Eigenstate thermalization hypothesis, time operator, and extremely quick relaxation of fidelity

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Abstract

The eigenstate thermalization hypothesis (ETH) states that for nonintegrable systems, each energy eigenstate accurately gives microcanonical expectation values for a class of observables. In this paper, we explore the ETH in terms of the time-energy uncertainty and an intrinsic thermal nature shared by the majority of quasi eigenstates of operationally defined ‘time operator’: First, we show that the energy eigenstates are superposition of uncountably many quasi eigenstates of suitably defined ‘time operator’. Majority of such quasi eigenstates are thermal for thermodynamic isolated quantum many-body systems and approximately orthogonal in terms of extremely short relaxation time of the fidelity. In this manner, our scenario provides a theoretical explanation of ETH.

1. Introduction

A considerable research attention has been devoted to the foundation of statistical mechanics on the basis of intrinsic thermal nature of individual pure states. The long-standing fundamental problems involve deriving the principle of equal weight and to explain the mechanism of irreversible thermalization in terms of isolated quantum many-body systems [1–5].

In particular, typicality shows that a pure state uniform randomly sampled with respect to the Haar measure from an appropriate energy shell well represents the microcanonical ensemble [6–9], and provides a simple scenario to justify the principle of equal weight: Fix a set of observables, then, a majority of the pure states in the Hilbert space are similar to each on another in terms of the expectation values. Thus, we may superpose them with an almost arbitrary weight, and this includes the case of equal weight.

Several different approaches have been studied, including through a restriction on the macroscopic observables [10, 11], the general evaluation of relaxation time [12–15], the Eigenstate thermalization hypothesis (ETH) [1, 16–21], and dynamical experiments in autonomous cold atomic systems [22–24]. Of these, we focus on the foundation of ETH in terms of the time-energy uncertainty by noting that the energy eigenstates are globally distributed in the basis of suitably defined ‘time operator’ as detailed below (1) and in section 2. Before this, in the rest of this paragraph, we recall the basic knowledges of ETH. The ETH claims that each energy eigenstate well represents the microcanonical ensemble for nonintegrable systems, i.e. their expectation values for a class of observables agree well with the microcanonical averages. By requiring this property and non-degeneracy condition, arbitrary initial pure states equilibrate for the long time average of expectation values of the fixed observables. The ETH has been discussed in terms of the nonintegrability [1, 25, 26], partly because the relaxation property is considered to be sensitive to the presence of integrals of motion. On the other hand, [25–27] orrdhave indicated that most energy eigenstates of integrable systems are thermal. Such an intrinsic thermal nature, shared by most energy eigenstates of integrable systems is often called weak ETH. By considering the observables of a small subsystem, the ETH resembles typicality, although there is still a possibility that the deviations from microcanonical ensemble average of typical states and energy eigenstates are quantitatively different. We will numerically evaluate the deviations later in section 3.

Let us try to understand the mechanism of ETH in terms of typicality. Our starting point is to seek a relevant basis \(\{|\phi_n\rangle\}\) that is thermal, and where each energy eigenstate is a superposition of sufficient number of
orthonormal states:

\[ |E_n\rangle = \sum_{m=1}^{d} c_{mn}|\phi_m\rangle, \]

(1)

Here, \( d = \text{dim} \mathcal{H}_{(E,E+\Delta E)} \) is the dimension of the energy shell, and \( c_{mn} = O\left(\frac{1}{\sqrt{d}}\right) \).

In this paper, we present a scenario where the ETH holds by explaining that the quasi eigenstates \( |\Psi(t)\rangle \) of ‘time operator’ \( \hat{T} \) form the relevant basis by considering a thermodynamic system where \( |\Psi(t)\rangle \) thermalizes and stays in equilibrium. Note that the ‘time operator’ is constructed in (2) via spectral decomposition, and is approximately canonical conjugate to the Hamiltonian up to a constant owing to a long time cutoff. However, proper definitions of ‘time operator’ remains still controversial in general. Instead of inquiring into the best definition, we explore a foundation of ETH by introducing ‘time operator’ as (2) and its quasi eigenstates, which are approximately orthogonal through the extremely quick relaxation of the fidelity [12–15]. Note that such quasi orthogonality is analogous to that of the coherent state, and is used in quantum non-demolition measurement [28, 29]. In particular, we show that each energy eigenstate can be expressed as a superposition of many mutually almost orthogonal pure states that are considered thermal. Note that [20] quantified the degree of superposition with the use of Shannon entropy, which is basis dependent and maximized to guarantee ETH. Subsequently, [21] addressed the issue to specify a class of observables, such as local and extensive quantities, that satisfy ETH in terms of mutually unbiased basis with respect to the Hamiltonian. Mutually unbiasedness can be regarded as a generalization of the concept of the canonical conjugate, which is significant for our argument, and thus [20, 21] are related to the present study. The main difference is that in this article, the quasi eigenstates of the ‘time operator’ are unbiased with respect to the Hamiltonian. However, we do not attempt to apply ETH to the ‘time operator’ itself, and instead, we explain that the vast majority of the quasi eigenstates of ‘time operator’ are regarded as thermal with the use of the typicality [6–9] by considering observables of a subsystem and the assumption of equilibration. Then, the energy eigenstates are regarded as thermal.

The remainder of this paper is organized as follows. In section 2, we express the energy eigenstates in terms of quasi eigenstates of ‘time operator’, and explore their orthogonality and thermal nature of quasi eigenstates of the ‘time operator’. In section 3, we numerically verify the approximate orthogonality and thermal nature of quasi eigenstates and the ETH. Section 4 is devoted to a summary.

2. Time-evolved states

The time-energy uncertainty \( \Delta E \Delta t \geq \frac{\hbar}{2} \) suggests that each energy eigenstate has vanishing fluctuation \( \Delta E = 0 \), and is considered as a superposition of various eigenstates of a suitably defined ‘time operator’. Here, we address this issue, and investigate the properties of quasi eigenstates of ‘time operator’. Suppose that the Hamiltonian \( \hat{H} = \sum_{n=1}^{d} E_n |\phi_n\rangle \langle \phi_n| \) has a thermodynamic density of the states \( \Omega(E) = e^{\lambda(E)} \), i.e. the entropy is additive for large system size \( N \) and \( \phi(x) \) is concave. We also assume that the eigenergies do not degenerate \( E_n = E_m \) (\( n \neq m \)). These assumptions are used to obtain the orthogonality and completeness of time evolved states. We randomly choose a state \( |\Psi\rangle = \sum_{n=1}^{d} c_{n}|E_n\rangle \) from an energy shell \( \mathcal{H}_{(E,E+\Delta E)} \), and consider the state \( |\Psi(t)\rangle = e^{-i\hat{T}t}|\Psi\rangle \) at time \( t \).

To define the ‘time operator’, we consider a typical superposition of energy eigenstates. As a typical superposition, we choose \( c_{n} = \frac{1}{\sqrt{d}} \), as the mean value of the coefficients with respect to the Haar measure is calculated by \( |c_{n}| = \frac{1}{\sqrt{d}} \). Here, \( \langle \cdot \rangle \) denotes the average over \( c_{n} \). Note that for random sampling of a state, the absolute values of the coefficients are \( \frac{1}{\sqrt{d}} \) plus a small fluctuation. By considering this the small fluctuation, our main point — namely, to express the energy eigenstates as a superposition of sufficiently many thermal states—remains unchanged, although the factor \( \sqrt{d} \) should be replaced by \( \frac{1}{\sqrt{d}} \) in (6) and the commutation relation (3) is slightly modified. We set \( c_{n} \) as real, since the phase factor at \( t = 0 \) can be absorbed into the definition of energy eigenstates \( |E_n\rangle \). Then, we can show that \( \langle \Psi(t)|\Psi(s)\rangle \approx 0 \) for \( |t-s| \) larger than the time resolution \( \tau = \frac{2\pi \hbar}{\Delta E_{\text{eff}}} \) [12], which is explained later.

Let us formally define the ‘time operator’ as

\[ \hat{T} = \frac{d}{T} \int_{0}^{T} t|\Psi(t)\rangle \langle \Psi(t)| dt, \]

(2)

where we consider a large but finite time \( T \) (c.f. infinitesimally small cut-off [30, 31]). It is well-known that ‘time operator’, which is canonical conjugate to the Hamiltonian, does not exist as an observable [32–35], partly because the Hamiltonian is bounded below. On the other contrary, ‘time operator’ defined by (2) approximately satisfies the commutation relation up to a boundary constant, just as in the case of ‘phase operator’ [34].
\[ [\hat{H}, \hat{T}] = -i\hbar \frac{d}{dt} + i\hbar \Delta(\Psi(T)) \langle \Psi(T) \rangle \]  

(3) with \( \lim_{\tau \to \infty} \hat{I}_\tau = \sum_{n=1}^{d} |E_n\rangle \langle E_n| \).

It is straightforward to formally prove (3) with the use of \( \hat{H} |\Psi(t)\rangle = i\hbar \frac{d}{dt} |\Psi(t)\rangle \), and a partial integral

\[ [\hat{H}, \hat{T}] = \frac{d}{T} \int_0^T t \left( i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle \langle \Psi(t) | + i\hbar |\Psi(t)\rangle \frac{\partial}{\partial t} \langle \Psi(t) | \right) \]

\[ = i\hbar \frac{d}{T} \langle t |\Psi(t)\rangle \langle \Psi(t) | \rangle^T \]

\[ = -i\hbar \frac{1}{T} \int_0^T e^{-\frac{1}{\beta} E_{n+1}\Delta t} \sum_{n,m} |E_m\rangle \langle E_n| \]

\[ = -i\hbar \hat{I}_\tau + i\hbar \Delta(\Psi(T)) \langle \Psi(T) |, \]

(4) where \( \hat{I}_\tau = \frac{1}{T} \int_0^T e^{-\frac{1}{\beta} E_{n+1}\Delta t} \sum_{n,m} |E_m\rangle \langle E_n| \) converges to \( \hat{I} = \sum_{n=1}^{d} |E_n\rangle \langle E_n| \) as \( T \to \infty \) from the nondegeneracy assumption. In our case, the diagonal basis \( |\Psi(t)\rangle \) of (2) are approximate eigenstates of \( \hat{T} \), as the orthogonality holds with time resolution \( \tau \), which is in marked contrast to the case of mechanical observables.

We explain how the time resolution \( \tau \) is extremely short for thermodynamic systems.

We can express the energy eigenstates by the inverse Fourier transform:

\[ |E_n\rangle = \lim_{T \to \infty} \frac{\sqrt{d}}{T} \int_0^T e^{\frac{\pi i}{\beta} E_n \Delta t} |\Psi(t)\rangle dt. \]  

(5) Equation (5) shows that the energy eigenstates are superpositions of continuously many quasi eigenstates of 'time operator', which approximately satisfies the orthogonality(8) [12]. In the next section, we numerically verify how the quasi eigenstate \( |\Psi(t)\rangle \) typically well represents the microcanonical state, and is considered as a relevant basis to discuss the foundation of ETH. Here, we analytically explore the quasi orthogonality and equilibrium nature of the basis |\Psi(t)\rangle. First, we recall the calculation of the fidelity detailed in [12] (see also [14, 15]). By using

\[ |\Psi(t)\rangle = \frac{1}{\sqrt{d}} \sum_{n=1}^{d} e^{-\frac{\pi i}{\beta} E_n \Delta t} |E_n\rangle, \]

the inner product of states at time \( t \) and \( s \) satisfies

\[ \langle \Psi(t) | \Psi(s) \rangle = 1 \]  

(7) for \( t = s \)

\[ \langle \Psi(t) | \Psi(s) \rangle = \frac{1}{d} \sum_{n=1}^{d} e^{-\frac{\pi i}{\beta} E_n (t-s)} \]

\[ \approx \frac{1}{d} \int_{E} e^{\frac{\pi i}{\beta} E (t-s)} \Omega(E')dE' \]

\[ \approx \frac{1}{d\Delta E} \int_{E} e^{-\frac{\pi i}{\beta} E (t-s)} \Omega(E')dE' \]

\[ \approx \frac{1}{\Delta E} \frac{1 - e^{-\frac{\beta}{\Delta E} (t-s) \Delta E}}{e^{-\frac{\beta}{\Delta E} x} - e^{-\frac{\beta}{\Delta E} (t-s) \Delta E}} \]

\[ \approx e^{\frac{\pi i}{\beta} E (t-s) \Delta E} \frac{1}{\Delta E} \Delta E (t-s) \]

(8) where the discrete sum is evaluated as integral using the density of the states \( \Omega(E') \), which renders the spectrum of the eigenenergy continuous, and the dynamics are supposed to be irreversible. At this stage, the recurrence phenomena in case of extremely long time is omitted. Such a continuous approximation accurately holds as shown in figure 1. We expand the density of the states as

\[ \log \Omega(E + \Delta E - x) = \log \Omega(E + \Delta E) - \beta x - \frac{\beta^2}{2\Delta E} x^2 + O\left(\frac{1}{N}\right) \]

with the inverse temperature

\[ \beta = \frac{d}{dE} \log \Omega(E)|_{E+\Delta E}, \]  

the heat capacity \( C_V \), and the system size \( N \). We set the Boltzmann constant to unity, and introduced an effective energy widths \( \Delta E = \frac{d}{\Omega(E + \Delta E)} \) and \( \Delta E_{\text{eff}} \leq \Delta E \). In particular, \( \Delta E_{\text{eff}} \) is chosen so that the linear approximation of \( \log \Omega(E + \Delta E - x) = \log \Omega(E + \Delta E) - \beta x - \frac{\beta^2}{2\Delta E} x^2 + O\left(\frac{1}{N}\right) \) holds in \( [E + \Delta E - \Delta E_{\text{eff}}, E + \Delta E] \). Here, we evaluate \( \Delta E_{\text{eff}} \) from the condition that the absolute value of the first order term \( \beta \lambda \) is much larger than that of the second order \( \frac{\beta^2}{2\Delta E} x^2 \) for \( \lambda = \Delta E_{\text{eff}} \), which yields \( C_V \gg \beta \Delta E_{\text{eff}} \). For thermodynamic density of the states, the energy width \( \Delta E_{\text{eff}} \) is considered to be of the same order as \( \frac{1}{N} \). As the heat capacity is proportional to the system size, we can accurately calculate the integral up to the first order of \( x \).
For $|t - s| \geq \tau$ with the time resolution $\tau = \frac{2\pi h}{\Delta E}$, the inner product (8) becomes considerably small [12], which is $O\left(\frac{1}{\sqrt{\tau}}\right)$ and the states $|\Psi(t)\rangle$ and $|\Psi(s)\rangle$ are almost orthogonal. Note also that because of the unitarity, the short-term expansion of the fidelity $|\langle \Psi[e^{-\frac{i\hat{H}t}}]|\Psi\rangle|^2 = 1 - \text{Var}[\hat{H}] t^2 + O(t^4)$ suggests that the decay rate of the fidelity is determined by the energy fluctuation $\text{Var}[\hat{H}] = \langle \Psi| (\hat{H} - \langle \Psi|\hat{H}|\Psi\rangle)^2 |\Psi\rangle$, which is compatible to our evaluation of $\tau$. It is also well-known that for long-term regime, the fidelity shows power-law decay by Paley-Wiener theorem for Fourier-Laplace transformation. Meanwhile, an exponential decay is observed for the time scale of interest to us.

We now discuss some properties of the 'time operator'. The operator

$$|\Psi(t)\rangle \langle \Psi(t)| = \frac{1}{d} \sum_{n=1}^{d} \sum_{j=1}^{d} e^{-i(E_n - E_j) t} |E_n\rangle \langle E_j|$$

(9)

can be regarded as a projection onto $|\Psi(t)\rangle$, i.e. $|\Psi(t)\rangle \langle \Psi(t)| |\Psi(s)\rangle$ is proportional to $|\Psi(t)\rangle$, and is nonnegligible for $|t - s| \leq \tau$ from the quasi orthogonality. Note that the projection operator (9) can be used for measurement of time as projection to $|\Psi(t)\rangle$: Given a state $|\Psi(t)\rangle$ with unknown $t$, such projection determines $t$ with an accuracy $\tau$. By repeating this thought experiment many times with randomly distributed $t$, we actually obtain the spectral fluctuation. The projection operator (9) also satisfies the completeness

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |\Psi(t)\rangle \langle \Psi(t)| dt = \sum_{n=1}^{d} |E_n\rangle \langle E_n|.$$  

(10)

3. Numerical simulation

First, we explore the quasi orthogonality for quasi eigenstates $|\Psi(t)\rangle$. Then, we also verify the validity of ETH, and investigate the thermal nature of time-evolved states. Regarding the quasi orthogonality, further details of calculation are shown in [12]. For the sake of simplicity and concreteness, we first consider one-dimensional (1D) Ising model in a magnetic field [26, 36] $B = (\alpha, 0, \gamma)$ where $\gamma_i$ is the $z$-component at the $i$-th site. The Hamiltonian for the N-th site is

$$\hat{H} = -J \sum_{j=1}^{N-1} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z + \alpha \sum_{j=1}^{N} \hat{\sigma}_j^x + \sum_{j=1}^{N} \gamma_j \hat{\sigma}_j^z.$$  

(11)

For $N = 9$ and $N = 10$, we choose an energy shell $\mathcal{H}_{E, E + \Delta E}$ as a subspace spanned by eigenstates $|E_n\rangle$ with (a) $151 \leq n \leq 200$ and (b) $251 \leq n \leq 300$. We set the parameters to $J = 1$ and $\alpha = 1$. On the contrary, we also explored various choices of $\gamma$, such as uniform case $\gamma_j = 0.5$ ($\gamma = 0$ corresponds to the integrable case), randomly distributed case $\gamma_i \in [0, \Delta]$ with $\Delta = 0.5$, 1. The eigenenergies were in increasing order $E_1 \leq E_2 \leq E_3 \leq ...$. For the case where $\gamma_j = 0.5$, the inverse energy width, and the effective energy fluctuation were (a) $\beta = 0.2$, $\Delta E = 0.908814$, $\hat{\Delta} E = 0.8523$ and (b) $\beta = 0.12$, $\Delta E = 0.722721$, $\hat{\Delta} E = 0.683753$. In these cases, we used $\Delta E_{\text{eff}} = \Delta E$, as the linearization of the entropy $\Omega(E + \Delta E - x)$ did hold for the entire shell.

We show the time evolution of the fidelity $F(t) = |\langle \Psi\Psi(t)\rangle|^2$ for the case (a) in figure 1. The result for the case (b) was similar. We compared the exact $F(t)$ (blue curve) and approximation

$$\frac{1 - e^{-2\Delta E_{\text{eff}} t}}{1 - e^{-2\Delta E_{\text{eff}} t} - 2e^{-\Delta E_{\text{eff}} t} \cos \Delta E_{\text{eff}} t}$$

(red broken line) calculated using equation (8), where we set $\hbar = 1$. Note that the relaxation time $\tau$ was quite general [12] and was the same order as the Boltzmann time.
2.7ββ for macroscopic systems [13], which is extremely short at room temperature τ~ 10^{-12}s. Therefore, we can conclude the bases |ψ(t)⟩ (t ≥ 0) in the expansion (5) are mutually orthogonal.

We then verified the thermalization of the quasi eigenstates of Ê, i.e. the basis |ψ(t)⟩ well represented the microcanonical state for most t ∈ [0, ∞]. For this purpose, it was necessary to calculate the expectation values of a class of observables  for |ψ(t)⟩ and compare with those of the microcanonical ensemble. Theoretically, |ψ(t)⟩ describes thermal equilibrium for most t according to the typicality [6, 7] and the unitarity of the time evolution. Numerically, we investigated the expectation values of arbitrary observables defined on the left–most m sites  for  most t ∈ [0, ∞]. Thus, we calculate the Hilbert-Schmidt distance \( \Delta \hat{ρ}(t) = ||\hat{ρ}_m(t) - \hat{ρ}_0|| \) between the reduced density matrices \( \hat{ρ}_m(t) = \text{Tr}_{N - m}[\hat{ρ}(t)] \) for the partial trace for the right-most \( N - m \) sites.

In figure 2(a), we show the time dependence of the deviation from equilibrium \( \Delta \hat{ρ}(t) \) for 1 ≤ m ≤ 3 (the inset shows the time average of the deviation for 1 ≤ m ≤ 5) and the energy shell \( [E_{151}, E_{200}] \). Note that the deviation \( \Delta \hat{ρ}(t) \) is roughly upper bounded by \( \frac{d'}{\sqrt{d'}} \) with \( d' = 2^m \). We can calculate the variance of \( \text{Tr}_{N - m}|\Psi⟩⟨\Phi| \) as

\[
\text{Var}(\text{Tr}_{N - m}|\Psi⟩⟨\Phi|) = \text{Var}\left(\frac{1}{d(d + 1)} \left( \sum_{|i⟩ ∈ Δ_s} c_i^* |E_i^{(1)}⟩ |E_i^{(1)}⟩ \right) \right)
\]

for a typical state |ψ⟩ = \( \sum_{|i⟩ ∈ Δ_s} c_i |E_i^{(1)}⟩ |E_i^{(1)}⟩ \) under the assumption of weak coupling. Here, ‘ stands for the uniform average over coefficients \( c_i \) [31], \( E_1^{(1)} \) and \( E_2^{(1)} \) are the local eigenenergies of the left most m sites and the right most \( N - m \) sites, and \( Δ_s = \{ j | E - E_j^{(1)} ≤ E_j^{(1)} ≤ E - E_j^{(1)} + ΔE \} \) denotes the set of excitation numbers of the right-most \( N - m \) sites. By upper-bounding \( \sum_{|i⟩ ∈ Δ_s} 1 ≤ d_s^2 \) with the dimension of the right-most \( N - m \) sites \( d_s \), we can evaluate the variance as smaller than \( \frac{d_s}{2} \).

Aside from the Hilbert-Schmidt distance from equilibrium, we numerically calculated the dependences of bipartite entanglement entropies [27] of the superposition |ψ(t)⟩ and energy eigenstates |En⟩ on the subsystem.
In this paper, we showed that each energy eigenstate can be seen as a typical state using the quasi eigenstates. The agreement may get better when the size $m$ is big. We confirmed that the entanglement entropy is quantitatively similar for $\Delta = 1$ as well. The error bars for the superposition states are slightly shifted along the horizontal direction, and curves are intended only for visual observation.

We also calculated the distance between $\hat{\rho}_m(t)$ and the averaged state in figure 2(b). To quantitatively compare the figures 2(a) and (b), we defined $\Delta^{(m)}$ as the sample average of distances $||\hat{\rho}_m(t) - \hat{\rho}_0||$ for $151 \leq m \leq 200$ and the time average of $||\hat{\rho}_m(t) - \hat{\rho}_0||$ for $0 \leq t \leq 100$. In figure 4, we show the dependences of $\Delta^{(m)}$ (red curve) on the subsystem size $m$ for the case of uniform magnetic field $\gamma = 0.5$, and randomly sampled $\gamma$ from $[0, 0.5]$. The deviations $\Delta^{(m)}$ and $\Delta^{(t)}$ agreed well one another all three cases. This means that ETH holds with the same accuracy as the thermal property of $|\Psi(t)\rangle$ for these cases. The agreement may get better when the finiteness of the energy width can be negligible for the calculation of $\Delta^{(m)}$.

We also investigated the XY spin-chain model, the Hamiltonian for which was $\hat{H} = -\sum_{j=1}^{N} (J + \Delta) \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + (J - \Delta) \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y + \alpha \sum_{j=1}^{N} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z$ and confirmed that the dependences of typical states $\Delta^{(m)}$ and eigenenergy states $\Delta^{(t)}$ on subsystem size were similar. For example, we show the case of $J = 1, \Delta = 0.2$, and $\alpha = \gamma = 0.5$ in figure 3(d). For the integrable case $\gamma = 0$ [26], we calculated the distance $\Delta^{(m)}$ between $\gamma$ and the averaged state $\hat{\rho}_0$ for $1 \leq m \leq 5$. The distance was greater than the case where $\gamma = 0.5$ roughly by a factor of 1.5 both for $N = 9$ and $N = 10$.

**4. Summary**

In this paper, we showed that each energy eigenstate can be seen as a typical state using the quasi eigenstates $|\Psi(t)\rangle$ of ‘time operator’ $\hat{T}$. From operational point of view, we can consider the measurement of ‘time operator’ as an estimation of unknown parameter $t$ of a given $|\Psi(t)\rangle$. Note that it would be possible to formally define the ‘phase operator’ $\hat{\theta}$ as an approximatively canonical conjugate to the number operator $\hat{N}$ in a similar way by using the gauge transformation instead of the unitary evolution: Given a state $e^{i\hat{N}\theta}|\Psi\rangle$ with unknown $\theta$, we can estimate $\theta$ by measuring so-obtained phase operator.

On the contrary, the subtlety of the non-existence of $\hat{T}$ as an observable that is rigorously conjugate to the Hamiltonian amounts to the approximate orthogonality of quasi eigenstates: There is a minimum time resolution $\tau$ given by the Boltzmann time [12–14], both for integrable and nonintegrable systems. Our point is that the quasi eigenstate of the ‘time operator’ is time evolved state $|\Psi(t)\rangle$ and, thus, for thermodynamic systems where equilibration occurs, $|\Psi(t)\rangle$ is considered to be in equilibrium for most values of $t$ according to the typicality [6–9] such that its typical superposition is also expected to be thermal. This strongly suggests that the ETH for diagonal elements holds as long as most of the time-evolved states $|\Psi(t)\rangle$ well reproduces microcanonical expectation values for a class of observables. We numerically verified this argument in two ways by comparing the bipartite entanglements of superposition states $|\Psi(t)\rangle$ and energy eigenstates $|E_0\rangle$, and those of the averaged errors $\Delta^{(m)}$ and $\Delta^{(t)}$ for nonintegrable systems. The entanglement entropies of energy eigenstates were nearly identical to those of the superposition states. On the contrary, the agreement of averaged errors of the reduced states indicates that energy eigenstates and superposition states yield similar expectation values for the observables of the subsystem.
In the presence of strong spatial disorder, the ETH breaks down [37–39]. Exploring the case of non-thermal case including the many-body localization possibly in more than one dimensions is an important future problem.

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Figure 4. Dependence of the deviations from averaged state $\Delta \hat{\rho}$ and $\Delta \hat{\tau}$ on subsystem size $m$ for the reduced density matrices of quasi eigenstates and energy eigenstates. (a) The case of uniform magnetic field along the $z$ direction with $\gamma_i = 0.5$. We also explored random values of $\gamma_i$, sampled from (b)$[0, 0.5]$ and (c)$[0, 1]$. (d) The case of the XY model with $f = 1$, $\Delta f = 0.2$, and $\alpha = \gamma = 0.5$. 
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