Groups with a solvable subgroup of prime-power index

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Abstract
In this paper we describe some properties of groups $G$ that contain a solvable subgroup of finite prime-power index (Theorem 1 and Corollaries 2–3). We prove that if $G$ is a non-solvable group that contains a solvable subgroup of index $p^\alpha$ (for some prime $p$), then the quotient $G/\text{Rad}(G)$ of $G$ over the solvable radical is asymptotically small in comparison to $p^\alpha!$ (Theorem 4).

Keywords Solvable groups · Hall $\pi$-subgroups · Solvable radical · Fermat primes · Mersenne primes

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1 Introduction
In this paper we explore some properties of groups with solvable subgroups of $p$-power index for some prime $p$. In the case of finite groups, our condition is equivalent to requiring that the group contains a solvable Hall $p'$-subgroup. There are several well-known results concerning the solvability of finite groups assuming the existence of certain Hall $p'$-subgroups. The most famous of these is Hall’s Theorem that states that a finite group is solvable if and only if it contains Hall $p'$-subgroups for all $p$ [10]. Furthermore, in a finite solvable group, for a fixed $p$, the Hall $p'$-subgroups are

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conjugate (see [9] or [18, 9.1.7–9.1.8]). Later Wielandt proved that a finite group is solvable if it possesses three solvable subgroups whose indices are pairwise relatively prime (see for instance [19, Lemma 11.25]). In particular, if a finite group $G$ contains solvable Hall $p'$-subgroups for three different primes, then $G$ is solvable. Arad and Ward extended Wielandt’s result to show that if $G$ contains a Hall $2'$-subgroup and a Hall $3'$-subgroup, then $G$ is solvable [1,3]. More solvability criteria in terms of some (solvable) subgroups can be found in [2,6,11,12,14,16,20].

Fermat primes are prime numbers of the form $2^m + 1$ with $m \geq 1$. To simplify notation, we set

$$\pi_0 = \{2, 7, 13\} \cup \{p \mid p \text{ is a Fermat prime}\}. \quad (1)$$

For $m \geq 2$, primes of the form $2^m - 1$ are said to be Mersenne primes.

Our first theorem implies, for certain primes $p$, the solvability of finite groups $G$ under the condition that $G$ has a solvable Hall $p'$-subgroup.

**Theorem 1** Let $G$ be a finite group, let $p$ be a prime number and assume that $G$ contains a solvable subgroup of index $p^\alpha$ for some $\alpha \geq 1$. Then $G$ contains a solvable Hall $p'$-subgroup and the following assertions are valid.

1. If $p \notin \pi_0$, then $G$ is solvable.
2. If $p \neq 7, 13$, then the Hall $p'$-subgroups of $G$ are conjugate.

The proof of Theorem 1 is based on Guralnick’s classification [8] of the finite simple groups containing a subgroup of prime-power index. From this classification, it is easy to obtain a classification of finite simple groups with a solvable subgroup of prime-power index (see Lemma 6). Guralnick’s classification and its consequences for finite simple and characteristically simple groups with a solvable subgroup of prime-power index will be explored in Sect. 2.

Theorem 1 yields the following corollaries.

**Corollary 2** If $p$ is a prime such that $p \notin \pi_0$ and $G$ is a not necessarily finite group with a solvable subgroup of $p$-power index, then $G$ is solvable.

A finite group $G$ is $p$-nilpotent if there exists a normal Hall $p'$-subgroup of $G$. It is an immediate consequence of the Feit–Thompson Theorem [7] that a finite 2-nilpotent group is solvable. For $p$ odd, a $p$-nilpotent group need not be solvable and part (3) of the following corollary gives a sufficient condition for the solvability of $p$-nilpotent groups. We denote by $\pi(G)$ the set of primes that divide the order of a finite group $G$.

**Corollary 3** Let $p$ and $q$ be different primes and let $G$ be a finite group.

1. If $\{p, q\} \neq \{2, 7\}$ and $G$ contains a solvable Hall $p'$-subgroup and a solvable Hall $q'$-subgroup, then $G$ is solvable.
2. If $G$ contains a solvable Hall $p'$-subgroup with $p \neq 3$ and a Hall $3'$-subgroup, then $G$ is solvable.
3. If $G$ is $p$-nilpotent, $q \notin \pi_0$ and $G$ contains a solvable Hall $\{p, q\}'$-subgroup, then $G$ is solvable.

Theorem 1 and Corollaries 2–3 are proved in Sect. 3.
If $G$ is a group and $H$ is a solvable subgroup of $G$ with $|G : H| = m = p^\alpha$ for some prime $p$, then, noting that $\text{Core}_G(H)$ is a solvable normal subgroup of $G$ and so is contained in $\text{Rad}(G)$ and considering the transitive $G$-action on the coset space modulo $H$, we obtain that

$$|G/\text{Rad}(G)| \leq |G/\text{Core}_G(H)| \leq m!.$$ 

A more careful analysis, in Sect. 4, of the structure of $G$ gives the following polynomial bound on the size of $G/\text{Rad}(G)$.

**Theorem 4** If $p$ is a prime, and $G$ is a (not necessarily finite) group with a solvable subgroup $H$ such that $|G : H| = p^\alpha = m$, then $|G/\text{Rad}(G)| \leq m^5$. Moreover, if $p \neq 13$, then $|G/\text{Rad}(G)| \leq m^4$.

### 2 Simple groups with solvable subgroups of prime-power index

An easy argument shows that if $2^k + 1$ is a prime number, then $k = 2^m$, and so a Fermat prime must be of the form $2^{2^m} + 1$ for some $m \geq 0$. At this moment only five Fermat primes are known, namely $2^1 + 1 = 3$, $2^2 + 1 = 5$, $2^4 + 1 = 17$, $2^8 + 1 = 257$, and $2^{16} + 1 = 65,537$. The question whether there are more Fermat primes or whether the number of Fermat primes is finite or infinite is open. Catalan’s conjecture proved by Mihăilescu [17] states that two consecutive natural numbers cannot both be proper powers except for 8 and 9. Thus the following lemma is valid.

**Lemma 5** If $q$ is a natural number such that $q$ and $q + 1$ are both prime-powers, then one of the following holds:

1. $q + 1$ is a power of 2 and $q$ is a Mersenne prime;
2. $q = 2^3 = 8$ and $q + 1 = 3^2 = 9$;
3. $q = 2^{-m}$ with some $m \geq 0$ and $q + 1$ is a Fermat prime.

The following Lemma is a consequence of Guralnick’s classification [8] of finite simple groups with a subgroup of prime-power index. Recall the definition of the set $\pi_0$ in Eq. (1).

**Lemma 6** Suppose that $T$ is a non-abelian simple group and $H < T$ is a solvable subgroup of prime-power index $p^\alpha$. Then $p \in \pi_0$ and one of the following holds:

1. $T = A_5$, $H = A_4$ and $p^\alpha = 5$;
2. $T = \text{PSL}_2(8)$, $H$ is the stabiliser of a line in $\mathbb{F}_8^2$, and $p^\alpha = 9$;
3. $T = \text{PSL}_2(q)$ with some odd prime $q \geq 5$, $H$ is the stabiliser of a line in $\mathbb{F}_q^2$, and $p^\alpha = q + 1$ is a 2-power;
4. $T = \text{PSL}_2(2^m)$ with some $m \geq 2$, $H$ is the stabiliser of a line in $\mathbb{F}_{2^m}^2$, $\alpha = 1$, and $p$ is a Fermat prime;
5. $T = \text{PSL}_3(2)$, $H$ is the stabiliser of a line or a plane in $\mathbb{F}_2^3$, and $p^\alpha = 7$;
6. $T = \text{PSL}_3(3)$, $H$ is the stabiliser of a line or a plane in $\mathbb{F}_3^3$, and $p^\alpha = 13$.
Proof Guralnick [8] classified subgroups of prime-power index in finite non-abelian simple groups. Our lemma follows by inspection of Guralnick’s list which contains five cases (a)–(e). In Guralnick’s case (a), $H = A_{n-1}$ which is solvable only for $n - 1 = 4$ (that is, $n = 5$); this gives item (1). In case (b), $T = \text{PSL}_n(q)$ and $H$ is the stabilizer of either a line or a hyperplane in $\mathbb{P}^n_q$. This stabilizer is a parabolic subgroup and it contains a section isomorphic to $\text{PSL}_{n-1}(q)$. Hence $H$ can only be solvable if $\text{PSL}_{n-1}(q)$ is solvable which happens only for $n - 1 = 1$ or for $n - 1 = 2$ and $q \in \{2, 3\}$. If $n - 1 = 1$, then we obtain items (2)–(4). In items (3)–(4), the fact that $q$ and $p$ are primes, respectively, follows from Lemma 5. If $n - 1 = 2$ and $q = 2, 3$, then we obtain items (5)–(6). In cases, (c)–(e) of Guralnick’s list, $H$ is non-solvable (in case (e), $A_5$ is a composition factor of $H$).

Corollary 7 If $T$, $H$, $p$, and $\alpha$ are as in Lemma 6, then the following hold.

1. $H$ is a maximal subgroup of $T$;
2. In cases (1)–(5), $4 \cdot |\text{Aut}(T)| \leq p^{4\alpha}$, while in case (6), $4 \cdot |\text{Aut}(T)| \leq 13^5$;
3. $H$ is a Hall $p'$-subgroup of $T$;
4. In cases (1)–(4), $T$ contains a unique conjugacy class of Hall $p'$-subgroups, while in cases (5)–(6), $T$ contains two conjugacy classes of Hall $p'$-subgroups.

Proof (1) The maximality of $H$ follows by noting that $H$ is the point stabiliser of a primitive action of $T$.

(2) In cases, (1), (2), (5) and (6) of Lemma 6, the statement follows by direct calculation observing that $|\text{Out}(A_5)| = |\text{Out}(\text{PSL}_3(2))| = |\text{Out}(\text{PSL}_3(3))| = 2$, while $|\text{Out}(\text{PSL}_2(8))| = 3$. In case (3), we have $|\text{Out}(\text{PSL}_2(q))| = 2$ and $q + 1 = 2^\alpha$ for some $\alpha \geq 2$, and so

$$4 \cdot |\text{Aut}(\text{PSL}_2(q))| \leq 4 \cdot \frac{q \cdot (q^2 - 1)}{2} \cdot 2 \leq 4 \cdot (2^\alpha)^3 \leq 2^{4\alpha} = |T : H|^4.$$ 

In case (4), $p = 2^{2m} + 1$ with some $m \geq 2$, $T = \text{PSL}_2(p - 1)$, and $|\text{Out}(\text{PSL}_2(p - 1))| = 2^m$. Consequently,

$$4 \cdot |\text{Aut}(\text{PSL}_2(p - 1))| = 4 \cdot (p - 1) \cdot p \cdot (p - 2) \cdot 2^m \leq p^4 = |T : H|^4.$$ 

(3) and (4) Let us show that $p \nmid |H|$. If $T = A_5$, $\text{PSL}_3(2)$, or $\text{PSL}_3(3)$, this follows by directly computing the order of $T$ and $H$. If $T = \text{PSL}_2(q)$, then $|T| = q \cdot (q + 1) \cdot (q - 1)/d$ and $H = q \cdot (q - 1)/d$ where $d = \text{gcd}(q + 1, 2)$. We claim that

$$\text{gcd}(|H|, |T : H|) = \text{gcd}(q \cdot (q - 1)/d, q + 1) = 1.$$ 

If $q$ is even, this is clear, since a prime divisor $r$ of $q + 1$ must be odd and so $r$ cannot divide $q$ or $q - 1$. If $q$ is odd, then $q + 1 = p^\alpha$ must be a power of 2. Since $q \geq 3$, $4 \mid q + 1$, which gives that $4 \nmid q - 1$. Hence $|H| = q \cdot (q - 1)/2$ is odd and this shows that $\text{gcd}(|H|, |T : H|)$ must also be odd. On the other hand, an odd prime $r$ which divides $q + 1$ does not divide $q \cdot (q - 1)$, and hence $\text{gcd}(|H|, |T : H|) = 1$. 

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In particular, in each of these cases, $H$ is a maximal subgroup of $T$ and $H$ is Hall $p'$-subgroup of $T$. Furthermore, in cases (1)–(4), the possible subgroups $H$ are stabilizers of a line in $\mathbb{F}_q^2$ and since $T$ is transitive on the set of such lines, these stabilizers form a single conjugacy class. In cases (5)–(6), $H$ is the stabilizer of either a line or a plane in $\mathbb{F}_q^3$. As $T$ is transitive on the set of lines and also on the set of planes, the stabilizers of lines form a conjugacy class, and another distinct conjugacy class is formed by the stabilizers of the planes.

The reason why the constant “4” appears in Corollary 7(2) is that it also appears in the particular case of Maróti’s result [15, Corollary 1.5] stated as Theorem 14 in Sect. 4.

**Lemma 8** Suppose that $k \geq 1$, $T_1, \ldots, T_k$ are finite, non-abelian, simple groups, and set

$$M = T_1 \times \cdots \times T_k.$$ Suppose that $K$ is a solvable subgroup of $M$ with $p$-power index for some prime $p$. Then $K$ is core-free, and there exists, for each $i \in \{1, \ldots, k\}$, a proper subgroup $H_i < T_i$ such that $|T_i : H_i| = p^{\alpha_i}$, the triple $(T_i, H_i, p^{\alpha_i})$ is as in one of the items of Lemma 6, and $K = H_1 \times \cdots \times H_k$. Furthermore, the following hold.

1. If $p \neq 2$, then the $T_i$ are pairwise isomorphic.
2. If $p \neq 2$ and $p \neq 3$, then $\alpha_i = 1$ for all $i$ and $|M : K| = p^k$; while if $p = 3$, then $\alpha_i = 2$ for all $i$ and $|M : K| = p^{2k}$.
3. $K$ is a Hall $p'$-subgroup of $M$.
4. If $p \neq 7, 13$, then $M$ contains a unique conjugacy class of Hall $p'$-subgroups.

**Proof** Let $\sigma_i : M \to T_i$ be the $i$-th coordinate projection that maps $(t_1, \ldots, t_k) \mapsto t_i$. For $i \in \{1, \ldots, k\}$, set $H_i = K \sigma_i$. Since $K$ is solvable, $H_i$ is solvable, and in particular, $H_i < T_i$. Since $\text{Core}_M(K)$ is a normal subgroup of $M$ and is contained in $K$, and as the normal subgroups of $M$ are subproducts of some of the $T_i$, we obtain that $\text{Core}_M(K) = 1$ and $K$ is core-free. Set $\overline{K} = \prod_i H_i$. Clearly, $K \leq \overline{K}$. As $|M : K| = p^m$ for some $m$, we have that $|M : \overline{K}|$ is a power of $p$, and hence, for all $i$, we have $|T_i : H_i| = p^{\alpha_i}$ for some $\alpha_i$. Thus the triple $(T_i, H_i, p^{\alpha_i})$ is as in one of the items of Lemma 6. Since $p \nmid |H_i|$, $p \nmid |\overline{K}|$, and so $K = \overline{K}$; that is, $K = H_1 \times \cdots \times H_k$. Now items (1)–(2) follow by inspection of the cases in Lemma 6. Let us verify claim (3). If $L$ is another Hall $p'$-subgroup of $M$, then the same argument shows that $L \cong L_1 \times \cdots \times L_k$ where the subgroup $L_i$ of $T_i$ satisfies the same conditions as $H_i$. Assuming that $p \neq 7, 13$, $T_i$ has a unique conjugacy class of subgroups isomorphic to $H_i$ (Corollary 7), and hence $L$ and $K$ are conjugate in $M$.

3 The structure of groups with a solvable subgroup of prime-power index

The next proposition characterizes the possible non-abelian composition factors in groups with a solvable subgroup of prime-power index.
**Proposition 9** Let $G$ be a possibly infinite group with a solvable subgroup $H$ such that $|G : H| = p^a$ with some prime $p$. Suppose that $G/\text{Core}_G(H)$ has a non-cyclic composition factor $T$. Then $p \in \pi_0$ and one of the following is valid:

1. $p = 2$ and $T \cong \text{PSL}_2(q)$ with some odd prime $q$ such that $q \geq 5$ and $q + 1$ is a power of $2$;
2. $p = 3$ and $T \cong \text{PSL}_2(8)$;
3. $p = 5$ and $T \cong A_5$;
4. $p = 7$ and $T \cong \text{PSL}_3(2)$;
5. $p = 13$ and $T \cong \text{PSL}_3(3)$;
6. $p$ is a Fermat prime and $T \cong \text{PSL}_2(2^m)$ with some $m \geq 2$ such that $p = 2^m + 1$.

**Proof** Suppose without loss of generality that $\text{Core}_G(H) = 1$. Then $G$ is a finite group, as it can be embedded into $S_{p^a}$. We proceed by induction on $|G|$. In the base case of the induction $G$ is simple and the assertion follows from Lemma 6. The induction hypothesis is that the assertion of the proposition is valid for groups of order less than $|G|$. Suppose that $G$ is not simple and let $N$ be a minimal normal subgroup of $G$. First we claim that the non-cyclic composition factors of $N$ are as in the relevant item of the proposition. If $N$ is abelian, then the claim is trivially true, so suppose that $N$ is non-abelian. As $|N : N \cap H| = |HN : H|$, the index of $N \cap H$ in $N$ is a power of $p$. Since $H$ is solvable, $H \not=N$, and in particular $N \cap H$ is a proper subgroup of $N$. By Lemma 8, the composition factors of $N$ must be as in the corresponding item of the proposition.

It remains to show that the non-cyclic composition factors of $G/N$ are as in the proposition. Since $|G/N : HN/N| = |G : HN|$, we find that $HN/N$ is a subgroup of $G/N$ with index $p^\beta$ for some $\beta \leq \alpha$. If $\beta = 0$, then $HN = G$ and $G/N = HN/N \cong H/(N \cap H)$ which is solvable. If $\beta \geq 1$, then $G/N$ satisfies the conditions of the induction hypothesis with the subgroup $HN/N$, and so a non-cyclic composition factor of $G/N$ must be as in the corresponding item of the proposition. \hfill \qed

We are now in a position to prove Theorem 1.

**Proof of Theorem 1** First note that $H$ is a finite solvable group and so it contains a solvable Hall $p'$-subgroup and this subgroup is also a solvable Hall $p'$-subgroup of $G$. If $p$ is a prime not contained in $\pi_0$ (defined in (1)), then Proposition 9 implies that all composition factors of $G$ must be cyclic, and so $G$ is solvable. This shows part (1).

(2) Let us now suppose that $p \neq 7, 13$ and prove that $G$ contains a unique conjugacy class of Hall $p'$-subgroups. Note that this assertion holds when $G$ is solvable (by Hall’s Theorem [18, 9.1.7]), or when $G$ is characteristically simple (Lemma 8). Our argument goes by induction on $|G|$, the base case being the case of solvable or characteristically simple groups.

Assume that $G$ is not solvable and is not characteristically simple. Let $K_1$ and $K_2$ be two Hall $p'$-subgroups of $G$ and suppose that $K_1$ is solvable. Let $M$ be a minimal normal subgroup of $G$. First, assume that $M$ is an elementary abelian $p$-group. Note that $K_1M/M$ and $K_2M/M$ are Hall $p'$-subgroups of $G/M$. Thus, by the induction hypothesis, $K_1M/M$ and $K_2M/M$ are conjugate in $G/M$; that is $(K_1M/M)^gM = K_2M/M$, for some $g \in G$, which implies that $K_1^gM = K_2M$. As $K_1$ and $K_2$ are
Let $H$ and $M$ be Hall $p'$-subgroups of $Y = K_1^g M = K_2 M$. Furthermore, $Y$ is a solvable group, and hence $K_1^g$ and $K_2$ are conjugate in $Y$ by Hall’s Theorem ([18, 9.1.7]). Therefore $K_1$ and $K_2$ are conjugate in $G$.

Next we suppose that $M$ is an elementary abelian $r$-group with $r \neq p$. Then, for $i = 1, 2$, the product $MK_i$ is a $p'$-subgroup of $G$, but, since the $K_i$ are Hall $p'$-subgroups, we must have $MK_i = K_i$, and so $M \leq K_i$. Furthermore, $K_1/M$ and $K_2/M$ are Hall $p'$-subgroups of $G/M$, and hence by the induction hypothesis, $(K_1/M)^g M = K_2/M$ for some $g \in G$. That is, $K_1^g = K_2$, and so $K_1$ and $K_2$ are conjugate in $G$, as required.

Finally, assume that $M$ is a non-abelian characteristically simple group. Then $M \cong T^k$ where $T$ is a finite simple group. Furthermore, $T$ is a non-cyclic composition factor of $G$, and hence the pair $(p, T)$ is as in one of the items (1)–(3), or (6) of Proposition 9. Note that $K_1 \cap M$ and $K_2 \cap M$ are proper solvable subgroups of $M$ with $p$-power index. By Lemma 8, both $K_1 \cap M$ and $K_2 \cap M$ are Hall $p'$-subgroups of $M$ and, since $p \neq 7, 13$, they are conjugate in $M$. Hence there exists $n \in N$ such that $(K_1 \cap M)^n = K_2 \cap M$. Swapping $K_1$ by $K_1^n$, we may assume without loss of generality that $K_1 \cap M = K_2 \cap M$ and let us call this group $Y$. If $K_1 = K_2$, then there is nothing more to prove, and so suppose that $K_1 \neq K_2$. Set $N = N_G(Y)$. Since $1 < Y < M$, the subgroup $N$ is proper in $G$, such that $K_1, K_2 \leq N$. Furthermore, $K_1$ and $K_2$ are Hall $p'$-subgroups of $N$ with $K_1$ being solvable. By the induction hypothesis $K_1$ and $K_2$ are conjugate in $N$, and in particular, they are conjugate in $G$.

**Remark 10** The examples in Lemma 6 show that the condition imposed on the prime $p$ in both assertions of Theorem 1 is necessary. Moreover, Theorem 1(1) improves the solvability criterion given by Carocca and Matos [5, Theorem A] for all $p \notin \pi_0$, because we only ask for the solvability of $H$ rather than its 2-nilpotency.

Let us now turn to the proof of Corollaries 2–3.

**The proof of Corollary 2** Since $H$ is a solvable subgroup of $G$ with prime-power index, $G/Core_G(H)$ is a finite group with a solvable subgroup $H/Core_G(H)$. Furthermore, $|G/Core_G(H) : H/Core_G(H)| = |G : H| = p^d$. By Proposition 9, the quotient $G/Core_G(H)$ is solvable. However, since $Core_G(H)$ is contained in $H$, it is solvable, and so the whole group $G$ is solvable.

**The proof of Corollary 3** (1) If $G$ contains a solvable Hall $p'$-subgroup and a solvable Hall $q'$-subgroup for two distinct primes $p$ and $q$, then a possible non-cyclic composition factor of $G$ would appear in two distinct lines of Proposition 9. The only isomorphism type that appears in two distinct lines of Proposition 9 is $T = PSL_2(7) \cong PSL_3(2)$, but these two lines correspond to $p = 2$ and $p = 7$, respectively, and this case was excluded. Thus such a group $G$ must be solvable.

(2) Let $H$ and $K$ be a solvable Hall $p'$-subgroup and a Hall $3'$-subgroup of $G$, respectively. If $p \notin \pi_0$, then Theorem 1(1) implies that $G$ is solvable. So we may suppose that $p \in \pi_0$. Since $H \cap K$ is a solvable Hall $\{3, p\}'$-subgroup of $G$, $H \cap K$ is a solvable Hall $p'$-subgroup of $K$. Furthermore, 3 does not divide $|K|$. If $K$ were non-solvable, then a non-cyclic composition factor $T$ of $K$ would be isomorphic to the group in one of the lines of Proposition 9. Now, inspection of the orders
of the possible groups $T$ shows in all cases that $3 \parallel |T|$, which is a contradiction. Consequently, $K$ is solvable. Finally, the solvability of $G$ follows from item (1).

(3) By assumption, $G = HP$, where $H$ is a normal Hall $p'$-subgroup of $G$ and $P$ is a Sylow $p$-subgroup of $G$. Since $H$ is a normal complement of $P$, it is sufficient to show that $H$ is solvable. Let $K$ be a solvable Hall $\{p, q\}'$-subgroup of $G$. Since $H$ is a normal $p'$-subgroup, $HK$ is a $p'$-subgroup of $G$. Further, since $H$ is a Hall $p'$-subgroup, $K \leq H$, and so $K$ is a solvable Hall $q'$-subgroup of $H$. As $q \notin \pi_0$, Theorem 1(1) gives that $H$ is solvable, and so $G = HP$ must also be solvable. \( \square \)

Remark 11

(1) Corollary 3(1) is no longer valid if the assumption of the solvability of both Hall $p'$-subgroups is dropped. For instance, if $q$ is a prime, $q \geq 7$, then the group $G = A_5 \times C_q$ contains a solvable Hall $5'$-subgroup $H_1 = A_4 \times C_q$ and a non-solvable Hall $q'$-subgroup $H_2 = A_5$, and $G$ is non-solvable.

(2) When $p = 2$, Corollary 3(2) coincides with solvability criterion given by Arad and Ward [3, Corollary 4.4].

(3) In Corollary 3(3), the normality of the $p$-complement is essential. For instance, consider $G = \operatorname{PSL}_2(11)$, $p = 5$ and $q = 11 \notin \pi_0$. The group $G$ is non-solvable with an 11-complement and a solvable Hall $\{5, 11\}'$-subgroup.

At the end of this section, we present two constructions that produce complex examples of groups that have solvable subgroups of prime power index.

Example 12

Suppose that $G$ is a group and $H$ is a solvable subgroup of $G$ with $|G : H| = p^\alpha$ for some prime $p$ and for $\alpha \geq 1$. Suppose that $K$ is a solvable subgroup of the symmetric group $S_\ell$ of degree $\ell$. Then the subgroup $H \wr K$ of the wreath product $W = G \wr K$ is solvable and its index is $p^{\ell \alpha}$. Note, in this construction, that $K$ is not assumed to be transitive. In particular, if $K = 1$, then $W = G^\ell$ and its subgroup of $p$-power index is $H^\ell$.

Example 13

Let $G$ be a group, let $H \leq G$ be a solvable subgroup with $|G : H| = p^\alpha$ for some prime $p$ and for $\alpha \geq 1$. Suppose that $K$ is a solvable group and $L \leq K$ such that $|K : L| = p^\beta$ with $\beta \geq 1$. Consider $G$ as a group acting on the right coset space $[G : H]$ and define the wreath product $W = K \wr G$ with respect to this action; hence $W = K^{p^\alpha} \rtimes G$. Then the subgroup $(L \times K^{p^\alpha - 1}) \rtimes H$ is a solvable subgroup of $W$ with index $p^{\alpha + \beta}$.

4 The bound on the order of $G/\operatorname{Rad}(G)$

A group $G$ such that $G/\operatorname{Rad}(G)$ is finite has a characteristic series (often referred to as the radical series and is used in computational group theory; see [13, 10.1.1] or [4, Section 3.3]) of subgroups.
where the terms of the series are defined as follows.

1) $\text{Rad}(G)$ is the solvable radical of $G$.

2) The subgroup $A$ is defined to be the complete inverse image in $G$ of the socle (that is, the product of the minimal normal subgroups) $\text{Soc}(G/\text{Rad}(G))$ of $G/\text{Rad}(G)$.

3) Thus $A/\text{Rad}(G)$ is the direct product of a uniquely defined set $\Delta$ of non-abelian simple groups. This set is permuted by $G$, by conjugation, and $B$ is the kernel of the $G$-action on $\Delta$.

4) Since the centraliser of $A/\text{Rad}(G)$ in $G/\text{Rad}(G)$ is trivial, the factor $B/A$ is isomorphic to a subgroup of the direct product of the outer automorphism groups of the simple factors of $A/\text{Rad}(G)$, and is therefore solvable. The factor $G/B$ can be regarded as a permutation group on the (generally small) set $\Delta$.

The following theorem follows from taking $d = 5$ in [15, Corollary 1.5].

**Theorem 14** If $X$ is a subgroup of $S_k$ such that $X$ has no composition factor isomorphic to $A_n$ with $n \geq 6$, then $|X| \leq 4^{k-1}$.

Let us now prove Theorem 4.

**Proof of Theorem 4** We may assume without loss of generality that $G$ is non-solvable. Then, Corollary 2 implies that $p \in \pi_0$.

Since $H$ is a solvable subgroup of prime-power index, $G/\text{Core}_G(H)$ is a finite group in which the subgroup $H/\text{Core}_G(H)$ is solvable of index $|G/\text{Core}_G(H)| = p^\alpha$. Set $m = p^\alpha$. In particular, $\text{Core}_G(H) \leq \text{Rad}(G)$ and $G/\text{Rad}(G)$ is a finite group with a solvable subgroup of index at most $m = p^\alpha$. Now, passing to the quotient $G/\text{Rad}(G)$, we can assume without loss of generality that $\text{Rad}(G) = 1$ and it is sufficient to study the order $|G|$.

Consider the radical series of $G$ defined in (2). Since $\text{Rad}(G) = 1$, it follows that $A$ is a product of the minimal normal subgroups of $G$ and these are all non-abelian. Set $\Delta = \{T_1, \ldots, T_k\}$ to be the simple factors of $A$. Then $A = T_1 \times \cdots \times T_k$ and $H \cap A$ is a solvable subgroup of $A$ of index $p^\beta$ with $1 \leq \beta \leq \alpha$. By Lemma 8, each $T_i$ contains a proper solvable subgroup $H_i$ of index $p^{\beta_i}$ such that $A \cap H = H_1 \times \cdots \times H_k$ and $|A : A \cap H| = p^{\beta_1 + \cdots + \beta_k} = p^\beta \leq p^\alpha$. Since $\beta_i \geq 1$ for all $i$, it also follows that $k \leq \beta \leq \alpha$. Moreover, Corollary 7 implies the inequality $4 \cdot |\text{Aut}(T_i)| \leq p^{5\beta_i}$, and in fact $4 \cdot |\text{Aut}(T_i)| \leq p^{4\beta_i}$ when $p \neq 13$. Furthermore, the quotient $B/A$ is isomorphic to a subgroup of $\text{Out}(T_1) \times \cdots \times \text{Out}(T_k)$ and $X = G/B$ is a subgroup of $\text{Sym}(\Delta)$.

By Proposition 9, $G$ has no composition factor isomorphic to $A_n$ with $n \geq 6$, and so $|X| \leq 4^{k-1}$ (Theorem 14).

Bounding the individual quotients of the radical series (2) and using the estimate in Corollary 7(2), we obtain that
\[ |G| \leq |X| \cdot \left( \prod_{i=1}^{k} |\text{Aut}(T_i)| \right) \leq \left( \prod_{i=1}^{k} 4 \cdot |\text{Aut}(T_i)| \right) \]
\[ \leq \left( \prod_{i=1}^{k} p^{5\beta_i} \right) \leq p^{5(\beta_1 + \cdots + \beta_k)} \leq p^{5\beta} \leq p^{5\alpha} = m^5. \]

If \( p \neq 13 \), then Corollary 7(2) implies that the constant “5” can be replaced by “4”.

\[ \square \]

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