MODELS OF $\mu_{p^2,K}$ OVER A DISCRETE VALUATION RING

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WITH AN APPENDIX BY XAVER CARUSO

Abstract. Let $R$ be a discrete valuation ring with residue field of characteristic $p > 0$. Let $K$ be its fraction field. We prove that any finite and flat $R$-group scheme, isomorphic to $\mu_{p^2,K}$ on the generic fiber, is the kernel in a short exact sequence which generically coincides with the Kummer sequence. We will explicitly describe and classify such models. In the appendix X. Caruso shows how to classify models of $\mu_{p^2,K}$, in the case of unequal characteristic, using the Breuil-Kisin theory.

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Introduction

**Notation and Conventions.** If not otherwise specified we denote by $R$ a discrete valuation ring (in the sequel d.v.r.) with residue field $k$ of characteristic $p > 0$. Moreover we write $S = \text{Spec}(R)$. If for $n \in \mathbb{N}$ there exists a distinguished primitive $p^n$-th root of unity $\zeta_n$ in a d.v.r. $R$, we call $\lambda(n) := \zeta_n - 1$. Moreover we suppose $\zeta_{i-1} = \zeta_i^p$ for $2 \leq i \leq n$. We will denote by $\pi \in R$ one fixed uniformizer of $R$. All the schemes will be assumed noetherian. All the cohomological groups are calculated in the fppf topology.

Let $K$ be a field of characteristic 0 which contains a primitive $p^n$-th root of unity: this implies $\mu_{p^n,K} \cong (\mathbb{Z}/p^n\mathbb{Z})_K$. We recall the following exact sequence

$$1 \rightarrow \mu_{p^n,K} \rightarrow G_{m,K} \overset{p^n}{\rightarrow} G_{m,K} \rightarrow 1,$$

so-called the Kummer sequence. The Kummer theory says that this sequence is universal, namely any $p^n$-cyclic Galois extension of $K$ can be deduced by the Kummer sequence. We stress that this sequence can be written also as follows

$$1 \rightarrow \mu_{p^n,K} \rightarrow G_{m,K} \overset{\theta_n}{\rightarrow} G_{m,K} \rightarrow 1$$

where $\theta_n((T_1, \ldots, T_n)) = (1 - T_1^p, T_1 - T_2^p, \ldots, T_{n-1} - T_n^p)$.

Let $k$ be a field of characteristic $p > 0$. The following exact sequence

$$0 \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow W_n(k) \overset{F-1}{\rightarrow} W_n(k) \rightarrow 0,$$

where $W_n(k)$ is the group scheme of Witt vectors of length $n$, is called the Artin-Schreier-Witt sequence. The Artin-Schreier-Witt theory implies that any $p^n$-cyclic Galois covering of $k$ can be deduced by the Artin-Schreier-Witt sequence.

Let now $R$ be a d.v.r. of unequal characteristic which contains a $p^n$-th root of unity. There exists a theory, the so called Kummer-Artin-Schreier-Witt theory, which unifies the above two theories. This means that there exists, for any $n$, a sequence

$$0 \rightarrow \mathbb{Z}/p^n\mathbb{Z} \rightarrow W_n \rightarrow W'_n \rightarrow 0,$$

with $W_n$ and $W'_n$ smooth, which coincides with the Kummer sequence over the generic fiber, and with the Artin-Schreier-Witt sequence over the special fiber.

The case $n = 1$ has been independently studied by Oort-Sekiguchi-Suwa ([16]) and Waterhouse ([25]). Later Green-Matignon ([6]) and Sekiguchi-Suwa([22]) have, independently, constructed explicitly a unifying exact sequence for $n = 2$. The case $n > 2$ is treated in [15] and [21]. The universality of these sequences is discussed in [20]. We remark that the explicit Kummer-Artin-Schreier-Witt theory has been one of the main tools used by Green-Matignon ([6]) to give a positive answer to the lifting problem for cyclic Galois covers of order $p^2$.

From now on $R$ is a d.v.r. with residue field of characteristic $p > 0$. In this paper we study finite and flat $R$-group schemes of order $p^2$ which are isomorphic to $\mu_{p^2,K}$ on the generic fiber, i.e. models of $\mu_{p^2,K}$. And we prove that, for any such a group scheme $G$, there exists an exact sequence

$$0 \rightarrow G \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow 0,$$

with $\mathcal{E}_1, \mathcal{E}_2$ smooth $R$-group schemes, which coincides with the Kummer sequence on the generic fiber. So if $R$ is of unequal characteristic and contains a primitive $p^2$-th root of unity, the sequence (2), with $n = 2$, is the special case when we consider the model $G = \mathbb{Z}/p^2\mathbb{Z}$. The sequence (3) is universal in the sense that any $G$-torsor over a local scheme is obtained pulling back this sequence (see [24, §2.3]). Moreover we will describe explicitly all such isogenies and their kernels and we will give a classification of models of $\mu_{p^2,K}$. The case of models of $\mu_{p,K}$ is well known.

We now explain more precisely the classification we have obtained. In the first section we recall the definition of a class of group schemes of order $p$, called $G_{\lambda,1}$ with $\lambda \in R \setminus \{0\}$, which are
In the rest of the paper, with a slight abuse of notation, for a group scheme \( \mu \in R \) it will be clear from the context what we mean and it should not cause any confusion.

Let \( \lambda \) sometimes simply write \( R \) flat \( \text{Ext}^{3} \) with \( 3 \) interpretation of this group in terms of Witt vectors. Moreover a few preliminary results are added.

In particular the set \( \{ \mu, \lambda \} \) must also satisfy the inequality \( (p - 1)v(\lambda) \leq v(p) \). We remark that for any of them there is an exact sequence
\[
0 \rightarrow G_{\lambda,1} \rightarrow G^{(\lambda)} \rightarrow G^{(\lambda')} \rightarrow 0
\]
where \( G^{(\lambda)} \) and \( G^{(\lambda')} \) are \( R \)-smooth models of \( G_{m,K} \). Moreover on the generic fiber this sequence coincides with Kummer sequence. We recall the following well known result.

\textbf{Proposition. 1.3.} If \( G \) is a finite and flat \( R \)-group scheme such that \( G_K \cong \mu_p,K \) then \( G \cong G_{\lambda,1} \) for some \( \lambda \in R \setminus \{0\} \).

One crucial remark is the following: any model of a diagonalizable group scheme of order \( p^2 \) is an abelian extension of \( G_{\mu,1} \) by \( G_{\lambda,1} \) for some \( \mu, \lambda \in R \setminus \{0\} \) (see 3.1). This leads us to study the group \( \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \). We stress we compute \( \text{Ext}^{3}_{\mu}(G, H) \) in the category of abelian fppf sheaves over \( S \) and this is not at all restrictive for our purposes by the above remark.

As one could imagine the above group of extensions is related to the group \( \text{Ext}^{3}_{\mu}(G^{[\mu]},G^{(\lambda)}) \). In the second section we essentially recall the theory of Sekiguchi-Suwa, which gives an interesting interpretation of this group in terms of Witt vectors. Moreover a few preliminary results are added.

In §3 we study the models of \( \mu_{p^2,K} \). First of all we investigate on \( \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \). We distinguish four cases:
\[
\begin{align*}
\text{Ext}^{3}_{\mu}(\mu_{p,S},\mu_{p,S}) : & \text{ with } \lambda \in \pi R \setminus \{0\} ; \\
\text{Ext}^{3}_{\mu}(\mu_{p,S},G_{\lambda,1}) : & \text{ with } \mu \in \pi R \setminus \{0\} ; \\
\text{Ext}^{3}_{\mu}(G_{\mu,1},\mu_{p,S}) : & \text{ with } \mu \in \pi R \setminus \{0\} ; \\
\text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) : & \text{ with } \mu, \lambda \in \pi R \setminus \{0\} .
\end{align*}
\]
The first three cases are easy and already known. The fourth case is more subtle. If \( \lambda' \in \pi R \), let \( S_{\lambda'} := \text{Spec}(R/\lambda' R) \) and let \( i_{\lambda'} \) be the closed immersion \( i_{\lambda'} : S_{\lambda'} \rightarrow S \). Let us suppose that \( \mu, \lambda \in \pi R \setminus \{0\} \). We define the group
\[
\text{rad}_{\mu,\lambda} := \left\{ (F(T), j) \in \text{Hom}_{S_{\lambda'},gr}(i_{\lambda'}^*G_{\mu,1},G_{m,S_{\lambda'}}) \times \mathbb{Z}/p\mathbb{Z} \text{ such that } F(T)\beta(1 + \mu T)^{-1} = 1 \in \text{Hom}_{S_{\lambda},gr}(i_{\lambda}^*G_{\mu,1},G_{m,S_{\lambda}}) \right\} / \langle 1 + \mu T \rangle
\]
In the rest of the paper, with a slight abuse of notation, for a group scheme \( G \) over \( R \) and \( \lambda \in \pi R \), we will sometimes simply write \( G \) to indicate its restriction over \( S_{\lambda'} \), instead of writing \( i_{\lambda'}^*G \). However it will be clear from the context what we mean and it should not cause any confusion.

Let \( (F(T), j) \in \text{rad}_{\mu,\lambda} \) and \( \tilde{F}(T) \in R[T] \) any its lifting. We will explicitly define in §3.2 a finite and flat \( R \)-group scheme \( E^{(\mu,\lambda,F,j)} \) which is extension of \( G_{\mu,1} \) by \( G_{\lambda,1} \). Its extension class \( [E^{(\mu,\lambda,F,j)}] \) in \( \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \) does not depend on the choice of the lifting. So we will sometimes denote it simply with \( [E^{(\mu,\lambda,F,j)}] \), omitting to specify the lifting. The above group scheme is the kernel of an isogeny of smooth group schemes of dimension 2. This isogeny is generically isomorphic to the Kummer sequence, in the form of (1). Using this notation, we give a description of \( \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \).

\textbf{Theorem. 3.19.} Let \( \mu, \lambda \in \pi R \setminus \{0\} \). If \( \text{char}(R) = 0 \), we suppose \( (p - 1)v(\lambda), (p - 1)v(\mu) \leq v(p) \).

Then there exists an exact sequence
\[
0 \rightarrow \text{rad}_{\mu,\lambda} \rightarrow \beta \rightarrow \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \rightarrow \ker \left( H^1(S,G_{\mu,1}^{\vee}) \rightarrow H^1(S,G_{\mu,1}^{\vee}) \right)
\]
where \( \beta \) is defined by
\[
(F,j) \mapsto [E^{(\mu,\lambda,F,j)}].
\]
In particular the set \( \{ [E^{(\mu,\lambda,F,j)}] ; (F,j) \in \text{rad}_{\mu,\lambda} \} \subseteq \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \) is a group isomorphic to \( \text{rad}_{\mu,\lambda} \).

The group \( \text{Ext}^{3}_{\mu}(G_{\mu,1},G_{\lambda,1}) \) has been described by Greither in [7] through a short exact sequence, different from the one of the previous theorem. An advantage of our description is that we individuate a big class of extensions. For instance in unequal characteristic this class "essentially" covers all the group schemes of order \( p^2 \). Indeed from 3.19 it follows that in such a case any group scheme of order \( p^2 \), up to an extension of \( R \), is of the form \( E^{(\mu,\lambda,F,j)} \) (see 3.21).

The following result is obtained combining the previous Theorem and the Sekiguchi-Suwa theory.
Proposition. 3.24 Let $\mu, \lambda \in \pi R \setminus \{0\}$. If $\text{char}(R) = 0$ we suppose $(p - 1)v(\mu), (p - 1)v(\lambda) \leq v(p)$. Then the group $\{[\mathcal{E}(\mu,\lambda;F,j)];(F,j) \in \mathcal{R}(\mu,\lambda)\}$ is isomorphic to

$$
\Phi_{\mu,\lambda} := \left\{ (a,j) \in (R/\lambda R)F^{-p-1} \times \mathbb{Z}/p\mathbb{Z} \text{ such that } pa - j\mu = \frac{p}{\mu^{p-1}} a^p \in R/\lambda^p R \right\}/\langle (\mu,0) \rangle,
$$

through the morphism

$$(a,j) \mapsto [\mathcal{E}(\mu,\lambda;F(T),j)],$$

where $F(T) = 1 + \sum_{i=1}^{p-1} \prod_{r=1}^{i} (a - kn)^{T^i}$.

It could be possible that two different elements of $\text{Ext}_{\mathcal{R}}^1(G_{\mu,1},G_{\lambda,1})$ are isomorphic just as group schemes. We study when this happens, in the case of models of $\mu_{p^n,K}$, and we prove the following theorem.

Theorem. 3.37 Let $G$ be a finite and flat $R$-group scheme such that $G_K \simeq \mu_{p^n,K}$. Then $G \simeq \mathcal{E}(\pi^m,\pi^n;F,1)$ for some $m,n \geq 0$, $F(T)$ a lifting of $F(T) = \sum_{k=0}^{p-1} \frac{a^{k^n}}{k^n} T^k$ with $(a,1) \in \Phi_{\pi^m,\pi^n}$. If $\text{char}(R) = 0$ then $m \geq n$ and $(p - 1)m \leq v(p)$, while if $\text{char}(R) = p$ then $m \geq pn$. Moreover $m,n$ and $a$ are isomorphic just as group schemes. We study when this happens, in the case of models of $\mu_{p^n,K}$, and we prove the following theorem.

The last section is devoted to determine, through the description of 3.24, the special fibers of the extensions $\{[\mathcal{E}(\mu,\lambda;F,T),j]\}$

We want to remark that sometimes we treat the cases of equal and unequal characteristic separately. But it could be possible to treat them always together. This relies on the fact that results of Sekiguchi and Suwa work over $\mathbb{Z}/p\mathbb{Z}$-algebras. However, for reader convenience, we prefer sometimes to separate the two cases; this could also allow the reader to appreciate how the equal characteristic case is in fact easier.

We finally stress that Breuil and Kisin (see [3], [8], [9]) have recently developed a theory which classifies finite and flat commutative group schemes over a complete d.v.r. of unequal characteristic with perfect residue field. This classification is obtained through some objects of linear algebra which are very easy to study. In the appendix of this paper X. Caruso shows, using this theory, how to obtain in a very efficient and elegant way a result similar to 3.37 (under the further assumptions $R$ complete, of unequal characteristic and $k$ perfect). Unfortunately with this purely algebraic approach one losses some important geometric aspects, namely the explicit description of these group schemes, the geometric meaning of parameters which appear in the classification and the existence of a universal sequence from which one can determine any torsor under these group schemes.

In principle our approach could be applied, for instance, also for the classification of models of $\mu_{p^n,K}$ with $n > 2$. But the technical problems which can arise could be very difficult. Over a complete d.v.r. of unequal characteristic with perfect residue field maybe the right way is just to mix the above two theories: one can obtain the parameters which classify these models using the very efficient theory of Breuil-Kisin and then one can try to understand geometrically these parameters through the Sekiguchi-Suwa theory, in such a way to obtain explicit descriptions of these models.

As recalled at the beginning of the introduction, if $\text{char}(R) = 0$ and $R$ contains a primitive root of unity, then $\mu_{p^n,K} \simeq (\mathbb{Z}/p^n\mathbb{Z})_K$. In such a case the explicit description of the models of $(\mathbb{Z}/p^n\mathbb{Z})_K$ presented in this paper has been used in [24] to study the degeneration of $(\mathbb{Z}/p^n\mathbb{Z})_K$-torsors from characteristic $0$ to characteristic $p$ and in particular the problem of extension of $(\mathbb{Z}/p^n\mathbb{Z})_K$-torsors.

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MODELS OF $\mu_{p^2,K}$

1. Models of $\mu_{p,K}$

1.1. Some group schemes of order $p$. For any $\lambda \in R$ we define the group scheme

$$G^{(\lambda)} := \text{Spec}(R[T, \frac{1}{1 + \lambda T}])$$

The $R$-group scheme structure is given by

$$
\begin{align*}
T &\to 1 \otimes T + T \otimes 1 + \lambda T \otimes T & \text{comultiplication} \\
T &\to 0 & \text{counit} \\
T &\to - \frac{T}{1 + \lambda T} & \text{coinverse}
\end{align*}
$$

We observe that if $\lambda = 0$ then $G^{(\lambda)} \simeq G_{a,S}$. It is possible to prove that $G^{(\lambda)} \simeq G^{(\mu)}$ if and only if $v(\lambda) = v(\mu)$ and the isomorphism is given by $T \to \frac{T}{\mu}$. Moreover it is easy to see that, if $\lambda \in \pi R \setminus \{0\}$, then $G^{(\lambda)}_k \simeq G_{a,k}$ and $G^{(\lambda)}_K \simeq G_{m,K}$. When $\lambda \in R \setminus \{0\}$ we can define the morphism $\alpha^{(\lambda)} : G^{(\lambda)} \to G_{m,S}$

given, on the level of Hopf algebras, by $T \mapsto 1 + \lambda T$: it is an isomorphism on the generic fiber. If $\nu(\lambda) = 0$ then $\alpha^{(\lambda)}$ is an isomorphism.

We now define some finite and flat group schemes of order $p$. If $R$ is of unequal characteristic we will assume that $\lambda \in R$ satisfies the condition

$$v(p) \geq (p - 1)v(\lambda).$$

Let $\lambda \in R \setminus \{0\}$. Then the morphism

$$\psi_{\lambda,1} : G^{(\lambda)} \to G^{(\lambda^p)}$$

$$T \to P_{\lambda,1}(T) := \frac{(1 + \lambda T)^p - 1}{\lambda^p}$$

is an isogeny of degree $p$. Let $G_{\lambda,1} := \text{Spec}(R[T]/P_{\lambda,1}(T))$ be its kernel. It is a commutative finite flat group scheme over $R$ of order $p$. It is easy to generalize this definition to obtain some group schemes of order $p^n$, the so called $G_{\lambda,n}$. We observe that $\alpha^{(\lambda)}$ is compatible with $\psi_{\lambda,1}$, i.e. the following diagram

$$
\begin{array}{ccc}
G^{(\lambda)} & \xrightarrow{\alpha^{(\lambda)}} & G_{m,S} \\
\downarrow{\psi_{\lambda,1}} & & \downarrow{\rho} \\
G^{(\lambda^p)} & \xrightarrow{\alpha^{(\lambda)}} & G_{m,S}
\end{array}
$$

is commutative. Then it induces a morphism $\alpha^{(\lambda)} : G_{\lambda,1} \to \mu_{p,S}$

which is an isomorphism on the generic fiber. And if $v(\lambda) = 0$ it is an isomorphism. Moreover it is possible to prove that

$$(G_{\lambda,1})_k \simeq \alpha_{p,k} \quad \text{if char}(R) = 0 \text{ and } v(p) > (p - 1)v(\lambda) > 0, \text{ or char}(R) = p \text{ and } v(\lambda) > 0;$$

$G_{\lambda,1}$ étale if char$(R) = 0$ and $(p - 1)v(\lambda) = v(p)$.

We also recall that

$$(4) \quad \text{Hom}_{S-gr}(G_{\lambda,1}, G_{\lambda',1}) \simeq \text{Hom}_{S-gr}(G^{(\lambda)}, G^{(\lambda')}) \simeq \begin{cases} 
\mathbb{Z}/p\mathbb{Z}, & \text{if } v(\lambda) \geq v(\lambda') \\
0, & \text{if } v(\lambda) < v(\lambda').
\end{cases}$$
If \( v(\lambda) \geq v(\lambda') \) the morphisms are given by
\[
\sigma_j : G_{\lambda,1} \longrightarrow G_{\lambda',1}
\]
\[
T \mapsto (1 + \lambda T)^{\frac{1}{\lambda'}} - 1.
\]
It follows easily that \( G_{\lambda,1} \simeq G_{\lambda',1} \) if and only if \( v(\lambda) = v(\lambda') \).

In the following, if \( \text{char}(R) = 0 \), any time we will speak about \( G_{\lambda,1} \) it will be assumed that \((p - 1)v(\lambda) \leq v(p)\). If \( R \) contains a distinguished primitive \( p \)-th root of unity \( \zeta_1 \) then, since \( v(p) = (p - 1)v(\lambda(1)) \), the above condition is equivalent to \( v(\lambda) \leq v(\lambda(1)) \). We recall that by definition \( \lambda(1) := \zeta_1 - 1 \).

1.2. Classification of models of \( \mu_{p,K} \).

**Definition 1.1.** Let \( H \) be a group scheme over \( K \). Any flat \( R \)-group scheme \( G \) such that \( G_K \simeq H \) is called a model of \( H \). If \( H \) is finite over \( K \) we also require \( G \) finite over \( R \).

**Remark 1.2.** It has been proved by Waterhoue and Weisfeiler, in [26, 2.5], that any smooth model of \( G_{\mu,1} \) isomorphic to \( G^{(\lambda)} \) for some \( \lambda \in R \setminus \{0\} \).

It follows from [26, 2.4] that \( \mu_{p,S} \) is the minimal model for \( \mu_{p,K} \), i.e. for any model \( G \) of \( \mu_{p,K} \) we have a model map, namely a morphism which is an isomorphism on the generic fiber,
\[
G \longrightarrow \mu_{p,S}.
\]
We now recall the classification of \( \mu_{p,K} \)-models, which is due to Waterhouse-Weisfeiler ([26, §2]).

**Proposition 1.3.** If \( G \) is a finite and flat \( R \)-group scheme such that \( G_K \simeq \mu_{p,K} \) then \( G \simeq G_{\lambda,1} \) for some \( \lambda \in R \setminus \{0\} \).

**Proof.** As remarked above we have an \( R \)-model map \( \varphi : G \longrightarrow \mu_{p,S} \). By [26, 1.4] it is a composition of a finite number of Néron blow-ups. Now, let us consider the group scheme \( G_{\mu,1} \). The trivial subgroup scheme \( H = e \) is the unique subgroup \( H \) of \( (G_{\mu,1})_k \) which gives a nontrivial blow-up. If \( G_{\mu,1} \) is not étale, which means \( \text{char}(R) = p \) or \( \text{char}(R) = 0 \) and \((p - 1)v(\mu) < v(p)\), it is easy to see that the Néron blow up of \( G_{\mu,1} \) at \( e \) is \( G_{\mu_{p,1}} \). In the étale case we obtain a not finite group scheme. So \( G \simeq G_{\lambda,1} \) with \( v(\lambda) \) equal to the number of (nontrivial) blow-ups from \( \mu_{p,S} \). □

We observe that if \( \text{char}(R) = 0 \) (resp. \( \text{char}(R) = p \)) there are a finite (resp. infinite) number of models of \( \mu_{p,K} \).

2. Preliminaries

The main point to classify models of \( \mu_{p,K} \) will be the study of extensions of \( G_{\mu,1} \) by \( G_{\lambda,1} \). As we will see in 3.1 any such an extension is abelian. Therefore, for simplicity, we restrict to study abelian extensions. Hence in the following, for any scheme \( X \) and \( H \) flat commutative group schemes over \( X \), we define \( \text{Ext}^1_X(G, H) \) as the right derived functor of \( H \longrightarrow \text{Hom}_{X, -gr}(G, H) \) on the abelian fppf-sheaves over \( X \) (see [2]). We remark that if \( H \) is affine over \( X \) then any element of \( \text{Ext}^1_X(G, H) \) is representable by a flat commutative group scheme over \( X \) (see [11, III 17.4]).

In this section we recall some results about some groups of extensions which will play a key role in the next section.

2.1. Sekiguchi-Suwa Theory. We here briefly recall some results about \( \text{Ext}^1_S(G^{(\mu)}, G^{(\lambda)}) \) with \( \mu, \lambda \in R \setminus \{0\} \). We now distinguish four cases:

\[
\begin{align*}
\text{Ext}^1_S(G_{m,S}, G_{m,S}) ; \\
\text{Ext}^1_S(G^{(\mu)}, G_{m,S}) , & \quad \text{with } \mu \in \pi R \setminus \{0\} ; \\
\text{Ext}^1_S(G_{m,S}, G^{(\lambda)}) , & \quad \text{with } \lambda \in \pi R \setminus \{0\} ; \\
\text{Ext}^1_S(G^{(\mu)}, G^{(\lambda)}) , & \quad \text{with } \mu, \lambda \in \pi R \setminus \{0\} .
\end{align*}
\]
The first three groups are trivial, see for instance [19, I 2.7, II 1.4]. The fourth case is more subtle and it has been studied in [14] and [22]. We now suppose $\mu, \lambda \in \pi R \setminus \{0\}$. For any $\chi' \in \pi R \setminus \{0\}$ set $S_{\chi'} = \text{Spec}(R/\chi' R)$. Let us now consider the exact sequence on the fppf site over $S$

$$0 \to \mathcal{G}^{(\lambda)} \overset{\alpha}{\to} \mathcal{G}_{m,S} \overset{i^*_\lambda}{\to} \mathcal{G}_{m, S_{\chi'}} \to 0,$$

where $i^*_\lambda$ denotes the closed immersion $S_{\chi'} := \text{Spec}(R/\lambda R) \hookrightarrow S$ (see [16, II 1.1]). We observe that by definitions we have that

$$\text{Hom}_{S_{\chi'}}(\mathcal{G}^{(\mu)}, \mathcal{G}_{m, S_{\chi'}}) = \{ F(T) \in (R/\lambda R)[T]^* | F(U)F(V) = F(U + V + \mu UV) \}.$$ 

Here there is the abuse of notation we mentioned in the introduction. Indeed if it is clear from the context, for a group scheme $G$ over $R$ we sometimes simply write $G$ to indicate its restriction over $S_{\chi'}$, instead of writing $i^*_\lambda G$.

If we apply the functor $\text{Hom}_{S_{\chi'}}(\mathcal{G}^{(\mu)}, \cdot)$ to the sequence (5) we obtain, in particular, a morphism

$$\text{Hom}_{S_{\chi'}}(\mathcal{G}^{(\mu)}, \mathcal{G}_{m, S_{\chi'}}) \overset{\alpha}{\to} \text{Ext}^1_{S_{\chi'}}(\mathcal{G}^{(\mu)}, \mathcal{G}^{(\lambda)}),$$

given by

$$F \mapsto [E(\mu, \lambda; \tilde{F})],$$

where $\tilde{F}(T) \in R[T]$ is a lifting of $F(T)$ and $E(\mu, \lambda; \tilde{F})$ is an $R$-smooth affine commutative group defined as follows:

$$E(\mu, \lambda; \tilde{F}) := \text{Spec}(R[T_1, T_2, \frac{1}{1 + \mu T_1}, \frac{1}{\lambda (\tilde{F}(T_1) + \lambda T_2)}]).$$

(1) comultiplication

$$T_1 \mapsto T_1 \otimes 1 + 1 \otimes T_1 + \mu T_1 \otimes T_1$$

$$T_2 \mapsto T_2 \otimes \tilde{F}(T_1) + \tilde{F}(T_1) \otimes T_2 + \lambda T_2 \otimes T_2 + \lambda \tilde{F}(T_1) \otimes \tilde{F}(T_1) - \tilde{F}(T_1 \otimes 1 + 1 \otimes T_1 + \mu T_1 \otimes T_1).$$

(2) counit

$$T_1 \mapsto 0$$

$$T_2 \mapsto \frac{1 - \tilde{F}(0)}{\lambda}$$

(3) coinverse

$$T_1 \mapsto - \frac{T_1}{1 + \mu T_1}$$

$$T_2 \mapsto \frac{1}{\lambda (\tilde{F}(T_1) + \lambda T_2)} - \frac{T_2}{\lambda (1 + \mu T_1)}$$

We moreover define the following homomorphisms of group schemes

$$\mathcal{G}^{(\lambda)} = \text{Spec}(R[T, (1 + \lambda T)^{-1}]) \to E(\mu, \lambda; \tilde{F})$$

by

$$T_1 \mapsto 0$$

$$T_2 \mapsto T + \frac{1 - \tilde{F}(0)}{\lambda}$$

and

$$E(\mu, \lambda; \tilde{F}) \to \mathcal{G}^{(\mu)} = \text{Spec}(R[T, \frac{1}{1 + \mu T}])$$

by

$$T \mapsto T_1.$$
It is easy to see that

$$0 \longrightarrow G^{(\lambda)} \longrightarrow \mathcal{E}^{(\mu, \lambda, F)} \longrightarrow G^{(\mu)} \longrightarrow 0$$

is exact. A different choice of the lifting $\tilde{F}(T)$ gives an isomorphic extension. We recall the following theorem.

**Theorem 2.1.** For any $\mu, \lambda \in \pi R \setminus \{0\}$, the map

$$\alpha : \text{Hom}_{S_{A_{g}}}^{\lambda}(G^{(\mu)}, G_{m, S_{A_{g}}}) \longrightarrow \text{Ext}_{S}^{1}(G^{(\mu)}, G^{(\lambda)})$$

is a surjective morphism of groups. And $\ker(\alpha)$ is generated by the class of $1 + \mu T$.

**Proof.** [14, §3]. □

**Remark 2.2.** The group scheme $\mathcal{E}^{(\mu, \lambda; 1)}$ is well defined also in the case $v(\mu) = 0$ or $v(\lambda) = 0$ and it is isomorphic to $G^{(\mu)} \times G^{(\lambda)}$.

We now define some spaces used by Sekiguchi and Suwa to describe $\tilde{\mathcal{W}}^{2}$.

**Remark 2.3.** For any ring $A$, let $W(A)$ be the ring of infinite Witt vectors. We define

$$\tilde{W}(A) = \left\{ (a_{0}, \ldots, a_{n}, \ldots) \in W(A) : \begin{array}{l}
  a_{i} \text{ is nilpotent for any } i \\
  a_{i} = 0 \text{ for all but a finite number of } i
\end{array} \right\}.$$ 

We recall the definition of the so-called Witt-polynomial: for any $r \geq 0$ it is

$$\Phi_{r}(T_{0}, \ldots, T_{r}) = T_{0}^{\frac{r}{2}} + pT_{1}^{r-1} + \cdots + p^{r}T_{r}.$$ 

Then the following maps are defined:

- **Verschiebung**

  $$V : W(A) \longrightarrow W(A)$$

  $$(a_{0}, \ldots, a_{n}, \ldots) \longmapsto (0, a_{0}, \ldots, a_{n}, \ldots)$$

- **Frobenius**

  $$F : W(A) \longrightarrow W(A)$$

  $$(a_{0}, \ldots, a_{n}, \ldots) \longmapsto (F_{0}(T_{0}), F_{1}(T_{0}), \ldots, F_{n}(T_{0}), \ldots),$$

  where the polynomials $F_{r}(T_{0}) = F_{r}(T_{0}, \ldots, T_{r}) \in \mathbb{Q}[T_{0}, \ldots, T_{r+1}]$ are defined inductively by

  $$\Phi_{r}(F_{0}(T_{0}), F_{1}(T_{0}), \ldots, F_{r}(T_{0})) = \Phi_{r+1}(T_{0}, \ldots, T_{r+1}).$$

If $p = 0 \in A$ then $F$ is the usual Frobenius. The ideal $\tilde{W}(A)$ is stable with respect to these maps.

For any morphism $G : \tilde{W}(A) \longrightarrow \tilde{W}(A)$ we will set $\tilde{W}(A)^{G} := \ker G$. And for any $a \in A$ we denote the element $(a, 0, 0, \ldots, 0, \ldots) \in W(A)$ by $[a]$. It is called the Teichmüller representant of $a$.

The following lemma will be useful later.

**Lemma 2.4.** Let $\lambda \in \pi R \setminus \{0\}$. If $\text{char}(R) = 0$ we suppose $(p - 1)v(\lambda) \leq pv(p)$. For any $a = (a_{0}, a_{1}, \ldots), b = (b_{0}, b_{1}, \ldots) \in \tilde{W}(R/\lambda R)^{F}$ then

$$a + b = (a_{0} + b_{0}, a_{1} + b_{1}, \ldots, a_{n} + b_{n}, \ldots)$$

**Proof.** We suppose that $a + b = (c_{0}, c_{1}, \ldots, c_{n}, \ldots)$. If $U_{t}$ and $V_{t}$ have weight $p^{t}$ then we have that $c_{r}(U, V)$ is isobaric of weight $p^{r}$. By definition of sum between Witt vectors we have

$$c_{r}(a, b) = a_{r} + b_{r} + c_{r}'((a_{0}, a_{1}, \ldots, a_{r-1}),(b_{0}, b_{1}, \ldots, b_{r-1}))$$

for some polynomial $c_{r}'(U_{0}, \ldots, U_{r-1}, V_{0}, \ldots, V_{r-1})$ isobaric of weight $p^{r}$. Hence $\deg(c_{r}') \geq p$.

Let $\tilde{a} = (\tilde{a}_{0}, \tilde{a}_{1}, \ldots), \tilde{b} = (\tilde{b}_{0}, \tilde{b}_{1}, \ldots) \in \tilde{W}(R)$ be liftings of $a$ and $b$, respectively. For any $r \geq 1$, up to changing $\tilde{a}$ with $\tilde{b}$, we can suppose that $v(\tilde{a}_{k}) = \min\{v(\tilde{a}_{i}), v(\tilde{b}_{i})| i = 0, \ldots, r - 1\}$, for some $0 \leq k \leq r - 1$. Since $\deg(c_{r}') \geq p$ then $v(c_{r}'(\tilde{a}, \tilde{b})) \geq pv(\tilde{a}_{k})$. We claim that, since $F(a) = 0$, then $pv(\tilde{a}_{k}) \geq v(\lambda)$. If $\text{char}(R) = p$ this is clear. We now suppose $\text{char}(R) = 0$. From [22, 1.2.1] we have
that \( pv(\tilde{a}_k) \geq \min\{v(p) + v(\tilde{a}_k), v(\lambda)\} \). This implies, by hypothesis on \( v(\lambda) \), \( pv(\tilde{a}_k) \geq v(\lambda) \). Hence \( c'_i(a, b) = 0 \in R/\lambda R \). So
\[
a + b = (a_0 + b_0, a_1 + b_1, \ldots, a_i + b_i, \ldots)
\]
□

We now recall the definition of the Artin-Hasse exponential series
\[
E_p(T) := \exp \left( \sum_{r \geq 0} \frac{T^{p^r}}{p^r} \right) = \prod_{r=0}^{\infty} \exp \left( \frac{T^{p^r}}{p^r} \right) \in \mathbb{Z}(p)[[T]].
\]
Sekiguchi and Suwa introduced a deformation of the Artin-Hasse exponential map in [22]. They defined \( E_p(U, \Lambda; T) \in \mathbb{Q}[U, \Lambda][[T]] \) by
\[
E_p(U, \Lambda; T) := (1 + \Lambda T) \prod_{r=1}^{\infty} (1 + \Lambda^{p^r} T^{p^r}) \left( \left( \frac{p}{\Lambda} \right)^{p^r} - (\frac{p}{\Lambda})^{p^r - 1} \right).
\]
They proved that \( E_p(U, \Lambda; T) \) has in fact its coefficients in \( \mathbb{Z}_p(U, \Lambda) \). It is possible to show ([22, 2.4]) that
\[
E_p(U, \Lambda; T) = \begin{cases} 
\prod_{(i,p)=1} E_p(U \Lambda^{i-1} T^{i-1})^{-1}, & \text{if } p > 2; \\
\prod_{(i,2)=1} E_p(U \Lambda^{i-1} T^{i})^{\frac{1}{2}} \left[ \prod_{(i,2)=1} E_p(U \Lambda^{2i-1} T^{2i}) \right]^{-1}, & \text{if } p = 2.
\end{cases}
\]
Let \( A \) be a \( \mathbb{Z}_p \)-algebra and \( a, \mu \in A \). We define \( E_p(a, \mu; T) \) as \( E_p(U, \Lambda; T) \) evaluated at \( U = a \) and \( \Lambda = \mu \).

If \( a = (a_0, a_1, a_2, \ldots) \in W(A) \) we define the formal power series
\[
E_p(a, \mu; T) = \prod_{k=0}^{\infty} E_p(a_k, \mu^{p^k}; T^{p^k}).
\]
We stress that if \( \mu \) is nilpotent in \( A \) then \( a \in \hat{W}(A) \) if and only if \( E_p(a, \mu; T) \) is a polynomial (see [22, 2.11]). The following result gives an explicit description of \( \text{Hom}_{\mathbb{S}_\lambda}^{-gr}(\mathcal{G}(\mu), \mathbb{G}_m, \mathbb{S}_\lambda) \).

**Theorem 2.5.** Let \( \mu, \lambda \in \pi R \setminus \{0\} \). The homomorphism
\[
\xi^0_{R/\lambda R} : \hat{W}(R/\lambda R)^{F - [\mu^{-1}]} \rightarrow \text{Hom}_{\mathbb{S}_\lambda}^{-gr}(\mathcal{G}(\mu), \mathbb{G}_m, \mathbb{S}_\lambda)
\]
\[
a \mapsto E_p(a, \mu; T)
\]
is bijective.

**Proof.** It is a particular case of [22, 2.19.1]. □

Moreover 2.1 and 2.5 give the following:

**Corollary 2.6.** For any \( \mu, \lambda \in \pi R \setminus \{0\} \) the map
\[
\alpha \circ \xi^0_{R/\lambda R} : \hat{W}(R/\lambda R)^{F - [\mu^{-1}]} / (1 + \mu T) \rightarrow \text{Ext}^1_{\mathbb{S}_\lambda}(\mathcal{G}(\mu), \mathcal{G}(\lambda))
\]
\[
a \mapsto [\mathcal{E}(\mu, \lambda, F)],
\]
where \( \hat{F}(T) \in R[T] \) is a lifting of \( E_p(a, \mu; T) \), is an isomorphism.

We now describe some natural maps through these identifications. Let \( \mu, \lambda \in R \setminus \{0\} \) and, if \( \text{char}(R) = 0 \), let us assume that \( (p - 1) \mu \leq v(p) \). Consider the isogeny
\[
\psi_{\mu, 1} : \mathcal{G}(\mu) \rightarrow \mathcal{G}(\mu^p).
\]
Let us first suppose that \( p > 2 \). Then we have, by [22, 1.4.1 and 3.8)], that, if \( p^2 \equiv 0 \pmod{\lambda} \),
\[
\psi_{\mu, 1}^* : \text{Hom}_{\mathbb{S}_\lambda}^{-gr}(\mathcal{G}(\mu^p), \mathbb{G}_m, \mathbb{S}_\lambda) \rightarrow \text{Hom}_{\mathbb{S}_\lambda}^{-gr}(\mathcal{G}(\mu), \mathbb{G}_m, \mathbb{S}_\lambda)
\]
is given by
\[ a \mapsto \frac{p}{\mu^{p-1}} a + V(a), \quad \text{if } \text{char}(R) = 0 \text{ and} \]
\[ a \mapsto V(a), \quad \text{if } \text{char}(R) = p. \]

For \( p = 2 \) the situation is slightly different. Let us define a variant of the Verschiebung as follows. Define polynomials
\[ \tilde{V}_r(T) = \tilde{V}_r(T_0, \ldots, T_r) \in \mathbb{Q}[T_0, \ldots, T_r] \]
inductively by \( \tilde{V}_0 = 0 \) and
\[ \Phi_r(\tilde{V}_0(T), \ldots, \tilde{V}_r(T)) = 2^{2r} \Phi_{r-1}(T_0, \ldots, T_{r-1}) \]
for \( r \geq 1 \). It has been proved in [22, 1.4.1] that, if \( a = (a_0, a_1, \ldots) \), then
\[ \tilde{V}(a) \equiv (0, 2a_0, 2a_0^2, \ldots, 2a_0^{2^{r-1}}, \ldots) \mod 2^2 \]
In particular \( \tilde{V}(a) \equiv 0 \mod 2 \). Then we have, by [22, 3.8], that (with possibly \( 2^2 \not\equiv 0 \mod \lambda \))
\[ \psi_{\mu,1} : \text{Hom}_{S_{\lambda}}(G^{(\mu)}, \mathbb{G}_{m,S_{\lambda}}) \to \text{Hom}_{S_{\lambda}}(G^{(\mu)}, \mathbb{G}_{m,S_{\lambda}}) \]
is given by
\[ a \mapsto \frac{2}{\mu} a + \tilde{V}(a), \quad \text{if } \text{char}(R) = 0 \text{ and} \]
\[ a \mapsto V(a), \quad \text{if } \text{char}(R) = p. \]

We now come back to the general case \( p \geq 2 \). Consider the morphism
\[ p : \text{Hom}_{S_{\lambda}}(G^{(\mu)}, \mathbb{G}_{m,S_{\lambda}}) \to \text{Hom}_{S_{\lambda}}(G^{(\mu)}, \mathbb{G}_{m,S_{\lambda}}) \]
\[ F(T) \to F(T)^p. \]
We observe that we have
\[ \psi_{\lambda,1} \circ \alpha = \alpha \circ p. \]
Let \( a \in (\overline{W}(R/\mathbb{Q}_p))^F - [\mu^{p-1}] \) and take any lifting \( \tilde{a} \in W(R) \). Using the identifications of 2.5 the morphism \( p \) above is given by
\[ a \mapsto p\tilde{a} \]
(see [22, 4.6]). We will sometimes simply write \( pa \).

2.2. Interpretation of \( \text{Ext}^1_S(G_{\mu,1}, \mathbb{G}_{m,S}) \). First of all, we briefly recall a useful spectral sequence. Let \( G, H \) be flat and commutative \( S \)-group schemes and let \( \text{Ext}^i_S(G, H) \) denote the fppf-sheaf on \( \text{Sch}_S \), associated to the presheaf \( X \to \text{Ext}^i_S(G \times_S X, H \times_S X) \). Then we have a spectral sequence
\[ E_2^{i,j} = H^i(S, \text{Ext}^j_S(G, H)) \Rightarrow \text{Ext}^{i+j}_S(G, H), \]
which in low degrees gives
\[ 0 \to H^1(S, \text{Hom}_{S_{\lambda}}(G, H)) \to \text{Ext}^1_S(G, H) \to H^0(S, \text{Ext}^1_S(G, H)) \to \]
\[ \to H^2(S, \text{Hom}_{S_{\lambda}}(G, H)) \to \text{Ext}^2_S(G, H). \]
Moreover \( H^1(S, \text{Hom}_{S_{\lambda}}(G, H)) \) is isomorphic to the subgroup of \( \text{Ext}^1_S(G, H) \) formed by the extensions \( E \) which split over some faithfully flat affine \( S \)-scheme of finite type (cf. [4, III §6, 3.6]). We will consider the case \( H = \mathbb{G}_{m,S} \) and \( G \) a finite flat group scheme. In this case, \( \text{Hom}_{S_{\lambda}}(G, \mathbb{G}_{m,S}) \) is the Cartier dual of \( G \), denoted by \( G^\vee \). Since, by [12, 6.2.2], \( \text{Ext}^1_S(G, \mathbb{G}_{m,S}) = 0 \) then we obtain that the natural morphism
\[ H^1(S, G^\vee) \to \text{Ext}^1_S(G, \mathbb{G}_{m,S}) \]
of (11) is in fact an isomorphism.
Proposition 2.7. Let $X$ be a normal integral scheme. For any finite and flat commutative group scheme $G$ over $X$,

$$i^*: H^1(X, G) \longrightarrow H^1(\text{Spec}(K(X)), G_{K(X)})$$

is injective, where $i: \text{Spec}(K(X)) \longrightarrow X$ is the generic point.

Proof. A sketch of the proof has been suggested to us by F. Andreatta. We recall that for any commutative group scheme $G'$ over a scheme $X'$ we have that $H^1(X', G')$ is a group and it classifies the $G'$-torsors over $X'$. Suppose there exists a $G$-torsor $f : Y \longrightarrow X$ such that $i^* f : i^* Y \longrightarrow \text{Spec}(K(X))$ is trivial. This means there exists a section $s$ of $i^* f$. We consider the scheme $Y_0$ which is the schematic closure of $s(\text{Spec}(K(X)))$ in $Y$. Then $f_{Y_0} : Y_0 \longrightarrow X$ is a finite birational morphism with $X$ a normal integral scheme. So, by Zariski’s Main Theorem ([10, 4.4.6]), we have that $f_{Y_0}$ is an open immersion, and so it is an isomorphism. So we have a section of $f$ and $Y$ is a trivial $G$-torsor. □

Remark 2.8. The hypothesis $G$ finite and $X$ normal are necessary. For the first it is sufficient to observe that any $G_{m,X}$-torsor is trivial on $\text{Spec}(K(X))$. For the second one, consider $X = \text{Spec}(k[x, y]/(x^p - y^{p+1})) = \text{Spec}(A)$, with $k$ any field of characteristic $p > 0$, and $Y$ the $\alpha_p$-torsor $\text{Spec}(A[T]/(T^p - y))$. Generically this torsor is trivial since we have $y = (\frac{x}{p})^p$. But $Y$ is not trivial since $y$ is not a $p$-power in $A$.

We deduce the following result which however it will not be used in the sequel.

Corollary 2.9. Let $X$ be a normal integral scheme. Let $f : Y \longrightarrow X$ be a morphism with a rational section and let $g : G \longrightarrow G'$ be a map of finite and flat commutative group schemes over $X$, which is an isomorphism over $\text{Spec}(K(X))$. Then

$$f^*g_* : H^1(X, G) \longrightarrow H^1(Y, G'_Y)$$

is injective.

Proof. By hypothesis $\text{Spec}(K(X)) \longrightarrow X$ factors through $f : Y \longrightarrow X$. If $i : \text{Spec}(K(X)) \longrightarrow X$, we have

$$i_* : H^1(X, G) \xrightarrow{\beta} H^1(X, G') \xrightarrow{f} H^1(Y, G'_Y) \longrightarrow H^1(\text{Spec}(K(X)), G_{K(X)}).$$

Therefore, by the previous proposition, it follows that

$$H^1(X, G) \longrightarrow H^1(Y, G'_Y)$$

is injective. □

Remark 2.10. The previous corollary can be applied, for instance, to the case $f = \text{id}_X$ or to the case $f : U \longrightarrow X$ an open immersion and $g = \text{id}_G$. Roberts ([13, p. 692]) has proved the corollary in the case $f = \text{id}_X$, with $X = \text{Spec}(A)$ and $A$ the ring of integers in a local number field.

3. Models of $\mu_{p^2,K}$

First of all we prove that any model of $\mu_{p^2,K}$ is an abelian extension of $G_{\mu,1}$ by $G_{\lambda,1}$ for some $\mu, \lambda \in R \setminus \{0\}$.

Lemma 3.1. Let $G$ be a finite and flat $R$-group scheme of order $p^2$ such that $G_K$ is diagonalizable. Then $G$ is an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ for some $\mu, \lambda \in R \setminus \{0\}$. Moreover the generic fiber of any extension $G$ of $G_{\mu,1}$ by $G_{\lambda,1}$ is multiplicative. In particular $G$ is commutative.

Proof. We have that $G_{K}$ is isomorphic to $\mu_{p^2,K}$ or to $\mu_{p,K} \times \mu_{p,K}$. We consider the factorization

$$0 \longrightarrow \mu_{p,K} \longrightarrow G_K \longrightarrow \mu_{p,K} \longrightarrow 0.$$ 

We take the schematic closure $G_1$ of $\mu_{p,K}$ in $G$. Then $G_1$ is a model of $\mu_{p,K}$. So by 1.3 it follows that $G_1 \simeq G_{\lambda,1}$ for some $\lambda \in R \setminus \{0\}$. Moreover $G/G_{\lambda,1}$ is a model of $\mu_{p,K}$, too. So, again by 1.3, we have $G/G_{\lambda,1} \simeq G_{\mu,1}$ for some $\mu \in R \setminus \{0\}$. The first assertion is proved.

Now let $G$ be an extension of $G_{\mu,1}$ by $G_{\lambda,1}$. We recall that $G_K$ is of multiplicative type if and only if $G_{\lambda,1}$ is diagonalizable. If $\text{char}(R) = 0$ then $G_{\lambda,1}$ is a constant group of order $p^2$. Since $\overline{K}$ is algebraically closed, it is diagonalizable. If $\text{char}(R) = p$ then $G_{\lambda,1}$ is isomorphic to $\mu_{p,K} \times \mu_{p,K}$ or $\mu_{p^2,K}$ by [4, III §6, 8.7]. □
So it seems natural to study the group $\text{Ext}_S^1(G_{\mu,1}, G_{\lambda,1})$. We observe that by the previous lemma it follows that it is not restrictive to consider only abelian extensions, as supposed at the beginning of §2. Before beginning the study of $\text{Ext}_S^1(G_{\mu,1}, G_{\lambda,1})$ we prove the following lemma.

**Lemma 3.2.** Let $E$ be an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ with $\mu, \lambda \in R \setminus \{0\}$. If $E$ is a model of $\mu p^2, K$ then necessarily $v(\mu) \geq v(\lambda)$.

**Proof.** Let us consider the map $G_{\mu,1} \rightarrow G_{\lambda,1}$ induced by $G_{\mu,1}$. We observe that $G_{\mu,1}$ is trivial on $G_{\lambda,1}$, therefore we have an induced map $p : G_{\mu,1} \rightarrow E$. Since $E_k \simeq \mu p^2, K$ we have, on the generic fiber, the factorization

$$p : (G_{\mu,1})_K \rightarrow (G_{\lambda,1})_K \hookrightarrow E_K,$$

where the first map is an isomorphism. So $p$ induces a model map $G_{\mu,1} \rightarrow G_{\lambda,1}$, hence from (4) it follows $v(\mu) \geq v(\lambda)$. □

### 3.1. Two exact sequences

The main tools which we will use to calculate the extensions of $G_{\mu,1}$ by $G_{\lambda,1}$ are two exact sequences. We recall them in this subsection. See (13) and (16) below.

Applying the functor $\text{Ext}$ to the following exact sequence of group schemes

$$(\Lambda) : 0 \rightarrow G_{\lambda,1} \xrightarrow{i} G(\lambda) \xrightarrow{\delta_1} G(\lambda^p) \rightarrow 0,$$

we obtain

$$0 \rightarrow \text{Hom}_{S^{\text{gr}}}(G_{\mu,1}, G(\lambda^p)) \xrightarrow{\delta'} \text{Ext}_S^1(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{i_*} \text{Ext}_S^1(G_{\mu,1}, G(\lambda^p)).$$

We remark that $\delta'$ is injective by (4). The map

$$\delta' : \text{Hom}_{S^{\text{gr}}}(G_{\mu,1}, G(\lambda^p)) \rightarrow \text{Ext}_S^1(G_{\mu,1}, G_{\lambda,1})$$

is defined by

$$\sigma_j \mapsto (\sigma_j)^*(\Lambda),$$

where $(\sigma_j)^*(\Lambda)$ is the extension class of the group scheme

$$\mathcal{E}^{(\mu,\lambda;1,j)} := \text{Spec}(R[T_1, T_2]/((1 + \mu T_1)^p - 1, (1 + \lambda T_2)^p - (1 + \mu T_1)^j)/\lambda^p),$$

with the maps

$$G_{\lambda,1} \rightarrow \mathcal{E}^{(\mu,\lambda;1,j)}$$

$$T_1 \mapsto 0$$

$$T_2 \mapsto T$$

and

$$\mathcal{E}^{(\mu,\lambda;1,j)} \rightarrow G_{\mu,1}$$

$$T \mapsto T_1.$$
So we have, using (4),
\[
\ker \alpha^n \simeq \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n})/(1 + \mu T).
\]

In the following we give a more explicit description of the main ingredients of the exact sequences (13) and (16).

### 3.2. Explicit description of \( \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n}) \)

Let \( \lambda \in \pi R \setminus \{0\} \). First we consider the simplest case: by [4, II \( \S \) 1, 2.11]

\[
\text{Hom}_{S^n - \text{gr}}(\mu_{p^n}, \mathbb{G}_{m,S^n}) = \{ T^i \in (R/\lambda R)[T, 1/T]; i \in \mathbb{Z}/p\mathbb{Z} \}.
\]

Now we study \( \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n}) \) in general.

**Proposition 3.3.** Let \( \mu, \lambda \in \pi R \setminus \{0\} \). We suppose \( (p - 1)v(\mu) \leq v(p) \) if \( \text{char}(R) = 0 \).

(i) The map
\[
i^*: \text{Hom}_{S^n - \text{gr}}(G^{(\mu)}, \mathbb{G}_{m,S^n}) \rightarrow \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n})
\]
induced by
\[
i: G_{\mu,1} \rightarrow G^{(\mu)},
\]
is surjective.

Moreover, in the case \( \text{char}(R) = 0 \), we also assume \( (p - 1)v(\lambda) \leq v(p) \).

(ii) We have the following isomorphism of groups
\[
(G^{(\mu)}_R)_{p^n} : (R/\lambda R)^{F^n - \mu^{p^n - 1}} \rightarrow \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n})
given by
\[
\sigma \mapsto \sigma^{(p^n)}(a, \mu, T).
\]

Moreover \( E_p(a, \mu, T) = 1 + \sum_{i=1}^{p-1} (\lambda - \mu) T^i \).

(iii) The restriction map
\[
i^*: \text{Hom}_{S^n - \text{gr}}(G^{(\mu)}, \mathbb{G}_{m,S^n}) \simeq \text{Witt}(R/\lambda R)^{F^n - \mu^{p^n - 1}} \rightarrow \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n}) \simeq (R/\lambda R)^{F^n - \mu^{p^n - 1}}
is given, in terms of Witt vectors, by
\[
a = (a_0, a_1, \ldots, 0, 0, 0, \ldots) \mapsto a_0 + \sum_{j=1}^{\infty} (-1)^j j! \left( \prod_{i=0}^{j-1} (\frac{p}{\mu^{p^n - 1}})^{j_1} \right) a_j
\]
if \( \text{char}(R) = 0 \),
\[
a = (a_0, a_1, \ldots, 0, 0, 0, \ldots) \mapsto a_0
\]
if \( \text{char}(R) = p \).

**Remark 3.4.** If \( \text{char}(R) = 0 \) and \( v(\lambda) \leq v(p) - (p - 1)v(\mu) \) then \( i^*(a) = a_0 \).

**Proof.** (i) An element of \( \text{Hom}_{S^n - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,S^n}) \) could be represented by a polynomial
\[
F(T) = \sum_{i=0}^{p-1} a_i T^i \in (R/\lambda R)[T]
\]
such that
\[
\text{(a)} \quad F(U)F(V) \equiv F(U + V + \mu UV) \mod \left( \frac{(1+U)^{p^n - 1}}{p}, \frac{(1+V)^{p^n - 1}}{p} \right)
\]
\[
\text{(b)} \quad F(T) \text{ is invertible in } (R/\lambda R)[T]/\left( \frac{(1+U)^{p^n - 1}}{p} \right).
\]
The condition (a) implies that \( F(U)F(V) = F(U + V + \mu UV) \), since all terms of \( F(U)F(V) \) and \( F(U + V + \mu UV) \) have degree at most \( p - 1 \) in \( U \) and \( V \). Moreover the finite group scheme \( (G_{\mu,1})_k \) is a closed subgroup scheme of \( G_{a,k} \) and therefore the Cartier dual \( (G_{\mu,1}^\vee)_k \) is infinitesimal. In particular, we have \( \text{Hom}_{k - \text{gr}}((G_{\mu,1})_k, \mathbb{G}_{m,k}) = \text{Hom}_{k - \text{gr}}(G_{\mu,1}, \mathbb{G}_{m,k}) = \{0\} \). Then \( F(T) \equiv 1 \mod \pi \). This implies that \( F(T) \) is invertible in \( (R/\lambda R)[T] \), since \( \pi \) is nilpotent in \( R/\lambda R \). Therefore we have that \( F(T) \) represents an element of \( \text{Hom}_{S^n - \text{gr}}(G^{(\mu)}, \mathbb{G}_{m,S^n}) \) and the map \( i^* \) is surjective.

(ii) By the exact sequence
\[
0 \rightarrow G_{\mu,1} \rightarrow G^{(\mu)} \xrightarrow{\psi_{\mu,1}} G^{(\mu)} \rightarrow 0
\]
over $S_{\lambda}$, we have the long exact sequence of cohomology

$$0 \rightarrow \text{Hom}_{S_{\lambda}-gr}(G^{(\mu)}, G_{m,S_{\lambda}}) \xrightarrow{\psi_{0,1}^{-1}} \text{Hom}_{S_{\lambda}-gr}(G^{(\nu)}, G_{m,S_{\lambda}}) \xrightarrow{i^*} \text{Hom}_{S_{\lambda}-gr}(G^{(\mu)}, G_{m,S_{\lambda}}) \rightarrow \ldots$$

We study separately three cases. From 2.4 we have that the restriction of the Teichmüller map

$$T : (R/\lambda R)^F \rightarrow \widehat{W}(R/\lambda R)^F,$$

given by

$$a \mapsto [a],$$

is a morphism of groups. Moreover, if we consider the isomorphism

$$\xi^{0}_{R/\lambda R} : \widehat{W}(R/\lambda R)^F \rightarrow \text{Hom}_{S_{\lambda}-gr}(G^{(\mu)}, G_{m,S_{\lambda}}),$$

we have

$$i^* \circ \xi^{0}_{R/\lambda R} \circ T = (\xi^{0}_{R/\lambda R})_p.$$

So $(\xi^{0}_{R/\lambda R})_p$ is a morphism of groups. We now prove that it is surjective.

As remarked in (i) any element of $\text{Hom}_{S_{\lambda}-gr}(G^{(\mu)}, G_{m,S_{\lambda}})$ can be represented by a polynomial $F(T)$ of degree at most $p - 1$ with coefficients in $R/\lambda R$. By [18, 3.5, 3.7] it satisfies $F(U)F(V) = F(U + V + \mu UV)$ if and only if $F(T) = 1 + \sum_{i=1}^{p-1} \prod_{j=0}^{i-1}(a - k\mu)^i$ for some $a \in R/\lambda R$ such that $\prod_{i=0}^{p-1}(a - k\mu) = 0$. But, since $p$ and $\mu^{p-1}$ are zero in $R/\lambda R$, the last condition is equivalent to $a^p = 0$. On the other hand, since $a^p = \mu^p = 0 \in R/\lambda R$, then $a^i\mu^{p-i} = 0$ for any $i = 0, \ldots, p$. Therefore by (6), $E_p(a, \mu; T)$ is a polynomial of degree at most $p - 1$. Moreover it satisfies $E_p(a, \mu; U)E_p(a, \mu; V) = E_p(a, \mu; U + V + \mu UV)$ and the coefficient of $T$ in $E_p(a, \mu; T)$ is $a$. Hence $E_p(a, \mu; T) = 1 + \sum_{i=0}^{p-1} \prod_{j=0}^{i}(a - k\mu)^i = F(T)$.

We now prove that $(\xi^{0}_{R/\lambda R})_p$ is injective. By (7), (8), (9), and (19) its kernel is

$$T((R/\lambda R)^F) \cap \left\{ \left[ \frac{p}{\mu^{p-1}}b + V(b); b \in \widehat{W}(R/\lambda R)^F \right] \right\}.$$

Let us now suppose that there exist $b = (b_0, b_1, \ldots) \in \widehat{W}(R/\lambda R)^F$ and $a \in (R/\lambda R)^F$ such that $\left[ \frac{p}{\mu^{p-1}}b + V(b) \right] = [a]$. It follows by the definition of Witt vector ring that

$$[\frac{p}{\mu^{p-1}}b] = \left[ \frac{p}{\mu^{p-1}}b_0, \ldots, (\frac{p}{\mu^{p-1}})^j b_j, \ldots \right],$$

and

$$[a] - V(b) = (a, -b_0, -b_1, \ldots).$$

Since $b \in \widehat{W}(R/\lambda R)$, there exists $r \geq 0$ such that $b_j = 0$ for any $j \geq r$. Moreover, comparing (20) and (21), it follows

$$\left( \frac{p}{\mu^{p-1}} \right)^{j+1} b_{j+1} = -b_j$$

for $j \geq 0$.

$$\frac{p}{\mu^{p-1}}b_0 = a.$$

Hence $b_j = a = 0$ for any $j \geq 0$. It follows that $(\xi^{0}_{R/\lambda R})_p$ is injective.

So if $\text{char}(R) = p$, or $\text{char}(R) = 0$, $p \geq 2$ and $(p - 1)v(\mu) < v(\lambda)$, since $p = 0 \in R/\lambda R$ then $F V = V F$. Hence, using the fact that $V$ is injective, it is straightforward to prove that

$$V^{-1}(\widehat{W}(R/\lambda R)^F - [\mu^{p-1}]) = \widehat{W}(R/\lambda R)^F - [\mu^{p-1}].$$

Moreover if $\text{char}(R) = 0$ then

$$v(p) - (p - 1)v(\mu) > v(p) - v(\lambda) \geq (p - 2)v(\lambda) \geq v(\lambda).$$
So it follows, using also (7), (9) and (19), that

\[(R/\lambda R)^{F-|\mu|} \to \widetilde{W}(R/\lambda R)^{F-|\mu|} / \{V(b) : b \in \widetilde{W}(R/\lambda R)^{F-|\mu|}\} \to \text{Hom}_{S_{\lambda^{-gr}}}(G_{\mu,1}, G_{m, S_{\lambda}})\]

are isomorphisms, where the first morphism is induced by the Teichmüller map, and the second one by \(\xi_{R/\lambda R}^0\). The composition is nothing else but \((\xi_{R/\lambda R}^0)^p\).

Now let us consider \(E_p(a, \mu; T) \in \text{Hom}_{S_{\lambda^{-gr}}}(G_{\mu,1}, G_{m, S_{\lambda}})\). We observe that

\[T^p = 0 \in (R/\lambda R[T])/(1 + \mu T)^p - 1.\]

If \(\text{char}(R) = p\) this is clear. While if \(\text{char}(R) = 0\) then \(T^p = -\frac{p}{\mu^p}T + (\frac{p}{\mu^p})^2 T^2 + \cdots + (\frac{p}{\mu^p})^{p-1} \cdot T^{p-1}\).

Therefore \(E_p(a, \mu; T)\) is a polynomial of degree \(\mu - 1\). Reasoning as in the previous case we have that \(E_p(a, \mu; T) = 1 + \sum_{i=1}^{\mu-1} \prod_{j=1}^{i-1} (a-k\mu) T^i = F(T)\).

Since \(T^2 = -\frac{p}{\mu} T \in R[G_{\mu,1}]\), it follows that \(\mu T^2 = 0 \in R/\lambda R[G_{\mu,1}]\). So \(a^2 = \mu a \in R/\lambda R\) implies, by (6),

\[E_p(a, \mu; T) = E_p(a T) = 1 + a T \in \text{Hom}_{S_{\lambda^{-gr}}}(G_{\mu,1}, G_{m, S_{\lambda}})\]

Thus \((\xi_{R/\lambda R}^0)_p(a) = 1 + a T\). We now prove that \((\xi_{R/\lambda R}^0)_p \) is a morphism of groups. Let \(a, b \in (R/\lambda R)^{F-|\mu|}\). We observe that, since \(v(\mu) < v(\lambda)\), this implies \(\mu | a^2\) and \(\mu | b^2\). Hence \(\mu | a b\). Then

\[(1 + a T)(1 + b T) = 1 + (a + b) T + a b T^2 = 1 + (a + b - \frac{2}{\mu} a b) T = 1 + (a + b) T.\]

It is easy to check that the above morphism is in fact an isomorphism.

(iii) First of all we remark that for any \(a = (a_0, a_1, \ldots) \in \widetilde{W}(R/\lambda R)^{F-|\mu|}\) we have

\[a = \sum_{j=0}^{\infty} V^j([a_j]).\]

It is clear that for any \(a \in R/\lambda R\) we have \(i^*([a]) = a\). While, by (7), (9) and (19), it follows that for any \(b \in \widetilde{W}(R/\lambda R)^{F-|\mu|}\)

\[i^*V(b) = -i^*\left(\frac{p}{\mu^{p-1}} b\right) \text{ if char}(R) = 0,\]

\[i^*V(b) = 0 \text{ if char}(R) = p.\]

Hence, if \(\text{char}(R) = 0\), \(i^*(V^j(b)) = (-1)^ji^*\left(\prod_{r=0}^{j-1} (\frac{p}{\mu^r}) b^r\right)\) for any \(j \geq 1\) and then it follows that

\[i^*(a) = i^* \left(\sum_{j=0}^{\infty} V^j([a_j])\right)\]

\[= \sum_{j=0}^{\infty} i^*(V^j([a_j]))\]

\[= a_0 + \sum_{j=1}^{\infty} (-1)^j \prod_{r=0}^{j-1} \left(\frac{p}{\mu^r}\right) a_j.\]

While, if \(\text{char}(R) = p\), we have

\[i^*(a) = i^* \left(\sum_{j=0}^{\infty} V^j([a_j])\right) = i^*([a_0]) = a_0.\]
3.3. Explicit description of $\delta$. Let $\mu, \lambda \in \pi R \setminus \{0\}$ and, if $\text{char}(R) = 0$, $(p - 1)v(\mu) \leq v(p)$. The map

$$
\delta : \text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S_{\lambda}}) \to \text{Ext}^1_{S}(G_{\mu,1}, G^{(\lambda)})
$$

can also be explicitly described. We have the following commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_{S_{\lambda}}(G^{(\mu)}, G_{m,S_{\lambda}}) & \xrightarrow{i^*} & \text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S_{\lambda}}) \\
\downarrow{\alpha} & & \downarrow{\delta} \\
\text{Ext}^1_{S}(G^{(\mu)}, G^{(\lambda)}) & \xrightarrow{i^*} & \text{Ext}^1_{S}(G_{\mu,1}, G^{(\lambda)})
\end{array}
$$

where the first horizontal map is surjective by 3.3(i). So, given

$$
F(T) \in \text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S_{\lambda}}),
$$

we can choose a representant in $\text{Hom}_{S_{\lambda}}(G^{(\mu)}, G_{m,S_{\lambda}})$ which we denote again by $F(T)$ for simplicity. Let $\tilde{F}(T) \in R[S]$ any its lifting. Then $\delta$ is defined by

$$
F(T) \mapsto [\tilde{E}^{(\mu,\lambda,F)}] := i^*(\{E^{(\mu,\lambda,F)}\} = i^*(\alpha(F(T))).
$$

We observe that $\tilde{E}^{(\mu,\lambda,F)}$ is the subgroup scheme of $E^{(\mu,\lambda,F)}$, defined as a scheme by

$$
\tilde{E}^{(\mu,\lambda,F)} = \text{Spec} \left(R[T_1, T_2, (\tilde{F}(T_1) + \lambda T_2)^{-1}/(1 + \mu T_1)p - 1]\right).
$$

The above extension does not depend on the choice of the lifting since the same is true for $[E^{(\mu,\lambda,F)}]$. We have therefore proved the following proposition.

**Proposition 3.5.** Let $\mu, \lambda \in \pi R \setminus \{0\}$ and, if $\text{char}(R) = 0$, $(p - 1)v(\mu) \leq v(p)$. Then $\delta$ induces an isomorphism

$$
\text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S_{\lambda}})/r_\lambda(\text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S})) \to \{[\tilde{E}^{(\mu,\lambda,F)}]; F \in \text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S_{\lambda}})\}
$$

where $\tilde{F}(T)$ is a lifting of $F(T)$.

We finally remark that, by (16), we also have that $\ker(\alpha^{(\lambda)}_*) \subseteq \text{Ext}^1_{S}(G_{\mu,1}, G^{(\lambda)})$ is nothing else but the group

$$
\{[\tilde{E}^{(\mu,\lambda,F)}]; F \in \text{Hom}_{S_{\lambda}}(G_{\mu,1}, G_{m,S_{\lambda}})\}.
$$

3.4. Description of $\text{Ext}^1_{S}(G_{\mu,1}, G_{\lambda,1})$. We finally have all the ingredients to give a description of the group $\text{Ext}^1_{S}(G_{\mu,1}, G_{\lambda,1})$. In the rest of the section we will suppose, if $\text{char}(R) = 0$, that $(p - 1)v(\mu) \leq v(p)$ and $(p - 1)v(\lambda) \leq v(p)$. We distinguish, for clarity, four cases

$$
\begin{align*}
\text{Ext}^1_{S}(G_{\mu,1}, G_{\lambda,1}); & \quad \text{with } \mu \in \pi R \setminus \{0\}; \\
\text{Ext}^1_{S}(G_{\mu,1}, G_{\lambda,1}); & \quad \text{with } \lambda \in \pi R \setminus \{0\};
\end{align*}
$$

The first three cases are easy. The first two cases have already been treated in [17] and the third one can be obtained with an argument identical to that one used for the proof of the first case. We report here the proofs.

**Proposition 3.6.** We have the following exact sequences.

(i) $0 \to \mathbb{Z}/p\mathbb{Z} \to \text{Ext}^1_{S}(G_{\mu,1}, G_{\mu,1}) \to H^1(S, \mathbb{Z}/p\mathbb{Z}) \to 0$;

(ii) if $v(\mu) > 0$, $0 \to \mathbb{Z}/p\mathbb{Z} \to \text{Ext}^1_{S}(G_{\mu,1}, G_{\mu,1}) \to H^1(S, G^{(\lambda)}_{\mu,1}) \to 0$;

(iii) if $v(\lambda) > 0$, $0 \to \text{Ext}^1_{S}(G_{\mu,1}, G_{\lambda,1}) \to H^1(S, \mathbb{Z}/p\mathbb{Z}) \to H^1(S, \mathbb{Z}/p\mathbb{Z})$. 

Proof. We set \( G = G_{\mu,1} \), with \( v(\mu) > 0 \), or \( G = G_{p,S} \). The Kummer sequence
\[
1 \to \mu_{p,S} \to G_{m,S} \overset{p}{\to} G_{m,S} \to 1
\]
yields, using the isomorphism (12), an exact sequence
\[
0 \to \text{Hom}_{S-gr}(G, G_{m,S}) \to \text{Ext}^1_S(G, \mu_{p,S}) \to H^1(S, G^\vee) \overset{p}{\to} H^1(S, G^\vee).
\]
This is a particular case of (13). Since \( G \) is annihilated by \( p \) then we have the exact sequence
\[
(22) \quad 0 \to \text{Hom}_{S-gr}(G, G_{m,S}) \to \text{Ext}^1_S(G, \mu_{p,S}) \to H^1(S, G^\vee) \to 0.
\]
The proofs of (i) and (ii) follow, since, by (4), we have
\[
\text{Hom}_{S-gr}(G_{\mu,1}, G_{m,S}) \simeq \text{Hom}_{S-gr}(\mu_{p,S}, G_{m,S}) \simeq \mathbb{Z}/p\mathbb{Z}.
\]
We now prove (iii). Combining the isomorphism (12) and the exact sequence (16) we obtain the exact sequence
\[
\text{Hom}_{S-gr}(G_{\mu,1}, G_{m,S}) \overset{r_{\lambda}}{\to} \text{Hom}_{S_{\lambda-gr}}(G_{\mu,1}, G_{m,S}) \overset{\delta}{\to} \text{Ext}^1_S(G_{\mu,1}, G^\vee(S_{\lambda})) \to H^1(S_{\lambda}, G^\vee_{\mu,1}).
\]
If \( \mu \) is invertible in \( R \) then \( G_{\mu,1} \simeq \mu_{p,S} \) and we obtain
\[
\text{Ext}^1_S(\mu_{p,S}, G^\vee_{\mu,1}) \simeq \ker(H^1(S, \mathbb{Z}/p\mathbb{Z}) \to H^1(S_{\lambda}, \mathbb{Z}/p\mathbb{Z}))
\]
since the reduction map \( r_{\lambda} : \text{Hom}_{S-gr}(\mu_{p,S}, G_{m,S}) \to \text{Hom}_{S_{\lambda-gr}}(\mu_{p,S}, G_{m,S}) \) is surjective. On the other hand let us consider the commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^1_S(\mu_{p,S}, G^\vee) & \overset{\psi_{\lambda,1}}{\to} & \text{Ext}^1_S(\mu_{p,S}, G^\vee_{\mu,1}) \\
\alpha^{(\lambda)} & \downarrow & \alpha^{(\mu)} \\
\text{Ext}^1_S(\mu_{p,S}, G_{m,S}) & \overset{p,\lambda}{\to} & \text{Ext}^1_S(\mu_{p,S}, G_{m,S})
\end{array}
\]
The morphism \( p,\lambda \) is trivial since \( \text{Ext}^1 \) is a bi-additive functor and \( \mu_{p,S} \) has order \( p \). From (17) it follows that \( \alpha^{(\mu)} : \text{Ext}^1_S(\mu_{p,S}, G^\vee_{\mu,1}) \to \text{Ext}^1_S(\mu_{p,S}, G_{m,S}) \) is injective, therefore \( \psi_{\lambda,1} \) is the zero map. Now, by (4) we have \( \text{Hom}_{S-gr}(\mu_{p,S}, G^\vee_{\mu,1}) = 0 \); hence, from the exact sequence (16) it follows
\[
\text{Ext}^1_S(\mu_{p,S}, G^\vee_{\mu,1}) \simeq \text{Ext}^1_S(\mu_{p,S}, G_{\lambda,1}).
\]
So we are done. \( \square \)

Remark 3.7. The proof of (i) works over any base, not necessarily a d.v.r.

Remark 3.8. By §3.1 the group \( \mathbb{Z}/p\mathbb{Z} \) which appears in the exact sequence (i) corresponds to the group of extensions \( \{E_j,S; j = 0, \ldots, p-1\} \) and that one of (ii) corresponds to \( \{E_{(\mu,\lambda,j)}; j = 0, \ldots, p-1\} \), with \( v(\mu) > 0 \) and \( v(\lambda) = 0 \).

Lemma-Definition 3.9. Let \( \mu, \lambda \in R \setminus \{0\} \). Let us assume \( F(T) \in (R/\lambda R)[T], j \in \mathbb{Z} \) and
(a) \( F(0) = 1 \),
(b) \( F(U)F(V) = F(U + V + \mu UV), \)
(c) \( F(T)^p \equiv (1 + \mu T)^j \mod (\lambda^p, (1+\mu T)^p - 1). \)

Let \( \tilde{F}(T) \in R[T] \) be any its lifting. Then
\[
E_{(\mu,\lambda;\tilde{F},j)} := \text{Spec} \left( R[T_1,T_2]/\left( \frac{(1 + \mu T_1)^p - 1}{\mu^p}, \frac{(\tilde{F}(T_1) + \lambda T_2)^p(1 + \mu T_1)^{-j} - 1}{\lambda^p} \right) \right),
\]
is a closed subgroup scheme of \( E_{(\mu,\lambda;\tilde{F})} \).

Proof. The proof is straightforward. \( \square \)

Remark 3.10. If \( F(T) = 1 \) the definition coincides with (15).
Remark 3.11. In the above definition the integer $j$ is uniquely determined by $F(T)$ if and only if $\lambda^p \downarrow \mu$.

Remark 3.12. Let $\lambda \in \pi R \backslash \{0\}$. We stress that (a) and (b) means that $F(T) \in \text{Hom}_{S_\lambda - gr}(G^{(\mu)}, \mathbb{G}_{m,S_\lambda})$.

The closed immersion $\mathcal{E}^{(\mu,\lambda,F,j)} \hookrightarrow \mathcal{E}^{(\mu,\lambda,F)}$ induces the following commutative diagrams of exact rows:

$$
\begin{array}{cccc}
0 & \longrightarrow & G_{\lambda,1} & \longrightarrow \mathcal{E}^{(\mu,\lambda,F,j)} & \longrightarrow G_{\mu,1} & \longrightarrow 0 \\
0 & \longrightarrow & G(\lambda) & \longrightarrow \mathcal{E}^{(\mu,\lambda,F)} & \longrightarrow G(\mu) & \longrightarrow 0 \\
0 & \longrightarrow & G_{\lambda,1} & \longrightarrow \mathcal{E}^{(\mu,\lambda,F,j)} & \longrightarrow G_{\mu,1} & \longrightarrow 0 \\
1 & \longrightarrow & \mu_{p,S} & \longrightarrow \mathcal{E}_{j,S} & \longrightarrow \mu_{p,S} & \longrightarrow 1 \\
\end{array}
$$

and

where $\alpha^{(\mu,\lambda,F)}$ is a model map. Since the extension class does not depend on the choice of the lifting $\tilde{F}$, in the following we will sometimes denote it simply with $[\mathcal{E}^{(\mu,\lambda,F,j)}]$, omitting to specify the lifting. We will use the same convention also for the extensions $[\mathcal{E}^{(\mu,\lambda,F)}]$.

Lemma 3.13. Let $\mu, \lambda \in R \backslash \{0\}$ and let $F(T) \in R/\lambda^p R[T]$ be as in the lemma-definition. The group scheme $\mathcal{E}^{(\mu,\lambda,F,j)}$ is the kernel of the isogeny

$$
\psi^j_{\mu,\lambda,F,G}: \mathcal{E}^{(\mu,\lambda,F)} \longrightarrow \mathcal{E}^{(\mu,p,\lambda,p,G)}
$$

$$
\begin{align*}
T_1 & \mapsto (1 + \mu T_1)^{p - 1} \\
T_2 & \mapsto (\tilde{F}(T_1) + \lambda T_2)^p (1 + \mu T_1)^{-j} - \tilde{G}((1 + \mu T_1)^{p-1})/\lambda^p
\end{align*}
$$

for some $G(T) \in R/\lambda^p R[T]$ which satisfies (a), (b), (c) of lemma-definition 3.9; and $\tilde{F}(T), \tilde{G}(T) \in R[T]$ are liftings of $F(T)$ and $G(T)$ respectively.

Proof. If $v(\lambda) = 0$ we can assume $\tilde{F}(T) = 1$. Therefore if we take $\tilde{G} = 1$ we are done. Let us assume $\lambda \in \pi R \backslash \{0\}$. By 3.12 we have $F(T) \in \text{Hom}_{S_\lambda - gr}(G^{(\mu)}, \mathbb{G}_{m,S_\lambda})$. If we restrict this morphism to $G_{\mu,1}$ we obtain an element of $\text{Hom}_{S_\lambda - gr}(G_{\mu,1}, \mathbb{G}_{m,S_\lambda})$, which we denote again $F(T)$ for simplicity. Now the condition (c) means $F(T)^p (1 + \mu T_1)^{-j} = 1 \in \text{Hom}_{S_{\lambda^p} - gr}(G_{\mu,1}, \mathbb{G}_{m,S_{\lambda^p}})$. Let us consider the exact sequence over $S_{\lambda^p}$

$$
0 \longrightarrow G_{\mu,1} \longrightarrow G^{(\mu)} \longrightarrow G^{(\mu,p)} \longrightarrow 0.
$$

Applying to this exact sequence the functor $\text{Hom}_{S_{\lambda^p} - gr}(\cdot, \mathbb{G}_{m,S_{\lambda^p}})$ we obtain

$$
\ker \left( i^* : \text{Hom}_{S_{\lambda^p} - gr}(G^{(\mu)}, \mathbb{G}_{m,S_{\lambda^p}}) \longrightarrow \text{Hom}_{S_{\lambda^p} - gr}(G_{\mu,1}, \mathbb{G}_{m,S_{\lambda^p}}) \right) = \psi_{\mu,1,*} \text{Hom}_{S_{\lambda^p} - gr}(G^{(\mu,p)}, \mathbb{G}_{m,S_{\lambda^p}}).
$$

Therefore condition (c) is equivalent to saying that there exists $G(T) \in \text{Hom}_{S_{\lambda^p} - gr}(G^{(\mu,p)}, \mathbb{G}_{m,S_{\lambda^p}})$ with the property that $F(T)^p (1 + \mu T_1)^{-j} = G((1 + \mu T_1)^{p-1}) \in \text{Hom}_{S_{\lambda^p} - gr}(G^{(\mu)}, \mathbb{G}_{m,S_{\lambda^p}})$. This implies the thesis.
We observe that we have the following commutative diagram of exact rows

\[
\begin{array}{c}
0 \xrightarrow{} \mathcal{E}(\mu, \lambda, \tilde{F}, j) \xrightarrow{\psi_{\lambda, 1}} \mathcal{E}(\mu, \lambda, \tilde{F}) \xrightarrow{\psi_{\mu, 1}} G(\lambda) & \xrightarrow{\psi_{\mu, 1}} G(\mu) & \xrightarrow{0} \\
0 \xrightarrow{} (G_{m, S})^2 \xrightarrow{\psi_{\lambda, 1}} (G_{m, S})^2 & \xrightarrow{0} & \\
\end{array}
\]

and the following commutative diagram of exact rows and columns

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & G_{\lambda, 1} & \mathcal{E}(\mu, \lambda, \tilde{F}, j) & G_{\mu, 1} & 0 \\
0 & G(\lambda) & \mathcal{E}(\mu, \lambda, \tilde{F}) & G(\mu) & 0 \\
0 & G(\lambda^p) & \mathcal{E}(\mu^p, \lambda^p, \tilde{G}) & G(\mu^p) & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

**Example 3.14.** Let us suppose that $R$ is a d.v.r of unequal characteristic which contains a primitive $p^2$-th root of unity $\zeta_2$. We define

\[
\eta = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \lambda^{(2)}. 
\]

We remark that $v(\eta) = v(\lambda(2))$. We put

\[
\tilde{F}(T) \equiv \sum_{k=1}^{p-1} \frac{(\eta T)^k}{k!} 
\]

We have $\tilde{F}(T) \equiv E_p(\eta T) \mod \lambda(1)$. It has been shown in [22, §5] that, using our notation,

\[
\mathbb{Z}/p^2\mathbb{Z} \cong \mathcal{E}(\lambda(1) \lambda(1) \tilde{F}(T), 1).
\]

A similar description of $\mathbb{Z}/p^2\mathbb{Z}$ was independently found by Green and Matignon ([6]). This gives the explicit Kummer-Artin-Schreier-Witt theory for cyclic covers of order $p^2$.

**Example 3.15.** Assume $\mu, \lambda \in R \setminus \{0\}$ and $j \neq 0 \mod p$. Then there exists a group scheme of type $\mathcal{E}(\mu, \lambda; F, j)$, with $F(T) \equiv 1 \mod \lambda$, if and only if $v(\mu) \geq p v(\lambda)$. Indeed the condition $F(T)^p \equiv (1 + \mu T)^j \mod \lambda^p \left(\frac{(1 + \mu T)^{p-1}}{\lambda^p}\right)$ reads as $1 \equiv (1 + \mu T)^j \mod \lambda^p \left(\frac{(1 + \mu T)^{p-1}}{\lambda^p}\right)$. This is the case if and only if $\mu \equiv 0 \mod \lambda^p$.

In the limit case $v(\mu) = p v(\lambda)$ we have an isomorphism

\[
\mathcal{E}(\lambda^p, \lambda; 1, 1) \xrightarrow{\quad} G_{\lambda, 2} := \text{Spec}(R[T]/\left(\frac{(1 + \lambda T)^{p^2} - 1}{\lambda^{p^2}}\right)) \cong G(\lambda),
\]

given, on the level of Hopf algebras, by

\[
T \mapsto T_2
\]

and the inverse by

\[
T_1 \mapsto \frac{(1 + \lambda T)^p - 1}{\lambda^p}, \quad T_2 \mapsto T.
\]

We observe that, if $\text{char}(R) = 0$, since $(p - 1)v(\mu) \leq v(p)$, then $v(\mu) = p v(\lambda)$ implies $p(p - 1)v(\mu) \leq v(p)$. The last one is in fact the condition which ensures that the flat $R$-group scheme $G_{\lambda, 2}$ is finite over $R$. 

Theorem 3.19. For any $(25)$

\[
F(T)^p(1 + \mu T)^{-1} = 1 \in \text{Hom}_{S_1 - gr}(G_{\mu,1}, G_{m,S_1}) \times \mathbb{Z}/p\mathbb{Z} \text{ such that }
\]

It is a subgroup of $(\text{Hom}_{S_1 - gr}(G_{\mu,1}, G_{m,S_1}) \times \mathbb{Z}/p\mathbb{Z}) / (1 + \mu T, 0)$. If $v(\lambda) = 0$ we set $rad_{\mu,\lambda} := \{1\} \times \mathbb{Z}/p\mathbb{Z}$.

Remark 3.17. If $v(\mu) = 0$ and $v(\lambda) > 0$ then the group is trivial by (18).

Let $\mu, \lambda \in R \setminus \{0\}$. We define

\[
\beta : rad_{\mu,\lambda} \rightarrow \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1})
\]

by

\[
(F(T), j) \mapsto [\mathcal{E}(\mu,\lambda,F(j))] = \delta(F)
\]

for any $(F, j) \in rad_{\mu,\lambda}$. Moreover by construction

\[
[[\mathcal{E}(\mu,\lambda,F,j)]_K] = [\mathcal{E}_j)_K] \in \text{Ext}^1_S(\mu_{p,K}, \mu_{p,K}).
\]

Let $(F_1, j_1), (F_2, j_2) \in rad_{\mu,\lambda}$. Then

\[
i_*(\beta(F_1, j_1) + \beta(F_2, j_2) - \beta(F_1 + F_2, j_1 + j_2)) = \delta(F_1) + \delta(F_2) - \delta(F_1 + F_2) = 0,
\]

since $\delta$ and $\delta$ are morphisms of groups. For an extension $[\mathcal{E}] \in \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1})$ we define $[\mathcal{E}]_K := [\mathcal{E}_K] \in \text{Ext}^1_K(\mu_{p,K}, \mu_{p,K})$. Then

\[
(\beta(F_1, j_1) + \beta(F_2, j_2) - \beta(F_1 + F_2, j_1 + j_2))_K = [\mathcal{E}_j)_K] + [\mathcal{E}_j)_K] - [\mathcal{E}_j)_K] = 0.
\]

The last equality follows from 3.6(i) and 3.8. By (24) we have

\[
\beta(F_1, j_1) + \beta(F_2, j_2) - \beta(F_1 + F_2, j_1 + j_2) \in \ker i_*
\]

and therefore, by (13) and (14),

\[
\beta(F_1, j_1) + \beta(F_2, j_2) - \beta(F_1 + F_2, j_1 + j_2) = (\sigma_j)^* \Lambda.
\]

for some $j \in \mathbb{Z}/p\mathbb{Z}$. By (25) it follows that

\[
((\sigma_j)^* \Lambda)_K = [\mathcal{E}_j)_K] = 0,
\]

therefore $j = 0$. So $\beta$ is a morphism of groups. The last assertion is clear.

We now give a description of $\text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1})$ with $\mu, \lambda \in \pi R \setminus \{0\}$. We recall that, till the end of the section, we are assuming that $(p-1)v(\mu), (p-1)v(\lambda) \leq v(p)$ if $\text{char}(R) = 0$.

Theorem 3.19. Let $\mu, \lambda \in \pi R \setminus \{0\}$. The following sequence

\[
0 \rightarrow rad_{\mu,\lambda} \rightarrow \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1}) \rightarrow \ker \left( H^1(S, G_{\mu,1}^\vee) \rightarrow H^1(S, G_{\mu,1}^\vee) \right)
\]
is exact. In particular $\beta$ induces an isomorphism

$$\text{rad}_{\mu \lambda} \simeq \{[E(\mu \lambda F, j)]; (F, j) \in \text{rad}_{\mu \lambda}\}.$$

**Proof.** Using (16) and 3.5, we consider the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \{\tilde{E}(\mu \lambda F)\} & \rightarrow & \text{Ext}_S^1(G_{\mu,1}, G(\lambda)) & \rightarrow & \text{Ext}_S^1(G_{\mu,1}, G_{m, S}) \\
& & \downarrow \tilde{\psi}_{\lambda,1} & & \downarrow \alpha(\lambda) & & \downarrow \psi_{\lambda,1} \\
0 & \rightarrow & \{E(\mu \lambda', F)\} & \rightarrow & \text{Ext}_S^1(G_{\mu,1}, G(\lambda')) & \rightarrow & \text{Ext}_S^1(G_{\mu,1}, G_{m, S}) \\
\end{array}
\]

The map $\tilde{\psi}_{\lambda,1}$, induced by $\psi_{\lambda,1} : \text{Ext}_S^1(G_{\mu,1}, G(\lambda)) \rightarrow \text{Ext}_S^1(G_{\mu,1}, G(\lambda'))$, is given by

$$\frac{[\tilde{E}(\mu \lambda F)]}{[\tilde{E}(\mu \lambda' F)]}.$$
We now define the morphism of groups $j \in r_{\lambda^p}(\text{Hom}_{S-gr}(G_{\mu,1}, \mathbb{Z}_m,S))$. So $i_F = j$. Hence $(F, 0) \mapsto (F, i_F) = (F, j)$. While if $\lambda^p \mid \mu$ then $\text{Hom}_{S-gr}(G_{\mu,1}, G^{(\lambda^p)}) = \mathbb{Z}/p\mathbb{Z}$ and $r_{\lambda^p}(\text{Hom}_{S-gr}(G_{\mu,1}, G_{m,S})) = 0$. Hence

$$\ker(\widetilde{\psi}_{\lambda,1_*}) = \left\{ F(T) \in \text{Hom}_{S_{\lambda-gr}}(G_{\mu,1}, \mathbb{Z}_m,S); F(T)^p = 1 \in \text{Hom}_{S_{\lambda-gr}}(G_{\mu,1}, \mathbb{Z}_m,S^p) \right\}.$$ 

Let us now take $(F, j) \in \text{rad}_{\mu,\lambda}$. This means that

$$F(T)^p = (1 + \mu T)^j = 1 \in \text{Hom}_{S_{\lambda-gr}}(G_{\mu,1}, \mathbb{Z}_m,S^p).$$

Therefore $F(T) \in \ker(\widetilde{\psi}_{\lambda,1_*})$ and $i_F = 0$. So $(F, j) \mapsto (F, i_F + j) = (F, j)$.

**Interpretation of $\beta$.** We now define the morphism of groups

$$\varrho : \ker(\widetilde{\psi}_{\lambda,1_*}) \rightarrow \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1})$$

$$F \mapsto \beta(F, i_F) = [\mathcal{E}(\mu,\lambda,F, i_F)].$$

We recall the definition of $\delta'$ given in (14):

$$\delta' : \text{Hom}_{S-gr}(G_{\mu,1}, G^{(\lambda^p)}) \rightarrow \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1})$$

is defined by $\delta'(\sigma_j) = \sigma_j^*(\Lambda)$. Then, under the isomorphism (29), we have

$$\beta = \varrho + \delta' : \ker(\widetilde{\psi}_{\lambda,1_*}) \times \text{Hom}_{S-gr}(G_{\mu,1}, G^{(\lambda^p)}) \rightarrow \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1}).$$

**Injectivity of $\beta$.** First of all we observe that $\tilde{\delta}$ factors through $\varrho$, i.e.

$$\tilde{\delta} = i_* \circ \varrho : \ker(\widetilde{\psi}_{\lambda,1_*}) \xrightarrow{\varrho} \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{i_\ast} \ker(\widetilde{\psi}_{\lambda,1_*}).$$

Indeed

$$i_* \circ \varrho(F) = i_\ast([\mathcal{E}(\mu,\lambda,F, i_F)]) = [\mathcal{E}(\mu,\lambda,F)] = \tilde{\delta}(F).$$

In particular, since $\tilde{\delta}$ is injective, $\varrho$ is injective, too.

We now prove that $\beta = \varrho + \delta'$ is injective, too. By (28),

$$i_* \circ \delta' = 0.$$

Now, if $(\varrho + \delta')(F, \sigma_j) = 0$, then $\varrho(F) = -\delta'(\sigma_j)$. So

$$\tilde{\delta}(F) = i_\ast(\varrho(F)) = i_\ast(-\delta'(\sigma_j)) = i_\ast(\delta'(\sigma_j)) = 0.$$

But $\tilde{\delta}$ is injective, so $F = 1$. Hence $\delta'(\sigma_j) = 0$. But by (13), also $\delta'$ is injective. Then $\sigma_j = 0$.

**Calculation of $\text{Im} \beta$.** We finally prove $\text{Im} (\varrho + \delta') = \ker(\alpha_\ast(\Lambda) \circ i_*)$. Since $\tilde{\delta} = i_* \circ \varrho \circ (\varrho + \delta') = 0$ and $i_* \circ \delta' = 0$ then

$$\alpha_\ast(\Lambda) \circ i_* \circ (\varrho + \delta') = 0.$$

Then $\text{Im}(\varrho + \delta') \subseteq \ker(\alpha_\ast(\Lambda) \circ i_*)$. On the other hand, if $[\mathcal{E}] \in \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1})$ is such that $\alpha_\ast(\Lambda) \circ i_*([\mathcal{E}]) = 0$, then, by (27), there exists $F \in \ker(\widetilde{\psi}_{\lambda,1_*})$ such that $i_*([\mathcal{E}]) = \tilde{\delta}(F) = i_\ast(\varrho(F))$. Hence, by the isomorphism (28), $[\mathcal{E}] \sim \varrho(F) \in \text{Im}(\delta')$. Therefore $\text{Im}(\varrho + \delta') = \ker(\alpha_\ast(\Lambda) \circ i_*)$. Moreover since $i_* : \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1}) \rightarrow \ker(\widetilde{\psi}_{\lambda,1_*})$ is surjective then $\text{Im}(\alpha_\ast(\Lambda)) = \text{Im}(\alpha_\ast(\Lambda) \circ i_*)$. Thus we have proved, using also (27), that the following sequence

$$0 \rightarrow \ker(\widetilde{\psi}_{\lambda,1_*}) \times \text{Hom}_{S-gr}(G_{\mu,1}, G^{(\lambda^p)}) \xrightarrow{\varrho + \delta'} \text{Ext}^1_S(G_{\mu,1}, G_{\lambda,1}) \xrightarrow{\alpha_\ast(\Lambda) \circ i_*}
\ker\left(H^1(S, G^{*}_{\mu,1}) \rightarrow H^1(S_{\lambda}, G^{*}_{\mu,1})\right)$$

is exact. Finally, by definitions, it follows that

$$\beta(\text{rad}_{\mu,\lambda}) = \{[\mathcal{E}(\mu,\lambda,F,j)]; (F,j) \in \text{rad}_{\mu,\lambda}\}.$$
Lemma-Definition 3.22. Let \( \mu, \lambda \in R \setminus \{0\} \). The inverse image of \( \{E_{i,K}; j \in \mathbb{Z}/p\mathbb{Z}\} \) by the canonical map \( \text{Ext}^1_2(G_{\mu,1}, G_{\lambda,1}) \to \text{Ext}^1_2(p(K), p(K)) \) is given by \( \{E^{(\mu,\lambda,F,j)}; (F,j) \in \text{rad}_{\mu,\lambda}\} \). Moreover any extension \( [E] \in \text{Ext}^1_2(G_{\mu,1}, G_{\lambda,1}) \) is of type \( [E^{(\mu,\lambda,F,j)}] \), up to an extension of \( R \).

Remark 3.21. If \( \text{char}(R) = 0 \), any \( R \)-group scheme of order \( p^2 \) is of type \( E^{(\mu,\lambda,F,j)} \), possibly after an extension of \( R \). Indeed, up to an extension of \( R \), the generic fiber of any \( R \)-group scheme is isomorphic to \( E_{i,K} \). Then the result follows from 3.1 and the previous corollary.

Proof. Let us first suppose \( \mu, \lambda \in \pi R \setminus \{0\} \). Let us consider the following commutative diagram with exact rows

\[
\begin{array}{c}
\text{rad}_{\mu,\lambda} \\
\downarrow \\
\text{Ext}^1_2(G_{\mu,1}, G_{\lambda,1}) \\
\downarrow \\
0 \quad \mathbb{Z}/p\mathbb{Z} \cong [E_{i,K}] \quad \text{Ext}^1_2(p(K), p(K)) \\
\downarrow \\
0 \\
\end{array}
\]

where the vertical maps are the restrictions to the generic fiber. The first statement follows, remarking that the second vertical map is injective by 2.7.

We now prove the second statement. Let \( [E] \in \text{Ext}^1_2(G_{\mu,1}, G_{\lambda,1}) \). Suppose that \( \alpha^*_2(i_*[E]) = [S'] \), with \( S' \to S \) a \( G_{\mu,1}' \)-torsor. If \( S' \to S \) is trivial by the first exact row of the above diagram we are done. Otherwise \( S_K' \) is integral and we consider the integral closure \( S'' = \text{Spec}(R'') \) of \( S \) in \( S_K' \). Since \( S_K' \to S' \), then \( S_K' \times_K S_K' \) is a trivial \( (G_{\mu,1})_K \)-torsor over \( S_K' \). By 2.7 we have that \( S' \times_S S'' \) is a trivial \( G_{\mu,1}' \)-torsor over \( S'' \). Let \( \bar{R} \) be the localization of \( R'' \) at a closed point over \( (\pi) \). Then \( \bar{S} = \text{Spec}(\bar{R}) \). So, if we make the base change \( f: \bar{S} \to S \), then \( \alpha^*_2(i_*([E \times_S \bar{S}])) = 0 \). Again by the above diagram, this implies that \( [E \times_S \bar{S}] \) is of type \( [E^{(\mu,\lambda,F,j)}] \).

The cases \( v(\mu) = 0 \) or \( v(\lambda) = 0 \) are similar, even simpler, using 3.6 and 3.8.

□

3.5. \( \text{Ext}^1_2(G_{\mu,1}, G_{\lambda,1}) \) and the Sekiguchi-Suwa theory. We now give a description of the group \( \{E^{(\mu,\lambda,F,j)}; (F,j) \in \text{rad}_{\mu,\lambda}\} \) through the Sekiguchi-Suwa theory. We begin with an object which will play a key role in such a description.

Lemma-Definition 3.22. Let \( \mu, \lambda \in R \setminus \{0\} \). We define

\[
\Phi_{\mu,\lambda} := \left\{ (a,j) \in (R/\lambda R)^{p^{\mu-1}} \times \mathbb{Z}/p\mathbb{Z} \mid \text{such that } pa - j\mu = \frac{p}{\mu^{p-1}} a^p \in R/\lambda^p R \right\} / \langle (\mu,0) \rangle
\]

if \( \text{char}(R) = 0 \), and

\[
\Phi_{\mu,\lambda} := \left\{ (a,j) \in (R/\lambda R)^{p^{\mu-1}} \times \mathbb{Z}/p\mathbb{Z} \mid \text{such that } j\mu \equiv 0 \mod \lambda^p \right\} / \langle (\mu,0) \rangle
\]

if \( \text{char}(R) = p \). It is a subgroup of \( ((R/\lambda R)^{p^{\mu-1}} \times \mathbb{Z}/p\mathbb{Z}) / \langle (\mu,0) \rangle \).

Proof. One should prove that it is a subgroup \( ((R/\lambda R)^{p^{\mu-1}} \times \mathbb{Z}/p\mathbb{Z}) / \langle (\mu,0) \rangle \). In particular the quotient is well defined. The proof is very elementary.

Remark 3.23. If \( v(\mu) < v(\lambda) \) then \( (a,j) \in \Phi_{\mu,\lambda} \) implies \( j = 0 \). If \( \text{char}(R) = p \) this is trivial. If \( \text{char}(R) = 0 \), by \( a^p = \mu^{p-1} a \in R/\lambda R \) we obtain

\[
pa - j\mu = \frac{p}{\mu^{p-1}} a^p = -j\mu \mod \lambda,
\]

which implies \( j = 0 \).

Proposition 3.24. Let \( \mu, \lambda \in R \setminus \{0\} \). Then the map

\[
\Phi_{\mu,\lambda} \to \{E^{(\mu,\lambda,F,j)}; (F,j) \in \text{rad}_{\mu,\lambda}\}
\]

\[
(a,j) \mapsto [E^{(\mu,\lambda,F,(a,j))}]
\]

MODELS OF \( \mu_{p^2,i} \)
is an isomorphism of groups.

Remark 3.25. It is clear that if \((0,j) \in \Phi_{\mu,\lambda}\), with \(j \neq 0\), then \(\mu \equiv 0 \mod \lambda^p\).

Proof. If \(v(\lambda) = 0\) clearly \(\Phi_{\mu,\lambda}\) is the group \(\{0\} \times \mathbb{Z}/p\mathbb{Z}\), while if \(v(\mu) = 0\) and \(v(\lambda) > 0\) it is the trivial group. So \(\Phi_{\mu,\lambda} \simeq \text{rad}_{\mu,\lambda} \simeq \{(\xi(\mu,\lambda,F,j))\}; \ (F,j) \in \text{rad}_{\mu,\lambda}\) in these two cases (see 3.8).

We now suppose \(\mu,\lambda \in \pi R \setminus \{0\}\). By 3.19 we have an isomorphism \(\text{rad}_{\mu,\lambda} \rightarrow \{(\xi(\mu,\lambda,F,j))\}; \ (F,j) \in \text{rad}_{\mu,\lambda}\) given by \((F,j) \mapsto (\xi(\mu,\lambda,F,j))\). We now prove that the map \(\Phi_{\mu,\lambda} \rightarrow \text{rad}_{\mu,\lambda}\), given by \((a,j) \mapsto (E_p(a,\mu, T), j)\) is an isomorphism. We distinguish some cases.

\(\text{char}(R) = p\). By 3.3, (7) and (23) it follows that \(\text{rad}_{\mu,\lambda}\) is isomorphic to
\[
\left\{(a,j) \in (R/\lambda R)^{F-\mu p-1} \times \mathbb{Z}/p\mathbb{Z}\mid \exists b \in \widehat{W}(R/\lambda^p R)^{F-\mu p-1}\right\}
\]

such that \(p[a] - j[\mu] = V(b) \in \widehat{W}(R/\lambda^p R)\). This group is in fact \(\Phi_{\mu,\lambda}\). Indeed, since \(F V = V F = p\), then \(j[\mu] = p[a] - V(b) = V([a^p] - b)\) if and only if \(j[\mu] = [j[\mu]] = 0\) and \(b = [a^p]\).

\(\text{char}(R) = 0\) and \(p > 2\). By 3.3, (7), (10) and (23) it follows that \(\text{rad}_{\mu,\lambda}\) is isomorphic to
\[
\left\{(a,j) \in (R/\lambda R)^{F-\mu p-1} \times \mathbb{Z}/p\mathbb{Z}\mid \exists b \in \widehat{W}(R/\lambda^p R)^{F-\mu p-1}\right\}
\]

such that \(p[a] - j[\mu] = \frac{p}{\mu p^{-1}} b + V(b) \in \widehat{W}(R/\lambda^p R)\). We prove that this group is in fact \(\Phi_{\mu,\lambda}\). Let \(a, j\) and \(b = (b_0, b_1, \ldots)\) be as above. First of all we observe that if \(v(\mu) < v(\lambda)\) then, by 3.2, \((\xi(\mu,\lambda; E_p(a,\mu, T), j))_K = E_{j,K} \simeq \mu_p \times \mu_p\). This implies, together to 3.23, that if \(v(\mu) < v(\lambda)\) we can assume \(j = 0\).

Moreover we remark that, by [22, 5.10], we have
\[
p[a] \equiv (pa, a^p, 0, \ldots) \mod p^2.
\]

We first assume \((p-1)v(\mu) \geq v(\lambda)\). We observe that \(j[\mu] \in \widehat{W}(R/\lambda^p R)^F\). Indeed if \(v(\mu) \geq v(\lambda)\) this is clear; while if \(v(\mu) < v(\lambda)\) we recall that \(j = 0\). Since also \(p[a], b \in \widehat{W}(R/\lambda^p R)^F\), then, \(V(b) = p[a] - j[\mu] = \frac{p}{\mu p^{-1}} b + V(b) \in \widehat{W}(R/\lambda^p R)^F\); hence by 2.4,
\[
(p[\mu^{-1}])b + V(b) = \left(\frac{p}{\mu p^{-1}} b_0, \left(\frac{p}{\mu p^{-1}}\right)^2 b_1 + b_0, \ldots, \left(\frac{p}{\mu p^{-1}}\right)^{p^i+1} b_{i+1} + b_i, \ldots\right) \in \widehat{W}(R/\lambda^p R).
\]

and
\[
\frac{p}{\mu p^{-1}} b + V(b) = \left(\frac{p}{\mu p^{-1}} b_0, \left(\frac{p}{\mu p^{-1}}\right)^2 b_1 + b_0, \ldots, \left(\frac{p}{\mu p^{-1}}\right)^{p^i+1} b_{i+1} + b_i, \ldots\right) \in \widehat{W}(R/\lambda^p R).
\]

Comparing the above equations it follows
\[
\left(\frac{p}{\mu p^{-1}}\right)^{p^i+1} b_{i+1} + b_i = 0 \quad \text{for } i \geq 1
\]
\[
\left(\frac{p}{\mu p^{-1}}\right)^{p} b_1 + b_0 = a^p
\]
\[
\frac{p}{\mu p^{-1}} b_0 = pa - j[\mu].
\]

Since there exists \(r \geq 0\) such that \(b_i = 0\) for any \(i \geq r\) then \(b_i = 0\) if \(i \geq 1\), \(b_0 = a^p\) and \(pa - j[\mu] = \frac{p}{\mu p^{-1}} a^p\).

While if \(v(\mu) < (p-1)v(\lambda)\) then \(v(p) - (p-1)v(\mu) > v(\lambda)\). Therefore \(\left(\frac{p}{\mu p^{-1}}\right)^{p^i} = 0 \in R/\lambda^p R\) for \(i > 0\). Thus we have
\[
\left(\frac{p}{\mu p^{-1}}\right) b + V(b) = \left(\frac{p}{\mu p^{-1}} b_0, 0, \ldots\right) + V(b) = \left(\frac{p}{\mu p^{-1}} b_0, b_0, b_1, \ldots\right).
\]

Since \(j = 0\) we have
\[
(pa, a^p, 0, \ldots) = \left(\frac{p}{\mu p^{-1}} b_0, b_0, b_1, \ldots\right).
\]
This gives $b_i = 0$ if $i \geq 0$, $b_0 = a^p$ and
\[ pa - j \mu = pa = \frac{p}{\mu^p - 1} a^p \]
as required.

\begin{itemize}
\item \textbf{char}(R) = 0 and $p = 2$ \quad Let us consider the isomorphism
\[
(C_R^0/\langle \text{char} \rangle)_p \times \text{id} : (R/\langle \text{char} \rangle)^{F - \mu^p - 1} \times \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Hom}_{\mathbb{S}_2}(G_{\mu,1}, \mathbb{G}_m) \times \mathbb{Z}/2\mathbb{Z}.
\]
We have just to prove that $((C_R^0/\langle \text{char} \rangle)_p \times \text{id})^{-1}(\text{rad}_{\mu, \lambda}) = \Phi_{\mu, \lambda}$.

Let $(F,j) \in \text{Hom}_{\mathbb{S}_2}(G_{\mu,1}, \mathbb{G}_m) \times \mathbb{Z}/2\mathbb{Z}$. Then $F(T) = 1 + aT$. Moreover the class of $(F,j)$ mod $\langle (1 + \mu T, 0) \rangle$ belongs to $\text{rad}_{\mu, \lambda}$ if and only if
\[
(1 + aT)^2 = (1 + \mu T)^2 = (1 + j \mu T) \in \text{Hom}_{\mathbb{S}_2}(G_{\mu,1}, \mathbb{G}_m).
\]
Recall that $T^2 = -\frac{2}{\mu} T \in R[G_{\mu,1}]$. Therefore (32) is satisfied if and only if
\[
2a - \frac{2}{\mu} a^2 = j \mu \in R/\lambda^2 R.
\]
This is equivalent to say the class of $(a, j)$ mod $\langle (\mu, 0) \rangle$ belongs to $\Phi_{\mu, \lambda}$.
\end{itemize}

We now find explicitly all the solutions $(a, j) \in (R/\langle \text{char} \rangle)^{F - \mu^p - 1} \times \mathbb{Z}/p\mathbb{Z}$ of the equation $pa - j \mu = a^p \in R/\lambda^p R$. By 3.24 this means finding explicitly all the extensions of type $E(\mu, \lambda; F, j)$. Let us consider the restriction map
\[
r : \{E(\mu, \lambda; F, j) ; (F,j) \in \text{rad}_{\mu, \lambda}\} \longrightarrow H_0^3(\mu, \lambda, \mu, p) \simeq \mathbb{Z}/p\mathbb{Z}.
\]
We remark that it coincides with the projection on the second component
\[
p_2 : \Phi_{\mu, \lambda} \longrightarrow \mathbb{Z}/p\mathbb{Z}.
\]

The proof of the following lemma is trivial.

\begin{lemma}
There is an extension of $G_{\mu,1}$ by $G_{\lambda,1}$ which is a model of $\mu_{p^2, K}$ if and only if $p_2$ is surjective.
\end{lemma}

We now describe the kernel of the above map.

\begin{lemma}
Under the assumptions of 3.24 we have
\[
\ker p_2 = \left\{(a,0) \in R/\lambda R \times \mathbb{Z}/p\mathbb{Z} \text{ s. t., for any lifting } \tilde{a} \in R,
\right. \left.
\begin{align*}
&v(\tilde{a}^p - \mu^{-1} \tilde{a}) \geq \max\{pv(\lambda) + (p - 1)v(\mu) - v(p), v(\lambda)\} \\
&\left.\left\langle (\mu, 0) \right.\right\rangle
\end{align*}
\right\}
\]
if $\text{char}(R) = 0$, and
\[
\ker p_2 = \left\{(R/\lambda R)^{F - \mu^p - 1} \times \{0\} \right\} / \langle (\mu, 0) \rangle
\]
if $\text{char}(R) = p$.
\end{lemma}

\begin{proof}
The case $\text{char}(R) = p$ is trivial. Let us now suppose $\text{char}(R) = 0$. Let $(a, 0) \in \ker p_2 \cap R/\lambda R \times \mathbb{Z}/p\mathbb{Z}$. By definition we have that
\[
pa = \frac{p}{\mu^p - 1} a^p \in R/\lambda^p R \quad \text{and} \quad a^p - \mu^{-1} a = 0 \in R/\lambda R.
\]
Let $\tilde{a} \in R$ be a lifting of $a$. Since $a^p = \mu^{-1} a \in R/\lambda R$ then $\tilde{a}^p - \mu^{-1} \tilde{a} = b$ with $v(b) \geq v(\lambda)$. Hence we have $pa = \frac{p}{\mu^p - 1} a^p \in R/\lambda^p R$ if and only if $\frac{p}{\mu^p - 1} b = 0 \in R/\lambda^p R$. This happens if and only if $v(\tilde{a}^p - \mu^{-1} \tilde{a}) \geq pv(\lambda) + (p - 1)v(\mu) - v(p)$.
\end{proof}
Example 3.28. Let \( \text{char}(R) = 0 \). If \( v(\mu) \geq v(\lambda) \), which is the case of models of \( \mu, \lambda \) by 3.2, then
\[
\ker p_2 = \left\{ (a, 0) \in R/\lambda R \times \mathbb{Z}/p\mathbb{Z} \mid \text{s. t., for any lifting} \, \tilde{a} \in R, \right. \\
\left. \quad pv(\tilde{a}) \geq \max\{pv(\lambda) + (p-1)v(\mu) - v(p), v(\lambda)\} \right\}
\]
and it is easy to show that \( p_2 \) is injective if and only if \( v(\lambda) \leq 1 \) or \( v(p) - (p-1)v(\mu) < p \).

We now compute \( \Phi_{\mu, \lambda} \).

Proposition 3.29. Let \( \mu, \lambda \in \pi R \setminus \{0\} \). In the case \( \text{char}(R) = 0 \) we will suppose that \( R \) contains a distinguished primitive \( p^2 \)-th root of unity \( \zeta_2 \) (and \( (p-1)v(\mu), (p-1)v(\lambda) \leq v(p) \) as usual).

Let \( \text{char}(R) = 0 \).

(i) If \( v(\mu) < v(\lambda) \), or \( v(\lambda) \leq v(\mu) < pv(\lambda) \) and \( pv(\mu) - v(\lambda) < v(p) \) then \( p_2 \) is trivial.

(ii) If \( v(\lambda) \leq v(\mu) < pv(\lambda) \) and \( pv(\mu) - v(\lambda) \geq v(p) \) then \( p_2 \) is surjective and \( \Phi_{\mu, \lambda} \) is isomorphic to the group
\[
\{(j \eta \frac{\mu}{\lambda(1)} + \alpha, j); (\alpha, 0) \in \ker(p_2) \text{ and } j \in \mathbb{Z}/p\mathbb{Z}\}
\]
For the definitions of \( \eta \) see 3.14.

(iii) If \( v(\mu) \geq pv(\lambda) \), then \( p_2 \) is surjective and \( \Phi_{\mu, \lambda} \) is isomorphic to the group
\[
\{(\alpha, j); (\alpha, 0) \in \ker(p_2) \text{ and } j \in \mathbb{Z}/p\mathbb{Z}\}.
\]

Let \( \text{char}(R) = p \).

(iv) \( p_2 \) is surjective if and only if \( v(\mu) \geq pv(\lambda) \). If \( p_2 \) is surjective then \( \Phi_{\mu, \lambda} \) is isomorphic to the group
\[
\{(\alpha, j); (\alpha, 0) \in \ker(p_2) \text{ and } j \in \mathbb{Z}/p\mathbb{Z}\}
\]

Remark 3.30. Let us suppose \( v(\lambda) \leq v(\mu) < pv(\lambda) \). Let \( (b, j) \in \Phi_{\mu, \lambda} \) with \( j \neq 0 \). By 3.25, then \( b \neq 0 \).

Let \( \tilde{b} \in R \) be any of its lifting. Then \( v(\tilde{b}) = v(\eta \frac{\mu}{\lambda(1)}) = v(\mu) - \frac{v(p)}{p} \). Indeed, by the proposition, we have \( \tilde{b} = j \eta \frac{\mu}{\lambda(1)} + \alpha \) for some \( \alpha \in R/\lambda R \) with \( v(\tilde{\alpha}) > v(\eta \frac{\mu}{\lambda(1)}) = v(\mu) - \frac{v(p)}{p} \), where \( \tilde{\alpha} \in R \) is any lifting of \( \alpha \).

Remark 3.31. We need the hypothesis \( \zeta_2 \in R \) just in the proof of (ii). If we knew an explicit solution \( a \) (if there exists) of the equation \( pa - jm = \frac{p^2}{p^2 - 1}a \in R/\lambda^p R \), also in the case \( (p-1)v(\mu) = (p-1)v(\lambda) = v(p) \) but \( \zeta_2 \not\in R \), then a statement as above holds. This solution would play the role of \( \eta \frac{\mu}{\lambda(1)} \).

Proof. i) If \( v(\mu) < v(\lambda) \) the result follows from 3.2. Let us now suppose \( v(\lambda) \leq v(\mu) < pv(\lambda) \) and \( pv(\mu) - v(\lambda) < v(p) \). Since the target of \( p_2 \) is \( \mathbb{Z}/p\mathbb{Z} \) then \( p_2 \) is either surjective or trivial. Let us suppose it is surjective. This is equivalent to saying that
\[
(33) \quad pa - jm = \frac{p}{p^2 - 1}a^p \in R/\lambda^p R
\]
has a solution \( a \in (R/\lambda R)^F \) with \( j \neq 0 \). Since \( v(\mu) < pv(\lambda) \) then \( a \neq 0 \). Let \( \tilde{a} \in R \) be a lifting of \( a \). Since \( v(\mu) < v(p) \) and \( v(\mu^p) < v(p\tilde{a}) \), then
\[
(34) \quad v(\mu) = v(p) - (p-1)v(\mu) + pv(\tilde{a}),
\]
From \( a \in (R/\lambda R)^F \) it follows \( pv(\tilde{a}) \geq v(\lambda) \). Hence, by (34), \( pv(\mu) - v(\lambda) \geq v(p) \), against the hypothesis.

ii) We know by 3.14 and 3.24 that
\[
(35) \quad \eta \frac{\mu}{\lambda(1)} = \frac{p}{p^2 - 1}\eta^p \in R/\lambda^p R.
\]
Therefore the first vertical map is an isomorphism. It is in fact an isomorphism where the exact rows come from \[4, \text{III} \S 6, 2.5\]. Moreover since \(\text{char} \mu = 0\), \(\text{char} \lambda = 0\), and \(\text{char} p = 0\), we have a classification of \(G\) extensions of \(G\). Without loss of generality we can suppose \(\eta \in (R/\lambda R)^p\), \((a, j) \in (R/\lambda R)\), and \(\alpha, j \in R\). This implies that \(p_2\) is surjective and, moreover, that \((\alpha, j) \in (R/\lambda R \times \mathbb{Z}/p\mathbb{Z})\) if and only if \((\alpha, 0) \in \ker(p_2)\). The result immediately follows from the explicit description of the map \(p_2\) in the case \(\text{char}(R) = p\).

\[\Box\]

**Example 3.32.** Assume \(\text{char}(R) = 0\), \(\zeta_2 \in R\), and \(v(\mu) = v(\lambda) > 0\). Then \(p_2\) is surjective if and only if \(v(\mu) = v(\lambda) = v(\lambda_1)\).

**Example 3.33.** Let us suppose \(\text{char}(R) = 0\), \(\zeta_2 \in R\), and \(v(\mu) = v(\lambda_1) \geq v(\lambda)\). We have \(G_{\mu,1} \simeq \mathbb{Z}/p\mathbb{Z}\). Without loss of generality we can suppose \(\mu = \lambda_1\). Then \(p_2\) is an isomorphism. Indeed in this case \(\ker(p_2) = 0\) by 3.28 and it is surjective by 3.29(ii). However it is possible to prove this fact in a more direct way. We have the following commutative diagram of exact rows

\[
\begin{array}{cccccc}
0 & \to & \text{H}_0^2(\mathbb{Z}/p\mathbb{Z}, G_{\lambda,1}) & \to & \text{Ext}_R^1(\mathbb{Z}/p\mathbb{Z}, G_{\lambda,1}) & \to & \text{H}_1^1(S, G^{(\lambda)}) & \to & 0 \\
0 & \to & \text{H}_0^2(\mathbb{Z}/p\mathbb{Z}, G_{\lambda,1}) & \to & \text{Ext}_R^1(\mathbb{Z}/p\mathbb{Z}, G_{\lambda,1}) & \to & \text{H}_1^1(K, G^{(\lambda)}) & \to & 0
\end{array}
\]

where the exact rows come from [4, III \S 6, 2.5]. Moreover since \(\mathbb{Z}/p\mathbb{Z}\) is constant and cyclic, using 4.4.2 and 4.2 and 23, VIII \S 4\] we have

\[\text{H}_0^2(\mathbb{Z}/p\mathbb{Z}, G_{\lambda,1}) \simeq G_{\lambda,1}(S), \quad \text{and} \quad \text{H}_0^2(\mathbb{Z}/p\mathbb{Z}, (G_{\lambda,1})_K) \simeq G_{\lambda,1}(K)\]

Therefore the first vertical map is an isomorphism. It is in fact \(p_2\). This finally implies that there is an isomorphism

\[\text{H}_0^2(\mathbb{Z}/p\mathbb{Z}, G_{\lambda,1}) \simeq \{[\mathcal{E}(\lambda_1, \lambda, F_j); j = 0, \ldots, p - 1]\},\]

where \(F_j(T) = \sum_{k=0}^{p-1} (jk)^k T^k\).

### 3.6. Classification of models of \(\mu_{p^2, K}\)

By the previous paragraphs we have a classification of \(G_{\mu,1}\) by \(G_{\lambda,1}\) whose generic fibre is isomorphic, as group scheme, to \(\mu_{p^2, K}\). But this classification is, a priori, too fine for our tasks. We want here to forget the structure of extension. We are only interested in the group scheme structure. We observe that it could happen that two non isomorphic extensions are isomorphic as group schemes. We here study when it happens.

We now recall that by 3.1, 3.20 and 3.24 any model of \(\mu_{p^2, K}\) is of the form \(\mathcal{E}(\mu, \lambda; F_j)\) such that \(j \neq 0\), \(F(T) \equiv \sum_{i=0}^{p-1} \alpha^i T^i \mod \lambda\) with \((a, j) \in \Phi_{\mu, \lambda}\). Moreover if \(\text{char}(R) = 0\) then \(v(p) \geq (p - 1)v(\mu) \geq (p - 1)v(\lambda)\) (see 3.2 for \(v(\mu) \geq v(\lambda)\)), while if \(\text{char}(R) = p\) then \(v(\mu) \geq pv(\lambda)\) (see 3.29(iv)). In these cases we have

\[
\Phi_{\mu, \lambda} = \left\{ (a, j) \in (R/\lambda R)^p \times \mathbb{Z}/p\mathbb{Z} \text{ such that } pa - j\mu = \frac{p}{\mu^{p-1}}a^p \in \mathbb{R}/\lambda^p \mathbb{R} \right\}
\]

if \(\text{char}(R) = 0\) and

\[
\Phi_{\mu, \lambda} = \left\{ (a, j) \in (R/\lambda R)^p \times \mathbb{Z}/p\mathbb{Z} \text{ such that } j\mu \equiv 0 \mod \lambda^p \right\}
\]
if \( \text{char}(R) = p \). For \( i = 1, 2 \) let us consider \( E^{(\mu_1, \lambda_1; F_i, j_i)} \), models of \( \mu_{p^2, K} \). First of all we remark that there is an injection

\[
\rho_K : \text{Hom}_{S-gr}(E^{(\mu_1, \lambda_1; F_1, j_1)}, E^{(\mu_2, \lambda_2; F_2, j_2)}) \rightarrow \text{Hom}_{K-gr}(E_{j_1, K}, E_{j_2, K})
\]

given by

\[
f \mapsto (\alpha^{(\mu_2, \lambda_2, \hat{G}_i)})_K \circ f_K \circ (\alpha^{(\mu_1, \lambda_1, \hat{F}_i)})^{-1}_K.
\]

We recall that

\[
\text{Hom}_{S-gr}(E_{j_1}, E_{j_2}) \simeq \text{Hom}_{K-gr}(E_{j_1, K}, E_{j_2, K})
\]

and the elements are the morphisms

\[
\psi_{r,s} : E_{j_1} \rightarrow E_{j_2},
\]

which, on the level of Hopf algebras, are given by

\[
\begin{align*}
T_1 & \mapsto T_1^{rj_1} \\
T_2 & \mapsto T_2^{rj_2}
\end{align*}
\]

for some \( r, s = 0, \ldots, p - 1 \). Moreover the map

\[
\text{Hom}_{S-gr}(E_{j_1}, E_{j_2}) \rightarrow \mathbb{Z}/p^2\mathbb{Z}
\]

\[
\psi_{r,s} \mapsto r + \frac{p}{j_1}s
\]

is an isomorphism. So \( \text{Hom}_{S-gr}(E^{(\mu_1, \lambda_1; F_1, j_1)}, E^{(\mu_2, \lambda_2; F_2, j_2)}) \) is a subgroup of \( \mathbb{Z}/p^2\mathbb{Z} \) through the map \( \rho_K \). We remark that the unique nontrivial subgroup of \( \text{Hom}(E_{j_1}, E_{j_2}) \) is \( \{\psi_{0,s}; s = 0, \ldots, p - 1\} \). Finally we have that any morphism \( E^{(\mu_1, \lambda_1; F_1, j_1)} \rightarrow E^{(\mu_2, \lambda_2; F_2, j_2)} \) is given by

\[
\begin{align*}
T_1 & \mapsto (1 + \mu_1 T_1) \frac{2\lambda}{\mu_2} - 1 \\
T_2 & \mapsto (F_1(T_1) + \lambda_1 T_2)^\gamma(1 + \mu_1 T_1)^s - F_2\left(1 + \mu_1 T_1\right)^\frac{2\lambda}{\mu_2},
\end{align*}
\]

for some \( r, s \in \mathbb{Z}/p\mathbb{Z} \). With abuse of notation we call it \( \psi_{r,s} \). We remark that the morphisms \( \psi_{r,s} : E^{(\mu_1, \lambda_1; F_1, j_1)} \rightarrow E^{(\mu_2, \lambda_2; F_2, j_2)} \) which are model maps correspond, by (35), to \( r \neq 0 \). In such a case \( \psi_{r,s} \) is a morphism of extensions, i.e. there exist morphisms \( \psi_1 : G_{\lambda_1, 1} \rightarrow G_{\lambda_2, 1} \) and \( \psi_2 : G_{\mu_1, 1} \rightarrow G_{\mu_2, 1} \) such that

\[
\begin{array}{ccccc}
0 & \rightarrow & G_{\lambda_1, 1} & \rightarrow & E^{(\mu_1, \lambda_1; F_1, j_1)} \\
& \downarrow & & \downarrow & \psi_1 \\
0 & \rightarrow & G_{\lambda_2, 1} & \rightarrow & E^{(\mu_2, \lambda_2; F_2, j_2)} \\
& \downarrow & & \downarrow & \psi_2 \\
0 & \rightarrow & G_{\mu_1, 1} & \rightarrow & E^{(\mu_1, \lambda_1; F_1, j_1)} \\
& & & \downarrow & \\
& & & \psi_{r,s} & \rightarrow \\
0 & \rightarrow & G_{\mu_2, 1} & \rightarrow & E^{(\mu_2, \lambda_2; F_2, j_2)} \\
& & & & 0
\end{array}
\]

commutes. More precisely \( \psi_1 \) is given by \( T \mapsto \frac{1 + \lambda_1 T}{\lambda_2} \) and \( \psi_2 \) by \( T \mapsto \frac{1 + \mu_2 T}{\mu_2} \).

**Proposition 3.34.** For \( i = 1, 2 \), if \( F_i(T) = \sum_{k=0}^{p-1} a_k T^k \) and \( \delta_i = E^{(\mu_i, \lambda_i; F_i, j_i)} \) are models of \( \mu_{p^2, K} \) we have

\[
\text{Hom}_{S-gr}(\delta_1, \delta_2) = \begin{cases} 
0, & \text{if } v(\mu_1) < v(\lambda_2); \\
\{\psi_{0,s}\} \simeq \mathbb{Z}/p^2\mathbb{Z}, & \text{if } v(\mu_2) \leq v(\mu_1), v(\lambda_2) \leq v(\lambda_1) \\
\{\psi_{0,s}\} \simeq \mathbb{Z}/p\mathbb{Z} & \text{mod } \lambda_2; \\
\{\psi_{0,s}\} \simeq \mathbb{Z}/p\mathbb{Z}, & \text{if } v(\mu_2) > v(\lambda_2); \\
\end{cases}
\]

Proof. It is immediate to see that \( \psi_{0,s} \in \text{Hom}_{S-gr}(\delta_1, \delta_2) \), with \( s \neq 0 \), if and only if \( v(\mu_1) \geq v(\lambda_2) \). We now see conditions for the existence of \( \psi_{r,s} \) with \( r \neq 0 \). If it exists, in particular, we have two
morphisms \( G_{\mu_1} \rightarrow G_{\mu_2} \) and \( G_{\lambda_1} \rightarrow G_{\lambda_2} \). This implies \( v(\mu_1) \geq v(\mu_2) \) and \( v(\lambda_1) \geq v(\lambda_2) \).

Moreover we have that
\[
F_1(T_1)^r(1 + \mu_1T_1)^r = F_2\left(\frac{(1 + \mu_1T_1)^{\frac{r}{r-j}} - 1}{\mu_2}\right) \in \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}).
\]

Since \( v(\mu_1) \geq v(\mu_2) \) we have
\[
F_1(T_1)^r = F_2\left(\frac{(1 + \mu_1T_1)^{\frac{r}{r-j}} - 1}{\mu_2}\right) \in \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}).
\]

If we define the morphism of groups
\[
\frac{[\mu_1]}{\mu_2} : \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}) \rightarrow \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2})
\]

then
\[
F(T_1)^r = F_2\left(\frac{[\mu_1]}{\mu_2}(T_1)^r\right) \in \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}).
\]

Therefore we have
\[
F(T_1)^r = (F_2\left(\frac{[\mu_1]}{\mu_2}(T_1)\right))^{\frac{r}{r-j}} \in \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}).
\]

Any element of \( \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}) \) has order \( p \). Let \( t \) be an inverse for \( r \) modulo \( p \). Then raising the equality to the \( t \)-th power we obtain
\[
F(T_1)^t = (F_2\left(\frac{[\mu_1]}{\mu_2}(T_1)^t\right))^{\frac{t}{t-j}} \in \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}).
\]

This means
\[
a_1 \equiv \frac{j_1 \mu_1}{j_2 \mu_2} \mod \lambda_2.
\]

Conversely it is clear that, if \( v(\mu_1) \geq v(\mu_2) \), \( v(\lambda_1) \geq v(\lambda_2) \) and
\[
F(T_1)^r = (F_2\left(\frac{[\mu_1]}{\mu_2}(T_1)^r\right))^{\frac{r}{r-j}} \in \text{Hom}_{S_{\lambda_2}}(G_{\mu_1}, G_{m,S_2}),
\]

then (36) defines a morphism of group schemes.

We have the following result which gives a criterion to determine the class of isomorphism, as a group scheme, of an extension of type \( \mathcal{E}(\mu, \lambda; F, j) \).

**Corollary 3.35.** For \( i = 1, 2 \), let \( F_i(T) = \sum_{k=0}^{p-1} \frac{r_i}{r_i-k} T^k \) and let \( \psi_i = \mathcal{E}(\mu_i, \lambda_i; F_i, j_i) \) be models of \( \mu_i^2, K \), with \( \tilde{T}_i \) liftings of \( F_i \). Then they are isomorphic if and only if \( v(\mu_1) = v(\mu_2) \), \( v(\lambda_1) = v(\lambda_2) \) and \( a_1 \equiv \frac{j_1 \mu_1}{j_2 \mu_2} \mod \lambda_2 \). Moreover if it happens then any model map between them is an isomorphism.

**Remark 3.36.** The last sentence is in fact true in more general: any model map between isomorphic finite and flat commutative \( R \)-group schemes is in fact an isomorphism (see [5, Cor. 3]).

**Proof.** By the proposition we have that a model map \( \psi_{r,s} : \mathcal{E}(\mu_1, \lambda_1; F_1, j_1) \rightarrow \mathcal{E}(\mu_2, \lambda_2; F_2, j_2) \) exists if and only if \( v(\mu_1) \geq v(\mu_2) \), \( v(\lambda_1) \geq v(\lambda_2) \) and \( a_1 \equiv \frac{j_1 \mu_1}{j_2 \mu_2} \mod \lambda_2 \). It is a morphism of extensions as remarked before the proposition. Let us consider the commutative diagram (37). Then \( \psi_{r,s} \) is an isomorphism if and only if \( \psi \) is an isomorphism for \( i = 1, 2 \). This is equivalent to requiring \( v(\mu_1) = v(\mu_2) \) and \( v(\lambda_1) = v(\lambda_2) \). This also proves the last assertion. \( \square \)
We conclude the section with the complete classification of $\mu_{p^2,K}$-models. The following theorem summarizes the above results.

**Theorem 3.37.** Let $G$ be a finite and flat $R$-group scheme such that $G_K \simeq \mu_{p^2,K}$. Then $G \simeq \mathcal{E}(\pi^m,\pi^n,F,1)$ for some $m,n \geq 0$, $F(T)$ a lifting of $F(T) = \sum_{k=0}^{p-1} \frac{a^k}{p^k} T^k$ with $(a,1) \in \Phi_{\pi^m,\pi^n}$. If $\text{char}(R) = 0$ then $m \geq n$ and $(p-1)m \leq v(p)$, while if $\text{char}(R) = p$ then $m \geq pn$. Moreover, again by 3.35, it follows that 3.26 and 3.29(iv), then $G$ is a lifting of $(\mathcal{E}(\pi^m,\pi^n,F,1))$.

**Proof.** By 3.1, 3.20 and 3.24 any model of $\mu_{p^2,K}$ is of type $\mathcal{E}(\pi^m,\pi^n,F,1)$ for some $m,n \geq 0$, and $\tilde{G}(T)$ a lifting of $G(T) = \sum_{k=0}^{p-1} \frac{a^k}{p^k} T^k$ with $(b,j) \in \Phi_{\pi^m,\pi^n}$ and $j \neq 0$. Moreover, by 3.2 and by definition of group schemes $G_{\pi^n,1}$, if $\text{char}(R) = 0$ then $m \geq n$ and $(p-1)m \leq v(p)$. While if $\text{char}(R) = p$, by 3.26 and 3.29(iv), then $m \geq pn$.

Now we prove $(\frac{b,j}{p^n}) \in \Phi_{\pi^m,\pi^n}$. If $\text{char}(R) = p$ this is trivial. We now assume $\text{char}(R) = 0$. Since $(b,j) \in \Phi_{\pi^m,\pi^n}$ then $b \in (R/\pi^n R)^F$ and

$$pb - j\pi^m = \frac{p}{\pi^{m-p-1}} b^p \mod \pi^{np}.$$ 

Clearly $\frac{b}{j} \in (R/\pi^n R)^F$. Moreover, multiplying (38) by $\frac{1}{j}$, we have

$$pb - j\pi^m \equiv \frac{p}{\pi^{m-p-1}} \left(\frac{b}{j}\right)^p \mod \pi^{np}.$$ 

Let $a := \frac{b}{j}$, $F(T) = \sum_{k=0}^{p-1} \frac{a^k}{p^k} T^k$ and $\tilde{F}(T)$ a lifting of $F(T)$. Then by 3.35 we can conclude that

$$\mathcal{E}(\pi^m,\pi^n,F,1) \simeq \mathcal{E}(\pi^m,\pi^n,F,1).$$ 

Moreover again by 3.35, it follows that $(a_i,1) \in \Phi_{\pi_{m_i},\pi_{n_i}}$ for $i = 1,2$, correspond to two isomorphic models of $\mu_{p^2,K}$ if and only if $m_1 = m_2$, $n_1 = n_2$ and $a_1 = a_2 \in R/\pi^{n_1} R$.

4. Reduction on the special fiber

In the following we study the special fibers of the extension classes of type $[\mathcal{E}(\mu,\lambda,F,j)]$ with $(F,j) \in \text{rad}_{\mu,\lambda}$. In this section, if $\text{char}(R) = 0$ we suppose that $R$ contains a distinguished primitive $p^2$-th root of unity $\zeta_2$. We remark that the special fibers could be in one of the following $\text{Ext}^1_k$

$$\text{Ext}^1_k(\mu_{p,k},\mu_{p,k}) \quad \text{Ext}^1_k(\alpha_{p,k},\mu_{p,k})$$ 

and moreover, if $\text{char}(R) = 0$,

$$\text{Ext}^1_k(\mu_{p,k},\mathbb{Z}/p\mathbb{Z}) \quad \text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z},\mu_{p,k})$$ 

We recall that we consider only commutative extension. We study separately the different cases which can occur.

4.1. Case $v(\mu) = v(\lambda) = 0$. By 3.6(i), 3.7 and 3.8 we have

$$0 \rightarrow [\mathcal{E}_{j,S}; j \in \mathbb{Z}/p\mathbb{Z}] \rightarrow \text{Ext}^1_S(\mu_{p,S},\mu_{p,S}) \rightarrow H^1(S,\mathbb{Z}/p\mathbb{Z}) \rightarrow 0$$

where the vertical maps are the restriction maps. Clearly the first vertical map is an isomorphism.

4.2. Case $v(\mu) > v(\lambda) = 0$. In such a case we have

$$\{[\mathcal{E}(\mu,\lambda,F,j)]; (F,j) \in \text{rad}_{\mu,\lambda}\} = \{[\mathcal{E}(\mu,\lambda,F,j)]; j \in \mathbb{Z}/p\mathbb{Z}\}.$$ 

It is immediate to see that any extension $[\mathcal{E}(\mu,\lambda,F,j)]$ is trivial on the special fiber.
4.3. Case $v(\lambda) > v(\mu) = 0$. In this case $\{E(\mu,\lambda;F,j); (F,j) \in rad_{p,\lambda}\}$ is trivial (see 3.17).

4.4. Case $v(\mu), v(\lambda) > 0$ and, if $\text{char}(R) = 0$, $v(\mu), v(\lambda) < v(\lambda(1))$. Then $(G_{0,1})_k \simeq (G_{1,1})_k \simeq \alpha_{p,k}$. First, we recall some results about extensions, over $k$, with quotient $\alpha_{p,k}$. See [4, II §3 n°4, III §6 n°7] for a reference.

**Theorem 4.1.** The exact sequence

$$0 \rightarrow \alpha_{p,k} \rightarrow \mathbb{G}_{a,k} \rightarrow \mathbb{F}_{a,k} \rightarrow 0$$

induces the following split exact sequence

$$0 \rightarrow \text{Hom}_{k-\mathbb{M}}(\alpha_{p,k}, \mathbb{G}_{a,k}) \rightarrow \text{Ext}_k^1(\alpha_{p,k}, \alpha_{p,k}) \rightarrow \text{Ext}_k^1(\alpha_{p,k}, \mathbb{G}_{a,k}) \rightarrow 0.$$ 

It is also known that

$$\text{Ext}_k^1(\mathbb{G}_{a,k}, \mathbb{G}_{a,k}) \simeq H_k^2(\mathbb{G}_{a,k}, \mathbb{G}_{a,k}) \rightarrow H_k^2(\alpha_{p,k}, \mathbb{G}_{a,k}) \simeq \text{Ext}_k^1(\alpha_{p,k}, \mathbb{G}_{a,k}).$$

is surjective, where $H_k^2(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ is freely generated as a right $k$-module by 2-cocycles $C_i = V_i + V_{i-1} - (U + V)^{i-1}$, for all $i \in \mathbb{N} \setminus \{0\}$, it follows that $H_k^2(\alpha_{p,k}, \mathbb{G}_{a,k})$ is freely generated as right $k$-module by the class of the cocycle $C_i = V_i + V_{i-1} - (U + V)^{i-1}$. So $\text{Ext}_k^1(\alpha_{p,k}, \mathbb{G}_{a,k}) \simeq k$.

Moreover it is easy to see that $\text{Hom}_k(\alpha_{p,k}, \mathbb{G}_{a,k}) \simeq k$. The morphisms are given by $T \mapsto aT$ with $a \in k$. By these remarks we have that the isomorphism

$$\text{Hom}_k(\alpha_{p,k}, \mathbb{G}_{a,k}) \times \text{Ext}_k^1(\alpha_{p,k}, \mathbb{G}_{a,k}) \rightarrow \text{Ext}_k^1(\alpha_{p,k}, \alpha_{p,k}),$$

deduced from 4.1, is given by

$$(\beta, \gamma C_1) \mapsto [E_{\beta,\gamma}].$$

The group scheme $E_{\beta,\gamma}$ is so defined:

$$E_{\beta,\gamma} := \text{Spec}(k[T_1, T_2]/(T_1^n, T_2^p - \beta T_1))$$

(1) comultiplication

$$T_1 \mapsto T_1 \otimes 1 + 1 \otimes T_1$$

$$T_2 \mapsto T_2 \otimes 1 + 1 \otimes T_2 + \gamma T_1^n \otimes 1 + 1 \otimes T_1^n - (T_1 \otimes 1 + 1 \otimes T_1)^p$$

(2) counit

$$T_1 \mapsto 0$$

$$T_2 \mapsto 0$$

(3) coinverse

$$T_1 \mapsto -T_1$$

$$T_2 \mapsto -T_2$$

or

$$T_1 \mapsto -T_1$$

$$T_2 \mapsto -\gamma T_1^2 - T_2$$

if $p = 2$.

In [22, 4.3.1] the following result was proved.

**Proposition 4.2.** Let $\mu, \lambda \in \pi R \setminus \{0\}$. Then $[E_{\lambda}^{[\mu,\lambda;E_0(\alpha,\mu,T)]}] \in H_k^2(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$ coincides with the class of

$$\sum_{k=1}^{\infty} \left( \frac{\tilde{F}_{k-1}}{\lambda} \right) \tilde{a}_k,$$

where $F - [\mu^{p-1}] = (\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_k, \ldots)$ and $\tilde{a} \in \widetilde{W}(R)$ is a lifting of $a \in \widetilde{W}(R/\lambda R)$. 

We deduce the following corollary about the extensions of \(\alpha_{p,k}\) by \(G_{a,k}\).

**Corollary 4.3.** Let \(\mu, \lambda \in \pi R \setminus \{0\}\). If \(\text{char}(R) = 0\) we assume \(v(\lambda(1)) > v(\mu)\). Then \(\tilde{\mathcal{E}}_{k}^{(\mu, \lambda; E_{p}(a, \mu; T))} \in H^{2}_{0}(\alpha_{p,k}, G_{a,k})\) coincides with the class of 
\[
\frac{(\lambda - \mu^{p-1})\tilde{a}}{\lambda}
\]
where \(\tilde{a} = (\tilde{a}_{0}, \tilde{a}_{1}, \ldots, \tilde{a}_{i}, \ldots) \in \tilde{W}(R)\) is a lifting of \(a \in \tilde{W}(R/\lambda R)\).

**Proof.** This follows from the fact that \(\tilde{\mathcal{E}}_{k}^{(\mu, \lambda; E_{p}(a, \mu; T))} \mapsto \tilde{\mathcal{E}}_{k}^{(\mu, \lambda; E_{p}(a, \mu; T))}\) through the map
\[
\Ext^{1}_{k}(G_{a,k}, G_{a,k}) \simeq H^{2}_{0}(a_{p,k}, G_{a,k}) \rightarrow H^{2}_{0}(a_{p,k}, G_{a,k}) \simeq \Ext^{1}_{k}(a_{p,k}, G_{a,k}).
\]

Let us take an extension class \(\mathcal{E}^{(\mu, \lambda; E_{p}(a, \mu; T), j)}\). Let \(\tilde{a} \in R\) be a lifting of \(a \in R/\lambda R\). We have on the special fiber
\[
\mathcal{E}_{k}^{(\mu, \lambda; \tilde{F}(T), T), j} = \Spec(k[T]/(T^{p} - \tilde{F}(T))^{p}(1 + \mu T)^{-j - 1}),
\]
where \(\tilde{F}(T) = \sum_{i=0}^{p-1} \tilde{a}(\tilde{a} - \mu) \cdots (\tilde{a} - (i-1)\mu) T^{i}\). If \(\text{char}(R) = p\) we have, more precisely,
\[
\mathcal{E}_{k}^{(\mu, \lambda; \tilde{F}(T), T), j} = \Spec(k[T]/(T^{p} - \tilde{F}(T))^{p}(1 + \mu T)^{-j - 1}),
\]
with \(j \mu \equiv 0 \mod \lambda^{p}\).

Let us now suppose \(\text{char}(R) = 0\). Let us consider \(E_{p}(\tilde{a}, \mu; T) \in R[[T]]\). We have, by 3.3(ii), \(E_{p}(\tilde{a}, \mu; T) \equiv \tilde{F}(T) \mod (\lambda T, (1 + \mu T)^{p-1})\). Thus, since
\[
(p - 1)v(\lambda) < v(p) \quad \text{and} \quad T^{p} \equiv 0 \mod (\frac{1}{\mu^{p}}, (1 + \mu T))^{p-1},
\]
we have
\[
E_{p}(\tilde{a}, \mu; T) \equiv \tilde{F}(T) \mod (\lambda^{p}, (1 + \mu T)^{p-1}).
\]
We now suppose \(p > 2\). Let us consider \(E_{p}(\tilde{a}, \mu; T) \in R[[T]]\). By (7), we have
\[
E_{p}((\frac{p}{\mu^{p-1}} \tilde{a})^{p} + \tilde{F}(\tilde{a}^{p}), \mu; T) \equiv E_{p}(\tilde{a}, \mu; T) \equiv (\frac{1}{\mu^{p-1}} (1 + \mu T)^{p} - 1) \mod \lambda^{p}.
\]
Moreover by definitions
\[
E_{p}((\frac{p}{\mu^{p-1}} \tilde{a})^{p} + \tilde{F}(\tilde{a}^{p}), \mu; T) \equiv E_{p}(\tilde{a}, \mu; T) \equiv (1 + \mu T)^{p} - 1 \mod T^{p}.
\]
Hence
\[
E_{p}(\tilde{a}, \mu; T) \equiv E_{p}(\tilde{a}, \mu; T) \equiv (1 + \mu T)^{p} - 1 \mod \lambda^{p}T^{p}.
\]
Therefore, since \(T^{p} \equiv 0 \mod (\frac{1}{\mu^{p}}, (1 + \mu T)^{p-1}),
\]
we have
\[
E_{p}(\tilde{a}, \mu; T) \equiv E_{p}(\tilde{a}, \mu; T) \equiv (1 + \mu T)^{p} - 1 \mod \lambda^{p}T^{p}.
\]
By [22, 2.9.1] we have
\[
E_{p}(\tilde{a}, \mu; T) \equiv E_{p}(\tilde{a}, \mu; T) \equiv (1 + \mu T)^{p} - 1 \mod \lambda^{p}T^{p}.
\]
We remark that, by the definition of sum between Witt vectors and the proof of 3.24,
\[
p[\tilde{a}] - j[\mu] - \frac{p}{\mu^{p-1}} \tilde{a}^{p} - V([\tilde{a}]) = [pa - j\mu - \frac{p}{\mu^{p-1}} \tilde{a}^{p}] + V(e)
\]
with $c \equiv 0 \pmod{\lambda^p}$. Therefore

$$E_p([p \tilde{a}] - j[v] - \left[ \frac{p}{\mu^{p-1} \tilde{a}^{p-1}} \right] - V([\tilde{a}^p]), \mu; T) = E_p([p \tilde{a} - j\mu - \frac{p}{\mu^{p-1} \tilde{a}^p}], \mu; T) = E_p(c, \mu^p; T)^p$$

$$
\equiv E_p([p \tilde{a} - j\mu - \frac{p}{\mu^{p-1} \tilde{a}^p}], \mu; T) \mod \left( \lambda^p \pi, \frac{1 + \mu T^p - 1}{\mu^p} \right),
$$

where the last congruence follows from $T^p \equiv 0 \pmod{(\pi, \frac{(1 + \mu T^p - 1)}{\mu^p})}$ and $c \equiv 0 \pmod{\lambda^p}$. The case $p = 2$ is similar using $E_p([\frac{p}{2} \tilde{a}^2] + V([\tilde{a}^2]) + V([\tilde{a}^2]), \mu; T) \in R[[T]]$.

The above discussion implies, using (39) and (40),

$$\frac{\tilde{F}(T_1)^p (1 + \mu T_1)^{-j} - 1}{\lambda^p} = E_p([p \tilde{a} - j\mu - \frac{p}{\mu^{p-1} \tilde{a}^p}], \mu; T_1) - 1 \equiv \frac{p \tilde{a} - j\mu - \frac{p}{\mu^{p-1} \tilde{a}^p}}{\lambda^p} \mod \left( \pi, \frac{1 + \mu T^p - 1}{\mu^p} \right).$$

On the other hand, with no restriction on the characteristic of $R$, $E_k(\mu, \lambda; E_p(aT), j) \mapsto \tilde{E}_k(\mu, \lambda; E_p(aT))$ through the map $\Ext_k^1(\alpha_{p,k}, \alpha_{p,k}) \mapsto \Ext_k^1(\alpha_{p,k}, G_{a,k})$.

Therefore $E_k(\mu, \lambda; E_p(aT), j) \simeq E_{\beta, \gamma}$ with $\beta = (-\frac{p-1}{\lambda^p} \tilde{a} - \frac{p}{\mu^{p-1} \tilde{a}} \tilde{a}) \mod \pi)$ and $\gamma = (\tilde{a}^p - \frac{p-1}{\lambda} \tilde{a} \mod \pi)$.

We have the following result.

**Proposition 4.4.** Let $\mu, \lambda \in \pi R$. If $\text{char}(R) = 0$ we also assume $v(\lambda(\lambda)) > v(\lambda), v(\mu)$. Then $[\tilde{E}_k(\mu, \lambda; E_p(aT), j)] \in \Ext_k^1(\alpha_{p,k}, \alpha_{p,k})$ coincides with the class of

$$\left( -\frac{\tilde{a} - j\mu - \frac{p}{\mu^{p-1} \tilde{a}^p}}{\lambda^p}, \tilde{a} - \frac{p}{\mu^{p-1} \tilde{a}^p} \tilde{a} \tilde{a} \tilde{C}_1 \right),$$

if $\text{char}(R) = 0$ and

$$\left( \frac{\mu \tilde{a} - j\mu - \frac{p}{\mu^{p-1} \tilde{a}^p}}{\lambda^p}, \tilde{a} - \frac{p}{\mu^{p-1} \tilde{a}^p} \tilde{a} \tilde{C}_1 \right),$$

if $\text{char}(R) = p$, where $\tilde{a} \in R$ is a lifting of $a \in R/R^\pi$.

**4.5. Case char(R)=0 and v(\lambda(\lambda)) = v(\mu) > v(\lambda) > 0.** In this situation we have

$$(G_{\mu,1}) \simeq \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad (G_{\lambda,1}) \simeq \alpha_{p,k}.$$

**Proposition 4.5.** Let $\mu, \lambda \in \pi R \setminus \{0\}$ be such that $v(\mu) = v(\lambda(\lambda)) > v(\lambda).$ Then $E_k(\mu, \lambda; E_p(aT), j)$ is the trivial extension.

**Proof.** We can suppose $\mu = \lambda(\lambda)$. From 3.33 it follows that any extension of type $[\tilde{E}(\lambda(\lambda), \lambda; F, j)]$ is uniquely determined by the induced extension over $K$. Since by 3.29 we have $F(T) \equiv E_p(j\eta T)$ mod $\lambda$ then $[\tilde{E}(\lambda(\lambda), \lambda; F, j)]$ is the image of $[\tilde{E}(\lambda(\lambda), \lambda; E_p(j\eta T), j)]$ through the morphism

$$\Ext^1(G_{\lambda(\lambda),1}, G_{\lambda(\lambda),1}) \mapsto \Ext^1(G_{\lambda(\lambda),1}, G_{\lambda(\lambda),1})$$

induced by the map $\mathbb{Z}/p\mathbb{Z} \simeq G_{\lambda(\lambda),1} \mapsto G_{\lambda,1}$ given by $T \mapsto \frac{\lambda(\lambda)}{\lambda(\lambda)}$. But the above morphism is the zero morphism on the special fiber. So we are done.

We remark that if $v(\lambda) \leq v(\lambda(\lambda))$ then $\eta \equiv 0 \pmod{\lambda}$, indeed in such a case $v(\lambda) \leq v(\lambda(\lambda)) = v(\eta)$.

**4.6. Case char(R)=0 and v(\lambda(\lambda)) = v(\lambda) > v(\mu) > 0.** We have

$$(G_{\mu,1}) \simeq \alpha_p \quad \text{and} \quad (G_{\lambda,1}) \simeq \mathbb{Z}/p\mathbb{Z}.$$

From [4, III §6 7.3] it follows that $\Ext^1_k(\alpha_{p,k}, \mathbb{Z}/p\mathbb{Z}) = 0$. 

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MODELS OF $\mu_{p^2,k}$
4.7. **Case char(R)=0 and v(λ(1))=v(µ)=v(λ).** We have

\[(G_λ,1)_k \cong \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad (G_λ,1)_k \cong \mathbb{Z}/p\mathbb{Z}.
\]

Without loss of generality we can suppose \(µ=λ=λ(1)\). The Artin Schreier sequence

\[0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_{a,k} \xrightarrow{F^{-1}} \mathbb{G}_{a,k} \to 0\]

induces the following exact sequence

\[\text{Hom}_{k-Gr}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k}) \xrightarrow{F^{-1}} \text{Hom}_{k-Gr}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k}) \to \text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to \to \text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k}) \xrightarrow{F^{-1}} \text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k})\]

There are canonical isomorphisms \(\text{Hom}_{k-Gr}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k}) \cong k\) and \(\text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k}) \cong k\) (see [4, III §6 4.3]). Therefore we have the exact sequence

\[0 \to \mathbb{Z}/p\mathbb{Z} \to \text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to k/(F-1)k \to 0.\]

We recall that \(\text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k}) = H^2_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_{a,k})\) is freely generated as a right \(k\)-module by the class of the cocycle \(C_1 = \frac{U^p+V^p-(U+V)^p}{p}\). The above sequence splits and we have the following isomorphism,

\[k/(F-1)k \times \mathbb{Z}/p\mathbb{Z} \to \text{Ext}^1_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}),\]

given by

\[(a,b) \mapsto [E_{a,b}],\]

where the group scheme \(E_{a,b}\) is so defined: let \(\bar{a} \in k\) a lifting of \(a\),

\[E_{a,b} := \text{Spec}(k[T_1, T_2]/(T_1^p - T_1, T_2^p - T_2 - \bar{a}T_1))\]

(1) comultiplication

\[T_1 \mapsto T_1 \otimes 1 + 1 \otimes T_1\]

\[T_2 \mapsto T_2 \otimes 1 + 1 \otimes T_2 + b \cdot \frac{T_1^p \otimes 1 + 1 \otimes T_1^p - (T_1 \otimes 1 + 1 \otimes T_1)^p}{p}\]

(2) counit

\[T_1 \mapsto 0\]

\[T_2 \mapsto 0\]

(3) coinverse

\[T_1 \mapsto -T_1\]

\[T_2 \mapsto -T_2\]

or

\[T_1 \mapsto -T_1\]

\[T_2 \mapsto -bT_2^2 - T_2\]

if \(p = 2\).

We now study the reduction on the special fiber of the group scheme \(\mathcal{E}^{(λ(1),λ(1);E_p(jηT),j)}\) with \(j \in \mathbb{Z}/p\mathbb{Z}\).

**Proposition 4.6.** For any \(j \in \mathbb{Z}/p\mathbb{Z}\), \([\mathcal{E}^{(λ(1),λ(1);E_p(jηT),j)}] = E_{0,j} \in \text{Ext}_k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})\).

**Proof.** As group schemes, \(\mathcal{E}^{(λ(1),λ(1);E_p(jηT),j)} \cong \mathbb{Z}/p^2\mathbb{Z}\), if \(j \neq 0\), and \(\mathcal{E}^{(λ(1),λ(1);0)} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\) otherwise. In particular \(\mathcal{E}^{(λ(1),λ(1);E_p(jηT),j)}\) has a scheme-theoretic section. It is easy to see that \(\mathcal{E}^{(λ(1),λ(1);E_p(jηT),j)} \cong E_{0,b}\) with

\[b = (-j)^{p-1} \mod \pi = j,\]
Since \( \frac{\eta_p}{n(1)} \equiv \frac{\lambda_p}{n(1)} \equiv 1 \mod \pi \) and \((p-1)! \equiv -1 \mod \pi\) (Wilson Theorem).

We finally remarks that the exact sequence (41) also reads as

\[
0 \rightarrow H^0_0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{Ext}^1_0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(\mathbb{K}, \mathbb{Z}/p\mathbb{Z}),
\]

where the isomorphism \( H^1(\mathbb{K}, \mathbb{Z}/p\mathbb{Z}) \simeq k/(F-1)k \) comes from the Artin-Schreier Theory. And we have the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & H^0_0(\mu_{p,K}, \mu_{p,K}) \\
\uparrow & & \uparrow \\
0 & \rightarrow & H^0_0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \\
\uparrow & & \uparrow \\
0 & \rightarrow & H^0_0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \\
\uparrow & & \uparrow \\
0 & \rightarrow & H^0_0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \\
\uparrow & & \uparrow \\
0 & \rightarrow & H^1(\mu_{p,K}, \mu_{p,K}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^1(\mathbb{K}, \mathbb{Z}/p\mathbb{Z}) \\
\end{array}
\]

This diagram shows that the above proposition is an expression of the theory unifying the Kummer and Artin-Schreier theories.

**Appendix A**

**Classification of integral models of \( \mu_{p^2,K} \) via Breuil-Kisin theory**

**by Xavier Caruso**

In this appendix, we show how the theory presented by Breuil in [3] and developed by Kisin in [8] gives us the possibility to obtain very quickly in some cases a statement analogous to 3.37 of this paper. Although our approach is certainly more efficient, it has at least two defects. First, it forces us to assume \( p > 2 \) and \( R \) complete of unequal characteristic with perfect residue field. Therefore, the situation that we will consider in this appendix is slightly less general than the one discussed in the paper. Second, we do not obtain an explicit description of models of \( \mu_{p^2,K} \), but instead we describe some objects of linear algebra which correspond to these models through Breuil-Kisin theory.

**Acknowledgment.** The author thanks Eike Lau for interesting remarks and discussions and for pointing out to him the fact that Kisin’s classification \( a \) priori requires \( p > 2 \).

**Statement of the main theorem.** Let us fix notation. Let \( p \) be an odd prime number and \( K \) a perfect field of characteristic \( p \). We denote by \( W = W(k) \) (resp. \( W_n = W_n(k) \)) the ring of Witt vectors (resp. of truncated Witt vectors) with coefficients in \( k \) and \( K_0 \) its fraction field of \( W \). For any integer \( n \), \( W_n[[u]] \) is endowed with a continuous (for the \( u \)-adic topology) rings endomorphism \( \varphi \) defined as the usual Frobenius on \( W_n \) and by \( \varphi(u) = u^p \). Let’s fix a totally ramified extension \( K \) of \( K_0 \) of degree \( e \) and an uniformizer \( \pi \) of \( K \). We denote by \( E(u) \) the minimal polynomial of \( \pi \) over \( K_0 \) and \( R \) the ring of integers of \( K \). This one corresponds to the d.v.r. considered as base ring in Tossici’s paper.

Let \( \text{Mod}_{W_2[[u]]}^\varphi \) denote the following category:

- objects are \( W_2[[u]] \)-modules \( \mathcal{M} \) with no \( u \)-torsion endowed with a continuous (for the \( u \)-adic topology) \( \varphi \)-semi-linear endomorphism (called Frobenius) \( \varphi_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \) whose image generates a sub-module containing \( E(u)\mathcal{M} \);
- morphisms are the \( W_2[[u]] \)-linear maps which commute with Frobenius.

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In [8], Kisin has constructed an anti-equivalence of categories between \( \text{Mod}^{p}_{W_{2}[[u]]} \) and the category of finite, flat and commutative \( R \)-group schemes annihilated by \( p^{2} \). If we compose with the Cartier duality we obtain an equivalence of category. For our aims, an important property of the latter equivalence will be the following: if \( \mathcal{M} \) is the object of \( \text{Mod}^{p}_{W_{2}[[u]]} \) associated to a group scheme \( G \), then \( \mathcal{M}[1/u] \) completely determines the Galois representation \( G(K) \) (where \( K \) is an algebraic closure of \( K \)), i.e. the generic fiber of \( G \). From this fact, it is easy to prove that \( G \) is a model of \( \mu_{p^{2},K} \) if and only if \( \mathcal{M}[1/u] \) is isomorphic to \( W_{2}((u)) \) endowed with the usual Frobenius. We are going to prove the following result, which is the exact analogue in our context of 3.37.

**Theorem A.1.** Let \( \mathcal{M} \) be the object \( \text{Mod}^{p}_{W_{2}[[u]]} \) associated to a finite flat \( R \)-group scheme whose generic fiber is isomorphic to \( \mu_{p^{2},K} \). Then, there exist \( n,m \in \mathbb{N} \), \( a \in k[[u]] \) satisfying \( \frac{n}{p^{m}} \geq m \geq n \geq 0 \) and

\[
\begin{align*}
(42) & \quad \varphi(a) \equiv 0 \pmod{u^n} \\
(43) & \quad u^{e^{-m(p-1)}}\varphi(a) - u^a \equiv F(u)u^m \pmod{u^{pn}}
\end{align*}
\]

together with two elements \( e_1 \) and \( e_2 \) in \( \mathcal{M} \) such that:

i) \( \mathcal{M} \) is generated over \( W_2[[u]] \) by \( e_1 \) and \( e_2 \) with the unique relation \( u^{n-n}e_1 = pe_2 \);

ii) Frobenius is given by \( \varphi(e_1) = u^{n(p-1)}e_1 \) and \( \varphi(e_2) = u^{m(p-1)}e_2 + u^{-n}\varphi(u) - u^{m(p-1)-n}a e_1 \).

Furthermore two triples \((n, m, a)\) have equal reduction \((n, m, (a \mod u^n))\) if and only if the associated groups are isomorphic.

Conversely, any triple \((n, m, a)\) satisfying (42) and (43) comes from a finite flat \( R \)-group scheme whose generic fiber is isomorphic to \( \mu_{p^{2},K} \).

The last assertion of the Theorem is easy: one just need to check that the \( \varphi \)-module \( \mathcal{M} \) defined by conditions i) and ii) is actually an object of \( \text{Mod}^{p}_{W_{2}[[u]]} \). From now on, we concentrate ourselves to the proof of the rest of the Theorem.

**Proof of existence.** Let \( \mathcal{M} \) be a \( \varphi \)-module over \( W_{2}[[u]] \) such that \( \mathcal{M}[1/u] \) is isomorphic to \( W_{2}((u)) \) endowed with the usual Frobenius. Let us denote by \( \mathcal{M}_{1} \) the kernel of the multiplication by \( p \) on \( \mathcal{M} \) and \( \mathcal{M}_{2} = \mathcal{M}/\mathcal{M}_{1} \). It is easy to verify that they are both modules over \( W_{1}[[u]] = k[[u]] \) with no \( u \)-torsion. Moreover they inherit endomorphisms \( \varphi\mathcal{M}_{1} \) and \( \varphi\mathcal{M}_{2} \), whose images still generate a module which contains \( E(u)\mathcal{M}_{1} = u^{n}\mathcal{M}_{1} \) and \( E(u)\mathcal{M}_{2} = u^{m}\mathcal{M}_{2} \) respectively. In the following, we will write \( \varphi \) for \( \varphi\mathcal{M}_{1} \), \( \varphi\mathcal{M}_{2} \), and \( \varphi\mathcal{M}_{2} \).

**Lemma A.2.** The module \( \mathcal{M}_{1} \) is free of rank 1 over \( k[[u]] \). Moreover, there exists a base \((e_1)\) of \( \mathcal{M}_{1} \) and an integer \( n \in [0, \frac{m}{p-1}) \) such that \( \varphi(e_1) = u^{n(p-1)}e_1 \).

**Proof.** Since \( k[[u]] \) is a discrete valuation ring, the fact that \( \mathcal{M}_{1} \) has no \( u \)-torsion implies that its freeness. Moreover, it is certainly of rank 1 because \( \mathcal{M}_{1}[1/u] \) is isomorphic to the kernel of the multiplication by \( p \) on \( W_{2}((u)) \), that is \( k((u)) \). Let \( e_1 \) be any basis of \( \mathcal{M}_{1} \). From above it follows that we can consider it as an element of \( k((u)) \). We can write \( x = u^{n}y \) where \( y \) is invertible in \( k[[u]] \). Then, if we set \( e_1 = u^{n} \), it is a basis of \( \mathcal{M}_{1} \) and we have \( \varphi(e_1) = u^{n(p-1)}e_1 \) as expected. \( \square \)

In the same way it is possible to prove that \( \mathcal{M}_{2} = k[[u]]e_2 \) with \( \varphi(e_2) = u^{m(p-1)}e_2 \) for some integer \( m \in [0, \frac{n}{p-1}] \). Let \( e_2 \) be any lifting of \( e_2 \). Clearly it is a generator of \( \mathcal{M}[1/u] \) as \( W_{2}((u)) \)-module. We deduce that

\[
(44) \quad e_1 = pu^{-\delta}\alpha e_2
\]

where \( \delta \) is an integer and \( \alpha \) is invertible in \( W_{2}[[u]] \). In fact, \( \alpha \) is defined modulo \( pW_{2}[[u]] \), so that we may (and will) consider it as an element of \( k[[u]] \). The fact that \( e_1 \) generates \( \mathcal{M}_{1} \) easily implies \( \delta \geq 0 \). Moreover, since \( \varphi(e_2) \equiv u^{m(p-1)}e_2 \pmod{p} \), applying \( \varphi \) to (44), we obtain

\[
\varphi(e_1) = pu^{-\delta}\varphi(\alpha)e_2 = pu^{m(p-1)-\delta}\varphi(\alpha)e_2.
\]
Therefore \( \varphi(e_1) = u^{(m-\delta)(p-1)} \varphi(a) - 1 \). Comparing with \( \varphi(e_1) = u^{n(p-1)} e_1 \), we obtain \( m - \delta = n \) and \( \varphi(a) = a \). The first condition gives \( \delta = m - n \) (and in particular \( m \geq n \)), while the second one implies \( a \in \mathbb{F}_p^* \). So, up to replacing \( e_1 \) by \( e_1 \), we may assume \( \alpha = 1 \).

We have just proved that \( \mathfrak{M} \) is generated by two vectors \( e_1 \) and \( e_2 \) related by (44) with \( \alpha = 1 \). This is exactly what appears in the statement of Theorem A.1. We also know that \( \varphi(e_1) = u^{n(p-1)} e_1 \). It still remains to precise the shape of \( \varphi(e_2) \). Let \( z \) denote the image of \( e_2 \in \mathfrak{M}/[1/u] \) through the isomorphism \( \mathfrak{M}/[1/u] \simeq W_2((u)) \). From \( \varphi(e_2) = u^{n(p-1)} e_2 \), we deduce that, up to multiplying \( e_2 \) by a \((p-1)\)-th root of unity, we can write \( z = u^m + pa \), with \( a \in k((u)) \). After some calculations, we obtain

\[
\varphi(e_2) = u^{n(p-1)} e_2 + [u^{-n} \varphi(a) - u^{m(p-1) - n} a] e_1 = u^{n(p-1)} e_2 + be_1.
\]

Hence RHS have to be in \( \mathfrak{M} \), which gives directly (42) (using \( m \geq n \)). Now, using \( E(u) \mathfrak{M} \subset \langle \varphi(e_1), \varphi(e_2) \rangle \) (where the notation \( \langle \cdots \rangle \) means the generated submodule), we find that there exist \( x, y \in W_2[[u]] \) such that

\[
E(u)e_2 = xu^{n(p-1)} e_1 + y(u^{n(p-1)} e_2 + be_1).
\]

Reducing modulo \( p \), we have \( y = u^{x - m(p-1)} + py' \) for some \( y' \in W_2[[u]] \). Since \( x \) and \( y' \) are defined modulo \( p \), one may consider them as elements of \( k[[u]] \). After some calculations, we get \( F(u) = bu^{n-p}\mu + xu^{m-m} + y(u^{p-1}m) \) where \( F(u) \) is defined by the equality \( E(u) = u^p + pF(u) \). As \( m \geq n \), we have \( (p-1)m \geq p - m \). This shows that the equality we obtained is equivalent to the congruence \( bu^{n-p}\mu \equiv F(u) \) (mod \( u^{m-n} \)). Replacing \( b \) by its expression, we finally obtain (43). It remains to be prove that \( a \) is an element of \( k[[u]] \) (a priori, we only know that it belongs to \( k((u)) \), which is clear from (42).

End of the proof. Let \( \mathfrak{M} \) and \( \mathfrak{M}' \) be two \( \varphi \)-modules presented as in Theorem A.1 with parameters \((n, m, a)\) and \((n', m', a')\) respectively. We want to prove that \( \mathfrak{M} \) and \( \mathfrak{M}' \) are isomorphic if and only if \( n = n' \), \( m = m' \) and \( a \equiv a' \fnodim \text{(mod } u^n\text{)} \). Let us assume that there exists an isomorphism \( f : \mathfrak{M} \to \mathfrak{M}' \). Since \( \varphi \) acts by multiplication by \( u^{n(p-1)} \) (resp. \( u^{n'(p-1)} \)) on \( \mathfrak{M} = \ker p_{\mathfrak{M}} \) (resp. \( \mathfrak{M}' = \ker p_{\mathfrak{M}'} \)), we get \( n = n' \). In fact, examining the action of \( \varphi \) on \( e_1 \) and \( e'_1 \) it is easy to see that there exists \( a \in \mathbb{F}_p^* \) such that \( f(e_1) = ae'_1 \). In the same way, regarding actions of \( \varphi \) on quotients \( \mathfrak{M}/[1/u] \) and \( \mathfrak{M}'/\mathfrak{M}'_1 \), we have \( m = m' \). Then, equalities \( u^{m-n} e_1 = pe_2 \) and \( u^{m-n} e'_1 = pe'_2 \) give \( f(e_2) = ae'_2 + xe_1 \) for some element \( x \in k[[u]] \). A little computation shows that the compatibility with \( \varphi \) implies

\[
\alpha u^{-n} \varphi(a') - \alpha u^{n(p-1) - n} a' + \varphi(x)u^{n(p-1)} - xu^{m(p-1)} + \alpha u^{-n} \varphi(a) - \alpha u^{n(p-1) - n} a,
\]

which gives \( \varphi(t) = u^{m(p-1)} t \) where we set \( t = a(a' - a) + u^n x \). Comparing \( u \)-adic evaluations of both sides, we see that any solution \( t \) has to be divisible by \( u^m \). As \( m \geq n \), we have \( a \equiv a' \fnodim \text{(mod } u^n\text{)} \) as wanted. Conversely, if \( a \equiv a' \fnodim \text{(mod } u^n\text{)} \), it is sufficient to set \( \alpha = 1 \) and \( x = \frac{a - a'}{u^n} \) to obtain an isomorphism \( f : \mathfrak{M} \to \mathfrak{M}' \).

Discussion about \( p = 2 \). In this appendix, we have assumed \( p > 2 \). The only restriction for that is the fact that Kisin’s equivalence of categories between \( \text{Mod}^{\mathfrak{p}}_{\mathfrak{W}_2([u])} \) and finite flat and commutative \( R \)-group schemes annihilated by \( p^2 \) requires (at least nowadays) \( p \neq 2 \).

Nevertheless, in [9], Kisin proved that the equivalence of categories still holds for \( p = 2 \) between \( R \)-group schemes with unipotent special fiber and a suitable subcategory of \( \text{Mod}^{\phi}_{\mathfrak{W}_2([u])} \). Using this result and the method of this appendix, one can classify all models of \( \mu_{p^2} K \) with unipotent special fiber, which we will call, for simplicity, \textit{unipotent} models. Actually, it is not so hard to find by hand non-unipotent models. Indeed, by Lemma 3.1 of thee paper, one knows that one model \( G \) of \( \mu_{p^2} K \) must be an extension of \( G_{\mu, 1} \) by \( G_{\lambda, 1} \). If \( G \) is not unipotent, it means that \( G_{\mu, 1} \) or \( G_{\lambda, 1} \) is itself not unipotent, that is \( \mu \) or \( \lambda \) is a unit. By Lemma 3.2 of the paper, it follows that \( \lambda \) is a unit, i.e. \( G_{\lambda, 1} \simeq \mu_{p^2} R \). This remaining case can be treated rather easily (cf. §3.4 of loc. cit.): in particular it does not use Sekiguchi-Suwa’s theory.

Moreover in a very recent note (still unpublished), Eike Lau proved that Kisin’s classification also works in general for \( p = 2 \). Consequently, the method of the appendix can finally be applied without restriction on the characteristic \( p \).
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