LATTICE COVERINGS AND GAUSSIAN MEASURES OF $n$-DIMENSIONAL CONVEX BODIES

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Abstract

Let $\| \cdot \|$ be the euclidean norm on $\mathbb{R}^n$ and $\gamma_n$ the (standard) Gaussian measure on $\mathbb{R}^n$ with density $(2\pi)^{-n/2}e^{-\|x\|^2/2}$. Let $\vartheta \simeq 1.3489795$ be defined by $\gamma_1([-\vartheta/2, \vartheta/2]) = 1/2$ and let $L$ be a lattice in $\mathbb{R}^n$ generated by vectors of norm $\leq \vartheta$. Then, for any closed convex set $V$ in $\mathbb{R}^n$ with $\gamma_n(V) \geq 1/2$ and for any $a \in \mathbb{R}^n$, $(a + L) \cap V \neq \emptyset$. The above statement can be viewed as a “nonsymmetric” version of Minkowski Theorem.

Let $U, V$ be a pair of convex sets in $\mathbb{R}^n$ containing the origin in the interior. Let us define $\beta(U, V)$ as the smallest $r > 0$ satisfying the following condition: to each sequence $u_1, \ldots, u_n \in U$ there correspond signs $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$ such that $\varepsilon_1 u_1 + \cdots + \varepsilon_n u_n \in rV$. Upper and lower bounds for $\beta(U, V)$ for various sets $U$ and $V$ (usually centrally symmetric) were investigated by several authors. We will mention some of their results once the appropriate notation is introduced, see also references in [3].

Let $L$ be a lattice in $\mathbb{R}^n$, i.e. an additive subgroup of $\mathbb{R}^n$ generated by $n$ linearly independent vectors. The quantities (again, usually defined for centrally symmetric sets)

$$\lambda_n(L, U) = \min\{r > 0 : \dim \text{span} \{L \cap rU\} = n\},$$

$$\mu(L, V) = \min\{r > 0 : L + rV = \mathbb{R}^n\}$$

are called the $n$th minimum and the covering radius of $L$ with respect to $U$ and $V$, respectively; sometimes $\mu(L, V)$ is called ”the $n$th covering minimum” and denoted $\mu_n(L, V)$. Let us define

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\[ \alpha(U, V) = \sup_L \frac{\mu(L, V)}{\lambda_n(L, U)} \]

where the supremum is taken over all lattices \( L \) in \( \mathbb{R}^n \). A standard elementary argument shows that \( \alpha(U, V) \leq \beta(U, V) \) (see e.g. Lemma 4 in [3]).

By \( B_n \) we shall denote the closed euclidean unit ball in \( \mathbb{R}^n \). Let \( E \) be an \( n \)-dimensional ellipsoid in \( \mathbb{R}^n \) with centre at zero and principal semiaxes \( \alpha_1, \ldots, \alpha_n \). The result of [4], that closed connected additive subgroups of nuclear locally convex spaces are linear subspaces, was essentially based on the fact that

\[ \alpha(B_n, E) = \frac{1}{2}(\alpha_1^2 + \cdots + \alpha_n^2)^{1/2}. \]

Then it was proved in [2] that

\[ \beta(B_n, E) = (\alpha_1^2 + \cdots + \alpha_n^2)^{1/2}. \]

Let \( K_n \) be the unit cube in \( \mathbb{R}^n \). Consider the rectangular parallelepiped

\[ P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_k| \leq \alpha_k \text{ for } k = 1, \ldots, n\} \]

where \( \alpha_1, \ldots, \alpha_n > 0 \). This paper was motivated by an attempt to give possibly best upper bounds for \( \alpha(B_n, P) \) and \( \beta(B_n, P) \) as functions of \( \alpha_1, \ldots, \alpha_n \) (for \( \beta(K_n, P) \), see [5] and [8] where it was, in particular, proved that \( \beta(K_n, K_n) = O(\sqrt{n}) \) as \( n \to \infty \); see also [1]). In particular, we were interested in the so-called Komlós conjecture which asserts that \( \beta(B_n, K_n) \) remains bounded as \( n \to \infty \).

Let us denote by \( \gamma_n \) the (standard) Gaussian measure on \( \mathbb{R}^n \) with density

\[ (2\pi)^{-n/2} e^{-\|x\|^2/2}, \]

where \( \|x\| \) is the euclidean norm of \( x \). Let \( \vartheta \) (\( \simeq 1.3489795 \)) be the positive number given by \( \gamma_1([-\vartheta/2, \vartheta/2]) = \frac{1}{2} \), i.e.

\[ \int_0^{\vartheta/2} e^{-t^2/2} dt = \frac{\sqrt{2\pi}}{4}. \]

By a \( \vartheta \)-coset in \( \mathbb{R}^n \) we shall mean a coset modulo a lattice \( L \) generated by vectors of Euclidean norm \( \leq \vartheta \), i.e. satisfying \( \lambda_n(L, B_n) \leq \vartheta \). The aim of this paper is to prove the following fact.

**Theorem.** If \( V \) is a closed convex set in \( \mathbb{R}^n \) with \( \gamma_n(V) \geq 1/2 \), then \( V \) intersects every \( \vartheta \)-coset.

**Corollary.** If \( V \) is as in the Theorem, then \( \alpha(B_n, V) \leq \vartheta^{-1} \). In particular \( \alpha(B_n, K_n) = O(\sqrt{\log n}) \) as \( n \to \infty \).

We point out that, in full generality, the Theorem is sharp and that, similarly, the first part of the Corollary can not be significantly improved. However, it is conceivable that \( \alpha(B_n, \cdot) \) may be replaced by \( \beta(B_n, \cdot) \) in the Corollary; see the Conjecture at the end of this paper.
For the proof we need the following.

**Lemma.** If $V$ is a closed convex set in $\mathbb{R}^n$ with $\gamma_n(V) \geq \frac{1}{2}$ and $M$ is a linear subspace of $\mathbb{R}^n$ of dimension $m$, then $\gamma_m(V \cap M) \geq \frac{1}{2}$.

**Remark 1.** An analysis of the proof shows that unless $V$ is a half space, or an infinite cylinder orthogonal to $M$, the inequality in the assertion of the Lemma is strict.

We need some preparation for the proofs of the Lemma and of the Theorem. For a convex set $V$ in $\mathbb{R}^n$ and $x \in \mathbb{R}$ denote

$$V_x = \{(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, \ldots, x_{n-1}, x) \in V\} \quad (1)$$

Recall now an inequality of Ehrhard (see [6], Thm. 3.2). If $A$, $B$ are non-empty convex Borel subsets of $\mathbb{R}^n$ and $0 \leq \lambda \leq 1$, then

$$\Phi^{-1}(\gamma_n(\lambda A + (1 - \lambda)B)) \geq \lambda \Phi^{-1}(\gamma_n(A)) + (1 - \lambda)\Phi^{-1}(\gamma_n(B)) \quad (2)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \quad x \in \mathbb{R}.$$  

is the (standard) Gaussian cumulative distribution function. It follows in particular that $g(x) = \Phi^{-1}(\gamma_{n-1}(V_x))$ is a concave function of $x$ on the interval $I = \{x : \gamma_{n-1}(V_x) > 0\}$. Consequently,

$$W = \{(x, y) \in \mathbb{R}^2 : x \in I \text{ and } y \leq g(x)\} \quad (3)$$

is a closed convex subset of $\mathbb{R}^2$. Note that $\gamma_1(W_x^c) = \gamma_1((-\infty, g(x)]) = \gamma_{n-1}(V_x^c)$ for $x \in \mathbb{R}$, where $W_x^c$ is defined analogously to $V_x^c$; in particular $\gamma_n(V) = \gamma_2(W)$.

**Proof of the Lemma.** Clearly it is enough to consider the case $m = n - 1$ and (by the rotationary invariance of the Gaussian measure) $M = \{(x_1, \ldots, x_n) : x_n = 0\}$. For $V$ with $\gamma_n(V) \geq \frac{1}{2}$ we construct $W \subset \mathbb{R}^2$ as above, the assertion of the Lemma is then equivalent to $\gamma_1(W_0^c) \geq \frac{1}{2}$ or $(0, 0) \notin W$. To conclude the argument it remains to note that $(0, 0) \notin W$, together with $W$ being closed and convex, would imply $\frac{1}{2} > \gamma_2(W) = \gamma_n(V)$, a contradiction.

**Remark 2.** For the proof of the Theorem we use the Lemma with $n = 2$ and $m = 1$, a special case that can be proved without appealing to the Ehrhard’s inequality (2). However, the proof of the Theorem itself does use Ehrhard’s inequality.

**Proof of the Theorem.** We use induction on $n$. For $n = 1$, the Theorem is rather trivial. So, suppose that for a certain $n \geq 2$ the Theorem is true for all dimensions strictly less than $n$. Take an arbitrary $\vartheta$-coset $H$ in $\mathbb{R}^n$ and a convex set $V$ in $\mathbb{R}^n$ disjoint with $H$. We are to prove that $\gamma_n(V) < \frac{1}{2}$.  

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Fix some \( u \in H \) and consider the lattice \( L = H - u \). By assumption, we have \( \lambda_n(L, B_n) \leq \vartheta \). Choose \( a_1, \ldots, a_n \in L \cap \vartheta B_n \) generating \( L \) and let \( M \) be the linear span of \( a_1, \ldots, a_{n-1} \). As before, we may assume that \( M = \{(x_1, \ldots, x_n) : x_n = 0\} \). Let \( H' \) be the orthogonal projection of \( H \) onto the \( n \)th coordinate axis of \( \mathbb{R}^n \) (i.e., onto the orthogonal complement of \( M \)). Clearly \( H' \) is a \( \vartheta \)-coset. Additionally, if \( x \in H' \), then, by our inductive hypothesis, \( \gamma_{n-1}(V_x) < \frac{1}{2} \) and so \( (x, 0) \not\in W \) (\( V_x, W \) have the same meaning here as in (1) and (3)). The case \( n = 1 \) of the Theorem yields now that \( \gamma_1(W \cap \{(x, 0) : x \in \mathbb{R}\}) < \frac{1}{2} \) and the Lemma implies then that \( \frac{1}{2} > \gamma_2(W) = \gamma_n(V) \), as required.

**Conjecture.** There exists some function \( f \) on \((0, 1)\) such that for each symmetric convex set \( V \) in \( \mathbb{R}^n \) one has \( \beta(B_n, V) \leq f(\gamma_n(V)) \).

**Remark 3.** Let \( T \) be a bounded linear operator from a Hilbert space \( H \) to a Banach space \( X \). We say that \( T \) is tight if the image of every connected additive subgroup of \( H \) is dense in its linear span in \( X \). If \( X \) is a Hilbert space, then \( T \) is tight if and only if it is a Hilbert-Schmidt operator; sufficiency was proved in [4], the proof of necessity can easily be obtained by standard methods. The argument of [4] together with the theorem proved above imply that \( \ell \)-operators are tight (for the definition of \( \ell \)-operators, see [7], p. 38). An interesting problem, closely connected with the Komlós conjecture, is to describe tight diagonal operators from \( l_2 \) to \( c_0 \).

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