AREA BOUNDS FOR MINIMAL SURFACES THAT PASS THROUGH A PRESCRIBED POINT IN A BALL

SIMON BRENDLE AND PEI-KEN HUNG

Abstract. Let $\Sigma$ be a $k$-dimensional minimal submanifold in the $n$-dimensional unit ball $B^n$ which passes through a point $y \in B^n$ and satisfies $\partial \Sigma \subset \partial B^n$. We show that the $k$-dimensional area of $\Sigma$ is bounded from below by $|B^k|(1 - |y|^2)^{k/2}$. This settles a question left open by the work of Alexander and Osserman in 1973.

1. Introduction

In this note, we study the area of a minimal submanifold in the unit ball in $\mathbb{R}^n$. For a minimal submanifold $\Sigma$ that passes through the center of the ball, it is well-known that the area of $\Sigma$ is bounded from below by the area of a flat $k$-dimensional disk:

**Theorem 1.** Let $\Sigma$ be a $k$-dimensional minimal submanifold in the unit ball $B^n$ which passes through the origin and satisfies $\partial \Sigma \subset \partial B^n$. Then $|\Sigma| \geq |B^k|$.

Theorem 1 is a direct consequence of the well-known monotonicity formula for minimal submanifolds. This technique is discussed, for example, in [5] and [6].

In 1973, Alexander and Osserman [2] studied a closely related problem. More precisely, they considered a minimal surface in the unit ball in $\mathbb{R}^3$ which passes through a prescribed point in the interior of the ball (not necessarily the center of the ball). In the special case of disk-type minimal surfaces, they were able to show that the area of the surface is bounded from below by the area of a flat disk. However, their argument does not work for minimal surfaces of other topological types, nor does it generalize to higher dimensions. In 1974, Alexander, Hoffman, and Osserman [1] proved an analogous inequality in higher dimensions, but only in the special case of area-minimizing surfaces.

In this note, we completely settle this question for minimal submanifolds of arbitrary dimension and codimension:

**Theorem 2.** Let $\Sigma$ be a $k$-dimensional minimal submanifold in the unit ball $B^n$ which passes through a point $y \in B^n$ and satisfies $\partial \Sigma \subset \partial B^n$. Then...
\(|\Sigma| \geq |B^k| \left(1 - |y|^2\right)^{\frac{k}{2}}\). Moreover, the inequality is strict unless \(\Sigma\) is a flat \(k\)-dimensional disk which is orthogonal to \(y\).

The proof of Theorem 2 relies on an application of the first variation formula for minimal submanifolds (cf. \([3], [6]\)) to a carefully chosen vector field in ambient space. In particular, our argument generalizes immediately to the varifold setting. A similar technique was used in \([4]\) to prove a sharp bound for the area of a free-boundary minimal surface in a ball. The main difficulty in this approach is to find the correct vector field. The vector field used in \([4]\) was obtained as the gradient of the Green’s function for the Neumann problem on the unit ball. By contrast, the vector field used in the proof of Theorem 2 is not a gradient field, and does not have any obvious geometric interpretation.

2. Proof of Theorem 2

Let us fix a point \(y \in B^n\). We define a vector field \(W\) on \(B^n \setminus \{y\}\) in the following way: For \(k > 2\), we define

\[
W(x) = -\frac{1}{k} \left( \left( \frac{1 - 2\langle x, y \rangle + |y|^2}{|x - y|^2} \right)^{\frac{k}{2}} - 1 \right) (x - y) + \frac{1}{k - 2} \left( \left( \frac{1 - 2\langle x, y \rangle + |y|^2}{|x - y|^2} \right)^{\frac{k-2}{2}} - 1 \right) y.
\]

For \(k = 2\), we define

\[
W(x) = -\frac{1}{2} \left( \frac{1 - 2\langle x, y \rangle + |y|^2}{|x - y|^2} - 1 \right) (x - y) + \frac{1}{2} \log \left( \frac{1 - 2\langle x, y \rangle + |y|^2}{|x - y|^2} \right) y.
\]

Note that \(1 - 2\langle x, y \rangle + |y|^2 \geq |x - y|^2 > 0\) for all points \(x \in B^n \setminus \{y\}\). This shows that \(W\) is indeed a smooth vector field on \(B^n \setminus \{y\}\).

Lemma 3. For every point \(x \in B^n\) and every orthonormal \(k\)-frame \(\{e_1, \ldots, e_k\} \subset \mathbb{R}^n\), we have

\[
\sum_{i=1}^{k} \langle D_{e_i} W, e_i \rangle \leq 1.
\]
Proof. We compute
\[
\sum_{i=1}^{k} \langle D_{e_i} W, e_i \rangle = 1 - \left( \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^2} \right)^{\frac{k}{2}} \\
+ \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^k} \sum_{i=1}^{k} \langle y, e_i \rangle \langle x - y, e_i \rangle \\
+ \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^{k+2}} \sum_{i=1}^{k} \langle x - y, e_i \rangle^2 \\
- \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^{k-2}} \sum_{i=1}^{k} \langle y, e_i \rangle^2 \\
- \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^k} \sum_{i=1}^{k} \langle x, e_i \rangle \langle y, e_i \rangle \\
= 1 - \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^{k+2}} \left( |x - y|^2 - \sum_{i=1}^{k} \langle x - y, e_i \rangle^2 \right) \\
- \frac{1 - 2 \langle x, y \rangle + \|y\|^2}{|x - y|^{k-2}} \sum_{i=1}^{k} \langle y, e_i \rangle^2 \\
\leq 1.
\]
Note that the preceding calculation is valid both for \( k > 2 \) and for \( k = 2 \). This proves the assertion.

**Lemma 4.** The vector field \( W \) vanishes along the boundary \( \partial B^n \).

**Proof.** Suppose that \( x \in \partial B^n \). Then \( 1 - 2 \langle x, y \rangle + \|y\|^2 = |x - y|^2 \). This directly implies \( W(x) = 0 \). Again, this conclusion holds both for \( k > 2 \) and for \( k = 2 \).

**Lemma 5.** We have
\[
W(x) = -\left(1 - \|y\|^2\right)^{\frac{k}{2}} \frac{x - y}{k |x - y|^k} + o\left(\frac{1}{|x - y|^{k-1}}\right)
\]
as \( x \to y \).

**Proof.** By definition of \( W(x) \), we have
\[
W(x) = -\left(1 - 2 \langle x, y \rangle + \|y\|^2\right)^{\frac{k}{2}} \frac{x - y}{k |x - y|^k} + o\left(\frac{1}{|x - y|^{k-1}}\right) \\
= -\left(1 - \|y\|^2\right)^{\frac{k}{2}} \frac{x - y}{k |x - y|^k} + o\left(\frac{1}{|x - y|^{k-1}}\right)
\]
as \( x \to y \). This proves the assertion.
We now describe the proof of Theorem 2. To that end, we assume that \( \Sigma \) is a minimal surface in \( B^n \) passing through the point \( y \). Since the vector field \( W \) vanishes along the boundary \( \partial \Sigma \subset \partial B^n \), we obtain
\[
\int_{\Sigma \setminus B_r(y)} (1 - \text{div}_\Sigma W) = |\Sigma \setminus B_r(y)| - \int_{\Sigma \cap \partial B_r(y)} \langle W, \nu \rangle
\]
by the divergence theorem. Here, \( \nu \) denotes the inward pointing unit normal to the region \( \Sigma \cap B_r(y) \) within the surface \( \Sigma \). In other words, the vector \( \nu \) is tangential to \( \Sigma \), but normal to \( \Sigma \cap \partial B_r(y) \). It is easy to see that
\[
\nu = -\frac{x - y}{|x - y|} + o(1)
\]
for \( x \in \Sigma \cap \partial B_r(y) \). Using Lemma 5, we obtain
\[
\langle W, \nu \rangle = (1 - |y|^2) \frac{1}{k} \frac{1}{r^{k-1}} + o\left(\frac{1}{r^{k-1}}\right)
\]
for \( x \in \Sigma \cap \partial B_r(y) \). Since
\[
|\Sigma \cap \partial B_r(y)| = |\partial B^k| \frac{r^{k-1}}{r^{k-1}} + o(r^{k-1}),
\]
we conclude that
\[
\lim_{r \to 0} \int_{\Sigma \cap \partial B_r(y)} \langle W, \nu \rangle = \frac{1}{k} |\partial B^k| (1 - |y|^2)^{\frac{1}{k}} = |B^k| (1 - |y|^2)^{\frac{1}{k}}.
\]
Combining (1) and (2) gives
\[
\lim_{r \to 0} \int_{\Sigma \setminus B_r(y)} (1 - \text{div}_\Sigma W) = |\Sigma| - |B^k| (1 - |y|^2)^{\frac{1}{k}}.
\]
On the other hand, by Lemma 3 we have the pointwise inequality
\[
1 - \text{div}_\Sigma W \geq 0.
\]
Putting these facts together, we obtain \( |\Sigma| - |B^k| (1 - |y|^2)^{\frac{1}{k}} \geq 0 \), as claimed.

Finally, we study the case of equality. Suppose that \( |\Sigma| - |B^k| (1 - |y|^2)^{\frac{1}{k}} = 0 \). In this case, we have
\[
1 - \text{div}_\Sigma W = 0
\]
for each point \( x \in \Sigma \setminus \{y\} \). Hence, if \( x \) is an arbitrary point on \( \Sigma \setminus \{y\} \) and \( \{e_1, \ldots, e_k\} \) is an orthonormal basis of \( T_x \Sigma \), then we have
\[
|x - y|^2 - \sum_{i=1}^k \langle x - y, e_i \rangle^2 = \sum_{i=1}^k \langle y, e_i \rangle^2 = 0.
\]
This implies that \( \Sigma \) is a flat \( k \)-dimensional disk which is orthogonal to \( y \). This completes the proof of Theorem 2.
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DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027

E-mail address: simon.brendle@columbia.edu

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027

E-mail address: pkhung@math.columbia.edu