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Differential Equations Arising from the 3-Variable Hermite Polynomials and Computation of Their Zeros

Cheon Seoung Ryoo

Abstract

In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. Finally, we investigate the zeros of the 3-variable Hermite polynomials by using computer.

Keywords: differential equations, heat equation, Hermite polynomials, the 3-variable Hermite polynomials, generating functions, complex zeros

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers see [1–15]. The special polynomials of two variables provided new means of analysis for the solution of a wide class of differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

In [1], the Hermite polynomials are given by the exponential generating function

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}.$$ 

We can also have the generating function by using Cauchy’s integral formula to write the Hermite polynomials as...
\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \frac{n!}{2^n} \int e^{2tx-t^2} dt \]

with the contour encircling the origin. It follows that the Hermite polynomials also satisfy the recurrence relation

\[ H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \]

Further, the two variables Hermite Kampé de Fériet polynomials \( H_n(x, y) \) defined by the generating function (see [3])

\[
\sum_{n=0}^\infty H_n(x, y) \frac{\mu^n}{n!} = e^{x^2+y^2}
\] (1)

are the solution of heat equation

\[
\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y), \quad H_n(x, 0) = x^n.
\]

We note that

\[ H_n(2x, -1) = H_n(x). \]

The 3-variable Hermite polynomials \( H_n(x, y, z) \) are introduced [4].

\[ H_n(x, y, z) = n! \sum_{k=0}^{[\frac{n}{3}]} \frac{\mu^k}{k! (n-3k)!} H_{n-3k}(x, y) \]

The differential equation and the generating function for \( H_n(x, y, z) \) are given by

\[
\left(3z \frac{\partial^3}{\partial x^3} + 2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n\right) H_n(x, y, z) = 0
\]

and

\[
e^{x^2+y^2+z^2} = \sum_{n=0}^\infty H_n(x, y, z) \frac{\mu^n}{n!}
\] (2)

respectively.

By (2), we get

\[
\sum_{n=0}^\infty H_n(x_1 + x_2, y, z) \frac{\mu^n}{n!} = e^{(x_1+x_2)^2+y^2+z^2}
\]

\[
= \sum_{n=0}^\infty x_2^\mu \sum_{n=0}^\infty H_n(x_1, y, z) \frac{\mu^n}{n!}
\]

\[
= \sum_{n=0}^\infty \left( \sum_{l=0}^{n} \binom{n}{l} H_l(x_1, y, z)x_2^{n-l}\right) \frac{\mu^n}{n!}
\] (3)
By comparing the coefficients on both sides of (3), we have the following theorem.

**Theorem 1.** For any positive integer \( n \), we have

\[
H_n(x_1 + x_2, y, z) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) H_l(x_1, y, z) x_2^{n-l}.
\]

Applying Eq. (2), we obtain

\[
\sum_{n=0}^{\infty} H_n(x, y, z_1 + z_2) \frac{t^n}{n!} = e^{t^2 + y^2 + (z_1 + z_2)t}.
\]

\[
= \sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} \frac{H_{n-3l}(x, y, z_1) z_2^l n!}{k!(n-3k)!} \right) \frac{t^n}{n!}.
\]

On equating the coefficients of the like power of \( t \) in the above, we obtain the following theorem.

**Theorem 2.** For any positive integer \( n \), we have

\[
H_n(x, y, z_1 + z_2) = n! \sum_{k=0}^{\infty} \frac{H_{n-3k}(x, y, z_1) z_2^k}{k!(n-3k)!}.
\]

Also, the 3-variable Hermite polynomials \( H_n(x, y, z) \) satisfy the following relations

\[
\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z),
\]

and

\[
\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z).
\]

The following elementary properties of the 3-variable Hermite polynomials \( H_n(x, y, z) \) are readily derived from (2). We, therefore, choose to omit the details involved.

**Theorem 3.** For any positive integer \( n \), we have

1. \( H_n(2x, -1, 0) = H_n(x) \).
2. \( H_n(x, y_1 + y_2, z) = n! \sum_{l=0}^{\infty} \frac{H_{n-2l}(x, y_1, z) y_2^l}{l!(n-2k)!} \).
3. \( H_n(x, y, z) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) H_l(x) H_{n-l}(-x, y + 1, z). \)
Theorem 4. For any positive integer \( n \), we have

1. \( H_n(x_1 + x_2, y_1 + y_2, z) = \sum_{i=0}^{n} \binom{n}{i} H_i(x_1, y_1, z) H_{n-i}(x_2, y_2) \).

2. \( H_n(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{i=0}^{n} \binom{n}{i} H_i(x_1, y_1, z) H_{n-i}(x_2, y_2, z_2) \).

The 3-variable Hermite polynomials can be determined explicitly. A few of them are

\[
H_0(x, y, z) = 1, \\
H_1(x, y, z) = x, \\
H_2(x, y, z) = x^2 + 2y, \\
H_3(x, y, z) = x^3 + 6xy + 6z, \\
H_4(x, y, z) = x^4 + 12x^2y + 12y^2 + 24xz, \\
H_5(x, y, z) = x^5 + 20x^3y + 60xy^2 + 60x^2z + 120yz, \\
H_6(x, y, z) = x^6 + 30x^4y + 180x^2y^2 + 120x^3z + 120x^2y + 720xyz + 360z^2, \\
H_7(x, y, z) = x^7 + 42x^5y + 420x^3y^2 + 840x^2y^3 + 210x^4z + 2520x^3yz + 2520x^2z^2 + 2520xz^2, \\
H_8(x, y, z) = x^8 + 56x^6y + 840x^4y^2 + 3360x^3y^3 + 1680y^4 + 336x^2z + 6720x^2yz + 10080x^2z^2 + 20160y^2z + 20160yz^2.
\]

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [7, 8, 12, 16–19]). In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. In addition, we investigate the zeros of the 3-variable Hermite polynomials using numerical methods. Using computer, a realistic study for the zeros of the 3-variable Hermite polynomials is very interesting. Finally, we observe an interesting phenomenon of ‘scattering’ of the zeros of the 3-variable Hermite polynomials.

2. Differential equations associated with the 3-variable Hermite polynomials

In this section, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials.

Let

\[
F = F(t, x, y, z) = e^{t^2 + xy^2 + xz^2} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \quad x, y, z, t \in \mathbb{C}.
\]
Then, by (4), we have

\[ F^{(1)} = \frac{\partial}{\partial t} F(t, x, y, z) = \frac{\partial}{\partial t} \left( e^{x^2 + y^2 + z^2 t} \right) = e^{x^2 + y^2 + z^2 t} \left( x + 2yt + 3zt^2 \right) \]

(5)

\[ F^{(2)} = \frac{\partial}{\partial t} F^{(1)}(t, x, y, z) = (2y + 6zt)F(t, x, y, z) + \left( x + 2yt + 3zt^2 \right) F^{(1)}(t, x, y, z) \]

(6)

Continuing this process, we can guess that

\[ F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{i=0}^{2N} a_i(N, x, y, z)t^iF(t, x, y, z), \quad (N = 0, 1, 2, \ldots). \]

(7)

Differentiating (7) with respect to \( t \), we have

\[ F^{(N+1)} = \frac{\partial F^{(N)}}{\partial t} = \sum_{i=0}^{2N} a_i(N, x, y, z)t^iF(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z)tF^{(1)}(t, x, y, z) \]

\[ = \sum_{i=0}^{2N} a_i(N, x, y, z)t^iF(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z)t(2yt + 3zt^2)F(t, x, y, z) \]

\[ = \sum_{i=0}^{2N} a_i(N, x, y, z)t^{i+1}F(t, x, y, z) + \sum_{i=0}^{2N} \left( a_i + i + 1 \right) a_i(N, x, y, z)t^iF(t, x, y, z) \]

\[ + \sum_{i=0}^{2N} 2ya_i(N, x, y, z)t^{i+1}F(t, x, y, z) + \sum_{i=0}^{2N} 3za_i(N, x, y, z)t^{i+2}F(t, x, y, z) \]

\[ = \sum_{i=0}^{2N-1} (i+1)a_{i+1}(N, x, y, z)t^iF(t, x, y, z) + \sum_{i=0}^{2N} xa_i(N, x, y, z)t^iF(t, x, y, z) \]

\[ + \sum_{i=1}^{2N+1} 2ya_{i-1}(N, x, y, z)t^iF(t, x, y, z) + \sum_{i=2}^{2N+2} 3za_{i-2}(N, x, y, z)t^iF(t, x, y, z) \]

Hence we have

\[ F^{(N+1)} = \sum_{i=0}^{2N-1} (i+1)a_{i+1}(N, x, y, z)t^iF(t, x, y, z) \]

\[ + \sum_{i=0}^{2N} xa_i(N, x, y, z)t^iF(t, x, y, z) \]

(8)

\[ + \sum_{i=1}^{2N+1} 2ya_{i-1}(N, x, y, z)t^iF(t, x, y, z) \]

\[ + \sum_{i=2}^{2N+2} 3za_{i-2}(N, x, y, z)t^iF(t, x, y, z). \]
Now replacing $N$ by $N + 1$ in (7), we find

$$F^{(N+1)} = \sum_{i=0}^{2N+2} a_i(N + 1, x, y, z) t^i F(t, x, y, z).$$  \hspace{1cm} (9)$$

Comparing the coefficients on both sides of (8) and (9), we obtain

$$a_0(N + 1, x, y, z) = a_1(N, x, y, z) + xa_0(N, x, y, z),$$

$$a_1(N + 1, x, y, z) = 2a_2(N, x, y, z) + xa_1(N, x, y, z) + 2ya_0(N, x, y, z),$$

$$a_{2N}(N + 1, x, y, z) = xa_{2N}(N, x, y, z) + 2ya_{2N-1}(N, x, y, z) + 3za_{2N-2}(N, x, y, z),$$

$$a_{2N+1}(N + 1, x, y, z) = 2ya_{2N}(N, x, y, z) + 3za_{2N-1}(N, x, y, z),$$

$$a_{2N+2}(N + 1, x, y, z) = 3za_{2N}(N, x, y, z),$$

and

$$a_i(N + 1, x, y, z) = (i + 1)a_{i+1}(N, x, y, z) + xa_i(N, x, y, z) + 2ya_{i-1}(N, x, y, z) + 3za_{i-2}(N, x, y, z), \hspace{1cm} (2 \leq i \leq 2N - 1).$$  \hspace{1cm} (11)$$

In addition, by (7), we have

$$F(t, x, y, z) = F^{(0)}(t, x, y, z) = a_0(0, x, y, z) F(t, x, y, z),$$

which gives

$$a_0(0, x, y, z) = 1.$$  \hspace{1cm} (13)$$

It is not difficult to show that

$$xF(t, x, y) + 2ytF(t, x, y, z) + 3zt^2F(t, x, y, z)$$

$$= F^{(1)}(t, x, y, z)$$

$$= \sum_{i=0}^{2} a_i(1, x, y, z) F(t, x, y, z)$$

$$= (a_0(1, x, y, z) + a_1(1, x, y, z) t + a_2(1, x, y, z) t^2) F(t, x, y, z).$$

Thus, by (14), we also find

$$a_0(1, x, y, z) = x, \quad a_1(1, x, y, z) = 2y, \quad a_2(1, x, y, z) = 3z.$$  \hspace{1cm} (15)$$

From (10), we note that

$$a_0(N + 1, x, y, z) = a_1(N, x, y, z) + xa_0(N, x, y, z),$$

$$a_0(N, x, y, z) = a_1(N - 1, x, y, z) + xa_0(N - 1, x, y, z),$$

$$a_0(N + 1, x, y, z) = \sum_{i=0}^{N} x^i a_i(N - i, x, y, z) + x^{N+1},$$

and
Theorem 5. Therefore, we obtain the following theorem.

\[ a_{2N+2}(N+1,x,y,z) = 3za_{2N}(N,x,y,z), \]
\[ a_{2N}(N,x,y,z) = 3za_{2N-2}(N-1,x,y,z), \ldots \]
\[ a_{2N+2}(N+1,x,y,z) = (3z)^{N+1}. \]  

(17)

Note that, here the matrix \( a_{i}(j,x,y) \) is given by

\[
\begin{pmatrix}
1 & x & 2y + x^2 & \cdots \\
0 & 2y & 4xy + 6z & \cdots \\
0 & 3z & 6xz + 4y^2 & \cdots \\
0 & 0 & 12yz & \cdots \\
0 & 0 & (3z)^2 & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \cdots (3z)^{N+1}
\end{pmatrix}
\]

Therefore, we obtain the following theorem.

**Theorem 5.** For \( N = 0, 1, 2, \ldots \), the differential equation

\[ F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t,x,y,z) = \left( \sum_{i=0}^{N} a_i(N,x,y,z) t^i \right) F(t,x,y,z) \]

has a solution

\[ F = F(t,x,y,z) = e^{t^2+y^2+z^2}, \]

where

\[ a_0(N+1,x,y,z) = \sum_{i=0}^{N} x^i a_i(N-i,x,y,z) + x^{N+1}, \]
\[ a_1(N+1,x,y,z) = 2a_0(N,x,y,z) + xa_1(N,x,y,z) + 2ya_0(N,x,y,z), \]
\[ a_{2N}(N+1,x,y,z) = xa_0(N,x,y,z) + 2ya_{2N-1}(N,x,y,z) + 3za_{2N-2}(N,x,y,z), \]
\[ a_{2N+1}(N+1,x,y,z) = 2ya_{2N}(N,x,y,z) + 3za_{2N-1}(N,x,y,z), \]
\[ a_{2N+2}(N+1,x,y,z) = (3z)^{N+1}. \]
and
\[ a_i(N + 1, x, y, z) = (i + 1)a_{i+1}(N, x, y, z) + xa_i(N, x, y, z) + 2ya_{i-1}(N, x, y, z), \quad (2 \leq i \leq 2N - 1). \]

From (4), we note that
\[ F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{k=0}^{\infty} \frac{H_{k+N}(x, y, z)}{k!} t^k, \quad (18) \]

By (4) and (18), we get
\[ e^{-mt} \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \left( \sum_{m=0}^{\infty} \frac{(-n)^m}{m!} \right) \left( \sum_{m=0}^{\infty} \frac{H_{m+N}(x, y, z)}{m!} \right) t^m, \quad (19) \]

By the Leibniz rule and the inverse relation, we have
\[ e^{-mt} \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{N}{k} (-n)^{m-k} H_{N+k}(x, y, z) \frac{t^m}{m!}. \quad (20) \]

Hence, by (19) and (20), and comparing the coefficients of \( \frac{t^m}{m!} \) gives the following theorem.

**Theorem 6.** Let \( m, n, N \) be nonnegative integers. Then
\[ \sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} H_{N+k}(x, y, z) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} H_{m+k}(x - n, y, z). \quad (21) \]

If we take \( m = 0 \) in (21), then we have the following corollary.

**Corollary 7.** For \( N = 0, 1, 2, \ldots \), we have
\[ H_N(x, y, z) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} H_k(x - n, y, z). \]

For \( N = 0, 1, 2, \ldots \), the differential equation
\[ F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \left( \sum_{i=0}^{N} a_i(N, x, y, z) t^i \right) F(t, x, y, z) \]

has a solution
\[ F = F(t, x, y, z) = e^{xt + yt^2 + zt^3}. \]

Here is a plot of the surface for this solution. In Figure 1(left), we choose \(-2 \leq z \leq 2, -1 \leq t \leq 1, x = 2, \) and \(y = -4.\) In Figure 1(right), we choose \(-5 \leq x \leq 5, -1 \leq t \leq 1, y = -3, \) and \(z = -1.\)

### 3. Distribution of zeros of the 3-variable Hermite polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the 3-variable Hermite polynomials \(H_n(x, y, z).\) By using computer, the 3-variable Hermite polynomials \(H_n(x, y, z)\) can be determined explicitly. We display the shapes of the 3-variable Hermite polynomials \(H_n(x, y, z)\) and investigate the zeros of the 3-variable Hermite polynomials \(H_n(x, y, z).\)

We investigate the beautiful zeros of the 3-variable Hermite polynomials \(H_n(x, y, z)\) by using a computer. We plot the zeros of the \(H_n(x, y, z)\) for \(n = 20,\) \(y = 1, -1, 1 + i, -1 - i,\) \(z = 3, -3, 3 + i, -3 - i\) and \(x \in \mathbb{C}\) (Figure 2). In Figure 2(top-left), we choose \(n = 20, y = 1,\) and \(z = 3.\) In Figure 2(top-right), we choose \(n = 20, y = -1,\) and \(z = -3.\) In Figure 2(bottom-left), we choose \(n = 20, y = 1 + i,\) and \(z = 3 + i.\) In Figure 2(bottom-right), we choose \(n = 20, y = -1 - i,\) and \(z = -3 - i.\)

In Figure 3(top-left), we choose \(n = 20, x = 1,\) and \(y = 1.\) In Figure 3(top-right), we choose \(n = 20, x = -1,\) and \(y = -1.\) In Figure 3(bottom-left), we choose \(n = 20, x = 1 + i,\) and \(y = 1 + i.\) In Figure 3(bottom-right), we choose \(n = 20, x = -1 - i,\) and \(y = -1 - i.\)

Stacks of zeros of the 3-variable Hermite polynomials \(H_n(x, y, z)\) for \(1 \leq n \leq 20\) from a 3-D structure are presented (Figure 3). In Figure 4(top-left), we choose \(n = 20, y = 1,\) and \(z = 3.\) In Figure 4(top-right), we choose \(n = 20, y = -1,\) and \(z = -3.\) In Figure 4(bottom-left), we choose \(n = 20, y = 1 + i,\) and \(z = 3 + i.\) In Figure 4(bottom-right), we choose \(n = 20, y = -1 - i,\) and \(z = -3 - i.\)
Our numerical results for approximate solutions of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ are displayed (Tables 1–3).

The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \leq n \leq 20$ structure are presented (Figure 5).

In Figure 5(left), we choose $y = 1$ and $z = 3$. In Figure 5(right), we choose $y = -1$ and $z = -3$.

Stacks of zeros of $H_n(x, -2, 4)$ for $1 \leq n \leq 40$, forming a 3D structure are presented (Figure 6). In Figure 6(top-left), we plot stacks of zeros of $H_n(x, -2, 4)$ for $1 \leq n \leq 20$. In Figure 6(top-right), we draw $x$ and $y$ axes but no $z$ axis in three dimensions. In Figure 6(bottom-left), we draw $y$
and z axes but no x axis in three dimensions. In Figure 6 (bottom-right), we draw x and z axes but no y axis in three dimensions.

It is expected that \( H_n(x, y, z), x \in \mathbb{C}, y, z \in \mathbb{R} \), has \( \text{Im}(x) = 0 \) reflection symmetry analytic complex functions (see Figures 2–7). We observe a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials \( H_n(x, y, z) \) for \( y, z \in \mathbb{R} \). We also hope to verify a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials \( H_n(x, y, z) \) for \( y, z \in \mathbb{R} \) (Tables 1 and 2). Next, we calculated an approximate solution satisfying \( H_n(x, y, z) = 0, x \in \mathbb{C} \). The results are given in Tables 3 and 4.
The plot of real zeros of the 3-variable Hermite polynomials $H_n(x, y, z)$ for $1 \leq n \leq 20$ structure are presented (Figure 7).

In Figure 7(left), we choose $x = 1$ and $y = 2$. In Figure 7(right), we choose $x = -1$ and $y = -2$.

Finally, we consider the more general problems. How many zeros does $H_n(x, y, z)$ have? We are not able to decide if $H_n(x, y, z) = 0$ has $n$ distinct solutions. We would also like to know the number of complex zeros $C_{H_n(x,y,z)}$ of $H_n(x, y, z), \text{Im}(x) \neq 0$. Since $n$ is the degree of the polynomial $H_n(x, y, z)$, the number of real zeros $R_{H_n(x,y,z)}$ lying on the real line $\text{Im}(x) = 0$ is then $R_{H_n(x,y,z)} = n - C_{H_n(x,y,z)}$, where $C_{H_n(x,y,z)}$ denotes complex zeros. See Tables 1 and 2 for...
| Degree $n$ | Real zeros | Complex zeros |
|-----------|------------|---------------|
| 1         | 1          | 0             |
| 2         | 0          | 2             |
| 3         | 1          | 2             |
| 4         | 2          | 2             |
| 5         | 1          | 4             |
| 6         | 2          | 4             |
| 7         | 3          | 4             |
| 8         | 2          | 6             |
| 9         | 3          | 6             |
| 10        | 4          | 6             |
| 11        | 3          | 8             |
| 12        | 4          | 8             |
| 13        | 3          | 10            |
| 14        | 4          | 10            |

Table 1. Numbers of real and complex zeros of $H_n(x, 1, 3)$.

| Degree $n$ | Real zeros | Complex zeros |
|-----------|------------|---------------|
| 1         | 1          | 0             |
| 2         | 2          | 0             |
| 3         | 1          | 2             |
| 4         | 2          | 2             |
| 5         | 3          | 2             |
| 6         | 2          | 4             |
| 7         | 3          | 4             |
| 8         | 4          | 4             |
| 9         | 3          | 6             |
| 10        | 4          | 6             |
| 11        | 5          | 6             |
| 12        | 6          | 6             |
| 13        | 5          | 8             |
| 14        | 6          | 8             |

Table 2. Numbers of real and complex zeros of $H_n(x, -1, -3)$. 

Differential Equations Arising from the 3-Variable Hermite Polynomials and Computation of Their Zeros

http://dx.doi.org/10.5772/intechopen.74355
tabulated values of $R_{\text{H}_n(x,y,z)}$ and $C_{\text{H}_n(x,y,z)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the 3-variable Hermite polynomials $H_n(x,y,z)$ which appear in mathematics and physics. The reader may refer to [2, 11, 13, 20] for the details.

| Degree $n$ | $x$ |
|-----------|-----|
| 1         | 0   |
| 2         |     |
| 3         | 1.8845 |
| 4         | 3.1286, 0.17159 |
| 5         | 4.5385 |
| 6         | -5.8406, -1.3476 |
| 7         | -7.1098, -2.1887, -0.3635 |
| 8         | -8.3241, -3.4645 |
| 9         | -9.4984, -4.6021, 1.1118 |
| 10        | -10.637, -5.7212, -1.5785, -0.61919 |
| 11        | -11.745, -6.8105, 2.8680 |
| 12        | -12.824, -7.8743, -3.8894, -0.99513 |

Table 3. Approximate solutions of $H_n(x, 1, 3) = 0, x \in \mathbb{R}$.

Figure 5. Real zeros of $H_n(x, y, z), 1 \leq n \leq 20$. 

The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the 3-variable Hermite polynomials $H_n(x,y,z)$ which appear in mathematics and physics. The reader may refer to [2, 11, 13, 20] for the details.
Figure 6. Stacks of zeros of $H_n(x, -2, 4)$ for $1 \leq n \leq 20$.

| degree $n$ | $x$                      |
|-----------|-------------------------|
| 1         | 0                       |
| 2         | -1.4142, 1.4142         |
| 3         | 3.3681                  |
| 4         | 0.16229, 5.0723         |
| 5         | -1.3404, 1.4745, 6.6661 |
| 6         | 2.9754, 8.1678          |
| 7         | 0.31213, 4.3783, 9.5946 |
Acknowledgements

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1A2B4006092).

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| degree n | x               |
|----------|-----------------|
| 8        | −1.2604, 1.5304, 5.7274, 10.959 |
| 9        | 2.8224, 7.0271, 12.270 |
| 10       | 0.44594, 4.0615, 8.2834, 13.535 |
| 11       | −1.1740, 1.5825, 5.2667, 9.5013, 14.760 |
| 12       | −1.4659, −0.87728, 2.7469, 6.4398, 10.685, 15.949 |

Table 4. Approximate solutions of \( H_n(x, -1, -3) = 0, x \in \mathbb{R} \).

Figure 7. Real zeros of \( H_n(x, y, z), 1 \leq n \leq 20 \).

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