Massless scattering at special kinematics as Jacobi polynomials

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Abstract
We study the scattering equations recently proposed by Cachazo, He and Yuan in the special kinematics where their solutions can be identified with the zeros of the Jacobi polynomials. This allows for a non-trivial two-parameter family of kinematics. We present explicit and compact formulas for the n-gluon and n-graviton partial scattering amplitudes for our special kinematics in terms of Jacobi polynomials. We also provide alternative expressions in terms of gamma functions. We give an interpretation of the common reduced determinant appearing in the amplitudes as the product of the squares of the eigenfrequencies of small oscillations of a system whose equilibrium is the solutions of the scattering equations.

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1. Introduction
Recently a new and elegant formula for the complete tree-level S-matrix of pure Yang–Mills and gravity in arbitrary dimensions has been given by Cachazo, He and Yuan (CHY) in [1]. It was later extended to a massless colored cubic scalar theory [2]. The formula was proven by Dolan and Goddard for gluon amplitudes in [3]. In [4], Mason and Skinner constructed a chiral infinite tension limit of the RNS superstring which was shown to compute the CHY formulas. The chiral infinite tension limit was later generalized to the pure spinor superstring by Berkovits in [5]. Further progress on the study of the CHY formulas also include [6–8].

The goal of this work is to study the system of [1] and more precisely the scattering equations [9, 10], since the latter play an important role in the context of scattering amplitudes. While the current interest in the scattering equations is certainly the work of CHY, historically they were first written down by Fairlie and Roberts in [11–13] and later in the work of Gross
and Mende on the high energy behavior of string theory [14]. The have also arisen in the context of twistor string theory in the work of Witten [15].

The general case is quite involved and we only focus on a particular kinematics. This would allow us to associate the solutions of the scattering equations with the zeros of the Jacobi polynomials. The aforementioned polynomials depend on two continuous parameters and so does our kinematics. After choosing polarizations for our system we will be able to obtain compact expressions for \( n \)-gluon and \( n \)-gravity scattering in arbitrary dimensions. The final result can be written as the product of a piece that depends on the kinematics and can be expressed as Jacobi polynomials and a piece that is related to the helicities.

One of the motivations besides mathematical curiosity and the desire to obtain explicit results for the amplitudes is the fact that systems whose equilibrium is associated with the zeros of classical polynomials usually admit a Lax pair representation, are completely integrable and in some cases can be explicitly solved (employing for example the projection method of Olshanetsky and Perelomov [16]). Probably the most representative example is the Calogero–Moser [17, 18] system whose equilibrium is the zeros of the Hermite polynomials.

We start in the next section by reviewing the work of [1] and setting up notation. We then move on to the derivation of our special kinematics and also choose appropriate helicities. We continue by evaluating numerically the gluon and gravity amplitudes and give several different but equivalent expressions for them. We then give an interpretation of the common reduced determinant appearing in both Yang–Mills and gravity amplitudes and finally we present our conclusions. In an appendix we summarize properties of the roots of the Jacobi polynomials.

### 2. Prolegomena

In the work of [9] it was pointed out the existence of polynomial equations that connect the space of kinematic invariants of \( n \) massless particles with momentum \( k_a \) in arbitrary spacetime dimensions with the positions \( \sigma_c \) of \( n \) points on a Riemann sphere through

\[
\sum_{b=1, b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0. \tag{1}
\]

Due to some remarkable properties that these equations possess they were proposed to play an important role in the scattering of massless particles and were called the scattering equations. These equations are invariant under \( \text{SL}(2, \mathbb{C}) \) transformations, which allows us to fix three of the \( \sigma s \) to arbitrary values.

The connection with the tree-level \( S \)-matrix of massless particles was made clear in the subsequent work of CHY [1] where it was proposed that the tree-level \( n \)-gluon partial amplitude of Yang–Mills is given by

\[
A_n = \sum_{\{\sigma\} \in \text{solutions}} \frac{1}{\sigma_{12} \cdots \sigma_{n1}} \frac{\text{Pf} \Psi(k, \epsilon, \sigma)}{\det \Phi}, \tag{2}
\]

and that of gravity by

\[
M_n = \sum_{\{\sigma\} \in \text{solutions}} \frac{\det \Psi(k, \epsilon, \sigma)}{\det \Phi}, \tag{3}
\]

where the sum runs over all the \((n - 3)!\) solutions of (1) and \( \sigma_{ab} = \sigma_a - \sigma_b \).

In order to explain the above notation we start by defining the \( 2n \times 2n \) antisymmetric matrix \( \Psi \) as

\[
\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \tag{4}
\]
with the $n \times n$ matrices $A, B, C$ defined as

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b, \\ 0 & a = b, \end{cases} \quad B_{ab} = \begin{cases} \epsilon_a \cdot \epsilon_b & a \neq b, \\ 0 & a = b, \end{cases} \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b, \\ -\sum_{c\neq a} \frac{\epsilon_a \cdot k_c}{\sigma_a - \sigma_c} & a = b, \end{cases}$$

where $\epsilon_a$ are the helicities of the external particles. We further define the $n \times n$ matrix $\Phi$ through

$$\Phi_{ab} = \begin{cases} \frac{k_a \cdot k_b}{(\sigma_a - \sigma_b)^2} & a \neq b, \\ -\sum_{c\neq a} \frac{k_a \cdot k_c}{(\sigma_a - \sigma_c)^2} & a = b. \end{cases}$$

Then, the reduce Pfaffian appearing in the numerator of (2) is defined to be $\text{Pf}'\Psi = \text{Pf'}(\Psi[i,j])$, where the matrix $\Psi[i,j]$ is derived from the matrix $\Psi$ by removing the $i$th and $j$th rows and the $i$th and $j$th columns with $1 \leq i < j \leq n$. The numerator of (3) is defined $\text{det}'\Psi = 4\text{det}\Psi[1]$, whereas the common denominator of (2) and (3) is given by $\text{det}'\Phi = \text{det}[^{(i,j)}\Phi[i,j]]$, where the minor $[^{(i,j)}\Phi[i,j]]$ is the determinant of the matrix $\Phi$ after removing rows $[i, j, k]$ and columns $[p, q, r]$.

In the following sections we occasionally use the short notation $k_{ab} = k_a \cdot k_b$ and $\epsilon_{ab} = \epsilon_a \cdot \epsilon_b$.

3. Derivation of the special kinematics

It turns out that we can uniquely choose our kinematics in such a way that the solutions to (1) are identified with the zeros of the Jacobi polynomials. We fix the $\text{SL}(2, \mathbb{C})$ invariance by choosing $\sigma_1 = 1$, $\sigma_2 = 1$, $\sigma_3 = \infty$. Then for $a \geq 4$, equation (1) gives

$$\sum_{b=4, b\neq a}^{n-1} \frac{k_{ab}}{\sigma_a - \sigma_b} = \frac{(k_{a2} - k_{a1}) + (k_{a2} + k_{a1})\sigma_a}{1 - \sigma_a^2}, \quad a \geq 4. \quad (8)$$

We now choose the special kinematics $k_{a1} = (1 + \beta)/2, k_{a2} = (1 + \alpha)/2, k_{ab} = 1$ for $a, b \geq 4$, so that the $(n - 3)$ variables $\sigma_a$ in (8) can be interpreted according to (A.4) as the $(n - 3)$ roots of the Jacobi polynomials $P_n^{\alpha, \beta}$. It is known that only $(n - 3)$ of the equations in (1) are independent, therefore the rest of the kinematics can be derived from the conservation of momentum. As a consistency check we can consider the scattering equations (1) for $a = 1$ and $a = 2$. Using (1) and (A.2) we get $k_{12} = (1 + \beta)P_{n-3}^{\alpha, \beta}(-1)/P_{n-3}^{\alpha, -1}(-1)$ and $k_{12} = -(1 + \alpha)P_{n-3}^{\alpha, \beta}(-1)/P_{n-3}^{\alpha, -1}(1)$ respectively. The two values match and equal to $k_{12} = (3 - n)(\alpha + \beta + n - 2)/2$ in accordance with the conservation of momentum. We summarize our results as given in table 1.

We now want to choose polarization vectors compatible with $\epsilon_a \cdot k_a = 0$ for every $a$ as well as conservation of momentum. We choose according to the criterion that in the limit $\sigma_3 \to \infty$ the quantity $\sigma_3^2\sigma_2^2\text{det}(\Psi[1])$ is invariant under $\sigma_i$ permutations with $i \geq 4$. Since the general case seems to be quite involved we also choose to further simplify our problem by making the choice as given in table 2 where $c_i$ are arbitrary constants. We assume spacetime dimension large enough compared to the number of particles.
The determinant of (9) is evaluated to
\[ n \text{ might be within reach. We have checked our result numerically for up to} \]
\[ \text{where the quantity in the numerator means the first three rows of (7). Then the} \]
\[ \text{In order to evaluate the reduced determinant we choose to remove the first three columns and} \]
\[ \text{the first three rows of (7). Then the} \]
\[ \text{4. Evaluation of the amplitudes} \]
\[ \text{In order to evaluate det} \]
\[ \text{We have found numerically (up to} \]
\[ \text{One observes that the determinant of (9) is independent of swapping any} \]
\[ \text{perms} \]
\[ \text{sum over all} \]
\[ \text{Helicity choice.} \]
\[ \text{Table 2. Helicity choice.} \]
\[ \text{Table 1. Our two-parameter special kinematics.} \]
\[ k_{11} = (3 - n)(\alpha + \beta + n - 2)/2 \]
\[ k_{12} = (n - 3)(a - 3 + \alpha)/2 \]
\[ k_{13} = (n - 3)(n - 3 + \alpha)/2 \]
\[ k_{1a} = (1 + \beta)/2 \]
\[ k_{2a} = (1 + \alpha)/2 \]
\[ k_{3a} = (6 - 2n - \alpha - \beta)/2 \]
\[ k_{ab} = 1 \quad a, b \geq 4, a \neq b \]
\[ \epsilon_{ij}, \epsilon_{13}, \epsilon_{23} \quad \text{Arbitrary} \]
\[ \epsilon_{14} = c_1 \quad a \geq 4 \]
\[ \epsilon_{24} = c_2 \quad a \geq 4 \]
\[ \epsilon_{34} = c_3 \quad a \geq 4 \]
\[ \epsilon_{ab} = c_4 \quad a, b \geq 4, a \neq b \]
\[ \text{The determinant of (9) is independent of swapping any } \sigma_i \text{ with any } \sigma_j. \]
\[ \text{To see that we have to swap row } (i - 3) \text{ with row } (j - 3) \text{ and column } (i - 3) \text{ with column } (j - 3). \]
\[ \text{The determinant of (9) is evaluated to} \]
\[ \left| \Phi \right|_{23}^{12} = \begin{pmatrix} 1 & \ldots & 1 \\ \sigma_a - \sigma_b & \ldots & \sigma_{a-1} - \sigma_{a+1} \end{pmatrix} \]
\[ \frac{1}{(n - 3)!} P_{n-3}^{(\alpha, \beta)}(-1) \]
\[ \text{where the quantity in the numerator means the (n - 3)th derivative. We do not have an analytic} \]
\[ \text{proof of the above result although a proof based on recurrence relations of Jacobi polynomials} \]
\[ \text{might be within reach. We have checked our result numerically for up to } n = 20. \]
\[ \text{The following sum over all } (n - 3)! \text{ permutations of } \{1, 2, \ldots, n - 3\} \text{ is} \]
\[ \sum_{\text{perms}} \frac{1}{(\sigma_4 + 1)(\sigma_5 + 1) \cdots (\sigma_{n+1} + 1)} = (-1)^{n+1} \frac{P_{n-3}^{(\alpha, \beta)}(x)}{p_{n-3}^{(\alpha, \beta)}(1)} \]
\[ \text{In order to evaluate } \text{det} \Psi \text{ we choose to eliminate the first and second row and column of (4).} \]
\[ \text{We have found numerically (up to } n = 20) \text{ that for odd } n \text{ the determinant vanishes whereas for even } n \text{ it is} \]
\[ \sigma_2^2 \sigma_3^2 \text{det} \left| \Phi \right|_{23}^{12} = \left( \frac{2(n - 3)!!}{((n - 4)!!)^2} \frac{P_{n-3}^{(\alpha, \beta)}(x)}{p_{n-3}^{(\alpha, \beta)}(1)} \right)^2 \]
with the helicity dependent part $H_n$ given by

$$H_n = \frac{c_n^{n/2-3}}{2(n-3)} + \alpha + \beta \left(\frac{2(n-3)(n-4)c_1c_2c_3}{(1+\alpha)(1+\beta)} - c_3c_4\epsilon_{12}\right) + c_n^{n/2-2} \left(\frac{c_2\epsilon_{13} + c_1\epsilon_{23}}{1+\alpha + c_n^{1/2-1}}\right).$$

(13)

As a consistency check one can verify the vanishing of $H_n$ under the replacement $\epsilon_a \rightarrow k_a$ as required by gauge invariance.

Putting all pieces together we arrive at the final expressions for the amplitudes. For odd $n$ the amplitudes vanish, whereas for even $n$ we find

$$A_n = \frac{(-1)^{n/2}2^{5-n}(n-2)!(n-3)!!P^{(\frac{n-1}{2})}}{P^{(\frac{n-3}{2})}}(x)H_n,$$

$$M_n = \frac{2^{-n}(n-2)!(n-3)!!P^{(\frac{n-1}{2})}}{P^{(\frac{n-3}{2})}}(-1)P^{(\frac{n+1}{2})}H_n^2,$$

(14)

We also provide two alternative expressions

$$A_n = \frac{2(n-3)!(n-3)!!\prod_{j=1}^{n/2}(2j+\alpha)}{\prod_{j=1}^{n/2}(2j+\beta)(2j+n-3+\alpha+\beta)}H_n,$$

$$M_n = -\frac{2^{5-n}(n-3)!!\prod_{j=1}^{n/2-2}(2j+\alpha)(2j+\beta)(2j+n-3+\alpha+\beta)}{\prod_{j=1}^{n/2}(2j+\alpha)(2j+\beta)(2j+n-3+\alpha+\beta)}H_n^2,$$

(15)

and

$$A_n = 2^{4-n/2}(n-3)!!\frac{\Gamma\left(\frac{n-1+\alpha}{2}\right)\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(\frac{n-2+\alpha+\beta}{2}\right)\Gamma\left(\frac{2n-5+\alpha+\beta}{2}\right)}H_n,$$

$$M_n = -2^{8-n}((n-3)!!)^2\frac{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n-1+\alpha}{2}\right)\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(\frac{n-2+\alpha+\beta}{2}\right)\Gamma\left(\frac{2n-5+\alpha+\beta}{2}\right)}H_n^2.$$

(16)

5. An interpretation of the reduced determinant

For our special kinematics the scattering equations (8) become

$$\sum_{j=1}^{n} \frac{1}{\sigma_i - \sigma_j} + \frac{1+\beta}{2(\sigma_i + 1)} + \frac{1+\alpha}{2(\sigma_i - 1)} = 0.$$

(17)

We now consider the $(n-3)$-body dynamical system described by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n-3} p_i^2 + U$$

(18)

with potential energy

$$U = \sum_{i<j} \ln |x_i - x_j| + \frac{1+\beta}{2} \sum_{i=1}^{n-3} \ln |x_i + 1| + \frac{1+\alpha}{2} \sum_{i=1}^{n-3} \ln |x_i - 1|.$$

(19)
One may think of (18)–(19) as a system of \( n \) particles, one of them at the fixed position \( x = -1 \) with ‘charge’ \( \frac{1+\beta}{2} \), the second at \( x = 1 \) with charge \( \frac{1+\alpha}{2} \), the third at infinity\(^2\) and the remaining \((n-3)\) which have unit charge are bound to move in the interval \((-1, 1)\).\(^3\) The particles of our system preserve their ordering.

Two observations follow. The equations of motion for the system (18)–(19) are

\[
\ddot{x}_i = -\sum_{j=1}^{n-3} \frac{1}{x_i - x_j} - \frac{1+\beta}{2(x_i + 1)} - \frac{1+\alpha}{2(x_i - 1)},
\]

therefore (18)–(19) has an equilibrium at the solutions of the scattering equations (17). Moreover the squares of the eigenfrequencies of small oscillations of (18)–(19) around the equilibrium position are given by the eigenvalues of the matrix

\[
\partial_i \partial_j U = \left( -\frac{1+\beta}{2(x_i + 1)^2} - \frac{1+\alpha}{2(x_i - 1)^2} - \sum_{j=1, j\neq j}^{n-3} \frac{1}{(x_i - x_j)^2} \right) \delta_{ij} + \frac{1}{(x_j - x_j)^2} (1 - \delta_{ij}),
\]

which is precisely (9). The generalization to general kinematics is obvious.

In [19] it was found that the zeros of Jacobi polynomials are related to the equilibrium of the \( BC_{n-1} \) Sutherland model and a Lax representation was given for the special case \( \alpha = \beta = 1 \).

We do not study further the system (18)–(19) here.

6. Concluding remarks

In this work we have presented the \( n \)-gluon and \( n \)-graviton partial amplitudes for a non-trivial two-parameter family of kinematics. After we have chosen convenient helicities for our systems we were able to write the gluon and gravity partial scattering amplitudes in an explicit and compact form. The key idea behind this calculation was the observation that the scattering equations (1) can be a special case of the equations that the zeros of the Jacobi polynomials satisfy.

Although the solutions of the scattering equation associated with the roots of Jacobi polynomials are in general complicated, the final result for the amplitude is surprisingly simple. This simplicity is due to several properties of the Jacobi polynomials that are expressed as cancellations in the evaluation of the amplitudes. It is possible that other polynomials could also lead to other kinematics and simple final expressions.

In [20] it was shown that for four-dimensional kinematics the scattering equations possess \((n-3)!\) solutions. This result was later extended to arbitrary dimensions in [9]. In our case this number follows naturally as all possible ways to permute the \((n-3)\) distinct solutions of the \((n-3)\)th order Jacobi polynomial. Another observation is that the final expressions are simpler than the individual pieces, since cancellations occurred when we combined all pieces together.

We have also associated the common reduced determinant appearing in (2) and (3) (and also in the colored ordered scalar theory in [2]) with a system whose equilibrium is the solutions of the scattering equations. This observations holds also for general kinematics. The simple result of the reduced determinant in our kinematics asks for a simple derivation which we do not have at the moment.

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1 From now on we drop the quotation marks around ‘charge’.
2 One may be bother by the fact that the interaction of the particle at infinity gives rise to a term proportionally to \( \ln(\infty) \) in the potential energy. This is not a problem though, since the sum of all interactions of the particle at infinity with the rest \((n-1)\) particles is exactly zero, due to the conservation of momentum.
3 We can specialize to the case \( \alpha > -1 \) and \( \beta > -1 \) where all the zeros of the Jacobi polynomials are simple and lie in the interval \((-1, 1)\).
Perhaps one of the most interesting observation of this work is the existence of a Lax pair for the system (18)–(19), even if this is only known for a special case. This hints to a possible integrable structure behind the amplitudes at least at tree level and the possibility to study the amplitudes as an $N$-body system. We hope to report on those issues in the future. It would also be interesting to find a $Y$-system description for our system possibly along the lines of [21, 22].

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Appendix. Properties of the roots of the Jacobi polynomials

In this appendix we review some useful properties of the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. The reader may also consult the classical [23] as well as [24].

The Jacobi polynomials obey the differential equation

$$\quad (1-x^2)y''(x) + (\beta - \alpha - (\alpha + \beta + 2)x)y'(x) + n(n + \alpha + \beta + 1)y(x) = 0.$$  

(A.1)

For $\alpha > -1$ and $\beta > -1$ the $n$th order polynomial has $n$ distinct roots that lie in the interval $(-1, 1)$.

Next we prove some identities associated to the roots of $P_n^{(\alpha,\beta)}(x)$. We start by expressing the Jacobi polynomials as

$$\quad \prod_{j=1}^{n} (x - x_j).$$

After taking the logarithm and then the derivative we arrive at

$$\quad \sum_{j=1}^{n} \frac{1}{x - x_j} = \frac{P_n^{(\alpha,\beta)'}(x)}{P_n^{(\alpha,\beta)}(x)}.$$  

(A.2)

or

$$\quad \sum_{j \neq i}^{n} \frac{1}{x - x_j} = \frac{(x - x_i)P_n^{(\alpha,\beta)'}(x) - P_n^{(\alpha,\beta)}(x)}{(x - x_i)P_n^{(\alpha,\beta)}(x)}.$$  

(A.3)

Taking the limit where $x$ approaches the root $x_i$ and applying de l’Hôpital’s rule twice we arrive at

$$\quad \sum_{j \neq i}^{n} \frac{1}{x_i - x_j} = \frac{\alpha - \beta + (\alpha + \beta + 2)x_i}{2(1 - x_i^2)}.$$  

(A.4)

Similarly, by differentiating (A.3), taking the limit where $x$ approaches the root $x_i$ and applying de l’Hôpital’s rule four times we arrive at

$$\quad 12(1 - x_i^2)^2 \sum_{j \neq i}^{n} \frac{1}{(x_i - x_j)^2} = 4(n - 1)(\alpha + \beta + n + 2) - (\alpha - \beta)^2$$

$$\quad - 2(\alpha - \beta)(\alpha + \beta + 6)x_i - (4n(\alpha + \beta + n + 1) + (\alpha + \beta + 2)(\alpha + \beta + 6))x_i^2.$$  

(A.5)
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