Kinematic model of the parallel approximation method

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Abstract. This article considers a model of the pursuit problem using the parallel approach method. The purpose of this article is to modify the method of parallel approach in order to take into account the case when the pursuer’s speed vector is not directed at the target at the pursuit beginning moment. In addition, in the model discussed in the article, the pursuer cannot instantly change the movement direction. That is, the condition is imposed that the pursuer’s trajectory curvature radius cannot be less than a certain value. The proposed method is based on the fact that the pursuer, choosing a step at the iteration stage, will try to follow the predicted trajectories. Based on the materials of the article, we have written the test program that calculates the pursuer’s trajectory, taking into account the conditions set out. The completed animated image visualizes the change in the coordinates of the pursuer, the target, and the predicted trajectories from time to time.

Keywords – target, pursuer, trajectory, approach, simulation.

1. Introduction

In the description of the problem of pursuit by the method of parallel convergence in the works of L. O. Petrosyan [1], Pontryagin L. S. [2], Krassovski N. N. and Subbotin A. I. [3] the direction of the pursuer’s velocity vector P and the direction of the target's velocity vector T intersect at one point K, which belongs to the Apollonian circle (Figure 1).

![Figure 1. The Apollonian circle](image)

For points P and T, the point K of the Apollonian circle is characterized by the fact that the ratio of lengths |PK|/|QK| = |V_p|/|V_T| is the ratio of the velocity modules of the pursuer and the target.

For quasi-discrete modeling of the points of the path of the pursuer \{P_i\}, the following iterative scheme can be proposed (Figure 2):
\[ P_i = P_{i-1} + V_F \cdot \Delta T \cdot \frac{K_{i-1} - P_{i-1}}{|K_{i-1} - P_{i-1}|}. \]

The radius of the Apollonian circles will be:

\[ R_i = \frac{V_T^2}{V_F^2 - V_T^2} \cdot |T_i - P_i|. \]

The centers of the Apollonian circles are calculated as follows:

\[ Q_i = T_i + \frac{V_T^2}{V_F^2 - V_T^2} \cdot (T_i - P_i). \]

The coordinates of the point \( K_i \) are the result of solving a system of equations with respect to a continuous parameter \( t \):

\[
\begin{align*}
(K_i - Q_i)^2 &= R_i^2, \\
K_i &= T_i + V_T \cdot \frac{T_{i+1} - T_i}{|T_{i+1} - T_i|} \cdot t.
\end{align*}
\]

This is one of the quasi-discrete models for constructing the pursuer’s trajectory. It requires that the directions of the pursuer's and target's motion vectors intersect at points belonging to the Apollonian circles.

2. Problem statement

If we consider the iterative scheme shown in figure 2, we see that at each iteration step, the lines connecting the pursuer and the target \( (P_i, T_i) \) are always parallel to each other. The sources [2], [3] provide proof of this fact. But we will bring it here.

Consider the segment \( [P_i, T_i] \), the coordinates of points \( P_i \) and \( T_i \) are equal (Fig. 2):

![Figure 2. Iterative scheme](image-url)
\[ P_i = P_{i-1} + V_p \frac{P_{i-1}K_{i-1}}{|P_{i-1}K_{i-1}|} \cdot \Delta T \]
\[ T_i = T_{i-1} + V_T \frac{K_{i-1}}{|T_{i-1}K_{i-1}|} \cdot \Delta T \]

Based on the fact that the pursuer and the target must come to the point \( K_{i-1} \) on the circle of Apollonian at the same time, we can conclude that:
\[
\frac{V_p}{|P_{i-1}K_{i-1}|} \cdot \Delta T = \frac{V_T}{|T_{i-1}K_{i-1}|} \cdot \Delta T = \varepsilon.
\]
further
\[ P_i T_i = T_i - P_i = (T_{i-1} - P_{i-1}) + \varepsilon \cdot T_{i-1}K_{i-1} - \varepsilon \cdot P_{i-1}K_{i-1} = (1 - \varepsilon) \cdot (T_{i-1} - P_{i-1}). \]

In other words, the vector \( P_i T_i \) is co-directed to the vector \( P_{i-1} T_{i-1} \) (Figure 2).

This model of calculating the pursuer’s trajectory does not allow modeling if the intersection point of the directions of the motion vectors does not belong to the Apollonian circle for a given pair of points of the pursuer and the target. The purpose of this article is to develop a method for solving this problem.

3. Theory

Interpretation of the iterative scheme

The iterative scheme shown in figure 2 can be interpreted differently. Figure 3 shows an iterative process that determines the coordinates of the point \( P_i \) with the known coordinates of the points \( P_{i-1}, T_{i-1}, T_i \) and the speeds of the pursuer and the target, \( V_p \) and \( V_T \), respectively.

![Figure 3. Interpreting the iteration scheme](image)

First, the unit vector is defined:
\[ \overrightarrow{\tau} = \frac{T_{i-1} - P_{i-1}}{|T_{i-1} - P_{i-1}|} \]

Point position \( T_i = T_{i-1} + V_T \cdot \Delta T \cdot \overrightarrow{\tau} \)-the sampling period , is predefined by the target’s behavior. Then the straight line that will connect the points \( P_i \) and \( T_i \) can be represented as \( L(\mu) = T_i + \mu \cdot \overrightarrow{\tau} \). Then the coordinates \( P_i \) of the next iteration step of the pursuer's trajectory can be interpreted as the point of a circle of radius \( V_p \cdot \Delta T \) centered at point \( P_{i-1} \) and a straight line \( L(\mu) \):
\[
\begin{cases}
    \left( L(\mu) - P_{i-1} \right)^2 = \left( V_p \cdot \Delta T \right)^2 \\
    L(\mu) = T_i + \mu \cdot \overrightarrow{\tau}
\end{cases}
\]

The solution of the above system of equations with respect to the parameter \( \mu \) will give a parameter value at which the following will be performed: \( P_i = T_i + \mu \cdot \overrightarrow{\tau} \).
This interpretation of the iterative parallel approach scheme allows us to proceed to the calculation of the pursuer’s trajectory, when at the moment the pursuit begins, the pursuer’s speed is not directed at a point on the Apollonian circle. In other words, not as shown in figure 1.

II. Description of a quasi-discrete parallel pursuit model

We propose to modify the iterative scheme of parallel convergence as follows. Let the pursuer’s velocity vector $P_{i-1}$ be directed arbitrarily at the approach moment, but not at a point on the corresponding pair of points $\{P_{i-1}, T_{i-1}\}$ of the Apollonius circle (Figure 4).

Due to the inertia of the pursuer, the minimum radius of curvature of the trajectory cannot be less than a certain value $r_c$. The pursuer’s point $P_{i-1}$ corresponds to the velocity vector $\vec{V}_{P_{i-1}}$ and the unit vector normal $\vec{n}_{i-1}, \vec{V}_{P_{i-1}} \cdot \vec{n}_{i-1} = 0$.

Next, find the center of the circle of radius $r_c$: $C_{i-1} = P_{i-1} + V_P \cdot \Delta T \cdot \vec{n}_{i-1}$. A tangent is drawn to the constructed circle from the point $T_{i-1}$ to find the point $P_{t_i-1}$ of the conjugation of the straight line and the circle (Fig. 4).

![Figure 4. The quasi-discrete parallel pursuit model](image)

The arc of the circle $\{P_{i-1}P_{t_i-1}\}$ and the segment $[P_{t_i-1}T_{i-1}]$ will be considered as a single composite curve line $f_{i-1}(s)$, where the parameter $s$ is the arc length of our parametric curve. The choice of arc length as a parameter is quite reasonable because the segments of a composite curve can serve not only a straight line segment and a circle arc, but also, for example, Bezier curves or cubic parabolas.

It should be noted that in our test program, written based on the materials of the article, the arc length starts from the point $T_{i-1}, f_{i-1}(0) = T_{i-1}$.

We make a parallel transfer of the line $f_{i-1}(s)$ to the vector $T_i - T_{i-1}$. The position of the point $T_i$ is known and is completely determined by the behavior of the target. As part of our task, we will consider the behavior of the goal to be completely deterministic.

The equation of the parallel line $f_i(s) = f_{i-1}(s) + T_i - T_{i-1}$ will be considered known and to find the point $P_i$ of the pursuer’s next step, it is necessary to solve the following system of equations and inequalities with respect to the parameter $s$:

\[
\begin{align*}
(f_i(s) - P_{i-1})^2 &= (V_p \cdot \Delta T)^2 \\
 f_i(s) &= f_{i-1}(s) + T_i - T_{i-1} \\
0 &\leq s < s_{i-1}
\end{align*}
\]

Where $s_{i-1}$ is the value of the parameter $s$ corresponding to the point $P_{i-1}$.

4. The results of the experiments
To implement the kinematic model of the pursuit problem by the parallel approach method, the computer mathematics package MathCAD 15 was chosen. Let's note some features of our test program.

The first thing we did in our program was to perform parametrization from the arc length for a composite curve at the start of the pursuit, consisting of an arc and a segment. To do this, we needed to get an ordered set of points \( \{x_i, y_i\} \). For each coordinate, using built-in MathCAD tools, we performed cubic spline interpolation from the formal parameter \( \delta \) and obtained the functions \( X(\delta), Y(\delta), i \in [0, N - 1], \delta_i \leq \delta \leq \delta_{i+1}, \) where \( \delta_i = i \), and \( N \) is the number of array elements \( \{x_i, y_i\} \).

Next, the Jacobian was compiled for transmission to the built-in solvers of ordinary differential equations of the MathCAD system:

\[
D(s, \delta) = \frac{1}{\sqrt{\left(\frac{dX}{d\delta}\right)^2 + \left(\frac{dY}{d\delta}\right)^2}}
\]

The solution obtained by the Runge-Kutta method of order 4 expresses the dependence of the arrays \( \{x_i, y_i\} \) on the arc length parameter \( s \). Thus, we believe that the equation of the base curve, from which we will make a parallel transfer, is obtained (Figure 5).

Next, we have to create a computational cycle in which the system of equations and inequalities was solved. This problem is reduced to the numerical solution of the equation by searching for zeros of the function using the secant method in a given range.

Figure 5 shows the results of modeling the test program. Figure 5 is supplemented with a link to an animated image, where you can view the process in dynamics using the parallel approach method.

5. Discussion of results

In our test program, we deliberately chose the target's trajectory as a straight line. In the results of these experiments, we found out the following. If at the movement beginning, the pursuer's speed is directed at
a point on the Apollonian circle, then the time to reach the goal will always be less than the time if the pursuit would be performed using the chase method with the same parameters. In our case, the described method is not optimal, but the prospects in terms of group pursuit with different speeds from different points, but with synchronous achievement of the goal, are unquestionable.

6. Conclusions

In this article, we consider a kinematic model of the pursuit problem on a plane by the chase method, when at the moment pursuit begins, the pursuer's speed is not directed at the target.

This method can be used when developing a geometric model of group pursuit with simultaneous achievement of a goal or goals. It can also be used in the development of models when the pursuer reaches the target at specified angles using the parallel approach method.

This method of modeling pursuit tasks using the parallel approach method can be used in the design of UAVs with Autonomous control.

Based on the proposed models and algorithms, test programs for calculating trajectories are written in the MathCAD computer mathematics system. Program texts are available on the author's resource: http://dubanov.exponenta.ru. Links to animated images produced based on the results of the programs are available on the resource: https://www.youtube.com/watch?v=qNXdykK21Z8.

The article is based on theoretical results obtained in the following sources [1-4]. The results of the work [5-10] are also taken into account.

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