Boundary Control of Traffic Congestion Modeled as a Non-stationary Stochastic Process

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Abstract—In this paper, we introduce a new conservation-based approach to model traffic dynamics, and apply the model predictive control (MPC) approach to manage the boundary traffic inflow and outflow, so that the traffic congestion is reduced. We establish an interface between the Simulation of Urban Mobility (SUMO) software and MATLAB to define a network of interconnected roads (NOIR) as a directed graph, and present traffic congestion management as a network control problem. By formally specifying the traffic feasibility conditions, and using the linear temporal logic, we present the proposed MPC-based boundary control problem as a quadratic programming with linear equality and inequality constraints. The success of the proposed traffic boundary control is demonstrated by simulation of traffic congestion control in Center City Philadelphia.

I. INTRODUCTION

Traffic congestion has result in miscellaneous problems [1]–[5]. Over the past years, researchers have proposed a large number of approaches to alleviate traffic congestion. We can divide the approaches into two categories: model-based approaches and model-free approaches.

An essential task in the model-based approach is to construct a virtual traffic model to represent the real traffic dynamics’ properties. Then, based on this traffic model, appropriate approaches can be applied to control the traffic dynamics or predict the traffic states. The cell transmission model (CTM), which is applied to spatially partition a network into road elements, is incorporated with mass conservation law to describe the traffic dynamics [6]. Ref. [7] incorporates the CTM model with the finite-state Markov decision process (MDP) to obtain the optimal movement phases at the traffic junctions. In addition, the macroscopic fundamental diagram (MFD) of the traffic flow model [8], which describes the relationship between traffic density and traffic flow, is a widely used approach to describe the traffic dynamics. Ref. [9], [10] integrate the cell transmission model (CTM) approach with the MFD model to improve the accuracy of the large-scale traffic coordination model. Moreover, Ref. [11] integrates the perimeter control with the MFD model to improve the traffic network capacity and mobility. Once the traffic model is generated, the control and optimization methods can be implemented to find the optimal solution. Ref. [12]–[14] adopt the model predictive control (MPC) approach to obtain the optimal solution for boundary traffic control.

Traffic congestion control can also be accomplished through a model-free approach. For example, Ref. [15] uses a data model to represent the traffic dynamics and incorporates adaptive predictive control to adjust the boundary input. Another model-free possibility for reducing congestion is through traffic signal optimization. Genetic Algorithm (GA) [16] and Markov Decision Process (MDP) [17] are applied in the traffic signal optimization problem to reduce the traffic delay. Also, Ref. [18]–[21] propose Deep Reinforcement Learning (DRL) algorithms to improve the applicability of the learning algorithm on the traffic signal optimization of the large-scale NOIR.

This paper presents further research based on our previous work [12]. In this paper, we continue to use mass conservative law to define the traffic coordination model and describe traffic dynamics. Also, we model the dynamics of the traffic coordinate as a non-stationary stochastic process. Comparing with our previous research, in this paper we modify the definition of the stochastic parameter to improve and perfect the traffic model to make it more realistic. In addition, in the process of determining the optimal boundary control solution, we consider and simulate the impact of the current number of road vehicles in the results. Similar to our previous work, the network of interconnected roads (NOIR) is generated by converting the real street map using...
the software Simulation of Urban Mobility (SUMO) and MATLAB. We accomplish the objective of traffic congestion alleviation by controlling the boundary inflow and outflow using the model predictive control (MPC) approach. In the case study, we integrate the proposed traffic model with the MPC approach and illustrate the results of traffic congestion management on the Center City district of Philadelphia (See. Fig. 1).

This paper is organized as follows: Section II gives an introduction about the notions existing in the NOIR and the linear temporal logic symbols used to describe traffic state conditions. The basic principles and definitions for the model construction and the traffic feasibility conditions for solving the cost function in boundary control are explained in Section III. The explanation about the traffic network dynamics and the traffic control approach are presented in Section IV and Section V, respectively. The simulation results of a case study using the proposed traffic model and control approach in Center City Philadelphia are illustrated in Section VI, followed by the conclusion in Section VII.

II. PRELIMINARIES

A. Graph Theory Notions

The network of inter-connected roads (NOIR) can be denoted as a graph \( G(V,E) \), where set \( V = \{1, \ldots, N\} \) defines \( N \) road elements and set \( E \subset V \times V \) determines the interconnections between the road elements. As shown in Fig. 1, every element \( i \in V \) is a unidirectional street and located between two consecutive junctions. We define that element \((j,i) \in E\) represents the connection directed from road element \( i \in V \) to road element \( j \in V \). Set \( V \) can be divided into three subsets: the inlet road elements set \( V_{in} = \{1, \ldots, N_{in}\} \), the outlet road elements set \( V_{out} = \{N_{in} + 1, \ldots, N_{out}\} \), and the interior road elements set \( V_I = \{N_{out} + 1, \ldots, N\} \). For every road element \( i \in V \), we define in-neighbor set \( I_i \) and out-neighbor set \( O_i \) as follows:

\[
I_i = \{(j,(i,j) \in E)\},
\]

\[
O_i = \{(j,(j,i) \in E)\},
\]

where the in-neighbor and out-neighbor road elements refer to the upstream adjacent road elements and the downstream adjacent road elements. Note that the in-neighbor set of every inlet road element and the out-neighbor set of every outlet road element are empty, i.e., \( I_i = \emptyset \), if \( i \in V_{in} \) and \( O_i = \emptyset \), if \( i \in V_{out} \).

B. Linear Temporal Logic

We use the linear temporal logic (LTL) to describe the properties and feasible conditions of the traffic dynamics model. A LTL formula normally consists of three components, the propositional variables, the logical operators, and the temporal modal operators. The propositional variable is the most fundamental element in the propositional logic whose value is either true or false. The logical operators, such as negative (\( \neg \)), disjunction (\( \lor \)) and conjunction (\( \land \)), can act on a single propositional variable or between multiple propositional variables to express a sophisticated logical formula accurately. The temporal modal operators including eventually (\( \Diamond \)), always (\( \Box \)), next (\( \Diamond \)), and until (\( U \)), define the temporal variables of LTL formulas [22].

Inspired by Metric Temporal Logic (MTL) [23], we extend the classic LTL by integrating the temporal modal operators with a distance function to restrict the logical formula into a finite domain. For example, assuming \( T = \{0, \ldots, N_T\} \) is a finite time domain, the formula \( \Diamond_T (\cdot) \) at sampling time \( k \) means that the statement expressed in the parenthesis is always true at every time \( k+t \), where \( t \in T \).

III. PROBLEM STATEMENT

In this paper, we consider the traffic dynamics as a physical model which satisfies the mass conservation law and describe the dynamics in every road element \( i \in V \) as follows:

\[
n_i[k+1] = n_i[k] + y_i[k] - z_i[k] + s_i[k]
\]

where \( k = 1, 2, \ldots \) denotes the discrete sampling time, \( n_i[k] \) is the number of existing cars at road element \( i \), and called traffic capacity. Also, \( y_i[k] \), \( z_i[k] \) and \( s_i[k] \) are the network traffic inflow, network traffic outflow and the external traffic flow, respectively.

The external traffic flow \( s_i[k] \), which is defined in Eq. (3), specifies the traffic exchange between the NOIR and the external environment. It is prescribed that, at every sampling interval \([t_k, t_{k+1}]\) the traffic could only drive into the NOIR through the inlet road elements, defined by \( V_{in} \), and depart from the NOIR through the outlet road elements, defined by \( V_{out} \). Therefore, we define \( s_i[k] \) by

\[
s_i[k] = \begin{cases} u_i[k] \geq 0 & i \in V_{in} \\ -v_i[k] \leq 0 & i \in V_{out} \\ 0 & i \in V_I \end{cases}
\]

at every discrete time \( k \), where \( u_i[k] \geq 0 \) is the number of cars entering the NOIR through \( V_{in} \), and \( v_i[k] \geq 0 \) is the number of cars leaving the NOIR through \( V_{out} \).

The network traffic inflow \( y_i[k] \) and traffic outflow \( z_i[k] \) given by Eq. (4) determine the traffic flow exchange between road elements within the NOIR at every discrete sampling time \( k \), and defined by

\[
y_i[k] = \begin{cases} 0 & i \in V_{in} \\ v_i[k] & i \in V_{out} \\ \Sigma_{j \in \delta^+} q_{i,j}[k] z_j[k] & i \in V_I \end{cases}
\]

\[
z_i[k] = \begin{cases} 0 & i \in V_{in} \\ u_i[k] & i \in V_{out} \\ p_i[k] (n_i[k] + y_i[k]) & i \in V_I \end{cases}
\]

where

\[
p_i[k] = \begin{cases} \frac{z_i[k]}{n_i[k] + y_i[k]} & \text{If } i \in V_I \text{ and } y_i + n_i = 0 \\ 1 & \text{If } i \in V_I \text{ and } y_i + n_i \neq 0 \end{cases}
\]

denotes the proportion of vehicles departing the road element \( i \in V \) within the sampling interval \([t_k, t_{k+1}]\). Also, we set that
vehicles entering the inlet or outlet road element will depart the corresponding road element within the sampling period. In addition, \( q_{i,j}[k] \) in Eq. (4a) is the fraction of cars driving from road element \( j \in \mathcal{V} \setminus \mathcal{V}_{out} \) to each of its downstream adjacent road elements \( i \in \mathcal{V} \) at every sampling time \( k \). Therefore, \( q_{i,j}[k] \) must satisfy the following equality constraint:

\[
\sum_{i \in \Omega_j} q_{i,j}[k] = 1. \tag{6}
\]

Note that the flow probability \( p_{i,k} \) and fraction probability \( q_{i,j}[k] \) reflect the uncertainty caused by human driver intentions. Therefore, our proposed model consistently incorporates the human intent into modeling of traffic coordination. In this paper, we assume that the flow probability \( p_{i,k} \) and fraction probability \( q_{i,j}[k] \) are known and they are generated uniformly and randomly at every road \( i \in \mathcal{V} \) and at every discrete time \( k = 0, 1, 2, \cdots \).

**Assumption 1.** In the process of building the traffic dynamics model, we assume that the time increment \( \Delta t = t_{k+1} - t_k \) is constant and sufficiently small such that the flow probability \( p_{i,k} \) and the fraction probability \( q_{i,j}[k] \) of each road element \( i \in \mathcal{V}_t \) satisfy the following conditions:

\[
p_{i,k} = \frac{z_i[k]}{n_i[k] + y_i[k]} \in (0, 1], \quad \forall j \in I_i, \quad q_{i,j}[k] \in [0, 1]. \tag{7a,b}
\]

In this paper, we offer an MPC-based boundary control to manage the traffic congestion in the NOIR. Therefore, boundary input \( s_t \) is determined at every road \( i \in \mathcal{V}_{in} \cup \mathcal{V}_{out} \) by solving a quadratic programming problem with the cost function and constraints that are described below:

**Traffic Coordination Cost:** We define

\[
C = \frac{1}{2} \sum_{j=k+1}^{k+N_r} \left( \sum_{i \in \mathcal{V}_{in} \cup \mathcal{V}_{out}} \frac{1}{n_i[k]} \sum_{j=1}^{\mathcal{V}_{in} \cup \mathcal{V}_{out}} \frac{1}{n_j[k]} \right) \tag{8}
\]

as the traffic coordination cost, where \( s_i = u_i \) for every inlet road element \( i \in \mathcal{V}_{in} \), \( s_i = v_i \) for every outlet road element \( i \in \mathcal{V}_{out} \), and \( \beta > 0 \) is a constant scaling parameter, which reflects the influence weight of the number of existing cars on the interior road element \( n_i \) on the cost function \( C \).

**State Feasibility Condition:** Traffic capacity is set to be a non-negative physical parameter. Moreover, we assume every road element in the NOIR holds a maximal capacity \( n_{max} \) and the traffic capacity of the road element cannot exceed this maximal value within the next \( N_r \) time steps at every sampling time \( k \). We define a LTL formula \( \pi_1 \) to express this state constraint at every interior road element \( i \in \mathcal{V}_t \) by

\[
\pi_1 := \bigwedge_{i \in \mathcal{V}_t} \Box T (n_i > 0 \land n_i < n_{max}). \tag{9}
\]

**Input Feasibility Condition:** At every sampling time \( k \), back-flow must be prohibited at every boundary road element \( i \in \mathcal{V}_{in} \cup \mathcal{V}_{out} \) within the next \( N_r \) sampling times. We express this feasibility condition using following LTL formula

\[
\pi_2 := \bigwedge_{i \in \mathcal{V}_{in} \cup \mathcal{V}_{out}} \Box T (u_i \geq 0 \land v_j \geq 0). \tag{10}
\]

**Input Optional Condition:** We assume that the demand for entering and leaving the NOIR is high and only \( d_0 \) amount of cars are permitted to cross the boundary of the NOIR within the next \( N_r \) sampling times at every discrete time \( k \). Therefore, the following LTL formula

\[
\pi_3 := \Box T \left( \sum_{i \in \mathcal{V}_{in}} u_i + \sum_{j \in \mathcal{V}_{out}} v_j = d_0 \right) \tag{11}
\]

must be satisfied.

**IV. Traffic Network Dynamics**

We substitute Eqs. (3) and (4) into Eq. (2) and simplify the traffic dynamics for every road element \( i \in \mathcal{V} \) as follows:

\[
\forall i \in \mathcal{V}_{in} \cup \mathcal{V}_{out}, \quad n_i[k+1] = n_i[k] \quad \tag{12a}
\]

\[
\forall i \in \mathcal{V}_t, n_i[k+1] = (1 - p_i[k]) \left( n_i[k] + \sum_{j \in I_i} q_{i,j}[k] z_j[k] \right). \tag{12b}
\]

Eq. (12) implies that the traffic capacity remains constant at every road \( i \in \mathcal{V}_{in} \cup \mathcal{V}_{out} \), but it is updated with time at every interior road elements \( i \in \mathcal{V}_t \). As a result, the network traffic dynamics are only defined over the interior road elements. To obtain the traffic dynamics, we define the state vector \( x = [n_{N_{max}+1} \cdots n_N]^T \in \mathbb{R}^{(N-N_{out}) \times 1} \), the inflow vector \( y \in \mathbb{R}^{(N-N_{out}) \times 1} \), and the outflow vector \( z \in \mathbb{R}^{(N-N_{out}) \times 1} \). Moreover, we define the outflow probability matrix \( P \in \mathbb{R}^{(N-N_{out}) \times (N-N_{out})} \), and the tendency probability matrix \( Q \in \mathbb{R}^{(N-N_{out}) \times (N-N_{out})} \) as follows:

\[
P[k] = \text{diag}(p_{N_{out}+1}[k], \cdots, p_{N}[k]), \tag{13a}
\]

\[
Q[k] = \left[ Q_{ij}[k] \right] = \left[ q_{i+N_{out}+1+j+N_{out}}[k] \right], \tag{13b}
\]

where \( Q_{ij} = q_{i+N_{out},j+N_{out}}[k] \) determines the fraction of departing vehicles driving from road element \( (j+N_{out}) \in \mathcal{V}_t \) towards road element \( (i+N_{out}) \in \mathcal{V}_t \) at discrete time \( k \).

By considering definitions of traffic inflow and outflow given in Eq. (4), the network inflow vector \( y \) and network outflow vector \( z \) are related to state vector \( x \) by

\[
y[k] = (1 - Q[k] P[k])^{-1} Q[k] P[k] x[k], \tag{14}
\]

\[
z[k] = (P[k] (1 - Q[k] P[k])^{-1} Q[k] P[k] + P[k]) x[k], \tag{15}
\]

at every sampling time \( k \), and the traffic dynamics can be expressed in the state space form by

\[
x[k+1] = A[k] x[k] + B[k] s[k]. \tag{16}
\]

In Eq. (16), \( s[k] = [s_i[k]] \in \mathbb{R}^{N_{out} \times 1} \), \( B[k] = b_{i,j}[k] \in \mathbb{R}^{(N-N_{out}) \times N_{out}} \), and \( A[k] \in \mathbb{R}^{(N-N_{out}) \times (N-N_{out})} \) refer to the
input vector, the input matrix, and the system matrix at every sampling time $t$, respectively, in which

$$s_i[t] = \begin{cases} u_i[t], & \text{If } i \in V_i = \{1, \cdots, N_i\} \\ v_i[t], & \text{If } i \in V_{out} = \{N_i + 1, \cdots, N_{out}\}, \end{cases}$$ (17a)

$$b_{ij}[t] = \begin{cases} 1, & j \in I_i + N_{out} \\ -1, & j \in O_i + N_{out} \end{cases},$$ (17b)

$$A[k] = (I - P[k])\left(I + (I - Q[k]P[k])^{-1}Q[k]P[k]\right)$$

$$= (I - P[k])\left(I - Q[k]P[k]\right)^{-1}. \quad (17c)$$

**Theorem 1.** The traffic dynamics (16) is bounded-input bounded-output (BIBO) stable, when the following premises are satisfied:

1. Vehicles can only enter the NOIR through an inlet road element and exit from the NOIR through an outlet road element.
2. The out-neighbors of an inlet road element are all interior road elements, i.e., if $i \in V_i$, then, $O_i \subset V_i$.
3. Vehicles entering the NOIR through an inlet road element departs the NOIR within a finite time period.
4. Every road element has at least one in-neighbor element or out-neighbor element, i.e., no road element is isolated in the NOIR ($\bigcup_{i \in V_i} \overline{\{I_i \cup O_i \neq \emptyset\}}$).

**Proof:** See the proof in the Appendix.

V. TRAFFIC CONGESTION CONTROL

We use MPC to determine the boundary control $s[k]$ at every discrete time $k$ by solving a quadratic programming problem with quadratic costs and linear constraints imposing the feasibility conditions into management of traffic coordination. To this end, according to Eq. (16), we predict the traffic dynamics within the next $N_r$ sampling times by obtaining the following finite-horizon predictive model:

$$X[k] = G[k]x[k] + H[k]U[k]$$ (18)

where

$$X[k] = \begin{bmatrix} x[k + 1] \\ \vdots \\ x[k + N_r] \end{bmatrix} \in \mathbb{R}^{(N - N_{out})N_r \times 1},$$ (19a)

$$G[k] = \begin{bmatrix} A[k] \\ \vdots \\ A^{N_r}[k] \end{bmatrix} \in \mathbb{R}^{(N - N_{out})N_r \times (N - N_{out})},$$ (19b)

$$H[k] = \begin{bmatrix} B[k] \\ A[k]B[k] \\ A^2[k]B[k] \\ \vdots \\ A^{N_r-1}[k]B[k] \\ A^{N_r-2}[k]B[k] \\ A^{N_r-3}[k]B[k] \\ \vdots \\ B[k] \end{bmatrix},$$ (19c)

$$x[k] = \begin{bmatrix} n_1[k] \\ \vdots \\ n_N[k] \end{bmatrix} \in \mathbb{R}^{(N - N_{out}) \times 1},$$ (19d)

$$U[k] = \begin{bmatrix} s[k] \\ \vdots \\ s[k + N_r - 1] \end{bmatrix} \in \mathbb{R}^{(N_{out}N_r) \times 1}. \quad (19e)$$

The cost function $C$, previously defined in (8), can be rewritten as follows:

$$C(U[k]) = \frac{1}{2}U^T[k]U[k] + \beta X^T[k]X[k] = \frac{1}{2}U^T[k]W_1[k]U[k] + W_2^T[k]U[k] + W_3[k],$$ (20)

where

$$W_1[k] = I_{N_{out}N_r} + \beta H^T[k]H[k],$$ (21a)

$$W_2^T[k] = \beta x^T[k]G^T[k]H[k],$$ (21b)

$$W_3[k] = \frac{1}{2}\beta x^T[k]G^T[k]G[k]x[k].$$ (21c)

Note that $W_3[k]$ can be removed from cost function (20) since $W_3[k]$ depends on $x[k]$ at every discrete time $k$, but it is independent of $U[k]$. Therefore,

$$C' = \frac{1}{2}U^T[k]W_1[k]U[k] + W_2^T[k]U[k]$$ (22)

is defined as the cost function of traffic coordination, and the optimal control variable

$$s^*[k] = [I_{N_{out}} \ 0_{N_{out} \times N_{out}(N_r - 1)}] U^*[k]$$ (23)

is assigned by determining $U^*[k]$ as the solution of the following optimization problem:

$$\min C' = \min \left( \frac{1}{2}U^T[k]W_1[k]U[k] + W_2^T[k]U[k] \right)$$ subject to

$$W_4[k]U[k] + W_5[k] \leq 0, \quad (24a)$$

$$W_6[k]U[k] + W_7[k] = 0, \quad (24b)$$

where

$$W_4[k] = \begin{bmatrix} -I_{N_{out}N_r} \\ H[k] \\ -H[k] \end{bmatrix},$$ (25a)

$$W_5[k] = \begin{bmatrix} 0_{N_{out}N_r^2} \\ 0_{N_{out}N_r^2} \otimes x[k] + G[k]x[k] \\ -G[k]x[k] \end{bmatrix},$$ (25b)

$$W_6[k] = I_{N_r} \otimes 1_{N_{out}},$$ (25c)

$$W_7[k] = -d_0I_{N_r \times 1}.$$ (25d)

Note that the inequality (24a) integrates the feasibility conditions (9) and (10), and the equality (24b) is identical to the input conditions defined in (11).

**Theorem 2.** The traffic capacity at every interior road element and discrete time $k$ satisfies the inequality equation

$$n_i[k] \geq 0, \quad \forall i \in V_I \cup k \geq 1,$$ (26)

if conditions

$$n_i[1] \geq 0, \quad \forall i \in V_I, \quad (27a)$$

$$\vdots$$

$$n_{N_r}[N_{out}] \geq 0, \quad \forall i \in V_I, \quad (27b)$$

$$\vdots$$

$$n_{N_r}[N_{out}] \geq 0, \quad \forall i \in V_I. \quad (27c)$$

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\[ u_j[k] \geq 0, \quad \forall j \in \mathcal{V}_n, \quad (27b) \]
\[ v_n[k] \geq 0, \quad \forall n \in \mathcal{V}_{out}, \quad (27c) \]

hold.

**Proof:** See the proof in [12].

**Remark 1.** Per Theorem 2, it is ensured that the traffic density is non-negative at every discrete time \( k \), if conditions (27) hold. Therefore, constraint Eq. (24a) simplifies to

\[
\begin{bmatrix}
-I_{N_t, N_{out}} \\
H[k]
\end{bmatrix}
\begin{bmatrix}
U[k] + 0_{N_t, N_{out} \times 1} \\
0_{(N - N_{out})N_t \times 1} \otimes \mathbf{s}_{\text{max}} + G[k]x[k]
\end{bmatrix} \leq 0.
\]

(28)

**VI. Simulation Results**

In this section, we illustrate the simulation results of the traffic control based on a part of real street map of the Philadelphia Center City which is composed of 259 road elements (See Fig. 1). In order to denote and manage the road elements in the map conveniently, we assign specific indices for every road element in the NOIR. According to the road types, the road elements set \( \mathcal{V} = \{1, \ldots, 259\} \) is partitioned to \( \mathcal{V}_{\text{in}} = \{1, \ldots, 20\} \), \( \mathcal{V}_{\text{out}} = \{21, \ldots, 42\} \), and \( \mathcal{V}_I = \{43, \ldots, 259\} \).

We assign the initial traffic capacity \( n_i[0] \) at every road element \( i \in \mathcal{V} \) in the NOIR uniformly and randomly. Then, the simulation is performed for 300 time steps, where sampling time \( k \) represents the continuous time interval \([t_k, t_{k+1})\) and \( \Delta t = t_{k+1} - t_k \) is constant at each sampling time. Moreover, at each sampling time \( k \), the flow probability matrix \( \mathbf{P}[k] \) and fraction probability matrix \( \mathbf{Q}[k] \) are randomly generated.

For simulation, we ignore the inter-vehicle distance and assume that the length of vehicle \( l_{\text{veh}} \) is equal to 4.5 m and obtain the maximum traffic capacity \( n_{i,\text{max}} \) for every interior road element \( i \in \mathcal{V}_I \) by

\[ n_{i,\text{max}} = \frac{N_{i,\text{lane}} \times l_i}{l_{\text{veh}}}, \quad (29) \]

where \( l_i \) is the length of the road element \( i \) obtained from the street map data and \( N_{i,\text{lane}} \) is the number of lanes in road element \( i \in \mathcal{V}_I \). Furthermore, we assume that 400 cars can cross the boundary of the NOIR during the interval \([t_k, t_{k+1})\) at every sampling time \( k \). Therefore, \( d_0 = 400 \) is used in equality constraint (25d).

Simulations are executed under different scaling parameters. Specifically, we use different values of \( \beta = 0, 0.5, \text{ and } 1 \) to weigh the cost function (20). The results are presented for two boundary inlet, boundary outlet, and interior road elements with index numbers and locations presented in Table I.

| Road Type | Road Element Index | Name and Location |
|-----------|--------------------|-------------------|
| Inlet     | 9                  | Chestnut St. (S19th-S20th) |
| Inlet     | 12                 | Sansom St. (S9th-S10th)  |
| Outlet    | 28                 | Filbert St. (S9th-S10th) |
| Outlet    | 33                 | Arch St. (N19th-N20th)   |
| Interior  | 150                | Market St. (N16th-N17th) |
| Interior  | 259                | Walnut St. (S15th-S16th) |

**TABLE I: Example road elements in NOIR**

Fig. 2: External traffic inflows of inlet road elements under different \( \beta \) values

We illustrate the variation of the optimal external traffic flow \( s_i \) at the example road elements under different values of \( \beta \) throughout the whole simulation time in Figs. 2 and Fig. 3. It is observed that the external traffic flow of the inlet road elements \( u_i \) and outlet road elements \( v_i \) hold similar properties at different values of \( \beta \), that is: the external traffic inflow and outflow reach the steady state condition after about 20 sampling times. For \( \beta = 0 \), the weight matrix \( \mathbf{W}_I[k] \) is diagonal at every discrete time \( k \), but \( \mathbf{W}_I[k] \) is not diagonal and \( \mathbf{W}_I^T[k] \) is not \( \mathbf{0} \) when \( \beta > 0 \) is selected. Therefore, we observe that the variations of external flow are different when \( \beta = 0 \) is selected. In addition, it is observed that that the plots have large fluctuations when \( \beta > 0 \) due to the variation of the number of existing cars in the interior road elements. Fig. 4 plots variations of \( n_{150}[k] \) and \( n_{239}[k] \) at different sampling times \( k \). Although, variation curves of traffic densities at interior road elements have a large fluctuation, we could still observe an stable tendency after a certain period of time.

The number of vehicles entering and exiting the NOIR during the whole simulation time are plotted in the Fig. 5 for \( \beta = 0 \). Consistent with the constraint Eq. (24b), the sum of vehicles crossing the border of the NOIR is equal to \( d_0 = 400 \) at every sampling time \( k \). Starting from a larger value, the amount of external traffic inflow \( u_i \) gradually decreases to a steady-state value at 200. It is seen that the external traffic outflow \( v_i \) symmetrically increases from a small value and reaches the steady state value at 200. The simulation results of external traffic flows for \( \beta = 0.5 \) and \( \beta = 1 \) shown in Fig. 6 illustrate a similar trend as when \( \beta = 0 \). Note the equilibrium (steady-state) condition, which implies that the traffic entering the NOIR is equal to the traffic exiting from the NOIR, is first observed at \( k \geq 20 \) where

\[ \forall k \geq 20, \quad \sum_{i \in \mathcal{V}_n} u_i[k] = \sum_{i \in \mathcal{V}_{out}} v_i[k] \approx 200 \]

(30)
VII. Conclusion

This paper introduces a physics-inspired approach based on the mass conservation law to model the traffic dynamics and implements the MPC method to control the boundary traffic inflow and outflow, so that the traffic congestion can be alleviated. Comparing with our previous research, the traffic dynamics model in this paper is more realistic. Simulation applied in a area of Philadelphia Center City demonstrates that the proposed traffic model and control approach can achieve the objective of traffic congestion alleviation successfully through controlling the boundary traffic flow. Our further research will focus on the integration of the Markov decision process (MDP) with the traffic dynamics model to control the traffic more efficiently and intelligently.

VIII. Acknowledgement

This work has been supported by the Department of Mechanical Engineering at Villanova University. The authors would like to gratefully acknowledge Dr. Sergey Nersesov for the useful comments on this paper, and the Mechanical Engineering PhD fellowship provided to Xun Liu which was made possible by a generous gift from Dr. Yongping Gu and Fei Gu.

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Property 3. For every $i \in \mathcal{V}_f$, $i \notin O_i$, therefore the diagonal entries of matrix $Q[k]$ are all 0.

Property 4. For $(i + N_{\text{out}}) \in \mathcal{V}_f$ and $O_{i+N_{\text{out}}} \cap \mathcal{V}_f = O_{i+N_{\text{out}}}$,
\[
\sum_{j=1}^{N-N_{\text{out}}} Q_{ji}[k] = 1,
\]
at each discrete time $k$.

Property 5. For $(i + N_{\text{out}}) \in \mathcal{V}_f$ and $O_{i+N_{\text{out}}} \cap \mathcal{V}_f \neq O_{i+N_{\text{out}}}$,
\[
\sum_{j=1}^{N-N_{\text{out}}} Q_{ji}[k] < 1,
\]
at each discrete time $k$.

Given the above introduced properties 1-5, the spectral radius of matrix $Q[k]$ is less than 1 [24], which indicates that the spectral radius of $Q[k]P[k]$ is less than the spectral radius of matrix $P[k]$. Therefore, the spectral radius of matrix $I - P[k]$ is less than the spectral radius of matrix $I - Q[k]P[k]$ which in turn implies that the spectral radius of matrix $A[k] = (I - P[k])(I - Q[k]P[k])^{-1}$ is less than 1 at every discrete time $k$.

Now, we can rewrite the traffic dynamics (16) as:
\[
x[k+1] = \Theta_k B[s[k]]
\]
where
\[
\Theta_k = [\Gamma_k \cdots \Gamma_1 \Gamma_0]
\]
\[
\Gamma_h = \prod_{j=h+1}^{k} A[j]
\]
for $h = 1, \cdots, k$, and $\Gamma_0 = 1$. $N_{\text{out}} \in \mathbb{R}^{(N-N_{\text{out}}) \times N-N_{\text{out}}}$ is an identity matrix. Because the initial traffic capacity vector $x[1]$ and control input vector $s[k]$ are bounded, we can say:
\[
x[1] \leq \varepsilon_{\text{max}} I_{N-N_{\text{out}}} x_1
\]
\[
B[s[k]] \leq \varepsilon_{\text{max}} I_{N-N_{\text{out}}} x_1
\]
where $\varepsilon_{\text{max}}$ is a sufficiently large value. Moreover, as shown in Eq. (32b), $\Gamma_h$ is the product of matrices $A[j]$. Therefore, we can draw the conclusion that the spectral radius of matrix $\Gamma_h$ must also be less than 1 at every discrete time $k$, since the spectral radius of matrix $A[k]$ is less than upper bound $r_u < 1$ at every sampling time $k$. Now, we calculate the norm of $||x[k+1]||^2$ as follow:
\[
x^T[k+1] x[k+1] \leq \varepsilon_{\text{max}} I_{N-N_{\text{out}}} \left( \sum_{l=0}^{k} \sum_{h=0}^{l} \Gamma_h^l \right) \varepsilon_{\text{max}} I_{N-N_{\text{out}}} x_1
\]
\[
\leq \varepsilon_{\text{max}}^2 (N-N_{\text{out}}) \sum_{l=0}^{k} r_u^l \leq \varepsilon_{\text{max}}^2 (N-N_{\text{out}}) \left( 1 - r_u \right).
\]
which implies that $||x[k+1]||^2$ is bounded at every discrete time $k$. Thus the BIBO stability of traffic dynamics (16) is proven.

APPENDIX

Proof of Theorem 1: Matrices $P[k]$ and $Q[k]$ defined in Eq. (13) hold the following properties:

Property 1. Because $p_i \in (0, 1)$ for each road $i \in \mathcal{V}_f$, entries in diagonal matrix $P[k]$ are all in the interval $(0, 1)$.

Property 2. Because $q_{i,j} \in [0, 1]$, for every road $j \in \mathcal{V}_f$ and its out-neighbor road $i \in O_j$, matrix $Q[k]$ is non-negative matrix.

Property 3. For every $i \in \mathcal{V}_f$, $i \notin O_i$, therefore the diagonal entries of matrix $Q[k]$ are all 0.

Property 4. For $(i + N_{\text{out}}) \in \mathcal{V}_f$ and $O_{i+N_{\text{out}}} \cap \mathcal{V}_f = O_{i+N_{\text{out}}}$,
\[
\sum_{j=1}^{N-N_{\text{out}}} Q_{ji}[k] = 1,
\]
at each discrete time $k$.

Property 5. For $(i + N_{\text{out}}) \in \mathcal{V}_f$ and $O_{i+N_{\text{out}}} \cap \mathcal{V}_f \neq O_{i+N_{\text{out}}}$,
\[
\sum_{j=1}^{N-N_{\text{out}}} Q_{ji}[k] < 1,
\]
at each discrete time $k$.

Given the above introduced properties 1-5, the spectral radius of matrix $Q[k]$ is less than 1 [24], which indicates that the spectral radius of $Q[k]P[k]$ is less than the spectral radius of matrix $P[k]$. Therefore, the spectral radius of matrix $I - P[k]$ is less than the spectral radius of matrix $I - Q[k]P[k]$ which in turn implies that the spectral radius of matrix $A[k] = (I - P[k])(I - Q[k]P[k])^{-1}$ is less than 1 at every discrete time $k$.

Now, we can rewrite the traffic dynamics (16) as:
\[
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for $h = 1, \cdots, k$, and $\Gamma_0 = 1$. $N_{\text{out}} \in \mathbb{R}^{(N-N_{\text{out}}) \times N-N_{\text{out}}}$ is an identity matrix. Because the initial traffic capacity vector $x[1]$ and control input vector $s[k]$ are bounded, we can say:
\[
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\]
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B[s[k]] \leq \varepsilon_{\text{max}} I_{N-N_{\text{out}}} x_1
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\[
x^T[k+1] x[k+1] \leq \varepsilon_{\text{max}} I_{N-N_{\text{out}}} \left( \sum_{l=0}^{k} \sum_{h=0}^{l} \Gamma_h^l \right) \varepsilon_{\text{max}} I_{N-N_{\text{out}}} x_1
\]
\[
\leq \varepsilon_{\text{max}}^2 (N-N_{\text{out}}) \sum_{l=0}^{k} r_u^l \leq \varepsilon_{\text{max}}^2 (N-N_{\text{out}}) \left( 1 - r_u \right).
\]
which implies that $||x[k+1]||^2$ is bounded at every discrete time $k$. Thus the BIBO stability of traffic dynamics (16) is proven.