Supplement to “Liquidity based modeling of asset price bubbles via random matching”

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This is a supplement to the paper [1]. The supplement is organized as follows. First, we prove Theorem 3.13 in [1] which provides the existence of the dynamical system \( D \) introduced in Definition 3.6 in [1]. Second, we show some properties of \( D \) which are summarized in Theorem 3.14 in [1]. In the following, we only state the basic setting and refer to [1] for definitions.

1 Setting

Let \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) be a probability space and \((\hat{\Omega}, \hat{\mathcal{F}})\) another measurable space. We define the product space

\[
(\Omega, \mathcal{F}) := (\tilde{\Omega} \times \hat{\Omega}, \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}).
\] (1.1)

Let \(\hat{\mathbb{P}}\) be a Markov kernel (or stochastic kernel) from \(\tilde{\Omega}\) to \(\hat{\Omega}\). Given \(\tilde{\omega} \in \tilde{\Omega}\), we set \(\hat{\mathbb{P}}(\tilde{\omega})\) with a slight notational abuse. We then introduce a probability measure \(\mathbb{P}\) on \((\Omega, \mathcal{F})\) as the semidirect product of \(\tilde{\mathbb{P}}\) and \(\hat{\mathbb{P}}\), that is,

\[
\mathbb{P}(\tilde{A} \times \hat{A}) := (\tilde{\mathbb{P}} \otimes \hat{\mathbb{P}})(\tilde{A} \times \hat{A}) = \int_{\hat{A}} \hat{\mathbb{P}}(\tilde{\omega}) d\tilde{\mathbb{P}}(\tilde{\omega}).
\] (1.2)

We fix an atomless probability space \((I, \mathcal{I}, \lambda)\) representing the space of agents and let \((I \times \Omega, I \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\) be a rich Fubini extension of \((I \times \Omega, I \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\). All agents in \(I\) can be classified according to their type. In particular, we let \(S = \{1, 2, ..., K\}\) be a finite space of types and say that an agent has type \(J\) if he is not matched. We denote by \(\hat{S} := S \times (S \cup \{J\})\) the extended type space. Moreover, we call \(\hat{\Delta}\) the space of extended type distributions, which is the set of probability distributions \(p\) on \(\hat{\Delta}\) satisfying \(p(k, l) = p(l, k)\) for any \(k\) and \(l\) in \(S\). This space is endowed with the topology \(\mathcal{T}_{\hat{\Delta}}\) induced by the topology of the space of matrices with \(|S|\) rows and \(|S| + 1\) columns. We consider \((n)_{n \geq 1}\) time periods and denote by \((\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)\) the matrix valued processes, with \((\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n) = (\eta^n_{kl}, \theta^n_{kl}, \xi^n_{kl}, \sigma^n_{kl}[r, s], \varsigma^n_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}\) for \(n \geq 1\), on \((\Omega, \mathcal{F}, \mathbb{P})\). For a detailed introduction of these processes we refer to Section 3 in [1]. Moreover, let \(\hat{p} = (\hat{p}^n)_{n \geq 1}\) be a stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \(\hat{\Delta}\), representing the evolution of the underlying extended type distribution. We assume that \(\hat{p}^0\) is deterministic.

Given the input processes \((\eta, \theta, \xi, \sigma, \varsigma)\) we denote by \(\hat{D}\) a dynamical system on \((I \times \Omega, I \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\) and

\[\hat{p} = (\hat{p}^n)_{n \geq 1}\] are the input processes.

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by $\Pi = (\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n \in \mathbb{N} \setminus \{0\}}$ the agent-type function, the random matching and the partner-type function, respectively, as introduced in Definition 3.6 in [1], which we recall in the following.

**Definition 1.1.** A dynamical system $\mathcal{D}$ defined on $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is a triple $\Pi = (\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n \in \mathbb{N} \setminus \{0\}}$ such that for each integer period $n \geq 1$ we have:

1. $\alpha^n : I \times \Omega \rightarrow S$ is the $\mathcal{I} \otimes \mathcal{F}$-measurable agent type function. The corresponding end-of-period type of agent $i$ under the realization $\omega \in \Omega$ is given by $\alpha^n(i, \omega) \in S$.

2. A random matching $\pi^n : I \times \Omega \rightarrow I$, describing the end-of-period agent $\pi^n(i)$ to whom agent $i$ is currently matched, if agent $i$ is currently matched. If agent $i$ is not matched, then $\pi^n(i) = i$. The associated $\mathcal{I} \otimes \mathcal{F}$-measurable partner-type function $g^n : I \times \Omega \rightarrow S \cup \{J\}$ is given by

$$
g^n(i, \omega) = \begin{cases} 
\alpha^n(\pi^n(i), \omega) & \text{if } \pi^n(i, \omega) \neq i \\
J & \text{if } \pi^n(i, \omega) = i,
\end{cases}
$$

providing the type of the agent to whom agent $i$ is matched, if agent $i$ is matched, or $J$ if agent $i$ is not matched.

Let the initial condition $\Pi^0 = (\alpha^0, \beta^0)$ of $\mathcal{D}$ be given. We now construct a dynamical system $\mathcal{D}$ defined on $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ with input processes $(\eta^n, \theta^n, \xi^n, \sigma^n, \zeta^n)_{n \geq 1}$. We assume that $\Pi^{n-1} = (\alpha^{n-1}, \pi^{n-1}, g^{n-1})$ is given for some $n \geq 1$, and define $\Pi^n = (\alpha^n, \pi^n, g^n)$ by characterizing the three sub-steps of random change of types of agents, random matchings, break-ups and possible type changes after matchings and break-ups as follows.

**Mutation:** For $n \geq 1$ consider an $\mathcal{I} \otimes \mathcal{F}$-measurable post mutation function

$$\hat{\alpha}^n : I \times \Omega \rightarrow S.$$

In particular, $\hat{\alpha}^n(i, \omega)$ is the type of agent $i$ after the random mutation under the scenario $\omega \in \Omega$.

The type of the agent to whom an agent is matched is identified by a $\mathcal{I} \otimes \mathcal{F}$-measurable function

$$\hat{g}^n : I \times \Omega \rightarrow S \cup \{J\},$$

given by

$$\hat{g}^n(i, \omega) = \hat{\alpha}^n(\pi^{n-1}(i), \omega)$$

for any $\omega \in \Omega$. In particular, $\hat{g}^n(\omega) := \hat{g}^n(\pi^n(i), \omega)$ is the type of the agent to whom an agent is matched under the scenario $\omega \in \Omega$. Given $\hat{p}^{n-1}$ and $\hat{\omega} \in \hat{\Omega}$, for any $k_1, k_2, l_1$ and $l_2$ in $S$, for any $r \in S \cup \{J\}$, for $\lambda$-almost every agent $i$, we set

$$\hat{p}^{n-1} \hat{\omega}^i (\hat{\omega}, r) = k_2, \hat{g}^n(\hat{\omega}, \cdot) = l_2|_{\alpha^n(\hat{\omega}, \cdot)} = k_1, g^{n-1}_i (\hat{\omega}, \cdot) = l_1, \hat{p}^{n-1}(\hat{\omega}, \cdot)) (\hat{\omega})
$$

$$= \eta_{k_1} \cdot \eta_{k_2} (\hat{\omega}, n, \hat{p}^{n-1}(\hat{\omega}, \hat{\omega})) \eta_{l_1} \cdot \eta_{l_2} (\hat{\omega}, n, \hat{p}^{n-1}(\hat{\omega}, \hat{\omega}))
$$

(1.3)

$$\hat{p}^{\omega} \hat{\omega}^i (\hat{\omega}, r) = k_2, \hat{g}^n(\hat{\omega}, \cdot) = r|_{\alpha^n(\hat{\omega}, \cdot)} = k_1, g^{n-1}_i (\hat{\omega}, \cdot) = J, \hat{p}^{n-1}(\hat{\omega}, \cdot)) (\hat{\omega})
$$

$$= \eta_{k_1} \cdot \eta_{k_2} (\hat{\omega}, n, \hat{p}^{n-1}(\hat{\omega}, \hat{\omega})) \delta_{J}(r),
$$

(1.4)

We then set

$$\hat{\beta}^n(\omega) = (\hat{\alpha}^n(\omega), \hat{g}^n(\omega)), \quad n \geq 1.$$
The post-mutation extended type distribution realized in the state of the world \( \omega \in \Omega \) is denoted by \( \bar{\tilde{p}}(\omega) = (\bar{\tilde{p}}^n(\omega)[k,l])_{k \in S, l \in S \cup J} \), where

\[
\bar{\tilde{p}}^n(\omega)[k,l] := \lambda(\{i \in I : \alpha^n(i,\omega) = k, g^n(i,\omega) = l\}).
\]  

\[ (1.5) \]

**Matching:** We introduce a random matching \( \tilde{\pi}^n : I \times \Omega \to I \) and the associated post-matching partner type function \( \tilde{g}^n \) given by

\[
\tilde{g}^n(i,\omega) = \begin{cases} 
\alpha^n(\tilde{\pi}^n(i,\omega),\omega) & \text{if } \tilde{\pi}^n(i,\omega) \neq i \\
J & \text{if } \tilde{\pi}^n(i,\omega) = i,
\end{cases}
\]

satisfying the following properties:

1. \( \tilde{g}^n \) is \( I \otimes F \)-measurable.

2. For any \( \tilde{\omega} \in \tilde{\Omega} \), \( k, l \in S \) and any \( r \in S \cup \{J\} \), it holds

\[
\tilde{P}^\omega(\tilde{g}^n(\tilde{\omega},\cdot) = r|\alpha^n(\tilde{\omega},\cdot) = k, \tilde{g}^n(\tilde{\omega},\cdot) = l)(\tilde{\omega}) = \delta_i(r).
\]

This means that

\[
\tilde{\pi}^n(i) = \pi^n_{\tilde{\omega}}(i) \quad \text{for any } i \in \{i : \pi^{-1}(i,\omega) \neq i\}.
\]

3. Given \( \tilde{\omega} \in \tilde{\Omega} \) and the post-mutation extended type distribution \( \bar{\tilde{p}}^n \) in \( \Omega \), an unmatched agent of type \( k \) is matched to a unmatched agent of type \( l \) with conditional probability \( \theta_{kl}(\tilde{\omega},n,\bar{\tilde{p}}^n) \), that is for \( \lambda \)-almost every agent \( i \) and \( \tilde{P}^\omega \)-almost every \( \tilde{\omega} \), we define

\[
\tilde{P}^\omega(\bar{\tilde{g}}^n(\tilde{\omega},\cdot) = l|\alpha^n(\tilde{\omega},\cdot) = k, \bar{\tilde{g}}^n(\tilde{\omega},\cdot) = J, \tilde{p}^n(\tilde{\omega},\cdot))(\tilde{\omega}) = \theta_{kl}(\tilde{\omega},\tilde{p}^n(\tilde{\omega},\tilde{\omega})).
\]  

\[ (1.6) \]

This also implies that

\[
\tilde{P}^\omega(\bar{\tilde{g}}^n(\tilde{\omega},\cdot) = J|\alpha^n(\tilde{\omega},\cdot) = k, \bar{\tilde{g}}^n(\tilde{\omega},\cdot) = J, \tilde{p}^n(\tilde{\omega},\cdot))(\tilde{\omega}) = 1 - \sum_{l \in S} \theta_{kl}(\tilde{\omega},\tilde{p}^n(\tilde{\omega},\tilde{\omega})) = b^k(\tilde{\omega},\tilde{p}^n(\tilde{\omega},\tilde{\omega})).
\]  

\[ (1.7) \]

The extended type of agent \( i \) after the random matching step is

\[
\tilde{\bar{\tilde{p}}}^n_i(\omega) = (\tilde{\bar{\tilde{p}}}^n(\omega),\tilde{g}^n_i(\omega)), \quad n \geq 1.
\]

We denote the post-matching extended type distribution realized in \( \omega \in \Omega \) by \( \tilde{\tilde{p}}^n(\omega) = (\tilde{\tilde{p}}^n(\omega)[k,l])_{k \in S, l \in S \cup J} \), where

\[
\tilde{\tilde{p}}^n(\omega)[k,l] := \lambda(\{i \in I : \tilde{\bar{\tilde{p}}}^n(i,\omega) = k, \tilde{g}^n(i,\omega) = l\}).
\]  

\[ (1.8) \]

**Type changes of matched agents with break-up:** We now define a random matching \( \pi^n \) by

\[
\pi^n(i) = \begin{cases} 
\tilde{\pi}^n(i) & \text{if } \tilde{\pi}^n(i) \neq i \\
i & \text{if } \tilde{\pi}^n(i) = i.
\end{cases}
\]  

\[ (1.9) \]

We then introduce an \( (I \otimes F) \)-measurable agent type function \( \alpha^n \) and an \( (I \otimes F) \)-measurable partner function \( g^n \) with

\[
g^n(i,\omega) = \alpha^n(\pi^n(i,\omega),\omega), \quad n \geq 1,
\]

for all \( (i,\omega) \in I \times \Omega \). Given \( \tilde{\omega} \in \tilde{\Omega} \), \( \tilde{p}^n \in \tilde{\Delta} \), for any \( k_1, k_2, l_1, l_2 \in S \) and \( r \in S \cup \{J\} \), for \( \lambda \)-almost every agent \( i \), and for \( \tilde{\bar{\tilde{p}}}^n \)-almost every \( \tilde{\omega} \), we set

\[
\tilde{P}^\omega(\alpha^n(\tilde{\omega},\cdot) = l_1, g^n(\tilde{\omega},\cdot) = r|\alpha^n(\tilde{\omega},\cdot) = k_1, \tilde{g}^n(\tilde{\omega},\cdot) = J)(\tilde{\omega}) = \delta_{k_1}(l_1)\delta_J(r).
\]  

\[ (1.10) \]
\[
= (1 - \xi_{k_1 k_2}(\bar{\omega}, n, \bar{\theta}(\bar{\omega}))) \sigma_{k_1 k_2}[1, l_2](\bar{\omega}, n, \bar{\theta}(\bar{\omega})), \tag{1.11}
\]

\[
P^2 (\alpha^n(\bar{\omega}, \cdot) = l_1, g^n(\bar{\omega}, \cdot) = J(\bar{\alpha}_n^{k_1}(\bar{\omega}, \cdot) = k_1, \bar{g}_n^{k_2}(\bar{\omega}, \cdot) = k_2, \bar{\theta}(\bar{\omega}, \cdot)) (\bar{\omega})
= \xi_{k_1 k_2}(\bar{\omega}, n, \bar{\theta}(\bar{\omega}, \bar{\omega})) \xi_{k_1 k_2}[1](\bar{\omega}, n, \bar{\theta}(\bar{\omega}, \bar{\omega})). \tag{1.12}
\]

The extended-type function at the end of the period is
\[\beta^n(\omega) = (\alpha^n(\omega), g^n(\omega)), \quad n \geq 1.\]

We denote the extended type distribution at the end of period \(n\) realized in \(\omega \in \Omega\) by \(\bar{\theta}(\omega) = (\bar{\theta}(\omega)[k, l])_{k \in S, l \in S U J}\), where
\[\bar{\theta}(\omega)[k, l] := \lambda\{i \in I : \alpha^n(i, \omega) = k, g^n(i, \omega) = l\}. \tag{1.13}\]

Furthermore, the definition of Markov conditionally independent (MCI) dynamical system is provided in Definition 3.8 in [1]. We work under the following assumption, which is Assumption 3.9 in [1].

**Assumption 1.2.** Let \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\) be the probability space introduced. We assume that there exists its corresponding hyperfinite internal probability space, which we denote from now on also by \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\) by a slight notational abuse.

As already pointed out in [1], the proofs of the results below follow by analogous arguments as in [2] which is possible due to the product structure of the space \(\Omega\) in (1.1) and the Markov kernel \(P\) in (1.2). As in [2] we use some concepts and notations from nonstandard analysis. Note here that an object with an upper left star means the transfer of a standard object to the nonstandard universe. For a detailed overview of the necessary tools of nonstandard analysis, we refer to Appendix D.2. in [2].

# 2 Proof of Theorem 3.13 in [1]

From now on, we fix the hyperfinite internal space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})\), along with the input functions \((\eta_{k l}, \theta_{k l}, \xi_{k l}, \sigma_{k l} \{r, s\}, \xi_{k l}[\{r\}])_{k, l, r, s} E \times S \times S \times S \times S\) from \(\hat{\Omega} \times \mathbb{N} \times \Delta\) to \([0, 1]\) introduced above. Given this framework we prove the existence of a rich Fubini extension \((I \times \hat{\Omega}, I \otimes \hat{\mathcal{F}}, \lambda \otimes \hat{P})\), on which a dynamical system \(D\) described in Definition 3.11 for such input probabilities is defined. More specifically, we are going to construct the space \(\hat{\Omega}\) and the probability measure \(\hat{P}\) such that \(\hat{\Omega} = \hat{\Omega} \times \hat{\Omega}\) and \(P = \hat{P} \times \hat{P}\) is a Markov kernel from \(\hat{\Omega}\) to \(\hat{\Omega}\).

We now present and prove Theorem 3.13 in [1]. The proof is based on Proposition 3.12 in [1], which focuses on the random matching step and shows the existence of a suitable hyperfinite probability space and partial matching, generalizing Lemma 7 in [2].

**Theorem 2.1.** Let Assumption 3.9 in [1] hold and \((\eta_{k l}, \theta_{k l}, \xi_{k l}, \sigma_{k l}[\{r, s\}], \xi_{k l}[\{r\}])_{k, l, r, s} E \times S \times S \times S \times S\) be the input functions from \(\hat{\Omega} \times \mathbb{N} \times \Delta\) defined in Section 3 in [1]. Then for any extended type distribution \(\bar{p} \in \hat{\Delta}\) and any deterministic initial condition \(\Pi^0 = (\alpha^0, \pi^0)\) there exists a rich Fubini extension \((I \times \hat{\Omega}, I \otimes \hat{\mathcal{F}}, \lambda \otimes \hat{P})\) on which a discrete dynamical system \(D = (\Pi^0)_{n=0}^\infty\) as in Definition 3.6 in [1] can be constructed with discrete time input processes \((\eta^n, \theta^n, \xi^n, \sigma^n, \xi^n)_{n \geq 1}\) coming from \((\eta_{k l}, \theta_{k l}, \xi_{k l}, \sigma_{k l}[\{r, s\}], \xi_{k l}[\{r\}])_{k, l, r, s} E \times S \times S \times S \times S\) as stated in Section 2 in [1]. In particular,
\[\Omega = \hat{\Omega} \times \hat{\Omega}, \quad \mathcal{F} = \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}, \quad P = \hat{P} \times \hat{P},\]
where \((\hat{\Omega}, \hat{\mathcal{F}})\) is a measurable space and \(\hat{P}\) a Markov kernel from \(\hat{\Omega}\) to \(\hat{\Omega}\). The dynamical system \(\mathcal{D}\) is also MCI according to Definition 3.8 in \([\text{?}]\) and with initial cross-sectional extended type distribution \(\hat{p}^0\) equal to \(\hat{p}^0\) with probability one.

**Proof.** At each time period we construct three internal measurable spaces with internal transition probabilities taking into account the following steps:

1. random mutation
2. random matching
3. random type changing with break-up.

Let \(M\) be a limited hyperfinite number in \(^*N\). Let \(\{n\}_{n=0}^M\) be the hyperfinite discrete time line and \((I, \mathcal{I}_0, \lambda_0)\) the agent space, where \(I = \{1, \ldots, M\}, \mathcal{I}_0\) is the internal power set on \(I\), \(\lambda_0\) is the internal counting probability measure on \(\mathcal{I}_0\), and \(M\) is an unlimited hyperfinite number in \(^*N\).

We start by transferring the deterministic functions \(\eta(0, \cdot), \theta(0, \cdot), \xi(0, \cdot), \sigma(0, \cdot), \varsigma(0, \cdot) : \hat{\Delta} \rightarrow [0, 1]\) to the nonstandard universe. In particular, we denote by \(*\theta^0_{kl}\) for any \(k, l \in S\) and by \(*f^0\) for \(f = \eta, \xi, \sigma, \varsigma\) the internal functions from \(*\hat{\Delta}\) to \([0, 1]\).

We also let \(\hat{\theta}^0_{kl}(\hat{\rho}) = *\theta^0_{kl}(\hat{\rho})\) and \(\hat{b}^0_k = 1 - \sum_{l \in S} \theta^0_{kl}(\hat{\rho})\) for any \(k, l \in S\) and \(\hat{\rho} \in *\hat{\Delta}\), with \(1 \in \mathbb{N}\).

We start at \(n = 0\). To do so, we introduce the trivial probability space over the single set \(\{0\}\) denoted by \(\langle \hat{\Omega}_0, \hat{\mathcal{F}}_0, \hat{Q}_0 \rangle\). Let \(\{A_{kl}\}_{(k,l) \in S}\) be an internal partition of \(I\) such that \(\frac{|A_{kl}|}{M} \simeq \hat{p}^0_{kl}\) for any \(k \in S\) and \(l \in S\cup\{J\}\), and that \(|A_{kk}|\) is even for any \(k, l \in S\) and \(|A_{kl}| = |A_{lk}|\) for any \(k, l \in S\). Let \(\alpha^0\) be an internal function from \((I, \mathcal{I}_0, \lambda_0)\) to \(S\) such that \(\alpha^0(i) = k\) if \(i \in \bigcup_{l \in S \cup \{J\}} A_{kl}\). Let \(\pi^0\) be an internal partial matching from \(I\) to \(I\) such that \(\pi^0(i) = i\) on \(\bigcup_{k \in S} A_{kj}\), and the restriction \(\pi^0|_{A_{kl}}\) is an internal bijection from \(A_{kl}\) to \(A_{lk}\) for any \(k, l \in S\).

Let \(\langle \hat{\Omega}, \hat{\mathcal{F}}, \hat{P} \rangle\) be the hyperfinite internal space. Since the intensities are supposed to be deterministic at initial time, the Markov kernel from \(\hat{\Omega}\) is trivial and we define the initial internal product probability space as

\[
\langle \Omega_0, \mathcal{F}_0, Q_0 \rangle := (\hat{\Omega} \times \hat{\Omega}_0, \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}_0, \hat{P} \otimes \hat{Q}_0).
\]

Suppose now that the dynamical system \(\mathcal{D}\) has been constructed up to time \(n - 1 \in \mathbb{N}\) for \(n \geq 1\), i.e., that the sequences \(\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=0}^{3n-3}\) and \(\{\alpha^l, \pi^l\}_{l=0}^{n-1}\) have been constructed. In particular, we assume to have introduced the spaces \((\hat{\Omega}_m, \hat{\mathcal{F}}_m)\) and the Markov kernel \(\hat{P}_m\) from \(\hat{\Omega}\) to \(\hat{\Omega}_m\) for any \(m = 1, \ldots, n - 3\), so that we can define \(\Omega_m := \hat{\Omega} \times \hat{\Omega}_m\) as a hyperfinite internal set with internal power set \(\mathcal{F}_m := \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}_m\) and \(Q_m := \hat{P} \times \hat{P}_m\) as an internal transition probability from \(\Omega^{m-1}\) to \((\Omega_m, \mathcal{F}_m)\), where

\[
\Omega^m := \hat{\Omega} \times \hat{\Omega}_m, \quad \hat{\Omega}^m := \hat{\Omega}_0 \times \prod_{j=1}^{m} \hat{\Omega}_j, \quad \hat{\mathcal{F}}^m := \hat{\mathcal{F}}_0 \otimes \left(\otimes_{j=1}^{m} \hat{\mathcal{F}}_j\right) \quad \text{and} \quad \mathcal{F}^m := \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}_m. \tag{2.1}
\]

\[\hat{\Delta} \to [0, 1]\]
In this setting, $\alpha^I$ is an internal type function from $I \times \Omega^{3l-1}$ to the space $S$, and $\pi^I$ an internal random matching from $I \times \Omega^l$ to $I$, such that

$$\alpha^I(i, (\bar{\omega}, \hat{\omega}^{3l-1})) = \alpha^I(i, \hat{\omega}^{3l-1}), \quad \text{for any } (\bar{\omega}, \hat{\omega}^{3l-1}) \in \Omega^{3l-1}$$

and

$$\pi^I(i, (\bar{\omega}, \hat{\omega}^l)) = \pi^I(i, \hat{\omega}^l), \quad \text{for any } (\bar{\omega}, \hat{\omega}^l) \in \Omega^l.$$

Given $\omega^l \in \Omega^l$ we denote by $\pi^I_{\omega^l} : I \rightarrow I$ the function given by

$$\pi^I_{\omega^l}(i) := \pi^I(i, (\bar{\omega}, \hat{\omega}^l)) = \pi^I(i, \hat{\omega}^l).$$

A similar notation will be used for $\alpha^I_{\omega^l} : I \rightarrow S$. We now have the following.

(i) **Random mutation step:**

We let $\Omega_{3n-2} := S^l$, which is the space of all internal functions from $I$ to $S$, and denote its internal product set by $\hat{\Omega}_{3n-2}$. For each $i \in I$ and $\omega^{3n-3} = (\bar{\omega}, \hat{\omega}^{3n-3}) \in \Omega^{3n-3}$, if $\alpha^{n-1}(i, \omega^{3n-3}) = \alpha^{n-1}(i, \hat{\omega}^{3n-3}) = k$, define a probability measure $\gamma_i \omega_{\hat{\omega}^{3n-3}}$ on $S$ by letting

$$\gamma_i \omega_{\hat{\omega}^{3n-3}}(l) := \theta_{kl}(\bar{\omega}, n, \hat{\omega}^{3n-3})$$

for each $l \in S$ with

$$\rho^n_{\omega_{3n-1}}[k, r] := \lambda_0(\{i \in I : \alpha^n_{\omega_{3n-3}}(i) = k, \alpha^n_{\omega_{3n-3}}(\pi^n_{\omega_{3n-3}}(i)) = r\}), \quad k, r \in S$$

and

$$\rho^n_{\omega_{3n-1}}[k, J] := \lambda_0(\{i \in I : \alpha^n_{\omega_{3n-3}}(i) = k, \pi^n_{\omega_{3n-3}}(i) = i\}), \quad k \in S.$$

Define a Markov kernel $\hat{\Omega}^{3n-3} \rightarrow \Omega_{3n-2}$ by letting $\hat{\Omega}^{3n-3} \rightarrow \Omega_{3n-2}$ by letting $\hat{\pi}^{3n-3} \rightarrow \Omega_{3n-2}$ by setting

$$\hat{\pi}^{3n-3}(i, \omega^{3n-3}) := \hat{\pi}^{3n-3}(i, \hat{\omega}^{3n-3}) := \hat{\omega}^{3n-2}(i)$$

and $\hat{g}^{3n} : (I \times \Omega^{3n-2}) \rightarrow S \cup \{J\}$ by

$$\hat{g}^{3n}(i, (\bar{\omega}, \hat{\omega}^{3n-2})) := \hat{g}^{3n}(i, \hat{\omega}^{3n-2}) := \begin{cases} \hat{\pi}^{3n}(\pi^{n-1}(i, \hat{\omega}^{3n-3}), \hat{\omega}^{3n-2}) & \text{if } \pi^{n-1}(i, \hat{\omega}^{3n-3}) \neq i \\ J & \text{if } \pi^{n-1}(i, \hat{\omega}^{3n-3}) = i. \end{cases}$$

Moreover, we introduce the notation

$$\hat{\alpha}^{3n-2}(\cdot) : I \mapsto S, \quad \hat{\alpha}^{3n-2}(i) := \hat{\alpha}^{3n}(i, \hat{\omega}^{3n-2}) = \hat{\alpha}^{3n}(i, \hat{\omega}^{3n-2})$$

for the type function. We then define $\pi_{\omega_{3n-3}}^{n-1}(\cdot) : I \rightarrow I$ and $\hat{g}^{n}_{\omega_{3n-2}} : I \rightarrow S \cup \{J\}$ analogously. Finally, we define the cross-internal extended type distribution after random mutation $\rho^n_{\omega_{3n-2}}$ by

$$\rho^n_{\omega_{3n-2}}[k, l] := \lambda_0(\{i \in I : \alpha^n_{\omega_{3n-2}}(i) = k, \hat{g}^{n}_{\omega_{3n-2}}(i) = l\}), \quad k, l \in S.$$

(ii) **Directed random matching:**

Let $(\Omega_{3n-1}, \hat{F}_{3n-1})$ and $\hat{\Omega}^{3n-2}$ be the measurable space and the Markov kernel, respectively, provided by Proposition 3.12 in [3], with type function $\alpha^n_{\omega_{3n-2}}(\cdot)$ and partial matching function $\pi_{\omega_{3n-3}}^{n-1}(\cdot)$, for fixed matching probability function $\theta(\cdot, n, \rho^n_{\omega_{3n-2}})$. Proposition 3.12 in [3] also provides the directed random matching

$$\pi^{n-1}_{\theta^n_{\omega_{3n-2}}}(\cdot, \hat{\alpha}^{n}_{\omega_{3n-2}}) \hat{\alpha}^{n}_{\omega_{3n-2}} \pi^{n-1}_{\omega_{3n-3}},$$
which is a function defined on \((\Omega_{3n-1}, \mathcal{F}_{3n-1})\) by
\[
\pi^n_i(\cdot, \tilde{\omega}_{3n-2}), \tilde{\alpha}^{n-1}_{3n-2}(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})): = \pi^n_i(\cdot, \tilde{\omega}_{3n-2}), \tilde{\alpha}^{n-1}_{3n-2}(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})).
\]
We then define \(\tilde{\pi}^n : (I \times \Omega^{3n-1}) \to I\) by
\[
\tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) := \tilde{\pi}^n(i, \tilde{\omega}^{3n-1}) := \pi^n_i(\cdot, \tilde{\omega}_{3n-2}), \tilde{\alpha}^{n-1}_{3n-2}(i, \tilde{\omega}_{3n-1})
\]
and
\[
\tilde{\varphi}^n(i, (\tilde{\omega}, \tilde{\omega}^{3n-1})) = \tilde{\varphi}^n(i, \tilde{\omega}^{3n-1}) := \begin{cases} 
\tilde{\alpha}^n(\tilde{\pi}^n(i, \tilde{\omega}^{3n-1}), \tilde{\omega}^{3n-2}) & \text{if } \tilde{\pi}^n(i, \tilde{\omega}^{3n-1}) \neq i \\
J & \text{if } \tilde{\pi}^n(i, \tilde{\omega}^{3n-1}) = i.
\end{cases}
\]
Define now the cross-internal extended type distribution after the random matching \(\tilde{\rho}^n_{3n-1}\) by
\[
\tilde{\rho}^n_{3n-1}[k, l] := \lambda_0(\{i \in I : \tilde{\alpha}^{n-1}_{3n-1}(i) = k, \tilde{\varphi}^n_{3n-1}(i) = l\}).
\]

(iii) Random type changing with break-up for matched agents:

Introduce \(\Omega_{3n} := (S \times \{0, 1\})^t\) with internal power set \(\mathcal{F}_{3n}\), where 0 represents “unmatched” and 1 represents “paired” each point \(\tilde{\omega}_{3n} = (\tilde{\omega}_1^{3n}, \tilde{\omega}_2^{3n}) \in \Omega_{3n}\) represents an internal function from \(I\) to \(S \times \{0, 1\}\). Define a new type function \(\alpha^n : (I \times \Omega^{3n}) \to S\) by letting \(\alpha^n(i, (\tilde{\omega}, \tilde{\omega}^{3n})) := \alpha^n(i, \tilde{\omega}^{3n}) = \tilde{\omega}_1^{3n}(i)\). Fix \((\tilde{\omega}, \tilde{\omega}^{3n-1}) \in \Omega^{3n-1}\) in \(\Omega^{3n-1}\).

For each \(i \in I\), we proceed in the following way.

1. If \(\tilde{\pi}^n(i, \tilde{\omega}^{3n-1}) = i\) \((i\text{ is not paired after the matching step at time } n)\), let \(\tau_{\tilde{\varphi}^{3n-1}}\) be the probability measure on the type space \(S \times \{0, 1\}\) that gives probability one to the type \((\tilde{\alpha}^n(i, \tilde{\omega}^{3n-2}), 0) = (\tilde{\alpha}^n(i, \tilde{\omega}^{3n-2}), 0)\) and zero to the rest

2. If \(\tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}^{3n-1})) = \pi^n(i, \tilde{\omega}^{3n-1}) = j \neq i\) \((i\text{ is paired after the matching step at time } n)\), \(\tilde{\alpha}^n(i, (\tilde{\omega}, \tilde{\omega}^{3n-2})) = \tilde{\alpha}^n(i, \tilde{\omega}^{3n-2}) = k, \tilde{\varphi}^n(i, (\tilde{\omega}, \tilde{\omega}^{3n-1})) = \pi^n(i, \tilde{\omega}^{3n-1}) = j\) and \(\tilde{\alpha}^n(j, (\tilde{\omega}, \tilde{\omega}^{3n-1})) = \tilde{\alpha}^n(j, \tilde{\omega}^{3n-1}) = l\), define a probability measure \(\tau_{\tilde{\varphi}^{3n-1}}\) on \((S \times \{0, 1\}) \times (S \times \{0, 1\})\) as
\[
\tau_{\tilde{\varphi}^{3n-1}}((k', 0), (l', 0)) := (1 - \xi_{kl}(\tilde{\omega}, n, \tilde{\rho}^n_{3n-1})) \alpha_{kl}(\tilde{\omega}, n, \tilde{\rho}^n_{3n-1}) \xi_{kl}[k', l'] \tilde{\omega}(n, \tilde{\rho}^n_{3n-1})
\]
and
\[
\tau_{\tilde{\varphi}^{3n-1}}((k', 1), (l', 1)) := \xi_{kl}(\tilde{\omega}, n, \tilde{\rho}^n_{3n-1}) \sigma_{kl}[k', l'] \tilde{\omega}(n, \tilde{\rho}^n_{3n-1})
\]
for \(k', l' \in S\), and zero for the rest.

Let \(A^n_{\tilde{\omega}^{3n-1}} = \{(i, j) \in I \times I : i < j, \tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}^{3n-1})) = \pi^n(i, \tilde{\omega}^{3n-1}) = j\}\) and \(B^n_{\tilde{\omega}^{3n-1}} = \{i \in I : \pi^n(i, (\tilde{\omega}, \tilde{\omega}^{3n-1})) = \pi^n(i, \tilde{\omega}^{3n-1}) = i\}\). Define a Markov kernel \(\tilde{\rho}^n_{3n-1}\) from \(\tilde{\omega}\) to \(\Omega^{3n}\) by
\[
\tilde{\rho}^n_{3n-1}((\tilde{\omega})) := \prod_{i \in B^n_{\tilde{\omega}^{3n-1}}} \tau_{\tilde{\varphi}^{3n-1}}(\tilde{\omega}) \otimes \prod_{(i, j) \in A^n_{\tilde{\omega}^{3n-1}}} \tau_{\tilde{\varphi}^{3n-1}}(\tilde{\omega}).
\]

Let
\[
\pi^n(i, (\tilde{\omega}, \tilde{\omega}^{3n})) = \pi^n(i, \tilde{\omega}^{3n})
\]
\[
:= \begin{cases} 
J & \text{if } \tilde{\pi}^n(i, \tilde{\omega}^{3n-1}) = J \text{ or } \tilde{\omega}_3^n(i) = 0 \text{ or } \tilde{\omega}_3^n(\tilde{\pi}^n(i, \tilde{\omega}^{3n-1})) = 0 \\
\pi^n(i, \tilde{\omega}^{3n-1}) & \text{otherwise,}
\end{cases}
\]
and
\[ g^n(i, (\hat{\omega}, \hat{\omega}^3n)) = g^n(i, \hat{\omega}^3n) := \begin{cases} \alpha^n(\pi^n(i, \hat{\omega}^3n), \hat{\omega}^3n) & \text{if } \pi^n(i, \hat{\omega}^3n) \neq i \\ J & \text{if } \pi^n(i, \hat{\omega}^3n) = i. \end{cases} \]

Define \( \hat{\rho}^n = \lambda_0(\alpha^n, \pi^n)^{-1} \).

By repeating this procedure, we construct a hyperfinite sequence of internal transition probability spaces \( \{\Omega_m, \mathcal{F}_m, Q_m\}_{m=0}^{3M} \) and a hyperfinite sequence of internal type functions and internal random matchings \( \{(\alpha^n, \pi^n)\}_{n=0}^{M} \). Moreover, define \( (\Omega^m, \mathcal{F}^m) \) as in \((2.1)\), and
\[ \hat{P}^m := \prod_{i=1}^{m} \hat{P}_i, \quad Q^m := \hat{P} \times \hat{P}^m, \]
where the product of the Markov kernels is \( \hat{\omega} \)-wise.

Let \( (I \times Q^3M, \mathcal{I} \otimes Q^3M, \lambda_0 \otimes Q^3M) \) be the internal product probability space of \( (I, \mathcal{I}, \lambda_0) \) and \( (Q^3M, \mathcal{F}^3M, Q^3M) \). Denote the Loeb spaces of \( (\Omega^3M, \mathcal{F}^3M, Q^3M) \) and the internal product \( (I \times Q^3M, \mathcal{I} \otimes Q^3M, \lambda_0 \otimes Q^3M) \) by \( (\hat{\Omega}^3M, \hat{\mathcal{F}}, \hat{\lambda}) \) and \( (I \times \hat{\Omega}^3M, \hat{\mathcal{I}} \otimes \hat{\mathcal{F}}, \hat{\lambda} \otimes \hat{\mathcal{P}}) \), respectively. For simplicity, let \( \Omega^3M \) be denoted by \( \Omega \) and \( \hat{\Omega}^3M \) by \( \hat{\Omega} \). Denote now \( Q^3M \) by \( \hat{P} \) and the Markov kernel \( P^3M \) by \( \hat{P} \).

The properties of a dynamical system as well as the independence conditions follow now by applying similar arguments as in the proof of Theorem 5 in \([1]\) for any fixed \( \hat{\omega} \in \hat{\Omega} \). The only difference is that in our setting the input processes for the random mutation step and the break-up step also depend on the extended type distribution. Furthermore, these arguments are similar to the ones in the proof of Lemma 3.2 and can be found there with all details. \( \square \)

### 3 Proof of Theorem 3.14 in \([1]\)

We now prove Theorem 3.14 in \([1]\) which is a generalization of the results in Appendix C in \([2]\). For \( n \geq 1 \) we define the mapping \( \Gamma^n \) from \( \hat{\Omega} \times \Delta \to \Delta \) by
\[
\Gamma^n_{kl}((\hat{\omega}, \hat{\rho})) = \sum_{k_1, l_1 \in S} (1 - \xi_{k_1 l_1}((\hat{\omega}, n, \hat{\rho}^n))) \sigma_{k_1 l_1}[k, l](\hat{\omega}, n, \hat{\rho}^n) \hat{p}_{k_1 l_1}^n,
\]
and
\[
\Gamma^n_{kJ}((\hat{\omega}, \hat{\rho})) = b_k((\hat{\omega}, n, \hat{\rho}^n)) \hat{p}_{k l}^n + \sum_{k_1, l_1 \in S} \xi_{k_1 l_1}(\hat{\omega}, n, \hat{\rho}^n) \varsigma_{k_1 l_1}[k]((\hat{\omega}, \hat{\rho}^n) \theta_{k_1 l_1}((\hat{\omega}, n, \hat{\rho}^n)) \hat{p}_{k_1 l_1}^n.
\]

with
\[
\hat{p}_{k l}^n = \sum_{k_1, l_1 \in S} \eta_{k_1 l_1}(\hat{\omega}, n, \hat{\rho}) \eta_{l_1}(\hat{\omega}, n, \hat{\rho}) \hat{p}_{k_1 l_1},
\]
and
\[
\hat{p}_{k l}^n = \sum_{l \in S} \hat{p}_{k l}(\hat{\omega}, n, \hat{\rho}).
\]
Lemma 3.1. Assume that the discrete dynamical system $\mathbb{D}$ defined in Definition 3.6 in [2] is Markov conditionally independent given $\tilde{\omega}$ as defined in Definition 3.8 in [4]. Then given $\tilde{\omega} \in \tilde{\Omega}$, the discrete time processes $\{\beta^n_i\}_{n=0}^\infty$, $i \in I$, are essentially pairwise independent on $(I \times \tilde{\Omega}, \mathcal{I} \otimes \tilde{\mathcal{F}}, \lambda \otimes \tilde{P}^{\tilde{\omega}})$. Moreover, for fixed $n = 1, ..., M$ also $(\tilde{\beta}^n_i)_{n=0}^\infty$ and $(\tilde{\beta}_i^n)_{n=0}^\infty$, $i \in I$, are essentially pairwise independent on $(I \times \tilde{\Omega}, \mathcal{I} \otimes \tilde{\mathcal{F}}, \lambda \otimes \tilde{P}^{\tilde{\omega}})$.

Proof. This can be proven by the same arguments used in the proof of Lemma 3 in [2].

We now derive a result which shows how to compute for a fixed $\tilde{\omega} \in \tilde{\Omega}$ the expected cross-sectional distributions $E^{\tilde{P}^{\tilde{\omega}}} [p^n]$, $E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n]$ and $E^{\tilde{P}^{\tilde{\omega}}} [\hat{p}^n]$.

Lemma 3.2. The following holds for any fixed $\tilde{\omega} \in \tilde{\Omega}$.

1. For each $n \geq 1$, $E^{\tilde{P}^{\tilde{\omega}}} [p^n] = \Gamma^n (\tilde{\omega}, E^{\tilde{P}^{\tilde{\omega}}} [p^{n-1}])$, with $\Gamma$ defined in (3.1).

2. For each $n \geq 1$, we have

$$E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n_{k,l}] = \sum_{k_1, l_1 \in S} \eta_{k_1, k}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}]) \eta_{l_1, l}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}]) E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}_{k_1, l_1}]$$

and

$$E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n_{k,l}] = \sum_{k \in S} \eta_{k_1, k}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}]) E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}_{k_1,k}].$$

3. For each $n \geq 1$, we have

$$E^{\tilde{P}^{\tilde{\omega}}} [p^n_{k,l}] = E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n_{k,l}] + \theta_{k_1}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n]) E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n_{k_1,l_1}]$$

and

$$E^{\tilde{P}^{\tilde{\omega}}} [p^n_{k,l}] = b_{k}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n]) E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^n_{k,l}].$$

Proof. Fix $\tilde{\omega} \in \tilde{\Omega}$ and $k, l \in S$. By Lemma 3.1 we know that the processes $(\beta^n_i)_{n=0}^\infty$, $i \in I$, are essentially pairwise independent. Then the exact law of large numbers in Lemma 1 in [2] implies that $\tilde{p}^{n-1}(\tilde{\omega}) = E^{\tilde{P}^{\tilde{\omega}}} [\lambda(\beta^{n-1})^{-1}]$ for $\tilde{P}$-almost all $\tilde{\omega} \in \tilde{\Omega}$. Thus equations (1.3) and (1.4) are equivalent to

$$\tilde{P}^{\tilde{\omega}} (\tilde{\alpha}_i^n = k_2, \tilde{g}_i^n = l_2 | \alpha_i^{n-1} = k_1, g_i^{n-1} = l_1) = \eta_{k_1, k_2}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}]) \eta_{l_1, l_2}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}])$$

and

$$\tilde{P}^{\tilde{\omega}} (\tilde{\alpha}_i^n = k_2, \tilde{g}_i^n = r | \alpha_i^{n-1} = k_1, g_i^{n-1} = J) = \eta_{k_1, k_2}(\tilde{\omega}, n, E^{\tilde{P}^{\tilde{\omega}}} [\tilde{p}^{n-1}]) \delta_r(r).$$

Therefore, for any $k_1, l_1 \in S$ we have

$$\tilde{P}^{\tilde{\omega}} (\tilde{\beta}_i^n = (k, J) | \beta_i^{n-1} = (k_1, l_1)) = 0$$

and

$$\tilde{P}^{\tilde{\omega}} (\tilde{\beta}_i^n = (k, l) | \beta_i^{n-1} = (k_1, J)) = 0.$$
Then with the same calculations as in the proof of Lemma 4 in [2] we get that

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{kl}^n] = \mathbb{E}^{\hat{P}}[\lambda(i \in I : \hat{\beta}_n(i) = (k, l))]
$$

$$
= \int \hat{P}^\omega(\hat{\beta}_n^\omega = (k, l)) d\lambda(i)
$$

$$
= \sum_{k_1, l_1 \in S} \eta_{k_1, k}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n^{n-1}]) \eta_{l_1, l}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n^{n-1}]) \mathbb{E}^{\hat{P}}[\hat{p}_{k_1, l_1}^{n-1}]
$$

(3.7)

and

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] = \sum_{k_1 \in S} \eta_{k_1, k}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n^{n-1}]) \mathbb{E}^{\hat{P}}[\hat{p}_{k_1,j}^{n-1}].
$$

(3.8)

By Lemma 3.1 we know that $\hat{\beta}_n$ is essentially pairwise independent. Again it follows by the exact law of large numbers that $\hat{p}_n(\hat{\omega}) = \mathbb{E}^{\hat{P}}[\hat{p}_n]$ for $\hat{P}$-almost all $\hat{\omega} \in \hat{\Omega}$. Then (1.9) and (1.7) are equivalent to

$$
\hat{P}^\omega(\hat{\omega} = \omega, n, \mathbb{E}^{\hat{P}}[\hat{p}_n])
$$

(3.9)

$$
\hat{P}^\omega(\hat{\omega} = \omega, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]).
$$

(3.10)

By the same calculations as in the proof of Lemma 4 in [2] we have

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] = \mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] + \theta_{k_1, k}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]) \mathbb{E}^{\hat{P}}[\hat{p}_{k_1,j}^n].
$$

(3.11)

and

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] = \mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] \mathbb{E}^{\hat{P}}[\hat{p}_{k_1,j}^n].
$$

(3.12)

By Lemma 3.1 $\hat{\beta}_n$ is essentially pairwise independent and thus $\hat{p}_n(\hat{\omega}) = \mathbb{E}^{\hat{P}}[\hat{p}_n]$ for $\hat{P}$-almost all $\hat{\omega} \in \hat{\Omega}$. Then (1.11) and (1.12) are equivalent to

$$
\hat{P}^\omega(\alpha_i^n = l_1, \bar{g}_i^n = l_2 | \alpha_i^n = k_1, \bar{g}_i^n = k_2) = (1 - \xi_{l_1, l_2}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n])) \sigma_{k_1, k_2}[l_1, l_2](\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n])
$$

and

$$
\hat{P}^\omega(\alpha_i^n = l_1, \bar{g}_i^n = J | \alpha_i^n = k_1, \bar{g}_i^n = k_2) = \xi_{k_1, k_2}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]) \sigma_{k_1, k_2}[l_1, l_2](\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]),
$$

respectively. Thus

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{kl}^n] = \sum_{k_1, l_1 \in S} (1 - \xi_{k_1, l_1}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n])) \sigma_{k_1, l_1}[k, l](\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]) \mathbb{E}^{\hat{P}}[\hat{p}_{k_1,l_1}^n]
$$

(3.13)

and

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] = \mathbb{E}^{\hat{P}}[\hat{p}_{k,j}^n] + \sum_{k_1, l_1 \in S} \xi_{k_1, l_1}(\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]) \sigma_{k_1, l_1}[k](\hat{\omega}, n, \mathbb{E}^{\hat{P}}[\hat{p}_n]) \mathbb{E}^{\hat{P}}[\hat{p}_{k_1,l_1}^n].
$$

(3.14)

By plugging (3.13) in (3.12) we get

$$
\mathbb{E}^{\hat{P}}[\hat{p}_{kl}^n]
$$
\begin{align*}
&= \sum_{k,i,l \in S} (1 - \xi_{k,i,l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n])) \sigma_{k,i,l}[k,l] \left(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]\right) \mathbb{E}^{\tilde{\omega}}[\tilde{p}^n_{k,i,l}] \\
&\quad + \sum_{k,i,l \in S} (1 - \xi_{k,i,l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n])) \eta_{k,i,l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]) \mathbb{E}^{\tilde{\omega}}[\tilde{p}^n_{k,i,l}],
\end{align*}
(3.15)

By using (3.12) and (3.13), it follows that

\begin{align*}
\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n_{k,l}] &= b_k(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]) \mathbb{E}^{\tilde{\omega}}[\tilde{p}^n_{k,l}] \\
&\quad + \sum_{k,i,l \in S} \xi_{k,i,l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]) \sigma_{k,i,l}[k] \left(\tilde{\omega},\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]\right) \mathbb{E}^{\tilde{\omega}}[\tilde{p}^n_{k,i,l}] \\
&\quad + \sum_{k,i,l \in S} \xi_{k,i,l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]) \eta_{k,i,l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^n]) \mathbb{E}^{\tilde{\omega}}[\tilde{p}^n_{k,i,l}],
\end{align*}
(3.16)

\[\square\]

**Lemma 3.3.** Assume that the discrete dynamical system \(\mathcal{D}\) defined in Definition 3.6 in [1] is Markov conditionally independent given \(\tilde{\omega} \in \tilde{\Omega}\) according to Definition Definition 3.8 in [1]. Then for fixed \(\tilde{\omega} \in \tilde{\Omega}\) the following holds:

1. For \(\lambda\)-almost all \(i \in I\), the extended type process \(\{\beta^n_i\}_{n=0}^\infty\) for agent \(i\) is a Markov chain on \((I \times \tilde{\Omega}, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbb{P}^{\tilde{\omega}})\) with transition matrix \(z^n\) at time \(n - 1\).

2. \(\{\beta^n\}_{n=0}^\infty\) is also a Markov chain with transition matrix \(z^n\) at time \(n - 1\).

**Proof.** Fix \(\tilde{\omega} \in \tilde{\Omega}\).

1. The Markov property of \(\{\beta^n_i\}_{n=0}^\infty\) on \((I \times \tilde{\Omega}, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbb{P}^{\tilde{\omega}})\) follows by using the same arguments as in the proof of Lemma 5 in [2], for \(\lambda\)-almost all \(i \in I\). We now derive the transition matrix with similar arguments as in [2]. By putting together (3.7), (3.8) and (3.15), we get

\begin{align*}
\mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}_{k,l}] &= \sum_{k',l',l \in S} \left(1 - \xi_{k,l,k'}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n})\right) \sigma_{k,l,k'}[k,l] \left(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n}\right) \eta_{k',l',l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}]) \\
&\quad \cdot \mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}_{k',l'}] \\
&\quad + \sum_{k',l',l \in S} \left(1 - \xi_{k,l,k'}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n})\right) \sigma_{k,l,k'}[k,l] \left(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n}\right) \theta_{k',l',l}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}]) \\
&\quad \cdot \mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}_{k',l'}].
\end{align*}

Thus we have

\begin{align*}
\tilde{z}^n_{(k',l')(k,l)}(\tilde{\omega}) &= \sum_{k_1,l_1 \in S} \left(1 - \xi_{k_1,l_1}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n})\right) \sigma_{k_1,l_1}[k,l] \left(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n}\right) \theta_{k,l_1,k_1}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n}) \\
&\quad \cdot \eta_{k_1,l_1}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}])
\end{align*}
(3.17)

and

\begin{align*}
\tilde{z}^n_{(k',l')(k,l)}(\tilde{\omega}) &= \sum_{k_1,l_1 \in S} \left(1 - \xi_{k_1,l_1}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n})\right) \sigma_{k_1,l_1}[k,l] \left(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n}\right) \eta_{k,l_1,k_1}(\tilde{\omega},n,\mathbb{E}^{\tilde{\omega}}[\tilde{p}^{n+1}])
\end{align*}
\[
\cdot \eta_{k,l}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n})} \cdot \eta_{l}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})}. \tag{3.18}
\]

Similarly, equations (3.7), (3.8) and (3.16) yield to
\[
\mathbb{E}^{\hat{p}_{k,l}^{n}} = \sum_{k' \in S} b_{k}(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n}) \eta_{k',k}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \mathbb{E}^{\hat{p}_{k',l}^{n-1}}
+ \sum_{k_{1},l_{1},k',l' \in S} \xi_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n})} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \mathbb{E}^{\hat{p}_{k',l'}^{n-1}}
+ \sum_{k_{1},l_{1},k' \in S} \xi_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n})} \theta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n})} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})}.
\]

Therefore, the transition probabilities from time \(n-1\) to time \(n\) can be written as
\[
z_{(k',l')}(\hat{\omega}) = \sum_{k_{1},l_{1} \in S} \xi_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n})} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \cdot \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})}. \tag{3.19}
\]
and
\[
z_{(k',l')(k,l)}(\hat{\omega}) = b_{k}(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n}) \eta_{k,l}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})}
+ \sum_{k_{1},l_{1} \in S} \xi_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{p}_{\tilde{n}}^{n})} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})}. \tag{3.20}
\]

2. The transition matrix of \(\{\beta^{n}\}_{n=0}^{\infty}\) at time \(n-1\) can be derived by using (3.17)-(3.20) and the Fubini property applied to \(\lambda \otimes \hat{P}_{\tilde{n}}\) for every fixed \(\hat{\omega} \in \hat{\Omega}\) as in the proof of Lemma 6 in [2]. We are now able to prove Theorem 3.14 in [1], which we present here.

**Theorem 3.4.** Assume that the discrete dynamical system \(\mathcal{D}\) introduced in Definition 3.6 in [4] is Markov conditionally independent given \(\hat{\omega} \in \hat{\Omega}\) according to Definition 3.8 in [4]. Given \(\hat{\omega} \in \hat{\Omega}\), the following holds:

1. For each \(n \geq 1\), \(\hat{E}^{\tilde{p}}^{n} = \Gamma^{n}(\hat{\omega}, \hat{E}^{\tilde{p}}^{n-1})\).

2. For each \(n \geq 1\), we have
\[
\mathbb{E}^{\hat{p}_{k,l}^{n}} = \sum_{k_{1},l_{1} \in S} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \eta_{k_{1},l_{1}}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \mathbb{E}^{\hat{p}_{k_{1},l_{1}}^{n-1}}
\]
and
\[
\mathbb{E}^{\hat{p}_{k,l}^{n}} = \sum_{k_{1} \in S} \eta_{k_{1},k}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \mathbb{E}^{\hat{p}_{k_{1},l}^{n-1}}.
\]

3. For each \(n \geq 1\), we have
\[
\mathbb{E}^{\hat{p}_{k,l}^{n}} = \mathbb{E}^{\hat{p}_{k,l}^{n}} + \theta_{k,l}^{(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n-1})} \mathbb{E}^{\hat{p}_{k,l}^{n}}
\]
and
\[
\mathbb{E}^{\hat{p}_{k,l}^{n}} = b_{k}(\hat{\omega}, n, \hat{E}^{\tilde{p}}^{n}) \mathbb{E}^{\hat{p}_{k,l}^{n}}.
\]

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4. For $\lambda$-almost every agent $i$, the extended-type process $\{\beta_i^n\}_{n=0}^\infty$ is a Markov chain in $\hat{S}$ on $(I \times \hat{\Omega}, \mathcal{I} \otimes \hat{\mathcal{F}}, \lambda \otimes \hat{P}\mathbb{\hat{\omega}})$, whose transition matrix $z^n$ at time $n - 1$ is given by

$$z_n^{(k',l')(k,l)}(\hat{\omega}) = \sum_{k_1, l_1, k_1' \in S} (1 - \xi_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}})) \sigma_{k_1l_1}[k, l](\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \theta_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}})$$

$$\cdot \eta_{k'k_1}(\hat{\omega}, n, \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^{n-1}]) \quad (3.21)$$

$$z_n^{(k',l')(k,l)}(\hat{\omega}) = \sum_{k_1, l_1, k_1' \in S} (1 - \xi_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}})) \sigma_{k_1l_1}[k, l](\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \eta_{k'k_1}(\hat{\omega}, n, \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^{n-1}])$$

$$\cdot \eta_{l_1l}(\hat{\omega}, n, \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^{n-1}]) \quad (3.22)$$

$$z_n^{(k',l')(k,l)}(\hat{\omega}) = \sum_{k_1, l_1 \in S} \xi_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \sigma_{k_1l_1}[k](\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \theta_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}})$$

$$\cdot \eta_{k'k_1}(\hat{\omega}, n, \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^{n-1}]) \quad (3.23)$$

$$z_n^{(k',l')(k,l)}(\hat{\omega}) = b_k(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \eta_{k'k}(\hat{\omega}, n, \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^{n-1}])$$

$$+ \sum_{k_1, l_1 \in S} \xi_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \sigma_{k_1l_1}[k](\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}}) \theta_{k_1l_1}(\hat{\omega}, n, \hat{P}\mathbb{\hat{\omega}})$$

$$\cdot \eta_{k'k_1}(\hat{\omega}, n, \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^{n-1}]). \quad (3.24)$$

5. For $\lambda$-almost every $i$ and every $\lambda$-almost every $j$, the Markov chains $\{\beta_i^n\}_{n=0}^\infty$ and $\{\beta_j^n\}_{n=0}^\infty$ are independent on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}\mathbb{\hat{\omega}})$.

6. For $\hat{P}\mathbb{\hat{\omega}}$-almost every $\hat{\omega} \in \hat{\Omega}$, the cross sectional extended type process $\{\beta^n\}_{n=0}^\infty$ is a Markov chain on $(I, \mathcal{I}, \lambda)$ with transition matrix $z^n$ at time $n - 1$, which is defined in (3.21) - (3.24).

7. We have $\hat{P}\mathbb{\hat{\omega}}$-a.s. that

$$\mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^n] = \hat{p}^n_k \quad \text{and} \quad \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^n_i] = \hat{p}^n_i \quad \text{and} \quad \mathbb{E}\hat{P}\mathbb{\hat{\omega}}[\hat{p}^n_{kl}] = \hat{p}^n_{kl}.$$

Proof. Fix $\hat{\omega} \in \hat{\Omega}$. Points 1. to 5. of Theorem 3.14 in [1] follow directly by Lemma 3.1, 3.2 and 3.3. Moreover, Points 6. and 7. can be proven by using the same arguments as in the proof of Theorem 4 in [2].

References

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