Higher Order SUSY in Quantum Mechanics and Integrability of Two-dimensional Hamiltonians

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To Memory of V.N. Popov

Abstract. The new method based on the SUSY algebra with supercharges of higher order in derivatives is proposed to search for dynamical symmetry operators in 2-dim quantum and classical systems. These symmetry operators arise when closing the SUSY algebra for a wide set of potentials. In some cases they are of 2-nd order in derivatives. The particular solutions are obtained also for potentials accepting symmetry operators of 4-th order. The investigation of quasiclassical limit of the SUSY algebra yields new classical integrals of motion for a certain type of systems which are polynomials of 4-th order in momenta. The general SUSY-inspired algorithm to construct classical systems with additional integrals of motion is outlined.

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1. Introduction

The existence of dynamical symmetries in quantum systems leads to their complete or partial integrability and therefore gives the opportunity of a more detailed study of their spectral properties.

We describe here the new method for searching the dynamical symmetry operators based on the construction of the isospectral Hamiltonians. The algebraic form of the isospectral transformations in Quantum Mechanics is realized by the SUSY algebra [1] - [3].

The standard one-dimensional Supersymmetrical Quantum Mechanics (SSQM) is generated by supercharge operators $Q^\pm$, which form the SUSY algebra (together with the Hamiltonian of supersystem $H$):

$$\{Q^+, Q^-\} = H;$$

$$[Q^\pm, H] = 0.$$

This algebra is represented by $2 \times 2$ - matrix supercharges

$$Q^- = (Q^+)^\dagger = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix}; \quad q^\pm = \mp \partial + W(x)$$

and by the superhamiltonian $H$ consisted of two Schrödinger operators:

$$H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix} = \begin{pmatrix} q^+q^- & 0 \\ 0 & q^-q^+ \end{pmatrix}, \quad h^{(i)} = -\partial^2 + V^{(i)}.$$

Eq.(2), in terms of components, leads to the intertwining relations for two Hamiltonians:

$$h^{(1)}q^+ = q^+h^{(2)}; \quad q^-h^{(1)} = h^{(2)}q^-.$$ 

Therefore the spectra of $h^{(1)}$ and $h^{(2)}$ almost coincide (up to zero - modes of the operators $q^\pm$) and their wave functions are connected by $q^\pm$ mappings.

The extension of one-dimensional SSQM with supercharge operators of higher order in derivatives was proposed in [4], [5]. The related SUSY algebra turns out to be a polynomial one, namely, the polynomial of the Schrödinger Hamiltonian appears in the right hand side of Eq.(1).

Multidimensional generalizations of SSQM with supercharges linear in derivative are known [6], [7]. In the present paper the two-dimensional SSQM generated by supercharge operators of second order in derivatives is investigated. In this case, in analogy to 1-dimensional SSQM we would expect that the corresponding SUSY algebra becomes polynomial in a superhamiltonian. However, actually we find that the SUSY algebra [1] for a wide set of potentials is closed by the diagonal operator of dynamical symmetry (see Sect.2). It corresponds to the additional integral of motion for the classical system (Sect.5). In some cases this symmetry operator is of second order in derivatives (Sect.3) and the system admits the $R$-separation of variables [8]. But, in general it is of 4-th order in derivatives. The particular solutions for corresponding potentials and coefficient functions of the symmetry operator are displayed in Sect.4. In Sect.5 the quasiclassical limit of the superalgebra is studied and classical integrals of motion of 4-th order were constructed (up to our knowledge, some of them are new (compare to [9]).
2. Two-dimensional Darboux transformations of second order in derivatives

Let us consider the intertwining relations (5) retaining the Planck constant $\hbar$ in the Hamiltonian and in the supercharge operators with components of general form:

$$q^+ = (q^-)^\dagger = \hbar^2 g_{ik}(\vec{x}) \partial_i \partial_k + \hbar C_i \partial_i + B.$$  

(6)

The form of metric $g_{ik}(\vec{x})$ is determined by the intertwining relations (5):

$$\partial_l g_{ik}(\vec{x}) + \partial_i g_{lk}(\vec{x}) + \partial_k g_{il}(\vec{x}) = 0.$$  

(7)

Its solutions can be easily calculated:

$$g_{11} = \tilde{\alpha} x_2^2 + \tilde{a}_1 x_2 + \tilde{b}_1;$$  

(8)

$$g_{22} = \tilde{\alpha} x_1^2 + \tilde{a}_2 x_1 + \tilde{b}_2;$$  

(9)

$$g_{12} = -\frac{1}{2}(2\tilde{\alpha} x_1 x_2 + \tilde{a}_1 x_1 + \tilde{a}_2 x_2) + \tilde{b}_3.$$  

(10)

Thus the senior in derivative part of supercharges belongs to the $E(2)$ - universal enveloping algebra [8]. We distinguish four different classes in second derivatives:

$$q^{(1)+} = \gamma \Delta + C_i \partial_i + B;$$  

(11)

$$q^{(2)+} = \alpha P_1^2 + \gamma \Delta + C_i \partial_i + B;$$  

(12)

$$q^{(3)+} = \alpha \{J, P_1\} + \gamma \Delta + C_i \partial_i + B;$$  

(13)

$$q^{(4)+} = \alpha J^2 + \beta P_1^2 + \gamma \Delta + C_i \partial_i + B,$$  

(14)

where $J$ and $\vec{P}$ are generators of rotations and translations, correspondingly, and $\alpha \neq 0$.

The coefficient functions of supercharge operator (5) must satisfy differential equations:

$$\hbar \partial_i C_k + \hbar \partial_k C_i + \hbar^2 \Delta g_{ik} - (V^{(1)} - V^{(2)})g_{ik} = 0;$$  

(15)

$$\hbar^2 \Delta C_i + 2\hbar \partial_i B + 2\hbar g_{ik} \partial_k V^{(2)} - (V^{(1)} - V^{(2)})C_i = 0;$$  

(16)

$$\hbar^2 \Delta B + \hbar^2 g_{ik} \partial_k \partial_i V^{(2)} + \hbar C_i \partial_i V^{(2)} - (V^{(1)} - V^{(2)})B = 0.$$  

(17)

From Eqs.(5) it is evident that the generalized superalgebra gives rise to the symmetry operator $\tilde{R}$ for the Hamiltonian $H$:

$$\{Q^+, Q^-\} = \tilde{R}; \quad [\tilde{R}, H] = 0.$$  

(18)

This operator determines a higher-order dynamical symmetry which, in general, can not be represented by the polynomial of the Hamiltonian $H$ and of the second-order symmetry operator $R$. As it will be shown in the next Section the latter possibility exists for the supercharges (11) of the class 1 only. However in all cases the closing of the SUSY algebra leads to the integrability of the corresponding dynamical system.

The general solutions of nonlinear Eqs. (15) - (17) for all four classes of the metric are not known. But in some cases the particular solutions can be found in the analytical form.
3. Particular solutions of class 1: dynamical symmetries of second order

Let us solve the system of differential Eqs. (15) - (17) for the supercharges with the diagonal metric \( g = \text{diag}(-1, -1) \) (\( \hbar = 1 \)):

\[
q^+ = -\Delta + C_i \partial_i + B; \quad q^- = (q^+)^\dagger.
\]

From Eqs. (15), (16) it follows that

\[
C^2 \equiv (C_1 + iC_2)^2 = \alpha z^2 + 8\beta z + \gamma; \quad z \equiv x_1 + ix_2;
\]

\[
V^{(2)} - B = \frac{1}{4}\alpha |z|^2 + z\bar{\beta} + \bar{z}\beta + \frac{1}{4} |C|^2 - \eta;
\]

\[
V^{(2)} - V^{(1)} = \partial_z C + \partial_{\bar{z}} \bar{C},
\]

where \( \alpha, \eta \) are real constants and \( \beta, \gamma \) may be complex ones. Eq. (17) can be written in the form:

\[
(C\partial_z + \bar{C}\partial_{\bar{z}})(B | C |^2) = G | C |^2.
\]

where

\[
G = \alpha + (\partial_z C)(\partial_{\bar{z}} \bar{C}) - \frac{\alpha}{2}(\bar{z}C + z\bar{C}) - 2(\bar{\beta}C + \beta \bar{C}).
\]

To solve (23) it is useful to change the variables \( \bar{z} \) to \( \tau_1, \tau_2 \):

\[
\tau_1 = \int \frac{dz}{C} + \int \frac{d\bar{z}}{C}; \quad i\tau_2 = \int \frac{dz}{C} - \int \frac{d\bar{z}}{C}.
\]

Then

\[
B = \frac{1}{2 |C|^2} \int G | C |^2 d\tau_1 + \frac{F(\tau_2)}{|C|^2},
\]

where \( F(\tau_2) \) is an arbitrary real function. Thus the irreducible SUSY algebra of second order in derivatives is realized for a broad variety of potentials.

For the class 1 the superalgebra (18) gives rise to the symmetry operator \( R \) of second order in derivatives:

\[
\tilde{R} = H^2 + R + 2\eta H,
\]

where \( R \) is diagonal operator:

\[
R = \begin{pmatrix}
R^{(1)} & 0 \\
0 & R^{(2)}
\end{pmatrix}.
\]

Its components

\[
R^{(1)} = 2(\alpha |z|^2 + 4(\bar{\beta}z + \beta \bar{z}))\partial_z \partial_{\bar{z}} - C^2 \partial^2_z - \bar{C}^2 \partial^2_{\bar{z}} - C(\partial_z C)\partial_z - \bar{C}(\partial_{\bar{z}} \bar{C})\partial_{\bar{z}} + (C\partial_z + \bar{C}\partial_{\bar{z}})(B + V^{(1)}) + B^2 - V^{(1)2} - 2\eta V^{(1)};
\]

\[
R^{(2)} = \alpha z^2 + 8\beta z + \gamma;
\]

\[
V^{(2)} - B = \frac{1}{4}\alpha |z|^2 + z\bar{\beta} + \bar{z}\beta + \frac{1}{4} |C|^2 - \eta;
\]

\[
V^{(2)} - V^{(1)} = \partial_z C + \partial_{\bar{z}} \bar{C},
\]
are symmetry operators for the Hamiltonians \( h^{(1)}, h^{(2)} \), respectively.

Let us find explicit expressions for the operators \( H, Q^\pm, R \). They depend on the values of \( \alpha \) and \( \beta \) in Eq.\((20)\).

i) If \( \alpha = 0, \beta \neq 0 \), the constant \( \gamma \) in \((20)\) can be excluded by the translation \( z \to z - (\gamma/8\beta) \). Then the variables \((25)\) read

\[
\tau_1 = \sqrt{\frac{z}{2\beta}} + \sqrt{\frac{z}{2\beta}}, \quad \tau_2 = i\left(\sqrt{\frac{z}{2\beta}} - \sqrt{\frac{z}{2\beta}}\right)
\]

and they are related to the conventional \cite{8} parabolic coordinates. It follows from Eq.\((26)\) that

\[
B = \frac{2\tau_1 - |\beta|^2 \tau_1^4 - 2|\beta|^2 \tau_1^2 \tau_2^2 - 4(\bar{z}^2 \tau_1^2 - \tau_1^2 \tau_2^2)\phi_1(\tau_2)}{\tau_1^2 + \tau_2^2}.
\]

Correspondingly, from Eqs.\((21), (22)\) we find the potentials \( V^{(1,2)} \):

\[
V^{(i)} = \frac{(-1)^i2\tau_1 + \beta^2 \tau_1^4 + F(\tau_2)}{\tau_1^2 + \tau_2^2} - \eta;
\]

and from Eqs.\((28), (29)\) – the components of the symmetry operator \( R \):

\[
R^{(i)} = \frac{4}{\tau_1^2 + \tau_2^2} \left( \frac{\tau_1^2 \partial_{\tau_1}^2 - \tau_2^2 \partial_{\tau_2}^2}{\tau_1^4 + \tau_2^4} \right) + \frac{4}{\tau_1^2 + \tau_2^2} \left( (-1)^i2\tau_1 + \beta^2 \tau_1^4 \tau_2^2 - \tau_1^2 \tau_2^2 F(\tau_2) \right) - \eta^2.
\]

In terms of variables \( \tau_1, \tau_2 \) the Laplacian

\[
\Delta = \frac{1}{|\beta|^2 (\tau_1^2 + \tau_2^2)} (\partial_{\tau_1}^2 + \partial_{\tau_2}^2)
\]

contains the same multiplier as in Eqs.\((30)\). Thus the spectral problem for both Hamiltonians, \( h\Psi = \mathcal{E}\Psi \), can be solved by the \( R \)-separation of variables \cite{8}. Namely, the corresponding eigenfunctions \( (h^{(1)} \text{ and } h^{(2)}) \) with a given energy \( \mathcal{E} \) can be expanded into the sum

\[
\Psi = \sum_n \nu_n \phi_{1n}(\tau_1) \phi_{2n}(\tau_2),
\]

where \( \nu_n \) are constants, and \( \phi_{1n}(\tau_1), \phi_{2n}(\tau_2) \) are solutions of 1-dimensional equations

\[
-\phi''_{1n}(\tau_1) + |\beta|^2 [-(E + \eta) \tau_1^2 + 2\tau_1 + |\beta|^2 \tau_1^4] \phi_{1n}(\tau_1) = \frac{\lambda_n^+}{4} \phi_{1n}(\tau_1);
\]

\[
-\phi''_{2n}(\tau_2) + |\beta|^2 [-(E + \eta) \tau_2^2 + F(\tau_2)] \phi_{2n} = -\frac{\lambda_n^+}{4} \phi_{2n}(\tau_2).
\]

The upper (lower) sign in Eq.\((32)\) corresponds to \( h^{(1)}(h^{(2)}) \), and \( \lambda_n^\pm \) are constants of separation, which play role of the spectral parameters for the symmetry operators \( R^{(1)} \) \( (R^{(2)}) \) respectively:

\[
R\phi_{1n}(\tau_1) \phi_{2n}(\tau_2) = (\lambda_n - \eta^2) \phi_{1n}(\tau_1) \phi_{2n}(\tau_2).
\]
Depending on the properties of the function $F(\tau_2)$ the degeneracy of energy levels, i.e. the dimension of the space $\{\lambda_i\}$ for a given $E$ can be finite or even infinite.

ii) Let us describe the second case $\beta = 0$, $\alpha > 0$, when the suitable coordinates,

$$\tau_1 = \frac{1}{\sqrt{\alpha}} \ln \left[ (z + \sqrt{z^2 + \frac{\gamma}{\alpha}}) (\bar{z} + \sqrt{\bar{z}^2 + \frac{\gamma}{\alpha}}) \right];$$

$$\tau_2 = -\frac{i}{\sqrt{\alpha}} \ln \frac{z + \sqrt{z^2 + \frac{\gamma}{\alpha}}}{\bar{z} + \sqrt{\bar{z}^2 + \frac{\gamma}{\alpha}}};$$

are connected to the conventional elliptic ones. In full analogy with the case i) one can obtain that:

$$B = \frac{1}{2(f_1 + f_2)} (2\partial_\tau f_1 - \frac{1}{2} f_1^2 - f_1 f_2 + F(\tau_2));$$

$$V^{(i)} = \frac{1}{2(f_1 + f_2)} ((-1)^i 2\partial_\tau f_1 + \frac{1}{2} f_1^2 + F(\tau_2)) - \eta;$$

$$R^{(i)} = \frac{4}{f_1 + f_2} (f_1 \partial_{\tau_2}^2 - f_2 \partial_{\tau_1}^2) + \frac{f_2((-1)^i 2\partial_\tau f_1 + f_1^2/2) - f_1 F(\tau_2)}{f_1 + f_2} - \eta^2,$$

where

$$f_1 = \frac{1}{4} \left( \alpha \exp(\sqrt{\alpha} \cdot \tau_1) + \frac{|\gamma|^2}{\alpha} \exp(-\sqrt{\alpha} \cdot \tau_1) \right);$$

$$f_2 = \frac{1}{4} \left( \tilde{\gamma} \exp(i\sqrt{\alpha} \tau_2) + \gamma \exp(-i\sqrt{\alpha} \cdot \tau_2) \right).$$

In terms of the coordinates $\tau_1$, $\tau_2$ the solution for eigenfunctions can be again decomposed into the sum (33), where now the functions $\phi_{1n}(\tau_1)$; $\phi_{2n}(\tau_2)$ satisfy the equations:

$$-\phi''_{1n}(\tau_1) + \frac{1}{4} \left( \mp \partial_\tau f_1 + \frac{1}{2} f_1^2 - (E + \eta) f_1 \right) \phi_{1n}(\tau_1) = \frac{\lambda^+_{n}}{4} \phi_{1n}(\tau_1);$$

$$-\phi''_{2n}(\tau_2) + \frac{1}{4} \left( \frac{F(\tau_2)}{2} - (E + \eta) f_2 \right) \phi_{2n}(\tau_2) = -\frac{\lambda^-_{n}}{4} \phi_{2n}(\tau_2)$$

and $\lambda^\pm_n$ are eigenvalues of the symmetry operator (37).

The analysis of the case $\beta = 0$, $\alpha < 0$ is similar and its result can be formulated as follows. It is necessary to replace real $(\tau_1, \tau_2)$ in Eqs. (33), (34) to imaginary $(-i\tau_2, -i\tau_1)$ and, respectively, $(f_1(\tau_1), f_2(\tau_2))$ in Eqs. (38), (39) to $(-f_2(\tau_2), -f_1(\tau_1))$. Then the relations for $B, V^{(1,2)}, R^{(1,2)}$ are given by Eqs. (33)-(37).

Note that for specially chosen $\alpha > 0$, $\beta = \gamma = 0$ the supercharge is factorized into the product of two conventional superoperators that corresponds to the separation of variables in the polar coordinates. Thus the supersymmetry, i.e. the intertwining relations between two Hamiltonians, leads inevitably to the hidden dynamical symmetry realized by the operator $R$ and moreover to the $R$- separation of variables in the case when a supercharge operator contains senior derivatives in the form of Laplacian.
4. Particular solutions of class 2: dynamical symmetries of fourth order

I. Let us consider the intertwining relations (3) for the supercharge (3) with the metric \( g = \text{diag}(1, -1) \):

\[ q^+ = h^2(\partial^2_1 - \partial^2_2) + hC_k\partial_k + B. \]  

(40)

From Eqs.(15)-(17) we obtain, that:

\[ C_\pm \equiv C_1 \mp C_2 = C_\pm(x_\pm); \]  

(41)

\[ B = \frac{1}{4}(C_+ C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)); \]  

(42)

\[ \partial_-(C_-F) = -\partial_+(C_+F); \quad F = F_1(x_+ + x_-) + F_2(x_+ - x_-), \]  

(43)

and, respectively, for potentials:

\[ V^{(1,2)} = \pm \frac{h}{2}(C'_+ + C'_-) + \frac{1}{8}(C^2_+ + C^2_-) + \]  

\[ \frac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)) + \text{const}, \]  

(44)

where \( x_\pm \equiv x_1 \pm x_2; \quad \partial_\pm = \partial/\partial x_\pm. \)

Eqs.(13) can be solved in certain cases:

1) Let \( C_- = 0 \), then

\[ C_+(x_+) = \frac{1}{\delta_1 \exp(\sqrt{\lambda} \cdot x_+) + \delta_2 \exp(-\sqrt{\lambda} \cdot x_+)}; \]  

(45)

\[ F_1 = \sigma_1\delta_1 \exp(\sqrt{\lambda} \cdot (x_- + x_+)) + \sigma_2\delta_2 \exp(-\sqrt{\lambda} \cdot (x_- + x_+)); \]  

(46)

\[ F_2 = \sigma_1\delta_2 \exp(\sqrt{\lambda} \cdot (x_+ - x_-)) + \sigma_2\delta_1 \exp(-\sqrt{\lambda} \cdot (x_+ - x_-)). \]  

(47)

Here and in what follows the Greek letters stand for constants. Depending on the sign \( \lambda \) the latter ones may be real or complex.

2) Let the function \( F \) allow the factorization: \( F = F_+(x_+) \cdot F_-(x_-) \). Then from Eq.(13) we obtain that:

\[ C_\pm = \frac{\nu_\pm}{F_\pm} \pm \frac{\gamma}{F_\pm} \int_{x_\pm}^x F_\pm dx_\pm, \]  

(48)

and there appear two possibilities:

\[ a) \quad F_\pm(x_\pm) = \epsilon_\pm x_\pm, \]  

(49)

\[ b) \quad F_\pm = \sigma_\pm \exp(\sqrt{\lambda} \cdot x_\pm) + \delta_\pm \exp(-\sqrt{\lambda} \cdot x_\pm). \]  

(50)

Note that the constant was omitted in the r.h.s. of Eq.(19) since the constant solutions \( F_\pm \) can be deduced from Eq.(17) with \( \lambda = 0 \).
In the case $a$) we obtain:

$$F_1(x_+ + x_-) = \frac{e_+e_-}{4}(x_+ + x_-)^2; \quad F_2(x_+ - x_-) = -\frac{e_+e_-}{4}(x_+ - x_-)^2;$$

and in the case $b$):

$$F_1(x_+ + x_-) = \sigma_+\sigma_- \exp(\sqrt{\lambda} \cdot (x_+ + x_-)) + \delta_+\delta_- \exp(-\sqrt{\lambda} \cdot (x_+ + x_-));$$

$$F_2(x_+ - x_-) = \sigma_+\delta_- \exp(\sqrt{\lambda} \cdot (x_+ - x_-)) + \sigma_-\delta_+ \exp(-\sqrt{\lambda} \cdot (x_+ - x_-))$$

(51) \hspace{1cm} (52)

In the cases 1), 2$a$) and 2$b$) potentials can be found by means of Eq. (44).

For this metric the components of the symmetry operator $\bar{R}$ (18) are given by:

$$\bar{R}^{(1,2)} = \frac{16h^2 \partial_1^2 \partial_2^2 + (\mp 4h\partial_2 C_+ - C_2^2)h^2 \partial_1^2 + 2(4B - C_+ C_-)h^2 \partial_1 \partial_2 + (4h\partial_1 B - hC_+ \partial_2 C_- + BC_-)h\partial_1 + (4h\partial_1 B - hC_- \partial_1 C_+ + BC_+)h\partial_2 + B^2 + h(C_- \partial_1 + C_+ \partial_2) + 4h^2 \partial_1 \partial_2 B.$$

(53)

II. Let us examine now the subclass of degenerate metric $g = diag(1, 0)$, when the supercharges take the form,

$$q^+ = h^2 \partial_1^2 + hC_k \partial_k + B.$$

(54)

It follows from Eqs. (15)-(17) that:

$$C_1 = -x_2 F'_1 + G_1; \quad C_2 = F_1;$$

$$V^{(1)} = \hbar(2G_1' - x_2 F''_1) + \frac{1}{4}x_2^2(F''_1)'' - x_2(F_1 G_1)' + K_1(x_1) + K_2(x_2);$$

$$V^{(2)} = \hbar x_2 G''_1 + \frac{1}{4}x_2^2(F''_1)'' - x_2(F_1 G_1)' + K_1(x_1) + K_2(x_2);$$

$$B = -\frac{\hbar}{2}(G'_1 + x_2 F''_1) + \frac{1}{2}G_1^2 - \frac{1}{2}x_2^2 F'_1 F''_1 + x_2 F_1 G'_1 - K_1(x_1),$$

(55) \hspace{1cm} (56) \hspace{1cm} (57) \hspace{1cm} (58)

where $F_1, G_1, K_1$ are arbitrary real functions of variable $x_1$, and $K_2(x_2)$ depends on $x_2$ only.

After substitution of Eqs. (55) - (58) into Eq. (17) we arrive to the equation for the coefficient functions:

$$-\frac{\hbar}{2}G''_1 + \frac{\hbar}{2}(G''_1)'' + 2G''_1 + G_1 K'_1 + 2G'_1 K_1 - F_1(F_1 G_1)' - G'_1 G_1^2 + x_2 \left[ \frac{\hbar}{2}F''_1 - \hbar(F'_1 G''_1 + 2G'_1 F''_1) - G_1(2G'_1 F''_1 + G''_1 F_1) - F'_1 K'_1 + \frac{1}{2}F_1(F''_1)'' - 2G''_1 F_1 - 2F''_1 W_1 \right] + x_2^2 \left[ \frac{1}{4}G_1(F''_1)'' + F'_1(F_1 G_1)'' + 3G'_1 F_1 F''_1 \right] - x_2^3 \left[ \frac{1}{4}F'_1(F''_1)'' + F''_1 F''_1 \right] + F_1 K'_2(x_2) = 0.$$

(59)
It means that \( K_2(x_2) \) is the polynomial of \( x_2 \) with constant coefficients:

\[
K_2(x_2) = m_0 - m_1 x_2 - \frac{1}{2} m_2 x_2^2 - \frac{1}{3} m_3 x_2^3 + \frac{1}{4} m_4 x_2^4. \tag{60}
\]

Therefore Eq. (59) is equivalent to four equations:

\[
- \bar{\hbar}^2 G'' + \bar{\hbar} (G''')'' + 2G''_1 + G_1 K' + 2G_1 K_1 - F_1 (F_1 G_1)' - G_1 G_1^2 = m_1 F_1; \tag{61}
\]

\[
\frac{\hbar^2}{2} F_1'^{(IV)} - \hbar (F_1'' + 2G''_1) - G_1 (2G_1 F_1' + G_1'' F_1) - F_1 K_1 + \frac{1}{2} F_1 (F_1^2)'' - 2G_1 K_1 = m_2 F_1; \tag{62}
\]

\[
\frac{1}{4} G_1 (F_1^2)'' + F_1 (F_1 G_1)'' + 3G_1 F_1 F_1'' = m_3 F_1; \tag{63}
\]

\[
\frac{1}{4} F_1 (F_1^2)'' + F_1 F_1'' = m_4 F_1. \tag{64}
\]

It happens to be possible to solve the system of differential Eqs. (61) - (64) in particular cases. These solutions will be presented for the quasiclassical limit \( \bar{\hbar} \to 0 \) in the next Sec.

5. Quasiclassical limit and integrable systems of class 2

The quantum dynamical symmetries which were found by the intertwining method in previous Sections have the natural analogs, integrals of motion – in the corresponding classical systems. For the class 1 these integrals of motion of 2-nd order are known [9].

Let us describe certain two-dimensional classical systems of class 2 with additional integrals of motion of the 4-th order in derivatives. Systems of this type are discussed in the book [9], where the list of known particular solutions for the coefficient function of integrals of motion was presented.

I. Let us define momenta \( p_\pm = -i \hbar \partial_\pm \) and take the limit \( \bar{\hbar} \to 0 \). Then for the Lorentz metric we derive from Eqs. (44), (53), that the classical Hamiltonian

\[
h_{cl} = 2(p_+^2 + p_-^2) + V_{cl},
\]

where

\[
V_{cl} = \frac{1}{8} (C_+^2 + C_-^2) + \frac{1}{4} (F_2 (x_+ - x_-) - F_1 (x_+ + x_-)) \tag{65}
\]

reveals the additional integral of motion

\[
I = 16p_+^2 p_-^2 + C_+^2 p_-^2 + C_-^2 p_+^2 - 2(F_1 + F_2) p_+ p_- + B^2. \tag{66}
\]
The functions $C_{\pm}, F, B$ were obtained for certain cases in Sect. 4.

II. Let us investigate what kind of integrable systems can be obtained from the SSQM with degenerate metric (see Sect. 4). At first, let us define classical functions of supercharge from Eq. (54):

$$q^+_{cd} = -p_1^2 + iC_k p_k + B; \quad q^-_{cd} = -p_1^2 - iC_k p_k + B.$$  

(67)

Thus the Hamiltonian

$$h_{cd} = p_k^2 + V_{cd},$$

where

$$V_{cd} = \frac{1}{4} x_2^2 (F_1^2)'' - x_2 (F_1 G_1)'' + K_1(x_1) + K_2(x_2),$$

(68)

has the integral of motion

$$I \equiv q^+_{cd} q^-_{cd} = p_1^4 + (C_1^2 - 2B)p_1^2 + C_2^2 p_2^2 + 2C_1 C_2 p_1 p_2 + B^2.$$  

(69)

The functions $C_k$ are defined according to Eq. (55) and other functions satisfy the system of equations (61) - (64) with $\bar{h} = 0$.

Let us display the non-trivial solutions which have been found by now ($k_i$ are constants):

1) $F_1 = k_1 \neq 0$. Then the constants $m_3 = m_4 = 0$, and the function $G_1(x_1)$ satisfies the equation

$$\int \frac{G_1^2 dG_1}{\sqrt{k - \frac{1}{2} m_2 G_1^4}} = x_1,$$  

(70)

in turn, the function $K_1(x_1)$ can be written in terms of $G_1(x_1)$:

$$K_1(x_1) = k_2 G_1^{-2} + \frac{1}{4} G_1^2 + k_1 m_1 G_1^{-2} \int G_1 dx'.$$  

(71)

For nonzero values of constants $k$ and $m_2$ the integral can be represented as a sum of elliptic functions of 1-st and 2-nd genus. However in two limits $G_1(x_1)$ can be an elementary function when a) $k > 0, \quad m_2 = 0$; b) $k = 0, \quad m_2 < 0$.

In the case a), after the redefinition of constants $k_1, k_2$ and an appropriate shift of $x_2$, we find from Eqs. (54), (59), (71), taking into account (70), that

$$V_{cd} = -\frac{k_1 k_2}{3} x_2 x_1^{-2/3} + \frac{1}{4} \left[ k_2^2 + \frac{3k_1 m_1}{k_2} \right] x_1^{2/3} - m_1 x_2 + \text{const};$$  

(72)

$$I = p_1^4 + \left[ \frac{1}{2} \left( \frac{k_2}{k_2} + \frac{3k_1 m_1}{k_2} \right) x_1^{2/3} - \frac{2k_1 k_2}{3} x_2 x_1^{-2/3} + k_1^2 \right] p_1^2 + k_1^2 p_2^2 + 2k_1 k_2 x_1^{1/3} p_1 p_2 + \left[ \frac{k_1 k_2}{3} x_2 x_1^{-2/3} + \frac{1}{4} \left( \frac{k_2^2}{k_2} - \frac{3k_1 m_1}{k_2} \right) x_1^{2/3} - \frac{k_2^2}{2} \right].$$  

(73)

In the case b) and other solvable cases such as 2) $G_1 = 0, \quad F_1 = k_1 x_1$ or $F_1 = k_1 x_1^2$; 3) $F_1 = k_1 x_1^3, \quad G_1 = k_2 x_1$; 4) $F_1 = k_1 x_1, \quad G_1 = k_2$; 5) $F_1 = 0$, the potentials are obtained as
cited in [9] (i.e. the potentials with separable variables), that is why they are not displayed here.

When comparing the potentials [9] (with the coefficient functions $C_{\pm}, F_1, F_2$ in the cases 1) and 2b) of Sec. 4) with the list in [9] we find that they are new and possess the dynamical symmetries of 4-th order.

In conclusion let us formulate the recipe that we have established for the construction of integrals of motion in Classical Mechanics. For a given classical Hamiltonian $h_{cl}^{(1)}$ we look for the complex function $q_{cl}^{\mp}(\vec{x}, \vec{p}) = (q_{cl})^\mp$, polynomial in momenta and such that its Poisson brackets with $h_{cl}^{(1)}$ have the form:

$$\{q_{cl}^-, h_{cl}^{(1)}\} = -if(\vec{x}, \vec{p})q_{cl}^-, \quad \{q_{cl}^+, h_{cl}^{(1)}\} = if(\vec{x}, \vec{p})q_{cl}^+, \quad (74)$$

where $f(\vec{x}, \vec{p})$ is an arbitrary real function. It is evident that then the classical factorizable integral of motion $I = q_{cl}^+q_{cl}^-$ exists.

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