Numerical Algorithms of the Two-dimensional Feynman–Kac Equation for Reaction and Diffusion Processes

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Abstract
This paper provides a finite difference discretization for the backward Feynman–Kac equation, governing the distribution of functionals of the path for a particle undergoing both reaction and diffusion (Hou and Deng in J Phys A Math Theor 51:155001, 2018). Numerically solving the equation with the time tempered fractional substantial derivative and tempered fractional Laplacian consists in discretizing these two non-local operators. Here, using convolution quadrature, we provide the first-order and second-order schemes for discretizing the time tempered fractional substantial derivative, which doesn’t require the assumption of the regularity of the solution in time; we use the finite difference method to approximate the two-dimensional tempered fractional Laplacian, and the accuracy of the scheme depends on the regularity of the solution on $\bar{\Omega}$ rather than the whole space. Lastly, we verify the predicted convergence orders and the effectiveness of the presented schemes by numerical examples.

Keywords Two-dimensional Feynman–Kac equation · Finite difference approximation · Convolution quadrature · Error estimates

1 Introduction

The random motion of a particle is a most fundamental and widely appeared natural phenomena. The so-called particle can be a really physical one or an abstract one, e.g., stock market. Stochastic processes $x_0(t)$ are mathematical models to describe this phenomena. In a long history, the Wiener process is the most studied and representative stochastic process, the second moment of which is a linear function of time $t$. In the past 20 years, it is found that anomalous dynamics are ubiquitous in the natural world. The anomalous stochastic processes are distinguished from Wiener process by the evolution of their second moments with respect to time $t$, that is, the second moment of the stochastic anomalous process is a non-linear function of $t$ [25].
Currently, stochastic anomalous processes are hot research topics, including modeling, theoretical analysis, and numerical methods. Obtaining the probability density function (PDF) of the statistical observables plays an important role in studying the stochastic processes; not only it is a useful technique to extract practical messages but also a key strategy to help understand the mechanism of stochastic processes. The functional is one of the most useful and representative statistical observables, defined as

$$A = \int_0^t U[x_0(\tau)]d\tau,$$

where $x_0(t)$ is a trajectory of particle, $U(x_0)$ is a prescribed function depending on specific applications, e.g., one can take $U(x_0) = 1$ in a given region and set it to be zero in the rest of the region for (1.1) to study the kinetics of chemical reactions in some given domain [2,7]. Some important progresses have been made for deriving the governing equation of the PDF of the functional $A$ in Fourier space. The earliest work [19] for this issue is for the distribution of the functional of Wiener process, completed by Kac in 1949 who was influenced by Feynman’s thesis of the derivation of Schrödinger’s equation. After that the related equations are usually named with the word: Feynman–Kac. For the stochastic process $x_0(t)$ described by a continuous time random walk (CTRW) with power law waiting time and jump length distributions, Barkai and his collaborators derive the governing equation of the distribution of the corresponding functional in Fourier space [7,29]. Because of the finite lifespan and the bounded physical space, sometimes the power law waiting time and jump length distributions of the CTRW need to be tempered [24]; then the tempered fractional Feynman–Kac equations are derived in [30], and the numerical methods for the equations are discussed in [10,11,15,27,34,35]. For $x_0(t)$ characterized by Langevin pictures, the readers can refer to [5,6] for the derivation of Feynman–Kac equations.

More recently, the reactions are introduced to the stochastic process $x_0(t)$, which means that the particles perform both diffusion and chemical reaction. With the exponentially tempered power law distributions of waiting time and jump length for the diffusion, the Feynman–Kac equation for the reaction and diffusion process is derived in [17], which includes the tempered fractional substantial derivative and the tempered fractional Laplacian. In this paper, we consider the numerical scheme of the two-dimensional backward tempered fractional Feynman–Kac equation [17] for reaction and diffusion processes with homogeneous Dirichlet boundary conditions [13], i.e.,

$$\frac{\partial}{\partial t}G(\rho, t, x_0) = K_0 D_t^{1-\alpha,\lambda,x_0}(\Delta + \gamma)^{\beta} G(\rho, t, x_0) + (r(x_0) + j \rho U(x_0)) G(\rho, t, x_0)$$

$$+ (\lambda^\alpha - 0) D_t^{1-\alpha,\lambda,x_0} - \lambda\left(G(\rho, t, x_0) - e^{\rho U(x_0) t} e^{\rho x_0 t}\right), \quad x_0 \in \Omega, 0 \leq t \leq T,$$

$$G(\rho, 0, x_0) = G_0(\rho, x_0), \quad x_0 \in \Omega,$n

$$G(\rho, t, x) = 0, \quad x \in \mathbb{R}^2 \setminus \Omega, \quad 0 \leq t \leq T.$$

Here, $G(A, t, x_0)$ is the PDF of the functional $A$ at time $t$ with the initial position $x_0$, and $\rho$ is the Fourier pair of $A$; $K$ is a positive constant, for convenience, we take $K = 1$ in this paper; $0 < \alpha < 1$, $J = \sqrt{-1}$ and $\lambda > 0$; $\Omega$ denotes a bounded domain; $r(x_0)$ stands for the reaction rate and satisfies $\sup_{x_0 \in \Omega} r(x_0) < 0$, where $\Omega$ denotes the closure of $\Omega$; $0 D_t^{1-\alpha,\lambda,x_0}$ is the tempered fractional substantial derivative, which is defined by
with \( P.V. \) denotes the principal value integral, function.  

\[ L \text{ time, such as G–L scheme and } \]

= a stable and consistent linear multi-step method \[ 16 \]. In this paper, we take \( 0D_t^{1-\alpha,\lambda,x_0} G(\rho, t, x_0) = \frac{1}{\Gamma(\alpha)} \left( \frac{\partial}{\partial t} + \lambda - r(x_0) - J \rho U(x_0) \right) \]

\[ \int_0^t e^{-\frac{(t-\tau)(\lambda-r(x_0)-J \rho U(x_0))}{(t-\tau)^{1-\alpha}}} G(\rho, \tau, x_0) d\tau. \]  

(1.3)

And \((\Delta + \gamma)^{\frac{\beta}{2}} \) denotes the two-dimensional tempered fractional Laplacian \[ 14 \], whose definition is \((\Delta + \gamma)^{\frac{\beta}{2}} G(x) = -c_{2,\beta} P.V. \int_{\mathbb{R}^2} \frac{G(x) - G(y)}{e^{\gamma|x-y|}|x-y|^{\beta} d\gamma} \) for \( \beta \in (0, 2) \), (1.4)  

with  

\[ c_{2,\beta} = \left\{ \begin{array}{ll}
\frac{2\pi}{\beta \Gamma(\frac{\beta}{2})} & \text{for } \lambda > 0 \text{ and } \beta \neq 1, \\
\frac{2^{1-\beta} \pi \Gamma(1-\beta/2)}{\beta \Gamma(\frac{\beta}{2})} & \text{for } \lambda = 0 \text{ or } \beta = 1,
\end{array} \right. \]

P.V. denotes the principal value integral, \( x, y \in \mathbb{R}^2 \), and \( \Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds \) is the Gamma function.

When numerically solving Eq. (1.2), two main problems need to be carefully dealt with. The first one is to discretize the tempered fractional substantial derivative (1.3), which is a time-space coupled operator and whose form depends on the initial position \( x_0 \); existing methods of discretizing it mainly require that the solution should be a \( C^2 \) or \( C^3 \)-function in time, such as G–L scheme and \( L_1 \) scheme \[ 10-12,15,31-33 \]. Here, we develop the first-order and second-order schemes based on convolution quadrature introduced by Lubich \[ 21,22 \]. Theorems 4.1 and 4.3 in Sect. 4 show that the convergence order only depends on the regularity of source term \( f \) instead of the exact solution \( G \). The second problem is to discretize (1.4); so far, the discretizations of (1.4) are mainly for the one-dimensional case \[ 34,35 \]; our previous work \[ 27 \] provides a finite difference scheme for the two-dimensional tempered fractional Laplacian \((\Delta + \gamma)^{\frac{\beta}{2}} \). Here, we modify the discretization according to \[ 34 \], so that the regularity requirement can be relaxed from the whole space to \( \Omega \), and the optimal convergence rate is achieved. Furthermore, we provide a method to construct a preconditioner when we solve relative linear system by Preconditioned Conjugate Gradient (PCG) method.

The framework of convolution quadrature \[ 18 \] can be briefly reviewed as follow. Firstly, one can define \( \tilde{\mathcal{B}}(\partial_t) v(t) = (\mathcal{B} * v)(t) \), where \( \partial_t \) denotes time differentiation and * stands for convolution; and let \( \tau \) denote the time step size. The convolution quadrature refers to an approximation of any function of the form \( \mathcal{B} * v \) as

\[ (\mathcal{B} * v)(t) = \int_0^t \mathcal{B}(t-s) v(s) ds \approx \tilde{\mathcal{B}}(\partial_\tau) v(t), \]

where

\[ \tilde{\mathcal{B}}(\partial_\tau) v(t) = \sum_{0 \leq j \tau \leq t} d_j v(t - j \tau), \quad t > 0, \]

and the quadrature weights \( \{d_j\}_{j=0}^\infty \) are computed from \( \tilde{\mathcal{B}}(z) \) denoting Laplace transform of \( \mathcal{B}(t) \), i.e., \( \sum_{j=0}^\infty d_j \zeta^j = \tilde{\mathcal{B}}(\delta(\zeta)/\tau) \). Here \( \delta(\zeta) \) is the quotient of the generating polynomials of a stable and consistent linear multi-step method \[ 16 \]. In this paper, we take \( \delta(\zeta) = 1 - \zeta \) and \( \delta(\zeta) = (1 - \zeta) + (1 - \zeta)^2/2 \) to, respectively, get the first-order and second-order schemes for the tempered fractional substantial derivative (1.3).
The remainder of the paper is organized as follows. In Sect. 2, we give some preliminaries needed in the paper and derive an equivalent form of Eq. (1.2). In Sect. 3, we combine the weighted trapezoidal rule and the bilinear interpolation to discretize the tempered fractional Laplacian and perform error analysis. In Sect. 4, we use convolution quadrature to discretize the tempered fractional substantial derivative, and then get the first-order and second-order schemes. In Sect. 5, we present the efficient computation of the linear system generated by the discretization. At last, we verify the predicted convergence rates by numerical experiments and conclude our paper.

2 Preliminaries and Equivalent form of Eq. (1.2)

This section introduces some preliminary knowledges and derives the equivalent formulation of Eq. (1.2).

2.1 Preliminaries

This subsection provides some definitions and properties needed in the paper. Firstly, define the discrete inner product and the discrete norm as

\[ (v, w) = h^2 \sum_{i=1}^{M} v_i \bar{w}_i, \quad \|v\|_2 = \sqrt{(v, v)}, \quad \|v\|_{L^\infty} = \max_{1 \leq i \leq M} |v_i|, \]

where \( v, w \in \mathbb{R}^M \) and \( M \in \mathbb{N}^+ \); denote as the continuous norms, where \(|v(x)|^2 = v(x)\bar{v}(x)\) and \(\bar{v}(x)\) means the conjugate of \(v(x)\). For convenience, denote \(G(t)\) and \(f(t)\) as \(G(\rho, t, x_0)\) and \(f(\rho, t, x_0)\), respectively, in the following. Furthermore, we recall some definitions of the tempered fractional integrals and derivatives.

**Definition 1** (Riemann–Liouville tempered fractional integral [4,8,20]) Suppose that the real function \(v(t)\) is piecewise continuous on \((a, b)\) and \(\alpha > 0, \lambda \geq 0, v(t) \in L[a, b]\). The Riemann–Liouville tempered fractional integral of order \(\alpha\) is defined to be

\[ aI_t^{\alpha, \lambda} v(t) = e^{-\lambda t} aI_t^{\alpha} (e^{\lambda t} v(t)) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} e^{-\lambda(t - \tau)} v(\tau) d\tau, \]

where \(aI_t^{\alpha} v(t)\) denotes the Riemann–Liouville fractional integral, i.e.

\[ aI_t^{\alpha} v(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} v(\tau) d\tau. \]

**Definition 2** (Riemann–Liouville tempered fractional derivative [3,8,20]) For \(n - 1 < \alpha < n, \ n \in \mathbb{N}^+, \lambda \geq 0\), the Riemann–Liouville tempered fractional derivative is defined by

\[ aD_t^{\alpha, \lambda} v(t) = e^{-\lambda t} aD_t^{\alpha} (e^{\lambda t} v(t)) = \frac{e^{-\lambda t}}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{e^{\lambda \tau} v(\tau)}{(t - \tau)^{\alpha-n+1}} d\tau, \]
where \( aD_t^\alpha v(t) \) denotes the Riemann–Liouville fractional derivative and it is described as
\[
aD_t^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{\alpha-n+1} \tau^n d\tau.
\]

**Remark 2.1** [15] When \( 0 < \alpha < 1 \), we have
\[
0D_t^\alpha 0I_t^\alpha v(t) = v(t) \quad \text{and} \quad 0I_t^\alpha 0D_t^\alpha v(t) = v(t).
\]

**Definition 3** (Caputo tempered fractional derivative [20,26,28]) For \( n-1 < \alpha < n, n \in \mathbb{N}^+ \), \( \lambda \geq 0 \), the Caputo tempered fractional derivative is defined as
\[
C_a D_t^{\alpha,\lambda} v(t) = e^{-\lambda t} C_a D_t^{\alpha} \left( e^{\lambda t} v(t) \right) = \frac{e^{-\lambda t}}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} \left( e^{\lambda \tau} v(\tau) \right) d\tau,
\]
where \( C_a D_t^\alpha v(t) \) denotes the Caputo fractional derivative and it is defined by
\[
C_a D_t^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} v(\tau) d\tau.
\]

**Proposition 1** [20] The Laplace transform of the Riemann–Liouville tempered fractional derivative is given by
\[
\hat{0D_t^{\alpha,\lambda} v(z)} = (z+\lambda)^\alpha \hat{v}(z) - \sum_{k=0}^{n-1} (z+\lambda)^k \left. \left( 0D_t^{\alpha-k} \left( e^{\lambda t} v(t) \right) \right) \right|_{t=0}.
\]
while the Laplace transform of the Caputo tempered fractional derivative is
\[
\hat{\tilde{C}_0 D_t^{\alpha,\lambda} v(z)} = (z+\lambda)^\alpha \hat{\tilde{v}}(z) - \sum_{k=0}^{n-1} (z+\lambda)^{\alpha-k-1} \left. \left( \frac{d_k}{dt^k} \left( e^{\lambda t} v(t) \right) \right) \right|_{t=0}.
\]

### 2.2 Equivalent form of Eq. (1.2)

According to (1.3) and Definition 1, we get
\[
0D_t^{1-\alpha,\lambda, \xi_0} G(\rho, t, \xi_0) = e^{-(\lambda-r(\xi_0)-J \rho U(\xi_0))t} 0D_t^{1-\alpha} \left( e^{(\lambda-r(\xi_0)-J \rho U(\xi_0))t} G(\rho, t, \xi_0) \right).
\] (2.1)

Combining (2.1) with (1.2), we have
\[
\left( \frac{\partial}{\partial t} G(\rho, t, \xi_0) + \lambda G(\rho, t, \xi_0) - (r(\xi_0) + J \rho U(\xi_0)) G(\rho, t, \xi_0) \right) e^{(\lambda-r(\xi_0)-J \rho U(\xi_0))t}
\]
\[
= 0D_t^{1-\alpha} \left( e^{(\lambda-r(\xi_0)-J \rho U(\xi_0))t} (\Delta + \gamma)^{\frac{\beta}{2}} G(\rho, t, \xi_0) \right) + \lambda e^{\lambda t} + \lambda \alpha 0D_t^{1-\alpha} \left( e^{(\lambda-r(\xi_0)-J \rho U(\xi_0))t} G(\rho, t, \xi_0) \right) - \lambda \alpha 0D_t^{1-\alpha} e^{\lambda t}.
\] (2.2)

It is easy to check that
\[
\left( \frac{\partial}{\partial t} G(\rho, t, \xi_0) + \lambda G(\rho, t, \xi_0) - (r(\xi_0) + J \rho U(\xi_0)) G(\rho, t, \xi_0) \right) e^{(\lambda-r(\xi_0)-J \rho U(\xi_0))t}
\]
\[
= \frac{\partial}{\partial t} \left( e^{(\lambda-r(\xi_0)-J \rho U(\xi_0))t} G(\rho, t, \xi_0) \right).
\]
Under the assumption that the solution to Eq. (1.2) is sufficiently regular, (2.2) can be rewritten as
\[ C_0 D_t^\alpha \left( e^{(\lambda - r(x_0) - J \rho U(x_0)) t} G(\rho, t, x_0) \right) = e^{(\lambda - r(x_0) - J \rho U(x_0)) t} \left( \Delta + \gamma \right)^{\frac{\beta}{2}} G(\rho, t, x_0) + \lambda_0 I_1^{1-\alpha} e^{\lambda t} + \lambda_0 e^{(\lambda - r(x_0) - J \rho U(x_0)) t} G(\rho, t, x_0) - \lambda_0 e^{\lambda t}. \]

According to Definition 3, we have the following equivalent form of Eq. (1.2)
\[ e^{(r(x_0) + J \rho U(x_0)) t} C_0 D_t^{\alpha, \lambda} \left( e^{-(r(x_0) + J \rho U(x_0)) t} G(\rho, t, x_0) \right) = (\Delta + \gamma)^{\frac{\beta}{2}} G(\rho, t, x_0) + f, \] where
\[ f = -\lambda_0 e^{(r(x_0) + J \rho U(x_0)) t} + \lambda_0 e^{-(\lambda - r(x_0) - J \rho U(x_0)) t} I_1^{1-\alpha} e^{\lambda t}. \]

So, to get an effective numerical scheme for Eq. (1.2) with nonhomogeneous initial condition, we need to homogenize the initial condition for Eq. (2.3), i.e.,
\[ G(\rho, t, x_0) = W(\rho, t, x_0) + G(\rho, 0, x_0) e^{-(\lambda - r(x_0) - J \rho U(x_0)) t}. \] (2.4)

So (2.3) can be rewritten as
\[ e^{(r(x_0) + J \rho U(x_0)) t} C_0 D_t^{\alpha, \lambda} \left( e^{-(r(x_0) + J \rho U(x_0)) t} W(\rho, t, x_0) \right) = (\Delta + \gamma)^{\frac{\beta}{2}} W(\rho, t, x_0) + f_w, \] where
\[ f_w = \lambda_0 G(\rho, 0, x_0) e^{-(\lambda - r(x_0) - J \rho U(x_0)) t} + (\Delta + \gamma)^{\frac{\beta}{2}} \left( G(\rho, 0, x_0) e^{-(\lambda - r(x_0) - J \rho U(x_0)) t} - \lambda_0 e^{(r(x_0) + J \rho U(x_0)) t} + \lambda_0 e^{-(\lambda - r(x_0) - J \rho U(x_0)) t} I_1^{1-\alpha} e^{\lambda t} \right). \] (2.6)

According to (2.3) and (2.6), we can get
\[ f_w(0) = 0. \]

Comparing Eq. (2.5) with Eq. (2.3), we only need to consider how to develop the numerical scheme for (2.3) with homogeneous initial condition, i.e., \( G_0(\rho, x_0) = 0 \), since the numerical scheme can also be applied to (2.5)–(2.6). For convenience, we rewrite (2.3) as
\[ I_t^{\alpha, \lambda} G(\rho, t, x_0) = (\Delta + \gamma)^{\frac{\beta}{2}} G(\rho, t, x_0) + f, \] (2.7)
where
\[ I_t^{\alpha, \lambda} G(\rho, t, x_0) = e^{(r(x_0) + J \rho U(x_0)) t} C_0 D_t^{\alpha, \lambda} \left( e^{-(r(x_0) + J \rho U(x_0)) t} G(\rho, t, x_0) \right) - \lambda_0 G(\rho, t, x_0). \] (2.8)

and \( f \) satisfies
\[ f(0) = 0. \] (2.9)

**Remark 2.2** Through the above derivation, it can be noted that one only needs to discretize Eq. (2.7) to approximate Eq. (1.2).
3 Space Discretization and Error Analysis

This section provides a finite difference discretization for the two-dimensional tempered fractional Laplacian on a bounded domain \( \Omega = (-l, l) \times (-l, l) \) with extended homogeneous Dirichlet boundary conditions: \( G(x, y) \equiv 0 \) for \( (x, y) \in \Omega^c \), which is based on our previous work [27] and modifies the regularity requirement according to [34]. Afterwards, we give the error analysis of the space semi-discrete scheme. Here, we set the mesh sizes \( h_1 = l/N_1 \) and \( h_2 = l/N_2 \); denote grid points \( x_i = ih_1 \) and \( y_j = jh_2 \), \( i, j \in \mathbb{Z} \), for \( -N_1 \leq i \leq N_1 \) and \(-N_2 \leq j \leq N_2 \); for convenience, let \( N_1 = N_2 = N \), then we can set \( h_1 = h_2 = h \).

3.1 Spatial Discretization

According to (1.4), we have

\[
-(\Delta + \gamma)^\beta G(x, y) = -c_{2, \beta} \text{P.V.} \int_{\mathbb{R}^2} \frac{G(\xi, \eta) - G(x, y)}{\vartheta(x, y, \xi, \eta)} d\xi d\eta, \tag{3.1}
\]

where

\[
\vartheta(x, y, \xi, \eta) = e^{r \sqrt{(\xi - x)^2 + (\eta - y)^2}} \left( \sqrt{(\xi - x)^2 + (\eta - y)^2} \right)^{2+\beta}.
\tag{3.2}
\]

To discretize \( (\Delta + \gamma)^\beta G(x_p, y_q) \) for any \(-N \leq p, q \leq N\), we first divide the integral domain into two parts for (3.1), i.e.,

\[
\int_{\mathbb{R}^2} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} d\xi d\eta = \int_{\Omega} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} d\xi d\eta + \int_{\mathbb{R}^2 \setminus \Omega} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} d\xi d\eta. \tag{3.3}
\]

It is easy to see that

\[
\int_{\mathbb{R}^2 \setminus \Omega} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} d\xi d\eta = -W_{p,q}^\infty G(x_p, y_q),
\]

where the fact \( G(\xi, \eta) \equiv 0 \) for \( (\xi, \eta) \in \mathbb{R}^2 \setminus \Omega \) is used and

\[
W_{p,q}^\infty = \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{\vartheta(x_p, y_q, \xi, \eta)} d\xi d\eta.
\]

Here, \( W_{p,q}^\infty \) can be calculated by the function ‘integral2.m’ in MATLAB.

Next, we formulate the first integral in (3.3) as

\[
\int_{\Omega} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} d\eta d\xi = \sum_{i = -N}^{N-1} \sum_{j = -N}^{N-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} d\eta d\xi, \tag{3.4}
\]

where \( \xi_i = ih \) and \( \eta_j = jh \). Denote \( \mathcal{I}_{p,q} = \{(p, q), (p-1, q), (p, q-1), (p-1, q-1)\} \) and

\[
\psi(x_p, y_q, \xi, \eta) = G(x_p + \xi, y_q + \eta) + G(x_p - \xi, y_q + \eta) + G(x_p - \xi, y_q - \eta) + G(x_p + \xi, y_q - \eta) - 4G(x_p, y_q).
\]
For (3.4), when \((i, j) \in \mathcal{I}_{p,q}\), we rewrite them as

\[
\int_{\xi_p}^{\xi_{p+1}} \int_{\eta_q}^{\eta_{q+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} \, d\eta d\xi + \int_{\eta_q}^{\eta_{q+1}} \int_{\xi_p}^{\xi_{p+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} \, d\eta d\xi
\]

\[
+ \int_{\xi_p}^{\xi_{p+1}} \int_{\eta_q}^{\eta_{q+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} \, d\eta d\xi
\]

\[
= \int_{\xi_p}^{\xi_{p+1}} \int_{\eta_q}^{\eta_{q+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} \, d\eta d\xi
\]

\[
= \int_{0}^{h} \int_{0}^{h} \frac{\psi(x_p, y_q, \xi, \eta)}{e^{\gamma \sqrt{\xi^2 + \eta^2}} \left(\sqrt{\xi^2 + \eta^2}\right)^{2+\beta}} \, d\eta d\xi,
\]

where we use

\[
\vartheta(x, y, x - \xi, y - \eta) = \vartheta(x, y, x + \xi, y - \eta)
\]

\[
= \vartheta(x, y, x - \xi, y + \eta) = \vartheta(x, y, x + \xi, y + \eta).
\]

Further denoting

\[
\phi_\sigma(\xi, \eta) = \frac{\psi(x_p, y_q, \xi, \eta)}{e^{\gamma \sqrt{\xi^2 + \eta^2}} \left(\sqrt{\xi^2 + \eta^2}\right)^{2+\beta}}, \quad \sigma \in (\beta, 2]
\]

and using the weighted trapezoidal rule, we have

\[
\int_{0}^{h} \int_{0}^{h} \phi_\sigma(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\sigma - 2 - \beta}{2}} \, d\eta d\xi
\]

\[
\approx \begin{cases} 
\frac{1}{4} \left( \lim_{(\xi, \eta) \to (0,0)} \phi_\sigma(\xi, \eta) + \phi_\sigma(\xi_0, \eta_1) + \phi_\sigma(\xi_1, \eta_1) + \phi_\sigma(\xi_1, \eta_0) \right) & \text{if } \sigma \in (\beta, 2); \\
W_{0,0}, \quad \sigma = 2,
\end{cases}
\]

where

\[
W_{0,0} = \int_{0}^{h} \int_{0}^{h} (\xi^2 + \eta^2)^{\frac{\sigma - 2 - \beta}{2}} \, d\eta d\xi.
\]

Assuming that \(u\) is smooth enough, for \(\sigma \in (\beta, 2)\), there exists

\[
\lim_{(\xi, \eta) \to (0,0)} \phi_\sigma(\xi, \eta) = 0;
\]

and further introduce a parameter

\[
k_\sigma = \begin{cases} 
1 & \sigma \in (\beta, 2), \\
\frac{1}{3} & \sigma = 2.
\end{cases}
\]

So, Eq. (3.7) can be rewritten as

\[
\int_{0}^{h} \int_{0}^{h} \phi_\sigma(\xi, \eta)(\xi^2 + \eta^2)^{\frac{\sigma - 2 - \beta}{2}} \, d\eta d\xi
\]
For (3.4), when \((i, j) \notin \mathcal{S}_{p,q}\), denote \(I_{p,q,p+i,q+j}\) as the approximation of
\[
\int_{\xi_{i+1}}^{\xi_{i+1}} \int_{\eta_{j+1}}^{\eta_{j+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} \, d\eta d\xi;
\]
we use the bilinear interpolation to approximate \(\frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)}\) in \([\xi_{i+1}, \xi_{i+1} + 1] \times [\eta_{j+1}, \eta_{j+1} + 1]\) and get
\[
I_{p,q,p+i,q+j} = \left( \frac{G(\xi_{i+1}, \eta_{j+1}) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi_{i+1}, \eta_{j+1})} \right) W_{i,j}^1 + \left( \frac{G(\xi_{i+1}, \eta_{j+1}) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi_{i+1} + 1, \eta_{j+1})} \right) W_{i+1,j}^2 + \left( \frac{G(\xi_{i+1}, \eta_{j+1} + 1) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi_{i+1}, \eta_{j+1} + 1)} \right) W_{i,j+1}^3 + \left( \frac{G(\xi_{i+1}, \eta_{j+1} + 1) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi_{i+1} + 1, \eta_{j+1} + 1)} \right) W_{i+1,j+1}^4,
\]
where
\[
W_{i,j}^1 = H_{i,j}^\xi - \xi_{i+1} H_{i+1,j}^\xi - \eta_{j+1} H_{i,j+1}^\eta + \xi_i + 1 \eta_{j+1} H_{i,j},
\]
\[
W_{i,j}^2 = - \left( H_{i+1,j}^\xi - \xi_{i+1} H_{i+1,j}^\xi - \eta_{j+1} H_{i+1,j+1}^\eta + \xi_i + 1 \eta_{j+1} H_{i+1,j} \right),
\]
\[
W_{i,j}^3 = - \left( H_{i,j+1}^\xi - \xi_{i+1} H_{i+1,j+1}^\xi - \eta_{j+1} H_{i+1,j}^\eta + \xi_i + 1 \eta_{j+1} H_{i+1,j+1} \right),
\]
\[
W_{i,j}^4 = H_{i,j+1}^\xi - \xi_i H_{i,j+1}^\xi - \eta_{j+1} H_{i+1,j+1}^\eta + \xi_i + 1 \eta_{j+1} H_{i+1,j+1}.
\]

and
\[
H_{i,j}^\xi = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} (\xi^2 + \eta^2)^{-\frac{\gamma + \beta}{2}} \, d\eta d\xi,
\]
\[
H_{i,j}^\eta = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \xi (\xi^2 + \eta^2)^{-\frac{\gamma + \beta}{2}} \, d\eta d\xi,
\]
\[
H_{i,j}^{\xi \eta} = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \eta (\xi^2 + \eta^2)^{-\frac{\gamma + \beta}{2}} \, d\eta d\xi,
\]
\[
H_{i,j}^{\eta \xi} = \frac{1}{h^2} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \xi (\xi^2 + \eta^2)^{-\frac{\gamma + \beta}{2}} \, d\eta d\xi.
\]

Then Eq. (3.4) becomes
\[
\sum_{i=-N}^{N-1} \sum_{j=-N}^{N-1} \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} \frac{G(\xi, \eta) - G(x_p, y_q)}{\vartheta(x_p, y_q, \xi, \eta)} \, d\eta d\xi \approx k_{\sigma} \frac{1}{4} (\phi_\sigma(\xi_0, \eta_1) + \phi_\sigma(\xi_1, \eta_1) + \phi_\sigma(\xi_1, \eta_0)) W_{0,0} + \left. \sum_{i=-N}^{N-1} \sum_{j=-N}^{N-1} I_{p,q,i,j} \right|_{(i,j) \notin \mathcal{S}_{p,q}}.
\]
To make the form of weight \( w_{p,q,i,j}^{\beta,\gamma} \) unified, according to (3.9), we denote

\[
\begin{align*}
W_{-1,-1}^1 &= W_{1,-1}^2 = W_{-1,1}^3 = W_{1,1}^4 = \frac{k_\sigma}{4} \frac{W_{0,0}}{(\sqrt{2}h)^\sigma}, \\
W_{-1,0}^1 &= W_{1,0}^2 = W_{-1,0}^3 = W_{1,0}^4 = W_{0,-1}^1 = W_{0,1}^2 \\
&= W_{0,1}^3 = W_{0,1}^4 = \frac{k_\sigma}{4} \frac{W_{0,0}}{(h)^\sigma}.
\end{align*}
\]

(3.13)

Let \( G_{p,q} = G(x_p, x_q) \). So, we have the discretization scheme

\[
-(\Delta + \gamma)^{\beta}_h G_{p,q} = \sum_{i=-N}^{N} \sum_{j=-N}^{N} w_{p,q,i,j}^{\beta,\gamma} G_{i,j},
\]

(3.14)

where

\[
\begin{align*}
&\frac{-}{(i=-N+1, j=-N+1)} \sum_{i=-N+1, j=-N+1} \frac{W_{i-p, -q}^1}{W_{i-p, -q}^1 + W_{i-p, -q}^4 + W_{i-p, -q}^4} e^{\gamma h \sqrt{(i(p))^2 + (j(q))^2}} \\
&+ \frac{W_{N-p, -q}^1}{W_{N-p, -q}^1 + W_{N-p, -q}^4 + W_{N-p, -q}^4} e^{\gamma h \sqrt{(N-p)^2 + (N-q)^2}} \\
&+ \frac{W_{1, -p, q}^1}{W_{1, -p, q}^1 + W_{1, -p, q}^4 + W_{1, -p, q}^4} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
&+ \frac{W_{1, p, q}^1}{W_{1, p, q}^1 + W_{1, p, q}^4 + W_{1, p, q}^4} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
&+ \frac{W_{1, -p, q}^1}{W_{1, -p, q}^1 + W_{1, -p, q}^4 + W_{1, -p, q}^4} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}}
\end{align*}
\]

\[
\left(\begin{array}{c}
\frac{1}{W_{1, -p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
\frac{1}{W_{1, p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
\frac{1}{W_{1, -p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
\frac{1}{W_{1, p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}}
\end{array}\right) \right), \quad i = p, j = q;
\]

\[
\begin{align*}
w_{p,q,i,j}^{\beta,\gamma} &= -C_{2,\beta} \left(\begin{array}{c}
\frac{1}{W_{1, -p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
\frac{1}{W_{1, p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
\frac{1}{W_{1, -p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}} \\
\frac{1}{W_{1, p, q}^1} e^{\gamma h \sqrt{(1-p)^2 + (1-q)^2}}
\end{array}\right) \right), \quad i = p, j = q;
\]

\[
\begin{align*}
&-N < i < N, j = -N; \\
&i = -N, -N < j < N; \\
&-N < i < N, j = N; \\
&i = N, -N < j < N; \\
&i = -N, j = -N; \\
&N < i < N, j = N; \\
&i = N, j = N; \\
&i = N, j = -N; \\
&N < i < N, j = -N; \\
&i = -N, j = N; \\
&i = -N, j = -N;
\end{align*}
\]

(3.15)
Remark 3.1 Here, we discretize the tempered fractional Laplacian satisfying homogeneous Dirichlet boundary conditions, so (3.14) can be rewritten as

\[-(\Delta + \gamma)^{\frac{\beta}{2}} G_{p,q} = \sum_{i=-N+1}^{N-1} \sum_{j=-N+1}^{N-1} w_{p,q,i,j}^{\beta,\gamma} G_{i,j}. \] (3.16)

3.2 Error Analysis for the Space Semi-discrete Scheme

First, we define an operator from a function to a vector \( \mathcal{V} : f \rightarrow \mathbf{f} \), where \( f \) denotes a function,

\[ f = (f_{-N+1,-N+1}, f_{-N+1,-N+2}, \ldots, f_{-N+1,N-1}, f_{-N+2,-N+1}, \ldots, f_{N-1,N-1})^T, \]

and \( f_{p,q} = f(x_p, y_q) \).

According to (2.7) and (3.16), the spatially semi-discrete scheme can be written as

\[ L_{\alpha,\lambda}^t G_{h,p,q}(t) = -(\Delta + \gamma)^{\frac{\beta}{2}} h G_{h,p,q}(t) + \mathbf{V} f_p(t) \quad \text{for} \quad -N < p, q < N, \] (3.17)

where \( G_{h,p,q}(t) \) is the numerical solution at \((x_p, y_q)\) of the spatially semi-discrete scheme. Denoting \( G_{h}(t) = (G_{h,-N+1,-N+1}(t), G_{h,-N+1,-N+2}(t), \ldots, G_{h,N-1,N-1}(t))^T \), then the spatially semi-discrete scheme can be rewritten as

\[ L_{\alpha,\lambda}^t G_{h}(t) = (\Delta + \gamma)^{\frac{\beta}{2}} h G_{h}(t) + \mathcal{V} f(t). \] (3.18)

According to Proposition 1 and taking the Laplace transform for (2.7), we get

\[ \tilde{L}_{\alpha,\lambda}^t \tilde{G} = (\Delta + \gamma)^{\frac{\beta}{2}} \tilde{G} + \tilde{f}, \]

where \( \tilde{G} \) and \( \tilde{f} \) denote the Laplace transforms of \( G \) and \( f \), respectively, and

\[ \tilde{L}_{\alpha,\lambda}^t = \left((z + \lambda - r(x_0) - J \rho U(x_0))^{\alpha} - \lambda \right) =: \omega(z, x_0) =: \omega. \]

So we obtain

\[ \tilde{G} = \left(\omega - (\Delta + \gamma)^{\frac{\beta}{2}} h \right)^{-1} \tilde{f} =: \tilde{E}(z) \tilde{f}. \] (3.19)

Similarly, taking the Laplace transform for (3.18), we get

\[ \omega_h \tilde{G}_{h}(t) = (\Delta + \gamma)^{\frac{\beta}{2}} h \tilde{G}_{h}(t) + \mathcal{V} \tilde{f}, \]

where

\[ \omega_h \tilde{G}_{h} = \text{diag}(\mathcal{V} \omega) \tilde{G}_{h} \]

and ‘diag’ denotes a diagonal matrix formed from its vector argument. The solution of spatially semi-discrete scheme is

\[ \tilde{G}_{h} = \left(\omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} h \right)^{-1} \mathcal{V} \tilde{f} =: \tilde{E}_{h}(z) \mathcal{V} \tilde{f}. \] (3.20)
The solution of (2.7) may therefore be obtained by the inverse Laplace transform of (3.19), with integration along a line parallel to and to the right of the imaginary axis. So, we need to choose a suitable integral contour to get the error estimate between (2.7) and (3.18). Let
\[
\frac{\pi}{2} \leq \theta < \frac{\pi}{2} + \theta_e,
\]
where
\[
\theta_e = \inf_{x_0 \in \Omega} \arccos \left( \frac{|\rho U(x_0)|}{\sqrt{r(x_0)^2 + (\rho U(x_0))^2}} \right). \tag{3.22}
\]

**Lemma 3.1** For any \( z \in \Sigma_\theta \) and \( x_0 \in \Omega \), \( \omega(z, x_0) \in \Sigma_\theta \) holds, where \( \Sigma_\theta = \{ z \in \mathbb{C} : |\arg z| \leq \theta \} \).

**Proof** We first introduce a notation \( \Gamma_\theta = \{ z \in \mathbb{C} : |\arg z| = \theta \} \cup \{0\} \). (3.21) and (3.22) yield that for any \( z \in \Sigma_\theta \) and \( x_0 \in \Omega \),
\[
(z - r(x_0) - J \rho U(x_0)) \in \Sigma_\theta.
\]
So we just need to prove \( ((z + \lambda)^\alpha - \lambda^\alpha) \in \Sigma_\theta \) for any \( z \in \Sigma_\theta \) to get \( \omega(z, x_0) \in \Sigma_\theta \). By simple calculations, there exists
\[
((z + \lambda)^\alpha - \lambda^\alpha) = \lambda^\alpha \left( \left( \frac{z}{\lambda} + 1 \right)^\alpha - 1 \right).
\]
So we just need to prove \( ((z + 1)^\alpha - 1) \in \Sigma_\theta \) for any \( z \in \Sigma_\theta \), which is equal to prove \( (z + 1)^\alpha \in \Sigma_\theta + 1 = \{ z \in \mathbb{C} : z - 1 \neq 0, |\arg(z - 1)| < \theta \} \). Therefore, we need to prove \( |(z + 1)^\alpha| \geq \bar{z} \), where \( \bar{z} \in \bar{\Gamma} = \{ z \in \mathbb{C} : z - 1 \in \Gamma_\theta \} \) and \( |\arg \bar{z}| = |\arg(z + 1)^\alpha| \). Assuming that \( |\arg(z + 1)| = \bar{\theta} \), we have \( |\arg \bar{z}| = \alpha \bar{\theta} \) and \( 0 < \bar{\theta} < \theta \). By simple calculations, we obtain
\[
|(z + 1)^\alpha| = |(z + 1)|^{\alpha} \geq \left( \frac{\cos \bar{\theta}}{\cos (\alpha \bar{\theta} - \bar{\theta})} \right)^{\alpha},
\]
where \( \bar{\theta} = \theta - \pi/2 \). Denote
\[
F(\bar{\theta}) = \frac{1}{|\bar{z}|} \left( \frac{\cos \bar{\theta}}{\cos (\alpha \bar{\theta} - \bar{\theta})} \right)^{\alpha} = (\cos \bar{\theta})^{\alpha - 1} \frac{\cos(\alpha \bar{\theta} - \bar{\theta})}{(\cos(\bar{\theta} - \bar{\theta}))^{\alpha}}.
\]
Taking the first derivative for \( F(\bar{\theta}) \), we get
\[
F'(\bar{\theta}) = \frac{\alpha(\cos \bar{\theta})^{\alpha - 1}}{(\cos(\bar{\theta} - \bar{\theta}))^{\alpha + 1}} \left( \sin \bar{\theta} \cos(\bar{\theta} - \bar{\theta}) - \sin(\bar{\theta} - \bar{\theta}) \cos(\alpha \bar{\theta} - \bar{\theta}) \right)
\]
\[
= \frac{\alpha(\cos \bar{\theta})^{\alpha - 1}}{(\cos(\bar{\theta} - \bar{\theta}))^{\alpha + 1}} \left( \sin \bar{\theta} \cos(\alpha \bar{\theta} - \bar{\theta}) - \sin(\alpha \bar{\theta} - \bar{\theta}) \cos(\bar{\theta} - \bar{\theta}) \right).
\]
Since \( 0 \leq \bar{\theta} < \theta = \pi/2 + \bar{\theta} \), we have \( \cos \bar{\theta} > 0 \) and \( \cos(\bar{\theta} - \bar{\theta}) > 0 \). Then, for \( F'(\bar{\theta}) \), there are the following discussions.
(1) When $\bar{\theta} - \tilde{\theta} > \alpha \bar{\theta} - \tilde{\theta} \geq 0$, we have
\[
\sin(\bar{\theta} - \tilde{\theta}) > \sin(\alpha \bar{\theta} - \tilde{\theta}) \geq 0; \quad \cos(\alpha \bar{\theta} - \tilde{\theta}) > \cos(\bar{\theta} - \tilde{\theta}) > 0.
\]
So there is
\[
F'(\bar{\theta}) > 0.
\]
(2) When $\bar{\theta} - \tilde{\theta} \geq 0 > \alpha \bar{\theta} - \tilde{\theta}$, we have
\[
\sin(\bar{\theta} - \tilde{\theta}) \geq \sin(\alpha \bar{\theta} - \tilde{\theta}) > 0; \quad \cos(\alpha \bar{\theta} - \tilde{\theta}), \cos(\bar{\theta} - \tilde{\theta}) > 0.
\]
So it holds that
\[
F'(\bar{\theta}) > 0.
\]
(3) When $0 > \bar{\theta} - \tilde{\theta} > \alpha \bar{\theta} - \tilde{\theta}$, we have
\[
\sin(\tilde{\theta} - \alpha \bar{\theta}) > \sin(\tilde{\theta} - \bar{\theta}) > 0; \quad \cos(\bar{\theta} - \tilde{\theta}) > \cos(\alpha \bar{\theta} - \tilde{\theta}) > 0.
\]
So there is
\[
F'(\bar{\theta}) > 0.
\]
(4) When $\bar{\theta} - \tilde{\theta} = \alpha \bar{\theta} - \tilde{\theta}$, that is $\bar{\theta} = 0$, there is
\[
F'(0) = 0.
\]
Finally, it leads to
\[
F(\bar{\theta}) \geq F(0) = 1
\]
and
\[
|(z + 1)^{\alpha}| \geq |\bar{z}|.
\]
Therefore, we obtain $\omega(z, x_0) \in \Sigma_\theta$ for any $z \in \Sigma_\theta$.  

Lemma 3.2 $\tilde{E}(z)$ is analytic and satisfies $\|\tilde{E}(z)\|_{L^2 \to L^2} \leq M/|z|^\alpha$ in $\Sigma_{\theta,K}$, where $\Sigma_{\theta,K} = \{z \in \mathbb{C} : |z| > \kappa, |\arg z| \leq \theta\}$. Similarly, $\tilde{E}_h(z)$ is analytic and satisfies $\|\tilde{E}_h(z)\|_{L^2 \to L^2} \leq M/|z|^\alpha$ in $\Sigma_{\theta,K}$.

Proof First, we prove $\frac{|z - r(x_0) - J \rho U(x_0)|}{|z|} \geq C$ for any $x_0 \in \tilde{\Omega}$. Let
\[
z = c_1 e^{J \eta_1}, \quad |\eta_1| < \theta;
\]
\[
r(x_0) + J \rho U(x_0) = c_2 e^{J \eta_2}, \quad |\eta_2| \geq \theta_\epsilon + \frac{\pi}{2}.
\]
Then we obtain
\[
\frac{|z - r(x_0) - J \rho U(x_0)|}{|z|} = \frac{|z - r(x_0) - J \rho U(x_0)|}{|z|} = \frac{|c_1 e^{J \eta_1} - c_2 e^{J \eta_2}|}{c_1 e^{J \eta_1}} = \left|1 - \frac{c_2}{c_1} e^{J (\eta_2 - \eta_1)}\right|.
\]
By using the relationship between the complex point $1$ and the line $z = |z| e^{J (\eta_2 - \eta_1)}$ in complex plane, we obtain
\[
\frac{|z - r(x_0) - J \rho U(x_0)|}{|z|} \geq \sin \left(\theta_\epsilon + \frac{\pi}{2} - \theta\right) \geq C, \quad (3.23)
\]
where $C$ is a positive constant. Let $z_e = z - r(x_0) - J \rho U(x_0)$. So there exists a constant $c_e > 0$ satisfying $|z_e| \geq c_e > 0$ and $|\arg z_e| \leq \theta$. Moreover, by the mean value theorem, we have

$$\frac{|(\lambda + z_e)^\alpha - \lambda^\alpha|}{|z_e|^\alpha} \geq C \frac{|(\lambda + iz_e)|^{\alpha-1} |z_e|}{|z_e|^\alpha} \geq C \frac{|z_e|^{1-\alpha}}{|(\lambda + iz_e)|^{1-\alpha}},$$

where $\iota \in (0, 1)$. Since $|z_e| \geq c_e > 0$, there exists

$$\frac{|(\lambda + z_e)^\alpha - \lambda^\alpha|}{|z_e|^\alpha} \geq C.$$  

(3.24)

According to (3.23) and (3.24), we arrive at

$$|\omega| = |(\lambda + z_e)^\alpha - \lambda^\alpha| \geq C |z_e|^\alpha \geq C |z|^\alpha.$$  

(3.25)

Finally, according to [35], we know that $-(\Delta + \gamma)^\frac{\beta}{2}$ with homogeneous Dirichlet boundary conditions is a positive definite and self-adjoint operator in $L^2(\Omega)$. Thus $-(\Delta + \gamma)^\frac{\beta}{2}$ generates an analytic semigroup [1,23], which implies that for any $\hat{\theta}$ and $M = M_{\hat{\theta}}$, we have the resolvent estimate

$$\left\| \left( zI - (\Delta + \gamma)^\frac{\beta}{2} \right)^{-1} \right\|_{L^2 \to L^2} \leq \frac{M}{|z|} \quad \text{for} \quad z \in \Sigma_{\hat{\theta}} = \{ z \in \mathbb{C} : |z| \neq 0, |\arg z| < \hat{\theta} \}.$$  

By Lemma 3.1, the condition $\Sigma_{\theta, \kappa} \subseteq \Sigma_{\theta}$ and (3.25), for fixed $\bar{x} \in \tilde{\Omega}$, we obtain

$$\left\| \left( \omega(z, \bar{x}) - (\Delta + \gamma)^\frac{\beta}{2} \right)^{-1} \right\|_{L^2 \to L^2} \leq \frac{M}{|z|^\alpha} \quad \text{for} \quad z \in \Sigma_{\theta, \kappa}.$$  

(3.26)

Let

$$\left( \omega(z, x_0) - (\Delta + \gamma)^\frac{\beta}{2} \right) G = F.$$  

(3.27)

Then we have

$$\left( \omega(z, \bar{x}) - (\Delta + \gamma)^\frac{\beta}{2} \right) G = F + (\omega(z, \bar{x}) - \omega(z, x_0))G.$$  

(3.28)

It is easy to get that

$$\| F + (\omega(z, \bar{x}) - \omega(z, x_0))G \|_{L^2} \leq \| F \|_{L^2} + \| (\omega(z, \bar{x}) - \omega(z, x_0))G \|_{L^2} \leq \| F \|_{L^2} + \sup_{x_0 \in \Omega} |(r(x_0) - r(\bar{x})) - J \rho (U(x_0) - U(\bar{x}))|^\alpha \| G \|_{L^2} \leq \| F \|_{L^2} + \| G \|_{L^2}.$$  

(3.29)

Combining (3.26), (3.28) and (3.29), we have

$$\| G \|_{L^2} \leq M |z|^{-\alpha} \| F \|_{L^2} + M |z|^{-\alpha} \| G \|_{L^2}.$$  

When $\kappa$ is large enough, namely, $|z|$ is large enough, we have $\| G \|_{L^2} \leq M |z|^{-\alpha} \| F \|_{L^2}$ and

$$\left\| \left( \omega(z, x_0) - (\Delta + \gamma)^\frac{\beta}{2} \right)^{-1} \right\|_{L^2 \to L^2} \leq \frac{M}{|z|^\alpha} \quad \text{for} \quad z \in \Sigma_{\theta, \kappa}.$$  

Next, to prove $\| \tilde{E}_h(z) \|_{L^2 \to L^2} \leq M |z|^\alpha$, we need to show the positive definiteness and self-adjoint of $(\Delta + \gamma)^\frac{\beta}{2}$, i.e., the matrix generated by discretizing the tempered fractional Laplacian is positive definite and symmetric. The detailed proof is similar to the one in [27].

The rest of the proof of $\tilde{E}_h(z)$ is similar to the case of $E(z)$.  

$\square$
By simple calculation, we get the following estimations of the first derivatives for \( \tilde{E}(z) \) and \( \tilde{E}_h(z) \).

**Lemma 3.3** \( \| \tilde{E}'(z) \|_{L^2 \to L^2} \leq M/|z|^\alpha + 1 \) in \( \Sigma_{\theta, \kappa} \), similarly, \( \| \tilde{E}_h'(z) \|_{L^2 \to L^2} \leq M/|z|^\alpha + 1 \) in \( \Sigma_{\theta, \kappa} \), where \( \Sigma_{\theta, \kappa} \) is defined in Lemma 3.2.

**Theorem 3.1** Denote \( \Delta + \gamma \) as a finite difference approximation of the tempered fractional Laplacian \( \Delta + \gamma \). Suppose that \( G(x) \in C^2(\tilde{\Omega}) \) is supported in an open set \( \Omega \subset \mathbb{R}^2 \).

Then, there are

\[
\left\| \mathcal{V}(\Delta + \gamma)^{\frac{\beta}{2}} G - (\Delta + \gamma)^{\frac{\beta}{2}} \tilde{f} \right\|_{L^\infty} \leq C h^{2-\beta},
\]

\[
\left\| \mathcal{V}(\Delta + \gamma)^{\frac{\beta}{2}} G - (\Delta + \gamma)^{\frac{\beta}{2}} \tilde{f} \right\|_{L^2} \leq C h^{2-\beta},
\]

for \( \beta \in (0, 2) \) with \( C \) being a positive constant depending on \( \beta \) and \( \gamma \).

**Proof** The proof is similar to the one in [27]. \( \square \)

Before giving the error estimate between (2.7) and (3.18), we introduce the following lemma.

**Lemma 3.4** For \( z \in \Sigma_\theta \) and the solution \( G(t, x) \in C^2(\tilde{\Omega}) \) for \( t \in (0, T) \), where \( \Sigma_\theta \) is defined in Lemma 3.1, there is the estimate

\[
\| \mathcal{V}(\Delta + \gamma)^{\frac{\beta}{2}} \tilde{E}(z) - \tilde{E}_h(z) \mathcal{V} \|_{L^2 \to L^2} \leq C h^{2-\beta} |z|^{-\alpha}.
\]

**Proof** According to (3.19) and (3.20), there are

\[
\left( \omega - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G} = \tilde{f}
\]

(3.30)

and

\[
\left( \omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G}_h = \mathcal{V} \tilde{f}.
\]

(3.31)

Performing \( \mathcal{V} \) on both sides of (3.30) leads to

\[
\mathcal{V} \left( \omega - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G} = \mathcal{V} \tilde{f}.
\]

(3.32)

Subtracting (3.32) from (3.31) results in

\[
\left( \omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G}_h - \mathcal{V} \left( \omega - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G} = 0.
\]

Then we obtain

\[
\left( \omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G}_h - \left( \omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \mathcal{V} \tilde{G}
\]

\[
+ \left( \omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \mathcal{V} \tilde{G} - \mathcal{V} \left( \omega - (\Delta + \gamma)^{\frac{\beta}{2}} \right) \tilde{G} = 0.
\]

Rearranging the terms leads to

\[
\left( \omega_h - (\Delta + \gamma)^{\frac{\beta}{2}} \right) (\tilde{G}_h - \mathcal{V} \tilde{G}) = (\Delta + \gamma)^{\frac{\beta}{2}} \mathcal{V} \tilde{G} - \mathcal{V} (\Delta + \gamma)^{\frac{\beta}{2}} \tilde{G},
\]

(3.33)
where the fact $\omega_h \mathcal{V} \tilde{G} = \mathcal{V} \omega \tilde{G}$ is used. According to Theorem 3.1, Lemma 3.2 and (3.33), we get

$$\| \mathcal{V} \tilde{G} - \tilde{G}_h \|^2 \leq \left\| \left( \omega_h - (\Delta + \gamma)^{-1}_h \right) \left( 2^{\gamma} \mathcal{V} \tilde{G} - (\Delta + \gamma)^{-1}_h \mathcal{V} \omega \tilde{G} \right) \right\|_{L^2}^2 \leq \left\| \omega_h - (\Delta + \gamma)^{-1}_h \right\|_{L^2}^2 \left\| (\Delta + \gamma)^{-1}_h \mathcal{V} \omega \tilde{G} - (\Delta + \gamma)^{-1}_h \mathcal{V} \omega \tilde{G} \right\|_{L^2}^2 \leq C h^{2-\beta} |z|^{-\alpha}. \quad (3.34)$$

Combining (3.19) and (3.20) leads to

$$\| \mathcal{V} \tilde{E}(z) \tilde{f} - \tilde{E}_h(z) \mathcal{V} f \|_{L^2} \leq C h^{2-\beta} |z|^{-\alpha}. \quad (3.35)$$

Taking $\| \tilde{f} \|_{L^2} = 1$ results in

$$\| \mathcal{V} \tilde{E}(z) - \tilde{E}_h(z) \mathcal{V} \|_{L^2} \leq C h^{2-\beta} |z|^{-\alpha},$$

which completes the proof. \qed

According to (3.19), (3.20) and Lemma 3.2, we have

$$G(t) = \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{zt} \tilde{E}(z) \tilde{f} dz \quad (3.35)$$

and

$$G_h(t) = \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{zt} \tilde{E}_h(z) \mathcal{V} \tilde{f} dz, \quad (3.36)$$

where $\Gamma_{\theta,\kappa} = \{ z \in \mathbb{C} : |z| \geq \kappa, |\arg z| = \theta \} \cup \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \}$.

**Theorem 3.2** Let $G$ be the solution of Eq. (2.7) satisfying $G(t, x) \in C^2(\Omega)$ for $t \in (0, T]$ and $G_h$ be the solution of Eq. (3.33). Suppose $f \in L^\infty(0, T; L^2(\Omega))$ with $\int_0^T (t - s)^{\alpha-1} \| f(s) \|_{L^2} ds < \infty$. Then, we have the following error estimate between the solutions of (2.7) and (3.18)

$$\| \mathcal{V} G(t) - G_h(t) \|_{L^2} \leq C h^{2-\beta} \int_0^T (t - s)^{\alpha-1} \| f(s) \|_{L^2} ds. \quad (3.37)$$

**Proof** According to (3.35) and (3.36), we have

$$\mathcal{V} G(t) - G_h(t) = \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{zt} \left( \mathcal{V} \tilde{E}(z) - \tilde{E}_h(z) \mathcal{V} \right) \tilde{f} dz. \quad (3.38)$$

From Lemma 3.4 and the property of convolution, there exists

$$\| \mathcal{V} G(t) - G_h(t) \|_{L^2} \leq \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{zt} \left( \mathcal{V} \tilde{E}(z) - \tilde{E}_h(z) \right) dz * f \|_{L^2} \leq C h^{2-\beta} \int_{\Gamma_{\theta,\kappa}} e^{-\sin(\theta-\pi/2)|z|} |z|^{-\alpha}d|z| * \| f \|_{L^2} \leq C h^{2-\beta} f^{\alpha-1} * \| f \|_{L^2},$$

which completes the proof. \qed

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4 Time Discretization and Error Analysis

In this section, we use the backward Euler (BE) method and the second-order backward difference (SBD) method to discretize the time tempered fractional substantial derivative and obtain the first-order and second-order schemes. Following that, we perform the error analyses for these two schemes.

4.1 BE Scheme and Error Analysis

First, let the time step size \( \tau = T/L \), \( L \in \mathbb{N}, t_i = i \tau, i = 0, 1, \ldots, L \) and \( 0 = t_0 < t_1 < \cdots < t_L = T \). Taking \( \delta(\zeta) = (1 - \zeta) \) and using convolution quadrature, for (2.8) we have the discretization scheme

\[
\mathcal{L}_t^{\alpha,\lambda} v(x_0, t_n) \approx \frac{1}{\tau^\alpha} \sum_{j=0}^{n} d_{n-j}^{\alpha,\lambda}(x_0) v_j(x_0), \tag{4.1}
\]

where

\[
v_j(x_0) = v(x_0, t_j)
\]

and

\[
\sum_{j=0}^{\infty} d_j^{\alpha,\lambda}(x_0) \zeta^j = (1 - \zeta + \tau \lambda - \tau r(x_0) - \tau J \rho U(x_0))^\alpha - (\tau \lambda)^\alpha. \tag{4.2}
\]

Here

\[
d_j^{\alpha,\lambda}(x_0) = \begin{cases} (1 + \tau \lambda - \tau r(x_0) - \tau J \rho U(x_0))^\alpha - (\tau \lambda)^\alpha, & j = 0; \\ - \alpha(1 + \tau \lambda - \tau r(x_0) - \tau J \rho U(x_0))^{\alpha-1}, & j = 1; \\ - (\alpha - j + 1) d_{j-1}^{\alpha,\lambda}(x_0) / j (1 + \tau \lambda - \tau r(x_0) - \tau J \rho U(x_0)), & j > 1. \end{cases}
\]

Remark 4.1 By simple calculations, \( \Re d_0^{\alpha,\lambda}(x_0) > 0 \) holds, where \( \Re z \) denotes the real part of \( z \). Here we denote \( d_j^{\alpha,\lambda}(x_p, y_q) \) as \( d_j^{\alpha,\lambda} \). The coefficients \( d_j^{\alpha,\lambda}(x_0) \) can also be calculated by Fast Fourier Transform (FFT).

Then the time semi-discrete scheme is as follows,

\[
\frac{1}{\tau^\alpha} \sum_{k=0}^{n} d_k^{\alpha,\lambda} G(t_n-k) = (\Delta + \gamma)^{\frac{\alpha}{2}} G(t_n) + f(t_n). \tag{4.3}
\]

Combining (4.3) with (3.14), we obtain the fully discrete scheme of Eq. (2.7), i.e., BE scheme

\[
\frac{1}{\tau^\alpha} \sum_{k=0}^{n} \text{diag}(\gamma^{\alpha} d_k^{\alpha,\lambda}) G_{h,k}^{n-k} = (\Delta + \gamma)^{\frac{\alpha}{2}} G_{h}^{n} + \gamma f^{n}, \tag{4.4}
\]

where \( f^{n} = f(t_n) \), \( G_{h,p,q}^{n} \) is the numerical solution at \( (x_p, y_q, t_n) \) for fully discrete scheme and

\[
G_{h}^{n} = (G_{h,-N+1,-N+1}^{n}, G_{h,-N+1,-N+2}^{n}, \ldots, G_{h,N-1,N-1}^{n})^T.
\]
Theorem 4.1 Let \( G_h \) and \( G_h^n \) be the solutions of Eqs. (3.18) and (4.4), respectively. If \( f \in L^\infty(0, T, L^2(\Omega)) \) with \( \int_0^T (t-s)^{\alpha-1}\|\mathcal{V}f'(s)\|_2 \, ds < \infty \) for \( t \in (0, T] \), then we have
\[
\|G_h(t_n) - G_h^n\|_2 \leq C \tau \int_0^{t_n} (t_n-s)^{\alpha-1}\|\mathcal{V}f'(s)\|_2 \, ds.
\]

Proof According to (3.36), we have
\[
G_h(t_n) = \int_{V_h} e^{\tau \zeta \tilde{E}_h(z)}1 \ast \mathcal{V}f'(z) \, dz,
\] (4.5)
where the fact \( f(t) = f(0) + 1 \ast f'(t) \), the property of convolution and (2.9) are used. To get the solution of (4.4), we need to multiply by \( \zeta^n \) and sum from 0 to \( \infty \), so
\[
\sum_{n=0}^{\infty} \left( \frac{1}{\tau^\alpha} \sum_{k=0}^{n} \text{diag}(\mathcal{V}d_{k}^{\alpha, \lambda}G_{h}^{n-k}) \right) \zeta^n = \sum_{n=0}^{\infty} \left( (\Delta + \mathcal{V}) \frac{\zeta}{\tau} G_{h}^{n} \zeta^n + \mathcal{V}f^n \zeta^n \right).
\]
According to (4.2), we obtain
\[
\left( \omega_h \left( \frac{1}{\tau} - \zeta \right) - (\Delta + \mathcal{V}) \frac{\zeta}{\tau} \right) \sum_{n=0}^{\infty} \zeta^n G_{h}^{n} = \sum_{n=0}^{\infty} \zeta^n \mathcal{V}f^n,
\]
which implies
\[
\sum_{n=0}^{\infty} \zeta^n G_{h}^{n} = \left( \omega_h \left( \frac{1}{\tau} - \zeta \right) - (\Delta + \mathcal{V}) \frac{\zeta}{\tau} \right)^{-1} \sum_{n=0}^{\infty} \zeta^n \mathcal{V}f^n = \tilde{E}_h \left( \frac{1}{\tau} - \zeta \right) \sum_{n=0}^{\infty} \zeta^n \mathcal{V}f^n.
\]
Thus
\[
G_{h}^{n} = \tau \sum_{j=0}^{n} E_{n-j} \mathcal{V}f^j,
\]
where
\[
\tilde{E}_h \left( \frac{1}{\tau} - \zeta \right) = \tau \sum_{j=0}^{\infty} E_{j} \zeta^j. \tag{4.6}
\]
For convenience, we denote
\[
Q(t', v) = \tau \cdot \text{diag} \left( \sum_{0 \leq j \leq t'} E_{j} \mathcal{V}v(t' - t_j) \right) = \text{diag} \left( (\tilde{E}_t \ast \mathcal{V}v)(t') \right),
\]
where \( \tilde{E}_t = \tau \sum_{j=0}^{\infty} E_j \delta_{t_j} \), with \( \delta_t \) the delta function concentrated at \( t \). By the fact \( f(t) = f(0) + 1 \ast f'(t) \), the property of convolution and (2.9), we have
\[
G_{h}^{n} = (\tilde{E}_t \ast \mathcal{V}f)(t_n) = \text{diag}(\tilde{E}_t \ast 1) \ast (\mathcal{V}f')(t_n) = (Q(t', 1) \ast \mathcal{V}f'(t'))(t_n). \tag{4.7}
\]
According to (4.5), (4.7) and the property of convolution, we have
\[
G_{h}(t_n) - G_{h}^{n} = \left( \frac{1}{2\pi i} \int_{\partial_{\zeta}} e^{\zeta t'} \tilde{E}_h(z)z^{-1} \, dz - Q(t', 1) \right) \ast \mathcal{V}f'(t'). \tag{4.8}
\]
To get the desired bound, we need to consider the error between \( \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{z\tau} \tilde{E}_h(z)z^{-1}dz \) and \( Q(t', 1) \) for \( t' \in [t_{n-1}, t_n) \), \( n \geq 1 \). As for \( n = 1 \), we have

\[
\begin{align*}
|Q(t', 1) - \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{z\tau} \tilde{E}_h(z)z^{-1}dz|_{l_2 \to l_2} \leq C t'(\alpha-1) \tau.
\end{align*}
\]

Then for \( n > 1 \), we have the estimate

\[
\begin{align*}
\left\| \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{z\tau} \tilde{E}_h(z)z^{-1}dz - \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{z\tau_{n-1}} \tilde{E}_h(z)z^{-1}dz \right\|_{l_2 \to l_2} \\
\leq C \tau \int_{\Gamma_{\theta,\kappa}} e^{-c|z'||z|^{-\alpha}}d|z| \leq C \tau t'(\alpha-1).
\end{align*}
\]

Following the above, we need to prove that

\[
\begin{align*}
|Q(t', 1) - \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{z\tau_{n-1}} \tilde{E}_h(z)z^{-1}dz|_{l_2 \to l_2} \leq C \tau t'(\alpha-1).
\end{align*}
\]

According to (4.6), we have for small \( \xi_{\tau} = e^{-\tau(\kappa+1)} \),

\[
\tau E_n = \frac{1}{2\pi J} \int_{|\xi|=\xi_{\tau}} \zeta^{-n-1} \tilde{E}_h \left( \frac{1-\xi}{\tau} \right) d\xi.
\]

Since \( \sum_{j=0}^{n-1} \zeta^{-j-1} = (\zeta^{-n} - 1)/(1 - \zeta) \), \( Q(t', 1) \) can be written as

\[
Q(t', 1) = \tau \sum_{j=0}^{n-1} E_j = \frac{1}{2\pi \tau J} \int_{|\xi|=\xi_{\tau}} \zeta^{-n} \tilde{E}_h \left( \frac{1-\xi}{\tau} \right) \left( \frac{1-\zeta}{\tau} \right)^{-1} d\zeta,
\]

where the fact \( \tilde{E}_h ((1 - \zeta)/\tau) / (1 - \zeta) \) is analytic for small \( \zeta \) is used. Taking \( \zeta = e^{-\tau t} \), we get

\[
Q(t', 1) = \frac{1}{2\pi J} \int_{\Gamma^\tau} e^{z\tau_{n-1}} \tilde{E}_h \left( \frac{1-e^{-\tau t}}{\tau} \right) \left( \frac{1-e^{-\tau t}}{\tau} \right)^{-1} dz,
\]

where \( \Gamma^\tau = \{ z = \kappa + 1 + Jy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau \} \). Next we deform the contour \( \Gamma^\tau \) to \( \Gamma_{\theta,\kappa}^\tau = \{ z \in \mathbb{C} : \kappa \leq |z| \leq \frac{\pi}{\tau \sin(\theta)}, \arg z = \theta \} \cup \{ z \in \mathbb{C} : |z| = \kappa, \arg z \leq \theta \} \), then \( Q(t', 1) \) can be rewritten as

\[
Q(t', 1) = \frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}^\tau} e^{z\tau_{n-1}} \tilde{E}_h \left( \frac{1-e^{-\tau t}}{\tau} \right) \left( \frac{1-e^{-\tau t}}{\tau} \right)^{-1} dz.
\]

Thus we have

\[
\frac{1}{2\pi J} \int_{\Gamma_{\theta,\kappa}} e^{z\tau_{n-1}} \tilde{E}_h(z)z^{-1}dz - Q(t', 1) = I + II.
\]
For $I$, according to Lemma 3.2, there exists the estimate

$$\|I\|_{l^2 \to l^2} \leq C \int_{I_{\theta,\kappa} \setminus I_{\theta,\kappa}^T} e^{-c|z|t'}|z|^{-\alpha-1}dz \leq C \tau t'(\alpha-1).$$  

(4.11)

For $II$, we obtain, with the mean value theorem, Lemma 3.3 and $1-e^{-\frac{z}{\tau}} = z + O(\tau z^2)$,

$$\|\tilde{E}_h(z)z^{-1} - \tilde{E}_h\left(\frac{1-e^{-\frac{z}{\tau}}}{\tau}\right)\|_{l^2 \to l^2} \leq C|z|^{-\alpha-2}|\tau z^2| \leq C|z|^{-\alpha}.$$  

(4.12)

Consequently,

$$\|II\|_{l^2 \to l^2} \leq C\tau \left|\int_{\pi}^{\tau} e^{-crt'}|r|^{-\alpha}dr\right| + C\tau \left|\int_{-\theta}^{\theta} e^{\kappa \cos(\eta)\theta_{n-1}}\kappa^{1-\alpha}d\eta\right| \leq C t'(\alpha-1)\tau + C\tau \kappa e^{\kappa t'} \leq C t'(\alpha-1)\tau,$$

where the fact $\kappa e^{\kappa t'} \leq \kappa T^{1-\alpha}e^{\kappa T t'(\alpha-1)}$ is used. Combining (4.9) and (4.10) leads to the desired result. $\square$

Theorem 4.2 Let $G$ be the solution of Eq. (2.7) satisfying $G(t, x) \in C^2(\tilde{\Omega})$ for $t \in (0, T]$ and $G^h_n$ be the solution of Eq. (4.4). If $f \in L^\infty(0, T, L^2(\Omega))$ satisfying

$$\int_0^t (t-s)^{\alpha-1}\|f(s)\|_{L^2}ds < \infty, \quad \int_0^t (t-s)^{\alpha-1}\|\nabla f'(s)\|_{l^2}ds < \infty$$

for $t \in (0, T]$, then

$$\|\nabla G(t_n) - G^h_n\|_{l^2} \leq C\tau \int_0^{t_n} (t_n-s)^{\alpha-1}\|\nabla f'(s)\|_{l^2}ds + Ch^{2-\beta} \int_0^{t_n} (t_n-s)^{\alpha-1}\|f(s)\|_{L^2}ds.$$

Proof Combining Theorems 3.2, 4.1 and triangle inequality, we can obtain the desired results. $\square$

4.2 SBD Scheme and Error Analysis

Taking $\delta(\zeta) = (1 - \zeta) + (1 - \zeta)^2/2$, we arrive at

$$\mathcal{L}_{t^\alpha,\lambda} u(x_0, t_n) \approx \frac{1}{\tau^\alpha} \sum_{j=0}^n d_{n-j}^{\alpha,\lambda}(x_0)u_j(x_0)$$  

(4.13)

and

$$\sum_{j=0}^\infty d_{j}^{\alpha,\lambda}(x_0)\zeta^j = ((1 - \zeta) + (1 - \zeta)^2/2 + \tau \lambda - \tau r(x_0) - \tau J \rho U(x_0))^\alpha - (\tau \lambda)^\alpha,$$

(4.14)
where \( d_{j}^{\alpha,\lambda}(x_0) \) can also be calculated by FFT. As for \( d_{0}^{\alpha,\lambda}(x_0) \), we have \( \Re d_{0}^{\alpha,\lambda}(x_0) > 0 \). Then the time semi-discrete scheme can be got as

\[
\frac{1}{\tau^\alpha} \sum_{k=0}^{n} d_{k}^{\alpha,\lambda} G(t_{n-k}) = (\Delta + \gamma)^{\frac{\alpha}{2}} G(t_{n}) + f(t_{n}).
\]  

(4.15)

Combining (4.15) with (3.14), we obtain the fully discrete scheme of Eq. (2.7), i.e., SBD scheme

\[
\frac{1}{\tau^\alpha} \sum_{k=0}^{n} \text{diag}(\mathcal{V} d_{k}^{\alpha,\lambda}) G_{n-k}^{h} = (\Delta + \gamma)^{\frac{\alpha}{2}} G_{n}^{h} + \mathcal{V} f_{n},
\]  

(4.16)

where the definitions of \( G_{n}^{h} \) and \( f_{n} \) are same as ones in (4.4).

**Theorem 4.3** Let \( G_{h} \) and \( G_{n}^{h} \) be the solutions of Eqs. (3.18) and (4.16), respectively. If \( f \in L^\infty(0, T, L^{2}(\Omega)) \) with \( \int_{0}^{t}(t-s)^{\alpha-1} \| \mathcal{V} f''(s) \|_{L^{2}} \, ds < \infty \) for \( t \in (0, T] \) and \( f'(0) \in L^{2}(\Omega) \), then we get

\[
\| G_{h}(t_{n}) - G_{n}^{h} \|_{L^{2}} \leq C \tau^{2} \left( t_{n}^{\alpha-1} \| \mathcal{V} f'(0) \|_{L^{2}} + \int_{0}^{t_{n}} (t_{n} - s)^{\alpha-1} \| \mathcal{V} f''(s) \|_{L^{2}} \, ds \right).
\]

**Proof** According to (3.36), we have

\[
G_{h}(t_{n}) = \frac{1}{2\pi J} \int_{\Gamma_{t,\kappa}} e^{\zeta t_{n}} \bar{E}_{h}(z) z^{-2} d z \mathcal{V} f'(0)
\]

\[
+ \left( \frac{1}{2\pi J} \int_{\Gamma_{t,\kappa}} e^{\zeta t} \bar{E}_{h}(z) z^{-2} d z \ast \mathcal{V} f''(t) \right) (t_{n}),
\]

(4.17)

where the fact \( f(t) = f(0) + tf'(0) + t \ast f''(t) \), the property of convolution and (2.9) are used. Multiplying both sides of (4.16) by \( \zeta^{n} \) and summing from 0 to \( \infty \) lead to

\[
\sum_{n=0}^{\infty} \zeta^{n} G_{h}^{n} = \bar{E}_{h} \left( \frac{(1 - \zeta) + (1 - \zeta)^{2}/2}{\tau} \right) \sum_{n=0}^{\infty} \zeta^{n} \mathcal{V} f_{n}.
\]

Thus

\[
G_{h}^{n} = \tau \sum_{j=0}^{n} E_{n-j} \mathcal{V} f^{j},
\]

where

\[
\bar{E}_{h} \left( \frac{(1 - \zeta) + (1 - \zeta)^{2}/2}{\tau} \right) = \tau \sum_{j=0}^{\infty} E_{j} \zeta^{j}.
\]

(4.18)

For convenience, we define

\[
Q(t', v) = \tau \cdot \text{diag} \left( \sum_{0 \leq t_{j} \leq t'} E_{j} \mathcal{V} v(t' - t_{j}) \right) = \text{diag} \left( (\bar{E}_{\tau} \ast \mathcal{V} v)(t') \right),
\]

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where $\hat{E}_t = \tau \sum_{j=0}^{\infty} E_j \delta_{t_j}$. Denoting $g(\cdot, t) = t$, then

$$G_h^n = (\hat{E}_t \ast \mathcal{Y} f)(t_n) = \text{diag}((\hat{E}_t \ast \mathcal{Y} g)(t_n)) \mathcal{Y} f'(0) + \text{diag}(\hat{E}_t \ast \mathcal{Y} g)(\mathcal{Y} f''(t_n)) + Q(t_n, g) \mathcal{Y} f'(0) + (Q(\cdot, g) \ast \mathcal{Y} f''(\cdot))(t_n),$$

(4.19)

where the fact $f(t) = f(0) + tf''(0) + t \ast f''(t)$, the property of convolution and (2.9) are used. According to (4.17) and (4.19), we have

$$G_h - G_h^n = \left( \frac{\tau}{2\pi J} \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-2} dz - Q(t_n, g) \right) \mathcal{Y} f'(0) + \left( \frac{\tau}{2\pi J} \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-2} dz - Q(t, g) \right) \ast \mathcal{Y} f''(t)(tn).$$

Now we need to consider the error between $\frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-2} dz$ and $Q(t', g)$ for $t' \in [t_{n-1}, t_n)$, $n \geq 1$. As for $n = 1$, we have

$$\left\| Q(t', g) - \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-2} dz \right\|_{l_2 \to l_2} \leq C t'(\alpha - 1) \tau^2.$$

Then for $n > 1$, by Taylor’s expansion, we have

$$\frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-2} dz = \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-2} dz + \frac{1}{2\pi J} (t' - t_n) \int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-1} dz + \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}} \int_{t_n}^{t'} (t' - s) e^{zs} \hat{E}_h(z)z^{-2} dz.$$

(4.20)

By simple calculation, we get

$$\left\| \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}} \int_{t_n}^{t'} (t' - s) e^{zs} \hat{E}_h(z)z^{-2} dz \right\|_{l_2 \to l_2} \leq C \tau^2 t'(\alpha - 1).$$

(4.21)

As for $Q(t', g)$, we have

$$Q(t', g) = \lim_{t' \to t_n} Q(t', g) + (t' - t_n) Q(t', 1).$$

(4.22)

First we consider the error between $\int_{\Gamma_{\theta, \kappa}} e^{zt_\kappa} \hat{E}_h(z)z^{-1} dz$ and $Q(t', 1)$. It can be noted that, for $\xi_t = e^{-\tau(\kappa + 1)}$,

$$\tau E_n = \frac{1}{2\pi J} \int_{|\xi| = \xi_t} \xi^{-n-1} \hat{E}_h \left( \frac{(1 - \xi) + (1 - \xi)^2/2}{\tau} \right) d\xi.$$

Hence

$$Q(t', 1) = \frac{\tau^{-1}}{2\pi J} \int_{|\xi| = \xi_t} \xi^{-n-1} \mu(\xi) \hat{E}_h \left( \frac{(1 - \xi) + (1 - \xi)^2/2}{\tau} \right).$$
Thus we have
\[
\left( \frac{1 - \zeta + (1 - \zeta)^2/2}{\tau} \right)^{-1} d\zeta,
\]
where we define \( \mu(\zeta) = \frac{1}{2}(3 - \zeta)\zeta \), and use the fact
\[
\sum_{j=0}^{n-1} \zeta^{-j-1} = \zeta^{-n-1} \frac{1 - \zeta^{-n}}{1 - \zeta} = \zeta^{-n-1} \frac{\mu(\zeta)}{(1 - \zeta) + (1 - \zeta)^2/2} - \frac{1}{1 - \zeta}
\]
and \( \tilde{E}_h \left( \frac{(1 - \zeta) + (1 - \zeta)^2/2}{\tau} \right) \) is analytic for small \( \zeta \). Taking \( \zeta = e^{-\tau t} \), and denoting \( z_\tau = \frac{(1-e^{-\tau t})+(1-e^{-\tau t})^2/2}{\tau} \), we have
\[
Q(t', 1) = \frac{1}{2\pi J} \int_{\Gamma_\tau} e^{z_t n} \mu(e^{-\tau t}) \tilde{E}_h (z_\tau) z_\tau^{-1} d\zeta,
\]
where \( \Gamma^\tau = \{ z = \kappa + 1 + Jy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau \} \). Next we deform the contour \( \Gamma^\tau \) to \( \Gamma_{\theta, \kappa}^\tau = \{ z \in \mathbb{C} : \kappa \leq |z| \leq \frac{\pi}{\tau \sin(\theta)}, |\arg z| = \theta \} \cup \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \} \), then \( Q(t', 1) \) can be rewritten as
\[
Q(t', 1) = \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}^\tau} e^{z_t n} \mu(e^{-\tau t}) \tilde{E}_h (z_\tau) z_\tau^{-1} d\zeta.
\]
Thus we have
\[
\frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}^\tau} e^{z_t n} \tilde{E}_h (z_\tau) z_\tau^{-1} d\zeta - Q(t', 1) = \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}^\tau} e^{z_t n} \tilde{E}_h (z_\tau) z_\tau^{-1} d\zeta
\]
\[
+ \frac{1}{2\pi J} \int_{\Gamma_{\theta, \kappa}^\tau} e^{z_t n} \left[ \tilde{E}_h (z_\tau) z_\tau^{-1} - \mu(e^{-\tau t}) \tilde{E}_h (z_\tau) z_\tau^{-1} \right] d\zeta = I + II.
\]
For \( I \), there exists the estimate
\[
\| I \|_{L^2_{t2}} \leq C \int_{\Gamma_{\theta, \kappa}^\tau} e^{-c|z'||z|-\alpha-1} d|z| \leq C t \tau'^{(\alpha-1)}.
\] (4.23)

For \( II \), we obtain, with the mean value theorem, Lemma 3.3, \( z_\tau = \frac{(1-e^{-\tau t})+(1-e^{-\tau t})^2/2}{\tau} = z + O(\tau^2z^3) \) and \( c|z| \leq |z_\tau| \leq C|z| \) when \( z \in \Gamma_{\theta, \kappa}^\tau \),
\[
\left\| \tilde{E}_h (z_\tau) z_\tau^{-1} - \mu(e^{-\tau t}) \tilde{E}_h (z_\tau) z_\tau^{-1} \right\|_{L^2_{t2}} \leq C|z|^{-\alpha-2}\tau^2z^3 + C \tau|z|^{-\alpha} \leq C \tau|z|^{-\alpha}.
\] (4.24)

Consequently,
\[
\|II\|_{L^2_{t2}} \leq C \tau \int_{\frac{\pi}{\tau \sin(\theta)}}^{\frac{\pi}{\tau \sin(\theta)}} e^{-c\tau|\eta|} d\eta \leq C \tau^{\alpha-1}\tau + C \tau \kappa e^{\kappa t} \leq C \tau^{\alpha(\alpha-1)} \tau,
\]
where the fact \( \kappa e^{\kappa t} \leq \kappa T^{1-\alpha} e^{\alpha T \tau'^{\alpha-1}} \) is used.

Next, we consider the error between \( \int_{\Gamma_{\theta, \kappa}^\tau} e^{z_t n} \tilde{E}_h (z_\tau) z_\tau^{-2} d\zeta \) and \( \lim_{t' \to t_n} Q(t', g) \). Using the fact that
\[
\sum_{j=0}^{n} j \varepsilon^j = (\zeta - \zeta^{n+1})/(1-\zeta)^2 - n\zeta^{n+1}/(1-\zeta) \text{ and } \tilde{E}_h \left( \frac{(1 - \zeta) + (1 - \zeta)^2/2}{\tau} \right) / \]
$$\frac{1}{(1-\zeta)\text{ and } \tilde{E}_h \left((1-\zeta) + (1-\zeta)^2/2\right)/\tau} / (1-\zeta)^2$$ are analytic for small $\zeta$, introducing $\mu_1(\zeta) = \zeta(3-\zeta)^2/4$ and taking $\zeta = e^{-\zeta t}$, we have

$$\lim_{t' \to t_n} \frac{1}{2\pi J} \int_{\Gamma_\tau} e^{z_{\tau}t} \mu_1(e^{-\zeta t}) \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} dz,$$

where $\Gamma_\tau = \{ z = \kappa + 1 + Jy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau \}. Next we deform the contour $\Gamma_\tau$ to $\Gamma_{\theta, k}^\tau = \{ z \in \mathbb{C} : \kappa \leq |z| \leq \frac{\pi}{\tau \sin(\theta)}, |\arg z| = \theta \} \bigcup \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \},$ then

$$\lim_{t' \to t_n} \frac{1}{2\pi J} \int_{\Gamma_{\theta, k}^\tau} e^{z_{\tau}t} \mu_1(e^{-\zeta t}) \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} dz.$$

Thus we have

$$\frac{1}{2\pi J} \int_{\Gamma_{\theta, k}^\tau} e^{z_{\tau}t} \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} dz - \lim_{t' \to t_n} \frac{1}{2\pi J} \int_{\Gamma_{\theta, k}^\tau} e^{z_{\tau}t} \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} dz + \frac{1}{2\pi J} \int_{\Gamma_{\theta, k}^\tau} e^{z_{\tau}t} \left( \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} - \mu_1(e^{-\zeta t}) \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} \right) dz = III + IV.$$

For $III$, there exists the estimate

$$\| III \|_{L_2 \to L_2} \leq C \int_{\Gamma_{\theta, k}^\tau} e^{-c|z| t} |z|^{-\alpha} d|z| \leq C \tau^2 \ell(\alpha - 1).$$

For $IV$, we obtain, with the mean value theorem, Lemma 3.3, $z_{\tau} = \frac{(1-e^{-\zeta t})+(1-e^{-\zeta t})^2/2}{\tau} = z + O(\tau^2 z^3)$ and $c|z| \leq |z_{\tau}| \leq C|z|$ for $z \in \Gamma^\tau_{\theta, k},$

$$\| \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} - \mu_1(e^{-\zeta t}) \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} \|_{L_2 \to L_2} \leq \| \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} - \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} \|_{L_2 \to L_2} + \| \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} - \mu_1(e^{-\zeta t}) \tilde{E}_h (z_{\tau}) z_{\tau}^{-2} \|_{L_2 \to L_2} \leq C|z|^{-\alpha} \tau^2 z^3 + C \tau^2 |z|^{-\alpha} \leq C \tau^2 |z|^{-\alpha}.$$

Consequently,

$$\| IV \|_{L_2 \to L_2} \leq C \tau^2 \left| \int_{\kappa}^{\pi/\tau \sin(\theta)} e^{-c t r} |r|^{-\alpha} dr \right| + C \tau^2 \left| \int_{-\theta}^{\theta} e^{\kappa \cos(\eta) t_n} e^{-k 1-\alpha} d\eta \right| \leq C \tau^2 \kappa \tau^2 e^{k t} \leq C \tau^2 \kappa \tau^2 e^{k t},$$

where the fact $\kappa e^{k t} \leq \kappa T^2 e^{\kappa T^2 t} < 1$ is used. So we complete the proof. \hfill \square

**Theorem 4.4** Let $G$ be the solution of Eq. (2.7) satisfying $G(t, x) \in C^2(\bar{\Omega})$ for $t \in (0, T]$ and $G_{h}^n$ be the solution of Eq. (4.16). If $f \in L^\infty(0, T, L^2(\Omega))$ satisfying

$$\int_0^t (t-s)^{\alpha-1} \| f(s) \|_{L^2} ds < \infty, \quad \int_0^t (t-s)^{\alpha-1} \| \nabla f''(s) \|_{L^2} ds < \infty$$

for $t \in (0, T]$ and $f'(0) \in L^2(\Omega)$, then

$$\| G(t_n) - G_{h}^n \|_{L^2} \leq C \tau^2 \left( t_n^{\alpha-1} \| G(t_n) \|_{L^2} + \int_0^{t_n} (t_n-s)^{\alpha-1} \| \nabla f''(s) \|_{L^2} ds \right) + Ch^2 \beta \int_0^{t_n} (t_n-s)^{\alpha-1} \| f(s) \|_{L^2} ds.$$
**Proof** Theorems 3.2, 4.3 and triangle inequality lead to the desired results. □

### 5 Efficient Computations

When discretizing the non-local operator, it generally gives rise to a full matrix, so an effective algorithm is needed to numerically solve (2.7) satisfying homogeneous Dirichlet boundary conditions, especially for high dimensional cases. In this section, we state how to reduce the complexity of our algorithm.

We first give a lemma about the property of $w^{\beta,\gamma}_{p,q,i,j}$ in (3.15).

**Lemma 5.1** Assume $-N < p_1, i_1, p_2, i_2 < N$ and $-N < q_1, j_1, q_2, j_2 < N$. If $(|p_1 - i_1|, |q_1 - j_1|) = (|p_2 - i_2|, |q_2 - j_2|)$ and $(|p_1 - i_1|, |q_1 - j_1|) \neq (0,0)$, then there is

$$w^{\beta,\gamma}_{p_1,q_1,i_1,j_1} = w^{\beta,\gamma}_{p_2,q_2,i_2,j_2}.$$  

**Proof** We first prove

$$W^1_{p,q} = W^2_{-p,q} = W^3_{p,-q} = W^4_{-p,-q}. \quad (5.1)$$

According to (3.13), for $|p| \leq 1$ and $|q| \leq 1$, (5.1) holds. By (3.10), there exists

$$W^1_{p,q} - W^2_{-p,q} = \left( H^\xi_{p,q} - \xi_{p+1} H^\eta_{p,q} - \eta_{q+1} H^\xi_{p,q} + \xi_{p+1} \eta_{q+1} H_{p,q} \right) + \left( H^\xi_{-p-1,q} - \xi_{p} H^\eta_{-p-1,q} - \eta_{q+1} H^\xi_{-p-1,q} + \xi_{p-1} \eta_{q+1} H_{-p-1,q} \right).$$

From (3.11), we have

$$H^\xi_{p,q} = -H^\xi_{-p-1,q}, \quad H^\eta_{p,q} = H^\eta_{-p-1,q}, \quad H^\xi_{p,q} = -H^\xi_{-p-1,q}, \quad H_{p,q} = H_{-p-1,q}. $$

Then there exists $W^1_{p,q} = W^2_{-p,q}$. Similarly, we have $W^1_{p,q} = W^3_{p,-q} = W^4_{-p,-q}$. Combining (3.15) with (5.1), the lemma can be proved. □

When numerically solving Eq. (2.7), the full-discretization scheme (4.4) or (4.16) can be written as the matrix form

$$A G_h^n = F,$$  

where

$$A = \frac{1}{h^\beta} A_s + \frac{1}{\tau^\alpha} A_t.$$  

Here, the elements of $A_s$ correspond to the discretization of the tempered fractional Laplacian; the elements of $A_t$ are with the discretization of the tempered fractional substantial derivative when $k = 0$ for Eqs. (4.4) or (4.16); the element of $F$ is composed of discretizing the source term $f$ defined by (2.7) and the tempered fractional substantial derivative when $k \neq 0$ for Eqs. (4.4) or (4.16); and

$$A_t = \text{diag}(\gamma d_0^{\alpha,\lambda}).$$

Next, we divide the matrix $A_s$ into

$$A_s = A_0 + A_d,$$  

where $A_0$ and $A_d$ are diagonal and full, respectively.
where

\[
A_0 = \begin{bmatrix}
0 & \cdots & w_{-N+1,-N+1} & \cdots & 0 \\
w_{-N+1,-N+2} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
w_{N-1,N-1} & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{bmatrix},
\]

\[
A_d = \begin{bmatrix}
w_{-N+1,-N+1} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
w_{N-1,N-1} & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{bmatrix}.
\]

Based on Lemma 5.1 and the structure of matrix \(A_0\), it is easy to find that \(A_0\) is a symmetric block Toeplitz matrix with Toeplitz block (BTTB) matrix. Being similar to [9], the memory requirement for the \((2N-1) \times (2N-1)\) matrix \(A_0\) can be reduced from \(O(N^2)\) to \(O(N)\) (\(N\) is the dimension of matrix).

When solving Eq. (5.2), we use the Krylov subspace iterative methods to reduce computational costs, such as the conjugate gradient (CG) method and the PCG method. In the iteration process, the \(Av\) needs to be calculated (\(v\) is a vector). By the above decomposition, one can calculate

\[
Av = (A_0v + A_dv)/h^\beta + (A_4v)/\tau^\alpha.
\]

Since \(A_0\) is a BTTB matrix, one can calculate \(A_0v\) by FFT and the computation costs can be reduced from \(O(N^2)\) to \(O(N \ln N) + O(N)\). To reduce the total number of iteration steps, one needs to consider how to construct a suitable preconditioner. Reference [9] builds a preconditioner for a BTTB matrix. However, matrix \(A\) isn’t a BTTB matrix (due to the entries on the main diagonal), so the preconditioner constructed in [9] can not be directly used. Instead, we denote

\[
\tilde{A} = A_0 + \sum_{(p,q) = (-N+1,-N+1)}^{(N-1,N-1)} \left( w_{p,q}^\beta \gamma + d_0^{\alpha,\lambda}(x_p, y_q) \right) \frac{1}{4(N-1)^2} I,
\]

where \(I\) is an identity matrix. It is easy to find that \(\tilde{A}\) is a BTTB matrix, so one can take a preconditioner of \(\tilde{A}\) as the one of \(A\). In numerical experiments, the effectiveness of the preconditioner is verified.

6 Numerical Experiments

In this section, we verify the theoretical results on convergence rate and the effectiveness of the scheme by solving (2.7) without the assumption on the regularity of the solution in time. Here, we consider the domain \(\Omega = (-1, 1) \times (-1, 1)\) and time \(T = 1\); \(l_2\) norm and \(l_\infty\) norm are used to measure the numerical errors.
6.1 Spatial Convergence Order

**Example 1** Choose $U(x_0) = 1$, $r(x_0) = -1$, and take the initial condition as
\[ G_0(\rho, x_0) = (1 - x^2)(1 - y^2) \quad x_0 \in \Omega; \]
the source term is
\[ f(t, x_0) = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu - \alpha)} e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (t^\nu - \alpha)(1 - x^2)(1 - y^2) \]
\[ - (\Delta + \gamma)^{\frac{\beta}{2}} \left( e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (t^\nu + 1)(1 - x^2)(1 - y^2) \right) \]
\[ - \lambda^2 e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (t^\nu + 1)(1 - x^2)(1 - y^2) \).

Then Eq. (1.2) has the exact solution
\[ G(\rho, t, x_0) = e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (t^\nu + 1)(1 - x^2)(1 - y^2) \quad x_0 \in \Omega. \]
By (2.4), there exists
\[ G(\rho, t, x_0) = W(\rho, t, x_0) + e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (1 - x^2)(1 - y^2). \]
So $W(\rho, t, x_0)$ solves
\[ \mathcal{L}_{t}^{\alpha, \lambda} W = (\Delta + \gamma)^{\frac{\beta}{2}} W - f_w(t, x_0), \]
where
\[ f_w(t, x_0) = f - (\Delta + \gamma)^{\frac{\beta}{2}} \left( e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (1 - x^2)(1 - y^2) \right) \]
\[ - \lambda^2 e^{-(\lambda-r(x_0)-\int \rho U(x_0))t} (1 - x^2)(1 - y^2). \]

Here, in order to reduce the effect of time-discrete errors on spatial convergence rates, we use SBD method to discrete the $\mathcal{L}_{t}^{\alpha, \lambda}$, i.e., (4.16). We choose $\nu = 1.5$, $\alpha = 0.3$ and $\lambda = 0.1$ to make $f_w$ satisfy the conditions of Theorem 4.3, which ensures the accuracy of the scheme. At the same time, we take $\tau = 1/640$ and $\sigma = 1 + \frac{\beta}{2}$. Table 1 shows the spatial convergence rates of solving Eq. (2.7); it can be noted that the results are consistent with the theoretical ones.

Table 2 shows the CPU time(s) and average iteration times of solving Eq. (2.7) when using CG method and PCG method. When the mesh size $h$ is small, PCG method has a significant advantage of time and average iteration times compared to CG method, which shows that our preconditioner is effective.

6.2 Temporal Convergence Order

**Example 2** Choose the exact solution given in Example 1 to verify the time convergence orders by BE and SBD methods. Here, in order to reduce the effect of spatial discrete errors on time convergence rates, we choose $h = 1/256$.

Firstly, we verify convergence orders of the BE scheme (4.4). We take $\nu = 0.8$ to satisfy the conditions needed in Theorem 4.1, and then let $\beta = 0.5$, $\gamma = 0$ and $\sigma = 1.25$. The results are shown in Table 3, which are consistent with our theoretical results. Afterwards, Table 4 gives the numerical errors and convergence rates of the SBD scheme (4.16) when $\nu = 1.8$, $\beta = 0.2$, $\gamma = 0$, and $\sigma = 2$. 

\( \beta = g \) gives the numerical errors and convergence rates of the SBD scheme (4.16) when $\nu = 5$, $\beta = 0.8$, $\gamma = 0$, and $\sigma = 2$. After this, Table 4 shows the results of solving Eq. (2.7); it can be noted that the results are consistent with the theoretical ones. Table 2 shows the CPU time(s) and average iteration times of solving Eq. (2.7) when using CG method and PCG method. When the mesh size $h$ is small, PCG method has a significant advantage of time and average iteration times compared to CG method, which shows that our preconditioner is effective.

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Table 1  Numerical errors and convergence rates with $\alpha = 0.3$, $\lambda = 0.1$, and $\sigma = 1 + \frac{\beta}{2}$

| $\beta$ | $l^\infty$ | $l^2$ | $l^\infty$ | $l^2$ | $l^\infty$ | $l^2$ |
|---------|-----------|--------|-----------|--------|-----------|--------|
| $0.5$   | $3.464E-03$ | $4.470E-03$ | $8.354E-03$ | $1.037E-02$ | $2.274E-02$ | $2.670E-02$ |
| Rate    | $1.3281$ | $1.3048$ | $1.0770$ | $1.0528$ | $1.0581$ | $0.6993$ |
| $0.8$   | $5.262E-04$ | $6.961E-04$ | $1.805E-03$ | $2.301E-03$ | $8.343E-03$ | $1.001E-02$ |
| Rate    | $1.3908$ | $1.3780$ | $1.1333$ | $1.1194$ | $1.1552$ | $0.7471$ |
| $1.0$   | $1.956E-04$ | $2.601E-04$ | $8.061E-04$ | $1.033E-03$ | $4.887E-03$ | $5.894E-03$ |
| Rate    | $1.4276$ | $1.4205$ | $1.1630$ | $0.7717$ | $0.7643$ | $0.5763$ |

Table 2  Performance of the CG and PCG methods

| $\gamma$ | $\beta$ | $l^\infty$ | $l^2$ | $l^\infty$ | $l^2$ | $l^\infty$ | $l^2$ |
|---------|---------|-----------|--------|-----------|--------|-----------|--------|
| $0.05$  | $0.5$   | $10.31$   | $9.86$ | $10.31$   | $11.64$ | $12.00$   | $12.00$ |
| PCG time (s) | $28.91$ | $29.47$ | $32.58$ | $36.98$ | $17.00$ | $17.00$ |
| Iterations | $90.86$ | $99.77$ | $103.64$ | $133.17$ | $22.00$ | $22.00$ |
| $0.05$  | $0.8$   | $490.63$  | $538.13$ | $583.30$  | $878.98$ | $805.63$  | $805.63$ |
| PCG time (s) | $1641.22$ | $2076.00$ | $2047.86$ | $3478.66$ | $2676.86$ | $2676.86$ |
| Iterations | $1641.22$ | $2076.00$ | $2047.86$ | $3478.66$ | $2676.86$ | $2676.86$ |
| $0.05$  | $1.2$   | $490.63$  | $538.13$ | $583.30$  | $878.98$ | $805.63$  | $805.63$ |
| PCG time (s) | $1641.22$ | $2076.00$ | $2047.86$ | $3478.66$ | $2676.86$ | $2676.86$ |
| Iterations | $1641.22$ | $2076.00$ | $2047.86$ | $3478.66$ | $2676.86$ | $2676.86$ |
Table 3  Numerical errors and convergence rates with $\beta = 0.5$, $\gamma = 0$, and $\sigma = 1.25$

| $\tau$ | 1/5    | 1/10   | 1/20   | 1/40   |
|--------|--------|--------|--------|--------|
| $\alpha = 0.3$ | $l^\infty$ | 7.34E-03 | 3.681E-03 | 1.846E-03 | 9.287E-04 |
|        | Rate   | 0.9967 | 0.9958 | 0.9910 |        |
| $\lambda = 0.5$ | $l^2$   | 8.104E-03 | 4.064E-03 | 2.040E-03 | 1.028E-03 |
|        | Rate   | 0.9958 | 0.9945 | 0.9885 |        |
| $\alpha = 0.5$ | $l^\infty$ | 1.284E-02 | 6.427E-03 | 3.204E-03 | 1.597E-03 |
|        | Rate   | 0.9986 | 1.0041 | 1.0048 |        |
| $\lambda = 0.5$ | $l^2$   | 1.414E-02 | 7.081E-03 | 3.532E-03 | 1.762E-03 |
|        | Rate   | 0.9770 | 0.9928 | 1.0008 |        |
| $\alpha = 0.7$ | $l^\infty$ | 1.998E-02 | 1.105E-03 | 5.100E-03 | 2.548E-03 |
|        | Rate   | 0.9749 | 0.9916 | 0.9999 |        |

Table 4  Numerical errors and convergence rates with $\beta = 0.2$, $\gamma = 0$, and $\sigma = 2$

| $\tau$ | 1/5    | 1/10   | 1/20   | 1/40   |
|--------|--------|--------|--------|--------|
| $\alpha = 0.3$ | $l^\infty$ | 1.702E-03 | 4.329E-04 | 1.071E-04 | 2.697E-05 |
|        | Rate   | 1.9747 | 2.0146 | 1.9898 |        |
| $\lambda = 0.8$ | $l^2$   | 1.845E-03 | 4.697E-04 | 1.164E-04 | 2.944E-05 |
|        | Rate   | 1.9738 | 2.0129 | 1.9829 |        |
| $\alpha = 0.5$ | $l^\infty$ | 2.872E-03 | 7.322E-04 | 1.804E-04 | 4.487E-05 |
|        | Rate   | 1.9718 | 2.0208 | 2.0075 |        |
| $\lambda = 0.8$ | $l^2$   | 3.106E-03 | 7.920E-04 | 1.953E-04 | 4.867E-05 |
|        | Rate   | 1.9713 | 2.0200 | 2.0043 |        |
| $\alpha = 0.8$ | $l^\infty$ | 4.208E-03 | 1.073E-03 | 2.633E-04 | 6.508E-05 |
|        | Rate   | 1.9720 | 2.0263 | 2.0165 |        |
| $\lambda = 0.8$ | $l^2$   | 4.542E-03 | 1.158E-03 | 2.843E-04 | 7.037E-05 |
|        | Rate   | 1.9717 | 2.0260 | 2.0147 |        |

Following that, we verify the temporal and spatial convergence orders by the unknown exact solution.

**Example 3**  Consider $U (x_0) = (x^2 + y^2)$ and $r (x_0) = -(x^2 + y^2)$. Take the initial condition $G_0 (\rho, x_0) = 0$, $x_0 \in \Omega$;

the source term is

$$f(t, x_0) = t^\nu.$$  

Since the exact solution is unknown, we use

$$e_h = \|G_{2h} - G_h\|, \quad e_\tau = \|G_{2\tau} - G_\tau\|$$

to measure the errors, where $G_h$ is the numerical solution under mesh size $h$ and $G_\tau$ is the numerical solution with time step size $\tau$.  

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Table 5  Numerical errors and convergence rates with $\alpha = 0.5$, $\lambda = 0.3$, and $\sigma = 1 + \beta/2$

| $h$ | 1/8 | 1/16 | 1/32 | 1/64 | 1/128 |
|-----|-----|------|------|------|-------|
| $\beta = 0.5$ | $l^\infty$ | 3.844E−02 | 3.694E−02 | 3.383E−02 | 2.981E−02 | 2.709E−02 |
| Rate | 0.0574 | 0.1268 | 0.1829 | 0.1379 | |
| $\lambda = 0.3$, and $\sigma = 1 + \beta/2$ | $l^2$ | 2.183E−02 | 1.689E−02 | 1.223E−02 | 8.459E−03 | 5.657E−03 |
| Rate | 0.3700 | 0.4662 | 0.5314 | 0.5803 | |
| $\gamma = 0.5$ | $l^\infty$ | 4.165E−02 | 4.162E−02 | 3.727E−02 | 3.229E−02 | 2.756E−02 |
| Rate | 0.0008 | 0.1595 | 0.2068 | 0.2286 | |
| $\beta = 0.8$ | $l^2$ | 2.259E−02 | 1.860E−02 | 1.389E−02 | 9.674E−03 | 6.393E−03 |
| Rate | 0.2803 | 0.4212 | 0.5219 | 0.5975 | |
| $\gamma = 0.5$ | $l^\infty$ | 4.003E−02 | 3.113E−02 | 2.515E−02 | 1.876E−02 | 1.327E−02 |
| Rate | 0.3629 | 0.3078 | 0.4230 | 0.4987 | |
| $\beta = 1.2$ | $l^2$ | 2.490E−02 | 1.894E−02 | 1.332E−02 | 9.674E−03 | 6.393E−03 |
| Rate | 0.2803 | 0.4212 | 0.5219 | 0.5975 | |
| $\gamma = 0.5$ | $l^\infty$ | 1.772E−02 | 1.504E−02 | 1.211E−02 | 9.288E−03 | 6.918E−03 |
| Rate | 0.3629 | 0.3078 | 0.4230 | 0.4987 | |
| $\beta = 1.5$ | $l^2$ | 1.480E−02 | 1.331E−02 | 1.089E−02 | 8.416E−03 | 5.761E−03 |
| Rate | 0.1527 | 0.2901 | 0.3713 | 0.4199 | |

Firstly, to verify the spatial convergence orders, we take $\nu = 1.2$, $\tau = 1/640$, $\alpha = 0.5$, $\lambda = 0.3$, and $\sigma = 1 + \beta/2$. The results are shown in Table 5. Since the regularity of the unknown solution does not meet the theoretical assumption, the convergence rates are lower. But the errors still converge, which verifies the effectiveness of our methods.

Next, we verify the temporal convergence orders, i.e., the BE scheme (4.4) and SBD scheme (4.16). Here, we take $\nu = 0.2$ and $\nu = 1.2$ to satisfy the conditions needed for Theorems 4.1 and 4.3, respectively, and then we let $h = 1/256$, $\beta = 0.5$, $\gamma = 0.05$, and $\sigma = 2$. The results are shown in Tables 6 and 7, respectively, which are consistent with our theoretical results.
### Table 7  Numerical errors and convergence rates with $\beta = 0.5$, $\gamma = 0.05$, and $\sigma = 2$

| $\tau$ | $1/10$ | $1/20$ | $1/40$ | $1/80$ | $1/160$ |
|---|---|---|---|---|---|
| $\alpha = 0.1$ | $l^\infty$ | 5.710E−04 | 1.533E−04 | 3.826E−05 | 9.483E−06 | 2.351E−06 |
| Rate | | 1.8966 | 2.0029 | 2.0123 | 2.0124 |
| $\lambda = 0.1$ | $l^2$ | 8.954E−04 | 2.336E−04 | 5.822E−05 | 1.446E−05 | 3.593E−06 |
| Rate | | 1.9386 | 2.0044 | 2.0098 | 2.0086 |
| $\alpha = 0.5$ | $l^\infty$ | 1.901E−03 | 4.165E−04 | 9.991E−05 | 2.486E−05 | 6.276E−06 |
| Rate | | 2.1905 | 2.0594 | 2.0066 | 1.9862 |
| $\lambda = 0.1$ | $l^2$ | 2.105E−03 | 5.095E−04 | 1.254E−04 | 3.119E−05 | 7.804E−06 |
| Rate | | 2.0470 | 2.0227 | 2.0069 | 1.9991 |
| $\alpha = 0.9$ | $l^\infty$ | 4.117E−03 | 9.535E−04 | 2.298E−04 | 5.692E−05 | 1.427E−05 |
| Rate | | 2.1102 | 2.0525 | 2.0137 | 1.9961 |
| $\lambda = 0.1$ | $l^2$ | 4.135E−03 | 9.653E−04 | 2.333E−04 | 5.768E−05 | 1.442E−05 |
| Rate | | 2.0989 | 2.0489 | 2.0158 | 2.0004 |

### 7 Conclusion

The model describing the functional distribution of the trajectory of the reaction and diffusion process was recently built [17], which is composed of tempered fractional substantial derivative in time and tempered fractional Laplacian in space. To develop the finite difference schemes for the two dimensional model, we use the convolution quadrature to approximate the tempered fractional substantial derivative and, respectively, get the first-order and second-order approximations, and the weighted trapezoidal rule and bilinear interpolation are used to deal with the tempered fractional Laplacian, which is based on our previous work and modifies the regularity requirement of the solution according to [34]. The error analyses of the designed schemes are strictly performed. Moreover, some techniques are introduced to effectively reduce the complexity of the algorithm. Finally, we verify the predicted convergence rates and the effectiveness of the proposed schemes by numerical experiments.

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