Boundary terms in the AdS/CFT correspondence for spinor fields

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Abstract

The requirement that the action be stationary for solutions of the Dirac equations in anti-de Sitter space with a definite asymptotic behaviour is shown to fix the boundary term (with its coefficient) that must be added to the standard Dirac action in the AdS/CFT correspondence for spinor fields.

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1 Introduction

The stationary phase method enables one to give an asymptotic expansion for the path integral

$$Z = \int [D\varphi] \exp i\frac{i}{\hbar} S[\varphi]$$

(1.1)

of a quantum system in the classical limit $\hbar \to 0$. One finds to leading order (and dropping the prefactors)

$$Z \sim \exp \frac{i}{\hbar} S_{cl}$$

(1.2)

where $S_{cl}$ is the value of the action functional $S[\varphi]$ evaluated on the classical history, i.e., on the solution of the classical equations of motion

$$\frac{\delta S}{\delta \varphi^i} = 0$$

(1.3)

fulfilling the given boundary conditions.

It is quite crucial in order for (1.2) to be correct that the action $S[\varphi]$ be indeed stationary on the classical path. This rather trivial observation has far-reaching consequences on spaces with boundaries since it enables one to discriminate among the boundary terms that may be added to the action. The point is that if a given choice of the action $S[\varphi]$ fulfills $\delta S = 0$ on the classical histories, it is in general not true that $\delta(S + B_\infty) = 0$ where $B_\infty = \int \partial_\mu j^\mu d^d x$ is a surface term, since one may have $\delta B_\infty \neq 0$. This criterion fixes for instance the form of the conserved charges in gauge theories. It was used in particular in [4] to establish that the action for Einstein gravity must be supplemented by the time-integral of the ADM mass at infinity in the case of asymptotically flat spaces.

The purpose of this paper is to show that the same criterion fixes the form of the boundary term that must be added to the standard Dirac action [2] in the AdS/CFT correspondence [4, 5, 6]. Besides its intrinsic interest, this problem illustrates quite well the general fact that the action is much more than just a mere mnemonic device whose only purpose is to concisely summarize the equations of motion. Although all Lagrangians $\mathcal{L}$ that differ by a local divergence $\partial_\mu j^\mu$ have identical Euler-Lagrange derivatives, it is only for a special subclass of these Lagrangians that the Euler-Lagrange equations are equivalent.
to the statement that the action $S = \int d^d x \mathcal{L}$ is stationary in the class of histories entering the variational principle (i.e., $\delta \int d^d x \mathcal{L}$ is equal to zero and not a non-vanishing boundary term).

We refer to the literature for background information on the AdS/CFT correspondence (see [6] for a recent review) and shall discuss here only the problem of boundary terms\[1\].

## 2 Dirac equations on $AdS_{d+1}$

We follow [2] and consider free spinor fields of mass $m$ on (Euclidean) $AdS_{d+1}$. We take Poincaré coordinates $x^\mu = (x^0, x^i) = (x^0, \mathbf{x})$ ($i = 1, \ldots, d$) such that $AdS_{d+1}$ is the domain $x^0 > 0$ with metric

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = (x^0)^{-2} (dx^0 dx^0 + d\tilde{s}^2).$$

In (2.1), the $d$-dimensional $d\tilde{s}^2$ is the flat metric

$$d\tilde{s}^2 = d\mathbf{x} \cdot d\mathbf{x}.$$ (2.2)

In the local Lorentz frame

$$e_\mu^a = (x^0)^{-1} \delta_\mu^a$$ (2.3)

the Dirac operator $\not{D}$ is given by [2]

$$\not{D} \equiv e_\mu^a \Gamma^a (\partial_\mu + \frac{1}{2} \omega^b_{\mu \Sigma} \Sigma^b) = x^0 \Gamma^0 \partial_0 + x^0 \Gamma \cdot \nabla - \frac{d}{2} \Gamma^0$$ (2.4)

where the matrices $\Gamma^a = (\Gamma^0, \Gamma^i) = (\Gamma^0, \mathbf{\Gamma})$ are the flat space gamma-matrices and where the notation $\partial_\mu = (\partial_0, \partial_i) = (\partial_0, \nabla)$ is used.

The Dirac equations read

$$(\not{D} - m) \psi = 0$$ (2.5)

for $\psi$ and

$$\bar{\psi} (-\not{D} - m) = 0$$ (2.6)

for the conjugate spinor $\bar{\psi}$. Without loss of generality, we assume $m$ to be positive.

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\[1\] The actual talk given in Tbilissi did not deal with the subject covered in this paper but with the AdS central charge $c = 3l/2G$ that arises in the asymptotic conformal algebra of $(2 + 1)$-AdS gravity.
The equations of motion can be solved near the boundary $x^0 = 0$ by the standard Frobenius procedure. One tries solutions of the form 
\[(x^0)^\rho \sum_{n=0}^{\infty} c_n(x^0)^n\] where $c_n(x^0)$ are $x^0$-independent spinors. For this series to solve the equations, $\rho$ must be given by
\[\rho = \frac{d}{2} \pm m \tag{2.7}\]
and the first coefficient $c_0(x^0)$ should be annihilated by $I - \Gamma^0$ or $I + \Gamma^0$ depending on which value of $\rho$ is taken. There are thus two types of solutions near $x^0 = 0$. Solutions of the first type are annihilated by the matrix $I + \Gamma^0$ to leading order and behave asymptotically as
\[\psi^{-}(x) = (x^0)^{\frac{d}{2} - m} \psi_0(x) + o((x^0)^{\frac{d}{2} - m}) \tag{2.8}\]
$(x^0 \to 0)$ where $\psi_0(x)$ is an eigenvector of $\Gamma^0$ for the eigenvalue $-1$ but is otherwise arbitrary,
\[\Gamma^0 \psi_0(x) = -\psi_0(x). \tag{2.9}\]
Solutions of the second type start with higher powers of $x^0$ and are annihilated by $I - \Gamma^0$ to leading order. They behave asymptotically as
\[\psi^{+}(x) = (x^0)^{\frac{d}{2} + m} \chi_0(x) + o((x^0)^{\frac{d}{2} + m}) \tag{2.10}\]
$(x^0 \to 0)$ where now $\chi_0(x)$ is an arbitrary eigenvector of $\Gamma^0$ for the eigenvalue $+1$,
\[\Gamma^0 \chi_0(x) = \chi_0(x). \tag{2.11}\]

Similarly, one finds for the conjugate spinors
\[\bar{\psi}^{-}(x) = \bar{\psi}_0(x)(x^0)^{\frac{d}{2} - m} + o((x^0)^{\frac{d}{2} - m}), \tag{2.12}\]
\[\bar{\psi}_0(x)\Gamma^0 = \bar{\psi}_0(x) \tag{2.13}\]
and
\[\bar{\psi}^{+}(x) = \bar{\chi}_0(x)(x^0)^{\frac{d}{2} + m} + o((x^0)^{\frac{d}{2} + m}), \tag{2.14}\]
\[\bar{\chi}_0(x)\Gamma^0 = -\bar{\chi}_0(x). \tag{2.15}\]

One may construct recursively the next terms in the solutions by using the Dirac equations. In order to formulate the boundary conditions below, we note that up to terms of order $o((x^0)^{\frac{d}{2}+m})$,
solutions (2.8) and (2.12) read

\[ \psi^-(x) = (x^0)^{\frac{d}{2} - m} \left[ I + \sum_{n=1}^{p} (x^0)^n \alpha_n \right] \psi_0(x) + o\left( (x^0)^{\frac{d}{2} + m} \right) \]

\[ \bar{\psi}^+(x) = (x^0)^{\frac{d}{2} - m} \bar{\psi}_0(x) \left[ I + \sum_{n=1}^{p} \beta_n (x^0)^n \right] + o\left( (x^0)^{\frac{d}{2} + m} \right) \]

where \( p \) is the biggest integer such that \( p \leq 2m \) and where \( \alpha_n \) and \( \beta_n \) are definite differential operators in \( d \) dimensions which can be computed from the Dirac equations (e.g., \( \alpha_1 \sim \Gamma \cdot \nabla \)) and whose explicit form will not be needed in the sequel.

The general solution near \( x^0 = 0 \) is obtained by superposition of \( \psi^-(x) \) and \( \psi^+(x) \). It is determined by the boundary data \( \psi_0(x) \) and \( \chi_0(x) \), which may be taken independently at this stage. The leading order of the solution is determined by \( \psi_0(x) \), which fulfills \( (I + \Gamma^0)\psi_0(x) = 0 \). The other boundary datum \( \chi_0(x) \) is (non trivially) relevant at lower orders. Thus, to leading order, the general solution of the Dirac equation near \( x^0 = 0 \) belongs to the eigenspace of \( \Gamma^0 \) with eigenvalue \( -1 \). The next order belongs to the eigenspace with eigenvalue \( +1 \). This "peeling off" property of Dirac fields is reminiscent of what happens for Rarita-Schwinger fields \([8]\).

If one demands that the solution be well-behaved in the volume of anti-de Sitter space \( \text{AdS}_{d+1} \) up to \( x^0 = \infty \) (which consists of a single boundary point since the metric along \( x \) vanishes in the limit \( x^0 \to \infty \) \([5, 6]\)), one finds that \( \chi_0(x) \) and \( \psi_0(x) \) cannot be taken independently. Rather, they must be related as

\[ \chi_0(k) = -i \frac{k \cdot \Gamma}{k} k^{2m} \frac{2m \Gamma\left( \frac{1}{2} - m \right)}{\Gamma(m + \frac{1}{2})} \psi_0(k) \tag{2.18} \]

where \( \chi_0(k) \) and \( \psi_0(k) \) are the respective Fourier transforms of \( \chi_0(x) \) and \( \psi_0(x) \), and where \( k = \sqrt{k^2} \). One must similarly impose

\[ \bar{\chi}_0(k) = i \bar{\psi}_0(k) \frac{k \cdot \Gamma}{k} k^{2m} \frac{2m \Gamma\left( \frac{1}{2} - m \right)}{\Gamma(m + \frac{1}{2})} \tag{2.19} \]

One may actually need \( (\log x^0) (x^0)^{\frac{d}{2} + m} \)-terms when \( m \) is a half-integer but we will not write explicitly these terms - also determined by the first terms in the expansion - when they are needed. The formulas \((2.18)\) and \((2.19)\) below assume \( m \neq \) a half-integer.

\[ ^2 \text{One may actually need \( (\log x^0) (x^0)^{\frac{d}{2} + m} \)-terms when \( m \) is a half-integer but we will not write explicitly these terms - also determined by the first terms in the expansion - when they are needed. The formulas \((2.18)\) and \((2.19)\) below assume \( m \neq \) a half-integer.} \]

\[ ^3 \text{This point is the compactification point at infinity of \( x^0 = 0 \) \([\mathbb{R}, \mathbb{C}]\). We shall assume that the data \( \psi_0(x) \) and \( \bar{\psi}_0(x) \) can be Fourier-transformed in \( x \) (e.g., are of compact support) and so vanish for \( |x| \to \infty \). For regularity, the solution must then vanish for \( x^0 \to \infty \).} \]
for the conjugate spinors.

These relations follow from the work of [2, 7], where the general solution that vanishes for $x^0 \to \infty$ is explicitly constructed. One finds

$$\psi(x) = (x^0)^{d+1 \over 2} \int {d^d k \over (2\pi)^d} e^{-ik \cdot x} [A(k) + B(k)]$$

(2.20)

where

$$A(k) = -i k \cdot \Gamma K_{m-\frac{1}{2}}(kx^0) \frac{k^{m+\frac{1}{2}} \psi_0(k)}{\Gamma(m + \frac{1}{2})2^{m-\frac{1}{2}}}$$

(2.21)

and

$$B(k) = K_{m+\frac{1}{2}}(kx^0) \frac{k^{m+\frac{1}{2}} \psi_0(k)}{\Gamma(m + \frac{1}{2})2^{m-\frac{1}{2}}}$$

(2.22)

Here $K_\nu(z)$ is the modified Bessel function which vanishes as $z \to \infty$. Similarly, one obtains [7]

$$\bar{\psi}(x) = (x^0)^{d+1 \over 2} \int {d^d k \over (2\pi)^d} e^{-ik \cdot x} [\bar{A}(k) + \bar{B}(k)]$$

(2.23)

with

$$\bar{A}(k) = i \frac{\bar{\psi}_0(k)k^{m+\frac{1}{2}}}{\Gamma(m + \frac{1}{2})2^{m-\frac{1}{2}}} k \cdot \Gamma K_{m-\frac{1}{2}}(kx^0)$$

(2.24)

and

$$\bar{B}(k) = \frac{\bar{\psi}_0(k)k^{m+\frac{1}{2}}}{\Gamma(m + \frac{1}{2})2^{m-\frac{1}{2}}} K_{m+\frac{1}{2}}(kx^0)$$

(2.25)

Using the small argument expansion of the Bessel functions,

$$K_\nu(z) = 2^{\nu-1} \Gamma(\nu)(z)^{-\nu}[1 + \ldots] - 2^{-\nu-1} \frac{\Gamma(1-\nu)}{\nu}(z)^{\nu}[1 + \ldots]$$

(2.26)

where the dots denote positive integer powers of $(z)^2$, one can verify that the solutions behave as indicated above for $x^0 \to 0$, and that $\chi_0(x)$ and $\tilde{\chi}_0(x)$ are related to $\psi_0(x)$ and $\bar{\psi}_0(x)$ as written in (2.18) and (2.19). The leading term in $\psi^-(x)$ comes from the first term in the expansion of $B(k)$; the leading term in $\psi^+(x)$ comes from the second term in the expansion of $A(k)$ (coefficient of $(x^0)^{m-\frac{1}{2}}$).

It follows that the general solution of the Dirac equations in the whole of AdS is determined by a single spinor field annihilated by $1+\Gamma_0$ on the boundary [2, 3]. Half of the boundary data are expressed in
terms of the other half if the solution is to be a solution everywhere. It should be stressed, however, that (2.18) and (2.19) are on-shell relations valid for solutions of the classical equations of motion but not for all competing histories considered in the variational principle. In fact, one can easily construct off-shell configurations which are regular in the whole of $AdS_{d+1}$ and which involve independent $\psi^-$ and $\psi^+$ in the vicinity of $x^0 = 0$.

3 Variational principle

Because the Dirac equations contain only first order derivatives, one cannot fix simultaneously all the components of $\psi$ and $\bar{\psi}$ at $x^0 = 0$. This would be like fixing both $q$’s and $p$’s at the boundary, and this is not permissible. Rather, one can only fix a complete set of “commuting” variables, e.g. the $q$’s or the $p$’s. For the AdS/CFT correspondence, it is appropriate to fix the components $\psi_0$ and $\bar{\psi}_0$, which are the sources for the Green functions in $d$ dimensions [2] and to leave $\chi_0(x)$ and $\bar{\chi}_0(x)$ free to vary. We shall thus consider in the variational principle all configurations which take the form

$$\psi(x) = \psi^-(x) + \psi^+(x),$$

$$\bar{\psi}(x) = \bar{\psi}^+(x) + \bar{\psi}^-(x)$$

where $\psi^-(x)$ and $\bar{\psi}^+(x)$ behave asymptotically ($x^0 \to 0$) as in (2.16) and (2.17) with given $\psi_0(x)$ and $\bar{\psi}_0(x)$, while $\psi^+(x)$ and $\bar{\psi}^-(x)$ behave asymptotically as (2.10) and (2.14) with coefficients $\chi_0(x)$ and $\bar{\chi}_0(x)$ that are free to vary. The $o\left[(x^0)^{\frac{d}{2}+m}\right]$ terms in $\psi(x)$ and $\bar{\psi}(x)$ are of course also allowed to vary and need not be such that the histories $\psi(x)$ and $\bar{\psi}(x)$ are solutions of the Dirac equations.

So, one does not (and cannot) impose any relationship between $\chi_0(x)$ and $\psi_0(x)$ or $\bar{\chi}_0(x)$ and $\bar{\psi}_0(x)$ in the variational principle. The relations (2.18) and (2.19) emerge on-shell.

If one varies the standard Dirac action

$$S_D = \int_{AdS} d^{d+1}x \sqrt{G} \psi \left[ \frac{1}{2} (\not{D} - \not{D}) - m \right] \psi$$

with respect to $\psi$ and $\bar{\psi}$ in the class of field configurations just defined, one finds, keeping all surface terms,

$$\delta S_D = B_\infty + \text{terms that vanish when the Dirac equations hold}$$

(3.4)
where $B_\infty$ is given by

$$B_\infty = -\frac{1}{2} \int d^d x [\bar{\psi}_0(x) \delta \chi_0(x) + \delta \bar{\chi}_0(x) \psi_0(x)]$$  \hfill (3.5)

(recall that $\sqrt{(d+1)G} = (x^0)^{-d-1}$). This term is itself the variation of a surface term at infinity,

$$B_\infty = -\delta C_\infty$$  \hfill (3.6)

with

$$C_\infty = \frac{1}{2} \int d^d x [\bar{\psi}_0(x) \chi_0(x) + \bar{\chi}_0(x) \psi_0(x)]$$  \hfill (3.7)

since $\delta \bar{\psi}_0(x) = 0$ and $\delta \psi_0(x) = 0$.

Because $B_\infty \neq 0$, the action $S_D$ is not stationary on the Dirac solutions in the class of competing histories defined by the boundary conditions that $\psi_0$ and $\bar{\psi}_0$ are fixed while $\chi_0$ and $\bar{\chi}_0$ are free to vary. However, the “improved action” obtained by adding $C_\infty$ to $S_D$,

$$S = S_D + C_\infty$$  \hfill (3.8)

is such that $\delta S = 0$ on-shell. It is therefore $S$ (and not $S_D$) that must be used in the AdS/CFT correspondence.

Three comments are in order:

1. Since $\bar{\psi}_0(x)$ and $\psi_0(x)$ are fixed, one may add any function of these variables to $S$ without changing the property that $\delta S = 0$. Such terms correspond to “phase transformations”. However, if one requires that the surface term (i) is local; (ii) does not contain derivatives (because the bulk part is first-order); and (iii) preserves the AdS symmetry, then, (3.7) appears to be the only choice ($\bar{\psi}_0 \psi_0 = 0$). Note in particular that the coefficient of $C_\infty$ in (3.7) is completely determined.

2. Up to terms which give delta-function contact terms in the correlators and which are thus unimportant \[11\] (even though they may contain powers of $\epsilon$ that blow up as $\epsilon \to 0$), one may rewrite the surface term (3.7) as

$$C_\infty = \lim_{\epsilon \to 0} \frac{1}{2} \int_{M_\epsilon} d^d x \sqrt{(d)G_\epsilon} \bar{\psi} \psi$$  \hfill (3.9)

where $M_\epsilon$ is a $d$-dimensional surface that approaches the boundary of $AdS_{d+1}$ as $\epsilon$ goes to zero, and where $(d)G_\epsilon$ is the determinant of the induced metric on $M_\epsilon$. This agrees with \[3\].
3. The kinetic term in the action (with $x^0$ being the “time”) shows that $\bar{\psi}_0(x)$ and $\chi_0(x)$ (respectively, $\bar{\chi}_0(x)$ and $\psi_0(x)$) form canonically conjugate pairs. So, one is indeed fixing the $q$’s and leaving the $p$’s free to vary at the boundary.

4 Correlation functions

We can now compute the correlation functions, following closely [2, 7]. To that end, one needs the value $S_{cl}$ of the classical action. As stressed in [2], the standard Dirac action vanishes on-shell and the whole contribution comes from the surface term. Replacing in $C_\infty$ the spinors $\chi_0(x)$ and $\bar{\chi}_0(x)$ by their on-shell values given by (2.18) and (2.19) yields

$$S_{cl} = \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} [\bar{\psi}_0(-k)\chi_0(k) + \bar{\chi}_0(k)\psi_0(-k)]$$

$$= \frac{i2^{-2m}\Gamma(\frac{d}{2} - m)}{\Gamma(m + \frac{1}{2})} \int \frac{d^dk}{(2\pi)^d} \bar{\psi}_0(k) \frac{k \cdot \Gamma}{k} k^{2m} \psi_0(-k) \quad (4.1)$$

If one rewrite the integral in position space, one gets

$$S_{cl} = -\frac{\Gamma(\frac{d+1}{2} + m)}{\pi \frac{d}{2} \Gamma(m + \frac{1}{2})} \int d^dx d^dy \bar{\psi}_0(x) \frac{\Gamma}{|x - y|^{d+2m+1}} \psi_0(y) \quad (4.2)$$

which shows that the 2-point function is

$$\Omega(x, y) \sim \frac{\Gamma}{|x - y|^{d+2m+1}} \quad (4.3)$$

as shown in [3, 4] and in agreement with the CFT on the boundary.

5 Conclusions

In this paper, we have shown that the surface term needed in the AdS/CFT correspondence for spinor fields can be understood from the requirement that the variational principle has a solution in the relevant class of field histories. This class of field histories is characterized by the boundary data at $x^0 = 0$: one fixes the components of
the spinor fields that are annihilated by \( I + \Gamma^0 \) but not the components with opposite “chirality”. These components come with a higher power of \( x^0 \) but play a non trivial role because they do contribute to the boundary terms. They are unrelated off-shell to the prescribed components of \( \psi \) and are free to vary in the variational principle, although they become functions of these prescribed components when the Dirac equations hold everywhere.

*Note added:* A different (but equivalent) justification of the boundary term has been given recently in [1]. It is based on the Hamiltonian formalism with \( x^0 \) viewed as the evolution parameter. It is equivalent to the requirement that the classical path be a true stationary point, since this imposes the form \( pq \) (rather than \(-pq\), say) to the kinetic term in the Hamiltonian action in the coordinate representation where the \( q \)'s are given at the boundary [2]. I am grateful to Kostas Sfetsos for pointing out reference [1] to me.

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