Superintegrable geodesic flows

on the hyperbolic plane

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Abstract

In the framework laid down by Matveev and Shevchishin, superintegrability is achieved with one integral linear in the momenta (a Killing vector) and two extra integrals of of any degree above two in the momenta. However these extra integrals may exhibit either a trigonometric dependence in the Killing coordinate (a case we have already solved) or a hyperbolic dependence and this case is solved here. Unfortunately the resulting geodesic flow is never defined on the two-sphere, as was the case for Koenigs systems (with quadratic extra integrals). Nevertheless we give some sufficient conditions under which the geodesic flow is defined on the hyperbolic plane.

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1 Introduction

Matveev and Shevchishin [5] have proposed an interesting approach to superintegrability on surfaces of revolution equipped with the hamiltonian

\[ H = h_x^2(P_x^2 + P_y^2), \quad h_x = D_x h(x). \] (1.1)

This system is integrable and to reach superintegrability we need some extra integral, let us say \( S_1 \), which implies in turn the existence of a second one, namely \( S_2 = \{ P_y, S_1 \} \).

Under appropriate hypotheses, they have shown that the \( y \) dependence of the extra integral (which was supposed to be cubic in the momenta) may be of three kinds:

1. In the so called “affine case” the extra integral can be quadratic:
   \[ S_1 = S_{1,0} + S_{1,1} y + S_{1,2} y^2. \]

2. In the second case the extra integral can be trigonometric
   \[ S_1 = \cos y S + \sin y T. \]

3. In the third case the extra integral can be hyperbolic
   \[ S_1 = \cosh y S + \sinh y T. \]

In fact a prior work by Koenigs [2], popularized and generalized in [3] and [4], involving a completely different analysis, had established that for a quadratic extra integral only the three issues stated above were possible. Unfortunately the geodesic flow of Koenigs superintegrable (SI) systems never meet \( S^2 \) as emphasized in [8].

We gave the solution for the affine case in [9] by reducing the problem to a linear ODE for which the general solution could be obtained explicitly.

We solved the trigonometric case in [10]: as foreseen by Matveev and Shevchishin in this case one runs into Zoll geometry (see [11] and [12] for this subject) and we could give sufficient conditions insuring that the geodesic flow is globally defined on \( M = S^2 \). This seems to be the most interesting situation.

The remaining hyperbolic case is the subject of this article, but, as we will see, despite the superintegrability of its geodesic flow it never meets \( S^2 \). Nevertheless, some sufficient conditions do give a quite large panel of solutions globally defined on \( \mathbb{H}^2 \).

This article has the following content: in Section 2 the main results are stated.

Then, in a First Part, we examine the case where \( \sharp(S) = 2n \geq 2 \), beginning with a presentation of the framework laid down by Matveev and Shevchishin in Section 3, followed by the proofs of Theorems 1 and 2 in Sections 4 and 5. In Section 6 and for quadratic extra integrals we relate our results with Koenigs ones.

We proceed to the Second Part dealing with integrals of order \( 2n + 1 \geq 3 \). In Section 7 the setting for this case is given and we prove Theorems 3 and 4 respectively in Sections 8 and 9.

The global issues are discussed in Section 10, where Theorem 5 is proved, and some concluding remarks are presented in Section 11.

The article ends up with Appendices A and B in which we prove several technical relations.
2 The results

As explained in the introduction we will consider the geodesic flow of the hamiltonian

\[ H = \Pi^2 + \frac{P_y^2}{\cosh^2 t}, \quad \Pi = \frac{P_t}{A(t)}. \]  

(2.1)

It does exhibit one linear integral \( P_y \), and to reach superintegrability we have to construct two extra integrals \( S_1 \) and \( S_2 \) which are polynomials in \( H \) and in \( P_y^2 \), of fixed degree, denoted by \( \sharp(S_1) = \sharp(S_2) \), in the momenta.

2.1 Integrals of even degree

Let us first consider the case where \( \sharp(S_1) = \sharp(S_2) = 2n \) with \( n \geq 1 \) and the set of integrals

\[ H, \quad P_\phi, \quad S_1 = \cosh y S + \sinh y T, \quad S_2 = \cosh y T - \sinh y S, \]  

(2.2)

where

\[ S = \sum_{k=0}^{n} \lambda_{2k-1}(\theta) H^{n-k} P_y^{2k}, \quad T = \Pi \sum_{k=0}^{n} \lambda_{2k}(t) H^{n-k} P_y^{2k}. \]  

(2.3)

We will prove:

**Theorem 1** The geodesic flow of the hamiltonian (2.1) is superintegrable if one takes

\[ A(t) = 1 + \sum_{k=1}^{2n-1} \frac{e_k \sinh t}{\sqrt{m_k \cosh^2 t - 1}}, \quad \forall k : \quad e_k = \pm 1, \]  

(2.4)

where all of the \( 2n - 1 \) real parameters \( m_k \) are restricted to \( m_k > 1 \).

**Theorem 2** The set of integrals

\[ H, \quad P_y, \quad S_\pm = S_1 \pm S_2, \]

generates a Poisson algebra, with the relations

\[ S_+ S_- = \sum_{k=0}^{2n} \sigma_k H^{2n-k} P_y^{2k}, \quad \{S_+, S_-\} = -2 \sum_{k=0}^{2n-1} (k + 1) \sigma_k H^{2n-k} P_y^{2k-1}, \]

(2.5)

and the coefficients \( \sigma_1 \) are explicitly given in terms of the coefficients \( m_k \), see [5,13].

2.2 Integrals of odd degree

In the case where \( \sharp(S_1) = \sharp(S_2) = 2n + 1 \) with \( n \geq 1 \), we will consider the set of integrals

\[ H, \quad P_y, \quad S_1 = \cosh y S + \sinh y T, \quad S_2 = \sinh y S + \cosh y T, \]  

(2.6)

where

\[ S = \Pi \sum_{k=0}^{n} \lambda_{2k-1}(\theta) H^{n-k} P_y^{2k}, \quad T = \sum_{k=0}^{n-1} \lambda_{2k}(\theta) H^{n-k-1} P_y^{2k+1}, \]  

(2.7)

and we get similar results:
Theorem 3  The geodesic flow of the Hamiltonian (2.1) is superintegrable if one takes

\[ A(t) = 1 + \sum_{k=1}^{2n} \frac{e_k \sinh t}{\sqrt{m_k \cosh^2 t - 1}}, \quad \forall k : \quad e_k = \pm 1, \tag{2.8} \]

where all of the \(2n\) real parameters \(m_k\) are restricted to \(m_k > 1\).

Theorem 4  The set of integrals

\[ H, \quad P_y, \quad S_{\pm} = S_1 \pm S_2, \]

generates a Poisson algebra, with the relations

\[ S_+ S_- = \sum_{l=0}^{2n} \sigma_l H^{2n-l} P_y^{2l}, \quad \{S_+, S_\pm\} = -2 \sum_{l=0}^{2n-1} (l + 1) \sigma_{l+1} H^{2n-1-l} P_y^{2l+1}, \tag{2.9} \]

and the coefficients \(\sigma_l\) are explicitly given in terms of the coefficients \(m_k\).

2.3 The global issues

They strongly depend on the degree of the integrals. We have:

Theorem 5  The superintegrable geodesic flows considered in this article:

a) Are never globally defined on \(S^2\).

b) Are globally defined on \(\mathbb{H}^2\) for quadratic integrals (Koenigs case) while for higher even degree integrals they are never globally defined.

c) Are globally defined on \(\mathbb{H}^2\), under appropriate restrictions on the parameters, for integrals of odd power in the momenta.

In all what follows the constraints \(m_k > 1\) will be assumed to hold.

I. INTEGRALS OF EVEN DEGREE

3 The setting

Let us turn ourselves to the general case where \(\sharp(S_1) = \sharp(S_2) = 2n \geq 2\). The Hamiltonian is

\[ H = \Pi^2 + \frac{P_y^2}{\cosh^2 t}, \quad \Pi = \frac{P_t}{A(t)}, \quad A(t) = 1 + \sum_{k=1}^{2n-1} \frac{e_k \sinh t}{\sqrt{m_k \cosh^2 t - 1}}, \tag{3.1} \]

and the extra integrals are given by

\[ S_1 = \cos y S + \sin y T, \quad S_2 = \cos y T - \sin y S, \tag{3.2} \]
where
\[ S = \sum_{k=0}^{n} \lambda_{2k-1}(t) H^{n-k} P_y^{2k}, \quad T = \prod_{k=0}^{n-1} \lambda_{2k}(t) H^{n-k-1} P_y^{2k+1}. \] (3.3)

The main problem is therefore to determine the following array of functions of \( t \)
\[ \lambda_{-2} = 0, \quad \lambda_{-1} = 1, \quad \left( \begin{array}{cccc} \lambda_1 & \lambda_3 & \ldots & \lambda_{2n-1} \\ \lambda_0 & \lambda_2 & \ldots & \lambda_{2(n-1)} \end{array} \right) = 0. \] (3.4)

The conventional values \( \lambda_{-2} = \lambda_{2n} = 0 \) are introduced to alleviate many formulae in the sequel.

### 4 Proof of Theorem 1

Let us begin with

**Proposition 1** \( S_1 \) and \( S_2 \) will be integrals iff the \( \lambda \)'s solve the differential system:
\[ \begin{array}{l}
\begin{align*}
0 \leq k \leq n : & \quad \cosh^2 t \lambda'_{2k-1} = -A \lambda_{2(k-1)} \\
& \quad \cosh^2 t \lambda'_{2k} = \lambda'_{2(k-1)} - \tanh t \lambda_{2(k-1)} - A \lambda_{2k-1},
\end{align*}
\end{array} \] (4.1)

taking into account the conventional values \( \lambda_{-2} = \lambda_{2n} = 0 \).

**Proof:** Both constraints \( \{H, S_1\} = 0 \) and \( \{H, S_2\} = 0 \) are seen to be equivalent to
\[ \{H, S\} = -2 \frac{P_y}{\cosh^2 t} T, \quad \{H, T\} = -2 \frac{P_y}{\cosh^2 t} S. \] (4.2)

Using the explicit form of \( S \) and \( T \) elementary computations give \( (4.1) \).

A simplifying approach to the differential system \( (4.1) \) makes use of generating functions, which encode all the array in a couple of objects:
\[ \mathcal{L}(t, \xi) = \sum_{k=0}^{n-1} \lambda_{2k}(t) \xi^k, \quad \mathcal{M}(t, \xi) = \sum_{k=0}^{n} \lambda_{2k-1}(t) \xi^k. \] (4.3)

Let us prove

**Proposition 2** The differential system given in Proposition 1 is equivalent, in terms of generating functions, to the set of partial differential equations
\[ \cosh^2 t (1 + \tau) \partial_t \mathcal{L} + \xi \tanh t \mathcal{L} + A \mathcal{M} = 0, \quad \partial_t \mathcal{M} - \tau A \mathcal{L} = 0, \] (4.4)

where \( \tau = -\frac{\xi}{\cosh^2 t} \).
Proof: Using relation (b) in (4.1) we have
\[
\cosh^2 t \partial_t \mathcal{L} = \sum_{k=0}^{n-1} \left( \lambda'_{2k-1} - \tanh t \lambda_{2k-1} - A \lambda_{2k-1} \right) \xi^k
\]
which becomes
\[
\sum_{l=0}^{n-1} \left( \lambda'_{2l} - \tanh t \lambda_{2l} \right) \xi^{l+1} - \left( \lambda'_{2(n-1)} - \tanh t \lambda_{2(n-1)} \right) \xi^n - A \left( \mathcal{M} - \lambda_{2n-1} \xi^n \right),
\]
and can be written
\[
\xi \partial_t \mathcal{L} - \xi \tanh t \mathcal{L} - A \mathcal{M} - \left( \lambda'_{2(n-1)} - \tanh t \lambda_{2(n-1)} - A \lambda_{2n-1} \right) \xi^n.
\]
So we have obtained
\[
\cosh^2 (1 + \tau) \partial_t \mathcal{L} + \xi \tanh t \mathcal{L} + A \mathcal{M} = \left( \lambda'_{2(n-1)} - \tanh t \lambda_{2(n-1)} - A \lambda_{2n-1} \right) \xi^n.
\]
The right hand member does vanish thanks to relation (b) for \( k = n \) in (4.1). Conversely, expanding this relation in powers of \( \xi \) one recovers relations (b) in (4.1).

Using relation (a) in (4.1) we have
\[
\partial_t \mathcal{M} = \sum_{k=1}^{n} \lambda'_{2k-1} \xi^k = -\frac{A}{\cosh^2 t} \sum_{l=0}^{n-1} \lambda_{2l} \xi^{l+1} = \tau A \mathcal{L},
\]
which was to be proved.

To get an explicit form of these functions we need:

**Definition 1** For \( n \geq 1 \) and \( k \in \{1, 2, \ldots, \nu\} \) let us introduce
\[
\forall k : \quad h_k(t) = e_k \sqrt{m_k \cosh t - 1}, \quad e_k = \pm 1, \quad m_k > 1,
\]
and let us define the \( H'_\nu(t, \xi) \) by the
\[
\mathcal{H'}(t, \xi) \equiv \prod_{k=1}^{\nu} (1 + \xi h_k(t)) = \sum_{k=0}^{\nu} H'_k(t) \xi^k.
\]

In Appendix A the reader will find useful relations for these functions to be used in the sequel.

In this first part with \( S = 2n \) we will take \( \nu = 2n - 1 \) and to simplify matters we will omit the upper index \( \nu = 2n - 1 \) of the functions \( H'_k \).

These functions allow to define the \( \lambda \)'s:

**Definition 2** Let us consider, for \( k \in \{0, 1, \ldots, n-1\} \)
\[
\lambda_{2k} = \frac{(-1)^{k+1}}{\cosh^{2k+1} t} \sum_{l=0}^{k} (-1)^l \binom{n - 1 - l}{n - 1 - k} \left( \mathcal{H}_{2l+1} + \sinh t \mathcal{H}_{2l} \right),
\]
5
for $k \in \{1, \ldots, n-1\}$\footnote{This set is empty for $n = 1$.}
\[
\lambda_{2k-1} = \frac{(-1)^k}{\cosh^2 t} \left\{ \sum_{l=0}^{k} (-1)^l \left( \frac{n-l}{n-k} \right) H_{2l} - \sinh t \sum_{l=0}^{k-1} \left( \frac{n-1-l}{n-k} \right) H_{2l+1} \right\}, \quad (4.13)
\]
and for $k = n$
\[
\lambda_{2n-1} = \frac{(-1)^n}{\cosh^2 t} \sum_{l=0}^{n-1} (-1)^l \left[ H_{2l} - \sinh t H_{2l+1} \right]. \quad (4.14)
\]

Let us compute the generating functions.

**Proposition 3** Defining
\[
\psi_{n,l} = \tau^l (1 + \tau)^{n-l}, \quad 0 \leq l \leq n, \quad (4.15)
\]
the generating functions are given by
\[
\mathcal{L}(t, \xi) = -\frac{1}{\cosh t} \sum_{l=0}^{n-1} (-1)^l \psi_{n-1,l} \left[ H_{2l+1} + \sinh t H_{2l} \right], \quad (4.16)
\]
and by
\[
\mathcal{M}(t, \xi) = \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l} - \sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}. \quad (4.17)
\]

**Proof:** From the definition of $\mathcal{L}$, given in (4.3), and upon use of the formulae in (4.12) we have
\[
-\cosh t \mathcal{L} = \sum_{k=0}^{n-1} \tau^k \sum_{l=0}^{k} (-1)^l \left( \frac{n-1-l}{n-1-k} \right) H_{2l+1} \left[ H_{2l+1} + \sinh t H_{2l} \right]. \quad (4.18)
\]
Reversing the order of the summations leads to
\[
-\cosh t \mathcal{L} = \sum_{l=0}^{n-1} (-1)^l \left[ H_{2l+1} + \sinh t H_{2l} \right] \sum_{k=l}^{n-1} \left( \frac{n-1-l}{k-l} \right) \tau^k, \quad (4.19)
\]
and since we have
\[
\sum_{k=l}^{n-1} \left( \frac{n-1-l}{k-l} \right) \tau^k = \sum_{K=0}^{n-1-l} \left( \frac{n-1-l}{K} \right) \tau^{K+l} = \tau^l (1 + \tau)^{n-1-l} = \psi_{n-1,l} \quad (4.20)
\]
the relation (4.16) is proved.

From the definition of $\mathcal{M}$, given in (4.3), and upon use of the formulae in (4.13) and (4.14) we have
\[
\mathcal{M} = \sum_{k=0}^{n} \tau^k \sum_{l=0}^{k} (-1)^l \left( \frac{n-l}{n-k} \right) H_{2l} - \sinh t \sum_{k=1}^{n} \tau^k \sum_{l=0}^{k-1} (-1)^l \left( \frac{n-1-l}{n-k} \right) H_{2l+1}. \quad (4.21)
\]
Reversing the summations gives
\[
\mathcal{M} = \sum_{l=0}^{n} (-1)^l H_{2l} \sum_{k=l}^{n} \binom{n-k}{l} \tau^k - \sinh t \sum_{l=0}^{n-1} (-1)^l H_{2l+1} \sum_{k=l+1}^{n} \binom{n-1-k}{l} \tau^k. \quad (4.22)
\]

Using the binomial theorem, as explained in (4.20), gives
\[
\sum_{k=l}^{n} \binom{n-k}{l} \tau^k = \psi_{n,l}, \quad \sum_{k=l+1}^{n} \binom{n-1-k}{l} \tau^k = \psi_{n,l+1}, \quad (4.23)
\]
which proves (4.17).

Now let us use Proposition 3 to prove Proposition 4

The generating functions \( \mathcal{L} \) and \( \mathcal{M} \) obtained in Proposition 3 are solutions of the partial differential equations given in Proposition 2, namely:
\[
\begin{align*}
(a) : \quad & \cosh^2 t (1 + \tau) \partial_t \mathcal{L} + \xi \tanh t \mathcal{L} + A \mathcal{M} = 0, \\
(b) : \quad & \partial \mathcal{M} - \tau A \mathcal{L} = 0.
\end{align*}
\quad (4.24)
\]
Proof of relation (a): Let us define the splitting
\[
\mathcal{L}_1 = -\frac{1}{\cosh t} \sum_{l=0}^{n-1} (-1)^l \psi_{n-1,l} H_{2l+1}, \quad \mathcal{L}_2 = -\tanh t \sum_{l=0}^{n-1} (-1)^l \psi_{n-1,l} H_{2l} \quad (4.25)
\]
and similarly
\[
\mathcal{M}_1 = \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l}, \quad \mathcal{M}_2 = -\sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}. \quad (4.26)
\]

In the sequel we will need the easily proved relations, valid for \( l = 0, 1, \ldots, n-1 \):
\[
\begin{align*}
(a) : \quad & \cosh t (1 + \tau) \partial_t \psi_{n-1,l} = -\sinh t \left[ 2l \psi_{n,l} + 2(n-l-1) \psi_{n,l+1} \right], \\
(b) : \quad & (1 + \tau) \psi_{n-1,l} = \psi_{n,l}, \\
(c) : \quad & \tau \psi_{n-1,l} = \psi_{n,l+1}. \quad (4.27)
\end{align*}
\]
Using relation (a) and the Proposition [A.3] in Appendix A, we get for \( \cosh^2 t (1 + \tau) \partial_t \mathcal{L}_1 \):
\[
\begin{align*}
& \sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l+1} + \sinh t \sum_{l=0}^{n-1} (-1)^l \left[ 2l \psi_{n,l} + 2(n-l-1) \psi_{n,l+1} \right] H_{2l+1} \\
& \quad - \sinh t \sum_{l=0}^{n-1} (-1)^l \left[ (2l + 1) H_{2l+1} + (2l - 2n) H_{2l-1} + \frac{(A - 1)}{\sinh t} H_{2l} \right] \psi_{n,l}. \quad (4.28)
\end{align*}
\]
All the terms involving $\psi_{n,l} H_{2l+1}$ add up to zero. The term involving $A - 1$ is nothing but $-(A - 1)M_1$ and the remaining terms
\[ \sinh t \left( \sum_{l=0}^{n-1} (-1)^l (2n - 2l - 2)\psi_{n,l+1}H_{2l+1} - \sum_{l=1}^{n} (-1)^l(2l - 2n)\psi_{n,l}H_{2l-1} \right) \]
add up to zero. Hence we have proved
\[ \cosh^2 t (1 + \tau) \partial_t L_1 = -(A - 1)M_1. \]
The relation (c) in (4.27) implies that $\xi \tanh t L_1 = -M_2$ so we conclude to
\[ \cosh^2 t (1 + \tau) \partial_t L_1 + \xi \tanh t L_1 = -AM_1 + M_1 - M_2. \]
Similarly one can prove
\[ \cosh^2 t (1 + \tau) \partial_t L_2 + \xi \tanh t L_2 = -AM_2 - M_1 + M_2. \]
Adding up (4.31) and (4.32) gives
\[ \cosh^2 t (1 + \tau) \partial_t L_1 + \xi \tanh t L_1 + AM = 0, \]
which is the relation (a) to be proved in Proposition 4.

**Proof of relation (b):** Using the easy relation
\[ \partial_t \psi_{n,l} = -\tanh t \left[ 2l \psi_{n,l} + 2(n - l)\psi_{n,l+1} \right], \quad 0 \leq l \leq n, \]
and Proposition A.3 we have
\[ \frac{1}{\tanh t} \partial_t M_1 = -\sum_{l=0}^{n-1} (-1)^l \left[ 2l \psi_{n,l} + 2(n - l)\psi_{n,l+1} \right] H_{2l} + \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} 2l H_{2l} \]
\[ + \sum_{l=1}^{n-1} (-1)^l \psi_{n,l} \left[ (2l - 2n - 1)H_{2l-1} + \frac{(A - 1)}{\sinh t} H_{2l-1} \right]. \]
The terms involving $\psi_{n,l} H_{2l}$ add up to zero. In the second line the missing term for $l = n$
\[ - H_{2(n-1)} + \frac{(A - 1)}{\sinh t} \]
does vanish using (A.4), so let us add it. Observing that
\[ \tau L_1 = -\frac{1}{\cosh t} \sum_{l=0}^{n-1} (-1)^l \tau \psi_{n-1,l} H_{2l+1} = -\frac{1}{\cosh t} \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}, \]
the factor of $A - 1$ becomes
\[ \frac{1}{\sinh t} \sum_{l=1}^{n} (-1)^l \psi_{n,l} H_{2l-1} = -\frac{1}{\sinh t} \sum_{L=0}^{n-1} (-1)^l \psi_{n,L+1} H_{2L+1} = \frac{1}{\tanh t} \tau L_1. \]
The remaining terms are

\[
- \sum_{l=0}^{n-1} (-1)^l 2(n-l) \psi_{n,l+1} H_{2l} + \sum_{l=1}^{n} (-1)^l \psi_{n,l} (2l - 2n - 1) H_{2(l-1)} = \\
= - \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l} - \sum_{l=0}^{n-1} (-1)^l \tau \psi_{n-1,l} H_{2l} = \frac{1}{\tanh t} \tau \mathcal{L}_2.
\]

So we have obtained

\[\partial_t M_1 = A \tau \mathcal{L}_1 - \tau \mathcal{L}_1 + \tau \mathcal{L}_2.\] (4.40)

Similarly one can prove

\[\partial_t M_2 = A \tau \mathcal{L}_2 + \tau \mathcal{L}_1 - \tau \mathcal{L}_2.\] (4.41)

Adding (4.40) and (4.41) ends up the proof of relation (b) in Proposition 4.

The generating functions \(\mathcal{L}\) and \(M\), given by (4.16) and (4.17), are solutions of the system of equations (4.24). It follows, from Proposition 2 that \(S_1\) and \(S_2\) are indeed integrals for the hamiltonian (3.1). Its geodesic flow is therefore superintegrable.

This concludes the proof of Theorem 1.

5 Proof of Theorem 2

5.1 Computing the moments

Let us define

\[S_+ = S_1 + S_2 = e^y (S + T), \quad S_- = S_1 - S_2 = e^{-y} (S - T).\] (5.1)

It follows that we can define the moments \(\sigma_k\) according to

\[S_+ S_- = S^2 - T^2 = \sum_{k=0}^{2n} \sigma_k H^{2n-k} P_y^{2k}.\] (5.2)

The moments are mere constants since \(S_+ S_-\) is an integral, and we have

**Proposition 5** The moments are given in terms of the \(\lambda\)'s by the relations

\[
0 \leq k \leq n : \quad \sigma_k = \sum_{l=0}^{k} S_{l,k-l} \quad (\sigma_0 = 1) \\
n + 1 \leq k \leq 2n : \quad \sigma_k = \sum_{l=k-n}^{n} S_{l,k-l}
\]

where

\[S_{l,k} = \lambda_{2l-1} \lambda_{2k-1} - \lambda_{2l} \lambda_{2(k-1)} + \frac{\lambda_{2(l-1)} \lambda_{2(k-1)}}{\cosh^2 t}.\] (5.4)
Proof: Elementary computations involving products of finite series. □

In order to get more insight into the moments let us prove:

Proposition 6 The generating function of the moments, defined by
\[
\Sigma(\xi) = \sum_{k=0}^{2n} \sigma_k \xi^k
\] (5.5)
is given in terms of the generating functions by
\[
\Sigma = \mathcal{M}^2 - \xi(1 + \tau)\mathcal{L}^2.
\] (5.6)

Proof: Using the relations given in (5.3) we have
\[
\Sigma(\xi) = \sum_{l=0}^{n} \xi^l \sum_{k=0}^{l} S_{k,l} + \sum_{k=0}^{2n} \xi^k \sum_{l=k}^{n} S_{k,l} - k
\]
(5.7)
Reversing the order of the summations gives
\[
\Sigma = \sum_{k=0}^{n} \xi^k \sum_{L=0}^{n} S_{k,L} \xi^L = \sum_{k=0}^{n} \xi^k \sum_{L=0}^{n} \left(\lambda_{2k-1} \lambda_{2L-1} - \lambda_{2k} \lambda_{2(L-1)} + \frac{\lambda_{2(k-1)} \lambda_{2(L-1)}}{\cosh^2 t}\right) \xi^L
\] (5.9)
Taking into account the conventional values: \( \lambda_{-4} = \lambda_{-2} = \lambda_{2n} = 0 \), let us compute the first term
\[
\sum_{k=0}^{n} \lambda_{2k-1} \xi^k \sum_{L=0}^{n} \lambda_{2L-1} \xi^L = \mathcal{M}^2
\] (5.10)
The second term is
\[
-\sum_{k=0}^{n-1} \lambda_{2k} \xi^k \sum_{L=1}^{n} \lambda_{2(L-1)} \xi^L = -\mathcal{L} \sum_{l=0}^{n-1} \lambda_{2l} \xi^{l+1} = -\xi \mathcal{L}^2
\] (5.11)
while the last term is
\[
\frac{1}{\cosh^2 t} \sum_{k=1}^{n} \lambda_{2(k-1)} \xi^k \sum_{L=1}^{n} \lambda_{2(L-1)} \xi^L = \frac{1}{\cosh^2 t} \sum_{K=0}^{n-1} \lambda_{2K} \xi^{K+1} \sum_{l=0}^{n} \lambda_{2l} \xi^{l+1} = -\xi \tau \mathcal{L}^2
\] (5.12)
which concludes the proof. □

We are now in position to establish
Proposition 7  We have the explicit relation

\[ \Sigma(\xi) = (1 - \xi) \prod_{k=1}^{2n-1} (1 - m_k \xi). \]  

(5.13)

Using the symmetric functions \((M)_k\) of the masses \(m_k\), defined by

\[ \prod_{k=1}^{2n-1} (1 - m_k \xi) = \sum_{k=0}^{2n-1} (-1)^k (M)_k \xi^k, \]

one can express the moments according to

\[ \sigma_0 = 1, \]

\[ \sigma_k = (-1)^k \left[ (M)_k + (M)_{k-1} \right], \quad 1 \leq k \leq 2n - 1, \]

(5.14)

\[ \sigma_{2n} = (M)_{2n-1} = \prod_{k=1}^{2n-1} m_k. \]

Proof: The second part of this Proposition is trivial: expanding \(\Sigma(\xi)\) in powers of \(\xi\) gives (5.14).

In order to prove (5.13) we need a new writing of the generating functions. Let us begin with

\[ \mathcal{M} = \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l} - \sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}. \]

(5.15)

For \(\xi < 0\) one can define \(\eta = \sqrt{\frac{\tau}{1 + \tau}}\) which gives

\[ \mathcal{M} = (1 + \tau)^n \left( \sum_{l=0}^{n-1} (-1)^l \eta^{2l} H_{2l} - \eta \sinh t \sum_{l=0}^{n-1} (-1)^l \eta^{2l+1} H_{2l+1} \right). \]

(5.16)

Defining

\[ \mathcal{H}_\pm = \mathcal{H}(t, \pm i\eta) = \prod_{k=1}^{2n-1} \left( 1 \pm i\eta h_k(t) \right), \quad h_k = e_k \sqrt{m_k \cosh^2 t - 1} \]

(5.17)

and using the relations given in (A.11) of Appendix A, we get

\[ \mathcal{M} = \frac{(1 + \tau)^n}{2} \left( (1 + i\eta \sinh t)\mathcal{H}_+ + (1 - i\eta \sinh t)\mathcal{H}_- \right). \]

(5.18)

Similarly we have

\[ \mathcal{L} = i \frac{(1 + \tau)^{n-1}}{2\eta \cosh t} \left( (1 + i\eta \sinh t)\mathcal{H}_+ - (1 - i\eta \sinh t)\mathcal{H}_- \right). \]

(5.19)
Plugging these relations into (5.6) we deduce
\[ \Sigma(\xi) = (1 + \eta^2 \sinh^2 t) (1 + \tau)^{2n} \mathcal{H}_+ \mathcal{H}_-, \] (5.20)
and the relations
\[ 1 + \eta^2 \sinh^2 t = \frac{1 - \xi}{1 + \tau}, \quad (1 + \tau)^{2n} \mathcal{H}_+ \mathcal{H}_- = (1 + \tau) \prod_{k=1}^{2n-1} (1 - \xi m_k), \] (5.21)
prove (5.13) for \( \xi < 0 \). Since \( \Sigma \) is a polynomial the result is valid for any complex \( \xi \). □

### 5.2 Poisson algebra

To establish the Poisson algebra structure we need to compute
\[ -\frac{1}{2} \{ S_+, S_- \} = \frac{1}{2} \frac{\partial}{\partial P_y} (S^2 - T^2) + \{ S, T \}. \] (5.22)
The first term is
\[ \sum_{k=1}^{2n} k \sigma_k H^{2n-k} P_y^{2k-1} + \frac{1}{\cosh^2 t} \sum_{k=0}^{2n} (2n - k) \sigma_k H^{2n-1} P_y^{2k+1}, \] (5.23)
so if we prove the relation
\[ \cosh^2 t \{ S, T \} + \sum_{k=0}^{2n} (2n - k) \sigma_k H^{2n-k-1} P_y^{2k+1} = 0, \] (5.24)
we will have proved Theorem 2
\[ \{ S_+, S_- \} = -2 \sum_{k=0}^{2n-1} (k + 1) \sigma_{k+1} H^{2n-k-1} P_y^{2k+1}. \] (5.25)

Let us proceed to

**Proposition 8** The relation (5.24) does hold true.

**Proof:** Let us compute the Poisson bracket
\[ \{ \mathcal{S}, \mathcal{T} \} = \sum_{k,l} P_y^{2(k+l)-1} \{ \lambda_{2k-1} H^{n-k}, \lambda_{2l-1} \Pi H^{n-l} \}. \] (5.26)
Introducing the notation \( \Psi_{k+l}^{2n} = H^{2n-k-1-1} P_y^{2(k+l)-1} \) we get for the right hand side \(^3\)
\[ \sum_{k,l} \Psi_{k+l}^{2n} (n-k) \lambda_{2k-1} \{ H, \lambda_{2l-1} \} \Pi + (n-k) \lambda_{2k-1} \lambda_{2l-1} \{ H, \Pi \} - \\
-(n-k) \lambda_{2(k-1)} \{ H, \lambda_{2l-1} \} \Pi - \lambda_{2(k-1)} \{ \Pi, \lambda_{2l-1} \} H. \] (5.27)

\(^3\)Any mute index is summed from 0 to \( n \).
The remaining brackets are all elementary and lead to

\[
\sum_{k,l} \Psi_{k,l}^{2n} \left( 2(n - k) \lambda_{2k-1} \frac{\lambda_2^{(l-1)}}{A} + 2(n - k) \frac{\lambda_2^{(k-1)} \lambda_{2(l-1)}}{\cosh^2 t} \right) \Pi^2 + \frac{2(n - k) \tanh t \lambda_2^{(k-1)} \lambda_{2(l-1)}}{\cosh^2 t} \frac{P_y^2}{A} + \frac{\lambda_2^{(k-1)} \lambda_{2(l-1)}}{\cosh^2 t} H^2.
\]

(5.28)

Getting rid of \(\Pi^2 = H - \frac{P_y^2}{\cosh^2 t}\) and upon multiplication by \(\cosh^2 t\) one obtains

\[
\sum_{k,l} \Psi_{k,l}^{2n} \left[ 2(n - k) \lambda_{2k-1} \frac{\cosh^2 t \lambda_{2(l-1)}}{A} + (2n - 2k + 1) \lambda_2^{(k-1)} \lambda_{2(l-1)} \right] + \sum_{k,l} \Psi_{k,l}^{2n} \left[ 2(n - k) \frac{\lambda_{2k-1}}{A} \left( -\lambda_{2(l-1)} + \tanh t \lambda_{2(l-1)} \right) - 2(n - k) \frac{\lambda_2^{(k-1)} \lambda_{2(l-1)}}{\cosh^2 t} \right].
\]

(5.29)

Let us consider the first term

\[
\sum_{k} 2(n - k) \lambda_{2k-1} \sum_{l=1}^{n} \Psi_{k+l}^{2n} \frac{\cosh^2 t \lambda_{2(l-1)}}{A} = \sum_{k} 2(n - k) \lambda_{2k-1} \sum_{L=0}^{n} \Psi_{k+L}^{2n} \frac{\cosh^2 t \lambda_{2L}}{A}.
\]

(5.30)

Adding it to the similar terms in the second line of (5.29) we get

\[
\sum_{k,l} \Psi_{k+l}^{2n} 2(n - k) \frac{\lambda_{2k-1}}{A} \left( \cosh^2 t \lambda_{2(l-1)}' - \lambda_{2(l-1)}' + \tanh t \lambda_{2(l-1)} \right) = -\sum_{k,l} \Psi_{k+l}^{2n} 2(n - k) \lambda_{2k-1} \lambda_{2l-1}.
\]

(5.31)

upon use of relation (b) in (4.1). Taking into account the \(k \leftrightarrow l\) symmetry we can write the remaining terms

\[
\sum_{k,l} \Psi_{k+l}^{2n} (2n - k - l + 1) \lambda_{2(k-1)} \lambda_{2(l-1)} - \sum_{k,l} \Psi_{k+l}^{2n} (2n - k - l) \left( \lambda_{2k-1} \lambda_{2l-1} \frac{\lambda_{2(k-1)} \lambda_{2(l-1)}}{\cosh^2 t} \right).
\]

(5.32)

The first term becomes

\[
\sum_{k,l} \Psi_{k+l}^{2n} (2n - K - l) \lambda_{2K} \lambda_{2(l-1)},
\]

(5.33)

recalling the definition of \(S_{k,l}\), given by (5.4), we conclude to

\[
cosh^2 t \{S, T\} = -\sum_{k,l} (2n - k - l) S_{k,l} H^{2n-k-l-1} P_y^{2(k+l)+1}.
\]

(5.34)

Setting \(L = l + k\) we have

\[
cosh^2 t \{S, T\} = -\sum_{k=0}^{n} \sum_{L=k}^{n+k} (2n - L) S_{k,L-k} H^{2n-L-1} P_y^{2L+1}.
\]

(5.35)
and reversing the summations gives

\[
\cosh^2 t \{S, T\} = -\sum_{L=0}^{n} (2n - L) H^{2n-L-1} P_y^{2L+1} \sum_{k=0}^{L} S_{k,L-k} - \sum_{L=n+1}^{2n} (2n - L) H^{2n-L-1} P_y^{2L+1} \sum_{k=L-n}^{n} S_{k,L-k}.
\]  

(5.36)

Thanks to relations (5.3) we have

\[
\cosh^2 t \{S, T\} = -\sum_{l=0}^{2n} (2n - l) \sigma_l H^{2n-l-1} P_y^{l+1},
\]  

(5.37)

which was to be proved. □

This concludes the proof of Theorem 2. □

6 Relation with Koenigs

For \( n = 1 \) the integrals are quadratic in the momenta. This case was first solved by Koenigs in [2]:

\[
\begin{align*}
H_K &= \frac{\sin^2 x}{1 - \rho \cos x} (P_x^2 + P_y^2), \\
S_1^K &= \cosh y \left( \frac{\rho}{2} H - \cos x P_y^2 \right) - \sinh y \sin x P_x P_y.
\end{align*}
\]  

(6.1)

As shown in [6], taking \( x \in (0, \pi) \) and \( y \in \mathbb{R} \) and defining \( e^\chi = \tan(x/2) \), we obtain

\[
\begin{align*}
H_K &= \frac{1}{1 + \rho \tanh \chi} \left( \frac{P_x^2}{\cosh^2 \chi} + \frac{P_y^2}{\cosh^2 \chi} \right), \quad (\chi, y) \in \mathbb{R}^2, \quad \rho \in (0, +1), \\
S_1^K &= \cosh y \left( \frac{\rho}{2} H + \tanh \chi P_y^2 \right) - \sinh y P_x P_y.
\end{align*}
\]  

(6.2)

Let us relate these results with our work. In Koenigs case the extra integral is quadratic so \( n = 1 \) and Theorem 1 gives for hamiltonian

\[
H = \Pi^2 + \frac{P_y^2}{\cosh^2 t}, \quad \Pi = \frac{P_t}{A},
\]  

(6.3)

where

\[
A = 1 + \frac{\sinh t}{\sqrt{m \cosh^2 t - 1}}, \quad (t, y) \in \mathbb{R}^2, \quad m > 1.
\]  

(6.4)

The first extra integral is

\[
S_1 = \cosh y \left( H + \lambda_1 P_y^2 \right) + \sinh y \lambda_0 \Pi P_y,
\]  

(6.5)
and Definition 2 gives

\[ \lambda_0 = -\frac{1}{\cosh t}(H_1 + \sinh t), \quad \lambda_1 = -\frac{1}{\cosh t}(1 - \sinh t H_1). \]  

(6.6)

Here we have \( H_1 = \sqrt{m \cosh^2 t - 1} \) so that

\[ S_1 = \cosh y \left( H + \left( \frac{\sinh t \sqrt{m \cosh^2 t - 1}}{\cosh^2 t} \right) P_y \right) - \sinh y \frac{\sinh t + \sqrt{m \cosh^2 t - 1}}{\cosh t} \Pi P_y. \]  

(6.7)

Let us describe the diffeomorphism relating \( H_K \) and \( H \):

**Proposition 9** The correspondence

\[ e^{\chi} = \frac{\sqrt{m \sinh t + \sqrt{m \cosh^2 t - 1}}}{\sqrt{m + 1}}, \quad \chi \in \mathbb{R}, \]  

(6.8)

implies the relations

\[ H_K = \frac{H}{\mu^2}, \quad S_1^K = S_1, \quad \mu = \sqrt{\frac{m}{m + 1}}. \]  

(6.9)

**Proof:** The relation \( g_K = \mu^2 g \) is equivalent to

\[ (a) : \sqrt{1 + \rho \tanh \chi} \cosh \chi = \mu \cosh t \quad \& \quad (b) : \sqrt{1 + \rho \tanh \chi} d\chi = \mu A dt. \]  

(6.10)

Taking \( \rho = \frac{2\sqrt{m}}{m + 1} \) the relation \( (a) \) is purely algebraic and has for solution

\[ e^{2\chi} = \frac{2m \cosh^2 t - m - 1 + 2\sqrt{m} \sinh t a(t)}{(\sqrt{m} + 1)^2}. \]

Taking its square root gives (6.8) and \( \frac{d\chi}{dt} = \frac{\sqrt{m} \cosh t}{\sqrt{m} \cosh^2 t - 1} \). A somewhat hairy computation gives then

\[ \sqrt{m + 1} \sqrt{1 + \rho \tanh \chi} = \frac{\sinh t + \sqrt{m \cosh^2 t - 1}}{\cosh t}, \]

which shows that the relation \( (b) \) is identically true. \( \square \)
II. INTEGRALS OF ODD DEGREE

7 The setting

Here $\sharp(S_1) = \sharp(S_2) = 2n + 1 \geq 3$. Let us recall that the hamiltonian is

$$H = \Pi^2 + \frac{P_y^2}{\cosh^2 t}, \quad \Pi = \frac{P_y}{A(t)}, \quad A(t) = 1 + \sum_{k=1}^{2n} \frac{e_k \sinh t}{\sqrt{m_k \cosh^2 t - 1}} \quad (7.1)$$

and the extra integrals

$$S_1 = \cosh y S + \sinh y T, \quad \quad S_2 = \{P_y, S_1\} = \cosh y T + \sinh y S. \quad (7.2)$$

The following array of functions of $t$:

$$\lambda_{-1} = 1, \quad \left( \begin{array}{cccc} \lambda_1 & \lambda_3 & \ldots & \lambda_{2n} \\ \lambda_0 & \lambda_2 & \ldots & \lambda_{2n-1} \end{array} \right), \quad \lambda_{2n+1} = 0,$$

allows to define the building blocks of $S_1$ and $S_2$:

$$S = \Pi \sum_{k=0}^{n} \lambda_{2k-1}(t) H^{n-k} P_y^{2k}, \quad \quad T = \sum_{k=0}^{n} \lambda_{2k}(t) H^{n-k} P_y^{2k+1}. \quad (7.3)$$

8 Proof of Theorem 3

Let us begin with:

**Proposition 10** $S_1$ and $S_2$ will be integrals if and only if the $\lambda$’s solve the differential system:

$$0 \leq k \leq n: \quad \begin{cases} (a) : & \cosh^2 t \lambda'_{2k} = -A \lambda_{2k-1}, \\ (b) : & \cosh^2 t \lambda'_{2k+1} = \lambda'_{2k-1} - \tanh t \lambda_{2k-1} - A \lambda_{2k}. \end{cases} \quad (8.1)$$

**Proof:** Similar to the proof of Proposition 1. \hfill \square

Let us define the generating functions defined by

$$\mathcal{L}(t, \xi) = \sum_{k=0}^{n} \lambda_{2k}(t) \xi^k, \quad \mathcal{M}(t, \xi) = \sum_{k=0}^{n} \lambda_{2k-1}(t) \xi^k, \quad \xi \in \mathbb{C}, \quad (8.2)$$

and let us prove

**Proposition 11** The differential system in Proposition 10 is equivalent to the partial differential system for the generating functions:

$$\cosh^2 t \partial_t \mathcal{L} + A \mathcal{M} = 0, \quad \cosh^2 t (1 + \tau) \partial_t \mathcal{M} + \xi \tanh t \mathcal{M} + \xi A \mathcal{L} = 0, \quad (8.3)$$

where $\tau = -\frac{\xi}{\cosh^2 t}$. 

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Proof: Using relation (a) in (8.20) we have
\[ \cosh^2 t \partial_t L = -A \sum_{k=0}^{n} \lambda_{2k-1} \xi^k = -A M. \] (8.4)

Conversely, expanding this relation in powers of \( \xi \) gives relation (a) in (8.20).

Using relation (b) in (8.20) we have
\[ \cosh^2 t \partial_t M = -\sum_{k=0}^{n-1} \lambda_{2k} + \sum_{k=0}^{n} \xi^{k+1} \left( \lambda'_{2k-1} - \tanh t \lambda_{2k-1} - A \lambda_{2k} \right), \] (8.5)
or in terms of the generating functions
\[ \cosh^2 t \partial_t M = \xi \left( \partial_t M - \tanh t M - A L \right) - \xi^{n+1} \left( \lambda'_{2n-1} - \tanh t \lambda_{2n-1} - A \lambda_{2n} \right), \] (8.6)
leading to
\[ \cosh^2 t (1 + \tau) \partial_t M + \xi \tanh t M + \xi A L = -\xi^{n+1} \left( \lambda'_{2n-1} - \tanh t \lambda_{2n-1} - A \lambda_{2n} \right). \] (8.7)
The right hand side of this relation does vanish due to relation (b) for \( k = n \). Conversely, expanding this relation in powers of \( \xi \) one recovers relation (b).

We will need again the functions of Definition 1, but in the case \( \sharp(S) = 2n \) considered in this second part, we will take \( \nu = 2n - 1 \) and to simplify matters we will omit the upper index \( \nu = 2n - 1 \) of the functions \( H_{2n-1}^k \) from the formulae.

These functions allow to define the \( \lambda \)'s:

**Definition 3** For \( k \in \{0, 1, \ldots, n - 1\} \), let us take for the functions defining \( S \):
\[ \lambda_{2k} = \frac{(-1)^{k+1}}{\cosh^{2k+1} t} \sum_{l=0}^{k} (-1)^l \binom{n-l}{n-k} \left[ \frac{H_{2l+1}^{2k+1} + \sinh t \frac{H_{2l+1}}{H_{2l}}} \right], \] (8.8)
and
\[ \lambda_{2n} = \frac{(-1)^{n+1}}{\cosh^{2n+1} t} \left[ \sum_{l=0}^{n-1} (-1)^l \frac{H_{2l+1}^{2n+1} + \sinh t \sum_{l=0}^{n} (-1)^l \frac{H_{2l}}{H_{2l+1}}} \right]. \] (8.9)
The functions needed for \( T \) are, for \( k \in \{1, 2, \ldots, n - 1\} \), given by
\[ \lambda_{2k-1} = \frac{(-1)^k}{\cosh^{2k} t} \left[ \sum_{l=0}^{k} (-1)^l \binom{n-l}{n-k} \frac{H_{2l} - \sinh t \sum_{l=0}^{k-1} (-1)^l \binom{n-l}{n-k} \frac{H_{2l+1}}{H_{2l}}} \right], \] (8.10)
and
\[ \lambda_{2n-1} = \frac{(-1)^n}{\cosh^{2n} t} \left[ \sum_{l=0}^{n-1} (-1)^l \frac{H_{2l} - \sinh t \sum_{l=0}^{n-1} (-1)^l \frac{H_{2l+1}}{H_{2l+1}}} \right]. \] (8.11)

Now it is possible to compute the generating functions \( L \) and \( M \) :
Proposition 12 Defining
\[ \psi_{n,l} = \tau^l (1 + \tau)^{n-l}, \quad 0 \leq l \leq n, \quad (8.12) \]
the generating functions are given by
\[ - \cosh t L (t, \xi) = \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l+1} + \sinh t \sum_{l=0}^{n} (-1)^l \psi_{n,l} H_{2l} \quad (8.13) \]
and by
\[ M(t, \xi) = \sum_{l=0}^{n} (-1)^l \psi_{n,l} H_{2l} - \sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}. \quad (8.14) \]

Proof: We have for the first generating function
\[ - \cosh t L = \sum_{k=0}^{n-1} \tau^k \sum_{l=0}^{k} (-1)^l \binom{n-l}{n-k} [H_{2l+1} + \sinh t H_{2l}] + + \tau^n \sum_{l=0}^{n-1} (-1)^l [H_{2l+1} + \sinh t H_{2l}]. \quad (8.15) \]
Reversing the summations in the first term we get
\[ - \cosh t L = \sum_{l=0}^{n-1} (-1)^l [H_{2l+1} + \sinh t H_{2l}] \sum_{k=l}^{n} \binom{n-l}{n-k} \tau^k, \quad (8.16) \]
and the first relation in (4.23) concludes the proof.

For the second generating function we have
\[ M = \sum_{k=0}^{n} \tau^k \sum_{l=0}^{k} (-1)^l \binom{n-l}{n-k} H_{2l} - \sinh t \sum_{k=1}^{n} \tau^k \sum_{l=0}^{k-1} (-1)^l \binom{n-1-l}{n-k} H_{2l+1}, \quad (8.17) \]
and the first term, thanks to the first relation in (4.23), gives
\[ \sum_{l=0}^{n} (-1)^l H_{2l} \sum_{k=l}^{n} \binom{n-l}{n-k} \tau^k = \sum_{l=0}^{n} (-1)^l \psi_{n,l} H_{2l}. \quad (8.18) \]
The second piece gives, thanks to the second relation in (4.23):
\[ - \sinh t \sum_{l=0}^{n-1} (-1)^l H_{2l+1} \sum_{k=l+1}^{n} \binom{n-1-l}{k-l-1} \tau^k = - \sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}, \quad (8.19) \]
which was to be proved. □

Let us proceed to
Proposition 13  The generating functions obtained in Proposition 12 are solutions of the partial differential equations given in Proposition 11, namely

\[
\begin{align*}
(a) : \quad & \cosh^2 t \partial_t \mathcal{L} + A\mathcal{M} = 0, \\
(b) : \quad & \cosh^2 t(1 + \tau) \partial_t \mathcal{M} + \xi \tanh t \mathcal{M} + \xi A \mathcal{L} = 0,
\end{align*}
\]

(8.20)

Proof of relation (a): Let us define the splittings

\[
\begin{align*}
\mathcal{L}_1 &= \frac{1}{\cosh t} \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l+1}, \quad \mathcal{L}_2 = -\tanh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l}, \\
\mathcal{M}_1 &= \sum_{l=0}^{n} (-1)^l \psi_{n,l} H_{2l}, \quad \mathcal{M}_2 = -\sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}.
\end{align*}
\]

(8.21)

and

\[
\begin{align*}
\mathcal{L}_1 &= \frac{1}{\cosh t} \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l+1}, \quad \mathcal{L}_2 = -\tanh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l}, \\
\mathcal{M}_1 &= \sum_{l=0}^{n} (-1)^l \psi_{n,l} H_{2l}, \quad \mathcal{M}_2 = -\sinh t \sum_{l=0}^{n-1} (-1)^l \psi_{n,l+1} H_{2l+1}.
\end{align*}
\]

Upon use of the easy relation

\[
\begin{align*}
\cosh^2 t \partial_t \psi_{n,l} &= -\sinh t \cosh t \left(2l \psi_{n,l} + 2(n-l) \psi_{n,l+1},
\right)
\end{align*}
\]

and of relation (A.3) in Appendix A, one gets

\[
\begin{align*}
\cosh^2 t \partial_t \mathcal{L}_1 &= \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l+1} + \sum_{l=0}^{n-1} (2l \psi_{n,l} + 2(n-l) \psi_{n,l+1}) H_{2l+1} - \\
&- \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} ((2l+1) H_{2l+1} - (2n - 2l + 1) H_{2l-1} + \frac{(A-1)}{\sinh t} (H)_{2l}.
\end{align*}
\]

(8.23)

The terms involving \( \psi_{n,l} H_{2l+1} \) add up to zero. The term involving \((A-1)\) is

\[
- \frac{(A-1)}{\sinh t} \sum_{l=0}^{n-1} (-1)^l \psi_{n,l} H_{2l} = -\frac{(A-1)}{\sinh t} \left( \mathcal{M}_1 - (-1)^n H_{2n} \right).
\]

(8.24)

The remaining terms compensate partially and we are left with

\[
\sum_{l=0}^{n-2} (-1)^l \psi_{n,l+1} H_{2l+1} = \frac{1}{\sinh t} \left( - \mathcal{M}_2 + (-1)^{n-1} \sinh t H_{2n-1} \right).
\]

(8.25)

Collecting all the terms we have obtained

\[
\cosh^2 t \partial_t \mathcal{L}_1 = -A\mathcal{M}_1 + \mathcal{M}_1 - \mathcal{M}_2 + (-1)^{n-1} \left( \sinh t H_{2n-1} - (A-1) H_{2n} \right),
\]

(8.26)

and the last term vanishes thanks to relation (A.4).

Similarly one can check that

\[
\cosh^2 t \partial_t \mathcal{L}_2 = -A\mathcal{M}_2 - \mathcal{M}_1 + \mathcal{M}_2.
\]

(8.27)
Adding these last two equations proves relation \((a)\).

**Proof of relation \((b)\):** Using the easy relations

\[
\begin{align*}
\cosh^2 t (1 + \tau) \partial_t \psi_{n,l} &= \xi \tanh t (2l\psi_{n,l-1} + 2(n - l)\psi_{n,l}), \\
\cosh^2 t (1 + \tau) \psi_{n,l-1} &= -\xi \psi_{n,l-1},
\end{align*}
\]

one gets

\[
\begin{align*}
\frac{\cosh^2 t}{\tanh t} (1 + \tau) \partial_t \mathcal{M}_1 &= \xi \sum_{i=0}^{n} (-1)^i \left( 2l\psi_{n,l-1} + 2(n - l)\psi_{n,l} \right) H_{2l} - \\
&\quad -\frac{\xi}{\sinh t} \sum_{i=1}^{n} (-1)^i \psi_{n,l-1} \left( 2lH_{2l} + (2l - 2n - 2)H_{2(l-1)} \right) - \\
&\quad -\frac{\xi(A - 1)}{\sinh t} \sum_{i=1}^{n} (-1)^i \psi_{n,l-1} H_{2l-1}.
\end{align*}
\]

The terms involving \(\psi_{n,l-1}H_{2l}\) add up to zero. The term involving \((A - 1)\) is

\[
\frac{\xi}{\sinh t} \sum_{i=0}^{n-1} (-1)^i \psi_{n,l} H_{2l+1} = -\frac{(A - 1)}{\tanh t} \mathcal{L}_1.
\]

The remaining terms add up to zero, so we have obtained

\[
\cosh^2 t (1 + \tau) \partial_t \mathcal{M}_1 = -\xi A \mathcal{L}_1 + \xi \mathcal{L}_1.
\]

Adding the relation \(\xi \tanh t \mathcal{M}_1 = -\xi \mathcal{L}_2\) we end up with

\[
\cosh^2 t (1 + \tau) \partial_t \mathcal{M}_1 + \xi \tanh t \mathcal{M}_1 = -\xi A \mathcal{L}_1 + \xi (\mathcal{L}_1 - \mathcal{L}_2).
\]

Similarly one can show the relation

\[
\cosh^2 t (1 + \tau) \partial_t \mathcal{M}_2 + \xi \tanh t \mathcal{M}_2 = -\xi A \mathcal{L}_2 - \xi (\mathcal{L}_1 - \mathcal{L}_2).
\]

Adding these last two equations proves relation \((b)\).

So we have proved that the generating functions \(\mathcal{L}\) and \(\mathcal{M}\), as defined in Proposition 12 are indeed solutions of the partial differential relations stated in Proposition 11 which insure that \(S_1\) and \(S_2\) are integrals for the hamiltonian \((7.1)\). Its geodesic flow is therefore superintegrable.

This concludes the proof of Theorem 3.

\[\square\]

9 The moments and the Poisson structure

To avoid agony for the reader, we will state the results without proofs since the techniques and the relations established in the first part are easily adapted to this case.
Let us define
\[ S_+ = S_1 + S_2 = e^y(S + T), \quad S_- = S_1 - S_2 = e^{-y}(S - T), \quad (9.1) \]
we have
\[ S_+ S_- = S^2 - T^2 = \sum_{k=0}^{2n+1} \sigma_k H^{2n+1-k} \frac{P^2_y}{y^k}. \quad (9.2) \]
The moments are given in terms of the \( \lambda \)'s by the relations
\[ \begin{align*}
0 \leq k \leq n : & \quad \sigma_k = \sum_{l=0}^{k} S_{l,k-l} \quad (\sigma_0 = 1) \\
\quad n + 1 \leq k \leq 2n + 1 : & \quad \sigma_k = \sum_{l=k-n-1}^{n} S_{l,k-l}
\end{align*} \quad (9.3) \]
where
\[ S_{l,k} = \lambda_{2l-1} \lambda_{2k-1} - \lambda_{2l} \lambda_{2(k-1)} - \frac{\lambda_{2l-1} \lambda_{2k-3}}{\cosh^2 t}. \quad (9.4) \]
Defining the generating function of the moments by
\[ \Sigma(\xi) = \sum_{k=0}^{2n+1} \sigma_k \xi^k, \quad (9.5) \]
it is related to the generating functions by
\[ \Sigma = (1 + \tau)M^2 - \xi L^2. \quad (9.6) \]
Its computation gives
\[ \Sigma(\xi) = (1 - \xi) \prod_{k=1}^{2n} (1 - \xi m_k). \quad (9.7) \]
The Poisson algebra structure follows from
\[ S_+ S_- = \sum_{k=0}^{2n+1} \sigma_k H^{2n+1-k} \frac{P^2_y}{y^k}, \quad (9.8) \]
\[ \{S_+, S_-\} = -2 \sum_{k=0}^{2n} (k + 1) \sigma_{k+1} H^{2n-k} \frac{P^2_{y+1}}{y^k}, \]
where the moments are given by
\[ \begin{align*}
\sigma_0 &= 1 \\
\sigma_l &= (-1)^l \left( (M)_l + (M)_{l-1} \right) \quad l \in \{1, \ldots, 2n\} \\
\sigma_{2n+1} &= - (M)_{2n} = - \prod_{k=1}^{2n} m_k,
\end{align*} \quad (9.9) \]
where the symmetric functions \((M)_k\) of the \(m_k\) are defined by

\[
\prod_{k=1}^{2n} (1 - \xi m_k) = \sum_{k=0}^{2n} (-1)^k (M)_k \xi^k.
\]  
(9.10)

This concludes the statements of Theorem 4. □

Let us turn ourselves to the global problems.

10 Global aspects

The metric to discuss for \(\nu \in \mathbb{N}\backslash\{0\}\) is

\[
g = A^2(t) \, dt^2 + \cosh^2 t \, dy^2,
\]  
(10.1)

where

\[
A(t) = 1 + \sum_{k=1}^{\nu} \frac{e_k \sinh t}{\sqrt{m_k \cosh^2 t - 1}}, \quad \forall k : e_k = \pm 1, \quad m_k > 1.
\]  
(10.2)

For integrals with \(\sharp(S_i) = 2n\) (resp. \(\sharp(S_i) = 2n + 1\)) one has to take \(\nu = 2n - 1\) (resp. \(\nu = 2n\)). For \(\nu = 1\) we recover Koenigs.

10.1 Proof of point a) in Theorem 5

It is necessary that the sectional curvature be at least continuous for \(t \in \mathbb{R}\). Since we have

\[
R = \frac{\sinh t A' - \cosh t A}{A^3},
\]  
(10.3)

it follows that \(A\) must not vanish, so we can take \(A > 0\) for \(t \in \mathbb{R}\).

Let us compute the area of the surface:

\[
\mu(M) = \int_{\mathbb{R}} \cosh t A(t) \, dt \int dy.
\]  
(10.4)

If one takes \(y \in \mathbb{R}\) the second integral diverges. But this is not mandatory since we can take \(y \in \mathbb{S}^1\) and the second integral gives \(2\pi\). In this case let us consider the integral over \(t\). Its convergence requires

\[
A(-\infty) = A(\infty) = 0 \quad \iff \quad 1 \pm \sum_{k=1}^{n} \frac{e_k}{\sqrt{m_k}} = 0
\]

which is impossible, hence \(\mu(M)\) is divergent excluding \(M = \mathbb{S}^2\). □
10.2 Conformal structure of the metric

To proceed we need a conformal writing of the metric. For this let us recall that \( \mathbb{H}^2 \) is embedded into \( \mathbb{R}^3 \) according to
\[
x_1^2 + x_2^2 - x_3^2 = -1, \quad (x_1, x_2) \in \mathbb{R}^2, \quad x_3 \geq 1.
\] (10.5)

The coordinates choice
\[
x_1 = \cosh \chi \sinh y, \quad x_2 = \sinh \chi, \quad x_3 = \cosh \chi \cosh y, \quad (\chi, y) \in \mathbb{R}^2
\] (10.6)
gives
\[
g_0(\mathbb{H}^2) \equiv dx_1^2 + dx_2^2 - dx_3^2 = d\chi^2 + \cosh^2 \chi dy^2.
\] (10.7)

So we can define the conformal factor \( \rho \) by
\[
A^2 dt^2 + \cosh^2 t dy^2 = \rho^2 (d\chi^2 + \cosh^2 \chi dy^2) = \rho^2 g_0(\mathbb{H}^2),
\] (10.8)
as well as
\[
\psi(t) \equiv \sum_{k=1}^{\nu} \arctan (h_k(t)), \quad h_k(t) = e_k \sqrt{m_k \cosh^2 t - 1},
\] (10.9)
and
\[
\Sigma^{(\nu)}(t) \equiv \cos \psi(t) - \sin \psi(t) \sinh t.
\] (10.10)

Let us prove

**Lemma 1** If \( \Sigma^{(\nu)} > 0 \) then the manifold on which the metric (10.8) is defined will be diffeomorphic to \( \mathbb{H}^2 \). A zero of \( \Sigma^{(\nu)} \) precludes any manifold.

**Proof:** The coordinate \( \chi \) and the conformal factor \( \rho \) are given by
\[
\rho D_t \chi = A, \quad \& \quad \rho \cosh \chi = \cosh t.
\] (10.11)

Dividing these two relations and integrating gives
\[
e^\chi = \frac{e^t + \tan(\psi(t)/2)}{1 - e^t \tan(\psi(t)/2)} = \frac{e^t \Sigma^{(\nu)}(t)}{(\cos(\psi(t)/2) - \sin(\psi(t)/2) \sinh t)^2},
\] (10.12)

The strict positivity of \( e^\chi \) is therefore equivalent to \( \Sigma^{(\nu)} > 0 \).

Computing \( D_t \chi \) and plugging it into the first relation in (10.12) gives \( \rho = \Sigma^{(\nu)} \).

The relation (10.12) allows to compute \( \cosh \chi \) and to check that the second relation in (10.11) gives anew \( \rho = \Sigma^{(\nu)} \).

So, if the conformal factor \( \rho = \Sigma^{(\nu)}(t) \) never vanishes then the manifold is \( M \cong \mathbb{H}^2 \). □
10.3 Proof of point b) in Theorem 5

The case of integrals quadratic in the momenta, due to Koenigs, was already settled in Section 6. In this case the manifold is $M \cong \mathbb{H}^2$.

Let us consider now the case of higher integrals in the momenta with $\sharp(S_i) = 2n$ for $n \geq 2$ in which case we have $\nu = 2n - 1$. As proved in Appendix B we have

$$\Sigma^{(2n-1)}(t) = \frac{1}{S_{2n-1}(\cosh t)^{2n-1}} \left( \sum_{l=0}^{n-1} (-1)^l H^{(2n-1)}_{2l}(t) - \sinh t \sum_{l=0}^{n-1} (-1)^l H^{(2n-1)}_{2l+1}(t) \right),$$

(10.13)

where

$$\prod_{k=1}^{2n-1} (1 + \xi \sqrt{m_k}) = \sum_{k=0}^{2n-1} \xi^k S_k, \quad \implies S_{2n-1} = \prod_{k=1}^{2n-1} \sqrt{m_k}.$$  

(10.14)

For $t \to \pm \infty$ we have the equivalents

$$\frac{1}{S_{2n-1}(\cosh t)^{2n-1}} \sum_{l=0}^{n-1} (-1)^l H^{(2n-1)}_{2l}(t) \sim (-1)^{n-1} \frac{A(+\infty)}{\cosh t}$$

(10.15)

and

$$-\frac{\sinh t}{S_{2n-1}(\cosh t)^{2n-1}} \sum_{l=0}^{n-1} (-1)^l H^{(2n-1)}_{2l+1} \sim (-1)^n \sinh t.$$  

(10.16)

Hence $\Sigma^{(2n-1)}(\pm \infty)$ are of opposite signs and this implies a zero of $\Sigma^{(2n-1)}(t)$, precluding any manifold.

10.4 Towards the proof of point c) in Theorem 5

Here we need some preparatory material. First we have $\sharp(S_i) = 2n + 1$, with $n \geq 1$, and so $\nu = 2n$.

We have $2n$ "masses"

$$\{m_1, \ldots, m_n, \tilde{m}_1, \ldots, \tilde{m}_n\}, \quad e_{n+k} = -e_k, \quad \forall k \in \{1, \ldots, n\}.$$  

This just means that we take $\sum_{k=1}^{2n} e_k = 0$. These choices give

$$A^{(2n)}(t) = 1 + A^{(2n)}(t), \quad A^{(2n)}(t) = \sum_{k=1}^{n} \sinh t \left( \frac{1}{h_k(t)} - \frac{1}{\tilde{h}_k(t)} \right),$$

(10.17)

where

$$h_k(t) = e_k \sqrt{m_k \cosh^2 t - 1}, \quad \tilde{h}_k(t) = e_k \sqrt{\tilde{m}_k \cosh^2 t - 1},$$

(10.18)

as well as

$$\psi^{(2n)}(t) = \sum_{k=1}^{n} \left( \arctan(h_k) - \arctan(\tilde{h}_k) \right).$$  

(10.19)
\[ \prod_{k=1}^{n} (1 + \xi h_k(t))(1 - \xi \tilde{h}_k(t)) = \sum_{l=0}^{2n} H_l^{2n}(t) \xi^l. \] (10.20)

Let us begin with

**Lemma 2** The inequality

\[ \left| \frac{1}{\sqrt{\mu_k}} - \frac{1}{\sqrt{\tilde{\mu}_k}} \right| < 1, \quad \mu_k = m_k - 1, \quad \tilde{\mu}_k = \tilde{m}_k - 1, \] (10.21)

implies that

\[ (\forall t \in \mathbb{R} : A^{(2n)}(t) > 0) \quad \& \quad A^{(2n)}(\pm \infty) > 0. \] (10.22)

**Proof:** Defining \( \tau = \tanh t \) and \( \mu_k = m_k - 1, \quad \tilde{\mu}_k = \tilde{m}_k - 1 \) gives

\[ A^{(2n)} = \sum_{k=1}^{n} \left( \frac{\tau}{\mu_k + \tau^2} - \frac{\tau}{\tilde{\mu}_k + \tau^2} \right), \quad \tau \in [-1, +1]. \] (10.23)

So we have

\[ |A^{(2n)}| \leq \sum_{k=1}^{n} \frac{|\mu_k - \tilde{\mu}_k|}{\sqrt{\mu_k + \tau^2} \sqrt{\tilde{\mu}_k + \tau^2} (\sqrt{\mu_k + \tau^2 + \sqrt{\mu_k + \tau^2}})}, \] (10.24)

from which we deduce

\[ |A^{(2n)}| \leq \sum_{k=1}^{n} \frac{|\mu_k - \tilde{\mu}_k|}{\sqrt{\mu_k} \sqrt{\tilde{\mu}_k} (\sqrt{\mu_k + \sqrt{\mu_k}})} \leq \sum_{k=1}^{n} \frac{|\mu_k - \sqrt{\mu_k}|}{\sqrt{\mu_k} \sqrt{\mu_k}} < 1 \quad \forall \tau \in [-1, +1]. \] (10.25)

It follows that \( A(t) \) will be strictly positive not only for \( t \in \mathbb{R} \) but also for \( t \to \pm \infty \). \( \square \)

The next step is

**Lemma 3** One has the following relations:

\[ l = 0 : \quad H_0^{2n} = H_0^{2(n-1)} \]
\[ l = 1 : \quad H_1^{2n} = H_1^{2(n-1)} + (h_n - \tilde{h}_n) H_0^{2(n-1)} \]
\[ 2 \leq l \leq 2(n - 1) : \quad H_l^{2n} = H_l^{2(n-1)} + (h_n - \tilde{h}_n) H_{l-1}^{2(n-1)} - h_n \tilde{h}_n H_{l-2}^{2(n-1)} \] (10.26)
\[ l = 2n - 1 : \quad H_{2n-1}^{2n-1} = + (h_n - \tilde{h}_n) H_{2(n-1)}^{2(n-1)} - h_n \tilde{h}_n H_{2n-3}^{2(n-1)} \]
\[ l = 2n : \quad H_{2n}^{2n} = - h_n \tilde{h}_n H_{2(n-1)}^{2(n-1)}. \]

**Proof:** Relation (10.20) implies

\[ (1 + \xi h_n)(1 - \xi \tilde{h}_n) \sum_{l=0}^{2(n-1)} H_l^{2(n-1)}(t) \xi^l = \sum_{l=0}^{2n} H_l^{2n}(t) \xi^l. \] (10.27)
Expanding both sides in powers of $\xi$ gives \( E \).

In Appendix B it is proved that

$$
\cos \psi(2n)(t) = \frac{1}{S_n(\cosh t)^{2n}} \sum_{l=0}^{n} (-1)^l H_{2k}^2(t),
$$

and

$$
\sin \psi(2n)(t) = \frac{1}{S_n(\cosh t)^{2n}} \sum_{l=0}^{n-1} (-1)^l H_{2k+1}^2(t),
$$

From these relations we deduce

**Lemma 4** One has the following recurrences

$$
\cos \psi(2n) = \frac{1}{\sqrt{m_n m_n}} \left( \frac{1 + h_n \tilde{h}_n}{\cosh^2 t} \cos \psi(2n-2) \cos \psi(2n-2) - \frac{(h_n - \tilde{h}_n)}{\cosh^2 t} \sin \psi(2n-2) \right),
$$

and

$$
\sin \psi(2n) = \frac{1}{\sqrt{m_n m_n}} \left( \frac{1 + h_n \tilde{h}_n}{\cosh^2 t} \sin \psi(2n-2) + \frac{(h_n - \tilde{h}_n)}{\cosh^2 t} \cos \psi(2n-2) \right).
$$

**Proof:** Use relations (10.28) and (10.29) and Lemma 3.

Similarly we have

**Lemma 5** Having defined

$$
\Sigma^{(2n)} = \cos \psi(2n) - \sinh t \sin \psi(2n),
$$

implies the recurrence relation

$$
\Sigma^{(2n)} = \frac{1}{\sqrt{m_n m_n}} \left( 1 + \frac{(h_n + \sinh t)(h_n - \sinh t)}{\cosh^2 t} \right) \Sigma^{(2n-2)} + (\tilde{h}_n - h_n) \sin \psi(2n-2).
$$

**Proof:** Use relations (10.28) and (10.29) and Lemma 4.

To prepare the final proof we need

**Lemma 6** For $n = 1$ we have for manifold $M \cong \mathbb{H}^2$.

**Proof:** We have

$$
\Sigma^{(2)}(t) = \frac{H_0^2 - H_1^2 - \sinh t H_1^2}{\sqrt{m_1 m_1 \cosh^2 t}} = \frac{1 + h_1 \tilde{h}_1 - (h_1 - \tilde{h}_1) \sinh t}{\sqrt{m_1 m_1 \cosh^2 t}} = \frac{(h_1 + \sinh t)(h_1 - \sinh t) + \cosh^2 t}{\sqrt{m_1 m_1 \cosh^2 t}} > 0 \quad \forall t \in \mathbb{R}.
$$
Since strict inequalities are not respected when taking limits, we must check that $\Sigma^{(2)}(\pm \infty) > 0$. This follows from

$$\Sigma^{(2)}(\pm \infty) = 1 + \frac{1}{\sqrt{m_1}} - \frac{1}{\sqrt{m_1}}. \quad (10.35)$$

Since $m_1 > 1$ and $\tilde{m}_1 > 1$, we have

$$-1 < \frac{1}{\sqrt{m_1}} - \frac{1}{\sqrt{m_1}} < +1 \implies \Sigma^{(2)}(\pm \infty) > 0. \quad (10.36)$$

Use of Lemma 1 concludes the proof. □

Before giving the proof of point c) in Theorem 5, let us give the precise hypotheses needed:

(h1) In relation (10.18) we take all $\epsilon_k = 1$.

(h2) For the “masses” we have $\forall k \in \{1, \ldots, n-1\}$: $m_k > \tilde{m}_k > 1$, $1 < m_n < \tilde{m}_n$.

(h3) And the bound

$$\sum_{k=1}^{n} \left| \frac{1}{\sqrt{\mu_k}} - \frac{1}{\sqrt{\tilde{\mu}_k}} \right| < 1, \quad \mu_k = m_k - 1, \quad \tilde{\mu}_k = \tilde{m}_k - 1.$$

Under these hypotheses let us prove:

### 10.5 Proof of point c) in Theorem 5

For $n = 1$ we have proved in Lemma 6 that $\Sigma^{(2)}$ never vanishes. Let us proceed by recurrence. Lemma 5 gives

$$\Sigma^{(2n)} = \frac{1}{\sqrt{m_n m_n}} \left( 1 + \frac{(h_n + \sinh t)(\tilde{h}_n - \sinh t)}{\cosh^2 t} \right) \Sigma^{(2n-2)} + (\tilde{h}_n - h_n) \sin \psi^{(2n-2)} \right). \quad (10.37)$$

Hypothesis h2 implies that $\tilde{h}_n - h_n$ is strictly positive, so if $\sin \psi^{(2n-2)} > 0$ we can conclude that $\forall t \in \mathbb{R}$ we have $\Sigma^{(2n)} > 0$.

Let us recall that

$$\psi^{(2n-2)}(t) = \sum_{k=0}^{n-1} (\arctan h_k(t) - \arctan \tilde{h}_k(t)) > 0, \quad (10.38)$$

with

$$D_t \psi^{(2n-2)} = \frac{A^{(2n-2)} - 1}{\cosh t}, \quad A^{(2n-2)} - 1 = \sum_{k=1}^{n-1} \sinh t \left( \frac{1}{h_k} - \frac{1}{\tilde{h}_k} \right). \quad (10.39)$$
Using these relations one can easily establish the bounds

\[ 0 < \psi^{(2n-2)}(t) \leq B_n \equiv \sum_{k=0}^{n-1} (\arctan \sqrt{\mu_k} - \arctan \sqrt{\tilde{\mu}_k}). \]  

(10.40)

The upper bound becomes

\[ B_n = \sum_{k=0}^{n-1} \left( \arctan \frac{1}{\sqrt{\mu_k}} - \arctan \frac{1}{\sqrt{\tilde{\mu}_k}} \right) = \sum_{k=0}^{n-1} \arctan \left( \frac{1}{\sqrt{\mu_k}} - \frac{1}{\sqrt{\tilde{\mu}_k}} \right). \]  

(10.41)

leading to

\[ B_n < \sum_{k=0}^{n-1} \arctan \left( \frac{1}{\sqrt{\mu_k}} - \frac{1}{\sqrt{\tilde{\mu}_k}} \right) \leq \sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{\mu_k}} - \frac{1}{\sqrt{\tilde{\mu}_k}} \right| < \sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{\mu_k}} - \frac{1}{\sqrt{\tilde{\mu}_k}} \right|. \]  

(10.42)

So we get \( 0 < \psi^{(2n-2)}_0 < 1 < \pi/2 \) implying the strict positivity of \( \sin \psi^{(2n-2)}(t) \).

Strict inequalities are not respected when taking limits, so we have to check that \( \Sigma^{(2n)}(\pm \infty) > 0 \). Since we have

\[ \Sigma^{(2n)}(\pm \infty) = A^{(2n)}(\pm \infty), \]  

(10.43)

the hypothesis (h3) and Lemma 2 conclude the proof. □

This ends up the proof of point c) in Theorem 5. □

11 Conclusion

Let us add the following remarks:

1. Once more let us point out that SI geodesic flows are not necessarily related to Zoll geometry. Conversely it is not known whether any Zoll metric of revolution, globally defined on \( S^2 \), produces a SI geodesic flow.

2. We have proved the existence of a solution for the differential systems in Propositions 1 and 10. However the problem of uniqueness is left open.

3. Let us recall an important open problem: do there exist SI geodesic flows beyond the hypotheses of Matveev and Shechishin?

4. Locally there is an intriguing “symmetry” between the trigonometric and the hyperbolic cases but globally they are drastically different:
   - In the trigonometric case we may find metrics defined on \( S^2 \),
   - In the hyperbolic case we may find metrics defined on \( \mathbb{H}^2 \)!
5. It is interesting to compare the two cases:

| trigonometric | hyperbolic |
|---------------|------------|
| $g = A^2(t)dt^2 + \sin^2 t\, dy^2$ | $g = A^2(t)dt^2 + \cosh^2 t\, dy^2$ |
| $t \in (0, \pi)$, $y \in S^1$ | $(t, y) \in \mathbb{R}^2$ |
| $A(t) = 1 + \sum_k e_k \sin t / \sqrt{1 - m_k \sin^2 t}$ | $A(t) = 1 + \sum_k e_k \sinh t / \sqrt{m_k \cosh^2 t - 1}$ |

$\#(S) = 2$ (Koenigs) : no manifold
$\#(S) = 2n \geq 4$ : no manifold no manifold
$\#(S) = 2n + 1 \geq 3$ : $\mathbb{Zol} \to S^2$ $\mathbb{H}^2$

6. Leaving aside Koenigs system, there seems to be a curse in the case $\#(S) = 2n$ for which no manifold seems to be allowed. There should be an explanation of this fact.

**Appendices**

**A Appendix A**

Let us recall the content of Definition 1. We have

$$\forall k \in \{1, 2, \ldots, \nu\} : h_k(t) = e_k \sqrt{m_k \cosh^2 t - 1} \quad \forall k : (e_k = \pm 1 \& m_k \geq 1),$$

as well as

$$A(t) = 1 + \sum_{k=1}^{\nu} \frac{\sinh t}{h_k(t)}. \quad (A.1)$$

The functions $H^\nu_k(t)$ are defined by the generating function

$$\mathcal{H}^\nu(t, \xi) = \prod_{k=1}^{\nu} (1 + \xi h_k(t)) = \sum_{k=0}^{\nu} H^\nu_k(t) \xi^k. \quad (A.2)$$

For convenience we will introduce the conventional values $H^\nu_{-1}(t) = H^\nu_{-2}(t) = H^\nu_{\nu+1}(t) \equiv 0$.

Let us prove some useful relations:

**Proposition 14** For all $k \in \{0, 1, \ldots, \nu\}$ one has

$$D_t H^\nu_k(t) = \tanh t \left( k H^\nu_k + (k - \nu - 2) H^\nu_{k-2} \right) + \frac{(A - 1)}{\cosh t} H^\nu_{k-1}, \quad (A.3)$$

as well as

$$\sinh t \ H^\nu_{\nu-1} = (A - 1) \ H^\nu_\nu. \quad (A.4)$$
**Proof:** The relation is trivial for $k = 0$, so let it be supposed that $k \geq 1$. Using the relations

$$h'_k = \tanh t \left( h_k + \frac{1}{h_k} \right), \quad (A.5)$$

we deduce

$$\frac{\partial_t \mathcal{H}}{\mathcal{H}} = \tanh t \left( \sum_{k=1}^{\nu} \frac{\xi h_k}{1 + \xi h_k} + \sum_{k=1}^{\nu} \frac{\xi}{h_k(1 + \xi h_k)} \right). \quad (A.6)$$

The first sum is merely

$$\sum_{k=1}^{\nu} \frac{\xi h_k}{1 + \xi h_k} = \frac{\xi \partial_\xi \mathcal{H}}{\mathcal{H}}, \quad (A.7)$$

while the second sum is transformed according to

$$\sum_{k=1}^{\nu} \frac{\xi(1 - \xi^2 h_k^2 + \xi^2 h_k^2)}{h_k(1 + \xi h_k)} = \sum_{k=1}^{\nu} \frac{\xi(1 - \xi h_k)}{h_k} + \xi^2 \sum_{k=1}^{\nu} \frac{\xi h_k}{1 + \xi h_k} = \frac{\xi(A - 1)}{\sinh t} - \nu \xi^2 + \xi^2 \frac{\xi \partial_\xi \mathcal{H}}{\mathcal{H}}. \quad (A.8)$$

Hence we have obtained

$$\partial_t \mathcal{H} = \tanh t \left( (1 + \xi^2) \xi \partial_\xi \mathcal{H} - \nu \xi^2 \mathcal{H} \right) + \xi \frac{(A - 1)}{\cosh t} \mathcal{H}. \quad (A.9)$$

Expanding in powers of $\xi$ gives (A.3) for $k$ from 1 to $\nu$. For $k = \nu + 1$ we get (A.4) while for $k = \nu + 2$ the relation obtained is trivial.

We will need also, for the functions

$$\mathcal{H}^\nu = \mathcal{H}^\nu(t, \pm i\eta), \quad (A.10)$$

the following:

**Proposition 15** For $\nu = 2n$ one has the relations

$$\frac{\mathcal{H}^{2n}_+ + \mathcal{H}^{2n}_-}{2} = \sum_{l=0}^{n} (-1)^l H^{2n}_{2i l} \eta^{2l}, \quad \frac{\mathcal{H}^{2n}_+ - \mathcal{H}^{2n}_-}{2i} = \sum_{l=0}^{n-1} (-1)^l H^{2n}_{2i l+1} \eta^{2l+1}. \quad (A.11)$$

**Proof:** Starting from (A.2) with $\xi \to \pm i\eta$, expanding in powers of $i\eta$ and separating the real and imaginary parts proves both relations. \hfill \Box

### B Appendix B

Let us come back to relation (A.2) for $\xi = i$. We have

$$\prod_{k=1}^{\nu} (1 + ih_k(t)) = \sum_{k=0}^{\nu} i^k H^\nu_k(t). \quad (B.1)$$
Noticing that \( 1 + i h_k(t) = \sqrt{m_k} \cosh t e^{i \text{arctanh} h_k(t)} \) the previous relation becomes

\[
\prod_{k=1}^{\nu} \sqrt{m_k} \cosh \nu t e^{i \psi(t)} = \sum_{k=0}^{\nu} t^k H_k^\nu(t), \quad \psi(t) = \sum_{k=1}^{\nu} \arctan (h_k(t)). \tag{B.2}
\]

Defining \( S_\nu = \prod_{k=1}^{\nu} \sqrt{m_k} \) and comparing the real and the imaginary gives \( \cos \psi(t) \) and \( \sin \psi(t) \) which lead, for \( \nu = 2n - 1 \), to

\[
\Sigma^{(2n-1)}(t) = \frac{1}{S_{2n-1}(\cosh t)^{2n-1}} \left( \sum_{l=0}^{n-1} (-1)^l H_{2l}^{2n-1}(t) - \sinh t \sum_{l=0}^{n-1} (-1)^l H_{2l+1}^{2n-1} \right). \tag{B.3}
\]

For \( \nu = 2n \) we get similarly the relation

\[
\Sigma^{(2n)}(t) = \frac{1}{S_{2n}(\cosh t)^{2n}} \left( \sum_{l=0}^{n} (-1)^l H_{2l}^{2n}(t) - \sinh t \sum_{l=0}^{n-1} (-1)^l H_{2l+1}^{2n} \right). \tag{B.4}
\]

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