Reduced class groups grafting relative invariants *

Haruhisa Nakajima
Department of Mathematics, Faculty of Science, JOSAI UNIVERSITY
Keyakidai, Sakado 350-0295, JAPAN

Abstract

Let \((X, T)\) be a regular stable conical action of an algebraic torus on an affine normal conical variety \(X\) defined over an algebraically closed field of characteristic zero. We define a certain subgroup of \(\text{Cl}(X/T)\) and characterize its finiteness in terms of a finite \(T\)-equivariant Galois descent \(\tilde{X}\) of \(X\). Consequently we show that the action \((X, T)\) is equidimensional if and only if there exists a \(T\)-equivariant finite Galois covering \(X \to \tilde{X}\) such that \((\tilde{X}, T)\) is cofree. Moreover the order of \(\text{Gal}(X/\tilde{X})\) is controlled by a certain subgroup of \(\text{Cl}(X)\). The present result extends thoroughly the equivalence of equidimensionality and cofreeness of \((X, T)\) for a factorial \(X\). The purpose of this paper is to evaluate orders of divisor classes associated to modules of relative invariants for a Krull domain with a group action. This is useful in studying on equidimensional torus actions as above. The generalization of R. P. Stanley’s criterion for freeness of modules of relative invariants plays an important role in showing key assertions.

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*Dedicated to the memory of late Professor Masayoshi Nagata.
1 Introduction

Let \((X,G)\) denote a regular action of an affine algebraic group \(G\) on an affine algebraic variety \(X\) over an algebraically closed field \(K\) of characteristic \(p \geq 0\). We say that \((X,G)\) is admitting an algebraic quotient, if the algebra \(\mathcal{O}(X)^G\) of invariants of \(G\) in the coordinate ring \(\mathcal{O}(X)\) of \(X\) is finitely generated over \(K\). The algebraic quotient \(X//G\) is defined by \(\mathcal{O}(X)^G\). Such an action \((X,G)\) is said to be equidimensional (resp. cofree), if the quotient morphism \(\pi_{X,G} : X \to X//G\) is equidimensional, i.e., closed fibers of \(\pi_{X,G}\) are pure \((\text{dim } X - \text{dim } X//G)\)-dimensional (resp. if \(\mathcal{O}(X)\) is \(\mathcal{O}(X)^G\)-free).

For a finite dimensional linear representation \(G \to GL(V)\) of a connected algebraic group \(G\) over the complex number field \(C\), the following conjecture is well known (e.g., Appendix to Chap. 4 of [9]) and relates to the subject of the present paper (e.g., [13]).

The Russian conjecture If \((V,G)\) is equidimensional, then it is cofree.

Especially for an algebraic torus \(G\), the Russian conjecture is solved affirmatively (cf. [20]), which is generalized in [12] to the case where \((V,G)\) is a conical factorial variety \(V = X\) with a stable conical action of an algebraic torus over \(K\) of characteristic zero. It should be noted that this result is not true for any conical normal variety \(X\) (cf. Example 5.7, i.e., the Russian conjecture for normal varieties does not hold). Here \(X\) (resp. \((X,G)\)) is said to be conical, if \(X\) is affine and \(\mathcal{O}(X)\) is a positively graded, i.e., a \(\mathbb{Z}_0\)-graded algebra defined over \(\mathcal{O}(X)_0 = K (\mathbb{Z}_0 = \mathbb{N} \cup \{0\})\) (resp. if the action of \(G\) preserves each homogeneous part of \(\mathcal{O}(X)\)). Moreover \((X,G)\) is said to be stable, if there is a non-empty open subset of \(X\) consisting of closed \(G\)-orbits.

The purpose of this paper is to study on the following problem which produces extensions of the results in [12].

Problem 1.1 Suppose that \(G\) is an algebraic torus and \((X,G)\) a stable conical action of \(G\) on a conical normal variety \(X\) defined over \(K\) of characteristic zero. If \((X,G)\) is equidimensional, then:

- Does there exist a \(G\)-equivariant finite Galois covering \(X \to \tilde{X}\) for a normal conical variety \(\tilde{X}\) with a conical \(G\)-action admitting the commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \tilde{X} \\
\downarrow_{\pi_{X,G}} & & \downarrow_{\pi_{\tilde{X},G}} \\
X//G & \longrightarrow & \tilde{X}//G
\end{array}
\]

such that \((\tilde{X},G)\) is cofree?

- Moreover can we choose \(X \to \tilde{X}\) in such a way that the order \(|\text{Gal}(X/\tilde{X})|\) of the Galois group of \(X \to \tilde{X}\) is a divisor of a power of the exponent of a subgroup of the divisor class group \(\text{Cl}(X)\) of \(X\)?

This problem is closely connected with relative invariants of algebraic tori, because their irreducible representations are of degree one. Relative invariants of finite groups and compact Lie groups are studied by R. P. Stanley (cf. [18, 19]), which inspires the author to extend R. P. Stanley’s criterion for freeness of modules of relative invariants.
Our study on Problem 1.1 in the present paper is based on the result in [11, 15] mentioned as above.

The main auxiliary part is given in Sect. 3. We explain the results in Sect. 3 in more detail. Let us consider a Krull domain R acted by an abstract group G as ring automorphisms and the $R^G$-modules of $\chi$-invariants in R for 1-cocycles $\chi$’s of G in the group $U(R)$ of units of R. We associate some qualified cocycles $\chi$ with the Weil divisor divisors $D(\chi)$ on R and the divisorial $R^G$-lattices $d_{(R^G,R)}(R_\chi)$. In Corollary 3.6, the orders $\text{ord}([D(\chi)])$ in $\text{Cl}(R)$ will be characterized by the equality

$$\text{ord}([D(\chi)]) = \min \left\{ \{ m \in N \mid R_{m\chi} \cong R^G \} \right\}$$

on the order of $[D(\chi)]$, if $R_{m(\chi)\chi} \cong R^G$ for some $m(\chi) \in N$. By the identity

$$d_{(R^G,R)}(R_{m\chi}) = m \cdot d_{(R^G,R)}(R_\chi)$$

for some cocycles, we will establish Theorem 3.12 as a main result in Sect. 3, which shows $\text{ord}([D(\chi)])$ is equal to $\text{ord}([d_{(R^G,R)}(R_\chi)])$ in $\text{Cl}(R^G)$. In Definition 3.13, the reduced class group $\text{UrCl}(R, G)$ (resp. $\text{Cl}(R, G)$) is defined to be a subgroup of $\text{Cl}(R)$ (resp. of $\text{Cl}(R^G)$) generated by certain $[D(\chi)]$’s (resp. certain $[d_{(R^G,R)}(R_\chi)]$’s). Then by Theorem 3.15 we have the equality of exponents

$$\exp(\text{UrCl}(R, G)) = \exp(\text{Cl}(R, G)) < \infty$$

(1.2)

of reduced class groups if $R_{m(\chi)\chi} \cong R^G$ for some qualified $\chi$’s with $m(\chi) \in N$.

Sect. 4, 5 are devoted to the study on Problem 1.1 and hence, without specifying, suppose that $(X, G)$ is a regular action of a connected affine algebraic group G on an affine normal variety over K of characteristic zero. The pseudo-reflections in the general linear group $GL(V)$ are recognized as elements which are inertial at minimal prime ideals in $O(V) = \text{Sym}(V^*)$ (cf. [11]). We define pseudo-reflections for $(X, G)$ and characterize the ramification indices of discrete valuations on X over $X//G$ in the case where $G^0$ is an algebraic torus, in terms of orders of pseudo-reflections (cf. [14]). The qualified cocycles treated in Sect. 3 are determined by pseudo-reflection subgroups (cf. Proposition 4.5). Moreover suppose that both X and $(X, G)$ are conical. Then, combining Proposition 4.5 with Theorem 3.15, by [12] we will obtain the main result of Sect. 4, 5, i.e., Theorem 5.2. Furthermore we define in Definition 5.4 the obstruction subgroup $\text{Obs}(X, G)$ of G for cofreeness of $(X, G)$ which excludes some characters $\chi$ such that $O(X)_\chi \neq O(X)^G$. Consequently, we solve affirmatively Problem 1.1 as follows (cf. Theorem 5.8):

**Theorem 1.2** Under the same circumstances as in Problem 1.1, $(X, G)$ is equidimensional if and only if there exists a (normal) closed subgroup $\text{Obs}(X, G)$ of G whose restriction to $\bar{X}$ is of order dividing a power of the exponent of the subgroup $\text{UrCl}(O(X), G)$ of $\text{Cl}(X)$ such that $(X//\text{Obs}(X, G), G)$ is cofree.

The order $|\text{Obs}(X, G)|_{\chi}$ of restrictions of $\text{Obs}(X, G)$ to $\bar{X}$ is effectively determined and is closely related to the equality (1.2). Clearly putting $\bar{X} := X//\text{Obs}(X, G)$, we have a commutative diagram (1.1) and $\text{Gal}(X/\bar{X}) = \text{Obs}(X, G)|_{\chi}$. Thus the following criterion for cofreeness is obtained:
Corollary 1.3 Under the same circumstances as in Problem 1.1, suppose that \((X,G)\) is equidimensional. Then \((X,G)\) is cofree if and only if \(|\text{Obs}(X,G)|_X| = 1\).

The if part of Corollary 1.3 is an immediate consequence of Theorem 1.2. There are many equidimensional \((X,G)\)'s which are not cofree (e.g., Example 5.7). Theorem 1.2 and Corollary 1.3 are regarded as quite generalizations of the main theorem in [12].

For a group homomorphism \(G \to \text{Aut}(Y)\) we denote by \(G|_Y\) the set \(\{\sigma|_Y \mid \sigma \in G\}\) where \(\sigma|_Y\) denotes the restriction of \(\sigma\) to \(Y\). Let \(\exp(G)\) be the exponent of a group \(G\) and \(\text{ord}(a)\) denote the order of \(a \in G\). For a subset (or an element) \(\Omega\) of \(G\), let \(\langle \Omega \rangle\) be the subgroup of \(G\) generated by \(\Omega\) and \(Z_G(\Omega)\) the centralizer of \(\Omega\) in \(G\). Let \(\text{tor}(A)\) denote the torsion part of an abelian group \(A\). The notations \(\mathbb{Z}\) and \(\mathbb{N}\) are standard (i.e., the set of all integers and that of all natural numbers, respectively). Let \(\mathbb{Z}_0\) denote the additive monoid consisting of all non-negative integers.

2 Preliminaries

Let \(\mathcal{Q}(A)\) be the total quotient ring of a commutative ring \(A\) and let \(\text{Ht}_1(A)\) be the set consisting of all prime ideals \(p\) of \(A\) of height one (\(\text{ht}(p) = 1\)). Consider a ring extension \(A \hookrightarrow B\) of integral domains. For \(i \in \mathbb{Z}_0\) set
\[
\text{Ht}_1^{(i)}(B, A) := \{ p \in \text{Ht}_1(B) \mid \text{ht}(p \cap A) = i \}
\]
and moreover \(\text{Ht}_1^{[j]}(B, A) := \bigcup_{i \geq j} \text{Ht}_1^{(i)}(B, A)\) for \(j \in \mathbb{N}\). Put
\[
\text{X}_\Omega(B) := \{ p \in \text{Ht}_1(B) \mid p \cap A = \Omega \}
\]
for any \(\Omega \in \text{Ht}_1(A)\). For a nonempty subset \(M\) of \(\mathcal{Q}(B)\), we define
\[
d_{(A,B)}(M) := \bigcap_{q \in \text{Ht}_1(A)} (A_q \otimes_A (A \cdot M)).
\]
Here \(A \cdot M\) denotes the \(A\)-submodule generated by \(M\) and each \(A_q \otimes_A (A \cdot M)\) is regarded as a subset of \(\mathcal{Q}(A) \otimes_A (A \cdot M) \hookrightarrow \mathcal{Q}(B)\). Especially in the case of \(A = B\), we simply use \(d_A(M)\) instead of the notation \(d_{(A,A)}(M)\). If \(B\) is a Krull domain and \(A = \mathcal{Q}(A) \hookrightarrow B\), then \(A\) is a Krull domain (e.g., [8]) and for any \(q \in \text{Ht}_1(A)\) the set \(X_q(B)\) is nonempty (cf. [7, 12]). In this case, for \(\mathfrak{P} \in X_q(B)\) let \(e(\mathfrak{P}, q)\) denote the reduced ramification index of \(\mathfrak{P}\) over \(q\) (e.g., [14]). In the case where \(A\) is a Krull domain, let \(v_{A,p}\) denote the discrete valuation of \(A\) defined by \(p \in \text{Ht}_1(A)\) and, for a subset \(M\) of \(\mathcal{Q}(A)\) generating a fractional ideal \(A \cdot M\) of \(A\) in \(\mathcal{Q}(A)\), let \(\text{div}_A(M)\) be the divisor on \(A\) associated with the divisorialization \(d_A(A \cdot M) = d_A(M)\) of \(A \cdot M\) on \(A\) (e.g., [1]). Let \(\text{Div}(A)\) and \(\text{Cl}(A)\) denote the (Weil) divisor group and the divisor class group of \(A\), respectively. For a (Weil) divisor \(D \in \text{Div}(A)\), we denote by \(I_A(D)\) the divisorial fractional ideal of \(A\) associated with \(D\). Our notation is standard (cf. [1, 4]).

Proposition 2.1 ([15]) Let \(R\) be a Krull domain and \(L\) a subfield of \(\mathcal{Q}(R)\) such that \(L = \mathcal{Q}(R \cap L)\). Suppose that \(\mathfrak{J}\) is a divisorial fractional ideal of \(R\). If \(\mathfrak{J} \cap L \neq 0\), then \(\mathfrak{J} \cap L\) is a fractional ideal of \(R \cap L\) and \(\text{div}_{R \cap L}(\mathfrak{J} \cap L)\) equals to \(D_{\mathfrak{J}}\) which is given by
\[
D_{\mathfrak{J}} := \sum_{q \in \text{Ht}_1(R \cap L)} \left(\max_{\mathfrak{P} \in X_q(R)} \left[ \frac{v_{R, \mathfrak{J}}(\mathfrak{P})}{e(\mathfrak{P}, q)} \right] \right) \cdot \text{div}_{R \cap L}(q),
\]
where \([a]^\#\) denotes \([-\lceil a \rceil]\) for any real number \(a\) and \(\lceil \cdot \rceil\) denotes the Gauss symbol.

**Corollary 2.2** Under the same circumstances as in Proposition 2.1, suppose moreover that \(v_{R,\mathfrak{q}}(\mathfrak{J}) \equiv 0 \mod e(\mathfrak{P}, \mathfrak{q})\) for any \(\mathfrak{q} \in \text{Ht}_1(R \cap L)\) and any \(\mathfrak{P} \in X_q(R)\). Then
\[
n \cdot \div_{R \cap L}(\mathfrak{J} \cap L) = \div_{R \cap L}(d_R(\mathfrak{J}^n) \cap L)
\]
for any \(n \in \mathbb{N}\).

**Proof.** Let \(\mathfrak{q} \in \text{Ht}_1(R \cap L)\) be any prime ideal. For a prime ideal \(\mathfrak{P}_0 \in X_q(R)\), the condition
\[
\max_{\mathfrak{P} \in X_q(R)} (v_{R,\mathfrak{P}}(\mathfrak{J})) = v_{R,\mathfrak{P}_0}(\mathfrak{J})
\]
holds if and only if
\[
\max_{\mathfrak{P} \in X_q(R)} (v_{R,\mathfrak{P}}(d_R(\mathfrak{J}^n))) = v_{R,\mathfrak{P}_0}(d_R(\mathfrak{J}^n)).
\]
By the assumption on discrete valuations of \(\mathfrak{J}\) in \(R\) and Proposition 2.1, we have
\[
v_{R \cap L, \mathfrak{q}}(d_R(\mathfrak{J}^n) \cap L) = \max_{\mathfrak{P} \in X_q(R)} \left( \frac{v_{R,\mathfrak{P}}(d_R(\mathfrak{J}^n))}{e(\mathfrak{P}, \mathfrak{q})} \right) = \frac{v_{R,\mathfrak{P}_0}(d_R(\mathfrak{J}^n))}{e(\mathfrak{P}, \mathfrak{q})} = n \cdot \max_{\mathfrak{P} \in X_q(R)} \left( \frac{v_{R,\mathfrak{P}}(\mathfrak{J})}{e(\mathfrak{P}, \mathfrak{q})} \right) = n \cdot v_{R \cap L, \mathfrak{q}}(\mathfrak{J} \cap L).
\]
The assertion follows from this. □

For a Krull domain \(R\) and a subfield \(L\) of \(\mathbb{Q}(R)\), the subgroup \(D_{(R, R \cap L)}(g_1, \ldots, g_n)\) of \(\text{Div}(R \cap L)\) is defined as follows:

**Lemma 2.3** Suppose that \(R\) is a Krull domain and let \(L\) be a subfield of \(\mathbb{Q}(R)\) such that \(L = \mathbb{Q}(R \cap L)\). Let \(\{g_1, \ldots, g_n\}\) be a finite set of nonzero elements in \(R\). Then the subgroup
\[
\left\{ \text{div}_{R \cap L} \left( \left( \frac{1}{\prod_{j=1}^{n} g_j^{i_j}} R \right) \cap L \right) \mid \forall i_1, \ldots, i_n \in \mathbb{Z}_0 \right\},
\]
which is denoted by \(D_{(R, R \cap L)}(g_1, \ldots, g_n)\), of \(\text{Div}(R \cap L)\) is finitely generated.

**Proof.** By Proposition 2.1, we see that, for any \(\mathfrak{q} \in \text{Ht}_1(R \cap L)\), the equivalence
\[
v_{R \cap L, \mathfrak{q}} \left( \left( \frac{1}{\prod_{j=1}^{n} g_j^{i_j}} R \right) \cap L \right) \neq 0 \iff \max_{\mathfrak{P} \in X_q(R)} \left[ - \frac{\sum_{j=1}^{n} i_j \cdot v_{R,\mathfrak{P}}(R \cdot g_j)}{e(\mathfrak{P}, \mathfrak{q})} \right]^\# \neq 0
\]
holds, where \([ \cdot ]^\#\) defined in Proposition 2.1. On the other hand, the set
\[
\bigcup_{j=1}^{n} \left\{ \mathfrak{q} \in \text{Ht}_1(R \cap L) \mid \exists \mathfrak{P} \in X_q(R) \text{ such that } v_{R,\mathfrak{P}}(R \cdot g_j) \neq 0 \right\},
\]
which is denoted by supp$(g_1, \ldots, g_n : R, R \cap L)$, has only finite elements. Since

$$\left\{ q \in H_1(R \cap L) \,igg|\, v_{R \cap L, q} \left( \left( \frac{1}{\prod_{j=1}^{n} g_j^{i_j}} R \right) \cap L \right) \neq 0 \right\} \subseteq \text{supp}(g_1, \ldots, g_n : R, R \cap L)$$

for any $i_1, \ldots, i_n \in \mathbb{Z}_0$, we see that

$$D(R, R \cap L)(g_1, \ldots, g_n) \subseteq \sum_{q \in \text{supp}(g_1, \ldots, g_n : R, R \cap L)} \mathbb{Z} \cdot \text{div}_{R \cap L}(q) \subseteq \text{Div}(R \cap L).$$

The assertion follows from this. □

Suppose that $R$ is an integral domain on which a group $G$ acts as automorphisms. Let $Z^1(G, U(R))$ be the group of all 1-cocycles of $G$ on the group $U(R)$ of units of $R$ whose group structure is given by addition. The trivial 1-cocycle is denoted by $\theta \in Z^1(G, U(R))$. For any $\chi \in Z^1(G, U(R))$, let

$$R_{\chi} = \{ x \in R \mid \sigma(x) = \chi(\sigma) \cdot x \ (\forall \sigma \in G) \}$$

which is regarded as an $R^G$-module. If $\Lambda$ and $\Gamma$ are subsets of $Z^1(G, U(R))$, let $-\Lambda$ be the set $\{ -\chi \in Z^1(G, U(R)) \mid \chi \in \Lambda \}$ and put

$$\Lambda + \Gamma := \{ \chi + \psi \in Z^1(G, U(R)) \mid \chi \in \Lambda, \psi \in \Gamma \}.$$

Denote by $Z^1(G, U(R))^R$ the set $\{ \chi \in Z^1(G, U(R)) \mid R_{\chi} \neq \{0\} \}$ and put

$$Z^1_R(G, U(R))_{(2)} := \left\{ \chi \in -Z^1(G, U(R))^R \right. \mid R_{-\chi} \not\subseteq \mathfrak{P} \ (\forall \mathfrak{P} \in H_1^R(1, R^G)) \left. \right\},$$

$$\overline{Z}^1_R(G, U(R)) := Z^1_R(G, U(R))_{(2)} \cap (-Z^1_R(G, U(R))_{(2)})$$

respectively. Set

$$Z^1(G, U(R))_R := \left\{ \chi \in Z^1(G, U(R))^R \mid \dim_{Q(R^G)} Q(R^G) \otimes_{R^G} R_{\chi} = 1 \right\}.$$

For $\chi \in Z^1(G, U(R))^R$, the condition that $\dim_{Q(R^G)} Q(R^G) \otimes_{R^G} R_{\chi} = 1$ holds if and only if the equality

$$((1/f) \cdot R) \cap Q(R^G) = ((1/f) \cdot R)^G$$

holds for some (or any) nonzero $f \in R_{\chi}$ (cf. Lemma 3.1 of [15]). Moreover in the case where $R$ is a Krull domain, put

$$Z^1_R(G, U(R))_e := \left\{ \chi \in Z^1(G, U(R))^R \right. \mid \exists f_\mathfrak{P} \in R_{\chi} \setminus \{0\} \text{ such that} \left. \right. v_{R, \mathfrak{P}}(f_\mathfrak{P}) \equiv 0 \ (\text{mod} \ (\mathfrak{P}, \mathfrak{P} \cap R^G)) \ (\forall \mathfrak{P} \in H_1^R(R, R^G)) \right\}.$$

**Lemma 2.4** The set $Z^1(G, U(R))^R$ has the following properties:

(i) For $\chi_i \in Z^1(G, U(R))^R$ ($i = 1, 2$), the condition $\chi_1 + \chi_2 \in Z^1(G, U(R))^R$ implies that $\chi_i \in Z^1(G, U(R))^R$.  


(ii) $Z^1(G, U(R))_R \supseteq Z^1(G, U(R))^R \cap (-Z^1(G, U(R))^R)$.

(iii) If $Q(R^G) = Q(R)^G$, then $Z^1(G, U(R))_R = Z^1(G, U(R))^R$.

**Proof.** (i): Let $f$ be a nonzero element of $R_{\chi^2}$. Then the map $R_{\chi^1} \ni x \mapsto x \cdot f \in R_{\chi^1+\chi^2}$ is $R^G$-monomorphism, which implies the assertion.

The remainder of the assertions follows from Lemma 3.1 of [15]. □

We immediately have the following lemma whose proof is easy and omitted:

**Lemma 2.5** The set $Z^1_R(G, U(R))_{(2)}$ (resp. $\tilde{Z}^1_R(G, U(R))$) is an additive submonoid (resp. subgroup) of $Z^1(G, U(R))$ containing the group $B^1(G, U(R))$ of $1$-st coboundaries of $G$ in $U(R)$. Especially in the case where $R$ is a Krull domain, the set $Z^1_R(G, U(R))_{(2)} \cap Z^1_R(G, U(R))_e$ (resp. $\tilde{Z}^1_R(G, U(R)) \cap Z^1_R(G, U(R))_e$) is a submonoid (resp. subgroup) of $Z^1(G, U(R))$ containing the group $B^1(G, U(R))$. □

**Lemma 2.6** Let $\chi$ be a $1$-cocycle in $Z^1(G, U(R))^R$.

(i) $R_{\chi} = d_{(R^G, R)}(R_{\chi})$ if and only if $d_{(R^G, R)}(R_{\chi}) \subseteq R$.

(ii) Suppose that $\chi \in Z^1(G, U(R))_R$ and $d_{(R^G, R)}(R_{\chi}) \subseteq R$ hold. Then $R_{\chi} \cong R^G$ as $R^G$-modules if and only if the following condition is satisfied: For a nonzero element $f$ of $R_{\chi}$, the $R^G$-module $d_{(R^G, R)} \left( (1/f) \cdot R \cap Q(R^G) \right)$ is principal.

**Proof.** (i): The “only if” part follows from the fact $d_{(R^G, R)}(R_{\chi}) \subseteq (Q(R^G) \otimes_{R^G} R)_\chi$.

(ii): By the choice of $\chi$ and Lemma 3.1 of [15], we see $d_{(R^G, R)} \left( (1/f) \cdot R \cap Q(R^G) \right) = d_{(R^G, R)} \left( (1/f) \cdot R_{\chi} \right) = (1/f) \cdot d_{(R^G, R)}(R_{\chi})$, which shows the assertion. □

## 3 Krull domains with group actions

In this section suppose that $R$ is a Krull domain acted by a group $G$ as automorphisms. Let $E^a(G, R)$ denote the subgroup

$$\sum_{q \in H^1_{(R^G)}} Z \cdot \left( \sum_{p \in X_p(R)} e(p, q) \cdot \text{div}_R(p) \right) \oplus \left( \sum_{p \in H^1_{(R, R^G)}} Z \cdot \text{div}_R(p) \right)$$

of $\text{Div}(R)$. Recall that a divisor $D$ is said to be effective (i.e., $D \geq 0$), if $I_R(D) \subseteq R$.

**Definition 3.1** An effective divisor $D \in \text{Div}(R)$, which is denoted by $D \geq 0$, is said to be minimal effective relative to $(R^G, R)$, if $D$ has a decomposition $D = D_1 + D_2$ for $0 \leq D_1 \in E^a(G, R)$ and $0 \leq D_2 \in \text{Div}(R)$, then $D_1$ must be equal to zero.

For a minimal effective divisor $D$ relative to $(R^G, R)$ on $R$, we immediately see $D = 0$ if and only if $D \equiv 0 \mod E^a(G, R)$. In the following lemma we define the minimal effective divisor $D(\chi)$ relative to $(R^G, R)$ for any cocycle $\chi \in Z^1(G, U(R))_R$.

The group $G$ acts on $\text{Div}(R)$ naturally and a divisor $D$ on $R$ is said to be $G$-invariant if $D$ is invariant under the action of $G$ on $\text{Div}(R)$, i.e., $D \in \text{Div}(R)^G$. 7
Lemma 3.2 Let $\chi$ be a cocycle in $Z^1(G, U(R))_R$. Then:

(i) For any nonzero $f, g \in R$, $\text{div}_R(f) \equiv \text{div}_R(g) \mod E^*(G, R)$.

(ii) If $\chi \in Z^1_R(G, U(R))_e$, then for any nonzero $f \in R$, we have

$$v_{R, \mathfrak{P}}(f) \equiv 0 \mod \left( e(\mathfrak{P}, \mathfrak{P} \cap q) \right) \quad (\forall \mathfrak{P} \in \text{Ht}^{(1)}_1(R, R^G)).$$

(iii) There exists a unique minimal effective divisor $D(\chi)$ on $R$ relative to $(R^G, R)$ such that, for a nonzero element $f \in R$,

$$E^*(G, R) \ni \text{div}_R(f) - D(\chi) \geq 0.$$

(iv) The divisor $D(\chi)$ is $G$-invariant and does not depend on the choice of $f \in R$. Moreover the divisorial ideal $I_R(D(\chi))$ associated with $D(\chi)$ contains $R_\chi$.

(v) In the case where $\chi \in Z^1_R(G, U(R))_e$, we have

$$v_{R, \mathfrak{P}}(I_R(D(\chi))) \equiv 0 \mod \left( e(\mathfrak{P}, \mathfrak{P} \cap q) \right) \quad (\forall \mathfrak{P} \in \text{Ht}^{(1)}_1(R, R^G)).$$

Moreover if $m \chi \in Z^1(G, U(R))_R$ for some $m \in \mathbb{N}$, then the divisors $D(\chi)$ and $D(m \chi)$ satisfy $m \cdot D(\chi) = D(m \chi)$ in $\text{Div}(R)$.

(vi) For any cocycles $\chi_1, \chi_2 \in Z^1(G, U(R))^R$ such that $\chi_1 + \chi_2 \in Z^1(G, U(R))_R$, the divisors $D(\chi_1)$ and $D(\chi_2)$ (for existence of $D(\chi_i)$, see (i) of Lemma 2.4) satisfy

$$D(\chi_1 + \chi_2) \equiv D(\chi_1) + D(\chi_2) \mod E^*(G, R).$$

(vii) Suppose that $D(\chi)$ is principal, i.e., $D(\chi) = \text{div}_R(g)$ for some $g \in R$. Then there is a cocycle

$$\psi \in (\chi + (Z^1(G, U(R))^R)) \cap Z^1(G, U(R))^R$$

such that $g \in R_\psi$. This cocycle $\psi$ has the properties that $\psi \in Z^1(G, U(R))_R$, $D(\psi) = D(\chi)$ and $D(\chi - \psi) = 0$.

Proof. Since the choice of $\chi$ implies $f/g = f'/g'$ for some nonzero $f', g' \in R^G$, the assertion (i) follows easily from this. The assertion (ii) is a consequence of (i) and the definition of $Z^1_R(G, U(R))_e$.

(iii): For any nonzero $f \in R$, put

$$D_2(f) = \sum_{\mathfrak{P} \in \text{Ht}^{(0)}_1(R, R^G) \cup \text{Ht}^{(1)}_1(R, R^G)} v_{R, \mathfrak{P}}(f) \cdot \text{div}_R(\mathfrak{P})$$

$$- \sum_{q \in \text{Ht}^{(1)}_1(R^G)} \left( \min_{Q \in X_q(R)} \left[ \frac{v_{R, D(f)}}{e(Q, q)} \right] \right) \cdot \left( \sum_{\mathfrak{P} \in X_q(R)} e(\mathfrak{P}, q) \cdot \text{div}_R(\mathfrak{P}) \right) \quad (3.1)$$

and $D_1(f) = \text{div}_R(f) - D_2(f)$, where $[ \cdot , \cdot ]$ denotes the Gauss symbol. Then we see $0 \leq D_1(f) \in E^*(G, R)$ and $D_2(f)$ is a minimal effective divisor on $R$ relative to $(R^G, R)$. For a nonzero $g \in R$, define $D_1(g)$ and $D_2(g)$ similarly as above. By (i) we see that

$$D_2(f) - D_2(g) \equiv -D_1(f) + D_1(g) \mod E^*(G, R)$$

$$\equiv 0 \mod E^*(G, R) \quad (3.2)$$
Theorem 3.3 ([15])

if and only if the following conditions are satisfied:

\[ v_{R, \mathfrak{P}}(I_R(D_2(f))) - v_{R, \mathfrak{P}}(I_R(D_2(g))) = c_q \cdot e(\mathfrak{P}, q) \quad (\forall \mathfrak{P} \in \chi_q(R)) \]

and depending only on \( q \). Exchanging \( f \) with \( g \), we may suppose that \( c_q \geq 0 \). By the definition (3.1) of \( D_2(f) \), we see \( v_{R, \mathfrak{P}}(I_R(D_2(f))) < e(\mathfrak{P}, q) \) for some \( \mathfrak{P} \in \chi_q(R) \), which requires \( c_q = 0 \). Consequently \( D_2(f) \) is just the minimal effective divisor desired in the assertion (iii).

(iv): By the congruence (3.2) the divisor \( D_2(f) \) does not depend on the choice of a nonzero \( f \in R_\chi \). The remainder follows from (iii).

(v): The congruence in (v) follows from (ii) and (3.1). Choose any \( q \) from \( \text{Ht}_1(R^G) \).

Then, by the choice of \( \chi \), the assertion (ii) and the definition of \( D(\chi) \), we see that \( v_{R, \mathfrak{P}}(I_R(D(\chi))) = 0 \) for some \( \mathfrak{P} \in \chi_q(R) \), which implies \( v_{R, \mathfrak{P}}(I_R(m \cdot D(\chi))) = 0 \). From this we see that \( m \cdot D(\chi) \) is a minimal effective divisor on \( R \) relative to \( (R^G, R) \). Since

\[ E^*(G, R) \ni m \cdot (\text{div}_R(f) - D(\chi)) = \text{div}_R(f^m) - m \cdot D(\chi) \geq 0, \]

the assertion follows from the uniqueness of \( D(m\chi) \) for \( m\chi \in Z^1(G, U(R))_R \) shown in the assertion (iv).

(vi): Note that \( \chi_i \in Z^1(G, U(R)) R \) (cf. (i) of Lemma 2.4) and \( D(\chi_i) \) are well defined. The congruence in (vi) follows easily from (iii) and the independence of \( D(\chi) \) on the choice of \( f \) stated in (iv).

(vii): As \( D(\chi) \) is \( G \)-invariant, we have a cocycle \( \psi \in Z^1(G, U(R))^R \) such that \( R_\psi \ni \psi \). Then \( D(\chi) \) is a minimal effective divisor relative to \( (R^G, R) \) with the property

\[ E^*(G, R) \ni \text{div}_R(g) - D(\chi) = 0 \quad (\geq 0). \]

Since \( \{0\} \neq R_\chi \subseteq I_R(D(\chi)) = R \cdot g \) (cf. (iv)), we see that \( \chi - \psi \in Z^1(G, U(R))^R \), which implies \( \psi, \chi - \psi \in Z^1(G, U(R))_R \) (cf. (vi)). Clearly by the uniqueness of \( D(\psi) \) (cf. (iii)), \( D(\chi) = D(\psi) \). By (vi) we have

\[ D(\psi) + D(\chi - \psi) \equiv D(\chi) \text{ mod } E^*(G, R). \]

This shows \( D(\chi - \psi) = 0 \). □

**Theorem 3.3 ([15])** For \( \chi \in Z^1(G, U(R)) \), the \( R^G \)-module \( R_\chi \) is \( R^G \)-free of rank one if and only if the following conditions are satisfied:

(i) \( \dim Q(R^G) \otimes_{R^G} R_\chi = 1. \)

(ii) There exists a nonzero element \( f \in R_\chi \) satisfying

\[ \forall q \in \text{Ht}_1(R^G) \Rightarrow \exists \mathfrak{P} \in \chi_q(R) \text{ such that } v_{R, \mathfrak{P}}(f) < e(\mathfrak{P}, q) \quad (3.3) \]

If these conditions are satisfied, \( R_\chi = R^G \cdot f \) for any nonzero element \( f \in R_\chi \) such that (3.3) holds.

**Corollary 3.4** Let \( \chi \) be a cocycle in \( Z^1(G, U(R))_R \). Then \( R_\chi \) is \( R^G \)-free if and only if

\[ \text{div}_R(f) \equiv D(\chi) \text{ mod } \bigoplus_{\mathfrak{P} \in \text{Ht}^2(G, R^G)} \mathbb{Z} \cdot \text{div}_R(\mathfrak{P}) \quad (3.4) \]
for some nonzero $f \in R_{\chi}$. Especially in the case where $\chi \in -Z^1_{R}(G, U(R))_{(2)}$, $R_{\chi} \cong R^G$ as $R^G$-modules if and only if $D(\chi) = \text{div}_R(f)$ for some nonzero $f \in R_{\chi}$. Moreover $R_{\chi} = R^G \cdot f$ holds, in both cases where these conditions are satisfied.

**Proof.** Suppose that the congruence (3.4) holds for some nonzero $f \in R_{\chi}$. Then the condition (3.3) holds for $f$ and hence, by Theorem 3.3 we see that $R_{\chi} \cong R^G$ as $R^G$-modules. Conversely suppose that $R_{\chi}$ is $R^G$-free. Then by Theorem 3.3 we can choose a nonzero element from $f \in R_{\chi}$ in such a way that $f$ satisfies the condition (3.3). By the definition of $D(\chi)$ (cf. $D_2(f)$ defined by (3.1)) we must have

$$D(\chi) = \sum_{\mathfrak{P} \in \text{Ht}^1_{R,R^G}(\chi)} v_{R, \mathfrak{P}}(f) \cdot \text{div}_R(\mathfrak{P}),$$

which shows that the congruence (3.4) holds.

The remainder of the assertion follows easily from Theorem 3.3 and these observations. \(\square\)

Let coh($\chi$) denote the cohomology class of a 1-cocycle $\chi \in Z^1(G, U(R))$. This induces a homomorphism coh : $Z^1(G, U(R)) \to H^1(G, U(R))$. The zero element of the additive group $H^1(G, U(R))$ is denoted by coh($\theta$) (recall that $\theta$ is the trivial 1-cocycle in $Z^1(G, U(R))$). For a $G$-invariant principal divisor $D = \text{div}_R(h)$ for some $h \in Q(R)$, define the cohomology class coh($D$) $\in H^1(G, U(R))$ which is the class of the 1-cocycle $G \ni \sigma \to \sigma(h)/h \in U(R)$. Clearly coh($D$) does not depend on the choice of $h$.

For $\chi \in Z^1(G, U(R))$, put $N \cdot \chi := \{n\chi \in Z^1(G, U(R)) \mid n \in N\}$. Recall that tor($A$) stands for the torsion part of an abelian group $A$.

**Proposition 3.5** Let $\chi$ be a cocycle in $Z^1_{R}(G, U(R)) \cap (-Z^1_{R}(G, U(R))_{(2)})$ such that $N \cdot \chi \subseteq Z^1(G, U(R))_{(2)}$. Then the following conditions are equivalent:

(i) $D(\chi)$ is a principal divisor satisfying the conditions as follows:

(a) coh($\chi$) $\neq$ coh($D(\chi)$) $\notin$ tor($H^1(G, U(R))$)$\setminus\{\text{coh}(\theta)\}$.

(b) $R_{m\chi} \cong R^G$ as $R^G$-modules for some $m \in N$.

(ii) For any $n \in N$, $R_{n\chi} \cong R^G$ as $R^G$-modules.

(iii) $R_{\chi} \cong R^G$ as $R^G$-modules.

**Proof.** The implication (ii) $\Rightarrow$ (i) follows immediately from Corollary 3.4 (in this case, coh($\chi$) = coh($D(\chi)$)).

(i) $\Rightarrow$ (iii) : Let $f$ be a nonzero element of $R$ such that div$_R(f) = D(\chi)$ and $g$ a nonzero element of $R_{\chi}$. Then, as $R \cdot f \supseteq R \cdot g$ (cf. the inequality in (iii) of Lemma 2.2), we express $g = f \cdot h$ for some $h \in R$. Since $m\chi \in -Z^1_{R}(G, U(R))_{(2)}$ (cf. Lemma 2.5), by the choice of $\chi$, (v) of Lemma 3.2 and Corollary 3.4, we see that

$$D(m\chi) = \text{div}_R(f^m) = \text{div}_R(w)$$

for some nonzero element $w \in R_{m\chi} \cong R^G$. Here note $R_{m\chi} = R^G \cdot w$ (cf. the last statement of Corollary 3.4). So, let $u \in U(R)$ be the unit satisfying $f^m = w \cdot u$. As $R_{m\chi} \ni g^m$, we have

$$f^m \cdot h^m = g^m = w \cdot v = f^m \cdot u^{-1} \cdot v$$
for some $v \in R_G$, which requires
\[
\text{coh}(\chi) - \text{coh}(D(\chi)) = \text{coh}(\text{div}_R(h)) \in \text{tor}(H^1(G, U(R))).
\]
Since $G \ni \sigma \to \sigma(h)/h \in U(R)$ is a coboundary (cf. (a) of (i)), we see $h \cdot t \in R_G$ for a unit $t \in U(R)$, which implies $t^{-1} \cdot f \in R_\chi$. Hence the assertion of (iii) follows from Corollary 3.4.

(iii) ⇒ (ii): By Corollary 3.4, we can choose a nonzero element $f \in R_\chi$ in such a way that $D(\chi) = \text{div}_R(f)$. For any $n \in \mathbb{N}$, from Lemma 3.2 we infer that $D(n\chi) = \text{div}_R(f^n)$, which implies that $R_{n\chi} \cong R^G$ as $R^G$-modules. \(\square\)

**Corollary 3.6** Under the same circumstances as in Proposition 3.5, suppose that there exists a number $m \in \mathbb{N}$ such that $R_{m\chi} \cong R^G$ as $R^G$-modules. Then the divisor class $[D(\chi)]$ in $\text{Cl}(R)$ is a torsion element. Suppose moreover that
\[
\text{coh}(i\chi) - \text{coh}(D(i\chi)) \not\in \text{tor}(H^1(G, U(R)))\backslash\{\text{coh}(\theta)\}
\]  
for any $i \in \mathbb{N}$ such that $D(i\chi)$ is a principal divisor. Then the following equality holds;
\[
\text{ord}([D(\chi)]) = \min\{q \in \mathbb{N} \mid R_{q\chi} \cong R^G \text{ as } R^G\text{-modules}\}
\]
where $\text{ord}([D(\chi)])$ is the order of $[D(\chi)]$ in the group $\text{Cl}(R)$.

**Proof.** Since $m\chi \in -Z^1_R(G, U(R))$, by the assumption that $R_{m\chi} \cong R^G$ and Corollary 3.4, we see $D(m\chi) = \text{div}_R(g)$ for some $g \in R_{m\chi}$, which shows $m \cdot [D(\chi)] = 0$ (cf. (v) of Lemma 3.2). We similarly have
\[
\text{ord}([D(\chi)]) \leq \min\{q \in \mathbb{N} \mid R_{q\chi} \cong R^G \text{ as } R^G\text{-modules}\}.
\]
Let $k$ denote the right hand side of this inequality. As $R_{k\chi} \cong R^G$, we have an element $h \in R_{k\chi}$ satisfying $D(k\chi) = \text{div}_R(h)$ (cf. Corollary 3.4). Let $t \in \mathbb{N}$ be a common multiplier of $k$ and $\text{ord}([D(\chi)])\chi$. Put $\psi = \text{ord}([D(\chi)])\chi$. Then, by (v) of Lemma 3.2, $D(t\chi) = (t/k) \cdot D(k\chi)$. Clearly $D(t\chi) = \text{div}_R(h^{t/k})$ and $h^{t/k} \in R_{t\chi}$. This implies that $R_{t/\text{ord}([D(\chi)])\psi} \cong R^G$ as $R^G$-modules (cf. Corollary 3.4). Then from the condition (3.5) and the implication (i) ⇒ (iii) of Proposition 3.5 we infer that $R_{t\psi} \cong R^G$ as $R^G$-modules, which completes the proof. \(\square\)

**Remark 3.7** Define
\[
Z^1(G, U(R))_{R,0} := \{\delta \in Z^1(G, U(R))_R \mid D(\delta) = 0\}.
\]
For $\chi \in Z^1(G, U(R))_R$ such that $D(i\chi)$ is principal, we have a cocycle $\psi \in Z^1(G, U(R))^R$ satisfying $D(i\chi) = \text{div}_R(g) = \text{div}(\psi)$ for some $g \in R_\psi$ and $D(i\chi - \psi) = 0$ (cf. (vii) of Lemma 3.2). Since
\[
\text{coh}(i\chi) - \text{coh}(D(i\chi)) = \text{coh}(i\chi) - \text{coh}(D(\psi)) = \text{coh}(i\chi - \psi),
\]
the condition (3.5) holds if the following equality holds:
\[
\{\text{coh}(\delta) \mid \delta \in (\mathbb{N} \cdot \chi + (-Z^1(G, U(R))^R)) \cap Z^1(G, U(R))_{R,0}\} \cap \text{tor}(H^1(G, U(R))) = \{\text{coh}(\theta)\}.
\]  
(3.6)
In general we denote by $H^1(G, U(R))_{R,0}$ the image of $Z^1(G, U(R))_{R,0}$ under the canonical map $\text{coh} : Z^1(G, U(R)) \to H^1(G, U(R))$. Clearly $B^1(G, U(R)) \subseteq Z^1(G, U(R))_{R,0}$. If the equality
\[ H^1(G, U(R))_{R,0} \cap \text{tor}(H^1(G, U(R))) = \{ \text{coh}(\theta) \} \]
holds, then (3.6) is satisfied.

**Lemma 3.8** For a cocycle $\chi \in Z^1(G, U(R))_R$, the $R^G$-module $R_\chi$ is $R^G$-isomorphic to an integral ideal $I$ of $R^G$, which is unique up to a multiplication of a non-zero element of $Q(R^G)$. Moreover we have an $R^G$-isomorphism $d_{(R^G, R)}(R_\chi) \to d_{R^G}(I)$ whose restriction induces $R_\chi \cong I$ and the divisor class $|d_{R^G}(I)|$ of $d_{R^G}(I)$ in $\text{Cl}(R^G)$ is uniquely determined by $\chi$.

**Proof.** For a nonzero $f \in R_\chi$ we see $R_\chi \cong (1/f)R \cap Q(R^G)$ as $R^G$-modules (cf. Lemma 3.1 of [15]) and by Proposition 2.1 see that the $R^G$-module $(1/f)R \cap Q(R^G)$ is a fractional ideal of $R^G$. The assertion follows immediately from this. \[ \square \]

**Definition 3.9** In the circumstances as in Lemma 3.8 we denote by $[R_\chi]$ the divisor class $|d_{R^G}(I)| \in \text{Cl}(R^G)$ of $d_{R^G}(I)$ for $\chi \in Z^1(G, U(R))_R$, where the ideal $I$ of $R^G$ is defined in Lemma 3.8.

**Proposition 3.10** There is a canonical embedding $d_{(R^G,R)}(R_\chi) \subseteq R$ for any cocycle $\chi \in Z^1(G, U(R))_R \cap Z^1(G, U(R))_{R,2}$.

**Proof.** Let $\chi$ be a cocycle in $Z^1(G, U(R))_{(2)}$. Then, for any $\mathfrak{P} \in \text{Ht}_1(R)$ such that $ht(\mathfrak{P} \cap R^G) \geq 2$, we can choose a nonzero element $g$ from $R_{-\chi}$ in such a way that $g \notin \mathfrak{P}$. Since $(R^G)_{\mathfrak{P} \cap R^G}$ is a Krull domain, we see
\[
(R^G)_{\mathfrak{P} \cap R^G} = \bigcap_{q \in \text{Ht}_1(R^G) \cap \mathfrak{P}} (R^G)_q \supseteq \bigcap_{q \in \text{Ht}_1(R^G) \cap \mathfrak{P}} (R^G)_q \otimes_{R^G} (g \cdot R_\chi) \supseteq g \cdot d_{(R^G,R)}(R_\chi).
\]
Thus by the choice of $g$, one sees $g \cdot R_\mathfrak{P} = R_\mathfrak{P} \supseteq (R^G)_{\mathfrak{P} \cap R^G} \supseteq g \cdot d_{(R^G,R)}(R_\chi)$, which requires $R_\mathfrak{P} \supseteq d_{(R^G,R)}(R_\chi)$ for any $\mathfrak{P} \in \text{Ht}_1(R)$ such that $ht(\mathfrak{P} \cap R^G) \geq 2$.

In general, we have
\[
R_\mathfrak{P} \supseteq R_\mathfrak{P} \otimes_R (R \cdot R_\chi) \supseteq (R \setminus \mathfrak{P})^{-1} R_\chi \left( = \left\{ \frac{a}{b} \mid a \in R_\chi, \ b \in R \setminus \mathfrak{P} \right\} \right) \supseteq R^G_{\mathfrak{P} \cap R^G} \otimes_{R^G} R_\chi
\]
for a prime ideal $\mathfrak{P}$ of $R$. Here we identify $R^G_{\mathfrak{P} \cap R^G} \otimes_{R^G} R_\chi$ (resp. $R_\mathfrak{P} \otimes_R (R \cdot R_\chi)$) with $(R^G \setminus (\mathfrak{P} \cap R^G))^{-1} R_\chi$ (resp. $(R \setminus \mathfrak{P})^{-1} R \cdot R_\chi$) in $Q(R)$. Since $X_q(R) \neq \emptyset$ for all $q \in \text{Ht}_1(R^G)$, by (3.7) we see that
\[
\bigcap_{q \in \text{Ht}_1(R^G)} ((R^G)_q \otimes_{R^G} R_\chi) = \bigcap_{q \in \text{Ht}_1(R^G)} ((R^G)_{q \cap R^G} \otimes_{R^G} R_\chi) = \bigcap_{q \in \text{Ht}_1(R^G)} ((R^G)_{q \cap R^G} \otimes_{R^G} R_\chi) \cap \bigcap_{q \in \text{Ht}_1(R^G)} ((R^G)_{q \cap R^G} \otimes_{R^G} R_\chi) \\
\subseteq \bigcap_{q \in \text{Ht}_1(R) \setminus \text{Ht}_1^2(R^G)} (R_\mathfrak{P} \otimes_R R).
\]
These localizations are regarded as modules of fractions in \(\mathbb{Q}(R)\) and intersections are defined in \(\mathbb{Q}(R)\). Thus we must have
\[
d_{(R^G, R)}(R_\chi) = \bigcap_{q \in H_1(R^G)} (R^q \otimes_{R^G} R_\chi) \subseteq \bigcap_{q \in H_1(R)} (R_q \otimes_R R) = R
\]
because \(R\) is a Krull domain. \(\Box\)

**Corollary 3.11** Let \(\chi\) be a cocycle in \(Z^1_R(G, U(R))\). Then, for a natural number \(n\), \(R_n\chi \cong R^G\) as \(R^G\)-modules if and only if \(n \cdot [R_\chi] = 0\) in \(\text{Cl}(R^G)\).

**Proof.** By the choice of \(\chi\), we see
\[
N \cdot \chi \subseteq Z^1(G, U(R))^R \cap (-Z^1(G, U(R))^R) \subseteq Z^1(G, U(R))_R.
\]
Let \(f\) be a nonzero element of \(R_\chi\). Applying Corollary 2.2 to the ideal \(R \cdot (1/f)\) with the aid of (ii) of Lemma 3.2, we see
\[
\text{div}_{R^G}\left(R \cdot \frac{1}{f} \cap \mathbb{Q}(R^G)\right) = n \cdot \text{div}_{R^G}\left(R \cdot \frac{1}{f} \cap \mathbb{Q}(R^G)\right)
\]
for any \(n \in N\) (cf. Lemma 3.1 of [15]). From Lemma 3.8, the identity (3.8) can be deduced to \([R_n\chi] = n \cdot [R_\chi]\) in \(\text{Cl}(R^G)\). By Lemma 2.5, Lemma 2.6 and Proposition 3.10, we immediately see that \(R_n\chi \cong R^G\) if and only if \([R_n\chi] = 0\) in \(\text{Cl}(R^G)\), which implies the assertion. \(\Box\)

Combining Corollary 3.6 with Corollary 3.11, we immediately have

**Theorem 3.12** Let \(\chi\) be a cocycle in \(Z^1_R(G, U(R))\). Suppose that the equality (3.5) holds for any \(i \in N\) such that \(D(i\chi)\) is a principal divisor. If \([R_\chi] \in \text{tor}(\text{Cl}(R^G))\), then
\[
\text{ord}([R_\chi]) in \text{Cl}(R^G) = \text{ord}([D(\chi)]) in \text{Cl}(R) (< \infty)
\]
which is equal to \(\min \{q \in N \mid R_q \chi \cong R^G\ as \ R^G\text{-modules}\}\). \(\Box\)

**Definition 3.13** Let \(\text{UrCl}(R, G)\) denote the subgroup of \(\text{Cl}(R)\) generated by
\[
\left\{[D(\chi)] \mid \chi \in Z^1_R(G, U(R)) \cap \tilde{Z}^1_R(G, U(R))\right\}
\]
where \([D(\chi)]\) denotes the divisor class of \(D(\chi) \in \text{Div}(R)\). Define \(\tilde{\text{Cl}}(R, G)\) to be the subgroup
\[
\left\{[R_\chi] \mid \chi \in Z^1_R(G, U(R)) \cap \tilde{Z}^1_R(G, U(R))\right\}
\]
of \(\text{Cl}(R^G)\).

Recall that \(\exp(N)\) denotes the *exponent* of a group \(N\). The next result follows from Remark 3.7 and Theorem 3.12.
Proof. Let $\chi$ be a cocycle in $Z^1(G, U(R))_R \cap \tilde{Z}^1_R(G, U(R))$ and $i$ a natural number such that $D(i\chi) = \text{div}_R(g)$ for a nonzero $g \in R$. Let $\psi$ be a cocycle of $G$ in $U(R)$ defined by $R_\psi \ni g$. Then $\psi \in Z^1_R(G, U(R))_R$, $D(\psi) = D(i\chi)$, $R_\psi = R^{G^i} \cdot g$, $i\chi - \psi \in Z^1_R(G, U(R))_R$ and $D(i\chi - \psi) = 0$ (cf. (vii) of Lemma 3.2). Since $R_{i\chi} = R_{i\chi - \psi} \cdot g$ (cf. (iv) of Lemma 3.2), $R_{i\chi} \cdot g \subseteq R_{i\chi + \psi}$ and $i\chi \in Z^1_R(G, U(R))_e \cap \tilde{Z}^1_R(G, U(R))$, by (v) of Lemma 3.2 and (3.1) we have

$$i\chi - \psi \in Z^1_R(G, U(R))_e \cap \tilde{Z}^1_R(G, U(R)).$$

Thus the condition (3.5) holds for $i\chi$ and Theorem 3.12 can be applied to $\chi$. By our assumption, $\text{UrCl}(R, G)$ and $\tilde{\text{Cl}}(R, G)$ are torsion groups. The exponent of a torsion abelian group generated by a subset $\Xi$ is determined by the ideal $\cap_{a \in \Xi} \text{ord}(a)\mathbb{Z}$ of $\mathbb{Z}$. The assertion follows from this observation and Theorem 3.12. $\Box$

Theorem 3.15 Suppose that

$$\text{coh}(Z^1(G, U(R))_R \cap \tilde{Z}^1_R(G, U(R))) \subseteq \sum_{\lambda \in \Lambda} \mathbb{Z}_0 \cdot \text{coh}(\lambda) \quad (3.9)$$

for a non-empty finite subset $\Lambda$ of $Z^1(G, U(R))^R$. Suppose that $\tilde{\text{Cl}}(R, G)$ is a torsion group. Then it is a finite group. Moreover if

$$\text{coh}(Z^1(G, U(R))_R \cap \tilde{Z}^1_R(G, U(R))) \subseteq \sum_{\lambda \in \Lambda} \mathbb{Z}_0 \cdot \text{coh}(\lambda)$$

in $H^1(G, U(R))$ does not contain a non-trivial torsion element, we have

$$\exp(\text{UrCl}(R, G)) = \exp(\tilde{\text{Cl}}(R, G)) < \infty.$$

Proof. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ and choose a nonzero $g_i$ from $R_{\lambda_i}$ for any $1 \leq i \leq n$. By (3.9), for $\chi \in Z^1_R(G, U(R))_e \cap \tilde{Z}^1_R(G, U(R))$, we can choose $j_i \in \mathbb{Z}_0$ and $u \in U(R)$ in such a way that $\chi = \sum_{i=1}^n j_i \lambda_i$ is a 1-coboundary of $G$ defined by $u^{-1}$. Then

$$d_{(R^G, R)}(R_\chi) \equiv d_{(R^G, R)} \left( \frac{u}{\prod g_i^{j_i}} \cdot R \cap Q(R^G) \right) = d_{(R^G, R)} \left( \frac{1}{\prod g_i^{j_i}} \cdot R \cap Q(R^G) \right)$$

as $R^G$-modules (cf. Lemma 3.1 of [15]). Consequently $\tilde{\text{Cl}}(R, G)$ is a subgroup of the image of $D_{(R, R^G)}(g_1, \ldots, g_n)$ (for definition, cf. Lemma 2.3) under the canonical homomorphism $\text{Div}(R^G) \to \text{Cl}(R^G)$. By Lemma 2.3 the group $\tilde{\text{Cl}}(R, G)$ is finitely generated. The remainder of the assertion of this theorem follows from Proposition 3.14. $\Box$
4 Affine normal varieties with group actions

Hereafter let $G$ be an affine algebraic group over an algebraically closed field $K$ of characteristic $p \geq 0$. Let $(X, G)$ be a regular action of $G$ on an affine normal variety $X$ defined over $K$. Let $\mathfrak{X}(G)$ be the additive group of all rational characters of $G$ and for a rational $G$-module $W$ and $\chi \in \mathfrak{X}(G)$, denote by $W_\chi$ the subspace $\{ x \in W \mid \sigma(x) = \chi(\sigma)x \ (\forall \sigma \in G) \}$ of $W$ consisting of all vectors of weight $\chi$. Put $\mathfrak{X}(G)^W := \{ \chi \in \mathfrak{X}(G) \mid W_\chi \neq \{0\} \}$. The cocycles of $G$ on $U(O(X))$ defined by regular functions are identified with rational characters: If $G$ is connected, by a result of M. Rosenlicht we see $Z^1(G, U(O(X)))^{O(X)} = \mathfrak{X}(G)^{O(X)}$ (e.g., [2, 7]), since $X$ is normal.

For any $\mathfrak{P} \in \text{Ht}^1(O(X), O(X)^G)$, put

$$I_G(\mathfrak{P}) := \{ \sigma \in G \mid \sigma(x) \equiv x \mod \mathfrak{P} \ (\forall x \in O(X)) \}$$

which is called the inertia group at $\mathfrak{P}$ under the action of $G$. Clearly $I_G(\mathfrak{P})$ contains the ineffective kernel Ker$$(G \to \text{Aut}(X))$$ of the action $(X, G)$ which is denoted by $L(X, G)$.

**Definition 4.1** The pseudo-reflection group of the action $(X, G)$, denoted by $\mathfrak{R}(X, G)$, is defined to be the subgroup of $G$ generated by $I_G(\mathfrak{P})$’s for all $\mathfrak{P} \in \text{Ht}^1(O(X), O(X)^G)$. Let $\mathfrak{R}(X, G)$ be the subgroup of $G$ generated by $L(X, G)$ and $I_G(\mathfrak{P})$’s for all $\mathfrak{P} \in \text{Ht}^1(O(X), O(X)^G)$ such that $\mathfrak{P}$ are non-principal. This is called the non-principal pseudo-reflection subgroup of the action $(X, G)$.

Obviously $G \triangleright \mathfrak{R}(X, G) \triangleright \mathfrak{R}(X, G) \triangleright L(X, G)$. By Sect. 1 of [15] the group $\mathfrak{R}(X, G)|_X$ $(\equiv \mathfrak{R}(X, G)/L(X, G)$) of restrictions to $X$ is finite under the assumption that $G^0$ is linearly reductive. In the case where $O(X)^G$ is finitely generated over $K$, let $X/G$ denote the affine variety defined by $O(X)^G$ and $\pi_{X, G} : X \to X/G$ denote the quotient morphism defined by the inclusion $O(X)^G \to O(X)$.

**Definition 4.2** The action $(X, G)$ admitting its quotient $X/G$, i.e., $O(X)^G$ is finitely generated over $K$, is defined to be cofree (resp. equidimensional), if $O(X)$ is a free $O(X)^G$-module (resp. $\pi_{X, G} : X \to X/G$ is equidimensional). Especially in the case where $G$ is linearly reductive, $(X, G)$ is defined to be isobaric cofree if the $\psi$-isotypical component $O(X)_{[\psi]} := \sum_{V \subseteq O(X), V \cong \psi} V \subseteq O(X)$ of the rational $G$-module $O(X)$ is zero or $O(X)^G$-free, for any irreducible representation $\psi$ of $G$.

For a conical action $(X, G)$ of a conical $X$ with a linearly reductive $G$, $(X, G)$ is cofree if and only if it is isobaric cofree. Here $X$ is said to be conical, if the coordinate ring $O(X)$ is a $\mathbb{Z}_Q$-graded algebra defined over $O(X)_0 = K$ and $(X, G)$ is said to be conical, moreover if the action of $G$ preserves each homogeneous part of $O(X)$. We immediately have

**Lemma 4.3** Suppose that $O(X)^G$ is finitely generated over $K$. If $O(X)^G \to O(X)$ is no-blowing-up of codimension one (e.g., PDE in p. 30 of [4]) (especially if $(X, G)$ is equidimensional), then $Z^1(G, U(R))^R \cap (-Z^1(G, U(R))_R = \tilde{Z}_R^1(G, U(R))$. □

**Proposition 4.4** Suppose that $G^0$ is linearly reductive. Let $N$ be a normal closed subgroup of $G$ such that $N|_X$ is finite. Then:
Proposition 4.5

(i) \( \mathfrak{R}(X//N,G) = N \cdot \mathfrak{R}(X,G) \) and \( N \triangleleft \mathfrak{R}(X//N,G) \triangleleft N \cdot \mathfrak{R}(X,G) \).

(ii) If both \( X \) and \( (X,G) \) are conical and the order \( |(N \cdot \mathfrak{R}(X,G))/\mathfrak{R}(X//N,G)| \) is a unit in \( K \), then the natural action \( (X//\mathfrak{R}(X//N,G),N \cdot \mathfrak{R}(X,G)) \) is cofree.

**Proof.** (i): Since \( N \) is normal in \( G \), we immediately see \( \mathfrak{R}(X//N,G) \cong \mathfrak{R}(X,G) \).

For any \( p \in \mathcal{H}_1(\mathcal{O}(X)^N,\mathcal{O}(X)^G) \) let \( \mathfrak{p} \) be a prime ideal of \( \mathcal{O}(X) \) lying over \( p \). Put \( H = \mathcal{I}_G(\mathfrak{p}) \). Clearly \( H \supseteq N \), \( \mathcal{I}_H(\mathfrak{p}) = \mathcal{I}_G(\mathfrak{p}) \) and \( H|_X \) is finite (cf. Sect. 1 of [14]).

From Exposé V of [5] we infer that the canonical morphisms

\[
\mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})} \rightarrow \mathcal{O}(X)^{\mathcal{I}_H(\mathfrak{p})}
\]

\[
\mathcal{O}(X)^H \rightarrow \mathcal{O}(X)^{\mathcal{I}_H(\mathfrak{p})}
\]

are étale at \( \mathfrak{p} \cap \mathcal{O}(X)^{\mathcal{I}_H(\mathfrak{p})} \). Consequently the monomorphism

\[
\mathcal{O}(X)^H \rightarrow \mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})}
\]

is unramified at \( \mathfrak{p} \cap \mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})} \). If \( \sigma \in H \), we see \( \sigma \cdot \mathcal{I}_H(\mathfrak{p}) \cdot (\sigma^{-1} = \mathcal{I}_H(\sigma(\mathfrak{p})) = \mathcal{I}_H(\tau(\mathfrak{p}))) = \mathcal{I}_H(\tau(\mathfrak{p})) \) for some \( \tau \in N \). Thus \( N \cdot \mathcal{I}_H(\mathfrak{p}) \) is a normal subgroup of \( H \) stabilizing \( \mathfrak{p} \cap \mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})} \). On the other hand, by the definition of \( H \) and Sect. 41 of [10], the residue class field of \( (\mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})})_{(p)} \cap \mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})} \) is purely inseparable over that of \( (\mathcal{O}(X)^{H})_{(p)} \cap \mathcal{O}(X)^{H} \), which requires that these fields coincide. Applying Nakayama Lemma to the finite unramified local morphism

\[
(\mathcal{O}(X)^H)_{(p)} \cap \mathcal{O}(X)^{H} \rightarrow (\mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})})_{(p)} \cap \mathcal{O}(X)^{N,\mathcal{I}_H(\mathfrak{p})},
\]

we must have \( H = N \cdot \mathcal{I}_H(\mathfrak{p}) = N \cdot \mathcal{I}_G(\mathfrak{p}) \), which shows the assertion.

(ii): Let \( p \in \mathcal{H}_1(\mathcal{O}(X)^N,\mathcal{O}(X)^G) \) be a principal ideal such that \( \mathcal{I}_G(\mathfrak{p})|_X \not\cong N \) is non-trivial. Then there exists a *homogeneous* element \( f_1 \) generating principally \( p \). Let \( \{Kf_1,Kf_2,\ldots,Kf_s\} \) be the \( N \cdot \mathfrak{R}(X,G) \)-orbit of \( Kf_1 \) consisting of \( s \) -subspaces of \( \mathcal{O}(X)^N \). Let \( q \) be a non-principal ideal in \( \mathcal{H}_1(\mathcal{O}(X)^N,\mathcal{O}(X)^G) \). For any \( \tau \in \mathcal{I}_G(\mathfrak{q}) \), we easily see that \( \tau \left( \prod_{i=1}^{s} f_i \right) = \prod_{i=1}^{s} f_i \). Thus \( p \cap \mathcal{O}(X)^{\mathfrak{R}(X//N,G)} = \mathcal{O}(X)^{\mathfrak{R}(X//N,G)} \prod_{i=1}^{s} f_i \) and

\[
(\sigma - 1)(\mathcal{O}(X)^{\mathfrak{R}(X//N,G)}) \subseteq \mathcal{O}(X)^{\mathfrak{R}(X//N,G)} \prod_{i=1}^{s} f_i
\]

for any \( \sigma \in \mathcal{I}_G(\mathfrak{p}) \). Since \( N \cdot \mathfrak{R}(X,G)/\mathfrak{R}(X//N,G) \) is generated by generalized reflections in \( \operatorname{Aut}(\mathcal{O}(X)^{\mathfrak{R}(X//N,G)}) \) in the sense of M. Hochster and J. A. Eagon [6], the \( \mathcal{O}(X)^{N \cdot \mathfrak{R}(X,G)} \)-module \( \mathcal{O}(X)^{\mathfrak{R}(X//N,G)} \) is free (e.g., Chapitre 5 of [3]). □

**Proposition 4.5** Suppose that \( \operatorname{char}(K) = p = 0 \), \( G^0 \) is an algebraic torus, the action \( (X,G) \) is stable and \( G \) equals to the centralizer \( Z_G(G^0) \) of \( G^0 \) in \( G \). Then

\[
\mathbb{X}(G) \cap Z_G^0(X,G) = \{ \chi \in \mathbb{X}(G) \mid \mathcal{O}(X) \chi \neq \{0\}, \chi(\mathfrak{R}(X,G)) = \{1\} \}.
\]
Proof. Let $\Psi \in H^1_G(\mathcal{O}(X), \mathcal{O}(X)^G)$ and $\chi \in \mathfrak{X}(G) \cap Z^1(G, U(\mathcal{O}(X)))^{G(X)}$. Suppose that there exists a nonzero $f_\Psi \in R_X$ such that $v_{\mathcal{O}(X),\Psi}(f_\Psi)$ is divisible by the ramification index $e(\mathcal{O}, \mathcal{O}(X)^G)$. Then, as $\mathcal{O}(X)^G/\mathcal{O}(X)^G$ is a discrete valuation ring, $f_\Psi = g \cdot w$ for some $g \in U(\mathcal{O}(X)_\Psi)$ and $w \in \mathcal{O}(X)^G/\mathcal{O}(X)^G$. For any $\tau \in I_G(\Psi)$, we have $\chi(\tau)g \cdot w = \tau(g) \cdot w$ and hence $\chi(\tau) \equiv 1 \pmod{\Psi}$, which induces $\chi(\tau) = 1$.

Conversely suppose that $\chi(I_G(\Psi)) = \{1\}$. For any nonzero $f \in R_X$, we see

$$v_{\mathcal{O}(X),\Psi}(f) = e(\mathcal{O}, \mathcal{O}(X)^G_G) \cdot v_{\mathcal{O}(X),\Psi}(\tau_\Psi \cdot \Psi \cap \mathcal{O}(X)^G_G)$$

as $e(\mathcal{O}, \mathcal{O}(X)^G_G) = e(\mathcal{O}, \mathcal{O}(X)^G_G)$ (cf. [14]). This shows the assertion. □

For any $m \in \mathcal{N}$, put

$$\text{tor}(m, G, H) := \{\sigma \in G \mid \sigma^m \in H\}$$

for a closed subgroup $H$ of $G$. When $G \triangleright H$ and $G/H$ is abelian, $\text{tor}(m, G, H)$ is a normal closed subgroup of $G$ containing $H$. Suppose that $G$ is connected. Put

$$\mathfrak{X}(G)_{\mathcal{O}(X)} := \{\chi \in \mathfrak{X}(G) \mid \mathcal{O}(X)_\chi \cdot \mathcal{O}(X)_{\neg \chi} \neq \{0\}\}$$

and define $\mathcal{K}(X, G) := \bigcap_{\chi \in \mathfrak{X}(G)_{\mathcal{O}(X)}} \text{Ker}(\chi)$. Then

$$\mathcal{O}(X)^{\mathcal{K}(X, G)} = K[\mathcal{O}(X)_{\chi} \mid \chi \in \mathfrak{X}(G)_{\mathcal{O}(X)}] = \bigoplus_{\chi \in \mathfrak{X}(G)_{\mathcal{O}(X)}} \mathcal{O}(X)_{\chi}$$

and if this $K$-algebra is finitely generated, the induced action $(X \mathcal{K}(X, G), G)$ is stable (cf. [12]). In the case where $G$ is an algebraic torus, $(X, G)$ is stable if and only if $X = X/\mathcal{K}(X, G)$.

Let $G \times \mathfrak{X}(G) \rightarrow K^*$ denote the canonical pairing, where $K^*$ denotes $U(K)$. The orthogonals set operation $\perp_G$ is defined naturally by this pairing, i.e., $Y^{\perp_G} := \bigcap_{\psi \in Y} \text{Ker}(\psi)$ for a subset $Y \subseteq \mathfrak{X}(G)$ and $N^{\perp_G} := \{\psi \in \mathfrak{X}(G) \mid \psi(N) = \{1\}\}$ for a subset $N \subseteq G$.

**Lemma 4.6** Suppose that $\text{char}(K) = p = 0$ and $G$ is a connected algebraic group. Let $\Gamma$ be a closed normal subgroup of $G$ containing $\mathcal{K}(X, G)$. Then:

(i) $\Gamma^{\perp_G} = \mathfrak{X}(G)_{\mathcal{O}(X)^G} \cap (\mathfrak{X}(G)_{\mathcal{O}(X)^R})^{\perp_G} = \Gamma$.

(ii) For any $m \in \mathcal{N}$, $\text{tor}(m, G, \Gamma) = m \cdot \mathfrak{X}(G)_{\mathcal{O}(X)^G} \subseteq \mathfrak{X}(G)$.

Especially, putting $\Omega(X, \Gamma) := \widetilde{Z}^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X)))^{\perp_G}$, we have

$$\widetilde{Z}^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))) \cong \widetilde{Z}^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))^{\Omega(X, \Gamma)}) = \mathfrak{X}(G)_{\mathcal{O}(X)^{\Omega(X, \Gamma)}}.$$

**Proof.** Since the canonical pairing

$$G/(\bigcap_{\psi \in \mathfrak{X}(G)} \text{Ker}(\psi)) \times \mathfrak{X}(G/(\bigcap_{\psi \in \mathfrak{X}(G)} \text{Ker}(\psi))) \rightarrow K^*$$

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of a diagonalizable group is non-degenerate, \((\Gamma \perp G) \perp G = \Gamma\) and \(((\mathfrak{X}(G_{O(X)}) \perp G = \mathfrak{X}(G_{O(X)}) \supseteq \Gamma \perp G).\) Especially

\[
K(X,G) \perp G = (((\mathfrak{X}(G_{O(X)}) \perp G) = \mathfrak{X}(G_{O(X)}) \supseteq \Gamma \perp G)
\]

and hence for \(\psi \in \Gamma \perp G\) we have \(\{0\} \neq O(X)_{\perp \psi} \subseteq O(X)^G\), which shows \(\Gamma \perp G \subseteq \mathfrak{X}(G_{O(X)})\). The converse inclusion is obvious. Thus the assertion in \((i)\) following.

\((ii)\) : Consider the canonical non-degenerate pairing \(G/\Gamma \times \mathfrak{X}(G/\Gamma) \to K^*\). Then by the definition of \(\text{tor}(m,G,\Gamma)\), we have

\[
(\text{tor}(m,G,\Gamma)/\Gamma) \perp G/\Gamma = \text{tor}(m,G/\Gamma,\{\Gamma\}) \perp G/\Gamma = m \cdot \mathfrak{X}(G/\Gamma)
\]

(e.g., Chap. 3 of [17]). This implies \(\text{tor}(m,G,\Gamma) \perp G = m \cdot \Gamma \perp G\) and the assertion follows from \((i)\).

Let \(\chi \in \tilde{Z}^1_{O(X)}(G,U(O(X)))\) be a cocycle. For \(\Omega \in \text{Ht}_1^[2](O(X)^U(G),O(X)^G)\), we have \(\mathfrak{P} \in X_{\Omega}(O(X))\) and, by the choice of \(\chi)\,

\[
O(X)_{\perp \chi} \in O(X)_{\perp \chi} \cap O(X)^U(G) \not\subseteq \mathfrak{P} \cap O(X)^U(G) = \Omega,
\]

which implies that

\[
\tilde{Z}^1_{O(X)}(G,U(O(X))) \subseteq \tilde{Z}^1_{O(X)^U(G)}(G,U(O(X)^U(G))) \subseteq \mathfrak{X}(O(X)^U(G)).
\]

Here \(Z^1(G,U(O(X)^U(G)))\) is naturally regarded as a subgroup of \(Z^1(G,U(O(X)))\). The last assertion can be shown by applying \((i)\) to \(\Gamma = \Omega_{(X,G)}\).

5 Equidimensional toric actions

Denote by \(\Delta_{(X,G)}\) the group \(\left(\tilde{Z}^1_{O(X)}(G,U(O(X))) \cap Z^1_{O(X)}(G,U(O(X)))\right) \perp G\) for a connected algebraic group \(G\) (for \(\perp G\), see the paragraph preceding Lemma 4.6). In the case where \(O(X)^{\Delta_{(X,G)}}\) is finitely generated, the natural action \((X/\Delta_{(X,G)},G)\) is stable.

**Proposition 5.1** Suppose that \(\text{char}(K) = p = 0\) and \(G\) is a connected algebraic group such that \(O(X)^{R_u(G)}\) is finitely generated over \(K\) as a \(K\)-algebra, where \(R_u(G)\) denote the unipotent radical of \(G\). Then \(\mathcal{C}(O(X),G)\) is finitely generated.

**Proof.** By our assumption the \(K\)-algebra \(O(X)^{\Delta_{(X,G)}}\) is finitely generated (cf. [9]), as \(G/R_u(G)\) is reductive. Since \(\Delta_{(X,G)} \supseteq K(X,G)\), we see that

\[
\mathfrak{X}(G)^{O(X)^{\Delta_{(X,G)}}} = \mathfrak{X}(G)^{O(X)^{\Delta_{(X,G)}}} = \Delta_{(X,G)} \perp G
\]

\[
= \tilde{Z}^1_{O(X)}(G,U(O(X))) \cap Z^1_{O(X)}(G,U(O(X))) \in \mathfrak{X}(O(X)^{\Delta_{(X,G)}})
\]

(cf. Lemma 4.6). Since \(O(X)^{\Delta_{(X,G)}}\) is generated by a finite set of relative invariants of \(G\) as a \(K\)-algebra, there is a finite subset \(\Lambda\) of \(\mathfrak{X}(G)^{O(X)^{\Delta_{(X,G)}}}\) satisfying

\[
\mathfrak{X}(G)^{O(X)^{\Delta_{(X,G)}}} \subseteq \sum_{\lambda \in \Lambda} \mathbb{Z} \cdot \lambda,
\]

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which shows that the condition (3.9) holds for $\mathcal{O}(X) = R$. By the proof of Theorem 3.15 the reduced class group $\text{Cl}(\mathcal{O}(X), G)$ is finitely generated. $\Box$

**Theorem 5.2** Under the same circumstances as in Proposition 5.1, the following four conditions are equivalent:

(i) The action $(X//\text{tor}(v, G, \Delta_{(X,G)}), G/\text{tor}(v, G, \Delta_{(X,G)}))$ is isobaric cofree for some $v \in \mathbb{N}$.

(ii) $\widetilde{\text{Cl}}(\mathcal{O}(X), G)$ is a finite group.

(iii) $\exp(\widetilde{\text{Cl}}(\mathcal{O}(X), G)) < \infty$.

(iv) $v := \exp(\widetilde{\text{Cl}}(\mathcal{O}(X), G)) < \infty$ and $(X//\text{tor}(v, G, \Delta_{(X,G)}), G/\text{tor}(v, G, \Delta_{(X,G)}))$ is isobaric cofree.

In the case where $U(\mathcal{O}(X)) = K^*$, these conditions are equivalent to

(v) $\exp(U(\mathcal{O}(X), G)) = \exp(\widetilde{\text{Cl}}(\mathcal{O}(X), G)) < \infty$.

Especially if both $X$ and $(X, G)$ are conical, these conditions are equivalent to

(vi) The natural action $(X//\Delta_{(X,G)}, G)$ is equidimensional.

**Proof.** The equivalence (ii) $\iff$ (iii) follows from Proposition 5.1. For a natural number $v$, by Lemma 4.6 we have

$$\mathcal{O}(X//\text{tor}(v, G, \Delta_{(X,G)})) = \bigoplus_{\chi \in \text{tor}(v, G, \Delta_{(X,G)})^{G}} \mathcal{O}(X)_\chi$$

$$= \bigoplus_{\chi \in v^{-1}\mathcal{O}(X)/\Delta_{(X,G)}} \mathcal{O}(X)_\chi$$

$$= \bigoplus_{\psi \in (\Delta_{(X,G)})^{G}} \mathcal{O}(X)_{v\psi}. \quad (5.2)$$

(iii) $\Rightarrow$ (i): Put $v = \exp(\widetilde{\text{Cl}}(\mathcal{O}(X), G))$ and let $\chi$ be a cocycle in $\text{tor}(v, G, \Delta_{(X,G)})^{G}$. Then there is a cocycle $\psi \in Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))) \cap Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X)))_e$ satisfying $v\psi = \chi$ (cf. (5.1) and Lemma 4.6). By the definition of $\widetilde{\text{Cl}}(\mathcal{O}(X), G)$, the divisor class $v \cdot [\mathcal{O}(X)_{\psi}] \in \text{Cl}(\mathcal{O}(X)^G)$ equals to zero. Applying Corollary 3.11 to $\chi$, we see that $\mathcal{O}(X)_\chi \cong \mathcal{O}(X)^G$ as $\mathcal{O}(X)^G$-modules. Thus $(X//\text{tor}(v, G, \Delta_{(X,G)}), G/\text{tor}(v, G, \Delta_{(X,G)}))$ is isobaric cofree (cf. (5.2)). The implication (iii) $\Rightarrow$ (iv) follows from this.

(i) $\Rightarrow$ (iii): Let $\psi \in Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))) \cap Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X)))_e$ be any cocycle. By Lemma 4.6 we have $v\psi \in \text{tor}(v, G, \Delta_{(X,G)})^{G}$ and see that $\mathcal{O}(X)_{v\psi} \cong \mathcal{O}(X)^G$ (cf. (i)). Then it follows from Corollary 3.11 that $v \cdot [\mathcal{O}(X)_{\psi}] = 0 \in \text{Cl}(\mathcal{O}(X))$. Thus $\widetilde{\text{Cl}}(\mathcal{O}(X), G)$ is a torsion group, which shows (iii) (cf. Proposition 5.1).

Suppose $\mathcal{O}(X)$ has only trivial units. Then $Z^1(G, U(\mathcal{O}(X))) \cong H^1(G, U(\mathcal{O}(X)))$. As $H^1(G, U(\mathcal{O}(X)))_{R,0} \subseteq \mathfrak{X}(G) \subseteq H^1(G, U(\mathcal{O}(X)))$, by Remark 3.7 the equivalence (ii) $\iff$ (v) follows from Theorem 3.15.
Suppose that both $X$ and $(X,G)$ are conical. The implication $(i) \Rightarrow (vi)$ is a consequence of the generic fiber theorem of graded version of flat local morphisms (e.g., [8]). So suppose that the condition $(vi)$ holds. Let
\[ \psi \in \tilde{Z}^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))) \cap Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X)))_e = (\Delta(X,G))^1 \]
be a cocycle. Since $(X/\Delta(X,G), G/\Delta(X,G))$ is a stable and equidimensional action of an algebraic torus on a conical normal variety, applying Sect. 4 of [12] to $\mathcal{O}(X)_\psi$, we can choose a natural number $r$ in such a way that
\[ \mathcal{O}(X)_{nr\psi} = \mathcal{O}(X/\Delta(X,G)_{nr\psi}) \cong \mathcal{O}(X)^G \]
for all $n \in \mathbb{N}$ as $\mathcal{O}(X)^G$-modules. Hence $\widetilde{\mathcal{C}}l(\mathcal{O}(X), G)$ is a torsion group (cf. Corollary 3.11), which shows $(ii)$. □

**Corollary 5.3** Under the same circumstances as in Proposition 5.1, suppose that both $X$ and $(X,G)$ are conical. Then the following conditions are equivalent:

(i) $\nu := \exp(UrCl(\mathcal{O}(X/K(X,G)), G)) < \infty$ and $(X/\mathcal{K}(X,G), G)$ is cofree.

(ii) $\exp(UrCl(\mathcal{O}(X/K(X,G)), G)) = \exp(\widetilde{\mathcal{C}}l(\mathcal{O}(X/K(X,G)), G)) < \infty$ and the inclusion $\mathcal{O}(X)^G \rightarrow \mathcal{O}(X)^{\mathcal{K}(X,G)}$ is no-blowing-up of codimension one (cf. p. 30 of [4]).

(iii) $\widetilde{\mathcal{C}}l(\mathcal{O}(X/K(X,G)), G)$ is a finite group and the inclusion $\mathcal{O}(X)^G \rightarrow \mathcal{O}(X)^{\mathcal{K}(X,G)}$ is no-blowing-up of codimension one.

(iv) $(X/K(X,G), G)$ is equidimensional.

**Proof.** In order to show this corollary, we may suppose that $\mathcal{K}(X,G) = \{1\}$ as
\[ \text{tor}(m, G, \mathcal{K}(X,G)) \supseteq \mathcal{K}(X,G) \]
for $m \in \mathbb{N}$. So $(X,G)$ is a faithful stable action of an algebraic torus $G$. The implication $(i) \Rightarrow (iv)$ and the equivalence $(iv) \Leftrightarrow “(X/\mathcal{K}(X,G), G) is equidimensional”$ follow from the finiteness of $\mathcal{K}(X,G)$. Each condition of $(i)$ - $(iv)$ satisfies the condition that the inclusion $\mathcal{O}(X)^G \rightarrow \mathcal{O}(X)^{\mathcal{K}(X,G)}$ is no-blowing-up of codimension one (cf. (PDE), p. 30 of [4]), which is assumed in the following proof. Then $\tilde{Z}^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))) = \mathfrak{X}(G)_{\mathcal{O}(X)} = \mathfrak{X}(G)$. On the other hand, by Proposition 4.5, $\mathcal{R}(X,G)^1 = Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X)))_e$, which shows
\[ \Delta(X,G) = \left( \tilde{Z}^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X))) \cap Z^1_{\mathcal{O}(X)}(G, U(\mathcal{O}(X)))_e \right)^1 = \mathcal{R}(X,G) \]
(cf. Lemma 4.6). Thus the equivalence of conditions in this corollary is a consequence of Theorem 5.2. □

Suppose that $G$ is an algebraic torus and $(X,G)$ is stable. Clearly $\mathcal{Q}(\mathcal{O}(X)^G) = \mathcal{Q}(\mathcal{O}(X))^G$. Moreover, in the case where $t_{(X,G)} := \exp(UrCl(\mathcal{O}(X), G))$ is finite, express
\[ t_{(X,G)} = t_{(X,G)} \cdot t_{(X,G)}^{\mathcal{R}} \]
(5.3)
as a product of natural numbers $\tilde{t}_{(X,G)}$, $t^\mathfrak{p}_{(X,G)}$ satisfying that

$$\text{GCD}(\tilde{t}_{(X,G)}, |\mathfrak{R}(X,G)|_X) = 1$$

and $|\mathfrak{R}(X,G)|_X$ is divisible in $\mathbb{Z}$ by any prime divisor of $t^\mathfrak{p}_{(X,G)}$. Let $F_{(X,G)}$ be the subgroup of $\mathfrak{R}(X,G)$ consisting all elements $\sigma \in \mathfrak{R}(X,G)$ such that prime divisors of $\text{ord}(\sigma|_X)$ are divisors of $t^\mathfrak{p}_{(X,G)}$. Obviously there exists a natural number $k$ such that

$$\text{Tor}(\tilde{t}^\mathfrak{p}_{(X,G)})^k, G, L_{(X,G)}) \cap \mathfrak{R}(X,G) = \text{Tor}(\tilde{t}^\mathfrak{p}_{(X,G)})^{k+j}, G, L_{(X,G)}) \cap \mathfrak{R}(X,G) = F_{(X,G)}$$

for any $j \in \mathbb{N}$.

**Definition 5.4** Under the same circumstances as above, define the obstruction subgroup for cofreeness of $(X, G)$, denoted by $\text{Obs}(X, G)$, as follows;

$$\text{Obs}(X, G) := \mathfrak{R}(X//H_{(X,G)}), G)$$

where $H_{(X,G)} := \text{Tor}(\tilde{t}_{(X,G)}, G, L_{(X,G)}) \cdot \text{Tor}(\tilde{t}^\mathfrak{p}_{(X,G)}), G, F_{(X,G)})$ (cf. Definition 4.1).

**Remark 5.5** Obviously $\text{Obs}(X, G)|_X$ is a finite group. If $X$ (i.e., $O(X)$) is factorial, then $t_{(X,G)} = 1$ and we see that $L_{(X,G)} = \text{Obs}(X, G)$, i.e., $\text{Obs}(X, G)|_X = \{1\}$. On the other hand in the case where $U(O(X)) = K^*$, unless $\text{Obs}(X, G)|_X = \{1\}$, $(X, G)$ is never isobaric cofree (cf. Corollary 3.11 and Theorem 5.2).

There are many examples of equidimensional actions of (connected) algebraic tori on conical normal varieties which are not cofree as follows:

**Example 5.6** Let $V = K^4$ be a 4-dimensional vector space over $K$ and let $M$ be a subgroup

$$\left\{ \begin{pmatrix} u & u^{-1} \\ v & v^{-1} \end{pmatrix} \mid (u, v) \in (K^*)^2 \right\} \cup \left\{ \tau = \begin{pmatrix} \zeta_3 & \zeta_3^{-1} \\ \zeta_3^{-1} & \zeta_3 \end{pmatrix} \right\}$$

of $GL(V^\vee)$ where $V^\vee$ the $K$-dual space of $V$ and $\zeta_3$ is a primitive 3-th root of 1 in $K$.

Put $X := V//\langle \tau \rangle$ and $G := M^0$ which acts naturally on $X$. Since $M \subseteq SL(V)$, we see that $\mathfrak{R}(V, M) = \{1\}$ and hence $\mathfrak{R}(X,G) = \{1\}$. Clearly $V//G \cong A^2_K$ and the action $(V,G)$ is cofree, which implies that $(X,G)$ is equidimensional. Applying Samuel’s Galois descent to the action $(V//G, \langle \tau \rangle)$, we have

$$\text{Cl}(O(X)^G) = \text{Cl}((O(V)^G)^{\langle \tau \rangle}) \cong \mathbb{Z}/3\mathbb{Z}.$$ 

Because $(X,G)$ is not cofree, we see $\{[\theta]\} \neq \tilde{\text{Cl}}(O(X),G) = \text{Cl}(O(X)^G)$ and $t_{(X,G)} = 3$ (recall $\theta$ denotes the trivial cocycle). Consequently

$$\text{Obs}(X, G)|_X \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}.$$
Example 5.7 Let $V = K^4$ and put

$$G := \begin{cases} 
(tu, tu^{-1}, t, t^{-3}) & (t, u) \in (K^*)^2 
\end{cases} \subseteq GL(V^*),$$

and $X := V//H$. Here the matrix representation is given by the basis $\{X_1, \ldots, X_4\}$. As in Example 5.6, we similarly see that $\mathfrak{R}(X, G) = \{1\}$. By [16] one sees that $\text{Cl}(\mathcal{O}(X)) \cong H_{X_4} \cong \mathbb{Z}/3\mathbb{Z}$. There is a finite dominant morphism $\pi : X \rightarrow Y = \mathbb{A}^3$, where $Y$ is defined by $K[X_1^4X_4, X_2^2X_4, X_3^3X_4]$ and $\pi$ is associated with $K[X_1^4X_4, X_2^2X_4, X_3^3X_4] \rightarrow \mathcal{O}(X)$. Clearly $G$ acts naturally on $Y$ and $\pi$ is $G$-equivariant. Since $(Y, G)$ is cofree, $(X, G)$ is stable and equidimensional. On the other hand we easily see that both $(V, G)$ and $(X, G)$ are not cofree. Hence $\text{Cl}(\mathcal{O}(X), G) \neq \{0\}$ and $\text{Cl}(\mathcal{O}(X)) \supset \text{UrCl}(\mathcal{O}(X), G) \cong \mathbb{Z}/3\mathbb{Z}$. Consequently

$$\text{Obs}(X, G)|_X \cong \mathbb{Z}/3\mathbb{Z}.$$

Obviously $X$ is regarded as an affine toric variety with a non-cofree equidimensional torus action commuting with its toric structure. In general, for any finitely generated abelian group $A$, there exists an affine toric variety whose class group is isomorphic to $A$ (cf. [16]).

Theorem 5.8 Suppose that $G$ is a connected algebraic torus and $p = 0$. Suppose that both $X$ and $(X, G)$ are conical. If $(X, G)$ is stable, then the following conditions are equivalent:

(i) The action $(X, G)$ is equidimensional.

(ii) The exponent $\exp(\text{UrCl}(\mathcal{O}(X), G))$ is finite and the action $(X//\text{Obs}(X, G), G) = (X//\text{Obs}(X, G), G/\text{Obs}(X, G))$ is cofree.

Proof. (ii) $\Rightarrow$ (i) : Since cofree conical actions are equidimensional and the group $\text{Obs}(R, G)|_X$ is finite, we immediately see that (ii) implies (i).

(i) $\Rightarrow$ (ii) : Suppose that $(X, G)$ is equidimensional. We use notation in Definition 5.4 and the paragraph preceding to Definition 5.4. By Proposition 4.4 we see that

$$\text{Obs}(X, G) \prec \mathfrak{R}(X//H_{(X, G)}, G) = H_{(X, G)} \cdot \mathfrak{R}(X, G)$$

and $(X//\text{Obs}(X, G), \mathfrak{R}(X//H_{(X, G)}, G))$ is cofree. Let $\chi \in \mathfrak{X}(G)$ be a character satisfying $\chi(H_{(X, G)} \cdot \mathfrak{R}(X, G)) = \{1\}$. Note $\chi \in \mathfrak{X}(G)_{\mathcal{O}(X)}$, as $\chi(L_{(X, G)}) = \{1\}$. It suffices to show that $\mathcal{O}(X)_\chi \cong \mathcal{O}(X)^G$ as $\mathcal{O}(X)^G$-modules. Let $\sigma$ be any element of $G$ such that $\sigma^\chi(x, G) \in \mathfrak{R}(X, G)$. Recall the expression (5.3). As the map

$$\mathfrak{R}(X, G) \ni \tau \rightarrow \tau^\chi(x, G) \in \mathfrak{R}(X, G)$$
induces an automorphism of $\mathfrak{R}(X,G)|_X$, the element $\sigma t_{(X,G)}^{R}$ belongs to the subgroup $\text{tor}(\hat{t}_{(X,G)},G,L_{(X,G)}) \cdot \mathfrak{R}(X,G)$. Since $\text{tor}(\hat{t}_{(X,G)},G,L_{(X,G)}) \ni \tau \to \tau t_{(X,G)}^{R} \in \text{tor}(\hat{t}_{(X,G)},G,L_{(X,G)})$ induces an automorphism of $\text{tor}(\hat{t}_{(X,G)},G,L_{(X,G)})|_X$ and $\mathfrak{R}(X,G)$ contains $L_{(X,G)}$, we can choose $\mu$ from $\text{tor}(\hat{t}_{(X,G)},G,L_{(X,G)})$ in such a way that $(\sigma \cdot \mu)t_{(X,G)}^{R} \in \mathfrak{R}(X,G)$. Then express $(\sigma \cdot \mu)t_{(X,G)}^{R} = \gamma_1 \cdot \gamma_2$ for some $\gamma_i \in \mathfrak{R}(X,G)$ such that $\text{GCD}(\text{ord}(\gamma_1|_X),t_{(X,G)}^{R}) = 1$ and any prime divisor of $\text{ord}(\gamma_2|_X)$ is a divisor of $|\mathfrak{R}(X,G)|_X|$. Clearly $\gamma_2 \in F_{(X,G)}$.

Since $F_{(X,G)} \supseteq L_{(X,G)}$ and $(\gamma_1) \ni \tau \to \tau t_{(X,G)}^{R} \in (\gamma_1)|_X$, there exists an element $\delta \in (\gamma_1)$ satisfying $(\sigma \cdot \mu \cdot \delta)t_{(X,G)}^{R} \in F_{(X,G)}$.

Consequently we see

$$\sigma \in \text{tor}(\hat{t}_{(X,G)},G,L_{(X,G)}) \cdot \text{tor}(t_{(X,G)}^{R},G,F_{(X,G)}) \cdot \mathfrak{R}(X,G) = H_{(X,G)} \cdot \mathfrak{R}(X,G).$$

Thus $\chi(\sigma) = 1$, which implies

$$\chi(\text{tor}(t_{(X,G)},\mathfrak{R}(X,G))) = \{1\}.$$ 

By the implication $(iv) \Rightarrow (i)$ of Corollary 5.3, we see that $O(X)_\chi \cong O(X)^G$ as $O(X)^G$-modules. □

**Proof of Theorem 1.2.** Suppose that $(X,G)$ is faithful and by Theorem 5.8 we may suppose that

$$t_{(X,G)} := \exp (\text{UrCl}(O(X),G)) = \exp \left(\tilde{\Cl}(O(X),G)\right) < \infty.$$ 

By Definition 5.4 we see that prime divisors of $|\text{Obs}(X,G)|$ are same as those of $t_{(X,G)}$, which implies that there exists a power of $t_{(X,G)}$ which is divisible by $|\text{Obs}(X,G)|$. □

Now, Corollary 1.3 follows immediately from the definition of $\tilde{\Cl}(O(X),G)$ and Theorem 5.8.

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