SOME INTEGRAL INEQUALITIES ON WEIGHTED RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. In this paper, we continue to study some applications with respect to a Reilly type integral formula associated with the \( \phi \)-Laplacian. Some inequalities of Brascamp-Lieb type and Colesanti type are provided.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold with the dimension \(n \geq 3\), where \(g = \langle , \rangle\) is the metric. It is well known that the \( \phi \)-Laplacian associated with \( \phi \) is given by
\[
\Delta_{\phi} v = e^\phi \text{div}(e^{-\phi} \nabla v) = \Delta v - \langle \nabla \phi, \nabla v \rangle, \quad \forall \ v \in C^\infty(M),
\]
which is symmetric with respect to the \( L^2(M) \) inner product under the weighted measure
\[
d\mu = e^{-\phi} dv_g.
\]
The \(m\)-dimensional Bakry-Émery Ricci curvature (see \cite{1, 3–5, 8, 9, 13}) associated with the above \( \phi \)-Laplacian is given by
\[
\text{Ric}_{\phi,m} = \text{Ric} + \nabla^2 \phi - \frac{1}{m-n} \text{d} \phi \otimes \text{d} \phi,
\]
where \( m \) is a real constant, and \( m = n \) if and only if \( \phi \) is a constant. Let \( m \to \infty \), then \( \text{Ric}_{\phi,\infty} = \text{Ric} + \nabla^2 \phi \). We define
\[
\text{Ric}_{\phi} = \text{Ric}_{\phi,\infty}.
\]
Thus, \( \text{Ric}_{\phi} \) can be seen as the \( \infty \)-dimensional Bakry-Émery Ricci curvature.

For the convenience, we still denote \( \Delta, \nabla \) by the Laplacian operator and gradient operator on \( M \), and \( \overline{\Delta}, \overline{\nabla} \), respectively, by the Laplacian operator and gradient operator on the boundary \( \partial M \). The mean curvature \( H \) of \( \partial M \) is given by \( H = \text{tr}_g(II) \), where \( II(X,Y) = g(\nabla_X \nu, Y) \) denotes the second fundamental form of \( \partial M \) with \( \nu \) the outward unit normal on \( \partial M \). For any positive twice differentiable function \( V \), we make the following conventions:
\[
\overline{\text{Ric}}_{\phi,m}^V = \frac{\Delta_{\phi} V}{V} g - \frac{1}{V} \nabla^2 V + \text{Ric}_{\phi,m},
\]
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\[ H_\phi = H - \phi \nu, \quad II^V = II - (\ln V)_\nu \eta \] and \( d\sigma \) denotes the measure induced on \( \partial M. \)

In [2], the following Reilly type integral formula associated with \( \phi \)-Laplacian has been proved:

**Theorem A.** Let \( V \) be a positive twice differentiable function on a given compact Riemannian manifold \( M \) with the boundary \( \partial M \). For any smooth function \( f \) and \( m \in (-\infty, 0) \cup [n, +\infty) \), we have

\[
0 \leq \int_{\partial M} \left[ -V II^V \left( V \nabla_{\nu} \frac{f}{V}, V \nabla_{\nu} \frac{f}{V} \right) - V^3 H_\phi \left( \left( \frac{f}{V} \right)_\nu \right)^2 
- 2V^2 \left( \frac{f}{V} \right)_\nu \left( \Delta_\phi f - \frac{\Delta_\phi V f}{V} \right) \right] d\sigma + \int_M V \left[ \frac{m-1}{m} \left( \Delta_\phi f - \frac{\Delta_\phi V f}{V} \right)^2 
- \widehat{\text{Ric}}_{\phi,m}^V \left( V \nabla_{\nu} \frac{f}{V}, V \nabla_{\nu} \frac{f}{V} \right) \right] d\mu, \tag{1.3}
\]

where the equality occurs if and only if

\[
\nabla^2 f - \frac{f}{V} \nabla^2 V = \frac{1}{n} \left( \Delta f - \frac{f}{V} \Delta V \right) g \tag{1.4}
\]

and

\[
\Delta f - \frac{f}{V} \Delta V + \frac{n}{\sqrt{(m-n)^2} V} \nabla_\phi, \nabla_{\nu} \frac{f}{V} = 0. \tag{1.5}
\]

Using the above theorem, we can obtain the following applications:

**Theorem 1.1.** Let \( M \) be a compact Riemannian manifold with a smooth boundary \( \partial M \) and \( V \widehat{\text{Ric}}_{\phi,m}^V \geq (m-1)K g \) for some positive constant \( K \), where \( m \geq n \). If \( \lambda \) is the eigenvalue of the problem \( \Delta_\phi f - \frac{\Delta_\phi V f}{V} = -\lambda \frac{f}{V} \), then the first nonzero eigenvalue \( \lambda_1 \) satisfies the following:

(1) In the Dirichlet case, if \( H_\phi \geq 0 \) on \( \partial M \), then \( \lambda_1 \geq mK \). Moreover, the equality is attained only when \( m = n, \phi \) is a constant and \( \partial M \) is totally geodesic;

(2) In the Neumann case, if \( II^V \geq 0 \) on \( \partial M \), then \( \lambda_1 \geq mK \). Moreover, the equality is attained only when \( m = n \) and \( \phi \) is a constant.

**Remark 1.1.** For a closed Riemannian manifold without boundary, we also obtain the similar results as in Theorem 1.1. On the other hand, we let \( L_\phi^V (f) = e^{\phi} \text{div}(e^{-\phi} V^2 \nabla_{\nu} \frac{f}{V}). \) It is easy to check that the differential operator \( L_\phi^V \) is linear and the problem \( \Delta_\phi f - \frac{\Delta_\phi V f}{V} = -\lambda \frac{f}{V} \) is equivalent to the eigenvalue problem of \( L_\phi^V (f) = -\lambda f \). In particular, if \( u = 1 \), then Theorem 1.1 reduces to the first eigenvalue estimates with respect to \( \phi \)-Laplacian associated with Bakry-Émery Ricci curvature.

Next, we give the following Brascamp-Lieb type inequalities:
Theorem 1.2. Let $M$ be a compact Riemannian manifold with a smooth boundary $\partial M$ and $\widetilde{\text{Ric}}^V_{\phi,m} > 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. Then for any $\varphi \in C^\infty(M)$, we have

1. In the Dirichlet case, if $H_\phi \geq 0$ and $\varphi = 0$ on $\partial M$, then
   \[ \frac{m}{m-1} \int_M V \varphi^2 d\mu \leq \int_M V \langle \widetilde{\text{Ric}}^V_{\phi,m} \rangle^{-1} \nabla \varphi, \nabla \varphi \rangle d\mu. \]  
   \[ (1.6) \]

2. In the Neumann case, if $\Pi V \geq 0$ on $\partial M$, then
   \[ \frac{m}{m-1} \int_M V \left( \varphi - \frac{\int_M V \varphi d\mu}{\int_M V d\mu} \right)^2 d\mu \leq \int_M V \langle \widetilde{\text{Ric}}^V_{\phi,m} \rangle^{-1} \nabla \varphi, \nabla \varphi \rangle d\mu. \]  
   \[ (1.7) \]

Remark 1.2. When $V = 1$, our Theorem 1.2 becomes partial results of Theorem 1.2 of Kolesnikov and Milman in [6].

Remark 1.3. Taking $\varphi = f$ in (1.6), we obtain that if $V \widetilde{\text{Ric}}^V_{\phi,m} \geq (m-1)Kg$ for some positive constant $K$, then
   \[ mK \leq \frac{\int_M V^2 |\nabla f|^2 d\mu}{\int_M V (f)^2 d\mu}, \]  
   \[ (1.8) \]
which show that $\lambda_1 \geq mK$ provided $H_\phi \geq 0$ and $f = 0$ on $\partial M$. It is exactly the result of (1) of Theorem 1.1.

As another application, we also give the following Colesant i type inequality:

Theorem 1.3. Let $M$ be a compact Riemannian manifold with a smooth boundary $\partial M$ and $\widetilde{\text{Ric}}^V_{\phi,m} \geq 0$, where $m \in (-\infty, 0) \cup [n, +\infty)$. If $\Pi V > 0$ on $\partial M$, then for any $\varphi \in C^\infty(\partial M)$, we have

\[ \int_M V d\mu \int_{\partial M} VH_\phi \varphi^2 d\sigma - \frac{m-1}{m} \left( \int_{\partial M} V \varphi d\sigma \right)^2 \leq \int_M V d\mu \int_{\partial M} V \langle (\Pi V)^{-1} \nabla \varphi, \nabla \varphi \rangle d\sigma. \]  
   \[ (1.9) \]

Remark 1.4. In particular, taking $\varphi = 1$ in (1.9), then we obtain

\[ \int_M V d\mu \int_{\partial M} VH_\phi d\sigma \leq \frac{m-1}{m} \left( \int_{\partial M} V d\sigma \right)^2, \]  
   \[ (1.10) \]
which is exactly the formula (1.11) of Theorem 1.4 in [2]. On the other hand, using the Cauchy inequality

\[ \left( \int_{\partial M} V d\sigma \right)^2 \leq \int_{\partial M} VH_\phi d\sigma \int_{\partial M} \frac{V}{H_\phi} d\sigma \]

in (1.10) gives

\[ \int_M V d\mu \leq \frac{m-1}{m} \int_{\partial M} \frac{V}{H_\phi} d\sigma, \]  
   \[ (1.11) \]
which is exactly the formula (1.9) of Theorem 1.3 in [2].
Next, we also can establish the following dual Colesanti type inequality:

**Theorem 1.4.** Let $M$ be a compact Riemannian manifold with a smooth boundary $\partial M$ and $\overline{\text{Ric}}_{\phi,m}^V \geq (m-1)Kg$ for some constant $K$, where $m \geq n$. If $H_\phi > 0$ on $\partial M$, then for any $\varphi \in C^\infty(\partial M)$, we have

$$
\int_{\partial M} VH\left(V\overline{\nabla}^2 \nu, V\overline{\nabla}^2 \nu \right) d\sigma \leq \int_{\partial M} \frac{1}{H_\phi} V\left(\overline{\Delta}_\phi \varphi - \overline{\Delta}_\phi V \varphi \right) + \frac{m-1}{2m} t^2 d\sigma,
$$

(1.12)

where $t$ is a parameter satisfying $t \leq mK$.

**Remark 1.5.** Obviously, our Theorem 1.4 generalizes Theorem 1.2 of Kolesnikov and Milman in [7].

2. Proof of results

2.1. Proof of Theorem 1.1. Using the divergence theorem, we have

$$
\int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V}{V} f \right)^2 d\mu = \int_M \left(\Delta_\phi f - \frac{\Delta_\phi V}{V} f \right) \left(\Delta_\phi f - f \Delta_\phi V \right) d\mu
$$

(2.1)

On the other hand, by virtue of

$$
\Delta_\phi f = \overline{\Delta}_\phi f + H_\phi f + f_{\nu\nu}
$$

and

$$
\Delta_\phi V = \overline{\Delta}_\phi V + H_\phi V + V_{\nu\nu},
$$

we obtain that on $\partial M$, it holds that

$$
\Delta_\phi f - \frac{\Delta_\phi V}{V} f = \overline{\Delta}_\phi f - \overline{\Delta}_\phi V f + V H_\phi \left(\frac{f}{V}\right) + f_{\nu\nu} - \frac{f}{V} V_{\nu\nu}.
$$

(2.2)

This shows that (2.1) can be written as

$$
\int_M V \left(\Delta_\phi f - \frac{\Delta_\phi V}{V} f \right)^2 d\mu = \int_{\partial M} \left[ V^2 \left(\frac{f}{V}\right) \nu \left(\Delta_\phi f - \frac{\Delta_\phi V}{V} f \right) + V^3 H_\phi \left(\frac{f}{V}\right)^2
$$

$$
+ V^2 \left(f_{\nu\nu} - \frac{f}{V} V_{\nu\nu}\right) \left(\frac{f}{V}\right) \nu \right] d\sigma - \int_M V^2 \left(\nabla \frac{f}{V}, \nabla \left(\Delta_\phi f - \frac{\Delta_\phi V}{V} f \right) \right) d\mu.
$$

(2.3)
Hence, putting (2.3) into (1.3) gives an alternative form of Reilly type integral formula:

**Proposition 2.1.** Let $V$ be a positive twice differentiable function on a given compact Riemannian manifold $M$ with the boundary $\partial M$. For any smooth function $f$ and $m \in (-\infty, 0) \cup [n, +\infty)$, we have

$$0 \leq \int_{\partial M} \left[ -V H_V \left( V \nabla \frac{f}{V} , V \nabla \frac{f}{V} \right) - \frac{1}{m} V^3 H_V \left( \left( \frac{f}{V} \right)_\nu \right)^2 \right.$$  

$$- \frac{m+1}{m} V^2 \left( \frac{f}{V} \right)_\nu \left( \Delta \phi \frac{f}{V} - \frac{A \phi V}{V} \right) + \frac{m-1}{m} V^2 \left( f_{\nu \nu} - \frac{f}{V} V_{\nu \nu} \right) \left( \frac{f}{V} \right)_\nu \right] d\sigma$$  

$$- \int_M \left[ \frac{m-1}{m} V^2 \langle \nabla \frac{f}{V}, \nabla \left( \Delta \phi \frac{f}{V} - \frac{A \phi V}{V} \right) \rangle + \hat{\text{Ric}}_\phi,m \left( V \nabla \frac{f}{V} , V \nabla \frac{f}{V} \right) \right] d\mu,$$

(2.4)

where the equality occurs if and only if (1.4) and (1.5) occur.

Now, we are in the position to complete the proof of Theorem 1.1. In the Dirichlet case, we let $f$ is a solution to

$$\Delta \phi f - \frac{A \phi V}{V} f = -\lambda \frac{f}{V} \text{ on } M, \quad f = 0 \text{ on } \partial M.$$

Since the above problem is equivalent to $e^\phi \text{div} \left( e^{-\phi} V^2 \nabla \frac{f}{V} \right) = -\lambda f$, the existence and uniqueness is due to the Fredholm alternative (for detail, see Remark 1.1 or page 512 in [10]). Hence, from (2.4), we obtain

$$0 \leq \int_{\partial M} \left[ -V H_V \left( \left( \frac{f}{V} \right)_\nu \right)^2 \right.$$  

$$+ \frac{m-1}{m} V^2 \left( f_{\nu \nu} - \frac{f}{V} V_{\nu \nu} \right) \left( \frac{f}{V} \right)_\nu \right] d\sigma$$  

$$- \int_M \left[ \frac{m-1}{m} V^2 \langle \nabla \frac{f}{V}, \nabla \left( \Delta \phi \frac{f}{V} - \frac{A \phi V}{V} \right) \rangle + \hat{\text{Ric}}_\phi,m \left( V \nabla \frac{f}{V} , V \nabla \frac{f}{V} \right) \right] d\mu$$

(2.5)

Since $f |_{\partial M} = 0$, it follows from (2.2) that $f_{\nu \nu} - \frac{f}{V} V_{\nu \nu} = -V H_V \left( \frac{f}{V} \right)_\nu$. Then (2.5) becomes

$$0 \leq - \int_{\partial M} V^3 H_V \left( \left( \frac{f}{V} \right)_\nu \right)^2 d\sigma + \frac{m-1}{m} (\lambda - mK) \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 d\mu$$

(2.6)

from $H_V \geq 0$, which shows that $\lambda \geq mKg$. When $\lambda = mK$, we have $\hat{\text{Ric}}_\phi,m = (m-1)K$ and all inequalities must be equal. If $m > n$, then (1.5) shows that

$$V \Delta f - f \Delta V + \frac{n}{m-n} V^2 \langle \nabla \phi, \nabla \frac{f}{V} \rangle = 0,$$
which is equivalent to
\[ e^{-\frac{n}{m-n}\phi} \left[ e^{\frac{n}{m-n}\phi} V^2 \left( \frac{f}{V} \right) \right]_i = \left[ V^2 \left( \frac{f}{V} \right) \right]_i + \frac{n}{m-n} V^2 \langle \nabla \phi, \nabla \frac{f}{V} \rangle \]
\[ = 0, \quad (2.7) \]
where we notice that \( V \Delta \frac{f}{V} = \Delta f - \frac{f}{V} \Delta V - 2 \langle \nabla V, \nabla \frac{f}{V} \rangle \). Multiplying both sides of (2.7) with \( \frac{f}{V} e^{\frac{n}{m-n}\phi} \) gives
\[ 0 = \int_M \frac{f}{V} \left[ e^{\frac{n}{m-n}\phi} V^2 \left( \frac{f}{V} \right) \right]_i \, dv_g \]
\[ = - \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 e^{\frac{n}{m-n}\phi} \, dv_g, \quad (2.8) \]
and then \( f = \theta V \), where \( \theta \) is a constant. This contradicts with that \( f \) is nontrivial, which shows that \( m = n \) and \( \phi \) must be constant. In this case, (1.4) implies
\[ \Delta f - \frac{\Delta \phi V}{V} f = -\lambda \frac{f}{V} \text{ on } M, \quad V \left( \frac{f}{V} \right)_\nu = 0 \text{ on } \partial M. \]
Then, from (2.4), we obtain
\[ 0 \leq - \int_{\partial M} II^V \left( V \nabla \frac{f}{V}, V \nabla \frac{f}{V} \right) \, d\sigma \]
\[ - \int_M \left[ \frac{m-1}{m} V^2 \left( \nabla \frac{f}{V}, \nabla \left( \Delta f - \frac{\Delta \phi V}{V} f \right) \right) + V \tilde{\text{Ric}}_{\phi,m} (V \nabla \frac{f}{V}, V \nabla \frac{f}{V}) \right] \, d\mu \]
\[ \leq \frac{m-1}{m} (\lambda - mk) \int_M V^2 \left| \nabla \frac{f}{V} \right|^2 \, d\mu \quad (2.10) \]
from \( II^V \geq 0 \), which shows that \( \lambda \geq mk \). Similarly, we can also prove that \( m = n \) and \( \phi \) must be constant.

2.2. **Proof of Theorem 1.2**

(1) In the Dirichlet case. We assume that \( \phi = 0 \) on \( \partial M \), and solve the following Dirichlet poisson equation:
\[ \Delta \phi f - \frac{\Delta \phi V}{V} f = \varphi \text{ on } M, \quad f = 0 \text{ on } \partial M. \]
Thus, it follows from (1.3) that
\[ \frac{m-1}{m} \int_M V \left( \Delta \phi f - \frac{\Delta \phi V}{V} f \right)^2 \, d\mu \geq \int_M V^3 \langle \tilde{\text{Ric}}_{\phi,m} (V \nabla \frac{f}{V}), V \nabla \frac{f}{V} \rangle \, d\mu, \quad (2.11) \]
which is equivalent to
\[ \frac{m-1}{m} \int_M V \varphi^2 \, d\mu \geq \int_M V^3 \langle \tilde{\text{Ric}}_{\phi,m} (V \nabla \frac{f}{V}), V \nabla \frac{f}{V} \rangle \, d\mu. \quad (2.12) \]
Using the divergence theorem, we have
\[
\int_M V \phi^2 \, d\mu = \int_M \phi (V \Delta_\phi f - f \Delta_\phi V) \, d\mu \\
= - \int_M V^2 (\nabla \phi, \nabla f) \, d\mu \\
\leq \left[ \int_M V^3 (\hat{\text{Ric}}_{\phi,m} (\nabla f/V), \nabla f/V) \, d\mu \right]^{1/2} \left[ \int_M V ((\hat{\text{Ric}}_{\phi,m})^{-1} \nabla \phi, \nabla \phi) \, d\mu \right]^{1/2}.
\]

Thus, putting (2.12) into (2.13) gives the desired estimate (1.6).

(2) In the Neumann case, for any \( \phi \in C^\infty(M) \) with \( \int_M V \phi \, d\mu = 0 \), there exists a function \( f \) such that
\[
\Delta_\phi f - \frac{\Delta_\phi V}{V} f = \phi \text{ on } M, \quad V \left( \frac{f}{V} \right)_\nu = 0 \text{ on } \partial M.
\]
Thus, by using (1.3), we also have that (2.12) holds. Using the Neumann condition, we also have that (2.13) is true and the estimate
\[
\frac{m}{m-1} \int_M V \phi^2 \, d\mu \leq \int_M V ((\hat{\text{Ric}}_{\phi,m})^{-1} \nabla \phi, \nabla \phi) \, d\mu
\]
follows by inserting (2.12) into (2.13). In particular, we let
\[
\tilde{\phi} = \phi - \frac{\int_M V \phi \, d\mu}{\int_M V \, d\mu}.
\]
Then we have \( \int_M V \tilde{\phi} \, d\mu = 0 \) and the estimate (1.7) follows.

2.3. **Proof of Theorem 1.3.** We consider the following Neumann problem:
\[
\Delta_\phi f - \frac{\Delta_\phi V}{V} f = C \text{ on } M, \quad V \left( \frac{f}{V} \right)_\nu = \phi \text{ on } \partial M,
\]
which implies that
\[
C = \frac{\int_{\partial M} V \phi \, d\sigma}{\int_M V \, d\mu}.
\]
Then, from (1.3) we obtain
\[
\frac{m-1}{m} C^2 \int_M V \, d\mu \geq \int_{\partial M} \left[ V V^2 (V \nabla f/V, V \nabla f/V) + VH_\phi \phi^2 \\
+ 2V \phi \left( \Delta_\phi f - \frac{\Delta_\phi V}{V} f \right) \right] \, d\sigma \\
= \int_{\partial M} \left[ V^3 (\nabla f/V, \nabla f/V) + VH_\phi \phi^2 \\
- 2V^2 (\nabla \phi, \nabla f/V) \right] \, d\sigma.
\]
Applying the inequality
\[
-2V^2 (\nabla \phi, \nabla f/V) \geq -V^3 (\nabla f/V, \nabla f/V) - V((1V)^{-1} \nabla \phi, \nabla \phi)
\]
in (2.15) yields
\[ m - 1 \int_M V \, d\mu \geq \int_{\partial M} \left[ VH\phi\varphi^2 - V(VH\phi \nabla\varphi, \nabla\varphi) \right] \, d\sigma, \tag{2.16} \]
which is equivalent to (1.9).

2.4. \textbf{Proof of Theorem 1.4}. We consider the following Dirichlet problem:
\[ \Delta \phi f - \frac{\Delta \phi V}{V} f + t \frac{f}{V} = 0 \text{ on } M, \quad f = \varphi \text{ on } \partial M, \]
where \( t \leq mK \). Then (2.4) shows that
\[ 0 \geq \int_{\partial M} \left[ V^3H\phi\left( \frac{f}{V} \right) \right]^2 + V^2 \left( \varphi - \Delta \phi \frac{f}{V} \right) \, d\sigma \]
and then (2.17) becomes
\[ 0 \geq \int_{\partial M} \left[ V^3H\phi\left( \frac{f}{V} \right) \right]^2 + V^2 \left( \varphi - \Delta \phi \frac{f}{V} \right) \, d\sigma. \tag{2.19} \]

Using the inequality \( ax^2 + bx \geq -\frac{b^2}{4a} \) with \( a > 0 \), we obtain
\[ V^3H\phi\left( \frac{f}{V} \right) \geq - \frac{1}{H\phi} V \left[ \Delta \phi \varphi - \Delta \phi \frac{f}{V} \right] \left( \frac{f}{V} \right) \, d\sigma. \tag{2.20} \]
Thus, putting (2.20) into (2.19) completes the proof of Theorem 1.4.
3. Further remarks

Let \( \Omega^n \subset \mathbb{H}^n (\mathbb{S}^n_+, \text{respectively}) \) be a compact domain with smooth boundary \( \partial \Omega \). Let \( V = \cosh r, K = -1 \) or \( V = \cos r, K = 1 \) for the case \( H^n \) or \( S^n_+ \), respectively, where \( r(x) = \text{dist}(x, p) \) is the distance function for the fixed point \( p \in \mathbb{H}^n (p \in \mathbb{S}^n_+, \text{respectively}) \), see \([12]\). Then we have

\[
\nabla^2 V = -KV g.
\] (3.1)

In this case, we obtain

\[
\hat{\text{Ric}}_V := \hat{\text{Ric}}_{0,n} = (\Delta V)g - \nabla^2 V + \text{Ric} = 0.
\]

Hence, Theorems 1.3 and 1.4 can be stated as follows:

**Theorem 3.1.** Let \( \Omega \subset \mathbb{H}^n (\mathbb{S}^n_+, \text{respectively}) \) be a compact domain with smooth boundary \( \partial \Omega \) and \( V \) defined as above. If \( II^V > 0 \) on \( \partial \Omega \), then for any \( \varphi \in C^\infty(\partial \Omega) \), we have

\[
\int_\Omega V \, dv_g \int_{\partial \Omega} V H \varphi^2 \, dv_\mathbb{H} - \frac{n-1}{n} \left( \int_{\partial \Omega} V \varphi \, dv_\mathbb{H} \right)^2 \leq \int_\Omega V \, dv_g \int_{\partial \Omega} V (II^V)^{-1} \nabla \varphi, \nabla \varphi \, dv_\mathbb{H}. \quad (3.2)
\]

**Theorem 3.2.** Let \( \Omega \subset \mathbb{H}^n (\mathbb{S}^n_+, \text{respectively}) \) be a compact domain with smooth boundary \( \partial \Omega \) and \( V \) defined as above. If \( H > 0 \) on \( \partial \Omega \), then for any \( \varphi \in C^\infty(\partial \Omega) \), we have

\[
\int_{\partial \Omega} II^V \left( \nabla \varphi, \nabla \varphi \right) \, dv_\mathbb{H} \leq \int_{\partial \Omega} \frac{1}{H} V \left( \Delta \varphi - \frac{\Delta V}{V} \varphi \right)^2 \, dv_\mathbb{H}. \quad (3.3)
\]

In particular, taking \( \varphi = 1 \) in (3.2), we have

\[
\int_\Omega V \, dv_g \int_{\partial \Omega} VH \, dv_\mathbb{H} \leq \frac{n-1}{n} \left( \int_{\partial \Omega} V \, dv_\mathbb{H} \right)^2, \quad (3.4)
\]

which is exactly the formula (5) of Theorem 1.1 in \([12]\). Therefore, our Theorem 3.1 generalizes Theorem 1.1 of Xia in \([12]\).

On the other hand, for a closed hypersurface \( x : (\Sigma, g) \to \mathbb{H}^{n+1} (\mathbb{S}^{n+1}_+, \text{respectively}) \) and \( V \) defined as above, the second fundamental form of \( x \) defined by

\[
II_x^V = II_x - (\ln V)_\nu g
\] (3.5)
satisfies \( II_x^V \geq 0 \). Moreover, in the case of \( \mathbb{H}^{n+1} \) we assume that \( II_x^V > 0 \) and \( \Sigma \) is horo-convex and in the case of \( \mathbb{S}^{n+1}_+ \) we assume that \( \Sigma \) is convex, then we have

\[
\hat{\text{Ric}}_x^V := (\Delta V)g - \nabla V + \text{Ric}_\Sigma > 0,
\] (3.6)

where \( \text{Ric}_\Sigma \) is the Ricci curvature of \( \Sigma \) (see page 515 in \([10]\)).

Hence, applying Theorem 1.2 on \( \Sigma \) yields the following:

**Theorem 3.3.** Let \( x : (\Sigma, g) \to \mathbb{H}^{n+1} (\mathbb{S}^{n+1}_+, \text{respectively}) \) be a closed hypersurface. In the case of \( \mathbb{H}^{n+1} \) we assume that \( II_x^V > 0 \) and \( \Sigma \) is horo-convex.
In the case of $S_{+}^{n+1}$ we assume that $\Sigma$ is convex. Then for any $\varphi \in C^{\infty}(\Sigma)$, the following inequality holds:

$$\frac{n}{n-1} \int_{\Sigma} V \left( \varphi - \frac{\int_{M} V \varphi \, d\mu}{\int_{M} V \, d\mu} \right)^{2} \, dv_{g} \leq \int_{\Sigma} V \langle (\widehat{\text{Ric}}_{x})^{-1} \nabla \varphi, \nabla \varphi \rangle \, dv_{g}. \quad (3.7)$$

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