A Characterization of Product BMO by Commutators

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1 Introduction

In this paper we establish a commutator estimate which allows one to concretely identify the product BMO space, $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, of Chang and R. Fefferman, as an operator space on $L^2(\mathbb{R}^2)$. The one parameter analogue of this result is a well–known theorem of Nehari [8]. The novelty of this paper is that we discuss a situation governed by a two parameter family of dilations, and so the spaces $H^1$ and BMO have a more complicated structure.

Here $\mathbb{R}^2_+$ denotes the upper half-plane and $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is defined to be the dual of the real-variable Hardy space $H^1$ on the product domain $\mathbb{R}^2_+ \times \mathbb{R}^2_+$. There are several equivalent ways to define this latter space and the reader is referred to [5] for the various characterizations. We will be more interested in the biholomorphic analogue of $H^1$, which can be defined in terms of the boundary values of biholomorphic functions on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ and will be denoted throughout by $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+$), cf [10].

In one variable, the space $L^2(\mathbb{R})$ decomposes as the direct sum $H^2(\mathbb{R}) \oplus \overline{H^2(\mathbb{R})}$ where $H^2(\mathbb{R})$ is defined as the boundary values of functions in $H^2(\mathbb{R})$ and $\overline{H^2(\mathbb{R})}$ denotes the space of complex conjugate of functions in $H^2(\mathbb{R})$. The space $L^2(\mathbb{R}^2)$, therefore, decomposes as the direct sum of the four spaces $H^2(\mathbb{R}) \otimes H^2(\mathbb{R})$, $\overline{H^2(\mathbb{R})} \otimes H^2(\mathbb{R})$, $H^2(\mathbb{R}) \otimes \overline{H^2(\mathbb{R})}$ and $\overline{H^2(\mathbb{R})} \otimes \overline{H^2(\mathbb{R})}$ where the tensor products are the Hilbert space tensor products. Let $P_{\pm,\pm}$ denote the orthogonal projection of $L^2(\mathbb{R}^2)$ onto the holomorphic/anti-holomorphic subspaces, in

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the first and second variables, respectively, and let $H_j$ denote the one–dimensional Hilbert transform in the $j^{th}$ variable, $j = 1, 2$. In terms of the projections $P_{\pm,\pm}$,

$$H_1 = P_{+,+} + P_{+,-} - P_{-,+} - P_{--} \quad \text{and} \quad H_2 = P_{+,+} + P_{-,+} - P_{+-} - P_{--,}.$$ 

The nested commutator determined by the function $b$ is the operator $[[M_b, H_1], H_2]$ acting on $L^2(\mathbb{R}^2)$ where, for a function $b$ on the plane, we define $M_b f := bf$.

In terms of the projections $P_{\pm,\pm}$, it takes the form

$$\frac{1}{4}[[[M_b, H_1], H_2] = P_{+,+}M_bP_{--,} - P_{+,-}M_bP_{+-} - P_{-,+}M_bP_{--} + P_{--,}M_bP_{-,+}. \tag{1.1}$$

Ferguson and Sadosky [4] established the inequality $\|[[M_b, H_1], H_2]\|_{L^2} \leq c\|b\|_{BMO}$. The main result is the converse inequality.

1.2. Theorem. There is a constant $c > 0$ such that $\|b\|_{BMO} \leq c\|[[M_b, H_1], H_2]\|_{L^2 \rightarrow L^2}$ for all functions $b$ in $BMO(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$.

As A. Chang and R. Fefferman have established for us, the structure of the space $BMO$ is more complicated in the two parameter setting, requiring a more subtle approach to this theorem, despite the superficial similarity of the results to the one parameter setting. The proof relies on three key ideas. The first is the dyadic characterization of the $BMO$ norm given in [1]. The second is a variant of Journé’s lemma, [6], (whose proof is included in the appendix.) The third idea is that we have the estimates, the second of which was shown in [4],

$$\|b\|_{BMO(rect)} \leq c\|[[M_b, H_1], H_2]\|_{L^2 \rightarrow L^2} \leq c'\|b\|_{BMO}.$$

An unpublished example of L. Carleson shows that the rectangular $BMO$ norm is not comparable to the $BMO$ norm, [3]. We may assume that the rectangular $BMO$ norm of the function $b$ is small. Indeed, this turns out to be an essential aspect of the argument.

From theorem 1.2 we deduce a weak factorization for the (biholomorphic) space $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$. The idea is that if the function $b$ has biholomorphic extension to $\mathbb{R}_+^2 \times \mathbb{R}_+$ then for functions $f, g \in L^2(\mathbb{R}^2)$,

$$\frac{1}{4}\left\langle [[M_b, H_1], H_2]f, g \right\rangle = \langle b, \overline{P_{--}f}P_{-,+}g \rangle.$$
So in this case, the operator norm of the nested commutator $[[M_b, H_1], H_2]$ is comparable to the dual norm

$$
\|b\|_* := \sup |\langle fg, b \rangle|
$$

where the supremum above is over all pairs $f, g$ in the unit ball of $H^2(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$. On the other hand, since $\|b\|_{BMO}$ and $\|[[M_b, H_1], H_2]\|_{L^2 \to L^2}$ are comparable, the dual norm above satisfies

$$
\|b\|_* \sim \sup |\langle h, b \rangle|
$$

where the supremum is over all functions $h$ in the unit ball of $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$.

1.3. Corollary. Let $h$ be in $H^1(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ with $\|h\|_1 = 1$. Then there exists functions $(f_j), (g_j) \subseteq H^2(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ such that $h = \sum_{j=1}^{\infty} f_j g_j$ and $\sum_{j=1}^{\infty} \|f_j\|_2 \|g_j\|_2 \leq c$.

We remark that the weak factorization above implies the analogous factorization for $H^1$ of the bidisk. Indeed, for all $1 \leq p < \infty$, the map $u_p : H^p(\mathbb{R}_+^2 \times \mathbb{R}_+^2) \to H^p(\mathbb{D}^2)$ defined by

$$(u_p f)(z, w) = \pi^{2/p} \left( \frac{2i}{1 - z} \right)^{2/p} \left( \frac{2i}{1 - w} \right)^{2/p} f(\alpha(z), \alpha(w)) \quad \alpha(\lambda) := i \frac{1 + \lambda}{1 - \lambda},$$

is an isometry with isometric inverse

$$(u_p^{-1} g)(z, w) = \pi^{-2/p} \left( \frac{1}{z + i} \right)^{2/p} \left( \frac{1}{w + i} \right)^{2/p} g(\beta(z), \beta(w)) \quad \beta(\lambda) := \lambda - i \frac{\lambda + i}{\lambda + i}.$$

The dual formulation of weak factorization for $H^1(\mathbb{D}^2)$ is a Nehari theorem for the bidisk. Specifically, if $b \in H^2(\mathbb{D}^2)$ then the little Hankel operator with symbol $b$ is densely defined on $H^2(\mathbb{D}^2)$ by the formula

$$
\Gamma_b f = P_{-\cdot} (\overline{b} f).
$$

By (1.1), $\|\Gamma_b\| = \|[M_b, H_1], H_2\|_{L^2 \to L^2}$ and thus, by theorem 1.2, $\|\Gamma_b\|$ is comparable to $\|b\|_{BMO}$ which, by definition, is just the norm of $b$ acting on $H^1(\mathbb{D}^2)$. So the boundedness of the Hankel operator $\Gamma_b$ implies that there is a function $\phi \in L^\infty(\mathbb{T}^2)$ such that $P_{+,+} \phi = b$.

Several variations and complements on these themes in the one parameter setting have been obtained by Coifman, Rochberg and Weiss [2].
The paper is organized as follows. Section 2 gives the one-dimensional preliminaries for the proof of theorem 1.2 and Section 3 is devoted to the proof of theorem 1.2. The appendix contains the variant of Journé’s lemma we use in our proof in Section 3. One final remark about notation. \( A \preceq B \) means that there is an absolute constant \( C \) for which \( A \leq CB \). \( A \approx B \) means that \( A \preceq B \) and \( B \preceq A \).

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## 2 Remarks on the one dimensional case

Several factors conspire to make the one dimensional case easier than the higher dimensional case. Before proceeding with the higher dimensional case, we make several comments about the one dimensional case, comments that extend and will be useful in the subsequent section.

Let \( H \) denote the Hilbert transform in one variable, \( P_+ = \frac{1}{2}(I + H) \) be the projection of \( L^2(\mathbb{R}) \) onto the positive frequencies, and \( P_- \) is \( \frac{1}{2}(I - H) \), the projection onto the negative frequencies. We shall in particular rely upon the following basic computation.

\[
\frac{1}{2}[M_b, H]b = P_- |P_- b|^2 - P_+ |P_+ b|^2. 
\]

The frequency distribution of \( |P_- b|^2 \) is symmetric since it is real valued. Thus,

\[
\|b\|_4^2 \leq \|P_- |P_- b|^2 - P_+ |P_+ b|^2\|_2 \\
\leq \|[M_b, H]\|_{2 \to 2} \|b\|_2
\]

Moreover, if \( b \) is supported on an interval \( I \), we see that

\[
\|b\|_2 \leq |I|^{1/4} \|b\|_4 \leq |I|^{1/4} \|[M_b, H]\|_{2 \to 2}^{1/2} \|b\|_2^{1/2}
\]

which is the \( BMO \) estimate on \( I \). We seek an extension of this estimate in the two parameter setting.

We use a wavelet proof of theorem 1.2, and specifically use a wavelet with compact frequency support constructed by Y. Meyer [7]. There is a Schwarz function \( w \) with these properties.

- \( \|w\|_2 = 1 \).
\( \hat{w}(\xi) \) is supported on \([2/3, 8/3]\) together with the symmetric interval about 0.

- \( P_{\pm} w \) is a Schwartz function. More particularly, we have
  \[
  |w(x)|, |P_{\pm} w(x)| \leq (1 + |x|)^{-n}, \quad n \geq 1.
  \]

Let \( \mathcal{D} \) denote a collection of dyadic intervals on \( \mathbb{R} \). For any interval \( I \), let \( c(I) \) denote its center, and define

\[
  w_I(x) := \frac{1}{\sqrt{|I|}} w\left(\frac{x - c(I)}{|I|}\right).
\]

Set \( w_i^\pm := P_{\pm} w_I \). The central facts that we need about the functions \( \{w_I : I \in \mathcal{D}\} \) are these.

First, that these functions are an orthonormal basis on \( L^2(\mathbb{R}) \). Second, that we have the Littlewood–Paley inequalities, valid on all \( L^p \), though \( p = 4 \) will be of special significance for us. These inequalities are

\[
\|f\|_p \approx \left\| \left[ \sum_{I \in \mathcal{D}} \left| \frac{|f, w_I|^2}{|I|} \right| 1_I \right]^{1/2} \right\|_p, \quad 1 < p < \infty.
\]

Third, that the functions \( w_I \) have good localization properties in the spatial variables. That is,

\[
|w_I(x)|, |w^+_I(x)| \leq |I|^{-1/2} \chi_I(x)^n, \quad n \geq 1,
\]

where \( \chi_I(x) := (1 + \text{dist}(x, I)/|I|)^{-1} \). We find the compact localization of the wavelets in frequency to be very useful. The price we pay for this utility below is the careful accounting of “Schwartz tails” we shall make in the main argument. Fifth, we have the identity below for the commutator of one \( w_I \) with a \( w_J \). Observe that since \( P_{+} \) is one half of \( I + H \), it suffices to replace \( H \) by \( P_{+} \) in the definition of the commutator.

\[
  w_{I, J} := [w_I, P_+] w_J
  = w_IW_J - P_+ w_I w_J^+
  = P_+ w_I w_J - P_+ w_I w_J^+
  = P_+ w_I w_J - P_+ w_I w_J^+
\]
From this we see a useful point concerning orthogonality. For intervals $I, I', J$ and $J'$, assume $|J| \geq 8|I|$ and likewise for $I'$ and $J'$. Then

\begin{equation}
\text{supp}(\hat{w}_{I,J}) \cap \text{supp}(\hat{w}_{I',J'}) = \emptyset, \quad |I'| \geq 8|I|.
\end{equation}

Indeed, this follows from a direct calculation. The positive frequency support of $\hat{w}_I\hat{w}_J$ is contained in the interval $[(3|I|)^{-1}, 8(3|I|)^{-1}]$. Under the conditions on $I$ and $I'$, the frequency supports are disjoint.

### 3 Proof of the main theorem

$BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ will denote the $BMO$ of two parameters (or product $BMO$) defined as the dual of (real) $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$.

The following characterization of the space $BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is due to Chang and R. Fefferman [1].

The relevant class of rectangles is $\mathcal{R} = \mathcal{D} \times \mathcal{D}$, all rectangles which are products of dyadic intervals. These are indexed by $R \in \mathcal{R}$. For such a rectangle, write it as a product $R_1 \times R_2$ and then define

$$v_R(x_1, x_2) = w_{R_1}(x_1)w_{R_2}(x_2).$$

A function $f \in BMO(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ iff

$$\sup_U \left( |U|^{-1} \sum_{R \in U} |\langle f, v_R \rangle|^2 \right)^{1/2} < \infty.$$

Here, the sum extends over those rectangles $R \in \mathcal{R}$ and the supremum is over all open sets in the plane of finite measure. Note that the supremum is taken over a much broader class of sets than merely rectangles in the plane. We denote this supremum as $\|f\|_{BMO}$.

In this definition, if the supremum over $U$ is restricted to just rectangles, this defines the “rectangular $BMO$” space, and we denote this restricted supremum as $\|f\|_{BMO(\text{rec})}$. 
Let us make a further comment on the \( BMO \) condition. Suppose that for \( R \in \mathcal{R} \), we have non-negative constants \( a_R \) for which
\[
\sum_{R \subset U} a_R \leq |U|,
\]
for all open sets \( U \) in the plane of finite measure. Then, we have the John–Nirenberg inequality
\[
\left\| \sum_{R \subset U} |R|^{-1} a_R 1_R \right\|_p \leq |U|^{1/p}, \quad 1 < p < \infty.
\]
See [1]. This, with the Littlewood–Paley inequalities, will be used several times below, and referred to as the John–Nirenberg inequalities.

### 3.1 The Principal Points in the Argument

We begin the principle line of the argument. The function \( b \) may be taken to be of Schwarz class. By multiplying \( b \) by a constant, we can assume that the \( BMO \) norm of \( b \) is 1. Set \( B_{2 \rightarrow 2} \) to be the operator norm of \([M_b, H_1], H_2\). Our purpose is to provide a lower bound for \( B_{2 \rightarrow 2} \). Let \( U \) be an open set of finite measure for which we have the equality
\[
\sum_{R \subset U} |\langle b, v_R \rangle|^2 = |U|.
\]
As \( b \) is of Schwarz class, such a set exists. By invariance under dilations by a factor of two, we can assume that \( \frac{1}{2} \leq |U| \leq 1 \). In several estimates below, the measure of \( U \) enters in, a fact which we need not keep track of.

An essential point is that we may assume that the rectangular \( BMO \) norm of \( b \) is at most \( \epsilon \). The reason for this is that we have the estimate \( \|b\|_{BMO(\text{rec})} \leq B_{2 \rightarrow 2} \). See [4]. Therefore, for a small constant \( \epsilon \) to be chosen below, we can assume that \( \|b\|_{BMO(\text{rec})} \leq \epsilon \), for otherwise we have a lower bound on \( B_{2 \rightarrow 2} \).

Associated to the set \( U \) is the set \( V \), defined below, which is an expansion of the set \( U \). In defining this expansion, it is critical that the measure of \( V \) be only slightly larger than the measure of \( U \), and so in particular we do not use the strong maximal function to define this.
expansion. In the definition of $V$, the parameter $\delta > 0$ will be specified later and $M_j$ is the one dimensional maximal function applied in the direction $j$. Define

$$V_{ij} := \{M_i1_{\{M_j1U>\delta\}} > \delta\},$$
$$V := V_{12} \cup V_{21},$$
$$\mu(R) := \sup\{\mu : \mu R \subset V\} \quad R \subset U.$$  

The quantity $\mu(R)$ measures how deeply a rectangle $R$ is inside $V$. This quantity enters into the essential Journé’s Lemma, see [6] or the variant we prove in the appendix.

In the argument below, we will be projecting $b$ onto subspaces spanned by collections of wavelets. These wavelets are in turn indexed by collections of rectangles. Thus, for a collection $\mathcal{A} \subseteq \mathcal{R}$, let us denote

$$b^A := \sum_{R \in \mathcal{A}} \langle b, v_R \rangle v_R.$$  

The relevant collections of rectangles are defined as

$$\mathcal{U} := \{R \in \mathcal{R} : R \subset U\},$$
$$\mathcal{V} = \{R \in \mathcal{R} - \mathcal{U} : R \subset V\},$$
$$\mathcal{W} = \mathcal{R} - \mathcal{U} - \mathcal{V}.$$  

For functions $f$ and $g$, we set $\{f, g\} := [[M_f, H_1], H_2]g$.

We will demonstrate that for all $\delta, \epsilon > 0$ there is a constant $K_\delta > 0$ so that

(i) $\|\{b^\mathcal{V}, b^\mathcal{U}\}\|_2 \leq \delta^{1/8}$

(ii) $\|\{b^\mathcal{W}, b^\mathcal{U}\}\|_2 \leq K_\delta \epsilon^{1/3}$

Furthermore, we will show that $1 \leq \|\{b^\mathcal{U}, b^\mathcal{U}\}\|_2$. Since $b = b^\mathcal{U} + b^\mathcal{V} + b^\mathcal{W}$, $\|b^\mathcal{U}\|_2 \leq 1$ and $\delta, \epsilon > 0$ are arbitrary, a lower bound on $B_{2 \to 2}$ will follow from an appropriate choice of $\delta$ and $\epsilon$. To be specific, one concludes the argument by estimating

$$1 \leq \|\{b^\mathcal{U}, b^\mathcal{U}\}\|_2$$
$$\leq \|\{b^\mathcal{U} + b^\mathcal{V}, b^\mathcal{U}\}\|_2 + \delta^{1/8}$$
$$\leq \|\{b^\mathcal{U} + b^\mathcal{V} + b^\mathcal{W}, b^\mathcal{U}\}\|_2 + \delta^{1/8} + K_\delta \epsilon^{1/3}$$
\[ B_{2 \to 2} \leq B_{2 \to 2} + \delta^{1/8} + K_3 \epsilon^{1/3} \]

Implied constants are absolute. Choosing \( \delta \) first and then \( \epsilon \) appropriately small supplies a lower bound on \( B_{2 \to 2} \).

The estimate \( 1 \leq \| \{ b^\prime, b^\prime \} \|_2 \) relies on the John–Nirenberg inequality and the two parameter version of (2.1), namely
\[
\frac{1}{4} \| [M_b, H_1], H_2 b \| = P_{+,+}|P_{+,+}b|^2 - P_{+-+-}|P_{+-+-}b|^2 + P_{-,+}|P_{-,+}b|^2
\]
This identity easily follows from the one-variable identities. Here \( P_{\pm,\pm} \) denotes the projection onto the positive/negative frequencies in the first and second variables. These projections are orthogonal and moreover, since \( \| P_{\pm,\pm}b \|^2 \) is real valued we have that
\[
\| P_{\pm,\pm}b \|^2 \geq \frac{1}{4} \| P_{\pm,\pm}b \|^2
\]
Therefore, \( \| b^\prime \|_2^4 \leq \| \{ b^\prime, b^\prime \} \|_2^2. \) It follows by the John–Nirenberg inequality that
\[
1 \leq \| b^\prime \|_2^2 = \left[ \sum_{R \in \mathcal{U}} | \langle b, v_R \rangle |^2 \right]^{1/2}
\leq \left\| \left[ \sum_{R \in \mathcal{U}} | \langle b, v_R \rangle |^2 1_R \right] \right\|_4^{1/2}
\leq \| b^\prime \|_4^4
\leq \| \{ b^\prime, b^\prime \} \|_2
\]

The estimate (i) relies on the fact that the one dimensional maximal function maps \( L^1 \) into weak \( L^1 \) with norm one. Thus, for all \( 0 < \delta < 1/2, \)
\[
| \{ M_2 1_{\{ M_1 1_{U>1-\delta} \}} \} | \leq 1 - \delta \]
\[
| \{ M_2 1_{\{ M_1 1_{U>1-\delta} \}} \} | \leq 1 - \delta \]
\[
\leq (1 - \delta)^{-2} |U| \leq (1 + 6\delta) |U|.
\]
Now, if \( R \in \mathcal{V} \), then \( R \subset V \) and since \( b \) has \( BMO \) norm one, it follows that
\[
|U| + \| b^\prime \|_2^2 = \sum_{R \in \mathcal{U} \cup \mathcal{V}} | \langle b, v_R \rangle |^2 \leq (1 + 6\delta) |U|.
\]
Hence \( \| b^\prime \|_2 \leq \delta^{1/2} \). Yet the \( BMO \) norm of \( b^\prime \) can be no more than that of \( b \), which is to say 1. Interpolating norms we see that \( \| b^\prime \|_4^4 \leq \delta^{1/4} \), and so
\[
\| \{ b^\prime, b^\prime \} \|_2 \leq \| b^\prime \|_4 \| b^\prime \|_4 \leq \delta^{1/4}.
\]
3.2 Verifying the Estimate (ii)

We now turn to the estimate (ii). Roughly speaking $\hat{b}'$ and $\hat{b}$ live on disjoint sets. But in this argument we are trading off precise Fourier support of the wavelets for imprecise spatial localization, that is the "Schwartz tails" problem. Accounting for this requires a careful analysis, invoking several subcases.

A property of the commutator that we will rely upon is that it controls the geometry of $R$ and $R'$. Namely, $\{v_{R'}, v_R\} \neq 0$ iff writing $R = R_1 \times R_2$ and likewise for $R'$, we have for both $j = 1, 2$, $|R'_j| \leq 4|R_j|$. This follows immediately from our one–dimensional calculations, in particular (2.4). We abbreviate this condition on $R$ and $R'$ as $R' \preceq R$ and restrict our attention to this case.

Orthogonality also enters into the argument. Observe the following. For rectangles $R^k, \tilde{R}_k, k = 1, 2, \tilde{R}_k \preceq R^k$, and for $j = 1$ or $j = 2$

\[
\langle v_{\tilde{R}_1^j}, v_{R_1^j} \rangle = 0 \text{ (3.1)}
\]

This follows from applying (2.5) in the $j$th coordinate.

Therefore, there are different partial orders on rectangles that are relevant to our argument. They are

- $R' < R$ iff $8|R'_j| \leq |R_j|$ for $j = 1$ and $j = 2$.
- For $j = 1$ or $j = 2$, define $R' <_j R$ iff $R' \preceq R$ and $8|R'_j| \leq |R_j|$ but $R' \not\preceq R$.
- $R' \simeq R$ iff $\frac{1}{4}|R_j| \leq |R'_j| \leq |R_j|$ for $j = 1$ and $j = 2$.

These four partial orders divide the collection $\{(R', R) : R' \in W, R \in U, R' \preceq R\}$ into four subclasses which require different arguments.

In each of these four arguments, we have recourse to this definition. Set $U_k$, for $k = 0, 1, 2, \ldots$ to be those rectangles in $U$ with $2^{-k-1} < \mu(U_k) \leq 2^k$.

Journé’s Lemma enters into the considerations. Let $U' \subset U_k$ be a collection of rectangles which are pairwise incomparable with respect to inclusion. For this latter collection, we have the inequality

\[
\sum_{R \in U'} |R| \leq K4^{2k/100} \left| \bigcup_{R \in U'} R \right|.
\]
See Journé [6], also see the appendix. This together with the assumption that \( b \) has small rectangular \( BMO \) norm gives us

\[
\|b^U_k\|_{BMO} \leq K_\delta 2^{k/100}\epsilon. \tag{3.3}
\]

This interplay between the small rectangular \( BMO \) norm and Journé’s Lemma is a decisive feature of the argument.

Essentially, the decomposition into the collections \( \mathcal{U}_k \) is a spatial decomposition of the collection \( \mathcal{U} \). A corresponding decomposition of \( \mathcal{W} \) enters in. Yet the definition of this class differs slightly depending on the partial order we are considering.

For \( R' \in \mathcal{W} \) and \( R \in \mathcal{U} \) the term \( \{v_{R'}, v^R\} \) is a linear combination of

\[
v_{R'}H_2H_1\overline{v_{R}}, \quad H_2(v_{R'}H_1\overline{v_{R}}), \quad (H_1v_{R'})(H_2\overline{v_{R}}), \quad H_1H_2(v_{R'}\overline{v_{R}}). \]

Consider the last term. As we are to estimate an \( L^2 \) norm, the leading operators \( H_1H_2 \) can be ignored. Moreover, the essential properties of wavelets used below still hold for the conjugates and Hilbert transforms of the same. These properties are Fourier localization and spatial localization. Similar comments apply to the other three terms, and so the arguments below applies to each type of term separately.

### 3.2.1 The partial order ‘<’

We consider the case of \( R' < R \) for \( R' \in \mathcal{W} \) and \( R \in \mathcal{U} \). The sums we considering are related to the following definition. Set

\[
b_{\text{trun}}(x) := \sup_{R' \in \mathcal{W}, R < R'} \left| \sum_{R \in \mathcal{U}_k} \langle b, v_R \rangle v_R(x) \right|. \]

Note that we consider the maximal truncation of the sum over all choices of dimensions of the rectangles in the sum. Thus, this sum is closely related to the strong maximal function \( M \) applied to \( b^U_k \), so that in particular we have the estimate below, which relies upon (3.3).

\[
\|b_{\text{trun}}^U\|_p \leq \epsilon 2^{k/100}, \quad 1 < p < \infty.
\]
(By a suitable definition of the strong maximal function $M$, one can deduce this inequality from the $L^p$ bounds for $M$.) We apply this inequality far away from the set $U$. For the set $W = \mathbb{R}^2 - \bigcup_{R \in U_k} \lambda R$, $\lambda > 1$, we have the inequality

$$
\|b_{U_{k_{\text{trun}}}}\|_{L^p(W)} \leq c 2^{k/100} \lambda^{-100}, \quad 1 < p < \infty.
$$

(3.4)

We shall need a refined decomposition of the collection $W$, the motivation for which is the following calculation. Let $W' \subset W$. For $n = (n_1, n_2) \in \mathbb{Z}^2$, set $W'(n) := \{ R' \in W' : |R'_j| = 2^{n_j}, j = 1, 2 \}$. In addition, let

$$
B(W', n) := \sum_{R' \in W'(n)} \sum_{R \in U_k \atop R' < R} \langle b, v_{R'} \rangle \langle b, v_R \rangle v_{R'} v_R.
$$

And set $B(W') = \sum_{n \in \mathbb{Z}^2} B(W', n)$.

Then, in view of (3.1), we see that $B(W', n)$ and $B(W', n')$ are orthogonal if $n$ and $n'$ differ by at least 3 in either coordinate. Thus,

$$
\left\| \sum_{n \in \mathbb{Z}^2} B(W', n) \right\|^2 \leq 3 \sum_{n \in \mathbb{Z}^2} \| B(W', n) \|^2.
$$

The rectangles $R' \in W(n)$ are all translates of one another. Thus, taking advantage of the rapid spatial decay of the wavelets, we can estimate

$$
\| B(W', n) \|^2 \leq \sum_{R' \in W(n)} \int \left| \frac{\langle b, v_{R'} \rangle}{\sqrt{|R'|}} \chi_{R'} \right|^2 (\chi_{R'} \ast 1_{R'}) b_{U_{k_{\text{trun}}}}^2 \, dx
$$

In this display, we let $\chi(x_1, x_2) = (1 + x_1^2 + x_2^2)^{-10}$ and for rectangles $R$, $\chi_R(x_1, x_2) = \chi(x_1|R_1|^{-1}, x_2|R_2|^{-1})$. Note that $\chi_R$ depends only on the dimensions of $R$ and not its location.

Continuing, note the trivial inequality $\int (\chi_R \ast f)^2 g \, dx \leq \int |f|^2 \chi_R \ast g \, dx$. We can estimate

$$
\| B(W') \|^2 \leq \sum_{R' \in W'} |\langle b, v_{R'} \rangle|^2 \left\{ |R'|^{-1} \int_{R'} M(b_{U_{k_{\text{trun}}}}^2) \, dx \right\}
$$

(3.5)

$$
\leq \left| \bigcup_{R' \in W'} R' \right| \sup_{R' \in W'} \text{avg}(R').
$$
Here we take \( \text{avg}(R') := |R'|^{-1} \int_{R'} M(|b'_{\text{trun}}|^2) \).

The terms \( \text{avg}(R') \) are essentially of the order of magnitude \( \epsilon^2 \) times a the scaled distance between \( R' \) and the open set \( U \). To make this precise requires a decomposition of the collection \( \mathcal{W} \).

For integers \( l > k \) and \( m \geq 0 \), set \( \mathcal{W}(l,m) \) to be those \( R' \in \mathcal{W} \) which satisfy these three conditions:

- First, \( \text{avg}(R') \leq \epsilon 2^{-4l} \) if \( m = 0 \) and \( \epsilon 2^{-4l+m-1} < \text{avg}(R') \leq \epsilon 2^{-4l+m} \) if \( m > 0 \).
- Second, there is an \( R \in \mathcal{U}_k \) with \( R' < R \) and \( R' \subset 2^{l+1}R \).
- Third, for every \( R \in \mathcal{U}_k \) with \( R' < R \), we have \( R' \not\subset 2^{l+1}R \). Certainly, this collection of rectangles is empty if \( l \leq k \).

We see that

\[
\left| \bigcup_{R' \in \mathcal{W}(l,m)} R' \right| \lesssim \min(2^{2lp}, 2^{-mp/2}), \quad 1 < p < \infty.
\]

The first estimate follows since the rectangles \( R' \in \mathcal{W}(l,m) \) are contained in the set \( \{M1_U \geq 2^{-2l-1}\} \). The second estimate follows from (3.4).

But then from (3.5) we see that for \( m > 0 \),

\[
\|B(\mathcal{W}(l,m))\|_2^2 \lesssim \epsilon 2^{-4l+m} \min(2^{2lp}, 2^{-mp/2}) \lesssim \epsilon 2^{-(m+l)/10}.
\]

In the case that \( m = 0 \), we have the bound \( 2^{2lp} \). This is obtained by taking the minimum to be \( 2^{2lp} \) for \( p = 5/4 \) and \( 0 < m < \frac{11}{8}l \). For \( m \geq \frac{11}{8}l \) take the minimum to be \( 2^{-mp/2} \) with \( p = 4 \).

This last estimate is summable over \( 0 < k < l \) and \( 0 < m \) to at most \( \lesssim \epsilon \), and so completes this case.

### 3.2.2 The Partial Orders ‘\(<_j\), \( j = 1, 2 \).

We treat the case of \( R' <_1 R \), while the case of \( R' <_2 R \) is same by symmetry. The structure of this partial order provides some orthogonality in the first variable, leaving none in the
second variable. Bounds for the expressions from the second variable are derived from a cognate of a Carleson measure estimate.

There is a basic calculation that we perform for a subset $W' \subset W$. For an integer $n' \in \mathbb{Z}$ define $W'(n') := \{ R' \in W' : |R'_1| = 2^{n'} \}$, and

$$B(W', n') := \sum_{R' \in W'(n')} \sum_{\substack{R \in \mathcal{U}_k \ R \subset R' \ \text{and} \ R < 1 \ R}} \langle b, v_{R'} \rangle \overline{v_R} v_{R'}.$$ 

Recalling (3.1), if $n'$ and $n''$ differ by more than 3, then $B(W', n')$ and $B(W', n'')$ are orthogonal.

Observe that for $R'$ and $R$ as in the sum defining $B(W', n)$, we have the estimate

$$(3.6) \quad |v_{R'}(x)\overline{v_R}(x)| \leq (|R||R'|)^{-1/2} \text{dist}(R', R)^{1000} \chi_{R'} * 1_{R'}(x), \quad x \in \mathbb{R}^2.$$ 

In this display, we are using the same notation as before, $\chi(x_1, x_2) = (1 + x_1^2 + x_2^2)^{-10}$ and for rectangles $R$, $\chi_R(x_1, x_2) = \chi(x_1|R_1|^{-1}, x_2|R_2|^{-1})$. In addition, dist$(R', R) := M1_R(c(R'))$, with $c(R')$ being the center of $R'$. [This “distance” is more properly the inverse of a distance that takes into account the scale of the rectangle $R$.]

Now define

$$(3.7) \quad \beta(R') := \sum_{\substack{R \in \mathcal{U} \ R \subset R' \ \text{and} \ R < 1 \ R}} |R|^{-1/2} |\langle b, v_{R'} \rangle| \text{dist}(R', R)^{1000}.$$ 

The main point of these observations and definitions is this. For the function $B(W') := \sum_{n' \in \mathbb{Z}} B(W', n')$, we have

$$\|B\|_2^2 \leq \sum_{n' \in \mathbb{Z}} \|B(W', n')\|_2^2 \leq \sum_{n' \in \mathbb{Z}} \int_{\mathbb{R}^2} \left[ \sum_{R' \in W'(n')} |\langle b, v_{R'} \rangle| \beta(R') |R'|^{-1/2} \chi_{R'} * 1_{R'} \right]^2 \ dx \leq \sum_{n' \in \mathbb{Z}} \int_{\mathbb{R}^2} \left[ \sum_{R' \in W'(n')} |\langle b, v_{R'} \rangle| \beta(R') |R'|^{-1/2} \chi_{R'} * 1_{R'} \right]^2 \ dx.$$ 

At this point, it occurs to one to appeal to the Carleson measure property associated to the coefficients $|\langle b, v_{R'} \rangle||R'|^{-1/2}$. This necessitates that one proves that the coefficients $\beta(R')$
satisfy a similar condition, which doesn’t seem to be true in general. A slightly weaker condition is however true.

To get around this difficulty, we make a further diagonalization of the terms $\beta(R')$ above. For integers $\nu \geq \nu_0$, $\mu \geq 1$ and a rectangle $R' \in \mathcal{W}$, consider rectangles $R \in \mathcal{U}_k$ such that

$$R' < R, \quad 2^{-\nu} \leq \text{dist}(R', R) \leq 2^{-\nu+1}, \quad 2^\mu |R'| = |R|.$$ 

[The quantity $\nu_0$ depends upon the particular subcollection $\mathcal{W}'$ we are considering.] We denote one of these rectangles as $\pi(R')$.

An important geometrical fact is this. We have $\pi(R') \subset 2^{2v+\mu+10}R' \times 2^{v+10}R'_2$. And in particular, this last rectangle has measure $\leq 2^{2v+\mu}|R'|$.

Therefore, there are at most $O(2^{2v})$ possible choices for $\pi(R')$. [Small integral powers of $2^v$ are completely harmless because of the large power of $\text{dist}(R', R)$ that appears in (3.7).]

Our purpose is to bound this next expression by a term which includes a power of $\epsilon$, a small power of $2^v$ and a power of $2^{-\mu}$. Define

$$S(\mathcal{W}', \nu, \mu) := \sum_{n' \in \mathbb{Z}} \int_{\mathbb{R}^2} \left[ \sum_{R' \in \mathcal{W}'(n')} \left| \frac{\langle b, v_{R'} \rangle \langle b, v_{\pi(R')} \rangle}{\sqrt{|R'||\pi(R')}|} \chi_{R'} \ast 1_{R'} \right|^2 \right] dx$$

$$\leq \sum_{n' \in \mathbb{Z}} \int_{\mathbb{R}^2} \left[ \sum_{R' \in \mathcal{W}'(n')} \left| \frac{\langle b, v_{R'} \rangle \langle b, v_{\pi(R')} \rangle}{\sqrt{|R'||\pi(R')}|} \right|^2 \right] dx$$

$$= \sum_{n' \in \mathbb{Z}} \sum_{R' \in \mathcal{W}'(n')} \left| \frac{\langle b, v_{R'} \rangle \langle b, v_{\pi(R')} \rangle}{\sqrt{|R'||\pi(R')}|} \right| \sum_{R'' \in \mathcal{W}'(n')} \sqrt{\frac{|R''|}{|\pi(R'')|}} \left| \langle b, v_{R''} \rangle \langle b, v_{\pi(R'')} \rangle \right|$$

The innermost sum can be bounded this way. First $\|b\|_{BMO} \leq \epsilon$, so that

$$\sum_{R'' \subset R'} |\langle b, v_{R''} \rangle|^2 \leq \epsilon^2 |R'|.$$ 

Second, by our geometrical observation about $\pi(R')$,

$$\sum_{R'' \subset R'} \left| \frac{|R''|}{|\pi(R'')|} \right| \left| \langle b, v_{\pi(R'')} \rangle \langle b, v_{\pi(R'')} \rangle \right|^2 \leq \epsilon^2 2^{2v|R'|}.$$
In particular, the factor $2^u$ does not enter into this estimate.

This means that

$$S(W, v, \mu) \leq \epsilon^2 2^{2v} \sum_{R' \in W'} \sqrt{\frac{|R'|}{\pi(R')}} |\langle b, v_{R'} \rangle \langle b, v_{\pi(R')} \rangle|$$

$$\leq \epsilon^2 2^{2v-\mu/2} \left[ \sum_{R' \in W'} |\langle b, v_{R'} \rangle|^2 \sum_{R \in U_k} |\langle b, v_{R} \rangle|^2 \right]^{1/2}$$

$$\leq \epsilon^2 2^{2v-\mu/2} \left| \bigcup_{R' \in W'} R' \right|^{1/2}$$

The point of these computations is that a further trivial application of the Cauchy–Schwartz
inequality proves that

$$\| B(W') \|_2 \leq \epsilon 2^{-100\nu_0} \left| \bigcup_{R' \in W'} R' \right|^{1/4}$$

where $\nu_0$ is the largest integer such that for all $R' \in W'$ and $R \in U_k$, we have $\text{dist}(R', R) \leq 2^{-\nu_0}$.

We shall complete this section by decomposing $W$ into subcollections for which this last
estimate summable to $\epsilon 2^{-k}$. Indeed, take $W_v$ to be those $R' \in W$ with $R' \not\subset 2^v R$ for all
$R \in U_k$ with $R' <_1 R$. And there is an $R \in U_k$ with $R' \subset 2^{v+1} R$ and $R' <_1 R$. Certainly, we
need only consider $v \geq k$.

It is clear that this decomposition of $W$ will conclude the treatment of this partial order.

### 3.2.3 The partial order ‘≃’

We now consider the case of $R' \simeq R$, which is less subtle as there is no orthogonality to exploit
and the Carleson measure estimates are more directly applicable. We prove the bound

$$\left\| \sum_{R' \in W} \sum_{R \in \mathcal{U}} \langle b, v_{R'} \rangle \langle b, v_{R} \rangle v_{R'} v_{R} \right\|_2 \leq K_{\delta} \epsilon^{1/3}. $$
The diagonalization in space takes two different forms. For \( \lambda \geq 2^k \) and \( R \in \mathcal{U}_k \) set \( \sigma(\lambda, R) \) to be a choice of \( R' \in \mathcal{W} \) with \( R' \simeq R \) and \( R' \subset 2\lambda R \). (The definition is vacuous for \( \lambda < 2^k \).) It is clear that we need only consider \( \simeq \lambda^2 \) choices of these functions \( \sigma(\lambda, \cdot) : \mathcal{U}_k \to \mathcal{W} \).

There is a \( L^1 \) estimate which allows one to take advantage of the spatial separation between \( R \) and \( \sigma(\lambda, R) \).}

\[
\left\| \sum_{R \in \mathcal{U}_k} \langle b, v_{\sigma(\lambda, R)} \rangle \overline{\langle b, v_R \rangle} v_{\sigma(\lambda, R)} v_R \right\|_1 \leq \lambda^{-100} \sum_{R \in \mathcal{U}_k} |\langle b, v_{\sigma(\lambda, R)} \rangle \overline{\langle b, v_R \rangle}| \leq \lambda^{-100} \left[ \sum_{R \in \mathcal{U}_k} |\langle b, v_{\sigma(\lambda, R)} \rangle|^2 \right]^{1/2} \left[ \sum_{R \in \mathcal{U}_k} |\langle b, v_R \rangle|^2 \right]^{1/2} \leq K_\delta \epsilon \lambda^{-90}.
\]

This estimate uses (3.3) and is a very small estimate.

To complete this case we need to provide an estimate in \( L^4 \). Here, we can be quite inefficient. By Cauchy–Schwartz and the Littlewood–Paley inequalities,

\[
\left\| \sum_{R \in \mathcal{U}_k} \langle b, v_{\sigma(\lambda, R)} \rangle \overline{\langle b, v_R \rangle} v_{\sigma(\lambda, R)} v_R \right\|_4 \leq \left\| \left[ \sum_{R \in \mathcal{U}_k} |\langle b, v_{\sigma(\lambda, R)} \rangle v_{\sigma(\lambda, R)}|^2 \right]^{1/2} \right\|_4 \left\| \left[ \sum_{R \in \mathcal{U}_k} |\langle b, v_R \rangle|^2 \right]^{1/2} \right\|_4 \leq \lambda.
\]

This follows directly from the \( BMO \) assumption on \( b \). Our proof is complete.

\section{A A Remark on Journé’s Lemma}

Let \( U \) be an open set of finite measure in the plane. Let \( \mathcal{R}^*(U) \) be those maximal dyadic rectangles in \( \mathcal{R} \) that are contained in \( U \). Define for each \( 0 < \delta < 1 \) and \( i, j \in \{1, 2\} \),

\[ V_{\delta, i, j} = \{ M_i 1_{\{M_j 1_U > \delta\}} > \delta \}, \]

and \( V_\delta = V_{\delta, 1, 2} \cup V_{\delta, 2, 1} \). For each \( R \in \mathcal{R}(U) \) set

\[ \mu_\delta(R) = \sup \{ \mu > 0 : \mu R \subset V_\delta \}. \]

The form of Journé’s Lemma we need is
A.1. Lemma. For each $0 < \delta, \epsilon < 1$ and each open set $U$ in the plane of finite measure,
\[
\sum_{R \in \mathcal{R}^*(U)} \mu_\delta(R)^{-\epsilon} |R| \leq |U|.
\]

Journé’s Lemma is the central tool in verifying the Carleson measure condition, and points to the central problem in two dimensions: That there can be many rectangles close to the boundary of an open set.

Among the references we could find in the literature \,[6, 9]\, the form of Journé’s Lemma cited and proved is relative to a strictly larger quantity than the one we use, $\mu_\delta(R)$ above. To define it, for any rectangle $R$, denote it as a product of intervals $R_1 \times R_2$. Set $M$ to be the strong maximal function. Then, for open set $U$ of finite measure and $R \in \mathcal{R}(U)$, set (taking $\delta = 1/2$ for simplicity)
\[
\nu(R) = \sup \{ \nu > 0 : \nu R_1 \times R_2 \subset \{ M1_U > 1/2 \} \}.
\]

Thus, one only measures how deeply $R$ is in the enlarged set in one direction. The Lemma above then holds for $\nu(R)$, with however a slightly sharper form of the sum than we prove here.

In addition, note that one measures the depth of $R$ with respect to a simpler set, $\{ M1_U > 1/2 \}$. We did not use this simplification in our proof as the strong maximal function $M$ does not act boundedly on $L^1$ of the plane.

There are however examples which show that that the quantity $\nu(R)$ can be much larger than $\mu(R)$. Indeed, consider a horizontal row of evenly spaced squares. For a square $R$ in the middle of this row, $\nu(R)$ will be quite big, while $\mu(R)$ will be about 1 for all $R$. Thus we give a proof of our form of Journé’s Lemma.

Proof of lemma A.1. We can assume that $1/2 \leq \delta < 1$ as the terms $\mu_\delta(R)$ decrease as $\delta$ increases. Fix $\mu \geq 1$. Set $\mathcal{S}$ to be those rectangles in $\mathcal{R}^*(U)$ with $\mu \leq \mu_\delta(R) \leq 2\mu$. It suffices to show that
\[
\sum_{R \in \mathcal{S}} |R| \leq (1 + \log \mu)^2 |U|.
\]

For then this estimate is summed over $\mu \in \{ 2^k : k \in \mathbb{Z} \}$.
In showing this estimate, we can further assume that for all $R, R' \in S$, writing $R = R_1 \times R_2$ and likewise for $R'$, that if for $j = 1, 2$, $|R_j| > |R'_j|$ then $|R_j| > 16\mu(1-\gamma)^{-1}|R'_j|$, where we set $\gamma = \delta^{1/3}$. This is done by restricting $\log_2|R_j|$ to be in an arithmetic progression of difference $\simeq \log \mu$. This necessitates the division of all rectangles into $\leq (1 + \log \mu)^2$ subclasses and so we prove the bound above without the logarithmic term.

We define a “bad” class of rectangles $B = B(S)$ as follows. For $j = 1, 2$, let $B_j(S)$ be those rectangles $R$ for which there are rectangles $R_1, R_2, \ldots, R_K \in S - \{R\}$, so that for each $1 \leq k \leq K$, $|R_k| > |R_j|$, and

$$ |R \cap \bigcup_{k=1}^K R^k| > \gamma|R|. $$

Thus $R \in B_j$ if it is nearly completely covered in the $j$th direction of the plane. Set $B(S) = B_1(S) \cup B_2(S)$. It follows that if $R \notin B(S)$, it is not covered in both the vertical and horizontal directions, hence

$$ |R \cap \bigcap_{R' \in S - \{R\}} (R')^c| \geq (1 - \gamma)^2|R|. $$

And, since all $R \subset U$, it follows that

$$ \sum_{R \in S - B} |R| \leq (1 - \gamma)^{-2}|U|. $$

Thus, it remains to consider separately the set of rectangles $B_1(S)$ and $B_2(S)$. Observe that for any collection $S'$, $B_j(S') \subset S'$ as follows immediately from the definition. Hence $B_1(B_2(B_1(S))) \subset B_1(B_1(S))$. And we argue that this last set is empty. As our definition of $V_\delta$ and $\mu(R)$ is symmetric with respect to the coordinate axes, this is enough to finish the proof.

We argue that $B_1(B_1(S))$ is empty by contradiction. Assume that $R \in B'$. Consider those rectangles $R'$ in $B_1(S)$ for which $(i) |R'_1| > |R_1|$ and $(ii) R' \cap R \neq \emptyset$. Then

$$ |R \cap \bigcup_{R' \in C} R'| \geq \gamma|R|. $$
Fix a one of these rectangles $R'$ with $|R'|$ being minimal. We then claim that $8\mu R' \subset \{M_1 1_{\{M_2 1_U > \delta\}} > \delta\}$, which contradicts the assumption that $\mu(R')$ is no more than $2\mu$.

Indeed, all the rectangles in $B_1(S)$ are themselves covered in the first coordinate axis. We see that the set $\{M_2 1_U > \gamma^3\}$ contains the rectangle $R''_1 \times \gamma^{-1}R_2$, in which $R_2$ is the second coordinate interval for the rectangle $R$ and $R''_1$ is the dyadic interval that contains $R'_1$ and has measure $8\mu(1 - \gamma)^{-1}|R'_1| \leq |R''_1| < 16\mu(1 - \gamma)^{-1}|R'_1|$.

But then the rectangle $\gamma^{-1}R''_1 \times \gamma^{-1}R_2$ is contained in $\{M_1 1_{\{M_2 1_U > \gamma^3\}} > \gamma^3\}$. And since $8\mu R'$ is contained in this last rectangle, we have contradicted the assumption that $\mu(R') < 2\mu$.

\[
\square
\]

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