Computation of the derivative of the Hurwitz $\zeta$–function and the higher Kinkelin constants

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I use the numerical values of the generalised Glaisher-Kinkelin-Bendersky (GKB) constants to give numerical values for the derivatives of the Hurwitz $\zeta$–function at negative integers, rather than the other way round. I point out that both Glaisher’s numerical approach and Bendersky’s recursion for the generalised gamma function were anticipated by Jeffery in 1862 who gave the value of the second constant as an example. I therefore propose that GKB become GKBJ.

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1. Introduction

In many calculations of the effective action, long combinations of derivatives of Hurwitz \( \zeta \)-functions occur and often it can be more useful to give their numerical values. Although this is a fairly standard topic, some further discussion might be of interest. One could of course use a built-in algorithm in a suitable CAS, but if this is not available, a direct analytical calculation is necessary, preferably not involving quadrature. Suitable forms have been found by Elizalde, [1], some time ago. My calculation will incidentally produce the same (asymptotic) expression but by a different route.

2. The basics

I approach the formula in a possibly circular way allowing me to introduce some other notions and, later, a few historical remarks. I start with the basic relation between \( \zeta'_H(-k, w) \) and \( \zeta'_H(0, 1) \), the Riemann \( \zeta \)-function, with (initially) \( w \) an integer,

\[
\zeta'(-k, w) - \zeta'(-k) = \log \Gamma_k(w),
\]

dropping the ‘H’. Here \( \Gamma_k(w) \) is Bendersky’s generalised gamma function defined (initially) by the sum

\[
\log \Gamma_k(w + 1) = \sum_{m=1}^{w} m^k \log m, \quad w \in \mathbb{Z}.
\]

There are two attitudes one can take towards (1). These are, in essence, whether it is to be read from right to left or left to right. In the first case, which is the one I mostly adopt here, the right-hand side, defined as the explicit sum (2), is analysed independently using finite calculus summation techniques leading to calculable quantities in which \( w \) can be considered continuous. This is the procedure of Bendersky, [2], and Jeffery, [3]. (They do not consider the relation with the \( \zeta \)-function.) Having an expression for the generalised gamma function then allows information about the \( \zeta \)-function to be extracted from (1).

On the other hand if the \( \zeta \)-function is assumed known then \( \Gamma_k \) can be taken as defined by (1) and its properties determined therefrom. I will not consider this point of view much here but it has the advantages of rapidity and elegance. An example will appear later.
I now invoke, without worrying too much about where it comes from, Adamchik’s formula for \( \zeta'(-k) \), [4],

\[
\zeta'(-k) = \frac{H_k B_{k+1}}{k+1} - \log A_k, \tag{3}
\]

where \( A_k \) are Bendersky’s generalisation of the Glaisher–Kinkelin constants defined by the asymptotic behaviour,

\[
\log A_k \equiv L_k = \lim_{w \to \infty} \log \Gamma_k(w+1) \big|_{w=\text{independent}}. \tag{4}
\]

\( H_k \) is a harmonic number.

One writes,

\[
\log \Gamma_k(w+1) = L_k + \Lambda_k(w+1) \tag{5}
\]

where \( \Lambda_k(w) \) is a function with the appropriate limiting behaviour. Bendersky derives an explicit asymptotic (divergent) series from the Euler–Maclaurin summation formula applied to \( \log \Gamma_k(w+1) \). It is not necessary to write it out yet.

Combining (1), (3) and (5) trivially yields the required computable formula,

\[
\zeta'(-k, w) = \frac{H_k B_{k+1}}{k+1} - L_k + \log \Gamma_k(w) = \frac{H_k B_{k+1}}{k+1} + \Lambda_k(w), \tag{6}
\]

which gives the asymptotic expansion obtained by Elizalde directly from integral forms of the \( \zeta \)-function.

I look upon (3) as a means of finding \( \zeta'(-k) \) from the \( A_k \), rather than the other way round, as is more usual. As a check of the numbers, if \( k \) is even \( \zeta'(-k) \) can be transformed into \( \zeta(2k+1) \) the values of which are readily available, to high accuracy.\(^2\)

Numerically one can treat the two lines in (6) separately. In the top line \( L_k \) is to be calculated by trial from (4) using the known asymptotic form of \( \Lambda \). (The method used by Jeffery, Glaisher and Bendersky). Less accurately, one can just substitute the asymptotic form directly into the second line. Of course, using an asymptotic form is not the best numerical procedure, but an accuracy of more than 10 places is easily attained this way. (The isolated case of \( w = 1 \) is not possible in the second approach.)

\(^2\)Or say Wolfram Alpha can be employed.
The more accurate computation follows from the first line in (6) where $L_k$ is found from (4) using the asymptotic form for $\Lambda_k(w+1)$ with a suitable choice for $w$. $\log \Gamma_k(w)$ is determined ‘exactly’ by (2). This method is therefore restricted to $w$ integral.

If $w$ is not integral, one has to use the second line in (6). If $w$ is small a direct application is not possible but one can translate $w$ to a large enough value by adding an integer and then employing the fundamental property of the generalised Gamma functions, [2], valid for any $x$,

$$
\Gamma_k(x+1) = x^k \Gamma_k(x),
$$

several times, if necessary

3. The details

I will now fill out the previous discussion by outlining Bendersky’s approach. I do this because his paper is not widely recognised. To begin with, it is helpful to write down the explicit expression for the asymptotic series, $\Lambda_k$, as given by Bendersky. One particular form of this is, and I write it out exactly as in [2] p.276, (with a few misprints corrected),

$$
\Lambda_{k+1}(x+1) = \frac{x^{k+2}}{k+2} \log x - \frac{x^{k+2}}{(k+2)^2} + \frac{1}{2} x^{k+1} \log x + \\
+ (k+1)! \sum_{r=1}^{k-1} \frac{B_{r+1}}{(r+1)! (k+1-r)!} \left( \log x - \frac{1}{k+1} - \frac{1}{k} + \ldots + \frac{1}{k+2-r} \right) + \\
+ x B_{k+1} \left( \log x - \frac{1}{k+1} - \frac{1}{k} + \ldots + \frac{1}{3} + \frac{1}{2} \right) + \frac{B_{k+2}}{k+2} \log x + \\
+ (k+1)! \sum_{s=1}^{\infty} (-1)^s \frac{B_{k+1+s}}{(k+1+s)!} \frac{(s-2)!}{x^{s-1}},
$$

which is easily coded.

Bendersky derives this, at general $k$, from an Euler–Maclaurin summation, after some amalgamation of terms. The details are of no immediate concern.

It is always illuminating to have some specific examples before one’s eyes and
I give the following,

\[
\Lambda_2(x + 1) = \frac{x(x + 1)(2x + 1)}{1.2.3} \log x - \frac{x^3}{9} + \frac{x}{12} - \frac{1}{360x} + \frac{1}{7560x^3} - \ldots
\]

\[
\Lambda_1(x + 1) = \left(\frac{x(x + 1)}{1.2} + \frac{1}{12}\right) \log x - \frac{x^2}{4} + \frac{1}{720x^2} - \frac{1}{5040x^4} + \ldots
\]

\[
\Lambda_0(x + 1) = \left(\frac{x + 1}{2}\right) \log x - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \ldots
\]

\[
\Lambda_{-1}(x + 1) = \log x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \ldots
\]

\[
\Lambda_{-2}(x + 1) = -\frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \ldots.
\]

I have included two ‘lower’ expansions which can be taken as part of the set. Explicitly,

\[
\Lambda_{-1}(x + 1) = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{x} - S_1
\]

\[
\Lambda_{-2}(x + 1) = \frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{x^2} - S_2
\]

where \(S_1\) and \(S_2\) are the corresponding ‘Glaisher–Kinkelin–Bendersky (GKB)’ constants and have the (known) values,

\[S_1 = -\Gamma'(1), \quad S_2 = \zeta(2)\].

We recognise in (9) some standard asymptotic (Stirling) series, \textit{e.g.} for,

\[
\Lambda_0(x + 1) = \log \frac{\Gamma(x + 1)}{\sqrt{2\pi}}, \quad \Lambda_{-1}(x + 1) = \frac{d}{dx} \log \Gamma(x + 1) = \psi(x + 1).
\]

and so on downwards to give the polygamma functions, up to a factor.

As is visually clear, one can run up and down the right-hand sides in (9) by integration and differentiation. The exact relationship is given by Bendersky who derived it by brute force from (8). He found,

\[
\Lambda_{k+1}(x + 1) = (k + 1) \int \Lambda_k(x + 1) \, dx + \frac{1}{k + 1} \phi_{k+1}(x + 1) + x H_k B_{k+1},
\]

(10)

where \(
\phi_n(x)
\) is a Bernoulli polynomial given in terms of the more usual polynomials, \textit{e.g.} Nörlund, [5], \(B_n(x)\), by,

\[
\phi_n(x) = \frac{1}{n+1} \left( B_{n+1}(x) - B_{n+1} \right),
\]

which equals the sum of the \(n\)th powers of the first \(x - 1\) integers.
Constants of integration can be considered to be absorbed into the GKB constants, $L_k$, which are found, in each case, by trial of varying $x$.

It is important to note that, although derived using an asymptotic series, the recursion, (10), is valid generally.

Another derivation of (10) is given later.

The actual numerical values of the GKB constants, $L_k$, can be determined from the definition, (5)

$$L_k = \log \Gamma(w + 1) - \Lambda_k(w + 1)$$

using (2) and a suitably chosen value for the integer, $w$. This choice is linked to the necessary truncation of the infinite series in (8) and leads to an accuracy of at least 29 places, e.g. for $w = 100$ and 20 terms of the sum retained.

Bendersky also develops recursion relations for the $L_k$ which involve just convergent series.

In order to find a more precise definition of $\log \Gamma_k$ than the asymptotic series, one starts from the observation that $\Gamma_0(x + 1) = \Gamma(x + 1)$, in terms of the ordinary Gamma function and takes this as the starting point of an upwards recursion based on (10). To take the simplest case, $\log \Gamma_1(x + 1)$ has to involve the integral of $\log(\Gamma(x + 1)/\sqrt{2\pi})$ and now, for exactness, a definite integral is used,

$$\log \Gamma_1(x + 1) = \int_0^x dx \log \frac{\Gamma(x + 1)}{\sqrt{2\pi}} + X(x),$$

where $X$ is to be found. Because (10) has to hold up to a constant, $C$,

$$X = \phi_1(x + 1) - C = \frac{1}{2}x(x + 1) - C,$$

which is determined by setting $x = 0$, so that,

$$C = X(0) = 0,$$

and, finally,

$$\log \Gamma_1(x + 1) = \int_0^x dx \log \Gamma(x + 1) + \phi_1(x + 1) - x L_0$$

$$= \int_0^x dx \log \Gamma(x + 1) + \frac{x(x + 1)}{2} - x \log \sqrt{2\pi}$$

(11)

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3 The particular values $\log \Gamma_k(1) = \log \Gamma_k(2) = 0$ have been used as boundary conditions.
defining $\Gamma_1$ in terms of the standard $\Gamma$. (Setting $x = 1$ yields a known result.)

Barnes, [6] p.281, refers to this equation (or its equivalent) as *Alexeiewsky’s theorem*.

Just to see how things fit together, I look at the next iteration according to (10),

$$
\log \Gamma_2(x + 1) = 2 \int_0^x dx \log \Gamma_1(x + 1) + \frac{1}{2} \phi_2(x + 1) + \frac{1}{6} x - 2xL_1
$$

$$
= 2 \int_0^x dx \int_0^x dx' \log \Gamma(x' + 1) + 2 \int_0^x dx (\phi_1(x + 1) - xL_0) +
$$

$$
+ \frac{1}{2} \phi_2(x + 1) + \frac{1}{6} x - 2xL_1
$$

$$
= 2 \int_0^x dx \int_0^x dx' \log \Gamma(x' + 1) + \frac{3}{2} \phi_2(x + 1) - 2xL_1 - x^2L_0.
$$

A continuation of this process produces Bendersky’s important general solution,

$$
\log \Gamma_k(x + 1) = k! I_k(x) + H_k \phi_k(x + 1) - \psi_k(x),
$$

(12)

with the definitions

$$
I_k(x) = \int_0^x dx' I_{k-1}(x') , \quad I_0(x) = \log \Gamma(x + 1)
$$

$$
\psi_k(x) = \sum_{r=0}^{k-1} \binom{k}{r} L_r x^{k-r}.
$$

(13)

Bendersky takes the convergent (12) as the definition of the generalised Gamma functions for all $x$ and any integer $k$ and uses it systematically to obtain their basic properties such as (7), encountered earlier. It would be out of place to give a summary here.

4. Earlier history

The quantities $\log \Gamma_k(x)$ in (2) were first considered by Kinkelin who concentrated on just $\Gamma_1(x)$ in [7],\(^4\) and derived its properties *ab initio* from the expression in terms of the ordinary Gamma function,

$$
\log \Gamma_1(x) = \int_0^x \log \Gamma(t) \, dt + \frac{x(x - 1)}{2} - \frac{1}{2} x \log 2\pi
$$

(14)

\(^4\) Kinkelin’s work is dated 1856, but was published in 1860. I have not been able to access his earlier paper.
which he obtained using Raabe’s formula and the fact that it reduces to the summation, (2), for \( x \) integral.

He computes what is essentially the constant \( L_1 \), not by the trial method but from derived convergent series. These series can also be found in Bendersky.

Glaisher, [8], gave a numerical treatment of \( L_1 \) using the trial method. In this, however, he was forestalled by the earlier work of Jeffery in a known paper [10], written in 1860. In a second, neglected paper, [3], written in 1862, Jeffery gives what seems to be the earliest calculation of the second GKB constant, \( L_2 \). Indeed he describes, inter alia, the construction of the complete general recursion system, (10), and I have taken equation (9) from his paper. It is clear he could have computed as many of the \( L_k \) as desired and so I propose that GKB be extended to GKBJ.

It is worth noting, with Jeffery, that the coefficient of \( \log x \) in \( \Lambda_k(x + 1) \) equals \( \sum_{i=1}^{x} i^k \) or \( B_{1+k}(x)/(1 + k) \), although he does not seem to prove this in generality. Jeffery’s approach is worth re–exposure. It does not overtly use the Euler–Maclaurin formula.

5. Jeffery’s treatment

For shortness, I define

\[
v_k(x) = \log \Gamma_k(x + 1) \tag{15}\]

From the original expression (2) algebra gives,

\[
\Delta_x v_k(x) = (x + 1)^k \log(x + 1),
\]

which defines a system of equations.

Extending \( x \) into the reals gives

\[
D\Delta v_k(x) = \Delta D v_k(x) = k(x + 1)^{k-1} \log(x + 1) + (x + 1)^{k-1}
\]

\[
= k\Delta v_{k-1}(x) + (x + 1)^{k-1},
\]

so that, by summation,

\[
Dv_k(x) = k\Delta^{-1} \Delta v_{k-1}(x) + \Delta^{-1}(x + 1)^{k-1} + \varpi_1(k)
\]

\[
= kv_{k-1}(x) + \phi_{k-1}(x + 1) + \varpi(k) \tag{16}
\]

\( ^5 \) A very brief biography can be found in [9].
The final periodic constant \( \varpi(k) \) is given by setting \( x = 0 \) which yields

\[
\varpi(k) = D v_k(0), \quad k = 2, 3, \ldots \\
\varpi(1) = D v_1(0) - 1
\] (17)

since \( v_k(0) = 0 = \phi_k(1) \). I will use (17) to find \( v_k(0) \) rather than the \( \varpi(k) \).

Jeffery, [10], gives only the numerical value \( \varpi(2) = -0.2475089541 \ldots \) which is \( 1/4 - 2L_1 \), in terms of the GKBJ constant for \( k = 1 \). This can be shown in the following somewhat lengthy way. How Jeffery works out the value is not clear to me.

Integrating (16),

\[
v_k(x) - C(k) = k \int_0^x dx v_{k-1}(x) + \frac{1}{k}(\phi_k(x + 1) - B_k x) + \varpi(k) x
\] (18)

where \( C(k) = 0 \), again on setting \( x = 0 \).

As a trivial check, set \( k = 1 \). Then

\[
v_1(x) = \int_0^x dx v_0(x) + (\phi_1(x + 1) - B_1 x) + \varpi(1) x
\]

\[
= \int_0^x dx v_0(x) + \frac{x(x + 1)}{2} + (\varpi(1) + \frac{1}{2}) x.
\]

Setting \( x \) to zero produces nothing while \( x = 1 \) gives, using Raabe’s formula,

\[
0 = \int_0^1 dx v_0(x) + \frac{3}{2} + \varpi(1)
\]

\[
= \log \sqrt{2\pi} + \varpi(1) + \frac{1}{2}
\]

therefore the relation to the GKBJ constant, \( L_0 \), is

\[
\varpi(1) = -L_0 - \frac{1}{2} = -\log \sqrt{2\pi} - \frac{1}{2}.
\]

Then

\[
v_1(x) = \int_0^x dx v_0(x) + (\phi_1(x + 1) - B_1 x) + \varpi(1) x
\]

\[
= \int_0^x dx v_0(x) + \frac{x(x + 1)}{2} - \log \sqrt{2\pi} x,
\] (19)

which is the same as (11), or (14).

Incidentally, from (17), one sees for that the series for \( v_1 \) in ascending powers of \( x \) begins with the term \((1/2 - \log \sqrt{2\pi})x\).
Relation (18) can be iterated. Thus

\[ v_k(x) = k \int_0^x dx' \left( (k - 1) \int_0^{x'} dx'' v_{k-2}(x'') + \frac{1}{k - 1} (\phi_{k-1}(x' + 1) - B_{k-1} x') + \omega(k - 1) x' \right) + \frac{1}{k} (\phi_k(x + 1) - B_k x) + \omega(k) x \]

\[ = k(k - 1) \int_0^x dx' \int_0^{x'} dx'' v_{k-2}(x'') + \frac{1}{k - 1} (\phi_k(x + 1) - B_k x) \]

\[- \frac{k}{2(k - 1)} B_{k-1} x^2 + \frac{k}{2} \omega(k - 1) x^2 + \frac{1}{k} (\phi_k(x + 1) - B_k x) + \omega(k) x. \]

The simplest, non–trivial, case is \( k = 2 \), when,

\[ v_2(x) = 2 \int_0^x dx' \int_0^{x'} dx'' v_0(x'') + (\phi_2(x + 1) - B_2 x) \]

\[ - B_1 x^2 + \omega(1) x^2 + \frac{1}{2} (\phi_2(x + 1) - B_2 x) + \omega(2) x. \]

Setting \( x = 1 \) enables an expression for the summation constant, \( \omega(2) \), to be found,

\[ 0 = 2 \int_0^1 dx' \int_0^{x'} dx'' v_0(x'') + (\phi_2(2) - B_2) \]

\[ - B_1 + \omega(1) + \frac{1}{2} (\phi_2(2) - B_2) + \omega(2) \]

\[- 2 \int_0^1 dx (1 - x) \log \Gamma(x + 1) + (\phi_2(2) - B_2) \]

\[- B_1 + \omega(1) + \frac{1}{2} (\phi_2(2) - B_2) + \omega(2) \]

\[ = 2 \int_0^1 dx (1 - x) \log \Gamma(x + 1) + \frac{7}{4} + \omega(1) + \omega(2). \]

Rewrite this in terms of \( \Gamma(x) \),

\[ -\frac{3}{2} + 2 \log \sqrt{2\pi} + \frac{7}{4} + \omega(1) + \omega(2) \]

\[ = -2 \int_0^1 dx x \log \Gamma(x) - \frac{3}{2} - \omega(1) + \omega(2). \]

In order now to relate the constant of integration, \( \omega(2) \), to the GKBJ asymptotic constant, \( L_1 \), I use Kinkelin’s general approach.
The derivation is based on the classic expression, or definition, of the gamma function as an infinite product. This gives,

\[ \log \Gamma(x) = \log w! + (x-1) \log w - \sum_{i=0}^{w-1} \log (x+i), \quad w \to \infty. \]  \hspace{1cm} (23)

Also needed is the asymptotic limit, assumed known, of the factorial (Stirling),

\[ \log w! \sim -w + w \log w + \frac{1}{2} \log w + L_0. \]  \hspace{1cm} (24)

The relevant integral in (22) is

\[ 2 \int_0^1 dx \log \Gamma(x) = \log w! - \frac{1}{3} \log w - 2 \sum_{i=0}^{w-1} \int_0^1 dx \log (x+i), \]  \hspace{1cm} (25)

and so one needs,

\[ 2 \sum_{i=0}^{w-1} \int_0^1 dx \log (x+i) = \sum_{i=0}^{w-1} \left( i^2 \left( \log i - \log (i+1) \right) + \log (i+1) + i - \frac{1}{2} \right). \]

Rorganising,

\[ \sum_{i=0}^{w-1} \left( i^2 \left( \log i - \log (i+1) \right) + \log (i+1) \right) = \sum_{i=1}^{w-1} i^2 \log i - \sum_{i=1}^{w} i^2 - 2i + 1 - 1 \log i \]

\[ = \sum_{i=1}^{w} 2i \log i - w^2 \log w, \]

giving, for this piece,

\[ 2 \sum_{i=0}^{w-1} \int_0^1 dx \log (x+i) = \sum_{i=1}^{w} 2i \log i - w^2 \log w + \frac{w^2}{2} - w. \]  \hspace{1cm} (26)

Combining this expression with (25) and (24) I find,

\[ 2 \int_0^1 dx \log \Gamma(x) = \]

\[ w \log w + \frac{\log w}{6} - \frac{1}{2} - \varpi(1) - \sum_{i=1}^{w} 2i \log i + w^2 \log w - \frac{w^2}{2}. \]

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Then (22) becomes

\begin{equation}
-\frac{3}{4} - \varpi(1) + \varpi(2) = \nonumber
w \log w + \frac{\log w}{6} - \frac{1}{2} - \varpi(1) - \sum_{i=1}^{w} 2i \log i + w^2 \log w - \frac{w^2}{2}
\end{equation}

or

\[ \sum_{i=1}^{w} i \log i = \left( \frac{1}{12} + \frac{1}{2} (w + w^2) \right) \log w + \frac{1}{8} - \frac{w^2}{4} - \frac{1}{2} \varpi(2) \]

showing that the constant, \( L_1 \) is, by definition,

\[ L_1 = \frac{1}{8} - \frac{1}{2} \varpi(2). \]

The initial conditions (17) also then show that the power series expansion of \( v_2(x) \equiv \log \Gamma_2(x + 1) \) starts with \((1/4 - 2L_1) x\) as stated, numerically, by Jeffery, [3] §43.

Kinkelin derives a constant, \( \varpi \), as a convergent series, from multiplication properties of \( \Gamma_1 \) and then shows that this is the constant in the asymptotic form. It can be found from the information already obtained if I assume Kinkelin’s equn.(22a) which is,

\[ \log \varpi = -\frac{1}{6} + \log \sqrt{2\pi} - 2 \int_{0}^{1} dx x \log \Gamma(x) \]

\[ = -\frac{2}{3} - \varpi(1) + \frac{3}{4} + \varpi(1) - \varpi(2) \]

\[ = \frac{1}{12} - \varpi(2) = 2L_1 - \frac{1}{6} \sim 0.33084228740 \]

and agrees with Kinkelin’s value, obtained by series summation.

Such a case by case approach, directly applied, is not algebraically efficient for the higher iterations. As two final examples I find that,

\[ \varpi(3) = -3L_2 \sim -0.091345371176, \quad \varpi(4) = -\frac{5}{72} - 4L_3 \sim 0.013180972097. \]
6. The general solution. Some integrals

I briefly return to Bendersky’s general solution, (12), which he takes as the definition of the function. The ingredients are in (13), the main part of which is the nested integral \( I_k \) which can be reduced to a single (fractional) integral of \( \log \Gamma \),

\[
I_k(x) = \frac{1}{(k-1)!} \int_0^x dt (x-t)^{k-1} \log(t+1).
\]  

(28)

By giving \( x \) particular values, expressions for this integral can be found. For example if \( x = 1 \) the left–hand side of (12) vanishes and so,

\[
k \int_0^1 dt (1-t)^{k-1} \log(t+1) = \psi_k(1) - H_k \phi_k(2)
\]

\[
= \sum_{r=0}^{k-1} \binom{k}{r} L_r - H_k \left[ \frac{1}{2} + \frac{1}{k+1} \left( 1 + \sum_{r=2}^{k} \binom{k+1}{r} B_r \right) \right]
\]

\[
= \sum_{r=0}^{k-1} \binom{k}{r} \left( \frac{H_r B_{r+1}}{r+1} - \zeta'(-r) \right) - H_k \left[ \frac{1}{2} + \frac{1}{k+1} \left( 1 + \sum_{r=2}^{k} \binom{k+1}{r} B_r \right) \right].
\]  

(29)

As a check take \( k = 2 \). Then simple arithmetic yields,

\[
2 \int_0^1 dt (1-t) \log(t) = -\zeta'(0) - 2\zeta'(-1) + \frac{1}{6},
\]

which agrees with the value given in [11] p.674. In fact equn.(109) of [11] contains (29). Other integration ranges can be accommodated.

I note that \( I_k, \) (28), is directly equivalent to the polygamma function of negative order, \( \psi^{(-n)} \), discussed by, e.g., Adamchik, [4]. See also Espinosa and Moll, [12].

More precisely, just changing \( \Gamma(t+1) \) to \( \Gamma(t) \),

\[
I_k(x) = \frac{1}{(k-1)!} \int_0^x dt (x-t)^{k-1} \log(t) + \frac{1}{(k-1)!} \int_0^x dt (x-t)^{k-1} \log t
\]

\[
= \psi^{(k-1)}(x) + \frac{1}{k!} \int_0^x ((x-t)^k - x^k) t^{-1} dt + x^k \log x
\]

\[
= \psi^{(k-1)}(x) + \frac{1}{k!} x^k \left( \sum_{n=1}^{k} \frac{(-1)^n}{n} \binom{k}{n} + \log x \right). 
\]

Furthermore, Proposition 2 in [4] for the polygamma function is equivalent to the general solution (12) together with the basic relation, (1).
7. Comments

Unless I am missing something, Jeffery’s method of section 5 does not easily give the general form for $v_k$. Furthermore, in the details of the algebra, cancellations occur which make $\varpi(k)$ a function of $L_{k-1}$ only, which ought to be derivable directly.

By contrast, Bendersky’s approach is a top down one in that a general recursion is found, somewhat out of the blue, for $\Lambda_k = v_k - L_k$. This leads, essentially by induction, to the form of $v_k$ for all $k$, (12), which is then taken as the definition of $v_k(x)$ for any $x$. Equation (12) straightaway shows that,

$$Dv_k(0) = H_k B_k - kL_{k-1}$$

as confirmed above in particular cases. A derivation of this expression that does not depend on the explicit form of the general solution for $v_k$ would be welcomed.

8. Alternative approach

I mentioned that looking at the basic equation (1) as the definition of $\Gamma_k(x)$ allows a rapid derivation of its properties and relations. As an example, I now derive the recursion (10) which follows almost immediately from the standard formula, used in [11] for a related purpose (see also [13] for a generalisation),

$$\frac{\partial}{\partial w} \zeta(s, w) = -s \zeta(s + 1, w), \quad (30)$$

so that, differentiating with respect to $s$,

$$\frac{\partial}{\partial w} \zeta'(s, w) = -\zeta(s + 1, w) - s \zeta'(s + 1, w).$$

Then, setting $s = -k$ with $k$ a positive integer, and integrating, gives the result,

$$(k + 1) \int_0^x \left( \zeta'(-k, t) - \frac{\zeta(-k, t)}{k + 1} \right) dt = \zeta'(-k - 1, x) - \zeta'(-k - 1).$$

where I have used $\zeta'(-k, 0) = \zeta'(-k)$, [11], Appx.C.

This recursion is equivalent to Bendersky’s, (10), as a few lines of algebra, utilising (3), confirms. Kurokawa and Oshiai, [14], unaware of the work of Bendersky, use this approach and also generalise it to Barnes $\zeta$–functions. I note that their definition of the generalised gamma function (‘higher depth’ gamma function) does not include the (constant) second term on the left–hand side of (1).
As a minor point, one of the integrations can be done using the well known relation,

\[
\zeta(-k - 1, x) - \zeta(-k - 1) = (k + 1) \int_0^x dt \zeta(-k, t), \quad k > 0,
\]

which follows trivially from (30). It expresses a standard relation between Bernoulli polynomials.

Continuing this theme of rewriting, Adamchik’s form of the GKBK constants is,

\[
L_k = -\zeta'(-k) - H_k \zeta(-k). \tag{31}
\]

But I take this no further since, I will give, at another time, an extended analysis of this approach and its generalisations as in [11], [15] and elsewhere.

9. Summary

I have shown that Glaisher’s numerics and Bendersky’s recursive approach to the generalised gamma functions were both anticipated by Jeffery whose work appeared a couple of years after Kinkelin’s paper, of which Jeffery seems unaware.

I have advocated the computation of the derivative of the Hurwitz \( \zeta \)-function at the negative integers via the generalised gamma function, rather then vice versa.

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