ORBITAL STABILITY OF PERIODIC STANDING WAVES FOR THE LOGARITHMIC KLEIN-GORDON EQUATION

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Abstract. The main goal of this paper is to present orbital stability results of periodic standing waves for the one-dimensional logarithmic Klein-Gordon equation. To do so, we first use compactness arguments and a non-standard analysis to obtain the existence and uniqueness of weak solutions for the associated Cauchy problem in the energy space. Second, we prove the orbital stability of standing waves using a stability analysis of conservative systems.

1. Introduction

Consider the Klein-Gordon equation with $p$–power nonlinearity,

\[(1.1) \quad u_{tt} - u_{xx} + u - \mu \log(|u|^p)u = 0.\]

Here, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a complex valued function, $\mu > 0$ and $p$ is a positive integer.

When $p = 2$, the problem (1.1) models a relativistic version of logarithmic quantum mechanics introduced in [5] and [6]. The parameter $\mu$ measures the force of the nonlinear interactions. It has been shown experimentally (see [15], [26] and [27]) that the nonlinear effects in quantum mechanics are very small, namely for $0 < \mu < 3.3 \times 10^{-15}$. Still, the model can be found in many branches of physics, e.g. nuclear physics, optics, geophysics (see [16]). In addition, Klein-Gordon equation with logarithmic potential has been also introduced in the quantum field theory as in [25]. This kind of nonlinearity appears naturally in inflation cosmology and in super-symmetric field theories (see [16]).

Along the last thirty years, the theory of stability of traveling/standing wave solutions for nonlinear evolution equation has increased into a large field that attracts the attention of both mathematicians and physicists. By considering $\mu = 1$ in equation (1.1), our purpose is to give a contribution in the stability theory by proving the first result of orbital stability of periodic waves of the form $u(x,t) = e^{ict}\varphi(x)$, $t > 0$, where $c$ is called frequency and $\varphi$ is a real, even and periodic function. So, if we substitute this kind of solution in equation (1.1), one has the following nonlinear ordinary differential equation

\[(1.2) \quad -\varphi''_c + (1 - c^2)\varphi_c - \log(|\varphi_c|^p)\varphi_c = 0,\]

where $\varphi_c$ indicates the dependence of the function $\varphi$ with respect to the parameter $c$. In a general setting, we can use a qualitative analysis of planar waves to determine the existence of special solutions related to the equation, namely: solitary waves and periodic waves. To do so, we see that equation (1.2) has three equilibrium points, a saddle point and two center points (around the center

2010 Mathematics Subject Classification. Primary 35A01, 34C25, 37K35.
Key words and phrases. Logarithmic Klein-Gordon, compactness method, periodic waves, orbital stability.
points we have strictly negative and positive periodic orbits). The saddle point is the origin and one sees an explicit solitary wave (which is unique up to translation) given by
\begin{equation}
\phi_c(x) = e^{\frac{1}{2} \frac{1-c^2}{r^2}} e^{-\frac{px^2}{4}}.
\end{equation}

Solution in (1.3) is very similar to the solitary wave concerning the one dimensional version of the logarithmic nonlinear Schrödinger equation given by
\begin{equation}
iu_t + \Delta u + \log(|u|^2)u = 0,
\end{equation}
where \( u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C} \) is a complex-valued function and \( n \geq 1 \).

Concerning the orbital stability of waves for the equation (1.4), we have interesting results. In fact, in [11] and [12] the authors have used variational techniques to get the orbital stability for solitary waves for the model (1.4) (considering equation (1.1) in higher dimensions, similar results have been determined by the same authors). Furthermore, the stability of solitary waves has also been treated in [7], where the authors used the general theory in [17] in the space of radial functions. In periodic context and \( n = 1 \), we have the work in [22], where the authors approached the abstract theory in [17] to deduce the orbital stability for the equation (1.4) in the even periodic Sobolev space \( H^1_{\text{per},e}([0,L]) \).

In [10], the authors have considered the Korteweg-de Vries equation with logarithmic nonlinearity, namely,
\begin{equation}
u_t + u_{xxx} + (u \log(|u|))_x = 0,
\end{equation}
where \( u = u(x,t) \) is a real-valued function with \( (x,t) \in \mathbb{R} \times \mathbb{R} \). The authors established results of linear stability by using numerical approximations and they showed that the Gaussian initial data do not spread out and preserve their spatial Gaussian decay in the time evolution of the linearized logarithmic KdV equation. Concerning the stability of periodic traveling waves for the equation (1.5), we can cite [14]. In both cases, the authors have determined their results of stability in a conditional sense since the uniqueness of solutions for the associated Cauchy problem is obtained by supposing that the evolution \( u \) satisfies \( \partial_x(\log(|u|)) \in L^\infty(0,T;L^\infty) \).

Next, we shall give an outline of our work. The logarithmic nonlinearity in equation (1.1) brings a rich set of difficulties since function \( x \in \mathbb{R} \mapsto x \log(|x|) \) is not differentiable at the origin. The lack of smoothness of the nonlinearity interferes in questions concerning the local solvability since it is not possible to apply a contraction argument to deduce the existence, uniqueness and continuous dependence with respect to the initial data. In all these works [10], [11], [12] and [13], the authors have determined a modified problem related to the model to overcome the absence of regularity. The modified solution (or approximate solution) converges in some sense to the solution of the original problem provided that convenient uniform estimates for the approximate solution are established. The construction of the approximate solution, which converges in the weak sense to the solution of the original problem, gives us the existence of weak solutions in a convenient Banach space.

In [16] the author have used the Galerkin approximation to deduce existence of global weak solutions related to the problem in (1.1) in bounded domains by assuming Dirichlet boundary conditions. In our work, the existence of periodic weak solutions follows the same spirit of [16] and we present new results concerning the uniqueness of weak periodic solutions without restrictions on the initial data. The authors in [4] have determined the uniqueness of smooth solutions for the one-dimensional version of (1.1) posed on the real line by assuming that the initial data is bounded
away from zero. Concerning the equation (1.4), a uniqueness result has been treated in [13] by combining energy estimates with a convenient Gronwall-type inequality and fact that the $L^2$–norm is a conserved quantity. The $L^2$–norm is not a conserved quantity when equation (1.1) is considered and it seems a hard task some kind of adaptation of the arguments contained in [13].

We prove the existence of global weak solutions in time, uniqueness and existence of conserved quantities in the following theorem:

**Theorem 1.1.** Let $p$ be a positive integer and consider $L > 0$. There exists a unique global (weak) solution to the problem (3.1) in the sense that

$$u \in L^\infty(0,T; H^1_{\text{per}}([0,L])), \quad u_t \in L^\infty(0,T; L^2_{\text{per}}([0,L])), \quad u_{tt} \in L^\infty(0,T; H^{-1}_{\text{per}}([0,L])),$$

and $u$ satisfies,

$$\langle u_{tt}(\cdot, t), \zeta \rangle_{H^1_{\text{per}}, H^{-1}_{\text{per}}} + \int_0^L \nabla u(\cdot, t) \cdot \nabla \zeta \, dx$$

$$+ \int_0^L u(\cdot, t) \zeta \, dx = \int_0^L u(\cdot, t) \log(|u(\cdot, t)|^p) \zeta \, dx \quad \text{a.e. } t \in [0,T],$$

for all $\zeta \in H^1_{\text{per}}([0,L])$. Furthermore, $u$ must satisfy

$$u(\cdot, 0) = u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1.$$ 

In addition, the weak solution satisfies the following conserved quantities:

$$\mathcal{E}(u(\cdot, t), u'(\cdot, t)) = \mathcal{E}(u_0, u_1) \quad \text{and} \quad \mathcal{F}(u(\cdot, t), u'(\cdot, t)) = \mathcal{F}(u_0, u_1),$$

a.e. $t \in [0,T]$. Here, $\mathcal{E}$ and $\mathcal{F}$ are defined by

$$\mathcal{E}(u(\cdot, t), u_t(\cdot, t)) := \frac{1}{2} \left[ \int_0^L |u_x(\cdot, t)|^2 + |u_t(\cdot, t)|^2 + \left( 1 + \frac{p}{2} - \log(|u(\cdot, t)|^p) \right) |u(\cdot, t)|^2 \, dx \right]$$

and

$$\mathcal{F}(u(\cdot, t), u_t(\cdot, t)) := \text{Im} \int_0^L \frac{u(\cdot, t) u_t(\cdot, t) \, dx}{u(\cdot, t)}$$

$$= \int_0^L \left[ \text{Re} \ u(\cdot, t) \text{ Im} \ u_t(\cdot, t) - \text{Im} \ u(\cdot, t) \ Re \ u_t(\cdot, t) \right] \, dx.$$ 

We now present the basic ideas to prove Theorem 1.1. In fact, we first present an approximate problem for the equation (1.4) defined on the subspace

$$V_m = [\omega_1, ..., \omega_m],$$

where $\{\omega_p\}_{p \in \mathbb{N}}$ is a complete orthonormal set in $L^2_{\text{per}}([0,L])$ which is orthogonal in $H^1_{\text{per}}([0,L])$. After that, we use Carathéodory’s Theorem to deduce the existence of an approximate solution $u_m$ for the approximate problem (see (3.1)) for all $m \in \mathbb{N}$. Thus, uniform bounds are required in order to get the existence of global weak solutions after passage to the limit. In addition, the approximate solution satisfies

$$\mathcal{E}(u_m(\cdot, t), u'_m(\cdot, t)) = \mathcal{E}(u_{0,m}, u_{1,m}) \quad \text{and} \quad \mathcal{F}(u_m(\cdot, t), u'_m(\cdot, t)) = \mathcal{F}(u_{0,m}, u_{1,m}).$$
However, since $u$ and $u_t$ are obtained as weak limits of the approximate solution $u_m$ and $u'_m$, respectively, we only guarantee the “conserved” inequalities

$$E(u(\cdot,t),u_t(\cdot,t)) \leq E(u_0,u_1) \quad \text{and} \quad F(u(\cdot,t),u_t(\cdot,t)) \leq F(u_0,u_1) \quad \text{a.e. } t \in [0,T].$$

In order to obtain that equalities in (1.9) occur, we employ the arguments in [20] (see Chapter 1, page 22) provided that a result of uniqueness for weak solutions can be established. The uniqueness of solutions for the model (1.1) is one of the cornerstones of our paper. In fact, as we have already mentioned above, the nonlinearity $x \mapsto x \log(|x|)$ is not locally Lipschitz in convenient Lebesgue measurable spaces. Thus, uniqueness of solutions cannot be determined using a difference of weak solutions with zero initial data combined with contraction arguments as is usual in evolution problems. To give a positive answer for the uniqueness of solutions, let $L > 0$ be fixed. We first establish the existence of $T_0 \in (0,\frac{L}{4})$ such that the weak solution $w$ for the inhomogeneous Cauchy problem

$$\begin{cases} w_{tt} - w_{xx} = f(x,t), & (x,t) \in \mathbb{R} \times [0,T_0]. \\ w(x,0) = 0, & w_t(x,0) = 0, \quad x \in \mathbb{R}. \\ w(x + L,t) = w(x,t) \quad \text{for all } t \in [0,T_0], \quad x \in \mathbb{R}, \end{cases}$$

satisfies

$$w(x,t) = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} f(y,\tau) \, dy \, d\tau, \quad (x,t) \in \mathbb{R} \times [0,T_0].$$

We use the characterization in (1.11) to determine our uniqueness result provided that $t \in [0,T_0]$. Moreover, since solution $w$ is global in time, we employ an interaction argument to get the uniqueness over $\mathbb{R} \times [0,T], \quad T > 0$. In [13], the same formula (1.11) has been used to obtain uniqueness of weak solutions for a similar problem as in (1.10) posed on $\mathbb{R}^3 \times [0,T]$.

With the results determined by Theorem 1.1 in hand, we are able to establish our orbital stability result given by:

**Theorem 1.2.** Consider $p = 1, 2, 3$ and $c \in I$ satisfying $\frac{\sqrt{p}}{2} < |c| < 1$. Let $\varphi_c$ be a periodic solution for the equation (1.2). The periodic wave $\tilde{u}(x,t) = e^{ict} \varphi_c(x)$ is orbitally stable by the periodic flow of the equation (1.2).

We present some important facts concerning Theorem 1.2. First, it is important to mention that we are interested in obtaining smooth periodic waves. Thus, the presence of a logarithmic type nonlinearity in the ODE (1.2) forces us to consider positive and periodic waves. A planar analysis concerning the equilibrium points of (1.2) gives us that the period of our periodic solutions must satisfy $L > \frac{2\pi}{\sqrt{p}}$. Additionally, it is important to mention that it is possible deduce (at least formally) periodic waves having small periods (see [4]). However, we are not capable to decide about the orbital stability in this case because the energy $E$ in (1.7) is not a smooth functional at these waves (in fact, the waves obtained with this property have two zeros over the interval $[0,L]$) and our stability analysis consists in proving that the periodic waves are minimizers of the energy $E$ with constraint $F$. 
In a general setting, let us consider the Hill operator
\begin{equation}
L_Q = -\frac{d^2}{dx^2} + Q(x),
\end{equation}
where \( Q \) is a smooth \( L \)–periodic function. In [22], the authors have presented a tool based on the classical Floquet theorem to establish a characterization of the first eigenvalues of \( L_Q \) by knowing one of its eigenfunctions. The key point of the work is that it is not necessary to know an explicit smooth solution \( \varphi = \varphi(x) \) which solves the general nonlinear differential equation
\begin{equation}
-\varphi'' + h(c, \varphi) = 0,
\end{equation}
where \( h \) is smooth in a open subset contained in \( \mathbb{R}^2 \). Moreover, in [22] it is possible to decide that the eigenvalue zero is simple without knowing an explicit periodic solution which solves equation (1.13). In [1], [2], [3], and references therein, the authors have determined explicit solutions to obtain the behavior of the non-positive spectrum for the Hill operator (1.12) and the orbital stability of periodic waves.

Following arguments in [8], [17] and [28], the first requirement for the stability of periodic waves related to the equation (1.1) concerns in proving the existence of an open interval \( I \subset \mathbb{R} \) and a smooth branch \( c \in I \mapsto \varphi_c \) which solves (1.2), all of them with the same period \( L > 0 \). In our study, if \( L \in \left( \frac{2\pi}{\sqrt{p}}, +\infty \right) \) we are capable to combine the approach in [22] and the Implicit Function Theorem to deduce a smooth curve of even periodic solutions defined over \( I \subset \mathbb{R} \). The second step to obtain the stability of periodic waves is to analyze the behavior of the non-positive spectrum of the linearized operator
\begin{equation}
L_{\varphi_c} = \begin{pmatrix}
L_{\text{Re}, \varphi_c} & 0 \\
0 & L_{\text{Im}, \varphi_c}
\end{pmatrix},
\end{equation}
where \( L_{\text{Re}, \varphi_c} \) and \( L_{\text{Im}, \varphi_c} \) are defined, respectively, as
\begin{equation}
L_{\text{Re}, \varphi_c} = \begin{pmatrix}
-\partial_x^2 + 1 - \log(||\varphi_c||^p) - p & -c \\
-c & 1
\end{pmatrix},
\end{equation}
and
\begin{equation}
L_{\text{Im}, \varphi_c} = \begin{pmatrix}
-\partial_x^2 + 1 - \log(||\varphi_c||^p) & c \\
c & 1
\end{pmatrix}.
\end{equation}
The approach in [22] can be used to conclude that the diagonal operator \( L_{\varphi_c} \) has only one negative eigenvalue which is simple, zero is double eigenvalue with
\[ \ker(L_{\varphi_c}) = \text{span}\{(\varphi_c', c\varphi_c', 0, 0), (0, 0, \varphi_c, -c\varphi_c)\}. \]
Moreover, the remainder of the spectrum is discrete and bounded away from zero.

Finally, the stability of periodic waves can be determined provided that the following condition holds
\[ \left( L_{\text{Re}, \varphi_c}^{-1} \begin{pmatrix}
\varphi_c' \\
\varphi_c
\end{pmatrix}, \begin{pmatrix}
\varphi_c' \\
\varphi_c
\end{pmatrix} \right)_{2,2} = \int_0^L \varphi_c^2 \, dx + c \frac{d}{dc} \left( \int_0^L \varphi_c^2 \, dx \right) = -d''(c) < 0. \]
In our case, we establish that \( d''(c) > 0 \) provided that \( c^2 \in \left( \frac{p}{4}, 1 \right) \) and \( p = 1, 2, 3. \)
This paper is organized as follows: in Section 2 we present some basic notations and results which will be useful in the whole paper. In Section 3 we study existence, uniqueness and existence of conservation laws related to the model. Finally, the orbital stability of periodic waves will be shown in Section 4.

2. Preliminary Results

In this section, some basic notation and results are presented in order to give a complete explanation of the arguments discussed in our paper. The arguments below can be found in [12].

The $L^2$-based Sobolev spaces of periodic functions are defined as follows: if $\mathcal{P} = C^\infty_{\text{per}}$ denotes the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ which are $C^\infty$ and periodic with period $L > 0$, collection $\mathcal{P}'$ of all continuous linear functionals from $\mathcal{P}$ into $\mathbb{C}$.

If $\Psi \in \mathcal{P}'$ we denote the evaluation of $\Psi$ at $\varphi \in \mathcal{P}$ by $\Psi(\varphi) = [\Psi, \varphi]$. For $k \in \mathbb{Z}$, consider $\Lambda_k(x) = e^{\frac{2\pi ikx}{L}}$, $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. The Fourier transform of $\Psi \in \mathcal{P}'$ is a function $\hat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by $\hat{\Psi}(k) = \frac{1}{L} \int_{[0,L]} \Psi(x) e^{\frac{-2\pi ikx}{L}} dx$, $k \in \mathbb{Z}$. $\hat{\Psi}(k)$ are called the Fourier coefficients of $\Psi$. As usual, a function $f \in L^p_{\text{per}}([0,L])$, $p \geq 1$ is an element of $\mathcal{P}'$ by defining

$$[f,g] = \frac{1}{L} \int_{0}^{L} f(x)g(x)dx, \quad g \in \mathcal{P}.$$  

We denote by $C_{\text{per}} := C^0_{\text{per}}$ the space of the continuous and $L$-periodic functions. For $s \in \mathbb{R}$, the Sobolev space $H^s_{\text{per}}([0,L]) := H^s_{\text{per}}$ is the set of all $f \in \mathcal{P}'$ such that

$$||f||^2_{H^s_{\text{per}}} := ||f||^2_s \equiv L \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 < \infty,$$

where $\hat{f}$ indicates the periodic Fourier transform defined as above.

Collection $H^s_{\text{per}}$ is a Hilbert space with inner product

$$(f,g)_{H^s_{\text{per}}} := (f,g)_s = L \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s \hat{f}(k)\overline{\hat{g}(k)}.$$

When $s = 0$, $H^0_{\text{per}}([0,L])$ is a Hilbert space which is isometrically isomorphic to a subspace of $L^2([0,L])$ and $(f,g)_{H^0_{\text{per}}} = (f,g)_{L^2_{\text{per}}} = \int_{0}^{L} f(x)g(x)dx$. Space $H^0_{\text{per}}$ will be denoted by $L^2_{\text{per}}$ and its norm will be $||\cdot||_{L^2_{\text{per}}}$. Of course, $H^s_{\text{per}} \subseteq L^2_{\text{per}}$, $s \geq 0$, and we have the Sobolev embedding $H^s_{\text{per}} \hookrightarrow C_{\text{per}}$, $s > \frac{1}{2}$. For $s \geq 0$, we denote $H^s_{\text{per, even}}([0,L])$ as the closed subspace of $H^s_{\text{per}}([0,L])$ constituted by even periodic functions. If $H$ is a Hilbert space, we denote by $\langle \cdot, \cdot \rangle_{H',H}$ the duality pair. In particular, if $H = L^2_{\text{per}}([0,L])$ we are able to write $H' = L^2_{\text{per}}([0,L])$ with $\langle \cdot, \cdot \rangle_{L^2_{\text{per}}' \times L^2_{\text{per}}} = \langle \cdot, \cdot \rangle_{L^2_{\text{per}}' \times L^2_{\text{per}}}$. 

Next propositions are technical results used in our manuscript.

**Proposition 2.1.** Consider $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying $|\alpha_1| \leq |\alpha_2|$. Then,

$$|\alpha_1 \log(|\alpha_1|) - \alpha_2 \log(|\alpha_2|)| \leq |1 + \log(|\alpha_2|)|| |\alpha_1 - \alpha_2|.$$
Proof. See [13, Chapter II, Lemma 2.4.3]. □

Proposition 2.2. Consider $\alpha > 0$ and $v_0 \in \left[0, \frac{1}{e}\right]$. Let $v \in L^\infty(0, T)$ be a non-negative function. If
\[
v(t) \leq v_0 - \alpha \int_0^t v(s) \log(v(s)) \, ds \quad \text{a.e.} \quad t \in [0, T],
\]
then,
\[
v(t) \leq (v_0)^{e^{-\alpha t}}
\]
with
\[
0 \leq t \leq \inf \left\{ \frac{\log(-\log(v_0))}{\alpha}, T \right\}.
\]
Proof. See [13, Chapter III, Corollary 2.1.2]. □

Important Remark: Since $u$ found in (1.1) is a complex valued function, the notion of stability will be considered over the complex space $X = H^1_{per}([0, L]) \times L^2_{per}([0, L])$. However, in some particular cases, it is convenient to consider real spaces $H^s_{per}([0, L])$, $s \geq 0$, instead of complex ones (see Section 4). In this paper, we will not distinguish whether this space is real or complex.

3. Existence and Uniqueness of Weak Solutions

Next, we establish results of existence and uniqueness of weak solutions. Let us consider $(u_0, u_1) \in H^1_{per}([0, L]) \times L^2_{per}([0, L])$. First, we use Galerkin’s method to prove the existence of weak solutions for the following Cauchy problem
\[
\begin{cases}
    u_{tt} - u_{xx} + u - \log(|u|^p)u = 0, & (x, t) \in \mathbb{R} \times [0, T]. \\
    u(x, 0) = u_0(x), & u_1(x, 0) = u_1(x), \quad x \in \mathbb{R}. \\
    u(x + L, t) = u(x, t) & \text{for all } t \in [0, T], \quad x \in \mathbb{R},
\end{cases}
\]
where $p \in \mathbb{N}$, $L > 0$ and $T > 0$. Regarding the uniqueness of weak solutions, we need to use a non-standard analysis to rewrite the weak solution into a convenient integral form. A consequence of this last fact enables us to deduce the existence of two conserved quantities $E$ and $F$ as in (1.7) and (1.8).

Remark 3.1. Once obtained the results contained in Theorem 1.1, we can apply the arguments in [20] to establish the following smoothness result. Indeed, if $u$ is a weak solution to the problem (3.1), one has
\[
u \in C^0([0, T]; L^2_{per}([0, L])) \cap C_s(0, T; H^1_{per}([0, L]))
\]
and
\[
u_t \in C^0([0, T]; H^{-1}_{per}([0, L])) \cap C_s(0, T; L^2_{per}([0, L]))
\]
Here, $C_s(0, T; H)$ indicates the space of weakly continuous functions in the Hilbert space $H$, that is, the set of $f \in L^\infty(0, T; H)$ such that the map $t \mapsto \langle f(t), v \rangle_{H^*, H}$ is continuous over $[0, T]$ for all $v \in H$.

Next, we present the proof of Theorem 1.1 by splitting it into two parts. The first one concerns the existence and uniqueness of global weak solutions.
Proposition 3.2. Let $p$ be a positive integer and consider $L > 0$. There exists a unique global weak solution to the problem \((3.1)\) in the sense as mentioned in Theorem 1.1.

Proof. We perform the proof of this result by showing two basics steps. The first one corresponds to the existence of weak solutions (which contains the construction of the approximate problem, a priori estimates and passage to the limit). After that, we show the uniqueness of weak solutions.

Step 1. Existence of weak solutions.

Here, the main point is to use Galerkin’s method to determine the existence of weak solutions related to the problem \((3.1)\). Firstly, we need to present the approximate solution $u_m$, $m \in \mathbb{N}$. After that, we obtain a priori estimate to give an uniform bound for the approximate solution in a convenient space. This last fact enables us to consider this solution for all values $T > 0$ and for $m$ large, we can pass to the limit to obtain a weak solution $u$.

Indeed, let $\{\omega_j\}_{j \in \mathbb{N}}$ be a sequence of periodic eigenfunctions satisfying

\[
\begin{aligned}
&\begin{cases}
-\partial^2_x \omega_j + \omega_j = \lambda_j \omega_j & \text{in } [0, L], \\
\omega_j(0) = \omega_j(L).
\end{cases}
\end{aligned}
\]

Using Fourier analysis, we can choose $\{\omega_j\}_{j \in \mathbb{N}}$ such that

- $\{\omega_j\}_{j \in \mathbb{N}}$ is a complete orthonormal set in $L^2_{\text{per}}([0, L])$;
- $\{\omega_j\}_{j \in \mathbb{N}}$ is a complete orthogonal set in $H^1_{\text{per}}([0, L])$;
- $\{\omega_j\}_{j \in \mathbb{N}}$ is smooth.

By \((3.2)\), it is possible to give a characterization of the eigenfunctions $\omega_j$ and eigenvalues $\lambda_j$, $j \in \mathbb{N}$, as

\[
\{\omega_j\}_{j \in \mathbb{N}} = \left\{ \frac{e^{2i \pi jx}}{L} \right\}_{j \in \mathbb{N}} \quad \text{and} \quad \{\lambda_j\}_{j \in \mathbb{N}} = \left\{ \left( \frac{2\pi j}{L} \right)^2 + 1 \right\}_{j \in \mathbb{N}}.
\]

Consider $V_m$ the subspace spanned by the $m$ first eigenfunctions $\omega_1, \omega_2, \ldots, \omega_m$. Let $u_m(t) \in V_m$ be the function defined as

\[
(3.3) \quad u_m(t) = \sum_{j=1}^{m} g_{jm}(t) \omega_j.
\]

We see that solution $u_m$ in \((3.3)\) solves the following approximate problem

\[
\begin{aligned}
&\begin{cases}
\langle u_m''(t), \omega_j \rangle_{L^2_{\text{per}}, L^2_{\text{per}}} + \langle \nabla u_m(t), \nabla \omega_j \rangle_{L^2_{\text{per}}, L^2_{\text{per}}} = \\
-\langle u_m(t), \omega_j \rangle_{L^2_{\text{per}}, L^2_{\text{per}}} + \langle u_m(t) \log(|u_m(t)|^p), \omega_j \rangle_{L^2_{\text{per}}, L^2_{\text{per}}}, \quad j \in \{1, 2, \ldots, m\}.
\end{cases}
\end{aligned}
\]

\[
(3.4) \quad \begin{cases}
u_m(0) = u_{0,m} = \sum_{j=1}^{m} \alpha_{jm} \omega_j, \quad \alpha_{jm} = \langle u_0, \omega_j \rangle_{L^2_{\text{per}}, L^2_{\text{per}}}, \\
u_m'(0) = u_{1,m} = \sum_{j=1}^{m} \beta_{jm} \omega_j, \quad \beta_{jm} = \langle u_1, \omega_j \rangle_{L^2_{\text{per}}, L^2_{\text{per}}}.
\end{cases}
\]
Thus, Caratheodory’s Theorem can be used to establish that \( u_m \) has an absolutely continuous solution \( u_m \), defined on an interval \([0, T_m)\), where \( 0 < T_m \leq T \). Next, we need to show good bounds for the solution \( u_m \) to guarantee that solution \( u_m \) can be defined for all values of \( T > 0 \).

Multiplying both sides of the first identity in (3.4) by \( \overline{g_j'(t)} \) and adding the final result in \( j = 1, 2, \ldots, m \), one has

\[
\frac{d}{dt} \left( \| u'_m(t) \|_{L^2_{\text{per}}}^2 \right) + \frac{d}{dt} \left( \| \nabla u_m(t) \|_{L^2_{\text{per}}}^2 \right) + \left( 1 + \frac{p}{2} \right) \frac{d}{dt} \left( \| u_m(t) \|_{L^2_{\text{per}}}^2 \right) = p \left[ \frac{d}{dt} \left( \int_0^L |u_m(t)|^2 \log(|u_m(t)|) \, dx \right) \right].
\]

Hence for all \( t \in [0, T] \), we obtain

\[
\| u'_m(t) \|_{L^2_{\text{per}}}^2 + \| \nabla u_m(t) \|_{L^2_{\text{per}}}^2 + \left( 1 + \frac{p}{2} \right) \| u_m(t) \|_{L^2_{\text{per}}}^2 = p \int_0^L |u_m(t)|^2 \log(|u_m(t)|) \, dx.
\]

(3.5)

Since \( u_{0,m} \xrightarrow{m \rightarrow \infty} u_0 \) in \( H^1_{\text{per}}([0, L]) \) and \( u_{1,m} \xrightarrow{m \rightarrow \infty} u_1 \) in \( L^2_{\text{per}}([0, L]) \), there exists a constant \( C_1 > 0 \) such that

\[
\| u_{1,m} \|_{L^2_{\text{per}}}^2 + \| \nabla u_{0,m} \|_{L^2_{\text{per}}}^2 + \left( 1 + \frac{p}{2} \right) \| u_{0,m} \|_{L^2_{\text{per}}}^2 \leq C_1 \text{ for all } m \in \mathbb{N}.
\]

Furthermore, the Sobolev embedding \( H^1_{\text{per}}([0, L]) \hookrightarrow L^\infty_{\text{per}}([0, L]) \) yields

\[
\left| p \int_0^L |u_{0,m}|^2 \log(|u_{0,m}|) \, dx \right| \leq C_2 \text{ for all } m \in \mathbb{N}.
\]

Define the constant \( C_3 := C_1 + C_2 \). We can write

\[
\| u'_m(t) \|_{L^2_{\text{per}}}^2 + \| \nabla u_m(t) \|_{L^2_{\text{per}}}^2 + \left( 1 + \frac{p}{2} \right) \| u_m(t) \|_{L^2_{\text{per}}}^2 \leq C_3 + \frac{p}{2} \int_0^L |u_m(t)|^2 \log(|u_m(t)|^2) \, dx \text{ for all } m \in \mathbb{N}.
\]

(3.6)

In order to estimate the right-hand-side of the inequality (3.6), we see that

\[
\int_0^L |u_m(t)|^2 \log(|u_m(t)|) \, dx \leq \int_0^L |u_m(t)|^3 \, dx \leq \| u_m(t) \|_{L^\infty_{\text{per}}} \| u_m(t) \|_{L^2_{\text{per}}}^2.
\]

(3.7)
Using the Sobolev embedding $H^1_{per}(0, L) \hookrightarrow L^\infty_{per}(0, L)$ and an application of Young's inequality we obtain the existence of a constant $C_4 > 0$ such that

$$\int_0^T |u_m(t)|^2 \log(|u_m(t)|) \, dx \leq C_4 \|u_m(t)\|_{H^1_{per}} \|u_m(t)\|_{L^2_{per}}^2$$

(3.8)

$$\leq C_4 \left[ \frac{\varrho \|u_m(t)\|_{H^1_{per}}^2}{2} + \frac{\|u_m(t)\|_{L^2_{per}}^4}{2\varrho} \right],$$

where $\varrho > 0$ is a fixed parameter.

Let $\varrho > 0$ be sufficiently small. By combining (3.6) and (3.8), there exists a constant $C_5 > 0$ such that

$$\|u'_m(t)\|_{L^2_{per}}^2 + \|\nabla u_m(t)\|_{L^2_{per}}^2 + \|u_m(t)\|_{L^2_{per}}^2 \leq C_5 + C_5 \left[ \|u_m(t)\|_{L^2_{per}}^2 \right]^2 \quad \text{for all } m \in \mathbb{N}.$$  

(3.9)

Next, let us denote

$$u_m(t) = u_m(0) + \int_0^t \frac{\partial u_m}{\partial \varsigma}(\varsigma) \, d\varsigma.$$ 

Hence,

$$\|u_m(t)\|_{L^2_{per}}^2 \leq 2\|u_{0,m}\|_{L^2_{per}}^2 + 2 \left\| \int_0^t \frac{\partial u_m}{\partial \varsigma}(\varsigma) \, d\varsigma \right\|_{L^2_{per}}^2 \leq C_6 + 2 \int_0^t \|u'_m(\varsigma)\|_{L^2_{per}}^2 \, d\varsigma,$$

where $C_6 > 0$ is a constant. Using inequality (3.9),

$$\|u_m(t)\|_{L^2_{per}}^2 \leq C_7 + C_7 \int_0^t \left[ \|u_m(\varsigma)\|_{L^2_{per}}^2 \right]^2 \, d\varsigma \quad \text{for all } m \in \mathbb{N},$$

(3.10)

where $C_7 > 0$ is a constant.

From Gronwall-Bellman-Bihari inequality, there exists $C_8(T) > 0$ such that

$$\|u_m(t)\|_{L^2_{per}}^2 \leq C_8(T) \quad \text{for all } m \in \mathbb{N}.$$ 

In addition, by (3.9) there exists a constant $C_9(T) > 0$ satisfying

$$\|u'_m(t)\|_{L^2_{per}}^2 + \|\nabla u_m(t)\|_{L^2_{per}}^2 + \|u_m(t)\|_{L^2_{per}}^2 \leq C_9(T)$$

(3.11)

for all $m \in \mathbb{N}$. By (3.11), we can deduce that the interval of existence of approximate solutions can be chosen as $[0, T]$ for all $T > 0$. Furthermore,

$$\|u_m\|_{L^\infty(0, T; H^1_{per})}^2 + \|u'_m\|_{L^\infty(0, T; L^2_{per})}^2 \leq C_9(T) \quad \text{for all } m \in \mathbb{N}.$$ 

(3.12)

The next step is to estimate $\|v''_m(t)\|_{H^{-1}_{per}}$. Consider $v \in H^1_{per}(0, L)$ such that $\|v\|_{H^1_{per}} \leq 1$. Decompose $v := v_1 + v_2$, where $v_1 \in \text{span}\{\omega_j\}_{j=1}^m$ and $v_2 \in \text{span}\{\omega_j\}_{j=m+1}^\infty$.

$$0 = \langle v_2, \omega_j \rangle_{L^2_{per}, L^2_{per}}, \quad j \in \{1, 2, \ldots, m\}.$$ 

(3.13)
Identities in (3.4) and (3.13) give us
\[
\langle u_m''(t), v \rangle_{L^2_{per}, L^2_{per}} = -\langle \nabla u_m(t), \nabla v \rangle_{L^2_{per}, L^2_{per}} - \langle u_m(t), v_1 \rangle_{L^2_{per}, L^2_{per}}
\]
and
\[
|\langle u_m''(t), v \rangle_{L^2_{per}, L^2_{per}}| \leq 2\|u_m(t)\|_{H^1_{per}}
\]
(3.14)
\[
+ \left( \int_0^L |u_m(t)| \log(|u_m(t)|^p) \, dx \right) \|v_1\|_{L^\infty_{per}}.
\]

On the other hand, we can use inequality (3.12) to show that \(\|u_m(t)\|_{H^1_{per}} \leq \sqrt{C_9(T)}\) for all \(m \in \mathbb{N}\). Next, the Sobolev embedding \(H^1_{per}([0, L]) \hookrightarrow L^\infty_{per}([0, L])\) and the orthogonality of \(\{\omega_\nu\}_{\nu \in \mathbb{N}}\) in \(H^1_{per}([0, L])\) enable us to conclude \(\|v_1\|_{L^\infty_{per}} \leq C_{10}\|v_1\|_{H^1_{per}} \leq C_{10}\|v\|_{H^1_{per}} \leq C_{10}, \ C_{10} > 0\) being a constant. In addition, there exists a constant \(C_{11}(T) > 0\) such that
\[
\int_0^L |u_m(t)| \log(|u_m(t)|^p) \, dx \leq C_{11}(T) \text{ for all } m \in \mathbb{N}.
\]
Taking \(C_{12}(T) := 2\sqrt{C_9(T)} + C_{10}C_{11}(T)\), we have from (3.14) and the final inequality
\[
\|u_m''(t), v\rangle_{H^{-1}_{per}, H^1_{per}} \leq C_{12}(T) \text{ for all } m \in \mathbb{N}.
\]
Since \(v\) is arbitrary one has
(3.15)
\[
\|u_m''\|_{L^\infty(0, T; H^{-1}_{per})} \leq C_{12}(T) \text{ for all } m \in \mathbb{N}.
\]

Inequalities (3.12) and (3.13) enable us to consider a subsequence (still denoted by \(\{u_m\}_{m \in \mathbb{N}}\) and a function \(u\) such that
\[
\begin{align*}
u_m \rightharpoonup u & \quad \text{weak in } \ L^2(0, T; H^1_{per}([0, L])), \\
u_m' \rightharpoonup u' & \quad \text{weak in } \ L^2(0, T; L^2_{per}([0, L])), \\
u_m'' \rightharpoonup u'' & \quad \text{weak in } \ L^2(0, T; H^{-1}_{per}([0, L])), \\
u_m \rightharpoonup u & \quad \text{weak-* in } \ L^\infty(0, T; H^1_{per}([0, L])), \\
u_m' \rightharpoonup u' & \quad \text{weak-* in } \ L^\infty(0, T; L^2_{per}([0, L])), \\
u_m'' \rightharpoonup u'' & \quad \text{weak-* in } \ L^\infty(0, T; H^{-1}_{per}([0, L])).
\end{align*}
\]
(3.16)

Moreover, using Aubin-Lions-Simon Theorem (see [3] Theorem II.5.16) for more details) and the fact that \(u_m\) is bounded in \(L^2(0, T; H^1_{per}([0, L]))\), we can choose \(\{u_m\}_{m \in \mathbb{N}}\) such that
\[
u_m \rightharpoonup u \quad \text{strongly in } \ L^2(0, T; L^2_{per}([0, L])).
\]
(3.17)
As a consequence of this last convergence we obtain \(u_m \rightharpoonup u \text{ a.e. in } [0, L] \times [0, T]\). Now, since the map \(x \mapsto x \log(|x|^p)\) is continuous, one has
\[
|u_m \log(|u_m|^p) - u \log(|u|^p)| \rightharpoonup 0 \quad \text{a.e. in } [0, L] \times [0, T].
\]
On the other hand, from the Sobolev embedding $H_{\text{per}}^{1}([0,L]) \hookrightarrow L_{\text{per}}^{\infty}([0,L])$ we obtain that $|u_m\log(|u_m|^p) - u\log(|u|^p)|$ is bounded in $L^{\infty}([0,L] \times [0,T])$. Next, taking into account the Lebesgue Dominated Convergence Theorem, we have

$$u_m\log(|u_m|^p) \to u\log(|u|^p) \quad \text{strongly in} \quad L^2(0,T;L^2_{\text{per}}([0,L])).$$

Finally, one can pass to the limit at the equation (3.18) to obtain the existence of weak solutions related to (3.1) in a standard form. We can easily check that the initial conditions are also satisfied.

Step 2. Uniqueness of Solution.

Let $u$ and $v$ be two weak solutions for the Cauchy problem (3.1). Define $w := u - v$. We see that

$$\langle w_{tt}(\cdot,t), \zeta \rangle_{H_{\text{per}}^{1}} + \int_{0}^{L} \nabla w(\cdot,t) \cdot \nabla \zeta \, dx + \int_{0}^{L} w(\cdot,t) \zeta \, dx$$

$$= \int_{0}^{L} [u(\cdot,t) \log(|u(\cdot,t)|^p) - v(\cdot,t) \log(|v(\cdot,t)|^p)] \zeta \, dx \quad \text{a.e.} \ t \in [0,T],$$

for all $\zeta \in H_{\text{per}}^{1}([0,L])$. Furthermore, $w(\cdot,0) = 0$ and $w_t(\cdot,0) = 0$. Our intention is to prove that $w \equiv 0$, that is, the Cauchy problem in (3.1) has an unique weak solution. To do so, we need to stop with the proof of the proposition for a while to present the following auxiliary lemma:

**Lemma 3.3.** Consider $0 < T_0 < \frac{L}{4}$ and $w$ an $L$-periodic function at the spatial variable. Suppose that $w(\cdot,0) = 0$ and $f \in L^2(0,T_0;L^2_{\text{per}}([0,L]))$. Furthermore, let us assume that

$$\int_{[0,L] \times [0,T_0]} w(\zeta _{tt} - \zeta _{xx}) \, dx \, dt = \int_{[0,L] \times [0,T_0]} f \zeta \, dx \, dt,$$

for all smooth real function $\zeta$ defined over $\mathbb{R} \times [0,T_0]$ which is periodic at the spatial variable for all $t \in [0,T_0]$ and it satifies $\zeta(x,T_0) = \zeta(x,0) = 0$ for all $x \in \mathbb{R}$. Thus for all $(x,t) \in \mathbb{R} \times [0,T_0]$ one has

$$w(x,t) = \frac{1}{2} \int_{0}^{t} \int_{x-t+\tau}^{x+t-\tau} f(y,\tau) \, dy \, d\tau.$$

**Proof.** First of all, it is important to mention that standard methods of partial differential equations enable us to deduce that the concept of weak solutions can be rewritten in order to use (3.19). Thus, let $(x,t) \in \mathbb{R} \times (0,T_0]$ be fixed. Define an $L$-periodic smooth real function $\chi^* : \mathbb{R} \to \mathbb{R}$ as

$$\chi^*(y) := \begin{cases} 
0, & y \in [0, \frac{2t}{5}], \\
0 \leq \chi^*(y) \leq 1, & y \in [\frac{2t}{5}, \frac{4t}{5}], \\
1, & y \in [\frac{4t}{5}, t], \\
\chi^*(2t - y), & y \in [t, 2t], \\
0, & y \in [2t, L].
\end{cases}$$

In what follows, let us assume that $\chi^*$ is a nondecreasing function in the interval $[\frac{2\varepsilon}{5}, \frac{4\varepsilon}{5}]$. Now, let $\varepsilon > 0$ be sufficiently small and define the auxiliary $L$-periodic function

$$
\chi_\varepsilon(y) := \begin{cases}
0, & y \in [0, \frac{2\varepsilon}{5}], \\
\frac{1}{\varepsilon}(\chi^*)'(\frac{y}{\varepsilon}), & y \in [\frac{2\varepsilon}{5}, \frac{4\varepsilon}{5}], \\
1, & y \in [\frac{4\varepsilon}{5}, 5], \\
\chi_\varepsilon(2t - y), & y \in [t, 2t]. \\
0, & y \in [2t, L].
\end{cases}
$$

We have that

$$
\chi_\varepsilon(y) \xrightarrow{\varepsilon \to 0^+} \begin{cases}
1, & y \in (0, 2t) \\
0, & y \in (2t, L)
\end{cases}
$$

and

$$
\chi'_\varepsilon(y) = \begin{cases}
0, & y \in [0, \frac{2\varepsilon}{5}], \\
\frac{1}{\varepsilon}((\chi^*)'(\frac{y}{\varepsilon}), & y \in [\frac{2\varepsilon}{5}, \frac{4\varepsilon}{5}], \\
0, & y \in [\frac{4\varepsilon}{5}, 2t - \frac{4\varepsilon}{5}], \\
-\chi'_\varepsilon(2t - y) = -\frac{1}{\varepsilon}((\chi^*)'(\frac{2t-y}{\varepsilon}), & y \in [2t - \frac{4\varepsilon}{5}, 2t - \frac{2\varepsilon}{5}], \\
0, & y \in [2t - \frac{2\varepsilon}{5}, 2t].
\end{cases}
$$

Letting

$$
\zeta_\varepsilon(y, \tau) := \chi_\varepsilon(y - \tau - x + t) \chi_\varepsilon(x + t - y - \tau), \quad (y, \tau) \in \mathbb{R} \times [0, T_0].
$$

We see that $\zeta_\varepsilon$ is a smooth and $L$-periodic function. Therefore, we deduce from (3.19),

$$
\iint_{[0,L] \times [0,T_0]} w[\zeta_\varepsilon(t) - (\zeta_\varepsilon)_{xx}] \, dy \, d\tau = \iint_{[0,L] \times [0,T_0]} f\zeta_\varepsilon \, dy \, d\tau.
$$

Next, if $0 < x - t < x + t < L$ and $t \in (0, T_0]$ we get

$$
\{[\text{supp} \, (\zeta_\varepsilon)] \cap [0, L] \times [0, T_0]\} \subset \{(y, \tau); \, x - t + \tau < y < x + t - \tau, \, 0 < \tau < t\},
$$

since

$$
t < \frac{L}{2} - t \iff t < \frac{L}{4} \quad \text{and} \quad t \leq T_0 < \frac{L}{4}.
$$

Using (3.20), one can see from Lebesgue Dominated Convergence Theorem that

$$
\lim_{\varepsilon \to 0^+} \iint_{[0,L] \times [0,T_0]} f\zeta_\varepsilon \, dy \, d\tau = \int_0^t \int_{x-t+\tau}^{x+t-\tau} f(y, \tau) \, dy \, d\tau.
$$

By considering other values of $(x, t)$, it is possible to deduce a similar situation as in (3.22) since function $\zeta_\varepsilon$ is periodic at the spatial variable.

On the other hand,

$$
\iint_{[0,L] \times [0,T_0]} w[(\zeta_\varepsilon)_{tt} - (\zeta_\varepsilon)_{xx}] \, dy \, d\tau = (I)_{\varepsilon} + (II)_{\varepsilon} + (III)_{\varepsilon},
$$
Assume that $T$ and $w$. The proof of the proposition is completed by combining (3.21), (3.22) and (3.23).

\begin{align*}
(III)_\varepsilon &= 4 \int_{0}^{L} \int_{x-t+\tau - \frac{4\varepsilon}{5}}^{x-t+\tau + \frac{4\varepsilon}{5}} \chi'(y - \tau - x + t) \chi'(x + t - y - \tau) \, w(y, \tau) \, dy \, d\tau \\
&= O(\varepsilon) - 2 \int_{0}^{\frac{4\varepsilon}{5}} (\chi^*(\kappa))^t \chi^*(\kappa) \, w (x + t - \varepsilon \kappa, 0) \, d\kappa.
\end{align*}

Since $w(\cdot, 0) = 0$ we deduce for $\varepsilon \to 0^+$ that

$$
\lim_{\varepsilon \to 0^+} \int_{[0,L] \times [0,T_0]} w[(\zeta_{\varepsilon})_{tt} - (\zeta_{\varepsilon})_{xx}] \, dy \, d\tau = 2w(x, t).
$$

The proof of the proposition is completed by combining (3.21), (3.22) and (3.23).

Next, we continue with the proof of the Proposition 3.2 and we are going to follow the arguments in [12]. Indeed, let us consider

$$
f := -(u - v) + [u \log(|u|) - v \log(|v|)] \in L^2(0, T; L^2_{\text{per}}([0, L])).$$

Assume that $T_0 \leq T$, where $0 < T_0 < \frac{L}{4}$. Since $w$ satisfies (3.18) with $w(\cdot, 0) = 0$ and $w_t(\cdot, 0) = 0$, we can apply the characterization of the functionals in the dual space of $H^1_{\text{per}}([0, L])$ in order to deduce that $w$ satisfies (3.19). This last fact enables us to apply Lemma 3.9 to obtain

$$
w(x, t) = -\frac{1}{2} \int_{0}^{t} \int_{x-t+\tau}^{x+t-\tau} [u(y, \tau) - v(y, \tau)] \, dy \, d\tau
$$

$$
+ \frac{p}{2} \int_{0}^{t} \int_{x-t+\tau}^{x+t-\tau} [u(y, \tau) \log(|u(y, \tau)|) - v(y, \tau) \log(|v(y, \tau)|)] \, dy \, d\tau,
$$

$$
= 2w \left( x, t - \frac{4t \varepsilon}{5} \right) + O(\varepsilon),
$$

(II) \quad 4 \int_{0}^{t} \int_{x-t+\tau + \frac{4\varepsilon}{5}}^{x-t+\tau - \frac{4\varepsilon}{5}} \chi'(y - \tau - x + t) \chi'(x + t - y - \tau) \, w(y, \tau) \, dy \, d\tau \\
= O(\varepsilon) - 2 \int_{0}^{\frac{4\varepsilon}{5}} (\chi^*(\kappa))^t \chi^*(\kappa) \, w (x - t - \varepsilon \kappa, 0) \, d\kappa.
\]
for all \((x, t) \in \mathbb{R} \times [0, T_0]\).

If \((y, \tau) \in \mathbb{R} \times [0, T_0]\) and \(|u(y, \tau)| \leq |v(y, \tau)|\), we can use Proposition 2.1 to get
\[
|u(y, \tau) \log(|u(y, \tau)|) - v(y, \tau) \log(|v(y, \tau)|)| \leq |1 + |\log(|v(y, \tau)||)| \cdot |u(y, \tau) - v(y, \tau)|.
\]

First, let us suppose \(|v(y, \tau)| \leq 1\). So, \(|v(y, \tau) - u(y, \tau)| \leq |v(y, \tau)| + |u(y, \tau)| \leq 2|v(y, \tau)|\) and
\[
|\log(|v(y, \tau)|)| \leq |\log(|w(y, \tau)|)| + \log 2.
\]

Now, if \(|v(y, \tau)| \geq 1\) then
\[
|\log(|v(y, \tau)|)| \leq 1 + |v(y, \tau)|.
\]

Using (3.25), (3.26) and (3.27), we have
\[
|u(y, \tau) \log(|u(y, \tau)|) - v(y, \tau) \log(|v(y, \tau)|)| \leq [2 + \log 2 + |v(y, \tau)| + |\log(|w(y, \tau)||)| \cdot |w(y, \tau)|.
\]

On the other hand, if we assume \(|v(y, \tau)| \leq |u(y, \tau)|\), then
\[
|u(y, \tau) \log(|u(y, \tau)|) - v(y, \tau) \log(|v(y, \tau)|)| \leq [2 + \log 2 + |u(y, \tau)| + |\log(|w(y, \tau)||)| \cdot |w(y, \tau)|.
\]

Inequalities (3.28) and (3.29) can be applied to establish
\[
|u(y, \tau) \log(|u(y, \tau)|) - v(y, \tau) \log(|v(y, \tau)|)| \leq [4 + \log 4 + |u(y, \tau)| + |v(y, \tau)| + 2|\log(|w(y, \tau)||)|] \cdot |w(y, \tau)|
\]
\[
\leq C_{13} \cdot [1 + |\log(|w(y, \tau)||)|] \cdot |w(y, \tau)|,
\]
where \(C_{13} > 0\) is a constant. The existence of \(C_{13} > 0\) is obtained since we have
\[
|u(y, \tau)| \leq \|u\|_{L^\infty([0,L]\times[0,T])} \quad \text{and} \quad |v(y, \tau)| \leq \|v\|_{L^\infty([0,L]\times[0,T])}.
\]

From (3.24) and (3.30), we have \((x, t) \in \mathbb{R} \times [0, T_0]\),
\[
|w(x, t)| \leq \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} |u(y, \tau) - v(y, \tau)| \, dy \, d\tau
\]
\[
+ \frac{pC_{13}}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} [1 + |\log(|w(y, \tau)||)|] \cdot |w(y, \tau)| \, dy \, d\tau
\]
\[
\leq (1 + pC_{13})T_0 \int_0^t \|w(\cdot, \tau)\|_{L^\infty_{per}} \, d\tau + pC_{13}T_0 \int_0^t \|w(\cdot, \tau) \log(|w(\cdot, \tau)||)\|_{L^\infty_{per}} \, d\tau.
\]

Let us define the constant \(C_{14} := \frac{(1+pC_{13})L}{4}\). Inequality (3.31) gives us
\[
\|w(\cdot, t)\|_{L^\infty_{per}} \leq C_{14} \int_0^t \|w(\cdot, \tau)\|_{L^\infty_{per}} \, d\tau + C_{14} \int_0^t \|w(\cdot, \tau) \log(|w(\cdot, \tau)||)\|_{L^\infty_{per}} \, d\tau,
\]
for all \( t \in [0, T_0] \). Hence, using (3.32) and the fact that \( w \) is bounded on \([0, L] \times [0, T] \), there exists \( T_1 \in (0, T_0) \) such that \( \|w(\cdot, t)\|_{L_{\text{per}}^\infty} < \frac{1}{e} \) for all \( t \in [0, T_1] \).

Next, define the function \( \tilde{F}(\beta) = |\beta| \log(|\beta|) \) for all \( \beta \in \mathbb{C} \). Since \( |\beta| < -\tilde{F}(\beta) \) for all \( |\beta| \in (0, \frac{1}{e}) \), one has

\[
\int_0^t \|w(\cdot, \tau)\|_{L_{\text{per}}^\infty(\mathbb{R})} \, d\tau \leq -\int_0^t \|w(\cdot, \tau)\|_{L_{\text{per}}^\infty} \log(\|w(\cdot, \tau)\|_{L_{\text{per}}^\infty}) \, d\tau.
\]

Thus, from (3.32), we see that

\[
\|w(\cdot, t)\|_{L_{\text{per}}^\infty} \leq -2C_{14} \int_0^t \|w(\cdot, \tau)\|_{L_{\text{per}}^\infty} \log(\|w(\cdot, \tau)\|_{L_{\text{per}}^\infty}) \, d\tau \quad \text{for all } t \in [0, T_1].
\]

Using Proposition 2.2, we finally conclude \( \|w(\cdot, t)\|_{L_{\text{per}}^\infty} = 0 \) a.e. \( t \in [0, T_1] \).

Let us prove the result over \([0, T] \). Consider the auxiliary functions

\[
\tilde{w}(\cdot, \cdot) := w(\cdot, \cdot + T), \quad \tilde{u}(\cdot, \cdot) := u(\cdot, \cdot + T) \quad \text{and} \quad \tilde{v}(\cdot, \cdot) := v(\cdot, \cdot + T).
\]

The function \( \tilde{w} \) is a weak solution to the Cauchy problem

\[
\begin{cases}
\tilde{w}_{tt} - \tilde{w}_{xx} = - (\tilde{u} - \tilde{v}) + p[\log(\tilde{u})\tilde{u} - \log(\tilde{v})\tilde{v}], & (x, t) \in \mathbb{R} \times (0, T_1). \\
\tilde{w}(x, 0) = 0, & x \in \mathbb{R}. \\
\tilde{w}_t(x, 0) = 0, & x \in \mathbb{R}.
\end{cases}
\]

So,

\[
\tilde{w}(x, t) = -\frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} [\tilde{u}(y, \tau) - \tilde{v}(y, \tau)] \, dy \, d\tau
\]

\[
+ \frac{p}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} [\tilde{u}(y, \tau) \log(\tilde{u}(y, \tau))] - \tilde{v}(y, \tau) \log(\tilde{v}(y, \tau))] \, dy \, d\tau,
\]

for all \((x, t) \in \mathbb{R} \times [0, T_1] \). Similar argument above gives us \( \|w(\cdot, t)\|_{L_{\text{per}}^\infty} = 0 \) a.e. \( t \in [T_1, 2T_1] \).

Therefore, we deduce the uniqueness for all values of \( T > 0 \) using an interaction argument. We have completed the proof of Proposition 3.2.

Now we prove that our weak solution obtained in Proposition 3.2 satisfies equalities in (1.6). Thus, Theorem 1.1 can be proven by combining both results.

**Proposition 3.4.** The weak solution obtained in Proposition 3.2 satisfies the following conserved quantities:

\[
E(u(\cdot, t), u'(\cdot, t)) = E(u_0, u_1) \quad \text{and} \quad F(u(\cdot, t), u'(\cdot, t)) = F(u_0, u_1) \quad \text{a.e. } t \in [0, T],
\]

where \( E \) and \( F \) are defined as in (1.7) and (1.8).

**Proof.** First of all, we prove that the conserved quantity (1.7) holds. Indeed, the construction of \( u_m \) and (3.5) gives us

\[
E(u_m(t), u'_m(t)) = E(u_{0,m}, u_{1,m}).
\]
Consider \( \vartheta \in C^0([0,T]) \), \( \vartheta \geq 0 \). Multiplying the identity (3.33) by the function \( \vartheta \) and integrating the result over interval \([0, T]\),

\[
\int_0^T \mathcal{E}(u_m(t), u'_m(t)) \vartheta(t) \, dt = \int_0^T \mathcal{E}(u_{0,m}, u_{1,m}) \vartheta(t) \, dt.
\]

Next, since \( u_{0,m} \xrightarrow{m \to \infty} u_0 \) in \( H^1_{\text{per}}([0,L]) \) and \( u_{1,m} \xrightarrow{m \to \infty} u_1 \) in \( L^2_{\text{per}}([0,L]) \), we get

\[
\int_0^T \mathcal{E}(u_{0,m}, u_{1,m}) \vartheta(t) \, dt \xrightarrow{m \to \infty} \int_0^T \mathcal{E}(u_0, u_1) \vartheta(t) \, dt.
\]

In addition, using (3.17),

\[
\int_0^T \vartheta(t) \int_0^L |u_m(x,t)|^2 \, dx \, dt \xrightarrow{m \to \infty} \int_0^T \vartheta(t) \int_0^L |u(x,t)|^2 \, dx \, dt
\]

and

\[
\int_0^T \vartheta(t) \int_0^L |u_m(x,t)|^2 \log(|u_m(x,t)|^p) \, dx \, dt \xrightarrow{m \to \infty} \int_0^T \vartheta(t) \int_0^L |u(x,t)|^2 \log(|u(x,t)|^p) \, dx \, dt.
\]

Next, the first limit in (3.16) enables us to deduce

\[ \sqrt{\vartheta} u_m \rightharpoonup \sqrt{\vartheta} u \text{ weakly in } L^2(0,T; H^1_{\text{per}}([0,L])). \]

So, Fatou’s Lemma applied to last weak convergence gives us

\[
\int_0^T \vartheta(t) \|u(t)\|^2_{H^1_{\text{per}}} \, dt \leq \liminf_{m \to \infty} \int_0^T \vartheta(t) \|u_m(t)\|^2_{H^1_{\text{per}}} \, dt.
\]

Similarly, using the second limit in (3.17), we get

\[
\int_0^T \vartheta(t) \|u'(t)\|^2_{L^2_{\text{per}}} \, dt \leq \liminf_{m \to \infty} \int_0^T \vartheta(t) \|u'_m(t)\|^2_{L^2_{\text{per}}} \, dt.
\]

From (3.34), (3.35), (3.36), (3.37), (3.38) and (3.39), we conclude the following inequality

\[
\int_0^T \mathcal{E}(u(\cdot,t), u'_t(\cdot,t)) \vartheta(t) \, dt \leq \int_0^T \mathcal{E}(u_0, u_1) \vartheta(t) \, dt.
\]

Therefore, using the fact that function \( \vartheta \) is arbitrary, we obtain

\[
\mathcal{E}(u(\cdot,t), u'_t(\cdot,t)) \leq \mathcal{E}(u_0, u_1) \text{ a.e. } t \in [0,T].
\]

On the other hand, consider \( s, t \) satisfying \( 0 < s < t < T \). Since \( u \) is a weak solution, one has

\[
u_{tt}(\cdot, \xi) - u_{xx}(\cdot, \xi) + u(\cdot, \xi) - u(\cdot, \xi) \log(|u(\cdot, \xi)|^p) = 0 \text{ in } H^1_{\text{per}}([0,L]),
\]
a.e. \( \xi \in [0,T] \). Furthermore,

\[
u_{tt} - u_{xx} + u - u \log(|u|^p) = 0 \text{ in } L^2(0,T; H^1_{\text{per}}([0,L])).
\]
Let us consider $n \in \mathbb{N}$, such that $n > \max \left\{ \frac{1}{s}, \frac{1}{t} \right\}$. Define function $\vartheta_n$ as

$$
\vartheta_n(x) = \begin{cases} 
0, & 0 \leq x \leq s - \frac{1}{n}, \\
1 + n(x - s), & s - \frac{1}{n} \leq x \leq s, \\
1, & s \leq x \leq t, \\
1 - n(x - t), & t \leq x \leq t + \frac{1}{n}, \\
0, & t + \frac{1}{n} \leq x \leq T.
\end{cases}
$$

Next, let $\{\rho_k\}_{k \in \mathbb{N}} \subset C^\infty_0(\mathbb{R})$ be the standard mollifier, where $\rho_k$ is an even function satisfying $\text{supp}(\rho_k) \subset [-\frac{1}{k}, \frac{1}{k}]$ for all $k \in \mathbb{N}$. Consider $k > \max \left\{ \frac{2n}{n-1}, \frac{2n}{T-n-1} \right\}$ and define function $\Theta_{n,k} := \vartheta_n[(\vartheta_nu') * \rho_k * \rho_k]$. Here, the symbol $*$ denotes the usual convolution in time-variable. We have the following property

$$
\text{supp}(\Theta_{n,k}) \subset \left[ s - \frac{1}{n}, t + \frac{1}{n} \right] + \left[ -\frac{2}{k'}, \frac{2}{k'} \right] \subset (0, T).
$$

One has

$$
(\vartheta_n * \rho_k) * \rho_k = (\vartheta_n') * \rho_k * \rho_k = (\vartheta_n * \rho_k) * \rho_k + (\vartheta_n') * \rho_k * \rho_k
$$

and

$$
\Theta_{n,k} = \vartheta_n[(\vartheta_nu') * \rho_k * \rho_k] - (\vartheta_n') * \rho_k * \rho_k \in L^2(0, T; H^1_{\text{per}}([0, L])).
$$

Hence, from the identity given in (3.41), we get

$$
\int_0^T \langle u'', \Theta_{n,k}(\xi) \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi - \int_0^T \langle u_{xx}(\cdot, \xi), \Theta_{n,k}(\xi) \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi
$$

$$
+ \int_0^T \langle u(\cdot, \xi), \Theta_{n,k}(\xi) \rangle_{L^2_{\text{per}}, L^2_{\text{per}}} d\xi = \int_0^T \langle \vartheta_n''(\cdot, \xi), \Theta_{n,k}(\xi) \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi
$$

$$
+ \int_0^T \langle (\vartheta_n')' * \rho_k, (\vartheta_nu') * \rho_k \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi
$$

$$
- \int_0^T \langle (\vartheta_n'') * \rho_k, (\vartheta_nu') * \rho_k \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi.
$$

In what follows, we will omit temporal variables. A similar procedure as in [20] Theorem 1.6] establishes a convenient expression to the first term in the left-hand side of (3.42) as

$$
\int_0^T \langle u'', \Theta_{n,k} \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi = \int_0^T \langle (\vartheta_nu')' * \rho_k, (\vartheta_nu') * \rho_k \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi
$$

$$
- \int_0^T \langle (\vartheta_n'') * \rho_k, (\vartheta_nu') * \rho_k \rangle_{H^{-1}_{\text{per}}, H^1_{\text{per}}} d\xi.
$$

In addition, since

$$
\vartheta_nu' \xrightarrow{k \to \infty} \vartheta_nu' \quad \text{in} \quad L^2(0, T; L^2_{\text{per}}([0, L]))
$$

and

$$
\vartheta_nu' \xrightarrow{k \to \infty} \vartheta_nu' \quad \text{in} \quad L^2(0, T; L^2_{\text{per}}([0, L])),
$$
we deduce, from (3.44) and (3.45),
\[
\lim_{k \to \infty} \int_0^T \langle (\partial_n^l u') * \rho_k, (\partial_n u') * \rho_k \rangle_{H^{-1}_{per},H^{1}_{per}} \, d\xi = \lim_{k \to \infty} \int_0^T \langle (\partial_n^l u') * \rho_k, (\partial_n u') * \rho_k \rangle_{L^2_{per},L^2_{per}} \, d\xi
\]
(3.46)
\[
= \int_0^T \langle \partial_n^l u', \partial_n u' \rangle_{L^2_{per},L^2_{per}} \, d\xi = \int_0^T \partial_n \partial_n^l \|u'\|_{L^2_{per}}^2 \, d\xi.
\]
Furthermore, since supp \([(\partial_n u') * \rho_k] \subset \left[ s - \frac{1}{n}, t + \frac{1}{n} \right] + \left[ -\frac{1}{k}, \frac{1}{k} \right] \subset (0,T), we obtain
\[
\Re \left( \int_0^T \langle (\partial_n u') * \rho_k, (\partial_n u') * \rho_k \rangle_{H^{-1}_{per},H^{1}_{per}} \, d\xi \right) = \frac{1}{2} \int_0^T \partial_\xi \left\| (\partial_n u') * \rho_k \right\|_{L^2_{per}}^2 \, d\xi = 0.
\]
(3.47)
From (3.43), (3.46) and (3.47), we see that
\[
\lim_{k \to \infty} \Re \left( \int_0^T \langle u', \Theta_{n,k} \rangle_{H^{-1}_{per},H^{1}_{per}} \, d\xi \right) = - \int_0^T \partial_n \partial_n^l \|u'\|_{L^2_{per}}^2 \, d\xi.
\]
Moreover,
\[
- \int_0^T \langle u_{xx}, \Theta_{n,k} \rangle_{H^{-1}_{per},H^{1}_{per}} \, d\xi + \int_0^T \langle u, \Theta_{n,k} \rangle_{L^2_{per},L^2_{per}} \, d\xi
\]
(3.49)
\[
= \int_0^T \langle u, \partial_n [(\partial_n u') * \rho_k] \rangle_{H^{1}_{per},H^{1}_{per}} \, d\xi
\]
\[
= \int_0^T \langle (\partial_n u) * \rho_k, (\partial_n u) * \rho_k \rangle_{H^{1}_{per},H^{1}_{per}} \, d\xi - \int_0^T \langle (\partial_n u) * \rho_k, (\partial_n^l u) * \rho_k \rangle_{H^1_{per},H^{1}_{per}} \, d\xi.
\]
On the left-hand-side of (3.49), it is possible to use a similar argument as in (3.48) to obtain
\[
\lim_{k \to \infty} \Re \left( - \int_0^T \langle u_{xx}, \Theta_{n,k} \rangle_{H^{-1}_{per},H^{1}_{per}} \, d\xi + \int_0^T \langle u, \Theta_{n,k} \rangle_{L^2_{per},L^2_{per}} \, d\xi \right)
\]
(3.50)
\[
= - \int_0^T \partial_n \partial_n^l \|u\|_{H^{1}_{per}}^2 \, d\xi.
\]
Next,
\[
\lim_{k \to \infty} \int_0^T \langle u \log(|u|^p), \Theta_{n,k} \rangle_{L^2_{per},L^2_{per}} \, d\xi = \int_0^T \partial_n^2 \langle u \log(|u|^p), u' \rangle_{L^2_{per},L^2_{per}} \, d\xi.
\]
(3.51)
Hence, collecting the results in (3.42), (3.48), (3.50) and (3.51), we have for $k$ large enough,

\[-\int_0^T \partial_n \vartheta_n \left( \| u' (\cdot, \xi) \|^2_{L^2_{\text{per}}} + \| u (\cdot, \xi) \|^2_{H^1_{\text{per}}} \right) d\xi - \frac{p}{2} \int_0^T \partial_n (\xi) \vartheta'_n (\xi) \left[ \int_0^L |u(x, \xi)|^2 dx \right] d\xi \]

\[= -\frac{p}{2} \int_0^T \partial_n (\xi) \vartheta'_n (\xi) \left[ \int_0^L |u(x, \xi)|^2 \log(|u(x, \xi)|^2) dx \right] d\xi. \quad (3.52)\]

Now for $\Lambda \in L^1([0, T])$ one has

\[-\int_0^T \partial_n \vartheta_n \Lambda \ d\xi \xrightarrow{n \to \infty} \frac{1}{2} \Lambda(t) - \frac{1}{2} \Lambda(s) \quad (3.53)\]

(see [20] pg. 25). Convergence in (3.53) combined with (3.52) enables us to conclude

\[\mathcal{E}(u(\cdot, t), u'(\cdot, t)) = \mathcal{E}(u(\cdot, s), u'(\cdot, s)), \quad 0 < s < t < T. \quad (3.54)\]

Since $u \in C^0([0, T]; L^2_{\text{per}}([0, L]))$, we obtain

\[\int_0^L |u(x, s)|^2 \ dx \xrightarrow{s \to 0^+} \int_0^L |u(x, 0)|^2 \ dx = \int_0^L |u_0(x)|^2 \ dx. \quad (3.55)\]

In addition,

\[\int_0^L |u(x, s)|^2 \log(|u(x, s)|^p) dx \xrightarrow{s \to 0^+} \int_0^L |u_0(x)|^2 \log(|u_0(x)|^p) dx. \quad (3.56)\]

On the other hand, the fact that $u \in C_s(0, T; H^1_{\text{per}}([0, L]))$ and Fatou’s Lemma imply that

\[\|u_0\|_{H^1_{\text{per}}}^2 = \|u(\cdot, 0)\|_{H^1_{\text{per}}}^2 \leq \liminf_{s \to 0^+} \|u(\cdot, s)\|_{H^1_{\text{per}}}^2. \quad (3.57)\]

Similarly, since $u' \in C_s(0, T; L^2_{\text{per}}([0, L]))$, we get

\[\|u_1\|_{L^2_{\text{per}}}^2 = \|u'(\cdot, 0)\|_{L^2_{\text{per}}}^2 \leq \liminf_{s \to 0^+} \|u'(\cdot, s)\|_{L^2_{\text{per}}}^2. \quad (3.58)\]

From relations (3.54), (3.55), (3.56), (3.57), and (3.58), we obtain that \(\mathcal{E}(u_0, u_1) \leq \liminf_{s \to 0^+} \mathcal{E}(u(\cdot, s), u'(\cdot, s)) = \mathcal{E}(u(\cdot, t), u'(\cdot, t)) \ a.e. \ t \in [0, T]. \quad (3.59)\]

Therefore, from (3.40) and (3.59) \(\mathcal{E}(u(\cdot, t), u'(\cdot, t)) = \mathcal{E}(u_0, u_1) \ a.e. \ t \in [0, T]. \) So, \(\mathcal{E}\) is a conserved quantity.

The next step is to prove the conserved quantity in (1.18). Consider a function $\tilde{\vartheta} \in \mathcal{D}([0, T])$. We recall that function $u_m$ satisfies the identity

\[u'''_m(t) - \Delta u_m(t) + u_m(t) \log(|u_m(t)|^p) = 0 \quad \text{in} \ V_m \ a.e. \ t \in [0, T]. \]

Since $\{\omega_\nu\}_{\nu \in \mathbb{N}} \subset H^3_{\text{per}}([0, L])$, we have

\[(u_m)_{tt} - (u_m)_{xx} + u_m - u_m \log(|u_m|^p) = 0 \quad \text{a.e.} \ (x, t) \in [0, L] \times [0, T]. \]

We claim that

\[\text{Im} \int_0^L u_m(x, t) u'_m(x, t) \ dx = \text{Im} \int_0^L u_{0,m}(x) u_{1,m}(x) \ dx \ a.e. \ t \in [0, T]. \quad (3.61)\]
Indeed, from (3.60) we have that (3.61) occurs since
\[(3.62) \quad \int_0^L \Re u_m(x,t) \Im u_m''(x,t) - \Im u_m(x,t) \Re u_m''(x,t) \, dx = 0 \text{ a.e. } t \in [0,T],\]
and identities (3.61) and (3.62) are equivalent.

Multiplying identity (3.61) by \(\tilde{\vartheta}\) and integrating the result over the interval \([0,T]\), we deduce for \(m\) large,
\[(3.63) \quad \int_0^T \tilde{\vartheta}(t) \left[ \Im \int_0^L \overline{u(x,t)} u'(x,t) \, dx \right] dt = \int_0^T \tilde{\vartheta}(t) \left[ \Im \int_0^L \overline{u_0(x)} u_1(x) \, dx \right] dt.\]

Finally, using (3.63) and since function \(\tilde{\vartheta}\) is arbitrary, we have that
\[\Im \int_0^L \overline{u(x,t)} u'(x,t) \, dx = \Im \int_0^L \overline{u_0(x)} u_1(x) \, dx \text{ a.e. } t \in [0,T].\]
Therefore, \(\mathcal{F}\) is also a conserved quantity.

Remark 3.5. A similar argument as determined in the proof of Proposition 3.4 to prove that \(\mathcal{E}\) is a conserved quantity enables us to deduce
\[\|u(\cdot,t_0)\|_{H^1_{\text{per}}}^2 \leq \liminf_{s \to t_0} \|u(\cdot,s)\|_{H^1_{\text{per}}}^2, \quad \|u_t(\cdot,t_0)\|_{L^2_{\text{per}}}^2 \leq \liminf_{s \to t_0} \|u_t(\cdot,s)\|_{L^2_{\text{per}}}^2,\]
and
\[\lim_{s \to t_0} \|u(\cdot,s)\|_{L^2_{\text{per}}}^2 = \|u(\cdot,t_0)\|_{L^2_{\text{per}}}^2.\]
Moreover,
\[\lim_{s \to t_0} \int_0^L \log(|u(x,s)|^p)|u(x,s)|^2 \, dx = \int_0^L \log(|u(x,t_0)|^p)|u(x,t_0)|^2 \, dx,\]
for all \(t_0 \in (0,T)\). Since \(\mathcal{E}\) is a conserved quantity, the arguments contained in Cazenave 13 Chapter II, Lemma 2.4.4 give us that
\[u \in C^0([0,T];H^1_{\text{per}}([0,L])) \quad \text{and} \quad u_t \in C^0([0,T];L^2_{\text{per}}([0,L])).\]

Existence and Uniqueness in \(H^1_{\text{per},e}([0,L])\).

Similar arguments can be used to show existence and uniqueness of weak solutions \(u\) related to the Cauchy problem (3.1) in a convenient subspace constituted by even periodic functions. Suppose \(p\) and \(L\) as above. Consider \(T > 0\), \(u_0 \in H^1_{\text{per},e}([0,L])\) and \(u_1 \in L^2_{\text{per},e}([0,L])\). We have the following result:

**Theorem 3.6.** There exists an unique weak solution \(u : \mathbb{R} \times [0,T] \to \mathbb{C}\) for the Cauchy problem (3.7). In addition, solution \(u\) must satisfy \(u \in L^\infty(0,T;H^1_{\text{per},e}([0,L]))\), \(u_t \in L^\infty(0,T;L^2_{\text{per},e}([0,L]))\) and \(u_{tt} \in L^\infty(0,T;(H^1_{\text{per},e}([0,L]))')\).
4. Orbital Stability of Standing Waves for the Logarithmic Klein-Gordon Equation

Now, we establish the orbital stability of periodic waves related to the equation (1.1). First, we present the existence of periodic standing waves \( u(x,t) = e^{ict}\varphi(x) \), \( t > 0 \), where \( c \in \mathbb{R} \) is called frequency of the wave and \( \varphi \) is a smooth periodic function. After that, we need some basic tools concerning the spectral analysis related to the linearized operators in (1.14) and (1.15). To do so, we recall the arguments in [22] which give us the spectral information for the associated Hill operator. Finally, we present the orbital stability of periodic standing waves using the arguments in [8], [17] and [28].

4.1. Existence of Periodic Solutions. Let us consider \( p \in \mathbb{Z}^+ \) and \( c \in \mathbb{R} \). We seek for periodic waves of the form \( u(x,t) = e^{ict}\varphi(x) \) where \( \varphi \) is a smooth \( L \)-periodic function. By substituting this kind of solution in (1.1) with \( \mu = 1 \), we obtain the following Euler-Lagrange equation,

\[
-\varphi'' + h(c,\varphi) = 0,
\]

(4.1)

where \( h \) is given by \( h(c,\varphi) = (1 - c^2)\varphi - \log(|\varphi|^p)\varphi \).

We suppose that the function \( \varphi \) is strictly positive. So, \( h \) is smooth in the open subset \( \mathcal{O} \subset \mathbb{R} = \mathbb{R} \times (0, +\infty) \). Equation (4.1) is conservative and periodic solutions are contained in the level curves of the energy

\[
\mathcal{H}(\varphi,\xi) = \frac{\xi^2}{2} + \left(\frac{1 - c^2}{2} + \frac{p}{4}\right)\varphi^2 - \frac{1}{2} \log(|\varphi|^p)\varphi^2,
\]

where \( \xi = \varphi' \). The function \( H \),

\[
H(c,\varphi) = \left(\frac{1 - c^2}{2} + \frac{p}{4}\right)\varphi^2 - \frac{1}{2} \log(|\varphi|^p)\varphi^2,
\]

satisfies \( \frac{\partial H}{\partial \varphi} = h \) and \( \lim_{\varphi \to 0^+} H(c,\varphi) = 0 \).

For a fixed \( c \in \mathbb{R} \), we see that function \( h(c,\cdot) \) has only three zeros, namely, \( -e^{\frac{1-c^2}{p}}, 0, e^{\frac{1-c^2}{p}} \), since \( h(c,\varphi) \to 0 \) if \( \varphi \to 0^+ \). In our analysis, we consider the consecutive roots \( r_1(c) = 0 \) and \( r_2(c) = e^{\frac{1-c^2}{p}} \). On \( \mathcal{O} \) the derivative of \( h \) with respect to parameter \( \varphi \) is given by

\[
\frac{\partial h}{\partial \varphi}(c,\varphi) = 1 - c^2 - p - \log(|\varphi|^p).
\]

So, \( \frac{\partial h}{\partial \varphi}(c, e^{\frac{1-c^2}{p}}) = -p < 0 \) and \( \lim_{\varphi \to 0^+} \frac{\partial h}{\partial \varphi}(c,\varphi) = \infty \). From standard ODE theory, the pair \((\varphi, \varphi') = \left( e^{\frac{1-c^2}{p}}, 0 \right)\) is a center point and \((\varphi, \varphi') = (0, 0)\) is a saddle point. Around the center point, we obtain periodic solutions for the equation (4.1). Furthermore, the function

\[
g_c(x) = e^{\frac{1}{2} + \frac{1-c^2}{p}} e^{-\frac{x^2}{4}}
\]
is a solitary wave for the equation (4.1). The pair \((\varrho_c, \varrho'_c)\) determines a closed curve \(C^\infty\) which contains \((r_2(c), 0)\) in its interior. In addition, \((\varrho_c, \varrho'_c)\) also satisfies the identity

\[
\mathcal{H}(\varrho_c, \varrho'_c) = -\frac{(\varrho'_c)^2}{2} + \left( \frac{1 - c^2}{2} + \frac{p}{4} \right) \varrho_c^2 - \frac{1}{2} \log(\varrho_c^p) \varrho_c^2 = 0 = \mathcal{H}(0, 0).
\]

In applications, the closed curve \(C^\infty\) is formed by either graphs of a homoclinic orbits or graphs of pairs heteroclinic orbits of (4.1). All the orbits (4.1) that live inside curve \(C^\infty\) are positive periodic orbits that turn around of \((r_2(c), 0)\) and they are contained in the level curves \(\mathcal{H}(\varphi, \xi) = B\). Here, \(B\) is a real constant satisfying

\[
0 = \mathcal{H}(0, 0) < B < \mathcal{H}\left( e^{\frac{1-c^2}{p}}, 0 \right) = \frac{p}{4} e^{\frac{2(1-c^2)}{p}}.
\]

We have the following result:

**Proposition 4.1.** For all \(c \in \mathbb{R}\), the equation

\[-\varphi'' + (1 - c^2)\varphi - \log(|\varphi|^p)\varphi = 0\]

has a positive \(L_c\)-periodic solution, where \(L_c \in \left(\frac{2\pi}{\sqrt{p}}, \infty\right)\). Moreover, the solution \(\varphi = \varphi_c\) and the period \(L_c\) are continuously differentiable with respect to the parameter \(c\).

**Proof.** See [22, Theorem 2.1].

Standard ODE theory gives us that if the pair \((\alpha_0, \alpha_1) \in \mathbb{R}^2, (\alpha_0, \alpha_1) \neq (r_2(c), 0)\) belongs to the interior of the closed curve \(C^\infty\), then there exists a unique periodic positive solution \(\varphi\) of (4.1) with \(\varphi(0) = \alpha_0\) and \(\varphi'(0) = \alpha_1\). Moreover, if

\[
e^{\frac{1-c^2}{p}} = r_2(c) < \alpha_0 < \varrho_c(0) = e^{\frac{1}{2} + \frac{1-c^2}{p}}\quad \text{and} \quad \alpha_1 = 0,
\]

we can use the symmetry of the problem in order to see that \(\varphi\) is an even function. In particular,

\[
\max_{x \in \mathbb{R}} \varphi(x) = \varphi(0)\quad \text{and} \quad \text{period satisfies} \quad L_c > \frac{2\pi}{\sqrt{p}}.
\]

**4.2. Spectral Analysis.** Let \(\varphi_c\) be the \(L_c\)-periodic solution obtained in Proposition 4.1. Define the linearized operator \(\mathcal{L}_{\varphi_c}\) around \((\varphi_c, ic\varphi_c) = (\varphi_c, c\varphi_c, 0, 0)\),

\[
\mathcal{L}_{\varphi_c} = \begin{pmatrix}
-\partial_x^2 + 1 - \log(|\varphi_c|^p) - p & -c & 0 & 0 \\
-c & 1 & 0 & 0 \\
0 & 0 & -\partial_x^2 + 1 - \log(|\varphi_c|^p) & c \\
0 & 0 & c & 1
\end{pmatrix},
\]

defined in \([L^2_{\text{per}}([0, L_c])]^4\) with the domain \([H^2_{\text{per}}([0, L_c]) \times L^2_{\text{per}}([0, L_c])]^2\). Our objective is to study the behavior of the non-positive spectrum of \(\mathcal{L}_{\varphi_c}\). First of all, we need to present some preliminary results which will be useful later. Consider the auxiliary operators

\[
\mathcal{L}_{\text{Re}, \varphi_c} = \begin{pmatrix}
-\partial_x^2 + 1 - \log(|\varphi_c|^p) - p & -c \\
-c & 1
\end{pmatrix}
\]

and

\[
\mathcal{L}_{\text{Im}, \varphi_c} = \begin{pmatrix}
-\partial_x^2 + 1 - \log(|\varphi_c|^p) & c \\
c & 1
\end{pmatrix}.
\]
Proposition 4.2. Let us consider the self-adjoint operator
\[(4.4) \quad L_{1,\varphi_c} := -\partial_x^2 + (1 - c^2) - p - \log(|\varphi_c|^p),\]
defined in \(L^2_{\text{per}}([0, L_c])\) with the domain \(H^2_{\text{per}}([0, L_c])\). Let \(L_{\text{Re},\varphi_c}\) be the operator in (4.2), defined in \([L^2_{\text{per}}([0, L_c])]^2\) with the domain \(H^2_{\text{per}}([0, L_c]) \times L^2_{\text{per}}([0, L_c])\). The real number \(\lambda \leq 0\) is a non-positive eigenvalue of \(L_{\text{Re},\varphi_c}\), if and only if, \(\gamma := \lambda \left(1 - \frac{c^2}{\lambda - 1}\right) \leq 0\) is a non-positive eigenvalue of \(L_{1,\varphi_c}\).

Proof. Indeed, let \(\lambda \leq 0\) be a non-positive eigenvalue for the operator \(L_{\text{Re},\varphi_c}\) whose eigenfunction is \((\xi_1, \xi_2) \in H^2_{\text{per}}([0, L_c]) \times L^2_{\text{per}}([0, L_c])\). Thus,
\[
L_{\text{Re},\varphi_c} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \lambda \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},
\]
So,
\[(4.5) \quad -\xi_1'' + \xi_1 - \log(|\varphi_c|^p)\xi_1 - p\xi_1 - c\xi_2 = \lambda\xi_1 \quad \text{and} \quad \xi_2 = -\frac{c}{\lambda - 1}\xi_1.
\]
We have that \(\gamma \leq 0\) since \(\lambda \leq 0\). Moreover, from (4.5), we have that
\[
L_{1,\varphi_c}(\xi_1) = -\xi_1'' + (1 - c^2)\xi_1 - p\xi_1 - \log(|\varphi_c|^p)\xi_1
\]
\[
= \lambda\xi_1 - c^2\xi_1 - \frac{c^2}{\lambda - 1}\xi_1 = \lambda \left(1 - \frac{c^2}{\lambda - 1}\right)\xi_1 = \gamma\xi_1.
\]
The converse of the result follows from similar arguments. \(\square\)

Remark 4.3. A similar result can be determined by comparing operator \[(4.6) \quad L_{2,\varphi_c} := -\partial_x^2 + (1 - c^2) - \log(|\varphi_c|^p),\]
with the operator \(L_{1m,\varphi_c}\) given by (4.3).

Previous proposition helps us to determine the non-positive spectrum of the linear operator \(L_{\varphi_c}\) by knowing the behavior of the non-positive spectra for the operators \(L_{1,\varphi_c}\) and \(L_{2,\varphi_c}\). In fact, operator \(L_{\varphi_c}\) is diagonal and thus, it is sufficient to analyze the non-positive spectra of the operators \(L_{\text{Re},\varphi_c}\) and \(L_{1m,\varphi_c}\). In addition, operators \(L_{1,\varphi_c}\) and \(L_{2,\varphi_c}\) are Hill operators since their potentials are periodic. In \([22]\), the authors have determined a method to establish the position of the zero eigenvalue related to the general Hill operator
\[(4.7) \quad L_s(y) = -y'' + g(s, \varphi(x))y,\]
where \(s\) belongs to a convenient open interval contained in \(\mathbb{R}\) and \(g\) is a smooth function which depends smoothly on the parameters \(s\) and \(\varphi\). We will describe the method in some few lines. Indeed, accordingly to \([21]\), the spectrum of \(L_s\) is formed by an unbounded sequence of real numbers
\[
\gamma_0 < \gamma_1 \leq \gamma_2 < \gamma_3 \leq \gamma_4 < \ldots < \gamma_{2n-1} \leq \gamma_{2n} < \ldots
\]
where equality means that \(\gamma_{2n-1} = \gamma_{2n}\) is a double eigenvalue. The spectrum of \(L_s\) is characterized by the number of zeros of the eigenfunctions if \(\tilde{q}\) is an eigenfunction associated to the eigenvalue \(\gamma_{2n-1}\) or \(\gamma_{2n}\), then \(\tilde{q}\) has exactly \(2n\) zeros in the half-open interval \([0, L_s]\). Moreover, since equation
\[(4.8) \quad -y'' + g(s, \varphi(x)) = 0\]
is a Hill type equation, we conclude from classical Floquet theory in [21] the existence of an 2-
dimensional basis \{\bar{y}, q\} formed by smooth solutions of the equation (4.8), where \(q\) is \(L_s\)-periodic. In addition, there exists a constant \(\theta \in \mathbb{R}\) such that
\[
\bar{y}(x + L_s) = \bar{y}(x) + \theta q(x) \quad \text{for all } x \in \mathbb{R}.
\]
Constant \(\theta\) in (4.9) measures how function \(\bar{y}\) is periodic. In fact, \(\bar{y}\) is periodic if, and only if, \(\theta = 0\).

**Definition 4.4.** The inertial index in \(\mathcal{L}_s\) of \(\mathcal{L}_s\) is a pair of integers \((n, z)\), where \(n\) is the dimension of the negative subspace of \(\mathcal{L}_s\) and \(z\) is the dimension of the null subspace of \(\mathcal{L}_s\).

We also need the concept of isoinertial family of self-adjoint operators.

**Definition 4.5.** A family of self-adjoint operators \(\mathcal{L}_s\), which depends on the real parameter \(s\), is called isoinertial if the inertial index in \(\mathcal{L}_s\) does not depend on \(s\).

Next result has been determined in [22] and [24] and it determines the behavior of the non-positive spectrum of the linear operator \(\mathcal{L}_s\) in (4.7) just by knowing it for a fixed value \(s_0\) in an open interval of \(\mathbb{R}\).

**Proposition 4.6.** Let \(\mathcal{L}_s\) be the Hill operator as in (4.7) defined in \(L^2_{\text{per}}([0, L_s])\) with the domain \(D(\mathcal{L}_s) = H^2_{\text{per}}([0, L_s])\). If \(\lambda = 0\) is an eigenvalue of \(\mathcal{L}_s\) for every \(s\) in an open interval of \(\mathbb{R}\) and the potential \(g(s, \varphi(x))\) is continuously differentiable in all variables, then the family of operators \(\mathcal{L}_s\) is isoinertial.

First, we shall calculate the inertial of \(\mathcal{L}_{1, \varphi_{c_0}}\) for a fixed value of the parameter \(s_0 := c_0 \in I = \mathbb{R}\). Let \(\varphi_{c_0}\) be a periodic function with period \(L_0 := L_{c_0}\) for the equation (1.2). The mentioned solution satisfies \((\varphi_{c_0}(0), \varphi_{c_0}'(0)) = (\alpha_0, 0)\), where \(\alpha_0 \in \left(e^{ \frac{1-c_0^2}{p}}, e^{ \frac{1-c_0^2}{p}}\right)\). Hence, \(\varphi_{c_0}\) is even, positive and \(L_0\)-periodic function. Moreover, \(\max_{x \in [0, L_0]} \varphi_{c_0}(x) = \varphi_{c_0}(0)\) and \(L_0 > \frac{2\pi}{\sqrt{p}}\).

Next, let us consider \(q = \varphi'_{c_0}\) and \(\bar{y}\) as the unique solution of the Cauchy problem
\[
\begin{cases}
-\bar{y}'' + [(1 - c_0^2) - p - \log(|\varphi_{c_0}|^p)]\bar{y} = 0 \\
\bar{y}(0) = -\frac{1}{\varphi_{c_0}'(0)} \\
\bar{y}'(0) = 0
\end{cases}
\]

and also, from (4.9), we see that
\[
\theta = \frac{\bar{y}(L_0)}{\varphi_{c_0}'(0)}.
\]
It is important to mention that if \(q = \varphi'_{c_0}\), the solution \(\bar{y}\) is obtained by a simple application of Lemma 2.1 in [23].

Next \(\varphi'_{c_0} \in \ker(\mathcal{L}_{1, \varphi_{c_0}})\), that is, this smooth function is an eigenfunction associated to the zero eigenvalue. Moreover, \(\varphi'_{c_0}\) has exactly two zeros in the half-open interval \([0, L_0]\). So, from Floquet’s Theory, we have three possibilities:

(i) \(\gamma_1 = \gamma_2 = 0 \Rightarrow \text{in}(\mathcal{L}_{1, \varphi_{c_0}}) = (1, 2)\).
(ii) \(\gamma_1 = 0 < \gamma_2 \Rightarrow \text{in}(\mathcal{L}_{1, \varphi_{c_0}}) = (1, 1)\).
(iii) \( \gamma_1 < \gamma_2 = 0 \) \( \Rightarrow \) \( \text{in}(L_{1,\varphi_{c_0}}) = (2, 1) \).

The method that we use to decide and calculate the inertial index is based on Lemma 2.1, Theorem 2.2 and Theorem 3.1 of [23]. This result can be stated as follows.

**Proposition 4.7.** Let \( \theta \) be the constant given by (4.10). Then the eigenvalue \( \lambda = 0 \) of \( L_{1,\varphi_{c_0}} \) is simple, if and only if, \( \theta \neq 0 \). Moreover, if \( \theta \neq 0 \) then \( \gamma_1 = 0 \) if \( \theta < 0 \) and \( \gamma_2 = 0 \) if \( \theta > 0 \).

The next step is to count exactly the number of negative eigenvalues for the operator \( L_{1,\varphi_{c_0}} \) and proving that zero is simple. For this purpose, we need to consider which is simple. The family of operators \( \{L_{\varphi,\theta}\} \) illustrate the method, we shall consider some specific values for \( c \) and \( \theta \).

| \( p \) | \( \varphi_{c_0}(0) \) | \( \varphi'_{c_0}(0) \) | \( \varphi''_{c_0}(0) \) | \( y(0) \) | \( \bar{L}_0 \) | \( \bar{y}(L_0) \) | \( \bar{y}'(L_0) \) | \( \theta \) |
|------|----------------|----------------|----------------|--------|-------|--------|--------|------|
| 1    | 2.5            | 0              | -0.4157        | 2.4054 | 6.3129 | 2.0454 | 0.2316 | -0.5571 |
| 2    | 1.5            | 0              | -0.0914        | 10.941 | 4.4425 | 10.941 | 0.0563 | -0.6158 |
| 3    | 1.5            | 0              | -0.6996        | 1.4294 | 3.6462 | 1.4294 | 0.1823 | -0.2606 |
| 4    | 1.5            | 0              | -1.3078        | 0.7646 | 3.1775 | 0.7646 | 0.2710 | -0.2073 |
| 5    | 1.5            | 0              | -1.9160        | 0.5219 | 2.8580 | 0.5219 | 0.3287 | -0.1716 |
| 6    | 1.5            | 0              | -2.5242        | 0.3962 | 2.6214 | 0.3962 | 0.3677 | -0.1457 |
| 7    | 1.5            | 0              | -3.7406        | 0.2673 | 2.2873 | 0.2673 | 0.4138 | -0.1106 |
| 8    | 1.5            | 0              | -4.9570        | 0.2017 | 2.0570 | 0.2017 | 0.4366 | -0.0881 |
| 9    | 1.5            | 0              | -11.034        | 0.0906 | 1.4754 | 0.0906 | 0.4333 | -0.0402 |

From previous table it is possible to see from Proposition 4.7 that \( \text{in}(L_{1,\varphi_{c_0}}) = (1, 1) \). So, zero is a simple eigenvalue and \( L_{1,\varphi_{c_0}} \) has only one negative eigenvalue which is simple (the first eigenvalue is always simple using Floquet’s Theorem). Since \( \{L_{1,\varphi_{c}}, c \in \mathbb{R}\} \) is isoinertial, we deduce from Proposition 4.6 that \( \text{in}(L_{1,\varphi_{c}}) = (1, 1) \) for all \( c \in \mathbb{R} \).

From Proposition 4.2 we have that the operator \( L_{Re,\varphi_{c}} \) has only one negative eigenvalue which is simple and zero is a simple eigenvalue whose eigenfunction is \( (\varphi'_{c}, c\varphi'_{c}) \). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues (so, it is bounded away from zero).

Concerning the operator \( L_{2,\varphi_{c}} \) as defined in (4.16), the procedure is quite similar. However, since \( \varphi_{c_0}(x) > 0 \) we deduce directly from Floquet’s Theory that zero is the first eigenvalue of \( L_{2,\varphi_{c}} \) which is simple. The family of operators \( \{L_{2,\varphi_{c}}, c \in \mathbb{R}\} \) is isoinertial and, therefore, \( \text{in}(L_{2,\varphi_{c}}) = (0, 1) \) for all \( c \in \mathbb{R} \). Hence, from Remark 4.3, operator \( L_{Im,\varphi_{c}} \) has no negative eigenvalues and zero is a simple eigenvalue whose eigenfunction is \( (\varphi_{c_{0}}, -c\varphi_{c_{0}}) \). Furthermore, the remainder of the spectrum is discrete and bounded away from zero.

Arguments above allow us to conclude that diagonal operator \( L_{\varphi_{c}} \) has only one negative eigenvalue which is simple and zero is double eigenvalue with

\[
\ker(L_{\varphi_{c}}) = \text{span}\{ (\varphi'_{c}, c\varphi'_{c}, 0, 0), (0, 0, \varphi_{c}, -c\varphi_{c}) \}.
\]

Next, we analyze the non-positive spectrum of the linearized operator \( L_{\varphi_{c}} \), defined in \( X_{\varphi} := [L^2_{per}\varepsilon([0, L_{c}])]^4 \) with the domain \( Z_{\varphi} := [H^2_{per}\varepsilon([0, L_{c}]) \times L^2_{per}\varepsilon([0, L_{c}])]^2 \). Indeed, first we see that eigenfunction \( (\varphi'_{c}, c\varphi'_{c}, 0, 0) \) does not belong to the kernel of \( L_{\varphi_{c}}|_{X_{\varphi}} \) because \( \varphi_{c} \) is odd. Therefore, we deduce that \( \ker(L_{\varphi_{c}}|_{X_{\varphi}}) = \text{span}\{ (0, 0, \varphi_{c}, -c\varphi_{c}) \} \). Moreover, since the eigenfunction for the
first eigenvalue of $L_{1,\varphi_c}$ is even (see Theorem 1.1 in [21]), we obtain that the number of negative eigenvalues of the linearized operator $L_{\varphi_c}$, defined in $X_c$ with the domain $Z_c$, remains equal to one.

4.3. The convexity of the function $d$. Let $\varphi_{c_0}$ be a smooth, even, positive and periodic solution with period $L_0 > \frac{2\pi}{\sqrt{p}}$ for the equation (1.2). Operator $L_{1,\varphi_{c_0}}$ in (4.1) has zero as a simple eigenvalue. So, from Proposition 1.7, $\theta \neq 0$ where $\theta$ is given in (4.10). From Theorem 3.3 in [22], we conclude the existence of a neighborhood $I$ of $c_0$ and a family of functions $\{\varphi_c\}_{c \in I}$ such that $\varphi_c$ is a solution to the equation (1.2) for all $c \in I$. Moreover, $\varphi_c$ is a smooth, even, positive and $L_c$-periodic function and the map $c \in I \mapsto \varphi_c \in H^2_{\text{per},e}([0,L])$ is smooth. Furthermore, $\text{in}(L_{Re,\varphi_c}) = (1,1)$ and $\text{in}(L_{lm,\varphi_c}) = (0,1)$ for all $c \in I$.

Now, in order to simplify the notation, we denote $L = L_0$. Consider $E$ and $F$ the two conserved quantities in (1.7) and (1.8). Define the function

$$d: \mathbb{R} \rightarrow \mathbb{R}$$
$$c \mapsto E(\varphi_c,c\varphi_c,0,0) - cF(\varphi_c,c\varphi_c,0,0).$$

Since $(\varphi_c,c\varphi_c,0,0)$ is a critical point of the functional $G = G_c = E - cF$, we deduce from (4.11)

$$d'(c) = -F(\varphi_c,c\varphi_c,0,0) = -\int_0^L c\varphi_c^2(x) \, dx.$$

So for all $c \in I$,

$$d''(c) = -\int_0^L \varphi_c^2 \, dx - c \frac{d}{dc} \left( \int_0^L \varphi_c^2 \, dx \right).$$

Our intention is to give a convenient expression for (4.12). In fact, recall that $\varphi_c > 0$ and

$$\varphi_c'' - (1 - c^2)\varphi_c + \log(\varphi_c^p)\varphi_c = 0 \quad \text{for all } c \in I.$$

Since $c \in I \mapsto \varphi_c \in H^2_{\text{per},e}([0,L])$ is a smooth function, we can define $\eta_c := \frac{d}{dc} (\varphi_c) \in H^2_{\text{per}}([0,L])$.

Deriving equation (4.13) with respect to the parameter $c$ to obtain

$$\eta_c'' + 2c\varphi_c - (1 - c^2)\eta_c + \log(\varphi_c^p)\eta_c + p\varphi_c\eta_c = 0.$$

Multiplying equation (4.14) by $\varphi_c$ and integrating the result over $[0,L]$, we have

$$\int_0^L \left[ \eta_c'' \varphi_c + 2c\varphi_c^2 - (1 - c^2)\eta_c \varphi_c + \log(\varphi_c^p)\eta_c \varphi_c + p\varphi_c \eta_c \right] \, dx = 0.$$

Equation above is equivalent to

$$\int_0^L \left[ \varphi_c \eta_c'' + 2c\varphi_c^2 - (1 - c^2)\eta_c \varphi_c + \log(\varphi_c^p)\eta_c \varphi_c + p\varphi_c \eta_c \right] \, dx = 0.$$

By combining expressions (4.13) and (4.15), we see that

$$\int_0^L \left[ 2c\varphi_c^2 + p\varphi_c \eta_c \right] \, dx = 0.$$

Hence,

$$2c \int_0^L \varphi_c^2 \, dx + \int_0^L \eta_c \varphi_c \, dx = 0.$$

(4.16)
From identity (4.16),
\[
\frac{d}{dc} \left( \int_0^L \varphi_c^2 \, dx \right) = -\frac{4c}{p} \int_0^L \varphi_c^2 \, dx.
\]

Therefore for all \( c \in I \),
\[
d''(c) = -\int_0^L \varphi_c^2 \, dx + \frac{4c^2}{p} \int_0^L \varphi_c^2 \, dx = \left( \frac{4c^2}{p} - 1 \right) \| \varphi_c \|_{L^2_{\text{per}}([0,L])}^2.
\]

The sign of \( d''(c) \) depends on the sign of the quantity \( \frac{4c^2}{p} - 1 \), where \( c \in I \). Finally, we have that
\[
d''(c) > 0 \iff |c| > \frac{\sqrt{p}}{2} \quad \text{and} \quad d''(c) < 0 \iff |c| < \frac{\sqrt{p}}{2}.
\]

4.4. **Orbital Stability of Standing Waves.** In this subsection, we prove results of orbital stability of standing waves related to the Logarithmic Klein-Gordon equation. To do so, we use classical methods based on ideas established in [8], [17] and [28] to get the stability over the complex space \( X := H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L]) \).

In what follows, let us consider \( p = 1, 2, 3 \) (the reason to consider these values of \( p \) will be explained later).

**Definition 4.8.** We say that \( \varphi_c \) is **orbitally stable by the periodic flow of the equation (1.1)**, where \( \varphi_c \) satisfies (4.2) if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if
\[
(u_0, u_1) \in X = H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L]) \text{ satisfies } \| (u_0, u_1) - (\varphi, ic\varphi) \|_X < \delta
\]

then \( \tilde{u} = (u, u_1) \) is a weak solution to equation (1.1) with \( \tilde{u}(-\cdot, 0) = (u_0, u_1) \) and
\[
\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}} \| \tilde{u}(-\cdot, t) - e^{i\theta}(\varphi(-\cdot + y), ic\varphi(-\cdot + y)) \|_X < \varepsilon.
\]

Otherwise, we say that \( \varphi_c \) is **orbitally unstable.**

Firstly, we will need some additional information about spectral proprieties of the operators \( \mathcal{L}_{Re, \varphi_c} \) and \( \mathcal{L}_{Im, \varphi_c} \), in (4.2) and (4.3) respectively. Indeed, as we have already determined in the last subsection one sees that \( \text{in}(\mathcal{L}_{Re, \varphi_c}) = (1,1) \) and \( \text{in}(\mathcal{L}_{Im, \varphi_c}) = (1,0) \) for all \( c \in I \). Since \( \text{in}(\mathcal{L}_{Im, \varphi_c}) = (1,0) \), we have that
\[
\left\langle \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) , \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right\rangle_{2,2} \geq 0 \quad \text{for all } \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \in X,
\]

where \( \langle \cdot, \cdot \rangle_{2,2} \) denotes the inner product in \( L^2_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L]) \) and \( \| \cdot \|_{2,2} \) corresponds to the respective induced norm. We have the following result:

**Proposition 4.9.** If
\[
\beta := \inf \left\{ \left\langle \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) , \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right\rangle_{2,2} ; \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \in X, \right. \right. \left. \left. \left\| \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right\|_{2,2} = 1, \left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) , \left( \begin{array}{c} \varphi_c \log(\varphi_c^p) \\ -c\varphi_c \end{array} \right) \right\rangle_{2,2} = 0 \right\}
\]
then $\beta > 0$.

Proof. First, from (4.18) we obtain $\beta \geq 0$. Let us suppose that $\beta = 0$. There exists a sequence

(4.19) \[ \left\{ \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right\}_{j \in \mathbb{N}} \subset X \]

such that

(4.20) \[ \left\langle \mathcal{L}_{1m,\varphi_c} \left( \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right), \left( \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right) \right\rangle_{2,2} \xrightarrow{i \to \infty} 0 \]

and for all $j \in \mathbb{N}$, one has

(4.21) \[ \left\| \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right\|_{2,2} = 1 \quad \text{and} \quad \left\langle \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix}, \left( \varphi_c \log(|\varphi|^p) - c\varphi_c \right) \right\rangle_{2,2} = 0. \]

Now, using the convergence in (4.20), we see that \( \left\{ \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right\}_{j \in \mathbb{N}} \) is uniformly bounded in $X$. There exists a subsequence of (4.19), still denoted by (4.19) and \( \begin{pmatrix} \psi^*_1 \\ \psi^*_2 \end{pmatrix} \in X \) such that

(4.22) \[ \left( \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right) \rightharpoonup \left( \begin{pmatrix} \psi^*_1 \\ \psi^*_2 \end{pmatrix} \right) \text{ weakly in } X. \]

On the other hand, since the embedding $H^1_{per}([0, L]) \hookrightarrow L^2_{per}([0, L])$ is compact and \( \{ \psi_{1,j} \} \) is uniformly bounded in $H^1_{per}([0, L])$, we deduce, up to a subsequence that

(4.23) \[ \psi_{1,j} \rightharpoonup \psi^*_1 \text{ in } L^2_{per}([0, L]). \]

The weak convergence in (4.22) and the strong convergence in (4.23) give us that second condition in (4.21) is satisfied for \( \begin{pmatrix} \psi^*_1 \\ \psi^*_2 \end{pmatrix} \).

From (4.20), we see that for all $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that if $j \geq j_0$, we have

(4.24) \[ \int_0^L \left( \psi'_{1,j} \right)^2 + \psi^2_{1,j} - \log(|\varphi|^p)\psi^2_{1,j} + 2c\psi_{1,j}\psi_{2,j} + \psi^2_{2,j} dx < \varepsilon. \]

that is, if $j \geq j_0$, we obtain from the fact \( \left\| \begin{pmatrix} \psi_{1,j} \\ \psi_{2,j} \end{pmatrix} \right\|_{2,2} = 1 \) that

(4.25) \[ 1 \leq \int_0^L \left( \psi'_{1,j} \right)^2 + \psi^2_{1,j} + \psi^2_{2,j} dx < \int_0^L \log(|\varphi|^p)\psi^2_{1,j} - 2c\psi_{1,j}\psi_{2,j} dx + \varepsilon. \]

Since in particular $1 < \int_0^L \log(|\varphi|^p)\psi^2_{1,j} - 2c\psi_{1,j}\psi_{2,j} dx + \varepsilon$, we obtain from the fact $\psi_{2,j} \rightharpoonup \psi^*_2$ weakly in $L^2_{per}([0, L])$ and (4.23) that

(4.26) \[ 1 \leq \int_0^L \log(|\varphi|^p)\psi^*_1 - 2c\psi^*_1\psi^*_2 dx + \varepsilon. \]
From (4.26), we conclude that \( \langle \psi_1^* \psi_2^* \rangle \neq 0 \). Consider \( \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \frac{1}{\| (\psi_1^* \psi_2^*) \|_{2,2}} \left( \begin{array}{c} \psi_1^* \\ \psi_2^* \end{array} \right) \). We obtain

\[
(4.27) \quad \left\| \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right\|_{2,2} = 1 \quad \text{and} \quad \left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \left( \begin{array}{c} \varphi_c \log(\varphi_c^p) \\ -c\varphi_c \end{array} \right) \right\rangle_{2,2} = 0.
\]

In addition, Fatou’s Lemma gives us

\[
0 \leq \left\langle \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right\rangle_{2,2} \leq \liminf_{j \to \infty} \left\langle \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \end{array} \right), \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \end{array} \right) \right\rangle_{2,2} = 0.
\]

Therefore, we obtain \( \left\langle \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right\rangle_{2,2} = 0 \) and as consequence, the minimum \( \beta \) is attained in \( \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \neq 0 \).

We are in position to use Lagrange’s Theorem to guarantee the existence of \((a, b) \in \mathbb{R}^2\) such that

\[
(4.28) \quad \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = a \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) + b \left( \begin{array}{c} \varphi_c \log(\varphi_c^p) \\ -c\varphi_c \end{array} \right).
\]

Taking the inner product of (4.28) with the function \( \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \), we get from (4.27) that \( a = 0 \).

On the other hand, since \( \ker(\mathcal{L}_{Im, \varphi_c}) = \text{span} \left\{ \left( \begin{array}{c} \varphi_c \\ -c\varphi_c \end{array} \right) \right\} \) and \( \mathcal{L}_{Im, \varphi_c} \) is a self-adjoint operator, we have the identity

\[
0 = \left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \mathcal{L}_{Im, \varphi_c} \left( \begin{array}{c} \varphi_c \\ -c\varphi_c \end{array} \right) \right\rangle_{2,2} = b \left\langle \left( \begin{array}{c} \varphi_c \log(\varphi_c^p) \\ -c\varphi_c \end{array} \right), \left( \begin{array}{c} \varphi_c \\ -c\varphi_c \end{array} \right) \right\rangle_{2,2}.
\]

Identity (1.2) allows us to deduce

\[
\left\langle \left( \begin{array}{c} \varphi_c \log(\varphi_c^p) \\ -c\varphi_c \end{array} \right), \left( \begin{array}{c} \varphi_c \\ -c\varphi_c \end{array} \right) \right\rangle_{2,2} = \int_0^L \varphi_c \log(\varphi_c^p) + c^2 \varphi_c \ dx > 0
\]

and so, we conclude that \( a = b = 0 \). There exists a constant \( c_1 \neq 0 \) such that

\[
\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = c_1 \left( \begin{array}{c} \varphi_c \\ -c\varphi_c \end{array} \right).
\]

Hence,

\[
\left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \left( \begin{array}{c} \varphi_c \log(\varphi_c^p) \\ -c\varphi_c \end{array} \right) \right\rangle_{2,2} \neq 0,
\]

which is a contradiction with (4.27). Therefore, \( \beta > 0 \). \( \square \)

In the next proposition we need to use Lemma 3.1 in [28]. In whole this subsection, we shall consider \(|c| > \sqrt{\frac{p}{2}}\).
Proposition 4.10. (i) If
\[ \gamma := \inf \left\{ \left\langle \mathcal{L}_{Re,\varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right), \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \right\rangle ; \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \in X, \right\} \]

\[ \left\| \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \right\|_{2,2} = 1, \left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right), \left( \begin{array}{c} c\varphi_c \\ \varphi_c \\
\end{array} \right) \right\rangle_{2,2} = 0 \}
\]

then \( \gamma = 0 \).

(ii) If
\[ \kappa := \inf \left\{ \left\langle \mathcal{L}_{Re,\varphi_c} \left( \psi_1, \psi_2 \right), \left( \psi_1, \psi_2 \right) \right\rangle ; \left( \psi_1, \psi_2 \right) \in X, \right\} \]

\[ \left\| \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \right\|_{2,2} = 1, \left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right), \left( \begin{array}{c} c\varphi_c \\ \varphi_c \\
\end{array} \right) \right\rangle_{2,2} = 0 \}
\]

then \( \kappa > 0 \).

Proof. (i). The fact that function \( \varphi_c \) is bounded gives us that \( \gamma \) is finite. Now, since
\[ \left\langle \left( \begin{array}{c} \varphi'_c \\ c\varphi'_c \\
\end{array} \right), \left( \begin{array}{c} \varphi'_c \\ c\varphi'_c \\
\end{array} \right) \right\rangle_{2,2} = 0 \] and \( \mathcal{L}_{Re,\varphi_c} \left( \begin{array}{c} \varphi'_c \\ c\varphi'_c \\
\end{array} \right) = 0 \),

it follows that \( \gamma \leq 0 \).

There exists a sequence \( \left\{ \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \\
\end{array} \right) \right\}_{j \in \mathbb{N}} \subset X \) such that
\[ \left\| \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \\
\end{array} \right) \right\|_{2,2} = 1 \] and \( \left\langle \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \\
\end{array} \right), \left( \begin{array}{c} \varphi_c \\ \varphi_c \\
\end{array} \right) \right\rangle_{2,2} = 0 \), for all \( j \in \mathbb{N} \).

Moreover,
\[ \left\{ \mathcal{L}_{Re,\varphi_c} \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \\
\end{array} \right), \left( \begin{array}{c} \psi_{1,j} \\ \psi_{2,j} \\
\end{array} \right) \right\rangle_{2,2} \xrightarrow{j \to \infty} \gamma. \]

Using a similar analysis as determined in Proposition 4.9, we guarantee the existence of \( \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \) such that
\[ \left\| \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \right\|_{2,2} = 1, \left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right), \left( \begin{array}{c} c\varphi_c \\ \varphi_c \\
\end{array} \right) \right\rangle_{2,2} = 0 \]

and
\[ \left\langle \mathcal{L}_{Re,\varphi_c} \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right), \left( \begin{array}{c} \psi_1 \\ \psi_2 \\
\end{array} \right) \right\rangle_{2,2} = \gamma. \]
Next, we apply the method of Lagrange multipliers in order to guarantee the existence of \((a_*, b) \in \mathbb{R}^2\) such that

\[
\mathcal{L}_{Re, \varphi_c} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = a_* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + b \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix}.
\]

Before analyzing identity (4.32), we will deduce the existence of \(\begin{pmatrix} M \\ N \end{pmatrix} \in X\) such that

\[
\mathcal{L}_{Re, \varphi_c} \begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix}.
\]

Indeed, the identity (4.33) is equivalent to

\[
\begin{pmatrix} -M'' + (1 - p)M - \log(\varphi_c^p)M - cN \\ -cM + N \end{pmatrix} = \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix}.
\]

Thus, \(-cN = -c^2M - c\varphi_c\) and

\[
\mathcal{L}_{1, \varphi_c}(M) = -M'' + (1 - c^2)M - pM - \log(\varphi_c^p)M = 2c\varphi_c.
\]

Since \(\text{ker}(\mathcal{L}_{1, \varphi_c}) = \text{span}\{\varphi_c'\}\) and \(\varphi_c \perp \varphi_c',\) we have that \(M = 2c\mathcal{L}_{1, \varphi_c}^{-1}(\varphi_c).\) On the other hand, by deriving equation (1.2) with respect to \(c,\) one has

\[
-\left[\frac{d}{dc}(\varphi_c)\right]'' + (1 - c^2)\frac{d}{dc}(\varphi_c) - 2c\varphi_c - \log(\varphi_c^p)\frac{d}{dc}(\varphi_c) - p\frac{d}{dc}(\varphi_c) = 0,
\]

that is,

\[
\mathcal{L}_{1, \varphi_c} \left(\frac{d}{dc}(\varphi_c)\right) = 2c\varphi_c \quad \text{and} \quad M = 2c\mathcal{L}_{1, \varphi_c}^{-1}(\varphi_c) = \frac{d}{dc}(\varphi_c).
\]

Therefore, we conclude

\[
\begin{pmatrix} M \\ N \end{pmatrix} = \begin{pmatrix} \frac{d}{dc}(\varphi_c) \\ \varphi_c + c\frac{d}{dc}(\varphi_c) \end{pmatrix}.
\]

Moreover, since \(|c| > \frac{\sqrt{p}}{2}\) from (4.17), we have

\[
\left\langle \mathcal{L}_{1, \varphi_c}^{-1} \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix}, \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix} \right\rangle_{2,2} = \left\langle \begin{pmatrix} M \\ N \end{pmatrix}, \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix} \right\rangle_{2,2}
\]

\[
= \int_0^L \varphi_c^2 dx + c\frac{d}{dc} \left(\int_0^L \varphi_c^2 dx\right) = -d''(c) < 0.
\]

We see that \(-p\) is the unique negative eigenvalue of the operator \(\mathcal{L}_{1, \varphi_c}\) which is for the eigenfunction \(\varphi_c.\) In fact, from (1.2),

\[
\mathcal{L}_{1, \varphi_c}(\varphi_c) = -\varphi_c'' + (1 - c^2)\varphi_c - \log(\varphi_c^p)\varphi_c - p\varphi_c = -p\varphi_c.
\]
From Proposition 4.2 we see that
\[
\lambda_0 := \lambda_0(c) = \frac{1 + c^2 - p - \sqrt{1 + 2c^2 + 2p + c^4 - 2c^2p + p^2}}{2} < 0
\]
is the unique negative eigenvalue to operator \(\mathcal{L}_{\text{Re}, \varphi_c}\) which is for the eigenfunction \(\left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right)\), where \(m = -\frac{c}{\lambda_0 - 1}\).

Thus,
\[
\left\langle \left(\begin{array}{c} c\varphi_c \\ \varphi_c \end{array}\right), \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right) \right\rangle_{2,2} = \left(\frac{\lambda_0 - 2)c}{\lambda_0 - 1}\right) \int_0^L \varphi_c^2 \, dx \neq 0. \tag{4.35}
\]

From (4.30)-(4.32) we get
\[
a_* = \left\langle \mathcal{L}_{\text{Re}, \varphi_c} \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right), \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right) \right\rangle_{2,2} = \gamma.
\]

The spectral properties related to the operator \(\mathcal{L}_{\text{Re}, \varphi_c}\) allow us to conclude \(\lambda_0 \leq \gamma\). Next, suppose that \(\lambda_0 = \gamma\). By taking the inner product of (4.32) with the function \(\left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right)\), we have
\[
\lambda_0 \left\langle \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right), \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right) \right\rangle_{2,2} = \left\langle \mathcal{L}_{\text{Re}, \varphi_c} \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right), \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right) \right\rangle_{2,2}
\]
\[
= a_* \left\langle \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right), \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right) \right\rangle_{2,2} + b \left\langle \left(\begin{array}{c} c\varphi_c \\ \varphi_c \end{array}\right), \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right) \right\rangle_{2,2},
\]
that is,
\[
0 = b \left\langle \left(\begin{array}{c} c\varphi_c \\ \varphi_c \end{array}\right), \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right) \right\rangle_{2,2}.
\]

Thus, from (4.35), we get \(b = 0\). So,
\[
\mathcal{L}_{\text{Re}, \varphi_c} \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right) = \lambda_0 \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right) = c_2 \left(\begin{array}{c} \varphi_c \\ m\varphi_c \end{array}\right),
\]
where \(c_2 \neq 0\) is a constant. This last fact contradicts (4.30) and therefore \(a_* \neq \lambda_0\).

Now, we suppose that \(\gamma = a_* \in (\lambda_0, 0)\). From (4.32) and spectral properties concerning to the operator \(\mathcal{L}_{\text{Re}, \varphi_c}\), we obtain
\[
(4.36) \quad \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array}\right) = b(\mathcal{L}_{\text{Re}, \varphi_c} - a_*I)^{-1} \left(\begin{array}{c} c\varphi_c \\ \varphi_c \end{array}\right),
\]
that is, \(b \neq 0\). Consider \(a \in (\lambda_0, 0)\) and the auxiliary function \(G\),
\[
G(a) = \left\langle (\mathcal{L}_{\text{Re}, \varphi_c} - aI)^{-1} \left(\begin{array}{c} c\varphi_c \\ \varphi_c \end{array}\right), \left(\begin{array}{c} c\varphi_c \\ \varphi_c \end{array}\right) \right\rangle_{2,2}.
\]
From (4.30) and (4.35), one has $G(a_*) = 0$. Since $L_{Re,\varphi_c}$ is a self-adjoint operator, we get for all $a \in (\lambda_0, 0)$,

$$G'(a) = \left\| (L_{Re,\varphi_c} - aI)^{-1} \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix} \right\|_{2,2}^2 > 0.$$ 

Moreover, from (4.31), we have

$$G(0) = \left\langle L_{Re,\varphi_c}^{-1} \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix}, \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix} \right\rangle_{2,2} < 0,$$

that is, $G(a) \neq 0$ for all $a \in (\lambda_0, 0)$. Hence, $a_* \notin (\lambda_0, 0)$ and therefore, $\gamma = a_* = 0$. From (27), the proof of (i) is now completed.

(ii). From item (i), we infer that $\kappa \geq 0$. Let us suppose that $\kappa = 0$. We can repeat the same argument as in Proposition 4.9 to deduce

$$\left\langle L_{Re,\varphi_c} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\rangle_{2,2} = 0,$$

with

$$\left\| \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right\|_{2,2} = 1, \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix} \right\rangle_{2,2} = 0,$$

and

$$\left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \log(\varphi_c^p)\varphi'_c + p\varphi'_c \\ c\varphi'_c \end{pmatrix} \right\rangle_{2,2} = 0.$$

Next, we can use the Lagrange multiplier theory to guarantee the existence of $(b_1, b_2, b_3) \in \mathbb{R}^3$ such that

$$L_{Re,\varphi_c} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = b_1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + b_2 \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix} + b_3 \begin{pmatrix} \log(\varphi_c^p)\varphi'_c + p\varphi'_c \\ c\varphi'_c \end{pmatrix}.$$ 

Taking the inner product of $L_{Re,\varphi_c} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ in (4.40) with $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in X$, we have from (4.37), (4.38) and (4.39) that $b_1 = 0$. On the one hand, since $L_{Re,\varphi_c}$ is self-adjoint, we get

$$0 = \left\langle L_{Re,\varphi_c} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \begin{pmatrix} \varphi'_c \\ c\varphi'_c \end{pmatrix} \right\rangle_{2,2} = b_3 \left\langle \begin{pmatrix} \log(\varphi_c^p)\varphi'_c + p\varphi'_c \\ c\varphi'_c \end{pmatrix}, \begin{pmatrix} \varphi'_c \\ c\varphi'_c \end{pmatrix} \right\rangle_{2,2}.$$ 

We also see that

$$\left\langle \begin{pmatrix} \log(\varphi_c^p)\varphi'_c + p\varphi'_c \\ c\varphi'_c \end{pmatrix}, \begin{pmatrix} \varphi'_c \\ c\varphi'_c \end{pmatrix} \right\rangle_{2,2} = \int_0^L \varphi'_c (\log(\varphi_c^p)\varphi'_c + p\varphi'_c + c^2\varphi'_c) \, dx$$

$$= \int_0^L \varphi'_c (\varphi'_c - \varphi''_c) \, dx = \int_0^L (\varphi'_c)^2 + (\varphi''_c)^2 \, dx > 0.$$ 

Thus $b_3 = 0$ and, therefore,

$$L_{Re,\varphi_c} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = b_2 \begin{pmatrix} c\varphi_c \\ \varphi_c \end{pmatrix}.$$
From (4.33), we deduce

$$L_{Re\phi_c} \left( \begin{array}{c} \frac{d}{dc}(\phi_c) \\ \phi_c + c\frac{d}{dc}(\phi_c) \end{array} \right) = \left( \begin{array}{c} c\phi_c \\ \phi_c \end{array} \right).$$

If we combine (4.41) and (4.42), there exists $b_4 \in \mathbb{R}$ such that

$$\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) - b_2 \left( \begin{array}{c} \frac{d}{dc}(\phi_c) \\ \phi_c + c\frac{d}{dc}(\phi_c) \end{array} \right) = b_4 \left( \begin{array}{c} \phi_c' \\ c\phi_c' \end{array} \right).$$

This last identity implies that

$$\left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \left( \begin{array}{c} c\phi_c \\ \phi_c \end{array} \right) \right\rangle - b_2 \left\langle \left( \begin{array}{c} M \\ N \end{array} \right), \left( \begin{array}{c} c\phi_c \\ \phi_c \end{array} \right) \right\rangle = b_4 \left\langle \left( \begin{array}{c} \phi_c' \\ c\phi_c' \end{array} \right), \left( \begin{array}{c} \phi_c' \\ c\phi_c' \end{array} \right) \right\rangle_{2,2} = 0.$$

Identities (4.34), (4.38) and (4.43) allow us to conclude $b_2 = 0$. So,

$$\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = b_4 \left( \begin{array}{c} \phi_c' \\ c\phi_c' \end{array} \right),$$

where $b_4 \neq 0$. Finally,

$$\left\langle \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \left( \log(\phi_c')\phi_c' + p\phi_c' \right) \right\rangle_{2,2} \neq 0.$$

So, we get a contradiction with (4.39). \qed

Next, we are going to use Propositions 4.9 and 4.10 in order to prove Theorem 1.2.

**Proof of Theorem 1.2** First, let $u$ be a weak solution related to the problem (3.1) with initial data $(u(\cdot, 0), u_t(\cdot, 0)) = (u_0, u_1) \in X$. Define

$$O_{\phi_c} := \left\{ e^{i\theta}(\phi_c(\cdot + y), ic\phi_c(\cdot + y)); (y, \theta) \in \mathbb{R} \times [0, 2\pi] \right\}$$

the orbit generated by $\phi_c$ and $v := u_t$. For $y \in [0, L]$, $\theta \in [0, 2\pi]$ and $t \geq 0$, consider the function $\Omega_t$ as

$$\Omega_t(y, \theta) := \|u_x(\cdot + y, t)e^{i\theta} - \phi_c'\|_{L^2_{per}}^2 + (1 - c^2)\|u(\cdot + y, t)e^{i\theta} - \phi_c\|_{L^2_{per}}^2 + \|v(\cdot + y, t)e^{i\theta} - ic\phi_c\|_{L^2_{per}}^2.$$

Since $c^2 \in \left( \frac{p}{1}, 1 \right)$ and $p = 1, 2, 3$ for $t \geq 0$ fixed, one see that the square root of $\Omega_t(y, \theta)$ *defines an equivalent norm* in $X$. We see that for all $t \geq 0$, function $\Omega_t$ is continuous on the compact set $[0, L] \times [0, 2\pi]$. There exists $(y, \theta) = (y(t), \theta(t))$ such that

$$\Omega_t(y(t), \theta(t)) = \inf_{(y, \theta) \in [0, L] \times [0, 2\pi]} \Omega_t(y, \theta) = [\rho_c(\bar{u}(\cdot, t), O_{\phi_c})]^2,$$
where \( \rho_c(\vec{u}(-,t), \mathcal{O}_{\varphi_c}) \) is the “distance” between function \( \vec{u} \) and the orbit \( \mathcal{O}_{\varphi_c} \) generated by \( \varphi_c \).

Moreover, the map

\[
t \mapsto \inf_{(y, \theta) \in [0,L] \times [0,2\pi]} \Omega_t(y, \theta)
\]

is continuous (see [8, Chapter 4, Lemma 2]).

Next, we consider the perturbation of the periodic wave \((\varphi_c, ic \varphi_c)\). Suppose that

\[\tag{4.45} u(x + y, t)e^{i\theta} := \varphi_c(x) + w(x, t) \quad \text{with} \quad w := A + iB \]

and

\[\tag{4.46} v(x + y, t)e^{i\theta} := ic\varphi_c(x) + z(x, t) \quad \text{with} \quad z := C + iD, \]

where \( t \geq 0, x \in \mathbb{R} \) and \( y = y(t) \) and \( \theta = \theta(t) \) are determined by \((4.44)\).

Denote the vector

\[
\vec{w} = \vec{w}_c = (w, z) = (\text{Re } w, \text{Im } z, \text{Im } w, \text{Re } z) = (A, D, B, C),
\]

where \((A, D, B, C) \in \mathbb{R}^4\). So, using the property of minimum \((y(t), \theta(t))\), we obtain from \((4.45)\) and \((4.46)\) that \( A, B, C \) and \( D \) must satisfy the compatibility relations

\[\tag{4.47} \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} \log(\varphi^p_c) \varphi'_c + p\varphi'_c \\ \varphi'_c \end{array} \right) \right)_{2,2} = 0 \]

and

\[\tag{4.48} \left( \begin{array}{c} B(\cdot, t) \\ C(\cdot, t) \end{array} \right), \left( \begin{array}{c} \log(\varphi^p_c) \varphi_c \\ -\varphi'_c \end{array} \right) \right)_{2,2} = 0, \]

for all \( t \geq 0 \). Next, we use the fact that \( \mathcal{E} \) and \( \mathcal{F} \) defined in \((1.7)\) and \((1.8)\) are invariant by translations and rotations to get

\[
\Delta G = G(u_0, u_1) - G(\varphi_c, ic \varphi_c) = G(w(\cdot, t) + \varphi_c, z(\cdot, t) + ic \varphi_c) - G(\varphi_c, ic \varphi_c),
\]

for all \( t \geq 0 \). Since \( G'(\varphi_c, ic \varphi_c) = G'(\varphi_c, c \varphi_c, 0, 0) = \vec{0} \), we can write

\[
\Delta G = \sum_{n=2}^4 \frac{G^{(n)}(\varphi_c, ic \varphi_c)}{n!} \cdot [\vec{w}(\cdot, t)]^n + O(w(\cdot, t))
\]

and for all \( t \geq 0 \), we deduce

\[
\Delta G = \frac{1}{2} \left\langle \mathcal{L}_{\text{Re}, \varphi_c} \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) \right\rangle_{2,2} \\
+ \frac{1}{2} \left\langle \mathcal{L}_{\text{Im}, \varphi_c} \left( \begin{array}{c} B(\cdot, t) \\ C(\cdot, t) \end{array} \right), \left( \begin{array}{c} B(\cdot, t) \\ C(\cdot, t) \end{array} \right) \right\rangle_{2,2} \\
+ \frac{p}{6} \int_0^L \frac{A(\cdot, t) B(\cdot, t)^2 - A(\cdot, t)^3}{\varphi_c} \, dx \\
+ \frac{p}{24} \int_0^L \frac{A(\cdot, t)^4 - 2A(\cdot, t)^2 B(\cdot, t)^2 - 3B(\cdot, t)^4}{\varphi_c^2} \, dx + O(w(\cdot, t)),
\]
where $|O(w(\cdot), t)| \leq O(\|\bar{w}(\cdot, t)\|^2_{X \times X})$. There exist positive constants $\beta_3$ and $\beta_4$ such that

$$\Delta g(t) = \Delta \tilde{g} \geq \frac{1}{2} \left( \mathcal{L}_{Re,\phi} \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) \right)_{2,2}$$

(4.49)

$$+ \frac{1}{2} \left( \mathcal{L}_{Im,\phi} \left( \begin{array}{c} B(\cdot, t) \\ C(\cdot, t) \end{array} \right), \left( \begin{array}{c} B(\cdot, t) \\ C(\cdot, t) \end{array} \right) \right)_{2,2}$$

$$- \beta_3 \|\bar{w}(\cdot, t)\|^2_{X \times X} - \beta_4 \|\bar{w}(\cdot, t)\|^4_{X \times X} - O(\|\bar{w}(\cdot, t)\|^5_{X \times X}),$$

for all $t \geq 0$.

Initially, let us suppose that $\mathcal{F}(\varphi_c, ic\varphi_c) = \mathcal{F}(u_0, u_1) = \mathcal{F}(u(\cdot, t), v(\cdot, t))$. So,

$$\left\langle \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} c\varphi_c \\ \varphi_c \end{array} \right) \right\rangle_{2,2} = \int_0^L [B(\cdot, t)C(\cdot, t) - A(\cdot, t)D(\cdot, t)] \, dx.$$  

(4.50)

Without loss of generality, we consider

$$\left\| \left( \begin{array}{c} c\varphi_c \\ \varphi_c \end{array} \right) \right\|_{2,2} = 1.$$  

(4.51)

Let us define the auxiliary functions,

$$P(\cdot, t) = \left\langle \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} c\varphi_c \\ \varphi_c \end{array} \right) \right\rangle_{2,2} \quad \text{and} \quad P(\cdot, t) = \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) - P(\cdot, t).$$

In view of the identity (4.51), one has $P(\cdot, t) \perp \left( \begin{array}{c} c\varphi_c \\ \varphi_c \end{array} \right)$. Moreover, we can use the compatibility condition (4.47) to deduce

$$P(\cdot, t) \perp \left( \begin{array}{c} \log(\varphi_c)v' \varphi_c' + p\varphi'_c \end{array} \right).$$

So, Proposition 4.10 can be used to obtain the existence of a constant $\kappa > 0$ such that

$$\left\langle \mathcal{L}_{Re,\phi} P(\cdot, t), P(\cdot, t) \right\rangle_{2,2} \geq \kappa\|P(\cdot, t)\|^2_{2,2}.$$  

(4.52)

Identity (4.50) combined with basic inequalities allows us to conclude

$$\|P(\cdot, t)\|^2_{2,2} = \left\| \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) \right\|^2_{2,2} - \left\langle \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} c\varphi_c \\ \varphi_c \end{array} \right) \right\rangle_{2,2}^2$$

(4.53)

$$\geq \left\| \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) \right\|^2_{2,2} - \frac{1}{4} \left[ \left\| \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) \right\|^2_{2,2} + \left\| \left( \begin{array}{c} B(\cdot, t) \\ C(\cdot, t) \end{array} \right) \right\|^2_{2,2} \right]^2.$$  

On the other hand,

$$\left\langle \mathcal{L}_{Re,\phi} \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right), \left( \begin{array}{c} A(\cdot, t) \\ D(\cdot, t) \end{array} \right) \right\rangle_{2,2} = \left\langle \mathcal{L}_{Re,\phi} P(\cdot, t), P(\cdot, t) \right\rangle_{2,2}$$

(4.54)

$$+ 2 \left\langle \mathcal{L}_{Re,\phi} P(\cdot, t), P(\cdot, t) \right\rangle_{2,2} + \left\langle \mathcal{L}_{Re,\phi} P(\cdot, t), P(\cdot, t) \right\rangle_{2,2}.$$
for all $t \geq 0$. Moreover,

$$
(4.55) \quad \left| 2 \left< L_{R_e, \varphi_c} P_\perp (\cdot, t), P_\parallel (\cdot, t) \right>_{2,2} \right| \leq \beta_5 \| \vec{w}(\cdot, t) \|^3_{X \times X}
$$

and

$$
(4.56) \quad \left| \left< L_{R_e, \varphi_c} P_\parallel (\cdot, t), P_\perp (\cdot, t) \right>_{2,2} \right| \leq \beta_6 \| \vec{w}(\cdot, t) \|^4_{X \times X},
$$

where $\beta_5$ and $\beta_6$ are positive constants. Gathering the results in (4.52), (4.53), (4.54), (4.55) and (4.56) we see that

$$
\left< L_{R_e, \varphi_c} \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right), \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right) \right>_{2,2} \geq \kappa \left\| \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right) \right\|^2_{2,2} - \beta_5 \| \vec{w}(\cdot, t) \|^3_{X \times X} - (\kappa + \beta_6) \| \vec{w}(\cdot, t) \|^4_{X \times X}.
$$

Using the definition of the operator $L_{R_e, \varphi_c}$, we get the existence of a constant $\beta_7 > 0$ such that

$$
(4.57) \quad \left< L_{R_e, \varphi_c} \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right), \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right) \right>_{2,2} \geq \int_0^L |A_x(\cdot, t)|^2 \, dx - \beta_7 \left\| \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right) \right\|^2_{2,2}.
$$

Inequalities (4.57) and (4.58) enable us to guarantee the existence of positive constants $\beta_8$, $\beta_9$ and $\beta_{10}$ such that

$$
(4.59) \quad \left< L_{R_e, \varphi_c} \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right), \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right) \right>_{2,2} \geq \beta_8 \left\| \left( \frac{A(\cdot, t)}{D(\cdot, t)} \right) \right\|^2_{X} - \beta_9 \| \vec{w}(\cdot, t) \|^3_{X \times X} - \beta_{10} \| \vec{w}(\cdot, t) \|^4_{X \times X}.
$$

In addition, the compatibility conditions in (4.48) and Proposition 4.3 give us the existence of a constant $\beta > 0$ such that

$$
\left< L_{I_m, \varphi_c} \left( \frac{B(\cdot, t)}{C(\cdot, t)} \right), \left( \frac{B(\cdot, t)}{C(\cdot, t)} \right) \right>_{2,2} \geq \beta \left\| \left( \frac{B(\cdot, t)}{C(\cdot, t)} \right) \right\|^2_{2,2}.
$$

Finally, from the definition of the operator $L_{I_m, \varphi_c}$ and similar arguments as above we obtain the existence of a constant $\beta_{11} > 0$ such that

$$
(4.60) \quad \left< L_{I_m, \varphi_c} \left( \frac{B(\cdot, t)}{C(\cdot, t)} \right), \left( \frac{B(\cdot, t)}{C(\cdot, t)} \right) \right>_{2,2} \geq \beta_{11} \left\| \left( \frac{B(\cdot, t)}{C(\cdot, t)} \right) \right\|^2_{X}.
$$

Therefore, by substituting (4.59) and (4.60) in (4.49), we have that $\Delta G(t) \geq h_1(\| \vec{w}(\cdot, t) \|_{X \times X})$ for all $t \geq 0$, where $h_1(x) := \eta_1 x^2 (1 - \eta_2 x - \eta_3 x^2 - O(x^3))$ is a smooth function and $\eta_1$, $\eta_2$ and $\eta_3$ are positive constants. We see that $h_1(0) = 0$ and $h_1(x) > 0$ for $x$ small enough. Consider $\varepsilon > 0$. Then, using the property that $E$ is continuous on the manifold

$$
S = S_c := \{ (u_0, u_1) \in X; \quad F(u_0, u_1) = F(\varphi_c, i \varepsilon \varphi_c) \},
$$

there exists $\delta = \delta(\varepsilon, c) > 0$ such that if $(u_0, u_1) \in S$ and $\| (u_0, u_1) - (\varphi_c, i \varepsilon \varphi_c) \|_X < \delta$, then for all $t \geq 0$, $h_1(\| \vec{w}(\cdot, t) \|_{X \times X}) \leq \Delta G(t) = \Delta G(0) < h_1(\varepsilon)$.
Since \( h_1 \) is an invertible function over \((0, \varepsilon)\), we have that
\[
\|\vec{w}(\cdot, t)\|_{X \times X} < \varepsilon \quad \text{for all } t \geq 0.
\]
Statement (4.61) is enough to prove the stability on the manifold \( S \) since
\[
\|(u(\cdot + y(t), t)e^{i\theta(t)} - \varphi_c, v(\cdot + y(t), t)e^{i\theta(t)} - ic\varphi_c)\|_X < \varepsilon,
\]
for all \( t \geq 0 \).

The next step is to prove the general case. Suppose a particular situation where \( c \in \left(\frac{\sqrt{p}}{2}, 1\right) \) is fixed. By repeating the same process above, there exist a constant \( \delta_1 = \delta_1(c) > 0 \) and a real function \( z_0 \) such that \( z_0(0) = 0 \), where \( z_0 \) is a strictly increasing for (small) positive values.

Moreover, for all
\[
\|s - c| < \delta_1 \text{ and } F(u_0, u_1) = F(\varphi_s, is\varphi_s),
\]
provided that \( \varepsilon > 0 \) small enough, the continuity of \( \mathcal{E} \) enables us to guarantee the existence of the parameter \( \delta_2 \), \( 0 < \delta_2 = \delta_2(c, \varepsilon) < \frac{\varepsilon}{2} \), such that if \( (u_0, u_1), (v_0, v_1) \in B_{\delta_2}(\varphi_c, ic\varphi_c) \subset X \), then
\[
|\mathcal{E}(u_0, u_1) - \mathcal{E}(v_0, v_1)| < z_0 \left(\frac{\varepsilon}{2}\right).
\]

Since the map \( s \in \mathbb{R} \mapsto \varphi_s \in H^2_{\text{pers}}([0, L]) \) is smooth, there exists \( \delta_3 > 0 \) such that if \( |s - c| < \delta_3 < \delta_1 \), then
\[
\frac{\sqrt{p}}{2} < s < 1 \quad \text{and} \quad \|F(\varphi_s, is\varphi_s) - (\varphi_c, ic\varphi_c)\|_X < \delta_2 < \frac{\varepsilon}{2}.
\]

Furthermore, since \( -\frac{d}{ds}[F(\varphi_s, is\varphi_s)] = d''(s) > 0 \) for all \( s \in \left(\frac{\sqrt{p}}{2}, 1\right) \), there exists \( \delta_4 > 0 \) such that if
\[
\|(u_0, u_1) - (\varphi_c, ic\varphi_c)\|_X < \delta_4 < \frac{\delta_2}{2},
\]
one has \( s^* = s^*(u_0, u_1) \in \mathbb{R} \), where \( |s^* - c| < \delta_3 \) and \( s^* \) satisfy \( F(u_0, u_1) = F(\varphi_{s^*}, is^*\varphi_{s^*}) \).

Consider \( \delta_1 > 0 \) as above and suppose that \( (u_0, u_1) \) verifies (4.65). Since \( |s^* - c| < \delta_3 \), we get condition (4.64) applied to \( s = s^* \), that is,
\[
\frac{\sqrt{p}}{2} < s^* < 1 \quad \text{and} \quad \|F(\varphi_{s^*}, is^*\varphi_{s^*}) - (\varphi_c, ic\varphi_c)\|_X < \delta_2 < \frac{\varepsilon}{2}.
\]

Thus, from (4.63), (4.64) and (4.66), we obtain
\[
|\mathcal{E}(u_0, u_1) - \mathcal{E}(\varphi_{s^*}, is^*\varphi_{s^*})| < z_0 \left(\frac{\varepsilon}{2}\right).
\]

Next, since \( |s^* - c| < \delta_3 < \delta_1 \) and \( F(u_0, u_1) = F(\varphi_{s^*}, is^*\varphi_{s^*}) \), we deduce from (4.62) and (4.67) that
\[
z_0(\|\vec{w}_{s^*}(\cdot, t)\|_{X \times X}) \leq \mathcal{G}_{s^*}(u_0, u_1) - \mathcal{G}_{s^*}(\varphi_{s^*}, is^*\varphi_{s^*}) < z_0 \left(\frac{\varepsilon}{2}\right).
\]

Hence, the fact that \( z_0 \) is a locally invertible function gives us
\[
\|\vec{w}_{s^*}(\cdot, t)\|_{X \times X} < \frac{\varepsilon}{2} \quad \text{for all } t \geq 0.
\]
Finally, if one combines (4.66) and (4.68),
\[
\|w_c(t)\|_{X \times X} \approx \inf_{y \in [0, L], \theta \in [0, 2\pi]} \|(u(\cdot + y, t)e^{i\theta}, v(\cdot + y, t)e^{i\theta}) - (\varphi_c, ic\varphi_c)\|_X \\
\leq \|w_s^*(t)\|_{X \times X} + \|(\varphi_s^*, is^s\varphi_s^*) - (\varphi_c, ic\varphi_c)\|_X < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for all } t \geq 0,
\]
which concludes the proof for the case \(c \in \left(\sqrt{\frac{p}{2}}, 1\right)\). A similar argument can be used to prove the stability for the case \(c \in \left(-1, -\sqrt{\frac{p}{2}}\right)\).

\[\square\]

4.5. Orbital Stability of Standing Waves in the space \(H^1_{\text{per, e}}([0, L]) \times L^2_{\text{per, e}}([0, L])\). We present a brief comment about results of stability/instability of standing waves for the Logarithmic Klein-Gordon equation (1.1) in the Hilbert space \(Y_e := H^1_{\text{per, e}}([0, L]) \times L^2_{\text{per, e}}([0, L])\) constituted by even periodic functions.

The abstract theory in [17] gives us a general method to study the orbital stability/instability to standing waves accordingly with Definition 4.8 for the abstract Hamiltonian systems of form
\[
U_t = JE'(U(t)),
\]
where \(J\) is a skew-symmetric linear operator and \(E\) is a convenient conserved quantity. In our case, if \(U = (u, u_t) := (\Re u, \Im u_t, \Im u, \Re u), J\) is taken to be
\[
J = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]
and \(E\) is the conserved quantity given by (1.7), we can follow the arguments determined on Sections 3 and 4 to obtain:
- the existence of a weak solution to the problem (3.1) for all \((u_0, v_0) \in H^1_{\text{per, e}}([0, L]) \times L^2_{\text{per, e}}([0, L]),\) as determined in Section 3.
- The existence of a smooth curve of positive solutions \(c \in I \mapsto \varphi_c \in H^2_{\text{per, e}}([0, L])\) which solves (1.2) all of them with the same period \(L > \frac{2\pi}{\sqrt{p}}\).
- \(\operatorname{in}(L_{\varphi_c}|_{X_e}) = (1, 1)\) for all \(c \in I\).
- \(d''(c) < 0\) if \(|c| < \frac{\sqrt{p}}{2}\) and \(d''(c) > 0\) if \(|c| > \frac{\sqrt{p}}{2}\).

So, by a direct application of the result in [17], we are in position to enunciate the following result.

**Theorem 4.11.** Consider \(c \in I\) satisfying \(|c| > \frac{\sqrt{p}}{2}, p = 1, 2, 3\). The periodic solution \(\varphi_c(x)\) is orbitally stable in \(Y_e\) by the periodic flow of the equation (1.1) accordingly Definition 4.8. If \(|c| < \frac{\sqrt{p}}{2}\), then \(u(x, t) = e^{ict} \varphi_c(x)\) is orbitally unstable by the periodic flow of the equation (1.1).

\[\square\]
Acknowledgements

F. N. is supported by Fundação Araucária and CNPq 304240/2018-4.

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