Consecutive Quadratic Residues And Quadratic Nonresidue Modulo $p$

N. A. Carella

Abstract: Let $p$ be a large prime, and let $k \ll \log p$. A new proof of the existence of any pattern of $k$ consecutive quadratic residues and quadratic nonresidues is introduced in this note. Further, an application to the least quadratic nonresidues $n_p$ modulo $p$ shows that $n_p \ll (\log p)(\log \log p)$.

1 Introduction

Given a prime $p \geq 2$, a nonzero element $u \in \mathbb{F}_p$ is called a quadratic residue, equivalently, a square modulo $p$ whenever the quadratic congruence $x^2 \equiv u \mod p$ is solvable. Otherwise, it called a quadratic nonresidue. A finite field $\mathbb{F}_p$ contains $(p + 1)/2$ squares $\mathcal{R} = \{u^2 \mod p : 0 \leq u < p/2\}$, including zero. The quadratic residues are uniformly distributed over the interval $[1, p - 1]$. Likewise, the quadratic nonresidues are uniformly distributed over the same interval.

Let $k \geq 1$ be a small integer. This note is concerned with the longest runs of consecutive quadratic residues and consecutive quadratic nonresidues (or any pattern)

$$u, \ u + 1, \ u + 2, \ldots, \ u + k - 1,$$

in the finite field $\mathbb{F}_p$, and large subsets $\mathcal{A} \subset \mathbb{F}_p$. Let $N(k, p)$ be a tally of the number of sequences $\mathcal{A}$.

Theorem 1.1. Let $p \geq 2$ be a large prime, and let $k = O(\log p)$ be an integer. Then, the finite field $\mathbb{F}_p$ contains $k$ consecutive quadratic residues (or quadratic nonresidues or any pattern). Furthermore, the number of $k$ tuples has the asymptotic formulas

\[
\begin{align*}
(i) \quad N(k, p) &= \frac{p}{2^k} \left(1 - \frac{1}{p}\right)^k \left(1 + O\left(\frac{1}{p}\right)\right), \quad \text{if } k \geq 1. \\
(ii) \quad N(k, p) &= \frac{p}{2^k} + O\left(k^2\right), \quad \text{if } k \geq 1.
\end{align*}
\]

The first expression is suitable for applications requiring any integer $k$, and the second expression is suitable for applications requiring small integers $k$. The proofs are assembled in Section 8. The main results are proved using a new counting technique, based on Lemma 5.2. It provides sharper error terms than the standard technique based on Lemma 5.1.

MSC2020: Primary 11A15, Secondary 11L40.

Keywords: Least quadratic nonresidue, Consecutive quadratic residues.
Theorem 1.2. Let \( p \geq 2 \) be a large prime, and let \( k = O(\log p) \) be an integer. Then, for any subset of consecutive elements \( A \subset \mathbb{F}_p \) of cardinality \( p^{1-\varepsilon/2} \ll \#A \) contains \( k \) consecutive quadratic residues (or quadratic nonresidues or any pattern), \( \varepsilon > 0 \) is an arbitrary small number. Furthermore, the number of \( k \) tuples has the asymptotic formulas

\[
(i) \quad N(k, p, A) = \frac{\#A}{2^k} \left(1 - \frac{1}{p}\right)^k \left(1 + O\left(\frac{1}{p}\right)\right), \quad \text{if } k \geq 1.
\]

\[
(ii) \quad N(k, p, A) = \frac{\#A}{2^k} + O(k), \quad \text{if } k \geq 1.
\]

Quadratic residues \( r \in \mathbb{F}_p \) (and quadratic non residues) in finite fields have orders \( \text{ord}_p(r) = 2 \). The analysis and results for \( k \) consecutive power residues or any pattern of \( d \) power residues have similar details, but are more complex as the orders of the elements increases. The other result consider an interesting application to the least quadratic nonresidue.

The other result considers an interesting application to the least quadratic nonresidue \( n_p \) modulo \( p \). The current unconditional result for the least quadratic nonresidues in the literature states that

\[
n_p \ll p^{1/4\sqrt{\varepsilon}+\varepsilon},
\]

where \( \varepsilon > 0 \), and the strongest conditional result for primitive character \( \chi \) states that

\[
n_{\chi}(p) \ll (\log p)^{1.37+o(1)},
\]

see [4, Corollary 2], and Conjecture 10.1. The following result is proved here.

**Theorem 1.3.** For any large prime \( p \geq 2 \), the least quadratic nonresidue is bounded by

\[
n_p \ll (\log p)(\log \log p).
\]

The implied constant should be small, perhaps \( \leq 20 \), see Table 1 in Section 11. In Section 10 several Lemmas are spliced together to prove this result. Section 3 to Section ?? cover the supporting materials and other optional topics.

## 2 Quadratic Symbol

**Definition 2.1.** Let \( p \geq 2 \) be a prime, and let \( u \in \mathbb{F}_p \). The quadratic symbol modulo \( p \) of a nonzero element \( u \) is defined by

\[
\left( \frac{u}{p} \right) = \begin{cases} 
1 & \text{if } u \text{ is a quadratic residue;} \\
-1 & \text{if } u \text{ is not a quadratic residue.}
\end{cases}
\]

In term of calculations, this can be determined via the Euler criterion

\[
\left( \frac{u}{p} \right) = u^{(p-1)/2} \equiv \pm 1 \mod p.
\]

**Lemma 2.1.** (Legendre) Let \( p \geq 2 \) and \( q \) be a pair of distinct primes. Then, the quadratic symbol satisfies the following properties.

\[
(i) \quad \left( \frac{a}{p} \right) \equiv a^{(p-1)/2} \mod p, \quad \text{Euler congruence equation.}
\]
Consecutive Quadratic Residues and Quadratic Nonresidues Modulo $p$

(ii) \( \left( \frac{ab}{p} \right) = \left( \frac{a}{p} \right) \left( \frac{b}{p} \right) \), completely multiplicative function.

(iii) \( \left( \frac{-1}{p} \right) = (-1)^{(p-1)/2} \), evaluation at \( a = -1 \).

(iv) \( \left( \frac{2}{p} \right) = (-1)^{(p^2-1)/8} \), evaluation at \( a = 2 \).

(v) \( \left( \frac{q}{p} \right) \left( \frac{p}{q} \right) = (-1)^{(p-1)(q-1)/4} \), quadratic reciprocity.

3 Some Quadratic Exponential Sums

Definition 3.1. Let \( p \geq 2 \) be a prime. The finite Fourier transform of a periodic function \( f : \mathbb{Z} \rightarrow \mathbb{C} \) of period \( p \) is defined by

\[
\hat{f}(s) = \frac{1}{\eta_p \sqrt{p}} \sum_{t \in \mathbb{F}_p} f(t) e^{i 2\pi st/p}, \tag{7}
\]

where \( \eta_p = 1 \) if \( p \equiv 1 \mod 4 \) or \( \eta_p = i \) if \( p \equiv 3 \mod 4 \).

Except for a normalizing factor, the standard Gauss sum is a finite Fourier transform of the nonprincipal character \( \chi : \mathbb{Z} \rightarrow \mathbb{C} \), namely,

\[
\tau_s(\chi) = \sum_{t \in \mathbb{F}_p} \chi(t) e^{i 2\pi ts/p}. \tag{8}
\]

Lemma 3.1. The quadratic character mod \( p \) is the unique fix point of the finite Fourier transform. Specifically

\[
\left( \frac{s}{p} \right) = \frac{1}{\eta_p \sqrt{p}} \sum_{t \in \mathbb{F}_p} \left( \frac{t}{p} \right) e^{i 2\pi st/p}, \tag{9}
\]

where \( \eta_p = 1 \) if \( p \equiv 1 \mod 4 \) or \( \eta_p = i \) if \( p \equiv 3 \mod 4 \).

Lemma 3.2. (Gauss) If \( p \geq 2 \) is a prime, then

\[
\sum_{u \in \mathbb{F}_p} e^{i 2\pi u^2/p} = \begin{cases} 
\sqrt{p} & \text{if } p \equiv 1 \mod p; \\
i\sqrt{p} & \text{if } p \equiv 3 \mod p.
\end{cases} \tag{10}
\]

Proof. It is widely available in the literature, exampli gratia, [7, Theorem 1.1.5], [15, Section 3.3], [20, Lemma 3.3].

Lemma 3.3. Let \( p \geq 2 \) be a prime, and let \( (x|p) \) be the quadratic character mod \( p \). If the element \( s \neq 0 \), then,

\[
\sum_{u \in \mathbb{F}_p} e^{i 2\pi u^2 s/p} = \left( \frac{s^{-1}}{p} \right) \eta_p \sqrt{p}, \tag{11}
\]

where \( \eta_p = 1 \) if \( p \equiv 1 \mod 4 \) or \( \eta_p = i \) if \( p \equiv 3 \mod 4 \).
Proof. Let \( \chi(n) = (x \mid p) \). Replace the characteristic function of quadratic residue, see Lemma 5.1, to obtain

\[
\sum_{u \in \mathbb{F}_p} e^{i2\pi u^2/p} = \sum_{u \in \mathbb{F}_p} (1 + \chi(u)) e^{i2\pi us/p} = \sum_{u \in \mathbb{F}_p} \chi(u)e^{i2\pi us/p}.
\]  
(12)

The change of variable \( z = us \) returns

\[
\sum_{u \in \mathbb{F}_p} e^{i2\pi u^2/p} = \chi(s^{-1}) \sum_{z \in \mathbb{F}_p} \chi(z)e^{i2\pi z/p} = \chi(s^{-1})\eta_p \sqrt{p}.
\]  
(13)

\[\square\]

Lemma 3.4. If \( p \geq 2 \) is a prime, and \( a \neq 0 \) is an integer such that \( \gcd(a,p) = 1 \), then

\[
\sum_{u \in \mathbb{F}_p} \left( ax^2 + bx + c \right) \equiv \begin{cases} 
\left( \frac{a}{p} \right) & \text{if } b^2 - 4ac \not\equiv 0 \mod p; \\
\left( \frac{a}{p} \right)(p-1) & \text{if } b^2 - 4ac \equiv 0 \mod p.
\end{cases}
\]  
(14)

Proof. Consult the literature, [7, Theorem 2.1.2], [17], and similar references. \(\square\)

4 Some Incomplete Exponential Sums

A classical application of the finite Fourier transform provides nontrivial upper bounds of incomplete character sums. The simplest one uses the quadratic symbol or plain character \( \chi \) modulo \( q \).

Lemma 4.1. (Polya-Vinogradov) For \( q \) is a large prime, and character \( \chi \) modulo \( q \),

\[
\sum_{n \leq x} \chi(n) \ll \sqrt{q} \log q.
\]  
(15)

Proof. Use the finite Fourier transform of \( \chi(n) \) as in Lemma 5.1 and the geometric series, and other means. \(\square\)

The distribution, and various properties of the implied constant has a vast literature, and it is a topic of current research, see [5], [12], et alii. Many improved upper bounds for some specific characters such as \( \chi(-1) = -1 \) or \( \chi(-1) = 1 \) are known. An explicit for the Burgess inequality is stated below.

Lemma 4.2. (30) Let \( p \geq 10^7 \) be a prime, and let \( \chi \) be character modulo \( p \). Let \( M \), and \( N \geq 1 \) be nonnegative integer and let \( r \geq 1 \). Then

\[
\sum_{M \leq n \leq N+M} \chi(n) \leq 2.7N^{1-1/r}p^{(r+1)/4r^2} \log p^{1/r}.
\]  
(16)

At \( r = 1 \) it reduces to the Polya-Vinogradov inequality, and as \( r \to \infty \), it becomes a trivial upper bound.

Lemma 4.3. Suppose that GRH is true. Then, for any nonprincipal character \( \chi \) modulo \( q \) and any large number \( x \),

\[
\sum_{n \leq x} \chi(n) \ll \sqrt{q} \log \log q.
\]  
(17)

Proof. This is done in [23, Theorem 2]. \(\square\)
Lemma 4.4. (Paley) There are infinitely many discriminant \( q \equiv 1 \mod q \) for which

\[
\sum_{n \leq x} \left( \frac{n}{q} \right) \gg \sqrt{q} \log \log q.
\] (18)

Proof. The original version appears in [26] and a recent version is given in [21, Theorem 9.24]. ■

5 Characteristic Functions For Quadratic Residues

The standard characteristic function of quadratic residues and quadratic nonresidues are induced by the quadratic symbol.

Lemma 5.1. Let \( p \geq 2 \) be a prime, and let \((x \mid p)\) be the quadratic character mod \( p \). If \( u \in \mathbb{F}_p \) is a nonzero element, then,

1. \( \Psi_2(u) = \frac{1}{2} \left( 1 + \left( \frac{u}{p} \right) \right) = \begin{cases} 1 & \text{if } u^{(p-1)/2} \equiv 1 \mod p, \\ 0 & \text{if } u^{(p-1)/2} \equiv -1 \mod p. \end{cases} \)

2. \( \Psi_2(u) = \frac{1}{2} \left( 1 - \left( \frac{u}{p} \right) \right) = \begin{cases} 1 & \text{if } u^{(p-1)/2} \equiv -1 \mod p, \\ 0 & \text{if } u^{(p-1)/2} \equiv 1 \mod p, \end{cases} \)

are the characteristic functions for quadratic residues and quadratic non residues modulo \( p \) respectively in the finite field \( \mathbb{F}_p \).

A new representation of the characteristic function for quadratic residues and quadratic nonresidues are introduced below.

Lemma 5.2. Let \( p \geq 2 \) be a prime, and let \( \tau \) be a primitive root mod \( p \). If \( u \in \mathbb{F}_p \) is a nonzero element, then,

1. \( \Psi_2(u) = \sum_{0 \leq n < (p-1)/2} \frac{1}{p} \sum_{0 \leq m \leq p-1} e^{2\pi i (\tau^{2n} - u)m/p} = \begin{cases} 1 & \text{if } u^{(p-1)/2} \equiv 1 \mod p, \\ 0 & \text{if } u^{(p-1)/2} \equiv -1 \mod p. \end{cases} \)

2. \( \Psi_2(u) = \sum_{0 \leq n < (p-1)/2} \frac{1}{p} \sum_{0 \leq m \leq p-1} e^{2\pi i (\tau^{2n+1} - u)m/p} = \begin{cases} 1 & \text{if } u^{(p-1)/2} \equiv -1 \mod p, \\ 0 & \text{if } u^{(p-1)/2} \equiv 1 \mod p. \end{cases} \)

Proof. (i) For a fixed \( u \neq 0 \), the finite field \( \mathbb{F}_p \) equation

\[
\tau^{2n} - u = 0
\] (19)

has a unique solution \( n = n_0 \in \{0, 1, 2, \ldots (p - 1)/2 - 1\} \) if and only if \( 0 \neq u = \tau^{2n_0} \) is a quadratic residue modulo \( p \). This, in turns, implies that the inner exponential sum collapses to \( p \). Specifically,

\[
\sum_{0 \leq m \leq p-1} e^{2\pi i (\tau^{2n} - u)m/p} = \begin{cases} p & \text{if } u^{(p-1)/2} \equiv 1 \mod p, \\ 0 & \text{if } u^{(p-1)/2} \equiv -1 \mod p. \end{cases} \] (20)

Otherwise, \( \tau^{2n} - u \neq 0 \), which implies that the inner exponential sum vanishes. ■
6 Estimate For The Sum $T(k, p)$

The calculations for the sum

$$T(k, p) = \sum_{0 \neq n \in \mathbb{F}_p} \prod_{0 \leq i \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n_i < (p-1)/2} 1 \right)$$

assumes the existence of a sequence of $k$ consecutive quadratic residues

$$u, \ u + 1, \ u + 2, \ \ldots, \ u + k - 1$$

in the finite field $\mathbb{F}_p$. However, it is valid for any pattern of quadratic residues and nonresidues.

**Lemma 6.1.** Let $p \geq 2$ be a large prime, let $k = O(\log p)$ be an integer, then,

(i) $T(k, p) = \frac{p - 1}{2^k} \left( 1 - \frac{1}{p} \right)^k$, if $k \geq 1$.

(ii) $T(k, p) = \frac{p}{2^k} + O(k)$, if $k \geq 1$.

**Proof.** (i) Routine calculations return

$$T(k, p) = \sum_{0 \neq n \in \mathbb{F}_p} \prod_{0 \leq i \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n_i < (p-1)/2} 1 \right)$$

$$= \sum_{0 \neq n \in \mathbb{F}_p} \left( \frac{1}{p} \left( \frac{p - 1}{2} \right)^k \right)$$

$$= \frac{p - 1}{2^k} \left( 1 - \frac{1}{p} \right)^k .$$

(ii) For small integer $k = O(\log p)$, the binomial series leads to

$$T(k, p) = \frac{p - 1}{2^k} \left( 1 - \frac{1}{p} \right)^k$$

$$= \frac{p - 1}{2^k} \left( 1 - \left( \frac{k}{1} \right) \frac{1}{p} + \left( \frac{k}{2} \right) \frac{1}{p^2} + \cdots + \left( -1 \right)^k \frac{1}{p^k} \right)$$

$$\leq \frac{p - 1}{2^k} \left( 1 + k \left( \frac{k}{k/2} \right) \frac{1}{p} \right)$$

$$= \frac{p}{2^k} + O(k),$$

since the central binomial coefficient $\binom{k}{k/2} \leq 2^k$. ■

7 The Estimates For The Sum $U(k, p)$

The exponential sums over finite fields $\mathbb{F}_p$ studied in Section 3 are used to estimate the sum

$$U(k, p) = \sum_{0 \neq u \in \mathbb{F}_p} \prod_{0 \leq i \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n_i < (p-1)/2} e^{i2\pi((r^{n_i} - u)_{m_r})_{m_r}} \right),$$

which is used to prove Theorem 1.1.
Lemma 7.1. Let $p \geq 2$ be a large prime, let $k \geq 1$ be an integer, and let $\tau$ be a primitive root mod $p$. If the elements $u + a_r = v_r^2 \neq 0$ are quadratic residues for $r = 0, 1, 2, ..., k - 1$, then,

(i) $U(k, p) = O \left( \frac{1}{2^k} \left( 1 + \frac{1}{p} \right)^k \right)$, if $k \geq 1$.

(ii) $U(k, p) = O \left( \frac{k}{2^k} \right)$, if $k \geq 1$.

Proof. (i) Rewrite the multiple finite sum (25) as a product

$$U(k, p) = \sum_{0 \neq u \in \mathbb{F}_p} \left( \frac{1}{p} \sum_{0 \leq m_0 < p-1} e^{i2\pi (\tau^{2m_0} - u - a_0)m_0} \right) \prod_{1 \leq r \leq k-1} \left( \frac{1}{p} \sum_{0 \leq m_r < (p-1)/2} e^{i2\pi (\tau^{2m_r} - u - a_r)m_r} \right)$$

$$= \sum_{0 \neq u \in \mathbb{F}_p} U_1 \times U_2.$$ 

Merging Lemma 7.2 and Lemma 7.3 (using Holder inequality is optional), return

$$U(k, p) = \sum_{0 \neq u \in \mathbb{F}_p} U_1 \cdot U_2$$

$$= \sum_{0 \neq u \in \mathbb{F}_p} \frac{e^{-i2\pi a_0/p}}{4p} \left( 1 + \frac{\eta_p^2}{p} \left( \frac{u^{-1}}{p} \right) \right) \cdot \left( \frac{p + 1}{2p} \right)^k - 1$$

$$= \frac{e^{-i2\pi a_0/p}}{4p} \left( \frac{p + 1}{2p} \right)^k \sum_{0 \neq u \in \mathbb{F}_p} \frac{1 + \eta_p^2 e^{-i2\pi a_0/p}}{4} \left( \frac{p + 1}{2p} \right)^k \sum_{0 \neq u \in \mathbb{F}_p} \left( \frac{u^{-1}}{p} \right)$$

$$= \frac{e^{-i2\pi a_0/p}}{4p} \left( \frac{p + 1}{2p} \right)^k (p - 1),$$

since

$$\sum_{0 \neq u \in \mathbb{F}_p} \left( \frac{u^{-1}}{p} \right) = 0.$$

The absolute value $|U(k, p)|$ of the last expression in (27) can be rewritten as in the Lemma.

(ii) Same as the proof in Lemma 6.1.(ii). \hfill ■

Lemma 7.2. Let $p \geq 2$ be a large prime, and let $\tau$ be a primitive root mod $p$. If the element $u + a_0 = v_0^2 \neq 0$ is a quadratic residue, then,

$$U_1 = \frac{1}{2} \sum_{0 \leq n < (p-1)/2} e^{i2\pi (\tau^{2n} - u - a_0)n} = \frac{e^{-i2\pi a_0/p}}{4p} \left( 1 + \eta_p^2 \left( \frac{u^{-1}}{p} \right) \right),$$

where $|\eta_p| = \pm 1.$
Proof. Rearrange the finite sum as

\[ U_1 = \frac{1}{p} \sum_{0 \leq n < (p-1)/2} \sum_{0 < m \leq p-1} e^{i2\pi((r^2n_u-a_0)m)} \]

\[ = \frac{e^{-i2\pi a_0/p}}{p} \sum_{0 < m \leq p-1} e^{-i2\pi um/p} \sum_{0 \leq n < (p-1)/2} e^{i2\pi mr^2n/p} \]

\[ = \frac{e^{-i2\pi a_0/p}}{2p} \sum_{0 < m \leq p-1} e^{-i2\pi um/p} \sum_{0 \leq n < p-1} e^{i2\pi mr^2n/p}. \tag{30} \]

Let \( s = \tau^n \). An application of Lemma 3.2 yields

\[ 1 + 2 \sum_{0 \leq n < p-1} e^{i2\pi mr^2n/p} = \sum_{0 \leq n < p-1} e^{i2\pi ms^2/p} = \left( \frac{m^{-1}}{p} \right) \eta_p \sqrt{p}. \tag{31} \]

Substituting this into the previous equation returns

\[ U_1 = \frac{e^{-i2\pi a_0/p}}{4p} \sum_{0 < m \leq p-1} e^{-i2\pi um/p} \left( -1 + \left( \frac{m^{-1}}{p} \right) \eta_p \sqrt{p} \right) \tag{32} \]

\[ = \frac{e^{-i2\pi a_0/p}}{4p} \left( 1 + \eta_p \sqrt{p} \sum_{0 < m \leq p-1} \left( \frac{m^{-1}}{p} \right) e^{-i2\pi um/p} \right) \]

\[ = \frac{e^{-i2\pi a_0/p}}{4p} \left( 1 + \eta_p^2 \left( \frac{1}{p} \right) \right), \]

where \(|\eta_p| = \pm 1. \]

Lemma 7.3. Let \( p \geq 2 \) be a large prime, and let \( \tau \) be a primitive root mod \( p \). If the elements \( u + a_r = v_r^2 \neq 0 \) are quadratic nonresidues, then,

\[ U_2 = \prod_{1 \leq r \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n < (p-1)/2} \sum_{0 < m_r \leq p-1} e^{i2\pi((r^2n_u-u-a_r)m_r)} \right) = \left( \frac{p + 1}{2p} \right)^{k-1}. \tag{33} \]

Proof. The hypothesis \( u + a_r = v_r^2 \) for \( r = 1, 2, \ldots, k-1 \), is used to determine the value of each incomplete exponential sum. Start with a complete exponential sum and break it up into two subsums:

\[ 1 = \frac{1}{p} \sum_{0 \leq n < (p-1)/2} \sum_{0 < m_r \leq p-1} e^{i2\pi((r^2n_u-u-a_r)m_r)} \tag{34} \]

\[ = \frac{1}{p} \sum_{0 \leq n < (p-1)/2} 1 + \frac{1}{p} \sum_{0 \leq n < (p-1)/2} \sum_{0 < m_r \leq p-1} e^{i2\pi((r^2n_u-u-a_r)m_r)} \]

\[ = \frac{p - 1}{2p} + \frac{1}{p} \sum_{0 \leq n < (p-1)/2} \sum_{0 < m_r \leq p-1} e^{i2\pi((r^2n_u-u-a_r)m_r)}. \]
Solving for the incomplete exponential sum on the right side of (34) yields

\[ \frac{p+1}{2p} = 1 - \frac{p-1}{2p}, \]

which is basically the trivial value. Taking the product of all the incomplete exponential sums yields

\[ U_2 = \left( \frac{1}{p} \sum_{0 \leq n_1 < (p-1)/2 \atop 0 < m_1 \leq p-1} e^{2\pi i (x^{2n_1} - u - a_1)m_1} \right) \times \cdots \times \left( \frac{1}{p} \sum_{0 \leq n_1 < (p-1)/2 \atop 0 < m_1 \leq p-1} e^{2\pi i (x^{2n_{k-1}} - u - a_{k-1})m_{k-1}} \right) = \frac{p+1}{2p} \times \cdots \times \frac{p+1}{2p} = \left( \frac{p+1}{2p} \right)^{k-1}. \]

(36)

\section{Consecutive Quadratic Residues And Nonresidues}

Consecutive quadratic nonresidues is one of the simplest configuration of a subset of two or more consecutive quadratic nonresidues. The earliest attempts are surveyed in [13], [14], etc. A more general result was proved by Carlitz [10, Theorem 3] using a counting technique based on Lemma 5.1. More precisely, the number of $k$ consecutive quadratic residue symbols or any pattern of quadratic residue and quadratic nonresidue symbols

\[ \left( \frac{u}{p} \right) = \epsilon_0, \quad \left( \frac{u+1}{p} \right) = \epsilon_1, \quad \cdots, \quad \left( \frac{u+k-1}{p} \right) = \epsilon_{k-1}, \]

(37)

where $\epsilon_n = \pm 1$, in the finite field $\mathbb{F}_p$ has the asymptotic formula

\[ N(k, p) = \frac{1}{2^k} \sum_{0 \leq u \leq p-1} \left( 1 + \left( \frac{u}{p} \right) \epsilon_0 \right) \left( 1 + \left( \frac{u+1}{p} \right) \epsilon_1 \right) \cdots \left( 1 + \left( \frac{u+k-1}{p} \right) \epsilon_{k-1} \right) = \frac{p}{2^k} + E(k, p), \]

(38)

see Lemma [5.1] for details on the characteristic functions. An explicit error term $E(k, p) = \pm (k+1)(3 + \sqrt{p})$ is proved in [23, Corollary 5]. A slightly different proof appears in a new survey [24, Theorem 5.6]. For $k \leq 2$, the exponential sums involved have exact evaluations, and there are no error terms. But, in general, for $k \geq 3$, with very few exceptions, the exponential sums are estimated, and have error terms of the forms $E(k, p) = O(k \sqrt{p})$. A new and sharper proof and counting technique based on Lemma 5.2 is given here.

Let $a_0, a_1, a_2, \ldots, a_{k-1}$ be a sequence of distinct and increasing integers. Let $p \geq 2$ be a large prime, and let $\tau \in \mathbb{F}_p$ be a primitive root. A pattern of $k$ consecutive quadratic
residues and quadratic nonresidues $u + a_0, u + a_1, u + a_2, \ldots, u + a_{k-1}$ exists if and only if the system of equations
\[
\tau^{2n_0} = u + a_0, \quad \tau^{2n_1} = u + a_1, \quad \tau^{2n_2} = u + a_2, \ldots, \quad \tau^{2n_{k-1}} = u + a_{k-1},
\]
has one or more solutions. A solution consists of a $k$-tuple $n_0, n_1, \ldots, n_{k-1}$ of integers such that $0 \leq n_i < (p-1)/2$ for $i = 0, 1, \ldots, k - 1$, and some $u \in \mathbb{F}_p$. Let
\[
N(k, p) = \# \{ u \in \mathbb{F}_p : \text{ord}_p(u + a_i) = 2 \}
\]
for $i = 0, 1, \ldots, k - 1$, denotes the number of solutions.

**Proof.** (Theorem 1.1): The total number of solutions is written in terms of characteristic function for quadratic residues, see Lemma 5.2 as
\[
N(k, p) = \sum_{0 \neq u \in \mathbb{F}_p} \Psi_2(u + a_0) \Psi_2(u + a_1) \cdots \Psi_2(u + a_{k-1})
\]
(41)
\[
= \sum_{0 \neq u \in \mathbb{F}_p} \prod_{0 \leq i \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n_i \leq (p-1)/2} \psi((\tau^{2n_i} - u - a_i) m_i) \right)
\]
\[
= T(k, p) + U(k, p).
\]
The term $T(k, p)$, which is determined by the indices $m_0 = m_1 = \cdots = m_{k-1} = 0$, has the form
\[
T(k, p) = \sum_{0 \neq u \in \mathbb{F}_p} \prod_{0 \leq i \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n_i \leq (p-1)/2} 1 \right),
\]
(42)
and the term $U(k, p)$, which is determined by the indices $m_0 \neq 0, m_1 \neq 0, \ldots, m_{k-1} \neq 0$, has the form
\[
U(k, p) = \sum_{0 \neq u \in \mathbb{F}_p} \prod_{0 \leq i \leq k-1} \left( \frac{1}{p} \sum_{0 \leq n_i \leq (p-1)/2} \psi((\tau^{2n_i} - u - a_i) m_i) \right).
\]
(43)
(i) Applying Lemma 6.1 to the term $T(k, p)$, and Lemma 7.1-i to the term $U(k, p)$, yield
\[
N(k, p) = T(k, p) + U(k, p)
\]
(44)
\[
= \frac{p - 1}{2^k} \left( 1 - \frac{1}{p} \right)^k + O \left( \frac{1}{2^k} \left( 1 + \frac{1}{p} \right)^k \right)
\]
\[
= \frac{p}{2^k} \left( 1 - \frac{1}{p} \right)^k \left( 1 + O \left( \frac{1}{p} \right) \right)
\]
\[
> 0,
\]
for all sufficiently large primes $p \geq 2$.
(ii) Applying Lemma 6.1-ii to the term $T(k, p)$, and Lemma 7.1 to the term $U(k, p)$, yield
\[
N(k, p) = T(k, p) + U(k, p)
\]
(45)
\[
= \frac{p}{2^k} + O \left( \frac{k}{2^k} \right)
\]
\[
= \frac{p}{2^k} + O \left( \frac{k}{2^k} \right)
\]
\[
> 0,
\]
for all sufficiently large primes $p \geq 2$. 

\[\square\]
9 Synopsis Of Upper Bounds For Quadratic Nonresidues

The mathematical literature has many estimates for the least quadratic nonresidue $n_p$ modulo $p$. A list of the most frequently encountered upper bounds is complied below.

(1) $n_p \leq p^{1/2} + 1$, derived using an ad hoc elementary argument, see [23, Theorem 3.9], [29], etc.

(2) $n_p \ll p^{1/2} \log p$, derived from the Polya-Vinogradov inequality.

(3) $n_p \leq p^{1/2} \sqrt{\varepsilon + \varepsilon}$, derived from the Polya-Vinogradov inequality for any $\varepsilon > 0$.

(4) $n_p \leq p^{1/4} \sqrt{\varepsilon + \varepsilon}$, derived from the Burgess inequality for any $\varepsilon > 0$, see Lemma 4.2 and [2].

(5) $n_p \ll p^{\varepsilon}$, the Vinogradov conjecture, where $\varepsilon > 0$, see [16], and the literature.

(6) $n_p \leq 2(\log p)^2$, derived from the GRH, see [1], [3].

10 The Least Quadratic Nonresidue

Two slightly different heuristics for the conjectured upper bound of the least quadratic nonresidue are given in [24] and [30]. These are summarized below.

**Conjecture 10.1.** For every large prime $p \geq 3$,

$$n_p \ll (\log p)(\log \log p).$$

The heuristic is based on the proportion of primes $p$ such that the $n$th prime $p_n$ is the least quadratic nonresidue modulo $p$. The proportion has a geometric distribution, and its proof is based on quadratic reciprocity and Dirichlet theorem for primes in arithmetic progressions. For example, the probability for each $n \geq 1$ is given by limit

$$P(n_p = p_n) = \lim_{x \to \infty} \frac{\# \{p \leq x : n_p = p_n \}}{\pi(x)} = \frac{1}{2^n}. \quad (47)$$

Note that the form of the main term in Theorem 1.1-ii implies that quadratic residues (or quadratic nonresidues) in a finite field $\mathbb{F}_p$ are independent or nearly independent random variables $X = X(p)$ with probability

$$P(\text{ord}_p(X) = 2) = \frac{1}{2} + O\left(\frac{1}{p^\varepsilon}\right), \quad (48)$$

where $\varepsilon > 0$ is an arbitrary small number.

On average, the expected value $\bar{n}$ of the least quadratic residue $n_p$ is quite small

$$\bar{n} = \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{2 \leq p \leq x} n_p = \sum_{n \geq 1} \frac{p_n}{2^n} = 3.67464 \ldots, \quad (49)$$

where $\pi(x) = \{p \leq x\}$, and $p_n$ is the $n$th prime, see [6, p. 253], and [19] for the generalized concept.
Proof. (Theorem 1.3) To obtain a reductio ad absurdum, suppose that there exists a prime \( p \geq 2 \) for which \( n_p > (\log p)(\log \log p) \). Let \( k = (\log p)(\log \log p) \). This implies that the finite field \( \mathbb{F}_p \) contains a sequence of \( k \) consecutive quadratic residues

\[
u, \; u + 1, \; u + 2, \; \ldots, \; u + k - 1.
\] (50)

This immediately implies that

\[
\left( \frac{u}{p} \right) = 1, \; \left( \frac{u + 1}{p} \right) = 1, \; \ldots, \; \left( \frac{u + k - 1}{p} \right) = 1.
\] (51)

By Theorem 1.1, the total number of such sequences of quadratic residues of length \( k \) is

\[
N(k, p) = \frac{p}{2^k} \left( 1 - \frac{1}{p} \right)^k \left( 1 + O\left( \frac{1}{p} \right) \right) \gg 1.
\] (52)

Taking logarithm, and simplifying return

\[
\log p - k \log 2 + k \log \left( 1 - \frac{1}{p} \right) + \log \left( 1 + O\left( \frac{1}{p} \right) \right) \gg 0.
\] (53)

Rearranging it, and replacing \( k = (\log p)(\log \log p) \) give

\[
\log p \gg k \log 2 + O\left( \frac{k}{p} \right) - \log \left( 1 + O\left( \frac{1}{p} \right) \right)
\] (54)

\[
\gg (\log p)(\log \log p) \log 2 + O\left( \frac{(\log p)(\log \log p)}{p} \right) + \log C_0,
\]

where \( C_0 \approx 1 \). Clearly, this is false. Hence, a finite field \( \mathbb{F}_p \) contains a quadratic nonresidues \( n_p \ll (\log p)(\log \log p) \). \( \blacksquare \)

The best upper bound for the parameter \( H \geq 0 \) for which a character \( \chi(n) \) modulo \( p \) is constant on the interval \([N, N + H]\) is \( H < 7.07p^{1/4}\log p \) for large primes, see [18, Theorem 1.1], and [31]. The above result in Theorem 1.3 provides an improved and effective upper bound \( H \ll (\log p)(\log \log p) \) for the parameter \( H \geq 0 \). While the result \( N(k, p) = p/2^k + O(k\sqrt{p}) \) in [25, Corollary 5] provides an effective lower bound \( H \gg \log p \).

**Corollary 10.1.** Let \( p \geq 2 \) be a large prime, let \( \chi \) be a nonprincipal character modulo \( p \), and let \( x \geq 1 \) be a large real number. Define the real value function

\[
f(x) = \sum_{0 \leq n \leq x} \chi(n).
\] (55)

Then, \( f : \mathbb{R} \rightarrow \mathbb{Z} \) satisfies the following properties.

(i) \( f(x) = f(x + p) \), is periodic of period \( p \).

(ii) \( f(x) \ll p^{1/2} \log p \), is of absolute bounded variation on the real line.

(iii) \( f(N) < f(N + 1) < \cdots < f(N + H) \), is monotonically increasing on a short interval \([N, N + H]\) if and only if \( H \ll (\log p)(\log \log p) \), for any \( N \geq 0 \).

(iv) \( f(N) > f(N + 1) > \cdots > f(N + H) \), is monotonically decreasing on a short interval \([N, N + H]\) if and only if \( H \ll (\log p)(\log \log p) \), for any \( N \geq 0 \).
11 Experimental Data

The numerical data [20, Table 1], an expanded version is duplicated below, suggests that the constant is \( c_p = n_p/(\log p)(\log \log p) \leq 20 \) for all primes \( p \geq 2 \).

Table 1: Numerical Data for the Least Quadratic Nonresidue Modulo \( p \).

| \( n \) | \( n_p = p_n \) | \( p \) | \( (\log p)(\log \log p) \) | \( c_p \) |
|---|---|---|---|---|
| 1 | 2 | 3 | 0.10 | 20.00 |
| 2 | 3 | 7 | 1.30 | 2.31 |
| 3 | 5 | 23 | 3.58 | 1.40 |
| 4 | 7 | 71 | 6.18 | 1.13 |
| 5 | 11 | 311 | 10.03 | 1.10 |
| 6 | 13 | 479 | 11.23 | 1.16 |
| 7 | 17 | 1559 | 14.67 | 1.16 |
| 8 | 19 | 5711 | 18.66 | 1.02 |
| 9 | 23 | 10559 | 20.63 | 1.11 |
| 10 | 29 | 18191 | 22.40 | 1.29 |
| 11 | 31 | 31391 | 24.20 | 1.28 |
| 12 | 37 | 422231 | 33.18 | 1.22 |
| 13 | 41 | 701399 | 40.00 | 1.03 |
| 14 | 43 | 366791 | 32.68 | 1.32 |
| 15 | 47 | 3818929 | 41.20 | 1.14 |

12 Computational Complexity Of Quadratic Residues And Square Roots

The identification of an element \( u \in \mathbb{F}_p \) as a quadratic residue (or quadratic nonresidue) has nearly linear deterministic time complexity \( O(\log p)(\log \log p)^c \) for some \( c > 0 \). The discovery of an algorithm of linear complexity is an open problem, see [6, p. 3] for details.

The determination of the roots of congruence \( x^2 - u \equiv 0 \pmod{p} \) has slightly higher time complexity depending on the following data.

(1) The residue class of the prime \( p \geq 3 \).

(2) The method used to compute a quadratic nonresidue: probabilistic, deterministic.

(3) The method used to compute the roots: probabilistic, deterministic.

The worst case has deterministic time complexity \( O(\log p)^4 \) bit operations. Many algorithms such as Cipolla algorithm, tonelli algorithm, Berlekamp algorithm etc, are explained in [6, p. 157], [11, p. 102], [9]. Some polynomials for computing the square roots are provided in [5].

13 Twin Quadratic Residues And Nonresidues

The precise numbers of pairs \( QQ, QN, NQ \) or \( NN \) of quadratic residues and quadratic nonresidues are proved in [7, Theorem 6.3.1]. The proof based on Lemma [72] is given below.
Theorem 13.1. Let \( p \geq 3 \) and \( a \in \mathbb{Z} \), \( \gcd(a, p) = 1 \). The number of pairs \( n \) and \( n + a \) such that \( (n|p) = \epsilon_0 \) and \( (n|p) = \epsilon_1 \) is exactly

\[
N(\epsilon_0, \epsilon_1, P) = \frac{1}{4} \left( p - 2 - \epsilon_0 \left( \frac{a}{p} \right) - \epsilon_1 \left( \frac{-a}{p} \right) - \epsilon_0 \epsilon_1 \right),
\]

where \( \epsilon_i = \pm 1 \).

Proof. Sum the product of the two characteristic functions over the finite field \( \mathbb{F}_p \):

\[
N(\epsilon_0, \epsilon_1, P) = \frac{1}{4} \sum_{n \in \mathbb{F}_p} \left( 1 + \left( \frac{n}{p} \right) \epsilon_0 \right) \left( 1 + \left( \frac{n + a}{p} \right) \epsilon_1 \right)
\]

\[
= \frac{1}{4} \left( p - 2 + \epsilon_0 \sum_{n \in \mathbb{F}_p, n \neq 0, n \neq -a} \left( \frac{n}{p} \right) + \epsilon_1 \sum_{n \in \mathbb{F}_p, n \neq 0, n \neq -a} \left( \frac{n + a}{p} \right) + \epsilon_0 \epsilon_1 \sum_{n \in \mathbb{F}_p, n \neq 0, n \neq -a} \left( \frac{n^2 + an}{p} \right) \right).
\]

Use Lemma 3.4 to evaluate

\[
\sum_{n \in \mathbb{F}_p, n \neq 0, n \neq -a} \left( \frac{n^2 + an}{p} \right) = -1
\]

and simplify the expression. \( \blacksquare \)

A new proof based on Lemma 5.2 yields the same result up to a small error term. In particular, at \( k = 2 \), Theorem 1.1 reduces to

\[
N(\epsilon_0, \epsilon_1, P) = N(2, p) = \frac{p}{2^k} + O(k) = \frac{p}{4} + O(1).
\]

14 Problems

14.1 Square Roots Problems

Exercise 14.1. Let \( n \geq 2 \) be a squarefree integer, let \( \mathcal{U} = \{ u \neq 0 : u \cdot u^{-1} \equiv 1 \mod n \} \) be the subset of units (invertible elements), and let \( \mathcal{Q} = \{ m^2 \mod n : m \geq 0 \} \) be the subset of squares in the finite ring \( \mathbb{Z}/n\mathbb{Z} \).

(a) Show that total number of units is \( \# \mathcal{U} = \varphi(n) = n \prod_{p|n} (1 - 1/p) \) units, where \( \varphi(n) \) is the number integers relatively prime number to \( n \).

(b) Show that total number of squares is \( \# \mathcal{Q} = \prod_{p|n} (p + 1)/2 \) squares modulo \( n \).

(c) Show that a square element \( s = r^2 \in \mathbb{Z}/n\mathbb{Z} \) has \( 2^{\omega(n)} \) square roots modulo \( n \), where \( \omega(n) \) is the number of prime divisors of \( n \). For example, \( \sqrt[n]{s} = r_0, r_1, \ldots, r_{m-1} \).

Exercise 14.2. Let \( n \geq 2 \) be an integer, and let \( \mathcal{Q}_n = \{ m^2 \mod n : m \geq 1 \} \) be the subset of squares. Show that the subset of squares \( \# \mathcal{Q} \cup \mathbb{Z}/n\mathbb{Z} \) is a multiplicative subgroup.

Exercise 14.3. Let \( n \geq 2 \) be an integer, and let \( \mathcal{Q} = \{ m^2 \mod n : m \geq 1 \} \) be the subset of squares. Find a formula for the total number of the subset of squares \( \# \mathcal{Q} \).
Exercise 14.4. Let $n = pq \geq 6$, where $p \geq 2$ and $q \geq 2$ are distinct primes, and let $m = \varphi(n)$ the number of units in the finite ring $\mathbb{Z}/n\mathbb{Z}$, and let $Q = \{m^2 \mod n : m \geq 1\}$ be the subset of squares. Verify these questions:

(a) A square $s = r^2 \in \mathbb{Z}/n\mathbb{Z}$ has $4 = 2^\omega(n)$ square roots $r_0, r_1, r_2, r_3$.

(b) If $p \equiv q \equiv 3 \mod 4$, then a single square root $r_i \in Q$ for some $i = 0, 1, 2, 3$.

Exercise 14.5. Let $n = pq \geq 6$, where $p \geq 2$ and $q \geq 2$ are distinct primes, and let $m = \varphi(n)$ the number of units in the finite ring $\mathbb{Z}/n\mathbb{Z}$, and let $Q = \{k^2 \mod n : k \geq 1\}$ be the subset of squares. Verify these questions:

(a) A square $s = r^2 \in \mathbb{Z}/n\mathbb{Z}$ has $4 = 2^\omega(n)$ square roots $r_0, r_1, r_2, r_3$.

(b) If $p \equiv q \equiv 1 \mod 4$, then square roots, $r_i \not \in Q$ for $i = 0, 1, 2, 3$.

Exercise 14.6. Let $n \geq 2$ be an integer, and let $Q = \{m^2 \mod n : m \geq 1\}$ be the subset of squares. Reference: Handbook of Cryptography.

(a) If $n = pq$, where $p$ and $q$ are primes, and $s \in Q$, then the inverse $s^{-1} \equiv s^{(p-1)(q-1)+1}/8 \mod n$.

(b) If $n = pqr$, where $p$, $q$, and $r$ are primes, and $s \in Q$, find a similar formula for then the inverse $s^{-1} \equiv s^{2((p-1)(q-1)(r-1)+1)/16} \mod n$.

14.2 Character Sums Problems

Exercise 14.7. Let $q \geq 2$ be an integer, let $\chi$ be a multiplicative nonprincipal character modulo $q$, and $L(s, \chi) = \sum_{n \geq 1} \chi(n)n^{-s}$. Show that

$$\sum_{n \leq x} \chi(n) = \frac{1}{i2\pi} \int_{c-i\infty}^{c+i\infty} L(s, \chi) \frac{x^s}{s} ds,$$

where $c > 1$ is a constant, and $x \in \mathbb{R} - \mathbb{Z}$ is a real number.

14.3 Sums Of Squares Problems

Exercise 14.8. Let $p \equiv 1 \mod 4$ be a prime, and let $Q = \{k^2 \mod p : k \geq 1\}$ be the subset of squares. Show that

$$\sum_{r \in Q} r = \frac{p(p-1)}{4}.$$

Exercise 14.9. Let $n \geq 1$ be an integer, and let $Q = \{k^2 \mod n : k \geq 1\}$ be the subset of squares. Classify and evaluate the sums of squares

$$\sum_{r \in Q} r?.$$

14.4 Distribution And Spacing Between Squares Problems

Exercise 14.10. Use the quadratic reciprocity and Dirichlet theorem for primes in arithmetic progressions to prove that the proportion of primes $p$ such that the $n$th prime $p_n$ is the least quadratic nonresidue modulo $p$ has a geometric distribution. For each $n \geq 1$, the probability is given by limit

$$P(n_p = p_n) = \lim_{x \to \infty} \frac{\#\{p \leq x : n_p = p_n\}}{\pi(x)} = \frac{1}{2n}.$$
Exercise 14.11. Use the geometric distribution for the proportion of primes \( p \) such that the \( n \)th prime \( p_n \) is the least quadratic nonresidue modulo \( p \) to compute the average least quadratic nonresidue

\[
\overline{n} = \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{2 < p \leq x} n_p = 3.67464 \ldots
\]

Exercise 14.12. Let \( n \geq 2 \) be a squarefree integer, and let \( \mathcal{Q} = \{m^2 \mod n : m \geq 1\} \) be the subset of squares. Show that the average spacing between squares

\[
\overline{S_n} = \frac{n}{\#\mathcal{Q}} = \frac{n}{\prod_{p|n}(p + 1)/2} = \frac{n2^{\omega(n)}}{\psi(n)}.
\]

where \( \omega(n) \) is the prime divisors counting function, and \( \psi(n)/n = \prod_{p|n}(1 + 1/p) \) is the Dedekind psi function, see Exercise 14.1.

Exercise 14.13. Let \( n \geq 2 \) be an integer, and let \( \mathcal{Q} = \{m^2 \mod n : m \geq 1\} \) be the subset of squares. Find an expression for the average spacing between squares

\[
\overline{S_n} = \frac{n}{\#\mathcal{Q}},
\]

in terms of the \( \omega(n) \) is the prime divisors counting function, and \( \sigma(n)/n = \prod_{p^v||n}(1 + 1/p + \cdots + 1/p^v) \) is the sum of divisors function, see Exercise 14.3.

14.5 Algorithm Problems

Exercise 14.14. Let \( m, n \in \mathbb{Z} \) be a pair of distinct integers. Construct a deterministic algorithm to compute a simultaneous quadratic nonresidue \( \eta \) modulo both \( n \) and \( m \). Hint: Try a pair of distinct prime \( p \) and \( q \) first, then generalize it.

Exercise 14.15. Determine the time complexity of computing the inverse \( s^{-1} \equiv a \mod p \) using the Euclidean algorithm. Hint: Consider Lame theorem.

Exercise 14.16. Determine the time complexity of computing the inverse \( s^{-1} \equiv a \mod p \) using the Fermat theorem \( s^{p-2} \mod p \). Hint: Consider the add-multiply algorithm.

14.6 Primitive Roots Quadratic Nonresidues Problems

Exercise 14.17. Given a prime \( p \geq 3 \), prove the followings.

(a) Show that a primitive root in a finite field \( \mathbb{F}_p \) must be a quadratic nonresidue, but a quadratic nonresidue must not be primitive root.

(b) A finite field \( \mathbb{F}_p \) has \( (p-1)/2 - \varphi(p-1) \) quadratic nonresidues which are not primitive roots.

(c) Verify that every quadratic nonresidues in the finite field \( \mathbb{F}_p \) is primitive root if and only if \( p = 2^{2^n} + 1 \) is prime.
14.7 Open Problems

Exercise 14.18. Given a large prime \( p \geq 3 \), let \( Q = \{ n^2 \mod p : n \geq 1 \} \) be the subset of squares, and let \( A, B \subset \mathbb{F}_p \) be a pair of nonempty subsets. The subsets are proper subsets and must have zero densities in \( \mathbb{F}_p \) to avoid trivial cases. Reference: [28], [32].

1. (Sarkozy conjecture) Prove or disprove the existence of an additive partition \( Q = A + B \).

2. Prove or disprove that every square is a sum of two squares: the existence of an additive partition \( Q = A + B \), where \( A, B \subset Q \) are a pair of nonempty proper subsets.

3. Prove or disprove the existence of a difference partition \( Q = A - B \).

Exercise 14.19. (Lehmer conjecture) Given a large prime \( p \geq 3 \), and let \( (x \mid p) \) be the quadratic symbol, and let \( a = r^2 \) and \( b = s^2 \) be a pair of distinct squares. Prove or disprove the claim that

\[
\sum_{n \leq p} \left( \frac{n+a}{p} \right) \left( \frac{n}{p} \right) \left( \frac{n+b}{p} \right) > 1
\]

exists for a finite number of primes \( p \geq 2 \). Reference: [13, p. 246].

References

[1] Ankeny, N. C. The least quadratic non residue. Ann. of Math. (2) 55 (1952), 65-72.

[2] Burgess, D. A. A note on the distribution of residues and nonresidues. J. London Math. Soc. 38 (1963) 253-256.

[3] Bach, E. Analytic methods in the analysis and design of number theoretic algorithms, ACM Distinguished Dissertations, MIT Press, MA, 1985.

[4] Bober, J. Goldmakher, L. Polya-Vinogradov and the least quadratic nonresidue. arXiv:1311.7556.

[5] Bober, J. Goldmakher, L. Granville, A. Koukoulopoulos, D. The frequency and the structure of large character sums. J. Eur. Math. soc. 20 (2018), no. 7, 1759-1818.

[6] Bach, Eric; Shallit, Jeffrey. Algorithmic number theory. Vol. 1. Efficient algorithms. Foundations of Computing Series. MIT Press, Cambridge, MA, 1996.

[7] Berndt, Bruce C.; Evans, Ronald J.; Williams, Kenneth S. Gauss and Jacobi sums. Canadian Math. Soc. Series of Monographs. A Wiley-Interscience Publication. New York, 1998.

[8] N. A. Carella. Formulas for the Square Roots Mod p. arxiv.org/ftp/arxiv/papers/1101/1101.4605.

[9] N. A. Carella. Quadratic Nonresidues And Applications. Technical Report 190, Pace University, 2003.

[10] Carlitz, L. Sets of primitive roots. Compositio Math. 13 (1956), 65-70.
Consecutive Quadratic Residues And Quadratic Nonresidues Modulo $p$

[11] Crandall, Richard; Pomerance, Carl. *Prime numbers. A computational perspective.* Second edition. Springer, New York, 2005.

[12] Fromm, E. Goldmakher, L. *Improving the Burgess bound via Polya-Vonogradov.* Proc. Amer. Math. Soc. 147 (2019), no. 2, 461-466.

[13] Guy, Richard K. *Unsolved problems in number theory.* Third edition. Problem Books in Mathematics. Springer-Verlag, New York, 2004.

[14] Hummel, Patrick. *On consecutive quadratic non-residues: a conjecture of Issai Schur.* J. Number Theory 103 (2003), no. 2, 257-266. Arxiv:0305298.

[15] Lemmermeyer, F. *Reciprocity Laws. From Euler to Eisenstein.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[16] Linnik, Yu. *A remark on the least quadratic nonresidue.* C. R. (Doklady) Acad Sci. URSS (N.S) 36 (1942), 119-120.

[17] Lidl, Rudolf; Niederreiter, Harald. *Finite fields.* Encyclopedia of Mathematics and its Applications, 20. Cambridge University Press, Cambridge, 1997.

[18] McGown, K. *On the constant in Burgess bound for the number of consecutive residues or non-residues.* Arxiv.1011.4490.

[19] G. Martin and P. Pollack. *The average least character non-residue and further variations on a theme of Erdos*, J. London Math. Soc. 87 (2013) 22-42. Arxiv.1112.1175.

[20] Kevin McGown, Enrique Trevino. *The least quadratic non-residue.* Preprint, July 18, 2019.

[21] Montgomery, Hugh L.; Vaughan, Robert C. *Multiplicative number theory. I. Classical theory.* Cambridge University Press, Cambridge, 2007.

[22] Montgomery, Hugh L.; Vaughan, Robert C. *Exponential sums with multiplicative coefficients.* Invnt. Math. 43 (1977), no. 1, 69-82.

[23] Niven, I. Zuckerman, H. Montgomery, H. *An Introduction to the Theory of Numbers.* Wiley & Sons, New York, 1991.

[24] Paul Pollack. *The smallest quadratic nonresidue modulo a prime,* Slides, August 29, 2012.

[25] Peralta, Rene. *On the distribution of quadratic residues and nonresidues modulo a prime number.* Math. Comp. 58 (1992), no. 197, 433-440.

[26] Paley, R. *A theorem on Characters,* J. London math. soc. 7 (1932), no. 1, 28-32.

[27] J.B. Rosser and L. Schoenfeld. *Approximate formulas for some functions of prime numbers,* Illinois J. Math. 6 (1962) 64-94.

[28] Igor E. Shparlinski. *Additive Decompositions Of Subgroups Of Finite Fields,* arxiv.org/pdf/1301.2872.

[29] Shapiro, Harold N. *Introduction to the theory of numbers.* Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1983.
[30] Trevino, Enrique. *The least quadratic non-residue and related problems*. Slides, April 17, 2015.

[31] Trevino, Enrique. *On the maximum number of consecutive integers on which the a character is constant*. Mosc. J. Comb. Number Theory 2 (2012), no. 1, 56-81.

[32] Vsevolod F. Lev, Jack Sonn. *Quadratic residues and difference sets*, arXiv:1502.06833.

ConsecutiveQuadraticNonresidue-11-17-20-57.tex.