Collective Josephson Vortex Dynamics in Long Josephson Junction Stacks

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Abstract

We investigate the collective phase dynamics in conventional long Josephson junction (LJJ) stacks and in layered superconductors, exhibiting intrinsic LJJ behaviors. Using a theoretical model which accounts for both the magnetic induction effect and the breakdown of local charge neutrality (i.e., charging effect), we show that the collective motion of Josephson vortices, including the dispersion of Josephson plasma mode and the Swihart-type velocity, in an intrinsic LJJ stack such as Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$ (BSCCO) is significantly modified from those in a conventional LJJ stack. In BSCCO, the strength of the charging effect $\alpha$ is small (i.e., $\alpha \sim 0.1 - 0.4$), but it leads to notable changes in collective phase dynamics, including changes to the stability condition. Also, we show that splitting of the supercurrent branch in the resistive state is due to collective motion of Josephson vortices. The width of spread of these sub-branches in the linear current-voltage regime depends on $\alpha$, suggesting another way to measure the charging effect in BSCCO.

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I. INTRODUCTION

Dynamics of magnetic vortices in a stack of long Josephson junctions (LJJ) in a magnetic field applied parallel to the junction layers have attracted much attention due to their intriguing applied and fundamental interests. The motion of Josephson vortices in a single junction system has been exploited in various devices. Collective motion of these vortices in a LJJ stack, in which layers of superconductor (S) and insulator (I) are arranged vertically as in Fig. 1, can be exploited in high frequency devices such as tunable submillimeter-wave oscillators and detectors. Here collective motion, including both in-phase and out-of-phase modes shown in Fig. 2, arises from mutual phase-locking of Josephson junctions caused by (magnetic) inductive coupling between screening currents flowing around adjacent Josephson vortices as they move under a bias current. The phase-locking establishes phase coherence across the Josephson junctions. A LJJ stack exhibiting this phase coherence leads to high power output and bandwidth, and it can serve as a model system for scientific studies.

The motion of Josephson vortices in LJJ stacks yields interesting phenomena: (i) Josephson plasma resonance (JPR) and (ii) supercurrent sub-branching. The experiments on both conventional LJJ stacks (e.g., Nb-Al/AlO$_x$-Nb multilayers) and layered superconductors (e.g., Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$ (BSCCO)) behaving as intrinsic LJJ stacks indicate that JPR can be tuned by magnetic field $B$ and can occur over a broad range of frequencies, from microwave to submillimeter-wave. Also, the supercurrent branch in the current-voltage (I-V) data splits into multiple sub-branches when a bias current exceeds some critical value. To explain the data, two theoretical models have been proposed: one is based on the inductive coupling (i.e., magnetic induction model) and the other is based on the coupling due to charge variation in the S layers (i.e., charging effect model).

The magnetic induction model assumes that the S layer thickness $d_S$ is much larger than the Debye (charge screening) length $r_D$ (i.e., $d_S \gg r_D$), as in conventional LJJ. In this case, charge variations (or electric field) at each S layers are screen out, yielding local charge neutrality. Consequently, the electric field does not lead to the longitudinal coupling
between the S layers. In this model, an applied magnetic field induces supercurrents along the S layers and results in the inductive interaction between adjacent S layers. The induction coupling strength is inversely proportional to the common S layer thickness. This model has been used to explain the experimental data for BSCCO. However, the underlying assumption is not justified in BSCCO since $d_S \sim 3\AA$ and $r_D \sim 2 - 3\AA$.

On the other hand, the charging effect model accounts for the nonequilibrium effect in atomic scale thick superconducting layers. When the S layers are so thin to be comparable to the Debye length (i.e., $d_S \sim r_D$), as in BSCCO, the breakdown of local charge neutrality yields the charging effect. The particle-hole imbalance may also occur since each superconducting layer cannot completely screen out the charge variation. Hence the presence of charge variations yields the interaction between the contiguous superconducting layers and leads to the coupling between the S layers. Recently the charging effect model, neglecting the magnetic induction effect, has been used to interpret the data for BSCCO.

Earlier studies including numerical simulations of a finite LJJ system, show that these two models can explain the data qualitatively, but considerable inconsistencies between the experimental and the theoretical results have been found. For example, transverse and longitudinal JPR are predicted by the magnetic induction model and the charging effect model, respectively, but the data indicate that both types of resonance occur. Recent experiments on HgI$_2$-intercalated BSCCO and BSCCO single crystals indicate that the supercurrent branch in the I-V data splits into multiple sub-branches in the resistive state when $B \sim H_o$ (i.e., low vortex density regime). An estimated value of $H_o$ for Nb-Al/AlO$_x$-Nb multilayers and BSCCO is roughly 0.001T and 0.2T, respectively. In the dense vortex regime (i.e., $B \gg H_o$), the I-V data exhibit characteristic kinks, and these kinks closely resemble the prediction made by Machida et al., using the magnetic induction model. However a closer examination of the data reveals some inconsistencies. These inconsistencies suggest that a better theoretical model is needed to describe the LJJ stacks.

In this paper, we investigate collective phase dynamics in conventional LJJ stacks and layered superconductors at low magnetic fields (i.e., $B \sim H_o$ in which Josephson vortices
are in every I layers as in Fig. 2) and at low temperatures (i.e., below the Abrikosov vortex lattice melting temperature), using a theoretical model accounting for both the induction effect and the charging effect. These two effects are equally important in BSCCO since \( r_D \sim d_S \), but the charging effect is neglected in many studies because its strength \( \alpha \) is small (e.g., \( \alpha \sim 0.1 - 0.4 \) in BSCCO). We show how the collective motion of Josephson vortices is modified by a weak charging effect. We outline two main results. First, the Josephson plasma dispersion relation, the Swihart velocity, and the stability condition for collective motion in BSCCO are considerably modified from those in Nb-Al/AlO\(_x\)-Nb multilayers. Second, the splitting of the supercurrent branch in the resistive state is due to collective motion of Josephson vortices, and the width of spread of these sub-branches in the linear I-V regime depends on \( \alpha \). These results are consistent with the experimental data described above.

The remainder of the paper is organized as follows. In Sec. II, a theoretical model, which accounts for both the magnetic induction effect and the charging effect, is derived by extending previous models. In Sec. III, the Josephson plasma dispersion relation and the Swihart velocity for the collective modes are computed from our model derived in Sec. II. In Sec. IV, we determine the stability condition for the mutually phase-locked modes, performing the linear stability analysis. In Sec. V, we show that the splitting of the supercurrent branch in the resistive state is due to the collective motion of Josephson vortices. Finally, in Sec. VI, we summarize our results and conclude.

II. THEORETICAL MODEL

In this section, we derive a theoretical model, extending previous approaches. A brief discussion of this model was published. Here we consider a system with a large number of LJJ (i.e., \( N \gg 1 \)) neglecting the boundary effect and present new results obtained from this model in later sections. To account for both the magnetic induction effect and the charging effect, we start with the gauge-invariant phase difference between the S layers \( \ell \) and \( \ell - 1 \),

\[
\varphi_{\ell,\ell-1} = \theta_{\ell} - \theta_{\ell-1} - \frac{2\pi}{\phi_0} \int_{\ell-1}^{\ell} A \cdot dl ,
\]  

(1)
where \( \theta \) is the phase of the superconducting order parameter, \( \phi_o = \hbar c/2e \) is the flux quantum, and \( \mathbf{A} \) is the vector potential in the I layers. In this paper, we employ the Cartesian coordinates and assume that the S and I layers are stacked along \( z \)-direction and the magnetic field is applied along the \( y \)-direction, as in Fig 1. For simplicity, the thicknesses of the S \( (d_S) \) and I \( (d_I) \) layers are taken to be uniform.

The magnetic induction effect due to the applied magnetic field (along the \( y \)-direction) yields a spatial variation of the phase difference (along the \( x \)-direction). An equation describing the magnetic inductive coupling between the S layers

\[
\frac{\phi_o}{2\pi} \frac{\partial \varphi_{\ell,\ell-1}}{\partial x} = s(B_{\ell+1,\ell} + B_{\ell-1,\ell-2}) + d' B_{\ell,\ell-1}
\]

is easily obtained by taking a spatial derivative of Eq. (1) and by using the expression for the supercurrent density

\[
J_\ell = \frac{\phi_o}{8\pi^2 \lambda^2} \left( \nabla \theta_\ell - \frac{2\pi}{\phi_o} \mathbf{A}_\ell \right).
\]

Here \( d' = d_I + 2\lambda \coth(d_S/\lambda) \) and \( s = -\lambda [\sinh(d_S/\lambda)]^{-1} \) are expressed in terms of the London penetration depth \( \lambda \). The magnetic field \( B_{\ell,\ell-1} \) in the I layer between two S layers \( \ell \) and \( \ell - 1 \) is parallel to the layers. Note that \( B_{\ell,\ell-1} \) differs from \( B \) since the magnetic field generated by the supercurrent in the S layers modifies the field in the I layer. Using Maxwell’s equation, we express the spatial derivative of the magnetic field as

\[
\frac{\partial B_{\ell,\ell-1}}{\partial x} = \frac{4\pi}{c} (J_c \sin \varphi_{\ell,\ell-1} - J_B + J_{T,\ell,\ell-1})
\]

where \( J_c \) is the Josephson critical current density, and \( J_B \) is a bias current density. Note that the magnetic field entering the I layers yields a triangular Josephson vortex lattice (JVL) when the bias current is either absent or small. The current density \( J^T \),

\[
J^T_{\ell,\ell-1} = \frac{\phi_o}{2\pi} \frac{\sigma}{\mathcal{D}} \frac{\partial \varphi_{\ell,\ell-1}}{\partial t} + \frac{\epsilon}{4\pi \mathcal{D}} \frac{\partial V_{\ell,\ell-1}}{\partial t},
\]

includes the quasiparticle and the displacement current contribution. Here \( \mathcal{D} = d' + 2s = d_I + 2\lambda \tanh(d_S/2\lambda) \) is the effective thickness of the block layer, \( \sigma \) is the quasiparticle conductivity,
\[ \frac{\phi_o}{8\pi^2} \frac{\partial^2 \varphi_{\ell,\ell-1}}{\partial x^2} = J_c \left[ s(\sin \varphi_{\ell+1,\ell} + \sin \varphi_{\ell-1,\ell-2}) + d' \sin \varphi_{\ell,\ell-1} \right] - D J_B + d' J_{T,\ell-1} + s(J_{T+1,\ell} + J_{T-1,\ell-2}) . \] (6)

Note that \( S = s/d' \) measures the induction coupling strength, and \( S = -0.5 \) in the strong coupling limit. The phase difference equation (Eq. (4) in Ref. 27) derived by Bulaevskii and Clem within the framework of Lawrence-Doniach model\(^{28}\) can be obtained from Eq. (6) when the time-dependent terms are neglected (i.e., \( J_{T,\ell-1} = 0 \)) and the relations \( (8\pi^2/\phi_o)J_c d' = (2/\lambda_c^2) + (1/\lambda_J^2) \) and \( (8\pi^2/\phi_o)J_c s = -1/\lambda_J^2 \) are used. Here \( \lambda_c \) is the magnetic penetration depth in the direction perpendicular to the S layers.

The presence of a nonequilibrium state leads to the interaction between the S layers. When the S layer thickness is comparable to the Debye screening length (i.e., \( r_D \sim d_S \)), the S layers are in a nonequilibrium state because the charge variations in these layers are not completely screened. This incomplete charge screening enhances the temporal variation of the phase difference. One can include this effect in the phase dynamics, modifying the usual AC Josephson relation, which is a time derivative of Eq. (6), to

\[ \frac{\phi_o}{2\pi} \frac{\partial \varphi_{\ell,\ell-1}}{\partial t} = V_{\ell,\ell-1} + \Phi_{\ell} - \Phi_{\ell-1} \] (7)

as a way to account for a nonzero gauge-invariant potential \( \Phi_{\ell} = \phi_{\ell} + (\hbar/2e)(\partial \theta_{\ell}/\partial t) \) generated inside the S layers. Here \( \phi_{\ell} \) is the electrostatic potential. The modified AC Josephson relation of Eq. (7) can be rewritten as

\[ \frac{\phi_o}{2\pi} \frac{\partial \varphi_{\ell,\ell-1}}{\partial t} = V_{\ell,\ell-1} - \alpha (V_{\ell+1,\ell} - 2V_{\ell,\ell-1} + V_{\ell-1,\ell-2}) - \Psi_{\ell} + \Psi_{\ell-1} , \] (8)

using the charge density \( \rho_{\ell} = -\langle \Phi_{\ell} - \Psi_{\ell} \rangle/4\pi r_D^2 \) and the Maxwell’s equation \( \epsilon \nabla \cdot E = 4\pi \rho \). Here \( \alpha = e r_D^2 / D d_S \) measures the strength of the charging effect, and \( \Psi_{\ell} \) measures the particle-hole imbalance in the S layer. For simplicity, we consider only the charging effect by setting \( \Psi_{\ell} = \Psi_{\ell-1} = 0 \), as it has been done in earlier studies.\(^{23}\) Note that the usual AC
Josephson relation is obtained from Eq. (8) when \( \alpha = 0 \). This indicates that the charging effect (i.e., \( \alpha \neq 0 \)) enhances the coupling between neighboring junctions. Using Eq. (5), we relate the time derivative of the phase difference to the current densities and obtain

\[
J_c \left[ \frac{1}{\omega_p^2} \frac{\partial^2 \varphi_{\ell, \ell-1}}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \varphi_{\ell, \ell-1}}{\partial t} - \frac{\alpha \beta}{\omega_p} \frac{\partial}{\partial t} \left( \varphi_{\ell+1, \ell} - 2 \varphi_{\ell, \ell-1} + \varphi_{\ell-1, \ell-2} \right) \right] = J_{T, \ell, \ell-1} - \alpha \left( J_{\ell+1, \ell} - 2 J_{\ell, \ell-1} + J_{\ell-1, \ell-2} \right). 
\]

Here \( \omega_p = c/(\sqrt{\epsilon} \lambda_c) \) is the plasma frequency, \( \beta = (4 \pi/c)(\sigma \lambda_c/\sqrt{\epsilon}) = 1/\sqrt{\beta c} \), and \( \beta \) is the McCumber parameter. Note that the spatial variation of \( \varphi_{\ell, \ell-1} \) is neglected in the charging effect model of Eq. (9). The terms of the order \( O(\alpha \beta) \) can be safely neglected since \( \alpha \) and \( \beta \) are small in the layered superconductors (i.e., \( \alpha \beta \ll 1 \)). For example, the experimental value for \( \alpha \) and \( \beta \) in BSCCO are roughly 0.1-0.4 and 0.2, respectively. Neglecting these small terms, we rewrite Eq. (9) as

\[
J_c \left( \frac{1}{\omega_p^2} \frac{\partial^2 \varphi_{\ell, \ell-1}}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \varphi_{\ell, \ell-1}}{\partial t} \right) \approx J_{T, \ell, \ell-1} - \alpha \left( J_{\ell+1, \ell} - 2 J_{\ell, \ell-1} + J_{\ell-1, \ell-2} \right). 
\]

As we shall see in Sec. III, the charging effect terms in Eq. (10) yield purely longitudinal Josephson plasma excitations.

A theoretical model, including both the magnetic induction effect and the charging effect, can be obtained easily by noting that the magnetic induction model of Eq. (6) and the charging effect model of Eq. (10) are coupled to each other via the current density \( J^T \). Combining Eqs. (6) and (10), we obtain the coupled sine-Gordon equations,

\[
\frac{\partial^2 \varphi_{\ell, \ell-1}}{\partial x^2} - \frac{1}{\lambda_c^2 D} \left( \frac{1}{\omega_p^2} \frac{\partial^2 \Delta_{\ell}}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \Delta_{\ell}}{\partial t} \right) \approx \alpha \frac{1}{\lambda_c^2 D} \frac{\partial^2 \Xi_{\ell}}{\partial t^2} = \frac{1}{\lambda_c^2 D} \left[ d' \sin \varphi_{\ell, \ell-1} + s(\sin \varphi_{\ell+1, \ell} + \sin \varphi_{\ell-1, \ell-2}) - D \frac{J_B}{J_c} \right], 
\]

where \( \Delta_{\ell} = d' \varphi_{\ell, \ell-1} + s(\varphi_{\ell+1, \ell} + \varphi_{\ell-1, \ell-2}) \) and \( \Xi_{\ell} = s(\varphi_{\ell-2, \ell-3} + \varphi_{\ell+2, \ell+1}) + (d' - 2s)(\varphi_{\ell-1, \ell-2} + \varphi_{\ell+1, \ell}) + 2(s - d') \varphi_{\ell, \ell-1} \). The third term on the left hand side of Eq. (11) (due to the charging effect) is the main modification from the earlier models. Hence, Eq. (11) becomes identical to the phase difference equation derived by Bulaevskii et al. (Eq. (11) in Ref. 17) when \( \alpha = 0 \). Using Eq. (11), we show below that a weak charging effect in BSCCO (i.e., \( \alpha \sim 0.1 - 0.4 \)) can yield significant changes to the phase dynamics.
III. JOSEPHSON PLASMA DISPERSION RELATION

We now determine the dispersion relation for the Josephson plasma and the Swihart velocity for the collective modes, using linear analysis: \( \varphi_{\ell,\ell-1} = \varphi_{\ell,\ell-1}^{(0)} + \varphi'_{\ell,\ell-1} \). Here \( \varphi_{\ell,\ell-1}' \) describes small fluctuations about \( \varphi_{\ell,\ell-1}^{(0)} \) describing uniform motion of Josephson vortices in the I layer between \( \ell \) th and \( \ell - 1 \) th S layers. \( \varphi_{\ell,\ell-1}^{(0)} \) is zero in the Meissner state, but in general, it depends on a magnetic field, allowing JPR to be tuned by the field. The effect of magnetic field on JPR can be accounted for more accurately via the field dependence of \( J_c \) and via imposing the boundary condition, \( (\phi_0/2\pi)(\partial \varphi_{\ell,\ell-1}/\partial x) = DB \), explicitly at both \( x = 0 \) and \( x = L_x \) (a junction length) in numerical simulations.

When the bias current \( J_B \) equals the Josephson current in each of the I layers (i.e., \( J_B = J_c \sin \varphi_{\ell,\ell-1}^{(0)} \)), we describe the motion of vortices in terms of a uniform motion \( \varphi_{\ell,\ell-1}^{(0)} \) and small perturbation \( \varphi'_{\ell,\ell-1} \) about \( \varphi_{\ell,\ell-1}^{(0)} \). The uniform phase motion is described by

\[
\frac{\partial^2 \varphi_{\ell,\ell-1}^{(0)}}{\partial x^2} - \frac{1}{\lambda_c^2 D} \left( \frac{1}{\omega_p^2} \frac{\partial^2 \Delta (0)}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \Delta (0)}{\partial t} \right) - \frac{\alpha}{\lambda_c^2 D} \frac{1}{\omega_p^2} \frac{\partial^2 \varphi_{\ell,\ell-1}^{(0)}}{\partial t^2} = 0 , \tag{12}
\]

while small fluctuations (i.e., \( \varphi'_{\ell,\ell-1} \)) about \( \varphi_{\ell,\ell-1}^{(0)} \) are described by

\[
\frac{\partial^2 \varphi_{\ell,\ell-1}'}{\partial x^2} = J_c \left[ d' \varphi'_{\ell,\ell-1} \cos \varphi_{\ell,\ell-1}^{(0)} + s(\varphi'_{\ell+1,\ell} \cos \varphi_{\ell+1,\ell}^{(0)} + \varphi'_{\ell-1,\ell} \cos \varphi_{\ell-1,\ell}^{(0)} - 2 \varphi_{\ell,\ell-1}^{(0)} \cos \varphi_{\ell-1,\ell-2}^{(0)}) \right] + d' J^T_{\ell,\ell-1} + s(J^T_{\ell+1,\ell} + J^T_{\ell-1,\ell-2}) , \tag{13}
\]

\[
J_c \left( \frac{1}{\omega_p^2} \frac{\partial^2 \varphi_{\ell,\ell-1}'}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \varphi_{\ell,\ell-1}'}{\partial t} \right) = J^T_{\ell,\ell-1} - \alpha (J^T_{\ell+1,\ell} - 2 J^T_{\ell,\ell-1} + J^T_{\ell-1,\ell-2}) . \tag{14}
\]

Equations (13) and (14) are coupled through the current density \( J^T_{\ell,\ell-1} \), suggesting that these equations can be simplified by expressing \( \varphi'_{\ell,\ell-1} \) and \( J^T_{\ell,\ell-1} \) as Fourier series in space for the \( z \)-direction: \( \varphi'_{\ell,\ell-1} = \sum_{m=1}^{2N+1} T_m e^{ik_m a} \) and \( J^T_{\ell,\ell-1} = \sum_{m=1}^{2N+1} J^T_m e^{ik_m a} \). \( k_m = m\pi/(N + 1) a \) represents the wavenumber for the collective mode along the \( z \)-direction, \( a = d_t + d_s \), \( m \) is the mode index, and \( N \) represents the number of Josephson junctions in a stack.

The Josephson plasma mode dispersion relation is determined easily by approximating that \( \varphi_{\ell,\ell-1}^{(0)} \approx \varphi^{(0)} \) and by combining Eqs. (13) and (14) into a single equation as

\[
\frac{\partial^2 T_m}{\partial t^2} + \omega_p^2 \left[ (k_x \lambda_c)^2 A_m B_m + A_m \cos \varphi^{(0)} \right] T_m \approx 0 , \tag{15}
\]
where $A_m = 1 + 4\alpha \sin^2(k_m a/2)$, $B_m = \{1 + 4[(-S/(1 + 2S))] \sin^2(k_m a/2)\}^{-1}$, and $S = s/d'$. Here, we set $\beta = 0$ for simplicity. From Eq. (15), we obtain the dispersion relation of

$$\omega(k_x, k_m) = \omega_p \left[ (k_x \lambda_c)^2 A_m B_m + A_m \langle \cos \varphi(0) \rangle_t \right]^{1/2}$$

(16)

for the collective mode. $\langle \cdots \rangle_t$ represents thermal averages. The dispersion relation of Eq. (16) naturally recovers both purely longitudinal\(_{12}\) and purely transverse plasma excitations\(_{12}\) at $k_x = 0$ and at $k_m = 0$, respectively. However, there are notable differences between our result of Eq. (16) and the results from other models\(_{12,13}\). Figure 3 illustrates the difference between the dispersion relation of our model and that of the magnetic induction model\(_{12}\) (Fig. 3(a)) and that of the charging effect model\(_{12}\) (Fig. 3(b)).

The changes in the dispersion relation due to the charging effect increase the characteristic velocity of the collective mode. The group velocity for the electromagnetic waves in these LJJ is easily determined from Eq. (16) by evaluating

$$\frac{d\omega}{dk_x} = \frac{\omega_p^2}{\omega} k_x \lambda_c^2 A_m B_m$$

(17)

within the linearized model. This group velocity, asymptotically (i.e., as $k_x \to \infty$), leads to the Swihart velocity

$$\bar{c}_m = c_o \left[ \frac{1 + 4\alpha \sin^2(k_m a/2)}{1 + 4\left[(-S/(1 + 2S)) \sin^2(k_m a/2)\right]} \right]^{1/2}$$

(18)

the effective maximum velocity for the collective mode $m$. Here $c_o = c/\sqrt{\varepsilon}$. Equation (18) recovers the result of the magnetic induction model ($\bar{c}_m^{MI}$) when $\alpha = 0$ (i.e., $\bar{c}_m^{MI} = \bar{c}_m(\alpha = 0)$), indicating that the charging effect yields the mode-dependent enhancement of the Swihart velocity from $\bar{c}_m^{MI}$. For example, the Swihart velocity is not enhanced for the $m = 1$ mode (i.e., $\bar{c}_1 = \bar{c}_1^{MI}$), but it is enhanced for the $m = N$ mode (i.e., $\bar{c}_N = (1 + 4\alpha)^{1/2}\bar{c}_N^{MI}$). This enhancement reflects the increase in the coupling strength between the junctions due to the charging effect and indicates that the threshold velocity $v_{th}$ ($=\bar{c}_N$) for emitting Cherenkov radiation\(_{14}\) (i.e., non-Josephson emission) is also increased. For example, $v_{th} = \bar{c}_N^{MI}$ when $\alpha = 0$, but $v_{th} = 1.34\bar{c}_N^{MI}$ when $\alpha = 0.2$. Evidence, indicating the need
to account for the charging effect, may be also found in the I-V data for BSCCO. Recent analysis of the I-V data in the dense vortex regime (i.e., $B \gg H_o$) indicates that a better agreement between the predicted and observed position of the kinks can be obtained if the Swihart velocity for $m > 1$ is slightly larger than $\tilde{c}_m^{MI}$. This suggests that accounting for the charging effect is important for quantitative understanding of the kinks in the I-V curves.

In Fig. 4, we compare the Josephson plasma mode dispersion for (a) Nb-Al/AlO$_x$-Nb multilayers and (b) BSCCO in the Meissner state (i.e., $\langle \varphi^{(0)} \rangle_t \approx 0$), using of Eq. (16). To illustrate the difference between the dispersion of collective mode for these two systems, we use the experimental values for the parameters $\alpha$ (i.e., charging effect strength) and $S$ (i.e., induction coupling strength). For the spectrum corresponding to the Nb-Al/AlO$_x$-Nb multilayers (Fig. 4(a)), we chose $\alpha = 0.0$ and $S \sim -0.47$ (assuming $\lambda \sim 900\,\text{Å}, d_I \sim 20\,\text{Å},$ and $d_S \sim 30\,\text{Å}$). Here we chose $\alpha = 0$ since the charging effect is negligible when $d_S$ is much larger than an atomic length. For the spectrum corresponding to BSCCO (Fig. 4(b)), we chose $\alpha = 0.2$ and $S \sim -0.49999$ (since $\lambda \sim 1500\,\text{Å}, d_I \sim 15\,\text{Å},$ and $d_S \sim 3\,\text{Å}$). Here $\alpha = 0.2$ is chosen. There are two notable differences between Figs. 4(a) and 4(b). First, due to a stronger inductive coupling (i.e., $S = -0.49999$ versus $-0.47$), the frequency $\omega/\omega_p$, for a fixed $k_x\lambda_c$, near $k_mA = 0$ in Fig. 4(b), decreases more sharply with $k_mA$ than that in Fig. 4(a). Second, due to the charging effect (i.e., $\alpha = 0.2$ versus 0.0), the collective mode frequency for $k_x\lambda_c = 0$ shows a dispersion as a function of $k_m$ in Fig. 4(b), indicating purely longitudinal excitations, while no dispersion is shown in Fig. 4(a), indicating the absence of these excitations. Note that the effect of finite, but small, $\beta$ is negligible, here.

**IV. STABILITY OF COLLECTIVE MODES**

In this section, we discuss the stability of uniform motion of collective modes (i.e., moving JVL) shown in Fig. 2 against small fluctuations. The structure of the moving JVLs, driven by a bias current, evolves as a function of its velocities. This evolution can be easily understood in terms of the stable-unstable transition for the collective modes.
We now carry out the linear analysis and determine the condition for maintaining stable uniform motion (i.e., the condition for bound oscillations of $\varphi'_{\ell,\ell-1}$) by computing the velocities at which the driven collective modes are stable. Here, instability of uniform motion arises when the amplitude of fluctuations grows exponentially as the collective modes propagate along the junction layers. Similar analysis, not including the charging effect, have been carried out to investigate the stability of moving JVL against lattice deformation. Also the effects of quantum and thermal fluctuations have been studied. We note that accounting for either the dynamic phase transition induced by the lattice displacements or the fluctuation effects are beyond the scope of the present analysis.

We proceed the analysis writing the spatial and the temporal dependence of phase fluctuations (i.e., $\varphi'_{\ell,\ell-1}$) of Eqs. (13) and (14) in Fourier space for the $z$-direction. Combining Eqs. (13) and (14) in Fourier space, we obtain

$$\lambda^2 D \frac{\partial^2 T_m}{\partial C_m \partial x^2} - \frac{A_m \partial^2 T_m}{\omega_p^2} - \left( \frac{\beta}{\omega_p} \frac{\partial T_m}{\partial t} + \cos \varphi^{(0)}_m T_m \right) = 0 \quad (19)$$

where $C_m = 1 + 2S \cos k_m a$. The uniform motion of the phase locked mode $\varphi^{(0)}_m$ with the wavenumber $k_m$ is given by $\varphi^{(0)}_m = \tilde{\omega}_m t + k_x x + \varphi_{o,m}$, where $\tilde{\omega}_m = (2\pi/\phi_o)V_m A_m$ is the Josephson frequency, $V_m$ is the average voltage, and $\varphi_{o,m}$ is a mode dependent constant. The induced field contribution to $\varphi^{(0)}_m$ from the Josephson effect is neglected. This contribution is negligible when the magnetic vortices are in every I layers, as the case for $B \sim H_o$. $\tilde{\omega}_m/k_x$ is the velocity of the collective mode $m$. We transform Eq. (13) into a familiar Mathieu equation in the following two steps: first, make a change of variables from $(x,t)$ to $\zeta_m(=\varphi^{(0)}_m)$; and second, let $T_m = \tilde{T}_m e^{-\Gamma_m \zeta_m/2}$. Here $\Gamma_m = \beta \tilde{\omega}_m \omega_p C_m / \Omega_m$ and $\Omega_m = A_m C_m \omega_p^2 - (k_x \lambda_c)^2 (D/d') \omega_p^2$. The stability condition is determined, solving

$$\frac{\partial^2 \tilde{T}_m}{\partial \zeta_m^2} + \left[ \tilde{\delta}_m^T + \tilde{\eta}_m^T \cos \zeta_m \right] \tilde{T}_m = 0 \quad (20)$$

where $\tilde{\delta}_m^T = -\Gamma_m^2/4$ and $\tilde{\eta}_m^T = C_m \omega_p^2/\Omega_m$ are the mode dependent (i.e., $k_m$) parametric constants. Solutions of Eq. (20) exhibit instability for certain values of $\tilde{\delta}_m^T$ and $\tilde{\eta}_m^T$, indicating that the collective mode becomes unstable against small fluctuations. We determine the stability condition, finding $\tilde{(\delta)_m^T, (\eta)_m^T}$ at which all solutions of Eq. (20) are bounded.
Note that a similar parametric instability (in the \( \bar{\delta}_m^T - \bar{\eta}_m^T \) space) occurs both in the magnetic induction model\(^3\) (i.e., \( \alpha = 0 \)) and in the charging effect model\(^3\) (i.e., \( k_x \lambda_c = 0 \)).

For finding the stability condition for the collective modes, it is useful to determine, first, the stability diagram of the Mathieu equation

\[
\frac{d^2 T}{d\zeta^2} + [\delta + \eta \cos \zeta] T = 0 ,
\]

(21)

and then, find the values of \( (\bar{\delta}_m^T, \bar{\eta}_m^T) \) corresponding to the stable region of this diagram. The boundary curves separating the region of bound (stable) and unbound (unstable) solutions can be obtained easily, solving Eq. (21) numerically following the procedure outlined in Ref. 35. The boundary curves for the periodic oscillations with the period \( 2\pi \) (i.e., \( \zeta = 2\pi \)) and \( 4\pi \) (i.e., \( \zeta = 4\pi \)) are obtained, imposing that the determinant \( \mathcal{E}_n \) for \( n = \infty \), derived from Eq. (21) writing \( T = \sum_{n=-\infty}^{n=\infty} C_n e^{i n \zeta} \) for \( \zeta = 2\pi \) and \( T = \sum_{n=-\infty}^{n=\infty} d_n e^{i n \zeta/2} \) for \( \zeta = 4\pi \), is zero (i.e., \( \mathcal{E}_n = 0 \)).\(^3\) Here \( \mathcal{E}_n \) is the determinant of a \((2n+1) \times (2n+1)\) matrix for a periodic solution with the period \( 2\pi \) (or a \(2n \times 2n\) matrix for a periodic solution with the period \( 4\pi \)).

The determinant \( \mathcal{E}_n \) can be computed using the recursion relation\(^3\)

\[
\mathcal{E}_{n+2} = (1 - \gamma_{n+2} \gamma_{n+1}) \mathcal{E}_{n+1} - \gamma_{n+2} \gamma_{n+1} (1 - \gamma_{n+2} \gamma_{n+1}) \mathcal{E}_n + \gamma_{n+2} \gamma_{n+1} \gamma_n^2 \mathcal{E}_{n-1}
\]

(22)

where \( \mathcal{E}_0 = 1, \mathcal{E}_1 = 1 - 2\gamma_0 \gamma_1, \mathcal{E}_2 = (1 - \gamma_1 \gamma_2)^2 - 2\gamma_0 \gamma_1 (1 - \gamma_1 \gamma_2), \) and \( \gamma_n = \eta/[2(\delta - n^2)] \) for a solution with the period-\( 2\pi \) and \( \mathcal{E}_1 = 1 - \gamma_1^2, \mathcal{E}_2 = (1 - \gamma_1 \gamma_2)^2 - \gamma_1^2, \mathcal{E}_3 = (1 - \gamma_1 \gamma_2 - \gamma_2 \gamma_3)^2 - \gamma_1^2 (1 - \gamma_2 \gamma_3)^2, \) and \( \gamma_n = 2\eta/[4\delta - (2n - 1)^2] \) for a solution with the period-\( 4\pi \).

The stability diagram for Eq. (21) is shown in Fig. 5. The unstable regions, where at least one solution is unbounded, are shaded, and the stable regions, where all solutions are bounded, are not shaded. The boundary curves separating these regions are periodic solutions with period \( 2\pi \) (dashed lines) and \( 4\pi \) (solid lines). These curves are obtained by calculating \( \mathcal{E}_n = 0 \) for \( n = 200 \). The filled squares represent the values of \( (\bar{\delta}_m^T, \bar{\eta}_m^T) \) satisfying the stability condition. Here we set \( \bar{\delta}_m^T = 0 \) (i.e., \( \beta = 0 \)) since \( \bar{\delta}_m^T \propto \beta^2 \) and the terms of this order \( \mathcal{O}(\beta^2) \) have been neglected due to small \( \beta \). For \( \bar{\delta}_m^T = 0 \), the following values satisfy the stability condition: the values shown in Fig. 5 are \( 0 < \bar{\eta}_m^T < 0.4540, \bar{\eta}_m^T \approx 3.7898, 10.6516, \ldots \).
and 20.9637, and the values not shown in Fig. 5 are \( T_{m} \approx 34.7142, 51.9022, 72.5278, 96.5910, 124.0918 \ldots \). For a large \( T_{m} \) satisfying the stability condition, we assume that \( \Omega_{m} \approx 0 \) since \( \Omega_{m} \rightarrow 0 \) as \( T_{m} \rightarrow \infty \). In this case, the velocity for uniform motion is given by

\[
\frac{\omega_{m}}{k_{x}} \approx \lambda_{c} \omega_{p} \left( \frac{1}{A_{m}C_{m}} \right)^{1/2},
\]

indicating that the presence of the charging effect yields the mode dependent modification to the stability condition. For example, \( \omega_{1}/k_{x} \approx \lambda_{c} \omega_{p} \) for \( m = 1 \) (i.e., rectangular lattice) but \( \omega_{N}/k_{x} \approx \lambda_{c} \omega_{p} \left[ (1+2S)/(1-2S)(1+4\alpha) \right]^{1/2} \) for \( m = N \) (i.e., triangular lattice). The velocity for the out-of-phase modes is reduced from the predicted value of the magnetic induction model (i.e., \( \alpha = 0 \)). Equation (23) indicates that moving Josephson vortices in a periodic array evolve from one stable mode to another as the vortex velocity increases. For example, as the vortex velocity exceeds \( \omega_{N}/k_{x} \), but less than \( \omega_{N-1}/k_{x} \), the moving triangular lattice \( (m = N) \) becomes unstable and the \( m = N - 1 \) mode becomes stable.

V. MULTIPLE SUB-BRANCHING OF SUPERCURRENT

In the resistive state, the supercurrent branch splits into multiple sub-branches as the bias current exceeds the Josephson current.\(^{7,8,9}\) Note that these supercurrent sub-branches differ from the observed multiple quasiparticle branches\(^{12,19,13}\) in the I-V data for LJJ stacks. This supercurrent sub-branching phenomenon, which appears clearly in the non-linear I-V regime, is attributed to the motion of Josephson vortices, but its origin is not understood clearly. Microwave induced voltage steps\(^{37}\) and geometric resonance\(^{38}\) are considered as other mechanisms, but we do not discuss them here. Instead, we argue that the splitting of the supercurrent branch is indeed due to the collective motion of Josephson vortices examining the low bias current regime where the I-V characteristics is linear. An analytic calculation is more tractable in this regime. Here, we illustrate qualitatively, rather than quantitatively, how the charging effect modifies the supercurrent sub-branches since the particle-hole imbalance effect\(^{14}\) neglected in this study may also need to be included for a quantitative comparison with the I-V data.
The current-voltage relations in the resistive state is obtained easily by noting that an AC voltage ripple with the Josephson frequency \( \omega_{\ell,\ell-1} \), in addition to the DC voltage, appears across the junction when a bias current \((J_B)\), greater than the critical current, is applied. This AC voltage ripple is due to the electron-pair tunneling current across the junction. Using the modified AC Josephson relation of Eq. (8), the time dependence of the phase difference between \( \ell \) th and \( \ell - 1 \) th S layers can be written as

\[
\varphi_{\ell,\ell-1}(t) \approx \varphi_{\ell,\ell-1}(0) + \frac{\phi_o V^s_{\ell,\ell-1}}{2\pi \bar{\omega}_{\ell,\ell-1}} \sin \omega_{\ell,\ell-1} t
\]  

(24)

where \( \bar{\omega}_{\ell,\ell-1} = (2\pi/\phi_o)[\langle V_{\ell,\ell-1} \rangle + \alpha(\langle V_{\ell+1,\ell} \rangle - 2\langle V_{\ell,\ell-1} \rangle + \langle V_{\ell-1,\ell-2} \rangle)] \) is the Josephson frequency, \( \langle V_{\ell,\ell-1} \rangle \) is the DC voltage (i.e., time averaged) across the superconductor layers \( \ell \) and \( \ell - 1 \). \( V^s_{\ell,\ell-1} \) is the amplitude of the AC voltage ripple. This time dependent phase difference of Eq. (24) yields a DC critical current response of

\[
J_c \sin \varphi_{\ell,\ell-1}(t) = -J_1 \left( \frac{\phi_o V^s_{\ell,\ell-1}}{2\pi \bar{\omega}_{\ell,\ell-1}} \right) \sin \varphi_{\ell,\ell-1}(0)
\]  

(25)

across the \( \ell \) th and \( \ell - 1 \) th S layers, indicating that the junction becomes resistive when \( J_B \) exceeds the DC critical current. Here \( J_1(x) \) is the first order Bessel function of the first kind. Equation (25) indicates that the current \( J_{\ell,\ell-1} = J_B - J_c \sin \varphi_{\ell,\ell-1}(t) \) between two adjacent S layers is not uniform along \( z \)-direction, even though a uniform bias current is applied. Hence, in this resistive state, we may reduce Eq. (11) to

\[
\left( \frac{1}{\omega_p^2} \frac{\partial^2 \Delta_t}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \Delta_t}{\partial t} \right) + \alpha \frac{\partial^2 \Xi_t}{\partial t^2} = \frac{1}{J_c} \left[ d'J_{\ell,\ell-1} + s(J_{\ell+1,\ell} + J_{\ell-1,\ell-2}) \right].
\]

(26)

Here, we neglected the spatial dependence (i.e., \( x \) variation) of \( \varphi_{\ell,\ell-1} \) for simplicity. To explicitly express the I-V relation for each collective mode, we now rewrite Eq. (26) in Fourier space for the \( z \)-direction as

\[
\frac{A_m}{\omega_p^2} \frac{\partial^2 \varphi_m}{\partial t^2} + \frac{\beta}{\omega_p} \frac{\partial \varphi_m}{\partial t} = \frac{J_m}{J_c}.
\]

(27)

Note that the first and the second term on the left hand side of Eq. (27) represent the capacitive and the resistive contribution of the junction, respectively.
We now average Eq. (27) over time. Since AC Josephson tunneling leads to a small voltage oscillation about the DC voltage, a further simplification of Eq. (27) can be made. The modified AC Josephson relation of $\frac{\partial \varphi_m}{\partial t} = \frac{2\pi}{\phi_0} V_m A_m$ indicates that the capacitive contribution vanishes when it is averaged over time (i.e., $\langle \partial^2 \varphi_m / \partial t^2 \rangle \propto \partial \langle V_m \rangle / \partial t \approx 0$).

This simplification leads to the current-voltage relation of

$$\langle V_m \rangle = \frac{\omega_p \phi_0}{2\pi} \frac{\bar{J}}{J_c A_m}.$$  

(28)

Here $\langle \cdots \rangle$ denotes the time average, and $\bar{J} = \langle J_m \rangle \sim J_B - J_c \langle \sin \varphi \rangle$. Since the collective modes for $m = 1, 2, \cdots N$ are identical to the modes for $m = 2N+1, 2N, \cdots N+2$, respectively, the number of sub-branches is the same as the number of junctions (i.e., $N$) in the stack.

In Fig. 6, we plot the I-V relation of Eq. (28) for (a) $\alpha = 0.0$ and (b) 0.1 to illustrate the effect of weak, but non-zero, charging effect. For clarity, we plot the curves for only the three collective modes (i.e., $m = 1, N/2$, and $N$) corresponding to the modes shown in Fig. 2. These I-V curves reveal two interesting points. First, the supercurrent splits into $N$ sub-branches, each corresponding to the collective mode, when the LJJ stack are in the resistive state. The $m = 1$ mode represents the high velocity mode (i.e., rectangular lattice), while the $m = N$ mode represents the low velocity mode (i.e., triangular lattice). Second, these $N$ sub-branches appear as a single curve when the charging effect is absent (i.e., $\alpha = 0$, see Fig. 6(a)), but they spread out when this effect is present (i.e., $\alpha \neq 0$, see Fig. 6(b)), suggesting that this can be used as another way to measure the charging effect. Since the width of this spread is related to the strength of the charging effect ($\alpha$), identification of each sub-branches is feasible at a low bias current. When $\alpha$ is small, as in BSCCO (i.e., $\alpha \sim 0.1 - 0.4$), observing the branch splitting in the linear I-V regime may be difficult but is still possible. The main difficulty is in observing the high velocity branches (i.e., $m \sim \mathcal{O}(1)$).

To observe these branches, a magnetic field, stronger than $B \sim H_o$, may be needed because of their stability conditions. Note that the appearance of these high velocity branches is expected when the interaction between the vortices is increased by the field, suggesting that a complete sub-branch structure may be more easily obtained from the I-V characteristics of
a LJJ stack with increasing microwave irradiation power (i.e., AC magnetic fields). These results are consistent with the data exhibiting supercurrent branch splitting.

VI. SUMMARY AND CONCLUSION

In summary, we investigated the collective phase dynamics in the conventional LJJ stacks and in layered superconductors, using a theoretical model which accounts for both the magnetic induction effect and the charging effect. These two coupling mechanisms are equally important in the intrinsic LJJ (e.g. BSCCO) due to the atomic length thick S layers. We showed that the collective phase dynamics in an intrinsic LJJ stack is modified from those in a conventional LJJ stack in two important ways. (i) The dispersion of Josephson plasma mode for BSCCO is significantly changed from the Nb-Al/AlO$_{x}$-Nb multilayers. Consequently, the Swihart velocity and the velocity of stable uniform motion for the out-of-phase collective modes in BSCCO increases and decreases, respectively, from the results of the magnetic induction model due to the presence of the charging effect. (ii) The supercurrent sub-branching in the resistive state is consistent with collective motion of Josephson vortices. The width of spread of these supercurrent sub-branches in the linear I-V regime depends on the strength of the charging effect. These results are consistent with the experimental data and illustrate the importance of accounting for the charging effect in BSCCO, even though its strength is weak ($\alpha \sim 0.1 - 0.4$). They also suggest that our model is useful for understanding the experimental data for JPR, non-Josephson emission, and the I-V characteristics in the resistive state. Since many applications of intrinsic LJJ stacks as high frequency devices exploit collective dynamics of Josephson vortices, these results indicate that our model is useful for future technological applications involving intrinsic LJJ stacks.

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**FIGURE CAPTIONS**

**Figure 1.** A stack of LJJ is shown schematically as alternating superconducting (S) and insulating (I) layers with thickness $d_S$ and $d_I$, respectively. $L_x$ denotes the dimension in $x$-direction. The magnetic field $B$ is applied in the plane of tunnel barriers and the bias current $I_B$ is applied along the vertical stack.

**Figure 2.** Mutual phase-locking of Josephson vortices (filled ovals) with the wave number $k_m = m\pi/(N+1)a$ is schematically illustrated. $N$ is the number of LJJ in the stack and $a = d_S + d_I$. For the in-phase mode ($m = 1$), the Josephson vortices form a rectangular lattice, but for the out-of-phase mode ($m = N$), they form a triangular lattice. The dotted lines are a guide to eyes for phase-locking, and the arrows indicate the direction of propagation.

**Figure 3.** The dispersion relation for (a) the longitudinal plasma excitations at $k_x = 0$ and (b) the transverse plasma excitations at $k_m = 0$ are plotted to illustrate the difference between (a) our model and the magnetic induction model, and (b) our model and the charging effect model. Here $\alpha = 0.2$ and $S = -0.49999$ are chosen.

**Figure 4.** The plasmon dispersion in the Meissner state (i.e., $\phi^{(0)}_{\ell,\ell-1} = 0$) is plotted as functions of $k_m a$ and $k_x \lambda_c$ for the parameters corresponding to (a) the Nb-Al$_x$/AlO$_y$-Nb multilayers ($\alpha = 0.0, S = -0.47$) and (b) Bi$_2$Sr$_2$CaCu$_2$O$_{8+y}$ ($\alpha = 0.2, S = -0.49999$).

**Figure 5.** Stability diagram of Mathieu’s equation. The unstable regions and the stable regions are shaded and not shaded, respectively. The periodic solutions of period $2\pi$ (dashed lines) and $4\pi$ (solid lines) represent the boundary between the stable and unstable regions. The filled squares represent the values of $(\delta^T_m, \tilde{\eta}^T_m)$ satisfying the stability condition.

**Figure 6.** The I-V curves for the supercurrent branch in the resistive state are plotted for (a) $\alpha = 0.0$ and (b) $0.1$ to illustrate the effect of nonzero $\alpha$ (i.e., charging effect). Here, only three curves corresponding to the collective modes shown in Fig. 2 are plotted for clarity.