Dynamic Optical Superlattices with Topological Bands

Stefan K. Baur,1 Monika H. Schleier-Smith,2,3,4 and Nigel R. Cooper1,3

1Cavendish Laboratory, J.J. Thomson Avenue, Cambridge CB3 0HE, United Kingdom
2Department of Physics, Stanford University, Stanford, California 94305, USA
3Fakultät für Physik, Ludwig-Maximilians-Universität, Schellingstrasse 4, 80799 München, Germany
4Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, 85748 Garching, Germany

(Dated: February 17, 2014)

We introduce an all-optical approach to producing high-flux synthetic magnetic fields for neutral atoms or molecules by designing intrinsically time-periodic optical superlattices. A single laser source, modulated to generate two frequencies, suffices to create dynamically modulated interference patterns which have topological Floquet energy bands. In particular, we propose a simple laser setup that realizes a tight-binding model with uniform flux per plaquette and well-separated interference bands. Our method relies only on the particles’ scalar polarizability and far detuned light.

In the quest to establish ultracold atoms as versatile quantum simulators of condensed-matter physics [1], a key challenge is to develop minimally invasive methods of mimicking the orbital effects of a magnetic field [2]. In solid-state systems the interplay of strong magnetic fields with Coulomb interactions gives rise to strongly correlated phases, notably the fractional quantum Hall effect [3]. Atomic systems offer the prospect of studying related phenomena in either bosonic or fermionic systems, under the influence of tunable interactions, with new tools such as high-resolution in situ imaging.

This goal has motivated intense theoretical and experimental effort at simulating, for neutral atoms, the Lorentz force experienced by a charged particle in a magnetic field [2]. Methods demonstrated to date have included subjecting quantum gases to rapid rotation [4] or imprinting geometric phases via spatially dependent couplings between internal states [6]. While the latter approach can in principle be extended to reach a high (net positive) flux density [7–10], only a select few atomic species offer a route to introducing the requisite optical couplings without significant spontaneous-emission-induced heating or atom loss [9,12].

A more broadly applicable approach to producing high-flux gauge fields is by modulating tight-binding optical lattices periodically in time [13–19]. Relying only on the optical dipole force, this approach is equally applicable to any polarizable atoms or molecules and can be implemented with arbitrarily far-detuned light. It has been demonstrated in pioneering experiments that periodically displace lattice sites [20,21]: such “shaking” can reverse the sign of tunneling matrix elements [13,14] or imprint Peierls-like phases indicative of broken time-reversal symmetry [16,20,22,23].

A related approach involves the modulation of on-site energies to produce a resonant photon-assisted hopping between orbitals of distinct lattice sites. Requiring only small modulation amplitude, this method is less susceptible to (multiphoton) heating processes. However, all existing proposals applying this method in periodic optical lattices with one resonant frequency lead to zero average flux. Recent success in producing large uniform flux by resonant photon-assisted hopping [24,25] is a technical feat, requiring not only multiple optical lattice lasers but also a strong magnetic field gradient [26].

In this Letter, we present an all-optical scheme for realizing a tight-binding Hamiltonian with uniform flux and well-separated topological bands. Underlying our proposal is a new approach to breaking time-reversal symmetry in far-detuned optical lattices, relying only on interference of light at two frequencies readily derived from a single laser. We introduce our approach with a minimal laser scheme yielding a Haldane-like model on a honeycomb lattice before proceeding to our principal proposal, which achieves uniform flux on a triangular lattice. We calculate the full Floquet band structures for both lattices and, by calculating their Chern numbers, demonstrate their non-trivial topology.

To realize each topological model, we design a dynamic interference pattern constituting a two-dimensional (2D) lattice that evolves periodically in time. First, using only a single frequency $\omega$ of light, we engineer a static lattice $V(r)$ whose unit cell comprises two sites (A and B) that are offset by an energy $\hbar \delta$. Whereas this energy offset suppresses tunneling between A and B, we reestablish the tunneling—with modified phases—by interfering the lattice beams with one additional laser field generated from the same source at a nearby frequency $\omega + \Omega$ and propagating normal to the 2D plane to form a modulating lattice $\tilde{V}$, oscillating at frequency $\Omega$. The combined effect of $V + \tilde{V}$, forms a dynamic superlattice which, for $\Omega \approx \delta$, induces chiral hopping. Since the beams at frequency $\omega$ contribute to both the static and the dynamic lattice, these are locked in register.

The two lattice models considered in this work are illustrated in Fig. 1(a). The A/B lattice sites are shown as shaded/white circles. Tunneling from A to A or B to B (dashed lines) occurs even within the static lattice, but tunneling from A to B (solid lines) is photon-assisted by the periodic modulations of the dynamic lattice. The full time-dependent potentials are illustrated in Fig. 1(b), where the shading indicates the depth of the static latt-
tice and arrows indicate the amplitude and phase of the modulating superlattice.

To engineer the structure and dynamics of a superlattice produced from a single laser source, we exercise independent control over the two polarization components of each of $N$ laser fields

$$ E_i = E \left( \cos \theta e^{i\alpha_i z} + \sin \theta (\mathbf{k}_i \times \hat{z}) \right) e^{i(k_i \cdot \mathbf{r} - \omega t - \varphi_i)} \tag{1} $$

forming the static lattice, as illustrated in Fig. 1(c). Here, each phase $\alpha_i$, labelled in orange (gray), denotes a retardation between in-plane ($p$) and out-of-plane ($s$) fields. Whereas the $s$ fields contribute only to the static lattice, the $p$ components additionally interfere with a coupling field $E = E_0 e^{i(kz - (\omega + \Omega) t)}$ to produce the modulating lattice $V$. For $N \leq 3$, as in our honeycomb lattice (I), the resulting dynamic superlattice is insensitive to the relative phases $\varphi_i$ of different lattice beams, which have no effect but to translate the entire structure. Our triangular lattice, formed from $N > 3$ wavevectors, offers greater control over the modulation pattern via additional phases $\varphi_i$, which can be stabilized by the method of Ref. 27. In Fig. 1 II(c), we indicate in bold these phases, chosen to produce a flux of $\pi/2$ through each plaquette.

Achieving non-zero fluxes through superlattice plaquettes requires designing a modulating potential $\tilde{V}$ that breaks the symmetry between clockwise and counterclockwise hopping. In particular, the modulation $\tilde{V}(\mathbf{R}, t) = V_R e^{i\phi_R + \Omega t}$ of the potential at lattice sites $\mathbf{R}$, with frequency $\Omega = \delta$, will: (i) restore photon-assisted tunneling between A and B sites (which are offset in energy by $h\delta$) providing a Peierls-like tunneling phase; and (ii) modify the amplitude of tunneling between degenerate sites (e.g. from an A site to a neighboring A site). While the Peierls-like phases determine the magnetic flux, the tunneling amplitudes set the energy scales for the band structure of each dynamic superlattice.

A useful pictorial rule to derive the photon-assisted tunneling matrix element between two sites is to draw the vectors corresponding to the complex numbers $V_0, V_1, ..., V_N$ in the complex plane at each of the two locations. When tunneling from a higher energy (B) site at $\mathbf{R}$ to lower energy (A) site at $\mathbf{R}'$, we construct the difference vector corresponding to $z = z_{\mathbf{R}} - z_{\mathbf{R}'}$. The phase $\theta$ and amplitude $A > 0$ of this complex number $z = A e^{i\theta}$ determine the effective tunneling matrix element $28$

$$ K_{\text{eff}} = e^{i\theta} J_1(A) \times K, \tag{2} $$

where $K$ is the bare tunneling matrix element for a time-independent Hamiltonian in which the two sites are degenerate, and we have assumed $K \ll \Omega, \delta$ 10. Likewise, for degenerate lattice sites (e.g. tunneling from A to A), the tunneling matrix element $J \ll \Omega, \delta$ for the static lattice becomes renormalized to

$$ J_{\text{eff}} = J_0(A) \times J. \tag{3} $$

Here, $J_{0,1}(x)$ denote Bessel functions. In the limit of small amplitude of the resonant modulation, $A \lesssim 1$, in which case $K_{\text{eff}} \approx K A e^{i\theta}/2$ and $J_{\text{eff}} \approx J$. Using the simple rules (2) and (3), we derive the effective time-independent tight-binding Hamiltonian for each of the schemes in Fig. 1(c), before proceeding to a full calculation of the topological band structure.

**Honeycomb Lattice**—Our honeycomb lattice is formed from three red-detuned travelling waves with wavevectors $k_1 = k(0,1,0), \ k_2 = -k(\sqrt{3},1,0), \ k_3 = k(\sqrt{3},-1,0)$, shown in Fig. 1II(c). The three beams interfere with each other to produce a static lattice $V(\mathbf{r}) = V_{||}(\mathbf{r}) + V_\perp(\mathbf{r})$, with $V_{||}(\mathbf{r}) = V_0 \sum_{i<j} \cos (K_{ij} \cdot \mathbf{r})$ and $V_\perp(\mathbf{r}) = -V_0 \sum_{i<j} \cos (K_{ij} \cdot \mathbf{r} + \alpha_{ij})$, where $K_{ij} = k_i - k_j, \ \alpha_{ij} \equiv \alpha_i - \alpha_j$, and $V_0,1 > 0$. $V_{||}$ and $V_\perp$ are
lattices formed by, respectively, the $p$ and $s$ polarization components of the fields $E_i$ (Eq. (1)): we fix the energy offset $\hbar \Omega$ between A and B sites by choosing the polarization orientation $\theta$ to tune the intensities, and using the ellipticity angles $\alpha_i$ to tune the displacements, of $V_{||}$ and $V_{⊥}$ relative to one another.

The superlattice modulation is created by the interference of the $p$-polarized lattice beams with a circularly polarized coupling field at frequency $\omega + \Omega$ propagating normal to the plane: i.e. $E^p = \hat{\sigma}_- E_0 e^{i[kz - (\omega + \Omega)t]}$, where $\hat{\sigma}_- = (\hat{x} - i \hat{y})/\sqrt{2}$. This leads to a dynamic superlattice potential

$$V = \sqrt{2} \sum_{i=1}^{3} \cos (k_i \cdot r + \Omega t + \gamma_i)$$

that breaks time-reversal symmetry due to the winding of the phases $\gamma_i = -\text{Arg} \left[ \hat{z} \cdot (\hat{\sigma}_- \times k_i) \right]$ with wavevector $k_i$. $\sqrt{2}$ modulates only the sites of the B sublattice, where the field $\sum_i E_i$ of the static lattice has a $\hat{\sigma}_+$ polarization component. (The A sites instead have $\hat{\sigma}_+$ polarization and thus, to lowest order, do not feel the time-varying drive potential.) The B sites are all driven with the same amplitude, but with a phase that increases in steps of $2\pi/3$ on moving around a supercell containing three B sites. The amplitudes and phases of the modulation of the B sites are shown as the arrows on the white sites in Fig. 2(a).

Using the prescription of Eq. 2 we find that the matrix elements for tunneling from a B to an A site acquire phases $0, 2\pi/3$ and $4\pi/3$ (equal to the phase of the drive on the B site), illustrated on the solid links in Fig. 2(a). These tunneling phases lead to the fluxes shown in Fig. 1 II(a), with a flux of $2\pi/3$ through B-A-B plaquettes. The model also has A-A couplings (not shown in Figs. 1 II(a) or Figs. 2 b)), but there is no flux through the corresponding A-B-A plaquettes.

To determine the bandstructure of this dynamic superlattice, with time-periodic Hamiltonian $H(t) = H(t + 2\pi/\Omega)$, we construct the Floquet states labelled by quasi-energy $\epsilon$, with $-\hbar \Omega/2 \leq \epsilon < \hbar \Omega/2$. The Floquet states are characterised by a conserved wavevector, $\mathbf{k}$, and give rise to two energy bands, there being two sites (A and B) in the magnetic unit cell. When driven on resonance ($\Omega = \delta$ in the tight-binding picture), the dispersion relation features a single unsplit Dirac cone and mass gap at the other Dirac point. The unsplit Dirac point can be made to acquire a mass gap by tuning $\Omega$ away from $\delta$. Depending on the sign of this detuning, one obtains a model with Berry curvature of either opposite or equal signs at the two Dirac cones. When the bands are split such that the Berry curvature has the same sign in the lowest band at both Dirac cones, the band is of non-trivial topology and has Chern number $\text{Ch} = 1$. The dispersion relations are shown in Fig. 3(a) for a case in which the bands are topological.

**Triangular Lattice.**— We now turn to the principal goal of this work: to obtain topological bands that are well separated in energy, we construct a triangular lattice with uniform flux.

The laser configuration we propose is illustrated in Fig. 1 II(c). A minimal set-up for the static lattice involves two retro-reflecting beams with wavevectors $k_{\pm 1} = \pm k(0,1,0), k_{\pm 2} = \mp k/2(\sqrt{3},1,0)$ (red arrows in Fig. 1 II(c)), which produce a potential of the form

$$V = -V_0 \sum_{i=1}^{2} \cos^2 (k_i \cdot r) + V_1 \cos (k_1 \cdot r) \cos (k_2 \cdot r) + \frac{\sqrt{2}}{3} \sum_{i=1}^{3} \cos (k_i \cdot r + \Omega t + \gamma_i),$$

where $V_0 > 0$. The A-B offset $\delta$ is controlled by $V_1$, which is tuned to a value $|V_1| \ll |V_0|$ by setting polarizations to $\alpha_1 = 0, \alpha_2 = \pi/2$, and $\theta = 1$. In order to obtain the approximate six-fold symmetric lattice geometry shown in Fig. 1 II(b), an optional third lattice beam (indicated by the dotted green arrows) can be included [31], adding a term $-V_0 \cos^2 (k_3 \cdot r)$ to the static lattice potential $V$. Interference of the fields $E_{\pm \pm \pm 2}$ with a linearly polarized beam at frequency $\omega + \Omega$ perpendicular to the plane generates a modulation

$$\bar{V} = \sqrt{2} \left[ \cos (k_1 \cdot r) \cos (\Omega t) + \cos (k_2 \cdot r) \cos \left( \Omega t + \frac{\pi}{2} \right) \right].$$

Here, the $(90^o)$ phase lag between the two terms breaks time-reversal symmetry. It is controlled by the position of one retro-reflecting mirror, which sets $\varphi_{-2} = \pi/2$ (relative to $\varphi_{+1,-1,+2} = 0$).

Lattice sites of the triangular lattice [Eq. (5)] are located at $R = m_1 R_1 + m_2 R_2$ with $R_1 = a/2(-1,\sqrt{3})$ and $R_2 = a(1,0)$ ($a = \lambda/\sqrt{3}$). On these lattice sites, the drive potential evaluates to

$$\bar{V}(R,t) = \frac{\sqrt{2}}{2} \left[ (-1)^{m_1} + i(-1)^{m_2} \right] e^{i\Omega t} + \text{c.c.}$$

Here, on A (B) sites, the amplitude and phase of the drive potential is proportional to $\pm 1 \pm i$ respectively, depicted

---

**Fig. 2:** Peierls-like phases for hopping between sites (along the direction indicated by the black arrows on the bonds) obtained from the tight-binding description for the honeycomb (a) and triangular lattice (b) models. The arrows on the lattice sites show the phase $\phi_{LR}$ of the drive potential on sites that are modulated. Gray (white) sites correspond to A (B) sites as defined in the main text.
as vectors in Fig. 2. This drive gives rise to the tunneling phases $0, \pi/2, \pi$ and $3\pi/2$ when hopping from B to A sites, which can be worked out using the rule Eq. 2.

The solution of the Floquet states shows a bandstructure very similar that of the (time-independent) triangular lattice tight-binding model with $\pi/2$ flux per plaquette: that is, the two bands are topological (with Chern numbers of $\pm 1$) and are separated by an energy gap $\Delta$ that is about twice the bandwidth $W$, as shown in Fig. 3(a).

Small deviations of the bandstructure from an ideal isotropic triangular lattice with uniform flux are visible in Fig. 3(a). These deviations come from a slight asymmetry between tunneling matrix elements for hopping from A to A and B to B sites that depends on the precise shape of the potential creating the superlattice. Note that even without the third in-plane beam (shown as dotted green lines in Fig. 1), one can create well separated Chern bands [see Fig. 4(b)].

Prospects.—Signatures of the topological nature of the band structure could be detected using a variety of proposed techniques [32, 43]. Even in our honeycomb lattice, where the band gap is small but the bands are well separated at each point in momentum space, the Berry curvature could be fully characterized via the semiclassical dynamics of a wavepacket undergoing Bloch oscillations [35, 36].

A consideration of the energy scales in the dynamical superlattices suggests that unwanted heating effects should be minimal. The bands have a natural energy scale set by the bare tunneling $J$ from A-to-A or B-to-B of the static lattice $V(r)$. For the honeycomb lattice, these next nearest neighbor tunnel couplings are small compared to the bare couplings $K$ from A-to-B in the absence of energy offset, $\delta = 0$. Thus the ratio $|K_{eff}|/J_{eff}$ is widely tunable even in the regime of weak driving amplitude $\tilde{V}_R/(\hbar\Omega) \ll 1$. For the triangular lattice, isotropic hopping amplitudes $|K_{eff}| \approx J_{eff}$ are achieved for moderate driving amplitude $\tilde{V}_R/(\hbar\Omega) \sim 1$, where we expect multi-photon heating to remain weak [41]. The modulation frequency $\Omega \approx E_R$ is small compared to the gap to higher bands, so these bands do not contribute. Furthermore, the absolute time-scales for tunneling are well within reach of current experiments. For example, for $^{6}$Li and a laser wavelength of $1 \mu m$, the narrow bandwidth of the triangular lattice corresponds to a tunneling rate $1/\tau \sim W/h = 130 - 590 Hz$.

Our species-independent artificial gauge fields can particularly benefit experiments with light alkali atoms, where alternative schemes involving Raman coupling of internal states cause rapid heating that precludes observing interesting many-body physics [11, 42]. Both lithium and potassium offer bosonic and fermionic isotopes with Feshbach resonances that enable exploration of the entire range of interaction strengths. Exposing these species to a high magnetic flux will open the door to studying novel strongly correlated states.
Another key feature of our scheme is that it allows for almost adiabatic loading into the lattice. A protocol for preparing a Chern insulator could start by loading a gas of fermions into the lowest band of the static lattice. Subsequently, the coupling laser is ramped on but initially kept off resonance, at a drive frequency $\Omega_1 \ll \delta$. Then, the drive frequency is swept to its final value $\Omega \approx \delta$. During this last step, the two coupled bands briefly touch at a single Dirac cone as they acquire a non-trivial topology. A Chern insulator would then be obtained by recompressing the gas, thus creating a band insulator in the center of a trapped atomic cloud.

This work was supported by EPSRC Grant No. EP/1010580/1 and the A. von Humboldt Foundation. We thank Ulrich Schneider and Immanuel Bloch for stimulating discussions.

[1] I. Bloch, J. Dalibard, and S. Nascimbene, Nat. Phys. 8, 267 (2012).
[2] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. 83, 1523 (2011).
[3] R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
[4] N. R. Cooper, Advances in Physics 539 (2008).
[5] A. L. Fetter, Rev. Mod. Phys. 81, 647 (2009).
[6] Y. J. Lin et al., Nature (London) 462, 628 (2009).
[7] D. Jaksch and P. Zoller, New J. Phys. 5, 56 (2003).
[8] E. J. Mueller, Phys. Rev. A 70, 041603 (2004).
[9] N. R. Cooper, Phys. Rev. Lett. 106, 175301 (2011).
[10] N. R. Cooper and J. Dalibard, Europhys. Lett. 95, 66004 (2011).
[11] R. Wei and E. J. Mueller, Phys. Rev. A 87, 042514 (2013).
[12] X. Cui et al., Phys. Rev. A 88, 011601 (2013).
[13] A. Eckardt, T. Jinasundera, C. Weiss, and M. Holthaus, Phys. Rev. Lett. 95, 200401 (2005).
[14] C. Sias et al., Phys. Rev. Lett. 100, 040404 (2008).
[15] A. R. Kolovsky, Europhys. Lett. 93, 20003 (2011).
[16] P. Hauke et al., Phys. Rev. Lett. 109, 145301 (2012).
[17] C. E. Creffield and F. Sols, Europhys. Lett. 101, 40001 (2013).
[18] M. Aidelsburger et al., Applied Physics B 1 (2013).
[19] C. J. Kennedy et al., Phys. Rev. Lett. 111, 225301 (2013).
[20] J. Struck et al., Phys. Rev. Lett. 108, 225304 (2012).
[21] C. V. Parker, L.-C. Ha, and C. Chin, Nat. Phys. 9, 769 (2013).
[22] J. Struck et al., Nat. Phys. 9, 738 (2013).
[23] M. Lebrat, Master’s thesis, ETH Zürich, 2013.
[24] M. Aidelsburger et al., Phys. Rev. Lett. 111, 185301 (2013).
[25] H. Miyake et al., Phys. Rev. Lett. 111, 185302 (2013).
[26] However, see [N.R. Cooper and J. Dalibard, Phys. Rev. Lett. 110, 185301 (2013)] for a method coupling three sub-band orbitals.
[27] G. Wirth, M. Olschlager, and A. Hemmerich, Nat. Phys. 7, 147 (2011).
[28] Here the phase $\theta$ is defined up to a gauge transformation.
[29] J. H. Shirley, Phys. Rev. 138, B979 (1965).
[30] See Supplemental Material at [URL will be inserted by publisher] for details of the band structure calculation.
[31] Interference with the other lattice beams can be avoided by slightly detuning the beam.
[32] E. Zhao et al., Phys. Rev. A 84, 063629 (2011).
[33] E. Alba et al., Phys. Rev. Lett. 107, 235301 (2011).
[34] C. E. Creffield and F. Sols, Phys. Rev. A 84, 023630 (2011).
[35] H. M. Price and N. R. Cooper, Phys. Rev. A 85, 033620 (2012).
[36] D. A. Abanin, T. Kitagawa, I. Bloch, and E. Demler, Phys. Rev. Lett. 110, 165304 (2013).
[37] N. Goldman et al., Proc. Natl. Acad. Sci. 110, 6736 (2013).
[38] A. Dauphin and N. Goldman, Phys. Rev. Lett. 111, 135302 (2013).
[39] H. M. Price and N. R. Cooper, Phys. Rev. Lett. 111, 220407 (2013).
[40] R. Barnett, Phys. Rev. A 88, 063631 (2013).
[41] A. Hemmerich, Phys. Rev. A 81, 063626 (2010).
[42] L. W. Cheuk et al., Phys. Rev. Lett. 109, 095302 (2012).
[43] S. K. Baur and N. R. Cooper, Phys. Rev. A 88, 033603 (2013).
Bloch wavefunctions. We then proceed to numerically
analyse lattice periodic and can be expanded in Floquet-
theory, one has
In particular, for our proposed triangular lattice geom-
Furthermore if an integer multiple $m$ of $G$ is a reciprocal
lattice vector
and we also define the Floquet Hamiltonian according to
Ref. \[29\] as
where $T = 2\pi/\omega$ is the oscillation period of the drive. In the
cases we are considering in this paper, the single-particle
Hamiltonians are of the general form
$$
\hat{H}(t) = \frac{\hat{p}^2}{2m} + V(\mathbf{r}) + F(\mathbf{r})e^{-i\Omega t} + F^*(\mathbf{r})e^{i\Omega t},
$$
Therefore we have $\hat{H}_0(\hat{p}) = \frac{\hat{p}^2}{2m} + V(\mathbf{r})$, $\hat{H}_1 = F(\mathbf{r})$, $\hat{H}_2 = F^*(\mathbf{r})$ and $\hat{H}_{m>1} = 0$. The time independent
potential $V(\mathbf{r})$ is invariant under lattice translations by
direct lattice vectors $\mathbf{R}_1, \mathbf{R}_2$. For the time dependent
potentials Eqs. (4), (6) of the main text, $F(\mathbf{r})$, characterizing phase and amplitude of the time dependent
drive, is a quasi-periodic function such that
$$
F(\mathbf{r} + \mathbf{R}_j) = e^{i\mathbf{G} \cdot \mathbf{R}_j} F(\mathbf{r}) \quad j=1,2
$$
for some wave-vector $\mathbf{G}$. This quasi-periodicity enables
us to perform a unitary (gauge) transformation in Floquet
space in order to obtain a lattice periodic Floquet
Hamiltonian (similar to what was done in \[2\] for the flux
lattice). For example with $(\hat{U})_{j,j'} = e^{i\mathbf{G} \cdot \delta_j j'}$, one
obtains the lattice periodic Floquet Hamiltonian
$$
\hat{H}'_F = \hat{U} \hat{H}_F \hat{U}^\dagger.
$$
The quasi-energies are then found by solving the eigenvalue
problem
$$
\left(
\begin{array}{ccc}
\cdots & \hat{H}(\hat{p} - \mathbf{G}) - \hbar \Omega & e^{i\mathbf{G} \cdot \mathbf{r}} F^*(\mathbf{r}) \\
\hat{H}(\hat{p}) - e^{-i\mathbf{G} \cdot \mathbf{r}} F(\mathbf{r}) & \cdots & e^{i\mathbf{G} \cdot \mathbf{r}} F^*(\mathbf{r}) \\
e^{-i\mathbf{G} \cdot \mathbf{r}} F(\mathbf{r}) & e^{i\mathbf{G} \cdot \mathbf{r}} F(\mathbf{r}) & \hat{H}(\hat{p} + \mathbf{G}) + \hbar \Omega
\end{array}
\right)
\left(
\begin{array}{c}
|\psi_{-1}\rangle \\
|\psi_{0}\rangle \\
|\psi_{1}\rangle \\
\vdots
\end{array}
\right)
= \epsilon
\left(
\begin{array}{c}
|\psi_{-1}\rangle \\
|\psi_{0}\rangle \\
|\psi_{1}\rangle \\
\vdots
\end{array}
\right)
$$
Supplementary material

\textbf{Full Floquet calculation.} We will now outline how one
can calculate the Floquet-Bloch spectrum of dynamic opti-
cal superlattices and show that it gives results consist-
tent with the tight-binding model description. For a time
periodic Hamiltonian $\hat{H}(t)$, one can find a set of time-
dependent quasi-stationary states
$$
|\psi(t)\rangle = e^{-i\epsilon t} \sum_j |\psi_j\rangle e^{-i\epsilon j}\Omega t \quad (S1)
$$
with quasi-energy $\epsilon \in \mathbb{R}$. These quasi-stationary states obey the eigenvalue equation
$$
\epsilon |\psi_j\rangle = \sum_j \left( \hat{H}_{j-j'} - j\hbar \Omega \delta_{jj'} \right) |\psi_j\rangle, \quad (S2)
$$
with
$$
\hat{H}_j = \frac{1}{T} \int_0^T dt e^{i\epsilon j}\Omega \hat{H}(t) \quad (S3)
$$
and we also define the Floquet Hamiltonian according to
Ref. \[29\] as
$$
(\hat{H}_F)_{j,j'} = \hat{H}_{j-j'} - \hbar \Omega \delta_{jj'} \quad (S4)
$$

Furthermore if an integer multiple $m$ of $G$ is a reciprocal
lattice vector
$$
m \mathbf{G} = n_1 \mathbf{K}_1 + n_2 \mathbf{K}_2, \quad (S9)
$$
an alternative and, as it turns out, computationally more
convenient choice for $\hat{U}$ is
$$
\hat{U} = e^{i\delta_j \mod m \mathbf{G} \cdot \delta_j j'} \quad (S10)
$$
In particular, for our proposed triangular lattice geometry,
one has $m = 2$ and for the honeycomb lattice $m = 3$. After
this gauge transformation, the Hamiltonian is lattice periodic and can be expanded in Floquet-
Bloch wavefunctions. We then proceed to numerically
calculate the band-structure of the lattice periodic Floquet
Hamiltonian, by projecting into the subspace of the resonantly coupled pair of Bloch bands and keep as many
Fourier modes as necessary to achieve convergence. Our
results for the triangular lattice are shown in Fig. 4 of the
main text. As can be seen in Fig. 4 (b) the three-
beam triangular lattice it is possible to achieve a lowest
band that resembles the lowest band of the corresponding
tight-binding model with tunneling matrix elements of
isotropic magnitude and flux 1/4 per triangular plaquette.

\textbf{Tight binding models.} Here we describe the effective
time independent tight-binding Hamiltonians discussed
in the main text in more detail. To lowest order, the

TABLE I: The first two columns show the amplitudes for hopping between A-A and B-B sites over the amplitude for A-B hopping for the driven triangular and honeycomb lattices described in the main text. Peierls-like phases for nearest-neighbor hoppings from B and A sites are also given (third column). Eqs. (3), (4) of the main text show how to relate $A$ and $\theta$ to tunneling matrix elements. In Fig. 2 of the main text it is shown how these Peierls-like phases are assigned to bonds.

| Geometry          | $A_{AA}/A_{AB}$ | $A_{BB}/A_{AB}$ | $\theta_{BA}$ |
|-------------------|------------------|------------------|--------------|
| Triangular lattice | $\sqrt{2}$       | $\sqrt{2}$       | $\frac{2\pi}{3}n, n = 0, \ldots, 3$ |
| Honeycomb lattice  | 0                | $\sqrt{3}$       | $\frac{2\pi}{3}n, n = 0, 1, 2$ |

A-A tunneling matrix elements are not modified since drive amplitude on A sites vanishes. B sites are driven with amplitude $\tilde{V}_R$ and with the phase pattern shown in Fig. 2 of the main text. For the honeycomb lattice, the effective tunneling amplitudes are related to the bare matrix elements via

$$J^{AA}_{\text{eff}} \approx J_{AA}$$
$$J^{BB}_{\text{eff}} \approx J_{BB} \times J_0(\sqrt{3}\tilde{V}_R/\Omega)$$
$$K^{BA}_{\text{eff}} \approx K_{BA} \times e^{i\theta_{BA}} J_1(\tilde{V}_R/\Omega).$$

Likewise, for the triangular lattice one finds the effective tunneling matrix elements (see Fig. S1)

$$J^{AA}_{\text{eff}} \approx J_{AA}$$
$$J^{BB}_{\text{eff}} \approx J_{BB} \times J_0(2\tilde{V}_R/\Omega)$$
$$K^{BA}_{\text{eff}} \approx K_{BA} \times e^{i\theta_{BA}} J_1(\sqrt{2}\tilde{V}_R/\Omega).$$

---

[1] J. H. Shirley, Phys. Rev. 138, B979 (1965).
[2] N. R. Cooper and J. Dalibard, Europhys. Lett. 95, 66004 (2011).