On energy functionals, Kähler-Einstein metrics, and the Moser-Trudinger-Onofri neighborhood

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Abstract. We prove that the existence of a Kähler-Einstein metric on a Fano manifold is equivalent to the properness of the energy functionals defined by Bando, Chen, Ding, Mabuchi and Tian on the set of Kähler metrics with positive Ricci curvature. We also prove that these energy functionals are bounded from below on this set if and only if one of them is. This answers two questions raised by X.-X. Chen. As an application, we obtain a new proof of the classical Moser-Trudinger-Onofri inequality on the two-sphere, as well as describe a canonical enlargement of the space of Kähler potentials on which this inequality holds on higher-dimensional Fano Kähler-Einstein manifolds.

Keywords: Energy functionals; Kähler-Einstein manifolds; Moser-Trudinger-Onofri inequality

1 Introduction. Our main purpose in this article is to give a new analytic characterization of Kähler-Einstein manifolds in terms of certain functionals defined on the infinite-dimensional space of Kähler forms. As a corollary of our approach we also obtain a new proof of the classical Moser-Trudinger-Onofri inequality on the two-sphere as well as an optimal extension of it to higher-dimensional Fano Kähler-Einstein manifolds.

A necessary condition for a manifold to admit a Kähler-Einstein metric is that its first Chern class be either positive, negative or zero. Aubin and Yau proved that this condition is also sufficient in the second case and Yau proved that the same is true also in the third case.

Yet additional geometric assumptions are necessary in the first case (in this case the manifold is called Fano): Matsushima proved that the group of automorphisms must be reductive, Futaki proved that a certain character on the algebra of holomorphic vector fields must be trivial, and Kobayashi and Lübke proved that the tangent

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bundle must be stable. Since then much work has been done on the subject (see for
eexample the recent expositions [Bi,F2,Sz,Th]).

In this article we will restrict attention to two closely related analytic criteria
relating the existence of Kähler-Einstein metrics to properties of certain energy func-
tionals (see the end of this section and Section 2 for notation and definitions) on the
space of Kähler forms $\mathcal{H}_c$. The first, introduced by Tian, can be thought of as a
“stability” criterion [T3]. It expresses the existence of a Kähler-Einstein metric as
equivalent to the properness of an energy functional:

**Theorem 1.1.** [T3,T4,TZ] Let $(M, J)$ be a Fano manifold and assume that
$\text{Aut}(M, J)$ is finite. Then the following are equivalent: (i) $(M, J)$ admits a Kähler-
Einstein metric, (ii) $E_0$ is proper on $\mathcal{H}^+_c$, (iii) $F$ is proper on $\mathcal{H}^+_c$.

The finiteness assumption
\(^1\) covers, for example, all Kähler-Einstein Fano surfaces except the product of two Riemann spheres, the projective plane $\mathbb{P}^2$, and $\mathbb{P}^2$ blown up at 3 non-collinear points [S1,T2,TY]. However, there is a slightly more technically involved version of Theorem 1.1, also due to Tian, which applies to all Kähler-Einstein Fano manifolds, that will be stated in Section 4 (Theorem 4.1).

The second analytic criterion, introduced by Bando and Mabuchi, can be thought
of as a “semi-stability” condition [BM]. Two related formulations appeared subse-
quently [Ba,DT]. It expresses the existence of “almost” Kähler-Einstein metrics as a
consequence of the lower boundedness of an energy functional:

**Theorem 1.2.** [BM,DT] Let $(M, J)$ be a Fano manifold. Assume that either $F$ or
$E_0$ is bounded from below on $\mathcal{H}^+_c$ and let $\epsilon > 0$. Then $(M, J)$ admits a Kähler metric
$\omega_\epsilon \in \mathcal{H}_c$ satisfying $\text{Ric} \omega_\epsilon > (1 - \epsilon) \omega_\epsilon$.

It is worth mentioning that a precise characterization of Fano manifolds for which
these functionals are bounded from below is still lacking. Also, examples of such
manifolds which are not Kähler-Einstein are yet to be given.

We point out that Theorem 1.1 and the version of Theorem 1.2 for the functional $F$
were originally stated with the assumptions on properness and boundedness made
on the whole space of Kähler forms $\mathcal{H}_c$ rather than on the subspace of forms of
positive Ricci curvature $\mathcal{H}^+_c$. However, the respective existence proofs only make use
of those assumptions on $\mathcal{H}^+_c$. Thus Theorem 1.1 implies that the properness of the
functionals on $\mathcal{H}^+_c$ implies their properness on $\mathcal{H}_c$. In addition, in Remark 4.4 we
prove that for any Fano manifold also the lower boundedness of the functionals on
$\mathcal{H}^+_c$ implies their lower boundedness on $\mathcal{H}_c$. Therefore it seems more natural to state

\(^1\) Since automorphisms of the complex structure preserve the first Chern class this assumption is
equivalent to the triviality of $\text{aut}(M, J)$ [Fu, Theorem 4.8].
Energy functionals and Kähler-Einstein metrics

Theorems 1.1 and 1.2 in the equivalent manner above. This will also be justified by the results of Section 5 (in particular Corollary 5.5).

Chen and Tian constructed a family of energy functionals $E_1, \ldots, E_n$, analogues of the ‘K-energy’ (‘Kähler energy’) $E_0$ corresponding to higher degree elementary symmetric polynomial expressions of the eigenvalues of the Ricci tensor $[CT]$. As with $E_0$ and $F$, Kähler-Einstein metrics are critical points of these functionals and it is therefore a natural idea to seek to extend Theorems 1.1 and 1.2 to $k = 1, \ldots, n$.

In this direction, an analogue of Theorem 1.1 for $k = 1$ was proved recently by Song and Weinkove $[SW]$. The main purpose of the present article is to prove the following two statements:

**Theorem 1.3.** Let $(M, J)$ be a Fano manifold and assume that $\text{Aut}(M, J)$ is finite. Let $k \in \{0, \ldots, n\}$. Then the following are equivalent: (i) $(M, J)$ admits a Kähler-Einstein metric, (ii) $E_k$ is proper on $\mathcal{H}^+_{c_1}$, (iii) $F$ is proper on $\mathcal{H}^+_{c_1}$.

**Theorem 1.4.** Let $(M, J)$ be a Fano manifold and let $k \in \{0, \ldots, n\}$. Assume that either $F$ or $E_k$ is bounded from below on $\mathcal{H}^+_{c_1}$ and let $\epsilon > 0$. Then $(M, J)$ admits a Kähler metric $\omega_\epsilon \in \mathcal{H}_{c_1}$ satisfying $\text{Ric} \omega_\epsilon > (1 - \epsilon) \omega_\epsilon$.

Our proofs carry over to Kähler-Einstein manifolds admitting holomorphic vector fields (for the more general statements the reader is referred to Sections 3 and 4). We remark that while Theorem 1.3 generalizes the work of Song and Weinkove, our methods provide a considerable simplification over the ones used there.

These theorems show that the functionals $E_k$ are, on the one hand, closely related to geometric stability, and, on the other hand, all equivalent in a suitable sense.$^2$

To prove these theorems we first observe that a certain formula of Bando and Mabuchi for the ‘Ricci energy’ $E_n$ extends naturally to all of the functionals $E_k$. The merit of this new formula (Proposition 2.6) is that it succinctly captures the relation between the different functionals. This shows in particular that the lower boundedness of $E_k$ implies the lower boundedness of $E_{k+1}$. We then interpret another observation of Bando and Mabuchi in order to close the loop and prove that the lower boundedness of $E_n$ on $\mathcal{H}^+_{c_1}$ implies that of $F$ on $\mathcal{H}_{c_1}$. This step is crucial in proving Theorem 1.4. In fact it proves more, namely, that the lower boundedness of any one of the functionals implies that of the rest (Corollary 4.2). Special cases of this fact have been observed previously $[CLW, DT, L2, P]$ (see Remark 4.4).

To prove Theorem 1.3 we consider the continuity method path (18) introduced by Aubin $[A2]$. As before, we show that the properness of $E_k$ implies the properness of $E_{k+1}$. Next, assuming $E_n$ is proper and using Theorem 1.4 we conclude that this path exists for all $t \in [0, 1)$. We show that on a fixed interval $[t_0, 1)$ each of the

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$^2$ In particular, after posting the first version of this article I became aware of the fact that Theorems 1.3 and 1.4 answer questions posed recently by Chen $[CLW]$. 

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functionals $E_k$ is uniformly bounded from above with $t_0$ depending only on $n$, and then conclude.

In Section 5 we observe that Proposition 2.6 allows to obtain without additional effort a strengthened version of the second main result of Song and Weinkove, the one concerning the nonnegativeness of the energy functionals with respect to a Kähler-Einstein base metric. We observe that our results, when combined with previous ones [BM,DT], provide for a new and entirely Kähler geometric proof of the Moser-Trudinger-Onofri inequality on the Riemann sphere. As a corollary of this approach we also characterize the functions for which this inequality continues to hold on higher-dimensional Kähler-Einstein Fano manifolds, thus extending the work of Ding and Tian. We call the set of all such functions the Moser-Trudinger-Onofri neighborhood of the space of Kähler potentials. It is a canonically defined set that strictly contains the space of Kähler potentials and lies within $C^\infty(M)$. This provides a higher dimensional analogue of the original Moser-Trudinger-Onofri inequality that is optimal in a certain sense and brings Ricci curvature into the picture (Theorem 5.4). Finally, we are able to show that the energy functionals $E_2, E_3, \ldots$ are not bounded from below on $H^{1,1}(\mathcal{K})$ (Corollary 5.5).

The results herein have applications also to the study of the Kähler-Ricci flow and geometric stability [R] that will appear in a subsequent article.

The article is organized as follows. In Section 2 we review the relevant background concerning energy functionals and present the formula for the functionals $E_k$ (Proposition 2.6) whose proof appears in the Appendix. In Section 3 we review results concerning the continuity method approach. The proofs of our main results are contained in Section 4. Section 5 concludes with our results on the lower boundedness of the functionals $E_k$ and on the generalized Moser-Trudinger-Onofri inequality.

**Setup and notation.** Let $(M,J)$ be a connected compact closed Kähler manifold of complex dimension $n$ and let $\Omega \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$ be a Kähler class with $\omega = \partial + \bar{\partial}$. Define the Laplacian $\Delta = -\partial \circ \bar{\partial}^* - \bar{\partial}^* \circ \partial$ with respect to a Riemannian metric $g$ on $M$ and assume that $J$ is compatible with $g$ and parallel with respect to its Levi-Civita connection. Let $g_{\text{Herm}} = 1/\pi \cdot g_{ij}(z) dz^i \otimes d\bar{z}^j$ be the associated Kähler metric, that is the induced Hermitian metric on $(T^1,0M,J)$, and let $\omega = \sqrt{-1}/\pi \cdot g_{ij}(z) dz^i \wedge d\bar{z}^j$ denote its corresponding Kähler form, a closed positive $(1,1)$-form on $(M,J)$ such that $g_{\text{Herm}} = 1/2g - \sqrt{-1} \omega$. Similarly denote by $g_\omega$ the Riemannian metric induced from $\omega$ by $g_\omega(\cdot,\cdot) = \omega(\cdot,J\cdot)$. For any Kähler form we let $\text{Ric}(\omega) = -\sqrt{-1}/2\pi \cdot \partial\bar{\partial} \log \det(g_{ij})$ denote the Ricci form of $\omega$. It is well-defined globally and represents the first Chern class $c_1 := c_1(T^1,0M,J) \in H^2(M,\mathbb{Z}) \cap H^{1,1}(M,\mathbb{C})$. One calls $\omega$ Kähler-Einstein if $\text{Ric}(\omega) = a\omega$ for some real $a$.

Denote by $\mathcal{D}_\Omega$ the space of all closed $(1,1)$-forms whose cohomology class is $\Omega$. For a Kähler form $\omega$ with $[\omega] = \Omega$ we will consider the space of strictly $\omega$-plurisubharmonic
functions
\[ \mathcal{H}_\omega = \{ \varphi \in C^\infty(M) : \omega_\varphi := \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}, \]
and the subspace \( \mathcal{H}_\Omega \subseteq D_\Omega \) of Kähler forms cohomologous to \( \Omega \). We denote by \( \mathcal{H}_\Omega^+ \subseteq \mathcal{H}_\Omega \) the subspace of those Kähler forms whose Ricci curvature is positive. Let \( \text{Aut}(M, J) \) denote the complex Lie group of automorphisms (biholomorphisms) of \((M, J)\) and denote by \( \text{aut}(M, J) \) its Lie algebra of infinitesimal automorphisms composed of real vector fields \( X \) satisfying \( \mathcal{L}_X J = 0 \). Let \( G \) be any compact real Lie subgroup of \( \text{Aut}(M, J) \), and let \( \text{Aut}(M, J)_0 \) denote the identity component of \( \text{Aut}(M, J) \). We denote by \( \mathcal{H}_\Omega(G) \subseteq \mathcal{H}_\Omega \) and \( \mathcal{H}_\Omega^+(G) \subseteq \mathcal{H}_\Omega^+ \) the corresponding subspaces of \( G \)-invariant forms.

2 Certain energy functionals on the space of Kähler forms. We call a real-valued function \( A \) defined on a subset \( \text{Dom}(A) \) of \( D_\Omega \times D_\Omega \) an energy functional if it is zero on the diagonal restricted to \( \text{Dom}(A) \). By a Donaldson-type functional, or exact energy functional, we will mean an energy functional that satisfies the cocycle condition \( A(\omega_1, \omega_2) + A(\omega_2, \omega_3) = A(\omega_1, \omega_3) \) with each of the pairs appearing in the formula belonging to \( \text{Dom}(A) \) [Do,M,T4]. We will occasionally refer to both of these simply as functionals and exact functionals, respectively. Note that if an exact functional is defined on \( U \times W \) with \( U \subseteq W \) then there exists a unique exact functional defined on \( W \times W \) extending it.

Let \( V := \int_M \omega^n = [\omega]^n([M]) \). The energy functionals \( I, J \), introduced by Aubin [A2], are defined for each pair \( (\omega, \omega_\varphi) := (\omega + \sqrt{-1} \partial \bar{\partial} \varphi) \in D_\Omega \times D_\Omega \) by

\[
I(\omega, \omega_\varphi) = V^{-1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \sum_{l=0}^{n-1} \omega^{n-1-l} \wedge \omega_\varphi^l = V^{-1} \int_M \varphi(\omega^n - \omega_\varphi^n), \tag{1}
\]

\[
J(\omega, \omega_\varphi) = \frac{V^{-1}}{n+1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \sum_{l=0}^{n-1} (n-l)\omega^{n-l-1} \wedge \omega_\varphi^l. \tag{2}
\]

One may also define them via a variational formula. Connect each pair \( (\omega, \omega_{\varphi_t}) := (\omega + \sqrt{-1} \partial \bar{\partial} \varphi_t) \) with a piecewise smooth path \( \{\omega_{\varphi_t}\}_{t \in [0,1]} \) (we regard this path as a function on \( M \times [0,1] \) and occasionally suppress the subscript \( t \)). Then we have for any such path

\[
(I - J)(\omega, \omega_{\varphi_1}) = -\frac{1}{V} \int_{M \times [0,1]} \varphi \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \omega_{\varphi_t}^{n-1} \wedge dt, \tag{3}
\]

\[
J(\omega, \omega_{\varphi_1}) = \frac{1}{V} \int_{M \times [0,1]} \varphi_t (\omega^n - \omega_{\varphi_t}^n) \wedge dt. \tag{4}
\]

On \( \mathcal{H}_\Omega \times \mathcal{H}_\Omega \) \( I, J \) and \( I - J \) are all nonnegative (and hence non-exact) and equivalent, namely,

\[
\frac{1}{n^2}(I - J) \leq \frac{1}{n(n+1)} I \leq \frac{1}{n} J \leq I - J \leq \frac{n}{n+1} I \leq nJ. \tag{5}
\]
Note that pulling-back both arguments of these functionals by an automorphism of $(M,J)$ does not change their value. It is important to understand the behavior of these functionals also outside the subspace $\mathcal{H}_\Omega$.

**Lemma 2.1.** Let $\omega \in \mathcal{H}_\Omega$. Then $I(\omega, \cdot)$ is unbounded from above on $\mathcal{H}_\Omega$ and, when $n > 1$, unbounded on $\mathcal{D}_\Omega$.

**Proof.** Fix a holomorphic coordinate patch

$$\psi : U \to \mathbb{C}^n, \psi(q) = z(q) := (z^1(q), \ldots, z^n(q)), \forall q \in U \subseteq M.$$ 

Let $a > 0$ be such that $\psi^{-1}(\{v \in \mathbb{C}^n : |v| < 3a\}) \subseteq U$. For the first statement, define $\tilde{\varphi}_b$ by letting $\tilde{\varphi}_b = b|z|^2$ on $\psi^{-1}(\{v \in \mathbb{C}^n : a < |v| < 2a\})$ and constant elsewhere on $U$ in such a way that it is continuous. Approximate $\tilde{\varphi}_b$ by smooth functions $\varphi_{b,m}$ that agree with it outside the set $\psi^{-1}(\{v \in \mathbb{C}^n : |v| \leq (a-\frac{1}{m}, a+\frac{1}{m}) \cup (2a-\frac{1}{m}, 2a+\frac{1}{m})\})$ and that satisfy $|\varphi_b - \varphi_{b,m}| < \frac{1}{m}$ on $U$. Given $a_2 > 0$ there exists $b$ and a corresponding $m$ such that $\varphi_{b,m} \in \mathcal{M}_\omega$ and $I(\omega, \varphi_{b,m}) > a_2$.

For the second statement, construct similarly functions, as above, now setting $\tilde{\varphi}_b = -b(|z_1|^2 + |z_2|^2)$ on $\psi^{-1}(\{v \in \mathbb{C}^n : a < |v| < 2a\})$. Again one may approximate using functions $\varphi_{b,m}$. Expanding $V^{-1} \int_M \sqrt{-1}\partial\varphi \wedge \partial\varphi \wedge \omega^{n-2} \wedge (a_2\omega + a_3\sqrt{-1}\partial\varphi_{b,m})$ for some $a_2, a_3 > 0$. We then see that given any $a_4 > 0$ there exists $b$ and a corresponding $m$ such that $I(\omega, \varphi_{b,m}) < -a_4$. □

We say that an exact functional $A$ is bounded from below on $U \subseteq \mathcal{H}_\Omega$ if for every $\omega$ such that $(\omega, \omega_\varphi) \in \text{Dom}(A)$ and $\omega_\varphi \in U$ holds $A(\omega, \omega_\varphi) \geq C_\omega$ with $C_\omega$ independent of $\omega_\varphi$. We say it is proper (in the sense of Tian) on a set $U \subseteq \mathcal{H}_\Omega(G)$ if for each $\omega \in \mathcal{H}_\Omega(G)$ there exists a smooth function $\tau_\omega : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{s \to \infty} \tau_\omega(s) = \infty$ such that $A(\omega, \omega_\varphi) \geq \tau_\omega((I-J)(\omega, \omega_\varphi))$ for every $\omega_\varphi \in U$. This is well-defined, in other words depends only on $|\omega|$ since the failure of $I-J$ to satisfy the cocycle condition is under control with respect to the two base metrics, $\omega, \omega_{\varphi_1}$ say, to wit,

$$(I-J)(\omega, \omega_{\varphi_2}) - (I-J)(\omega_{\varphi_1}, \omega_{\varphi_2}) = (I-J)(\omega, \omega_{\varphi_1}) - \frac{1}{V} \int_M \varphi_1(\omega_{\varphi_2} - \omega_{\varphi_1}^n),$$

with the last term controlled by the oscillation of $\varphi_1$. Properness of a functional implies it has a lower bound.

Define the following collection of energy functionals for each $k \in \{0, \ldots, n\}$

$$I_k(\omega, \omega_\varphi) = \frac{1}{V} \int_M \sqrt{-1}\partial\varphi \wedge \partial\varphi \wedge \sum_{l=0}^{k-1} \frac{k-l}{k+1} \omega^{n-l} \wedge \omega_\varphi^l,$$

$$= \frac{V^{-1}}{k+1} \int_M \varphi(k\omega^n - \sum_{l=1}^{k} \omega^{n-l} \wedge \omega_\varphi^l). \quad (6)$$
Note that $I_n = J$, $I_{n-1} = ((n+1)J - I)/n$.

Chen and Tian [CT] defined another such family

$$J_k(\omega,\omega_\phi) = V^{-1} \int_M \hat{\partial}(\omega^k_\phi \wedge \omega^{n-k} - \omega^k_\phi) \wedge dt, \quad k = 0, \ldots, n. \quad (7)$$

(Note that $J_{n-k-1}/(k+1)$ in their article corresponds to $J_k$ in this article.) The following computation relates these two families of functionals.

**Lemma 2.2.** The following relation holds on $\mathcal{H}_\Omega \times \mathcal{H}_\Omega$:

$$I_k(\omega,\omega_\phi) = J(\omega,\omega_\phi) - J_k(\omega,\omega_\phi).$$

**Proof.** Given a path $\{\omega_\phi\}_{t \in [0,1]}$ we compute the variational equation for $I_k$.

$$(k+1) \frac{d}{dt} I_k(\omega,\omega_\phi) = -\frac{1}{V} \int_M \sum_{l=0}^{k-1} \left( 2\hat{\partial}(\omega^k_\phi \wedge (k-l)\omega^{n-1-l} \wedge \omega^l_\phi)
+ \hat{\partial}(\omega^k_\phi \wedge (k-l)\omega^{n-1-l} \wedge \omega^{l-1}_\phi) \right)
= -\frac{1}{V} \int_M \hat{\partial}(\omega_\phi - \omega) \wedge (k-l)\omega^{n-1-l} \wedge \omega^{l-1}_\phi)
= -\frac{1}{V} \int_M \hat{\partial}(\omega_\phi - \omega) \wedge \sum_{l=0}^{k-1} (2(k-l)\omega^{n-1-l} \wedge \omega^l_\phi)
+ (\omega_\phi - \omega) \wedge (k-l)\omega^{n-1-l} \wedge \omega^{l-1}_\phi)
= -\frac{1}{V} \int_M \hat{\partial}(\omega_\phi - \omega) \wedge \sum_{l=0}^{k-1} (2(k-l)\omega^{n-1-l} \wedge \omega^l_\phi)
+ \sum_{l=1}^{k-1} (l(k-l)\omega^{n-1-l} \wedge \omega^l_\phi)
+ \sum_{l=1}^{k-2} ((k-l-1)(l+1)\omega^{n-1-l} \wedge \omega^l_\phi)
= - (k+1) \frac{1}{V} \int_M \hat{\partial}(\omega_\phi - \omega) \wedge \sum_{l=0}^{k-1} \omega^{n-1-l} \wedge \omega^l_\phi),$$

and putting $\sqrt{-1}\partial\bar{\partial}\phi = \omega_\phi - \omega$ we have

$$\frac{d}{dt} I_k(\omega,\omega_\phi) = V^{-1} \int_M \hat{\partial}(\omega^n + \omega^{n-k} \wedge \omega^k_\phi). \quad (8)$$
Combining with (7) and (4) we conclude.

Note that from the definitions it follows that

\[ 0 \leq I_k(\omega, \omega_\varphi) \leq J(\omega, \omega_\varphi), \quad \text{on } \mathcal{H}_\Omega \times \mathcal{H}_\Omega. \]  

(9)

As a corollary of Lemma 2.2 we have therefore \( 0 \leq J_k(\omega, \omega_\varphi) \leq J(\omega, \omega_\varphi) \) on \( \mathcal{H}_\Omega \times \mathcal{H}_\Omega \).

We point out that this upper bound improves [CT, Corollary 4.5] while the lower bound appears to be new. Also from (6)

\[ \frac{k+2}{k+1} I_{k+1} \geq \frac{k+1}{k} I_k, \quad \text{on } \mathcal{H}_\Omega \times \mathcal{H}_\Omega. \]  

(10)

Note that in particular \( I_{k+1} \geq I_k \) and so by Lemma 2.2 \( J_k \geq J_{k+1} \). We note in passing that this lemma also yields the following formula

\[ J_k(\omega, \omega_\varphi) = \frac{V^{-1}}{n+1} \int_M \sqrt{-1} \partial \phi \wedge \bar{\partial} \phi \wedge \left( \frac{n-k}{k+1} \sum_{l=0}^{k-1} (l+1) \omega_\varphi^{n-1-l} \wedge \omega_\varphi^l \right. \\
+ \sum_{l=k}^{n-1} (n-l) \omega_\varphi^{n-1-l} \wedge \omega_\varphi^l \right). \]  

(11)

The energy functionals \( E_k, \ k = 0, \ldots, n, \) are defined by

\[ E_k(\omega, \omega_\varphi) = \frac{V^{-1}}{n} \int_{M \times [0,1]} \Delta_{\varphi_\tau} \varphi_\tau \text{Ric} \left( \omega_\varphi^k \right) \wedge \omega_\varphi^{n-k} \wedge dt \\
- \frac{n-k}{k+1} \frac{V^{-1}}{n} \int_{M \times [0,1]} \varphi_\tau \left( \text{Ric} \left( \omega_\varphi^k \right) - \mu_k \omega_\varphi^{k+1} \right) \wedge \omega_\varphi^{n-k} \wedge dt, \]  

(12)

where \( \mu_k := \frac{e^{\text{Ric} \omega} \omega^{n-k} \text{Vol}(M)}{\omega^{n-k} \text{Vol}(M)} \). This gives rise to well-defined exact energy functionals [CT] (note that \( E_k/(k+1) \) in the aforementioned article corresponds to \( E_k \) in this article). The K-energy, \( E_0 \), was introduced by Mabuchi [M], while \( E_n \), which we refer to as the ‘Ricci energy’, was introduced by Bando and Mabuchi [BM].

For each \( \omega \in \mathcal{H}_\Omega \) these functionals (being exact) induce a (real) Lie group homomorphism \( \text{Aut}(M, J)_0 \to \mathbb{R} \) given by \( h \mapsto E_k(\omega, h^* \omega) \). The corresponding Lie algebra

3 Kähler-Einstein forms are the only critical points of these two functionals when \( \Omega = \mu c_1, \mu \in \{ \pm 1 \} \): For \( E_0 \) see [T4, p. 19] while for \( E_n \) the critical forms satisfy \( (\mu \text{Ric} \omega)^n = \omega^n \) and writing \( \mu \text{Ric} \omega = \omega + \sqrt{-1} \partial \bar{\partial} f \) we see that \( \mu \text{Ric} \omega > 0 \) at the minimum of \( f \). Since the smallest eigenvalue of a Hölder continuous matrix-valued function is also Hölder continuous [Al, p. 438] we conclude that \( \mu \text{Ric} \omega > 0 \) implying that \( f \) is constant by the uniqueness argument of Calabi (for a different proof see [Ma, §8]). However when \( c_1 = 0 \) there are nontrivial solutions of \( (\text{Ric} \omega)^n = 0 \) if the manifold is a product. For \( \mu = 1 \) critical points of \( E_k \) with nonnegative Ricci curvature are necessarily Kähler-Einstein [To].
homomorphism $\text{aut}(M,J) \to \mathbb{R}$ is given by $X \mapsto \left. \frac{d}{dt} \right|_{t=0} D_{\omega} (\omega, (\exp tX)^* \omega)$. This naturally extends to a complex Lie algebra homomorphism

$$X \mapsto F_k(X; \omega) := \left. \frac{d}{dt} \right|_{t=0} D_{\omega} (\omega, (\exp tX)^* \omega) - \sqrt{-1} \left. \frac{d}{dt} \right|_{t=0} D_{\omega} (\omega, (\exp tJX)^* \omega). \quad (13)$$

Changing $\omega$ within a fixed cohomology class does not change the homomorphism $[\text{CT}, \text{Ma}]$. This is an extension of the Bando-Calabi-Futaki Theorem, the case $k = 0$ $[\text{Be}, \text{C,F1}]$ (the construction was further generalized by Futaki $[\text{F3}]$). One calls these homomorphisms Futaki characters (or invariants). When $(M,J,\omega)$ is Fano Kähler-Einstein it follows from (12) that $F_k$ is trivial and hence $E_k(\omega, \omega) = 0$ if $\omega$ is Kähler-Einstein, since the set of Kähler-Einstein metrics is equal to an Aut$(M,J)$-orbit of $\omega$ $[\text{BM}]$.

Unless otherwise stated, from now and on we will assume that $(M,J)$ is Fano and let $\omega_\varphi \in \mathcal{H}_{c_1}$. Let $f_{\omega_\varphi} \in \mathcal{C}^\infty(M)$ denote the unique function satisfying $\sqrt{-1} \partial \overline{\partial} f_{\omega_\varphi} = \text{Ric}_{\omega_\varphi} - \omega_\varphi$ and $V^{-1} \int_M e^{f_{\omega_\varphi}} \omega_\varphi^n = 1$. Following Ding $[\text{D}]$, define an exact functional on $\mathcal{H}_{c_1} \times \mathcal{D}_{c_1}$ by

$$F(\omega, \omega_\varphi) = J(\omega, \omega_\varphi) - \frac{1}{V} \int_M \varphi \omega^n - \log \frac{1}{V} \int_M e^{f_{\omega_\varphi}} \omega_\varphi^n. \quad (14)$$

The critical points of this functional are the Kähler-Einstein metrics. We state the following relation between the functionals $E_0$ and $F$.

**Lemma 2.3.** $[\text{DT}]$ Let $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$. Then

$$F(\omega, \omega_\varphi) = E_0(\omega, \omega_\varphi) + \frac{1}{V} \int_M f_{\omega_\varphi} \omega_\varphi^n - \frac{1}{V} \int_M f_\omega \omega^n. \quad (13)$$

Note that

$$\frac{1}{V} \int_M f_{\omega_\varphi} \omega_\varphi^n \leq \frac{1}{V} \int_M e^{f_{\omega_\varphi}} \omega_\varphi^n - 1 = 0. \quad (14)$$

Note also that one may define a Lie algebra homomorphism corresponding to $F$ similarly to the construction for $E_k$ in (13). Lemma 2.3 implies that this homomorphism will coincide with $F_0$.

An equivalent form of the following was stated by Bando and Mabuchi $[\text{BM}, (1.5)]$.

**Lemma 2.4.** For every $(\omega, \omega_\varphi) \in \mathcal{H}_{c_1}^+ \times \mathcal{H}_{c_1}$ one has

$$E_n(\omega, \omega_\varphi) = F(\text{Ric}_\omega, \text{Ric}_{\omega_\varphi}).$$
Note that by exactness this formula completely determines $E_n$ on $\mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$, as remarked earlier.

**Proof.** Let $\{\varphi_t\}$ denote a smooth family of functions such that $\omega_{\varphi_t} = \omega$, $\omega_{\varphi_1} = \omega$. Write $\text{Ric} \omega_{\varphi} = \text{Ric} \omega + \sqrt{-1} \partial \bar{\partial} \log \omega^n_{\varphi}$. Then $f_{\text{Ric} \omega} = \log \frac{\omega^n_{\varphi}}{(\text{Ric} \omega)^n}$. Thus for each $t \in [0, 1]$,

$$F(\text{Ric} \omega, \text{Ric} \omega_{\varphi_t}) = J(\text{Ric} \omega, \text{Ric} \omega_{\varphi_t}) - \frac{1}{V} \int_M \log \frac{\omega^n_{\varphi_t}}{\omega^n_{\varphi_0}}(\text{Ric} \omega)^n.$$

Hence,

$$\frac{d}{dt} F(\text{Ric} \omega, \text{Ric} \omega_{\varphi_t}) = -V^{-1} \int_M (-\Delta_t \varphi_t)(\text{Ric} \omega_{\varphi_t})^n = \frac{d}{dt} E_n(\omega, \omega_{\varphi_t}),$$

from which we conclude by integration. $\square$

Bando and Mabuchi derived the following elegant formula.

**Proposition 2.5.** [BM, (1.8.1)] For every $(\omega, \omega_{\varphi}) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$,

$$E_n(\omega, \omega_{\varphi}) = E_0(\omega, \omega_{\varphi}) + J(\omega_{\varphi}, \text{Ric} \omega_{\varphi}) - J(\omega, \text{Ric} \omega).$$

We now show that Proposition 2.5 can be generalized as follows.

**Proposition 2.6.** Let $k \in \{0, \ldots, n\}$. For every $(\omega, \omega_{\varphi}) \in \mathcal{H}_{c_1} \times \mathcal{H}_{c_1}$,

$$E_k(\omega, \omega_{\varphi}) = E_n(\omega, \omega_{\varphi}) - J_k(\omega_{\varphi}, \text{Ric} \omega_{\varphi}) + J_k(\omega, \text{Ric} \omega),$$

$$= E_0(\omega, \omega_{\varphi}) + I_k(\omega_{\varphi}, \text{Ric} \omega_{\varphi}) - I_k(\omega, \text{Ric} \omega),$$

$$= ((1 - \frac{l}{k+1})E_0 + \frac{l}{k+1}E_n)(\omega, \omega_{\varphi}) + (I_k - \frac{l}{k+1}J)(\omega_{\varphi}, \text{Ric} \omega_{\varphi})$$

$$- (I_k - \frac{l}{k+1}J)(\omega, \text{Ric} \omega), \quad \forall l \in \{0, \ldots, k+1\}.$$

The proof appears in the Appendix. The functionals $E_k$ are thus seen to be described as ‘Kähler-Ricci’ energies, “interpolating” between the Kähler energy $E_0$ and the Ricci energy $E_n$. We note that there exist counterparts of the formulas presented in this section for other classes [R].
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One particularly visible consequence of Proposition 2.6 is the fact that the homomorphisms $F_k$ all coincide, a result first proved by Maschler [Ma, (17)] using an equivariant formulation and later by Liu [Li, §3] by a direct computation (see also [L1]). For other explicit expressions for the functionals $E_k$ see [CT,L1,P,SW].

3 Continuity method approach. Consider the path $\{\omega_{\varphi_t}\} \subseteq H_{c_1}$ given implicitly by

$$\omega_{\varphi_t}^n = e^{(t+1)f_\omega + c_1} \omega^n, \quad t \in [-1, 0],$$
$$\omega_{\varphi_t}^n = e^{f_\omega - t\varphi} \omega^n, \quad t \in [0, 1],$$

with the normalizations $\int_M e^{(t+1)f_\omega + c_1} \omega^n = V$ for $t \in [-1, 0]$ and $\int_M e^{f_\omega - t\varphi} \omega^n = V$ for $t \in [0, 1]$. Note that the first segment always exists by the proof of the Calabi-Yau Theorem [Y] while the second, when it exists, deforms the metric to a Kähler-Einstein metric [A2]:

$$\text{Ric} \omega_{\varphi_t} - \omega_{\varphi_t} = -(1 - t)\sqrt{-1} \partial \bar{\partial} \varphi_t, \quad t \in [0, 1].$$

We will make use of the following Proposition:

Proposition 3.1. [BM, Theorem 5.7] Assume that $(M, J)$ is Fano and let $G$ be a compact subgroup of $\text{Aut}(M, J)$. Assume that $E_0$ is bounded from below on $H^+_{c_1}(G)$ and let $\omega \in H_{c_1}(G)$. Then (18) has a unique smooth solution for each $t \in [0, 1)$.

Note that by Lemma 2.3 and (14) the same conclusion holds with $E_0$ replaced by $F$. In particular, Theorem 1.2 is a direct corollary of Proposition 3.1 combined with this observation (one obtains a version of Theorem 1.2 with the free choice of a subgroup $G$, although this, as opposed to the refinement of Theorem 1.1 that will be given in the next section, should not be considered as a gain in generality). We also note that one of the important ingredients in the proof of Proposition 3.1 is the fact that $(I - J)(\omega, \cdot)$ is nondecreasing along the continuity path (18) [BM, Theorem 5.1; T1, p. 232; T4, Lemma 6.25].

It is worth noting that Bando has shown that if $\omega \in H_{c_1}(G)$ satisfies $\text{Ric} \omega > (1 - \epsilon)\omega$, $\epsilon > 0$, then “flowing” it along the Ricci flow will produce another metric in $H_{c_1}(G)$ whose scalar curvature differs from $n$ by at most a fixed constant times $\epsilon$ [Ba]. Therefore, the existence of a lower bound for $E_0$ or for $F$ implies the existence of Kähler metrics in $H_{c_1}(G)$ whose scalar curvature is as close to a constant as desired (the original result of Bando extends to the $G$-invariant setting since its proof makes use of a Kähler-Ricci flow which, like the continuity method, preserves $H_{c_1}(G)$). These can be thought of as “almost Kähler-Einstein” metrics since a Kähler metric of constant scalar curvature in $H_{c_1}$ is necessary Kähler-Einstein.
Let $(M,J)$ be a Fano manifold and $G$ be a compact subgroup of $\text{Aut}(M,J)$. If $F$ or $E_0$ is proper on $\mathcal{H}^+_c(G)$ then $(M,J)$ admits a $G$-invariant Kähler-Einstein metric. Conversely, if $(M,J)$ admits a $G$-invariant Kähler-Einstein metric then $F$ and $E_0$ are proper on $\mathcal{H}^+_c(L_1)$. 

We remark that when $\text{aut}(M,J)$ is semisimple then $\mathcal{H}^+_c(G) \subseteq \mathcal{H}^+_c(L_1)$ [PSSW].

Let us turn to the proof of our main theorems and begin with Theorem 1.3. Assume that a Kähler-Einstein form $\omega$ exists. Then $F$ is proper on $\mathcal{H}^+_c$ by Theorem 1.1. By Lemma 2.3 and (14) so is $E_0$. From Proposition 2.6 we have

$$E_{k+1}(\omega,\omega_\varphi) = E_k(\omega,\omega_\varphi) + (I_{k+1} - I_k)(\omega_\varphi,\text{Ric}\omega_\varphi) - (I_{k+1} - I_k)(\omega,\text{Ric}\omega),$$

with $I_{k+1} \geq I_k$ on $\mathcal{H}_c \times \mathcal{H}_c$ as noted after (10). It follows that if $E_k$ is proper on $\mathcal{H}^+_c$, so is $E_{k+1}$. We conclude that $E_n$ is proper on $\mathcal{H}^+_c$.

Assume that $E_n$ is proper on $\mathcal{H}^+_c$. Then from Lemma 2.4 and the Calabi-Yau Theorem we see that $F$ is bounded from below on $\mathcal{H}_c$ and from Lemma 2.3 and (14) it follows that so is $E_0$. Therefore from Proposition 3.1, given $\omega \in \mathcal{H}_c$, the continuity path (18) extends for all $t < 1$.

From the properness and exactness of $E_n$ there exists a function $\tau_\omega$ as in Section 2 satisfying $E_n(\omega_\varphi,\omega_\varphi) \geq \tau_\omega(I(\omega,\omega_\varphi)) - E_0(\omega,\omega_\varphi)$. Hence it suffices now to show that $E_n(\omega_\varphi,\omega_\varphi)$ is uniformly bounded from above for all $t > t_0$ with $t_0$ depending only on $(M,J,\omega)$. We will then have that $I(\omega,\omega_\varphi)$ is uniformly bounded independently of $t \in [0,1)$. This will entail a uniform bound on $\parallel \varphi \parallel_{L^\infty}$ [A3, Proposition 7.35; T4, Lemma 6.19] and hence a uniform bound on $\parallel \varphi \parallel_{C^{2,\beta}(M,\omega_0)}$ for some $\beta \in (0,1)$. By the continuity method arguments therein one then concludes that a unique smooth solution exists at $t = 1$ that is a Kähler potential for a Kähler-Einstein form.

\[3\] A detailed exposition of this theorem will be found in a forthcoming article of Tian and Zhu.
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In fact we will find such a \( t_0 \) depending only on \( n \) for each \( E_k \). The computation that follows involves expressions similar to those that figure in the work of Song and Weinkove; using Proposition 2.6 considerably simplifies our calculations compared to the ones there.

Fix \( \tau \in [0, 1] \). First, from (19) and the definition of \( E_0 \) we have

\[
E_0(\omega_{\varphi_0}, \omega_{\varphi_\tau}) = \int_{[0, \tau]} \frac{d}{dt} E_0(\omega_{\varphi_0}, \omega_{\varphi_t}) dt
\]

\[
= \frac{1}{V} \int_M (1 - t) n \hat{\phi} \sqrt{-1} \partial \bar{\partial} \varphi_\tau \wedge \omega_\varphi^{n-1} \wedge dt
\]

\[
= - \int_{[0, \tau]} (1 - t) \frac{d}{dt} (I - J)(\omega, \omega_{\varphi_t}) dt
\]

\[
= -(1 - \tau)(I - J)(\omega, \omega_{\varphi_\tau})
\]

\[
+ (I - J)(\omega, \omega_{\varphi_0}) - \int_{[0, \tau]} (I - J)(\omega, \omega_{\varphi_t}) dt. \tag{20}
\]

From Proposition 2.6, (5) and (9) we therefore conclude that there exists a constant \( c_\omega \) depending only on \( (M, J, \omega) \) for which

\[
(n + 1) E_k(\omega_{\varphi_0}, \omega_{\varphi_\tau}) \leq -(1 - \tau)I(\omega, \omega_{\varphi_\tau}) + nI(\omega_{\varphi_\tau}, \text{Ric}_{\omega_{\varphi_\tau}}) + c_\omega. \tag{21}
\]

From (19)

\[
I(\omega_{\varphi_\tau}, \text{Ric}_{\omega_{\varphi_\tau}}) = (1 - \tau)^2 \frac{1}{V} \int_M \sqrt{-1} \partial \varphi_\tau \wedge \bar{\partial} \varphi_\tau \wedge \sum_{l=0}^{n-1} (\tau \omega_{\varphi_\tau} + (1 - \tau)\omega)^l
\]

\[
= (1 - \tau)^2 \frac{1}{V} \int_M \sqrt{-1} \partial \varphi_\tau \wedge \bar{\partial} \varphi_\tau \wedge \sum_{l=0}^{n-1} \sum_{j=0}^{l} \binom{l}{j} (1 - \tau)^j \omega_{\varphi_\tau}^{n-j-1} \wedge \omega^j
\]

\[
= (1 - \tau)^2 \frac{1}{V} \int_M \sqrt{-1} \partial \varphi_\tau \wedge \bar{\partial} \varphi_\tau \wedge \sum_{j=0}^{n-1} (1 - \tau)^j \sum_{l=j}^{n-1} \binom{l}{j} (1 - \tau)^j \omega_{\varphi_\tau}^{n-j-1} \wedge \omega^j.
\]

Note that

\[
(1 - \tau)^j \sum_{l=j}^{n-1} \binom{l}{j} \leq (1 - \tau)^j (n - 1) \binom{n - 1}{j}. \tag{22}
\]

We may choose \( t_1 \in [0, 1) \) depending only on \( n \) in such a way that for all \( \tau \in [t_1, 1] \) the expression on the right hand side of (22) is smaller than \( n \) for each \( j = 0, \ldots, n - 1 \). We conclude that

\[
I(\omega_{\varphi_\tau}, \text{Ric}_{\omega_{\varphi_\tau}}) \leq n(1 - \tau)^2 I(\omega, \omega_{\varphi_\tau}), \quad \forall \tau \in [t_1, 1]. \tag{23}
\]
Returning to (21) we then see that 

\[ E_k(\omega_{\phi_0}, \omega_{\phi_\tau}) \leq c_\omega/(n+1) \]

whenever \( \tau \in \{ \max \{ t_1, 1 - \frac{1}{n\tau} \}, 1 \} \). This concludes the proof of Theorem 1.3. \( \square \)

As a corollary of the proof we record the following fact.

**Corollary 4.2.** Let \((M, J)\) be a Fano manifold. If one of the functionals \(F, E_0, \ldots, E_n\) is bounded from below on \(\mathcal{H}^+_{c_1}\) so are the rest.

Combining Corollary 4.2 with Theorem 1.2 concludes the proof of Theorem 1.4. \( \square \)

We end this section with several remarks.

**Remark 4.3.** Our methods imply that the refined version of Theorem 1.1 (Theorem 4.1) also extends to each of the functionals \(E_k\).

**Remark 4.4.** Note that one may state Corollary 4.2 with \(\mathcal{H}^+_{c_1}\) replaced by \(\mathcal{H}^+_{c_1}\) for \(F, E_0\) and \(E_1\). Indeed, recall that once \(F\) is bounded from below on \(\mathcal{H}^+_{c_1}\) so are each of the \(E_k\) while a lower bound for \(E_n\) on \(\mathcal{H}^+_{c_1}\) implies a lower bound for \(F\) on \(\mathcal{H}^+_{c_1}\) (by Lemma 2.4) which, in turn, implies the same for \(E_0\) (using Lemma 2.3) and for \(E_1\) (using Proposition 2.6). Some special cases of Corollary 4.2 appeared previously, namely the fact that when \(F\) is bounded from below so is \(E_0\) \([\text{DT}]\) and vice versa \([\text{L2}]\), and the fact that when \(E_0\) is bounded from below so is \(E_1\) \([\text{P}]\) and vice versa \([\text{CLW}]\).

**Remark 4.5.** Assume that the functionals \(F\) and \(E_k, k \in \{0, \ldots, n\}\) are bounded from below on \(\mathcal{H}^+_{c_1}\) and for each \(\omega \in \mathcal{H}_{c_1}\) set \(l(\omega) = \inf_{\omega_\phi \in \mathcal{H}_{c_1}} F(\omega, \omega_\phi)\) and

\[
l_k(\omega) = \begin{cases} 
\inf_{\omega_\phi \in \mathcal{H}_{c_1}} E_k(\omega, \omega_\phi), & \text{for } k = 0, 1, \\
\inf_{\omega_\phi \in \mathcal{H}^+_{c_1}} E_k(\omega, \omega_\phi), & \text{for } k = 2, \ldots, n.
\end{cases}
\]

Then the following relations hold between the various lower bounds:

\[
l(\omega) + \frac{1}{V} \int_M f_\omega \omega^n = l_0(\omega) = l_k(\omega) + I_k(\omega, \text{Ric } \omega).
\]

This generalizes the relation between \(l\) and \(l_0\) \([\text{L2}]\) and between \(l\) and \(l_1\) \([\text{CLW}]\) that appeared recently; our proof, given below, appears considerably simpler.

**Proof.** By Lemma 2.3, (14) and Proposition 2.6

\[
l(\omega) + \frac{1}{V} \int_M f_\omega \omega^n \leq l_0(\omega) \leq l_k(\omega) + I_k(\omega, \text{Ric } \omega).
\]

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(For the second inequality we used (16) and the fact that $I_k(\omega,\text{Ric}\omega) \geq 0$ for $\omega \in \mathcal{H}_{c_1}^+$. On the other hand, note first that from (20) it follows that $\int_{[0,1]}(I - J)(\omega,\omega_{\varphi(t)})dt$ is bounded. As remarked in Section 3 the function $(I - J)(\omega,\omega_{\varphi(t)})$ is nondecreasing in $t$. Hence

$$(1 - \tau)(I - J)(\omega,\omega_{\varphi(t)}) \leq \int_{[\tau,1]}(I - J)(\omega,\omega_{\varphi(t)})dt,$$

and therefore [DT, p. 67]

$$\lim_{\tau \to 1^-} (1 - \tau)(I - J)(\omega,\omega_{\varphi(t)}) = 0. \quad (26)$$

Going back to (20) and using the identity $E_0(\omega,\omega_{\varphi_0}) + (I - J)(\omega,\omega_{\varphi_0}) = V^{-1} \int_M f_\omega \omega^n$ we have

$$\lim_{\tau \to 1^-} E_0(\omega,\omega_{\varphi(t)}) = \frac{1}{V} \int_M f_\omega \omega^n - \int_{[0,1]}(I - J)(\omega,\omega_{\varphi(t)})dt.$$

By a theorem of Ding and Tian we have [DT, Theorem 1.2]

$$l(\omega) = \lim_{t \to 1^-} F(\omega,\omega_{\varphi(t)}) = -\int_{[0,1]}(I - J)(\omega,\omega_{\varphi(t)})dt. \quad (27)$$

Combining with (25) we conclude that

$$l_0(\omega) = \lim_{t \to 1^-} E_0(\omega,\omega_{\varphi(t)}) = l(\omega) + \frac{1}{V} \int_M f_\omega \omega^n.$$

Finally, using (9), (5), (23) and (26) it follows that $\lim_{t \to 1^-} I_k(\omega,\text{Ric}\omega) = 0$. Therefore, using Proposition 2.6 (16) again we have $l_0(\omega) \geq l_k(\omega) + I_k(\omega,\text{Ric}\omega). \quad \Box$

Remark 4.6. Note that from Proposition 2.6 it follows that if $F$ is proper on $\mathcal{H}_{c_1}$ (equivalently on $\mathcal{H}_{c_1}^+$) with $F(\omega,\omega_\varphi) \geq \tau_\omega((I - J)(\omega,\omega_\varphi))$ then we have the inequality $E_k(\omega,\omega_\varphi) \geq \tau_\omega((I - J)(\omega,\omega_\varphi)) - I_k(\omega,\text{Ric}\omega)$ on $\mathcal{H}_{c_1}^+$ (and for $k = 0, 1$ on $\mathcal{H}_{c_1}$). On the determination of explicit functions $\tau_\omega$ we refer to [PSSW,T3,T4].

5 Boundedness of energy functionals and the Moser-Trudinger-Onofri inequality. In this section we suppose that a Kähler-Einstein metric $\omega$ exists. First, we state the following fundamental theorem:

Theorem 5.1. [BM, Theorem A, Corollary 8.3; Ba, Theorem 1] Let $(M, J, \omega)$ be a Kähler-Einstein Fano manifold. Then $E_0(\omega,\omega_\varphi) \geq 0$ for all $\omega_\varphi \in \mathcal{H}_{c_1}$ and $E_n(\omega,\omega_\varphi) \geq 0$ for all $\omega_\varphi \in \mathcal{H}_{c_1}^+$ with equality if and only if $\omega_\varphi = h^*\omega$ with $h \in \text{Aut}(M,J)_0$.
Building on these results, Song and Weinkove proved: (i) The first statement holds with $E_0$ replaced by $E_1$ (see also [P]), and (ii) the second statement holds with $E_n$ replaced by $E_k$ for each $k \in \{2, \ldots, n-1\}$. Proposition 2.6 provides a much simplified proof of these two facts. Moreover, it allows to improve on (ii). Let

$$A_k := \{\omega \varphi \in H_{c_1} : E_k(\omega, \omega \varphi) \geq 0\}. \quad (28)$$

Then we have shown that

$$A_k \supseteq B_k := \{\omega \varphi \in H_{c_1} : I_k(\omega \varphi, \text{Ric} \omega \varphi) \geq 0\}. \quad (29)$$

For example, for $k = 1$ this gives $A_1 = H_{c_1}$, when $k = 2$ we have

$$A_2 \supseteq B_2 \supseteq \{\omega \varphi \in H_{c_1} : \text{Ric} \omega \varphi + 2\omega \varphi \geq 0\},$$

for $k = 3$

$$A_3 \supseteq B_3 \supseteq \{\omega \varphi \in H_{c_1} : \text{Ric} \omega \varphi + \omega \varphi \geq 0\},$$

and for arbitrary $k$ one may readily obtain an explicit bound (depending on $k$) on the set $B_k$, and hence on $A_k$, in terms of a lower bound on the Ricci curvature, using the definition (6).

Let $\omega_{FS,c}$ denote the Fubini-Study form of constant Ricci curvature $c$ on $(S^2, J)$, the Riemann sphere, given locally by

$$\omega_{FS,c} = \sqrt{-1} \frac{dz \wedge d\bar{z}}{c \pi (1 + |z|^2)^2}.$$ 

Here $V = \int_{S^2} \omega_{FS,c} = c(\mu) / c = 2 / c$. For $c = 1 / 2\pi$ it is induced from restricting the Euclidean metric on $\mathbb{R}^3$ to the radius 1 sphere. Denote by $W^{1,2}(S^2)$ the space of functions on $S^2$ that are square-summable and so is their gradient (with respect to some Riemannian metric). The Moser-Trudinger-Onofri inequality states:

**Theorem 5.2. [Mo,O,Tr]** For $\omega = \omega_{FS,2/V}$ and any function $\varphi$ on $S^2$ in $W^{1,2}(S^2)$ one has

$$\frac{1}{V} \int_{S^2} e^{-\varphi + \frac{1}{2} \int_{S^2} \varphi \omega} \omega \leq e^{\frac{1}{2} \int_{S^2} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi}. \quad (30)$$

Equality holds if and only if $\omega \varphi$ is the pull-back of $\omega$ by a M"{o}bius transformation.

An alternative proof of this inequality has been given by Ding and Tian for those functions $\varphi$ that belong to the subspace $H_{\omega} \subseteq W^{1,2}(S^2)$. The proof uses the properties of $F$. We now note that our work provides a new and succinct proof of the original Moser-Trudinger-Onofri inequality entirely within the framework of exact energy functionals. This is the first proof that does not use symmetrization/rearrangement
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arguments. Other proofs of this inequality have been given by Onofri [O], Hong [H], Osgood-Phillips-Sarnak [OPS], Beckner [B], Carlen and Loss [CL1,CL2], Ghigi [G] (for more background we refer to Chang [Ch]).

Proof. By Theorem 5.1 and Proposition 2.6 \( E_1(\omega, \cdot) \geq 0 \) on \( \mathcal{H}_\Omega \). Given \( \varphi \in \mathcal{C}^\infty(S^2) \) there exists \( \psi \in \mathcal{H}_\omega \) such that \( \text{Ric}_\omega \psi = \omega \varphi \) by solving the Poisson equation on \( S^2 \). Thus by Lemma 2.4 \( F(\omega, \cdot) \geq 0 \) on \( D_\Omega \). Using the definition of \( F \), for any smooth function \( \varphi \) we obtain (30). Since \( \mathcal{C}^\infty(S^2) \) is dense in \( W^{1,2}(S^2) \) we conclude. \( \Box \)

Ding and Tian showed that a restricted analogue of this inequality holds also for higher dimensional manifolds:

Theorem 5.3. [DT] Let \((M,J)\) be a Fano manifold and let \( \omega \in \mathcal{H}_{c_1} \). Assume that \( F \) is bounded from below on \( \mathcal{H}_{c_1} \) and let \( a = -\inf_{\mathcal{H}_{c_1}} F(\omega, \cdot) \). Then for each \( \varphi \in \mathcal{H}_{\omega} \) holds

\[
\frac{1}{V} \int_M e^{-\varphi + \frac{1}{2} \int_M \varphi \omega^n} \omega^n \leq e^{F(\omega, \omega) + a}.
\] (31)

If \((M,J,\omega)\) is Kähler-Einstein then \( a = 0 \).

Recall Jensen’s inequality \( \frac{1}{V} \int_M e^{-\varphi + \frac{1}{2} \int_M \varphi \omega^n} \omega^n \geq 1 \) [HLP]. Now observe that in higher dimensions, due to Lemma 2.1, inequality (31) cannot be extended to all of \( \mathcal{C}^\infty(M) \). A natural question is therefore: On a Kähler-Einstein manifold, what is the largest neighborhood of \( \mathcal{H}_\omega \) inside \( \mathcal{C}^\infty(M) \) on which (31) does hold? Naturally, we call such a neighborhood the Moser-Trudinger-Onofri neighborhood of \( \mathcal{H}_\omega \), and put

\[
\text{MTO}_n = \{ \varphi \in \mathcal{C}^\infty(M) : \varphi \text{ satisfies } (31) \text{ on the Fano manifold } (M,J), \dim \mathbb{C} M = n \}.
\] (32)

Using Lemma 2.4, we have the following characterization of the Moser-Trudinger-Onofri neighborhood. By abuse of notation we do not distinguish here between the set \( \text{Ric}(A_n) \) in \( \mathcal{H}_\Omega \) and the corresponding set in \( \mathcal{H}_\omega \).

Theorem 5.4. Let \((M,J,\omega)\) be a Kähler-Einstein Fano manifold. Then \( \varphi \in \mathcal{C}^\infty(M) \) satisfies the generalized Moser-Trudinger-Onofri inequality (31) if and only if there exists a function \( \psi \in \mathcal{C}^\infty(M) \) such that \( \text{Ric}_\omega \psi = \omega \varphi \) and \( \omega \psi \in A_n \). That is, \( \text{MTO}_n = \text{Ric}(A_n) \supset \mathcal{H}_\omega \).

Recall that \( B_n \subseteq A_n \) and that we have bounds on \( B_n \) in terms of the Ricci curvature. Therefore, Theorem 5.4 shows that in higher dimensions the Moser-Trudinger-Onofri inequality is related to Ricci curvature and holds on a set strictly larger than the space of Kähler potentials. It would be interesting to improve the bounds both on \( A_n \) and on \( B_n \).

We now state another corollary of our arguments.
Corollary 5.5. Let \((M,J,\omega)\) be a Fano manifold. Then the Ricci energy \(E_n\) is unbounded from below on \(\mathcal{H}_{c_1}\) if and only if \(n > 1\).

Before concluding, we remark that Theorem 5.3 can be strengthened using the results obtained here. The same applies to later extensions of this inequality \([\text{PSSW}, \text{T4}]\) and will figure in a subsequent article.

Appendix. In this appendix we prove Proposition 2.6. First, in order to establish formula (16) we show that the variations of both sides of the equation agree.

\[-(k + 1)V \frac{d}{dt} I_k(\omega_\varphi, \text{Ric} \omega_\varphi) = \frac{d}{dt} \int_M f_{\omega_\varphi} \sqrt{-1} \partial \bar{\partial} f_{\omega_\varphi} \wedge \sum_{l=0}^{k-1} (k-l) \omega_\varphi^{n-1-l} \wedge (\text{Ric} \omega_\varphi)^l\]

\[= \frac{d}{dt} \int_M f_{\omega_\varphi} (\text{Ric} \omega_\varphi - \omega_\varphi) \wedge \sum_{l=0}^{k-1} (k-l) \omega_\varphi^{n-1-l} \wedge (\text{Ric} \omega_\varphi)^l\]

\[= \frac{d}{dt} \int_M f_{\omega_\varphi} \left( k \omega_\varphi^n + \sum_{l=1}^{k} \omega_\varphi^{n-l} \wedge (\text{Ric} \omega_\varphi)^l \right)\]

(33)

\[= \int_M f_{\omega_\varphi} \left( k \omega_\varphi^n + \sum_{l=1}^{k} \omega_\varphi^{n-l} \wedge (\text{Ric} \omega_\varphi)^l \right)\]

(34)

\[+ \int_M f_{\omega_\varphi} \sqrt{-1} \partial \bar{\partial} \varphi \wedge \left( \sum_{l=1}^{k} (n-l) \omega_\varphi^{n-l-1} \wedge (\text{Ric} \omega_\varphi)^l - kn \omega_\varphi^{n-1} \right)\]

(35)

\[- \int_M f_{\omega_\varphi} \sqrt{-1} \partial \bar{\partial} \Delta \varphi \sum_{l=1}^{k} l \omega_\varphi^{n-l} \wedge (\text{Ric} \omega_\varphi)^{l-1}.\]

First, we write (33) as

\[\int_M \dot{f}_{\omega_\varphi} \left( k \omega_\varphi^n + \sum_{l=1}^{k} \omega_\varphi^{n-l} \wedge (\text{Ric} \omega_\varphi)^l \right) =: \iota_1 + \mu_1.\]

We will evaluate (34) and (35) by substituting once again \(\sqrt{-1} \partial \bar{\partial} f_{\omega_\varphi} = \text{Ric} \omega_\varphi - \omega_\varphi\). For (34) we get

\[\int_M \dot{\varphi} (\text{Ric} \omega_\varphi - \omega_\varphi) \wedge \left( k n \omega_\varphi^{n-1} + \sum_{l=1}^{k} \omega_\varphi^{n-l-1} \wedge (\text{Ric} \omega_\varphi)^l \right)\]

\[=: \int_M \dot{\varphi} (k n \omega_\varphi^n - k n \omega_\varphi^{n-1} \wedge \text{Ric} \omega_\varphi)

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\[
- (n - 1)\omega^{n-1}_{\varphi} \wedge \text{Ric} \omega_{\varphi} + \sum_{l=2}^{k} \omega^{n-l}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^l
+ (n - k)\omega^{n-k-1}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^{k+1}
\]

\[
= \int_M \phi \left( - (n - k) + (k + 1)n - k \omega^{n}_{\varphi} - (k + 1)n \omega^{n-1}_{\varphi} \wedge \text{Ric} \omega_{\varphi}
+ \sum_{l=1}^{k} \omega^{n-l}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^l + (n - k)\omega^{n-k-1}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^{k+1} \right)
\]

\[
=: (\kappa_1 + \lambda_1 + \nu_2) + \lambda_2 + \mu_2 + \kappa_2.
\]

For (35) we get

\[
\int_M \Delta \phi \omega_{\varphi} - \text{Ric} \omega_{\varphi} \wedge \sum_{l=1}^{k} \omega^{n-l}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^{l-1}
\]

\[
= \int_M \Delta \phi \left( \omega^{n}_{\varphi} + \sum_{l=1}^{k-1} \omega^{n-l}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^l - k \omega^{n-k}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^k \right)
\]

\[
= \int_M \Delta \phi \left( \sum_{l=1}^{k} \omega^{n-l}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^l - (k + 1)\omega^{n-k}_{\varphi} \wedge (\text{Ric} \omega_{\varphi})^k \right)
\]

\[
=: \mu_3 + \kappa_3.
\]

Noting that \( \dot{\omega}_{\varphi} = -\Delta \phi \omega_{\varphi} - \varphi + c \) with \( c \) a constant yields \( \nu_1 + \nu_2 = -keV \) and \( \mu_1 + \mu_2 + \mu_3 = keV \). Note that \( \kappa_1 + \kappa_2 + \kappa_3 = -(k + 1)V \frac{d}{dt}E_k(\omega, \omega_{\varphi}) \) and \( \lambda_1 + \lambda_2 = (k + 1)V \frac{d}{dt}E_0(\omega, \omega_{\varphi}) \). This completes the proof of (16).

Formulas (15) and (17) now follow: first use (16) with \( k = n \) to express \( E_0 \) in terms of \( E_n \) and \( J \), and then substitute this expression back into (16) and apply Lemma 2.2.

Let us note that one way one could arrive at the formula would be to use the expression for \( (k+1)I_k - kI_{k-1} \) (see (6)) and Lemma 2.3 together with the observation

\[
\frac{d}{dt}((k + 1)E_k - kE_{k-1})(\omega, \omega_{\varphi}) = -\frac{1}{V} \int_M \dot{\phi} \omega_{\varphi}^{n} - \frac{d}{dt} \left( \frac{1}{V} \int_M \omega_{\varphi} (\text{Ric} \omega_{\varphi})^k \wedge \omega_{\varphi}^{n-k} \right).
\]

\[
(36)
\]

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