ANALYTICAL ON SHELL QED RESULTS: 3-LOOP VACUUM POLARIZATION, 4-LOOP $\beta$-FUNCTION AND THE MUON ANOMALY

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Abstract

We present the results of analytical calculations of the 3-loop contributions to the asymptotic photon vacuum polarization function, in the on shell scheme, and of the 4-loop contributions to the on shell QED $\beta$-function. These are used to evaluate various 4-loop and 5-loop contributions to the muon anomaly. Our analytical contributions to $(g - 2)_\mu$ differ significantly from previous numerical results. A very recent numerical re-evaluation of 4-loop muon-anomaly contributions has yielded results much closer to ours.

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1. There are several reasons for the increase of interest in analytical multi-loop renormalization group calculations, observed during the past decade. Firstly, the widespread use of the computer symbolic manipulation systems SCHOONSCHIP [1], REDUCE [2], and FORM [3] stimulates analytical calculations in high-energy physics. Secondly, the appearance of elegant methods [4] for the evaluation of massless three-loop diagrams within dimensional regularization, and their generalization to the massive case [5], have made higher-order calculations possible. Thirdly, the detailed study of theoretically interesting and experimentally important QED and QCD predictions entails the consideration of the effects of high-order radiative corrections.

Consequently, several important results have been obtained in gauge theories, for example: the analytical 4-loop corrections to the QED $\beta$-function in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, and in the momentum (MOM) scheme (namely, the $\Psi$-function) [3]; the 4-loop corrections to $\sigma_{\text{tot}} (e^+e^- \rightarrow \text{hadrons})$ in QCD [7]; the 3-loop QED corrections to the photon vacuum polarization function in the $\overline{\text{MS}}$ scheme [8]; the 2-loop corrections to the relation between the on shell (OS) and $\overline{\text{MS}}$ scheme quark masses in QCD [9]; the 2-loop contributions to OS renormalization constants in QCD and QED [10, 11]; the finite parts of 3-loop OS charge renormalization [5].

These results have been obtained with different computer programs, written in different symbolic manipulation languages. Specifically, the results of Refs. [6]–[8] have been obtained with the help of the SCHOONSCHIP program MINCER [12], the calculations of Refs. [5, 9, 10] were made with the help of several REDUCE packages, whilst the recent analysis of Ref. [11] was carried out with the help of the FORM program SHELL2 [13]. Therefore, in view of the importance of the results obtained, it is necessary not only to study carefully the final results, but to carry out independent cross-checks of the systems and the computer programs involved in the above-mentioned investigations.

In this work we combine various methods of computer symbolic manipulation [1–3], and of evaluating multiloop Feynman integrals [4, 5], to calculate analytically the 3-loop QED expression for the photon vacuum polarization function at large $q^2$ in the OS scheme, the 4-loop corrections to the QED $\beta$-function in the OS scheme, and certain 4-loop and 5-loop contributions to the muon anomalous magnetic moment $a_\mu = (g/2 - 1)_\mu$. A comparison of our analytical results with recent numerical calculations of $a_\mu$ [14] is presented.

2. We will start by deriving the basic renormalization group relations between QED $\beta$-functions and photon vacuum polarization functions in different renormalization schemes. This problem has been previously analysed in Refs. [3, 15] at the 3-loop level, and in Refs. [8, 10] at the 4-loop level. Here we will apply the methods of Ref. [3] to the 4-loop level.

Let us introduce some basic definitions. Firstly, we define the QED $\beta$-function in the $\overline{\text{MS}}$ scheme as

$$\frac{\partial \ln \overline{\pi}}{\partial \ln \mu} \bigg|_{\alpha_B \text{-fixed} \atop \epsilon \to 0} = \beta(\overline{\pi}) = \sum_{n=1}^\infty \tilde{\beta}_n \left(\frac{\overline{\pi}}{\pi}\right)^n,$$

where the renormalized $\overline{\text{MS}}$ scheme coupling constant $\overline{\pi}$ is related to the bare one $\alpha_B$ by $\overline{\pi}(\mu) = \overline{Z}_3(\overline{\pi})\alpha_B$, $\mu$ is the mass unit introduced within dimensional regularization in $D = 4 - 2\epsilon$ space–time dimensions, and $\overline{Z}_3(\overline{\pi})$ is the photon wavefunction renormalization constant in the $\overline{\text{MS}}$ scheme. The similar expression for the $\beta$-function in the OS scheme
reads
\[
\frac{\partial \ln \alpha}{\partial \ln m} = \beta(\alpha) = \sum_{n=1}^{\infty} \beta_n \left( \frac{\alpha}{\pi} \right)^n,
\]
where \( \alpha = Z_3 \alpha_B \) is the OS scheme coupling constant (the physical fine structure constant), \( m \) is the electron pole mass, and \( Z_3 \) is the photon wavefunction renormalization constant in the OS scheme\(^1\).

Using Eqs. (1) and (2), one can obtain the following relation:
\[
\frac{\beta(\alpha)}{\beta(\alpha)}_{\mu=m} = \frac{\partial \ln \pi}{\partial \ln \alpha}_{\mu=m} = 1 + \frac{\partial}{\partial \ln \alpha} \left( \frac{Z_3(\mu=m)}{Z_3} \right).
\]

It can be shown that the ratio \( \frac{\alpha(\mu=m)}{\alpha} = \frac{Z_3(\mu=m)}{Z_3} \) has a vanishing \( O(\alpha) \) contribution\(^5\), i.e.
\[
\frac{\alpha(\mu=m)}{\alpha} = 1 + g_2 \left( \frac{\alpha}{\pi} \right)^2 + g_3 \left( \frac{\alpha}{\pi} \right)^3 + O(\alpha^4).
\]

Let us now, following the lines of Refs.\(^{16, 17}\), expand the coefficients \( \beta_n, \overline{\beta}_n, \) and \( g_n \), in powers that correspond to the numbers of electron loops of the corresponding diagrams contributing to the photon vacuum polarization function. Thus we have
\[
\begin{align*}
\beta_1 &= \beta_1[1] N, \quad \overline{\beta}_1 = \overline{\beta}_1[1] N \\
\beta_2 &= \beta_2[1] N, \quad \overline{\beta}_2 = \overline{\beta}_2[1] N \\
\beta_3 &= \beta_3[1] N + \beta_3[2] N^2, \quad \overline{\beta}_3 = \overline{\beta}_3[1] N + \overline{\beta}_3[2] N^2 \\
\beta_4 &= \beta_4[1] N + \beta_4[2] N^2 + \beta_4[3] N^3 \\
\overline{\beta}_4 &= \overline{\beta}_4[1] N + \overline{\beta}_4[2] N^2 + \overline{\beta}_4[3] N^3 \\
g_2 &= g_2[1] N, \quad g_3 = g_3[1] N + g_3[2] N^2,
\end{align*}
\]

where \( N \) is (for formal purposes) the number of types of lepton.

Substituting Eqs. (5) into Eq. (4) and Eq. (3), we get
\[
\begin{align*}
\beta_1 &= \beta_1[1] N, \quad \overline{\beta}_1 = \overline{\beta}_1[1] N \\
\beta_2 &= \beta_2[1] N, \quad \overline{\beta}_2 = \overline{\beta}_2[1] N \\
\beta_3 &= \beta_3[1] N + \beta_3[2] N^2, \quad \overline{\beta}_3 = \overline{\beta}_3[1] N + \overline{\beta}_3[2] N^2 \\
\beta_4 &= \beta_4[1] N + \beta_4[2] N^2 + \beta_4[3] N^3 \\
\overline{\beta}_4 &= \overline{\beta}_4[1] N + \overline{\beta}_4[2] N^2 + \overline{\beta}_4[3] N^3 \\
g_2 &= g_2[1] N, \quad g_3 = g_3[1] N + g_3[2] N^2,
\end{align*}
\]

Using the concept of the QED invariant charge
\[
\alpha_{\text{inv}}(x, \alpha) = \frac{\alpha}{1 + (\alpha/\pi) \Pi(x, \alpha)} = \frac{\bar{\alpha}}{1 + (\bar{\alpha}/\pi) \bar{\Pi}(x, \bar{\alpha})},
\]
with \( \Pi(x, \alpha) = \sum_{n=1}^{\infty} \Pi_n(x) (\alpha/\pi)^{n-1}, \quad \Pi(x, \bar{\alpha}) = \sum_{n=1}^{\infty} \Pi_n(x) (\bar{\alpha}/\pi)^{n-1}, \quad x = -q^2/m^2, \)

one can find similar relations between the 3-loop expressions for the photon vacuum polarization function.

\(^1\)The \( \beta \)-functions and coefficients of Eqs. (1) and (2) are normalized as in Refs.\(^{[1, 22, 27]}\), not as in Refs.\(^{[3, 8, 19]}\).
polarization function in the OS scheme, namely \( \Pi(x, \alpha) \), and that in the \( \overline{\text{MS}} \) scheme, renormalized at \( \mu = m \), namely \( \Pi(x, \mu) \). Substituting the large-\( x \) expansions

\[
\begin{align*}
\Pi_1 &= (a_1 + b_1 \ln x) , \\
\Pi_2 &= (a_2 + b_2 \ln x) , \\
\Pi_3 &= (a_3 + b_3 \ln x + c_3 \ln^2 x) \\
\Pi_3 &= (\overline{a}_3 + \overline{b}_3 \ln x + \overline{c}_3 \ln^2 x)
\end{align*}
\]

in Eq. (7), and taking into account Eq. (4), we get

\[
\begin{align*}
\Pi_1 &= \Pi_1 \\
\Pi_2 &= \Pi_2 - g_2 \\
\Pi_3 &= \Pi_3 - g_3 .
\end{align*}
\]

This means in turn that

\[
\begin{align*}
a_1 &= \overline{a}_1 , \quad b_1 = \overline{b}_1 \\
b_2 &= \overline{b}_2 , \quad b_3 = \overline{b}_3 \\
b_3 &= \overline{b}_3 , \quad c_3 = \overline{c}_3 .
\end{align*}
\]

These identities are supported by the results of explicit calculations. (Compare the OS scheme results, summarized in Ref. [16], with those of \( \overline{\text{MS}} \), obtained in Ref. [8].) For the 2-loop and 3-loop contributions to the constant terms of \( \Pi(x, \alpha) \) and \( \overline{\Pi}(x, \mu) \) we have

\[
\begin{align*}
a_2 &= \overline{a}_2 - g_2 \\
a_3 &= \overline{a}_3 - g_3 \\
a_3 &= \overline{a}_3 - g_3 .
\end{align*}
\]

In Eqs. (10) and (11) we used notations for the coefficients of the photon vacuum polarization function similar to those defined in Eq. (5). Substituting Eqs. (11) into Eqs. (6), we get the following relations:

\[
\begin{align*}
\beta_3^{[2]} &= \beta_3^{[2]} - \beta_1^{[1]} \left( \overline{a}_3^{[1]} - a_3^{[1]} \right) \\
\beta_4^{[2]} &= \beta_4^{[2]} - 2\beta_1^{[1]} \left( \overline{a}_3^{[1]} - a_3^{[1]} \right) \\
\beta_4^{[3]} &= \beta_4^{[3]} - 2\beta_1^{[1]} \left( \overline{a}_3^{[2]} - a_3^{[2]} \right) ,
\end{align*}
\]

which can be also obtained from the analysis of Ref. [16].

3. Let us now consider the results of our calculations. The analytical expressions for the 2-loop and 3-loop coefficients of the photon vacuum polarization function in the \( \overline{\text{MS}} \) scheme, \( \overline{\Pi}(x, \mu) \), were presented in Ref. [8]. We are interested in the following gauge-invariant results [8]:

\[
\begin{align*}
\overline{a}_2^{[1]} &= \frac{55}{48} - \zeta(3) \\
\overline{a}_3^{[1]} &= -\frac{143}{288} - \frac{37}{24} \zeta(3) + \frac{5}{2} \zeta(5) \\
\overline{a}_3^{[2]} &= -\frac{3701}{2592} + \frac{19}{18} \zeta(3) .
\end{align*}
\]

(13)
They were obtained in the Feynman gauge in the process of carrying out the work described in Ref. [18] with the help of the SCHOONSCHIP program MINCER [12]. In the course of our work these results were confirmed by independent calculations made in an arbitrary covariant gauge with the help of the REDUCE program SLICER [19], especially written for this purpose.

The coefficients \( g_2^{[1]} \), \( g_3^{[1]} \), and \( g_3^{[2]} \), were calculated in Ref. [5] from 2-loop and 3-loop massive propagator-like integrals with zero external momentum, with the help of the REDUCE program RECURSOR. They read

\[
\begin{align*}
g_2^{[1]} &= \frac{15}{16}, \\
g_3^{[1]} &= \frac{77}{576} + \frac{5}{4} \zeta(2) - 2 \zeta(2) \ln 2 + \frac{1}{192} \zeta(3) \\
g_3^{[2]} &= -\frac{695}{648} + \frac{2}{3} \zeta(2) + \frac{7}{64} \zeta(3).
\end{align*}
\]

These numbers were also confirmed in an arbitrary covariant gauge, by means of the specially written FORM program SHELL3 [20].

Using the results of Eqs. (13) and (14), we get the following expressions for the constant contributions to the photon vacuum polarization function in the OS scheme:

\[
\begin{align*}
a_2^{[1]} &= \frac{5}{24} - \zeta(3) \\
a_3^{[1]} &= -\frac{121}{192} - \frac{5}{4} \zeta(2) + 2 \zeta(2) \ln 2 - \frac{99}{64} \zeta(3) + \frac{5}{2} \zeta(5) \\
a_3^{[2]} &= -\frac{307}{864} - \frac{2}{3} \zeta(2) + \frac{545}{576} \zeta(3).
\end{align*}
\]

The result for \( a_2^{[1]} \) is in agreement with that known for many years [21]. The expression for \( a_3^{[2]} \) coincides with that recently obtained in Ref. [22], which is known to be consistent with numerical calculations [14] of 4-loop corrections to the muon anomalous magnetic moment \( a_\mu \) [17, 22].

The result for \( a_3^{[1]} \) is new. It is obtained by combining results from Refs. [8] that have been exhaustively checked by our two new programs [19, 20]. Using SLICER [19], we calculated the bare photon self-energy, \( \Pi_B \), to three loops at large \( -q^2 \). Then we used SHELL3 [20] to calculate \( \Pi_B \) to three loops at \( q^2 = 0 \). The constant \( Z_3 \) that gives the OS renormalized result \( \Pi(0, \alpha) = 0 \) also gave the asymptotic OS constants of Eq. (15) directly, without any reference to the \( \overline{\text{MS}} \) scheme.

Note the presence in \( a_3^{[1]} \) of \( \zeta(2) \) and \( \zeta(2) \ln 2 \) terms, which are typical of calculations in the OS scheme. The origin of the \( \zeta(2) \ln 2 \) term is well understood: it comes from a scalar massive integral that also contributes to the 2-loop correction to the fermion anomalous magnetic moment \( \xi \) [3, 4]. Notice also that the \( \zeta(5) \) contribution to the 3-loop expression of \( \Pi(x, \alpha) \), like the 2-loop \( \zeta(3) \) term, is independent of the choice of scheme. (Compare Eq. (15) with Eq. (13).) This fact can be understood by using the methods of Ref. [4].

It should be stressed that our numerical value of \( a_3^{[1]} \), namely

\[
a_3^{[1]} = 0.3268745 \ldots \quad (\text{this work}),
\]

strongly differs from the numerical result

\[
a_3^{[1]} = 1.356(41) + O\left(\frac{m_e}{m_\mu}\right) \quad (\text{Ref. [14]})
\]
obtained from the analysis of the results of numerical calculation of the 4-loop corrections to \( a_{\mu} \) [14] by the methods of Ref. [23], taking into account the correct analytical expression for the \( a_3^{[3]} \) term.

4. By using our new analytical expression for the \( a_3^{[1]} \) term, we can also obtain the analytical expression for the set of 4-loop diagrams contributing to \( a_{\mu} \) formed by inserting the sum of 15 one-particle irreducible sixth order single-electron-loop vacuum-polarization diagrams in a second-order muon vertex. (See Fig. 1.) Indeed, using the methods of Ref. [23], which agree with those of Ref. [24], we get

\[
\begin{align*}
a_{\mu} (\text{Fig. 1}) &= \left( -a_3^{[1]} I_0 - b_3^{[1]} I_1 - 2b_3^{[1]} I_0 \ln\left( \frac{m}{m_{\mu}} \right) + O\left( \frac{m_e}{m_{\mu}} \right) \right) \left( \frac{\alpha}{\pi} \right)^4, \\
&= \left[ -0.290987 + O\left( \frac{m_e}{m_{\mu}} \right) \right] \left( \frac{\alpha}{\pi} \right)^4 \quad \text{(this work)} \quad (19)
\end{align*}
\]

whilst the numerical calculations of Ref. [14] gave

\[
a_{\mu} (\text{Fig. 1}) = -0.7945(202) \left( \frac{\alpha}{\pi} \right)^4 \quad \text{(Ref. [14]).} \quad (20)
\]

The discrepancy between Eqs. (19) and (20) is much greater than any revealed by the analytical calculations in Ref. [20] of some 8th order contributions to the electron anomaly.

The discrepancy between Eqs. (16) and (17) also leads to a substantial change to the 10th order contributions to \( a_{\mu} \) of the diagrams of Fig. 2. Indeed, using the methods of Ref. [23], one can get the following expression for them [14]:

\[
\begin{align*}
a_{\mu} (\text{Fig. 2}) &= \left[ \frac{25}{18} a_3^{[1]} + \frac{1}{12} - \frac{1}{24} \zeta(2) + \frac{5}{16} - \frac{2}{3} a_3^{[1]} \ln\left( \frac{m}{m_{\mu}} \right) \\
&\quad - \frac{1}{24} \ln^2\left( \frac{m}{m_e} \right) + O\left( \frac{m_e}{m_{\mu}} \right) \right] \left( \frac{\alpha}{\pi} \right)^5. \quad (21)
\end{align*}
\]

Taking the value of the \( a_3^{[1]} \) term from Eq. (15), we are now able to obtain the following totally analytical expression for Eq. (21):

\[
\begin{align*}
a_{\mu} (\text{Fig. 2}) &= \left[ -\frac{3409}{3456} - \frac{16}{9} \zeta(2) + \frac{25}{9} \zeta(2) \ln 2 - \frac{275}{128} \zeta(3) + \frac{125}{36} \zeta(5) \\
&\quad + \left( \frac{161}{288} + \frac{5}{6} \zeta(2) - \frac{4}{3} \zeta(2) \ln 2 + \frac{33}{32} \zeta(3) - \frac{5}{3} \zeta(5) \right) \ln\left( \frac{m}{m_e} \right) \\
&\quad - \frac{1}{24} \ln^2\left( \frac{m}{m_e} \right) + O\left( \frac{m_e}{m_{\mu}} \right) \right] \left( \frac{\alpha}{\pi} \right)^5. \quad (22)
\end{align*}
\]

\(^2\)The analysis of Ref. [14] gives \( a_3 = a_3^{[1]} + a_3^{[2]} = 1.041(41) + O(m_e/m_{\mu}).\)
The corresponding numerical result reads

\[
a_\mu (\text{Fig. 2}) = \left[ -1.3314 + O \left( \frac{m_e}{m_\mu} \right) \right] \left( \frac{\alpha}{\pi} \right)^5 \text{ (this work)}.
\]

(23)

It should be compared with the similar expression obtained in Ref. [16] from Eq. (21), after taking into account the ‘old’ \(a_3^{[1]}\) value from Eq. (17), namely

\[
a_\mu (\text{Fig. 2}) = \left[ -3.560(89) + O \left( \frac{m_e}{m_\mu} \right) \right] \left( \frac{\alpha}{\pi} \right)^5 \text{ (Ref. [16])}.
\]

(24)

Therefore, one way of resolving the discrepancy between Eqs. (16) and (17) might be an accurate numerical calculation of the diagrams of Fig. 2.

5. Our next step is the determination of the 4-loop correction to the QED \(\beta\)-function in the OS scheme, in analytical form. We first remark that the results of the analytical calculations of the 4-loop corrections to the QED \(\beta\)-function in the \(\overline{\text{MS}}\) scheme, namely to the \(\overline{\beta}\)-function defined by Eq. (1), are [9]

\[
\overline{\beta}_1 = \frac{2}{3} N, \quad \overline{\beta}_2 = \frac{1}{2} N
\]

\[
\overline{\beta}_3 = -\frac{1}{16} N - \frac{11}{72} N^2
\]

\[
\overline{\beta}_4 = -\frac{23}{64} N + \left( \frac{95}{432} - \frac{13}{18} \zeta(3) \right) N^2 - \frac{77}{1944} N^3.
\]

(25)

Using now Eqs. (5) and (6), and the results of Eq. (14), we get

\[
\beta_3 = -\frac{1}{16} N - \frac{7}{9} N^2.
\]

(26)

At \(N = 1\), Eq. (26) coincides with the result \(\beta_3 = -121/144\) obtained in Ref. [27].

The 4-loop coefficient \(\beta_4\) has the following form:

\[
\beta_4 = -\frac{23}{64} N + \left( \frac{1}{24} - \frac{5}{3} \zeta(2) + \frac{8}{3} \zeta(2) \ln 2 - \frac{35}{48} \zeta(3) \right) N^2
\]

\[
+ \left( \frac{901}{648} - \frac{8}{9} \zeta(2) - \frac{7}{48} \zeta(3) \right) N^3,
\]

(27)

where the powers of \(N\) serve merely to distinguish contributions with corresponding numbers of electron loops. (With leptons of different mass, the finite parts of OS charge renormalization involve dilogarithms [4], and the OS expansions of Eq. (8) involve logarithms [17] of mass ratios; both are absent from the \(\overline{\text{MS}}\) scheme.) Setting \(N = 1\), we obtain

\[
\beta_4 = \frac{5561}{5184} - \frac{23}{9} \zeta(2) + \frac{8}{3} \zeta(2) \ln 2 - \frac{7}{8} \zeta(3),
\]

(28)

which may also be obtained from the \(m\)-independence of \(\Lambda_{\overline{\text{MS}}QED}^\beta\), as given in Ref. [4]. The order \(N\) term in Eq. (27) is scheme independent. The \(N^3\) contribution coincides with the result of Ref. [22]. The \(N^2\) contribution is new. In view of the results described in Sections 3 and 4 it differs from the one obtained previously [3] taking into account the ‘old’ numerical value of the \(a_3^{[1]}\) term. (Compare Eq. (16) with Eq. (17).)
Let us now compare the numerical behaviour of the 4-loop approximations of the \( \beta \)-functions in the \( \overline{\text{MS}} \) and OS schemes with the corresponding function in the MOM scheme, namely with the \( \Psi \)-function, which we here normalize in correspondence with Eqs. (1) and (2), as follows:

\[
\Psi(\alpha_{\text{MOM}}) = \frac{\partial \ln \alpha_{\text{MOM}}}{\partial \ln \alpha} \beta(\alpha) = \frac{\partial \ln \alpha_{\text{MOM}}}{\partial \ln \alpha} \beta(\alpha),
\]

where \( \alpha_{\text{MOM}} \) is the invariant charge of Eq. (7), at fixed \( q^2 = -\lambda^2 \). The expressions for the functions \( \beta(\alpha) \) and \( \Psi(\alpha) \) can be obtained from the results of Ref. [6]:

\[
\beta(\alpha) = 0.667 \left( \frac{\alpha}{\pi} \right) + 0.5 \left( \frac{\alpha}{\pi} \right)^2 - 0.215 \left( \frac{\alpha}{\pi} \right)^3 - 1.047 \left( \frac{\alpha}{\pi} \right)^4 \ldots
\]

(30)

\[
\Psi(\alpha) = 0.667 \left( \frac{\alpha}{\pi} \right) + 0.5 \left( \frac{\alpha}{\pi} \right)^2 + 0.100 \left( \frac{\alpha}{\pi} \right)^3 - 1.202 \left( \frac{\alpha}{\pi} \right)^4 \ldots
\]

(31)

In the OS scheme the analogous series reads

\[
\beta(\alpha) = 0.667 \left( \frac{\alpha}{\pi} \right) + 0.5 \left( \frac{\alpha}{\pi} \right)^2 - 0.840 \left( \frac{\alpha}{\pi} \right)^3 - 1.142 \left( \frac{\alpha}{\pi} \right)^4 \ldots
\]

(32)

Notice that now, in all three cases, the 4-loop coefficients are negative and have the same order of magnitude. This is a welcome result, in view of the natural expectation that the physical conclusions should be less scheme-dependent after taking into account higher-order perturbative corrections.

Finally, we use the new numerical value of the 4-loop coefficient of the OS \( \beta \)-function of Eq. (32), to obtain a slight modification of the estimate presented in Ref. [16] for the renormalization-group constrained [23] 10th order contributions to \( a_\mu \). Our result reads:

\[
a^{(10)}_{\mu}(\text{R.G.}) = [B_5 + 82.92(78)] \left( \frac{\alpha}{\pi} \right)^5 \quad \text{(this work)},
\]

(33)

where \( B_5 \) is the unknown 5-loop mass-independent contribution (expected to be positive) and the uncertainty in the numerical value of the renormalization-group determined mass-dependent term derives largely from the uncertainty in \( B_4 = -2.503(55) \) [14]. Our result differs little from that of Ref. [16], namely \( a^{(10)}_{\mu}(\text{R.G.}) = [B_5 + 86.57(78)](\alpha/\pi)^5 \), obtained using Eq. (17).

6. In conclusion: the 4-loop and 5-loop muon-anomaly contributions of Eqs. (19), (23) and (33) follow from our new results for the 3-loop OS vacuum-polarization coefficient \( a_3^{[1]} \) of Eq. (15) and the 4-loop OS \( \beta \)-function coefficient of Eq. (27), each given in analytical form, for the first time. In the course of deriving \( a_3^{[1]} = 0.3268745 \ldots \) we have devised two entirely new 3-loop programs, SLICER [14] and SHELL3 [20], which substantially validate the programs MINCER [12] and RECURSOR [5], respectively, and hence confirm the large discrepancy between our result and the numerical estimate \( a_3^{[1]} \approx 1.356 \) that follows from the 8th order muon-anomaly contribution of Eq. (20), obtained by Kinoshita et al. [14]. In this connection, we remark that there exists an impressive body of work [30] to support our belief that the methods of dimensional regularization, used here and in Refs. [5, 8, 12], yield final results free of any infra-red pathology. Our value for \( a_3^{[1]} \) does not affect the result of Ref. [10] for the on-shell constant \( b_4 \), whose \( N = 1 \) value has recently been verified [28]. A formula involving \( a_3 \) has been given for \( b_5 \) [29]. This is in error by the omission of a term involving the 5-loop coefficient in the expansion of Eq. (31), whose value is unknown.
Note added: After completing this work, we communicated our result of Eq. (19) to Professor Kinoshita, who undertook a re-evaluation of the diagrams of Fig. 1, obtaining a numerical result substantially different from that given in Eq. (20) and considerably closer to ours.

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