Compact Approximate Taylor Methods for Systems of Conservation Laws

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Abstract
A new family of high order methods for systems of conservation laws is introduced: the Compact Approximate Taylor (CAT) methods. As in the approximate Taylor methods proposed by Zorío et al. (J Sci Comput 71(1):246–273, 2017) the Cauchy–Kovalevskaya procedure is circumvented by using Taylor approximations in time that are computed in a recursive way. The difference is that here this strategy is applied locally to compute the numerical fluxes what leads to methods that have \((2p+1)\)-point stencil and order of accuracy \(2p\), where \(p\) is an arbitrary integer. Moreover we prove that they reduce to the high-order Lax–Wendroff methods for linear problems and hence they are linearly \(L^2\)-stable under a \(CFL^{-1}\) condition. In order to prevent the spurious oscillations that appear close to discontinuities two shock-capturing techniques have been considered: a flux-limiter technique and WENO reconstruction for the first time derivative (WENO-CAT methods). We follow [25] in the second approach. A number of test cases are considered to compare these methods with other WENO-based schemes: the linear transport equation, Burgers equation, the 1D compressible Euler equations, and the ideal Magnetohydrodynamics equations are considered. Although CAT methods present an extra computational cost due to the local character, this extra cost is compensated by the fact that they still give good solutions with CFL values close to 1.

Keywords Conservation laws · Finite difference methods · Approximate Lax–Wendroff procedure · Compact schemes

1 Introduction
Lax–Wendroff methods for linear systems of conservation laws are based on Taylor expansions in time in which the time derivatives are transformed into spatial derivatives using the equations \[11,12,23,26\]. The spatial derivatives are then discretized by means of centered high-order differentiation formulas. This procedure allows to derive numerical methods of
order $2p$, where $p$ is an arbitrary integer, using a centered $(2p+1)$-point stencil that are $L^2$-stable under a $CFL - 1$ condition (a review on these methods will be presented in Sect. 2).

This paper focuses on the extension of Lax–Wendroff methods to nonlinear systems of conservation laws. Many authors have developed numerical methods that use this approach for the time discretization as an alternative to multistep or multistage one-step methods like the SSP Runge–Kutta schemes (see [8]); this is the case of the original finite volume ENO schemes (see [9]). This approach was also followed by E.F. Toro and collaborators in the design of the so-called ADER (arbitrary high order schemes utilizing higher order derivatives) methods: see [17,20,21]). The computation of time derivatives in these methods is based on the modified generalized Riemann problem introduced by Toro [22]. A Lax–Wendroff second order evolution Galerkin method for multidimensional hyperbolic systems was also introduced in [15]. More recently, in [16] this procedure has been used together with WENO reconstructions for the spatial discretization. The main benefit compared to RK time discretizations is that only one WENO reconstruction is needed at each spatial cell per time step.

The main difficulty to extend Lax–Wendroff methods to nonlinear problems comes from the transformation of time derivatives into spatial derivatives. A first strategy to do this is given by the Cauchy–Kovalevskaya (CK) procedure, in which the PDE is used to replace time derivatives by spatial derivatives. The main drawback of this procedure comes from the fact that it leads to expressions whose number of terms grows exponentially what implies high computational costs and difficult implementations. In the context of ADER methods, this difficulty has been circumvented in ADER-WENO methods (see [5]) by replacing the CK procedure by local space-time problems that are solved with a Galerkin method. The so-called $P_N P_M$ methods introduced in [4] that generalize ADER-WENO and DG methods also follow this approach. These methods can be applied both on structured and unstructured meshes with $CFL - 1$ condition for stability.

An alternative to both CK and local space-time problems has been proposed recently in [25] based on an Approximate Taylor (AT) method: the time derivatives are approximated using high-order centered differentiation formulas combined with Taylor approximations in time that are computed in a recursive way. Nevertheless AT schemes are not proper generalizations of Lax–Wendroff methods: they have $(4p+1)$-point stencils and worse linear stability properties than the original Lax–Wendroff methods. Nevertheless, they can be stabilized by using one WENO reconstruction per spatial cell and time step, like in [16] and the resulting methods give good results under a $CFL - 0.5$ condition typically. These methods are easy to implement in Cartesian uniform meshes and show a good performance.

The goal of this paper is to obtain a family of numerical methods for nonlinear systems of conservation laws based on an approximate Taylor procedure that constitute a proper generalization of Lax–Wendroff methods, i.e. that reduce to the standard high-order Lax–Wendroff methods when the flux is linear. By construction, such a family of methods would be linearly $L^2$-stable under a $CFL - 1$ condition. Of course, as it happens in the linear case, such a family of methods would lead to spurious oscillations near discontinuities and a numerical technique would be required to get rid of them: flux limiters [10], essentially non-oscillatory reconstructions like ENO [9] or WENO [18], MOOD approach (see [3]), order-adaption techniques, etc.

In order to design such numerical methods, a compact variant of the AT procedure is introduced here: first, the conservative expression of the high-order Lax–Wendroff methods is considered; then, the derivatives appearing in the expression of the numerical flux are computed using $2p$-point differentiation formulas. This strategy leads to Compact Approximate Taylor (CAT) methods that have $(2p+1)$-point stencil and order of accuracy $2p$. Moreover,
these methods are easy to implement and highly parallelizable. Nevertheless, the number of operations to perform a step is bigger than the original AT methods: this is due to the fact that the approximation of the time derivatives is local in the sense that they depend both on the point and on the stencil. However, this extra cost is compensated by better stability properties. Two shock-capturing techniques are considered here: a flux-limiter technique and the use of a WENO reconstruction per cell and time step, as in [25] and [16].

The paper is organized as follows: in Sect. 2 a review of high order Lax–Wendroff methods for the linear transport equation is presented, including the study of the order, a heuristic study of the $L^2$-stability, and a discussion about the computation and properties of the coefficients. Section 3 is devoted to their extension to nonlinear problems: first the AT technique is recalled and then CAT methods are presented. We show that they reduce to Lax–Wendroff methods when applied to a linear problem and we analyze the order of accuracy. In Sect. 4 the techniques considered to cure the spurious oscillations near the discontinuities are presented. In Sect. 5 CAT methods are compared in a number of test cases with WENO-RK methods and AT methods. The linear transport equation, Burgers equation, the 1D compressible Euler and the ideal Magnetohydrodynamics equations are considered. Future developments and conclusions are drawn in Sect. 6.

### 2 The High-Order Lax–Wendroff Method for Linear Problems

Let us first consider the linear scalar equation:

$$u_t + a u_x = 0.$$  \(1\)

We consider the numerical method:

$$u^{n+1}_i = u^n_i + \sum_{k=1}^{m} \frac{(-1)^k c^k}{k!} \sum_{j=-p}^{p} \delta_{p,j}^k u^n_{i+j},$$  \(2\)

where \(\{x_i\}\) are the nodes of a uniform mesh of step \(\Delta x\); \(u^n_i\) is an approximation of the point value of the solution at \(x_i\) at the time \(n \Delta t\), where \(\Delta t\) is the time step; \(p \geq 1\) is a natural number; \(c = a \Delta t / \Delta x\); and \(\delta_{p,j}^k\) are the coefficients of the centered interpolatory formula of numerical differentiation based on a \((2p + 1)\)-point stencil, i.e. the unique formula of the form

$$\frac{p f}{(k)}(x_i) \simeq D_{p,i}^k(f, \Delta x) = \frac{1}{\Delta x^k} \sum_{j=-p}^{p} \delta_{p,j}^k f(x_{i+j}),$$  \(3\)

such that

$$p f^{(k)}(x_i) = \frac{1}{\Delta x^k} \sum_{j=-p}^{p} \delta_{p,j}^k f(x_{i+j}), \quad \forall f,$$

where \(p f\) is the Lagrange interpolation polynomial characterized by

$$p f(x_{i+j}) = f(x_{i+j}), \quad j = -p, \ldots, p.$$  

Here \(f^{(k)}\) represents the \(k\)-th derivative of a one-variable function \(f\) and \(f^{(0)} = f\).

The expression of the numerical method is obtained by applying a Taylor expansion in time, and replacing time derivatives by space derivatives through the identities

$$\partial_t^k u = (-1)^k a^k \partial_x^k u, \quad k = 1, 2 \ldots$$  \(4\)
### 2.1 Formulas of Numerical Differentiation

Besides (3) the following family of interpolatory formulas based on a $2p$-point stencil will be used in this work:

\[
f^{(k)}(x_i + q \Delta x) \simeq A_{p,i}^{k,q}(f, \Delta x) = \frac{1}{\Delta x^k} \sum_{j=-p}^{p} \gamma_{p,j}^{k,q} f(x_{i+j}),
\]

i.e. $A_{p,i}^{k,q}(f, \Delta x)$ is the numerical differentiation formula that approximates the $k$-th derivative at the point $x_i + q \Delta x$ using the values of the function at the $2p$ points $x_{i-p+1}, \ldots, x_{i+p}$.

Observe that the coefficients, like in (3), do not depend on $i$.

Given a variable $w$, the following notation will be used:

\[
D_{p,i}^{k}(w, \Delta x) = \frac{1}{\Delta x^k} \sum_{j=-p}^{p} \delta_{p,j}^{k} w_{i+j},
\]

\[
A_{p,i}^{k,q}(w, \Delta x) = \frac{1}{\Delta x^k} \sum_{j=-p}^{p} \gamma_{p,j}^{k,q} w_{i+j},
\]

to indicate that the formulas are applied to the approximations of $w, w_i$, and not to its exact point values $w(x_i)$. In cases where there are two or more indexes, the symbol $\ast$ will be used to indicate with respect to which the differentiation is applied. For instance:

\[
\partial_{x}^{k} u(x_i + q \Delta x, t_n) \simeq A_{p,i}^{k,q}(u^{n}_{\ast}, \Delta x) = \frac{1}{\Delta x^k} \sum_{j=-p}^{p} \gamma_{p,j}^{k,q} u^{n}_{i+j},
\]

\[
\partial_{t}^{k} u(x_i, t_n + q \Delta t) \simeq A_{p,n}^{k,q}(u^{*}_{i}, \Delta t) = \frac{1}{\Delta t^k} \sum_{r=-p}^{p} \gamma_{p,r}^{k,q} u^{n+r}_{i}.
\]

Using this notation, the algorithm (2) writes as follows:

\[
u_{i}^{n+1} = u_{i}^{n} + \sum_{k=1}^{m} (-1)^{k} a^{k} \Delta x^{k} D_{p,i}^{k}(u_{\ast}^{n}, \Delta x).
\]

Let us discuss some properties of the coefficients of the numerical differentiation formulas (3) and (5) and some relations between them that will be used in that follows. Since the coefficients are independent of $\Delta x$ and $i$, we can consider, without loss of generality, the case $i = 0, x_0 = 0, \Delta x = 1$:

\[
f^{(k)}(0) \simeq D_{p,0}^{k}(f, 1) = \sum_{j=-p}^{p} \delta_{p,j}^{k} f(j), \tag{7}
\]

\[
f^{(k)}(q) \simeq A_{p,0}^{k,q}(f, 1) = \sum_{j=-p}^{p} \gamma_{p,j}^{k,q} f(j). \tag{8}
\]

Since (7) is exact for polynomials of degree $\leq 2p$, by applying the formula to $x^s, s = 0, \ldots, 2p$ at $x = 0$, we get that the coefficients have to satisfy the equalities

\[
\sum_{j=-p}^{p} j^k \delta_{p,j}^{k} = k!, \quad \sum_{j=-p}^{p} j^s \delta_{p,j}^{k} = 0, \quad s \neq k, \quad 0 \leq s, k \leq 2p. \tag{9}
\]
Analogously:
\[
\sum_{j=-p+1}^{p} j^{k} \gamma_{p,j}^{k,0} = k! \quad \sum_{j=-p+1}^{p} j^{s} \gamma_{p,j}^{k,0} = 0, \quad s \neq k, \quad 0 \leq s, k \leq 2p - 1. \quad (10)
\]

\[
\sum_{j=-p+1}^{p} \gamma_{p,j}^{k,q} = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (11)
\]

As it is well known, the coefficients \(\delta_{p,j}^{k}\) are related to the Lagrange basis polynomials
\[
F_{p,j}(x) = \prod_{r=-p,r \neq j}^{p} \frac{(x - r)}{(j - r)}, \quad -p \leq j \leq p, \quad (12)
\]
through the equalities:
\[
\delta_{p,j}^{k} = F_{p,j}^{(k)}(0), \quad (13)
\]
which allow us to write the Taylor expansion of \(F_{p,j}\) centered at \(x = 0\) as follows:
\[
F_{p,j}(x) = \sum_{k=0}^{2p} \frac{\delta_{p,j}^{k}}{k!} x^{k}. \quad (14)
\]

**Proposition 1** The coefficients \(\delta_{p,j}^{k}\) of the formula (7), satisfy:
\[
\delta_{p,j}^{k} = (-1)^{k} \delta_{p,-j}^{k}; \quad (15)
\]
\[
\delta_{p,0}^{k} = 0 \text{ if } k \text{ is odd}; \quad (16)
\]
\[
\sum_{j=-p}^{p} \delta_{p,j}^{k} j^{(2p+1)} = 0 \text{ if } k \text{ is even}; \quad (17)
\]
\[
\sum_{j=-p}^{p} \delta_{p,j}^{k} j^{(2p+2)} = 0 \text{ if } k \text{ is odd}. \quad (18)
\]

**Proof** (15) is deduced from the equality
\[
F_{p,-j}(x) = F_{p,j}(-x); \quad (19)
\]
Using (15) we get (16), (17) and (18) are deduced from (15) and (16). \(\square\)

**Proposition 2** For \(k \geq 1\) the following relations hold:
\[
\delta_{p,p}^{k} = \gamma_{p,p}^{k-1,1/2}; \quad (19)
\]
\[
\delta_{p,j}^{k} = \gamma_{p,j}^{k-1,1/2} - \gamma_{p,j+1}^{k-1,1/2}, \quad j = -p + 1, \ldots, p - 1; \quad (20)
\]
\[
\delta_{p,-p}^{k} = -\gamma_{p,-p+1}^{k-1,1/2}. \quad (21)
\]

**Proof** Let us consider the formulas
\[
f^{(k-1)}(1/2) \simeq A_{p,0}^{k-1,1/2}(f, 1) = \sum_{j=-p+1}^{p} \gamma_{p,j}^{k-1} f(j), \quad (22)
\]
\[
f^{(k-1)}(-1/2) \simeq A_{p,-1}^{k-1,1/2}(f, 1) = \sum_{j=-p+1}^{p} \gamma_{p,j}^{k-1} f(j - 1), \quad (23)
\]
that are exact for polynomials of degree \( \leq 2p - 1 \). Let us consider now the formula

\[
f^{(k)}(0) \simeq \frac{1}{\Delta x} \left( A_{p,0}^{k-1,1/2}(f, \Delta x) - A_{p,-1}^{k-1,1/2}(f, \Delta x) \right).
\]

(24)

If \( f \) is a polynomial of degree \( 2p \), then (22) and (23) are exact for \( f \), furthermore

\[
A_{p,0}^{k-1,1/2}(f, 1) - A_{p,-1}^{k-1,1/2}(f, 1) = f^{(k-1)}(1/2) - f^{(k-1)}(-1/2) = f^{(k)}(0),
\]

where we have used that the formula

\[
g'(0) \simeq g(1/2) - g(-1/2).
\]

is exact for polynomials of degree 1. Therefore, (24) coincide with (7). The proof is finished by writing (24) in the form

\[
f^{(k)}(0) \simeq \sum_{j=-p+1}^{p} \sum_{j=-p+1}^{p} \gamma_{s,q}^{k-s,j} f(p)
\]

and matching the weights.

**Proposition 3**

Given \( 1 \leq k \leq 2p - 1, 0 \leq s \leq k \):

\[
\sum_{j=-p+1}^{p} \gamma_{s,q}^{k-s,j} f(p) = \gamma_{p,l}^{k,q}, \quad l = -p + 1, \ldots, p.
\]

(25)

**Proof** The proof is similar to the one of the preceding in Proposition 2: consider the formula

\[
f^{(k)}(q) \simeq \sum_{j=-p+1}^{p} \gamma_{s,q}^{k-s,j} f^{(k-s)}(j),
\]

with

\[
f^{(k-s)}(j) = \sum_{l=-p+1}^{p} \gamma_{s,q}^{k-s,j} f(l);
\]

check that it is exact for polynomials of degree \( 2p - 1 \); write it in the form:

\[
f^{(k)}(q) \simeq \sum_{l=-p+1}^{p} \left( \sum_{j=-p+1}^{p} \gamma_{s,q}^{k-s,j} \gamma_{p,l}^{k,q} \right) f(l);
\]

and match its weights with those of (8). \qed

### 2.2 Conservative Form of (2)

From the proof of Proposition 2 we deduce an alternative form for (3):

\[
f^{(k)}(x_i) \simeq \frac{1}{\Delta x} \left( A_{p,i}^{k-1,1/2}(f, \Delta x) - A_{p,i-1}^{k-1,1/2}(f, \Delta x) \right).
\]

(26)

Using this form in (6), the numerical method (2) can be written as:

\[
u_{i+1}^{p+1} = u_{i}^{p} + \frac{\Delta t}{\Delta x} \left( F_{i-1/2}^{p} - F_{i+1/2}^{p} \right),
\]

(27)
with
\[ F_{i+1/2}^p = \sum_{k=1}^{2p} (-1)^{k-1} a_k^p \Delta t^{k-1} \frac{k!}{k} A_{p,i}^{k-1.1/2} (u^n, \Delta x). \] (28)

Using (11) it is straightforward to verify that \( F_{i+1/2}^p \) is a consistent numerical flux, what proves that (2) is a conservative method.

### 2.3 Computation of the Coefficients: An Iterative Algorithm

Notice that (9) constitutes a \((2p+1) \times (2p+1)\) linear system with a Vandermonde matrix that can be used to compute \( \delta_{p,j}^k \). Nevertheless, as it is well-known, this system is ill-conditioned, so that it is recommendable to compute them by using an alternative algorithm: we adapt the recursive algorithm proposed in [6]. The following notation is adopted:

\[ \delta_{p,j}^k = 0 \text{ if } k > 2p \text{ or } k < 0. \]

Let us derive some recurrence formulas to compute the coefficients:

1. \( \delta_{p,j}^k \) for \( j = 0, \ldots, p - 1 \).
   From (12), we obtain
   \[ F_{p,j}(x) = \frac{x^2 - p^2}{j^2 - p^2} F_{p-1,j}(x). \] (29)
   Using then the Taylor expansions (14) in (29) we get
   \[ \delta_{p,j}^k = \frac{1}{p^2 - k^2} \left[ p^2 \delta_{p-1,j}^{k-1} - k(k - 1) \delta_{p-1,j}^{k-2} \right]. \] (30)

2. \( \delta_{p,j}^k \) with \( j = p \).
   Substituting \( j=p \) in (12), we get
   \[ F_{p,p}(x) = \frac{1}{(2p)(2p - 1)} (x^2 + x - p(p - 1)) F_{p-1,p-1}(x), \] (31)
   and, using (14), we obtain:
   \[ \delta_{p,p}^k = \frac{1}{2p(2p - 1)} \left[ k(k - 1) \delta_{p-1,p-1}^{k-2} + k \delta_{p-1,p-1}^{k-1} - p(p - 1) \delta_{p-1,p-1}^k \right]. \] (32)

3. \( \delta_{p,j}^k \) for \( j = -p, \ldots, -1 \). (15) is used.

The algorithm is computed only once in increasing order of \( p \). The coefficients \( \gamma_{p,j}^{k,q} \) are computed using the algorithms described in [6,7] and \( \gamma_{p,j}^{k,1/2} \) is obtained from \( \delta_{p,j}^{k+1} \).

### 2.4 Order of Accuracy

**Proposition 4** The formula of numerical differentiation (3) has order of accuracy \( \alpha_k - k \), with

\[ \alpha_k = \begin{cases} 2p + 1 & \text{if } k \text{ is odd,} \\ 2p + 2 & \text{if } k \text{ is even.} \end{cases} \]
Table 1 Order of the formula (3)

| k  | p  | 2  | 3  | 4  |
|----|----|----|----|----|
| 1  | 2  | 4  | 6  | 8  |
| 2  | 4  | 6  | 6  | 8  |
| 3  | 2  | 4  | 6  |    |
| 4  | 2  | 4  | 6  |    |
| 5  | 2  | 4  |    |    |
| 6  | 2  | 4  |    |    |
| 7  | 2  |    |    |    |
| 8  | 2  |    |    |    |

Proof Let \( f \) be a function of class \( C^{\alpha_k+1} \). Applying Taylor expansions and properties (9) and (17), we obtain:

\[
\frac{1}{\Delta x^k} \sum_{j=-p}^{p} \delta_{p,j}^k f(x_{i+j}) = f^{(k)}(x_i) + \varphi_k \frac{\Delta x^{\alpha_k-k}}{\alpha_k!} f^{(\alpha_k)}(x_i) + O(\Delta x^{\alpha_k-k+1}), \tag{33}
\]

where

\[
\varphi_k = \sum_{j=-p}^{p} \delta_{p,j}^k j^\alpha_k. \tag{34}
\]

Table 1 shows the order of (3) for different values of \( p \) and \( k \).

Proposition 5 The discretization error of the numerical method (2) is of order \( O(\Delta t^{m+1} + \Delta x^{2p+1}) \).

Proof Let \( u \) be a smooth enough solution of (1). Using Proposition 4 we obtain

\[
u(x_i, t_{n+1}) - u(x_i, t_n) = \sum_{k=1}^{m} \frac{(-1)^k c^k}{k!} \frac{\Delta t^k}{\alpha_k!} \partial_x^k u(x_i, t_n) + O(\Delta x^{\alpha_k+1}) + O(\Delta t^{m+1})
\]

\[
= u(x_i, t_{n+1}) - u(x_i, t_n) - \sum_{k=1}^{m} \frac{\Delta t^k}{k!} \partial_x^k u(x_i, t_n)
\]

\[
- \sum_{k=1}^{m} \varphi_k \frac{(-1)^k c^k}{k!} \frac{\Delta x^{\alpha_k}}{\alpha_k!} \partial_x^{\alpha_k} u(x_i, t_n) + O(\Delta t^{m+1})
\]

\[
= \frac{1}{(m+1)!} \partial_x^{m+1} u(x_i, t_n) \Delta t^{m+1}
\]

\[
+ \left( \sum_{k=0}^{p-1} \frac{\varphi_{2k+1} c^{2k+1}}{(2p+1)(2k+1)!} \right) \partial_x^{2p+1} u(x_i, t_n) \Delta x^{2p+1} + O(\Delta t^{m+2} + \Delta x^{2p+2}),
\]

\( \square \)
where (4) has been used.

As a consequence, the order of accuracy of (2) is \( \min(m, 2p) \). Therefore, the optimal combination of these parameters is \( m = 2p \). From now on, we shall assume that this relation holds.

### 2.5 Modified Equation and Stability

Taking into account that \( m = 2p \) and (18), the local discretization error is as follows:

\[
\begin{align*}
    u(x_i, t_{n+1}) - u(x_i, t_n) &= \sum_{k=1}^{m} \frac{(-1)^k c^k}{k!} \sum_{j=-p}^{p} \delta_{p,j} u(x_{i+j}, t_n) \\
    &= \frac{1}{(2p+1)!} \partial^2_{t}^{p+1} u(x_i, t_n) \Delta t^{2p+1} + \frac{1}{(2p+2)!} \partial^2_{x}^{2p+2} u(x_i, t_n) \Delta x^{2p+2} \\
    &+ \left( \sum_{k=0}^{p-1} \frac{\varphi_{2k+1} c^{2k+1}}{(2p+1)!(2k+1)!} \right) \partial^2_{x}^{2p+1} u(x_i, t_n) \Delta x^{2p+1} \\
    &- \left( \sum_{k=1}^{p} \frac{\varphi_{2k} c^{2k}}{(2p+2)!(2k)!} \right) \partial^2_{x}^{2p+2} u(x_i, t_n) \Delta x^{2p+2} + O(\Delta x^{2p+3}).
\end{align*}
\]

Using (34) and (14) we get:

\[
\sum_{k=0}^{p-1} \frac{\varphi_{2k+1} c^{2k+1}}{(2k+1)!} = \sum_{j=-p}^{p} \left( \sum_{k=0}^{p-1} \frac{\delta_{p,j} c^{2k+1}}{(2k+1)!} \right) j^{2p+1} = \frac{1}{2} \sum_{j=-p}^{p} \left( \sum_{l=1}^{2p} \left( \frac{\delta_{p,j} c^{l}}{l!} - \frac{\delta_{p,-j} c^{l}}{l!} \right) c^{l} \right) j^{2p+1} = \frac{1}{2} \sum_{j=-p}^{p} (F_{p,j}(c) - F_{p,-j}(c)) j^{2p+1} = \frac{1}{2} \left[ \sum_{j=-p}^{p} F_{p,j}(c) j^{2p+1} - \sum_{j=-p}^{p} F_{p,j}(-c) j^{2p+1} \right] = \frac{1}{2} [q(c) - q(-c)],
\]

where \( q(c) \) is the polynomial of degree \( \leq 2p \) that interpolates the points \( \{(-p, (-p)^{2p+1}), \ldots, (0, 0), \ldots, (p, p^{2p+1})\} \).

Since \( q \) is clearly an odd function, we finally obtain:

\[
\sum_{k=0}^{p-1} \frac{\varphi_{2k+1} c^{2k+1}}{(2k+1)!} = q(c).
\]

(35)

Reasoning in a similar way, we get:

\[
\sum_{k=1}^{p} \frac{\varphi_{2k} c^{2k}}{(2k)!} = r(c),
\]

(36)
where \( r \) is the polynomial of degree \( \leq 2p \) that interpolates the points 
\[ \{(−p, (−p)^{2p+2}), \ldots, (0, 0), \ldots, (p, p^{2p+2})\}. \]

Using now (4), (35), and (36), the local discretization error can be written as follows:

\[
u(x_i, t_{n+1}) - u(x_i, t_n) - \sum_{k=1}^{m} \left(-1\right)^k \frac{c^k}{k!} \sum_{j=-p}^{p} \delta_{p,j} u(x_{i+j}, t_n) = h_1(c) \frac{(2p+1)!}{(2p+1)!} \Delta x^{2p+1} u(x_i, t_n) \Delta x^{2p+1} + \mathcal{O}(\Delta^{2p+3}),
\]

with

\[
h_1(c) = q(c) - c^{2p+1}, \quad h_2(c) = r(c) - c^{2p+2}.
\]

Therefore, the numerical method solves with order \( \mathcal{O}(\Delta^{2p+2}) \) the following modified equation

\[
u_t + au_x = \mu_1 \partial_x^{2p+1} u - \mu_2 \partial_x^{2p+2} u,
\]

where

\[
\mu_1 = \frac{h_1(c)}{(2p+1)! \Delta t} \Delta x^{2p+1}, \quad \mu_2 = \frac{h_2(c)}{(2p+2)! \Delta t} \Delta x^{2p+2}.
\]

Following the heuristic theory proposed in [24], to study the stability in the small wave-number limit we look for an elementary solution \( u(x, t) \) of (40) of the form

\[
u(x, t) = e^{\alpha t} \cdot e^{ikx},
\]

where \( \alpha \) is complex number. The following equality has to be satisfied:

\[
au + ikau = \mu_1 (-1)^p i k^{2p+1} u + \mu_2 (-1)^p k^{2p+2} u.
\]

Therefore:

\[
\alpha = -\mu_2 (-1)^p k^{2p+2} - (ka - \mu_1 (-1)^p k^{2p+1} i).
\]

The numerical method is thus expected to be stable if the real part is negative, i.e.

\[
\mu_2 (-1)^p \leq 0,
\]

or, equivalently

\[
h_2(c)(-1)^p \leq 0.
\]

\( h_2 \) is an even polynomial of degree \( 2p + 2 \) such that

\[
\lim_{c \to \pm \infty} h_2(c) = -\infty.
\]

Moreover, 0 is a double root of \( h_2 \) and \( \pm 1, \ldots, \pm p \) are single roots. Analyzing the change of signs of \( h_2 \), it can be easily checked that

\[
h_2(c) \leq 0, \quad \forall c \in [0, 1] \quad \text{if } p \text{ even},
\]

\[
h_2(c) \geq 0, \quad \forall c \in [0, 1] \quad \text{if } p \text{ odd},
\]
and thus (41) is satisfied if \( c \in [0, 1] \) (see Fig. 1).

This argument shows that the method is expected to be stable at least for small wave-numbers under the standard CFL condition \( c \leq 1 \). In fact, in [24] (see also [14]) it has been shown that (41) is a necessary condition for stability in the von Neumann sense. The analysis of the sufficiency of this condition is out of the scope of this article. Nevertheless the numerical experiments seem to confirm that the method is \( L^2 \)-stable under the CFL-1 condition.

3 Extension to Nonlinear Problems

3.1 Approximate Taylor Method

Following [25], instead of using the Cauchy–Kovaleskaya process to extend (27)–(28) to nonlinear problems

\[ u_t + f(u)_x = 0, \tag{42} \]

we use the equalities

\[ \partial_t^k u = -\partial_x^k \partial_t^{k-1} f(u). \tag{43} \]

To derive the expression of the numerical method, let us suppose that approximations

\[ \tilde{f}_i^{(k-1)} \approx \partial_t^{k-1} f(u)(x_i, t_n), \]

are available. Then,

\[ \partial_t^k u(x_i, t_n) \approx \tilde{u}_i^{(k)} = -D^1_{p_{k-1}, j} \tilde{f}_i^{(k-1)} + \frac{1}{\Delta x} \sum_{j=-p_{k-1}}^{p_{k-1}} \delta^1_{p_{k-1}, j} \tilde{f}_{i+j}^{(k-1)}, \]

being

\[ \sum \delta^1_{p_{k-1}, j} \tilde{f}_{i+j}^{(k-1)} \]
\[ p_k = \lceil (p - k/2) \rceil, \]  
\[ (44) \]

where, \( \lceil \cdot \rceil \) denotes the ceiling function.

Using these approximations to approximate the Taylor expansion, we obtain the method
\[ u_i^{n+1} = u_i^n + \sum_{k=0}^{2p} \frac{\Delta t^k}{k!} \tilde{u}_i^{(k)}. \]  
\[ (45) \]

Equivalently, using (26), the numerical method can be written in conservative form (27) with numerical flux
\[ F_{i+1/2}^p = \sum_{k=1}^{2p} \frac{\Delta t^{k-1}}{k!} A_{pk-1,i}^{0.1/2} (\tilde{f}^{(k-1)}_* \Delta x), \]  
\[ (46) \]

being
\[ A_{pk-1,i}^{0.1/2} (\tilde{f}^{(k-1)}_* \Delta x) = \sum_{j=-pk+1}^{pk-1} Y_{pk-1,j}^{0.1/2} \tilde{f}^{(k-1)} i+j. \]  
\[ (47) \]

Now, to compute the approximations \( \tilde{f}^{(k)}_* \), new Taylor expansions in time are used recursively as follows:

- Compute \( \tilde{f}_i^{(0)} = f(u_i^n) \).
  - For \( k = 1 \ldots 2p \):
    * Compute
      \[ u_i^{(k)} = -D_{pk-1,i}^1 (\tilde{f}^{(k-1)}_* \Delta x). \]
    * Compute
      \[ \tilde{f}_i^{k,n+r} = f \left( u_i^n + \sum_{l=1}^{k} \frac{(r \Delta t)^l}{l!} \tilde{u}_i^{(l)} \right), \quad r = -p, \ldots, p. \]
    * Compute
      \[ \tilde{f}_i^{(k)} = D_{pk,n}^k (\tilde{f}^{k,*}_i \Delta t), \]
      where
      \[ D_{pk,n}^k (\tilde{f}^{k,*}_i \Delta t) = \frac{1}{\Delta t^k} \sum_{r=-p}^{p} \delta_{pk+r} \tilde{f}^{k,n+r}_i. \]

Observe that Taylor expansions are used to approximate \( f(u(x_i, t_n + r \Delta t)) \) and once these approximations have been computed, the centered formula of numerical differentiation (3) is used to approximate the temporal derivatives.

This method is order \( 2p \), but it is not a generalization of (2) in the sense that this latter method is not recovered if \( f(u) = au \). To see this, consider \( p = 1 \) and \( f(u) = au \): it can be easily checked that (45) writes as follows
\[ u_i^{n+1} = u_i^n - \frac{c}{2}(u_{i+1}^n - u_{i-1}^n) - \frac{c^2}{8}(u_{i+2}^n + 2u_i^n + u_{i-2}^n), \]  
\[ (48) \]

which is different from the standard Lax–Wendroff method: (45) is a \((4p + 1)\)-point method whose stability properties are worse than those of the standard Lax–Wendroff method. (see [12]).
3.2 Compact Approximate Taylor Method

In order to prevent the increase of the stencil observed for Approximate Taylor methods, we consider a modification based on the conservative form of the method. The numerical flux \( F_{i+1/2}^p \) will be computed using only the approximations

\[
u_{i-p+1}^n, \ldots, u_{i+p}^n,
\]

so that the values used to update \( u_{i}^{n+1} \) are only those of the centered \((2p+1)\)-point stencil, like in the linear case. In fact, we will show that this modification is a proper generalization of the Lax–Wendroff method for linear problems.

In order to be able to compute the numerical fluxes using only (49), for every \( i \) we will compute local approximations of

\[\partial_{t}^{k-1} f (u(x_{i-p+1}, t^n), \ldots, \partial_{t}^{k-1} f (u(x_{i+p}, t^n)),\]

that will be represented by

\[\tilde{f}^{(k-1)}_{i,j} \approx \partial_{t}^{k-1} f (u(x_{i+j}, t^n), \quad j = -p + 1, \ldots, p.\]

These approximations are local in the sense that \( i_1 + j_1 = i_2 + j_2 \), does not imply that \( f^{(k-1)}_{i_1,j_1} = f^{(k-1)}_{i_2,j_2} \). Once these approximations have been computed, the numerical flux is given by

\[
F_{i+1/2}^p = \sum_{k=1}^{2p} \frac{\Delta t^{k-1}}{k!} A_{p,0}^{1/2} (\tilde{f}_{i,*}^{(k-1)}, \Delta x),
\]

with

\[
A_{p,0}^{1/2} (\tilde{f}_{i,*}^{(k-1)}, \Delta x) = \sum_{j=-p+1}^{p} \gamma_{p,j}^{1/2} \tilde{f}_{i,j}^{(k-1)}.
\]

Now, given \( i \), to compute the approximations \( \tilde{f}_{i,j}^{(k-1)} \), new Taylor expansions in time are used recursively as follows:

- **Compute** \( \tilde{f}_{i,j}^{(0)} = f (u_{i+j}^n), \quad j = -p + 1, \ldots, p.\)
  - For \( k = 1 \ldots 2p:\)
    * Compute
      \[
      \tilde{u}_{i,j}^{(k)} = -A_{p,0}^{1/2} (\tilde{f}_{i,*}^{(k-1)}, \Delta x),
      \]
      where
      \[
      A_{p,0}^{1/2} (\tilde{f}_{i,*}^{(k-1)}, \Delta x) = \frac{1}{\Delta x} \sum_{r=-p+1}^{p} \gamma_{p,r}^{1/2} \tilde{f}_{i,r}^{(k-1)}.
      \]
    * Compute
      \[
      \tilde{f}_{i,j}^{k,n+r} = f \left( u_{i+j}^n + \sum_{l=1}^{k} \frac{(r \Delta t)^l}{l!} \tilde{u}_{i,j}^{(l)} \right), \quad j, r = -p + 1, \ldots, p.
      \]
Theorem 1 The compact approximate Taylor method reduces to (2) when \( f(u) = au \).

Proof For \( k > 1 \) we have:

\[
\tilde{f}_{i,j}^{(k-1)} = \frac{1}{\Delta t^{k-1}} \sum_{r=-p+1}^{p} \gamma_{p,r}^{k-1,0} \tilde{f}_{i,j}^{k-1,n+r}
\]

\[
= \frac{a}{\Delta t^{k-1}} \sum_{r=-p+1}^{p} \gamma_{p,r}^{k-1,0} \left( u_{i,j}^{n} + \sum_{l=1}^{k-1} \frac{(r \Delta t)^l}{l!} \tilde{u}_{i,j}^{(l)} \right)
\]

\[
= \frac{a}{\Delta t^{k-1}} \left( \left( \sum_{r=-p+1}^{p} \gamma_{p,r}^{k-1,0} \right) u_{i,j}^{n} + \sum_{l=1}^{k-1} \frac{\Delta t^l}{l!} \left( \sum_{r=-p+1}^{p} \gamma_{p,r}^{k-1,0,l} \right) \tilde{u}_{i,j}^{(l)} \right)
\]

\[
= au_{i,j}^{(k-1)},
\]

where (10) has been used. On the other hand:

\[
\tilde{u}_{i,j}^{(k)} = -\frac{1}{\Delta x} \sum_{r=-p+1}^{p} \gamma_{p,r}^{1,j} \tilde{f}_{i,r}^{(k-1)}
\]

\[
= -\frac{a}{\Delta x} \sum_{r=-p+1}^{p} \gamma_{p,r}^{1,j} \tilde{u}_{i,r}^{(k-1)}
\]

\[
= \frac{a^2}{\Delta x^2} \sum_{r=-p+1}^{p} \gamma_{p,r}^{1,j} \sum_{s=-p+1}^{p} \gamma_{p,s}^{1,r} \tilde{u}_{i,s}^{(k-2)}
\]

\[
= \frac{a^2}{\Delta x^2} \sum_{s=-p+1}^{p} \left( \sum_{r=-p+1}^{p} \gamma_{p,r}^{1,j} \gamma_{p,s}^{1,r} \right) \tilde{u}_{i,s}^{(k-2)}
\]

\[
= \frac{a^2}{\Delta x^2} \sum_{s=-p+1}^{p} \gamma_{p,s}^{2,j} \tilde{u}_{i,s}^{(k-2)},
\]

where (25) has been used. By recurrence:

\[
\tilde{u}_{i,j}^{(k)} = \frac{(-1)^k a^k}{\Delta x^k} \sum_{r=-p+1}^{p} \gamma_{p,r}^{k,j} u_{i+r}^{n}
\]

(52)
Next,

\[
A_{p,0}^{0,1/2}(\tilde{f}_{i,*}^{(k-1)}, \Delta x) = \frac{1}{\Delta x} \sum_{j=-p+1}^{p} \gamma_{p,j}^{0,1/2} \tilde{f}_{i,j}^{(k-1)}
\]

\[
= \frac{a}{\Delta x} \sum_{j=-p+1}^{p} \gamma_{p,j}^{0,1/2} \tilde{u}_{i,j}^{(k-1)}
\]

\[
= (-1)^{k-1} \frac{a^k}{\Delta x^k} \sum_{j=-p+1}^{p} \gamma_{p,j}^{0,1/2} \sum_{r=-p+1}^{p} \gamma_{p,r}^{k-1,j} u_{i+r}^n
\]

\[
= (-1)^{k-1} \frac{a^k}{\Delta x^k} \sum_{r=-p+1}^{p} \gamma_{p,j}^{k-1,1/2} u_{i+r}^n
\]

\[
= (-1)^{k-1} a^k A_{p,i}^{k-1,1/2}(u^n_{*, \Delta x}),
\]

where (25) has been used. Finally,

\[
F_{i+1/2}^p = \sum_{k=1}^{2p} \Delta t^{k-1} k! A_{p,0}^{0,1/2}(\tilde{f}_{i,*}^{(k-1)}, \Delta x)
\]

\[
= \sum_{k=1}^{2p} (-1)^{k-1} \frac{a^k \Delta t^{k-1}}{k!} A_{p,i}^{k-1,1/2}(u^n_{*, \Delta x}),
\]

what is the numerical flux (28) corresponding to (2), as we wanted to prove. \(\square\)

As a consequence, we obtain that the compact approximate Taylor method is linearly stable (in the \(L^2\) sense) under the usual CFL condition

\[
\max_i |f'(u_i)| \frac{\Delta t}{\Delta x} \leq 1. \tag{53}
\]

**Theorem 2** The compact approximate Taylor method is order \(2p\).

**Proof** Let us perform a step of the method starting from the point values at time \(t^n\), \(u(x_i, t_n)\), of a smooth enough exact solution. We assume that \(\Delta t/\Delta x\) remains constant.

First we have:

\[
\tilde{u}_{i,j}^{(1)} = -A_{p,0}^{1,j}(\tilde{f}_{i,*}^{(0)}, \Delta x) = -\partial_x f(u(x_{i+j}, t_n) + O(\Delta x^{2p-1})
\]

\[
= \partial_t u(x_{i+j}, t_n) + O(\Delta x^{2p-1}).
\]

Next:

\[
\tilde{f}_{i,j}^{1,n+r} = f(u(x_{i+j}, t_n) + \tilde{u}_{i,j}^{(1)} r \Delta t) = f(P_{i,j}^1(r \Delta t)) + O(\Delta x^{2p}),
\]

where

\[
P_{i,j}^1(s) = u(x_{i+j}, t_n) + s \partial_t u(x_{i+j}, t_n),
\]

\(\square\) Springer
is the first order Taylor polynomial in time of $u$ in $(x_{i+j}, t_n)$. Then

$$ f^{(1)}_{i,j} = A^{1,0}_{p,n}(f^{1}_{i,j}, \Delta t) $$

$$ f^{(1)}_{i,j} = \Delta t \sum_{r=-p+1}^{p} \gamma_{p,j}^{1,0} f_{i,j}^{1,0+r} $$

$$ f^{(1)}_{i,j} = \Delta t \sum_{r=-p+1}^{p} \gamma_{p,j}^{1,0} f(P^{1}_{i,j}(r \Delta t)) + O(\Delta x^{2p}) $$

$$ f^{(1)}_{i,j} = \Delta t \sum_{r=-p+1}^{2p-1} \gamma_{p,j}^{1,0} \frac{1}{k!} d^k(f \circ P^{1}_{i,j})(t_n) r^k \Delta t^k + O(\Delta x^{2p-1}) $$

$$ f^{(1)}_{i,j} = \frac{1}{\Delta t} \sum_{k=0}^{2p-1} \frac{1}{k!} d^k(f \circ P^{1}_{i,j})(t_n) \Delta x^k \sum_{r=-p+1}^{p} \gamma_{p,j}^{1,0} r^k + O(\Delta x^{2p-1}) $$

$$ f^{(1)}_{i,j} = d^1(f \circ P^{1}_{i,j})(t_n) + O(\Delta x^{2p-1}) $$

$$ f^{(1)}_{i,j} = \partial f(u)(x_{i+j}, t_n) + O(\Delta x^{2p-1}), \quad k = 1, \ldots, 2p - 1. \quad (54) $$

Using this equality we get:

$$ u(x_i, t_{n+1}) - u(x_i, t_n) + \frac{\Delta t}{\Delta x} \left( F^p_{i+1/2} - F^p_{i-1/2} \right) $$

$$ u(x_i, t_{n+1}) - u(x_i, t_n) + \frac{\Delta t}{\Delta x} \sum_{k=1}^{2p} \frac{\Delta x^k}{k!} \left( A^{0,1/2}_{p,0} (f^{(k-1)}_{i+1,\ast}, \Delta x) - A^{0,1/2}_{p,0} (f^{(k-1)}_{i-1,\ast}, \Delta x) \right) $$

$$ u(x_i, t_{n+1}) - u(x_i, t_n) + \frac{1}{\Delta x} \sum_{k=1}^{2p} \frac{\Delta x^k}{k!} \left( A^{0,1/2}_{p,i} (\partial_{\tilde{t}^{k-1}} f(u), \Delta x) - A^{0,1/2}_{p,i-1} (\partial_{\tilde{t}^{k-1}} f(u), \Delta x) \right) $$

$$ + O(\Delta x^{2p+1}) $$

$$ u(x_i, t_{n+1}) - u(x_i, t_n) + \frac{1}{\Delta x} \sum_{k=1}^{2p} \frac{\Delta x^k}{k!} D^{1}_{p,i} (\partial_{\tilde{t}^{k-1}} f(u), \Delta x) + O(\Delta x^{2p+1}) $$

$$ u(x_i, t_{n+1}) - u(x_i, t_n) + \frac{1}{\Delta x} \sum_{k=1}^{2p} \frac{\Delta x^k}{k!} \partial_{\tilde{t}^{k-1}} f(u)(x_i, t_n) + O(\Delta x^{2p+1}) $$

$$ u(x_i, t_{n+1}) - u(x_i, t_n) - \frac{1}{\Delta x} \sum_{k=1}^{2p} \frac{\Delta x^k}{k!} \partial_{\tilde{t}^{k}} u(x_i, t_n) + O(\Delta x^{2p+1}) $$

$$ = O(\Delta x^{2p+1}). \quad \square $
Remark In the Approximate Taylor method proposed in [25] the derivatives \( \tilde{u}_i^{(k+1)} \) are computed by applying the 2-point 1-point centered differentiation formula for first derivatives to \( f_i^{(k)} \), where \( p_k \) is given by (44): notice that \( p_k \) decreases as \( k \) increases. The same reduction of the stencil used to compute \( \tilde{u}_i^{(k)} \) could be applied here, what would allow us to reduce the number of computations while preserving the overall order of accuracy. Nevertheless, the resulting method will not be an extension of the linear Lax–Wendroff method. On the other hand, the CPU reduction will be not significant.

3.3 Example: Fourth Order Compact Approximate Taylor Method

Since the method is conservative, we will only show in detail how to compute the numerical flux (50) and, to do this, it is enough to specify how to compute

\[
\kappa_{i+1/2}^k = A_{2,0}^{0,1/2} (f_i, \Delta x), \quad k = 1, 2, 3, 4. \tag{55}
\]

The procedure is as follows:

- \( \kappa_{i+1/2}^1 \): First the assignment

\[
\tilde{f}_{i,j}^{(0)} = f(u_{i+j}^n), \quad j = -1, \ldots, 2
\]

is done and then:

\[
\kappa_{i+1/2}^1 = A_{2,0}^{0,1/2} (f_{i,*}, \Delta x) = \frac{-\tilde{f}_{i,-1}^{(0)} + 7\tilde{f}_{i,0}^{(0)} + 7\tilde{f}_{i,1}^{(0)} - \tilde{f}_{i,2}^{(0)}}{12}.
\]

- \( \kappa_{i+1/2}^2 \): The first order time derivatives of \( u \) at the nodes \( i-1, \ldots, i+2 \) are approximated by applying the corresponding differentiation numerical formula to \( \tilde{f}_{i,j}^{(0)} \):

\[
\tilde{u}_{i,-1}^{(1)} = -A_{2,0}^{1,-1} (f_{i,*}, \Delta x) = \frac{-11/6\tilde{f}_{i,-1}^{(0)} - 3\tilde{f}_{i,0}^{(0)} + 3/2\tilde{f}_{i,1}^{(0)} - 1/3\tilde{f}_{i,2}^{(0)}}{\Delta x},
\]

\[
\tilde{u}_{i,0}^{(1)} = -A_{2,0}^{1,0} (f_{i,*}, \Delta x) = \frac{1/3\tilde{f}_{i,-1}^{(0)} + 1/2\tilde{f}_{i,0}^{(0)} - \tilde{f}_{i,1}^{(0)} + 1/6\tilde{f}_{i,2}^{(0)}}{\Delta x},
\]

\[
\tilde{u}_{i,1}^{(1)} = -A_{2,0}^{1,1} (f_{i,*}, \Delta x) = \frac{-1/6\tilde{f}_{i,-1}^{(0)} + \tilde{f}_{i,0}^{(0)} - 1/2\tilde{f}_{i,1}^{(0)} - 1/3\tilde{f}_{i,2}^{(0)}}{\Delta x},
\]

\[
\tilde{u}_{i,2}^{(1)} = -A_{2,0}^{1,2} (f_{i,*}, \Delta x) = \frac{-1/3\tilde{f}_{i,-1}^{(0)} - 3/2\tilde{f}_{i,0}^{(0)} + 3\tilde{f}_{i,1}^{(0)} - 11/6\tilde{f}_{i,2}^{(0)}}{\Delta x}.
\]

Next first order Taylor expansions are used to approximate the values of the flux sixteen space-time local nodes: for \( r = -1, \ldots, 2 \)

\[
\begin{align*}
\tilde{f}_{i,-1}^{1,r} &= f (a_{i-1}^n + r \Delta t \tilde{u}_{i,-1}^{(1)}), \\
\tilde{f}_{i,0}^{1,r} &= f (a_{i,0}^n + r \Delta t \tilde{u}_{i,0}^{(1)}), \\
\tilde{f}_{i,1}^{1,r} &= f (a_{i+1}^n + r \Delta t \tilde{u}_{i,1}^{(1)}), \\
\tilde{f}_{i,2}^{1,r} &= f (a_{i+2}^n + r \Delta t \tilde{u}_{i,2}^{(1)}).
\end{align*}
\]
Then, the first order time derivatives of the flux at the nodes $i-1, \ldots, i+2$ are approximated by applying the corresponding differentiation numerical formula to $\tilde{f}_{i,j}^{1}$:

$$
\tilde{f}_{i,-1}^{(1)} = A_{2,n}^{1,0}(\tilde{f}_{i,-1}^{1,*}, \Delta t) = \frac{-1/3 \tilde{f}_{i,-1}^{1,n-1} - 1/2 \tilde{f}_{i,-1}^{1,n} + \tilde{f}_{i,-1}^{1,n+1} - 1/6 \tilde{f}_{i,-1}^{1,n+2}}{\Delta t},
$$

$$
\tilde{f}_{i,0}^{(1)} = A_{2,n}^{1,0}(\tilde{f}_{i,0}^{1,*}, \Delta t) = \frac{-1/3 \tilde{f}_{i,0}^{1,n-1} - 1/2 \tilde{f}_{i,0}^{1,n} + \tilde{f}_{i,0}^{1,n+1} - 1/6 \tilde{f}_{i,0}^{1,n+2}}{\Delta t},
$$

$$
\tilde{f}_{i,1}^{(1)} = A_{2,n}^{1,0}(\tilde{f}_{i,1}^{1,*}, \Delta t) = \frac{-1/3 \tilde{f}_{i,1}^{1,n-1} - 1/2 \tilde{f}_{i,1}^{1,n} + \tilde{f}_{i,1}^{1,n+1} - 1/6 \tilde{f}_{i,1}^{1,n+2}}{\Delta t},
$$

$$
\tilde{f}_{i,2}^{(1)} = A_{2,n}^{1,0}(\tilde{f}_{i,2}^{1,*}, \Delta t) = \frac{-1/3 \tilde{f}_{i,2}^{1,n-1} - 1/2 \tilde{f}_{i,2}^{1,n} + \tilde{f}_{i,2}^{1,n+1} - 1/6 \tilde{f}_{i,2}^{1,n+2}}{\Delta t}.
$$

Finally:

$$
\kappa_{i+1/2}^2 = A_{2,0}^{0,1/2}(\tilde{f}_{i,*}^{1}, \Delta x) = \frac{-\tilde{f}_{i,-1}^{1} + 7\tilde{f}_{i,0}^{1} + 7\tilde{f}_{i,1}^{1} - \tilde{f}_{i,2}^{1}}{12}.
$$

- $\kappa_{i+1/2}^3$: the second order time derivatives at the nodes are approximated by

$$
\tilde{u}_{i,-1}^{(2)} = -A_{2,0}^{-1}(\tilde{f}_{i,*}^{1}, \Delta x) = \frac{11/6 \tilde{f}_{i,-1}^{1} - 3 \tilde{f}_{i,0}^{1} + 3/2 \tilde{f}_{i,1}^{1} - 1/3 \tilde{f}_{i,2}^{1}}{\Delta x},
$$

$$
\tilde{u}_{i,0}^{(2)} = -A_{2,0}^{1,0}(\tilde{f}_{i,*}^{1}, \Delta x) = \frac{-1/3 \tilde{f}_{i,-1}^{1} + 1/2 \tilde{f}_{i,0}^{1} - \tilde{f}_{i,1}^{1} + 1/6 \tilde{f}_{i,2}^{1}}{\Delta x},
$$

$$
\tilde{u}_{i,1}^{(2)} = -A_{2,0}^{1,1}(\tilde{f}_{i,*}^{1}, \Delta x) = \frac{-1/6 \tilde{f}_{i,-1}^{1} + \tilde{f}_{i,0}^{1} - 1/2 \tilde{f}_{i,1}^{1} - 1/3 \tilde{f}_{i,2}^{1}}{\Delta x},
$$

$$
\tilde{u}_{i,2}^{(2)} = -A_{2,0}^{1,2}(\tilde{f}_{i,*}^{1}, \Delta x) = \frac{-1/3 \tilde{f}_{i,-1}^{1} - 3/2 \tilde{f}_{i,0}^{1} + 3 \tilde{f}_{i,1}^{1} - 11/6 \tilde{f}_{i,2}^{1}}{\Delta x}.
$$

Second order Taylor expansions are used to compute the fluxes at the sixteen nodes in the space-time mesh: for $r = -1, \ldots, 2$

$$
\tilde{f}_{i,-1}^{2,r} = f \left( u_{i,-1}^{n} + r \Delta t \tilde{u}_{i,-1}^{(1)} + \frac{r^2 \Delta t^2}{2} \tilde{u}_{i,-1}^{(2)} \right),
$$

$$
\tilde{f}_{i,0}^{2,r} = f \left( u_{i}^{n} + r \Delta t \tilde{u}_{i,0}^{(1)} + \frac{r^2 \Delta t^2}{2} \tilde{u}_{i,0}^{(2)} \right),
$$

$$
\tilde{f}_{i,1}^{2,r} = f \left( u_{i+1}^{n} + r \Delta t \tilde{u}_{i,1}^{(1)} + \frac{r^2 \Delta t^2}{2} \tilde{u}_{i,1}^{(2)} \right),
$$

$$
\tilde{f}_{i,2}^{2,r} = f \left( u_{i+2}^{n} + r \Delta t \tilde{u}_{i,2}^{(1)} + \frac{r^2 \Delta t^2}{2} \tilde{u}_{i,2}^{(2)} \right).
$$
Next, compute
\[
\tilde{f}_{i-1}^{(2)} = A_{2,0}^2 (\tilde{f}_{i-1}^*, \Delta t) = \frac{\tilde{f}_{i-1}^{2,n-1} - 2\tilde{f}_{i-1}^{2,n} + \tilde{f}_{i-1}^{2,n+1}}{\Delta t^2},
\]
\[
\tilde{f}_{i,0} = A_{2,0}^2 (\tilde{f}_{i,0}^*, \Delta t) = \frac{\tilde{f}_{i,0}^{2,n-1} - 2\tilde{f}_{i,0}^{2,n} + \tilde{f}_{i,0}^{2,n+1}}{\Delta t^2},
\]
\[
\tilde{f}_{i,1} = A_{2,0}^2 (\tilde{f}_{i,1}^*, \Delta t) = \frac{\tilde{f}_{i,1}^{2,n-1} - 2\tilde{f}_{i,1}^{2,n} + \tilde{f}_{i,1}^{2,n+1}}{\Delta t^2},
\]
\[
\tilde{f}_{i,2} = A_{2,0}^2 (\tilde{f}_{i,2}^*, \Delta t) = \frac{\tilde{f}_{i,2}^{2,n-1} - 2\tilde{f}_{i,2}^{2,n} + \tilde{f}_{i,2}^{2,n+1}}{\Delta t^2}.
\]

And finally:
\[
\kappa_{i+1/2}^3 = A_{2,0}^{0.1/2} (f_i^*, \Delta x) = \frac{-\tilde{f}_{i-1}^{(2)} + 7\tilde{f}_{i,0}^{(2)} + 7\tilde{f}_{i,1}^{(2)} - \tilde{f}_{i,2}^{(2)}}{12}.
\]

- \(\kappa_{i+1/2}^4\): the third order time derivatives at the nodes are approximated by

\[
\tilde{u}_{i-1}^{(3)} = -A_{2,0}^{1,-1} (\tilde{f}_{i,*}^2, \Delta x) = -\frac{11/6 \tilde{f}_{i-1}^{(2)} - 3 \tilde{f}_{i,0}^{(2)} + 3/2 \tilde{f}_{i,1}^{(2)} - 1/3 \tilde{f}_{i,2}^{(2)}}{\Delta x},
\]
\[
\tilde{u}_{i,0}^{(3)} = -A_{2,0}^{1,0} (\tilde{f}_{i,*}^2, \Delta x) = -\frac{1/3 \tilde{f}_{i-1}^{(2)} + 1/2 \tilde{f}_{i,0}^{(2)} - \tilde{f}_{i,1}^{(2)} + 1/6 \tilde{f}_{i,2}^{(2)}}{\Delta x},
\]
\[
\tilde{u}_{i,1}^{(3)} = -A_{2,0}^{1,1} (\tilde{f}_{i,*}^2, \Delta x) = -\frac{-1/6 \tilde{f}_{i-1}^{(2)} + \tilde{f}_{i,0}^{(2)} - 1/2 \tilde{f}_{i,1}^{(2)} - 1/3 \tilde{f}_{i,2}^{(2)}}{\Delta x},
\]
\[
\tilde{u}_{i,2}^{(3)} = -A_{2,0}^{1,2} (\tilde{f}_{i,*}^2, \Delta x) = -\frac{1/3 \tilde{f}_{i-1}^{(2)} - 3/2 \tilde{f}_{i,0}^{(2)} + 3 \tilde{f}_{i,1}^{(2)} - 11/6 \tilde{f}_{i,2}^{(2)}}{\Delta x}.
\]

Compute the approximations of the fluxes: for \(r = -1, \ldots, 2\)

\[
\tilde{f}_{i-1}^{3,r} = f \left( u_{i-1}^n + r\Delta t \tilde{u}_{i-1}^{(1)} + \frac{r^2\Delta t^2}{2} \tilde{u}_{i-1}^{(2)} + \frac{r^3\Delta t^3}{6} \tilde{u}_{i-1}^{(3)} \right),
\]
\[
\tilde{f}_{i,0}^{3,r} = f \left( u_i^n + r\Delta t \tilde{u}_{i,0}^{(1)} + \frac{r^2\Delta t^2}{2} \tilde{u}_{i,0}^{(2)} + \frac{r^3\Delta t^3}{6} \tilde{u}_{i,0}^{(3)} \right),
\]
\[
\tilde{f}_{i,1}^{3,r} = f \left( u_i^n + r\Delta t \tilde{u}_{i,1}^{(1)} + \frac{r^2\Delta t^2}{2} \tilde{u}_{i,1}^{(2)} + \frac{r^3\Delta t^3}{6} \tilde{u}_{i,1}^{(3)} \right),
\]
\[
\tilde{f}_{i,2}^{3,r} = f \left( u_{i+1}^n + r\Delta t \tilde{u}_{i,2}^{(1)} + \frac{r^2\Delta t^2}{2} \tilde{u}_{i,2}^{(2)} + \frac{r^3\Delta t^3}{6} \tilde{u}_{i,2}^{(3)} \right).
\]
Next, compute:

\[
\tilde{f}_{i-1}^{(3)} = A_{2,n}^{3,0} (\tilde{f}_{i-1}^{3,*}, \Delta t) = \frac{-\tilde{f}_{i-1}^{3,n-1} + 3 \tilde{f}_{i-1}^{3,n} - 3 \tilde{f}_{i-1}^{3,n+1} + \tilde{f}_{i-1}^{3,n+2}}{\Delta t^3},
\]

\[
\tilde{f}_{i,0}^{(3)} = A_{2,n}^{3,0} (\tilde{f}_{i,0}^{3,*}, \Delta t) = \frac{-\tilde{f}_{i,0}^{3,n-1} + 3 \tilde{f}_{i,0}^{3,n} - 3 \tilde{f}_{i,0}^{3,n+1} + \tilde{f}_{i,0}^{3,n+2}}{\Delta t^3},
\]

\[
\tilde{f}_{i,1}^{(3)} = A_{2,n}^{3,0} (\tilde{f}_{i,1}^{3,*}, \Delta t) = \frac{-\tilde{f}_{i,1}^{3,n-1} + 3 \tilde{f}_{i,1}^{3,n} - 3 \tilde{f}_{i,1}^{3,n+1} + \tilde{f}_{i,1}^{3,n+2}}{\Delta t^3},
\]

\[
\tilde{f}_{i,2}^{(3)} = A_{2,n}^{3,0} (\tilde{f}_{i,2}^{3,*}, \Delta t) = \frac{-\tilde{f}_{i,2}^{3,n-1} + 3 \tilde{f}_{i,2}^{3,n} - 3 \tilde{f}_{i,2}^{3,n+1} + \tilde{f}_{i,2}^{3,n+2}}{\Delta t^3}.
\]

Finally:

\[
\kappa_{i+1/2}^4 = A_{2,n}^{0,1/2} (\phi_{i,*}, \Delta x) = \frac{-\tilde{f}_{i-1}^{(2)} + 7 \tilde{f}_{i,0}^{(2)} + 7 \tilde{f}_{i,1}^{(2)} - 2 \tilde{f}_{i,2}^{(2)}}{12}.
\]

If \( f(u) = au \), then:

\[
F_{i+1/2}^2 = \frac{a}{12} (-u_{i+1}^n + 7u_i^n + 7u_{i+1}^n - u_{i+2}^n) \]

\[
+ \frac{a^2 \Delta t}{24 \Delta x} (-u_{i+1}^n + 15u_i^n - 15u_{i+1}^n + u_{i+2}^n) \tag{56}
\]

\[
+ \frac{a^3 \Delta t^2}{12 \Delta x^2} (u_{i+1}^n - u_i^n - u_{i+1}^n + u_{i+2}^n)
\]

\[
+ \frac{a^4 \Delta t^3}{24 \Delta x^3} (u_{i-1}^n - 3u_i^n + 3u_{i+1}^n - u_{i+2}^n), \tag{57}
\]

which coincides with the numerical flux of the fourth order Lax–Wendroff in conservative form.

### 4 Shock-Capturing Techniques

Although the Compact Approximate Taylor methods are linearly stable in the \( L^2 \) sense under the usual CFL = 1 condition, they may produce strong oscillations close to a discontinuity of the solution. The goal of this section is to modify the numerical method to avoid these oscillations. Two different techniques are considered here:

#### 4.1 Flux Limiter-Cat Methods

We consider the numerical method (27) with

\[
F_{i+1/2} = (1 - \varphi_{i+1/2}) F_{i+1/2}^L + \varphi_{i+1/2} F_{i+1/2}^2,
\]

where \( F_{i+1/2}^L \) is a first order robust numerical flux, \( F_{i+1/2}^2 \) is given by (50) with \( p = 1 \), and \( \varphi_{i+1/2} \) is a TVD centered flux limiter function, see [10,13,23]. We consider here

\[
\varphi_{i+1/2} = \varphi(r_{i+1/2}),
\]

\( \varphi \) Springer
where $\varphi$ is the van Albada second version flux limiter:

$$\varphi(r) = \max \left( 0, \frac{2r}{1 + r^2} \right),$$

(60)

and

$$r_{i+1/2} = \frac{\Delta u_{pw}}{\Delta loc} = \begin{cases} 
\frac{u_{i}^n - u_{i-1}^n}{u_{i+1}^n - u_{i}^n} & \text{if } a_{i+1/2} > 0, \\
\frac{u_{i+2}^n - u_{i+1}^n}{u_{i+1}^n - u_{i}^n} & \text{if } a_{i+1/2} < 0, 
\end{cases}$$

(61)

where $a_{i+1/2}$ is an estimate of the wave speed, for instance the one corresponding to Roe’s method:

$$a_{i+1/2} = \begin{cases} 
\frac{f(u_{i+1}^n) - f(u_i^n)}{u_{i+1}^n - u_i^n} & \text{if } u_i^n \neq u_{i+1}^n; \\
f’(u_i^n) & \text{otherwise.}
\end{cases}$$

4.2 WENO-CAT Methods

Following [16] WENO reconstructions of the flux are used to stabilize the method. The only differences with the algorithm described in Sect. 3.2 are the computation of $\tilde{u}_{i,j}^{(1)}$, that is now performed as follows:

$$\tilde{u}_{i,j}^{(1)} = - \frac{\hat{f}_{i,j+1/2} - \hat{f}_{i,j-1/2}}{\Delta x},$$

where $\hat{f}_{i+1/2}$ denotes the WENO flux splitting, reconstructions at $x_{i+1/2}$ of the flux function described in [18]. The expression of the numerical flux is then given by:

$$F_{i+1/2}^p = \hat{f}_{i+1/2} + \sum_{k=2}^{2p} \frac{\Delta t^{k-1}}{k!} A_{p,0}^{0.1/2} (\tilde{r}_{i,*}^{(k-1)}, \Delta x).$$

(61)

4.3 Systems of Conservation Laws

For systems of conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0,$$

(62)

where $\mathbf{u} = [u_1, \ldots, u_M]^T$, $\mathbf{f}(\mathbf{u}) = [f_1(u_1, \ldots, u_M), \ldots, f_M(u_1, \ldots, u_M)]^T$, the expression of CAT methods is given again by (50) and (51) just using bold characters for vectors.

Concerning the shock-capturing techniques, the implementation of the flux-limiter technique for systems of $M$ conserved variables, is done by computing (59) for every component $u_k$, with $k = 1, \ldots, M$, as follows:

First compute:

$$r_{k,i+1/2}^L = \frac{u_{k,i}^n - u_{k,i-1}^n}{u_{k,i+1}^n - u_{k,i}^n}, \quad r_{k,i+1/2}^R = \frac{u_{k,i+2}^n - u_{k,i+1}^n}{u_{k,i+1}^n - u_{k,i}^n},$$

next, compute

$$\varphi_k(r) = \min \{ \varphi(r_{k,i+1/2}^L), \varphi(r_{k,i+1/2}^R) \},$$

(63)
where \( \varphi_k(r) \) is the flux limiter for the \( u_k \) component. See [23] for more details.

Finally, WENO reconstructions are computed in conserved variables using the procedure described in [18].

5 Numerical Experiments

In this section the following numerical methods

- **LW-CATq**: Compact Approximate Taylor method of order \( q \) (space and time);
- **FL-CAT2**: Compact Approximate Taylor method of order 2 with flux limiter technique. The first order methods considered are Lax-Friedrich for scalar problems and HLL for systems;
- **WENO\( s \)-CATq**: Compact Approximate Taylor method of order \( q \) with WENO reconstructions of order \( s \) to compute \( \tilde{u}_{t,i}^{(1)} \);
- **WENO\( s \)-RK\( q \)**: WENO method of order \( s \) for the space discretization and TVD-RK\( q \) for the time discretization, see [18];
- **WENO\( s \)-LWAq**: Approximate Taylor method of order \( q \) with WENO reconstructions of order \( s \) to compute \( \tilde{u}_{t,i}^{(1)} \), see [25];

will be applied to different 1d scalar conservation laws and systems: transport and Burgers equations, Euler and ideal Magnetohydrodynamics equations (MHD).

5.1 Linear Transport Equation

We consider first (1) with \( a = 1 \), in the spatial interval \([0, 1]\), with initial condition

\[
    u(x, 0) = \begin{cases} 
    1 & 0 \leq x < 7/10, \\
    2 & 2/10 \leq x < 7/10, \\
    1 & 7/10 \leq x < 1, 
    \end{cases} \tag{64}
\]

periodic boundary conditions, a uniform mesh with \( N = 80 \) points and \( t = 1s \) is considered. The LW-CAT method (that, in this case, coincides with the Lax–Wendroff method) is applied for \( p = 1, \ldots, 5 \).

Numerical simulations are shown in Fig. 2: the \( L^2 \) stability of the scheme and the appearance of oscillations near the discontinuities can be observed. Next, we apply to the same problem to LW-CAT4, FL-CAT2, WENO5-CAT4, WENO5-RK3, and WENO5-LWA5 methods. A general view is shown in Fig. 3 together with a enlarged view of the area of interest. As it can be observed, the results given by WENO5-CAT4, WENO5-RK3 and WENO5-LWA5 are almost identical. Nevertheless, as it will be seen in the next test problem, WENO5-CAT4 still gives good results for CFL close to one, what is not the case for WENO5-RK3 or WENO5-LWA5.

Using the algorithm in Sect. 3.2 CAT methods are easily extended to any \( 2p \) order, nevertheless, this involves a significant increase of flops (number of operations required), that should be considered, see Table 2.

Finally, we consider (1) in the spatial interval \([0, 2]\) with initial condition,

\[
    u(x, 0) = 0.25 \sin(\pi x), \tag{65}
\]

and periodic boundary conditions. Table 3 shows the error and the empirical order for LW-CAT2, LW-CAT4, LW-CAT6, and Table 4 for WENO5-RK3 and WENO5-LWA5 which
Fig. 2 Transport equation with initial condition (64), CFL = 0.9 and t = 1 s: numerical results obtained with LW-CATq, q = 2, 4, 6, 8, 10

Fig. 3 Transport equation with initial condition (64), CFL = 0.5 and t = 1 s. Left-up: general view. a–d: enlarged view of interest areas

Table 2 Average rate flops to increase from LW-CAT2 to LW-CAT2p, for p = 2, 3, 4, 5 using the scalar transport equation with initial conditions (64)

| Rate flops | Order |
|-----------|-------|
| LW-CAT2   | 1.61  |
| LW-CAT4   | 2.51  |
| LW-CAT6   | 3.69  |
| LW-CAT8   | 5.16  |
| LW-CAT10  | 5.16  |
### Table 3  
Linear transport equation with initial condition (65), \( CFL = 0.5 \) and \( t = 1s \): \( L^1 \) errors and accuracy order for LW-CAT2p, \( p = 1, 2, 3 \)

| \( \Delta x \)   | LW-CAT2 Error \( \| \cdot \|_1 \) | Order \( \| \cdot \|_1 \) | LW-CAT4 Error \( \| \cdot \|_1 \) | Order \( \| \cdot \|_1 \) | LW-CAT6 Error \( \| \cdot \|_1 \) | Order \( \| \cdot \|_1 \) |
|-----------------|---------------------|-----------------|---------------------|-----------------|---------------------|-----------------|
| 0.1053          | 3.68e−02            |                 | 1.40e−02            |                 | 7.88e−03            |                 |
| 0.0526          | 6.84e−03            | 2.43            | 3.50e−05            | 8.64            | 4.25e−08            | 7.50            |
| 0.0263          | 1.70e−03            | 2.00            | 2.19e−06            | 4.00            | 6.49e−10            | 6.03            |
| 0.0132          | 4.27e−04            | 2.00            | 1.36e−07            | 4.00            | 9.89e−12            | 6.04            |
| 0.0066          | 1.06e−04            | 2.00            | 8.55e−09            | 4.00            | 1.53e−13            | 6.01            |
| 0.0033          | 2.66e−05            | 2.00            | 5.34e−10            | 4.00            | 2.64e−15            | 5.96            |

### Table 4  
Linear transport equation with initial condition (65), \( CFL = 0.5 \) and \( t = 1s \): \( L^1 \) errors and accuracy order for WENO5-RK3 and WENO5-LWA5

| \( \Delta x \)   | WENO5-RK3 Error \( \| \cdot \|_1 \) | Order \( \| \cdot \|_1 \) | WENO-LWA5 Error \( \| \cdot \|_1 \) | Order \( \| \cdot \|_1 \) |
|-----------------|---------------------|-----------------|---------------------|-----------------|
| 0.1053          | 2.03e−03            |                 | 5.44e−05            |                 |
| 0.0526          | 6.06e−05            | 5.06            | 1.65e−06            | 5.04            |
| 0.0263          | 1.87e−06            | 5.02            | 5.04e−08            | 5.04            |
| 0.0132          | 5.83e−08            | 5.00            | 1.51e−09            | 5.05            |
| 0.0066          | 1.82e−09            | 5.00            | 4.41e−11            | 5.10            |
| 0.0033          | 5.65e−11            | 5.01            | 1.15e−12            | 5.25            |

coincides in all cases with the theoretical one. For smooth solutions WENO-CATq reduce to the corresponding LW-CATq, so that the accuracy test is not necessary.

**Remark**  
In order to achieve fifth order accuracy in time for WENO5-RK3 we set \( \Delta t = h^{5/3} \).

### 5.2 Burgers Equation

We consider Burgers equation, i.e. (62) with

\[
f(u) = \frac{u^2}{2}.
\]

When CAT methods are applied to approximate a discontinuous solution of this nonlinear problem, the oscillations appearing close to the shocks tend to grow and to spoil the numerical solution. Nevertheless, it is still possible to apply these methods by reducing the \( CFL \) parameter (the reduction increases with \( p \)): for instance, Fig. 4 shows the results obtained with CAT-LW2p, \( p = 1, 2, 3, 4 \) and \( CFL = 0.8, 0.4, 0.2, 0.1 \), respectively, with initial conditions (64), periodic boundary conditions, a grid of 80-point mesh and \( t = 1.2s \).

Next, the same test problem is solved using LW-CAT4, FL-CAT2, WENO5-RK3 and WENO5-LWA5 methods. Using \( CFL = 0.5 \) and \( t = 2s \), we obtain numerical solutions without spurious oscillations for all the methods (except for LW-CAT4). Figure 5 shows a general view of solutions and the van Albada flux limiter function on every inter cell used for FL-CAT2. In order to show solutions for \( CFL \) close to 1, the same test is solved using \( N = 250, CFL = \{0.5, 0.9\} \) and \( t = \{1.2, 12\} \)s. From Fig. 6 we can conclude:
Fig. 4 Burgers equation with initial condition (64), $CFL = 0.8, 0.4, 0.2, 0.1, 0.05$ and $t = 1.2 \text{s}$: numerical results obtained with LW-CAT$q$, $q = 2, 4, 6, 8$. Left: general view. Right: enlarged view.

Fig. 5 Burgers equation with initial condition (64), $CFL = 0.5$ and $t = 1.2 \text{s}$. Up: general view. Down: flux limiter function $\phi_{i+1/2}$ for FL-CAT2.

- $CFL \leq 0.5$
  - LW-CAT4 shows oscillations near the discontinuities, but it is stable.
  - FL-CAT2 is very diffusive near the discontinuities, due to the selected first order accurate flux limiter function.
  - WENO5-CAT4, WENO5-LWA4 and WENO5-RK3 show good results, stable and essentially the same values.

- $CFL > 0.5$
  - LW-CAT4: the amplitude of oscillations increases near the discontinuities. However, they remain stable.
  - FL-CAT2: conversely to the previous $CFL$ condition, it shows acceptable solutions near the discontinuities.
  - WENO5-CAT4, WENO5-LWA5 and WENO5-RK3: slight oscillations appear near the discontinuities at the beginning of the simulations. Nevertheless, as the time increases, these oscillations tend to diminish and the result remains acceptable and
stable for WENO5-CAT4, while the solutions given by WENO5-LWA5 is very diffusive and the one given by WENO5-RK3 is overdamped.

Although FL-CAT2 shows better results for bigger $CFL$, it fails in smooth regions close to critical points and for systems (as it will be seen in Euler equations).

In order to study the order of convergence, we consider again initial condition (65) and periodic boundary conditions. A reference solution at time $t = 0.5s$ (when the solution is still smooth) is obtained with WENO5-RK3 using a fine grid of 1400 nodes. The errors and the empirical order are shown in Table 5: the numerical results verify the theoretical analysis.

### 5.3 1D Euler Equations

We solve the 1D Euler equations for gas dynamics

$$u_t + f(u)_x = 0,$$  \hspace{1cm} (66)
Table 6 1D Euler equations with initial condition (70), $CFL = 0.5$ and $t = 0.5s$: $L^1$ errors and accuracy order for LW-CAT2p, $p = 1, 2, 3$

| $\Delta x$ | LW-CAT2 Error $\parallel \cdot \parallel_1$ | LW-CAT4 Error $\parallel \cdot \parallel_1$ | LW-CAT6 Error $\parallel \cdot \parallel_1$ |
|------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
|            | Order $\cdot \parallel \cdot \parallel_1$   | Order $\cdot \parallel \cdot \parallel_1$   | Order $\cdot \parallel \cdot \parallel_1$   |
| 0.1053     | 3.34e$-03$                                   | 8.57e$-04$                                   | 5.49e$-04$                                   |
| 0.0526     | 8.82e$-03$                                   | 9.93e$-05$                                   | 3.53e$-05$                                   |
| 0.0263     | 2.28e$-04$                                   | 7.31e$-06$                                   | 1.01e$-06$                                   |
| 0.0132     | 5.69e$-05$                                   | 4.81e$-07$                                   | 1.94e$-08$                                   |
| 0.0066     | 1.35e$-05$                                   | 3.02e$-08$                                   | 3.21e$-10$                                   |
| 0.0033     | 2.71e$-06$                                   | 1.78e$-09$                                   | 4.99e$-12$                                   |

with

$$
\mathbf{u} = \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u (E + p) \end{bmatrix},
$$

(67)

where $\rho$ is the density, $u$ the velocity, $E$ the total energy per unit volume, and $p$ the pressure. We assume an ideal gas with the equation of state,

$$
p(\rho, e) = (\gamma - 1) \rho e,
$$

(68)

being $\gamma$ the ratio of specific heat capacities of the gas taken as 1.4 and $e$ is the internal energy per unit mass given by:

$$
E = \rho (e + 0.5u^2).
$$

(69)

We consider the spatial interval $[0, 2]$ with the initial condition:

$$
\rho(x, 0) = 0.75 + 0.5 \sin(\pi x),
\rho u(x, 0) = 0.25 + 0.5 \sin(\pi x),
E(x, 0) = 0.75 + 0.5 \sin(\pi x),
$$

(70)

and periodic boundary conditions. For this test we take $CFL = 0.5$ and $t = 0.5s$. We use a fine grid with 1400-point mesh to compute LW-CAT8 as a reference solution. The results in Table 6 support the theoretically obtained accuracy.

Two tests involving discontinuities are considered:

- The Sod Shock tube problem. The initial condition is given by:

$$
(\rho, u, p) = \begin{cases} 
(1, 0, 1) & \text{if } x < 1/2, \\
(0.125, 0, 0.1) & \text{if } x > 1/2.
\end{cases}
$$

Here, $x \in [0, 1]$, $CFL = 0.5$, $t = 0.25s$, and outflow- boundary conditions are considered at both sides. For details of this problem see [19]. We compare FL-CAT2, WENO5-CAT4, WENO5-LWA5 and WENO5-RK3 using 450 points. A reference solution is computed with the algorithm HE-E1RPEXACT by Toro, see [23]. The flux limiter function is computed for every variable.

While all numerical solutions show stable and similar values over smooth regions (see Fig. 7), the quality is different in the interest regions (a,b,c,d): an enlarged view of them can be seen in Fig. 8. By using $CFL = 0.5$ we observe that the solution given by FL-CAT2
Fig. 7 Sod shock tube problem, $CFL = 0.5$ and $t = 0.25\, s$. Left-up: general view of numerical solutions for density $\rho$ and $\phi_{i+2}$ used in FL-CAT2. Left-down: general view of numerical solutions for the internal energy $E$ and $\phi_{i+2}$ used in FL-CAT2. Right-up: general view of numerical solutions for velocity $u$ and $\phi_{i+2}$ used in FL-CAT2. Right-down: general view of numerical solutions for the pressure $p$.

Fig. 8 Sod shock tube problem, $CFL = 0.5$ and $t = 0.25\, s$: a general view and enlarge view of the numerical results for $\rho$ close to regions $a$–$d$.

is the most diffusive one, meanwhile, WENO5-CAT4, WENO5-LWA5 and WENO5-RK3 plots essentially the same results. Choosing the $CFL = 0.9$, notorious differences in the solutions are found, mostly in the approximate Taylor solutions. WENO-CAT4 and WENO-RK3 remains similar solutions to those obtained with $CFL = 0.5$, which is not the case for WENO-LWAT5, see Fig. 9.
Fig. 9  Sod shock tube problem, CFL = 0.5 and \( t = 0.25 \) s: a general view and enlarge view of the numerical solutions for internal energy \( e \) close to regions a–d

- The Shu–Osher problem. The initial condition is given by:

\[
(\rho, u, p) = \begin{cases} 
(3.8571, 2.6293, 10.3333) & \text{if } x < -4, \\
(1 + 0.2 \sin(5x), 0, 1) & \text{if } x > -4.
\end{cases}
\]

We consider the spatial interval \( x \in [-5, 5] \), CFL = 0.5 and time \( t = 1 \) s. For details see [18] test 8. We compare FL-CAT2, WENO5-CAT4, WENO5-LWA5 and WENO5-RK3 using 450-point mesh and a reference solution computed with WENO5-RK3 method using a 2500-point mesh. For this test, all solutions are closely similar and near to the reference solution with the exception of FL-CAT2, see Fig. 10.

5.4 1D MHD Equations

Finally we consider the 1D ideal Magnetohydrodynamics (MHD) system of equations whose expression is the following:

\[
\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \tag{71}
\]

with

\[
\mathbf{u} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ E \end{bmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} \rho v_x \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ 0 \\ v_x B_y - v_y B_x \\ v_x B_z - v_z B_x \\ v_x (E + p^*) - B_x (\mathbf{v} \cdot \mathbf{B}) \end{bmatrix}, \tag{72}
\]

where \( \rho \) is the mass density, \( \mathbf{v} = [v_x, v_y, v_z]^T \) and \( \mathbf{B} = [B_x, B_y, B_z]^T \) are the velocity and magnetic fields respectively, \( E \) is the total energy per unit volume, and \( p^* \) the total pressure.

We assume an ideal gas with the equation of state
Fig. 10 Shu–Osher problem, \(CFL = 0.5\) and \(t = 1\)s: Left: general view of numerical solutions for density. Right-up: enlarged view. Right-down: enlarged view

\[
p(\rho, e) = (\gamma - 1)\rho e,
\]

where \(p\) is the hydrostatic pressure; \(\gamma\), the adiabatic constant; and \(e\), the internal energy per unit mass related to the total energy by the equation

\[
E = \frac{1}{2} \rho |v|^2 + \frac{1}{2} |B|^2 + \rho e.
\]

Finally, the total pressure \(p^*\) is given by \(p + p_m\), where

\[
p_m = \frac{1}{2} |B|^2
\]

is the magnetic pressure. The spectral structure of (72) has been analyzed in [2].

Following [2], two tests for the MHD equations involving discontinuous weak solutions are considered:

- The Brio-Wu shock tube problem. The initial condition is given by:

\[
(\rho, v, B, p) = \begin{cases} 
(1, 0, 0, 0, 0, 0.75, 1, 0, 1) & \text{if } x < 0, \\
(0.125, 0, 0, 0, 0, 0.75, -1, 0, 0.1) & \text{if } x > 0.
\end{cases}
\]

We consider the spatial interval \([-1, 1]\), a \(\gamma = 2\), 800-point mesh, Dirichlet boundary conditions, \(CFL = 0.8\), and time \(t = 0.2\)s. A reference solution is computed using the HLL method with a 20000-point mesh. The solution of this test presents a compound wave consisting of an intermediate shock followed by a slow rarefaction wave. Plots of the numerical solutions for \(\rho, v_x, B_y\), and \(p\), using WENO5-CAT4, WENO5-RK3 and WENO5-LWA5 methods are shown in Fig. 11. From the solutions we can observe that all methods give similar solutions. The numerical solutions given by all of the methods present oscillations that remain bounded: see Fig. 11. While the numerical results for the density are similar, WENO5-RK3 is more oscillatory for \(v_x\) in some areas and WENO5-LWA5 produces non-smooth behaviors near shocks caused by the choice \(CFL = 0.8\): see Figs. 12 and 13.
Fig. 11 Brio-Wu shock tube problem, $CFL = 0.8$ and $t = 0.2$ s. Numerical solutions for $\rho$, $v_x$, $B_y$, $p$

Fig. 12 Brio-Wu shock tube problem, $CFL = 0.8$ and $t = 0.2$ s. Enlarged view for $\rho$

- High Mach shock tube problem. The initial condition is given by:

$$(\rho, v, B, p) = \begin{cases} 
(1, 0, 0, 0, 0, 1, 0, 1000) & \text{if } x < 0, \\
(0.125, 0, 0, 0, 0, -1, 0, 0.1) & \text{if } x > 0.
\end{cases}$$

In this case we consider the spatial interval $[-1, 1]$, a 400-point mesh, $\gamma = 2$, Dirichlet boundary conditions, $CFL = \{0.5, 0.8\}$, and time $t = 0.12$s. The reference solution is computed as in the previous test. From plots of solutions of $\rho$, $v_x$, $B_y$, $p$ using WENO5-CAT4, WENO5-RK3, and WENO5-LWA5 methods we observe that, with $CFL = 0.5$, acceptable and stable solutions are obtained for all of the methods: see Fig. 14. Using
Fig. 13  Brio-Wu shock tube problem, $CFL = 0.8$ and $t = 0.2\,\text{s}$. Enlarged view for $v_x$

Fig. 14  High mach shock problem, $CFL = 0.5$ and $t = 0.012\,\text{s}$. Numerical solutions for $\rho$, $v_x$, $B_y$, $p$

$CFL = 0.8$ WENO5-LWA5 is not stable and WENO5-RK3 is more oscillatory than WENO-CAT although discontinuities are captured slightly better: Fig. 15 shows a general view of solutions for $\rho$, $v_x$, $B_y$, $p$. Enlarged views of $\rho$ and $B_y$ are shown in Fig. 16.
Fig. 15 High mach shock problem, $CFL = 0.8$ and $t = 0.012\,s$. General view of the numerical solutions provided by WENO5-CAT4 and WENO5-RK3 for $\rho$, $u_x$, $B_y$ and $p$.

Fig. 16 High mach shock tube problem, $CFL = 0.8$ and $t = 0.012\,s$. Left: enlarged views for $\rho$. Right: enlarged views for $B_y$.

6 Conclusions

In this work, first a review of high order Lax–Wendroff methods for the linear transport equation has been presented, including the study of the order, a heuristic study of the $L^2$-stability, and the computation and properties of the coefficients. Next, an extension to nonlinear conservation laws has been introduced with arbitrary even order $2p$ of accuracy, the so-called Compact Approximate Taylor (CAT) methods. Unlike previous applications of Taylor methods to conservation laws, CAT methods have $(2p + 1)$-point centered stencils, like Lax–Wendroff methods for linear problems. Moreover, since they inherit the stability properties...
of Lax–Wendroff methods, they are linearly $L^2$-stable under a CFL-1 condition. In order to prevent the spurious oscillations that appear close to discontinuities two shock-capturing techniques have been considered: a flux-limiter technique (FL-CAT methods) and WENO reconstruction for the first time derivative (WENO-CAT methods). We follow [25] in the second approach.

These new methods have been compared in a number of test cases with WENO-RK methods (Finite Differences WENO reconstructions in space, TVD-RK in time) and with the WENO-LW methods introduced in [25] (Finite Differences WENO reconstruction for the first time derivative, Approximate Taylor in time). The linear transport equation, Burgers equation, the 1D compressible Euler and the MHD equations have been considered. For $CFL \leq 0.5$ all the numerical methods work correctly, and the results obtained with all the methods using WENO reconstructions are similar, while the FL-CAT method is more diffusive as expected. Nevertheless, CAT methods are more expensive in computational time and number of operations due to its local character (FL-CAT is less expensive than WENO-CAT as reconstructions are avoided). However, the extra computational cost of CAT methods is compensated by the fact that they still give good solutions with CFL values close to 1.

Future developments include:

- Parallel implementation.
- Use of fast WENO reconstructions: see [1]
- Order adaptive CAT methods based on smooth indicators.
- Application to systems of balance laws.
- Extension to multidimensional problems.

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