Imitation of 2d quantum field theory by means of REM like models.

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Abstract

An imitation of 2d field theory is formulated by means of a model on the hierarchic tree (with branching number close to one) with the same potential and the free correlators identical to those of 2d ones. Such a model possesses some features of original models for certain scale invariant theories. For the case of 2d conformal models it is possible to derive exact results. The renormalization group equation for the free energy is a reaction-diffusion equation, which is noise-free KPZ equation with an additional linear term. For the case of Liouville model and strings these models on trees may be naturally expressed via the Random Energy Model. This correspondence is used to identify the phase structure of strings for analytical continuation of DDK expressions. A phase transition is found for spherical strings a bit below three dimensions.

1 Hierarchic tree with branching number close to 1.

One of the most fruitful ideas in physics is the idea of universality. In fact, that the only hope due to which rather artificial models of the present theoretical physics can successfully capture the relevant aspects of the nature. We believe, that at the critical point the statistical mechanical system omits the secondary details. Usually this concerns the Hamiltonian in the $d$ dimensional Euclidean space.

It is proposed while keeping the Hamiltonian fixed to simplify the space geometry as much as possible retaining two point correlators and three point (for isosceles triangles) correlators. If the action of initial theory consists of the Laplacian and a potential, our model feels the space dimension through
the behavior of Green function
\[ G(x, x') \sim \frac{1}{r(x, x')^{d-2}} \]  
\[ (1) \]

The total volume is
\[ (\frac{L}{a})^d \]  
\[ (2) \]

where \( L \) and \( a \) are infrared and ultraviolet cutoffs, \( r(x, x') \) is the distance. The Euclidean geometry has too many constructions. One can rotate a point around some center and circumscribe a close circle. Let us now consider some metric space with properties:

A. For every pair of points there is a distance \( r(x, x') \).
B. There is some measure at every point \( d\mu_s(x) \) with the total measure \( \int d\mu_s = R^d \).
C. One can construct a quadratic form with corresponding asymptotic (1) for the Green function.

We start out to construct statistical mechanical models on the simplest space, that supports points A-C. We hope, that due to the universality these models will acquire some properties of models in d-dimensional space. To realize this program we will use some ideas from the theory of Random Energy Model (REM) [1-5]. In ref. [5] a relation of 2d quantum Liouville model to REM and to the Directed Polymer (DP) on Cayley tree was established.

Our present analysis shows, that the connection with REM is not a peculiar property of Liouville model and works well also for other conformal models. Besides, using similar ideas we intend to construct general 2d quantum models in the ultrametric space and thereby generalize the above-mentioned connection between the quantum field theoretical models and those defined on the hierarchical lattices. First, an ultrametric space (surface) with fields, located on the surface will be constructed. We define the following three geometrical objects: the distance between surface points, the surface measure and the volume measure for the ball, delimited by surface. These constructions are enough to define free field action with correlators that are identical to those in the corresponding two-dimensional space. One can add also the interaction term to the action. The practical merit of the proposed approach is that under certain conditions (for example, Coulomb gas approach to conformal theories) the theories on the ultrametric space can be solved much easier, as compared to the Euclidean 2d space.
Let us define the ultrametric (UM) space with some measure for the (spherical) area \( d\mu_s(V, X) \), volume (ball) measure \( d\mu_v(V, X) \), total surface \( e^V \) and total volume \( e^V - 1 \). It is commonly known from the mean field theory that the surface is of the same order of magnitude as the volume. We can construct this UM space as a limit of hierarchic lattices. Consider a tree with a constant number of branching \( q \) in each node and number \( N \) for hierarchies. The number of end points is \( q^N \), the number of branches - \((q^N - 1)/(q - 1)\).

We consider a set of end points as a surface of sphere, the set of branches making a volume (of the ball). Each point on the surface is connected with the origin (zero level of the hierarchy) via a single path, which consists of links. We determine a measure \( d\mu_s(V, X) = 1 \) for each end point, then \( d\mu_l(V, l) = (q - 1) \) for each link. This is mathematically correct in the limit \( N \to \infty \). Instead of integer \( q \) consider the limit \( q \to 1, N \ln q \to V_0 \). Now we have for the total area \( \mu_s = \int d\mu_s(V_0, X) = e^{V_0} \) and for the total volume \( \mu_v = \int d\mu_l(V_0, l) = e^{V_0} - 1 \). \( \mu_v \equiv \sum_l d\mu_l \), where the sum is over the links originated from the point \( X \). Later we will manipulate only with \( d\mu_l \). We choose \( q \to 1 \) to have an equation \( \mu_l + 1 = \mu_s \).

We determine UM distance between two points \( x, y \) on this surface \( V_0 - v \equiv V_0 - \frac{n}{N} \), where \( n \) is the number of the hierarchic level, on which \( x \) and \( y \) had the last common node on trajectories to their point from the origin. The maximum UM distance between two points \( v \) on the surface is \( V_0 \) (an ordinary distance as a function of \( V \) will be defined lately). Now a scalar field \( \phi(x) \) is defined on our surface. For determination of kinetic energy, which should be a quadratic form with the Laplacian as the kernel in the conventional space, let us consider the expansion

\[
\phi(x) = f_0 + \int_0^1 dv f(v, l)
\]  

(3)

Here \( f(v, l) \) is determined on the links. The integration in (4) is made along the trajectory of point \( X \). Since the measures on both the threads of (4) coincide (\( \int d\mu_l(V, l) = 1 + \int dV d\mu_l(V, l) \)), the Jacobian is equal to one. Now determine the kinematics part of the action for the field \( \phi(x) \)

\[
\frac{1}{2} \int_0^{V_0} dV d\mu_l(V, l) f(V, l)^2
\]

(4)

Then the partition under the potential \( U(\phi) \)

\[
\int df \exp\{- \int_0^{V_0} dV \int d\mu_l(V, l) \frac{1}{2} f(x, V)^2\}
\]
\[
\exp\{\int d\mu_s(V_0, X)U(\phi(X))\} \tag{5}
\]

We have for the correlator

\[
<\phi(X)\phi(X')> = v
\tag{6}
\]

where \(v\) is the UM distance between the points \(X, X'\). For usual 2d models with

\[
\int d\phi_0 d\phi \exp\{-\frac{1}{8\pi}dx^2\nabla\phi(x)^2\} \exp\{\int dxU(\phi(x))\} \tag{7}
\]

the total surface area is equal to \(R^2\), and the correlators read as

\[
<\phi(X)\phi(X')> = \ln \frac{L^2}{r^2} \tag{8}
\]

In Eq. (7) it is possible to take n-component fields instead of the one-component one \(\phi(x)\). We can determine the distance from the equality \(V = \ln r^2\). Then our correlators coincide (at any rate \(r \gg 1\)). It is possible to construct a quantum field theory in this case. Our constructions for the measure and distance are sufficient. One should bear in mind only, that the volume measure inside the sphere \(V\) is \(e^V - 1\).

We are going to discretize (5), then derive the iteration equations for imitations of 2d and 3d cases. For the case of spin models on hierarchic trees it is well known that it is possible to write simple iteration equations (similar to those in Ref. [6]) for any value of \(q\). For the case (5) we formulated iteration equations in Section 2 for 2-d imitation and in Section 3 for 3-d case. The resulting equation are similar to the KPZ equation [7], but there is an additional linear term. Instead of our abstract approach \((q = 1\) trees\) it is possible to consider a branching diffusion process like [4] and formulate 2-d models imitation on its basis. This approach is constructed in Section 4.

It is possible to derive the majority of results of 2-d conformal theory by means of Coulomb gas representation [8]. Here the free field action is modified by an imaginary linear term in the action. Correlators of fluctuating field in a critical theories with interaction are equivalent to correlators (from the exponentional of a such free field) in modified free field picture. In Section 5 we derived the three point correlators in our approach and the results agree with those in the 2-d case [8].
For the case of exponential function for $U(x)$ potential (2d Liouville model in the Euclidean space) there is a strict result [9]-[10] that the thermodynamics of our model on hierarchic tree is independent of $q$ and is similar to REM. In Section 6 a qualitative derivation for a REM with complex replica numbers (using the results of [11] for real number of replicas) is given and the solution of REM at complex temperatures [12]-[13] is obtained then those results are used to identify phase structure of strings, using analytical continuation of DDK formulas [14]-[16]. In Section 6 the main results of the work are discussed.

2 Iteration equations for the 2-d case.

The advantage of representation (5) is that we are in a position to calculate the partition function through iterations. This is well known for models on hierarchical lattices [6]. Let us for some large number $K$, divide $V$ into $K$ parts $V/K$ and determine a hierarchical tree with $K$ levels and branching number

$$q \equiv \exp\left\{\frac{V}{K}\right\}$$

(9)

Similar to [5] it is easy to define the partition $Z$ via iterations. In case of some large number $K$ we derive

$$I_1(x) = \sqrt{\frac{K}{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}Ky^2 + U(x+y)\} dy$$

$$I_{i+1}(x) = \sqrt{\frac{K}{2\pi}} \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}Ky^2\} [I_i(x+y)q] dy$$

$$Z = \lim_{K \to \infty} [I_K(0)]^q$$

(10)

As for the determination of partition function, we need only the equation (10). Our choice $q \to 1$ is a reasonable simplification. It is possible to construct perturbative field theory, calculate diagrams. To solve analytically equation (10) it is convenient to consider the other (opposite) case, $q \to \infty$. Only in this case the analytical solution proves possible. The point is, that for the bulk structure of theory the value of $q$ is irrelevant. For example, in asymptotic expansion of free energy

$$F(N) = F_0N + F_1 \ln N + F_2 \ldots$$

(11)
only the last term depends on the choice of \( q \) (in case of Directed Polymer \( N \sim V \)).

We found how to construct the simplified version of any 2d theory on a hierarchical lattice.

Let us consider carefully equation (10) in the limit of large \( K \) and introduce variable \( w(v, x) \equiv I_{\frac{V}{K}}(x) \). We consider the limit \( \frac{V}{K} \ll 1 \). For the differential \( dv \) we have an expression \( \frac{V}{K} \equiv dv \). Let us also assume

\[
q - 1 = \frac{V}{K} \equiv dv
\]

(12)

Using the expression \( x^q \approx x(1 + \log x(q - 1)) \) it is easy to obtain

\[
\frac{dw}{dv} = w \ln w + \frac{1}{2} \Delta w
\]

\[
w(0, x) = \exp(-U(x))
\]

(13)

After the replacement \( w = \exp(u(t, x)) \) we arrive at

\[
\frac{du}{dv} = \frac{1}{2} \Delta u + \frac{1}{2} (\nabla u)^2 + u
\]

\[
u(0, x) = U(x)
\]

(14)

where \( U(x) \) is the potential in Eq.(7). Having an expression for \( u(v, x) \), we obtain for the free energy

\[
\ln Z = u(V, 0)
\]

(15)

We have a noise-free KPZ equation (14) with additional linear term for the free energy.

There are two interesting solution of Eq. (14) at large values \( v \). If the couplings in the polynomial potential are \( O(1) \), it is reasonable at large values of \( v \) to consider the solution:

\[
u(v, x) = \text{const} \exp(v)
\]

(16)

If one considers the couplings \( \sim \frac{1}{\exp(v)} \) in the potential \( U(x) \), then the solution

\[
u(v, x) = \text{const} \exp(v) + u_s(x), u_s(x) \sim 1.
\]

(17)
\[
\frac{1}{2} \Delta u_s + \frac{1}{2} (\nabla u_s)^2 + u_s = 0 \quad (18)
\]
corresponds to the perturbative regime. This equation gives the effective potential at the stable point of renormalization group. One can rewrite of Eq. (18) in another form for \( z \equiv \frac{du_s}{dv} \):

\[
\frac{dz}{du_s} + z + 2 \frac{2}{z} u_s \quad (19)
\]

In analogy to Eqs. (10), (13) it is possible also to derive the correlators. To calculate the correlator \(< \exp(i\alpha \phi(x) - i\alpha \phi(y)) >\), where the hierarchic distance between points \(x, y\) is \(v_0\), one should distinguish during the iteration between the links located on the paths that connect the origin with the points \(x, y\). Thus we also consider the equation

\[
\frac{df(v, x, \alpha)}{dv} = f \ln w + \frac{1}{2} \Delta f
\]

\[
f(0, x, \alpha) = \exp(U(x) + i\alpha x) \quad (20)
\]

Then for the generating function \(f_0(v, x)\) of correlator one must solve again Eq. (20) with the boundary conditions at the point \(v_0\)

\[
f_0(v_0, x) = f(v_0, x, \alpha)f(v_0, x, -\alpha)/w(v_0, x) \quad (21)
\]

We obtain an expression for the correlator:

\[
< \exp(i\alpha \phi(x) - \alpha \phi(y)) > = \frac{f_0(\infty, 0)}{w(\infty, 0)} \quad (22)
\]

In this way we can calculate two point correlators, as well as other multipoint correlators.

### 3 High dimensions

The same approach may be used for the case of \(d > 2\). In d-d space one has for the volume \(\sim a^d\). If we identify it with our \(q^L\), then \(a = q^{d/L}\). To have
exact expression for the correlator, \( f_l \) are defined on the branches, \( f_0, f_1 \) at the origin. Here the free field action is defined as

\[
\phi(x) = f_0 + f_1 + \sum_{v,l} f(v,l)
\]  

(23)

The summation in (23) is along the trajectory of point X. Now determine the kinematical part of the action for \( \phi(x) \) field

\[
A = \frac{1}{2}[f_1^2 + \sum_{v,l} \exp(-\alpha v) f(v,l)^2/\alpha]
\]  

(24)

If one takes \( \alpha = \frac{d-2}{d} \) for the combined field, then

\[
< \phi(x) \phi(x') > = \exp(\alpha v) \sim \left[ \frac{L}{r(x,x')} \right]^{-(d-2)}
\]  

(25)

where \( L \) is the infrared cutoff. Now (10) transforms into:

\[
I_1(x) = \sqrt{\frac{K}{2V\alpha \pi}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{K}{2V\alpha} y^2 + U(x+y) \right\} dy
\]

\[
I_{i+1}(x) = \sqrt{\frac{K}{2V\alpha e^{\alpha V(K-i+1)/K} \pi}} \int_{-\infty}^{\infty} e^{-\frac{K}{2V\alpha} \exp[\alpha V(K-i+1)/K] y^2} [I_i(x+y)]^q dy
\]

\[
Z = \lim_{K \to \infty} \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} y^2 + U(y) \right\} dy [I_K(y)]^q
\]  

(26)

To calculate \( I_K(x) \equiv w(V,x) \equiv \exp(u(V,x)) \) we have to solve the equation like (14)

\[
\frac{du}{dv} = \frac{1}{2} d(d-2) \exp[d(d-2)v] \Delta u + \frac{1}{2} (\nabla u)^2 + u
\]

\[
u(0,x) = U(x)
\]  

(27)

It is important to investigate the version of Generalized Random Energy Model (GREM) corresponding to (27), when our model is defined on hierarchic lattice with large branching number \( q \). Here the physics at \( d = 2 \) and \( d > 2 \) is quite different and one cannot use the methods of [9] for the latter case.
4 Branching diffusion

Instead of the structure of $q = 1$ trees it is possible to consider an ensemble of hierarchic trees having again a small branching number after the averaging. An analogous process has been introduced in [4], and here we give only a little modification to choose proper boundary conditions for a desired potential. There is an origin and a branch from it. Branch appears from the original branches during the period of time $dt$ with the length $dt$. Alternatively, with the probability $1 - dt$ the old branch is elongated by the length $dt$ and a random variable $f_i$ is introduced with the variance

$$< f_i^2 >= dt/2$$

(28)

After some period of time $t$ the number of endpoints is $\exp(t)$. Every branch has one or more random variables. The fields at the endpoints of tree are again defined as a sum of random variables along the trajectory. The variable $w(t, x)$ is defined as

$$w(t, x) = < \exp(\sum_y U(y + x)) >$$

(29)

Here the summation $y$ is along all endpoints of a tree after period of time $t$. Let us define $w(t + dt, x)$. One should keep in view at the determination of $w(t + dt, x)$ that during $dt$ time either there appears a new branch (the contribution of this process is equal to $dtw(t, x)^2$), or, if stays the old branch has an increment of $f_i$, the contribution is $(1 - dt)w(t, x + f_i)$. Combining the contributions we easily derive the KPP equation [9] for a $w$:

$$\frac{dw(t, x)}{dt} = w^2(t, x) - w(t, x) + \frac{1}{2}\Delta w$$

(30)

The derivation of this equation has been done in [9] where the case $w(0, x) = \exp(-e^x)$ has been considered. It is similar to (13), with the only difference that the nonlinear term is replaced by $w^2 - w$. We believe that critical properties of both the approaches (13),(30) are the same. The point is that for $q = 1$ tree approach there is no any averaging in an ensemble and thus it is easier to deal with the perturbative expansion. Of course our equation (13) is also a reaction diffusion-equation. Another interesting problem is an imitation of d-d reaction diffusion equations on $q = 1$ space.
5 Coulomb gas representation for 2d conformal fields.

We can apply these ideas to conformal theories using the Coulomb-gas formalism with the background charge $\alpha_0$ [8]. If we correctly defined the zero mode of Laplacian and the correlator has correct dimension, we have good chances to imitate the 2d situation. We have an action

$$\frac{1}{8\pi} \int d^2 w \sqrt{g} \phi \Delta \phi + i 2\sqrt{2} \alpha_0 R \phi$$

(31)

Here the field $\phi(w)$ is defined on the sphere, $R$ is curvature, $\Delta$ is a Laplacian. One defines the screening charges from the condition $\alpha_{\pm 1}^2 - 2\alpha_0 \alpha_{\pm 1} = 1$. To calculate the correlator $\prod_k \exp\{i\sqrt{2}\alpha_k \phi\}$ with screening charges $Q^n_m$ one has to consider

$$Z = \int D_g \phi e^{\frac{1}{8\pi} \int d^2 w \sqrt{g} \phi \Delta \phi + i 2\sqrt{2} \alpha_0 R \phi}$$

$$\prod_k \exp\{i\sqrt{2}\alpha_k \phi\}(\int d^2 w \exp(i\sqrt{2}\alpha_+ \phi)^m(\int d^2 w \exp(i\sqrt{2}\alpha_- \phi)^n)$$

(32)

The zero mode integration gives for nonzero correlator the constraint

$$\sum_i \alpha_i + m \alpha_+ + n \alpha_- = 2\alpha_0$$

(33)

in 2d case [8].

To have finite set of $\alpha_i$ from (35) we should put a constraint

$$p' \alpha_+ + p \alpha_- = 0$$

(34)

which is the definition of minimal models. The deficiency of our approach is that we cannot find connection between $\alpha_0$ and conformal charge $c$.

While calculating (26) in UM space, we omit the $\alpha_0$ term. We again, as in 2d case, consider normal ordered operator product for $\exp[i\sqrt{2}\alpha E_i]$

$$< z^m(\sqrt{2}\alpha_+) z^n(\sqrt{2}\alpha_-) \exp[i\sqrt{2}\alpha E_1] \exp[i\sqrt{2}\alpha E_2] >$$

(35)

Here the average is over the normal distribution on our hierarchic tree. As $m$ and $n$ are integers, it is possible to perform the integration via $E_i$ directly.
How one can derive the expression for the pair correlator? All the \( n + m \) charges with the total charge \( 2\alpha_0 - \alpha \) are located near the point 1, or the point 2, so we have for (35)

\[
< z^m (\sqrt{2}\alpha_+) z^n (\sqrt{2}\alpha_-) \exp[i \sqrt{2}\alpha E_1] \exp[i \sqrt{2}\alpha E_2] > \sim \\
\exp[\alpha(2\alpha_0 - \alpha)v] = \exp[\alpha(2\alpha_0 - \alpha) \ln \frac{L^2}{r^2}]
\]

(36)

For 3 point correlators we now consider expressions like

\[
< z^m (\sqrt{2}\alpha_+) z^n (\sqrt{2}\alpha_-) \exp[i \sqrt{2}\alpha_1 E_1] \exp[i \sqrt{2}\alpha_2 E_2] \exp[i \sqrt{2}\alpha_3 E_3] > \\
v_{13} = v_{23} > v_{12} = v
\]

(37)

We assume that the screening charges are near the points 1 and 2. Then the distance from the point 3 to any charge is equal to \( R \) and we immediately obtain for the dependence of (37) on \( R \):

\[
R^{-2\alpha_3(2\alpha_0 - \alpha_3)}
\]

(38)

Let us consider the dependence of (32) on \( r = L \exp[\frac{u}{T}] \).

As a result of direct interaction of charges \( \alpha_1, \alpha_2 \) we obtain:

\[
r^{2\alpha_1\alpha_2}
\]

(39)

If all our \( n + m \) charges are located at the distance \( r \) from both the points 1, 2, then their interaction energy with charges 1, 2 is proportional to

\[
2(\alpha_1 + \alpha_2)(n\alpha_+ + m\alpha_-)
\]

(40)

For the energy of self-interaction of the screening charges we have:

\[
[(n\alpha_+ + m\alpha_-)^2 - n\alpha_+^2 - m\alpha_-^2] \\
= (n\alpha_+ + m\alpha_-)^2 - (n + m) - 2\alpha_0(n\alpha_+ + m\alpha_-)
\]

(41)

The integration over the coordinates of \( n + m \) charges gives \( \exp[3v] \), so eventually we have:

\[
r^{-\alpha_3(2\alpha_0 - \alpha_3)} r^{2\alpha_1\alpha_2 + (n\alpha_+ + m\alpha_-)^2 - 2\alpha_0(n\alpha_+ + m\alpha_-)}
\]

(42)
It is easy to check that this expression is equivalent to standard expression from the conformal field theory

\[ r^{2\alpha_3(2\alpha_0-\alpha_3)}r^{\alpha_1^2-2\alpha_1\alpha_0+\alpha_2^2-2\alpha_2\alpha_0-\alpha_3^2+2\alpha_3\alpha_0} \]  

Combining two expressions, we derive eventually for the correlator:

\[ R^{-2\alpha_3(2\alpha_0-\alpha_3)}r^{2\alpha_1\alpha_2+(n\alpha_++m\alpha_-)^2-2\alpha_0(n\alpha_++m\alpha_-)} \]

Let us put in [43] \( r_{12} = r_{13} = r_{23} \) in (43) and consider REM instead of the Directed Polymer. In case of REM we have that \( <\phi(X)\phi(X')> = \delta(x - x') \). It is possible to investigate the phase structure of (originally DP) correlator in this way.

6 REM at complex temperatures with complex numbers of replicas and strings.

6.1 REM version of strings

Using similar ideas we will connect a string partition (after integration by zero mode) with finite replica REM and investigate the phase structure. Recall some results from string theory for a string in \( d = c \) space with spherical surfaces [14-16]. It is known for the partition that after integration by zero mode

\[ Z \sim \int D\phi e^{\int d^2w\sqrt{g}\Delta\phi + QR\phi} \left( \int d^2w \sqrt{\hat{g}e^{\alpha\phi}} \right)^{-\frac{Q}{\alpha}} \]

where

\[ c = 1 - 12\alpha_0^2, Q = 2\sqrt{2+\alpha_0^2}, \alpha = -\frac{\sqrt{25-c}}{\sqrt{12}} + \frac{\sqrt{1-c}}{\sqrt{12}} \]

\[ \frac{Q}{\alpha} = \frac{1}{12}[c - 25 - \sqrt{(25-c)(1-c)}] \]

Here \( \phi(w) \) is a field on two dimensional sphere with coordinates \( w \), curvature \( R \). If we continue those formulas for \( c > 1 \), the coefficients become complex.
We see, that $Z \sim <z^\mu>$, where the averaging is over the normal distribution of field $\phi(w)$.

If the conjecture about the equivalence of 2-d model (49) and corresponding $q=1$ model is correct (it could be checked numerically), then one can solve explicitly DP problem in $Q \to \infty$ limit [9]-[10]. In this limit it is easy to prove [10], that the Directed Polymer has the same thermodynamic limit ($F_0$), as a simple REM.

If in (45) our variables $\phi(w)$ are distributed according to the normal law

$$e^{\frac{1}{8}\int d^2w \sqrt{g_0} \Delta \phi + QR}$$

with non-diagonal quadratic form, in the case of REM all $\phi(w)$ are independent variables with a normal distribution. If we replace the model (45),(46) with models on $q=1$ trees, then according to [9]-[10] these models are equivalent to REM. Thus it is worthwhile to consider REM instead of the set (45)-(46).

To construct an equivalent scheme of REM let us introduce infrared and ultraviolet cutoffs $L$ and $a$. Then the physical number of degrees is

$$M = \frac{L^2}{a^2}$$

(47)

Now define the distribution of $\phi(w)$ over all points $w$, using the free field action from (45):

$$\rho(\phi_0) \equiv <\delta(\phi_0 - \phi(w))><\phi(w)\sim \exp(-\frac{\phi_0^2}{2G(0)})$$

(48)

where the averaging is over the distribution

$$\rho(\phi(w)) \sim e^{\frac{1}{16}\int d^2w \sqrt{g_0} \Delta \phi + QR}$$

(49)

and

$$G(0) = 2 \ln \frac{L}{a}$$

(50)

We can replace our system with a collection of $M$ independent variables $E_i \sim \phi(w)$ with the distribution (43) instead of (44). Our goal is to calculate

$$Z = <z^{\mu_1+i\mu_2}>, z = \sum_i e^{-(\beta_1+i\beta_2)E_i}$$

(51)
It is possible to solve the system rigorously. Here we are giving a qualitative derivation, (checked by our exact calculations).

Note that

\[ N = G(0), \beta_c = \sqrt{\frac{2 \ln M}{G(0)}} \]  

(52)

### 6.2 Solution of REM at complex replica numbers

Let us consider Eq. (51) for positive integer values of \( \mu \), where the averaging is made over the distribution (52) for each \( E_i \).

There are only two competing terms in the sum and two corresponding phases. The first one is paramagnetic (PM) phase, originated from the cross terms in \( z^\mu \) expansion

\[ Z = M^\mu < e^{-\beta E_{i1} - \beta E_{i2} - \cdots - \beta E_{i\mu}} > \]
\[ \ln Z = \mu \ln M + N \frac{\beta^2 \mu^2}{2} = N \frac{(\beta_c^2 + \beta^2)\mu}{2} \]  

(53)

The second one, the correlated paramagnetic (CPM)\[11\] is originated from the diagonal terms in (53) like \( e^{-\beta \mu E_i} \)

\[ Z = \langle \sum_{i=1}^{M} e^{-\mu \beta E_i} \rangle > \]
\[ \ln Z = \ln M + \frac{N \beta^2 \mu^2}{2} = \frac{N(\beta_c^2 + \beta^2 \mu^2)}{2} \]  

(54)

Let us consider the continuation of (54) at \( \mu < 1 \). At critical \( \beta_c \) its entropy \( \ln Z - \beta \frac{d \ln Z}{d \beta} \) disappears. Let as assume for \( \ln Z \) in this region an expression proportional to \( \beta \) (it is natural for a system with zero entropy) and \( \mu \). The continuity of \( \ln Z \) gives for the spin-glass (SG) phase

\[ \ln Z = N \mu \beta_c \beta \]  

(55)

If we pass to complex temperatures \[14\]-\[15\], then (51) transforms to (it is easy to check this directly for integer \( \mu \))

\[ \ln Z = N \frac{(\beta_c^2 + \beta_1^2 - \beta_2^2)\mu}{2} \]  

(56)
In Eq. (55) one has to replace $\beta$ by $\beta_1$, then

$$\ln Z = N\mu \beta_c \beta_1$$  \hspace{1cm} (57)

For complex temperatures there is the fourth, Lee-Young-Fisher (LYF) phase [12]. Its derivation is not direct. The point is that for noninteger values of $\mu$

$$Z \sim < |z|^\mu >$$  \hspace{1cm} (58)

After this trick it is easy to derive the CPM expression. The principal terms are now given by terms $e^{-2\beta_1 E_i}$

$$\ln Z = \frac{N(\beta_c^2 + 8\beta_1^2)\mu}{4}$$  \hspace{1cm} (59)

Now continue our four expressions to complex values of $\mu$.

For PM phase

$$\ln Z = N\frac{(\beta_c^2 + \beta_1^2 - \beta_2^2)\mu_1}{2}$$  \hspace{1cm} (60)

For SG phase

$$\ln Z = N\mu_1 \beta_c \beta_1$$  \hspace{1cm} (61)

For LYF phase

$$\ln Z = \frac{N(\beta_c^2 + 8\beta_1^2)\mu_1}{4}$$  \hspace{1cm} (62)

For CPM

$$\ln Z = \frac{N[\beta_c^2 + \beta_1^2(\mu_1^2 - \mu_2^2)]}{2}$$  \hspace{1cm} (63)

The imaginary parts of $\ln Z$ in (61)-(63) were ignored.

To find the borders between four phases first has to obtain the correct phase at $\mu \to 0$ limit, then compare its expression for $|\ln Z|$ with the corresponding one for CPM phase. It is known that LYF phase exists at

$$\beta < \frac{\beta_c}{2}$$  \hspace{1cm} (64)
and PM one at $\beta < \beta_c$. For complex temperatures one has a condition for SG phase

$$\beta_1 > \beta_c + \beta_2$$  \hspace{1cm} (65)

The last point. The rigorous derivation gives that LYF for noninteger $\mu_1$ exists only at

$$\mu_1 > -2$$  \hspace{1cm} (66)

6.3 REM results for the phase structure of string.

One can apply this (though qualitative but rather strict) result to strings. Identifying $-\frac{a}{\sqrt{2}} \rightarrow \beta$, $\frac{Q}{a} \rightarrow \mu$ and using (2) we derive

$$\beta_1 = \frac{\sqrt{25 - c}}{\sqrt{24}}, \beta_2 = \frac{\sqrt{c - 1}}{\sqrt{24}}, \mu_1 = \frac{1}{12} [25 - c], \mu_2 = \sqrt{(25 - c)(c - 1)} \frac{1}{12}$$  \hspace{1cm} (67)

For other string topologies one has

$$\mu \rightarrow (1 - g)\mu$$  \hspace{1cm} (68)

Note that $y = \frac{25 - c}{24}$. LYF phase exists only in the torus case at $25 > c > 19$. At $19 > c > 1$ the system with torus topology and higher is in SG phase. For sphere topology $19 > c > 1$, it is in CPM at

$$1 + 4y(2y - 1)^2 > 4y^2,$$  \hspace{1cm} (69)

otherwise in SG phase. We see phase transition at $d \approx 2.98$.

For $25 > c > 19$ spherical topology case CPM is at

$$1 + 4y(2y - 1)^2 > \frac{1}{2} + 4y^2,$$  \hspace{1cm} (70)

otherwise at LYF phase. What can we say about string physics base on the REM picture?

The most interesting case is the spherical one. When the value of $c$ increased to pass over the $c = 1$ barrier, nothing happens in REM picture, the system is still in CPM phase, as for in the $c < 1$ case. The PM or CPM phases are ordinary physical phases, so one could try to succeed here with
the same level of reliability, as for $c < 1$ strings outside of the minimal series. To reveal interesting (unitary) theories explicitly one should solve the directed polymer at finite replica number including finite size corrections and correlators.

For the high topologies we have omitted modular space dependence of partition. It will be interesting to check our conclusion about different physical phases for different topologies numerically. But at least for spherical case the REM analysis seems quite reliable.

7 Conclusions

We generalized the results of [5], obtained for the 2d model with an exponential potential, for the case of other critical models, that may be formulated by means of Laplasian (as a single differential operator) and a desired potential of any dimensions $d \geq 2$. The models can be formulated on hierarchic trees with constant branching number $q$. Main observation of this work is that for the special case of $q \to 1$ it is possible to construct a field theory that is similar to renormalized field theories in continuous spaces.

We hope, that the bulk structure, the two and three point correlators (for isosceles triangles) are the same, as those in d-d critical models. This hypothesis is correct for the case of Liouville 2d model, as well as for the (free) case of Coulomb gas representation [8].

We have also analyzed the phase structure of strings by consideration of an analytical continuation of DDK formulas by analogy with REM and solution of complex replica REM at complex temperatures.

It is possible to check our hypothesis about the equivalence of our models on $q = 1$ trees with some segment of d-d field theory by means of direct numerical calculation of Eq. (14), (27), for example for, the field version of 3d Ising model with proper choice of potential $U$.

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