Parabolic equations with divergence-free drift in space $L_l^1 L_q^q$

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Abstract

In this paper we study the fundamental solution $\Gamma(t,x;\tau,\xi)$ of the parabolic operator $L_t = \partial_t - \Delta + b(t,x) \cdot \nabla$, where for every $t$, $b(t,\cdot)$ is a divergence-free vector field, and we consider the case that $b$ belongs to the Lebesgue space $L^l(0,T;L^q(\mathbb{R}^n))$. The regularity of weak solutions to the parabolic equation $L_t u = 0$ depends critically on the value of the parabolic exponent $\gamma = \frac{2}{l} + \frac{q}{q}$. Without the divergence-free condition on $b$, the regularity of weak solutions has been established when $\gamma \leq 1$, and the heat kernel estimate has been obtained as well, except for the case that $l = \infty, q = n$. The regularity of weak solutions was deemed not true for the critical case $L^\infty(0,T;L^n(\mathbb{R}^n))$ for a general $b$, while it is true for the divergence-free case, and a written proof can be deduced from the results in [7]. One of the results obtained in the present paper establishes the Aronson type estimate for critical and supercritical cases and for vector fields $b$ which are divergence-free. We will prove the best possible lower and upper bounds for the fundamental solution one can derive under the current approach. The significance of the divergence-free condition enters the study of parabolic equations rather recently, mainly due to the discovery of the compensated compactness. The interest for the study of such parabolic equations comes from its connections with Leray’s weak solutions of the Navier-Stokes equations and the Taylor diffusion associated with a vector field where the heat operator $L_t$ appears naturally.

key words: Aronson estimate, divergence-free vector field, Harnack inequality, parabolic equation, weak solution

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1 Introduction

We consider parabolic equations of second order with singular divergence-free drift
\[ \partial_t u(t,x) - \sum_{i,j=1}^n \partial_x (a_{ij}(t,x) \partial_x u(t,x)) + \sum_{i=1}^n b_i(t,x) \partial_x u(t,x) = 0, \tag{1} \]
where \((a_{ij})\) is a symmetric matrix-valued and Borel measurable function on \(\mathbb{R}^n\).

Throughout the article, we always assume that there exists a number \(\lambda > 0\) such that
\[ \lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \frac{1}{\lambda} |\xi|^2 \tag{E} \]
for every \(\xi \in \mathbb{R}^n\), and that \(b = (b_i)\) is a divergence-free vector field, i.e.
\[ \sum_{i=1}^n \partial_x b_i(t,x) = 0 \tag{S} \]
in the sense of distributions for all \(t\). Here we only deal with the case that \(b\) belongs to Lebesgue spaces \(L^1(0,T;L^q(\mathbb{R}^n))\) (or \(L^1 L^q\) for short) for \(l, q \in [1, \infty]\), and we will denote
\[ \Lambda := \|b\|_{L^1 L^q} = \left( \int_0^T \left( \int_{\mathbb{R}^n} |b(t,x)|^q dx \right)^{\frac{1}{q}} \right)^{\frac{1}{t}}. \]

Equation (1) has been well studied without the divergence-free condition (S). A classical monograph on such equation is \([?]\) by Ladyzhenskaia et al. If \(b\) is assumed to be in \(L^1(0,T;L^q(\mathbb{R}^n))\) with
\[ \gamma = \frac{2}{l} + \frac{n}{q} \leq 1, \quad l \in [2, \infty) \text{ and } q \in (n, \infty], \]
then there is a unique weak solution with Hölder regularity. If \(\gamma < 1\), in \([?]\), Aronson proved that there exist Gaussian upper and lower bounds for the fundamental solution, from which the Hölder continuity of solutions can be deduced. We call such estimate on the fundamental solution the Aronson estimate. The reason for such conditions on \(\gamma\) can be easily seen from the natural scaling property of parabolic equations. Under the following scaling transformation
\[ u^{(\rho)}(t,x) = u(\rho^2 t, \rho x), \quad a^{(\rho)}(t,x) = a(\rho^2 t, \rho x), \quad b^{(\rho)}(t,x) = \rho b(\rho^2 t, \rho x) \]
for \(\rho > 0\), if \(u\) is a solution to (1) with ellipticity constant \(\lambda\), then \(u^{(\rho)}\) is still a solution to the parabolic equation with \((a^{(\rho)}, b^{(\rho)})\) and condition (E) is still satisfied for the same \(\lambda\). However, for the drift, we have \(\|b^{(\rho)}\|_{L^1 L^q} = \rho^{1-\gamma} \|b\|_{L^1 L^q}\). So when \(\gamma = 1\), it is scaling invariant and this is called the critical case. If \(\gamma < 1\) (resp. \(\gamma > 1\), we call them subcritical (resp. supercritical). Since the Harnack inequality has its constant depending on \(n, \lambda\) and \(\Lambda\), in the supercritical case, we are unable to obtain the Harnack inequality uniformly in small scales. But in the critical and subcritical cases, we still can control solutions in small scales to obtain the Harnack inequality, hence obtain Hölder continuity. However, an exceptional case is \(L^q(0,T;L^q(\mathbb{R}^n))\), which is critical, but the Harnack inequality fails. The simple reason is that the energy estimate fails in this case.
In this article we will assume that $b$ is a divergence-free vector field, which is significant for applications to the study of weak solutions to incompressible fluid equations. There have been many works concerning this problem. In [?], assuming that $b \in L^\infty_t(W^{-1,q}_x)$, Osada established the fundamental solution estimate following the idea of Nash [?]. Then in [?], Zhang obtained the exponential decay upper bound for the fundamental solution when $m \in (1, 2]$ and $b$ satisfies the following entropy condition

$$\int_0^T \int_{\mathbb{R}^n} |b|^m \phi^2 \, dx \, dt \leq C \int_0^T \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \, dt$$  \hspace{1cm} (2)$$

for every smooth function $\phi$ on $[0, T] \times \mathbb{R}^n$ with compact support in space. Such condition can be traced back to [?], in which the previous entropy condition was first introduced for the time independent case in order to construct a semigroup theory. The Sobolev embedding allows us to deduce from the entropy condition (2) that $b \in L^\infty_t(0, T; L^\infty_x(\mathbb{R}^n))$. Therefore, the entropy condition is effectively scaling invariant when $m = 2$, i.e. it is a critical case, while it is supercritical if $m \in (1, 2)$. For the critical case, in an interesting paper [?] Semenov further developed a more general condition

$$\int_0^T \int_{\mathbb{R}^n} |b| \phi^2 \, dx \, dt \leq C_1 \int_0^T \left[ \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}^n} |\phi|^2 \, dx \right]^{\frac{1}{2}} \, dt$$

$$+ \int_0^T (C_2 + C(t)) \int_{\mathbb{R}^n} |\phi|^2 \, dx \, dt$$  \hspace{1cm} (3)$$

and obtained the existence and uniqueness of Hölder continuous weak solutions for the parabolic equation. For the supercritical case, Zhang [?] considered the following entropy condition:

$$\int_0^T \int_{\mathbb{R}^n} |b|(|\ln(1 + |b|)|^q \phi^2 \, dx \, dt \leq C \int_0^T \int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \, dt$$  \hspace{1cm} (4)$$

and he proved the existence of a bounded weak solution in this case. In fact, such weak solution must be Hölder continuous in space for each fixed time $t$. In [?, ?], assuming that $b \in L^\infty_t(\text{BMO}^{-1}_x)$ together with some extra technical conditions, the Hölder continuity of weak solution is obtained using Moser-De Giorgi’s scheme. In a recent paper [?], only assuming that $b \in L^\infty_t(\text{BMO}^{-1}_x)$, the Aronson estimate is proved. Further, the uniqueness and Hölder continuity of the weak solution is also proved. A diffusion process to the same operator $L_t$ can be constructed using the Aronson estimate. All these results are for divergence-free vector fields $b$ which belong to the critical case. Under supercritical conditions, recently in [?], assuming that $b \in L^q_{t,x} \cap L^p_{x,t}$ with $q \in (\frac{n}{2} + 1, n + 2]$, Ignatova, Kukavica and Ryzhik proved a weak Harnack inequality. The constant in the weak Harnack inequality explodes as the radius of the parabolic ball goes to zero, hence it fails to yield the Hölder continuity of weak solutions.

The strongest regularity result, in this respect, is still the Aronson estimate, which was established in [?]. The original proof of Aronson used the Harnack inequality. Later in [?, ?, ?], Stroock et al. proved the Aronson estimate directly by using Nash’s scheme [?]. Under the divergence-free condition, the dual operator is of the same
form up to a sign on $b$, so that Nash’s scheme demonstrates its full power as well for divergence-free vector fields. Inspired by Stroock et al. [?, ?, ?], our main result in this paper is the upper and lower bounds of fundamental solutions for the critical and supercritical cases $b \in L^1(0, T; L^q(\mathbb{R}^n))$ with $\gamma = \frac{2}{q} + \frac{q}{4} \in [1, 2)$ for divergence-free drift $b$.

In fact we will establish several heat kernel estimates for a range of critical and supercritical conditions on divergence-free drifts $b$. We will separate them into upper bound and lower bound. Since the divergence-free condition on drift $b$ prevents the formation of local blow up, we have the following upper bound which is of exponential decay.

**Theorem 1.** Suppose conditions (E), (S) hold and $b \in L^1(0, T; L^q(\mathbb{R}^n))$ for some $n \geq 3$, $l > 1$, $q > \frac{4}{\gamma}$ such that $1 \leq \gamma < 2$. In addition, we assume that $a$ and $b$ are smooth with bounded derivatives. Let $\Gamma(t, x; \tau, \xi)$ be the fundamental solution of (1). Then

$$\Gamma(t, x; \tau, \xi) \leq \frac{C}{(t - \tau)^{n/2}} \exp \left( m(t - \tau, x - \xi) \right), \quad (5)$$

where

$$m(t, x) = \min_{\alpha \in \mathbb{R}^n} (C(|\alpha|^2 t + |\alpha|^\mu \Lambda^\nu t^\gamma) + \alpha \cdot x)$$

with $\Lambda = \|b\|_{L^1(0, T; L^q(\mathbb{R}^n))}$, $\mu = \frac{2}{2 - \gamma + \frac{\nu}{\gamma}}$, $\nu = \frac{2 - \gamma}{2 - \gamma + \frac{\nu}{\gamma}}$ and $C = C(l, q, n, \lambda)$.

The point here is that the upper bound above only depends on $n$, $\lambda$ and $\Lambda$, but not on the bounds of the derivatives of $a$ or $b$. One restriction of Nash’s scheme is that its iteration procedure requires the bound on $b$ to be uniform in time, i.e. $l = \infty$. So in general, we use Moser’s iteration scheme instead and use cut-off functions.

When $\gamma = 1$, the upper bound (5) is in fact a Gaussian function, which further implies a Gaussian lower bound and regularity theory. Here we only write down the result in the case when $b \in L^\infty(0, T; L^q(\mathbb{R}^n))$, because it is the marginal case for which the regularity result is missing if $b$ is not assumed to be divergence-free. More explicitly, we prove the following

**Theorem 2.** The fundamental solution $\Gamma(t, x; \tau, \xi)$ to (1) satisfying conditions (E) and (S), with $b \in L^\infty(0, T; L^q(\mathbb{R}^n))$ possesses the following heat kernel estimate

$$\frac{1}{C(t - \tau)^{n/2}} \exp \left( -\frac{C|x - \xi|^2}{t - \tau} \right) \leq \Gamma(t, x; \tau, \xi) \leq \frac{C}{(t - \tau)^{n/2}} \exp \left( -\frac{|x - \xi|^2}{C(t - \tau)} \right)$$

for $t > \tau$, where $\Lambda = \|b\|_{L^\infty(0, T; L^q(\mathbb{R}^n))}$ and $C$ depends only on $n \geq 3$, $\lambda$ and $\Lambda$.

This result shows that the divergence-free condition brings extra regularity to weak solutions. For its discussion in other papers, we refer to [?, ?, ?]. Under the assumption that $b \in L^\infty(0, T; L^q(\mathbb{R}^n))$, the entropy condition (3) in [?] is satisfied, and the above two-side Gaussian estimate can therefore be deduced from Semenov’s main results too. Here we give a simpler and direct proof.

In the supercritical case $\gamma \in (1, 2)$, using upper bound (5), we still can derive a lower bound. Actually, we know the fundamental solution is conservative, and in fact, for fixed $t > \tau$, $\Gamma(t, x; \tau, \xi)$ is a probability density in $x$ (and in $\xi$ as well due to the divergence-free condition). Because the upper bound decays exponentially, we
can find a radius $\tilde{R}(t)$ such that $\Gamma$ has a lower bound inside the ball of radius $\tilde{R}(t)$. However, we can not hope too much for the lower bound for supercritical case. Using current techniques, we are able to establish the following theorem.

**Theorem 3.** Assume that $a$ and $b$ are smooth with bounded derivatives. Suppose conditions (E), (S) hold and $b \in L^1(0,T;L^q(\mathbb{R}^n))$ for some $n \geq 3$, $l \geq 2$, $q \geq 2$ such that $1 < \gamma < 2$. For any $\kappa > 0$, there is a constant $C > 0$ depending only on $\kappa, l, q, n, \lambda$ and $\Lambda = \|b\|_{L^1(0,T;L^q(\mathbb{R}^n))}$ such that

$$\Gamma(t,x;0,\xi) \geq \exp \left[ -C_l^{(\frac{2}{l}+1)(1-\gamma)} \left( \ln \frac{1}{t} \right)^{n+2} \right]$$

(6) for $x, \xi \in B(0, \kappa\tilde{R}(t))$ and small enough $t$, where $\tilde{R}(s) = C_d^{2-\gamma}/2 \ln \frac{1}{s}$ for $s > 0$.

Such form of lower bounds also appear in [?], but in a rather different setting of Dirichlet forms. Although we only deal with the lower bound in the cone $B(0,R(t))$ for small $t > 0$, by the Chapman-Kolmogorov equation, we can extend this lower bound to the whole space. So this form of lower bound is actually the essence of lower bounds in heat kernel estimates, and it determines the local behavior of solutions to the parabolic equation. In all cases the upper bound looks stronger than the lower bounds, and the lower bound in the supercritical case still fails to yield Hölder continuity of weak solutions. Therefore, in the supercritical case, the regularity problem for this kind of linear parabolic equations remains an open problem.

The rest of this paper is organized as follows. Section 2 explains how equation (1) is related to its dual equation and we prove some lemmas which will be used later. In Section 3, we will prove an Aronson type estimate for fundamental solutions under a class of critical and supercritical conditions on $b$. In particular, in the case $L^\infty(0,T;L^n(\mathbb{R}^n))$, the estimate will be Gaussian type. In Section 4, as an application of the Aronson estimate, we can prove the existence of a unique Hölder continuous weak solution for the critical case $L^\infty(0,T;L^n(\mathbb{R}^n))$ by an approximation approach. In an appendix, we prove Nash’s continuity theorem for the completeness of the paper because we will need this result in section 4.

### 2 Several technical facts

One important feature of the divergence-free condition is that the adjoint equation of (1) essentially has the same form up to a sign. Hence their fundamental solutions share essentially the same property. Consider equation (1) on $[0,T] \times \mathbb{R}^n$ and denote its fundamental solution as $\Gamma(t,x;\tau,\xi)$ with $0 \leq \tau < t \leq T$, $x, \xi \in \mathbb{R}^n$. For the adjoint equation

$$\partial_t u(t,x) - \sum_{i,j=1}^n \partial_{x_i} (a_{ij}(T-t,x) \partial_{x_j} u(t,x)) + \sum_{i=1}^n b_i(T-t,x) \partial_{x_i} u(t,x) = 0$$

(7)

and its fundamental solution $\Gamma^*_\tau(t,x;\tau,\xi)$, we have $\Gamma(t,x;\tau,\xi) = \Gamma^*_\tau(T-\tau,\xi;T-t,x)$.

We will need the following elementary facts in the proof of Aronson type estimates. The first one is the Poincaré-Wirtinger inequality for the Gaussian measures
[? Corollary 1.7.3]. In the sequel, we shall write \( C^1_b(\mathbb{R}^n) \) to be the space of functions with bounded continuous first order derivative.

**Lemma 4.** Let \( \mu \) be the standard Gaussian measure on \( \mathbb{R}^n \), i.e. \( \mu(dx) = \mu(x)dx \) with \( \mu(x) = \frac{1}{(2\pi)^{n/2}} \exp \left( -\frac{|x|^2}{2} \right) \). Then for every \( p \geq 1 \)
\[
\int_{\mathbb{R}^n} |f(x) - \bar{f}|^p \mu(dx) \leq M(p) \left( \frac{\pi}{2} \right)^p \int_{\mathbb{R}^n} |\nabla f(x)|^p \mu(dx),
\]
for any \( f \in C^1_b(\mathbb{R}^n) \), where \( \bar{f} = \int_{\mathbb{R}^n} f(x) \mu(dx) \) and
\[
M(p) = \int_{-\infty}^{\infty} |\xi|^p \frac{1}{(2\pi)^{1/2}} \exp \left( -\frac{|\xi|^2}{2} \right) d\xi.
\]

Further, setting \( \mu_r(x) = \frac{1}{r^n} \exp \left( -\frac{r^2|\xi|^2}{2} \right) \) and \( \bar{f}_r = \int_{\mathbb{R}^n} f(x) \mu_r(dx) \), one has
\[
\int_{\mathbb{R}^n} |f(x) - \bar{f}_r|^p \mu_r(dx) \leq M(p) \left( \frac{\pi}{2} \right)^p \left( \frac{r}{2\pi} \right)^{p/2} \int_{\mathbb{R}^n} |\nabla f(x)|^p \mu_r(dx).
\]

Actually, Lemma 4 can be extended to any function with weak derivative such that both sides of the Poincaré-Wirtinger inequality are well defined using a truncation and approximation argument. Also we will need the following lemma on a Riccati differential inequality.

**Lemma 5.** Suppose a non-positive valued function \( u \) is continuous and differentiable on \( [\frac{T}{2}, T] \), where \( T > 0 \) is a constant. If \( u \) satisfies the Ricatti differential inequality
\[
u'(t) \geq -\alpha + \beta u(t)^2 \quad (10)
\]
for \( t \in [\frac{T}{2}, T] \), where \( \alpha, \beta > 0 \) are two constants, then
\[
u(T) \geq \min \left\{ -\alpha T - 2 \sqrt{\frac{\alpha}{\beta}}, -\frac{8}{3\beta T} \right\}.
\]

**Proof.** If \( u(T) \geq -\alpha T - 2 \sqrt{\frac{\alpha}{\beta}} \), then the proof is done. Otherwise, integrating the differential inequality (10) from \( T/2 \) to \( T \), we have \( u(T) - u(t) \geq -\frac{\alpha T}{2} \) for any \( t \in [\frac{T}{2}, T] \). In other words, we have
\[
u(t) \leq u(T) + \frac{\alpha T}{2} \leq -\alpha T - 2 \sqrt{\frac{\alpha}{\beta}} + \frac{\alpha T}{2}
\]
which in turn yields that \( u(t) \leq -2 \sqrt{\frac{\alpha}{\beta}} \). Notice that \( u(t) \) is negative on \( [\frac{T}{2}, T] \) and therefore \( u(t)^2 \geq 4 \frac{\alpha}{\beta} \). Hence differential inequality (10) implies that
\[
u'(t) \geq \beta \left( -\frac{\alpha}{\beta} + u(t)^2 \right) \geq \beta \left( -\frac{1}{4} u(t)^2 + u(t)^2 \right) = \frac{3\beta}{4} u(t)^2
\]
for every \( t \in [\frac{T}{2}, T] \). Dividing both sides by \( u(t)^2 \) and integrating from \( t \in [\frac{T}{2}, T] \) to \( T \), we obtain that
\[
\frac{1}{u(T)} \leq -\frac{3\beta T}{8} + \frac{1}{u(t)} \leq -\frac{3\beta T}{8}.
\]
In particular \( u(T) \geq -\frac{8}{3\beta T} \) and the proof is complete. \( \Box \)
If \( \alpha \) is a function depending on \( t \), which is integrable and non-negative, then we still can derive a lower bound on \( u \).

**Lemma 6.** Let \( T > 0 \). Suppose a non-positive function \( u \) is continuous on \([\frac{T}{2}, T]\), and satisfies the following integral inequality

\[
\alpha(t) - \beta u(t) + \int_{t}^{T} \alpha(s) ds < -C_1 + C_2 =: -C_3,
\]

where \( \alpha \) is non-negative and integrable in \([\frac{T}{2}, T]\), and \( \beta > 0 \) is a constant, then

\[
u(T) \geq -\int_{\frac{T}{2}}^{T} \alpha(t) dt - C \beta^{-1} T^{-1}
\]

for some \( C > 0 \).

**Proof.** Let \( C_1 > C_2 \) be a constant to be determined later, where \( C_2 = \int_{\frac{T}{2}}^{T} \alpha(t) dt \). Suppose \( u(T) < -C_1 \). Then for any \( t \in [\frac{T}{2}, T] \) it holds that

\[
u(t) \leq u(T) + \int_{t}^{T} \alpha(s) ds < -C_1 + C_2 =: -C_3,
\]

where \( C_3 > 0 \) since \( C_1 > C_2 \). So \( u(t) \) is negative, which implies that \( u(t)^2 \geq C_3^2 \). Now we integrate (11) to deduce that

\[
u(t) \leq u(T) + \int_{t}^{T} \alpha(s) ds - \int_{t}^{T} \beta u(s) ds < -\int_{t}^{T} \beta u(s) ds,
\]

which implies that \( u(t) \leq -\beta C_3 (T - t) \) for all \( t \in [\frac{T}{2}, T] \). Repeating the procedure of using the old bound of \( u(t) \) and (12) to obtain a new bound, we deduce that

\[
u(t) \leq -\beta^{m-1} C_3^2 (T - t)^{m-1} \prod_{k=1}^{m} \left( \frac{1}{2^{m-k}} \right)^T \leq -C \beta C_3 (T - t)^{\prod_{k=1}^{m} \left( 2^{m-k} \right)} \leq -C \beta C_3 (T - t) \prod_{k=1}^{m} \left( 2^{\frac{m-k}{2}} \right)
\]

after \( m \) times. Since \( \inf_{t} \prod_{k=1}^{m} \left( 2^{\frac{m-k}{2}} \right) = C_4 > 0 \), the right-hand side can be arbitrarily small at time \( T \) if \( \beta C_3 (T - \frac{T}{2}) C_4 > 1 \), which contradicts to the fact that \( u \) is finite. So if we take \( C_3 > \frac{2}{\beta T C_4} \), i.e. \( C_1 = C_2 + C \beta^{-1} T^{-1} \) for some constant \( C \), then \( u(T) \geq -C_1 \). \( \square \)

### 3 Aronson type estimates

In this section, we will prove one of our main results, which is an *a priori* estimate of the fundamental solution to equation (1). We will assume that \( a \in C^\infty([0, T] \times \mathbb{R}^n) \) and \( b \in C^\infty_n([0, T], C_0^\infty(\mathbb{R}^n)) \) so that there exists a unique regular fundamental solution.
3.1 The upper bound

The idea here is to estimate the $h$-transform of the fundamental solution, which was first used by E. B. Davies [7]. But here we will use Moser’s approach instead of Nash’s to prove the upper bound because it has the potential of applicability to more general cases where $b \in L^1((0,T;L^q(\mathbb{R}^n)))$ satisfying

$$1 \leq \frac{2}{l} + \frac{n}{q} < 2$$

for $n \geq 3, l > 1$ and $q > \frac{n}{2}$.

Given a function $\psi$ on $\mathbb{R}^n$ which is smooth and has bounded derivatives, we define the operator

$$A^\psi_t u(x) = \exp(-\psi(x)) \sum_{i,j=1}^n \partial_i (a_{ij}(t,x) \partial_j [\exp(\psi(x)) u(x)])$$

$$- \exp(-\psi(x)) \sum_{i=1}^n b_i(t,x) \partial_i [\exp(\psi(x)) u(x)].$$

Then its corresponding fundamental solution is

$$\Gamma^{\psi}(t,x;\tau,\xi) = \exp(-\psi(x)) \Gamma(t,x;\tau,\xi) \exp(\psi(\xi)).$$

For any $f \in C^\infty_0(\mathbb{R}^n)$, we define a linear operator $\Gamma^\psi_{\tau,t} : C^\infty_0(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ as

$$\Gamma^\psi_{\tau,t} f(x) = \int_{\mathbb{R}^n} f(\xi) \Gamma^\psi(t,x;\tau,\xi) d\xi$$

$$= \int_{\mathbb{R}^n} f(\xi) \exp(-\psi(x)) \Gamma(t,x;\tau,\xi) \exp(\psi(\xi)) d\xi.$$ 

It is easy to observe that the adjoint operator of $\Gamma^\psi_{\tau,t}$ can be identified as the following linear operator

$$\Gamma^{\psi^\perp}_{\tau,t} f(x) = \int_{\mathbb{R}^n} f(\xi) \exp(-\psi(\xi)) \Gamma(\tau,\xi;\tau,t,x) \exp(\psi(x)) d\xi,$$

and they satisfy

$$\langle \Gamma^\psi_{\tau,t} f, g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \Gamma^{\psi^\perp}_{\tau,t} g \rangle_{L^2(\mathbb{R}^n)}. \quad (13)$$

Lemma 7. Suppose $(a,b)$ satisfies conditions (E), (S) and (A). Given $\alpha \in \mathbb{R}^n$, and $\psi(x) = \alpha \cdot x$, set

$$f_t(x) = \Gamma^\psi_{0,t} f(x) = \int_{\mathbb{R}^n} f(\xi) \Gamma^\psi(t,x;0,\xi) d\xi$$

for $f \in C^\infty_0(\mathbb{R}^n)$. Then there exists a constant $C$ depending on $(n,l,q)$ such that

$$\|f_t\|^2_{L^2} \leq \exp \left( \frac{2|\alpha|^2}{\lambda} t + 2C\lambda^{-\frac{1}{\gamma+1}} |\alpha|^\frac{2}{\gamma+1} \Lambda^\mu r^\nu \right) \cdot \|f\|^2_{L^2}, \quad (14)$$

where $\theta = \frac{n}{q} - 1$, $\mu = \frac{2}{2-\gamma+\nu}$, $\nu = \frac{2-\gamma}{2-\gamma+\nu}$, $\gamma = \frac{2}{l} + \frac{n}{q}$ and $\Lambda = \|b\|_{L^1((0,T;L^q(\mathbb{R}^n)))}$. 

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Proof. We begin with the fact that $f_t$ satisfies

$$\frac{d}{dt} \|f_t\|_{L^2_t}^2 = 2 \langle A^\psi f_t, f_t \rangle_{L^2(\mathbb{R}^n)}.$$ 

It follows that

$$\frac{1}{2} \left( \|f_t\|_{L^2_t}^2 - \|f\|_{L^2_t}^2 \right) = \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} \left( \sum_{i,j=1}^n a_{ij}(s,x) \partial_x^i [\exp(\psi(x)) f_t(x)] \partial_x^j [\exp(-\psi(x)) f_t(x)] \right) dxds$$

$$- \int_0^t \int_{\mathbb{R}^n} \left( \sum_{i=1}^n b_i(s,x) \partial_x [\exp(\psi(x)) f_t(x)] \right) dxds$$

$$= \int_0^t \int_{\mathbb{R}^n} (\nabla f_t(x) \cdot a(s,x), \nabla f_t(x)) dxds - \int_0^t \int_{\mathbb{R}^n} (\nabla f_t(x) \cdot a(s,x)) dxds + \int_0^t \int_{\mathbb{R}^n} (\nabla f_t(x) \cdot a(s,x), f_t(x)) dxds$$

$$- \int_0^t \int_{\mathbb{R}^n} (b(s,x), f_t^2(x)) dxds - \int_0^t \int_{\mathbb{R}^n} (b(s,x), f_t(x) f_t(x)) dxds.$$

Since $b$ is divergence-free, we have for any $s$

$$\int_{\mathbb{R}^n} \langle b(s,x), \nabla f_s(x) \rangle f_s(x) dx = 0.$$

The third and fourth terms cancel each other and condition (E) gives

$$\frac{1}{2} \left( \|f_t\|_{L^2_t}^2 - \|f\|_{L^2_t}^2 \right)$$

$$= \int_0^t \int_{\mathbb{R}^n} (\nabla f_t(x) \cdot a(s,x), \nabla f_t(x)) dxds - \int_0^t \int_{\mathbb{R}^n} (\nabla f_t(x) \cdot a(s,x)) dxds$$

$$- \int_0^t \int_{\mathbb{R}^n} (b(s,x), f_t^2(x)) dxds$$

$$\leq \int_0^t \int_{\mathbb{R}^n} (\nabla f_t(x) \cdot a(s,x)) dxds - \int_0^t \int_{\mathbb{R}^n} (b(s,x), f_t^2(x)) dxds.$$

For the last term, one obtains the following estimate

$$\left| \int_0^t \int_{\mathbb{R}^n} \langle b(s,x), \nabla f_t(x) \rangle f_t^2(x) dxds \right| \leq \int_0^t \|\nabla f_t(x)\|_{L^1_t} \|f_t^{1+\theta}\|_{L^{2+\theta}_t} \|f_t^{1-\theta}\|_{L^{2-\theta}_t} dxds$$

$$= \int_0^t \|\nabla f_t(x)\|_{L^1_t} \|f_t^{1+\theta}\|_{L^{2+\theta}_t} \|f_t^{1-\theta}\|_{L^{2-\theta}_t} dxds,$$

where

$$\theta = \frac{n}{q} - 1, \quad (1 + \theta) r_1 = \frac{2n}{n - 2}, \quad (1 - \theta) r_2 = 2.$$

By Sobolev’s embedding and Young’s inequality, we can further control it as follows
\[
\left| \int_0^t \int_{\mathbb{R}^n} \langle b(s,x), \alpha \rangle f_s^2(x) \, dx \, ds \right| \\
\leq \int_0^t C|\alpha||b(s,\cdot)||f_s||L_2^2\rangle \|\nabla f_s\|^{1+\theta}_{L_2^2} \, ds \\
= \int_0^t \left( \frac{2}{\lambda} \right)^{\frac{1}{1+\theta}} C|\alpha||b(s,\cdot)||f_s||L_2^2\rangle \|\nabla f_s\|^{1+\theta}_{L_2^2} \, ds \\
\leq \int_0^t \frac{1-\theta}{2} \left( \frac{2}{\lambda} \right)^{\frac{1}{1+\theta}} \left( C|\alpha||b|^{2+\gamma}_{L_2} \right)^{\frac{1}{2+\gamma}} \|f_s\|^{2+\gamma}_{L_2^2} + \frac{1+\theta}{2} \lambda \|\nabla f_s\|^{2+\gamma}_{L_2^2} \, ds \\
\leq \int_0^t \left( \frac{1}{\lambda} \right)^{\frac{1}{1+\theta}} \left( C|\alpha||b|^{2+\gamma}_{L_2} \right)^{\frac{1}{2+\gamma}} \|f_s\|^{2+\gamma}_{L_2^2} + \frac{\lambda}{2} \|\nabla f_s\|^{2+\gamma}_{L_2^2} \, ds.
\]
Combining all the estimates above, one has
\[
\|f_t\|^{2+\gamma}_{L_2^2} \leq \|f_0\|^{2+\gamma}_{L_2^2} + 2 \int_0^t \left( \frac{\alpha^2}{\lambda} + C \left( \frac{1}{\lambda} \right)^{\frac{1}{1+\theta}} \left( C|\alpha||b|^{2+\gamma}_{L_2} \right)^{\frac{1}{2+\gamma}} \right) \|f_s\|^{2+\gamma}_{L_2^2} \, ds.
\]
Recall $\frac{2}{\gamma} + \frac{2}{\theta} = \gamma$ with $1 \leq \gamma < 2$ and $\Lambda = \|b\|_{L^\infty([0,T],L^\infty(\mathbb{R}^n))}$. Hölder’s inequality implies that
\[
\int_0^t \|b(s,\cdot)\|^{2+\gamma}_{L^2} \, ds = \int_0^t \|b(s,\cdot)\|^{\frac{2+\gamma}{2}}_{L^2} \, ds \leq \left( \int_0^t \|b(s,\cdot)\|^{2+\gamma}_{L^2} \, ds \right)^{\frac{2}{2+\gamma}} \left( \int_0^t \|b(s,\cdot)\|^{2+\gamma}_{L^2} \, ds \right)^{\frac{2}{2+\gamma}}
\]
\[
= \Lambda^{\frac{2}{2+\gamma}} \frac{2+\gamma}{\gamma} \frac{2}{2+\gamma},
\]
where we set $\frac{2}{\gamma} = 0$ if $l = \infty$. For simplicity, we denote $\mu = \frac{2}{2-\gamma+\frac{\gamma}{\theta}}$ and $\nu = \frac{2-\gamma}{2-\gamma+\frac{\gamma}{\theta}}$.

Hence, by Grönwall’s inequality and
\[
\|f_t\|^{2+\gamma}_{L_2^2} \leq \|f_0\|^{2+\gamma}_{L_2^2} + 2 \int_0^t \left( \frac{\alpha^2}{\lambda} + C \left( \frac{1}{\lambda} \right)^{\frac{1}{1+\theta}} \left( C|\alpha||b|^{2+\gamma}_{L_2} \right)^{\frac{1}{2+\gamma}} \right) \|f_s\|^{2+\gamma}_{L_2^2} \, ds,
\]
we deduce that
\[
\|f_t\|^{2+\gamma}_{L_2^2} \leq \exp \left( 2 \int_0^t \left( \frac{\alpha^2}{\lambda} + C \left( \frac{1}{\lambda} \right)^{\frac{1}{1+\theta}} \left( C|\alpha||b|^{2+\gamma}_{L_2} \right)^{\frac{1}{2+\gamma}} \right) \|f_s\|^{2+\gamma}_{L_2^2} \right) \|f_0\|^{2+\gamma}_{L_2^2} \\
\leq \exp \left( 2 \left( \frac{\alpha^2}{\lambda} \right)^{\frac{2}{2+\gamma}} + 2C \left( \frac{1}{\lambda} \right)^{\frac{1}{1+\theta}} \left( C|\alpha||b|^{2+\gamma}_{L_2} \right)^{\frac{2}{2+\gamma}} \right) \|f_0\|^{2+\gamma}_{L_2^2}.
\]
Now the proof is complete.

\begin{lemma}
Suppose that $(a,b)$, $\psi$ and $f_t$ are defined as in Lemma 7. For any $p \geq 1$ and any smooth non-negative function $\eta$ on $[0,T]$ satisfying $\eta(0) = 0$, we have
\[
\|f_t^p \eta^\sigma\|^{2+\gamma}_{L_2^2} \leq C|\alpha|^2 \eta^2 \|f_t^p \eta^\sigma\|^{2+\gamma}_{L_2^2} + C(|\alpha| p)^{\frac{2}{1+\theta}} \|b\|^{2+\gamma}_{L_2^2} \|f_t^p \eta^\sigma\|^{\frac{2}{1+\theta}}_{L_2^2}.
\]
\end{lemma}
Next we multiply both sides by $\eta^{2\sigma}$ and integrate on $[0, T]$ to obtain

$$
\int_0^T \eta^{2\sigma}(t) \int_{\mathbb{R}^n} \partial_t f_i(x) f_i(x)^2 p - 1 \, dx \, dt
$$

where $\chi = \frac{n+2}{n}$, $\sigma = \frac{1}{2} - \epsilon$ and $C > 0$ is a constant depending only on $l, q, n, \lambda$.

**Proof.** For any $p \geq 1$, we have

$$
\frac{d}{dt} \| f_i \|_{L^2_p}^2 = 2p (A_t f_i, f_i^2 p - 1)_{L^2(\mathbb{R}^n)}.
$$

Condition (S) implies that

$$
\int_{\mathbb{R}^n} \langle b(t, x), \nabla f_i(x) \rangle f_i^2 p - 1(x) \, dx = 0
$$

for any $t$, and hence the last term vanishes. Set $g_t = f_i^p$ for simplicity, then the left-hand side becomes

$$
\int_0^T \eta^{2\sigma}(t) \int_{\mathbb{R}^n} \partial_t f_i(x) f_i(x)^2 p - 1 \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} \eta^{2\sigma}(t) \frac{1}{2p} \partial_t (g_t^2(x)) \, dx \, dt
$$

\[= \frac{1}{2p} \eta^{2\sigma}(t) g_t^2(x) \bigg|_0^T - \int_0^T \int_{\mathbb{R}^n} \frac{\sigma}{p} g_t^2(x) (\partial_t \eta(t)) \eta^{2\sigma - 1}(t) \, dx \, dt.
\]

Multiplying by $p$ on both sides of equation (15), we obtain

$$
\int_0^T \frac{1}{2} \eta^{2\sigma}(t) g_t^2(x) \, dx \bigg|_0^T - \int_0^T \int_{\mathbb{R}^n} \sigma g_t^2(x) (\partial_t \eta(t)) \eta^{2\sigma - 1}(t) \, dx \, dt
$$

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From this relation, it is easy to see

\[
= p \int_0^T \eta^{2\sigma}(t) \left( \langle \alpha \cdot a(t, x), \alpha \rangle g^2_n(x) dx \right) dt \\
- \frac{(2p-1)}{p} \int_0^T \eta^{2\sigma}(t) \left( \langle \nabla g_n(x) \cdot a(t, x), \nabla g_n(x) \rangle \right) dx \, dt \\
- (2p-2) \int_0^T \eta^{2\sigma}(t) \left( \langle \alpha \cdot a(t, x), \nabla g_n(x) \rangle g_n(x) \right) dx \, dt \\
- p \int_0^T \eta^{2\sigma}(t) \left( \langle b(t, x), \alpha \rangle g^2_n(x) dx \right) dt \\
= I_1 - I_2 - I_3 - I_4.
\]

Now we estimate each term individually as follows

\[
I_1 \leq \int_0^T \eta^{2\sigma}(t) \frac{|\alpha|^2}{\lambda} p \|g_n\|_{L^2}^2 dt, \\
-I_2 - I_3 \leq \int_0^T \eta^{2\sigma}(t) \frac{|\alpha|^2}{\lambda} (p-1) p \|g_n\|_{L^2}^2 dt - \int_0^T \eta^{2\sigma}(t) \lambda \|\nabla g_n\|_{L^2}^2 dt, \\
|I_4| = \left| p \int_0^T \eta^{2\sigma}(t) \left( \langle b(t, x), \alpha \rangle g^2_n(x) dx \right) \right| \\
\leq \int_0^T \int_{\mathbb{R}^n} p |b(t, x)||g_n^\sigma| |g|^{2-\gamma}(|\alpha| |\eta|) dx \, dt \\
\leq |\alpha| p \|b\|_{L^2} \|g_n^\sigma\|_{L^2} \|g_n^\gamma\|_{L^2} \|g_n^\eta\|_{L^2}^{2-\gamma},
\]

since \( \sigma \gamma = 2\sigma - 1 \) and

\[
\frac{1}{l} + \frac{\gamma}{s} + \frac{2 - \gamma}{2} = 1, \quad \frac{1}{q} + \frac{\gamma}{r} + \frac{2 - \gamma}{2} = 1.
\]

From this relation, it is easy to see

\[
\frac{2}{s} + \frac{n}{r} = \frac{n}{2},
\]

which yields the interpolation inequality

\[
\|f\|_{L^2 L^4} \leq C \|f\|_{L^2 L^2}^{1-\beta} \|\nabla f\|_{L^2 L^2}^\beta, \quad \beta = \frac{n}{2} - \frac{n}{r}.
\]

Together with Young’s inequality, we deduce the following estimate

\[
|I_4| \leq \varepsilon \|g_n^\sigma\|_{L^2 L^4}^2 + C(\varepsilon)(|\alpha| p)^{\frac{2}{2-\gamma}} \|b\|_{L^2} \|g_n^\gamma\|_{L^2} \|g_n^\eta\|_{L^2}^{2-\gamma} \\
\leq \frac{\lambda}{4} \left( \|g_n^\sigma\|_{L^2 L^4}^2 + \|\nabla g_n^\sigma\|_{L^2 L^2}^2 \right) + C(\varepsilon)(|\alpha| p)^{\frac{2}{2-\gamma}} \|b\|_{L^2} \|g_n^\gamma\|_{L^2} \|g_n^\eta\|_{L^2}^{2-\gamma}.
\]
Combining these together, we conclude that
\[
\left. \int_{\mathbb{R}} \frac{1}{2} \eta^{2\sigma}(t)g_t^2(x) \, dx \right|_0^T - \int_0^T \int_{\mathbb{R}^n} \sigma g_t^2(x)(\partial_t \eta(t)) \eta^{2\sigma-1}(t) \, dx dt
\]
\[
\leq \int_0^T \eta^{2\sigma}(t)\frac{|\alpha|^2}{\lambda} p^2 \|g_t\|^2_{L^2_t L^2_x} \, dt - \int_0^T \eta^{2\sigma}(t)\lambda \|
abla g_t\|^2_{L^2_t L^2_x} \, dt
\]
\[
+ \frac{\lambda \wedge 1}{4} \int_0^T (\|g_t\|_{L^2_t L^2_x} + \|\nabla g_t\|_{L^2_t L^2_x})^2 \, dt + C(|\alpha| p)^{\frac{2}{\gamma}} \|b\|^2_{L^4_t L^2_x} \|g_t\|^{\frac{1}{\gamma}} \|\eta\|_{L^2_t L^2_x}^{\frac{1}{\gamma}}
\]
If we set \(\eta(0) = 0\), then the inequality above implies that
\[
\frac{1}{2} \int_{\mathbb{R}} \eta^{2\sigma}(t)g_t^2(x) \, dx \leq \frac{\lambda \wedge 1}{4} \int_0^T (\|g_t\|_{L^2_t L^2_x} + \|\nabla g_t\|_{L^2_t L^2_x})^2 \, dt + C(|\alpha| p)^{\frac{2}{\gamma}} \|b\|^2_{L^4_t L^2_x} \|g_t\|^{\frac{1}{\gamma}} \|\eta\|_{L^2_t L^2_x}^{\frac{1}{\gamma}} \int_{\mathbb{R}} \sigma g_t^2(x) \, dx dt,
\]
and the same is true if we replace \(T\) by any \(t \in [0, T]\). Hence
\[
\frac{1}{2} \int_{\mathbb{R}} \eta^{2\sigma}(t)g_t^2(x) \, dx \leq \frac{\lambda \wedge 1}{4} \int_0^T (\|g_t\|_{L^2_t L^2_x} + \|\nabla g_t\|_{L^2_t L^2_x})^2 \, dt + C(|\alpha| p)^{\frac{2}{\gamma}} \|b\|^2_{L^4_t L^2_x} \|g_t\|^{\frac{1}{\gamma}} \|\eta\|_{L^2_t L^2_x}^{\frac{1}{\gamma}} \int_{\mathbb{R}} \sigma g_t^2(x) \, dx dt.
\]
Applying the interpolation inequality (16) with \(s = r = \chi = \frac{a+2}{n}\), we deduce that
\[
\|g_t\eta^\sigma\|_{L^{2s}_t L^{2r}_x}^2 \leq C(|\alpha|^2 p^2 \|g_t\|_{L^2_t L^2_x} + C(|\alpha| p)^{\frac{2}{\gamma}} \|b\|^2_{L^4_t L^2_x} \|g_t\|^{\frac{1}{\gamma}} \|\eta\|_{L^2_t L^2_x}^{\frac{1}{\gamma}} \int_{\mathbb{R}} \sigma g_t^2(x) \, dx dt,
\]
and the proof is complete. \(\square\)

Now we can use the Moser’s iteration to prove Theorem 1.

Proof of Theorem 1. Define open intervals \(I_k = \left(\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} \right) T, T\) and choose \(\eta_k\) as cut-off functions such that \(\eta_k = 1\) on \(I_k\), \(\eta_k = 0\) on \(I_{k+1}\), and \(|\partial_t \eta_k| \leq 4^k T^{-1}\). Denote \(L^p_{t,x}\) the \(L^p\) space on the space-time domain \(I_k \times \mathbb{R}^n\). Then
\[
\|g_t\|_{L^2_{t,x}}^2 \leq \|g_t\|_{L^2_{t,x}}^2
\]
\[
\leq C(|\alpha|^2 p^2 \|g_t\|_{L^2_t L^2_x} + C(|\alpha| p)^{\frac{2}{\gamma}} \|b\|^2_{L^4_t L^2_x} \|g_t\|^{\frac{1}{\gamma}} \|\eta_k\|_{L^2_t L^2_x}^{\frac{1}{\gamma}} \int_{\mathbb{R}} \sigma g_t^2(x) \, dx dt
\]
\[
\leq C(|\alpha|^2 p^2 \|g_t\|_{L^2_t L^2_x} + C(|\alpha| p)^{\frac{2}{\gamma}} \|b\|^2_{L^4_t L^2_x} \|g_t\|_{L^2_{t,x}}^2 + C \frac{4^k}{T} \|g_t\|_{L^2_{t,x}}^2.
\]
\[
\leq C \left( |\alpha|^2 p^2 + p \vec{\gamma} |b| \|L\|_{L^1} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right) \|g_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}.
\]

Recall that \(g_t = f_t^p\). Let \(p_0 = 1\) and \(p_k = \chi^k = (\frac{\alpha + 2}{2})^k\) for \(k = 1, 2, \ldots\). Then

\[
\|f_t^{p_{k-1}}\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}^2 \leq C \left( |\alpha|^2 p_{k-1}^2 + p_{k-1} \bar{\gamma} |b| \|L\|_{L^1} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right) \|f_t^{p_{k-1}}\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}^2,
\]

or equivalently,

\[
\|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n} \leq C^{\frac{1}{\chi^k-1}} \left( |\alpha|^2 p_{k-1}^2 + p_{k-1} \bar{\gamma} |b| \|L\|_{L^1} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right)^{\frac{1}{\chi^k-1}} \|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}.
\]

Iterate the procedure above to get that

\[
\|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n} \leq \left( \prod_{k=1}^{m} C^{\frac{1}{\chi^k-1}} \left( |\alpha|^2 p_{k-1}^2 + p_{k-1} \bar{\gamma} |b| \|L\|_{L^1} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right)^{\frac{1}{\chi^k-1}} \right) \|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}.
\]

Since \(p_k = (\frac{\alpha + 2}{2})^k \leq 2^k\) and \(-\gamma \leq 1\), we have

\[
\|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n} \leq \left( \prod_{k=1}^{m} C^{\frac{1}{\chi^k-1}} \left( |\alpha|^2 p_{k-1}^2 + p_{k-1} \bar{\gamma} |b| \|L\|_{L^1} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right)^{\frac{1}{\chi^k-1}} \right) \|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}
\leq \left( \prod_{k=1}^{m} C^{\frac{1}{\chi^k-1}} \left( |\alpha|^2 + \Lambda^{\frac{2}{\gamma}} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right)^{\frac{1}{\chi^k-1}} \right) \|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}
\leq C(|\alpha|^2 + \Lambda^{\frac{2}{\gamma}} |\alpha|^{\frac{2}{\gamma}} + \sigma \frac{4k}{T} \right)^{\frac{1}{\chi^k-1}} \|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}
= C(|\alpha|^2 T + \Lambda^{\frac{2}{\gamma}} T + |\alpha|^{\frac{2}{\gamma}} T + \sigma)^{\frac{4k+1}{4k-1}} \|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n}.
\]

We already proved inequality (14), which implies

\[
\|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n} \leq T^\frac{1}{\chi} \exp \left( C(|\alpha|^2 T + |\alpha|^{\frac{2}{\gamma}} \Lambda^\mu T^\nu) \right) \|f_t\|_{L_{\alpha \beta}^2}.
\]

Inserting this into (17), we derive that

\[
\|f_t\|_{L_{\alpha \beta}^2 \times \mathbb{R}^n} \leq C(|\alpha|^2 T + \Lambda^{\frac{2}{\gamma}} T + \sigma)^{\frac{m+2}{m-1}} T^{-\frac{\gamma}{\gamma+1}} \times \exp \left( C(|\alpha|^2 T + |\alpha|^{\frac{2}{\gamma}} \Lambda^\mu T^\nu) \right) \|f_t\|_{L_{\alpha \beta}^2}.
\]

Notice that \(1 - \theta = 2 - \frac{\alpha}{q} = 2 - \gamma + \frac{2}{\gamma}\) and recall that \(\mu = \frac{2}{2-\gamma+\frac{2}{\gamma}}\). Hence

\[
|\alpha|^{\frac{2}{\gamma}} \Lambda^{\frac{2}{\gamma}} T = (|\alpha|^{\frac{2}{\gamma}} \Lambda^\mu T^\nu)^{\frac{2-\gamma+\frac{2}{\gamma}}{2-\gamma+\frac{2}{\gamma}}},
\]

and \((|\alpha|^2 T + \Lambda^{\frac{2}{\gamma}} T + \sigma)^{\frac{m+2}{m-1}}\) can be regarded as a polynomial of parameters \((|\alpha|^2 T, |\alpha|^{\frac{2}{\gamma}} \Lambda^\mu T^\nu)\), which can be dominated by

\[
C \exp \left( C(|\alpha|^2 T + |\alpha|^{\frac{2}{\gamma}} \Lambda^\mu T^\nu) \right).
\]
So we have
\[ \|f_T\|_{L_x^\infty} \leq CT^{-\frac{\mu}{2}} \exp \left( C(|\alpha|^2T + |\alpha|^2 \Lambda \mu V^T) \right) \|f\|_{L_x^\infty}. \]

By duality, i.e. equation (13)
\[ \|f_T\|_{L_x^\infty} \leq CT^{-\frac{\mu}{2}} \exp \left( C(|\alpha|^2T + |\alpha|^2 \Lambda \mu V^T) \right) \|f\|_{L_x^1}. \]

Using the Chapman-Kolmogorov equation, one has
\[ \|f_{2T}\|_{L_x^\infty} \leq CT^{-\frac{\mu}{2}} \exp \left( C(|\alpha|^2T + |\alpha|^2 \Lambda \mu V^T) \right) \|f\|_{L_x^1}. \]

Recall that
\[ f_{2T}(x) = \int_{\mathbb{R}^n} f(\xi) \exp(-\psi(x))\Gamma(2T, x; 0, \xi) \exp(\psi(\xi)) \, d\xi \]
for any \( f \in C_0^\infty(\mathbb{R}^n) \) and that \( \psi(x) = \alpha \cdot x \). Replacing \( 2T \) by \( t \) and dividing both sides by \( \exp(-\psi(x)) \exp(\psi(\xi)) \), then we have a pointwise upper bound on \( \Gamma \) as follows
\[ \Gamma(t, x; 0, \xi) \leq \frac{C}{t^{n/2}} \exp \left( C(|\alpha|^2t + |\alpha|^2 \Lambda \mu V^T + \alpha \cdot (x - \xi)) \right) \]
for any \( \alpha \in \mathbb{R}^n \), where \( C \) depends only on \( (t, q, n, \lambda) \). Set \( m(t, x) = \min_{\alpha \in \mathbb{R}^n} (C(|\alpha|^2t + |\alpha|^2 \Lambda \mu V^T + \alpha \cdot x) \). Taking the minimum of the right-hand side over all \( \alpha \in \mathbb{R}^n \), we can conclude that
\[ \Gamma(t, x; 0, \xi) \leq \frac{C}{t^{n/2}} \exp(m(t, x - \xi)). \]

Finally, we shift \( \Gamma(t - \tau, x, 0, \xi) \) by \( \tau \) to obtain estimate (5). Now the proof is complete.

We may give an elementary and explicit estimate for the function \( m \) appearing in the theorem we just proved, which also gives a more explicit form of this upper bound.

**Corollary 9.** Under the same assumptions and notations as in Theorem 1. If \( \mu \equiv \frac{2}{2 - \gamma + \rho} > 1 \), the fundamental solution has upper bound
\[ \Gamma(t, x; \tau, \xi) \leq \begin{cases} \frac{C}{(t - \tau)^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x - \xi|^2}{t - \tau} \right) \right) & \frac{|\lambda|^2}{\mu^{n/4} - 1} < 1 \\ \frac{C}{(t - \tau)^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x - \xi|^2}{t - \tau} \right) \right) & \frac{|\lambda|^2}{\mu^{n/4} - 1} \geq 1 \end{cases} \]

(18)

where \( \Lambda = \|b\|_{L^1(0, T; L^q(\mathbb{R}^n))}, C_1 = C_1(l, q, n, \lambda), C_2 = C_2(l, q, n, \lambda, \Lambda) \). If \( \mu = 1 \), which implies \( q = \infty \), we can solve for \( m(t, x) \) explicitly and obtain
\[ \Gamma(t, x; \tau, \xi) \leq \frac{C_1}{(t - \tau)^{n/2}} \exp \left( -\frac{(C_1 \Lambda (t - \tau)^{\nu} - |x - \xi|^2)^2}{4C_1 (t - \tau)} \right). \]

(19)
Proof. Clearly, it is enough to estimate function $m(t,x)$. In this proof, we denote $C_1$ as a constant depending only on $(l,q,n,\lambda)$ and $C_2$ a constant depending on $(l,q,n,\lambda,\Lambda)$. Their values may be different throughout the proof. Notice that $\mu \geq 1$. When $\mu > 1$, by taking $\alpha = -\frac{|x|}{4C_2 t}$, we have

$$m(t,x) \leq \frac{C_1|x|^2}{16C_2^2 t} + \frac{C_1\Lambda^\mu|x|^\mu}{4\mu C_2^2 t^{\mu-\nu}} - \frac{|x|^\mu}{4C_2 t} + \frac{C_1\Lambda^\mu|x|^2}{4\mu C_2^2 t^{\mu-1}} \cdot \frac{|x|^\mu - |x|^2}{4C_2 t} \leq -\frac{|x|^2}{8C_2 t}$$

if $\frac{|x|^\mu - |x|^2}{4C_2 t} < 1$. When $\frac{|x|^\mu - |x|^2}{4C_2 t} \geq 1$, we take $\alpha = -\frac{1}{4C_2 t} (\frac{|x|^\mu}{|x|}) x$. Then one has

$$m(t,x) \leq \frac{C_1|x|^2}{16C_2^2 t^{\mu-1}} + \frac{C_1\Lambda^\mu|x|^\mu}{4\mu C_2^2 t^{\mu-1}} - \frac{|x|^\mu}{4C_2 t^{\mu-1}} + \frac{C_1\Lambda^\mu|x|^2}{4\mu C_2^2 t^{\mu-1}} - \frac{|x|^\mu}{4C_2 t^{\mu-1}} \leq -\frac{|x|^\mu}{2C_2 t^{\mu-1}}.$$

Now consider the case that $\mu = 1$. To obtain $m(t,x) = \min_{\alpha \in \mathbb{R}} (C_1(|\alpha|^2 + |\alpha|\Lambda^\nu) + \alpha \cdot x)$, it is easy to see that $\alpha$ must be in opposite direction of $x$, i.e. $\frac{\alpha}{|\alpha|} = -\frac{1}{|x|} x$. So we only need to find the minimum of the polynomial $C_1t|\alpha|^2 + (C_1\Lambda^\nu - |x|)|\alpha|$, which is obtained at $|\alpha| = -\frac{C_1\Lambda^\nu - |x|}{2C_1 t}$ and the value is

$$m(t,x) = -\frac{(C_1\Lambda^\nu - |x|)^2}{4C_1 t}.$$

Now the proof is complete. \hfill \Box

Recall that in dimension three, any Leray-Hopf weak solution to the Navier-Stokes equations satisfies

$$u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)).$$

Clearly $L^2(0,T;H^1(\mathbb{R}^3)) \subset L^2(0,T;L^6(\mathbb{R}^3))$, thus $\gamma = \frac{3}{2}$ for both function spaces. Notice that by interpolation, $u \in L^l(0,T;L^q(\mathbb{R}^3))$ for any $l \in [2,\infty]$ and $q \in [2,6]$ satisfying $\frac{1}{l} + \frac{3}{q} = \frac{3}{2}$. This is an interesting case for which we have the following theorem.

**Theorem 10.** Suppose $n = 3$, and conditions (E) and (S) hold for $b \in L^l(0,T;L^q(\mathbb{R}^3))$ satisfying $\frac{1}{l} + \frac{3}{q} = \frac{3}{2}$. Then the fundamental solution $\Gamma$ to (1) has the upper bound

$$\Gamma(t,x;\tau,\xi) \leq \begin{cases} \frac{C_1}{(l-\gamma)^{l-\gamma}} \exp \left( -\frac{l}{C_2} \left( \frac{|x|^2}{l-\gamma} \right) \right) & \frac{|\xi|^l}{l-\gamma} \leq 1 \\ \frac{C_1}{(l-\gamma)^{l-\gamma}} \exp \left( -\frac{l}{C_2} \left( \frac{|\xi|^l}{l-\gamma} - \frac{1}{l-\gamma} \right) \right) & \frac{|\xi|^l}{l-\gamma} \geq 1 \end{cases}$$

(20)

where $\Lambda = \|b\|_{L^l(0,T;L^q(\mathbb{R}^3))}$, $C_1 = C_1(l,q,n,\Lambda)$, $C_2 = C_2(l,q,\lambda,\Lambda)$. Here we set $\frac{|\xi|^l}{l-\gamma} = \frac{|x|}{l}$ when $l = \infty$.

Another example is $L^\infty(0,T;L^q(\mathbb{R}^n))$, which is not covered in the classical paper [2] by Aronson and [3]. Here with the divergence-free condition on $b$, we have obtained a Gaussian upper bound, which yields the first half of Theorem 2.
\textbf{Theorem 11.} The fundamental solution to (1) satisfying conditions (E) and (S), with \( b \in L^n(0,T;L^n(\mathbb{R}^n)) \) has the Gaussian upper bound

\[
\Gamma(t,x;\tau,\xi) \leq \frac{C_1}{(t-\tau)^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x-\xi|^2}{t-\tau} \right)^{\frac{1}{\nu}} \right),
\]

where \( \Lambda = \|b\|_{L^n(0,T;L^n(\mathbb{R}^n))}, \ C_1 = C_1(n,\lambda), \ C_2 = C_2(n,\lambda,\Lambda) \).

Since we know that \( \int B(0,1) \Gamma(t,x;\tau,\xi) d\xi = 1 \) and we have proved the upper bound in Corollary 9, which is of exponential decay in space, we can derive a lower bound for \( \Gamma \) in the following form. This proposition will be used in the proof of a pointwise lower bound later.

\textbf{Proposition 12.} Take the fundamental solution of (1) satisfying conditions (E) and (S), for any \( \delta \in (0,1) \) and \( t-\tau \) small enough, we have

\[
\int_{B(x,\hat{R}(t-\tau))} \Gamma(t,x;\tau,\xi) d\xi \geq \delta,
\]

where \( \hat{R}(\cdot) \) is a function defined as follows

\[
\hat{R}(t) = \begin{cases} 
Ct^{1/2} & \text{if } \gamma = 1 \\
Ct^{(2-\gamma)/2} \ln \frac{1}{t} & \text{if } \gamma > 1,
\end{cases}
\]

\( B(x,r) \) is the ball of radius \( r \) and center \( x \), and \( C \) depends only on \((\delta,1,q,n,\lambda,\Lambda)\).

\textbf{Proof.} Firstly, when \( \mu > 1 \), we have

\[
\Gamma(t,x;\tau,\xi) \leq h_1(t-\tau,x-\xi) + h_2(t-\tau,x-\xi),
\]

where

\[
h_1(t,x) = \frac{C_1}{t^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x|^2}{t} \right) \right), \quad h_2(t,x) = \frac{C_1}{t^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|x|^\mu}{t^\nu} \right)^{\frac{1}{\mu-1}} \right).
\]

Thus it is enough to prove that

\[
\int_{B(x,\hat{R}(t-\tau))^c} h_1(t-\tau,x-\xi) + h_2(t-\tau,x-\xi) d\xi \leq 1 - \delta.
\]

Without lose of generality, we can assume \( \tau = 0 \) and \( x = 0 \). By the following change of variable

\[
\int_{B(0,RC^{1/2},1/2)} \frac{C_1}{t^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|\xi|^2}{t} \right) \right) d\xi = C \int_{B(0,R)} \exp (-|\xi|^2) d\xi,
\]

we have

\[
\int_{B(x,R_1(t))^c} \frac{C_1}{t^{n/2}} \exp \left( -\frac{1}{C_2} \left( \frac{|\xi|^2}{t} \right) \right) d\xi \leq \frac{1-\delta}{2}
\]

with \( R_1(t) = Ct^{1/2} \) for some sufficiently large constant \( C > 0 \). For the second term, since \( \frac{\mu}{\nu} = \frac{2-q}{2} \leq \frac{1}{2} \), it follows that...
\[
\int_{B(0,Rt(\mu-1)/v,\mu/v)} C_1 \exp \left( -\frac{1}{C_2} \left( \frac{\|\xi\|}{t^\nu} \right)^{\frac{\mu}{2}} \right) d\xi = \frac{C}{t^{n(\frac{1}{2} - \frac{v}{\mu})}} \int_{B(0,Rt)} \exp \left( -\frac{\|\xi\|}{t^\nu} \right) d\xi.
\]

Setting \(\Phi(R) = \int_{B(0,Rt)} \exp \left( -\frac{\|\xi\|}{t^\nu} \right) d\xi\), then one has \(\Phi(R) \leq Ce^{-R}\) by \(\frac{\mu}{\mu-1} > 1\).

So setting \(R = C(1 - (\frac{1}{2} - \frac{v}{\mu}) \ln t)\) for some \(C\), we obtain that

\[
\int_{B(0,R)\gamma^v} \exp \left( -\frac{\|\xi\|}{t^\nu} \right) d\xi \leq Ce^{-R} \leq \frac{n(\frac{1}{2} - \frac{v}{\mu})}{C_1} \cdot \frac{1 - \delta}{2},
\]

and therefore

\[
\int_{B(0,R(\gamma+1)/\gamma)} C_1 \exp \left( -\frac{1}{C_2} \left( \frac{\|\xi\|}{t^\nu} \right)^{\frac{\mu}{\nu}} \right) d\xi \leq \frac{1 - \delta}{2}
\]

with

\[
R_2(t) = Ct^\nu/\mu (1 + (\frac{1}{2} - \frac{v}{\mu}) \ln \frac{1}{t}) = Ct^{(2-\gamma)/2} (1 + (\frac{1}{2} - \frac{2-\gamma}{2}) \ln \frac{1}{t})
\]

for some constant \(C\). When \(t\) is small enough, \(R_1(t) \leq R_2(t)\) and we obtain the radius \(\hat{R}(t) = R_2(t)\).

When \(\mu = 1\), we have \(v = \frac{2-\gamma}{\nu} \leq \frac{1}{2}\) and, by using the elementary inequality \((a-b)^2 + b^2 \geq \frac{b^2}{4}\),

\[
\int_{B(0,R)\gamma^v} \frac{C_1}{t^{n/2}} \exp \left( -\frac{(C_1 t^\nu - \|\xi\|)^2}{4C_1 t} \right) d\xi
\]

\[
= \frac{C}{t^{n(\frac{1}{2} - \gamma/v)}} \int_{B(0,R)} \exp \left( -\frac{(C_1 t - \|\xi\|)^2}{C_1 t^1-2v} \right) d\xi
\]

\[
\leq \frac{C}{t^{n(\frac{1}{2} - \gamma/v)}} \int_{B(0,R)} \exp \left( -\frac{(C_1 t - \|\xi\|)^2}{C_1} \right) d\xi
\]

\[
\leq \frac{C}{t^{n(\frac{1}{2} - \gamma/v)}} \int_{B(0,R)} \exp \left( -\frac{\|\xi\|^2}{2C_1} + C_1^2 \Lambda^2 \right) d\xi.
\]

Let \(\Phi(R) = \int_{B(0,Rt)} \exp \left( -\frac{\|\xi\|^2}{2} \right) d\xi\). Then \(\Phi(R) \leq Ce^{-R}\) for some universal constant. So we still take

\[
\hat{R}(t) = C t^{(2-\gamma)/2} (1 + (\frac{1}{2} - \frac{2-\gamma}{2}) \ln \frac{1}{t})
\]

to obtain

\[
\int_{B(0,\hat{R}(t))\gamma^v} C_1 \exp \left( -\frac{(C_1 t^\nu - \|\xi\|)^2}{4C_1 t} \right) d\xi \leq 1 - \delta.
\]

Clearly, when \(\gamma = 1\), \(\hat{R}(t)\) is just \(C t^{1/2}\). Since we only need \(\hat{R}(t)\) for small \(t\), and under this condition \(\ln \frac{1}{t} \gg 1\). Thus taking \(\hat{R}(t) = C t^{(2-\gamma)/2} \ln \frac{1}{t}\) when \(\gamma > 1\) will do.

Remark. Although estimate (21) seems better than (18), actually it can be shown that this observation will not affect the result. So based on Corollary 9, this \(\hat{R}(t)\) is the smallest cone radius such that we can derive a lower bound of this form inside the cone.
3.2 The lower bound

Let $T > 0$ and $x \in \mathbb{R}^n$. To prove the lower bound, Nash’s idea is to consider the quantity

$$G_r(t, x) = \int_{\mathbb{R}^n} \ln(\Gamma(T, x; T - t, \xi)) \mu_r(d\xi) = \int_{\mathbb{R}^n} \ln(\Gamma_r(t, 0, x)) \mu_r(d\xi)$$

for $t \in [0, T]$, where $\mu_r(x) = \frac{1}{r^n} \exp \left( -\frac{|x|^2}{r^2} \right)$ as defined in Lemma 4. Then Jensen’s inequality implies that $G_r(t, x) \leq 0$. We will write it as $G(t, x)$ if $r = 1$. If we have $G_r(T, x) > -C$ for some positive constant $C$, then we can derive a lower bound for $\Gamma(T, x; 0, \xi)$. Consider the time derivative of $G_r(t, x)$

$$G_r'(t, x) = \int_{\mathbb{R}^n} \partial_t \ln(\Gamma(T, x; T - t, \xi)) \mu_r(d\xi)$$

$$= \int_{\mathbb{R}^n} \left\langle \frac{2\pi}{r}, a(T - t, \xi) \cdot \nabla_\xi \ln\Gamma(T, x; T - t, \xi) \right\rangle \mu_r(d\xi)$$

$$+ \int_{\mathbb{R}^n} \left\langle \nabla_\xi \ln\Gamma(T, x; T - t, \xi), a(T - t, \xi) \cdot \nabla_\xi \ln\Gamma(T, x; T - t, \xi) \right\rangle \mu_r(d\xi)$$

$$+ \int_{\mathbb{R}^n} \left\langle b(T - t, \xi), \nabla_\xi \ln\Gamma(T, x; T - t, \xi) \right\rangle \mu_r(d\xi). \quad (22)$$

In the following subsections, we will estimate $G_r(t, x)$ under varies conditions on $b$ and hence obtain a lower bound of $\Gamma$.

We will separate the critical and supercritical cases. In critical case $\gamma = 1$, we will only consider the case where $l = \infty, q = n$, which is the only case that regularity theory is missing. Since in the critical case, $\|b\|_{L^r}^r$ is invariant under scaling, we do not need to worry about explicitly how the constant depends on $\Lambda$. Hence we will only need to estimate $G(1, x)$ and obtain the estimate of $G(t, x)$ for all $t$ by scaling.

In supercritical case, in order to use scaling, we need to find out how the constants appearing in lower bounds depend on $\Lambda$, and therefore it is not a good idea to use the scaling argument. So we will alter the strategy to estimate $G_r(t, x)$ for all $t$ directly.

3.2.1 Critical case $L^\infty(0, T; L^n(\mathbb{R}^n))$

In the critical case $L^\infty(0, T; L^n(\mathbb{R}^n))$, we can obtain the following Gaussian lower bound and hence complete the proof of Theorem 2.

**Lemma 13.** For any $x \in B(0, 2)$, there is a constant $C > 0$ depending only on $n, \lambda, \Lambda = \|b\|_{L^r(0, T; L^r(\mathbb{R}^n))}$, such that

$$G(1, x) = \int_{\mathbb{R}^n} \ln(\Gamma(1, x; 0, \xi)) \mu(d\xi) \geq -C, \quad (23)$$

and hence

$$\Gamma(2, x; 0, \xi) \geq e^{-2C}, \quad x, \xi \in B(0, 2). \quad (24)$$

**Proof.** If we fix $x \in B(0, 2)$, $T = 1$ and $r = 1$ in equation (22), then condition (E) and $b \in L^\infty(0, T; L^n(\mathbb{R}^n))$ implies

$$G'(t, x) \geq -\frac{C}{\lambda} \|\nabla_\xi \ln\Gamma(1, x; 1 - t, \xi)\|_{L^2(\mu)} + \lambda \|\nabla_\xi \ln\Gamma(1, x; 1 - t, \xi)\|_{L^2(\mu)}$$

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\[-C\|b(1-t)\|_{L^\infty(\mu)} \|\nabla_{\xi} \ln \Gamma(1, x; 1 - t, \xi)\|_{L^2(\mu)} \]
\[\geq -C(n, \lambda, \Lambda) + \frac{\lambda}{2} \|\nabla_{\xi} \ln \Gamma(1, x; 1 - t, \xi)\|_{L^2(\mu)}^2 \]
\[\geq -C(n, \lambda, \Lambda) + C(\lambda) \|\ln \Gamma(1, x; 1 - t, \cdot) - G(t, x)\|_{L^2(\mu)},\]

where the last step follows the Poincaré-Wirtinger inequality (8) for the Gaussian measure. The rest of the argument of the proof follows exactly the same as in [7]. For completeness, we include the full proof here. Since \(G(t, x) \leq 0\), for any \(K > 0\), using \((a - b)^2 \geq \frac{a^2}{2} - b^2\) we have

\[
\|\ln \Gamma(1, x; 1 - t, \cdot) - G(t, x)\|_{L^2(\mu)}^2 \geq \int_{\ln \Gamma(1, x; 1 - t, \xi) \geq -K} (\ln \Gamma(1, x; 1 - t, \xi) - G(t, x))^2 \mu(d\xi) \]
\[= \int_{\ln \Gamma(1, x; 1 - t, \xi) \geq -K} (\ln \Gamma(1, x; 1 - t, \xi) - G(t, x) - K)^2 \mu(d\xi) \]
\[\geq \frac{1}{2} \int_{\ln \Gamma(1, x; 1 - t, \xi) \geq -K} \ln \Gamma(1, x; 1 - t, \xi) + K - G(t, x))^2 \mu(d\xi) - K^2 \]
\[\geq \frac{1}{2} \int_{\ln \Gamma(1, x; 1 - t, \xi) \geq -K} G(t, x)^2 \mu(d\xi) - K^2 \]
\[= \frac{1}{2} G(t, x)^2 \mu \{\ln \Gamma(1, x; 1 - t, \xi) \geq -K\} - K^2.\]

By the upper bound in Theorem 11, we can find a \(R > 0\) depending only on \(n, \lambda, \Lambda\) such that

\[
\int_{|\xi| > R} \Gamma(1, x; 1 - t, \xi) d\xi < \frac{1}{4}, \quad \text{for all} \ t \in (0, 1], x \in B(0, 2).\]

Also the upper bound implies that there exists a \(M > 0\) depending only on \(n, \lambda\) such that \(\Gamma(1, x; 1 - t, \xi) \leq M\) for \(t \in [\frac{1}{2}, 1]\). So one has

\[
\frac{3}{4} \leq \int_{|\xi| \leq R} \Gamma(1, x; 1 - t, \xi) d\xi \leq |B(0, R)| e^{-K} + M(2\pi)^{n/2} e^{R^2/2} \mu \{\xi \in \mathbb{R}^n : \ln \Gamma(x, 1; \xi, 1 - t) \geq -K\}.
\]

Now we can choose a large enough \(K\) depending on \(R\) such that

\[
\mu \{\xi \in \mathbb{R}^n : \ln \Gamma(x, 1; \xi, 1 - t) \geq -K\} \geq \frac{1}{2M(2\pi)^{n/2} e^{R^2/2}}.
\]

Combining these all together, we have

\[
\|\ln \Gamma(1, x; 1 - t, \cdot) - G(t, x)\|_{L^2(\mu)}^2 \geq -K^2 + \frac{1}{4M(2\pi)^{n/2} e^{R^2/2}} G(t, x)^2,
\]
and therefore

\[
G'(t, x) \geq -C_1 + C_2 G(t, x)^2.
\]

Together with the fact that \(G \leq 0\), it follows from Lemma 5 that

\[
G(1, x) \geq \min \left\{ -C_1 - 2 \sqrt{\frac{C_1}{C_2}} , -\frac{8}{3C_2} \right\} = -C.
\]
Finally the Chapman-Kolmogorov equation and Jensen’s inequality yield that
\[
\ln \Gamma(2, x; 0, \xi) = \ln \left( \int_{\mathbb{R}^n} (2\pi)^{n/2} e^{-|z|^2/2} \Gamma(2, x; 1, z) \Gamma(1, z; 0, \xi) \, dz \right)
\]
\[
= \ln \left( \int_{\mathbb{R}^n} (2\pi)^{n/2} e^{-|z|^2/2} \Gamma(2, x; 1, z) \Gamma(1, z; 0, \xi) \mu(dz) \right)
\]
\[
\geq \ln \left( \int_{\mathbb{R}^n} \Gamma(2, x; 1, z) \Gamma(1, z; 0, \xi) \mu(dz) \right)
\]
\[
\geq \int_{\mathbb{R}^n} \ln \Gamma(2, x; 1, z) \mu(dz) + \int_{\mathbb{R}^n} \ln \Gamma(1, z; 0, \xi) \mu(dz)
\]
\[
\geq -2C,
\]
where the estimate of \( \int_{\mathbb{R}^n} \ln \Gamma(2, x; 1, z) \mu(dz) \geq -C \) is simply a shift in time and \( \int_{\mathbb{R}^n} \ln \Gamma(1, z; 0, \xi) \mu(dz) \geq -C \) can be obtained by analyzing the dual operator with the same argument.

**Proof of Theorem 2.** Recall the scaling invariant property, i.e. for any \( \rho > 0 \) and \( z \in \mathbb{R}^n \),
\[
\Gamma(\rho^2 t, \rho x + z; 0, \rho \xi + z) = \rho^{-n} \Gamma^{(a, b, c)}(t, x; 0, \xi)
\]
(27)
where \( a_{\rho z}(t, x) = a(\rho^2 t, \rho x + z) \), \( b_{\rho z}(t, x) = \rho b(\rho^2 t, \rho x + z) \) and \( \Gamma^{(a, b)} \) is the fundamental solution associated with \( (a, b) \). The transformation \( (a, b) \to (a_{\rho z}, b_{\rho z}) \) preserves the ellipticity constant \( \lambda \) of \( a \) and more importantly the \( L^\infty \) norms of \( b \). So we may apply Lemma 13 to \( \Gamma^{(a_{\rho z}, b_{\rho z})} \) to deduce that
\[
\Gamma(2t, x; 0, \xi) \geq \frac{e^{-2C}}{\rho^{n/2}}, \quad |\xi - x| < 4t^{1/2}.
\]
(28)
Next, to obtain a lower bound for all \( x, \xi \in \mathbb{R}^n \), we use the Chapman-Kolmogrov equation. Suppose \( |x - \xi|^2 \in [k, k + 1) \), we set \( \xi_m = \xi + \frac{m}{k+1}(x - \xi) \) and \( B_m = B(\xi_m, 1/k^{1/2}) \) for \( 0 \leq m \leq k + 1 \). For any \( z_m \in B_m \), we will have \( |z_m - z_{m-1}| < \frac{1}{k^{1/2}} \) and hence \( \Gamma(\frac{2m}{k+1}, z_m, \frac{2m}{k+1}, z_{m-1}) \geq \frac{1}{k^{n/2}} e^{-2C} \). By the Chapman-Kolmogrov equation we obtain that
\[
\Gamma(2, x; 0, \xi) \geq \int_{B_1} \cdots \int_{B_k} \prod_{m=1}^{k} \Gamma(\frac{2m}{k+1}, z_m, \frac{2m+1}{k+1}, z_{m-1}) \, dz_1 \cdots \, dz_k
\]
\[
\geq (\frac{1}{k^{n/2}} e^{-2C})^{k+1} (|B(1)|^{k-n/2})^k.
\]
Choose a constant \( C' \) such that \( e^{-C'} \leq |B(1)|^{k-n/2} \) to obtain
\[
\Gamma(2, x; 0, \xi) \geq e^{-2C} e^{-C'|x - \xi|^2}.
\]
Finally, we use scaling again to obtain that
\[
\Gamma(t, x; 0, \xi) \geq \frac{1}{C't^{n/2}} \exp \left( -C \frac{|x - \xi|^2}{t} \right).
\]
Now the proof is complete. \( \square \)
3.2.2 Supercritical cases

Now we consider the supercritical case \(1 < \gamma < 2\). We will estimate \(G_r(t,x)\) directly without using scaling argument.

**Lemma 14.** Suppose \(q \geq 2\), \(l \geq 2\), \(1 < \gamma < 2\) and \(\hat{R}(t)\) is defined as in Proposition 12. For any \(\kappa > 0\), \(x \in B(0, \kappa \hat{R}(t))\) and \(t > 0\) small enough, there is a constant \(C > 0\) depending only on \(\kappa, l, q, n, \lambda, \Lambda = \|b\|_{L^1((0,T;L^q(\mathbb{R}^n))}\) such that

\[
G_r(t,x) \geq -C(\lambda)\left(\frac{t}{r} + r^{-\eta/q} \hat{R}^2 \Lambda^2\right) - C\left(\frac{r}{l}\right)^{n/2+1} \exp\left(\frac{\pi |\hat{R}(t)|^2}{Cr}\right). \tag{29}
\]

**Proof.** We fix a \(T > 0\). By the definition of \(G_r(t,x)\), for \(0 < t_1 < t_2 \leq T\), we can deduce that

\[
G_r(t_2,x) - G_r(t_1,x) = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \frac{\partial}{\partial s} \Gamma(T,x;T-s,\xi) \mu_r(d\xi) ds \\
= \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left\langle \frac{2\pi}{r} a(T-s,\xi) \cdot \nabla_{\xi} \ln \Gamma(T,x;T-s,\xi), a(T-s,\xi) \cdot \nabla_{\xi} \ln \Gamma(T,x;T-s,\xi) \right\rangle \mu_r(d\xi) ds \\
+ \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left\langle b(T-s,\xi), \nabla_{\xi} \ln \Gamma(T,x;T-s,\xi) \right\rangle \mu_r(d\xi) ds \\
= -\int_{t_1}^{t_2} \frac{2\pi}{r} \|\xi\|_{L^2(\mu_r)} \|\nabla_{\xi} \ln \Gamma(T,x;T-s,\cdot)\|^2_{L^2(\mu_r)} ds \\
+ \int_{t_1}^{t_2} \lambda \|\nabla_{\xi} \ln \Gamma(T,x;T-s,\cdot)\|^2_{L^2(\mu_r)} ds \\
- \int_{t_1}^{t_2} \|b(T-s,\cdot)\|_{L^q(\mathbb{R}^n)} \|\mu_r^{\frac{1}{2}} \|_{L^{q/2}(\mathbb{R}^n)} \|\nabla_{\xi} \ln \Gamma(T,x;T-s,\cdot)\|_{L^2(\mu_r)} ds \\
\geq -\int_{t_1}^{t_2} \frac{C(\lambda)}{r^2} \|\xi\|^2_{L^2(\mu_r)} + C(\lambda) \|b(T-s,\cdot)\|^2_{L^q(\mathbb{R}^n)} \|\mu_r^{\frac{1}{2}}\|^2_{L^{q/2}(\mathbb{R}^n)} ds \\
+ \frac{\lambda}{2} \int_{t_1}^{t_2} \|\nabla_{\xi} \ln \Gamma(T,x;T-s,\cdot)\|^2_{L^2(\mu_r)} ds.
\]

Here we set \(\frac{2q}{q-2} = \infty\) when \(q = 2\). Since \(l \geq 2\), we have

\[
\int_0^T \|b(T-s,\cdot)\|^2_{L^q(\mathbb{R}^n)} ds < \infty.
\]

For the last term in the equation above, we use the Poincaré-Wirtinger inequality and obtain that

\[
\int_{t_1}^{t_2} \|\nabla_{\xi} \ln \Gamma(T,x;T-s,\cdot)\|^2_{L^2(\mu_r)} ds \geq C r^{-1} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |\ln \Gamma(T,x;T-s,\xi) - G_r(s,x)|^2 \mu_r(d\xi) ds.
\]
Since \( G_r(t,x) \leq 0 \), using \((a - b)^2 \geq \frac{a^2}{r^2} - b^2\), the right-hand side can be estimated as

\[
\int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left| \ln \Gamma(T,x;T-s,\xi) - G_r(s,x) \right|^2 \mu_r(d\xi) ds
\]

\[
\geq \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left| \ln \Gamma(T,x;T-s,\xi) - G_r(s,x) - 1 \right|^2 \mu_r(d\xi) ds
\]

\[
\geq \frac{1}{2} \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \left( \ln \Gamma(T,x;T-s,\xi) - G_r(s,x) - 1 \right)^2 \mu_r(d\xi) ds
\]

By Proposition 12, for any \( x \in B(0, \kappa \hat{R}(t)) \), we have

\[
\int_{B(0,(C+\kappa)\hat{R}(t))} \Gamma(T,x;T-t,\xi) d\xi \geq \frac{1}{2},
\]

which implies that

\[
\mu_r \{ \ln \Gamma(T,x;T-t,\xi) \geq -1 \} \geq \frac{C r^{n/2}}{T^{n/2}} \exp \left( \frac{\pi |\hat{R}(T)|^2}{Cr} \right) + |B(0,(C+\kappa)\hat{R}(t))| e^{-1} \geq \frac{1}{2}
\]

for \( t \in \left[ \frac{T}{r}, T \right] \). So if we take \( T > 0 \) small enough such that \(|B(0,(C+\kappa)\hat{R}(t))| e^{-1} \leq \frac{1}{4}\) for all \( t \in [0,T] \), then

\[
\mu_r \{ \ln \Gamma(T,x;T-t,\xi) \geq -1 \} \geq C \left( \frac{T}{r} \right)^{n/2} \exp \left( - \frac{\pi |\hat{R}(T)|^2}{Cr} \right).
\]

Also it is easy to calculate that \( \|\xi\|^2 \mathbb{L}^2(\mu_r) = r \) and

\[
\left\| \mu_r^{1/2} \mathbb{L}^{2/2}(\mathbb{R}^n) \right\| \left\| \mu_r^{1/2} \mathbb{L}^{2/2}(\mathbb{R}^n) \right\| = C(q) r^{-n/q}.
\]

We now can conclude that

\[
G_r(t_2,x) - G_r(t_1,x) \geq - \int_{t_1}^{t_2} C(\lambda) r^{n/q} \|b(T-s,x)\|^2 \mathbb{L}^2(\mathbb{R}^n) + Cr^{-1} ds
\]

\[
+ Cr^{-1} \left( \frac{T}{r} \right)^{n/2} \exp \left( - \frac{\pi |\hat{R}(T)|^2}{Cr} \right) \int_{t_1}^{t_2} G_r(s,x)^2 ds,
\]

for \( \frac{T}{2} \leq t_1 < t_2 \leq T \). By Lemma 6 and \( l \geq 2 \), we have

\[
G_r(T,x) \geq -C(\lambda) \left( \frac{T}{r} + r^{-n/q} T^{\frac{n}{2}+1} \right) - C \left( \frac{T}{r} \right)^{n/2+1} \exp \left( \frac{\pi |\hat{R}(T)|^2}{Cr} \right).
\]

\[\blacksquare\]
Proof of Theorem 3. For $x, \xi \in B(0, \kappa R(t))$, by using the Chapman-Kolmogorov equation we obtain that

\[
\ln \Gamma(2T, x; 0, \xi) = \ln \int_{\mathbb{R}^n} \Gamma(2T, x; T, z) \Gamma(T, z; 0, \xi) \, dz
\]

\[
\geq \ln \int_{\mathbb{R}^n} r^{n/2} \Gamma(2T, x; T, z) \Gamma(T, z; 0, \xi) \mu_r(\xi) \, dz
\]

\[
\geq \frac{n}{2} \ln r + \int_{\mathbb{R}^n} \ln \Gamma(2T, x; T, z) \Gamma(T, z; 0, \xi) \mu_r(\xi) \, dz
\]

\[
= \frac{n}{2} \ln r + \int_{\mathbb{R}^n} \ln \Gamma(2T, x; T, z) \mu_r(\xi) \, dz + \int_{\mathbb{R}^n} \ln \Gamma(T, z; 0, \xi) \mu_r(\xi) \, dz
\]

\[
\geq \frac{n}{2} \ln r - C(\lambda) \left( \frac{T}{r} + r^{-n/q} T^{\frac{2}{q+1}} \Lambda^2 \right) - C(\frac{r}{T})^{n/2+1} \exp \left( \frac{\pi |R(T)|^2}{Cr} \right),
\]

i.e.

\[
\Gamma(2T, x; 0, \xi) \geq r^2 \exp \left[ -C(\frac{T}{r} + r^{-n/q} T^{\frac{2}{q+1}} \Lambda^2) - C(\frac{r}{T})^{n/2+1} \exp \left( \frac{\pi |R(T)|^2}{Cr} \right) \right].
\]

Now we can take maximum of the right-hand side over all positive $r$. Recall $\tilde{R}(t) = Ct(2-\gamma)/2 \ln \frac{1}{t}$, if we take $r = \tilde{R}(T)^2$, then the right-hand side becomes

\[
T^{\frac{2}{q+1}} (\ln \frac{1}{T})^2 \exp \left[ -CT^{\theta_1} (\ln \frac{1}{T})^{-2} - CT^{\theta_2} (\ln \frac{1}{T})^{-2n/q} - CT^{\theta_3} (\ln \frac{1}{T})^{n+2} \right],
\]

where $\theta_1 = \gamma - 1$, $\theta_2 = 1 - \frac{\gamma}{q} - \frac{n}{q}(2-\gamma)$, and $\theta_3 = (\frac{q}{2} + 1)(1-\gamma) < 0$. Clearly $\theta_3 = \min \{\theta_1, \theta_2, \theta_3\} < 0$. Because we are considering only for small $t$, the dominant term will be

\[
\Gamma(2T, x; 0, \xi) \geq \exp \left[ -CT^{\theta_3} (\ln \frac{1}{T})^{(n+2)} \right],
\]

and the proof is complete.

We can use the Chapman-Kolmogorov equation to obtain a positive lower bound on the whole space, but we prefer to omit the details of computations.

Remark 15. Using full power of the Poincaré-Wirtinger inequality and following similar arguments as above, we can actually drop the assumptions that $q \geq 2$ and $l \geq 2$. We only need to assume that $1 \leq \gamma < 2$ to obtain a lower bound.

4 Application of the Aronson type estimate

Assume that $b \in L^n(0, T; L^\alpha(\mathbb{R}^n))$. In the previous section, we have proved the Aronson estimate

\[
\frac{1}{C t^{n/2}} \exp \left[ -C \frac{|x - \xi|^2}{t} \right] \leq \Gamma(t, x; 0, \xi) \leq \frac{C}{t^{n/2}} \exp \left[ -\frac{|x - \xi|^2}{C t} \right]
\]

where $C$ only depends on $(n, \lambda, \Lambda)$. As an application of the Aronson type estimate, we will prove the uniqueness of Hölder continuous weak solutions.
We denote the parabolic ball as \( Q((t_0, x_0), R) = (t_0 - R^2, t_0) \times B(x_0, R) \) and

\[
\text{Osc}_{Q((t_0, x_0), R)} u = \max_{Q((t_0, x_0), R)} u - \min_{Q((t_0, x_0), R)} u.
\]

Firstly, we will need Nash’s continuity theorem.

**Theorem 16.** Suppose \( u \in C^{1,2}(Q((t_0, x_0), R)) \) is a solution, then for any \( \delta \in (0, 1) \), there are \( \alpha \in (0, 1] \) and \( C > 0 \) depending only on \( (\delta, n, \lambda, \Lambda) \) such that

\[
|u(t_1, x_1) - u(t_2, x_2)| \leq C \left( \frac{|t_1 - t_2|^{1/2} \lor |x_1 - x_2|}{R} \right)^\alpha \text{Osc}_{Q((t_0, x_0), R)} u
\]

for any \((t_1, x_1), (t_2, x_2) \in Q((t_0, x_0), \delta R)\).

The proof of this theorem is in the Appendix. Applying this theorem to the fundamental solution, we have the following corollary.

**Corollary 17.** There exist \( \alpha \in (0, 1] \) and \( C > 0 \) depending only on \( (n, \lambda, \Lambda) \) such that for any \( \delta > 0 \), we have

\[
|\Gamma(t_1, x_1; 0, \xi_1) - \Gamma(t_2, x_2; 0, \xi_2)| \leq C \left( \frac{|t_1 - t_2|^{1/2} \lor |x_1 - x_2| \lor |\xi_1 - \xi_2|}{\delta} \right)^\alpha
\]

for all \((t_1, x_1, \xi_1), (t_2, x_2, \xi_2) \in [\delta^2, \infty) \times \mathbb{R}^n \times \mathbb{R}^n\) with \(|x_1 - x_2| \lor |\xi_1 - \xi_2| \leq \delta\).

Now we are well prepared for proving the uniqueness and Hölder continuity of the weak solutions. Given any \((a, b)\) satisfying conditions (E), (S) and \( b \) belonging to \( L^m(0, T; \mathbb{R}^n) \), by mollification we can find a sequence of smooth \((a_m, b_m)\) such that they satisfy conditions (E), (S) and \( \|b_m\|_{L^m_T L^q_x} \leq \|b\|_{L^m_T L^q_x} \). Moreover, \( b_m \) are compactly supported in space, \( a_m \to a \) in \( L^p_T \mathbb{R}^n(0, T) \times \mathbb{R}^n \) for any \( 1 \leq p < \infty \) and \( b_m \to b \) in \( L^p_T \mathbb{R}^n \) for any \( 1 \leq p < \infty \). Denote their corresponding fundamental solution as \( \Gamma^m \), then they have a uniform Aronson estimate, and hence the family of the associated fundamental solutions are equi-continuous in \( [\delta^2, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \) according to Corollary 17. Thus, by the Arzela-Ascoli theorem, there is a sub-sequence of \( \{\Gamma^m\} \) converging locally uniformly to some \( \Gamma \). Moreover, \( \Gamma \) still satisfies the same Aronson estimate, Hölder continuity and Chapman-Kolmogorov equation.

**Theorem 18.** Suppose equation (1) satisfies conditions (E), (S), and consider \( b \in L^m(0, T; L^p(\mathbb{R}^n)) \). Given initial value \( f \in L^2(\mathbb{R}^n) \), there exists a unique Hölder continuous weak solution \( u(t, x) \in L^m(0, T; L^p(\mathbb{R}^n)) \cap L^2(0, T; H^1(\mathbb{R}^n)) \) which satisfies \( u(0, x) = f(x) \). Moreover, \( \frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\mathbb{R}^n)) \) and

\[
u(t, x) = \int_{\mathbb{R}^n} \Gamma(t, x, \tau, \xi) u(\tau, \xi) d\xi,
\]

where \( \Gamma(t, x, \tau, \xi) \) is the fundamental solution of the heat equation.

**Proof.** Given \( f \in L^2(\mathbb{R}^n) \), denote \( u^m(t, x) = \Gamma^m f(x) \) the solution corresponding to \((a_m, b_m)\) and \( u(t, x) = \Gamma f(x) \). Since \( \Gamma^m \to \Gamma \) point-wise, the dominated convergence implies \( u^m \to u \) point-wise as well. Notice that we have energy inequality

\[
\|u^m(t, \cdot)\|_{L^2_x}^2 + \lambda \int_0^t \|\nabla u^m(s, \cdot)\|_{L^2_x}^2 ds \leq \|f\|_{L^2}^2,
\]

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which implies that \( u^m \) are weakly compact in \( L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n)) \). So its weak limit must be \( u \) as defined above. Since \( u^m \) satisfies the following identity

\[
\int_0^T \int_{\mathbb{R}^n} u^m(t,x) \frac{\partial}{\partial t} \varphi(t,x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \langle \nabla \varphi(t,x) \cdot a_m(t,x), \nabla u^m(t,x) \rangle \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \nabla \varphi(t,x) \cdot b_m(t,x) u^m(t,x) \, dx \, dt = 0
\]

for \( \varphi \in C_0^\infty([0,T] \times \mathbb{R}^n) \), by taking \( m \to \infty \) we obtain

\[
\int_0^T \int_{\mathbb{R}^n} u(t,x) \frac{\partial}{\partial t} \varphi(t,x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \langle \nabla \varphi(t,x) \cdot a(t,x), \nabla u(t,x) \rangle \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \nabla \varphi(t,x) \cdot b(t,x) u(t,x) \, dx \, dt = 0,
\]

which implies that \( u \) is a weak solution. Also we have

\[
\left| \int_0^T \int_{\mathbb{R}^n} \langle \nabla \varphi(t,x) \cdot a(t,x), \nabla u(t,x) \rangle \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \nabla \varphi(t,x) \cdot b(t,x) u(t,x) \, dx \, dt \right| \leq (\frac{1}{\lambda} + \Lambda) \| \varphi \|_{L^2(0,T,H^1)} \| u \|_{L^2(0,T;H^1)},
\]

and so

\[
\left| \int_0^T \int_{\mathbb{R}^n} u(t,x) \frac{\partial}{\partial t} \varphi(t,x) \, dx \, dt \right| \leq (\frac{1}{\lambda} + \Lambda) \| \varphi \|_{L^2(0,T,H^1)} \| u \|_{L^2(0,T;H^1)}.
\]

Now we obtain \( \frac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(\mathbb{R}^n)) \). This allows us to take \( \varphi = u \) to have the energy estimate

\[
\| u(t, \cdot) \|_2^2 + \lambda \int_0^t \| \nabla u(s, \cdot) \|_2^2 \, ds \leq \| f \|_2^2,
\]

and therefore we obtain the uniqueness of the weak solution in \( L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n)) \). Suppose \( \Gamma^m \to \Gamma' \), which will define another \( u' \) and it satisfies all the results above. So \( u = u' \) implies \( \Gamma = \Gamma' \), and we also have the uniqueness of the fundamental solution.

\[\square\]

**Appendix**

Here we prove the Nash’s continuity theorem. The proof is inspired by [?], which was originally written in probability language and relies heavily on the strong Markov property of the diffusion process. Here we rewrite it using a PDE approach instead.

We still assume \( (a, b) \) to be smooth and consider the Dirichlet problem on \([0,T] \times B(x_0, R)\) for any fixed \( x_0 \) and \( R > 0 \).

\[
\partial_t u - \text{div}(a \cdot \nabla u) + b \cdot \nabla u = 0 \quad \text{in } (0,T) \times B(x_0,R) \tag{30}
\]
with $u(0,x) = f(x)$ and $u(t,x) = 0$ for $x \in \partial B(x_0,R)$. Clearly there is a unique regular fundamental solution $\Gamma^{s_0,R}(t,x;\tau,\xi)$ with $x,\xi \in B(x_0,R)$. So for any $f \in C^0_c(B(x_0,R))$ satisfying $f \geq 0$,

$$\Gamma^{s_0,R}_t f(x) = \int_{B(x_0,R)} \Gamma^{s_0,R}(t,x;0,\xi)f(\xi) d\xi$$

is the unique strong solution to Dirichlet problem. We will prove the following lower bound for $\Gamma^{s_0,R}(t,x;\tau,\xi)$, which is also interesting by its own.

**Theorem 19.** For any $\delta \in (0,1)$, there exists a constant $C = C(\delta,n,\lambda,\Lambda)$ such that

$$\Gamma^{s_0,R}(t,x;\tau,\xi) \geq \frac{1}{C(t-\tau)^{n/2}} \exp \left(-\frac{C|x-\xi|^2}{t-\tau}\right)$$

for any $t - \tau \in (0,R^2]$ and $x,\xi \in B(x_0,\delta R)$.

**Proof.** Without loss of generality, we will take $\tau = 0$. For any $t > 0$, given $f \in C^0_c(B(x_0,R))$ satisfying $f \geq 0$, consider $w(s,x) = \Gamma_t f(x) - \Gamma^{s_0,R}_t f(x) - M$ for $s \in [0,t]$ where

$$M = \sup_{s \in [0,t], z \in B(x_0,R)^c} \Gamma_s f(z).$$

Then we notice that $w$ solves (30) in $[0,t] \times B(x_0,R)$ with the initial-boundary condition that $w(0,x) \leq 0$ for $x \in B(x_0,R)$ and $w(s,x) \leq 0$ for $s \in (0,t], x \in \partial B(x_0, R)$. So the maximum principle implies that $w \leq 0$ in $(0,t] \times B(x_0,R)$, which means $\Gamma^{s_0,R}_t f(x) \geq \Gamma_t f(x) - M$. Since this is true for any $f \in C^0_c(B(x_0,\delta R))^+$ with $\delta \in (0,1)$, we have

$$\Gamma^{s_0,R}(t,x;0,\xi) \geq \Gamma(t,x;0,\xi) - \sup_{s \in [0,t], z \in B(x_0,R)^c, y \in B(x_0,\delta R)} \Gamma(s,z;0,y)$$

$$\geq \frac{1}{Ct^{n/2}} \exp \left(-\frac{C|x-\xi|^2}{t}\right) - \sup_{s \in [0,t]} \frac{C}{s^{n/2}} \exp \left(-\frac{(1-\delta)^2 R^2}{C s}\right)$$

for any $x,\xi \in B(x_0,\delta R)$. Consider the second term, set $\bar{t} = t/R^2$ and $\bar{s} = s/R^2$

$$\sup_{s \in [0,t]} \frac{C}{s^{n/2}} \exp \left(-\frac{(1-\delta)^2 R^2}{C s}\right)$$

$$= \frac{1}{2C\bar{t}^{n/2}} \exp \left(-\frac{C|x-\xi|^2}{\bar{t}}\right) \sup_{\bar{s} \in [0,\bar{t}]} \frac{2C^2 \bar{t}^{n/2}}{\bar{s}^{n/2}} \exp \left(-\frac{(1-\delta)^2 R^2}{C \bar{s}} + C|x-\xi|^2/\bar{s}\right)$$

$$= \frac{1}{2C\bar{t}^{n/2}} \exp \left(-\frac{C|x-\xi|^2}{\bar{t}}\right) \sup_{\bar{s} \in [0,\bar{t}]} \frac{2C^2 \bar{t}^{n/2}}{\bar{s}^{n/2}} \exp \left(-\frac{(1-\delta)^2 R^2}{C \bar{s}} - C|x-\xi|^2/\bar{t}\bar{R}^2\right).$$

If $|x-\xi|^2 \leq (1-\delta)^2 R^2/2\bar{t}$ and $t \leq R^2$, it implies

$$\sup_{\bar{s} \in [0,\bar{t}]} \frac{2C^2 \bar{t}^{n/2}}{\bar{s}^{n/2}} \exp \left(-\frac{(1-\delta)^2 R^2}{C \bar{s}} + C|x-\xi|^2/\bar{t}\bar{R}^2\right) \leq \sup_{\bar{s} \in [0,\bar{t}]} \frac{2C^2}{\bar{s}^{n/2}} \exp \left(-\frac{(1-\delta)^2 R^2}{2C \bar{s}}\right),$$

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where \( \frac{2c^2}{\rho^2} \exp\left(-\frac{(1-\delta^2)^2}{2\rho^2} \right) \to 0 \) as \( \delta \to 0 \). So we can take \( \tilde{\epsilon} \) small enough so that \( RHS \leq 1 \) and hence we have

\[
\Gamma^{s_0, R}(t, x; 0, \xi) \geq \frac{1}{2C_{\rho^2/2}} \exp\left(-C \frac{|x-\xi|^2}{t} \right)
\]

where \( \max\{t, |x-\xi|^2\} \leq \varepsilon^2 R^2 \) for some small \( \varepsilon \) depending on \( (\delta, n, \lambda, \Lambda) \).

Now we use the Chapman-Kolmogorov equation to extend this to any \( x, \xi \in B(x_0, \delta R) \) and \( t \in (0, R^2] \). First consider \( |x-\xi| \geq \varepsilon R \) and any \( t \), we set \( \xi_m = \xi + \sum_{k=1}^{m} (x-\xi), B_m = B(x_0, \delta R) \cap B(\xi_m, \frac{|x-\xi|}{k+1}), t_m = \frac{mt}{k+1} \). Then for any \( z_m \in B_m \), we have \( |z_m - z_m-1| \leq \frac{3|x-\xi|}{k+1} \). So, to obtain \( |z_m - z_m-1| \leq \varepsilon R \) and \( |t_m - t_{m-1}| \leq \varepsilon^2 R^2 \), we just need to choose \( k \geq \frac{1}{\varepsilon^2} \). Now one has

\[
\Gamma^{s_0, R}(t, x; 0, \xi) \geq \int_{B_1} \cdots \int_{B_k} \prod_{m=0}^{k} \Gamma^{s_0, R}(t_{m+1}, z_{m+1}; t_m, z_m) d\tilde{z}_1 \cdots d\tilde{z}_k
\]

\[
\geq C \frac{|x-\xi|^n}{k+1} \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) \frac{(k+1)^{n/2}}{2C_{\rho^2/2}} \exp\left(-C \frac{|x-\xi|^2}{(k+1)t} \right)
\]

\[
\geq C \frac{|x-\xi|^n}{k+1} \frac{1}{t^{n/2}} \exp\left(-C \frac{|x-\xi|^2}{t} \right)
\]

The only case left now is the case where \( |x-\xi| \leq \varepsilon R \) and \( t \geq \varepsilon^2 R^2 \). Set \( t_m \) as before, then

\[
\Gamma^{s_0, R}(t, x; 0, \xi) \geq C \frac{\varepsilon R}{k+1} \left( \begin{array}{c} n+1 \\ 2 \end{array} \right) \frac{(k+1)^{n/2}}{2C_{\rho^2/2}} \exp\left(-C \frac{(k+1)|x-\xi|^2}{t} \right)
\]

\[
\geq \frac{1}{C_{\rho^2/2}} \exp\left(-C \frac{|x-\xi|^2}{t} \right),
\]

and the proof is complete. \( \square \)

Now we give the proof of Nash’s continuity theorem. First consider a non-negative solution on a parabolic ball \( u \in C^{1,2}([t_0 - R^2, t_0] \times B(x_0, R)) \), clearly we have

\[
u(t, x) \geq \int_{B(x_0, R)} \Gamma^{s_0, R}(t, x; t_0 - R^2, \xi) u(t_0 - R^2, \xi) d\xi
\]

by the maximum principle. Then by Theorem 19

\[
\int_{B(x_0, \delta_2 R)} u(t, x) \geq \frac{1}{C |B(x_0, \delta_2 R)|} \int_{B(x_0, \delta_1 R)} u(t_0 - R^2, \xi) d\xi
\]

(31)

for any \( (t, x) \in [t_0 - \varepsilon^2 R^2, t_0] \times B(x_0, \delta_2 R), \delta_1, \delta_2 \in (0, 1) \) and \( C \) depending only on \( \delta_1, \delta_2, n, \lambda, \Lambda \). This estimate is called the super-mean value property.
Lemma 20. Suppose \( u \in C^{1,2}(Q((t_0, x_0), R)) \) is a solution, then for any \( \delta \in (0, 1) \), there is a \( \theta = \theta(\delta, n, \lambda, \Lambda) \in (0, 1) \) such that

\[
\text{Osc}_{Q((t_0, x_0), \delta R)} u \leq \theta \text{Osc}_{Q((t_0, x_0), R)} u.
\]

Proof. Let

\[
M(r) = \max_{Q((t_0, x_0), r)} u, \quad m(r) = \min_{Q((t_0, x_0), r)} u,
\]

and consider \( M(R) - u \) and \( u - m(R) \), which are non-negative solutions. Inequality (31) implies that

\[
M(R) - M(\delta R) \geq \frac{1}{C|B(x_0, \delta R)|} \int_{B(x_0, \delta R)} M(R) - u(0 - R^2, \xi) d\xi
\]

and

\[
m(\delta R) - m(R) \geq \frac{1}{C|B(x_0, \delta R)|} \int_{B(x_0, \delta R)} u(0 - R^2, \xi) - m(R) d\xi.
\]

The sum of above two inequalities gives us

\[
[M(R) - m(R)] - [M(\delta R) - m(\delta R)] \geq \frac{1}{C} [M(R) - m(R)],
\]

which completes the proof. \( \square \)

Proof of Theorem 16. Denote \( l = |t_1 - t_2|^{1/2} \vee |x_1 - x_2| \). If \( \frac{l}{R} \geq 1 - \delta \), then it is easy to find \( C \) and the proof is done. If \( \frac{l}{R} < 1 - \delta \), we choose integer \( K \) such that \((1 - \delta)^{K+1} \leq \frac{l}{R} < (1 - \delta)^K\). Assume \( t_1 \leq t_2 \). Then

\[
|u(t_1, x_1) - u(t_2, x_2)| \leq \text{Osc}_{Q((t_2, x_2), (1-\delta)^K R)} u \leq \theta^{K-1} \text{Osc}_{Q((t_2, x_2), (1-\delta) R)} u
\]

\[
\leq \theta^{K-1} \text{Osc}_{Q((t_0, x_0), R)} u = \theta^{-2} (\theta^{K+1}) \text{Osc}_{Q((t_0, x_0), R)} u.
\]

Now we can find \( \alpha \) such that \( \theta = ((1 - \delta) \wedge \theta)\alpha \), which implies \( \theta^{K+1} \leq (1 - \delta)^{(K+1)}\alpha \leq \left(\frac{1}{R}\right)^\alpha \). The proof is complete. \( \square \)

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