Attraction of principal heteroclinic cycles

Sofia B. S. D. Castro†
sdcastro@fep.up.pt

Alexander Lohse∗†‡
alexander.lohse@math.uni-hamburg.de

∗ Corresponding author.
† Faculdade de Economia and Centro de Matemática, Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal.
‡ Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany.

Abstract

We provide a counterexample for a conjecture on the stability of principal heteroclinic cycles, i.e. subcycles consisting of connections tangent to the eigenspaces of the strongest expanding eigenvalues at the equilibria, which was stated in P. Ashwin and P. Chossat (1998) Attractors for robust heteroclinic cycles with continua of connections, Journal of Nonlinear Science 8, 103–129. This contributes to a clear distinction between heteroclinic dynamics when the jacobian matrix at equilibria has only real eigenvalues and when some eigenvalues are complex.

Keywords: heteroclinic cycle, heteroclinic network, asymptotic stability

AMS classification: 34C37, 37C80, 37C75

1 Introduction

Heteroclinic dynamics (involving equilibria and trajectories connecting them) appear in various real-life systems ranging from fluid dynamics to Lotka-Volterra-type models in a persistent way. When 1-dimensional connecting trajectories from an equilibrium $\xi_i$ to an equilibrium $\xi_j$ are contained in a higher-dimensional unstable manifold of $\xi_i$, does the connection tangent to the eigendirection of the greatest expanding eigenvalue attract the biggest proportion of initial conditions near the cycle? Answering this type of question is crucial when it comes to identifying (the most relevant) attractors in a system with a possibly large invariant heteroclinic set. If not, some competition may arise among the various 1-dimensional connections in the unstable manifold of $\xi_i$. Such a situation is described by Kirk and Silber [10] who construct a heteroclinic network (a connected union of finitely many heteroclinic cycles), consisting of two heteroclinic cycles. They show that, depending on a combination of the magnitudes of the eigenvalues, the cycles
take turns in attracting the biggest proportion of the initial conditions close to the network.

Ashwin and Chossat [1] conjecture that, under some hypotheses not satisfied by the example in [10], the answer to the question above is affirmative. The authors of [1] prove a special case of their conjecture when the cycle is homoclinic (connections are from one equilibrium to itself) and the eigenvalues are complex.

One of the simplest heteroclinic objects is a cycle with two hyperbolic equilibria and connections in both directions between these equilibria. We construct an equivariant vector field in \( \mathbb{R}^4 \) supporting such a heteroclinic cycle and satisfying the hypotheses of [1]. There is a 1-dimensional connection from one equilibrium to the other and a 2-dimensional connection in the opposite direction. Our construction is a modification of an example in Castro and Lohse [4]. We prove that the conjecture of [1] does not hold for this cycle.

Our result shows that heteroclinic dynamics involving equilibria at which the linearised dynamics are governed by complex eigenvalues is substantially different from heteroclinic dynamics when only real eigenvalues play a part. Several authors have contributed to the study of the stability of heteroclinic cycles and connections (see, for instance, Hofbauer [7], Melbourne [15], Krupa and Melbourne [12, 13] or Podvigina and Ashwin [16]) either by establishing asymptotic stability of heteroclinic objects or by defining and providing conditions to determine intermediate notions of stability. However, when two connections between a pair of equilibria are available, it is important to determine which one is the preferred one. The conjecture in [1] solves this problem in some instances. Our counterexample shows there are unanswered questions when only real eigenvalues are present in the linearization at the equilibria. These we leave for further research.

We finish this section with a brief description of the essential definitions and concepts. The following section constructs the vector field which we show to be a counterexample for the conjecture in [1]. The final section concludes and points towards relevant open questions in the context of stability in heteroclinic dynamics.

**Background:** We use the term heteroclinic cycle as in [1] where all the precise definitions and further detail can be found. We assume the reader is somewhat familiar with robust heteroclinic cycles in a symmetric context, for a comprehensive overview we refer to Krupa [11]. In what follows we consider dynamics induced by an ODE

\[ \dot{x} = f(x), \]  

(1)
where \( x \in \mathbb{R}^n \) and \( f \) is smooth and \( \Gamma \)-equivariant for some finite group \( \Gamma \subset O(n) \).

Given two hyperbolic equilibria \( \xi_i \) and \( \xi_j \) of system (1), a connecting trajectory between them exists in \( W^u(\xi_i) \cap W^s(\xi_j) \) if this intersection is non-empty. A heteroclinic cycle is a sequence of such connecting trajectories among a set of finitely many distinct equilibria \( \xi_1, \ldots, \xi_m \) such that \( \xi_{m+1} = \xi_1 \). The heteroclinic cycle is the union of the equilibria and the connections.

In what follows \( C_{ij} = W^u(\xi_i) \cap W^s(\xi_j) \) denotes the set of trajectories connecting two equilibria \( \xi_i \) and \( \xi_j \). At an equilibrium \( \xi_i \) the set of principal connections is \( C^p_{ij} = W^{pu}(\xi_i) \cap W^{s}(\xi_j) \), where \( W^{pu}(\xi_i) \) is the invariant manifold of trajectories tangent to the generalized eigenspace of the strongest expanding eigenvalue at \( \xi_i \). A principal cycle is comprised only of principal connections.

We use attractor in the sense of Milnor, as in Definition 3 of [1], that is, a Milnor attractor is a compact invariant set whose basin of attraction has positive Lebesgue measure.

2 The conjecture and its counterexample

The conjecture of [1] states that for a closed heteroclinic cycle among equilibria \( \xi_1, \ldots, \xi_m \) satisfying (Ha)-(Hd) and (3) below, generically, the principal heteroclinic cycle is an attractor. The hypotheses are:

(Ha) for any non-empty connection \( C_{ij} \) there exists an isotropy subgroup \( \Sigma \) such that all trajectories in \( C_{ij} \) have isotropy \( \Sigma \),

(Hb) for any \( \Sigma \) in (Ha) \( \xi_j \) is a sink for the flow restricted to \( \text{Fix}(\Sigma) \),

(Hc) the heteroclinic cycle contains all unstable manifolds of its equilibria,

(Hd) the eigenspaces tangent to connections \( C_{ij} \) and \( C_{jk} \) at \( \xi_j \in \text{Fix}(\Delta_j) \) lie within a single \( \Delta_j \)-isotypic component of \( \mathbb{R}^n \).

(3) Let \( -\bar{c}_j \) be the weakest contracting eigenvalue and \( \bar{e}_j \) be the strongest expanding eigenvalue at \( \xi_j \). Then \( \prod_{j=1}^m \bar{c}_j > \prod_{j=1}^m \bar{e}_j \).

Our counterexample consists in the following modification of a vector field
generating a \((B_+^+, B_+^-)\) network\(^4\) from \([4]\),

\[
\begin{align*}
\dot{x}_1 &= x_1 + \sum_{i=1}^{4} b_{1i} x_i^2 + c_1 x_1^3 \\
\dot{x}_2 &= x_2 + x_1 \sum_{i=1}^{4} b_{2i} x_i^2 + d_2 x_1 x_2 \\
\dot{x}_3 &= x_3 + x_3 \sum_{i=1}^{4} b_{3i} x_i^2 + c_3 x_3^2 x_4 + d_3 x_1 x_3 \\
\dot{x}_4 &= x_4 + x_4 \sum_{i=1}^{4} b_{4i} x_i^2 + c_4 x_3 x_2^4 + d_4 x_1 x_4
\end{align*}
\]

(2)

where all constants are real and chosen conveniently below.

This vector field is equivariant under the action of the group \(\Gamma \cong \mathbb{Z}_2^2\) generated by

\[
\kappa_2. (x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, x_4),
\]

\[
\kappa_{34}. (x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4).
\]

The isotypic decomposition of \(\mathbb{R}^4\) with respect to \(\Gamma\) is

\[
\mathbb{R}^4 = L_1 \oplus L_2 \oplus P_{34},
\]

(3)

where \(L_i\) is the \(i\)-th coordinate axis and \(P_{ij} = L_i \oplus L_j\).

We choose coefficients in an open set such that the system (2) possesses two equilibria \(\xi_a = (x_a, 0, 0, 0), \xi_b = (x_b, 0, 0, 0) \in L_1\) such that \(x_a < 0 < x_b\);

(ii) in \(P_{12}\), \(\xi_a\) is a saddle and \(\xi_b\) is a sink; furthermore, there is a connection \(C_{ab} \subset P_{12}\);

(iii) in \(S_{123} := L_1 \oplus P_{34}\), \(\xi_a\) is a sink and \(\xi_b\) is a saddle; furthermore, there is a continuum of connections \(C_{ba} \subset L_1 \oplus P_{34}\);

(iv) the principal connection \(C_{ba}^0\) from \(\xi_b\) to \(\xi_a\) lies in \(P_{13}\).

To achieve such a choice we proceed as follows, collecting the conditions in Table \([4]\). Choose \(b_{11} \neq 0\) and \(b_{11}^2 - 4c_1 > 0\) so that \(x_a \neq -x_b\). In \(L_1\), \(x_a\) and \(x_b\) satisfy

\[1 + b_{11} x_{a/b} + c_1 x_{a/b}^2 = 0 \iff x_{a/b} = \frac{-b_{11} \pm \sqrt{b_{11}^2 - 4c_1}}{2c_1}.
\]

Choose \(c_1 < 0\) so that

\[x_a = \frac{-b_{11} + \sqrt{b_{11}^2 - 4c_1}}{2c_1} < 0 \quad \text{and} \quad x_b = \frac{-b_{11} - \sqrt{b_{11}^2 - 4c_1}}{2c_1} > 0,
\]

\(^4\)In \([4]\) another definition of heteroclinic cycle and network is used which is why the heteroclinic object is called a network. In the context of the present article it is a heteroclinic cycle.
proving (i).

To prove (ii), choose coefficients so that the flow far from the origin points inwards towards the origin. For example, $b_{22} < 0$ besides the choice already made of $c_1 < 0$. Define a region $D \subset P_{12}$ bounded by the horizontal axis and the two vertical lines $x_1 = x_{a/b}$. If we choose $b_{12} > 0$ then $\dot{x}_1 > 0$ if $x_1 \in \{\xi_a, 0, \xi_b\}$ and (see the next proof of stability below) the unstable manifold of $\xi_a$ and the stable manifold of $\xi_b$ are as pictured in Figure 1.

![Figure 1: The flow in $D \subset P_{12}$.](image)

Since the origin is a source, it suffices to show there are no equilibria in $D$ to ensure the existence of $C_{ab}$. Equilibria in $P_{12}$, outside the axes, satisfy

$$
\begin{cases}
x_1 + b_{11}x_1^2 + b_{12}x_2^2 + c_1x_1^3 = 0 \\
1 + b_{21}x_1^2 + b_{22}x_2^2 + d_2x_1 = 0
\end{cases}
\iff
\begin{cases}
x_2 = -\frac{1}{b_{22}} (x_1 + b_{11}x_1^2 + c_1x_1^3) \\
x_2 = -\frac{1}{b_{22}} (1 + b_{21}x_1^2 + d_2x_1)
\end{cases}.
$$

The right-hand side of the first equation is a cubic with zeros at the origin, $\xi_{a/b}$. The right-hand side of the second equation is a parabola with a positive minimum value provided $b_{21} > 0$. There is always a solution for this system of equations but, by choosing $b_{22}$ close to zero, the parabola is pulled upwards thus guaranteeing that it occurs outside $D$. See Figure 2.

To prove the stability claims in (ii) and (iii), note that at the equilibria the Jacobian matrix is diagonal with the following entries:

$$
b_{11}x_{a/b} + 2c_1x_{a/b}, \quad 1 + b_{21}x_{a/b} + d_2x_{a/b}, \\
1 + b_{31}x_{a/b} + d_3x_{a/b}, \quad 1 + b_{41}x_{a/b} + d_4x_{a/b}.
$$

At $\xi_a$ and $\xi_b$, the eigenvalue along $L_1$ is, respectively,

$$
b_{11}x_a + 2c_1x_a^2 = x_a\sqrt{b_{11}^2 - 4c_1} < 0 \quad \text{and} \quad b_{11}x_b + 2c_1x_b^2 = -x_b\sqrt{b_{11}^2 - 4c_1} < 0.
$$
The eigenvalue along $L_2$ can be written at the equilibria as

$$1 - \frac{b_{21}}{c_1} + \left( d_2 - \frac{b_{21}b_{11}}{c_1} \right) x_{a/b}. $$

Since $x_a < 0 < x_b$, by choosing $d_2 - \frac{b_{21}b_{11}}{c_1} < 0$ and large in absolute value, we achieve the desired signs:

$$1 - \frac{b_{21}}{c_1} + \left( d_2 - \frac{b_{21}b_{11}}{c_1} \right) x_a > 0 \quad \text{and} \quad 1 - \frac{b_{21}}{c_1} + \left( d_2 - \frac{b_{21}b_{11}}{c_1} \right) x_b < 0. $$

Analogous calculations for the eigenvalues along $L_3$ and $L_4$ at the equilibria show that if $d_3 - \frac{b_{31}b_{11}}{c_1} > 0$ and $d_4 - \frac{b_{41}b_{11}}{c_1} > 0$ and large in absolute value, both eigenvalues are negative at $\xi_a$ and positive at $\xi_b$.

To finish proving (iii) and ensure the existence of $C_{ba}$ we choose coefficients $b_{33}, b_{44}, c_3, c_4 < 0$ so that infinity is repelling and proceed as in the proof of the existence of $C_{ab}$ to ensure connections in $P_{13}$ and $P_{14}$.

Next, we observe that the set of points $Z$ in $S_{134}$ where $\dot{x}_1 = 0$ is determined by $x_1 + b_{11}x^2 + c_1x^3 = -b_{13}x_3^2 - b_{14}x_4^2$. In planes of constant $x_1$ this is an ellipse, see Figure 3. Outside of $Z$ we have $\dot{x}_1 < 0$ (inside we have $\dot{x}_1 > 0$) provided $b_{13}, b_{14} < 0$, so that trajectories on $W^u(\xi_b)$, which is tangent to the affine plane spanned by $x_3$ and $x_4$, have decreasing $x_1$ component close to $\xi_b$ and therefore move in the direction of $\xi_a$. Since any equilibria in $S_{134}$ lie in $Z$ then all the trajectories in $W^u(\xi_b)$ approach $\xi_a$.

To prove (iv), choose coefficients so that at $\xi_b$ the eigenvalue along $x_3$ is
Figure 3: The set $Z$ (thick lines) in $S_{134}$.

greater than that along $x_4$, that is, so that

$$1 + b_{31}x_b^2 + d_3x_b > 1 + b_{41}x_b^2 + d_4x_b \iff x_b > \frac{d_4 - d_3}{b_{31} - b_{41}},$$

which is trivially verified if the right-hand side is negative.

The conditions imposed so far correspond to C1–C15 in Table 1. It is clear that C1–C11 are compatible and define an open set in the space of all coefficients. Writing

$$C_{13} - C_{12} = d_4 - d_3 - \frac{b_{41}}{c_1}(b_{41} - b_{31}),$$

relates these conditions to C14 and C15, ensuring the compatibility of C11–C15. In particular, $C_{13} - C_{12} > 0$.

**Theorem 2.1.** The system (2) with the choice of coefficients in Table 1 satisfies (Ha)–(Hd) and (3) and the principal cycle is not an attractor.

**Proof.** We verify the hypotheses:

(Ha) The connection $C_{ab}$ has isotropy $\{\text{Id}, \kappa_{34}\}$ and $C_{ba}$ has isotropy $\{\text{Id}, \kappa_2\}$.

(Hb) This is clear by (ii) in the construction above.

(Hc) The 2-dimensional unstable manifold of $\xi_b$ is just the set of connections $C_{ba}$. Analogously, the 1-dimensional unstable manifold of $\xi_a$ is the connection $C_{ab}$.

(Hd) At $\xi_a$ and $\xi_b$, the eigenspace tangent to $C_{ab}$ is $L_2$ and that tangent to $C_{ba}$ is $P_{34}$. 

7
(3) Direct inspection shows that
\[
\bar{e}_a = 1 - \frac{b_{21}}{c_1} + \left( d_2 - \frac{b_{21}}{c_1} b_{11} \right) x_a
\]
\[
\bar{e}_b = 1 - \frac{b_{31}}{c_1} + \left( d_3 - \frac{b_{31}}{c_1} b_{11} \right) x_b
\]
\[
\bar{c}_b = 1 - \frac{b_{21}}{c_1} + \left( d_2 - \frac{b_{21}}{c_1} b_{11} \right) x_b,
\]
whereas we have
\[
\bar{c}_a = 1 - \frac{b_{31}}{c_1} + \left( d_3 - \frac{b_{31}}{c_1} b_{11} \right) x_a \quad \text{or} \quad \bar{c}'_a = 1 - \frac{b_{41}}{c_1} + \left( d_4 - \frac{b_{41}}{c_1} b_{11} \right) x_a.
\]

Using \( \bar{c}_a \) we obtain \( \prod_{j=a,b} \bar{e}_j > \prod_{j=a,b} \bar{e}_j \) if and only if
\[
(c_1 - b_{31})(d_2 c_1 - b_{21} b_{11}) < (c_1 - b_{21})(d_3 c_1 - b_{31} b_{11}).
\]

The choice of \( \bar{c}'_a \) produces the analogous inequality by replacing the index 3 by 4.

Under these circumstances the principal cycle consists of the two equilibria together with \( C_{ab} \) and the trajectory \( C^p_{ba} = C_{ba} \cap P_{13} \). In order to apply Theorem A.1(ii) in [4], we observe that \( \rho, \tilde{\rho} > 1 \) follows from condition (3) above. We write the condition \( \delta > 0 \) in [4] equivalently in the coefficients of (2) as follows
\[
(1 + d_3 x_a + b_{31} x_a^2)(1 + d_4 x_b + b_{41} x_b^2) - (1 + d_3 x_b + b_{31} x_b^2)(1 + d_4 x_a + b_{41} x_a^2) > 0.
\]
This is equivalent to
\[
d_4 - d_3 + (b_{41} - b_{31})(x_a + x_b) + (d_3 b_{41} - d_4 b_{31}) x_a x_b > 0,
\]
where the only quantities not already chosen to be positive are: \( x_a + x_b \) has the sign of \( b_{11} \) and can be chosen positive, and \( d_3 b_{41} - d_4 b_{31} \) which we choose to be negative so that its product with \( x_a x_b < 0 \) is positive.

Then by Theorem A.1(ii) in [4] the (non-principal) cycle consisting of the equilibria together with \( C_{ab} \) and the trajectory \( C_{ba} \cap P_{14} \) attracts a set of positive measure of nearby initial conditions, while the principal cycle does not.\(^2\) Note that even though none of these cycles is simple\(^3\) in the sense of

---

\(^2\)For the reader familiar with the concept of local stability index introduced by Podvigina and Ashwin [16] and its relation with determining the stability of compact invariant sets both in [10] and in [14], Theorem A.1(ii) in [4] establishes that the local stability indices of both connections in the principal cycle are equal to \( -\infty \), whereas those of the two connections in the non-principal cycle are equal to \( +\infty \) and negative but finite, respectively. Since having at least one stability index greater than \( -\infty \) is equivalent to being an attractor, this proves our claim.

\(^3\)The cycles are not simple because \( P_{13} \) and \( P_{14} \) are not fixed point spaces.
Table 1: List of conditions imposed on the coefficients of (2) in the construction of the vector field and the proof of Theorem 2.1. The first column assigns a label to each condition and the third column states the purpose of the restriction. The middle column contains the conditions on the coefficients.

Krupa and Melbourne [13], Theorem 3.10 in [6] guarantees that the stability properties are the same as those in Theorem A.1 in [4].

Note that C11 implies that in C17 we have \(d_2c_1 - b_{21}b_{11} > 0\). Also, C12 implies that in C17 we have \(d_3c_1 - b_{31}b_{11} < 0\). Then C17 can be satisfied by setting \(c_1 - b_{31} < 0\) and \(c_1 - b_{21} < 0\). Observe also that C16 is compatible with C14 and C15. Hence, the subset of coefficients satisfying the inequalities in Table 1 is open.
3 Concluding remarks

Our construction of the counterexample above illustrates the fact that dynamics ruled by real eigenvalues are quite different from those ruled by complex eigenvalues. Although this is not surprising, it is worth noting. To further explore this effect, it would be interesting to check the conjecture in examples such as that given by Kirk et al. [9], of a heteroclinic network with some two-dimensional connections and complex eigenvalues at one of the equilibria. Its asymptotic stability stands in contrast to a recent result in Podvigina et al. [17], stating that heteroclinic networks with 1-dimensional connections and only real eigenvalues cannot be asymptotically stable.

Given our counterexample, a good understanding of the dynamics near heteroclinic cycles for which only real eigenvalues exist requires more than the knowledge and comparison of the magnitude of eigenvalues. Such heteroclinic cycles appear naturally in the context of population dynamics and game theory when restricting the state space to a finite-dimensional simplex, see Hofbauer and Sigmund [8] and references therein. The flow-invariance of the edges of the simplex creates heteroclinic cycles with connections along the edges and only real eigenvalues. These are called edge cycles/networks by Field [5].

The work of Rodrigues [18] provides a first step in one of the directions opened by the fact that the conjecture of [1] does not hold true: that of finding minimal conditions such that it does. Our example is excluded from the result in [18] by their assumption that all connections are 1-dimensional. This also points towards another interesting possibility of broadening the scope of the conjecture in [1], i.e. asking when principal subcycles of heteroclinic networks (rather than cycles) are attracting. Of course, attention must then be paid to the distinction between cycle and network, which is not obvious as mentioned above. Much in this context is still open to understanding.

Acknowledgements: The authors are grateful to P. Ashwin for some helpful comments.

The first author was, in part, supported by CMUP (UID/MAT/00144/2013), funded by the Portuguese Government through the Fundação para a Ciência e a Tecnologia (FCT) with national (MEC) and European structural funds through the programs FEDER, under the partnership agreement PT2020.

Both authors benefitted from DAAD-CRUP funding through “Ação Integrada Luso-Alemã A10/17”, which on the German side is funded as project 57338573 PPP Portugal 2017 by the German Academic Exchange Service (DAAD), sponsored by the Federal Ministry of Education and Research (BMBF).
References

[1] P. Ashwin and P. Chossat (1998) Attractors for robust heteroclinic cycles with continua of connections, *Journal of Nonlinear Science* 8, 103–129.

[2] W. Brannath (1994) Heteroclinic networks on the tetrahedron. *Nonlinearity* 7, 1367–1384.

[3] S.B.S.D. Castro, I.S. Labouriau and O. Podvigina (2010) A heteroclinic network in mode interaction with symmetry, *Dynamical Systems: an international journal* 25 (3), 359–396.

[4] S.B.S.D. Castro and A. Lohse (2014) Stability in simple heteroclinic networks in $\mathbb{R}^4$, *Dynamical Systems: an International Journal* 29 (4), 451–481.

[5] M.J. Field (2017) Patterns of desynchronization and resynchronization in heteroclinic networks, *Nonlinearity* 30, 516–557.

[6] L. Garrido-da-Silva and S.B.S.D. Castro (2017) Stability of quasi-simple heteroclinic cycles, *Dynamical Systems: an International Journal*, to appear.

[7] J. Hofbauer (1994) Heteroclinic cycles in ecological differential equations, *Tatra Mountains Math. Publ.* 4, 105–116

[8] J. Hofbauer and K. Sigmund, (1998) Evolutionary Games and Population Dynamics, Cambridge University Press, Cambridge.

[9] V. Kirk, E. Lane, C. Postlethwaite, A.M. Rucklidge and M. Silber (2010), A mechanism for switching near a heteroclinic network, *Dynamical Systems: an International Journal* 25 (3), 323–349.

[10] V. Kirk and M. Silber (1994) A competition between heteroclinic cycles, *Nonlinearity* 7, 1605–1621.

[11] M. Krupa (1997) Robust Heteroclinic Cycles, *Journal of Nonlinear Science* 7, 129–176.

[12] M. Krupa and I. Melbourne (1995) Asymptotic stability of heteroclinic cycles in systems with symmetry, *Ergod. Theory Dyn. Sys.* 15, 121–147.

[13] M. Krupa and I. Melbourne (2004) Asymptotic stability of heteroclinic cycles in systems with symmetry II, *Proc. Royal Soc. Edin.* 134, 1177–1197.
[14] A. Lohse (2014) Attraction properties and non-asymptotic stability of simple heteroclinic cycles and networks in $\mathbb{R}^4$, PhD Thesis, University of Hamburg, Germany: http://ediss.sub.uni-hamburg.de/frontdoor.php?source_opus=6795

[15] I. Melbourne (1991) An example of a non-asymptotically stable attractor, *Nonlinearity* 4, 835–844.

[16] O. Podvigina and P. Ashwin (2011) On local attraction properties and a stability index for heteroclinic connections, *Nonlinearity* 24, 887–929.

[17] O. Podvigina, S.B.S.D. Castro and I.S. Labouriau (2018) Stability of a heteroclinic network and its cycles: a case study from Boussinesq convection, *Dynamical Systems: an International Journal*, to appear.

[18] A.A.P. Rodrigues (2017) Attractors in complex networks, *Chaos* 27, 103105