On point-like interaction of three particles: two fermions and another particle. II

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Abstract

This work continues [1] where the construction of Hamiltonian $H$ for the system of three quantum particles is considered. Namely the system consists of two fermions with mass 1 and another particle with mass $m > 0$. In the present paper, like in [1], we study the part $T_{l=1}$ of auxiliary operator $T = \oplus_{l=0}^{\infty} T_l$ involving the construction of the resolvent for the operator $H$. In this work together with the previous one two constants $0 < m_1 < m_0 < \infty$ were found such that: 1) for $m > m_0$ the operator $T_{l=1}$ is selfadjoint but for $m \leq m_0$ it has the deficiency indexes $(1, 1)$; 2) for $m_1 < m < m_0$ any selfadjoint extension of $T_{l=1}$ is semibounded below; 3) for $0 < m < m_1$ any selfadjoint extension of $T_{l=1}$ has the sequence of eigenvalues $\{\lambda_n < 0, n > n_0\}$ with the asymptotics

$$\lambda_n = \lambda_0 e^{\delta n} + O(1), \quad n \to \infty,$$

where $\lambda_0 < 0, \delta > 0, n_0 > 0$ and there is not other spectrum on the interval $\lambda < \lambda_{n_0}$.

1 Introduction and main results

This paper is continuation of work [1] (see also [2–5]) devoted to construction of Hamiltonian for the system of three point-like interacting particles: two fermions with mass 1 and a different particle with mass $m > 0$. In the mentioned papers the construction of Hamiltonian begins with the introduction of the symmetric operator

$$H_0 = -\frac{1}{2} \left( \frac{1}{m} \Delta_y + \Delta_{x_1} + \Delta_{x_2} \right) \quad (1.1)$$

where $x_1, x_2 \in \mathbb{R}^3$ are the positions of the fermions, $y \in \mathbb{R}^3$ is the position of another particle, $\Delta_y, \Delta_{x_1}, \Delta_{x_2}$ are Laplacians w.r.t. these variables. The operator $H_0$ is given on the set $D(H_0) \subset L^2(\mathbb{R}^3) \otimes L^2_{\text{asym}}(\mathbb{R}^3 \times \mathbb{R}^3)$ consisting

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of smooth functions $\Psi(y, x_1, x_2)$ rapidly decreasing at infinity, antisymmetric w.r.t. variables $x_1, x_2$ and satisfying the following conditions:

$$\Psi(y, x_1, x_2)\big|_{y=x_i} = 0, \quad i = 1, 2. \quad (1.2)$$

The operator $H_0$ has non-zero deficiency indexes and “true” Hamiltonian is contained among its selfadjoint extensions. Usually a one-parametric family $\{H_\varepsilon, \varepsilon \in \mathbb{R}\}$ of such extensions is considered, the so-called Ter-Martirosian–Scornyakov’s extensions (see [6, 7]). However, for some values of parameter $m$ this extension is not selfadjoint.

In such cases a problem of description of selfadjoint extensions for $H_\varepsilon$ arises. It is shown in [4, 5] that this problem is equivalent to the one for some auxiliary symmetric operator $T$ acting in the space $L_2(\mathbb{R}^3)$ according to the formula:

$$(Tf)(k) = 2\pi^2 \left( \sqrt{1 - \left(\frac{\mu}{2}\right)^2} k^2 + 1 \right) f(k)$$

$$+ \int_{\mathbb{R}^3} \frac{f(p) dp}{k^2 + p^2 + \mu(p, k) + 1}, \quad k \in \mathbb{R}^3$$

($\mu = 2/(m + 1)$). This operator is defined on the set

$$D(T) = \{ f \in L_2(\mathbb{R}^3) : |k|f(k) \in L_2(\mathbb{R}^3) \}. \quad (1.4)$$

In addition, every selfadjoint extension of $T$ generates by specific way a self-adjoint extension of $H_\varepsilon$ (see [3]) and so all extensions of $H_\varepsilon$ are formed. In particular any eigenvalue $\lambda$ of selfadjoint extension $\hat{T}$ of the operator $T$ generates a negative eigenvalue of the corresponding extension of $H_\varepsilon$ that is equal to

$$- \left(\frac{\lambda}{\varepsilon}\right)^2. \quad (1.5)$$

The operator $T$ commutes with the operators of representation of rotation group $O_3$:

$$(\tau_g f)(k) = f(g^{-1}k), \quad g \in O_3 \quad (1.6)$$

acting in the space $L_2(\mathbb{R}^3)$. Thus the operator $T$ is decomposed in the direct sum of the operators $T_l$, $l = 0, 1, \ldots$ acting each in the subspace

$$\mathcal{H}_l = L_2(\mathbb{R}^3 \cup r^2 dr) \otimes L_2^l(S) \subset L_2(\mathbb{R}^3) \quad (1.7)$$

correspondingly. Here $L_2^l(S) \subset L_2(S)$ is the subspace of functions on a unit sphere $S \subset \mathbb{R}^3$ where the irreducible representation of the group $O_3$ with weight $l$ acts (subspace of the spherical functions of weight $l$, see [7]). The operator $T_l$ is tensor product of operators:

$$T_l = M_l \otimes E_l, \quad (1.8)$$
where $E_l$ is unit operator in $L_2^2(S)$ and the symmetric operator $M_l$ acts in the space $L_2^{2}(\mathbb{R}^{+}, r^2 \, dr)$ by formula

$$
(M_l \varphi)(r) = 2\pi^2 \left( \sqrt{1 - \left( \frac{\mu}{2} \right)^2} r^2 + 1 + \int_{-1}^{1} P_l(x) \, dx \right) \int_{0}^{\infty} \frac{(r')^2 \varphi(r') \, dr'}{\left( r'^2 + r^2 + \mu^2 + r^2 \right)^{1/2}}
$$

(1.9)

where $\{P_l(x), l = 0, 1, \ldots\}$ are polynomials of Legendre with conditions: $P_l(1) = 1$. The domain of $M_l$ is

$$
D(M_l) = \{ \varphi \in L_2^{2}(\mathbb{R}^{+}, r^2 \, dr) : r \varphi(r) \in L_2^{2}(\mathbb{R}^{+}, r^2 \, dr) \}. 
$$

(1.10)

Note that every selfadjoint extension of $T$ commuting with the operators $\tau_g$ (see (1.6)) generates a selfadjoint extension of each operator $M_l$ and vice versa any collection of selfadjoint extensions of all operators $M_l$ reduces to the one for the operator $T$ (commuting with $\tau_g$, $g \in O_3$).

In [1] the operators $M_{l=0}$ and $M_{l=1}$ were studied in detail and it was proved that the operator $M_{l=0}$ is selfadjoint and semibounded below for all values of $m$ while for the operator $M_{l=1}$ there exists a constant $m_0$ such that for $m > m_0$ this operator is selfadjoint and bounded below but for $m < m_0$ the operator $M_{l=1}$ has deficiency indexes $(1, 1)$. In this paper these results are complemented by the next assertion:

**Theorem 1.1.** There exists a constant $0 < m_1 < m_0$ so that:

1. for $m \in [m_1, m_0)$ the operator $M_{l=1}$ is bounded below,

$$
\inf_{\|\varphi\| = 1} (M_{l=1} \rho, \varphi) = k > -\infty,
$$

(1.11)

and the part of spectrum of any selfadjoint extension $M_{l=1}^\beta$ of the operator $M_{l=1}$ (the parameter $\beta$, $|\beta| = 1$, is introduced below) lying below $k$ consists of not more than one eigenvalue;

2. for $m \in (0, m_1)$ the operator $M_{l=1}$ is not bounded below and any its selfadjoint extension $M_{l=1}^\beta$ has the sequence of eigenvalues $\{\lambda_n, n > n_0\}$ going off to $-\infty$ with the asymptotics

$$
\lambda_n = \lambda_0 e^{\alpha n} + O(1), \quad n \to \infty
$$

(1.12)

where $\lambda_0 < 0$ and $n_0$ depends on the extension, but $\delta > 0$ is the same for all extensions;

3. for every selfadjoint extension $M_{l=1}^\beta$ there exists a constant $\kappa = \kappa(\beta) < 0$ so that the part of spectrum of $M_{l=1}^\beta$ lying below $\kappa$ consists only of the elements of the sequence (1.12).
Remark 1.1. Evidently, an eigenvalue \( \lambda \) of any selfadjoint extension of \( M_{l=1} \) is a triple eigenvalue of the corresponding extension of the operator \( T_{l=1} \) and hence of some extensions of the operator \( \mathcal{T} \). From here it follows that there exist selfadjoint extensions of \( H_\varepsilon \) having a sequence of eigenvalues of form
\[
-\left( \frac{\lambda_n}{\varepsilon} \right)^{-2}, \quad n > n_0,
\]
as it follows from (1.5) (\( \lambda_n \) are defined by (1.12)). The sequence (1.13) approaches zero for \( n \to \infty \) i.e. the Efimov’s effect appears.

2 Some reminders and beginning of the proof of Theorem 1.1

Here we explain some facts from the previous works (see [1, 5]).

1. The operator \( M_{l=1} \) can be represented in form
\[
M_{l=1} = M_0 + M_1
\]
where \( M_1 \) is a bounded selfadjoint operator and \( M_0 \) is the symmetric operator acting by the formula
\[
(M_0 \varphi)(r) = 2\pi^2 \left( \sqrt{1 - \left( \frac{\mu}{2} \right)^2} r \varphi(r) + \int_{-1}^{1} P_1(x) \, dx \right) \int_{0}^{\infty} \frac{(r')^2 \varphi(r') \, dr'}{(r' + r^2 + \mu x)^2}
\]
on the domain \( D(M_0) = D(M_{l=1}) \).

Below we shall study mainly the operator \( M_0 \) returning from time to time to the operator \( M_{l=1} \).

2. Denote by \( \mathcal{J} \) the set of functions \( F(z) \) defined in the closed strip
\[
I = \{ z \in \mathbb{C}^1 : 0 \leq \Im z \leq 1 \},
\]
continuous and analytical inside \( I \), satisfying the condition
\[
\sup_{0 \leq t \leq 1} \int_{-\infty}^{\infty} |F(s + it)|^2 \, ds < \infty.
\]
The contraction of the function \( F \in \mathcal{J} \) on the lines
\[
l_+ = \{ z = s, s \in \mathbb{R}^1 \}, \quad l_- = \{ z = s + i, s \in \mathbb{R}^1 \}
\]
limiting the strip \( I \) is denoted by \( F_+(s) \) and \( F_-(s) \) correspondingly. Denote by \( j \in L^2(\mathbb{R}^1) \) the set of functions \( f(s) \in L^2(\mathbb{R}^1) \) which have the form
\[
f(s) = F_+(s), \quad s \in \mathbb{R}^1
\]
for some (and unique) function $F \in J$. This function $F = F(f)$ is called 
*superstructure* on $f$.

Consider unitary Mellin’s map

$$\omega : L_2(\mathbb{R}_1^+, r^2 dr) \rightarrow L_2(\mathbb{R}_1^1, ds) : \varphi \rightarrow f = f(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-is+1/2} \varphi(r) \, dr$$

(2.6)

and the inverse map:

$$\omega^{-1} : f \rightarrow \varphi = \varphi(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty r^{is-3/2} f(s) \, ds.$$  (2.7)

The map $\omega$ transfers the domain $D(M_0)$ of the operator $M_0$ in the set $j : \omega(D(M_0)) = j$. In addition the operator $M_0$ is passed by $\omega$ to a unitary equivalent operator $\tilde{M}_0$:

$$\tilde{M}_0 = \omega M_0 \omega^{-1}$$  (2.8)

acting in the space $L_2(\mathbb{R}_1^1, ds)$ with the help of the formula

$$(\tilde{M}_0 f)(s) = (N(z) F(z))_-(s) = N_-(s) F_-(s), \quad f \in D(\tilde{M}_0)$$  (2.9)

where $F = F(f)$ is superstructure on $f$ and $N(z)$ is analytical function defined in the strip $I$:

$$N(z) = 2\pi^2 \left( \sqrt{1 - \left( \frac{\mu}{2} \right)^2} - \Lambda(z) \right)$$  (2.10)

where

$$\Lambda(z) = \int_0^1 x \, dx \frac{\text{sh}(v(x)(z - i/2))}{\cos(v(x)) \cdot \text{sh}((\pi/2)(z - i/2))}$$  (2.11)

and $v(x) = \arcsin \mu x / 2$.

Note that the function $N(z)$ is invariant w.r.t. the reflection of the strip $N(z)$ into itself:

$$z \rightarrow z^* = -z + i.$$  (2.12)

3. It is shown in [1] that the operator $\tilde{M}_0$ is selfadjoint for those values of $\mu = 2/(m + 1)$ which lie in the interval $(0, \mu_0)$ where $\mu_0$ is a unique root of the equation

$$\sqrt{1 - \left( \frac{\mu}{2} \right)^2} = \Lambda(0) \equiv q_0(\mu).$$  (2.13)
For \( \mu \geq \mu_0 \) the function \( N(z) \) has two simple zeros \( z_{\pm} \in I \) and the operator \( \tilde{M}_0 \) is not selfadjoint; it has deficiency indexes \((1, 1)\). Further, for \( \mu \in (\mu_0, \mu_1) \), where \( \mu_1 > \mu_0 \) is a unique root of the equation

\[
\sqrt{1 - (\frac{\mu}{2})^2} = \Lambda \left( \frac{i}{2} \right) \equiv q_1(\mu), \quad (2.14)
\]

the zeros \( z_{\pm} \) lie on the unit interval of imaginary axis \( \tau = (0, i) \):

\[
z_{\pm} = \frac{i}{2} \pm t_0, \quad t_0 = t_0(\mu) \in \left(0, \frac{1}{2}\right), \quad (2.15)
\]

For \( t_0 = 0 \) these zeros coincide, forming a unique zero of \( N(z) \) with multiplicity 2. For \( \mu \in (\mu_1, 2) \) the zeros \( z_{\pm} \) lie on the line

\[
l_{1/2} = \{z = s + i/2, s \in \mathbb{R}^1\}
\]

and have the form

\[
z_{\pm} = \frac{i}{2} \pm s_0, \quad s_0 = s_0(\mu) \in (0, \infty). \quad (2.16)
\]

The graphs of functions \( \sqrt{1 - (\mu/2)^2} \), \( q_0(\mu) \) and \( q_1(\mu) \) are presented in Figure 1.

![Figure 1](image-url)

4. Note that

\[
\min_{z \in \tau} \Lambda(z) = \Lambda(i/2) = \max_{z \in l_{1/2}} \Lambda(z) \quad (2.17)
\]

and consequently, for \( \mu \in (\mu_0, \mu_1) \), for \( \Lambda(i/2) \leq \sqrt{1 - (\mu/2)^2} \),

\[
N(z)\big|_{l_{1/2}} = N\left(\frac{i}{2} + s\right) \geq 0. \quad (2.18)
\]

On the other hand, as it follows from the construction in [1] the quadratic form \( (\tilde{M}_0 f, f) \) can be represented as

\[
(\tilde{M}_0 f, f) = \int_{-\infty}^{\infty} \left| F\left(\frac{i}{2} + s\right) \right|^2 N\left(\frac{i}{2} + s\right) ds, \quad f \in D(\tilde{M}_0) \quad (2.19)
\]
where \( F = F(f) \) is superstructure on \( f \). From (2.18) and (2.19) it follows that the operator \( \tilde{M}_0 \) (and also \( M_0 \)) is non-negative:

\[
(\tilde{M}_0 f, f) \geq 0, \quad (M_0 \varphi, \varphi) \geq 0, \quad f \in D(\tilde{M}_0), \quad \varphi \in D(M_0).
\]  

(2.20)

Returning to the symmetric operator \( M_1 = 1 \) we find from (2.1) and (2.20) that

\[
\inf_{\|\varphi\|=1} (M_1 \varphi, \varphi) \geq \inf_{\|\varphi\|=1} (\tilde{M}_0 \varphi, \varphi) \equiv k > -\infty.
\]

(2.21)

It is known that if a symmetric operator with the deficiency indexes \((1, 1)\) is bounded below by a constant \( k \), the part of spectrum for any its selfadjoint extension that lies below \( k \) consists of not more than one eigenvalue. Setting \( m_1 = 2/\mu_1 - 1 \) and \( m_0 = 2/\mu_0 - 1 \) we arrive to the first assertion of Theorem 1.1.

5. For \( \mu > \mu_0, \mu \neq \mu_1 \), the domain \( D(\tilde{M}_0^*) \) of the operator \( \tilde{M}_0^* \) consists of the functions of the form

\[
g(s) = f(s) + \frac{C_+}{s - z_+} + \frac{C_-}{s - z_-}
\]

where \( f \in D(\tilde{M}_0) \) and \( C_\pm \) are arbitrary constants. In the case \( \mu = \mu_1 \), when \( z_+ = z_- = z_0 \)

\[
g(s) = f(s) + \frac{C_1}{s - z_0} + \frac{C_2}{(s - z_0)^2}.
\]

(2.23)

The meromorphic function \( G(z), z \in I \) of the form

\[
G(z) = F(z) + \frac{C_+}{z - z_+} + \frac{C_-}{z - z_-}
\]

(2.24)

in the case (2.22) or

\[
G(z) = F(z) + \frac{C_1}{z - z_0} + \frac{C_2}{(z - z_0)^2}
\]

(2.25)

in the case (2.23) is called the superstructure on \( g \). Here the function \( F(z) \) is the superstructure on \( f \). The operator \( M_0^* \) acts on \( D(\tilde{M}_0^*) \) as before by the formula

\[
(\tilde{M}_0^* g)(s) = N(z) G(z) \big|_{z=s+i} = N_+(s) G_+(s)
\]

(2.26)

where \( G = G(g) \) is superstructure on \( g \). The set of functions of the form (2.24) is denoted by \( J^* \). It is easy to prove the following assertion:

**Proposition 2.1.** The meromorphic function \( G(z), z \in I \) with the simple poles \( z_+, z_- \) belongs to the class \( J^* \) if there exists some bounded open set \( U \in I \) containing \( z_+ \) and \( z_- \) so that

\[
\sup_{0 \leq t \leq 1} \int_{-\infty}^{\infty} \chi_{I \setminus U}(s + it) \left| G(s + it) \right|^2 ds < \infty
\]

(2.27)
where $\chi_{I \setminus U}$ is characteristic function of the set $I \setminus U$. And conversely if $G \in J^*$, the inequality $\eqref{2.27}$ is true for any bounded open set $U \in I$ containing $z_+ \pm 1$ and $z_- \pm 1$.

3 Selfadjoint extensions of the operator $M$ for $m \in (0, m_1)$ and its negative spectrum

Above we have studied the operator $\tilde{M}_0$ and also $M_{l=1}$ for the case $m \in (m_1, m_0)$. We now turn to the case $m \in (0, m_1)$. At first we shall construct the eigenvectors of the operator $\tilde{M}_0^*$ with the eigenvalues $\lambda$ lying outside the positive semiaxis $(0, \infty)$:

$$\lambda = |\lambda|e^{i\theta}, \quad 0 < \theta < 2\pi.$$  \hspace{1cm} (3.1)

These eigenfunctions $g^\lambda$ satisfy the equation

$$(\tilde{M}_0^* g^\lambda)(s) = \lambda g^\lambda(s)$$  \hspace{1cm} (3.2)

which, proceeding to the superstructure $G^\lambda$ on $g^\lambda$, takes the form

$$N_-(s)G^\lambda_+(s) = \lambda G^\lambda_+(s).$$  \hspace{1cm} (3.3)

For the solution of this equation and further investigation of the functions $G^\lambda$ we perform a new reduction of our picture, namely we change the variables:

$$w = e^{2\pi z}, \quad z \in I.$$  \hspace{1cm} (3.4)

While the variable $z$ runs over the strip $I$, the variable $w$ varies in the set $\bar{\Pi}$ consisting of the area $\Pi = C^1 \setminus [0, \infty)$ and two coasts of the cut $[0, \infty)$ — the upper one, $\hat{l}_+$, and the lower one, $\hat{l}_-$. In addition, the boundary of the strip $I$ consisting of two lines, $l_+$ and $l_-$, proceeds to the boundary of $\Pi$: $l_\pm \rightarrow \hat{l}_\pm$. For any function $H(z)$ defined in the strip $I$, we denote its values w.r.t. the variable $w$ by $\hat{H}(w)$:

$$\hat{H}(w) = H\left(\frac{\ln w}{2\pi}\right).$$  \hspace{1cm} (3.5)

Here $\ln w$ means the branch of the logarithmic function continuous in $\bar{\Pi}$ with imaginary part included in $(0, 2\pi]$. The boundary values of $\hat{H}(w)$ on the coasts $l_\pm$ are denoted by $H_\pm(t)$ or $H_\pm(t \pm i0)$. The Hilbert space $L_2(\mathbb{R}_1, ds)$ of the functions on the line $l_+$ passes to the space $L_2(\mathbb{R}_1^+, dt/t)$ of the functions on the coast $\hat{l}_+$. The functions $F(z) \in J$ pass to the functions $\hat{F}(w)$ continuous in $\bar{\Pi}$, analytical in $\Pi$ and satisfying a condition similar to $\eqref{2.4}$:

$$\sup_{0 \leq \psi \leq 2\pi} \int_0^\infty \frac{|\hat{F}(re^{i\psi})|^2}{r} dr < \infty.$$  \hspace{1cm} (3.6)
The set of such functions is denoted by \( \hat{J} \). Analogously the functions \( G \in J^* \) pass to the meromorphic functions \( \hat{G} \) on \( \bar{\Pi} \) with the simple poles \( w_{\pm} \) and satisfying a condition similar to (2.2)

\[
\sup_{0 \leq \psi \leq 2\pi} \chi_{\Pi \setminus U}(re^{i\psi}) \left| \frac{\hat{G}(re^{i\psi})}{r} \right|^2 dr < \infty.
\]  

(3.7)

Here \( w_{\pm} = \exp\{2\pi z_{\pm}\} = -\exp\{\pm 2\pi s_0\} \), \( U \in \bar{\Pi} \) is a bounded area containing the points \( w_{\pm} \) and \( \chi_{\Pi \setminus U} \) is characteristic function of \( \Pi \setminus U \). The set of such functions \( \hat{G} \) is denoted by \( \hat{J}^* \). The domains of the operators \( \hat{M}_0 \) and \( \hat{M}_0^* \) pass to the sets of functions \( \hat{f} \) and \( \hat{g} \) of the form

\[
\hat{f}(t) = \hat{F}_+(t + i0) \equiv F(w)|_{I_+}, \quad \hat{F} \in \hat{J},
\]

\[
\hat{g}(t) = \hat{G}_+(t + i0) \equiv G(w)|_{I_+}, \quad \hat{G} \in \hat{J}^*.
\]

(3.8)

As before, the functions \( \hat{F} = \hat{F}(\hat{f}) \) and \( \hat{G} = \hat{G}(\hat{g}) \) are called the superstructures on \( \hat{f} \) and \( \hat{g} \) correspondingly. The operators \( \hat{M}_0 \) and \( \hat{M}_0^* \) pass due to the change (3.4) to the operators \( \hat{M}_0 \) and \( \hat{M}_0^* \) acting by formula

\[
(\hat{M}_0 \hat{f})(t + i0) = \hat{N}-(t - i0)\hat{F}_-(t - i0), \quad \hat{f} \in D(\hat{M}_0)
\]

(3.9)

and the same for \( \hat{M}_0^* \).

The equation (3.3) for \( G^\lambda \) now reads for \( \hat{G}^\lambda \)

\[
(\hat{N}^*_\lambda \hat{G}_\lambda)(t - i0) = \lambda^* \hat{G}_\lambda(t + i0)
\]

(3.10)

where

\[
\hat{N}^*_\lambda(w) = \frac{1}{2\pi^2 \sqrt{1 - (\mu/2)^2}} \hat{N}(w) = 1 - \frac{1}{\sqrt{1 - (\mu/2)^2}} \hat{\Lambda}(w)
\]

and

\[
\lambda^* = \frac{\lambda}{2\pi^2 \sqrt{1 - (\mu/2)^2}}.
\]

We are going to write the explicit solution of equation (3.10) but first we need to introduce some quantity.

1. Denote by \( h(w) \) the following function on \( \bar{\Pi} \)

\[
h(w) = \frac{1}{2\pi} \ln w - s_0 - \frac{i}{2}
\]

(3.11)

It is obvious that \( h(w_+) = 0 \) and \( w_+ \) is a unique zero of \( h(w) \).

2. Introduce then a function \( a(x) \), \( x \in (0, \infty) \), by

\[
a(x) = \hat{N}_+(x) \frac{h_+(x)}{h_-(x)}
\]

(3.12)

This function is smooth and distinct from zero everywhere. Indicate the following properties of this function.
Lemma 3.1. The increment of \( \arg a(x) \) for the change of \( x \) from zero to infinity is equal to zero:

\[
\Delta_{x=0}^{x=\infty} \arg(a(x)) = 0. \tag{3.13}
\]

The function \( \ln(a(x)) \) has the following asymptotics as \( x \to \infty \):

\[
\ln(a(x)) = -\frac{2\pi i}{\ln x} + O\left(\frac{1}{(\ln x)^2}\right). \tag{3.14}
\]

Here \( \ln u, u \in \mathbb{C} \setminus (-\infty, 0) \) is the branch of the logarithmic function defined on the complex plane with the cut \((-\infty, 0)\) with imaginary part constrained between \(-i\pi\) and \(i\pi\).

The proof of this lemma is contained in Appendix.

Corollary 3.1. From (3.13) and (3.14) it follows that for any \( w \in \Pi \) there exists the limit

\[
\lim_{n \to \infty} \left( \frac{1}{2\pi i} \int_0^n \frac{\ln(a(x))}{x-w} \, dx + \ln(\ln n) \right) \equiv K(w) \tag{3.15}
\]

(a regularized integral) which we shall write in the form

\[
K(w) = \frac{1}{2\pi i} \int_0^\infty \frac{\ln(a(x))}{x-w} \, dx, \quad w \in \Pi. \tag{3.16}
\]

It is evident that \( K(w) \) is analytical function in \( \Pi \), then because of the smoothness of \( \ln(a(x)) \) it can be continued to the both coasts \( \tilde{l}_{\pm} \) of the cut (by the well-known Plemel–Privalov's lemma, see [10]). In addition, the known equalities of Sohotsky are fulfilled for the limiting values \( K_{\pm}(t \pm i0) \):

\[
K_{\pm}(t \pm i0) = \frac{1}{2\pi i} \left[ P \int_0^\infty \frac{\ln(a(x))}{x-t} \, dx \pm i\pi \ln(a(t)) \right] \tag{3.17}
\]

where \( P \int_0^\infty \ldots dx \) means the principal value of regularized integral (see [10]).

Now we can write the solution of equation (3.10).

Lemma 3.2. For any complex value \( \lambda = |\lambda| \exp\{i\theta\}, 0 < \theta < 2\pi \) lying outside semiaxis \( 0, \infty \) there exists unique (up to a constant factor) solution of the equation (3.10) belonging to the class \( \mathcal{F}^* \). It has the form

\[
G^\lambda(w) = \frac{w^{(\theta-i\ln|\lambda^*|)/2\pi}}{h(w)(w-w_-)} \exp \left\{ \frac{1}{2\pi i} \int_0^\infty \frac{\ln(a(x))}{x-w} \, dx \right\}. \tag{3.18}
\]
The proof of this lemma is postponed to Appendix.

As a consequence of the lemma we find for each selfadjoint extension of the operator $M_0$ its negative eigenvalues ($\theta = \pi$). Namely these eigenvalues form the two-sided geometric progression

$$\lambda_n^0 = \lambda_0^0 \exp \left\{ \frac{\pi n}{8\theta} \right\}, \quad n = 0, \pm 1, \pm 2, \ldots \quad (3.19)$$

where $\lambda_0^0 < 0$ depends on the extension.

For the proof note that any vector $\hat{g} \in D(\hat{M}_0^\ast)$ can be represented in the form (see [9])

$$\hat{g}(t) = \hat{f}(t) + C_1 g^{\lambda=i}(t) + C_2 g^{\lambda=-i}(t) \quad (3.20)$$

where $\hat{f} \in D(\hat{M}_0^0)$ and $C_1, C_2$ are arbitrary constants. In the case when

$$\|g^{\lambda=i}\| = \|g^{\lambda=-i}\| \quad (3.21)$$

(and this equality is fulfilled in our case, see below) any selfadjoint extension of $\hat{M}_0^0$ is obtained by the restriction of the operator $\hat{M}_0^\ast$ on the set of functions having the form (3.20) with

$$C_2 = \beta C_1 \quad (3.22)$$

where $\beta, |eta| = 1$ is the parameter defining the extension (see [9]). We denote the extension by $\hat{M}_0^\beta$. In particular, the eigenfunction $\hat{g}^\lambda \in D(\hat{M}_0^\beta)$ of the operator $\hat{M}_0^\beta$ with $\lambda < 0$ has the form

$$\hat{g}^\lambda(t) = \hat{f}(t) + C(\hat{g}^{\lambda=i}(t) + \beta \hat{g}^{\lambda=-i}(t)) \quad (3.23)$$

where $\hat{f} \in D(\hat{M}_0^\beta)$, $C$ depends on $\lambda$. Passing on to the superstructures in equality (3.23) we get that

$$\hat{G}^\lambda(w) = \hat{F}(w) + C(\hat{G}^{\lambda=i}(w) + \beta \hat{G}^{\lambda=-i}(w)), \quad \hat{F} \in \hat{J}. \quad (3.24)$$

From here we obtain the following relation for the residues of the functions $\hat{G}^\lambda$, $\hat{G}^{\lambda=\pm i}$ in poles $w_{\pm}$

$$\text{res}_{w_+} \hat{G}^\lambda = C(\text{res}_{w_+} \hat{G}^{\lambda=i} + \beta \text{res}_{w_+} \hat{G}^{\lambda=-i}), \quad \text{res}_{w_-} \hat{G}^\lambda = C(\text{res}_{w_-} \hat{G}^{\lambda=i} + \beta \text{res}_{w_-} \hat{G}^{\lambda=-i}). \quad (3.25)$$

Calculating these residues with the help of (3.15) we get

$$w_+^{1/2-i\ln|\lambda|/2\pi} = C(w_+^{1/4} + \beta w_+^{3/4}), \quad (3.26)$$

$$w_-^{1/2-i\ln|\lambda|/2\pi} = C(w_-^{1/4} + \beta w_-^{3/4})$$
and hence
\[
\left( \frac{w_+}{w_-} \right)^{-i \ln |\lambda| / 2\pi} = \frac{w_+^{-1/4} + \beta w_+^{1/4}}{w_-^{-1/4} + \beta w_-^{1/4}} \equiv \gamma(\beta, s_0).
\] (3.27)

The ratio \( w_+/w_- \) is equal to \( \exp \{ 4\pi s_0 \} \) and the value \( \gamma(\beta, s_0) \) is equal to \( (1 + i\beta \exp(\pi s_0)) / (\exp(\pi s_0) + \beta i) \). It is easy to check that \( |\gamma(\beta, s_0)| = 1 \) i.e. \( \gamma(\beta, s_0) = \exp \{ i\eta(\beta, s_0) \} \), \( 0 < \eta(\beta, s_0) < 2\pi \). Then we find that the equality (3.27) is fulfilled for all \( \lambda \) from the sequence (3.19) and only for these \( \lambda \). In (3.19) \( \lambda_0^0 \) is equal to
\[
\lambda_0^0 = \frac{-e^{-\eta(\beta, s_0)}}{2s_0}. \quad (3.28)
\]

**Lemma 3.3.** For any selfadjoint extension \( \hat{M}_0^\beta \) its negative part of spectrum consist of the eigenvalues (3.19).

This lemma will be proved in Appendix.

### 4 Completion of the proof of Theorem 1.1

Here we consider the case \( m < m_1 \); assertions 2 and 3 of Theorem 1.1 are related to this case.

It is easily seen that any selfadjoint extension \( M_0^\beta \) of \( M_0 \) generates the one for the operator \( M_{l=1}^\beta \):
\[
M_{l=1}^\beta = M_0^\beta + M_1 \quad (4.1)
\]
and that all extensions of \( M_{l=1} \) are obtained in this way.

Now we prove the existence of the sequence of the negative eigenvalues for \( M_{l=1}^\beta \) having the form (1.12). Let \( c = \|M_1\| \) be the norm of the bounded operator \( M_1 \). Denote by \( O_n \) the circle in the complex plane with the center \( \lambda_0^0 \) and radius \( 2c \) and let \( \Delta_n = (\lambda_0^0 - 2c, \lambda_0^0 + 2c) \) be the interval being cut on the real axis by this circle. It is obvious that for large enough \( n > n_0 \) the successive intervals \( \Delta_n \), \( \Delta_{n+1} \) do not intersect.

Denote
\[
h_\Delta^0 = \int_{\Delta_n} dh^0(\lambda) \quad \text{and} \quad h_\Delta = \int_{\Delta_n} dh(\lambda) \quad (4.2)
\]
where \( \{h^0(\lambda), \lambda \in \mathbb{R}^1\} \), \( \{h(\lambda), \lambda \in \mathbb{R}^1\} \) are spectral families of subspaces for the operators \( M_0^\beta \) and \( M_{l=1}^\beta \) correspondingly (see [9]).

Let \( P_\Delta^0 \) and \( P_\Delta \) be orthogonal projectors on these subspaces.

From inequality which will be proved below
\[
\|P_\Delta^0 - P_\Delta\| < 1 \quad (4.3)
\]
it follows that the dimensions of the subspaces $h_{\Delta_n}^0$ and $h_{\Delta_n}$ are equal (see \[\text{9}\]). This fact and Lemma \[3.3\] imply that the operator $M_{l=1}^{\beta}$ has a unique simple eigenvalue $\lambda_n$ in the interval $\Delta_n$ and $\Delta_n$ does not contain any other points of the spectrum of $M_{l=1}^{\beta}$. In addition

$$\lambda_n = \lambda_n^0 + \delta_n, \quad |\delta_n| < 2c, \quad n \geq n_0$$  \hspace{1cm} (4.4)

which gives the second assertion of Theorem 1.1.

To prove the last assertion of Theorem 1.1 we consider the intervals

$$\kappa_n = (\lambda_{n+1}^0 + 2c, \lambda_n^0 - 2c)$$

lying between two successive intervals $\Delta_n$ and $\Delta_{n+1}$, and spectral subspaces $h_{\kappa_n}^0$ and $h_{\kappa_n}$ similar to $h_{\Delta_n}^0$, $h_{\Delta_n}$. From the inequality

$$\|P_{\kappa_n}^0 - P_{\kappa_n}\| < 1$$  \hspace{1cm} (4.5)

it follows again that the dimensions of $h_{\kappa_n}^0$ and $h_{\kappa_n}$ are equal. Here $P_{\kappa_n}^0$ and $P_{\kappa_n}$ are the projectors on $h_{\kappa_n}^0$ and $h_{\kappa_n}$ respectively. Thus due to Lemma 3.3 the operator $M_{l=1}^{\beta}$ does not have the points of spectrum on the interval $\kappa_n$ for $n \geq n_0$.

**The proof of inequality (4.3)**

The difference $P_{\Delta_n}^0 - P_{\Delta_n}$ can be represented through the resolvents of the operators $M_0^{\beta}$ and $M_{l=1}^{\beta}$ (see \[13\])

$$P_{\Delta_n}^0 - P_{\Delta_n} = \frac{1}{2\pi i} \int_{O_n} (R_{M_0^{\beta}}(z) - R_{M_{l=1}^{\beta}}(z)) \, dz. \hspace{1cm} (4.6)$$

Use the identity

$$R_{M_{l=1}^{\beta}}(z) = (E + R_{M_0^{\beta}}(z)M_1)^{-1}R_{M_0^{\beta}}(z). \hspace{1cm} (4.7)$$

For $z \in O_n$ the nearest point of the spectrum of $M_0^{\beta}$ is $\lambda_n$ and we get

$$\|R_{M_0^{\beta}}(z)\| \leq \frac{1}{2c}, \quad z \in O_n$$

and hence

$$\|R_{M_0^{\beta}}(z)M_1\| \leq \frac{1}{2}$$

Then

$$(E + R_{M_0^{\beta}}(z)M_1)^{-1} = E + T$$

where $T = \sum_{k=1}^{\infty} ((-1)^k (R_{M_0}(z)M_1))^k$ and $\|T\| \leq 1$. 

Finally
\[
\| R_{M_{l=1}}^\beta (z) - R_{M_0^\beta} (z) \| = \| TR_{M_0^\beta} (z) \| \leq \frac{1}{2c}, \quad z \in O_n. \tag{4.8}
\]
Now (4.8) follows from (4.8) and (4.6).

The proof of (4.5)

For every interval \( \kappa_n, n \geq n_0 \) we construct the enveloping rectangle \( u_n \) with vertical sides \( c \) (see Figure 2).

\[ O_{n+1} \quad \lambda_{n+1} \quad u_n \quad \lambda_n \quad O_n \]

As before
\[
\| P_{\kappa_n}^0 - P_{\kappa_n} \| < \frac{1}{2\pi} \int_{\partial u_n} \| (R_{M_0^\beta} (z) - R_{M_{l=1}}^\beta (z)) \| \, dz. \tag{4.9}
\]

For \( z \) which lies on the vertical side of \( u_n \), as above,
\[
\| R_{M_0^\beta} (z) - R_{M_{l=1}}^\beta (z) \| < \frac{1}{2c}. \tag{4.10}
\]

For \( z \) which lies on the horizontal side of \( u_n \) at the distance \( y \) from the border of this side the norm of \( R_{M_0^\beta} (z) \) is equal to \( 1/(2c + y) \), and
\[
\| R_{M_0^\beta} (z) - R_{M_{l=1}}^\beta (z) \| < \frac{c}{(c + y)(2c + y)} \tag{4.11}
\]
as it follows from the calculations similar to the previous one. From (4.10) and (4.11) we obtain that the right part of (4.9) does not exceed
\[
\frac{1}{2\pi} \left( 1 + 4 \int_{0}^{x_n/2} \frac{c \, dy}{(c + y)(2c + y)} \right) < \frac{5}{2\pi} < 1.
\]
The inequality (4.5) is proved as well as Theorem 1.1.
Appendix

A.1 The proof of Lemma 3.1

Consider first the function $\hat{N}^*(x)$. This function has the asymptotics for some $\alpha > 0$

$$
\hat{N}^*(x) = \begin{cases} 
1 + O(x^\alpha), & x \to 0, \\
1 + O(x^{-\alpha}), & x \to \infty,
\end{cases} \quad (A.1)
$$

and the increment of its argument on $(0, \infty)$ is equal to

$$
\Delta_{x=0}^{x=\infty} \arg \hat{N}^*(x) = -2\pi \quad (A.2)
$$

Assertion (A.1) follows from (2.10) and (2.11) for the function $N(z)$. Indeed we see that

$$
N^*(s) = 1 + O(e^{\pm \alpha s}) \quad \text{as} \quad s \to \mp \infty \quad (A.3)
$$

for some $\alpha > 0$. After the change (3.5) we get (A.1).

To prove (A.2) we note that

$$
\Delta_{s=-\infty}^{s=\infty} \arg N^*_-(s) + \Delta_{s=-\infty}^{s=\infty} \arg N^*_+(s) = -4\pi \quad (A.4)
$$

due to the principle of argument. By (2.12),

$$
\Delta_{s=-\infty}^{s=\infty} \arg N^*_-(s) = \Delta_{s=-\infty}^{s=\infty} \arg N^*_+(s) = -2\pi \quad (A.5)
$$

and we obtain (A.2). Thus

$$
\ln \hat{N}^*_-(x) = O(x^\alpha), \quad x \to 0, \quad (A.6)
$$

$$
\ln \hat{N}^*_+(x) = -2\pi i + O(x^{-\alpha}), \quad x \to \infty. \quad (A.7)
$$

Now consider the function $h^+(x)/h^-(x)$.

As before we find that

$$
\frac{h^+(x)}{h^-(x)} = \begin{cases} 
1 + O\left(\frac{1}{\ln x}\right), & x \to 0, \\
1 - \frac{2\pi i}{\ln x} + O\left(\frac{1}{(\ln x)^2}\right), & x \to \infty,
\end{cases} \quad (A.9)
$$

and

$$
\Delta_{x=0}^{x=\infty} \arg \frac{h^+(x)}{h^-(x)} = 2\pi. \quad (A.8)
$$

The assertion (A.7) follows immediately from (3.11) for $h(w)$. The equality (A.8) can be established by the following representation

$$
\frac{h^+(x)}{h^-(x)} = \frac{u - i/2}{u + i/2} \quad (A.9)
$$
where the function $u = u(x) = (\ln x)/2\pi - s_0$ is real and monotonically changes from $-\infty$ to $+\infty$ when $x$ varies from 0 to $\infty$. The linear-fractional function $(u-i/2)/(u+i/2)$ transfers the real axis to the unit circle with counter-clockwise direction. This implies (A.8). Thus
\[
\frac{\ln h_+(x)}{h_-(x)} = \frac{2\pi i}{\ln x} + O\left(\frac{1}{(\ln x)^2}\right) + 2\pi i.
\] (A.10)

From (A.6) and (A.10) both assertions of Lemma 3.1 follow.

A.2 The proof of Lemma 3.2

We first show that the function $G(w)$ (see (3.18)) satisfies equation (3.10). Using Sohortsy’s formula (see [11]) we get
\[
G_\pm(t \pm i0) = \frac{t^{\theta-i\ln |\lambda^*|} (\lambda^*)^{\varepsilon(\pm)} h_\pm(t \pm i0)(t - w_-)}{h_\pm(t \pm i0)(t - w_+)} \exp\left\{\frac{1}{2\pi i} \int_0^\infty \frac{\text{Ln} a(x)}{x - t} dx \pm \frac{1}{2} \text{Ln} a(t)\right\}
\] (A.11)

where $\varepsilon(+)=0$, $\varepsilon(-)=1$. From (A.11) we find that
\[
a(t)G_\pm(t - i0) = \lambda^* G_\pm(t + i0) \frac{h_+(t + i0)}{h_-(t - i0)}
\]

and thus obtain (3.10).

It follows obviously from (3.18) that the meromorphic function $G^\lambda(w)$ has two simple poles in $w_{\pm}$. We prove now that it satisfies condition (3.7).

From (3.18) we find that for $w = r \exp\{i\psi\} \in \Pi$
\[
|G^\lambda(w)|^2 = \frac{r^{\theta/\pi} |\lambda^*|^{\psi/\pi}}{|h(w)|^2 |w - w_-|} \exp\left\{\frac{1}{2\pi i} \int_0^\infty \frac{\text{Ln} a(x)}{x - w} - \frac{\text{Ln} \bar{a}(x)}{x - \bar{w}} dx\right\}
\] (A.12)

where we use that $\text{Ln} a = \text{Ln} \bar{a}$.

Proposition A.1. For $|G^\lambda(w)|^2$ the following representation is true ($w = r \exp\{i\psi\}$)
\[
|G^\lambda(w)|^2 = \frac{(2\pi)^2 r^{\theta/\pi} |\lambda^*|^{\psi/\pi}}{|w - w_-| |w - w_+|} \times \exp\left\{\frac{1}{2\pi i} \int_{-\infty}^0 (\text{Ln} |\hat{N}^*(s)|) \left(\frac{1}{s - w} - \frac{1}{s - \bar{w}}\right) ds\right\} \times \begin{cases} 1, & 0 < \psi < \pi, \\ 1 & |\hat{N}^*(w)| |\hat{N}^*(\bar{w})|, & \pi < \psi < 2\pi. \end{cases}
\] (A.13)
In the case \( \psi = \pi \) and \( \bar{w} \in (w_-, w_+) \) the expression for \( |G^\lambda(w)|^2 \) reads:

\[
|G^\lambda(w)|^2 = \frac{(2\pi)^2 r^{\theta/\pi}|\lambda^*|}{|w - w_+||w - w_-||N^*(w)|}.
\] (A.14)

**The proof.**

Write the expression standing in the exponent in (A.12) in the form:

\[
\frac{1}{2\pi i} \left[ \int_{l_-^n} \left( \frac{\text{Ln} \hat{N}^*(x) - \text{Ln} h_+(x)}{x - w} - \frac{\text{Ln} h_-(x)}{x - \bar{w}} \right) dx + \int_{l_+^n} \left( \frac{\text{Ln} \hat{N}^*(x) - \text{Ln} h_-(x)}{x - \bar{w}} - \frac{\text{Ln} h_-(x)}{x - w} \right) dx \right] + 2\text{Ln} (\text{ln}n) + o(1) \quad \text{for} \; n \to \infty.
\] (A.15)

Further we introduce a new complex variable \( \zeta \) and consider in the plane of this variable two contours \( \Gamma_{\pm}^n \) depicted in Figure 3.

![Figure 3: Each of the contours \( \Gamma_{\pm}^n \) consists of three parts: \( l_{\pm}^n \) are the intervals of length \( n \) on the coasts of the cut \( \mathcal{V} \); \( m_{\pm}^n \) are the same intervals on the coasts of the cut \( \bar{\mathcal{V}} \); and \( C_{\pm}^n \) are two semicircles of radius \( n \) with center at the origin. The contours are bypassed clockwise.

Denote by \( V(\zeta, w) \)

\[
V(\zeta, w) = \frac{\text{Ln} \hat{N}^*(\zeta) - \text{Ln} h(\zeta)}{\zeta - w} - \frac{\text{Ln} h(\zeta)}{\zeta - \bar{w}}
\] (A.16)

and rewrite the integrals in (A.15) in the form

\[
\frac{1}{2\pi i} \left[ \int_{l_-^n} V(\zeta, w) \, d\zeta + \int_{l_+^n} V(\zeta, \bar{w}) \, d\zeta \right]
\] (A.17)
where the interval \( l^n \) is passed from 0 to \( n \) whence \( l^n + \) is passed in the opposite direction. Then we decompose the sum \( (A.17) \) as follows:

\[
\frac{1}{2\pi i} \left[ \int_{l^n_-} V(\zeta, w) \, d\zeta + \int_{l^n_+} V(\zeta, \bar{w}) \, d\zeta \right]
\]

\[
= \frac{1}{2\pi i} \left[ \int_{l^n_-} \cdots d\zeta + \int_{l^n_+} \cdots d\zeta + \int_{C^n_-} \cdots d\zeta + \int_{C^n_+} \cdots d\zeta + \int_{m^n_-} \cdots d\zeta + \int_{m^n_+} \cdots d\zeta \right].
\]

In addition the left part of \( (A.18) \) is equal to the sum of residues of the integrands at the points \( w = r \exp\{i\psi\} \) and \( \bar{w} = r \exp\{i(2\pi - \psi)\} \):

\[
\frac{1}{2\pi i} \left[ \int_{l^n_-} \cdots d\zeta + \int_{l^n_+} \cdots d\zeta \right]
\]

\[
= \begin{cases} 
\ln h(w) + \ln h(\bar{w}), & 0 < \psi < \pi, \\
\ln h(w) + \ln h(\bar{w}) - \ln \hat{N}^*(w) - \ln \hat{N}^*(\bar{w}), & \pi < \psi < 2\pi.
\end{cases}
\]

It is easy to calculate that

\[
\frac{1}{2\pi i} \left[ \int_{C^n_-} \cdots d\zeta + \int_{C^n_+} \cdots d\zeta \right] = 2\ln \ln n - 2\ln(2\pi) + o(1), \quad n \to \infty.
\]

Thus for \( \psi \neq \pi \)

\[
\frac{1}{2\pi i} \left[ \int_0^n \frac{\ln a(x)}{x-w} \, dx - \int_0^n \frac{\ln a(x)}{x-\bar{w}} \, dx \right] + 2\ln \ln n + o(1)
\]

\[
= \frac{1}{2\pi i} \left[ \int_{l^n_-} V(\zeta, w) \, d\zeta + \int_{l^n_+} V(\zeta, \bar{w}) \, d\zeta \right] + 2\ln \ln n
\]

\[
= - \frac{1}{2\pi i} \left[ \int_{m^n_-} V(\zeta, w) \, d\zeta + \int_{m^n_+} V(\zeta, \bar{w}) \, d\zeta \right] + 2\ln 2\pi
\]

\[
+ \begin{cases} 
\ln (h(w)h(\bar{w})), & 0 < \psi < \pi, \\
\ln (h(w)h(\bar{w})) - \ln (\hat{N}^*(w)\hat{N}^*(\bar{w})), & \pi < \psi < 2\pi.
\end{cases}
\]
Passing on to the limit \( n \to \infty \) we get

\[
\frac{1}{2\pi i} \left[ \int_{0}^{\infty} \frac{\ln \alpha(x)}{x - w} \, dx - \int_{0}^{\infty} \frac{\ln \bar{\alpha}(x)}{x - \bar{w}} \, dx \right]
\]

\[
= - \frac{1}{2\pi i} \left[ \int_{-\infty}^{0} (V(s - i0, w) - V(s + i0, \bar{w})) \, ds \right] + 2\ln 2\pi
\]

\[
+ \begin{cases} 
\ln(h(w)h(\bar{w})), & 0 < \psi < \pi, \\
\ln(h(w)h(\bar{w})) - \ln(\hat{N}^*(w)\hat{N}^*(\bar{w})), & \pi < \psi < 2\pi.
\end{cases}
\]

Finally for \( \psi \neq \pi \)

\[
- \frac{1}{2\pi i} \left[ \int_{-\infty}^{0} (V(s - i0, w) - V(s + i0, \bar{w})) \, ds \right] \tag{A.21}
\]

\[
= - \frac{1}{2\pi i} \int_{-\infty}^{0} \ln |\hat{N}^*(s)| \left( \frac{1}{s - w} - \frac{1}{s - \bar{w}} \right) + \frac{1}{2} \int_{-w}^{w} \left( \frac{1}{s - w} - \frac{1}{s - \bar{w}} \right) ds.
\]

This equality is obtained with the help of representation

\[
\ln \hat{N}^*(s \pm i0) = \ln |\hat{N}^*(s)| + i \arg \hat{N}^*(s \pm i0)
\]

where

\[
\arg \hat{N}^*(s \pm i0) = \begin{cases} 
0, & w_- < s < 0, \\
\pm\pi, & w_+ < s < w_-, \\
\pm2\pi, & -\infty < s < w_+,
\end{cases}
\]

and similarly for \( \ln h(s \pm i0) = \ln |h(s)| = i \arg h(s \pm 0) \)

\[
\arg h(s \pm i0) = \begin{cases} 
0, & w_+ < s < 0, \\
\pm\pi, & -\infty < s < w_+.
\end{cases}
\]

In addition the terms in \( (V(s - i0, w) - V(s + i0, w)) \) containing \( \ln(h(s)) \) mutually cancel and the terms with \( \arg \hat{N}^*(s \pm i0) \) for \( s < w_+ \) cancel with the terms containing \( \arg h(s \pm i0) \). Finally

\[
\frac{1}{2} \int_{-w}^{w} \left( \frac{1}{s - w} - \frac{1}{s - \bar{w}} \right) \, ds = \frac{1}{2} \ln \left( \frac{w_- - w}{w_+ - w} \right) + \frac{1}{2} \ln \left( \frac{w_- - \bar{w}}{w_+ - \bar{w}} \right). \tag{A.22}
\]

Substituting (A.20), (A.21) and (A.22) into (A.12) and noting that \( \bar{h}(w) = h(\bar{w}) \) we obtain (A.13) for \( \psi \neq \pi \). For the case \( \psi = \pi \) when \( w = \bar{w} \in (-\infty, 0) \) and
w \neq w_+ the formula \[ (A.14) \] be obtained with the help of similar arguments. Proposition \[ A.1 \] is proved.

Passing on to the limits \( \psi \to 0 \) and \( \psi \to 2\pi \) in \[ (A.13) \], \[ (A.14) \] we get that

\[
|G_\lambda(t + i0)|^2 = |g_\lambda(t)|^2 = \frac{(2\pi)^2 t^{\theta/\pi}}{|t - w_+| |t - w_-|} \tag{A.23}
\]

and

\[
|G_\lambda(t - i0)|^2 = \frac{(2\pi)^2 t^{\theta/\pi} |\lambda^*|}{|t - w_+| |t - w_-| |\hat{N}^z(t)|^2}. \tag{A.24}
\]

In particular, from \[ (A.23) \] we find that

\[
\|g_\lambda=\|^2 = \int_0^\infty \frac{(2\pi)^2 t^{1/2} dt}{|t - w_+| |t - w_-| t} \tag{A.25}
\]

and

\[
\|g_\lambda=-i\|^2 = \int_0^\infty \frac{(2\pi)^2 t^{3/2} dt}{|t - w_+| |t - w_-| t}. \tag{A.26}
\]

Changing \( t \to 1/t \) and taking into account that \( w_+ w_- = 1 \) we achieve the equality mentioned above:

\[
\|g_\lambda=i\| = \|g_\lambda=-i\|.
\]

Let \( U \subset \Pi \) be a bounded area containing the points \( w_\pm \). It can proved that there exists a constant \( A = A(U) \) such that for all \( w, \bar{w} \in U \),

\[
\left| \int_{-\infty}^0 \ln |\hat{N}^z(s)| \left( \frac{1}{s - w} - \frac{1}{s - \bar{w}} \right) ds \right| < A(U).
\]

From this estimate and \[ (A.13) \], \[ (A.14) \] it follows that

\[
\sup_{0 \leq \psi \leq 2\pi} \int_0^\infty \frac{|G_\lambda(re^{i\psi})|^2}{r} \chi_{\Pi \setminus U}(re^{i\psi}) dr < \infty, \quad U \subset \Pi
\]

i.e. \( G_\lambda \in \hat{J}^* \) for all complex \( \lambda \) with \( \arg \lambda \neq 0 \). We now show that \( G_\lambda \) is a unique solution of the equation \[ (3.10) \] belonging to \( \hat{J}^* \). Let us assume that there is another solution \( H_\lambda \in \hat{J}^* \) of equation \[ (3.10) \]. Consider the ratio

\[
\frac{H_\lambda(w)}{G_\lambda(w)} = \Phi(w) \tag{A.27}
\]
(it follows from (A.13), (A.14) that $G^\lambda(w) \neq 0$ for all $w \neq 0$). From equation (3.10) it follows that $\Phi_+(t + i0) = \Phi_-(t - i0)$. Thus $\Phi$ is the analytic function on the complex plane $\mathbb{C}^1$ with deleted point $w = 0$ and can be decomposed in Loran’s series (see [13])

$$\Phi(w) = \sum_{n=-\infty}^{\infty} c_n w^n \quad \text{(A.28)}$$

where at least one coefficient $c_{n_0} \neq 0$ for $n_0 \neq 0$.

Consider the ring $\mathfrak{T} = \{0 < \tau_0 < |w| < \tau_1 < \infty\}$ containing the points $w_{\pm}$. For large enough $R$

$$\min_{\tau_1 < |w| < R} \left| G^\lambda(w) \right|^2 > \frac{\text{const}}{R^{2-\theta/\pi}}$$

and

$$\min_{1/R < |w| < \tau_0} \left| G^\lambda(w) \right|^2 > \frac{\text{const}}{R^{\theta/\pi}}.$$

Let $U$ be an area so that $U \subset \mathfrak{T}$ and $w_{\pm} \in U$. Then

$$\sup_{0 < \psi < 2\pi} \int_0^\infty \frac{\chi_{\Pi \setminus U}(re^{i\psi})|H^\lambda(re^{i\psi})|^2}{r} \, dr \quad \text{(A.29)}$$

$$> \frac{1}{2\pi} \int_0^{2\pi} d\psi \int_0^\infty \frac{\chi_{\Pi \setminus U}(re^{i\psi})|H^\lambda(re^{i\psi})|^2}{r} \, dr$$

$$> \frac{1}{2\pi} \min_{\tau_1 < |w| < R} \left| G^\lambda(w) \right|^2 \int_0^{2\pi} d\psi \int_{\tau_1}^R \frac{|\Phi(re^{i\psi})|^2}{r} \, dr$$

$$+ \frac{1}{2\pi} \min_{1/R < |w| < \tau_0} \left| G^\lambda(w) \right|^2 \int_0^{2\pi} d\psi \int_{1/R}^{\tau_0} \frac{|\Phi(re^{i\psi})|^2}{r} \, dr$$

$$> \frac{\text{const}}{R^{2-\theta/\pi}} \sum_{n>0} |c_n|^2 \int_{\tau_1}^R r^{2n-1} \, dr + \frac{\text{const}}{R^{\theta/\pi}} \sum_{n<0} |c_n|^2 \int_{1/R}^{\tau_0} r^{-2n-1} \, dr.$$

Consider two cases:

1) $c_{n_0} \neq 0$ for $n_0 > 0$.

Then the last part of (A.29) can be bounded from below by the value $\sim \frac{\text{const}}{R^{\theta/\pi}}$.
2) If \( c_{n_0} \neq 0 \) for \( n_0 < 0 \), \( A.29 \) is bounded below by \( \sim R^{2-\theta/\pi} \).

Since \( 0 < \theta < 2\pi \), both estimates increase to infinity as \( R \to \infty \). Consequently,

\[
\sup_{0<\psi<2\pi} \int_0^\infty \chi_{\Pi U}(re^{i\psi}) \frac{|H(\lambda, re^{i\psi})|^2}{r} \, dr = \infty.
\]

(A.30)

Thus \( H^\lambda \in J^* \) and the uniqueness of solution \( G^\lambda \) is proved. The proof of Lemma 3.2 is completed.

### A.3 The proof of Lemma 3.3

Here we construct the resolvent of the operator \( \hat{M}_0^\beta \):

\[
R_{\hat{M}_0^\beta}(\lambda) = \left(M_0^\beta - \lambda E\right)^2
\]

for negative values of \( \lambda = -|\lambda| \). The equation

\[
\hat{M}_0^\beta q(t) - \lambda q(t) = f(t), \quad q \in D(\hat{M}_0^\beta), \quad f \in L^2\left(\mathbb{R}, \frac{dt}{t}\right)
\]

determining the resolvent: \( q = R_{\hat{M}_0^\beta} f \) after passing to the superstructure \( Q = Q(q) \) takes the form

\[
\hat{N}^*(t)Q_-(t - i0) - \lambda^* Q(t + i0) = f(t).
\]

(A.31)

Introduce the function

\[
B^\lambda(w) = (w - w)G^\lambda(w)
\]

\[
= \frac{w^{1/2-i \ln |\lambda|/2\pi}}{h(w)} \exp \left\{ \frac{1}{2\pi i} \int_0^\infty \frac{\text{Ln} a(x)}{x-w} \, dx \right\}, \quad \lambda = |\lambda|e^{i\pi}
\]

which (like \( G^\lambda(w) \)) satisfies equation (3.40). Then we represent the solution of the equation (A.31) in the form

\[
Q^\lambda(w) = B^\lambda(w)S^\lambda(w)
\]

where the function \( S^\lambda(w) \) satisfies the following equation:

\[
S^\lambda(t - i0) - S^\lambda(t + i0) = \frac{1}{\lambda^*} \left(B^\lambda_+(t + i0)\right)^{-1} f(t).
\]

We choose the following solution of this equation

\[
S^\lambda(w) = -\frac{1}{2\pi i \lambda^*} \int_0^\infty \frac{(B^\lambda_+(x))^{-1} f(x)}{x-w} \, dx + \frac{C_1}{w - w_-}
\]

(A.34)
where $C_1$ will be determined below.

Thus the solution of equation (A.31) has the form

$$Q^\lambda(w) = -L^\lambda(w) + C_1 G^\lambda(w)$$  \hspace{1cm} (A.35)

where

$$L^\lambda(w) = \frac{B^\lambda(w)}{2\pi i \lambda^*} \int_0^\infty \frac{(B^\lambda_+(x))^{-1} f(x)}{x - w} \, dx.$$  \hspace{1cm} (A.36)

Using the previous calculations we find that

$$\left| B^\lambda(w) \right|^2 = (2\pi)^2 r \lambda^* |\psi|^\pi \times \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^0 \text{Ln} |\tilde{N}^\ast(s)| \left( \frac{1}{s - w} - \frac{1}{s - \bar{w}} \right) \, ds \right\} \left| w - w_- \right| \left| w - w_+ \right|.$$  \hspace{1cm} (A.37)

where $w = r \exp\{i\psi\}$ and $0 < \psi < \pi$. A similar representation of $|B^\lambda(w)|^2$ takes place for $\pi \leq \psi \leq 2\pi$. From (A.37) we find that for any bounded area $U \subset \Pi$ containing $w_\pm$ the following inequality is true

$$\sup_{w \in \Pi \setminus U} \frac{|B^\lambda(w)|^2}{r} < \infty.$$  \hspace{1cm} (A.38)

For $(B^\lambda(w))^{-1}$ a representation similar to (A.37) takes place and it implies

$$(B^\lambda_+(w))^{-1} f(t) \equiv b(t) \in L^2(\mathbb{R}^1, dt).$$  \hspace{1cm} (A.39)

We now show that $Q^\lambda \in J^\ast$. Indeed from (A.35) and (A.36) it is seen that $Q^\lambda$ is a meromorphic function with two simple poles at points $w_\pm$ and we have to show that the condition (3.7) is fulfilled. It is sufficient to check it for $L^\lambda(w)$.

$$\int_0^\infty \frac{\chi_{\Pi \setminus U}(r e^{i\psi}) |L^\lambda(r e^{i\psi})|^2}{r} \, dr.$$  \hspace{1cm} (A.40)

Proposition A.2. For any function $b \in L^2(\mathbb{R}^1, dt)$

$$\sup_{\psi} \int_0^\infty \left| \int_0^\infty \frac{b(x) \, dx}{x - r e^{i\psi}} \right|^2 \, dr < \infty.$$  \hspace{1cm} (A.41)
The proof. Write
\[ \int_0^\infty dr \int_0^\infty \frac{b(x) dx}{x - r e^{i\psi}} = \int_0^\infty dr \int_0^\infty \frac{b(x)b(y) dx dy}{(x - r e^{i\psi})(y - r e^{-i\psi})}. \] (A.41)

We can easily calculate that
\[ \int_0^\infty \frac{dr}{(x - r e^{i\psi})(y - r e^{-i\psi})} = -\ln(y/x) - 2(\pi - \psi)i \frac{ye^{i\psi} - xe^{-i\psi}}{ye^{i\psi} - xe^{-i\psi}} \] (A.42)

and thus (A.41) is equal to
\[ -\int_0^\infty \int_0^\infty \frac{\ln(y/x) - 2(\pi - \psi)i}{ye^{i\psi} - xe^{-i\psi}} b(x)b(y) dx dy. \]

After unitary Mellin’s transformation
\[ \omega : L_2(\mathbb{R}^1_+) \rightarrow L_2(\mathbb{R}^1_+), \quad b(x) \rightarrow \tilde{b}(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-is-1/2} b(x) dx \]

(A.41) reads
\[ -\frac{1}{2\pi} \int_{-\infty}^\infty |\tilde{b}(s)|^2 ds \int_0^\infty \frac{\ln \xi - 2(\pi - \psi)i}{\xi e^{i\psi} - e^{-i\psi}} \xi^{-is-1/2} d\xi. \] (A.43)

It is easy to calculate that
\[ -\frac{1}{2\pi} \int_0^\infty \frac{\ln \xi - 2(\pi - \psi)i}{\xi e^{i\psi} - e^{-i\psi}} \xi^{-is-1/2} d\xi = \frac{2\pi}{(e^{-2\pi s} + 1) e^{2\pi s}}. \] (A.44)

Since the right-hand side of (A.44) is bounded, (A.40) follows from (A.43) and (A.44). The proposition is proved and hence \( Q^\lambda \in \mathcal{J}^* \).

Since the function \( q^\lambda(t) = Q^\lambda(t + i0) \) belongs to \( D(\hat{M}_0^\delta) \), it admits representation similar to (3.20) together with (3.22). Going over to the superstructures we obtain
\[ Q^\lambda(w) = F(w) + C_0(\tilde{G}^{\lambda - i} + \beta \tilde{G}^{\lambda + i}) \] (A.45)

where \( F \in \hat{\mathcal{J}} \). From this and (A.35) we get the following relations for the residues of the functions \( Q^\lambda \), \( L^\lambda \), \( \tilde{G}^{\lambda \pm i} \) and \( G^\lambda \), at the points \( w_{\pm} \):
\[ \text{res}_{w_{\pm}} Q^\lambda = C_1 \text{res}_{w_{\pm}} G^\lambda + \text{res}_{w_{\pm}} L^\lambda = C_0 \left( \text{res}_{w_{\pm}} G^{\lambda + i} + \beta \text{res}_{w_{\pm}} G^{\lambda - i} \right). \]
Thus we get the following equations w.r.t. constants $C_1, C_0$:

$$C_1 \text{ res}_w G^\lambda - C_0 \left( \text{ res}_w G^{\lambda=+i} + \beta \text{ res}_w G^{\lambda=-i} \right) = -\text{ res}_w L^\lambda,$$

(A.46)

$$C_1 \text{ res}_w G^\lambda - C_0 \left( \text{ res}_w G^{\lambda=-i} + \beta \text{ res}_w G^{\lambda=+i} \right) = -\text{ res}_w L^\lambda.$$

We see that the resolvent $R_{\lambda}^{\beta \hat{M}}(\lambda)$ exists iff the system (A.46) is solvable for any right part or in other words for those $\lambda < 0$ for which the homogeneous system has only trivial solution: $C_1 = C_0 = 0$. The other $\lambda$’s, $\lambda < 0$, form the negative spectrum of the operator $\hat{M}_0^\beta$ and for these $\lambda$ the homogeneous system has non-zero solution $(C_1, C_0)$. One can check that in this case $C_1 \neq 0$ and the homogeneous system is equivalent to the system (3.25), which has a solution iff $\lambda$ belongs to the sequence (3.19). Thus Lemma 3.3 is proved.

**Concluding remarks**

1. We have established Efimov’s effect ($\lambda_n \to 0, n \to -\infty$), Thomas’s effect ($\lambda_n \to -\infty, n \to \infty$) for the operator $M_0^\beta$. As we see it implies Thomas’s effect for the operator $M_{l=1}^\beta$. However nothing is known about Efimov’s effect for that operator.

2. The methods of this work (as well as [1]) can be used for the investigation of the operators $T_l, l > 1$, if we know the number and the position of zeros for the function $N_l(z)$, $z \in I$.

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