SYMBOLIC BLOWUP ALGEBRAS OF MONOMIAL CURVES IN $\mathbb{A}^3$ DEFINED BY ARITHMETIC SEQUENCE

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Abstract. In this paper, we consider monomial curves in $\mathbb{A}^3_k$ parameterized by $t \to (t^{2q+1}, t^{2q+1+m}, t^{2q+1+2m})$ where $\gcd(2q+1, m) = 1$. The symbolic blowup algebras of these monomial curves is Gorenstein (\cite{10}, \cite{11}). From the results in \cite{17}, it follows that the defining ideal of $R_s(p)$ is given by the Pfaffians of a skew-symmetric matrix. In this paper we give a proof which mainly involves the use of Gröbner basis. We also are able to describe all the symbolic powers $p^{(n)}$ for all $n \geq 1$.

1. Introduction

Let $k$ be a field and let $\mathbb{A}^n_k$ (or $\mathbb{A}^n$) be the affine $n$-space over $k$. One well known question is: Is every affine irreducible curve $Y$ in $\mathbb{A}^n_k$ a set theoretic complete intersection of $(n-1)$ hypersurfaces? In other words, if $T = k[x_1, \ldots, x_n]$ and $I(Y) = \{f \in T | f(a) = 0 \text{ for all } a \in Y\}$, then does there exist $n-1$ elements $f_1, \ldots, f_{n-1} \in T$ such that $I(Y) = \sqrt{(f_1, \ldots, f_{n-1})}$? The answer to this question is quite difficult and depends upon the characteristic of the field $k$. The oldest known result in this direction is a result of L. Kronecker where he showed that $I(Y)$ can be generated set theoretically by $(n+1)$-equations \cite{18}.

Later, several researchers showed that there exist interesting examples of algebraic subsets $Y \subseteq \mathbb{A}^n$ which can be set theoretically defined by $(n-1)$-equations. One remarkable result which appeared in 1978 was by R. Cowsik and M. Nori. They showed that if $k$ is a perfect field such that $\text{char}(k) = p$ and if $I$ is a radical ideal of codimension one, then $I$ is a set theoretic complete intersection [3, Theorem 1]. Later in 1979, for $\text{char}(k) = 0$, H. Bresinsky showed that all monomial curves in $\mathbb{A}^3_k$ are set theoretic complete intersection [1]. In [2] the author extended his ideas to monomial curves in $\mathbb{A}^4_k$. In [25] the author extends the ideas in [24] to study symbolic powers of monomial curves in $\mathbb{A}^4$. In 1990, D. Patil gave more general class of monomial curves in $\mathbb{A}^n$ which are set theoretic complete intersection [23, Theorem 1.1].

In 1981, R. Cowsik gave a new direction to this problem. He showed that if $(R, m)$ a regular local ring and $p$ is a prime ideal such that $\dim(R/p) = 1$, then Noetherianness of the symbolic Rees Algebra $R_s(p) := \bigoplus_{n \geq 0} p^{(n)}$ implies that $p$ is a set theoretic complete intersection [4]. However, the converse need not be true [12]. Motivated by Cowsik’s result, in 1987, Huneke gave necessary and sufficient conditions for $R_s(p)$ to be Noetherian when $\dim R = 3$ [16]. Huneke’s result was generalised in 1991 for $\dim R \geq 3$ by M. Morales [22]. All these results paved a new way to study the famous problem of set theoretic complete intersection. In the last twenty-five years several researchers have worked on the symbolic Rees Algebra $R_s(p)$. However, there are still many unanswered questions related both the problems, i.e., set theoretic complete intersection and Noetherianness of symbolic Rees algebra. A few problems will be listed at the

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end of this paper. A good survey article with some open questions on set theoretic complete intersection is by G. Lyubeznik [19].

In this paper, we consider the monomial curve in $\mathbb{A}^3$ parameterized by $(t^{n_1}, t^{n_2}, t^{n_3})$ where $n_i = 2q + 1 + (i - 1)m$ and $gcd(n_1, m) = 1$. Throughout this paper $R = k[[x_1, x_2, x_3]]$, $S = k[[t]]$ and $p = \ker(\phi)$, where $\phi$ is the homomorphism defined by $\phi(x_i) = t^{n_i}$ for $1 \leq i \leq 3$. We say that $p$ is the defining ideal of the monomial curve parameterized by $(t^{n_1}, t^{n_2}, t^{n_3})$. It is well known that these curves are a set-theoretic complete intersection ([1], [27], [23]). The symbolic powers of curves are also of interest ([6], [15], [17], [24], [29]). In this paper we describe all the symbolic powers $p^{(n)}$ (Theorem 5.9).

The symbolic Rees algebra of these monomial curves has been studied by several authors in the past. For example see [27], [14], [10], [17] and [11]. The Cohen-Macaulay and Gorenstein property of these monomial curves has also been studied [10] and [11]. In [17], the authors are able to explicitly describe when $R_s(p)$ is Cohen-Macaulay and when it is Gorenstein.

We give a different and simple proof for the Cohen-Macaulayness and Gorensteinness of the blowup algebras $R_s(p)$ and $G_s(p) := \oplus_{n \geq 0} p^{(n)}/p^{(n+1)}$. One crucial point here is that though the ideals we are interested in are binomial ideals in $k[[x_1, x_2, x_3]]$, we can to get our main result, by dealing with the corresponding monomial ideals in the polynomial ring $k[x_2, x_3]$. These ideals are defined in Section 2. Powers and quotients of these monomial ideals are much easier to compute and do not depend on any sophisticated results. Hence, any student with a good knowledge of basic Commutative Algebra can follow the proof. Our proofs may give a different approach and help to answer some questions related to symbolic powers. All basic results used in this paper can be found in [20].

We now describe the organisation of this paper. In Section 2, we state some basic results. We also state a result which gives a way to compute lengths of modules in the polynomial ring $k[x_2, x_3]$. In Section 3, we prove the Cohen-Macaulayness of the filtration $F = \{I_n\}_{n \geq 0}$ where the ideals $I_n \subseteq k[x_2, x_3]$ are defined in Section 2. In Section 4, we compute length of $I_{n-1}/(I_n : x_3^2)$ which is useful in the computations in the next section. In Section 5, we compute lengths various modules over $k[[x_1, x_2, x_3]]$ which are needed to prove the Cohen-Macaulay and Gorenstein property of the blowup algebras $R_s(p)$ and $G_s(p)$. In Section 6, we prove our main results. In Section 7 we suggest list a few questions which might be of interest to the reader.

2. Preliminaries

It is well known that the generators for $p$ are the $2 \times 2$ minors of the matrix $\begin{pmatrix} x_1 & x_2 & x_3^q \\ x_2 & x_3 & x_1^{m+q} \end{pmatrix}$ [13]. In particular, if $g_1 = x_1^{m+q}x_2 - x_3^{q+1}$, $g_2 = x_1^{m+q+1} - x_2x_3^q$ and $g_3 = x_2^2 - x_1x_3$, then $p = (g_1, g_2, g_3)$.

The following result is well known ([11, Corollary 4.4]). We state it for the sake of completion.

**Lemma 2.1.** $R_s(p)$ is a Noetherian ring.

**Proof.** Let $f_1 := g_3 = x_2^2 - x_2x_3 \in p$ and $f_2 = -x_1^{2(m+q)+1} - x_1^{m+q-1}x_2^3x_3^{q-1} + 3x_1^{m+q}x_2x_3^q - x_3^{2q+1}$. Then $x_3 \cdot f_2 = -g_1^2 + x_1^{m+q-1}g_2g_3 \in p^2 \subseteq p^{(2)}$. 


As $x_3^n$ is nonzerodivisor on $R/p^{(2)}$ for all $n$, $f_2 \in p^{(2)}$. Moreover,

$$\ell \left( \frac{R}{(x_1, f_1, f_2)} \right) = \ell \left( \frac{R}{(x_1, x_2, x_3^{2q+1})} \right) = 2(2q + 1) = 2 \cdot e(x_1; R/p).$$

By Huneke’s criterion [16, Theorem 3.1], $R_s(p)$ is Noetherian.

Let $T = k[x_1, x_2, x_3]$. The following lemma gives us a way to compute the length of an $R$-module in terms of the length of the corresponding $T$-module.

**Lemma 2.2.** [7, Lemma 2.8] Let $m = (x_1, x_2, x_3)T$ and $M$ a finitely generated $T$-module such that $\text{Supp}(M) = \{m\}$. Then

$$\ell_R(M \otimes_T R) = \ell_T(M).$$

Let $J_1 = \{g_1, g_2, g_3\}$ and $J_2 = \{f_2\}$. We define

$$J_1T := (g_1, g_2, g_3)T, \quad J_2T := (f_2)T, \quad I_nT := \sum_{a_1 + 2a_2 = n} (J_1^{a_1}T)(J_2^{a_2}T). \quad (2.3)$$

As $R$ is a flat $T$-module, $I_nR = I_nT \otimes_T R$.

Let $T' = k[x_2, x_3] \cong T/x_1T$. For $i = 1, 2$ put

$$J_1 := J_1T' = (x_2^2, x_2x_3^q, x_3^{q+1}), \quad J_2 := J_2T' = (x_3^{2q+1}), \quad I_n := I_nT' = \sum_{a_1 + 2a_2 = n} J_1^{a_1}J_2^{a_2}. \quad (2.4)$$

**Proposition 2.5.** Let $n \geq 1$. Then

1. $I_nR \subseteq p^{(n)}$.
2. $(I_n + (x_1))T$ is an homogeneous ideal.
3. $(I_n + (x_1))T$ is an $m$-primary ideal.

**Proof.** (1) As $J_1p$ and $J_2 = (f_2) \subseteq p^{(2)}$, for all $a_1, a_2 \in \mathbb{Z}_{\geq 0}$,

$$J_1^{a_1}J_2^{a_2} \subseteq p^{a_1}(p^{(2)})^{a_2} \subseteq p^{(a_1 + 2a_2)}. \quad (2.6)$$

Summing over all $a_1 + 2a_2 = n$ and applying (2.6) to (2.3) we get (1).

(2) As $(J_1 + (x_1))T = (J_1, (x_1))T$ and $(J_2 + (x_1))T = (J_2, x_1)T$ are homogenous ideals and $(I_n + (x_1))T = (\sum_{a_1 + 2a_2 = n} J_1^{a_1}J_2^{a_2}, x_1)T$ we get (2).

(3) By (2.3), $J_1^nT \subseteq I_nT$ and $(J_1^n + (x_1))T = ((x_2^2, x_2x_3^q, x_3^{q+1})^n, x_1)T$ which implies that $mT = (\sqrt{J_1^n + (x_1)})T \subseteq (\sqrt{I_n + (x_1)})T \subseteq mT$. \qed

We state a result on monomial ideals which follows from [8, Proposition 1.14] and will be consistently used in all the proofs which involve monomial ideals.

**Proposition 2.7.** Let $I = (u_1, \ldots, u_r)$ and $J = (v)$ be monomial ideals in a polynomial ring over a field $k$. Then $I : J = (\{u_i/gcd(u_i, v) : i = 1, \ldots, r\})$. 
3. The associated graded ring corresponding to the filtration $\mathcal{F} := \{I_n\}_{n \geq 0}$.

Throughout this section we will work with the ring $T' = \mathbb{k}[x_2, x_3]$ and the ideals $J_1$, $J_2$ and $I_n$. Our goal is to compute $\ell(T'/(I_n + (x_2)^2))$ and $\ell(T'/(I_n + (x_2^2, x_3^{2q+1})))$. To attain our goal, we show that $x_2^2, x_3^{2q+1}$ is a regular sequence in $G(\mathcal{F})$, where $G(\mathcal{F}) := \oplus_{n \geq 0} I_n/I_{n+1}$ is the associated graded ring corresponding to the filtration $\mathcal{F} := \{I_n\}_{n \geq 0}$.

**Theorem 3.1.** $G(\mathcal{F})$ is Cohen-Macaulay.

**Proof.** We first show that $(I_n : x_2^2) = I_{n-1}$ for all $n \geq 2$. Clearly $x_2^2 I_{n-1} \subseteq J_1 I_{n-1} \subseteq I_n$. Write $J_1^{a_1} = \sum_{i=0}^{a_1-1} x_2^{2(a_1-i)} x_3^{q(i)} (x_2, x_3)^i + x_3^{a_1} (x_2, x_3)^{a_1}$. Then

$$
(I_n : x_2^2) = \left( \left( \sum_{a_1+2a_2=n} J_1^{a_1} J_2^{a_2} : x_2^2 \right) \right) = \sum_{a_1+2a_2=n} (J_1^{a_1} J_2^{a_2} : x_2^2)
$$

$$
= \sum_{a_1+2a_2=n} \sum_{i=0}^{a_1-1} (x_2^{2(a_1-i)} x_3^{q(i)+(2q+1)a_2} (x_2, x_3)^i : x_2^2) + (x_3^{a_1+(2q+1)a_2} (x_2, x_3)^{a_1} : x_2^2)
$$

$$
= \sum_{a_1+2a_2=n} \sum_{i=0}^{a_1-1} (x_2^{2(a_1-i-1)} x_3^{q(i)+(2q+1)a_2} (x_2, x_3)^i) + (x_3^{a_1+(2q+1)a_2} (x_2, x_3)^{a_1-1})
$$

$$
\subseteq I_{n-1}. \quad (3.2)
$$

Let $\overline{\cdot}$ denote the image in $R/(x_2^2)$. Then

$$
\frac{G(\mathcal{F})}{(x_2^2)^N} \cong \bigoplus_{n \geq 0} \frac{I_n}{I_{n+1} + x_2^2 I_{n-1}} = G(\mathcal{F}).
$$

To show that $\overline{x_3^{2q+1}}$ is a regular element in $G(\mathcal{F})$, we need to verify that

$$
((I_{n+2} + x_2^2 I_{n+1}) : (x_3^{2q+1})) = I_n + x_2^2 I_{n-1}. \quad (3.3)
$$

Let $m \geq 1$. Then

$$
\frac{(J_1^{m+2} : x_3^{2q+1})}{x_2^m (x_2, x_3)}
$$

$$
= \sum_{i=0}^{m+2} \frac{(x_2^{2(m+2-i)} x_3^{q(i)} (x_2, x_3)^i : x_3^{2q+1}) + \sum_{i=3}^{m+2} (x_2^{2(m+2-i)} x_3^{q(i-3)} (x_2, x_3)^i - (x_2^{2q} x_3^{i-3}))}{x_2^m (x_2, x_3)}
$$

$$
\subseteq ((x_2^{2m}) + \sum_{i=3}^{m+2} (x_2^{2(m+2-i)} x_3^{q(i-3)+1} (x_2, x_3)^i - (x_2^{2q} x_3^{i-3}))(x_2^{2q}, x_2 x_3, x_3^{q+1})
$$

$$
= J_1^m. \quad (3.4)
$$
Hence
\[(I_{n+2} + x_2^2 I_{n+1}) (x_3^{2q+1})\]
\[= \sum_{a_1+2a_2=n+2} (J_1^{a_1} J_2^{a_2} : x_3^{2q+1}) + \sum_{a_1+2a_2=n+1} (x_2^2 J_1^{a_1} J_2^{a_2} : x_3^{2q+1})\]
\[\subseteq \sum_{a_1+2a_2=n+2, a_2 \neq 0} (J_1^{a_1} J_2^{a_2-1}) + (J_1^{n+2} : x_3^{2q+1}) + \sum_{a_1+2a_2=n+1, a_2 \neq 0} (x_2^2 J_1^{a_1} J_2^{a_2-1}) + (x_2^2 J_1^{n+1} : x_3^{2q+1})\]
\[\subseteq I_n + x_2^2 I_{n-1}.\] [by (3.4)]
\[\vdash\]

The other inclusion is easy to verify. \[\square\]

We are now ready to prove the main result of this section.

**Proposition 3.6.** For all \(n \geq 1\),

\[
\ell \left( \frac{T'}{(I_n + (x_2^q) T')^T} \right) = \ell \left( \frac{T'}{(I_n T')^T} \right) - \ell \left( \frac{T'}{(I_{n-1} T')^T} \right),
\]

\[
\ell \left( \frac{T'}{(I_n + (x_2^2, x_3^{2q+1}) T')^T} \right) = \ell \left( \frac{T'}{(I_n T')^T} \right) - \ell \left( \frac{T'}{(I_{n-1} T')^T} \right) - \ell \left( \frac{T'}{(I_{n-2} T')^T} \right) + \ell \left( \frac{T'}{(I_{n-3} T')^T} \right).
\]

**Proof.** The proof follows from [7, Proposition 2.4] and Theorem 3.1. \[\square\]

4. **The inductive step**

In this section we describe the generators of \(I_{n-1}\) modulo \((I_n : x_3^q)\). This will be used in our computations in the next section.

**Lemma 4.1.** For all \(n \geq 1\),

1. \((I_n : x_3^q) \subseteq I_{n-1}\).
2. \(I_{n-1} = \begin{cases} \sum_{a_2=0}^{n-2} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1)a_2} (x_2, x_3^q) + (I_n : x_3^q) & \text{if } 2 \nmid (n-1) \\ \left( x_3^{(2q+1) \left( \frac{n-1}{2} \right) } \right) + \sum_{a_2=0}^{n-3} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1)a_2} (x_2, x_3^q) + (I_n : x_3^q) & \text{if } 2\mid (n-1) \end{cases}\).
Proof. (1) One can verify that
\[
(I_n : x_3^q) = \sum_{a_1 + 2a_2 = n; a_2 \neq 0} (J_1^{a_1} J_2^{a_2} : x_3^q) + (J_1^n : x_3^q)
\]
\[
= \sum_{a_1 + 2a_2 = n; a_2 \neq 0} (x_3^{q+1}) J_1^{a_1} J_2^{a_2-1} + (x_2^{2n}) + \sum_{i=1}^{n} (x_2^{2(n-i)} x_3^{q(i-1)} (x_2, x_3^q)^i)
\]
\[
\subseteq I_{n-1}.
\]

(2) As \(I_1 = J_1 = x_2(x_2, x_3^q) + (x_3^{q+1}) \subseteq x_2(x_2, x_3^q) + (I_2 : x_3^q), (2)\) is true for \(n = 1\). Let \(n > 1\). For all \(a_1 \geq 1\),
\[
(J_1^{a_1} = J_1 J_2^{a_1-1}
\]
\[
\subseteq (x_2(x_2, x_3^q)) (x_2^{2a_1-3}(x_2, x_3^q) + (I_1 : x_3^q)) \quad \text{[by induction hypothesis]}
\]
\[
= x_2^{2a_1-1}(x_2, x_3^q) + x_2(x_2, x_3^q) (I_1 : x_3^q) + x_3^{q+1} x_2^{a_1-3}(x_2, x_3^q) + x_3^{q+1}(I_1 : x_3^q)
\]
\[
= x_2^{2a_1-1}(x_2, x_3^q) + (I_{a_1+1} : x_3^q)
\]
\[
(2a_1-2, q+1) x_3^q = x_2^{2(a_1-1)} x_3^{2q+1} \subseteq J_1^{a_1-1} J_2^{a_2} \subseteq I_{a_1-1+2} = I_{a_1+1}
\]
\[
(x_{2}^{q+1} x_2^{a_1-3})(x_2, x_3^q) x_3^q = x_2^{2(a_1-2)} x_2(x_2, x_3^q) x_3^{2q+1} \subseteq J_1^{a_1-1} J_2^{a_2} \subseteq I_{a_1-1+2} = I_{a_1+1}
\]
\[
x_3^{q+1}(I_1 : x_3^q) \subseteq (I_{a_1+1} : x_3^q)
\]

Hence
\[
I_n
\]
\[
= \begin{cases} 
\sum_{a_2=0}^{n-2} J_1^{n-1-2a_2} J_2^{a_2} & \text{if } 2 \not| (n-1) \\
J_2^{(n-1)/2} + \sum_{a_2=0}^{n-3} J_1^{n-1-2a_2} J_2^{a_2} & \text{if } 2 | (n-1) 
\end{cases}
\]
\[
\subseteq \begin{cases} 
\sum_{a_2=0}^{n-2} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1) a_2} (x_2, x_3^q) + x_3^{(2q+1) a_2} (I_{n-2a_2} : x_3^q) & \text{if } 2 \not| (n-1) \\
x_3^{(2q+1) a_2} + \sum_{a_2=0}^{n-3} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1) a_2} (x_2, x_3^q) + x_3^{(2q+1) a_2} (I_{n-2a_2} : x_3^q) & \text{if } 2 | (n-1) 
\end{cases} \quad \text{[by (4.2)]}
\]
\[
\subseteq \begin{cases} 
\sum_{a_2=0}^{n-2} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1) a_2} (x_2, x_3^q) + (I_1 : x_3^q) & \text{if } 2 \not| (n-1) \\
x_3^{(2q+1) a_2} + \sum_{a_2=0}^{n-3} x_2^{2(n-1-2a_2)-1} x_3^{(2q+1) a_2} (x_2, x_3^q) + (I_1 : x_3^q) & \text{if } 2 | (n-1) 
\end{cases}
\]

This implies that \(I_n \subseteq \text{RHS}\). The other inclusion follows from (1) and checking element-wise. \(\Box\)
Proposition 4.3. For all \( n \geq 1 \),
\[
\dim_k \left( \frac{I_{n-1}}{(I_n : x_3^q)} \right) = n.
\]

Proof. Put \( m' = (x_2, x_3) \). Then \( x_3^2 m' I_{n-1} \subseteq J_1 I_{n-1} \subseteq I_n \). Hence by Lemma 4.1(2) we get
\[
\dim_k \left( \frac{I_{n-1}}{(I_n : x_3^q)} \right) \leq \begin{cases} 
2(n/2) & \text{if } 2 \nmid n-1 \\
1 + 2(n-1)/2 & \text{if } 2 \mid n-1
\end{cases} = n.
\]
To complete the proof we need to show that we have \( n \) linearly independent elements. As all the elements in \( I_{n-1}/(I_n : x_d) \) has the degree of \( x_2 \) different, they form a linearly independent set. \( \square \)

5. Cohen-Macaulayness of \( R/(p^{(n)} + (f_k)) \)

Let \((f_1) := (f_1, f_2)\), where \( f_1 \) and \( f_2 \) are defined in Lemma 2.1. The main step in proving the Cohen-Macaulayness of \( R_s(p) \) is to show that or all \( n \geq 1 \) and \( k = 1, 2 \), the rings \( R/(p^{(n)} + (f_k)) \) are Cohen-Macaulay \( [9] \). This was proved by Goto for \( q = 1 \) and \( n \leq 3 \) \( [9, \text{Proposition 7.6}] \). We prove it for all \( q \geq 1 \) and for all \( n \). As a consequence, we prove that \( p^{(n)} = \mathcal{I}_n R \) and \( (p^{(n)}T') = I_n T' \) for all \( n \geq 1 \).

We first compute the length of the \( T'/I_n \). Next, we prove an interesting result which gives the the equality of the lengths of the various modules (Theorem 5.2, Theorem 5.6).

Proposition 5.1. For all \( n \geq 1 \),
\[
\ell \left( \frac{T'}{I_n} \right) = (2q + 1) \left( \frac{n + 1}{2} \right).
\]

Proof. We prove induction on \( n \). If \( n = 1 \), then
\[
\ell \left( \frac{T'}{I_1} \right) = \ell \left( \frac{k[x_2, x_3]}{(x_2^2, x_2 x_3, x_3^q + 1)} \right) = 2q + 1.
\]
Now let \( n > 1 \). From the exact sequence
\[
0 \rightarrow \frac{T'}{(I_n : x_3^q)} \xrightarrow{x_3^q} \frac{T'}{I_n} \xrightarrow{T'} \frac{T'}{I_n + (x_3^q)} \rightarrow 0
\]
we get
\[
\ell \left( \frac{T'}{I_n} \right)
= \ell \left( \frac{T'}{I_n + (x_3^q)} \right) + \ell \left( \frac{T'}{(I_n : x_3^q)} \right)
= \ell \left( \frac{T'}{(x^{2n}, x_3^q)} \right) + \ell \left( \frac{T'}{I_{n-1}} \right) + \ell \left( \frac{I_{n-1}}{(I_n : x_3^q)} \right) \quad \text{[Lemma 4.1(1)]}
= 2qn + (2q+1) \left( \frac{n}{2} \right) + n \quad \text{[by induction hypothesis and Proposition 4.3]}
= (2q+1) \left( \frac{n+1}{2} \right).
\]
\( \square \)
**Theorem 5.2.** For all \( n \geq 1 \),
\[
e \left( x_1; \frac{R}{p(n)} \right) = \ell \left( \frac{R}{p(n) + (x_1)} \right) = \ell_R \left( \frac{R}{\mathcal{I}_n, x_1)R} \right) = \ell_{T'} \left( \frac{T'}{\mathcal{I}_n, x_1)T'} \right) = \ell \left( \frac{T'}{I_n} \right) = (2q + 1) \left( \frac{n + 1}{2} \right).
\]

**Proof.** From Proposition 2.5(1), \( \mathcal{I}_n R \subseteq p(n) \). Hence,
\[
e \left( x_1; \frac{R}{p(n)} \right) = \ell_R \left( \frac{R}{p(n) + (x_1)} \right) \quad \text{[as } R/p(n) \text{ is Cohen-Macaulay]}
\leq \ell_R \left( \frac{R}{(\mathcal{I}_n, x_1)R} \right). \quad (5.3)
\]

By Proposition 2.5(3), for any prime \( q \neq m, (\mathcal{I}_n, x_1)T_q = T \). Hence \( \text{Supp}_T \left( \frac{T}{\mathcal{I}_n, x_1)T} \right) = \{m\} \) and we get
\[
\ell_R \left( \frac{R}{(\mathcal{I}_n, x_1)R} \right) = \ell_{T'} \left( \frac{T'}{\mathcal{I}_n, x_1)T'} \right) \quad \text{[Lemma 2.2]}
= \ell_{T'} \left( \frac{T'}{I_n} \right) \quad \text{[(2.4)]}
= (2q + 1) \left( \frac{n + 1}{2} \right) \quad \text{[Proposition 5.1]}
= e \left( x_1; \frac{R}{p(n)} \right) \ell_{R_p} \left( \frac{R_p}{p^n R_p} \right)
= e \left( x_1; \frac{R}{p(n)} \right) \ell_{R_p} \left( \frac{R_p}{p(n) R_p} \right) \quad \text{[since } p(n) R_p = p^n R_p \text{]}
= e \left( x_1; \frac{R}{p(n)} \right) \quad \text{[by [20, Theorem 14.7]]}. \quad (5.4)
\]

Thus equality holds in (5.3) and (5.4) which proves the theorem. \( \square \)

**Notation 5.5.** Let \( f_1 = f_1, f_2 = f_1, f_2, (x_2) = (x_2^2) \) and \( (x_3) = (x_2^2, x_3^{2q+1}) \).

**Theorem 5.6.** Let \( k = 1, 2 \). Then for all \( n \geq 1 \),
\[
e \left( x_1; \frac{R}{p(n) + (f_k)} \right) = \ell_R \left( \frac{R}{p(n) + (x_1, f_k)} \right) = \ell_{T'} \left( \frac{T'}{\mathcal{I}_n + (f_k)T'} \right) = \ell \left( \frac{T'}{I_n + (x_{k+1})} \right).
\]

In particular, \( \frac{R}{p(n) + (f_k)} \) is Cohen-Macaulay.

**Proof.** From Proposition 2.5(1) \( (\mathcal{I}_n, x_1, f_k)R \subseteq (p(n), x_1, f_k)R \). Hence
\[
e \left( x_1; \frac{R}{p(n) + (f_k)} \right) \leq \ell_R \left( \frac{R}{p(n) + (x_1, f_k)} \right) \quad \text{[20, Theorem 14.10]}
\leq \ell_R \left( \frac{R}{(\mathcal{I}_n, x_1, f_k)R} \right). \quad (5.7)
\]

Since \( (\mathcal{I}_n, x_1)T \subseteq (\mathcal{I}_n, f_k, x_1)T \) by Proposition 2.5(3), for any prime \( q \neq m, ((\mathcal{I}_n, f_k, x_1)T)_q = T \). This implies that \( \text{Supp}_T \left( \frac{T}{\mathcal{I}_n, f_k, x_1)T} \right) = \{m\} \). Hence we get
we get

\[ \ell_R \left( \frac{R}{(I_n, x_1, f_k)R} \right) = \ell_T \left( \frac{T}{(I_n, x_1, f_k)T} \right) = \ell_{T'} \left( \frac{T'}{I_n + (x_{k+1})} \right) \]

[Lemma 2.2]

\[ \ell_T \left( \frac{T}{(I_n, x_1, f_k)T} \right) = \ell_{T'} \left( \frac{T'}{(I_n-1)T'} \right) \]

[Proposition 3.6]

\[ \ell_{T'} \left( \frac{T'}{(I_{n-1})T'} \right) - \ell_{T'} \left( \frac{T'}{(I_{n-2})T'} \right) \]

[Theorem 5.2]

\[ e \left( \frac{R}{p^{(n)} + (f_k)} \right) \]

[7, Corollary 2.6]. (5.8)

Hence equality holds in (5.7) and (5.8) which proves the theorem.

We end this section by explicitly describing the generators of \( p^{(n)} \) for all \( n \geq 1 \).

**Theorem 5.9.** For all \( n \geq 1 \), \( p^{(n)} = I_nR; p^{(n)} = \sum_{a_1 + 2a_2 = n} p^{a_1}(p^{(2)})^{a_2} \) and \( p^{(n)}T' = I_n = I_nT' \).

**Proof.** By Theorem 5.2 we get \( p^{(n)} + (x_1) = I_nR + (x_1) \). Hence \( p^{(n)} = I_nR + x_1(p^{(n)} : (x_1)) \). As \( x_1 \) is a nonzerodivisor on \( R/p^{(n)} \), \( (p^{(n)} : (x_1)) = p^{(n)} \). By Nakayama’s lemma, \( p^{(n)} = I_nR \).

For all \( n \geq 3 \), applying Proposition 2.5(1) we get

\[ p^{(n)} = I_nR = \sum_{a_1 + 2a_2 = n} J_1^{a_1} J_2^{a_2} R \subseteq \sum_{a_1 + 2a_2 = n} p^{a_1}(p^{(2)})^{a_2} \subseteq p^{(n)}. \]

Hence equality holds. The last equality follows from Theorem 5.2.

\[ \square \]

6. **Cohen-Macaulayness and Gorensteinness of symbolic blowup algebras**

In this section we show that both symbolic blowup algebras \( G_s(p) \) and \( R_s(p) \) are Cohen-Macaulay and Gorenstein. From Proposition 3.7 in [11] it follows that \( G_s(p) \) is Cohen-Macaulay. We use the results in this paper to prove it.

**Theorem 6.1.**

1. \( G_s(p) \) is Cohen-Macaulay.
2. \( G_s(p) \) is Gorenstein.

**Proof.** (1) By [11, Corollary 3.2], it is enough to show that \( x_1^t, f_1^t, f_2^t \) is a regular sequence in \( G_s(p) \). For all \( n \geq 1 \), \( x_1^t \) is a nonzerodivisor on \( R/p^{(n)} \) for all \( t \). Hence \( x_1^t \) is a regular element in \( G_s(p) \) and

\[ G_s left( fraction{p + x_1 R}{x_1 R} right) \cong \bigoplus_{n \geq 0} \frac{p^{(n)} + x_1 R}{p^{(n+1)} + x_1 R} \cong \bigoplus_{n \geq 0} \frac{p^{(n)}}{p^{(n+1)} + x_1 p^{(n)}} \cong \frac{G_s(p)}{x_1^t G_s(p)}. \]
To show that $f_1^*$ is a regular sequence we need to show that $(p^{(n)} + x_1 R : f_1) = p^{(n)} + x_1 R$. Now

$$\ell \left( \frac{R}{(p^{(n+1)}, x_1) : (f_1)} \right) = \ell \left( \frac{R}{(p^{(n+1)}, x_1)} \right) - \ell \left( \frac{R}{(p^{(n+1)}, x_1, f_1)} \right)$$

$$= \ell \left( \frac{T'}{I_{n+1}} \right) - \ell \left( \frac{T'}{I_{n+1} + (x_2^2)} \right) \quad \text{[Theorem 5.6]}$$

$$= \ell \left( \frac{T'}{I_{n+1} : x_2^2} \right)$$

$$= \ell \left( \frac{T'}{I_n} \right) \quad \text{[proof of Theorem 3.1]}$$

$$= \ell \left( \frac{R}{(p^{(n)}, x_1)} \right) \quad \text{[Theorem 5.6]}.$$  

Similarly, one can show that $(p^{(n+1)}, x_1, f_1) = (p^{(n-1)}, x_1, f_1)$ which will imply that $f_2^*$ is a nonzerodivisor on $G_s(p)/(x_1^*, f_1^*)$. This proves (1).

(2) As $G(pR_n)$ is a polynomial ring, it is Gorenstein. Hence the result follows from Theorem 5.6 and [9, Corollary 5.8].

**Theorem 6.2.** ([11, Theorem 4.1], [17, Theorem 2]

(1) $\mathcal{R}_s(p) = R[pt, J_2t^2] = R[pt, f_2t^2]$.

(2) $\mathcal{R}_s(p)$ is Cohen-Macaulay.

(3) $\mathcal{R}_s(p)$ is Gorenstein.

**Proof.** (1) The proof follows from Theorem 5.9.

(2) By Theorem 5.6, $\frac{R}{p^{(n)} + (f_2)}$ is Cohen-Macaulay for all $n \geq 1$. Hence, by [9, Theorem 6.7], $\mathcal{R}_s(p)$ is Cohen-Macaulay.

(3) By [9, Lemma 6.1], the a-invariant of $(G_s(p))$, $a(G_s(p)) = -(2)$. By [9, Theorem 6.6], and Theorem 6.1, $\mathcal{R}_s(p)$ is Gorenstein.

\[ \square \]

7. A FEW QUESTIONS

In this section we state a few related questions which are of interest.

**Question 7.1.** In [21] S. Goto and M. Morimoto studied the monomial curves $(t^{n^2+2n+2}, t^{n^2+2n+1}, t^{n^2+n+1})$. They showed that for $n = p^r$ $(r \geq 1)$, $R_s(p)$ is Noetherian but not Cohen-Macaulay if char $k = p$. In [12], the authors raised the following question. If char $k = p' \neq p$, is $R_s(p)$ Noetherian? In [28], W. Vasconcelos showed that if char $k = 0$ and $n = 2$, then $R_s(p)$ Noetherian. If $p$ is a prime and $n = p > 2$, then is $R_s(p)$ Noetherian?

**Question 7.2.** Let $k$ be a field and let $p$ be the prime ideal defining the monomial curve $(t^a, t^b, c^c)$ in $\mathbb{A}^3$ where $a$, $b$ and $c$ are pairwise coprime. In [5], D. Cutkosky gave a geometric meaning to symbolic primes. He found some interesting examples of monomial curves for which $R_s(p)$ is finitely generated. Using the criteria in D. Cutkosky’s paper, H. Srinivasan showed that if $a = 6$, then $R_s(p)$ is finitely generated [26].
In [16], C. Huneke showed that $R_s(p)$ is finitely generated if $a = 4$. Can we find all possible $(a, b, c)$ such that $R_s(p)$ is finitely generated?

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