New Approach of Polyominoes Using James Diagram with Beta Number

E. F. Mohoomed
Mustansiriyah University, College of Education- Iraq

Email: emanfatel@uomustansiriyah.edu.iq

Abstract. James abacus diagram used in number theory to describe integer partitions with the special numbers of columns $e, e \geq 2$. One way of producing it is by using decreasing sequence of integer number called beta number. In this work, we proved the graphical representation of the beta number of any partition to polyomino by using James abacus diagram and $e$-core of James abacus. New identities of Fibonacci to count the number of ways for $2 \times 1$ dominoes to tiling a James abacus diagram chain by using monomino and domino.

Keywords: Tiling, Polyominoes, $\beta$-number, James diagram, Fibonacci number.

1. Introduction

A polyomino is a plane geometric figure shaped by going along with one or more equivalent squares edge to edge. It is a polyform whose cells are squares. It might be viewed as a limited subset of the general square tiling with an associated inside. Polyominoes have been utilized as a part of mainstream riddles since no less than 1907, and the identification of pentominoes is dated to antiquity [1]. Many results with the bits of 1 to 6 squares were initially distributed in Fairy Chess Review between the years 1937 to 1957, under the name of "dissection issues." The name polyomino was imagined by Solomon W. Golomb in 1953 and it was advanced by Martin Gardner in his November 1960 "Numerical Games segment" in Scientific American [2]. Identified with polyominoes are polyiamonds, framed from equilateral triangles; polyhexes, shaped from normal hexagons; and another plane polyforms. Polyominoes have been summed up to higher measurements by joining solid shapes to frame polycubes, or hypercubes to shape polyhypercubes. In the same way as other riddles in recreational science, polyominoes raise numerous combinatorial issues. The most essential is listing polyominoes of a given size. No equation has been found aside from unique classes of polyominoes. Various appraisals are known, and there are calculations for figuring them. Polyominoes with gaps are awkward for a few purposes, for example, tiling issues. In a few settings polyominoes with openings are rejected, permitting just essentially associated polyominoes [3,4,5]. Sever studies used James abacus to enumeration the polyominoes [6,7]. Benjamin use some of identity Fibonacci to find the number of ways to tile a graph consist of $n$ vertices such that every independent vertices and vertex which followed covered by domino $2 \times 1$ square and rest vertices covered by $1 \times 1$ square [8,9]. This study will proposer new method to fined polyominoes by using James abacus diagram and enumerating of polyominoes beta.
which is a represents of connected beta number sequence if $e = 2$. Further, Further, tilling with James abacus for beta number creating.

2. Preliminaries

In this section we will redefine polyominoes as James abacus diagram. The first step is to study the structure of James abacus diagram.

James abacus is a graphical representation of non-increasing sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_b)$ called partition of positive integer $t$ by using beta number sequence $\{\beta_1, \beta_2, \ldots, \beta_b\}$ where $b$ is the number of parts partition. To define James abacus diagram we must first determine the value of $e$ where $e > 2$ such that $e$ represent to the number of the columns in the diagram which labelled from 0 to $e - 1$. The beta number positions are located across the columns on the James abacus diagram which are labelled from left to right and continues to lower from top starting with 0, the beta number positions $ne, ne + 1, ne + 2, \ldots, ne + e - 1$ are located in row $n$ in the James abacus diagram. The process to define James abacus diagram is more easily if we using a graph. We can think of our diagram in the fourth quadrant of a standard two dimension plane. Knowing that James abacus is have at most $b$ beta number, any beta number position can be represented as unit square in two dimension plane called beta number sequence.

Definition 2.1. Beta number is a sequence of four vertices with length 1 such that any two vertices are collinear. The vertices are a coordinates in two dimension plane, thus

$$\beta\text{-number} = \{(c, -n), (c + 1, -n), (c, -n - 1), (c + 1, -n - 1)\} \quad 0 \leq c \leq e - 1, 0 \leq n \leq \left\lfloor \frac{\beta_1}{e} \right\rfloor - 1.$$

These vertices connected by horizontal or vertical vertex edges with length 1.

Remark 2.2 Two edges are connected if they are shared with the same vertex.

Definition 2.3 Path is a finite sequence of horizontal and vertical connected edges.

3. Beta number sequence

Let and $\beta_h$ are beta numbers square. If $\beta_k$ is a unit square in the fourth quadrant of standard two dimension plan with four coordinates $(c, -n) (c + 1, -n) (c, -n - 1) (c + 1, -n - 1)$. Then $\beta_h$ is a unit square in the fourth quadrant of standard two dimension plan with one of the following coordinates

1. $(c + 1, -n), (c + 2, -n), (c + 1, -n - 1), (c + 2, -n - 1)$. In this case $\beta_k$ and $\beta_h$ denoted connected beta numbers, see Figure 2.a.
2. $(c, -n - 1), (c, -n - 2), (c + 1, -n - 1), (c + 1, -n - 2)$. In this case $\beta_k$ and $\beta_h$ denoted connected beta numbers, see Figure 2.b.
3. $(c, -n), (c, -n + 1), (c + 1, -n), (c + 1, -n + 1)$ where $n < -1$. In this case $\beta_k$ and $\beta_h$ denoted connected beta numbers, see Figure 2.c.
4. $(c + 1, -n), (c, -n), (c + 1, -n - 1), (c, -n - 1)$ where $c > 1$. In this case $\beta_k$ and $\beta_h$ denoted connected beta numbers, see Figure 2.d.
5. $(c + y, -n), (c + y + 1, -n), (c + y, -n - 1), (c + y + 1, -n - 1)$ where $y$ is a positive integer. In this case $\beta_k$ and $\beta_h$ denoted disconnected beta numbers, see Figure 2.e.
6. $(c - y, -n), (c - y - 1, -n), (c - y, -n - 1), (c - y - 1, -n - 1)$
where $0 \leq c - y - 1$ and $y$ is a positive integer. In this case $\beta_k$ and $\beta_h$ denoted disconnected beta numbers, see Figure 2.f.

7. $(c, -n - y - 1), (c, -n - y - 2), (c + 1, -n - y - 1), (c + 1, -n - y - 2)$ where $y$ is a positive integer. In this case $\beta_k$ and $\beta_h$ denoted disconnected beta numbers, see Figure 2.z.

8. $(c + x, -n + x), (c + x + 1, -n + x), (c + x + 1, -n + x - 1)$ where $x$ is a positive integer, $0 \leq c + x$ and $x \leq n$. In this case $\beta_k$ and $\beta_h$ denoted disconnected beta numbers, see Figure 2.j.

9. $(c \pm v, -n \pm \delta), (c \pm v + 1, -n \pm \delta), (c \pm v, -n \pm \delta - 1), (c \pm v + 1, -n \pm \delta - 1)$ where $c > v$, $\delta \leq n$. In this case $\beta_k$ and $\beta_h$ denoted disconnected beta numbers, see Figure 2.w.

Figure 1. Connected and disconnected beta numbers

4. Some combinatorial result

Definition 4.1 A sequence of beta numbers are connected if there are path between each two of the beta numbers.

Definition 4.2 The James abacus which configuration of connected beta number sequence called James abacus connected.

Definition 4.3 polyominoe is an James abacus connected. Or polyominoe is a graphical represent of connected beta numbers sequence, see Figure 2

Figure 2: James abacus where beta number sequence is $\{0, 1, 2, 3, 6, 7, 8, 9, 10, 11, 14, 15\}$ and $e = 4$.

After redefine the polyominoes as James abacus connected. In the next section we use some Some combinatorial result to solve one of the polyominoes problem if the polyominoes is an 2-abacus connected.

During this section we found new combinatorial method to determined the number of polyominoes beta which is a represents of n-connected beta number sequence if $e = 2$. The polyominoes beta
numbers sequence will be distributed in two columns \((c_1, c_2)\).

4.1 Enumerating of fixed polyominoes beta of s-connected beta number sequence if \(e = 2\).

Theorem 4.1.1 Let \(s\)-connected beta numbers \(s - j\)-beta number sequence. Then the enumerating of fixed polyominoes beta of \(s\)-connected beta number sequence if \(e = 2\) is

\[
1 + 2(s - 3) + 2\binom{s-j}{j}.
\]

Where \(j\)-beta number sequence location in \(c_1\) and \(s - j\)-beta number sequence location in \(c_2\) for \(s, j\) are positive integer.

Proof: For any polyominoes beta of \(s\)-connected beta number sequence with \(e = 2\) there are two sub sequences. First sub sequence have and location in first column of polyominoes beta \((c_1)\). Second sub sequence have \(s - j\)-beta numbers and location in second column of polyominoes beta \((c_2)\), where \(s\) and \(j\) are positive integer.

Step one
In this step there is a bijection from \(j\)-beta numbers sequence to \(s - j\)-beta numbers sequence

- If \(j = 1\) then \(j\)-beta numbers sequence will be connected with one beta number location in \(c_1\) and have \(s - j\) selection to connected with \(s - j\)-beta numbers sequence. Thus the enumeration of fixed polyominoes beta of \(s\)-connected beta numbers sequence with \(e = 2\) in this case equal to \(\binom{s-j}{1}\).
- If \(j = 2\) then \(j\)-beta numbers sequence will be connected with two with beta numbers of \(s - j\) selection to connected. Thus the enumeration of fixed polyominoes beta of \(s\)-connected beta numbers sequence with \(e = 2\) in this case equal to \(\binom{s-j}{2}\).
- In general the enumeration of fixed polyominoes beta of \(s\)-connected beta numbers sequence with \(e = 2\) such that \(c_i\) have \(j\) beta numbers and \(c_2\) have \(s - j\) beta numbers equal to .

Step two
In this step there is a bijection from \(s - j\)-beta numbers sequence to \(j\)-beta numbers sequence then we have same result. Thus the enumeration of fixed polyominoes beta of \(s\)-connected beta numbers sequence with \(e = 2\) if have bijection between \(j\) beta numbers sequence and \(s - j\) beta numbers sequence equal to \(2 \binom{s-j}{j}\).

Step three
- Larger beta number of \(j\) beta numbers sequence will be connection with smaller beta number of \(s - j\) beta numbers sequence
  1. If \(j = 1\) then we have same case in step one.
  2. If \(j = s - 1\) then we have same case in step one.
  3. If \(1 < j < s - 1\) then in this case the enumeration of fixed polyominoes beta of \(s\)-connected beta numbers sequence with \(e = 2\) and \(j \leq s - j\) equal to \((s - 3)\).
- Larger beta number of \(s - j\) beta numbers sequence will be connection with smaller beta number of \(j\) beta numbers sequence
  1. If \(s - j = 1\) then we have same case in step one.
  2. If \(s - j = s - 1\) then we have same case in step one.
  3. If \(1 < s - j < s - 1\) then the enumeration of fixed polyominoes beta of \(s\)-connected beta numbers sequence with \(e = 2\) and \(j > s - j\) equal to \((s - 3)\).
Thus polyominoes beta of s-connected beta numbers sequence with $e = 2$ such that smaller with larger beta number of other subsequence of connected equal to $2(s - 3)$.

Step four

In this step all beta number of s-connected beta number sequence are location in one column. Hence the enumeration of fixed of s-connected beta numbers sequence with $e = 2$ equal to

$$1 + 2(s - 3) + 2^{(s-1)}.$$ 

Remark: 4.1.2 Let $\beta_h = me + n, \beta_l = we + y$ are two beta numbers of partition $\mu$. Then

1. $|\beta_h - \beta_l| = 1$ if $\beta_h - \beta_l = 1$ and $m = w$.
2. $|\beta_h - \beta_l| = e$ if $\beta_h - \beta_l = e$ and $n = y$. 

Where $m,n,w,y$ are integers such that $0 \leq h, l \leq b$ and $h < l$ where $b$ is number of parts partition.

Definition 4.1.3 Two beta $\beta_h, \beta_l$ of the beta sequence $\{\beta_1, \beta_2, \ldots, \beta_b\}$ for partition $\mu = (\mu_1, \mu_2, \ldots, \mu_b)$ are connected if it has achieved one of the following condition

1. $|\beta_h - \beta_l| = 1$.
2. $|\beta_h - \beta_l| = e$.

Such that $1 \leq h$ and, $i \leq b$ where $b$ is the number of parts partition $\mu$.

Definition 4.1.4 Three beta $\beta_h, \beta_l, \beta_y$ of the beta sequence $\{\beta_1, \beta_2, \ldots, \beta_b\}$ for partition $\mu = (\mu_1, \mu_2, \ldots, \mu_b)$ are connected if it has achieved one of the following

1. $|\beta_h - \beta_l| = 1$ and $|\beta_l - \beta_y| = 1$.
2. $|\beta_h - \beta_y| = e$ and $|\beta_y - \beta_l| = 1$.
3. $|\beta_h - \beta_y| = 1$ and $|\beta_y - \beta_l| = e$.
4. $|\beta_h - \beta_y| = e$ and $|\beta_y - \beta_l| = e$.

Such that $1 \leq h$ and, $i \leq b$ where $b$ is the number of parts partition $\mu$.

Definition 4.1.5. A path between two beta $\beta_h, \beta_l$ of the beta sequence $\{\beta_1, \beta_2, \ldots, \beta_b\}$ for partition $\mu = (\mu_1, \mu_2, \ldots, \mu_b)$ is a finite sequence $q_\beta$ of beta numbers such that any beta belong to $q_\beta$ except $\beta_h, \beta_l$ must connected with two elements in $q_\beta$ and each $\beta_h, \beta_l$ are connected with one beta in $q_\beta$. Such that $1 \leq h$ and, $i \leq b$ where $b$ is the number of parts partition $\mu$ s show in figure 3.
Theorem and corollary proved

\[ \gamma_j \geq 1, \gamma_j \leq C \]

Based on definition 8, 9 and 10 there are path between any two beta number in first row. Based on definition 4.1.1 and James abacus diagram of any beta number sequence is a polyomino for special number of columns \( e \). Next section we will prove James abacus diagram of any beta number sequence is a polyomino by using e-core idea.

4.2 e-core

e-core of any partition \( \mu \) can be obtained by pushing all of the beta numbers in James abacus diagram up to their highest possible beta number in each column. The following theorem and corollary proved any e-core is one or more polyominos.

Theorem 4.2.1 Let \( \gamma = (\gamma_1^{r_1}, \gamma_2^{r_2}, \ldots, \gamma_m^{r_m}) \) is partition of integer \( \Gamma \) and \( \tau_j \geq e \) then e-core of \( \gamma \) is polyomino.

Proof:

e-core is configuration of partition \( \tau_j \) by moving all beta number as high as possible in each column of James abacus. Based on definition 4.1.1 the beta numbers in each runner are connected beta, Since \( \tau_j \geq e \) then all beta number in 1,2,...-1 (beta number in first row) are connected beta number. Based on definition 9 and 10 there are path between any two beta number in e-core diagram. Thus e-core of James abacus diagram is polyomino beta if \( \tau_j \geq e \).

Corollary 4.2.2 Let \( \gamma = (\gamma_1^{r_1}, \gamma_2^{r_2}, \ldots, \gamma_m^{r_m}) \) is partition of integer \( \Gamma \) and \( \tau_j \geq e \) then e-core of \( \gamma \) is polyomino.

Proof:

Let \( C_j \) be the number of columns which headed by position \( j \)

1. If \( g \leq j \leq y \) and \( C_j \) have \( k_j \) beta such that \( 0 \leq g \leq e - 1 \) and \( 0 \leq y \leq e - 1 \) where \( g \) and \( y \) are position integers then \( \rho_p = \{ek^y + C_y, ek^{y-1} + C_{y-1}, \ldots, ek^0 + C_0\} \) is polyomino for each sub partition of \( \rho \) where \( \rho \) is e-core partition. Then \( \rho \) is a polyomino beta.
2. If \( j \) is random numbers then \( C_j = C_j \cup C_j \cup \ldots \) such that \( \bar{g} \leq j \leq \bar{v} \) and \( \bar{g} \leq j \leq \bar{v} \) where \( \bar{g}, \bar{j}, \bar{g}, \bar{v} \) and \( \bar{y} \) are positive integers then 
\[
\rho_p = \rho_{\bar{y}} \cup \rho_{\bar{y}} \cup \ldots \\
\rho = \{ek\bar{y} + C_y, ek\bar{y}^{-1} + C_y, \ldots, ek\bar{y} + C_g\} \cup \{ek\bar{y} + C_g, ek\bar{y}^{-1} + C_g, \ldots, ek\bar{y} + C_g\}
\]

5. Creating tiling with James abacus for \( \beta \)-number

In this section we will use James abacus diagram to found polyomino by using beta number. Polyomino is a finite subset of square tiling Where the squares are connected by the edges. By using James diagram we can found encoding for any polyomino as partition such that any beta number represented one of the polyomino square. Polyominoes will be classified according to the number of beta number (head position). Each successive beads are located on the same column or same row in the James diagram. In Figure 5 we use James abacus diagram to found polyomino by using beta number. (a) \( \mu = (3^2, 2) \), \( \beta = (2, 4, 5, 6) \) and \( e = 2 \). (b) \( \mu = (3^2, 2, 1^2), \beta = \{1, 2, 4, 6, 7\} \) and \( e = 3 \). (c) \( \mu = (1^5, 0) \), \( \beta = \{0, 2, 3, 4, 5, 6\} \) and \( e = 2 \). (d) \( \mu = (3^2, 2, 1^2, 0^2), \beta = \{0, 2, 3, 4, 5, 6\} \) and \( e = 2 \). (e) \( \mu = (8, 6, 5^4, 4, 2) \), \( \beta = \{2, 4, 7, 8, 9, 10, 11, 15\} \), \( e = 3 \). (f) \( \mu = (3^3, 2, 1^4, 0) \), \( \beta = \{0, 2, 3, 4, 5, 7, 9, 10, 11\} \) and \( e = 3 \) every one of previous shapes in Figure 5 have general form. \( \beta = \{me + n, (m + 1) e + n, (m + 1) e + n + 1, (m + 2) e + n + 1\} \) is a general form of (a). \( \beta = \{me + n, me + n + 1, (m + 1) e + n, (m + 2) e + n + 1\} \) where \( n > 0 \) is a general form of (b). \( \beta = \{me + n, (m + 1) e + n, (m + 1) e + n + 1, (m + 2) e + n, (m + 2) e + n + 1\} \) where \( n > 0 \) is a general form of (c). \( \beta = \{me + n, me + n + 1, (m + 1) e + n, (m + 2) e + n, (m + 3) e + n, (m + 4) e + n, \ldots\} \) is a general form of (d). \( \beta = \{me + n, (m + 1) e + n, (m + 1) e + n + 1, (m + 2) e + n + 1, (m + 3) e + n + 1, (m + 4) e + n + 1, (m + 5) e + n + 2\} \) is a general form of (e). \( \beta = \{me + n, (m + 1) e + n, (m + 1) e + n + 2, (m + 1) e + n, \ldots\} \) is a general form of (f).

![Figure 5](image-url)

**Figure 5.** The 6 polyomino (a) tetromino (b) pentomino (c) hexomino (d) heptomino

The number of way there to tiling a region with a fixed set of tiles is an interesting problem in polyominoes. We bid to solve this problem in special case by using James abacus diagram. In [8] there are tiling of a rectangle by rectangles called simple tiling. There are construct rectangle tiling from three parts called left simple tiling, middle simple tiling and right simple tiling. In this work we construct previous three parts by using James diagram with connected sequence of beta number and its properties. The general partition of the three tiling are:

- right simple: \( y = (y_1^x, y_2^x, \ldots, y_m^x) \) where \( y_j = (m + 1)e \). \( y_{j-a} = (m + a + 1)e - \sum_{v=j}^{j-a+1} \lambda_v \) and \( \lambda_v = k, k + 1, \ldots, j = k + j - 1 \) as shown in Figure 6.
Figure 6. right simple where \( \gamma = (20, 14^2, 9^3, 5^4, 2^5, 0^{13}) \) and \( e = 7 \).

• \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{j+1}) \) where \( \gamma_{j+1} = 0, \gamma_{j-a+1} = \gamma_{j-a+2} + k + a - 1, \tau_{j+1} = (m + 1)e - 1, \tau_{j-a+1} = e - k - a + 1 \) and \( a = 1, 2, \ldots, j - 1 \). As shown in Figure 7.

Figure 7. right simple where \( \gamma = (28^6, 26^5, 23^4, 19^3, 14^2) \) and \( e = 7 \).

• Left simple: this tiling consist only of empty columns.

For any \( m, r, k \) such that \( m = j = r - 1, k + j \leq e \).

Our primary focus will be on the number of ways to tile an James diagram chain. If we imagine every James diagram positions is a cell. So James diagram have \( er \) cells. Based on [7] James abacus consist of \( i \) chains. Let \( S \) is a sequence of empty position (-) and the bead (o). We assume that position of bead (length 2) will tile with two square and empty bead of length 1 will tile with one square. \( S^{J-n} \)is the sum of all empty positions \((n)\) and bead positions \(J-n\). Then

\[
S^{j-1} = S_n^j + S_{j-n}^j.
\]

For any sub sequence \( S \) we need \( 2K+D \) square to tile it, where \( K \) is the number of bead and \( D \) is the number of empty bead. Thus we have \( f_n \) ways to tiling a \( S \) or any sub sequence \( S \).

Table 1 Illustrates the values of \( f_n \) for some \( n \). \( f_n \) begins like the Fibonacci numbers and it will continue to grow like Fibonacci number.

Example 5.1

1- if \( S = \{ - - - - \} \), then we need 4 square (length 4)
2- if \( S = \{ o \} \), then we need 1 domino (length 2)
3- if \( S = \{ - , o \} \), then we need 1 square 1 domino (length 3)
4- if \( S = \{ - , - , o , - , o \} \), then we need 3 square 2 domino (length 7)
Table 1. Illustrates the values of \( f_n \) for some \( n \).

Table 1 Illustrates the values of \( f_n \) for some \( n \). \( f_n \) begins like the Fibonacci numbers and it will continue to grow like Fibonacci number and it will continue to grow like Fibonacci number. Let \( f_0 \) count the tiling of the empty sequence. thus

\[
f_n = f_{n-1} + f_{n-2}
\]

We late \( f_0 = 1 \) count the empty tiling of the o-board. Hence

\[
f_n = F_{n-1}
\]

Theorem 5.2 Let \( n = K + 2D \) be the number of tiling \( S \) then

\[
f_n = \sum_{n=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-D}{D}
\]

Proof

Let \( D \) the number of dominoes and \( n - 2D \) the number of squares then, the number of tiling \( K + 2D \) is \( \binom{n-2D+D}{D} \). Thus, the number of way to tile \( n \)-board by dominoes and squares are

\[
f_n = \sum_{n=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-D}{D}
\]

6. Conclusion

In this paper we proved the New graphical representation of polyominos by using James abacus diagram and Beta number. Further, the concept of the partition was used to provide a representation for each connected Beta number. In addition, The \( e \)-core was used to representation any polyominos. Then, we creating a tilling with James abacus for \( \beta \)-number. Finally, A New identities of Fibonacci to count the number of ways for \( 2 \times 1 \) dominoes to tiling a James abacus diagram chain by using monomino and domino was development.

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Reference

[1] Papadimitriou F. 2020 The Geometric Basis of Spatial Complexity. In Spatial Complexity, Springer, Cham. p 39-50
[2] Bloom G, Golomb S 1977 Applications of numbered undirected graphs. Proceedings of the IEEE. 65(4), 562-570.
[3] Barequet, G, Shalah, M, Zheng Y 2019 An improved lower bound on the growth constant of polyiamonds. Journal of Combinatorial Optimization, 37(2), 424-438.
[4] Rhoads G 2005 Planar tilings by polyominoes, polyhexes, and polyiamonds. Journal of Computational and Applied Mathematics, 174(2), p 329-353.
[5] Massé A, Carufel, J, Goupil A 2018 Non saturated polyhexes and polyiamonds. In GASCom, p 116-123.
[6] Mohommed E, Ibrahim H, Ahmad N 2017 Enumeration of n-connected ominoes inscribed in an abacus. JP Journal of Algebra and application, 39(6), 843-874.
[7] Mohommed, E 2017 A family of classes in nested chain abacus and related generating functions (Doctoral dissertation, Universiti Utara Malaysia).
[8] Benjamin A, Quinn J 1999 Recounting Fibonacci and Lucas Identities. The College Mathematics Journal, 30(5), 359-366.
[9] Benjamin A, Crouch J, Sellers 2019 Unified tiling proofs of a family of Fibonacci identities. Fibonacci Quarterly, 57(1), P 29-31
[10] Chung F and Gilbert E Graham R Shearer J van Lint J 1982 Tiling rectangles with rectangles, journal Mathematics Magazine, 55(5), p 286-291.