A NOTE ON SYZYGIES AND NORMAL GENERATION FOR TRIGONAL CURVES

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Abstract. Let \( C \) be a trigonal curve of genus \( g \geq 5 \) and let \( T \) be the unique trigonal line bundle inducing a map \( \pi : C \rightarrow \mathbb{P}^1 \). This note provides a short and easy proof of the normal generation for the residual line bundle \( K_C \otimes T^{-1} \) for curves of genus \( g \geq 7 \). Moreover, we compute the minimal free resolution of the embedded curve \( C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})^*) \) for the residual line bundle \( K_C \otimes T^{-n} \) for \( n \geq 1 \) and \( g \geq 3n + 4 \).

1. Introduction

In this short note we show how to determine the minimal free resolution of a trigonal curve lying on a rational normal surface scroll following [Sch86]. This was studied in a general context (arithmetic Cohen-Macaulay and non-arithmetic Cohen-Macaulay divisors on rational normal scrolls) by [Nag99] or [Par14], but we could not find any application for trigonal curves\(^1\). In particular, for a trigonal curve \( C \) of genus \( g \geq 5 \) with trigonal bundle \( T \) the shape of the minimal free resolution implies normal generation for line bundles of the form \( K_C \otimes T^{-n} \) (by our knowledge, the latter seems to be unknown for \( n = 1 \) and small genus or for \( n \geq 2 \)). We present a short and selfcontained proof (see also Remark 1.3).

We summarize known result about normal generation of line bundles on trigonal curves. Let \( C \) be a curve of genus \( g \). A line bundle \( L \) on \( C \) is said to be normally generated if \( L \) is very ample and the embedded curve \( C \subset \mathbb{P}(H^0(C, L)^*) \) is projectively normal (equivalently, the natural maps \( S^mH^0(C, L) \rightarrow H^0(C, L^m) \) are surjective for all \( m \geq 0 \)). In their seminal paper [GL86], Green and Lazarsfeld prove that a very ample line bundle \( L \) with \( h^1(C, L) \geq 2 \) of degree \( \deg(L) = 2g - 2 \cdot h^1(C, L) - c \) is normally generated.

**Theorem.** [GL86, Theorem 2] There exists an explicit constant \( N(c) \) such that if \( C \) is a curve of Clifford index \( c \) and of genus \( g > N(c) \), then every very ample line bundle \( L \) with \( h^1(C, L) \geq 2 \) of degree \( \deg(L) = 2g - 2 \cdot h^1(C, L) - c \) is normally generated.

\(^1\)A trigonal curve is a non-hyperelliptic smooth curve \( C \) of genus \( g \) with a line bundle \( T \) inducing a \( 3:1 \) morphism to \( \mathbb{P}^1 \). For \( g \geq 5 \), the line bundle \( T \) is unique.

\(^2\)For \( g \geq 4 \) the Clifford index is defined as \( \min\{\deg L - 2 \cdot h^0(C, L) - 2 : L \in \text{Pic}(C), h^i(C, L) \geq 2, i = 0, 1\} \).
They furthermore showed the existence of non-special line bundles which are not normally generated, and classified not normally generated line bundles with $h^1(C, L) = 1$.

**Example 1.1.** For a trigonal curve $C$ (⇒ Clifford index 1) let $T$ be the trigonal bundle. We set $L = K_C \otimes T^{-1}$ in the above theorem. Then $K_C \otimes T^{-1}$ is normally generated if $g > N(1) = 16$. We could not find any further result in the literature concerning the normal generation of the line bundle $K_C \otimes T^{-1}$ for trigonal curves of genus $16 \geq g \geq 7$ (the low genus cases are trivial by a Bézout argument). We close the remaining gap in the genus with our theorem and furthermore describe the whole minimal free resolution of $C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-1})^*)$.

Given a projective variety $Y \subset \mathbb{P}^n$, we can resolve the structure sheaf of $Y$ by free $\mathcal{O}_{\mathbb{P}^n}$-modules, that is, a minimal free resolution

$$\mathcal{O}_Y \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_m \leftarrow 0,$$

where $F_i = \bigoplus_j \mathcal{O}_{\mathbb{P}^n}(-j)^{\beta_{ij}}$. We call the numbers $\beta_{ij}$ the Betti numbers of $Y$. The shape of the minimal free resolution will be encoded in the so-called Betti table $(\beta_{ij})_{ij}$ (see [Eis95] for an introduction).

Our main theorem is the following (see also Theorem 3.4 for its generalization).

**Theorem 1.2.** Let $C$ be a trigonal curve of genus $g \geq 7$ and let $T$ be the unique trigonal bundle inducing the map $\pi : C \to \mathbb{P}^3$. The Betti table of a minimal free resolution of $C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-1})^*) = \mathbb{P}^{g-3}$ has the following shape

$$
\begin{array}{cccccc}
0 & 1 & \cdots & g-6 & g-5 & g-4 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & \ast & \cdots & \ast & \ast & 0 \\
2 & 0 & \ast & \cdots & 0 & 0 & 0 \\
3 & 0 & 0 & \cdots & 0 & \ast & \ast \\
\end{array}
$$

where $\ast$ indicates nonzero entries. In particular, the embedded curve is projectively normal.

**Remark 1.3.** Given a trigonal curve $C$ of genus $g$ and a positive number $n$, we provide a sharp range for $g$ and $n$ where $K_C \otimes T^{-n}$ is very ample in Section 3.1. Hence the embedded curve $C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})^*)$ is smooth. One can show that projective normality of $K_C \otimes T^{-n}$ follows already from the classical Castelnuovo lemma [Cas89].

We note that trigonal curves $C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})^*)$ are extremal in the sense of Castelnuovo’s bound and refer to [ACGH85, Chapter III, §2, pp 120] for a detailed analysis. Furthermore the statement of Theorem 1.2 also follows from [Nag99, Theorem 2.4 and Corollary 2.5] using the smoothness assumption. We recall that we present a new self-contained proof where our method of the proof is exactly valid in the range for $g$ and $n$ where $K_C \otimes T^{-n}$ is very ample.

In [MS86], Martens and Schreyer classified non-special not normally generated line bundles on trigonal curves. They used the extrinsic geometry of the embedding $C \subset \mathbb{P}(H^0(C, L)^*)$ depending on the Maroni-invariant (see Section 2.1 for the definition).
Remark 1.4. In [LN13], Lange and Newstead determined all vector bundles on trigonal curves which compute all higher Clifford indices (see for the definition e.g., [LN10]). Their main theorem [LN13, Theorem 4.7] relies on the normal generation of $K_C \otimes T^{-1}$ and is thus true for all trigonal curves of genus $g \geq 5$ by our main theorem.

Remark 1.5. The nonzero Betti numbers in Theorem 1.2 are:

\[ \beta_{j,j+1} = j \cdot \binom{g-4}{j+1}, \quad \text{for } 1 \leq j \leq g-5 \quad (\text{first row}) \]
\[ \beta_{j,j+2} = (g - 5 - j) \cdot \binom{g-4}{j-1}, \quad \text{for } 1 \leq j \leq g-6 \quad (\text{second row}) \]
\[ \beta_{j,j+3} = (j - g + 6) \cdot \binom{g-4}{j}, \quad \text{for } g-5 \leq j \leq g-4 \quad (\text{third row}) \]

These are exactly the numbers as in [Nag99, Theorem 2.4].

In Section 2 we present the proof of Theorem 1.2 and introduce the necessary background. In Section 3 we give a bound for the very ampleness of $K_C \otimes T^{-n}$ in terms of $n$ and the genus of the trigonal curve $C$. At the end we proof the generalization of Theorem 1.2.

2. Proof of the main theorem

Our proof follows the strategy of [Sch86]:

- $C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})^*)$ lies on a rational normal surface $X$ swept out by $T$,
- resolve $\mathcal{O}_C$ as an $\mathcal{O}_X$-module by direct sums of line bundles on $X$ and take the minimal free resolution of each of these line bundles as $\mathcal{O}_{\mathbb{P}^g-3}$-modules,
- a mapping cone construction induces a minimal free resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^g-3}$-module.

2.1. A rational normal surface associated to the trigonal bundle. Let $C$ be a trigonal curve of genus $g \geq 5$. An important invariant associated to any trigonal curve - classically used to describe its Brill–Noether locus - is the so-called Maroni-invariant. It is defined as follows. Let $\pi : C \to \mathbb{P}^1$ be the map associated to the trigonal bundle $T$. The direct image $\pi_* K_C$ of the canonical bundle on $C$ splits into a direct sum of line bundles

$$\pi_* K_C = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$

where $m$ is the Maroni-invariant. It is known that

\[ 0 < \frac{g - 4}{3} \leq m \leq \frac{g - 2}{2}, \tag{2.1} \]

and $a = g - 2 - m$. Furthermore, $m + 2$ and $a + 2$ describe the two jumps of the function $f : \mathbb{N} \to \{0, 1, 2\}$ with $f(n) := h^1(C, T^{n-1}) - h^1(C, T^n)$. Hence,

\[ m + 2 = \min\{n \in \mathbb{N} : h^0(C, T^n) > n + 1\}. \tag{2.2} \]

Maroni [Mar46] introduced $m$ as a geometrical invariant determining a smooth rational normal scroll which is swept out by the trigonal bundle in the canonical space $\mathbb{P}(H^0(C, K_C)^*)$. 


We will explain this procedure for our purpose in the following setting. We assume that \( g \geq 6 \). Let \( C \subset \mathbb{P}^{g-3} := \mathbb{P}(H^0(C, K_C \otimes T^{-1})) \) be embedded by the very ample linear series \( |K_C \otimes T^{-1}| \) (see Section 3.1 and note that for \( g = 5 \), \( K_C \otimes T^{-1} \) maps \( C \) to a plane quintic with one node). We consider the scroll swept out by the unique pencil \( |T| \), that is, \( X = \bigcup_{D \in |T|} D \subset \mathbb{P}^{g-3} \)

where \( D \) is the linear span of the divisor \( D \) defined by the linear forms \( H^0(C, K_C \otimes T^{-1})(-D)) \rightarrow H^0(C, K_C \otimes T^{-1}) \).

Since \( H^0(C, K_C \otimes T^{-2}) \) is \((g - 4)\)-dimensional (by (2.1) and (2.2) for \( g \geq 6 \)), the linear span \( D \) is a line and \( X \) is a rational normal surface of degree \( g - 4 \). In particular, \( X \) is the image of a projective bundle \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2)) \) where \( e_1 = g - 3 - m \) and \( e_2 = m - 1 \geq 0 \). We will explain this fact in the next section.

2.2. Resolution of the curve on the rational normal surface. Let \( C \) be a trigonal curve with Maroni-invariant \( m \). Let \( X \) be the rational normal surface of degree \( g - 4 \) as above and let \( \pi : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2) \rightarrow \mathbb{P}^1 \) be the rank two bundle on \( \mathbb{P}^1 \) with \( e_1 = g - 3 - m \) and \( e_2 = m - 1 \). By [Har81], if \( e_1, e_2 > 0 \), then \( j : \mathbb{P}(\mathcal{E}) \rightarrow X \subset \mathbb{P}(\mathcal{E}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathbb{P}^{g-3} \)

is an isomorphism. Otherwise it is a resolution of singularities. Since \( R^i j_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} = 0 \), it is convenient to consider \( \mathbb{P}(\mathcal{E}) \) instead of \( X \) for cohomological considerations. Note that this may happen for \( g = 6, 7 \) and \( m = 1 \).

It is furthermore known that the Picard group Pic(\( \mathbb{P}(\mathcal{E}) \)) is generated by the ruling \( R = [\pi^* \mathcal{O}_{\mathbb{P}^1}(1)] \) and the hyperplane class \( H = [j^* \mathcal{O}_{\mathbb{P}^{g-3}}(1)] \) with intersection products \( H^2 = e_1 + e_2 = g - 4 \), \( H \cdot R = 1 \), \( R^2 = 0 \).

Since \( C \subset X \) and \( T \) is base point free, we consider \( C \subset \mathbb{P}(\mathcal{E}) \) as a subvariety of the projectivised bundle. Note that \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H) \otimes \mathcal{O}_C = K_C \otimes T^{-1} \) and \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R) \otimes \mathcal{O}_C = T \).

Since the curve \( C \) is a codimension one subvariety of \( \mathbb{P}(\mathcal{E}) \), the ideal sheaf \( \mathcal{I}_{C/\mathbb{P}(\mathcal{E})} \) is generated by an element in \( H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(aH - bR)) \). Since the ruling on \( C \) has degree \( 3 \) (\( a = 3 \)) and the degree of \( C \subset \mathbb{P}^{g-3} \) is \( \deg(C) = 2g - 5 \) (\( b = g - 7 \)), the resolution of \( \mathcal{O}_C \) as an \( \mathcal{O}_{\mathbb{P}(\mathcal{E})} \)-module is

\[
\begin{align*}
0 & \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + (g - 7)R) \longrightarrow 0
\end{align*}
\]

In the next step we will resolve the line bundles in the above resolution in terms of \( \mathcal{O}_{\mathbb{P}^{g-3}} \)-modules. Therefore we recall the definition of Eagon–Northcott type resolutions whereby we restrict to our case.

We have a natural multiplication map \( H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R)) \otimes H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H - R)) \rightarrow H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H)) \),
and the equations of the rational normal surface $X$ are given by the $2 \times 2$-minors of the matrix $\Phi$. We define
\[
F := H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(H-R)) \otimes \mathcal{O}_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}^{-4},
\]
and regard $\Phi$ as a map $\Phi : F \to G$ (where we identify $G = G^*$).

For $b \geq -1$ we can resolve $\mathcal{O}_{\mathbb{P}(E)}(aH+bR)$ as an $\mathcal{O}_{\mathbb{P}^3}(a)$-module by an Eagon-Northcott type complex $\mathcal{C}^b \otimes \mathcal{O}_{\mathbb{P}^3}(a)$ where the $j^{th}$ term of $\mathcal{C}^b$ is defined as
\[
\mathcal{C}^b_j = \begin{cases} 
\bigwedge^j F \otimes S_{b-j} \mathcal{O}_{\mathbb{P}^3}(-j) & \text{for } 0 \leq j \leq b \\
\bigwedge^{j+1} F \otimes D_{j-b-1} G^* \otimes \mathcal{O}_{\mathbb{P}^3}(-j-1) & \text{for } j \geq b+1
\end{cases}
\]
and whose differentials $\delta_j : \mathcal{C}^b_j \to \mathcal{C}^b_{j-1}$ are given by the multiplication by
\[
\Phi \in H^0(\mathbb{P}(E), F^* \otimes G) \quad \text{for } j \neq b+1
\]
\[
\bigwedge^2 \Phi \in H^0(\mathbb{P}(E), \lambda^2 F^* \otimes \lambda^2 G) \quad \text{for } j = b+1
\]
(see [BE75]). The resolution in (2.3) induces a double complex of Eagon–Northcott type resolutions
\[
(2.4) \quad \mathcal{C}^0 \leftarrow \mathcal{C}^{g-7} \otimes \mathcal{O}_{\mathbb{P}^3}(-3).
\]

2.3. Mapping cone construction and examples. Since all maps in the double complex in (2.4) are minimal, the mapping cone $[\mathcal{C}^0 \leftarrow \mathcal{C}^{g-7} \otimes \mathcal{O}_{\mathbb{P}^3}(-3)]$ of this double complex induces a minimal free resolution of $C \subset \mathbb{P}^3$ (see [Eis95]).

Example 2.1. For a trigonal curve of genus $g = 7$ the double complex is
\[
\begin{array}{cccc}
0 & \mathcal{O}_C & \mathcal{O}_{\mathbb{P}(E)} & \mathcal{O}_{\mathbb{P}(E)}(-3H) & 0 \\
& \mathcal{O}_{\mathbb{P}^4} & \mathcal{O}_{\mathbb{P}^4}(-3) & \\
& \bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}^4}(-2) & \bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}^4}(-5) & \\
& \bigwedge^3 F \otimes D G^* \otimes \mathcal{O}_{\mathbb{P}^4}(-3) & \bigwedge^3 F \otimes D G^* \otimes \mathcal{O}_{\mathbb{P}^4}(-6) & \\
0 & 0 & 0 & \\
\end{array}
\]

In the mapping cone construction we sum up the terms on the diagonal in the double complex (with the right differentials). This yields
\[
\begin{array}{cccc}
0 & \mathcal{O}_C & \mathcal{O}_{\mathbb{P}^4} & \mathcal{O}_{\mathbb{P}^4}(-2) & \mathcal{O}_{\mathbb{P}^4}(-3) & \mathcal{O}_{\mathbb{P}^4}(-5) & \mathcal{O}_{\mathbb{P}^4}(-6) & 0 \\
& \bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}^4}(-3) & \bigwedge^3 F \otimes D G^* \otimes \mathcal{O}_{\mathbb{P}^4}(-3) & \bigwedge^3 F \otimes D G^* \otimes \mathcal{O}_{\mathbb{P}^4}(-6) & 0, \\
\end{array}
\]
and the Betti table of the minimal free resolution of $C \subset \mathbb{P}^4$ is

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 3 | 2 |
| 2 | 0 | 1 | 0 |
| 3 | 0 | 0 | 3 | 2 |

For $g \geq 8$ the construction of Section 2.2 yields the following double complex

$$
\begin{array}{cccc}
0 & \rightarrow & \mathcal{O}_C & \rightarrow \mathcal{O}_{\mathbb{P}(E)} & \rightarrow \mathcal{O}_{\mathbb{P}(E)}(-3H + (g - 7)R) & \rightarrow 0 \\
\uparrow & & \uparrow & & \uparrow & \\
\mathcal{O}_{\mathbb{P}_g(-3)} & \rightarrow & S_{g-7}G \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-3) & \rightarrow & F \otimes S_{g-8}G \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-4) \\
\uparrow & & \uparrow & & \uparrow & \\
\bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-2) & \rightarrow & F \otimes S_{g-8}G \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-4) \\
\uparrow & & \uparrow & & \uparrow & \\
\bigwedge^3 F \otimes DG^* \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-3) & \rightarrow & \vdots \\
\end{array}
$$

and the mapping cone induces the minimal free resolution

$$
0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}_g(-3)} \leftarrow \bigwedge^2 F \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-2) \oplus S_{g-7}G \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-3) \leftarrow \bigwedge^3 F \otimes DG^* \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-3) \oplus F \otimes S_{g-8}G \otimes \mathcal{O}_{\mathbb{P}_g(-3)}(-5) \leftarrow \ldots
$$

The shape of the minimal free resolution of $C$ as an $\mathcal{O}_{\mathbb{P}_g(-3)}$-module is given as stated in the main theorem. Indeed, we note that the second last map in $\mathcal{O}_{\mathbb{P}_g(-3)}(-3)$ is of degree 2 and occurs in the $(g-6)^{th}$ step in the mapping cone construction. The theorem follows.

**Example 2.2.** We end this section with a trigonal curve $C$ of genus 6. The minimal free resolution of $C \subset \mathbb{P}^3 = \mathbb{P}(H^0(C, K_C \otimes T^{-1})^*)$ has the following shape

| 0 | 1 | 2 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 2 | 0 | 0 |
| 3 | 0 | 2 |
| 4 | 0 | 0 | 1 |

We can still apply the above method to compute a minimal free resolution. Note that $C$ has quartic generators induced by $\mathcal{O}_{\mathbb{P}^3}(-3)$. 
3. Higher residuals of $T$ with respect to the canonical bundle

In this section we extend Theorem 1.2 to trigonal curves of genus $g$ embedded by line bundles of the form $K_C \otimes T^{-n}$ for $n \geq 1$ and $g \geq 3n + 4$. We fix the notation of this section. For a trigonal curve $C$ of genus $g \geq 5$ let $T$ be the unique trigonal bundle and let $m$ be the Maroni-invariant (and $a = g - 2 - m$).

3.1. Very ampleness of $K_C \otimes T^{-n}$. We have the following two lemmata.

**Lemma 3.1.** For $n > m$ the line bundle $K_C \otimes T^{-n}$ does not separate points of the morphism $C \dashrightarrow \mathbb{P}^1$ induced by $T$. In particular, $K_C \otimes T^{-n}$ is not very ample.

**Proof.** We may assume that $n = m + 1$ since $H^0(C, K_C \otimes T^{-n_1}) \subset H^0(C, K_C \otimes T^{-n_2})$ for $n_1 \geq n_2$. Let $D = p + q + r \in |T|$ be a divisor. There is a short exact sequence

$$0 \to T^{m+1} \to T^{m+1}(D) \to T^{m+1}(D)|_D \to 0.$$ 

The long exact sequence

$$0 \to H^0(C, T^{m+1}) \to H^0(C, T^{m+2}) \to \Gamma(T^{m+1}(D)|_D) \to H^1(C, T^{m+1}) \to H^1(C, T^{m+2}) \to 0$$

is induced by the global section functor. Since $m + 2 = \min\{n \in \mathbb{N} : h^0(C, T^n) > n + 1\}$, the difference is

$$h^0(C, T^{m+2}) - h^0(C, T^{m+1}) \geq 2.$$ 

Hence, by the long exact sequence $h^1(C, T^{m+1}) - h^1(C, T^{m+2}) \leq 1$ and for $p, q \in \text{Supp}(D)$

$$h^0(C, K_C \otimes T^{-m-1}) - h^0(C, K_C \otimes T^{-m-1}(-p - q)) \leq h^0(C, K_C \otimes T^{-m-1}) - h^0(C, K_C \otimes T^{-m-2})$$

$$= h^1(C, T^{m+1}) - h^1(C, T^{m+2}) \leq 1.$$ 

Thus, we cannot find a section separating the point $p$ and $q$. The lemma follows. □

**Lemma 3.2.** [Har77, V, Theorem 2.17] or [MS86, Lemma 1, p. 176] For $n < m$ the line bundle $K_C \otimes T^{-n}$ on $C$ is very ample. The line bundle $K_C \otimes T^{-m}$ is generated by global sections and separates points of the morphism $C \dashrightarrow \mathbb{P}^1$ induced by $T$.

**Remark 3.3.** For $g \geq 3m + 3$ the line bundle $K_C \otimes T^{-m}$ is very ample since $h^0(C, T^m(p + q)) = h^0(C, T^m) = m + 1$. Indeed, by [Mar46] or [MS86, Proposition 1] the Brill–Noether locus $W^3_{3m+2}(C)$ is empty for $g \geq 3m + 3$ and that implies $h^0(C, T^m(p + q)) = m + 1$. This can only happen in few cases since $g \leq 3m + 4$ always holds by the inequalities (2.1). Furthermore, the bound $g \geq 3m + 3$ is satisfied for our assumptions in the next section.

3.2. The minimal free resolution of $C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})^*)$. We fix an integer $n \geq 1$. In order to ensure that the linear system $K_C \otimes T^{-n}$ on a curve $C$ of genus $g$ is very ample, we have to assume that $n$ is less than or equal to the minimal possible Maroni-invariant of a curve of genus $g$. By (2.1), this yields the bound $n \leq \frac{g - 4}{3}$, or equivalently

$$g \geq 3n + 4.$$
We use the same strategy as in the previous section to show that the curve \( C \) is projectively normal in \( \mathbb{P}(H^0(C, K_C \otimes T^{-n})) \). We have the following generalisation of Theorem 1.2.

**Theorem 3.4.** Let \( n \geq 1 \) be an integer. Let \( C \) be a trigonal curve of genus \( g \geq 3n + 4 \) and let \( T \) be the unique trigonal bundle. The Betti table of a minimal free resolution of \( C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})) = \mathbb{P}^{g-2n-1} \) has the following shape

|   | 0  | 1  | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | \ldots | 2  | 0  |
|---|----|----|--------|--------|--------|--------|--------|--------|--------|----|----|
| 0 | 1  | 0  | \ldots | 0      | \ldots | 0      | \ldots | 0      | \ldots | 0  | 0  |
| 1 | 0  | *  | \ldots | *      | \ldots | *      | \ldots | *      | \ldots | 0  | 0  |
| 2 | 0  | *  | \ldots | 0      | \ldots | 0      | \ldots | 0      | 0      | 0  | 0  |
| 3 | 0  | 0  | \ldots | 0      | \ldots | *      | \ldots | *      | \ldots | *  | *  |

where * indicates nonzero entries. In particular, the embedded curve is projectively normal.

The nonzero Betti numbers are given as in [Nag99, Theorem 2.4] for \( c = g - 2n - 2 \) and \( p = g - 3n - 3 \).

**Proof.** By Section 3.1, let \( C \subset \mathbb{P}(H^0(C, K_C \otimes T^{-n})) =: \mathbb{P}^{g-2n-1} \) be a smooth embedded curve of genus \( g \) and of degree \( \deg(C) = 2g - 3n - 2 \). Note that \( g - 2n - 1 \geq n + 3 \geq 4 \).

Let

\[ X = \bigcup_{D \in |T|} \overline{D} \subset \mathbb{P}^{g-2n-1} \]

be the rational normal scroll swept out by the trigonal bundle. Since \( n + 1 < m + 2 \),

\[ h^0(C, K_C \otimes T^{-n}) - h^0(C, K_C \otimes T^{-n-1}) = 2 \]

by (2.2). Hence, \( X \) is a rational normal surface of degree \( g - 2n - 2 \). Let \( \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(g - 2 - m - n) \oplus \mathcal{O}_{\mathbb{P}^1}(m - n) \) be a rank 2 bundle. Then the rational normal surface \( X \) is the image of \( \mathbb{P}(\mathcal{E}) \) as in Section 2.2. Since \( T \) is base point free, we consider \( C \subset \mathbb{P}(\mathcal{E}) \). We denote again \( H \) and \( R \) the generators of the Picard group of \( \mathbb{P}(\mathcal{E}) \) as in Section 2.2 such that \( H^2 = g - 2n - 2, H.R = 1 \) and \( R^2 = 0 \) as well as \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(H) \otimes \mathcal{O}_C = K_C \otimes T^{-n} \) and \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(R) \otimes \mathcal{O}_C = T \).

The resolution of \( \mathcal{O}_C \) as an \( \mathcal{O}_{\mathbb{P}(\mathcal{E})} \)-module is

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_C & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-aH + bR) & \to & 0
\end{array}
\]

for integers \( a, b \) since \( C \) is a divisor on \( \mathbb{P}(\mathcal{E}) \). We have

\[
(aH - bR).R = 3 \quad \text{and} \quad (aH - bR).H = \deg(C) = 2g - 3n - 2.
\]

Hence, \( a = 3 \) and \( b = g - 3n - 4 \geq 0 \) and thus the \( \mathcal{O}_{\mathbb{P}(\mathcal{E})} \)-modules in

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_C & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})} & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-3H + (g - 3n - 4)R) & \to & 0
\end{array}
\]

can be resolved as \( \mathcal{O}_{\mathbb{P}^{g-2n-1}} \)-modules by Eagon–Northcott type resolutions (see Section 2.2). As in Section 2.3, the mapping cone of the induced double complex

\[
[\mathcal{E}^0 \leftarrow \mathcal{E}^{3-4} \otimes \mathcal{O}_{\mathbb{P}^{g-2n-1}}(-3)]
\]
is a minimal free resolution of $C \subset \mathbb{P}^{g-2n-1}$. The shape of its Betti table is given as in Theorem 3.4. Indeed, the length of $\mathcal{C}^0$ is $g-2n-3$ and the degree 2 syzygies in $\mathcal{C}^{g-3n-4}$ are in the step $j = b + 1 = g - 3n - 3$. Since $\mathcal{C}^{g-3n-4}$ is homologically shifted by 1 in the mapping cone construction, these syzygies appear in step $g - 3n - 2$ as stated in the theorem.

\[ \square \]

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