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Spin structures on generalized real Bott manifolds

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Abstract: In this paper, we give a necessary and sufficient condition for a generalized real Bott manifold to have a spin structure in terms of column vectors of the associated matrix. We also give an interpretation of this result to the associated acyclic \(\omega\)-weighted digraphs. Using this, we obtain a family of real Bott manifolds that does not admit spin structure.

Key words: Generalized Bott manifold, small cover, acyclic digraph

1. Introduction

A generalized real Bott tower of height \(k\) is a sequence of real projective bundles

\[
B_k \longrightarrow B_{k-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow \{pt\}
\]  

(1.1)

where \(B_i\) is the projectivization of the Whitney sum of \(n_i + 1\) real line bundles over \(B_{i-1}\). This notion is introduced by Choi et al. [3] as a generalization of the notion of a Bott tower given in [8]. The manifold \(B_k\) is called a real Bott manifold when \(n_i = 1\) for each \(i\) and a generalized real Bott manifold, otherwise. The manifold \(B_k\) can be realized as a small cover over \(\prod_{i=1}^k \Delta^{n_i}\) where \(\Delta^{n_i}\) is the \(n_i\)-simplex [10, Corollary 4.6]. It is also known that every small cover over a product of simplices is a generalized real Bott manifold [3, Remark 6.5].

Let \(P\) be a simple convex polytope of dimension \(n\) with the facet set \(\mathcal{F}(P) = \{F_1, \cdots, F_m\}\). For every small cover \(M\) over \(P\), there is an associated \((n \times (m - n))\) matrix \(A = [a_{ij}]\) with entries in \(\mathbb{Z}_2\) which can be used to reconstruct \(M\) (see Section 2). Moreover, the mod 2 cohomology ring structure of \(M\) depends only on the face poset of \(P\) and the matrix \(A\). More precisely, let \(\mathbb{Z}_2[\mathcal{F}(P)]\) be the Stanley-Reisner ring of \(P\), that is, the quotient of the polynomial ring \(\mathbb{Z}_2[x_1, \cdots, x_m]\) with the ideal \(I\) generated by the square free monomials \(x_{i_1} \cdots x_{i_r}\) for which \(F_{i_1} \cap \cdots \cap F_{i_r}\) is empty. There is a graded ring isomorphism between \(H^* (M, \mathbb{Z}_2)\) and \(\mathbb{Z}_2[\mathcal{F}(P)]/J\) where \(J\) is the homogeneous ideal generated by the monomials \(x_i + \sum_{j=1}^{m-n} a_{ij}x_{n+j}\), [4, Theorem 4.14].

Here the degree of \(x_i\) is 1. In [4, Corollary 6.8], Davis and Januskiewicz show that the total Stiefel-Whitney
class of \(P\) is given by
\[
w(M) = \left( \prod_{i=1}^{m-n} (1 + x_{n+i}) \right) \cdot \left( \prod_{i=1}^{n} (1 + \sum_{j=1}^{m-n} a_{ij}x_{n+j}) \right) \mod I.
\] (1.2)

Therefore, the coefficient of \(x_i\) in the first Stiefel-Whitney class of \(M\) is one more than the sum of the entries of the \((i-n)\)-th column of \(A\) when \(i > n\) and zero, otherwise. Hence, the small cover \(M\) is orientable if and only if the sum of the entries of the \(i\)-th column of the matrix \(A\) is congruent to 1 modulo 2 for each \(i \geq 1\) [12, Theorem, 1.7]. Since \(A\) is a matrix over \(\mathbb{Z}_2\), the sum of the entries of the \(j\)-th column of \(A\) is equivalent to the dot product of the column vector with itself. Therefore, the small cover \(M\) is orientable if and only if \(A_j \cdot A_j \equiv 1 \mod 2\), where \(A_j\) denote the \(j\)-th column vector of \(A\).

In Section 2, we observe that a small cover \(M\) has a spin structure when \(A_i \cdot A_i \equiv 3 \mod 4\) and \(A_i \cdot A_j \equiv 0 \mod 2\) for all \(1 \leq i < j \leq m - n\) (Corollary 2.4). It turns out that when \(P\) is a product of simplices of dimensions greater than 1, the converse is also true (Corollary 3.2). In other words, when each \(B_i\) is a projectivization of the Whitney sum of \(3\) or more line bundles, the generalized Bott manifold \(B_k\) has a spin structure if and only if \(A_i \cdot A_i \equiv 3 \mod 4\) and \(A_i \cdot A_j \equiv 0 \mod 2\) for all \(1 \leq i < j \leq m - n\). In Theorem 3.1, we give a criterion for an arbitrary generalized Bott manifold \(B_k\) to have a spin structure. It is equivalent to the criterion given in [6].

In [7, Lemma 2.1], Gasior gives a formula for the second Stiefel-Whitney class of \(M(A)\) in terms of the second Stiefel-Whitney classes of \(M(A_{ij})\), where \(A_{ij}\) is an \(n \times n\) matrix whose \(k\)-th column is \(A_k\) if \(k = i, j\) and 0 otherwise, called an elementary component. After reducing the problem to elementary components, the author gives a necessary and sufficient condition on existence of a spin structure on them in [7, Theorem 1.2] which can also be obtained as a corollary of Theorem 3.1. Moreover, Proposition 3.7 is a generalization of this result to the generalized real Bott manifolds.

It is well-known that real Bott manifolds can be classified by acyclic digraphs [3]. In [5, Theorem 4.5], Dsouza gives a necessary and sufficient condition on the associated digraph for a given real Bott manifold to have a spin structure. In [9], Güçlükan İlhan and Gürbüz give that for every generalized Bott manifold \(B_k\), there is an associated acyclic digraph \(D_{B_k}\) on labeled vertices \(\{v_1, \ldots, v_k\}\) where each edge from a vertex \(v_i\) has a vector weight in \(\mathbb{Z}_2^{n_i}\). In Section 4, we generalize the condition given by Dsouza and Uma to a condition on \(D_{B_k}\) for the associated generalized Bott tower \(B_k\) to have a spin structure (Theorem 4.2).

The Wu formula implies that \(w_3(M) = 0\) whenever \(w_1(M)\) and \(w_2(M)\) are zero. Therefore, the result of Section 3 gives us sufficient conditions for \(w_3(M)\) to be zero. In Section 5, we obtain a formula for \(w_3(M)\) when \(M\) is a small cover over a product of simplices of dimensions greater than or equal to 3. As a corollary, we give necessary conditions for the vanishing of the third Stiefel-Whitney class of \(M\). We obtain similar results for \(w_4\) and we classify small covers over a product of simplices of dimensions greater than or equal to 4 whose first four Stiefel-Whitney classes are zero.

2. Small covers
Let \(P\) be an \(n\)-dimensional simple convex polytope and \(\mathcal{F}(P) = \{F_1, F_2, \ldots, F_m\}\) be the set of facets of \(P\). A small cover over \(P\) is an \(n\)-dimensional smooth closed manifold \(M\) with a locally standard \(\mathbb{Z}_2^n\)-action whose orbit space is \(P\). Two small covers \(M_1\) and \(M_2\) over \(P\) are said to be Davis-Jansukiewicz equivalent.
if there is a weakly \( \mathbb{Z}_2^n \)-equivariant homeomorphism between \( M_1 \) and \( M_2 \) covering the identity on \( P \). The Davis-Januskiewicz classes of small covers over \( P \) are given by the characteristic functions.

A characteristic function \( \lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n \) over \( P \) is a \( \mathbb{Z}_2^n \)-coloring function satisfying the following nonsingularity condition:

\[
F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset \quad \Rightarrow \quad \langle \lambda(F_{i_1}), \ldots, \lambda(F_{i_n}) \rangle = \mathbb{Z}_2^n.
\]

In [3], Davis and Janueskiewicz construct a small cover \( M(\lambda) \) associated to a given characteristic function \( \lambda \) as the quotient space of the space \( (P \times \mathbb{Z}_2^n) \) and the equivalence relation defined by

\[
(p, g) \sim (q, h) \quad \text{if} \quad p = q \quad \text{and} \quad g^{-1}h \in \langle \lambda(F_{i_1}), \ldots, \lambda(F_{i_k}) \rangle
\]

where the intersection \( \bigcap_{j=1}^k F_{i_j} \) is the minimal face containing \( p \) in its relative interior.

**Theorem 2.1** [3, Proposition 1.8] For every small cover \( M \) over \( P \), there is a characteristic function \( \lambda \) with \( \mathbb{Z}_2^n \)-homeomorphism \( M(\lambda) \to M \) covering the identity on \( P \).

The group \( GL(n, \mathbb{Z}_2) \) acts freely on the set of characteristic functions over \( P \) by composition. Moreover, the orbit space of this action is in one-to-one correspondence with the Davis-Januszkiewicz equivalence classes of small covers over \( P \). Fix a basis \( e_1, \ldots, e_n \) for \( \mathbb{Z}_2^n \) and reorder facets of \( P \) in such a way that \( \bigcap_{i=1}^n F_i \neq \emptyset \).

By the above theorem, for a given small cover \( M \) over \( P \), there is an \( (n \times (m - n)) \)-matrix \( A = [a_{ij}] \) such that \( M \) and \( M(\lambda) \) are Davis-Januszkiewicz equivalent where

\[
\lambda(F_{i}) = \begin{cases} 
eq e_i, & i \leq n \\ \sum_j a_{ij} e_j, & i > n. \end{cases}
\]

**Theorem 2.2** (Theorem 4.14, [3]) The mod 2 cohomology ring of \( M \) is \( \mathbb{Z}[P]/J \), where \( J \) is the homogeneous ideal generated by the monomials \( x_i + \sum_{j=1}^{m-n} a_{ij} x_{n+j} \).

Let \( w_i(M) \) and \( w(M) \) denote the \( i \)-th and the total Stiefel-Whitney classes of \( M \), respectively. By Corollary 6.8 in [3], the total-Stiefel Whitney class of a small cover over \( M \) is given by the equation (1.2). Let \( A_j \) denote the \( j \)-th column vector of \( A \). Then the first Stiefel-Whitney class of \( M \) is given by the following formula

\[
w_1(M) = \sum_{i=1}^{m-n} \left( 1 + \sum_j a_{ji} \right) x_{i+n} = \sum_{i=1}^{m-n} (1 + A_i \cdot A_i) \cdot x_{i+n}
\]

since \( a_{ji}^2 = a_{ji} \). Hence, \( M \) is orientable if and only if \( A_i \cdot A_i \equiv 1 \) (mod 2) for all \( 1 \leq i \leq m - n \). By comparing the degree 2-terms in each side of the equation (1.2), one obtains a similar formula for the second Stiefel-Whitney class of \( M \).
Proposition 2.3 The second Stiefel-Whitney class of $M$ is
\[
w_2(M) = \sum_{i=1}^{m-n} \alpha_i \cdot x_{i+n}^2 + \sum_{1 \leq i < j \leq m-n} \beta_{ij} \cdot x_{i+n} \cdot x_{j+n} \quad \text{(mod 2)}
\]  
(2.1)

where $\alpha_i = \left(1 + A_i \cdot A_i\right)$ and $\beta_{ij} = (1 + A_i \cdot A_i)(1 + A_j \cdot A_j) + A_i \cdot A_j$.

Proof The coefficient of $x_{i+n}^2$ in the equation (1.2) equals the coefficient of $y^2$ in $(1 + y)\left(\prod_j (1 + a_jy)\right)$, which is the $(k_i + 1)$-th power of $1 + y$, where $k_i$ is the number of 1s in $A_i$. Since the entries of $A_i$ are either 0 or 1, the number of 1’s in $A_i$ is equal to $A_i \cdot A_i$. Hence, the coefficient of $x_{i+n}^2$ in (1.2) is $(1 + A_i \cdot A_i)$.

To find $\beta_{ij}$, first note that $|\{t \mid a_{ti} = a_{tj} = 1\}| = A_i \cdot A_j$. Therefore, $\beta_{ij}$ is equal to the coefficient of $y_iy_j$ in the product
\[
(1 + y_i)^{(A_i \cdot A_i - A_i \cdot A_j + 1)}(1 + y_j)^{(A_j \cdot A_j - A_i \cdot A_j + 1)}(1 + y_i + y_j)^{A_{ij}}.
\]
Hence, we have
\[
\beta_{ij} = (A_i \cdot A_i - A_i \cdot A_j + 1)(A_j \cdot A_j + 1) + A_{ij}(A_j \cdot A_j) = (1 + A_i \cdot A_i)(1 + A_j \cdot A_j) - A_i \cdot A_j.
\]

Since we work with $\mathbb{F}_2$ coefficients, the result follows. \qed

Corollary 2.4 Let $M$ be a small cover over $P$ with an associated reduced matrix $A$. If $A_i \cdot A_i \equiv 3 \pmod{4}$ and $A_i \cdot A_j \equiv 0 \pmod{2}$ for all possible $i < j$ then $M$ has a spin structure.

3. Existence of spin structure

In this section, we give a necessary and sufficient condition for the existence of spin structure for generalized Bott manifolds. Let $B_k$ be a generalized real Bott manifold given in (1.1). One can realize $B_k$ as a small cover over $P = \prod_{i=1}^{k} \Delta^{n_i}$, where $\sum_{i=1}^{k} n_i = n$. The facets of $P$ is given by the following set
\[
\mathcal{F} = \{F^i_j = \Delta^{n_1} \times \cdots \times \Delta^{n_{i-1}} \times f^i_j \times \Delta^{n_{i+1}} \times \cdots \times \Delta^{n_k} \mid 1 \leq i \leq k, \ 0 \leq j \leq n_i\},
\]
where $\{f^i_0, \ldots, f^i_{n_i}\}$ is the set of facets of the simplex $\Delta^{n_i}$. Note that $P$ has $(n+k)$-facets and the intersection $\bigcap_{j \neq 0} F^i_j$ is nonempty. Hence, $B_k$ can be represented by a $(n \times k)$ matrix $A = [a_{ij}]$ by choosing $F_i = F^i_j$ for $l = n_1 + \cdots + n_{i-1} + j$ and $1 \leq j \leq n_i$ and $F_i = F^i_0$ for $l = n + i$. Following [2, 3], one can see $A$ as a $(k \times k)$ vector matrix $A = [v_{ij}]$ where $v_{ij} \in \mathbb{Z}_2^{n_i}$. Here $v_{ij}$ is the column vector whose $l$-th entry is $a_{n_1 + \cdots + n_{i-1} + 1, l}$.

Note that facets in $\mathcal{F} \setminus \{F^i_j, \cdots, F^k_j\}$ intersect at a vertex for every $0 \leq j_i \leq n_i$ and $1 \leq i \leq k$. Moreover, a family of facets containing the set $\{F^0_0, \cdots, F^i_{n_i}\}$ has an empty intersection for any $1 \leq i \leq k$. Let $A_{l_1 \cdots l_k}$ be a $(k \times k)$ matrix whose $j$-th row is the $l_j$-th row of $A$ for $1 \leq l_i \leq n_i$ and $1 \leq i \leq k$. In [3], using these facts, it is shown that the characteristic function corresponding to $A$ satisfies the nonsingularity condition if
and only if every principal minor of $A_{i_1 \ldots i_k}$ is 1 for all $1 \leq l_i \leq n_i$ and $1 \leq i \leq k$. This forces $(v_{ij})_t = 1$ for all $1 \leq i \leq k$ and $1 \leq t \leq n_i$.

Note that the Stanley-Reisner ring of $P$ is

$$\mathbb{Z}_2[x_{i0}, \ldots, x_{in_1}, \ldots, x_{kn}] / I$$

where $I$ is the homogeneous ideal generated by monomial products $x_{i0} \cdots x_{in_1}$, $1 \leq i \leq k$. In this notation, $x_{ij}$ corresponds to $x_{n_1 + \ldots + n_i - 1 + j}$ when $1 \leq j \leq n_i$ and to $x_{n+i}$ when $j = 0$ in the equation (1.2). Therefore, the second Stiefel-Whitney class of $B_k$ is equal to

$$w_2(M) = \sum_{i=1}^{k} \alpha_i \cdot x_{i0}^2 + \sum_{1 \leq i < j \leq k} \beta_{ij} \cdot x_{i0} \cdot x_{j0}$$

modulo $I$ where $\alpha_i$ and $\beta_{ij}$ are as given in Proposition 2.3. From now on, we assume that $n_i = 1$ for $1 \leq i \leq l$ and $n_i > 1$, otherwise. This means that the only relations involving the monomials of degree 2 are

$$x_{i0}^2 = \sum_{j \neq i} v_{ij} \cdot x_{i0} \cdot x_{j0}$$

for $1 \leq i \leq l$ (here, the vector $v_{ij} \in \mathbb{Z}_2$ is considered a scalar). Therefore, we have

$$w_2(M) = \sum_{i=1}^{k} \alpha_i \cdot x_{i0}^2 + \sum_{1 \leq i < j \leq k} \beta_{ij} \cdot x_{i0} \cdot x_{j0} + \sum_{i < j \leq l} (\beta_{ij} + v_{ij} \cdot \alpha_i + v_{ji} \cdot \alpha_j) \cdot x_{i0} \cdot x_{j0} + \sum_{i < l+1 \leq j \leq k} (\beta_{ij} + v_{ij} \cdot \alpha_i) \cdot x_{i0} \cdot x_{j0}$$

(3.1)

**Theorem 3.1** The generalized real Bott manifold $B_k$ has a spin structure if and only if the following conditions are satisfied:

i) $A_i \cdot A_i \equiv 1 \pmod{2}$ when $i \leq l$ and $A_i \cdot A_i \equiv 3 \pmod{4}$; otherwise,

ii) $A_i \cdot A_j \equiv 0 \pmod{2}$ for all $l \leq i < j \leq k$,

iii) $A_i \cdot A_j$ and $\frac{v_{ij} \cdot (A_i \cdot A_i + 1) + v_{ji} \cdot (A_j \cdot A_j + 1)}{2}$ have the same parity when $1 \leq i < j \leq l$.

iv) $A_i \cdot A_j$ and $\frac{v_{ij} \cdot (A_i \cdot A_i + 1)}{2}$ have the same parity when $1 \leq i < l+1 \leq j \leq k$.

**Proof** The manifold $B_k$ has a spin structure if and only if it is orientable and $w_2$ vanishes. Recall that the manifold $B_k$ is orientable if and only if $A_i \cdot A_i$ is congruent to 1 modulo 2. In this case, $\beta_{ij} \equiv A_i \cdot A_j$ modulo 2. Then the theorem follows from the equation (3.1) and the fact that $\frac{(1+A_i \cdot A_j)}{2}$ have the same parity with $A_i \cdot A_j + 1 \frac{1}{2}$. □
It is well-known that the vector matrix \( A \) is equivalent to an upper triangular one in which the entries of the diagonal vectors are all 1 via conjugation by a permutation matrix [3, Lemma 5.1]. Under this assumption, the above theorem is equivalent to the [6, Theorem 4.7]. In [6, Theorem 4.7], the ordering in the product is chosen so that the last \( k-l \) of the simplices have dimension 1. Moreover, \( T_s \) and \( T_{rs} \) in [6, Theorem 4.7] are equivalent to \( (A_s \cdot A_s) \) and \( A_r \cdot A_s \), respectively and the orientability condition is equivalent to \( A_s \cdot A_s \equiv 1 \) (mod 2).

By Theorem 3.1, it follows that the converse of Corollary 2.4 is also true when \( l = 0 \).

**Corollary 3.2** The generalized real Bott manifold with \( l = 0 \) has a spin structure if and only if \( A_i \cdot A_i \equiv 3 \) (mod 4) and \( A_i \cdot A_j \equiv 0 \) (mod 2) for all \( 1 \leq i < j \leq k \), where \( A \) is the reduced matrix.

**Example 3.3** Let \( P = \Delta^2 \times \Delta^3 \times \Delta^5 \) and \( B \) be a 3-step generalized Bott manifold corresponding to the reduced matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

Then \( B \) has a spin structure by the above Corollary.

The following corollary also follows from the Proposition 5.1 of [13].

**Corollary 3.4** If a generalized real Bott manifold over \( P = \prod_{t=1}^{k} \Delta^{n_t} \) with \( l = 0 \) admits a spin structure, then \( n_j \equiv 3 \) (mod 4) for some \( j \).

**Proof** Let \( P = \prod_{t=1}^{k} \Delta^{n_t} \) and \( M \) be a small cover over \( P \) with an associated vector matrix \( A \). If \( B \) is a vector matrix obtained by conjugating \( A \) via permutation matrix \( P_\sigma \) then \( A_i \cdot A_i = B_{\sigma(i)} \cdot B_{\sigma(i)} \) and \( A_i \cdot A_j = B_{\sigma(i)} \cdot B_{\sigma(j)} \). Therefore, we can assume that \( A \) is an upper triangular vector matrix in which the entries of the diagonal vectors are all 1. Then we have \( A_1 \cdot A_1 = n_1 \). So if \( M \) has a spin structure, \( n_1 \equiv 3 \) (mod 4).

When \( l = k \), we have the following result.

**Corollary 3.5** The real Bott manifold \( B_k \) has a spin structure if and only if

i) \( A_i \cdot A_i \equiv 1 \) (mod 2), \( 1 \leq i \leq k \),

ii) \( A_i \cdot A_j \) and \( \frac{v_{ij} \cdot (A_i \cdot A_i + 1) + v_{ji} \cdot (A_j \cdot A_j + 1)}{2} \) have the same parity when \( 1 \leq i < j \leq l \).
The above corollary is equivalent to Theorem 3.2 in [5] where $A$ is assumed to be upper-triangular. In particular, Theorem 1.2 in [7] directly follows from the corollary.

**Example 3.6** Let $P = I \times \Delta^2 \times \Delta^2$. Then a small cover $B_3$ over $P$ corresponds to a vector matrix

$$A = \begin{bmatrix}
1 & a_{12} & a_{13} \\
 a_{21} & 1 & a_{23} \\
 a_{31} & a_{32} & 1 \\
a_{41} & a_{42} & 1 \\
a_{51} & a_{52} & 1 \\
\end{bmatrix}.$$  

If $B_3$ has a spin structure, then $a_{12} + a_{42} + a_{52} = 1$ and $a_{13} + a_{23} + a_{33} = 1$ by part i of Theorem 3.1 and $a_{12}a_{13} + a_{23} + a_{33} + a_{42} + a_{52} \equiv 0 \pmod{2}$ by part ii of Theorem 3.1. By substituting the first two equations to the last one, we get $a_{12} = a_{13} = 0$ and hence $a_{42} + a_{52} = a_{23} + a_{33} = 1$. On the other hand, at least one of the vectors $\begin{pmatrix} a_{12} \\ a_{52} \end{pmatrix}$ and $\begin{pmatrix} a_{23} \\ a_{33} \end{pmatrix}$ must be zero by the nonsingularity condition. Hence, there is no small cover over $I \times \Delta^2 \times \Delta^2$ with a spin structure when $n \geq 2$.

It is well-known that when $n_i$’s are all even, there is no orientable small cover over $P$ [1]. Hence, small covers over $P$ have no spin structures when all the $n_i$’s are even. In the next section, we generalize the above example to have a nonexistence result for every small cover over $P = I \times \Delta^{2n_1} \times \cdots \times \Delta^{2n_k}$ for $k \geq 2$. When $k = 1$, a small cover over $I \times \Delta^4$ does not have a spin structure since $A_2 \cdot A_2$ is either $4t$ or $4t + 1$. However, the small cover over $P = I \times \Delta^{4t+2}$ corresponding to a characteristic function $\lambda$ which sends $F_0^1$ to $e_1$ and $F_0^2$ to $e_1 + e_2 + \cdots + e_{4t+3}$ has a spin structure.

Given a dimension function $\omega : \{1, 2, \ldots, n\} \to \mathbb{N}$, let $I_\omega$ be the identity vector matrix associated to $\omega$, i.e. the $(i, j)$-entry of $I_\omega$ is 1 when $\omega(1) + \cdots + \omega(j - 1) + 1 \leq i \leq \omega(1) + \cdots + \omega(j)$, and 0, otherwise. To generalize Theorem 1.2 in [7] to our case, we denote the matrix $A - I_\omega$, where $\omega(i) = n_i$ by $B$.

**Proposition 3.7** The generalized real Bott manifold with an associated matrix $B$ has a spin structure if and only if for all $1 \leq i < j \leq k$, the generalized Bott manifold corresponding to $B_{ij}$ has a spin structure, where $B_{ij}$ is the vector matrix whose $l$-th column is $B_l$ if $l = i, j$ and 0, otherwise.

4. $\omega$-weighted digraph interpretation

In [2], Choi shows that there is a bijection between the set of real Bott manifolds and acyclic digraphs with $n$-labeled vertices which sends $B_k$ to a graph whose adjacency matrix is $A - I_k$. In [5, Theorem 4.5], Dsouza and Uma give an interpretation of existence of a spin structure for real Bott manifolds in terms of associated digraphs. In this section, we generalize [5, Theorem 4.5] to small covers over a product of simplices.

**Definition 4.1** Given a dimension function $\omega : V \to \mathbb{N}$, a digraph with vertex set $V$ is called $\omega$-vector weighted if every edge $(u, v)$ is assigned a nonzero vector $w(u, v)$ in $\mathbb{Z}_2^{\omega(u)}$.

Let $G$ be a $\omega$-vector weighted digraph. For convenience, we take the weight of $(u, v)$ to be the zero vector in $\mathbb{Z}_2^{\omega(u)}$ when there is no edge from $u$ to $v$. If $(u, v)$ is an edge of $G$, then $u$ is called an in-neighbor of
v and v is called an out-neighbor of u. Let \( N_G^-(v) \) and \( N_G^+(v) \) denote the set of in-neighbors and out-neighbors of v in G. We define in-degree \( \deg^-(v) \) and out degree \( \deg^+(v) \) of v as follows:

\[
\deg^-(v) = \sum_{u \in N_G^-(v)} w(u, v) \cdot w(u, v)
\]

\[
\deg^+(v) = \sum_{z \in N_G^+(v)} w(v, z) \cdot w(v, z).
\]

We can consider a digraph as an \( \omega \)-weighted digraph with \( \omega(i) = 1 \) for each i. In this case, the notion of in-degree and out-degree of a vertex of a \( \omega \)-weighted digraph agrees with those of digraphs. An adjacency matrix \( A\omega(G) \) of an \( \omega \)-weighted digraph G with labeled vertices \( v_1, \ldots, v_n \) is defined to be an \((n \times n)\) \( \omega \)-vector matrix whose \((i, j)\)-th entry is \( w(v_i, v_j) \). An \( \omega \)-vector weighted digraph is called acyclic if it does not contain any directed cycle.

As shown in [9], there is a one-to-one correspondence between the set of small covers over the product \( P = \Delta^n \times \cdots \times \Delta^n \) and the set of acyclic \( \omega \)-weighted digraphs where \( \omega : \{v_1, \ldots, v_k\} \rightarrow \mathbb{N} \) is defined by \( \omega(v_i) = n_i \). The correspondence is obtained by sending a small cover with an associated matrix A to a \( \omega \)-weighted digraph whose adjacency matrix is \( A - I_\omega \). For a given small cover \( B \) over \( P \), we denote the associated acyclic \( \omega \)-weighted digraph by \( D_B \). Recall that the dot product of a vector v over \( \mathbb{Z}_2 \) with itself is equal to the number of nonzero coordinates of v. Therefore, \( A_i \cdot A_j \) is equal to \( A_\omega(D_B)_i \cdot A_\omega(D_B)_j + \omega(i) \) when \( i = j \) and \( A_\omega(D_B)_i \cdot (A_\omega(D_B)_j + w(v_i, v_j) \cdot w(v_i, v_j) + w(v_j, v_i) \cdot w(v_j, v_i), otherwise. Moreover, \( A_\omega(D_B)_i \cdot A_\omega(D_B)_j \) is equal to \( \deg^-(v_i) \). Let \( M_{ij} \) be the sum of \( \omega(u, v_i) \cdot \omega(u, v_j) \) where u runs in the set of in-neighbor of both \( v_i \) and \( v_j \). Then \( A_\omega(D_B)_i \cdot A_\omega(D_B)_j = M_{ij} \).

**Theorem 4.2** The generalized real Bott manifold \( B \) with associated \( w \)-weighted digraph \( D_B \) has a spin structure if and only if the following conditions are satisfied:

i) Indegree of a vertex v of \( D_B \) is even if \( \omega(v) = 1 \) and is congruent to \(-\omega(v) + 3 \) modulo 4, otherwise,

ii) \( M_{ij} \) is even if \( v_i \) is neither in-neighbor nor out-neighbor \( v_j \) with \( i \neq j \),

iii) \( M_{ij} \) and \( \frac{w(v_i, v_j) \cdot \deg^-(v_i)}{2} \) have the same parity when \( v_i \) is an in-neighbor of \( v_j \) with \( \omega(v_i) = 1 \),

iv) \( M_{ij} \) and \( w(v_i, v_j) \cdot w(v_i, v_j) \) have the same parity when \( v_i \) is an in-neighbor of \( v_j \) with \( \omega(v_i) > 1 \).

**Proof** If \( v_i \) is neither in-neighbor nor out-neighbor \( v_j \), conditions iii and iv of Theorem 3.1 is equivalent to the statement that \( A_i \cdot A_j \) is even for \( i \neq j \). In this case, we also have \( M_{ij} = A_i \cdot A_j \). Otherwise, either \( v_i \) or \( v_j \) is an in-neighbor of the other one. Since \( M_{ij} = M_{ji} \), without loss of generality, we can assume that \( v_i \) is. Then \( A_i \cdot A_j = M_{ij} + w(v_i, v_j) \cdot w(v_i, v_j) \). Therefore, when \( \omega(v_i) = 1 \), combining conditions iii and iv of Theorem 3.1, one obtains condition iii above. When \( \omega(v_i) > 1 \), iv can be obtained by combining parts ii and iv of Theorem 3.1.
Example 4.3  Let $P = \Delta^2 \times \Delta^3 \times \Delta^3 \times \Delta^3$, and $B$ be a 4-step generalized Bott manifold corresponding to the reduced matrix

$$A = \begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}.$$ 

Then $\omega : \{1, 2, 3, 4\} \to \mathbb{N}$ with $\omega(1) = 2$, $\omega(2) = \omega(3) = \omega(4) = 3$ and an $\omega$-weighted digraph corresponding to $B$ is as given below. Since $\deg(v_1) = 2$, $B$ has no spin structure by part i of the above theorem.

Corollary 4.4  A small cover over $P = I \times \Delta^{2n_1} \cdots \times \Delta^{2n_k}$ does not have a spin structure when $k \geq 2$.

Proof  Let $M$ be a small cover over $P$, and $G$ be the associated acyclic $\omega$-weighted digraph. Assume for a contradiction that $M$ has a spin structure. The underlying digraph of $G$ has a source, say $v_i$. Since indegree of $v_i$ is zero, the weight of $v_i$ must be 1. Let $v_j$ be a source of the digraph obtained by removing $v_i$ from the underlying digraph and, $v_k$ be a source of the digraph obtained by removing $v_i$ and $v_j$. Then the in-degrees of vertices $v_j$ and $v_k$ are $w(v_i, v_j)$ and $w(v_i, v_k) + w(v_j, v_k) \cdot w(v_j, v_k)$, respectively. By part i of the above theorem, both of them must be odd. In particular $w(v_i, v_j) = 1$ and, $w(v_i, v_k)$ and $w(v_j, v_k) \cdot w(v_j, v_k)$ have different parities. On the other hand, $M_{jk} = w(v_i, v_k)$ as a dot product of $j$-th and $k$-th column of the adjacency matrix. Since $M_{jk}$ and $w(v_j, v_k) \cdot w(v_j, v_k)$ have different parities, $v_j$ cannot be an in-neighbor of $v_k$, by part iv of the above theorem. This means that $w(v_j, v_k)$ is the zero vector. Hence, by part ii, $M_{jk}$ must be even and hence $w(v_i, v_k) = 0$. Contradiction.

5. Higher Stiefel-Whitney classes

It is well-known that the Stiefel-Whitney classes $w_i$ of a smooth manifold satisfy the Wu formula [11]

$$Sq^i(w_j) = \sum_{t=0}^{i} {j + t - i - 1 \choose t} w_{i-t} w_{j+t}.$$
where $Sq^i$ denotes the Steenrod squares. Therefore, for any $i \leq j$ with $i + j = m$, one has

$$\binom{j - 1}{i}w_m = Sq^i(w_m) + \sum_{t=0}^{i-1} \binom{j + t - i - 1}{t}w_{i-t}w_{j+t}.$$  

Substituting $m = 3$ and $i = 1$ gives $w_3 = Sq^1(w_2) + w_1w_2$. This means that whenever $w_1$ and $w_2$ are both zero, so is $w_3$. Therefore, the following result directly follows from Corollary 3.2.

**Proposition 5.1** The first three Stiefel-Whitney classes of a small cover over a product of simplices of dimensions greater than or equal to 2 are zero if and only if $A_i \cdot A_i \equiv 3 \pmod{4}$ and $A_i \cdot A_j \equiv 0 \pmod{2}$ for all $i \neq j$ where $A$ is the associated reduced matrix.

Now we show that the conditions of the above proposition are not necessary for $w_3(M)$ to be zero. For this, let $k_S(A)$ denote the size of the set $\{t | a_{ts} = 1 \text{ for all } s \in S\}$ for any $S \subseteq \{1, 2, \ldots, k\}$. We write $k_S$ instead of $k_S(A)$ when it is clear from the context. Note that $k_{\{i\}} = A_i \cdot A_i$ and $k_{\{i,j\}} = A_i \cdot A_j$.

**Theorem 5.2** The third Stiefel-Whitney class of a small cover $M$ over $P = \prod_{i=1}^{k} \Delta^{n_i}$ modulo 1 is equal to

$$w_3(M) = \sum_{1 \leq i \leq k} \binom{k(i)}{3} + 1 \cdot x_{i0}^3 + \sum_{i,j} P(i,j)x_{i0}x_{j0} + \sum_{i_1 < i_2 < i_3} Q(i_1, i_2, i_3)x_{i_10}x_{i_20}x_{i_30}$$

where

$$P(i,j) = \binom{k(i)}{2} \cdot (k(j) + 1) - k(i) \cdot k_{\{i,j\}},$$

$$Q(i_2, i_2, i_3) = \binom{3}{p=1} (k_{\{i_p\}} + 1) + \sum_{p=1}^{3} (k_{\{i_p\}} + 1) \cdot k_{\{i_1, i_2, i_3\} - \{i_p\}}.$$  

**Proof** One can easily find the coefficient of $x_{i0}^3$ as in the Stiefel-Whitney classes of smaller dimensions. The coefficient of $x_{i0}^2x_{j0}$ is equal to the coefficient of $y_1^2y_2$ in the polynomial

$$(1 + y_1)^{k(i)} - k_{\{i,j\}} + 1(1 + y_2)^{k(j)} - k_{\{i,j\}} + 1(1 + y_1 + y_2)^{k_{\{i,j\}}}$$

as before. We can pick $y_2$ either from the factor $(1 + y_2)^{k_{\{i,j\}} - k_{\{i,j\}} + 1}$ or from the factor $(1 + y_1 + y_2)^{k_{\{i,j\}}}$. If we chose it from the second one, we have to choose $y_1^2$ from $(1 + y_1)^{k(i)} - k_{\{i,j\}} + 1(1 + y_1 + y_2)^{k_{\{i,j\}} - 1}$. Therefore, we have

$$P(i,j) = (k_{\{i\}} - k_{\{i,j\}} + 1) \binom{k_{\{i\}}}{2} + k_{\{i,j\}} \binom{k_{\{i\}}}{2}$$

$$= \left(\binom{k_{\{i\}}}{2} \cdot (k_{\{j\}} + 1) - k_{\{i,j\}}\right).$$

The coefficient of the monomial $x_{i_10}x_{i_20}x_{i_30}$ in $w_3(M)$ is equal to the coefficient of $y_1y_2y_3$ in the product

$$\left(\prod_{j=1}^{3} (1 + y_j)^{k_{\{i_{j}\}}} - \sum_{p \neq j} k_{\{i_p, i_{j}\}} + k_{\{i_1, i_2, i_3\}} + 1\right) \cdot \left(\prod_{p \neq q} (1 + y_p + y_q)^{k_{\{p, i_q\}} - k_{\{i_1, i_2, i_3\}}}\right) \cdot (1 + y_1 + y_2 + y_3)^{k_{\{i_1, i_2, i_3\}}}.$$  

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Now we can choose \( y_3 \) from either of the factors \((1+y_3)_{k_{i_3}} - \sum_{p \neq 3} k_{\{p\}i_3} + k_{\{i_1,i_2,i_3\}} + 1 \), \((1+y_1+y_3)_{k_{i_1,i_3}} - k_{\{i_1,i_2,i_3\}} \), \((1+y_2+y_3)_{k_{i_2,i_3}} - k_{\{i_1,i_2,i_3\}} \) or \((1+y_1+y_2+y_3)_{k_{\{i_1,i_2,i_3\}}} \). Therefore, we have

\[
Q(i_1,i_2,i_3) = \left( k_{\{i_3\}} + k_{\{i_1,i_2,i_3\}} + 1 - \sum_{p \neq 3} k_{\{p\}i_3} \right) \cdot \left( (1+k_{\{i_1\}})(1+k_{\{i_2\}}) + k_{\{i_1,i_2\}} \right)
\]

\[
+ \left( \sum_{p=1}^{2} (k_{\{p\}i_3} - k_{\{i_2,i_3\}}) \cdot \left(k_{\{i_p\}}(1+k_{\{i_1,i_2\}} - \{i_p\}) + k_{\{i_1,i_2\}} \right) \right)
\]

\[
+ k_{\{i_1,i_2,i_3\}} \cdot (k_{i_1}k_{i_2} + k_{i_1,i_2} - 1).
\]

By algebraically manipulating terms, one can easily obtain the desired formula for \( Q(i_1,i_2,i_3) \).

Note that the above theorem is also true for small covers over an arbitrary simple convex polytope when the cohomology classes are represented appropriately. Moreover, one can easily find a formula for the third Stiefel-Whitney class of a small cover over a product of simplices as in the equation (3.1) by taking the relations coming from \( I \) into account. Here, we focus on the case where the dimension of simplices are all greater than equal to 3 in which \( I \) does not contain any relation of dimension 3 to obtain a simple formula.

**Corollary 5.3** Let \( M \) be a small cover over \( P = \prod_{i=1}^{k} \Delta^{n_i} \) with \( n_i \geq 3 \). Then \( w_3(M) = 0 \) if and only if the following conditions hold:

i) \( k_{\{i\}} \not\equiv 2 \) (mod 4),

ii) If \( k_{\{i\}} \) or \( k_{\{j\}} \) is odd then \( k_{\{i,j\}} \equiv 1 \) (mod 2) if and only if either \( k_{\{i\}} \equiv 0 \) (mod 4) and \( k_{\{j\}} \equiv 1 \) (mod 4) or vice versa,

iii) If \( k_{\{i_1\}} \equiv k_{\{i_2\}} \equiv k_{\{i_3\}} \equiv 0 \) (mod 4) for \( i_1 < i_2 < i_3 \) then \( k_{\{i_1,i_2\}} + k_{\{i_1,i_3\}} + k_{\{i_2,i_3\}} \equiv 1 \) (mod 2).

**Proof** Since \( I \) does not contain a monomial of degree less than or equal to 3 when \( P = \prod_{i=1}^{k} \Delta^{n_i} \) with \( n_i \geq 3 \), \( w_3(M) \) is zero if and only if \( \left( \frac{k_{\{i\}} + 1}{3} \right) \equiv 0 \) (mod 2) for all \( i \), \( P(i,j) \equiv 0 \) (mod 2) for all \( i \neq j \) and \( Q(i_1,i_2,i_3) \equiv 0 \) (mod 2) for all \( i_1 < i_2 < i_3 \). Here the first condition is equivalent to condition i. If neither \( k_{\{i\}} \) nor \( k_{\{j\}} \) is divisible by 4 then \( P(i,j) \equiv P(j,i) \equiv 0 \) (mod 2) if and only if \( k_{\{i,j\}} \equiv 0 \) (mod 2). Let \( k_{\{i\}} \equiv 0 \) (mod 4). Then \( P(i,j) \equiv 0 \) (mod 2) for all \( j \neq i \). Moreover, \( P(j,i) \equiv \left( \frac{k_{\{j\}} + 1}{2} \right) - k_{\{j\}}k_{\{i,j\}} \) is even if and only if either \( k_{\{j\}} \equiv 0 \) (mod 4) or \( k_{\{j\}} \equiv 1 \) (mod 4) and \( k_{\{i,j\}} \equiv 1 \) (mod 2), or \( k_{\{j\}} \equiv 3 \) (mod 4) and \( k_{\{i,j\}} \equiv 0 \) (mod 2). Therefore, when condition i holds, \( P(i,j) \equiv P(j,i) \equiv 0 \) (mod 2) if and only if \( M \) satisfies condition ii.

Now suppose that conditions i and ii hold. If \( k_{\{i_1\}}, k_{\{i_2\}} \) and \( k_{\{i_3\}} \) are all divisible by 4, then we have

\[
Q(i_1,i_2,i_3) \equiv 1 + k_{\{i_2,i_3\}} + k_{\{i_1,i_3\}} + k_{\{i_1,i_2\}} \pmod{2}
\]

and hence, \( Q(i_1,i_2,i_3) \equiv 0 \pmod{2} \) if and only if iii holds for the triple \( (i_1,i_2,i_3) \). Now suppose that at least one of them is not divisible by 4. WLOG, assume that \( k_{\{i_1\}} \not\equiv 0 \pmod{4} \). Then \( (1+k_{\{p\}})k_{\{i_1,p\}} \equiv 1 \pmod{4} \).
2) if and only if \( k_{\{p\}} \equiv 0 \, (\text{mod } 4) \) and \( k_{\{i_1\}} \equiv 1 \, (\text{mod } 4) \). Therefore, we have

\[
Q(i_1, i_2, i_3) \equiv (1 + k_{\{i_2\}})k_{\{i_1, i_3\}} + (1 + k_{\{i_3\}})k_{\{i_1, i_2\}} \equiv 0 \quad (\text{mod } 2).
\]

This proves the theorem. \(\square\)

Since \( k_i = \deg^-(v_i) + \omega(i) \) and \( k_{ij} = M_{ij} + w(v_i, v_j) \cdot w(v_i, v_j) + w(v_j, v_i) \cdot w(v_j, v_i) \), we have the following.

**Corollary 5.4** Let \( D_M \) be an \( \omega \)-weighted acyclic digraph associated to a small cover \( M \) over \( P = \prod_{i=1}^{k} \Delta^{n_i} \) with \( n_i \geq 3 \). Then \( w_3(M) = 0 \) if and only if the following conditions hold for vertices of \( D_M \):

\begin{enumerate}
  \item[i)] \( \deg^-(v_i) + \omega(i) \not\equiv 2 \, (\text{mod } 4) \),
  \item[ii)] If \( \deg^-(v_i) + \omega(i) \) or \( \deg^-(v_j) + \omega(j) \) is odd then \( M_{ij} + w(v_i, v_j) \cdot w(v_i, v_j) + w(v_j, v_i) \cdot w(v_j, v_i) \equiv 1 \, (\text{mod } 2) \) if and only if either \( \deg^-(v_i) + \omega(i) \equiv 0 \, (\text{mod } 4) \) and \( \deg^-(v_j) + \omega(j) \equiv 1 \, (\text{mod } 4) \) or vice versa,
  \item[iii)] If \( \deg^-(v_{i_1}) + \omega(i_1) \equiv \deg^-(v_{i_2}) + \omega(i_2) \equiv \deg^-(v_{i_3}) + \omega(i_3) \equiv 0 \, (\text{mod } 4) \), then

\[
\sum_{p \neq q} \left( M_{ip, iq} + w(v_{ip}, v_{iq}) \cdot w(v_{ip}, v_{iq}) + w(v_{iq}, v_{ip}) \cdot w(v_{iq}, v_{ip}) \right) \equiv 1 \, (\text{mod } 2).
\]
\end{enumerate}

As shown above, when \( M \) is a generalized Bott manifold, the Stiefel-Whitney classes of \( M \) of dimensions less than or equal to 3 can be written in terms of the dot products of columns of the associated reduced vector matrix \( A \). It is natural to ask whether this is true for all dimensions. The following theorem gives an affirmative answer to this question.

**Theorem 5.5** The fourth Stiefel-Whitney class of a small cover \( M \) over \( P = \prod_{i=1}^{k} \Delta^{n_i} \) modulo \( I \) is equal to

\[
w_4(M) = \sum \left( \frac{k(i) + 1}{4} \right) x_0^4 + \sum P_1(i, j)x_0^3x_0 + \sum P_2(i, j)x_0^2x_0^2 + \sum Q(i_1, i_2, i_3)x_0^2x_0^2x_0 + \sum R(i_1, i_2, i_3, i_4)x_0^2x_0^2x_0x_0
\]

where
\[ P_1(i, j) = \binom{k(i)}{3} \cdot (k(j) + 1) - \binom{k(i)}{2} \cdot k(i, j), \]
\[ P_2(i, j) = \binom{k(i)}{2} \cdot (k(j) + 1) - k(i)k(j)k(i, j) + \binom{k(i)}{2}, \]
\[ Q_{i_1, i_2, i_3} = \binom{k(i_1)}{2} \left( (k(i_2) + 1)(k(i_3) + 1) - k(i_1) \right) - k(i_1) \left( \sum_{p \neq 1} k_{i_1, i_p} (k_{i_2, i_3} - (i_p) + 1) \right) + k(i_{i_1, i_2}) - k(i_{i_1, i_2}), \]
\[ R_{i_1, i_2, i_3, i_4} = \left( \prod_{p=1}^{4} (k(i_p) + 1) \right) - \sum_{p \neq q} \left( (k(i_p) + 1)(k(i_q) + 1) - \frac{k(i_p, i_q)}{2} \right) \cdot k(i_{i_1, i_2, i_3, i_4} - (i_p, i_q)). \]

**Proof** Since the rest can be found similarly, we only provide a proof for the formula for \( Q_{i_1, i_2, i_3}. \) Here \( Q_{i_1, i_2, i_3} \) is equal to the coefficient of \( y_1y_2y_3 \) in (5.3). One can choose \( y_3 \) from either of the factors \( 1 + y_3 = k(i_3) - \sum_{p \neq 3} k_{i_p, i_3} + k_{i_1, i_2, i_3} + 1, \) \( 1 + y_3 = k(i_1, i_2, i_3) - k(i_1, i_2, i_3), \) \( 1 + y_3 = k(i_1, i_2, i_3) - k(i_1, i_2, i_3), \) or \( (1 + y_1 + y_2 + y_3) = k(i_1, i_2, i_3). \) Therefore, we have

\[ Q_{i_1, i_2, i_3} = \left( k(i_3) + k_{i_1, i_2, i_3} + 1 - \sum_{p \neq 3} k_{i_p, i_3} \right) \cdot \left( \binom{k(i_1)}{2} \cdot (k(i_2) + 1) - k(i_1)k(i_{i_1, i_2}) \right) + (k(i_{i_1, i_2}) - k(i_{i_1, i_2})) \cdot \left( \binom{k(i_1)}{2} \cdot (k(i_2) + 1) - k(i_1)k(i_{i_1, i_2}) \right) + (k(i_{i_2, i_3}) - k(i_{i_2, i_3})) \cdot \left( \binom{k(i_1)}{2} \cdot (k(i_2) - k(i_1)k(i_{i_1, i_2}) \right) + k(i_{i_1, i_2, i_3}) \cdot \left( \binom{k(i_1)}{2} \cdot (k(i_2) - k(i_1)k(i_{i_1, i_2}) - 1) \right). \]

Since the sum of the first factors of each term in the RHS of the equation is \( k(i_3) + 1, \) the result easily follows.

**Corollary 5.6** Let \( M \) be a small cover over \( P = \prod_{i=1}^{k} \Delta^{n_i} \) with \( n_i \geq 4. \) Then \( w_4(M) = 0 \) if and only if the following conditions hold:

\( i) \) \( k(i) \equiv 0, 1, 2 \) or 7 (mod 8),

\( ii) \) \( k(i, j) \) must satisfy the following table:

\[ \begin{array}{c|cccc}
\hline
& & & & \\
& & & & \\
\end{array} \]
that remaining restrictions on \( k_{i,j} \equiv 2 \) and \( \theta_{i,j} \equiv 1 \) for all \( i, j \). Since \( \theta_{i,j} \equiv 0 \) (mod 2), \( P_{l}(i,j) = 0 \) (mod 2) yields \( k_{i,j} \equiv 0 \) (mod 4) and hence we have \( k_{i,j} \equiv 0 \) (mod 2) when \( \theta_{i,j} = 0 \). Similarly, when \( \theta_{i,j} = 1 \), \( P_{l}(i,j) = 0 \) (mod 2) gives \( k_{i,j} \equiv 1 \) (mod 2) and hence we have \( k_{i,j} \equiv 1 \) (mod 4) when \( \theta_{i,j} = 1 \).

When \( \theta_{i,j} = 0 \) for all \( i \in \{i_1, i_2, i_3, i_4\} \), \( R(i_1, i_2, i_3, i_4) = 1 \) if any of \( k_{i_1,i_2} = 1 \) (mod 4). Hence \( k_{i_1,i_2} \equiv 0 \) (mod 4) and hence \( k_{i_1,i_2} \equiv 0 \) (mod 4).

Under these assumptions, \( \theta_{i,j} = 7 \) for one of the \( i_1, i_2 \) or \( i_3 \). \( Q(i_1, i_2, i_3) = 0 \) (mod 2). When \( \theta_{i_1,i_2,i_3} = (0, q_2, q_2) \) and \( \theta_{i_1,i_2,i_3} = (0, q_1, q_2) \) where \( 0 \leq p_i \leq 2 \) and \( 0 \leq q_i \leq 1 \) give the all the remaining restrictions on \( k_{i_1,i_2,i_3} = 1 \) and proves the only if part of the theorem. One can easily check that under these restrictions, \( w_4(M) = 0 \).

Whenever \( m \) is not a power of 2, the Wu formula can be used to express \( w_m \) in terms of lower Stiefel-
Whitney classes and their Steenrod squares. Hence, one can conclude that whenever the lower dimensional Stiefel-Whitney classes are zero then so is \(w_m\) for \(m \neq 2^p\) for any \(p\). Hence, we have the following result.

**Corollary 5.7** Let \(M\) be a small cover over \(P = \prod_{i=1}^{k} \Delta^{n_i}\) with \(n_i \geq 4\) with an associated matrix \(A\). Then the first seven Stiefel-Whitney classes of \(M\) are zero if and only if \(A_i \cdot A_i \equiv 7 \pmod{8}\), \(A_i \cdot A_j \equiv 0 \pmod{4}\) and \(k_{\{i,j,l\}} = |\{t|a_{it} = a_{jt} = a_{lt} = 1\}| \equiv 0 \pmod{2}\) for all \(i < j < l\).

**Proof** By Proposition 5.1 and the above argument, it suffices to show that if \(A_i \cdot A_i \equiv 7 \pmod{8}\), \(A_i \cdot A_j \equiv 0 \pmod{4}\) and \(k_{\{i,j,l\}} = |\{t|a_{it} = a_{jt} = a_{lt} = 1\}| \equiv 0 \pmod{2}\) for all \(i < j < l\) then \(w_4(M) = 0\). This directly follows from Theorem 5.5.

When \(m\) is a power of 2, for all \(i + j = m\), \((i^{-1})\) is always even and hence one can not use the Wu formula to find \(w_m\). Considering the results of the paper, we believe that for each \(m = 2^t\), \(k_{\{S\}}\)'s where \(S\) is a subset of size \(t\) of \(\{1, 2, \cdots, k\}\) will appear as a coefficient of \(w_m(M)\) and we conjecture the following.

**Conjecture 5.8** Let \(M\) be a small cover over \(P = \prod_{i=1}^{k} \Delta^{n_i}\) with \(n_i \geq 2^t\) with an associated matrix \(A\). Then the first \(2^{t+1} - 1\) Stiefel-Whitney classes of \(M\) are zero if and only if for any \(S \subseteq \{1, 2, \cdots, k\}\) of size less than or equal to \(t + 1\), \(k_S = |\{i| a_{si} = 1\} for any s \in S}\) is congruent to \(-1\) modulo \(2^{t+1}\) when \(|S| = 1\) and is congruent to \(0\) modulo \(2^{t+1} - |S|\), otherwise.

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