Tensor power decomposition. $B_n$ case

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Abstract. Recurrent properties of fusion coefficients in a tensor power decomposition $(L^ω)^{⊗p} = \sum_ν m(ν, p) L^ν$ are studied for classical Lie algebras $B_n$ and the spinor fundamental $B_n$-module $L^ω$. We find the set of recurrent relations for fusion coefficients $m(ν, p)$. The general structure of a solution for these recurrent relations is formulated using Weyl symmetry properties for singular elements of $(L^ω)^{⊗p}$. We prove that such a solution is unique and defines explicitly coefficients $m(ν, p)$ as functions of coordinates $ν_i$ and the tensor power $p$.

1. Introduction

For non-simply laced classical Lie algebra $B_n$ we study properties of structure constants for tensor products of irreducible modules $L_{B_n}$. More precisely, we consider tensor powers $(L^ω)^{⊗p}$ of spinor fundamental module $L^ω_{B_n}$ and are interested in coefficients $m(ν, p)$ that fix the decomposition $(L^ω)^{⊗p} = \sum_ν m(ν, p) L^ν$. We find the set of recurrent relations for fusion coefficients $m(ν, p)$. The general structure of a solution for these recurrent relations is formulated using Weyl symmetry properties for singular elements of $(L^ω)^{⊗p}$. We prove that such a solution is unique and defines explicitly coefficients $m(ν, p)$ as functions of coordinates $ν_i$ and the tensor power $p$.

A major problem is to find an expression for these coefficients, i.e. to describe their explicit dependence on $ν$ and $p$.

There are numerous combinatorial studies of the problem (see [1] and [2] and references therein). On this way important general results were obtained. At the same time the combinatorial multiplicity formulas mostly describe the procedures necessary to evaluate multiplicity coefficients rather than explicit dependence of these coefficients on the parameters such as weight coordinates and power.

In this paper we demonstrate how Weyl symmetry can be used to solve this problem. Two different auxiliary tools are constructed. Firstly we obtain the set of recurrent relations for coefficients $m(ν, p)$. It is convenient to pass from the module decomposition formula to the corresponding singular element decomposition: $Ψ((L^ω)^{⊗p}) = \sum_ν m(ν, p) Ψ(L^ν)$. The r.h.s. of this relation is a set of singular weights corresponding to submodules $L^ν$. We formulate a set of restrictions imposed on $m(ν, p)$ by the Weyl symmetry. As an intermediate result we are able to predict an expression for the multiplicity function. Using the structural induction method we prove that the proposed function comply with the set of recurrent relations and that the obtained solution is unique.
2. Basic instruments and notation

We consider simple Lie algebras $\mathfrak{g} = B_n = so(2n + 1)$. The simple roots in $e$-basis are $S = \{\alpha_i = e_i - e_{i+1}, \alpha_n = e_n \mid i = 1, \ldots, n - 1\}$. Denote by $\Delta$ the root system and by $\Delta^+$ — the positive root system. The root space of $B_n$ is dual to the space of its Cartan subalgebra, $h^\ast$. Let $W$ be the Weyl group attributed to $\Delta$. The fundamental weights $\omega_i$ generate the fundamental Weyl chamber and let $\mathcal{C}$ be its closure. Highest weight modules are described by weight diagrams $N^\mu = N(L^\mu)$ and formal characters $\text{ch}(L^\mu)$. The latter are elements of the group algebra $\mathcal{E}$ attributed to $P_{B_n}$. In $\mathcal{E}$ the Weyl formula [3]

$$\text{ch}(L^\mu) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}$$

connects formal characters with singular elements of modules

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho},$$

$$\text{ch}(L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}.$$ 

Here $R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$ is the Weyl denominator, $\rho = \sum \omega_i$ and the sign factor is $\epsilon(w) := (-1)^{|\text{length}(w)|}$.

We are especially interested in the n-th fundamental weight $\omega = \frac{\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n}{2}$ (we shall omit its index) and the corresponding highest weight module $L^{\omega}$. Consider the tensor power $(L^{\omega})^{\otimes p}$ for $p \in \mathbb{Z}_+$. Our aim is to find explicit dependence of multiplicities $m(p, \nu)$ in the decomposition

$$(L^{\omega})^{\otimes p} = \sum_{\nu} m(p, \nu) L^{\nu}$$

on coordinates of $\nu$ and the power $p$. The same coefficients describe the decomposition of the singular element and the formal algebra decomposition

$$\Psi((L^{\omega})^{\otimes p}) = \sum_{\nu} m(p, \nu) \Psi^{(\nu)} = \sum_{\nu} m(p, \nu) \sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho) - \rho} = \sum_{\xi \in P} M(p, \xi) \epsilon^\xi,$$

that defines the multiplicity function

$$M(w(\xi + \rho) - \rho, \nu)_{w \in W} = \epsilon(w) m(\xi, p).$$

3. Bratteli diagrams, inductive classes and recurrent relations

Brattely diagrams provide an effective tool to study recurrent properties of tensor decompositions.

3.1. Bratteli diagram

Let us remind that Bratteli diagram $\mathcal{B}$ is an infinite graph $\mathcal{B} = (V, E)$ defined by the vertex set $V = \bigcup_{p \geq 0} V_p$ and the edge set $E = \bigcup_{p \geq 1} E_p$ partitioned into disjoint subsets $V_p$ and $E_p$ such that

(i) $V_0 = \{v_0\}$ is a single vertex,
(ii) $V_p$ and $E_p$ are finite sets,
(iii) there exist a range map $r$ and a source map $s$ from $E$ to $V$ such that $r(E_p) = V_p$, $s(E_p) = V_{p-1}$ and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v) \neq \emptyset$ for any $v \in V$ and $v \in V \setminus V_0$.

The pair $(V_p, E_p)$ (as well as the set $V_p$ itself) is called the $p$-th level of the diagram $\mathcal{B}$. A finite or infinite sequence of edges $\{e_p \mid e_p \in E_p\}_{p=1,2,\ldots}$ such that $r(e_p) = s(e_{p+1})$ is called a finite or infinite path respectively.
3.2. Tensor product decomposition in terms of Bratteli diagram

Attribute a vertex \( v = v_{\nu,p} \) in \( V_p \) to each irreducible module \( L' \) in the decomposition (2). Let \( V_p \) be the set of all such vertices \( V_p = \{ v = v_{\nu,p} | m(\nu,p) \neq 0 \} \). Let \( a_i \) be the coordinates of a weight in a basis \( X_i = \{ x_i = \frac{1}{n} e_i \} \).

Notice that we have a one-to-one correspondence between the submodules \( L' \) in (2) and vertices in \( V_p \). The vertex set \( V = \bigcup_{p \geq 0} V_p \) is thus the disjoint union of sets of weight vectors belonging to \( \mathfrak{h}^* = \mathbb{R}^n \).

Tensor product induces an action on \( \mathcal{B} \) described by the edge set \( E = \bigcup_{p \geq 1} E_p \) and the maps \( r \) and \( s \). Each edge \( e(p + 1) \in E_{p+1} \) connects vertices \( v_{\nu,p} \in V_p \) and \( \hat{v}_{\xi,p+1} \in V_{p+1} \) so that \( r(e(p + 1)) = \hat{v} \) and \( s(e(p + 1)) = v \). The product action is encoded by the number and type of vertices in \( V_{p+1} \) connected to \( v \). It is convenient to interpret \( \mathcal{B} \) as a graph embedded in \( \mathbb{R}^{n+1} \) with levels \( V_p \) identified with the set of vectors in \( \mathbb{R}^n \).

To each vertex \( v_{\nu,p} \in V(p) \) corresponds a positive integer \( l(\nu, p) \) — the number of paths with \( p \) steps leading from \( v_0 \) to \( v_{\nu,p} \). The direct consequence of these definitions is the equality \( l(\nu, p) = m(\nu, p) \).

3.3. Inductive classes and structural induction

Inductive class \( \mathcal{X} \) is a class that is defined by two sets of specifications, called initial specifications and generating specifications [4]. The initial specifications define the initial elements; the latter constitute a class, say \( \mathcal{S} \), often called the basis of \( \mathcal{X} \). The generating specifications define a (not necessarily finite) class, say \( \mathcal{M} \), of modes of combination. With each mode \( c \) a fixed number is associated called the degree. Mode of degree \( n \) is a map \( c : \mathcal{X} \times \mathcal{X} \times \ldots \times \mathcal{X} \to \mathcal{X} \). Any element of \( \mathcal{X} \) can be reached by an effective process which starts with initial elements and at each step applies mode of combination to arguments already constructed.

Let \( \mathcal{X} \) be an inductive class and \( P(x) \) — a property of \( x \). To demonstrate that for all \( x \in \mathcal{X} \) the property \( P(x) \) holds we are to show that:

- For all initial elements \( x_0 \in \mathcal{X} \) the property \( P(x_0) \) holds.
- For \( x_1, x_2, \ldots, x_n \in \mathcal{X} \) and any mode such that \( c(x_1, x_2, \ldots, x_n) = x \) the properties \( P(x_1), P(x_2), \ldots, P(x_n) \) induce \( P(x) \)

3.4. Bratteli diagram as an inductive class

Now we consider vertices of a diagram \( \mathcal{B} \) as elements of an inductive class \( \mathcal{X} \). The initial specification is \( \mathcal{S} = V_0 \). Since we connect vertices according to the tensor product rule the properties of multiplicity functions \( M(\nu, p) \) indicate how generating specifications are to be formulated.

In [5] it was shown that the multiplicity function \( M(\nu, p) \) is subject to the relation

\[
\sum_{\xi \in P} M(\xi, p + 1) e^\xi = \mathcal{N}(\mathcal{L}(\omega)) \Psi(\otimes^p \omega).
\]

This means that we have the following set of recurrent relations for \( M(\nu, p) \):

\[
M(\{a_i\}, p + 1) = \sum_{b_i = \pm 1} M(\{a_i + b_i\}, p),
\]

where \( \{a_i\}_{i=1}^n \) are the \( X \)-coordinates of \( \xi \).

We shift the origin of \( P \) to the weight \((-p)\) and consider the shifted Weyl chamber \( \overline{C}_s \) as a sublattice in \( P \) generated by the fundamental weights \( \omega_i \) with coefficients in \( \mathbb{Z}_{\geq 0} \).

Now let us remind some properties of the singular element. Singular weights vanish outside the orbit of the highest weight \( p \omega \) and they vanish also on the boundaries of \( \overline{C}_s \). We use also the
weight polytope $T(\nu)$ which is defined as the convex hull of the Weyl orbit of $\nu$. Define $I(\nu)$ as a set of weights in the interior of $T(\nu)$. In these terms we can reformulate the recurrent relations (6) for the multiplicity coefficients $m_{\{a_i\}}(p)$:

$$m(\{a_i\}, p + 1) = \sum_{\{b_i = \pm 1\} | (a_i + b_i) \in \mathbb{C}_s \cap I(p \omega)} m(\{a_i + b_i\}, p)$$  \hspace{1cm} (7)$$

In our case path numbers in $B$ are identified with multiplicity coefficients $m(\nu, p) = l(\nu, p)$. Thus $l(\nu, p)$ also satisfy the set of recurrent relations (7).

It follows that generating specifications for $B$ require that the vertex $v_{\xi, p + 1} = v_{\{a_i\}, p + 1} \in V_{p + 1}$ is connected with vertices $v_{\nu, p} = v_{\{a_i + b_i\}, p} \in V_p$ with $(a_i + b_i) \in \mathbb{C}_s \cap I(p \omega)$.
3.5. Multiplicity function and structural induction

Let assume, that $\hat{M}$ is a postulated expression for $M$ and $\hat{m}(\nu, p)$ is the corresponding multiplicity coefficient.

We are to prove that $\hat{m}(x) = l(x)$ is valid for any element of the inductive class $X = \mathfrak{B}$ (in other words we want to demonstrate that the proposed $\hat{m}(x)$ counts the number of paths from $x_0 = v_0$ to $x = \nu$ ). It is sufficient to prove that $\hat{M}$ satisfies relation (6). In the basis of $\mathfrak{B}$ we have $\hat{m}(x_0) = m(x_0) = l(x_0) = 1$. In the induction step we assume that $\hat{m}(x_i) = l(x_i)$ for all $\{x_i\}$. Then from the equalities $\hat{m}(x) = \sum_i \hat{m}(x_i)$, $l(x) = \sum_i l(x_i)$ and $\hat{m}(x_i) = l(x_i)$ it follows that $\hat{m}(x) = l(x)$.

Thus to prove that $\hat{m}$ is the solution of tensor product decomposition problem it is enough to prove that $\hat{M}$ satisfies recurrent relation (6).

3.6. Postulated expression for $\hat{M}(\nu, p)$

According to the Weyl symmetry properties of a singular element $\Psi^{(\otimes p, \omega)}$ the multiplicity function $\hat{M}(\nu, p)$ should meet the following requirements:

- $\hat{M}(\nu, p) = 0$ outside the orbit of the highest weight of $L^{\otimes p, \omega}$;
- $\hat{M}(\nu, p) = 0$ on the boundaries of $C_\nu$;
- $\hat{M}(\nu, p)$ is anti-invariant with respect to $W$.

Using these requirements we are able to construct $\hat{M}(\nu, p)$ as a rational function of $\nu$ and $p$ multiplied by an arbitrary constant $A$. To prove that $\hat{M}(\nu, p)$ is a solution of tensor product decomposition problem we apply structural induction. According to notes presented above it is sufficient to check the following:

(i) The condition $\hat{M}(p\omega, p\omega) = 1$ must be satisfied. This will fix the value of $A$.
(ii) $\hat{M}(\nu, p)$ must satisfy the relation $\hat{M}(\{a_i\}, p + 1) = \sum_{b_i = \pm 1} \hat{M}(\{a_i + b_i\}, p)$.

In the next Section we shall describe the induction step for $L^{\otimes n}_{\omega_{B_n}}$-case in details.

4. Multiplicity function for $L^{\otimes n}_{\omega_{B_n}}$

Applying the algorithm described in 3.6 and the induction basis requirement we obtain the postulated solution for $B_n$:

$$\hat{M}(\{a_i\}, p) = \prod_{k=0}^{n-1} 2^{2k} (p^2 + a_{k+1} + 2n - 1)! \left(\frac{p^2 - a_{k+1} + 2n - 1}{2}\right) \prod_{i=1}^{n} a_i \prod_{i<j} (a_i - a_j^2)$$

Let us denote by $V(a_1, \ldots, a_n)$ the generalized Vandermonde determinant

$$V(a_1, \ldots, a_n) = \begin{vmatrix}
a_1 & a_2 & \ldots & a_n \\
a_1^3 & a_2^3 & \ldots & a_n^3 \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{(2n-1)} & a_2^{(2n-1)} & \ldots & a_n^{(2n-1)}
\end{vmatrix}$$

and put $x = \frac{p^2 + 2n}{2}$. In these terms the recurrent relation takes the form

$$\prod_{k=0}^{n-1} (p + 2k + 1) V(a_1, \ldots, a_n) = \sum_{b_i = \pm 1} \prod_{i=1}^{n} (x + \frac{a_i b_i}{2}) V(a_1 - b_1, \ldots, a_n - b_n)$$ (8)
Performing the $p$-power decomposition in the l.h.s. as $\sum_{j=1}^{n} l_j p^j$
with
\[ l_{n-f} = \sum_{0 \leq k_1 < \cdots < k_f \leq n-1} (2k_1 + 1)(2k_2 + 1) \cdots (2k_f + 1) V(a_1, \ldots, a_n) \]
and $x$-power decomposition in the r.h.s as $\sum_{j=1}^{n} r_j x^j$
with
\[ r_{n-f} = \sum_{b_i=\pm 1} \sum_{0 \leq k_1 < \cdots < k_f \leq n-1} \frac{a_k b_{k_1} \cdots a_k b_{k_f}}{2^f} V(a_1 - b_1, \ldots, a_n - b_n). \] \[ (9) \]
We want to rearrange the double sum of determinants to obtain a relation like $r_j = m_j V(a_1, \ldots, a_n)$. Then the recurrent relation reduces to a set of equalities:
\[ l_{n-f} = m_n C_n^{m-f} \frac{n!}{2^{n-f}} + m_{n-1} C_n^{m-f-1} \frac{n!}{2^{n-f-1}} + \cdots + m_{n-f} \frac{1}{2^{n-f}} \] \[ (10) \]
with $C_n^k = \frac{n!}{k!(n-k)!}$.
Let us firstly rewrite expression (9) in the following form
\[ \sum_{0 \leq k_1 < \cdots < k_f \leq n-1} \frac{a_k b_{k_1} \cdots a_k b_{k_f}}{2^f} \sum_{b_i=\pm 1} b_{k_1} \cdots b_{k_f} V(a_1 - b_1, \ldots, a_n - b_n). \]
There are $2^n$ determinants in the sum, each having the factor $b_{k_1} \cdots b_{k_f} = \pm 1$. Each pair of determinants that differ only by one column can be substituted by a new one. Applied successfully, this procedure gives the following values for elements of the a determinant in $i$-th row and $j$-th column ($i, j = 1 \ldots n$):
\[ (a_j + 1)^{2i-1} + (a_j - 1)^{2i-1} = \sum_{l \in 2Z+1, 0 \leq l \leq (2i-1)} 2C_{2i-1}^j (a_j)^l \] \[ (11) \]
\[ (a_j + 1)^{2i-1} - (a_j - 1)^{2i-1} = \sum_{l \in 2Z, 0 \leq l \leq (2i-1)} 2C_{2i-1}^j (a_j)^l \] \[ (12) \]
The sum (12) appears only for $a_j = a_{k_1}, a_{k_2}, \ldots, a_{k_f}$. Extracting these expressions from the factor $\frac{a_k b_{k_1} \cdots a_k b_{k_f}}{2^f}$ and multiplying $j$-th column of the determinant by $a_j$, ($j = k_1, \ldots, k_f$) we come to the determinant $\tilde{V}([a_{q_1}], [a_{k_1}])$: 

\[
\begin{vmatrix}
\vdots & a_{q_1} & a_{k_1} & a_{q_2} & a_{k_2} & a_{k_f} & a_{q_{n-f}} \\
\vdots & O_3(a_{q_1}) & N_3(a_{k_1}) & O_3(a_{q_2}) & N_3(a_{k_2}) & O_3(a_{q_{n-f}}) & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & O_{2n-1}(a_{q_1}) & N_{2n-1}(a_{k_1}) & O_{2n-1}(a_{q_2}) & N_{2n-1}(a_{k_2}) & O_{2n-1}(a_{q_{n-f}}) & \vdots
\end{vmatrix}
\]

where
\[ N_{(2i-1)}(a_j) = \sum_{l \in 2Z, 0 \leq l \leq (2i-1)} C_{2i-1}^j (a_j)^l+1 \quad j = k_1, \ldots, k_f, \]
\[ O_{(2i-1)}(a_j) = \sum_{l \in 2Z+1, 0 \leq l \leq (2i-1)} C_{2i-1}^j (a_j)^l \quad j = q_1, \ldots, q_{n-f}. \]
Both $N_{(2i-1)}(a_j)$ and $O_{(2i-1)}(a_j)$ are polynomials of the highest degree $2i - 1$ in the i-th row and thus satisfy the properties of a Vandermonde determinant $V(a_1, \ldots, a_n)$. There are $C_{n}^{i}$
determinants with \( f \) columns containing polynomials \( N_{(2i-1)}(a_j) \) and \( n - f \) columns containing polynomials \( O_{(2i-1)}(a_j) \). The determinants differ by selections of \( a_{k_1}, \ldots, a_{k_f} \) from \( a_1, a_2, \ldots, a_n \). It turns out that in each of these determinants we can extract the part proportional to \( V(a_1, \ldots, a_n) \):

\[
\tilde{V}(\{a_{q_j}\}, \{a_{k_j}\}) = 2^n V(a_1, \ldots, a_n) + \delta(\{a_{q_j}\}, \{a_{k_j}\}).
\]

Finally, when we take the sum over different selections \( a_{k_1}, \ldots, a_{k_f} \), the remainders \( \delta(\{a_{q_j}\}, \{a_{k_j}\}) \) will also combine into a part proportional to \( V(a_1, \ldots, a_n) \):

\[
\sum_{0 \leq k_1 < \cdots < k_f \leq n-1} \frac{1}{2^f} \tilde{V}(\{a_{q_j}\}, \{a_{k_j}\}) = 2^{n-f} C_n^f V(a_1, \ldots, a_n) + T_f(n) V(a_1, \ldots, a_n)
\]

here \( T_f(n) \) is a polynomial on \( n \). This gives us the following values \( m_{n-f} = 2^{n-f} C_n^f + T(n) \). Being substituted into the expression (10) they give an identity. Recurrent relations are thus checked.

5. Conclusions

Using structural induction method we have demonstrated that tensor power decomposition for the spinor fundamental \( B_n \)-module \( L_{\omega} \) the multiplicity coefficients can be written down as explicit (rational) functions of highest weight coordinates and power \( p \).

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