CONTINUUM EIGENMODES IN SOME LINEAR STELLAR MODELS

CHRISTOPHER J. WINFIELD

Abstract. We apply parallel approaches in the study of continuous spectra to adiabatic stellar models. We seek continuum eigenmodes for the LAWE formulated as both finite difference and linear differential equations. In particular, we apply methods of Jacobi matrices and methods of subordinancy theory in these respective formulations. We find certain pressure-density conditions which admit positive-measured sets of continuous oscillation spectra under plausible conditions on density and pressure. We arrive at results of unbounded oscillations and computational or, perhaps, dynamic instability.

Introduction

The problem of radial, adiabatic (isentropic) stellar oscillations is well studied in cases where discrete eigenmodes are calculated in the frameworks of linearized differential equations and in parallel applications of operator theory (see, for instance, [2, 3, 9, 22]). However, continuous eigenvalues, although previously suggested in the general literature, are typically absent, either excluded from particular physical models or explicitly avoided to assure dynamic or numerical stability [1, 14, 17, 23] (see also the Appendix of [6]). In this article we investigate some cases where continuous spectra are present which have ramifications on the associated motion - and perhaps on the perturbation method itself. Our study involves linearized perturbations of the differential system

\[
\frac{d^2}{dt^2} r = \frac{-G\mathcal{M}(r)}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m} \\
\frac{1}{\rho} = \frac{4\pi}{3} \frac{\partial r^3}{\partial m}.
\]

Here, \( \mathcal{M}(r) \) is total mass measured from the origin to distance \( r \) from the center of a spherically symmetric star, and \( P \) and \( \rho \) are pressure and density, respectively, depending on \( r \). Moreover, the differential mass \( dm \) is interpreted as that of a spherical mass shell (or mass element) with mean radius \( r \) about

1991 Mathematics Subject Classification. 47N50, 85-08, 85-A30.

1In the general literature (cf. [9]), the problem is often referred to as the LAWE - the linear adiabatic wave equation.
the origin. We suppose, $0 < r \leq R^*$, $0 \leq M \leq M^*$, where $R^*$ and $M^*$ are fixed (arbitrary) positive constants; and, we consider only shell models where $r$ is confined to intervals of the form $[R^* - \delta, R^*]$ for some fixed, but arbitrary, $0 < \delta < R^*$.

In our study we adopt a background model where hydrostatic equilibrium (HSE) holds approximately, which of course is a standing assumption of the LAWE. We consider various adiabatic, polytropic and non-polytropic, equations of state (EOS): By 'polytropic' we mean that pressure is a power function of density (not to be confused with 'polytrope', a solution of the Lane-Emden equation). We find that rigorous analysis has indeed been done [3, 4] along these lines with conditions that ensure pure-point spectra and, hence, discrete radial motion analogous to standing waves. These conditions include those where HSE holds exactly on neighborhoods of shell boundaries. Most of our models likewise impose some exact HSE conditions, but, in contrast, only as a boundary condition at the outer surface. For our classes of polytropic EOS we instead obtain continuous spectra for operators defined on radial intervals away from the origin with mass density functions satisfying $\rho(r) \propto (R^* - r)^a$, $a > 1$ (in stark contrast to Section 4, [23] which places extreme mass concentration at the stellar center).

The general approach of this article is as follows: Our study of eigenvalues first arises after linearization and discretization of the system (0.1) whereby a change of variables converts a finite difference equation into an eigenvalue problem $(A - \lambda)\vec{X} = \vec{0}$, a problem typically left to computation involving large but finite matrices $A$ (cf. [6, 9]). We find, however, that the adiabatic case admits tri-diagonal $A$'s which lend themselves to analysis of Jacobi matrices after passing to an infinite system: Here, we impose non-uniform partitions which we let cluster at the stellar surface $r = R^*$. In this context one turns to a large body of mathematical work on the infinite (cf. [16, 21]) Jacobi matrix. Indeed, cases of continuous spectra as well as discrete spectra are well known whereby unbounded oscillations modes can be discerned in this application by the nature of the spectra. We then pass to second-order ordinary differential equation models, applying various results arising from subordinancy theory [12, 18] and (Weyl) limit-point case Sturm-Liouville operators [8, 20, 24]. The author is not aware if such oscillation behavior is actually observed in stars. Perhaps, the properties studied here are merely artifacts of the perturbation methods and yet prove absent in more complete, non-linear methods or in simple models which (say) incorporate non-adiabatic or stochastic processes.

This article is organized as follows: In Section 1 we formulate our finite difference equation and place it in a Jacobi-matrix format. In Section 2 we apply results from certain Hilbert-Schmidt and trace-class operators to arrive at general results on spectra and solutions. In Section 3 we study non-polytropic cases where absolutely continuous spectra are present and further develop some cases where pure-point spectra also appear. In Section 4 we reformulate and study
the matrix equation in some polytropic cases. Then, in Section 5 we present the differential equation form of the LAWE in Sturm-Liouville (SL) form. Finally, in Sections 6 and 7 we revisit polytropic and non-polytropic cases, respectively, in the SL context.

1. THE DIFFERENCE EQUATION

To begin this section we outline a derivation of our finite difference equation and the perturbation scheme involved which follow from [6, 9]. For the system (0.1) we suppose that \( m = \mathcal{M}(r) \) is a strictly increasing function and treat \( m \) as an independent variable. Then, one replaces \( r, P, \) and \( \rho \) by the perturbed quantities \( r(m) + \delta r(m, t), P(r, \rho) + \delta P(r, t), \) and \( \rho(r) + \delta \rho(r, t) \), respectively. One perturbs the system about HSE where

\[
4\pi r^2 \frac{\partial P}{\partial m} = -\frac{G\mathcal{M}(r)}{r^2}
\]

and replace \( \delta r(m, t) \rightarrow e^{i\omega t}\delta r(m) \). Here, \( \frac{\partial}{\partial m} \overset{\text{def}}{=} \frac{1}{4\pi r^2 \rho(r) \frac{\partial}{\partial r}}. \)

Upon discretization, we introduce mass elements \( M(I) \overset{\text{def}}{=} M(I) - M(I - 1) \) and set \( \mathcal{M}(I) \overset{\text{def}}{=} \sum_{j=1}^{I} M(j) \) (vacuous sums are zero) and, after a separation of variables, obtain

\[
-\omega^2 \delta r(I) = 4G\mathcal{M}(I) \frac{1}{r^3(I)} \frac{\partial}{\partial m} \left( \frac{\delta P(I) - \delta P(I - 1)}{M(I)} \right)
\]

where

\[
\delta P(I) = (\Gamma P)(I) \cdot (\mathcal{R}_1(I) \cdot X(I) - \mathcal{R}_2(I) \cdot X(I + 1)),
\]

\[
\mathcal{R}_i(I) \overset{\text{def}}{=} \left( \frac{\rho(I)}{M(I)} \right) \left( \frac{4\pi r^2 (I + i - 1)}{\sqrt{M(I + i - 1)}} \right) : i = 1, 2,
\]

\[
\Gamma \overset{\text{def}}{=} \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S \quad \text{(entropy} \ S \text{held fixed) and,}
\]

\[
X(I) \overset{\text{def}}{=} \delta r(I) \sqrt{M(I)}.
\]

By setting

\[
\delta P(I) - \delta P(I - 1) = ((\Gamma P\mathcal{R}_1)(I) + (\Gamma P\mathcal{R}_2)(I - 1)) \cdot X(I)
\]

\[
-((\Gamma P\mathcal{R}_2)(I) \cdot X(I + 1) - (\Gamma P\mathcal{R}_1)(I - 1) \cdot X(I - 1)
\]

and adopting some notation of [6], we arrive at our eigenvalue problem

\[
\lambda X(I) = G_1(I)X(I - 1) + G_2(I)X(I) + G_3(I)X(I + 1)
\]

\footnote{For simplicity, we do not precisely follow their mass-division numerical schemes (see also [13]) which is irrelevant to our study, given monotonically decreasing \( M(I) \).}
where \( \lambda \overset{\text{def}}{=} -\omega^2 \leq 0 \),

\[
G_3(I) \overset{\text{def}}{=} 16\pi^2 \left( \frac{r^2}{\sqrt{M}} \right) (I) \left( \frac{\Gamma P \rho}{M} \right) (I) \left( \frac{r^2}{\sqrt{M}} \right) (I + 1) \\
= 16\pi^2 \left( \frac{\Gamma P \rho r^4}{M^2} \right) (I) \frac{r^2(I + 1)}{r^2(I)} \sqrt{\frac{M(I)}{M(I + 1)}}
\]

\( G_1(I) \overset{\text{def}}{=} G_3(I - 1) : I \geq 2 \), and

\[
-G_2(I) + \left( \frac{4G_2r^3}{r^3} \right) (I) \overset{\text{def}}{=} \frac{4\pi^2[(\Gamma P \rho r^4)(I) - (\Gamma P \rho r^4)(I - 1)]r^2(I)}{M(I)}
\]

\[
= \left( \frac{\Gamma P \rho}{M} (I) + \frac{\Gamma P \rho}{M} (I - 1) \right) \left( \frac{4\pi^2}{\sqrt{M}} \right)^2 (I)
\]

\[
= G_3(I) \frac{r^2(I) \sqrt{M(I + 1)}}{r^2(I + 1) \sqrt{M(I)}} + G_1(I) \frac{r^2(I) \sqrt{M(I - 1)}}{r^2(I - 1) \sqrt{M(I)}}
\]

In our sign convention, \( \lambda > 0 \) would correspond to exponential time dependence.\(^3\)

We note that this model is also simplified by an assumption of constant entropy and that the equations decouple from dependence on temperature, opacity, and luminosity, the admission of which would lead to larger systems and much greater complexity (cf. Section 9, [15]). Moreover, we note that \( P \) can represent pressure from various sources - not simply mechanical - and can involve temperature and chemical variation. Indeed, such effects could conceivably be incorporated in a quasi-static EOS with \( P \) having dependence on \( r \) as well as \( \rho \).

2. **Matrix Works and Spectra**

We may reformulate the difference equation (1.3) in matrix form as

\[
A \vec{X} = -\omega^2 \vec{X}
\]

\[
A = \begin{pmatrix}
  a_1 & c_1 & 0 & 0 & \ldots \\
  c_1 & a_2 & c_2 & 0 & \ldots \\
  0 & c_2 & a_3 & c_3 & \ldots \\
  0 & 0 & c_3 & a_4 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
c_I = G_3(I), \quad a_I = G_2(I)
\]

where we set \( \vec{X} = (X(1), X(2), \ldots) \) for \( X(I) = \delta r(I) \sqrt{M(I)} \). More generally, we let \( \vec{X} \) denote the function (or vector) defined on \( \mathbb{Z}^+ \) given by the corresponding assignments (components) \( X(I) : I = 1, 2, \ldots \) and likewise define \( \vec{P}, \vec{\delta r}, \vec{\rho} \), etc.

\(^3\)We do not elaborate on the interpretation of negative \( \omega^2 \) but we defer to discussions found in Sections 8.7 and 8.8 of [15].
In our analysis we consider $A$ as a bounded, self-adjoint operator on the space of vectors $\vec{X} \in \ell^2(\mathbb{Z}^+)$, noting that $\text{supp}_j |a_j|$, $\text{supp}_i |c_j| < \infty$. In turn, we may regard $\delta r$ as an element of a weighted $\ell^2$ space since

\[
\sum_{I=1}^{\infty} (X(I))^2 = \sum_{I=1}^{\infty} (\delta r(I))^2 M(I)
\]

where, for a star of finite mass, we have $\mathfrak{M}^* \overset{\text{def}}{=} \sum_{I=1}^{\infty} M(I) < \infty$. What we then find from a simple $\ell^p$-space fact is the following

**Proposition 2.1.** For a finite-mass star, the perturbation $\delta r(I)$ is unbounded as $I \to \infty$ if the associated solution $\vec{X}$ is not in $\ell^2(\mathbb{Z}^+)$. For ease of exposition we suppose for now that for some real constant $z$

\[
\lim_{I \to \infty} G_2(I) = z; \quad \lim_{I \to \infty} G_3(I) = 1.
\]

Now, let $J_0$ denote the tri-diagonal matrix in the form (2.1) but with $c_i = 1$, $a_i = 0$ $\forall i$ and let constants denote scalar multiplication. It is immediate that $A - z - J_0$ is compact, whereby the essential spectrum of $A$ is that of $z + J_0$ (see [21]). More precise information is obtained in the literature depending on whether $A - z - J_0$ is of Hilbert-Schmidt class, whereby

\[
\sum_{I=1}^{\infty} |G_2(I) - z|^2 + \sum_{I=1}^{\infty} |G_3(I) - 1|^2 < \infty,
\]

of trace class, whereby

\[
\sum_{I=1}^{\infty} |G_2(I) - z| + \sum_{I=1}^{\infty} |G_3(I) - 1| < \infty,
\]

or of slightly more rapid convergence, such as

\[
\sum_{I=1}^{\infty} I \cdot |G_2(I) - z| + \sum_{I=1}^{\infty} I \cdot |G_3(I) - 1| < \infty.
\]

We note the following: The mode of convergence of $G_1(I), G_2(I)$ will depend on $P, \rho$ and $M$ and how they are inter-related; these conditions may be imposed via mass conservation together with, as we will show, an equation of state. Such modes can imply various spectral properties of $A$ which in turn have implications on associated solutions $ AX = \lambda X$ (cf. [12, 20]). Although this approach does not typically yield precise estimates of $X(I)$ (especially when mixtures of spectral types are possible), asymptotic (Jost) estimates in some cases do indeed obtain and, in turn, lead to estimates of $\delta r(I)$.

Our main results depend on the following well known results which we list together in
Theorem 2.2. (Previously Known) Suppose \( r(I) : I = 1, 2, \ldots \) is a monotonically increasing sequence where \( \lim_{I \to \infty} r(I) = R < \infty \) and let \( M(I) \in \ell^1(\mathbb{Z}^+) \) with \( M(I) > 0 \) \( \forall I \). Then,

i) if (2.2) holds, then \( \sigma_{\text{ess}} \), essential spectrum, satisfies
\[
\sigma_{\text{ess}}(A) = [z - 2, z + 2];
\]

ii) if (2.3) holds, then the pure-point spectra \( E_k : k = 1, 2, \ldots \) such that \( |E_k - z| > 2 \) (perhaps none, finitely or infinitely many) of \( A \) satisfy
\[
\sum_k (|E_k - z| - 2)^{3/2} < \infty;
\]

iii) if (2.4) holds, then \( \sigma_{\text{ac}}(A) = [z - 2, z + 2] \) and, the pure point spectrum \( \sigma_{\text{pp}} \) satisfies
\[
\sigma_{\text{pp}}(A) \cap (z - 2, z + 2) = \emptyset
\]
which is to say that no \( \ell^2 \) eigenvalues exist in \((z - 2, z + 2)\);

iv) finally, if (2.5) holds, then \( z \pm 2 \in \sigma_{\text{pp}}(A) \) hold along with iii).

We make several remarks: Statement i) follows from a routine application (see [21]) of the Weyl Invariance theorem, noting that \( A - z - J_0 \) is compact.

Statement ii) is one of several criteria (see Theorem 1 [16] or Theorem 1.10.1 [21]) which together are equivalent to \( A - z - J_0 \) being of Hilbert-Schmidt class.

Statements iii) and iv) follow from Theorem A.6 [11] (see also [20]). The absolutely continuous part \( f(x) \, dx \) of the implied spectral measure is supported on \([z - 2, z + 2] \), the positivity of which is demonstrated by a finite lower bound on a weighted Lebesgue integral of \( \log|f(x)| \) (the so-called Quasi-Szegö condition; indeed, in case iii) an even stricter integral condition (the Szegö condition) on \( f(x) \) also holds (see [16, 21]).

We introduce notation to indicate various modes of convergence of sequences:
The expression \( a(I) \xrightarrow{O_p} L \) means that \( \lim_{I \to \infty} a(I) = L \) and that for \( b(I) = a(I) - L \) the sequence satisfies \( b(I) \in \ell^p(\mathbb{Z}^+) \). For example, a geometric sequence \( a(I) = C \eta^I \) with fixed \( C \) and \( |\eta| < 1 \) satisfies \( a(I) \xrightarrow{O_p} |a||_p \). We will need to establish

**Proposition 2.3.** For sequences \( \bar{a} \) and \( \bar{b} \) suppose for some \( p \geq 1 \) that \( a(I) \xrightarrow{O_p} L_a \) and \( b(I) \xrightarrow{O_p} L_b \) for some constants \( L_a, L_b \). Then, \( a(I)b(I) \xrightarrow{O_p} L_aL_b \). Moreover, \( f(a(I)) \xrightarrow{O_p} f(L_a) \) for any Lipshitz function \( f \).

**Proof.** If either \( L_a \) or \( L_b \) are zero, the first result is clear since both \( \bar{a} \) and \( \bar{b} \) are bounded. If both \( L_a \) or \( L_b \) are non-zero, then by scaling arguments, it will suffice to prove the result for \( L_a = L_b = 1 \). We write
\[
(2.6) \quad a(I)b(I) - 1 = (a(I) - 1)(b(I) + 1) + (b(I) - 1) - (a(I) - 1).
\]
From standard $\ell^p$ inequalities we find that the left-hand side of (2.6) defines a sequence in $\ell^p(\mathbb{Z}^+)$. The last statement is clear since $|f(a_I) - f(L)|^p \leq (c|a_I - L|)^p$ for some constant $c \geq 0$. □

Our examples in the discrete-systems case will involve mass distributions of the form

(2.7) \[ M(I + 1) = \eta^\gamma M(I); \quad r(I) = R_s \cdot (1 - \eta^I) \]

for fixed $0 < \eta < 1$ and $\gamma > 0$, in which case

\[ \mathfrak{M}(I) = (1 - \eta^\gamma)\mathfrak{M}_s \sum_{j=0}^I \eta^j = \mathfrak{M}_s (1 - \eta^{(I+1)}) . \]

Here $\gamma = 1$ corresponds to constant density (to order $O(\eta^I)$). It will be convenient to introduce

(2.8) \[ \Lambda_* \overset{\text{def}}{=} \frac{G \mathfrak{M}_s}{R_s^3}; \quad \kappa \overset{\text{def}}{=} \frac{4 + \zeta}{(\eta^\gamma - \frac{\gamma}{2}) + \eta^\gamma} > 0 \]

for $\zeta > -4$, amounting to scaling and translation parameters, to state

**Theorem 2.4.** Given a mass distribution of the form (2.7), suppose that for some fixed $\zeta > -4$

(2.9) \[ \left( \frac{\Gamma\rho}{M^2} \right)(I) \overset{O_{\ell^2}}{\to} \frac{1 + \zeta/4}{4\pi^2 R_s^2} \cdot (1 + \eta^{-\gamma}) \Lambda_*. \]

Then, the essential spectrum of $A$ is given by $\sigma_{\text{ess}}(A) = \mathcal{I}$ for

(2.10) \[ \mathcal{I} \overset{\text{def}}{=} \left[ (-\zeta - 2\kappa)\Lambda_*, (-\zeta + 2\kappa)\Lambda_* \right] \]

and the eigenvalues $\lambda_k$ satisfy

(2.11) \[ |\lambda_k + \zeta\Lambda_*| \overset{O_{\ell^2}}{\to} 2\kappa\Lambda_* \]

if there are infinitely many. Moreover, if the convergence (2.9) is in $O_{\ell^2}$ mode, the solutions space corresponding to each $\lambda \in \mathcal{I}^o$, the interior of $\mathcal{I}$, is of dimension two and contains no non-trivial $\ell^2$ solutions.

**Proof.** By inclusion of $\ell^p$ spaces we have $r(I) \overset{O_{\ell^p}}{\to} R_s \neq 0$ so that $r^{-3}(I) \overset{O_{\ell^p}}{\to} R_s^{-3}$ by way of the Mean Value Theorem. Therefore,

\[ \frac{\mathfrak{M}(I)}{r^3(I)} \overset{O_{\ell^2}}{\to} \frac{\mathfrak{M}_s}{R_s^3} . \]

Then, since

(2.12) \[ \frac{r^2(I)\sqrt{M(I+1)}}{r^2(I+1)\sqrt{M(I)}} \overset{O_{\ell^2}}{\to} \eta^{\gamma/2}; \quad \frac{r^2(I)\sqrt{M(I-1)}}{r^2(I-1)\sqrt{M(I)}} \overset{O_{\ell^2}}{\to} \eta^{-\gamma/2} , \]

---

4See equation (8.20) and following discussion [6] for physical interpretation of quantities $\omega^2\Lambda^*$. 

7
it is not difficult to show that

\[(2.13) \quad G_3(I) \overset{O(\zeta)}{\longrightarrow} \frac{(4 + \zeta)\Lambda_\ast}{\eta^{\gamma/2} + \eta^{-\gamma/2}}; \quad G_2(I) \overset{O(\zeta)}{\longrightarrow} -\zeta \Lambda_\ast \]

Now, the result follows by applying Proposition 2.3 and Theorem 2.2 and a scaling argument. The remaining claim follows as above, mutatis-mutandis with (2.13) in \(O_{r_+}\) mode. \(\square\)

We remark: The spectrum depends on the partition \(\{r(I)| I = 1, 2, \ldots\};\) and, if the convergence of (2.13) is in \(O_{r_+}\) mode, the solutions \(\vec{X}_\lambda\) to \(AX = \lambda X\) for \(\lambda \in \mathcal{I}\) are complex linear combinations of vectors \(\vec{Y}_\pm\) (Jost solutions [21]) satisfying

\[(2.14) \quad \vec{Y}_\pm(I) = e^{\pm i\theta I}(1 + o(1))\]

(as \(I \to \infty\)) for \(\theta = \arccos [(\lambda + \zeta \Lambda_\ast)/(2\Lambda_\ast \kappa)]\).

3. Some Non-polytropic Cases

We find natural examples where the assignments \(\vec{M}\) and \(\vec{r}\) determine \(\vec{\rho}\) by mass conservation. Here, and until otherwise specified, we will suppose that HSE holds exactly at the surface \(r = R_\ast\). However, we do not at this point assign an equation of state nor any explicit dependence of \(P\) on \(\rho\). In the general case we suppose it plausible that \(\Gamma P\) depends on \(\rho\) and other parameters which, in turn, may vary with \(r\) as well (see [15] and sections 4.2b and 8.7 of [9]). We will denote \(\mathcal{D} : \mathcal{D}(I) = (\Gamma P\rho)(I)\) which will call a pressure-density distribution; and, we will call a correspondence between the assignments \(\vec{M}, \vec{r}\), such as (2.7) a mass distribution. We will also say that \(\mathcal{D}\) is admissible if (0.1) holds and if (1.1) holds but in the limit as \(I \to \infty\), each in the discrete sense. We state

**Proposition 3.1.** Let \(\mathcal{D}\) be an admissible pressure-density distribution such that

\[(3.1) \quad \frac{\Gamma(I) \rho(I)}{M(I)} \overset{O(\zeta)}{\longrightarrow} \frac{1 + \zeta/4}{\pi R_\ast^4 \cdot (1 + \eta^{-\gamma})}\]

Then, \(\sigma_{ac}(A) = \mathcal{I}\) for \(\mathcal{I}\) as in (2.10). Moreover, (1.10) has no non-trivial bounded solutions \(\delta \bar{r}\) for \(-\omega^2 \in \mathcal{I} \cap (-\infty, 0]\) if \(\Gamma\) satisfies

\(\Gamma(I) = c\nu^I\)

(as \(I \to \infty\)) for any positive constant \(c\).

Some such EOS may be of the form \(P = \tau \rho(r) + l(r)\) for some constant \(\tau\) and function \(l(r)\) tending sufficiently rapidly to 0 as \(r \to R_\ast^+\).

**Proof.** The hypothesis (3.1) assures that

\(\left(\frac{P}{M}\right)(I) \overset{O(\zeta)}{\longrightarrow} \frac{G2\mathcal{M}_\ast}{4\pi R_\ast^4} = \frac{\Lambda_\ast}{4\pi R_\ast^4}\)
and that (2.9) holds in the desired mode. The result then follows from Proposition 2.3 and Theorem 2.4, with choice of \( \zeta \) determined by \( c \), since the asymptotic estimates of the LHS of (3.1) lead to

\[
4 + \zeta = c \cdot K
\]

(3.2)

\[
K \overset{\text{def}}{=} (1 + \eta^{-\gamma})/(\eta^{-1} - 1).
\]

□

For later reference we make the following

**Remark 3.3.** For the choice of \( \vec{D} \) in Proposition 3.1 the elements \( G_1(I), G_2(I) \) equal their respective limits as in (2.13) up to order \( O(\eta^{I}) \) (as \( I \to \infty \)) so that

\[
A_0 \overset{\text{def}}{=} A - \kappa \Lambda^* J_0 + \zeta \Lambda^* \text{ satisfies, more precisely, } [A_0]_{j,k} = O(\eta^{I}) \text{ as } j \to \infty \text{ for } |j - k| \leq 1.
\]

It turns out that we may indeed choose \( \vec{D} \) so that condition (2.2) is satisfied, for suitable \( \vec{M} \) and \( \vec{r} \), as we demonstrate in

**Theorem 3.2.** Let \( \vec{M}, \vec{r} \) determine a mass distribution as in (2.7) for \( \gamma > 1 \). Then, there is an admissible pressure-density distribution \( \vec{D} \) by which the results of Proposition 3.1 hold for some interval \( I \).

**Proof.** We find

\[
\frac{r(I + 1) - r(I)}{M(I + 1)} = \frac{R_* \eta^{I}(1 - \eta)}{2M \cdot (1 - \eta) \eta^{I(I + 1)}}.
\]

Since \( r^2(I) \overset{O(I)}{\to} R_*^2 \) we have from (0.1) and L'Hôpital's Rule that \( \rho(I) = O(\eta^{(\gamma - 1)I}) \) (as \( I \to \infty \)). We may choose \( \Gamma \) so that for given \( \zeta > -4 \)

\[
\Gamma(I) = \frac{1 + \zeta/4}{\pi(1 + \eta^{-\gamma}) R_*^2 \rho(I)} \eta^{I(I + 1)} (1 + O(\eta^{I})) \text{ (as } I \to \infty \text{)}
\]

(3.4)

to apply Proposition 3.1. □

Example (3.4) satisfies \( \Gamma(I) = O(\eta^{I}) \) (as \( I \to \infty \)) and thus vanishes to order \( O(R_* - r) \) as \( r \to R_*^\gamma \). Physical models associated with such non-polytropic cases may well incorporate certain gas and chemical (indeed isentropic) properties: Some related discussion can be found in Appendix B.3 of [22], Section 8.9 of [9] and Section 58 of [17].

We will construct an admissible pressure-density distribution that results in an infinite number of eigenvalues. Our construction follows one by [10] whereby \( pp \) spectra result in a special case satisfying \( G_1(I) \equiv 1 \) and \( G_2(I) \overset{O(\eta^{I})}{\to} 0 \) (as \( I \to \infty \)). We outline the results of their construction below as

**Proposition 3.3.** (Previously Known.) One can construct the diagonal elements \( G_2(I) \) of \( A \) along with a sequence of vectors \( \vec{X}_m \) such that the following hold for

\[
\Phi \overset{\text{def}}{=} A - J_0,:\]

...
i) \( G_2(I) = I^{-\alpha} \) for \( I \in \bigcup_m B_m \) for some \( 1/2 < \alpha < 1 \), fixed, where the sets \( B_m \) are bounded, disjoint, discrete intervals with distance greater than 2 between each other with \( \min B_m > K m^{p+1} \) for some positive constant \( K \) uniformly in \( m \);

ii) the \( \vec{X}_m \)'s satisfy \( ||\vec{X}_m||_{\ell^\infty} = 1 \) with \( \text{supp} \vec{X}_m \subseteq B_m \) whose components vanish at the endpoints;

iii) for the standard inner product on \( \ell^2(\mathbb{Z}^+) \), the vectors \( \vec{X}_m \) are mutually orthogonal and

\[
m^{-p} \lesssim \sum_I |\vec{X}_m(I + 1) - \vec{X}_m(I)|^2 \lesssim 1,
\]

so that \(- \langle \vec{X}_m, (J_0 + 2)\vec{X}_m \rangle \gtrsim m^{-p} \), for some \( p \) satisfying \( 1/3 < p < \alpha(p + 1)/2 \);

iv) \( \langle \vec{X}_m, \mathcal{G}\vec{X}_m \rangle \gtrsim m^{p-\alpha(p+1)} \); 

v) \( \langle \vec{X}_m, \vec{X}_m \rangle \approx m^p \); 

vi) and, \( \sigma_{\text{ess}}(A) = [-2,2] \).

Several remarks are in order: Here \( f(x) \lesssim g(x) \) means \( f(x) = O(g(x)) \) as \( x \to \infty \) and \( f(x) \asymp g(x) \) means that \( f(x) \lesssim g(x) \) and \( g(x) \lesssim f(x) \) both hold; thus, the implied bounds above are uniform in \( m \). There results, by variational arguments (Theorem 9.2 [10]), an infinite sequence of eigenvalues \( \{E_k\} \), such that \( |E_k| \overset{O^{3/2}}{\to} 2 \), but the convergence is not in \( O_1 \). Indeed, given any \( q < 3/2 \), \( \alpha \) and \( p \) may be chosen so that the sequence \( \{E_k\} \) diverges in \( O_q \) mode. Such may also be constructed so that \( \{E_k\} \) clusters about both \( \pm 2 \), but negative eigenvalues \( E_k \) better serve for our eigenfrequencies \( \omega = \sqrt{-E_k} \).

We find that similar results arise in our problem (1.2). We start with an admissible pressure-density distribution \( \mathfrak{D} \) as above but with

\[
\Gamma(I) = (c - b \cdot \mathfrak{G}(I))\eta^I
\]

for \( \mathfrak{D} \) as in Proposition 3.3 and for constants \( 1/2 < \alpha < 1 \) and \( c, b > 0 \). Recalling the various parameters as in (2.8) and (3.2), we state

\textbf{Theorem 3.4.} There is an admissible pressure-density distribution resulting in an operator \( A \) as in (2.1) for which \( \sigma_{pp}(A) \) contains an infinite sequence \( E_k \) : \( k = 1, 2, \ldots \) tending to \( -(\zeta + 2\kappa)\Lambda_\ast \), an endpoint of \( \sigma_{\text{ess}}(A) = I \), from values less. Moreover, for each \( k \) there is an eigenvector associated to \( E_k \) for which the corresponding perturbation \( \delta r \) is of class \( \ell^\infty(\mathbb{Z}^+) \).
Proof. It will be convenient to prove the case for $\Lambda \kappa = 1$, choosing $\Gamma$ as in (3.5), and letting $c = \frac{4}{\pi}$ and $b = \frac{4}{\pi}$ from which we obtain $\zeta = 0$,

$$G_3(I) = 1 - \frac{\kappa}{4}\Theta(I) + O(\eta^I)$$

and

$$G_2(I) = \frac{1}{4\Lambda_*} \left( \eta^{\gamma/2} \Theta(I) + \eta^{-\gamma/2} \Theta(I - 1) \right) + O(\eta^I).$$

Since $\Theta$ is constant on $B_m$, we have from i) of Proposition 3.3 that

$$\left< \tilde{X}_m, \Theta(I - 1) \tilde{X}_m \right> = \left< \tilde{X}_m, \Theta(I) \tilde{X}_m \right>.$$  

If we denote $G_2 \overset{\text{def}}{=} \text{diag}(G_2(1), G_2(2), \ldots)$, we find

$$\left< \tilde{X}_m, G_2 \tilde{X}_m \right> = \left< \tilde{X}_m, \frac{1}{\kappa\Lambda_*} \Theta \tilde{X}_m \right> = 1 \cdot \left< \tilde{X}_m, \Theta \tilde{X}_m \right>,$$

modulo $O(m^p \eta^m)$. Let us set $\tilde{A} \overset{\text{def}}{=} A - J_0 - G_2$ which is also self-adjoint. Considering Remark 3.3 we find that the elements along the main diagonal of $\tilde{A}$ are of order $O(\eta^I)$ and those along super- and sub-main diagonals are appraised via $|\tilde{A}(I, I+1)| \lesssim I^{-\alpha} \eta^I$. With $\alpha > 1/2$ we conclude from the Weyl Invariance theorem that $\sigma_{\text{ess}}(A) = [-2, 2]$. Here,

$$\left| \left< \tilde{X}_m, \tilde{A} \tilde{X}_m \right> \right| \lesssim m^{-\alpha(p+1)} \sum_I |\tilde{X}_m(I+2) - \tilde{X}_m(I)|^2 + \eta^m \cdot ||\tilde{X}||^2$$

$$\lesssim m^{-\alpha(p+1)} \sum_I |\tilde{X}_m(I+1) - \tilde{X}_m(I)|^2 + \eta^m m^p$$

Hence, we find that there are positive constants $\kappa_1$ and $\kappa_2$ so that

$$\left| \left< \tilde{X}_m, \tilde{A} \tilde{X}_m \right> \right| < \kappa_1 m^{-\alpha(p+1)} + \kappa_2 \eta^m m^p$$

holds uniformly for all sufficiently large $m$. It follows as in Proposition 3.3 that

$$- \left< \tilde{X}_m, (J_0 + G_2 + 2) \tilde{X}_m \right> > C m^p m^{-\alpha(p+1)}$$

for some positive constant $C$.

We have only to follow a part of the cited theorem [10], mutatis-mutandis: There are positive constants $M$ and $C_M$ such that

$$- \left< \tilde{X}_m, (A + 2) \tilde{X}_m \right> > C_M ||\tilde{X}_m||^2 m^{-\alpha(p+1)}$$

for all $m \geq M$. Variational arguments now apply to show that supported in each $B_m : m \geq M$ is a corresponding eigenvector with eigenvalue $E_m < -2$ such that $|E_m + 2| \geq \frac{C_M}{m^{\alpha(p+1)}}$, concluding the present case.

If we keep $b$ as above and vary $c$, we may apply the above construction but with $A$ replaced by $A + \zeta \Lambda$, for an appropriate constant $\zeta$ (see Proposition 3.1). Considering Remark 3.3, $\tilde{A}$ likewise yields elements of order $O(\eta^I)$ along the
super- and sub-main diagonal elements. The result follows but with a shift of the spectral values by \(-\zeta \Lambda\), where \(\zeta\) is determined by \(c\). The more general case now follows by scaling arguments.

We note the following about the results above: Such a construct shows that a small variation of \(\Gamma\) can change the discrete spectrum \(\sigma_{\text{disc}}(A)\). Moreover, as a sequence of eigenvalues converges to particular value near the stellar surface, this value does not necessarily correspond to a value in \(\sigma_{\text{disc}}\). We see that \(G(I)\) in (3.5) vanishes as \(r \to R^*_s\) to order \(O(\ln(R^*_s - r)^{-\alpha})\) and, hence, the entire expression for \(\Gamma\) still vanishes to order \(O(R^*_s - r)\) as in example (3.4). Yet, the additional term \(-b\mathfrak{S}\) has a demonstrable effect on the spectrum of \(A\), even for small \(b > 0\), yielding oscillation states localized near \(r = R^*_s\).

4. Polytropic Cases: Modified Difference Equation

We now apply our methods to some polytropic states in the discrete setting. In order to apply the Jacobi spectral matrix methods we resort to another change of variables. Here, we allow the oscillation frequency \(\omega\) to vary with position, while freezing time: We impose \(\delta r_t = e^{it\omega(r)\delta r(r)}\), supposing that the linear perturbation \(\delta\) and the operation \(\frac{d}{dt}\) commute (albeit in a formal sense; cf. [22]). Such modes satisfy \(\partial_t \delta r_t |_{t=0} = \partial_t \delta r(r) : j = 0, 1, 2\), leaving (1.2) still valid. Then, with

\[
W \stackrel{\text{def}}{=} \text{diag}(\omega(1), \omega(2), \ldots, \omega(I), \ldots),
\]

we obtain an infinite matrix equation of the form

\[
-W^2 \vec{X} = A \vec{X}
\]

As we develop below, with an appropriate choice of \(W\), depending on \(\eta\) and \(\lambda\), (4.2) leads to the form (1.2) but with \(LHS = -\omega^2(I)\delta r(I)\). Regarding the spatial dependence of \(\omega\), we will call the values \(\omega(I)\) local frequencies.

We now motivate our formulation (4.2) by way of a certain modification of the Jacobi matrix: We will study tri-diagonal matrices \(A\) given by

\[
G_3(I) = \mu_1 \eta^{-I}; \quad G_2(I) = \theta - \beta_1 \eta^{-I}
\]

(self-adjoint) where

\[
\mu_j \stackrel{O_j}{\to} \mu; \quad \beta_j \stackrel{O_j}{\to} \beta
\]

for some positive numbers \(\eta, \theta, \beta, \mu\) such that \(\eta < 1\).
We now reformulate our matrix equation (2.1): First, we consider square, $n \times n$ matrices (for $n > 3$, say) of the form

$$(4.4) \quad A(x, y) = \begin{pmatrix}
  a_1 & c_1x & 0 & 0 & \ldots & 0 \\
  b_1y & a_2 & c_2x^2 & 0 & \ldots & 0 \\
  0 & b_2y^2 & a_3 & c_3x^3 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \ldots & 0 & b_{n-2}y^{n-2} & a_{n-1} & c_{n-1}x^{n-1} \\
  0 & 0 & \ldots & 0 & b_{n-1}y^{n-1} & a_n
\end{pmatrix}$$

Lemma 4.1. Let $A$ be a square, tri-diagonal matrix of the form (4.4). Then, there is a diagonal matrix

$$D(x, y) = \text{diag}(d_1(x, y), d_2(x, y), \ldots, d_n(x, y))$$

for which the following hold:

$$(4.5) \quad B(x, y) = D(y, x)A(x, y)D(x, y)$$

is a tri-diagonal matrix whose elements $[B]_{i,j}$ satisfy

$$(4.6) \quad [B]_{i,j} = \begin{cases}
  [A(1, 1)]_{i,j} & \text{for } i \neq j \\
  \frac{a_j}{(xy)^{\lfloor j/2 \rfloor}} & \text{for } i = j
\end{cases}$$

and, the $d_j(x, y)$’s are rational monomials in the variables $x$ and $y$.

Here $\lfloor \cdot \rfloor$ denotes the integer part of the argument.

Proof. Our choice for diagonal elements of $D(x, y)$ are as follows:

$$d_j(x, y) = \begin{cases}
  1 & \text{for } j = 1 \\
  \prod_{k=1}^{m} \left(\frac{y^{2k-2}}{x^{2k-1}}\right)^{\lfloor j/2 \rfloor} & \text{for } j = 2m, \text{ even} \\
  \prod_{k=1}^{m} \left(\frac{x^{2k-2}}{y^{2k-1}}\right)^{\lfloor j/2 \rfloor} & \text{for } j = 2m + 1, \text{ odd}
\end{cases}$$

We verify our results by considering row and column operations applied to $A(x, y)$ according to (4.3). For $j = 2m \geq 2$, even, we find

$$d_j(x, y)d_{j+1}(y, x) = \prod_{k=1}^{m} \left(\frac{y^{2k-2}}{x^{2k-1}}\right)^{\lfloor j/2 \rfloor} \left(\frac{x^{2k-1}}{y^{2k}}\right)^{\lfloor j/2 \rfloor} = \prod_{k=1}^{m} \frac{1}{y^{2k}} = \frac{1}{y^j}$$

and

$$d_j(x, y)d_{j-1}(y, x) = \left(\frac{y^{2m-2}}{x^{2m-1}}\right) \prod_{k=1}^{m-1} \left(\frac{y^{2k-2}}{x^{2k-1}}\right)^{\lfloor j/2 \rfloor} \left(\frac{x^{2k-1}}{y^{2k}}\right) = \frac{1}{x^{j-1}}$$

and

$$d_j(x, y)d_j(x, y) = \prod_{k=1}^{m} \left(\frac{y^{2k-2}}{x^{2k-1}}\right)^{\lfloor j/2 \rfloor} \left(\frac{x^{2k-2}}{y^{2k}}\right)^{\lfloor j/2 \rfloor} = \frac{1}{(xy)^m} = \frac{1}{(xy)^{j/2}}.$$
For \( j = 2m + 1 \), odd, 

\[
d_j(x, y) = \prod_{k=1}^{m} \left( \frac{y^{2k-1}}{x^{2k}} \right) \left( \frac{x^{2k-1}}{y^{2k}} \right) = \frac{1}{(xy)^m} = \frac{1}{(xy)^{(j-1)/2}}
\]

Hence, we obtain

\[
[B]_{j+1,j} = [A]_{j+1,j}d_j(x, y) = b_j \\
[B]_{j,j+1} = [A]_{j,j+1}d_j(y, x) = c_j \\
[B]_{j,j} = [A]_{j,j}d_j(x, y) = \frac{d_j}{(xy)^{j/2}}.
\]

\[\square\]

We note that the results do not depend on the dimension \( n \) and thus hold for the infinite case with elements \( d_j(x, y), [B]_{j,j} : j \in \mathbb{N} \) defined as above.

Now, setting \( \alpha(I) = [I/2], \lambda_I = -\lambda + \beta_I \eta^{\alpha(I)-1} \) (for parameter \( \lambda \leq 0 \), \( \omega(I) = \sqrt{\lambda_I \eta^{-\alpha(I)}} \), and \( x = y = \eta^{-1} \) in (4.4), we use Lemma 4.5 now to convert equation (4.2) into a Jacobi-matrix eigenvalue problem of the form (4.7)

\[
\lambda \vec{Y} = T \vec{Y},
\]

for a certain tri-diagonal \( T \). With \( W = H^{-1}L \) we denote the following:

\[
L \overset{\text{def}}{=} \text{diag} \left( \sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_j}, \ldots \right);
\]

\[
H \overset{\text{def}}{=} \text{diag} \left( \eta^{\alpha(1)}, \eta^{\alpha(2)}, \ldots, \eta^{\alpha(j)}, \ldots \right)
\]

so that \(-W^2X = H^{-2}(L^2X)\). We then set \( A \overset{\text{def}}{=} HAH \overset{\text{def}}{=} T - D \) for \( H \) as in (4.2) where

\[
T \overset{\text{def}}{=} \begin{pmatrix}
\theta & \mu_1 & 0 & 0 & 0 & 0 & \cdots \\
\mu_1 & \theta \eta & \mu_2 & 0 & 0 & 0 & \cdots \\
0 & \ddots & \ddots & \ddots & 0 & 0 & \cdots \\
\vdots & 0 & \mu_j & \theta \eta^{2[j/2]} & \mu_{j+1} & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\
\end{pmatrix}
\]

and

\[
D \overset{\text{def}}{=} \text{diag} \left( \beta_1/\eta, \beta_2/\eta, \beta_3/\eta, \beta_4/\eta, \ldots, \beta_j \eta^{2\alpha(j)-j}, \ldots \right).
\]

Since \( \vec{Y} = H^{-1}\vec{X} \), equation (4.2) leads to

\[
H^{-1}(\lambda \vec{Y} - D \vec{Y}) = H^{-1}(T - D)\vec{Y},
\]

yielding (4.7).

We are ready to state
Proposition 4.2. The absolutely continuous spectrum of the operator $T$ as in (4.7) is $[-2\mu/\eta, 2\mu/\eta]$ for $\mu$ as in (4.3), the interior of which contains no pure-point spectra.

Proof. This is a consequence of part iii) of Theorem 2.2 applied to (4.7). □

It is worthwhile to point out that the nearly periodic nature of the elements of $D$ leads to the presence of spectral gaps under an appropriate formulation. If we replace $\omega$ by $\tilde{\omega}$, where $\tilde{\omega}(I) = \sqrt{-\lambda \eta^{-\alpha(I)}}$ and replace $W^2$ by $\lambda H^{-2}$, equation (4.7) can be recasted as $\lambda \tilde{Y} = A \tilde{Y}$. We thereby convert equation (1.2) into a weighted eigenvalue problem, now with $LHS = -\lambda \eta^{-2\alpha(I)} \delta r(I)$. We further note that $A$ is a perturbation of a Jacobi matrix with periodic coefficients and, in our search for negative eigenvalues, make the following

Remark 4.10. The absolutely continuous spectrum of $B = -A = -T + B$ is given by

$$\sigma_{ac}(B) = [E^-, E_1] \cup [E_2, E^+]$$

where $E^\pm = \frac{\beta(1+\eta) \pm \sqrt{\lambda^2 \eta^2 + 2(1-\eta^2)}}{2\eta}$ (resp.), $E_1 = \beta$, and $E_2 = \frac{\beta}{\eta}$. Moreover, the pure-point spectra is restricted to the spectral gaps (i.e. $\sigma_{pp}(B) \subset \mathbb{R} \setminus \sigma_{ac}(B)$) and $\sigma_{ess}(B)$ is purely absolutely continuous.

See Theorem 7.11 along with equations (7.71) of [25]; or, see Chapter 5 of [21]. Although the $\tilde{\omega}(I)$ of Remark 4.10 do not depend on the $\beta_I$, the associated spectral sets still depend on $\beta$. Moreover, we see that the essential spectrum still depends on the choice of partition (determined by $\eta$) in either case. Thus, out of convenience, we continue with the formulation (4.7).

Recall that for a polytropic state there are positive constants $K, \Gamma$ so that $P = K \cdot (\rho(r))^{\Gamma}$. With mass distributions given by (2.7) for $\gamma > 1$, we now consider distributions $\tilde{D}$ that are admissible, thereby approximating polytropes near the boundary $r = R_*$. That is, for a polytopic state, we suppose

$$\left( \frac{P}{M} \right)(I) = \frac{G M_*}{(4\pi R_*^3)} + O(\eta^{(1+\epsilon)}l) \quad \text{(as } I \to \infty \text{ for some constant } \epsilon > 0)$$

Under these conditions such a $\tilde{D}$ will be called an almost polytrope. Using a version of the Mean Value Theorem, it is not difficult to show, for instance, that (4.11) and (4.12) both hold for our mass distributions if $\rho(r) \propto (R_* - r)^{\gamma-1}$ with $\gamma = \frac{1}{1-\frac{1}{\gamma}} > 1$ (taking $\epsilon = \gamma - 1 > 0$).

For ease of exposition, we start with an almost polytrope $\tilde{D}$ where $\gamma \geq 2$. Noting $(M/\rho)(I) = O(\eta^l)$ as $I \to \infty$, we introduce

$$\mu_1 = \eta^l G_3(I); \quad \beta_1 = \eta^l G_2(I - 4\Lambda_\epsilon).$$
Then, since \( \frac{G_3(I)}{G_2(I)} = \eta + O(\eta^{2I}) \) we obtain from (1.4) and (2.12), along with (4.11) and (4.12), that

\[
G_3(I) = \Lambda_\ast \Gamma \cdot \frac{\eta^{1-\gamma/2}}{1-\eta} \eta^{-I} (1 + O(\eta^{2I})) \\
G_2(I) = 4\Lambda_\ast - \Lambda_\ast \Gamma \cdot \frac{\eta + \eta^{1-\gamma}}{1-\eta} \eta^{-I} (1 + O(\eta^{2I}))
\]

(as \( I \to \infty \)) and find that \( \mu_I \) and \( \beta_I \) thereby satisfy

\[
\mu_I - \frac{\Lambda_\ast \Gamma \cdot \eta^{1-\gamma/2}}{1-\eta} \lesssim \eta^I; \quad \beta_I - \Lambda_\ast \Gamma \cdot \left( \frac{\eta + \eta^{1-\gamma}}{1-\eta} \right) \lesssim \eta^I.
\]

With \( \beta \overset{\text{def}}{=} \lim_{I \to \infty} \beta_I \text{ and } \mu \overset{\text{def}}{=} \lim_{I \to \infty} \mu_I \), the estimates of \( \mu_I \), and \( \beta_I \) result in

**Theorem 4.3.** For an almost polytrope \( \vec{\mathcal{O}} \) with \( 1 < \Gamma \leq 2 \) (equivalently, \( \gamma \geq 2 \)) let \( A \) and \( A \) be defined by \( G_3(I), \mu_I, G_2(I), \) and \( \beta_I \) as in (4.13). Then, the result of Proposition 4.2 holds for \( A \). Furthermore, if \( \lambda \in \sigma_{ac}(A) \cap (-\infty, 0] \), then any \( \delta r \) associated with a non-trivial solution vector \( \vec{X} \) of (4.2) satisfies

\[
\delta r(I_k) = \vec{Y}(I_k) = C_\ast \cdot \eta^{(\alpha(I_k) - \gamma I_k/2)} \geq \eta^{-I_k(\gamma - 1)/2},
\]

(as \( k \to \infty \)) resulting in

\[
\limsup_{I \to \infty} \lambda_I = -\lambda + \beta \eta > 0
\]

Finally, the unboundedness of \( \omega(I) \) follows since

\[
\liminf_{I \to \infty} \lambda_I = -\lambda + \beta \eta > 0
\]

and, hence, \( \omega(I) \gtrsim \eta^{-I/2} \) (as \( I \to \infty \)).

Following the above construction, we treat an alternative type of polytropic distribution where density vanishes at the surface but \( \gamma \) is nearly 1 (whereby \( \rho \) is nearly constant slightly away from surface): We suppose only condition (4.11) and not necessarily (4.12), thereby imposing mass conservation but not excluding the almost polytrope. In this case we set

\[
\left( \frac{P \rho}{M^2} \right)(I) = C_\ast \cdot \eta^{(\alpha(I) - \gamma I_k/2)} (1 + O(\eta^I))
\]
for constants $C_\ast > 0$ and $\gamma, \Gamma > 1$ with $0 < (\gamma - 1)/(\Gamma - 1) \leq 1$. We now denote 
\[ \nu = \eta^{2 - (\gamma - 1)/(\Gamma - 1)} \]
and set
\[ H = \text{diag} \left( \nu^{[1/2]}, \nu^{[2/2]}, \nu^{[3/2]}, \ldots, \nu^{[I/2]}, \ldots \right) \]
so as to recast our eigenvalue problem into the form (4.2) with $G_2, G_3$ so that 
\[ \tilde{g}(I) \stackrel{\text{def}}{=} \nu^I G_3(I) = 16\pi^2 \eta^{-\gamma/2} C_\ast \cdot \Gamma + O(\eta^I) \]
\[ \tilde{h}(I) \stackrel{\text{def}}{=} \nu^I G_2(I) = \nu^I 4A_\ast - 16\pi^2 R_4^4 C_\ast \Gamma \cdot (1 + \eta^{-\gamma} \nu) + O(\nu^I \eta^I). \]

For convenience we now set
\[ \tilde{\beta} = - \lim_{j \to \infty} \tilde{h}(j) = 16\pi^2 R_4^4 C_\ast \Gamma \cdot (1 + \eta^{-\gamma} \nu) \]
(4.19)
\[ \tilde{\mu} = \lim_{j \to \infty} \tilde{g}(j) = 16\pi^2 R_4^4 \eta^{-\gamma/2} C_\ast \cdot \Gamma \]
(4.20)
to state

**Theorem 4.4.** Suppose the mass distribution is as in (2.7) with $1 < \gamma \leq \Gamma$ with a polytropic $\vec{\mathcal{D}}$ satisfying (4.12) and (4.17) for some constant $\Gamma > 1$. Then the conclusions of Theorem 4.3 likewise hold, but for 
\[ \sigma_{ac}(\mathcal{A}) = [-2\tilde{\mu}/\nu, 2\tilde{\mu}/\nu] \]
with $\tilde{\mu}$ as in (4.20).

**Proof.** For our choices of $\gamma, \Gamma$ we have $0 < \nu < 1$ so that we may apply the decomposition $-HAH = \mathcal{T} - \mathcal{D}$ as in (4.8) and (4.9), and likewise apply the analysis of Proposition 4.2 and Theorem 2.4 but with $\theta = 0$, $\eta$ replaced by $\nu$, and obvious substitutions for parameters $\beta_j, \beta, \mu_j, \mu$ via (4.20) and (4.19). In particular, we have $\nu \leq \eta$ so that results concerning $\vec{\delta}r$ and $\vec{\omega}$ from the redefinition (4.18) of $H$ follow as in (4.15) and (4.16). \[ \square \]

We end this section by noting the following about our polytropic and non-polytropic cases, as well as the almost polytrope: The essential spectra can admit intervals of arbitrary length as we are allowed to choose arbitrarily small $\eta$ (or $\nu$) in choosing our partitions via $\vec{M}$ and $\vec{r}$. This suggests perhaps that unbounded intervals of such spectra may be present in some such cases as values of our discrete parameters $m$ and $r$ pass to a continuous interval. This expectation is indeed borne out in the remaining sections of this article.

5. **Differential Equation Model**

In the continuous case we study the perturbed mass distribution $\delta r(r)$ by way of equation (8.6) of [9] (see also [1])
\[ \frac{d}{dr} \left( \Gamma P r^4 \frac{d}{dr} \xi \right) - \left( \rho r^4 \frac{d}{dr} \right) \left( 3\Gamma - 4 \right) P \right] \xi = \sigma^2 \rho r^4 \xi \]
where \( \xi(r) = \delta r(r)/r \). The analysis that we apply requires no special boundary conditions, but we will check our models against a so-called regularity condition given by
\[
(5.2) \quad \lim_{r \to R_*} (P\Gamma)(r)(3\xi(r) + r\xi'(r)) = 0
\]
at the (finite) boundary \( r = R_* > 0 \) \([4, 17]\). Regarding HSE, we note that \( \xi \) are assumed to be perturbations about 0 of
\[
(5.3) \quad \frac{d}{dr} P(r) + \frac{G \cdot \rho(r)}{r^2}.
\]
We (re-)introduce notation in accord with some of our references: We set \( x = r, y = \xi, p = \Gamma P x^4, \lambda = \sigma^2 \) (switching sign convention), \( q = -x^3 \frac{d}{dx} [(3\Gamma - 4) P] \), and \( w = \rho x^4 \), whereby equation \( (5.1) \) takes the SL form
\[
(5.4) \quad - (p(x)y'(x))' + q(x)y(x) = \lambda w(x)y(x)
\]
Here we take \( \prime \) to mean the full derivative with respect to the independent variable as indicated.

By way of the Liouville transform \([5]\) we may further convert \( (5.1) \) to the canonical form
\[
(5.5) \quad -Y''(X) + Q(X)Y(X) = \lambda Y(X)
\]
where, for some fixed positive constants \( R_{\delta} < R_* < \infty \),
\[
(5.6) \quad X(x) \overset{\text{def}}{=} \int_{R_{\delta}}^{x} \sqrt{w(t)/p(t)} \, dt \text{ and } \quad Y(X)(x) \overset{\text{def}}{=} (p(x)w(x))^{1/4}y(x)
\]
\[
Q(X)(x) \overset{\text{def}}{=} \frac{q(x)}{w(x)} - \left( \frac{p(x)}{(w(x))^{3}} \right)^{1/4} \left( \left( \frac{p(x)}{w(x)} \right)^{1/2} \left( \left( (p(x)w(x))^{1/4} \right) ' \right) ' \right)
\]
with the standing assumptions that \( q, p, w, 1/w, \sqrt{w/p} \) are continuous on \([R_{\delta}, R_*]\).

We introduce notation and terminology here to accommodate more precise estimates needed in the following sections: It will be convenient to denote \( q_0(x) \overset{\text{def}}{=} Q(X)(x), q_1(x) \overset{\text{def}}{=} \frac{q(x)}{w(x)} \) and \( q_2(x) \overset{\text{def}}{=} q_0(x) - q_1(x) \). From the invertibility of \( (5.6) \) we may, with obvious notation, likewise decompose \( Q_0(X) = Q_1(X) + Q_2(X) \). The phrase "near \( R_* \)" ("near \( \infty \)") regarding an estimate or bound will mean such will hold on some interval of the form \([x_0, R_*]\) (resp. \([x_0, \infty)\)).

Our strategy in the next sections will be as follows: We apply results of subordinancy theory where general boundedness of (non-trivial) solutions yields non-\( L^2 \) behavior (counter-intuitively); in turn, the later property yields unbounded oscillation \( \xi \). Along these lines, we make frequent use of the results discussed below.
Theorem 5.1. (Already Known) Suppose \( Q(X) = V_1(X) + V_2(X) \) defined on \((0, \infty)\) where \( V_1 \in L^1 \) and \( V_2 \in C^1 \) where \( V_2' \in L^1 \) with \( \lim_{X \to \infty} V_2(X) = 0 \). Then, given \( \lambda > 0 \), to every solution \( Y \) of (5.5) there are constants \( \alpha \) and \( \beta \) so that
\[
Y(X) = \alpha u_+(X) + \beta u_-(X) + o(1)
\]
\[
Y'(X) = i\alpha u_+(X) - i\beta u_-(X) + o(1)
\]
(as \( X \to \infty \)) where \( u_\pm(X) = \exp(\pm i \int_{X_0}^X \sqrt{\lambda^2 - V_2(t)} \, dt) \) (resp.) for some sufficiently large, fixed \( X_0 > 0 \).

This is Theorem B.1 [20] (also see Theorem 6 [24]). We then find unbounded oscillations for a star of finite mass \( M_\ast \) and radius \( R_\ast \) \((0 < M_\ast, R_\ast < \infty)\) as we apply

Proposition 5.2. If a solution \( Y \) of (5.5) is not square integrable on the domain \([0, X(R_\ast))\), then the corresponding \( \delta r(r) = \xi(r)r \) satisfies
\[
\limsup_{r \to R_\ast^-} |\delta r(r)| = \infty.
\]

Proof. We compute
\[
\int_0^{X(x)} |Y(t)|^2 \, dt = \int_{R_\delta}^{X} \xi^2(s)s^4 \rho(s) \, ds
\]
\[
= \int_{R_\delta}^{X} s^2 \rho(s)(\delta r(s))^2 \, ds
\]
\[
\leq \sup_{s \in (R_\delta, X)} (\delta r(s))^2 \cdot \int_{R_\delta}^{X} \rho(s)s^2 \, ds.
\]
We have LHS of (5.7) is unbounded as \( x \to R_\ast \) while \( \int_{R_\delta}^{X} \rho(s)s^2 \, ds \leq M_\ast/(4\pi) \) so that the desired result is clear. \( \square \)

6. A Polytropic Outer Shell

As in Section 4 we consider an EOS of the form \( P = K(\rho(x))^b \) with \( \rho(x) = (R_\ast - x)^a \) some fixed \( a, b > 0 \). Likewise, we have that \( \Gamma = b > 1 \) is constant on an interval \([R_\delta, R_\ast]\) for some fixed, positive \( R_\delta \) \(\Delta R_\delta \leq R_\ast - \delta < R_\ast \). After a change of variables we find that for \( a(b-1) > 1 \) our eigenvalue problem amounts to certain \( L^1 \) perturbations of a simple operator whereby the existence of absolutely continuous spectra is clear by the Kato-Rosenblum Theorem [19] (see also Chapter 11, [18]). For the case \( a(b-1) > 2 \) we will further elaborate on the behavior of solutions.

We introduce the following notation: Let \( \mathcal{F}_\lambda \) be the set of solutions to (5.5) on \([0, \infty)\); let \( S \) be the defined by \( S \triangleq \{ \lambda > 0 \mid Y \in \mathcal{F}_\lambda \Rightarrow \sup_{[0, \infty)} |Y(X)| < \infty \} \); and,
Proof. Here $p(x) = bK(R_* - x)^{ab}x^4$ and $w(x) = (R_* - x)^a x^4$ and a computation shows that

$$q_1(x) = K\frac{ab(-4 + 3b) (R_* - x)^{ab-a-1}}{x}$$

and

$$q_2(x) = -KR_*^2b(R_* - x)^{ab-a-2} \frac{16x^2}{\Omega(x/R_*)}$$

where

$$(6.1) \quad \Omega(u) = 32 - 32(2 + ab)u + (32 + 4a(-1 + 7b) + a^2(-1 + 2b + 3b^2))u^2.$$ 

In the case $ab - a = 2$ we have that $q_0(x) - k_{a,b} \lesssim (R_* - x)$ near $R_*$ for $k_{a,b} = q_0(R_*);$ hence, $(q_0(x) - k_{a,b}) W(x)$ is bounded on $[R_0,R_*).$ For $1 < ab - a < 2$ we set $k_{a,b} = 0$ as we find that $q_0(x) W(x)$ is absolutely integrable on $[R_0,R_*).$ \qed

We notice that $\Omega(u) = 0$ has a root $u = 1$ when $a = 4/(3b - 1),$ but we will not use this fact considering other restrictions on $a$ and $b.$

We now perform a change of variables

$$(6.2) \quad X = \int_{R_0}^x W(t) \, dt = \int_{R_0}^x (R_* - t)^{(a-ab)/2}/\sqrt{bK} \, dt$$

for positive $a,b > 1$ with $a(b-1) > 2$ whereby $X$ takes values of $[0,\infty).$ Here $Q(X)$ is smooth and bounded and $\lim_{X \to \infty} Q_1(X) = 0.$ We state
Theorem 6.3. For the EOS as in Proposition 6.2 with $a$ and $b$ as in (6.2) we have the following:

i) The result of Theorem 6.1 holds;

ii) the regularity condition (5.2) holds; and,

iii) for each $\lambda > 0$ every corresponding non-trivial $\delta r(x)$ is unbounded.

Proof. We integrate over $(R_\delta, R_\ast)$ the quantity

$$Q(X)(x)dX(x) = q_0(x)W(x)dx$$

with $q_0(x)W(x) \propto (R_\ast - x)^{ab - a} - 2$ whereby Theorem 6.1 applies to prove i).

Addressing ii), we write (6.3)

$$y(x) = \frac{Y(X)(x)}{(pw)^{1/4}(x)} = \frac{Y(X)(x)}{x (\rho(x)bP(x))^{1/4}}.$$

For any solution in $\bar{T}_\lambda$ we have that $Y$ and $Y'$ are bounded and find

$$(PT')(x)y(x) \lesssim (R_\ast - x)^{ab - a} \lesssim (R_\ast - x)^{(a+3)/2}$$

$$(PT')(x)y'(x) \lesssim (R_\ast - x)^{ab - a - 1} \lesssim (R_\ast - x)^{(a+1)/2}$$

near $R_\ast$ so that (5.2) holds.

Finally, by Theorem 5.1 $Y(X) \approx 1$ near $\infty$ so that result iii) follows from Proposition 5.2. □

We remark: The requirement that $a(b-1) > 1$ excludes those states studied in [3] which amount to $0 < a < 5$ and $b = (1 + a)/a$, whereby $a(b - 1) = 1$.

7. A Non-polytropic Case

We now consider an example where $\Gamma$ is not necessarily constant. We will consider a case where we replace $P(x)$ by $P(\rho, x)$ where

$$P(\rho, x) = T(x)\rho + L(x)$$

for a non-increasing function $T$. The function $T$ can be considered a function of temperature as in some physical models, where temperature varies with position.

We compute $q_1(x)$ and $q_2(x)$ for $\rho$ as in Proposition 6.2 and $P$ as in (7.1) with

$$T(x) = K_0 \cdot (R_\ast - x)^{ab - a}$$

and $L(x) = L_0 \cdot (R_\ast - x)^c$ for fixed $K_0, L_0 > 0$, $a \geq 1$, $b \geq 1$, and $c > 0$. With $w(x) = x^4\rho(x)$ and $P(x) = T(x)\rho(x)x^4$ we have that $q_2(x)$ is the same as that of Proposition 6.2, but with $bk$ replaced by $K_0$. We compute

$$q_1(x) = \frac{abK_0 \cdot (R_\ast - x)^{-1 - a + ab} + 4cL_0 \cdot (R_\ast - x)^{-1 - a + c}}{x}$$

$$q_2(x) = -\frac{K_0R_\ast^2(R_\ast - x)^{-2 - a + ab} \Omega(x/R_\ast)}{16x^2}$$

where, of course, $\Omega$ is the polynomial given by (5.1).
We comment: With the EOS (7.1) we admit some cases of unbounded \( Q(X) \) such as the case \( c \leq ab - 1 \) whereby \( q_0(x) \propto (R_* - x)^{-1-a+c} \) near \( x = R_* \) and, hence, \( \lim_{x \to R_*^-} q_0(x) = -\infty \) for \( c < a + 1 \). We can treat such unbounded \( q_0 \) by applying results from [24] (see also [26]) for which we will find the following estimates useful:

\[
(7.2) \quad Q'_1(X)(x) = \left( \frac{q'_1}{W} \right) (x) \propto (R_* - x)^{-(c-a)+(ab-a)/2}
\]

\[
Q''_1(X)(x) = \left( \frac{d}{dx}Q'_1(X)(x) \right) \propto (R_* - x)^{-(c-a)+(ab-a)}
\]

\[
Q'_2(X)(x) = \left( \frac{q'_2}{W} \right) (x) \propto (R_* - x)^{-(c-a)+(ab-a)/2}
\]

near \( R_* \). As for physical motivation, we note that when \( b > 1 \) and \( c > a \), we have that \( P/\rho \to 0 \) as \( x \to R_* \), corresponding to some related models as discussed in Section 8.3 of [9]. Here pressure can be attributed to that of perfect gas, given by \( T \cdot \rho \) where temperature \( T \) vanishes at the surface, and another source, given by \( L(x) \) - such as radiation, for instance. In the case \( c = a + 1 \leq ab \) we can also assign \( L_0 \) or perhaps \( K_0 \) so that HSE holds at \( x = R_* \). Moreover, we note that our general results do not depend on the domain length \( \delta > 0 \) where, by continuity arguments, \( (5.3) \) can be made arbitrarily small.

**Theorem 7.1.** For the EOS (7.1) with \( a, b \geq 1 \) suppose that either

i) \( ab \geq a + 2 \) and \( c > a + 1 \); or,

ii) \( ab \geq a + 3 \) and \( c > a \).

Then the absolutely continuous spectrum contains \((0, \infty)\) and, the differential equation (5.5) has no solutions of \( L^2(dX) \) class near \( X = \infty \) for any such \( \lambda \).

**Proof.** We note that in either case i) or ii) the result follows by Theorem 6.1 when \( c > ab - 1 \). Moreover, for the remainder of the proof we may suppose that \( c \leq a \) since the argument for \( c > ab \) is the same as that for \( c = ab \).

We start with case i). We first consider the subcase \( c \geq (ab + a)/2 \) where we find \( Q \in L^1(dX) \) near \( \infty \) so that we may apply Theorems 5 and 6 of [24]. In the subcase \( a + 1 < c \leq (ab + a)/2 \) we have \( \lim_{X \to \infty} Q(X) = \lim_{x \to R_*^-} q_0(x) = 0 \) and that \( Q(X) \leq 0 \) is bounded so that we may again apply Theorems 5 and 6 of [24].

We now consider case ii), needing only to suppose \( c \leq a + 1 \). From (7.2) we find that \( Q_2(X), Q'(X) \) and \( Q'_2(X) \) are locally absolutely continuous. We also
have the following estimates near \( x = R_+ \):

\[
(7.3) \quad \frac{1}{\sqrt{1 - Q_1(X)(x)}} W(x) \asymp (R_+ - x)^{-\frac{(ab-a)/2 - (c-a)/2 + 1}{2}} \notin L^1;
\]
\[
\frac{Q_2'(X)(x)}{\lambda - Q_1(X)(x)} W(x) \asymp (R_+ - x)^{(ab-a) - (c-a) - 2} \in L^1;
\]
\[
\frac{Q_2''(X)(x)}{(\lambda - Q_1(X)(x))^{3/2}} W(x) \asymp (R_+ - x)^{-3 + 2(ab-a) - (c-a)/2} \in L^1;
\]
\[
\frac{(Q_1'(X)(x))^2}{(\lambda - Q_1(X)(x))^{9/2}} W(x) \asymp (R_+ - x)^{-3 + (ab-a) - (c-a)/2} \in L^1.
\]

We may therefore apply Theorems 9 and 2 of [24].

**Remark 7.4.** We note the results of Theorem 7.1 hold for other combinations of \( a, b > 1 \) and \( c > a \) provided \( ab - a > 2 \) and the exponent of the RHS of estimate (7.3) is no greater than \(-1\).

We analyze our model, demonstrating unbounded \( \delta r \) while satisfying the regularity condition via

**Theorem 7.2.** The following hold for all non-trivial solutions in \( \mathcal{S}_\lambda \) for the equation as in Theorem 7.1

i) The regularity condition is satisfied; and,

ii) the corresponding \( \delta r \) is unbounded as \( r \to R_+ \).

**Proof.** To prove item i) we use (6.3) and apply estimates for \( Y \) and \( Y' \) according to various cases in the proof of Theorem 7.1.

For \( c > a + 1 \) we may apply Theorem 6.1 to obtain

\[ Y(X), Y'(X) \asymp 1 \]

near \( \infty \) and the result follows as in Theorem 6.2. For \( a < c \leq a + 1 \) we have from Lemma 12 [24] that

\[ Y(X) \lesssim (1 + |Q_1(X)|)^{-1/4}; \quad Y'(X) \lesssim (1 + |Q_1(X)|)^{1/4} \]

near \( \infty \) so that from (5.9), the chain rule, and the product rule

\[
y(x) \lesssim (R_+ - x)^{\frac{1-ab-c}{4}}; \quad y'(x) \lesssim (R_+ - x)^{\frac{c-1-3ab}{4}} + (R_+ - x)^{-\frac{e^{-ab-3}}{4}}
\]

near \( R_+ \). The limit (5.2) is obtained for \( y = \xi \) by the estimates

\[
(7.5) \quad (PT) (x) \lesssim (R_+ - x)^{ab}
\]
\[
(PT y)(x) \lesssim (R_+ - x)^{\frac{3(ab-a) + 3a + 1 - c}{4}} \lesssim (R_+ - x)^2
\]
\[
(PT y')(x) \lesssim (R_+ - x)^{\frac{c+1-ab-1}{4}} + (R_+ - x)^{\frac{3ab-3-c}{4}} \lesssim (R_+ - x)^{3/4}
\]

which clearly vanish as \( x \to R_+^- \).

Item ii) follows from Theorem 7.1 and Proposition 5.2

\[ \square \]
Remark 7.6. In the estimates 7.5 of the case \( c < a+1 \) we only suppose \( ab - a > 2 \) and, hence, the results of Theorem 7.2 likewise hold for cases as discussed in Remark 7.4.

References

[1] C. Aerts, J. Christesen-Dalsgaard, D.W. Kurtz, Asteroseismology, Springer, 2010.
[2] Horst Beyer, "Spectrum of adiabatic oscillations" J. Math. Phys., 36, (9), (1995), 4792-4814.
[3] Horst Beyer, "Spectrum of radial adiabatic oscillations" J. Math. Phys., 36, (9), (1995), 4815-4825.
[4] Horst R. Beyer and Bernd G. Schmidt, "Newtonian stellar oscillations," Astron. Astrophys. 296, (1995), 722-726.
[5] Gerrett Birkhoff, Gian-Carlo Rota, Ordinary Differential Equations, Wiley, 1989.
[6] John I. Castor, On the calculation of linear, nonadiabatic pulsations of stellar models, APJ 166, (1971), 109-129.
[7] Robert F. Christy, "The Calculation of Stellar Pulsation," Rev. Mod. Phys., 36, (1964), 555-571.
[8] E. A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1955.
[9] John P. Cox, "Theory of Stellar Oscillation," Princeton University Press, Princeton, NJ (1980).
[10] D. Damanik, B. Simon, "Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegő asymptotics," Invent. Math. 165, (2006), 1-50.
[11] D. Damanik, B. Simon, "Jost functions and Jost solutions for Jacobi matrices, II. Decay and analyticity," IMRN 2006, (2006), 1-32.
[12] D. J. Gilbert and D. B. Pearson, "On subordinancy and analysis of the spectrum of one-dimensional Schrödinger operators", J. Math. Anal. Appl. 128 (1987), 30-56.
[13] L. G. Henyey, J. E. Forbes, and N. L. Gould, "A new method for automatic computation of stellar evolution", Ap. J. 139, (1964), 306-317.
[14] Zdenek Kopal,"Radial Oscillations of the limiting models of polytropic gas spheres," Proc. Nat. Acad. Sci., 34 (8), (1948), 377 -384.
[15] Rudolf. Kippenhahn and A. Weigert, Stellar Structure and Evolution, Springer-Verlag, Berlin (1990).
[16] R. Killip and B. Simon, "Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegő asymptotics," Invent. Math. 165, (2006), 1-50.
[17] Paul Ledoux and Théodore Walraven, "Variable stars," Handbuch der Physik, Volume 51, (1958), 353-604.
[18] D. B. Pearson, Quantum Scattering and Spectral Theory, Academic Press, (1988).
[19] M. Reed and B. Simon, Methods of Modern Mathematical Physics, vols. I and III, Academic Press, (1980).
[20] B. Simon, "Bounded eigenfunctions and absolutely continuous spectra one-dimensional Schrödinger operators," Proc. AMS Volume 124, 11, (1996).
[21] B. Simon, Szegő’s Theorem and its Descendants: spectral theory for L^2 perturbations of orthogonal polynomials, Princeton University Press, (2011).
[22] Paul Smeyers and Tim Van Hoolst Linear Isentropic Oscillations of Stars: Theoretical Foundations, Springer-Verlag Berlin Heidelberg, (2010).
[23] T.E. Sterne, "Modes of Radial Oscillation", MNRAS, vol. 97, 582S, (1937).
[24] G. Stolz, "Bounded solutions and absolute continuity of Sturm-Liouville operators" J. Math. Anal. Appl. 169, (1992), 210-228.

[25] G. Teschl, "Jacobi Operators and Completely Integrable Nonlinear Lattices," Mathematical Surveys and Monographs, AMS, Providence, RI, vol. 72 (2000).

[26] J. Walter, "Absolute continuity of the essential spectrum of $-\frac{d^2}{dx^2} + q(t)$ without monotony of $q$" Math. Z. 129, (1972), 83-94.

University of Alaska - Fairbanks, Fairbanks, AK
E-mail address: cjwinfield01@alaska.edu