DYNAMICAL DEGREES OF (PSEUDO)-AUTOMORPHISMS
FIXING CUBIC HYPERSURFACES

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Abstract. We give a way to construct group of pseudo-automorphisms of rational varieties of any dimension that fix pointwise the image of a cubic hypersurface of \( \mathbb{P}^n \). These group are free products of involutions, and most of their elements have dynamical degree \( > 1 \). Moreover, the Picard group of the varieties obtained is not big, if the dimension is at least 3.

We also answer a question of E. Bedford on the existence of birational maps of the plane that cannot be lifted to automorphisms of dynamical degree \( > 1 \), even if we compose them with an automorphism of the plane.

1. Introduction

A birational map \( \varphi : \mathbb{P}^n \to \mathbb{P}^n \) (or a Cremona transformation) is a rational map given by

\[
(x_0 : \cdots : x_n) \mapsto (P_0(x_0, \ldots, x_n) : \cdots : P_n(x_0, \ldots, x_n)),
\]

where all \( P_i \) are homogeneous polynomials of the same degree, which admits an inverse of the same type. Choosing all \( P_i \) without common component, the degree \( \deg(\varphi) \) of \( \varphi \) is by definition the degree of the polynomials \( P_i \), or equivalently the degree of the pull-back of hyperplanes of \( \mathbb{P}^n \) by \( \varphi \).

The (first) dynamical degree of \( \varphi \) is the number

\[
\lim_{n \to \infty} \left( \deg(\varphi^n) \right)^{1/n},
\]

which always exists, since \( \deg(\varphi^{a+b}) \leq \deg(\varphi^a) \cdot \deg(\varphi^b) \) for any \( a, b \geq 0 \). It is moreover invariant under conjugation.

There is a sequence of articles which provide families of examples of birational maps of \( \mathbb{P}^2 \) with dynamical degree \( > 1 \), lifting to automorphisms of a smooth rational surface obtained by blowing-up a finite number of points (among them, see [BedKim06], [BedDil06], [McM07], [BedKim09], [BedKim10], [Dil11], [DesGri10]...).

The general way of producing examples is to start by a simple birational map (quadratic involution, automorphism of the affine plane,...), and to compose it with a linear automorphism of \( \mathbb{P}^2 \) to impose that the base-points of the inverse "come back" to the base-points of the map after a certain number of iteration of the map.

This approach was generalised in dimension 3, in [BedKim11], to provide pseudo-automorphisms of projective 3-folds of dynamical degree \( > 1 \), starting from a special family of quadratics elements of Bir(\( \mathbb{P}^3 \)). Recall that a pseudo-automorphism of \( X \) is a birational self-map \( \varphi \in \text{Bir}(X) \) such that \( \varphi \) and \( \varphi^{-1} \) do not contract any codimension 1 set; it is the same as automorphism for smooth projective surfaces.

1991 Mathematics Subject Classification. 37F10, 32H50, 14J50, 14E07.
Supported by the SNSF grant no PP00P2_128422 /1.
Other examples of pseudo-automorphisms of dynamical degree > 1 of rational projective varieties of any dimension were given in [PerZha11], using actions of Weyl groups on blow-ups of \( \mathbb{P}^n \) at a finite number of points.

In this article, we give another way of constructing examples, which also works in any dimension. This produces large groups of pseudo-automorphisms, where almost all elements have dynamical degree > 1. Moreover, the rank of the Picard group of the varieties obtained does not need to be very large, contrary to what happens in dimension 2, or to the examples of [BedKim11] and [PerZha11].

We recall the following construction, defined in [Giz94, page 42, Example 3] over the name of \( \mathbb{R}^p \).

\textbf{Definition 1.1.} Let \( Q \subset \mathbb{P}^n \) be a cubic hypersurface, and let \( p \in Q \) be a smooth point. We define an involution \( \sigma_{p,Q} \in \text{Bir}(\mathbb{P}^n) \) which fixes pointwise \( Q \) by the following: if \( L \) is a general line of \( \mathbb{P}^n \) passing through \( p \), we have \( \sigma_{p,Q}(L) = L \) and the restriction of \( \sigma_{p,Q} \) to \( L \) is the involution that fixes \( (L \cap Q) \setminus \{p\} \).

From the geometric definition, we can easily get an algebraic definition by polynomials (see [Giz94] or Section 2). We will show that any \( \sigma_{p,Q} \) lifts to an automorphism of a smooth variety obtained by blowing-up codimension 2 subsets of \( \mathbb{P}^n \). Taking a finite number of points on the same cubic gives a huge group of pseudo-automorphisms of a rational \( n \)-fold:

\textbf{Theorem 1.} Let \( Q \subset \mathbb{P}^n \) be a cubic hypersurface, and let \( p_1, \ldots, p_k \in Q \) be distinct smooth points. For \( i = 1, \ldots, k \), we write \( \Gamma_i = \{ x \in Q \mid \text{the line through } x \text{ and } q_i \text{ is tangent to } Q \text{ at } x \} \).

Denote by \( \pi : X \rightarrow \mathbb{P}^n \) the following birational morphism: it first blows-up all points \( p_1, \ldots, p_k \), then blows-up the strict transform of \( \Gamma_1 \), then the strict transform of \( \Gamma_2 \) and so on until blowing-up the strict transform of \( \Gamma_k \).

If \( p_i \notin \Gamma_j \) for any \( i \neq j \), then \( \sigma_{p_1,Q}, \ldots, \sigma_{p_k,Q} \) lift to pseudo-automorphisms \( \hat{\sigma}_1, \ldots, \hat{\sigma}_k \) of \( X \), that generate a free product
\[
G = \ast_{i=1}^k \langle \hat{\sigma}_i \rangle,
\]
having the following properties:

1. Any element of \( G \) of finite order is conjugate to \( \hat{\sigma}_i \) for some \( i \) (and has dynamical degree 1);
2. Any element conjugate to \( (\hat{\sigma}_i \hat{\sigma}_j)^m \) for \( i \neq j \) and \( m \geq 1 \) is of infinite order, and its dynamical degree is equal to 1 if and only if \( d = 3 \);
3. Any other element has dynamical degree > 1.
4. Each element of \( G \) fixes pointwise the lift of the cubic \( Q \) on \( X \).

\textbf{Corollary 1.2.} For any \( n \geq 3 \), there exist a rational smooth \( n \)-fold \( X \) with \( \text{rk Pic}(X) = 7 \) admitting a group of pseudo-automorphisms \( G \) isomorphic to the free group with 2 generators, such that all elements of \( G \setminus \{1\} \) have dynamical degree > 1. Moreover, \( G \) fixes pointwise an hypersurface isomorphic to a general smooth cubic of \( \mathbb{P}^n \).

\textbf{Remark 1.3.} Any pseudo-automorphism of a smooth projective surface \( X \) with \( \text{rk Pic}(X) \leq 10 \) has dynamical degree 1. All previously known examples of smooth rational varieties admitting pseudo-automorphisms with dynamical degree > 1 had Picard rank bigger than 10.
Section 2 is devoted to the proof of Theorem 1 and of its corollary.

**Question 1.4.** What is the minimal rank of a smooth projective rational 3-fold admitting an automorphism of dynamical degree > 1?

Restricting to dimension $n = 2$, Theorem 1 gives the existence of group of automorphisms of rational surfaces with many elements of dynamical degree > 1. The rank of the Picard group is however quite large, at least 16. This is because the varieties $\Gamma_i$ are in fact union of 4 distinct points.

As we said above, the usual way to construct automorphisms of projective rational surfaces with dynamical degree > 1 is to take a birational map of small degree, and then to compose it with an automorphism so that all base-points of the inverse are sent after some iterations onto base-points of the map. This approach gives rise to the following question of Eric Bedford, also stated and studied by Julie Déserti and Julien Grivaux in [DesGri10].

**Question 1.5.** Does there exist a birational map of the projective plane $\varphi$ of degree > 1 such that for all $\tau \in \text{Aut}(\mathbb{P}^2)$ the map $\tau \varphi$ is not birationally conjugate to an automorphism of dynamical degree > 1?

Here, by ”conjugate to an automorphism”, we mean the existence of a birational map $\nu: \mathbb{P}^2 \rightarrow X$, where $X$ is a projective smooth surface, such that $\nu(\tau \varphi)\nu^{-1} \in \text{Aut}(X)$.

It is know that if $\varphi$ is a birational map of degree 2, there exists an automorphism $\tau$ such that $\tau \varphi$ is conjugate to an automorphism of dynamical degree > 1. We recall this fact in Section 3. The same was also proved by different authors for some special maps $\varphi$ of degree 3. Using the involutions $\sigma_{p,Q}$, we prove in Section 3 that the same holds for a general map of degree 3.

The possible map $\varphi$ of Question 1.5 cannot thus be a general cubic transformation, or a transformation of degree 2. Section 4 is devoted to the proof of the following result, showing that the map can be of degree 6, and answering the question of Bedford, Déserti and Grivaux:

**Theorem 2.** Let $\chi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birational map given by

$$
\chi: (x : y : z) \dashrightarrow (x z^5 + (y z^2 + x^3)^2 : y z^5 + x^3 z^3 : z^6).
$$

For any automorphism $\tau \in \text{Aut}(\mathbb{P}^2)$, the birational map $\tau \chi \in \text{Bir}(\mathbb{P}^2)$ is not conjugate to an automorphism of a smooth projective rational surface.

**Question 1.6.** Does there exists a transformation of degree < 6 having the above property?

**Acknowledgements.** The author thanks Julie Déserti for interesting discussions on the topic during the preparation of the article.

2. **The Maps $\sigma_{p,Q}$, Their Lifts and the Groups Generated by These**

Let us describe algebraically the map $\sigma_{p,Q}$ introduced in Definition 1.1.

For this, choose homogeneous coordinates $(x_1 : x_2 : \cdots : x_n : y)$ on $\mathbb{P}^n$ and assume, up to a change of coordinates, that $p$ is equal to $(0 : \cdots : 0 : 1)$. The equation of $Q$ is thus $y^2 P_1 + y P_2 + P_3$, where $P_1, P_2, P_3 \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous of degree 1, 2, 3. The involution $\sigma_{p,Q}$ sends a point $(x_1 : x_2 : \cdots : x_n : y)$ onto

$$(\text{if } x_1 P_1 + x_2 P_2 + x_n P_3) : \cdots : \text{if } x_1 P_1 + x_2 P_2 + x_n P_3 : P_2 y + 2 P_3).$$
The point $p$ is a base-point of multiplicity 2. The subscheme $\Gamma_p \subset Q \subset \mathbb{P}^n$ of codimension 2 given by $P_2 + 2yP_3 = 0$ and $yP_2 + 2P_3 = 0$ is also contained in the base-locus, and $\sigma_{p,Q}$ is defined on $\mathbb{P}^n \setminus \Gamma_p$.

The cone $V_p \subset \mathbb{P}^n$ given by $(P_2)^2 - 4P_3P_1$ is contracted onto $\Gamma_p$ by $\sigma_{p,Q}$, and the hypersurface given by $P_2 + 2yP_1$ is contracted onto the point $p$.

Note that $\Gamma_p$ is also given by the intersection of $Q$ with the hypersurface of equation $P_2 + 2yP_1 = 0$, or with the cone $V_p$, and corresponds to the points $q \in Q$ such that the line passing through $p$ and $q$ is tangent to $Q$ at $q$, as defined in the introduction.

**Proposition 2.1.** Denote by $\pi_p: X_p \to \mathbb{P}^n$ the blow-up of $p$, by $\pi_\Gamma: X \to X_p$ the blow-up of the strict transform of $\Gamma_p$ and write $\pi = \pi_p \circ \pi_\Gamma: X \to \mathbb{P}^n$. The lift $\tilde{\sigma}_p = \pi^{-1}\sigma\pi$ of $\sigma_{p,Q}$ is an automorphism of $X$.

Writing $H \subset \text{Pic}(X)$ the pull-back of an hyperplane of $\mathbb{P}^n$ by $\pi$, $E \subset \text{Pic}(X)$ the pull-back of $\pi_p^{-1}(p)$ by $\pi_\Gamma$ and $F$ the exceptional divisor of $\pi_\Gamma$, $(H,E,F)$ is a basis of a sub-$\mathbb{Z}$-module of $\text{Pic}(X)$ invariant by $\tilde{\sigma}_p$ (this sub-module is equal to $\text{Pic}(X)$ if and only if $\Gamma_p$ is irreducible). Moreover, the action of $\tilde{\sigma}_p$ relative to this basis is

$$
\begin{bmatrix}
    3 & 2 & 4 \\
    -2 & -1 & -4 \\
    -1 & -1 & -1
\end{bmatrix}.
$$

**Proof.** We can view $X_p$ in $\mathbb{P}^n \times \mathbb{P}^n$ as

$$
X_p = \left\{ \left( (x_1 : x_2 : \cdots : x_n : y), (z_1 : \cdots : z_n) \right) \mid x_iz_j = x_jz_i \text{ for } 1 \leq i, j \leq n \right\},
$$

where $\pi_p: X_p \to \mathbb{P}^n$ is given by the projection on the first factor. The variety $X_p$ is covered by open subsets $U_1, \ldots, U_n$, where $U_i$ is the set where $z_i \neq 0$.

Each $U_i$ is isomorphic to $\mathbb{A}^{n-1} \times \mathbb{P}^1$. For $i = 1$, the isomorphism is given by:

$$
\mathbb{A}^{n-1} \times \mathbb{P}^1 \xrightarrow{\sim} U_1 \\
((t_2, \ldots, t_n), (\alpha : \beta)) \mapsto \left( (\alpha : \alpha t_2 : \cdots : \alpha t_n : \beta), (1 : t_2 : \cdots : t_n) \right).
$$

The lift of $\sigma$ preserves $U_1$ and restricts to the following birational map

$$
((t_1, \ldots, t_n), (\alpha : \beta)) \dashrightarrow ((t_1, \ldots, t_n), (-\alpha R_2 + 2\beta R_1 : \beta R_2 + 2\alpha R_3)),
$$

where $R_i = P_i(1, t_2, \ldots, t_n)$. On this chart, $\Gamma_p$ is given by $\alpha R_2 + 2\beta R_1 = 0$ and $R_2\beta + 2\alpha R_3 = 0$. Blowing-up the corresponding ideal, we obtain the variety $W_1 \subset \mathbb{A}^n \times \mathbb{P}^1$ given by

$$
W_1 = \left\{ \left( (t_1, \ldots, t_n), (\alpha : \beta), (u : v) \right) \mid u(R_2\beta + 2\alpha R_3) = -v(\alpha R_2 + 2\beta R_1) \right\},
$$

and the blow-up is given by the projection $W_1 \to \mathbb{A}^n \times \mathbb{P}^1$ on the first two factors. The map $\tilde{\sigma}$ corresponds in these coordinates to

$$
((t_1, \ldots, t_n), (\alpha : \beta), (u : v)) \mapsto ((t_1, \ldots, t_n), (u : v), (\alpha : \beta)),
$$

and is thus an automorphism (the same calculation holds on the other charts).

Since $E$ is exchanged with the strict transform of the hypersurface of equation $P_2 + 2yP_1$, which has degree 2, passes through $p$ with multiplicity 1, and also through a general point of $\Gamma_p$, $E$ is sent onto $2H - E - F$. Moreover, $H$ is exchanged with hyperplanes of degree 3 having multiplicity 2 at $p$ and multiplicity one at a general point of $\Gamma_p$. This shows that $H$ is exchanged with $3H - 2E - F$. The fact that $\tilde{\sigma}_p$ is an involution gives the last column of the matrix, i.e. that $F$ is exchanged with
4H − 4E − F, which corresponds to the cone \((P_2)^2 + 4P_1P_3 = 0\), which has degree 4, passes through \(p\) with multiplicity 4, and through \(\Gamma_p\) with multiplicity 1. 

We now generalise the construction by taking many points on the same cubic hypersurface.

**Proposition 2.2.** Let \(Q \subset \mathbb{P}^n\) be a cubic hypersurface, and let \(p_1, \ldots, p_k \in Q\) be distinct smooth points. For \(i = 1, \ldots, k\) we write \(\sigma_i\) the map \(\sigma_{p_i, Q}\) given in Definition 1.1 and by \(\Gamma_i = \Gamma_{p_i} \subset \mathbb{P}^n\) the codimension 2 subset associated. We assume that for \(p_i \notin \Gamma_j\) for any \(i \neq j\).

Denote by \(\pi : X \rightarrow \mathbb{P}^n\) the following birational morphism: it first blows-up all points \(p_1, \ldots, p_k\), then blows-up the strict transform of \(\Gamma_1\), then the strict transform of \(\Gamma_2\) and so on until blowing-up the strict transform of \(\Gamma_k\).

Then, \(\sigma_1, \ldots, \sigma_k\) lift to pseudo-automorphisms \(\hat{\sigma}_1, \ldots, \hat{\sigma}_k\) of \(X\).

**Proof.** Applying Proposition 2.1 \(\sigma_i\) lifts to an automorphism of a variety obtained by blowing-up first \(p_i\) and then the strict transform of \(\Gamma_i\). Since \(p_j \notin \Gamma_i\) for \(j \neq i\), all these points correspond to points of the strict transform of \(Q\), and are thus fixed by the lift of \(\sigma_i\). We can thus blow-up all the points \(p_j\) with \(j \neq i\) and \(\sigma_i\) again lifts to an automorphism. The strict transforms of the lifts of \(\Gamma_j\) for \(j \neq i\) are again contained in the strict transform of \(W\) and thus fixed pointwise by the automorphism. We blow-up all these schemes and lift \(\sigma_i\) to an automorphism of a variety \(X_i\) obtained.

Note that \(X_1 = X\), but that in general \(X_i\) is not the same as \(X_j\) for \(i \neq j\), since the varieties \(\Gamma_i\) and \(\Gamma_j\) intersect: the choice in the order of the sets blown-up is important. The maps \(X_j \rightarrow X_i\) induced by the blow-ups are pseudo-isomorphisms, they are isomorphisms outside the pull-back of the sets where \(\Gamma_i\) and \(\Gamma_j\) intersect, which have codimension \(\geq 2\). This implies that \(\hat{\sigma}_1 \in \text{Aut}(X_1)\) and that all others \(\hat{\sigma}_i\) are pseudo-automorphisms of \(X\). 

The following proposition describes the group generated by these pseudo-automorphisms, and the dynamical properties of its elements. It yields – with Proposition 2.2 – the proof of Theorem 1.1

**Proposition 2.3.** Let \(\hat{\sigma}_1, \ldots, \hat{\sigma}_k \in \text{Bir}(X)\) be pseudo-automorphisms as in Proposition 2.2. These elements generate a free product

\[ G = \ast_{i=1}^k < \hat{\sigma}_i >, \]

and we have the following description of elements of \(G\):

1. Any element of finite order is conjugate to a \(\hat{\sigma}_i\) and has dynamical degree 1;
2. Any element conjugate to \((\hat{\sigma}_i\hat{\sigma}_j)^m\) for \(i \neq j\) and \(m \geq 1\) is of infinite order, and its dynamical degree is equal to 1;
3. Any other element has dynamical degree \(> 1\).

**Proof.** Let \(H \in \text{Pic}(X)\) be the pull-back of an hyperplane of \(\mathbb{P}^n\), denote by \(E_1, \ldots, E_k \in \text{Pic}(X)\) the total pull-back of the divisors obtained by blowing-up the \(p_i\), and by \(F_1, \ldots, F_k \in \text{Pic}(X)\) the exceptional divisors associated to \(\Gamma_1, \ldots, \Gamma_k\), using the
satisfying the following properties.

$$\sigma_i(H) = \begin{cases} 3H & -2E_i - F_i, \\ 2H & -E_i - F_i, \\ 4H & -4E_i - F_i, \\ E_j & \text{for } i \neq j, \\ F_j & \text{for } i \neq j. \end{cases}$$

Writing $\nu_i = \sigma_i(H) - H = 2H - 2E_i - F_i$, we get

$$\sigma_i(H) = H + \nu_i, \quad \sigma_i(\nu_i) = -\nu_i, \quad \sigma_i(\nu_j) = \nu_j + 2\nu_i \text{ for } i \neq j.$$

Let us choose any element $\varphi = \sigma_{a_1} \cdots \sigma_{a_k}$, where $a_1, \ldots, a_r \in \{1, \ldots, k\}$, $a_i \neq a_{i+1}$ for $i = 1, \ldots, r - 1$. By induction on $r$, we prove that $\varphi(H) = H + \sum_{i=1}^{k} a_i \nu_i$, satisfying the following properties:

1. $a_1, \ldots, a_k$ are non-negative integers;
2. $a_i > a_i$ for $i \neq a_r$;
3. if $r > 1$, then $a_{a_{r-1}} > a_i$ for $i \notin \{a_r, a_{r-1}\}$;
4. $\sum_{i=1}^{k} a_i \geq (\frac{\beta}{4})^t$, where $t = \# \{i \mid i \geq 3, a_i \neq a_{i-2}\}$.

When $r = 1$, the result is obvious since $\varphi(H) = H + \nu_{a_1}$. We assume the result true for $r - 1$ and prove it for $r$. We have $\varphi(H) = \sigma_{a_r}(V)$, where $V = \sigma_{a_{r-1}} \circ \cdots \circ \sigma_1(H) = H + \sum_{i=1}^{k} \beta_i \nu_i$ and all $\beta_i$ satisfy the properties above. In particular,

$$\sum_{i \neq a_r} \beta_i \geq \beta_{a_{r-1}} > \beta_{a_r}.$$

Applying (4), we get

$$\begin{align*}
\alpha_i &= \beta_i \text{ for } i \neq a_r, \\
a_{a_r} &= 1 - \beta_{a_r} + 2 \sum_{i \neq a_r} \beta_i > \sum_{i \neq a_r} \beta_i \geq \beta_{a_{r-1}} > \beta_{a_r},
\end{align*}$$

which proves the first three assertions. To prove (iv), we compute

$$\sum_{i=1}^{k} a_i = 1 - 4\beta_{a_r} + 3 \sum_{i=1}^{k} \beta_i.$$

We always have $\sum_{i=1}^{k} a_i > \sum_{i=1}^{k} \beta_i$. It suffices thus to prove that if $r \geq 3$ and $a_r \neq a_{r-2}$, then $\sum_{i=1}^{k} a_i \geq \frac{\beta}{4} \sum_{i=1}^{k} \beta_i$.

The fact that $r \geq 3$ and $a_r \neq a_{r-2}$ gives $\beta_{a_r} < \beta_{a_{r-1}}$ and $\beta_{a_r} < \beta_{a_{r-2}}$, and thus implies

$$\sum_{i=1}^{k} a_i - \frac{\beta}{4} \sum_{i=1}^{k} \beta_i = 1 - 4\beta_{a_r} + (3 - \frac{\beta}{4}) \sum_{i=1}^{k} \beta_i,$$

$$= 1 - \frac{\beta}{4} \beta_{a_r} + \frac{\beta}{4} \sum_{i \neq a_r} \beta_i,$$

which is positive since $2\beta_{a_r} < \beta_{a_{r-1}} + \beta_{a_{r-2}} \leq \sum_{i \neq a_r} \beta_i$.

Now that (i) – (iv) have been proved, we show how they imply the result. First, assertions (i) and (ii) show that $G$ is the free product of the groups $< \sigma_1 > \cong \mathbb{Z}/2\mathbb{Z}$. 

Second, any non-trivial element of the group is conjugated to \( \varphi = \sigma_{a_1} \cdots \sigma_{a_r} \), where \( a_1, \ldots, a_r \in \{1, \ldots, k\}, a_i \neq a_{i+1} \) for \( i = 1, \ldots, r-1 \) and \( a_r \neq a_1 \).

The element \( \varphi \) has finite order if and only if \( r = 1 \). If \( r > 1 \), we compute its dynamical degree by computing \( \deg(\varphi^n) \) for \( n \in \mathbb{N} \). The degree here is the degree as a birational map of \( \mathbb{P}^3 \), which is the degree of the system \( \pi(\varphi^{-n}(H)) \). Since each \( \nu_i \) corresponds to a divisor of degree 2, we get

\[
\deg(\varphi^n) = 1 + 2 \sum_{i=1}^{k} \alpha_i \text{ if } \varphi^{-n}(H) = H + \sum_{i=1}^{k} \alpha_i \nu_i.
\]

The assertions above imply that if the set \( \{a_1, \ldots, a_r\} \) has at least three elements, \( \deg(\varphi^n) \geq \frac{\deg(\mathbb{P}^3)}{4} \), so the dynamical degree of \( \varphi \) is strictly bigger than 1. The only case where the dynamical degree 1 could be one is when \( \varphi = (\hat{\sigma}_i \hat{\sigma}_j)^m \) for \( i \neq j \) and \( m \geq 1 \). It remains to prove that in this case, the dynamical degree is 1; and we only have to consider the case \( m = 1 \). The submodule of \( \text{Pic}(\mathcal{X}) \) generated by \( H, \nu_i, \nu_j \) is invariant by \( \varphi \), and the action relative to this basis is

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 2 \\
0 & 0 & 1
\end{pmatrix}.
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 2 & -1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
3 & 3 & -2 \\
1 & 2 & -1
\end{pmatrix},
\]

which has only one eigenvalue, equal to 1. This achieves the proof.

\[\square\]

**Remark 2.4.** Note that the dynamical degree of any element of the free group \( G \) generated above is easy to compute.

(i) As we observed in the above proof, the dynamical degree of \( \sigma_i \cdot \sigma_j \), for \( i \neq j \), is the biggest eigenvalue of

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 2 \\
0 & 0 & 1
\end{pmatrix}.
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 2 & -1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
3 & 3 & -2 \\
1 & 2 & -1
\end{pmatrix},
\]

whose characteristic polynomial is \( (x - 1)^3 \). This dynamical degree is thus 1.

(ii) We can do a similar calculation with \( \sigma_i \cdot \sigma_j \cdot \sigma_k \) where \( i, j, k \) are pairwise distinct. The dynamical degree is the the highest real eigenvalue of

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & -1 & 2 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 2 & -1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
9 & 15 & 10 & -6 \\
3 & 6 & 3 & -2 \\
1 & 2 & 2 & -1
\end{pmatrix},
\]

which is \( 9 + 4\sqrt{5} \approx 17.944272 \).

(iii) All other dynamical degrees can be computed in the same way.

**Remark 2.5.** With the descriptions above, it is easy to take explicit cubic hypersurfaces, for example smooth ones, and to compute explicitly the locus to blow-up and the involutions.

Now that Theorem 1 is proved, we finish the section with the proof of its corollary.

**Proof of Corollary 1.2.** In any dimension \( n \geq 3 \), we take a smooth cubic hypersurface \( Q \subset \mathbb{P}^n \), and choose three distinct general points \( p_1, p_2, p_3 \) such that the line through two of them intersects the cubic into another point. These points in \( Q \) satisfy then the conditions of Theorem 1 and yields pseudo-automorphisms \( \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3 \).
of the variety $X$ obtained by blowing-up $p_1, p_2, p_3$ and the varieties $\Gamma_1, \Gamma_2, \Gamma_3$ associated. These latter being irreducible, the rank of $\text{Pic}(X)$ is exactly 7 (a fact which is false in dimension 2).

Because $<\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3>$ is the free product $\star_{i=3}^k <\hat{\sigma}_i>$, the group generated by $\alpha = \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3$ and $\beta = \hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_3\hat{\sigma}_2$ is the free group over two generators. Moreover, none of the non-trivial elements of the group is conjugate to an element of length $<3$, so each element has dynamical degree $>1$. □

3. The involutions on $\mathbb{P}^2$ and the blow-up

In this section, we deal with dimension 2.

3.1. Degree 2. We will say that two birational maps $\varphi, \varphi'$ are projectively equivalent if $\varphi = \alpha\varphi'\beta$ for some $\alpha, \beta \in \text{Aut}(\mathbb{P}^2)$. In Question 1.5, we can only study equivalence classes, since $\alpha\varphi\beta$ is conjugate to $\varphi(\beta\alpha^{-1})$. We can also replace $\varphi$ with $\varphi^{-1}$.

There are three equivalence classes of birational maps of $\mathbb{P}^2$ of degree 2. Each such map has three base-points of multiplicity 1, which are not collinear, and the classes correspond to

(i) three points $p_1, p_2, p_3$ that belong to $\mathbb{P}^2$ as proper points;
(ii) two points $p_1, p_2$ that belong to $\mathbb{P}^2$ as proper points, the point $p_3$ is infinitely near to $p_1$;
(iii) one point $p_1$ is a proper point of $\mathbb{P}^2$, $p_2$ is infinitely near to $p_1$ and $p_3$ is infinitely near to $p_2$.

There are many known examples of quadratic maps of type (i) that are conjugate to automorphisms of projective surfaces and have dynamical degree $>1$. See for example [BedKim06] or [BedKim09]. For examples of type (iii), see [BedKim10].

In fact, in [Dil11], all possible types of quadratic maps preserving cubics and being conjugate to automorphisms of projective surfaces and have dynamical degree $>1$ are constructed. They depend on an orbit data $[n_1, n_2, n_3]$, which provides all types (i), (ii), (iii) depending on the number of $n_i$ equal.

All these examples yield the following:

Lemma 3.1. If $\varphi$ is a birational map of $\mathbb{P}^2$ of degree 2, there exists an automorphism $\tau \in \text{Aut}(\mathbb{P}^2)$ such that $\tau\varphi$ is conjugate to an automorphism of a smooth projective rational surface with dynamical degree $>1$.

3.2. Degree 3. It is also possible to describe equivalence classes of elements of $\text{Bir}(\mathbb{P}^2)$ of degree 3. There are in fact 32 algebraic families, corresponding to the type of base-points (if some are collinear, or if some infinitely near,...), or equivalently to the curves contracted (see [CerDes08, Table of page 176]). The family of biggest dimension (dimension 2) consists of cubic maps $\varphi$ having five proper base-points, no 3 being collinear. All others have dimension $<1$.

We will prove that for a general cubic map $\varphi \in \text{Bir}(\mathbb{P}^2)$, there exists $\tau \in \text{Aut}(\mathbb{P}^2)$ such that $\tau\varphi$ is conjugate to an automorphism of positive entropy. To do this, we will use involutions $\sigma_{p,Q}$ associated to a point $p$ of a smooth cubic $\Gamma$ (see Definition 1.1). Recall that $\sigma_{p,Q}$ preserves a general line $L$ passing through $p$ and restricts on it to the unique involution that fixes the two points $(L \cap \Gamma) \setminus \{p\}$.

The base-points of $\sigma_{p,Q}$ are described by the following lemma:
Lemma 3.2. [Bla08, Proposition 12] \( C \subset \mathbb{P}^2 \) be a smooth cubic curve, let \( p \in C \) and let \( \sigma_{p,Q} \) be the element defined in Definition 1.1. The following occur:

1. The degree of \( \sigma_{p,Q} \) is 3, and \( \sigma_{p,Q}^2 = 1 \), i.e. \( \sigma_{p,Q} \) is a cubic involution.

2. The base-points of \( \sigma_{p,Q} \) are the point \( p \) — which has multiplicity 2 — and the four points \( p_1, p_2, p_3, p_4 \) such that the line passing through \( p \) and \( p_i \) is tangent at \( p_i \) to \( C \).

3. If \( p \) is not an inflexion point of \( C \), all the points \( p_1, \ldots, p_4 \) belong to \( \mathbb{P}^2 \). Otherwise, only three of them belong to \( \mathbb{P}^2 \), and the fourth is the point in the blow-up of \( p \) that corresponds to the tangent of \( C \) at \( p \).

Since \( \sigma_{p,Q} \) is an involution, the blow-up \( \pi: X \to \mathbb{P}^2 \) of its five base-points conjugates it to an automorphism of \( X \). We now describe the action of this automorphism on the Picard group of \( X \).

Lemma 3.3. Let \( C \subset \mathbb{P}^2 \) be a smooth cubic curve, let \( p \in C \) and let \( \sigma_{p,Q} \) be the element defined in Definition 1.1. Let \( p_1, p_2, p_3, p_4 \) be the base-points of \( \sigma_{p,Q} \) of multiplicity one (see Proposition 3.2), and let \( \pi: X \to \mathbb{P}^2 \) be the blow-up of the five base-points.

Denote by \( L \subset \text{Pic}(X) \) the pull-back of a general line of \( \mathbb{P}^2 \), by \( E_i \) the divisor corresponding to the point \( p_i \), and by \( E \) the divisor corresponding to \( p \), which is the total pull-back on \( X \) of the curve contracted on \( p \) (if \( p \) is an inflexion point, \( E \) corresponds to a reducible curve). The set \( (L, E, E_1, \ldots, E_4) \) is an orthogonal basis of \( \text{Pic}(X) \); the element have self-intersection \((1, -1, -1, -1, -1)\) and the action of \( \hat{\sigma} = \pi^{-1}\sigma_{p,Q}\pi \in \text{Aut}(X) \) on the Picard group is

\[
\begin{bmatrix}
3 & 2 & 1 & 1 & 1 & 1 \\
-2 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & 0 \\
-1 & -1 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

Proof. Only the action of \( \hat{\sigma} \) is not clear. By Lemma 3.2 the map \( \sigma_{p,Q} \) is a cubic involution, and its base-points are \( p \) with multiplicity 2, and \( p_1, \ldots, p_4 \) with multiplicity 1. This implies that \( \hat{\sigma}(L) = 3L - 2E - E_1 - E_2 - E_3 - E_4 \). Because \( \sigma_{p,Q} \) preserve the pencil of lines passing through \( p \), we have \( \hat{\sigma}(L - E) = L - E \). The lift of this pencil on \( X \) gives a conic bundle \( X \to \mathbb{P}^1 \), with four singular fibres, each one being the union of \( E_i \) and \( L - E - E_i \) for \( i = 1, \ldots, 4 \). This implies that the set \( \{ E_i, L - E - E_i \} \) is invariant for \( i = 1, \ldots, 4 \). Computing the intersection with \( L \) and \( \hat{\sigma}(L) \) shows that \( \hat{\sigma}(E_i) = L - E - E_i \), for \( i = 1, \ldots, 4 \). This achieves the proof.

Proposition 3.4. Let \( C \subset \mathbb{P}^2 \) be a smooth cubic curve, let \( p \in C \) and let \( \sigma_{p,Q} \) be the element defined in Definition 1.1. There exists an automorphism \( \tau \) of \( \mathbb{P}^2 \), acting via a translation of order 3 on \( C \), such that \( \tau \sigma_{p,Q} \) is conjugate to an automorphism of a smooth projective rational surface \( Y \), with dynamical degree > 1.

Proof. Denote as above by \( p_1, \ldots, p_4 \) the base-points of \( \sigma_{p,Q} \) of multiplicity 1. Recall (Lemma 3.2) that \( p_1, p_2, p_3, p_4 \) are the points of \( C \) such that the tangent of \( C \) at \( p_i \) passes through \( p \); if \( p \) is an inflexion point, one of the points is the point infinitely near to \( p_1 \) corresponding to the tangent direction of \( C \).
We change coordinates on $\mathbb{P}^2$ and put the curve $C$ into its Hessian form, which is the equation

$$x^3 + y^3 + z^3 + \lambda xyz = 0$$

for some $\lambda \in \mathbb{C}$ satisfying $\lambda^3 \neq -27$. Let $H \subset \mathbb{P}^2$ be the group generated by

$$(x : y : z) \mapsto (y : z : x),$$

$$(x : y : z) \mapsto (x : \omega y : \omega^2 z),$$

where $\omega$ is a 3-rd root of unity. One directly sees that $H$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and preserves $C$. Moreover, the action of any of the non-trivial elements of $H$ on $C$ is fixed-point-free. We obtain thus an isomorphism of $H$ with the 3-torsion of the group of translations $C \subset \text{Aut}(C)$.

Let us denote by $\pi : Y \to \mathbb{P}^2$ the blow-up of the orbit of $\{p, p_1, \ldots, p_4\}$ by $\tau$. If one of the $p_i$ is infinitely near to $p$, then its orbit consists of points infinitely near to the orbit of $p$. As before, we denote by $L \subset \text{Pic}(Y)$ the pull-back of a general line of $\mathbb{P}^2$, by $E_i$ the divisor corresponding to the point $p_i$, and by $E$ the divisor corresponding to $p$, which is the total pull-back on $X$ of the curve contracted on $p$. The automorphism $\tau$ lifts to an automorphism $\hat{\tau} = \pi^{-1}\tau\pi \in \text{Aut}(Y)$, which sends $E_i$ onto the divisor corresponding to $\tau(p_i)$, and sends $E$ onto the divisor corresponding to $\tau(p)$. The group $\text{Pic}(X)$ is generated by $L$ and by $\{\hat{\tau}^i(E), \hat{\tau}^i(E_1), \ldots, \hat{\tau}^i(E_4)\}_{i=0}^{2}$. The birational involution $\sigma_{p,Q} \in \text{Bir}(\mathbb{P}^2)$ lifts to an automorphism of the surface obtained by blowing-up $p, p_1, \ldots, p_4$; because this one fixes each of the other points blown-up (which belong to the curve $C$), it lifts to an automorphism $\hat{\sigma}$ of $Y$. This shows that $\tau\sigma_{p,Q}$ is conjugate by $\pi^{-1}$ to the automorphism $\hat{\tau}\hat{\sigma}$ of $Y$, and it suffices to show that its dynamical degree is $> 1$ to conclude. This amounts to find a real eigenvalue of the action of $\hat{\sigma}$ on $\text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ which is bigger than 1. The action of $\hat{\sigma}$ on $\text{Pic}(Y)$ is given by Lemma \ref{lem:action} and the action of $\hat{\tau}$ fixes $L$ and permutes the exceptional divisors according to the action of $\tau$ on the points.

Because $\tau$ does not fix any point of $C$, the divisors $E, \hat{\tau}(E), \hat{\tau}^2(E)$ are distinct. This is also true for the divisors $E_i, \hat{\tau}(E_i), \hat{\tau}^2(E_i)$ for $i = 1, \ldots, 4$. Note that $\tau(p_i) = p_j$ is also impossible, because it would imply that $\tau$ sends the line tangent to $C$ at $p_i$ onto the line tangent to $C$ at $p_j$, and thus $\tau$ would fix $p$. It remains to study two possible cases:

1) **There exists an $i \in \{1, \ldots, 4\}$ such that $\tau(p) = p_i$ or $\tau^2(p) = p_i$.**

Replacing $\tau$ by $\tau^2$ and renumbering the $p_i$, if needed, we can assume that $\tau(p) = p_1$.

We see that $\tau(p_1) = \tau^2(p)$ is distinct from $\tau^i(p_j)$ for $j \geq 2$. The sub-$\mathbb{Z}$-module of $\text{Pic}(Y)$ generated by

$$L, E, \hat{\tau}(E) = E_1, \hat{\tau}(E_1), \sum_{i=2}^{4} E_i, \sum_{i=2}^{4} \hat{\tau}(E_i), \sum_{i=2}^{4} \hat{\tau}^2(E_i)$$

is invariant, and the action of $\hat{\tau}$ and $\hat{\sigma}$, relatively to this basis, are given by
Corollary 3.5. Let $\varphi \in \text{Bir}(\mathbb{P}^2)$ be a birational map of degree 3.

1. If all base-points of $\varphi$ and $\varphi^{-1}$ are proper points of the plane, there exists an automorphism $\tau \in \text{Aut}(\mathbb{P}^2)$ such that $\tau \varphi$ is conjugate to an automorphism of a smooth projective rational surface with dynamical degree $> 1$.

2. In the algebraic set of birational maps of $\mathbb{P}^2$ of degree 3, the set of maps having this property is a dense subset with complement of codimension 1.
existence of $\sigma, \gamma \in p, p$ degree 4 obtained by blowing-up geometrically, by looking at the automorphism group of the del Pezzo surface of $P$. The biggest dimension is when all points are on $P$. The set of cubic birational maps is then parametrised by one point of $P$. The same base-points differ only by the post-composition with an automorphism. It is exactly the set where the map and its inverse have only proper base-points. It has the dimension of $(P, p)$.

Remark 3.6. i) By Proposition 3.4, the same result holds for map projectively equivalent to $\sigma_{p, Q}$ where $p$ is an inflexion point of a smooth cubic $Q$. These are the maps having 4 proper base-points $p, p_1, p_2, p_3$ of multiplicity 2, 1, 1, 1 such that $p_1, p_2, p_3$ are collinear, and a point $p_5$ infinitely near to $p$.

ii) It also holds for other special cubics maps: some with two proper base-points (see [BedDil06]), with one proper base-point (see [BedKim10]), or some maps of degree 3 with exactly two proper base-points (see [BedDil06] and [DesGr10]).
iii) If a birational map \( \varphi \in \text{Bir}(\mathbb{P}^2) \) of degree 3 has all its base-points which are proper but 3 are collinear, then it is projectively equivalent to

\[
\psi_\lambda : (x : y : z) \mapsto (yz(y - z + (\lambda - 1)x) : xy(z - \lambda y) : xz(z - \lambda y))
\]

for some \( \lambda \in \mathbb{C} \setminus \{0, 1\} \). In particular, \( \varphi^{-1} \) has only 4 proper base-points.

**Question 3.7.** Does there exists a \( \lambda \in \mathbb{C} \setminus \{0, 1\} \) such that for any \( \tau \in \text{Aut}(\mathbb{P}^2) \) the map \( \tau \psi_\lambda \) is not conjugate to an automorphism of a projective surface (with dynamical degree \( > 1 \))?

4. **THE EXAMPLE**

4.1. **Actions on infinitely near points.** Before proving Theorem 2 we need some general tools.

Let \( X, Y \) be two smooth projective rational surfaces, and let \( \psi : X \to Y \) be a birational map. If \( p \) is a point of \( X \) or a point infinitely near, which is not a base-point of \( \psi \), we define a point \( \psi^*(p) \), which will also be a point of \( Y \) or a point infinitely near. For this, take a minimal resolution

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi_1} & X \\
\downarrow \psi & & \downarrow \psi \\
Y & \xleftarrow{\pi_2} & \end{array}
\]

where \( \pi_1, \pi_2 \) are sequences of blow-ups. Because \( p \) is not a base-point of \( \psi \), it corresponds, via \( \pi_1 \), to a point of \( Z \) or infinitely near. Using \( \pi_2 \), we view this point on \( Y \), again maybe infinitely near, and call it \( \psi^*(p) \). Observe that this point is not a base-point of \( \psi^{-1} \), and that \( (\psi^{-1})^*(\psi^*(p)) = p \).

**Remark 4.1.** If \( p \) is not a base-point of \( \phi \in \text{Bir}(X) \) and \( \phi(p) \) is not a base-point of \( \psi \in \text{Bir}(X) \), we have \((\psi\phi)^*(p) = \psi^*(\phi^*(p)) \). If \( p \) is a general point of \( X \), then \( \phi^*(p) = \phi(p) \in X \).

**Example 4.2.** If \( p = (1 : 0 : 0) \in \mathbb{P}^2 \) and \( \psi \in \text{Bir}(\mathbb{P}^2) \) is the map \((x : y : z) \mapsto (yz + x^2 : xz : z^2) \), the point \( \psi^*(p) \) is not equal to \( p = \psi(p) \), but is infinitely near to it.

The following easy lemma will be important for the proof of Theorem 2.

**Lemma 4.3.** Let \( \varphi \in \text{Bir}(\mathbb{P}^2) \) be a birational map and let \( p \) be a point of \( \mathbb{P}^2 \) (or infinitely near). If there exists an integer \( N \geq 0 \) such that \( p \) is a base-point of \( \varphi^{-k} \) for any \( k \geq N \) but is not a base-point of \( \varphi^k \) for any \( k \geq N \), then \( \varphi \) is not conjugate to an automorphism of a smooth projective surface.

**Proof.** We prove first that \((\varphi^k)^*(p)\) and \((\varphi^l)^*(p)\) are two distinct points of \( \mathbb{P}^2 \) (or infinitely near), for any \( k, l \geq N \) with \( k \neq l \). Otherwise, assuming that \( l > k \), the equality \((\varphi^k)^*(p) = (\varphi^l)^*(p)\) implies that \( \varphi^{-l} \) is defined at \((\varphi^k)^*(p)\) (because it is defined at \((\varphi^l)^*(p)\), and that \((\varphi^{-l})^*((\varphi^k)^*(p)) = p \). In particular, \( \varphi^{-l} \) is defined at \( p \), and \((\varphi^{-l})^*(p) = p \), which means that \((\varphi^{(k-l)m})^*(p) = p \) for any \( m \geq 0 \). This is incompatible with the fact that \( p \) is a base-point of \( \varphi^{-i} \) for any \( i \geq N \).

The set \( \{(\varphi^k)^*(p)\}_{k=N}^\infty \) is thus an infinite set of points that belong to \( \mathbb{P}^2 \), as proper or infinitely near points. Suppose now that there exists a birational map \( \alpha : \mathbb{P}^2 \to S \), where \( S \) is a smooth projective surface, that conjugates \( \varphi \) to an
automorphism \( \varphi = \alpha \varphi \alpha^{-1} \) of \( S \). The map \( \alpha \) having only a finite number of base-points, there exists \( M \geq N \) such that no one of the points \( \{(\varphi^k)\bullet (p)\}_{k=M}^\infty \) is a base-point of \( \alpha \). Writing \( p_k = \alpha^*((\varphi^k)\bullet (p)) \) for any \( k \geq M \), we obtain a family of distinct points \( \{p_k\}_{k=M}^\infty \) such that \( \hat{\varphi}(p_k) = p_{k+1} \) for each \( k \geq M \). Writing \( p_{k-m} = \hat{\varphi}^{-m}(p_k) \) for any \( m \geq 0 \), we obtain an orbit \( \{p_k\}_{k \in \mathbb{Z}} \) of the automorphism \( \hat{\varphi} \), so that \( p_k \neq p_l \) for \( k \neq l \). Increasing maybe \( M \), we can assume that \( p_k \) is not a base-point of \( \alpha^{-1} \) for any \( k \geq M \) and any \( k \leq -M \). This implies that \( (\varphi^M)^\bullet (p) \) is not a base-point of the map \( \varphi^{-2M} = \alpha^{-1} \varphi^{2M} \alpha \); indeed, \( \alpha^*((\varphi^M)^\bullet (p)) = p_m \), and \( (\hat{\varphi}^{-2m})^\bullet (p_m) = p_{-m} \), which is not a base-point of \( \alpha^{-1} \).

We obtain a contradiction with the fact that \( p \) is a base-point of \( \varphi^M \) but not of \( \varphi^{-M} \).

The section is devoted to the proof of Theorem 2. We will always denote by \( \chi \in \text{Bir}(\mathbb{P}^2) \) the birational map

\[
\chi: (x : y : z) \mapsto (xz^5 + (yz^2 + x^3)^2 : yz^5 + x^3z^3 : z^6)
\]

which restricts on the affine plane where \( z = 1 \) to the automorphism

\[
(x, y) \mapsto (x + (y + x^3)^2, y + x^3),
\]

being the composition of \((x, y) \mapsto (x + y^2, y)\) with \((x, y) \mapsto (x, y + x^3)\).

4.2. Basic description of the map \( \chi \). The proof of Theorem 2 will rely on the study of the base-points of \( \chi \), and of its inverse, which is

\[
\chi^{-1}: (x : y : z) \mapsto (xz^5 - y^2z^4 : yz^5 - (xz - y^2)^3 : z^6).
\]

It will use two main properties: both \( \chi \) and \( \chi^{-1} \) have only one proper base-point, but the geometry of the base-points of the two maps are different (see below for more details). Note that many other examples can be constructed in the same way, the map \( \chi \) is only the simplest one having the above properties.

4.3. Base-points of \( \chi \). As all birational maps of \( \mathbb{P}^2 \) which contract only one curve, \( \chi \) has only one proper base-point, namely \( p_1 = (0 : 1 : 0) \), and all its base-points are "in tower" (see [Bla09, Lemma 2.2]). This means that the 8 base-points of \( \chi \), that we denote by \( p_1, \ldots, p_8 \), are such that \( p_i \) is infinitely near to \( p_{i-1} \) for \( i = 2, \ldots, 8 \). We denote by \( \pi: X \to \mathbb{P}^2 \) the blow-up of the 8 base-points, and write \( C \subset X \) the strict transform of the line \( C \subset \mathbb{P}^2 \) of equation \( z = 0 \), which is the only curve of \( \mathbb{P}^2 \) contracted by \( \chi \). We denote by \( E_1 \subset X \) the strict transform of the curve obtained by blowing-up \( p_i \). The intersection form on \( X \) corresponds to the dual diagram of Figure (this can be checked directly in local charts or by the decomposition of \( \chi \) into two simple maps as above).

![Diagram](image)

**Figure 1.** The configuration of the curves \( E_1, \ldots, E_8, C \) on the surface \( X \). Two curves are connected by an edge if their intersection is positive (and here equal to 1). The self intersections correspond to the shape of the vertices.
Let us write $\varphi = \tau \chi$, where $\tau$ is an automorphism of $\mathbb{P}^2$. Because $\pi$ is the blow-up of the base-points of $\chi$, which are also the base-points of $\varphi$, the map $\eta = \varphi \pi$ is a birational morphism $X \to \mathbb{P}^2$, which is the blow-up of the base-points of $\varphi^{-1}$. In fact, Figure 2 is the minimal resolution of $\varphi$.

\[
\begin{array}{c}
\xrightarrow{\pi} X \\
\xleftarrow{\eta} \mathbb{P}^2 \\
\end{array}
\]

**Figure 2.** The minimal resolution of indeterminacies of $\varphi$.

The line $C \subset \mathbb{P}^2$ being the only curve of $\mathbb{P}^2$ contracted by $\varphi$, the morphism $\eta$ contracts $C$, and the union of 7 other irreducible curves, which are among the curves $E_1, \ldots, E_8$. The configuration of Figure 1 shows that $\eta$ contracts the curves $C, E_2, E_3, E_4, E_5, E_6, E_7$ following this order.

We study now the blow-up $\eta: X \to \mathbb{P}^2$ in the same way as we did for $\pi$. We denote by $q_1, \ldots, q_8$ the base-points of $\varphi^{-1}$ (or equivalently the points blown-up by $\eta$), so that $q_1 = (1 : 0 : 0)$, and $q_i$ is infinitely near to $q_{i-1}$ for $i = 2, \ldots, 8$. We denote by $D \subset \mathbb{P}^2$ the line which is contracted by $\varphi^{-1}$ (which is the image by $\tau$ of the line of equation $z = 0$), and write $D \subset X$ the strict transform by $\eta^{-1}$ of the curve $D$. We then denote by $F_i$ the strict transform of the curve obtained by blowing-up $q_i$. Because of the order of the curves contracted by $\eta$, we get equalities between $C, E_1, \ldots, E_8$ and $D, F_1, \ldots, F_8$, according to Figure 3.

\[
E_6 = F_1 \\
E_8 = D \\
E_7 = F_2 \\
E_5 = F_3 \\
E_3 = F_4 \\
E_1 = F_5 \\
E_2 = F_6 \\
C = F_7
\]

**Figure 3.** The configuration of the curves $E_i, F_i, C, D$ on the surface $X$.

In particular, we see that the configuration of the points $p_1, \ldots, p_8$ is not the same as the one of the points $q_1, \ldots, q_8$. Saying that a point $a$ is proximate to a point $b$ if $a$ is infinitely near to $b$ and belongs to the strict transform of the curve obtained by blowing-up $b$, the configuration of the points $p_i$ and of the points $q_i$ are given in Figure 4.

**Figure 4.** The configuration of the points $p_1, \ldots, p_8$ and of the points $q_1, \ldots, q_8$. An arrow corresponds to the relation "is proximate to".
4.4. The proof of the theorem.

*Proof of Theorem.* We write as above $\varphi = \tau \chi$, where $\tau$ is an automorphism of $\mathbb{P}^2$, and recall that $p_1, \ldots, p_3$ are the base-points of $\varphi$, and $q_1, \ldots, q_8$ are the base-points of $\varphi^{-1}$. Our aim is to show that $p_3$ is a base-point of $\varphi^i$ and not of $\varphi^{-i}$, for any $i > 0$. This will imply that $\varphi$ is not conjugate to an automorphism of a smooth projective rational surface, by Lemma 4.3.

Denote by $k$ the lowest positive integer such that $p_1$ is a base-points of $\varphi^{-k}$; if no such integer exists, we write $k = \infty$.

For any integer $i$ such that $1 \leq i < k$, the point $p_1$ is not a base-point of $\varphi^{-i}$, hence the maps $\varphi$ and $\varphi^{-i}$ have no common base-point. This implies that the set of base-points of the map $\varphi^{i+1} = \varphi \circ \varphi^{-i}$ is the union of the base-points of $\varphi^i$ and of the points $(\varphi^{-i})^*(p_j)$ for $j = 1, \ldots, 8$. The map $\varphi^{-i}$ being defined at $p_1$, the point $(\varphi^{-i})^*(p_j)$ is proximate to the point $(\varphi^{-i})^*(p_k)$ if and only if $p_j$ is proximate to $p_k$.

Proceeding by induction on $i$, we obtain the following assertions:

1. For any integer $i$ with $1 \leq i \leq k$, the set of base-points of $\varphi^i$ is equal to
   $$\{(\varphi^{-m})^*(p_j) \mid j = 1, \ldots, 8, m = 0, \ldots, i - 1\}.$$

2. For any integer $l$ with $0 \leq -l < k$, the configuration of the points $\{(\varphi^l)^*(p_j)\}_{j=1}^8$ is given by
   $$\begin{array}{c}
   (\varphi^l)^*(p_1) \quad (\varphi^l)^*(p_2) \quad (\varphi^l)^*(p_3) \quad (\varphi^l)^*(p_4) \\
   (\varphi^l)^*(p_5) \quad (\varphi^l)^*(p_6) \quad (\varphi^l)^*(p_7) \quad (\varphi^l)^*(p_8)
   \end{array}$$

In particular, $p_3$ is a base-point of $\varphi^i$ for any $i$ satisfying $1 \leq i \leq k$.

If $k = \infty$, this implies that $p_3$ is a base-point of $\varphi^i$ for any $i > 0$, and by definition of $k$, the point $p_1$ is not a base-point of $\varphi^{-i}$ for any $i > 0$, so neither is $p_3$. We can thus assume that $k$ is a positive number.

Observe that $q_1$ is not a base-point of $\varphi^i$ for $1 \leq i \leq k - 1$. Indeed, otherwise $q_1$ would be equal to $(\varphi^{-m})^*(p_j)$ for some $m, j$ satisfying $0 \leq m \leq k - 2$ and $1 \leq j \leq 8$. This would imply that $p_j$ is a base-point of $\varphi^{m+1}$, which is impossible. We see thus that $\varphi^{-1}$ has no common base-point with $\varphi^i$ for $1 \leq i \leq k - 1$. In particular, the set of common base-points of $\varphi^{-1}$ and $\varphi^k$ is equal to
   $$B = \{(\varphi^{-(k-1)}^*)((p_j))_{j=1}^8 \cap \{q_j\}_{j=1}^8.$$

Because $p_1$ is a base-point of $\varphi^{-k}$ and not of $\varphi^{-(k-1)}$, the point $(\varphi^{-(k-1)})^*(p_1)$, which is a base-point of $\varphi^k$, is also a base-point of $\varphi^{-1}$. In particular, the set $B$ is not empty. The configuration of the two sets of points $\{\varphi^{-(k-1)}^*)((p_j))_{j=1}^8$ and $\{q_j\}_{j=1}^8$ implies that $q_1 = (\varphi^{-(k-1)}^*)((p_1))$. Moreover, either $B = \{q_1\}$ or $B = \{q_1, q_2\}$. Indeed, the point $q_3$ is proximate to $q_1$ but not to $q_2$, whereas $((\varphi^{-(k-1)}^*)((p_3))$ is proximate to $((\varphi^{-(k-1)}^*)((p_1))$ and $((\varphi^{-(k-1)}^*)((p_2))$.

The point $(\varphi^{-(k-1)}^*)((p_3))$ is therefore a point infinitely near to $q_1$, which corresponds, on the blow-up $\eta: X \to \mathbb{P}^2$, to a point that belong, as proper or infinitely near point, to one of the curves $F_1$ or $F_2$, equal respectively to $F_7$ and $F_6$ (see Figure 3). Applying $\varphi^{-1}$ to it corresponds to apply $\pi \eta^{-1}$, so $(\varphi^{-k})(p_3)$ is a point that is infinitely near to $p_6$, and thus to $p_3$. Because $p_3$ is not a base-point of $\varphi^{-1}$, the point $p_3$ is not a base-point of $\varphi^{-(k+1)}$ and $(\varphi^{-k})^*(p_3)$ is infinitely near to $(\varphi^{-k})^*(p_3)$. In particular, $(\varphi^{-2k})^*(p_3)$ is infinitely near to $(\varphi^{-k})^*(p_3)$, which is infinitely near to $p_3$. Continuing like this, we get the following assertion:

2. For any $i \geq 1$, the point $p_3$ is not a base-point of $\varphi^{-i}$.
It remains to show that $p_3$ is a base-point of $\varphi^i$ for each $i \geq 1$ to get the result. To do this, we will need the following assertion:

(3) For any $i \geq 1$, the point $q_3$ is not a base-point of $\varphi^i$.

Note that (3) could be proved in the same way as (2), reversing the order of $\varphi$ and $\varphi^{-1}$. We quickly recall the way to deduce it. Note that $q_3$ is not a base-point of $\varphi^i$ for $1 \leq i \leq k-1$, because $q_1$ is not a base-point of $\varphi^i$ (see above). Since $q_3$ does not belong to $B$, which is the set of common base-points of $\varphi^{-1}$ and $\varphi^k$, the point $q_3$ is not a base-point of $\varphi^k$. The point $q_3$ is infinitely near to $q_1 = (\varphi^{-(k-1)})^*(p_1)$, in the second neighbourhood. The point $(\varphi^{k-1})^*(q_3)$ is thus infinitely near to $p_1$. On the blow-up $\pi: X \to \mathbb{P}^2$, the point $(\varphi^{k-1})^*(q_3)$ corresponds thus to a point that belongs, as a proper or infinitely point, to $\mathcal{E}_1$ or $\mathcal{E}_2$, equal respectively to $\mathcal{F}_4$ and $\mathcal{F}_7$. The point $(\varphi^k)^*(q_3)$ is then a point infinitely near to $q_4$, and then to $q_3$. As before, the fact that $q_3$ is not a base-point of $\varphi^i$ for $i = 1, \ldots, k$ and that $(\varphi^k)^*(q_3)$ is infinitely near to $q_3$ implies that $q_3$ is not a base-point of $\varphi^i$ for any $i \geq 0$, proving Assertion (3).

It remains to see that Assertion (3) implies that $p_3$ is a base-point of $\varphi^i$ for any $i \geq 1$. For $i = 1$, this is obvious. For $i > 1$, we decompose $\varphi^i$ into $\varphi^{i-1} \circ \varphi$, we decompose $\pi: X \to \mathbb{P}^2$ into $\pi = \pi_{12} \circ \pi_{38}$, where $\pi_{12}: Y \to \mathbb{P}^2$ is the blow-up of $p_1, p_2$ and $\pi_{38}: X \to Y$ is the blow-up of $p_3, \ldots, p_8$, and do the same with $\eta$. This yields the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^2 & \xrightarrow{\pi} & \mathbb{P}^2 \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & \mathbb{P}^2 \\
\downarrow & & \downarrow \\
\mathbb{P}^2 & \xrightarrow{\varphi^{i-1}} & \mathbb{P}^2.
\end{array}
\]

Note that $\eta_{38}$ contracts $\mathcal{F}_8, \ldots, \mathcal{F}_3$ onto the point $q_3 \in X_2$, which is not a base-point of $\varphi^{i-1} \circ \eta_{12}$. Let us take the system of conics of $\mathbb{P}^2$ passing through $p_1, p_2, p_3$ and denote by $\Lambda$ the lift of this system on $Y$, which is a system of smooth curves passing through $q_3$ with movable tangents, having dimension 2. The strict transform on $X$ of $\Lambda$ is a system of curves intersecting $\mathcal{E}_3$ at a general movable point. The map $\eta_{38}$ contracts the curves $\mathcal{C}, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_1, \mathcal{E}_5$. The curve $\mathcal{E}_3$ being contracted and being not the last one, the image of the system by $\eta_{38}$ passes through $q_3$ with a fixed tangent (corresponding to the point $q_4$). The point $q_3$ being not a base-point of $\varphi^{i-1} \circ \eta_{12}$, the image of the system $\Lambda \subset Y$ by $\varphi^{i-1} \circ \eta \circ (\pi_{38})^{-1}$ has a fixed tangent at the point $(\varphi^{i-1} \circ \eta)(q_3)$. This shows that $p_3$ is a base-point of $\varphi^{i-1} \circ \eta \circ (\pi_{38})^{-1}$, and thus of $\varphi^i$. □

References

[BedDil06] E. Bedford, J. Diller, *Dynamics of a two parameter family of plane birational maps: maximal entropy*. J. Geom. Anal. 16 (2006), no. 3, 409-430.

[BedKim06] E. Bedford, K. Kim, *Periodicities in linear fractional recurrences: degree growth of birational surface maps*. Michigan Math. J. 54 (2006), no. 3, 647-670.

[BedKim09] E. Bedford, K. Kim, *Dynamics of rational surface automorphisms: linear fractional recurrences*. J. Geom. Anal., 19(3):553–583, 2009.
[BedKim10] E. Bedford, K. Kim, Continuous families of rational surface automorphisms with positive entropy. Math. Ann., 348(3):667–688, 2010.

[BedKim11] E. Bedford, K. Kim, Pseudo-Automorphisms of 3-space: Periodicities and Positive Entropy in Linear Fractional Recurrences http://arxiv.org/abs/1101.1614v2

[Bla08] J. Blanc, On the inertia group of elliptic curves in the Cremona group of the plane. Michigan Math. J. 56 (2008), no. 2, 315-330.

[Bla09] J. Blanc, The correspondence between a plane curve and its complement. J. Reine Angew. Math. 633 (2009), 1-10.

[Bla09b] J. Blanc, Linearisation of finite abelian subgroups of the Cremona group of the plane. Groups Geom. Dyn. 3 (2009), no. 2, 215266.

[CerDes08] D. Cerveau, J. Déserti, Transformations birationnelles de petit degré http://arxiv.org/abs/0811.2325v2

[DésGri10] J. Déserti, J. Grivaux, Automorphisms of rational surfaces with positive topological entropy. Indiana Univ. Math. J. (to appear). http://arxiv.org/abs/1004.0656v5

[Dil11] J. Diller, Cremona transformations, surface automorphisms, and plane cubics. Michigan Math. J. 60 (2011), no. 2, 409–440.

[Giz94] M.H. Gizatullin, The decomposition, inertia and ramification groups in birational geometry, Algebraic Geometry and its Applications, Aspects of Mathematics, E, vol. 25 (1994), 39-45.

[McM07] C. T. McMullen, Dynamics on blowups of the projective plane. Publ. Math. Inst. Hautes tudes Sci. 105 (2007), 49–89.

[PerZha11] F. Perroni, D-Q. Zhang, Pseudo-Automorphisms of positive entropy on the blowups of products of projective spaces, http://arxiv.org/abs/1111.3546v1

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