LEFSCHETZ FIBRATIONS ON COMPACT STEIN MANIFOLDS

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Abstract. Here we prove that a compact Stein manifold \( W^{2n+2} \) of dimension \( 2n+2 > 4 \) admits a Lefschetz fibration over the disk \( D^2 \) with Stein fibers, such that the monodromy of the fibration is a symplectomorphism induced by compositions of “generalized Dehn twists” along imbedded \( n \)-spheres on the generic fiber. Also, the open book on the boundary \( \partial W \), which is determined by the fibration, is compatible with the contact structure induced by the Stein structure. This generalizes the Stein surface case of \( n = 1 \), previously proven by Loi-Piergallini and Akbulut-Ozbagci.

1. Introduction

In [AO] (see also [LP]), it is proved that every compact Stein surface admits a positive allowable Lefschetz fibration over \( D^2 \) with bounded fibers (PALF in short), and conversely every 4-dimensional positive Lefschetz fibration over \( D^2 \) with bounded fibers is a Stein surface. Here we prove the following, which can be thought as a generalization of this results to higher dimensions:

Theorem 1.1. Any compact Stein manifold \( W^{2n+2} \) of dimension \( 2n+2 > 4 \) admits a Lefschetz fibration over \( D^2 \) with compact Stein fibers, such that the monodromy of the fibration is a symplectomorphism induced by compositions of “generalized Dehn twists” along imbedded \( n \)-spheres on the generic fiber. Moreover, the corresponding open book on \( \partial W \) supports the contact structure induced by the Stein structure on \( W \).

The converse of this theorem follows from Eliashberg’s theorem [E], which says that any compact smooth manifold \( W^{2n+2} \) of dimension \( 2n+2 > 4 \) has handles of index \( \leq n + 1 \) if and only if it has a Stein structure. So we can assume that \( W^{2n+2} \) is obtained from a subcritical Stein manifold by attaching handles with index \( n + 1 \). Also a theorem of Cieliebak [C] says that any subcritical Stein manifold decomposes as \( W_0^{2n} \times D^2 \), where \( W_0 \) is a Stein manifold. We use this decomposition as the starting point of our proof. We should point out that our proof does not apply to the case of Stein surfaces (\( n=1 \) case), where one needs a different approach of [AO]. By using [K], in dimension 6 the above theorem can be strengthen to: A compact manifold \( W^6 \) is Stein if and only if it admits a Lefschetz fibration over \( D^2 \) with fibers having nonempty boundary (Remark 6.5).

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Before proceeding with the proof of Theorem 1.1 in Section 6 we will first recall some definitions and necessary results in Sections 2-5. For more details and the proofs of the statements, we refer the reader to [K] for Lefschetz fibrations, to [E] for Stein manifolds, to [Ge, Gi] for contact structures and open books, and to [MS] for symplectic geometry.

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2. Characterization of Lefschetz fibrations

Let $W^{2n+2}$ be a compact oriented smooth manifold of dimension $2n + 2$ for $n \geq 1$.

**Definition 2.1.** A smooth map $\pi : W \to D^2 \subset \mathbb{C}$ is called a Lefschetz fibration if $\pi$ has finitely many critical points, such that in a neighborhood of each critical point $W$ admits a coordinate neighborhood with complex coordinates $w = (z_1, z_2, ..., z_{n+1})$, consistent with the given orientation of $W$ where $\pi(p)$ has the representation:

$$\pi(w) = z_0 + z_1^2 + z_2^2 + ... + z_{n+1}^2.$$

By definition, $\pi$ has finitely many critical values, say $\{\lambda_1, \lambda_2, ..., \lambda_\mu\}$. Consider the base disk as $D^2 = \{z \in \mathbb{C} : |z| \leq 2\}$. Composing $\pi$ with an orientation preserving diffeomorphism of $D^2$, we can assume that the critical values are $\mu$ roots of unity. Such a map $\pi$ is called normalized. Let $a$ be a regular value and $X = \pi^{-1}(a)$ be a regular fiber.

We want to understand the handlebody decomposition of $W$ associated to $\pi$. Define a Morse function $F : W \to [0, 2] \subset \mathbb{R}$ given by $F(x) = |\pi(x)|^2$. By using [AF], it is easy to verify that outside of the set

$$F^{-1}(0) \cup F^{-1}(2),$$

$F$ has only nondegenerate critical points of index $n + 1$. Since $|\lambda_i| = 1$ for all $i = 1, 2, ..., \mu$, the map $\pi$ has no critical values on the set $D_t = \{z \in \mathbb{C} : |z| \leq t\}$ for $t < 1$ and hence

$$F^{-1}([0, t]) = \pi^{-1}(D_t) \cong X \times D^2 \quad \text{for} \quad t < 1.$$

On the other hand, for $t > 1$, $\pi^{-1}(D_t)$ is diffeomorphic to the manifold obtained from $X \times D^2$ by attaching $\mu$ handles of index $n + 1$, via the attaching maps

$$\Phi_j : S^n \times D^{n+1} \to \partial(X \times D^2) = X \times S^1, \quad j = 1, 2, ..., \mu.$$

Let $\Phi_j : \epsilon^{n+1} \to \nu$ be the framing of the $j$-th handle, where $\epsilon^k$ denotes the trivial bundle of rank $k$, and $\nu$ denotes the normal bundle of the attaching sphere $\Phi_j(S^n \times \{0\})$ in $\partial(X \times D^2)$. The proofs of the following facts can be found in [K].

**Fact 2.2.** The embeddings $\Phi_j$ may be chosen so that for each $j = 1, 2, ..., \mu$ there exists $z_j$ such that $\Phi_j(S^n \times \{0\}) \subset \pi^{-1}(z_j) \cong X$.

So, set $\phi_j : S^n \to X$ to be the embedding defined by restricting $\Phi_j$ to $S^n \times \{0\}$. Let $\nu_1$ denote the normal bundle of $S^n \cong \phi_j(S^n)$ in $X$ corresponding to the embedding $\phi_j$, and consider $\nu$ as the normal bundle of $S^n$ in $F^{-1}(1 - \delta)$. Clearly, $\nu \cong \nu_1 \oplus \epsilon$. Let $\tau$ denote the tangent bundle of $S^n$.

**Fact 2.3.** For each $j = 1, 2, ..., \mu$, there exists a bundle isomorphism $\phi'_j : \tau \to \nu_1$ such that the framing $\Phi'_j$ of the $(n+1)$-handle corresponding to $\lambda_j$ coincides with $\phi'_j$. That is, $\Phi'_j$ is given by the composition

$$\epsilon^{n+1} \xrightarrow{\cong} \tau \oplus \epsilon \xrightarrow{\phi'_j \oplus \text{id}} \nu_1 \oplus \epsilon \xrightarrow{\cong} \nu.$$

**Definition 2.4 ([K]).** $S^n \cong \phi_j(S^n)$ is called a vanishing cycle of $\pi$. The bundle isomorphism $\phi'_j : \tau \to \nu_1$ is called a normalization of $\phi_j$. The pair $(\phi_j, \phi'_j)$ is called a normalized vanishing cycle.

The following theorem is one of the key points in the proof of the main result:
Theorem 2.5 (K). The normalized Lefschetz fibration $\pi : W \to D^2$ is uniquely determined by a sequence of vanishing cycles $(\phi_1, \phi_2, ..., \phi_\mu)$ and a sequence of their normalizations $(\phi'_1, \phi'_2, ..., \phi'_\mu)$.

Next we want to understand the geometric monodromy of a Lefschetz fibration. A Dehn twist $\delta^k$ is a diffeomorphism from the closed cotangent unit disk bundle $T^*S^n(1)$ to itself constructed in the following way. We write points in $S^n(1)$ as $(q, p) \in \mathbb{R}^{2(n+1)}$ with $|q| = 1$, $q \perp p$ and $|p| \leq 1$. We define

$$
\delta^k(q, p) = \begin{pmatrix}
\cos g_k(|p|) & |p|^{-1} \sin g_k(|p|) \\
-|p| \sin g_k(|p|) & \cos g_k(|p|)
\end{pmatrix}
\begin{pmatrix}
q \\
p
\end{pmatrix}
$$

where $g_k$ is a smooth function that increases monotonically from $\pi k$ to $2\pi k$ on some interval, and outside this interval $g_k$ is identically equal to $\pi k$ or $2\pi k$.

Definition 2.6. For $k \in \mathbb{N}$, the map $\delta^k$ is called a $k$-fold right-handed Dehn twist, and the map $\delta^{-k}$ is called a $k$-fold left-handed Dehn twist.

Throughout the paper we will take $k = 1$ and denote $\delta^1 = \delta$. Now consider the pair $(\phi, \phi')$ where $\phi : S^n \to X$ is an embedding of $S^n$ into a 2n-dimensional manifold $X$ and $\phi' : \tau \to \nu_1$ is a bundle isomorphism between the tangent bundle $\tau$ of $S^n$ and the normal bundle $\nu_1$ of $\phi(S^n)$ in $X$. Let $TS^n(1) \subset \tau$ denote the closed tangent unit disk bundle of $S^n$. By the tubular neighborhood theorem and the isomorphism $T^*S^n(1) \cong TS^n(1)$, we can apply $\delta$ to a tubular neighborhood of $\phi(S^n)$ in $X$, and we can extend $\delta$, by the identity, to a self-diffeomorphism of $X$ which we denote by

$$
\delta_{(\phi, \phi')} : X \xrightarrow{\approx} X.
$$

Up to smooth isotopy $\delta_{(\phi, \phi')} \in \text{Diff}(X)$ depends only on the smooth isotopy class of the embedding $\phi$ and the bundle isomorphism $\phi'$.

Definition 2.7. $\delta_{(\phi, \phi')} : X \to X$ is called the Dehn twist with center $(\phi, \phi')$.

Now suppose $\pi : W \to D^2$ is a Lefschetz fibration with regular fiber $X$ and $p \in W$ is a critical point of $\pi$ whose index is $n + 1$. The part of $W$ which lies over the boundary of a small disk about $\pi(p)$ is a fiber bundle over $S^3$ with fiber $X$. Such a fiber bundle is diffeomorphic to $X \times [0, 1]/(x, 1) \sim (h(x), 1)$ where $h : X \to X$ is a diffeomorphism which is uniquely defined up to smooth isotopy. The map $h$ is called the geometric monodromy of $\pi$ associated with the critical value $\pi(p)$. Assuming $p$ is the only critical point with critical value $\pi(p)$, we can construct an embedding $\phi : S^n \to X$ and its normalization $\phi' : \tau \to \nu_1$ as before.

Theorem 2.8 (DK, K). The Dehn twist $\delta_{(\phi, \phi')} : X \to X$ is (up to isotopy) equal to the geometric monodromy $h$ of the fibering $\pi : W \to D^2$ about $f(p)$.

By using the notation introduced above and writing $\delta_j = \delta_{(\phi_j, \phi'_j)}$ we have

Theorem 2.9 (K). The normalized Lefschetz fibration $\pi : W \to D^2$ is determined (not uniquely) by its total geometric monodromy given by the composition

$$
\delta_\mu \circ \cdots \circ \delta_2 \circ \delta_1 \in \text{Diff}(X).
$$
Remark 2.10. With respect to the coordinates \((q, p)\) on \(\mathbb{R}^{2(n+1)}\) consider the canonical 1-form \(\lambda_{can} = p \cdot dq\) on \(T^*S^n(1) \subset \mathbb{R}^{2(n+1)}\). We compute
\[
\delta^*\lambda_{can} = \lambda_{can} + |p|d(g_1(|p|)).
\]
which implies that the difference \(\delta^*\lambda_{can} - \lambda_{can}\) is exact. Therefore, \(\delta\) is a symplectomorphism of the symplectic manifold \((T^*S^n(1), d\lambda_{can})\). As a result, if a regular fiber \(X\) of a Lefschetz fibration \(\pi : W \rightarrow D^2\) equipped with a symplectic structure \(\omega\), then its total geometric monodromy is a symplectomorphism of \((X, \omega)\). That is,
\[
\delta_\mu \circ \cdots \circ \delta_2 \circ \delta_1 \in \text{Symp}(X, \omega).
\]

3. Contact structures and open book decompositions

Let \(M\) be an oriented smooth \((2n + 1)\)-manifold. A contact structure on \(M\) is a global 2\(n\)-plane field distribution \(\xi\) which is totally non-integrable. Non-integrability condition is equivalent to the fact that locally \(\xi\) can be given as the kernel of a 1-form \(\alpha\) such that \(\alpha \wedge (d\alpha)^n > 0\). If \(\alpha\) is globally defined (in such a case, it is called a contact form), then one can define the Reeb vector field of \(\alpha\) to be the unique global nowhere-zero vector field \(R\) on \(M\) satisfying the equations: \(\iota_Rd\alpha = 0\), \(\alpha(R) = 1\) where \(\iota\) denotes the interior product: \(\iota_Rd\alpha = R \cdot d\alpha\). A \(k\)-dimensional \((k \leq n)\) submanifold \(Y\) of \((M, \xi)\) is called isotropic if \(TY \subset \xi\). An \(n\)-dimensional isotropic submanifold is called Legendrian. A contact vector field on \(M\) is a vector field whose flow preserves the contact structure \(\xi\). Recall a vector field \(v\) is called a contact vector field if it satisfies
\[
\mathcal{L}_v \alpha = f\alpha
\]
for some (possibly zero) function \(f : M \rightarrow \mathbb{R}\). Note that the Reeb vector field \(R\) is contact. One useful fact which we will use later is that Legendrian submanifolds stay Legendrian under isotopies given by the flows of the contact vector fields. Two contact structures \(\xi_0, \xi_1\) on \(M\) are said to be isotopic if there exists a 1-parameter family \(\xi_t\) of contact structures joining them. We say that two contact manifolds \((M_1, \xi_1)\) and \((M_2, \xi_2)\) are contactomorphic if there exists a diffeomorphism \(f : M_1 \rightarrow M_2\) such that \(f_*(\xi_1) = \xi_2\). Now assume that \(M\) is closed. The following definitions are taken from [Gi].

Definition 3.1. An open book (decomposition) \((B, \Theta)\) on \(M\) is given by a codimension 2 submanifold \(B \hookrightarrow M\) with trivial normal bundle, and a fiber bundle \(\Theta : (M - B) \rightarrow S^1\). The neighborhood of \(B\) should have a trivialization \(B \times D^2\), where the angle coordinate on the disk agrees with the map \(\Theta\). The manifold \(B\) is called the binding, and for any \(t_0 \in S^1\) a fiber \(X = \Theta^{-1}(t_0)\) is called a page of the open book.

Definition 3.2. A contact structure \(\xi = \text{Ker } (\alpha)\) on \(M\) is said to be supported by an open book \((B, \Theta)\) of \(M\), if

(i) \((B, \alpha|_B)\) is a contact manifold.
(ii) For every \(t \in S^1\), the page \(X = \Theta^{-1}(t)\) is a symplectic manifold with symplectic form \(d\alpha\).
(iii) If \(X\) denotes the closure of a page \(X\) in \(M\), then the orientation of \(B\) induced by its contact form \(\alpha|_B\) coincides with its orientation as the boundary of \((X, d\alpha)\).

Such a contact form is said to be adapted to (or compatible with) \((B, \Theta)\).
In the case of dimension three \((n = 1)\), the compatibility is defined in a slightly different way, and it is known by [Gi] that a positive stabilization of a compatible open book gives another open book supporting the same contact structure up to isotopy. For the precise statements see [Et].

Next we will generalize the stabilization process to higher dimensions \((n \geq 2)\). To this end, we need an abstract description of open books: If \((\bar{X}, \partial \bar{X})\) is an open book supporting the same contact structure up to isotopy. For the precise way, and it is known by [Gi] that a positive stabilization of a compatible open book gives \((0, 1)\)-sphere in the binding \((\partial \bar{X})\), this end, we need an abstract description of open books: If \((\bar{X}, \partial \bar{X})\) is an another open book supporting the same contact structure up to isotopy. For the precise way, and it is known by [Gi] that a positive stabilization of a compatible open book gives \((0, 1)\)-sphere in the binding \((\partial \bar{X})\), this end, we need an abstract description of open books: If \((\bar{X}, \partial \bar{X})\) is an another open book supporting the same contact structure up to isotopy. 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Remark 3.5. The isomorphism $TS \to T\tilde{X}/TS$ in Definition 3.4 exists because $S$ is Lagrangian in $T\tilde{X}$ (see the proof of Lemma 6.1 for detailed discussion). Note that $(\tilde{X}, h)$ satisfies the conditions in Theorem 3.3 (take $\beta = \alpha$), and we have $h' \in \text{Symp}(\tilde{X}', \omega)$ as $\delta_{(\phi', \phi)} \in \text{Symp}(X', \omega)$ by Remark 2.10. Moreover, we can extend the Lioville vector field of $\alpha$ transverse to $\partial \tilde{X}$ over the Weinstein handle $H$ (see [W]) and obtain a Lioville vector field of $\omega$ transverse to $\partial \tilde{X}'$. Hence, there exists a contact structure $\xi'$ on $M$ supported by $(\tilde{X}', h')$ by Theorem 3.3. The new contact manifold $(M, \xi')$ is conjectured to be contactomorphic to $(M, \xi)$. The latter is actually proved in many cases: It was shown for dimension three ($n = 1$ case). Also for $n \geq 2$ it is known that $(M, \xi')$ is contactomorphic to $(M, \xi)$ if $L$ is a boundary parallel Lagrangian $n$-disk in $X$ (see [Gi]).

4. Characterization of Stein manifolds

A Stein manifold is a complex manifold (necessarily noncompact) which admits proper holomorphic embeddings in $\mathbb{C}^N$ for sufficiently large $N$. A complex manifold $W$ is Stein if and only if it admits an “exhausting strictly plurisubharmonic” function, which is essentially characterized as being a proper function $\psi : W \to R$ that is bounded below and can be assumed a Morse function, whose level sets $\psi^{-1}(c)$ are “strictly pseudoconvex” (away from critical points), where $\psi^{-1}(c)$ is oriented as the boundary of the complex manifold $\psi^{-1}(-\infty, c]$ (see [E] for details). A compact, complex manifold $W$ with boundary is called a Stein domain if it admits a strictly plurisubharmonic function such that the boundary $\partial W$ is a level set. Therefore, the phrase “compact Stein manifold” should be understood as Stein domain. If $J$ is the underlying integrable almost complex structure on $W$, then $-d(d\psi \circ J)$ defines an exact symplectic (indeed Kähler!) form on $W$. We will consider Stein manifolds as triples of the form $(W, J, \psi)$. We have the following topological characterization is given by Eliashberg:

Theorem 4.1. ([E]) Let $(W, J')$ be a compact almost complex $2(n + 1)$-manifold with a handle decomposition whose handles have index $\leq n + 1$. Then $W$ admits a Stein domain structure $(W, J, \psi)$ where $J$ is homotopic to $J'$, and $\psi$ is a suitable plurisubharmonic function inducing the handle decomposition. Conversely, any Stein domain of dimension $2(n + 1)$ admits a handle decomposition whose handles have index $\leq n + 1$.

Two Stein structures $(J_0, \psi_0), (J_1, \psi_1)$ on the same smooth manifold $W$ are called Stein homotopic if there exists a continuous family of Stein structures $J_t$ with properly homotopic exhausting strictly plurisubharmonic functions $\psi_t$. Two Stein manifolds $(W, J, \psi)$ and $(W', J', \psi')$ are called deformation equivalent if there exists a diffeomorphism $f : W \to W'$ such that $J$ and $f^*J'$ are Stein homotopic on $W$. For the equivalence we write $(W, J, \psi) \cong (W', J', \psi')$. A Stein manifold (or domain) of dimension $2n + 2$ is called subcritical if it admits a handle decomposition with handles of index $\leq n$. Subcritical Stein manifolds admit the following splitting:

Theorem 4.2 ([C]). A subcritical Stein manifold is deformation equivalent to

$$(W_0 \times \mathbb{C}, J_0 \times i, \psi_0 + |z|^2)$$

where $(W_0, J_0, \psi_0)$ is a Stein manifold and $z$ is the coordinate on $\mathbb{C}$.
Remark 4.3. If $(W^{2n+2}, J, \psi)$ is a compact subcritical Stein manifold then
\[(W, J, \psi) \cong (W_0 \times D^2, J_0 \times i, \psi_0 + |z|^2)\]
where $(W_0, J_0, \psi_0)$ is a compact Stein manifold of dimension $2n$. Here we consider $D^2$
as the standard unit disk in $\mathbb{C}$. Note that the exact Kähler form on $W = W_0 \times D^2$
(compatible with $J$) can be considered as
\[\omega = -d(d\psi \circ J_0 \times i) = \omega_0 + rdr \wedge d\theta = \omega_0 + dx \wedge dy\]
where $\omega_0 = -d(d\psi_0 \circ J_0)$ is the exact Kähler form on $W_0$, and $(r, \theta)$ are the polar coordinates, and $(x, y)$ are the cartesian coordinates on the unit disk $D^2 \subset \mathbb{C}$ (i.e., $z = re^{i\theta}$).

5. Handle attaching in almost complex category

In this section we will recall the handle attaching process in the category of almost complex manifolds as explained in [F]. To attach a handle in this category we need a handle attaching triple, in short called “HAT”. Fix the coordinates on
\[\mathbb{C}^{n+1} \cong \mathbb{C}^k \oplus \mathbb{C}^{n+1-k} \cong \mathbb{R}^k \oplus i\mathbb{R}^k \oplus \mathbb{R}^{n+1-k} \oplus i\mathbb{R}^{n+1-k}\]
as $x = (x_1, ..., x_k), y = (y_1, ..., y_k), x' = (x_1', ..., x'_{n+1-k}), y' = (y_1', ..., y'_{n+1-k})$. The standard handle of index $k$ is the set
\[H^k = \mathbb{D}^k \times \mathbb{D}^1_{1+\varepsilon} \times \mathbb{D}^{n+1-k} \times \mathbb{D}^{n+1-k}_{\varepsilon} = \{ ||x|| \leq \varepsilon, ||y|| \leq 1 + \varepsilon, ||x'|| < \varepsilon, ||y'|| < \varepsilon \}\]
where $\varepsilon$ is a positive number. Consider the $k$-disk $D := 0 \times \mathbb{D}^1_k \times 0 \times 0 \subset H^k$, and its boundary $(k-1)$-sphere $S := \partial D$. Let $\nu$ be the normal bundle to $D$ in $H^k$ along $S$ and $\nu = \nu_0 \oplus \nu_1 \oplus \nu_2$ be its canonical decomposition corresponding to the first, and two last factors of $H^k$, respectively. Let $v$ be the outward unit normal vector field to $S$ in $D$, and $\tau$ be the tangent bundle to $D$ restricted to $S$. So,
\[\tau = T(D)|_{T(S)} = T(S) \oplus <v>\]

From now on we restrict our discussion to compact Stein manifolds. Let $(W, J, \psi)$ be a Stein domain with contact boundary $(\partial W, \xi)$, where $\xi$ is formed by complex tangencies with respect to $J$. Fix an inward transversal vector field $w$ to $\partial W$ in $W$ such that $J(w)$ is tangent to $\partial W$. For a map $f : S \to \partial W$, let $Df : \tau \to T(W)$ be a homomorphism with
\[Df|_{T(S)} = df, \quad Df(v) = w.\]

Consider the homomorphism $\overline{Df} : \nu_0 \to T(W)$ given by $\overline{Df} = -J \circ Df \circ i$.

A handle attaching triple (HAT) is $(f, \beta, \varphi)$ where $f : S \hookrightarrow \partial W$ is an embedding, $\beta : \nu \to \nu_f := T(\partial W)|_{T(S)}/df(T(S))$ is a nondegenerate real homomorphism, and also $\varphi : T(H^k)|_S \to T(W)$ is a complex homomorphism, which is homotopic to $Df \oplus \beta$. A HAT is called special if the image $f(S)$ is tangent to $\xi$ (i.e., isotropic), $\beta = \overline{Df} \oplus \theta$ for some complex monomorphism $\theta : \nu_1 \oplus \nu_2 \to T(W)$. An isotopy of HAT’s is an isotopy $f_t : S \to \partial W$ covered by homotopies $\beta_t : \nu \to \nu_f$ and $\varphi_t : T(H^k)|_S \to T(W)$. We have:

Theorem 5.1. Attaching a handle with isotopic HAT’s yields diffeomorphic smooth manifolds with homotopic almost complex structures.

The following results are proved in [E]:
Theorem 5.2. Suppose \( f : S^n \to X \) is a generic fiber \( X^{2n} \) and the monodromy \( \Lambda \) is a Lefschetz fibration with a generic fiber \( X^{2n} \) and the monodromy \( h \in \text{Symp}(X,\omega) \) for some symplectic structure \( \omega \) on \( X \). Let \( L \) be an embedded Lagrangian sphere in the boundary open book \( (X,\omega) \). Let \( W' \) be a manifold obtained from \( W \) by attaching a handle of index \( n+1 \) along \( L \). Then \( W' \) admits a Lefschetz fibration with a generic fiber \( X^{2n} \) and the monodromy \( h' = \delta \circ h \in \text{Symp}(X,\omega) \) where \( \delta \) is the right-handed Dehn twist along \( L \).

Proof. Let \( \phi : S^n \to W \) be the embedding of the attaching sphere of the handle attachment resulting \( W' \). Since \( \phi(S^n) = L \) is Lagrangian in the page \((X,\omega)\), we have
\[
\omega|_{T\phi(S^n)} = \omega|_{TL} = 0.
\]
The proof is based on the fact that the normal bundle \( N_X(L) \) of \( L \) in \( X \) is isomorphic to the tangent bundle \( TL \). To define an explicit isomorphism, we use an almost complex structure \( J : TX \to TX \) compatible with \( \omega \) (for the existence of \( J \) see Proposition 2.61 in [MS]). More precisely, we define the bundle isomorphism
\[
\Lambda_J : TL \to N_X(L) \quad \text{as} \quad \Lambda_J(u) = Ju.
\]
(Here the compatibility implies that \( \omega(u, Ju) > 0 \) and so \( Ju \notin TL \) since \( \omega|_{TL} = 0 \). Therefore, \( Ju \in N_X(L) = TX/TL \).) But then we immediately get the composition
\[
\phi' := \Lambda_J \circ \phi_* : TS^n \to N_X(\phi(S^n))
\]
which is a normalization of the embedding \( \phi \) (according to Definition 2.4). Therefore, the pair \((\phi,\phi')\) is a normalized vanishing cycle, and so \( W' \) admits a Lefschetz fibration by Theorem 2.5. Observe that a generic fiber of the new Lefschetz fibration (on \( W' \)) is still \( X \). However, by Theorem 2.3, the new monodromy is \( h' = \delta \circ h \) where \( \delta \) is the right-handed Dehn twist with center \((\phi,\phi')\) as claimed.

Remark 5.4. For our purpose, we now focus on HAT’s of index \( k = n+1 \). By Theorem 5.2 if a HAT \((f,\beta,\varphi)\) has index \( n+1 \), we may assume that it is special, and so the image \( f(S) \approx S^n \) is an embedded Legendrian \( n \)-sphere in the contact manifold \((\partial W,\xi)\). Also we have \( \nu_1 = \nu_2 = 0 \), and so \( \nu = \nu_0, \beta = \overline{Df} \). Moreover, since \( T(f(S)) \subset \xi \) and \( \xi \) is complex tangencies in \( \partial W \), \( \overline{Df}(\nu_0) \) is transversal to \( Df(\tau) \) and the map
\[
Df \oplus \overline{Df} : T(H^{n+1})|_S = \tau \oplus \nu_0 \to T(W)
\]
is a complex bundle isomorphism onto its image, and \( \varphi \approx Df \oplus \overline{Df} \) where the homotopy respects the complex structures. Observe that we have a complex bundle isomorphism
\[
T(S) \oplus iT(S) = T(S) \otimes C \to \xi|_{T(f(S))} = T(f(S)) \oplus JT(f(S)).
\]
respecting the direct sums (i.e., sending \( T(S) \) to \( T(f(S)) \) and \( iT(S) \) to \( JT(f(S)) \)).
Proposition 6.2. Let $(W^{2n+2}, J, \psi)$ be a Stein domain which admits a Lefschetz fibration with a generic fiber $X^{2n}$, and the monodromy $h \in \text{Diff}(X)$. Suppose that the boundary open book $(X, h)$ supports the contact structure on $\partial W$ induced by the Stein structure on $W$. Let $(W', J', \psi')$ be the Stein domain obtained from $(W, J, \psi)$ by attaching a special HAT $(f, \beta, \varphi)$ of index $n+1$, where $f(S^n)$ lies in the interior of a page $X$. Then $W'$ admits a Lefschetz fibration with a generic fiber $X$ and the monodromy $h' = \delta \circ h \in \text{Diff}(X)$ where $\delta$ is the right-handed Dehn twist along $f(S^n)$ with the normalization determined by the embedding $f(S^n)$. Moreover, the new boundary open book $(X, h')$ supports the contact structure on $\partial W'$ induced by the Stein structure on $W'$.

Proof. Let $\chi$ be the Liouville vector field $\nabla \psi$ defined on a collar neighborhood of $\partial W$. Note that $\chi$ transversely points out from $\partial W$, and $\mathcal{L}_\chi \omega = \omega$ where $\omega$ is the exact Kähler form $\omega = -d(d\psi \circ J)$. Then $\alpha = \iota_\chi \omega$ is a contact form for the contact structure $\xi$ on $\partial W$ induced by the Stein structure on $W$ (i.e., $\xi = \text{Ker}(\alpha)$ and $\alpha \wedge (d\alpha)^n \neq 0$). On $\partial W$, by Cartan formula we have

$$d\alpha = d\iota_\chi \omega = \mathcal{L}_\chi \omega - \iota_\chi d\omega = \omega.$$ 

Since $(X, h)$ supports $\xi$, all the pages of the boundary open book $(X, h)$ are equipped with the exact symplectic structure $d\alpha$.

By assumption the HAT $(f, \beta, \varphi)$ is special, hence the image $f(S^n) \subset X$ is a Legendrian submanifold in $(\partial W, \xi = \text{Ker}(\alpha))$. Since $d\alpha$ is a symplectic structure on $X$, and from

$$d\alpha(u, v) = \mathcal{L}_v \alpha(v) + \mathcal{L}_u \alpha(u) + \alpha([u, v])$$

we see that $d\alpha(u, v) = 0$ for $u, v \in Tf(S^n)$ (see [B, Chapter III] for discussion of integrable submanifolds of contact structures). This shows that $f(S^n)$ is Lagrangian on the page $(X, d\alpha)$, and so Lemma 6.1 implies that $W'$ admits a Lefschetz fibration with a generic fiber $X$ and the monodromy $h' = \delta \circ h$, where $\delta$ is the right-handed Dehn twist with center $(f, f')$ with the normalization $f' = \Lambda f \circ f$ determined by the embedding $f$ and an almost complex structure $J_1$ on $X$ compatible with $d\alpha$ (as in the proof of Lemma 6.1).

It remains to show that the open book $(X, h')$ supports the contact structure on $\partial W'$ induced by the Stein structure on $(W', J', \psi')$. This essentially follows from the fact that, the extended Stein structure on $W'$ is compatible with the one on $W$ ([W, E, CE]). For example, the first condition of compatibility (Definition 3.2) holds because the handle is attached away from the binding, and so the new contact form $\alpha' = \iota_{\chi'}(\omega')$, where $\chi' = \nabla \psi'$ and $\omega' = -d(d\psi' \circ J')$, coincides with $\alpha$ near the binding.

$\square$

For the next result we need some notions introduced in [EG] and also in [CE]: Let $(X, d\lambda)$ be an exact symplectic manifold with the Liouville vector field $\chi$ expanding $d\lambda$, that is, $\iota_\chi d\lambda = \lambda$ (which implies that $\mathcal{L}_\chi d\lambda = \lambda$). The triple $(X, d\lambda, \chi)$ is called a Liouville manifold. A Liouville manifold $(X, d\lambda, \chi)$ is called (symplectically) convex if the expanding vector field $\chi$ is complete and there exists an exhaustion $X = \bigcup_{k=1}^{\infty} X^k$ by compact domains $X^k \subset X$ with smooth boundaries along which $\chi$ points outward. If $\chi^{-t} : X \to X$ ($t > 0$) denotes the contracting flow of $\chi$, then the core of the convex Liouville manifold $(X, d\lambda, \chi)$ is defined as
A convex Liouville manifold \((X, d\lambda, \chi)\) is said to have cylindrical end if \(\chi\) has no zeros outside a compact set. A contact manifold \((M, \eta)\) is called strongly symplectically filled by a convex Liouville manifold \((X, d\lambda, \chi)\) with cylindrical end if \(M = \partial X, \eta = \text{Ker}(\lambda|_{\partial X})\), \(\chi\) points outward along \(\partial X\), and the negative half of the symplectization \((M \times \mathbb{R}, d(e^t\lambda))\) symplectically embeds into \(X\) so that its complement in \(X\) is \(\text{Core}(X, d\lambda, \chi)\) and the embedding matches the positive \(t\)-direction of \(\mathbb{R}\) with \(\chi\). \((M, \lambda)\) is also called an ideal contact boundary of the convex Liouville manifold \((X, d\lambda, \chi)\) with cylindrical end. Finally, we remark that any Stein manifold \((X, J, \psi)\) induces a convex Liouville manifold \((X, -d(\psi \circ J), \nabla \psi)\), see [CE] for more details.

**Proposition 6.3.** Let \((X^{2n}, d\lambda, \chi)\) be a convex Liouville manifold with cylindrical end with the ideal contact boundary \((\partial X, \eta)\). Suppose that \(W^{2n+2}\) is a manifold which admits a Lefschetz fibration with a generic fiber \(X\) and the monodromy \(h \in \text{Symp}(X, d\lambda)\), and also that the boundary open book \((X, h)\) supports a contact structure \(\xi = \text{Ker}(\alpha)\) on \(\partial W\) for some contact form \(\alpha\) such that \(\alpha|_X = \lambda\). Let \(L\) be an immersed Legendrian submanifold of a page \((X, d\lambda)\) with only finitely many transverse double points and no other self intersections of any other types. Suppose that all the double points of \(L\) lie in \(X \setminus \text{Core}(X, d\lambda, \chi)\) and the trajectory of \(\chi\) connecting each double point to the boundary \(\partial X\) does not intersect \(L\) in \(X \setminus \text{Core}(X, d\lambda, \chi)\) other than the double point. Then \(W\) admits another Lefschetz fibration with the following properties:

- a new generic fiber is a convex Liouville manifold \((X', d\lambda', \chi')\) with the ideal contact boundary such that \(X \subset X'\) and the Liouville structure on \(X'\) is compatible with the one on \(X\), i.e., \(\lambda'|_X = \lambda\) and \(\chi'|_X = \chi\);
- the new monodromy \(h' \in \text{Symp}(X', d\lambda')\);
- the new boundary open book \((X', h')\) supports \(\xi\), and
- there exists an embedded Lagrangian submanifold \(L'\) of a page \((X', d\lambda')\) which is obtained from \(L\) by desingularizing all the double points.

Moreover, if the convex Liouville structure \((d\lambda, \chi)\) on \(X\) is given by some Stein structure \((J, \psi)\), then \(X'\) also admits a Stein structure \((J', \psi')\) inducing the convex Liouville structure \((d\lambda', \chi')\) such that \((J', \psi')|_X\) and \((J, \psi)\) are deformation equivalent.

**Proof.** Suppose \(X_0 \approx X\) is the page of the open book \((X, h)\) containing \(L\). The argument used in the proof of Proposition 6.2 implies that \(L\) is Lagrangian in \((X_0, d\lambda)\). It suffices to consider a single double point, because the following can be applied to each double point separately. Let \(p\) be the only transverse double point of \(L\).

By assumption, \(p\) lies in the symplectization of \((\partial X_0, \text{Ker}(\lambda))\) which is symplectically embedded in \((X_0, d\lambda)\). Also there exists a trajectory \(\gamma\) of \(\chi\) connecting \(p\) to a point \(q \in \partial X_0\) such that \(\gamma \cap L = \{p\}\). Therefore, we can find a neighborhood \(N_\gamma \approx D^{2n}\) of \(\gamma\) in \(X_0\) which can be considered as the disjoint union of all nearby trajectories. Note that \(N_\gamma\) meets \(\partial X_0\) along some \((2n - 1)\)-dimensional disk \(D\) which is a neighborhood \(q \in \partial X_0\).

Recall (e.g. [MS]) that in any symplectic \(2n\)-manifold, there exist a \(2n\)-dimensional open (closed) ball \(B\), called a Darboux ball, around any point and a symplectomorphism from
Therefore, we can find a closed Darboux ball $B \approx D^{2n}$ in $X_0$ such that $p \in B$ and $B \cap L = L_1 \cup L_2$ where $L_1, L_2$ are Lagrangian disks in $(B, \omega|_B)$ intersecting transversely at $p$. Note that, for each $i = 1, 2$, $S_i := L_i \cap \partial B$ is an isotropic $(n-1)$-sphere in $(\partial W, \xi)$ as each $L_i$ is Legendrian in $(\partial W, \xi)$. By taking $B$ small enough we may assume that it is completely contained in $N_\gamma$.

Now, we isotope the part $D = \partial X_0 \cap N_\gamma$ of the binding $\partial X_0$ in $N_\gamma$ until the new page $\tilde{X}_0$ does not contain the double point $p$. We perform the isotopy such that the image of the disk $D$ always stays transversal to the Liouville direction $\chi$, and the isotopy ends in the Darboux ball $B$ (after the image of $D$ crosses $p$) so that $S_1$ and $S_2$ sits on the new binding $\partial \tilde{X}_0$ as illustrated in Figure 1. Note that this process does not change the diffeomorphism type of the pages and the monodromy of the original open book $(X, h)$ since we may assume that the isotopy takes place in a tubular neighborhood of the binding $\partial X$ in $\partial W$. Here we consider that the above isotopy is changing all the pages accordingly so that they still have the common binding $\partial \tilde{X}_0$.

![Figure 1](image)

**Figure 1.** $L$ immersed in $X_0$ and isotoping the binding transversally.

By Proposition 12.2 of [CE], the isotopy we made transforms $(X_0, d\lambda, \chi)$ to another convex Liouville manifold $(\tilde{X}_0, d\tilde{\lambda}, \tilde{\chi})$ thorough a diffeotopy respecting the symplectic structures. We note that $\tilde{X}_0 \subset X_0$ (indeed, $\tilde{X}_0 \approx X_0$!), $\tilde{\lambda} = \chi|_{\tilde{X}_0}$, and $\tilde{\chi} = \chi|_{\tilde{X}_0}$.

Hence, we get another open book $(\tilde{X}, h)$ which supports $\xi$ (as $\tilde{\lambda} = \alpha|_{\tilde{X}}$) such that $S_1$ and $S_2$ are isotropic $(n-1)$-spheres embedded in the contact binding $(\partial \tilde{X}, \tilde{\eta} = \xi|_{\partial \tilde{X}})$.

At this point we consider not the whole $L$ but only its part $L \setminus (L_1 \cup L_2)$. Let $D_1, D_2$ be the boundary parallel Lagrangian $n$-disks in $X_0$ such that $\partial D_i = S_i$ for $i = 1, 2$. We positively stabilize the boundary open book $(\tilde{X}, h)$ twice along $D_1$ and $D_2$ as illustrated in Figure 2(a), where each $C_i$ is the (Lagrangian) core $n$-disk of the Weinstein handle $H_i \approx D^n \times D^n$ in each stabilization.

By Remark 3.3 the new open book $(X', h')$, where $X' = \tilde{X} \cup H_1 \cup H_2$, still supports the contact structure $\xi$ on $\partial W$. Here $h' = \delta_2 \circ \delta_1 \circ h$ where $\delta_i$ is the right-handed Dehn twist along the Lagrangian sphere $C_i \cup S_i D_i$. As in Definition 3.4 there exists an exact symplectic form on $X'$. In fact, the new page is an another convex Liouville manifold...
(X', dλ', χ') such that λ' and χ' on X' restrict to ˜λ and ˜χ on ˜X, respectively (see [W] and [CE]). On the new symplectic page X₀' we construct a Lagrangian L' by replacing L₁ ∪ L₂ by C₁ ∪ C₂ (Figure 2-(b)):

\[ L' := (L \setminus (L₁ ∪ L₂)) \cup_{S₁ ∪ S₂} (C₁ ∪ C₂). \]

So far we have constructed a new open book (X', h') supporting ξ on the boundary ∂W, and have shown that from a given immersed L in X₀ ≈ X one can construct L' which can be embedded into X₀' ≈ X'. However, we want to construct a new Lefschetz fibration on W, so we need to change the given Lefschetz fibration W → D₂ correspondingly. To this end, we take (modify) the Weinstein handle Hᵢ of each stabilization (given above) as

\[ H'_i = H_i \times D^2 = D^n \times D^n \times D^2 \]

so that all the fibers over D² gain an handle of index n and become X'. Here the attaching sphere of Hᵢ is still Sᵢ ⊂ ∂W, indeed we are only thickening the handle Hᵢ.

Moreover, the new right-handed Dehn twist δᵢ introduced by each stabilization will be considered as the result of attaching a (Lefschetz) handle ˜Hᵢ = Dⁿ⁺¹ × Dⁿ⁻¹ × D² along the Lagrangian sphere Cᵢ ∪ Sᵢ ∪ Dᵢ ≈ Sⁿ. One should note that the normalization of the vanishing cycle Cᵢ ∪ Dᵢ is determined as in the proof of Lemma 6.1. By trivially extending the fibration W → D² over the handles H₁', H₂', we may assume that W ∪ H₁' ∪ H₂' also admits a Lefschetz fibration. The new monodromy (still denoted by h) is obtained by extending h identically over H₁ and H₂. When we attach ˜H₁ and ˜H₂ to W ∪ H₁' ∪ H₂', the new monodromy becomes h' = δ₂ ◦ δ₁ ◦ h by Theorem 2.9. Also we have h' ∈ Symp(X', dλ') by Remark 2.10. Finally, note that the core disk of ˜H₁ and the cocore disk of H₁' intersect each other transversally once, so \{H₁', ˜H₁\} is a canceling pair which means that the manifold

\[ W ∪ H₁' ∪ H₂' ∪ ˜H₁ ∪ ˜H₂ \]

is still diffeomorphic to W. Hence, we proved that W admits another Lefschetz fibration with the properties claimed.

Now, suppose that the convex Liouville structure (dλ, χ) on X is given by some Stein structure (J, ψ) on X. Consider the compact Stein manifold (X₀, J, ψ) (for the other pages one can proceed in a similar way). Since ˜X₀ ⊂ X₀, we can restrict J and ψ (and so −d(dψ ◦ J) = dλ) on ˜X₀. Note that the restriction ψ|X₀ is an exhausting J-convex (i.e.,
strictly plurisubharmonic) function on $\tilde{X}_0$. By a theorem of Grauert [Gr], $\tilde{X}_0$ admits a Stein structure. Indeed, the Stein structure on $\tilde{X}_0$ is the restriction of the one on $X_0$. Here we should note that if a page $X$ is gaining a region (rather then loosing) during the isotopy moving the binding, then the Stein structure on $\tilde{X}$ is the extension of the one on $X$. Hence, we may assume that every page $\tilde{X}$ of the open book $(\tilde{X}, \tilde{h})$ still equips with a Stein structure $(\tilde{J}, \tilde{\psi})$ which is deformation equivalent to $(J, \psi)$ on $X$.

Next, we observe that the positive stabilizations can be performed in the almost complex category. Indeed, we can glue each handle $H_i$ above to the Stein manifold $(\tilde{X}, \tilde{J}, \tilde{\psi})$ using a special HAT. By Theorem 5.3, this will extend the Stein structure $(\tilde{J}, \tilde{\psi})$ on $\tilde{X}$ to a Stein structure $(J', \psi')$ on $X'$ inducing the convex Liouville structure $(d\lambda', \chi')$.

\[\square\]

**Proof of Theorem 1.1.** Let $(W, J, \psi)$ be a compact Stein manifold of dimension $2n+2$ with $n > 1$. By Theorem 4.1, we know that $W^{2n+2}$ is topologically characterized by

$$W^{2n+2} \cong B^{2n+2} + \{\text{handles of index } \leq n + 1\}.$$ 

Assume that $W^{2n+2}$ is obtained by a single $(n+1)$-handle attachment to a subcritical Stein manifold. Note that by induction on the number of $(n+1)$-handles, it is enough to consider the case of a single handle. By Theorem 4.2, every subcritical Stein manifold “splits”, and so Remark 4.3 implies that

$$(W, J, \psi) \cong (W_0 \times D^2, J_0 \times i, \psi_0 + |z|^2) + H^{n+1}$$

where $(W_0, J_0, \psi_0)$ is a compact Stein $2n$-manifold, and $H^{n+1}$ is a handle (of index $n+1$) attached in the almost complex category. Let $(f, \beta, \varphi)$ be a HAT corresponding to $H^{n+1}$ which can be assumed to be special by Theorem 5.2. Suppose $H^{n+1}$ is (topologically) glued via the attaching map

$$\Phi : S^n \times D^{n+1} \to \partial(W_0 \times D^2) = W_0 \times S^1 \cup \partial W_0 \times D^2.$$ 

Here, we consider $S \subset \mathbb{C}^{n+1}$ as $S^n$, the bundle $\nu_0$ as $D^{n+1}$ (using tubular neighborhood theorem), and also we forget $\varphi$. Note that the embedding $f : S \to f(S)$ of the Legendrian $n$-sphere $f(S)$ corresponds to the embedding $\phi := \Phi|_{S^n \times \{0\}}$.

By a sequence of isotopies, we want to arrange so that $\phi(S^n)$ misses $W_0 \times p$ for some $p \in S^1$. Being a Stein manifold, $W_0 = W_0^{2n}$ has an $n$ dimensional spine $\Delta^n$, and since $\dim(\phi(S^n)) + \dim(\Delta^n \times p) = 2n < 2n + 1 = \dim(\partial(W_0 \times D^2))$, we can isotope $\Phi$ so that its restriction $\phi := \Phi|_{S^n \times \{0\}}$ misses $\Delta^n \times p$ for some $p \in S^1$. Therefore, $\Phi$ itself misses $W_0 \times p$ (here we consider $\Phi(S^n \times D^{n+1})$ and $W_0$ as small tubular neighborhoods of $\Phi(S^n \times \{0\})$ and $\Delta^n \times p$, respectively). This isotopy is not necessarily an isotopy of HAT’s but we want to keep $\phi(S^n)$ Legendrian for our purposes. So, we will either not isotope $\phi(S^n)$, or when we isotope it, to keep Legendrian, we will isotope it by contact amorphisms. We will show under such isotopies we can make $\phi(S^n)$ miss a page.

By Remark 4.3, the induced exact symplectic form on $W = W_0 \times D^2$ is given by $\omega = \omega_0 + dx \wedge dy$. Let $\xi$ be the contact structure on $\partial W$ induced by the Stein structure on $W$. Let $\chi_0 := \nabla \psi_0$ be the gradient vector field of $\psi_0$ with respect to the Riemannian metric $g_0(\cdot, \cdot) := \omega_0(\cdot, J_0 \cdot)$. $\chi_0$ is a Liouville vector field of $\omega_0$ (i.e., $\mathcal{L}_{\chi_0} \omega_0 = \omega_0$). Therefore,
the 1-form $\alpha_0 := \iota_{\chi_0} \omega_0$ is a primitive of $\omega_0$ (i.e., $d\alpha_0 = \omega_0$). Indeed, the restriction $\alpha_0|_{\partial W_0}$ defines the contact structure on $\partial W_0$ induced by the Stein structure on $W_0$. Let $R_0$ be the Reeb vector field of $\alpha_0$. By a direct computation, one can see that the vector field

$$\chi := \chi_0 + (x/2)\partial x + (y/2)\partial y$$

is a Liouville vector field for $\omega$ (we abbreviate $\partial/\partial x$ by $\partial x$, and $\partial/\partial y$ by $\partial y$). Therefore

$$\alpha := \iota_{\chi} \omega - \iota_{\chi_0} \omega_0 + [(x/2)\partial x + (y/2)\partial y] \wedge (dx \wedge dy) = \alpha_0 + (x/2)dy - (y/2)dx$$

is a contact form for $\xi$. Note that in polar coordinates $\alpha = \alpha_0 + (r^2/2)d\theta$. Now observe

$$\partial(W_0 \times D^2) = W_0 \times S^1 \cup \partial W_0 \times D^2$$

defines a “trivial” open book $(W_0, \text{id})$ for $\partial(W_0 \times D^2)$ with the trivial monodromy. This open book supports $\xi$ because $\alpha|_{\partial W_0} = \alpha_0$ is a contact form on the binding $\partial W_0 \times \{(0,0)\}$ and $d\alpha = d\alpha_0 = \omega_0$ is an exact symplectic form on every page $W_0$ (recall Definition 3.2).

Next we consider the following four useful vector fields in $\partial(W_0 \times D^2)$:

(2) $Z_1 = \partial x - (y/2)R_0$, $Z_2 = \partial y + (x/2)R_0$, $Z_3 = \chi_0 + (\theta/2)\partial \theta$, $Z_4 = \partial \theta$

where the first two vector fields are defined on the tubular neighborhood $\partial W_0 \times D^2$ of the binding $\partial W_0 \times \{(0,0)\}$, and $\theta$ is the circle coordinate. It is easy to check that:

$$\mathcal{L}_Z \alpha = \begin{cases} 0 & \text{if } i = 1, 2, 4 \\ \alpha & \text{if } i = 3 \end{cases}$$

So they are all contact vector fields on $\partial(W_0 \times D^2)$.

Now recall $W^{2n+2} = (W_0 \times D^2) + H^{n+1}$, where the handle $H^{n+1}$ is attached along a Legendrian sphere $\phi : S^n \hookrightarrow \partial(W_0 \times D^2)$. For any constants $a_1, a_2$, the vector field $Z = a_1 Z_1 + a_2 Z_2$ is also contact and everywhere transverse to the binding $\partial W_0 \times \{(0,0)\}$. By using $Z$, we can isotope the binding to any of its nearby copies $\partial W_0 \times q$ by contactomorphisms, where $q \in int(D^2)$. In particular, by choosing a regular value $q = (a_1, a_2) \in D^2$ of the composition

$$S^n \supset U \xrightarrow{\phi} \partial W_0 \times D^2 \xrightarrow{\pi_2} D^2$$

where $\pi_2$ is the projection and $U = \phi^{-1}(\partial W_0 \times int D^2)$ and isotoping $\phi(S^n)$ along $Z$, we can make $\phi(S^n)$ transversal to $\partial W_0 \times \{(0,0)\}$. Hence we can assume that the Legendrian attaching sphere $\phi(S^n)$ is transversal to the binding of the open book of $W_0 \times S^1 \cup \partial W_0 \times D^2$ in its neighborhood. Now by picking a regular value $p \in S^1$ for the restriction of the projection $\pi_2 : \phi(S^n) \cap (W_0 \times S^1) \to S^1$ we can assume that the intersection

$$L := \phi(S^n) \cap (W_0 \times p)$$

is a properly imbedded $n-1$ dimensional submanifold $(L, \partial L) \subset (W_0, \partial W_0)$ meeting the binding along an $n-2$ dimensional submanifold $L' = \partial L$. Here for simplicity we identified $W_0 \times p$ by $W_0$, will continue to use this identification.

**Proposition 6.4.** $L$ is contained in a properly imbedded ball $B_2^{2n} \subset W_0$, that is $(L, L') \subset (B_1^{2n}, B_2^{2n-1}) \subset (W_0, \partial W_0)$, were $B_2^{2n-1} = B_1^{2n} \cap \partial W_0$ is also a ball.
Proof. Let $\Delta^n \subset W_0^{2n}$ be its spine. By the general position in $W_0^{2n} \times p$, we can $\epsilon$-isotope $\Delta$ to a nearby copy $\tilde{\Delta}$ which is disjoint from $L^{n-1}$.

Next we claim that the inclusion maps $L' \hookrightarrow \partial W_0$ and $L \hookrightarrow W_0$ are null homotopic. Since $\tilde{\Delta} \subset W_0$ is the spine, $L$ homotopies into the binding $\partial W_0$. Hence the map induced by inclusion $\pi_i(L) \to \pi_i(W_0)$ factors through inclusions

$$\pi_i(L) \xrightarrow{\alpha} \pi_i(\partial W_0) \xrightarrow{\beta} \pi_i(W_0)$$

Also, notice that the composition $L \subset \phi(S^n) \subset D^{n+1} \subset W$ is null homotopic (where $D^{n+1}$ is the core of the handle attached by $\phi$). So the image of $\alpha$ lies in the kernel $\pi_i(\partial W_0) \to \pi_i(W)$, which is the image the boundary map $\pi_i(W, \partial W) \xrightarrow{\partial} \pi_i(\partial W_0)$. The inclusions $\partial W_0 \subset W_0 \subset W$ give the exact sequence:

$$\ldots \to H_i(W_0, \partial W_0) \to H_i(W, \partial W_0) \to H_i(W, W_0) \to \ldots$$

For $i \le n$, $H_i(W_0, \partial W_0) \cong H^{2n-i}(W_0) \cong H^{2n-i}(\Delta) = 0$. Also for $i \le n$ we have $H_i(W, W_0) = 0$ (since $W$ is obtained from $W_0$ by attaching $n+1$ cells). So $H_i(W, \partial W_0) = 0$ for $i < n$. Hence $\pi_i(W, \partial W_0) = 0$ for $i < n$. So for $i \le n - 1$, $\pi_i(L) \to \pi_i(W_0)$ is the zero map. We claim the map $\pi_{n-1}(L) \to \pi_{n-1}(W_0)$ is also the zero map; this follows from the
following commutative diagram, where vertical isomorphisms are Hurewicz maps, and the bottom horizontal sequence is the special case of (3).

\[
\begin{array}{ccc}
\pi_{n-1}(\partial W_0) & \xrightarrow{\beta} & \pi_{n-1}(W_0) \\
\uparrow \cong & & \uparrow \\
\pi_n(W_0, \partial W_0) & \rightarrow & \pi_n(W, \partial W_0) \\
\downarrow \cong & & \downarrow \\
H_n(W_0, \partial W_0) & \rightarrow & H_n(W, \partial W_0) \rightarrow 0
\end{array}
\]

Hence the inclusion \( L^{n-1} \hookrightarrow W_0 \) is null homotopic. It remains to prove \( (L')^{n-2} \hookrightarrow \partial W_0 \) is null homotopic. For this we use \( \pi_i(W, \partial W_0) = 0 \) for \( i < n \implies \pi_i(\partial W_0) \rightarrow \pi_i(W) \) is an isomorphism for \( i \leq n - 2 \). Also for the same reason as above, the following composition of inclusions is the zero map:

\[
\pi_i(L') \rightarrow \pi_i(\partial W_0) \rightarrow \pi_i(W)
\]

Hence \( \pi_i(L') \rightarrow \pi_i(\partial W_0) \) is zero when \( i \leq n - 2 \), so \( L' \hookrightarrow \partial W_0 \) is null homotopic.

When \( n > 2 \) we get the conclusion of the proposition by applying the smooth version of Stallings’s “Egulfing theorem” (e.g. [HZ]). The special case \( n = 2 \) can be checked directly by hand, since any null-homotopic 1-manifold in a 4-manifold lies in a ball. \( \square \)

**Figure 5.**

Next we will show that after isotopies along the contact vector fields given in (2), we can make \( \phi(S^n) \) miss a page \( W_0 \times p \). By Proposition 6.4, \( L = \phi(S^n) \cap (W_0 \times p) \) is contained in a 2n-ball \( B \subset W_0 \times p \), which lies in some tubular neighborhood of the binding \( \partial W_0 \times \{(0, 0)\} \) in \( \partial(W_0 \times D^2) \). First, we isotope \( \phi(S^n) \) using \( Z_3 \) and \( Z_4 \) until the final image \( \tilde{L} \) of \( L \) is completely contained in \( \text{int}(\partial W_0 \times D^2) \). Here we use \( Z_3 \) to bring \( L \subset \phi(S^n) \) closer to the binding, while \( Z_4 \) is used to bring it back to the same page where \( L \) lies. Note that the flows of \( Z_3 \) and \( Z_4 \) maps pages to pages, and so \( L \) is the new intersection of \( \phi(S^n) \) with \( W_0 \times p \). Since \( L \subset B \), there is a smaller 2n-ball \( \tilde{B} \) in the page \( W_0 \times p \) such that \( \tilde{L} = \phi(S^n) \cap (W_0 \times p) \subset \tilde{B} \subset N := \text{int}(\partial W_0 \times D^2) \).
Now, for $0 < r < 1$ consider the disk $D_r = \{(x, y) \in D^2 \mid x^2 + y^2 \leq r^2\} \subset D^2$ and the neighborhood $N_r = \partial W_0 \times D_r$ of the binding. The above argument implies that $\tilde{L} \subset \tilde{B} \subset N_{r_0}$ for some $0 < r_0 < 1$. Let $(a_1, a_2) \in D^2$ be a point which corresponds to the angular coordinate $p \in S^1$. We isotope $\phi(S^n)$ along the contact vector field $Z = a_1 Z_1 + a_2 Z_2$ until it completely crosses the binding. Observe now that the page corresponding to $p$ does not intersect $\phi(S^n)$ (see Figure 6).

![Figure 6.](image)

Observe that $\partial(W_0 \times S^1) - W_0 \times p \cong W_0 \times (0, 1) \cong W_0 \times \mathbb{R}$. Therefore, we can consider $\phi(S^n)$ as a Legendrian submanifold of the contact manifold

$$(W_0 \times \mathbb{R}, \text{Ker}(\alpha_0 + dz))$$

where $z$ is the coordinate on $\mathbb{R}$ and $W_0$ is an exact symplectic manifold with the exact symplectic form $\omega_0 = d\alpha_0$. Next, we will project $\phi(S^n)$ onto $W_0$ by using the map $\Pi : W_0 \times \mathbb{R} \to W_0$, which forgets the $z$-coordinate. The following definitions and results are given in [EES1] and [EES2]. Let $c$ denote a Reeb chord of $\phi(S^n)$, that is, a trajectory of the Reeb vector field $\partial z$ starting and ending at points on $\phi(S^n)$. Then $p := \Pi(c)$ is a double point of $\Pi(\phi(S^n))$. We say that $\phi(S^n)$ is chord generic if the only self intersections of the Lagrangian immersion $\Pi(\phi(S^n))$ are transverse double points. Note that this is an open and dense condition. Indeed, Lemma 2.7 of [EES2] implies that, by an arbitrarily small Legendrian isotopy we can isotope $\phi(S^n)$ so that it becomes chord generic.

Therefore, we may assume that $\phi(S^n)$ is chord generic, so $L := \Pi(\phi(S^n))$ is a Lagrangian immersed sphere in $(W_0, \omega_0)$ with finitely many transverse double points and no other self intersections of any other types. Let $d_1, d_2, \ldots, d_k$ be the transverse double points of $L$, and $c_i$ be the Reeb chord corresponding to each $d_i$, that is, $d_i = \Pi(c_i)$.

Observe that $L$ lies on a Stein page $(W_0, J_0, \psi_0) \times \{pt\}$ of the open book $(W_0, \text{id})$ which supports the contact structure $\xi = \text{Ker}(\alpha)$. $(W_0, \text{id})$ is the boundary open book of the trivial Lefschetz fibration $W_0 \times D^2 \to D^2$ (with trivial monodromy). We want to apply
Proposition 6.3 to our case. To this end, let \( p \in \{d_1, d_2, ..., d_k\} \) be any double of \( L \) on \( W_0 \). Let \( \mathcal{D} \subset W_0 \) be a Legendrian \( n \)-disk passing through \( p \). By Legendrian Neighborhood Theorem (see [MS] (p.99, 105), also [Ge] (p.71, 72)), there exists a contactomorphism between a neighborhood of \( \mathcal{D} \) in \( (W_0 \times \mathbb{R}, \text{Ker}(\alpha_0 + dz)) \) and a neighborhood of the zero section of the 1-jet bundle

\[
(T^*(\mathcal{D}) \times \mathbb{R}, \text{Ker}(\sum_{j=1}^{n} q_j dp_j + dz))
\]

where \( p_1, p_2, ..., p_n \) are the coordinates on the zero section \( \{q_j = 0, \text{ for all } j\} \), and \( z \) is the \( \mathbb{R} \)-coordinate (for simplicity, we’ll use \( z \) again). Since the contactomorphism maps \( \mathcal{D} \) to the zero section and maps the Reeb direction \( \partial z \) to the Reeb direction \( \partial z \), we may assume that there exists a neighborhood \( N(\mathcal{D}) \approx D^{2n} \) of \( \mathcal{D} \) in \( W_0 \times \{pt\} \), and on \( N(\mathcal{D}) \times \mathbb{R} \) there are coordinates \( (p_1, p_2, ..., p_n, q_1, q_2, ..., q_n, z) \) such that \( p = (0, ..., 0, 0, ..., 0, 0) \), and

\[
\mathcal{D} = \{q_j = 0, \text{ for all } j\}, \quad \alpha = \sum_{j=1}^{n} q_j dp_j + dz.
\]

Notice on \( N(\mathcal{D}) \times \mathbb{R} \), there are 2n linearly independent contact vector fields, namely,

\[
X_1 = \partial p_1, ..., X_n = \partial p_n, Y_1 = \partial q_1 - p_1 \partial z, ..., Y_n = \partial q_n - p_n \partial z.
\]

Using these vector fields we can locally isotope the embedding \( \phi(S^n) \) (through Legendrian spheres) in \( N(\mathcal{D}) \times \mathbb{R} \) along any contact vector field direction of the form

\[
a_1 X_1 + ... + a_n X_n + b_1 Y_1 + ... + b_n Y_n
\]

where \( a_i \) and \( b_i \) are arbitrary constants for each \( i \).

Now, let \( p' \in N(\mathcal{D}) \subset W_0 \) be a point near the double point \( p \) such that

- \( p' \in W_0 \setminus \text{Core}(W_0, -d(\psi_0 \circ J_0), \nabla \psi_0) \), and
- there exists a trajectory of \( \nabla \psi_0 \) connecting \( p' \) to a point on the boundary of \( W_0 \) such that the trajectory intersects \( L \) only at \( p' \).

Note that, by general position and the fact that \( \text{int Core}(W_0, -d(\psi_0 \circ J_0), \nabla \psi_0) = \emptyset \), such a \( p' \) always exits in \( N(\mathcal{D}) \). Say in \( (p_i, q_i, z) \)-coordinates, \( p' = (a_1, ..., a_n, b_1, ..., b_n, 0) \). Then we isotope the embedding \( \phi(S^n) \) in \( N(\mathcal{D}) \times \mathbb{R} \) along \( a_1 X_1 + ... + a_n X_n + b_1 Y_1 + ... + b_n Y_n \) so that the double point \( p \) of \( L \) is replaced with the new one, namely, \( p' \). Repeating this process for each double point of \( L \), we may assume that the projection \( L \) satisfies the conditions in Proposition 6.3. Hence, Proposition 6.3 and Remark 2.10 implies that:

- There exists a Lefschetz fibration on \( W_0 \times D^2 \) whose generic fiber is a Stein manifold \( X^{2n} \) with the exact symplectic form \( \omega_1 \), and whose monodromy is given by
  \[
h = (\delta_{k_1} \circ \delta_{k_2}) \circ ... \circ (\delta_{11} \circ \delta_{12}) \in \text{Symp}(X, \omega_1)
\]
  where each pair \( \{\delta_{i_1}, \delta_{i_2}\} \) of right-handed Dehn twists is the result of a pair of positive stabilizations used to remove each double point \( d_i \) of \( L \),
- The boundary open book \( (X, h) \) still supports \( \xi \) and so there exists a contact form \( \alpha_1 \) for \( \xi \) such that \( d\alpha_1 = \omega_1 \), and
- There exists an embedded Lagrangian submanifold \( L' \) of a page \( (X, \omega_1) \) which is obtained from \( L \) by desingularizing all the double points.
Before applying Proposition 6.3 we should have noted that $L$ can be considered as the final image of $\phi(S^n)$ under a Legendrian isotopy along the Reeb vector field $\partial z$, and so $L$ is a Legendrian immersed (it has double points) submanifold of $(\partial(W_0 \times D^2), \xi)$. Now, let $U$ denote the union of tubular neighborhoods of the double points of $L$ in $L$. Then by the construction in the proof of Proposition 6.3 $L \setminus U$ is a subset of $L'$, and $L'$ is the union of $L \setminus U$ with the union $V$ of the core disks of the handles in the stabilizations.

Each core disk in $V$ can be considered as the image of the corresponding disk in $U$ under a Legendrian isotopy along the new Reeb vector field of the new contact form $\alpha_1$ of $\xi$. As a result, $L'$ is an embedded Legendrian sphere in $(\partial(W_0 \times D^2), \xi)$ which is Legendrian isotopic to $\phi(S^n)$. (Here we can also consider this as follows: projecting $\phi(S^n)$ to $L \subset W_0$ shrinks all the Reeb chords. By replacing each disk in $U$ by the corresponding core disk in $V$, we are actually forming each Reeb chord back which eventually gives $L' \subset X$ that is Legendrian isotopic to $\phi(S^n)$ in $(\partial(W_0 \times D^2), \xi)$.)

So far we have proved that for the special HAT $(f, \beta, \varphi)$ the embedding

$$f : S^n \rightarrow f(S^n) = \phi(S^n)$$

is Legendrian isotopic to an another embedding, say

$$f' : S^n \rightarrow f'(S^n) = L',$$

which lies, as a Lagrangian sphere, on a page $(X, \omega_1)$ of the open book $(X, h)$ supporting $\xi$. Proposition 6.2 implies that if we attach a handle (of index $n + 1$) to $W_0 \times D^2$ using a special HAT with attaching sphere $L'$, the Lefschetz fibration (on $W_0 \times D^2$) extends to the resulting manifold. However, we want the resulting manifold to be $W$ (which we started with). Therefore, we should keep track of not only how we changed the attaching sphere $f(S^n)$ but also how the whole attaching region of $(f, \beta, \varphi)$ is changing.

Let $f_t$ denote the isotopy connecting $f_0 = f$ to $f_1 = f'$. As $f_t$ is constructed using the flows of various contact vector fields, it also preserves the contact distribution $\xi$. Indeed, as in Remark 5.3, the special HAT $(f, \beta, \varphi)$ and $f_t$ determines a homotopy between the complex bundle isomorphisms

$$T(S^n) \oplus iT(S^n) = T(S^n) \otimes \mathbb{C} \longrightarrow \xi|_{T(f(S^n))} = T(f(S^n)) \oplus JT(f(S^n))$$

and

$$T(S^n) \oplus iT(S^n) = T(S^n) \otimes \mathbb{C} \longrightarrow \xi|_{T(f'(S^n))} = T(f'(S^n)) \oplus JT(f'(S^n)).$$

This homotopy can be extended along the direction transverse to $\xi$ and determines a homotopy $\beta_t = \overline{Df_t} : \nu_0 \rightarrow T(\partial W)$ between $\beta = \beta_0$ and $\beta' := \beta_1$ given by

$$\beta_t = \overline{Df_t} = -J \circ Df_t \circ i.$$

Finally, $f_t$ and $\beta_t$ together determine a homotopy

$$\varphi_t = Df_t \oplus \overline{Df_t} : T(H^{n+1})|_{S^n} = \tau \oplus \nu_0 \rightarrow T(W)$$

between the complex bundle isomorphisms $\varphi = \varphi_0$ and $\varphi' := \varphi_1$.

To summarize, we showed that the Legendrian isotopy $f_t : S^n \rightarrow \partial W$ is covered by the homotopies $\beta_t : \nu_0 \rightarrow T(\partial W)$ and $\varphi_t : T(H^{n+1})|_{S^n} \rightarrow T(W)$ which means that the special HAT’s $(f, \beta, \varphi)$ and $(f', \beta', \varphi')$ are isotopic. Therefore, by Theorem 5.1 and Theorem 5.3, attaching a handle to $W_0 \times D^2$ using $(f', \beta', \varphi')$ yields a Stein manifold.
such that there is a diffeomorphism \( F : W \rightarrow W' \) and \( F^*J' \) is homotopic to \( J \). Indeed, \((W, F^*J', F^*\psi')\) and \((W, J, \psi)\) are deformation equivalent, and hence the induced contact structures \( \xi' \) and \( \xi \) on their boundaries are isotopic (if \( J_t \) is a homotopy connecting \( J_0 = J \) to \( J_1 = F^*J' \), then \( \xi_t = T(\partial W) \cap J_t(\partial W) \) is the required isotopy). On the other hand, as we mentioned above that \( W' \) admits a Lefschetz fibration whose boundary open book supports the contact structure \( \xi' \) by Proposition 6.2. Hence, \( W \) admits a Lefschetz fibration such that the boundary open book supports the contact structure \( \xi \), a generic fiber is the symplectic manifold \((X, \omega_1)\) and the monodromy of the fibration is a symplectomorphism given by

\[
h' = \delta \circ (\delta_{k1} \circ \delta_{k2}) \circ \cdots \circ (\delta_{11} \circ \delta_{12}) \in \text{Symp}(X, \omega_1)
\]

where \( \delta \) is the right-handed Dehn twist whose center is determined by the Lagrangian embedding \( \delta'(S^n) \) in \((X, \omega_1)\). Observe that \( X \) is still Stein because it is obtained from \( W_0^{2n} \) by attaching handles of index \( n \). Indeed the Stein structure on \( X \) is compatible with the one on \( W_0 \). □

As mentioned in the beginning of the paper there is a slight strengthening of the main theorem in dimension 6:

**Remark 6.5.** Let \( W^6 \) be a compact smooth manifold which admits a Lefschetz fibration \( \pi : W \rightarrow D^2 \) with a generic fiber \( X^4 \), by assumption \( \partial X \neq \emptyset \). Therefore, there exists a handle decomposition of \( X^4 \) consisting of handles with indices 0,1,2, and 3 only. By the arguments of [K] summarized in Section 2, we know that \( W^6 \) is obtained from \( X^4 \times D^2 \) by attaching a finite number of “Lefschetz” handles of index 3. By thickening the handles of \( X^4 \), we get a handle decomposition of \( X^4 \times D^2 \) consisting of handles with indices 0,1,2, and 3. This shows that \( W^6 \) admits a handle decomposition whose handles have index 0,1,2, and 3. Hence, there exists a Stein structure on \( W^6 \) by [E]. The converse follows from Theorem 1.1 above.

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