On Spherically Symmetric Motions of the Atmosphere Surrounding a Planet Governed by the Compressible Euler Equations

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Abstract

We consider spherically symmetric motions of inviscid compressible gas surrounding a solid ball under the gravity of the core. Equilibria touch the vacuum with finite radii, and the linearized equation around one of the equilibria has time-periodic solutions. To justify the linearization, we should construct true solutions for which this time-periodic solution plus the equilibrium is the first approximation. But this leads us to difficulty caused by singularities at the free boundary touching the vacuum. We solve this problem by the Nash-Moser theorem.

Key Words and Phrases. Compressible Euler equations, Spherically symmetric solutions, Vacuum boundary, Nash-Moser theorem

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1 Introduction

We consider spherically symmetric motions of atmosphere governed by the compressible Euler equations:

\[
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{2}{r} \rho u = 0, \quad \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) + \frac{\partial P}{\partial r} = -\frac{g_0 \rho}{r^2} \quad (R_0 \leq r)
\]

(1)
and the boundary value condition
\[ \rho u |_{r=R_0} = 0. \] (2)

Here \( \rho \) is the density, \( u \) the velocity, \( P \) the pressure. \( R_0 (>0) \) is the radius of the central solid ball, and \( g_0 = G_0 M_0 \), \( G_0 \) being the gravitational constant, \( M_0 \) the mass of the central ball. The self-gravity of the atmosphere is neglected.

In this study we always assume that
\[ P = A \rho^\gamma, \] (3)

where \( A \) and \( \gamma \) are positive constants, and we assume that \( 1 < \gamma \leq 2 \).

Equilibria of the problem are given by
\[ \bar{\rho}(r) = \begin{cases} A_1 \left( \frac{1}{r} - \frac{1}{R} \right)^{\frac{1}{\gamma-1}} & (R_0 \leq r < R) \\ 0 & (R \leq r) \end{cases} \]

where \( R \) is an arbitrary number such that \( R > R_0 \) and \( A_1 = \left( \frac{(\gamma-1)g_0}{\gamma A} \right)^{\frac{1}{\gamma-1}} \).

**Remark** The total mass \( M \) of the equilibrium is given by
\[ M = 4\pi A_1 \int_{R_0}^{R} \left( \frac{1}{r} - \frac{1}{R} \right)^{\frac{1}{\gamma-1}} r^2 dr. \]

\( M \) is an increasing function of \( R \). Of course \( M \to 0 \) as \( R \to R_0 \). But as \( R \to +\infty \), we see
\[ M \to 4\pi A_1 \int_{R_0}^{\infty} r^{\frac{2\gamma-3}{\gamma-1}} dr = \begin{cases} +\infty & \text{if } \gamma \geq 4/3 \\ M^*(< \infty) & \text{if } \gamma < 4/3, \end{cases} \]
where
\[ M^* = \frac{4\pi A_1(\gamma-1)}{4-3\gamma} R_0^{\frac{4-3\gamma}{4-3\gamma}}. \]

Hence if \( \gamma \geq 4/3 \) there is an equilibrium for any given total mass, but if \( \gamma < 4/3 \) the possible mass has the upper bound \( M^* \). Anyway, given the total
mass $M$, a conserved quantity, in $(0, +\infty)$ or $(0, M^*)$, then the radius $R$ or the configuration of the equilibrium is uniquely determined.

Let us fix one of these equilibria. We are interested in motions around this equilibrium.

Here let us glance at the history of researches of this problem. Of course there were a lot of works on the Cauchy problem to the compressible Euler equations. But there were gaps if we consider density distributions which contain vacuum regions.

As for local-in-time existence of smooth density with compact support, [13] treated the problem under the assumption that the initial density is non-negative and the initial value of

$$
\omega := \frac{2\sqrt{A\gamma}}{\gamma - 1} \frac{r^{\gamma-1}}{\rho^{\gamma-1}}
$$

is smooth, too. By the variables $(\omega, u)$ the equations are symmetrizable continuously including the region of vacuum. Hence the theory of quasi-linear symmetric hyperbolic systems can be applied. The discovery of the variable $\omega$ can go back to [12], [15]. However, since

$$
\omega \propto \left(\frac{1}{r} - \frac{1}{R}\right)^{\frac{1}{2}} \sim \text{Const.} (R - r)^{\frac{1}{2}} \quad \text{as } r \to R - 0
$$

for equilibria, $\omega$ is not smooth at the boundary $r = R$ with the vacuum. Hence the class of “tame” solutions considered in [13] cannot cover equilibria.

On the other hand, possibly discontinuous weak solutions with compactly supported density can be constructed. The article [14] gave local-in-time existence of bounded weak solutions under the assumption that the initial density is bounded and non-negative. The proof by the compensated compactness method is due to [4]. Of course the class of weak solutions can cover equilibria, but the concrete structures of solutions were not so clear.

Therefore we wish to construct solutions whose regularities are weaker than solutions with smooth $\omega$ and stronger than possibly discontinuous weak solutions. The present result is an answer to this wish. More concretely speaking, the solution $(\rho(t, r), u(t, r))$ constructed in this article should be continuous on $0 \leq t \leq T, R_0 \leq r < \infty$ and there should be found a continuous curve $r = R_F(t), 0 \leq t \leq T$, such that $|R_F(t) - R| \ll 1, \rho(t, r) > 0$ for
\(0 \leq t \leq T, R_0 \leq r < R_F(t)\) and \(\rho(t, r) = 0\) for \(0 \leq t \leq T, R_F(t) \leq r < \infty\).

The curve \(r = R_F(t)\) is the free boundary at which the density touches the vacuum. It will be shown that the solution satisfies

\[
\rho(t, r) = C(t)(R_F(t) - r)^{\frac{1}{\gamma - 1}}(1 + O(R_F(t) - r))
\]
as \(r \to R_F(t) - 0\). Here \(C(t)\) is positive and smooth in \(t\). This situation is “physical vacuum boundary” so-called by [7] and [3]. This concept can be traced back to [10], [11], [20]. Of course this singularity is just that of equilibria.

The major difficulty of the analysis comes from the free boundary touching the vacuum, which can move along time. So it is convenient to introduce the Lagrangian mass coordinate

\[
m = 4\pi \int_{R_0}^{r} \rho(t, r')r'^2dr',
\]
to fix the interval of independent variable to consider. Taking \(m\) as the independent variable instead of \(r\), the equations turn out to be

\[
\frac{\partial \rho}{\partial t} + 4\pi \rho^2 \frac{\partial}{\partial m}(r^2u) = 0,
\]

\[
\frac{\partial u}{\partial t} + 4\pi r^2 \frac{\partial P}{\partial m} = -\frac{g_0}{r^2} \quad (0 < m < M),
\]

where

\[
r = \left( R_0^3 + \frac{3}{4\pi} \int_0^m \frac{dm}{\rho} \right)^{1/3}.
\]

We note that

\[
\frac{\partial r}{\partial t} = u, \quad \frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}.
\]

Let us take \(\bar{r} = \bar{r}(m)\) as the independent variable instead of \(m\), where \(\bar{r}(m)\) is the inverse function \(\bar{m}^{-1}(m)\) of the function

\[
\bar{m} : r \mapsto 4\pi \int_{R_0}^{r} \bar{\rho}(r')r'^2dr'.
\]

Then, since

\[
\frac{\partial}{\partial m} = \frac{1}{4\pi \bar{r}^2 \bar{\rho} \frac{\partial}{\partial \bar{r}}}, \quad \rho = \left( 4\pi r^2 \frac{\partial r}{\partial m} \right)^{-1} = \bar{\rho} \left( \frac{r^2 \frac{\partial r}{\partial m}}{\bar{r}^2 \frac{\partial \bar{r}}{\partial m}} \right)^{-1},
\]

\[\text{[4]}\]
we have a single second-order equation

\[
\frac{\partial^2 r}{\partial t^2} + \frac{1}{\bar{\rho} r^2} \frac{\partial}{\partial r} \left( \bar{P} \left( r^2 \frac{\partial r}{\partial r} \right)^{-\gamma} \right) + \frac{g_0}{r^2} = 0.
\]

The variable \( \bar{r} \) runs on the interval \([R_0, R]\) and the boundary condition is

\[r|_{\bar{r}=R_0} = R_0.\]

Without loss of generality, we can and shall assume that

\[R_0 = 1, \quad g_0 = \frac{1}{\gamma - 1}, \quad A = \frac{1}{\gamma}, \quad A_1 = 1.\]

Keeping in mind that the equilibrium satisfies

\[\frac{1}{\bar{\rho}} \frac{\partial \bar{P}}{\partial \bar{r}} + \frac{g_0}{\bar{r}^2} = 0,\]

we have

\[
\frac{\partial^2 r}{\partial t^2} - \frac{1}{\bar{\rho} r^2} \frac{\partial}{\partial r} \left( \bar{P} \left( 1 - \left( \frac{r^2 \partial r}{\bar{r}^2} \right)^{-\gamma} \right) \right) + \frac{1}{\gamma - 1} \left( \frac{1}{\bar{r}^2} - \frac{r^2}{\bar{r}^4} \right) = 0.
\]

Introducing the unknown variable \( y \) for perturbation by

\[r = \bar{r}(1 + y),\]

we can write the equation as

\[
\frac{\partial^2 y}{\partial t^2} - \frac{1}{\rho r} \frac{(1 + y)^2}{(r^2)} \frac{\partial}{\partial r} \left( P G \left( y, r \frac{\partial y}{\partial r} \right) \right) - \frac{1}{\gamma - 1} \frac{1}{r^3} H(y) = 0,
\]

where

\[G(y, v) := \left( 1 + y \right)^{-2\gamma} \left( 1 + y + v \right)^{-\gamma} = \gamma (3y + v) + [y, v],\]

\[H(y) := \left( 1 + y \right)^2 - \frac{1}{(1 + y)^2} = 4y + [y].\]
and we have used the abbreviations $r, \rho, P$ for $\bar{r}, \bar{\rho}, \bar{P}$.

**Notational Remark** Here and hereafter $[X]_q$ denotes a convergent power series, or an analytic function given by the series, of the form $\sum_{j \geq q} a_j X^j$, and $[X, Y]_q$ stands for a convergent double power series of the form $\sum_{j+k \geq q} a_{jk} X^j Y^k$.

We are going to study the equation (5) on $1 < r < R$ with the boundary condition

$$y|_{r=1} = 0.$$  

Of course $y$ and $\frac{\partial y}{\partial r}$ will be confined to

$$|y| + \left| r \frac{\partial y}{\partial r} \right| < 1.$$  

Here let us propose the main goal of this study roughly. Let us fix an arbitrarily large positive number $T$. Then we have

**Main Goal** For sufficiently small $\varepsilon > 0$ there is a solution $y = y(t, r; \varepsilon)$ of (5) in $C^\infty([0, T] \times [1, R])$ such that

$$y(t, r; \varepsilon) = \varepsilon y_1(t, r) + O(\varepsilon^2).$$

The same estimates $O(\varepsilon^2)$ hold between the higher order derivatives of $y$ and $\varepsilon y_1$.

Here $y_1(t, r)$ is a time-periodic function specified in Section 2, which is of the form

$$y_1(t, r) = \sin(\sqrt{\lambda t} + \theta_0) \cdot \bar{\Phi}(r),$$

where $\lambda$ is a positive number, $\theta_0$ a constant, and $\bar{\Phi}(r)$ is an analytic function of $1 \leq r \leq R$.

Once the solution $y(t, r; \varepsilon)$ is given, then the corresponding motion of gas particles can be expressed by the Lagrangian coordinate as

$$r(t, m) = \bar{r}(m)(1 + y(t, \bar{r}(m); \varepsilon))$$
$$= \bar{r}(m)(1 + \varepsilon y_1(t, \bar{r}(m)) + O(\varepsilon^2)).$$

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The curve \( r = R_F(t) \) of the free vacuum boundary is given by

\[
R_F(t) = r(t, M) = R + \varepsilon R \sin(\sqrt{\lambda} t + \theta_0)\Phi(R) + O(\varepsilon^2).
\]

The free boundary \( R_F(t) \) oscillates around \( R \) with time-period \( 2\pi/\sqrt{\lambda} \) approximately.

The solution \((\rho, u)\) of the original problem \((1)(2)\) is given by

\[
\rho = \bar{\rho}(\bar{r})(1 + y(t, \bar{r}(m); \varepsilon))^{-1}, \quad u = \frac{\partial y}{\partial t}
\]

implicitly by

\[
\bar{r} = \bar{r}(m), \quad y = y(t, \bar{r}(m); \varepsilon)
\]

\[
\frac{\partial y}{\partial \bar{r}} = \partial_r y(t, \bar{r}(m); \varepsilon), \quad \frac{\partial y}{\partial t} = \partial_t y(t, \bar{r}(m); \varepsilon),
\]

where \( m = m(t, r) \) for \( 1 \leq r \leq R_F(t) \). Here \( m(t, r) \) is given as the inverse function \((f(t))^{-1}(r)\) of the function

\[
f(t) : m \mapsto r(t, m) = \bar{r}(m)(1 + y(t, \bar{r}(m); \varepsilon)).
\]

We note that

\[
R_F(t) - r(t, m) = R(1 + y(t, R)) - \bar{r}(m)(1 + y(t, \bar{r}(m))
\]

implies

\[
\frac{1}{\kappa}(R - \bar{r}) \leq R_F(t) - r \leq \kappa(R - \bar{r})
\]

with \( 0 < \kappa - 1 \ll 1 \), since \(|y| + |\partial_r y| \leq \varepsilon C\). Therefore

\[
y(t, \bar{r}(m)) = y(t, R) + O(R_F(t) - r),
\]

and so on. Hence we get the “physical vacuum boundary”. (See Remark to Theorem 1.)

We shall give a precise statement of the main result in Section 3 and give a proof of the main result in Sections 4, 5. We shall apply the Nash-Moser theory. The reason is as follows.

The equation \((5)\) looks like as if it is a second-order quasi-linear hyperbolic equation, and one might expect that the usual iteration method in a suitable
Sobolev spaces, e.g., $H^s$, or something like them, could be used. But it is not the case. Actually the linear part of the equation is essentially the d’Alembertian operator

$$\frac{\partial^2}{\partial t^2} - \triangle = \frac{\partial^2}{\partial t^2} - x \frac{\partial^2}{\partial x^2} - \frac{N}{2} \frac{\partial}{\partial x} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \xi^2} - \frac{N - 1}{\xi} \frac{\partial}{\partial \xi}$$

in the variables $x, \xi$ such that

$$\frac{R - r}{R} \sim x = \frac{\xi^2}{4},$$

and the nonlinear terms are smooth functions of $y$ and $\partial y/\partial r$. (See (13) and (15).) Here the term $\partial y/\partial r$ apparently looks like as if to be the first order derivative. If it was the case, the usual iteration in the Picard’s scheme applied to the wave equation would work, since the inverse of the d’Alembertian operator may recover the regularity up to one order of derivative, that is, roughly speaking, the d’Alembertian may pull back $C^1([0, T], L^2)$ to $C^1([0, T], H^1)$. But indeed the apparently first order derivative $\partial y/\partial r$ performs like

$$\frac{r}{R} \frac{\partial y}{\partial r} \sim \frac{\partial y}{\partial x} \propto \frac{\partial y}{\partial \xi} \sim - \frac{\partial^2 y}{\partial \xi^2}$$

near $r = R$ or $\xi = 0$, like the second order derivative. So, since the inverse of the d’Alembertian recovers the regularity up to only one order of derivative, the usual iteration for non-linear wave equations may cause troubles of the loss of regularities, which occur at the free vacuum boundary $r = R$. This is the reason why we like to apply the Nash-Moser theory to our problem.

2 Analysis of the linearized problem

The linearization of the equation (5) is clearly

$$\frac{\partial^2 y}{\partial t^2} + Ly = 0,$$

where

$$Ly := - \frac{1}{\rho r} \frac{d}{dr} \left( P \gamma \left( 3y + r \frac{dy}{dr} \right) \right) - \frac{1}{\gamma - 1} \frac{1}{r^3} (4y)$$

$$= - \left( \frac{1}{r} - \frac{1}{R} \right) \frac{d^2 y}{dr^2} + \left( - \frac{4}{\gamma - 1} \frac{1}{r} - \frac{\gamma - 1}{r^2} \right) \frac{dy}{dr} + \frac{3\gamma - 4}{\gamma - 1} \frac{y}{r^3}. \quad (7)$$
In order to analyze the eigenvalue problem $\mathcal{L}y = \lambda y$, we introduce the independent variable $z$ by

$$z = \frac{R - r}{R}$$

and the parameter $N$ by

$$\frac{\gamma}{\gamma - 1} = \frac{N}{2} \quad \text{or} \quad \gamma = 1 + \frac{2}{N - 2}. \quad (9)$$

Then we can write

$$R^3 \mathcal{L}y = -z \frac{d^2 y}{dz^2} - \left( \frac{N}{2} - 4z \right) \frac{dy}{dz} + \frac{8 - N}{2} \frac{y}{(1 - z)^2}.$$  \quad (10)

The variable $z$ runs over the interval $[0, 1 - 1/R]$, the boundary $z = 0$ corresponds to the free boundary touching the vacuum, and the boundary condition at $z = 1 - 1/R$ is the Dirichlet condition $y = 0$.

Although the boundary $z = 1 - 1/R$ is regular, the boundary $z = 0$ is singular. In order to analyze the singularity, we transform the equation $\mathcal{L}y = \lambda y$, which can be written as

$$-z \frac{d^2 y}{dz^2} - \left( \frac{N}{2} \frac{1}{1 - z} - \frac{4}{1 - z} \right) \frac{dy}{dz} + \frac{8 - N}{2} \frac{y}{(1 - z)^2} = \lambda R^3 (1 - z) y,$$

to an equation of the formally self-adjoint form

$$-\frac{d}{dz} p(z) \frac{dy}{dz} + q(z) y = \lambda R^3 \mu(z) y.$$  

This can be done by putting

$$p = z^{N/2} (1 - z)^{\frac{N - 2}{2}},$$

$$q = \frac{8 - N}{2} z^{\frac{N - 2}{2}} (1 - z)^{\frac{N - 2}{2}},$$

$$\mu = z^{\frac{N - 2}{2}} (1 - z)^{\frac{10 - N}{2}}.$$  

Using the Liouville transformation, we convert the equation

$$-\frac{d}{dz} p(z) \frac{dy}{dz} + q(z) y = \lambda R^3 \mu(z) y + f$$
to the standard form
\[-d^2\eta\over d\xi^2 + Q\eta = \lambda R^3\eta + \hat{f}.\]

This can be done by putting
\[
\xi = \int_0^z \sqrt{\frac{\mu}{p}} dz = \int_0^z \sqrt{\frac{1 - \zeta}{\zeta}} d\zeta = z(1 - z) + \tan^{-1}\sqrt{\frac{z}{1 - z}}, \tag{11}
\]
\[
\eta = (\mu p)^{1/4} y = z^{\frac{2}{N + 1}} (1 - z)^{\frac{2N - 1}{N + 1}} y,
\]
\[
\hat{f} = p^{1/4} \mu^{-3/4} f = z^{\frac{2N}{N + 1}} (1 - z)^{\frac{2N - 11}{N + 1}} f,
\]
and
\[
Q = \frac{p}{\mu} \left( \frac{q}{p} + \frac{1}{4} \left( \frac{p'}{p} + \frac{\mu'}{\mu} \right)' - \frac{1}{16} \left( \frac{p'}{p} + \frac{\mu'}{\mu} \right)^2 + \frac{1}{4p} \left( \frac{p'}{p} + \frac{\mu'}{\mu} \right) \right)
= \frac{1}{z(1 - z)^3} \left( \frac{(N - 1)(N - 3)}{16} + \frac{7 - 2N}{2} z + 2z^2 \right).
\]

Putting
\[
\xi_R := \int_0^{1 - \frac{1}{\pi}} \sqrt{\frac{1 - z}{z}} dz,
\]
we see that the variable \(\xi\) runs over the interval \([0, \xi_R]\). Since \(z \sim \frac{\xi^2}{4}\) as \(\xi \to 0\), we see
\[
Q \sim \frac{(N - 1)(N - 3)}{4} \frac{1}{\xi^2}
\]
as \(\xi \to 0\). But \(\gamma < 2\) implies \(N > 4\) and \(\frac{(N - 1)(N - 3)}{4} > \frac{3}{4}\). Hence the boundary \(\xi = 0\) is of the limit point type. See, e.g., [17], p.159, Theorem X.10. The exceptional case \(\gamma = 2\) or \(N = 4\) will be considered separately. Anyway the potential \(Q\) is bounded from below on \(0 < \xi < \xi_R\) provided that \(N > 3\). Thus we have

**Proposition 1** The operator \(T_0, \mathcal{D}(T_0) = C_0^\infty(0, \xi_R), T_0\eta = -\eta_{\xi\xi} + Q\eta, \) in \(L^2(0, \xi_R)\) has the Friedrichs extension \(T\), a self-adjoint operator whose spectrum consists of simple eigenvalues \(\lambda_1 R^3 < \lambda_2 R^3 < \cdots < \lambda_n R^3 < \cdots \to +\infty\).
In other words, the operator \( S_0, \mathcal{D}(S_0) = C_0^\infty(0, 1 - \frac{1}{R}) \), \( S_0y = L_2y \) in
\[
\mathfrak{X} := L^2((0, 1 - \frac{1}{R}), \mu dz(= z^{\frac{N-2}{2}}(1 - z)^{\frac{10-N}{2}}dz)),
\]
has the Friedrichs extension \( S \), a self-adjoint operator with the eigenvalues \((\lambda_n)_n \).

We note that the domain of \( S \) is
\[
\mathcal{D}(S) = \{y \in \mathfrak{X} \mid \exists \phi_n \in C_0^\infty(0, 1 - \frac{1}{R})
\text{ such that } \phi_n \to y \text{ in } \mathfrak{X} \text{ and } Q[\phi_m - \phi_n] \to 0 \text{ as } m, n \to \infty,
\text{ and } L_2y \in \mathfrak{X} \text{ in distribution sense.}\}
\]

Here
\[
Q[\phi] := \int_0^{1-\frac{1}{R}} \left| \frac{d\phi}{dz} \right|^2 z^{\frac{N}{2}}(1 - z)^{\frac{8-N}{2}}dz
\]
\[
= \int_0^{1-\frac{1}{R}} \frac{z}{1-z} \left| \frac{d\phi}{dz} \right|^2 \mu(z)dz.
\]

Moreover we have

**Proposition 2** If \( N \leq 8 \) (or \( \gamma \geq 4/3 \)), the least eigenvalue \( \lambda_1 \) is positive.

**Proof** Suppose \( N \leq 8 \). Clearly \( y \equiv 1 \) satisfies
\[
-\frac{d}{dz} \frac{dy}{dz} + qy = q = \frac{8 - N}{2} z^{\frac{N-2}{2}}(1 - z)^{\frac{10-N}{2}}.
\]
Therefore the corresponding \( \eta_1(\xi) \) given by
\[
\eta_1 = z^{\frac{N-1}{4}}(1 - z)^{\frac{9-N}{4}}
\]
satisfies
\[
-\frac{d^2 \eta_1}{d\xi^2} + Q\eta_1 = \hat{q} = \frac{8 - N}{2} z^{\frac{N-1}{4}}(1 - z)^{-\frac{N+3}{4}}.
\]
It is easy to see \( \eta_1 = \frac{d\eta_1}{d\xi} = 0 \) at \( \xi = 0 \). Let \( \phi_1(\xi) \) be the eigenfunction of
\(-d^2/d\xi^2 + Q \) associated with the least eigenvalue \( \lambda_1 \). We can assume that
\( \phi_1(\xi) > 0 \) for \( 0 < \xi < \xi_R \), \( \phi_1(\xi_R) = 0 \), and \( \frac{d\phi_1}{d\xi} < 0 \) at \( \xi = \xi_R \). Then integrations by parts give

\[
\lambda_1 \int_0^{\xi_R} \phi_1 \eta_1 d\xi = \int_0^{\xi_R} \left( -\frac{d^2\phi_1}{d\xi^2} + Q\phi_1 \right) \eta_1 d\xi \\
= -\frac{d\phi_1}{d\xi} \eta_1 \bigg|_{\xi=\xi_R} + \int_0^{\xi_R} \left( \frac{d\phi_1}{d\xi} \frac{d\eta_1}{d\xi} + Q\phi_1 \eta_1 \right) d\xi \\
> \int_0^{\xi_R} \left( \frac{d\phi_1}{d\xi} \frac{d\eta_1}{d\xi} + Q\phi_1 \eta_1 \right) d\xi \\
= \int_0^{\xi_R} \phi_1 \left( -\frac{d^2\eta_1}{d\xi^2} + Q\eta_1 \right) d\xi \\
= \int_0^{\xi_R} \phi_1 \hat{q} d\xi \geq 0.
\]

Remark When \( N = 8 \), \( \eta_1 \) satisfies \( -\frac{d^2\eta}{d\xi^2} + Q\eta = 0 \), but does not satisfy the boundary condition \( \eta|_{\xi=\xi_R} = 0 \). Hence it is not an eigenfunction of zero eigenvalue, and \( \lambda_1 > 0 \) even if \( N = 8 \).

For the sake of convenience of the further analysis, let us rewrite the linear part \( \mathcal{L} \) by introducing a new variable

\[
\tilde{x} = \frac{\xi^2}{4} = \frac{1}{4} \left( \sqrt{z(1-z)} + \tan^{-1} \sqrt{\frac{z}{1-z}} \right)^2.
\]

Clearly \( \tilde{x} = z + [z]_2 \) and the change of variables \( z \mapsto \tilde{x} \) is analytic on \( 0 \leq z < 1 \) and its inverse \( \tilde{x} \mapsto z \) is analytic on \( 0 \leq \tilde{x} < \tilde{x}_\infty := \pi^2/16 \). Since

\[
\frac{d}{dz} = \sqrt{\frac{\tilde{x}}{z}} \sqrt{1-z} \frac{d}{d\tilde{x}}, \\
\frac{d^2}{dz^2} = \frac{1-z}{z} \left( \frac{\tilde{x}}{d\tilde{x}^2} + \frac{1}{2} \left( 1 - \sqrt{\frac{\tilde{x}}{z(1-z)\sqrt{1-z}}} \right) \frac{d}{d\tilde{x}} \right), \\
\sqrt{\frac{\tilde{x}}{z}} = 1 + [\tilde{x}]_1,
\]

we can write

\[
R^3 \mathcal{L} y = -\left( \tilde{x} \frac{d^2y}{d\tilde{x}^2} + \frac{N}{2} \frac{dy}{d\tilde{x}} \right) + \ell_1(\tilde{x})\frac{dy}{d\tilde{x}} + \ell_0(\tilde{x})y,
\]
where $\ell_1(\tilde{x})$ and $\ell_0(\tilde{x})$ are analytic on $0 \leq \tilde{x} < \tilde{x}_\infty$. Putting

$$x = \frac{R^3\tilde{x}}{4} = \frac{R^3}{4}\left(\sqrt{z(1 - z)} + \tan^{-1}\sqrt{\frac{z}{1 - z}}\right)^2$$

we can write

$$\mathcal{L}y = -\triangle y + L_1(x)x\frac{dy}{dx} + L_0(x)y,$$

where

$$\triangle = x\frac{d^2}{dx^2} + \frac{N}{2}\frac{d}{dx}$$

and $L_1(x)$ and $L_0(x)$ are analytic on $0 \leq x < x_\infty := R^3\tilde{x}_\infty = \pi^2 R^3 / 16$. While $r$ runs over the interval $[1, R]$, $x$ runs over $[0, x_R]$, where $x_R := R^3\xi^2 R^3 / 4(< x_\infty)$. The Dirichlet condition at the regular boundary is $y|_{x=x_R} = 0$.

**Remark** Since $x = R^3\xi^2 / 4$, we have

$$\triangle = x\frac{d^2}{dx^2} + \frac{N}{2}\frac{d}{dx} = \frac{1}{R^6}\left(\frac{d^2}{d\xi^2} + \frac{N - 1}{\xi}\frac{d}{d\xi}\right).$$

Thus $\triangle$ is the radial part of the Laplacian in the $N$-dimensional Euclidean space $\mathbb{R}^N$ provided that $N$ is an integer. But we do not assume that $N$ is an integer in this study.

Let us fix an eigenvalue $\lambda = \lambda_n$ and an associated eigenfunction $\Phi(x)$ of $\mathcal{L}$. Then

$$y_1(t, x) = \sin(\sqrt{\lambda}t + \theta_0)\Phi(x)$$

is a time-periodic solution of the linearized problem

$$\frac{\partial^2 y}{\partial t^2} + \mathcal{L}y = 0, \quad y|_{x=x_R} = 0.$$ 

Moreover we claim that $\Phi(x)$ is an analytic function of $|x| < x_0$. To verify it, we use the following

**Lemma 1** We consider the differential equation

$$x\frac{d^2 y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0,$$
where
\[ b(x) = \beta + [x]_1, \quad c(x) = [x]_0, \]
and we assume that \( \beta \geq 2 \). Then 1) there is a solution \( y_1 \) of the form
\[ y_1 = 1 + [x]_1, \]
and 2) there is a solution \( y_2 \) such that
\[ y_2 = x^{-\beta+1}(1 + [x]_1) \]
provided that \( \beta \notin \mathbb{N} \) or
\[ y_2 = x^{-\beta+1}(1 + [x]_1) + h y_1 \log x \]
provided that \( \beta \in \mathbb{N} \). Here \( h \) is a constant which can vanish in some cases.

For a proof, see [2], Chapter 4. Applying this lemma with \( \beta = N/2 \) to the equation
\[ x \frac{d^2 y}{dx^2} + \left( \frac{N}{2} - L_1(x)x \right) \frac{dy}{dx} + (\lambda - L_0(x))y = 0, \]
we get the assertion, since \( y_2 \sim x^{-\frac{N-2}{2}} \) cannot belong to \( \mathcal{X} = L^2(x^{\frac{N-2}{2}} dx) \) for \( N \geq 4 \), even if \( N = 4 \), which was the exceptional case in the preceding discussion of the limit point type.

### 3 Statement of the main result

We rewrite the equation (5) by using the linearized part \( \mathcal{L} \) defined by (7) as
\[ \frac{\partial^2 y}{\partial t^2} + \left( 1 + G_I(y, r \frac{\partial y}{\partial r}) \right) \mathcal{L} y + G_{II}(r, y, r \frac{\partial y}{\partial r}) = 0, \quad (15) \]
where
\[ G_I(y, v) = (1 + y)^2 \left( 1 + \frac{1}{\gamma} \partial_v G_2(y, v) \right) - 1, \]
\[ G_{II}(r, y, v) = \frac{P}{pr^2} G_{II0}(y, v) + \frac{1}{\gamma - 1} \frac{1}{r^3} G_{II1}(y, v), \]
\[ G_{II0}(y, v) = (1 + y)^2 (3 \partial_v G_2 - \partial_y G_2)v, \]
\[ G_{II1}(y, v) = (1 + y)^2 \left( \frac{1}{\gamma} (\partial_v G_2)((4 - 3\gamma)y - \gamma v) + G_2 \right) - H + 4y(1 + y)^2. \]

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Here
\[ G_2 := G - \gamma(3y + v) = [y, v]_2, \]
\[ \partial_v G_2 := \frac{\partial}{\partial v} G_2 = \partial G - \gamma = [y, v]_1, \]
\[ \partial_y G_2 := \frac{\partial}{\partial y} G_2 = \partial G - 3\gamma = [y, v]_1. \]

We have fixed a solution \( y_1 \) of the linearized equation \( y_{tt} + \mathcal{L}y = 0 \) (see (14)), and we seek a solution \( y \) of (5) or (15) of the form
\[ y = \varepsilon y_1 + \varepsilon w, \]
where \( \varepsilon \) is a small positive parameter.

**Remark** The following discussion is valid if we take
\[ y_1 = \sum_{k=1}^{K} c_k \sin(\sqrt{\lambda_{n_k}} t + \theta_k) \cdot \Phi_k(x), \]
(14)
where \( \Phi_k \) is an eigenfunction of \( \mathcal{L} \) associated with the eigenvalue \( \lambda_{n_k} \) and \( c_k \) and \( \theta_k \) are constants for \( k = 1, \cdots, K \).

Then the equation which governs \( w \) turns out to be
\[ \frac{\partial^2 w}{\partial t^2} + \left( 1 + \varepsilon a(t, r, w, r \frac{\partial w}{\partial r}, \varepsilon) \right) \mathcal{L}w + \varepsilon b(t, r, w, r \frac{\partial w}{\partial r}, \varepsilon) = \varepsilon c(t, r, \varepsilon), \]
(16)
where
\[ a(t, r, w, \Omega, \varepsilon) = \varepsilon^{-1} G_I(\varepsilon(y_1 + w), \varepsilon(v_1 + \Omega)), \]
\[ b(t, r, w, \Omega, \varepsilon) = \varepsilon^{-1} G_I(\varepsilon(y_1 + w), \varepsilon(v_1 + \Omega)) \mathcal{L}y_1 + \varepsilon^{-2} G_{II}(\varepsilon(y_1 + w), \varepsilon(v_1 + \Omega)) \]
\[ -\varepsilon^{-1} G_I(\varepsilon y_1, \varepsilon v_1) \mathcal{L}y_1 - \varepsilon^{-2} G_{II}(\varepsilon y_1, \varepsilon v_1), \]
\[ c(t, r, \varepsilon) = \varepsilon^{-1} G_I(\varepsilon y_1, \varepsilon v_1) \mathcal{L}y_1 + \varepsilon^{-2} G_{II}(\varepsilon y_1, \varepsilon v_1). \]

Here \( v_1 \) stands for \( r \partial y_1 / \partial r \).

The main result of this study can be stated as follows:
Theorem 1 For any \( T > 0 \), there is a sufficiently small positive \( \varepsilon_0(T) \) such that, for \( 0 < \varepsilon \leq \varepsilon_0(T) \), there is a solution \( w \) of (16) such that \( w \in C^\infty([0, T] \times [1, R]) \) and

\[
\sup_{j+k \leq n} \left\| \frac{\partial^j}{\partial t^j} \left( \frac{\partial}{\partial r} \right)^k w \right\|_{L^\infty([0, T] \times [1, R])} \leq C_n \varepsilon,
\]

or a solution \( y \in C^\infty([0, T] \times [1, R]) \) of (5) or (15) of the form

\[
y(t, r) = \varepsilon y_1(t, r) + O(\varepsilon^2),
\]

or a motion which can be expressed by the Lagrangian coordinate as

\[
r(t, m) = \bar{r}(m)(1 + \varepsilon y_1(t, \bar{r}(m)) + O(\varepsilon^2))
\]

for \( 0 \leq t \leq T, 0 \leq m \leq M \).

Remark The corresponding density distribution \( \rho = \rho(t, r) \), where \( r \) is the original Euler coordinate, satisfies

\[
\rho(t, r) > 0 \text{ for } 1 \leq r < R_F(t), \quad \rho(t, r) = 0 \text{ for } R_F(t) \leq r,
\]

where

\[
R_F(t) := r(t, M) = R + \varepsilon R \sin(\sqrt{\lambda} t + \theta_0) \Phi(0) + O(\varepsilon^2).
\]

Since \( y(t, r) \) is smooth on \( 1 \leq r \leq R \), we have

\[
\rho(t, r) = C(t)(R_F(t) - r)^{\frac{1}{\gamma - 1}} \left( 1 + O(R_F(t) - r) \right)
\]

as \( r \to R_F(t) - 0 \). Here \( C(t) \) is positive and smooth in \( t \).

Our task is to find the inverse image \( \mathfrak{P}^{-1}(\varepsilon c) \) of the nonlinear mapping \( \mathfrak{P} \) defined by

\[
\mathfrak{P}(w) = \frac{\partial^2 w}{\partial t^2} + (1 + \varepsilon a) \mathcal{L} w + \varepsilon b.
\]

Let us note that \( \mathfrak{P}(0) = 0 \). This task, which will be done by applying the Nash-Moser theorem, will require a certain property of the derivative of \( \mathfrak{P} \):

\[
D \mathfrak{P}(w) h = \frac{\partial^2 h}{\partial t^2} + (1 + \varepsilon a_1) \mathcal{L} h + \varepsilon a_{21} r \frac{\partial h}{\partial r} + \varepsilon a_{20} h,
\]
where
\[ a_1 = a(t, r, w, r \frac{\partial w}{\partial r}, \varepsilon), \]
\[ a_{20} = \frac{\partial a}{\partial w} Lw + \frac{\partial b}{\partial w}, \]
\[ a_{21} = \frac{\partial a}{\partial \Omega} Lw + \frac{\partial b}{\partial \Omega}. \]

Here \( \Omega \) is the dummy of \( r \frac{\partial w}{\partial r} \). The following observation will play a crucial role in energy estimates later.

**Lemma 2** We have
\[ a_{21} = \frac{\gamma}{\rho} (1+y)^{-2\gamma+2}(1+y+v)^{-\gamma-2} \left( (\gamma+1) \frac{\partial^2 Y}{\partial r^2} + \frac{4\gamma}{r} \frac{\partial Y}{\partial r} + \frac{2\varepsilon(\gamma-1)}{1+y} \left( \frac{\partial Y}{\partial r} \right)^2 \right), \]
where
\[ y = \varepsilon(y_1 + w), \quad v = r \frac{\partial y}{\partial r}, \quad Y = y_1 + w. \]

**Proof** It is easy to see
\[ a_{21} = (\partial_v G_I)LY + \varepsilon^{-1} \partial_v G_{II} \]
\[ = (\partial_v G_I) \left( -\frac{\gamma P}{\rho r} (3Y + V)' \right) + \varepsilon^{-1} \frac{P}{pr^2} \partial_v G_{II} + \frac{1}{\gamma - 1} \frac{1}{r^3} [U], \]
where
\[ [U] = \gamma (\partial_v G_I)(3Y + V) + \partial_v G_I(-4Y) + \varepsilon^{-1} \partial_v G_{II}. \]
Using
\[ \partial_v G_I = (1+y)^2 \frac{1}{\gamma} \partial_v^2 G_2, \]
we can show that \([U] = 0\). Then a direct calculation leads us to the conclusion. ■

### 4 Proof of the main result

We use the variable \( x \) defined by (12) instead of \( r \). We note that
\[ x = R^2 (R-r) + [R-r]_2, \]
\[ \frac{\partial}{\partial r} = -R^2 (1+[x]_1)) \frac{\partial}{\partial x}. \]
Therefore a function of $1 \leq r \leq R$ which is infinitely many times continuously differentiable is also so as a function of $0 \leq x \leq x_R$.

We are going to apply the Nash-Moser theorem formulated by R. Hamilton ([5], p.171, III.1.1.1):

Let $\mathcal{E}_0$ and $\mathcal{E}$ be tame spaces, $U$ an open subset of $\mathcal{E}_0$ and $\mathcal{P} : U \to \mathcal{E}$ a smooth tame map. Suppose that the equation for the derivative $D\mathcal{P}(w)h = g$ has a unique solution $h = V\mathcal{P}(w)g$ for all $w$ in $U$ and all $g$, and that the family of inverse $V\mathcal{P} : U \times \mathcal{E} \to \mathcal{E}_0$ is a smooth tame map. Then $\mathcal{P}$ is locally invertible.

In order to apply the Nash-Moser theorem, we consider the spaces of functions of $t$ and $x$:

$\mathcal{E} := \{y \in C^\infty([-2\tau_1, T] \times [0, x_R]) \mid y(t,x) = 0 \text{ for } -2\tau_1 \leq t \leq -\tau_1\}$,

$\mathcal{E}_0 := \{w \in \mathcal{E} \mid w|_{x=x_R} = 0\}$.

Here $\tau_1$ is a positive number. Let $U$ be the set of all functions $w$ in $\mathcal{E}_0$ such that $|w| + |\partial w/\partial x| < 1$ and suppose that $|\varepsilon| \leq \varepsilon_1$, $\varepsilon_1$ being a small positive number. Then we can consider that the nonlinear mapping $\mathcal{P}$ maps $U(\subset \mathcal{E}_0)$ into $\mathcal{E}$, since the coefficients $a, b$ are smooth functions of $t, x, \varepsilon w$ and $\varepsilon \partial w/\partial x$.

Let us assume that $\varepsilon c(t, x) = 0$ for $-2\tau_1 \leq t \leq -\tau_1$ after changing the value of $c$ for $-2\tau_1 \leq t < 0$. To fix the idea, we replace $c(t, x)$ by $\alpha(t)c(t, x)$ with a cut off function $\alpha \in C^\infty(\mathbb{R})$ such that $\alpha(t) = 1$ for $t \geq 0$ and $\alpha(t) = 0$ for $t \leq -\tau_1$. Then $\mathcal{P}^{-1}(\varepsilon c)$ is a solution of (16) on $t \geq 0$.

We should show that the Fréchet space $\mathcal{E}$ is tame for some gradings of norms. For $y \in \mathcal{E}$, $n \in \mathbb{N}$, let us define

$$\|y\|_n^{(\infty)} := \sup_{0 \leq j + k \leq n} \left\|(-\partial^2/\partial t^2)^j (-\Delta)^k y\right\|_{L^\infty([-2\tau_1, T] \times [0, x_R])}.$$  

Then we can claim that $\mathcal{E}$ turns out to be tame by this grading ($\| \cdot \|_n^{(\infty)}$) (see [5], p.136, II.1.3.6 and p.137, II 1.3.7). In fact, even if $N$ is not an integer, we can define the Fourier transformation $Fy(\xi)$ of a function $y(x)$ for $0 \leq x < \infty$ by

$$Fy(\xi) := \int_0^\infty K(\xi x)y(x)x^{N-1}dx.$$
Here \( K(X) \) is an entire function of \( X \in \mathbb{C} \) given by
\[
K(X) = 2(\sqrt{X})^{-\frac{\nu}{2} + 1}J_{\frac{\nu}{2} - 1}(4\sqrt{X}) = 2\Phi_{\frac{\nu}{2} - 1}(X),
\]
\( J_\nu \) being the Bessel function. Then we have
\[
F(-\triangle y)(\xi) = 4\xi \cdot Fy(\xi)
\]
and the inverse of the transformation \( F \) is \( F \) itself. See, e.g. [18]. Then it is easy to see \( \mathcal{E} \) endowed with the grading \((\| y \|_n^{(\infty)})_n\) is a tame direct summand of the tame space
\[
\hat{\mathcal{E}} := L_1^\infty(\mathbb{R} \times [0, \infty), d\tau \otimes \xi^{\frac{\nu}{2} - 1} d\xi, \log(1 + \tau^2 + 4\xi))
\]
through the Fourier transformation
\[
\mathcal{F}y(\tau, \xi) = \frac{1}{\sqrt{2\pi}} \int e^{-\sqrt{-1}\tau t} Fy(t, \cdot)(\xi) dt
\]
and its inverse applied to the space \( \hat{\mathcal{E}}_0 := C_0^\infty((-2T - 2\tau_1, 2T) \times [0, x_R + 1)) \), into which functions of \( \mathcal{E} \) can be extended (see, e.g. [1], p.88, 4.28 Theorem, -the existence of ‘total extension operator’) and the space
\[
\hat{\mathcal{E}} := \hat{C}_0^\infty(\mathbb{R} \times [0, \infty)) := \{y|\forall j\forall k \lim_{L \to \infty} \sup_{|t| \geq L, x \geq L} |(-\partial^2_t)^j(-\triangle)^k y| = 0\},
\]
for which functions of \( \hat{\mathcal{E}} \) are restrictions. Actually, if we denote by \( \epsilon : \mathcal{E} \to \hat{\mathcal{E}}_0 \) the extension operator, and by \( \tau : \hat{\mathcal{E}} \to \mathcal{E} \) the restriction operator, then the operators \( \mathcal{F} \circ \epsilon : \mathcal{E} \to \hat{\mathcal{E}} \) and \( \tau \circ \mathcal{F} : \hat{\mathcal{E}} \to \mathcal{E} \) are tame and the composition \( (\tau \circ \mathcal{F}) \circ (\mathcal{F} \circ \epsilon) \) is the identity of \( \mathcal{E} \). For the details, see the proof of [5], p.137, II.1.3.6. Theorem.

On the other hand, let us define
\[
\| y \|_n^{(2)} := \left( \sum_{0 \leq j + k \leq n} \int_{-\tau_1}^T \left( -\frac{\partial^2}{\partial t^2} \right)^j (-\triangle)^k y \|_X^2 dt \right)^{1/2}.
\]
Here \( X = L^2((0, x_R); x^{\frac{\nu}{2} - 1} dx) \) and
\[
\| y \|_X := \left( \int_0^{x_R} |y(x)|^2 x^{\frac{\nu}{2} - 1} dx \right)^{1/2}.
\]
We have
\[ \sqrt{\frac{N}{2}} \| y \|_x \leq \| y \|_{L^\infty} \leq C \sup_{j \leq \nu} \| (-\triangle)^j y \|_x, \]
by the Sobolev imbedding theorem (see Appendix A), provided that \( 2\sigma > N/2 \). The derivatives with respect to \( t \) can be treated more simply. Then we see that the grading \( (\| \cdot \|_n^{(2)})_n \) is tamely equivalent to the grading \( (\| \cdot \|_n^{(\infty)})_n \), that is, we have
\[ \frac{1}{C} \| y \|_n^{(2)} \leq \| y \|_n^{(\infty)} \leq C \| y \|_n^{(2)} \]
with \( 2s > 1 + N/2 \). Hence \( \mathcal{E} \) is tame with respect to \( (\| \cdot \|_n^{(2)})_n \), too. The grading \( (\| \cdot \|_n^{(2)})_n \) will be suitable for estimates of solutions of the associated linear wave equations.

Note that \( \mathcal{E}_0 \) is a closed subspace of \( \mathcal{E} \) endowed with these gradings.

Now we verify the nonlinear mapping \( \mathcal{P} = \mathcal{P}(w) \) is tame for the grading \( (\| \cdot \|_n^{(\infty)})_n \).

To do so, we write
\[ \mathcal{P}(w) = F(t, x, Dw, D^2 w, w_{tt}), \]
where \( D = \partial / \partial x \), \( F \) is a smooth function of \( t, x, Dw, D^2 w, w_{tt} \) and linear in \( D^2 w, w_{tt} \). According to [5] (see p.142, II.2.1.6 and p.145, II.2.2.6), it is sufficient to prove the linear differential operator \( w \mapsto Dw = \partial w / \partial x \) is tame. But it is clear because of the following result.

**Proposition 3** For any \( m \in \mathbb{N} \) and for any \( y \in C^\infty[0, 1] \) we have the formula
\[ \triangle^m Dy(x) = x^{2/2 - m - 1} \int_0^x \triangle^{m+1} y(x')(x')^{2/2 + m} \, dx'. \]
As a corollary it holds that, for any \( m, k \in \mathbb{N} \),
\[ \| (-\triangle)^m D^k y \|_{L^\infty} \leq \frac{1}{\prod_{j=0}^{k-1} (\frac{N}{2} + m + j)} \| (-\triangle)^{m+k} y \|_{L^\infty}. \]

**Proof** It is easy by integration by parts in induction on \( m \) starting from the formula
\[ Dy(x) = x^{-N/2} \int_0^x \triangle y(x')(x')^{N/2 - 1} \, dx'. \]
\[ \blacksquare \]

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In parallel with the results of [5] (see p.144, II.2.2.3. Corollary and p.145, II.2.2.5. Theorem), we should use the following two propositions. Proofs for these propositions are given in Appendix B.

**Proposition 4** For any positive integer \( m \), there is a constant \( C \) such that

\[
|\triangle^m (f \cdot g)|_0 \leq C(|\triangle^m f|_0|g|_0 + |f|_0|\triangle^m g|_0),
\]

where \( |\cdot|_0 \) stands for \( \| \cdot \|_{L^\infty} \).

**Proposition 5** Let \( F(x, y) \) be a smooth function of \( x \) and \( y \) and \( M \) be a positive number. Then for any positive integer \( m \), there is a constant \( C > 0 \) such that

\[
|\triangle^m F(x, y(x))|_0 \leq C(1 + |y|_m)
\]

provided that \( |y|_0 \leq M \), where we denote

\[
|y|_m = \sup_{0 \leq j \leq m} \|(-\triangle)^j y\|_{L^\infty}.
\]

Summing up, we can claim that

\[
\|\mathcal{P}(w)\|^{(\infty)}_n \leq C(1 + \|w\|^{(\infty)}_n),
\]

provided that \( \|w\|_2^{(\infty)} \leq M \).

Therefore the problem is concentrated to estimates of the solution and its higher derivatives of the linear equation

\[
D\mathcal{P}(w)h = g,
\]

when \( w \) is fixed in \( \mathcal{E}_0 \) and \( g \) is given in \( \mathcal{E} \). Let us investigate the structure of the linear operator \( D\mathcal{P}(w) \).

First we note that

\[
\frac{\gamma P}{\rho} = \frac{1}{r} - \frac{1}{R} = \frac{x}{R^3}(1 + [x]_1).
\]

Therefore it follows from Lemma 2 that there exists a smooth function \( \hat{a}(t, x) \) such that

\[
\varepsilon a_{21} r \frac{\partial}{\partial r} = \varepsilon \hat{a}(t, x) x \frac{\partial}{\partial x}.
\]
Let us put
\[ b_1 := (1 + \varepsilon a_1) L_1(x) + \varepsilon \dot{a}, \]
\[ b_0 := (1 + \varepsilon a_1) L_0(x) + \varepsilon a_{20}, \]
taking into account the observation in Section 2, (13).

Then we can write
\[ D \mathfrak{P}(w) h = \frac{\partial^2 h}{\partial t^2} - (1 + \varepsilon a_1) \triangle h + b_1(t, x) \frac{\partial h}{\partial x} + b_0(t, x) h. \]

We note that \( b_1, b_0 \) depend only on \( w, \partial w/\partial x, \partial^2 w/\partial x^2 \). Then we can claim

**Lemma 3** If a solution of \( D \mathfrak{P}(w) h = g \) satisfies
\[ h|_{x=x_R} = 0, \quad h|_{t=0} = \frac{\partial h}{\partial t}|_{t=0} = 0, \]
then \( h \) enjoys the energy inequality
\[ \| \partial_t h \|_x + \| \dot{D} h \|_x + \| h \|_x \leq C \int_0^T \| g(t') \|_x dt', \]
where \( \dot{D} = \sqrt{x} \partial/\partial x \) and \( C \) depends only on \( N, R, T, \quad A := \| \varepsilon \partial_t a_1 \|_{L^\infty} + \sqrt{2} \| \varepsilon \dot{D} a_1 + b_1 \|_{L^\infty} \) and \( B := \| b_0 \|_{L^\infty} \), provided that \( |\varepsilon a_1| \leq 1/2 \).

**Proof** We consider the energy
\[ E(t) := \int_{-\infty}^{x_R} \left( (\partial_t h)^2 + (1 + \varepsilon a_1)(\dot{D} h)^2 \right) x^{N-1} dx. \]

Mutiplying the equation by \( \partial_t h \), and integrating by parts under the boundary condition, we get
\[ \frac{1}{2} \frac{dE}{dt} = \int_{-\infty}^{x_R} \left( \frac{1}{2} \partial_t (\varepsilon a_1)(\dot{D} h)^2 - \dot{D}(\varepsilon a_1)(\dot{D} h)(\partial_t h) + \right. \]
\[ \left. - \sqrt{x} b_1(\dot{D} h)(\partial_t h) - b_0 h(\partial_t h) + g(\partial_t h) \right) x^{N-1} dx, \]
which implies
\[ \frac{1}{2} \frac{dE}{dt} \leq AE + B \left| \int_{-\infty}^{x_R} h(\partial_t h)x^{N-1} dx \right| + E^{1/2}\| g(t) \|_x. \]
On the other hand, using the initial condition, we see that $U(t) := \|h\|_\mathcal{X}^2$ enjoys

$$\frac{1}{2} \frac{dU}{dt} = \int_0^x h(\partial_t h)x^{N-1} dx \leq U^{1/2} E^{1/2},$$

$U(0) = 0$.

Hence $U(t) \leq \int_0^t E^{1/2}$ and

$$\left| \int_0^x h(\partial_t h)x^{N-1} dx \right| \leq E^{1/2}(t) \int_0^t E^{1/2}.$$

Summing up, we have

$$\frac{1}{2} \frac{dE}{dt} \leq AE + BE(t)^{1/2} \int_0^t E^{1/2} + E^{1/2}\|g(t)\|_\mathcal{X},$$

$E(0) = 0$.

By the Gronwall's lemma, we can derive the inequality

$$E^{1/2}(t) \leq C \int_0^t \|g(t')\|_\mathcal{X} dt'.$$

$\blacksquare$

A tame estimate of the inverse $D\mathfrak{P}(w)^{-1} : g \mapsto h$ will be discussed in the next section. This will completes the proof of the main result.

## 5 Tame estimate of solutions of linear wave equations

We consider the wave equation

$$\frac{\partial^2 h}{\partial t^2} + \mathcal{A}h = g(t, x), \quad (0 \leq t \leq T, 0 \leq x \leq 1),$$

where

$$\mathcal{A}h = -b_2 \Delta h + b_1 \tilde{D} h + b_0 h,$$

$$\Delta = x \frac{d^2}{dx^2} + \frac{N}{2} \frac{d}{dx}, \quad \tilde{D} = x \frac{d}{dx}.$$
We denote $\vec{b} = (b_2, b_1, b_0)$. The given function $\vec{b}(t, x)$ is supposed to be in $C^\infty([0, T] \times [0, 1])$ and we assume that $|b_2(t, x) - 1| \leq 1/2$. The function $g(t, x)$ belongs to $C^\infty([0, T] \times [0, 1])$ and we suppose that

$$g(t, x) = 0 \quad \text{for} \quad 0 \leq t \leq \tau_1,$$

where $\tau_1$ is a positive number. Let us consider the initial boundary value problem (IBP):

$$\frac{\partial^2 h}{\partial t^2} + A h = g(t, x),$$

$$h|_{x=1} = 0,$$

$$h|_{t=0} = \frac{\partial h}{\partial t} \bigg|_{t=0} = 0.$$

Then (IBP) admits a unique solution $h(t, x)$ thanks to the energy estimate, and $h(t, x) = 0$ for $0 \leq t \leq \tau_1$ because of the uniqueness. Moreover, since the compatibility conditions are satisfied, the unique solution turns out to be smooth. A proof can be found e.g. in [6], Chapter 2. To satisfy ourselves, we shall give a brief sketch of a proof of the existence of smooth solutions in Appendix C. We are going to get estimates of the higher derivatives of $h$ by them of $g$ and the coefficients $b_2, b_1, b_0$.

### 5.1 Notations

Let us introduce the following notations:

For $m, n \in \mathbb{N}$ and for functions $y = y(x)$ of $x \in [0, 1]$, we put

$$\langle y \rangle_{2m} := \| \Delta^m y \|, \quad \| y \| := \| y \|_x := \left( \int_0^1 |y(x)|^2 x^{N-1} dx \right)^{1/2},$$

$$\langle y \rangle_{2m+1} := \| \hat{D} \Delta^m y \|, \quad \hat{D} = \sqrt{x} \frac{d}{dx},$$

$$\| y \|_n := \left( \sum_{0 \leq \ell \leq n} \langle y \rangle_{2\ell}^2 \right)^{1/2},$$

$$| y |_n := \max_{0 \leq \ell \leq n} \| \hat{D}^\ell y \|_{L^\infty(0,1)}.$$
For $n \in \mathbb{N}$, a fixed $T > 0$, and for functions $y = y(t, x)$ of $(t, x) \in [0, T] \times [0, 1]$, we put

$$
\|y\|_T^n := \left( \sum_{j+k \leq n} \int_0^T \| \partial_t^j y \|_{L^2}^2 dt \right)^{1/2},
$$

$$
|y|_T^n := \max_{j+k \leq n} \| \partial_t^j \dot{D}^k y \|_{L^\infty([0,T] \times [0,1])}.
$$

Here $\partial_t = \partial/\partial t$.

Let us say that a grading of norms $(p_n)_{n \in \mathbb{N}}$ is interpolation admissible if for $\ell \leq m \leq n$ it holds

$$
p_m(f) \leq C p_n(f) \frac{m-\ell}{n-\ell} p_\ell(f) \frac{n-\ell}{n-m}.
$$

It is well known that, if and only if

$$
p_n(f)^2 \leq C p_{n+1}(f)p_{n-1}(f)
$$

for any $n \geq 1$, $(p_n)_n$ is interpolation admissible. If $(p_n)_n$ and $(q_n)_n$ are interpolation admissible, and if $(i, j)$ lies on the line segment joining $(k, \ell)$ and $(m, n)$, then

$$
p_i(f)q_j(g) \leq C(p_k(f)q_\ell(g) + p_m(f)q_n(g)).
$$

(For a proof, see [5], p.144, 2.2.2. Corollary.)

It is well-known that $(| \cdot |_n)_n$ and $(| \cdot |^T_n)_n$ are interpolation admissible, since $\dot{D} = \partial/\partial \xi$, where $x = \xi^2/4$.

Moreover $(\| \cdot \|_n)_n$ and $(\| \cdot \|^T_n)_n$ are interpolation admissible. To verify it, it is sufficient to note that

$$
y = \sum_{k=1}^\infty c_k \phi_k \in C_0^\infty[0,1)
$$

enjoys

$$
(y)_\ell = \left( \sum_k \lambda_k^\ell |c_k|^2 \right)^{1/2}.
$$

Here $(\lambda_k)_k$ are eigenvalues of $-\Delta$ with the Dirichlet boundary condition at $x = 1$ and $(\phi_k)_k$ are associated eigenfunctions. We note that $(\dot{D}\phi_n/\sqrt{\lambda_n})_n$
is a complete orthonormal system of $\mathfrak{X}$ and $(\dot{D}y|\dot{D}\phi)_{\mathfrak{X}} = (-\triangle y|\phi)_{\mathfrak{X}}$ if $y \in C^\infty[0, 1)$.

Then it is clear by the Schwartz inequality that 

$$(y)^2 \leq (y)_{n+1}(y)_{n-1}$$

for $y \in C^\infty_0[0, 1)$. Since $(y)_j \leq (y)_{j'}$ for $j \leq j'$, $y \in C^\infty_0[0, 1)$, we have 

$$(y)_\ell \leq \|y\|_\ell \leq C \cdot (y)_\ell$$

and 

$$\|y\|^2 \leq C\|y\|_{n-1}\|y\|_{n+1}$$

at least for $y \in C^\infty_0[0, 1)$. By using a continuous linear extension of functions on $[0, 1]$ to functions on $[0, 2]$ with supports in $[0, 3/2)$, we can claim that this inequality holds for any $y \in C^\infty[0, 1]$ with a suitable change of the constant C. We refer to [16], Chapter 3, Section 4, Theorem 3.11. It is sufficient to note the following

**Proposition 6** If $\alpha(x) \in C^\infty(\mathbb{R})$ is fixed, then there is a constant $C$ depending on $\alpha$ such that 

$$\|\alpha y\|_n \leq C\|y\|_n.$$

A proof can be found in Appendix B. Hence $(\| \cdot \|_n)_n$ and $(\| \cdot \|_n^T)_n$ are interpolation admissible.

## 5.2 Goal of this Section

Our goal is:

**Lemma 4** Assume that $|b_2 - 1| \leq 1/2$, $|\tilde{b}|_2 \leq M$ and $\|g\|_T^T \leq M$. Then there is a constant $C_n = C_n(T, M, N)$ such that if $h$ is the solution of (IBP) then 

$$\|h\|_{n+2}^T \leq C_n(1 + \|g\|_{n+1}^T + |\tilde{b}|_{n+3}).$$

We see that $\|y\|_{2m}^T$ is equivalent to 

$$\|y\|_{m}^{(2)} = \left( \sum_{j+k \leq m} \int_0^T ((\partial_t^{2j}y)_{2k})^2 dt \right)^{1/2}$$

for $y \in C^\infty([0, T] \times [0, 1])$. In fact it is sufficient to note the following
Proposition 7  For any $y \in C^\infty([0,1])$ we have
\[
\|\dot{\Delta}^m y\|_x \leq C(\|\Delta^m y\|_x + \|\Delta^{m+1} y\|_x).
\]

A proof can be found in Appendix B.

Therefore the conclusion of the Lemma reads:
\[
\|h\|_m^{(2)} \leq C(1 + \|g\|_m^{(2)} + \|w\|_{m+3+s}^{(2)})
\]
with $2s > 1 + N/2$, provided that $\|g\|_1^{(2)} \leq M$ and $\|w\|_{3+s}^{(2)} \leq M$, since $\vec{b}$ is a smooth function of $w, Dw, D^2 w, \partial^2_t w$ in our context so that $|\vec{b}|_{n+3}^T \leq C(1 + |w|^T_{n+7})$, provided that $|w|^T_4 \leq M$. This says that $(w, g) \mapsto h$ is tame with respect to the grading $(\|\cdot\|_n^{(2)})_n$.

Let us sketch a proof of this Lemma.

5.3 Elliptic a priori estimates

By tedious calculations we have
\[
[\Delta^m, \mathcal{A}] y := \Delta^m \mathcal{A} y - \mathcal{A} \Delta^m y = \sum_{j+k=m} (b_{1k}^{(m)} \dot{\Delta}^j y + b_{0k}^{(m)} \Delta^j y),
\]
where $\dot{D} = xd/dx$ and
\[
\begin{align*}
b_{10}^{(m)} & = -2mDb_2, \\
b_{00}^{(m)} & = -m((2m-1)\Delta + (m-1)(1-N)D)b_2 + m(1+2\dot{D})b_1,
\end{align*}
\]
where $D = d/dx$ and $b_{1k}^{(m)}, b_{0k}^{(m)}, k \geq 1$ are determined by
\[
\begin{align*}
b_{11}^{(1)} & = 2Db_0 + (\Delta - (N-2)D)b_1, \\
b_{01}^{(1)} & = \Delta b_0,
\end{align*}
\]

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and the recurrence formula

\[ b_{1k}^{(m+1)} = b_{1k}^{(m)} + (\triangle - (N - 2)D) b_{1,k-1}^{(m)} + 2Db_{0,k-1}^{(m)} \quad \text{for} \quad k \geq 2, \]

\[ b_{11}^{(m+1)} = b_{11}^{(m)} - 4m^2(\triangle + \frac{3 - N}{2}D) Db_2 + \]

\[ + ((4m + 1)\triangle - (2mN - 6m + N - 2)D)b_1 + 2Db_0, \]

\[ b_{0k}^{(m+1)} = b_{0k}^{(m)} + (1 + 2\tilde{D}) b_{1k}^{(m)} + \triangle b_{0,k-1}^{(m)} \quad \text{for} \quad k \geq 2, \]

\[ b_{01}^{(m+1)} = b_{01}^{(m)} - m\triangle((2m - 1)\triangle + (m - 1)(1 - N)D)b_2 + \]

\[ + m(3 + 2\tilde{D})\triangle b_1 + \triangle b_0 + (1 + 2\tilde{D})b_{11}^{(m)}. \]

We have used the following calculus formula:

\[ D\tilde{D} = \triangle - \left(\frac{N}{2} - 1\right)D, \quad \triangle \tilde{D} - \tilde{D} \triangle = \triangle, \]

\[ \triangle(Q\tilde{D}P) = Q\tilde{D}\triangle P + (1 + 2\tilde{D})Q \cdot \Delta P + (\triangle - (N - 2)D)Q \cdot \tilde{D}P, \]

\[ \triangle(QP) = Q\Delta P + 2(DQ)\tilde{D}P + (\triangle Q)P. \]

Then it follows that

\[ \|b_{0k}^{(m)}\|_{L^\infty} \leq C|\vec{b}|_{2k+3} \quad \|b_{1k}^{(m)}\|_{L^\infty} \leq C|\vec{b}|_{2k+2} \]

and therefore

\[ \|\triangle^m[A]y\| \leq C \sum_{j+k=m} (|\vec{b}|_{2k+2}\|y\|_{2j+1} + |\vec{b}|_{2k+3}\|y\|_{2j}). \]

Since

\[ \triangle^m[A, \Delta] = [\triangle^{m+1}, A] - [\triangle^m, A] \triangle, \]

it follows that

\[ \|\triangle^m[A, \Delta]y\| \leq CA_m, \]

where

\[ A_m := \sum_{j+k=m+1} (|\vec{b}|_{2k+2}\|y\|_{2j+1} + |\vec{b}|_{2k+3}\|y\|_{2j}). \]
Remark This estimate is very rough and may be far from the best possible. But it is enough for our purpose. To derive this estimate, we have used the following observations:

Let \( M_k \) denote the set of all functions of the form
\[
\sum_{\alpha=2,1,0} \sum_{i+j \leq k} C_{\alpha ij} \Delta^i D^j b_\alpha,
\]
where \( C_{\alpha ij} \) are constants. Then it can be shown that \( \Delta f, Df \) and \( D\tilde{f} (=\Delta f + (-\frac{N}{2} + 1) Df) \) belong to \( M_{k+1} \) if \( f \) belongs to \( M_k \). Using this, we can claim inductively that
\[
b(\alpha_{m+1}^k) \in M_{k+1} \text{ and } b(\alpha_0^k) \in \tilde{M}_{k+1} \text{ for any } m, k \leq m + 1.
\]

Differentiating \([\Delta^m, A]y\), we get
\[
\dot{D}[\Delta^m, A]y = \sum_{k+j=m} (\dot{b}_{2k}^{(m)} \Delta^{j+1} y + \dot{b}_{1k}^{(m)} \dot{D} \Delta^j y + \dot{b}_{0k}^{(m)} \Delta^j y),
\]
where
\[
\begin{align*}
\dot{b}_{2k}^{(m)} &= \sqrt{2} b_{1k}^{(m)}, \\
\dot{b}_{1k}^{(m)} &= (-\frac{N}{2} + 1 + \dot{D})b_{1k}^{(m)} + b_{0k}^{(m)}, \\
\dot{b}_{0k}^{(m)} &= \dot{D}b_{0k}^{(m)}.
\end{align*}
\]
Using
\[
\dot{D} \Delta^m [\Delta, A] = \dot{D}[\Delta^{m+1}, A] - \dot{D}[\Delta^m, A] \Delta,
\]
we have
\[
\|\dot{D} \Delta^m [\Delta, A]y\| \leq CA_m^2,
\]
where
\[
A_m^2 := \sum_{j+k=m+1} (|\tilde{b}|_{2k+2} \|y\|_{2j+2} + |\tilde{b}|_{2k+3} \|y\|_{2j+1} + |\tilde{b}|_{2k+4} \|y\|_{2j}).
\]

Since \( A_{m-1} \leq A_m^2 \leq 2A_m \leq 2A_m^2 \), we can claim that
\[
\begin{align*}
\|[\Delta, A]y\|_{2m} &\leq CA_m, \\
\|[\Delta, A]y\|_{2m+1} &\leq CA_m^2.
\end{align*}
\]
Now
\[ \begin{aligned}
\triangle y &= -\frac{1}{b_2} (Ay - b_1 \dot{D}y - b_0 y) \\
\end{aligned} \]
implies
\[ \| \triangle y \| \leq C(\| Ay \| + \| y \|_1), \]
and
\[ \| y \|_2 \leq C(\| Ay \| + \| y \|_1). \]
Moreover
\[ \begin{aligned}
\dot{D} \triangle y &= -\frac{1}{b_2} \left( \dot{D} Ay + (-\dot{D} b_2 + \sqrt{x} b_1) \triangle y + \right. \\
&\left. + \left( -\frac{N}{2} + 1 + \dot{D} b_1 + b_0 \right) \dot{D} y + (\dot{D} b_0) y \right) \\
\end{aligned} \]
implies
\[ \| \dot{D} \triangle y \| \leq C(\| \dot{D} Ay \| + \| y \|_2), \]
and
\[ \| y \|_3 \leq C(\| Ay \|_1 + \| y \|_1). \]
Using the estimates of $[\triangle, A]$, we can show inductively that, for $n \geq 2$,
\[ \| y \|_{n+2} \leq C(\| Ay \|_n + \| y \|_1 + K(n)), \]
where
\[ K(n) = \begin{cases} 
A_m & \text{for } n = 2m + 2, \\
A_m^* & \text{for } n = 2m + 3
\end{cases} \]
By interpolation we have
\[ K(n) \leq C(\| \vec{b} \|_2 \| y \|_{n+1} + |\vec{b}|_n \| y \|). \]
Therefore we have

**Proposition 8** Suppose $|b_2 - 1| \leq 1/2$ and $|\vec{b}|_2 \leq M$. Then
\[ \| y \|_{n+2} \leq C(\| Ay \|_n + \| y \|_1 + |\vec{b}|_{n+3} \| y \|). \]
5.4 Estimates for evolutions

Hereafter we denote generally by \( H \) a solution of the boundary value problem

\[
\frac{\partial^2 H}{\partial t^2} + AH = G(t, x), \quad H|_{x=1} = 0
\]

such that \( H(t, x) = 0 \) for \( 0 \leq t \leq \tau_1 \). Thus \( \partial_j^t H|_{t=0} = 0 \) for any \( j \in \mathbb{N} \). The time derivative \( H_j = \partial_j^t H \) satisfies

\[
\frac{\partial^2 H_j}{\partial t^2} + AH_j = G_j, \quad H_j|_{x=1} = 0,
\]

where

\[
G_j := \partial_j^t G - [\partial_j^t, A]H.
\]

We put \( G_0 = G \). Hereafter we always assume that \( |b_2 - 1| \leq 1/2 \) and \( |\vec{b}|^T_2 \leq M \).

Note that we have the energy estimate

\[
\|\partial_t H\| + \|H\|_1 \leq C \int_0^t \|G(t')\| dt'
\]

for \( 0 \leq t \leq T \). (See Lemma 3.)

**Remark** In this subsection \( H \) and \( G \) do not mean the particular functions defined in Section 1.

We put

\[
Z_n(H) := \sum_{j+k=n} \|\partial_j^t H\|_k.
\]

First we claim

**Proposition 9** For \( n \in \mathbb{N} \), we have

\[
Z_{n+2}(H) \leq C(Z_{n+1}(\partial_t H) + \|G\|_n + \|H\|_1 + |\vec{b}|_{n+3}\|H\|).
\]

**Proof** By definition we have

\[
Z_{n+2}(H) = Z_{n+1}(\partial_t H) + \|H\|_{n+2}.
\]
By Proposition 8 we have
\[
\|H\|_{n+2} \leq C(\|AH\|_n + \|H\|_1 + |\vec{b}_{n+3}\|H|)
\]
\[
= C(\|\partial_t^2 H - G\|_n + \|H\|_1 + |\vec{b}_{n+3}\|H|)
\]
\[
= C(\|\partial_t^2 H\|_n + \|G\|_n + \|H\|_1 + |\vec{b}_{n+3}\|H|)
\].

Note that \(\|\partial_t^2 H\|_n \leq Z_{n+1}(\partial_t H)\). ■

This implies by induction the following

**Proposition 10** For \(n \in \mathbb{N}\) we have
\[
Z_{n+2}(H) \leq C(\|\partial_t^{n+1} H\|_1 + \sum_{j+k=n} \|G_j\|_k + \sum_{j+k=n} (\|\partial_t^j H\|_1 + |\vec{b}_{k+3}\|\partial_t^j H|)).
\]

Applying the energy estimate on \(\|\partial_t^{n+1} H\|_1\), we get

**Proposition 11** We have
\[
Z_{n+2}(H)(t) \leq C\left(\left(\int_0^t F_n(t')^2 dt'\right)^{1/2} + F_n(t)\right),
\]
where
\[
F_n(t) = \int_0^t \|\partial_t^{n+1} G(t')\|dt' + |\vec{b}_n^T\|_n^{1/2} + \sum_{j+k=n} \|G_j\|_k + \sum_{j+k=n} (\|\partial_t^j H\|_1 + |\vec{b}_{k+3}\|\partial_t^j H|).
\]

**Proof** The energy estimate reads
\[
\|\partial_t^{n+1} H\|_1 \leq C \left(\int_0^t \|\partial_t^{n+1} G\| + \int_0^t \|[\partial_t^{n+1} A] H\|\right)
\]
But
\[
\int_0^t \|[\partial_t^{n+1} A] H\| \leq C \sum_{\alpha+\beta=n+1, \alpha \neq 0} |\partial_t^\alpha \vec{b}_0^T| \left(\int_0^t \|\partial_t^\beta H\|_2^2\right)^{1/2}
\]
\[
\leq C' \left(|\vec{b}_1^T| \left(\int_0^t Z_{n+2}(H)^2\right)^{1/2} + |\vec{b}_n^T| \|H\|_2^2\right)
\]
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by interpolation. Then Proposition 10 implies

\[ Z_{n+2}(H)(t) \leq C\left( \left( \int_0^t Z_{n+2}(H)^2 \right)^{1/2} + F_n(t) \right). \]

We can apply the Gronwall’s lemma to this inequality. ■

Integrating the conclusion of Proposition 11, we see

\[ \|H\|_{T_n+2}^T = \left( \sum_{j+k\leq n+2} \int_0^T \|\partial_j^T H\|_k^2 dt \right)^{1/2} \]

\[ = \left( \int_0^T (\|H\|^2 + \|\partial_t H\|^2 + \|H\|_1^2 + \sum_{0\leq \nu\leq n} Z_{\nu+2}(H)^2) dt \right)^{1/2} \]

\[ \leq \left( \int_0^T (\|H\|^2 + \|\partial_t H\|^2 + \|H\|_1^2) dt + C \sum_{0\leq \nu\leq n} \int_0^T F_{\nu}^2 \right)^{1/2} \]

\[ \leq C\left( \|G\|_{T_{n+1}}^T + |\vec{b}|_{T_{n+1}}^T \|H\|_2^2 + \sum_{j+k\leq n} \left( \int_0^T \|G_j\|_k^2 \right)^{1/2} + \right. \]

\[ + \left. \sum_{0\leq j\leq n} \sup_{0\leq t\leq T} \|\partial_j^T H\|_1 + |\vec{b}|_{2n+1}^T \|H\|_{n+1}^T + |\vec{b}|_{2n+3}^T \|H\|_{T_n} \right) \]

by interpolation. Hereafter we suppose that \( n \geq 1 \). Then by interpolation we have

\[ |\vec{b}|_{n+1}^T \|H\|_2^2 \leq C(|\vec{b}|_{2n+1}^T \|H\|_{n+1}^T + |\vec{b}|_{2n+3}^T \|H\|_T) \]

and therefore we can claim

**Proposition 12** We have

\[ \|H\|_{T_{n+2}}^T \leq C \left( \|G\|_{T_{n+1}}^T + \sum_{j+k\leq n} \left( \int_0^T \|G_j\|_k^2 \right)^{1/2} + \right. \]

\[ + \left. \sum_{0\leq j\leq n} \sup_{0\leq t\leq T} \|\partial_j^T H\|_1 + \|H\|_{n+1}^T + |\vec{b}|_{n+3}^T \|H\|_T \right). \]  \( (21) \)

Let us estimate the second and third terms in the right-hand side of (21).
We have
\[
\sum_{j+k=\nu} \left( \int_0^T \|G_j\|_k^2 \right)^{1/2} \leq C(\|G\|_\nu^T + \|H\|_{\nu+1}^T + |\vec{b}|_{\nu+3}^T \|H\|_T^T) \tag{22}
\]

**Proof** It is sufficient to estimate
\[
\int_0^T \|\partial^j_t, \mathcal{A}(\vec{b})\|_k^2 dt.
\]

But
\[
[\partial^j_t, \mathcal{A}(\vec{b})]H = \sum_{\alpha+\beta = j, \alpha \neq 0} \binom{j}{\alpha} \mathcal{A}(\partial^\alpha_t \vec{b}) \partial^\beta_t H,
\]
and
\[
\|\mathcal{A}(\partial^\alpha_t \vec{b}) \partial^\beta_t H\|_k \leq C(\|\partial^\beta_t H\|_{k+2} + |\partial^\alpha_t \vec{b}|_{k+3} \|\partial^\beta_t H\|),
\]
since
\[
\|\mathcal{A}(\vec{b})y\|_k \leq C(\|y\|_{k+2} + |\vec{b}|_{k+3} \|y\|).
\]

(The estimate of \(\|\mathcal{A}y\|_n\) can be derived by the discussion of the preceding subsection, keeping in mind that \(\nabla^m \mathcal{A} = \mathcal{A} \nabla^m + [\nabla^m, \mathcal{A}]\).) By interpolation, we have, for \(\alpha + \beta + k = \nu, \alpha \neq 0\),
\[
\left( \int_0^T \|\mathcal{A}(\partial^\alpha_t \vec{b}) \partial^\beta_t H\|_k^2 \right)^{1/2} \leq C(\|H\|_{\beta+k+2}^T + |\vec{b}|_{\alpha+k+3}^T \|H\|_T^T)
\leq C'(\|H\|_{\nu+1}^T + |\vec{b}|_{\nu+3}^T \|H\|_T^T).
\]

\[\blacksquare\]

Next we have
\[
\sup_{0 \leq t \leq T} \|\partial^j_t H\|_1 \leq C(\|G\|_j^T + \|H\|_{j+1}^T + |\vec{b}|_{j+3}^T \|H\|_T^T). \tag{23}
\]

**Proof** By the energy estimate, we have
\[
\|\partial^j_t H\|_1 \leq C \int_0^T \|G_j\|.
\]

Here we can use the estimate of
\[
\int_0^T \|[\partial^j_t, \mathcal{A}]H\|^2
\]

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given in the proof of the preceding proposition with $k = 0, n = j$. □

Substituting (22), (23) to (21) for $h = H$, we have

$$\|h\|_{n+2}^T \leq C(\|g\|_{n+1}^T + \|\bar{b}\|_{n+3}^T \|h\|^T).$$  \hfill (24)

Noting that

$$\|h\|^T \leq C\|g\|^T \leq CM,$$

we have

$$\|h\|_{n+2}^T \leq C(\|h\|_{n+1}^T + \|g\|_{n+1}^T + \|\bar{b}\|_{n+3}),$$  \hfill (25)

which implies inductively that

$$\|h\|_{n+2}^T \leq C(1 + \|g\|_{n+1}^T + \|\bar{b}\|_{n+3}),$$  \hfill (26)

provided that $\|g\|^T \leq M$ and $\|\bar{b}\|^T \leq M$.

This completes the proof of Lemma 4.

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Appendix

A. The Sobolev imbedding theorem

For the sake of self-containedness, we prove the Sobolev imbedding theorem for our framework. (The statement is well-known if $N$ is an integer.) Let $y \in C^\infty[0, 1]$ and $m \in \mathbb{N}$, we denote

$$(y) := \|(-\Delta)^m y\|_x.$$

For $y \in X = L^2((0, 1), x^\frac{N}{2}-1dx)$ we have the expansion

$$y(z) = \sum_{n=1}^\infty c_n \phi_n,$$

where $(\phi_n)_n$ is the orthonormal system of eigenfunctions of the operator $T = -\Delta$ with the Dirichlet boundary condition at $x = 1$. Then, for $m \in \mathbb{N}$ and for $y \in C^\infty[0, 1)$, we have

$$(-\Delta)^m y(x) = \sum_{n=1}^\infty c_n \lambda_n^m \phi_n(x)$$
and

\[(y)_m = \left( \sum_n |c_n|^2 \lambda_n^{2m} \right)^{1/2}.\]

**Lemma A.1.** Let \(j_{\nu,n}\) be the \(n\)-th positive zero of the Bessel function \(J_\nu\), where \(\nu = N - 1\). Then we have

\[\lambda_n = (j_{\nu,n}/2)^2 \sim \frac{\pi^2}{4} n^2 \text{ as } n \to \infty.\]

**Proof** By the Hankel’s asymptotic expansion (see [19]), the zeros of \(J_\nu\) can be determined by the relation

\[\tan\left(r - \left(\frac{\nu}{2} + \frac{1}{4}\right) \pi\right) = \frac{2}{\nu^2 - \frac{1}{4}} r(1 + O(r^{-2})).\]

Then we see

\[j_{\nu,n} = \left(n_0 + n + \frac{\nu}{2} + \frac{3}{4}\right) \pi + O\left(\frac{1}{n}\right) \text{ as } n \to \infty,\]

for some \(n_0 \in \mathbb{Z}\.\]

**Lemma A.2.** There is a constant \(C = C(N)\) such that

\[|\phi_n(x)| \leq C n^{\frac{N-1}{2}} \text{ for } 0 \leq x \leq 1.\]

**Proof** Note that \(\phi_n(x)\) is a normalization of \(\Phi_\nu(\lambda_n x)\), where

\[\Phi_\nu\left(\frac{r^2}{4}\right) = J_\nu (r) \left(\frac{r}{2}\right)^{-\nu}.\]

Since \(|\Phi_\nu(x)| \leq C\) for \(0 \leq x < \infty\), it is sufficient to estimate \(\|\Phi_\nu(\lambda_n x)\|_x\).

Using the Hankel’s asymptotic expansion in the form

\[J_\nu(r) = \sqrt{\frac{2}{\pi r}} \left(\cos \left(r - \frac{\nu}{2} \pi - \frac{\pi}{4}\right) \left(1 + O\left(\frac{1}{r}\right)\right) + \right.\]

\[- \left. \frac{1}{r} \sin \left(r - \frac{\nu}{2} \pi - \frac{\pi}{4}\right) \left(\frac{\nu^2 - 1}{2} + O\left(\frac{1}{r}\right)\right)\right),\]

we see that

\[\|\Phi_\nu(\lambda_n x)\|^2_\lambda = (\lambda_n)^{-\nu} \int_0^{j_{\nu,n}} J_\nu(r)^2 rdr = (\lambda_n)^{-\nu} \left(\frac{1}{\pi} j_{\nu,n} + O(1)\right)\]

\[= (\lambda_n)^{-\nu} \cdot \frac{2}{\pi} (\lambda_n^{1/2} + O(1)) \sim \frac{2}{\pi} (\lambda_n)^{-\nu - \frac{1}{2}}.\]
Then Lemma A.1 implies that
\[
\|\Phi_{\nu}(\lambda_n x)\|_{X}^{-1} \sim \text{Const.} n^{\nu + \frac{1}{2}}.
\]

\[\blacksquare\]

**Lemma A.3.** If \(y \in C_0^\infty[0, 1]\) and \(0 \leq j \leq m\), then \((\langle y \rangle)_j \leq (\langle y \rangle)_m\).

**Proof** For \(y = \sum c_n \phi_n\), we have
\[
(\langle y \rangle)^2 = \sum |c_n|^2 \lambda_n^{2j} = (\lambda_1)^{2j} \sum |c_n|^2 (\lambda_n/\lambda_1)^{2j} \leq (\lambda_1)^{2j} \sum |c_n|^2 (\lambda_n/\lambda_1)^{2m} = \lambda_1^{2j-2m} (\langle y \rangle)^2.
\]

According to [19] (see Section 15-6, p.208), we know that \(j_\nu, 1\) is an increasing function of \(\nu > 0\) and \(j_{\frac{1}{2}, 1} = \pi\). Therefore, \(\lambda_1 \geq (\pi/2)^2 > 1\) for \(N \geq 2\) and which implies \((\langle y \rangle)_j \leq (\langle y \rangle)_m\)

**Lemma A.4.** If \(2s > N/2\), then there is a constant \(C = C(s, N)\) such that \(\|y\|_{L^\infty} \leq C(\langle y \rangle)_s\)

for any \(y \in C_0^\infty[0, 1]\).

**Proof** Let \(y = \sum c_n \phi_n\), then Lemmas A.1 and A.2 imply that
\[
|y(x)| \leq \sum |c_n||\phi_n(x)| \leq C \sum n^{\frac{s-1}{2}}
\leq C \sqrt{\sum |c_n|^2 \lambda_n^{2s}} \sqrt{\sum n^{N-4s-1}}.
\]

Since \(N - 4s < 0\), the last term in the above inequality is finite. Therefore we get the required estimate. \[\blacksquare\]

Now, for \(R > 0\), we denote by \(X(0, R)\) the Hilbert space of functions \(y(x)\) of \(0 \leq x \leq R\) endowed with the inner product
\[
(\langle y_1 \rangle, y_2)_{X(0, R)} = \int_0^R y_1(x) y_2(x) x^{N-1} \, dx.
\]

Moreover, for \(m \in \mathbb{N}\), we denote by \(X^{2m}(0, R)\) the space of functions \(y(x)\) of \(0 \leq x \leq R\) for which the derivatives \((-\Delta)^j y \in X\) exist in the sense of distribution for \(0 \leq j \leq m\). And we use the norm
\[
\|y\|_{X^{2m}(0, R)} := \left( \sum_{0 \leq j \leq m} \|(-\Delta)^j y\|_{X(0, R)}^2 \right)^{1/2}.
\]
Let us denote by $\mathfrak{X}_0^{2m}(0, R)$ the closure of $\mathcal{C}^\infty_0[0, R)$ in the space $\mathfrak{X}^{2m}(0, R)$. There is a continuous linear extension $\Psi : \mathfrak{X}^{2m}(0, 1) \to \mathfrak{X}_0^{2m}(0, 2)$ such that

$$\|y\|_{\mathfrak{X}^{2m}(0, 1)} \leq \|\Psi y\|_{\mathfrak{X}^{2m}(0, 2)} \leq C\|y\|_{\mathfrak{X}^{2m}(0, 1)}.$$ 

See [16], p.186, Theorem 3.11, keeping in mind Propositions 6, 7. Then, by Lemmas A.3 and A.4, the Sobolev imbedding theorem holds for $y \in \mathfrak{X}_0^{2s}(0, 2)$. Say, if $2s > N/2$, there is a constant $C$ such that

$$\|y\|_{L^\infty} \leq C\|y\|_{\mathfrak{X}_0^{2s}}$$

for $y \in \mathfrak{X}_0^{2s}(0, 2)$. Thus the same imbedding theorem holds for $y \in C^\infty[0, 1] \subset \mathfrak{X}^{2s}(0, 1)$ through the above extension. The conclusion is that, if $2s > N/2$, there is a constant $C = C(s, N)$ such that

$$\|y\|_{L^\infty} \leq C \sup_{0 \leq j \leq s} \|(-\Delta)^j y\|_{\mathfrak{X}}$$

for any $y \in C^\infty[0, 1]$.

**B. Nirenberg-Moser type inequalities**

Let us prove Propositions 4, 5, 6 and 7.

**Proof of Proposition 4**

First, it is easy to verify the formula

$$\dot{D}^k Dy(x) = x^{-\frac{N+k}{2}} \int_0^x \dot{D}^k \Delta y(x')(x')^{\frac{N+k}{2}-1} dx',$$  \hspace{1cm} (B.1)$$

where $k \in \mathbb{N}$,

$$\dot{D} := \sqrt{x} \frac{d}{dx} \quad \text{and} \quad D := \frac{d}{dx}.$$ 

Since $\Delta = \dot{D}^2 + \frac{N-1}{2}D$, (B.1) implies

$$|\dot{D}^k Dy|_0 \leq \frac{2}{N + k} |\dot{D}^{k+2} y|_0 + \frac{N-1}{N + k} |\dot{D}^k D y|_0.$$ 

Here and hereafter $|\cdot|_0$ stands for $\|\cdot\|_{L^\infty}$. Thus we have

$$|\dot{D}^k D y|_0 \leq \frac{2}{k + 1} |\dot{D}^{k+2} y|_0.$$ 

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Repeating this estimate, we get
\[ |\dot{D}^k D^j y|_0 \leq \left( \frac{2}{k+1} \right)^j |\dot{D}^{k+2j} y|_0. \]  
(B.2)

On the other hand, since \( \dot{D}^2 = \triangle - \frac{N-1}{2} D \) and \( D \triangle - \triangle D = D^2 \), we have
\[ \dot{D}^{2\mu} = \sum_{k=0}^{\mu} C_{k\mu} \triangle^{-k} D^k \]  
(B.3)

with some constants \( C_{k\mu} = C(k, \mu, N) \). Then it follows from (B.3) and Proposition 3 that
\[ |\dot{D}^{2\mu} D^j y|_0 \leq C|\triangle^{\mu+j} y|_0. \]  
(B.4)

Since
\[ \triangle = \dot{D}^2 + \frac{N-1}{2} D \quad \text{and} \quad \dot{D}^2 - \dot{D}^2 D = D^2, \]

it is easy to see that there are constants \( C_{km} = C(k, m, N) \) such that
\[ \triangle^m = \sum_{k=0}^{m} C_{km} \dot{D}^{2(m-k)} D^k. \]  
(B.5)

Applying the Leibnitz' rule to \( D \) and \( \dot{D} \), we see
\[ \triangle^m (f \cdot g) = \sum C_{k\ell jm} (\dot{D}^{2(m-k)-\ell} D^{\ell-j} f) \cdot (\dot{D}^\ell D^j g) \]  
(B.6)

with some constants \( C_{k\ell jm} \). The summation is taken for \( 0 \leq j \leq k \leq m, 0 \leq \ell \leq 2(m-k) \). By estimating each term of the right-hand side of (B.6), we can obtain the assertion of Proposition 4. In fact, we consider the term
\[ (\dot{D}^\ell D^j f) \cdot (\dot{D}^\ell D^j g) \]
provided that \( \ell + \ell + 2(j' + j) = 2m \). By (B.2) and (B.4) we have
\[ |\dot{D}^\ell D^j g|_0 \leq C|\dot{D}^{\ell+2j} g|_0 \leq C' |\dot{D}^{2m} g|_0 \frac{\xi^{j'} + \xi^j}{2m} |g|_0 \frac{\xi^1 + \xi^{j'}}{2m} \leq C'' |\dot{D}^\ell \dot{\triangle}^m g|_0 \frac{\xi^{j''} + \xi^{j'}}{2m} |g|_0 \frac{\xi^1 + \xi^{j'}}{2m} \]
for some positive constants \( C, C' \) and \( C'' \). Here we have used the Nirenberg interpolation for \( \dot{D} = \partial/\partial \xi \), where \( x = \xi^2/4 \). The same estimate holds for \( |\dot{D}^\ell D^j f|_0 \). Therefore we have
\[ |(\dot{D}^\ell D^j f) \cdot (\dot{D}^\ell D^j g)|_0 \leq C|\dot{\triangle}^m f|_0 \frac{\xi^{1+j''} + \xi^{1+j'} g}{2m} |\dot{\triangle}^m g|_0 \frac{\xi^1 + \xi^{j'}}{2m} |g|_0 \frac{\xi^{1+j'}}{2m} \leq C(|\dot{\triangle}^m f|_0 |g|_0 + |f|_0 |\dot{\triangle}^m g|_0), \]
since $X^\theta Y^{1-\theta} \leq X + Y$.

**Proof of Proposition 5**

Suppose $F(x, y)$ is a smooth function of $x$ and $y$. Let us consider the composed function $U(x) := F(x, y(x))$. We claim that

$$|\Delta^m U|_0 \leq C(1 + |y|_m)$$

provided that $|y|_0 \leq M$. In fact,

$$\Delta^m U = \sum C_{km} \hat{D}^{2(m-k)} D^k U$$

consists of several terms of the following form:

$$\left( \hat{D}_x^K \left( \frac{\partial}{\partial y} \right)^L \hat{D}_x^{k_1} \left( \frac{\partial}{\partial y} \right) \ell F \right) \cdot (\hat{D}^{K_1} y) \cdots (\hat{D}^{K_L} y) \cdot (\hat{D}^{\mu_1} D^{k_1} y) \cdots (\hat{D}^{\mu_\ell} D^{k_\ell} y),$$

where

$$k + k_1 + \cdots + k_\ell = \kappa,$$

$$K + K_1 + \cdots + K_L + \mu_1 + \cdots + \mu_\ell = 2(m - \kappa).$$

Therefore

$$K_1 + \cdots + K_L + (\mu_1 + 2k_1) + \cdots (\mu_\ell + 2k_\ell) \leq 2m.$$ 

Applying the Nirenberg interpolation to $\hat{D}$ and using (B.4), we have

$$|\hat{D}^{K_1} y|_0 \leq C |y|_{2m}^{K_1} |y|_0^{1-K_1}.$$ 

Similarly,

$$|\hat{D}^{\mu_1} D^{k_1} y|_0 \leq C |\hat{D}^{\mu_1+2k_1} y|_0 \leq C' |y|_{2m}^{\mu_1+2k_1} |y|_0^{1-\mu_1-2k_1},$$

and so on. Then our claim follows obviously.

- We note that by (B.2), (B.4) and (B.5) we have

$$\frac{1}{C} |\hat{D}^{2j} f|_0 \leq |\Delta^j f|_0 \leq C |\hat{D}^{2j} f|_0.$$ 

(B.7)
Proof of Proposition 6

It can be verified that
\[
\triangle^m(\alpha y) = \sum_{j+k=m} (\alpha_{1k}^{(m)} \hat{D} \triangle^j y + \alpha_{0k}^{(m)} \triangle^j y),
\]
where \(\alpha_{1k}^{(m)}\) and \(\alpha_{0k}^{(m)}\) are determined by the recurrence formula
\[
\alpha_{1k}^{(m+1)} = \alpha_{1k}^{(m)} + (\triangle - (N-2)\hat{D})\alpha_{1,k-1}^{(m)} + 2\hat{D}\alpha_{0,k-1}^{(m)},
\]
\[
\alpha_{0k}^{(m+1)} = (1 + 2\hat{D})\alpha_{1k}^{(m)} + \alpha_{0k}^{(m)} + \triangle\alpha_{0,k-1}^{(m)},
\]
starting from
\[
\alpha_{10}^{(0)} = 0, \quad \alpha_{00}^{(0)} = \alpha.
\]
Here we have used the convention \(\alpha_{1k}^{(m)} = \alpha_{0k}^{(m)} = 0\) for \(k < 0\) or \(k > m\). Of course \(\alpha_{10}^{(m)} = 0\) for any \(m\). Therefore we see that \(\|\triangle^m(\alpha y)\|_0 \leq C\|y\|_{2m}\).

Differentiating the formula, we get
\[
\dot{\hat{D}}\triangle^m(\alpha y) = \sum_{j+k=m} (\dot{\alpha}_{2k}^{(m)} \triangle^{j+1} y + \dot{\alpha}_{1k}^{(m)} \hat{D} \triangle^j y + \dot{\alpha}_{0k}^{(m)} \triangle^j y),
\]
where
\[
\dot{\alpha}_{2k}^{(m)} = \sqrt{x}\alpha_{1k}^{(m)},
\]
\[
\dot{\alpha}_{1k}^{(m)} = \left(-\frac{N}{2} + 1 + \hat{D}\right)\alpha_{1k}^{(m)} + \alpha_{0k}^{(m)},
\]
\[
\dot{\alpha}_{0k}^{(m)} = \hat{D}\alpha_{0k}^{(m)}.
\]
It is clear that \(\|\dot{\hat{D}}\triangle^m(\alpha y)\|_0 \leq C\|y\|_{2m+1}\), since \(\alpha_{20}^{(m)} = 0\) for any \(m\). ■

Proof of Proposition 7

It is sufficient to prove that
\[
\|\dot{\hat{D}}y\| \leq C(\|y\| + \|\triangle y\|),
\]
where and hereafter we denote \(\| \cdot \| = \| \cdot \|_x\).
If \( w \) satisfies the Dirichlet boundary condition \( w(1) = 0 \), then
\[
\| \dot{D}w \|^2 = (-\Delta w \mid w) \leq \| \Delta w \| \| w \|.
\]
Therefore we have
\[
\| \dot{D}y \|^2 \leq \| \Delta y \| (\| y \| + |y(1)|).
\]
On the other hand we have
\[
\sqrt{\frac{2}{N}} |y(1)| \leq \| y \| + \sqrt{\frac{2}{N - 2}} \| \dot{D}y \|.
\]
In fact, since
\[
y(1) = y(x) + \int_x^1 \frac{1}{\sqrt{x}} \dot{D}y(x')dx',
\]
we have
\[
|y(1)|^2 \leq |y(x)|^2 + \frac{2}{N - 2} \| \dot{D}y \|^2 x^{-\frac{N}{2}+1}
\]
for \( x > 0 \). Integrating this, we get the above estimate of \( |y(1)| \). Hence we have, for any \( \epsilon > 0 \),
\[
\| \dot{D}y \|^2 \leq C\| \Delta y \| (\| y \| + \| \dot{D}y \|)
\leq C\left( \frac{1}{2\epsilon} \| \Delta y \|^2 + \frac{\epsilon}{2} (\| y \| + \| \dot{D}y \|)^2 \right)
\leq C\left( \frac{1}{2\epsilon} \| \Delta y \|^2 + \epsilon \| y \|^2 + \epsilon \| \dot{D}y \|^2 \right).
\]
Taking \( \epsilon \) to be small, we get the desired estimate. ■

C. Existence of the smooth solution to the linear wave equation

Let us give a proof of the existence of the smooth solution to the initial boundary value problem (IBP):
\[
\frac{\partial^2 h}{\partial t^2} + \mathcal{A}h = g(t, x), \quad h|_{x=1} = 0,
\]
\[
h|_{t=0} = \frac{\partial h}{\partial t} |_{t=0} = 0.
\]
We assume that \( g(t, x) = 0 \) for \( 0 \leq t \leq \tau_1 \).

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Existence. The existence of the solution can be proved by applying the Kato’s theory developed in [8]. In fact, we consider the closed operator

\[ A(t) = \begin{bmatrix} 0 & -1 \\ \mathcal{A}(t) & 0 \end{bmatrix} \]

in \( H := \mathcal{X}^1_0 \times \mathcal{X} \) densely defined on

\[ \mathcal{D}(\mathcal{A}(t)) = \mathcal{G} := \mathcal{X}^2_{(0)} \times \mathcal{X}^1. \]

Here \( \mathcal{X} = L^2((0, 1); x^{\frac{\alpha}{2}-1}dx), \mathcal{X}^1 = \{ y \in \mathcal{X}| \hat{D}y \in \mathcal{X} \}, \mathcal{X}^1_0 = \{ y \in \mathcal{X}^1 | y|_{x=1} = 0 \}, \mathcal{X}^2 = \{ y \in \mathcal{X}^1 | \Delta y \in \mathcal{X} \} \) and \( \mathcal{X}^2_{(0)} = \mathcal{X}^2 \cap \mathcal{X}^1_0 = \{ y \in \mathcal{X}^2 | y|_{x=1} = 0 \} \). The problem (IBP) is equivalent to

\[ \frac{du}{dt} + A(t)u = f(t), \quad u|_{t=0} = 0, \]

with

\[ u = \begin{bmatrix} h \\ \frac{\partial h}{\partial t} \end{bmatrix} \quad \text{and} \quad f(t) = \begin{bmatrix} 0 \\ g(t, \cdot) \end{bmatrix}. \]

We can write

\[ A(t)y = -x^{-\frac{\alpha}{2}+1} \frac{d}{dx} ax^{\frac{\alpha}{2}} \frac{dy}{dx} + b\hat{D}y + cy, \]

where

\[ a = b_2, \quad b = b_1 + Db_2, \quad c = b_0. \]

Then

\[ (A(t)y|v)_x = (a(t)\hat{D}y|\hat{D}v)_x + ((b\hat{D} + c)y|v)_x \]

for \( y \in \mathcal{X}^2_{(0)} \) and \( v \in \mathcal{X}^1_0 \). The inner product

\[ (y|v)_x = (a(t)\hat{D}y|\hat{D}v)_x + (y|v)_x \]

introduces an equivalent norm \( \| \cdot \|_t \) in \( \mathcal{X}^1_0 \) provided that \( |1 - a| \leq 1/2, \|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty} \leq M_0 \). Then we have

\[ -(A(t)u|u)_{\mathcal{H}} = (u_2|u_1)_x - ((b\hat{D} + c)u_1|u_2)_x \leq \beta\|u\|^2_{\mathcal{H}}, \]

where

\[ (u|\phi)_{\mathcal{H}} = (u_1|\phi_1)_t + (u_2|\phi_2)_x \]

\[ = (a(t)\hat{D}u_1|\hat{D}\phi_1)_x + (u_1|\phi_1)_x + (u_2|\phi_2)_x \]
and $\beta$ depends only upon $M_0$. $\|u\|_{\mathcal{B}} = \sqrt{(u|u)_{\mathcal{B}}}$ is equivalent to $\|u\|_{\mathcal{B}}$ and depends on $t$ smoothly in the sense of [8], Proposition 3.4. From the above estimate it follows that $\mathcal{A}(t)$ is a quasi-accretive generator in the norm $\| \cdot \|_{\mathcal{B}}$.

In fact the following argument is standard: the equation

$$(\lambda + \mathcal{A}(t))u = f$$

is reduced to an elliptic equation

$$(\lambda^2 + \mathcal{A}(t))u_1 = \lambda f_1 + f_2,$$

which admits a solution $u_1 \in \mathcal{X}^2(0)$ for given $f_3 := \lambda f_1 + f_2 \in \mathcal{X}$, provided that $\lambda^2 > \|b\|^2_{L^\infty} + \|c\|_{L^\infty} + \frac{1}{4}$; then

$$Q[u] := \lambda^2\|u\|_X^2 + (a(t)\dot{D}u|\dot{D}u)_X + ((b\dot{D} + c)u|u)_X \geq \frac{1}{4}\|u\|_{X^1}^2,$$

and for given $f_3 \in \mathcal{X}$ there is a $u_1 \in \mathcal{X}^1(0)$ such that $Q(u_1, v) = (f|v)_X$ for any $v \in \mathcal{X}^1(0)$; thus $(\lambda + \mathcal{A}(t))^{-1} \in \mathcal{B}(\mathcal{H})$ and $\|\lambda + \mathcal{A}(t)\|^{-1}_{\mathcal{B}(\mathcal{H})} \leq (\lambda - \beta)^{-1}$.

Therefore by [8], Proposition 3.4, $(\mathcal{A}(t))_t$ is a stable family of generators. Hence by [8], Theorem 7.1 and 7.2, we can claim that there exists a solution $u \in C^1([0, T]; \mathcal{H}) \cap C([0, T]; \mathcal{G})$, which gives the desired solution $h \in C^2([0, T]; \mathcal{X}) \cap C^1([0, T]; \mathcal{X}^1) \cap C([0, T]; \mathcal{X}^2)$, since $g \in C^\infty([0, T] \times [0, 1])$.

**Regularity.** We want to show that $h \in C^\infty([0, T] \times [0, 1])$. To do so, we apply the Kato’s theory developed in [8], Section 2. We consider the spaces

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 = \mathcal{X}^1_0 \times \mathcal{X} \times \mathbb{R},$$
$$\hat{\mathcal{H}}_j = \mathcal{X}^j_{(0)} \times \mathcal{X}^j \times \mathbb{R},$$
$$\mathcal{G} = \mathcal{G}_1 = \mathcal{X}^2(0) \times \mathcal{X}^1_0 \times \mathbb{R},$$
$$\hat{\mathcal{G}}_j = \mathcal{G} \cap \hat{\mathcal{H}}_j = \mathcal{X}^{j+1}_{(0)} \times \mathcal{X}^j_0 \times \mathbb{R}.$$ 

Here $\mathcal{X}^k$ is $\{y|y|^k = \left(\sum_{0 \leq \ell \leq k}(y_\ell^2)^{1/2}\right)^{1/2} < \infty\}$ and so on. Introducing the closed operator

$$\hat{\mathcal{A}}(t) = \begin{bmatrix} 0 & -1 & 0 \\ A(t) & 0 & -g(t) \\ 0 & 0 & 0 \end{bmatrix}$$
in $\hat{\mathcal{H}}$ densely defined on

$$\mathcal{D}(\hat{\mathcal{A}}(t)) = \hat{\mathcal{G}},$$

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we can convert (IBP) to
\[ \frac{du}{dt} + \hat{A}(t) = 0, \quad u|_{t=0} = \phi_0, \]
where
\[ \phi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]
Since \( g \in C^\infty \), the stability of \((\hat{A}(t))\) is reduced to that of \((A(t))\) by the perturbation theorem (\[9\], Proposition 1.2). Therefore \((\hat{A}(t))\) is a stable family of generators in \( \hat{H} \). Since the coefficients of the differential operator \( A \) are in \( C^\infty \), we see \( D(\hat{A}(t)) \cap \hat{H} = \hat{G} \) and
\[ \frac{d^k}{dt^k} \hat{A}(t) \in L^\infty([0, T]; B(\hat{G}_{j+1}, \hat{H}_j)) \]
for all \( j, k \). Moreover we have ‘ellipticity’, i.e., for each \( t, j \) \( u \in D(\hat{A}(t)) \) and \( \hat{A}(t)u \in \hat{H}_j \) implies \( u \in \hat{H}_{j+1} \) with
\[ \|u\|_{\hat{H}_{j+1}} \leq C(\|\hat{A}(t)u\|_{\hat{H}_j} + \|u\|_{\hat{H}_j}). \]
In fact this condition is reduced to the fact that if \( y \in X^2 \) and \( A(t)y \in X^j \) then \( y \in X^{j+2} \) and
\[ \|y\|_{j+2} \leq C(\|A(t)y\|_{j} + \|y\|_{1}). \]
See Proposition 8. Thus we can apply \[9\] Theorem 2.13, say, if \( \phi_0 \in D_m(0) \), then the solution \( u \) satisfies
\[ \bigcap_{j+k=m} C^k([0, T]; \hat{H}_j), \]
which implies
\[ \bigcap_{j+k=m} C^k([0, T]; X^{j+1}_{(0)}). \]
Recall \( \phi_0 = (0, 0, 1)^T \) and the space of compatibility \( D_m(0) \) is characterized by
\[
\begin{align*}
D_0(0) &= \hat{H}, \\
S^0(0) &= I, \\
D_{j+1}(0) &= \{ \phi \in D_j(0)|S^k(0)\phi \in \hat{G}_{j+1-k}, 0 \leq k \leq j \}, \\
S^{j+1}(0)\phi &= -\sum_{k=0}^{j} \binom{j}{k} \left( \frac{d}{dt} \right)^{j-k} \hat{A}(0)S^k(0)\phi.
\end{align*}
\]
See [9], (2.40), (2.41). Since $g = 0$ for $0 \leq t \leq \tau_1$, we have

$$\left( \frac{d}{dt} \right)^n \hat{\mathbf{A}}(0) = 
\begin{bmatrix}
0 & 0 & 0 \\
\left( \frac{d}{dt} \right)^n \mathbf{A}(0) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$}

Thus it is easy to see $\phi_0 \in D_j(0)$ and $S^j(0)\phi = 0$ for $j \geq 1$ inductively on $j$. (Note that $S^0(0)\phi_0 = \phi_0$.) Hence for any positive integer $m$ we have $\phi_0 \in D_m(0)$ and obtain the desired regularity of the solution $h$.

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