SCATTERING OF THE FOCUSING ENERGY-CRITICAL NLS WITH INVERSE SQUARE POTENTIAL IN THE RADIAL CASE

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ABSTRACT. We consider the Cauchy problem of the focusing energy-critical nonlinear Schrödinger equation with an inverse square potential. We prove that if any radial solution obeys the supreme of the kinetic energy over the maximal lifespan is below the kinetic energy of the ground state solution, then the solution exists globally in time and scatters in both time directions.

1. Introduction.

1.1. Review of the focusing energy critical NLS. The Cauchy problem of the focusing energy-critical nonlinear Schrödinger equation (NLS) in $\mathbb{R}^d (d \geq 3)$ can be formulated as

\[
\begin{cases}
(i \partial_t + \Delta) u = -|u|^{\frac{4}{d-2}} u \\
 u(0, x) = u_0(x)
\end{cases}
\]

(NLS)

Cazenave and Weissler [7] developed the well-posedness theory of (NLS) with small initial data. The main result is that if $\|u_0\|_{\dot{H}^1}$ is sufficiently small, then $u(t, x)$ is a global solution and scatters in both time directions. The global well-posedness and scattering theory of (NLS) with large initial data was established in recent years. It is believed that the ground state solution $W_0(x) = (1 + \frac{|x|^2}{d(d-2)})^{\frac{d-2}{2}}$ contributes the threshold of scattering. Assuming $u_0$ is radial and on the maximal lifespan $I$ of the (NLS) solution satisfies

\[
\sup_{t \in I} \|u(t)\|_{\dot{H}^1} < \|W_0\|_{\dot{H}^1},
\]

(1.1)

Kenig and Merle [13] first achieve the global well-posedness and scattering result when $d = 3,4,5$. For nonradial initial data, see Killip and Visan [19] for $d \geq 5$, and Dodson [10] for $d=4$. The case $d=3$ is currently open. Above results also hold when

\[
\|u_0\|_{\dot{H}^1} < \|W_0\|_{\dot{H}^1} \text{ and } E_0(u_0) < E_0(W_0),
\]

(1.2)

which is stronger than (1.1), where $E_0(\cdot)$ is defined in (1.3).
1.2. Energy critical NLS with inverse square potential. In this article, we consider the initial value problem of the focusing NLS with inverse square potential

\[
\begin{aligned}
(i\partial_t - \mathcal{L}_a)u &= -|u|^{\frac{4}{d-2}}u \\
u(0, x) &= u_0(x) \in H^1,
\end{aligned}
\]

where \( \mathcal{L}_a = -\Delta + \frac{a}{|x|^2} \) and \( a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{2}\right)^2 \). This problem is energy critical as the energy

\[
E_a(u(t)) = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t)|^2 + \frac{a}{2|x|^2} |u(t)|^2 - \frac{1}{2} |u(t)|^2 \, dx
\]

is invariant under the scaling \( u(t, x) \mapsto \lambda^{\frac{2-d}{2}} u(t, \frac{x}{\lambda}) \). Compared with the free NLS, the model (NLS\(_a\)) breaks the translation in space symmetry. For non-radial problems, this is a major obstacle to overcome, see [18, 38]. On the time of existence, both the mass and energy are conserved

\[
M(u(t)) = \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M(u_0), \quad E_a(u(t)) = E_a(u_0).
\]

**Definition 1.1** (Solution, [18]). Let \( t_0 \in I \subset \mathbb{R} (I \text{ is an interval}) \) and \( u_0 \in \dot{H}^1 \). A function \( u : I \times \mathbb{R}^d \to \mathbb{C} \) is a solution to NLS\(_a\) if \( u(t) \in C_t H^1_a \cap S(J \times \mathbb{R}^d) \) for any compact \( J \subset I \) and obeys the Duhamel formula

\[
u(t) = e^{-it-t_0} \mathcal{L}_a u_0 + i \int_{t_0}^t e^{-i(t-t-s)} \mathcal{L}_a (|u(s)|^{\frac{4}{d-2}} u(s)) \, ds,
\]

for all \( t \in I \). We refer \( I \) as the life-span of \( u \). We say \( u \) is a maximal life-span solution if the solution can not be extended to any strictly larger interval. If \( I = \mathbb{R} \), \( u \) is a global solution.

From equivalence of Sobolev norms and Strichartz estimate, if the initial data \( u_0 \) is sufficiently small in \( \dot{H}^1_a \), the solution \( u(t) \) to (NLS\(_a\)) is global and scatters followed by the same argument in [7]. Global well-posedness and scattering result for large solutions was first obtained in [18] for the non-radial defocusing (NLS\(_a\)) in \( \mathbb{R}^3 \), then extended to higher dimension in [38]. For defocusing (NLS\(_a\)), any \( \dot{H}^1_a \) bounded initial data leads to global scattering solution. For focusing (NLS\(_a\)), [18] also classifies the blow-up solutions.

**Proposition 1.** Fix \( d \geq 3 \) and \( a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{2}\right)^2 \). Let \( u_0 \in \dot{H}^1 \) be such that \( E_a(u_0) < E_{a,0}(W_{a,0}) \) and \( \|u_0\|_{\dot{H}^1_a} \geq \|W_{a,0}\|_{\dot{H}^1_{a,0}} \). Assume also that either \( xu_0 \in L^2(\mathbb{R}^d) \) or \( u_0 \in \dot{H}^1 \) is radial. Then the corresponding solution \( u \) to NLS\(_a\) blows up at finite time.

Here \( W_a(x) \) is the ground state solution [18]:

\[
\mathcal{L}_a W_a = |W_a|^{\frac{4}{d-2}} W_a,
\]

and

\[
W_a(x) = |d(d-2)\beta^2|^{\frac{d-2}{4}} \left( \frac{|x|^{\beta-1}}{1 + |x|^{2\beta}} \right)^{\frac{d-2}{2}},
\]

where \( 0 > a > -\left(\frac{d-2}{2}\right)^2 \) and \( \beta = \sqrt{1 + \frac{4a}{(d-2)^2}} \). Clearly, \( W_a(x) \) is radial and non-negative, and it is a global non-scattering solution to (NLS\(_a\)).
In [35], the author proved scattering of the focusing energy critical NLS\(_{a}\) under the condition
\[
\|u_0\|_{\dot{H}^1} < \|W_{a\wedge 0}\|_{\dot{H}^1_{a\wedge 0}}, \quad E_a(u_0) < E_{a\wedge 0}(W_{a\wedge 0}). \tag{1.6}
\]
In this paper, scattering is obtained under weaker assumption (1.7). Our main result reads as follows.

**Theorem 1.2.** Fix \(a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2\) and \(3 \leq d \leq 6\). Let \(u\) be a solution of NLS\(_{a}\) with radial initial data \(u_0\) and maximal lifespan \(I_u\) such that
\[
\sup_{t \in I_u} \|u(t)\|_{\dot{H}^1} < \|W_{a\wedge 0}\|_{\dot{H}^1_{a\wedge 0}}, \tag{1.7}
\]
then \(u\) is a global solution and \(\|u\|_{S(\mathbb{R})} < \infty\). In addition, the solution scatters in both time direction, i.e. there exists \(u_{\pm} \in \dot{H}^1\) such that
\[
\lim_{t \to \pm \infty} \|u - e^{-itL_a} u_{\pm}\|_{\dot{H}^1} = 0.
\]

**Remark 1.** The construction of the local solution depends on equivalence of Sobolev norms (Lemma 2.1) and the fractional product rule (Lemma 2.2) associated with \(L_a\), thus we require \(a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2\), see Section 2.4 or [18, 38].

Note that when \(a > 0\), the Sharp constant \(\frac{\|W_{a\wedge 0}\|_{\dot{H}^1}}{\|W_0\|_{\dot{H}^1}}\) in the Gagliardo-Nirenberg inequality (Proposition 2) is never obtained. In this case, we pursue scattering below the threshold of \(W_0\) as in (1.7).

Besides what mentioned above, other Dispersive PDEs with inverse square potential have drawn great attention of many authors. See [27], for energy critical nonlinear wave equation with inverse square potential; [17, 25, 40, 41], for energy sub-critical and mass super-critical NLS\(_{a}\); [8], for Hartree equation with inverse square potential. Dispersive PDEs involving inverse square potential also comes from models of Physics, for a quick review see, [5, 18] and the references therein.

The proof of Theorem 1.2 follows from the general outline in [13, 19]. We also incorporate some development of (NLS\(_{a}\)) discussed in [18].

Let \(K > 0\), we define
\[
L(K) = \sup_u \{\|u(t)\|_{S(I)} : \sup_{t \in I} \|u(t)\|_{\dot{H}^1} \leq K\},
\]
where the supremum is taken for all solutions \(u\) to NLS\(_{a}\) with maximal lifespan \(I\).

Define
\[
K_c = \inf \{K : L(K) = \infty\}. \tag{1.8}
\]
From the small data theory of NLS\(_{a}\) (Theorem 2.9), there exists \(\eta_0 > 0\) such that if \(\|u_0\|_{\dot{H}^1} < \eta_0\), then \(u\) exists globally and \(\|u\|_{S(\mathbb{R})} \lesssim \eta_0\). Note that \(L(K)\) is a continuous function with respect to \(K\) from the local theory and stability result. Hence, \(K_c > 0\). For any solution \(u\) to (NLS\(_{a}\)) with \(\sup_{t \in I} \|u(t)\|_{\dot{H}^1} < K_c\), then the maximal lifespan \(I = \mathbb{R}\) and \(\|u\|_{S(\mathbb{R})} < \infty\). In other words, \(L(K) = \infty\) if \(K \geq K_c\); \(L(K) < \infty\) if \(K < K_c\). Hence, Theorem 1.2 is equivalent to
\[
K_c = \|W_{a\wedge 0}\|_{\dot{H}^1_{a\wedge 0}}. \tag{1.9}
\]

The scattering result is achieved by way of contradiction. Assume otherwise \(K_c < \|W_{a\wedge 0}\|_{\dot{H}^1_{a\wedge 0}}\). There exists a sequence of blow-up solutions \(u_n\). At this stage, no further information of \(u_n\) can be extracted directly. We thus decompose their initial data \(u_n(0)\) into linear profiles at the \(\dot{H}^1\) level. From each linear profile, one
can build a corresponding maximal lifespan solution of NLS$_a$ which is denoted as nonlinear profile. Together with the stability result, the blow up solutions guarantee the existence of “bad” profiles: nonlinear profiles which blow up in scattering norm. Then the kinetic energy decoupling (3.11) shows that there exists exactly one nonlinear profile: the “bad” profile. (Here, we also introduce the weighted Sobolev space in [23, 24]. Applying the Duhamel’s formula combined with cut-off functions, we are able to complete two weak convergence lemmas, Lemma 2.14 and 2.15. Those lemmas are crucial in the proof of the kinetic energy decoupling.) Up to scaling symmetry, the “bad” nonlinear profile converges. Next, one can construct the “minimal blowup” solution, see Theorem 3.2. This solution has precompactness property in $\dot{H}^1$ up to scaling symmetry. However, Virial type argument and conservation laws exclude the existence of such “minimal blow-up” solution. This contradiction confirms the scattering result of NLS$_a$ and proves Theorem 1.1.

The rest of the paper is organized as follows: Section 2 starts with reviewing some basic estimates and inequalities adapted to the operator $L_a$ or $e^{-itL_a}$, coercivity of energy and energy trapping. In Section 2.4, we prove and review the local well-posedness, stability, and local theory of NLS$_a$. Section 2.5 reviews the linear profile decomposition for the linear flow $e^{-itL_a}$ associated to bounded radial $\dot{H}^1$ functions. Section 3 demonstrates that the failure of Theorem 1.1 guarantees the existence of the minimal blow-up solution(Theorem 3.2). Lastly, in Section 4, we prove the Rigidity theorem showing that the minimal blow-up solution does not exist. This completes the proof of Theorem 1.2.

2. Preliminaries.

2.1. Notation.

| $L_a = -\Delta + \frac{a}{|x|^2}$ | $2* = \frac{2d}{d-2}$ |
|---|---|
| $X \lesssim Y$ or $X = O(Y)$ if $X \leq CY$ for some constant $C > 0$ | $X \lesssim_Z Y$ if $X \leq C(Z)Y$ for some constant $C(Z) > 0$ depending on $Z$ |
| $A \land B = \min\{A, B\}$ | $\langle x \rangle = \sqrt{1 + |x|^2}$ |
| $\sigma = \frac{d-2}{2} - \sqrt{\left(\frac{d-2}{2}\right)^2 + a}$ | $\beta = \sqrt{1 + \frac{4a}{(d-2)^2}}$ |
| $\|f\|_p = \|f\|_{L^p_c(\mathbb{R}^d)}$ | $\|f\|_{q, r(I)} = \|f\|_{L^q_r(I \times \mathbb{R}^d)}$ |
| $\|\phi\|_{B^1} = \|\sqrt{L_a}f\|_2$ | $u(t, x)|_{[\lambda]} = \lambda^{\frac{d-2}{2}} u(t, \frac{x}{\lambda})$ |
| $\dot{X}^1(I) = L_i^{\frac{2(d+2)}{d+4}} \dot{H}^{\frac{1}{2}, \frac{2d(d+2)}{d^2+d+4}}(I \times \mathbb{R}^d)$ | $\dot{X}^0(I) = L_i^{\frac{2(d+2)}{d+4}} L_x^{\frac{2d(d+2)}{d^2+d+4}}(I \times \mathbb{R}^d)$ |
| $S^0(I) = L_i^2L_x^2 \cap L_i^\infty L_x^2(I \times \mathbb{R}^d)$ | $S^1(I) = \{u : I \times \mathbb{R}^d : \nabla u \in S^0(I)\}$ |
| $S(I) = L_i^{\frac{4}{d-2}}(I \times \mathbb{R}^d)$ | $N^0(I) := L_i^2 L_x^{\frac{2d}{d-2}} + L_i^1 L_x^2(I \times \mathbb{R}^d)$ |

2.2. Preliminary estimates.

**Lemma 2.1** (Equivalence of Sobolev norms, [16]). Fix $d \geq 3$, $a \geq -(\frac{d-2}{2})^2$, and $0 < s < 2$. If $1 < p < \infty$ satisfies $\frac{d+s}{d} < \frac{1}{p} < \min\{1, \frac{d-s}{d}\}$, then

$$\|(-\Delta)^{\frac{s}{2}} f\|_p \lesssim_{s, d, p, a} \|L_a^{\frac{s}{2}} f\|_p \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d \setminus \{0\}). \quad (2.1)$$

If $\max\{\frac{s}{2}, \frac{d}{2}\} < \frac{1}{p} < \min\{1, \frac{d-s}{d}\}$, which ensures already that $1 < p < \infty$, then

$$\|L_a^{\frac{s}{2}} f\|_p \lesssim_{s, d, p, a} \|(-\Delta)^{\frac{s}{2}} f\|_p \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d \setminus \{0\}). \quad (2.2)$$
If $a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2$, then
\[
\|\nabla f\|_p \sim \|\sqrt{L_a} f\|_p \quad \text{for all} \quad \frac{2d}{d+2} \leq p \leq \frac{2d(d+2)}{d^2+4}.
\]

**Lemma 2.2** (Fractional product rule, [16]). Fix $a > -\left(\frac{d-2}{2}\right)^2 + \left(\frac{d-2}{d+2}\right)^2$. Then for all $f, g \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ we have
\[
\|\sqrt{L_a} (fg)\|_p \lesssim \|\sqrt{L_a} f\|_{p_1} \|g\|_{p_2} + \|f\|_{q_1} \|\sqrt{L_a} g\|_{q_2},
\]
for any exponents satisfying $\frac{2d}{d+2} \leq p, p_1, p_2, q_1, q_2 \leq \frac{2d(d+2)}{d^2+4}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$. Recall the Littlewood–Paley projections, Bernstein estimates and square function estimate introduced in [16]. Let
\[
I_{\leq N} = P_{\leq N} f = \phi_N(\sqrt{L_a}), \quad I_N = P_N f = \psi_N(\sqrt{L_a}).
\]
where $\phi_N(r) = \phi(r/N)$, $\psi_N(r) = \phi_N(r) - \phi_N(2r)$ and $\phi$ is a smooth cutoff function such that $\phi(r) = 1$ for $0 \leq r \leq 1$ and $\phi(r) = 0$ for $r \geq 2$.

**Lemma 2.3** (Bernstein estimates, [16]). For $1 < p \leq q \leq \infty$ when $a \geq 0$ or $r_0 < p \leq q < r_0' = \frac{d}{q}$ when $-(\frac{d-2}{2})^2 \leq a < 0$, the following hold:
(1) The operators $P_{\leq N}$ and $P_N$, are bounded on $L^p$.
(2) The operators $P_{\leq N}$ and $P_N$, map $L^p$ to $L^q$ with norm $O(N^{\frac{d}{2} - \frac{d}{q}})$.
(3) For any $s \in \mathbb{R}$, $N^s \|P_N f\|_p \sim \|(C_s) \frac{P}{p} f\|_p$.

**Lemma 2.4** (Square function, [16]). Fix $0 \leq s < 2$. For $1 < p < \infty$ when $a \geq 0$ or $r_0 < p < r_0' := \frac{d}{q}$ when $-(\frac{d-2}{2})^2 \leq a < 0$ and any $f \in C_c(\mathbb{R}^d \setminus \{0\})$,
\[
\|\left(\sum_{N \in \mathbb{Z}^d} N^{2s} |P_N f|^{\frac{2s}{s}}\right)^{\frac{s}{2}} \|_p \sim \|\mathcal{L}_{\frac{a}{2}} f\|_p.
\]

Strichartz estimates for the linear flow(propagator) $e^{-it\mathcal{L}_{\alpha}}$ were first proved by Burq, Planchon, Stalker, and Tahvildar-Zadeh in [5].

**Theorem 2.5** (Strichartz estimates, [5, 39]). Fix $a > -\left(\frac{d-2}{2}\right)^2$. The solution $u$ to
\[
i\partial_t u - \mathcal{L}_a u = F
\]
on an interval $I \ni t_0$ obeys
\[
\|u\|_{L^{q,r}(I)} \lesssim \|u(t_0)\|_2 + \|F\|_{\dot{L}^{\frac{d}{q},\frac{d}{r}}(I)},
\]
whenever $\frac{2}{q} + \frac{d}{r} = \frac{2}{q} + \frac{4}{d} = \frac{d}{2}$, $2 \leq q, \tilde{q} \leq \infty$.

Note that the double endpoint Strichartz estimate is not available for NLS$_a$ in [5]. Later, Zhang and Zheng [39] confirmed the double endpoint Strichartz estimate.

We record one local smoothing result for the linear propagator $e^{-it\mathcal{L}_{\alpha}}$.

**Lemma 2.6** (Local smoothing, [18]). Fix $a > -\left(\frac{d-2}{2}\right)^2$ and let $w = e^{-it\mathcal{L}_{\alpha}} w_0$. Then
\[
\left\| \int_{\mathbb{R}^d} \frac{\left|\nabla w(t,x)\right|^2}{R(R^{1-\frac{1}{2}})^{\frac{d}{2}}} \, dx \, dt \right\|_{L^q} \lesssim \|w_0\|_2 \|\nabla w_0\|_2 + R^{-1} \|w_0\|_2^2,
\]
\[
\left\| \int_{|x-z| \leq R} \frac{1}{R} \left|\nabla w(t,x)\right|^2 \, dx \, dt \right\|_{L^q} \lesssim \|w_0\|_2 \|\nabla w_0\|_2 + R^{-1} \|w_0\|_2^2,
\]
uniformly for $z \in \mathbb{R}^d$ and $R > 0$. 
Consequently, we have the following estimate of the linear flow $e^{-it\mathcal{L}_a}w_0$ on compact domains.

**Corollary 1** ([35]). Denote $B_{T,R} = \{|t-\tau| \leq T, |x-z| \leq R\}$. Fix $a > -(\frac{d-2}{2})^2 + (\frac{d-2}{4})^2$ and let $w_0 \in \dot{H}^1_a(\mathbb{R}^d)$, where $3 \leq d \leq 6$. Then

$$\|\nabla e^{-it\mathcal{L}_a}w_0\|_{\frac{d+2}{2} - \frac{d(d+2)}{4(d-2)}(B_{T,R})} \leq \frac{T^{(d-2)^2}}{R^{(d+2)^2 - (d+2)^2}} \|e^{-it\mathcal{L}_a}w_0\|_{\frac{d+2}{2} S(B(R))} \|w_0\|_{\dot{H}^1_a}$$

uniformly in $w_0$ and the parameters $R, T > 0$, $\tau \in \mathbb{R}$, and $z \in \mathbb{R}^d$.

**Proof.** We first estimate the low frequency. By equivalent of Sobolev norms $(\frac{2d}{d+2} \leq \frac{2d}{d+2} \leq \frac{2d}{d+2})$ when $3 \leq d \leq 6$, the inequality fails if $d > 6$), Hölder inequality and Bernstein inequality,

$$\|\nabla e^{-it\mathcal{L}_a}P^a_\leq_N w_0\|_{\frac{d+2}{2} - \frac{d(d+2)}{4(d-2)}(B_{T,R})} \leq N^{\frac{d-2}{2}} T^{\frac{d^2+4}{d+2}} \|e^{-it\mathcal{L}_a}P^a_\leq_N w_0\|_{S(B(R))} \|w_0\|_{\dot{H}^1_a}$$

For the high frequency, from Hölder inequality and Strichartz estimate, we get

$$\|\nabla e^{-it\mathcal{L}_a}P^a_{> N} w_0\|_{\frac{d+2}{2} - \frac{d(d+2)}{4(d-2)}(B_{T,R})} \leq R^{\frac{d-2}{2}} \|\nabla e^{-it\mathcal{L}_a}P^a_{> N} w_0\|_{\frac{d^2+4}{d+2}(B_{T,R})} \|e^{-it\mathcal{L}_a}P^1_{> N} w_0\|_{\frac{6-d}{2} N^4(B_{T,R})}$$

$$\leq R^{\frac{d-2}{2}} \|\nabla e^{-it\mathcal{L}_a}P^a_{> N} w_0\|_{\frac{d^2+4}{d+2}(B_{T,R})} \|w_0\|_{\dot{H}^1_a}.$$ 

Lemma 2.6 and Bernstein inequality yield that

$$\|\nabla e^{-it\mathcal{L}_a}P^a_{> N} w_0\|_{2,2(B_{T,R})} \leq R \|P^a_{> N} w_0\|_{L^2_x} \|\nabla P^a_{> N} w_0\|_{L^2_x} + \|P^a_{> N} w_0\|_{L^2_x}$$

$$\leq (RN^{-1} + N^{-2})\|w_0\|_{\dot{H}^1_a}^2.$$ 

Hence,

$$\|\nabla e^{-it\mathcal{L}_a}P^a_{> N} w_0\|_{\frac{d+2}{2} - \frac{d(d+2)}{4(d-2)}(B_{T,R})} \leq R^{\frac{d-2}{2}} (RN^{-1} + N^{-2}) \|w_0\|_{\dot{H}^1_a}$$

$$\leq (R^{\frac{3d-6}{8}} N^{\frac{2-d}{8}} + R^{\frac{d-2}{2}} N^{\frac{2-d}{2}})|w_0|_{\dot{H}^1_a}.$$ 

The conclusion follows from optimizing the choice of $N$. 

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2.3. **Coercivity of energy and energy trapping.** The sharp Sobolev constant $C_{GN}$ of the embedding $\|f\|_{2^*} \leq C_{GN} \|f\|_{\dot{H}^1}$ was achieved in [18] when $a \neq 0$. When $a = 0$, this was obtained by Aubin [1] and Talenti [30].

**Proposition 2** (Sharp Gagliardo-Nirenberg inequality, [1, 18, 30]). Fix $d \geq 3$ and $a > -(\frac{d-2}{2})^2$.

(i) If $-(\frac{d-2}{2})^2 < a < 0$, then

$$\|f\|_{2^*} \leq \|W_a\|_{2^*} \|W_a\|_{\dot{H}^1_a}^{-1} \|f\|_{\dot{H}^1}.$$  \hspace{1cm} (2.6)

Moreover, equality holds if and only if $f(x) = \alpha W_a(\lambda x)$ for some $\alpha \in \mathbb{C}$ and some $\lambda > 0$.

(ii) The inequality (2.6) is valid also when $a = 0$; however, equality now holds if and only if $f(x) = \alpha W_0(\lambda x + y)$ for some $\alpha \in \mathbb{C}$, some $y \in \mathbb{R}^d$, and some $\lambda > 0$. 

Lemma 2.7

Introduced by Kenig-Merle [13] for the free NLS. These properties are applications of the threshold of the ground state solution. The energy trapping argument was first introduced by Kenig-Merle [13] for the free NLS. These properties are applications of the Sharp Gagliardo-Nirenberg inequality.

2.4. (Energy trapping)

Stability result of NLS by the Sharp Gagliardo-Nirenberg inequality

Proof. We clearly have $E_a(f) \leq \frac{1}{2} L_p^{2_+} \|f\|_{H^1_{x,t}}^2$ by definition. It remains to show $(\frac{1}{2} - \frac{1}{2}) \|f\|_{H^1_{x,t}}^2 \leq E_a(f)$. Recall that $\|W_{a^d}^\perp\|^2_{H^1_{x,t}} = \|W_{a^d}^\perp\|^2_{H^1_{x,t}}$ and $\|f\|_{H^1_{x,t}} \leq \|W_{a^d}^\perp\|^2_{H^1_{x,t}}$, by the Sharp Gagliardo-Nirenberg inequality $\|f\|_{2^+} \leq (\frac{\|W_{a^d}^\perp\|^2_{H^1_{x,t}}}{\|f\|_{2^+}}) \|f\|_{H^1_{x,t}}$, we get

\[
E_a(f) \geq \|f\|_{H^1_{x,t}}^2 \left( \frac{1}{2} - \frac{1}{2} \frac{\|W_{a^d}^\perp\|^2_{H^1_{x,t}}}{\|f\|_{2^+}} \right)^{2^*} \|f\|_{2^+}^{2^*-2} \geq \left( \frac{1}{2} - \frac{1}{2} \right) \|f\|_{H^1_{x,t}}^2.
\]

Hence, (2.8) holds.

Lemma 2.8 (Energy trapping). Let $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a solution of NLS with initial data $u_0$. If $\sup_{t \in I} \|u(t)\|_{H^1_{x,t}}^2 \leq (1 - \delta_0) \|W_{a^d}^\perp\|^2_{H^1_{x,t}}$ for some $1 > \delta_0 > 0$, then there exists $\delta_1 > 0$ (depending on $\delta_0$ and $d$) such that for any $t \in I$,

\[
\begin{cases}
\left( \frac{1}{2} - \frac{(1-\delta_0)^2}{2^*} \right) \|u(t)\|_{H^1_{x,t}}^2 \leq E_a(u(t)) \leq \frac{1}{2} \|u(t)\|_{H^1_{x,t}}^2, \\
\|u(t)\|_{H^1_{x,t}} \sim E_a(u(t)) \sim \|u_0\|_{H^1_{x,t}}, \\
\delta_1 \|u(t)\|_{H^1_{x,t}}^2 \leq \|u(t)\|_{2^*}^2 - \|u(t)\|_{2^*}^2.
\end{cases}
\]

Proof. From the sharp Gagliardo-Nirenberg inequality, we have

\[
E_a(u(t)) \geq \|u(t)\|_{H^1_{x,t}}^2 \left( \frac{1}{2} - \frac{1}{2^*} \right) \|W_{a^d}^\perp\|^2_{H^1_{x,t}} \|u(t)\|_{2}^{2^*-2} \geq \left( \frac{1}{2} - \frac{(1-\delta_0)^2}{2^*} \right) \|u(t)\|_{H^1_{x,t}}^2,
\]

this and energy conservation yield the result.

2.4. Local well-posedness and stability of NLS. In $\mathbb{R}^3$, the local theory and stability result of NLS has been proved in [18]. The higher dimensional case ($4 \leq d \leq 6$) was considered in [35].
Theorem 2.9 ([35]). Fix $a > -(\frac{d-2}{2})^2 + \frac{(d-2)^2}{4(d+2)}$. Given $A \geq 0$, there exists $\eta = \eta(A)$ so that the following holds: Suppose $u_0 \in H^1_a$ obeys
\[ \| \sqrt{\mathcal{L}_a} u_0 \| \leq A \quad \text{and} \quad \| e^{it\mathcal{L}_a} u_0 \|_{S(I)} \leq \eta \]
for some time interval $I \ni 0$. Then there is a unique solution $u$ to NLS$_a$ on the time interval $I$ such that
\[ \| \sqrt{\mathcal{L}_a} u \|_{C_t L_t^2 \cap X^0(I)} \leq A \quad \text{and} \quad \| u \|_{S(I)} \lesssim \eta. \]

Theorem 2.10 (Stability, [18, 35]). Fix $a > -(\frac{d-2}{2})^2 + \frac{(d-2)^2}{4(d+2)}$. Let $I$ be a compact time interval and let $\tilde{u}$ be an approximate solution to NLS$_a$ on $I \times \mathbb{R}^d$ in the sense that
\[ (i\partial_t - \mathcal{L}_a) \tilde{u} = -|\tilde{u}|^{\frac{4}{d-2}} \tilde{u} + e \]
for some function $e$. For some positive constants $E$ and $L$, assume that
\[ \| \tilde{u} \|_{L_t^\infty H_x^1(I \times \mathbb{R}^d)} \leq E, \quad \| \tilde{u} \|_{S(I)} \leq L. \]

Let $t_0 \in I$ and let $u_0 \in H^1_a$. Assume that
\[ \| u_0 - \tilde{u}(t_0) \|_{H^1_a} + \| \sqrt{\mathcal{L}_a} e \|_{N^0(I)} \leq \varepsilon \tag{2.10} \]
for some $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(E, L)$. Then, there exists a unique solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to NLS$_a$ with initial data $u_0$ at time $t = t_0$ satisfying
\[ \begin{cases} \| \sqrt{\mathcal{L}_a} (u - \tilde{u}) \|_{S^0(I)} \leq C(E, L) \varepsilon \\ \| \sqrt{\mathcal{L}_a} u \|_{S^0(I)} \leq C(E, L) \end{cases} \tag{2.11} \]

We recall the local theory of NLS$_a$. The proof is similar to the free NLS by the Strichartz estimate of NLS$_a$ and equivalence of Sobolev norms.

Theorem 2.11 (Local theory, [6, 13, 18]). Given $u_0 \in H^1_a$ and $t_0 \in \mathbb{R}$, there exists a unique maximal life-span solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to NLS$_a$ with initial condition $u(t_0) = u_0$. In addition, we have:

- Local existence: $I$ is an open interval such that $t_0 \in I$.
- Scattering: If $u$ is a solution of NLS$_a$ on $I \times \mathbb{R}^d$ with $|0, \infty) \subseteq I$ and $\| u \|_{S(I)} < \infty$, then there exists $u(t)$ scatters forward in time. Similar result holds when $(-\infty, 0] \subseteq I$.
- Small data theory: There exists $\eta_0 > 0$ such that if $\| u_0 \|_{H^1_a} \leq \eta_0$, then the solution $u$ to NLS$_a$ is global in time with $\| u \|_{S(\mathbb{R})} \lesssim \| u_0 \|_{H^1_a}$ and scatters in both time directions.
- Large data theory: For large initial data $u_0$, there exists an open interval $I_0 \ni 0$ such that the hypothesis of local well-posedness is satisfied i.e.,
\[ \| e^{-it\mathcal{L}_a} u_0 \|_{S(t_0)} \leq \eta. \]
- Blow-up criterion: If $\sup I < \infty$, then $\| u \|_{S(I)} = \infty$. Also, a corresponding result holds for $\inf I > -\infty$.
- Uniqueness: If $\tilde{u} \in C_t H^1(I_1)$ solves NLS$_a$ with $\tilde{u}(t_0) = u$, then $I_1 \subset I$ and $\tilde{u}(t) = u(t)$ for all $t \in I_1$.
- Continuity of the flow: If $\tilde{u}$ solves NLS$_a$ on $I$ with initial data $\tilde{u}_0$ and $\sup_{t \in I} \| \tilde{u}(t) \|_{H^1_a} + \| u \|_{S(I)} \leq A$ for some $A > 0$, then $\exists \varepsilon_0(A), C(A) > 0$ such that for any $u \in H^1_a$ with $\| \tilde{u}_0 - u_0 \|_{H^1_a} = \varepsilon < \varepsilon_0$, the solution $u$ of NLS$_a$ with initial data $u_0$ is defined on $I$ and satisfies $\| u \|_{S(I)} \leq C$ and $\sup_{t \in I} \| \tilde{u}(t) - u(t) \|_{H^1_a} \leq C\varepsilon$. 


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2.5. Linear profile decomposition. In this subsection, we recall the linear profile decomposition for $e^{-it\Delta_a}$ associated to a sequence of bounded radial functions in $H_a^1$. In $\mathbb{R}^3$, the corresponding result in the non-radial case has been proved in [18]. From the refined Strichartz estimate (Lemma 2.13) for all $\mathbb{R}^d (d \geq 3)$, it is laborious to generalize their result to higher dimensions.

The linear profile decomposition gives us the insight that bounded functions in $H_a^1$ can be decomposed into summation of linear profiles and a tail term. The linear profiles have decoupling properties, the tail term decays in certain Strichartz space norm, and different symmetries in the linear profiles are orthogonal. We will see the application in Section 3 proving the existence of minimal blow-up solution. This treatment was first introduced by Bahouri and Gérard [2] for energy critical NLW and Keraani [15] in the setting of NLS. Variants of linear profile decomposition for the dispersive PDEs were discussed in [18, 20, 21, 22, 27, 36, 37].

We only use the linear profile decomposition for radial functions in $H_a^1$. Compared with the non-radial case, the major difference is the loss of the translation symmetry in the profile.

**Theorem 2.12** (Radial $H_a^1$ linear profile decomposition, [18]). Let $\{f_n\}$ be a bounded sequence in $H_a^1$. After passing to a subsequence, there exist $J^* \in \{0, 1, 2, \ldots, \infty\}$, non-zero profiles $\{\phi^j_n\}_{j=1}^{J^*}$, $\{\lambda^j_n\}_{j=1}^{J^*} \subset (0, \infty)$, and $\{t^j_n\}_{j=1}^{J^*} \subset \mathbb{R}$ such that for each finite $0 \leq J \leq J^*$, we have the decomposition

$$f_n = \sum_{j=1}^{J} \phi^j_n + w^j_n$$

with $\phi^j_n = (e^{-it^j_n\Delta_a} \phi^j)_{\lambda^j_n}$ and $w^j_n \in \dot{H}_a^1$ (2.12)

satisfying

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|e^{-it^j_n\Delta_a}w^j_n\|_{S(\mathbb{R})} = 0,$$

$$\lim_{n \to \infty} \{\|f_n\|^2_{H_a^1} - \sum_{j=1}^{J} \|\phi^j_n\|^2_{H_a^1} - \|w^j_n\|^2_{H_a^1}\} = 0,$$

$$\lim_{n \to \infty} \{\|f_n\|^2_{2^*} - \sum_{j=1}^{J} \|\phi^j_n\|^2_{2^*} - \|w^j_n\|^2_{2^*}\} = 0.$$ (2.15)

Moreover, for all $j \neq k$ we have the asymptotic orthogonal property

$$\frac{\lambda^j_n}{\lambda^k_n} + \frac{|\lambda^k_n|^2}{\lambda^j_n} + \frac{|t^j_n(\lambda^j_n)^2 - t^k_n(\lambda^k_n)^2|}{\lambda^j_n\lambda^k_n} \to \infty \text{ as } n \to \infty.$$ (2.16)

In addition, we may assume that for each $j$ either $t^j_n \equiv 0$ or $t^j_n \to \pm \infty$.

From the Refined Strichartz Estimate, it is routine to obtain an inverse Strichartz inequality which generates the linear profile decomposition by iteration. We refer the reader to [18, 20] for details.

**Lemma 2.13** (Refined Strichartz Estimate). Let $f \in \dot{H}_a^1$ and $a > -(d-2)^2 + (d-2)^2$, then for all $3 \leq d \leq 6$,

$$\|e^{-it\Delta_a} f\|_{S(I)} \lesssim (\sup_{N \in \mathbb{Z}^2} \|e^{-it\Delta_a} f_N\|_{S(I)})^{\frac{4}{d-2}} \|f\|_{\dot{H}_a^1}^{\frac{d-2}{d-2}}.$$ (2.17)

**Proof.** When $d = 3$, one can refer to [18]. When $d \geq 4$, one can check [35].
2.6. **Weak convergence lemma.** The decoupling of Kinetic energy (3.11) requires the following weak convergence lemmas (Lemma 2.14 and 2.15). We remind the reader that similar results for the free NLS were discussed by Kenig-Merle in [14] for $\dot{H}^{\frac{2}{d}}$ critical NLS and Merle-Vega [26] for mass critical NLS.

Recall the weighted Sobolev space:

$$\| f \|_{H^{2,2}} = \sum_{j=0}^{2} \| (x)^j \nabla^{2-j} f \|_2.$$  

And the following inequality holds:

$$\| e^{it\Delta} f \|_{L^{\frac{2(d+2)}{d+6}}} \lesssim \| e^{it\Delta} f \|_{H^{2,2}} \lesssim (1 + |t|^2) \| f \|_{H^{2,2}},$$  

where in the last inequality we used $\| e^{it\Delta} g \|_{H^{2,2}} \lesssim (1 + |t|^2) \| g \|_{H^{2,2}}$, which follows from the energy estimate type argument, see [23, 24].

**Lemma 2.14.** Under the assumption in Theorem 2.12. For a bounded sequence $\{f_n\} \subset H^1$, $e^{-it\mathcal{L}_a} f_n \rightharpoonup 0$ in $S(\mathbb{R})$, then $f_n \rightharpoonup 0$ weakly in $\dot{H}^1$.

**Proof.** Since $f_n$ is bounded in $\dot{H}^1$, $f_n \rightharpoonup f_0$ weakly in $\dot{H}^1$ up to a sub-sequence. We proceed with showing $f_n \rightharpoonup 0$ weakly in $L^2$ first, and $f_0 = 0$ in the end. Indeed, choose $\phi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$, then $1_{\{0 \leq |x| \leq 1\}} e^{-it\mathcal{L}_a} \phi \equiv 0$ and

$$0 = \lim_{n \to \infty} \langle e^{-it\mathcal{L}_a} f_n, 1_{\{0 \leq |x| \leq 1\}} e^{-it\mathcal{L}_a} \phi \rangle = \lim_{n \to \infty} \langle f_n, \phi \rangle,$$

where the first convergence follows from $e^{-it\mathcal{L}_a} f_n \rightharpoonup 0$ in $S(\mathbb{R})$. Thus, $f_n \rightharpoonup 0$ weakly in $L^2$. As $f_n \rightharpoonup f_0$ weakly in $\dot{H}^1$, for any given $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\langle f_0, \varphi \rangle_{\dot{H}^1} = \lim_{n \to \infty} \langle f_n, \varphi \rangle_{\dot{H}^1} = \lim_{n \to \infty} \langle f_n, -\Delta \varphi \rangle = 0.$$

From density argument and uniqueness of the weak limit, $f_0 = 0$.

It remains to show $e^{-it\mathcal{L}_a} \phi$ is uniformly bounded in $S([0, 1])$.

Let $\chi_0$ be a smooth cutoff function supported on $|x| \leq 2$ and $\chi_0(x)$ equals 1 when $|x| \leq 1$. We denote $\chi = 1 - \chi_0$. Hölder and Strichartz inequality yield

$$\| e^{-it\mathcal{L}_a} \phi \|_{S([0, 1])} \lesssim \| \chi_0 e^{-it\mathcal{L}_a} \phi \|_{L^\infty_t L^2_x ([0, 1])} + \| \chi e^{-it\mathcal{L}_a} \phi \|_{L^2_t L^{\frac{2(d+2)}{d+6}}_x ([0, 1])} \lesssim 1 + \| \chi e^{-it\mathcal{L}_a} \phi \|_{L^2_t L^{\frac{2(d+2)}{d+6}}_x ([0, 1])}.$$  

Denote $U = e^{-it\mathcal{L}_a} \phi$, then $\chi U$ solves the following equaiton

$$\begin{cases}
i \partial_t (\chi U) + \Delta (\chi U) = \frac{a}{|x|^2} \chi U + 2\nabla \chi \nabla U + \Delta \chi U \\
\chi U |_{t=0} = \chi \phi
\end{cases}$$

The Duhamel’s formula yields

$$\chi U = e^{it\Delta} [\chi \phi] - i \int_0^t e^{i(t-s)\Delta} \left[ \frac{a}{|x|^2} \chi U + 2\nabla \chi \nabla U + \Delta \chi U \right] ds.$$  

From Minkowski inequality and (2.18),

$$\| \chi U \|_{L^\infty_t L^{\frac{2(d+2)}{d+6}}_x ([0, 1])} \lesssim e^{it\Delta} [\chi \phi] \|_{L^\infty_t L^{\frac{2(d+2)}{d+6}}_x ([0, 1])} + \| e^{it\Delta} [\frac{a}{|x|^2} \chi U + 2\nabla \chi \nabla U + \Delta \chi U] \|_{L^\infty_t L^{\frac{2(d+2)}{d+6}}_x ([0, 1])}.$$
Indeed,
\[ \| e^{it\Delta} [\chi \phi] \|_{L_t^\infty L_x^{2(d+2)/(d-2)}} \lesssim \| e^{it\Delta} [\chi \phi] \|_{L_t^\infty H^{2,2}([0,1])} \lesssim \| \chi \phi \|_{L_t^\infty H^{2,2}([0,1])}. \]
By (2.18), Strichartz estimate and equivalence of Sobolev norms,
\[ \| e^{it\Delta} aU \|_{L_t^\infty L_x^{2(d+2)/(d-2)}} \lesssim \| \chi U \|_{L_t^\infty H^{2,2}([0,1])} \lesssim \| U \|_{L_t^\infty L_x^2((0,1))} + \| \nabla U \|_{L_t^\infty L_x^2((0,1))} + \| \chi \Delta U \|_{L_t^\infty L_x^2((0,1))} \]
\[ \lesssim \| \phi \|_2 + \| \sqrt{\mathcal{L}_a} U \|_{L_t^\infty L_x^2((0,1))} + \| \chi \mathcal{L}_a U \|_{L_t^\infty L_x^2((0,1))} + \| \chi U \|_{L_t^\infty L_x^2((0,1))} \]
\[ \lesssim \| \phi \|_2 + \| \sqrt{\mathcal{L}_a} \phi \|_2 + \| \mathcal{L}_a \phi \|_2. \]

Then,
\[ \| e^{it\Delta} [\nabla \chi \nabla U] \|_{L_t^\infty L_x^{2(d+2)/(d-2)}} \lesssim \| \nabla \chi \nabla U \|_{L_t^\infty H^{2,2}([0,1])} \lesssim \| \nabla \chi \nabla U \|_{L_t^\infty L_x^{2(d+2)/(d-2)}} \]
\[ \lesssim \| \nabla U \|_{L_t^\infty L_x^2((0,1))} + \| \nabla \chi \Delta U \|_{L_t^\infty L_x^2((0,1))} + \| \nabla \chi \partial_t U \|_{L_t^\infty L_x^2((0,1))} \]
\[ \lesssim \| \phi \|_2 + \| \sqrt{\mathcal{L}_a} \phi \|_2 + \| \mathcal{L}_a \phi \|_2 + \| (\mathcal{L}_a)^{\frac{1}{2}} \phi \|_2. \]

Indeed, form equivalence of Sobolev norms,
\[ \| \nabla \chi \partial_t^2 U \|_{L_t^\infty L_x^2((0,1))} \lesssim \| \nabla \chi \partial_t(\mathcal{L}_a U) \|_{L_t^\infty L_x^2((0,1))} + \| \nabla \chi \partial_t(\frac{U}{|x|^2}) \|_{L_t^\infty L_x^2((0,1))} \]
\[ \lesssim \| (\mathcal{L}_a)^{\frac{1}{2}} \phi \|_2 + \| \sqrt{\mathcal{L}_a} \phi \|_2 + \| \phi \|_2. \]

Similarly,
\[ \| e^{it\Delta} [\Delta \chi U] \|_{L_t^\infty L_x^{2(d+2)/(d-2)}} \lesssim \| \phi \|_2 + \| \sqrt{\mathcal{L}_a} \phi \|_2 + \| \mathcal{L}_a \phi \|_2. \]

Thus, \( \| e^{-it\mathcal{L}_a} \phi \|_{L_t^\infty L_x^{2(d+2)/(d-2)}} \) is uniformly bounded and so is \( \| e^{-it\mathcal{L}_a} \phi \|_{S((0,1)^d)} \).

Lemma 2.15. Under the assumption in Theorem 2.12. For \( 1 \leq j < J \), the sequence \( e^{it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]} \rightarrow 0 \) weakly in \( \dot{H}^1 \) as \( n \rightarrow \infty \).

Proof. Given \( L \geq 1 \) and \( 1 \leq j < J \), note that the sequence \( e^{it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]} \) is bounded in \( \dot{H}^1 \). Thus, passing to a sub-sequence, \( e^{it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]} \rightarrow h \) weakly in \( \dot{H}^1 \) and it remains to show \( h \equiv 0 \). From equivalence of Sobolev norms, applying (2.12) on \( w_n^L \) yields
\[ \| h \|_{\dot{H}^1}^2 \sim \lim_{n \rightarrow \infty} \left< \sqrt{\mathcal{L}_a} e^{-it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]}, \sqrt{\mathcal{L}_a} h \right> \]
\[ = \sum_{m=L+1}^J \lim_{n \rightarrow \infty} \left< \sqrt{\mathcal{L}_a} e^{-it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]}, \sqrt{\mathcal{L}_a} e^{-it\mathcal{L}_a} h \right> \]
\[ + \lim_{n \rightarrow \infty} \left< \sqrt{\mathcal{L}_a} e^{it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]}, \sqrt{\mathcal{L}_a} h \right> \]
\[ = \sum_{m=L+1}^J \lim_{n \rightarrow \infty} \left< \sqrt{\mathcal{L}_a} e^{-it\mathcal{L}_a}((w_n^j)_{[(\lambda_n^j)^{-1}]}, \sqrt{\mathcal{L}_a} e^{-it\mathcal{L}_a} h \right> \]
Thus, for $J^3$. Existence of minimal blow-up solutions.

(2.19)

From (2.16) and changing of variables (for a complete justification, one can refer to the proof of (3.14)), for $J \geq m \geq L + 1 > j$,

$$
\lim_{n \to \infty} \left( \sqrt{\mathcal{L}_a e^{-it\mathcal{L}_a}} \langle (w_n^j)_{|\lambda_n^j|^{-1}}, \sqrt{\mathcal{L}_a} h \rangle \right) = 0.
$$

From changing of variable on time and (2.13),

$$
\lim_{J \to \infty} \lim_{n \to \infty} \| e^{-it\mathcal{L}_a} e^{-it\mathcal{L}_a} \langle (w_n^j)_{|\lambda_n^j|^{-1}} \rangle \|_{S(\mathbb{R})} = \lim_{J \to \infty} \lim_{n \to \infty} \| e^{-it\mathcal{L}_a} w_n^j \|_{S(\mathbb{R})} = 0.
$$

Thus, $e^{-it\mathcal{L}_a} \langle (w_n^j)_{|\lambda_n^j|^{-1}} \rangle \to 0$ weakly in $\dot{H}^1$ when $n, J \to \infty$ by Lemma 2.14. Thus, for $J$ sufficiently large,

$$
\left| \left\langle \sqrt{\mathcal{L}_a e^{it\mathcal{L}_a}} \langle (w_n^j)_{|\lambda_n^j|^{-1}}, \sqrt{\mathcal{L}_a} h \rangle \right\rangle \right| = o_n(1) \| h \|_{\dot{H}^1}^2.
$$

This and (2.19) verify that $h \equiv 0$ for $J$ large and the proof is complete. 

3. Existence of minimal blow-up solutions.

**Lemma 3.1 (Palais-Smale Condition).** Let $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of solutions to NLS$_a$ and $t_n \in I_n$. Suppose that

$$
\begin{align*}
\lim_{n \to \infty} \sup_{t \in I_n} \| u_n(t) \|_{\dot{H}^1_a}^2 &= K_a < \| W_{a^\perp} \|_{\dot{H}^1_{a^\perp}}^2, \\
\lim_{n \to \infty} \| u_n \|_{S(t=0)} &= \lim_{n \to \infty} \| u_n \|_{S(t=t_n)} = \infty
\end{align*}
$$

then there exists $\{\lambda_n\} \subset \mathbb{R}^+$ such that $\{\langle u_n(t_n) \rangle_{|\lambda_n|} \}$ is precompact in $\dot{H}^1_a$.

**Proof.** By the time translation invariance, we may assume that $t_n \equiv 0$, thus

$$
\lim_{n \to \infty} \| u_n \|_{S(t=0)} = \lim_{n \to \infty} \| u_n \|_{S(t=\infty)} = \infty.
$$

Apply the linear profile decomposition on $u_{n,0} = u_n(0)$, then after passing to a subsequence, there exist $J^* \in \{1, 2, \ldots, \infty\}$, non-zero profiles $\{\phi^j\}_{j=1}^J \subset \dot{H}^1(\mathbb{R}^d)$, $\{\lambda_n^j\}_{j=1}^J \subset (0, \infty)$, and $\{t_n^j\}_{j=1}^{J^*} \subset \mathbb{R}$ such that for each finite $1 \leq J \leq J^*$, we have the decomposition

$$
u_{n,0} = \sum_{j=1}^J \phi_n^j + w_n^j \text{ with } \phi_n^j = \langle e^{-it\mathcal{L}_a} \phi^j \rangle_{|\lambda_n^j|} \text{ and } w_n^j \in \dot{H}^1_a
$$

satisfying

$$
\lim_{J \to J^*} \lim_{n \to \infty} \| e^{-it\mathcal{L}_a} w_n^j \|_{S(\mathbb{R})} = 0,
$$

$$
\lim_{n \to \infty} \left\{ \| u_{n,0} \|_{\dot{H}^1_a}^2 - \sum_{j=1}^J \| \phi_n^j \|_{\dot{H}^1_a}^2 - \| w_n^j \|_{\dot{H}^1_a}^2 \right\} = 0,
$$

When $t_n^j \equiv 0$, we define $v_j : I_j \times \mathbb{R}^d \to \mathbb{C}$ as the maximal life-span $(I_j)$ solution to NLS$_a$ with initial data $\phi^j$; when $t_n^j \to \pm \infty$, define $v_j : I_j \times \mathbb{R}^d \to \mathbb{C}$ as the maximal life-span $(I_j)$ solution to NLS$_a$ which scatters to $e^{-it\mathcal{L}_a} \phi^j$ as $t \to \pm \infty$.

Let

$$
v_n^j = v_j(\langle \lambda_n^j \rangle^{-2} t + t_n^j, x)_{|\lambda_n^j|};
$$

then it is a maximal life-span $(I_n^j = \{ t \in \mathbb{R}, (\lambda_n^j)^{-2} t + t_n^j \in I_j \})$ solution. Above construction yields

$$
\lim_{n \to \infty} \| v_n^j(0) - \phi_n^j \|_{\dot{H}^1_a} = 0.
$$
From (3.5) and small data theory, there exists $J_0 \geq 1$ such that for all $j \geq J_0$, 
\[ \|\phi_j\|_{\dot{H}^1} \leq \eta_0 \] and $v^j_n$ are global in time solutions with 
\[ \|v^j_n\|_{L^\infty_t H^1_x(\mathbb{R})} + \|v^j_n\|_{S(\mathbb{R})} \lesssim \|\phi\|_{\dot{H}^1}. \]

While for $j < J_0$, there exists at least one nonlinear profile such that 
\[ \|v^j_n\|_{L^\infty_t H^1_x(I_n^j)} \geq K_c. \] (3.7)

If not, $\|v^j_n\|_{L^\infty_t H^1_x(I_n^j)} < K_c$ for all $j$, the definition of $K_c$ shows that $v^j_n$ exists globally and $\|v^j_n\|_{S(\mathbb{R})} \lesssim 1$ contradicting the following which we postpone in the end

\[ \text{there exists } j_0 < J_0 \text{ such that } \|v^{j_0}_n\|_{S(I^n_{j_0})} = \infty. \] (3.8)

Next, we follow the argument in [19] to show that only one nonlinear profile$(v^{j_0}_n)$ actually survives. We may re-index such that for all $j \leq J_1 < J_0$, $\lim_{n \to \infty} \|v^j_n\|_{S(I_n^j)} = \infty$; for all $j > J_1$, $\lim_{n \to \infty} \|v^j_n\|_{S(\mathbb{R})} < \infty$. For each $m, n \geq 1$, let $1 \leq j(m, n) \leq J_1$ and define an interval $K_m^n = (0, \tau]$ by
\[ \sup_{1 \leq j \leq J_1} \|v_j^n\|_{S(K_m^n)} = \|v_{j(m, n)}^n\|_{S(K_m^n)} = m. \] (3.9)

Note that $J_1$ is fixed, the pigeonhole principle yields that for infinitely many $m$, $j(m, n) = j_0(1 \leq j_0 \leq J_1)$ for infinitely many $n$. From (3.7),
\[ \lim_{m \to \infty} \lim_{n \to \infty} \|v^n_m\|_{L^\infty_t H^1_x(K_m^n)} \geq K_c. \] (3.10)

From the construction of $K_m^n$ in (3.9), same argument in the proof of (3.8) shows that
\[ \lim_{j \to J_1} \lim_{n \to \infty} \|u^n_j - u^n\|_{L^\infty_t H^1_x(K_m^n)} = 0. \]

Thus, $u^n_j$ defined in (3.15) is a good approximation solution of $u^n_m$ on $K_m^n$. Moreover, on $K_m^n$, the Kinetic energy decoupling for $u^n_m$ holds (we postpone the proof in the end; for the free NLS, see Lemma 3.3 in [19]): for all $J \geq 1$ and $m \geq 1$,
\[ \lim_{n \to \infty} \sup_{t \in K_m^n} \left( \|u^n_j'\|^2_{\dot{H}^1_x} - \sum_{j=1}^J \|v^j_n\|^2_{\dot{H}^1_x} - \|u^n_m\|^2_{\dot{H}^1_x} \right) = 0. \] (3.11)

Thus,
\[ K_c \geq \lim_{n \to \infty} \|u^n_m\|^2_{L^\infty_t H^1_x(K_m^n)} = \lim_{j \to J_1} \lim_{n \to \infty} \left\{ \sup_{t \in K_m^n} \left( \sum_{j=1}^J \|v^j_n\|^2_{\dot{H}^1_x} \right) + \|u^n_m\|^2_{\dot{H}^1_x} \right\}. \]

In view of (3.10), this shows that $J_1 = 1$ and $\|w^1_n\|_{\dot{H}^1_x} \to 0$. Up to a sub-sequence, $t^1_n \to t^* \in (-\infty, \infty, 0)$. If $t^* = \infty$, from Strichartz estimate and Monotone convergence theorem,
\[ \|e^{-it\mathcal{L}_a} u_n,0\|_{S(\mathbb{R} \geq 0)} \lesssim \|e^{-it\mathcal{L}_a} \phi^1\|_{S(\mathbb{R} \geq t^1_n)} + \|w^1_n\|_{\dot{H}^1_x} \to 0. \]

Theorem 2.9 implies that $\lim_{n \to \infty} \|u_n\|_{S(\mathbb{R} \geq 0)} = 0$ which contradicts (3.2), thus $t^* \neq \infty$. Similarly, $t^* \neq -\infty$.

If $t^* = 0$, pick $\lambda_n = (\lambda^1_n)^{-1}$, then $(u_n,0)_{[\lambda_n]} = e^{-it\mathcal{L}_a} \phi^1 + (w^1_n)_{[\lambda_n]}$. As $\|(w^1_n)_{[\lambda_n]}\|_{\dot{H}^1_x} = \|w^1_n\|_{\dot{H}^1_x} \to 0$ and $t^1_n \to 0$, $(u_n,0)_{[\lambda_n]} \to \phi^1$ in $\dot{H}^1_a$. Thus, $\{(u_n,0)_{[\lambda_n]}\}$ is precompact in $\dot{H}^1_a$. This completes the proof of this lemma.
Proof of the existence of the “bad profile” (3.8). Assume it fails, then \( I_n^j = \mathbb{R} \) and \( \|v_n^j\|_{\dot{S}(\mathbb{R})} < \infty \) for all \( j \). Strichartz estimate then yields that \( \|v_n^j\|_{\dot{X}^1(\mathbb{R})} < \infty \).

From density argument, given \( \varepsilon > 0 \), there exists \( \psi_n^j \in C^\infty_c \) such that
\[
\|v_n^j - \psi_n^j\|_{\dot{X}^1(\mathbb{R})} < \varepsilon.
\]

From change of variable in time, we obtain
\[
\|v_n^j(t, x) - \psi_n^j((\lambda_n^j)^{-2}t + t_n^j, x)\|_{\dot{X}^1(\mathbb{R})} < \varepsilon.
\] (3.12)

We list two properties associated with the nonlinear profile \( v_n^j \).

- **Boundedness of \( v_n^j \):**
\[
\|v_n^j\|_{\dot{X}^1(\mathbb{R})} \lesssim \begin{cases} 
\|\phi_n^j\|_{H^1_0}, & \text{if } \|\phi_n^j\|_{H^1_0} \leq \eta_0 \\
1, & \text{if } \|\phi_n^j\|_{H^1_0} > \eta_0
\end{cases}
\] (3.13)

This follows from the small data theory and Strichartz estimate as discussed above.

- **Orthogonality of \( v_n^j \) and \( v_n^k \):**
\[
\|\nabla v_n^j \nabla v_n^k\|_{\frac{d+2}{d+4}} + \|v_n^j v_n^k\|_{\frac{d+2}{d+4}} + \|v_n^j v_n^k\|_{\frac{d+2}{d+4}} = o_n(1).
\] (3.14)

To ease notation, we define \( T_n^j f(t, x) = f((\lambda_n^j)^{-2}t + t_n^j, x) \).

Indeed, by (3.12) and change of variable, we have
\[
\|v_n^j v_n^k\|_{\frac{d+2}{d+4}} \lesssim \|v_n^j\|_{S(\mathbb{R})} \|v_n^k\|_{S(\mathbb{R})} + \|v_n^j\|_{S(\mathbb{R})} \|v_n^k - T_n^j \psi_n^j\|_{S(\mathbb{R})}
\]
\[
+ \|T_n^j \psi_n^j T_n^k \psi_n^k\|_{\frac{d+2}{d+4}}
\]
\[
\lesssim \varepsilon + \|T_n^j \psi_n^j T_n^k \psi_n^k\|_{\frac{d+2}{d+4}}.
\]

In addition, if \( \lambda_n^j(\lambda_n^k)^{-1} + (\lambda_n^k)^{-1}\lambda_n^j \to \infty \), by Hölder inequality (equip \( \psi_n^j, \psi_n^k \) with \( L_{t,x}^{\infty} \) norm), we get
\[
\|T_n^j \psi_n^j T_n^k \psi_n^k\|_{\frac{d+2}{d+4}} \leq \min\{|(T_n^j)^{\frac{1}{2}} \psi_n^j \|_{\frac{d+2}{d+4}}, |(T_n^k)^{\frac{1}{2}} \psi_n^k \|_{\frac{d+2}{d+4}}\}
\]
\[
\lesssim \min\{|(\lambda_n^j)^{\frac{1}{2}} \|_{X^{(d+4)(\lambda_n^j)^{-2}, \frac{d+2}{d+4}}}, |(\lambda_n^k)^{\frac{1}{2}} \|_{X^{(d+4)(\lambda_n^k)^{-2}, \frac{d+2}{d+4}}}\} = o_n(1).
\]

Thus, we may assume \( \lambda_n^j(\lambda_n^k)^{-1} - \lambda_0 \in (0, \infty) \). (2.16) then yields that
\[
|t_n^j(\lambda_n^j)^{-2} - t_n^k(\lambda_n^k)^{-2}| \to \infty.
\]
As \( \lambda_n^j(\lambda_n^k)^{-1} \to \lambda_0 \), \( |t_n^j - (\lambda_n^j)^2(\lambda_n^k)^{-2}t_n^k| \to \infty \). This implies that
\[
|T_n^j \psi_n^j T_n^k \psi_n^k\|_{\frac{d+2}{d+4}} \lesssim \|\psi_n^j((\lambda_n^j)^{-2}t + t_n^j - (\lambda_n^j)^2(\lambda_n^k)^{-2}t_n^k, x)\|_{\frac{d+2}{d+4}} = o_n(1).
\]

Similarly, other terms are \( o_n(1) \).

We define an approximate solution \( u_n^j \) to NLS by
\[
u_n^j = \sum_{j=1}^{J} v_n^j + e^{-it\mathcal{L}_a} w_n^j.
\] (3.15)

We will prove the following three claims, which demonstrates that for large \( n \) and \( J \), \( u_n^j \) is a close approximate solution to NLS with finite \( S(\mathbb{R}) \) norm in the sense of Theorem 2.10. Then \( u_n \) is also a global solution with finite \( S(\mathbb{R}) \) norm which contradicts (3.2) and completes the proof of (3.8). Indeed, the proof of the following
Proof of Claim 1: From the construction of $u_n(0)$ in (3.3) and $u_n^j$, by (3.6), as $n \to \infty$, we have
\[\|u_n^j(0) - u_n(0)\|_{\dot{H}^1} \leq \sum_{j=1}^J \|v_n^j(0) - \phi_n^j\|_{\dot{H}^1} \to 0.\]

Proof of Claim 2: From the linear profile decomposition, we see that it suffices to show
\[\lim_{n \to \infty} \|\sum_{j=1}^J v_n^j\|_{X^1(\mathbb{R})} \lesssim_{K_c, \delta} 1\] uniformly for finite $J \leq J^*$. By (3.13) and (3.14) and equivalence of Sobolev norms, we get
\[\|\sum_{j=1}^J v_n^j\|_{X^1(\mathbb{R})}^2 \lesssim \sum_{j=1}^J \|v_n^j\|_{X^1(\mathbb{R})}^2 = \left\| \sum_{j=1}^J \|\nabla v_n^j\|_{d+2, \frac{d(d+2)}{d+4}}^2 \right\|_{\frac{d(d+2)}{d+4}} \lesssim \sum_{j=1}^J \|v_n^j\|_{X^1(\mathbb{R})}^2 + C(J) \sum_{j \neq k} \|\nabla v_n^j \nabla v_n^k\|_{d+2, \frac{d(d+2)}{d+4}} \lesssim \sum_{j=1}^J \|v_n^j(0)\|_{\dot{H}^1}^2 + 1 + o_n(1) \lesssim \sum_{j=1}^J \|\phi_n^j\|_{\dot{H}^1}^2 + 1 \lesssim_{K_c, \delta} 1.\]

Proof of Claim 3: Define $F(z) = -|z|^\frac{4}{d-2} z$, then
\[(i\partial_t + \mathcal{L}_a)u_n^j - F(u_n^j) = \sum_{j=1}^J F(v_n^j) - F(u_n^j) = \sum_{j=1}^J F(v_n^j) - F\left(\sum_{j=1}^J v_n^j\right) + F\left(\sum_{j=1}^J v_n^j\right) - F(u_n^j).\]

Thus, it suffices to show
\[\lim_{J \to \infty} \lim_{n \to \infty} \left\| \sum_{j=1}^J F(v_n^j) - F\left(\sum_{j=1}^J v_n^j\right) \right\|_{N^1} = 0, \tag{3.17}\]
and
\[\lim_{J \to \infty} \lim_{n \to \infty} \left\| F(u_n^j - e^{-it\mathcal{L}_a} u_n^j) - F(u_n^j) \right\|_{N^1} = 0. \tag{3.18}\]

We first prove (3.17). Note that
\[\left\| \nabla \left( \sum_{j=1}^J F(v_n^j) - F\left(\sum_{j=1}^J v_n^j\right) \right) \right\| \lesssim J \sum_{j \neq k} \|v_n^j\|_{\frac{4}{d-2}} \|\nabla v_n^j\| \tag{3.19}\]
By equivalence of Sobolev norms, Strichartz estimate and (3.14), we get
\[
\left\| \sum_{j=1}^{J} F(u_n^j) - F(\sum_{j=1}^{J} u_n^j) \right\|_{X_1^2} \lesssim \sum_{j \neq k} \left\| \nabla u_n^j \right\|_{H^k}^{\frac{4}{d+2}} \left\| \frac{\partial u_n^j}{\partial x} \right\|_{H^{\frac{d+2}{d+2}}} = o_n(1).
\]
Next, we prove (3.18). Note that
\[
\left\| \nabla \left(F(u_n^j - e^{-itL_n} u_n^{j'}) - F(u_n^j)\right) \right\| \lesssim \left\| \int (u_n^j) \right\|_{H^{\frac{4}{d+2}}} + \left\| e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^j \right\|
\]
\[
+ \left\| e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^j \right\|_{H^{\frac{4}{d+2}}}.
\]
Similarly, from equivalence of Sobolev norms, Strichartz estimate and Claim 2, we have
\[
\left\| F(u_n^j - e^{-itL_n} u_n^{j'}) - F(u_n^j) \right\|_{X_1^2} \lesssim \left\| u_n^j \right\|_{X_1} \left( \left\| u_n^j \right\|_{H^\frac{4}{d+2}} \left\| e^{-itL_n} u_n^{j'} \right\|_{S(\mathbb{R})} + \left\| e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \right)
\]
\[
+ \left\| e^{-itL_n} u_n^{j'} \right\|_{X_1} \left( \left\| e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} + \left\| e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^j \right\|_{H^{\frac{4}{d+2}}} \right).
\]
By (3.4), it suffices to prove
\[
\lim_{j \to \infty} \lim_{n \to \infty} \left\| u_n^j \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} = 0.
\]
Indeed, by Claim 2,
\[
\left\| u_n^j \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \leq \left\| u_n^j \right\|_{H^{\frac{4}{d+2}}} \left\| e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^j \right\|_{H^{\frac{4}{d+2}}}
\]
\[
\lesssim \left\| \sum_{j=1}^{J} u_n^j + e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^j \right\|_{H^{\frac{4}{d+2}}}
\]
From equivalence of Sobolev norms, Strichartz estimate and (3.4),
\[
\left\| e^{-itL_n} u_n^{j'} \right\|_{H_1^2} \lesssim \left\| e^{-itL_n} u_n^{j'} \right\|_{S(\mathbb{R})} \left\| e^{-itL_n} u_n^{j'} \right\|_{H_1^2}
\]
\[
\lesssim \left\| e^{-itL_n} u_n^{j'} \right\|_{S(\mathbb{R})} \left\| u_n^{j'} \right\|_{H_1^2} = 0.
\]
It remains to show
\[
\lim_{j \to \infty} \lim_{n \to \infty} \left\| \sum_{j=1}^{J} u_n^j \right\|_{H^{\frac{4}{d+2}}} \left\| \nabla e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} = 0.
\]
We need the following argument which can be proved same as Claim 2:
Given \( \eta > 0 \), there exists \( J' = J'(\eta) \) such that
\[
\lim_{n \to \infty} \sum_{j=J'}^{J} u_n^j \right\|_{\dot{X}_1^2} < \eta \quad \text{uniformly in } J \geq J'.
\]
Then,
\[
\lim_{n \to \infty} \left\| \sum_{j=J'}^{J} \nabla e^{-itL_n} u_n^{j'} \right\|_{H^{\frac{4}{d+2}}} \lesssim \lim_{n \to \infty} \left\| \nabla e^{-itL_n} u_n^{j'} \right\|_{\dot{X}_1^2} \lesssim \eta.
\]
It is further reduced to prove
\[
\lim_{j \to \infty} \lim_{n \to \infty} \|v_n^j \nabla e^{-it\mathcal{L}_a} u_n^j\|_{\frac{d-2}{2} \frac{d(d+2)}{2d-4+2}} = 0, \quad \text{for all } 1 \leq j \leq J'.
\] (3.20)

We approximate \(v_n^j \) by \(C_c^{\infty}(\mathbb{R} \times \mathbb{R}^d) \) function \(\psi_n^j \) obeying (3.12) with support in \([-T, T] \times \{x \leq R\} \). By Corollary 1 and (3.4), we have that
\[
\|v_n^j \nabla e^{-it\mathcal{L}_a} u_n^j\|_{\frac{d-2}{2} \frac{d(d+2)}{2d-4+2}} \leq \|\psi_n^j\|_{S(R)} \|e^{-it\mathcal{L}_a} u_n^j\|_{\dot{H}^1} + \|\psi_n^j\|_{L^\infty} \|\nabla e^{-it\mathcal{L}_a} u_n^j\|_{\frac{d-2}{2} \frac{d(d+2)}{2d-4+2}}
\]
and
\[
\leq \varepsilon + T^{\frac{(d-2)^2}{4d+8+16d}} R^{\frac{2d+8+16d}{2d+8+16d}} \|e^{-it\mathcal{L}_a} u_n^j\|_{\frac{d-2}{2} \frac{d(d+2)}{2d-4+2}} \|u_n^j\|_{\dot{H}^1}^\frac{2}{d-2}
+ T^{\frac{(d-2)^2}{4d+8+16d}} R^{\frac{2d+8+16d}{2d+8+16d}} \|e^{-it\mathcal{L}_a} u_n^j\|_{S(R)} \|u_n^j\|_{\dot{H}^1}^\frac{4d-8}{2d-4+2}.
\]

By taking the limit and choosing \(\varepsilon\) small, we obtain (3.20). Hence, Claim 3 holds. There exists at least one profile such that \(\|v_n^j\|_{S(t_n^{(1/p)})} = \infty\).

Proof of the Kinetic energy decoupling (3.11). For \(t \in \mathbb{K}_n^{m} \), (3.15) yields
\[
\|u_n^j(t)\|_{\dot{H}^1}^2 - \sum_{j=1}^{J} \|v_n^j(t)\|_{\dot{H}^1}^2 - \|u_n^j\|_{\dot{H}^1}^2 = \sum_{j \neq k} \langle v_n^j(t), v_n^k(t) \rangle_{\dot{H}^1} + \sum_{j=1}^{J} 2\text{Re} \langle e^{-it\mathcal{L}_a} u_n^j, v_n^j(t) \rangle_{\dot{H}^1}.
\]

It suffices to show that for all \(\{t_n\} \subset \mathbb{K}_n^{m}\) and \(j, k \in \{1, 2, \cdots, J\}\) with \(j \neq k\),
\[
\lim_{n \to \infty} \left(e^{-it_n\mathcal{L}_a} u_n^j, v_n^j(t_n)\right)_{\dot{H}^1} = 0,
\]
(3.21)
\[
\lim_{n \to \infty} \left(v_n^j(t_n), v_n^k(t_n)\right)_{\dot{H}^1} = 0.
\]
(3.22)

We first rewrite \(\left(e^{-it\mathcal{L}_a} u_n^j, v_n^j(t_n)\right)_{\dot{H}^1}\) equivalently as
\[
\left(\sqrt{\mathcal{L}_a} e^{-it_n\mathcal{L}_a} (u_n^j(t_n))_{(\lambda_n^j)^{-2} + \mathcal{L}_a}, \sqrt{\mathcal{L}_a} v_n^j(t_n)_{(\lambda_n^j)^{-2} + \mathcal{L}_a}\right).
\]

Note that \(t_n(\lambda_n^j)^{-2} + t_n^j \in \mathcal{P}\) (maximal lifespan of \(v_n^j\)) if \(t_n \in \mathbb{K}_n^{m}\). From the diagonalisation argument, one can assume that \(t_n(\lambda_n^j)^{-2} + t_n^j \to T^j \in \mathbb{R}\) for each \(j\).

We split our discussion depending on the behaviour of \(T^j\).

If \(T^j \in \mathcal{P}\), then \(v_n^j(t_n^j(\lambda_n^j)^{-2} + t_n^j) \to v^j(T^j)\) in \(\dot{H}^1\) by the local theory. Thus,
\[
\lim_{n \to \infty} \left(e^{-it\mathcal{L}_a} u_n^j, v_n^j(t_n)\right)_{\dot{H}^1} = \lim_{n \to \infty} \left(\sqrt{\mathcal{L}_a} e^{-it_n\mathcal{L}_a} (u_n^j(t_n))_{(\lambda_n^j)^{-2} + \mathcal{L}_a}, \sqrt{\mathcal{L}_a} v^j(T^j)\right)
= \lim_{n \to \infty} \left(\sqrt{\mathcal{L}_a} e^{it_n\mathcal{L}_a} (u_n^j(t_n))_{(\lambda_n^j)^{-2} + \mathcal{L}_a}, \sqrt{\mathcal{L}_a} e^{i\mathcal{L}_a T^j} v^j(T^j)\right)
\]

This is Lemma 2.15 and equivalence of Sobolev norms yield (3.21).

If \(T^j = \sup \mathcal{P}\), then \(T^j = +\infty\) and \(v^j\) scatters forward to \(e^{-it\mathcal{L}_a} \phi\) for some \(\phi \in \dot{H}^1\). Suppose otherwise that \(T^j\) is finite, then either \(t_n^j \to -\infty\) or \(t_n^j \equiv 0\). As \(T^j = \sup \mathcal{P}\) is finite, the local theory then yields
\[
\lim_{n \to \infty} \|v_n^j(t_n)\|_{S([0, t_n])} = \lim_{n \to \infty} \|v^j(t)\|_{S([t_n^j, t_n^j(\lambda_n^j)^{-2} + t_n^j])} = \infty;
\]
however, this contradicts \(t_n \in \mathbb{K}_n^{m}\). Similarly, from Lemma 2.15 and equivalence of Sobolev norms, we have
\[
\lim_{n \to \infty} \left(e^{-it\mathcal{L}_a} u_n^j, v_n^j(t_n)\right)_{\dot{H}^1}
\]
Theorem 3.2 (Existence of minimal blow-up solution). Suppose Theorem 1.2 fails. Then there exist a critical value 
\[ 0 < K_c < \| W_{a,0} \|_{H^1_{a,0}}^2 \] and a solution 
\( u : [0, T^*_a) \times \mathbb{R}^d \to \mathbb{C} \) to NLS\(_a\) with 
\[
\sup_{t \in [0, T^*_a)} \| u(t) \|_{H^1_a}^2 = K_c < \| W_{a,0} \|_{H^1_{a,0}}^2, \quad \text{and} \quad \| u \|_{S[0, T^*_a)} = \infty. \tag{3.23}
\]
Moreover, there exists \( \lambda(t) : [0, T^*_a) \to \mathbb{R}^+ \) such that the set 
\( \{ u(t)(\lambda(t)) : t \in [0, T^*_a) \} \)

is precompact in \( H^1_a \). An analogous result holds backward in time.

Proof. Assume Theorem 1.2 fails. From the definition of \( K_c \), there exists a sequence of solutions \( u_n : I_n \times \mathbb{R}^d \to \mathbb{C} \) to NLS\(_a\) such that

\[
\begin{align*}
&\lim_{n \to \infty} \sup_{t \in I_n} \| u_n(t) \|_{H^1_a}^2 = K_c < \| W_{a,0} \|_{H^1_{a,0}}^2, \\
&\lim_{n \to \infty} \| u_n \|_{S(t \geq t_n)} = \lim_{n \to \infty} \| u_n \|_{S(t \leq t_n)} = \infty. \tag{3.24}
\end{align*}
\]

By picking \( t_n \in I_n \) such that \( \| u_n(t) \|_{S(t \geq t_n)} = \| u_n(t) \|_{S(t \leq t_n)} \). From the time translation invariance, we may assume \( t_n = 0 \). By Lemma 3.1, passing to a subsequence, there exists \( \{ \lambda_n \} \subset \mathbb{R}^+ \) such that

\[ u_n(0)_{|\lambda_n} \to u_0 \text{ in } H^1_a. \]

Define \( u : I_a \times \mathbb{R}^d \to \mathbb{C} \) with \( u(0) = u_0 \) as the maximal life-span solution to NLS\(_a\), the stability result and (3.24) yield that for any compact interval \( G \subset I_u \)

\[
\begin{align*}
&\lim_{n \to \infty} \sup_{t \in G} \| u_n(t/\lambda_n^2) - u(t) \|_{L^\infty_c \dot{H}^1_{a,0}(G)} = 0, \\
&\| u(t) \|_{S(t \geq 0)} = \| u(t) \|_{S(t \leq \infty)} = \infty, \\
&\sup_{t \in G} \| u(t) \|_{H^1_a}^2 = K_c < \| W_{a,0} \|_{H^1_{a,0}}^2.
\end{align*}
\]

Indeed, for any sequence \( \{ s_n \} \subset I_u \), \( \| u(t) \|_{S(t \geq s_n)} = \| u(t) \|_{S(t \leq s_n)} = \infty \). Lemma 3.1 shows that for some sequence \( \{ \lambda_n \} \subset \mathbb{R}^+ \), \( u(s_n)_{|\lambda_n} \to v \) in \( H^1_a \) up to a subsequence.

As \( 0 < K_c < \| W_{a,0} \|_{H^1_{a,0}}^2 \), there exists \( \delta_0 > 0 \) such that \( K_c \leq (1 - \delta_0) \| W_{a,0} \|_{H^1_{a,0}}^2 \).

From (2.9), \( E_a(u(t)) \geq (\frac{1}{2} - \frac{(1 - \delta_0) \pi^2}{2^*}) \| u(t) \|_{H^1_a}^2 \). From the energy conservation and taking supreme on time,

\[ E_a(u(t)) \geq (\frac{1}{2} - \frac{(1 - \delta_0) \pi^2}{2^*}) K_c. \]
$E_n(u(t)) \leq \frac{1}{2} \|u(t)\|^2_{\dot{H}^1_a}$ then yields that for any $t \in I_u$,
\[\|u(t)\|^2_{\dot{H}^1_a} \geq (1 - \frac{2(1 - \delta_0)\pi^2}{2^*})K_c. \tag{3.25}\]
Let $\delta_2 = \frac{1}{16} (1 - \frac{2(1 - \delta_0)\pi^2}{2^*})$, we are ready to construct the scale function $\lambda(t)$.

Fix $t \in [0, T_u^+]$, we define
\[\lambda(t) = \sup\{\lambda : \int_{|x| \leq \frac{1}{\lambda}} |\sqrt{\mathcal{L}_a} u(t)|^2 dx = \delta_2 K_c\}. \tag{3.26}\]
Clearly, for any $t \in [0, T_u^+]$,
\[0 < \lambda(t) < \infty. \tag{3.27}\]
We claim that there exists $C \geq 1$ such that
\[C^{-1} \lambda(s_n) \leq \lambda_n \leq C \lambda(s_n). \tag{3.28}\]
From the definition of $\lambda(t)$ and a change of variable,
\[\delta_2 K_c = \int_{|x| \leq \frac{1}{\lambda(s_n)}} |\sqrt{\mathcal{L}_a} u(s_n)|^2 dx = \int_{|x| \leq \frac{\lambda_n}{\lambda(s_n)}} |\sqrt{\mathcal{L}_a} u(s_n)|^2 dx.\]
If $\frac{\lambda_n}{\lambda(s_n)} \to \infty$, as $u(s_n)|_{\lambda_n} \to v$ in $\dot{H}^1_a$, we would have $\|u(s_n)\|^2_{\dot{H}^1_a} \to \|v\|^2_{\dot{H}^1_a} = \delta_2 K_c$, which contradicts (3.25). If $\frac{\lambda_n}{\lambda(s_n)} \to 0$, as $u(s_n)|_{\lambda_n} \to v$ in $\dot{H}^1_a$, we would get $\delta_2 K_c = 0$, which contradicts $K_c > 0$. Hence, (3.28) holds. Together with $u(s_n)|_{\lambda_n} \to v$ in $\dot{H}^1_a$, this imply that $u(s_n)|_{\lambda(s_n)}$ converges strongly in $\dot{H}^1_a$. As $\{s_n\}$ is arbitrarily taken,
\[\{u(t)|_{\lambda(t)} : t \in [0, T_u^+]\} \text{ is precompact in } \dot{H}^1_a. \tag{3.29}\]
The proof is complete.

4. Rigidity Theorem. Suppose that Theorem 1.2 fails and let $u : [0, T_u^+] \times \mathbb{R}^d \to \mathbb{C}$ be a minimal blow-up solution proposed in Theorem 3.2. The Arzelà–Ascoli theorem, Theorem 3.2 yields that for any $\eta > 0$ there exists $C(\eta) > 0$ such that
\[\int_{|x| \geq \frac{C(\eta)}{t_\eta}} |\nabla u(t)|^2 + \frac{\alpha}{|x|^2} |u(t)|^2 + |u(t)|^2 dx \leq \eta, \tag{4.1}\]
uniformly for $t \in [0, T_u^+]$. From coercivity of energy,
\[\|u(t)\|^2_{\dot{H}^1_a} \sim K_c \quad \text{uniformly for } t \in [0, T_u^+]. \tag{4.2}\]

To exclude the existence of the minimal blow-up solution $u$ as in Theorem 3.2, we split our discussion into the following two cases: Case 1. $\lambda(t) \geq \lambda_0$ for some $\lambda_0 > 0$: if $T_u^+ = \infty$, we essentially use the energy trapping argument of the focusing problem and a weighted Virial argument; if $T_u^+ < \infty$, we prove such minimal blow-up solution $u$ will be in $L^2$ with 0 mass and contradicts $u$ being non-trivial. Case 2. there exists $\{t_n\}$ such that $\lambda(t_n) \to 0$, then $t_n \to T_u^+$ (otherwise one can easily obtain a contradiction), we can construct a non-trivial solution with the precompactness property and the scale function is bounded below which contradicts previous case.

The proofs are influenced by [13]. It is possible to generalize the result to the non-radial case by the arguments in [19, 18] in certain dimension.

Claim 4.1. There are no solutions to NLS$_a$ of the form given in Theorem 3.2 with $T_u = \infty$ and $\lambda(t) \geq \lambda_0$ for some $\lambda_0 > 0$. 

Proof. By way of contradiction, we assume that \( u : [0, \infty) \times \mathbb{R}^d \to \mathbb{C} \) is such a solution. For some \( R > 0 \) to be chosen later, we define a weighted mass quantity

\[
V_R(t) = \int_{\mathbb{R}^d} \phi_R(x)|u(t, x)|^2 \, dx,
\]

where \( \phi_R(x) = R^2 \phi(\frac{|x|^2}{R^2}) \) and \( \phi \) is a smooth radial cutoff function such that \( \phi(x) = 0 \) when \( |x| \geq 2 \) and \( \phi(x) = |x| \) when \( |x| \leq 1 \). Taking partial derivatives on time, we have

\[
\partial_t V_R(t) = 2\text{Im} \int_{\mathbb{R}^d} \overline{u(t)} \nabla u(t) \cdot \nabla \phi_R \, dx;
\]

\[
\partial_{tt} V_R(t) = 4\text{Re} \int_{\mathbb{R}^d} (\phi_R)_{jk}(x) u_j(t) \bar{u}_k(t) \, dx - \frac{4}{3} \int_{\mathbb{R}^d} (\Delta \phi_R)|u(t)|^2 \, dx
\]

\[
- \int_{\mathbb{R}^d} (\Delta \Delta \phi_R)|u(t)|^2 \, dx + 4a \int_{\mathbb{R}^d} |\nabla \phi_R||u(t)|^2 \, dx.
\]

From H"older inequality, Sobolev embedding, and (4.2), for any \( t \in [0, \infty) \),

\[
|\partial_t V_R(t)| \lesssim \|\phi'(\frac{|x|^2}{R^2})||x||u(t)||_{L^2} \|u(t)||_{H^1} \lesssim K_c R^2.
\]

By Hardy inequality, Sobolev embedding, and (4.1) with \( R = R(\eta) \) large enough, we obtain

\[
\partial_{tt} V_R(t) = 8 \int_{\mathbb{R}^d} |\nabla u(t)|^2 + \frac{a}{|x|^2} |u(t)|^2 - |u(t)|^2^* \, dx
\]

\[
+ O\left(\int_{R \leq |x| \leq 2R} |\nabla u(t)|^2 + \frac{a}{|x|^2} |u(t)|^2 + |u(t)|^2^* \, dx \right) \gtrsim \|u(t)||_{H^1} - \eta,
\]

where \( \int_{\mathbb{R}^d} |\nabla u(t)|^2 + \frac{a}{|x|^2} |u(t)|^2 - |u(t)|^2^* \, dx \lesssim \|u(t)||_{H^1}^2 \) follows from the energy trapping. By taking \( \eta \) small enough depending on \( K_c \) and (4.2), we indeed have

\[
\partial_{tt} V_R(t) \gtrsim K_c 1, \quad \text{uniformly for } t \in [0, \infty).
\]

Thus, from the fundamental theorem of calculus,

\[
T \lesssim K_c \left| \int_0^T \partial_t V_R(t) \, dt \right| \lesssim K_c R^2.
\]

By taking \( T \to \infty \), we get a contradiction and the proof of this claim is complete. \( \square \)

Claim 4.2. There are no solutions to NLS\(_u\) of the form given in Theorem 3.2 with \( T_u^+ < \infty \) and \( \lambda(t) \geq \lambda_0 \) for some \( \lambda_0 > 0 \).

Proof. By way of contradiction, we assume that \( u(t, x) : [0, T_u^+) \times \mathbb{R}^d \to \mathbb{C} \) is such a solution. For some \( R > 1 \) to be chosen later, we define another weighted mass quantity

\[
F_R(t) = \int_{\mathbb{R}^d} \phi(\frac{x}{R})|u(t)|^2 \, dx,
\]

where \( \phi \) is a smooth radial cutoff function such that \( \phi(x) = 0 \) when \( |x| \geq 2 \) and \( \phi(x) = 1 \) when \( |x| \leq 1 \). From taking partial derivatives on time, H"older, Hardy inequality, and (4.2), we get

\[
|\partial_t F_R(t)| = \frac{2}{R} \left| \text{Im} \int_{\mathbb{R}^d} \overline{u(t)} \nabla u(t) \cdot (\nabla \phi)(\frac{x}{R}) \, dx \right| \lesssim \|u(t)||_{H^1}^2 \lesssim K_c.
\]

The Mean Value Theorem yields that for any \( t, T \in [0, T_u^+) \),

\[
|F_R(t) - F_R(T)| \lesssim K_c |t - T|.
\] (4.3)
To conclude the proof, we need the following
\[ \lim_{T \to T_u^+} F_R(T) = 0. \]  

Indeed, for any \( 1 > r_0 > 0 \), from changing of variables, Hölder inequality, and Sobolev embedding,
\[
F_R(T) = \int_{|x| \leq r_0} \phi\left(\frac{x}{R}\right)|u(T)|^2 \, dx + \int_{|x| > r_0} \phi\left(\frac{x}{R}\right)|u(T)|^2 \, dx
\]
\[
\leq r_0^2 \|u(T)\|_2^2 + \|\phi\left(\frac{x}{R}\right)\|_2 \|u(T)\|_{L_x^\infty(|x| > \lambda(T) r_0)}^2
\]
\[
\leq r_0^2 \|u(t)\|_{H^1_x}^2 + R^2 \|u(T)\|_{L_x^\infty(|x| > \lambda(T) r_0)}^2.
\]

Note that \( \lim_{T \to T_u^+} \lambda(T) = \infty \). If otherwise, there exists \( \{t_n\} \subset [0, T_u^+] \) such that \( \lambda(t_n) \to \lambda_0 \in \mathbb{R}^+ \). From Lemma 3.1, \( u(t_n)\|_{L_x^\infty(|x| > \lambda(T) r_0)} \) holds by first taking \( R \to 0 \). In addition, \( \|u(t)\|_{L_x^\infty(|x| > \lambda(T) r_0)} \) yields that \( R^2 \|u(T)\|_{L_x^\infty(|x| > \lambda(T) r_0)}^2 \to 0 \) for any fixed \( r_0 > 0 \) as \( T \to T_u^+ \). (4.4) holds by first taking \( T \to T_u^+ \) and then \( r_0 \to 0 \).

By taking \( T \to T_u^+ \) and \( R \to 0 \) in (4.3) and using (4.4), we have
\[
\int_{\mathbb{R}^d} |u(t)|^2 \, dx \leq K_c |t - T_u^+|.
\]

Let \( t \to T_u^+ \), from conservation of mass, we get \( u(t) = 0 \) for all \( t \in [0, T_u^+] \) which contradicts \( \sup_{t \in T_u^+} \|u(t)\|_{H^1_x}^2 = K_c > 0 \).

\[ \square \]

Claim 4.3. There are no solutions to NLS\(_a\) of the form given in Theorem 3.2.

Proof. By way of contradiction, we assume that \( u(t, x) : [0, T_u^+] \times \mathbb{R}^d \to C \) is such a solution. From Claim 4.2 and 4.1 and previous discussion, it suffices to consider a positive sequence of time \( \{t_n\} \) such that \( t_n \to T_u^+ \) and \( \lambda(t_n) \to 0 \). Taking a sub-sequence if necessary, we could assume \( \lambda(t_n) \leq 2 \inf_{t \in [0, t_n]} \lambda(t) \). Theorem 3.2 then yields
\[ u(t_n)_{\|\lambda(t_n)\|} \to v_0 \text{ in } H^1_x. \]  

Note that \( v_0 \neq 0 \), otherwise \( \|u(t_n)\|_{H^1_x} \to 0 \) contradicts \( \sup_{t \in T_u^+} \|u(t)\|_{H^1_x}^2 = K_c > 0 \) and \( \|u_0\|_{H^1_x}^2 \sim \|u(t)\|_{H^1_x}^2 \) by (2.9).

Let \( v_n(t) \) and \( v(t) \) be the solutions to NLS\(_a\) with initial data \( u(t_n)_{\|\lambda(t_n)\|} \) and \( v_0 \). From (4.5) and the local theory, \( \lim_{n \to \infty} T_{v_n}^- \geq T_v^- \), \( v_n(t) \to v(t) \) in \( H^1_x \) for each \( t \in (-T_v^-, 0) \), and \( v_n(t) = u(t_n + t/\lambda(t_n)^2)_{\|\lambda(t_n)\|} \) for \( t_n + t/\lambda(t_n)^2 \geq 0 \). In addition, \( 0 \leq t_n + t/\lambda(t_n)^2 \leq t_n \), \( \forall t \in (-T_v^-, 0), n \) large enough.

If not, there exists \( t_n \lambda(t_n)^2 \to t_0 < T_v^- \), and \( v_n(-t_n \lambda(t_n)^2) = u_0_{\|\lambda(t_n)\|} \to v(-t_0) \) in \( H^1_x \) and \( v(t) \equiv 0 \) as \( \lambda(t) \to 0 \) contradicting \( v_0 \neq 0 \).

Consider the following sequence of solutions
\[
u(t_n + \frac{t}{\lambda(t_n)^2})_{\|\lambda(t_n+t/\lambda(t_n)^2)\|} = v_n(t)_{\|\lambda(t_n+t/\lambda(t_n)^2)\|}
\]
which is well defined for all \( t \in (-T_v^-, 0) \) and \( n \) large enough. From \( \lambda(t_n) \leq 2 \inf_{t \in [0, t_n]} \lambda(t) \), \( \frac{\lambda(t_n+t/\lambda(t_n)^2)}{\lambda(t_n)} \geq \frac{1}{2} \). Passing to a sub-sequence if necessary, we get
\[ \frac{\lambda(t_n + t/\lambda(t_n)^2)}{\lambda(t_n)} \rightarrow \lambda^*(t) \geq \frac{1}{2} \quad \text{and} \quad v_n(t) \rightarrow v_0(t) \quad \text{in} \quad \dot{H}^1. \]

Same as the proof that \(v_0 \neq 0\), we get \(z_0 \neq 0\). This implies that \(\lambda^*(t) < \infty\). Thus, \(v(t)|_{\lambda^*(t)}\) has the precompactness property with scale function \(\lambda^*(t) \geq \frac{1}{2}\). This contradicts Claim 4.2 and 4.1 and completes the proof of the main theorem. \(\square\)

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