DISTRIBUTION OF SIMILAR CONFIGURATIONS IN SUBSETS OF $\mathbb{F}_q^d$

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Abstract. Let $\mathbb{F}_q$ be a finite field of order $q$ and $E$ be a set in $\mathbb{F}_q^d$. The distance set of $E$ is defined by $\Delta(E) := \{\|x - y\| : x, y \in E\}$, where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_d^2$. Iosevich, Koh and Parshall (2018) proved that if $d \geq 2$ is even and $|E| \geq 9q^{d/2}$, then

$$\mathbb{F}_q = \frac{\Delta(E)}{\Delta(E)} = \left\{ \frac{a}{b} : a \in \Delta(E), b \in \Delta(E) \setminus \{0\} \right\}.$$ 

In other words, for each $r \in \mathbb{F}_q^*$ there exist $(x, y) \in E^2$ and $(x', y') \in E^2$ such that $\|x - y\| \neq 0$ and $\|x' - y'\| = r\|x - y\|$.

Geometrically, this means that if the size of $E$ is large, then for any given $r \in \mathbb{F}_q^*$ we can find a pair of edges in the complete graph $K_{|E|}$ with vertex set $E$ such that one of them is dilated by $r \in \mathbb{F}_q^*$ with respect to the other. A natural question arises whether it is possible to generalize this result to arbitrary subgraphs of $K_{|E|}$ with vertex set $E$ and this is the goal of this paper.

In this paper, we solve this problem for $k$-paths ($k \geq 2$), simplexes and $4$-cycles. We are using a mix of tools from different areas such as enumerative combinatorics, group actions and Turán type theorems.

1. Introduction

Many problems in discrete geometry ask whether certain structure exists in a set of sufficiently large size. The most renowned result of this type belongs to Szemerédi [14] which claims that a subset of natural numbers with positive upper density contains arbitrarily long arithmetic progressions.

The Erdös distinct distances problem asks for the smallest possible number of distinct distances determined by a finite subset of $\mathbb{R}^d$, $d \geq 2$. In [3], Guth and Katz solved the Erdös distance problem for $d = 2$. For $d \geq 3$ the conjecture remains open with the best results due to Solymosi and Vu ([12]). They proved that the number of distinct distances in a well-distributed set of $n$ points in $\mathbb{R}^d$ is $\Omega(n^{2/d-1/d^2})$ which is close to the best known upper bound $O(n^{2/d})$ and they have a further improvement $\Omega(n^{0.5794})$ in the case $d = 3$. The continuous analog of this problem is called Falconer’s conjecture, which asks for the smallest Hausdorff dimension of a subset $E$ of $\mathbb{R}^d$ ($d \geq 2$) such that the Lebesgue measure of the distance set

$$\Delta(E) := \{|x - y| : x, y \in E\}$$

is positive.

The Erdös-Falconer distance problem in vector spaces over finite fields asks for the smallest possible size of $\Delta(E) := \{|\|x - y\| : x, y \in E\}$.
where \( \|a\| := \alpha_1^2 + \cdots + \alpha_d^2 \) for \( \alpha = (\alpha_1, \ldots, \alpha_d) \) and \( E \subset \mathbb{F}_q^d \), \( d \geq 2 \). This problem was introduced by Bourgain, Katz and Tao in \([2]\). Here \( \mathbb{F}_q \) denotes the finite field with \( q \) elements and \( \mathbb{F}_q^d \) is the \( d \)-dimensional vector over this field.

In \([3]\), Iosevich and Rudnev proved that if \( E \subset \mathbb{F}_q^d \), \( d \geq 2 \), with \( |E| > 2q^{\frac{d+1}{2}} \), then \( \Delta(E) = \mathbb{F}_q \). Moreover, they proved that one cannot in general obtain \( cq \) distances if \( |E| \ll q^{\frac{d}{2}} \).

In \([4]\), Hart, Iosevich, Koh and Rudnev proved that the size of the critical exponent can be increased to \( \frac{d+1}{2} \) in odd dimensions, thus showing that the result in \([3]\) is best possible in this setting. In \([1]\), Bennett et al. proved that if \( d = 2 \) and \( |E| \gg q^{\frac{d}{2}} \), then \( |\Delta(E)| > cq \). It is reasonable to conjecture that if \( |E| \gg q^4 \), then \( |\Delta(E)| > \frac{q}{2} \), but this conjecture still remains open. The exponent \( \frac{d}{2} \) cannot be improved. Indeed, let \( q = p^2 \), where \( p \) is a prime and let \( E = \mathbb{F}_p^d \subset \mathbb{F}_q^d \). Then \( |E| = q^d \), yet \( \Delta(E) = \mathbb{F}_p \). In \([4]\), Hart et al. proved that if \( q \) is a prime and \( d \geq 4 \), the sharpness of \( \frac{d}{2} \) can be demonstrated using Lagrangian subspaces.

Iosevich, Koh and Parshall (see \([6]\)) showed that if \( d \geq 2 \) is even and \( E \subset \mathbb{F}_q^d \) with \( |E| \geq 9q^2d \), then

\[
\mathbb{F}_q = \frac{\Delta(E)}{\Delta(E)} = \left\{ \frac{a}{b} : a \in \Delta(E), \ b \in \Delta(E) \setminus \{0\} \right\}.
\]

If the dimension \( d \) is odd and \( E \subset \mathbb{F}_q^d \) with \( |E| \geq 6q^2d \), then

\[
(\mathbb{F}_q)^2 \subset \frac{\Delta(E)}{\Delta(E)},
\]

where \( (\mathbb{F}_q)^2 := \{a^2 : a \in \mathbb{F}_q\} \) is the set of quadratic residues in \( \mathbb{F}_q \).

We shall write \( \mathbb{F}_q^* \) for the set of all non-zero elements in \( \mathbb{F}_q \).

As the main idea to deduce \((1.1)\) and \((1.2)\), the authors \([6]\) first observed that for any \( r \in \mathbb{F}_q^* \), we have

\[
r \in \frac{\Delta(E)}{\Delta(E)} \quad \text{if} \quad \sum_{t \in \mathbb{F}_q} \nu(t)\nu(rt) > \nu^2(0),
\]

where \( \nu(t) := \{(x, y) \in E \times E : \|x - y\| = t\} \).

Next, using the discrete Fourier analysis, they estimated a lower bound of \( \sum_{t \in \mathbb{F}_q} \nu(t)\nu(rt) \) and an upper bound of \( \nu^2(0) \). Finally, the required size condition on the sets \( E \) was obtained by comparing them. Although the method of the proof led to the optimal threshold result on the problem for the quotient set of the distance set, it has two drawbacks below, as mentioned by Pham \([11]\).

- The proof is too sophisticated and requires large amount of computation.
- It is not clear from the proof that how many quadruples \( (x, y, z, w) \in E^4 \) contribute to producing each element \( r \in \mathbb{F}_q^* \) such that \( \frac{\|x - y\|}{\|z - w\|} = r \), namely, \( r \in \frac{\Delta(E)}{\Delta(E)} \).

As a way to overcome the above issues, Pham \([11]\) utilized the machinery of group actions in two dimensions and obtained a lower bound of \( V(r) \) for any square number \( r \) in \( \mathbb{F}_q^* \), where \( V(r) := \{(x, y, z, w) \in E^4 : \|x - y\|/\|z - w\| = r\} \).

As a consequence, he provided a short proof to deduce the following lower bound of \( V(r) \) in two dimensions.
Theorem 1.1 (Theorem 1.2, [11]). Let $E \subset \mathbb{F}_q^2$ and suppose that $|E| \geq Cq$ with $q \equiv 3 \pmod{4}$. Then, for any non-zero square number $r$ in $\mathbb{F}_q^*$, we have
\[ V(r) \geq \frac{c|E|^4}{q}. \]
In particular, we have $\frac{\Delta(E)}{\Delta(F)} \supseteq (\mathbb{F}_q)^2 := \{a^2 : a \in \mathbb{F}_q\}$.

Pham’s approach, based on the group action, is powerful in the sense that it gives a relatively simple proof and an information about a lower bound of $V(r)$. However, his result, Theorem 1.1, is limited to two dimensions with $-1$ square in $\mathbb{F}_q$, and it gives us no information about $V(r)$ for a non-square $r$ in $\mathbb{F}_q^*$.

In [7], Iosevich, Koh and Rakhamnov improved and extended Pham’s result to all dimensions for arbitrary finite fields. In particular, they worked with the non-degenerate quadratic distances which generalize the usual distance.

Equality (1.1) immediately implies that for each $r \in \mathbb{F}_q$ one can find $(x, y) \in E^2$ and $(x', y') \in E^2$ such that $\|x - y\| \neq 0$ and $\|x' - y'\| = r\|x - y\|$. In other words, if $r \in \mathbb{F}_q^*$ and $E \subset \mathbb{F}_q^d$ with $|E| \geq 9q^2$, then one can find a pair of edges in the complete graph $K_{|E|}$ with vertex set $E$ such that one of them is dilated by $r$ with respect to the other.

A natural question arises whether it is possible to generalize this result to other subgraphs of the complete graph $K_{|E|}$ with vertex set $E$ and this is the goal of this paper. In this paper, we are about to generalize this result for $k$-paths ($k \geq 2$), 4-cycles and simplexes.

For clarity, let’s discuss the case of 2-paths. Firstly, we assume that $p$ is a prime such that $p \equiv 3 \pmod{4}$ since it implies $\|x\| = 0$ iff $x = (0, 0)$. Then we prove that for each $r \in \mathbb{F}_p^*$ and $E \subset \mathbb{F}_p^d$ with $|E| \gg p$ there are two copies of 2-path with vertices in $E$, i.e. $(x_1, x_2, x_3) \in E^3$ and $(y_1, y_2, y_3) \in E^3$ such that
\[ x_i \neq x_j, \ y_i \neq y_j \quad \text{for} \quad i \neq j \]
and
\[ \|y_1 - y_2\| = r\|x_1 - x_2\|, \quad \|y_2 - y_3\| = r\|x_2 - x_3\|. \]

We notice that condition (1.3) simply means that 2-paths $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ are nondegenerate, i.e. $\|x_i - x_j\| \neq 0$ and $\|y_i - y_j\| \neq 0$ for $i \neq j$ which easily follows from the fact that $x_i \neq x_j, y_i \neq y_j$ for $i \neq j$ and $p \equiv 3 \pmod{4}$. Condition (1.4) says that the 2-path $(y_1, y_2, y_3)$ is dilated by $r$ with respect to $(x_1, x_2, x_3)$ (see Figure 2).

Now we pause to state some basic definitions which we need throughout the paper.
Definition 1.2. Let $E$ be a set in $\mathbb{F}_p^2$ and $r \in \mathbb{F}_p^*$. We say that $(x_1, \ldots, x_{k+1}) \in E^{k+1}$, $(y_1, \ldots, y_{k+1}) \in E^{k+1}$ is a pair of $k$-paths in $E$ with dilation ratio $r$ if $\|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|$ for $i \in [k]$ and $x_i \neq x_j$, $y_i \neq y_j$ for $1 \leq i < j \leq k + 1$.

Definition 1.3. Let $E$ be a set in $\mathbb{F}_p^2$, $r \in \mathbb{F}_p^*$ and $k \geq 3$. We say that $(x_1, \ldots, x_k) \in E^k$, $(y_1, \ldots, y_k) \in E^k$ is a pair of $k$-cycles in $E$ with dilation ratio $r$ if $\|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|$ for $i \in [k-1]$, $\|y_k - y_1\| = r\|x_k - x_1\|$ and $x_i \neq x_j$, $y_i \neq y_j$ for $1 \leq i < j \leq k$.

Definition 1.4. Let $E$ be a set in $\mathbb{F}_p^d$, $r \in \mathbb{F}_p^*$ and $d \geq 2$. We say that $(x_1, \ldots, x_{d+1}) \in E^{d+1}$, $(y_1, \ldots, y_{d+1}) \in E^{d+1}$ is a pair of $d$-simplexes in $E$ with dilation ratio $r$ if $\|y_i - y_{j}\| = r\|x_i - x_j\|$ for $i, j \in [d+1]$ and $x_i \neq x_j$, $y_i \neq y_j$ for $1 \leq i < j \leq d + 1$.

Now we proceed to the formulation of the problem. Let $r \in \mathbb{F}_p^*$ and $E \subset \mathbb{F}_p^d$ ($d \geq 2$). One can ask how large $E$ needs to be to guarantee the existence of a pair of $k$-paths ($k$-cycles or $d$-simplexes) in $E$ with dilation ratio $r$. We solve this problem for 2-paths using refined combinatorial and graph-theoretic methods and using the arithmetic structure of spheres in $\mathbb{F}_p^d$ we extend this result to 4-cycles. Then using the arithmetic structure of orthogonal groups over finite fields and tools from group actions we prove the result for 2-simplexes and then generalize it for $d$-simplexes. In this setting, the study of the distance set problem reduces to the investigation $O_d(\mathbb{F}_p)$ the orthogonal group of $d \times d$ matrices with entries in $\mathbb{F}_p$ and the $L^d$, $L^{d+1}$-norms of the counting function

$$\lambda_{r, \theta}(z) := |\{(u, v) \in E^2 : u - \sqrt{r}\theta v = z\}|,$$

where $\theta \in O_d(\mathbb{F}_p)$, $z \in \mathbb{F}_p^d$ and $r \in (\mathbb{F}_p)^2 \setminus \{0\}$ the set of nonzero quadratic residues in $\mathbb{F}_p$.

With this setup, our main results are following:

The first theorem asserts the existence of a pair of 2-paths with dilation ratio $r \in \mathbb{F}_p^*$.

Theorem 1.5. If $r \in \mathbb{F}_p^*$, $p$ is a prime such that $p \equiv 3 \pmod{4}$ and $E \subset \mathbb{F}_p^2$ with $|E| > (\sqrt{3} + 1)p$, then

$$\left|\{(x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \frac{\|y_i - y_{i+1}\|}{\|x_i - x_{i+1}\|}, i \in [2], x_i \neq x_j, y_i \neq y_j, i \neq j\}\right| > 0.$$

Figure 2. Pair of 2-paths with dilation ratio $r \in \mathbb{F}_p^*$.

Once we prove this, we proceed to the case of 4-cycles.

Theorem 1.6. If $r \in \mathbb{F}_p^*$, $p$ is a prime such that $p \equiv 3 \pmod{4}$ and $E \subset \mathbb{F}_p^2$ with $|E| > 4\sqrt{3p^2}$, then

$$\left|\{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in E^8 : \frac{\|y_i - y_{i+1}\|}{\|x_i - x_{i+1}\|}, i \in [3], x_i \neq x_j, y_i \neq y_j, i \neq j\}\right| > 0.$$
The following theorem considers the case of 2-simplexes (3-cycles).

**Theorem 1.7.** If \( r \in (\mathbb{F}_p)^2 \setminus \{0\}, \) \( p \) is an odd prime and \( E \subset \mathbb{F}_p^2 \) with \( |E| \geq 3p \), then

\[
\left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \begin{array}{c}
\|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \ i \in [2], \\
\|y_3 - y_1\| = r\|x_3 - x_1\|, \\
x_i \neq x_j, \ y_i \neq y_j, \ i \neq j
\end{array} \right\} > 0
\]

The following theorem generalizes the Theorem 1.7 to higher dimensions. For example, it tells us that if \( E \subset \mathbb{F}_p^3 \) such that \( |E| \gg p^d \), then there exists a pair of 3-simplexes (tetrahedrons) in \( E \) with dilation ratio \( r \in (\mathbb{F}_p)^2 \setminus \{0\} \).

**Theorem 1.8.** If \( r \in (\mathbb{F}_p)^2 \setminus \{0\}, \) \( p \) is an odd prime and \( E \subset \mathbb{F}_p^d \) such that \( |E| \geq (d+1)p^2 \), then

\[
\left\{ (x_1, \ldots, x_{d+1}, y_1, \ldots, y_{d+1}) \in E^{2d+2} : \begin{array}{c}
\|y_i - y_j\| = r\|x_i - x_j\|, \ i, j \in [d+1], \\
x_i \neq x_j, \ y_i \neq y_j, \ i \neq j
\end{array} \right\} > 0.
\]

The following theorem gives the lower bound for the number of paths of length \( k \geq 3 \) in graphs in terms of edge density \( e(G)/(\nu(G))^2 \). This result has been established in 1959 by Mulholland and Smith (see [10]).

**Theorem 1.9.** If \( G = (V, E) \) is a simple graph with \( n \) vertices, then

\[
\left| \left\{ (v_1, v_2, \ldots, v_{k+1}) \in V^{k+1} : v_iv_{i+1} \in E, \ i \in [k] \right\} \right| \geq \frac{(2e(G))^k}{n^{k-1}},
\]

where \( e(G) \) is the number of edges of graph \( G \).
Generalization of this inequality for bipartite graphs is called Sidorenko’s conjecture. This conjecture still remains open but it has been proved for certain family of bipartite graphs such as paths, trees, cycles of even length, complete bipartite graphs, etc. Li and Szegedy in 2011 (see [8]) introduced the idea of using entropy to prove some cases of Sidorenko’s conjecture. Later, Szegedy (see [13]) applied these ideas to prove that an even wider class of bipartite graphs have Sidorenko’s property.

Using Theorem 1.9 we obtain the lower bound for the number of all pairs of $k$-paths ($k \geq 3$) in $E$ with dilation ratio $r \in \mathbb{F}^*_p$. We would like to notice that here by all we mean degenerate and nondegenerate pairs.

**Theorem 1.10.** If $r \in \mathbb{F}^*_p$, $p$ is a prime such that $p \equiv 3 \pmod{4}$ and $E \subset \mathbb{F}^2_p$ with $|E| > 2p$, then

$$\left| \left\{ (x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}) \in E^{2k+2} : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \quad x_i \neq x_{i+1}, \quad i \in [k] \right\} \right| > \frac{|E|^{2k+2}}{(3p)^k}.$$ 

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## 2. Preliminaries

We recall some definitions. Let $\mathbb{F}_p$ be the finite field of order $p$ and $\mathbb{F}^d_p$ be a $d$-dimensional vector space over $\mathbb{F}_p$.

Let $O_d(\mathbb{F}_p)$ denote the group of orthogonal $d \times d$ matrices with entries in $\mathbb{F}_p$.

Let $SO_d(\mathbb{F}_p)$ denote the subgroup of the elements of $O_d(\mathbb{F}_p)$ with determinant 1.

Let $\mathbb{F}^*_p$ denote the set of nonzero elements in $\mathbb{F}_p$.

Let $(\mathbb{F}_p)^2$ denote the set of nonzero elements in $\mathbb{F}_p$.

Let $\mathbb{F}^2_p$ denote the set of quadratic residues in $\mathbb{F}_p$.

Consider the map $\|\cdot\| : \mathbb{F}^d_p \to \mathbb{F}_p$ defined by $\|\alpha\| := \alpha_1^2 + \cdots + \alpha_d^2$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$.

This mapping is not a norm since we do not impose any metric structure on $\mathbb{F}^d_p$, but it does share the important feature of the Euclidean norm: it is invariant under orthogonal matrices.
Lemma 2.2, 2.3 and 2.4. First of all, we need to prove the following.

If \( t \in \mathbb{F}_p \), then let \( S_t \) denote the sphere of radius \( t \) in \( \mathbb{F}_p^d \); thus \( S_t = \{ x \in \mathbb{F}_p^d : \| x \| = t \} \).

If \( X \) is a finite set, then let \( |X| \) denote the size (the cardinality) of \( X \).

If \( n \in \mathbb{N} \), then \( [n] \) denote the set \( \{1, \ldots, n\} \).

For any two sets \( A \) and \( B \), let \( A \cup B \) denote the disjoint union of \( A \) and \( B \).

We write \( X \gg Y \) to mean that there is some constant \( C > 0 \) so that \( X \geq CY \).

We use \( X \gg_d Y \) as shorthand for the inequality \( X \geq C_d Y \) for some constant \( C_d > 0 \) depending only on \( d \).

If \( A \subset \mathbb{F}_p^d \), then let \( \mathbb{1}_A(x) \) to denote the indicator function of \( A \); thus

\[
\mathbb{1}_A(x) = \begin{cases} 
1, & \text{if } x \in A \\
0, & \text{if } x \notin A.
\end{cases}
\]

If \( E \subset \mathbb{F}_p^d \) and \( t_1, \ldots, t_k \in \mathbb{F}_p \) with \( k \geq 1 \), then define the following incidence function:

\[
u_k(t_1, \ldots, t_k) := \left\{ \left( x_1, \ldots, x_{k+1} \right) \in E^{k+1} : \| x_i - x_{i+1} \| = t_i, \ i \in [k] \right\}.
\]

Basically, this function counts the number of walks of length \( k \) with step length \( t_1, t_2, \ldots, t_k \) in the complete graph \( K_{|E|} \) with vertex set \( E \).

Likewise, if \( E \subset \mathbb{F}_p^d \) and \( t_1, t_2, t_3, t_4 \in \mathbb{F}_p \), then define the following incidence function:

\[
\mu(t_1, t_2, t_3, t_4) := \left\{ \left( x_1, x_2, x_3, x_4 \right) \in E^4 : \begin{array}{l}
\| x_1 - x_2 \| = t_1, \| x_2 - x_3 \| = t_2, \\
\| x_3 - x_4 \| = t_3, \| x_4 - x_1 \| = t_4
\end{array} \right\}.
\]

This function counts the number of closed walks of length 4 with step length \( t_1, t_2, t_3, t_4 \) in the complete graph \( K_{|E|} \) with vertex set \( E \).

For \( r \in \mathbb{F}_p^* \) and \( k \geq 1 \) we will consider the following sums:

\[
\sum_{t_1, \ldots, t_k \in \mathbb{F}_p^*} \nu_k(t_1, \ldots, t_k) \nu_k(rt_1, \ldots, rt_k),
\]

\[
\sum_{t_1, \ldots, t_4 \in \mathbb{F}_p^*} \mu(t_1, t_2, t_3, t_4) \mu(rt_1, rt_2, rt_3, rt_4).
\]

Moreover, for each \( k \geq 1 \) we will consider the following sets:

\[
S_k(r) := \left\{ \left( x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1} \right) \in E^{2k+2} : \| y_i - y_{i+1} \| = r \| x_i - x_{i+1} \|, \ x_i \neq x_{i+1}, \ i \in [k] \right\},
\]

\[
C(r) := \left\{ \left( x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \right) \in E^8 : \begin{array}{l}
\| y_i - y_{i+1} \| = r \| x_i - x_{i+1} \|, \ x_i \neq x_{i+1}, \ i \in [3], \\
\| y_4 - y_1 \| = r \| x_4 - x_1 \|, \ x_4 \neq x_1
\end{array} \right\}.
\]

One can show that the above sums are equal to the size of the defined sets, i.e.

\[
\sum_{t_1, \ldots, t_k \in \mathbb{F}_p^*} \nu_k(t_1, \ldots, t_k) \nu_k(rt_1, \ldots, rt_k) = |S_k(r)|,
\]

\[
\sum_{t_1, t_2, t_3, t_4 \in \mathbb{F}_p^*} \mu(t_1, t_2, t_3, t_4) \mu(rt_1, rt_2, rt_3, rt_4) = |C(r)|.
\]

We need the estimates for the size of the sets \( S_1(r) \), \( S_2(r) \) and \( C(r) \), which are given in Lemma 2.2, 2.3 and 2.4. First of all, we need to prove the following
Lemma 2.1. Let p be a prime number such that $p \equiv 3 \pmod{4}$. If $u, v \in \mathbb{F}_p^2 \setminus \{(0, 0)\}$ and $r \in (\mathbb{F}_p)^2 \setminus \{0\}$ such that $||u|| = ||v||$, then there exists unique $\theta \in SO_2(\mathbb{F}_p)$ such that $u = \sqrt{r}\theta v$, where $\sqrt{r}$ is an element of $\mathbb{F}_p$ whose square is $r$.

Proof. Suppose that $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and consider matrices $U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$, $V = \begin{pmatrix} v_1 & -v_2 \\ v_2 & v_1 \end{pmatrix}$. It’s clear that $U^t U = ||u||I_2$, $V^t V = ||v||I_2$, where $I_2$ is an identity $2 \times 2$ matrix. Since $U \begin{pmatrix} 1 \\ 0 \end{pmatrix} = u$ and $V \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v$, hence $UV^{-1}v = u$. Thus $u = \sqrt{r}\theta v$, where $\theta = \frac{1}{\sqrt{r}}UV^{-1}$. It is not difficult to check that $\theta \in SO_2(\mathbb{F}_p)$ and it is unique since the matrix $h \in SO_2(\mathbb{F}_p)$ such that $h \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is unique. \hfill $\Box$

Lemma 2.2. The following inequality holds:

$$|S_1(r)| \geq \left(\frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}\right)|E|^4 - \frac{2|E|^3}{p} - (p + 1)|E|^2.$$

Proof. This inequality follows from (2.7) in [6] by taking $d = 2$. \hfill $\Box$

Lemma 2.3. The following inequality holds:

$$|S_2(r)| \geq |E|^{-2}|S_1(r)|^2.$$  

Remark. We shall prove the Lemma 2.3 in two ways. The first proof is algebraic and relies on the group actions approach. However, the second proof is done by means of graph theory. It is worth noting that graph-theoretic proof gives a better result since it establishes the inequality (2.1) for all $r \in \mathbb{F}_p^*$, but algebraic method proves it only for $r \in (\mathbb{F}_p)^2 \setminus \{0\}$.

The first proof of Lemma 2.3 Assume that $r \in (\mathbb{F}_p)^2 \setminus \{0\}$ and if we apply Lemma 2.1 then we obtain:

$$|S_2(r)| = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \left| |y_i - y_{i+1}| - r|x_i - x_{i+1}| \right|, x_i \neq x_{i+1}, i \in [2] \right\}$$

$$= \sum_{\theta_1, \theta_2} \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \left| y_i - y_{i+1} - \sqrt{r}\theta_i(x_i - x_{i+1}) \right|, x_i \neq x_{i+1}, i \in [2] \right\}$$

$$= \sum_{\theta_1, \theta_2} \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \left| y_i - \sqrt{r}\theta_i x_i = y_i - y_{i+1} - \sqrt{r}\theta_i x_{i+1} \right|, x_i \neq x_{i+1}, i \in [2] \right\}$$

$$= \sum_{\theta_1, \theta_2} \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \left| y_1 - \sqrt{r}\theta_1 x_1 = y_2 - \sqrt{r}\theta_1 x_2 = y_3 - \sqrt{r}\theta_1 x_3 = a, y_2 - \sqrt{r}\theta_2 x_2 = y_3 - \sqrt{r}\theta_2 x_3 = b \right|, x_1 \neq x_2, x_2 \neq x_3 \right\}$$

$$= \sum_{\theta_1, \theta_2} \sum_{\begin{subarray}{c} y_2 - \sqrt{r}\theta_1 x_2 = a \\ y_2 - \sqrt{r}\theta_2 x_2 = b \end{subarray}} \sum_{\begin{subarray}{c} y_1 - \sqrt{r}\theta_1 x_1 = a \\ x_1 \neq x_2 \end{subarray}} \sum_{\begin{subarray}{c} y_3 - \sqrt{r}\theta_1 x_3 = b \\ x_3 \neq x_2 \end{subarray}} \sum_{\begin{subarray}{c} y_3 - \sqrt{r}\theta_2 x_3 = b \end{subarray}} 1_{E}(x_2) 1_{E}(y_2) 1_{E}(y_1) 1_{E}(y_3)$$
which completes the proof. □

Hence

One can easily check that I and II can be written as follows:

\[
I = \sum_{\theta_1, a} \sum_{y_1, y_2} \mathbb{I}_E(x_1) \mathbb{I}_E(y_1) \mathbb{I}_{\{a\}}(y_2 - \sqrt{r\theta_1}x_2) \\
= \sum_{\theta_1} \left| \left\{ (x_1, y_1) \in E^2 : y_1 - \sqrt{r\theta_1}x_1 = y_2 - \sqrt{r\theta_1}x_2, x_2 \neq x_2 \right\} \right|,
\]

\[
II = \sum_{\theta_2, b} \sum_{y_3, y_4} \mathbb{I}_E(x_3) \mathbb{I}_E(y_3) \mathbb{I}_{\{b\}}(y_2 - \sqrt{r\theta_2}x_3) \\
= \sum_{\theta_2} \left| \left\{ (x_3, y_3) \in E^2 : y_3 - \sqrt{r\theta_2}x_3 = y_2 - \sqrt{r\theta_2}x_2, x_1 \neq x_2 \right\} \right|.
\]

If we let \(\lambda(r, \theta, x_2, y_2) := \left| \left\{ (x, y) \in E^2 : y - \sqrt{r\theta}x = y_2 - \sqrt{r\theta}x_2, x \neq x_2 \right\} \right|\), then we obtain

\[
|S_2(r)| = \sum_{x_2, y_2 \in E} \left( \sum_{\theta} \lambda(r, \theta, x_2, y_2) \right)^2.
\]

Applying Cauchy-Schwarz inequality, we obtain

\[
\left( \sum_{x_2, y_2 \in E} \sum_{\theta} \lambda(r, \theta, x_2, y_2) \right)^2 \leq |E|^2 \sum_{x_2, y_2 \in E} \left( \sum_{\theta} \lambda(r, \theta, x_2, y_2) \right)^2.
\]

Hence

\[
|S_2(r)| \geq |E|^{-2} \left( \sum_{x_2, y_2 \in E} \sum_{\theta} \lambda(r, \theta, x_2, y_2) \right)^2.
\]

We shall show that the inner sum is \(|S_1(r)|\). Indeed,

\[
\sum_{x_2, y_2 \in E} \sum_{\theta \in SO_2(\mathbb{F}_p)} \lambda(r, \theta, x_2, y_2) \\
= \sum_{x_2, y_2 \in E} \sum_{\theta \in SO_2(\mathbb{F}_p)} \left| \left\{ (x, y) \in E^2 : y - \sqrt{r\theta}x = y_2 - \sqrt{r\theta}x_2, x \neq x_2 \right\} \right| \\
= \sum_{x_2, y_2 \in E} \left| \left\{ (x, y) \in E^2 : \|y - y_2\| = r\|x - x_2\|, x \neq x_2 \right\} \right| \\
= \left| \left\{ (x, x_2, y_2) \in E^4 : \|y - y_2\| = r\|x - x_2\|, x \neq x_2 \right\} \right| = |S_1(r)|,
\]

which completes the proof. □
The second proof of Lemma 2.4. Consider the graph $G = (V, E)$ defined as follows: let’s introduce the following notations:

$$L := E \times E = \{(x, y) : x, y \in E\},$$

$$R := S_1(r) = \{(x_1, x_2, y_1, y_2) \in E^4 : \|y_1 - y_2\| = r\|x_1 - x_2\|, x_1 \neq x_2\}.$$  

Define the vertex set $V$ to be $L \cup R$ and the edge set $E$ we will define in the following way: each vertex $(x_1, x_2, y_1, y_2) \in R$ we join with $(x_1, y_1) \in L$ and $(x_2, y_2) \in L$ and we notice that $(x_1, y_1) \neq (x_2, y_2)$.

For any vertex $v \in V$, let $d_v$ denote it’s degree, i.e.

$$d_v := |\{e \in E : e \text{ is incident with } v\}|.$$  

From degree sum formula, we obtain

$$\sum_{v \in L} d_v + \sum_{v \in R} d_v = 2|E|.$$  

If $v \in R$, then $d_v = 2$ and $|E| = 2|R| = 2|S_1(r)|$ and hence

$$\sum_{v \in L} d_v = 2|S_1(r)|.$$  

Applying Cauchy-Schwarz, we obtain

$$\sum_{v \in L} d_v^2 \geq |L|^{-1} \left( \sum_{v \in L} d_v \right)^2 = 4|E|^{-2}|S_1(r)|^2.”$$  

We see that $d_v = |\{u \in R : uv \in E\}|$ because the graph $G$ is simple (without loops and multiple edges), then

$$\sum_{v \in L} d_v^2 = |\{(u, u', v) \in R \times R \times L : uv \in E, u'v \in E\}|.$$  

We will show that the quantity on the RHS of (2.3) is equal to $4|S_2(r)|$.

Indeed, let’s consider a function

$$f : \{(u, u', v) \in R \times R \times L : uv \in E, u'v \in E\} \rightarrow S_2(r)$$

defined as follows: each element of the domain is $(u, u', v) \in R \times R \times L$ with $uv \in E, u'v \in E$ and hence $u = (a, b, a', b') \in S_1(r), u' = (c, d, c', d') \in S_1(r)$. The vertex $v \in L$ being incident with both $u = (a, b, a', b')$ and $u' = (c, d, c', d')$ implies that

$$((v = (a, a')) \vee (v = (b, b'))) \wedge ((v = (c, c')) \vee (v = (d, d'))).$$

Therefore, we have the following 4 cases:

- if $v = (a, a') = (c, c')$, then $(u, u', v) \equiv ((a, b, a', b'), (a, d, a', d'), (a, a')) \overset{f}{\rightarrow} (b, a, d, b', a', d').$
- if $v = (a, a') = (d, d')$, then $(u, u', v) \equiv ((a, b, a', b'), (c, a, c', a'), (a, a')) \overset{f}{\rightarrow} (b, a, c, b', a', c').$
- if $v = (b, b') = (c, c')$, then $(u, u', v) = ((a, b, a', b'), (b, d, b', d'), (b, b')) \overset{f}{\rightarrow} (a, b, d, a', b', d').$
Lemma 2.5. Let \( (2.6) \) where \( x \) edge with \((graph)\) \( G \) introduce the following notations: We will consider the graph \( G \).

Proof. Lemma 2.4.

The following inequality holds:

\[ |\{(u, u', v) \in R \times R \times L : uv \in E, u'v \in E\}| = 4|S_2(r)|. \]  

(2.4)

Comparing (2.4) with (2.3) and (2.2), we obtain the desired inequality

\[ |S_2(r)| \geq |E|^{-4}|S_1(r)|^2. \]

\[ \square \]

Lemma 2.4. The following inequality holds:

\[ |C(r)| \geq |E|^{-4}|S_2(r)|^2. \]

Proof. We will consider the graph \( G = (V, E) \) which is defined in the following way: let’s introduce the following notations:

\[ L := E \times E \times E \times E, \]

\[ R := S_2(r). \]

So we define the vertex set \( V \) of graph \( G \) to be \( L \cup R \). We will define edge set \( E \) of the graph \( G \) as follows: each vertex \( v \in R \) which has form \((x_1, x_2, x_3, y_1, y_2, y_3) \in L\).

From degree sum formula, we obtain

\[ \sum_{v \in L} d_v + \sum_{v \in R} d_v = 2|E|, \]

where \( d_v := \deg(v) \). If \( v \in R \), then \( d_v = 1 \) and \( |E| = |S_2(r)| \). Hence we have

\[ \sum_{v \in L} d_v = |S_2(r)|. \]

Applying Cauchy-Schwarz inequality, we obtain

\[ \sum_{v \in L} d_v^2 \geq |E|^{-4}|S_2(r)|^2. \]

(2.5)

However,

\[ \sum_{v \in L} d_v^2 = |\{(u, u', v) \in R \times R \times L : uv \in E, u'v \in E\}| = |C(r)|. \]

Comparing (2.6) with (2.5) we obtain the desired inequality

\[ |C(r)| \geq |E|^{-4}|S_2(r)|^2. \]

\[ \square \]

We also need to know the size of the sphere \( S_t \) in \( \mathbb{F}_q^d \) which is given in the following

Lemma 2.5. Let \( S_t \) denote the sphere of radius \( t \in \mathbb{F}_q \) in \( \mathbb{F}_q^d \). If \( d \geq 2 \) is even, then

\[ |S_t| = q^{d-1} + \lambda(t)q^{d-2} \eta \left( (-1)^{\frac{d}{2}} \right), \]

where \( \eta \) is the quadratic character of \( \mathbb{F}_q^* \), \( \lambda(t) = -1 \) for \( t \in \mathbb{F}_q^* \), and \( \lambda(0) = q - 1 \).

Proof. This follows from Theorem 6.26 in [9].

\[ \square \]
The trivial inequality between counting functions \( \nu_2(\cdot, \cdot) \) and \( \nu_1(\cdot) \) which comes in handy is given in the following

**Lemma 2.6.** If \( t_1, t_2 \in \mathbb{F}_p \), then the following inequality holds:

\[
\nu_2(t_1, t_2) \leq |E|\nu_1(t_1).
\]

**Proof.** By definition of the counting function \( \nu_2(\cdot, \cdot) \), we obtain

\[
\nu_2(t_1, t_2) = |\{(x_1, x_2, x_3) \in E^3 : \|x_1 - x_2\| = t_1, \|x_2 - x_3\| = t_2\}|
\]

\[
= \sum_{x_1, x_2, x_3 \in \mathbb{F}_p^2} \mathbb{1}_{E}(x_1) \mathbb{1}_{E}(x_2) \mathbb{1}_{E}(x_3) \mathbb{1}_{S_1}(x_1 - x_2) \mathbb{1}_{S_2}(x_2 - x_3)
\]

\[
\leq \sum_{x_1, x_2, x_3 \in \mathbb{F}_p^2} \mathbb{1}_{E}(x_1) \mathbb{1}_{E}(x_2) \mathbb{1}_{E}(x_3) \mathbb{1}_{S_1}(x_1 - x_2)
\]

\[
= \sum_{x_3 \in \mathbb{F}_p^2} \mathbb{1}_{E}(x_3) \sum_{x_1, x_2 \in \mathbb{F}_p^2} \mathbb{1}_{E}(x_1) \mathbb{1}_{E}(x_2) \mathbb{1}_{S_1}(x_1 - x_2)
\]

\[
= |E|\nu_1(t_1). \quad \square
\]

The following lemma gives us the order of the group of orthogonal matrices over finite field.

**Lemma 2.7.** If \( F \) is any field and \( O_n(F) \) is a group of orthogonal \( n \times n \) matrices with entries in \( F \), then for any odd prime \( p \) we have:

\[
|O_{2n+1}(\mathbb{F}_p)| = 2p^{n^2} \prod_{i=1}^{n}(p^{2i} - 1),
\]

\[
|O_{2n}^+(\mathbb{F}_p)| = 2p^{n(n-1)}(p^n - 1) \prod_{i=1}^{n-1}(p^{2i} - 1),
\]

\[
|O_{2n}^-(\mathbb{F}_p)| = 2p^{n(n-1)}(p^n + 1) \prod_{i=1}^{n-1}(p^{2i} - 1).
\]

**Proof.** This statement can be found on page 141 in [15]. \( \square \)

3. **Proof of Theorem 1.5.**

From the definition of a set \( S_2(r) \), it is clear that it also contains pairs of 2-paths in \( E \) with dilation ratio \( r \) which are degenerate. For instance, degenerate 2-paths come if one takes \( x_1 = x_3 \) or \( y_1 = y_3 \). That is why we need to rule out all degenerate cases and in order to implement it we need to consider the following sets:

\[
\mathcal{A} := \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \; i \in [2], \; x_1 \neq x_2, \; x_2 \neq x_3, \; x_1 = x_3 \right\},
\]

\[
\mathcal{B} := \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \; i \in [2], \; x_1 \neq x_2, \; x_2 \neq x_3, \; y_1 = y_3 \right\},
\]
\begin{equation}
C := \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \ i \in [2], \ x_1 \neq x_2, \ x_2 \neq x_3, \ x_1 \neq x_3, \ y_1 \neq y_3 \right\}.
\end{equation}

It is easy to see that
\begin{equation}
A \cup B \cup C = S_2(r).
\end{equation}
From the definition of sets \( A, B, C \) follows that
\begin{equation}
A \cap C = B \cap C = A \cap B \cap C = \emptyset.
\end{equation}
One can see that the pairs of 2-paths in \( E \) with dilation ratio \( r \) is exactly the set \( C \) since \( y_1 \neq y_2 \) follows from the fact that \( \|y_1 - y_2\| = r\|x_1 - x_2\|, \ x_1 \neq x_2 \) and \( \|x\| = 0 \) iff \( x = (0, 0) \) since \( p \equiv 3 \ (\text{mod} \ 4) \). The same reasoning holds for \( y_2 \neq y_3 \).
Applying inclusion–exclusion principle to (3.1) and taking into account (3.2), we obtain
\begin{equation}
|C| = |S_2(r)| - |A| - |B| + |A \cap B|.
\end{equation}
Now we can explicitly compute the size of \( |A|, |B| \) and \( |A \cap B| \). Indeed,
\begin{align*}
|A \cap B| &= \left| \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \ i \in [2], \ x_1 \neq x_2, \ x_2 \neq x_3, \ x_1 = x_3, \ y_1 = y_3 \right\} \right| \\
&= \left| \left\{ (x_1, x_2, y_1, y_2) \in E^4 : \|y_1 - y_2\| = r\|x_1 - x_2\|, \ x_1 \neq x_2 \right\} \right| \\
&= |S_1(r)|.
\end{align*}
Now we proceed to the size of \( A \):
\begin{align*}
|A| &= \left| \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \ i \in [2], \ x_1 \neq x_2, \ x_2 \neq x_3, \ x_1 = x_3 \right\} \right| \\
&= \left| \left\{ (x_1, x_2, y_1, y_2, y_3) \in E^6 : \|y_1 - y_2\| = \|y_2 - y_3\| = r\|x_1 - x_2\|, \ x_1 \neq x_2 \right\} \right| \\
&= \sum_{t \in \mathbb{F}_p} \sum_{x_1, x_2, y_1, y_2, y_3} \mathbbm{1}_{E(x_1)} \mathbbm{1}_{E(x_2)} \mathbbm{1}_{E(y_1)} \mathbbm{1}_{E(y_2)} \mathbbm{1}_{E(y_3)} \mathbbm{1}_{S_1(x_1 - x_2)} \mathbbm{1}_{S_{rt}(y_1 - y_2)} \mathbbm{1}_{S_{rt}(y_2 - y_3)} \\
&= \sum_{t \in \mathbb{F}_p} \sum_{x_1, x_2} \mathbbm{1}_{E(x_1)} \mathbbm{1}_{E(x_2)} \mathbbm{1}_{S_1(x_1 - x_2)} \sum_{y_1, y_2, y_3} \mathbbm{1}_{E(y_1)} \mathbbm{1}_{E(y_2)} \mathbbm{1}_{E(y_3)} \mathbbm{1}_{S_{rt}(y_1 - y_2)} \mathbbm{1}_{S_{rt}(y_2 - y_3)} \\
&= \sum_{t \in \mathbb{F}_p} \nu_1(t) \nu_2(rt, rt).
\end{align*}
In an analogous way one can compute the size of \( B \) and we obtain the following equalities:
\begin{equation}
|A \cap B| = |S_1(r)|, \quad |A| = \sum_{t \in \mathbb{F}_p} \nu_1(t) \nu_2(rt, rt),
\end{equation}
\begin{equation}
|B| = \sum_{t \in \mathbb{F}_p} \nu_1(rt) \nu_2(t, t),
\end{equation}
If we plug (3.4) into (3.3), then we obtain

\[(3.5) \quad |C| = |S_2(r)| + |S_1(r)| - \sum_{t \in \mathbb{F}_p^*} \nu_1(t) \nu_2(rt, rt) - \sum_{t \in \mathbb{F}_p^*} \nu_1(rt) \nu_2(t, t).\]

The upper bounds for the third and fourth terms in (3.5) can be obtained from Lemma 2.6. Indeed, since \(\nu_2(rt, rt) \leq |E| \nu_1(rt)\), then

\[(3.6) \quad \sum_{t \in \mathbb{F}_p^*} \nu_1(t) \nu_2(rt, rt) \leq |E| \sum_{t \in \mathbb{F}_p^*} \nu_1(t) \nu_1(rt) = |E||S_1(r)|.

In an analogous way, we obtain

\[(3.7) \quad \sum_{t \in \mathbb{F}_p^*} \nu_1(rt) \nu_2(t, t) \leq |E| \sum_{t \in \mathbb{F}_p^*} \nu_1(t) \nu_1(rt) = |E||S_1(r)|.

Using estimates (3.6) and (3.7) in (3.5), we obtain

\[|C| \geq |S_2(r)| + |S_1(r)| - 2|E||S_1(r)|.

Applying Lemma 2.3 we obtain that

\[(3.8) \quad |C| \geq |E|^{-2}|S_1(r)|^2 + |S_1(r)| - 2|E||S_1(r)| = |E|^{-2}|S_1(r)| (|S_1(r)| + |E|^2 - 2|E|^3).

The following lemma shows the positivity of (3.8) which implies the proof of Theorem 1.5. \(\Box\)

**Lemma 3.1.** If \(|E| > (\sqrt{3} + 1)p\), then

\[|E|^{-2}|S_1(r)| > 0 \quad \text{and} \quad |S_1(r)| + |E|^2 - 2|E|^3 > 0.

**Proof.** Let’s start with the first inequality. Applying Lemma 2.2 we obtain

\[|E|^{-2}|S_1(r)| \geq \left(\frac{1}{p} + \frac{1}{p^2 - \frac{1}{p^3}}\right)|E|^2 - \frac{2|E|}{p} - (p + 1)\]

\[= \frac{1}{p^3} \left(p^2 + p - 1\right)|E|^2 - 2p^2|E| - p^3(p + 1)\]

\[= p^2 + p - 1 \left(\left|E\right| - \frac{p^2 - \sqrt{p^6 + 2p^5 + p^4 - p^3}}{p^2 + p - 1}\right)\]

\[\times \left(\left|E\right| - \frac{p^2 + \sqrt{p^6 + 2p^5 + p^4 - p^3}}{p^2 + p - 1}\right).

It suffices to show that the expression in the second parenthesis is positive since it immediately implies that the expression in the first parenthesis is also positive.

If \(|E| > (\sqrt{3} + 1)p\), then
hence \( \phi \) means that the expression in the first parenthesis is positive and hence \( x \) increases on \( \big( 3 + 1 \big) \), which completes the proof of Theorem 1.5.

The function \( \phi : [3, +\infty) \to \mathbb{R} \) defined by \( \phi(x) = \frac{x^3 + 2x^2}{x^3 + x^2 - x} \) is decreasing and hence \( \phi(x) \leq \phi(3) = \frac{15}{11} \).

Therefore, we obtain
\[
|E| - \frac{p^2 + \sqrt{p^6 + 2p^5 + p^4 - p^3}}{p^2 + p - 1} > p \left( (\sqrt{3} + 1) - \frac{15}{11} \right) > 0.
\]

We have shown that
\[
|E|^{-2}|S_1(r)| > 0.
\]

Now we proceed to the second inequality. Let’s use the Lemma 2.2 and we obtain
\[
|S_1(r)| + |E|^2 - 2|E|^3
\geq \frac{|E|^2}{p^3} \left( (p^2 + p - 1)|E|^2 - (2p^3 + 2p^2)|E| - p^4 \right)
= \frac{(p^2 + p - 1)|E|^2}{p^3} \left( |E| - \frac{p^3 + p^2 + \sqrt{2p^6 + 3p^5}}{p^2 + p - 1} \right)
\times \left( |E| - \frac{p^3 + p^2 - \sqrt{2p^6 + 3p^5}}{p^2 + p - 1} \right).
\]

Again, it suffices to show that the expression in the first parenthesis is positive.

If \( |E| > (\sqrt{3} + 1)p \), then
\[
|E| - \frac{p^3 + p^2 + \sqrt{2p^6 + 3p^5}}{p^2 + p - 1} > p \left( (\sqrt{3} + 1) - \frac{(\sqrt{3} + 1)p^3 + p^2}{p^3 + p^2 - p} \right).
\]

The function \( \varphi : [3, +\infty) \to \mathbb{R} \) defined by \( \varphi(x) = \frac{(\sqrt{3} + 1)x^3 + 2x^2}{x^3 + x^2 - x} \) decreases on \([3, x_0]\) and increases on \([x_0, +\infty)\), where \( x_0 \approx 3.32 \). Since \( \lim_{x \to +\infty} \varphi(x) = \sqrt{3} + 1 \), then \( \varphi(x) \leq \sqrt{3} + 1 \). It means that the expression in the first parenthesis is positive and hence
\[
|S_1(r)| + |E|^2 - 2|E|^3 > 0. \quad \square
\]

Combining Lemma 3.1 with (3.8), we obtain that if \( |E| > (\sqrt{3} + 1)p \), then
\[
|C| = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in E^6 : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \; i \in [2], \; x_1 \neq x_2, \; x_2 \neq x_3, \; x_1 \neq x_3, \; y_1 \neq y_3 \right\} > 0,
\]
which completes the proof of Theorem 1.5.
4. Proof of Theorem 1.6

It is clear that \( C(r) \) also contains pairs of 4-cycles in \( E \) with dilation ratio \( r \) which are degenerate. For example, degenerate pairs are pairs if one takes \((x_1 = x_3) \vee (x_2 = x_4) \vee (y_1 = y_3) \vee (y_2 = y_4)\).

Therefore, we need to rule out these degenerate cases and show that their size is less than the size of \( C(r) \). That is why we will consider the following sets:

\[
A_{13} := \{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in E^8 : \begin{array}{l}
|y_i - y_{i+1}| = r|x_i - x_{i+1}|, x_i \neq x_{i+1}, i \in [3], \\
|y_4 - y_1| = r|x_4 - x_1|, x_4 \neq x_1, x_1 = x_3
\end{array}\},
\]

\[
A_{24} := \{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in E^8 : \begin{array}{l}
|y_i - y_{i+1}| = r|x_i - x_{i+1}|, x_i \neq x_{i+1}, i \in [3], \\
|y_4 - y_1| = r|x_4 - x_1|, x_4 \neq x_1, x_4 = x_2
\end{array}\},
\]

\[
B_{13} := \{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in E^8 : \begin{array}{l}
|y_i - y_{i+1}| = r|x_i - x_{i+1}|, x_i \neq x_{i+1}, i \in [3], \\
|y_4 - y_1| = r|x_4 - x_1|, x_4 \neq x_1, y_1 = y_3
\end{array}\},
\]

\[
B_{24} := \{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in E^8 : \begin{array}{l}
|y_i - y_{i+1}| = r|x_i - x_{i+1}|, x_i \neq x_{i+1}, i \in [3], \\
|y_4 - y_1| = r|x_4 - x_1|, x_4 \neq x_1, y_2 = y_4
\end{array}\}.
\]

We will also define the set which is exactly the family of pairs of 4-cycles in \( E \) with dilation ratio \( r \).

\[
F := \{(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in E^8 : \begin{array}{l}
|y_i - y_{i+1}| = r|x_i - x_{i+1}|, i \in [3], \\
x_i \neq x_j, y_i \neq y_j, i \neq j \in [4]
\end{array}\}.
\]

It is clear that we have the following set equality:

\[
C(r) = F \cup (A_{13} \cup A_{24} \cup B_{13} \cup B_{24}).
\]

Hence, we have

\[
|C(r)| = |F| + |A_{13} \cup A_{24} \cup B_{13} \cup B_{24}|
\]

One can trivially estimate the term \( |A_{13} \cup A_{24} \cup B_{13} \cup B_{24}| \) as follows:

\[
|A_{13} \cup A_{24} \cup B_{13} \cup B_{24}| \leq |A_{13}| + |A_{24}| + |B_{13}| + |B_{24}|
\]

Comparing (4.2) with (4.1), we obtain the following lower bound for the size of \( F \):

\[
|F| \geq |C(r)| - |A_{13}| - |A_{24}| - |B_{13}| - |B_{24}|
\]

Lemma 2.4 gives us the lower bound for \( |C(r)| \) and our current goal is to find the appropriate upper bound for the size of \( A_{13}, A_{24}, B_{13} \) and \( B_{24} \). We would like to point out that we have performed the same approach in the proof of Theorem 1.5 and we found the right the upper bounds for \( |A| \) and \( |B| \) relying on the trivial estimates provided by Lemma 2.6.

However, this approach is fruitless in the case of 4-cycles. More precisely, if we apply Lemma 2.6 to estimate the size of \( A_{ij} \) and \( B_{ij} \) we will not be able to get a nontrivial exponent for the size of \( E \).

Fortunately, this barrier can be overcome if we know the arithmetic structure of the sphere \( S_t \) in \( \mathbb{F}_q^d \) and this information is given by Lemma 2.5.
where \((\frac{a}{b})\) is a Legendre symbol. Since \(d = 2\), then Lemma 2.5 gives us that
\[
|S_t| = p + 1 \quad \text{for} \quad t \in \mathbb{F}_p^*.
\]

The following Lemma gives us the correct upper bound for the size of \(A_{13}, A_{24}, B_{13}\) and \(B_{24}\).

**Lemma 4.1.** The following inequality holds:
\[
|S_2(r)| \leq |A_{13}|, |A_{24}|, |B_{13}|, |B_{24}| \leq (p + 1)|S_2(r)|.
\]

**Proof.** For concreteness, we will prove this inequality only for \(|A_{13}|\). The remaining inequalities can be proven in an analogous way.

From the definition of \(A_{13}\) it follows that \(|A_{13}| = |\tilde{A}_{13}|\), where
\[
\tilde{A}_{13} := \left\{ (x_1, x_2, x_4, y_1, y_2, y_3, y_4) \in E^7 : \begin{array}{l}
|y_1 - y_2| = |y_2 - y_3| = r|x_1 - x_2|, \\
x_1 \neq x_2, \ x_1 \neq x_4
\end{array} \right\}.
\]

Consider the function \(f : \tilde{A}_{13} \to S_2(r)\) defined as
\[
(x_1, x_2, x_4, y_1, y_2, y_3, y_4) \mapsto (x_4, x_1, x_2, y_4, y_1, y_2).
\]

We notice that \(f\) is surjective. Indeed, for each \((x, y, z, x', y', z') \in S_2(r)\) its preimage under \(f\) is \((y, z, x, y', z', y', x') \in \tilde{A}_{13}\). It immediately implies that \(|S_2(r)| \leq |\tilde{A}_{13}|\) and hence \(|S_2(r)| \leq |A_{13}|\).

Now we proceed to the proof of the RHS inequality in (4.5). It suffices to show that for each \(y \in S_2(r)\) the inequality \(|f^{-1}\{y\}| \leq p + 1\) holds. Let’s fix an arbitrary \(y \in S_2(r)\), then \(y = (y_1, y_2, y_3, y_4, y_5, y_6)\) and consider it’s preimage, i.e. the set \(f^{-1}\{y\}\). We already know that \(f^{-1}\{y\} \neq \emptyset\) due to the surjectivity of \(f\). Consider an element \(x_0 := (y_2, y_3, y_1, y_5, y_6, y_4)\) and one can check that \(x_0 \in f^{-1}\{y\}\).

Choose arbitrary element \(x \in f^{-1}\{y\}\), then \(x = (y_2, y_3, y_1, y_5, y_6, \alpha, y_4)\) with \(|\alpha - y_4| = r|y_1 - y_2|\). We see that \(|y_1 - y_2| \neq 0\) since \(x_0 \in \tilde{A}_{13}\).

We have shown that for arbitrary \(x \in f^{-1}\{y\}\) we have \(x - x_0 = (\alpha - y_5) \cdot \vec{e}_6\) with \(|\alpha - y_4| = t\), where \(\vec{e}_6 = (0, 0, 0, 0, 0, 1, 0)\) and \(t := r|y_1 - y_2| \neq 0\).

That can be written as the following set containment:
\[
f^{-1}\{y\} \subseteq \{x_0 + \vec{e}_6 \cdot (\alpha - \pi_5(y)) : \alpha \in \pi_4(y) + S_t\},
\]
where \(\pi_j(y)\) is the \(j\)th coordinate of \(y\) and \(\pi_4(y) + S_t\) means the translation of sphere \(S_t\) by \(\pi_4(y)\). Containment (4.6) implies that
\[
|f^{-1}\{y\}| \leq \{x_0 + \vec{e}_6 \cdot (\alpha - \pi_5(y)) : \alpha \in \pi_4(y) + S_t\}.
\]

However, the RHS term in (4.7) is at most
\[
|\pi_4(y) + S_t| = |S_t|.
\]

Comparing (4.7) and (4.8) with (4.4), we obtain
\[
|f^{-1}\{y\}| \leq p + 1,
\]
which completes the proof of Lemma 4.1.

Plugging inequalities (4.5) to the inequality (4.3), we obtain the following lower bound for the size of $\mathcal{F}$:

$$|\mathcal{F}| \geq |C(r)| - 4(p+1)|S_2(r)|.$$

Applying Lemma 2.3 we obtain

$$|\mathcal{F}| \geq |E| - 4\left|S_2(r)\right| - 4(p+1)|E|^4,$$

(4.9)

We shall show that $|\mathcal{F}| > 0$ if $|E| \gg p^{\frac{3}{2}}$.

**Lemma 4.2.** If $|E| > 4\sqrt{3}p^{\frac{3}{2}}$, then

$$|E| - 4\left|S_2(r)\right| - 4(p+1)|E|^4 > 0.$$

**Proof.** Lemma 3.1 claims that if $|E| > (\sqrt{3} + 1)p$, then $|E|^{-2}|S_1(r)| > 0$. Applying Lemma 2.3, we obtain that $|E|^{-4}|S_2(r)| > 0$.

Lemma 2.2 implies that

$$|S_1(r)| > \frac{|E|^4}{6p},$$

(4.10)

since

$$\frac{|E|^4}{6p} > \frac{2|E|^3}{p} - (p+1)|E|^2$$

provided that $|E| > (\sqrt{3} + 1)p$.

Inequality (4.10) combined with Lemma 2.3 implies that

$$|S_2(r)| - 4(p+1)|E|^4 > \frac{|E|^6}{9p^2} - 4(p+1)|E|^4$$

$$= \frac{|E|^4}{9p^2} \left(|E|^2 - 36p^2(p+1)\right).$$

One can check that if $|E| > 4\sqrt{3}p^{\frac{3}{2}}$, then $|E|^2 > 36p^2(p+1)$. Moreover, the first inequality $|E|^{-4}|S_2(r)| > 0$ also holds if $|E| > 4\sqrt{3}p^{\frac{3}{2}}$ since $4\sqrt{3}p^{\frac{3}{2}} > (\sqrt{3} + 1)p$. □

Combining Lemma 4.2 with (4.9), we obtain that if $|E| > 4\sqrt{3}p^{\frac{3}{2}}$, then $|\mathcal{F}| > 0$ which completes the proof of Theorem 1.6.
5. Proof of Theorem 1.7

In this section we will obtain a nontrivial estimate for size of \( E \subset \mathbb{F}_p^2 \) such that it contains a pair of 3-cycles in \( E \) with dilation ratio \( r \in (\mathbb{F}_p)^2 \setminus \{0\} \). The approach will be somewhat different and it relies on the introducing certain counting function and investigating its \( L^3 \)-norm.

If \( r \in (\mathbb{F}_p)^2 \setminus \{0\} \), \( z \in \mathbb{F}_p^2 \) and \( \theta \in O_2(\mathbb{F}_p) \), then consider the counting function defined as follows:

\[
\lambda_{r,\theta}(z) := |\{(u,v) \in E^2 : u - \sqrt{r}\theta v = z\}|
\]

and the cube of its \( L^3 \)-norm

\[
\|\lambda_{r,\theta}(z)\|_3^3 := \sum_{\theta, z} \lambda_{r,\theta}^3(z).
\]

One can see that

\[
\lambda_{r,\theta}^3(z) = |\{(u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : u_i - \sqrt{r}\theta v_i = z, \ i \in [3]\}|.
\]

Hence, we obtain

\[
\|\lambda_{r,\theta}(z)\|_3^3 = \sum_{\theta \in O_2(\mathbb{F}_p)} \sum_{z \in \mathbb{F}_p^2} |\{(u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : u_i - \sqrt{r}\theta v_i = z, \ i \in [3]\}|
\]

\[
= \sum_{\theta \in O_2(\mathbb{F}_p)} |\{(u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : u_1 - \sqrt{r}\theta v_1 = u_2 - \sqrt{r}\theta v_2 = u_3 - \sqrt{r}\theta v_3\}|.
\]

If we introduce the following notation:

\[
\Lambda_{\theta}(r) := \{(u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : u_1 - \sqrt{r}\theta v_1 = u_2 - \sqrt{r}\theta v_2 = u_3 - \sqrt{r}\theta v_3\},
\]

then \( \|\lambda_{r,\theta}(z)\|_3^3 = \sum_{\theta} |\Lambda_{\theta}(r)| \).

Consider the following subset of \( \Lambda_{\theta}(r) \), where \( v_i \)'s are pairwise distinct:

\[
N_{\theta}(r) := \left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \frac{u_i - u_j}{v_i - v_j}, \ i \neq j \in [3] \right\}.
\]

Applying inclusion-exclusion principle one can compute the size of \( N_{\theta}(r) \) explicitly. Indeed,

\[
|N_{\theta}(r)| = |\Lambda_{\theta}(r)| - \sum_{1 \leq k < l \leq 3} \left| \left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \frac{u_i - u_j}{v_i - v_j}, \ i \neq j \in [3], \ v_k = v_l \right\} \right|
\]

\[
+ 2 \left| \left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \frac{u_i - u_j}{v_1 - v_2} = \frac{u_i - u_j}{v_2 - v_3}, \ i \neq j \in [3], \ v_2 = v_3 \right\} \right|.
\]

One can show that

\[
\left| \left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \frac{u_i - u_j}{v_i - v_j}, \ i \neq j \in [3], \ v_k = v_l \right\} \right| = \sum_{z \in \mathbb{F}_p^2} \lambda_{r,\theta}^2(z).
\]

We will prove (5.3) only for \((k,l) = (1,2)\) since the remaining two cases can be done in an analogous way.
Indeed, if \((k, l) = (1, 2)\), then
\[
\left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \begin{array}{l}
u_i - u_j = \sqrt{\theta}(v_i - v_j), \\
i \neq j \in [3], v_1 = v_2 \end{array} \right\}
\]
\[
= \left\{ (u_1, u_3, v_1, v_3) \in E^4 : u_1 - \sqrt{\theta}v_1 = u_3 - \sqrt{\theta}v_3 \right\}
\]
\[
= \sum_{z \in \mathbb{F}_p^2} \left| \left\{ (u_1, u_3, v_1, v_3) \in E^4 : u_1 - \sqrt{\theta}v_1 = u_3 - \sqrt{\theta}v_3 = z \right\} \right|
\]
\[
= \sum_{z \in \mathbb{F}_p^2} \left| \{ (u_1, v_1) \in E^2 : u_1 - \sqrt{\theta}v_1 = z \} \times \{ (u_3, v_3) \in E^2 : u_3 - \sqrt{\theta}v_3 = z \} \right|
\]
\[
= \sum_{z \in \mathbb{F}_p^2} \lambda^2_{r,\theta}(z).
\]

One can verify that
\[
(5.4) \quad \left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \begin{array}{l}
u_i - u_j = \sqrt{\theta}(v_i - v_j), \\
i \neq j \in [3], v_1 = v_2 \end{array} \right\} = |E|^2.
\]

Therefore, if we plug (5.3) and (5.4) into (5.2), we obtain
\[
(5.5) \quad |N_{\theta}(r)| = |\Lambda_{\theta}(r)| - 3 \sum_{z \in \mathbb{F}_p^2} \lambda^2_{r,\theta}(z) + 2|E|^2.
\]

Summing (5.5) over all \( \theta \in O_2(\mathbb{F}_p) \), we obtain
\[
(5.6) \quad \sum_{\theta} |N_{\theta}(r)| = \sum_{\theta} |\Lambda_{\theta}(r)| - 3 \sum_{\theta, z} \lambda^2_{r,\theta}(z) + 2|E|^2 \times |O_2(\mathbb{F}_p)|
\]
\[
= \sum_{\theta, z} \lambda^3_{r,\theta}(z) - 3 \sum_{\theta, z} \lambda^2_{r,\theta}(z) + 2|E|^2 \times |O_2(\mathbb{F}_p)|.
\]

One can check that for arbitrary \( \theta \in O_2(\mathbb{F}_p) \) the following set containment holds:
\[
(5.7) \quad N_{\theta}(r) \subset \mathcal{T},
\]
where
\[
\mathcal{T} := \left\{ (u_1, u_2, u_3, v_1, v_2, v_3) \in E^6 : \begin{array}{l}
\|u_i - u_j\| = r \|v_i - v_j\|, v_i \neq v_j, \\
u_i \neq u_j, i \neq j \in [3] \end{array} \right\}.
\]

One can see that \( \mathcal{T} \) is exactly the family of pairs of 3-cycles in \( E \) with dilation ratio \( r \in (\mathbb{F}_p)^2 \setminus \{0\} \).

Containment (5.7) implies that
\[
|\mathcal{T}| \geq \frac{1}{|O_2(\mathbb{F}_p)|} \sum_{\theta} |N_{\theta}(r)|.
\]

Applying (5.6), we obtain
\[
(5.8) \quad |\mathcal{T}| \geq \frac{1}{|O_2(\mathbb{F}_p)|} \left( \sum_{\theta, z} \lambda^3_{r,\theta}(z) - 3 \sum_{\theta, z} \lambda^2_{r,\theta}(z) \right).
\]
In other words, we managed to obtain the lower bound for the size of \( T \) in terms of \( L_2 \), \( L_3 \)-norms of \( \lambda_{r,\theta}(z) \) and the size of \( O_2(\mathbb{F}_p) \). More precisely, inequality (5.8) can be rewritten in the following equivalent way:

\[
|T| \geq \frac{1}{|O_2(\mathbb{F}_p)|}\left( \|\lambda_{r,\theta}(z)\|_3^2 - 3\|\lambda_{r,\theta}(z)\|_2^2 \right).
\]

It remains to show that the RHS in (5.9) is positive. The lower bound for \( L_3 \)-norm of \( \lambda_{r,\theta}(z) \) can be obtained by means of Hölder’s inequality. Indeed,

\[
|T| \geq F \left( \sum_{z \in \mathbb{F}_p^2} \lambda_{r,\theta}^3(z) \right)^{\frac{4}{3}} \left( \sum_{z \in \mathbb{F}_p^2} 1 \right)^{\frac{2}{3}}.
\]

From the definition of \( \lambda_{r,\theta}(z) \) it follows that

\[
\sum_{z \in \mathbb{F}_p^2} \lambda_{r,\theta}(z) = |E|^2
\]

and taking this into account we can rewrite (5.10) in the following way:

\[
\sum_{z \in \mathbb{F}_p^2} \lambda_{r,\theta}^3(z) \geq |E|^6 p^4.
\]

Summing inequality (5.11) over all \( \theta \in O_2(\mathbb{F}_p) \), we obtain the following lower bound for \( L_3 \)-norm:

\[
\|\lambda_{r,\theta}(z)\|_3^3 = \sum_{\theta, z} \lambda_{r,\theta}^3(z) \geq \frac{|E|^6}{p^4},
\]

since \( |O_2(\mathbb{F}_p)| \geq p \).

The inequality (5.9) can be written as follows:

\[
|T| \geq \frac{1}{|O_2(\mathbb{F}_p)|}\left( \sum_{\theta, z} \lambda_{r,\theta}^3(z) - \frac{3}{2} \sum_{\theta, z} \lambda_{r,\theta}^2(z) \right)
\]

\[
= \frac{1}{|O_2(\mathbb{F}_p)|}\left( \frac{1}{2} \sum_{\theta, z} \lambda_{r,\theta}^3(z) - \frac{3}{2} \sum_{\lambda \geq 6} \lambda_{r,\theta}^2(z) \right) + \frac{1}{2} \sum_{\theta, z} \lambda_{r,\theta}^3(z) - 3 \sum_{\lambda < 6} \lambda_{r,\theta}^2(z)\right).
\]

We see that I is nonnegative. Indeed,

\[
I \geq \frac{1}{2} \sum_{\lambda \geq 6} \lambda_{r,\theta}^3(z) - 3 \sum_{\lambda \geq 6} \lambda_{r,\theta}^2(z) = \sum_{\lambda \geq 6} \frac{\lambda_{r,\theta}^2(z)}{2} (\lambda_{r,\theta}(z) - 6) \geq 0.
\]

Moreover, we have the following lower bound for II:

\[
II \geq \frac{1}{2} \sum_{\theta, z} \lambda_{r,\theta}^3(z) - 108 \sum_{\lambda < 6} 1
\]

\[
\geq \frac{1}{2} \sum_{\theta, z} \lambda_{r,\theta}^3(z) - 108 \times |\mathbb{F}_p^2| \times |O_2(\mathbb{F}_p)|
\]

\[
\geq \frac{|E|^6}{2p^3} - 324p^3.
\]
Combining lower bounds for I and II, we obtain

\[ I + II \geq \frac{|E|^6}{2p^3} - 324p^3. \] (5.14)

We know that \(|O_2(\mathbb{F}_p)| > 0\), then combining equations (5.13) and (5.14), we obtain

\[ |\mathcal{T}| \geq \frac{1}{|O_2(\mathbb{F}_p)|} \left( \frac{|E|^6}{2p^3} - 324p^3 \right). \]

It implies that if \(|E| \geq 3p\), then \(\frac{|E|^6}{2p^3} - 324p^3 > 0\) and thus \(|\mathcal{T}| > 0\) and it completes the proof of Theorem 1.7.

6. Proof of Theorem 1.8.

In this section we will extend Theorem 1.7 to the case of \(d\)-simplexes. The approach will be analogous to the one which we implemented in the course of the proof of Theorem 1.7 but with minor modifications. We will investigate the size of \(O_d(\mathbb{F}_p)\) and \(L^{d+1}\)-norm of the counting function \(\lambda_{r,\theta}(z)\) which was defined in (5.1).

We will show that if \(E \subset \mathbb{F}_p^d\) and \(|E| \gg_d p^\frac{d}{2}\), then \(|\mathcal{P}| > 0\), where

\[ \mathcal{P} := \left\{(u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}) \in E^{2d+2} : \|u_i - u_j\| = r\|v_i - v_j\|, \; v_i \neq v_j, \; u_i \neq u_j, \; i \neq j \in [d+1] \right\}. \]

For the counting function introduced in (5.1) we consider it’s \(L^{d+1}\)-norm:

\[ \|\lambda_{r,\theta}(z)\|_{d+1} := \sum_{\theta,z} \lambda_{r,\theta}^{d+1}(z) \] (6.1)

\[ = \sum_{\theta,z} \left| \left\{(u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}) \in E^{2d+2} : u_1 - \sqrt{r}\theta v_1 = \cdots = u_{d+1} - \sqrt{r}\theta v_{d+1} = z \right\} \right| \]

\[ = \sum_{\theta} \left| \left\{(u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}) \in E^{2d+2} : u_i - u_j = \sqrt{r}\theta (v_i - v_j), \; 1 \leq i < j \leq d+1 \right\} \right|. \]

We let \(\Lambda_\theta(r)\) denote the set

\[ \left\{(u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}) \in E^{2d+2} : u_i - u_j = \sqrt{r}\theta (v_i - v_j), \; 1 \leq i < j \leq d+1 \right\}. \]

Therefore, (6.1) can be rewritten in the following way:

\[ \|\lambda_{r,\theta}(z)\|_{d+1} := \sum_{\theta} |\Lambda_\theta(r)|. \] (6.2)

In \(\Lambda_\theta(r)\) we extract the subset where \(v_i \neq v_j\) for \(i \neq j\).

\[ N_\theta(r) := \left\{(u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}) \in E^{2d+2} : u_i - u_j = \sqrt{r}\theta (v_i - v_j), \; v_i \neq v_j, \; i \neq j \in [d+1] \right\}. \]

Remark. One can check that if \(u = \sqrt{r}\theta v\) for \(\theta \in O_d(\mathbb{F}_p)\), then \(\|u\| = r\|v\|\).
This remark immediately implies that for each $\theta \in O_d(\mathbb{F}_p)$ we have $N_\theta(r) \subseteq \mathcal{P}$.

Hence, we have

\begin{equation}
|\mathcal{P}| \geq \frac{1}{|O_d(\mathbb{F}_p)|} \sum_\theta |N_\theta(r)|.
\end{equation}

(6.3)

For each pair $(k, l)$ such that $1 \leq k < l \leq d + 1$, we define the following set:

$A_{kl} := \{ (u_1, \ldots, u_{d+1}, v_1, \ldots, v_{d+1}) \in E^{2d+2} : u_i - u_j = \sqrt{r\theta(v_i - v_j)}, 1 \leq i < j \leq d + 1, v_k = v_l \}.$

One can see that the following set equality holds:

$\Lambda_\theta(r) \setminus \bigcup_{1 \leq k < l \leq d+1} A_{kl} = N_\theta(r).$

Applying Bonferroni inequality, we obtain

\begin{equation}
|N_\theta(r)| \geq |\Lambda_\theta(r)| - \sum_{1 \leq k < l \leq d+1} |A_{kl}|.
\end{equation}

(6.4)

One can show that for each such pair $(k, l)$, we have

\begin{equation}
|A_{kl}| = \sum_{z \in \mathbb{F}_p^d} \lambda_{r, \theta}^d(z).
\end{equation}

(6.5)

Plugging (6.5) into inequality (6.4), we obtain

\begin{equation}
|N_\theta(r)| \geq |\Lambda_\theta(r)| - \left(\frac{d+1}{2}\right) \sum_{z \in \mathbb{F}_p^d} \lambda_{r, \theta}^d(z).
\end{equation}

(6.6)

Summing (6.6) over all $\theta \in O_d(\mathbb{F}_p)$, we obtain the following inequality:

\begin{equation}
\sum_\theta |N_\theta(r)| \geq \sum_\theta |\Lambda_\theta(r)| - \left(\frac{d+1}{2}\right) \sum_{\theta, z} \lambda_{r, \theta}^d(z).
\end{equation}

(6.7)

Taking into account (6.2) and (6.7) the inequality (6.3) can be written as

\begin{equation}
|\mathcal{P}| \geq \frac{1}{|O_d(\mathbb{F}_p)|} \left( \|\lambda_{r, \theta}(z)\|_{d+1}^{d+1} - \left(\frac{d+1}{2}\right) \|\lambda_{r, \theta}(z)\|_d^d \right).
\end{equation}

(6.8)

We can derive the lower bound for the $\|\lambda_{r, \theta}(z)\|_{d+1}^{d+1}$ by means of Hölder’s inequality. Indeed,

\begin{equation}
\sum_{z \in \mathbb{F}_p^d} \lambda_{r, \theta}(z) \leq \left( \sum_{z \in \mathbb{F}_p^d} \lambda_{r, \theta}^{d+1}(z) \right)^{\frac{1}{d+1}} \times \left( \sum_{z \in \mathbb{F}_p^d} 1 \right)^{\frac{d}{d+1}}.
\end{equation}

(6.9)

From the definition of $\lambda_{r, \theta}(z)$ follows that $\sum_{z \in \mathbb{F}_p^d} \lambda_{r, \theta}(z) = |E|^2$.

Therefore, (6.9) implies that

\begin{equation}
\sum_{z \in \mathbb{F}_p^d} \lambda_{r, \theta}^{d+1}(z) \geq \frac{|E|^{2d+2}}{p^d}.
\end{equation}

(6.10)

Summing (6.10) over all $\theta \in O_d(\mathbb{F}_p)$, we obtain the following lower bound for the $L^{d+1}$-norm of $\lambda_{r, \theta}(z)$:
\[ (6.11) \quad \| \lambda_{r,\theta}(z) \|_{d+1}^{d+1} \geq |O_d(F_p)| \times \frac{|E|^{2d+2}}{p^{d^2}}. \]

We can write the inequality (6.8) as follows:

\[ |\mathcal{P}| \geq \frac{1}{|O_d(F_p)|} \left( \| \lambda_{r,\theta}(z) \|_{d+1}^{d+1} - \left( \frac{d+1}{2} \right) \sum_{\lambda \geq d(d+1)} \lambda^d_{r,\theta}(z) + \sum_{\lambda < d(d+1)} \lambda^d_{r,\theta}(z) \right). \]

We notice that the expression in the parenthesis can be written as \( I + II \), where

\[ I := \frac{1}{2} \sum_{\theta, z} \lambda^{d+1}_{r,\theta}(z) - \left( \frac{d+1}{2} \right) \sum_{\lambda \geq d(d+1)} \lambda^d_{r,\theta}(z), \]

\[ II := \frac{1}{2} \sum_{\lambda \geq d(d+1)} \lambda^{d+1}_{r,\theta}(z) - \left( \frac{d+1}{2} \right) \sum_{\lambda < d(d+1)} \lambda^d_{r,\theta}(z). \]

It is not difficult to show that \( I \geq 0 \). Indeed,

\[ I = \frac{1}{2} \sum_{\theta, z} \lambda^{d+1}_{r,\theta}(z) - \left( \frac{d+1}{2} \right) \sum_{\lambda \geq d(d+1)} \lambda^d_{r,\theta}(z) \geq \frac{1}{2} \sum_{\lambda \geq d(d+1)} \lambda^d_{r,\theta}(z) \left( \lambda_{r,\theta}(z) - d(d+1) \right) \geq 0. \]

Using (6.11) one can obtain the following lower bound for \( II \):

\[ II \geq \frac{|O_d(F_p)|}{2} \times \frac{|E|^{2d+2}}{p^{d^2}} - (d(d+1))^d \left( \frac{d+1}{2} \right) \times |O_d(F_p)| \times |F_{p^d}| \]

\[ = \frac{|O_d(F_p)|}{2} \left( \frac{|E|^{2d+2}}{p^{d^2}} - (d^2 + d)^{d+1}p^d \right). \]

Combining inequalities (6.12), (6.13) and since \( |O_2(F_p)| > 0 \), we have

\[ |\mathcal{T}| \geq \frac{1}{2} \left( \frac{|E|^{2d+2}}{p^{d^2}} - (d^2 + d)^{d+1}p^d \right). \]

It is easy to verify that if \( |E| \geq (d+1)p^d \), then \( |\mathcal{T}| > 0 \) and this completes the proof of Theorem 1.8.
7. Proof of Theorem 1.10

Basically, using Cauchy-Schwarz inequality we were able to derive lower bounds for $|S_2(r)|$ and $|C(r)|$ in Lemma 2.3 and 2.4 in terms of $|S_1(r)|$. However, this technique fails for $|S_3(r)|$ since paths of length 3 have an odd number of edges. In this section, we will show how to obtain the lower bound for $|S_k(r)|$ in terms of $|S_1(r)|$ by means of Theorem 1.9.

Let $r \in \mathbb{F}_p^*$, $p$ be a prime such that $p \equiv 3 \pmod{4}$ and $E \subset \mathbb{F}_p^2$.

Define the graph $G = (V, E)$ as follows: let $V := E \times E \equiv \{(x, x') : x, x' \in E\}$. If $(x, x'), (y, y') \in V$, then we connect them via an edge iff $(x, x') \neq (y, y')$ and $\|y - x'\| = r\|y - x\|$.

![Figure 6. Joining vertices $(x, x')$ and $(y, y')$ with an edge.](image)

Fix $(x, x') \in V$ and consider the degree of this vertex:
\[
\deg((x, x')) = |\{(y, y') \in E^2 : (y, y') \text{ is incident with } (x, x')\}| = |\{(y, y') \in E^2 : \|y' - x'\| = r\|y - x\|, (x, x') \neq (y, y')\}|.
\]

Therefore, we have
\[
\sum_{(x, x') \in V} \deg((x, x')) = \sum_{x, x' \in E} |\{(y, y') \in E^2 : \|y' - x'\| = r\|y - x\|, (x, x') \neq (y, y')\}| = |\{(x, y, y', x') \in E^4 : \|y' - x'\| = r\|y - x\|, (x, x') \neq (y, y')\}| = |\{(x, y, x', y') \in E^4 : \|y' - x'\| = r\|y - x\|, x \neq y\}| = |S_1(r)|.
\]

**Remark.** In the penultimate equality we have used the fact that $\|x\| = 0$ iff $x = (0, 0)$ since $p \equiv 3 \pmod{4}$.

By degree sum formula it follows that $e(G) = |S_1(r)|/2$. Therefore, we constructed the graph with $|E|^2$ vertices and $|S_1(r)|/2$ edges.

The number of paths of length $k \geq 3$ in our graph $G = (V, E)$ is equal to the size of $S_k(r)$. Indeed,
\[
\left|\{(v_1, v_2, \ldots, v_{k+1}) \in V^{k+1} : v_i v_{i+1} \in E, i \in [k]\}\right| = \left|\{(x_1, y_1), (x_2, y_2), \ldots, (x_{k+1}, y_{k+1}) \in V^{k+1} : (x_i, y_i) \sim (x_{i+1}, y_{i+1}), i \in [k]\}\right|
\]
\[
\left\{(x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}) \in E^{2k+2} : \|y_i - y_{i+1}\| = r\|x_i - x_{i+1}\|, \ x_i \neq x_{i+1}, \ i \in [k]\right\}
= |S_k(r)|.
\]

Combining this with Theorem 1.9, we have

\[
|S_k(r)| \geq \frac{(2e(G))^k}{n^{k-1}} \iff |S_k(r)| \geq \frac{|S_1(r)|^k}{|E|^{2k-2}}.
\]

Lemma 2.2 implies that if \(|E| > 2p\), then

\[
|S_1(r)| > \frac{|E|^4}{3p}.
\]

Therefore, comparing (7.1) with (7.2), we obtain

\[
|S_k(r)| > \frac{|E|^{2k+2}}{(3p)^k},
\]

which proves Theorem 1.10.
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