Rayleigh–Taylor instability for nonhomogeneous incompressible fluids with Navier-slip boundary conditions

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This paper is concerned with the Rayleigh–Taylor instability for the nonhomogeneous incompressible Navier–Stokes equations with Navier-slip boundary conditions around a steady state in an infinite slab, where the Navier-slip coefficients do not have defined sign and the slab is horizontally periodic. Motivated by Jiang et al. (Sci. China Math., 2013), we extend the result from Dirichlet boundary condition to Navier-slip boundary conditions. Our results indicate the factor that “heavier density with increasing height” still plays a key role in the instability under Navier-slip boundary conditions.

KEYWORDS
incompressible flows, Navier-slip boundary conditions, Navier–Stokes equations, Rayleigh–Taylor instability

MSC CLASSIFICATION
76N10; 35Q30; 35Q35

1 | INTRODUCTION

In this paper, we focus on the instability of the following nonhomogeneous incompressible Navier–Stokes equations with gravity in an infinite slab domain \( \Omega = 2\pi L_T \times (0, 1) \):

\[
\begin{cases}
\rho_t + \mathbf{v} \cdot \nabla \rho = 0, \\
\rho \mathbf{v}_t + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mu \Delta \mathbf{v} - \rho \mathbf{e}_2, \\
\text{div} \mathbf{v} = 0.
\end{cases}
\]

where the unknowns \((\rho, \mathbf{v}, p)\) denote density, velocity, and pressure of the fluid, respectively. The constant \(\mu > 0\) stands for the coefficient of shear viscosity, \(\mathbf{e}_2 = (0, 1)\) is the vertical unit vector, and \(-\mathbf{e}_2\) describes the gravity.

The Navier-slip boundary condition being considered is given as follows:

\[
\begin{cases}
\mathbf{v} \cdot \mathbf{n} = 0, \\
2\mu \mathbb{D}(\mathbf{v}) \cdot \mathbf{n} \cdot \mathbf{t} = k(x) \mathbf{v} \cdot \mathbf{r},
\end{cases}
\]

on \(\Sigma_1 \bigcup \Sigma_0\),

where \(\mathbb{D}(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + \nabla^T \mathbf{v})\), \(\mathbf{n}\) is the outward normal vector of the boundary, \(\mathbf{t}\) is the tangential vector, and \(\Sigma_1\) and \(\Sigma_0\) are the upper and lower boundary, respectively, that is, \(\Sigma_1 = 2\pi L_T \times \{1\}\), \(\Sigma_0 = 2\pi L_T \times \{0\}\), where \(2\pi L_T\) stands for the 1D torus of length \(2\pi L\). In addition, \(k(x)\) is a scalar function describing the slip effect on the boundary. In this paper, \(k(x)\) will be taken to be constants \(k_0\) and \(k_1\) on \(\Sigma_0\) and \(\Sigma_1\), respectively, which do not have a defined sign.

We first look for a smooth steady-state \((\bar{\rho}, \bar{\mathbf{v}}, \bar{p})\) to the system (1), where the density profile \(\bar{\rho} := \rho(y)\) satisfies

\[
\bar{\rho} \in C^\infty([0, 1]), \quad \inf_{y \in (0, 1)} \bar{\rho} > 0,
\]
and the pressure \( \bar{p} \) is determined by the following equality:

\[
\nabla \bar{p} = -\bar{\rho} g e_2. \tag{4}
\]

Because we are interested in Rayleigh–Taylor (RT) instability, we assume that the steady density satisfies

\[
\bar{\rho}'(y_0) > 0, \quad \text{for some } y_0 \in (0, 1). \tag{5}
\]

This condition means that there is a neighborhood of \( y_0 \), such that \( \bar{\rho} \) increases with \( y \), that is, a heavy fluid is on top of the light one.

Now, we define the perturbation as

\[
(\varphi, \mathbf{u}, q) := (\rho - \bar{\rho}, \mathbf{v} - \mathbf{0}, p - \bar{p}).
\]

Then, \((\varphi, \mathbf{u}, q)\) satisfies the following equations:

\[
\begin{cases}
\varphi_t + \mathbf{u} \cdot \nabla (\varphi + \bar{\rho}) = 0, \\
(\varphi + \bar{\rho}) \mathbf{u}_t + (\varphi + \bar{\rho}) \mathbf{u} \cdot \nabla \mathbf{u} + \nabla q + \rho g e_2 = \mu \Delta \mathbf{u}, & \text{in } \Omega, \\
\text{div} \mathbf{u} = 0,
\end{cases}
\]

with the corresponding initial data and boundary conditions turning to

\[
\begin{cases}
(\varphi, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0), & \text{in } \Omega, \\
\mathbf{u}_t = 0, & \text{on } \Sigma_1 \cup \Sigma_0, \\
\partial_u \mathbf{u}_1 = \frac{k_1}{\mu} \mathbf{u}_1, & \text{on } \Sigma_1, \\
\partial_u \mathbf{u}_1 = -\frac{k_2}{\mu} \mathbf{u}_1, & \text{on } \Sigma_0.
\end{cases}
\]

Linearizing system (6), one gets

\[
\begin{cases}
\varphi_t + \bar{\rho}^2 \mathbf{u}_2 = 0, \\
\bar{\rho} \mathbf{u}_t + \nabla q + \rho g e_2 = \mu \Delta \mathbf{u}, & \text{in } \Omega, \\
\text{div} \mathbf{u} = 0.
\end{cases}
\]

To analyze our problem, we would like to apply the growing normal mode method, for which the readers can refer to Ding et al.,\(^1\) for instance.

Precisely, we first assume a growing mode ansatz of solutions

\[
(\varphi, \mathbf{u}, q)(x, y; t) = e^{\lambda t}(\bar{\varphi}, \bar{\mathbf{v}}, \bar{p})(x, y)
\]

to the linearized system (8) with some constant \( \lambda > 0 \). Substituting these ansatzes into (8) deduces a system for the new unknowns \((\bar{\varphi}, \bar{\mathbf{v}}, \bar{p})\) as follows

\[
\begin{cases}
\lambda \bar{\varphi} + \bar{\rho}^2 \bar{v}_2 = 0, \\
\lambda \bar{\rho} \bar{v}_t + \nabla \bar{p} + \rho g e_2 = \mu \Delta \bar{\mathbf{v}}, & \text{in } \Omega, \\
\text{div} \bar{\mathbf{v}} = 0.
\end{cases}
\]

Eliminating \( \bar{p} \) in (9), one arrives at

\[
\begin{cases}
\lambda^2 \bar{\rho} \bar{v}_t + \lambda \bar{\rho} \bar{v}_x = \lambda \mu \Delta \bar{\mathbf{v}} + g \bar{\rho} \bar{v}_2 e_2, \\
\text{div} \bar{\mathbf{v}} = 0,
\end{cases}
\]

which is endowed with the following initial data and boundary conditions:

\[
\begin{cases}
(\bar{\varphi}, \bar{\mathbf{v}})|_{t=0} = (\rho_0, \mathbf{u}_0), & \text{in } \Omega, \\
\bar{v}_2 = 0, & \text{on } \Sigma_1 \cup \Sigma_0, \\
\partial_u \bar{v}_1 = \frac{k_1}{\mu} \bar{v}_1, & \text{on } \Sigma_1, \\
\partial_u \bar{v}_1 = -\frac{k_2}{\mu} \bar{v}_1, & \text{on } \Sigma_0.
\end{cases}
\]
Second, for any frequency $\xi \in \mathbb{R}$, $\xi \neq 0$, we rewrite the unknowns in (10) and (11) in terms of $(\phi, \psi, \pi)(y) : (0, 1) \to \mathbb{R}$ with

\[
\begin{align*}
\tilde{v}_1(x, y) &= -i\phi(y)e^{i\xi y}, \\
\tilde{v}_2(x, y) &= \psi(y)e^{i\xi y}, \\
p(x, y) &= \pi(y)e^{i\xi y},
\end{align*}
\]

from which we can infer that the new unknowns $(\phi, \psi, \pi)$ satisfy the following ordinary differential equations (ODEs)

\[
\begin{align*}
-\lambda^2 \delta \phi + \lambda \xi \pi - \lambda \mu(\xi^2 \phi - \phi'') &= 0, \\
\lambda^2 \delta \psi + \lambda \mu \pi' + \lambda \mu(\xi^2 \psi - \psi'') &= g \delta \psi, \\
\xi \phi + \psi' &= 0,
\end{align*}
\]

with the corresponding boundary conditions

\[
\begin{align*}
\psi(0) &= \psi(1) = 0, \\
\phi'(1) &= \frac{\delta_k}{\mu} \phi(1), \\
\phi'(0) &= -\frac{\delta_k}{\mu} \phi(0).
\end{align*}
\]

Third, eliminating $\pi$ in (12), one yields a fourth-order ODE of $\psi(y)$:

\[
-\lambda^2 \left[ \xi^2 \delta \psi - (\delta \psi')' \right] = \lambda \mu \left( \psi^{(4)} - 2\xi^2 \psi'' + \xi^4 \psi \right) - g \xi^2 \delta \psi,
\]

and the corresponding boundary conditions

\[
\begin{align*}
\psi(0) &= \psi(1) = 0, \\
\psi''(1) &= \frac{\delta_k}{\mu} \psi'(1), \\
\psi''(0) &= -\frac{\delta_k}{\mu} \psi'(0).
\end{align*}
\]

In conclusion, problems (7) and (8) are finally reduced to the fourth-order ODE systems (14) and (15).

The main results of this paper are stated as follows:

**Theorem 1.1.** (Linear instability) The steady-state $(\bar{\rho}, 0, \bar{\rho})$ is linearly unstable, provided that the steady density profile $\bar{\rho}$ satisfies (3) and (5). Precisely, there exists exponentially growing solution to the linearized perturbed problems (7) and (8), such that $\|[(\rho, u, q)(t)]_{L^2(\Omega)} \to \infty$, as $t \to \infty$.

**Theorem 1.2.** (Nonlinear instability) The steady-state solution $(\bar{\rho}, 0, \bar{\rho})$ is nonlinearly unstable in the Hardama sense, provided that the steady density profile $\bar{\rho}$ satisfies (3) and (5). Precisely, there exist positive constants $\Lambda, \varepsilon, m_0$, and a pair $(\delta_0, \bar{\mathbf{u}}_0) \in L^2(\Omega) \times H^2(\Omega)$, such that for any $\delta \in (0, \varepsilon)$, there is a unique global strong solution $(\rho, \mathbf{u})$ to the nonlinear perturbed problems (6) and (7) on $[0, T)$ with the initial data $(\rho_0, \mathbf{u}_0) := (\delta_0, \delta \bar{\mathbf{u}}_0)$, but

\[
\|\rho(T^8)\|_{L^2(\Omega)}, \|\mathbf{u}(T^8)\|_{L^2(\Omega)} \geq \varepsilon,
\]

for some escape time $T^8 := \frac{1}{\Lambda} \ln \frac{2\varepsilon}{m_0 \delta} \in (0, T)$.

Before proving these theorems, let us recall some results on the RT instability problems. The RT instability is a kind of well-known instability in fluid dynamics, which is driven by the gravity when the upper fluid is heavier than the lower one. In 1883, Rayleigh\(^2\) first considered the linear instability for an incompressible fluid. One hundred twenty years later, in 2003, Hwang and Guo\(^3\) studied 2D nonhomogeneous incompressible inviscid fluid in strip domain with zero normal velocity on the boundary and proved the RT instability in the Hardama sense. However, when the viscosity is taken into account, there is no direct variational structure for constructing exponentially growing solution. In 2011, Guo and Tice\(^4\) introduced a general method for these problem; they turned to study the modified variational problem first and then went back to the original problem by fixed-point theory. Motivated by Guo and Tice\(^4\), Jiang et al.\(^5\) in 2013 studied the RT instability of nonhomogeneous incompressible viscous fluid at the presence of the uniform gravity field in $\mathbb{R}^3$. Whereafter, F. Jiang and S. Jiang made a breakthrough in 3D bounded domain case\(^6\) and analyzed both the instability and stability for given different steady density profile $\bar{\rho}$.

When fluids are electrically conducting at the presence of magnetic field, the RT instability arises and the growth of the instability will be influenced by the magnetic field due to the Lorentz force. Thus, some authors pay considerable
attention to the inhibition of the magnetic field on the RT instability. In 1954, Kruskal and Schwarzschild first proved that a horizontal magnetic field has no effect on the development of the linear RT instability.\(^\text{7}\) Afterwards, Hide\(^\text{8}\) investigated the influence of a vertical magnetic field. To our knowledge, there is some critical magnetic number \(B_c\), such that when the vertical background magnetic field is less than the critical magnetic number, then the magnetic field has no effect on the RT instability.\(^\text{9}\) While the vertical background magnetic field is larger than the critical magnetic number, then the magnetic field has an inhibitory on the RT instability.\(^\text{10}\) Therefore, this physical phenomenon has been verified mathematically in some cases.

In 2018, Ding et al.\(^\text{1}\) found that there exists a critical viscosity coefficient for distinguishing stability from instability when considering the homogeneous fluid with Navier-slip boundary conditions in 2D strip domain.

As for the Navier-slip boundary conditions, which was first proposed by C. Navier\(^\text{11}\) in 1827, it describes the phenomenon that fluid moves along the boundary. Mathematically, it can be depicted as in (2). Compared with the classical Dirichlet boundary condition, Navier-slip boundary conditions are more realistic in some situations, but the mathematical literature is less. Early in 1973, V. Solonnikov and V. Ščilov\(^\text{12}\) gave the first rigorous mathematical analysis to Navier–Stokes equations with Navier-slip boundary conditions; they focused on linear stationary equations in 3D with the coefficient \(k(x) = 0\) and the external force \(f \in L^2\). In 1980s, G. Mulone and F. Salem\(^\text{13,14}\) considered the Navier–Stokes equations with Navier-slip boundary conditions in a 3D bounded domain. They proved the well-posedness for the corresponding stationary problem and the existence of weak solution for the evolutionary problem. Afterwards, J. Kelliher\(^\text{15}\) established the existence, uniqueness, and regularity for 2D bounded domain case in 2006, where the domain consisted of a finite number of connected components with the slip coefficient \(k(x) \in L^\infty(\partial\Omega)\). Based on the above results of weak solution, in 2010, H. B. da Veiga\(^\text{16}\) improved the regularity of the weak solutions, up to the boundary. Later, C. Amrouche and co-authors studied stationary and evolutionary problems\(^\text{17-19}\) in \(L^p\) with \(p \in (1, \infty)\), proving the existence of weak and strong solutions in a 3D bounded domain with smooth boundary. It should be noted that the noncompact infinite slab case is not included in any result mentioned above. In 2018, Ding and Li proved the existence and uniqueness of strong solution of incompressible fluid with Navier boundary conditions in 2D infinite slab.\(^\text{20}\) For the references on the vanishing viscosity limit of Navier–Stokes equation with Navier boundary conditions, we refer the readers to other studies\(^\text{21-23}\) and the references therein. The main aim of this paper is to investigate the slip effect from the boundary on the RT instability.

The rest of the paper is arranged as follows. In Section 2, we give some notations and list some useful inequalities. In Sections 3 and 5, we will give the proofs of Theorems 1.1 and 1.2, respectively. In Section 4, we deduce the energy estimates, which is a preparation for proving the nonlinear instability in Section 5.

## 2 | PRELIMINARY

For simplicity, we denote \(L^2(0, 1)\) and \(H^k(0, 1)\) by \(L^2\) and \(H^k\). Without confusion, we will also write \(L^p(\Omega)\) and \(H^k(\Omega)\) by \(L^p\) and \(H^k\), respectively. The integral form \(\int_{\Omega} f \, dx\, dy\) will be simply denoted by \(\int f\). In addition, the scalar function and vector function will be denoted by \(f\) and \(\mathbf{f}\) for distinction, such as \(\mathbf{f} = (f_1, f_2)\). The product functional space \((X)^2\) will also be denoted by \(X\), for example, the vector function \(\mathbf{u} \in (H^1)^2\) will be still denoted by \(\mathbf{u} \in H^1\). The usual notations will be used as in general unless with extra statements.

For convenience, we list a few lemmas that will be used in this paper, without the proofs. The readers interested in the proof could refer to Li and Ding\(^\text{20}\) for the details.

**Lemma 2.1.** (Poincaré inequality in \(\Omega\)) There exists constant \(C > 0\), such that for any \(\mathbf{u} \in V = \{\mathbf{v} \in H^1 | \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0\) on \(\partial\Omega\}\), the following inequality holds:

\[
\|\mathbf{u}\|_{L^2(\Omega)} \leq C \|\partial_\nu \mathbf{u}\|_{L^2(\Omega)}. \tag{17}
\]

**Lemma 2.2.** (\(L^4\) estimate in \(\Omega\)) There exists constant \(C > 0\), such that for any \(\mathbf{u} \in W = \{\mathbf{v} \in V \cap H^2 | \text{satisfies (2)}\}_2\), there holds

\[
\|\mathbf{u}\|^2_{L^4(\Omega)} \leq C \|\mathbf{u}\|_{L^2(\Omega)} \|
abla \mathbf{u}\|_{L^2(\Omega)}. \tag{18}
\]

**Lemma 2.3.** (\(L^\infty\) estimate in \(\Omega\)) There exists constant \(C > 0\), such that for any \(\mathbf{u} \in W\), one has

\[
\|\mathbf{u}\|_{L^\infty(\Omega)}^2 \leq C \|\mathbf{u}\|_{L^2(\Omega)} \|\mathbf{u}\|_{H^2(\Omega)}. \tag{19}
\]
Lemma 2.4. \((L^2 \text{ estimate of gradient in } \Omega)\) There exists constant \(C > 0\), such that for any \(u \in W\), there holds
\[
\|\nabla u\|_{L^2(\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \|\nabla u\|_{H^1(\Omega)}.
\] (20)

Lemma 2.5. \((L^4 \text{ estimate of gradient in } \Omega)\) There exists constant \(C > 0\), such that for any \(u \in W\), there holds
\[
\|\nabla u\|_{L^4(\Omega)}^2 \leq C \|u\|_{L^2(\Omega)} \|\nabla u\|_{H^1(\Omega)}^3.
\] (21)

Proof. Using estimate (19), the proof of (21) is similar to that of (20). \(\square\)

3 | THE PROOF OF THEOREM 1.1

In this section, we will make effort to construct a solution for (14) and (15) by variational method, which at once deduces a solution for ODE systems (12) and (13). Then, an exponentially growing solution will be given for linearized perturbed equations (8) and (7), and Theorem 1.1 follows.

However, similar to Guo and Tice,\(^4\) the appearance of \(\lambda\) both quadratically and linearly in Equation (14) breaks the natural variational structure of problems (14) and (15). To circumvent this obstacle, for any spatial frequency \(\xi \neq 0\), we introduce a family \((s > 0)\) of modified variational problem
\[
\alpha(s) := \inf_{\psi \in \mathcal{A}} E(\psi, s),
\] (22)
with the energy functional
\[
E(\psi, s) := s \left[ \int_0^1 \mu |\psi''|^2 dy - \left( k_1 |\psi'(1)|^2 + k_0 |\psi'(0)|^2 \right) \right] + s\mu \int_0^1 \left( 2\xi^2 |\psi'|^2 + \xi^4 \psi^2 \right) dy - \int_0^1 g\xi^2 \tilde{\rho} \psi^2 dy,
\] (23)
and the corresponding admissible set
\[
\mathcal{A} = \left\{ \psi \in H_0^1 \cap H^2 \left| J(\psi) := \int_0^1 \tilde{\rho} \left( \xi^2 \psi^2 + |\psi'|^2 \right) dy = 1 \right\}.
\] (24)

In addition, to emphasize the dependence on \(\xi\), we sometimes write (22) as
\[
\alpha(s, \xi) = \inf_{\psi \in \mathcal{A}} E(\psi, s, \xi).
\]
In order to recover the corresponding variational form of problems (14) and (15), we will show later in this section that there exists a fixed point \(s_0 > 0\), so that \(\alpha(s_0) = -s_0^2\). The first proposition is devoted to proving the well-definedness of (22).

Proposition 3.1. Suppose that the steady density satisfies (3). Then, for any \(\xi \neq 0\) and \(s \in (0, +\infty)\), \(E(\psi, s)\) achieves its minimum on \(\mathcal{A}\).
Proof. For any $\psi \in A$, because $J(\psi) = 1$, by Cauchy inequality, one has

$$E(\psi) \geq - \int_0^1 g \xi^2 \tilde{\rho} \psi'^2 dy + s \int_0^1 \left\{ \mu |\psi''|^2 - \left[ ((k_0 + k_1)y - k_0) |\psi'(y)|^2 \right] \right\} dy$$

$$\geq - g \|\tilde{\rho}''\|_{L^\infty} + s \int_0^1 \left\{ \mu |\psi''|^2 - \sum_{i=0}^1 k_i |\psi'(y)|^2 - \frac{1}{2} \sum_{i=2}^1 k_i (y - k_0) \psi'' \right\} dy$$

$$\geq - g \|\tilde{\rho}''\|_{L^\infty} - sC_0 \int_0^1 |\psi'|^2 dy \geq - g \|\tilde{\rho}''\|_{L^\infty} - sC_0 \frac{1}{\tilde{\rho}} \|\tilde{\rho}\|_{L^\infty},$$

(25)

where $C_0 := \max_{0 \leq y \leq 1} \left( |k_0 + k_1| + \mu^{-1}((k_0 + k_1)y - k_0)^2 \right)$.

Because $\tilde{\rho}$ has a positive lower bound, it follows from (25) that $E(\psi)$ is bounded below on $A$, and so it has an infimum. Denote that

$$M := \inf_{\psi \in A} E(\psi).$$

In view of the definition of infimum, there exists a minimizing sequence $\{\psi_n\}_{n=1}^\infty \in A$, such that

$$\lim_{n \to +\infty} E(\psi_n) = M.$$

Without loss of generality, we suppose that

$$E(\psi_n) \leq M + 1.$$  

(26)

On one hand, because $J(\psi_n) = 1$, there holds

$$\int_0^1 \psi_n^2 dy \leq \xi^{-2} \|\tilde{\rho}\|_{L^\infty} \quad \text{and} \quad \int_0^1 |\psi_n'|^2 dy \leq \|\tilde{\rho}\|_{L^\infty},$$

(27)

which imply that $\{\psi_n\}_{n=1}^\infty$ is a bounded sequence in $H^1_0$.

On the other hand, similar to (25), it follows from the definition of $E(\psi)$ that

$$\mu \int_0^1 |\psi''|^2 dy = \frac{E(\psi_n)}{s} + \int_0^1 g \xi^2 \tilde{\rho} \psi'^2 dy + \sum_{i=0}^1 k_i |\psi_i'(i)|^2 - \mu \xi^2 \int_0^1 (2 |\psi_n'|^2 + \xi^2 \psi_n^2) dy$$

$$\leq \frac{M + 1 + g \|\tilde{\rho}''\|_{L^\infty}}{s} + 2C_0 \int_0^1 |\psi_n'|^2 dy + \frac{\mu}{2} \int_0^1 |\psi_n''|^2 dy,$$

(28)

which, together with (27), indicates that

$$\int_0^1 |\psi''|^2 dy \leq 2\mu^{-1} \left[ s^{-1} \left( M + 1 + g \|\tilde{\rho}''\|_{L^\infty} \right) + 2C_0 \|\tilde{\rho}\|_{L^\infty} \right].$$

(29)

In conclusion, for any fixed $s \in (0, \infty)$, $\{\psi_n\}_{n=1}^\infty$ is a bounded sequence in $H^1_0 \cap H^2$, and hence, there exists a convergent subsequence, still denoted by $\{\psi_n\}_{n=1}^\infty$, such that $\psi_n \rightharpoonup \psi$ weakly in $H^2$ and $\psi_n \to \psi$ strongly in $H^1_0$. Then $J(\psi) = \lim_{n \to +\infty} J(\psi_n) = 1$, which implies $\psi \in A$.

In addition, using the weak lower semi-continuity and the convergence of $\{\psi_n\}_{n=1}^\infty$, one gets

$$E(\psi) = s \mu \int_0^1 |\psi''|^2 dy + s(2\mu \xi^2 - (k_1 + k_0)) \int_0^1 |\psi'|^2 dy$$

$$+ \xi^2 \int_0^1 (\xi^2 \mu - g \tilde{\rho}') \psi'^2 dy - s \int_0^1 ((k_1 + k_0)y - k_0) \psi' \psi'' dy$$

$$\leq \liminf_{n \to +\infty} E(\psi_n) = \lim_{n' \to -\infty} E(\psi_{n'}) = M,$$

(30)

which means that $\psi$ is the minimizer of $E(\cdot)$ on $A$. The proof of this proposition is finished. □
In what follows, we will prove that \( a(s) \) is negative for any \( s \in (0, \infty) \), where \( \infty \) will be defined in Proposition 3.4. For convenience, we denote

\[
\begin{align*}
E_0(\psi) &= \int_0^1 \mu|\psi''|^2dy - (k_1|\psi'(1)|^2 + k_0|\psi'(0)|^2), \\
E_1(\psi) &= \mu \int_0^1 (2|\psi'|^2 + \xi^2 \psi^2)dy, \\
G(\psi) &= E_0(\psi) + \xi^2 E_1(\psi), \\
E_2(\psi) &= \int_0^1 g \xi^2 \rho \psi^2 dy.
\end{align*}
\]

Then, (23) can be rewritten as \( E(\psi) = sG(\psi) - E_2(\psi) \).

Similar to Ding et al.,\(^1\) firstly, we define the critical viscosity

\[
\mu_c := \sup_{\psi \in A} \frac{k_1|\psi'(1)|^2 + k_0|\psi'(0)|^2}{\int_0^1 |\psi''|^2dy},
\]

which, according to Ding et al.,\(^1\) can be explicitly expressed by

\[
\mu_c = \begin{cases} 
0, & k_0 \leq 0 & k_1 \leq 0; \\
\frac{k_0}{k_1}, & k_0 = k_1 := k > 0; \\
\frac{k_0 + k_1 + \sqrt{k_0^2 + k_1^2 + 2k_0k_1}}{6}, & \text{otherwise.}
\end{cases}
\]

It is clear that \( E_0(\psi) \geq 0 \) for any \( \psi \in A \), provided \( \mu \geq \mu_c \). Conversely, there exists \( \psi \in A \) such that \( E_0(\psi) < 0 \), if \( \mu < \mu_c \).

Further, according to Ding et al.,\(^1\) if \( \mu < \mu_c \), there exists a critical frequency defined by

\[
0 < \xi^2_c := \sup_{\psi \in A} \frac{k_1|\psi'(1)|^2 + k_0|\psi'(0)|^2 - \int_0^1 \mu|\psi''|^2dy}{\mu \int_0^1 (2|\psi'|^2 + \xi^2 \psi^2)dy} = \sup_{\psi \in A} \frac{-E_0(\psi)}{E_1(\psi, \xi_c)}.
\]

And, under the assumption of \( \mu < \mu_c \), one can point out from the definition of \( \xi_c \) that

\[
0 < \xi^2_c \leq \frac{k_1|\psi'(1)|^2 + k_0|\psi'(0)|^2 - \int_0^1 \mu|\psi''|^2dy}{\mu \int_0^1 2|\psi'|^2dy} \leq \frac{C_0 \int_0^1 |\psi'|^2dy}{2\mu} = \frac{C_0}{2\mu},
\]

which indicates that \( 0 < |\xi| \leq \sqrt{\frac{C_0}{2\mu}} \). Here, \( C_0 \) is the same number defined as in (25). In conclusion, for any \( \mu > 0 \), the functional \( G(\psi) \) is positive on \( A \), provided \( |\xi| > \xi_c \).

Now, we define the solvable domain for frequency

\[
\mathbb{A}^\delta := \{ \xi \in \mathbb{R} | |\xi| \in (a, b) := (|\xi_c|, b) \subset (0, +\infty) \},
\]

where \( 0 < a < b < +\infty \). Next, we will show that for any \( \xi \in \mathbb{A}^\delta \), there exists a fixed point \( s_0 > 0 \) such that \( a(s_0) = -s_0^2 \). To this end, several propositions will be devoted to study some properties of \( a(s) \) in what follows.

**Proposition 3.2.** For any \( \xi \in \mathbb{A}^\delta \) and \( s \in (0, +\infty) \), there exist positive numbers, \( C_1, C_2 \), depending on \( \mu, a, b, \rho, g, k_0, k_1 \), such that \( a(s) \leq sc_2 - C_1 \).

**Proof.** In view of (5), there is a point \( y_0 \in (0, 1) \), such that \( \rho'(y_0) > 0 \). Thus, there exists an open neighborhood of \( y_0 \)

\[
B_{y_0}^\delta := \{ y \in (0, 1) | |y - y_0| < \delta \} \subset (0, 1)
\]

with \( \delta \) small sufficiently, such that \( \rho'(y) > \rho'(y_0)/2 \) in \( B_{y_0}^\delta \). Now, define

\[
\bar{\psi}(y) := f(y - y_0),
\]
where \( f(r) \in C^\infty_0(\mathbb{R}) \) is a cut-off function satisfying \( f(r) = 1 \) for \( |r| < \delta/2 \) and \( f(r) = 0 \) for \( |r| \geq 3\delta/4 \). Then, \( \tilde{\psi}(y) \in H_0^1 \cap H^2 \) and 
\[
E_2(\tilde{\psi}) = \int_0^1 g \xi^2 \tilde{\psi}^2 dy > g \xi^2 \tilde{\psi}(y_0) \delta/4 > 0.
\]

So, for any \( \xi \in L^\infty \), there holds that
\[
\frac{E_2(\tilde{\psi})}{J(\tilde{\psi})} = \frac{g \xi^2 \int_0^1 \tilde{\psi}'^2 dy}{\int_0^1 \tilde{\psi}^2 dy} \geq \frac{ga^2 \int_0^1 \tilde{\psi}'^2 dy}{\int_0^1 \tilde{\psi}^2 dy} := C_1,
\]
and that
\[
\frac{G(\tilde{\psi})}{J(\tilde{\psi})} = \frac{\mu \int_0^1 (|\psi''|^2 + 2\xi^2 |\psi'|^2 + \xi^4 \tilde{\psi}^2) dy - (k_1 |\tilde{\psi}'(1)|^2 + k_0 |\tilde{\psi}'(0)|^2)}{\int_0^1 \tilde{\psi}^2 dy} \leq \frac{\mu \int_0^1 (2b^2 |\psi'|^2 + b^4 \tilde{\psi}^2) dy + C_0 \left\| \tilde{\psi} \right\|_{L^\infty(0,1)}}{\int_0^1 \tilde{\psi}^2 dy} := C_2,
\]
where \( C_1 \) and \( C_2 \) are positive constants depending on \( a, b, \tilde{\rho}, \mu, k_0, k_1 \).

In conclusion, recalling that
\[
\alpha(s) = \inf_{\psi \in H_0^1 \cap H^2, \psi \neq 0} \frac{E(\psi)}{J(\psi)} = \left( \frac{sG(\psi) - E_2(\psi)}{\int_0^1 \tilde{\psi}^2 dy} \right) \leq sC_2 - C_1.
\]

for any \( \xi \in L^\infty \).

**Proposition 3.3.** For any \( \xi \in L^\infty \), \( \alpha(s) \) is a locally Lipschitz continuous function on \((0, \infty)\), that is, \( \alpha(s) \in C^{0,1}_{\text{loc}}(0, \infty) \).

**Proof.** For any \([c, d] \subset (0, \infty)\) and \( s \in [c, d] \), in view of Proposition 3.1, there exists a minimizer \( \psi_s \in \mathcal{A} \), such that \( \alpha(s) = E(\psi_s, s) \). Then, by the definition of \( G(\psi_s, s) \), one has
\[
G(\psi_s, s) = \frac{E(\psi_s, s)}{s} + g \int_0^1 \xi^2 \tilde{\psi}'^2 dy \leq \frac{E(\psi_s, s)}{s} + g \frac{\|\tilde{\psi}'\|_{L^\infty}}{c} := K,
\]
where \( K \) is a positive number depending on \( c, d, \mu, \tilde{\rho}, g, k_0, k_1 \).

Similarly, for any \( s_1, s_2 \in [c, d] \), there exist \( \psi_{s_1}, \psi_{s_2} \in \mathcal{A} \), such that \( \alpha(s_1) = E(\psi_{s_1}, s_1) \) and \( \alpha(s_2) = E(\psi_{s_2}, s_2) \). Furthermore, note that
\[
\alpha(s_1) = E(\psi_{s_1}, s_1) \leq E(\psi_{s_1}, s_1) = E(\psi_{s_1}, s_2) + (s_1 - s_2)G(\psi_{s_1}) \leq \alpha(s_2) + K|s_1 - s_2|.
\]

Hence, one deduces
\[
|\alpha(s_1) - \alpha(s_2)| \leq K|s_1 - s_2|,
\]
which implies that \( \alpha(s) \in C^{0,1}_{\text{loc}}(0, \infty) \). \( \Box \)

By virtue of Proposition 3.2, there exists a constant \( s' > 0 \) such that
\[
\alpha(s) < 0, \text{ for any } s \in (0, s'),
\]
which implies the following proposition

**Proposition 3.4.** For any \( \xi \in L^\infty \), denote \( \mathbb{C} := \inf\{ r \mid \alpha(r) > 0 \} \). Then \( \alpha(s) < 0 \) holds for any \( s \in (0, \mathbb{C}) \).
Next, we will prove the well-definedness of \(-\lambda^2 = a(\lambda)\) with \(\lambda \in (0, \infty)\) by intermediary theorem.

**Proposition 3.5.** For any \(\xi \in \mathbb{H}_s\), there exists \(\lambda \in (0, \infty)\), such that

\[
-\lambda^2(\xi) = a(\lambda(\xi)) = \inf_{\psi \in A} E(\psi, \lambda(\xi)).
\]  

**Proof.** For any \(s_1, s_2 \in (0, \infty), s_1 < s_2\), and denote that \(\psi_{s_i}\) is the minimizer of functional \(E(\psi, s_i)\) on \(A, i = 1, 2\), Then, it follows from \(G(\cdot) > 0\) that

\[
a(s_1) = E(\psi_{s_1}, s_1) \leq E(\psi_{s_2}, s_1) < E(\psi_{s_2}, s_2) = a(s_2),
\]  

This indicates that \(a(s)\) is strictly monotonically increasing on \(s \in (0, \infty)\).

Now, define

\[
\Phi(s) = \frac{s^2}{-a(s)}.
\]

On one hand, by Proposition 3.2, one has

\[
\lim_{s \to 0^+} -a(s) \geq \lim_{s \to 0^+} -(sC_2 - C_1) = C_1 > 0,
\]  

which implies that

\[
\lim_{s \to 0^+} \Phi(s) = \lim_{s \to 0^+} \frac{s^2}{-a(s)} = 0.
\]

On the other hand, from the definition of \(\mathbb{H}\), it is clear that

\[
\lim_{s \to \infty} \Phi(s) = \lim_{s \to \infty} \frac{s^2}{-a(s)} = +\infty.
\]

By virtue of the continuity and monotonicity of \(a(s)\), together with the fact that \(a(s) < 0\) on \((0, \infty)\), we infer that \(\Phi(s)\) is continuous and monotonically increasing on \((0, \infty)\), and hence it follows from intermediary theorem that there exists an unique fixed-point \(s_0 \in (0, \infty)\), such that \(\Phi(s_0) = 1\), in other words, \(-s_0^2 = a(s_0)\). Taking \(\lambda = s_0\), we get \(-\lambda^2 = a(\lambda)\). This proposition follows. \(\square\)

It should be noted that the variational problem (38) achieves its minimizer on \(A\) according to Proposition 3.1. Now we are on the position to show that the minimizer of problem (38) is also a solution to a boundary value problem equivalent to problems (14) and (15), where \(\lambda > 0\).

**Proposition 3.6.** If \(\psi \in A\) is a minimizer of problem (38) and denote \(-\lambda^2 := E(\psi, \lambda)\), then \(\psi\) is a solution to boundary value problems (14) and (15).

**Proof.** For any \(t, r \in \mathbb{R}\) and \(\psi_0 \in H_0^1 \cap H^2\), define \(j(t, r) := J(\psi + t\psi_0 + r\psi)\). Then, \(j(t, r)\) is a smooth function of \((t, r)\) with \(j(0, 0) = 1\). Note that

\[
\partial_t j(0, 0) = 2 \int_0^1 \tilde{\rho} (\tilde{\zeta}^2 \psi \psi_0 + \psi' \psi_0') \, dy, \quad \partial_r j(0, 0) = 2 \int_0^1 \tilde{\rho} (\tilde{\zeta}^2 \psi^2 + |\psi'|^2) \, dy = 2 \neq 0.
\]

By implicit function theorem, there exists a smooth function \(r = r(t)\) near \(t = 0\) satisfying \(r(0) = 0\) and \(j(t, r(t)) = 1\).

Because \(\psi\) is the minimizer of \(E(\cdot)\) on \(A\), one deduces that the single-variable smooth function \(e(t) = E(\psi + t\psi_0 + r(t)\psi)\) reaches its minimum at \(t = 0\), which, by Fermat's Lemma, implies that \(e'(0) = 0\), that is,

\[
e'(0) = 2r'(0)E(\psi) + 2\lambda \left[ \int_0^1 \mu \psi'' \psi_0' dy - k_1 \psi'(1)\psi_0'(1) - k_0 \psi'(0)\psi_0'(0) \right] - 2 \int_0^1 \tilde{\rho} \tilde{\zeta}^2 \tilde{\rho} \psi \psi_0 \, dy + 2\lambda \mu \int_0^1 (2\tilde{\zeta}^2 \psi' \psi_0' + \tilde{\zeta}^4 \psi \psi_0) \, dy = 0.
\]
To get \( r'(0) \), we differentiate the equation \( j(t, r(t)) = 1 \) at \( t = 0 \) and yields

\[
2r'(0)J(\psi) + 2 \int_0^1 \tilde{\rho} (\tilde{\xi}^2 \psi \psi_0 + \psi' \psi_0') \, dy = 0,
\]

which implies that

\[
r'(0) = - \int_0^1 \tilde{\rho} (\tilde{\xi}^2 \psi \psi_0 + \psi' \psi_0') \, dy.
\]  (45)

Substituting (45) into (44) and using \( E(\psi) = -\lambda^2 \), one sees that

\[
-\lambda^2 \left[ \int_0^1 \tilde{\rho} (\tilde{\xi}^2 \psi \psi_0 + \psi' \psi_0') \, dy \right] = \lambda \left[ \int_0^1 \mu \psi'' \psi_0'' \, dy - (k_1 \psi'(1)\psi_0'(1) + k_0 \psi'(0)\psi_0'(0)) \right] + \lambda \mu \int_0^1 (2\tilde{\xi}^2 \psi' \psi_0' + \tilde{\xi}^4 \psi \psi_0) \, dy - \int_0^1 g \xi^2 \tilde{\rho}' \psi_0 \, dy.
\]  (46)

Taking \( \psi_0 \in C^\infty(0, 1) \) in (46) infers that the minimizer \( \psi \) satisfies (14) in the weak sense. Then, by standard bootstrap arguments, one can deduce that \( \psi \) is smooth.

Now, it remains to show that the minimizer \( \psi \) satisfies boundary conditions (15). In fact, because \( \psi \in \mathcal{A} \), it is clear that \( \psi(0) = \psi(1) = 0 \). Moreover, because \( \psi_0 \in H^1_{0} \cap H^2 \), in view of (14), applying integration by parts to (46), one obtains that

\[
\mu \psi''(1) - k_1 \psi'(1)\psi_0'(1) = \left( \mu \psi''(0) + k_0 \psi'(0) \right) \psi_0'(0).
\]

As \( \psi_0 \) being arbitrary, we deduce that

\[
\psi''(1) = \frac{k_1}{\mu} \psi'(1), \quad \psi''(0) = - \frac{k_0}{\mu} \psi'(0).
\]  (47)

which implies that \( \psi \) satisfies boundary conditions (15).

In conclusion, \( \psi \) is the solution to (14) with boundary condition (15). \( \square \)

So far, we have proved that there exists smooth solution \( \psi(y) \) with the corresponding eigenvalue \( \lambda > 0 \) for problems (14) and (15). To ensure the validity of Fourier synthesis in constructing exponential growing mode solution to (8) to (7), we still need some properties of function \( \lambda(\xi) \).

**Proposition 3.7.** The eigenvalue function \( \lambda(\xi) : \mathbb{H}^k \to (0, \infty) \) defined by (38) is continuous and bounded.

**Proof.** Because the definition of \( -\lambda^2 = \inf_{\psi \in A} E(\psi, \lambda) \) is equivalent to \( \lambda^2 = \sup_{\psi \in A} (-E(\psi, \lambda)) \), it follows from (25) that

\[
\lambda^2 \leq -E(\psi) + 1 \leq g \left\| \frac{\partial^2}{\partial \xi^2} \right\|_{L^\infty} + \lambda C_0 \left\| \frac{1}{\tilde{\rho}} \right\|_{L^\infty} + 1,
\]

which indicates the boundedness of \( \lambda(\xi) \) on \( \mathbb{H}^k \).

We now turn to the proof of the continuity, which is similar to Proposition 2.5 in Jiang et al.\(^5\) For the reader’s convenience, we give the details here to make it more clear. For any fixed \( \xi_0 \in \mathbb{H}^k \), let \( \xi \in \mathbb{H}^k \) with \( \kappa = |\xi|^2 - |\xi_0|^2 \), then \( |\xi| \to |\xi_0| \) as \( \kappa \to 0 \). The first step is to show that

\[
\lim_{|\xi| \to |\xi_0|} \alpha(\xi, s) = \alpha(\xi_0, s), \quad \text{for any } s \in (0, \infty).
\]  (48)

In view of Proposition 3.1, for any \( \xi \in \mathbb{H}^k \), there exists \( \psi_\xi \in \mathcal{A} \), such that

\[
\alpha(\xi, s) = \int_0^1 s \mu \left( |\psi_\xi''|^2 + 2\tilde{\xi}^2 |\psi_\xi'|^2 + \tilde{\xi}^4 |\psi_\xi|^2 \right) - s \sum_{i=0}^1 k_i |\psi_\xi'(i)|^2 - \int_0^1 g \xi^2 \tilde{\rho}' \tilde{\xi}^2.
\]  (49)
Substituting $|\xi|^2 = |\xi_0|^2 + \kappa$ into (49), we get

$$\alpha(\xi, s) = \int_0^1 s \mu \left( |\psi''(\xi)|^2 + 2 \xi_0^2 |\psi'(\xi)|^2 + \xi_c^2 s^2 \right) - s \sum_{i=0}^1 k_i |\psi'(\xi)|^2 - \int_0^1 g_{\xi_0}^2 \rho' |\psi|^2 + \kappa \int_0^1 \left[ s \mu \left( 2 |\psi'(\xi)|^2 + 2 \xi_0^2 |\psi'(\xi)|^2 + \kappa |\psi|^2 \right) - g_{\xi_0}^2 \rho' |\psi|^2 \right]$$

$$\geq \alpha(\xi_0, s) + \kappa f(\kappa, \psi_c),$$

where

$$f(\kappa, \psi_c) := \int_0^1 \left[ s \mu \left( 2 |\psi'(\xi)|^2 + 2 \xi_0^2 |\psi'(\xi)|^2 + \kappa |\psi|^2 \right) - g_{\xi_0}^2 \rho' |\psi|^2 \right].$$

Because $\psi \in A$, there exists constant $c$, depending on $\rho, g, a, b$ and $s$, such that $|f(\kappa, \psi_c)| \leq c$.

Similarly, we also have

$$\alpha(\xi_0, s) \geq \alpha(\xi, s) - \kappa f(-\kappa, \psi_{\xi_0}),$$

and there exists a constant $c$ depending on $\rho, g, a, b$ and $s$, such that

$$|f(-\kappa, \psi_{\xi_0})| \leq c.$$

Thus,

$$\kappa f(\kappa, \psi_c) \leq \alpha(\xi, s) - \alpha(\xi_0, s) \leq \kappa f(-\kappa, \psi_{\xi_0}).$$

The boundedness of $f(\kappa, \psi_c)$ and $f(-\kappa, \psi_{\xi_0})$ implies that, as $\kappa \to 0$, one has

$$\lim_{|\kappa| \to 0} \alpha(\xi, s) = \alpha(\xi_0, s), \quad \text{for any } s \in (0, \infty).$$

Now, denote $\dot{\lambda}(\xi, s) = \sqrt{-\alpha(\xi, s)}$, then it follows from (53) that

$$\lim_{|\kappa| \to 0} \dot{\lambda}(\xi, s) = \dot{\lambda}(\xi_0, s) \quad \text{for any } s \in (0, \infty).$$

In the second step, by virtue of Proposition 3.5, for any $\xi \in \mathbb{H}^e$, there exists a unique $s_\xi \in (0, \infty)$, such that

$$s_\xi = \dot{\lambda}(\xi, s_\xi) \equiv \dot{\lambda}(\xi).$$

Then, the main aim of this proposition is to prove

$$\lim_{|\xi| \to |\xi_0|} \dot{\lambda}(\xi, s_\xi) = \dot{\lambda}(\xi_0, s_\xi).$$

In view of (54), for any $\varepsilon > 0$, there exists a constant $\delta > 0$, as $|\xi| - |\xi_0| < \delta$, so that

$$\left| \dot{\lambda}(\xi, s_\xi) - \dot{\lambda}(\xi_0, s_\xi) \right| < \varepsilon.$$ 

Moreover, because that for any $\xi \in \mathbb{H}^e$, $\alpha(\xi, s)$ is monotonically increasing on $s \in (0, \infty)$, refer to (39), one also has that $\dot{\lambda}(\xi, s)$ is decreasing with respect to $s \in (0, \infty)$.

Now, using (55), (57), and the monotonicity of $\dot{\lambda}$ with respect to $s$, we are able to prove (56). If $s_\xi \leq s_{\xi_0}$, then

$$\dot{\lambda}(\xi_0, s_\xi) - \varepsilon < \dot{\lambda}(\xi, s_\xi) \leq \dot{\lambda}(\xi, s_\xi) = s_\xi \leq s_{\xi_0} = \dot{\lambda}(\xi_0, s_\xi) < \dot{\lambda}(\xi_0, s_{\xi_0}) + \varepsilon.$$ 

On the contrary, if $s_\xi \geq s_{\xi_0}$, then

$$\dot{\lambda}(\xi_0, s_{\xi_0}) - \varepsilon < \dot{\lambda}(\xi_0, s_{\xi_0}) = s_{\xi_0} \leq s_\xi = \dot{\lambda}(\xi, s_\xi) \leq \dot{\lambda}(\xi, s_{\xi_0}) < \dot{\lambda}(\xi_0, s_{\xi_0}) + \varepsilon.$$
In conclusion, (56) follows from the above two inequalities, and the proof of this proposition is completed. \(\square\)

Now, we are able to construct solutions \((\phi, \psi, \pi)\) to ODE systems (12) and (13).

**Proposition 3.8.** For any \(\xi \in \mathbb{R}^d\), there exist solutions \((\phi, \psi, \pi)\) to systems (12) and (13) with the corresponding eigenvalue \(\lambda(\xi) > 0\). Furthermore, \((\phi, \psi, \pi)(y) \in H^k\) for any \(k \in \mathbb{N}\).

*Proof.* In view of Propositions 3.1 and 27, we have constructed solution \(\psi(\xi, y)\) to problems (14) and (15) with \(\psi \in \mathcal{A} \cap H^k\), for any \(k \in \mathbb{N}\). According to (12) and (12), we have

\[
\phi(\xi, y) = -\xi^{-1} \psi', \quad \pi(\xi, y) = -\xi^{-1} (\lambda \bar{\rho} \phi + \mu \xi^2 \phi - \mu \phi''),
\]

which gives rise to solutions \((\phi, \psi, \pi)\) to (12) and also indicates that \((\phi, \psi, \pi)(y) \in H^k\). In addition, boundary conditions (13) follows directly from (15) and (12). \(\square\)

**Remark 3.9.** In view of the definition of functional \(E(\psi)\) and the definition of \(\lambda(\xi)\) in (38), it is clear that the function \(\lambda(\xi)\) with respect to \(\xi\) is an even function on \(\mathbb{R}^d\). In the meantime, the associated \(\psi(\xi)\) constructed in Proposition 27 is also an even function with respect to \(\xi\) defined on \(\mathbb{R}^d\). Subsequently, in view of Proposition 3.8, the corresponding function \(\pi(\xi, y)\) is also an even function on \(\xi \in \mathbb{R}^d\), while \(\phi(\xi, y)\) is an odd function on \(\xi \in \mathbb{R}^d\).

The next proposition provides \(H^k\)– estimates for the solution \((\phi, \psi, \pi)(y)\) with \(\xi\) varying. To emphasize the dependence on \(\xi\), we denote the solution by \((\phi, \psi, \pi)(\xi) := (\phi, \psi, \pi)(\xi, y)\).

**Proposition 3.10.** For any \(\xi \in \mathbb{R}^d\), let \((\phi, \psi, \pi)(\xi)\) with the corresponding function \(\lambda(\xi)\) be solution to problems (12) and (13), constructed as above. Then, for any \(k \in \mathbb{N}\), there exist positive constants \(A_k, B_k, C_k\) depending on \(a, b, \bar{\rho}, \mu, k_0, k_1, g\), such that

\[
\|\psi(\xi)\|_{H^k} \leq A_k, \quad \|\phi(\xi)\|_{H^k} \leq B_k, \quad \|\pi(\xi)\|_{H^k} \leq C_k.
\]

*Proof.* Throughout the proof, \(\bar{C}\) is a generic positive number depending on \(a, b, \bar{\rho}, \mu, k_0, k_1, g\).

Firstly, because \(\psi \in \mathcal{A}\), one gets \(\|\psi\|_{L^2} > 0\). Furthermore, there exists constant \(A_1\), such that

\[
\|\psi(\xi)\|_{H^k} \leq A_1.
\]

Secondly, Proposition 3.5 implies that there exists \(\psi(\xi) \in \mathcal{A}\), such that

\[
-\lambda^2(\xi) = \alpha(\xi, \lambda) = E(\lambda(\xi), \psi).
\]

Rewriting (58), similar to (25), one can deduce that

\[
\lambda \mu \int_0^1 |\psi''|^2 dy = -\lambda^2 + \lambda \left[ \sum_{i=0}^1 k_i |\psi'(i)|^2 - \mu \int_0^1 \xi^2 (2|\psi'|^2 + \xi^2 \psi^2) dy \right] + \int_0^1 g \xi^2 \bar{\rho} \psi^2 dy
\]

\[
\leq -\lambda^2 + \frac{\lambda \mu}{2} \int_0^1 |\psi''(\xi)|^2 dy + \lambda C_0 \left\| \frac{1}{\bar{\rho}} \right\|_{L^\infty(0,1)} + g \left\| \frac{\bar{\rho}'}{\bar{\rho}} \right\|_{L^\infty(0,1)}.
\]

According to Proposition 3.7, \(\lambda\) has positive bounds from upper and lower, and hence

\[
\|\psi(\xi)\|_{H^k} \leq A_2.
\]

Thirdly, equation (14) can be rewritten as

\[
\psi''(\xi) = \frac{\lambda \mu (2\xi^2 \psi'' - \xi^4 \psi) + g \xi^2 \bar{\rho} \psi' - \lambda^2 [\xi^2 \bar{\rho} \psi - (\bar{\rho} \psi')']}{\lambda \mu} = \frac{(2\lambda \mu \xi^2 + \lambda \xi^4) \psi'' + \lambda \xi^2 \bar{\rho} \psi' + \xi^2 (g \bar{\rho}' - \lambda \mu \xi^2 - \lambda^2 \bar{\rho}) \psi}{\lambda \mu}.
\]
which infers that

\[ \|\psi^{(4)}(\xi)\|_{L^2} \leq \tilde{C}. \]

Applying Gagliardo–Nirenberg interpolation inequality, one has

\[ \left\| \psi'' \right\|_{L^2} \leq \left\| \psi'' \right\|_{L^2}^{\frac{1}{2}} \left\| \psi^{(4)} \right\|_{L^2}^{\frac{1}{2}} \leq \tilde{C}. \]

In conclusion, one deduces

\[ \left\| \psi(\xi) \right\|_{H^r} \leq A_4. \] (61)

Differentiating equation (60) with respect to \( y \) and using (61), we find, by induction on \( k \), that \( \|\psi(y)\|_{H^k} \leq A_k \), for any \( k \in \mathbb{N} \). Note that \( \phi, \pi \) can be expressed by \( \psi \) and some constants, the rest inequalities then follows.

In the next proposition, we will construct the exponentially growing solutions of linearized problems (8) and (7).

**Proposition 3.11.** Under the assumptions of Theorem 1.1, let

\[ \Lambda = \sup_{|\xi| \in A \cap \mathbb{R}^+ \mathbb{Z}} \lambda(\xi), \] (62)

then there exist a positive constant \( \Lambda^* \in (2\Lambda/3, \Lambda] \) and a real-valued solution \((\phi, u, q)\) to the linearized problems (7) and (8) defined on the horizontally periodic domain \( \Omega \), such that

1. For any \( k \in \mathbb{N} \),

\[ \|(\phi, u, q)(0)\|_{H^k} < +\infty; \] (63)

2. For any \( t > 0 \), \( (\phi, u, q) \in H^k \), and

\[ \|(\phi, q)(t)\|_{H^k} = e^{\lambda^* t} \|(\phi, q)(0)\|_{H^k}, \] (64)

\[ \|u_i(t)\|_{H^k} = e^{\lambda^* t} \|u_i(0)\|_{H^k}, \quad i = 1, 2. \] (65)

3. Moreover,

\[ \text{div} u(0) = 0, \quad \|u_1(0)\|_{L^2} \|u_2(0)\|_{L^2} > 0 \] (66)

**Proof.** Denote

\[ w(\xi, y) = -id\phi(\xi, y)e_1 + \psi(\xi, y)e_2, \] (67)

where \((\phi, \psi)\) with an associated growth rate \( \lambda(\xi) \) is constructed in Proposition 3.8 for any given \( \xi \in A^\mathbb{N} \). Recalling the definition of \( \Lambda \), there exist \( \xi \in A^\mathbb{N} \cap L^{-1} \mathbb{Z} \), such that

\[ \Lambda^* := \lambda(\xi) = \lambda(-\xi) \in (2\Lambda/3, \Lambda]. \]

In view of Remark 3.9, the following real-value functions

\[ \phi(x, y; t) = -e^{\lambda^* t} \tilde{\phi}(y)w_2(\xi, y) \left[ e^{iy} + e^{-iy} \right], \] (68)

\[ u(x, y; t) = \Lambda^* e^{\lambda^* t} \left[ w(\xi, y)e^{iy} + w(-\xi, y)e^{-iy} \right], \] (69)

\[ q(x, y; t) = e^{\lambda^* t} \pi(\xi, y) \left[ e^{iy} + e^{-iy} \right]. \] (70)

constitute a horizontally periodic, real-value solution to linearized problems (7) and (8) with

\[ \|(\phi, u, q)(0)\|_{H^k} \leq M_k, \quad \|u_2(0)\|_{L^2} > 0. \] (71)

where \( M_k \) are positive numbers depending on \( A_k, B_k, C_k \) in Proposition 3.10. In addition, \( \|u_1(0)\|_{L^2} \|u_2(0)\|_{L^2} > 0 \) is clear, since that \( \psi \in A \) and \( \xi \phi + \psi' = 0 \), and thus (71) follows.
It remains to prove (64) and (65). We take \( q(x, y; t) \) for an example, and the others can be done similarly. Recalling the expression of \( q(x, y; t) \), one sees that

\[
\|q(t)\|_{H^k} = 2\Lambda^* e^{\Lambda^*t}\|\pi(x, y)\|_{H^k} = e^{\Lambda^*t}\|q(0)\|_{H^k}. \tag{72}
\]

The proof of this proposition is completed.

\[\square\]

4 | ENERGY ESTIMATES FOR THE NONLINEAR PERTURBED PROBLEM

As a preparation to prove nonlinear instability in next section, referring to other studies,\(^5,24,25\) we establish some energy estimates for nonlinear perturbation equations in this section.

Suppose that \((\rho, u, q)\) is a strong solution to nonlinear problems (6) and (7). Then, for convenience, we denote

\[
\mathcal{E}(t) := \mathcal{E}(\rho, u)(t) = \sqrt{\|\rho(t)\|_{H^1}^2 + \|u(t)\|_{H^2}^2}, \tag{73}
\]

\[
\mathcal{E}_0 := \mathcal{E}(\rho_0, u_0) = \sqrt{\|\rho_0\|_{H^1}^2 + \|u_0\|_{H^2}^2}, \tag{74}
\]

and assume that there exists a constant \(\delta_0 \in (0, 1)\) depending only on \(\mu, \bar{\rho}, \hat{\rho}, k_0, k_1\), such that

\[
\mathcal{E}(t) \leq \delta_0. \tag{75}
\]

In this section, \(C\) is denoted as a generic positive number depending only \(\mu, \bar{\rho}, \hat{\rho}, k_0, k_1\).

For \(x \in \Omega\), we define \(X \in \Omega\) as below

\[
\begin{cases}
\frac{dX(x,t)}{dt} = v(X(x, t), t), \\
X(x, 0) = x,
\end{cases}
\]

where \(v\) is the velocity of the fluid given in (1). Then, recalling (1)_1, one sees that

\[
\frac{d}{dt}\rho(X(x, t), t) = \rho_t + \frac{dX}{dt} \cdot \nabla \rho = \rho_t + v \cdot \nabla \rho = 0,
\]

which implies

\[
\rho(X(x, t), t) = \rho(X(x, 0), 0) = \rho_0(x). \tag{76}
\]

Denote that

\[
\alpha := \inf_{x \in \Omega} \{\rho_0(x) > 0,\} \quad \beta := \sup_{x \in \Omega} \{\rho_0(x)\} < \infty. \tag{77}
\]

Then, for any \(t \in (0, T)\), it follows from (76) that

\[
0 < \alpha \leq \rho(x, t) \leq \beta < \infty, \tag{78}
\]

in other words,

\[
\alpha - \bar{\rho} \leq \rho(x, t) \leq \beta - \bar{\rho}, \tag{79}
\]

which will be used to show the boundedness of density \(\rho\) and \(\varrho\) in the rest of this section.

4.1 Estimates for \(\|\varrho\|_{L^2}\) and \(\|u\|_{L^2}\)

Multiplying (6)_1 and (6)_2 by \(\varrho\) and \(u\), respectively, integrating by parts over \(\Omega\), and using (6)_3, we have

\[
\frac{1}{2} \frac{d}{dt} \int \varrho^2 + \int \varrho' u_2 = 0, \tag{80}
\]

\[
\frac{1}{2} \frac{d}{dt} \int \rho|u|^2 + \mu \int |\nabla u|^2 - \int \sum_{i=0}^{1} k_i |u_i(x, i)|^2 \ dx = -\int g u_2. \tag{81}
\]
Adding them up gives
\[
\frac{1}{2} \frac{d}{dt} \int (\rho^2 + \rho|\mathbf{u}|^2) + \mu \int |\nabla \mathbf{u}|^2 \\
= -\int (\tilde{\rho'} + g) \rho \mathbf{u}_t + \int_{2 \Omega T} (k_1 |u_1(x, 1)|^2 + k_0 |u_1(x, 0)|^2) dx \quad := I_1 + I_2.
\] (82)

As for \( I_1 \), using Hölder inequality and Cauchy inequality, together with \( \tilde{\rho}(\cdot) \in C^\infty_0(0, 1) \), one gets
\[
I_1 \leq (\| \tilde{\rho}' \|_{L^\infty} + g) \| \rho \|_{L^2} \| \mathbf{u} \|_{L^2} \leq \frac{(\| \tilde{\rho}' \|_{L^\infty} + g)}{2} \| (\rho, \mathbf{u}) (t) \|_{L^2}^2.
\] (83)

To estimate \( I_2 \), similar to (25), one also has
\[
I_2 < 2C_0 \| \mathbf{u} \|_{L^2}^2 + \frac{\mu}{2} \| \nabla \mathbf{u} \|_{L^2}^2.
\] (84)

Substituting (83) and (84) into (82) and using Poincaré inequality (17), we deduce
\[
\frac{d}{dt} \left( \| (\rho, \sqrt{\rho} \mathbf{u}) (t) \|_{L^2}^2 + \mu \| \mathbf{u} \|_{H^1}^2 \right) \leq C_1 \left( \| (\rho, \sqrt{\rho} \mathbf{u}) (t) \|_{L^2}^2 \right),
\] (85)

which, together with Gronwall inequality and (74), gives
\[
\left\| (\rho, \sqrt{\rho} \mathbf{u}) (t) \right\|_{L^2}^2 + \mu \int_0^t \| \mathbf{u}(s) \|_{H^1}^2 ds \leq \beta \delta_0^2 e^{C_1 t}.
\] (86)

Furthermore, using (78), one yields
\[
\| \rho(t) \|_{L^2}^2 + \| \mathbf{u}(t) \|_{L^2}^2 + \int_0^t \| \mathbf{u}(s) \|_{H^1}^2 ds \leq C \delta_0^2 e^{C_1 t}.
\] (87)

### 4.2 Estimates for \( \| \mathbf{u} \|_{H^1} \) and \( \| \mathbf{u} \|_{H^2} \)

First, multiplying (6) by \( \mathbf{u} \) and integrating by parts over \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt} \int \mu |\nabla \mathbf{u}|^2 + \int \rho |\mathbf{u}_t|^2 \\
= \int 2 \sum_{i=0}^{1} k_i u_1(x, i) \partial_i u_1(x, i) \ dx - \int \rho g \partial_i u_2 - \int \rho \mathbf{u} \cdot \nabla \mathbf{u} \\
: = J_1 + J_2 + J_3.
\] (88)

Similar to (83) and (84), we also have
\[
J_1 \leq \epsilon \| \sqrt{\rho} \mathbf{u} \|_{L^2}^2 + C_\epsilon \| \mathbf{u} \|_{H^1}^2 + \epsilon \| \nabla \mathbf{u} \|_{L^2}^2,
\] (89)

\[
J_2 \leq \int \left| \frac{\rho g}{\sqrt{\rho}} \right| \left| \sqrt{\rho} \mathbf{u} \right| \leq \epsilon \| \sqrt{\rho} \mathbf{u} \|_{L^2}^2 + C_\epsilon \| \rho \|_{L^2}^2.
\] (90)

For \( J_3 \), using Cauchy inequality and (78), we obtain
\[
J_3 \leq \| \sqrt{\rho} \mathbf{u} \|_{L^2}^2 \| \sqrt{\rho} \mathbf{u} \cdot \nabla \mathbf{u} \|_{L^2} \leq \epsilon \| \sqrt{\rho} \mathbf{u} \|_{L^2}^2 + C_\epsilon \| \mathbf{u} \|_{H^1}^2 \| \mathbf{u} \|_{L^2}^2.
\] (91)
Second, note that \((u, q)\) satisfies
\[
\begin{cases}
-\mu \Delta u + \nabla q = -\rho u_t - \rho u \cdot \nabla u - \rho g e_2, & \text{in } \Omega, \\
\text{div}(u) = 0, & \text{in } \Omega, \\
u_2(x, 1) = u_2(x, 0) = 0, & \text{in } x \in 2\pi L T, \\
\partial_x u_1(x, 1) = \frac{1}{\mu} u_1(x, 1), & \text{in } x \in 2\pi L T, \\
\partial_x u_1(x, 0) = -\frac{k}{\mu} u_1(x, 0), & \text{in } x \in 2\pi L T.
\end{cases}
\] (92)

Then, it follows from the Stokes estimates, reference to theorem A.1 in,\(^1\) that
\[
\|u\|_{H^2} \leq \|\sqrt{\rho} u\|_{L^2} + \|\sqrt{\rho} u \cdot \nabla u\|_{L^2} + g\|\rho\|_{L^2} + \|u\|_{L^2}.\] (93)

Substituting (89)-(91) into (88) and adding the result to (93), we deduce
\[
\frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2} \|\sqrt{\rho} u\|_{L^2}^2 + \frac{1}{8C_2} \|u\|_{H^2}^2 \leq \|(\rho, u)\|_{L^2}^2 + C_{\epsilon} \|u\|_{H^2}^2 + \|\rho \cdot \nabla u\|_{L^2}^2. \] (94)

To control \(\|\rho \cdot \nabla u\|_{L^2}^2\), by Lemmas 2.3 and 2.4, together with (75), we get
\[
\|\rho \cdot \nabla u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 \|\rho\|_{L^2} \leq C(\|u\|_{H^2}(\|u\|_{H^2})(\|u\|_{L^2} \|u\|_{H^2})) \\
\leq C\|u\|_{L^2}^2 \|u\|_{L^2} \leq C\delta^2 \|u\|_{L^2}^2. \] (95)

Substituting (95) into (94) gives
\[
\frac{\mu}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{H^2}^2 \leq C(\|\rho, u\|_{L^2}^2 + C_{\epsilon} \|u\|_{H^2}^2 + \|\nabla u\|_{L^2}^2). \] (96)

Third, because \(u = v - 0\) satisfies (1)_2, one has
\[
\rho u_t + \rho u \cdot \nabla u + \nabla p = \mu \Delta u - \rho g e_2,
\]
differentiating which with respect to \(t\) gives
\[
\rho u_{tt} + \rho u \cdot \nabla u_t - \mu \Delta u_t + \nabla p_t = -\rho_t (u_t + u \cdot \nabla u + g e_2) - \rho u_t \cdot \nabla u. \] (97)

Then, testing (97) by \(u_t\), integrating by parts over \(\Omega\) and using (6), one yields
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t(t)|^2 + \mu \int |\nabla u_t|^2 - \int_{2\pi L T} (k_1 |\partial_t u_1(x, 1)|^2 + k_0 |\partial_t u_1(x, 0)|^2) dx \\
= \int \nabla \cdot (\rho u_t) (u_t + u \cdot \nabla u + g e_2) \cdot u_t - \int \rho u_t \cdot \nabla u \cdot u_t,
\] (98)

which implies that
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t(t)|^2 + \mu \int |\nabla u_t|^2 \\
\leq \int \sum_{j=0}^7 k_j |\partial_t u_1(x, i)|^2 dx + \int [2\rho |u_t| |u_t| |\nabla u_t| + \rho |u_t| |u_t| |\nabla u_t|^2 + \rho |u_t|^2 |u_t| |\nabla^2 u_t| \\
+ \rho |u_t|^2 |\nabla u_t| + \rho |u_t|^2 |\nabla u_t| + g\rho |u_t| |\nabla u_t|] := \sum_{j=1}^7 K_j. \] (99)
Now, we estimate each term $K_i$ above, in which (17)-(21), Hölder inequality, Young inequality, and Poincaré inequality may be used.

\[
K_1 \leq 2C_0 \|\mathbf{u}\|^2_{L^2} + \frac{\mu}{2} \|\nabla \mathbf{u}\|^2_{L^2} \leq C \|\sqrt{\rho} \mathbf{u}\|^2_{L^2} + \frac{\mu}{2} \|\nabla \mathbf{u}\|^2_{L^2},
\]

\[
K_2 \leq \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|_{L^2} \leq \left( \|\mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{u}\|_{L^2}^{1/2} \right) \left( \|\mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{u}\|_{L^2}^{1/2} \right) \|\nabla \mathbf{u}\|_{L^2}.
\]

Consequently, integrating (101) with respect to the time variable on $(0, t)$ and we obtain

\[
\frac{d}{dt} \left\| \rho \sqrt{\nabla \mathbf{u}}(\mathbf{u}, \nabla \mathbf{u}) \right\|_{L^2}^2 + \|\mathbf{u}\|^2_{H^2} + \|\mathbf{u}\|^2_{H^2} \leq C \left\| \rho \mathbf{u} \right\|_{L^2}^2.
\]
Furthermore, recalling that \((u, q)\) solves the following Stokes equation

\[ -\mu \Delta u + \nabla q = -\rho u_t - \rho u \cdot \nabla u - \rho \varepsilon, \tag{104} \]

Then, by virtue of Stokes estimate and \(L^4\) estimate, we have

\[ \|u\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \leq \|\rho u_t\|_{L^2}^2 + \|\rho u \cdot \nabla u\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 \]
\[ \leq \|u\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|\rho u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \]
\[ \leq C \left( \|u\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|\rho u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \right) \]
\[ \leq C \left( \|\rho u\|_{L^2}^2 + \|\varepsilon\|_{L^2}^2 + \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \right) + \epsilon \|u\|_{L^2}^2, \tag{105} \]

which, together with \(\mathcal{E}(t) \leq \delta\), implies that

\[ \|u\|_{L^2}^2 + \|\nabla q\|_{L^2}^2 \leq C \|u, \nabla u\|_{L^2}^2. \tag{106} \]

Finally, add (106) to (103), for any \(t \in (0, T]\), we have

\[ \|u\|_{L^2}^2 + \|\phi, u_t, \nabla q\|_{L^2}^2 + \int_0^t \left( \|u_t(s)\|_{L^2}^2 + \|u(s)\|_{L^2}^2 \right) \, ds \]
\[ \leq C \left( \mathcal{E}_0^2 + \int_0^t \|\phi, u(s)\|_{L^2}^2 \, ds \right). \tag{107} \]

### 4.3 Estimates for \(\|\phi\|_{H^1}\)

Similar to the preparation part in Section 4, for any \(x \in \Omega\), we define the streamline function \(X = X(x, t)\) by

\[
\begin{cases}
\frac{dX(x,t)}{dt} = u(X(x,t), t), \\
X(x, 0) = x,
\end{cases}
\]

so that

\[
\frac{d}{dt} \phi(X(x, t), t) = \phi_t(X(x, t), t) + \frac{dX(x,t)}{dt} \cdot \nabla \phi(X(x,t), t) \\
= \phi_t + u(X(x,t), t) \cdot \nabla \phi(X(x,t), t) \\
= -u_2(X(x,t), t)\phi'(X_2(x,t)). \tag{108}
\]

Integrating it over \((0, t)\), we have

\[
\phi(X(x, t), t) = \phi_0(X(x, 0)) - \int_0^t u_2(X(x, s), s)\phi'(X_2(x, s)) \, ds. \tag{109}
\]

Then, there holds that

\[
\|\phi_t\|_{H^1} \leq \|\phi(x)\|_{H^1} + \|\phi'(x)\|_{L^4} \int_0^t \|u_2(x, s)\|_{H^1} \, ds \]
\[ \leq \|\phi_0\|_{H^1} + CT \sup_{0 \leq s \leq T} \|u(s)\|_{H^1}. \tag{110} \]

It remains to estimate \(\|u(s)\|_{H^1}\). In fact, applying Gronwall inequality to (107) yields

\[
\sup_{0 \leq s \leq T} \|u(s)\|_{H^1}^2 \leq C(T)\mathcal{E}_0^2. \tag{111}
\]

Then, one obtains

\[ \|\phi(t)\|_{H^1} \leq C(T)\mathcal{E}_0^2. \tag{112} \]
In conclusion, it follows from (107) and (112) that

\[
\mathcal{E}^2(t) + \| \mathbf{v}(t) \|^2_{L^2} + \int_0^t \left( \| \mathbf{v}(s) \|^2_{H^1} + \| \mathbf{u}(s) \|^2_{H^2} \right) \, ds \leq C(T)\mathcal{E}_0^2.
\]

(113)

Compared with classical well-posedness results of nonhomogeneous Navier–Stokes equations in Choe and Kim and Simon, the boundary integral terms produced by the Navier-slip boundary condition in this paper bring some new difficulties in the energy estimates. However, these difficulties can be overcome by using the technique applied in Li and Ding. Thus, based on estimate (113), the global well-posedness of the nonlinear systems (6) and (7) follows. We state this result below without the proof.

**Proposition 4.1.** Suppose that the steady state satisfies (3). Then for any given initial data \((\rho_0, \mathbf{u}_0) \in (H^1 \cap L^\infty) \times H^2\) satisfying (74), (77), \(\nabla \cdot \mathbf{u}_0 = 0\), and also being compatible with the boundary conditions (2), the nonlinear problems (6) and (7) have a global strong solution \((\rho, \mathbf{u}, \nabla \mathbf{q}) \in C([0, T] ; H^1 \times H^2 \times L^2)\), such that

\[
\mathcal{E}^2(t) + \| \mathbf{v}(t) \|^2_{L^2} + \int_0^t \left( \| \mathbf{v}(s) \|^2_{H^1} + \| \mathbf{u}(s) \|^2_{H^2} \right) \, ds \leq C(T)\mathcal{E}_0^2,
\]

where the positive number \(C(T)\) depends only on \(g, \mu, \alpha, \beta, \bar{\rho}\), and also \(T\).

5 | THE PROOF OF NONLINEAR INSTABILITY

In this section, we will prove the nonlinear instability by the bootstrap argument proposed by Guo and Strauss in their study. To be precise, we will show that there exists a constant \(\epsilon > 0\) such that for any \(\delta_0\), though small enough, and the initial data is smaller then \(\delta_0\) in some suitable sense, the nonlinear systems (6) and (7) admits a strong solution \(\mathbf{u}^\delta\) and an escape time \(T^\delta > 0\) such that \(\| \mathbf{u}(T^\delta) \|_{L^2} > \epsilon\).

To this end, we first give the following elementary inequality, which will be used in the proof of Proposition 5.2.

**Lemma 5.1.** Let \(\mathbf{w} \in H^1_0 \cap H^2\), where \(H^1_0 = \{ \mathbf{w} \in H^1 \mid \text{div} \mathbf{w} = 0, w_2 = 0 \text{ on } \Sigma_1 \cup \Sigma_0 \}\), then it holds that

\[
\int g\bar{\rho}|w_2|^2 + \Lambda^2 \sum_{i=0}^1 \int_{2T\Omega} |w_1(x, i)|^2 \, dx \leq \Lambda^2 \int \bar{\rho}|\mathbf{w}|^2 + \Lambda \mu \int |\nabla \mathbf{w}|^2.
\]

(115)

**Proof.** Because the proof is similar and simpler to the Step 1 of Lemma 4.1 in Jian et al., we omit the details here. ☐

According to Proposition 3.11, there exists constant \(A^* \in (2A/3, A]\), such that the exponentially increasing functions

\[
(\rho^\delta, \mathbf{u}^\delta) = e^{\lambda^\delta t}(\bar{\rho}_0, \bar{\mathbf{u}}_0) \in H^2 \times (H^1_0 \cap H^2) \text{ for each } t > 0
\]

satisfy the linearized systems (8) and (7) with an associated pressure \(q^\delta = e^{\lambda^\delta t}\bar{q}_0\), where \(\bar{q}_0 \in H^1\) and \((\bar{\rho}_0, \bar{\mathbf{u}}_0) \in H^2 \times (H^1_0 \cap H^2)\) satisfy

\[
\| \bar{\rho}_0 \|_2 \| \bar{u}_{02} \|_2 \| \bar{u}_{01} \|_2 \geq 0, \quad \mathcal{E}(\bar{\rho}_0, \bar{\mathbf{u}}_0) = \sqrt{\| \bar{\rho}_0 \|_{H^1}^2 + \| \bar{\mathbf{u}}_0 \|_{H^2}^2} = 1.
\]

(117)

Here \(\bar{u}_{0i}\) stands for the \(i\)th component of \(\bar{\mathbf{u}}_0\) for \(i = 1, 2\).

Denote \((\rho^\delta_0, \mathbf{u}^\delta_0) := \delta(\bar{\rho}_0, \bar{\mathbf{u}}_0)\), and \(C_5 := \| (\bar{\rho}_0, \bar{\mathbf{u}}_0) \|_{L^2}\). Keeping in mind that

\[
\inf_{x \in \Omega} \{ \bar{\rho}(x) \} > 0
\]

and the embedding \(H^2 \hookrightarrow L^\infty\), we can choose a sufficiently small \(\delta_1 \in (0, 1)\), such that

\[
\inf_{x \in \Omega} \{ \bar{\rho}(x) \} \leq \inf_{x \in \Omega} \{ \delta^\delta_0(x) + \bar{\rho}(x) \} \text{ for any } \delta \in (0, \delta_1).
\]
Hence, by virtue of Proposition 4.1, the perturbed problems (6) and (7) admits a strong solution \((\rho^\delta, u^\delta) \in C^0([0, T_{max}^\ast], H^1 \times H^2)\) with an associated pressure \(Vq^\delta \in C^0([0, T_{max}^\ast], L^2)\), satisfying the initial data \((\rho_0^\delta, u_0^\delta)\) with \(E((\rho^\delta, u^\delta)) = \delta\). Moreover, we have

\[
0 < \frac{\inf_{x \in \Omega} \{\bar{\rho}(x)\}}{2} \leq \inf_{x \in \Omega} \{\bar{\rho}^\delta(t, x) + \bar{\rho}\}
\]

for any \(t \in [0, T_{max}^\ast]\), where \(C_\delta\) is the constant from the embedding \(H^2 \hookrightarrow L^\infty\).

Now we choose the value of \(\delta_0 \in (0, 1)\) as small as \((75)\). Let \(\sigma = \min\{\delta_0, \delta_1, \varepsilon_0\}\), and \(\delta \in (0, \sigma)\), define

\[
T^\delta := \frac{1}{\Lambda^*} \ln \frac{2\varepsilon_0}{\delta} > 0
\]

that is, \(\delta e^{\Lambda^* T^\delta} = 2\varepsilon_0\).

where \(\varepsilon_0\) is a constant independent of \(\delta\), satisfying \(\varepsilon_0 \in (0, 1)\), which will be defined in \((143)\).

\[
T^* := \sup \left\{ t \in I_{T_{max}^\ast} \mid E((\rho^\delta, u^\delta)(t)) \leq \delta_0 \right\} > 0,
\]

\[
T^{**} := \sup \left\{ t \in I_{T_{max}^\ast} \mid \| (\rho^\delta, u^\delta)(t) \|_{L^2} \leq 2\delta C_5 e^{\Lambda^* T^\delta} \right\} > 0.
\]

Then \(T^*\) and \(T^{**}\) may be finite, and furthermore,

\[
E \left( (\rho^\delta, u^\delta) \left( T^* \right) \right) = \delta_0, \quad \text{if} \quad T^* < \infty,
\]

\[
\| (\rho^\delta, u^\delta) \left( T^{**} \right) \|_{L^2} = 2\delta C_5 e^{\Lambda^* T^{**}}, \quad \text{if} \quad T^{**} < T_{max}^\ast.
\]

Now, we denote \(T_{min} := \min\{T^\delta, T^*, T^{**}\}\), then for all \(t \in I_{T_{min}}\), we deduce from the estimate \((114)\) and the definitions of \(T^*\) and \(T^{**}\) that

\[
E^2 \left( (\rho^\delta, u^\delta)(t) \right) + \|u^\delta_t(t)\|_{L^2}^2 + \int_0^t \|\nabla u^\delta\|_{L^2}^2 \, dt \leq C_4 \delta^2 E^2((\bar{\rho}_0, \bar{\rho})), \quad \text{where} \quad C_4 \text{ is independent of } \delta.
\]

Let \((\rho^\delta, u^\delta) = (\delta \rho^\delta, \delta^\frac{1}{2} u^\delta)\). Noting that \((\delta \rho^\delta, \delta^\frac{1}{2} u^\delta) \in C^0((0, +\infty), H^1 \times H^2)\) is also a linear solution to \((8)\) and \((7)\) with the initial data \((\rho_0^\delta, u_0^\delta) \in H^1 \times H^2\) and an associated pressure \(q^\delta = \delta q^\delta \in C^0((0, +\infty), L^2)\), we find that \((\rho^\delta, u^\delta)\) satisfies the following error equations:

\[
\begin{cases}
\rho^\delta + \rho^\delta \Delta u^\delta = -u^\delta \cdot \nabla \rho^\delta, \\
(\rho^\delta + \rho^\delta) u^\delta_t - \mu \Delta u^\delta + \nabla q^\delta = f^\delta - g \rho^\delta e_2, \\
\text{div} u^\delta = 0,
\end{cases}
\]

where

\[
q^\delta := q^\delta - q^a \in C^0(I_{T_{min}}, H^1)\text{ and } f^\delta := -(\rho^\delta + \rho^\delta) u^\delta \cdot \nabla u^\delta - \rho^\delta u^\delta_t.
\]

The initial and boundary conditions become

\[
\begin{aligned}
\left. (\rho^\delta, u^\delta) \right|_{t=0} &= 0, \\
\left. u^\delta_t(x, 0) = u^\delta_t(x, 1) \right|_{t=0} &= 0, \\
\partial_x u^\delta(x, 1) &= \frac{k_2}{\mu} u^\delta_t(x, 1), \\
\partial_x u^\delta(x, 0) &= -\frac{k_2}{\mu} u^\delta_t(x, 0),
\end{aligned}
\]

with compatibility conditions read as

\[
\left. u^\delta_t(x, 0) \right|_{t=0} = \left. u^\delta_t(x, 1) \right|_{t=0} = 0, \quad \text{div} u^\delta \big|_{t=0} = 0,
\]
In the following, we will establish the error estimate for \((\phi^d, u^d)\) in \(L^2\)-norm.

**Proposition 5.2.** There is a constant \(C_7\), such that for all \(t \in \tilde{T}_{\min}\),

\[
\| (\phi^d, u^d) (t) \|_{L^2}^2 \leq C_7 \delta^3 e^{3 \Lambda \gamma t}.
\] (125)

**Proof.** Recalling that \((\phi^d, u^d) = (\phi^g, u^g) - (\phi^h, u^h)\), in view of the regularity of \((\phi^g, u^g)\) and \((\phi^h, u^h)\), we can deduce from (124) that for a.e. \(t \in \tilde{T}_{\min}\),

\[
\frac{d}{dt} \int (\phi^g + \tilde{\rho}) |u^d|^2 = 2 < (\phi^g + \tilde{\rho}) u^d, u^d > - \int \phi^g |u^d|^2 \leq 2 \int (f^d - g\phi^g e_2) u^d - 2 \left( \int |\nabla u^d|^2 - \int k_i |\partial_i u^d(x, i)|^2 \, dx \right) - \int \phi^g |u^d|^2,
\] (126)

and \(\| \sqrt{\phi^g + \tilde{\rho} u^d} \|_{L^2} \in C^0(\tilde{T}_{\min})\), referring to Remark 6 of Cho and Kim. Noting that

\[
\frac{d}{dt} \int \tilde{\rho} |u^d|^2 = 2 \int \tilde{\rho} u^d \partial_i u^d, \quad \text{thus, adding up the equalities (126) and (127)}, \quad \text{using (124), it gives}
\]

\[
\frac{d}{dt} \left[ \int (\phi^g + \tilde{\rho}) |u^d|^2 - 2 g\tilde{\rho} |u^d|^2 \right] + 2 \mu \int |\nabla u^d|^2 - \int k_i |\partial_i u^d(x, i)|^2 \, dx \) \]

\[
= \int \left( 2f^d + 2g\phi^g \cdot \nabla \phi^g e_2 - \phi^g u^d \right) \cdot u^d.
\] (128)

Integrating (128) with respect to time variable from 0 to \(t\), we get

\[
\| \sqrt{\phi^g + \tilde{\rho} u^d} (t) \|_{L^2}^2 + 2 \int_0^t \left( \mu \|\nabla u^d\|_{L^2}^2 - k_i |\partial_i u^d(x, i)|^2 \, dx \right) \, d\tau \]

\[
= \int g\tilde{\rho} |u^d|^2 + R_1 + R_2(t),
\] (129)

where

\[
R_1 = \int (\phi^g + \tilde{\rho}) |u^d|^2 \, dx\]

and

\[
R_2(t) = \int_0^t \left( 2f^d + 2g\phi^g \cdot \nabla \phi^g e_2 - \phi^g u^d \right) \cdot u^d \, d\tau.
\]

The estimate of the above two terms \(R_1\) and \(R_2(t)\) follows from Jiang et al. For convenience, we state the conclusion without proofs

\[
R_1 + R_2(t) \leq \delta^3 e^{3 \Lambda \gamma t},
\] (130)

which, together with (129), yields that

\[
\| \sqrt{\phi^g + \tilde{\rho} u^d} (t) \|_{L^2}^2 + 2 \int_0^t \left( \mu \|\nabla u^d\|_{L^2}^2 - k_i |\partial_i u^d(x, i)|^2 \, dx \right) \, d\tau \]

\[
\leq \int g\tilde{\rho} |u^d|^2 + C \delta^3 e^{3 \Lambda \gamma t}.
\] (131)

In addition, it follows from Lemma 5.1 that

\[
\int g\tilde{\rho} |u^d|^2 \leq \Lambda^2 \int (\phi^g + \tilde{\rho}) |u^d|^2 + \Lambda \mu \int |\nabla u^d|^2 - \Lambda \sum_{i=0}^1 k_i \int |u^d(x, i)|^2 \, dx.
\] (132)
Recalling that \( u^d \in C^0(\bar{\Omega}_T, H^2) \) and \( \nabla u^d(t) \big|_{t=0} = 0 \), using Newton–Leibniz’s formula and Cauchy inequality, we rewrite and estimate the last two terms in the right hand side as follows:

\[
\Lambda (\mu \int |\nabla u^d|^2 - \sum_{i=0}^1 k_i \int_{2\pi T} u_i^d(x, i)^2 \, dx ) \\
\leq \Lambda^2 \int_0^t \left( \mu \|\nabla u^d\|_{L^2}^2 - \sum_{i=0}^1 k_i \|u_i^d(x, i)\|_{L^2}^2 \right) \, dr \\
+ \int_0^t \left( \mu \|\nabla u^d\|_{L^2}^2 - \sum_{i=0}^1 k_i \|\partial_t u_i^d(x, i)\|_{L^2}^2 \right) \, dr \\
+ \int_0^t \left[ \sum_{i=0}^1 \int_{2\pi T} k_i (\Lambda u_i^d(x, i) - \partial_t u_i^d(x, i))^2 \, dx \right] \, dr.
\] (133)

Putting the above three inequalities together gives

\[
\left\| \sqrt{\phi^d + \bar{\rho}u^d(t)} \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left( \mu \|\nabla u^d\|_{L^2}^2 - \sum_{i=0}^1 k_i \|u_i^d(x, i)\|_{L^2}^2 \right) \, dr \\
+ \frac{1}{2} \Lambda \left( \mu \int |\nabla u^d|^2 - \sum_{i=0}^1 k_i \int_{2\pi T} |u_i^d(x, i)|^2 \, dx \right) \\
\leq \Lambda^2 \int_0^t \left( (\phi^d + \bar{\rho}) \|u^d\|_{L^2}^2 + C\delta^3 e^{3\Lambda t} \right) + \frac{3}{2} \int_0^t \left( \mu \|\nabla u^d\|_{L^2}^2 - \sum_{i=0}^1 k_i \|u_i^d(x, i)\|_{L^2}^2 \right) \, dr \\
+ \frac{3}{2} \int_0^t \left[ \sum_{i=0}^1 \int_{2\pi T} k_i (\Lambda u_i^d(x, i) - \partial_t u_i^d(x, i))^2 \, dx \right] \, dr.
\] (134)

On the other hand, by virtue of Cauchy inequality, we get

\[
\frac{d}{dt} \left\| \sqrt{\phi^d + \bar{\rho}u^d} \right\|_{L^2}^2 = 2 \int \left( \phi^d + \bar{\rho} \right) u^d \cdot u_i^d + \int \phi_i^d |u_i^d|^2 \\
\leq \frac{1}{\Lambda} \left( \left\| \sqrt{\phi^d + \bar{\rho}u_i^d} \right\|_{L^2}^2 + \Lambda \left\| \sqrt{\phi^d + \bar{\rho}u_i^d} \right\|_{L^2}^2 + \int \phi_i^d |u_i^d|^2 \right). \] (135)

Utilizing (116), (119), (123), and embedding inequality, the last term can be estimated as

\[
\int \phi_i^d |u_i^d|^2 = - \int (u^d \cdot \nabla \phi^d + \bar{\rho} u_i^d) |u_i^d|^2 = \int (2\phi^d \cdot \nabla u^d - \bar{\rho} u_i^d u^d) \cdot u^d \leq \delta^3 e^{3\Lambda t}. \] (136)

Furthermore, integrating (135) in time from 0 to t, together with (134) and using Young inequality, we have

\[
\left\| \sqrt{\phi^d + \bar{\rho}u^d} \right\|_{L^2}^2 + \alpha \mu \left\| \nabla u_i^d \right\|_{L^2}^2 + \int_0^t \mu \left\| \nabla u_i^d \right\|_{L^2}^2 \, dr \\
\leq \Lambda \int_0^t \left( \left\| \sqrt{\phi^d + \bar{\rho}u^d} \right\|_{L^2}^2 + \Lambda \mu \left\| \nabla u_i^d \right\|_{L^2}^2 \right) \, dr + \Lambda \left\| \sqrt{\phi^d + \bar{\rho}u^d} \right\|_{L^2}^2 + C\delta^3 e^{3\Lambda t}. \] (137)

Summing up the previous three estimates together yields

\[
\frac{d}{dt} \left\| \sqrt{\phi^d + \bar{\rho}u^d(t)} \right\|_{L^2}^2 + \left( \left\| \sqrt{\phi^d + \bar{\rho}u^d(t)} \right\|_{L^2}^2 + \Lambda \mu \left\| \nabla u_i^d(t) \right\|_{L^2}^2 \right) \\
\leq \Lambda \left[ \left\| \sqrt{\phi^d + \bar{\rho}u^d(t)} \right\|_{L^2}^2 + \int_0^t \left( \left\| \sqrt{\phi^d + \bar{\rho}u^d(t)} \right\|_{L^2}^2 + \Lambda \mu \left\| \nabla u_i^d \right\|_{L^2}^2 \right) \, dr \right] + C\delta^3 e^{3\Lambda t}. \] (138)
Therefore, applying Gronwall’s inequality to (138), one obtains

$$\left\| \sqrt{\dot{\phi} + \ddot{\rho}} u^d(t) \right\|_{L^2}^2 + \int_0^t \left( \left\| \sqrt{\dot{\phi} + \ddot{\rho}} u^d_\tau \right\|_{L^2}^2 + \left\| \nabla u^d \right\|_{L^2}^2 \right) \, d\tau \lesssim C \delta^3 e^{\lambda^* t},$$

(139)

for all $t \leq T_{\text{min}}$, where the constant $C$ depends on $k_0, k_1, \mu, \ddot{\rho}, T_{\text{min}}, \Lambda$. Together with (119) and (137), we deduce that

$$\left\| u^d(t) \right\|_{H^1}^2 + \int_0^t \left( \left\| u^d_\tau \right\|_{L^2}^2 + \left\| \nabla u^d \right\|_{L^2}^2 \right) \, d\tau \lesssim \delta^3 e^{\lambda^* t}.$$  

(140)

Finally, using the estimates (123), (140), and embedding inequality, we can deduce from the Equation (124) that

$$\left\| \phi^d(t) \right\|_{L^2} \lesssim \int_0^t \left\| \phi^d_\tau \right\|_{L^2} \, d\tau \lesssim \int_0^t \left( \left\| u^d \right\|_{H^1} + \left\| u^d \cdot \nabla \phi^d \right\|_{L^2} \right) \, d\tau \lesssim \int_0^t \left( \delta^3 \frac{e^{\lambda^* t}}{\tau} + \delta^2 e^{2\lambda^* t} \right) \, d\tau \lesssim \delta^3 e^{\lambda^* t},$$

(141)

in which we have used the following fact

$$\delta e^{\lambda^* t} \lesssim \delta e^{\lambda^* t^2} = \epsilon_0 < 1, \text{ for any } t \in T_{\text{min}}.$$  

(142)

Therefore, consolidating (141) with (139), the desired estimate (125) follows. This completes the proof of this proposition.

\[ \square \]

**Proof of Theorem 1.2.** Now, with (116), (120), (123), and (125) in hands, referring to Jiang et al.\(^9\) for details, we can conclude that

$$T^g = T_{\text{min}}, \text{ provided } \epsilon_0 = \min \left\{ \delta_0 \frac{C_2^2}{4\sqrt{C_4}}, \frac{m_0^2}{2C_7^2} \frac{m_0^2}{8C_7} \right\},$$

(143)

where we have defined that $m_0 = \min \{ \left\| \tilde{\theta}_0 \right\|_{L^2}, \left\| \tilde{\theta}_0 \right\|_{L^2}, \left\| \tilde{\theta}_0 \right\|_{L^2} \} > 0$ due to (117).

Because $T^g = T_{\text{min}}$, (125) holds for $t = T^g$. Therefore, we can use (125) and (143) with $t = T^g$ to obtain that

$$\left\| \phi^d(T^g) \right\|_{L^2} \gtrsim \left\| \phi^d_0(T^g) \right\|_{L^2} - \left\| \phi^d(T^g) \right\|_{L^2} \gtrsim \delta e^{\lambda^* T^g} \left\| \tilde{\theta}_0 \right\|_{L^2} - \sqrt{C_7} \delta^{3/2} e^{3\lambda^* T^g/2} \gtrsim 2\epsilon_0 \left\| \tilde{\theta}_0 \right\|_{L^2} - 2^{3/2} \sqrt{C_7 \epsilon_0^{3/2}} \gtrsim 2m_0 \epsilon_0 - 2^{3/2} \sqrt{C_7 \epsilon_0^{3/2}} \gtrsim m_0 \epsilon_0.$$

Similarly, it is easy to get that

$$\left\| u^d_i(T^g) \right\|_{L^2} \gtrsim 2m_0 \epsilon_0 - 2^{3/2} \sqrt{C_7 \epsilon_0^{3/2}} \gtrsim m_0 \epsilon_0,$$

where $u^d_i(T^g)$ denotes the $i$th component of $u^d(T^g)$ for $i = 1, 2$. This completes the proof of Theorem 1.2 by defining $\epsilon := m_0 \epsilon_0$.

\[ \square \]

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CONFLICT OF INTEREST
This work does not have any conflicts of interest.

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