Optimal control with bounded entries principle of Pontryagin

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Abstract. The generally, systems control the saturation results in their dynamics, either in their control signal, in their states or in both. This paper presents the theory and wording of the Pontryagin principle, which has been applied as this type of continuous time problems, to obtain control laws that optimize the index in question. Likewise, a problem is presented and solved with which the application of the Pontryagin principle is exposed and finally the conclusions are presented.

1. Introduction

When designing optimal control systems, it is very important to take into account the saturation that may occur in control actions or states, which may be due to different factors, such as maximum speed of a motor, maximum height of a tank, maximum voltage in an electric current converter, etc. To deal with this type of problem, it is deduced from the maximum principle of Pontryagin, the admissible control action, in such a way that a closed loop control law is obtained that minimizes a performance index associated with the optimization problem (minimum time, minimum energy, linear quadratic regulator) [1].

What proposes an optimal control problem is that given an initial condition, it is desired to construct an optimal control path such that the objective function subject to a certain restriction is minimized. The theorem or principle that is studied in this document (Pontryagin) can be used to obtain this trajectory in a simple way [2].

2. Pontryagin minimum principle

The obtaining of the necessary conditions of optimization for the problem of optimal control is offered by the Pontryagin principle, some of them are described in summary form through the Hamiltonian of a vector function \( f(x(t), u(t), t) \), defined as Equation (1):

\[
H(t, x(t), u(t)) := \lambda(t)^T f(t, x(t), u(t))
\]  

With \( \lambda(t) \) an n-dimensional vector function. The conditions of the Pontryagin principle reduce the calculation of an optimal control to the solution of a system of differential equations with two boundary points and an optimization problem [3].

2.1. Formulation of the Pontryagin principle

For the formulation of the optimal control problem, we start from the representation in the state space of a system like the one shown below in the Equation (2):
\[
\dot{x} = f(x(t), u(t), t)
\]

With the respective performance index (cost function) given by Equation (3):

\[
J = \phi(x(t_f), t_f) + \int_0^{t_f} L(x(t), u(t), t)dt
\]

Additionally, a restriction is presented, where the initial and final states of the system are obliged to satisfy the boundary conditions Equation (4):

\[
x(t_0) = x_0, t_f \quad \text{and} \quad x(t_f) = t_f
\]

Which are free. The important steps in obtaining optimal control for the previous system are [4]:

- The Hamiltonian formulation Equation (5):
  \[
  H(x(t), u(t), \lambda(t), t) = L(x(t), u(t), t) + \dot{\lambda}(t)f(x(t), u(t), t)
  \]

- The three relationships for control, state and co-state Equation (6) to Equation (8):
  \[
  \frac{\partial H}{\partial u} = 0 \quad \text{(6)}
  \]
  \[
  \frac{\partial H}{\partial \lambda} = \dot{x}^*(t) \quad \text{(7)}
  \]
  \[
  -\frac{\partial H}{\partial x} = \dot{\lambda}^*(t) \quad \text{(8)}
  \]

To solve the optimal values \(x^*(t), u^*(t)\) and \(\lambda^*(t)\), respectively, together with the general limit condition Equation (9).

\[
\left[\frac{\partial \phi}{\partial x} - \lambda(t)\right]_{t_f} \delta x_f + \left[H + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f = 0
\]

The physical system to be controlled in an optimal way has some restrictions on its inputs (controls), internal variables (states) and/or outputs due to considerations mainly regarding safety, cost and other inherent limitations.

The control action \(u(t)\) can be defined, indicating that the magnitude of this must be less than a saturation value, with which the condition will park for an optimal control system, given by: \(\frac{\partial H}{\partial u(t)} = 0\) it must be replaced by the more general condition [4] resulting Equation (10).

\[
H(x^*(t), u^*(t), \lambda^*(t)) \leq H(x^*(t), u^*(t) + \delta u(t), \lambda^*(t))
\]

When an optimal control system has limitations within its dynamics, the \(\delta u(t)\) variation cannot be considered as arbitrary, since saturation could occur and therefore an optimization problem would not be occurring. For now, only restrictions on the control signal are considered, but if you also have limitations for the states, the equations of state condition and co-state would also present variations of the same type [5,6].
In order to have an optimal solution the Hamiltonian should be the minimum possible with respect to the admissible control actions \( u(t) \in U \) in the optimal values of the state \( x^*(t) \) and co-state \( \lambda^*(t) \), therefore this condition can be written as Equation (11) [7]:

\[
\min_{u(t) \in U} H(x^*(t), u(t), \lambda^*(t))
\] (11)

Where \( U \) is the set of admissible values that the control action can take \( u \). Figure 1 shows the representation of the control signal limited by a boundary, where the admissible region for said signal is shown (restriction of the control signal).

Figure 1. Bounded control signal.

After having the state space model of the system, the cost index or function and the boundary conditions proposed, we proceed to find the solution to the problem, which is reduced to solving the set of equations of state and co-state given by Equation (12) [8]:

\[
\begin{align*}
\dot{x}^*(t) &= \left(\frac{\partial H}{\partial x}\right) \\
\dot{\lambda}^*(t) &= -\left(\frac{\partial H}{\partial x}\right)
\end{align*}
\] (12)

With boundary conditions Equation (13):

\[
\left[\frac{\partial \phi}{\partial x} - \lambda(t)\right]_{t_f} \delta x_f + \left[H + \frac{\partial S}{\partial x}\right]_{t_f} \delta t_f = 0
\] (13)

3. Numerical results

It is considered a problem where a fleet of reconﬁgurable general-purpose robots is sent to Mars at time \( t_0 \) to help build a Martian base. They can be used for two things [9,10]:

1. The robots can be replicated.
2. Robots can make habitats for humans.

The number of robots at time \( t \) is denoted by \( x(t) \) Equation (14), and the number of habitats per \( n(t) \) Equation (15). It is desired to optimize (maximize) the size of the Martian base in the time interval \( T = [0, t_f] \).

\[
\begin{align*}
\dot{x}(t) &= u(t)x(t), \quad x(0) = x > 0 \\
\dot{n}(t) &= (1 - u(t)) \cdot x(t), \quad n(0) = 0 \\
0 &\leq u(t) \leq 1, \quad \rightarrow U = [0,1]
\end{align*}
\] (14) (15)
Here, \(u(t)\) denotes the percentage of \(x(t)\) used for reproduction [11]. So, the objective is to find the control path \(u(t)\) that maximizes the function \(n(t_f) = \int_0^{t_f} (1 - u(t))x(t)\,dt\).

We consider \(x(t)\) as the state and \(u(t)\) as the control input. The Hamiltonian is Equations (16) to Equation (19):

\[
H(x, u, \lambda) = (1 - u)x + \lambda xu
\]

\[
\dot{x}^* = \left(\frac{\partial H}{\partial \lambda}\right) = x^*(t)u^*(t), \quad x^*(0) = x
\]

\[
\dot{\lambda}^*(t) = -\left(\frac{\partial H}{\partial x}\right) = -1 + u^*(t) - \lambda(t)u^*(t)
\]

\[
\min_{u(t)\in U} H(x^*(t), u(t), \lambda^*(t)) = \min_{u(t)\in U} (x^*(t) + x^*(t) \ast (\lambda(t) - 1)u)
\]

Therefore, we have that: \(x^*(t) > 0\) for \(t \in [0, t_f]\) Equation (20):

\[
u^*(t) = \begin{cases} 0, & \text{si } \lambda(t) < 1 \\ 1, & \text{si } \lambda(t) \geq 1 \end{cases}
\]

Now we have that for \(t = t_f\), since \(\lambda(t_f) = 0\), for \(t\) near \(t_f\), we have that \(u^*(t) = 0\) and therefore \(\dot{\lambda}(t) = -1\). and therefore \(t = t_f - 1, \lambda(t) = 1\) and this happens when the control input changes \(u^*(t) = 1\). So, for \(t \leq t_f - 1\) Equations (21) to Equation (23) [12]. Figure 2 shows results.

\[
\dot{\lambda}(t) = -\lambda(t), \quad \lambda(t_f - 1) = 1
\]

\[
\lambda(t) = e^{(t_f-1)t}
\]

\[
u^*(t) = \begin{cases} 1, & \text{si } 0 \leq t \leq t_f - 1 \\ 0, & \text{si } t_f - 1 \leq t \leq t_f \end{cases}
\]

Figure 2. Control and status trajectories.
4. Conclusion
When there are limitations on the control action of an optimal control system, the stationary condition of said system can be seen as the minimum search for the Hamiltonian function, and to obtain the optimal solution, the Hamiltonian should be the minimum possible with respect to the control actions $u(t)$, in the $x^*(t)$ and $λ^*(t)$ values.

In conclusion, an optimal control path is to use all the robots to replicate from time $0$ to $t = T - 1$, and then use all the robots to build habitats. If $T < 1$, then robots should only build habitats. In general, if the Hamiltonian is linear in $u$, the maximum or minimum of the Hamiltonian can only be reached in the limits of $U$. The result of the control path is known as bang-bang control.

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