
\]

The integrality of \( f_{-1, \epsilon_2} \) follows from the fact that \( H_1 f_{-1, \epsilon_2} \in M(3, 2, 1) \) and the Sturm bound is then \( \frac{2}{3} \). The integrality of \( f_{-2, \epsilon_1} \) follows from the fact that \( H_1^2 f_{-2, \epsilon_1} \in M(3, 5, \chi) \) and the Sturm bound is then \( \frac{5 \times 8}{12} < 4 \).

To deal with the rest two reduced modular forms, note first that for \( i = 1, 2 \)

\[
H_1^2 H_2 f_{-3, \epsilon_i} \in M(3, 11, \chi).
\]

The Sturm bound in both cases is \( \frac{11 \times 8}{12} < 8 \). By Corollary 6.5, we see that all of the three modular forms

\[
2 f_{-3, \epsilon_1}, \quad 2 f_{-3, \epsilon_2}, \quad f_{-3, \epsilon_1} + f_{-3, \epsilon_2}
\]

have integral coefficients. On the one hand, if \( 3 \mid n \), then \( s(n)a_{-3, \epsilon_i}(n) \in \mathbb{Z} \) for \( i = 1, 2 \). On the other hand, if \( 3 \nmid n \), then one of \( a_{-3, \epsilon_1}(n) \) and \( a_{-3, \epsilon_2}(n) \) is zero. But the sum of these two coefficient is integral, so both of them are integral.

Therefore, when \( N = 3 \), the Miller basis exists.

We expect that similar arguments can handle any specific \( N \).

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