Weighted Efficient Domination for \((P_5 + kP_2)\)-Free Graphs in Polynomial Time

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Abstract

Let \( G \) be a finite undirected graph. A vertex dominates itself and all its neighbors in \( G \). A vertex subset \( D \subseteq V \) is an efficient dominating set (e.d. for short) of \( G \) if every vertex of \( G \) is dominated by exactly one vertex of \( D \). The Efficient Domination (ED) problem, which asks for the existence of an e.d. in \( G \), is known to be \( \mathbb{NP} \)-complete even for very restricted graph classes such as for claw-free graphs, for chordal graphs and for \( 2P_3 \)-free graphs (and thus, for \( P_7 \)-free graphs). We call a graph \( F \) a linear forest if \( F \) is cycle- and claw-free, i.e., its components are paths. Thus, the ED problem remains \( \mathbb{NP} \)-complete for \( F \)-free graphs, whenever \( F \) is not a linear forest. Let WED denote the vertex-weighted version of the ED problem asking for an e.d. of minimum weight if one exists.

In this paper, we show that WED is solvable in polynomial time for \((P_5 + kP_2)\)-free graphs for every fixed \( k \), which solves an open problem, and, using modular decomposition, we improve known time bounds for WED on \((P_4 + P_2)\)-free graphs, \((P_6, S_{1,2,2})\)-free graphs, and on \((2P_3, S_{1,2,2})\)-free graphs and simplify proofs. For \( F \)-free graphs, the only remaining open case is WED on \( P_6 \)-free graphs.

Keywords: Weighted efficient domination; \( F \)-free graphs; linear forests; \( P_k \)-free graphs; polynomial time algorithm; robust algorithm.

1 Introduction

Let \( G = (V,E) \) be a finite undirected graph. A vertex \( v \in V \) dominates itself and its neighbors. A vertex subset \( D \subseteq V \) is an efficient dominating set (e.d. for short) of \( G \) if every vertex of \( G \) is dominated by exactly one vertex in \( D \). Note that not every graph has an e.d.; the Efficient Dominating Set (ED) problem asks for the existence of an e.d. in a given graph \( G \). If a vertex weight function \( \omega : V \rightarrow \mathbb{N} \) is given, the Weighted Efficient Dominating Set (WED) problem asks for a minimum weight e.d. in \( G \) if there is one \( G \) or for determining that \( G \) has no e.d.

For a set \( \mathcal{F} \) of graphs, a graph \( G \) is called \( \mathcal{F} \)-free if \( G \) contains no induced subgraph isomorphic to a member of \( \mathcal{F} \). For two graphs \( F \) and \( G \), we say that \( G \) is \( F \)-free if \( G \) is \{\( F \)\}-free. We denote by \( G + H \) the disjoint union of graphs \( G \) and \( H \). Let \( P_k \) denote a chordless path with \( k \) vertices, and let \( 2P_k \) denote \( P_k + P_k \), and correspondingly for \( kP_2 \). The claw is the 4-vertex tree with three vertices of degree 1.

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Many papers have studied the complexity of ED on special graph classes - see e.g. [2] for references. In particular, ED remains \( \text{NP}\)-complete for \( 2P_3\)-free graphs, for chordal graphs, for line graphs and thus for claw-free graphs. A linear forest is a graph whose components are paths; equivalently, it is a graph which is cycle-free and claw-free. The \( \text{NP}\)-completeness of ED on chordal graphs and on claw-free graphs implies: If \( F \) is not a linear forest, then ED is \( \text{NP}\)-complete on \( F\)-free graphs. This motivates the analysis of ED/WED on \( F\)-free graphs for linear forests \( F \).

In this paper, we show that WED is solvable in polynomial time for \( (P_5 + kP_2)\)-free graphs for every fixed \( k \), which solves an open problem, and, using modular decomposition, we improve known time bounds for WED on \( (P_4 + P_2)\)-free graphs, \( (P_6, S_{1,2,2})\)-free graphs, and on \( (2P_3, S_{1,2,2})\)-free graphs and simplify proofs (see [2] [4] for known results). For \( F\)-free graphs, the only remaining open case is WED on \( P_6\)-free graphs.

Various of our algorithms are robust in the sense of [7], that is, a robust algorithm for a graph \( C \) works on every input graph \( G \) and either solves the problem correctly or states that \( G \notin C \). We say that the algorithm is weakly robust if it either gives the optimal WED solution for the input graph \( G \) or states that \( G \) has no e.d. or is not in the class.

2 Basic Notions and Results

2.1 Some Basic Notions

All graphs considered in this paper are finite, undirected and simple (i.e., without loops and multiple edges). For a graph \( G \), let \( V(G) \) or simply \( V \) denote its vertex set and \( E(G) \) or simply \( E \) its edge set; throughout this paper, let \( |V| = n \) and \( |E| = m \). We can assume that \( G \) is connected (otherwise, WED can be solved separately for its components); thus, \( m \geq n - 1 \). A graph is nontrivial if it has at least two vertices. For a vertex \( v \in V \), \( N(v) = \{ u \in V \mid uv \in E \} \) denotes its (open) neighborhood, and \( N[v] = \{ v \} \cup N(v) \) denotes its closed neighborhood. A vertex \( v \) sees the vertices in \( N(v) \) and misses all the others. The anti-neighborhood of vertex \( v \) is \( A(v) = V \setminus N[v] \).

For a vertex set \( U \subseteq V \), its neighborhood is \( N(U) = \{ x \mid x \notin U, \exists y \in U, xy \in E \} \), and its anti-neighborhood \( A(U) \) is the set of all vertices not in \( U \) missing \( U \).

The degree of a vertex \( x \) in a graph \( G \) is \( d(x) := |N(x)| \). Let \( \delta(G) \) denote the minimum degree of any vertex in \( G \).

A vertex \( u \) is universal for \( G = (V, E) \) if \( N[u] = V \). Independent sets, complement graph, and connected components are defined as usual. Unless stated otherwise, \( n \) and \( m \) will denote the number of vertices and edges, respectively, of the input graph.

2.2 A General Approach for the WED Problem

For a graph \( G = (V, E) \) and a vertex \( v \in V \), the distance levels with respect to \( v \) are

\[
N_i(v) = \{ w \in V \mid \text{dist}(v, w) = i \}
\]

for all \( i \in N \). If \( v \) is fixed, we denote \( N_i(v) \) by \( N_i \). Let \( R := V \setminus (\{ v \} \cup N_1 \cup N_2) \), and let \( G_v := G[N_2 \cup R] \) where vertices in \( N_2 \) get weight \( \infty \). Obviously, we have: \( G \) has a finite weight e.d. \( D_v \) with \( v \in D_v \) if and only if \( G_v \) has a finite weight e.d. \( D \), and \( D_v = \{ v \} \cup D \).

In some cases, for every vertex \( v \in V \), the WED problem can be efficiently solved on \( G_v \), say in time \( t(m) \) with \( t(m) \geq m \).
If graph $G = (V, E)$ has an e.d. $D$ then for any vertex $v \in V$, either $v \in D$ or one of its neighbors is in $D$. Thus, if $deg_G(v) = \delta(G)$, one has to consider the WED problem on $G_x$ for $\delta(G) + 1$ vertices $x \in N[v]$. Thus we obtain:

**Lemma 1.** If for a graph class $\mathcal{C}$ and input graph $G = (V, E)$ in $\mathcal{C}$, WED is solvable in time $t(m)$ on $G_v$ for all $v \in V$ then WED is solvable in time $O(\delta(G) \cdot t(m))$ for graph class $\mathcal{C}$.

### 2.3 Linear Forests

As already mentioned, if $F$ is a linear forest such that one of its components contain $2P_3$, or two of its components contain $P_3$, the WED problem is $\mathbb{NP}$-complete for $F$-free graphs. For $2P_2$-free graphs and more generally, for $kP_2$-free graphs, it is known that the number of maximal independent sets is polynomial \cite{1, 8} and can be enumerated efficiently \cite{8}. Since every e.d. is a maximal independent set, WED can be solved in polynomial time for $kP_2$-free graphs.

In Section 2 we show that WED is solvable in polynomial time for $(P_5 + kP_2)$-free graphs for every fixed $k$. Thus, the only remaining open case is the one of $P_6$-free graphs; our approach used for $(P_5 + kP_2)$-free graphs shows that if WED is polynomial for $P_6$-free graphs then it is polynomial for $(P_6 + kP_2)$-free graphs as well.

### 2.4 Modular Decomposition for the WED Problem

A set $H$ of at least two vertices of a graph $G$ is called *homogeneous* if $H \neq V(G)$ and every vertex outside $H$ is either adjacent to all vertices in $H$, or to no vertex in $H$. Obviously, $H$ is homogeneous in $G$ if and only if $H$ is homogeneous in the complement graph $\overline{G}$. A graph is *prime* if it contains no homogeneous set. A homogeneous set $H$ is *maximal* if no other homogeneous set properly contains $H$. It is well known that in a connected graph $G$ with connected complement $\overline{G}$, the maximal homogeneous sets are pairwise disjoint and can be determined in linear time using the so called *modular decomposition* (see, e.g., \cite{4}). The *characteristic graph* $G^*$ of $G$ is the graph obtained from $G$ by contracting each of the maximal homogeneous sets $H$ of $G$ to a single representative vertex $h \in H$, and connecting two such vertices by an edge if and only if they are adjacent in $G$. It is well known (and can be easily seen) that $G^*$ is a prime graph.

For a disconnected graph $G$, the WED problem can be solved separately for each component. If $\overline{G}$ is disconnected, then obviously, $D$ is an e.d. of $G$ if and only if $D$ is a single universal vertex of $G$. Thus, from now on, we can assume that $G$ and $\overline{G}$ are connected, and thus, maximal homogeneous sets are pairwise disjoint. Obviously, we have:

**Lemma 2.** Let $H$ be a homogeneous set in $G$ and $D$ be an e.d. of $G$. Then the following properties hold:

(i) $|D \cap H| \leq 1$.

(ii) If $H$ has no vertex which is universal for $H$ then $|D \cap H| = 0$.

Thus, the WED problem on a connected graph $G$ for which $\overline{G}$ is connected can be easily reduced to its characteristic graph $G^*$ by contracting each homogeneous set $H$ to a single representative vertex $h$ whose weight is either $\infty$ if $H$ has no universal vertex or the minimum weight of a universal vertex in $H$ otherwise. Obviously, $G$ has an e.d. $D$ of finite weight if and only if $G^*$ has a corresponding e.d. of the same weight. Thus, we obtain:
Theorem 1. Let \( \mathcal{G} \) be a class of graphs and \( \mathcal{G}^* \) the class of all prime induced subgraphs of the graphs in \( \mathcal{G} \). If the (W)ED problem can be solved for graphs in \( \mathcal{G}^* \) with \( n \) vertices and \( m \) edges in time \( O(T(n,m)) \), then the same problem can be solved for graphs in \( \mathcal{G} \) in time \( O(T(n,m) + m) \).

The modular decomposition approach leads to a linear time algorithm for WED on \( 2P_2 \)-free graphs (see [2]) and to a very simple \( O(\delta(G)m) \) time algorithm for WED on \( P_5 \)-free graphs (a simplified variant of the corresponding result in [2]); the modular decomposition approach is also described in [3].

### 3 WED in Polynomial Time for \( (P_5 + kP_2) \)-Free Graphs

In this section we solve an open problem from [2]. Let \( G \) be a \( (P_5 + P_2) \)-free graph and assume that \( G \) is not \( P_5 \)-free; otherwise, WED can be solved in time \( O(\delta(G)m) \) as described in [2]. Let \( v_1, v_2, v_3, v_4, v_5 \) induce a \( P_5 \) \( H \) in \( G \) with edges \( v_1v_2, v_2v_3, v_3v_4, v_4v_5 \), let \( X = N(H) = \{ x \mid x \notin V(H), \exists i (xv_i \in E) \} \) denote the neighborhood of \( H \) and let \( Y \) denote the anti-neighborhood \( A(H) \) of \( H \) in \( G \). Since \( G \) is \( (P_5 + P_2) \)-free, we have:

Claim 1. \( Y \) is an independent set.

Assume that \( G \) has an e.d. \( D \). Then:

Claim 2. \(|(V(H) \cup X) \cap D| \leq 5\).

Proof of Claim 2. Obviously, \(|V(H) \cap D| \leq 2\). If \(|V(H) \cap D| = 2\) then \(|X \cap D| = 0\) or \(|X \cap D| = 1\) since \( D \) is an e.d. If \(|V(H) \cap D| = 1\) then \(|X \cap D| \leq 3\). Finally, if \(|V(H) \cap D| = 0\) then \(|X \cap D| \leq 5\) which shows Claim 2.

Let \( D = D_1 \cup D_2 \) be the partition of \( D \) into \( D_1 = D \cap (V(H) \cup X) \) and \( D_2 = D \cap Y \).

Claim 3. \( D_2 = A(D_1) \cap Y \).

Proof of Claim 3. Since the anti-neighborhood \( Y \) of \( H \) is an independent set, every vertex in \( Y \) can only be dominated by itself or by a vertex from \( D \cap X \). Thus, Claim 3 holds.

This leads to the following simple algorithm for checking whether \( G \) has an e.d. \( D \):

1. Check whether \( G \) is \( P_5 \)-free; if yes, apply the corresponding algorithm for WED on \( P_5 \)-free graphs (which works in time \( O(\delta(G)m) \)), otherwise let \( H \) be a \( P_5 \) in \( G \). Determine \( X = N(H) \) and \( Y = A(H) \). If \( Y \) is not independent then \( G \) is not \( (P_5 + P_2) \)-free. Otherwise do the following:

2. For every independent set \( S \subseteq V(H) \cup X \) with \(|S| \leq 5\), check whether \( S \cup (A(S) \cap Y) \) is an e.d.

3. If there is such a set then take one of minimum weight, otherwise output “\( G \) has no e.d.”.

Obviously, the algorithm is correct and its running time is at most \( O(n^5m) \).

For every fixed \( k \), the approach for \( (P_5 + P_2) \)-free graphs can be generalized to \( (P_5 + kP_2) \)-free graphs: Assume inductively that WED can be solved in polynomial time for \( (P_5 + (k-1)P_2) \)-free graphs. Thus, if the given graph \( G \) is \( (P_5 + (k-1)P_2) \)-free, we can use the assumption,
Corollary 1. For every fixed \(k\), WED is solvable in polynomial time for \((P_5 + kP_2)\)-free graphs.

The approach can be easily generalized to \((H + kP_2)\)-free graphs whenever WED is solvable in polynomial time for \(H\)-free graphs. However, WED remains \(\mathbb{NP}\)-complete for \((H + kP_2)\)-free graphs whenever WED is \(\mathbb{NP}\)-complete for \(H\)-free graphs. If WED is solvable in polynomial time for \(P_5\)-free graphs then it is solvable in polynomial time for \((P_5 + kP_2)\)-free graphs for every fixed \(k\).

4 WED for \((P_4 + P_2)\)-Free Graphs in Time \(O(\delta(G)m)\)

In this section we slightly improve the time bound \(O(nm)\) for WED \([2]\) to \(O(\delta(G)m)\) and simplify the proof in \([2]\). According to Lemma \([1]\) for a vertex \(v \in V\) with minimal degree \(\delta(G)\), we check for all \(x \in N[v]\) whether \(G_x\) has an e.d. \(D_x\). We first collect some properties assuming that \(G\) is \((P_4 + P_2)\)-free and has an e.d. \(D_v\). As before, let \(G_v := G[N_v \cup R]\); we can assume that \(G_v\) is prime. We are looking for an e.d. of \(G_v\) with finite weight and assume that \(D_v \setminus \{v\}\) is such an e.d. Since \(G\) is \((P_4 + P_2)\)-free, we have:

Claim 4. \(G[R]\) is a cograph.

Let \(R_1, \ldots, R_\ell\) denote the connected components of \(G[R]\). Note that an e.d. of a connected cograph \(H\) has only one vertex, namely a universal vertex of \(H\). Thus:

Claim 5. For all \(i \in \{1, \ldots, \ell\}\), \(|D_v \cap R_i| = 1\), and in particular, if \(d \in D_v \cap R_i\) then \(d\) is universal for \(R_i\).

For all \(i \in \{1, \ldots, \ell\}\), let \(D_v \cap R_i = \{d_i\}\). Let \(U_i\) be the set of universal vertices in \(R_i\). Thus, if \(U_i = \emptyset\) then \(G\) has no e.d., and if \(U_i = \{d_i\}\) then necessarily \(d_i \in D_v\). From now on, assume that for every \(i \in \{1, \ldots, \ell\}\), \(|U_i| \geq 2\). We first claim that \(\ell > 1\): Since \(G_v\) is prime and in case \(\ell = 1\), \(G_v\) has an e.d. (of finite weight) if and only if \(G_v\) contains a universal vertex \(z \in R_1\) for \(G_v\), it follows:

Claim 6. For prime \(G_v\) with e.d. \(D_v\) of finite weight, \(\ell > 1\) holds.

Since for finding an e.d. in \(G_v\), every \(R_i\) can be reduced to the set \(U_i\) of its universal vertices (since the non-universal vertices in \(R_i\) cannot dominate all \(R_i\) vertices), we can assume that for all \(i \in \{1, \ldots, \ell\}, R_i\) is a clique. If \(|N_2| = 1\) then, since \(G_v\) is prime, for all \(i \in \{1, \ldots, \ell\}, |R_i| \leq 2\) and thus, \(G_v\) is a tree (in particular: If there are \(i, j \in \{1, \ldots, \ell\}, i \neq j, |R_i| = |R_j| = 1\) then \(G_v\) has no e.d., if there is exactly one \(i \in \{1, \ldots, \ell\}\) with \(|Z_i| = 1\) then this determines the \(D_v\) vertices in \(Z\) and if for all \(i \in \{1, \ldots, \ell\}, |R_i| = 2\) then one has to choose the e.d. with smallest weight in the obvious way). From now on, let \(|N_2| \geq 2\). If for all \(z \in R\), either \(z\) has a join or a co-join to \(N_2\) then \(N_2\) would be homogeneous in \(G_v\) - contradiction. Thus, from now on we have:

Claim 7. There is a vertex \(z \in R\) having a neighbor and a non-neighbor in \(N_2\).
Since $G$ is $(P_4 + P_2)$-free, we have:

**Claim 8.** If $x \in N_2$ has a neighbor in $R_i$ then for all $j \neq i$, it has at most one non-neighbor in $R_j$.

In particular, this means:

**Claim 9.** If $x \in N_2$ is adjacent to $d_i \in R_i \cap D_v$ then for all $j \neq i$, $x$ has exactly one non-neighbor in $Z_j$ which is the $D_v$-vertex in $R_j$.

**Claim 10.** If a vertex $z \in R_i$ has a non-neighbor $x \in N_2$ and $z \notin D_v$ then for all $j \neq i$, $x$ has exactly one non-neighbor in $R_j$, namely $x d_j \notin E$ for $d_j \in R_j \cap D_v$.

**Proof of Claim 10.** Let $z \in R_i$ have non-neighbor $x \in N_2$, and $z \notin D_v$, i.e., $z \neq d_1$. Then, since $G$ is $(P_4 + P_2)$-free, $x d_1 \in E$. By Claim 9, $x$ has exactly one non-neighbor in $R_j$ for each $j \in \{2, \ldots, \ell\}$ (which is the corresponding $D_v$ vertex in $R_j$). □

**Algorithm $(P_4 + P_2)$-Free-WED-$G_v$:**

**Given:** Graph $G = (V, E)$ and prime graph $G_v = G[N_2 \cup R]$ as constructed above with vertex weights $w(x)$; for all $x \in N_2$, $w(x) = \infty$.

**Output:** An e.d. $D_v$ of $G_v$ of finite minimum weight, if $G_v$ has an e.d., or the statement that $G$ is not $(P_4 + P_2)$-free or $G_v$ does not have any e.d. of finite weight.

(0) Initially, $D_v := \emptyset$.

(1) Check if $G[R]$ is a cograph. If not then $G$ is not $(P_4 + P_2)$-free - STOP. Else determine the connected components $R_1, \ldots, R_{\ell}$ of $G[R]$. If $\ell = 1$ then $G_v$ has no e.d. of finite weight - STOP.

(2) For all $i \in \{1, \ldots, \ell\}$, determine the set $U_i$ of universal vertices in $R_i$. If for some $i$, $U_i = \emptyset$ then $G_v$ has no e.d. - STOP. From now on, let $R_i := U_i$. If $U_i = \{d_i\}$ then $D_v := D_v \cup \{d_i\}$.

(3) If $|N_2| = 1$ then check whether $G_v$ is a tree and solve the problem in the obvious way. If $|N_2| > 1$, choose a vertex $z \in R$ with a neighbor $w \in N_2$ and a non-neighbor $x \in N_2$, say $z \in R_i$.

(3.1) Check if $z \in D_v$ leads to an e.d. (by using neighbor $w$ of $z$ and Claim 9).

(3.2) Check if $z \notin D_v$ leads to an e.d. (by using the non-neighbors of $x$ in $R_j$, $j \neq i$ and Claim 10).

(3.3) If there is no e.d. in both cases (3.1) and (3.2) then either $G$ is not $(P_4 + P_2)$-free or has no e.d. of finite weight - STOP.

**Theorem 2.** Algorithm $(P_4 + P_2)$-Free-WED-$G_v$ is correct and runs in time $O(m)$.

**Proof.** Correctness. The correctness follows from Claims 4 - 10

Time bound. The linear time bound is obvious. □

**Corollary 2.** WED is solvable in time $O(\delta(G)m)$ for $(P_4 + P_2)$-free graphs.
5 WED for Some Subclasses of $P_6$-Free Graphs

Recall that the complexity of WED for $P_6$-free graphs is open. In this section we consider WED for some subclasses of $P_6$-free graphs. Let $G = (V, E)$ be a prime $P_6$-free graph, let $v \in V$ and let $N_1, N_2, \ldots$ be the distance levels of $v$. Then we have:

$$N_k = \emptyset \ \text{for all} \ k \geq 5 \ \text{and} \ N_4 \ \text{is an independent vertex set.} \quad (1)$$

Assume that $G$ admits an e.d. $D_v$ of finite weight with $v \in D_v$. Let $G_v := G[N_2 \cup N_3 \cup N_4]$; we can assume that $G_v$ is prime. As before, $D_v \cap (N_1 \cup N_2) = \emptyset$; set $w(x) = \infty$ for $x \in N_2$.

Thus, vertices of $N_2$ have to be dominated by vertices of $D_v \cap N_3$. We claim:

At most one vertex in $D_v \cap N_3$ has neighbors in $N_4$. \quad (2)

**Proof.** Assume that there are two vertices $d_1, d_2 \in N_3 \cap D_v$ with neighbors in $N_4$, say $x_i \in N_4$ with $d_i x_i \in E$ for $i = 1, 2$. Let $b_i \in N_2$ with $b_i d_i \in E$ for $i = 1, 2$. Since $D_v$ is an e.d., $b_1 \neq b_2$ and $x_1 \neq x_2$ and $d_1$ misses $b_2, x_2$ while $d_2$ misses $b_1, x_1$. Since $N_4$ is independent, $x_1 x_2 \notin E$ holds. Now, if $b_1 b_2 \in E$, then $x_1, d_1, b_1, b_2, d_2, x_2$ induce a $P_6$ in $G$, and if $b_1 b_2 \notin E$, there is a $P_6$ as well (together with $N_1$ vertices), a contradiction. \hfill $\square$

### 5.1 WED for $(P_6, S_{1,2,2})$-free graphs in time $O(\delta(G)m)$

In this subsection we improve the time bound $O(n^2 m)$ for WED [2] to $O(\delta(G)m)$ and simplify the proof in [2]. Let $G = (V, E)$ be a connected $(P_6, S_{1,2,2})$-free graph, let $v \in V$ and let $N_1, N_2, \ldots$ be the distance levels of $v$. We claim:

$$D_v \cap N_4 = \emptyset. \quad (3)$$

**Proof.** Assume to the contrary that there is a vertex $d \in D_v \cap N_4$. Let $c \in N_3$ be a neighbor of $d$, let $b \in N_2$ be a neighbor of $c$ and let $a \in N_1$ be a neighbor of $b$. Then $b$ has to be dominated by a $D_v$-vertex $d' \in N_3$, and since $D_v$ is an e.d., $cd' \notin E$ and $dd' \notin E$ but now, $v, a, b, c, d, d'$ induce an $S_{1,2,2}$, a contradiction. \hfill $\square$

Thus, set $w(x) := \infty$ for all $x \in N_4$. By [3], $D_v \subseteq N_3 \cup \{v\}$. Claim [2] means that $D_v$ vertices in $N_3$ have either a join or a co-join to $N_4$. Thus for finding an e.d. of $G_v$, we can delete all vertices in $N_3$ which have a neighbor and a non-neighbor in $N_4$. Reducing $G_v$ in this way gives $G'_v$; again, we can assume that $G'_v$ is prime. Now, $N_4$ is a module and thus, $|N_4| \leq 1$. Let $N_4 = \{z\}$ if $N_4$ is nonempty. Let $Q_1, \ldots, Q_\ell$ denote the connected components of $G[N_3]$. We claim:

No component $Q_i$ in $G[N_3]$ contains two vertices of $D_v$. \quad (4)

**Proof.** Assume to the contrary that $Q_1$ contains $d_1, d_2 \in D_v$, $d_1 \neq d_2$. Let $x \in N_2$ be a neighbor of $d_1$, and let $P$ denote a path in $Q_1$ connecting $d_1$ and $d_2$, i.e., either $P = (d_1, x_1, x_2, d_2)$ or $P = (d_1, x_1, x_2, x_3, d_2)$. Let $a$ be a common neighbor of $x$ and $v$. If $P = (d_1, x_1, x_2, d_2)$ then $x$ is not adjacent to $x_2$ since $G$ is $S_{1,2,2}$-free (otherwise $v, a, x, d_1, x_2, d_2$ induce an $S_{1,2,2}$) and since $G$ is $P_6$-free, $x$ is adjacent to $x_1$ (otherwise $v, a, x, d_1, x_1, x_2$ induce a $P_6$) but now $v, a, x, x_1, x_2, d_2$ induce a $P_6$ - contradiction. If $P = (d_1, x_1, x_2, x_3, d_2)$, the arguments are similar. \hfill $\square$

Thus, by [4], if $D_v$ is an e.d. of $G_v$ then $|D_v \cap Q_i| = 1$ for all $i$, $1 \leq i \leq \ell$, and the corresponding $D_v$-vertex is universal for $Q_i$. Thus, we can restrict $Q_i$ to its universal vertices.
$U_i$ (which means that now, $Q_i$ is a clique; if $U_i = \emptyset$ then $G_v$ has no e.d.) In case $\ell = 1$ this means that if $G_v$ has an e.d., $G_v$ must have a universal vertex (since a $D_v$-vertex being universal for $Q_1$ must also be universal for $N_2 \cup N_4$) which is impossible since $G_v$ is prime. This implies $\ell > 1$. If $|Q_i| = 1$ then the corresponding vertex in $Q_i$ is a forced vertex for $D_v$ and has to be added to $D_v$. We claim:

$$N_2 \text{ vertices cannot distinguish more than one } Q_i, i \in \{1, \ldots, \ell\}. \quad (5)$$

**Proof.** Since $G$ is $S_{1,2,2}$-free, no vertex in $N_2$ can distinguish two components $Q_i, Q_j$ in $N_3$. To show (5), assume to the contrary that there are components $Q_1, Q_2$ in $G_v$ with $c_1, d_1 \in Q_1$ and $c_2, d_2 \in Q_2$ which are distinguished by vertices $x_1, x_2 \in N_2$ such that $x_1d_1 \in E$, $x_1c_1 \notin E$, and $x_2d_2 \in E$, $x_2c_2 \notin E$. Since no vertex in $N_2$ can distinguish two components $Q_i, Q_j$, $x_1 \neq x_2$ holds, and since $G$ is $S_{1,2,2}$-free, $x_1c_2 \notin E$ and $x_1d_2 \notin E$, and by symmetry also $x_2c_1 \notin E$ and $x_2d_1 \notin E$, but now $c_1, d_1, x_1, x_2, d_2, c_2$ induce a $P_6$ if $x_1x_2 \in E$ or a $P_6$ together with $N_1$ vertices if $x_1x_2 \notin E$ - contradiction. □

First assume $N_4 \neq \emptyset$, i.e., $N_4 = \{z\}$. For every $i \in \{1, \ldots, \ell\}$, let $Q_i^+$ denote the neighbors of $z$ in $Q_i$ and let $Q_i^-$ denote the non-neighbors of $z$ in $Q_i$. By (3), at most one $Q_i$ has more than two vertices, say $|Q_i| \leq 2$ for all $i \in \{2, \ldots, \ell\}$ since in this case, $Q_i^+$ and $Q_i^-$ are modules. Since $D_v$ is an e.d., there is a vertex $d \in D_v$ with $dz \in E$; say $d \in Q_1^+$. Let $b \in N_2$ with $bd \in E$ and $a \in N_1$ with $ab \in E$. Now for $j \neq i$, every neighbor $x \in Q_j^+$ of $z$ must see $b$ since otherwise $v, a, b, d, z, x$ induce a $P_6$, and every non-neighbor $y \in Q_j^-$ of $z$ must miss $b$ since otherwise $v, a, b, d, z, y$ induce an $S_{1,2,2}$ but if $Q_j$ contains both $x$ and $y$ then $v, a, b, d, x, y$ induce an $S_{1,2,2}$ - contradiction. Thus, we have:

At most one $Q_i$ has more than one vertex. \quad (6)

Say $|Q_i| = 1$ for all $i \in \{2, \ldots, \ell\}$. If $N_4 = \emptyset$, this holds as well.

This leads to the following algorithm for WED with time bound $O(m)$ for every $v$:

**Algorithm** ($P_6, S_{1,2,2}$)-Free-WED-$G_v$:

**Given:** Connected graph $G = (V, E)$ and prime graph $G_v = G[N_2 \cup N_3 \cup N_4]$ as constructed above with vertex weights $w(x)$; for all $x \in N_2 \cup N_4$, $w(x) = \infty$.

**Output:** An e.d. $D_v$ of $G_v$ of finite weight, if $G_v$ has such an e.d., or the statement that $G$ is not ($P_6, S_{1,2,2}$)-free or $G_v$ does not have any e.d. of finite weight.

(0) Initially, $D_v := \emptyset$.

(1) Check if $G[N_3] = \emptyset$; if not then $G$ is not $P_6$-free - STOP. Else determine the connected components $Q_1, \ldots, Q_\ell$ of $G[N_3]$. If $\ell = 1$ then $G_v$ has no e.d. of finite weight - STOP.

(2) For all $i \in \{1, \ldots, \ell\}$, determine the set $U_i$ of universal vertices in $Q_i$. If $U_i = \emptyset$ then $G_v$ has no e.d. - STOP. From now on, let $Q_i := U_i$. If $U_i = \{d_i\}$ then $D_v := D_v \cup \{d_i\}$. Delete all vertices $x \in N_3$ which have a neighbor and a non-neighbor in $N_4$. Contract $N_4$ to one vertex $z$ if $N_4 \neq \emptyset$.

(3) For all $|Q_i| = 1$, add its vertex to $D_v$ and delete its neighbors from $N_2$. If there is an $i \in \{1, \ldots, \ell\}$ with $|Q_i| > 1$, say $|Q_1| > 1$, then check whether there is a vertex $d \in Q_1$ which has exactly the remaining $N_2$ vertices as its neighborhood in $N_2$ and sees $z$ for $N_4 = \{z\}$. 


(4) Finally check whether \( D_v \) is an e.d. of \( G_v \) - if not then either \( G \) is not \( (P_6, S_{1,2,2}) \)-free or has no e.d. (containing \( v \)) of finite weight.

**Theorem 3.** Algorithm \((P_6, S_{1,2,2})\)-Free-WED-\( G_v \) is correct and runs in time \( O(m) \).

**Proof.** *Correctness.* The correctness follows from the previous claims and considerations.

*Time bound.* The linear time bound is obvious. \( \square \)

**Corollary 3.** WED is solvable in time \( O(\delta(G)m) \) for \((P_6, S_{1,2,2})\)-free graphs.

### 5.2 WED for \( P_6 \)-free graphs of diameter 3

In this subsection, we reduce the WED problem on \( P_6 \)-free graphs in polynomial time to such graphs having diameter 3. Let \( D \) be an e.d. of \( G \). By Theorem 1 we can assume that \( G \) is prime. As before, we check for every vertex \( v \in V \) if \( v \in D \) leads to an e.d. of \( G \). For this purpose, let \( N_i \), \( i \geq 1 \), again be the distance levels of \( v \). Recall that by \( 1 \), \( N_k = \emptyset \) for \( k \geq 5 \) and \( N_4 \) is an independent vertex set, and by \( 2 \), at most one vertex in \( D_v \cap N_3 \) has neighbors in \( N_4 \).

Recall that \( A(x) \) denotes the anti-neighborhood of \( x \). Thus, if \( N_4 \neq \emptyset \) then check for every vertex \( x \in N_3 \) whether \( \{v, x\} \cup (A(x) \cap N_4) \) is an e.d. in \( G \); since \( N_4 \) is independent, vertices in \( N_4 \) not dominated by \( x \) must be in \( D_v \). This can be done in polynomial time for all \( v \) with \( N_4 \neq \emptyset \).

Now we can assume that the diameter of \( G \) is at most 3, i.e., for every \( v \in V \), the distance level \( N_4 \) is empty.

**Corollary 4.** If WED is solvable in polynomial time for \( P_6 \)-free graphs of diameter 3 then WED is solvable in polynomial time for \( P_6 \)-free graphs.

### 6 WED for \((2P_3, S_{1,2,2})\)-Free Graphs in Time \( O(\delta(G)n^3) \)

In this section we improve the time bound \( O(n^5) \) for WED \( 2 \) to \( O(\delta(G)n^3) \) and simplify the proof in \( 2 \). Let \( G = (V, E) \) be a connected \((2P_3, S_{1,2,2})\)-free graph, let \( v \in V \) and let \( N_1, N_2, \ldots \) be the distance levels of \( v \). Since \( G \) is \( 2P_3 \)-free, we have \( N_k = \emptyset \) for \( k \geq 6 \). Let \( R := V \setminus \{v\} \cup N_1 \cup N_2 \). Assume that \( G \) admits an e.d. \( D_v \) of finite weight with \( v \in D_v \). Let \( G_v := G[N_2 \cup N_3 \cup N_4 \cup N_5] \), i.e. \( G_v = G[N_2 \cup R] \); we can assume that \( G_v \) is prime. Since \( D_v \) is an e.d., \( R \neq \emptyset \). Let \( Q_1, \ldots, Q_{\ell} \), \( \ell \geq 1 \), denote the connected components of \( G[R] \). Clearly, \( D_v \cap Q_i \neq \emptyset \) for every \( i \). Let \( D_v \setminus \{v\} = \{d_1, \ldots, d_k\} \), and assume that \( k \geq 2 \) (otherwise, \( G_v \) would have a universal vertex which is impossible for a prime graph). Since \( G \) is \( S_{1,2,2} \)-free and \( D_v \) is an e.d., we have:

Every \( x \in N_2 \) seeing a vertex \( d_i \in D_v \) misses \( N[d_j] \cap R, j \neq i \). \hspace{1cm} (7)

We claim:

For every \( i = 1, \ldots, k, N[d_i] \cap R \) is a clique. \hspace{1cm} (8)

**Proof.** Suppose that \( N[d_1] \cap R \) is not a clique, i.e., there are neighbors \( x, y \in R \) of \( d_1 \) with \( xy \notin E \). Let \( b \in N_2 \) be a neighbor of \( d_2 \). By \( 7 \), \( b \) misses \( x \) and \( y \) but now, \( a, b, d_2, x, d_1, y \) induce \( 2P_3 \), a contradiction. \( \square \)
Next we claim:

If $k \geq 3$ then $G[N_3]$ is the disjoint union of cliques.  \hfill (9)

Proof. Suppose that $k \geq 3$ and there is an edge $uw \in E$ for $u \in N(d_2) \cap R$ and $w \in N(d_3) \cap R$. Let $x \in N_2$ with $xd_1 \in E$ and $a \in N_1$ with $ax \in E$. Then by (7), $v, a, x, d_2, u, w$ induce $2P_3$, a contradiction. \hfill \Box

Thus, for $k \geq 3$, every $Q_i$ is a clique containing exactly one $D_v$ vertex:

$$|D_v \cap Q_i| = 1.$$  \hfill (10)

If $Q_i$ is a single vertex $q_i$ then $q_i$ is forced and has to be added to $D_v$. From now on assume that for all $i$, $|Q_i| \geq 2$. By (7), we have:

If $z \in N_2$ sees $Q_i$ then it misses all $Q_j, j \neq i$. \hfill (11)

Let $S_i$ denote the set of vertices in $N_2$ distinguishing vertices in $Q_i$. Since $Q_i$ is not a module, $S_i \neq \emptyset$ for all $i \in \{1, \ldots, k\}$. Let $U_i$ denote the vertices in $Q_i$ which have a join to $S_i$; $U_i \neq \emptyset$ since $S_i$ vertices must have a $D_v$ neighbor in $Q_i$. We claim:

For all $i \in \{1, \ldots, k\}$, $|U_i| = 1$. \hfill (12)

Proof. Assume to the contrary that $|U_1| > 1$. If $x \in S_1$ then by (7), $xd_1 \in E$, i.e., $d_1 \in U_1$. Now, a vertex distinguishing $U_1$ would be in $S_1$ but vertices in $U_1$ have a join to $S_1$ and thus cannot be distinguished which is a contradiction to the assumption that $G_v$ is prime. \hfill \Box

The other case when $k \leq 2$, i.e., $|D_v \setminus \{v\}| \leq 2$, can be easily done via the adjacency matrix of $G$: For any pair $x, y \in R$, $x \neq y$, with $xy \notin E$, check whether all other vertices in $G_v$ are adjacent to exactly one of them; this can be done in time $O(n^2)$.

This leads to the following:

Algorithm $(2P_3, S_{1,2,2})$-Free-WED-$G_v$:

Given: Connected graph $G = (V, E)$ and prime graph $G_v = G[N_2 \cup R]$ as constructed above with vertex weights $w(x)$; for all $x \in N_2$, $w(x) = \infty$.

Output: An e.d. $D_v$ of $G_v$ of finite weight if $G_v$ has such an e.d., or the statement that $G$ is not $(2P_3, S_{1,2,2})$-free or $G_v$ does not have any e.d. of finite weight.

(0) Initially, $D_v := \emptyset$.

(1) Determine $N_1, N_2$ and $R$. If $R = \emptyset$ then $G_v = G[N_2 \cup R]$ has no e.d. - STOP. Else determine the connected components $Q_1, \ldots, Q_\ell$ of $R$.

(2) If $G[R]$ is not the disjoint union of cliques $Q_1, \ldots, Q_\ell$, $\ell \geq 3$, then check whether $G_v$ has a finite weight e.d. with two vertices, and determine an e.d. with minimum weight. If not, $G_v$ has no e.d. of finite weight - STOP.

(3) (Now $G[R]$ is the disjoint union of cliques $Q_1, \ldots, Q_\ell$, $\ell \geq 3$) If $Q_i = \{d_i\}$ then $d_i$ is forced - $D_v := D_v \cup \{d_i\}$. If $|Q_i| > 1$ then determine the set $S_i$ of vertices distinguishing $Q_i$, and determine the set $U_i$ of vertices in $Q_i$ having a join to $S_i$. If $U_i = \emptyset$ then $G_v$ has no e.d. - STOP. Otherwise, $U_i = \{d_i\}$ and $d_i$ is forced - $D_v := D_v \cup \{d_i\}$.
(4) Finally check whether \( D_v \) is an e.d. of finite weight of \( G_v \) - if not then either \( G \) is not \((2P_3, S_{1,2,2})\)-free or has no e.d. of finite weight.

**Theorem 4.** Algorithm \((2P_3, S_{1,2,2})\)-Free-WED-\(G_v\) is correct and runs in time \(O(n^3)\).

**Proof.** Correctness. The correctness follows from the previous claims and considerations.

**Time bound.** The time bound is obvious since step (1) can be done in time \(O(m)\), step (2) can be done in time \(O(n^3)\), and steps (3) and (4) can be done in time \(O(m)\).

**Corollary 5.** \(WED\) is solvable in time \(O(\delta(G)n^3)\) for \((2P_3, S_{1,2,2})\)-free graphs.

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