Remarks on the time–energy uncertainty relation

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Abstract

We analyze the Heisenberg and Mandelstam–Tamm time–energy uncertainty relations and we show that contrary to the position–momentum uncertainty relation, these relations can not be considered as universally valid.

The famous Heisenberg uncertainty relations [1] play an important and significant role in the understanding of the quantum world and in explanations of its properties. There is a mathematically rigorous derivation of the position–momentum uncertainty relation but this same can not be said about time–energy uncertainty relation. Nonetheless the time–energy uncertainty relation is considered by many authors as having the same status as the position–momentum uncertainty relation and it is often used as the basis for drawing far–reaching conclusions regarding the prediction of the behavior of some physical systems in certain situations in various areas of physics and astrophysics and from time to time such conclusions were considered as the crucial. So, the time–uncertainty relation still requires its analysis and checking whether it is correct and well motivated by postulates of quantum mechanics. We present here an analysis of the Heisenberg and Mandelstam–Tamm time–energy uncertainty relations and show that the time–energy uncertainty relation can not be considered as universally valid.

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The uncertainty principle belongs to the one of characteristic and the most important consequences of the quantum mechanics. The most known form of this principle is the Heisenberg uncertainty principle [1] for the position and momentum, which can be written as follows (see e.g. [2]),

\[ \Delta \phi x \cdot \Delta \phi p_x \geq \hbar, \] (1)

where \( \Delta \phi x \) and \( \Delta \phi p_x \) are the standard (root–mean–square) deviations: In the general case for an observable \( F \) the standard deviation is defined as follows

\[ \Delta \phi F = \| (F - \langle F \rangle_\phi) \|, \]

where \( \langle F \rangle_\phi = \langle \phi | F | \phi \rangle \) is the average (or expected) value of an observable \( F \) in a system whose state is represented by the normalized vector \( |\phi\rangle \in H \), provided that \( |\langle \phi | F | \phi \rangle| < \infty \). Equivalently:

\[ \Delta \phi F \equiv \sqrt{\langle F^2 \rangle_\phi - \langle F \rangle^2_\phi}. \]

In Eq. (1) \( F \) stands for position and momentum operators \( x \) and \( p_x \) as well as for their squares. The observable \( F \) is represented by hermitian operator \( F \) acting in a Hilbert space \( \mathcal{H} \) of states \( |\phi\rangle \).

In general, the relation (1) results from basic assumptions of the quantum theory and from the geometry of Hilbert space [3]. Analogous relations hold for any two observables, say \( A \) and \( B \), represented by non–commuting hermitian operators \( A \) and \( B \) acting in the Hilbert space of states (see [4]), such that \([A, B] \) exists and \( |\phi\rangle \in \mathcal{D}(AB) \cap \mathcal{D}(BA) \), (\( \mathcal{D}(O) \) denotes the domain of an operator \( O \) or of a product of operators):

\[ \Delta \phi A \cdot \Delta \phi B \geq \frac{1}{2} |\langle [A, B] \rangle_\phi|, \] (2)

where the equality takes place if \( (B - \langle B \rangle_\phi) |\phi\rangle = i\lambda (A - \langle A \rangle_\phi) |\phi\rangle \), (here, \( \lambda = \lambda^* \)), or if \( |\phi\rangle \) is an eigenvector for operators \( A \) or \( B \), (see, eg. [3]). The derivation of inequality (2) is the rigorous one.

Heisenberg in [1] postulated also the validity of the analogous relation to (1) for the time and energy (see also [5]). This relation was a result of his heuristic considerations and it is usually written as follows

\[ \Delta \phi t \cdot \Delta \phi E \geq \frac{\hbar}{2}. \] (3)

The more rigorous derivation of this relation was given by Mandelstam and Tamm [6] and now it is known as the Mandelstam–Tamm time–energy uncertainty relation. Their derivation is reproduced in [2] and goes as follows: In the general relation (2) the operator \( B \) is replaced by the selfadjoint non–depending on time Hamiltonian \( H \) of the system considered and \( \Delta \phi B \) is
replaced by $\Delta_\phi H$ and then identifying the standard deviation $\Delta_\phi H$ with $\Delta_\phi E$ one finds that

$$\Delta_\phi A \cdot \Delta_\phi E \geq \frac{1}{2} |\langle [A, H] \rangle_\phi|,$$

(4)

where it is assumed that $A$ does not depend upon the time $t$ explicitly, $|\phi\rangle \in \mathcal{D}(HA) \cap \mathcal{D}(AH)$, and $[A, H]$ exists. The next step is to use the Heisenberg representation and corresponding equation of motion which allows to replace the average value of the commutator standing in the right–hand side of the inequality (4) by the derivative with respect to time $t$ of the expected value of $A$,

$$\langle [A, H] \rangle_\phi \equiv i\hbar \frac{d}{dt} \langle A \rangle_\phi.$$

(5)

Using this relation one can replace the inequality (4) by the following one,

$$\Delta_\phi A \cdot \Delta_\phi E \geq \frac{\hbar}{2} \left| \frac{d}{dt} \langle A \rangle_\phi \right|.$$

(6)

(Relations (4) — (6) are rigorous). Next authors \[2, 6\] and many others divide both sides of the inequality (6) by the term $\left| \frac{d}{dt} \langle A \rangle_\phi \right|$, which leads to the following relation

$$\frac{\Delta_\phi A}{\left| \frac{d}{dt} \langle A \rangle_\phi \right|} \cdot \Delta_\phi E \geq \frac{\hbar}{2},$$

(7)

or, using

$$\tau_A \overset{\text{def}}{=} \frac{\Delta_\phi A}{\left| \frac{d}{dt} \langle A \rangle_\phi \right|},$$

(8)

to the final result known as the Mandelstam–Tamm time–energy uncertainty relation,

$$\tau_A \cdot \Delta_\phi E \geq \frac{\hbar}{2},$$

(9)

where $\tau_A$ is usually considered as a time characteristic of the evolution of the statistic distribution of $A$ \[2\]. The time–energy uncertainty relation (9) and the above described derivation of this relation is accepted by many authors analyzing this problem or applying this relation (see, e.g. \[7, 8, 9, 10\] and many other papers). On the other hand there are some formal controversies regarding the role and importance of the parameter $\tau_A$ in (9) or $\Delta t$ in (3). These controversies are caused by the fact that in the quantum mechanics
the time $t$ is a parameter. Simply it can not be described by the hermitian operator, say $T$, acting in the Hilbert space of states (that is time can not be an observable) such that $[T, H] = i\hbar \mathbb{I}$ if the Hamiltonian $H$ is bounded from below. This observation was formulated by Pauli [11] and it is know as "Pauli’s Theorem" (see, eg. [7, 12]). Therefore the status of the relations (3) and relations (1), (2) is not the same regarding the basic principles of the quantum theory (see also discussion, e.g., in [13, 14, 15, 16]).

The Mandelstam–Tamm uncertainty relation (9) is also not free of controversies. People applying and using the above described derivation of (9) in their discussions of the time-energy uncertainty relation made use (consciously or not) of a hidden assumption that right hand sides of Es. (4), (6) are non–zero, that is that there does not exist any vector $|\phi_{\beta}\rangle \in \mathcal{H}$ such that $\langle[A, H]\rangle_{\phi_{\beta}} = 0$, or $d/dt\langle A\rangle_{\phi_{\beta}} = 0$. Although in the original paper of Mandelstam and Tamm [6] there is a reservation that for the validity of the formula of the type (9) it is necessary that $\Delta_{\phi_{\beta}} H \neq 0$ (see also, e.g. [17, 18]), there are not an analogous reservations in [2] and in many other papers.

Basic principles of mathematics require that before dividing the both sides of Eq. (6) by $|\frac{d}{dt}\langle A\rangle_{\phi}|$, one should check whether $\frac{d}{dt}\langle A\rangle_{\phi}$ is different from zero or not. Let us do this now: Let $\Sigma_{H} \subset \mathcal{H}$ be a set of eigenvectors $|\phi_{\beta}\rangle$ of $H$ for the eigenvalues $E_{\beta}$. We have $H|\phi_{\beta}\rangle = E_{\beta}|\phi_{\beta}\rangle$ for all $|\phi_{\beta}\rangle \in \Sigma_{H}$ and therefore for all $|\phi_{\beta}\rangle \in \Sigma_{H} \cap \mathcal{D}(A)$ (see (5)),

$$\langle[A, H]\rangle_{\phi_{\beta}} = i\hbar \frac{d}{dt}\langle A\rangle_{\phi_{\beta}} \equiv 0. \quad (10)$$

Similarly,

$$\Delta_{\phi_{\beta}} H = \sqrt{\langle |H^{2}|\rangle_{\phi_{\beta}} - (\langle |H|\rangle_{\phi_{\beta}})^{2}} \overset{\text{def}}{=} \Delta_{\phi_{\beta}} E \equiv 0, \quad (11)$$

for all $|\phi_{\beta}\rangle \in \Sigma_{H}$. This means that in all such cases the non–strict inequality (6) takes the form of the following equality

$$\Delta_{\phi_{A}} A \cdot 0 = \frac{\hbar}{2} \cdot 0. \quad (12)$$

In other words, one can not divide the both sides of the inequality (6) by $|\frac{d}{dt}\langle A\rangle_{\phi}| \equiv 0$ for all $|\phi_{\beta}\rangle \in \Sigma_{H}$, because in all such cases the result is an undefined number and such mathematical operations are unacceptable. It should be noted that although the authors of the publications [2, 17] knew that the property (11) occurs for the vectors from the set $\Sigma_{H}$, it did not prevent them to divide both sides of inequality equality (6) by $|\frac{d}{dt}\langle A\rangle_{\phi}|$. 

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that is by \( \frac{d}{dt} (A) \phi \equiv 0 \), without taking into account (11) and without any explanations. What is more, this shows that there is no reason to think of \( \tau_A \) as infinity in this case as it was done, e.g., in [2, 17]. In general, the problem is that usually the set \( \Sigma_H \) of the eigenvectors of the Hamiltonian \( H \) is a linearly dense (complete) set in the state space \( \mathcal{H} \). Hence the conclusion that such relations as (7) and then (9) can not be considered as correct and rigorous seems to be justified. Summing up, we have proved that contrary to the uncertainty relations (1) and (2), the relations of type (3) and (9) can not hold on linearly dense sets in the state space \( \mathcal{H} \) and therefore such relations can not be considered as the universally valid.

This conclusion agrees with the intuitive understanding of stationary states. The stationary states are represented by eigenvectors of the Hamiltonian \( H \) of the system considered and if it is known that the system is in a stationary state represented, say, by the state vector \( |\phi_\beta \rangle \) then one is sure that at any time \( t \) (and during any time interval \( \Delta t = t_2 - t_1, \) where \( t_1 < t_2 < \infty \)) the energy is equal \( E_\beta \) or that \( \Delta E = E_\beta(t_2) - E_\beta(t_1) \equiv 0. \)

Similar picture one meets when \( |\phi \rangle = |\phi_\alpha \rangle \) is an eigenvector for \( A \). (This case was also noticed in [17]). Then also for any \( |\phi_\alpha \rangle \in \Sigma_A \cap \mathcal{D}(H) \), (where by \( \Sigma_A \) we denote the set of eigenvectors \( |\phi_\alpha \rangle \) for \( A \) ), \( \frac{d}{dt} (A) \phi \equiv 0 \) and \( \Delta \phi A \equiv 0. \)

Thus, instead of (12) one once more has \( 0 \cdot \Delta \phi H = \frac{\hbar}{2} \cdot 0, \) and once again dividing both sides of this inequality by zero has no mathematical sense. Now note that the relations (1), (2) are always satisfied for all \( |\phi \rangle \in \mathcal{H} \) fulfilling the conditions specified before Eq. (2), and in contrast to this property, we have proved that the Mandestam–Tamm relation (7) can not be true not only on the set \( \Sigma_H \subset \mathcal{H} \), whose span is usually dense in \( \mathcal{H} \), but also on the set \( \Sigma_A \subset \mathcal{H} \).

A detailed analysis of relations (3), (9) suggests that they may be in conflict with one of the basic postulates of Quantum Mechanics: Namely, with the projection (reduction) postulate. It is because the projection postulate leads to the Quantum Zeno Effect [19] (see also, e.g., [20, 21, 22, 23]), that is it makes possible to force the system to stay in a given state as a result of continuous or quasi–continuous observations verifying if the system is in this given state. It is possible if time interval separating the successive measurements (observations) are separated by suitable short time intervals \( \Delta t \) such that \( \Delta t \to 0 \) when the number of observations increases [20 21 22 23]. In general the duration of each of these measurements must be shorter than the time interval separating them, and in turn, the uncertainty of the time
t can not be larger then the duration of these measurements. Therefore the conclusion that the relation (3) should make impossible to observe the Quantum Zeno Effect seems to be legitimate. Contrary to such a conclusion there are experimental tests verifying and confirming this effect [24]. The state of the system is characterized by a set of quantum numbers and one of these numbers is the energy of the system in the state considered. Therefore if the quantum system is forced to stay in the given state by continuously or quasi-continuously checking it if it is in this state, then quantum numbers characterizing this state (including the energy) also remain unchanged. This means that there is $\Delta E = 0$ and $\Delta t \to 0$ in such a case and thus there is a conflict with relations (1), (2).

As it was mentioned earlier there is a reservation in [6] that derivation of (9) does not go for eigenvectors of $H$ (Then $\Delta H = 0$). In fact it can be only applied for eigenvectors corresponding to the continuous part of the spectrum of $H$. As an example of possible applications of the relation (9) unstable states modeled by wave-packets of such eigenvectors of $H$ are considered in [6], where using (9) the relation connecting half-time $\tau_{1/2}$ of the unstable state, say $|\varphi\rangle$, with the uncertainty $\Delta_\varphi H$ was found: $\tau_{1/2} \cdot \Delta_\varphi H \geq \frac{\pi}{4} \hbar$. In general, when one considers unstable states such a relation and the similar one appear naturally [25, 26, 27] but this is quite another situation then that described by the relations (11), (12). The other example is a relation between a life-time $\tau_\varphi$ of the system in the unstable state, $|\varphi\rangle$, and the decay width $\Gamma_\varphi$: In such cases we have $\tau_\varphi \cdot \Gamma_\varphi = \hbar$ but there are not any uncertainties of the type $\Delta E$ and $\Delta t$ in this relation (see, e.g., [25]). Note that in all such cases the vector $|\varphi\rangle$ representing the unstable state can not be the eigenvector of the Hamiltonian $H$. It should be noted here that even in the case of unstable states one should be very careful using the relation (9): For example in the case of unstable states $|\phi\rangle$ modeled by the Breit-Wigner energy density distribution $\omega_{BW}(E) = \frac{N}{2\pi} \Theta(E - E_{\min}) \frac{\Gamma_0}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2}$, where $\Theta(E)$ is the unit step function and $N$ is the normalization constant, the average values $\langle H \rangle_\phi = \int_{E_{\min}}^{E_{\max}} E \omega_{BW}(E) dE$ and $\langle H^2 \rangle_\phi = \int_{E_{\min}}^{E_{\max}} E^2 \omega_{BW}(E) dE$ have not definite values and hence $\Delta_\varphi H$ is undefined which means that the relation (9) does not work in this case.

Concluding: The above discussion of relations (3) and (9) and the detailed analysis of the derivation of the relation (9) suggests that these time-energy uncertainty relations are not well founded and can not be considered as universally valid. Therefore when using these relations as the basis for
predictions of the properties and of a behavior of some systems in physics or astrophysics (including cosmology — see, e.g., [10, 28]) one should be very careful interpreting and applying results obtained. In general in some problems the use of the relation (9) may be reasonable (see, e.g. the case of unstable states) but then it should not be interpreted analogously to the relations (1), (2).

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