Representation of the Dirac delta function in $\mathcal{C}(\mathbb{R}^\infty)$ in terms of infinite-dimensional Lebesgue measures

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Abstract: A representation of the Dirac delta function in $\mathcal{C}(\mathbb{R}^\infty)$ in terms of infinite-dimensional Lebesgue measures in $\mathbb{R}^\infty$ is obtained and some its properties are studied in this paper.

MSC 2010 subject classifications: Primary 28xx; Secondary 28C10.
Keywords and phrases: The Dirac delta function, infinite-dimensional Lebesgue measure.

1. Introduction

The Dirac delta function (\(\delta\)-function) was introduced by Paul Dirac at the end of the 1920s in an effort to create the mathematical tools for the development of quantum field theory. He referred to it as an improper functional in Dirac (1930). Later, in 1947, Laurent Schwartz gave it a more rigorous mathematical definition as a spatial linear functional on the space of test functions $D$ (the set of all real-valued infinitely differentiable functions with compact support). Since the delta function is not really a function in the classical sense, one should not consider the value of the delta function at $x$. Hence, the domain of the delta function is $D$ and its value for $f \in D$ is $f(0)$. Khuri (2004) studied some interesting applications of the delta function in statistics.

The purpose of the present paper is an introduction of a concept of the Dirac delta function in the class of all continuous functions defined in the infinite-dimensional topological vector space of all real valued sequences $\mathbb{R}^\infty$ equipped with Tychonoff topology and a representation of this functional in terms of infinite-dimensional Lebesgue measures in $\mathbb{R}^\infty$.

The paper is organized as follows:

In Section 2 we present a concept of ordinary and standard Lebesgue measures in $\mathbb{R}^\infty$ introduced in [1]. In Section 3 we present a concept of uniform distribution in infinite-dimensional rectangles for calculation of Riemann integrals for continuous functions over such rectangles(cf. [2]). In Section 4 we present Change of Variables Formula for $\alpha$-ordinary Lebesgue measure in $\mathbb{R}^\infty$.

*The research for this paper was partially supported by Shota Rustaveli National Science Foundation’s Grant no FR/116/5-100/14
established in [3]. In Section 5 we give a representation of the Dirac delta function in $C(R^\infty)$ in terms of infinite-dimensional Lebesgue measures and consider some properties of this functional.

2. On ordinary and standard Lebesgue measures in $R^\infty$

The problem of the existence of an analog of the Lebesgue measure for the vector space of all real-valued sequences $R^\infty = \prod_{i=1}^{\infty} R$ equipped with Tychonoff topology was discussed in [1].

R. Baker [4] firstly introduced the notion of “Lebesgue measure” in $R^\infty$ as follows: a measure $\lambda$ being the completion of a translation invariant Borel measure in $R^\infty$ is called a “Lebesgue measure” in $R^\infty$ if for any measurable rectangle $\prod_{i=1}^{\infty} (a_i, b_i), -\infty < a_i < b_i < +\infty$ with $0 \leq \prod_{i=1}^{\infty} (b_i - a_i) < +\infty$, the following equality

$$\lambda\left(\prod_{i=1}^{\infty} (a_i, b_i)\right) = \prod_{i=1}^{\infty} (b_i - a_i)$$

holds, where

$$\prod_{i=1}^{\infty} (b_i - a_i) := \lim_{n \to \infty} \prod_{i=1}^{n} (b_i - a_i).$$

Subsequently, R. Baker [5] extended his notion of “Lebesgue measure” in $R^\infty$ as follows: a measure $\lambda$ being the completion of a translation invariant Borel measure on $R^\infty$ is called a “Lebesgue measure” if for any measurable rectangle $\prod_{i=1}^{\infty} R_i, R_i \in B(R)$ with $0 \leq \prod_{i=1}^{\infty} m(R_i) < \infty$, the following equality

$$\lambda\left(\prod_{i=1}^{\infty} R_i\right) = \prod_{i=1}^{\infty} m(R_i)$$

holds, where $m$ denotes a linear Lebesgue measure in $R$.

To propose a new concept of “Lebesgue measure” in $R^\infty$, in [1] main attention has been attracted to the following two simple facts:

**Fact 2.1.** Let $\mu$ be a probability measure defined on a measure space $(E, S)$. Then the product measure $\mu^N$ defined on $(E^N, S^N)$ has the following essential property: if $f$ is any permutation of $N$ and $A_f((x_k)_{k \in N}) = (x_{f(k)})_{k \in N}$ for $(x_k)_{k \in N} \in E^N$, then $\mu^N(A_f(X)) = \mu^N(X)$ for every $X \in S^N$.

**Fact 2.2.** The $n$-dimensional Lebesgue measure $\ell_n$ in $R^n$ has the following property: if $f$ is any permutation of $\{1, \ldots, n\}$ and

$$A_f((x_k)_{1 \leq k \leq n}) = (x_{f(k)})_{1 \leq k \leq n} ((x_k)_{1 \leq k \leq n} \in R^n),$$

then $\ell_n(A_f(X)) = \ell_n(X)$ for every $X \in B(R^n)$. 

In view of these facts one can say that Baker’s measures [4], [5] have no essential property of a product - measure to be an invariant under the group of all canonical permutations \(^1\) of \(\mathbb{R}^\infty\).

Indeed, if we consider the following infinite-dimensional rectangular set

\[
X = \prod_{k=1}^{\infty} [0, e^{\frac{(-1)^k}{k}}],
\]

then for every non-zero real number \(a\) there exists a permutation \(f_a\) of \(N\) such that \(\lambda(A_{f_a}(X)) = a\), where \(\lambda\) is any Baker’s measure [4], [5].

To introduce new concepts of the Lebesgue measure in \(\mathbb{R}^\infty\), the following definitions were introduced in [1]:

**Definition 2.1.** Let \(\beta_j \in [0, +\infty]^N\). We say that a number \(\beta \in [0, +\infty]\) is an ordinary product of numbers \(\beta_j \in N\) if

\[
\beta = \lim_{n \to \infty} \prod_{i=1}^{n} \beta_i.
\]

An ordinary product of numbers \((\beta_j)_{j \in \mathbb{N}}\) is denoted by \((O) \prod_{i \in \mathbb{N}} \beta_i\).

**Definition 2.2.** Let \(\beta_j \in [0, +\infty]^N\). A standard product of the family of numbers \((\beta_i)_{i \in \mathbb{N}}\) is denoted by \((S) \prod_{i \in \mathbb{N}} \beta_i\) and defined as follows:

\[
(S) \prod_{i \in \mathbb{N}} \beta_i = 0 \text{ if } \sum_{i \in \mathbb{N}} \ln(\beta_i) = -\infty, \text{ where } N^- = \{i : \ln(\beta_i) < 0\}, \quad \text{and} \quad (S) \prod_{i \in \mathbb{N}} \beta_i = e^{\sum_{i \in \mathbb{N}} \ln(\beta_i)} \text{ if } \sum_{i \in \mathbb{N}} \ln(\beta_i) \neq -\infty.
\]

Let \(\alpha = (n_k)_{k \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N\). We set

\[
F_0 = [0, n_0] \cap N, \quad F_1 = [n_0 + 1, n_0 + n_1] \cap N, \quad \ldots, \quad F_k = [n_0 + \cdots + n_k + 1, n_0 + \cdots + n_k] \cap N, \quad \ldots.
\]

**Definition 2.3.** We say that a number \(\beta \in [0, +\infty]\) is an ordinary \(\alpha\)-product of numbers \((\beta_i)_{i \in \mathbb{N}}\) if \(\beta\) is an ordinary product of numbers \(\prod_{i \in F_k} \beta_i\). An ordinary \(\alpha\)-product of numbers \((\beta_i)_{i \in \mathbb{N}}\) is denoted by \((O, \alpha) \prod_{i \in \mathbb{N}} \beta_i\).

**Definition 2.4.** We say that a number \(\beta \in [0, +\infty]\) is a standard \(\alpha\)-product of numbers \((\beta_i)_{i \in \mathbb{N}}\) if \(\beta\) is a standard product of numbers \(\prod_{i \in F_k} \beta_i\). A standard \(\alpha\)-product of numbers \((\beta_i)_{i \in \mathbb{N}}\) is denoted by \((S, \alpha) \prod_{i \in \mathbb{N}} \beta_i\).

**Definition 2.5.** Let \(\alpha = (n_k)_{k \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N\). Let \((\alpha)\mathcal{OR}\) be the class of all infinite-dimensional measurable \(\alpha\)-rectangles \(R = \prod_{i \in \mathbb{N}} R_i, R_i \in \mathcal{B}(\mathbb{R}^{n_i})\) for which an ordinary product of numbers \((m^{n_i}(R_i))_{i \in \mathbb{N}}\) exists and is finite. We say that a measure \(\lambda\) being the completion of a translation-invariant Borel measure is an ordinary \(\alpha\)-Lebesgue measure in \(\mathbb{R}^\infty\) (or, shortly, \(O(\alpha)\text{LM}\)) if for every \(R \in (\alpha)\mathcal{OR}\) we have

\[
\lambda(R) = (O) \prod_{k \in \mathbb{N}} m^{n_k}(R_k).
\]

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\(^1\)Let \(f\) be any permutation of \(N\). A mapping \(A_f : \mathbb{R}^\infty \to \mathbb{R}^\infty\) defined by \(A_f((x_k)_{k \in \mathbb{N}}) = (x_{f(k)})_{k \in \mathbb{N}}\) for \((x_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty\) is called a canonical permutation of \(\mathbb{R}^\infty\).

\(^2\)We set \(\ln(0) = -\infty\)
Definition 2.6. Let \( \alpha = (n_k)_{k \in \mathbb{N}} \in (N \setminus \{0\})^N \). Let \((\alpha)\mathcal{SR}\) be the class of all infinite-dimensional measurable \( \alpha \)-rectangles \( R = \prod_{i \in N} R_i (R_i \in \mathcal{B}(\mathbb{R}^{n_i})) \) for which a standard product of numbers \( (m^{n_i}(R_i))_{i \in N} \) exists and is finite. We say that a measure \( \lambda \) being the completion of a translation-invariant Borel measure is a standard \( \alpha \)-Lebesgue measure in \( \mathbb{R}^\infty \) (or, shortly, \( S(\alpha)LM \)) if for every \( R \in (\alpha)\mathcal{SR} \) we have
\[
\lambda(R) = \langle S \rangle \prod_{k \in N} m^{n_k}(R_k).
\]

Proposition 2.1. ([1], Proposition 1, p. 212) Note that for every \( \alpha = (n_k)_{k \in \mathbb{N}} \in (N \setminus \{0\})^N \) the following strict inclusion
\[
(\alpha)\mathcal{OR} \subset (\alpha)\mathcal{SR}
\]
holds.

The presented approach gives us a possibility to construct such translation-invariant Borel measures in \( \mathbb{R}^\infty \) which are different from the Baker measures [5] in the sense that it does not apply the metric properties of \( \mathbb{R}^\infty \). It is an adaptation of a construction from general measure theory which allows us to construct interesting examples of analogs of a Lebesgue measure on the entire space.

Let \( (E, S) \) be a measurable space and let \( \mathcal{R} \) be any subclass of the \( \sigma \)-algebra \( S \). Let \( (\mu_B)_{B \in \mathcal{R}} \) be such a family of \( \sigma \)-finite measures that for \( B \in \mathcal{R} \) we have \( \text{dom}(\mu_B) = S \cap \mathcal{P}(B) \), where \( \mathcal{P}(B) \) denotes the power set of the set \( B \).

Definition 2.7. A family \( (\mu_B)_{B \in \mathcal{R}} \) is called to be consistent if
\[
(\forall X)(\forall B_1, B_2)(X \in S \& B_1, B_2 \in \mathcal{R} \to \mu_{B_1}(X \cap B_1 \cap B_2) = \mu_{B_2}(X \cap B_1 \cap B_2)).
\]

The following assertion plays a key role for construction of new translation-invariant measures.

Lemma 2.1. ([1], Lemma 1, p. 213) Let \( (\mu_B)_{B \in \mathcal{R}} \) be a consistent family of \( \sigma \)-finite measures. Then there exists a measure \( \mu_\mathcal{R} \) on \( (E, S) \) such that
(i) \( \mu_\mathcal{R}(B) = \mu_B(B) \) for every \( B \in \mathcal{R} \);
(ii) if there exists a non-countable family of pairwise disjoint sets \( \{B_i : i \in I\} \subseteq \mathcal{R} \) such that \( 0 < \mu_{B_i}(B_i) < \infty \), then the measure \( \mu_\mathcal{R} \) is non-\( \sigma \)-finite;
(iii) if \( G \) is a group of measurable transformations of \( E \) such that \( G(\mathcal{R}) = \mathcal{R} \) and
\[
(\forall B)(\forall X)(\forall g)((B \in \mathcal{R} \& X \in S \cap \mathcal{P}(B) \& g \in G) \to \mu_{g(B)}(g(X)) = \mu_B(X)),
\]
where \( \mathcal{P}(B) \) denotes a power set of the set \( B \), then the measure \( \mu_\mathcal{R} \) is \( G \)-invariant.

Lemma 2.2. ([1], Lemma 2, p. 216) Let \( \alpha = (n_i)_{i \in N} \in (N \setminus \{0\})^N \). We set \( \mathcal{R} = (\alpha)\mathcal{OR} \). Suppose that \( R = \prod_{i \in N} R_i \in \mathcal{R} \) for which \( R_i \in \mathcal{B}(\mathbb{R}^{n_i}) \) for \( i \in N \).
For \( X \in \mathcal{B}(\mathbb{R}) \), we set \( \mu_R(X) = 0 \) if

\[
(\text{O}) \prod_{i \in N} m^{n_i}(R_i) = 0,
\]

and

\[
\mu_R(X) = (\text{O}) \prod_{i \in N} m^{n_i}(R_i) \times \left( \prod_{i \in N} \frac{m^{n_i}(R_i)}{m^{n_i}(R_i)} \right)(X)
\]

otherwise, where \( \frac{m^{n_i}(R_i)}{m^{n_i}(R_i)} \) is a Borel probability measure defined on \( R_i \) as follows

\[
(\forall X) \left( X \in \mathcal{B}(R_i) \rightarrow \frac{m^{n_i}(R_i)}{m^{n_i}(R_i)}(X) = \frac{m^{n_i}(Y \cap R_i)}{m^{n_i}(R_i)} \right).
\]

Then the family of measures \( (\mu_R)_{R \in \mathbb{R}} \) is consistent.

**Lemma 2.3.** ([1], Lemma 2. p. 217) Let \( \alpha = (n_i)_{i \in N} \in (N \setminus \{0\})^N \). We set \( \mathcal{R} = (\alpha)S\mathcal{R} \). Suppose that \( R = \prod_{i \in N} R_i \in \mathcal{R} \) for which \( R_i \in \mathcal{B}(\mathbb{R}^{n_i}) \) for \( i \in N \) and \( R \in (\alpha)S\mathcal{R} \).

For \( X \in \mathcal{B}(\mathbb{R}) \), we set \( \mu_R(X) = 0 \) if

\[
(\text{S}) \prod_{i \in N} m^{n_i}(R_i) = 0,
\]

and

\[
\mu_R(X) = (\text{S}) \prod_{i \in N} m^{n_i}(R_i) \times \left( \prod_{i \in N} \frac{m^{n_i}(R_i)}{m^{n_i}(R_i)} \right)(X)
\]

otherwise, where \( \frac{m^{n_i}(R_i)}{m^{n_i}(R_i)} \) is a Borel probability measure defined in \( R_i \) as follows

\[
(\forall X) \left( X \in \mathcal{B}(R_i) \rightarrow \frac{m^{n_i}(R_i)}{m^{n_i}(R_i)}(X) = \frac{m^{n_i}(Y \cap R_i)}{m^{n_i}(R_i)} \right).
\]

Then the family of measures \( (\mu_R)_{R \in \mathbb{R}} \) is consistent.

Next two theorems are corollaries of Lemmas 2.12–2.13.

**Theorem 2.1.** ([1], Theorem 1. p. 217) For every \( \alpha = (n_i)_{i \in N} \in (N \setminus \{0\})^N \), there exists a Borel measure \( \mu_{\alpha} \) in \( \mathbb{R}^\infty \) which is \( O(\alpha)LM \).

**Theorem 2.2.** ([1], Theorem 1. p. 218) For every \( \alpha = (n_i)_{i \in N} \in (N \setminus \{0\})^N \), there exists a Borel measure \( \nu_{\alpha} \) in \( \mathbb{R}^\infty \) which is \( S(\alpha)LM \).

Let \( \mu_1 \) and \( \mu_2 \) be two measures defined on the measurable space \((E, \mathcal{S})\).

**Definition 2.8** ([6], p. 124). We say that the \( \mu_1 \) is absolutely continuous with respect to the \( \mu_2 \), in symbols \( \mu_1 \ll \mu_2 \), if

\[
(\forall X) (X \in \mathcal{S} \& \mu_2(X) = 0 \rightarrow \mu_1(X) = 0).
\]
Definition 2.9 ([6], p. 126). Two measures \( \mu_1 \) and \( \mu_2 \) for which both \( \mu_1 \ll \mu_2 \) and \( \mu_2 \ll \mu_1 \) are called equivalent, in symbols \( \mu_1 \equiv \mu_2 \).

We have the following assertion.

Theorem 2.3. ([1], Theorem 5, p. 217) For every \( \alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N \), we have \( \nu_\alpha \ll \mu_\alpha \) and the measures \( \nu_\alpha \) and \( \mu_\alpha \) are not equivalent.

Remark 2.1. Note that the \( \mu_\alpha \) coincides with Baker’s measure [5] for \( \alpha = (1, 1, \ldots) \). By Lemmas 2.12 and 2.13 we can get the construction of Baker’s measure [4]. In this direction we must consider a class \( \mathcal{R}_B \) of all measurable rectangles \( \prod_{i=1}^\infty (a_i, b_i), -\infty < a_i < b_i < +\infty \) for which \( 0 \leq (O) \prod_{i \in \mathbb{N}} (b_i - a_i) < +\infty \). Since \( \mathcal{R}_B \) is translation-invariant and the family of measures \( (\mu_R)_{R \in \mathcal{R}_B} \) is consistent as a subfamily of the consistent family of measures constructed in Lemma 2.12, we claim that Baker’s measure [4] coincides with the measure \( \lambda_{\mathcal{R}_B} \).

Definition 2.10. Let \( \alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N \) such that \( n_i = n_j \) for every \( i, j \in \mathbb{N} \). We set \( f_i = (a_{1}^{(i)}, \ldots, a_{n_0}^{(i)}) \) for every \( i \in \mathbb{N} \) (see, notations introduced before Definition 2.5). Let \( f \) be any permutation of \( N \) such that for every \( i \in \mathbb{N} \) there exists \( j \in N \) such that \( f(a_k^{(i)}) = a_k^{(j)} \) for \( 1 \leq k \leq n_0 \). Then a map \( A_f : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \) defined by \( A_f((z_k)_{k \in \mathbb{N}}) = (z_{f(k)})_{k \in \mathbb{N}} \) for \( (z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty \), is called a canonical \( \alpha \)-permutations of \( \mathbb{R}^\infty \).

A group of transformations generated by all \( \alpha \)-permutations and shifts of \( \mathbb{R}^\infty \), is denoted by \( \mathcal{G}_\alpha \).

Corollary 2.1. For every \( \alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N \) for which \( n_i = n_j (i, j \in \mathbb{N}) \), the measure \( \nu_\alpha \) is \( \mathcal{G}_\alpha \)-invariant.

One can easily get the validity of the following propositions.

Proposition 2.2. ([1], Proposition 2, p. 219) For every \( \alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N \) there exists \( \beta \in (\mathbb{N} \setminus \{0\})^N \) such that \( \mu_\alpha \) and \( \mu_\beta \) are different.

Proposition 2.3. ([1], Proposition 2, p. 220) For every \( \alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N \) there exists \( \beta \in (\mathbb{N} \setminus \{0\})^N \) such that \( \nu_\alpha \) and \( \nu_\beta \) are different.

3. On uniformly distributed sequences of increasing family of finite sets in infinite-dimensional rectangles

Let \( s_1, s_2, s_3, \ldots \) be a uniformly distributed in an interval \( [a, b] \) (see, for example [7]. Setting \( Y_n = \{s_1, s_2, s_3, \ldots, s_n\} \) for \( n \in \mathbb{N} \), the \( (Y_n)_{n \in \mathbb{N}} \) will be such an increasing sequence of finite subsets of the \( [a, b] \) that, for any subinterval \( [c, d] \) of the \( [a, b] \), the following equality

\[
\lim_{n \to \infty} \frac{\#(Y_n \cap [c, d])}{\#(Y_n)} = \frac{d - c}{b - a}
\]

will be valid.

This remark raises the following
Definition 3.1. An increasing sequence \((Y_n)_{n \in N}\) of finite subsets of the \([a, b]\) is said to be equidistributed or uniformly distributed in an interval \([a, b]\) if, for any subinterval \([c, d]\) of \([a, b]\), we have

\[
\lim_{n \to \infty} \frac{\#(Y_n \cap [c, d])}{\#(Y_n)} = \frac{d-c}{b-a}
\]

Definition 3.2. Let \(\prod_{k \in N}[a_k, b_k] \in \mathcal{R}\). A set \(U\) is called an elementary rectangle in the \(\prod_{k \in N}[a_k, b_k]\) if it admits the following representation

\[
U = \prod_{k=1}^{m} [c_k, d_k] \times \prod_{k \in N \setminus \{1, \ldots, m\}} [a_k, b_k],
\]

where \(a_k \leq c_k < d_k \leq b_k\) for \(1 \leq k \leq m\).

It is obvious that

\[
\lambda(U) = \prod_{k=1}^{m} (d_k - c_k) \times \prod_{k=m+1}^{\infty} (b_k - a_k),
\]

for the elementary rectangle \(U\).

Definition 3.3. An increasing sequence \((Y_n)_{n \in N}\) of finite subsets of the infinite-dimensional rectangle \(\prod_{k \in N}[a_k, b_k] \in \mathcal{R}\) is said to be uniformly distributed in the \(\prod_{k \in N}[a_k, b_k]\) if for every elementary rectangle \(U\) in the \(\prod_{k \in N}[a_k, b_k]\) we have

\[
\lim_{n \to \infty} \frac{\#(Y_n \cap U)}{\#(Y_n)} = \frac{\lambda(U)}{\lambda(\prod_{k \in N}[a_k, b_k])}.
\]

Theorem 3.1. ([2], Theorem 3.1, p. 328) Let \(\prod_{k \in N}[a_k, b_k] \in \mathcal{R}\). Let \((x^{(k)}_n)_{n \in N}\) be uniformly distributed in the interval \([a_k, b_k]\) for \(k \in N\). We set

\[
Y_n = \prod_{k=1}^{n} \left(\cup_{j=1}^{n} x^{(k)}_j \right) \times \prod_{k \in N \setminus \{1, \ldots, n\}} \{x^{(k)}_0\}.
\]

Then \((Y_n)_{n \in N}\) is uniformly distributed in the rectangle \(\prod_{k \in N}[a_k, b_k]\).

Definition 3.4. Let \(\prod_{k \in N}[a_k, b_k] \in \mathcal{R}\). A family of pairwise disjoint elementary rectangles \(\tau = (U_k)_{1 \leq k \leq n}\) of the \(\prod_{k \in N}[a_k, b_k]\) is called Riemann partition of the \(\prod_{k \in N}[a_k, b_k]\) if \(\cup_{1 \leq k \leq n} U_k = \prod_{k \in N}[a_k, b_k]\).

Definition 3.5. Let \(\tau = (U_k)_{1 \leq k \leq n}\) be Riemann partition of the \(\prod_{k \in N}[a_k, b_k]\). Let \(\ell(Pr_i(U_k))\) be a length of the \(i\)-th projection \(Pr_i(U_k)\) of the \(U_k\) for \(i \in N\). We set

\[
d(U_k) = \sum_{i \in N} \frac{\ell(Pr_i(U_k))}{2^i(1 + \ell(Pr_i(U_k)))}.
\]

It is obvious that \(d(U_k)\) is a diameter of the elementary rectangle \(U_k\) for \(k \in N\) with respect to Tikhonov metric \(\rho\) defined as follows

\[
\rho((x_k)_{k \in N}, (y_k)_{k \in N}) = \sum_{k \in N} \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}
\]
for \((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\infty}\).

A number \(d(\tau)\), defined by

\[
d(\tau) = \max\{d(U_k) : 1 \leq k \leq n\}
\]
is called mesh or norm of the Riemann partition \(\tau\).

**Definition 3.6.** Let \(\tau_1 = (U^{(1)}_k)_{1 \leq k \leq n}\) and \(\tau_2 = (U^{(2)}_j)_{1 \leq j \leq m}\) be Riemann partitions of the \(\prod_{k \in \mathbb{N}} [a_k, b_k]\). We say that \(\tau_2 \leq \tau_1\) if

\[
(\forall j)(1 \leq j \leq m) \to (\exists i_0)(1 \leq i_0 \leq n \text{ and } U^{(2)}_j \subseteq U^{(1)}_{i_0}).
\]

**Definition 3.7.** Let \(f\) be a real-valued bounded function defined on the \(\prod_{i \in \mathbb{N}} [a_i, b_i]\). Let \(\tau = (U_k)_{1 \leq k \leq n}\) be a Riemann partition of the \(\prod_{i \in \mathbb{N}} [a_i, b_i]\) and \((t_k)_{1 \leq k \leq n}\) be a sample such that, for each \(k, t_k \in U_k\). Then

(i) a sum \(\sum_{k=1}^n f(t_k)\lambda(U_k)\) is called Riemann sum of the \(f\) with respect to Riemann partition \(\tau = (U_k)_{1 \leq k \leq n}\) together with sample \((t_k)_{1 \leq k \leq n}\);

(ii) a sum \(S_\tau = \sum_{k=1}^n m_k\lambda(U_k)\) is called the upper Darboux sum with respect to Riemann partition \(\tau\), where \(m_k = \sup_{x \in U_k} f(x)(1 \leq k \leq n)\);

(ii) a sum \(s_\tau = \sum_{k=1}^n m_k\lambda(U_k)\) is called the lower Darboux sum with respect to Riemann partition \(\tau\), where \(m_k = \inf_{x \in U_k} f(x)(1 \leq k \leq n)\).

**Definition 3.8.** Let \(f\) be a real-valued bounded function defined on \(\prod_{i \in \mathbb{N}} [a_i, b_i]\). We say that the \(f\) is Riemann-integrable on \(\prod_{i \in \mathbb{N}} [a_i, b_i]\) if there exists a real number \(s\) such that for every positive real number \(\epsilon\) there exists a real number \(\delta > 0\) such that, for every Riemann partition \((U_k)_{1 \leq k \leq n}\) of the \(\prod_{i \in \mathbb{N}} [a_i, b_i]\) with \(d(\tau) < \delta\) and for every sample \((t_k)_{1 \leq k \leq n}\), we have

\[
\left| \sum_{k=1}^n f(t_k)\lambda(U_k) - s \right| < \epsilon.
\]

The number \(s\) is called Riemann integral and is denoted by

\[
(R) \int_{\prod_{i \in \mathbb{N}} [a_i, b_i]} f(x) \, d\lambda(x).
\]

**Definition 3.9.** A function \(f\) is called a step function on \(\prod_{i \in \mathbb{N}} [a_i, b_i]\) if it can be written as

\[
f(x) = \sum_{k=1}^n c_k \chi_{U_k}(x),
\]

where \(\tau = (U_k)_{1 \leq k \leq n}\) is any Riemann partition of the \(\prod_{i \in \mathbb{N}} [a_i, b_i]\), \(c_k \in \mathbb{R}\) for \(1 \leq k \leq n\) and \(\chi_A\) is the indicator function of the \(A\).

**Theorem 3.2.** ([3], Theorem 3.2, p.331) Let \(f\) be a continuous function on \(\prod_{i \in \mathbb{N}} [a_i, b_i]\) with respect to Tikhonov metric \(\rho\). Then the \(f\) is Riemann-integrable on \(\prod_{i \in \mathbb{N}} [a_i, b_i]\).

Let denote by \(\mathcal{C}(\prod_{i \in \mathbb{N}} [a_i, b_i])\) a class of all continuous (with respect to Tikhonov topology) real-valued functions on \(\prod_{i \in \mathbb{N}} [a_i, b_i]\).
Theorem 3.3. ([2], Theorem 3.4, p. 336) For \( \prod_{i \in \mathbb{N}} [a_i, b_i] \in \mathcal{R} \), let \((Y_n)_{n \in \mathbb{N}}\) be an increasing family of its finite subsets. Then \((Y_n)_{n \in \mathbb{N}}\) is uniformly distributed in the \( \prod_{k \in \mathbb{N}} [a_k, b_k] \) if and only if for every \( f \in C(\prod_{k \in \mathbb{N}} [a_k, b_k]) \) the following equality
\[
\lim_{n \to \infty} \frac{\sum_{y \in Y_n} f(y)}{\#(Y_n)} = \frac{\int_{\prod_{i \in \mathbb{N}} [a_i, b_i]} f(x) d\lambda(x)}{\lambda(\prod_{i \in \mathbb{N}} [a_i, b_i])}
\]
holds.

4. Change of variable formula for the \( \alpha \)-ordinary Lebesgue measure in \( R^N \)

Let \( R^n (n > 1) \) be an \( n \)-dimensional Euclidean space and let \( \mu_n \) an \( n \)-dimensional standard Lebesgue measure on \( R^n \). Further, let \( T \) be a linear \( \mu_n \)-measurable transformation of \( R^n \).

It is obvious that \( \mu_n T^{-1} \) is absolutely continuous with respect to \( \mu_n \), and there exists a non-negative \( \mu_n \)-measurable function \( \Phi \) on \( R^n \) such that
\[
\mu_n(T^{-1}(X)) = \int_X \Phi(y) d\mu_n(y)
\]
for every \( \mu_n \)-measurable subset \( X \) of \( R^n \).

The function \( \Phi \) plays the role of the Jacobian \( J(T^{-1}) \) of the transformation \( T^{-1} \)(or, rather the absolute value of the Jacobian)(see, e.g., [6]) in the theory of transformations of multiple integrals. It is clear that \( J(T^{-1}) \) coincides with a Radon-Nikodym derivative \( \frac{d\mu_n T^{-1}}{d\mu_n} \), which is unique a.e. with respect to \( \mu_n \).

It is clear that
\[
\frac{d\mu_n T^{-1}}{d\mu_n}(x) = \lim_{k \to \infty} \frac{\mu_n(T^{-1}(U_k(x)))}{\mu_n(U_k(x))}(\mu_n - \text{a.e.}),
\]
where \( U_k(x) \) is a spherical neighborhood with the center in \( x \in R^n \) and radius \( r_k > 0 \) so that \( \lim_{k \to \infty} r_k = 0 \). The class of such spherical neighborhoods generate so-called Vitali differentiability class of subsets which allows us to calculate the Jacobian \( J(T^{-1}) \) of the transformation \( T^{-1} \).

If we consider a vector space of all real-valued sequences \( R^N \)(equipped with Tychonoff topology), then we observe that for the infinite-dimensional Lebesgue measure \( [4] \) (or \([5]\)) defined in \( R^N \) there does not exist any Vitali system of differentiability, but in spite of non-existence of such a system the inner structure of this measure allows us to define a form of the Radon-Nikodym derivative defined by any linear transformation of \( R^N \). In order to show it, let consider the following

Example 4.1. Let \( \mathcal{R}_1 \) be the class of all infinite dimensional rectangles \( R \in B( R^N ) \) of the form
\[
R = \prod_{i=1}^{\infty} R_i, \quad R_i = [a_i, b_i], \quad -\infty < a_i \leq b_i < +\infty,
\]
such that
\[ 0 \leq \prod_{i=1}^{\infty} (b_i - a_i) < \infty. \]

Let \( \tau_1 \) be the set function on \( \mathcal{R}_1 \) defined by
\[ \tau_1(R) = \prod_{i=1}^{\infty} (b_i - a_i). \]

R. Baker [4] proved that the functional \( \lambda_1 \) defined by
\[ \langle \forall X \rangle \left( X \in \mathcal{B}(R^N) \rightarrow \lambda_1(X) = \inf \left\{ \sum_{j=1}^{\infty} \tau_j(R_j) : R_j \in \mathcal{R}_1 \& X \subseteq \bigcup_{j=1}^{\infty} R_j \right\} \right) \]
is a quasi-finite translation-invariant Borel measure in \( R^N \).

The following change of variable formula has been established in [4] (cf. p. 1029): Let \( T^n : R^n \rightarrow R^n, n > 1, \) be a linear transformation with Jacobian \( \Delta \neq 0, \) and let \( T^N : R^N \rightarrow R^N \) be the map defined by
\[ T^N(x) = (T^n(x_1, \ldots, x_n), x_{n+1}, x_{n+2}, \ldots), \quad x = (x_i)_{i \in N} \in R^N. \]

Then for each \( E \in \mathcal{B}(R^N), \) we have
\[ \lambda_1(T^N(E)) = |\Delta|\lambda_1(E). \]

**Theorem 4.1.** Let \( \alpha = (n_i)_{i \in N} \) be the sequence of non-zero natural numbers and \( \mu_\alpha \) is \( O(\alpha)LM. \) Further, let \( T^{n_i} : R^{n_i} \rightarrow R^{n_i}, i \geq 1, \) be a family of linear transformation with Jacobians \( \Delta_i \neq 0 \) and \( 0 < \prod_{i=1}^{\infty} \Delta_i < \infty. \) Let \( T^N : R^N \rightarrow R^N \) be the map defined by
\[ T^N(x) = (T^{n_1}(x_1, \ldots, x_{n_1}), T^{n_2}(x_{n_1+1}, \ldots, x_{n_1+n_2}), \ldots), \]
where \( x = (x_i)_{i \in N} \in R^N. \) Then for each \( E \in \mathcal{B}(R^N), \) we have
\[ \mu_\alpha(T^N(E)) = \left( \prod_{i=1}^{\infty} \Delta_i \right) \mu_\alpha(E). \]

**Remark 4.1.** Theorem 4.5 is change of variable formula for the \( \alpha \)-ordinary Lebesgue measure. It extends change of variable formula for Baker’s measure considered in Example 4.1. Indeed, let \( T^n : R^n \rightarrow R^n, n > 1, \) be a linear transformation with Jacobian \( \Delta \neq 0. \) Let \( n_1 = n \) and \( n_i = 1 \) for \( i > 1, \) that is \( \alpha = (n, 1, 1, \ldots). \) Further, we set \( T^{n_1} = T^n \) and \( T^{n_k} = I, \) where \( I : R \rightarrow R \) is an identity transformation of \( R \) defined by \( I(x) = x \) for \( x \in R. \)

Let a map \( T^N : R^N \rightarrow R^N \) be defined by
\[ T^N(x) = (T^n(x_1, \ldots, x_n), x_{n+1}, x_{n+2}, \ldots), \quad x = (x_i)_{i \in N} \in R^N. \]

Then, by Theorem 4.5, for \( T^N \) and for each \( E \in \mathcal{B}(R^N), \) we have
\[ \lambda_1(T^N(E)) = \mu_\alpha(T^N(E)) = |\Delta|\mu_\alpha(E) = |\Delta|\lambda_1(E). \]
5. Concept of the Dirac delta function in $\mathcal{C}(\mathbb{R}^\infty)$

**Lemma 5.1.** *(Intermediate value theorem)* Let $f$ be a continuous function on $\prod_{k \in \mathbb{N}}[a_k, b_k]$. Suppose that $\max(f(x) : x \in \prod_{k \in \mathbb{N}}[a_k, b_k]) = M$ and $\min(f(x) : x \in \prod_{k \in \mathbb{N}}[a_k, b_k]) = m$. Let $u \in [m, M]$. Then there is $c \in \prod_{k \in \mathbb{N}}[a_k, b_k]$ such that $f(c) = u$.

**Proof.** Let $(y_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}}[a_k, b_k]$ be such sequence that $f((y_k)_{k \in \mathbb{N}}) = M$. Let $Z^* = (z_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}}[a_k, b_k]$ be such a sequence that $f((z_k)_{k \in \mathbb{N}}) = m$.

Let consider a function $g(t) = f((z_k)_{k \in \mathbb{N}} + t((y_k)_{k \in \mathbb{N}} - (z_k)_{k \in \mathbb{N}}))$ on $[0, 1]$. This function is well defined because $(z_k)_{k \in \mathbb{N}} + t((y_k)_{k \in \mathbb{N}} - (z_k)_{k \in \mathbb{N}})$ is in $\prod_{k \in \mathbb{N}}[a_k, b_k]$ for each $t \in [0, 1]$. It is obvious that $g_{\max} = g(1) = M$ and $g_{\min} = g(0) = m$. Using Intermediate value theorem for a real valued function $g$ on $[0, 1]$ there is $t_0 \in [0, 1]$ such that $g(t_0) = u$. Setting $c = (z_k)_{k \in \mathbb{N}} + t_0((y_k)_{k \in \mathbb{N}} - (z_k)_{k \in \mathbb{N}})$, we end the proof of the lemma.

Let $\lambda$ be Baker measure in $\mathbb{R}^\infty$. For $\epsilon > 0$, we set

$$a_k(\epsilon) = e^{-\frac{1}{2\epsilon}}$$

and

$$\Delta_\epsilon = \prod_{k=1}^\infty [-a_k(\epsilon), a_k(\epsilon)].$$

Note that the diameter of the set $\Delta_\epsilon$ is calculated by

$$\text{diam}(\Delta_\epsilon) = \sum_{i \in \mathbb{N}} \frac{|2a_k(\epsilon)|}{2^i(1 + |2a_k(\epsilon)|)} = \sum_{i \in \mathbb{N}} \frac{e^{-\frac{1}{2\epsilon}}}{2^i(1 + e^{-\frac{1}{2\epsilon}})}.$$ 

**Lemma 5.2.** $\lim_{\epsilon \to 0^+} \text{diam}(\Delta_\epsilon) = 0$.

**Proof.**

For $\sigma > 0$ there is $n_\sigma \in \mathbb{N}$ such that

$$\sum_{i=n_\sigma}^\infty 2^{-i} < \frac{\sigma}{2}.$$ 

Since $\lim_{\epsilon \to 0^+} \frac{e^{-\frac{1}{2\epsilon}}}{2^i(1 + e^{-\frac{1}{2\epsilon}})} = 0$ for each $k \in \mathbb{N}$, we deduce that

$$\lim_{\epsilon \to 0^+} \sum_{i=1}^{n_\sigma} \frac{e^{-\frac{1}{2\epsilon}}}{2^i(1 + e^{-\frac{1}{2\epsilon}})} = 0.$$ 

The latter relation means that there is $\rho_\sigma > 0$ such that

$$\sum_{i=1}^{n_\sigma} \frac{e^{-\frac{1}{2\epsilon}}}{2^i(1 + e^{-\frac{1}{2\epsilon}})} < \frac{\sigma}{2}.$$
for all $\epsilon$ with $0 < \epsilon < \rho_\sigma$.

Finally, for each $\sigma > 0$, $\rho_\sigma$ is such a positive number that

$$\text{diam}(\Delta_\epsilon) = \sum_{i \in \mathbb{N}} \frac{|2a_k(\epsilon)|}{2^i(1 + |2a_k(\epsilon)|)} = \sum_{i \in \mathbb{N}} \frac{e^{-\frac{1}{2\epsilon^2}}}{2^i(1 + e^{-\frac{1}{2\epsilon^2}})} \leq \sum_{i=1}^{n_\sigma} \frac{e^{-\frac{1}{2\epsilon^2}}}{2^i(1 + e^{-\frac{1}{2\epsilon^2}})} + \sum_{i=n_\sigma}^{\infty} 2^{-i} \leq \frac{\sigma}{2} + \frac{\sigma}{2} = \sigma$$

for each $\epsilon$ with $0 < \epsilon < \rho_\sigma$.

This ends the proof of the lemma.

For $y \in \mathbb{R}^\infty$ we set $\Delta_\epsilon(y) = \Delta_\epsilon + y$.

Since Tychonoff metric is translation invariant, by virtue of Lemma 5.2 we deduce that

$$\lim_{\epsilon \to O^+} \text{diam}(\Delta_\epsilon(y)) = 0.$$

Note also that if $\epsilon^{(i)} > 0$ for $i \in \mathbb{N}$ and $\lim_{i \to \infty} \epsilon^{(i)} = 0$ then the equality

$$\cap_{i \in \mathbb{N}} \Delta_{\epsilon^{(i)}}(y) = \{y\}$$

holds true for each $y \in \mathbb{R}^\infty$.

**Lemma 5.3.** Let $f$ be a continuous function on $\mathbb{R}^\infty$. Then the following formula

$$\lim_{\epsilon \to O^+} \frac{1}{\lambda(\Delta_\epsilon(y))} \int_{\Delta_\epsilon(y)} f(x) d\lambda(x) = f(y)$$

holds true for all $y \in \mathbb{R}^\infty$.

**Proof.** If consider the restriction of $f$ on $\Delta_\epsilon(y)$ is also continuous. Let denote by $M_\epsilon(y)$ and $m_\epsilon(y)$ maximum and minimum of the function of $f$ on $\Delta_\epsilon(y)$. Hence we have

$$m_\epsilon \times \lambda(\Delta_\epsilon(y)) \leq \int_{\Delta_\epsilon(y)} f(x) d\lambda(x) \leq M_\epsilon \times \lambda(\Delta_\epsilon(y))$$

for each $\epsilon > 0$. Equivalently, we have

$$m_\epsilon \leq \frac{1}{\lambda(\Delta_\epsilon(y))} \int_{\Delta_\epsilon(y)} f(x) d\lambda(x) \leq M_\epsilon$$

for each $\epsilon > 0$.

By Lemma 5.1, there is $y_\epsilon \in \Delta_\epsilon(y)$ such that

$$\frac{1}{\lambda(\Delta_\epsilon(y))} \int_{\Delta_\epsilon(y)} f(x) d\lambda(x) = f(y_\epsilon).$$

When one takes the limit when $\epsilon \to O^+$, then $y_\epsilon$ tends to $y$, and so

$$\lim_{\epsilon \to O^+} \frac{1}{\lambda(\Delta_\epsilon(y))} \int_{\Delta_\epsilon(y)} f(x) d\lambda(x) = \lim_{\epsilon \to O^+} f(y_\epsilon) = f(y).$$
We have
\[
\lambda(\Delta_\epsilon) = \prod_{k=1}^{\infty} (2a_k(\epsilon)) = e^{-\sum_{k=1}^{\infty} \frac{1}{2k^2}}.
\]

We set \( \eta_\epsilon(x) = e^{\sum_{k=1}^{\infty} \frac{1}{2k^2}} \) if \( x \in \Delta_\epsilon \) and \( \eta_\epsilon(x) = 0 \), otherwise. \( \eta_\epsilon(x) \) is called a nascent delta function.

The Dirac delta function \( \delta(x) \), formally is defined by
\[
\delta(x) = \lim_{\epsilon \to 0^+} \eta_\epsilon(x),
\]
which, of course, has no any reasonable sense.

Let \( f \) be a continuous real-valued function on \( \mathbb{R}^\infty \). We define a Dirac delta integral as follows
\[
(\delta) \int_{\mathbb{R}^\infty} \delta(x)f(x)d\lambda(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_\epsilon(x)f(x)d\lambda(x).
\]

We define a Dirac delta functional \( \delta : C(\mathbb{R}^\infty) \to \mathbb{R} \) by
\[
\delta(f) = (\delta) \int_{\mathbb{R}^\infty} \delta(x)f(x)d\lambda(x).
\]

The following assertion is valid.

**Theorem 5.1.** The Dirac delta functional \( \delta \) is a linear functional such that \( \delta(f) = f(0) \) for each \( f \in C(\mathbb{R}^\infty) \), where \( 0 \) denotes the zero of \( \mathbb{R}^\infty \).

**Proof.** We have
\[
\delta(f) = (\delta) \int_{\mathbb{R}^\infty} \delta(x)f(x)d\lambda(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_\epsilon(x)f(x)d\lambda(x) =
\]
\[
\lim_{\epsilon \to 0^+} \int_{\Delta_\epsilon} \left[ e^{\sum_{k=1}^{\infty} \frac{1}{2k^2}} \times \chi_{\Delta_\epsilon}(y) + 0 \times \chi_{\mathbb{R}^\infty \setminus \Delta_\epsilon}(y) \right] f(x)d\lambda(x) =
\]
\[
\lim_{\epsilon \to 0^+} \int_{\Delta_\epsilon} e^{\sum_{k=1}^{\infty} \frac{1}{2k^2}} f(x)d\lambda(x) = \lim_{\epsilon \to 0^+} \frac{1}{\lambda(\Delta_\epsilon)} \int_{\Delta_\epsilon} f(x)d\lambda(x).
\]

By Lemma 5.3 we know that
\[
\lim_{\epsilon \to 0^+} \frac{1}{\lambda(\Delta_\epsilon)} \int_{\Delta_\epsilon} f(x)d\lambda(x) = f(0).
\]

For \( \alpha, \beta \in \mathbb{R} \) and \( f, g \in C(\mathbb{R}^\infty) \), we have
\[
\delta(\alpha f + \beta g) = (\delta) \int_{\mathbb{R}^\infty} \delta(x)(\alpha f(x) + \beta g(x))d\lambda(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_\epsilon(x)(\alpha f(x) + \beta g(x))d\lambda(x) =
\]
\[
\alpha \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_\epsilon(x)f(x)d\lambda(x) + \beta \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_\epsilon(x)g(x)d\lambda(x) = \alpha f(0) + \beta g(0) = \alpha \delta(f) + \beta \delta(g).
\]
This ends the proof of the theorem.

Distributions are a class of linear functionals that map a set of all test functions (conventional and well-behaved functions) onto the set of real numbers. In the simplest case, the set of test functions considered is $D(R^\infty)$, which is the set of smooth (infinitely differentiable) functions $\varphi : R^\infty \to R$. Then, a distribution $d$ is a linear mapping $D(R^\infty) \to R$. Instead of writing $d(\varphi)$, where $\varphi$ is a test function in $D(R^\infty)$, it is conventional to write $\langle d, \varphi \rangle$.

A simple example of a distribution is the Dirac delta functional $\delta$, defined by

$$\delta(\varphi) = \langle \delta, \varphi \rangle = \varphi(0).$$

We have proved that Dirac delta functional $\delta$ is given by the Dirac delta integral as follows

$$\delta(\varphi) = (\delta) \int_{R^\infty} \delta(x) \varphi(x) d\lambda(x).$$

There are straightforward mappings from both locally integrable functions and probability distributions to corresponding distributions, as discussed below. However, not all distributions can be formed in this manner.

Suppose that $f : R^\infty \to R$ is a locally integrable function, and let $\varphi : R^\infty \to R$ be a test function in $D(R^\infty)$. We can then define a corresponding distribution $T_f$ by

$$\langle T_f, \varphi \rangle = \int_{R^\infty} f(x) \varphi(x) \lambda(x).$$

This integral is a real number which depends linearly and continuously on $f$. This suggests the requirement that a distribution should be a linear and continuous functional on the space of test functions $D(R^\infty)$, which completes the definition. In a conventional abuse of notation, $f$ may be used to represent both the original function $f$ and the distribution $T_f$ derived from it. Similarly, if $\mu$ is a Radon measure on $R^\infty$ and $f$ is a test function, then a corresponding distribution $T_\mu$ may be defined by

$$\langle T_\mu, \varphi \rangle = \int_{R^\infty} \varphi d\mu.$$

This integral depends continuously and linearly on $\varphi$, so that $T_\mu$ is a distribution. If $\mu$ is an absolutely continuous measure with respect to Baker measure $\lambda$ with density $f$, then this definition is the same as the one for $T_f$, but if $\mu$ is not absolutely continuous it gives a distribution that is not associated with a function. For example, if $P$ is the point-mass measure on $R^\infty$ that assigns $P$ measure one to the singleton set $0$ and measure zero to sets that do not contain zero, then

$$\int_{R^\infty} \varphi dP = (\delta) \int_{R^\infty} \delta(x) \varphi(x) d\lambda(x) = \varphi(0),$$

so $T_P = \delta$ is the Dirac delta functional.
It is well known that the $n$-dimensional Dirac delta function satisfies the following scaling property for a non-zero scalar $\alpha$:

$$(\delta) \int_{\mathbb{R}^n} \delta(\alpha x) \, dx = |\alpha|^{-n}$$

and so

$$\delta(\alpha x) = |\alpha|^{-n} \delta(x).$$

We have the direct generalization of that property in the case of infinite dimension.

**Theorem 5.2.** The infinite dimensional Dirac delta function satisfies the following scaling property for a non-zero scalar $\alpha$:

$$(\delta) \int_{\mathbb{R}^\infty} \delta(\alpha x) \, d\lambda(x) = |\alpha|^{-\infty}.$$  

**Proof.** We have

$$(\delta) \int_{\mathbb{R}^\infty} \delta(\alpha x) \, d\lambda(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_\epsilon(\alpha x) \, d\lambda(x) =$$

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \left[ \epsilon \sum_{k=1}^{\infty} \frac{1}{2^k} \times \chi_{\Delta_\epsilon}(\alpha x) + 0 \times \chi_{\mathbb{R}^\infty \setminus \Delta_\epsilon}(\alpha x) \right] \, d\lambda(x) =$$

$$\lim_{\epsilon \to 0^+} \left[ \epsilon \sum_{k=1}^{\infty} \frac{1}{2^k} \Delta_\epsilon e^{\frac{1}{\epsilon}} \right] \, d\lambda(x) =$$

$$\lim_{\epsilon \to 0^+} \frac{1}{\lambda(\Delta_\epsilon)} \int_{\frac{1}{\epsilon} \Delta_\epsilon} \, d\lambda(x) = \lim_{\epsilon \to 0^+} \frac{\lambda(\frac{1}{\epsilon} \Delta_\epsilon)}{\lambda(\Delta_\epsilon)}.$$  

Notice that $\lambda(\frac{1}{\epsilon} \Delta_\epsilon) = 0$ if $|\alpha| > 1$, $= e^{\sum_{k=1}^{\infty} \frac{1}{2^k}}$ if $|\alpha| = 1$ and $= +\infty$ if $|\alpha| < 1$.

Hence, the latter equality can be rewriten as follows

$$(\delta) \int_{\mathbb{R}^\infty} \delta(\alpha x) \, d\lambda(x) = |\alpha|^{-\infty}.$$  

This ends the proof of the theorem.

**Theorem 5.3.** The infinite dimensional Dirac delta function is an even distribution, in the sense that

$$(\delta) \int_{\mathbb{R}^\infty} \delta(-x) f(x) \, d\lambda(x) = (\delta) \int_{\mathbb{R}^\infty} \delta(x) f(x) \, d\lambda(x)$$

for $f \in C(\mathbb{R}^\infty)$, which is homogeneous of degree $-1$.

The validity of Theorem 5.6 follows from the fact asserted that $-\Delta_\epsilon = \Delta_\epsilon$ for $\epsilon > 0$ and the invariance of $\lambda$ with respect to a transformation $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $T(x) = -x$. 
Theorem 5.4. (sifting property) The following equality

\[
(\delta) \int_{\mathbb{R}^\infty} \delta(x - T) f(x) d\lambda(x) = f(T)
\]

holds for \( f \in C(\mathbb{R}^\infty) \).

Proof. By virtue of Lemma 5.3, we have

\[
(\delta) \int_{\mathbb{R}^\infty} \delta(x - T) f(x) d\lambda(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \eta_{\epsilon}(x - T) f(x) d\lambda(x) = \\
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \left[ e^{\sum_{k=1}^{\infty} \frac{1}{2^k} \times \chi_{\Delta_{\epsilon}}(x - T)} + 0 \times \chi_{\mathbb{R}^\infty \setminus \Delta_{\epsilon}}(x - T) \right] f(x) d\lambda(x) = \\
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} \left[ e^{\sum_{k=1}^{\infty} \frac{1}{2^k} \times \chi_{\Delta_{\epsilon} + T}(x)} + 0 \times \chi_{\mathbb{R}^\infty \setminus (\Delta_{\epsilon} + T)}(x) \right] f(x) d\lambda(x) = \\
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^\infty} e^{\sum_{k=1}^{\infty} \frac{1}{2^k} \times \chi_{\Delta_{\epsilon} + T}} f(x) d\lambda(x) = \lim_{\epsilon \to 0^+} e^{\sum_{k=1}^{\infty} \frac{1}{2^k}} \int_{\Delta_{\epsilon} + T} f(x) d\lambda(x) = \\
\lim_{\epsilon \to 0^+} \frac{\int_{\Delta_{\epsilon} + T} f(x) d\lambda(x)}{\lambda(\Delta_{\epsilon} + T)} = f(T).
\]

Theorem 5.5. For \( \epsilon > 0 \), let \((Y_n(\epsilon))_{n \in \mathbb{N}} \) be an increasing family of finite subsets of \( \Delta_\epsilon \) which is uniformly distributed in the \( \Delta_\epsilon \). Let \( f \in C(\mathbb{R}^\infty) \). Then the following formula

\[
\lim_{\epsilon \to 0^+} \lim_{n \to \infty} \frac{\sum_{y \in Y_n(\epsilon)} f(y)}{\#(Y_n(\epsilon))} = f(0)
\]

holds true.

Proof. By Theorem 3.12 we have

\[
\lim_{n \to \infty} \frac{\sum_{y \in Y_n(\epsilon)} f(y)}{\#(Y_n(\epsilon))} = \frac{\int_{\Delta_\epsilon} f(x) d\lambda(x)}{\lambda(\Delta_\epsilon)}.
\]

By Lemma 5.3 we get

\[
\lim_{\epsilon \to 0^+} \frac{\int_{\Delta_\epsilon} f(x) d\lambda(x)}{\lambda(\Delta_\epsilon)} = f(0),
\]

which implies that

\[
\lim_{\epsilon \to 0^+} \lim_{n \to \infty} \frac{\sum_{y \in Y_n(\epsilon)} f(y)}{\#(Y_n(\epsilon))} = f(0).
\]

This ends the proof of the theorem.
Corollary 5.1. For $\epsilon > 0$, let $(Y_n(\epsilon))_{n \in \mathbb{N}}$ be an increasing family of finite subsets of $\Delta$, which is uniformly distributed in the $\Delta$. Let $\delta$ be Dirac delta functional defined in $C(\mathbb{R}^\infty)$. Then the following equality

$$
\delta(f) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\sum_{y \in Y_n(\epsilon)} f(y)}{\#(Y_n(\epsilon))}
$$

holds true for each $f \in C(\mathbb{R}^\infty)$.

Acknowledgment. The representation of the Dirac delta function in terms of the Baker measure can be extended also in terms of an arbitrary ordinary or standard infinite-dimensional Lebesgue measure in $\mathbb{R}^\infty$.

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