Anisotropic resistivity tensor from disk geometry magneto-transport

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Magneto-transport measurements on two dimensional van der Waals heterostructures have recently shown signatures of uniaxial anisotropy. Such measurements are almost exclusively performed in a Hall bar geometry which makes it difficult to extract the full resistivity tensor. The goal of this paper is to theoretically analyze anisotropic magneto-transport in a homogeneous disk geometry and to provide a closed form expression for the electrical potential anywhere on the disk if the current source and drain are located somewhere on the circumference. This expression can then be used to experimentally extract the full resistivity tensor.

I. INTRODUCTION

Two dimensional van der Waals (vdW) heterostructures host a broad range of interesting physical phenomena\cite{1}, including anisotropic magneto-transport. With rare exceptions\cite{2,3}, the transport measurements are performed in a Hall bar geometry, making it difficult to extract the full resistivity tensor particularly if the transport principal axis is misaligned with the current flow. For example, the heterostructures can be subject to an unintentional strain, in which case the misalignment is not directly controlled in an experiment. Moreover, the orientation of the transport principal axis can be carrier concentration (filling) dependent as was recently shown\cite{4} in numerical solutions of the Boltzman equation for twisted bilayer graphene subject to heterostrain, even if the strain tensor and the transport relaxation time are momentum and filling independent. For open Fermi surfaces, the magneto-resistance is expected to grow with the magnetic field without saturation along one of the principal axis, but to saturate with increasing $B$ along the perpendicular principal axis\cite{5}. Direct measurement of the full anisotropic resistivity tensor in the vdW heterostructures as a function of filling and $B$ would therefore help in understanding the complex transport phenomena in these materials.

In this paper we analyze the solution to the magneto-transport equations in a uniform disk of radius $a$. Inside the disk the anisotropic conductivity tensor is assumed to be homogeneous, while outside the disk there is no conduction. Thus,

$$\sigma = D(x,y) \left( \sigma_+ \hat{x} \hat{x} + \sigma_- \hat{y} \hat{y} + \sigma_H \left( \hat{x} \hat{y} - \hat{y} \hat{x} \right) \right),$$  \hspace{1cm} (1)

where $D(x,y) = \Theta(a^2 - x^2 - y^2)$ and $\Theta$ is the Heaviside step function, restricting the conduction to the interior of the circle. Without loss of generality, we also assume that the principal axes are aligned with the $x$ and $y$ axes of the coordinate system and adopt the dyadic product to represent the conductivity tensor. Here $\sigma_\pm$ are the two components of the longitudinal conductivity along the principal axes and $\sigma_H$ is the Hall conductivity.

We express the longitudinal conductivities as $\sigma_\pm = \bar{\sigma} \pm \Delta \sigma$ and without loss of generality take the $x$-axis to be along the axis with larger resistivity i.e. $\Delta \sigma / \bar{\sigma} < 0$. The analysis below then provides an expression in the form of a rapidly convergent series which can be used to extract the resistivity tensor for a point current source/drain at $r_{S,D} = a(\cos \theta_{A,B}, \sin \theta_{A,B})$. The expression for the electrical potential at $x, y$ reads

$$V(x, y; r_{S}, r_{D}) =$$

$$I \frac{\sqrt{\sigma_+ \sigma_-}}{\pi \sigma_+ \sigma_- + \sigma_H^2} \left( \sum_{n=0,2,4,...} \ln \left( \frac{1 + e^{-2i\theta_{A,B} \Omega^2 + 4n} - e^{-i\theta_{A,B} \Omega 2n}}{1 + e^{-2i\theta_{A,B} \Omega^2 + 4n} - e^{-i\theta_{A,B} \Omega 2n}} \right) + \sum_{n=1,3,5,...} \ln \left( 1 + e^{2i\theta_{A,B} \Omega^2 + 4n} - e^{i\theta_{A,B} Z \Omega 2n} \right) \right)$$

$$+ \frac{I}{\pi \sigma_+ \sigma_- + \sigma_H} \left( 1 + e^{-2i\theta_{A,B} \Omega^2 + 4n} - e^{-i\theta_{A,B} \Omega 2n} \right) \bar{\sigma} + \frac{I}{\pi \sigma_+ \sigma_- + \sigma_H} \left( 1 + e^{-2i\theta_{A,B} \Omega^2 + 4n} - e^{-i\theta_{A,B} \Omega 2n} \right) \sigma_H$$

$$+ \sum_{n=1,3,5,...} \arg \left( 1 + e^{-2i\theta_{A,B} \Omega^2 + 4n} - e^{-i\theta_{A,B} \Omega 2n} \right) - \arg \left( 1 + e^{-2i\theta_{A,B} \Omega^2 + 4n} - e^{-i\theta_{A,B} \Omega 2n} \right) .$$  \hspace{1cm} (2)

where the $x, y$ position enters via the complex variable $Z = X + iY = \frac{x}{\sqrt{1+\frac{x^2}{a^2}}} + i \frac{y}{\sqrt{1-\frac{y^2}{a^2}}}$ and the
parameters $\alpha_+ = \frac{a}{\pi} \left(\frac{1}{\sqrt{1 + \frac{\sigma}{\bar{\sigma}}} + \frac{1}{\sqrt{1 - \frac{\sigma}{\bar{\sigma}}}}\right)$ and $\Omega = \sqrt{\frac{1}{\sqrt{1 - \frac{\sigma}{\bar{\sigma}}} - \sqrt{1 + \frac{\sigma}{\bar{\sigma}}}} \frac{1}{\sqrt{1 + \frac{\sigma}{\bar{\sigma}}} + \sqrt{1 - \frac{\sigma}{\bar{\sigma}}}}}$. The function arg is the argument of a complex number. Note that because $|\Delta \sigma| < \bar{\sigma}$, the parameter $0 \leq \Omega < 1$ and therefore the above sum converges (the convergence is rapid unless $\Omega$ is very close to 1). Illustrative contour plots of $V(x, y; r_S, r_D)$ for several parameters are shown in Fig. 1.

Although the above expression is obtained for a point current source/drain, the linearity of the differential equation whose solution it is allows direct determination to 1). The solution to the above inhomogeneous linear partial differential equation gives $V$ as a function of $r$.

Expressing the longitudinal conductivities as

$$\sigma_{\pm} = \bar{\sigma} \pm \Delta \sigma,$$

(6)

it will be convenient to rescale the coordinate axes so that Eq. (5) becomes

$$- \frac{I}{\sqrt{\sigma_+ \sigma_-}} \left(\delta(X - X_A)\delta(Y - Y_A) - \delta(X - X_B)\delta(Y - Y_B)\right).$$

(9)

The new domain, specified by $D \left(\sqrt{1 + \frac{\Delta \sigma}{\bar{\sigma}}} X, \sqrt{1 - \frac{\Delta \sigma}{\bar{\sigma}}} Y\right)$, is given by $\Theta(a^2 - (1 + \frac{\Delta \sigma}{\bar{\sigma}}) X^2 - (1 - \frac{\Delta \sigma}{\bar{\sigma}}) Y^2)$, i.e. it is an ellipse. If $\Delta \sigma/\bar{\sigma} > 0$, the ellipse is elongated along the $Y$-direction, if $\Delta \sigma/\bar{\sigma} < 0$, then the ellipse is elongated along the $X$-direction. Without loss of generality we can choose the $x$-axis to be along the axis with larger resistivity, i.e. it will be assumed from now on that

$$\Delta \sigma/\bar{\sigma} < 0.$$

(10)

The equation (9) can be expressed using complex coordinates

$$Z = X + iY,$$

(11)

when, after some simplification, it becomes

$$-2 \frac{I}{\sqrt{\sigma_+ \sigma_-}} \left(\delta(X - X_A)\delta(Y - Y_A) - \delta(X - X_B)\delta(Y - Y_B)\right).$$

(12)
clear that inside the ellipse where \( D = 1 \), the solution can be written in terms of a sum of a function of \( Z \) and a function of \( \bar{Z} \). The boundary conditions are determined from the right hand side and the derivatives of the boundary function \( D \).

### A. Zhukovsky conformal mapping of the ellipse to annulus

It will be convenient to perform a conformal map transforming the boundary of the ellipse to the boundary of the circle. This can be done using the Zhukovsky transformation

\[
Z = \alpha_+w + \frac{\alpha_-}{w},
\]

\[
w = u + iv,
\]

where \( u(X,Y) \) and \( v(X,Y) \) are purely real. To determine the coefficients \( \alpha_+ \) and \( \alpha_- \) we demand that

\[
\left( 1 + \frac{\Delta \sigma}{\bar{\sigma}} \right) X_0^2 + \left( 1 - \frac{\Delta \sigma}{\bar{\sigma}} \right) Y_0^2 = a^2,
\]

implies

\[
u_0^2 + \bar{v}_0^2 = 1,
\]

i.e. if \( X_0 \) and \( Y_0 \) lie on the ellipse, then \( u_0 \) and \( v_0 \) are forced to lie on the unit circle. From Eq.(13), we have

\[
X_0 + iY_0 = \alpha_+(u_0 + iv_0) + \bar{\alpha}_-(\bar{u}_0 - \bar{v}_0),
\]

because, being on unit circle, \( 1/(u_0 + iv_0) = u_0 - iv_0 \). Therefore,

\[
X_0 = (\alpha_+ + \bar{\alpha}_-)u_0,
\]

\[
Y_0 = (\alpha_+ - \bar{\alpha}_-)v_0.
\]

So, from (15)

\[
\left( 1 + \frac{\Delta \sigma}{\bar{\sigma}} \right) (\alpha_+ + \bar{\alpha}_-)^2 u_0^2 + \left( 1 - \frac{\Delta \sigma}{\bar{\sigma}} \right) (\alpha_+ - \bar{\alpha}_-)^2 v_0^2 = a^2,
\]

which implies

\[
\alpha_\pm = \frac{a}{2} \left( \frac{1}{\sqrt{1 + \frac{\Delta \sigma}{\bar{\sigma}}}} \pm \frac{1}{\sqrt{1 - \frac{\Delta \sigma}{\bar{\sigma}}}} \right),
\]

This fixes the conformal map. Having established that the ellipse in the \( (X,Y) \)-plane maps onto the unit circle in the \( (u,v) \)-plane, we wish to know where does the interior of the ellipse map. To this end, seek such \( w = \Omega \) that would give

\[
\alpha_+\Omega = \frac{\alpha_-}{\Omega} \in \Re \epsilon,
\]

for \( \Delta \sigma/\bar{\sigma} < 0 \). This gives

\[
\Omega = \sqrt{\frac{\sqrt{1 - \frac{\Delta \sigma}{\bar{\sigma}}} - \sqrt{1 + \frac{\Delta \sigma}{\bar{\sigma}}}}{\sqrt{1 - \frac{\Delta \sigma}{\bar{\sigma}}} + \sqrt{1 + \frac{\Delta \sigma}{\bar{\sigma}}}}}.
\]

So, letting \( w = \Omega e^{i\phi} \) where \( \phi \) is the polar angle in the \( u,v \)-plane and using (22) results in

\[
\alpha_+\Omega e^{i\phi} + \frac{\alpha_-}{\Omega e^{i\phi}} = a\frac{\sqrt{-2\frac{\Delta \sigma}{\bar{\sigma}}}}{\sqrt{1 - \left( \frac{\Delta \sigma}{\bar{\sigma}} \right)^2}} \cos \phi.
\]

This means that the circle of radius \( \Omega \) in \( u,v \)-plane maps onto the line segment connecting the foci of the ellipse in the \( X,Y \)-plane. For \( \Delta \sigma/\bar{\sigma} < 0 \), the foci lie on the \( x \)-axis. Therefore, the ellipse in \( X,Y \)-plane, including its interior, maps onto an annulus in the \( u,v \)-plane with the outer radius 1 and the inner radius \( \Omega \) as illustrated in the Figure 2.
Using Cauchy-Riemann conditions, it can be readily shown that

\[
\frac{\partial w \partial \bar{w}}{\partial Z \partial \bar{Z}} = J \left( \frac{u}{X}, \frac{v}{Y} \right),
\]

(26)

where \( J \left( \frac{u}{X}, \frac{v}{Y} \right) \) is the Jacobian determinant. Eq. (12) therefore gives

\[
-\left( \frac{\partial}{\partial u} D \frac{\partial V}{\partial u} + \frac{\partial}{\partial v} D \frac{\partial V}{\partial v} \right) + \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \left( \frac{\partial D}{\partial v} \frac{\partial V}{\partial u} - \frac{\partial D}{\partial u} \frac{\partial V}{\partial v} \right) = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{\delta(X - X_A)\delta(Y - Y_A) - \delta(X - X_B)\delta(Y - Y_B)}{J \left( \frac{u}{X}, \frac{v}{Y} \right)}.
\]

(27)

But, by the properties of the Dirac \( \delta \) function under coordinate transformation, it follows that

\[
-\left( \frac{\partial}{\partial u} D \frac{\partial V}{\partial u} + \frac{\partial}{\partial v} D \frac{\partial V}{\partial v} \right) + \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \left( \frac{\partial D}{\partial v} \frac{\partial V}{\partial u} - \frac{\partial D}{\partial u} \frac{\partial V}{\partial v} \right) = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \delta(u - u_A)\delta(v - v_A) - \delta(u - u_B)\delta(v - v_B),
\]

(28)

where \( D = \Theta \left( 1 - u^2 - v^2 \right) \). Now, because \( X_{A,B}, Y_{A,B} \) lie on the ellipse, \( u_{A,B}, v_{A,B} \) must lie on the unit circle.
B. Polar coordinates in the \( u, v \)-plane

Switching to polar coordinates in the \( u, v \)-plane

\[
\rho = \sqrt{u^2 + v^2},
\]

\[
\phi = \tan^{-1} \frac{v}{u},
\]

gives

\[
\frac{\partial}{\partial u} = \cos \phi \frac{\partial}{\partial \rho} - \sin \phi \frac{\partial}{\partial \phi},
\]

\[
\frac{\partial}{\partial v} = \sin \phi \frac{\partial}{\partial \rho} + \cos \phi \frac{\partial}{\partial \phi}.
\]

Therefore, the derivatives of the boundary function are

and the differential equation (28) becomes

\[
- \left( \frac{\partial}{\partial \rho} \left( D \frac{\partial V}{\partial \rho} \right) + D \frac{\partial^2 V}{\partial \rho^2} \right) + \sigma_H \delta(\rho - 1) \frac{\partial V}{\partial \phi} = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \delta(\rho - 1) (\delta(\phi - \theta_A) - \delta(\phi - \theta_B)).
\]

C. Homogeneous solution and the boundary conditions

A general solution of Eq. (37) for \( \rho < 1 \) where the terms containing \( \delta(\rho - 1) \) vanish can be written as

\[
V(\rho, \phi) = \sum_{m=1}^{\infty} \left( A_{|m|} \left( \frac{\rho^m}{\Omega^m} + \frac{\Omega^m}{\rho^m} \right) \cos m\phi + B_{|m|} \left( \frac{\rho^m}{\Omega^m} - \frac{\Omega^m}{\rho^m} \right) \sin m\phi \right).
\]

This form satisfies the homogeneous differential equation and is continuous and differentiable across the line cut joining the foci. To see this, notice that the points on the circle of radius \( \Omega \) in the \( x, y \) plane which are related by the mirror reflection about the \( v = 0 \) axis should be identified as the same points. In other words, \( \rho = \Omega \) and \( \phi \), and \( \rho = \Omega \) and \(-\phi\) map onto the same physical point in the \( X, Y \) and therefore \( x, y \) plane. We therefore want the potential at \( \Omega^+ \) and \( \phi \) to either be the same at \(-\phi\) which is accomplished by \( \left( \frac{\rho^m}{\Omega^m} + \frac{\Omega^m}{\rho^m} \right) \cos m\phi \), or we want it to vanish at \( \Omega^+ \) with a continuous slope. Vanishing at \( \Omega \) is accomplished by \( \frac{\rho^m}{\Omega^m} - \frac{\Omega^m}{\rho^m} \), and the reason why only \( \sin m\phi \) can multiply it is that multiplying it by \( \cos m\phi \) would introduce a cusp across the line segment.

Integrating both sides of Eq. (37) over an infinitesimal interval straddling \( \rho = 1 \) gives the boundary condition

\[
\left. \frac{\partial V}{\partial \rho} \right|_{\rho=1} + \sigma_H \left. \frac{\partial V}{\partial \phi} \right|_{\rho=1} = \frac{I}{\sqrt{\sigma_+ \sigma_-}} (\delta(\phi - \theta_A) - \delta(\phi - \theta_B)).
\]

Substituting Eq. (38) into the above results in

\[
\left. \frac{\partial V(\rho, \phi)}{\partial \rho} \right|_{\rho=1} = \sum_{m=1}^{\infty} \left( mA_{|m|} \left( \frac{1}{\Omega^m} - \Omega^m \right) \cos m\phi + mB_{|m|} \left( \frac{1}{\Omega^m} + \Omega^m \right) \sin m\phi \right),
\]

\[
\left. \frac{\partial V(\rho, \phi)}{\partial \phi} \right|_{\rho=1} = \sum_{m=1}^{\infty} \left( -mA_{|m|} \left( \frac{1}{\Omega^m} + \Omega^m \right) \sin m\phi + mB_{|m|} \left( \frac{1}{\Omega^m} - \Omega^m \right) \cos m\phi \right),
\]

and the differential equation (37) becomes

\[
\sum_{m=1}^{\infty} \left( mA_{|m|} \left( \frac{1}{\Omega^m} - \Omega^m \right) \cos m\phi + mB_{|m|} \left( \frac{1}{\Omega^m} + \Omega^m \right) \sin m\phi \right) + 
\]

\[
\left. \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \sum_{m=1}^{\infty} \left( -mA_{|m|} \left( \frac{1}{\Omega^m} + \Omega^m \right) \sin m\phi + mB_{|m|} \left( \frac{1}{\Omega^m} - \Omega^m \right) \cos m\phi \right) \right. 
\]

\[
\left. \frac{I}{\sqrt{\sigma_+ \sigma_-}} \sum_{m=1}^{\infty} \left( \cos m\phi \left( \cos m\theta_A - \cos m\theta_B \right) + \sin m\phi \left( \sin m\theta_A - \sin m\theta_B \right) \right) \right.,
\]
where the following identity was used for the right hand side

\[
\delta (\phi - \theta_{A,B}) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} e^{-im\theta_{A,B}}
\]

\[
= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \left( \cos m\phi \cos m\theta_{A,B} + \sin m\phi \sin m\theta_{A,B} \right).
\]

Matching the coefficients of \(\cos m\phi\) and \(\sin m\phi\) and solving for \(A_{[m]}\) and \(B_{[m]}\) gives

\[
A_{[m]} = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{1}{\pi} \frac{1}{1 + \frac{\sigma_H}{\sigma_+ \sigma_-}} \frac{1}{m} \left( \frac{\cos m\theta_A - \cos m\theta_B}{\Omega^{-m} - \Omega^m} - \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \frac{\sin m\theta_A - \sin m\theta_B}{\Omega^{-m} + \Omega^m} \right)
\]

\[
B_{[m]} = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{1}{\pi} \frac{1}{1 + \frac{\sigma_H}{\sigma_+ \sigma_-}} \frac{1}{m} \left( \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \frac{\cos m\theta_A - \cos m\theta_B}{\Omega^{-m} - \Omega^m} + \frac{\sin m\theta_A - \sin m\theta_B}{\Omega^{-m} + \Omega^m} \right)
\]

Thus,

\[
V(\rho, \phi) = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{1}{\pi} \frac{1}{1 + \frac{\sigma_H}{\sigma_+ \sigma_-}} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\cos m\theta_A - \cos m\theta_B}{\Omega^{-m} - \Omega^m} - \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \frac{\sin m\theta_A - \sin m\theta_B}{\Omega^{-m} + \Omega^m} \right) \left( \frac{\rho^m}{\Omega^m} + \frac{\Omega^m}{\rho_m} \right) \cos m\phi
\]

\[
+ \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{1}{\pi} \frac{1}{1 + \frac{\sigma_H}{\sigma_+ \sigma_-}} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\sin m\theta_A - \sin m\theta_B}{\Omega^{-m} + \Omega^m} + \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \frac{\cos m\theta_A - \cos m\theta_B}{\Omega^{-m} - \Omega^m} \right) \left( \frac{\rho^m}{\Omega^m} - \frac{\Omega^m}{\rho_m} \right) \sin m\phi
\]

\[- (A \rightarrow B).
\]

D. Summing over the angular momenta

The sum over \(m\) converges slowly. In order to convert it into a rapidly convergent sum, we first Taylor expand the denominators involving \(\Omega^m\) and \(\Omega^{-m}\), in powers of \(\Omega\) as

\[
V(\rho, \phi) = \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{1}{\pi} \frac{1}{1 + \frac{\sigma_H}{\sigma_+ \sigma_-}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(\Omega^{1+2n})^m}{m} \left( \cos m\theta_A - (-1)^n \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \sin m\theta_A \right) \left( \frac{\rho^m}{\Omega^m} + \frac{\Omega^m}{\rho_m} \right) \cos m\phi
\]

\[
+ \frac{I}{\sqrt{\sigma_+ \sigma_-}} \frac{1}{\pi} \frac{1}{1 + \frac{\sigma_H}{\sigma_+ \sigma_-}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{(\Omega^{1+2n})^m}{m} \left( (-1)^n \sin m\theta_A + \frac{\sigma_H}{\sqrt{\sigma_+ \sigma_-}} \cos m\theta_A \right) \left( \frac{\rho^m}{\Omega^m} - \frac{\Omega^m}{\rho_m} \right) \sin m\phi
\]

\[- (A \rightarrow B).
\]

Then the resulting sum over \(m\) is related to the geometric series by integration, and since \(\Omega < 1\), the sum over \(n\) will be rapidly convergent. Therefore, for \(C = A, B\), we have

\[
\sum_{m=1}^{\infty} e^{im\phi} e^{im\theta_C} \frac{\Omega^{m(1+2n)}}{m} \left( \frac{\rho^m}{\Omega^m} \pm \frac{\Omega^m}{\rho_m} \right) = - \ln \left( 1 - e^{i\theta_C} \frac{\rho}{\Omega} \Omega^{1+2n} \right) \mp \ln \left( 1 - e^{i\theta_C} \frac{\Omega}{\rho} \Omega^{1+2n} \right).
\]

Adding longitudinal and Hall contributions finally gives Eq. \ref{eq:final}.

III. DISCUSSION

such a case the continuity equation reads

\[
\nabla \cdot \mathbf{j} = \sum_{j=1}^{n_S} I_j^S \delta (\mathbf{r} - \mathbf{r}_{A,j}) - \sum_{j=1}^{n_D} I_j^D \delta (\mathbf{r} - \mathbf{r}_{B,j}),
\]

where \(n_S\) is the number of point sources and \(n_D\) is the number of point drains, and \(\sum_{j=1}^{n_S} I_j^S = \sum_{j=1}^{n_D} I_j^D = I\). The resulting expression is
\[ V(x, y; \{r_{A,j}\}, \{r_{B,j}\}) = \]
\[ \frac{1}{\pi} \frac{\sqrt{\sigma_+ \sigma_-}}{\sigma_+ + \sigma_- + \sigma_H^2} \sum_{j=1}^{n_D} I_j^D \left( \sum_{n=0,2,4,...}^{\infty} \ln \left| 1 + e^{-2i\theta_{B,j} \Omega z^{2+4n}} - e^{-i\theta_{B,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right| + \sum_{n=1,3,5,...}^{\infty} \ln \left| 1 + e^{2i\theta_{B,j} \Omega z^{2+4n}} - e^{i\theta_{B,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right| \right) \]
\[ - \frac{1}{\pi} \frac{\sqrt{\sigma_+ \sigma_-}}{\sigma_+ + \sigma_- + \sigma_H^2} \sum_{j=1}^{n_S} I_j^S \left( \sum_{n=0,2,4,...}^{\infty} \ln \left| 1 + e^{-2i\theta_{A,j} \Omega z^{2+4n}} - e^{-i\theta_{A,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right| + \sum_{n=1,3,5,...}^{\infty} \ln \left| 1 + e^{2i\theta_{A,j} \Omega z^{2+4n}} - e^{i\theta_{A,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right| \right) \]
\[ + \frac{1}{\pi} \frac{\sigma_H}{\sigma_+ + \sigma_- + \sigma_H^2} \left( \sum_{n=0,2,4,...}^{\infty} \sum_{j=1}^{n_D} I_j^D \arg \left( 1 + e^{-2i\theta_{B,j} \Omega z^{2+4n}} - e^{-i\theta_{B,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right) - \sum_{j=1}^{n_S} I_j^S \arg \left( 1 + e^{-2i\theta_{A,j} \Omega z^{2+4n}} - e^{-i\theta_{A,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right) \right) \]
\[ + \sum_{n=1,3,5,...}^{\infty} \sum_{j=1}^{n_D} I_j^D \arg \left( 1 + e^{2i\theta_{B,j} \Omega z^{2+4n}} - e^{i\theta_{B,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right) - \sum_{j=1}^{n_S} I_j^S \arg \left( 1 + e^{2i\theta_{A,j} \Omega z^{2+4n}} - e^{i\theta_{A,j} \frac{Z}{\alpha_+} \Omega^{2n}} \right) \right) . \tag{51} \]

The expression for extended source/drain can be found by treating \( I_j^{S/D} \) as infinitesimal and then converting the Riemann sum into an integral. The obtained expression can now be used to fit measurements with multiple voltage probes for arbitrary current source and drain placed on the perimeter of the disk.

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