An exact solution to the Bertsch problem
and the non-universality of the Unitary Fermi Gas

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group point of view and compute the effective range dependence of the Bertsch parameter ξ exactly. To this end we
construct an effective block-diagonal two-body separable interaction with the Fermi momentum as a cut-off
which reduces the calculation to the mean field level. The interaction is separable in momentum space and is
determined by Tabakin’s inverse scattering formula. For a vanishing effective range we get ξ = 176π − 17 π = 0.56.
By using phase-equivalent similarity transformations we can show that there is a class of exact solutions with
any value in the range 0.56 ≥ ξ ≥ −1/3.

Keywords: Unitary Fermi Gas, Inverse Scattering, Separable potential

During the 10th Conference on Advances in Many-Body Theory, that took place in Seattle in 1999, G. F. Bertsch posed
the following challenge (see [1]):

What are the ground state properties of the many-body system composed of spin-1/2 fermions interacting via a zero
range, infinite scattering-length contact interaction? It may be assumed that the interaction has no two-body bound states.
Also, the zero range is approached with finite ranged forces. We analyze the universality of the Unitary Fermi Gas in its
ground state from a Wilsonian renormalization group point of view and compute the effective range dependence of the
Bertsch parameter ξ exactly. To this end we construct an effective block-diagonal two-body separable interaction with
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ξ = 176π − 17 π = 0.56. By using phase-equivalent similarity transformations we can show that there is a class of exact solutions
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where ξ is the Bertsch parameter which is expected to be a universal number, and in the r0 ≠ 0 case a universal function
of the combination r0kF.

This apparently simple problem provides an example of a strongly correlated fermion system and has been a major the-
etorical and experimental challenge over the last two decades. Experimental measurements on ultracold atomic gases for a
vanishing effective range, r0 = 0, yield [4] (ξ = 0.41(1)), [5] (ξ = 0.39(2)). [6] (ξ = 0.376(5)) and [7] (ξ = 0.37(1)). In
Ref. [2] over 50 different calculations based on different many body simulations and experiments are listed and, with a few
exceptions ξ ~ 0.37 − 0.40. The numerical resemblance suggests, as it was tacitly expected in those studies, that this is a
universal quantity which is uniquely determined by the conditions spelled out originally by Bertsch’s challenge. Here, we
provide an exact solution of the problem and show, contrary to this general and widespread belief, that more information
is needed than assumed hitherto.

From a theoretical point of view the strategy to deal with the Unitary Fermi Gas has been a two-step approach: one
first tunes a two-body interaction to fulfill the Bertsch scattering condition, Eq. (1), and then uses it to solve the many-
body problem. These two stages are usually discussed separately in the literature. A prototype calculation is the one pur-
sued recently by Conduit and Schonberg [8] where, for any kF, they have tuned a potential to Eqs. (2) and considered a
Monte Carlo simulation up to N = 294 particles and verify the expected δ(N−1) trend to the extrapolated N = ∞ ther-

modynamic limit. They find δ(N) = 0.388(1), the most precise determination to date, and consider also finite effective range
values in the interval, r0kF ∈ [−2, 2].

In the present letter we advocate for a different strategy. Indeed, there is a well-known and inherent ambiguity
associated to this approach; one can undertake a phase-equivalent

\[ f = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta_l} \sin \delta_l}{k} P_l(\cos \theta) \equiv \frac{1}{-\alpha_0 + \frac{1}{2} r_0 k^2 - i k}, \] (1)

with \( P_l(\cos \theta) \) Legendre polynomials and \( \delta_l(k) \) the phase shifts,

\[ \delta_0(k) = \cot^{-1} \left( -\frac{1}{\alpha_0 k} + \frac{1}{2} r_0 k \right), \]
\[ \delta_l(k) = 0 \quad \text{for} \quad l \geq 1. \] (2)

For \( \alpha_0 \to -\infty \) and \( r_0 \to 0 \) one has \( \delta_0(k) = \pi/2 \). In this limit, for a Fermi system with two (spin) species the Fermi momentum \( k_F \) is the only dimensionful quantity and hence the total energy per particle should be proportional to the energy of the free Fermi Gas:

\[ \frac{E}{N} \sim \xi \frac{3 k_F^2}{10 M}, \] (3)
unitary transformation of the potential \( \mathcal{V} \) (see e.g. [10] for a review). Here we propose to take advantage of this arbitrariness by building an interaction in block-diagonal form where the separation scale in the momentum, \( \Lambda \), is actually chosen to coincide with the Fermi momentum, \( \Lambda = k_F \). This has the important consequence that the mean field result already yields the exact many-body solution and a suitable choice permits to construct an analytic solution without violating any of the conditions originally spelled out by Bertsch.

For our purposes it is convenient to formulate the two-body scattering problem for two identical particles of mass \( M \) in momentum space for the kinematics

\[
(P + \bar{p}/2, \bar{P} - \bar{p}/2) \rightarrow (\bar{P} + \bar{p}'/2, \bar{P} - \bar{p}'/2)
\]

where \( \bar{P} \) is the (conserved) CM momentum and \( \bar{p} \) and \( \bar{p}' \) the relative momenta before and after the collision respectively. The Lippmann-Schwinger equation, which in operator form reads \( T(E) = V + V G_0(E) T(E) \) with \( G_0(E) = (E - H_0)^{-1} \), becomes \([11]\),

\[
\langle \bar{p}' | T(E) | \bar{p} \rangle = \langle \bar{p}' | V | \bar{p} \rangle + \int \frac{d^3 q}{(2 \pi)^3} \frac{\langle \bar{p}' | V | \bar{q} \rangle \langle \bar{q} | T(E) | \bar{p} \rangle}{E - q^2/2 \mu + i \varepsilon}
\]

where \( \mu = M/2 \) is the reduced mass. In the partial wave basis, \( \bar{p} \)

\[
\langle \bar{p}' | T(E) | \bar{p} \rangle = \frac{8 \pi^2}{\mu} \sum_{l m} Y_{l m}(\bar{p}) Y_{l m}(\bar{p}') T_l(p', p, E),
\]

where \( Y_{l m}(\bar{p}) \) spherical harmonics and similarly for the potential \( V_l \). In terms of the mean of the \( K \)-matrix fulfilling, \( T_l = K_l/(1 + i \sqrt{2 \mu E K_l}) \) which half-shell-off fulfills

\[
K_l(p', p) = V_l(p', p) + \frac{2}{\pi} \int_0^\infty dq q^2 \frac{V_l(p', q)}{p^2 - q^2} K_l(q, p).
\]

where \( f \) stands for the principal value integral and \( K_l(p', p) = K_l(p', p, E = p^2/2 \mu) \). The relation with the phase-shifts in Eq. \([1]\) follows from \(-4 \pi f = \langle \bar{p}' | T(E) | \bar{p} \rangle \) and is given by

\[
\tan \delta_l(p) = -K_l(p, p).
\]

Clearly, the conditions in Eq. \([2]\), are fulfilled by taking

\[
V_l(p, p') = 0, \quad \text{for} \quad l \geq 1.
\]

The problem is then to find the s-wave interaction \( V_0(p'|p) \) from Eq. \([2]\), as we will discuss shortly, after reviewing our many-body setup.

Following the conventional strategy, once our effective interaction \( V_0(p', p) \) has been tuned to the Bertsch renormalization condition, Eq. \([2]\), we turn now to the many-body problem. We will work first at lowest order in perturbation theory which corresponds to the mean field (Hartree-Fock) level, since this already provides an upper variational estimate for any \( V_0(p', p) \). For a two-fermion species the energy per particle at the Hartree-Fock level, is given by \([12]\)

\[
\frac{E}{N} = \frac{3k_F^3}{10M} + \frac{2}{\pi M} \int_0^{k_F} k^2 dk \left( 1 - \frac{3k^2}{2k_F} + \frac{k^3}{2k_F^2} \right) V_0(k, k) + O(V^2).
\]

According to the standard variational argument, first order perturbation theory provides an upper bound for the true ground state. If we have \( H = H_0 + V \) and \( H_0 \psi_n(0) = E_n(0) \psi_n(0) \), then for any normalized state \( \psi \) we have \( E_0 \leq \langle \psi | H_0 + V | \psi \rangle \) so that taking \( \varphi = \psi_n(0) \) a Slater determinant leading to Eq. \(\langle 3\rangle \). \( E_0 \leq E_n(0) + \langle \psi_n(0) | V | \psi_n(0) \rangle \). The neglected higher order corrections correspond to transitions \( \bar{p} \rightarrow \bar{p}' \) above the Fermi-level, \( |p \pm \bar{p}/2| \leq k_F \leq |\bar{p} \pm \bar{p}'/2| \) which requires \( p' > k_F > p \). Note, however, that if we have \( V_0(p', p) = 0 \) for \( p' > k_F > p \) higher order corrections vanish identically.

The main ingredient in our construction is thus to separate the two-body (relative) Hilbert space into two orthogonal (and decoupled) subspaces \( \mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_Q \) which are below or above some given \( \Lambda \) respectively. This can be denoted by projection operators \( P \) and \( Q \), fulfilling \( P^2 = P = P^2 = Q \) and \( PQ = QP = 0 \) and \( P + Q = 1 \). This separation endows the Hamiltonian \( H \) with a block structure, which can equivalently be transformed by a unitary transformation \( U \) into a block-diagonal form

\[
H = \begin{pmatrix} PHP & P QH \end{pmatrix} \begin{pmatrix} 0 & 0 \\ QHP & QHQ \end{pmatrix} U^+ (10)
\]

where \( H \) and \( H \) describe the low energy and high energy dynamics respectively. Of course, we can split the Hamiltonian as \( H = T + V \), with \( T \) and \( V \) kinetic and potential energies respectively. For the case when \( U \) commutes with the kinetic energy, \( [U, T] = 0 \), an equivalent decomposition holds for the potential \( V \) in terms of \( \Lambda \) in the \( P \)-space and \( \Lambda \) in the \( Q \)-space. Our idea is to assume already the former decomposition to the two-body problem from the start and to consider the following potential in the momentum \( P \)-space

\[
\Lambda(p', p) = \theta(\Lambda - p') \theta(\Lambda - p) v(p', p).
\]

The effective interaction \( \Lambda(p', p) \) depends explicitly on the separation scale or cut-off \( \Lambda \). It corresponds to a self-adjoint operator, \( \Lambda(p', p) = \Lambda(p, p')^* \), acting in a reduced model Hilbert space with \( p, p' \leq \Lambda \). Due to the fact that the transformation is unitary we get that the phase-shift associated with \( \Lambda(p', p) \) is just

\[
\delta_{\Lambda, \Lambda}(p) = \delta_{\Lambda}(p) \theta(\Lambda - p),
\]

in the \( P \)-model space. Note that for \( \Lambda = k_F \), the \( \Lambda \)-space becomes irrelevant in the many body problem, so that

\[
\xi = 1 + \int_0^{k_F} k^2 dk \left( 1 - \frac{3k^2}{2k_F} + \frac{k^3}{2k_F^2} \right) V_{k_F}(k, k), (13)
\]

We stress that for this choice of two body potential with only the s-wave contribution this equation is exact. We are only left with the determination of a suitable function \( v(p', p) \).

\[1\] There are many inequivalent ways how this procedure can be carried out as a result of a finite number of steps. A particular implementation is to achieve Block-Diagonalization in a continuous way in terms of flow equations \([13]\). We will exploit this freedom below.
Within the class of solutions given by Eq. (11) we can still exploit the arbitrariness to choose an interaction which will provide an analytical solution to the Bertsch’s problem. Here, we will search for a separable interaction solution of the form

$$V_0(p', p) = \pm g(p') g(p).$$

The ± sign specifies a repulsive and an attractive interaction respectively. This approach will work up to a value of $p < \Lambda$ where the phase shift $\delta_\Lambda(p)$ does reproduce Eq. (2). For a separable potential of the form of Eq. (14) the solution of the Lippmann-Schwinger equation reads [14][15]

$$p \cot \delta_0(p) = -\frac{1}{V_0(p, p)} \left[ 1 - \frac{2}{\pi} \int_0^\infty dq \frac{q^2}{p^2 - q^2} V_0(q, q) \right] = -\frac{1}{\alpha_0} + \frac{1}{2} r_0 p^2 \quad (15)$$

For separable potentials the inverse scattering problem may be solved in quadrature by applying the Tabakin’s formula devised in 1969 [16] (for a review see e.g. [17]). In our case the attractive solution without bound state will be the pertinent one, reading

$$|g(k)|^2 = \frac{\sin \delta(k)}{k} \exp \left[ -\int_0^\infty \frac{\delta(k')}{k' - k} dk' \right], \quad (16)$$

where $\delta(-k) = -\delta(k)$ and the real and positive $g(k)$ is taken provided $\sin \delta(k) > 0$ or $0 \leq \delta(k) < \pi$, a condition fulfilled by Eq. (2) for any value of $\alpha_0$ and $r_0$. In our case, according to Eq. (12) we have a limited integration interval $-\Lambda \leq k \leq \Lambda$. The case with $\alpha_0 \to -\infty$ and $r_0 = 0$ can be worked out explicitly, yielding for $p > 0$

$$g(p) = \frac{\theta(\Lambda - p)}{\sqrt{\Lambda^2 - p^2}}. \quad (17)$$

For $\Lambda = k_F$, the potential satisfying exactly the conditions is

$$V_{k_F}(k', k) = -\frac{\theta(k_F - k') \theta(k_F - k)}{\sqrt{k_F^2 - k'^2} \sqrt{k_F^2 - k^2}}, \quad (18)$$

which as expected depends on $k_F$. It can be readily checked that for $k > 0$ one has

$$\delta_0(k) = \frac{\pi}{2} \theta(k_F - k). \quad (19)$$

A direct evaluation of the integral in Eq. (9) yields

$$\xi = \frac{176}{9\pi} - \frac{17}{3} = 0.558, \quad (20)$$

contrary to the “universal” value $\xi = 0.37 - 0.40$ [2]. The case $\alpha_0 \to -\infty$ and $r_0 \neq 0$ can also be computed analytically from Tabakin’s formula, Eq. (16), in terms of dilogarithmic functions. The result is depicted in Fig. 1, contradicting Conduit and Schonberg [8].

We note, however, that the previous result, while exact is not unique. As already mentioned, one can still undertake a further phase-equivalent unitary transformation within the $P-$space of the potential $V \to U^* V U$ which preserves the essential feature that the interaction does not allow transitions above the Fermi surface, but reshuffles the $v(p', p)$ function.

A simple way to generate a continuous one-parameter unitary transformation according to the previous requirements is by means of the Similarity Renormalization Group (SRG) method introduced by Wilson and Glazek [18] (for a review see e.g. [19]). Defining the Hamiltonian $H_s = T + V_s$ at the operator level the SRG equation with the Wilson, $G_s = T$, generator reads

$$\frac{dV_s}{ds} = \left[ [T, V_s], T + V_s \right]. \quad (21)$$

This evolution equation monotonously minimizes the Frobenius norm of the potential $||V_s||^2 = tr(V_s^2)$, since $dr(V_s^2)/ds < 0$ and for $s \to \infty$ provides $\lim_{s \to \infty} [T, V_s] = 0$, i.e. the potential energy operator becomes diagonal in momentum space and hence on-shell [20][21]. Here, one has for $v(p, p')$ in Eq. (11) and in our case ($\Lambda = k_F$) the following equation,

$$\frac{d v_s(p, p')}{ds} = -\frac{\theta(p^2 - p'^2)^2}{\pi} v_s(p, p') + \frac{2}{\pi} \int_0^{k_F} dq \frac{q^2}{(p^2 + p'^2 - 2q^2)^3} v_0(p, q) v_0(q, p'). \quad (22)$$

where $s = 1/\lambda^2$ and $\lambda$ is the similarity cutoff. The flow equation generates a set of isospectral interactions that approaches a diagonal form as $s \to \infty$ (or $\lambda \to 0$). Only in few cases have the SRG integro-differential equations been solved.
analytically \cite{22}. Their numerical treatment requires introducing a finite momentum grid, so that results in the continuum are taken as a limiting procedure \cite{14} \cite{15}. Taking the \( v_s(k,k) \) into the mean field energy one obtains a phase-equivalent flow equation for the Bertsch parameter, \( \xi \). Defining \( \phi(x) = 1 - 3x/2 + x^3/2 \), Eq. (22) yields the inequality
\[
\frac{d\xi}{ds} = \frac{80}{3\pi} \left( \frac{2}{\pi} \right)^2 \int_0^{\Lambda} dq \int_0^{k_F} dk k^2 \left[ \left( \frac{k}{k_F} \right)^2 - \left( \frac{q}{k_F} \right)^2 \right] \times \left[ \phi(k) - \phi(q) \right] |v_s(k,q)|^2 \leq 0, \tag{23}
\]
since \( \phi(x) \) is a decreasing function, \( \phi'(x) = -3(1-x^2)/2 < 0 \), and thus \( (x^2 - y^2) |\phi(x) - \phi(y)| < 0 \) in \( 0 < x, y < 1 \). This inequality actually shows that the Bertsch parameter is not determined uniquely from the s-wave phase-shift and hence \( \xi \) is not universal.

In the on-shell limit, \( s \to \infty (\lambda \to 0) \), \( v_s(p',p) \) becomes diagonal, and one has thus \( d\xi/ds \to 0 \). The limiting value of Eq. (21) was determined in terms of the scattering phase-shifts \cite{14} \cite{15}. Adapted to our Eq. (22) and in the absence of bound states, the limit becomes a fixed point. If \( k' \neq k \) then
\[
\lim_{s \to \infty} v_s(k,k) = -\frac{\delta_0(k)}{k}, \quad \lim_{s \to \infty} v_s(k',k) = 0, \tag{24}
\]
which is asymptotically stable \cite{14} \cite{15} and the solutions are attracted to this one. Hence, for \( \delta_0(k) = \pi/2 \) and computing a trivial integral we finally get a fixed point solution
\[
\lim_{s \to \infty} \xi_s = 1 - \frac{4}{3} = -\frac{1}{3}. \tag{25}
\]
This corresponds to an unstable system. Thus, the previous argument shows that regardless of the initial function \( v(p',p) \) at \( s = 0 \) with a given value of \( \xi_s \), there is a phase-equivalent potential where \( \xi_s < 0 \).

The fixed point solution only depends on the choice \( \Lambda = k_F \). In the case, \( k_F < \Lambda \), the mean field result is only an upper bound, \( \xi \leq \xi_{MF} \). The flow equations for \( v_s(p',p) \) and \( \xi_{MF} \) read as Eq. (22) and Eq. (23) respectively with the replacement \( \int_0^{k_F} dq \to \int_0^{\Lambda} dq \), so that the same inequality holds. Thus, one has \( \xi_s \leq \xi_{MF} \to -1/3 \). In the general case, with finite \( \alpha_0 \neq 0 \) and \( r_0 \neq 0 \), see Eq. (2), the sign of \( \lim_{s \to \infty} \xi_s \) depends on their particular values. For instance for \( r_0 = 0 \), the intercept \( \xi_s = 0 \) happens for \( \alpha_0 k_F = -7.5378 \) and for \( \alpha_0 \to -\infty \) one has \( r_0 k_F = 2.038 \).

As we have mentioned above, these solutions, while exact, are not unique; we can still carry out a phase-equivalent transformation and change the value of \( \xi_s \). In particular, from the SRG equations on a momentum grid \cite{14} \cite{15} we can cover continuously all values from the starting one to the final one \cite{2}. This is shown in Fig. 2 as a function of \( \lambda/k_F \).

In particular, we could tune the SRG-scale \( \lambda \) to obtain from the potential \( V_{BM}^*(k',k) \) given by Eq. (18) with \( \xi = 0.558 \) the “universal” value \( \xi = 0.37 - 0.40 \) obtained in many calculations and experiments \cite{22}. We find that \( \xi = 0.37 \) happens for \( \lambda/k_F = 0.9 \), see Fig. 2. It is of course tempting to analyze the effective range behaviour at the scale \( \lambda/k_F = 0.9 \). This is done in Fig. 4 and compared again with the recent Monte Carlo calculation of Conduit and Schonberg \cite{8}. As we see the lack of universality of \( \xi \) is reinforced for finite \( r_0 \) even after tuning the \( r_0 = 0 \) value.

Besides illustrating the lack of universality our findings provide quite different values showing that the numerical resemblance among the many calculations and experiments is due to a common, yet unknown, feature among them which was not spelled out in the famous Bertsch’s problem and deserves an explanation. We can think of several reasons for not reproducing neither the Monte Carlo calculation nor the experimental data on ultracold atoms which agree among themselves. Firstly, Monte Carlo calculations have only been carried out for local potentials. The solution of the inverse scattering problem exists \cite{17} and will be discussed elsewhere. Second, the potentials experienced between neutral atoms are van der Waals-like and hence local. Thus, we conjecture that locality is the additional condition underlying the observed universality. Work along these lines is in progress.

\footnote{We take \( N = 50 \) points and Gauss-Legendre points. From the discretized form of Eq. (13), we get \( \xi = -0.55806 \) using Eq. (18) and \( \xi = 0.298993 \) for Eq. (24). The SRG equations become increasingly stiff for large \( N \)’s and small \( \xi \)’s \cite{14} \cite{15} \cite{21}. We estimate an error in \( \xi \) about 0.03, which is compatible with the expected flat behaviour at the fixed point.}
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