Research Article

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Karush-Kuhn-Tucker optimality conditions for a class of robust optimization problems with an interval-valued objective function

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Abstract: In this article, we study the nonlinear and nonsmooth interval-valued optimization problems in the face of data uncertainty, which are called interval-valued robust optimization problems (IVROPs). We introduce the concept of nondominated solutions for the IVROP. If the interval-valued objective function $f$ and constraint functions $g_i$ are nonsmooth on Banach space $E$, we establish a nonsmooth and robust Karush-Kuhn-Tucker optimality theorem.

Keywords: Karush-Kuhn-Tucker optimality, robust optimization, generalized convexity, interval-valued function, nondominated solution

MSC 2010: 49J15, 49J52, 58C06

1 Introduction

In the last few decades, the methodology for solving optimization problems has been widely applied to many research fields (see [1–13]). It is well known that the Karush-Kuhn-Tucker (KKT) optimality conditions play an important role in the study of optimization theory (see [14–32]). If the objective functions of optimization problems are taken as real numbers, they are categorized as deterministic optimization problems. In fact, the objective functions of optimization problems may be assumed as random variables with known distributions. Birge and Louveaux [33], Prékopa [34], and Vajda [35] provided the mainstream of these topics and gave many useful techniques for solving these problems.

However, more and more researchers have recently started to study the interval-valued optimization problems because of their important applications (see [36–42]). In this article, we shall investigate the following interval-valued robust optimization problem (IVROP):

\[
(IVROP) \quad \begin{align*}
\inf & \quad f(x) = [f^L(x), f^U(x)] \\
\text{s.t.} & \quad g_i(x, v_i) \leq 0, \quad \forall v_i \in V_i, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

where objective function $f$ is a locally Lipschitz interval-value function. If the constraint functions $g_i$ are nonlinear and nonsmooth on the Banach space $E$ for each $i \in \{1, 2, \ldots, m\}$, we shall establish the robust KKT necessary optimality conditions.

In order to highlight the generality of our IVROPs, we present several particular cases.

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Case 1. If the interval-valued function is reduced to a real-valued function, i.e., \( f^I(x) = f^U(x) \) for all \( x \in E \), then problem (1.1) reduces to the following problem:

\[
\begin{align*}
\text{(ROP)} & \quad \inf_{x \in K} f(x) \\
\text{subject to} & \quad K = \{ x \in E \mid g(x, v_i) \leq 0, \ \forall v_i \in V_i, \ i = 1, 2, \ldots, m \},
\end{align*}
\]  

(1.2)

which is called robust optimization problem (ROP). The KKT necessary optimality conditions of this kind of ROP have been studied by many scholars, for more details, we refer to [43–47].

Case 2. For each \( i = 1, 2, \ldots, m \), if constraint functions are independent of \( v_i \), then problem (1.1) reduces to the following interval-valued optimization problem:

\[
\begin{align*}
\text{(IVOP)} & \quad \inf f(x) = [f^L(x), f^U(x)] \\
\text{subject to} & \quad K = \{ x \in E \mid g(x) \leq 0, \ i = 1, 2, \ldots, m \}.
\end{align*}
\]  

(1.3)

Under the assumptions that each \( g(x) \) is convex and continuously differentiable, Ishibuchi and Tanaka in [36], and Inuiguchi and Kume in [37], and Wu in [38,39] studied the KKT necessary optimality conditions of the interval-valued optimization.

2 Preliminaries

In this section, we first recall some useful notions and well-known results in nonsmooth analysis and nonlinear analysis. Let \( E \) be a given real reflexive Banach space. \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( E^* \) and \( E \), where \( E^* \) is the dual space of \( E \). A functional \( \phi : E \rightarrow \mathbb{R} \) is called locally Lipschitz, if for every \( u \in E \) there exists a neighborhood \( U \) of \( u \) and a constant \( L_u > 0 \) such that

\[ |\phi(v_i) - \phi(v_2)| \leq L_u \|v_1 - v_2\|, \quad \forall v_1, v_2 \in U. \]

Assume that \( \phi : E \rightarrow \mathbb{R} \) is a locally Lipschitz functional. We denote by \( \phi^\prime(u; v) \) Clarke’s generalized directional derivative of \( \phi \) at \( u \in E \) in the direction \( v \in E \), that is,

\[ \phi^\prime(u; v) = \limsup_{\lambda \to 0^+} \frac{\phi(u + \lambda v) - \phi(u)}{\lambda}. \]  

(2.1)

We shall also denote the one-side directional derivative of \( \phi \) at \( u \) by \( \phi^\prime(u; v) \), i.e.,

\[ \phi^\prime(u; v) = \lim_{t \to 0^+} \frac{\phi(u + tv) - \phi(u)}{t}, \]  

(2.2)

whenever this limit exists.

Definition 2.1. A functional \( \phi : E \rightarrow \mathbb{R} \) is said to be upper semicontinuous (u.s.c. for short) at \( u \) if for every sequence \( \{u_n\} \) in \( E \) converging to \( u \), one has

\[ \phi(u) \geq \limsup_{n \to \infty} \phi(u_n). \]

A functional \( \phi \) is said to be upper semicontinuous on a subset \( K \) of \( E \), if it is upper semicontinuous at every point of \( K \).

Definition 2.2. The generalized gradient of a locally Lipschitz functional \( \phi : E \rightarrow \mathbb{R} \) at a point \( u \in E \) is denoted by

\[ \partial \phi(u) = \{ \xi \in E^* \mid \phi^\prime(u; v) \geq \langle \xi, v \rangle \text{ for all } v \in E \}. \]  

(2.3)
We point out that for each \( u \in E \), we have \( \partial \phi(u) \neq \emptyset \) (see [48]).

The next theorem provides some basic properties for Clarke’s generalized directional derivatives and Clarke’s generalized gradients (for details, see Clarke [48], Proposition 2.1.2).

**Theorem 2.3.** Let \( \phi : E \to \mathbb{R} \) be locally Lipchitz of rank \( L_u \) near the point \( u \in E \). Then,

(i) \( \partial \phi(u) \) is a convex, weak* compact subset of \( E^* \) and

\[
\| \xi \|_{E^*} \leq L_u \quad \text{for all} \quad \xi \in \partial \phi(u);
\]

(ii) the function \( v \mapsto \phi^*(u; v) \) is finite convex, positively homogeneous, subadditive and satisfies

\[
\phi^*(u; v) \leq L_u \| v \|_{E};
\]

(iii) for each \( v \in E \), one has

\[
\phi^*(u; v) = \sup \{ \langle \xi, v \rangle \mid \xi \in \partial \phi(u) \};
\]

(iv) \( \phi^*(u; v) \) is upper semicontinuous on \( E \times E \) as a function of \( u \), \( v \), i.e., for all \( u, v \in E \), \( \{u_n\} \subset E \), \( \{v_n\} \subset E \) with \( u_n \to u \) and \( v_n \to v \) as \( n \to \infty \), such that

\[
\limsup_{n \to \infty} \phi^*(u_n; v_n) \leq \phi^*(u; v).
\]

Now, we introduce the invex set and invex function on Banach space \( E \).

**Definition 2.4.** A set \( K \subset E \) is said to be invex with respect to \( \eta : E \times E \to E \), if

\[
\forall x, y \in K, \lambda \in [0, 1] \Rightarrow y + \lambda \eta(x, y) \in K.
\]

**Definition 2.5.** Let \( K \subset E \) be an invex set with respect to \( \eta : E \times E \to E \). We say that \( f : K \to \mathbb{R} \) is preconcave if

\[
f(y + \lambda \eta(x, y)) \geq \lambda f(x) + (1 - \lambda) f(y), \quad \forall x, y \in K, \lambda \in [0, 1].
\]

**Remark 2.6.** If \( \eta(x, y) = x - y \), then an invex set with respect to \( \eta \) will be reduced to a convex set, while a preconcave function will be reduced to a concave function.

Now, we give the well-known Gorden’s alternative theorem.

**Theorem 2.7.** (Gorden’s alternative theorem [46]) Let \( f_i : X \to \mathbb{R} \) for \( i = 1, \ldots, m \) be convex functions, \( X \) be a convex subset of Banach space \( E \). Define

\[
F = \{ x \in E^* \mid f_i(x) < 0, \quad \text{for} \quad i = 1, \ldots, m \},
\]

\[
G = \{ y \in \mathbb{R}^m, y_i > 0 \quad \text{for at least one} \quad i \mid y f(x) \geq 0 \quad \text{for all} \quad x \in E \},
\]

where \( f(x) = (f_1(x), \ldots, f_m(x)) \) and \( y f(x) = \sum_{i=1}^m y_i f_i(x) \). Then, \( F \neq \emptyset \) if and only if \( G \neq \emptyset \).

Finally, we shall introduce some concepts of interval-valued functions and the nondominated solutions of interval-valued optimization problems on Banach space \( E \).

Let us denote by \( \text{Cl}([\mathbb{R}]^\ast) \) the class of closed intervals in \( \mathbb{R} \), i.e., \( A \in \text{Cl}([\mathbb{R}]^\ast) \) means that \( A = [a^L, a^U] \), where \( a^L \) and \( a^U \) present the lower and upper bounds of the closed interval \( A \) in \( \mathbb{R} \), respectively. Let \( A, B \in \text{Cl}([\mathbb{R}]^\ast) \), then

(1) \( A + B = \{ a + b \mid a \in A \text{ and } b \in B \} = [a^L + b^L, a^U + b^U] \);

(2) \( -A = \{-a \mid a \in A\} = [-a^U, -a^L] \);

(3) \( A - B = A + (-B) = [a^L - b^U, a^U - b^L] \);

(4) \( A \times B = [\min_{a, b}, \max_{a, b}] \), where \( \min_{a, b} = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\} \) and \( \max_{a, b} = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\} \).
Especially, each \( x \in \mathbb{R} \) can be regarded as a closed interval \( A_x = [x, x] \) and \( xA = [xa | a \in A] \). In fact, interval functions have other properties, and for more details one can refer to [37–39].

A mapping \( f : E \to \text{Cl}(\mathbb{R}) \) is called an interval-valued function, i.e., \( f(x) = [f^L(x), f^U(x)] \), where \( f^L \) and \( f^U \) are two real valued continuous functions defined on \( E \) with \( f^L(x) \leq f^U(x) \), \( \forall x \in E \). First, we consider the following formulation of interval-valued optimization problems on Banach space \( E \):

\[
\begin{align*}
\text{(IVOP)} \\
\min \ f(x) = [f^L(x), f^U(x)] \\
\text{subject to } x \in U \subset E,
\end{align*}
\]

where \( f(x) \) is an interval-valued function. We need to make clear the meaning of minimization problems (IVOP). For the purpose, we introduce a partial order relationship in \( \mathbb{R} \) as:

- \( \preceq \) if and only if \( a^L \leq b^L \) and \( a^U \leq b^U \),
- \( \preceq \) if and only if \( a^L < b^L \) and \( a^U < b^U \).

Now, we define the concept of solutions for interval-valued optimization problems, which was introduced by Wu in [38,39].

**Definition 2.8.** Let \( x^* \) be a feasible solution of the IVOP, i.e., \( x^* \in U \) (see (2.8)). We say that \( x^* \) is a nondominated solution of the IVOP if and only if there exists no \( x \in U \) such that

\[
\begin{align*}
f(x) & \implies f(x^*).
\end{align*}
\]

In this case, \( f(x^*) \) is called the nondominated objective value of \( f \). We denote by \( \min(f, U) \) the set of all nondominated objective values of \( f \).

**Example 2.1.** Let us consider the following interval-valued optimization problem:

\[
\begin{align*}
\text{(P1)} \min \ f(x) &= f(x_1, x_2) = [f^L(x_1, x_2), f^U(x_1, x_2)] \\
\text{s.t. } x_1 + x_2 &\geq 2, \\
x_1 &\geq -2, \\
x_2 &\geq -2,
\end{align*}
\]

where functions \( f^L : \mathbb{R}^2 \to \mathbb{R} \), \( f^U : \mathbb{R}^2 \to \mathbb{R} \) as:

\[
\begin{align*}
f^L(x_1, x_2) &= -20e^{-0.2\left((\frac{x_1^2 + x_2^2}{2})^{0.2} - e^{cos(2\pi(x_1-2)) + cos(2\pi(x_2-2))}\right)} + 22.71289, \\
f^U(x_1, x_2) &= -20e^{-0.2\left((\frac{x_1^2 + x_2^2}{2})^{0.2} - e^{cos(2\pi(x_1-2)) + cos(2\pi(x_2-2))}\right)} + x + 28.71289.
\end{align*}
\]

It is not hard to see that

\[
f^L(2, 2) \leq f^L(x_1, x_2) \quad \text{and} \quad f^U(2, 2) \leq f^U(x_1, x_2),
\]

and for all feasible solutions \((x_1, x_2)\) there exists no \((x_1, x_2)\) such that (Figure 1)

\[
f(x_1, x_2) \preceq f(2, 2).
\]

This means that \( \bar{x} = (\bar{x}_1, \bar{x}_2) = (2, 2) \) is a **nondominated** solution of problem (P1).

**Example 2.2.** Let us consider another interval-valued optimization problem:

\[
\begin{align*}
\text{(P2)} \min \ g(x) &= g(x_1, x_2) = [g^L(x_1, x_2), g^U(x_1, x_2)] \\
\text{s.t. } \pi &\geq x_1 \geq -\pi, \\
\pi &\geq x_2 \geq -\pi,
\end{align*}
\]
where functions $g^L : \mathbb{R}^2 \to \mathbb{R}$ and $g^U : \mathbb{R}^2 \to \mathbb{R}$ as:

$$
\begin{align*}
g^L(x_1, x_2) &= \sin(x_1)\sin(x_2), \\
g^U(x_1, x_2) &= \sin(x_1)e^{1 - \cos(x_2)^2} + \cos(x_2)e^{1 - \sin(x_2)^2} + (x_1 - x_2)^2 + 200.
\end{align*}
$$

It is not hard to see that problem (P2) merely has nondominated solutions (Figure 2) $(x_1 = -1.58214, x_2 = -3.13025)$ and $(x_1 = -1.5708, x_2 = 1.5708)$, i.e., $\min(f, U) = \{(x_1 = -1.58214, x_2 = -3.13025), (x_1 = -1.5708, x_2 = 1.5708)\}$.

### 3 KKT nonsmooth robust optimality conditions with interval-valued functions

In this section, we will give a nonsmooth KKT optimality theorem for IVROPs (1.1).

An interval-valued function $f(x) = [f^L(x), f^U(x)]$ is called locally Lipschitz on $U \subseteq E$, if $f^L(x)$ and $f^U(x)$ are locally Lipschitz functionals on $U$. In the sequel, we assume that the interval-valued function $f(x)$ is locally Lipschitz and $V_i, i = 1, 2, ..., m$ are sequentially compact subsets of topological spaces $T_i$, which are involved in problem (1.1).
Assumptions:

(A1) $g_i: E \times V_i \to \mathbb{R}$ is upper semicontinuous on $E \times V_i$ for all $i = 1, 2, \ldots, m$;

(A2) $g_i$ is Lipschitz in $x$, uniformly for all $v$ in $V_i$, i.e., there exists a constant $L_i > 0$, such that for any $x, y \in E$ and for all $v \in V_i$, one has

$$|g_i(x, v) - g_i(y, v)| \leq L_i\|x - y\|_E; \quad (3.1)$$

(A3) $g_{i0}^0(x, v; \cdot) = g'_{i0}(x, v; \cdot)$, where $g_{i0}^0(x, v; \cdot)$ and $g'_{i0}(x, v; \cdot)$ are Clarke’s generalized directional derivative and one-side directional derivative of $g_i$ with respect to $x$, respectively; and

(A4) the generalized gradient $\partial g_i(x, v)$ with respect to $x$ is weak* upper semicontinuous in $(x, v)$.

For each $i = 1, 2, \ldots, m$, define $\phi_i: E \to \mathbb{R}$ by

$$\phi_i(x) = \max \{g_i(x, v) | v \in V_i\}. \quad (3.2)$$

The compactness of $V_i$ implies that $\phi_i$ is well defined on $E$. By (A2), we readily get that $\phi_i$ is Lipschitz on $E$ (with constant $L_i$). On the other hand, we define the constraint set of the (IVROP) by

$$S = \{x \in E | g_i(x, v) \leq 0, \forall v \in V_i, i = 1, 2, \ldots, m\}. \quad (3.3)$$

From the definition of function $\phi_i$, we have

$$S = \{x \in E | \phi_i(x) \leq 0, i = 1, 2, \ldots, m\}. \quad (3.4)$$

For any $x^* \in S$, let us decompose $I = \{1, 2, \ldots, m\}$ into two index sets $I_1(x^*)$ and $I_2(x^*)$ with $I = I_1(x^*) \cup I_2(x^*)$ and $I_1(x^*) \cap I_2(x^*) = \emptyset$, where $I_1(x^*) = \{i \in I | \phi_i(x^*) = 0, i = 1, 2, \ldots, m\}$ and $I_2(x^*) = I \setminus I_1(x^*)$. For each $i \in I_1(x^*)$, we define $V_i(x^*) = \{v \in V_i | g_i(x^*, v) = \phi_i(x^*) = 0\}$.

We also define an Extended Nonsmooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) at $x^* \in S$ as follows:

$$\exists d \in E \text{ such that } g_{i0}^0(x, v; d) < 0, \forall v \in V_i, \forall i \in I_1(x^*). \quad (3.5)$$

Now, we are going to prove the following KKT necessary optimality theorem of IVROPs.

**Theorem 3.1.** Suppose each $V_i$ is an invex set with respect to $\eta_i: V_i \times V_i \to T_i$ and $g_i(x, \cdot)$ is preconcave on $V_i$, for each $x \in E$, $i \in I$. If (A1)–(A4) hold and $x^* \in S$ is a nondominated solution of the IVROP, then there exist $v_i \in V_i(x^*), \lambda_0^L \geq 0, \lambda_0^U \geq 0$ and $\lambda_i \geq 0, i = 1, 2, \ldots, m$, such that

$$0 \in \lambda_0^L \partial f^L(x^*) + \lambda_0^U \partial f^U(x^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(x^*, v_i),$$

$$0 = \lambda_i g_i(x^*, v_i) \quad \text{for all } i = 1, 2, \ldots, n,$$

$$(\lambda_0^L, \lambda_0^U, \lambda_1, \ldots, \lambda_m) \neq (0, 0, 0, \ldots, 0).$$

Furthermore, if the ENMFCQ at $x^*$ holds, one has

$$0 \in \partial f^L(x^*) + \lambda_0^U \partial f^U(x^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(x^*, v_i),$$

$$0 = \lambda_i g_i(x^*, v_i), \quad \text{for all } i = 1, 2, \ldots, n.$$

or

$$0 \in \lambda_0^L \partial f^L(x^*) + \partial f^U(x^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(x^*, v_i),$$

$$0 = \lambda_i g_i(x^*, v_i), \quad i = 1, 2, \ldots, m.$$
Lemma 3.2. Let $U$ be a subset of $E$ and let $V$ be a sequentially compact subset of topological space $T$ and invex set with respect to $\eta : V \times V \rightarrow T$. And let $g : U \times V \rightarrow \mathbb{R}$ satisfy conditions $(A_1)$–$(A_6)$. So, we can also define maximum function $\phi : U \rightarrow \mathbb{R}$ by

$$\phi(x) = \max_{v \in V} g(x, v).$$

In addition, assume that the function $g(x, \cdot)$ is preconcave on $V$ for each $x \in E$. Then, the following statements hold:

(i) The set $V(x) = \{v \in V \mid g(x, v) = \phi(x)\}$ is invex with respect to $\eta$ and sequentially compact in $T$.

(ii) The set

$$\partial \phi(x, V(x)) = \{\xi \mid \exists v \in V(x) \text{ such that } \xi \in \partial \phi(x, v)\}$$

is convex and weak* compact.

(iii) $\partial \phi(x) = \{\xi \mid \exists v \in V(x) \text{ such that } \xi \in \partial \phi(x, v)\}$.

Proof.

(i) $V(x)$ is invex with respect to $\eta$ and sequentially compact in $T$.

Let $x \in U$ be fixed and $V$ an invex set with respect to $\eta$. Therefore, for any $v^1, v^2 \in V(x) \subset V$ and $\lambda \in [0,1]$, we have $v^2 + \lambda \eta(v^1, v^2) \in V$ and $g(x, v^1) = g(x, v^2) = \phi(x)$. Since the function $g(x, \cdot)$ is preconcave, we obtain that

$$\phi(x) \geq g(x, v^2 + \lambda \eta(v^1, v^2)) \geq \lambda g(x, v^2) + (1 - \lambda) g(x, v^2) = \phi(x).$$

Therefore, $g(x, v^2 + \lambda \eta(v^1, v^2)) = \phi(x)$, which implies $v^2 + \lambda \eta(v^1, v^2) \in V(x)$.

In order to prove that $V(x)$ is sequentially compact in $T$, let us consider a sequence $\{v^k\} \subset V(x) \subset V$. By $(A_1)$ and $g(x, v^k) = \phi(x)$ for all $k \geq 1$, we have

$$\phi(x) = \limsup_{k \to \infty} g(x, v^k) \leq g(x, v) \leq \phi(x).$$

Hence, $v \in V(x)$.

(ii) **The set $\partial \phi(x, V(x))$ is convex and weak* compact.**

According to the definition of $\partial \phi(x, V(x))$, for any $\xi^1, \xi^2 \in \partial \phi(x, V(x)), \lambda \in [0,1]$, there exist $v^1, v^2 \in V(x)$ such that $\xi^1 \in \partial \phi(x, v^1)$ and $\xi^2 \in \partial \phi(x, v^2)$. By (i), we get that the set $V(x)$ is invex with respect to $\eta$. Therefore,

$$v^2 + \lambda \eta(v^1, v^2) \in V(x),$$

and

$$\phi(x) = g(x, v^2 + \lambda \eta(v^1, v^2)) = g(x, v^1) = g(x, v^2).$$

From $(A_3)$ and $g(x, \cdot)$ being preconcave, we have for any $d \in E$

$$g^\ast(x, v^2 + \lambda \eta(v^1, v^2); d) = g^\ast(x, v^2 + \lambda \eta(v^1, v^2); d)$$

$$= \lim_{t \to 0^+} \frac{g(x + td, v^2 + \lambda \eta(v^1, v^2)) - g(x, v^2 + \lambda \eta(v^1, v^2))}{t}$$

$$\geq \lim_{t \to 0^+} \frac{\lambda g(x + td, v^1) + (1 - \lambda) g(x + td, v^2) - \lambda g(x, v^1) - (1 - \lambda) g(x, v^2)}{t}$$

$$= \lambda \lim_{t \to 0^+} \frac{g(x + td, v^1) - g(x, v^1)}{t} + (1 - \lambda) \lim_{t \to 0^+} \frac{g(x + td, v^2) - g(x, v^2)}{t}$$

$$= \lambda g^\ast(x, v^1; d) + (1 - \lambda) g^\ast(x, v^2; d)$$

$$= \lambda g^\ast(x, v^1; d) + (1 - \lambda) g^\ast(x, v^2; d)$$

$$\geq \lambda \langle \xi^1, d \rangle + (1 - \lambda) \langle \xi^2, d \rangle$$

$$= \langle \lambda \xi^1 - (1 - \lambda) \xi^2, d \rangle.$$
which implies
\[ \lambda \xi^1 + (1 - \lambda) \xi^2 \in \partial_x g(x, v^2 + \eta(v^1, v^2)), \]
i.e., \( \lambda \xi^1 + (1 - \lambda) \xi^2 \in \partial_x g(x, V(x)) \). So, \( \partial_x g(x, V(x)) \) is convex.

In the following, we shall use Alaoglu’s theorem to prove that \( \partial_x g(x, V(x)) \) is weak* compact. We first prove that the set \( \partial_x g(x, V(x)) \) is weak* closed. Let \( \{ \xi^k \} \subset \partial_x g(x, V(x)) \) be a sequence weak* converging to \( \xi \) in \( E^* \). Then, there exists \( \{ v^k \} \subset V(x) \) such that \( \xi^k \in \partial_x g(x, v^k) \). By (i), we may assume that \( \{ v^k \} \) converges to \( v \in V(x) \). It follows from condition (A4) that \( \xi \in \partial_x g(x, v) \). Hence, \( \partial_x g(x, V(x)) \) is weak* closed.

Next, we will prove that the set \( \partial_x g(x, V(x)) \) is bounded. By (ii), we have for any \( \xi \in \partial_x g(x, v) \) and for any \( v \in V \),
\[
\langle \xi, d \rangle \leq |g^*_x(x, v; d)| \leq \limsup_{h \to 0, t \to 0} \frac{|g(x + h + td, v) - g(x + h, v)|}{t} \leq L\|d\|_E, \quad \forall d \in E.
\]
Hereafter, \( \| \cdot \| \) stands for the norm in the Banach space \( E \). Consequently, we obtain
\[
\| \xi \| \leq L, \forall \xi \in \partial_x g(x, v) \quad \text{and} \quad \forall v \in V.
\]
Therefore, we have shown that the set \( \partial_x g(x, V(x)) \) is weak* closed and bounded in \( E^* \). By Alaoglu’s theorem, we obtain that \( \partial_x g(x, V(x)) \) is weak* compact.

(iii) \( \partial_x g(x, V(x)) = \partial f(x) \).

We first prove the inclusion
\[
\partial_x g(x, V(x)) \subset \partial f(x).
\]
For any \( \xi \in \partial_x g(x, V(x)) \), there exists \( v \in V(x) \) such that \( \xi \in \partial_x g(x, v) \). So, \( f(x) = g(x, v) \) and \( \forall d \in E \), we have
\[
\langle \xi, d \rangle \leq g^*_x(x, v; d) = g^*_x(x, v; d) = \lim_{t \to 0^+} \frac{g(x + td, v) - g(x, v)}{t} \leq \limsup_{t \to 0^+} \frac{\phi(x + td) - \phi(x)}{t} \leq \phi^*(x; d).
\]
Therefore, for any \( v \in V(x) \), one has
\[
\langle \xi, d \rangle \leq \phi^*(x; d) \quad \text{for all} \quad \xi \in \partial_x g(x, v),
\]
which implies
\[
\partial_x g(x, V(x)) \subset \partial f(x).
\]
Conversely, for any \( \xi \in \partial f(x) \)
\[
\phi^*(x; d) \geq \langle \xi, d \rangle, \quad \forall d \in E.
\]
By the assumptions (A1) and Theorem 2.1 of [48] we have
\[
\phi^*(x; d) = \phi^*(x; d) = \max \{ g^*_x(x, v; d) \mid v \in V(x) \} = \max \{ \langle \eta, d \rangle \mid \eta \in \partial_x g(x, v), v \in V(x) \}.
\]
(3.7)
Then
\[
\max \{ \langle \eta, d \rangle \mid \eta \in \partial_x g(x, v), v \in V(x) \} = \phi^*(x; d) \geq \langle \xi, d \rangle, \quad \forall d \in E.
\]
Consequently,
\[
\max \{ \langle \eta - \xi, d \rangle \mid \eta \in \partial_x g(x, v), v \in V(x) \} \geq 0, \quad \forall d \in E.
\]
Particularly, we get
\[
\inf_{|d| \leq 1, v \in V(x)} \max \{ \langle \eta - \xi, d \rangle \} \geq 0.
\]
By (ii), we know set $\partial g(x, V(x))$ is convex and weak* compact in $E^*$. Applying Sion’s minimax theorem (see [49]), we deduce the existence of an element $\bar{\eta} \in \partial g(x, V(x))$ such that
\[
\inf_{|d| \leq 1} \langle \bar{\eta} - \xi, d \rangle = \inf_{|d| \leq 1} \max_{\eta \in \partial g(x, V(x))} \langle \eta - \xi, d \rangle \geq 0.
\]
Therefore,
\[
0 \leq \inf_{|d| \leq 1} \langle \bar{\eta} - \xi, d \rangle = -\sup_{|d| \leq 1} (\xi - \bar{\eta}, d) = -\|\bar{\eta} - \xi\| \leq 0,
\]
which implies that $\bar{\eta} = \xi$, so we conclude $\partial \phi(x) \subset \partial g(x, V(x))$. This completes the proof. $\square$

Next, we shall consider the following nonsmooth interval-valued optimization problem:

\begin{equation}
\text{(IVOP)} \quad \begin{array}{ll}
\text{Minimize} & f(x) = [f^L(x), f^U(x)] \\
\text{subject to} & \phi_i(x) \leq 0, \quad i = 1, 2, \ldots, m.
\end{array}
\end{equation}

For this problem, we have

**Lemma 3.3.** Assume that $\phi_i$ ($i \in \{1, 2, \ldots, m\}$) are locally Lipschitz on $E$. If $x^* \in S$ is a nondominated solution of the IVOP, then the following system (S2):

\begin{align}
(f^L)^*(x^*; d) &< 0, \\
(f^U)^*(x^*; d) &< 0, \\
\phi_i^*(x^*; d) &< 0, \quad \forall i \in I_i(x^*)
\end{align}

has no solution $d \in E$.

**Proof.** Arguing by contradiction. Let us assume that system (S2) has at least one solution, i.e., there exists $d_0 \in E$ such that

\begin{align}
(f^L)^*(x^*; d_0) &< 0, \\
(f^U)^*(x^*; d_0) &< 0, \\
\phi_i^*(x^*; d_0) &< 0, \quad \forall i \in I_i(x^*)
\end{align}

Then, we have
\[
\limsup_{t \to 0^+} \frac{f^L(x^* + t d_0) - f^L(x^*)}{t} \leq \limsup_{w \to x^*, t \to 0^+} \frac{f^L(w + t d_0) - f^L(w)}{t} = (f^L)^*(x^*; d_0) < 0,
\]
which implies that there exists a sequence $t_k > 0$ with $\lim_{k \to \infty} t_k = 0$, such that
\[
\lim_{k \to \infty} \frac{f^L(x^* + t_k d_0) - f^L(x^*)}{t_k} = \limsup_{t \to 0^+} \frac{f^L(x^* + t d_0) - f^L(x^*)}{t} < 0.
\]
Consequently, there exists an integer number $N_i > 0$ such that
\[
f^L(x^* + t_k d_0) < f^L(x^*), \quad \forall k \geq N_i.
\]
Similarly, we get that
\[
\lim_{k \to \infty} \frac{f^U(x^* + t_k d_0) - f^U(x^*)}{t_k} = \limsup_{t \to 0^+} \frac{f^U(x^* + t d_0) - f^U(x^*)}{t} < 0
\]
and
\[
\lim_{k \to \infty} \frac{\phi_i(x^* + t_k d) - \phi_i(x^*)}{t_k} \leq \limsup_{w \to x^*, t \to 0^+} \frac{\phi_i(w + t d) - \phi_i(w)}{t} = \phi_i^*(x^*; d) < 0, \quad \forall i \in I_i(x^*).
\]
From (3.12) and (3.13), we can easily obtain that there exist two integer numbers \( N > 0, N_0 > 0 \) such that
\[
f^U(x^* + t_k d_0) < f^U(x^*), \quad \forall k \geq N_2
\]
and
\[
\phi_i(x^* + t_k d_0) < \phi_i(x^*) = 0, \quad \forall i \in I_k(x^*), \forall k \geq N_3.
\]
Moreover, \( \forall i \in I_k(x^*), \phi_i(x^*) < 0 \). By using the local Lipschitz continuity of \( \phi_i(x^*) \), there exists an integer number \( N_i > 0, \) such that
\[
\phi_i(x^* + t_k d_0) < 0, \quad \forall i \in I_k(x^*), \forall k \geq N_i.
\]
Therefore, \( k \geq \max \{N, N_0, N_3, N_i\} \), we get that \( x^* + t_k d_0 \) satisfies constraint conditions, which contradicts definition of the nondominated solution of the IVOP.

□

Proof of Theorem 3.1. Let \( f(x) = [f^L(x), f^U(x)] \) be an interval-valued function. If \( x^* \) is a nondominated solution of IVROP, for any \( x \in S \) there exists no \( x \) such that
\[
f(x) <_{LU} f(x^*). \tag{3.14}
\]

Note that, if \( I_k(x^*) = \emptyset \), then for all \( i = 1, 2, \ldots, m \) we have \( \phi_i(x^*) < 0 \) and the theorem holds for any \( \lambda_0^L = 1 \) (or \( \lambda_0^U = 1 \)) and \( \lambda_i = 0, i = 1, 2, \ldots, m \). Without loss of generality, we may assume that the set \( I_k(x^*) \neq \emptyset \).

Since the functions \( d \mapsto (f^L)^+(x^*; d), d \mapsto (f^U)^+(x^*; d) \) and \( d \mapsto \phi_i^+(x^*; d) \) are convex, according to Lemma 3.2 and Gordan’s alternative theorem (Theorem 2.7) in [46] that there exist \( \lambda_0^L \geq 0, \lambda_0^U \geq 0 \) and \( \lambda_i \geq 0, i \in I_k(x^*) \), not all zero such that
\[
\lambda_0^L (f^L)^+(x^*; d) + \lambda_0^U (f^U)^+(x^*; d) + \sum_{i \in I_k(x^*)} \lambda_i \phi_i^+(x^*; d) \geq 0, \quad \forall d \in E. \tag{3.15}
\]

Consequently, from Theorem 2.3(iii), we have
\[
\lambda_0^L \max \{\langle \xi_0^L, d \rangle | \xi_0^L \in \partial f^L(x^*) \} + \lambda_0^U \max \{\langle \xi_0^U, d \rangle | \xi_0^U \in \partial f^U(x^*) \} \\
+ \sum_{i \in I_k(x^*)} \lambda_i \max \{\langle \xi_i, d \rangle | \xi_i \in \partial \phi_i(x^*) \} \geq 0, \quad \forall d \in E.
\]

Set
\[
\Gamma = \{d \in E | \|d\| \leq 1\}.
\]
Hence,
\[
\max_{\xi_0^L \in \partial f^L(x^*)} \max_{\xi_0^U \in \partial f^U(x^*)} \max_{\xi_i \in \partial \phi_i(x^*)} \left\{ \lambda_0^L \xi_0^L + \lambda_0^U \xi_0^U + \sum_{i \in I_k(x^*)} \lambda_i \xi_i, d \right\} \geq 0, \quad \forall d \in \Gamma, \tag{3.16}
\]
which implies
\[
\inf_{d \in \Gamma} \max_{\xi_0^L \in \partial f^L(x^*)} \max_{\xi_0^U \in \partial f^U(x^*)} \max_{\xi_i \in \partial \phi_i(x^*)} \left\{ \lambda_0^L \xi_0^L + \lambda_0^U \xi_0^U + \sum_{i \in I_k(x^*)} \lambda_i \xi_i, d \right\} \geq 0. \tag{3.17}
\]

On the other hand, the set \( \partial f^L(x^*), \partial f^U(x^*) \) and \( \partial \phi_i(x^*) \) are convex and weak* compact. By Sion’s minimax theorem (see [49]), there exist elements \( \xi_0^L \in \partial f^L(x^*), \xi_0^U \in \partial f^U(x^*) \) and \( \xi_i \in \partial \phi_i(x^*) \) for all \( i \in I_k(x^*) \) such that
\[
\inf_{d \in I} \left( \lambda_0^L \tilde{z}_0^L + \lambda_0^U \tilde{z}_0^U + \sum_{i \in I(x')} \lambda_i \tilde{z}_i, d \right) \geq 0.
\]

Note that
\[
0 \leq \inf_{d \in I} \left( \lambda_0^L \tilde{z}_0^L + \lambda_0^U \tilde{z}_0^U + \sum_{i \in I(x')} \lambda_i \tilde{z}_i, d \right) = - \sup_{d \in I} \left( \lambda_0^L \tilde{z}_0^L + \lambda_0^U \tilde{z}_0^U + \sum_{i \in I(x')} \lambda_i \tilde{z}_i, -d \right) = - \left\| \lambda_0^L \tilde{z}_0^L + \lambda_0^U \tilde{z}_0^U + \sum_{i \in I(x')} \lambda_i \tilde{z}_i \right\| \leq 0.
\]

Therefore, \( \left\| \lambda_0^L \tilde{z}_0^L + \lambda_0^U \tilde{z}_0^U + \sum_{i \in I(x')} \lambda_i \tilde{z}_i \right\| = 0 \), which implies that
\[
0 \in \lambda_0^L \partial f^L(x^*) + \lambda_0^U \partial f^U(x^*) + \sum_{i \in I(x')} \lambda_i \partial g_i(x^*, \nu_i).
\] (3.18)

Furthermore, by Lemma 3.2(iii)
\[
\partial \phi_i(x^*) = \{ \xi_i \mid \exists \nu \in V(x^*) \}, \quad i = 1, 2, \ldots, m.
\]
So, for \( i \in I(x^*) \) there exists \( \nu_i \in V(x^*) \) such that
\[
0 \in \lambda_0^L \partial f^L(x^*) + \lambda_0^U \partial f^U(x^*) + \sum_{i \in I(x')} \lambda_i \partial g_i(x^*, \nu_i).
\]

Thus, by letting \( \lambda_i = 0 \) for all \( i \in I(x^*) \)
\[
0 \in \lambda_0^L \partial f^L(x^*) + \lambda_0^U \partial f^U(x^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(x^*, \nu_i),
\]
\[
0 = \lambda_i g_i(x^*, \nu_i), \quad i = 1, 2, \ldots, m.
\]

If we assume that the ENMFCQ at \( x^* \) holds, we are going to show that (3.14) holds. We argue by contradiction, if it is not true, i.e., \( \lambda^L = 0 \) and \( \lambda^U = 0 \), then \( \lambda_i \geq 0, i \in I(x^*) \) not all zero and
\[
0 \in \sum_{i \in I(x')} \lambda_i \partial g_i(x^*, \nu_i).
\]
Consequently, we have
\[
\sum_{i \in I(x')} \lambda_i g_i(x^*, \nu_i; d) \geq 0, \quad \forall d \in E,
\]
which contradicts condition (ENMFCQ). Hence, we may assume that \( \lambda^L = 1 \) or \( \lambda^U = 1 \), and so
\[
0 \in \partial f^L(x^*) + \lambda_0^U \partial f^U(x^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(x^*, \nu_i);
\]
\[
\text{(or } 0 \in \lambda_0^L \partial f^L(x^*) + \partial f^U(x^*) + \sum_{i=1}^{m} \lambda_i \partial g_i(x^*, \nu_i) \text{)}
\]
\[
0 = \mu_i g_i(x^*, \nu_i), \quad i = 1, 2, \ldots, m.
\]
This completes the proof. \( \square \)

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