Creep Deformations of Curvilinear Anisotropic Media: Finite Element Modeling

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Abstract
This paper proposes a method for preliminary analysis of layer daynamics, it composition and assessment of deep-seated rock. Since physical knowledge is insufficient or the cannot be experimentally obtained, the phase of development and industrial launch of wells, mines, and open pits is nesessary. The method based on modeling the stress-strain state of rocks includes block-curvilinear anisotropy and creep. We develop methods that qualitatively describe the non-standard geometric shapes of deep rock layers. A block-curvilinear description of the rock model is given. We also offer the method for the effective numerical solution to the stress-strain state considering block-curvilinear anisotropy and creep. The software application is designed in the Scientific and Educational Center Supercomputer Engineering Modeling and Development of Software Systems at Bauman Moscow State Technical University (BMSTU). This software application helps to create, describe, solve and analyze both rocks and other mathematical models, as well as creep equation.

Key words: rock, stress-strain state, block-curvilinear description, creep, anisotropy, numerical methods, software application

1. Introduction
In the field of prospecting for minerals, there are a significant number of problems associated with preliminary modeling and analyzing the terrain, in which geological exploration operations is carried out. One of the most important problems is calculating the stress-strain state (hereinafter SSS) of deep-seated rocks, the knowledge of which cannot be obtained by fast and cost-intensive methods. For the work related to the modeling of deep rocks, it is required to understand the soil layer nature and images. A set of explosive measures are carried out using trinitrotoluene cartridges placed at the required underground depths. The seismic data obtained are included into the table. This data is sometimes enough to determine where it is worth making prospective measures for mineral extraction, but there are complex varieties of both topographical relief and soil in which it is impossible to give an unambiguous answer to the question of where to find minerals. The process of modeling rock layer is used in such cases. The calculation of layer SSS is carried out in order to identify rock composition and dynamics. We analyze the substance content with some properties of oil, gas, coal and other precious materials, whose properties are similar to those listed above. We develop a geometric model based on 3D seismic migration data. The migration data is a set of vertical rock sections. A mesh of surface primitives (triangle and quadrilateral) is created. Base primitives are approximated by surfaces of different types. Curved clusters are highlighted in the resulting geometry using Voronoi diagram. The resulting geometry is used to build a tetrahedral mesh with a quadratic approximation. The finite element method is applied to solve the SSS problem and determine the both displacements and stresses under the pressure on the stone block from the side of its edges.

More detailed information can be found in the sources [1] – [21].
1.1 The goal of the work is developing numerical approximation algorithms for describing curvilinear surfaces and designing numerical methods for solving the SSS problem of rocks as well as curvilinear transversely isotropic materials with the creep equations.

1.2 Mathematical formulation of the problem
Consider a three-dimensional boundary value problem with the creep condition:
\[ \nabla \cdot \sigma + \rho f = 0; \]  
(1)
\[ \sigma = \mathbf{C} (\varepsilon - \varepsilon^c - \varepsilon^\theta); \]  
(2)
\[ \varepsilon^c = F(\varepsilon^c, \sigma); \]  
(3)
\[ \varepsilon = def(\mathbf{u}) = \frac{1}{2} (\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^T); \]  
(4)
\[ \varepsilon^c_{t=0} = 0; \]  
(5)
\[ \sigma \cdot n_\Sigma = s^e; \]  
(6)
\[ \mathbf{u}_{\Sigma \mathbf{u}} = \mathbf{u}^e, \]  
(7)
where (1) – equilibrium equation of a continuous medium; (2) – constitutive relations taking into account creep deformations; (3) – ratios of creep deformations; (4) – Cauchy relation; (5) – initial condition; (6), (7) – boundary conditions; the symbols are used: \( \nabla \) – nabla-operator; \( \sigma \) – stress tensor; \( \rho \) – variable density of the rock, depends on the specific type of rock (limestone, sandstone, clay, etc.); \( f = -g e_z \) – gravity density vector; \( g \) – acceleration of gravity; \( \mathbf{C} \) – Yong’s elastic tensor; \( \varepsilon \) – small deformations tensor; \( \varepsilon^c \) – creep strain tensor; \( \varepsilon^\theta = \alpha (\theta - \theta_0) \) – thermal strain tensor; \( F(\varepsilon^c, \sigma) \) – tensor function that is twice continuously differentiable in some region \( G \subset \mathbb{R}^{12} \) describes the model of thermal creep deformations; \( \mathbf{u} \) – displacement vector; \( \mathbf{u}^e \) – is a given displacement vector; \( n \) – outward normal vector; \( s^e \) – stress vector on a part of the body surface \( \Sigma_\sigma \); \( e_z \) – oriented in the direction of gravity basis vector; \( \alpha \) – thermal expansion tensor; \( \otimes \) – tensor product; \( (\cdot) \) – scalar product.

2. Numerical methods for describing the geometry of a complex curvilinear form
To describe the curved geometry of a rock the following methods are used:
To describe triangular primitives, the following parameters are used:
1) cubic Bezier splines; 2) cubic Bezier patches; 3) triangular analogs of the Koons surface; 4) rational Bezier patches with weights.
For describing quad primitives, the following parameters are used:
1) bicubic Bezier surfaces; 2) bicubic Koons surfaces; 3) bicubic Bezier surfaces at 4 centers; 4) rational bicubic Bezier surfaces on 4 centers with weights; 4) modified bicubic Bezier surfaces.

![Figure 1](image1.png)

**Figure 1.** a) – cubic Bezier patch; b) – bicubic Bezier surface; c) – bicubic Koons surfaces

3. Results of numerical methods
The model of rock layers is approximated by different methods. Qualitative results are presented in Figures 2 and 3. It should be noted that the constructed geometric model is based on the bicubic Bezier surface (Figure 1b) or the classical cubic Koons surface (Figure 1c) is in many ways superior to the previous graphical model based only on cubic Bezier curves used in [1].
Figure 2. Model from classic bicubic Bezier surfaces and cubic Bezier patches with an edge division factor equals 6, curvature factor equals 0.40333

4. Comparative analysis of results
The algorithm for representing geometry through classical bicubic Bezier surfaces and Bezier patches gives the best approximation result (Figure 2). It is worth noting that the multiplicity division of each element (curvilinear edge of a primitive) improves the smoothing quality of objects. Also, a comparison is made by the time spent for the calculation of local and support nodes of 4693 primitives with various approximation models used in the work. To determine the program costs, the gprof profiler is used in [6]. Table 1 and Table 2 present the results.

| Total program run time in percentages | The current sum of seconds | The number of seconds counted by only this function | Calls number | Function name         |
|---------------------------------------|---------------------------|--------------------------------------------------|--------------|-----------------------|
| 0.00                                  | 250.01                    | 0.01                                             | 46167        | processSmoothing      |
| 0.00                                  | 249.15                    | 0.02                                             | 107190       | getBezierKoonsPatchCoord |
| 0.01                                  | 249.26                    | 0.01                                             | 133088101    | Bezier                |

$T_{maxquadr} = 0.00200009\, s., T_{maxtriangle} = 0.00099995\, s.$

Table 2. Results of approximation by bicubic Bezier surfaces and Bezier patches

| Total program run time in percentages | The current sum of seconds | The number of seconds counted by only this function | Calls number | Function name         |
|---------------------------------------|---------------------------|--------------------------------------------------|--------------|-----------------------|
| 0.00                                  | 254.31                    | 0.00                                             | 46167        | processSmoothing      |
| 0.00                                  | 254.31                    | 0.00                                             | 108471       | getBezierPatchCoord   |
| 0.01                                  | 253.52                    | 0.01                                             | 133328269    | Bezier                |

$T_{maxquadr} = 0.00200011\, s., T_{maxtriangle} = 0.00100011\, s.$

5. Explicit Runge – Kutta – Felberg method 4-5 orders
Integrating by this method are calculated 6 intermediate points at one step. We solve 6 problems to find the strain tensor $\varepsilon$ and stress tensor $\sigma$ in our section. The implementation of the method involves the construction of an inequality to control the accuracy of calculations that does not lead to additional computational costs with the correct choice of the error.
For the initial step \((m = 0)\), based on the initial condition \(\varepsilon^c|_{t=0} = 0\), we obtain following system:

\[
\begin{align*}
\nabla \cdot \mathbf{e}^{(0)} + \rho \mathbf{f} &= -\nabla \cdot \mathbf{\sigma}^{\theta}, \\
\mathbf{e}^{(0)} &= 4 \mathbf{C} \cdot \varepsilon^{(0)}, \\
\varepsilon^{(0)} &= \mathbf{F}(\varepsilon^{c}, \sigma), \\
\varepsilon^{(0)} &= \mathbf{def}(\mathbf{u}^{(0)}), \\
\mathbf{e}^{(0)} \cdot \mathbf{n}|_{\Sigma} &= S_b^{(0)} - \sigma^{\theta} \cdot \mathbf{n}|_{\Sigma}, \\
\mathbf{u}^{(0)}|_{\Sigma} &= \mathbf{u}_b^{(0)}. \\
\end{align*}
\]

(8)

Considering (8), at the initial step for tensors \(\varepsilon\) and \(\sigma\), we receive:

\[
\begin{align*}
\varepsilon^{(0)} &= 0; \\
\sigma^{(0)} &= \varepsilon^{(0)} + \sigma^{\theta};
\end{align*}
\]

(9)

(10)

Then, at the \(m\)'s step \((m \in \mathbb{N}_{N+1})\), we develop the following procedure for the numerical method:

1. Calculation of strain tensors and creep stresses:

\[
\begin{align*}
\varepsilon^{(m)} &= \varepsilon^{(m-1)} + \sum_{i=1}^{6} y_i K_i \Delta \tau_m F(\sigma^{m-1}); \\
\sigma^{(m)} &= -4 \mathbf{C} \cdot \varepsilon^{(m)}. \\
\end{align*}
\]

(11)

2. Solution to the boundary value problem:

\[
\begin{align*}
\nabla \cdot \mathbf{e}^{(m)} + \rho \mathbf{f} &= -\nabla \cdot (\mathbf{e}^{(m)} + \sigma^{\theta}), \\
\mathbf{e}^{(m)} &= 4 \mathbf{C} \cdot \varepsilon^{(m)}, \\
\varepsilon^{(m)} &= \mathbf{def}(\mathbf{u}^{(m)}), \\
\mathbf{e}^{(m)} \cdot \mathbf{n}|_{\Sigma} &= S_{b}^{(m)} - (\sigma^{(m)} + \sigma^{\theta}) \cdot \mathbf{n}|_{\Sigma}, \\
\mathbf{u}^{(m)}|_{\Sigma} &= \mathbf{u}_b^{(m)}. \\
\end{align*}
\]

(12)

3. Calculation of the stress tensor:

\[
\sigma^{(m)} = \varepsilon^{(m)} + \sigma^{(m)} + \sigma^{\theta}. \\
\]

(13)

When choosing an acceptable method error, one should take into account the likelihood of an unpredictable increase in the accumulated error. It should be noted that Runge – Kutta – Felberg method of the 4-5th order algorithm (hereinafter RKF-45) is one of the best among the methods of the Runge – Kutta type of 5th order of accuracy [2].

6. Comparative analysis of the results while solving the problem of SSS of the beam by the analytical method, the Euler method and the RKF of the 4-5th order method

Figure 3 presents how the value of the component \(\varepsilon_{11}\) of the strain tensor \(\varepsilon\) varies with the number of time steps. We can see below that the numerical solution found by the RKF-45 method is nearer to the analytical solution in comparison with the solution found by Euler method.

![Figure 3. Analytical solution, Euler solution and RKF-45 solution. Comparative analysis](image-url)
7. Creep function

$F(\varepsilon^c, \sigma)$ is twice continuously differentiable in some region $G \subset \mathbb{R}^{12}$. This tensor function describes the thermal creep strain rate model.

Burger model of the tensor function:

$$\varepsilon = \frac{1}{\gamma_M} \frac{1}{\eta_M} + \frac{1}{\gamma_V} \left(1 - \exp \left(\frac{-\gamma_V t}{\eta_V}\right)\right).$$  \hspace{1cm} (14)

where $\gamma$ and $\eta$ – elastic modules and viscosity coefficients; indices $M$ and $V$ refer to Maxwell and Voigt models [4].

8. FEM for solving the problem of rock SSS

The finite element formulation of the problem is as follows:

$$\sum_{K \in \mathcal{N}_a} (v_K)^T A_K u_{s(K)}^G - f_K = 0.$$ \hspace{1cm} (15)

$$A_K = \int_{K} B_K^T C B_K dV.$$ \hspace{1cm} (16)

$$f_K = \int_{\partial K \cap \Sigma_d} (\phi_K)^T S_{gh} d\Sigma + \int_{K} B_K^T C dV, B_K(x).$$ \hspace{1cm} (17)

$$f_K = D \Phi_K(x).$$ \hspace{1cm} (18)

We arrive at the resolving system of linear algebraic equations (hereinafter SLAE) of the FEM scheme:

$$A^G u_{s}^{G} = f^G.$$ \hspace{1cm} (19)

where $A^G$ — matrix: $A^G \in L(\mathbb{R}, N - q, N - q), q$ – numbers of the columns excluded from the matrix; $f^G$ — variable pressure values (in this problem, it is pressure on the walls of a rock block), $f^G \in \mathbb{R}^{N-p}$, $p$ — the line numbers that are excluded from the column; $u_{s}^{G}$ — SLAE solution (19).

9. Numerical results

As we can see below the figures display diagrams of the stress tensor components $\sigma_z$ of rock clusters and show the dynamic of changes in stress components depending on the anisotropy bases direction.

Figure 4. 3D diagram of normal stresses along the OX axis is in its own coordinate system (component $\sigma_{11}$ of the stress tensor $\sigma$)
Figure 5. 3D diagram of shear stresses on the XY plane is in its own coordinate system

Figure 6. 3D diagram of the stress tensor intensity $\sigma$

10. Conclusion
Based on the results obtained we can concluded that the creation, approximation and further calculation of such parameters stresses, displacements, temperatures, pressures, at the nodes of the tetrahedron mesh on surface primitives should be done using the model described by a pair which consist of a classic bicubic Bezier surface and a Bezier patches. The result of the calculation largely depends on the degree of approximation quality of the curvilinear region (Figure 2) as well as the software costs for the implementing these methods do not significantly differ from the software costs of other methods (Table 1 and Table 2).

Using the RKF method of 4-5th orders, it is possible to accelerate the solution of the SSS problem by 27%, for achieving the required solution accuracy, as well as reducing the number of algorithm iterations by 3.12 times. Whereas solving this problem, the condition is established for the solution accuracy. The number of time steps is formed for each numerical algorithm, which automatically based on this condition – the problem will be solved with residual if it is stable and has all the necessary initial and boundary conditions.
Figures 4 - 6 present the diagrams of rock SSS according to which the rock layers prove to move depending on the forces and pressures acting on them over time. In other words, it becomes possible to analyze the dynamics and nature of the rock layers beddings.

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