Research Article

On Harmonic Functions Defined by Derivative Operator

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Let $S_\Delta$ denote the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$, where $h(z) = z + \sum_{k=2}^\infty a_kz^k$, $g(z) = \sum_{k=1}^\infty b_kz^k(|b_1| < 1)$. In this paper, we introduce the class $M_\Delta(n, \lambda, \alpha)$ of functions $f = h + \bar{g}$ which are harmonic in $U$. A sufficient coefficient of this class is determined. It is shown that this coefficient bound is also necessary for the class $M_\Delta(n, \lambda, \alpha)$. We call $h$ the analytic part and $g$ the coanalytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\Delta$ is that $|h'(z)| > |g'(z)|$ in $\Delta$; see [2].

Denote by $S_\Delta$ the class of functions $f = h + \bar{g}$ that are harmonic, univalent, and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_\Delta$, we may express the analytic functions $h$ and $g$ as

\[ h(z) = z + \sum_{k=2}^\infty a_kz^k, \quad g(z) = \sum_{k=1}^\infty b_kz^k, \quad |b_1| < 1. \] (1.1)
Observe that \( S_\mathcal{A} \) reduces to \( S \), the class of normalized univalent analytic functions, if the coanalytic part of \( f \) is zero. Also, denote by \( S_\mathcal{A}^* \) the subclasses of \( S_\mathcal{A} \) consisting of functions \( f \) that map \( U \) onto starlike domain.

For \( f = h + \overline{g} \) given by (1.1), we define the derivative operator introduced by authors (see [1]) of \( f \) as

\[
\mathfrak{D}_n h(z) = \mathfrak{D}_n h(z) + (-1)^n \mathfrak{D}_n \overline{g}(z), \quad n, \lambda \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad z \in U, \tag{1.2}
\]

where \( \mathfrak{D}_n h(z) = z + \sum_{k=2}^\infty k^n C(\lambda, k) a_k z^k \), \( \mathfrak{D}_n g(z) = \sum_{k=1}^\infty k^n C(\lambda, k) b_k z^k \), and \( C(\lambda, k) = \binom{k+\lambda-1}{\lambda-1} \).

We let \( M_\mathcal{A}(n, \lambda, \alpha) \) denote the family of harmonic functions \( f \) of the form (1.1) such that

\[
\text{Re} \left\{ \frac{\mathfrak{D}_{n+1} f(z)}{\mathfrak{D}_n f(z)} \right\} > \alpha, \quad 0 \leq \alpha < 1,
\]

where \( \mathfrak{D}_n f \) is defined by (1.2).

If the coanalytic part of \( f = h + \overline{g} \) is identically zero, then the class \( M_\mathcal{A}(n, \lambda, \alpha) \) turns out to be the class \( R^n_1(a) \) introduced by Al-Shaqsi and Darus [1] for the analytic case.

Let \( M_\mathcal{K}(n, \lambda, \alpha) \) denote that the subclass of \( M_\mathcal{A}(n, \lambda, \alpha) \) consists of harmonic functions \( f_n = h + \overline{g_n} \) such that \( h \) and \( g_n \) are of the form

\[
h(z) = z - \sum_{k=2}^\infty |a_k| z^k, \quad g_n(z) = (-1)^n \sum_{k=1}^\infty |b_k| z^k. \tag{1.4}
\]

It is clear that the class \( M_\mathcal{A}(n, \lambda, \alpha) \) includes a variety of well-known subclasses of \( S_\mathcal{A} \). For example, \( M_\mathcal{A}(0, 0, \alpha) \equiv S_\mathcal{A}^* (\alpha) \) is the class of sense-preserving, harmonic, univalent functions \( f \) which are starlike of order \( \alpha \) in \( U \), that is, \( \arg(\text{re}^{\theta}(z)) > \alpha \), and \( M_\mathcal{A}(1, 0, \alpha) \equiv M_\mathcal{K}(0, 1, \alpha) \equiv \mathcal{A}_\mathcal{K}(\alpha) \) is the class of sense-preserving, harmonic, univalent functions \( f \) which are convex of order \( \alpha \) in \( U \), that is, \( \arg((\text{re}^{\theta}) f(\text{re}^{\theta})) > \alpha \). Note that the classes \( S_\mathcal{A}^* \) and \( \mathcal{A}_\mathcal{K}(\alpha) \) were introduced and studied by Jahangiri [3]. Also we notice that the class \( M_\mathcal{K}(n, 0, \alpha) \) is the class of Salagean-type harmonic univalent functions introduced by Jahangiri et al. [4]; and \( M_\mathcal{K}(0, \lambda, \alpha) \) is the class of Ruscheweyh-type harmonic univalent functions studied by Murugusundaramoorthy and Vijaya [5].

In 1984, Clunie and Sheil-Small [2] investigated the class \( S_\mathcal{A} \) as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on \( S_\mathcal{A} \) and its subclasses such that Silverman [6], Silverman and Silvia [7], and Jahangiri [3, 8] studied the harmonic univalent functions. Jahangiri and Silverman [9] prove the following theorem.

**Theorem 1.1.** Let \( f = h + \overline{g} \) given by (1.1). If

\[
\sum_{k=2}^\infty k(|a_k| + |b_k|) \leq 1 - |b_1|, \tag{1.5}
\]

then \( f \) is sense-preserving, harmonic, and univalent in \( U \) and \( f \in S_\mathcal{A}^* \) consists of functions in \( S_\mathcal{A} \) which are starlike in \( U \).

The condition (1.5) is also necessary if \( f \in \mathcal{C}H \equiv M_\mathcal{A}(0, 0, 0) \).

In this paper, we will give sufficient condition for functions \( f = h + \overline{g} \), where \( h \) and \( g \) are given by (1.1) to be in the class \( M_\mathcal{A}(n, \lambda, \alpha) \); and it is shown that this coefficient condition is
also necessary for functions in the class $M_{\overline{U}}(n, \lambda, \alpha)$. Also, we obtain distortion theorems and characterize the extreme points for functions in $M_{\overline{U}}(n, \lambda, \alpha)$. Closure theorems and application of neighborhood are also obtained.

## 2. Coefficient bounds

We begin with a sufficient coefficient condition for functions in $M_{\overline{U}}(n, \lambda, \alpha)$.

**Theorem 2.1.** Let $f = h + \overline{g}$ be given by (1.1). If

$$
\sum_{k=1}^{\infty} [(k - \alpha)|a_k| + (k + \alpha)|b_k|]k^nC(\lambda, k) \leq 2(1 - \alpha),
$$

(2.1)

where $a_1 = 1$, $n, \lambda \in \mathbb{N}_0$, $C(\lambda, k) = \binom{k+1}{\lambda-1}$, and $0 \leq \alpha < 1$, then $f$ is sense-preserving, harmonic, univalent in $U$, and $f \in M_{\overline{U}}(n, \lambda, \alpha)$.

**Proof.** If $z_1 \neq z_2$, then

$$
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| \geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right|
$$

$$
= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right|
$$

$$
> 1 - \left| \frac{\sum_{k=1}^{\infty} [(k + \alpha)k^nC(\lambda, k)/(1 - \alpha)] |b_k|}{1 - \sum_{k=2}^{\infty} [(k - \alpha)k^nC(\lambda, k)/(1 - \alpha)] |a_k|} \right| \geq 0,
$$

(2.2)

which proves univalence. Note that $f$ is sense-preserving in $U$. This is because

$$
|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1}
$$

$$
> 1 - \sum_{k=2}^{\infty} \frac{(k - \alpha)k^nC(\lambda, k)}{1 - \alpha} |a_k|
$$

$$
\geq \sum_{k=1}^{\infty} \frac{(k + \alpha)k^nC(\lambda, k)}{1 - \alpha} |b_k|
$$

$$
> \sum_{k=1}^{\infty} \frac{(k + \alpha)k^nC(\lambda, k)}{1 - \alpha} |b_k||z|^{k-1} \geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.
$$

(2.3)

Using the fact that $\text{Re} \omega > \alpha$ if and only if $|1 - \alpha + \omega| \geq |1 + \alpha - \omega|$, it suffices to show that

$$
|(1 - \alpha)\mathcal{R}_f(z) + \mathcal{R}_f(z) - (1 + \alpha)\mathcal{R}_f(z) - \mathcal{R}_f(z)| \geq 0.
$$

(2.4)
Substituting $\mathcal{D}_n^m f(z)$ in (2.4) yields, by (2.1), we obtain

$$
\left|(1 - \alpha)\mathcal{D}_n^m f(z) + \mathcal{D}_n^{m+1} f(z)\right| - \left|(1 + \alpha)\mathcal{D}_n^m f(z) - \mathcal{D}_n^{m+1} f(z)\right|
$$

$$
= \left|(2 - \alpha)z + \sum_{k=1}^{\infty}(k + 1 - \alpha)k^n C(\lambda, k)a_k z^k - (-1)^n \sum_{k=1}^{\infty}(k - 1 + \alpha)k^n C(\lambda, k)b_k z^k\right|
$$

$$
- \left|- \alpha z + \sum_{k=1}^{\infty}(k - 1 - \alpha)k^n C(\lambda, k)a_k z^k - (-1)^n \sum_{k=1}^{\infty}(k + 1 - \alpha)k^n C(\lambda, k)b_k z^k\right|
$$

$$
\geq 2(1 - \alpha)\left|1 - \sum_{k=2}^{\infty}\frac{(k - \alpha)k^n C(\lambda, k)}{1 - \alpha} a_k \right| + \sum_{k=1}^{\infty}\frac{(k + \alpha)k^n C(\lambda, k)}{1 - \alpha} |b_k| \left|\frac{z}{1 - \alpha}\right|^{k-1} z^k
$$

(2.5)

$$
\geq 2(1 - \alpha)\left(1 - \sum_{k=2}^{\infty}\frac{(k - \alpha)k^n C(\lambda, k)}{1 - \alpha} a_k \right) - \sum_{k=1}^{\infty}\frac{(k + \alpha)k^n C(\lambda, k)}{1 - \alpha} b_k
$$

This last expression is nonnegative by (2.1), and so the proof is complete. \(\Box\)

The harmonic function

$$
f(z) = z + \sum_{k=2}^{\infty}\frac{1 - \alpha}{k - \alpha} k^n C(\lambda, k)y_k z^k + \sum_{k=1}^{\infty}\frac{1 - \alpha}{k + \alpha} k^n C(\lambda, k)x_k z^k
$$

(2.6)

where $n, \lambda \in \mathbb{N}_0$ and $\sum_{k=2}^{\infty}|x_k| + \sum_{k=1}^{\infty}|y_k| = 1$ show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.6) are in $M_{w}(n, \lambda, \alpha)$ because

$$
\sum_{k=1}^{\infty}\left|\frac{k - \alpha}{1 - \alpha} a_k + \frac{k + \alpha}{1 - \alpha} b_k\right| k^n C(\lambda, k) = 1 + \sum_{k=2}^{\infty}|y_k| + \sum_{k=1}^{\infty}|x_k| = 2.
$$

(2.7)

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h + g_n$, where $h$ and $g_n$ are of the form (1.4).

**Theorem 2.2.** Let $f_n = h + g_n$ be given by (1.4). Then $f_n \in M_{w}(n, \lambda, \alpha)$ if and only if

$$
\sum_{k=1}^{\infty}\left|\frac{(k - \alpha)}{1 - \alpha} |a_k| + \frac{(k + \alpha)}{1 - \alpha} |b_k|\right| k^n C(\lambda, k) \leq 2(1 - \alpha),
$$

(2.8)

where $a_1 = 1$, $n, \lambda \in \mathbb{N}_0$, $C(\lambda, k) = (k + 1 - \lambda)^{-1}$, and $0 \leq \alpha < 1$.

**Proof.** Since $M_{w}(n, \lambda, \alpha) \subset M_{w}(n, \lambda, \alpha)$, we only need to prove the “if and only if” part of the theorem. To this end, for functions $f_n$ of the form (1.4), we notice that the condition (1.3) is equivalent to

$$
\text{Re}\left\{\frac{(1 - \alpha)z - \sum_{k=2}^{\infty}(k - \alpha)k^n C(\lambda, k)a_k z^k - (-1)^{2\alpha}\sum_{k=1}^{\infty}(k + \alpha)k^n C(\lambda, k)b_k z^k}{z - \sum_{k=2}^{\infty}k^n C(\lambda, k)a_k z^k + (-1)^{2\alpha}\sum_{k=1}^{\infty}k^n C(\lambda, k)b_k z^k}\right\} \geq 0.
$$

(2.9)
The above required condition (2.9) must hold for all values of $z$ in $U$. Upon choosing the values of $z$ on the positive real axis, where $0 \leq z = r < 1$, we must have

$$
1 - \frac{\sum_{k=2}^{\infty} (k-\alpha)k\alpha C(\lambda, k)a_k r^{k-1} - \sum_{k=1}^{\infty} (k + \alpha)k\alpha C(\lambda, k)b_k r^{k-1}}{\sum_{k=2}^{\infty} k\alpha C(\lambda, k)a_k r^{k-1} + \sum_{k=1}^{\infty} k\alpha C(\lambda, k)b_k r^{k-1}} \geq 0. \tag{2.10}
$$

If the condition (2.8) does not hold, then the numerator in (2.10) is negative for $r$ sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.8) is negative. This contradicts the required condition for $f_n \in M_{\mathcal{M}}(n, \lambda, \alpha)$ and so the proof is complete. 

### 3. Distortion bounds

In this section, we will obtain distortion bounds for functions in $M_{\mathcal{M}}(n, \lambda, \alpha)$.

**Theorem 3.1.** Let $f_n \in M_{\mathcal{M}}(n, \lambda, \alpha)$. Then for $|z| = r < 1$, one has

$$
|f_n(z)| \leq (1 + |b_1|)r + \frac{1}{2^n(\lambda + 1)} \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right) r^2, \tag{3.1}
$$

$$
|f_n(z)| \geq (1 - |b_1|)r - \frac{1}{2^n(\lambda + 1)} \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right) r^2.
$$

**Proof.** We only prove the left-hand inequality. The proof for the right-hand inequality is similar and will be omitted. Let $f_n \in M_{\mathcal{M}}(n, \lambda, \alpha)$. Taking the absolute value of $f_n$, we obtain

$$
|f_n(z)| = \left| z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k z^k \right|
$$

$$
\geq (1 - |b_1|)r - \sum_{k=1}^{\infty} (|a_k| + |b_k|) r^k
$$

$$
\geq (1 - |b_1|)r - \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k
$$

$$
\geq (1 - |b_1|)r - \frac{1 - \alpha}{(2 - \alpha)2^n(\lambda + 1)} \left( \sum_{k=2}^{\infty} (k - \alpha)k\alpha C(\lambda, k) \frac{1 - \alpha}{1 - \alpha} |a_k| + \frac{(k + \alpha)k\alpha C(\lambda, k)}{1 - \alpha} |b_k| \right) r^2
$$

$$
\geq (1 - |b_1|)r - \frac{1 - \alpha}{(2 - \alpha)2^n(\lambda + 1)} \left( \sum_{k=2}^{\infty} (k - \alpha)k\alpha C(\lambda, k) \frac{1 - \alpha}{1 - \alpha} |a_k| + \frac{(k + \alpha)k\alpha C(\lambda, k)}{1 - \alpha} |b_k| \right) r^2
$$

$$
\geq (1 - |b_1|)r - \frac{1 - \alpha}{(2 - \alpha)2^n(\lambda + 1)} \left( 1 - \frac{1 + \alpha}{1 - \alpha} |b_1| \right) r^2. \tag{3.2}
$$

The functions

$$
f(z) = z + |b_1|z^2 + \frac{1}{2^n(\lambda + 1)} \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right) z^2,
$$

$$
f(z) = (1 - |b_1|)z - \frac{1}{2^n(\lambda + 1)} \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right) z^2 \tag{3.3}
$$

for $|b_1| \leq (1 - \alpha)/(1 + \alpha)$ show that the bounds given in Theorem 3.1 are sharp. 

□
The following covering result follows from the left-hand inequality in Theorem 3.1.

Corollary 3.2. If the function \( f_n = h + \overline{g_n} \), where \( h \) and \( g \) given by (1.4) are in \( M_{\overline{F}}(n, \lambda, \alpha) \), then

\[
\left\{ w : |w| < \frac{2^{n+1}(\lambda + 1) - 1 - (2^n(\lambda + 1) - 1)\alpha}{2^n(\lambda + 1)(2 - \alpha)} - \frac{2^{n+1}(\lambda + 1) - 1 - (2^n(\lambda + 1) + 1)\alpha}{2^n(\lambda + 1)(2 - \alpha)}|b_1| \right\} \subset f_n(U).
\] (3.4)

4. Convolution, convex combination, and extreme points

In this section, we show that the class \( M_{\overline{F}}(n, \lambda, \alpha) \) is invariant under convolution and convex combination of its member.

For harmonic functions \( f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k \) and \( F_n(z) = z - \sum_{k=1}^{\infty} b_k z^k \), the convolution of \( f_n \) and \( F_n \) is given by

\[
(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k B_k z^k.
\] (4.1)

Theorem 4.1. For \( 0 \leq \beta \leq \alpha < 1 \), let \( f_n \in M_{\overline{F}}(n, \lambda, \alpha) \) and \( F_n \in M_{\overline{F}}(n, \lambda, \beta) \). Then \( f_n * F_n \in M_{\overline{F}}(n, \lambda, \alpha) \subset M_{\overline{F}}(n, \lambda, \beta) \).

Proof. We wish to show that the coefficients of \( f_n * F_n \) satisfy the required condition given in Theorem 2.2. For \( F_n \in M_{\overline{F}}(n, \lambda, \beta) \), we note that \( |A_k| \leq 1 \) and \( |B_k| \leq 1 \). Now, for the convolution function \( f_n * F_n \), we obtain

\[
\sum_{k=2}^{\infty} \frac{(k - \beta) \lambda^k C(\lambda, k)}{1 - \beta} |a_k||A_k| + \sum_{k=1}^{\infty} \frac{(k + \beta) \lambda^k C(\lambda, k)}{1 - \beta} |b_k||B_k| 
\leq \sum_{k=2}^{\infty} \frac{(k - \beta) \lambda^k C(\lambda, k)}{1 - \beta} |a_k| + \sum_{k=1}^{\infty} \frac{(k + \beta) \lambda^k C(\lambda, k)}{1 - \beta} |b_k|
\leq \sum_{k=2}^{\infty} \frac{(k - \alpha) \lambda^k C(\lambda, k)}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(k + \alpha) \lambda^k C(\lambda, k)}{1 - \alpha} |b_k|\leq 1,
\] (4.2)

since \( 0 \leq \beta \leq \alpha < 1 \) and \( f_n \in M_{\overline{F}}(n, \lambda, \alpha) \). Therefore \( f_n * F_n \in M_{\overline{F}}(n, \lambda, \alpha) \subset M_{\overline{F}}(n, \lambda, \beta) \).

We now examine the convex combination of \( M_{\overline{F}}(n, \lambda, \alpha) \).

Let the functions \( f_n(z) \) be defined, for \( j = 1, 2, \ldots, \), by

\[
f_{n,j}(z) = z - \sum_{k=2}^{\infty} |a_{k,j}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,j}| z^k.
\] (4.3)

Theorem 4.2. Let the functions \( f_{n,j}(z) \) defined by (4.3) be in the class \( M_{\overline{F}}(n, \lambda, \alpha) \) for every \( j = 1, 2, \ldots, m \). Then the functions \( t_j(z) \) defined by

\[
t_j(z) = \sum_{j=1}^{m} c_j f_{n,j}(z), \quad 0 \leq c_j \leq 1
\] (4.4)

are also in the class \( M_{\overline{F}}(n, \lambda, \alpha) \), where \( \sum_{j=1}^{m} c_j = 1 \).
Hence the theorem follows.

**Proof.** According to the definition of $t_j$, we can write

$$t_j(z) = z - \sum_{k=2}^{\infty} \left( \sum_{j=1}^{m} c_j a_{k,j} \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m} c_j b_{n,j} \right) z^k. \quad (4.5)$$

Further, since $f_{n_j}(z)$ are in $M_{\mathcal{F}^n}(n, \lambda, \alpha)$ for every $j = 1, 2, \ldots$, then by (2.8), we have

$$\sum_{k=1}^{\infty} \left[ (k - a) \left( \sum_{j=1}^{m} c_j |a_{k,j}| \right) + (k + a) \left( \sum_{j=1}^{m} c_j |b_{k,j}| \right) \right] k^n C(\lambda, k) \right]$$

$$= \sum_{k=1}^{m} c_j \left( \sum_{k=1}^{\infty} \left[ (k - a) |a_{n,j}| + (k + a) |b_{n,j}| \right] k^n C(\lambda, k) \right)$$

$$\leq \sum_{j=1}^{m} c_j 2(1 - \alpha) \leq 2(1 - \alpha). \quad (4.6)$$

Hence the theorem follows. □

**Corollary 4.3.** The class $M_{\mathcal{F}^n}(n, \lambda, \alpha)$ is closed under convex linear combination.

**Proof.** Let the functions $f_{n_j}(z)$ ($j = 1, 2$) defined by (4.1) be in the class $M_{\mathcal{F}^n}(n, \lambda, \alpha)$. Then the function $\Psi(z)$ defined by

$$\Psi(z) = \mu f_{n_1}(z) + (1 - \mu) f_{n_2}(z), \quad 0 \leq \mu \leq 1 \quad (4.7)$$

is in the class $M_{\mathcal{F}^n}(n, \lambda, \alpha)$. Also, by taking $m = 2$, $t_1 = \mu$, and $t_2 = (1 - \mu)$ in Theorem 4.1, we have the corollary.

Next we determine the extreme points of closed convex hulls of $M_{\mathcal{F}^n}(n, \lambda, \alpha)$ denoted by $\text{clco}M_{\mathcal{F}^n}(n, \lambda, \alpha)$.

**Theorem 4.4.** Let $f_n$ be given by (1.4). Then $f_n \in M_{\mathcal{F}^n}(n, \lambda, \alpha)$ if and only if

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_n(z)), \quad (4.8)$$

where $h_1(z) = z$, $h_k(z) = z - ((1 - \alpha) / (k - \alpha)) k^n C(\lambda, k)) z^k$, $k = 2, 3, \ldots$, $g_n(z) = z + (-1)^n ((1 - \alpha) / (k + \alpha)) k^n C(\lambda, k)) z^k$, $k = 1, 2, 3, \ldots$, and $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$, $X_k \geq 0$, $Y_k \geq 0$. In particular, the extreme points of $M_{\mathcal{F}^n}(n, \lambda, \alpha)$ are \{h_k\} and \{g_n\}.

**Proof.** For the functions $f_n$ of the form (4.8), we have

$$f_n(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_n(z))$$

$$= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1 - \alpha}{(k - \alpha) k^n C(\lambda, k)} X_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{1 - \alpha}{(k + \alpha) k^n C(\lambda, k)} Y_k z^k. \quad (4.9)$$

Then

$$\sum_{k=2}^{\infty} \frac{(k - \alpha) k^n C(\lambda, k)}{1 - \alpha} |a_k| + \sum_{k=1}^{\infty} \frac{(k + \alpha) k^n C(\lambda, k)}{1 - \alpha} |b_k| = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1, \quad (4.10)$$

and so \( f_n \in \text{clco} M_{n, \alpha}(n, \alpha) \).

Conversely, suppose that \( f_n \in \text{clco} M_{n, \alpha}(n, \alpha) \). Setting

\[
X_k = \frac{(k - \alpha)k^n C(\lambda, k)}{1 - \alpha} |a_k|, \quad 0 \leq X_k \leq 1, \; k = 2, 3, \ldots,
\]

\[
Y_k = \frac{(k + \alpha)k^n C(\lambda, k)}{1 - \alpha} |b_k|, \quad 0 \leq Y_k \leq 1, \; k = 1, 2, 3, \ldots,
\]

and \( X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \). Therefore, \( f_n \) can be written as

\[
f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k z^k
\]

\[
= z - \sum_{k=2}^{\infty} \frac{(1 - \alpha)X_k}{(k - \alpha)k^n C(\lambda, k)} z^k + (-1)^n \sum_{k=1}^{\infty} \frac{(1 - \alpha)Y_k}{(k + \alpha)k^n C(\lambda, k)} z^k
\]

\[
= z + \sum_{k=2}^{\infty} \left( h_k(z) - z \right) X_k + \sum_{k=1}^{\infty} \left( g_n(z) - z \right) Y_k
\]

\[
= \sum_{k=2}^{\infty} h_k(z) X_k + g_n(z) Y_k + z \left( 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \right)
\]

\[
= \sum_{k=1}^{\infty} (h_k(z) X_k + g_n(z) Y_k), \text{ as required.}
\]

\[\square\]

Using Corollary 4.3 we have \( \text{clco} M_{n, \alpha}(n, \alpha) = M_{n, \alpha}(n, \alpha) \). Then the statement of Theorem 4.4 is really for \( f \in M_{n, \alpha}(n, \alpha) \).

5. An application of neighborhood

In this section, we will prove that the functions in a neighborhood of \( M_{n, \alpha}(n, \alpha) \) are starlike harmonic functions.

Following [10], we defined the \( \delta \)-neighborhood of a function \( f \in \mathcal{H} \) by

\[
\mathcal{N}_\delta(f) = \left\{ F(z) = z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k : \sum_{k=2}^{\infty} k \left[ |a_k - A_k| + |b_k - B_k| \right] + |b_1 - B_1| \leq \delta \right\},
\]

where \( \delta > 0 \).

**Theorem 5.1.** Let

\[
\delta = \frac{(2 - \alpha)2^n (\lambda + 1) - 1 + \alpha - [(2 - \alpha)2^n (\lambda + 1) - 1 - \alpha] |b_1|}{(2 - \alpha)2^n (\lambda + 1)}.
\]

Then \( \mathcal{N}_\delta(M_{n, \alpha}(n, \alpha)) \subset \mathcal{H} \).
Proof. Suppose $f_n \in M_p(\pi, \alpha, \beta)$. Let $F_n = H + G_n \in \mathcal{A}_0(f_n)$, where $H = z - \sum_{k=1}^{\infty} A_k z^k$ and $G_n = (-1)^n \sum_{k=1}^{\infty} B_k z^k$. We need to show that $F_n \in \mathcal{C}H$. In other words, it suffices to show that $F_n$ satisfies the condition $\mathcal{T}(F) = \sum_{k=2}^{\infty} k \left( |A_k| + |B_k| \right) + |B_1| \leq 1$. We observe that

$$
\mathcal{T}(F) = \sum_{k=2}^{\infty} k \left[ |A_k| + |B_k| + |B_1| \right]
= \sum_{k=2}^{\infty} k \left[ |A_k - a_k + a_k| + |B_k - b_k + b_k| + |B_1 - b_1 + b_1| \right]
= \sum_{k=2}^{\infty} k \left[ |A_k - a_k| + |B_k - b_k| \right] + \sum_{k=2}^{\infty} k \left[ |a_k| + |b_k| \right] + |B_1 - b_1| + |b_1|
= \left( \sum_{k=2}^{\infty} k \left[ |A_k - a_k| + |B_k - b_k| \right] + |B_1 - b_1| \right) + \sum_{k=2}^{\infty} k \left[ |a_k| + |b_k| \right] + |b_1|
= \delta + |b_1| + \sum_{k=2}^{\infty} k \left[ |a_k| + |b_k| \right]
= \delta + |b_1| + \frac{1 - \alpha}{(2 - \alpha)^2(\lambda + 1)} \sum_{k=2}^{\infty} \left[ 2 - \alpha \right] \left[ \frac{2 - \alpha}{1 - \alpha} \left| a_k \right| + \frac{2 + \alpha}{1 - \alpha} \left| b_k \right| \right] 2^n(\lambda + 1)
\leq \delta + |b_1| + \frac{1 - \alpha}{(2 - \alpha)^2(\lambda + 1)} \sum_{k=2}^{\infty} \left[ \frac{k - \alpha}{1 - \alpha} \left| a_k \right| + \frac{k + \alpha}{1 - \alpha} \left| b_k \right| \right] k^n C(\lambda, k)
\leq \delta + |b_1| + \frac{1 - \alpha}{(2 - \alpha)^2(\lambda + 1)} \left( 1 - \frac{1 + \alpha}{1 - \alpha} |b_1| \right).
$$

Now this last expression is never greater than one if

$$
\delta \leq 1 - |b_1| - \frac{1 - \alpha}{(2 - \alpha)^2(\lambda + 1)} \left( 1 - \frac{1 + \alpha}{1 - \alpha} |b_1| \right)
= \frac{(2 - \alpha)^2(\lambda + 1) - 1 + \alpha - (2 - \alpha)^2(\lambda + 1) - 1 - \alpha) |b_1|}{(2 - \alpha)^2(\lambda + 1)}.
$$

\[ \square \]

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