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**Weighted $L^p$ estimates for the area integral associated to self-adjoint operators**

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**Abstract.** This article is concerned with some weighted norm inequalities for the so-called horizontal (i.e., involving time derivatives) area integrals associated to a non-negative self-adjoint operator satisfying a pointwise Gaussian estimate for its heat kernel, as well as the corresponding vertical (i.e., involving space derivatives) area integrals associated to a non-negative self-adjoint operator satisfying in addition a pointwise upper bounds for the gradient of the heat kernel. As applications, we obtain sharp estimates for the operator norm of the area integrals on $L^p(\mathbb{R}^n)$ as $p$ becomes large, and the growth of the $A_p$ constant on estimates of the area integrals on the weighted $L^p$ spaces.

1. Introduction

1.1. Background

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi = 0$. Let $\varphi_t(x) = t^{-n} \varphi(x/t)$, $t > 0$, and define the Lusin area integral by

$$S_{\varphi}(f)(x) = \left( \int \int_{|x-y|<t} |f * \varphi_t(y)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}. \quad (1.1)$$

A celebrated result of Chang–Wilson–Wolff [7] says that for all $w \geq 0$, $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and all $f \in \mathcal{S}(\mathbb{R}^n)$, there is a constant $C = C(n, \varphi)$ independent of $w$ and $f$ such that

$$\int_{\mathbb{R}^n} S_{\varphi}^2(f) w \, dx \leq C \int_{\mathbb{R}^n} |f|^2 Mw \, dx, \quad (1.2)$$

where $Mw$ denotes the Hardy–Littlewood maximal operator of $w$.

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The fact that $\varphi$ has compact support is crucial in the proof of Chang, Wilson and Wolff. In [8], Chanillo and Wheeden overcame this difficulty, and they obtained weighted $L^p$ inequalities for $1 < p < \infty$ of the area integral, even when $\varphi$ does not have compact support, including the classical area function defined by means of the Poisson kernel.

From the theorem of Chang, Wilson and Wolff, it was already observed in [19] that Fefferman and Pipher obtained sharp estimates for the operator norm of a classical Calderón–Zygmund singular integral, or the classical area integral for $p$ tending to infinity, e.g.,

$$\|S\varphi (f)\|_{L^p (\mathbb{R}^n)} \leq CP^{1/2} \|f\|_{L^p (\mathbb{R}^n)},$$  

(1.3)

as $p \to \infty$.

1.2. Assumptions, notation and definitions

In this article, our main goal is to provide an extension of the result of Chang–Wilson–Wolff to study some weighted norm inequalities for the area integrals associated to non-negative self-adjoint operators, whose kernels are not smooth enough to fall under the scope of [7,8,37]. The relevant classes of operators is determined by the following condition:

**Assumption.** (H1) Assume that $L$ is a non-negative self-adjoint operator on $L^2 (\mathbb{R}^n)$, the semigroup $e^{-tL}$, generated by $-L$ on $L^2 (\mathbb{R}^n)$, has the kernel $p_t (x, y)$ which satisfies the following Gaussian upper bound if there exist $C$ and $c$ such that for all $x, y \in \mathbb{R}^n, t > 0$,

$$|p_t (x, y)| \leq \frac{C}{t^{n/2}} \exp \left(-\frac{|x - y|^2}{ct}\right).$$  

(GE)

Such estimates are typical for elliptic or sub-elliptic differential operators of second order (see for instance, [16] and [18]).

For $f \in \mathscr{S} (\mathbb{R}^n)$, define the (so-called vertical) area functions $S_P$ and $S_H$ by

$$S_P f (x) = \left(\iint_{|x - y| < t} |t \nabla_y e^{-t\sqrt{L}} f (y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$  

(1.4)

$$S_H f (x) = \left(\iint_{|x - y| < t} |t \nabla_y e^{-t^2 L} f (y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$  

(1.5)

as well as the (so-called horizontal) area functions $s_P$ and $s_H$ by

$$s_P f (x) = \left(\iint_{|x - y| < t} |t \sqrt{Le} e^{-t\sqrt{L}} f (y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$  

(1.6)

$$s_H f (x) = \left(\iint_{|x - y| < t} |t^2 Le^{-t^2 L} f (y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}.$$  

(1.7)

It is well known (cf. e.g., [22,35]) that when $L = \Delta$ is the Laplacian on $\mathbb{R}^n$, the classical area functions $S_P$, $S_H$, $s_P$ and $s_H$ are all bounded on $L^p (\mathbb{R}^n), 1 < p < \infty$. For a general non-negative self-adjoint operator $L$, $L^p$-boundedness of the area functions $S_P$, $S_H$, $s_P$ and $s_H$ associated to $L$ has been studied extensively—see for examples [1–3,12,36] and [38], and the references therein.
1.3. Statement of the main results

Firstly, we have the following weighted $L^p$ estimates for the area functions $s_p$ and $s_h$.

**Theorem 1.1.** Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bounds (GE). If $w \geq 0$, $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then

(a) \[ \int_{\mathbb{R}^n} s_h(f) w \, dx \leq C(n, p) \int_{\mathbb{R}^n} |f|^p M w \, dx, \quad 1 < p \leq 2, \]

(b) \[ \int_{\{s_h(f) > \lambda\}} w \, dx \leq \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f| M w \, dx, \quad \lambda > 0, \]

(c) \[ \int_{\mathbb{R}^n} s_h(f) w \, dx \leq C(n, p) \int_{\mathbb{R}^n} |f|^p (Mw)^{p/2} w^{-(p/2-1)} \, dx, \quad 2 < p < \infty. \]

Also, estimates (a), (b) and (c) hold for the operator $s_p$.

To study weighted $L^p$-boundedness of the (so-called vertical) area integrals $S_P$ and $S_H$, one assumes in addition the following condition:

**Assumption.** (H2) Assume that the semigroup $e^{-tL}$, generated by $-L$ on $L^2(\mathbb{R}^n)$, has the kernel $p_t(x, y)$ which satisfies a pointwise upper bound for the gradient of the heat kernel. That is, there exist $C$ and $c$ such that for all $x, y \in \mathbb{R}^n, t > 0$,

\[ |\nabla_x p_t(x, y)| \leq \frac{C}{t^{(n+1)/2}} \exp\left(-\frac{|x - y|^2}{ct}\right). \quad (G) \]

Then the following result holds.

**Theorem 1.2.** Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy conditions (GE) and (G). If $w \geq 0$, $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then (a), (b) and (c) of Theorem 1.1 hold for the area functions $S_P$ and $S_H$.

Let us now recall a definition of $A_p$ weights [21]. We say that a weight $w$ is in the the Muckenhoupt class $A_p$, $1 < p < \infty$, if

\[ \|w\|_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty. \]

$\|w\|_{A_p}$ is usually called the $A_p$ constant (or characterization or norm) of the weight. The case $p = 1$ is understand by replacing the right hand side by $(\text{ess inf}_{x \in Q} w(x))^{-1}$ which is equivalent to the one defined above. Observe the duality relation:

\[ \|w\|_{A_p} = \|w^{-p'}\|_{A_p}^{p-1}. \]

Following the Fefferman–Pipher’s method, we can use Theorems 1.1 and 1.2 to establish the $L^p$ estimates of the area integrals as $p$ becomes large.

**Theorem 1.3.** (i) Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy condition (GE). There exists a constant $C$ such that for all $w \in A_1$, the following estimate holds:

\[ \|s_h f\|_{L^p_w(\mathbb{R}^n)} \leq C \|w\|_{A_1}^{1/2} \|f\|_{L^2_w(\mathbb{R}^n)}. \quad (1.8) \]

This inequality implies that as $p \to \infty$,

\[ \|s_h f\|_{L^p(\mathbb{R}^n)} \leq C p^{1/2} \|f\|_{L^p(\mathbb{R}^n)}. \quad (1.9) \]

Also, estimates (1.8) and (1.9) hold for the operator $s_p$. 

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Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy conditions $(GE)$ and $(G)$. Then estimates (1.8) and (1.9) hold for the operators $S_P$ and $S_H$, respectively.

The next result we will prove is the following.

**Theorem 1.4.** (i) Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy condition $(GE)$. There exists a constant $C$ such that for all $w \in A_p$, the following estimate holds for all $f \in L^p_w(\mathbb{R}^n)$, $1 < p < \infty$:

$$
\|s_h f\|_{L^p_w(\mathbb{R}^n)} \leq C \|w\|_{A_p}^{\beta_p + 1/(p-1)} \|f\|_{L^p_w(\mathbb{R}^n)},
$$

where $\beta_p = \max\{1/2, 1/(p-1)\}$.

Also, estimate (1.10) holds for the operator $s_p$.

(ii) Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy conditions $(GE)$ and $(G)$. Then estimate (1.10) holds for the operators $S_P$ and $S_H$, respectively.

We should mention that Theorems 1.1 and 1.2 are of some independent of interest, and they provide an immediate proof of weighted $L^p$ estimates of the area functions $s_h$, $s_p$, $S_P$ and $S_H$ on $L^p_w(\mathbb{R}^n)$, $1 < p < \infty$ and $w \in A_p$ (see Lemma 5.1 below). In the proofs of Theorems 1.1 and 1.2, the main tool is that each area integral is controlled by $g^*_\mu,\psi$ pointwise:

$$
Tf(x) \leq C g^*_\mu,\psi(f)(x), \quad x \in \mathbb{R}^n,
$$

where $T$ is of $S_P$, $S_H$, $s_p$ and $s_h$, and $g^*_\mu,\psi$ is defined by

$$
g^*_\mu,\psi(f)(x) = \left( \int_{\mathbb{R}^n+1} \left( \frac{t}{t + |x - y|} \right)^n \mu |\Psi(t\sqrt{L}) f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \mu > 1
$$

with some $\Psi \in \mathcal{S}(\mathbb{R})$. The idea of using $g^*_\mu,\psi$ to control the area integrals is due to Calderón and Torchinsky [6] (see also [8] and [37]). Note that the singular integral $g^*_\mu,\psi$ does not satisfy the standard regularity condition of a so-called Calderón–Zygmund operator, thus standard techniques of Calderón–Zygmund theory [8,37] are not applicable. The lacking of smoothness of the kernel was indeed the main obstacle and it was overcome by using the method developed in [11,17], together with some estimates on heat kernel bounds, finite propagation speed of solutions to the wave equations and spectral theory of non-negative self-adjoint operators.

For the classical area function $S\phi$ in (1.1), the result of Theorem 1.4 was recently improved by Lerner in [27], i.e., there exists a constant $C = C(S\phi, n, p)$ such that for all $w \in A_p$, $1 < p < \infty$,

$$
\|S\phi\|_{L^p_w(\mathbb{R}^n)} \leq C \|w\|_{A_p} \max\left\{ \frac{1}{2}, \frac{1}{p-1} \right\},
$$

and the estimate (1.13) is the best possible for all $1 < p < \infty$. However, we do not know whether one can deduce the same bounds (1.13) for the $L^p_w$ operator norms of the area functions $s_h$, $s_p$, $S_P$ and $S_H$, and they are of interest in their own right.

The layout of the paper is as follows. In Sect. 2 we recall some basic properties of heat kernels and finite propagation speed for the wave equation, and build the necessary kernel.
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estimates for functions of an operator, which is useful in the proof of weak-type $(1, 1)$ estimate for the area integrals. In Sect. 3 we will prove that the area integral is controlled by $g_{\mu, \Psi}^*$ pointwise, which implies Theorems 1.1 and 1.2 for $p = 2$, and then we employ the Fefferman–Pipher’s method to show Theorem 1.3 to obtain sharp estimates for the operator norm of the area integrals on $L^p(\mathbb{R}^n)$ as $p$ becomes large. In Sect. 4, we will give the proofs of Theorems 1.1 and 1.2. Finally, in Sect. 5 we will prove our Theorem 1.4, which gives the growth of the $A_p$ constant on estimates on the weighted $L^p$ spaces.

Throughout the article, the letter “$c$” and “$C$” will denote (possibly different) constants that are independent of the essential variables.

2. Notation and preliminaries

Let us recall that, if $L$ is a self-adjoint positive definite operator acting on $L^2(\mathbb{R}^n)$, then it admits a spectral resolution

$$L = \int_0^\infty \lambda \, dE(\lambda).$$

For every bounded Borel function $F : [0, \infty) \to \mathbb{C}$, by using the spectral theorem we can define the operator

$$F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda).$$

(2.1)

This is of course, bounded on $L^2(\mathbb{R}^n)$. In particular, the operator $\cos(t \sqrt{L})$ is then well-defined and bounded on $L^2(\mathbb{R}^n)$. Moreover, it follows from Theorem 3 of [13] that the corresponding heat kernels $p_t(x, y)$ of $e^{-tL}$ satisfy Gaussian bounds ($GE$), then there exists a finite, positive constant $c_0$ with the property that the Schwartz kernel $K_{\cos(t \sqrt{L})}$ of $\cos(t \sqrt{L})$ satisfies

$$\text{supp} K_{\cos(t \sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq c_0 t\}.$$  

(2.2)

See also [5, 9] and [33]. The precise value of $c_0$ is inessential and throughout the article we will choose $c_0 = 1$.

By the Fourier inversion formula, whenever $F$ is an even, bounded, Borel function with its Fourier transform $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t \sqrt{L})$. More specifically, we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t \sqrt{L}) \, dt,$$

(2.3)

which, when combined with (2.2), gives

$$K_F(\sqrt{L})(x, y) = (2\pi)^{-1} \int_{|t| \geq |x-y|} \hat{F}(t) K_{\cos(t \sqrt{L})}(x, y) \, dt, \quad \forall x, y \in \mathbb{R}^n.$$  

(2.4)

Lemma 2.1. Let $\varphi \in C_0^\infty(\mathbb{R})$ be even, $\text{supp} \varphi \subseteq (-1, 1)$. Let $\Phi$ denote the Fourier transform of $\varphi$. Then for every $\kappa = 0, 1, 2, \ldots$, and for every $t > 0$, the kernel $K_{(t^2L)^\kappa \Phi(t \sqrt{L})}(x, y)$ of the operator $(t^2L)^\kappa \Phi(t \sqrt{L})$ which was defined by the spectral theory, satisfies

$$\text{supp} K_{(t^2L)^\kappa \Phi(t \sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$  

(2.5)
and
\[ \left| K_{(t^2 L)^k} \Phi(t \sqrt{L})(x, y) \right| \leq C t^{-n} \] (2.6)
for all \( x, y \in \mathbb{R}^n \).

**Proof.** The proof of this lemma is standard (see [34] and [23]). We give a brief argument of this proof for completeness and convenience for the reader.

For every \( \kappa = 0, 1, 2, \ldots \), we set \( \Psi_{\kappa} = (t \xi)^{2\kappa} \Phi(t \xi) \). Using the definition of the Fourier transform, it can be verified that
\[ \hat{\Psi}_{\kappa, t}(s) = (-1)^\kappa 2 \pi t^{-1} \Psi_k \left( \frac{s}{t} \right), \]
where we have set \( \Psi_k(s) = \frac{d^{2\kappa}}{ds^{2\kappa}} \varphi(s) \). Observe that for every \( \kappa = 0, 1, 2, \ldots \), the function \( \Psi_{\kappa, t} \in \mathcal{S}(\mathbb{R}) \) is an even function. It follows from formula (2.4) that
\[ K_{(t^2 L)^k} \Phi(t \sqrt{L})(x, y) = (-1)^\kappa \int_{|st| \geq |x-y|} \frac{d^{2\kappa}}{ds^{2\kappa}} \varphi(s)K_{\cos(st \sqrt{L})}(x, y) \, ds. \] (2.7)

Since \( \varphi \in C_0^\infty(\mathbb{R}) \) and \( \text{supp} \, \varphi \subset (-1, 1) \), (2.5) follows readily from this.

Note that for any \( m \in \mathbb{N} \) and \( t > 0 \), we have the relationship
\[ (I + tL)^{-m} = \frac{1}{(m - 1)!} \int_0^\infty e^{-tsL} e^{-s} s^{m-1} \, ds \]
and so when \( m > n/4 \),
\[ \left\| (I + tL)^{-m} \right\|_{L^2 \rightarrow L^\infty} \leq \frac{1}{(m - 1)!} \int_0^\infty \left\| e^{-tsL} \right\|_{L^2 \rightarrow L^\infty} e^{-s} s^{m-1} \, ds \leq Ct^{-n/4} \]
for all \( t > 0 \). Now \( \left\| (I + tL)^{-m} \right\|_{L^1 \rightarrow L^2} = \left\| (I + tL)^{-m} \right\|_{L^2 \rightarrow L^\infty} \leq Ct^{-n/4} \), and so
\[ \left\| (t^2 L)^k \Phi(t \sqrt{L}) \right\|_{L^1 \rightarrow L^\infty} \leq \left\| (I + t^2 L)^{2m} (t^2 L)^k \Phi(t \sqrt{L}) \right\|_{L^2 \rightarrow L^2} \left\| (I + t^2 L)^{-m} \right\|_{L^2 \rightarrow L^\infty}^2 \]
The \( L^2 \) operator norm of the last term is equal to the \( L^\infty \) norm of the function \( (1 + t^2 |x|)^{2m} (t^2 |s|)^k \Phi(t |\sqrt{|x|}|) \) which is uniformly bounded in \( t > 0 \). This implies that (2.6) holds. The proof of this lemma is concluded. \( \square \)

**Lemma 2.2.** Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be even function with \( \int \varphi = 1, \text{supp} \, \varphi \subset (-1/10, 1/10) \). Let \( \Phi \) denote the Fourier transform of \( \varphi \) and let \( \Psi(s) = s^{2n+2} \Phi^3(s) \). Then there exists a positive constant \( C = C(n, \Phi) \) such that the kernel \( K_{\Psi(t \sqrt{L})(1-\Phi(r \sqrt{L}))}(x, y) \) of \( \Psi(t \sqrt{L})(1 - \Phi(r \sqrt{L})) \) satisfies
\[ \left| K_{\Psi(t \sqrt{L})(1-\Phi(r \sqrt{L}))}(x, y) \right| \leq C \frac{r}{t^{n+1}} \left( 1 + \frac{|x-y|^2}{t^2} \right)^{-n+1/2} \] (2.8)
for all \( t > 0, r > 0 \) and \( x, y \in \mathbb{R}^n \).

**Proof.** One writes \( \Psi(s) = \Psi_1(s) \Phi^2(s) \), where \( \Psi_1(s) = s^{2n+2} \Phi(s) \). Then we have \( \Psi(t \sqrt{L}) = \Psi_1(t \sqrt{L}) \Phi^2(t \sqrt{L}) \). It follows from Lemma 2.1 that \( |K_{\Phi(t \sqrt{L})}(z, y)| \leq Ct^{-n} \)
Using the unitarity of \( \cos(\xi t) \) (\( t \leq 2 \)). Note that if \( |z - y| \geq t \), then \( (1 + |x - y|/t) \leq 2(1 + |x - z|/t) \). Hence,

\[
\left| K_{\psi(t\sqrt{L})}(1 - \Phi(\sqrt{L})) (x, y) \right| = \left( 1 + \frac{|x - y|}{t} \right)^{n+1} K_{\psi(t\sqrt{L})}(1 - \Phi(\sqrt{L})) (x, y) \leq C \int_{\mathbb{R}^n} \left| K_{\psi(t\sqrt{L})}(1 - \Phi(\sqrt{L})) (x, z) K_{\Phi(t\sqrt{L})}(z, y) dz \right| \leq C \int_{\mathbb{R}^n} \left| K_{\psi(t\sqrt{L})}(1 - \Phi(\sqrt{L})) (x, z) \right| \left( 1 + \frac{|x - z|}{t} \right)^{n+1} dz.
\]

By symmetry, we will be done if we show that

\[
\int_{\mathbb{R}^n} \left| K_{\psi(t\sqrt{L})}(1 - \Phi(\sqrt{L})) (x, z) \right| \left( 1 + \frac{|x - z|}{t} \right)^{n+1} dx \leq C \left( \frac{R}{t} \right)^{n+1}. \tag{2.9}
\]

Let \( G_{r,t}(s) = \Psi_1(t \cdot |s|)(1 - \Phi(\sqrt{L})) \). Since \( G_{r,t}(s) \) is an even function, apart from a \( (2\pi)^{-1} \) factor we can write

\[
G_{r,t}(s) = \int_{-\infty}^{+\infty} \hat{G}_{r,t}(\xi) \cos(s\xi) \, d\xi,
\]

and by (2.3),

\[
\Psi_1(t\sqrt{L})(1 - \Phi(\sqrt{L})) \Phi(t\sqrt{L}) = \int_{-\infty}^{+\infty} \hat{G}_{r,t}(\xi) \cos(\xi t\sqrt{L}) \Phi(t\sqrt{L}) d\xi.
\]

(2.10)

By (2.2) and Lemma 2.1, it can be seen that \( K_{\cos(\xi t\sqrt{L})\Phi(t\sqrt{L})} (x, z) = 0 \) if \( |x - z| > t + |\xi| \).

Using the unitarity of \( \cos(\xi t\sqrt{L}) \), estimates (2.5) and (2.6), we have

\[
\int_{\mathbb{R}^n} \left| K_{\cos(\xi t\sqrt{L})\Phi(t\sqrt{L})} (x, z) \right| dx = \int_{\mathbb{R}^n} \left| \cos(\xi t\sqrt{L}) \left( K_{\Phi(t\sqrt{L})} (\cdot, z) \right) (x) \right| dx \leq (t + |\xi|)^{n/2} \left\| \cos(\xi t\sqrt{L}) \left( K_{\Phi(t\sqrt{L})} (\cdot, z) \right) \right\|_{L^2(\mathbb{R}^n)} \leq (t + |\xi|)^{n/2} K_{\Phi(t\sqrt{L})} (\cdot, z) \right\|_{L^2(\mathbb{R}^n)} \leq \left( 1 + \frac{|\xi|}{t} \right)^{n/2}.
\]

This, in combination with (2.10), gives

\[
\text{LHS of (2.9)} \leq C \int_{-\infty}^{+\infty} \left| \hat{G}_{r,t}(\xi) \right| \left( 1 + \frac{|\xi|}{t} \right)^{2n+4} d\xi \leq Ct \left( \int_{-\infty}^{+\infty} \left| \hat{G}_{r,t}(t\xi) \right|^2 (1 + |\xi|)^{4n+4} d\xi \right)^{1/2} \leq C \| \delta_{t^{-1}} G_{r,t} \|_{W^{2n+2,2}(\mathbb{R})},
\]

(11.1)

where \( \delta_{t^{-1}} \) denote dilation by \( t^{-1} \). Next we estimate the term \( \| \delta_{t^{-1}} G_{r,t} \|_{W^{2n+2,2}(\mathbb{R})} \). Note that \( \delta_{t^{-1}} G_{r,t} = \Psi_1(s)(1 - \Phi(\frac{s}{2t})) \), \( \Phi(0) = \tilde{\varphi}(0) = \int \varphi = 1 \) and \( \Phi = \tilde{\varphi} \in \mathcal{S}(\mathbb{R}) \), also
\( \Psi_1(s) = s^{2n+2} \Phi(s) \). We have

\[
\| \delta_{t^{-1}} G_{r,t} \|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\Psi_1(s)|^2 |1 - \Phi(s)\|_{L^2}^2 ds \\
\leq C \| \Phi' \|_{L^\infty}^2 \int_{\mathbb{R}} |\Psi_1(s)|^2 \left( \frac{t s}{t} \right)^2 ds \\
\leq C \left( \frac{r}{t} \right)^2.
\]

(2.12)

Moreover, observe that for any \( k \in \mathbb{N} \), \( \left| \frac{d^k}{ds^k} (1 - \Phi(s)) \right| = (\frac{r}{t})^k \| \Phi \|_{L^\infty} \leq C \left( \frac{r}{t} \right) s^{1-k} \).

By Leibniz's rule, we obtain

\[
\frac{d^{2n+2}}{ds^{2n+2}} \delta_{t^{-1}} G_{r,t}(s) \|_{L^2(\mathbb{R})} \\
= \frac{d^{2n+2}}{ds^{2n+2}} \left( \Psi_1(s) \left( 1 - \Phi(s) \right) \right) \|_{L^2(\mathbb{R})} \\
\leq \sum_{m+k=2n+2} \left| \frac{d^m}{ds^m} \left( s^{2n+2} \Phi(s) \right) \frac{d^k}{ds^k} \left( 1 - \Phi(s) \right) \right|_{L^2(\mathbb{R})} \\
\leq C \left( \frac{r}{t} \right)^{2n+2} \sum_{m=0} \left| \frac{d^{m-(2n+1)}}{ds^{m}} \left( s^{2n+2} \Phi(s) \right) \right|_{L^2(\mathbb{R})} \\
\leq C \left( \frac{r}{t} \right).
\]

(2.13)

From estimates (2.12) and (2.13), it follows that \( \| \delta_{t^{-1}} G_{r,t} \|_{W^{2n+2,2}(\mathbb{R})} \leq C \left( \frac{r}{t} \right) \). This, in combination with (2.11), shows that the desired estimate (2.9) holds, and concludes the proof of Lemma 2.1.

Finally, for \( s > 0 \), we define

\[
\mathbb{F}(s) := \left\{ \psi : \mathbb{C} \rightarrow \mathbb{C} \text{ measurable} : \left| \psi(z) \right| \leq C \frac{|z|^s}{(1 + |z|^{2s})} \right\}.
\]

Then for any non-zero function \( \psi \in \mathbb{F}(s) \), we have that \( \left( \int_0^\infty |\psi(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \). It follows from the spectral theory in [39] that for any \( f \in L^2(\mathbb{R}^n) \),

\[
\left( \int_0^\infty \| \psi(t \sqrt{L}) f \|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2} = \left( \int_0^\infty \left\{ \psi(t \sqrt{L}) \psi(t \sqrt{L}) f, f \right\} \frac{dt}{t} \right)^{1/2} \\
= \left( \left( \int_0^\infty |\psi(t \sqrt{L}) f|^2 \frac{dt}{t} \right)^{1/2} \right)^2 \\
\leq \kappa \| f \|_{L^2(\mathbb{R}^n)},
\]

(2.14)

\( \kappa = \left( \int_0^\infty |\psi(t)|^2 dt / t \right)^{1/2} \), an estimate which will be often used in the sequel.

3. An auxiliary \( g^*_\mu, \psi \) function

3.1. The \( g^*_\mu, \psi \) function

Let \( \varphi \in C_0^\infty(\mathbb{R}) \) be even function with \( \int \varphi = 1 \), \( \text{supp} \varphi \subset (-1/10, 1/10) \). Let \( \Phi \) denote the Fourier transform of \( \varphi \) and let \( \Psi(s) = s^{2n+2} \Phi^3(s) \) (see Lemma 2.2 above). We define the \( g^*_\mu, \psi \) function by
by the time derivatives of heat kernels \( p_t \).

Lemma 3.1. The proof of (3.3) is simple. Indeed, the subordination formula

\[
\frac{\partial^k}{\partial t^k} p_t(x, y) \leq C_k t^{-n} \left( 1 + \frac{|x - y|}{t} \right)^{-(n+2\kappa+1)}, \quad \forall t > 0
\]

for all \( t > 0 \), and \( x, y \in \mathbb{R}^n \). For the proof of (3.2), see [16] and [30, Theorem 6.17].

Note that in the absence of regularity on space variables of \( p_t(x, y) \), estimate (3.2) plays an important role in our theory.

**Lemma 3.1.** Let \( L \) be a non-negative self-adjoint operator such that the corresponding heat kernels \( p_t(x, y) \) of the semigroup \( e^{-tL} \) satisfy Gaussian bounds (GE). Then for every \( \kappa = 0, 1, \ldots, \), the operator \((t \sqrt{L})^{2k} e^{-t \sqrt{L}}\) satisfies

\[
|K_{(t \sqrt{L})^{2k} e^{-t \sqrt{L}}}(x, y)| \leq C_k t^{-n} \left( 1 + \frac{|x - y|}{t} \right)^{-(n+2\kappa+1)}, \quad \forall t > 0
\]

for almost every \( x, y \in \mathbb{R}^n \).

**Proof.** The proof of (3.3) is simple. Indeed, the subordination formula

\[
e^{-t \sqrt{L}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{-1/2} e^{-\frac{u^2}{4t} L} du
\]

allows us to estimate

\[
|K_{(t \sqrt{L})^{2k} e^{-t \sqrt{L}}}(x, y)| \leq C_k \int_0^\infty e^{-u} \sqrt{\frac{t^2}{u}} \left( \frac{t^2}{u} \right)^{-n/2} \exp \left( -\frac{u|x - y|^2}{ct^2} \right) u^k \, du
\]

\[
\leq C_k t^{-n} \int_0^\infty e^{-u} u^{n/2+\kappa-1/2} \exp \left( -\frac{u|x - y|^2}{ct^2} \right) \, du
\]

\[
\leq C_k t^{-n} \left( 1 + \frac{|x - y|}{t} \right)^{-(n+2\kappa+1)}
\]

for every \( t > 0 \) and almost every \( x, y \in \mathbb{R}^n \).

**Lemma 3.2.** Let \( L \) be a non-negative self-adjoint operator such that the corresponding heat kernels \( p_t(x, y) \) of the semigroup \( e^{-tL} \) satisfy (GE) and (G). Then for every \( \kappa = 0, 1, \ldots, \), the operator \((t^{2\kappa+1} \nabla(L^\kappa e^{-t^2L}))\) satisfies

\[
|K_{t^{2\kappa+1} \nabla(L^\kappa e^{-t^2L})}(x, y)| \leq Ct^{-n} \exp \left( -\frac{|x - y|^2}{ct^2} \right), \quad \forall t > 0
\]

for almost every \( x, y \in \mathbb{R}^n \).
Proof. Note that $t^{2k+1} \nabla (L^k e^{-t^2 L}) = t \nabla e^{-\frac{t^2}{2} L} \circ (t^2 L)^k e^{-\frac{t^2}{2} L}$. Using (3.2) and the pointwise gradient estimate (G) of heat kernel $p_t(x,y)$, we have

$$\left| K_{t^{2k+1} \nabla (L^k e^{-t^2 L})}(x,y) \right| = \left| \int_{\mathbb{R}^n} K_{t \nabla e^{-\frac{t^2}{2} L}}(x,z) K_{(t^2 L)^k e^{-\frac{t^2}{2} L}}(z,y) dz \right|$$

$$\leq C t^{-2n} \int_{\mathbb{R}^n} \exp \left( \frac{|x-z|^2}{c t^2} \right) \exp \left( -\frac{|z-y|^2}{c t^2} \right) dz$$

$$\leq C t^{-n} \exp \left( -\frac{|x-y|^2}{c t^2} \right)$$

for every $t > 0$ and almost every $x, y \in \mathbb{R}^n$. \qed

Now we start to prove the following Propositions 3.3 and 3.4.

**Proposition 3.3.** Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy condition $(GE)$. Then for $f \in S(\mathbb{R}^n)$, there exists a constant $C = C_{n,\mu,\Psi}$ such that the area integral $s_p$ satisfies the pointwise estimate:

$$s_p f(x) \leq C g_{\mu,\Psi}(f)(x). \quad (3.4)$$

Estimate (3.4) also holds for the area integral $s_h$.

**Proposition 3.4.** Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy conditions $(GE)$ and $(G)$. Then for $f \in S(\mathbb{R}^n)$, there exists a constant $C = C_{n,\mu,\Psi}$ such that the area integral $S_p$ satisfies the pointwise estimate:

$$S_p f(x) \leq C g_{\mu,\Psi}(f)(x). \quad (3.5)$$

Estimate (3.5) also holds for the area integral $S_H$.

**Proofs of Propositions 3.3 and 3.4.** Let us begin to prove (3.5). By the spectral theory [39], for every $f \in S(\mathbb{R}^n)$ and every $\kappa \in \mathbb{N}$,

$$f = C_{\Psi} \int_0^\infty (t^2 L)^k e^{-t^2 L} \Psi(t \sqrt{L}) f \frac{dt}{t},$$

with $C_{\Psi}^{-1} = \int_0^\infty t^{2k} e^{-t^2} \Psi(t) dt / t$, and the integral converges in $L^2(\mathbb{R}^n)$. Recall the subordination formula:

$$e^{-t \sqrt{L}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{-1/2} e^{-\frac{u^2}{4uL}} du.$$ 

One writes

$$s \nabla e^{-s \sqrt{L}} f(y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{-1/2} s \nabla e^{-\frac{u^2}{4uL}} f(y) du$$

$$= C_{\Psi} \sqrt{\pi} \int_0^\infty \int_0^\infty e^{-u} u^{-1/2} s t^{2k} \nabla \left( L^k e^{-\left(\frac{t^2}{4u}+t^2\right)L} \right) \Psi(t \sqrt{L}) f(y) dt \frac{du}{t}.$$ \quad (3.6)
Fix $\kappa = \frac{\mu(\mu-1)}{2} + 1$. Using Lemma 3.2 and the Hölder inequality, we can estimate (3.6) as follows:

$$ \left| s \nabla e^{-s\sqrt L} f(y) \right| $$

$$ \leq C \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} e^{-u} u^{-1/2} s t^{2k} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2 - \kappa - n/2} e^{-|y-z|^2/\left(2\frac{s^2}{4u} + t^2\right)} \left| \Psi(t\sqrt L) f(z) \right| \frac{dz dt du}{t} $$

$$ \leq C A \cdot B, $$

where

$$ A^2 = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \left| \Psi(t\sqrt L) f(z) \right|^2 e^{-u} u^{-1/2} s t^{2k} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2 - \kappa - n/2} e^{-|y-z|^2/\left(2\frac{s^2}{4u} + t^2\right)} \frac{dz dt du}{t} $$

and

$$ B^2 = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} e^{-u} u^{-1/2} s t^{2k} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2 - \kappa - n/2} e^{-|y-z|^2/\left(2\frac{s^2}{4u} + t^2\right)} \frac{dz dt du}{t} $$

$$ = C \int_0^\infty \int_0^\infty \int_0^\infty e^{-u} u^{-1/2} s t^{2k} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2 - \kappa} e^{-r^2 r_{n-1}^2 dr dt du} $$

$$ \leq C \int_0^\infty \int_0^\infty e^{-u} v^{2k} \left( 1 + v^2 \right)^{-1/2 - \kappa} \frac{du dv}{v} $$

$$ \leq C. $$

Note that in the first equality of the above term $B$, we have changed variables $|y - z| \to r\left(\frac{s^2}{4u} + t^2\right)^{1/2}$ and $t \to v\left(s^2/u\right)^{1/2}$. Hence,

$$ |s \nabla e^{-s\sqrt L} f(y)|^2 $$

$$ \leq C \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} \left| \Psi(t\sqrt L) f(z) \right|^2 e^{-u} u^{-1/2} s t^{2k} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2 - \kappa - n/2} e^{-|y-z|^2/\left(2\frac{s^2}{4u} + t^2\right)} \frac{dz dt du}{t}. $$

Therefore, we put it into the definition of $S_p$ to obtain

$$ S^2_p(f)(x) = \int_0^\infty \int_{|x-y| < s} \left| s \nabla e^{-s\sqrt L} f(y) \right|^2 \frac{dy ds}{s^{n+1}} $$

$$ \leq C \int_{\mathbb{R}^n+1} \left| \Psi(t\sqrt L) f(z) \right|^2 $$

$$ \times \left( \int_0^\infty \int_0^\infty \int_{|x-y| < s} e^{-u} u^{-1/2} s t^{2k+n} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2 - \kappa - n/2} e^{-|y-z|^2/\left(2\frac{s^2}{4u} + t^2\right)} \frac{dz dt du}{t^{n+1}} \right). $$
We will be done if we show that
\[
\int_0^\infty \int_0^\infty \int_{|v-y|<s} e^{-u_{1/2}st^2 + n \left( \frac{s^2}{4u} + t^2 \right)^{-1/2} - \frac{n}{2}} dy ds du \leq C \left( \frac{t}{t+|v|} \right)^{n\mu},
\]
where we set $x - z = v$, and we will prove estimate (3.7) by considering the following two cases.

**Case 3.5.** $|v| \leq t$. In this case, it is easy to show that
\[
\text{LHS of (3.7)} \leq C \int_0^\infty \int_0^\infty \int_{|v-y|<s} e^{-u_{1/2}st^2 + n \left( \frac{s^2}{4u} + t^2 \right)^{-1/2} - \frac{n}{2}} dy ds du \\
\leq C \int_0^\infty \int_0^\infty e^{-u_{1/2}st^2} \left( \frac{s^2}{2u} + 1 \right)^{-1/2} - \frac{n}{2} ds du \\
\leq C.
\]

But $|v| \leq t$, so
\[
\left( \frac{t}{t+|v|} \right)^{n\mu} \geq C_{n,\mu}.
\]
This implies that (3.7) holds when $|v| \leq t$.

**Case 3.6.** $|v| > t$. In this case, we break the integral into two pieces:
\[
\int_0^\infty \int_0^{\infty} \cdots + \int_0^\infty \int_0^{\infty} \cdots =: I + II.
\]
For the first term, note that $|y| \geq |v| - |v - y| > |v|/2$. This yields
\[
I \leq C \int_0^\infty \int_0^{\infty} \int_{|v-y|<s} e^{-u_{1/2}st^2 + n \left( \frac{s^2}{4u} + t^2 \right)^{-1/2} - \frac{n}{2}} dy ds du \\
\leq C \left( \frac{t}{|v|} \right)^{n\mu} \int_0^\infty \int_0^{\infty} e^{-u_{1/2}st^2 + n \left( \frac{s^2}{4u} + t^2 \right)^{-1/2} - \frac{n}{2}} ds du \\
\leq C \left( \frac{t}{|v|} \right)^{n\mu} \int_0^\infty \int_0^{\infty} e^{-u_{1/2}st^2 + n\mu} \left( \frac{s^2}{4u} + t^2 \right)^n ds du \\
\leq C \left( \frac{t}{|v|} \right)^{n\mu} \int_0^\infty \int_0^{\infty} e^{-u_{1/2}st^2 + n\mu} \left( \frac{s^2}{4u} + t^2 \right)^n ds du \\
\leq C \left( \frac{t}{|v|} \right)^{n\mu} \int_0^\infty \int_0^{\infty} e^{-u_{1/2}st^2 + n\mu} \left( \frac{s^2}{4u} + t^2 \right)^n ds du \\
\leq C \left( \frac{t}{|v|} \right)^{n\mu} \int_0^\infty \int_0^{\infty} e^{-u_{1/2}st^2 + n\mu} \left( \frac{s^2}{4u} + t^2 \right)^n ds du.
\]
where we used condition $\kappa = \lfloor \frac{n(\mu - 1)}{2} \rfloor + 1$ in the last inequality. Since $|v| > t$, so $I \leq C_{n,\mu} \left( \frac{t}{t+|v|} \right)^{n\mu}$. 
For the term $II$, we have

$$II \leq C \int_0^\infty \int_0^\infty \int_0^\infty e^{-u} u^{-1/2} s t^{2k+n} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2-k-n/2} e^{-r^2/(c^2 + r^2)} r^{n-1} dr ds du \frac{dsdu}{sn+1}$$

$$\leq C \int_0^\infty \int_0^\infty e^{-u} u^{-1/2} s t^{2k+n} \left( \frac{s^2}{4u} + t^2 \right)^{-1/2-k} dsdu \frac{dsdu}{sn+1}$$

$$\leq C \int_0^\infty e^{-u} u^{-1/2} (\sqrt{ut})^{1-n} (s^2 + 1)^{-1/2-k} dsdu \frac{dsdu}{sn}$$

$$\leq C \left( \frac{t}{|v|} \right)^{2k+n}$$

$$\leq C \left( \frac{t}{t + |v|} \right)^{n\mu} ,$$

since $\kappa = \left[ \frac{n(\mu - 1)}{2} \right] + 1$ and $|v| > t$.

From the above Cases 1 and 2, we have obtained estimate (3.7), and then the proof of estimate (3.5) is complete. The similar argument as above gives estimate (3.5) for the area integral $S_H$, and this completes the proof of Proposition 3.4.

For the area functions $s_P$ and $s_h$, we can use a similar argument to show Proposition 3.3 by using either estimate (3.2) or Lemma 3.1 instead of Lemma 3.2 in the proof of estimate (3.5), and we skip it here.

3.2. Weighted $L^2$ estimate of $g^*_\mu,\psi$

**Theorem 3.7.** Let $\mu > 1$. Then there exists a constant $C = C_n,\mu,\psi$ such that for all $w \geq 0$ in $L^1_{loc}(\mathbb{R}^n)$ and all $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} g^*_\mu,\psi(f)^2 w dx \leq C \int_{\mathbb{R}^n} |f|^2 M w dx. \quad (3.8)$$

**Proof.** The proof essentially follows from [7] and [8] for the classical area function. Note that by Lemma 2.1, the kernel $K_{\Psi(t\sqrt{L})}$ of the operator $\Psi(t\sqrt{L})$ satisfies $\text{supp } K_{\Psi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}$. By (3.1), one writes

$$\int_{\mathbb{R}^n} g^*_\mu,\psi(f)^2 w dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x - y|} \right)^{n\mu} |\Psi(t\sqrt{L}) f(y)|^2 \frac{dy dt}{t^{n+1}} w(x) dx$$

$$= \int_{\mathbb{R}^{n+1}_+} |\Psi(t\sqrt{L}) f(y)|^2 \left( \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - y|} \right)^{n\mu} dx \right) \frac{dy dt}{t} . \quad (3.9)$$

For $k$ an integer, set

$$A_k = \left\{ (y, t) \in \mathbb{R}^n \times (0, +\infty) : 2^{k-1} \left\| \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - y|} \right)^{n\mu} dx \right\| \leq 2^k \right\} .$$
Then

\[ \text{RHS of (3.9)} \leq \sum_{k \in Z} 2^k \int_{\mathbb{R}^{n+1}_+} |\Psi(t \sqrt{L}) f(y)|^2 \chi_{A_k}(y, t) \frac{dy dt}{t}. \quad (3.10) \]

We note that if \((y, t) \in A_k\), then since \(\mu > 1\),

\[ 2^{k-1} \leq \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - y|} \right)^{n\mu} dx \leq CMw(y). \]

Now if \(|y - z| < t\), then \(t + |x - y| \approx t + |x - z|\). Thus if \(|y - z| < t\) and \((y, t) \in A_k\),

\[ 2^{k-1} \leq \frac{C}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - z|} \right)^{n\mu} dx \leq CMw(z). \]

In particular, if \((y, t) \in A_k\) and \(|y - z| < t\), then \(z \in E_k = \{ z : Mw(z) \geq C2^k \}\). Now since \(\text{supp} \Psi(t \sqrt{L})(y, z) \subset \{ (y, z) \in \mathbb{R}^n \times \mathbb{R}^n : |y - z| \leq t \}\), for \((y, t) \in A_k\),

\[ \Psi(t \sqrt{L}) f(y) = \int_{|y - z| < t} K\Psi(t \sqrt{L})(y, z) f(z) dz = \int_{\mathbb{R}^n} K\Psi(t \sqrt{L})(y, z) f(z) \chi_{E_k}(z) dz. \]

Therefore,

\[ \text{RHS of (3.9)} \leq \sum_{k \in Z} 2^k \int_{\mathbb{R}^{n+1}_+} |\Psi(t \sqrt{L}) (f \chi_{E_k})(y)|^2 \chi_{A_k}(y, t) \frac{dy dt}{t} \]

\[ \leq \sum_{k \in Z} 2^k \int_{\mathbb{R}^{n+1}_+} |\Psi(t \sqrt{L}) (f \chi_{E_k})(y)|^2 \frac{dy dt}{t} \]

\[ = \sum_{k \in Z} 2^k \int_0^\infty \|\Psi(t \sqrt{L}) (f \chi_{E_k})\|^2_{L^2(\mathbb{R}^n)} \frac{dt}{t} \]

\[ = C_\Psi \sum_{k \in Z} 2^k \| f \chi_{E_k} \|^2_{L^2(\mathbb{R}^n)} \]

with \(C_\Psi = \int_0^\infty |\Psi(t)|^2 dt / t < \infty\), and the last inequality follows from the spectral theory (see [39]). By interchanging the order of summation and integration, we have

\[ \int_{\mathbb{R}^n} g_{\mu, \Psi}(f)^2 w dx \leq C \sum_{k \in Z} 2^k \int_{\mathbb{R}^n} |f|^2 \chi_{E_k} dx \]

\[ \leq C \int_{\mathbb{R}^n} |f|^2 \left( \sum_{k \in Z} 2^k \chi_{E_k} \right) dx \]

\[ \leq C \int_{\mathbb{R}^n} |f|^2 M w dx. \]

This concludes the proof of the theorem. \(\square\)

As a consequence of Propositions 3.3, 3.4 and Theorem 3.7, we have the following analogy for the area function of the result of Chang, Wilson and Wolff.

**Corollary 3.8.** Let \(T\) be one of the area integrals \(s_h\), \(s_p\), \(S_p\) and \(S_H\). Under assumptions of Theorems 1.1 and 1.2, there exists a constant \(C\) such that for all \(w \geq 0\) in \(L^1_{loc}(\mathbb{R}^n)\) and all \(f \in \mathcal{S}(\mathbb{R}^n)\),

\[ \int_{\mathbb{R}^n} |Tf|^2 w dx \leq C \int_{\mathbb{R}^n} |f|^2 M w dx. \]
3.3. Proof of Theorem 1.3

Let $T$ be of the area functions $s_h, s_p, S_p$ and $S_H$. For $w \in A_1$, we have $Mw(x) \leq \|w\|_{A_1} w(x)$ for a.e. $x \in \mathbb{R}^n$. According to Corollary 3.8,

$$
\int_{\mathbb{R}^n} T(f)^2 w dx \leq C \int_{\mathbb{R}^n} |f|^2 Mw dx \leq C \|w\|_{A_1} \int_{\mathbb{R}^n} |f|^2 w dx.
$$

This implies (1.8) holds.

For (1.9), we follow the method of Rubio de Francia and García–Cuerva (see pp. 356–357, [19]). Let $p > 2$ and take $f \in L^p(\mathbb{R}^n)$. Then from duality, we know that there exist some $\varphi \in L^{(p/2)'}(\mathbb{R}^n)$, with $\varphi \geq 0$, $\|\varphi\|_{L^{(p/2)'}(\mathbb{R}^n)} = 1$, such that

$$
\|Tf\|_{L^p(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} |Tf|^2 \varphi dx.
$$

Set

$$
v = \varphi + \frac{M\varphi}{2\|M\|_{L^{(p/2)'}(\mathbb{R}^n)}} + \frac{M^2\varphi}{(2\|M\|_{L^{(p/2)'}(\mathbb{R}^n)})^2} + \cdots.
$$

Here $\|M\|_{L^{(p/2)'}(\mathbb{R}^n)}$ denotes the operator norm of the Hardy–Littlewood maximal operator on $L^{(p/2)'}(\mathbb{R}^n)$. Then $\|v\|_{L^{(p/2)'}(\mathbb{R}^n)} \leq 2$ and $\|v\|_{A_1} \leq 2\|M\|_{L^{(p/2)'}(\mathbb{R}^n)} \equiv O(p)$ as $p \to \infty$. Therefore

$$
\|Tf\|_{L^p(\mathbb{R}^n)}^2 \leq \int_{\mathbb{R}^n} |Tf|^2 \varphi dx \\
\leq \int_{\mathbb{R}^n} |Tf|^2 v dx \\
\leq C \|v\|_{A_1} \int_{\mathbb{R}^n} |f|^2 v dx \\
\leq Cp \|f\|_{L^p(\mathbb{R}^n)}^2.
$$

This proves (1.9), and then the proof of this theorem is complete. $\square$

Note that in Theorem 1.3, when $L = \Delta$ is the Laplacian on $\mathbb{R}^n$, it is well known that estimate (1.9) of the classical area integral on $L^p(\mathbb{R}^n)$ is sharp, in general (see, e.g., [19]).

4. Proofs of Theorems 1.1 and 1.2

Note that from Propositions 3.3 and 3.4, the area functions $S_H, S_p, s_h$ and $s_p$ are all controlled by the $(g_{\mu, \psi})$ function. In order to prove Theorems 1.1 and 1.2, it suffices to show the following result.

**Theorem 4.1.** Let $L$ be a non-negative self-adjoint operator such that the corresponding heat kernels satisfy Gaussian bounds (GE). Let $\mu > 3$. If $w \geq 0$, $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then

(a) $\int_{[g_{\mu, \psi}(f) > \lambda]} w dx \leq \frac{c(n)}{\lambda} \int_{\mathbb{R}^n} |f|Mw dx, \quad \lambda > 0,$

(b) $\int_{\mathbb{R}^n} g_{\mu, \psi}(f)^p w dx \leq c(n, p) \int_{\mathbb{R}^n} |f|^p Mw dx, \quad 1 < p \leq 2,$

(c) $\int_{\mathbb{R}^n} g_{\mu, \psi}(f)^p w dx \leq c(n, p) \int_{\mathbb{R}^n} |f|^p (Mw)^{p/2} w^{-(p/2-1)} dx, \quad 2 < p < \infty.$
4.1. Weak-type \((1, 1)\) estimate

We first state a Whitney decomposition. For its proof, we refer to Chapter 6, [35].

Lemma 4.2. Let \(F\) be a non-empty closed set in \(\mathbb{R}^n\). Then its complement \(\Omega\) is the union of a sequence of cubes \(Q_k\), whose sides are parallel to the axes, whose interiors are mutually disjoint, and whose diameters are approximately proportional to their distances from \(F\). More explicitly:

(i) \(\Omega = \mathbb{R}^n \setminus F = \bigcup_{k=1}^{\infty} Q_k\).

(ii) \(Q_j \cap Q_k = \emptyset\) if \(j \neq k\).

(iii) There exist two constants \(c_1, c_2 > 0\), (we can take \(c_1 = 1\), and \(c_2 = 4\)), so that

\[
    c_1 \text{diam}(Q_k) \leq \text{dist}(Q_k, F) \leq c_2 \text{diam}(Q_k).
\]

Note that if \(\Omega\) is an open set with \(\Omega = \bigcup_{k=1}^{\infty} Q_k\) a Whitney decomposition, then for every \(\varepsilon : 0 < \varepsilon < 1/4\), there exists \(N \in \mathbb{N}\) such that no point in \(\Omega\) belongs to more than \(N\) of the cubes \(Q^{*}_k\), where \(Q^{*}_k = (1 + \varepsilon)Q_k\).

Proof of (a) of Theorem 4.1. Since \(g^\mu \psi(f) \leq g^\mu \psi(f)\) whenever \(\mu' \geq \mu\), it is enough to prove (a) of Theorem 4.1 for \(3 < \mu < 4\). Since \(g^\mu \psi\) is subadditive, we may assume that \(f \geq 0\) in the proof (if not we only need to consider the positive part and the negative part of \(f\)).

For \(\lambda > 0\), we set \(\Omega = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}\). By [20] it follows that

\[
    \int_{\Omega} \omega dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| M\omega dx. \tag{4.1}
\]

Let \(\Omega = \bigcup Q_j\) be a Whitney decomposition, and define

\[
    h(x) = \begin{cases} 
        f(x), & x \notin \Omega \\
        \frac{1}{|Q_j|} \int_{Q_j} f(x) dx, & x \in Q_j
    \end{cases}
\]

\[
    b_j(x) = \begin{cases} 
        f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(x) dx, & x \in Q_j \\
        0, & x \notin Q_j
    \end{cases}
\]

Then \(f = h + \sum j b_j\), and we set \(b = \sum j b_j\). As in [35], we have \(|h| \leq C\lambda\) a.e. By (4.1), it suffices to show

\[
    w\{x \notin \Omega : g^\mu \psi(f)(x) > \lambda\} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| M\omega dx. \tag{4.2}
\]

By Chebychev’s inequality and Theorem 3.7,

\[
    w\{x \notin \Omega : g^\mu \psi(h)(x) > \lambda/2\} \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} g^\mu \psi(h)^2 (w\chi_{\mathbb{R}^n \setminus \Omega}) dx 
\]

\[
    \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} |h|^2 M(w\chi_{\mathbb{R}^n \setminus \Omega}) dx 
\]

\[
    \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |h| M(w\chi_{\mathbb{R}^n \setminus \Omega}) dx
\]

since \(|h| \leq C\lambda\) a.e. By definition of \(h\), the last expression is at most

\[
    \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| M\omega dx + \sum_j \frac{C}{\lambda} \int_{Q_j} \left( \frac{1}{|Q_j|} \int_{Q_j} |f(z)| dz \right) M(w\chi_{\mathbb{R}^n \setminus \Omega})(x) dx. \tag{4.3}
\]
From the property (iii) of Lemma 4.2, we know that for \( x, z \in Q_j \) there is a constant \( C \) depending only on \( n \) so that \( M(w\chi_{\mathbb{R}^n\setminus\Omega})(x) \leq CM(w\chi_{\mathbb{R}^n\setminus\Omega})(z) \). Thus (4.3) is less than

\[
\frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, Mw \, dx + \sum_j \frac{C}{|Q_j|} \int_{Q_j} \left( \frac{1}{|Q_j|} \int_{Q_j} |f(z)| Mw(z) \, dz \right) \, dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, Mw \, dx.
\]

This gives

\[
w\{ x \notin \Omega : g_{\mu,\psi}^*(h)(x) > \lambda/2 \} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, Mw \, dx.
\]

Therefore, estimate (4.2) will follow if we show that

\[
w\{ x \notin \Omega : g_{\mu,\psi}^*(b)(x) > \lambda/2 \} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, Mw \, dx. \quad (4.4)
\]

To prove (4.4), we follow an idea of [17] to decompose \( b = \sum_j b_j = \sum_j \Phi_j(\sqrt{L})b_j + \sum_j \left( 1 - \Phi_j(\sqrt{L}) \right) b_j \), where \( \Phi_j(\sqrt{L}) = \Phi \left( \frac{\ell(Q_j)}{32} \sqrt{L} \right) \), \( \Phi \) is the function as in Lemma 2.2 and \( \ell(Q_j) \) is the side length of the cube \( Q_j \). See also [11]. So, it reduces to show that

\[
w\{ x \notin \Omega : g_{\mu,\psi}^* \left( \sum_j \Phi_j(\sqrt{L})b_j \right)(x) > \lambda/4 \} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, Mw \, dx \quad (4.5)
\]

and

\[
w\{ x \notin \Omega : g_{\mu,\psi}^* \left( \sum_j \left( 1 - \Phi_j(\sqrt{L}) \right) b_j \right)(x) > \lambda/4 \} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, Mw \, dx. \quad (4.6)
\]

By Chebychev’s inequality and Theorem 3.7 again, we have

\[
\text{LHS of (4.5)} \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} \left| g_{\mu,\psi}^* \left( \sum_j \Phi_j(\sqrt{L})b_j \right) \right|^2 (w\chi_{\mathbb{R}^n\setminus\Omega}) \, dx \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} \left| \sum_j \Phi_j(\sqrt{L})b_j \right|^2 M(w\chi_{\mathbb{R}^n\setminus\Omega}) \, dx.
\]

Note that \( \Phi_j(\sqrt{L}) = \Phi \left( \frac{\ell(Q_j)}{32} \sqrt{L} \right) \), it follows from Lemma 2.1 that \( \text{supp} \, \Phi_j(\sqrt{L})b_j \subset 17Q_j/16 \) and \( |K_{\Phi_j(\sqrt{L})}(x, y)| \leq C/\ell(Q_j)^n \). Hence, the above inequality is at most

\[
\frac{C}{\lambda^2} \sum_j \int_{\mathbb{R}^n} \left| \Phi_j(\sqrt{L})b_j \right|^2 M(w\chi_{\mathbb{R}^n\setminus\Omega}) \, dx.
\]
This, together with Lemma 2.1 and the definition of \( b \), yields

\[
\text{LHS of (4.5)} \leq \frac{C}{\lambda^2} \sum_j \int_{Q_j} \left( \ell(Q_j)^{-n} \int_{Q_j} |b(y)|dy \right)^2 M(w\chi_{\mathbb{R}^n \setminus \Omega})(x) dx
\]
\[
\leq \frac{C}{\lambda^2} \sum_j \int_{Q_j} \left( \frac{1}{|Q_j|} \int_{Q_j} |f(y)|dy \right)^2 M(w\chi_{\mathbb{R}^n \setminus \Omega})(x) dx
\]
\[
\leq \frac{C}{\lambda} \sum_j \frac{1}{|Q_j|} \int_{Q_j} \int_{Q_j} |f(y)| M(w\chi_{\mathbb{R}^n \setminus \Omega})(y) dy dx
\]
\[
\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f|M w dx.
\]

This proves the desired estimate (4.5).

Next we turn to estimate (4.6). It suffices to show that

\[
\sum_j \int_{\mathbb{R}^n \setminus \Omega} g_{\mu,\Psi}^*(\left(1 - \Phi_j(\sqrt{L})\right)b_j) w dx \leq C \int_{\mathbb{R}^n} |f|M w dx.
\]

Further, the above inequality reduces to prove the following result:

\[
\int_{\mathbb{R}^n \setminus \Omega} g_{\mu,\Psi}^*(\left(1 - \Phi_j(\sqrt{L})\right)b_j) w dx \leq C \int_{Q_j} |f|M w dx.
\] (4.7)

Let \( x_j \) denote the center of \( Q_j \). Let us estimate \( \Psi(t\sqrt{L}) \left(1 - \Phi_j(\sqrt{L})\right)b_j(y) =: \psi_{t\sqrt{L}}(\sqrt{L})b_j(y) \) by considering two cases: \( t \leq \ell(Q_j)/4 \) and \( t > \ell(Q_j)/4 \).

**Case 4.3.** \( t \leq \ell(Q_j)/4 \). In this case, we use Lemma 2.1 to obtain

\[
\left| \psi_{t\sqrt{L}}(\sqrt{L})b_j(y) \right| \leq \left| \Psi(t\sqrt{L})b_j(y) \right| + \left| \Psi(t\sqrt{L})\Phi_j(\sqrt{L})b_j(y) \right|
\]
\[
\leq \left| \int_{Q_j} K_{\Psi_{t\sqrt{L}}} (y, z)b(z) dz \right|
\]
\[
+ \left| \int_{\mathbb{R}^n \setminus Q_j} K_{\Psi_{t\sqrt{L}}} (y, z) \left( \int_{Q_j} K_{\Phi_j(\sqrt{L})} (z, x)b(x) dx \right) dz \right|
\]
\[
\leq C \|b_j\|_1 t^{-n}.
\]

**Case 4.4.** \( t > \ell(Q_j)/4 \). Using Lemma 2.2, we have

\[
\left| \psi_{t\sqrt{L}}(\sqrt{L})b_j(y) \right| \leq \int_{\mathbb{R}^n} K_{\Psi_{t\sqrt{L}}} \left(1 - \Phi_j(\sqrt{L})\right)(y, z) |b_j(z)| dz
\]
\[
\leq C \|b_j\|_1 \ell(Q_j)t^{-n-1}.
\]

From the property (iii) of Lemma 4.2, we know that if \( x \notin \Omega \), then \(|x - x_j| > (\sqrt{n} + 1/2)\ell(Q_j)\). By Lemma 2.1, we have \( \Psi(t\sqrt{L}) \left(1 - \Phi_j(\sqrt{L})\right)b_j(y) = 0 \) unless
\[ |y - x_j| \leq t + (1/32 + \sqrt{n}/2)\ell(Q_j). \] Note that for \( x \notin \Omega \), 0 < \( t \leq \ell(Q_j)/4 \) and
\[ |y - x_j| \leq t + (1/32 + \sqrt{n}/2)\ell(Q_j), \quad |x - y| \geq |x - x_j| - |y - x_j| > \frac{\sqrt{n}+1/2}{\sqrt{n}+1/2} |x - x_j|. \]

Denote \( F_j := \{ y : |y - x_j| < (9/32 + \sqrt{n}/2)\ell(Q_j) \} \). Then for \( x \notin \Omega \) and \( \mu > 3 \), we have
\[
\left( \int_0^{\ell(Q_j)/4} \int_{F_j} |\Psi_{jt}(\sqrt{L})b_j(y)|^2 \left( \frac{t}{t + |x - y|} \right)^{n\mu/(n + 1)} \, dy \, dt \right)^{1/2} \leq C \| b_j \|_{L^1(\Omega_1)} \ell(Q_j)^{-n}\]
\[
\leq C \frac{\| b_j \|_{L^1(\Omega_1)}}{|x - x_j|^n} \ell(Q_j)^{-n}\]
\[
\leq C \frac{\| b_j \|_{L^1(\Omega_1)}}{|x - x_j|^n} \ell(Q_j)^{-n}\]
\[
\leq C \| f \chi_{Q_j} \|_{L^1(\Omega_1)} \ell(Q_j)^{-n}\]

where \( \tau(x) = 1/(1 + |x|)^{\mu/2} \). On the other hand, for 3 < \( \mu < 4 \), if \( n > 1 \),
\[
\left( \int_{\ell(Q_j)/4}^{\ell(Q_j)} \int_{E_{jt}} |\Psi_{jt}(\sqrt{L})b_j(y)|^2 \left( \frac{t}{t + |x - y|} \right)^{n\mu/(n + 1)} \, dy \, dt \right)^{1/2} \leq C \frac{\| b_j \|_{L^1(\Omega_1)}}{|x - x_j|^n} \ell(Q_j)^{-n}\]
\[
\leq C \frac{\| b_j \|_{L^1(\Omega_1)}}{|x - x_j|^n} \ell(Q_j)^{-n}\]
\[
\leq C \frac{\| b_j \|_{L^1(\Omega_1)}}{|x - x_j|^n} \ell(Q_j)^{-n}\]
\[
\leq C \| f \chi_{Q_j} \|_{L^1(\Omega_1)} \ell(Q_j)^{-n}\]
where $P(x) = 1/(1 + |x|)^{n+1}$.

Finally, since $t/(t + |x - y|) \leq 1$, so
\[
\left( \int_{|x-x_j|/4}^{\infty} \int_{E_j \setminus (x-x)\{/4} |\Psi_{jt}(\sqrt{L})b_j(y)| \left( \frac{t}{t + |x - y|} \right)^{n\mu} dy dt \right)^{1/2} \leq C \|b_j\|_1 \ell(Q_j) \left( \int_{|x-x_j|/4}^{\infty} t^{-2n-3} dt \right)^{1/2}
\]
\[
\leq C |f|_{X^s_j} \star P_{\ell(Q_j)}(x).
\]

Therefore, if $x \notin \Omega$, and $n > 1$, then $g_{\mu, \Psi}^* \left( \left( 1 - \Phi_j(\sqrt{L}) \right) b_j \right) (x) \leq C |f|_{X^s_j} \star P_{\ell(Q_j)}(x)$. And
\[
\int_{\mathbb{R}^n \setminus \Omega} g_{\mu, \Psi}^* \left( \left( 1 - \Phi_j(\sqrt{L}) \right) b_j \right) w dx \leq C \int_{\mathbb{R}^n \setminus \Omega} |f|_{X^s_j} \star P_{\ell(Q_j)} w dx
\]
\[
\leq C \int_{Q_j} |f| (P_{\ell(Q_j)} \ast w) dx
\]
\[
\leq C \int_{Q_j} |f| M w dx.
\]

If $n = 1$ we get the same thing, but with $P$ replaced by $\sigma$. This concludes the proof of (4.2), and then proves (a) of Theorem 4.1. Hence by interpolation, (b) of Theorem 4.1 is obtained.

\[\square\]

4.2. Estimate for $2 < p < \infty$

We now prove (c) of Theorem 4.1 by duality. Let $h(x) \geq 0$ and $h \in L^{(p/2)'}(w dx)$. By Theorem 3.7, we have
\[
\int_{\mathbb{R}^n} g_{\mu, \Psi}^* (f)^2 h w dx \leq C \int_{\mathbb{R}^n} |f|^2 M(h w) dx \leq C \int_{\mathbb{R}^n} |f|^2 M(w) M_w(h) dx,
\]
where
\[
M_w(h)(x) = \sup_{t > 0} \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} h w dx \right).
\]
Applying the H"older inequality with exponents $p/2$ and $(p/2)'$, we obtain the bound
\[
C \left( \int_{\mathbb{R}^n} |f|^p (M_w)^{p/2} w^{-(p/2-1)} dy \right)^{2/p} \left( \int_{\mathbb{R}^n} M_w(h)^{(p/2)'} w dy \right)^{(p-2)/p}.
\]

However, since $M_w$ is the centered maximal function, we have
\[
\int_{\mathbb{R}^n} M_w(h)^{(p/2)'} w dx \leq C_{n,p} \int_{\mathbb{R}^n} h^{(p/2)'} w dx,
\]
by a standard argument based on the Besicovitch covering lemma. Since $h$ is arbitrary, we obtain our result.
5. Proof of Theorem 1.4

In order to prove Theorem 1.4, we first prove the following result.

**Lemma 5.1.** Let $T$ be of the area functions $s_h$, $s_p$, $S_H$ and $g^{*}_{\mu, \Psi}$ with $\mu > 3$. Under assumptions of Theorems 1.1, 1.2 and 4.1, for $w \in A_p$, $1 < p < \infty$, we have

$$\| Tf \|_{L^p_w(\mathbb{R}^n)} \leq C \| f \|_{L^p_w(\mathbb{R}^n)}$$

where constant $C$ depends only on $p$, $n$ and $w$.

**Proof.** Let $T$ be of the area functions $s_h$, $s_p$, $S_H$ and $g^{*}_{\mu, \Psi}$ with $\mu > 3$. Note that if $w \in A_1$, then $Mw \leq Cw$ a.e. By Theorems 1.1, 1.2 and 4.1, $T$ is bounded on $L^p_w(\mathbb{R}^n)$, $1 < p < \infty$, for any $w \in A_1$, i.e.,

$$\| Tf \|_{L^p_w(\mathbb{R}^n)} \leq C \| f \|_{L^p_w(\mathbb{R}^n)}.$$

By extrapolation theorem, these operators are all bounded on $L^p_w(\mathbb{R}^n)$, $1 < p < \infty$, for any $w \in A_p$, and estimate (5.1) holds. For the detail, we refer the reader to Proposition 3.21 in [15].

Going further, we introduce some definitions. Given a weight $w$, set $w(E) = \int_E w(x)dx$. The non-increasing rearrangement of a measurable function $f$ with respect to a weight $w$ is defined by (cf. [10])

$$f^*_w(t) = \sup_{w(E) = t} \inf_{x \in E} |f(x)| \quad (0 < t < w(\mathbb{R}^n)).$$

If $w \equiv 1$, we use the notation $f^*(t)$.

Given a measurable function $f$, the local sharp maximal function $M^\#_\lambda f$ is defined by

$$M^\#_\lambda f(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} (f - c) \chi_Q^* (\lambda |Q|) \quad (0 < \lambda < 1).$$

This function was introduced by Strömberg [32], and motivated by an alternate characterization of the space $BMO$ given by John [25].

**Lemma 5.2.** For any $w \in A_p$ and for any locally integrable function $f$ with $f^*_w(+\infty) = 0$ we have

$$\| Mf \|_{L^p_w(\mathbb{R}^n)} \leq C \| w \|_{A_p}^{\gamma_{p,q}} \| M^\#_\lambda (|f|^q) \|_{L^{p/q}_w(\mathbb{R}^n)}^{1/q} \quad (1 < p < \infty, 1 \leq q < \infty),$$

where $\gamma_{p,q} = \max\{1/q, 1/(p-1)\}$, $C$ depends only on $p$, $q$ and on the underlying dimension $n$, and $\lambda_n$ depends only on $n$.

For the proof of this lemma, see Theorem 3.1 in [26].

**Proposition 5.3.** Let $g^{*}_{\mu, \Psi}$ be a function with $\mu > 3$ in (3.1). Then for any $f \in C_0^\infty(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$,

$$M^\#_\lambda (g^{*}_{\mu, \Psi}(f)^2) (x) \leq C Mf(x)^2,$$

where $C$ depends on $\lambda$, $\mu$, $\Psi$ and $n$. 
Proof. Given a cube $Q$, let $T(Q) = \{(y, t) : y \in Q, 0 < t < l(Q)\}$, where $\ell(Q)$ denotes the side length of $Q$. For $(y, t) \in T(Q)$, using (2.5) of Lemma 2.1 we have

$$\Psi(t \sqrt{\ell}) f(y) = \Psi(t \sqrt{\ell}(f \chi_Q)(y). \tag{5.3}$$

Now, fix a cube $Q$ containing $x$. For any $z \in Q$ we decompose $g_{\mu, \Psi}(f)^2$ into the sum of

$$I_1(z) = \int \int_{T(2Q)} |\Psi(t \sqrt{\ell}) f(y)|^2 \left( \frac{t}{t + |z - y|} \right)^{n\mu} \frac{dydt}{t^{n+1}},$$

and

$$I_2(z) = \int \int_{\mathbb{R}^n_+ \setminus T(2Q)} |\Psi(t \sqrt{\ell}) f(y)|^2 \left( \frac{t}{t + |z - y|} \right)^{n\mu} \frac{dydt}{t^{n+1}}.$$ 

From Theorem 4.1, we know that for $\mu > 3$, $g_{\mu, \Psi}(f)$ is of weak type $(1, 1)$. Then using (5.3), we have

$$(I_1)^*(\lambda |Q|) \leq \left( g_{\mu, \Psi}(f \chi_Q) \right)^*(\lambda |Q|)^2 \leq \left( \frac{C}{\lambda |Q|} \int_{6Q} |f| \right)^2 \leq CMf(x)^2. \tag{5.4}$$

Further, for any $z_0 \in Q$ and $(y, t) \notin T(2Q)$, by the Mean Value Theorem,

$$|(t + |z - y|)^{-n\mu} - (t + |z_0 - y|)^{-n\mu}| \leq C\ell(Q)(t + |z - y|)^{-n\mu-1}.$$ 

From this and (5.3), using Lemma 2.1 again and $\mu > 3$, we have

$$|I_2(z) - I_2(z_0)| \leq C\ell(Q) \int \int_{\mathbb{R}^n_+ \setminus T(2Q)} |\Psi(t \sqrt{\ell}) f(y)|^2 \left( \frac{1}{t + |z - y|} \right)^{n\mu+1} \frac{dydt}{t^{n+1}} \leq \frac{C}{2^{k+1}} \left( \frac{2^{k+1} \ell(Q)}{|2^{k+1} \ell(Q)^n|} \right)^{n\mu} \left( \int_{2^{k+1} \ell(Q)}^{\infty} t^{n\mu-3n-1} dt \right) \left( \int_{6Q}^{|f|} \right)^2 \leq CMf(x)^2.$$ 

Combining this estimate with (5.4) yields

$$\inf_c \left( (g_{\mu, \Psi}(f)^2 - c) \chi_Q \right)^*(\lambda |Q|) \leq ((I_1 + I_2 - I_2(z_0)) \chi_Q)^*(\lambda |Q|) \leq (I_1)^*(\lambda |Q|) + CMf(x)^2 \leq CMf(x)^2,$$

which proves the desired result. \qed

Then we have the following result.
Theorem 5.4. Let $T$ be of the area functions $s_h$, $s_p$, $s_P$, $S_H$ and $g_{\mu,\Psi}^*$ with $\mu > 3$. Under assumptions of Theorems 1.1, 1.2 and 4.1, for $w \in A_p$, $1 < p < \infty$, if $\|f\|_{L_p^w(\mathbb{R}^n)} < \infty$, then

$$
\left( \int_{\mathbb{R}^n} (M(Tf))^p w \, dx \right)^{1/p} \leq C \|w\|_{A_p}^{\beta_p} \left( \int_{\mathbb{R}^n} (M(f))^p w \, dx \right)^{1/p},
$$

(5.5)

where $\beta_p = \max\{1/2, 1/(p-1)\}$, and a constant $C$ depends only on $p$ and $n$.

Proof. Suppose $T = g_{\mu,\Psi}^*$. From Lemma 5.1, we know that $g_{\mu,\Psi}^*$ is bounded on $L_p^w(\mathbb{R}^n)$ when $w \in A_p$. Therefore, assuming that $\|f\|_{L_p^w(\mathbb{R}^n)}$ is finite, we clearly obtain that $(g_{\mu,\Psi}^*)^w(\mathbb{R}^n) = 0$. Letting $g_{\mu,\Psi}^*(f)$ instead of $f$ in (5.2) with $q = 2$ and applying Proposition 5.3, we get

$$
\left( \int_{\mathbb{R}^n} (M(g_{\mu,\Psi}^*(f))^p w \, dx \right)^{1/p} \leq C \|w\|_{A_p}^{\beta_p} \left( \int_{\mathbb{R}^n} (M(f))^p w \, dx \right)^{1/p}.
$$

Under assumptions of Theorems 1.1 and 1.2, it follows that the area functions $s_h$ and $S_H$ are all controlled by $g_{\mu,\Psi}^*$ pointwise. So we have the estimate (5.5) for $s_h$, $s_p$, $S_P$ and $S_H$. Then the proof of this theorem is complete. $\square$

Proof. In [4], Buckley proved that for the Hardy–Littlewood maximal operator,

$$
\|M\|_{L_p^w(\mathbb{R}^n)} \leq C \|w\|_{A_p}^{1/(p-1)} \quad (1 < p < \infty),
$$

(5.6)

and this result is sharp.

From (5.6) and Theorem 5.4, there exists a constant $C = C(T, n, p)$ such that for all $w \in A_p$,

$$
\|T\|_{L_p^w(\mathbb{R}^n)} \leq C \|w\|_{A_p}^{1/p + \max\left\{1, \frac{1}{p-1}\right\}} \quad (1 < p < \infty),
$$

(5.7)

where $T$ is of the area functions $s_h$, $s_p$, $S_P$ and $S_H$. This proves Theorem 1.4. $\square$

Remark. (i) Note that when $L = \Delta$ is the Laplacian on $\mathbb{R}^n$, it is well known that the exponents $\beta_p$ of (5.5) in Theorem 5.4 is best possible, in general (see, e.g., Theorem 1.5, [26]).

Sharp weighted optimal bounds for singular integrals has been studied extensively, see for examples, [14, 24, 28, 29, 31] and the references therein.

(ii) Finally, for $f \in \mathcal{S}(\mathbb{R}^n)$, we define the (so called vertical) Littlewood–Paley–Stein functions $\mathcal{G}_P$ and $\mathcal{G}_H$ by

$$
\mathcal{G}_P(f)(x) = \left( \int_0^\infty \left| t \nabla e^{-t\sqrt{\Delta}} f(x) \right|^2 \frac{dt}{t} \right)^{1/2},
$$

as well as the (so-called horizontal) Littlewood–Paley–Stein functions $g_P$ and $g_H$ by

$$
g_P(f)(x) = \left( \int_0^\infty \left| t \sqrt{\Delta} e^{-t\sqrt{\Delta}} f(x) \right|^2 \frac{dt}{t} \right)^{1/2},
$$

$$
g_H(f)(x) = \left( \int_0^\infty \left| t^2 \sqrt{\Delta} e^{-t\sqrt{\Delta}} f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.
$$

One then has the analogous statement as in Theorems 1.1, 1.2, 1.3 and 1.4 replacing $s_p, s_h, S_P, S_H$ by $g_P, g_H, \mathcal{G}_P, \mathcal{G}_H$, respectively.
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