Exact solution of the Dirac equation for a Coulomb and a scalar Potential in the presence of an Aharonov-Bohm and a magnetic monopole fields

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Abstract

In the present article we analyze the problem of a relativistic Dirac electron in the presence of a combination of a Coulomb field, a $1/r$ scalar potential as well as a Dirac magnetic monopole and an Aharonov-Bohm potential. Using the algebraic method of separation of variables, the Dirac equation expressed in the local rotating diagonal gauge is completely separated in spherical coordinates, and exact solutions are obtained. We compute the energy spectrum and analyze how it depends on the intensity of the Aharonov-Bohm and the magnetic monopole strengths.
I. INTRODUCTION

The Dirac equation is a system of four coupled partial differential which describes the relativistic electron and other spin 1/2 particles. Despite the remarkable effort made during the last decades in order to find exact solutions for the relativistic Dirac electron the amount of solvable configurations is relatively scarce, being the Coulomb problem perhaps the most representative example and also one of the most discussed and analyzed problems in relativistic quantum mechanics. Among the different approaches available in the literature for discussing the Dirac-Coulomb problem in the presence of other interactions like the Aharonov-Bohm field or any other electromagnetic potential we have the quaternionic approach proposed by Hautot, the Stäckel separation method developed by Bagrov et al., the algebraic method of separation of variables, the shift operator method, and the algebraic method proposed by Komarov and Romanova.

Recently, Lee Van Hoang et al. have solved the Dirac-Coulomb problem when an Aharonov-Bohm and a Dirac magnetic monopole fields are present. The authors use, for tackling the problem, a two dimensional complex space which results after applying the Kustaanheimo-Stiefel transformation on the three space variables, reducing in this way the Kepler problem to an oscillator problem. This idea lies on the utilization of a SU(2) dynamical algebra for computing the resulting energy spectrum, which, like the spinor solution, is expressed in terms of intrinsic coordinates appearing after using the complex space. The utilization of different techniques for studying the Dirac-Coulomb field in the presence of an Aharonov-Bohm field or a magnetic monopole could give rise to the idea that this problem is not soluble without introducing new variables or additional conserved quantities. Here it is shown that using the algebraic method of separation of variables it is possible to solve the Dirac equation in the presence of a Coulomb field and a scalar 1/r potential with an Aharonov-Bohm and a magnetic monopole fields. The advantage of this approach is that does not require the introduction of non bijective quadratic transformations, also it becomes clear the role played by the Dirac magnetic monopole as well as the Aharonov-Bohm field.
in the angular dependence of the spinor $\Psi(\vec{r})$ solution of the Dirac equation.

The article is structured as follows: In Sec. II, applying a pairwise scheme of separation, we separate variables in the Dirac equation expressed in the local rotating frame, we separate the radial dependence from the angular one. In Sec. III, the angular dependence is solved in terms of Jacobi Polynomials. In Sec IV, the separated radial equation is solved and the energy spectrum is computed. In Sec. V, we discuss the influence of the Aharonov-Bohm field and the magnetic monopole on the energy spectrum.

II. SEPARATION OF VARIABLES

In this section we proceed to separate variables in the Dirac equation when a Coulomb field, a scalar $1/r$ potential as well as a Dirac magnetic monopole and a Aharonov-Bohm field are present. For this purpose, we write in spherical coordinates the covariant generalization of the Dirac equation

\[
\left\{ \tilde{\gamma}^\mu (\partial_\mu - \Gamma_\mu - i A_\mu) + M + \tilde{V}(r) \right\} \Psi = 0 \tag{1}
\]

where $\tilde{\gamma}^\mu$ are the curved gamma matrices satisfying the relation, $\{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \}_+ = 2g^{\mu\nu}$, and $\Gamma_\mu$ are the spin connections with $\tilde{V}(r)$ as the scalar $1/r$ field

\[
\tilde{V}(r) = -\frac{\alpha'}{r} \tag{2}
\]

where $\alpha'$ is a constant, and the vector potential $A_\mu$ reads

\[
A_\mu = A_\mu^{(\text{mon})} + A_\mu^{(\text{Coul})} + A_\mu^{(\text{AB})} \tag{3}
\]

where the components of the Coulomb potential $A_\mu^{(\text{Coul})}$ take the form

\[
A_0^{(\text{Coul})} = V(r) = -\frac{\alpha}{r}, \quad A_i^{(\text{Coul})} = 0, \quad i = 1, 2, 3 \tag{4}
\]

the Aharonov Bohm potential $A_\mu^{(\text{AB})}$ reads

\[
A^{(\text{AB})} = \frac{F}{r \sin \theta} \hat{e}_\varphi \tag{5}
\]
and the Dirac monopole field $A^{(\text{mon})}_\mu$ is

$$A^{(\text{mon})}_\mu = g \frac{(1 - \cos \vartheta)}{r \sin \vartheta} \hat{e}_\varphi$$

where, following the Dirac prescription for quantizing the magnetic charge, $g$ takes integer or half integer values

$$g = \frac{j}{2}, \ j = 0, \pm 1, \pm 2, \ldots$$

(7)

If we choose to work in the fixed Cartesian gauge, the spinor connections are zero and the \(\tilde{\gamma}\) matrices take the form

\[
\begin{align*}
\tilde{\gamma}^0 &= \gamma^0 = \bar{\gamma}^0, \\
\tilde{\gamma}^1 &= \left[ (\gamma^1 \cos \varphi + \gamma^2 \sin \varphi) \sin \vartheta + \gamma^3 \cos \vartheta \right] = \bar{\gamma}^1, \\
\tilde{\gamma}^2 &= \frac{1}{r} \left[ (\gamma^1 \cos \varphi + \gamma^2 \sin \varphi) \cos \vartheta - \gamma^3 \sin \vartheta \right] = \frac{\bar{\gamma}^2}{r}, \\
\tilde{\gamma}^3 &= \frac{1}{r \sin \vartheta} (-\gamma^1 \sin \varphi + \gamma^2 \cos \varphi) = \frac{\bar{\gamma}^3}{r \sin \vartheta}
\end{align*}
\]

(8)

where $\gamma^\alpha$ are the standard Minkowski gamma matrices, and the Dirac equation in the fixed Cartesian tetrad frame \(8\) takes the form

\[
\left\{ \tilde{\gamma}^0 (\partial_t + iV(r)) + \tilde{\gamma}^1 \partial_r + \frac{\tilde{\gamma}^2}{r} \partial_\vartheta + \frac{\tilde{\gamma}^3}{r \sin \vartheta} (\partial_\varphi - iF - i(1 - \cos \vartheta)g) + M + \bar{V}(r) \right\} \Psi_{\text{Cart}} = 0
\]

(9)

where we have introduced the spinor $\Psi_{\text{Cart}}$, solution of the Dirac equation \(9\) in the fixed tetrad gauge. In order to separate variables in the Dirac equation, we are going to work in the diagonal tetrad gauge where the gamma matrices $\tilde{\gamma}_d$ take the form

$$\tilde{\gamma}^0_d = \gamma^0, \ \tilde{\gamma}^1_d = \gamma^1, \ \tilde{\gamma}^2_d = \frac{1}{r} \gamma^2, \ \tilde{\gamma}^3_d = \frac{1}{r \sin \vartheta} \gamma^3$$

(10)

Since the curvilinear matrices $\tilde{\gamma}^\mu$ and $\tilde{\gamma}_d$ satisfy the same anticommutation relations, they are related by a similarity transformation, unique up to a factor. In the present case we choose this factor in order to eliminate the spin connections in the resulting Dirac equation. The transformation $S$ can be written as

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{i\varpi} & 0 & 0 \\
0 & 0 & e^{-i\varpi} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(11)
\[
S = \frac{1}{r (\sin \vartheta)^{1/2}} \exp\left(-\frac{\varphi}{2} \gamma^1 \gamma^2\right) \exp\left(-\frac{\varphi}{2} \gamma^3 \gamma^1\right) a = S_0 a
\]  \hspace{1cm} (11)

where \(a\) is a constant non singular matrix given by, \(a = \frac{1}{2} (\gamma^1 \gamma^2 - \gamma^1 \gamma^3 + \gamma^2 \gamma^3 + I)\) which applied on the gamma’s acts as follows

\[
\begin{align*}
\gamma^1 a \gamma^{-1} &= \gamma^3, \\
\gamma^2 a \gamma^{-1} &= \gamma^1, \\
\gamma^3 a \gamma^{-1} &= \gamma^2,
\end{align*}
\]  \hspace{1cm} (12)

the transformation \(S\) acts on the curvilinear \(\tilde{\gamma}\) matrices, reducing them to the rotating diagonal gauge as follows

\[
S^{-1} \tilde{\gamma}^\mu S = g^\mu \gamma^\mu = \tilde{\gamma}_d^\mu \quad \text{(no summation)}
\]  \hspace{1cm} (13)

then, the Dirac equation in spherical coordinates, with the radial potential \(V(r)\), in the local rotating frame reads

\[
\left\{\gamma^0 (\partial_t + i V(r)) + \gamma^1 \partial_r + \gamma^2 \frac{\partial}{\partial \vartheta} + \frac{\gamma^3}{r \sin \vartheta} (\partial_\varphi - i F - i (1 - \cos \vartheta) g) + M + \tilde{V}(r)\right\} \Psi_{\text{rot}} = 0
\]  \hspace{1cm} (14)

where we have introduced the spinor \(\Psi_{\text{rot}}\), related to \(\Psi_{\text{Cart}}\) by the expression

\[
\Psi_{\text{Cart}} = S \Psi_{\text{rot}} = S_0 a \Psi_{\text{rot}}
\]  \hspace{1cm} (15)

and \(\gamma^\mu\) are the standard Dirac flat matrices.

Applying the algebraic method of separation of variables, it is possible to write eq. (14) as a sum of two first order linear differential operators \(\hat{K}_1, \hat{K}_2\) satisfying the relation

\[
\left[\hat{K}_1, \hat{K}_2\right] = 0, \quad \left\{\hat{K}_1 + \hat{K}_2\right\} \Phi = 0
\]  \hspace{1cm} (16)

\[
\hat{K}_2 \Phi = k \Phi = -\hat{K}_1 \Phi
\]  \hspace{1cm} (17)

then, if we separate the time and radial dependence from the angular one, we obtain

\[
\hat{K}_2 \Phi = -i \left[\gamma^2 \partial_\vartheta + \frac{\gamma^3}{\sin \vartheta} (\partial_\varphi - i F - i (1 - \cos \vartheta) g)\right] \gamma^0 \gamma^1 \Phi = k \Phi
\]  \hspace{1cm} (18)
\[ \hat{K}_1 \Phi = -ir \left[ \gamma^0 \partial_t + \gamma^1 \partial_r + M + i \gamma^0 V(r) + \tilde{V}(r) \right] \gamma^0 \gamma^1 \Phi = -k \Phi \tag{19} \]

with

\[ \Psi_{\text{rot}} = \gamma^0 \gamma^1 \Phi \tag{20} \]

where the operator \( \hat{K}_1 \) as well as \( \hat{K}_2 \) have been chosen to be Hermitean, therefore the constant of separation \( k \) appearing in (18) and (13) is real. Notice that if we drop out the Aharonov-Bohm and the magnetic charge contributions in eq. (18) we obtain the Brill and Wheeler\(^{10}\) angular momentum \( \hat{K} \), and separation constant \( k \) takes integer values.

Now we proceed to decouple the equation (18) governing the angular dependence of the Dirac spinor In order to simplify the resulting equations, we choose to work with the auxiliary spinor \( \bar{\Phi} \) related to \( \Phi \) as follows

\[ \bar{\Phi} = a \Phi \tag{21} \]

Since the operators \( \hat{K}_2 \) and \( \hat{K}_1 \) commute with the projection of the angular momentum \(-i \partial_\theta\), with eigenvalues \( m \), we have that equation (18) takes the form

\[ \left[ \gamma^1 \partial_\theta - i \frac{\gamma^2}{\sin \vartheta} (m + F + (1 - \cos \vartheta) g t) \right] \gamma^0 \gamma^3 \bar{\Phi} = -ik \bar{\Phi} \tag{22} \]

In order to reduce the equation (22) to a system of ordinary differential equations, we choose to work in the following representation for the gamma matrices.\(^{13}\)

\[ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{23} \]

Then, substituting (23) into (22) we obtain,

\[ \left[ \sigma^2 \partial_\theta - i \frac{\sigma^1}{\sin \vartheta} (m + F + (1 - \cos \vartheta) g t) \right] \bar{\Phi}_1 = -ik \bar{\Phi}_1 \tag{24} \]

\[ \left[ -\sigma^2 \partial_\theta + i \frac{\sigma^1}{\sin \vartheta} (m + F + (1 - \cos \vartheta) g t) \right] \bar{\Phi}_2 = -ik \bar{\Phi}_2 \tag{25} \]

with

\[ \bar{\Phi} = e^{i m \varphi} \begin{pmatrix} \bar{\Phi}_1 \\ \bar{\Phi}_2 \end{pmatrix} \tag{26} \]
III. SOLUTION OF THE ANGULAR EQUATIONS

In this section we are going to solve the systems of equations (24) and (25). It is not difficult to see that the spinor $\bar{\Phi}_2$ can be written as $G(r, t)\sigma^3\bar{\Phi}_1$, where $G(r, t)$ is an arbitrary function, this property allows us to consider only the system (24). Using the standard Pauli matrices, we have that (24) reduces to

$$k \sin \vartheta \xi_1 - (m - F - (1 - \cos \vartheta)g)\xi_2 + \sin \vartheta \frac{d\xi_2}{d\vartheta} = 0$$

(27)

$$-k \sin \vartheta \xi_2 + (m - F - (1 - \cos \vartheta))\xi_1 + \sin \vartheta \frac{d\xi_1}{d\vartheta} = 0$$

(28)

with

$$\bar{\Phi}_1 = Qe^{im\varphi} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

(29)

where $Q$ is a function depending on the variables $t, r$, to be determined after a complete separation of variables. In order to solve the coupled system of equations (27)-(28) we make the following ansatz,

$$\xi_1 = (\sin \frac{\vartheta}{2})^a (\cos \frac{\vartheta}{2})^bf(\vartheta), \ \xi_2 = (\sin \frac{\vartheta}{2})^c (\cos \frac{\vartheta}{2})^dq(\vartheta)$$

(30)

where $a, b, c,$ and $d$ are constants to be fixed in order to obtain solutions of the governing equations for $f(\vartheta)$ and $q(\vartheta)$ in terms of orthogonal special functions. Substituting (30) into (27) and (28) we obtain,

$$-kq(x) - \left(\frac{m}{2} - \frac{F}{2} + \frac{a}{2} + \frac{1}{2}\right)f(x) + (1 - x)\frac{df(x)}{dx} = 0$$

(31)

$$kf(x) + \left(\frac{d}{2} - g - \frac{m}{2} - \frac{F}{2}\right)q(x) + (1 + x)\frac{dq(x)}{dx} = 0$$

(32)

where we have made the change of variable $x = \cos \vartheta$, and we have simplified the resulting equations (31) and (32) by imposing $c = m - F$, and $b = m - F - 2g$. If we set $a = m - F + 1$, and $d = m - F + 1 - 2g$,and make the change of variables $u = (1 - x)/2$, we obtain that the coupled system of equations (31)-(32) takes the form
\[- kq(u) - (m - F + \frac{1}{2})f(u) - u \frac{df(u)}{du} = 0 \quad (33)\]

\[ kf(u) + (m - F - 2g + \frac{1}{2})q(u) + (u - 1) \frac{dq(u)}{du} = 0 \quad (34)\]

from (33) and (34) we obtain

\[ u(1 - u) \frac{d^2q(u)}{du^2} + [(m - F + \frac{1}{2}) - 2(m - F - g + \frac{1}{2})u] \frac{dq(u)}{du} + \]
\[ + [k^2 - (m - F + \frac{1}{2})(m - F - 2g + \frac{1}{2})]q(u) = 0 \quad (35)\]

\[ u(1 - u) \frac{d^2f(u)}{du^2} + [(m - F + \frac{3}{2}) - 2(m - F - g + 1)u] \frac{df(u)}{du} + \]
\[ + [k^2 - (m - F + \frac{1}{2})(m - F - 2g + \frac{1}{2})]f(u) = 0 \quad (36)\]

the solution of the equation (35) can be expressed in terms of the hypergeometric function $F(a, b; c; u)$ as follows

\[ q(u) = c_1 F(a, b; c; u) \quad (37)\]

where $c_1$ is a constant, and $a, b,$ and $c$ are

\[ a = m - F - g + \frac{1}{2} - \sqrt{k^2 + g^2} \quad (38)\]

\[ b = m - F - g + \frac{1}{2} + \sqrt{k^2 + g^2} \quad (39)\]

\[ c = m - F + \frac{1}{2} \quad (40)\]

then, with the help of (34) and (37) we find that $f(u)$ reads

\[ f(u) = c_1 \frac{k}{m - F + \frac{1}{2}} F(a, b; c + 1; u) \quad (41)\]

Since we are looking for normalizable solutions according to the product

\[ 2\pi \int_0^{2\pi} \Phi^{\mu*}_k \Phi^\mu_k \ d\theta = \delta_{kk'} \quad (42)\]
we have that the series associated with the hypergeometric function \( F(a, b; c; u) \) should be truncated reducing it to polynomials. This one is possible if \( a = -n \) or \( b = -n \) in (37) where \( n \) is a no negative integer value. Then using the relation between the Jacobi Polynomials and the functions \( F(a, b; c; x) \)

\[
P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} F(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2})
\]

we have that (41) and (37) reduce respectively to

\[
f(x) = c' P_n^{(m - F + 1/2, m - F - 2g - 1/2)}(x)
\]

\[
q(x) = c' \left( \frac{g + \sqrt{k^2 + g^2}}{k} \right)^{m - F} P_n^{(m - F - 1/2, m - F - 2g + 1/2)}(x)
\]

where \( c' \) is a constant, and \( n \) reads

\[
n = -m + F + g - \frac{1}{2} + \sqrt{k^2 + g^2}
\]

Then, the components \( \xi_1 \) and \( \xi_2 \) of the spinor \( \Phi \) (29) can be written as

\[
\xi_1 = c' (\sin \frac{\vartheta}{2})^{m-F+1} (\cos \frac{\vartheta}{2})^{m-F-2g} P_n^{(m-F+1/2, m-F-2g-1/2)}(\cos \vartheta)
\]

\[
\xi_2 = c' \left( \frac{g + \sqrt{k^2 + g^2}}{k} \right)^{m-F} (\sin \frac{\vartheta}{2})^{m-F} (\cos \frac{\vartheta}{2})^{m-F-2g+1} P_n^{(m-F-1/2, m-F-2g+1/2)}(\cos \vartheta)
\]

Notice that the orthogonality relation for the Jacobi Polynomials

\[
\int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx =
\]

\[
\frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{\Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)} \delta_{mn}
\]

imposes some restrictions on the values of \( m, F \) and \( g \) in (47) and (48). In fact, from (49) we have that \( \alpha > -1, \beta > -1 \) and consequently we have that

\[
m - F + \frac{1}{2} > 0, \ m - F - 2g + 1/2 > 0
\]

affords the required condition of orthogonality. However, some values considered in the inequalities given by (50) fail in fulfilling a condition which should be also took into account
in selecting the possible values of \( m, F \) and \( g \). All the expectation values associated with the separating operators should exist. In fact, if we consider the operator \( \hat{K}_2 \) defined by the relation (18) we have that the integrals \( \int d\vartheta \xi_1 \partial_\vartheta \xi_1 \) as well as \( \int d\vartheta \xi_2 \partial_\vartheta \xi_2 \) should be finite. This one imposes some restrictions on \( m, F \) and \( g \). Looking at the convergency when \( \vartheta \to 0 \), and \( \vartheta \to \pi \) we obtain the relations

\[
m - F > 0, \quad m - F - 2g > 0
\]

which are weaker than those given in (50).

In order to be able to consider other relations among the parameters \( m, F \) and \( g \) different from (46) we are going to consider a second solution of the equation differential equation (35)

\[
q(u) = u^{1-c}(1-u)^{c-a-b}F(1-a, 1-b; 2-c; u) \tag{52}
\]

then, using the recurrence relations for the hypergeometric functions, we find that a solution for the system (33)-(34) reads

\[
f(u) = c_2 u^{-1/2-m+F}(1-u)^{-m+F+2g+1/2}F(1-a, 1-b; 1/2 - m + F; u) \tag{53}
\]

\[
q(u) = c_2 u^{-1/2-m+F}(1-u)^{-m+F+2g+1/2}F(1-a, 1-b; 3/2 - m + F; u) \tag{54}
\]

where \( c_2 \) is a constant. From (53) and (54) we obtain that \( \xi_1 \) and \( \xi_2 \) take the form

\[
\xi_1 = c(sin \frac{\vartheta}{2})^{-m+F}(cos \frac{\vartheta}{2})^{-m+F+2g+1}P_n(-1/2-m+F,2g-m+F+1/2)(cos \vartheta) \tag{55}
\]

\[
\xi_2 = c \frac{k}{\sqrt{k^2 + g^2}}(sin \frac{\vartheta}{2})^{1-m+F}(cos \frac{\vartheta}{2})^{-m+F+2g}P_n(1/2-m+F,2g-m+F-1/2)(cos \vartheta) \tag{56}
\]

with \( c \) as a constant of normalization, and in the present case \( n \) is

\[
n = m - F - g - \frac{1}{2} + \sqrt{k^2 + g^2} \tag{57}
\]

Following the same reasoning used for deriving (51), we have that the expressions for \( \xi_1 \) and \( \xi_2 \) given by (55) and (56) are valid when
\[- m + F > 0, \ 2g - m + F > 0 \tag{58}\]

A third possible solution of the equation \((35)\) can be written as
\[
q(u) = (1 - u)^{c-a-b} F(c - a, c - b; c; u) \tag{59}\]

then, substituting \((59)\) into \((34)\) we find that \(f(u)\) and \(q(u)\) take the form:
\[
f(u) = c_3(1 - u)^{-m + F + 2g + 1/2} F(1 + g + \sqrt{k^2 + g^2}, 1 + g - \sqrt{k^2 + g^2}; m - F + \frac{3}{2}; u) \tag{60}\]
\[
q(u) = -c_3 \frac{m - F + 1/2}{k}(1 - u)^{-m + F + 2g - 1/2} F(g + \sqrt{k^2 + g^2}, g - \sqrt{k^2 + g^2}; m - F + \frac{1}{2}; u) \tag{61}\]

where \(c_3\) is a constant. Using the relation between the hypergeometric function \(F(a, b; c; x)\) and the Jacobi Polynomials \(P_n^{(α, β)}(x)\) given by eq. \((43)\) we arrive at
\[
ξ_1 = c (\sin \frac{θ}{2})^{-m-F+1}(\cos \frac{θ}{2})^{-m+F+2g+1} P_n^{(m-F+1/2, -m+F+2g+1/2)}(\cos θ) \tag{62}\]
\[
ξ_2 = -c \frac{n + 1}{k} (\sin \frac{θ}{2})^{-m-F}(\cos \frac{θ}{2})^{-m+F+2g} P_{n+1}^{(m-F-1/2, -m+F+2g-1/2)}(\cos θ) \tag{63}\]

where \(c\) is a constant of normalization and \(n\) is given by
\[
n = \sqrt{k^2 + g^2} - g \tag{64}\]

In this case, we have to impose the following restrictions on the values of \(m, F\) and \(g\).
\[
m - F > 0, \ -m + F + 2g > 0 \tag{65}\]

Finally, considering as solution of the equation \((35)\) the expression
\[
q(u) = u^{1-c} F(a - c + 1, b - c + 1; 2 - c; u) \tag{66}\]

then, in the present case the functions \(f(u)\) and \(q(u)\) read
\[
f(u) = u^{-1/2 - m + F} F(-g - \sqrt{k^2 + g^2}, -g + \sqrt{k^2 + g^2}; \frac{1}{2} - m + F; u) \tag{67}\]
\[ q(u) = \frac{k}{2 - m + F} u^{1/2 - m + F} F(-g - \sqrt{k^2 + g^2} + 1, -g + \sqrt{k^2 + g^2} + 1; \frac{3}{2} - m + F; u) \quad (68) \]

\[ \xi_1 = c_4 \left( \sin \frac{\vartheta}{2} \right)^{-m + F} \left( \cos \frac{\vartheta}{2} \right)^{m - F - 2g} P_n^{(-1/2 - m + F, 1/2 + m - F - 2g)} \cos \vartheta \quad (69) \]

\[ \xi_2 = c_4 \frac{k}{n} \left( \sin \frac{\vartheta}{2} \right)^{1 - m + F} \left( \cos \frac{\vartheta}{2} \right)^{m - F - 2g + 1} P_{n-1}^{(1/2 - m + F, -1/2 - m - F - 2g)} \cos \vartheta \quad (70) \]

with \( n \) given by

\[ n = \sqrt{k^2 + g^2} + g \quad (71) \]

and \( c_4 \) is a constant of normalization. The solutions (69) and (74) are well behaved according to the Dirac inner product as well as to the expectation value of the operator of angular momentum (18) if

\[ -m + F > 0, \ m - F - 2g > 0 \quad (72) \]

then, we have that the results (64) and (74) can be gathered as follows

\[ n = \sqrt{k^2 + g^2} - |g| \quad (73) \]

when the condition on \( m, F \) and \( g \)

\[ F + g + |g| > m > g - |g| + F \quad (74) \]

is satisfied.

Regarding the eigenvalues \( m \) of the projection of the angular momentum operator \(-i\partial_{\varphi}\) we have that since the transformation (11), relating the Dirac spinors \( \Psi_{\text{rot}} \) and \( \Psi_{\text{Cart}} \) in the local (rotating) and the Cartesian tetrad frames transforms after a rotation as follows

\[ S_z(\varphi + 2\pi) = -S_z(\varphi) \quad (75) \]

and the spinor \( \Psi_{\text{Cart}} \) is single valued, then we obtain

\[ \Psi_{\text{rot}}(\varphi + 2\pi) = -\Psi_{\text{rot}}(\varphi) \quad (76) \]

and therefore \( m \) takes half integer values

\[ m = N + \frac{1}{2}, \ N = 0, \pm 1, \pm 2... \quad (77) \]
IV. SOLUTION OF THE RADIAL EQUATION

Now, we are going to solve the system of equation (19), governing the radial dependence of the spinor $\Psi_{\text{rot}}$ solution of the Dirac equation. This equation can be written in the form,

$$
\left( -\gamma^3 \partial_t + \gamma^0 \partial_r + (M + \tilde{V}(r))\gamma^0 \gamma^3 - i\gamma^3 V(r) + i\frac{k}{r} \right) \bar{\Phi} = 0 \quad (78)
$$

Using the representation for the gamma matrices given by (23), and the fact that eq. (78) commutes with the energy operator $i\partial_t$ with eigenvalues $E$, we obtain the following system of equations

$$
\begin{align*}
\left( d_r + \frac{k}{r} \right) \Phi_1 - \sigma^3 (E - V(r) - M - \tilde{V}(r)) \Phi_2 &= 0 \quad (79) \\
\left( -d_r + \frac{k}{r} \right) \Phi_2 - \sigma^3 (E - V(r) + M + \tilde{V}(r)) \Phi_1 &= 0 \quad (80)
\end{align*}
$$

From (24)-(25), (79)-(80) and the fact that the Dirac equation (14) commutes with $-i\partial_\phi$ and $i\partial_t$, we have that the spinor $\Phi$ can be written as follows

$$
\bar{\Phi} = c_0 e^{i(m\phi - Et)} \begin{pmatrix}
\xi_1 A(r) \\
\xi_2 A(r) \\
c\xi_1 B(r) \\
-c\xi_2 B(r)
\end{pmatrix} \quad (81)
$$

where $c$ is a constant, and $A(r)$ and $B(r)$ satisfy the system of equations

$$
\begin{align*}
\left( d_r + \frac{k}{r} \right) A(r) - (E - V(r) - \tilde{V}(r) - M) B(r) &= 0 \quad (82) \\
\left( -d_r + \frac{k}{r} \right) B(r) - (E - V(r) + \tilde{V}(r) + M) A(r) &= 0 \quad (83)
\end{align*}
$$

notice that in this way we have fixed the values of the functions $Q(r,t) = e^{-iEt} A(r)$, and $G = B(r)/A(r)$ appearing during the process of separation of variables. Substituting into (82) and (83) the form of the scalar and the Coulomb potentials we get
\[
\left( \frac{d}{dr} + \frac{k}{r} \right) A(r) - \left( E - M + \frac{1}{r}(\alpha + \alpha') \right) B(r) = 0 \quad (84)
\]

\[
\left( \frac{d}{dr} - \frac{k}{r} \right) B(r) + \left( E + M - \frac{1}{r}(\alpha - \alpha') \right) A(r) = 0 \quad (85)
\]

Introducing the notation

\[
A = E + M, \quad B = M - E, \quad \hat{\alpha} = \alpha + \alpha', \quad \beta = \alpha' - \alpha \quad (86)
\]

and the new variable \( \rho \), related to \( r \) as follows

\[
\rho = Dr = \sqrt{M^2 - E^2}r = \sqrt{AB}r \quad (87)
\]

we have that the system of equations (84)-(85) reduces to

\[
\left( \frac{d}{d\rho} + \frac{k}{\rho} \right) A(r) - \left( -B \frac{D}{R} + \frac{\hat{\alpha}}{\rho} \right) B(r) = 0 \quad (88)
\]

\[
\left( \frac{d}{d\rho} - \frac{k}{\rho} \right) B(r) + \left( A \frac{D}{R} - \frac{\hat{\beta}}{\rho} \right) A(r) = 0 \quad (89)
\]

We shall look for solutions of the system (88)-(89) in the form of power series\(^{16,17}\)

\[
A(\rho) = e^{-\rho} \sum_{\nu=0}^{\infty} \rho^{s+\nu} a_{\nu} \quad (90)
\]

\[
B(\rho) = e^{-\rho} \sum_{\nu=0}^{\infty} \rho^{s+\nu} b_{\nu} \quad (91)
\]

Substituting (90)-(91) into (88)-(89) we find

\[
(s + k)a_0 - \hat{\alpha}b_0 = 0 \quad (92)
\]

\[
(s - k)b_0 - \hat{\beta}a_0 = 0 \quad (93)
\]

and

\[
[(s + \nu) + k] a_{\nu} - \hat{\alpha}b_{\nu} + \frac{B}{D} b_{\nu-1} - a_{\nu-1} = 0 \quad (94)
\]
\[(s + \nu) - k \] b_{\nu} - \hat{\beta} a_{\nu} + \frac{A}{D} a_{\nu-1} - b_{\nu-1} = 0 \quad (95)\]

From (92)-(93) it follows that

\[s = \sqrt{k^2 - \alpha^2 + \alpha'^2} \quad (96)\]

where we have dropped out (96) the negative root because we are looking for wavefunctions regular at the origin of coordinates. From (94)-(95) we have that

\[
\begin{cases}
\left(\sqrt{\frac{A}{B}} - \hat{\beta}\right) a_{\nu} = \left(\sqrt{\frac{A}{B}} - [(s + \nu) - k]\right) b_{\nu} \\
\end{cases}
\quad (97)\]

The series (96) and (97) will have a good behavior at infinity if they terminate for a finite value N. Putting \(a_{N+1} = b_{N+1} = 0\) in (94)-(95) with \(a_N \neq 0\) and \(b_N \neq 0\), we arrive at

\[
\frac{b_N}{a_N} = \frac{A}{D} \quad (98)\]

Substituting (98) into (97), and taking into account (86) we get

\[(s + N)\sqrt{M^2 - E^2} = E\alpha + \alpha'M \quad (99)\]

where \(s\) is given by (96)

Finally, we obtain the energy spectrum:

\[
E = M \left\{-\frac{\alpha\alpha'}{(s + N)^2 + \alpha'^2} \pm \sqrt{\left(\frac{\alpha\alpha'}{(s + N)^2 + \alpha'^2}\right)^2 - \frac{\alpha'^2 - (s + N)^2}{[(s + N)^2 + \alpha'^2]^2}}\right\} \quad (100)\]

Here two particular cases could be considered: a) \(\alpha' = 0\) which corresponds to the Coulomb potential. In this case the energy spectrum reduces to:

\[
E = M \left[1 + \frac{\alpha^2}{(s + N)^2}\right]^{-1/2} , \quad s = \sqrt{k^2 - \alpha^2} \quad (101)\]

where the negative root has been dropped out because it is not compatible with the relation (99) A second possibility is given by b) \(\alpha = 0\), which is the scalar \(Vr(r) = -\alpha'/r\) potential. In this case the energy spectrum takes the form

\[
E = \pm M \left[1 - \frac{\alpha^2}{(s + N)^2}\right]^{1/2} , \quad s = \sqrt{k^2 + \alpha'^2} \quad (102)\]

notice that in the present case states with negative energy are possible, here we do not have critical behavior like in the Coulomb case.
V. DISCUSSION OF THE RESULTS

In this section we are going to discuss the influence of the Aharonov-Bohm potential and the Dirac magnetic monopole charge on the energy spectrum. Here we have mention that the Aharonov-Bohm as well as the magnetic monopole contributions are present in the expression (100) via the factor $s$ given by (96), since the explicit form of $k$ depends on the relation among $m$, $F$ and $g$. In fact, we have that when the inequalities (65) or (72) are valid, the expression for $s$ takes the form

$$s = \sqrt{(n + |g|)^2 - g^2 - \alpha^2 + \alpha'^2} \quad (103)$$

and no contribution of the Aharonov Bohm potential is observed in (100). The values of $m$ for which (103) takes place are given by the expression (74).

It is worth mentioning that when the magnetic monopole contribution is absent, the inequalities given by the expression (74) never take place, and consequently, the expression for $s$ given by eq. (103) is not applicable. In this case the energy spectrum can be computed by substituting the expression

$$s = \sqrt{(n + |m - F| + 1/2)^2 - g^2 - \alpha^2 + \alpha'^2} \quad (104)$$

into (100). Analogously, the energy spectrum when $F \neq 0$ and $g \neq 0$ can be obtained after substituting into (100) the following value of $s$

$$s = \sqrt{(n + m - F - g + 1/2)^2 - g^2 - \alpha^2 + \alpha'^2} \quad (105)$$

when $m - F > 0$ and $m - f - 2g > 0$. Otherwise, when $m - F < 0$ and $m - F - 2g < 0$, the value of $s$ to substitute into (100) reads

$$s = \sqrt{(n - m + F + g + 1/2)^2 - g^2 - \alpha^2 + \alpha'^2} \quad (106)$$

Perhaps the most interesting and puzzling result of the present paper is the non dependence of the energy spectrum on the Aharanov-Bohm potential for a range of values given by the inequality (74). Despite this phenomenon was already pointed by Hoang et al there are
some discrepancies between their results and those ones present in this paper. Basically the problems lie on the criteria for establishing the boundary conditions and the normalizability of the wave functions. It is worth mentioning that since the Aharonov-Bohm contribution $F$ can take non integer values, then the parameters $\alpha$ and $\beta$ in the Jacobi Polynomials $P_{n}^{(\alpha,\beta)}(x)$ can be negative provided that $\alpha > -1$, and $\beta > -1$. Obviously if the Aharonov Bohm potential is absent, from the results presented in Sec. 3 we have that $\alpha \geq 0$ and $\beta \geq 0$ (or in the case when $F$ is an integer). Hoang et al consider that $\alpha$ and $\beta$ are always positive restricting in the way the range of validity of the solutions. A second point to remark is that we not only impose the normalizability of the wave functions but also the existence of the expectation value of the angular momentum operator, which is equivalent to say that $\int \Phi^{\dagger} \hat{K} \Phi d\theta d\varphi < \infty$, in this way we have to impose that the spinor components $\xi_{1}$ and $\xi_{2}$ presented in Sec. 3 should satisfy $\int \xi_{1,2} \partial_{\theta} \xi_{1,2} < \infty$. Regarding the assertion made in\cite{7} about the existence of quantum states forbidden for the Dirac particle in the presence of the Coulomb plus the Dirac monopole potentials, we have that the boundary conditions imposed on the wave function in the present article avoid such a anomalous behavior and consequently the spinor $\Phi$ is well defined for any value of the parameters $m$, $F$ and $g$.

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