$B^0_{(s)} - \bar{B}^0_{(s)}$ Mixing and $b$ Hadron Lifetimes from Lattice QCD

Jonathan Flynn and C.-J. David Lin
Department of Physics and Astronomy, the University of Southampton, Southampton SO17 1BJ, UK

Abstract. We discuss neutral $B_{s,d}$ meson mixing and $b$-hadron lifetimes from the perspective of recent lattice calculations. In particular, we consider matrix elements which can be combined with measured $\Delta M_{s,d}$ to constrain $|V_{td}|$, the lifetime ratios $\tau(\Lambda_b)/\tau(B^0_d)$ and $\tau(B^-)/\tau(B^0_d)$ and the lifetime difference, $\Delta \Gamma_{B_s}/\Gamma_{B_s}$, in the neutral $B_s$ meson system.
1. Introduction

In this workshop, the lifetimes and mixings working group identified a ‘wish list’ of hadronic quantities whose accurate theoretical determination is crucial for understanding current and future $b$-physics experimental results. We discuss several of these quantities from the perspective of recent lattice calculations:

- matrix elements which can be combined with measured $\Delta M_{s,d}$ to constrain $|V_{td}|$
- the lifetime ratios $\tau(B_s)/\tau(B_d^0)$ and $\tau(B^-)/\tau(B_d^0)$
- the lifetime difference, $\Delta \Gamma_{B_s}/\Gamma_{B_s}$, in the neutral $B_s$ meson system.

For a full survey of recent $b$-physics results from the lattice see the reviews in [1] [3].

2. $\Delta M_d$ and $\Delta M_s/\Delta M_d$

The mass difference of the $B_d-\bar{B}_d$ system, $\Delta M_d$, constrains the poorly known CKM matrix element $|V_{td}|$. $\Delta M_d$ has been experimentally measured to good accuracy. On the theory side, the main uncertainty comes from the long-distance strong-interaction effects in the matrix element

$$\mathcal{M}_{B_d}(\mu) = \langle \bar{B}_d|Q_{L_d}(\mu)|B_d \rangle,$$

which appears in the Standard Model prediction for $\Delta M_d$ to leading order in an expansion in $1/M_W$ [4,5]. In Eq. (1), $Q_{L_d}$ is the four-quark operator $[\bar{b}\gamma^\mu(1-\gamma^5)d][\bar{b}\gamma^\mu(1-\gamma^5)d]$ and $\mu$ is the renormalisation scale.

An alternative approach, in which many theoretical uncertainties cancel, is to consider the ratio, $\Delta M_s/\Delta M_d$, where $\Delta M_s$ is the mass difference in the neutral $B_s-\bar{B}_s$ system. In the Standard Model, one has

$$\frac{\Delta M_s}{\Delta M_d} = \frac{|V_{ts}|^2}{|V_{td}|^2} \left( \frac{M_{B_d}}{M_{B_s}} \right) \left( \frac{\langle B_s|Q_{L_s}|B_s \rangle}{\langle B_d|Q_{L_d}|B_d \rangle} \right) = \frac{|V_{ts}|^2}{|V_{td}|^2} \left( \frac{M_{B_s}}{M_{B_d}} \right) \xi^2,$$

(2)

where $Q_{L_s}$ is the same operator as $Q_{L_d}$ with $d$ replaced by $s$ and where the renormalisation-scale dependence of these operators cancels in the ratio. Because the unitarity of the CKM matrix implies $|V_{ts}| \simeq |V_{td}|$ and because $|V_{cb}|$ can be accurately obtained from semileptonic $B$ to charm decays, a measurement of $\Delta M_s/\Delta M_d$ determines $|V_{td}|$. This is experimentally very challenging because of the rapid oscillations in the $B_s^0-\bar{B}_s^0$ system. Nevertheless, the experimental lower bounds on $\Delta M_s/\Delta M_d$ already yield interesting constraints on the $b-d$ unitarity triangle [3].

The matrix elements in Eqs. (1) and (2) are traditionally parameterised by

$$\mathcal{M}_{B_q}(\mu) = \langle \bar{B}_q|Q_{L_q}(\mu)|B_q \rangle = \frac{8}{3} M_{B_q}^2 f_{B_q}^2 B_{B_q}(\mu),$$

(3)

with $q = d$ or $s$, where the $B$-parameter, $B_{B_q}$, measures deviations from vacuum saturation, corresponding to $B_{B_q} = 1$, and $f_{B_q}$ is the leptonic decay constant. One also usually introduces a renormalisation-group invariant and scheme-independent parameter $\hat{B}_{B_q}$, which to NLO in QCD is given by

$$\hat{B}_{B_q}^{\text{NLO}} = C_B(\mu) B_{B_q}(\mu),$$

(4)
where $C_B$ is the two-loop Wilson coefficient calculated in the same scheme as the matrix element [7,10].

Because $M_{B_s}$ and $M_{B_d}$ are measured experimentally, one needs to calculate $\xi$ and $f_{B_d}\sqrt{M_{B_d}}$ non-perturbatively to determine $|V_{td}|$ from experimental results for $\Delta M_s/\Delta M_d$ and $\Delta M_d$.

There are three recent quenched lattice calculations of $\xi$ [§]. The APE [13] and UKQCD [12] (with preliminary results reported previously in [14]) Collaborations use relativistic formulations of quarks and obtain $\xi$ at the physical $B$ meson mass by extrapolating from heavy-meson masses around that of the $D$. Giménez and Martinelli (denoted as GM in the following) [15] calculate this quantity in the static limit where the $b$ quark mass is taken to infinity. The results from these three groups are:

$$
\xi = \begin{cases} 
1.16(7) & \text{(APE)} \\
1.16(2^{+2}_{-3}) & \text{(UKQCD)} \\
1.17(3) & \text{(GM)},
\end{cases}
$$

(5)

where the first error is statistical and the second is systematic. In order to avoid the situation where discretisation errors, which can be significant in lattice calculations involving propagating heavy quarks, are out of control, both APE and UKQCD perform numerical simulations with several meson masses straddling the $D$ and then extrapolate to the $B$ meson mass using HQET-inspired relations. APE use a fully $O(a)$ ($a$ is the lattice spacing) improved fermion action [16] in which the leading discretisation error is $O(a^2)$. UKQCD use a mean-field improved fermion action [17], and the leading discretisation error here is formally $O(a\alpha_s)$, although it might be numerically smaller. However, it should be noted that both groups have $O(a\alpha_s)$ discretisation errors in $\xi$, because the four-quark operators $Q_{Lq}$ are not fully $O(a)$ improved in these calculations. These $O(a\alpha_s)$ errors are absent in the decay constants, and hence $f_{B_d}/f_{B_s}$, as calculated by APE, because they use the improved lattice axial current in the calculation.

In Eq. (5), UKQCD’s result for $\xi$ has a much smaller statistical error than that obtained by APE, although these two groups have very similar statistics in the Monte-Carlo simulations. This is because UKQCD calculate $\xi$ by performing heavy-quark-mass extrapolations in the ratios $f_{B_s}\sqrt{M_{B_s}}/f_{B_d}\sqrt{M_{B_d}}$ and $B_{B_s}/B_{B_d}$, in which the $1/m_Q$ corrections cancel significantly, especially for the decay constants [12,14]. This procedure determines $\xi$ to a better statistical accuracy than that calculated by extrapolating $f_{B_s}\sqrt{M_{B_s}}$, $f_{B_d}\sqrt{M_{B_d}}$, $B_{B_s}$, and $B_{B_d}$ individually.

The above three studies suggest that $\xi$ has small systematic uncertainties arising from discretisation effects and heavy-quark-mass extrapolation[¶], within the quenched approximation. The UKQCD result includes a systematic error[¶] not estimated by APE.

§ The $SU(3)$ breaking ratio $M_{B_s}/M_{B_d}$ has also been studied in [11,12] by using a different method, in which one does not need to calculate $\xi$. However, results from this method have large uncertainties.

∥ As mentioned above, the authors of [12] look at the $1/m_Q$ corrections for $(f_{B_s}\sqrt{M_{B_s}})/(f_{B_d}\sqrt{M_{B_d}})$ and $B_{B_s}/B_{B_d}$ and find that they are very small for both quantities.

¶ This systematic error looks small on $\xi$. However, it should be noted that it is the error in $\xi$’s deviation from unity.
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and GM. This is due to the uncertainty in the determination of the lattice spacing (see below for details), which introduces a variation of the strange-quark mass.

Quenched Chiral Perturbation Theory ($q\chi$PT) predicts that quenching errors in $B_{Bs}/B_{Bd}$ are small if the couplings in the theory are constrained by large-$N_c$ arguments [18]. Recent numerical studies with two flavours of dynamical quarks show little variation in $f_{Bs}/f_{Bd}$ compared to its quenched value [19, 22]. Therefore, quenching effects should not be significant in $\xi$. For more detailed discussions on this issue, please refer to [23].

APE [13] and UKQCD [12] also calculate $f_{Bd}\sqrt{\hat{B}_{Bd}^{\text{nlo}}}$ in quenched approximation. The results are:

$$f_{Bd}\sqrt{\hat{B}_{Bd}^{\text{nlo}}} = \begin{cases} 206(28)(7) \text{ MeV} & \text{(APE)} \\ 211(21)(28) \text{ MeV} & \text{(UKQCD)} \end{cases},$$

where the first error is statistical and the second is systematic.

The systematic error in APE’s result for $f_{Bd}\sqrt{\hat{B}_{Bd}^{\text{nlo}}}$ reflects the typical size of the $O(a^2)$ discretisation effects in $f_{Bd}$, a very small ($\sim 2\%$) uncertainty in the inverse lattice spacing and the uncertainty in matching the lattice-regularised four-quark operators onto the NDR-MS scheme, with these three errors taken in quadrature.

In addition to the $O(a^2)$ and $O(\alpha_s)$ discretisation errors in $f_{Bd}$, UKQCD investigate the uncertainties in the procedure of operator matching, heavy-quark-mass extrapolations and a $\pm 7\%$ uncertainty in the inverse lattice spacing. The systematic error in UKQCD’s result is obtained by taking these uncertainties in quadrature.

The overall systematic error on $f_{Bd}\sqrt{\hat{B}_{Bd}^{\text{nlo}}}$ obtained by UKQCD is much larger than that obtained by APE (see Eq. (6)), mainly because of the estimate of the error in the determination of the inverse lattice spacing, $a^{-1}$. APE use $a^{-1}$ set by the method described in [24] for their central value, and the ones set by $M_\rho$ and $M_\phi$ to get the systematic uncertainty. This method results in a 2% systematic error. UKQCD determine $a^{-1}$ in conjunction with the strange-quark mass by $f_K$ and $M_K$, and then vary $a^{-1}$ by $\pm 7\%$, a range which covers the typical variations of $a^{-1}$ set by gluonic and light-hadron spectral quantities with the same action used in their calculation [25]. This procedure introduces a 9% effect on their result for $f_{Bd}\sqrt{\hat{B}_{Bd}^{\text{nlo}}}$.

Both APE and UKQCD do not attempt to quantify the discretisation errors in the $B$ parameters, as this requires one to consider the mixing with dimension-seven four-fermion operators and is therefore very involved. Nevertheless, $B_{Bq}$ are ratios between closely related matrix elements, hence one expects that these errors might cancel significantly. As mentioned above, the fermion actions used by APE and UKQCD have different discretisation errors. However, as displayed in Table 1 and Figure 1, their results of the $B$ parameters show very good agreement. Furthermore, UKQCD perform the calculations at two lattice spacings and observe that the change in $B_{Bq}$
Table 1. Comparison of the latest lattice results for $B_{B_q}$ parameters renormalised in NDR-\MS scheme at $m_b$. The first error on each result is statistical, while the second one is systematic. For a more complete list of these results, including the earlier calculations, please refer to [1–6].

| Group               | $B_{B_d}(m_b)$ | $B_{B_s}(m_b)$ |
|---------------------|----------------|----------------|
| APE ($a^{-1} \approx 2.7$ GeV) [13] | 0.93(8)$^{+0}_{-1}$ | 0.92(3)$^{+0}_{-1}$ |
| UKQCD ($a^{-1} \approx 2.7$ GeV) [12] | 0.92(4)$^{+3}_{-0}$ | 0.91(2)$^{+3}_{-0}$ |
| UKQCD ($a^{-1} \approx 2.0$ GeV) [12] | 0.90(4)$^{+3}_{-0}$ | 0.92(2)$^{+3}_{-0}$ |
| BBS ($a^{-1} \rightarrow \infty$) [11] | 0.95(12) | 0.95(12) |
| NRQCD ($a^{-1} \approx 2.3$ GeV) [26, 27] | 0.84(2)$^{(8)}$ | 0.87(1)$^{(9)}$ |

Figure 1. Comparison between APE and UKQCD’s results of renormalisation group invariant $B$ parameters at different meson masses (from [13]). The inverse lattice spacing in this plot is approximately 2.7 GeV. $M_P$ is the heavy-meson mass in lattice units (i.e., one should multiply it by 2.7 to obtain the mass in GeV in this case). The open symbols are the results from numerical simulations, and the closed symbols are those extrapolated to the $B_d$ meson mass.

is insignificant. Finally, both groups obtain $B_{B_q}$ compatible with those in a slightly earlier work by Bernard, Blum and Soni (BBS) [11], who use a less improved fermion action but extrapolate their results to the continuum limit (See Table [1]). Recently, these $B$ parameters have also been calculated using NRQCD to describe the heavy quarks [26, 27], with the heavy-quark masses in numerical simulations around that of the $b$. The central values of these results are 2 standard-deviations lower than those obtained by APE and UKQCD. However, as Table [1] shows, when the systematic errors are taken into account, all these results are compatible.
UKQCD also normalise $f_{B_d} \sqrt{B_{B_d}^{\text{nlo}}}$ by $f_{D_s}$ because some systematic errors cancel, and there have been preliminary experimental results of $f_{D_s}$ \cite{28,29}. They obtain \cite{12}:

$$\frac{f_{B_d}}{f_{D_s}} \sqrt{B_{B_d}^{\text{nlo}}} = 0.89(7)^{+6}_{-6},$$

where the first error is statistical and the second one is systematic.

While quenching errors in $B_{B_q}$ are predicted to be within a few percent by q$\chi$PT with reasonable ranges of the couplings in the theory \cite{18}, they can be significant for the decay constants. Recent numerical studies with two flavours of dynamical quarks \cite{19,22} show visible increases on the decay constants. Again, please refer to \cite{23} for more detailed discussions on quenching errors.

Finally, there are on-going analyses \cite{1,30,31} in which results from the static-limit and relativistic-heavy-quark calculations are combined. In such analyses, $f_{B_d} \sqrt{B_{B_d}^{\text{nlo}}}$ and $B_{B_q}$ are obtained by interpolations between the charm-mass region and the limit of infinite heavy-quark mass.

### 3. Spectator Effects on $b$-Hadron Lifetimes

The combination of operator product and heavy quark expansions predicts that the ratio of lifetimes of two hadrons, $H_1$ and $H_2$, each containing a single $b$-quark, is given by \cite{32},

$$\frac{\tau(H_1)}{\tau(H_2)} = 1 + \frac{\mu_\pi^2(H_1) - \mu_\pi^2(H_2)}{2m_b^2} + c_G \frac{\mu_G^2(H_1) - \mu_G^2(H_2)}{m_b^2} + O(1/m_b^3).$$

Here the leading 1 on the right hand side arises from a universal first term describing free $b$-quark decay. There is no term of order $1/m_b$ in the expansion. At order $1/m_b^2$, the $\mu_\pi^2$ and $\mu_G^2$ terms are matrix elements in the heavy quark effective theory of the kinetic energy and chromomagnetic moment operators respectively between the corresponding hadron states. They can be fixed from hadron mass formulas and mass splittings. The coefficient $c_G$ is calculated in \cite{33,34}. Therefore, for the cases of the ratios of $B^-$ and $B^0_d$ lifetimes or $\Lambda_b$ and $B^0_d$ lifetimes, we have calculated values to compare to experimental results:

$$\frac{\tau(B^-)}{\tau(B^0_d)} = \begin{cases} 1 + O(1/m_b^3) & \tau(\Lambda_b) = 0.98 + O(1/m_b^3) \\ 1.066(20) \ [37] & 0.794(53) \ [37] \end{cases}$$

For the $B$-meson ratio, there is no difficulty. However, for the $\Lambda_b$ to $B$ ratio, the $O(1/m_b^3)$ terms apparently must account for almost all of the deviation from one.

Some work has addressed the question of whether “spectator effects” in the $O(1/m_b^3)$ terms can be responsible for such large deviations. In the operator product expansion, the effects of the diagrams in figure \ref{fig:diagrams} lead to the appearance of $\Delta B = 0$ four-quark operators of the form $\bar{b} \Gamma_i q q \Gamma_j b$. These diagrams are the first time the effects of spectator quarks enter explicitly. Moreover, the diagrams are one-loop, compared to the two-loop...
diagrams which generate $O(1/m_b^2)$ terms in the expansion. There is thus a hope that these spectator effects could be large. The four quark operators are classified as

\begin{align}
O^u_1 &= b_L \gamma_\mu q_L \bar{q}_L \gamma^\mu b_L \\
T^u_1 &= b_L \gamma_\mu T^a q_L \bar{q}_L \gamma^\mu T^a b_L \\
O^q_1 &= b_R \gamma_\mu \bar{q}_L q_R b_R \\
T^q_1 &= b_R T^a q_L \bar{q}_L T^a b_R
\end{align}

where $q = u, d, s$. The spectator contribution to the decay rate of a hadron $H$ depends on $\langle H | O^{\text{spec}} | H \rangle$, where

\begin{equation}
O^{\text{spec}} = F_u(z) O^u_1 + G_u(z) T^u_1 + \sum_{i=1,2; q = d, s} (F_{iq}(z) O^q_i + G_{iq}(z) T^q_i) .
\end{equation}

Here $z = m_c^2/m_b^2$ and the $F$’s and $G$’s are coefficient functions known at leading order \cite{32}.

Given the coefficient functions, the remaining ingredient is the evaluation of the four quark matrix elements. Lattice QCD simulations of these \cite{38, 40} are reported here (for sum rule calculations see \cite{41, 42}). To date, these calculations are still exploratory for the $\Lambda_b$ meson, but the results are encouraging and the calculations deserve repeating.

It is convenient to parameterise the matrix elements. For mesons we use factors $B_i$ and $\epsilon_i$,

\begin{equation}
\frac{1}{2m_B} \langle B | O^u_1 | B \rangle = f_B^2 m_B B_1, \quad \frac{1}{2m_B} \langle B | T^u_1 | B \rangle = \frac{f_B^2 m_B}{8} \epsilon_1 ,
\end{equation}

for $i = 1, 2$. From vacuum saturation or large $N_c$ arguments the expectation is that $B_i \simeq 1$ and the $\epsilon_i$ are small. For the $\Lambda_b$ baryon, heavy quark symmetry implies that only two matrix elements need to be considered (up to $1/m_b$ corrections) \cite{32}. The corresponding parameters, $L_1$ and $L_2$ are defined by\cite{32}:

\begin{align}
\frac{1}{2m_B} \langle \Lambda_b | O^u_1 | \Lambda_b \rangle &= \frac{f_B^2 m_B}{8} L_1 , \quad \frac{1}{2m_B} \langle \Lambda_b | T^u_1 | \Lambda_b \rangle = \frac{f_B^2 m_B}{8} L_2 .
\end{align}

+ The conversion to the parameters used in \cite{32} is $\bar{B} = -6L_1$ and $r = -2L_2/L_1 - 1/3$. 

Figure 2. Spectator contributions: diagrams like those on the left, where the box denotes a $|\Delta B| = 1$ transition, produce 4-quark operators in the operator product expansion. $\Gamma_i$ denotes some combination of Dirac and colour matrices. See \cite{32}.
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In terms of these parameters the lifetime ratios can be expressed as,

\[ \frac{\tau(B^-)}{\tau(B^0_d)} = 1 + a_1 \epsilon_1 + a_2 \epsilon_2 + a_3 B_3 + a_4 B_4, \]  
(15)

\[ \frac{\tau(\Lambda_b)}{\tau(B^0_d)} = 0.98 + b_1 \epsilon_1 + b_2 \epsilon_2 + b_3 L_1 + b_4 L_4, \]  
(16)

where tiny \( B_{1,2} \) terms are neglected in the second equation. The coefficients \( a_i \) and \( b_i \) are known perturbatively,

\[ \begin{align*}
  a_1 &= -0.697 & b_1 &= -0.175 \\
  a_2 &= 0.195 & b_2 &= 0.195 \\
  a_3 &= 0.020 & b_3 &= 0.030 \\
  a_4 &= 0.004 & b_4 &= -0.252
\end{align*} \]

These values are quoted in the \( \overline{\text{MS}} \) scheme at a scale \( \mu = m_b \) and are to be combined with matrix elements evaluated in the same scheme.

For the \( B \) meson, the lattice calculations [38] have been done in the quenched approximation with a static \( b \) quark at an inverse lattice spacing of \( a^{-1} = 2.9 \) GeV (corresponding to an input lattice coupling parameter \( \beta = 6.2 \)). The results have been extrapolated from the light quark masses actually simulated (for technical reasons, one cannot simulate with realistically light quark masses) to the chiral limit. For the \( \Lambda_b \) meson, the matrix element involves a more complicated set of lattice quark propagator contractions, as shown in figure 3, and the calculation is still exploratory [39, 40]. It is also done with a static \( b \) quark, but on a coarser lattice \( (a^{-1} = 1.1 \) GeV or \( \beta = 5.7 \)) and using a stochastic technique to calculate the light quark propagators [43]. No chiral limit is taken in this case, the results are quoted for pion masses given by \( a m_\pi = 0.52(3) \) (case A) and \( a m_\pi = 0.74(4) \) (case B). The results are:

\[ \begin{align*}
  B_1 &= 1.06(8) & \epsilon_1 &= -0.01(3) \\
  B_2 &= 1.01(6) & \epsilon_2 &= -0.02(2)
\end{align*} \]

for the \( B \) and

\[ \begin{align*}
  L_1 &= \begin{cases} 
    -0.31(3) & \text{A} \\
    -0.22(4) & \text{B}
  \end{cases} \\
  L_2 &= \begin{cases} 
    0.23(2) & \text{A} \\
    0.17(2) & \text{B}
  \end{cases}
\end{align*} \]

Figure 3. Lattice quark propagator contractions needed to compute spectator effects on \( B \) and \( \Lambda_b \) lifetimes.
for the $\Lambda_b$. Combining these with the calculated coefficients $a_i$ and $b_i$ leads to,

$$\frac{\tau(B^-)}{\tau(B_d^0)} = 1.03(2)(3), \quad \frac{\tau(\Lambda_b)}{\tau(B_d^0)} = \begin{cases} 0.91(1) & A \\ 0.93(1) & B \end{cases}$$

(20)

We see that the answer is close to unity as expected for the $B$ meson ratio. For the $\Lambda_b$ to $B$ ratio, about 40% of the deviation from unity required to match experiment is reproduced. Given the exploratory nature of the $\Lambda_b$ calculation, this indicates that $1/m_b^3$ spectator effects could be large enough to suppress the $\Lambda_b$ lifetime compared to the $B$ meson lifetime and further lattice studies are warranted. Moreover, next-to-leading order calculations of the coefficient functions are in progress [44]. Experience with the $B_s$ lifetime difference, where the next-to-leading effects are significant, provides further motivation.

4. Width Difference of $B_s$ Mesons

The decay width difference of $B_s$ and $\bar{B}_s$ mesons arises from the forward off-diagonal matrix element of a time ordered product of weak effective Hamiltonians mediating $b$ quark decay. Applying the heavy quark expansion leads to an expression of the form [45, 46],

$$\Delta \Gamma_{B_s} = \frac{G_F^2 m_b^2}{12 \pi M_{B_s}} |V_{cb} V_{cs}|^2 \left( G(z) \langle Q_{L_s}(m_b) \rangle - G_S(z) \langle Q_S(m_b) \rangle + \hat{\delta}_{1/m} \sqrt{1 - 4z} \right)$$

(21)

where $z = m_c^2/m_b^2$ and angle brackets denote matrix elements between $B_s$ and $\bar{B}_s$, $\langle O \rangle = \langle \bar{B}_s | O | B_s \rangle$. The four-quark $\Delta B = 2$ operators $Q_{L_s}$ and $Q_S$ arise at leading order in the heavy quark expansion, while $\hat{\delta}_{1/m}$ denotes $1/m_b$ corrections involving further operator matrix elements. The coefficients $G$ and $G_S$ are known at NLO in QCD [16]. $Q_{L_s}$ is the same single operator that contributes to the mass difference, $\Delta M_{B_s}$, while

$$Q_S = (\bar{b}s)_{S-P}(\bar{b}s)_{S-P}. \quad (22)$$

Notice that $\langle Q_{L_s} \rangle$ is actually the matrix element $\mathcal{M}_{B_s}$ defined in Eq. (3). Using the standard parameterisation,

$$\langle Q_S(\mu) \rangle = -\frac{5}{3} f_{B_s}^2 M_{B_s}^2 \frac{M_{B_s}^2}{(m_b + m_s)^2} B_S(\mu), \quad (23)$$

and Eq. (3), one can write,

$$\frac{\Delta \Gamma_{B_s}}{\Gamma_{B_s}} = \frac{G_F^2 m_b^2}{12 \pi} |V_{cb} V_{cs}|^2 \tau_{B_s} f_{B_s}^2 M_{B_s} \times \left( \frac{8}{3} G(z) B_{B_s} + \frac{5}{3} G_S(z) \frac{M_{B_s}^2}{(m_b + m_s)^2} B_S + \delta_{1/m} \sqrt{1 - 4z} \right). \quad (24)$$

The coefficients $G(\mu)$ and matrix element parameters $B_{B_s}$ and $B_S$ depend on a renormalisation scale $\mu$ such that the result is in principle $\mu$ independent. Results quoted below have $\mu$ set to the $b$-quark pole-mass, $m_b$, taking $m_b = 4.6$ GeV. Note that different authors make different choices for the definitions of the quark masses appearing in the defining equation for $B_S$. Here they are the pole masses.
Input from lattice QCD is needed for the values of $f_{B_s}$ and the parameters $B_{B_s}$ and $B_S$. From the lattice viewpoint, it is convenient to trade uncertainty in determining $f_{B_s}$ for the appearance of extra CKM factors by considering $\Delta \Gamma_{B_s}/\Delta M_{B_s}$. The mass difference is proportional to $\langle Q_{L_s} \rangle$, so that the ratio $\Delta \Gamma_{B_s}/\Delta M_{B_s}$ depends on the quantity

$$\mathcal{R}(m_b) = \frac{\langle Q_S \rangle}{\langle Q_{L_s} \rangle} = -\frac{5}{8} \frac{B_S(m_b)}{B_{B_s}(m_b)} \frac{M_{B_s}^2}{(m_b + m_s)^2}. \tag{25}$$

Systematic errors from uncertainty in determining the lattice spacing and from quenching effects should be included in this dimensionless ratio of similar matrix elements. In the absence of an experimental measurement for $\Delta M_{B_s}$, one further uses the ratio $\xi$, as defined in Eq. [2], which is quite well determined by lattice calculations (see section 2). In this way one arrives at an expression [47]

$$\frac{\Delta \Gamma_{B_s}}{\Gamma_{B_s}} = \frac{4\pi}{3} \frac{M_{B_s} m_q^2}{M_{B_s} m_W^2} \left| \frac{V_{tb} V_{ts}^*}{V_{td} V_{ts}^*} \right|^2 \frac{\tau_{B_s} \Delta M_{B_s}}{\eta_B(m_b) S_0(x_t)} \xi^2 \left( G(z) - G_S(z) \mathcal{R}(m_b) + \delta_{1/m} \right). \tag{26}$$

The quantities $\eta(m_b)$ and $S_0(x_t)$, where $x_t = m_t^2/m_W^2$, come from the expression for $\Delta M_{B_s}$ and are known factors [7, 46]. At leading order in $1/m_b$ the non-perturbative contribution is isolated in $\mathcal{R}(m_b)$. At order $1/m_b$, one needs both the explicit $\delta_{1/m}$ piece together with the implicit $m_b$ dependence from the matrix elements of $Q_{L_s,S}$ in the ratio $\mathcal{R}$.

Three groups have recent calculations giving values for $\mathcal{R}$. They use different lattice formalisms, but give very consistent results for this dimensionless ratio. Becirevic et al. [17] use heavy quarks at around the charm mass and extrapolate to the $b$. The Hiroshima-KEK group [20] use lattice NRQCD, simulating directly at the $b$ quark mass. Finally, Giménez and Reyes [30, 49] work in the lattice static quark theory, and hence include $1/m_b$ corrections to the matrix elements as a systematic error. All groups have results from quenched simulations, but Giménez and Reyes [30] also have results with two flavours of degenerate sea quarks.

$$\mathcal{R}(m_b) = \begin{cases} 
-0.93(3)^{(0)} & \text{extrap } c \to b \ [47] \\
-0.91(5)(17) & \text{NRQCD } [20] \\
-0.95(7)(9) & \text{static } b \ [30, 49] \\
-0.97(5)(15) & \text{static } b, n_f = 2 \ [30] 
\end{cases} \tag{27}$$

The variation of about $\pm 3\%$ in the central values for $\mathcal{R}$ is smaller than the corresponding uncertainty of about $\pm 10\%$ in the values of $B_{B_s}$ and $B_S$.

Since the ratio $G(z)/G_S(z)$ is $3–4\%$ [10], the contribution from $\langle Q_S \rangle$ dominates that from $\langle Q_{L_s} \rangle$ in $\Delta \Gamma_{B_s}/\Gamma_{B_s}$. However, the $1/m_b$ corrections, currently estimated using factorisation, cancel significantly against the $\langle Q_S \rangle$ term [50], so that two terms of order $0.1–0.15$ combine to give a prediction $\Delta \Gamma_{B_s}/\Gamma_{B_s} \approx 0.05$ from equation Eq. (28) with a large uncertainty. Moreover, using results for $B_{(B_s S)}$ and $f_{B_s}$ in Eq. (24) instead of Eq. (26) gives $\Delta \Gamma_{B_s}/\Gamma_{B_s}$ closer to $0.1$ [50]: here there is extra uncertainty from the lattice determination of $f_{B_s}$, although the $\delta_{1/m}$ uncertainty dominates. To make
progress, better input for the matrix elements in $\delta_{1/m}$ is vital, together with better knowledge of $f_{B_s}$; both may be addressed by future lattice calculations. Note also that the ‘ratio’ form of Eq. (26) assumes that $\Delta M_{B_s}$ (and $\Delta M_{B_d}$) are given by their standard model expressions. $\Delta \Gamma_{B_s}$ by itself is dominated by tree level physics and so is expected to be less sensitive to new physics than the mass differences. This would favour working with the expression in Eq. (24) once $f_{B_s}$ is better known.

Acknowledgments

We thank Damir Becirevic, Laurent Lellouch and Chris Sachrajda warmly for discussions and acknowledge PPARC grant PPA/G/O/1998/00525 for support. We are grateful to Laurent Lellouch and Chris Sachrajda for reading the manuscript carefully.

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