INTERIOR AND BOUNDARY REGULARITY FOR THE NAVIER-STOKES EQUATIONS IN THE CRITICAL LEBESGUE SPACES

Hongjie Dong* and Kunrui Wang

Division of Applied Mathematics
Brown University
182 George Street, Providence, RI 02912, USA

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ABSTRACT. We study regularity criteria for the d-dimensional incompressible Navier-Stokes equations. We prove if \( u \in L^t_t L^d_x((0,T) \times \mathbb{R}^d_+) \) is a Leray-Hopf weak solution vanishing on the boundary, then \( u \) is regular up to the boundary in \((0,T) \times \mathbb{R}^d_+\). Furthermore, with a stronger uniform local condition on the pressure \( p \), we prove \( u \) is unique and tends to zero as \( t \to \infty \) if \( T = \infty \).

This generalizes a result by Escauriaza, Seregin, and Šverák [14] to higher dimensions and domains with boundary. We also study the local problem in half unit cylinder \( Q^+ \) and prove that if \( u \in L^t_t L^d_x(Q^+) \) and \( p \in L^{2-1/d}_x(Q^+) \), then \( u \) is H"older continuous in the closure of the set \( Q^+(1/4) \).

1. Introduction. In this paper we discuss the incompressible Navier-Stokes equations in \( d \) spatial dimension with unit viscosity and zero external force:

\[
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \quad \text{div} \ u = 0
\]

for \( x \in \Omega \) and \( t > 0 \) with the initial condition

\[
u(0,x) = a(x) \quad \text{in} \ \Omega.
\]

Here \( u \) is the velocity and \( p \) is the pressure. We consider three kinds of domains: the whole space \( \Omega := \mathbb{R}^d \), the half space \( \Omega := \mathbb{R}^d_+ \), and the half ball \( \Omega := B^+ \).

For both \( \Omega = \mathbb{R}^d_+ \) and \( B^+ \), we assume that \( u \) satisfies the zero Dirichlet boundary condition:

\[
u = 0 \quad \text{on} \ \{x_d = 0\} \cap \partial \Omega.
\]

For the three dimensional Navier-Stokes equations, the global existence of strong solutions has long been an outstanding fundamental problem in fluid dynamics and is still widely open. The local solvability is well known under the assumption that the initial data \( a \) is sufficiently regular (see [24, 19, 48, 26]). The local solution is indeed unique and smooth in both spatial and time variables.

This paper is concerned with another important type of solutions called Leray-Hopf weak solutions. See Section 2.1 for the notation and definition. In the innovative works of Leray and Hopf, it is proved that for any divergence-free vector field

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\]

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* Corresponding author: Hongjie Dong.
a ∈ L_2, there exists at least one Leray-Hopf weak solution of the Cauchy problem (1)-(2) on (0, ∞) × \mathbb{R}^d. Nevertheless, the uniqueness and regularity of Leray-Hopf weak solutions remain open. There have been extensive literatures on conditional results under various criteria, among which the most well-known condition is the so-called Ladyzhenskaya-Prodi-Serrin condition: for some T > 0,

u ∈ L^t_T L^r_x(\mathbb{R}^{d+1}),

where the pair (r, q) satisfies

2/r + d/q ≤ 1, \quad q ∈ (d, \infty].

Prodi [36] and Serrin [45] proved the uniqueness of Leray-Hopf weak solutions under the conditions (4) and (5). Later the smoothness was obtained by Ladyzhenskaya [27]. For further results, we hereby refer the reader to [18, 46, 47, 5] and references therein. We are intrigued by the borderline case (r, q) = (∞, 3) which is not included in (5). This subtle case is of much interest since we cannot obtain a proof from usual methods using the local smallness of certain norms of u, which are invariant under the natural scaling

u(t, x) → \lambda u(\lambda^2 t, \lambda x), \quad p(t, x) → \lambda^2 p(\lambda^2 t, \lambda x).

For d = 3, this case was studied by Escauriaza, Seregin, and Šverák in a remarkable paper [14]. The main result of [14] is the following theorem.

**Theorem 1.1** (Escauriaza, Seregin, and Šverák). Let d = 3. Suppose that u is a Leray-Hopf weak solution of the Cauchy problem (1)-(3) in (0, T) × \mathbb{R}^3 and u satisfies the condition (4) with (r, q) = (∞, 3). Then u ∈ L^5_5((0, T) × \mathbb{R}^3), and hence it is smooth and unique in (0, T) × \mathbb{R}^3.

Before we explain Theorem 1.1, we shall recall another important concept involved in the proof, the partial regularity of weak solutions. The study of partial regularity of the Navier-Stokes equations was originated by Scheffer in a series of papers [37, 38, 39]. In three space dimensions, he established various partial regularity results for weak solutions satisfying the so-called local energy inequality. Later in a celebrated paper [4], Caffarelli, Kohn, and Nirenberg first introduced the notion of suitable weak solutions. They called a pair \((u, p)\) a suitable weak solution if \(u\) has finite energy norm, \(p\) belongs to the Lebesgue space \(L^3_{5/4}\), and \((u, p)\) is a pair of weak solution to the Navier-Stokes equations and satisfies a local energy inequality. They proved that for any suitable weak solution \((u, p)\), there exists an open subset in which the velocity field \(u\) is Hölder continuous, and the complement of it has zero 1D Hausdorff measure. In [31], with zero external force and assuming \(p \in L^3_{3/2}\), Lin gave a more direct and concise proof for Caffarelli, Kohn, and Nirenberg’s result. A detailed treatment was later given by Ladyzhenskaya and Seregin in [28]. See also Kenig-Koch [25] for an alternative approach of profile decomposition. Thereafter, the partial regularity result for the 3D time-dependent Navier-Stokes equations was extended up to the flat boundary by Seregin [41] and to the \(C^2\) boundary by Seregin, Shilkin, and Solonnikov [44]. The key step in the proofs of partial regularity results is to establish certain \(\varepsilon\)-regularity criteria. That is, intuitively speaking, if some scale invariant quantities are small then the solution is locally regular. Such results played a crucial part in the proof of [14].

In the higher dimensional case, interior partial regularity was first proved in Dong-Strain [12] for the 6D stationary incompressible Navier-Stokes equations and in Dong-Gu [9] for the 4D time-dependent equations. For the higher dimensional
boundary partial regularity cases, Dong and Gu [10] studied 4D time-dependent and 6D stationary Navier-Stokes equations. They proved that in both cases, the singular points sets have zero 2D Hausdorff measure up to the boundary. For the 4D time-dependent case, they obtained two boundary $\varepsilon$-regularity criteria [10, Theorems 1.1 and 1.2]. In Sections 3 and 4, we will extend [10, Theorem 1.2] to higher dimensions assuming the boundedness of certain norms of $u$ and $p$ and later use those criteria as tools to prove the main results of this paper.

Back to Theorem 1.1, the proofs in [14] are highly nontrivial and rely on the $\varepsilon$-regularity criteria in the light of [4], [31], and [28]. These regularity criteria may break down when the dimension increases, which inspires us to search for a way to modify and generalize the argument. Another main ingredient of the proof is a backward uniqueness theorem of heat equations with bounded lower order coefficients in the half space (see [15]). We will also use this part of argument in the proof of our theorems. Under an additional assumption on the pressure, there are some extensions of Theorem 1.1 to the half space case and the bounded domain case; we refer the reader to [42] and [34] for some results in this direction. See also [16, 51, 3, 2, 35, 1] and the references therein for other related results. In particular, in [3] Barker and Seregin extended the result in [14] to the boundary case when the solution is in $L^\infty_t L^d_x((0,T) \times \mathbb{R}^d)$. It is worth mentioning that the $\varepsilon$-regularity criteria are crucially used in all these references.

As for the extension to the higher dimensional Navier-Stokes equations, in [8], Dong and Du used Schoen’s trick to establish another regularity criterion similar to Theorem 1.1 and extended this result to $\mathbb{R}^d$ for $d \geq 3$. Unfortunately, there is a gap in the proof of [8, Lemma 3.2]. We borrow the idea from [8], that is, to find a priori $L^\infty$ bound only depends on the $L^\infty_t L^d_x$ norm and the dimension. Instead of using Schoen’s trick, we prove two $\varepsilon$-regularity criteria adapted from [10, Theorem 1.2] to provide a unified approach to obtain results in the spirit of Theorem 1.1 for both whole space $\mathbb{R}^d$ and half space $\mathbb{R}^d_+$ for $d \geq 4$.

We now state the main results of the article. The notation in Theorems 1.2, 1.3, and 1.4 is introduced in Section 2.

**Theorem 1.2.** Let $d \geq 4$ be an integer. Suppose $(u, p)$ is a pair of Leray-Hopf weak solution to the Cauchy problem in $(0,T) \times \mathbb{R}^d$ and $u$ satisfies the following condition for some $K > 0$,

$$
\|u\|_{L^\infty_t L^d_x((0,T) \times \mathbb{R}^d)} \leq K.
$$

Then $u \in L^{d+2}_{d+2}((0,T) \times \mathbb{R}^d)$, hence the solution is smooth and unique in $(0,T) \times \mathbb{R}^d$. Moreover, if $T = \infty$, we have

$$
\lim_{t \to \infty} \|u(t, \cdot)\|_{L^\infty_x} = 0.
$$

**Theorem 1.3.** Let $d \geq 4$ be an integer. Let $\mathbb{R}^d_+ := \{ x \in \mathbb{R}^d : x_d > 0 \}$ and $T \in (0,\infty]$. Suppose $(u, p)$ is a pair of Leray-Hopf weak solution to the Cauchy problem in $(0,T) \times \mathbb{R}^d_+$ and $u$ satisfies the following condition for some $K > 0$,

$$
\|u\|_{L^\infty_t L^d_x((0,T) \times \mathbb{R}^d_+)} \leq K.
$$

Let $u$ satisfy the vanishing boundary condition

$$
u(t, x) = 0 \quad \text{on } x_d = 0, \ 0 \leq t \leq T.
$$

We consider two types of local conditions on the pressure term $p$:

i) If we assume

$$
p \in L^{2-1/d}_{d,loc}([0,T] \times \mathbb{R}^d_+),
$$


then $u$ is regular up to the boundary in $(0, T) \times \mathbb{R}^d_+$. 

ii) If we assume a stronger uniform local condition on $p$: for any $z_0 \in (0, T) \times \mathbb{R}^d_+$, 
\[
\|p\|_{L^2_{-1/d}(Q(z_0,1) \cap (0,T) \times \mathbb{R}^d_+)} \leq K,
\]
then $u \in L^d_{d+2}((0,T) \times \mathbb{R}^d_+)$, hence the solution is smooth and unique. In this case, if $T = \infty$, we have 
\[
\lim_{t \to \infty} \|u(t, \cdot)\|_{L^\infty_{-d/1}(\mathbb{R}^d_+)} = 0.
\]

**Theorem 1.4.** Let $d \geq 4$ be an integer. Suppose $(u,p)$ is a pair of Leray-Hopf weak solution to the Navier-Stokes problem (1) in $(-1,0) \times \mathbb{R}^d_+$. Let $Q^+ := \{z = (t,x) : x \in \mathbb{R}^d, |x| < 1, x_d > 0, -1 < t < 0\}$. Assume $(u,p)$ satisfies the conditions:
\[
u \in L^t L^s_{d}(Q^+), \quad p \in L^{2-1/d}(Q^+),
\]
and the boundary condition
\[
u(t,x) = 0 \quad \text{on } x_d = 0, \quad |x| \leq 1, \quad -1 \leq t \leq 0.
\]
Then $\nu$ is Hölder continuous in the closure of the set
\[
Q^+ (1/4) := \{z = (t,x) : x \in \mathbb{R}^d, \quad |x| < 1/4, \quad x_d > 0, \quad -(1/4)^2 < t < 0\}.
\]

Theorem 1.2 provides an alternative approach to prove the main result in [8]. As for Theorem 1.3, we consider two different types of local integrability conditions on the pressure, which we believe that are necessary for the half-space and also for local problems in Theorem 1.4. In fact, stronger assumptions such as assuming the pressure belongs to certain Sobolev spaces have been made in previous papers [42, 34, 41, 44]. To justify the generalized energy inequality in the definition of suitable weak solutions, it also requires the local integrability conditions. We note that by using the argument in the proofs of [4, Sec. 2C], [7, Lemma 2.9], or [21, Lemma 4.2], it is possible to slightly relax such integrability conditions. However, in order not to overburden this paper we do not pursue to find an optimal condition for the pressure here. On the other hand, it is an interesting question whether the uniform condition (9) can be substantially relaxed in the half space setting. Also we remark that from the proof below, it is clear that the result in Theorem 1.4 also holds for suitable weak solutions to (1) in $Q^+$, instead of Leray Hopf weak solutions in $(-1,0) \times \mathbb{R}^d_+$. 

Next we give a brief description of our argument for the main theorem. By adding conditions (7)-(9), we extend [10, Theorem 1.2] to an $\varepsilon$-regularity criterion which reads that if certain scale invariant quantities are small then the solution is locally Hölder continuous. As in [14], we start with proof by contradiction and blow up the solution $(u,p)$ near a singular point at the first blow-up time. We can show the scale invariant quantities are uniformly bounded along a blow-up sequence $(u_k, p_k)$, hence this implies there exists a pair of limiting suitable weak solution $(u_\infty, p_\infty)$ to the Navier-Stokes equations. Furthermore, outside of a large cylinder, we can show the scale invariant quantities are indeed uniformly small for all $(u_k, p_k)$’s. Thus we can use the $\varepsilon$-regularity criterion to get local Hölder continuity and uniform local $L_\infty$ bound for $u_k$’s. Together with $L_\sigma$-convergence, we can show the local boundedness of $u_\infty$ as well as $u_\infty(0,\cdot) = 0$ by reversing the blow-up procedure. Then by applying the backward uniqueness theorem proved in [15] to the vorticity equation, we can see that $\text{curl } u_\infty = 0$ in the outside region for all time, which further implies that $u_\infty \equiv 0$ by using the spatial analyticity of strong solutions and the weak-strong uniqueness of the Navier-Stokes equations. The rest part of the proof follows the
approach in [8]. Utilizing the ε-regularity criteria proved in Sections 3 and 4, we first show there exists a $u_{k_0}$ that is regular around the origin, hence this contradicts with the assumption that $u$ blows up near a singular point. Next we bound the sup norm of $u$ to conclude $u \in L_{d+2}((0,T) \times \Omega)$. For $T = \infty$, a key observation is that $u$ is in $L_4((0,\infty) \times \mathbb{R}^d)$, which implies the smallness of its $L_4$ norm in $(T,\infty) \times \mathbb{R}^d$ for large $T$ and furthermore the smallness of the scale invariant quantities on any cylinder beyond time $T$. Again we can apply the ε-criteria to get a uniform $L_\infty$ bound on the scaled solutions beyond time $T$. We finally prove the decay with respect to the time by scaling back to the original $u$.

We remark that even though we state and prove the above theorems for $d \geq 4$, with a minor modification of the exponents in the scale invariant quantities we defined in Section 2.3, we can give an alternative proof of the more physically relevant 3D case which was proved before in [14, 42]. Since we do not use compactness argument, in our opinion, our method is more robust. This is probably one of the main points to consider those higher-dimensional or fractional-order Navier-Stokes equations. In a subsequent paper [13], we will further apply the method developed in this paper to the 3D Navier-Stokes equations, and obtain several boundary regularity results, which extend the previous interior results by Vasseur [49] and other people.

The remaining part of the paper is organized as follows. We introduce notation and terminologies in Section 2. Sections 3 and 4 are devoted to ε-regularity criteria in the whole space and half space, respectively. We use a three-step approach to obtain the ε-regularity criteria for both the whole space and the half space. In the first step, we give some estimates of the scale invariant quantities, which are by now standard and essentially follow the arguments in [31, 7]. In the second step, we establish a decay estimate of certain scale invariant quantities by using an iteration argument based on the estimates we proved in the first step. In the third step, we apply parabolic regularity to get an estimate of $L_2^ {\frac{1}{2}}$-mean oscillations of $u$, which yields the Hölder continuity of $u$ according to Campanato’s characterization of Hölder continuous functions. The main difference between the two cases lies in the treatment of the pressure term. In the interior case, the pressure can be decomposed into a sum of a harmonic function and a term controlled by $u$ using the Calderón-Zygmund estimate. In the boundary case, we need the additional assumption (9) on the pressure to use classical $L_p$ estimates for linear Stokes system to get a subtler control of the pressure. In Section 5, we present the proof of Theorem 1.3 via the blow-up procedure mentioned previously. Theorem 1.2 follows a similar argument of 1.3 so we omit the details. Theorem 1.4 is another application of the ε-regularity criteria we proved in Sections 3 and 4. We briefly describe the proof of Theorem 1.4 in Section 6.

2. Preliminaries. Let $T > 0$, $\Omega$ be a domain in $\mathbb{R}^d$, $\Gamma \subset \partial \Omega$, and $\Omega_T := (0,T) \times \Omega$ with the parabolic boundary

$$\partial_p \Omega_T = [0,T) \times \partial \Omega \cup \{t = 0\} \times \Omega.$$ 

We denote $\dot{C}_0^\infty(\Omega, \Gamma)$ the space of divergence-free infinitely differentiable vector fields which vanishes near $\Gamma$. Let $\dot{J}(\Omega, \Gamma)$ and $\dot{J}^2(\Omega, \Gamma)$ be the closures of $\dot{C}_0^\infty(\Omega, \Gamma)$ in the spaces $L_2(\Omega)$ and $W_2^1(\Omega)$, respectively.

2.1. Leray-Hopf weak solutions. By a Leray-Hopf weak solution of (1)-(2) in $\Omega_T$, we mean a vector field $u$ such that:
\[ \text{i) } u \in L_\infty(0, T; \dot{J}(\Omega, \partial \Omega)) \cap L_2(0, T; \dot{J}_2^1(\Omega, \partial \Omega)); \\
\text{ii) the function } t \mapsto \int_\Omega u(t, x) \cdot w(x) \, dx \text{ is continuous on } [0, T] \text{ for any } w \in L_2(\Omega); \\
\text{iii) the equation (1) holds weakly in the sense that for any } w \in C_0^\infty(\Omega_T), \\
\int_{\Omega_T} (-u \cdot \partial_t w - u \otimes u : \nabla w + \nabla u : \nabla w) \, dx \, dt = 0; \\
\text{iv) The energy inequality:} \\
\frac{1}{2} \int_{\Omega_T} |u(t, x)|^2 \, dx + \int_{\Omega_T} |\nabla u(s, x)|^2 \, dx \, ds \leq \frac{1}{2} \int_{\Omega_T} |u(x)|^2 \, dx \\
\text{holds for any } t \in [0, T], \text{ and we have} \\
\|u(t, \cdot) - a(\cdot)\|_{L_2^2} \to 0 \quad \text{as } t \to 0. \]

When } \Omega = \mathbb{R}^d \text{ or } \mathbb{R}^d_+, \text{ for any } a \in \dot{J}(\Omega, \partial \Omega), \text{ there exists at least one Leray-Hopf weak solution to the Cauchy problem (1)-(2) on } (0, \infty) \times \Omega. \text{ See [29] and [22].}

2.2. Suitable weak solutions. The definition of suitable weak solutions was introduced in [4]. We say a pair } (u, p) \text{ is a suitable weak solution of the Navier-Stokes equations on the set } \Omega_T \text{ vanishing on } (0, T) \times \Gamma \text{ if} \\
\text{i) } u \in L_\infty(0, T; J(\Omega, \Gamma)) \cap L_2(0, T; J_2^1(\Omega, \Gamma)) \text{ and } p \in L_2(\Omega_T); \\
\text{ii) } u \text{ and } p \text{ satisfy equation (1) in the sense of distribution.} \\
\text{iii) For any } t \in (0, T) \text{ and for any nonnegative function } \psi \in C_0^\infty(\Omega_T) \text{ vanishing in a neighborhood of the boundary } \{t = 0\} \times \Omega \text{ and } (0, T) \times (\partial \Omega \setminus \Gamma), \text{ the integrals in the following local energy inequality are summable and the inequality holds true:} \\
\text{ess sup}_{0 \leq s \leq t} \int_{\Omega} |u(s, x)|^2 \psi(s, x) \, dx + 2 \int_{\Omega_T} |\nabla u|^2 \psi \, dx \, ds \\
\leq \int_{\Omega_T} \{ |u|^2 (\psi_t + \Delta \psi) + (|u|^2 + 2p)u \cdot \nabla \psi \} \, dx \, ds. \tag{11} \\
\]

2.3. Scale invariant quantities. In this paper, we write a point in } [0, T] \times \mathbb{R}^d \text{ as } z = (t, x) = (t, x_1, x_2, \ldots, x_d) = (t, x', x_d), \text{ where } x' = (x_1, x_2, \ldots, x_{d-1}). \text{ We shall use the following notation for balls, half spheres, spheres, half spheres, parabolic cylinders, half parabolic cylinders, and parabolic boundaries:} \\
B(\hat{x}, r) = \{x \in \mathbb{R}^d \mid |x - \hat{x}| < r\}, \quad B_r = B(r) = B(0, r), \quad B = B(1); \\
B^+(\hat{x}, r) = \{x \in B(\hat{x}, r) \mid x = (x', x_d), x_d > \hat{x}_d\}, \quad B^+_r = B^+(r) = B^+(0, r), \quad B^+ = B^+(1); \\
S(\hat{x}, r) = \{x \in \mathbb{R}^d \mid |x - \hat{x}| = r\}, \quad S_r = S(r) = S(0, r), \quad S = S(1); \\
S^+(\hat{x}, r) = \{x \in S(\hat{x}, r) \mid x = (x', x_d), x_d > \hat{x}_d\}, \quad S^+_r = S^+(r) = S^+(0, r), \quad S^+ = S^+(1); \\
Q(\hat{x}, r) = (\hat{t} - r^2, \hat{t}) \times B(\hat{x}, r), \quad Q_r = Q(r) = Q(0, r), \quad Q = Q(1); \\
Q^+(\hat{x}, r) = (\hat{t} - r^2, \hat{t}) \times B^+(\hat{x}, r), \quad Q^+_r = Q^+(r) = Q^+(0, r), \quad Q^+ = Q^+(1); \\
\partial_p Q(\hat{x}, r) = [\hat{t} - r^2, \hat{t}] \times S(\hat{x}, r) \cup \{t = \hat{t} - r^2\} \times B(\hat{x}, r), \quad \partial_p Q^+(\hat{x}, r) = [\hat{t} - r^2, \hat{t}] \times S^+(\hat{x}, r) \cup \{t = \hat{t} - r^2\} \times B^+(\hat{x}, r), \quad \partial_p Q^+(\hat{x}, r).
where \( \hat{z} = (\hat{t}, \hat{x}) \) and \( \hat{x}_d \) is the \( d \)-th coordinate of \( \hat{x} \).

For the remaining part of the paper, we restrict our discussion to the following domains (except for the local problem in Section 6):

\[
\Omega = \mathbb{R}^d \text{ or } \mathbb{R}_+^d, \quad \Omega_T = (0, T) \times \Omega,
\]

\[
\Omega(\hat{z}, r) = B(\hat{z}, r) \cap \Omega, \quad \omega(\hat{z}, r) = Q(\hat{z}, r) \cap \Omega_T.
\]

In particular, we denote \( \mathbb{R}_{T+}^d = (0, T) \times \mathbb{R}^d \).

We notice that these quantities are all invariant under the natural scaling (6).

We recall the following local problem in Section 6:

\[
\Omega = \mathbb{R}^d \text{ or } \mathbb{R}_+^d, \quad \Omega_T = (0, T) \times \Omega,
\]

\[
\Omega(\hat{z}, r) = B(\hat{z}, r) \cap \Omega, \quad \omega(\hat{z}, r) = Q(\hat{z}, r) \cap \Omega_T.
\]

We denote mean values of summable functions as follows:

\[
[u]_t,r(t) = \frac{1}{|\Omega(\hat{z}, r)|} \int_{\Omega(\hat{z}, r)} u(t, x) \, dx, \quad (u)_t,r = \frac{1}{|\omega(\hat{z}, r)|} \int_{\omega(\hat{z}, r)} u(z) \, dz,
\]

where \(|A|\) as usual denotes the Lebesgue measure of the set \( A \).

Now we introduce the following important quantities:

i) When \( \Omega = \mathbb{R}^d \),

\[
A(r, z_0) = \sup_{t_0 - t \leq r} \frac{1}{\rho^{d-2}} \int_{B(x_0, r)} |u|^2 \, dx,
\]

\[
E(r, z_0) = \frac{1}{\rho^{d-2}} \int_{Q(z_0, r)} |\nabla u|^2 \, dz,
\]

\[
C(r, z_0) = \frac{1}{\rho^{d+2/d-2}} \int_{Q(z_0, r)} |u|^{2(2d-1)/d} \, dz,
\]

\[
D(r, z_0) = \frac{1}{\rho^{d+2/d-2}} \int_{Q(z_0, r)} |p - [p]_{z_0,r}(t)|^{(2d-1)/d} \, dz.
\]

ii) When \( \Omega = \mathbb{R}_+^d \) or \( Q^+ \),

\[
A^+(r, z_0) = \sup_{t_0 - t \leq r} \frac{1}{\rho^{d-2}} \int_{\Omega(x_0, r)} |u|^2 \, dx,
\]

\[
E^+(r, z_0) = \frac{1}{\rho^{d-2}} \int_{\omega(z_0, r)} |\nabla u|^2 \, dz,
\]

\[
C^+(r, z_0) = \frac{1}{\rho^{d+2/d-2}} \int_{\omega(z_0, r)} |u|^{2(2d-1)/d} \, dz,
\]

\[
D^+(r, z_0) = \frac{1}{\rho^{d+2/d-2}} \int_{\omega(z_0, r)} |p - [p]_{z_0,r}(t)|^{(2d-1)/d} \, dz.
\]

We notice that these quantities are all invariant under the natural scaling (6).

In the later part of the paper, we use notation \( A^+ \) to represent either \( A \) or \( A^+ \) depending on \( \Omega = \mathbb{R}^d \) or \( \Omega = \mathbb{R}_+^d \) when there is no confusion and similarly for \( E^+, C^+, D^+, Q^+ \), etc. We omit \( z_0 \), the argument for center, from the above expressions and write \( A(r), E(r), C(r), \) and \( D(r) \) when there is no ambiguity.

2.4. Strong solutions and spatial analyticity. We recall the following local solvability of (1)-(2) (see, for instance, [24, 19, 48, 26, 52]), and spatial analyticity of strong solutions (see, for instance, [20, 11, 23, 33]).

**Proposition 1.** For any divergence-free initial data \( a \in L_p(\Omega) \) with \( p \geq d \), where \( \Omega = \mathbb{R}^d \) or \( \mathbb{R}_+^d \), the Cauchy problem (1)-(2) has a unique strong solution \( u \in C([0, \delta]; L_p(\Omega)) \) for some \( \delta > 0 \). Moreover, \( u \) is infinitely differentiable and spatial analytic for \( t \in (0, \delta) \).
In the following two sections, we will show the Hölder continuity of $u$ given the scale invariant quantities defined previously are sufficiently small. The main difference between the interior estimate and the boundary estimate results from the different estimates of quantities $D$ and $D^+$.

### 3. Interior Hölder continuity estimate

In this section, we consider $\Omega = \mathbb{R}^d$. We take a three-step approach to prove the following $\varepsilon$-regularity criterion in the whole space.

**Theorem 3.1.** Let $(u, p)$ be a suitable weak solution of (1)-(2) in $(0, T) \times \mathbb{R}^d$ satisfying (7). There exists a universal constant $\varepsilon_0$ satisfying the following property. Assume that for a point $z_0 \in \mathbb{R}^{d+1}_T$ we have

$$A(\rho_0, z_0) + E(\rho_0, z_0) + D(\rho_0, z_0) \leq \varepsilon_0$$

for some small $\rho_0 > 0$. Then $u$ is Hölder continuous near $z_0$ and we have

$$\rho_0 \|u\|_{L^\infty(Q_{\rho_0/2}(z_0))} + \rho_0^{1+\alpha} [u]_{C^{\alpha/2, \alpha}(Q_{\rho_0/2}(z_0))} \leq N(d),$$

where $\alpha = 1/(2(2d-1))$.

### 3.1. Step 1

We present several inequalities of the scale invariant quantities. We will make use of the following interpolation inequality from [8, Lemma 2.1] substantially.

**Lemma 3.2.** For any function $u \in W^{1,2}_2(B_r)$, $r > 0$, and $q \in [2, 2d/(d-2)]$, we have

$$\int_{B_r} |u|^q \, dx \leq N(q, d) \left( \int_{B_r} |\nabla u|^2 \, dx \right)^{\frac{q}{2} \left(\frac{2}{q-1}\right)} \left( \int_{B_r} |u|^2 \, dx \right)^{\frac{q}{2} - \frac{q}{2} \left(\frac{2}{q-1}\right)} + N(q, d)^{-d \left(\frac{d}{q-1}\right)} \left( \int_{B_r} |u|^2 \, dx \right)^{\frac{2}{d}}.$$  \hspace{1cm} (12)

**Proof.** Without loss of generality, we assume $r = 1$. For $q \in [2, 2d/(d-2)]$, we use Hölder’s inequality inside the unit ball $B$,

$$\|u\|_{L^q(B)} \leq \|u\|_{L^2(B)}^{\frac{q-2}{2}} \|u\|_{L^{\frac{2d}{d-2}}(B)}^{\frac{2}{d-2}}.$$  \hspace{1cm} (13)

which together with Sobolev embedding theorem gives (12). The lemma is proved.

The next lemma is an application of the interpolation inequality proved in Lemma 3.2.

**Lemma 3.3.** For $\alpha \in [0, 1]$ and $p \in [2 + \frac{4\alpha}{d}, d + (4-d)\alpha]$, suppose $r > 0$, $Q(z_0, r) \subset \mathbb{R}^{d+1}_T$, and $u$ satisfies the condition (7). Then we have

$$\int_{Q(z_0, r)} |u|^p \, dz \leq N(A(r, z_0) + E(r, z_0))^{\frac{d+2\alpha}{d-2}}.$$  \hspace{1cm} (14)

In particular, taking $\alpha = 1$ and $p = 4 - 2/d$ we have

$$C(r, z_0) \leq N(A(r, z_0) + E(r, z_0))^{\frac{d+2+2/d}{d-2}}.$$
Due to Hölder’s inequality, we assume $r = 1$. Because $p \in \left[2 + \frac{4\alpha}{d}, d + (4 - d)\alpha\right]$, we know
\[
pd - 2d - 4\alpha \geq 0 \quad \text{and} \quad q := \frac{2d(d - p + 2\alpha)}{d^2 - dp + 4\alpha} \in \left[2, \frac{2d}{d - 2}\right].
\]
By using Hölder’s inequality and (12) with this $q$, we have
\[
\int_B |u|^p \, dx \leq \left(\int_B |u|^\frac{2d(d - p + 2\alpha)}{d^2 - dp + 4\alpha} \, dx\right)\left(\int_B |u|^d \, dx\right)^{\frac{d}{d - p + 2\alpha}} \leq N \left(\int_B |\nabla u|^2 \, dx\right)^{\alpha} \left(\int_B |u|^2 \, dx\right)^{\frac{(1 - \alpha)d + 4\alpha - p}{2 - \alpha}} + N \left(\int_B |u|^2 \, dx\right)^{\frac{d - p + 2\alpha}{d - 2}},
\]
where we used (7) in the last inequality. Integrating in time yields the desired result.

**Lemma 3.4.** Let $(u, p)$ be a pair of suitable weak solution of (1). For $\rho > 0$ and $Q(z_0, \rho) \subset \mathbb{R}^{d+1}_+$, we have
\[
A(\rho/2) + E(\rho/2) \leq N \left( C(\rho)^{\frac{d}{d - 1}} + C(\rho)^{\frac{d}{2(d - 1)}} + C(\rho)^{\frac{d}{2(d - 1)}} D(\rho)^{\frac{d}{d - 1}} \right),
\]
where $N$ is independent of $z_0$ and $\rho$.

**Proof.** By a scaling argument, we may assume $\rho = 1$. In the energy inequality (11), we put $t = t_0$ and choose a suitable smooth cut-off function $\psi$ such that
\[
\psi \equiv 0 \quad \text{in} \quad \mathbb{R}^{d+1}_0 \setminus Q(z_0, 1), \quad 0 \leq \psi \leq 1 \quad \text{in} \quad \mathbb{R}^{d+1}_0,
\]
\[
\psi \equiv 1 \quad \text{in} \quad Q(z_0, 1/2), \quad |\nabla \psi| + |\partial_t \psi| + |\nabla^2 \psi| \leq N \quad \text{in} \quad \mathbb{R}^{d+1}_0.
\]
By using (11), we get
\[
A(1/2) + 2E(1/2) \leq N \int_{Q(z_0, 1)} |u|^2 \, dz + N \int_{Q(z_0, 1)} (|u|^2 + 2|p - [p]_{x_0, 1}) |u| \, dz.
\]
Due to Hölder’s inequality, we can obtain
\[
\int_{Q(z_0, 1)} |u|^2 \, dz \leq N(C(1))^{\frac{d}{2(d - 1)}}, \quad \int_{Q(z_0, 1)} |u|^3 \, dz \leq N(C(1))^{\frac{2d}{2(d - 1)}},
\]
and
\[
\int_{Q(z_0, 1)} |p| |u| \, dz \leq \left(\int_{Q(z_0, 1)} |u|^\frac{2d}{d - 1} \, dz\right)^{\frac{d - 1}{2(d - 1)}} \left(\int_{Q(z_0, 1)} |p - [p]_{x_0, 1}|^{\frac{2d}{d - 1}} \, dz\right)^{\frac{d}{2(d - 1)}} \leq C(1)^{\frac{d}{2(d - 1)}} D(1)^{\frac{d}{d - 1}}.
\]
The conclusion of Lemma 3.4 follows immediately.

**Lemma 3.5.** Let $(u, p)$ be a pair of suitable weak solution of (1) when $d \geq 4$. For constants $\gamma \in (0, 1/2], \rho > 0$, and $Q(z_0, \rho) \subset \mathbb{R}^{d+1}_+$, we have
\[
A(\gamma \rho) + E(\gamma \rho) \leq N \left[ \gamma^2 A(\rho) + \gamma^{-d+1} \left( (A(\rho) + E(\rho))^{\frac{d}{d - 1}} + (A(\rho) + E(\rho))^{\frac{1}{2}} D(\rho)^{\frac{d}{d - 1}} \right) \right].
\]
Recall that

By using the equation

The test function has the following properties:

Therefore (14) yields

In the energy inequality (11), we choose \( \psi = \Gamma \phi \), where \( \phi \in C_0^\infty((t_0 - 1, t_0 + 1) \times B(x_0, 1)) \) is a suitable smooth cut-off function satisfying

\[
0 \leq \phi \leq 1 \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \quad \phi \equiv 1 \quad \text{in } Q(z_0, 1/2),
\]

\[
|\nabla \phi| \leq N, \quad |\nabla^2 \phi| \leq N, \quad |\partial_t \phi| \leq N \quad \text{in } \mathbb{R} \times \mathbb{R}^d.
\]

By using the equation

\[
\Delta \Gamma + \Gamma_t = 0,
\]

we have

\[
\begin{aligned}
\text{ess sup}_{t_0-1 \leq t \leq t_0} & \int_{B(x_0,1)} |u(t,x)|^2 \Gamma(t,x) \phi(t,x) \, dx + 2 \int_{Q(z_0,1)} |\nabla u|^2 \phi \, dz \\
& \leq \int_{Q(z_0,1)} |u|^2 \left( \Gamma \phi_t + \Gamma \Delta \phi + 2 \nabla \phi \nabla \Gamma \right) \\
& \quad + \left( |u|^2 + 2 |p - [p]_{x_0,1}| \right) u \cdot (\Gamma \nabla \phi + \phi \nabla \Gamma) \, dz.
\end{aligned}
\]

The test function has the following properties:

(i) For some constant \( c > 0 \), on \( Q(z_0, \gamma) \) it holds that

\[
\Gamma \phi = \Gamma \geq c \gamma^{-d}.
\]

(ii) For any \( z \in Q(z_0, 1) \), we have

\[
|\Gamma(z) \phi(z)| \leq N \gamma^{-d}, \quad |\nabla \Gamma(z) \phi(z)| + |\Gamma(z) \nabla \phi(z)| \leq N \gamma^{-d-1}.
\]

(iii) For any \( z \in Q(z_0, 1) \), we have

\[
|\Gamma(z) \phi_t(z)| + |\Gamma(z) \Delta \phi(z)| + |\nabla \Gamma(z) \nabla \phi(z)| \leq N.
\]

Therefore (14) yields

\[
A(\gamma) + E(\gamma) = \gamma^{-d+2} \text{ess sup}_{t_0-1 \leq t \leq t_0} \int_{B(x_0, \gamma)} |u(t,x)|^2 \, dx + \gamma^{-d+2} \int_{Q(z_0, \gamma)} |\nabla u|^2 \, dz
\]

\[
\leq N \gamma^2 \int_{Q(z_0,1)} |u|^2 \, dz + N \gamma^{-d+1} \int_{Q(z_0,1)} \left( |u|^2 + 2 |p - [p]_{x_0,1}| \right) |u| \, dz.
\]

Recall that \( d \geq 4 \). Applying Lemma 3.3 with \( \alpha = 1 \) and \( p = 3 \) we have

\[
\int_{Q(z_0,1)} |u|^3 \, dz \leq N \left( A(1, z_0) + E(1, z_0) \right)^{\frac{d-1}{4(d-1)}},
\]

and again applying Lemma 3.3 with \( \alpha = \frac{d}{4(d-1)} \) and \( p = \frac{2d-1}{d-1} \), we have

\[
\begin{aligned}
\int_{Q(z_0,1)} |p - [p]_{x_0,1}| |u| \, dz \\
& \leq \left( \int_{Q(z_0,1)} |u|^\frac{2d-1}{d-1} \, dz \right)^{\frac{d-1}{2d-1}} \left( \int_{Q(z_0,1)} |p - [p]_{x_0,1}|^{\frac{2d-1}{d-1}} \, dz \right)^{\frac{d}{2d-1}} \\
& \leq N \left( A(1, z_0) + E(1, z_0) \right)^{\frac{3}{2} \frac{d}{d-1}} D(1, z_0)^{\frac{d}{d-1}}.
\end{aligned}
\]

The lemma is proved. \( \square \)
Lemma 3.6. Let \((u, p)\) be a pair of Leray-Hopf weak solution of (1). For constants 
\(\gamma \in (0, 1/2), \rho > 0\), and \(Q(x_0, \rho) \subset \mathbb{R}^{d+1}_T\), we have
\[
D(\gamma \rho) \leq N(d)\xi^{-d-2/d+2C(\rho)} + \gamma^{4-3/d} D(\rho).
\]
(15)

Proof. Let \(r = \gamma \rho\) and \(\eta(x)\) be a smooth cut-off function supported in \(B(1), 0 \leq \eta \leq 1\) and \(\eta \equiv 1\) on \(B(2/3)\). In the sense of distribution, for a.e. \(t \in (t_0 - \rho^2, t_0)\), one has
\[
\Delta p = D_{ij}(u_i u_j).
\]

We consider the decomposition
\[
p = p_{x_0, \rho} + h_{x_0, \rho},
\]
where \(p_{x_0, \rho}\) is the Newtonian potential of
\[
D_{ij}(u_i u_j)((x - x_0)/\rho)).
\]

Then \(h_{x_0, \rho}\) is harmonic in \(B(x_0, 2\rho/3)\).

By using the Calderón-Zygmund estimate, we have
\[
\int_{Q(x_0, \rho)} |p_{x_0, \rho}|^{2d-1} \frac{dz}{\rho} \leq \int_{Q(x_0, \rho)} |p_{x_0, \rho}|^{2d-1} \frac{dz}{\rho} \leq N \int_{Q(x_0, \rho)} |u|^{2(2d-1)} \frac{dz}{\rho}.
\]
(16)

From the fact that any Sobolev norm of harmonic function \(h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}\) in a smaller ball can be estimated by any of its \(L_2\) norm in \(B(x_0, 2\rho/3)\), one obtains
\[
\int_{B(x_0, r)} |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{2d-1} \frac{dx}{\rho} \leq N \rho \frac{d+2d-1}{2d-1} \left( \frac{\rho}{\rho} \right) \int_{B(x_0, \rho)} |h_{x_0, \rho} - [p]_{x_0, \rho}|^{2d-1} \frac{dx}{\rho}.
\]
(17)

Integrating (17) in \(t \in (t_0 - \rho^2, t_0)\), we obtain
\[
\int_{Q(x_0, r)} |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{2d-1} \frac{dz}{\rho} \leq N \left( \frac{\rho}{\rho} \right) \frac{d+2d-1}{2d-1} \int_{Q(x_0, \rho)} \left[ |p - [p]_{x_0, \rho}|^{2d-1} + |p_{x_0, \rho}|^{2d-1} \right] \frac{dz}{\rho}.
\]
(18)

where we used (16) in the last inequality. We combine (18), (16), and use the triangle inequality to have
\[
\int_{Q(x_0, r)} |p - [p]_{x_0, \rho}|^{2d-1} \frac{dz}{\rho} \leq N \int_{Q(x_0, r)} |p - [h_{x_0, \rho}]_{x_0, r}|^{2d-1} \frac{dz}{\rho}
\]
\[
\leq N \int_{Q(x_0, r)} \left( |h_{x_0, \rho} - [h_{x_0, \rho}]_{x_0, r}|^{2d-1} + |p_{x_0, \rho}|^{2d-1} \right) \frac{dz}{\rho}.
\]
\[
\leq N \left( \frac{\rho}{\rho} \right) \frac{d+2d-1}{2d-1} \int_{Q(x_0, \rho)} \left| p - [p]_{x_0, \rho} \right|^{2d-1} + \frac{dz}{\rho}.
\]
The lemma is proved. □

**Corollary 1.** Let \((u, p)\) be a pair of suitable weak solution of \((1)\). For some \(z_0 \in \mathbb{R}_T^{d+1}\), suppose there exist \(\rho_0 > 0\) and \(C_1 > 0\), such that \(Q(z_0, \rho_0) \subset \mathbb{R}_T^{d+1}\) and \(C(\rho, z_0) \leq C_1\) for all \(\rho \in (0, \rho_0]\) and \(D(\rho_0, z_0) \leq C_1\). Then we can find \(C = C(C_1, d) > 0\) such that \(D(\rho, z) \leq C\) for all \(z \in Q(z_0, 1/2)\) and \(\rho \in (0, \rho_0/2]\).

**Proof.** Note that \(Q(z, \rho_0/2) \subset Q(z_0, \rho_0)\) for \(z \in Q(z_0, 1/2)\), hence \(D(\rho_0/2, z) \leq ND(\rho_0, z_0)\). By fixing \(\gamma\) small enough that \(N\gamma^{-3/d} \leq 1/2\) and using \((15)\), we have

\[
D(\gamma \rho_0/2, z) \leq \frac{1}{2} D(\rho_0/2, z) + C_2,
\]

where \(C_2 = N\gamma^{-d-2/d+2} C_1\). Inductively, for any integer \(k\) we have

\[
D(\gamma^k \rho_0/2, z) \leq \frac{1}{2^k} D(\rho_0/2, z) + C_2 \sum_{j=0}^{k-1} \frac{1}{2^j} \leq C_1 + 2C_2 := C_3.
\]

Now for any \(\gamma^{k+1} \rho_0 \leq \rho < \gamma^k \rho_0\), we can control \(D(\rho/2, z)\) by

\[
D(\rho/2, z) \leq \left(\frac{\gamma^k \rho_0}{\gamma^{k+1} \rho_0}\right)^{d+2/d-2} D(\gamma^k \rho_0/2, z) \leq \gamma^{-d-2/d+2} C_3 := C.
\]

The corollary is proved because \(C\) is independent of \(k\). □

**3.2. Step 2.** We will find some decay rates for the scale invariant quantities with respect to the radius assuming the quantities are initially small.

**Lemma 3.7.** There exists a universal constant \(\varepsilon_0 > 0\) satisfying the following property. Suppose that for some \(z_0 = (t_0, x_0)\) and \(\rho_0 > 0\), it holds that \(Q(z_0, \rho_0) \subset \mathbb{R}_T^{d+1}\) and

\[
A(\rho_0, z_0) + E(\rho_0, z_0) + D(\rho_0, z_0) \leq \varepsilon_0.
\]

Then fixing any \(\alpha_0 \in (0, 2)\), there exists \(N > 0\) such that for any \(\rho \in (0, \rho_0/2]\) and \(z \in Q(z_0, \rho_0/2)\), the following estimate holds uniformly

\[
A(\rho, z) + E(\rho, z) + C(\rho, z)^{\frac{d-2}{d+2/d-2}} + D(\rho, z) \leq N \varepsilon_0^{\alpha_0/2} \left(\frac{\rho}{\rho_0}\right)^{\alpha_0},
\]

where \(N\) is a positive constant depending on \(\alpha_0\), but independent of \(\varepsilon_0\), \(\rho_0\), \(\rho\), and \(z\).

**Proof.** For any \(z \in Q(z_0, \rho_0/2)\), by \((19)\) and

\[
Q(z, \rho_0/2) \subset Q(z_0, \rho_0) \subset \mathbb{R}_T^{d+1},
\]

we get

\[
A(\rho_1, z) + E(\rho_1, z) + D(\rho_1, z) \leq N \varepsilon_0,
\]

where \(\rho_1 = \rho_0/2\). By Lemma 3.3,

\[
C(\rho_1, z) \leq (N \varepsilon_0)^{\frac{d-2+2/d}{d-2}}.
\]

Next we fix an auxiliary parameter \(\alpha \in (\alpha_0, 2)\). By a scaling argument, we first discuss a special case when \(\rho_0^\alpha = N \varepsilon_0 < 1\). In this case, we can prove the following decay rates inductively:

\[
A(\rho_k) + E(\rho_k) \leq \rho_k^\alpha, \quad C(\rho_k)^{\frac{d-2}{d+2/d-2}} \leq \rho_k^\alpha, \quad D(\rho_k) \leq \rho_k^\alpha,
\]

\[\]
where \( \rho_{k+1} = \rho_k^{1+\beta} = \rho_1^{(1+\beta)^k} \) and \( \beta \) is a small number to be specified. For \( k = 1 \), the statement follows from (21), (22), and our assumption that \( \rho_1^\alpha = N\varepsilon_0 \). Next by choosing \( \gamma = \rho_1^\beta \) and \( \rho = \rho_k \) in (13) and (15), we have

\[
A(\rho_{k+1}) + E(\rho_{k+1}) \leq N \left[ \rho_k^{2\beta+\alpha} + \rho_k^{(-d+1)\beta} \left( \frac{d+1}{d-2} \rho_k^{-\alpha} + \rho_k^{(\frac{1}{2} + \frac{d}{d-2})\alpha} \right) \right],
\]

\[
D(\rho_{k+1}) \leq N \left[ \rho_k^{(-d-\frac{3}{2}+2)\beta + (1+\frac{d}{d-2})\alpha} + \rho_k^{(4-\frac{d}{2})\beta+\alpha} \right].
\]

We choose \( \beta \) satisfying

\[
\beta < \min \left\{ \frac{\alpha}{(d-2)(d+\alpha-1)}, \frac{\alpha}{2(2d-1)(d+\alpha-1)}, \frac{2\alpha}{d(\alpha + d + \frac{2}{d} - 2)(2d-2)} \right\}
\]

Then all the exponents on the right-hand sides are greater than \((1 + \beta)\alpha\).

Now we can find \( \xi > 0 \) depending on \( \beta \) such that

\[
A(\rho_{k+1}) + E(\rho_{k+1}) \leq N\rho_{k+1}^{\alpha+\xi}, \quad D(\rho_{k+1}) \leq N\rho_{k+1}^{\alpha+\xi},
\]

where \( N \) is a constant independent of \( k \) and \( \xi \). By taking \( \varepsilon_0 \) small enough such that \( N\rho_{k+1}^{\xi} < N(N\varepsilon_0)^{\xi/2} < 1 \), we obtain

\[
A(\rho_{k+1}) + E(\rho_{k+1}) \leq \rho_{k+1}^{\alpha}, \quad D(\rho_{k+1}) \leq \rho_{k+1}^{\alpha}.
\]

By induction, we have justified (23) for the case when \( \rho_1^\alpha = N\varepsilon_0 \).

For convenience, we additionally assume that the parameter \( \beta \) satisfying

\[
\alpha_0 < \min \left\{ \frac{1}{1+\beta}(\alpha - (d-2)\beta), \frac{1}{(1+\beta)} \left( \alpha - \left( \frac{d}{d-2} \right) \beta \right) \right\}.
\]

There always exists feasible \( \beta \) because \( \alpha > \alpha_0 \).

Now for any \( \rho \in (0, \rho_0/2) \), we can find a positive integer \( k \) such that \( \rho_{k+1} \leq \rho < \rho_k \). Then

\[
A(\rho) + E(\rho) \leq (\rho_k/\rho_{k+1})^{d-2} \left( A(\rho_k) + E(\rho_k) \right) \leq \rho_k^{\alpha-(d-2)\beta} = \rho_{k+1}^{\alpha-(d-2)\beta} \leq \rho_0^{\alpha_0},
\]

\[
D(\rho) \leq (\rho_k/\rho_{k+1})^{d+\frac{2}{d}+2} D(\rho_k) \leq \rho_k^{\alpha-(d+\frac{2}{d}-2)\beta} = \rho_{k+1}^{\alpha-(d+\frac{2}{d}-2)\beta} \leq \rho_0^{\alpha_0}.
\]

Hence we have proved (20) when \( \rho_1^\alpha = N\varepsilon_0 \). For the general case, we use the scale invariant property of the quantities and apply the previous results with an additional scaling factor \( N\varepsilon_0^{\alpha_0/\alpha} \rho_0^{-\alpha_0} \) on the right-hand side.

\( \square \)

3.3. **Step 3.** In this final step, we first use parabolic \( L_p \) estimates to further improve the decay rate and then conclude the result by using Campanato’s characterization of Hölder continuity.

**Lemma 3.8.** Suppose \( f(\rho_0) \leq C_0 \). If there exist \( \alpha > \beta > 0 \) and \( C_1, C_2 > 0 \) such that for any \( 0 < r < \rho \leq \rho_0 \), it holds that

\[
f(r) \leq C_1 \left( \frac{r}{\rho} \right)^\alpha f(\rho) + C_2 \rho^\beta,
\]

then there exist constants \( C_3, C_4 > 0 \) depending on \( C_0, C_1, C_2, \alpha, \beta \), such that

\[
f(r) \leq C_3 \left( \frac{r}{\rho_0} \right)^\beta f(\rho_0) + C_4 r^\beta
\]

for \( 0 < r \leq \rho_0 \).

**Proof.** See, for instance, [17, Chapter III, Lemma 2.1].

\( \square \)
By Lemma 3.7 we know, for any small \( \delta_1 > 0 \), the following estimates are true for all \( \rho \in (0, \rho_0/2) \) sufficiently small and \( z_1 := (t_1, x_1) \in Q(z_0, \rho_0/2):\)

\[
\int_{Q(z_1, \rho)} |u|^{4-\frac{4}{d}} \, dz \leq N \rho^{d+\frac{2}{d} - \delta_1}, \tag{24}
\]

\[
\int_{Q(z_1, \rho)} |p - [p]_{z_1, \rho}|^{2-\frac{4}{d}} \, dz \leq N \rho^{d+\frac{2}{d} - \delta_1}. \tag{25}
\]

Let \( v \) be the unique weak solution to the heat equation

\[\partial_t v - \Delta v = 0 \quad \text{in } Q(z_1, \rho),\]

with the boundary condition \( v = u \) on \( \partial_p Q(z_1, \rho) \). Let \( 0 < r < \rho \). By the Poincaré inequality with zero mean value and using the fact that the \( C^{1/2, 1} \) semi-norm of a caloric function in a smaller cylinder is controlled by any \( L_p \) norm of it in a larger cylinder. We have

\[
\int_{Q(z_1, r)} |v - (v)_{z_1, r}|^{2-\frac{4}{d}} \, dz \leq r^{d+4-\frac{4}{d}} \|v - (u)_{z_1, \rho}\|_2^{2-\frac{4}{d}}.
\]

Apply Lemma 3.7, we get

\[
\int_{Q(z_1, \rho)} \left| \nabla w_1 \right|^{2-\frac{4}{d}} \, dz \leq N \rho^{d+\frac{4}{d} - \delta_1}, \tag{27}
\]

which together with (24) and (25) yields

\[
\int_{Q(z_1, \rho)} |v|^{2-\frac{4}{d}} \, dz \leq N \rho^{d+\frac{2}{d} - \delta_1}.
\]

Using (26), (28), and the triangle inequality, we have

\[
\int_{Q(z_1, r)} |u - (u)_{z_1, r}|^{2-\frac{4}{d}} \, dz \leq \int_{Q(z_1, r)} |v - (v)_{z_1, r}|^{2-\frac{4}{d}} \, dz + \int_{Q(z_1, r)} |w - (w)_{z_1, r}|^{2-\frac{4}{d}} \, dz \leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{4}{d}} \int_{Q(z_1, \rho)} |v - (u)_{z_1, \rho}|^{2-\frac{4}{d}} \, dz + N \rho^{d+2+\frac{4}{d} - \delta_1},
\]

\[
\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{4}{d}} \left( \int_{Q(z_1, \rho)} |u - (u)_{z_1, \rho}|^{2-\frac{4}{d}} \, dz + \int_{Q(z_1, \rho)} |w|^{2-\frac{4}{d}} \, dz \right) + N \rho^{d+2+\frac{4}{d} - \delta_1},
\]

\[
\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{4}{d}} \int_{Q(z_1, \rho)} |u - (u)_{z_1, \rho}|^{2-\frac{4}{d}} \, dz + N \rho^{d+2+\frac{4}{d} - \delta_1}.
\]
Applying Lemma 3.8 and choosing $\delta_1 = \frac{1}{2d}$, we obtain
\[ \int_{Q(z_1, r)} |u - (u)_{z_1, r}|^{2-\frac{4}{r}} \, dz \leq N_r^{-d+2+\frac{1}{2d}} \]
for any $r \in (0, \rho_0/4)$ and $z_1 \in Q(z_0, \rho_0/4)$. We then conclude that $u$ is Hölder continuous near $z_0$ by Campanato’s characterization of Hölder continuity.

4. Boundary Hölder continuity estimate. In this section, we consider the case when $\Omega = \mathbb{R}^d_+$. We again use a three-step approach to prove an $\varepsilon$-regularity criterion near boundary. The main difference from the interior estimate is the iteration dealing with the pressure term.

**Theorem 4.1.** Let $(u, p)$ be a suitable weak solution of (1)-(2) in $(0, T) \times \mathbb{R}^d_+$ satisfying (7). There exists a universal constant $\hat{\varepsilon}_0$ satisfying the following property. Assume that for a point $\hat{z} = (\hat{t}, \hat{x})$, where $\hat{x} = (x', 0)$, and for some $\rho_0 > 0$ we have $Q^+(\hat{z}, \rho_0) \subset (0, T) \times \mathbb{R}^d_+$ and
\[ A^+(\rho_0, \hat{z}) + E^+(\rho_0, \hat{z}) + D^+(\rho_0, \hat{z}) \leq \hat{\varepsilon}_0. \]
Then $u$ is Hölder continuous near $\hat{z}$ and we have
\[ \rho_0 \|u\|_{L_\infty(Q^+_{2\rho_0}(\hat{z}))} + \rho_0^{1+\alpha} [u]_{C^{0, \alpha}(Q^+_{\rho_0}(\hat{z}))} \leq N(d), \]
where $\alpha \in (0, 1)$ is a constant depending only on $d$.

4.1. Step 1. We present several inequalities for the scale invariant quantities.

**Lemma 4.2.** Suppose function $u \in W^1_q(B^+_r)$ with $r > 0$ vanish on the boundary $x_d = 0$. For any $q \in [2, 2d/(d-2)]$, we have
\[ \int_{B^+_r} |u|^q \, dx \leq N(q, d) \left( \int_{B^+_r} |\nabla u|^2 \, dx \right)^{d(q/4-1/2)} \left( \int_{B^+_r} |u|^2 \, dx \right)^{q/2-d(q/4-1/2)}. \]
Proof. Modify the proof of Lemma 3.2 using the Poincaré inequality with odd extension for functions vanishing on the flat boundary
\[ \int_{B^+_r} |u|^2 \, dx \leq N_r^2 \int_{B^+_r} |\nabla u|^2 \, dx \]
to absorb the second term on the right-hand side. $\square$

We recall the following two important lemmas which are useful in handling the estimates for the pressure $p$.

**Lemma 4.3.** Let $\mathcal{D} \subset \mathbb{R}^d$ be a domain with smooth boundary and $T > 0$ be a constant. Let $1 < m < +\infty$, $1 < n < +\infty$ be two fixed numbers. Assume that $g \in L^1_T L^m_m(\mathcal{D}_T)$. Then there exists a unique function pair $(v, p)$, which satisfies the following equation:
\[
\begin{aligned}
\partial_t v - \Delta v + \nabla p &= g & \text{ in } \mathcal{D}_T, \\
\nabla \cdot v &= 0 & \text{ in } \mathcal{D}_T, \\
|p|_{\mathcal{D}}(t) &= 0 & \text{ for a.e. } t \in [0, T], \\
v &= 0 & \text{ on } \partial_p \mathcal{D}_T.
\end{aligned}
\]
Moreover, $v$ and $p$ satisfy the following estimate:
\[ \|v\|_{W^{1,2}_{n,m}(\mathcal{D}_T)} + \|p\|_{W^{0,1}_{n,m}(\mathcal{D}_T)} \leq C \|g\|_{L^1_T L^m_m(\mathcal{D}_T)}, \]
where the constant $C$ only depends on $m, n, T$, and $\mathcal{D}$. 

\[ \text{THE NAIVER-STOKES EQUATIONS} \]
Lemma 4.4. Let $1 < m \leq 2, 1 < n \leq 2$, and $m \leq s < +\infty$ be constants and $g \in L^s_nL^m_s(Q^+)$. Assume that the functions $v \in W^{0,1}_{n,m}(Q^+)$ and $p \in L^s_nL^m_s(Q^+)$ satisfy the equations:

\[
\begin{align*}
\partial_t v - \Delta v + \nabla p &= g \quad \text{in } Q^+, \\
\nabla \cdot v &= 0 \quad \text{in } Q^+, 
\end{align*}
\]

and the boundary condition

\[v = 0 \quad \text{on } \{y \mid y = (y',0), |y'| < 1\} \times [-1,0].\]

Then, we have $v \in W^{1,2}_n(Q^+(1/2)), p \in W^{0,1}_{n,m}(Q^+(1/2))$, and

\[
\|v\|_{W^{1,2}_n(Q^+(1/2))} + \|p\|_{W^{0,1}_{n,m}(Q^+(1/2))} 
\leq C \left(\|g\|_{L^s_nL^m_s(Q^+)} + \|v\|_{W^{0,1}_{n,m}(Q^+)} + \|p\|_{L^s_nL^m_s(Q^+)}\right),
\]

where the constant $C$ only depends on $m, n,$ and $s$.

We refer the reader to [32] for the proof of Lemma 4.3, and to [40, 43] for the proof of Lemma 4.4.

The following three lemmas are analogous to Lemma 3.3, 3.4, 3.5, and the proofs are similar.

Lemma 4.5. For $\alpha \in [0,1]$, $p \in [2 + 4\alpha/d, d + (4 - d)\alpha]$, suppose $r > 0$, $\hat{x} \in \partial\Omega$, $\omega(\hat{z}, r) = Q^+(\hat{z}, r)$, and $u$ satisfies the condition (7). Then we have

\[
\frac{1}{r^{d+2-p}} \int_{Q^+(\hat{z}, r)} |u|^p dz \leq N \left(A^+(r, \hat{z}) + E^+(r, \hat{z})\right)^{\frac{d-p+2\alpha}{d-2}}.
\]

In particular, taking $\alpha = 1$ and $p = 4 - 2/d$ we have

\[
C^+(r, \hat{z}) \leq N \left(A^+(r, \hat{z}) + E^+(r, \hat{z})\right)^{\frac{d+2/2}{d}}.
\]

Lemma 4.6. Let $(u, p)$ be a pair of suitable weak solution of (1). For $\hat{x} \in \partial\Omega$ and $\omega(\hat{z}, r) = Q^+(\hat{z}, r)$, we have

\[
A^+(\rho/2) + E^+(\rho/2) 
\leq N \left(C^+(\rho)^{\frac{d}{d+1}} + C^+(\rho)^{\frac{2d}{d+1} - 1} + C^+(\rho)^{\frac{d}{d+1} - 1} D^+(\rho)^{\frac{d}{d+1}}\right),
\]

where $N$ is independent of $\hat{z}$ and $\rho$.

Lemma 4.7. Let $(u, p)$ be a pair of suitable weak solution of (1) with $d \geq 4$. For constant $\gamma \in (0, 1/2]$, $\rho > 0$ and $\hat{x} \in \partial\Omega$, $\omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho)$, we have

\[
A^+(\gamma \rho) + E^+(\gamma \rho) \leq N \left[\gamma^2 A^+(\rho) \\
+ \gamma^{-d+1} \left((A^+(\rho) + E^+(\rho))^{\frac{d}{d+2}} + (A^+(\rho) + E^+(\rho))^{\frac{d}{d+2}} D^+(\rho)^{\frac{d}{d+2}}\right)\right].
\]

At last, we present an estimate for quantity $D^+(\rho)$, which is essentially different from Lemma 3.6 for the interior case.

Lemma 4.8. Let $(u, p)$ be a pair of Leray-Hopf weak solution of (1) and $u$ satisfies condition (7). Let $\gamma \in (0, 1/4]$ and $\rho > 0$ be constants. Suppose that $\hat{x} \in \partial\Omega$ and $\omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho)$. Then given any small $\delta_2 > 0$, we have

\[
D^+(\gamma \rho) \leq N \left[\gamma^{-d-2/d+2}(A^+(\rho) + E^+(\rho))^{1+\frac{2}{d+2}} \\
+ \gamma^{4-3/d-\delta_2} \left(D^+(\rho) + A^+(\rho)^{1-\frac{2}{d+2}} + E^+(\rho)^{1-\frac{2}{d+2}}\right)\right],
\]
where $N$ is a constant independent of $\gamma, \rho,$ and $\hat{z}$, but may depend on $\delta_2$.

Proof. Without loss of generality, by shifting the coordinate we may assume that $\hat{z} = (0, 0)$. By the scale-invariant property, we may also assume $\rho = 1$. We fix a domain $\tilde{B} \subset \mathbb{R}^d$ with smooth boundary so that

$$B^+(1/2) \subset \tilde{B} \subset B^+,$$

and denote $\tilde{Q} = (-1, 0) \times \tilde{B}$. Using Hölder’s inequality, Lemma 4.2 with $q = \frac{2d(d+2)}{d^2+4}$, and (7), we get

$$\left( \int_{B^+} |u \cdot \nabla u| \frac{d^{(2d-1)} + d}{d^2+4} \, dx \right)^{\frac{d^2+4}{2d^2+4d}} \leq \left( \int_{B^+} |\nabla u|^2 \, dx \right)^{\frac{d-2}{d+2}} \left( \int_{B^+} |u|^d \, dx \right)^{\frac{2}{d+2}}. (32)$$

Integrating in $t$, we have $u \cdot \nabla u \in L^{\frac{d}{2d-1}, \frac{d(2d-1)}{d^2+2d-1}} (\tilde{Q})$. By Lemma 4.3, there is a unique solution

$$v \in W^{1,2, \frac{d}{2d-1}, \frac{d(2d-1)}{d^2+2d-1}} (\tilde{Q}) \quad \text{and} \quad p_1 \in W^{0,1, \frac{d}{2d-1}, \frac{d(2d-1)}{d^2+2d-1}} (\tilde{Q})$$

to the following initial boundary value problem:

$$\begin{cases}
\partial_t v - \Delta v + \nabla p_1 = -u \cdot \nabla u & \text{in } \tilde{Q}, \\
\nabla \cdot v = 0 & \text{in } \tilde{Q}, \\
[p_1]_{\tilde{B}}(t) = 0 & \text{for a.e. } t \in [0,T], \\
v = 0 & \text{on } \partial_{\tilde{G}} \tilde{Q}.
\end{cases}$$

Moreover, we have

$$\|v| + |\nabla v| + |p_1| + |\nabla p_1| \|_{L^{\frac{d}{2d-1}, \frac{d(2d-1)}{d^2+2d-1}} (\tilde{Q})} \leq N \|u \cdot \nabla u\|_{L^{\frac{d}{2d-1}, \frac{d(2d-1)}{d^2+2d-1}} (\tilde{Q})}$$

$$\leq N \left( \int_{B^+} |\nabla u|^2 \, dx \right) \left( \int_{B^+} |u|^d \, dx \right)^{\frac{2}{d+2}} d/(2d-1),$$

where in the last inequality we used (32).

We set $w = u - v$ and $p_2 = p - p_1 - [p]_{0,1/2}$. Then $w$ and $p_2$ satisfy

$$\begin{cases}
\partial_t w - \Delta w + \nabla p_2 = 0 & \text{in } \tilde{Q}, \\
\nabla \cdot w = 0 & \text{in } \tilde{Q}, \\
w = 0 & \text{on } [-1,0] \times \{\partial \tilde{B} \cap \partial \Omega\}.
\end{cases}$$
By Lemma 4.4 and the triangle inequality, fixing some \( s > 0 \) large enough to be specified later, we have \( p_2 \in W^{0,1}_{2d-1/d, s}(Q^+(1/4)) \) and
\[
\|\nabla p_2\|_{L^2_{1-d}L^s_{(Q^+(1/4))}} \leq N\|\nabla u\| + \|\nabla u\| + |p_2|_{L^2_{1-d}L^s_{(Q^+(1/2))}} \leq N\|\nabla u\| + |p - [p]_{0,1/2}|_{L^2_{1-d}L^s_{(Q^+(1/2))}} + N\|v\| + \|p_1\|_{L^2_{1-d}L^s_{(Q^+(1/2))}}.
\]
Together with (33), we obtain
\[
\|\nabla p_2\|_{L^2_{1-d}L^s_{(Q^+(1/4))}} \leq N\|\nabla u\| + |p - [p]_{0,1/2}|_{L^2_{1-d}L^s_{(Q^+(1/2))}} + N\left(\int_{B^+} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B^+} |u|^2 \, dx \right)^{\frac{1}{2}} dh/(2d-1).
\]
Recall that \( 0 < \gamma \leq 1/4 \). Then by using the Sobolev-Poincaré inequality, the triangle inequality, (33), (34), and Hölder’s inequality, we bound \( D^+(\gamma) \) by
\[
N_\gamma^{d+2/d-2} \int_{-\gamma^2}^{0} \left(\int_{B^+} |\nabla p|_{2d-1} \, dx \right)^{\frac{d+2/d-1}{d^2}} \, dt 
\leq N_\gamma^{d+2/d-2} \int_{-\gamma^2}^{0} \left(\int_{B^+} |\nabla p_1|_{2d-1} \, dx \right)^{\frac{d+2/d-1}{d^2}} \, dt 
+ \left(\int_{B^+} |\nabla p_2|_{2d-1} \, dx \right)^{\frac{d+2/d-1}{d^2}} \, dt 
\leq N_\gamma^{-d-2/d+2} E^+(1) A^+(1) \frac{2d-1}{s} \int_{-\gamma^2}^{0} \left(\int_{B^+} |\nabla p_2| \, dx \right)^{\frac{2d-1}{s}} \, dt 
+ N_\gamma^{d+2/d+2} E^+(1) A^+(1) \frac{2d-1}{s} \left( D^+(1) + A^+(1)^{\frac{1}{d}} + E^+(1)^{\frac{1}{d}} \right).
\]
By making \( s \) large such that \( \frac{2d-1}{s} < \delta_2 \), we finish the proof.

\[\square\]

**Corollary 2.** Let \((u, p)\) be a pair of suitable weak solution of (1) with \( d \geq 4 \). Suppose there exist \( \hat{z} \in \partial\Omega \), \( C > 0 \), and \( \rho_0 > 0 \) such that \( \omega(\hat{z}, \rho) = Q^+(\hat{z}, \rho) \), \( C^+(\rho_0, z^*) \leq C_1 \) for all \( \rho \in (0, \rho_0] \) and \( D^+(\rho_0, \hat{z}) \leq C_1 \). Then we can find \( C := C(C_1) > 0 \) such that \( D^+(\rho, z^*) \leq C \) for all \( z^* \in Q(\hat{z}, \rho_0/2) \cap \{ x_d = 0 \} \) and \( \rho \in (0, \rho_0/4) \).

**Proof.** Because \( Q^+(z^*, \rho_0/2) \subset Q^+(\hat{z}, \rho_0) \) for \( z^* \in Q(\hat{z}, \rho_0/2) \cap \{ x_d = 0 \} \), we have
\[
D^+(\rho_0/2, z^*) \leq ND^+(\rho_0, \hat{z}).
\]
From (29) and (31) with $\delta = 1$, we can get an estimation for $D^+$:

$$
D^+ \left( \frac{\rho}{4}, z^* \right) \leq N \left[ \gamma^{-d-2+2} \left( A^+ \left( \frac{\rho}{4}, z^* \right) + E^+ \left( \frac{\rho}{4}, z^* \right) \right)^{1+\frac{2}{d+2}} + \gamma^{3-3/d} \left( D^+ \left( \frac{\rho}{4}, z^* \right) + A^+ \left( \frac{\rho}{4}, z^* \right)^{1-\frac{2}{d+2}} + E^+ \left( \frac{\rho}{4}, z^* \right)^{1-\frac{2}{d+2}} \right) \right]
$$

$$
\leq N(C_1) \left[ \gamma^{-d-2+2} \left( 1 + D^+ \left( \frac{\rho}{2}, z^* \right)^{\frac{d+2}{d+1}} \right)^{1+\frac{2}{d+2}} + \gamma^{3-3/d} \left( D^+ \left( \frac{\rho}{2}, z^* \right) + \left( 1 + D^+ \left( \frac{\rho}{2}, z^* \right)^{\frac{d+2}{\tau}} \right)^{1-\frac{2}{d+2}} \right) \right]
$$

$$
\leq N(C_1) \left[ \gamma^{-d-2+2} \left( 1 + D^+ \left( \frac{\rho}{2}, z^* \right)^{\frac{d+2}{d+1}} \right)^{1+\frac{2}{d+2}} + \gamma^{3-3/d} \left( D^+ \left( \frac{\rho}{2}, z^* \right) + 1 \right) \right].
$$

(35)

For any $\varepsilon > 0$ and $\delta := 1 - \frac{d^2 - 2d + 2}{2d^2 - 5d + 2}$, which is positive when $d \geq 4$, again by Young’s inequality we get

$$
D^+(\rho/2, z^*)^{1-\delta} = (\varepsilon D^+(\rho/2, z^*))^{1-\delta} \leq (1 - \delta)\varepsilon D^+(\rho/2, z^*) + \delta \varepsilon^{-\frac{1-\delta}{1-\delta}}.
$$

We can choose $\gamma$ and $\varepsilon$ small such that $N\gamma^{-3/d} < 1/8$ and $N\gamma^{-d-2+2}(1 - \delta)\varepsilon < 1/8$. The two inequalities above implies that (35) can be written into such form:

$$
D^+(\gamma\rho/4, z^*) \leq \frac{1}{2} D^+(\rho/2, z^*) + C.
$$

The rest of the proof is a handy modification of Corollary 1. 

4.2. Step 2. We will find some decay rates for the scale invariant quantities with respect to the radius of the cylinder assuming the quantities are initially small.

**Lemma 4.9.** There exists a universal constant $\hat{\varepsilon}_0 > 0$ satisfying the following property. Suppose that for some $\hat{z} = (\hat{x}, \hat{t})$, where $\hat{x} = (x', 0)$, and for some $\rho_0 > 0$, it holds that $\omega(\hat{z}, \rho_0) = Q^+(\hat{z}, \rho_0)$ and

$$
A^+(\rho_0, \hat{z}) + E^+(\rho_0, \hat{z}) + D^+(\rho_0, \hat{z})^{1/\tau} \leq \hat{\varepsilon}_0,
$$

(36)

where $\tau = 1 - \frac{1}{d} + \varepsilon$, $\varepsilon \sim \mathcal{O} \left( \frac{1}{d} \right)$. Then fixing any $\alpha_0 \in (0, 2)$, we can find $N > 0$ such that for any $\rho \in (0, \rho_0/4)$ and $z^* \in Q^+(\hat{z}, \rho_0/4)$, the following estimate holds uniformly

$$
A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{4}{d + 2}} + D^+(\rho, z^*)^{\frac{2}{d + 2}} + D^+(\rho, z^*) \leq N\varepsilon_0^{\alpha_0} \left( \frac{\rho}{\rho_0} \right)^{\alpha_0^{2/\tau}}.
$$

(37)

where $N$ is a positive constant depending on $\alpha_0$, but independent of $\hat{\varepsilon}_0$, $\rho_0$, $\rho$, and $z^*$.

**Proof.** We divide the proof into two parts. In the first part, we only consider $z^*$ on the boundary $Q(\hat{z}, \rho_0/2) \cap \{x_d = 0\}$. In the second part, we use an iteration argument to close the proof for general $z^* \in Q^+(\hat{z}, \rho_0/4)$.
i) First we assume $z^* \in Q(\hat{z}, \rho_0/2) \cap \{x_d = 0\}$. In this case we will prove a slightly stronger estimate than (37):

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*)^{\frac{d-2}{d+\alpha}} + D^+(\rho, z^*)^{1/\tau} \leq N\hat{\varepsilon}_0^{\alpha_0/2}\left(\frac{\rho}{\rho_0}\right)^{\alpha_0}. \quad (38)$$

By (36) and

$$Q^+(z^*, \rho_0/2) \subset Q^+(\hat{z}, \rho_0),$$

we get

$$A^+(\rho_1, z^*) + E^+(\rho_1, z^*) + D^+(\rho_1, z^*)^{1/\tau} \leq N\hat{\varepsilon}_0,$$  \quad (39)

where $\rho_1 = \rho_0/2$. By Lemma 4.5,

$$C^+(\rho_1, z^*) \leq (N\hat{\varepsilon}_0)^{\frac{d-2+2/\tau}{d+\alpha}}. \quad (40)$$

Next we fix an auxiliary parameter $\alpha \in (\alpha_0, 2)$. By a scaling argument, we first discuss a special case when $\rho_1^{\alpha} = N\hat{\varepsilon}_0 < 1$. In this case, we can prove the following decay rates inductively:

$$A^+(\rho_k) + E^+(\rho_k) \leq \rho_k^{\alpha}, \quad C^+(\rho_k)^{\frac{d-2}{d+\alpha}} \leq \rho_k^{\alpha}, \quad D^+(\rho_k) \leq \rho_k^\tau,$$  \quad (41)

where $\rho_{k+1} = \rho_k^{1+\beta} = \rho_k^{(1+\beta)^k}$, and $\beta$ is a small number to be specified. For $k = 1$, the statement follows from (39) and (40). Next by choosing $\gamma = \rho_k^\beta$ and $\rho = \rho_k$ in (30), we have

$$A^+(\rho_{k+1}) + E^+(\rho_{k+1}) \leq N\left[\rho_k^{2\beta+\alpha} + \rho_k^{(d+1)\beta}\left(\frac{d+1}{d-\frac{2}{\alpha}} + \frac{1}{d+\alpha} + \frac{2}{d-\frac{2}{\alpha}}\right)^\alpha\right].$$

We choose $\beta$ satisfying

$$\beta < \min \left\{ \frac{\alpha}{(d-2)(d+\alpha-1)}, \frac{\left(\frac{d+1}{d-\frac{2}{\alpha}} - \frac{1}{d+\alpha} + \frac{2}{d-\frac{2}{\alpha}}\right)\alpha}{\frac{d}{d+1} - 1} \right\} \sim O\left(\frac{1}{d^2}\right).$$

Then all the exponents on the right-hand sides are greater than $(1 + \beta)\alpha$.

To estimate the remaining term $D^+(\rho_{k+1})$, we apply Lemma 4.8 but with different step size. Let $\beta_1 = (1 + \beta)^{n_0+1} - 1$, where $n_0$ is an integer to be specified later. Instead of plugging in the result of one last previous step, we plug in (31) with $\gamma = \rho_{k-n_0}^{\beta_1}$ and $\rho = \rho_{k-n_0}$, and we have

$$D^+(\rho_{k+1}) \leq N\left[\rho_{k-n_0}^{(-d-2)/d+2)\beta_1 + 1 + \frac{2}{d(d-2)}\alpha} + \rho_{k-n_0}^{(4-3/d-\delta_2)\beta_1}\left(\rho_{k-n_0}^{\alpha\tau} + \rho_{k-n_0}^{\alpha(1-\beta_1)}\right)\right]. \quad (42)$$

Our goal is to choose an appropriate $\beta_1$ such that all three exponents on right-hand side of (42) are greater than $(1 + \beta_1)\alpha\tau$. We hence obtain an upper bound and a lower bound for $\beta_1$:

$$\beta_1 > \frac{\left(\tau - 1 - \frac{1}{d}\right)\alpha}{4 - \frac{3}{d} - \alpha\tau} = \frac{\varepsilon\alpha}{4 - \frac{3}{d} - \alpha\tau},$$

$$\beta_1 < \frac{\left(1 + \frac{2}{d(d-2)} - \tau\right)\alpha}{d + \frac{3}{d} - 2 + \alpha\tau} = \frac{\left(\frac{1}{2d} + \frac{2}{d(d-2)} - \varepsilon\right)\alpha}{d + \frac{3}{d} - 2 + \alpha\tau} \sim O\left(\frac{1}{d^2}\right).$$

To ensure such $\beta_1$ exists, we make $\varepsilon \sim O\left(\frac{1}{d^2}\right)$ small such that the upper bound is greater than the lower bound of $\beta_1$. As long as $\beta$ is small enough, there exists an integer $n_0$ such that $\beta_1 = (1 + \beta)^{n_0+1} - 1$ satisfies the conditions above.
Hence the distance of $z$ by the triangle inequality, we have $\omega$, which always exists a feasible $\beta$ satisfying $\rho$, and that $\rho$, $\alpha_0 < \min \left\{ \frac{1}{1 + \beta}(\alpha - (d - 2)\beta), \frac{1}{(1 + \beta)\tau} \left( \alpha \tau - \left( d + 2 \right) \frac{\beta}{2} \right) \right\}$. There always exists a feasible $\beta$ because $\alpha > \alpha_0$.

Now for any $\rho \in (0, \rho_0/2)$, we can find a positive integer $k$ such that $\rho_{k+1} \leq \rho < \rho_k$. Then

$$A^+(\rho) + E^+(\rho) \leq \left( \frac{\rho_k}{\rho_{k+1}} \right)^{d-2} (A^+(\rho_k) + E^+(\rho_k))$$

$$\leq \rho_k^{\alpha-(d-2)\beta} = \rho_{k+1}^{\alpha-(d-2)\beta} \leq \rho_{k+1}^\alpha,$$

$$D^+(\rho) \leq \left( \frac{\rho_k}{\rho_{k+1}} \right)^{d+2-2} D^+(\rho_k)$$

$$\leq \rho_k^{\alpha\tau-(d+2-2)\beta} = \rho_{k+1}^{\alpha\tau-(d+2-2)\beta} \leq \rho_{k+1}^{\alpha_0 \tau}.$$.

Hence we have proved the statement of the lemma when $\rho_0 = N^\alpha \rho_0 - \alpha_0$. For general $\rho_0 > 0$, we use the scale invariant property of the quantities and yield similar results with an additional scaling factor $N^\alpha \rho_0^{-\alpha_0}$ on the right-hand side. Hence (38) is true.

ii) To deal with $z^* \in Q^+(\hat{z}, \rho_0/4)$, we need to discuss two cases as comparing $x^*_d$, the distance of $z^*$ to the boundary, with $\rho$, the radius of the cylinder.

When $\rho \geq x^*_d$, we denote the projection of $z^*$ on the boundary by $\tilde{z}^*$. Because $\omega(z^*, \rho) \in Q^+(\tilde{z}^*, 2\rho)$, by definition we have

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*) \frac{d-2}{4-2d}$$

$$\leq N \left( A^+(2\rho, \tilde{z}^*) + E^+(2\rho, \tilde{z}^*) + C^+(2\rho, \tilde{z}^*) \frac{d-2}{4-2d} \right).$$

By the triangle inequality, we have

$$\int_{\omega(z^*, \rho)} |p - [p]_{x^*_d, \rho}(t)|^{2-1/d} dz \leq N \int_{Q^+(\tilde{z}^*, 2\rho)} |p - [p]_{\tilde{z}^*, 2\rho}(t)|^{2-1/d} dz.$$.

Hence $B^+(\rho, z^*) \leq N D^+(2\rho, \tilde{z}^*)$. From part (i) we know

$$A^+(2\rho, \tilde{z}^*) + E^+(2\rho, \tilde{z}^*) + C^+(2\rho, \tilde{z}^*) \frac{d-2}{4-2d} + D^+(2\rho, \tilde{z}^*)^{1/\tau} \leq N \xi^\alpha_0 \left( \frac{\rho}{\rho_0} \right)^{\alpha_0}.$$.

The three inequalities above together imply that

$$A^+(\rho, z^*) + E^+(\rho, z^*) + C^+(\rho, z^*) \frac{d-2}{4-2d} + D^+(\rho, z^*)^{1/\tau} \leq N \xi^\alpha_0 \left( \frac{\rho}{\rho_0} \right)^{\alpha_0}.$$.
When $\rho < x_d^*$, we have $\omega(z^*, \rho) = Q(z^*, \rho) \subset Q(z^*, x_d^*) \subset Q^+(z^*, 2x_d^*)$. By the proof above, we have

$$A^+(x_d^*, z^*) + E^+(x_d^*, z^*) = C^+(x_d^*, z^*)^\frac{d-2}{2(1+\rho)} + D^+(x_d^*, z^*)^{1/\tau} \leq N\varepsilon_0^{\alpha_0/2} \left( \frac{x_d^*}{\rho_0} \right)^{\alpha_0}.$$ 

With $\varepsilon_0$ small such that $N\varepsilon_0^{\alpha_0/2} \left( \frac{x_d^*}{\rho_0} \right)^{\alpha_0} < \varepsilon_0^{1/\tau}$ where $\varepsilon_0$ is from Lemma 3.7, we can apply the interior result of Lemma 3.7 to obtain

$$A^+(\rho, z^*) + E^+(\rho, z^*) = C^+(\rho, z^*)^\frac{d-2}{2(1+\rho)} + D^+(\rho, z^*) \leq N\varepsilon_0^{\alpha_0^{2/\tau}} \left( \frac{\rho}{\rho_0} \right)^{\alpha_0^{2/\tau}}.$$ 

The proof is complete.

4.3. **Step 3.** In this final step, we first use parabolic $L_p$ estimates to further improve the decay rate and then conclude the result by using Campanato’s characterization of Hölder continuity.

By (37) from the previous step, we know the following estimates are true for all $\rho > 0$ sufficiently small and $z^* = (t^*, x^*) \in Q(\hat{z}, \rho_0/4) \cap \{ x_d = 0 \}$:

$$\int_{Q^+(z^*, \rho)} |v|^{d+\frac{2}{\tau}} \, dz \leq N\rho^{d+\frac{2}{\tau} - \frac{1}{2} + \varepsilon}, \quad (43)$$

$$\int_{Q^+(z^*, \rho)} |p - [p]_{x^*, \rho}|^{2-\frac{4}{\tau}} \, dz \leq N\rho^{d+\frac{1}{\tau} + \varepsilon}, \quad (44)$$

where $\varepsilon \sim \mathcal{O} \left( \frac{1}{\rho^2} \right)$.

Let $v$ be the unique weak solution to the heat equation

$$\partial_t v - \Delta v = 0 \quad \text{in } Q^+(z^*, \rho)$$

with the boundary condition $v = u$ on $\partial_p Q^+(z^*, \rho)$.

Let $0 < r < \rho$. By the Poincaré inequality with zero boundary condition and using the fact that the $L_\infty$ norm of the gradient of a caloric function in a small half cylinder is controlled by any $L_p$ norm of it in a larger half cylinder, we have

$$\int_{Q^+(z^*, r)} |v|^{2-\frac{4}{\tau}} \, dz \leq Nr^{d+4-\frac{4}{\tau}} \| \nabla v \|_{L_\infty(Q^+(z^*, r))}$$

$$\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{4}{\tau}} \int_{Q^+(z^*, \rho)} |v|^{2-\frac{4}{\tau}} \, dz. \quad (45)$$

Denote $w = u - v$. Then $w$ satisfies the inhomogeneous heat equation

$$\partial_t w_i - \Delta w_i = -\partial_j(u_i u_j) - \partial_i(p - [p]_{x^*, \rho}) \quad \text{in } Q^+(z^*, \rho)$$

with the zero boundary condition. By the classical $L_p$ estimate for the heat equation, we have

$$\| \nabla w \|_{L_{2-\frac{4}{\tau}}(Q^+(z^*, \rho))} \leq N \left( \| w \|_{L_{2-\frac{4}{\tau}}(Q^+(z^*, \rho))} + N \| p - [p]_{x^*, \rho} \|_{L_{2-\frac{4}{\tau}}(Q^+(z^*, \rho))} \right),$$

which together with (43) and (44) yields

$$\int_{Q^+(z^*, \rho)} |\nabla w|^{2-\frac{4}{\tau}} \, dz \leq N\rho^{d+\frac{1}{\tau} + \varepsilon}.$$ 

By the Poincaré inequality with zero boundary condition, we get

$$\int_{Q^+(z^*, \rho)} |w|^{2-\frac{4}{\tau}} \, dz \leq N\rho^{d+2+\varepsilon}. \quad (46)$$
Using (45), (46), and the triangle inequality, we have
\[
\int_{Q^+(z^*, r)} |u|^2 \frac{1}{d} dz \leq N \int_{Q^+(z^*, r)} |v|^2 \frac{1}{d} dz + N \int_{Q^+(z^*, r)} |w|^2 \frac{1}{d} dz
\]
\[
\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q^+(z^*, \rho)} |v|^2 \frac{1}{d} dz + N r^{d+2+\varepsilon}
\]
\[
\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q^+(z^*, \rho)} (|u|^2 \frac{1}{d} + |w|^2 \frac{1}{d}) dz + N r^{d+2+\varepsilon}
\]
\[
\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q^+(z_0, \rho)} |u|^2 \frac{1}{d} dz + N \rho^{d+2+\varepsilon}.
\]
Applying Lemma 3.8, we obtain
\[
\int_{Q^+(z^*, r)} |u|^2 \frac{1}{d} dz \leq N r^{d+2+\varepsilon}
\] (47)
for any \( r \leq \rho_0 \) and \( z^* \in Q(\hat{z}, \rho_0/4) \cap \{ x_d = 0 \} \).

Consider any \( \hat{z} = (\hat{t}, \hat{x}) \in Q^+(\hat{z}, \rho_0/8) \). Let \( z^* = (\tilde{t}, \tilde{x}, 0) \) be the projection of \( \hat{z} \) on the boundary. Note that \( z^* \in Q(\hat{x}, \rho_0/8) \cap \{ x_d = 0 \} \). We consider two cases either the radius of the parabolic ball around \( z^* \) is smaller or larger than \( \tilde{x}_d \).

**Case 1.** \( \tilde{x}_d \leq r \). In this case, we have \( \omega(\hat{z}, r) \subset Q^+(z^*, 2r) \). Thus by (47), we have
\[
\int_{\omega(\hat{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz \leq N \int_{Q^+(z^*, 2r)} |u|^2 \frac{1}{d} dz \leq N r^{d+2+\varepsilon}.
\]

**Case 2.** \( r < \tilde{x}_d \). For \( r < \rho \leq \tilde{x}_d \), from the proof of Theorem 3.1, we have
\[
\int_{Q(\hat{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz
\]
\[
\leq N \left( \frac{r}{\rho} \right)^{d+4-\frac{1}{d}} \int_{Q(\hat{z}, \rho)} |u - (u)_{\tilde{z}, \rho}|^{2-1/d} dz + N \rho^{d+2+\varepsilon}.
\] (48)

We apply Lemma 3.8 to (48) to get
\[
\int_{Q(\hat{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz
\]
\[
\leq N \left( \frac{r}{\tilde{x}_d} \right)^{d+2+\varepsilon} \int_{Q(\tilde{z}, \tilde{x}_d)} |u - (u)_{\tilde{z}, \tilde{x}_d}|^{2-1/d} dz + N \tilde{x}_d^{d+2+\varepsilon}.
\] (49)

By Case 1, we have
\[
\int_{Q(\tilde{z}, \tilde{x}_d)} |u - (u)_{\tilde{z}, \tilde{x}_d}|^{2-1/d} dz \leq N \tilde{x}_d^{d+2+\varepsilon}.
\]
Plug this into (49) to get
\[
\int_{Q(\hat{z}, r)} |u - (u)_{\tilde{z}, r}|^{2-1/d} dz \leq N \tilde{x}_d^{d+2+\varepsilon}.
\]

By Campanato’s characterization of Hölder continuity near a flat boundary (see, for instance, [30, Lemma 4.11]), we can conclude that \( u \) is Hölder continuous in a neighborhood of \( \hat{z} \).
5. Proof of Theorems 1.2 and 1.3. In this section, we start with a construction on a sequence of suitable weak solutions which converges to a limiting solution. Let \((u, p)\) be a pair of Leray-Hopf weak solution of the Cauchy problem (1)-(2) on \(\mathbb{R}^d\) or \(\mathbb{R}^d_+\). Because of the local strong solvability for smooth data and the weak-strong uniqueness (see, for instance, [50]), we know that \(u\) is regular for \(t \in (0, T_0)\) for some \(T_0 \in (0, T]\). Suppose \(T_0\) is the first blowup time of \(u\), and

\[
Z_0 = (T_0, X_0) = (T_0, X_{0,1}, X_{0,2}, \ldots, X_{0,d}) = (T_0, X'_0, X_{0,d})
\]

is a singular point. We take a decreasing sequence \(\{\lambda_k\}\) converging to 0 and rescale the pair \((u, p)\) at time \(T_0\). Define

\[
u_k(t, x) = \lambda_k u(T_0 + \lambda_k^2 t, X_0 + \lambda_k x), \quad p_k(t, x) = \lambda_k^2 p(T_0 + \lambda_k^2 t, X_0 + \lambda_k x)
\]

for each \(k = 1, 2, \ldots\). We will show that each \((u_k, p_k)\) is a suitable weak solution of (1)-(2) and \(u_k\) is smooth for \(t \in (-\lambda_k^{-2} T_0, 0)\).

To prove this, the first observation is the property of uniform boundedness of the scale invariant quantities after rescaling. In this section, we use the ambiguous notation as we mentioned before in the preliminaries. By \(\mathbb{R}^d_+\) we mean either \(\mathbb{R}^d\) or \(\mathbb{R}^d_+\) depending on which domain we are talking about: the whole space or the half space. The same goes for \(A(\cdot), E(\cdot), C(\cdot), D(\cdot), \) etc. To specified the dependence of these quantities with respect to the solutions, we use the notation

\[
C^{(+)}(r, z_0, u_k, p_k), \quad D^{(+)}(r, z_0, u_k, p_k), \quad \text{etc.,}
\]

even though \(C^{(+)}\) does not explicitly depend on \(p_k\) and \(D^{(+)}\) does not explicitly depend on \(u_k\).

**Lemma 5.1.** Under the assumptions in Theorem 1.3, there exists \(N > 0\) such that for any \(z_0 \in (-\infty, 0] \times \mathbb{R}^{d+}_+(\cdot)\) and \(0 < r \leq 1\),

\[
\limsup_{k \to \infty} C^{(+)}(r, z_0, u_k, p_k) \leq N, \quad (50)
\]

and

\[
\limsup_{k \to \infty} D^{(+)}(r, z_0, u_k, p_k) \leq N. \quad (51)
\]

**Remark 1.** If we merely assume (8), then \(N\) may depend on the choice of \(Z_0\). If we assume the stronger condition (9), then \(N\) is independent the choice of \(Z_0\).

**Proof.** For \(z_0 \in (-\infty, 0] \times \mathbb{R}^{d+}_+(\cdot)\), denote

\[
z^k_0 = (t^k_0, x^k_0) = (T_0 + \lambda_k^2 t_0, X_0 + \lambda_k x_0).
\]

For convenience, we first assume \(\Omega = \mathbb{R}^d\) to prove (50). Since \(C\) and \(D\) are invariant, we have

\[
C(r, z_0, u_k, p_k) = C(\lambda_k r, z^k_0, u, p) \quad \text{and} \quad D(r, z_0, u_k, p_k) = D(\lambda_k r, z^k_0, u, p).
\]

Using Hölder’s inequality we have

\[
C(r, z_0, u, p) = \frac{1}{r^{d+1/2d-2}} \int_{Q(z_0, r)} |u|^{2(2d-1)/d} dz \leq N \|u\|^{2(2d-1)\over L_\infty L_{2/3}(Q(z_0, r))}. \quad (52)
\]

Substituting \(r\) with \(\lambda_k r\) and \(z_0\) with \(z^k_0\), we have

\[
C(\lambda_k r, z^k_0, u, p) \leq N \|u\|^{2(2d-1)\over L_\infty L_{2/3}(Q(z^k_0, \lambda_k r))} \leq N,
\]

where in the last inequality we used (7). This part of proof can easily be adapted to the case \(\Omega = \mathbb{R}^d_+\).
To prove (51), we need to consider several cases separately:

i) \( \Omega = \mathbb{R}^d \), by using the Calderón-Zygmund estimate, one has

\[
\|p\|_{L^p_{\text{loc}}L^{2p/(d+2)}_t((0,T) \times \mathbb{R}^d)} \leq K.
\]

Since \( \frac{d}{2} \geq 2 - \frac{1}{d} \), following the same reasoning in (52) we can reach (51).

ii) \( \Omega = \mathbb{R}^d_+ \) and \( X_{0,d} > 0 \), that is the half-space case when \( Z_0 \) does not lie on the boundary. The domain of \( u_k \) will expand to the whole space, therefore \( z_0 \in (-\infty,0] \times \mathbb{R}^d \). When \( X_{0,d} \geq 1/4 \), i.e., \( Z_0 \) is away from the boundary, from (8) or (9) we know \( D(1/4,Z_0,u,p) \leq N \). When \( k \) is large, \( Q(z_0^k,\lambda_k r) \subset Q(Z_0,1/4) \).

By Corollary 1, we know \( D(r,z_0,u_k,p_k) = D(\lambda_k r, z_0^k,u,p) \) is uniformly bounded. When \( X_{0,d} < 1/4 \), i.e., \( Z_0 \) is close to the boundary, recall the notation

\[
\omega(Z_0,1) = Q(Z_0,1) \cap (0,T) \times \mathbb{R}^d_+.
\]

Denote \( \tilde{Z}_0 = (X_0',0,T_0) \) to be the projection of \( Z_0 \) on the boundary and \( \tilde{z}_0^k \) to be projection of \( z_0^k \). Note when \( k \) is large, \( \lambda_k r \ll X_{0,d} \), hence

\[
Q(z_0^k,\lambda_k r) \subset Q(Z_0,X_{0,d}) \subset Q^+(\tilde{Z}_0,2X_{0,d}) \subset Q^+(\tilde{Z}_0,1/2) \subset \omega(Z_0,1).
\]

From the proofs of Corollaries 1 and 2, we know that

\[
D(r,z_0,u_k,p_k) = D(\lambda_k r, z_0^k,u,p) \leq ND(X_{0,d}, Z_0,u,p) + C,
\]

(53)

\[
D^+(2X_{0,d}, \tilde{Z}_0,u,p) \leq ND^+(1/2, \tilde{Z}_0,u,p) + C \leq ND^+(1,Z_0,u,p) + C \leq C.
\]

(54)

We use (9) in the last inequality. Moreover, we have

\[
\int_{Q(Z_0,X_{0,d})} |p - (p)_{Z_0,X_{0,d}}|^2dz \leq N \int_{Q^+(\tilde{Z}_0,2X_{0,d})} |p - (p)_{\tilde{Z}_0,2X_{0,d}}|^2dz,
\]

(55)

which implies

\[
D(X_{0,d}, Z_0,u,p) \leq ND^+(2X_{0,d}, \tilde{Z}_0,u,p).
\]

Together with (53) and (54), we again deduce that \( D(r,z_0,u_k,p_k) \) is uniformly bounded.

iii) \( \Omega = \mathbb{R}^d_+ \) and \( X_{0,d} = 0 \), that is the half-space case when \( Z_0 \) lies on the boundary. The domain of \( u_k \) will expand to the half space, therefore \( z_0 \in (-\infty,0] \times \mathbb{R}^d_+ \). We compare the radius of the cylinder against the distance from \( x_0 \) to the boundary. When \( r \geq x_0,d \), we have

\[
\omega(z_0^k,\lambda_k r) \subset Q^+(z_0^k,2\lambda_k r) \subset Q^+(Z_0,1)
\]

when \( k \) is large. By (9), (55), and the proof of Corollary 2, we have

\[
D^+(r,z_0,u_k,p_k) = D^+(\lambda_k r, z_0^k,u,p)
\]

\[
\leq ND^+(2\lambda_k r, z_0^k,u,p)
\]

\[
\leq ND^+(1,Z_0,u,p) + C \leq C.
\]

When \( r \leq x_{0,d} \), we have

\[
\omega(z_0^k,\lambda_k r) = Q(z_0^k,\lambda_k r) \subset Q(z_0^k,\lambda_k x_{0,d}) \subset Q^+(z_0^k,2\lambda_k x_{0,d}) \subset Q^+(Z_0,1)
\]
when \( k \) is large. From (9), (55), and the proofs of Corollaries 1, and 2, we know that
\[
D(r, z_0, u_k, p_k) = D(\lambda_k r, z_0^k, u, p) \\
\leq ND(\lambda_k x_{0,d}, z_0^k, u, p) + C \\
\leq ND^+(2\lambda_k x_{0,d}, z_0^k, u, p) + C \\
\leq ND^+(1, Z_0, u, p) + C \leq C.
\]

Therefore, we have proved that \( D^+(r, z_0, u_k, p_k) \) is uniformly bounded by the \( L^\infty L^\infty_d \) condition in (7) and the local pressure condition in (9).

Next we want to show, up to passing to a subsequence, \( \{(u_k, p_k)\}_{k=1}^\infty \) converge to a limiting solution \((u_\infty, p_\infty)\). We modify [8, Proposition 3.5] and state the results on \( \mathbb{R}^d \) in next proposition. These results can be easily extended to \( \mathbb{R}^d \). To make the statement concise, we hereby introduce the following notation: \( L_{p,\text{uniform}}(\Omega_T) \), which means that the \( L_p \) norm in \( Q(z_0, 1) \cap \Omega_T \) for any \( z_0 \in \Omega_T \) are uniformly bounded independent of the choice of \( z_0 \).

**Proposition 2.** i) There is a subsequence of \((u_k, p_k)\), which is still denoted by \((u_k, p_k)\), such that
\[
u_k \rightarrow u_\infty \quad \text{in} \quad C([t_0 - 1/4^2, t_0]; L_{q_1}(B(x_0, 1/4))),
\]
\[
p_k \rightarrow p_\infty \quad \text{weakly in} \quad L_{2-1/2}(Q(z_0, 1/4))
\]
for any \( z_0 \in (-\infty, 0] \times \mathbb{R}^d \) and \( q_1 \in [1, d) \).

ii) Furthermore, \((u_\infty, p_\infty)\) is a suitable weak solution of (1) in \((-\infty, 0) \times \mathbb{R}^d\), and
\[
u_\infty \in L^{q_1}_{q_2} L^{p_\infty}_{d}(Q(z_0, 1)), \quad p_\infty \in L_{2-1/2, \text{uniform}}((T_1, 0) \times \mathbb{R}^d)
\]
for any \( T_1 > 0 \) and \( q_2 \in [1, \infty) \).

**Proof.** First we fix a \( z_0 \in (-\infty, 0] \times \mathbb{R}^d \). By the previous Lemma 5.1, \( p_k \)'s have a uniform bound of the \( L_{2-1/2}(Q(z_0, 1)) \) norm, so there is a subsequence, which is still denoted by \( \{p_k\} \), such that (57) holds. Similarly,
\[
\|u_k\|_{L^{q_1}_{q_2} L^{p_\infty}_{d}(Q(z_0, 1))} \leq \|u_\infty\|_{L^{q_1}_{q_2} L^{p_\infty}_{d}(Q(z_0, 1))} \leq N,
\]
where \( N \) is independent of \( k \). By Lemmas 3.4 and 5.1, we have
\[
A(1/2, z_0, u_k, p_k) + E(1/2, z_0, u_k, p_k) \leq N.
\]

From (7) and the weak continuity of Leray-Hopf weak solutions, we can conclude that
\[
\|u_k(t, \cdot)\|_{L_d(B(z_0, 1/2))} \leq N
\]
for each \( t \in [-\lambda_k^{-2}T_0, 0) \). By using Lemma 3.2 with \( q = 2d/(d-2) \) and \( r = 1/2 \), we have
\[
\|u_k\|_{L^{2d/(d-2)}_d(Q(z_0, 1/2))} \leq N,
\]
which together with (58) and Hölder’s inequality yields
\[
\|u_k\|_{L_d(Q(z_0, 1/2))} \leq N, \quad \|u_k \cdot \nabla u_k\|_{L_{4/3}(Q(z_0, 1/2))} \leq N.
\]

Following the coercive estimate for the Stokes system (see, for instance, [32]) we have
\[
\partial_t u_k, \quad D^2 u_k, \quad \nabla p_k \in L_{4/3}(Q(z_0, 1/4))
\]
with uniform norms. Therefore, we can find a subsequence still denoted by \( \{u_k\} \) such that
\[
u_k \rightarrow u_\infty \text{ in } C([t_0 - 1/4^2, t_0]; L_{4/3}(B(x_0, 1/4))).
\]
This together with (58) gives (56) by using Hölder’s inequality. To finish the proof of Part i), it suffices to use a Cauchy diagonal argument. Part ii) then follows from Part i) and the fact that \( p_k \)'s have a uniform bound of the \( L_{2-\delta} \) norm in \( Q(z_0, 1) \), which is independent of the choice of \( z_0 \).

\[\square\]

**Corollary 3.** When \( d \geq 4 \), for any \( \varepsilon > 0 \), \( r > 0 \), and \( T_1 \geq 1 \), we can find \( R \geq 1 \) such that, for any \( z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}^d \setminus B_{R+1}^+) \),
\[
\limsup_{k \to \infty} C'(r, z_0, u_k, p_k) \leq \varepsilon.
\]

**Proof.** For simplicity let us assume \( \Omega = \mathbb{R}^d \), as the other case is similar. Due to Proposition 2 ii), for any \( r > 0 \), and \( T_1 \geq 1 \), we can find \( R \) large such that
\[
\frac{1}{r^{d+2/d-2}} \int_{Q_{z_0, r}^+} \left| u_\infty \right|^2 \, dz
\]
is sufficiently small. Thus by Hölder’s inequality when \( d \geq 4 \), for any \( z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}^d \setminus B_{R+1}),
\[
\frac{1}{r^{d+2/d-2}} \int_{Q_{z_0, r}^+} \left| u_\infty \right|^{2(2-1/d)} \, dz
\]
is sufficiently small. This together with Proposition 2 i) proves (59). \[\square\]

**Lemma 5.2.** When \( d \geq 4 \), for any \( \varepsilon_1 > 0 \) and \( T_1 \leq 1 \), we can find \( R \geq 1 \) and \( r_0 > 0 \) such that, for any \( z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}^d \setminus B_{R+2}^+) \),
\[
\limsup_{k \to \infty} \left( A^+(r_0, z_0, u_k, p_k) + E^+(r_0, z_0, u_k, p_k) + C^+(r_0, z_0, u_k, p_k) + D^+(r_0, z_0, u_k, p_k) \right) \leq \varepsilon_1.
\]

**Proof.** This lemma is to improve the boundedness property of \( C \) and \( D \) we achieved in Lemma 5.1 to smallness. We first prove the interior case. From Lemma 5.1, we have
\[
\limsup_{k \to \infty} D(r, z_0, u_k, p_k) \leq N
\]
for any \( z_0 \in (-\infty, 0] \times \mathbb{R}^d \). Now let \( r = 1 \). From Corollary 3, for any \( \varepsilon_0 > 0 \), we can find \( R > 0 \) such that for any \( z_0 \in (-T_1 - 1, 0] \times (\mathbb{R}^d \setminus B_{R+2}),
\[
\limsup_{k \to \infty} C(1, z_0, u_k, p_k) \leq \varepsilon_0.
\]
By Lemma 3.6, we immediately know that there exists \( \gamma > 0 \) such that
\[
\limsup_{k \to \infty} D(\gamma, z_0, u_k, p_k) \leq \frac{N \varepsilon_0}{\gamma^{d+2/d-2}} + N \gamma^{4-3/d}.
\]
Then
\[
\limsup_{k \to \infty} C(\gamma, z_0, u_k, p_k) \leq \frac{\varepsilon_0}{\gamma^{d+2/d-2}}.
\]
Using Lemma 3.4 we can get
\[
\limsup_{k \to \infty} (A(\gamma/2, z_0, u_k, p_k) + E(\gamma/2, z_0, u_k, p_k)) \leq N \left( \frac{\varepsilon_0}{\gamma^{d+2/d-2}} \right)^{1/4}.
\]
We add (62), (61), and (63) together to obtain
\[
\limsup_{k \to \infty} \left( \frac{\varepsilon_0}{\gamma^{d+2/d-2}} \right)^{1/4} + \frac{N\varepsilon_0}{\gamma^{d+2/d-2}} + N\gamma^{4-3/d}. \tag{64}
\]
For any \( \varepsilon_1 > 0 \), fix some \( \gamma \) such that
\[
N\gamma^{4-3/d} < \varepsilon_1/2. 
\]
Letting \( r_0 = \gamma/2 \), by (64) we have proved (60) for the interior case.

For the boundary case, we do a similar discussion on different shapes of \( \omega(z_k^0, \lambda_k r) \) as we did in the proof of Lemma 5.1. We hereby omit the repeated proof.

Next we show that \( u_\infty \) is identically equal to zero. We modify the proof of [8, Proposition 5.3] by replacing Schoen’s trick with the H{"o}lder continuity proved in Sections 3 and 4. We state and prove the following proposition for \( \Omega = \mathbb{R}^d_+ \). The results readily generalize to problem on \( \Omega = \mathbb{R}^d \).

**Proposition 3.** Under the assumptions of Theorem 1.3, let \( (u_\infty, p_\infty) \) be the suitable weak solution constructed in this section. Then
\[
u_\infty(t, \cdot) \equiv 0 \quad \forall t \in (-\infty, 0).
\]

**Proof.** Let \( \hat{\varepsilon}_0 \) be the constant in Theorem 4.1. Note that we can assume \( \hat{\varepsilon}_0 \) is smaller than \( \varepsilon_0 \) in Theorem 3.1. Fix some \( T_1 \geq 1 \). Owing to Lemma 5.2, we can find \( R \geq 1 \) and \( r_0 > 0 \) such that for any \( z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}^d_+ \setminus B_{R+1}^+) \),
\[
\limsup_{k \to \infty} (A^+(r_0, z_0, u_k, p_k) + E^+(r_0, z_0, u_k, p_k) + D^+(r_0, z_0, u_k, p_k)) \leq \hat{\varepsilon}_0.
\]
Thus Theorems 3.1 and 4.1 yields that
\[
\limsup_{k \to \infty} |u_k(z_0)| \leq N(d)
\]
for a.e. \( z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}^d_+ \setminus B_{R+2}^+) \). By Proposition 2, we obtain
\[
|u_\infty(z_0)| \leq N(d)
\]
for a.e. \( z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}^d_+ \setminus B_{R+2}^+) \). Upon using the regularity results for linear Stokes systems, one can estimate higher derivatives
\[
|D^ju_\infty(z_0)| \leq N(d, j) \tag{65}
\]
for any \( j \geq 1 \) and a.e. \( z_0 \in [-T_1 - 1, 0] \times (\mathbb{R}^d_+ \setminus B_{R+3}^+) \).
We now claim $u_\infty(0, \cdot) = 0$ by adapting the argument in the proof of [14, Theorem 1.4]. For any $x_0 \in \mathbb{R}^d_+$, by using (56),
\[
\int_{\Omega(x_0, 1/4)} |u_\infty(0, x)| \, dx
\]
\[
\leq \int_{\Omega(x_0, 1/4)} |u_k(0, x) - u_\infty(0, x)| \, dx + \int_{\Omega(x_0, 1/4)} |u_k(0, x)| \, dx
\]
\[
\leq \|u_k - u\|_{C([-1/4^2, 0]; L^1(\Omega(x_0, 1/4)))} + N(d)\|u_k(0, \cdot)\|_{L^d(\Omega(x_0, 1/4))}
\]
\[
= \|u_k - u\|_{C([-1/4^2, 0]; L^1(\Omega(x_0, 1/4)))} + N(d)\|u(T_0, \cdot)\|_{L^d(\Omega(\lambda_k x_0/4 + x_0, \lambda_k/4))}.
\]
The right-hand side of the above inequality goes to zero as $k \to \infty$, which proves the claim.

Because of (65), the vorticity $\omega = \text{curl} \ u_\infty$ satisfies the differential inequality
\[
|\partial_t \omega - \Delta \omega| \leq N(|\omega| + |\nabla \omega|)
\]
on $(-T_1, 0) \times (\mathbb{R}^d_+ \setminus B_{R+3}^+)$. We apply the half-space backward uniqueness theorem proved in [14, 15] on the open half space $\{x_d > R + 3\}$ to reach
\[
\omega(z) = 0 \quad \text{on} \quad (-T_1, 0) \times \{x_d > R + 3\}. \tag{66}
\]

Now we fix a $t_0 \in (-T_1, 0)$. Take an increasing sequence $\{t_k\}_{k=0}^\infty \subset (-T_1, 0)$ converging to $t_0$. For each $k$, we consider equation (1) with initial data $u_\infty(t_k, \cdot)$. By Proposition 1, one can locally find a strong solution
\[
v_k \in C([t_k, t_k + \delta_k]; L^d(\mathbb{R}^d_+))
\]
for some small $\delta_k$, and $v_k(t, \cdot)$ is spatial analytic for $t \in (t_k, t_k + \delta_k)$. We may assume that $t_k + \delta_k < t_0$. By the weak-strong uniqueness, $v_k \equiv u_\infty$ for $t \in [t_k, t_k + \delta_k)$. Therefore, $\omega(t, \cdot)$ is also spatial analytic for $t \in (t_k, t_k + \delta_k)$. Because of (66), we get
\[
\omega(z) = 0 \quad \text{on} \quad (t_k, t_k + \delta_k) \times \mathbb{R}^d_+,
\]
which implies that $u_\infty \equiv 0$ in the same region. In particular, we can take a sequence $\{s_k\}$ such that $t_k < s_k < t_k + \delta_k$. Then $\{s_k\}$ converges to $t_0$ and $u_\infty(s_k, \cdot) \equiv 0$. This together with the weak continuity of $u_\infty$ yields that $u_\infty(t_0, \cdot) \equiv 0$. Since $t_0 \in (-T_1, 0)$ and $T_1 \geq 1$ are both arbitrary, we complete the proof of the theorem. \(\Box\)

We now show the proof of Theorem 1.3 which also works for Theorem 1.2. Again this proof is a modification of [8, Section 5] by replacing Schoen’s trick with the Hölder continuity proved in Sections 3 and 4.

**Proof of Theorem 1.3.** We prove the theorem in four steps.

**Step 1.** In this step, we assume condition (8) holds. First we show that $u$ is regular for $t \in (0, T]$. Owning to Propositions 2 and 3,
\[
u_k \to 0 \in C([-1/4^2, 0]; L^2(2^{-1/d})(B^+(1/4))).
\]
Hence
\[
\limsup_{k \to \infty} C_+(r, z_0, u_k, p_k) = 0
\]
for $r \in (0, 1/4)$. Also recall that $D^+(r, z_0, u_k, p_k)$ has a uniform bound for $r \in (0, 1)$. Following the proof of Lemma 5.2 we have: for any $\varepsilon > 0$, there is a $r_0 > 0$ small and a positive integer $k_0$ such that, for any $z_0 \in (-2, 0) \times B^+(2),$
\[
A^+(r_0, z_0, u_{k_0}, p_{k_0}) + E^+(r_0, z_0, u_{k_0}, p_{k_0}) + D^+(r_0, z_0, u_{k_0}, p_{k_0}) \leq \varepsilon.
\]
We choose \( \varepsilon \) sufficiently small and apply Theorem 4.1 to get, for some \( \tilde{r} > 0 \),

\[
\sup_{(-\tilde{r}^2,0) \times B^+(\tilde{r})} |u_{k_0}| < \infty,
\]

which implies that

\[
\sup_{\omega(\tilde{Z}_0,\lambda_{k_0})} |u| < \infty.
\]

This contradicts the assumption that \( \tilde{Z}_0 = (T_0, X_0) \) is a blowup point. Therefore, \( u \) is regular up to the boundary for \( t \in (0, T] \).

**Step 2.** From this step on, we assume condition (9) holds. We bound the sup norm of \( u \) in this step. Fix some small \( \delta \in (0, T) \). Since

\[
\|u\|_{L^2_\delta L^2_\delta((0,T) \times \mathbb{R}^d_+)} \leq N, \quad \|p\|_{L^{2-1/d}_{\delta},\text{unif}((0,T) \times \mathbb{R}^d_+)} \leq N,
\]

by the same reasoning as at the beginning of the proof of Proposition 3, we see that there exists a large \( R \geq 1 \) such that

\[
\sup_{[\delta,T] \times (\mathbb{R}^d_+ \setminus B^+(R))} |u| \leq N. \tag{67}
\]

Next we estimate the sup norm of \( u \) in \([\delta,T] \times B^+(R)\). Fix a \( z_0 = (t_0, x_0) \) in \([\delta,T] \times B^+(R)\). In the construction of \( u_k \), we replace \((T_0, X_0)\) by \((t_0, x_0)\). By the same reasoning as in the first step, for some \( \tilde{r} = \tilde{r}(t_0, x_0) > 0 \), we have

\[
\sup_{\omega(z_0,\tilde{r})} |u| \leq N(t_0, x_0).
\]

Utilizing the compactness of \([\delta,T] \times \bar{B}^+(R)\), we see that

\[
\sup_{[\delta,T] \times \bar{B}^+(R)} |u| \leq N.
\]

This together with (67) yields

\[
\sup_{[\delta,T] \times \mathbb{R}^d_+} |u| \leq N. \tag{69}
\]

**Step 3.** Since we can choose the \( \delta \) in step 2 to be arbitrarily small, we can then utilize the local strong solvability of (1) to find some \( T_1 > \delta \) such that \( u \in L^{d+2}_{d+2}((0,T_1) \times \mathbb{R}^d_+) \). From Step 2, for \( t \in [T_1, T] \) the solution is uniformly bounded and belongs to \( L^1_\delta L^2_\delta((T_1,T) \times \mathbb{R}^d_+) \), thus \( u \in L^{d+2}_{d+2}((0,T) \times \mathbb{R}^d_+) \). The uniqueness follows from the Ladyzhenskaya-Prodi-Serrin criterion.

**Step 4.** Now it remains to prove (10). We use a scaling argument. Let \( \lambda > 0 \) be a constant to be specified later. We define

\[
u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x).
\]

Then \((u_\lambda, p_\lambda)\) is also a Leray-Hopf weak solution of (1) in \((0,\infty) \times \mathbb{R}^d_+\), and \( u_\lambda \) satisfies (7) with the same constant \( K \) due to the scale invariant property. Due to the weak continuity of Leray-Hopf weak solutions,

\[
\|u_\lambda(t,\cdot)\|_{L^4(\mathbb{R}^d_+)} \leq N \tag{68}
\]

for each \( t \in (0, \infty) \). By using Lemma 3.2 with \( q = 2d/(d-2) \) and \( r = \infty \), we have

\[
\|u_\lambda\|_{L^q_\delta L^{r\infty}_{2d/(d-2)}((0,\infty) \times \mathbb{R}^d_+)} \leq N. \tag{69}
\]

Putting together (68) and (69) and using Hölder’s inequality yield

\[
\|u_\lambda\|_{L^4((0,\infty) \times \mathbb{R}^d_+)} \leq N.
\]
Thus there exists $T > 0$ such that $\|u_\lambda\|_{L_4([T, \infty) \times \mathbb{R}^d_+)}$ is sufficiently small, which together with Hölder’s inequality implies the smallness of $C^+(1, z_0, u_\lambda, p_\lambda)$ for $z_0 \in (T, \infty) \times \mathbb{R}^d_+$. Again let $\hat{\varepsilon}_0$ denote the constant in Theorem 4.1. Following the argument in Lemma 5.2 we can find a large $T = T_\lambda$ and $r_0 > 0$ such that
\[
A^+(t_0, z_0, u_\lambda, p_\lambda) + E^+(t_0, z_0, u_\lambda, p_\lambda) + D^+(t_0, z_0, u_\lambda, p_\lambda) \leq \hat{\varepsilon}_0
\]
for any $z_0 \in [T, \infty) \times \mathbb{R}^d_+$. Owing to Theorems 3.1 and 4.1, we conclude
\[
\sup_{[T, \infty) \times \mathbb{R}^d_+} |u_\lambda(z)| \leq N,
\]
where $N = N(d)$ is independent of $\lambda$. Therefore,
\[
\sup_{[\lambda^2T, \infty) \times \mathbb{R}^d_+] |u(z)| \leq N/\lambda.
\]
Sending $\lambda \to \infty$ yields the desired result. \hfill \Box

6. Proof of Theorem 1.4. In this section, we prove the local result mentioned in Theorem 1.4. Most part of the proof remains the same with last section. We omit some repeated details in similar proofs from last section. We again start with the blow-up procedure: Suppose $(u, p)$ is a pair of Leray-Hopf weak solution (1)-(2) in $(-1, 0) \times \mathbb{R}^d_+$ and we study the local problem in $Q^+$. Correspondingly, we modify the notation $\omega(z, r) = Q(z, r) \cap Q^+$. Suppose $T_0$ is the first blowup time of $u$ in $Q^+_{1/4}$, and
\[
Z_0 = (T_0, X_0) = (T_0, X_{0,1}, X_{0,2}, \ldots, X_{0,d}) = (T_0, X'_0, X_{0,d})
\]
is a singular point in $\overline{Q^+_{1/4}}$. Take a decreasing sequence $\{\lambda_k\}$ converging to 0 and rescale the pair $(u, p)$ at time $T_0$. For each $k = 1, 2, \ldots$, by defining
\[
u_k(t, x) = \lambda_k u(T_0 + \lambda_k^2 t, X_0 + \lambda_k x), \quad p_k(t, x) = \lambda_k^2 p(T_0 + \lambda_k^2 t, X_0 + \lambda_k x),
\]
$(u_k, p_k)$ is a suitable weak solution of (1)-(2) for $t \in (-\lambda_k^{-2}T_0, 0)$. The first observation is the property of uniform boundedness of $C^+$ and $D^+$ after the rescaling.

**Lemma 6.1.** Under the conditions in Theorem 1.4, there exists $N > 0$ such that for any $z_0 \in (-\infty, 0] \times \mathbb{R}^d_+$ and $0 < r \leq 1$, we have
\[
\limsup_{k \to \infty} C^+(r, z_0, u_k, p_k) \leq N,
\]
and
\[
\limsup_{k \to \infty} D^+(r, z_0, u_k, p_k) \leq N. \tag{70}
\]

**Proof.** Following the proof of Lemma 5.1, we easily have
\[
C^+(\lambda_k r, z_0^k, u, p) \leq N\|u\|_{L_4L_2(\omega(z_0^k, \lambda_k r))} \leq N.
\]
To prove (70), we consider three cases on whether $Z_0$ is on the flat boundary and the shape of the $\omega(z_0^k, \lambda_k r)$:

i) Consider when $Z_0 \in Q^+_{1/4}$, i.e., $X_{0,d} > 0$. Based on the blow-up procedure, the $u_k$’s are defined on a larger and larger domain which eventually expands to the whole space, so we can always assume $u_k$ has definition on $Q(z_0, r)$ when $k$ is large. Denote $Z_0 = (X'_0, 0, T_0)$ to be the projection of $Z_0$ on the flat boundary and $\hat{z}_0^k$ to be the projection of $z_0^k$. Note when $k$ is large, $\lambda_k r \ll X_{0,d}$, we have
\[
Q(z_0^k, \lambda_k r) \subset Q(Z_0, X_{0,d}) \subset Q^+(\hat{Z}_0, 2X_{0,d}) \subset Q^+(\hat{Z}_0, 1/2) \subset Q^+.
\]
From Corollaries 1 and 2, we know that
\[ D(r, z_0, u_k, p_k) = D(\lambda_k r, z_0^k, u, p) \]
\[ \leq ND(X_{0,d}, Z_0, u, p) + C \]
\[ \leq ND^+(2X_{0,d}, \hat{Z}_0, u, p) + C \leq ND^+(1/2, \hat{Z}_0, u, p) + C \]
\[ \leq N\|p\|^{2-1/d}_{L_{2-1/d}(Q^+)} + C. \]

ii) Consider when \( Z_0 \in Q_{1/4}^+, X_{0,d} = 0, \) and \( r \geq x_{0,d}. \) When \( k \) is large, we have
\[ \omega(z_0^k, \lambda_k r) \subset Q^+(z_0^k, 2\lambda_k r) \subset Q^+(Z_0, 1/2) \subset Q^+. \]

From the proof of Corollary 2, we have
\[ D^+(r, z_0, u_k, p_k) = D^+(\lambda_k r, z_0^k, u, p) \]
\[ \leq ND^+(2\lambda_k r, z_0^k, u, p) \]
\[ \leq ND^+(1/2, Z_0, u, p) + C \]
\[ \leq N\|p\|^{2-1/d}_{L_{2-1/d}(Q^+)} + C. \]

iii) Consider when \( Z_0 \in Q_{1/4}^+, X_{0,d} = 0, \) and \( r < x_{0,d}. \) When \( k \) is large, we have
\[ \omega(z_0^k, \lambda_k r) \subset Q(z_0^k, \lambda_k x_{0,d}) \subset Q^+(Z_0, 1/2) \subset Q^+. \]

From the proofs of Corollaries 1 and 2 we have
\[ D^+(r, z_0, u_k, p_k) = D(\lambda_k r, z_0^k, u, p) \]
\[ \leq ND(\lambda_k x_{0,d}, z_0^k, u, p) + C \]
\[ \leq ND(2\lambda_k x_{0,d}, z_0^k, u, p) + C \leq ND^+(1/2, Z_0, u, p) + C \]
\[ \leq N\|p\|^{2-1/d}_{L_{2-1/d}(Q^+)} + C. \]

Therefore, \( D^+(r, z_0, u_k, p_k) \) is uniformly bounded by \( \|p\|^{2-1/d}_{L_{2-1/d}(Q^+)} \) plus some constant.

After we proved the boundedness, we can again prove the results of Proposition 2, Corollary 3, Lemma 5.2 and Proposition 3 in the same fashion as previous. Here we show the proof of Theorem 1.4 in the boundary case \( X_{0,d} = 0. \) The interior case is similar.

**Proof of Theorem 1.4.** Owing to Propositions 2 and 3, we have
\[ u_k \to 0 \text{ in } C([-1/4^2, 0]; L_{2(2-1/d)}(B^+(1/4))). \]

Hence
\[ \limsup_{k \to \infty} C^+(r, z_0, u_k, p_k) = 0 \]
for \( r \in (0, 1/4). \) Also recall that \( D^+(r, z_0, u_k, p_k) \) has a uniform bound for \( r \in (0, 1/4). \) Following the proof of Lemma 5.2 we have: for any \( \varepsilon > 0, \) there is a \( r_0 > 0 \) small and a positive integer \( k_0 \) such that, for any \( z_0 \in (-1/4^2, 0] \times B^+(1/4), \)
\[ A^+(r_0, z_0, u_{k_0}, p_{k_0}) + E^+(r_0, z_0, u_{k_0}, p_{k_0}) + D^+(r_0, z_0, u_{k_0}, p_{k_0}) \leq \varepsilon, \]
which implies
\[ A^+ (\lambda_k r_0, z_0, u, p) + E^+ (\lambda_k r_0, z_0, u, p) + D^+ (\lambda_k r_0, z_0, u, p) \leq \varepsilon. \]
We choose \( \varepsilon \) sufficiently small and apply Theorem 4.1 to deduce that \( u \) is Hölder continuous around \( Z_0 \) since \( z_0 \) is an arbitrary point around \( Z_0 \). We reach a contradiction and finish the proof.

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E-mail address: Hongjie_Dong@brown.edu
E-mail address: Kunrui_Wang@alumni.brown.edu