On the solutions of Knizhnik-Zamolodchikov system

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Abstract

We consider the Knizhnik-Zamolodchikov system of linear differential equations. The coefficients of this system are rational functions. We prove that under some conditions the solution of KZ system is rational too. We give the method of constructing the corresponding rational solution. We deduce the asymptotic formulas for the solution of KZ system when \( \rho \) is integer.

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1 Introduction

We consider the differential system
\[ \frac{\partial W(z_1, z_2, ..., z_n)}{\partial z_j} = \rho A_j(z_1, z_2, ..., z_n)W, \quad 1 \leq j \leq n, \quad (1.1) \]
where \( A_j(z_1, z_2, ..., z_n) \) and \( W(z_1, z_2, ..., z_n) \) are \( n \times n \) matrix functions. We suppose that \( A_j(z_1, z_2, ..., z_n) \) has the form
\[ A_j(z_1, z_2, ..., z_n) = \sum_{k=1, k \neq j}^{n} \frac{P_{j,k}}{z_j - z_k}, \quad (1.2) \]
where \( z_k \neq z_\ell \) if \( k \neq \ell \). Here the matrices \( P_{j,k} \) are connected with the matrix representation of the symmetric group \( S_n \) and are defined by formulas (2.1)-(2.4). We note that the well-known Knizhnik- Zamolodchikov equation has the form (1.1),(1.2) (see [3]). This system has found applications in several areas of mathematics and physics (see [3],[5]). In the first part of the paper (section 2) we prove the following assertion:

*The fundamental solution of KZ system (1.1),(1.2) is rational, when \( \rho \) is integer.*

We give an effective method of constructing this rational solution. Our method is elementary and is based on the results of linear algebra. The more complicated approach to constructing rational solutions is given by G.Felder and A.Veselov [4]. In our previous papers [7],[9],[10] we constructed rational solutions only for the cases \( \rho = \pm 1 \). In sections 3-6 we use the variables which were introduced by A.Varchenko [13]:
\[ u_1 = z_1 - z_2, \quad u_2 = \frac{z_2 - z_3}{z_1 - z_2}, \quad ..., \quad u_{n-1} = \frac{z_{n-1} - z_n}{z_{n-2} - z_{n-1}}, \quad u_n = z_1 + z_2 + ... + z_n. \quad (1.3) \]

In terms of the variables \( u_j \) the system KZ takes the form
\[ \frac{\partial W(u_1, u_2, ..., u_n)}{\partial u_j} = \rho H_j(u_1, u_2, ..., u_n)W, \quad 1 \leq j \leq n. \quad (1.4) \]

We investigate the form of \( H_j(u_1, u_2, ..., u_n) \) in a more detailed manner than it is done in paper [13]. In particular we prove that \( H_1(u_1, u_2, ..., u_n) \) depends only on \( u_1 \) and \( H_1(u_1, u_2, ..., u_n), (2 \leq j \leq n) \) does not depend on \( u_1 \). Using the results of sections 1-4 we deduce the asymptotic formula for the solutions of
KZ system (1.1) when $\rho$ is integer and $u_j \to 0$. The corresponding result for the irrational $\rho$ is well-known [13]. In the last section of the paper we consider the examples ($n=3$ and $n=4$). As a by-product we obtain the following result:

The hypergeometric function $F(a,b,c)$ is rational when

$$a = -\rho, \quad b = -3\rho, \quad c = 1 - 2\rho, \quad (\rho \text{ is integer}). \quad (1.5)$$

## 2 Rational solutions of KZ system

1. We consider the natural representation of the symmetric group $S_n$ (see [2]). By $(i; j)$ we denote the permutation which transposes $i$ and $j$ and preserves all the rest. The $n \times n$ matrix which corresponds to $(i; j)$ is denoted by

$$P(i, j) = \{p_{k, \ell}(i, j)\}_{k, \ell=1}^{n}. \quad (i \neq j). \quad (2.1)$$

The elements $\{p_{k, \ell}(i, j)\}$ are equal to zero except

$$\{p_{k, \ell}(i, j)\} = 1, \quad \text{when either} \quad k = i, \ell = j \quad \text{or} \quad k = j, \ell = i, \quad (2.2)$$

$$\{p_{k, \ell}(i, j)\} = 1, \quad \text{when} \quad k \neq i, j. \quad (2.3)$$

2. We begin with the first equation of the KZ system (1.1), (1.2):

$$\frac{\partial W(z)}{\partial z_1} = \rho A_1(z) W(z), \quad (2.4)$$

where $z = [z_1, z_2, ..., z_n]$. In this section we assume that $\rho$ is integer.

**Theorem 2.1.** Let $\rho$ be integer. Then the fundamental matrix solution $W(z)$ of system (2.4) can be written in the form

$$W(z) = \sum_{k=2}^{n} \sum_{j=1}^{m} \frac{L_{k,j}(\xi)}{(z_1 - z_k)^j} + Q(z), \quad (2.5)$$

where $\xi = [z_2, ..., z_n]$, $m = |\rho|$ and $L_{k,j}(\xi)$ are $n \times n$ matrix functions, $Q(z)$ is a matrix polynomial in respect to $z_1$ and

$$\deg Q(z) = m(n-1), \quad \text{if} \quad \rho > 0; \quad \deg Q(z) = m, \quad \text{if} \quad \rho < 0. \quad (2.6)$$

**Proof.** Changing the variables $z_1 = 1/x$ we obtain

$$\frac{\partial V}{\partial u} = B(u, \xi)V, \quad (2.7)$$

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where

\[ V(u, \xi) = W(1/u, \xi), \quad B(u, \xi) = (-\rho/u) \sum_{k=2}^{n} P_{1,k}/(1 - uz_k). \quad (2.8) \]

From relation (2.8) we deduce the representation

\[ B(u, \xi) = -\rho \sum_{p=-1}^{\infty} u^p T_p, \quad (2.9) \]

where

\[ T_p = \sum_{k=2}^{n} P_{1,k} z_k^{p+1}, \quad p \geq -1. \quad (2.10) \]

We investigate the case when \( V(u, \xi) \) can be written in the form

\[ V(u, \xi) = \sum_{j=s}^{\infty} u^j G_j(\xi), \quad G_s \neq 0, \quad |u| < r_0. \quad (2.11) \]

Here \( G_j(\xi) \) are \( n \times n \) matrix functions. It follows from (2.7), (2.9) and (2.11) that

\[ [(q + 1) + \rho T_{-1}] G_{q+1} = -\rho \sum_{j+\ell=q} T_j G_\ell, \quad j \geq 0. \quad (2.12) \]

According to (2.10) the matrix \( T_{-1} \) has the following structure

\[ T_{-1} = (n - 2) I_n + S, \quad (2.13) \]

where \( n \times n \) matrix \( S \) is defined by the equality

\[ S = \begin{bmatrix} 2 - n & e \\ e^T & 0 \end{bmatrix}, \quad e = [1, 1, \ldots, 1]. \quad (2.14) \]

Here \( e^T \) is transposed \( e \), i.e. \( e^T = \text{col}[1, 1, \ldots, 1] \). Let us write the eigenvalues of \( T_{-1} \):

\[ \lambda_{-1} = n - 1, \quad \lambda_{-2} = n - 2, \quad \lambda_{-3} = -1. \quad (2.15) \]

The corresponding \( 1 \times n \) eigenvectors of \( T_{-1} \) have the forms

\[ U_1 = \text{col}[1, 1, \ldots, 1], \quad U_{2,j} = \text{col}[0, a_{1,j}, a_{2,j}, \ldots, a_{n-1,j}], \quad (2.16) \]
We note, that vectors $U_{2,j}$, $(1 \leq j \leq n - 2)$ are linearly independent and satisfy the condition

$$a_{1,j} + a_{2,j} + ... + a_{n-1,j} = 0.$$  

(2.18)

We shall construct $n$ linear independent solutions $Y_k(z), (1 \leq k \leq n)$ of system (2.4). We begin with the case $m = -\rho > 0$.

**Step 1.** We assume that

$$G_k = 0, \quad k \leq m(n-2) - 1.$$  

(2.19)

In this case we have (see (2.12))

$$[(n-2)I_n - T_{-1}]G_{m(n-2)} = 0.$$  

(2.20)

According to (2.16) the vector $G_{m(n-2)}(j) = U_{2,j}$ satisfies the relation (2.20). From formula (2.12) we find the coefficients $G_p(j), (m(n-2) < p < m(n-1))$.

In case $p = m(n-1)$ the matrix $[(n-1)I_n - T_{-1}]$ is not invertible because the equality

$$[(n-1)I_n - T_{-1}]U_1 = 0$$  

(2.21)

is true. Now we shall use the relations

$$P_{1,k}U_1 = U_1, \quad 2 \leq k \leq n; \quad (U_2,j, U_1) = 0, \quad 1 \leq j \leq n - 2.$$  

(2.22)

It follows from (2.22) that the right side of equality (2.12), when $q + 1 = m(n-1)$, is orthogonal to $U_1$. Hence the corresponding equation (2.12), when $q + 1 = m(n-1)$, has a solution $G_{m(n-1)}(j)$. To find the matrix coefficients $L_{k,p}(j)$ we consider the block matrices

$$L_k(j) = \text{col}[L_{k,1}(j), L_{k,2}(j), ..., L_{k,m}(j)], \quad 2 \leq k \leq n,$$  

(2.23)

$$L(j) = \text{col}[L_2(j), L_3(j), ..., L_n(j)].$$  

(2.24)

$$G(j) = \text{col}[G_1(j), G_2(j), ..., G_{m(n-1)}(j)],$$  

(2.25)

and use the equality

$$RL_j = G_j.$$  

(2.26)

The $mn(n-1) \times mn(n-1)$ matrix $R$ has the following form:

$$R = [R_1, R_2, ..., R_{n-1}],$$  

(2.27)
where the \( n \times n \) blocks \( R_k \) of the \( mn(n-1) \times mn \) matrix \( R \) are defined by the relations

\[
\{R_k\}_{p,s} = 0_{n,n}, \quad p < s; \quad \{R_k\}_{p,s} = z_k^s \left( \frac{p-1}{s-1} \right) I_n \quad p \geq s. \tag{2.28}
\]

Here \( 0_{n,n} \) is the \( n \times n \) zero matrix, \( 1 \leq k \leq n-1 \), \( 1 \leq p \leq m(n-1) \), \( 1 \leq s \leq m \). The introduced matrix \( R \) is the block confluent Vandermonde matrix. Hence the matrix \( R \) is invertible and we have

\[
L_j = R^{-1} G_j. \tag{2.29}
\]

In paper \([\cdot]\) we proved the assertion (necessary condition)

*If the solution \( Y_j(z) \) of system (2.4) is rational then relation (2.26) is true.*

In such a way we constructed \((n-2)\) linearly independent vector functions \( Y_j(z) \) \( (1 \leq j \leq n-2) \) of the form

\[
Y_j(z) = \sum_{k=2}^{n} \sum_{p=1}^{m} \frac{L_{k,p}(j)(\xi)}{(z_1 - z_k)^p}. \tag{2.30}
\]

Later we shall show that \( Y_j \) are solutions of system (2.4).

**Step 2.** Now we assume that

\[
G_k = 0, \quad -m \leq k \leq m(n-1) - 1. \tag{2.31}
\]

In this case we have

\[
G_{m(n-1)} = U_1. \tag{2.32}
\]

It follows from the relation

\[
P_{1,k} U_1 = U_1 \tag{2.33}
\]

that the vector function

\[
Y_{n-1}(z) = \prod_{k=2}^{n} (z_1 - z_k)^{-m} U_1 \tag{2.34}
\]

is a solution of system (2.4).

**Step 3.** To construct the solution \( Y_n(z) \) we consider the case when

\[
G_{-m} = U_3. \tag{2.35}
\]
Using relations (2.12) we find the coefficients \( G_p, (-m \leq p \leq m(n - 2) - 1) \). In the case \( p = m(n - 2) \) the matrix \([m(n - 2)I_n - mT_{-1}]\) is not invertible. We represent the right side of (2.12) when \( q + 1 = m(n - 2) \) in the form \( U_2 + \beta U_3 \), where \( U_2 \) is the fixed concrete vector. We remark that the vector \( U_2 \) is a linear combination of the vectors \( U_2(j) \). The vector \( U_3 \) is orthogonal to \( U_2(j) \). Hence the equation

\[
[m(n - 2)I_n - mT_{-1}]G_{m(n-2)} = \beta U_3 \tag{2.36}
\]

has a solution \( G_{m(n-2)} \). We consider the case when

\[
L_{n,j} = 0, \quad (1 \leq j \leq m). \tag{2.37}
\]

We note that in case (2.35) equality (2.29) takes the form

\[
L = R^{-1}G, \tag{2.38}
\]

where

\[
G = \text{col}[G_1, G_2, ..., G_{m(n-2)}], \quad R = [R_1, R_2, ..., R_{n-2}]. \tag{2.39}
\]

By relations (2.12) and (2.38) we find \( G_p, (-m \leq p \leq 0) \) and \( L_{k,j}, \quad (2 \leq k \leq n - 1, \quad 1 \leq j \leq m) \). In such a way we construct the vector function

\[
Y_n(z) = \sum_{k=2}^{n-1} \sum_{p=1}^{m} \frac{L_{k,p}(\xi)}{(z_1 - z_k)^p} + Q(z_1, \xi). \tag{2.40}
\]

where \( Q(z_1, \xi) \) is a polynomial of degree \( m \) in respect to \( z_1 \).

**Step 4.** To prove that the constructed vector functions \( Y_k(z), (1 \leq k \leq n - 2) \) are solutions of system (2.4) we consider the vector functions

\[
M_k(z_1, \xi) = \frac{\partial Y_k}{\partial z_1} - A_1(z_1, \xi)Y_k. \tag{2.41}
\]

The vector functions \( M_k(z_1, \xi) \) are rational in respect to \( z_1 \) with the poles in the points \( z_1 = z_j, (2 \leq j \leq n) \). It follows from (2.29) that

\[
M_k(z_1, \xi) = O(\left|z_1\right|^{-p}), \quad z_1 \to \infty, \tag{2.42}
\]

where \( p = m(n - 1) + 2 \). We introduce the polynomials

\[
q(z_1, \xi) = \prod_{s=2}^{n} (z_1 - z_s)^m, \quad q_j(z_1, \xi) = \prod_{s=2, s \neq j}^{n} (z_1 - z_s)^m. \tag{2.43}
\]
Using relations (2.42) and (2.43) we deduce that
\[ q(z_1, \xi)M_k(z_1, \xi) = R_k(z_1, \xi) + \sum_{k=2}^{n} \frac{(-m)(I - P_{1,k})}{z_1 - z_k}L_{k,m}c_k \] (2.44)

Here \( R_k(z_1, \xi) \) is a polynomial with respect to \( z_1 \) and
\[ c_k = q_k(z_k, \xi) = \prod_{2 \leq s \leq n, s \neq k} (z_k - z_s)^m. \] (2.45)

Comparing (2.42) and (2.44) we have
\[ R_k(z_1, \xi) = 0, \sum_{k=2}^{n} (I - P_{1,k})L_{k,m}c_k = 0. \] (2.46)

By \( X_k \) we denote the vectors such that
\[ P_{1,k}X_k = -X_k, \quad X_k \neq 0, \quad 2 \leq k \leq n. \] (2.47)

From (2.47) we infer that
\[ (I_n - P_{1,k})L_{k,m} = \alpha_kX_k. \] (2.48)

The vectors \( X_k \) (\( 2 \leq k \leq n \)) are linearly independent. Hence according to (2.46) we have
\[ (I_n - P_{1,k})L_{k,m} = 0. \] (2.49)

It follows directly from (2.44), (2.46) and (2.49) that \( M_k(z_1, \xi) = 0 \) (\( 2 \leq k \leq n \)). Consequently, the vector functions \( Y_k(z) \) (\( 2 \leq k \leq n \)) are solutions of system (2.4).

It was shown before that the vector function \( Y_{n-1}(z) \) is a solution of system (2.4).

**Step 5.** Now we shall prove that the vector function \( Y_n(z) \) (see (2.40)) is a solution of system (2.4). To do it we introduce the rational vector function
\[ M_n(z_1, \xi) = \frac{\partial Y_n}{\partial z_1} - A_1(z_1, \xi)Y_n. \] (2.50)

From (2.37) and (2.38) we infer the equality
\[ z_1^{m(n-2)+1}M_n(z_1, \xi) \to \tilde{u}, \quad \text{when} \quad z_1 \to \infty. \] (2.51)
Here vector $\tilde{u}$ is a fixed vector which satisfies condition (2.18). We introduce the polynomials
\[
 r(z_1, \xi) = \prod_{s=2}^{n-1} (z_1 - z_s)^m, \quad r_j(z_1, \xi) = \prod_{s=2, s\neq j}^{n-1} (z_1 - z_s)^m. \tag{2.52}
\]

In view of (2.50) and (2.52) the equality
\[
 r(z_1, \xi) M_n(z_1, \xi) = R_n(z_1, \xi) + \sum_{k=2}^{n-1} (-m)(I - P_{1,k}) L_{k,m} d_k \tag{2.53}
\]
is true. Here $R_k(z_1, \xi)$ is a polynomial with respect to $z_1$ and
\[
 d_k = r_k(z_k, \xi) = \prod_{2 \leq s \leq n-1, s \neq k} (z_k - z_s)^m. \tag{2.54}
\]

According to (2.51) the relations
\[
 R_n(z_1, \xi) = 0, \quad \sum_{k=2}^{n-1} (-m)(I - P_{1,k}) L_{k,m} d_k = \tilde{u} \tag{2.55}
\]
hold. From (2.48) and (2.55) we infer
\[
 \sum_{k=2}^{n-1} \alpha_k d_k X_k = \tilde{u}. \tag{2.56}
\]

We took into account that $X_1$ is orthogonal to $\tilde{u}$. If we consider the case $L_{p,j} = 0, \quad (1 \leq j \leq m, \quad p \neq n)$, we obtain the analogue of (2.56):
\[
 \sum_{k=2, k \neq p}^{n} \tilde{\alpha}_k d_k X_k = \tilde{u}. \tag{2.57}
\]

Using the independence of the system $X_k \quad (2 \leq k \leq n)$ and comparing (2.56) and (2.57) we receive the equalities $\alpha_p = 0, \quad (2 \leq p \leq n)$. Hence we have
\[
 \tilde{u} = 0. \tag{2.58}
\]

It follows from (2.53), (2.55) and (2.58) that $M_n(z_1, \xi) = 0$ and the vector function $Y_n(z)$ is a solution of system (2.4). It is easy to see that the constructed solutions $Y_k(z)$ are linearly independent.
Now we use the following proposition [10]:

**Proposition 2.1.** If the matrix function $W_j(z_1, ..., z_j, z_{j+1}, ..., z_n)$ is a solution of the equation

$$\frac{\partial}{\partial z_j} W_j(z) = \rho A_j(z) W_j(z), \quad (1 \leq j \leq n)$$  \hspace{1cm} (2.59)

then the matrix function $W_{j+1}(z) = P_{j,j+1} W(z_1, ..., z_j, z_{j+1}, ..., z_n)$ is a solution of the equation

$$\frac{\partial}{\partial z_{j+1}} W_{j+1}(z) = \rho A_{j+1}(z) W_{j+1}(z), \quad (1 \leq j \leq n - 1)$$  \hspace{1cm} (2.60)

**Proof.** The proposition follows directly from (2.1-(2.3)) and the relations

$$P_{j,j+1}P_{j,j+1} = P_{j+1,i}; \quad j \neq i, \quad j \neq i + 1,$$  \hspace{1cm} (2.61)

$$P_{j,j+1}P_{j,j+1} = P_{j+1,j+1}.$$  \hspace{1cm} (2.62)

**Corollary 2.1.** Every equation of system (1.1) has a fundamental rational solution when $\rho$ is integer.

We introduce the fundamental $n \times n$ matrix solution of system (2.4):

$$W(z) = [Y_1(z), Y_2(z), ..., Y_n(z)].$$  \hspace{1cm} (2.63)

Now we fix some point $z_{1,0}$ and consider the new fundamental matrix solution

$$W_1(z) = W(z) W^{-1}(z_{1,0}, \xi).$$  \hspace{1cm} (2.64)

of system (2.4), which satisfies the condition

$$W_1(z_{1,0}, \xi) = I_n.$$  \hspace{1cm} (2.65)

In the next section we shall use the normalized fundamental solution $W_1(z)$. 
3 Common solution, consistency

1. In section 2 we have considered only one equation (2.4) of KZ system (1.1). Using this result we construct a common rational solution of KZ system (1.1). We note that the KZ system is consistent, i.e. the relations

\[ \rho \left( \frac{\partial A_i}{\partial z_j} - \frac{\partial A_j}{\partial z_i} \right) + \rho^2 [A_i, A_j] = 0 \]  (3.1)

are valid.

The correctness of (3.1) follows from the properties of \( P_{i,j} \):

\[ P_{i,j} = P_{j,i}, \]  (3.2)

\[ [P_{i,j} + P_{j,k}, P_{i,k}] = 0; \quad (i, j, k \text{ are distinct}), \]  (3.3)

\[ [P_{i,j}, P_{k,l}] = 0; \quad (i, j, k, \ell \text{ are distinct}), \]  (3.4)

The following assertion can be easily verified (see [6], Ch. 12.).

**Proposition 3.1.** Let the \( n \times n \) matrix function \( W_1(z_1, z_2) \) satisfy the equation

\[ \frac{\partial}{\partial z_1} W_1 = B_1(z_1, z_2) W_1 \]  (3.5)

and the condition

\[ W_1(z_{1,0}, z_2) = I_n, \]  (3.6)

where \( z_{1,0} \) is a fixed point. If the relation

\[ \frac{\partial B_1}{\partial z_2} - \frac{\partial B_2}{\partial z_1} + [B_1, B_2] = 0 \]  (3.7)

is valid, then the matrix function

\[ U(z_1, z_2) = \frac{\partial W_1}{\partial z_2} - B_2(z_1, z_2) W_1 \]  (3.8)

satisfies equation (3.5).

From condition (3.6) and equality (3.8) we deduce the relation

\[ U(z_{1,0}, z_2) = -B_2(z_{1,0}, z_2). \]  (3.9)

Using Proposition 3.1 and relation (3.9) we have -

\[ \frac{\partial W_1}{\partial z_2} - B_2(z_1, z_2) W_1 = -W_1(z) B_2(z_{1,0}, z_2). \]  (3.10)
Now we introduce the $n \times n$ matrix function $W_2(z_{1,0}, z_2)$ which satisfies the equation
\[
\frac{\partial W_2}{\partial z_2} - B_2(z_{1,0}, z_2)W_2 = 0 \quad (3.11)
\]
and the condition
\[
W_2(z_{1,0}, z_{2,0}) = I_n, \quad (3.12)
\]
where $z_{1,0}$ and $z_{2,0}$ are fixed points. It follows from (3.11) and (3.12) the assertion.

**Proposition 3.2.** The $n \times n$ matrix function
\[
W(z_1, z_2) = W_1(z_1, z_2)W_2(z_{1,0}, z_2) \quad (3.13)
\]
satisfies the equations
\[
\frac{\partial W}{\partial z_k} - B_k(z_1, z_2)W = 0; \quad k = 1, 2 \quad (3.14)
\]
and the condition
\[
W(z_{1,0}, z_{2,0}) = I_n, \quad (3.15)
\]
We introduce the notation
\[
\mu_i = (z_{1,0}, z_{2,0}, ..., z_i, 0), \quad \xi_i = (z_i, ..., z_n), \quad (3.16)
\]
where $z_{1,0}, z_{2,0}, ..., z_{n,0}$ are fixed points. Using Theorem 2.1, Proposition 2.1 and Proposition 3.2 we obtain the main result in this section.

**Theorem 3.1.** The $n \times n$ matrix function
\[
W(z) = W_1(z)W_2(z_1, \xi_2) \times \cdots \times W_n(\mu_{n-1}, z_n) \quad (3.17)
\]
is a fundamental rational solution of KZ system (1.1). Here
\[
\frac{\partial}{\partial z_j} W_i(\mu_{i-1}, \xi_i) = \rho A_i(\mu_{i-1}, \xi_i)W_i(\mu_{i-1}, \xi_i) \quad (3.18)
\]
and
\[
W_i(\mu_{i-1}, \xi_i) = I_n. \quad (3.19)
\]
4 KZ equations in terms of new variables

Following A.Varchenko [13] we change the variables

\[ u_1 = z_1 - z_2, \quad u_k = \frac{z_k - z_{k+1}}{z_{k-1} - z_k}, \quad (2 \leq k \leq n - 1), \]  

(4.1)

\[ u_n = z_1 + z_2 + ... + z_n. \]  

(4.2)

The KZ system takes the following form

\[ \frac{\partial W}{\partial u_j} = \rho H_j(u) W, \quad (1 \leq j \leq n), \]  

(4.3)

where \( u = (u_1, u_2, ..., u_n) \). In this section we are going to investigate the form of \( H_j(u) \) in a more detailed manner than it was done in paper [13]. We represent relations (4.1) and (4.2) in the form

\[ z_k - z_{k+1} = u_1 \cdot u_2 \cdot ... \cdot u_k, \quad (1 \leq k \leq n - 1), \quad z_1 + z_2 + ... + z_n = u_n. \]  

(4.4)

We write equality (4.4) in the matrix form

\[ SZ = U, \]  

(4.5)

where

\[ Z = \text{col}[z_1, z_2, ..., z_n], \quad U = \text{col}[u_1, u_1 \cdot u_2, ..., u_1 \cdot u_2 \cdot ... \cdot u_{n-1}, u_n]. \]  

(4.6)

The elements \( s_{k,k} \) of the \( n \times n \) matrix \( S \) are equal to zero except

\[ s_{k,k} = 1, \quad s_{k,k+1} = -1, \quad s_{n,k} = 1. \]  

(4.7)

It is easy to check that the matrix \( S^{-1} \) has the following form

\[ S^{-1} = \frac{1}{n} \begin{bmatrix} n - 1 & n - 2 & n - 3 & \ldots & 1 & 1 \\ -1 & n - 2 & n - 3 & \ldots & 1 & 1 \\ -1 & -2 & n - 3 & \ldots & 1 & 1 \\ -1 & -2 & -3 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \ldots & n + 1 & 1 \\ -1 & -2 & -3 & \ldots & -n + 1 & 1 \end{bmatrix}. \]  

(4.8)
Using the formula
\[
\frac{\partial W}{\partial u_j} = \sum_{k=1}^{n} \frac{\partial z_k}{\partial u_j} \frac{\partial W}{\partial z_k}
\] (4.9)
we have
\[
H_j(u) = \sum_{k=1}^{n} \frac{\partial z_k}{\partial u_j} A_k(u).
\] (4.10)
We note that \(\frac{\partial z_k}{\partial u_j}\) are defined by the relation
\[
\frac{\partial Z}{\partial u_j} = S^{-1} \frac{\partial U}{\partial u_j}.
\] (4.11)
It follows from (4.8) and (4.11) that
\[
\frac{\partial Z}{\partial u_n} = \frac{1}{n} \text{col}[1, 1, ..., 1].
\] (4.12)
In view of (1.2) and (4.12) the relation
\[
H_n(u) = \sum_{k=1}^{n} A_k(u) = 0
\] (4.13)
is true. It means that
\[
\frac{\partial W}{\partial u_n} = 0.
\] (4.14)
*The last relation is well-known (see [3],[13]).*
We introduce the triangular \(n\times n\) matrix
\[
T = \begin{bmatrix}
1 & 1 & ... & 1 \\
0 & 1 & ... & 1 \\
... & ... & ... & ... \\
0 & 0 & ... & 1
\end{bmatrix}
\] (4.15)
and the vector
\[
Y = TU = \text{col}[y_1, y_2, ..., y_n].
\] (4.16)
The matrices \(T\) and \(S\) are connected by the equality
\[
S^{-1} = T - \frac{1}{n} C,
\] (4.17)
where

\[ C = \begin{bmatrix} 1 & 2 & \cdots & n-1 & n-1 \\ 1 & 2 & \cdots & n-1 & n-1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & \cdots & n-1 & n-1 \end{bmatrix} \]  

(4.18)

In view of (4.17) and (4.18) the equality (4.10) can be written in the form

\[ H_j(u) = \sum_{k=1}^{n} \frac{\partial y_k}{\partial u_j} A_k(u). \]  

(4.19)

Formulas (4.4), (4.15), (4.16) and (4.19) imply the following assertion.

**Proposition 4.1** The matrix functions \( H_j \) do not depend on \( u_1 \), when \( 2 \leq j \leq n \).

Let us consider separately \( H_1 \). In this case we have

\[ \frac{\partial U}{\partial u_1} = \text{col}[v_1, v_2, \ldots, v_n], \]  

(4.20)

where

\[ v_1 = 1; \quad v_k = u_2 \cdots u_k, \quad (2 \leq k \leq n - 1); \quad v_n = 0. \]  

(4.21)

According to (4.15), (4.16) and (4.20) the equalities

\[ \frac{\partial y_k}{\partial u_1} = \sum_{j=k}^{n-1} v_j \quad (1 \leq k \leq n - 1); \quad \frac{\partial y_n}{\partial u_1} = 0 \]  

(4.22)

are true. Taking into account the relations

\[ \frac{\partial y_k}{\partial u_1} - \frac{\partial y_j}{\partial u_1} = \frac{z_k - z_j}{u_1}, \quad k > j, \]  

(4.23)

we deduce that

\[ H_1 = \sum_{k>j} P_{kj}/u_1. \]  

(4.24)

Thus we have proved the assertion.

**Proposition 4.2** The matrix function \( H_1 \) in KZ system (1.1) depends only on \( u_1 \) and has form (4.24).

Let us consider the case when \( 2 \leq k \leq n \). In this case we have

\[ \frac{\partial U}{\partial u_k} = \text{col}[v_{1,k}, v_{2,k}, \ldots, v_{n,k}], \]  

(4.25)
where
\[ v_{p,k} = 0, \quad (1 \leq p \leq k - 1); \quad v_{p,k} = u_1 v_p / u_k, \quad (k \leq p \leq n - 1); \quad \]
\[ v_{n,k} = 0, \quad k < n. \quad (4.27) \]

Using again (4.15), (4.16) and (4.20) we receive the relations
\[ \frac{\partial y_s}{\partial u_k} = \sum_{p=\max(k,s)}^{n-1} v_{p,k} \quad (1 \leq k \leq n - 1); \quad \frac{\partial y_n}{\partial u_k} = 0, \quad (1 \leq k < n). \quad (4.28) \]

We introduce the denotation
\[ \alpha_{s,j,k} = \left( \frac{\partial y_s}{\partial u_k} - \frac{\partial y_j}{\partial u_k} \right) / (z_s - z_j) \quad s > j, \quad (4.29) \]

Relations (4.4), (4.29), and (4.28) (4.29) imply that
\[ \alpha_{s,j,k} = \sum_{p=\max(j,k)}^{s-1} v_p / (u_k \sum_{p=j}^{s-1} v_p). \quad (4.30) \]

Hence the following formula
\[ H_k = \sum_{s>j} \alpha_{s,j,k}(u) P_{s,j} \quad (4.31) \]
is valid. From formulas (4.26) and (4.30) we infer that
\[ \alpha_{s,j,k} = 1 / u_k, \quad s > j \geq k, \quad (4.32) \]
\[ \alpha_{s,j,k} = 1 + o(1), \quad k = j + 1, \quad s \geq j + 2, \quad (4.33) \]
\[ \alpha_{s,j,k} = o(1), \quad k > j + 1. \quad (4.34) \]
The equality \( g(u) = o(u) \) means that the function \( g(u) \) is regular in the neighborhood of \( u = 0 \) and \( g(0) = 0 \). Using formulas (4.31) -(4.34) we deduce the theorem.

**Theorem 4.1** The following relations are true
\[ H_1 = \Omega_1 / u_1, \quad H_n(u) = 0, \quad (4.35) \]
\[ H_s = \Omega_s / u_s + P_{s-1} + o(u), \quad 2 \leq s \leq n - 1, \quad (4.36) \]
where
\[ P_r = \sum_{j > r} P_{j, r}, \quad \Omega_s = P_s + P_{s+1} + \ldots + P_n. \] (4.37)

**Remark 4.1.** The formula \( H_n(u) = 0 \) and the first term of asymptotic (4.36) were found by A Varchenko [13]. The second term of asymptotic (4.36) and explicit formulas (4.24) and (4.31) are new.

**Remark 4.2.** In this section we consider arbitrary \( \rho \) (not only integer).

**Corollary 4.1.** The fundamental solution \( W(u) \) of KZ system (1.1) can be represented in the form
\[ W(u) = W_1(u_1)W(u_2, \ldots, u_n), \] (4.38)
where \( W_1(u_1) \) and \( W(u_2, \ldots, u_n) \) are fundamental solutions of the system
\[ \frac{dW_1}{du_1} = \rho H_1(u_1)W_1(u_1) \] (4.39)
and the system
\[ \frac{\partial W_j}{\partial u_j} = \rho H_j(u_2, \ldots, u_n)W_2(u_2, \ldots, u_n), \quad 2 \leq j \leq n - 1. \] (4.40)
respectively.

## 5 Eigenvalues and eigenvectors of \( \Omega_s \)

According to (4.35) the matrices \( \Omega_s \) are coefficients of the main parts of the asymptotic \( H_s \). In the next section we show that the eigenvalues and eigenvectors of \( \Omega_s \) define the asymptotic of the solution \( W(u) \) of the KZ system. Therefore in this section we find the eigenvalues and eigenvectors of \( \Omega_s \) in an explicit form. To do it we use the following result.

**Theorem 5.1.** The matrices \( \Omega_s \) have the structure
\[ \Omega_s = \begin{bmatrix} N_{s-1}I_{s-1} & 0 & \omega_s \\ 0 & \omega_s \end{bmatrix}, \quad (1 \leq s \leq n - 1), \] (5.1)
where the \((n - s + 1) \times (n - s + 1)\) matrix \( \omega_s \) and the number \( N_s \) are defined by the relations
\[ \omega_s = \begin{bmatrix} N_s & 1 & 1 & \ldots & 1 \\ 1 & N_s & 1 & \ldots & 1 \\ 1 & 1 & N_s & \ldots & 1 \\ 1 & 1 & 1 & \ldots & N_s \end{bmatrix}, \] (5.2)
\[ N_s = (n - s)(n - s - 1)/2, \quad 1 \leq s \leq n - 1. \quad (5.3) \]

**Proof.** It follows from (4.37) that non-diagonal elements of \( \Omega_s \) and the right part of (5.1) coincide. According to (5.1) and (5.2) we have

\[ P_{ij} \Omega_s P_{ij} = \Omega_s \quad \text{if either} \quad 1 \leq i, j \leq s - 1 \quad \text{or} \quad s \leq i, j \leq n, \quad (5.4) \]

The relations (4.37) imply that \( \Omega_s \) consists of \((n - s)(n - s + 1)/2\) addend of \( P_{ij} \). It means that the left upper blocks of \( \Omega_s \) and the right side of (5.1) have the same diagonal elements \( N_{s-1} \). Using again (4.37) we prove that all other diagonal elements of \( \Omega_s \) are equal to \( N_s \). The theorem is proved. From the block structure of \( \Omega_s \) we deduce the assertion.

**Corollary 5.1.** The matrices \( \Omega_s \), \( (1 \leq s \leq n - 1) \) have the common system of the eigenvectors

\[ v_k = \text{col}\left[0,0,\ldots,0,\overset{n-k-1}{\overbrace{-k}},1,1,\ldots,1\right], \quad (1 \leq k \leq n - 1), \quad (5.5) \]

\[ v_n = \text{col}\left[1,1,\ldots,1\right]. \quad (5.6) \]

The eigenvalues of \( \Omega_s \) corresponding to the vectors \( v_k \) are defined by the relations

\[ \lambda_{k,s} = N_{s-1}, \quad n \geq k > n - s + 1, \quad (5.7) \]

\[ \lambda_{k,s} = N_s - 1, \quad 1 \leq k \leq n - s + 1. \quad (5.8) \]

**Corollary 5.2.** All the eigenvalues of \( \Omega_s \) are integer and non-negative.

6 **Asymptotic of solutions of the KZ system**

By \( D \) we denote the domain

\[ z_1 > z_2 > \ldots > z_n. \quad (6.1) \]

It means that (see (4.1))

\[ u_1 > u_2 > \ldots > u_n. \quad (6.2) \]

The following proposition was obtained by A. Varchenko [13].

**Theorem 6.1.** Let us assume that \( \rho \) is irrational. Then:
1) For every vector $v_k$ there exists a unique solution $\psi_k$ of the KZ system in $D$ such that
\[
\psi_k(u) = \prod_{s=1}^{n-1} u_s^{\rho \lambda_k,s} (v_k + o(u)), \quad (1 \leq k \leq n), \quad (6.3)
\]

2) The solutions $\psi_k(u)$ form a basis in the space of the solutions of the KZ system on $D$.

We deduce the following addition to Varchenko Theorem.

**Theorem 6.2.** Let us consider KZ system (1.1), where $\rho$ is integer. The assertions 1) and 2) of Varchenko theorem 6.1 are true.

**Proof.** It follows from Theorem 3.1 that there exists the fundamental rational solution of KZ system (1.1). Then according to the results from ([8],section 2) this solution can be represented in form (6.3). The theorem is proved.

**Remark 6.1.** When $\rho$ is integer the representation (6.3) is true not only in the domain $D$ but in a neighborhood of $u = 0$.

### 7 Examples

**Example 7.1.** Let us consider the simplest case $n=3$. Using (4.31), (4.35) and (4.39), (4.40) we have
\[
\frac{\partial W}{\partial u_1} = \rho \left( \frac{\Omega_1}{u_1} \right) W, \quad (7.1)
\]
\[
\frac{\partial W}{\partial u_2} = \rho \left( \frac{P_{3,2}}{u_2} + \frac{P_{3,1}}{1 + u_2} \right) W, \quad (7.2)
\]
where
\[
\Omega_1 = P_{1,2} + P_{1,3} + P_{23} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \quad (7.3)
\]

According to Corollary 4.1 we can represent $W(u)$ in the form
\[
W(u) = W_1(u_1)W_2(u_2), \quad (7.4)
\]
where $W_1(u_1)$ and $W_2(u_2)$ are fundamental solutions of system (7.1) and (7.2) respectively. It is easy to see that
\[
W_1(u_1) = u_1^{\rho \Omega_1}. \quad (7.5)
\]
We note that the eigenvalues of $\Omega_1$ are integers ($\lambda_1 = 3$, $\lambda_2 = \lambda_3 = 0$). Hence the matrix $W_1(u_1)$ is rational when $\rho$ is integer. We introduce the matrix function

$$F(y) = W_2(y)y^\rho(1+y)^\rho, \quad y = u_2. \quad (7.6)$$

Relations (7.2) and (7.6) imply that

$$\frac{dF}{dy} = \rho\left(\frac{P_{3,2} + I}{y} + \frac{P_{3,1} + I}{1+y}\right)F. \quad (7.7)$$

We consider the constant vectors

$$w_1 = \text{col}[0, 1, -1], \quad w_2 = \text{col}[1, -2, 1]. \quad (7.8)$$

It is easy to see that

$$(P_{3,2} + I)w_1 = 0, \quad (P_{3,1} + I)w_1 = -w_2, \quad (7.9)$$

$$(P_{3,2} + I)w_2 = 3w_1 + 2w_2, \quad (P_{3,1} + I)w_2 = 2w_2, \quad (7.10)$$

We represent $F(y)$ in the form

$$F(y) = \phi_1(y)w_1 + \phi_2(y)w_2 \quad (7.11)$$

and substitute it in (7.7). In view of (7.7) and (7.11) we have

$$\phi_1' = \frac{3\rho}{y}\phi_2, \quad \phi_2' = \rho\left(\frac{2\phi_2}{y} + \frac{2\phi_2}{1+y} - \frac{\phi_1}{1+y}\right). \quad (7.12)$$

It follows from (7.12) that

$$y(1+y)\phi_1''(y) + [1+y - 2\rho(1+2y)]\phi_1' + 3\rho^2\phi_1 = 0. \quad (7.13)$$

By introducing $\psi(y) = \phi_1(-y)$ we reduce equation (7.13) to Gauss hypergeometric equation [1]:

$$y(1-y)\psi''(y) + [\gamma - (\alpha + \beta + 1)y]\psi'(y) - \alpha\beta\psi(y) = 0, \quad (7.14)$$

where

$$\alpha = -\rho, \quad \beta = -3\rho, \quad \gamma = 1 - 2\rho. \quad (7.15)$$

Let $\psi_1$ and $\psi_2$ be linearly independent solutions of equation (7.14). Then the vector functions

$$Y_k(u_2) = [\psi_k(-u_2)w_1 - \frac{3\rho}{u_2}\psi_k'(-u_2)w_2]u_2^\rho(1+u_2)^{-\rho}, \quad k = 1, 2 \quad (7.16)$$

20
are the solutions of system (7.2). It is easy to check that the vector function

$$Y_3(u_2) = u_2^\rho (1 + u_2)^\rho \text{col}[1, 1, 1]$$  \hspace{1cm} (7.17)

is the solution of system (7.2) too. Hence we deduced the assertion

**Proposition 7.1.** The fundamental solution $W(u)$ of system (7.1), (7.2) is defined by relations (7.4) and (7.5), where

$$W_2(u_2) = [Y_1(u_2), Y_2(u_2), Y_3(u_2)].$$  \hspace{1cm} (7.18)

**Remark 7.1.** Substitution (7.11) was prompted by book ([3], section 4.2). As a by-product we deduce from Proposition 7.1 the following result.

**Proposition 7.2.** The solutions of Gauss hypergeometric equation (7.14) are rational functions if

$$\alpha = -\rho, \quad \beta = -3\rho, \quad \gamma = 1 - 2\rho,$$

where $\rho$ is integer.

**Remark 7.2.** The KZ system is connected with the generalized hypergeometric equations of several variables (see [5]). Hence the Theorem 2.1 can be applied to the generalized hypergeometric equations of several variables.

**Example 7.2.** Let us consider separately the case $n=3, \rho = -1$. Using results of paper [7] we have

$$\phi_1 = \frac{1}{1 - y}, \quad \phi_2 = \frac{1}{y^2} - \frac{1}{y}.$$  \hspace{1cm} (7.20)

The corresponding hypergeometric equation (7.14) has the form

$$y(1 - y)\psi''(y) + (3 - 5y)\psi'(y) - 3\psi(y) = 0,$$

\hspace{1cm} (7.21)

**Example 7.3.** Now we consider the case $n=4$. We have

$$\frac{\partial W}{\partial u_1} = \rho \left( \frac{\Omega_1}{u_1} \right) W,$$

\hspace{1cm} (7.22)

$$\frac{\partial W}{\partial u_2} = \rho \left( \frac{\Omega_2}{u_2} + \frac{P_{1,3}}{1 + u_2} + \frac{P_{1,4}(1 + u_3)}{1 + u_2 + u_2u_3} \right) W,$$

\hspace{1cm} (7.23)

$$\frac{\partial W}{\partial u_3} = \rho \left( \frac{P_{4,3}}{u_3} + \frac{P_{4,2}}{1 + u_3} + \frac{P_{4,1}u_2}{1 + u_2 + u_2u_3} \right) W.$$  \hspace{1cm} (7.24)
where

\[ \Omega_1 = \begin{bmatrix}
3 & 1 & 1 & 1 \\
1 & 3 & 1 & 1 \\
1 & 1 & 3 & 1 \\
1 & 1 & 1 & 3 \\
\end{bmatrix}, \quad (7.25) \]

\[ \Omega_2 = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}. \quad (7.26) \]

**Remark 7.3.** For the case \( n = 4, \quad \rho = -1 \) we constructed the solution \( W(z) \) of system (2.4) in the explicit form. Using formula (3.17) we can obtain in the explicit form the solution of KZ system (1.1) when \( n = 4, \quad \rho = -1 \).

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