Quantum Mechanics of Consecutive Measurements

Christoph Adami\textsuperscript{1,2}
\textsuperscript{1}Keck Graduate Institute, Claremont, CA 91711
\textsuperscript{2}California Institute of Technology 216-76, Pasadena, CA 91125

I analyze consecutive projective measurements in quantum mechanics in terms of quantum information theory and a no-collapse picture of measurement. I show that the entropy of all detectors that consecutively measured a quantum state is given by the entropy of the last measurement, while the amount of information that any of the consecutive measurements has about the quantum state is equal to the entropy of the preparation, that is, the first measurement of the pure state. Because the entropy of consecutive measurements cannot decrease, the entropy of the last detector tends to its maximum irrespective of the preparation. The formalism leads to a succinct description of the quantum Zeno and anti-Zeno effects.

Consecutive projective measurements on the same quantum system are interesting for the foundations of quantum mechanics because they probe directly the assumptions behind the collapse picture of quantum measurement. The quantum Zeno effect, for example, even though it is usually described within a collapse picture, does not require it \textsuperscript{1}and can be described in a purely unitary picture of quantum measurement \textsuperscript{2}. The physics of consecutive measurements is also important for a consistent formulation of covariant quantum mechanics \textsuperscript{3,4}, which does not allow for a time variable to define the order of (possibly non-commuting) projections.

Consecutive measurements can be seen to challenge our understanding of quantum measurement in an altogether different manner, however. According to standard theory, measuring a known (that is, prepared) quantum state in a basis that is at an angle with respect to the preparation results in a calculable amount of noise in the detector that can be quantified by the detector's entropy. This calculation assumes that the entropy of the quantum system before measurement vanishes, i.e., that the measurement collapses the wavefunction of a previously pure state. A quantum information-theoretic treatment of the no-collapse model, such as described in Refs. \textsuperscript{5,6}, does not have to make this assumption, however. Instead, one can show that it is the entropy of the quantum system given the preparation, that is, given the results of the measurement that prepared the system, that vanishes. Yet, because the marginal (unconditional) entropy can be non-zero at the same time, a collapse does not have to be assumed.

It is the purpose of this letter to show that a quantum information-theoretic no-collapse description of quantum measurement is not only internally consistent, but also reproduces standard quantum results such as the quantum Zeno and anti-Zeno effects very succinctly. In particular, I show that the entropy of all the detectors that interacted with a quantum system in a series of consecutive projective measurements is given by the entropy of the last detector only, while the entropy of the last detector given all previous measurement results is the difference between the entropy of the last and next to last measurements only. Meanwhile, the history of all measurements, that is, of all the quantum systems that ever interacted with that system, is imprinted on the entropy of the last detector.

We start with a quantum system expressed in terms of a measurement device $A$’s basis as
\begin{equation}
|Q\rangle = \sum_i \alpha_i |a_i\rangle,
\end{equation}
as we are about to measure $Q$ in that basis. Compared to all the following measurements, device $A$ is special because it prepares the quantum state $Q$ in the known basis states $|a_k\rangle$. This is done via a unitary operator acting on the joint state of $Q$ and the default state of $A$ (the ancilla) so that
\begin{equation}
U |QA\rangle = \sum_i \alpha_i |a_i\rangle.
\end{equation}
The von Neumann measurement operator $U$ can be written generally in terms of the projectors $P_i$ and the “ladder” operators $L_m$, defined via $L_m |0\rangle = |m\rangle$
\begin{equation}
U = \sum_i P_i \otimes L_i,
\end{equation}
where $P_i = |a_i\rangle\langle a_i|$. The probability to observe $A$ in its eigenstate $|i\rangle$ is
\begin{equation}
p_i = \langle i | \rho_A | i \rangle = |\langle Q|A\rangle|^2 = |\alpha_i|^2,
\end{equation}
so the entropy of the ancilla $A$ (the detector $A$) is

$$S(A) = H[p_i] = -\sum_i p_i \log p_i,$$

(5)

where $I$ introduced the notation $H[p_i] = -\sum_i p_i \log p_i$, and the constant $H_A$ to remind us that the von Neumann entropy of $A$ is a classical Shannon entropy. In the following, I discuss the physics of consecutive measurements of the pure quantum state $Q_A$. An extension to consecutive measurements of a mixed state is straightforward (see, e.g., [7]). Also, without loss of generality I assume that all measurements are fully resolving and that all detectors can resolve the same number of states.

We now entangle $Q_A$ with another detector; an ancilla $B$ in a different basis, specifically $|a_i\rangle = \sum_j U_{ij}|b_j\rangle$ so that $\langle b_j|a_i\rangle = U_{ij}$, where $U$ is unitary. Unitarity implies that

$$\sum_j |U_{ij}|^2 = 1.$$

Then

$$|QAB\rangle = \sum_{ij} \alpha_i U_{ij}|b_j, i, j\rangle,$$

(7)

where $|j\rangle$ are the $B$ ancilla basis states.

Because the system $QAB$ is pure, we have automatically $S(Q) = S(AB)$. It is easy to show that

$$\rho_{AB} = \sum_{i'j'} \alpha_{i'} \alpha_{j'}^* U_{i'j'}|i'j\rangle\langle ij|,$$

(8)

$$\rho_A = \sum_i |\alpha_i|^2 |i\rangle\langle i|,$$

(9)

$$\rho_B = \sum_{ij} |\alpha_{ij}|^2 |U_{ij}|^2 |j\rangle\langle j|,$$

(10)

and

$$\rho_Q = \sum_{ij} |\alpha_{ij}|^2 |U_{ij}|^2 |b_j\rangle\langle b_j|.$$  

(11)

The entropy of the preparer is still $H_A = H[p_i]$, while we also obtain $S(Q) = S(B) = H_B$, from comparing the density matrices $\rho_B$ and $\rho_Q$, with

$$H_B = -\sum_j q_j \log q_j,$$

(12)

where $q_j = \sum_i p_i |U_{ij}|^2 = \sum_i p_i q_{ij}$, and introducing the conditional probability $q_{ij} = |U_{ij}|^2$. The entropy Venn diagram that illustrates the entanglement of $Q$, $A$, and $B$ is shown in Fig. 1.

Note that because of the relationship $q_j = \sum_i p_i q_{ij}$, the entropy $H[q_j] \geq H[p_i]$, that is, the entropy of the detector must increase or stay constant.

Let us perform another measurement of the system, in terms of an ancilla $C$ so that $|c_i\rangle = U_{ij}|e_i\rangle$. We then find

$$|QABC\rangle = \sum_{ijk} \alpha_i U_{ij} U_{jk}^* |c_k\rangle |j\rangle |i\rangle.$$  

(13)

Tracing this expression over $C$ recovers $\rho_{AB}$ from Eq. (8) as it should, as ignoring $C$ is the same as undoing the $C$ measurement [8]. Tracing over $B$ gives

$$\rho_{AC} = \sum_{i'j'} \alpha_{i'} \alpha_{j'}^* U_{i'j'} U_{ij}^* |i'\rangle\langle i| \times \sum_k |U_{jk}'|^2 |k\rangle\langle k|.$$  

(15)

Tracing out $C$ now gives again $\rho_A = \sum_i |\alpha_i|^2 |i\rangle\langle i|$, whereas tracing over $A$ gives

$$\rho_C = \sum_{ijk} |\alpha_{ij}|^2 |U_{ij}|^2 |U_{jk}'|^2 |k\rangle\langle k|.$$  

(16)

Due to the symmetry between $|c_k\rangle$ and $|k\rangle$ in (15), we can conclude that now $S(Q) = S(C)$, and we also know that $S(ABC) = S(Q)$ from the purity of $QABC$, so that $S(C) = S(ABC)$. $S(B)$ is unchanged from (12), so we continue to use the same symbol $H_B$. Introducing

$$q_k' = \sum_{ij} |\alpha_{ij}|^2 |U_{ij}|^2 |U_{jk}'|^2 \equiv \sum_{ij} p_i q_{ij} q_k,'$$

(17)

we can write

$$S(C) = H[q_k'] = H_C.$$  

(18)

The entropy diagram in Fig. 1 shows how the entropies are distributed among the quantum system, the preparer, and the two detectors $B$ and $C$.

We can ask how the entropy evolves as more detectors measure the same system. While we are limited in our ability to depict these relationships in diagrams, we can still do so by grouping sets of detectors together. For example, detectors $A$ and $B$ will always show the same relationship to each other, depicted in Fig. 1, no matter how many measurements are performed subsequently. For an arbitrary chain of detectors, the entropy diagram will be a series of concentric circles as in Fig. 1, up to the last one. This can be expressed mathematically as follows.
FIG. 2: Entropy Venn diagram for three consecutive measurements. Only the conditional entropy of the quantum system $Q$ is indicated by dashed lines, but not its intersection with the measurement devices. The sum of all entropies must vanish.

Because all the von Neumann entropies can be expressed as Shannon entropies of the corresponding probability distribution of the detector, I will write $H(X_i)$ for the quantum entropy of detector $X_i$. The relationship between a joint entropy and the conditional entropy of the components is expressed by the chain rule for entropies [10], written here for random variables $X_1 \cdots X_n$

$$H(X_1 \cdots X_n) = \sum_{i=1}^n H(X_i | X_{i-1} \cdots X_1)$$ (19)

where I defined $H(X_1 | X_0) \equiv H(X_1)$, as $X_1$ is the first detector. Generalizing from the previous, we find that

$$H(X_i | X_{i-1} \cdots X_1) = H(X_i | X_{i-1}) = H(X_i) - H(X_{i-1}).$$ (20)

$$H(X_1 \cdots X_n) = H(X_1) = H(Q),$$ (22)

$$I(X_n : X_{n-1}) = H(X_{n-1}),$$ (23)

where $I(X_i : X_j)$ is the information detector $X_i$ conveys about detector $X_j$. As the entropy increases for every subsequent detector, we expect $H(X_n) \to \log N$ as $n \to \infty$ ($N$ is the number of discrete states of $Q$ or any of the detectors), that is, the detector’s state becomes more and more random.

The result that any quantum system will appear random after a sufficient number of projections appears counter intuitive. In particular, a prepared system measured in its eigenbasis will surely result in a detector with entropy $H(X) = 0$. However, this reasoning does not take into account the full history of the quantum state. Instead, the entropy that appears to vanish in a consecutive measurement in the same basis is really the conditional entropy $H(X_n | X_{n-1}) = H(X_n) - H(X_{n-1}).$ Indeed, the conditional entropy of any detector given the state of any other previous measurement result is the difference between the entropies of the respective detectors

$$H(X_j | X_k) = \begin{cases} H(X_j) - H(X_k) & \text{X after } X_k, \\ 0 & \text{else}. \end{cases}$$ (24)

Thus, even though we can ignore all earlier measurements (except for the one just preceding) when interpreting the state of a detector (see Eqs. [22][23]), all earlier measurements leave their trace due to the chain of conditional probabilities created by the measurements. Writing $p(X_n = j_n) \equiv p(j_n)$, we find

$$p(j_n) = \sum_{i,j_1 \cdots j_{n-1}} p(j_n | j_{n-1}) \cdots p(j_2 | j_1)p(j_1 | A = i)p_i.$$ (25)

Let us study the amount of information that each detector carries about the quantum state preparation. From the formulas above we can deduce that the information carried by the set of all detectors about the preparation (that is, the first detector) is equal to the information the last detector has, which is equal to the entropy of the preparation. By the chain rule for informations [9] (here, we treat the preparer as the variable $X_0$) we obtain using Eqs. [22][23],

$$I(X_1 \cdots X_n : A) = \sum_{i=1}^n I(X_n : A | X_{i-1} \cdots X_1)$$ (26)

$$= I(X_n : A) = H(A).$$ (27)

In fact, it is easy to show that the information shared by the last measurement with any previous measurement $X_k$ is given by the entropy of that previous measurement:

$$I(X_n : X_k) = H(X_k) \quad (n > k).$$ (28)

Note that this implies that the last measurement allows me to predict (in principle) the state of any measurement that went before it, no matter how distant (because the shared entropy between the two measurements is equal to the entire entropy of the prior measurement). This is in direct contradiction to a collapse picture, where every measurement erases the memory of past measurements. This is also implicit in the unbroken chain of conditional probabilities in Eq. [25] that connect all the consecutive measurements of a quantum system to each other. Such a chain is in contradiction to the collapse picture, where each measurement “wipes the slate clean”.

We can illustrate the physics of consecutive quantum measurements with qubits in the following example. Taking $|\alpha|^2 = p$, the entropy of detector $A$ is given by the
solid line in Figure 3. Adding a detector $B$ at $\theta = \pi/8$ results in the entropy $H(B)$ (dotted line). Detector $C$, at $\theta = \pi/8$ with respect to $B$ gives the dash-dotted line. Any measurement in a basis different from that of the previous measurement increases the detector entropy.

This formalism can be directly applied to demonstrate the quantum Zeno and anti-Zeno effects. Instead of a time-varying quantum state controlled by quantum measurements in the same basis, we can equivalently study a time-varying quantum state controlled by quantum measurements in random bases. A similar result was derived for the dephasing of photon polarization in Ref. [11].

In summary, I described consecutive quantum measurements in a no-collapse picture, and recovered several well-known results. The formalism implies that the entropy of a quantum measurement must be interpreted as the difference between the entropy of the system after measurement and before, in contradiction to the standard view that the entropy of the prepared system vanishes. Because this view also implies that the probability of obtaining a particular measurement result can, in principle, be obtained from an unbroken chain of conditional probabilities that links every measurement that ever occurred on a quantum system to each other, it makes fundamentally different predictions about quantum experiments than collapse pictures do. This reinterpretation of consecutive projective quantum measurements allows for a consistent, paradox-free interpretation because it does not presuppose the reality of the prepared state. Instead, the prepared state itself is entropic, owing to the chain of measurements that went before it.

I would like to thank N.J. Cerf and S.J. Olson for discussions. This work was supported by the Army Research Offices grant # DAAD19-03-1-0207.

---

[1] D. Home and M. A. B. Whitaker, Annals of Physics 258, 237 (1997).
[2] A. Peres, Quantum Theory: Concepts and Methods (Kluwer Academic, Dordrecht, 1995).
[3] M. Reisenberger and C. Rovelli, Phys. Rev. D 65, 125016 (2002).
[4] S. J. Olson and J. P. Dowling (2007), eprint quant-ph/0701200.
[5] N. J. Cerf and C. Adami (1996), quant-ph/9605002.
[6] N. J. Cerf and C. Adami, Physica D 120, 62 (1998).
[7] N. J. Cerf and C. Adami (1996), quant-ph/9611032.
[8] M. O. Scully and K. Drühl, Phys. Rev. A 25, 2208 (1982).
[9] T. M. Cover and J. A. Thomas, Elements of information theory (John Wiley, New York, 1991).
[10] L. Ekroot and T. M. Cover, IEEE Transactions on Information Theory 39, 1418 (1993).
[11] A. Kofman, G. Kurizki, and T. Opatrný, Phys. Rev. A 63, 042108 (2001).
[12] B. Kaulakys and V. Gontis, Phys. Rev. A 56, 1131 (1997).
[13] M. Lewenstein and K. Rzążewski, Phys. Rev. A 61, 022105 (2000).
[14] A. Luis, Phys. Rev. A 67, 062113 (2003).