Four Deviations Suffice for Rank 1 Matrices

Rasmus Kyng∗
kymg@seas.harvard.edu
Harvard University

Kyle Luh†
kluhr@cmsa.fas.harvard.edu
Harvard University

Zhao Song
zhaos@seas.harvard.edu
Harvard University

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Abstract

We prove a matrix discrepancy bound that strengthens the famous Kadison-Singer result of Marcus, Spielman, and Srivastava. Consider any independent scalar random variables $\xi_1, \ldots, \xi_n$ with finite support, e.g. $\{\pm 1\}$ or $\{0, 1\}$-valued random variables, or some combination thereof. Let $u_1, \ldots, u_n \in \mathbb{C}^m$ and

$$\sigma^2 = \left\| \sum_{i=1}^n \text{Var}[\xi_i](u_i u_i^*)^2 \right\|.$$

Then there exists a choice of outcomes $\varepsilon_1, \ldots, \varepsilon_n$ in the support of $\xi_1, \ldots, \xi_n$ s.t.

$$\left\| \sum_{i=1}^n \mathbb{E}[\xi_i]u_i u_i^* - \sum_{i=1}^n \varepsilon_i u_i u_i^* \right\| \leq 4\sigma.$$

A simple consequence of our result is an improvement of a Lyapunov-type theorem of Akemann and Weaver.

1 Introduction

Discrepancy theory is an area of combinatorics that studies how well continuous objects can be approximated by discrete ones. It lies at the heart of numerous problems in mathematics and computer science [Cha00]. Although closely tied to probability theory, direct randomized approaches rarely yield the best bounds. In a classical formulation in discrepancy theory, we have $n$ sets on $n$ elements, and would like to two-color the elements so that each set has roughly the same number of elements of each color. Using a simple random coloring, it is an easy consequence of Chernoff’s bound that there exists a coloring such that the discrepancy in all $n$ sets is $O(\sqrt{n \log n})$ [AS16]. However, in a celebrated result, Spencer showed that in fact there is a coloring with discrepancy at most $6\sqrt{n}$ [Spe85].

Recently, there has been significant success in generalizing Chernoff/Hoeffding/Bernstein/Bennett-type concentration bounds for scalar random variables to matrix-valued random variables [Rud99, Rud99, Rud99].

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Consider the following matrix concentration bound, which is a direct consequence of a matrix Hoeffding bound.

**Theorem 1.1** ([Tro12]). Let \( \xi_i \in \{\pm 1\} \) be independent, symmetric random signs and \( A_1, \ldots, A_n \in \mathbb{C}^{m \times m} \) be positive semi-definite matrices. Suppose \( \max_i \|A_i\| \leq \epsilon \) and \( \|\sum_{i=1}^n A_i\| \leq 1 \). Then,

\[
P \left( \left\| \sum_{i=1}^n \xi_i A_i \right\| \geq t \sqrt{\epsilon} \right) \leq 2 m \exp(-t^2/2).
\]

A consequence of this theorem is that with high probability

\[
\left\| \sum_{i=1}^n \xi_i A_i \right\| = O(\sqrt{\log m}) \sqrt{\epsilon}.
\]  

(1)

The Kadison-Singer theorem of [MSS15b] is essentially equivalent to the following statement (which can readily be derived from the bipartition statement in [MSS15b]).

**Theorem 1.2** ([MSS15b]). Let \( u_1, \ldots, u_n \in \mathbb{C}^m \) and suppose \( \max_{i \in [n]} \|u_i u_i^*\| \leq \epsilon \) and \( \sum_{i=1}^n u_i u_i^* = I \). Then, there exists signs \( \xi_i \in \{\pm 1\} \) s.t.

\[
\left\| \sum_{i=1}^n \xi_i u_i u_i^* \right\| \leq O(\sqrt{\epsilon}).
\]

Thus, for rank 1 matrices the theorem improves on the norm bound in Equation (1), by a factor \( \sqrt{\log m} \), in a manner analogous to the improvement of Spencer’s theorem over the bound based on the scalar Chernoff bound.

For random signings of matrices one can establish bounds in some cases that are much stronger than Theorem 1.1.

**Theorem 1.3** ([Tro12]). Let \( \xi_i \in \{\pm 1\} \) be independent random signs, and let \( A_1, \ldots, A_n \in \mathbb{C}^{m \times m} \) be Hermitian matrices. Let \( \sigma^2 = \|\sum_{i=1}^n \text{Var}[\xi_i A_i^2]\| \). Then,

\[
P \left( \left\| \sum_{i=1}^n \mathbb{E}[\xi_i] A_i - \sum_{i=1}^n \xi_i A_i \right\| \geq t \cdot \sigma \right) \leq 2 m \exp(-t^2/2).
\]

From this theorem we deduce that with high probability

\[
\left\| \sum_{i=1}^n \mathbb{E}[\xi_i] A_i - \sum_{i=1}^n \xi_i A_i \right\| = O(\sqrt{\log m}) \sigma.
\]  

(2)

Of course, this implies that there exists a choice of signs \( \varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\} \) such that

\[
\left\| \sum_{i=1}^n \mathbb{E}[\xi_i] A_i - \sum_{i=1}^n \varepsilon_i A_i \right\| = O(\sqrt{\log m}) \sigma.
\]

Our main result demonstrates that for rank-1 matrices, there exists a choice of signs with a stronger guarantee.
Theorem 1.4 (Main Theorem). Consider any independent scalar random variables $\xi_1, \ldots, \xi_n$ with finite support. Let $u_1, \ldots, u_n \in \mathbb{C}^m$ and

$$\sigma^2 = \left\| \sum_{i=1}^{n} \text{Var}[\xi_i](u_i u_i^*)^2 \right\|.$$

Then there exists a choice of outcomes $\varepsilon_1, \ldots, \varepsilon_n$ in the support of $\xi_1, \ldots, \xi_n$

$$\left\| \sum_{i=1}^{n} \mathbb{E}[\xi_i] u_i u_i^* - \sum_{i=1}^{n} \varepsilon_i u_i u_i^* \right\| \leq 4\sigma.$$

For rank 1 matrices our theorem improves on the norm bound in Equation (2), by a factor $\sqrt{\log m}$. Note for example, if $\xi_i$ is \{±1\}-valued, then $\text{Var}[\xi_i] = 1 - \mathbb{E}[\xi_i]^2$.

Specializing to centered random variables, we obtain the following corollary, which in a simple way generalizes the Kadison-Singer theorem, although with a slightly worse constant.

Corollary 1.5. Let $u_1, \ldots, u_n \in \mathbb{C}^m$ and

$$\sigma^2 = \left\| \sum_{i=1}^{n} (u_i u_i^*)^2 \right\|.$$

There exists a choice of signs $\varepsilon_i \in \{±1\}$ such that

$$\left\| \sum_{i=1}^{n} \varepsilon_i u_i u_i^* \right\| \leq 4\sigma.$$

Notice that if $\max_i \|u_i u_i^*\| \leq \epsilon$ and $\sum_{i=1}^{n} u_i u_i^* = I$, then $\sigma^2 \leq \epsilon$, and we obtain Theorem 1.2 as consequence of the corollary.

Our Theorem 1.4 has multiple advantages over Theorem 1.2. It allows us to show existence of solutions close to the mean of arbitrarily biased ±1 random variables, instead of only zero mean distributions. If some variable $\xi_i$ is extremely biased, its variance is correspondingly low, as $\text{Var}[\xi_i] = 1 - \mathbb{E}[\xi_i]^2$.

If only a small subset of the rank 1 matrices obtain the $\epsilon$ norm bound, and the rest are significantly smaller in norm, then we can have $\sigma^2 \approx \epsilon^2$, so we prove a bound of $O(\epsilon)$ instead of the $O(\sqrt{\epsilon})$ bound of the Kadison-Singer theorem. On the other hand, when the problem is appropriately scaled, we always have $\sigma \geq \epsilon^2$, so the gap between the two results can never be more than a square root. Additionally, note that although the Kadison-Singer theorem requires $\sum_{i=1}^{n} u_i u_i^* = I$, a multi-paving argument\(^1\) can be used to instead relax this to $\| \sum_{i=1}^{n} u_i u_i^* \| \leq 1$.

Approximate Lyapunov Theorems. Marcus, Spielman and Srivastava resolved the Kadison-Singer problem by proving Weaver’s conjecture [Wea04], which was shown to imply the Kadison-Singer conjecture. In [AW14], Akemann and Weaver prove a generalization of Weaver’s conjecture [Wea04].

Theorem 1.6 ([AW14]). Let $u_1, \ldots, u_n \in \mathbb{C}^m$ such that $\| \sum_{i=1}^{n} u_i u_i^* \| \leq 1$ and $\max_i \|u_i u_i^*\| \leq \epsilon$. For any $t_i \in [0, 1]$ and $1 \leq i \leq n$, there exists a set of indices $S \subset \{1, 2, \ldots, n\}$ such that

$$\left\| \sum_{i \in S} u_i u_i^* - \sum_{i=1}^{n} t_i u_i u_i^* \right\| = O(\epsilon^{1/8}).$$

\(^1\) This was pointed out to us by Tarun Kathuria.
Due to the classical Lyapunov theorem [Lya40] and its equivalent versions [Lin66], in their study of operator algebras, Akemann and Anderson [AA91] refer to a result as a Lyapunov theorem if the result states that for a convex set \( C \), the image of \( C \) under an affine map is equal to the image of the extreme points of \( C \). Theorem 1.6 is an approximate Lyapunov theorem as it can be interpreted as saying that the image of \([0, 1]^n\) under the map

\[
f : (t_1, \ldots, t_n) \to \sum_{i=1}^n t_i u_i u_i^*,
\]

can be approximated by the image of one of the vertices of the hypercube \([0, 1]^n\). A corollary of our main result, Theorem 1.4, is the following strengthening of Theorem 1.6. This result greatly improves the \( \epsilon \) dependence of the original Lyapunov-type theorem and provides a small explicit constant.

**Corollary 1.7.** Let \( u_1, \ldots, u_n \in \mathbb{C}^m \) such that \( \| \sum_{i=1}^n u_i u_i^* \| \leq 1 \) and \( \max_i \| u_i u_i^* \| \leq \epsilon \). For any \( t_i \in [0, 1] \) and \( 1 \leq i \leq n \), there exists a set of indices \( S \subset \{1, 2, \ldots, n\} \) such that

\[
\left\| \sum_{i \in S} u_i u_i^* - \sum_{i=1}^n t_i u_i u_i^* \right\| = 2 \epsilon^{1/2}.
\]

The corollary follows immediately from Theorem 1.4 by choosing as the \( \xi_i \) a set of independent \( \{0, 1\}\)-valued random variables with means \( t_i \in [0, 1] \). Note then that \( \text{Var}[\xi_i] = t_i(1 - t_i) \leq 1/4 \), and so by the assumptions \( \max_i \| u_i u_i^* \| \leq \epsilon \) and \( \sum_{i=1}^n u_i u_i^* \preceq I \), we have \( \sigma^2 \leq \epsilon/4 \), and we obtain Theorem 1.2 as consequence of the corollary.

### 1.1 Related work and open questions

Our work is an extension of the the Kadison-Singer theorem of [MSS15b]. This result has a rich history of connections with theoretical computer science, and, we hope, may represent a step toward getting polynomial time algorithms for finding Kadison-Singer partitions.

Expanders are sparse graphs that exhibit good connectivity. One view on this is that an expander graph approximates a complete graph in some sense, e.g. the spectrum of the adjacency matrix or Laplacian matrix associated with the graph resembles that of the complete graph. Ramanujan graphs [Mar73, LPS88] are sparse \( d \)-regular graphs whose adjacency matrix eigenvalues essentially optimally approximates those of a complete graph among all \( d \)-regular graphs for a given degree \( d \). These initial constructions of Ramanujan graphs relied on number-theoretic techniques and could not show existence of Ramanujan for all degrees and sizes.

Using the method of interlacing families of polynomials, [MSS15a, MSS18], finally showed the existence of **bipartite**\(^2\) Ramanujan graphs of all degrees and of all sizes. The existential result of [MSS15a] was turned into a polynomial time algorithm in [Coh16]. The restriction to bipartite Ramanujan graphs in these papers arose from the fact that interlacing families most naturally control only the largest eigenvalue of a matrix, while Ramanujan graphs require both the largest and the smallest eigenvalue to be controlled. [MSS15a] overcame this obstacle by studying only bipartite graphs, whose adjacency matrix eigenvalues are symmetric around zero and hence controlling the largest eigenvalue\(^3\) implies also controlling the smallest. Our techniques for simultaneously controlling the smallest and largest eigenvalues of a matrix are very different from earlier interlacing

\(^2\) Which approximates the complete bipartite graph.

\(^3\) More precisely, it is the largest non-trivial eigenvalue that is being controlled. The largest eigenvalue of the adjacency matrix of a \( d \)-regular graph is the trivial eigenvalue of \( d \).
family-based methods, and an intriguing open question is whether similar ideas can be used to construct non-bipartite Ramanujan graphs of all degrees and sizes.

[ST04] gave nearly-linear time algorithms for producing sparse graphs (with non-uniform weights) that closely approximate a given input graph, in the sense that the spectral behavior of the Laplacian matrices associated with the graphs is very similar. A greatly simplified and stronger result was given in [SS11], still with a nearly linear time algorithm. When used to construct a sparse graph that spectrally approximates a complete graph of size $n$, the result of [SS11] has a worse average degree than a corresponding Ramanujan graph by a factor $\Theta(\log n)$. Later, this was improved to within a factor $O(1)$, still with a polynomial time algorithm, in [BSS12]. Finally, a series of papers turned this latter result into a nearly-linear time algorithm [LS18, LS17]. The resolution of the Kadison-Singer theorem was motivated by [BSS12]⁴. A major open question in the area is whether a polynomial time algorithm for finding explicit solutions to the Kadison-Singer problem exists, and we hope that our strengthened form of the result may make the problem more tractable.

The Kadison-Singer theorem and our result can be interpreted as analogous to Spencer’s theorem in a matrix setting, in recent years that has been tremendous progress in obtaining polynomial time algorithms for Spencer’s theorem and related problems in other settings [Ban10, LM15, Rot17, BDG16, BG17, BDGL18, NDTTJ18]. Techniques developed in these papers have also found algorithmic uses in approximation algorithms beyond typical discrepancy statements [Rot13]. While interlacing families have notably been used to to bound integrality gaps for ATSP [AG14] algorithmic results based on these have been limited, with [Coh16] providing the most notable example of a polynomial time algorithm based on interlacing families. It seems likely, however, that a deeper understanding of the limits of matrix rounding could have many applications in approximation algorithms.

### 1.2 Our techniques

Our proof is based on the method of interlacing polynomials introduced in [MSS15a, MSS15b]. One difficulty in applying the method of interlacing polynomials is the inability to control both the largest and smallest eigenvalues of a matrix simultaneously. Various techniques and restrictions are used to overcome this problem in [MSS15a, MSS15b] (studying bipartite graphs, assuming isotropic position). We develop a seemingly more natural approach for simultaneously controlling both the largest and smallest roots of the matrices we consider. We study polynomials that can be viewed as expected characteristic polynomials, but are more easily understood as the expectation of a product of multiple determinants. The using a product of two determinants helps us bound the upper and lower eigenvalues both at the same time.

We introduce an analytic expression for the expected polynomials in terms of linear operators that use second order derivatives. This has the advantage of allowing us to gain stronger control over the movement of roots of polynomials under the linear operators we apply than those used by [MSS15b]. This is because movement of the roots now depends on the curvature of our polynomials in favorable way. Interestingly, linear operators containing second order derivatives also appear in the work of [AG14] but for different reasons. This lets us reuse one of their lemmas for bounding position of the roots of a real stable polynomial.

### 2 Preliminaries

We gather several basic linear algebraic and analytic facts in the following sections.

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⁴Gil Kalai suggested the connection between that result and the Kadison-Singer problem.
2.1 Linear Algebra

Lemma 2.1. Let \( x \in \mathbb{R}^n \). Then
\[
det(I - tx^*) = 1 - tx^*x.
\]

Fact 2.2 (Jacobi’s Formula). For \( A(t) \in \mathbb{R}^{n \times n} \) a function of \( t \),
\[
\frac{d}{dt} \det(A(t)) = \det(A(t)) \text{Tr} \left[ A^{-1}(t) \frac{d}{dt} A(t) \right].
\]

2.2 Real Stability

Definition 2.3. A multivariate polynomial \( p(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n] \) is stable if it has no zeros in the region \( \{(z_1, \ldots, z_n) : \text{Im}(z_i) > 0 \text{ for all } 1 \leq i \leq n\} \). \( p \) is real stable if \( p \) is stable and the coefficients of \( p \) are real.

Lemma 2.4 (Corollary 2.8, [AOGSS18]). If \( p \in \mathbb{R}[z_1, \ldots, z_n] \) is real stable, then for any \( c > 0 \), so is
\[
(1 - c\partial_{z_i}^2)p(z_1, \ldots, z_n)
\]
for all \( 1 \leq i \leq n \).

Lemma 2.5 (Proposition 2.4, [BB08]). If \( A_1, \ldots, A_n \) are positive semidefinite symmetric matrices, then the polynomial
\[
\det \left( \sum_{i=1}^n z_i A_i \right)
\]
is real stable.

We also need that real stability is preserved under fixing variables to real values (see [Wag11, Lemma 2.4(d)]).

Proposition 2.6. If \( p \in \mathbb{R}[z_1, \ldots, z_m] \) is real stable and \( a \in \mathbb{R} \), then \( p|_{z_1=a} = p(a, z_2, \ldots, z_m) \in \mathbb{R}[z_2, \ldots, z_m] \) is real stable.

2.3 Interlacing Families

We recall the definition and consequences of interlacing families from [MSS15a].

Definition 2.7. We say a real rooted polynomial \( g(x) = C \prod_{i=1}^{n-1} (x - \alpha_i) \) interlaces the real rooted polynomial \( f(x) = C' \prod_{i=1}^n (x - \beta_i) \) if
\[
\beta_1 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1} \leq \beta_n.
\]
Polynomials \( f_1, \ldots, f_k \) have a common interlacing if there is a polynomial \( g \) that interlaces each of the \( f_i \).

The following lemma relates the roots of a sum of polynomials to those of a common interlacer.

Lemma 2.8 (Lemma 4.2, [MSS15a]). Let \( f_1, \ldots, f_k \) be degree \( d \) real rooted polynomials with positive leading coefficients. Define
\[
f_0 := \sum_{i=1}^k f_i.
\]
If \( f_1, \ldots, f_k \) have a common interlacing then there exists an \( i \) for which the largest root of \( f_i \) is upper bounded by the largest root of \( f_0 \).
**Definition 2.9.** Let \( S_1, \ldots, S_n \) be finite sets. For every \( S \in \mathcal{F} \), we let \( f_S(x) \) be a real rooted polynomial of degree \( d \) with positive leading coefficient. For a choice of assignment \( s_1, \ldots, s_n \in S_1 \times \cdots \times S_n \), let \( f_{s_1, \ldots, s_n}(x) \) be a real rooted degree \( d \) polynomial with positive leading coefficient. For a partial assignment \( s_1, \ldots, s_k \in S_1 \times \cdots \times S_k \) for \( k < n \), we define

\[
f_{s_1, \ldots, s_k} := \sum_{s_{k+1} \in S_{k+1}, \ldots, s_n \in S_n} f_{s_1, \ldots, s_k, s_{k+1}, \ldots, s_n}.
\]

Note that this is compatible with our definition of \( f_\emptyset \) from Definition 2.7. We say that the polynomials \( \{f_{s_1, \ldots, s_n}\} \) form an interlacing family if for all \( k = 0, \ldots, n - 1 \) and all \( s_1, \ldots, s_k \in S_1 \times \cdots \times S_k \), the polynomials have a common interlacing.

The following lemma relates the roots of the interlacing family to those of \( f_\emptyset \).

**Lemma 2.10** (Theorem 4.4, [MSS15a]). Let \( S_1, \ldots, S_n \) be finite sets and let \( \{f_{s_1, \ldots, s_n}\} \) be an interlacing family. Then there exists some \( s_1, \ldots, s_n \in S_1 \times \cdots \times S_n \) so that the largest root of \( f_{s_1, \ldots, s_n} \) is upper bounded by the largest root of \( f_\emptyset \).

Finally, we recall a relationship between real-rootedness and common interlacings which has been discovered independently several times [DG94, Fel80, CS07].

**Lemma 2.11.** Let \( f_1, \ldots, f_k \) be univariate polynomials of the same degree with positive leading coefficient. Then \( f_1, \ldots, f_k \) have a common interlacing if and only if \( \sum_{i=1}^k \alpha_i f_i \) is real rooted for all nonnegative \( \alpha_i \) such that \( \sum_i \alpha_i = 1 \).

## 3 Expected Characteristic Polynomial

Instead of working with the random polynomial \( \det(xI - \sum_{i=1}^n \xi_i u_i^* u_i^*) \), we consider

\[
\det \left( x^2 I - \left( \sum_{i=1}^n \xi_i u_i^* u_i^* \right)^2 \right) = \det \left( xI - \sum_{i=1}^n \xi_i u_i^* u_i^* \right) \det \left( xI + \sum_{i=1}^n \xi_i u_i^* u_i^* \right).
\]

Observe that the largest root \( \lambda_{\text{max}} \) of this polynomial is

\[
\lambda_{\text{max}} \left( \det \left( x^2 I - \left( \sum_{i=1}^n \xi_i u_i^* u_i^* \right)^2 \right) \right) = \left\| \sum_{i=1}^n \xi_i u_i^* u_i^* \right\|.
\]

We gather some results that will allow us to extract an analytic expression for the expected characteristic polynomial.

**Lemma 3.1.** For positive semidefinite (PSD) matrices \( M, N \in \mathbb{R}^{m \times m} \), \( v \in \mathbb{R}^m \) and \( \xi \) a random variable with zero mean and variance \( \tau^2 \),

\[
\mathbb{E}_{\xi} \left[ \det(M - \xi vv^*) \det(N + \xi vv^*) \right] = \left( 1 - \frac{1}{2} \frac{d^2}{dt^2} \right)_{t=0} \det(M + t\tau vv^*) \det(N + t\tau vv^*). \tag{5}
\]

**Proof.** We can assume that both \( M \) and \( N \) are positive definitive and hence invertible. The argument for the the positive semi-definite case follows by a continuity argument (using Hurwitz’s theorem from complex analysis, see also [MSS15b]).
We show that the two sides of (5) are equivalent to the same expression. Beginning on the left hand side,
\[
\mathbb{E}[^\{\det(M - \xi vv^*)\} \det(N + \xi vv^*)] \\
= \det(M) \det(N) \mathbb{E}[\det(I - \xi M^{-1/2}vv^* M^{-1/2}) \det(I + \xi N^{-1/2}vv^* N^{-1/2})] \\
= \det(M) \det(N) \mathbb{E}[1 + \xi b^* b - \xi a^* a - \xi^2 a^* a b^* b] \\
= \det(M) \det(N)(1 - \tau^2 a^* a b^* b)
\]
where \( a := M^{-1/2}v \) and \( b := N^{-1/2}v \). For the right hand side of (5),
\[
\left(1 - \frac{1}{2} \frac{d^2}{dt^2}\right)_{t=0} \det(M + t\tau vv^*) \det(N + t\tau vv^*) \\
= \det(M) \det(N) \left(1 - \frac{1}{2} \frac{d^2}{dt^2}\right)_{t=0} \det(I + t\tau a^* a) \det(I + t\tau b^* b) \\
= \det(M) \det(N)(1 - \tau^2 a^* a b^* b)
\]
where the last line follows from Lemma 2.1. \( \square \)

Centering our random variables and applying the previous lemma leads to the following corollary for non-centered random variables.

**Corollary 3.2.** Let \( M, N \in \mathbb{R}^{m \times m} \) be arbitrary PSD matrices, \( v \in \mathbb{R}^m \) and \( \xi \) a random variable with expectation \( \mu \) and variance \( \tau^2 \).

\[
\mathbb{E}[\det(M - (\xi - \mu)vv^*) \det(N + (\xi - \mu)vv^*)] = \left(1 - \frac{1}{2} \frac{d^2}{dt^2}\right)_{t=0} \det(M + t\tau vv^*) \det(N + t\tau vv^*).
\]

We can now derive an expression for the expected characteristic polynomial.

**Proposition 3.3.** Let \( u_1, \ldots, u_n \in \mathbb{R}^m \). Consider independent random variables \( \xi_i \) with means \( \mu_i \) and variances \( \tau_i^2 \). Let \( Q \in \mathbb{R}^{m \times m} \) be a symmetric matrix.

\[
\mathbb{E}[^\{\det \left( x^2 I - \left( Q + \sum_{i=1}^n (\xi_i - \mu_i) u_i u_i^T \right)^2 \right) \}] = \prod_{i=1}^n \left(1 - \frac{\partial^2}{2}\right)_{z_i=0} \det \left( x I - Q + \sum_{i=1}^n z_i \tau_i u_i u_i^* \right) \\
\times \det \left( x I + Q + \sum_{i=1}^n z_i \tau_i u_i u_i^* \right),
\]

and this is a real rooted polynomial in \( x \).

**Proof.** For each \( i \), let \( \beta_i \) denote the maximum value of \( |\xi_i| \) among outcomes in the (finite) support of \( \xi_i \). We begin by restricting the domain of our polynomials to \( x > \|Q\| + 2 \sum_{i=1}^n \beta_i \|u_i u_i^*\| \). We then proceed by induction. Our induction hypothesis will be that for \( 0 \leq k \leq n \),

\[
\mathbb{E}[^\{\det \left( x^2 I - \left( Q + \sum_{i=1}^n (\xi_i - \mu_i) u_i u_i^* \right)^2 \right) \}] \\
= \mathbb{E}[^\{\prod_{i=1}^n \left(1 - \frac{\partial^2}{2}\right)_{z_i=0} \det \left( x I - Q + \sum_{i=k+1}^n (\xi_i - \mu_i) u_i u_i^* + \sum_{j=1}^k z_j \tau_j u_j u_j^* \right) \\
\times \det \left( x I + Q + \sum_{i=k+1}^n (\xi_i - \mu_i) u_i u_i^* + \sum_{j=1}^k z_j \tau_j u_j u_j^* \right) \}]
\]

(6)
The base case, $k = 0$ is trivially true as we get the same formula on both sides after recalling that for any matrix $Y$
\[
\det(x^2I - Y^2) = \det(xI - Y)\det(xI + Y).
\]

By our assumption that $x > \|Q\| + 2\sum_{i=1}^n \beta_i\|u_iu_i^*\|$, we have
\[
xI - \sum_{i=k+2}^n (\xi_i - \mu_i)u_iu_i^* + \sum_{j=1}^k z_j\tau_ju_ju_j^*
\]
is PSD for any realization of $\xi_{k+2}, \ldots, \xi_n$ and in a neighborhood of zero for each $z_j$. Applying Corollary 3.2 to the right hand side of (6), yields
\[
\mathbb{E}\left[\det\left(x^2I - \left(Q + \sum_{i=1}^n (\xi_i - \mu_i)u_iu_i^*)^2\right)\right]\right)
\]
\[
= \mathbb{E}\left[k+1\prod_{i=1}^n \left(1 - \frac{\partial^2}{2}\right)\right]_{\xi_i=0} \det(xI - Q - \sum_{i=k+2}^n (\xi_i - \mu_i)u_iu_i^* + \sum_{j=1}^k z_j\tau_ju_ju_j^*)
\]
\[
\times \det(xI + Q + \sum_{i=k+2}^n (\xi_i - \mu_i)u_iu_i^* + \sum_{j=1}^k z_j\tau_ju_ju_j^*)
\]
which completes the induction. To extend the proof to all $x$, we remark that we have shown the equivalence of two polynomials in the interval $x > \|Q\| + 2\sum_{i=1}^n \beta_i\|u_iu_i^*\|$, and two polynomials that agree on an interval are identical.

Real-rootedness of the right hand side follows by Lemma 2.5, Lemma 2.4, and that by Proposition 2.6 restriction to $z = 0$ preserves real-stability, and a univariate real stable polynomial is real rooted. \hfill \square

## 4 Defining the Interlacing Family

The next proposition establishes that it suffices to bound the largest root of $q_0$.

**Proposition 4.1.** There exists a choice of outcomes $\varepsilon_1, \ldots, \varepsilon_n$ in the finite support of $\xi_1, \ldots, \xi_n$, s.t.
\[
\left\|\sum_{i=1}^n \varepsilon_iu_iu_i^* - \sum_{i=1}^n \mu_iu_iu_i^*\right\|
\]
is less than the largest root of
\[
\mathbb{E}_{\xi_1, \ldots, \xi_n} \left[\det\left(x^2I - \left(\sum_{i=1}^n (\xi_i - \mu_i)u_iu_i^*)^2\right)\right]\right).
\]
where $\mathbb{E}[\xi_i] = \mu_i, \forall i \in [n]$.

**Proof.** For a vector of independent random variables $(\xi_1, \ldots, \xi_n)$ with finite support, let $p_{i,x}$ be the probability that $\xi_i = x$. For $s = (\varepsilon_1, \ldots, \varepsilon_n)$ in the support of $\xi_1, \ldots, \xi_n$, we define
\[
q_s(x) := \prod_{i=1}^n p_{i,\varepsilon_i} \det\left(x^2I - \left(\sum_{i=1}^n (\varepsilon_i - \mu_i)u_iu_i^*)^2\right)\right).
\]
Let $t$ be a vector of $k$ outcomes in the support of $\xi_1, \ldots, \xi_k$, i.e. a partial assignment of assignment of outcomes. Then we consider the conditional expected polynomial

$$q_t(x) := \left( \prod_{i=1}^{k} p_{t_i} t_i \right)_{\xi_{k+1}, \ldots, \xi_n} \mathbb{E} \left[ \det \left( x^2 I - \left( \sum_{i=1}^{k} (t_i - \mu_i) u_i u_i^* + \sum_{j=k+1}^{n} (\xi_j - \mu_j) u_j u_j^* \right)^2 \right) \right]$$

which coincides with (3). We show that $q_s$ is an interlacing family. Let $r_{k+1}^{(1)} \ldots r_{k+1}^{(l)}$ be the outcomes in the support of $\xi_{k+1}$. For a given $t$ and $r \in \left\{ r_{k+1}^{(1)}, \ldots, r_{k+1}^{(l)} \right\}$, let $(t, r)$ denote the vector $(t_1, \ldots, t_k, r)$. By Lemma 2.11, it suffices to show that for any choice of non-negative numbers $\alpha_1, \ldots, \alpha_l$ such that $\sum_j \alpha_j = 1$, the polynomial

$$\sum_j \alpha_j q_{t, r_{k+1}^{(j)}}(x)$$

is real rooted. We can consider these $\alpha$’s as a probability distribution and define $p_{k+1, r_{k+1}^{(j)}} = \alpha_j$. Therefore,

$$q_t(x) = \left( \prod_{i=1}^{k} p_{t_i} t_i \right)_{\xi_{k+1}, \ldots, \xi_n} \mathbb{E} \left[ \det \left( x^2 I - \left( \sum_{i=1}^{k} (t_i - \mu_i) u_i u_i^* + \sum_{j=k+1}^{n} (\xi_j - \mu_j) u_j u_j^* \right)^2 \right) \right]$$

$$= \sum_j \left( \prod_{i=1}^{k} p_{t_i} t_i \right)_{p_{k+1, r_{k+1}^{(j)}}} \mathbb{E}_{\xi_{k+2}, \ldots, \xi_n} \left[ \det \left( x^2 I - \left( \sum_{i=1}^{k} (t_i - \mu_i) u_i u_i^* - (r_{k+1}^{(j)} - \mu_{k+1} u_{k+1} u_{k+1}^* - \sum_{j=k+2}^{n} (\xi_j - \mu_j) u_j u_j^* \right)^2 \right) \right]$$

$$= \sum_j \alpha_j q_{t, r_{k+1}^{(j)}}(x).$$

By Proposition 3.3, $q_t$ is real rooted, which completes the proof that $q_s$ is an interlacing family. Finally, by Lemma 2.10, there exists a choice of $t$ so that the largest root of $q_t$ is upperbounded by $q_{\emptyset}$.

\[ \square \]

5 Largest Root of the Expected Characteristic Polynomial

We use the barrier method approach to control the largest root [MSS15b].

**Definition 5.1.** For a multivariate polynomial $p(z_1, \ldots, z_n)$, we say $z \in \mathbb{R}^n$ is above all the roots of $p$ if for all $t \in \mathbb{R}^n_+$,

$$p(z + t) > 0.$$  

We use $\text{Ab}_p$ to denote the set of points that are above all the roots of $p$.

We use the same barrier function as in [BSS12, MSS15b].

**Definition 5.2.** For a real stable polynomial $p$ and $z \in \text{Ab}_p$, the barrier function of $p$ in direction $i$ at $z$ is

$$\Phi^i_p(z) := \frac{\partial_z p(z)}{p(z)}.$$
We will also make use of the following lemma that controls the deviation of the roots after applying a second order differential operator. The lemma is a slight variation of Lemma 4.8 in [AG14].

**Lemma 5.3.** Suppose that $p$ is real stable and $z \in \text{Ab}_p$. If $\Phi_p^j(z) < \sqrt{2}$, then $z \in \text{Ab}_{(1 - 1/2^j)p}$. If additionally for $\delta > 0$,

$$\frac{1}{\delta} \Phi_p^j(z) + \frac{1}{2} \Phi_p^j(z)^2 \leq 1,$$

then and for all $i$,

$$\Phi_{(1 - 1/2^j)p}^i(z + \delta \cdot 1_j) \leq \Phi_p^i(z).$$

We provide a proof in the Appendix A for completeness.

We can now bound the largest root of the expected characteristic polynomial for subisotropic vectors.

**Proposition 5.4.** Let $\xi_1, \ldots, \xi_n$ be independent scalar random variables with finite support, with $E[\xi_i] = \mu_i$, and let $\tau_i^2 = E[(\xi_i - \mu_i)^2]$. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $\sum_{i=1}^n \tau_i^2 (v_i v_i^*)^2 \preceq I$. Then the largest root of

$$p(x) := E_{\xi_1, \ldots, \xi_n} \left[ \det \left( x^2 I - \left( \sum_{i=1}^n (\xi_i - \mu_i)v_i v_i^* \right)^2 \right) \right]$$

is at most 4.

**Proof.** Note that we must have

$$\max_{i \in [n]} \tau_i v_i^* v_i \leq 1,$$

since otherwise $\sum_{i=1}^n \tau_i^2 (v_i v_i^*)^2 \preceq I$ is false.

Let

$$Q(x, z) = \left( \det \left( x^2 I + \sum_{i=1}^n z_i \tau_i v_i v_i^* \right) \right)^2.$$

For $t > 0$, define $\delta_i = t \tau_i v_i^* v_i$. For $t < \alpha(t)$ a parameter to be chosen later, we evaluate our polynomial to find that

$$Q(\alpha, -\delta_1, \ldots, -\delta_n) = \left( \det \left( \alpha I - \sum_{i=1}^n \delta_i \tau_i v_i v_i^* \right) \right)^2$$

$$= \left( \det \left( \alpha I - t \sum_{i=1}^n \tau_i^2 (v_i v_i^*)^2 \right) \right)^2$$

$$= \left( \det \left( \alpha I - t \sum_{i=1}^n \tau_i^2 (v_i v_i^*) v_i v_i^* \right) \right)^2$$

$$\geq \left( \det \left( (\alpha - t)I \right) \right)^2$$

$$> 0,$$

where the second last step follows by $\sum_{i=1}^n \tau_i^2 (v_i v_i^*) v_i v_i^* \preceq I$. 

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This implies that \((\alpha, -\delta) \in \mathbb{R}^{n+1}\) is above the roots of \(Q(x, z)\). We can upper bound \(\Phi_Q^i(\alpha, -\delta)\) via Fact 2.2 as follows

\[
\Phi_Q^i(\alpha, -\delta) = \frac{\partial_z Q}{Q} \bigg|_{x=\alpha, z=-\delta} = 2 \det \left( xI + \sum_{i=1}^n z_i \tau_i v_i v_i^r \right) \partial_z, \det \left( xI + \sum_{i=1}^n z_i \tau_i v_i v_i^r \right)^2 \bigg|_{x=\alpha, z=-\delta} = 2 \Tr \left( \left( xI + \sum_{i=1}^n z_i \tau_i v_i v_i^r \right)^{-1} \tau_i v_i v_i^r \right) = 2 \Tr \left( (\alpha I - t \sum_{i=1}^n \tau_i^2 (v_i v_i^*)^2)^{-1} \tau_i v_i v_i^r \right) \leq 2 \Tr \left( (\alpha - t)^{-1} \tau_i v_i v_i^r \right) = \frac{2 \tau_i v_i^* v_i}{\alpha - t},
\]

where the fifth step follows by \(\sum_{i=1}^n \tau_i^2 (v_i v_i^*) v_i v_i^* \preceq I\).

Choosing \(\alpha = 2t\) and \(t = 2\), and recalling Condition (7) we get

\[
\Phi_Q^i(\alpha, -\delta) \leq \frac{2 \tau_i v_i^* v_i}{\alpha - t} \leq 1 < \sqrt{2}.
\]

By Lemma 2.4, \((1 - \frac{1}{2} \partial^2_{z_i})Q\) is real stable. By Lemma 5.3, and \((4, z) \in \Ab_{(1 - \frac{1}{2} \partial^2_{z_i})Q}\) and hence \((4, z + \delta_1^i) \in \Ab_{(1 - \frac{1}{2} \partial^2_{z_i})Q}\). Also

\[
\frac{1}{\delta_i} \Phi_Q^i(\alpha, -\delta) + \frac{1}{2} \Phi_Q^i(\alpha, -\delta)^2 \leq \frac{1}{t \tau_i v_i^* v_i} \frac{2 \tau_i v_i^* v_i}{t} + \frac{1}{2} \left( \frac{2 \tau_i v_i^* v_i}{t} \right)^2 \leq \frac{4}{t^2} = 1.
\]

Therefore, by Lemma 5.3, for all \(j\)

\[
\Phi_{(1 - \frac{1}{2} \partial^2_{z_i})Q}(4, z + \delta_1^i) \leq \Phi_Q^j(4, z).
\]

Repeating this argument for each \(i \in [n]\) demonstrates that \((4, 0, \ldots, 0)\) lies above the roots of

\[
\prod_{i=1}^n \left( 1 - \frac{\partial^2_{z_i}}{2} \right) \left( \det \left( xI + \sum_{i=1}^n z_i \tau_i u_i u_i^* \right) \right) \left( \det \left( xI + \sum_{i=1}^n z_i \tau_i u_i u_i^* \right) \right).
\]

After restricting to \(z_i = 0\) for all \(i\), we then conclude by Proposition 3.3 is equivalent to a bound on the largest root of the expected characteristic polynomial. \(\square\)

Having developed the necessary machinery, we now prove our main theorem.
Proof of Theorem 1.4. Define $v_i = \frac{u_i}{\sqrt{\sigma}}$. Then, $\| \sum_{i=1}^{n} \tau_i^2 (v_i v_i^*)^2 \| = 1$. Applying Proposition 5.4 and Proposition 4.1, we conclude that there exists a choice of outcomes $\varepsilon_i$ such that

$$\left\| \sum_{i=1}^{n} (\varepsilon_i - \mu_i) v_i v_i^* \right\| \leq 4.$$

From this, we conclude that

$$\left\| \sum_{i=1}^{n} (\varepsilon_i - \mu_i) u_i u_i^* \right\| \leq 4\sigma.$$  

□
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A Omitted Proofs

A.1 Proof of Lemma 5.3

Choosing \( c = 1/2 \) in Lemma A.1 gives a proof of Lemma 5.3.

**Lemma A.1** (Generalization of Lemma 4.8 in [AG14]). Suppose that \( p(z_1, \cdots, z_m) \) is real stable and \( z \in \text{Ab}_p \). For any \( c \in [0, 1] \).

If \( \Phi_p^i(z) < \sqrt{1/c} \), then \( z \in \text{Ab}_{1-c\partial_j^2 p} \). If additionally for \( \delta > 0 \),

\[
c \cdot \left( \frac{2}{\delta} \Phi_p^i(z) + (\Phi_p^i(z))^2 \right) \leq 1,
\]

then, for all \( i \in [m] \),

\[
\Phi_{1-c\partial_j^2 p}^i(z + \delta \mathbf{1}_j) \leq \Phi_p^i(z).
\]

**Proof.** We write \( \partial_i \) instead of \( \partial_{z_i} \) for ease of notation. By Lemma 2.4, \( (1-c\partial_j^2)p \) is real stable. Recall the definitions of \( \Phi_p^i(z) \) and \( \Psi_p^i(z) \)

\[
\Phi_p^i(z) = \frac{\partial_i \det(M)}{\det(M)} = \frac{\partial_i p}{p} \quad \text{and} \quad \Psi_p^i(z) = \frac{\partial_j^2 \det(M)}{\det(M)} = \frac{\partial_j^2 p}{p}.
\]

Consider a non-negative vector \( t \). By Lemmas 4.6 and 4.5 in [AG14], we have

\[
\Phi_p^i(z + t) \leq \Phi_p^i(z), \quad \Psi_p^i(z + t) \leq \Phi_p^i(z + t)^2 \leq \Phi_p^i(z)^2. \tag{8}
\]

Thus, \( \Psi_p^i(z + t) \leq \Phi_p^i(z)^2 < 1/c \) so \( c \cdot \partial_j^2 p(z + t) < p(z + t) \), i.e. \( (1-c\partial_j^2)p(z + t) > 0 \). Thus \( z \in \text{Ab}_{1-c\partial_j^2 p} \).

Next, we write \( \Phi_{1-c\partial_j^2 p}^i \) in terms of \( \Phi_p^i \) and \( \Psi_p^i \) and \( \partial_j \Psi_p^i \).

\[
\Phi_{1-c\partial_j^2 p}^i = \frac{\partial_i (p - c \cdot \partial_j^2 p)}{p - c \cdot \partial_j^2 p} = \frac{\partial_i ((1 - c \cdot \Psi_p^i)p)}{(1 - c \cdot \Psi_p^i)p} = \frac{(1 - c \cdot \Psi_p^i)(\partial_i p)}{(1 - c \cdot \Psi_p^i)p} + \frac{(\partial_i (1 - c \cdot \Psi_p^i))p}{(1 - c \cdot \Psi_p^i)p} = \Phi_p^i - \frac{c \cdot \partial_i \Psi_p^i}{1 - c \cdot \Psi_p^i}.
\]

We would like to show that \( \Phi_{1-c\partial_j^2 p}^i(z + \delta \mathbf{1}_j) \leq \Phi_p^i(z) \). Equivalently, it is enough to show that

\[
-\frac{c \cdot \partial_i \Psi_p^i(z + \delta \mathbf{1}_j)}{1 - c \cdot \Psi_p^i(z + \delta \mathbf{1}_j)} \leq \Phi_p^i(z) - \Phi_p^i(z + \delta \mathbf{1}_j).
\]

By convexity, it is enough to show that

\[
-\frac{c \cdot \partial_i \Psi_p^i(z + \delta \mathbf{1}_j)}{1 - c \cdot \Psi_p^i(z + \delta \mathbf{1}_j)} \leq \delta \cdot (-\partial_j \Phi_p^i(z + \delta \mathbf{1}_j)).
\]
By monotonicity, $\delta \cdot (-\partial_j \Phi_p^j(z + \delta 1_j)) > 0$ so we may divide both sides of the above inequality by this term and obtain that the above is equivalent to

\[
\frac{-c \cdot \partial_j \Psi_p^j(z + \delta 1_j)}{-\delta \cdot \partial_j \Phi_p^j(z + \delta 1_j)} \cdot \frac{1}{1 - c \Psi_p^j(z + \delta 1_j)} \leq 1,
\]

where we also used $\partial_j \Phi_p^j = \partial_i \Phi_p^i$. By Lemma 4.10 in [AG14], $\frac{\partial_j \Psi_p^j}{\partial \Phi_p^i} \leq 2\Phi_p^i$. So, we can write,

\[
\frac{2c}{\delta} \Phi_p^j(z + \delta 1_j) \cdot \frac{1}{1 - c \cdot \Psi_p^j(z + \delta 1_j)} \leq 1.
\]

By Equation (8), with $t = \delta 1_j$,

\[
\Phi_p^j(z + \delta 1_j) \leq \Phi_p^j(z), \quad \Psi_p^j(z + \delta 1_j) \leq \Phi_p^j(z + \delta 1_j)^2 \leq \Phi_p^j(z)^2.
\]

So, it is enough to show that

\[
\frac{2c}{\delta} \Phi_p^j(z) \cdot \frac{1}{1 - c \cdot \Phi_p^j(z)^2} \leq 1.
\]

Using $\Phi_p^j(z) < 1$ and $c \in [0, 1]$ we know that $1 - c \cdot \Phi_p^j(z)^2 < 1$. We can multiply both sides with $1 - c \cdot \Phi_p^j(z)^2$ and we obtain that it suffices to have

\[
c \cdot \left(\frac{2}{\delta} \Phi_p^j(z) + \Phi_p^j(z)^2\right) \leq 1,
\]

which is true by assumption. 

\[\square\]