Stationary superstatistics distributions of trapped run-and-tumble particles

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We present a novel analysis of the stationary distributions of run-and-tumble particles trapped in external potentials in terms of a thermophoretic potential, that appears when trapped active motion is mapped to trapped passive Brownian motion in a fictitious inhomogeneous thermal bath. We elaborate on the meaning of the non-Boltzmann-Gibbs stationary distributions that emerge as a consequence of the persistent motion of active particles. These stationary distributions are interpreted as a class of distributions in nonequilibrium statistical mechanics known as superstatistics. Our analysis provides an original insight on the link between the intrinsic nonequilibrium nature of active motion and the well-known concept of local equilibrium used in nonequilibrium statistical mechanics, and contributes to the understanding of the validity of the concept of effective temperature. Particular cases of interest, regarding specific trapping potentials used in other theoretical or experimental studies, are discussed. We point out as an unprecedented effect, the emergence of new modes of the stationary distribution as a consequence of the coupling of persistent motion in a trapping potential that varies highly enough with position.

Keywords: Active Motion, Diffusion Theory, Inhomogeneous Source of Heat

I. INTRODUCTION

Self-propelled or active particles are open systems driven out of thermal equilibrium by complex internal mechanisms that locally convert energy from the environment into active motion [1,2]. A variety of microorganisms suspended in temperate aqueous environments use internal motors that allow them to move, undulate, or rotate flagella or cilia at small Reynolds numbers to self-propel leading to a variety of patterns of motion that are of great interest to statistical physicists. On the other hand, different phoretic mechanisms, such as thermophoresis and diffusiophoresis, have been ingeniously devised to endow passive Brownian particles with active motion.

Two main features characterize active motion, first the tendency of particles to move at a characteristic speed as a consequence of self-propulsion, the other, the persistence or tendency to maintain the direction of motion for long enough intervals of time. Though it is clear that active motion corresponds to the class of intrinsically out-of-equilibrium phenomena, it is often difficult to give a measure for the departure from equilibrium [3]. Notwithstanding this, today there is a great effort to give a thermodynamic description of active matter [3,6]. On this course, the concept of effective temperature, $T_{\text{eff}}$, has resulted valuable and whose validity has been explored theoretically and experimentally in a variety of systems in non-equilibrium situations [6-18], particularly in the dilute regime [19]. More recently, it has been shown that the effective temperature in a two-component active Janus particles can be considered as a control parameter (in the sense of a thermodynamics variable) for the observed kinetics and phase behavior [20].

The physical intuition behind the concept of effective temperature relies on the fulfillment of the fluctuation-dissipation relation. The active particles trajectories, obtained from experiments [10,13], numerical simulations [11,16] or analytical results [12,14], have allowed to conclude that an effective temperature emerges from the internal fluctuations of the active motion, which are of no thermal origin and that effectively emulate thermal fluctuations. Indeed, $T_{\text{eff}}$ can be determined by using a tracer as thermometer that probes a non-equilibrium complex media through a diffusion process. Experimental realizations of this idea are numerous, for instance the motion of a bead in a bath of bacteria can exhibit effective temperatures as large as two or three orders of magnitude the room temperature [21].

Active particles freely moving at constant speed $v$ with characteristic persistence time $\alpha^{-1}$ exhibit normal diffusion in the long-time regime, the diffusion constant being $D_{\text{eff}} = v^2/\alpha$. In this regime the motion of an active particle can equivalently thought as the motion of a passive one in a fictitious hotter environment with effective temperature $T_0 = D_{\text{eff}}/k_B\mu$, $\mu$ and $k_B$ being the mobility and the Boltzmann constant respectively [12,22]. For instance, dilute suspensions of self-propelled particles in sedimentation processes can be considered as passive Brownian particles in a hotter source of heat with an effective temperature that scales linearly with the persistence time [13].

In systems of confined active particles, either by impenetrable walls or by external potentials [3,12,10,23-25], the situation is quite different. Indeed, the effects of persistence are conspicuously revealed in the long time regime, if the persistence length is larger than the characteristic length of confinement. Notoriously, the stationary distribution of the particle positions shows an accumulation of particles around the boundaries of con-
finement (see for instance Ref. [26] for trajectories of worker termites in a circular arena, [27] for confined colloidal rollers in a circular disc, or Ref. [24] for active Brownian particles confined in an acoustic trap) corresponding evidently to non-Boltzmann-Gibbs distributions [28, 29]. Such non-Boltzmann-Gibbs distributions have also been observed experimentally in optically trapped passive Brownian particles coupled to a bath of active ones [30], which manifestly exhibits the intrinsic nonequilibrium nature of the bath.

Is it possible to describe the non-Boltzmann-Gibbs stationary distributions of one-dimensional, non-interacting active particles trapped in an external potential $U(x)$, in terms of passives ones in a fictitious environment? The answer is in the positive if the concept of effective temperature (homogeneous) is extended to non-equilibrium (inhomogeneous) environment. For this purpose, we found that for spatially-independent parameters that characterize active motion (situation that can be thought as an effective homogeneous medium of “nutrients”), an effective inhomogeneous temperature, $T(x)$, can be defined. This allows us to go further an interpret these distributions as a class of the distributions that appear in nonequilibrium statistical mechanics known as superstatistics [31, 32].

Our theoretical analysis is based on a mathematical framework used by several authors and corresponds to a generalization of the telegrapher process that describes the run-and-tumble dynamics of active motion [22, 28, and references therein]. Such a framework is presented in section III for the sake of completeness. In section III we apply the ideas developed to a simple instance, namely, a single run-and-tumble particle with constant active parameters trapped in an external potential $U(x)$. We present in section IV our final comments and concluding remarks.

II. THEORETICAL CONSIDERATIONS

A rich variety of patterns of motion of biological microorganisms are observed in nature, and many different mathematical models have been introduced to describe their dynamics. A recurrently mathematical model used in biology, that takes into account the persistence of motion —characteristic of the active one— corresponds to the random walk and its variants [33, 34]. One pattern of motion that has received particular attention is the one called run-and-tumble [34], observed, for instance in the motion of bacteria such as Escherichia Coli [35]. The organism use cilia synchronization to move approximately in straight line and with constant speed for a random period of time (of the order of seconds), called the running period. Immediately after a running period, cilia get desynchronized for short periods of time (tenths of seconds) at which the bacteria tumbles. Cilia get synchronized again and the particle start a running period in a randomly chosen direction. Run-and-tumble dynamics can be considered as a paradigm for non-Brownian diffusive motion and has been used to explore some central aspects of nonequilibrium dynamics, such as the origins of motility induced phase separation in systems without detailed balance [36].

A simplified model of this dynamics consists of a particle running at a constant speed $v$, allowed to randomly change its direction of motion at a constant tumbling rate $\alpha$. In one-dimension only two directions of motion are possible, if the speed of the particle and the rate of change of direction are constants, the process is well described by the so called telegrapher’s equation which captures in an exact manner this dynamics [37]. A biased motion can be analyzed straightforwardly if the values of the particle speed and/or the transitions rates depends on the direction of motion, namely, $v_L$, $\alpha_L$ when the particle moves to the right and $v_R$, $\alpha_R$ when it moves to the left. This set of parameters embodies the description of the one-dimensional run-and-tumble dynamics. A generalization of this model considers the coupling of the particle’s motion to the environment, as for instance, drifting toward chemical attractants in a process called chemotaxis [22], or alternatively, in a mean field description, it considers the coupling to the local population density that takes into account many body effects [28]. In any case such a coupling makes $\alpha$ and $v$ to depend on the particle’s position $x$. Thus the probability densities of being at $x$ at the instant $t$ and moving to the right, $P_R(x,t)$, and to the left, $P_L(x,t)$, satisfy the equations

\[
\frac{\partial}{\partial t} P_R(x,t) + \frac{\partial}{\partial x} v_R(x)P_R(x,t) = \frac{1}{2} \left[ \alpha_L(x)P_L(x,t) - \alpha_R(x)P_R(x,t) \right], \tag{1a}
\]

\[
\frac{\partial}{\partial t} P_L(x,t) - \frac{\partial}{\partial x} v_L(x)P_L(x,t) = \frac{1}{2} \left[ \alpha_R(x)P_R(x,t) - \alpha_L(x)P_L(x,t) \right], \tag{1b}
\]

where the $v$’s and $\alpha$’s are positive functions of position. Equations (1) can be written in an equivalent form in terms of the coarse-grained probability density $P(x,t) = P_R(x,t) + P_L(x,t)$ and the corresponding probability current $J(x,t) = v_R(x)P_R(x,t) - v_L(x)P_L(x,t)$,
both related by the continuity equation
\[ \frac{\partial}{\partial t} P(x, t) + \frac{\partial}{\partial x} J(x, t) = 0. \] (2)

\( P(x, t) \) gives the probability density of finding a particle in a position \( x \) at a time \( t \) independently of the direction of motion, while \( J(x, t) \) gives the net flux of an ensemble of particles at \( x \) and \( t \), which satisfies the equation

\[ \frac{\partial}{\partial t} J(x, t) - v_{rel}(x) \frac{\partial}{\partial x} J(x, t) + \left[ \alpha(x) + \gamma(x) \right] J(x, t) = \alpha(x) \left[ \nu_{\text{drift}}(x) P(x, t) - \frac{\partial}{\partial x} D(x) P(x, t) \right]. \] (3)

\section*{A. The diffusive limit: Effective temperature}

The simplified situation of run-and-tumble particles moving freely in unconfined space, corresponds to the case when the coupling of the particle to the medium is such that the medium serves as a uniform source that keeps the tumbling rates, and the right and left velocities, equal and space independent, i.e. \( \alpha_R(x) = \alpha_L(x) = \alpha \), and \( v_R(x) = v_L(x) = v \). We refer to this bath as the \textit{uniform source of activity}, and can be understood in an analogy with the uniform temperature bath that causes Brownian motion. For systems in one dimension, this consideration leads to the simplest model of \textit{persistent motion} taken into account by the one-dimensional Telegrapher’s equation \[ 37 \text{--} 39, \] which can be obtained straightforwardly from Eqs. (2) and (3). This generalizes the diffusion equation in that it properly accounts for the finite speed signal propagation that results into a non-Gaussian probability density functions of the particle positions \( P(x, t) \). Such a diffusion process is non-stationary and is characterized by ballistic motion in the short-time regime and normal diffusion in the long-time limit, where \( P(x, t) \) asymptotically approximates the Gaussian solution of the diffusion equation, and a uniform effective diffusion coefficient is apparent, namely \( D_{\text{free}} = v^2/\alpha \) \[ 13 \text{--} 37 \]. Thus, assuming uniform swimming velocities and tumbling rates leads, into the long-time limit, to a uniform effective temperature

\[ T_0 = \frac{v^2}{\alpha \mu k_B}. \] (9)

if a kind of Einstein relation is assumed, where \( \mu \) is a uniform parameter that plays the roll of a mobility that describes the coupling of the particle to the fictitious heat bath at uniform temperature \( T_0 \).

\section*{B. Stationary solutions}

We are interested in the physical situations at which stationary solutions, \( P_{st}(x) \), exist, as happens for instance if the particles are trapped either by impenetrable walls or by some external forces. It is well known that the persistence effects of run-and-tumble particles makes
the particles to explore the container walls if the persistence length is larger or comparable with the confinement length. This is the case also if confinement is due to energetic constraints, for instance when particles are trapped by an external potential. In this case, stationary solutions are obtained from Eqs. (2) and (3) by setting \( J = 0 \), that leads to

\[
V_{\text{drift}}(x) P_{\text{st}}(x) - \frac{d}{dx} D(x) P_{\text{st}}(x) = 0, \quad (10)
\]

Notice that position dependence of the \( v \)'s and the \( \alpha \)'s is a necessary, but not sufficient condition for the existence of stationary solutions. Further, the relations (14) and (15) lead to a mapping between the stationary solutions of run-and-tumble particles and the corresponding ones of a drift-diffusion equation of Brownian motion in inhomogeneous media, analogous to the one presented in Ref. [28]. This mapping will be exploited to give a precise interpretation of confined active motion as a non-equilibrium feature as will be unveiled in the following sections.

The solutions of (11) that describe flux-free steady states can be written by considering the non-locality of the ratio \( V_{\text{drift}}(x)/D(x) \) as

\[
P_{\text{st}}(x) = \frac{N}{D(x)} \exp \left\{ \int^x dx' \frac{V_{\text{drift}}(x')}{{D(x')}} \right\}, \quad (11)
\]

where \( N \) is the required normalization factor that has units of Length/Time and the symbol \( \int^x dx' f(x') \) denotes the antiderivative or indefinite integral of \( f(x) \). Clearly, such solutions do not comply with the well known Boltzmann-Gibbs factor that describes thermodynamic equilibrium, and therefore solution (11) can be interpreted as a superstatics class of \( D(x) \)-distributions as the classes proposed in Ref. [22]. Notice however that, if the parameters of active motion satisfy certain conditions, to say, if the coupling between the particle motion and the environment are devised such that \( D(x) = D \), independent of the particle’s position, and \( V_{\text{drift}}(x) = -G(x) \), with \( G(x) \) an arbitrary function of \( x \), then \( P_{\text{st}}(x) \) given in (11) can be written as the Boltzmann-Gibbs-like weight

\[
P_{\text{B-G}}(x) = Z^{-1}(D) \exp \{-G(x)/D\} \quad (12)
\]

where \( Z(D) = \int_{-\infty}^{\infty} dx' \exp \{-G(x')/D\} \) is reminiscent of the single-particle partition function of the canonical ensemble and \( D \) an homogeneous global parameter that can be related to an effective thermodynamic quantity, like the effective temperature \( T_0 \) as has been demonstrated experimentally and theoretically [4, 12, 13, 28, 41]. The corresponding free energy, according to stochastic thermodynamics [42], is given by \( F = -D \ln Z(D) \). If this is the case, the fluctuation-dissipation theorem is expected to be valid and the time reversal symmetry of the microscopic dynamics is guaranteed, and the ratio of the transition rate from position \( x \) to \( x' \), \( W(x, x') \), to the inverse process \( W(x', x) \), is given by the well known equilibrium result \( \exp \{-[G(x') - G(x)]/D\} \).

Therefore, we can conclude that the intrinsic nature of the non-Boltzmann-Gibbs equilibrium solutions of active motion, resides in the fact that non such homogeneous, global parameter can be identified, thus leading to a situation that is intrinsically out of thermodynamic equilibrium. This statement is made clear by recognizing that the solution given in (11) can be interpreted, formally, as the stationary solution of an overdamped, passive Brownian particle that diffuses with constant mobility \( \mu \), in an medium of inhomogeneous temperature \( T(x) \) and under the influence of an external potential \( U(x) \), if the following identifications are made

\[
T(x) = D(x)/\mu k_B, \quad U'(x) = -V_{\text{drift}}(x)/\mu. \quad (13a)
\]

In the context of Brownian passive particles, it is well known that inhomogeneous temperature profiles have profound consequences on the local stability of the stationary solutions, known as the Landauer effect [47, 48]. Certainly, a hot layer can change the relative stability points of a particle diffusing in a bistable potential. Thus, by properly choosing a spatial inhomogeneity of temperature [namely Eq. (3)] it is possible to obtain a desired stationary distribution of particles, as for example to mimic the persistence effects of active motion on the statistical properties of diffusing trapped passive particles.

Two comments are in place, on the one hand, the physical assumptions underlying the establishment of relations (13), indicates that two elements of non-equilibrium can be identified, the immediate one refers to the inhomogeneity of the effective temperature, the other, more subtle, refers to the uniformity of the mobility \( \mu \). Indeed, this last element contrasts with the case when only conditions of local equilibrium are assumed, for which then, the mobility of an overdamped Brownian particle is space-dependent and constrained to the spatial dependence of temperature in order to satisfy a local fluctuation-dissipation relation [48, 49]. Though precise information regarding the dissipative coupling of the particle’s motion to the medium is required to avoid any loose assumption, in the present analysis a space-independent mobility is assumed.

### III. APPLICATION TO A SPECIFIC CASE: TRAPPED RUN-AND-TUMBLE PARTICLES IN AN EXTERNAL POTENTIAL

Though the coupling of the particle’s motion to the medium is in general complex, as in chemotactic behavior, here we analyze the simple situation for which a particle swims at constant speed \( v \) in a viscous fluid at low Reynolds numbers, such that an overdamped dynamics is valid. We also assume that the particle tumbles symmetrically at constant rate \( \alpha \) and that the motion is restricted by an external trapping potential \( U(x) \). Under
these considerations the effective right and left swimming speeds become space-dependent and given by

\[ v_R(x) = v - \mu U'(x), \]  

\[ v_L(x) = v + \mu U'(x), \]  

where \( \mu \) is the mobility of the particle in the fluid, which for simplicity is assumed to be space independent and finite. With these considerations we have that Eqs. (13) can be rewritten as

\[ \nu_{\text{drift}}(x) = -\mu U'(x), \]  

\[ T(x) = T_0 \left\{ 1 - \frac{\mu^2}{v^2} [U'(x)]^2 \right\}, \]  

where the effective temperature \( T_0 \) has been introduced in Eq. (9). It is clear that in the diffusive limit, \( v \to \infty \), \( \alpha \to \infty \) such that \( v^2/\alpha = \mu_B T_0 \) is kept constant [12], we recover the uniform effective temperature given by [9], since in such a limit the ratio of the drift velocity (caused by the external potential) to the particle swimming velocity vanishes. Also in this limit, the persistence length, \( L_{\text{pers}} \equiv v/\alpha \), that characterizes the length scale of fluctuations, goes to zero, thus satisfying a kind of fluctuation-dissipation relation that guarantees equilibrium states characterized by an uniform, effective temperature \( T_0 \) [10].

In the persistent regime on the other hand, the motion of the particle results sharply bounded by the external potential. Indeed, the particle cannot swim beyond a characteristic distance \( x_{\text{max}} \) from the local stable point of the trapping potential and where the self-propulsive force \( -\mu U'(x_{\text{max}}) \) equals that of the trapping force \( -U'(x) \), i.e. \( x_{\text{max}} \) is defined by the solution of the equation \( \mu U'(x_{\text{max}}) = v \). Thus, in the case of a even-symmetric potential \( U(-x) = U(x) \) with a global minimum at the origin, the particle moves in the region of space defined by \( [-x_{\text{max}}, x_{\text{max}}] \). Notice now that the inhomogeneous temperature (10b) takes its maximum value, \( T_0 \), just at the positions corresponding to the minima of the trapping potential \( U(x) \), and vanishes at the positions \( \pm x_{\text{max}} \). The medium, being “hotter” at the potential minima, push out the particles from the corresponding stable positions of \( U(x) \) towards the new stable positions given by the local minima of \( U_{\text{eff}}(x) \) on the interval \( [-x_{\text{max}}, x_{\text{max}}] \), thus changing the system stable points of the trapping potential for those at the boundaries. A similar effect was analysed by Landauer for the case of a simple hotter layer in between the local minima of a bistable potential [17] and extended by van Kampen for a general potential and a general temperature inhomogeneity [18]. This gives a precise picture that pinpoint the nonequilibrium nature of the stationary distribution of trapped run-and-tumble particles.

The probability distribution (11) can be written as

\[ P_{\text{st}}(x) = Z^{-1} \exp \left\{ -\int^x dx' \frac{U'(x')}{k_B T(x')} \right\}, \]  

where the effective potential

\[ U_{\text{eff}}(x) = U(x) + k_B T(x) \]  

takes into consideration the appearance of the fictitious thermophoretic force \( -k_B T(x) \) [19], due to the spatial inhomogeneity of the effective temperature and explicitly given by \( -k_B T(x) = (2k_B T_0 \mu^2/\nu^2) U'(x) U''(x) \). The stationary distribution (17) is conceptually a reminiscence of the distributions in nonequilibrium statistical mechanics known as superstatistics, for which the stationary distributions of a Brownian particle diffusing in an given random inhomogeneous medium [13, 46] arise as a superposition of locally Boltzmann-Gibbs factors [32]. This is not exactly the case for the stationary distribution given by Eq. (17), for which its strong nonlocal character allows to interpret the probability density at a given position \( x \) as a product of locally Boltzmann-Gibbs factors. In the diffusive limit, the Boltzmann-Gibbs statistics is recovered as particular case. The normalizing constant, \( Z \), corresponds to the system partition function which is given explicitly by

\[ Z = \int dx \exp \left\{ -\int dx' \frac{U'_\text{eff}(x')}{k_B T(x')} \right\}. \]  

Though no homogeneous effective temperature exists, a local free energy density \( F(x) \) can be defined through the relation

\[ Z = \exp \left\{ -\int dx' \frac{F(x')}{k_B T(x')} \right\}, \]  

that takes into account the non-local effects due to the inhomogeneity of the medium. The detailed discussion of this aspect will be presented elsewhere.

The effects of the thermophoretic potential, \( k_B T(x) \), that incorporates in an effective manner the persistence of active motion, are conspicuous revealed in the stability nature of the equilibrium positions of \( U_{\text{eff}}(x) \). In the diffusive limit, the effective temperature (10b) becomes spatially uniform and therefore the equilibrium positions of \( U_{\text{eff}}(x) \) corresponds to those of \( U(x) \). In this situation, the particles accumulate around the stable positions (global minima) of the trapping potential \( U(x) \). Furthermore, in the same limit, the Boltzmann-Gibbs distribution \( P_{B-G}(x) \), that describe thermodynamic equilibrium, emerges from expression (17) [20]. As the effects of persistence become more apparent, new equilibrium positions, beside those of \( U(x) \), appear in the system. These new equilibrium positions are explicitely given as the solutions of the equation \( U''(x) = \alpha/(2\mu) \), which explicitly exhibit the dependence on the ratio of the inverse of the persistence time and the mobility. The explicit appearance of the second derivative requires a rapidly enough varying external potential in order to have new equilibrium positions other than those of \( U(x) \).

On the other hand, the sharp accumulation of particles at the confining boundaries, being the nonequilibrium hallmark of dilute active systems, appears just in
the persistent regime, when the persistence length $l_{\text{pers}}$ is larger or of the order of the characteristic length of confinement. The departure from the well-known equilibrium Boltzmann-Gibbs distribution can be identified by rewriting Eq. (17) as

$$P_{\text{st}}(x) = Z^{-1} \exp \left\{ \frac{E_{\text{eff}}(x)}{k_B T_0} \right\} \times \exp \left\{ -\sum_{l=1}^{\infty} \left( \frac{\mu}{v} \right)^{2l} \int dx \left[ \frac{U'(x')}{k_B T_0} \right]^{2l+1} \right\}, \tag{21}$$

where the Boltzmann-Gibbs factor, with the effective potential $U_{\text{eff}}(x)$, is explicitly factorized and expression (10) has been explicitly used. Clearly, the stationary Boltzmann-Gibbs distribution is recovered in the diffusive limit. In other extreme limit, when the thermophoretic potential dominates, the stationary distribution that describes the sharp accumulation at the boundaries goes asymptotically as

$$P_{\text{st}}(x) \propto \frac{v}{k_B T_0 \mu} \left( 1 - \frac{\mu^2}{v^2} [U'(x)]^2 \right)^{-1}. \tag{22}$$

Though the stationary distribution given by (17) does not correspond to the one of Boltzmann and Gibbs, we maintain the use of the Boltzmann-Gibbs entropy $S = -k_B \int dx P_{\text{st}}(x) \ln P_{\text{st}}(x)$, and with the expression for the flux-free stationary distribution in the form of (21), we can give an interpretation of the process considered under the point of view of the stochastic thermodynamics. The Free energy $F = -k_B T_0 \ln Z$ can be written as

$$F = E_{\text{eff}} - T_0 S, \tag{23}$$

where the effective internal energy, $E_{\text{eff}}$, is defined by

$$E_{\text{eff}} = \langle U(x) \rangle + \frac{1}{2} m_{\text{eff}} v^2 \left( 1 - \langle \frac{V_{\text{drift}}^2}{v^2} \rangle \right) + U'(x_{\text{max}}) \sum_{l=1}^{\infty} \left( \frac{v_m}{v} \right)^{2l} \left( \int dx' \left[ \frac{U'(x')}{U'(x_{\text{max}})} \right]^{2l+1} \right), \tag{24}$$

with $m_{\text{eff}} = 2/\mu a$ and $\langle \cdot \rangle$ denotes the average using the stationary distribution $P_{\text{st}}(x)$. Notice that the first two terms in the right hand side of the last equation contribute to standard mechanical energy since the second term, being positive definite, can be interpreted as an effective kinetic energy. The last term gives the contribution from the effects of persistence in series expansion in powers of the dimensionless parameter $v_m/v$, $v_m$ given by $\mu U'(x_{\text{max}})$. In the diffusive limit we have simply that $E_{\text{eff}} = \langle U(x) \rangle_{\text{B-G}}$, where $\langle \cdot \rangle_{\text{B-G}}$ denotes the average taken with the Boltzmann-Gibbs distribution.

**Numerical simulations.** As stated in the previous section, the stationary behavior of trapped active motion, contained in expression (11) and particularly in (17), can be regarded as the stationary behavior of a passive Brownian particle that moves in an inhomogeneous temperature medium. Thus, for the cases of interest, stationary realizations of the particle trajectories can be obtained directly from the Langevin equation

$$\frac{d}{dt} x(t) = -\mu U'(x) + \sqrt{T(x)} \xi(t), \tag{25}$$

if initial conditions are compatible with the stationary solution (17). In Eq. (25), $\xi(t)$ denotes Gaussian-white noise, with vanishing mean $\langle \xi(t) \rangle = 0$ and autocorrelation function $\langle \xi(t) \xi(t') \rangle = 2\mu k_B \delta(t - t')$ and $T(x)$, the inhomogeneous medium temperature (16) that encodes the features of active motion. Ensemble calculations over realizations of these trajectories lead to all the observable quantities of interest, in particular those quantities of interest in stochastic thermodynamics (see for instance the review [42]).

We apply the ideas developed up to now to particular realizations of the potential $U(x)$ that had been considered before in the literature, both theoretically and experimentally, namely, the linear potential in the sedimentation process of active particles [12, 14, 24], the harmonic potential trapping active particles [12, 24], and the diffusion of active particles in a double well potential.

### A. Sedimentation: Linear Potential $U(x) = mgx$

The simplest physical realization for run-and-tumble particles in an external potential, corresponds to the case when $U(x)$ is linear with distance, i.e., when the particles are subject to a constant force, as is the case of active particles that swim above a hard wall in the presence of the gravitational force $-mg$. The probability density of finding a particle at height $x$ above the wall has been found in one dimension for run-and-tumble particles [22, 28], an in higher dimensions in Ref. [12], further, this situation has been realized experimentally in three dimensions for active Brownian particles [13].

From the relations (16) we have that the drift velocity $V_{\text{drift}} = -\mu mg$ corresponds to the sedimentation velocity $-v_{\text{sed}}$, and the effective temperature $T(x) = T_0(1 - v_{\text{sed}}^2/v^2)$ is spatially homogeneous. The familiar exponential decrease of the probability density with the distance from the wall is recovered from Eq. (17), to say

$$P_{\text{st}}(x) = \frac{mg}{k_B T_0(1 - v_{\text{sed}}^2/v^2)} e^{-mgx/k_B T_0(1 - v_{\text{sed}}^2/v^2)}, \tag{26}$$

The uniformity of the effective temperature allows us to interpret Eq. (26) as the probability density of a Brownian particle diffusing under the effects of the gravitational force in a medium of homogeneous temperature.
with \( v_{\text{sed}} < v \). By self-propelling, active particles develop higher speeds to overcome \( v_{\text{sed}} \). If \( v_{\text{sed}}/v \ll 1 \) the effective temperature corresponds to the maximum value \( T_0 \), while the temperature gets arbitrarily close to zero as \( v \) gets arbitrarily close to \( v_{\text{sed}} \), if this is the case, the particles accumulate all over the wall.

### B. Power law trapping potentials: The Harmonic External Potential

Experiments that analyze the effects of trapping potentials on the diffusion of active particles have been realized. In some approximation the trapping potential can be approximated by a harmonic potential, for which analytical calculations can be devised, however, the effects of steeper potential are also of interest since they might approximate better the experimental trapping potential.

The analysis of the diffusion of active particles confined by a harmonic potential has been considered theoretically \([12, 50]\), and more recently experimentally in a two-dimensional systems of active Brownian particles confined by transverse acoustic forces of a single-beam transducer \([24]\) and in a two dimensional system of an optically trapped passive Brownian particle coupled to a bath of active particles \([31]\).

In this section we consider run-and-tumble particles trapped in the following one-dimensional harmonic trapping potential

\[
U(x) = \frac{1}{2} \kappa_1 x^2, \tag{27}
\]

where \( \kappa_1 \) is a constant that characterizes the intensity of the trapping potential. For this potential the local effective temperature \([16, 19]\) is

\[
\mathcal{T}(x) = T_0 \left[ 1 - x^2/x_{\text{max}}^2 \right], \tag{28}
\]

where \( x_{\text{max}} = v/\mu \kappa_1 \) is a length scale related to confinement, which in this case coincides with the maximum displacement an active particle can go, given that moves with speed \( v \). At the center of the trap, the local temperature acquires its maximum value, given by the effective temperature of free diffusion \( T_0 \), and vanishes at \( x = \pm x_{\text{max}} \). In this case the thermophoretic force results linear in the displacement \( \mathcal{T} \) and repulsive from the center of the trap, and can be written as \((2\kappa_1^2/\mu) x \). This force opposes to the one due to the harmonic potential giving rise to the effective force \(-\kappa_1 (1 - 2\mu \kappa_1/\alpha)x \). The dimensionless parameter

\[
\beta_0 = \mu \kappa_1/\alpha, \tag{29}
\]

which is equivalent to the ratio \( l_{\text{pers}}/x_{\text{max}} \), measures the competition between the effects of confinement and persistence and drives the system into qualitatively different equilibrium distributions. Notice that \( \beta_0 \) is equivalent to the ratio of the energy \( \kappa_1 l_{\text{pers}}^2 \), to the effective thermal energy \( k_B T_0 \). Thus, the effective potential can be written as \( U_{\text{eff}}(x) = \frac{1}{2} \kappa_1 x^2 (1 - 2\beta_0) + k_B T_0 \), and from it, we deduce the following bifurcation scheme: for \( 2\beta_0 < 1 \), \( P_{\text{st}}(x) \) is unimodal around the center of the trap, which corresponds to the unique stable equilibrium position of \( U_{\text{eff}}(x) \); for \( 2\beta_0 = 1 \) the effects of active motion cancel out those of the trapping potential, thus no net force acts on the particle giving rise to a uniform distribution inside the whole interval \([-x_{\text{max}}, x_{\text{max}}]\) \([22]\), while for \( 2\beta_0 > 1 \) the equilibrium position becomes unstable and the distribution \( P_{\text{st}}(x) \) exhibits peaks at the borders \( \pm x_{\text{max}} \) due to the net pushing out force from the center. This bifurcation is directly shown in the corresponding equilibrium solution \([21]\)

\[
P_{\text{st}}(x) = Z^{-1} \left[ 1 - x^2 \beta_0^2 l_{\text{pers}}^2 \right]^{(1-2\beta_0)/2\beta_0}, \tag{30}
\]

where the partition function \( Z \) is given by

\[
Z = \sqrt{\pi \Gamma \left[ \frac{1}{\beta_0} \right]} \frac{l_{\text{pers}}}{\Gamma \left[ \frac{1}{2} + \frac{1}{\beta_0} \right]} \beta_0 \tag{31}
\]

which depends only on \( \beta_0 \), and \( \Gamma(z) \) denotes the gamma function. It can be shown straightforwardly that in the diffusive limit, which can be stated equivalently as \( \beta_0 \to 0, l_{\text{pers}} \to 0 \) such that \( \beta_0 l_{\text{pers}}^2 \) is finite, the Boltzmann-Gibbs equilibrium distribution \( P_{\text{B-G}}(x) \) is recovered, i.e.

\[
P_{\text{B-G}}(x) = \frac{\beta_0}{2\pi l_{\text{pers}}^2} \exp \left\{ -\beta_0 \frac{x^2}{2l_{\text{pers}}^2} \right\} \tag{32}
\]

The stationary distribution \([30]\), corresponds to the class of probability distributions in statistical physics for which the Boltzmann-Gibbs distribution is recovered as a particular limit, in this case, in the limit \( \beta_0 \to 0 \). Indeed, the distribution given in Eq. \([30]\) is of the kind of the so-called \( q \)-Gaussian distribution \([51]\), which is of the form

\[
\exp_q(-x^2) = \left[ 1 - \frac{1-q}{2-q} \left( \frac{1}{2} \right)^{2-q} x^2 \right]^{\frac{1}{1-q}} \tag{33}
\]

from which the usual Gaussian function is recovered as \( q \to 1 \). Notice that the definition of the \( q \)-exponential used in Tsallis statistics differs slightly from the one given in \([33]\), but in terms of this one we have

\[
P_{\text{st}}(x) = Z^{-1} \exp_q \left\{ -x^2/(2l_{\text{pers}}^2) \right\}, \tag{34}
\]

where the \( q \)-parameter is directly related to \( \beta_0 \) through

\[
q = \frac{4\beta_0 - 1}{2\beta_0 - 1}. \tag{35}
\]

This last result is one of the main points of the paper, which belongs to the few systems for which the
The function of a βinhomogeneous thermal bath with temperature profile trajectories of passive Brownian particles diffusing in an ensemble average of these trajectories for different realizations of x is given by the prescription (25), by computing the stationary probability distribution (30) from the ensemble average ⟨δ(x - x(t))⟩_st over a set of stationary trajectories of passive Brownian particles diffusing in an inhomogeneous thermal bath with temperature profile T(x) given by Eq. (28). In Fig. 1 we present some of these trajectories for different realizations of x(t) for a given value of the parameter β_0. In Fig. 2 we compare the analytical stationary distribution (30) (solid lines) for different values of β_0, with the ones obtained from the numerical solutions of the Langevin equation (25) (symbols).

We introduce the quantity
\[ \sigma(\beta_0) = \frac{d}{d(\ln \beta_0)} \ln \left[ \frac{1}{t_{\text{pers}}(\beta_0)} \right], \]
(36)
which can be interpreted, by analogy with the thermodynamic relation βE = ∂ ln Z⁻¹/∂ ln β, as the ratio of the internal energy [which in the overdamped limit is given solely by the average of the potential energy, E = ⟨U(x)⟩, to the effective thermal energy of the system k_B T_0, i.e. \( \sigma(\beta_0) = E/k_B T_0 \)]. Eq. (36) can be written in terms of the digamma function, \( \Psi(x) = [\ln \Gamma(x)]' \), as
\[ \sigma(\beta_0) = \frac{1}{2\beta_0} \left[ \Psi \left( \frac{1}{2} + \frac{1}{2\beta_0} \right) - \Psi \left( \frac{1}{2\beta_0} \right) \right]. \]
(37)
As is apparent from the Fig. 3, \( \sigma(\beta_0) \) is a sigmoid-like function of \( \beta_0 \). This characteristic make it suitable as a measure of the departure from the Boltzmann-Gibbs distribution. Certainly, in the limit of negligible persistence, i.e. \( \beta_0 \ll 1 \), one can substitute the Gamma functions that appear in Eq. (31) by their Stirling’s approximation to get \( \sigma(\beta_0) \approx 1/2 \) as is shown in Fig. 3. This result can be interpreted as the fulfillment of the equipartition theorem. In contrast, it can be shown that in the limit of large persistence, \( \beta_0 \gg 1 \), \( \sigma(\beta_0) \) saturates to the value 1, which characterizes the non-Boltzmann-Gibbs distribution for which the fulfillment of the equipartition theorem breaks down, we get \( E = k_B T_0 \).

1. Steeper Power Law Trapping Potentials

Owing to the fact that the harmonic potential (27) satisfies that \( |U'(x)|^2 \) is proportional to \( U(x) \) itself, the thermophoretic force [computed from Eq. (16)] opposes the force derived from the trapping potential with the same dependence on the position, linear in this case, which lead to the stationary solution (30). It is clear that more subtle effects should appear if steeper potentials are considered. We briefly point out some aspects for a power law potential of the form
\[ U(x) = \frac{\kappa_n}{2n} x^{2n}, \quad n = 1, 2, \ldots, \]
(38)
where \( \kappa_n \) is a parameter with units of energy over [length]^{2n} that indicates the strength of the potential. In analogy with the parameter (29), we introduce the parameter \( \beta_{0,n} \) given by
\[ \beta_{0,n} = \frac{\kappa_n}{2n} \left[ \frac{t_{\text{pers}}}{x_{\text{max}}} \right]^{2n-1}. \]
(39)
that again measures the competence between the confinement length, $x_{\max} = (v/\mu\kappa_n)^{1/(2n-1)}$, and the persistence length $l_{\text{pers}}$. A novel effect appears as consequence of this competence. Namely, it can be shown that for the potential as given in (38), new unstable equilibrium positions for the effective potential in Eq. (18) emerge for values of $\beta_0 n$ larger or equal to $[2(2n-1)]^{-(2n-1)}$ as long as $n > 1$. The appearance of this unstable positions lead to multimodal stationary distributions.

For the sake of clarity and for reasons that will be clear in the following section, where the symmetric double well potential is considered, we focus our analysis in the case $n = 2$, the so-called quartic potential,

$$U(x) = \kappa_2 x^4/4,$$  
(40)

for which $\beta_{0,2} = \mu \kappa_2 v^2/\alpha^3 = (l_{\text{pers}}/x_{\max})^3$. The value of $\beta_{0,2}$ for which the unstable equilibrium positions of the effective potential start to appear is $6^{-3}$. This value also marks a qualitative difference of the effective potential, namely, it has a vanishing slope at the boundaries. For larger values of $\beta_{0,2}$ the effective potential exhibits the mentioned unstable equilibrium (maxima) positions, however the center of the trap is still the stable equilibrium position of the effective potential. Thus, the particles are “pushed away” from the unstable positions accumulating, on the one hand, at the boundaries, and on the other, at the center of the trap leading to a multimodal probability distribution. This feature changes abruptly for $\beta_{0,2} > 4^{-3}$, since in this regime the effective potential has its minima at the borders.

In complete analogy with Eq. (36), we consider the quantity

$$\sigma(\beta_{0,2}) = \frac{d}{d(\ln \beta_{0,2})} \ln [l_{\text{pers}}(\beta_{0,2} Z)^{-1}]$$  
(41)

as a measure of the departure from the stationary distribution of Boltzmann-Gibbs, however, in contrast with the case of the harmonic potential, no analytical expression of the partition function for arbitrary $\beta_{0,2}$ exists. Notwithstanding this, it is straightforward to show that in the diffusive limit $\sigma(\beta_{0,2}) \rightarrow 1/4$, while in the asymptotic limit, $\beta_{0,2} \rightarrow \infty$, we have that $\sigma(\beta_{0,2}) \rightarrow 1/3$ as is shown in the Fig. 4.

C. The Symmetric Double-Well Potential

A more interesting case corresponds to that one when the trapping potential has a richer energy landscape, as those that exhibit more than one stable state. Here, we report on the simplest case when the trapping potential has only two degenerate stable states. Thus, we focus our analysis in the effects of persistence of active particles trapped by the symmetric double-well potential

$$U(x) = \Delta U \left[\frac{x^4}{L^4} - \frac{x^2}{L^2} + 1\right],$$  
(42)

leaving for a further analysis the effects of active motion on the asymmetric one as the one considered in Ref. 47 by Landauer. In Eq. (42) $\Delta U$ and $L$ are two positive parameters that characterize the external potential, the former one denotes the energy height of the barrier, while the last one, corresponds to half the distance between the two stable states that correspond to the minima of the external potential (42) located at $x = \pm L$, respectively.

FIG. 3. (Color online) $\sigma$ as function of $\beta_0$. It saturates in the persistent regime ($\beta_0 \gg 1$) to the value 1, while in the diffusive regime ($\beta_0 \ll 1$), it characterizes the Boltzmann-Gibbs distribution with the value $\frac{1}{2}$. The square marks the value of $\sigma$ at $\beta_0 = 1/2$ at which the bifurcation occurs. The probability density changes from unimodal with maximum at the trap center in the diffusive regime to a distribution sharply peaked at the boundaries.

FIG. 4. (Color online) $\sigma$ as function of $\beta_{0,2}$ is shown for the case of the quartic potential (40). It saturates in the persistent regime ($\beta_{0,2} \gg 1$) at the value $\frac{1}{4}$, while in the diffusive regime ($\beta_{0,2} \ll 1$) $\sigma$ characterizes the Boltzmann-Gibbs distribution with the value $\frac{1}{4}$.
Besides the characteristic lengths \( l_{\text{pers}} \) and \( L \), there is a third one,

\[
\mathcal{L} = \frac{8}{3\sqrt{3}} \frac{\Delta U \mu}{v},
\]

(43)

that gives an estimate of the average length the active particle travels from the center of the potential, when an energy \( \Delta U \) is available to swim at a swimming force \( v/\mu \).

For convenience we introduce the following two independent parameters

\[
\chi = \frac{\mathcal{L}}{L},
\]

(44a)

\[
\eta = \frac{l_{\text{pers}}}{L}.
\]

(44b)

Small values of \( \chi \) refer to either a shallow energy barrier or potential wells far apart, or both. Besides these, we introduce a third parameter

\[
\beta_{\text{dw}} = \frac{4}{L^2 \alpha} \frac{\Delta U}{\chi \eta},
\]

(45)

which characterizes the departure from the equilibrium regime and allows the transition between the Boltzmann-Gibbs distribution and its corresponding stationary superstatistics distribution in the active (persistence) regime. Indeed, if the effects of persistence of active motion are negligible as occurs in the diffusive regime, we have that \( \beta_{\text{dw}} \to 0 \) and the stationary distribution corresponds to the one of Boltzmann-Gibbs

\[
P_{\text{B-G}}(x) = \mathcal{Z}^{-1} \exp \left\{ -\frac{3\sqrt{3}}{8} \frac{\chi}{\eta} \left[ \frac{x^4}{L^4} - \frac{x^2}{L^2} + 1 \right] \right\},
\]

(46)

which corresponds to the bimodal distribution that is symmetric at the stable positions of the external potential \( \pm L \). In this limit (\( \beta_{\text{dw}} \to 0 \)) we are interested in the quantity analogous to the one given in Eq. (36), in this case given by

\[
\varsigma(\chi/\eta) = \frac{d}{d \ln(\chi/\eta)} \ln(L \mathcal{Z}^{-1}),
\]

(47)

which is shown in Fig. 5 as function of \( (\chi/\eta)^{-1} \). For large values of \( \chi/\eta \), i.e. \( \Delta U \gg k_B T_0 \), the system is reminiscent of a system of passive Brownian particles that get trapped at the minima of the potential and \( \varsigma(\chi/\eta) \to \frac{2}{3} \).

In the opposite regime \( \Delta U \ll k_B T_0 \), the effects of the energy barrier are negligible and the stationary distribution corresponds to that of a trapped passive Brownian particle diffusing in a quartic potential leading to \( \varsigma(\chi/\eta) = \frac{4}{3} \). The transition between these limit values is not monotonic as is clear from the figure.

As has been discussed above, in the stationary regime, the nonequilibrium nature of trapped active motion can be mapped into a fictitious inhomogeneous thermal bath characterized by the temperature profile given by

\[
T(x) = T_0 \left\{ 1 - \beta_{\text{dw}}^2 \frac{x^2}{l_{\text{pers}}^2} \left( \frac{x^2}{l_{\text{pers}}^2} \eta^2 - 1 \right)^2 \right\},
\]

(48)

which reduces to the homogeneous effective temperature \( T_0 \) as \( \beta_{\text{dw}} \to 0 \). The temperature profile (48) has a richer structure (see Fig. 6) in comparison with profiles associated with external potentials with only one stable state. By simple inspection it can be seen that the temperature profile (48) reaches its maximum value \( T_0 \) at the equilibrium positions of the external potential either stable or unstable, namely at \( x = \pm L \) and at the origin \( x = 0 \), respectively. Additionally, \( T(x) \) has two local minima at the positions \( x = \pm L/\sqrt{3} \), where the local temperature acquires the value \( T_0 \left\{ 1 - \chi^2 \right\} \).

We focus our analysis to the case for which the inequality \( \chi < 1 \) is valid, firstly because this condition guaranties a positive definite local temperature and secondly it avoids the unnecessary difficulty of defining the flux in the regions of space where this conditions is not valid. In simple words, this condition assures that the particle is active enough to overcome the energy barrier \( [v > (4/3)^{3/2} \mu \Delta U/L] \) and makes it able to freely swim.

FIG. 5. (Color online) Susceptibility \( \varsigma \) as in Eq. (47), as function of the dimensionless persistence length \( (\chi/\eta)^{-1} \). For \( (\chi/\eta)^{-1} \ll 1 \) the particle is trapped in any of the potential wells, in the opposite regime \( (\chi/\eta)^{-1} \gg 1 \) the quartic dependence of the potential is the dominant part of the trapping.
between the two boundary points $\pm x_{\text{max}}$, given by

$$x_{\text{max}} = \frac{L}{\sqrt{3}} \left[ \chi^{1/3} \left( 1 + \sqrt{1 - \chi^2} \right)^{-1/3} + \chi^{-1/3} \left( 1 + \sqrt{1 - \chi^2} \right)^{1/3} \right], \quad (49)$$

at which the net force on the particle vanishes. As $\chi \to 1$ from below, the local effective temperature vanishes at $\pm L/\sqrt{3}$ as also does the net force on the particle.

The thermophoretic force induced by the effective local temperature [18] is explicitly given by the product of the swimming force $v/\mu$ times a factor that takes into account the inhomogeneity of the fictitious medium, namely

$$2 \frac{v}{\mu} \beta_{\text{dw}} \frac{d}{\text{pers}} \left( \frac{x^2}{\text{pers}^2} \eta^2 - 1 \right) \left( 3 \frac{x^2}{\text{pers}^2} \eta^2 - 1 \right), \quad (50)$$

which, for finite $\beta_{\text{dw}}$, pushes the particles away from the positions of maximum temperature towards either the boundaries $\pm x_{\text{max}}$ or towards $\pm L/\sqrt{3}$, at which $\mathcal{T}(x)$ has local minima. Thus there is a competition between the thermophoretic force that tends to accumulate the particles at these positions (such a situation is pictorially shown in Fig. 2 for a temperature profile characterized by $\chi = 0.9$) and the force due to the external potential $U(x)$ that tends to accumulate the particles at $x = \pm L$.

An straightforward stability analysis of $U_{\text{eff}}(x)$ shows that for a given value of $\chi$, the landscape of the effective potential changes as $\beta_{\text{dw}}$ is varied across the threshold value

$$\beta_{\text{dw}}^* = \frac{1}{4}. \quad (51)$$

Indeed, for $\beta_{\text{dw}} < \beta_{\text{dw}}^*$, the positions $\pm L$ correspond to global minima of $U_{\text{eff}}(x)$ (dashed-dotted green line in Fig. 7 with $\chi = 0.9$), that coincides with the minima of $U(x)$. These positions, however, become local maxima for $\beta_{\text{dw}} > \beta_{\text{dw}}^*$, and $U_{\text{eff}}(x)$ acquires local minima at the positions $\pm x_{\text{max}}$ and at the positions $\pm x_{\text{meta}}$, which emerge as metastable states (dashed red line in the same figure), where

$$x_{\text{meta}} = \frac{L}{\sqrt{3}} \left[ 1 + \frac{1}{2} \beta_{\text{dw}}^{-1} \right]^{1/2}. \quad (52)$$

At the critical value $\beta_{\text{dw}} = \beta_{\text{dw}}^*$ the positions $\pm L$ are inflexion points and $\pm x_{\text{max}}$ become the global minima of $U_{\text{eff}}(x)$ (blue-solid line in Fig. 7).

The dimensionless probability density function in the stationary state, $x_{\text{max}}P_{\text{st}}(x)$, as function of $x/x_{\text{max}}$, is shown for the values of $\chi = 0.9$ and 0.1 in Figs. 8 and 9 respectively. This is computed numerically from the formulas (17) and (42) (shown in solid lines) since no closed analytical expression is available and it is compared with the corresponding distribution obtained from the numerical simulations (symbols) using the stationary Brownian dynamics as has been explained in section III. In each figure, the mentioned distributions are shown for different values of the parameter $\beta_{\text{dw}}$. For the case $\chi = 0.9$ (see Fig. 8), the thermophoretic force pushes the particles away from the center of the trap, thus the probability density at the center is small. At the threshold value $\beta_{\text{dw}} = \beta_{\text{dw}}^*$ (blue-solid line), the probability density grows monotonically from the center of the potential toward the boundaries, where the particles accumulate. For subcritical values, $\beta_{\text{dw}} < \beta_{\text{dw}}^*$, the effects of persistent motion are diminished and the particles accumulate at $x = \pm L$.
value, while the thin- and thick-dashed-red lines, numerical solution of Eq. (25).

FIG. 8. (Color online) The dimensionless stationary probability density function at the boundaries is diminished gradually until particles start to accumulate at $\pm L$. In the diffusive limit, the region close to the boundaries is visited much less frequently (see dashed-dotted green lines in Fig. 8 for $\beta_{dw} = 1/3$ and $\beta_{dw} = 1/2$). If $\beta_{dw}$ is further decreased as before, the probability distribution of the particle positions tends to the equilibrium distribution characterized by the Boltzmann-Gibbs factor. In contrast to the cases for which $\chi \lesssim 1$, the peaking of the probability distribution at $x_{meta}$ is subtle, while the peaking at the boundaries is conspicuous in the supercritical regime (see dashed red line in Fig. 9 for $\beta_{dw} = 1/2$).

FIG. 9. (Color online) The dimensionless stationary probability density function at the boundaries is diminished gradually until particles start to accumulate at $\pm L$. In the diffusive limit, the region close to the boundaries is visited much less frequently (see dashed-dotted green lines in Fig. 8 for $\beta_{dw} = 1/3$ and $\beta_{dw} = 1/2$). If $\beta_{dw}$ is further decreased as before, the probability distribution of the particle positions tends to the equilibrium distribution characterized by the Boltzmann-Gibbs factor. In contrast to the cases for which $\chi \lesssim 1$, the peaking of the probability distribution at $x_{meta}$ is subtle, while the peaking at the boundaries is conspicuous in the supercritical regime (see dashed red line in Fig. 9 for $\beta_{dw} = 1/2$).

We conclude the analysis of run-and-tumble particles trapped by the symmetric double-well potential (42) by computing the analogous quantity introduced in the previous section [Eq. (50)] given now by

$$\sigma(\beta_{dw}, \chi) = \frac{d}{d(\ln \beta_{dw})} \ln \left\{ t_{pers} \mathcal{Z}^{-1}(\beta_{dw}, \chi) \right\},$$

(53)

where the explicit dependence on $\chi$ has been pointed out and $\mathcal{Z}(\beta_{dw}, \chi)$ denotes the rescaled partition function $\exp \{-S(\beta_{dw}, \chi)\} \mathcal{Z}(\beta_{dw}, \chi)$, where $S(\beta_{dw}, \chi)$ is a shift that makes the argument of the exponential in the partition function (19) positive. In Fig. 10 the numerically computed susceptibility $\sigma(\beta_{dw}, \chi)$ as function of $\beta_{dw}$ is shown for $\chi = 0.1$, $0.2$, $0.5$ and $0.9$. A nonmonotonous dependence on $\beta_{dw}$ is observed. In the diffusive limit, for which the stationary distribution corresponds to that of Boltzmann-Gibbs, the susceptibility marks the value $1$ independently of the value of $\chi$. As the effects of persis-
Non-Boltzmann-Gibbs

FIG. 10. (Color online) $\sigma(\beta_{\text{dw}}, \chi)$ as defined in Eq. (53), for the case of the double well potential \([14]\), as shown as function of $\beta_{\text{dw}}$ for the values $\chi = 0.1, 0.2, 0.5$ and 0.9. In the diffusive regime ($\beta_{\text{dw}} \ll 1$), $\sigma \approx \frac{1}{\chi}$ characterizes the equilibrium stationary state that corresponds to the Boltzmann-Gibbs distribution. The persistence regime ($\beta_{\text{dw}} \gg 1$), $\sigma \approx \frac{1}{\chi}$ characterizes the no Boltzmann-Gibbs stationary distribution. These characteristic limiting values have been obtained analytically from Eq. (53).

The peaking of the distribution at the boundaries, as such, the details of trapping potential at the center potential can be neglected making the quartic potential the dominant part. Therefore it is expected that $\sigma(\beta_{\text{dw}}, \chi)$ goes to $\frac{1}{\chi}$ as occurs in Fig. 4. The peaking of the distribution at the boundaries in this regime makes computationally difficult such a calculation.

IV. FINAL COMMENTS AND CONCLUDING REMARKS

As is well known, the coupling between the diffusion process of active particles of spatially-independent active-motion traits and the inhomogeneity induced by the external potential, makes explicit the nonequilibrium aspect of active motion revealed in the non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position. In this paper we have established a single-parameter mapping between these non-Boltzmann-Gibbs stationary distributions of the particle position.

The extension of the present analysis to higher dimensions is not straightforward, indeed, neither the active Brownian nor the run-and-tumble model of active motion have explicit analytical solutions for arbitrary trapping potentials, thus leaving the determination of the mapping between the trapped active motion to passive Brownian motion in an inhomogeneous thermal bath as an open problem. On the other hand, the existence...
of the homogeneous effective temperature in the diffusive limit of two-dimensional active motion has been discussed in Ref. 29 which, as expected, is related with the equilibrium solution of Boltzmann-Gibbs in the zero-current stationary state. The existence of such homogeneous temperature has been shown to exists in a two-dimensional trapped system, namely for active Ornstein-Uhlenbeck particles trapped by an harmonic potential 53. Further, the analysis presented in this paper can be generalized to the case of one-dimensional run-and-tumble particles diffusing within a finite interval with reflecting boundary conditions (hard walls) at the borders 54, for which the high accumulation of particles at the boundaries can be mapped to passive Brownian motion in an inhomogeneous thermal bath under the same boundary conditions. Finally, it would be of broad interest to find a generalization of the Boltzmann-Gibbs entropy functional from which 91 is derived via the maximization of entropy principle, as has been shown from many non-Boltzmann-Gibbs distributions that occur in complex systems 55.

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