On RSOS Models Associated to Lie Algebras and RCFT

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ABSTRACT

RSOS models based on the Lie algebras $B_m$, $C_m$ and $D_m$ are derived from the braiding of conformal field theory. This gives the first systematic derivation of these models earlier described by Jimbo et al. The general two field Boltzmann weights associated to any RCFT are described, giving in particular the off critical thermalized Boltzmann weights. Crossing properties are discussed and are shown to agree with the general theory which connects these with toroidal modular transformations. The soliton systems based on these lattice models are described and are conjectured based on the mass formulae and the spins of the integrals of motions to describe perturbations of the RCFT $G_k \times G_{k+1}$, where $G$ is the corresponding Lie algebra.

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In ref. [1] the author has suggested a generic way of constructing solvable interaction round the face lattice models, and their associated fusion soliton systems, using directly the results of conformal field theory. The Boltzmann weights are related to the braiding matrices of the rational conformal field theory (RCFT). In ref. [2] such two block braiding matrices were computed for the general two block RCFT using the analytic properties of the correlation functions. Our aim in this note is to further establish this correspondence by revisiting the $B_m$, $C_m$ and $D_m$ RSOS models described in ref. [3], and to rederive them as fusion IRF models based on RCFT, along with describing the associated soliton systems which solve perturbations of the coset RCFT $\frac{G_k \times G_1}{G_{k+1}}$ where $G$ is any of the algebras.

Let us return to the expression for the two block braiding derived for a general RCFT in ref. [2]. It was found there that the braiding matrix in the channel $\langle \phi(z)\phi_s(0)\phi_t(1)\phi_u(\infty) \rangle$ is given by

$$B = \sigma \begin{pmatrix} d & \rho \\ \rho & -d^{-1} \end{pmatrix},$$  

where

$$d_i = e^{-\pi i (\Delta + \Delta_i - 2 \Delta_1)}, \quad d = \sqrt{d_1/d_2},$$

$$\sigma = \sqrt{\frac{\sin(\pi \beta + \pi \gamma) \sin(\pi \beta + \pi \delta)}{\sin(\pi \alpha) \sin(\pi \beta) d_1 d_2}},$$

$$\rho = \sqrt{-\frac{\sin(\pi \gamma) \sin(\pi \delta)}{\sin(\pi \beta + \pi \gamma) \sin(\pi \beta + \pi \delta)}},$$

where the exponents $\alpha$, $\beta$, $\gamma$ and $\delta$ are defined by

$$\alpha = \Delta_2^s - \Delta_1^s + 1,$$

$$\beta = \Delta_2^t - \Delta_1^t,$$
\[ \gamma = \Delta_1^s + \Delta_1^t + \Delta_1^u - \Delta - \Delta_s - \Delta_t - \Delta_u, \]  
\[ \delta = \Delta_1^s + \Delta_1^t + \Delta_2^u - \Delta - \Delta_s - \Delta_t - \Delta_u, \]
and the dimensions are defined by \( \Delta = \Delta(\phi), \Delta^a = \Delta(\phi^a) \) where \( \phi^a \) is the field appearing in the \( a \) channel, \( \phi \cdot \phi_a = \phi_1^a + \phi_2^a \), where \( a \) is \( s, t \) or \( u \).

Now suppose that the fields \( \phi_s \) and \( \phi_t \) are the same, or that we are interested in the braiding of two identical fields. This is the case relevant for the model \( IRF(O, x, x) \), where \( x = \phi_s = \phi_t \) in the notation of ref. [1]. Note that all the four exponents can now be expressed in terms of \( \alpha \) and the crossing parameter \( \lambda = \pi(\Delta_2^u - \Delta_1^u)/2 \),

\[ \beta = \alpha - 1, \quad \delta = 1 - \alpha + \frac{\lambda}{\pi}, \quad \gamma = 1 - \alpha - \frac{\lambda}{\pi}. \]

Substituting into eq. (1), the braiding matrix may be written as

\[ B = \text{"phase"} \times \frac{1}{\{\alpha\}} \left( -e^{-i\pi\alpha} \sqrt{\{1 - \alpha - \tilde{\lambda}\}\{1 - \alpha + \tilde{\lambda}\}} \right), \]

and we denoted \( \tilde{\lambda} = \lambda/\pi \) and \( \{x\} = \frac{\sin(\pi x)}{\sin \lambda} \).

Recall now from ref. [1] that from the braiding matrix of the RCFT one builds the Boltzmann weight of the solvable fusion interaction round the face (IRF) model \( IRF(O, x, x) \) as follows. One defines the Hecke algebra element associated to the braiding matrix by

\[ H_i = H \left( \begin{array}{c} \phi \\ p \end{array} \begin{array}{c} q \\ \phi_u \end{array} \right) = e^{-i\lambda} - B_{pq}, \]

where \( B_{pq} \) is the braiding matrix, eq. (10) and \( p \) and \( q \) are the primary fields which label the conformal blocks in the \( s \) channel. The \( H_i \) obey the usual relations for
the Hecke algebra,

\[ H_i H_{i+1} - H_i = H_{i+1} H_i - H_{i+1}, \]

\[ H_i H_j = H_j H_i, \quad \text{for } |i-j| \geq 2, \tag{12} \]

\[ H_i^2 = (2 \cos \lambda) H_i, \]

from which it follows that we may build the solvable face transfer matrix,

\[ X_i(u) = \sin(\lambda - u) + \sin u \cdot H_i, \tag{13} \]

where \( u \) is the spectral parameter. The Hecke algebra implies that the face transfer matrix \( X_i(u) \) obeys the star–triangle equation (STE),

\[ X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_i(u), \]

\[ X_i(u)X_j(v) = X_j(v)X_i(u), \tag{14} \]

from which it follows that the transfer matrices for different values of the spectral parameter \( u \) commute, enabling the exact solution of the model.

Substituting the braiding matrix eq. (10) into the expression for the face transfer matrices, we find the Boltzmann weights of the lattice model IRF(\( \mathcal{O}, x, x \)) which are as follows. In case there is only one block:

\[ p \phi_p^\phi = \{ \tilde{\lambda}(1-u) \}, \tag{15} \]

For the two block case, it is found

\[ p \phi_p^\phi = \{ \alpha + \tilde{\lambda}u \} / \{ \alpha \}, \quad q \phi_q^\phi = \{ \alpha - \tilde{\lambda}u \} / \{ \alpha \}, \]

\[ p \phi_p^\phi = \sqrt{ \{ \alpha + \tilde{\lambda} \} \{ \alpha - \tilde{\lambda} \} / \{ \alpha \} } \]

(16)

where \( \epsilon = \pm 1 \) labels complex conjugate solutions and \( p = \phi_1^u \) and \( q = \phi_2^u \) are
the two intermediate fields in the $u$ channel. It is straightforward to verify that the Boltzmann weights of the model $\text{IRF}(SU(N), N, N)$ given in ref. [3, 1, 2] agree precisely with the above general formula, eqs. (15-16), when one substitutes the RCFT $SU(N)_k$.

Now, it is possible to contemplate the generalization of the above Boltzmann weights off criticality. This is done by simply redefining the symbol $\{x\}$ to be

$$\{x\} = \frac{\Theta_1(\pi x, p)}{\Theta_1(\lambda, p)},$$

(17)

where the parameter $p$ labels the distance from criticality and the theta function is defined by

$$\Theta_1(u, p) = 2p^\lambda \sin u \prod_{n=1}^\infty [1 - 2p^{2n} \cos(2u) + p^{4n}](1 - p^{2n}).$$

(18)

In the critical limit $p \rightarrow 0$ we recover the previous definition of $\{x\}$ and thus obtain the same Boltzmann weights as before. This is the expression for the off critical Boltzmann weights by merely substituting the new definition of $\{x\}$ into eqs. (15-16). It should be possible to verify that this Boltzmann weights obey the STE for all values of $p$ and thus define a thermalized solvable lattice model. In particular, in the case of $SU(N)$ we recover the thermalized Boltzmann weights previously given in ref. [3].

Let us turn now to models associated to the other Lie algebras. These are the restricted solid on solid (RSOS) lattice models first described in ref. [3], which are associated with the Lie algebras $B_m$, $C_m$ and $D_m$ in the usual Cartan notation. We wish to revisit these models in light of the connection with RCFT put forwards in ref. [1]. The Boltzmann weights of these RSOS models are [3]

$$d+\mu \begin{pmatrix} d \\ d+2\mu \end{pmatrix} = \frac{[\lambda - u][\omega - u]}{[\lambda][\omega]}, \quad \text{for } \mu \neq 0,$$

(19)
\[ d_{\mu+\nu} = \left( \frac{\lambda - u}{[\lambda]} \right) \left( \frac{d_{\mu+\nu} + \omega}{d_{\mu+\nu}} \right)^{1/2}, \quad \text{for } \mu \neq \pm \nu, \quad (20) \]

\[ d_{\mu+\nu} = \left( \frac{\lambda - u}{[\lambda]} \right) \left( \frac{d_{\mu+\nu} + \omega}{d_{\mu+\nu}} \right)^{1/2} \delta_{\mu\nu}, \quad \text{for } \mu \neq \pm \nu, \quad (21) \]

\[ d_{\mu+\nu} = \left( \frac{u}{[\lambda]} \right) \left( \frac{d_{\mu-\nu} + \omega - \lambda + u}{d_{\mu-\nu} + \omega} \right) \left( g_{d\mu} g_{d\nu} \right)^{1/2} + \delta_{\mu\nu} \frac{[\lambda - u][d_{\mu+\nu} + \omega + u]}{[\lambda][d_{\mu+\nu} + \omega]}, \quad \text{for } \mu \neq \pm \nu, \quad (22) \]

\[ d_{\mu} = \left( \frac{\lambda + u}{[\lambda] 2\lambda} \right) - \left( \frac{u}{[\lambda] 2\lambda} \right) J_{d0}. \quad (23) \]

Here \( d \) stands for an arbitrary integrable highest weight of the respective algebra, at the level \( k \), where \( k \) is some integer. \( \mu \) and \( \nu \) are arbitrary elements of the set \( \Sigma \) which is defined as, \( \Sigma = \{0, \pm e_1, \pm e_2, \ldots, \pm e_m\} \) for \( B_m \) and \( \Sigma = \{\pm e_1, \pm e_2, \ldots, \pm e_m\} \) for \( C_m \) and \( D_m \), and where \( e_i \) are orthonormal set of unit vectors in the canonical basis of the algebras. We have used the symbol \([x] = \Theta_1(x; p)\) where the theta function was defined in eq. (18). At criticality, \( p = 0 \), and we find \([x] = \Theta_1(x, 0) \propto \sin(x)\). Here \( d_{\mu\nu} \) stands for \( d_{\mu+\nu} = \omega(d + \rho, \mu - \nu) \) and \( d_{\mu} = d_{\mu0} \).

Further,

\[ g_{d\mu} = \sigma \frac{s(d_{\mu} + \omega)}{s(d_{\mu})} \prod_{\kappa \neq \pm u, 0} \frac{[d_{\mu\kappa} + \omega]}{[d_{\mu\kappa}]}, \quad \text{for } \mu \neq 0, \quad g_{d0} = 1. \quad (24) \]

\[ J_{d0} = \sum_{\kappa \neq 0} \frac{[d_{k} + \frac{1}{2}\omega - 2\lambda]}{[d_{k} + \frac{1}{2}\omega]} g_{dk}. \quad (25) \]

The parameters are \( \sigma = 1, \lambda = (2m - 1)\omega/2 \) and \( s(z) = [z] \) for the \( B_m \) model, \( \sigma = -1, \lambda = (m+1)\omega/2 \) and \( s(z) = [2z] \) for the \( C_m \) model and \( \sigma = 1, \lambda = (m-1)\omega \)
and $s(z) = 1$ for the $D_m$ model. For all algebras $\lambda = g\omega/2$, where $g$ is the dual Coxeter number.

Now, note that the RSOS models based on $B_m, C_m$ and $D_m$ may be interpreted as fusion interaction round the face lattice models. The only difference between the restricted and unrestricted SOS models is that $\omega$ becomes fixed to the value $\omega = \frac{\pi}{k+g}$ and the representations that can appear are only the ones which are integrable representations at the level $k$. Importantly, the admissibility condition for the models is simply the fusion rules with respect to the vector representation whose highest weight is $\lambda = e_1$. In the notation of ref. [1] these are the fusion IRF models $IRF(B_m, v, v)$, $IRF(C_m, v, v)$ and $IRF(D_m, v, v)$. We wish to verify that the Boltzmann weights are indeed the specialization of the general ones described in ref. [1].

The product of the vector representation with itself contains three representations,

$$v^2 = 1 + s + a,$$

where 1 is the singlet, $a$ is the anti–symmetric tensor and $s$ is the symmetric tensor. Let us now compute the crossing parameters which are given by [1] $\zeta_i = \pi(\Delta_{i+1} - \Delta_i)/2$ where $\Delta_i$ is the conformal dimension of the $i$th field in the operator product expansion of $v$ with itself; arranged in the order 1, $\lambda_2$ and $2\lambda_1$ (this is so that the symmetry of the representation will be alternating). The dimension of a WZ field with highest weight $\lambda$ is

$$\Delta_\lambda = \frac{\lambda(\lambda + 2\rho)}{2(k + g)},$$

where $\rho$ is half the sum of positive roots and the value of the dual Coxeter number is $g = 2m - 2$ for $D_m$, $g = 2m - 1$ for $B_m$ and $g = m + 1$ for $C_m$. A straightforward calculation shows that the two crossing parameters of the models are

$$\lambda = \frac{\pi\Delta_{\lambda_2}}{2} = \frac{\pi g}{k + g}, \quad \omega = \frac{\pi(\Delta_{2\lambda_1} - \Delta_{\lambda_2})}{2} = \frac{\pi}{k + g},$$

and that $\lambda$ indeed has the values described above.
According to the general theory $\lambda$ is the crossing parameter of the model and the crossing multiplier should be the toroidal $S$ matrix [1]. We wish to check that this is so. Let us consider then the crossing properties of the amplitudes. It is known that these Boltzmann weights obey the crossing property [4],

$$w \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) | u \rangle = w \left( \begin{array}{cc} b & c \\ d & a \end{array} \right) | \lambda - \mu \rangle \left[ \frac{\psi(a)\psi(c)}{\psi(b)\psi(d)} \right]^\frac{1}{2},$$

where $\lambda$ plays the role of the crossing parameter and the crossing multiplier $\psi(a)$ is given by

$$\psi(d) = \prod_{k=1}^{m} s(d_k) \prod_{1 \leq i < j \leq m} [d_{ij}][d_{i-j}].$$

(30)

Now, according to the general theory of fusion IRF models ref. [1] the crossing multiplier should be identical to the torus modular matrix. We can see that this is indeed the case for all the models by making use of the quantum Weyl dimension formula (see e.g. [5])

$$\frac{S_{d,0}}{S_{0,0}} = \prod_{\alpha > 0} \frac{\sin[\pi(d + \rho, \alpha)/(k + g)]}{\sin[\pi(\rho, \alpha)/(k + g)]}.$$  

(31)

Remembering that the positive roots of $C_m$ are $e_i \pm e_j$ for $i < j$, and $2e_i$; $e_i \pm e_j$ and $e_i$ for $B_m$ and $e_i \pm e_j$ for $D_m$, it is concluded that eq. (30) is precisely the Weyl quantum dimension formula (up to the denominator, which is an irrelevant constant) and so it is established that indeed,

$$\psi(a) \propto S_{a,0},$$

(32)

for all the three algebras. We also need to identify $\omega = \frac{\pi}{k+g}$ which is precisely what we found, eq. (28). This is also the value that allows for the restriction of the SOS models as remarked above and so we find a complete agreement on this value from three points of view. This establishes the correct crossing properties of the Boltzmann weights of the $B_m$, $C_m$ and $D_m$ IRF models.
We wish to show now that the Boltzmann weights of the RSOS models eqs. (19-23) indeed agree with the ones derived, in general, from conformal field theory in ref. [1]. The $BCD$ models are three field cases. Namely, the primary field $v$ has three fields in its operator product, $v^2 = \phi_0 + \phi_1 + \phi_2$ which are, as before, $\phi_0 = 1$, $\phi_1 = \phi_{\lambda_2}$ and $\phi_2 = \phi_{2\lambda_1}$. It follows that the face transfer matrix is given by the expression [1],

$$X_i(u) = f_0(u)P_i^0 + f_1(u)P_i^1 + f_2(u)P_i^2;$$

(33)

where $f_i(u)$ are the three eigenvalues of the face transfer matrix which are

$$f_0(u) = \frac{\sin(\lambda + u) \sin(\omega + u)}{\sin \lambda \sin \omega}, \quad f_1(u) = \frac{\sin(\lambda - u) \sin(\omega + u)}{\sin \lambda \sin \omega},$$

$$f_2(u) = \frac{\sin(\lambda - u) \sin(\omega - u)}{\sin \lambda \sin \omega},$$

(34)

and the projection operators $P_i^a$ are independent of the spectral parameter $u$ and are defined by

$$P_i^a = \prod_{q \neq a} \frac{(B_i - \lambda_q)}{(\lambda_a - \lambda_q)},$$

(35)

where $B_i$ is the conformal braiding matrix whose eigenvalues are

$$\lambda_a = (-1)^a e^{i\pi(\Delta_a - 2\Delta_v)}.$$  

(36)

Note now that the projection operator

$$P^a \left( \begin{array}{cc} d & d + \mu \\ d + \mu & d + 2\mu \end{array} \right) = \delta_{a,2} e^{i\omega}. $$

(37)

This follows immediately from the fact that the relevant correlation function $\langle [d][v][v][d+2\mu]\rangle$ has only one intermediate block in the $s$ channel which is labeled by the field
$[d + \mu]$. It follows that this correlation function is given by a rational function of $z$ which is determined by the dimension of the fields, and so the braiding matrix is

$$B = e^{i\pi (\Delta_2 - 2\Delta_v)} = e^{i\omega},$$

and eq. (37) follows. From eq. (33) it is concluded that the face transfer matrix obeys

$$X \left( \begin{array}{c} d \\ d+\mu \\ d+2\mu \\ d+\nu \end{array} \right) = \frac{\sin(\lambda - u) \sin(\omega - u)}{\sin \lambda \sin \omega},$$

which agrees precisely with the Boltzmann weights described above, eq. (19).

Next consider the projection operators appearing in the block $P^a \left( \begin{array}{c} d \\ \alpha \\ \beta \\ d+\mu+\nu \end{array} \right)$ where $\alpha, \beta = d + \mu, d + \nu$. Evidently, this projection operator vanishes for $a = 0$. Further, the relevant correlation function is a two blocks case, and thus use can be made of the preceding formulae, eq. (15-16). The relevant correlation function is $\langle [d][v][d+\mu+\nu] \rangle$ and it has exactly two blocks in the $s$ channel labeled by the blocks $p = [d+\mu]$ or $p = [d+\nu]$. We may thus apply directly the previous formulae eq. (16) and it follows that the braiding matrix is, indeed, given by eq. (10), where we substitute $\omega$ instead of $\lambda$ and where the parameter $\alpha$ computed from eq. (5) assumes the value

$$\alpha = \Delta_{d+\mu} - \Delta_{d+\nu} = d_{\mu\nu},$$

where $d_{\mu\nu}$ was defined prior to eq. (24). Substituting the eigenvalues of this braiding matrix into the expression for the face transfer matrix eq. (33) we find the same expression as for the general two block case, but now in the $1-2$ channel,

$$B = \text{”phase” } \times \frac{1}{\{d_{\mu\nu}/\pi\}} \left( -e^{-id_{\mu\nu}} \sqrt{\{(d_{\mu\nu} + \omega)/\pi\}\{(d_{\mu\nu} - \omega)/\pi\}} \right) e^{id_{\mu\nu}}. \quad (41)$$
The face transfer matrix thus assumes the form,

\[
\begin{vmatrix}
  d \\
  d+\mu
\end{vmatrix}
\begin{vmatrix}
  d+\mu \\
  d+\mu+\nu
\end{vmatrix} = \frac{[\lambda - u][d_{\mu\nu} - u]}{[\lambda][d_{\mu\nu}]},
\]

\begin{equation}
(42)
\end{equation}

\[
\begin{vmatrix}
  d \\
  d+\nu
\end{vmatrix}
\begin{vmatrix}
  d+\nu \\
  d+\mu+\nu
\end{vmatrix} = \frac{[\lambda - u][u]}{[\lambda][\omega]} \cdot \left(\frac{[d_{\mu\nu} + \omega][d_{\mu\nu} - \omega]}{[d_{\mu\nu}]^2}\right)^{\frac{1}{2}},
\]

\begin{equation}
(43)
\end{equation}

where we have rescaled the spectral parameter \( u \to \frac{u}{\lambda} \) and used again \([x] = \sin(x)\).

Lo and behold the expression for the face transfer matrix derived from conformal field theory, eq. (42-43), is identical precisely to the BCD Boltzmann weights given above, eq. (19-23), thus providing a first systematic derivation of these weights previously found in ref. [3]. Finally, the last two Boltzmann weights appearing in eq. (22-23) involve the correlation function, \( \langle dvvd \rangle \). The number of blocks in this correlation function is the dimension of the set \( \Sigma \), \(|\Sigma|\), and is very large. It follows that it is difficult to calculate directly these Boltzmann weights from RCFT except for very small ranks, and this calculation will have to await the development of the proper generalization to any number of blocks of the two block calculation of ref. [2]. We may infer the conformal braiding matrix for this case by calculating the extreme UV limit of the BCD Boltzmann weights, \( u \to i\infty \). We find the result

\[
B_{pq} = \frac{[\omega]}{[d_{u-v} + \omega]} e^{-i(d_{u-v} + \omega - \lambda)} \left[ (g_{d\mu}g_{d\nu})^{\frac{1}{2}} - \delta_{\mu\nu} \right],
\]

\begin{equation}
(44)
\end{equation}

and for \( p = q = d \),

\[
B_{dd} = \frac{2[\omega]}{[2\lambda]} e^{i\lambda}.
\]

\begin{equation}
(45)
\end{equation}

Note that the generalization of the two fields face transfer matrix to arbitrary temperature, eq. (17) is consistent with the thermalized Boltzmann weights, eq.
(19-23), and indeed we find the same result. This is an important check that the thermalization of the general fusion IRF model suggested above is indeed correct and agrees with all the known examples.

Let us turn now to integrable soliton systems based on these lattice models. The vacua of the theory are labeled by the primary fields of the RCFT $G_k$ where $G$ is one of the Lie algebras $B_m$, $C_m$ or $D_m$. The kinks of the theory interpolate between neighboring vacua $a$ and $b$ provided that the pair $(a, b)$ obey the admissibility condition which is, in this case, that $b - a \in \Sigma$ or that $b$ appears in the operator product of $a$ with the vector representation. The two particle scattering matrices for the kink $v$ correspond to the process $(a|v|b) + (b|v|c) \rightarrow (a|v|d) + (d|v|c)$, where we labeled by $(a|v|b)$ the kink $v$ interpolating between the $a$ and $b$ vacua. As follows from the general theory of ref. [1], the $S$ matrices of these kink scattering processes are given by

$$S \left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| \theta \right) = F(\theta) \left[ \frac{S_{b,0}S_{c,0}}{S_{a,0}S_{d,0}} \right]^{\theta/2} \left( \begin{array}{cc} a & b \\ c & d \end{array} \bigg| \lambda \theta \right),$$

(46)

where $i\pi\theta$ is the relative rapidity of the incoming particles, $S$ is the torus modular matrix, $w$ is the Boltzmann weight of the associated RSOS lattice model and $F(\theta)$ is an overall function to be determined. As discussed in ref. [1] the $S$ matrix eq. (46) obeys the factorization equation for integrable soliton systems as is guaranteed from the fact that the Boltzmann weights obey the STE. Further, the $S$ matrix will be crossing invariant and unitary provided that the function $F(\theta)$ obeys the functional equations

$$F(\theta) = F(1 - \theta)$$

$$F(\theta)F(-\theta) = \frac{1}{\rho(\theta)\rho(-\theta)},$$

(47)

where the unitary factor $\rho(\theta)$ is

$$\rho(\theta) = \frac{\sin[\lambda(1 - \theta)]\sin[\omega - \lambda\theta]}{[\lambda][\omega]}.$$
The minimal solution of this set of functional equations is

$$F(\theta) = f_\frac{2}{g}(\theta)f_1 - \frac{2}{g}(\theta) \times Z(\theta)Z(1 - \theta),$$

where

$$f_\alpha(\theta) = \frac{\sin(\pi u/2 + \pi \alpha/2)}{\sin(\pi u/2 - \pi \alpha/2)};$$

is the Koberle Sweica amplitude. The piece $Z(\theta)Z(1 - \theta)$ is a $Z$ factor that does not have any poles or zeros in the physical sheet. A minimal solution for it is

$$Z(\theta) = \exp \left(2 \int_0^\infty \frac{dx}{x} \frac{\sinh(g \theta x/2)}{\sinh[(g + k)x] \sinh(g x)} \right).$$

It is noteworthy that the entire solution for $F(\theta)$ is simply a folding of the solution for the $SU(N)$ case ref. [6] as $F(\theta) = K(\theta)K(1 - \theta)$, where $K(\theta)$ is the $SU(g)_k$ solution [6].

$F(\theta)$, eq. (49), has a unique simple pole in the physical strip at $\theta_b = \frac{\omega}{\lambda}$ (except for $B_1$ where the pole lies outside the physical strip and there are no bound states), along with the crossing channel pole at $1 - \theta_b$. The mass of the first bound state is thus

$$M_b = 2M_v \cos\left(\frac{\pi \omega}{2\lambda}\right).$$

Note that $\lambda = g\omega/2$ where $g$ is the dual Coxeter number for all the algebras and, thus, the mass of the first bound state can be written as,

$$M_b = 2M_v \cos\left(\frac{\pi}{g}\right).$$

Crucially, the mass ratio of the first bound state, and indeed the entire piece of $F(\theta)$ which contains the poles in the physical sheet, is independent of the level $k$. 
It follows that for all values \( k \), the masses of the particles in the theory and their integrals of motions are the same. This holds also for the \( SU(N) \) case, along with the mass ratio eq. (53). We expect this to be true for all the other Lie algebras (i.e., the simply laced \( E_6, E_7 \) and \( E_8 \) along with the non–simply laced \( G_2 \) and \( F_4 \)).

To compute the particle spectrum and the spins of the integrals of motion it is thus enough to inspect the \( k = 1 \) soliton scattering amplitudes. For the simply laced algebras the scattering amplitudes reduce to the purely diagonal scattering amplitudes associated with the ADE algebras (for a review and references, consult, e.g., ref. [7]). The entire \( S \) matrix is given in the \( D_m, k = 1 \) case by \( S_{vv}(\theta) = f_{2/g}(\theta)f_{1-2/g}(\theta) \), for the scattering of two vector solitons.

For the non–simply laced algebras (i.e., \( B_m, C_m, G_2 \) and \( F_4 \)) the mass ratios and the integrals of motion reduce to those of the corresponding classical Toda theory refs. [8,9,10]. Note, that the amplitudes described in these references are a factor in the amplitudes we find for the RSOS theories, but that, however, the full amplitude is not purely diagonal even for \( k = 1 \) and is truly an RSOS theory. Curiously, in the Toda case it is claimed [9,10] that the classical results are not valid and need to be renormalized through perturbation theory. Here this does not occur and the RSOS theories described here thus give the first realization of the non–simply laced classical Toda system results. The masses and integrals of motions are the same as those of the classical Toda systems, as well. This, indeed, agrees with our derived mass formula eq. (53). The reminder of the soliton amplitudes may be found by the bootstrap procedure or alternatively, the fusion of IRF models.

The particles in the theory are in a one to one correspondence with the nodes of the Dynkin diagram of the respective algebras and are labeled by the fundamental weights. Each soliton in the theory is thus labeled by some \( \lambda_i \) where \( \lambda_i \) is the \( i \)th fundamental weight. The values of the integrals of motion \( \gamma_s^s \) are in a one to one correspondence with the eigenvectors of the Cartan matrix, which are labeled by the exponent set of the algebra \( s \). The Perron–Frobenius vector, which is the
eigenvector with the maximal eigenvalue, gives the masses of the corresponding solitons, i.e., the mass of the $\lambda_a$ soliton is $\gamma_a^1$. The $\lambda_a$ soliton mediates the vacua connected by the fusion with respect to the $\lambda_a$ representation, i.e., it corresponds to the solvable lattice model IRF($G, \lambda_a, \lambda_a$), and can be found by the fusion procedure. The vector amplitude described here is fundamental for the $B_m$ and $C_m$ cases, i.e., all the other solitons are composite particles of the vector soliton. The masses are (for $B_m$ and $C_m$),

$$M_a = \sin \left( \frac{\pi a}{g} \right),$$  \hspace{1cm} (54)

where $a$ labels the representations ($a = 1$ is the vector, and $a = 2$ is the bound state described above, agreeing with eq. (53)). In the case of $D_m$ the fundamental amplitudes are actually the spinor and anti-spinor representations, where the vector is the bound state of these. Via a bootstrap of the vector amplitude one can get all the amplitudes for all the representations except for the spinor ones. Unfortunately, to find the spinor amplitudes for $D_m$ with $k > 1$, it behooves us to calculate the Boltzmann weights of the solvable lattice model IRF($D_m, x, x$) where $x$ is either the spinor or the anti-spinor, which is quite a challenge, as they involve a large number of blocks. The masses of the $D_m$ solitons are given by

$$M_s = M_{\bar{s}} = 1, \quad M_a = 2 \sin \left( \frac{\pi a}{g} \right),$$  \hspace{1cm} (55)

where we labeled the vector as $a = 1$ and the anti-symmetric tensors which are its composites by $a = 2, 3, \ldots, m - 2$. This mass formula agrees, again, with eq. (53).

The spins of the integrals of motions in the theory are given by the exponents of the corresponding Lie algebra modulo the Coxeter number (or the dual one). In the case of $D_m$ the spins are 1, 3, 5, \ldots, $2m - 1, m \mod 2(m - 1)$. For $C_n$ and $B_n$ the spins are given by all odd integers. These are exactly the same spins of the integrals of motion which are encountered for the $W$ invariant coset theories $\frac{G_k \times G_k}{G_{k+1}}$, where $G$ is the corresponding Lie algebra, as perturbed by the operator $\Phi_{ad}^{0,0}$ (i.e. a singlet in the upper Lie algebras and an adjoint in the lower). We thus
conjecture that the RSOS scattering matrices described in this note correspond to
the soliton spectrum and scattering amplitudes of these perturbed conformal field
theories.

We hope that this work further illuminates the various connections between
solvable lattice models, conformal field theory and soliton scattering theories.

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