THE SPLIT FEASIBILITY PROBLEM WITH POLYNOMIALS

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Abstract. This paper discusses the split feasibility problem with polynomials. The sets are semi-algebraic, defined by polynomial inequalities. They can be either convex or nonconvex, either feasible or infeasible. We give semidefinite relaxations for representing the intersection of the sets. Properties of the semidefinite relaxations are studied. Based on that, a semidefinite relaxation algorithm is given for solving the split feasibility problem. Under a general condition, we prove that: if the split feasibility problem is feasible, we can get a feasible point; if it is infeasible, we can obtain a certificate for the infeasibility. Some numerical examples are given.

1. Introduction

The split feasibility problem (SFP) can be stated as follows: for two given sets $C \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^m$, and a given matrix $A \in \mathbb{R}^{m \times n}$, find a point $x^*$ such that

\begin{equation}
 x^* \in C, \quad Ax^* \in Q.
\end{equation}

Here, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space over the real field. The SFP was originally introduced by Censor and Elfving [5] for modeling phase retrieval problems. It can serve as a unified tool for modeling many different inverse problems, such as image reconstruction, signal processing and intensity-modulated radiation therapy problems. We refer to [3, 4, 5, 7] and the references therein for related work on split feasibility problems. Later, the SFP was generalized to the multiple-sets split feasibility problem (MSFP), which was introduced by Censor et al. [6].

The MSFP can be stated similarly as follows: for given sets $C_1, \ldots, C_r \subset \mathbb{R}^n$, $Q_1, \ldots, Q_t \subset \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$, find a point $x^*$ such that

\begin{equation}
 x^* \in \bigcap_{i=1}^r C_i \text{ such that } Ax^* \in \bigcap_{j=1}^t Q_j.
\end{equation}

In particular, if $t = r = 1$, the MSFP collapses to the SFP.

In the prior existing literature, $C, Q, C_i, Q_j$ are often assumed to be nonempty closed convex sets. The projection type methods have been widely used for solving the SFPs and MSFPs. We refer to the work Byrne [3, 4], Censor et. al. [5, 6, 7, 8] and others [9, 10, 26, 32, 33, 36].

The CQ algorithm, proposed by Byrne [3, 4], is a classic method for solving the SFP. Many other methods for solving SFPs and MSFPs can be viewed as variations of it. The CQ method has the basic iteration form:

\begin{equation}
 x^{k+1} = P_C(x^k - \gamma A^T(I - P_Q)Ax^k),
\end{equation}

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where $\gamma \in (0, 2/\rho(A^T A))$ is a parameter and $\rho(A^T A)$ denotes the largest eigenvalue of $A^T A$. The $P_C$ (resp., $P_Q$) stands for the projection onto the set $C$ (resp., $Q$). Usually, the performance of CQ type methods depends on the initial point $x^0$, the choice of $\gamma$ and the geometry of the sets. Moreover, they also require the sets $C$, $Q$, $C_1$, $Q_j$ to be convex. Although a lot of progresses have been made, there still exist computational challenges for CQ type methods.

In this paper, we focus on the split feasibility problem with polynomials, i.e., $C, Q$ are semi-algebraic sets given as

$$C := \{ x \in \mathbb{R}^n | f_i(x) \geq 0, \ i = 1, ..., r \},$$

$$Q := \{ y \in \mathbb{R}^m | g_j(y) \geq 0, j = 1, ..., t \}.$$  

In the above, the functions $f_i(x)$, $g_j(y)$ are real multivariate polynomials. The problem can also be formulated as a MSFP, with

$$C_i := \{ x \in \mathbb{R}^n | f_i(x) \geq 0 \}, \quad Q_j := \{ y \in \mathbb{R}^m | g_j(y) \geq 0 \}.$$

In this paper, the sets $C$, $Q$, $C_i$ and $Q_j$ are not necessarily assumed to be convex.

Since the sets are defined by polynomial inequalities, we propose semidefinite programming (SDP) relaxation methods for solving the SFP, using Lasserre type moment relaxations. The properties of the SDP relaxations are studied. Under a general condition, we show that: if the SFP is feasible (i.e., it has at least one solution), then we can compute a point $x^*$ satisfying (1.1); if the SFP is infeasible (i.e., there is no point $x^*$ satisfying (1.1)), then we can obtain a certificate for the infeasibility.

The paper is organized as follows. In Section 2, we give semidefinite relaxations for the sets in the SFP. To do this, we need some tools from polynomial optimization [14, 15, 16, 19, 20].

### Notation.

The symbol $\mathbb{N}$ stands for the set of nonnegative integers, and $\mathbb{R}$ for the set of real numbers. For $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the smallest integer not smaller than $s$. For $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, denote

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n.$$  

The $x_i$ (resp., $\alpha_i$) denotes the $i$-th entry of $x$ (resp., $\alpha$). For a degree $d > 0$, denote

$$\mathbb{N}^d := \{ \alpha \in \mathbb{N}^n | |\alpha| \leq d \}.$$  

Denote by $[x]_d$ the column vector of all monomials in $x$ and of degrees at most $d$ (they are ordered in the graded lexicographical ordering), i.e.,

$$[x]_d := [1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_{n-1} x_n^{d-1}, x_n^d]^T.$$  

The symbol $\mathbb{R}[x]$ denotes the ring of polynomials in $x$ with real coefficients, and $\mathbb{R}[x]_d$ is the space of real polynomials in $x$ with degrees at most $d$. For a polynomial $p$, $\text{deg}(p)$ stands for its total degree. For a symmetric matrix $X$, $X \succeq 0$ means $X$ is positive semidefinite. For a vector $x$, $\|x\|$ denotes its Euclidean norm. In the space $\mathbb{R}^n$, $e$ denotes the vector of all ones, while $e_i$ denotes the $i$-th unit vector in the
canonical basis. The 0 denotes the zero vector. Denote by $I$ the identity matrix, when the dimension is clear in the context.

2.1. Localizing matrices. The set $\mathbb{R}^{n_d}$ is the space of all real vectors that are labeled by $\alpha \in \mathbb{N}_d^n$. That is, every $y \in \mathbb{R}^{n_d}$ can be labeled as

$$y = (y_\alpha)_{\alpha \in \mathbb{N}_d^n}.$$ 

In some literature, e.g., [23], such $y$ is called truncated multi-sequences (tms) of degree $d$. For a polynomial $f \in \mathbb{R}[x]_{2k}$, the product $f(x)[x]_s[x]_s^T$ is a symmetric matrix polynomial of length $(n^*_s)$, where $s = \lceil k - \deg(f)/2 \rceil$. Expand it as

$$f(x)[x]_s[x]_s^T = \sum_{\alpha \in \mathbb{N}_{2k}} x_\alpha F_\alpha,$$

for symmetric matrices $F_\alpha$. For $y \in \mathbb{R}^{n_{2k}}$, define the matrix

(2.1) $$L_{f}^{(k)}[y] := \sum_{\alpha \in \mathbb{N}_{2k}} y_\alpha F_\alpha.$$

It is called the $k$th localizing matrix generated by $f$ and $y$. Clearly, for any fixed $f$, $L_{f}^{(k)}[y]$ is linear in $y$; for any fixed $y$, $L_{f}^{(k)}[y]$ is linear in $f$ (with fixed degree).

**Fact 2.1.** If $f(u) \geq 0$ and $y = [u]_{2k}$, then

$$L_{f}^{(k)}[y] = f(u)[u]_s[u]_s^T \succeq 0.$$ 

**Example 2.2.** For the case $n = 2$, $k = 2$ and $f = 1 - x_1^2 - x_2^2$, we have

$$L_{f}^{(2)}[y] = \begin{bmatrix}
y_{00} - y_{20} - y_{02} & y_{10} - y_{30} - y_{12} & y_{01} - y_{21} - y_{03} \\
y_{10} - y_{30} - y_{12} & y_{20} - y_{40} - y_{22} & y_{11} - y_{31} - y_{13} \\
y_{01} - y_{21} - y_{03} & y_{11} - y_{31} - y_{13} & y_{02} - y_{22} - y_{04}
\end{bmatrix}.$$ 

2.2. Semidefinite relaxations. The split feasibility problem is to find a point $x^*$ such that

(2.2) $$x^* \in C := \bigcap_{i=1}^t C_i, \quad Ax^* \in Q := \bigcap_{j=1}^t Q_j,$$

where $A \in \mathbb{R}^{m \times n}$ is given and

$$C_i = \{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \}, \quad Q_j = \{ y \in \mathbb{R}^m \mid g_j(y) \geq 0 \},$$

for polynomials $f_i(x)$, $g_j(y)$. For convenience, denote

(2.3) $$h_j(x) := g_j(Ax).$$

Each $h_j$ is a polynomial in $x$. Let

(2.4) $$H := \{ x \in \mathbb{R}^n \mid h_j(x) \geq 0, j = 1, \ldots, t \}.$$ 

The SFP is equivalent to finding a point

$$x^* \in C \cap H.$$ 

Denote the degrees

(2.5) $$d_{f,i} := \lceil \deg(f_i)/2 \rceil, \quad d_{h,j} := \lceil \deg(h_j)/2 \rceil,$$

$$d := \max_{i,j} \{ d_{f,i}, d_{h,j} \}.$$
Fact 2.3. For all $k \geq d$ and for all $u \in C \cap H$, we have
\[ f_i(u), h_j(u) \geq 0 \]
for all $i, j$. Hence,
\[ f_i(x)[x]^{k-d_{f,i}}([x]^{k-d_{f,i}})^T \succeq 0, \quad h_j(x)[x]^{k-d_{h,j}}([x]^{k-d_{h,j}})^T \succeq 0. \]
This implies that if $y = [u]_{2k}$ and $u \in C \cap H$, then
\[ L_{f_i}^{(k)}[y] \succeq 0, \quad L_{h_j}^{(k)}[y] \succeq 0. \]
Let $f_0 = 1$, then $f_0(x)[x]_1^{k} \succeq 0$ for all $x \in \mathbb{R}^n$. So,
\[ L_{f_i}^{(k)}[y] \succeq 0 \]
for all $y = [u]_{2k}$. It is called the moment matrix of $y$.

Note that
\[ C \cap H = \left\{ x \in \mathbb{R}^n \mid f_1(x) \geq 0, \ldots, f_r(x) \geq 0, h_1(x) \geq 0, \ldots, h_s(x) \geq 0 \right\}. \]
So, $C \cap H$ is always contained in the set (note $f_0 = 1$)
\[ S_k := \left\{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^{n_k}, y_0 = 1, \begin{array}{l} x = (y_{c_1}, \ldots, y_{c_n}), \quad \exists y \in \mathbb{R}^{n_k}, y_0 = 1, \end{array} \right. \]

(2.6)
\[ L_{f_i}^{(k)}[y] \succeq 0 (i = 0, \ldots, r), \quad L_{h_j}^{(k)}[y] \succeq 0 (j = 1, \ldots, s) \]
for all $k \geq d$. Each $S_k$ is the projection of a set in $\mathbb{R}^{n_k}$ that is defined by linear matrix inequalities. It is a semidefinite relaxation of $C \cap H$, because $C \cap H \subseteq S_k$ for all $k \geq d$. It holds the nesting containment relation \[ S_k \supseteq S_{k+1} \supseteq \cdots \supseteq C \cap H. \]

(2.7)

2.3. Some basic properties. The first one is about the feasibilities and infeasibilities between $C \cap H$ and the semidefinite relaxations $S_k$ in (2.6).

Proposition 2.4. Let $C, H$ be the sets in the above. If the intersection $C \cap H \neq \emptyset$, then the semidefinite relaxation $S_k \neq \emptyset$ for all $k \geq d$. Therefore, if $S_k = \emptyset$ for some $k \geq d$, then $C \cap H = \emptyset$, i.e., the split feasibility problem is infeasible.

Proof. For all $u \in C \cap H$ and $k \geq d$, the tms $y := [u]_{2k}$ satisfies the linear matrix inequalities (for all $i, j$)
\[ L_{f_i}^{(k)}[y] \succeq 0, \quad L_{h_j}^{(k)}[y] \succeq 0. \]

Note that $y_0 = 1$ and $u = (y_{c_1}, \ldots, y_{c_n})$, so $u \in S_k$. Therefore, $C \cap H \neq \emptyset$ implies $S_k \neq \emptyset$ for all $k \geq d$. Consequently, if there exists $k \geq d$ such that $S_k = \emptyset$, then we must have $C \cap H = \emptyset$. \[ \square \]

Remark 2.5. In Proposition 2.4 we do not require the set $C$ or $H$ to be convex. Moreover, neither $f_i$ nor $h_j$ is assumed to be concave.

Next, we give conditions for the equality $S_k = C \cap H$. A typical one is the sos-convexity/concavity.
Definition 2.6. (12) A polynomial \( f(x) \in \mathbb{R}[x] \) is called sos-convex if there exists a matrix polynomial \( P(x) \in \mathbb{R}[x]^{\ell \times n} \) such that (\( \ell \) may be different from \( n \))

\[
\nabla^2 f(x) := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^n = P(x)^T P(x).
\]

Similarly, \( f(x) \) is called sos-concave if \(-f(x)\) is sos-convex.

A polynomial \( p \in \mathbb{R}[x] \) is said to be sos if \( p = p_1^2 + \cdots + p_k^2 \), for some real polynomials \( p_1, \ldots, p_k \in \mathbb{R}[x] \). We refer to [12] for sos polynomials.

Theorem 2.7. Let \( C, H, S_k \) be the sets in the above. Assume that the polynomials \( f_i(x) \) and \( h_j(x) \) are all sos-concave. Then, \( S_k = C \cap H \) for all \( k \geq d \).

Proof. In [27], we have already seen that \( C \cap H \subseteq S_k \) for all \( k \geq d \). We need to prove the reverse containment \( C \cap H \supseteq S_k \). Choose an arbitrary point \( u \in S_k \), with \( u = (y_{e_1}, \ldots, y_{e_n}) \) and \( y \) satisfying the conditions in (2.6). We show \( u \in C \cap H \) as follows. Consider the new polynomial:

\[
\hat{f}_i(x) := -f_i(x) + f_i(u) + \nabla f_i(u)^T (x-u).
\]

It is a polynomial in \( x \), for fixed \( u \). Note that

\[
\hat{f}_i(u) = 0, \quad \nabla \hat{f}_i(u) = 0.
\]

By Lemma 8 of [12], we know that \( \hat{f}_i \) is an sos polynomial, say,

\[
\hat{f}_i = p_1(x)^2 + \cdots + p_N(x)^2,
\]

for some real polynomials \( p_1, \ldots, p_N \in \mathbb{R}[x] \). Define the linear functional

\[
(2.8) \quad \mathcal{R}_y : \mathbb{R}[x]_{2k} \to \mathbb{R}, \quad \mathcal{R}_y(x^\alpha) = y_\alpha,
\]

for all \( \alpha \in \mathbb{N}^n_{2k} \). Since \( L^{(k)}_1[y] \geq 0 \) (note \( f_0 = 1 \)), we can see that

\[
\mathcal{R}_y(\hat{f}_i) = \sum_{\ell=1}^N vec(p_\ell)^T \left( L^{(k)}_1[y] \right) vec(p_\ell) \geq 0.
\]

Here, \( vec(p_\ell) \) denote the coefficient vector of \( p_\ell \). So,

\[
\mathcal{R}_y(\hat{f}_i) = -\mathcal{R}_y(f_i) + f_i(u) \geq 0,
\]

because \( \mathcal{R}_y(\nabla f_i(u)^T (x-u)) = 0 \). Also note that \( \mathcal{R}_y(f_i) \) is the \((1,1)\)-entry of the matrix \( L^{(k)}_f[y] \). Thus, \( L^{(k)}_f[y] \geq 0 \) and the above imply that

\[
f_i(u) \geq \mathcal{R}_y(f_i) \geq 0.
\]

This is true for all \( i = 1, \ldots, r \). In the same way, we can prove that \( h_j(u) \geq 0 \) for all \( j \). Therefore, \( u \in C \cap H \) for all \( u \in S_k \), i.e., \( S_k \subseteq C \cap H \).

Remark 2.8. In Theorem 2.7, we do not assume that the sets \( C, H \) have nonempty interiors. In other words, even if either \( C \) or \( H \) has empty interior, the conclusion \( C \cap H = S_k \) is still true, under the sos-concavity assumption on \( f_i, h_j \). This is stronger than the results in [12].

When the polynomials \( f_i, h_j \) are not sos-concave, there still exist general conditions ensuring \( C \cap H = S_k \). We refer to [11, 12, 17, 18] for related work.
3. An algorithm for solving the SFP

Let $C, Q$ be the sets as in (1.3)-(1.4), defined by polynomials $f_i, g_j$, with a given matrix $A \in \mathbb{R}^{m \times n}$ and $h_j(x) = g_j(Ax)$. The set $H$ is as in (2.4). In this section, the sets $C, Q, H$ are not assumed to be convex, and none of $f_i, h_j$ is assumed to be concave. The split feasibility problem is to find a point $x^* \in C$ such that $Ax^* \in Q$, which is equivalent to $x^* \in C \cap H$.

In this section, we propose an algorithm for solving the SFP, based on the semidefinite relaxations in (2.6). If it is feasible, we want to get a point $x^* \in C \cap H$; if it is infeasible, we want to obtain a certificate for the infeasibility.

Let $d$ be the degree as in (2.6). Choose a generic vector $\xi \in \mathbb{R}^{N_{2d}}$ such that $\|\xi\| \leq \frac{1}{2}$. Consider the polynomial optimization problem

$$(3.1) \quad \text{min} \quad c(x) := \|x\| d + \xi^T [x] 2d,$$

s.t. $f_1(x) \geq 0, \ldots, f_r(x) \geq 0,$

$h_1(x) \geq 0, \ldots, h_t(x) \geq 0.$

Because $\|\xi\| \leq \frac{1}{2}$, the objective $c(x)$ is a coercive function, i.e., the sublevel set $\{x \in \mathbb{R}^n : c(x) \leq M\}$ is compact for all $M$, so (3.1) must have a global minimizer whenever the SFP is feasible. For solving (3.1), the Lasserre type moment relaxation of order $k (\geq d)$ is

$$(3.2) \quad \text{min} \quad \sum_{|\beta| \leq d} y_{2\beta} + \sum_{|\alpha| \leq 2d} \xi_{\alpha} \alpha,$$

s.t. $L^{(k)}_i[y] \geq 0 (1 \leq i \leq r),$

$L^{(k)}_h[y] \geq 0 (1 \leq j \leq t),$

$L^{(k)}_i[y] \geq 0, y_0 = 1, y \in \mathbb{R}^{N_{2k}}.$

The set of $y$ satisfying (3.2) is the same as the set of $y$ satisfying (2.6). So, (3.2) is a semidefinite relaxation for solving the SFP.

Algorithm 3.1. Let $f_i, h_j$ be as above. Set $k := d$.

Step 1 Solve the semidefinite program (3.2). If it is infeasible, then the SFP is infeasible and stop; otherwise, solve it for an optimizer $y^*$ if it exists and then go to Step 2.

Step 2 Let $u^k := (y^k_1, \ldots, y^k_e)$. If $f_i(u^k) \geq 0$ and $h_j(u^k) \geq 0$ for all $i, j$, then $u^k$ is a solution of the SFP; otherwise, let $k := k + 1$ and go to Step 1.

The conclusion about infeasibility in Step 1 is justified by Proposition 2.4 while the conclusion of Step 2 is very straightforward. When the SFP is feasible, the semidefinite program (3.2) does not necessarily have an optimizer. To guarantee the solvability of (3.2), we need the following assumption.

Assumption 3.2. There exist sos polynomials $a_0, a_1, \ldots, a_r, b_1, \ldots, b_t$ and a real number $R > 0$ such that

$$(3.3) \quad R - \|x\| d^2 = a_0 + a_1 f_1 + \cdots + a_r f_r + b_1 h_1 + \cdots + b_t h_t$$

and $\deg(a_i f_i), \deg(b_j h_j) \leq 2d$ for all $i, j$.

Assumption 3.2 implies that $\|x\| d^2 \leq R$ for all $x \in C \cap H$, so the intersection $C \cap H$ is bounded. The reverse is not necessarily true. However, when $C \cap H$ is bounded, say, $\|x\| d^2 \leq R$, we can add the polynomial $R - \|x\| d^2$ to the set $\{f_1, \ldots, f_r\}$, while the intersection $C \cap H$ is not changed. If the degree bounds
on $a_i, b_j$ are removed, Assumption 3.2 becomes the classical archimedean condition [14, 24]. The convergence of Algorithm 3.1 is summarized in the following theorem.

**Theorem 3.3.** Let $C, Q, f_i, h_j$ be as above. If Assumption 3.2 holds, then we have the properties:

(i) If the SFP is infeasible, then the semidefinite program (3.2) must be infeasible for all $k$ big enough.

(ii) If the SFP is feasible and $\xi$ is generically chosen such that $\|\xi\| \leq \frac{1}{2}$, then $\xi$ has an optimizer for all $k \geq d$ and the sequence $\{u^k\}_{k=d}^\infty$ converges to a solution $x^*$ of the SFP.

**Proof.** (i) For convenience of notation, let $f_{r+j} = h_j$ for $j = 1, \ldots, t$. Then

$$C \cap H = \{x \in \mathbb{R}^n : f_i(x) \geq 0, i = 1, \ldots, r + t\}.$$ 

If the SFP is infeasible, the intersection $C \cap H$ is empty. By the Positivstellensatz [2], there exist sos polynomials $s_\delta (\delta \in \{0, 1\}^{r+t})$ such that

$$-2 = \sum_{\delta \in \{0, 1\}^{r+t}} s_\delta f_1^{\delta_1} \cdots f_{r+t}^{\delta_{r+t}},$$

Denote the polynomial

$$p := 1 + \sum_{\delta \in \{0, 1\}^{r+t}} s_\delta f_1^{\delta_1} \cdots f_{r+t}^{\delta_{r+t}},$$

which is the constant polynomial $-1$. Clearly, $p$ is positive on $C \cap H$. Denote

$$F := (f_1, \ldots, f_{r+t}).$$

Define the $k$th quadratic module of $F$ ($f_0 = 1$)

$$\text{Qmod}_k(F) := \left\{ \sum_{i=0}^{r+t} s_i f_i \mid s_i \text{ is sos, } \deg(s_i f_i) \leq 2k \right\}.$$ 

It is a convex cone. By Putinar’s Positivstellensatz [24], we have $p \in \text{Qmod}_k(F)$ (i.e., $-1 \in \text{Qmod}_k(F)$ because $p = -1$), when $k$ is sufficiently large. The dual optimization problem of (3.2) can be shown to be (cf. [14])

$$\begin{equation}
(3.4) \quad \max \gamma \quad \text{s.t.} \quad c - \gamma \in \text{Qmod}_k(F).
\end{equation}$$

Every feasible $\gamma$ in (3.4) is a lower bound for the objective value of (3.2), whenever $y$ is feasible. This is the so-called weak duality. For all $k$ sufficiently large such that $-1 \in \text{Qmod}_k(F)$, the problem (3.4) is unbounded from above, by which the weak duality implies that (3.2) is infeasible.

(ii) Let $\mathcal{A}_y$ be the linear functional defined as in (2.8). By the equality (3.3) in Assumption 3.2, we can get (note $f_0 = 1$, $y_0 = 1$)

$$R - \sum_{|\beta| \leq d} y_{2\beta} = \mathcal{A}_y(R - \|x\|_d^2) = \sum_{i=0}^r \mathcal{A}_y(a_i f_i) + \sum_{j=1}^t \mathcal{A}_y(b_j h_j).$$

Since $a_i = p_1^2 + \cdots + p_t^2$ is sos, one can verify that

$$\mathcal{A}_y(a_i f_i) = \sum_{j=1}^t \mathcal{A}_y(p_j^2 f_i) = \sum_{j=1}^t \text{vec}(p_j)^T \left( L_{f_i}^{(k)}(y) \right) \text{vec}(p_j) \geq 0,$$
because $L_k^{(k)}[y] \geq 0$. (The $\text{vec}(p_j)$ denotes the coefficient vector of $p_j$.) Similarly, we can show that $\mathcal{A}_y(b_j h_j) \geq 0$, because $L^{(k)}_{h_j}[y] \geq 0$. So, $R \geq \sum_{|\beta| \leq d} y_{2\beta}$. Moreover, for all $x^\theta$ with $|\theta| \leq k - d$,
\[
Rx^{2\theta} - \|x^{\theta}[x]\|^2 = \sum_{i=0}^r (a_i x^{2\theta}) f_i + \sum_{j=1}^t \mathcal{A}_y(b_j x^{2\theta}) h_j.
\]
By the same argument, we can show that
\[
(3.5) \quad R y^{2\theta} \geq \sum_{|\beta| \leq d} y_{2\beta + 2\theta}
\]
for all $y$ that is feasible in (3.2). The diagonal entries of $L_{1}^{(k)}[y]$ are precisely $y_{2\beta}$ with $|\beta| \leq k$. Since $L_{1}^{(k)}[y] \geq 0$, $y_{2\beta} \geq 0$ for all $|\beta| \leq k$ and $\|y\|^2$ is bounded by the trace of $L_{1}^{(k)}[y]$. Applying (3.5) recursively, one can see that the norm of $y$ can be bounded by a constant that is only depending on $R, k$. So, the feasible set of (3.2) is compact, and hence (3.2) must have an optimizer.

Since $\|\xi\| \leq \frac{1}{2}$, the objective $c(x)$ is a coercive polynomial. When $\xi$ is generically chosen, the objective $c(x)$ has a unique optimizer over the set $C \cap H$. By Corollary 3.5 of [23] or Theorem 3.3 of [21], the sequence $\{u^{k}\}_{k=0}^{\infty}$ must converge to the unique optimizer of (3.1), which is clearly a solution to the SFP. \hfill \Box

**Remark 3.4.** Under some general conditions, Algorithm [3.7] must terminate in finitely many steps. To be more precisely, under the standard constraint qualification, the second order sufficiency and the strict complementarity conditions for the optimization problem (3.1), the hierarch of Lasserre type moment relaxations (3.2) must have finite convergence. We refer to [22] for more details about the finite convergence and refer to [21] about how to detect convergence.

In particular, Algorithm [3.7] terminates at the first iteration with $k = d$, when the polynomials $f_i, h_j$ are sos-concave. This is because $C \cap H = S_d$, by Theorem [2.7], and hence the set of $u = (y_1, \ldots, y_n)$, with $y$ satisfying all the constraints in (3.2), is the same as the feasible set of (3.7).

4. **NUMERICAL EXPERIMENTS**

This section reports numerical experiments for solving split feasibility problems with polynomials. Algorithm [3.1] is applied to solve them. It can be implemented conveniently by the software GloptiPoly [13], which calls SeDuMi [25] for solving the semidefinite programs. The computation is implemented in MATLAB R2014 on a laptop with Intel(R) Core(TM) i5-3337U CPU (1.80 GHz). The computational results are displayed in four decimal digits. For convenience of expression, we give the sets $C, Q$ by inequalities of the form $-f_i(x) \leq 0$ or $-g_j(y) \leq 0$. If there is an equality $f(x) = 0$, it can be equivalently expressed as: $-f(x) \leq 0, f(x) \leq 0$.

The classical CQ algorithm for solving SFPs has the iterative formula
\[
x^{k+1} = P_C(x^k - \gamma A^T(I - P_Q)Ax^k),
\]
for $k = 0, 1, 2, \ldots$, where $0 < \gamma < 2/\rho(A^TA)$ is a parameter and $P_C, P_Q$ denote projections onto the sets $C, Q$ respectively. To implement the CQ algorithm,
one needs to compute the projection \( P_C(u) \), which is equivalent to solving the optimization problem

\[
P_C(u) = \arg \min_{z \in C} \frac{1}{2} \| z - u \|^2.
\]

The same is true for \( P_Q \). This might require a large amount of computations. In practice, people often apply the relaxed CQ algorithm \([33]\), in which only the projections onto hyperplane or half spaces are required. A typical iterative formula for relaxed CQ algorithm is:

\[
x_{k+1} = P_C(x_k - \gamma A^T(I - P_Q)x_k),
\]

where \( C_k \) and \( Q_k \) are half spaces passing through \( x_k \) that contain the sets \( C \) and \( Q \), respectively. In Example 4.1 we choose \( \gamma = 1.8/\rho(A^T A) \). The stopping criterion is: \( f_i(x_k) < 10^{-5} \) and \( g_j(Ax_k) < 10^{-5} \), or \( k \geq k_{\text{max}} = 10^6 \). The performance of the relaxed CQ algorithm usually depends on the initial point \( x_0 \) and the geometry of the SFP. Moreover, the CQ type algorithms are not applicable when the sets \( C, Q \) are not convex. In contrast, our Algorithm 3.1 does not assume the sets are convex; it can also detect infeasibility.

First, we show an example of comparing the relaxed CQ algorithm and Algorithm 3.1.

**Example 4.1.** Consider the sets \( C, Q \):

\[
C = \left\{ x \in \mathbb{R}^3 : \frac{1}{2} (x - e)^T B_1 (x - e) + b_1^T (x - e) \leq 0 \right\},
\]

\[
Q = \left\{ y \in \mathbb{R}^2 : \frac{1}{2} (y - Ae)^T B_2 (y - Ae) + b_2^T (y - Ae) \leq 0 \right\},
\]

where \( a \) is a parameter

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 6 \\ 2 & 6 & 7 \end{bmatrix}, B_2 = \begin{bmatrix} a & 1 \\ 1 & 2 \end{bmatrix}, b_1 = \begin{bmatrix} 18 \\ 28 \\ 38 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.
\]

We want to find a point \( x^* \in C \) such that \( Ax^* \in Q \). Clearly, \( e = (1, 1, 1)^T \) is such a point. The matrix \( B_1 \) is positive semidefinite, so \( C \) is convex. When \( a \geq \frac{1}{2} \), \( B_2 \) is positive semidefinite and \( Q \) is also convex. For different values of \( a \), we compare the performance of Algorithm 3.1 and the relaxed CQ algorithm. We choose the initial point as \((-50, 50, 50)^T\), which is not close to \((1, 1, 1)^T\). The numerical results are reported in Table 1. The time is in seconds. For \( a \leq 500 \), the relaxed CQ algorithm can get a solution faster. However, for larger values of \( a \), e.g., \( a = 5000 \) and \( a = 20000 \), Algorithm 3.1 is faster. The time consumed by Algorithm 3.1 does not change much as a increases, while the time by the relaxed CQ algorithm increases fast. The semidefinite relaxation method behaves more stably.
Example 4.2. Consider the sets $C, Q$:

$$C = \left\{ x \in \mathbb{R}^5 \left| \begin{array}{l} x_1^2 + x_2^2 - x_3^2 + x_4x_5 - 3 \leq 0, \ x_1(x_1 - 1) = 0, \\ x_1^4 + x_2^4 - 2 \leq 0, \ 3x_2 + 2 \leq 0 \end{array} \right. \right\},$$

$$Q = \left\{ y \in \mathbb{R}^4 \left| \begin{array}{l} \frac{1}{2} y^T B_1 y + b_1^T y - 1 \leq 0, \\ \frac{1}{2} y^T B_2 y + b_2^T y - 2 \leq 0 \end{array} \right. \right\},$$

where

$$B_1 = \begin{bmatrix} 1 & 4 & 6.5 & 6 \\ 4 & 2 & 0.5 & 2.5 \\ 6 & 0.5 & 10 & 2.5 \\ 6 & 2.5 & 2.5 & 9 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 18 & 12 & 7 & 19.5 \\ 12 & 2.5 & 7.5 \\ 7 & 2.5 & 10 & 14 \\ 19.5 & 7.5 & 14 & 18 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 5 & 8 & 3 & 6 \\ 1 & 0 & 4 & 2 & 5 \\ 6 & 9 & 7 & 0 & 1 \\ 0 & 2 & 1 & 0 & 3 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 5 \end{bmatrix}.$$

The sets $C, Q$ are nonconvex. By Algorithm 3.1, we got a feasible point

$$(0.0000, -0.6667, 0.8611, -0.2181, -0.3341)^T.$$

It took about 2.95 seconds. The $CQ$ type methods are not applicable because of the nonconvexity of $C, Q$.

Example 4.3. Consider the sets (R is a parameter)

$$C = \left\{ x \in \mathbb{R}^3 \left| -f_1(x) := x_1^4 + x_2^4 + x_3^4 + 2x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 - 4x_1 - 4x_2 - 4x_3 + 1 \leq 0 \right. \right\},$$

$$Q = \left\{ y \in \mathbb{R}^3 \left| (y_1 - 2)^2 + (y_2 - 2)^2 + (y_3 - 2)^2 - R \leq 0 \right. \right\}.$$

The function $-f_1(x)$ is convex over $\mathbb{R}^3$, because

$$-\nabla^2 f_1(x) = \begin{bmatrix} 12x_1^2 + 4x_2^2 + 2x_3^2 & 8x_1x_2 & 4x_1x_3 \\ 8x_1x_2 & 4x_1^2 + 12x_2^2 + 2x_3^2 & 4x_2x_3 \\ 4x_1x_3 & 4x_2x_3 & 2x_1^2 + 2x_2^2 + 12x_3^2 \end{bmatrix}$$

is positive semidefinite for all $x \in \mathbb{R}^3$. (The diag$(w)$ denotes the diagonal matrix whose diagonal is $w$.) The sets $C, Q$ are both convex. Consider the matrix $A = I$. This SFP is equivalent to finding $x^* \in C \cap Q$. The smallest $R$ for $C \cap Q \neq \emptyset$ is $R_0 \approx 2.0623$, which is the square of the distance between $(2, 2, 2)$ and $C$. For different values of $R$, we apply Algorithm 3.1 to solve the SFP. The results are shown in Table 3.

Example 4.4. Let $C$ be the same set as in Example 4.3 and $Q$ be

$$Q := \left\{ y \in \mathbb{R}^3 \left| (y_1 - 2)^2 + (y_2 - 2)^2 + a(y_3 - 2)^2 - 1.5 \leq 0 \right. \right\},$$

where $a$ is a parameter. The matrix $A$ is also the identity. When $a = 1$, the SFP is infeasible because $1.5 < R_0$. However, the feasibility changes as $a$ varies.

In Table 3 we list some values of $a$ such that the SFP is feasible/infeasible. The maximum value for a such that $C \cap Q \neq \emptyset$ is around 0.2786, which can be found by maximizing a subject to the constraints in $C, Q$. 

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Consider the sets \( C, Q \):

\[
C = \left\{ x \in \mathbb{R}^3 \mid \begin{array}{l}
5x_1^6 + 3x_2^2x_3 + x_4^2 + x_5^3 + 8x_6 + 1 \leq 0, \\
x_1^6 + 5x_2^2x_3^2 - 8x_1x_2 + 3x_2x_3^2 + x_4^2 + x_5^2 - 1 \leq 0
\end{array} \right\},
\]

\[
Q = \left\{ y \in \mathbb{R}^2 \mid \begin{array}{l}
\frac{1}{2}\| y^T B_1 y + b_1^T y \| \leq 0, \\
\frac{1}{2}\| y^T B_2 y + b_2^T y \| - 1 \leq 0
\end{array} \right\},
\]

with

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1.5 & 5 \\ 1.5 \\ 5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5 & 0.5 \\ 0.5 & 2 \\ 2 & 8 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}.
\]

The sets \( C, Q \) are nonconvex. By Algorithm 3.1, we know the SFP is feasible, and we got a solution \((0.0138, -0.0128, -0.1270)^T\). It took about 2 seconds.

**Example 4.6.** Consider the sets (\( R \) is a parameter)

\[
C = \left\{ x \in \mathbb{R}^2 \mid f(x) := x_1^3 - 10x_1^2x_2 + 8x_2^2x_3 - 6x_1^2x_2^2 + 5x_1^3
\]

\[
-7x_2^2 + 3x_1x_2^2 - 7x_2^3 + 2x_1^2 - 1 \leq 0
\}

\[
Q = \left\{ y \in \mathbb{R}^2 \mid y_1^2 + y_2^2 - R \leq 0\right\}.
\]

The matrix \( A = I \). The set \( C \) is nonconvex and unbounded. We apply Algorithm 3.1 to solve the SFP, with different values of \( R \). The numerical results are stated in the following Table 4.

**Example 4.7.** Let \( A = I \) and \( C, Q \) be the sets (\( a \) is a parameter):

\[
C = \left\{ x \in \mathbb{R}^2 \mid \begin{array}{l}
\frac{1}{9}x_1^3 + \frac{1}{4}x_2^3 - 1 \leq 0, \\
x_1^2 + x_2^2 - 1 \geq 0
\end{array} \right\},
\]

\[
Q = \left\{ y \in \mathbb{R}^2 \mid y_2 - y_1 \leq 0, \ y_1 \leq 2, \ y_2 \geq a\right\}.
\]

The set \( C \) is nonconvex. The intersection \( C \cap Q \) is nonconvex for \( a < \sqrt{2}/2 \). The SFP is infeasible for \( a > \sqrt{36/13} \approx 1.6641 \). By Algorithm 3.1, for different values of \( a \), we solve the SFP. The computational results are in Table 5.
### Table 4. Computational results for Example 4.6

| $R$ | feasibility | a solution $x^*$ |
|-----|-------------|-----------------|
| 100.0 | feasible | $(-0.0845, 0.4690)$ |
| 10.0 | feasible | $(-0.0126, 0.4793)$ |
| 1.0 | feasible | $(-0.2343, 0.4292)$ |
| 0.5 | feasible | $(-0.2128, 0.4370)$ |
| 0.2 | infeasible | none |
| 0.1 | infeasible | none |

### Table 5. Computational results for Example 4.7

| $a$ | $C \cap Q$ | feasibility | a solution $x^*$ |
|-----|-------------|-------------|-----------------|
| $-2.0$ | nonconvex | feasible | $(0.4520, -0.8920)$ |
| $-1.5$ | nonconvex | feasible | $(0.4059, -0.9139)$ |
| $-1.0$ | nonconvex | feasible | $(-0.0922, -0.0922)$ |
| 0.0 | nonconvex | feasible | $(1.0000, 0.0000)$ |
| $\sqrt{2}/2$ | convex | feasible | $(0.7071, 0.7071)$ |
| 1.8 | convex | infeasible | none |

### 5. Conclusions

This paper discusses the split feasibility problem with polynomials. The sets are semi-algebraic sets, defined by polynomial inequalities. But they are allowed to be nonconvex or even infeasible. Semidefinite relaxations are proposed for representing the intersection of the sets. We gave conditions that guarantee these relaxations are exact for representing the intersection. Based on these relaxations, Algorithm 3.1 is proposed for solving the split feasibility problem. Its convergence is proved. Under a general condition, we prove that: if the SFP is feasible, we are able to compute a feasible solution; if it is infeasible, we can obtain a certificate for the infeasibility.

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