Pion Deuteron Scattering and Chiral Expansions

M. Pavón Valderrama\textsuperscript{1,}\textsuperscript{*} and E. Ruiz Arriola\textsuperscript{1,}\textsuperscript{†}

\textsuperscript{1} Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, E-18071 Granada, Spain.

(Dated: March 31, 2022)

We discuss convergence issues as well as short distance constraints on the deuteron wave functions based on chiral perturbation theory relevant to pion deuteron scattering. Non-analytical terms arise in the multiple scattering series for the pion-deuteron scattering length limiting the accuracy of the calculations. This result resembles similar findings in the structure of the NN interaction. The effects are found not to be numerically large. The iso-scalar $\pi N$ scattering length from the iso-vector one and the $\pi d$ scattering length yields values compatible with the experimental number and with much smaller errors.

PACS numbers: 03.65.Nk,11.10.Gh,13.75.Cs,21.30.Fe,21.45.+v

Keywords: pion-deuteron scattering; chiral symmetries; nuclear forces; multiple scattering; perturbation theory

I. INTRODUCTION

The relevance of pion dynamics in low energy hadronic and nuclear processes can hardly be exaggerated.\textsuperscript{1} Low energy theorems based on chiral symmetry provide a quantitative and model independent insight in low energy reactions involving pions and photons and nucleons due to the clear scale separation. The bound state character of nuclei makes such a description more complex from a theoretical viewpoint because many relevant scales enter into the problem, but one expects simplifications to occur in the limit of weak binding and low energies as it is the case for the deuteron. Actually, the possibility of computing Pion-Deuteron scattering (for a review see e.g.\textsuperscript{2} and references therein) in a model independent way was one of the original motivations to introduce Effective Field Theory (EFT) approaches\textsuperscript{3} based on the chiral symmetry of QCD and to derive the corresponding low energy theorem\textsuperscript{4}. This approach has been vigorously extended in the last decade to the calculation of low energy reactions for finite nuclei (for comprehensive reviews see e.g. Ref.\textsuperscript{5,6,7}). This is in fact a rather complicated process since there are many corrections and physical effects which add up to the final result\textsuperscript{8} and it is a challenge for chiral approaches to make reliably calculations based on a priori estimates of the accuracy. The use of the original power counting suggested by Weinberg to extract the $\pi N$ scattering length from the known $\pi p$-proton and $\pi d$-deuteron ones was suggested in Ref.\textsuperscript{10}. This power counting has been modified in Ref.\textsuperscript{11} to account for the long distance enhancement since the deuteron wavefunctions extend far beyond the pion Compton wavelength. Their result resembles the single and double scattering terms of the widely used older approaches\textsuperscript{2,8,9}

where the deuteron inverse moments, $\langle r^{-n}\rangle$, play an essential role. Re-scattering, deuteron recoil and binding effects can be re-summed to all orders in a rather elegant formula\textsuperscript{12} in the isospin limit (for a review see e.g. Ref.\textsuperscript{13}). Nucleon recoil effects have also been shown to be small\textsuperscript{14}. Pionic deuterium has been discussed in Ref.\textsuperscript{15}. Traditionally, these matrix elements have been evaluated using potential model calculations for the deuteron wave functions, which generally produce divergent results for $\langle r^{-n}\rangle$ (with $n > 2$). Perturbative wave functions based on OPE potentials yield divergences already for $\langle r^{-1}\rangle$\textsuperscript{16,17} and only recently has it been realized that non-perturbative calculations have a quite different convergent behaviour\textsuperscript{17,18,19}.

Following insightful previous works\textsuperscript{20,21}, we have developed in a series of papers\textsuperscript{17,22,23,24,25} a framework for the treatment of the NN interaction in a model independent way based on long distance correlations among physical observables. In our approach the long distance chiral NN- One Pion Exchange (OPE) and Two Pion Exchange (TPE) potentials computed within perturbation theory\textsuperscript{26,27,28}, are iterated to all orders in the Schrödinger equation very much in the spirit of the original Weinberg approach\textsuperscript{8} although some additional subtleties are encountered\textsuperscript{24,25,26}. Due to the power like singular character of the chiral potentials at the origin, a radial short distance cut-off is imposed which ultimately is removed by renormalization fixing some low energy observables or deuteron properties. This procedure turns out not to correspond to the imposition of a strict power counting in the sense that corrections to physical observables might be estimated \textit{a priori} by dimensional arguments, and hence departs from the original EFT program. Instead, one can show that naive perturbative expansions of renormalized scattering amplitudes do indeed display divergences which can be traced to given non-analyticities in the expansion parameter of the solution with the long distance potential\textsuperscript{24}. More specifically, if the long distance potential is written as $V(r) = V_{\text{OPE}}(r) + \lambda V_{\text{TPE}}(r) + \ldots$ one has that the scatter-
ing amplitude at low energies fulfills $A - A_{\text{OPE}} = O(\lambda^n)$ with $0 < \alpha < 1$ for singular potentials which fulfill the condition that $V_{\text{OPE}}(r) \gg V_{\text{TPE}}(r)$ at large distances. The value of the exponent $\alpha$ depends on the particular structure of the potentials involved. This weird fractional power counting is uncomfortable from a theoretical viewpoint and admittedly non conventional but does not contradict the physical requirement that effects which are small at large distances become parametrically small at low energies. This happens to be so even if at short distances both singular potentials fulfill the opposite relation $V_{\text{OPE}}(r) \ll V_{\text{TPE}}(r)$. As a consequence standard perturbation theory cannot be applied. The singularity of the potentials just provides a non-analytical enhancement of the perturbation. Thus, there is a lack of systematics a priori but one can always check the corrections to be small numerically as it turns out to be the case for the OPE and TPE potentials. For long distance potentials, which are local, as it is the case for the OPE and TPE potentials, severe restrictions on the possible counterterms are derived from the physical requirement that the wave function be small in the short distance and unknown region as the cut-off is removed. Calculations based on finite cut-offs do not show these problems but then the model independence is manifestly lost since the particular regularization acts itself as a short distance model.

With these new insights in mind we want to address in this paper the issue of $\pi d$ scattering and, more specifically, the calculation of the corresponding scattering length within the multiple scattering formalism to all orders as suggested by Deloff [12] where, as already mentioned, the inverse moments of the deuteron wave function play a role. In our OPE analysis of the deuteron [17] we showed that after renormalization the first inverse moment is finite due to the peculiar short distance behaviour of the deuteron wave function and because the potential at short distances presents a $1/r^3$ singularity. The analytical structure at short distances was determined to high orders. A similar observation was made by Nogga and Hanhart Ref. [13] using the momentum space wave functions of Ref. [23]. Recently, Platter and Phillips [19] have analyzed this matrix element as well as the second inverse moment and have shown that a direct treatment in coordinate space allows naturally for a rather clean extrapolation of the renormalized matrix element.

In the present work we discuss and extend these developments to the TPE case on the light of the multiple scattering formalism to all orders (Sect. II). Let us remind that the truncation of such an expansion where the coefficients are in fact the inverse deuteron moments is motivated by the smallness of the $\pi N$ scattering lengths. We show that higher order inverse moments are indeed convergent when the deuteron wave functions are built by fully iterating the long distance potentials (See Sect. III). The finite values are computed in Sect. IV. However, to any given approximation of the long distance potential there is always a finite value of the order of the moment above which none of them converges. In contrast, the re-summation of re-scattering effects provides convergent results and indeed exhibit a logarithmic enhancement of the multiple scattering expansion parameter (Sect. V). We use these approaches to examine the determination of the iso-scalar $\pi N$ scattering length within several schemes. Boost and finite range corrections are discussed qualitatively in Sect. VI. Finally, in Sect. VII we come to the conclusions.

II. THE MULTIPLE SCATTERING EXPANSION AND SHORT DISTANCE SENSITIVITY

In a remarkable paper Deloff [12] has analyzed $\pi d$ scattering at threshold by solving the Faddeev equation and has worked out the scattering length within a zero range model based on a boundary condition for the relative pion-nucleon wave function. In field operator language this corresponds to a contact $NN\pi\pi$ interaction, which can be regulated in this context by a radial regulator. In this approximation, intrinsic finite size effects of the $\pi N$ interaction are not included (see e.g. [8, 9] for a recent discussion on these and other effects).

The result he found in Ref. [12] for the scattering length is quite simple

$$a_{\pi d} = \frac{2}{1 + m_{2N}} \int_0^\infty dr \left[ u(r)^2 + w(r)^2 \right] A_{\pi d}(r)$$

(1)

where $u(r)$ and $w(r)$ are S- and D-wave deuteron wave functions satisfying the coupled channel $^3S_1 - ^3D_1$ set of equations

$$-u''(r) + U_s(r)u(r) + U_{sd}(r)w(r) = -\gamma^2 u(r),$$

$$-w''(r) + U_{sd}(r)u(r) + \left[ U_d(r) + \frac{6}{r^2} \right] w(r) = -\gamma^2 w(r),$$

(2)

and with asymptotic conditions

$$u(r) \to A_S e^{-\gamma r},$$

$$w(r) \to A_D e^{-\gamma r} \left( 1 + \frac{3}{\gamma r} + \frac{3}{(\gamma r)^2} \right),$$

(3)

where $\gamma = \sqrt{MB} = 0.231605 \text{fm}^{-1}$ is the deuteron wave number, $A_S$ is the normalization factor such that

$$\int_0^\infty dr \left[ u(r)^2 + w(r)^2 \right] = 1, \quad (4)$$

and the asymptotic D/S ratio parameter is defined by $\eta = A_D/A_S$. For conventions and numerical values of parameters we use Ref. [14, 23] throughout the paper.

The function $A_{\pi d}(r)$ in Eq. (1), which will be called Deloff function for short, is given by

$$A_{\pi d}(r) = \frac{\tilde{b}_0 + (\tilde{b}_0 + \tilde{b}_1)(\tilde{b}_0 - 2\tilde{b}_1)/r}{1 - \tilde{b}_1/r - (\tilde{b}_0 + \tilde{b}_1)(\tilde{b}_0 - 2\tilde{b}_1)/r^2}, \quad (5)$$
with \( \tilde{b}_i = (1 + m/M) b_i \), being \( b_0 \) and \( b_1 \) the \( \pi N \) scattering lengths according to the standard decomposition (assuming isospin symmetry)

\[
\mathcal{F}_{\pi N} = b_0 + b_1 \vec{t} \cdot \vec{r}.
\] (6)

Recoil and deuteron binding effects are taken into account by making the simple replacements 12

\[
\tilde{b}_i \to \hat{b}_i = \left( 1 + \frac{m}{M} \right) b_i, \quad 1/r \to e^{-\kappa r}/r.
\] (7)

where \( \kappa = \sqrt{2m/(m + M)} = 0.117261 \text{ fm}^{-1} \). The previous Eq. (1) sums up all multiple scattering effects due to zero range \( \pi N \) interactions. It does not include Fermi motion, higher partial waves contributions nor finite range \( \pi N \) corrections. We will comment on these corrections at the end of this paper. When expanded for small \( b_0 \) and \( b_1 \) one gets the result

\[
a_{\pi d} = \frac{2}{1 + \frac{2m}{M}} \left[ b_0 + (b_0^2 - 2b_0^2) \left( \frac{1}{r} \right) \right. \\
+ (b_0^3 - 2b_0^2b_0 - 2b_0^3) \left( \frac{1}{r^2} \right) \\
\left. + (b_0^4 - 4b_0^2b_0^2 + 2b_0^4) \left( \frac{1}{r^3} \right) + \ldots \right].
\] (8)

which has been used quite often truncated to second order 11. Here, the deuteron wave function average is defined as

\[
\langle r^n \rangle = \int_0^\infty dr \frac{u(r)^2 + w(r)^2}{r^n}
\] (9)

The multiple scattering expansion is motivated by the smallness of the \( s \)-wave \( \pi N \) scattering lengths. Actually, such an expansion finds theoretical support from Chiral Perturbation Theory since at lowest order one has the Weinberg-Tomozawa (WT) relations 11

\[
b_0^W = \frac{1}{3} (a_1 + 2a_3) = 0
\] (10)

\[
b_1^W = \frac{1}{3} (a_3 - a_1) = \frac{m_\pi}{8\pi(1 + m_\pi/M)f_{\pi}^2},
\] (11)

which compare rather well with the experimental numbers extracted from pionic hydrogen 30

\[
b_0 = -0.22 \pm 0.43 \times 10^{-2} m_\pi^{-1}
\] (12)

\[
b_1 = -9.05 \pm 0.42 \times 10^{-3} m_\pi^{-1}.
\] (13)

Higher order corrections to the current algebra relations have been computed via standard ChPT methods 31. A picture of the Deloff function can be seen in Fig. 1 for the WT values as well as the previous values from Ref. 30.

On the other hand, the measured \( \pi^-d \) scattering length is 32, 33

\[
a_{\pi^-d} = [-252 \pm 5(\text{stat.}) \pm 5(\text{syst.})] \times 10^{-4} m_\pi^{-1}.
\] (14)

From the viewpoint of the Chiral nuclear approach, the issue that we address here is to determine the compatibility of all scattering lengths using both the multiple scattering expansion as well as the deuteron wave functions based on chiral potentials.

In the case of the zero range \( \pi N \) interaction the convergence of the multiple scattering expansion, Eq. (8), would require in particular that any of the inverse moments of the deuteron wave function must be finite at the origin. However, potential model wave functions based on regular potentials, i.e. \( r^2 U(r) \to 0 \) for \( r \to 0 \), are dominated by the centrifugal term at short distances and hence satisfy the regularity conditions at the origin,

\[
u(r) \sim r, \quad w(r) \sim r^5,
\] (15)

reflecting their \( L = 0 \) and \( L = 2 \) angular momentum character respectively. So, it is clear that negative moments fail to converge starting at third order where a logarithmic divergence takes place. In addition, the short distance behaviour of the deuteron wave function becomes relevant for \( r^{-2} \) and, in fact, potential models indeed exhibit this sensitivity. The third and higher inverse moments are divergent. This strong short distance dependence looks very weird and counter intuitive, since we are looking at pion-deuteron scattering at zero energy, where the wavelength of the incoming and outgoing pion is much larger than any of the other length scales of the problem. So, we regard this effect as a mathematical artifact of the expansion and the potential model wave functions, and not as a genuine physical feature. It certainly does not agree with the philosophy underlying Effective Field Theories, namely that low energy physics does not depend on short distance details.

On the other hand the full formula does not present this problem because in spite of going to a finite limit at long distances, \( A_{\pi d}(r) \to b_0 \), there is a linear short distance...
suppression of the operator, \(A_{\pi d}(r) \sim -r\), as a result of the re-summation and in agreement with the EFT expectations. These considerations suggest that there may be problems with the convergence of multiple scattering expansion, which is ultimately motivated by the weak s-wave \(\pi N\) interaction at threshold and which fits quite naturally within Chiral Perturbation Theory. We will show below that the problem is related to the implicit assumption of analyticity in the \(\pi N\) scattering lengths.

III. SHORT DISTANCE CONSTRAINTS ON MATRIX ELEMENTS AND SINGULAR POTENTIALS

One relevant question is whether the multiple scattering expansion so widely used can still be undertaken and to what order in the case of the zero range \(\pi N\) interaction. Obviously, for this to happen at a given finite order, say \(\langle r^{-k} \rangle < \infty\), we must have

\[
\langle r^{-k} \rangle < \infty \quad \text{if} \quad U(r) \sim r^{-2k-2+\epsilon}, \tag{16}
\]

where \(\epsilon > 0\), and which implicitly requires that short distances cannot be dominated by the centrifugal term. The only way how this may happen is that the potential becomes more singular than the centrifugal barrier, which behaves as \(1/r^2\). Indeed for a potential which diverges like a given power \(U(r) \sim r^{-2n}\) the wave function has the power behaviour \(u(r) \sim r^{n/2}\) (up to some exponential or oscillatory function depending on the attractive or repulsive character of the potential at short distances) \(^1\). Thus, the maximal value for which the inverse moments converge fulfill

\[
\langle r^{-k} \rangle < \infty \quad \text{if} \quad U(r) \sim r^{-2k-2+\epsilon}. \tag{17}
\]

It is remarkable that chiral potentials do indeed exhibit these short distance singularities required by finiteness on the inverse moments of the deuteron wave function. Actually, a chiral expansion of the potential reads \(^2\)

\[
U(r) = \frac{Mm^3}{f^2} F^{(0)}(mr) + \frac{Mm^5}{f^4} F^{(2)}(mr) + \frac{m^6}{f^6} F^{(3)}(mr) + \ldots \tag{18}
\]

One can rewrite the expansion as

\[
U(r) = \frac{M}{f^2} G^{(0)}(mr) + \frac{M}{f^4} G^{(2)}(mr) + \frac{1}{f^4b} G^{(3)}(mr) + \ldots \tag{19}
\]

where the functions \(G^{(n)}(mr)\) have a finite limit for vanishing argument. Thus in the short distance limit

\[
U(r) \to \frac{M}{f^2} G^{(0)}(0) + \frac{M}{f^4} G^{(2)}(0) + \frac{1}{f^4b} G^{(3)}(0) + \ldots \tag{20}
\]

Note that in this limit the pion mass dependence disappears \(^2\). In the deuteron case the coefficients in the former expressions become matrices \(^2\) (see Appendix A).

Thus, in the absence of the long distance potential none of the inverse moments is finite, while at LO the moments \(\langle r^{-1} \rangle\) and \(\langle r^{-2} \rangle\) are finite, and at NLO and NNLO \(\langle r^{-1} \rangle\), \(\langle r^{-2} \rangle\) and \(\langle r^{-3} \rangle\) are also finite.

Given the fact that the short distance behaviour of the chiral potentials does not depend on the pion mass, the short distance contribution to the inverse moments is dominated by the corresponding short distance scale, \(R\), so one has

\[
\langle r^{-k} \rangle_{\text{short}} \sim R^{-k}, \tag{21}
\]

provided the integral is convergent. Thus, if we use the OPE exchange potential, \(R \sim M/f^2\) we see that inverse moments do indeed become large in the limit \(R \to 0\). As shown in our previous work on the deuteron \(^1\), long distance perturbation theory mistreats the behaviour of the wave function at short distances, introducing a very unnatural strong short distance dependence. Actually the first order contribution to the deuteron wave function diverges. As a consequence, in the OPE potential, \(\langle r^{-1} \rangle\) also diverges as first noted in Ref. \(^1\) using the PDS subtraction scheme. This was the reason to choose always the regular solutions of the fully iterated potential at the origin for the OPE case \(^1\). The divergence persists also when the TPE exchange potential is treated in perturbation theory on the OPE distorted wave basis as a zeroth order approximation, since as pointed out in Ref. \(^2\), the deuteron perturbative wave functions diverges strongly at the origin.

As we see, the multiple scattering expansion of the pion-deuteron scattering length involves negative moments which become more convergent and hence more insensitive to short distance details when the NN potential is more accurately described at shorter distances. This finiteness of negative moments to occur it is essential that the singular potentials be fully iterated since the short distance behaviour is highly non perturbative. This gives us some confidence on the treatment of the singular potentials and the renormalization process adopted in our previous works \(^1\) \(\cite{24}\). As we have also stressed in the introduction, the successive improvements

\(^1\) In the limit \(n \to \infty\) this includes also the possibility of an infiniteley repulsive hard core potential.

\(^2\) The short distance however does not coincide with the chiral limit; in the latter case less singular subleading powers are obtained.
on the potential are parametrically small although in a non-analytical way for the low energy NN observables. Note that this is not the case for the inverse moments. Nevertheless, we will see that a similar non-analytical behaviour occurs in the perturbative treatment of the pion-deuteron scattering length via a multiple scattering expansion.

IV. INVERSE MOMENTS FOR FULLY ITERATED CHIRAL POTENTIALS

The calculation of the inverse moments for the deuteron wave functions for the OPE and TPE potentials is in principle straightforward. In the OPE case the first inverse moment was calculated in Ref. 17 and then in Ref. 18. These numbers have been checked in Ref. 19. In the later reference also the second inverse moment (\(r^{-2}\)) has been estimated. Here we confirm and extend these results to the first three finite inverse moments in the TPE case.

One technical aspect to consider in the present calculation corresponds to the short distance contribution of the matrix elements. As an illustration we plot in Fig. 2 the integrand corresponding to the inverse square moment, \((u^2 + w^2)/r^2\) for both the OPE and the TPE potentials. As we see, a substantial contribution to the integral is dominated by the short distance region making the convergence at short distances numerically unreliable. In the OPE case this can be fixed by using the analytical solutions found in our previous work 17 and computing the integral analytically in the short distance region. In the appendix A we analyze the problem for the OPE and TPE potentials in a natural way which exploits the locality of the chiral potentials in a coordinate space method.

The calculation of the inverse moments for the \(u\) and \(w\) has been estimated. Here we confirm and reproduce these numbers and computing the inverse moments analytically as well as the determination of these estimates analytically. Further details are presented in appendix A.

 Numerically, for the chiral constants deduced in Ref. 34 (Set IV in our work) we find the expressions,\begin{align*}
\langle 1/r \rangle_{\text{TPE}} &= \frac{\langle \frac{1}{r} \rangle_{SS} + 2\eta \langle \frac{1}{r} \rangle_{SD} + \eta^2 \langle \frac{1}{r} \rangle_{DD}}{1 + 2\eta 1_{SD} + \eta^2 1_{DD}} \quad (23)
\end{align*}

And also\begin{align*}
\langle e^{-\gamma r} / r \rangle_{\text{TPE}} &= \frac{3.08822 - 221.425 \eta + 4807.6 \eta^2}{3.39582 - 189.582 \eta + 4175.2 \eta^2} \\
\langle e^{-2\gamma r} / r^2 \rangle_{\text{TPE}} &= \frac{3.66165 - 287.55 \eta + 6189.54 \eta^2}{3.39582 - 189.582 \eta + 4175.2 \eta^2} \\
\langle e^{-3\gamma r} / r^3 \rangle_{\text{TPE}} &= \frac{5.14335 - 417.313 \eta + 8914.07 \eta^2}{3.39582 - 189.582 \eta + 4175.2 \eta^2} \quad (25)
\end{align*}

The numerical coefficients depend solely on the TPE potential, \(\gamma\) and \(\kappa\) for which we take the values 0.231605 and 0.117261 fm\(^{-1}\) respectively. Our results for the computed moments are summarized in Table II. Errors on the short distance cut-off scale linearly, quadratically and cubically for \(r^{-3}\), \(r^{-2}\) and \(r^{-1}\) respectively (see appendix A).

3 This is in contrast to momentum space methods where computer space limitations may suggest a seeming convergence on the momentum space cut-off but still far from the infinite cut-off limit result (see e.g. Fig.5 in Ref. 13).
Besides the fact that the non-perturbative inclusion of long distance chiral potentials to all orders provides convergent inverse moments, one of the features one observes from inspection of table III is the size of the changes induced when going from OPE to TPE potentials. Whereas the TPE effects have moderate effects on the \( \langle r^{-1} \rangle \) moment as compared to the OPE value, almost a factor of two reduction for \( \langle r^{-2} \rangle \) is obtained. This is to be expected as the TPE potential modifies the short distance region. Moreover, we get a large reduction of the second inverse moment since \( \langle r^{-2} \rangle_{\text{OPE}} = 0.41 \text{fm}^{-2} \) while \( \langle r^{-2} \rangle_{\text{TPE}} = 0.25 - 0.25 \text{fm}^{-2} \) very close to the value 0.286 - 0.345fm\(^{-2}\) quoted in Ref. [11]. As we also found in our previous work [24] the Set IV [32] agrees best with the NijmII and Reid93 potential values.

Finally, let us note that, as already argued above, the more singular the potential at short distances the more convergent the matrix elements, and in particular short distance contributions to the TPE potential are much smaller than those in the OPE case. For instance for the second inverse moment in the region below \( r_c \) = 0.2fm one gets \( \langle r^{-2} \rangle_{\text{OPE}} = 0.08 \text{fm}^{-2} \) while \( \langle r^{-2} \rangle_{\text{TPE}} = 0.003 \text{fm}^{-2} \). This also suggests that finite cut-off effects in matrix elements become smaller when the long distance potential is improved at lower distances.

### V. DIVERGENT INVERSE MOMENTS AND NON-ANALYTICAL BEHAVIOUR

As we have also discussed, the use of the Deloff function (with or without binding and recoil) produces finite numbers regardless on the short distance behaviour of the deuteron wave function assuming it can be normalized. On the other hand when the multiple scattering expansion is undertaken divergences appear at some stage. Although this appears to be a bit puzzling, the mathematical reason why this is happening is indeed quite simple. If we use for definiteness the WT values for the \( \pi N \) scattering lengths, the denominator has complex conjugated poles located

\[
\rho = -\frac{(1 \pm i\sqrt{7})m}{16\pi f^2} + \ldots
\]

(26)

where binding and recoil corrections have been neglected for clarity. In this approximation the radius of convergence becomes \( |\rho| = m/4\pi\sqrt{2}f^2 \) which is a rather small distance and goes to zero in the chiral limit. For finite values of \( b_1 \) one can only expand above that region. The relevant scale is given by \( b_1 \sim -0.2 \text{fm} \) so that in the limit of small \( b_1 \) we become sensitive to the short distance behaviour. Numerically the radius of convergence is given by \( |\rho_c| = 0.15 \text{fm} \), so the the large \( r \) expansion converges only for \( r > |\rho_c| \). Thus, if we cut-off the integrand below such a value we get a convergent expansion but at

\[
|\rho e^{\kappa r}| = \frac{\sqrt{2}m}{(1 + m/M) [8\pi f^2(1 + m/M) - mk]}
\]

---

4 A similar factor of two was also found in the effective range parameter for the \( ^1S_0 \) phase shift when going from OPE to TPE.

---

5 The full expression is

\[
|\rho e^{\kappa r}| = \frac{\sqrt{2}m}{(1 + m/M) [8\pi f^2(1 + m/M) - mk]}
\]
the same time an important piece of physics is neglected. The short distance enhancement can be seen by displaying at the function $A_{\pi d}(r)$ in Fig. 4. The bump takes place at $r \sim 0.2$ fm so it is indeed true that a relevant contribution comes from short distances.

The kind of non-analyticity that appears can be easily illustrated with the asymptotic deuteron wave function

$$u(r) = \sqrt{2} \gamma e^{-\gamma r}, \quad w(r) = 0,$$

which corresponds to the pion-less theory. Even though the first term in the multiple scattering expansion in Eq. (27) does not converge, the full formula given by Eq. (27) yields a finite analytical result. Neglecting $m_x/M$ corrections and taking $b_0 = 0$ for illustration purposes one gets

$$a_{\pi d} = 4b_1^2 \gamma I(-b_1 \gamma)$$

(28)

where

$$I(t) = \int_0^\infty e^{-tx_x}(x-x_1)(x-x_2)$$

$$= e^{-tx_1}x_1 \Gamma(0, -tx_1) - e^{-tx_2}x_2 \Gamma(0, -tx_2).$$

(29)

In the case $b_0 = 0$ we have $x_{1,2} = (1 \pm i \sqrt{7})/2$ and $\Gamma(0, z) = f^\infty_0 e^{-t/t} dt$ is the incomplete gamma function which has a logarithmic branch cut at $z = 0$. Thus, there is a branch cut singularity at $b_1 = 0$, the leading term being

$$a_{\pi d} \sim -4b_1^2 \gamma \log(-b_1 \gamma) = -3.06 \times 10^{-2} m^{-1}$$

(30)

This example shows explicitly the kind of enhancement that one might expect. The full result with the short distance wave function and using the WT $\pi N$ scattering lengths $a_{\pi d} = -2.815 \cdot 10^{-2} m^{-1}$, a quite reasonable value taking into account the poor quality of the wave function. This value overshoots the real value due to the fact that the short range wave function, Eq. (27) does not vanish at the origin. For more regular functions such as those of potential models we expect that logarithmic short distance enhancement takes place precisely at the fourth and higher orders where the divergence of the inverse moment becomes manifest. The chiral TPE deuteron wave functions present the logarithmic enhancement at fifth order.

In Fig. 5 we plot the integrand for a variety of wave functions. A good feature of the use of the Deloff function is the irrelevance of short distance behaviour as compared to the multiple scattering expansion where there is an enhancement of short distances. One important lesson we learn from this exercise is that the appearance of non-analytical enhancements found in non-perturbative treatments of the NN interaction extends also to the computation of matrix elements. Strict power counting simply does not hold, although the corrections are parametrically small. Numerical results for the pion-less, OPE and TPE potential cases using the Deloff formula are presented in Table III for the central values of the $\pi N$ scattering lengths.

As we see, the multiple scattering series provides cancellations which are independent on the short distance constraints for the inverse moments discussed in Sect. III. One might think that they might be correlated through the chiral expansion, in the sense of a perturbative re-ordering of the truncated multiple scattering series by re-expanding the convergent inverse moments. This is unlikely, since the kind of non-analyticities appearing in the multiple scattering series as a function of $b_0$ and $b_1$ and those in the inverse moments as a function of the chiral potential parameters are of quite different nature.

It is worth displaying numerically the convergence of the multiple scattering series for both the OPE and the TPE deuteron wave functions. For illustration purposes we have taken the Deloff function ignoring both binding and recoil corrections (similar features are observed if these corrections are taken into account). We obtain

$$a_{\pi d|\text{OPE}} = 130.42 \hat{b}_0 + 62.32 (\hat{b}_0^2 - 2 \hat{b}_1^2) + 54.84 (\hat{b}_0^3 - 2 \hat{b}_0 \hat{b}_1^2 - 2 \hat{b}_1^3) + O(b^4 \log b)$$

$$-0.47013 \quad -2.744$$

(31)

$$a_{\pi d|\text{TPE}} = 130.42 \hat{b}_0 + 58.25 (\hat{b}_0^2 - 2 \hat{b}_1^2) + 37.01 (\hat{b}_0^3 - 2 \hat{b}_0 \hat{b}_1^2 - 2 \hat{b}_1^3) + 35.97 (\hat{b}_0^4 - 4 \hat{b}_0^2 \hat{b}_1^2 + 2 \hat{b}_1^4) + O(b^5 \log b),$$

(32)

where the results given in the under-braces are in units of $10^{-2} m^{-1}$ (as the pion-deuteron scattering length), and the $\hat{b}_i$’s are in fm. As we see, the bulk of the contribution is given by the double scattering term. Using the full OPE and TPE potentials and the full $A_{\pi d}(r)$ operator with no recoil and binding we get

$$a_{\pi d|\text{OPE}} = -2.873(3) \cdot 10^{-2} m^{-1}$$

(33)

$$a_{\pi d|\text{TPE}} = -2.77(2) \cdot 10^{-2} m^{-1},$$

(34)

while the analytic contributions in Eq. (32) sum up to $-2.847$ and $-2.75$ respectively. As we see, the non-
analytical effects are not dramatic, which was not completely obvious a priori and to a certain extend are compatible with the uncertainties in the TPE case. Obviously, very precise estimates might be sensitive to the non-analytical pieces. In any case, it would be rather interesting to compute the non-analytical contributions per se.

Finally, we plot in Fig. 4 the correlation between the iso-scalar and iso-vector $\pi N$ scattering lengths $b_0$ and $b_1$ deduced from direct application of the Deloff formula in several schemes where errors from the $\pi - d$ scattering length and chiral potential parameters are taken into account. The main source of uncertainty in all cases turns out to be $a_{\pi d}$. If we take the iso-vector $b_1$ value from Ref. 30 we obtain in the TPE case including both binding and recoil corrections the result

$$b_0 = -0.3(1) \cdot 10^{-2} m^{-1} \pi,$$  

which is compatible with the measured value but about an order of magnitude more accurate.

VI. REMARKS ON BOOST AND FINITE RANGE CORRECTIONS

One of the effects we have not taken into account in our discussions has to do with the boost corrections due to the fact that the CM $\pi - d$ and $\pi - N$ systems do not coincide, so the nucleons inside the deuteron recoil differently as the deuteron. Although it has been argued that these effects are small 14 it is interesting to reanalyze the issue on the light of the present investigation. Finite range $\pi N$ corrections have also been computed in Refs. 8, 9 and the leading contribution involves similar operators as in the boost corrections case, so we will refer mainly to the latter case in the following. Actually, the effect can be estimated perturbatively yielding an $O(Q^4)$ correction according to the modified counting proposed in Ref. 11.

$$a_{\pi-d}|_{\text{Boost}} = -\frac{1}{1 + m_\pi^2/2M \pi M \langle p^2 \rangle} \frac{m_\pi^2}{8 \pi M^3 \pi f_\pi^2 (g_A^2 - 8 M c_2)}$$  

(36)
\[ \langle p^2 \rangle = \int_0^\infty dr \left[ u'(r)^2 + u'(r)^2 + 6 \frac{u'(r)^2}{r^2} \right] \quad (37) \]

The operator \( \langle p^2 \rangle \) is weakly non-local and strictly speaking this number is infinite for any singular potential. Actually, the divergence behaves as \( 1/\sqrt{r_c} \) for the OPE potential and as \( 1/r^3 \) for the TPE potential. The large values of \( \langle p^2 \rangle \) have been discussed in finite cut-off calculations in momentum space and attributed to the inner knots of the wave function, hence favoring the use of relatively small cut-offs. This appears to be a serious difficulty, but resembles the one faced already for the inverse moments; one expects non-analytical effects in the couplings in front of \( \langle p^2 \rangle \). This suggests further lines of research, in particular invoking physically motivated boost re-summations or including nonlocal effects in the deuteron wave functions. As it is known, the higher order contributions to the NN potential, \( O(Q^4) \) according to Weinberg’s counting, contain nonlocal pieces. In momentum space and up to NNLO the long distance part of the potential depends on the momentum transfer \( q \) only and not on the total momentum \( k \). Essential nonlocalities, i.e. contributions of the form \( V(q, k) = L(q) k^2 \) with \( L(q) \) a non-polynomial function, depend weakly on the total momentum and appear first at N\(^3\)LO \( \sqrt{1/M^2} \) i.e. also \( O(Q^4) \) due to relativistic \( 1/M^2 \) one loop contributions. In coordinate space this weak non-locality corresponds to a modification of the kinetic energy term in the form of a general self-adjoint Sturm-Liouville operator, \(-u''(r) \rightarrow -\langle p(r)u'(r) \rangle'\), with a singular \( p(r) \) function at the origin and exponentially decaying at long distances. It is at present unclear how these non-locally modified wave functions might influence the pion-deuteron scattering length and further work along these lines should be pursued. A more promising perspective consists of resumming boost corrections non-perturbatively \(^6\). Obviously, to achieve a definite conclusion on this issue one should reexamine the \( \pi NN \) system within a full quantum mechanical Faddeev approach including chiral potentials between the nucleons which is nontrivial. We leave such a study for future developments.

VII. CONCLUSIONS

In this paper we have analyzed a particular re-summation of re-scattering effects to pion-deuteron scattering length corresponding to the fixed center approximation and where recoil and binding effects may be taken into account to all orders. This re-summation has the important feature of providing a physically motivated suppression of the deuteron matrix elements at the origin. This is most welcome since it agrees qualitatively with the naive expectation that low energy processes should not depend strongly on short distance details. The finiteness of the multiple scattering expansion of the full expression truncated to a given order requires a series of short distance constraints which are remarkably fulfilled by the regular solutions of the deuteron wave functions corresponding to chiral potentials. This is so precisely because the potentials become singular and are iterated to all orders. Nevertheless, further insensitivity at short distance can be gained by using a resummation formula proposed by Deloff. We have noted that divergences in multiple scattering expansion arise because non-analyticities in the chiral expansion appear. This is a rather general feature which goes beyond just the example of pion-deuteron scattering addressed in this paper and will also occur in other low energy reactions. The found non-analyticities are not large numerically due to the good short distance behaviour of the deuteron wave functions. Finally, we have compared several ways of extracting the poorly known iso-scalar \( \pi N \) scattering lengths from the known iso-vector \( \pi N \) and the pion-deuteron scattering length, yielding, as expected, a compatible but more accurate result.

Acknowledgments

We thank L. Platter, D. R. Phillips, M. Döring and A. Deloff for useful correspondence. We also thank Avarham Gal for pointing out a mistake in a previous version of the paper. This research was supported by DGI and FEDER funds, under contract FIS2004- and by the Junta de Andalucía grant no. FM-225 and EURIDICE grant number HPRN-CT-2003-00311.

APPENDIX A: SHORT DISTANCE CONTRIBUTIONS OF MATRIX ELEMENTS INVOLVING INVERSE MOMENTS

In this appendix we quote analytical results for the short distance contribution to the inverse matrix elements. Most integrals are sufficiently converging by letting the short distance cut-off to go to zero. The \( 1/r^2 \) for the OPE case is a bit special since the integrand presents many oscillations which make numerical extrapolation unreliable. As pointed out in Ref. \(^{19}\), it is much better to use the analytical expressions deduced in Ref. \(^{17}\).
1. Short distance OPE

At short distances the OPE potential behaves as

\[
U_s^{\text{OPE}}(r) \rightarrow \frac{R_s}{r^6}, \\
U_{sd}^{\text{OPE}}(r) \rightarrow \frac{R_{sd}}{r^6}, \\
U_d^{\text{OPE}}(r) \rightarrow \frac{R_d}{r^6}
\]

where \( R_d = 4R, R_{sd} = 2\sqrt{2}R \) and \( R = 3g_A^2 M/32\pi f^2 = 1.07764 \text{fm} \). Going to the diagonal basis the solution can be written as

\[
u(r) \rightarrow \sqrt{\frac{2}{3}} v_A(r) - \frac{1}{\sqrt{3}} v_R(r), \\
w(r) \rightarrow \frac{1}{\sqrt{3}} v_A(r) + \sqrt{\frac{2}{3}} v_R(r),
\]

where \( v_A(r) = \left( \frac{r}{R} \right)^{3/4} \left[ C_1 R e^{-4\sqrt{2} r} + C_2 R e^{-4\sqrt{2} r} \right] \),

\[v_A(r) = \left( \frac{r}{R} \right)^{3/4} \left[ C_1 A e^{-4\sqrt{2} r} + C_2 A e^{-4\sqrt{2} r} \right], \tag{A3}\]

and \( v_R(r) = \left( \frac{r}{R} \right)^{3/4} \left[ C_1 R e^{-4\sqrt{2} r} + C_2 R e^{-4\sqrt{2} r} \right] \).

The constants \( C_1 R, C_2 R, C_1 A \) and \( C_2 A \) have been fixed in Ref. [17] from matching the numerical solution to a short distance expansion (the regularity condition \( C_1 R = 0 \) is imposed). The integral can be computed for short distances yielding

\[
\frac{R^2}{A^2} \int_0^{r_c} dr \frac{u^2 + w^2}{r^2} \bigg|_{\text{OPE}} = \frac{1}{384} C_{1A}^2 \left\{ -91264 \pi + 1536 \sqrt{x_c} + 140 x_c^2 + 22816 \sqrt{x_c} \cos \left( \frac{8}{\sqrt{x_c}} \right) \right. \\
\left. \quad -1085 x_c^2 \cos \left( \frac{8}{\sqrt{x_c}} \right) + 2660 x_c \sin \left( \frac{8}{\sqrt{x_c}} \right) + 182528 \sin \left( \frac{8}{\sqrt{x_c}} \right) \right\} \\
+ \frac{1}{192} C_{1A} C_{2A} \left\{ 2660 x_c \cos \left( \frac{8}{\sqrt{x_c}} \right) + 182528 \left( \frac{8}{\sqrt{x_c}} \right) + 31 \sqrt{x_c} (736 + 35 x_c) \sin \left( \frac{8}{\sqrt{x_c}} \right) \right\} \\
+ \frac{1}{384} C_{2A}^2 \left\{ 91264 \pi + 1536 \sqrt{x_c} + 140 x_c^2 - 22816 \sqrt{x_c} \cos \left( \frac{8}{\sqrt{x_c}} \right) \right. \\
\left. \quad +1085 x_c^2 \cos \left( \frac{8}{\sqrt{x_c}} \right) - 2660 x_c \sin \left( \frac{8}{\sqrt{x_c}} \right) - 182528 \left( \frac{8}{\sqrt{x_c}} \right) \right\} + \mathcal{O}(x_c^2) \tag{A4}\]

with \( x_c = r_c/R \) and \( \text{Si} \) and \( \text{Ci} \) are the sine and cosine integral functions respectively defined as

\[
\text{Si}(z) = \int_z^\infty \frac{\sin t}{t} \, dt, \\
\text{Ci}(z) = \int_z^\infty \frac{\sin t}{t} \, dt
\]

The convergence on \( r_c \) is shown in Fig. 5. We see that 0.06fm\(^{-2}\), i.e. about 20% of the result, comes from the region below 0.2fm. We have checked that up to this region the result is rather stable with the given terms.

2. Short distance TPE

The short distance behaviour of the TPE has been determined in Ref. [24]. The potential at short distances and \( e_i = M C_i \) are the low energy chiral couplings appearing in \( \pi N \) scattering. Diagonalizing the corresponding

\[
\begin{align*}
U_s^{\text{TPE}}(r) & \rightarrow \frac{R_s}{r^6} \\
U_{sd}^{\text{TPE}}(r) & \rightarrow \frac{R_{sd}}{r^6} \\
U_d^{\text{TPE}}(r) & \rightarrow \frac{R_d}{r^6}
\end{align*}
\]

where

\[
\begin{align*}
(R_s)^4 & = \frac{3g_A^2}{128 f^4 \pi^2} (4 - 3g_A^2 + 24 \bar{c}_3 - 8 \bar{c}_4) \\
(R_{sd})^4 & = -\frac{3v^2 g_A^2}{128 f^4 \pi^2} (-4 + 3g_A^2 - 16 \bar{c}_4) \\
(R_d)^4 & = \frac{9g_A^2}{32 f^4 \pi^2} (-1 + 2g_A^2 + 2 \bar{c}_3 - 2 \bar{c}_4)
\end{align*}
\]

(A7)
Note that the potential is negative definite. In the diagonal basis one has
\[
\begin{pmatrix}
u_+ \\ v_-
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \sin \theta \\ -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
u_+ \\ v-
\end{pmatrix}
\]
where the short distance eigen functions are
\[
v_+ (r) = \left( \frac{r}{R_+} \right)^\frac{1}{2} \left\{ C_{+,s} \sin \left[ \frac{1 R^2}{2 r^2} \right] + C_{+,c} \cos \left[ \frac{1 R^2}{2 r^2} \right] \right\}
\]
\[
v_- (r) = \left( \frac{r}{R_-} \right)^\frac{1}{2} \left\{ C_{-,s} \sin \left[ \frac{1 R^2}{2 r^2} \right] + C_{-,c} \cos \left[ \frac{1 R^2}{2 r^2} \right] \right\}
\]

Higher order contributions could in principle be computed similarly to what was done in the OPE case \[\text{[7]}\] although such refinements will not be needed here. Matching our numerical solutions to these short distance solutions at \(r_c = 0.1\text{fm}\) we get (for Set IV of chiral constants)

\[\begin{align*}
C_{+,c} &= 6.174 - 257.329\eta \\
C_{+,s} &= -1.585 + 38.745\eta \\
C_{-,c} &= 2.451 - 118.84\eta \\
C_{-,s} &= 2.522 - 84.40\eta
\end{align*}\]

for the long distance normalization such that \(u(r) \to e^{-\gamma r}\). The integral for the inverse moment \(\langle r^{-3} \rangle\) can be computed analytically using that
\[
\int_0^{r_c} u^2 + w^2 dr = \int_0^{r_c} \frac{v_+^2 + v_-^2}{r^n} dr
\]
and the appropriate normalization. For the third inverse moments the integral can be written in terms of Fresnel integrals. The result is depicted in Fig. 6. As we see, the short distance contribution for the moments is rather small. The contributions from the region between the origin and \(r_c = 0.2\text{fm}\) are given by

\[\begin{align*}
\langle r^{-1/2}\rangle_T &= 0.0004 \text{fm}^{-1} \\
\langle r^{-2}\rangle_T &= 0.003 \text{fm}^{-2} \\
\langle r^{-3}\rangle_T &= 0.025 \text{fm}^{-3}
\end{align*}\]
[8] T. E. O. Ericson, B. Loiseau, and A. W. Thomas, Phys. Rev. C66, 014005 (2002), hep-ph/0009312.
[9] M. Doring, E. Oset, and M. J. Vicente Vacas, Phys. Rev. C70, 045203 (2004), nucl-th/0402086.
[10] S. R. Beane, V. Bernard, T. S. H. Lee, and U. G. Meissner, Phys. Rev. C57, 424 (1998), nucl-th/9708035.
[11] S. R. Beane, V. Bernard, E. Epelbaum, U.-G. Meissner, and D. R. Phillips, Nucl. Phys. A720, 399 (2003), hep-ph/0206219.
[12] A. Deloff, Phys. Rev. C64, 065205 (2001), nucl-th/0104067.
[13] A. Deloff, Fundamentals in hadronic atom theory. World Scientific (2003) 352 p.
[14] V. Baru, C. Hanhart, A. E. Kudryavtsev, and U. G. Meissner, Phys. Lett. B589, 118 (2004), nucl-th/0402027.
[15] U. G. Meissner, U. Raha and A. Rusetsky, Eur. Phys. J. C 41 (2005) 213 [arXiv:nucl-th/0501073].
[16] B. Borasoy and H. W. Griesshammer, Int. J. Mod. Phys. E12, 65 (2003).
[17] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C72, 054002 (2005), nucl-th/0504067.
[18] A. Nogga and C. Hanhart, Phys. Lett. B634, 210 (2006), nucl-th/0511011.
[19] L. Platter and D. R. Phillips (2006), nucl-th/0605024.
[20] D. W. L. Sprung, W. van Dijk, E. Wang, D. C. Zheng, P. Sarriguren, and J. Martorell, Phys. Rev. C49, 2942 (1994).
[21] S. R. Beane, P. F. Bedaque, M. J. Savage, and U. van Kolck, Nucl. Phys. A700, 377 (2002), nucl-th/0104030.
[22] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Lett. B580, 149 (2004), nucl-th/0306069.
[23] M. Pavon Valderrama and E. Ruiz Arriola, Phys. Rev. C70, 044006 (2004), nucl-th/0405057.
[24] M. P. Valderrama and E. R. Arriola (2005), nucl-th/0506047.
[25] M. Pavon Valderrama and E. Ruiz Arriola (2005), nucl-th/0507075.
[26] N. Kaiser, R. Brockmann, and W. Weise, Nucl. Phys. A625, 758 (1997), nucl-th/9706045.
[27] J. L. Friar, Phys. Rev. C60, 034002 (1999), nucl-th/9901082.
[28] M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar, and J. J. de Swart, Phys. Rev. Lett. 82, 4992 (1999), nucl-th/9901054.
[29] A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C72, 054006 (2005), nucl-th/0506005.
[30] H. C. Schroder et al., Phys. Lett. B469, 25 (1999).
[31] N. Fettes and U.-G. Meissner, Nucl. Phys. A693, 693 (2001), hep-ph/0110130.
[32] H. C. Schroder et al., Eur. Phys. J. C21, 473 (2001).
[33] P. Hauser et al., Phys. Rev. C58, 1809 (1998).
[34] D. R. Entem and R. Machleidt, Phys. Rev. C68, 041001 (2003), nucl-th/0304018.
[35] P. Buttiker and U.-G. Meissner, Nucl. Phys. A668, 97 (2000), hep-ph/9908247.
[36] D. R. Entem and R. Machleidt, Phys. Rev. C66, 014002 (2002), nucl-th/0202039.