FOLIATIONS ON THE OPEN 3-BALL BY COMPLETE SURFACES

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Abstract. When is a manifold a leaf of a complete closed foliation on the open unit ball? We give some answers to this question.

1. Introduction and statement of results

This paper is concerned with the topology of leaves of foliations. The concept of a foliation appeared in 1940’s as a geometric approach to solutions of differential equations, and it is now widespread among various areas such as complex analysis, exterior differential systems and contact topology (e.g. [6, 11]). Recall that a codimension \( q \) \( C^r \) foliation \( \mathcal{F} \) on an \( n \)-dimensional smooth manifold \( M \) is a decomposition \( \{ L_\lambda \}_{\lambda \in \Lambda} \) of \( M \) into a disjoint union of injectively immersed connected \((n - q)\)-dimensional submanifolds \( L_\lambda \) satisfying the following local triviality: each point of \( M \) has a neighborhood \( U \) such that \( \mathcal{F} \) restricted to \( U \) is \( C^r \) diffeomorphic to the family \( \{ \mathbb{R}^{n-q} \times \{ y \} \}_{y \in \mathbb{R}^q} \) of parallel \((n - q)\)-dimensional planes in \( \mathbb{R}^n \). \( L_\lambda \) is called a leaf of \( \mathcal{F} \). Note that, by collecting all the vectors tangent to leaves, the foliation can alternatively be defined as an integrable subbundle of \( TM \).

In 1975, Sondow [16] posed a basic question: when is a manifold a leaf? This question is natural (because it is regarded as a generalization of the classical embedding problem in differential topology) and important (because it may be related to the study of the topology of integral manifolds of differential equations). Thus, since then the question has been investigated extensively in various possible settings (see e.g. [7, 8, 9, 10, 12, 13]).

The purpose of this paper is to consider the question in an interesting new setting. Let \( \mathcal{F} \) be a foliation on a Riemannian manifold \((M, g)\). A leaf \( L \) of \( \mathcal{F} \) is called closed if \( L \) is a closed subset of \( M \) (this is equivalent to say that \( L \) is properly embedded), and complete if \( L \) is complete with respect to the induced Riemannian metric \( g|_L \). A foliation \( \mathcal{F} \) is said to be closed (resp. complete) if all leaves of \( \mathcal{F} \) are closed (resp. complete). Now, our setting is as follows: We fix, as the manifold which supports foliations, the

\[ \text{2020 Mathematics Subject Classification. Primary 57R30; Secondary 53C12.} \]

\[ \text{Key words and phrases. foliation, complete leaf, uni-leaf foliation.} \]

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open unit ball $\mathbb{B}^n$ of the Euclidean space $\mathbb{R}^n$ with the induced Euclidean metric — the simplest incomplete open manifold. And, foliations we try to construct on $\mathbb{B}^n$ should be complete and closed. Newness of this setting is to treat complete closed foliations on incomplete open manifolds. As one can imagine, in order to construct such foliations, one must “turbulize” all the leaves along the (ideal) boundary of $\mathbb{B}^n$.

The motivation of this work comes from recent deep works of Alarcón, Globevnik and Forstnerič [1, 2, 3]. They consider holomorphic foliations on the open ball of $\mathbb{C}^n$. Our work is, in a sense, a real smooth ($C^\infty$) version of theirs. Since holomorphic objects are very ‘rigid’, constructions of complete holomorphic foliations are much harder than those of real ones. The advantage of our approach is that, by forgetting holomorphic rigidity, one can concentrate on overcoming purely topological difficulties. In fact, on some topic we have thus succeeded in constructing infinitely many examples of foliations that are new in the literature (see Theorem 1.2 below).

Now, the first result of this paper is the following, whose holomorphic version has been obtained by Alarcón and Globevnik [1, 3]. (Note that our result is independent of theirs, because of the difference of codimension. The codimension of our foliation is 1 (the most cramped codimension, see §6), while the real codimension of their foliations are at least 2.)

**Theorem 1.1.** For any connected open orientable smooth surface $\Sigma$, there is a codimension 1 complete closed smooth foliation on $\mathbb{B}^3$ with a leaf diffeomorphic to $\Sigma$.

**Remark.** In [9], Hector and Bouma showed the same statement on $\mathbb{R}^3$. In [10], Hector and Peralta-Salas generalized it in higher dimensions.

**Remark.** The corresponding result to Theorem 1.1 for Sondow’s original question (i.e. the realization of manifolds as leaves of foliations on compact manifolds) was first obtained by Cantwell and Conlon [7]. For recent developments in this area, see e.g. [4, 13].

**Remark.** A non-orientable surface cannot be a leaf of a foliation of $\mathbb{B}^3$. In fact, if it can, the foliation must be transversely non-orientable. The existence of such a foliation contradicts the simply-connectedness of $\mathbb{B}^3$.

Our next concern is a uni-leaf foliation. Here, we call a foliation $\mathcal{F}$ uni-leaf if all the leaves of $\mathcal{F}$ are mutually diffeomorphic.

**Example.** A complete closed uni-leaf foliation on $\mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ can easily be constructed as follows. Begin with the standard
foliation $\mathcal{H}$ on $\mathbb{B}^2$ defined by $dy = 0$. Then, obviously all leaves of $\mathcal{H}$ are diffeomorphic to the real line and closed in $\mathbb{B}^2$. Let $h$ be a diffeomorphism of $\mathbb{B}^2$ defined by $h(r, \theta) = \left( r, \theta + \tan \frac{\pi r^2}{2} \right)$, where $(r, \theta)$ are the polar coordinates. Then, $h$ sends any leaf $\ell$ of $\mathcal{H}$ to a complete curve $h(\ell)$ in $\mathbb{B}^2$, because each end of $h(\ell)$ spirals asymptotically on $\partial \mathbb{B}^2$. Hence, $h(\mathcal{H})$ is a foliation we have desired. Note that, since $h$ is real analytic ($C^\infty$), so is $h(\mathcal{H})$.

So, let us consider uni-leaf foliations on $\mathbb{B}^3$. For a connected open orientable surface $\Sigma$, let $\mathcal{E}$ be the set of ends of $\Sigma$ with the usual topology and $\mathcal{E}^*$ the closed subset of $\mathcal{E}$ consisting of nonplanar ends. It is known ([15]) that the pair $(\mathcal{E}, \mathcal{E}^*)$ and the genus determine the homeomorphism type of $\Sigma$. It is also known that two smooth surfaces are diffeomorphic if and only if they are homeomorphic.

Now, we will introduce a new concept. We assume that the genus $g$ of $\Sigma$ is either 0 or $\infty$. Let $e$ be a point of $\mathcal{E}$. Let $Z$ be empty if $g = 0$ and a countably infinite subset of $\mathcal{E} - \mathcal{E}^* - \{e\}$ if $g = \infty$. Suppose further that every point of $Z$ is an isolated point of $\mathcal{E}$ and the derived set of $Z$ in $\mathcal{E}$ is $\mathcal{E}^*$. In this situation we say that the 4-tuple $(\mathcal{E}, \mathcal{E}^*, Z, e)$ satisfies the self-similarity property if the following condition holds: there exist two copies $(\mathcal{E}^+, \mathcal{E}^{++}, Z^+, e^+)$, $(\mathcal{E}^-, \mathcal{E}^{--}, Z^-, e^-)$ of $(\mathcal{E}, \mathcal{E}^*, Z, e)$ and a homeomorphism $h : \mathcal{E}^+ \lor_{e^+ = e^-} \mathcal{E}^- \to \mathcal{E}$ such that $h(e^+) = e$ and $h(Z^+ \sqcup Z^-) = Z$ (hence, $h(\mathcal{E}^{++} \lor_{e^+ = e^-} \mathcal{E}^{--}) = \mathcal{E}^*$), where $\lor$ is the wedge sum.

Remark. This concept is not the same as the usual self-similarity in fractal geometry. Ours is a kind of pointed self-similarity, meaning that we fix a basepoint once and for all and then only consider subspaces containing the basepoint and mappings preserving the basepoint.

Example. (1) Let $C$ be a Cantor set embedded in $S^2$. Then, all ends of the surface $\Sigma_C = S^2 - C$ are planar, and the endset $\mathcal{E}$ of $\Sigma_C$ is identified with $C$ and hence has the self-similarity property. In fact, $\mathcal{E}$ can be expressed as $\mathcal{E} = \mathcal{E}^+ \lor_e \mathcal{E}^-$, where $\mathcal{E}^+$ and $\mathcal{E}^-$ are subsets of $\mathcal{E}$ both homeomorphic to $C$ such that $\mathcal{E}^+ \cap \mathcal{E}^- = \{e\}$ for some $e \in \mathcal{E}$. In this case, $Z$ is empty.

(2) Let $J$ be the orientable open surface with one end and infinite genus (so called Jacob’s ladder). We take in $J$ a discrete infinite subset $S$ and put $\Sigma_J = J - S$. Then, the endset $\mathcal{E}$ of $\Sigma_J$ consists of isolated planar ends $e_n$ $(n \in \mathbb{N})$ each of which corresponds with a point of $S$ and one nonplanar end $e$ to which $e_n$’s converge. Thus, $\mathcal{E} = \{e\} \cup \{e_n\}_{n \in \mathbb{N}}$ and $\mathcal{E}^* = \{e\}$. See Fig.1. Set $Z = \{e_n\}_{n \in \mathbb{N}}$. Then, the 4-tuple $(\mathcal{E}, \mathcal{E}^*, Z, e)$ satisfies the self-similarity property. Indeed, it suffices to define: $\mathcal{E}^* = \{e\} \cup \{e_{2n-1}\}_{n \in \mathbb{N}}$.\[
\[ E^- = \{ e \} \cup \{ e_{2n} \}_{n \in \mathbb{N}}, \quad E^{**} = E^{-*} = \{ e \}, \quad Z^\pm = E^\pm - \{ e \}, \quad e^+ = e^- = e, \text{ and} \]
h is the identity.

Figure 1.

We remark that there are many other surfaces whose endsets have the self-similarity property. We will give them at the end of §4.

The next theorem shows that infinitely many surfaces can be realized as leaves of uni-leaf foliations on \( \mathbb{B}^3 \).

**Theorem 1.2.** Let \( \Sigma \) be a connected open orientable smooth surface with genus \( g \) either 0 or \( \infty \), and let \( (E, E^*) \) be its endset pair. Suppose that there exist a point \( e \) of \( E \) and a subset \( Z \) of \( E - E^* - \{ e \} \) such that

1. \( Z \) is empty if \( g = 0 \) and countably infinite if \( g = \infty \),
2. every point of \( Z \) is an isolated point of \( E \),
3. the derived set of \( Z \) in \( E \) is \( E^* \), and
4. \( (E, E^*, Z, e) \) satisfies the self-similarity property.

Then, there exists a codimension 1 complete closed smooth uni-leaf foliation of \( \mathbb{B}^3 \) having \( \Sigma \) as a leaf.
Remark. In the holomorphic situation, the existence of a uni-leaf foliation on the 2-dimensional holomorphic ball is known in the case where the leaf $\Sigma$ is a disk $\{z \in \mathbb{C} \mid |z| < 1\}$ (see [2]). It seems that the problem remains open whether other Riemann surfaces can be leaves of some holomorphic uni-leaf foliations.

The following two arguments are crucial to proving our theorems:

1. to build a kind of barrier in $\mathbb{B}^n$ (called a **labyrinth** in [2]) in order to force all leaves to become complete. The existence of a holomorphic labyrinth is a profound result. On the other hand, we find that a real labyrinth is quite easy to be built (see §2).

2. to show that the self-similarity property of the endset of a surface is a sufficient condition for the surface to be a leaf of some uni-leaf foliation. This assertion is proved by a careful construction of a rather complicated submersive function on some domain of $\mathbb{R}^3$ (see §4).

We close this section with two more remarks.

**Remark.** All foliations in this paper are $C^\infty$. For foliations in Theorems 1.1 and 1.2 the authors have an idea of raising the differentiability to $C^\omega$ by using $C^\omega$ approximation of $C^\infty$ Morse functions and diffeomorphisms. But, at present, they have not written up the proof in full precision yet.

**Remark.** All foliations in this paper are closed, hence their holonomy pseudogroups are always trivial.

### 2. Constructing complete foliations on the ball

The content of this section is a real smooth version of the argument developed in [2].

Let $\{r_k\}$ and $\{s_k\}$ be sequences of real numbers satisfying $0 < s_1 < r_1 < s_2 < r_2 < \cdots \to 1$, and let $B_k$ (resp. $S_k$) be the closed ball (resp. the sphere) in $\mathbb{R}^n$ centered at the origin with radius $r_k$ (resp. $s_k$). Put $\Gamma_k = S_k - U_\varepsilon(p_k)$, where $0 < \varepsilon \ll s_1$, $p_k = (0, \cdots, 0, (-1)^ks_k)$ and $U_\varepsilon(p_k)$ is the open $\varepsilon$-neighborhood of $p_k$ in $\mathbb{R}^n$. A path $\gamma : [0, \infty) \to M$ in an open manifold $M$ is called **divergent** if $\gamma(t)$ leaves any compact set of $M$ as $t \to \infty$. Then, the following is evident.

**Lemma 2.1.** Every divergent smooth path in $\mathbb{B}^n$ avoiding $\bigcup_{k \geq k_0} \Gamma_k$ (for some $k_0$) has infinite length.

Put $P_k = \mathbb{R}^{n-1} \times [-k, k]$. Let $\Omega$ be an open set of $\mathbb{R}^n$ diffeomorphic to $\mathbb{B}^n$ such that its image projected to the last coordinate of $\mathbb{R}^n$ is unbounded.
Then, we can choose an exhaustive sequence \( \{ C_k \}_{k \in \mathbb{N}} \) of subsets of \( \Omega \) satisfying the following properties: (i) \( C_k \) is diffeomorphic to the closed ball, (ii) \( C_k \subset \text{Int} \ C_{k+1} \), (iii) \( C_k \subset P_{k+1} \) and (iv) \( C_k - P_k \neq \emptyset \).

**Lemma 2.2.** There exists a diffeomorphism \( \Phi \) from \( \Omega \) to \( \mathbb{B}^n \) such that for all \( k \in \mathbb{N} \)

1. \( \Phi(C_k) = B_k \) and
2. \( \Phi(P_k \cap C_k) \cap \Gamma_k = \emptyset \).

**Proof.** We will enlarge the domain of definition inductively. First, define \( \Phi \) on \( C_1 \) so that it satisfies (1) and (2). This is possible because, by (iv) above, \( C_1 - P_1 \) is non-empty and \( \Gamma_1 \subset \text{Int} \ B_1 \). Next, suppose \( \Phi \) has already been defined on \( C_k \) so as to satisfy (1) and (2) for \( \ell \leq k \). Since \( \Gamma_{k+1} \subset B_{k+1} - B_k \) and, by (iv) above, \( C_{k+1} - P_{k+1} \) is non-empty, it is possible to extend the definition of \( \Phi \) on \( C_{k+1} \) so that \( \Phi(C_{k+1} - P_{k+1}) \supset \Gamma_{k+1} \) and that \( \Phi(C_{k+1} - C_k) = B_{k+1} - B_k \). Then, we see that the resulting \( \Phi \) satisfies (1) and (2). Since \( \{ C_k \}_{k \in \mathbb{N}} \) is an exhausting sequence in \( \Omega \), this inductive procedure gives a diffeomorphism from \( \Omega \) to \( \mathbb{B}^n \), as desired. \( \square \)

**Lemma 2.3.** Let \( \Phi \) be as in Lemma 2.2. Then, for all \( k \in \mathbb{N} \), \( \Phi(P_k \cap \Omega) \cap \Gamma_k = \emptyset \).

**Proof.** Since \( \Gamma_k \subset B_k = \Phi(C_k) \), a point \( p \) of \( \Omega \) satisfies \( \Phi(p) \in \Gamma_k \) only if \( p \in C_k \). Hence, by Lemma 2.2 (2), \( \Phi(P_k \cap \Omega) \cap \Gamma_k = \Phi(P_k \cap C_k) \cap \Gamma_k = \emptyset \). \( \square \)

Let \( \mathcal{G} \) be a closed foliation on \( \Omega \) (i.e. every leaf of \( \mathcal{G} \) is a closed subset of \( \Omega \)). Then, the direct image \( \mathcal{F} = \Phi(\mathcal{G}) \) is a closed foliation on \( \mathbb{B}^n \). Here, we consider the following property (P) for \( \mathcal{G} \):

(P) For any leaf \( L \) of \( \mathcal{G} \) there exists \( k \in \mathbb{N} \) such that \( L \subset P_k \).

We recall that a leaf \( F \) of \( \mathcal{F} \) is complete if and only if every divergent smooth path in \( F \) has infinite length. The next lemma gives a sufficient condition for the completeness of the leaves of \( \mathcal{F} \).

**Lemma 2.4.** If \( \mathcal{G} \) satisfies the property (P), then all leaves of \( \mathcal{F} \) are complete.

**Proof.** Suppose \( \mathcal{G} \) satisfies (P) and let \( F \) be any leaf of \( \mathcal{F} \). Put \( L = \Phi^{-1}(F) \). Since \( L \) is a leaf of \( \mathcal{G} \), by (P) there exists \( k_L \in \mathbb{N} \) such that \( L \subset P_{k_L} \). Then, noticing Lemma 2.3, for any \( k \geq k_L \), we have \( F \cap \Gamma_k = \Phi(L) \cap \Gamma_k \subset \Phi(P_{k_L} \cap \Omega) \cap \Gamma_k \subset \Phi(P_k \cap \Omega) \cap \Gamma_k = \emptyset \). Therefore, \( F \) does not intersect \( \bigcup_{k \geq k_L} \Gamma_k \), hence, in particular, neither does any smooth path on \( F \). This with Lemma 2.1 implies the completeness of the leaves of \( \mathcal{F} \). \( \square \)
We summarize the result obtained in this section as follows.

**Proposition 2.5.** Let $\mathcal{G}$ be a closed foliation on an open subset $\Omega$ of $\mathbb{R}^n$ diffeomorphic to $\mathbb{B}^n$ such that

1. $\text{pr}_n(\Omega)$ is unbounded, where $\text{pr}_n : \mathbb{R}^n \to \mathbb{R}$ is the projection to the $n$-th coordinate, and
2. each leaf $L$ of $\mathcal{G}$ verifies the property $(P) : \text{pr}_n(L)$ is bounded.

Then, there exists a diffeomorphism $\Phi$ from $\Omega$ to $\mathbb{B}^n$ such that $\Phi(\mathcal{G})$ is a complete closed foliation on $\mathbb{B}^n$.

3. **Realizing open surfaces as leaves**

First, we prepare an elementary lemma:

**Proposition 3.1.** Let $W$ be a smooth open manifold and $\mathcal{C}$ be a countable family of injective smooth paths $c_k : [0, \infty) \to W$ ($k \in \mathbb{N}$) such that

1. $c_k$ is divergent for each $k$.
2. they are pairwise disjoint and the family $\mathcal{C}$ is locally finite.

Then, $W - \bigcup_k c_k([0, \infty))$ is diffeomorphic to $W$.

The proof consists in “pushing to infinity” each point $c_k(0)$ along the path $c_k$. We give it here for the reader’s convenience.

**Proof.** Let $\dim W = n$ and give $W$ an arbitrary Riemannian metric. Put $N = D^{n-1} \times [-1, \infty)$ (where $D^{n-1}$ is the closed unit disk in $\mathbb{R}^{n-1}$ centered at the origin). Take a nonnegative bounded smooth function $\lambda : N \to \mathbb{R}$ such that $\lambda = 1$ near $\partial N$ and that for $(p, t) \in N$, $\lambda(p, t) = 0$ if and only if $p = 0$ and $t \in [0, \infty)$. Define a smooth vector field $V$ on $N$ by $V = \lambda \frac{\partial}{\partial t}$ and let $\varphi : N \times [0, \infty) \to N$ be the (local) flow generated by $V$. Then, the map $g : N \to N - (\{0\} \times [0, \infty))$ defined as $g(p, t) = \varphi((p, -1), t + 1)$ is a diffeomorphism which is the identity near $\partial N$. Now, for each $k$, take a neighborhood $N_k$ of $c_k([0, \infty))$ and a diffeomorphism $u_k : N \to N_k$ so that

1. $N_k$’s are pairwise disjoint and the family is locally finite, 
2. $u_k(0, t) = c_k(t)$ for $t \in [0, \infty)$, and 
3. the diameter of $u_k(D^{n-1} \times \{t\})$ tends to 0 as $t \to \infty$. We then obtain a desired diffeomorphism $h : W \to W - \bigcup_k c_k([0, \infty))$ by setting $h = u_k \circ g \circ u_k^{-1}$ on $N_k$ for any $k$, and is the identity everywhere else. 

Using almost the same argument, we can also show the following

**Proposition 3.2.** Let $M$ be a smooth manifold, $P$ a subset of $M$, and $N$ a neighborhood of $P$ in $M$. Put $W = M \times \mathbb{R}$. Let $\ell_p = \{p\} \times (-\infty, a_p)$ ($p \in
$P, a_p \in \mathbb{R}$) be a family of vertical lines in $W$. If the union $\bigcup_{p \in P} \ell_p$ is closed in $W$, then $W - \bigcup_{p \in P} \ell_p$ is diffeomorphic to $W$ by a fiber $(\{m\} \times \mathbb{R})_{m \in M}$-preserving diffeomorphism which is the identity outside $N \times \mathbb{R}$.

Proof. It suffices to push $\bigcup_{p} \ell_p$ to $-\infty$ with respect to the $\mathbb{R}$-factor. To be precise, take a nonnegative bounded smooth function $\mu : W \to \mathbb{R}$ which vanishes exactly on $\bigcup_{p} \ell_p$ and is constantly 1 outside $N \times \mathbb{R}$. We consider the flow $\psi : W \times \mathbb{R} \to W$ on $W$ generated by the vector field $\mu \frac{\partial}{\partial z}$, where $z$ is the coordinate of $\mathbb{R}$. Take a smooth function $\lambda : M \to \mathbb{R}$ such that $\lambda(p) > a_p$ for $p \in P$. Then, the map $h : W \to W - \bigcup_{p} \ell_p$ defined by $h(m, t) = \psi((m, \lambda(m)), t - \lambda(m))$ is a desired diffeomorphism. □

Note that in this proposition (i) the local finiteness need not be assumed and (ii) the pushing-to-infinity operation can be carried out for an arbitrary small neighborhood $N$ of $P$.

Proof of Theorem 1.1. Any connected open orientable surface $\Sigma$ can be constructed as follows: First, remove from $\mathbb{R}^2$ a closed totally disconnected set $X$. ($X \cup \{\infty\}$ will be the endset $E$ of $\Sigma$, where $\infty$ is the point of infinity of the one-point compactification of $\mathbb{R}^2$.) Next, take an at most countable set $Z$ in $\mathbb{R}^2 - X$ in such a way that for any compact set $K$ in $\mathbb{R}^2 - X$ the intersection $Z \cap K$ is finite. (The set of accumulation points of $Z$ in $\mathbb{R}^2 \cup \{\infty\}$ will be $E^*$.) Then, for each point $q$ of $Z$, choose a small compact neighborhood $U_q$ of $(q, 0)$ in $\mathbb{R}^2 - X$ so that they are pairwise disjoint, and in each $U_q$ perform a surgery to make a genus. The resulting surface is $\Sigma$. Observe that the whole of the above construction can be carried out in $(\mathbb{R}^2 - X) \times \mathbb{R}$.

To do so, for each $q \in Z$ choose a small compact neighborhood $V_q$ of $(q, 0)$ in $(\mathbb{R}^2 - X) \times \mathbb{R}$, and perform ambient surgeries on $(\mathbb{R}^2 - X) \times \{0\}$ inside each $V_q$. Thus, we obtain $\Sigma$ as a properly embedded submanifold of $(\mathbb{R}^2 - X) \times \mathbb{R}$. Note that $\Sigma$ separates $(\mathbb{R}^2 - X) \times \mathbb{R}$ into two connected components.

We then take a Morse function $f : (\mathbb{R}^2 - X) \times \mathbb{R} \to \mathbb{R}$ so that

(1) $f(x, y, z) = z$ for $(x, y, z) \in (\mathbb{R}^2 - X) \times [(-\infty, -1] \cup [1, \infty])$, and that

(2) 0 is a regular value of $f$ with $f^{-1}(0) = \Sigma$.

The existence of such $f$ follows from the above construction of $\Sigma$. We let $\text{Crit}(f)$ denote the set of critical points of $f$, (which is a countably infinite set if $\Sigma$ has nonplanar ends). Now, we take an increasing sequence $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots$ of codimension 0 compact submanifolds in $\mathbb{R}^2 - X$ such that $\bigcup_{i=1}^{\infty} K_i = \mathbb{R}^2 - X$. For each $p \in \text{Crit}(f)$, we can construct an
injective smooth path \( c_p : [0, \infty) \to (\mathbb{R}^2 - X) \times \mathbb{R} \) as follows. Suppose \( p \in (K_i - K_{i-1}) \times \mathbb{R} \). Then,

1. \( c_p(0) = p \),
2. \( c_p \) intersects neither \( \Sigma \) nor \( (\mathbb{R}^2 - X) \times \{\pm 1\} \),
3. \( c_p \) does not intersect \( K_{i-1} \times \mathbb{R} \),
4. for each \( j \geq i \), \( c_p \) intersects \( \partial K_j \times \mathbb{R} \) transversely at exactly one point,
5. \( c_p(t) \) converges to a point in \( (X \cup \{\infty\}) \times \{\pm 1/2\} \) as \( t \to \infty \), and
6. if \( p \neq q \), then \( c_p([0, \infty)) \) and \( c_q([0, \infty)) \) are disjoint.

We denote by \( M \) the space obtained from \( (\mathbb{R}^2 - X) \times \mathbb{R} \) by removing \( \bigcup_{p \in \text{Crit}(f)} c_p([0, \infty)) \). Then, by applying Proposition 3.1 for \( W = (\mathbb{R}^2 - X) \times \mathbb{R} \) and \( C = \{c_p\} \) we see that \( M \) is diffeomorphic to \( (\mathbb{R}^2 - X) \times \mathbb{R} \).

Next, define an open subset \( \Omega \) of \( \mathbb{R}^3 \) to be the union of \( M \) and \( \mathbb{R}^2 \times (2, \infty) \). Note that, by the above argument, \( \Omega \) is diffeomorphic to \( ((\mathbb{R}^2 - X) \times \mathbb{R}) \cup [\mathbb{R}^2 \times (2, \infty)] \). Thus, by Proposition 3.2 we can push \( X \times (-\infty, 2] \) to \( -\infty \) with respect to the second coordinate and obtain that \( \Omega \) is diffeomorphic to \( \mathbb{B}^3 \).

Next, we extend the domain of our Morse function \( f \) to \( \Omega \) by defining \( f \) to be the projection to the second factor on \( \mathbb{R}^2 \times (2, \infty) \). We let \( \mathcal{G} \) denote the foliation on \( \Omega \) whose leaves are connected components of the level sets of \( f \). Then, \( \mathcal{G} \) has no singularities because all the critical points of \( f \) are removed from \( \Omega \). It is also obvious that all leaves of \( \mathcal{G} \) are closed in \( \Omega \). By the construction, we see that \( \mathcal{G} \) satisfies the conditions (1) and (2) of Proposition 2.5. Therefore, we conclude that \( \mathcal{G} \) is diffeomorphic to a complete closed foliation on \( \mathbb{B}^3 \) containing \( \Sigma \) as a leaf. Theorem 1.1 is proved.

\[ \square \]

4. Uni-Leaf Foliations

In this section we consider the question: which manifold is a leaf of a complete closed uni-leaf foliation on the open unit ball? This question has first been asked by Alarcón and Forstnerič [2] in the holomorphic category. They have shown that for any integer \( n > 1 \), there exists a complete closed holomorphic uni-leaf foliation of the open unit ball in \( \mathbb{C}^n \) with disks as leaves. We work in the real smooth category and prove Theorem 1.2.

In order to clarify the flow of the proof of our theorem, we first treat two simple cases: \( \Sigma_C \) and \( \Sigma_J \) given in §1. We will realize each of them as a leaf of a complete closed smooth uni-leaf foliation of \( \mathbb{B}^3 \). (Then, the full proof of Theorem 1.2 will be understood as an elaboration of these cases.)
Example. Let $C$, $\Sigma_C$, $\mathcal{E}$, $\mathcal{E}^\pm$ and $e$ be as in Example (1) in §1. Through the identification of $S^2 - \{e\}$ with $\mathbb{R}^2$, we regard $\Sigma_C$, $\mathcal{E} - \{e\}$ and $\mathcal{E}^\pm - \{e\}$ as subsets of $\mathbb{R}^2$. Now, put

$$\Omega = \mathbb{R}^3 - (\mathcal{E}^+ - \{e\}) \times [-1, \infty) - (\mathcal{E}^- - \{e\}) \times (-\infty, 1].$$

Then, by Proposition 3.2, $\Omega$ is diffeomorphic to $\mathbb{R}^3$. We denote by $\mathcal{G}$ the foliation on $\Omega$ obtained by restricting the foliation $\{pr_3^{-1}(z)\}_{z \in \mathbb{R}}$ on $\mathbb{R}^3$. Then, all leaves of $\mathcal{G}$ are diffeomorphic to $\Sigma_C$. In fact, it is obvious when $|z| \leq 1$. In the case when $z > 1$ (resp. $z < -1$), we have $pr_3^{-1}(z) \cap \Omega$ is diffeomorphic to $\mathbb{R}^2 - (\mathcal{E}^+ - \{e\})$ (resp. $\mathbb{R}^2 - (\mathcal{E}^- - \{e\})$). But, since we are assuming that $\mathcal{E}^\pm$ are homeomorphic to $\mathcal{E}$, we have that $pr_3^{-1}(z) \cap \Omega$ is diffeomorphic to $\Sigma_C$ also in this case. Finally, since $\mathcal{G}$ satisfies the conditions (1) and (2) of Proposition 2.5, $\mathcal{G}$ is diffeomorphic to a complete closed unileaf foliation on $\mathbb{B}^3$, as desired.

Example. We first embed Jacob’s ladder $J$ in $\mathbb{R}^3$ as follows. Let $H = \mathbb{R}^2 \times \{0\}$ and $0 < \varepsilon \ll 1$. For each $n \in \mathbb{Z} - \{0, \pm 1\}$, choose a small neighborhood $U_n$ of $(n, 0)$ in $\mathbb{R}^2$. We put $W_n^+ = U_n \times (-1 - \varepsilon, n + \varepsilon)$ ($n \geq 2$) and $W_n^- = U_n \times (n - \varepsilon, 1 + \varepsilon)$ ($n \leq -2$). Inside each $W_n^\pm$ we perform an ambient surgery on $H$ to make a genus. Thus, we obtain a new surface embedded in $\mathbb{R}^3$ and diffeomorphic to $J$. Hereafter, we identify this surface with $J$. Next, we put $\ell_n^+ = \{(n, 0)\} \times [-1, \infty)$ ($n \geq 2$) and $\ell_n^- = \{(n, 0)\} \times (-\infty, 1]$ ($n \leq -2$). We can take a Morse function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the following conditions (by isotoping $J$ suitably in $\bigcup W_n^+ \cup \bigcup W_n^-$ if necessary):

1. $0$ is a regular value of $f$ and $f^{-1}(0) = J$,
2. $f(x, y, z) = z$ outside the union of $W_n^+$’s and $W_n^-$’s.
3. The critical points of $f$ are $A_n^+ = (n, 0, -1)$, $B_n^+ = (n, 0, n)$ ($n \geq 2$) and $A_n^- = (n, 0, n)$, $B_n^- = (n, 0, 1)$ ($n \leq -2$), whose critical values are their $z$-coordinates. If we pass through $A_n^+$ or $A_n^-$ (resp. $B_n^+$ or $B_n^-$), in the direction of increasing values of $f$, then the level set of $f$ is modified so that a genus is created (resp. erased).
4. Inside each $W_n^\pm$, $f$ is a standard Morse function admitting a cancelling pair of critical points.
5. On each $\ell_n^\pm$, $f$ is strictly increasing with respect to $z$.

Let $\mathcal{H}$ denote the foliation (with singularity) on $\mathbb{R}^3$ with the level sets of $f$ as leaves. Then, every regular leaf $f^{-1}(z)$ of $\mathcal{H}$ is diffeomorphic to $J$. In fact, if $|z| < 1$, it is obvious. If $|z| > 1$, $f^{-1}(z)$ loses “half” the number of infinite genus in comparison with $f^{-1}(0)$. But it still possesses infinite genus, hence, is diffeomorphic to $J$. We can also observe that any singular leaf of $\mathcal{H}$ has
infinite genus. Now, denote by $\Omega$ the set obtained from $\mathbb{R}^3$ by removing the infinite family of half lines $\ell_n^\pm$. Then, it follows from Proposition 3.1 that $\Omega$ is diffeomorphic to $\mathbb{R}^3$. Since all the critical points are removed, $\mathcal{H}$ restricted to $\Omega$ becomes a nonsingular foliation, say $\mathcal{G}$. We will check the topology of leaves of $\mathcal{G}$. First, we recall that the diffeomorphism type of $\Sigma_j$ is characterized by being orientable and having infinite genus, one non-planar end (say $e$) and countably infinite isolated planar ends converging to $e$. Note that each leaf $L = f^{-1}(z) \cap \Omega$ of $\mathcal{G}$ is obtained from the leaf $H = f^{-1}(z)$ of $\mathcal{H}$ by removing countably infinite discrete points $H \cap \ell_n^\pm$. Therefore, if $H$ is a regular leaf (hence diffeomorphic to $J$), then $L$ is diffeomorphic to $\Sigma_j$. In the case when $H$ is a singular leaf, we have to notice that by the removal of one singular point from $H$, two punctures (i.e. planar ends) are produced on $L$. But, anyway, planar ends of $L$ are countably infinite and anyone of them is isolated. Therefore, $L$ is diffeomorphic to $\Sigma_j$ also in this case. Consequently, all the leaves of $\mathcal{G}$ are diffeomorphic to $\Sigma_j$. As a final step, we take a diffeomorphism $\Psi : \mathbb{R}^3 \to V$, where $V = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2)|z| < 1\}$, such that $\Psi$ leafwise preserves the one-dimensional foliation $dx = dy = 0$. If we set $\hat{\Omega} = \Psi(\Omega)$, we can see that the foliation $\Psi(\mathcal{G})$ on $\hat{\Omega}$ satisfies the property (P) in §2, while $\hat{\Omega}$ is unbounded in the direction of $z$. Consequently, by Proposition 2.5, we obtain a complete closed uni-leaf foliation on $B^3$ with all leaves diffeomorphic to $\Sigma_j$.

Now, we will proceed to

Proof of Theorem 1.2. Let $\Sigma$ be a connected open orientable smooth surface and $(\mathcal{E}, \mathcal{E}^*)$ the endset pair of $\Sigma$. We use the notation in §1. We assume that there exist $e$ and $Z$ as in Theorem 1.2 such that $(\mathcal{E}, \mathcal{E}^*, Z, e)$ satisfies the self-similarity property. (Here, we should recall that if $\Sigma$ has no genus, then $\mathcal{E}^*$ and $Z$ are empty.) Put $X = \mathcal{E} - \{e\}$ and $X^\pm = \mathcal{E}^\pm - \{e^\pm\}$. Via $h$, we regard $\mathcal{E}^\pm$, $X^\pm$ and $Z^\pm$ as subsets of $\mathcal{E}$, $X$ and $Z$ respectively.

Now, we will start the construction of the uni-leaf foliation. We embed $\mathcal{E}$ into the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$ of $\mathbb{R}^2$ in such a way that $e$ is mapped to $\infty$. From now on, we identify $e$ with $\infty$ and regard $X$ and $Z$ as subsets of $\mathbb{R}^2$. We consider two cases separately.

The case where $Z$ is empty. In this case, the construction is a verbatim repetition of the one in the case of $\Sigma_C$ ($S^2$ minus a Cantor set): Namely, put

$$\Omega = \mathbb{R}^3 - X^+ \times [-1, \infty) - X^- \times (-\infty, 1]$$

and let $\mathcal{G}$ denote the foliation on $\Omega$ obtained by restricting the foliation $\{pr_3^{-1}(z)\}_{z \in \mathbb{R}}$ on $\mathbb{R}^3$. Then, by Proposition 3.2, $\Omega$ is diffeomorphic to $\mathbb{R}^3$. 
and, by the self-similarity condition, all leaves of \( \mathcal{G} \) are diffeomorphic to \( \Sigma \). Finally, by Proposition 2.5, we can conclude that \( \mathcal{G} \) is diffeomorphic to a complete closed uni-leaf foliation on \( \mathbb{R}^3 \).

**The case where \( Z \) is countably infinite.** Since \( X - Z \) is closed in \( \mathbb{R}^2 \), similarly as in §3 for each point \( q \) of \( Z \), we can choose a small compact neighborhood \( V_q \) of \((q, 0)\) in the open 3-manifold \((\mathbb{R}^2 - (X - Z)) \times \mathbb{R}\) so that they are pairwise disjoint. Then, perform an ambient surgery on \((\mathbb{R}^2 - (X - Z)) \times \{0\}\) to make a genus inside each \( V_q \). Thus, we obtain a new surface as a properly embedded submanifold of \((\mathbb{R}^2 - (X - Z)) \times \mathbb{R}\). Let \( \hat{\Sigma} \) denote this surface. We see that the endset pair of \( \hat{\Sigma} \) is \((E - Z, E^*)\). We may assume that for each \( q \in Z \) the intersection of \( \hat{\Sigma} \) and \( \{q\} \times \mathbb{R} \) is a single point.

By the self-similarity property, \( X \) and \( Z \) are respectively expressed as the disjoint union of two subsets as follows: \( X = X^+ \cup X^- \) and \( Z = Z^+ \cup Z^- \), with the property that \( X^\pm \) and \( Z^\pm \) are respectively homeomorphic to \( X \) and \( Z \). We put \( A^+ = (X^+ - Z^+) \times [-1, \infty) \), \( A^- = (X^- - Z^-) \times (-\infty, 1] \) and

\[
O = \mathbb{R}^3 - A^+ - A^-.
\]

Note that \( X^+ - Z^+ \) and \( X^- - Z^- \) are closed in \( \mathbb{R}^2 \), and hence \( A^+ \) and \( A^- \) are closed in \( \mathbb{R}^3 \). We number the elements of \( Z \) arbitrarily: \( Z^+ = \{q_n \mid n = 2, 3, 4, \cdots\} \) and \( Z^- = \{q_n \mid n = -2, -3, -4, \cdots\} \). We then take a Morse function \( f : O \to \mathbb{R} \) so that

1. \( f^{-1}(0) = \hat{\Sigma} \),
2. \( \text{Crit}(f) \) consists of the following points:
   \[
   (q_n, -1 - \frac{1}{n}, n) \text{ and } (q_n, n) \text{ for each } n = 2, 3, 4 \cdots,
   \]
   \[
   (q_n, n) \text{ and } (q_n, 1 - \frac{1}{n}) \text{ for each } n = -2, -3, -4 \cdots.
   \]
3. For each \( p \in \text{Crit}(f) \), the value \( f(p) \) is the \( z \)-coordinate of \( p \),

Let \( W^+_q = D^+_q \times I^+_q \ (q \in Z^+) \) be a compact product neighborhood of the segment \( \{q\} \times [-1 - \frac{1}{n}, 1] \) in \( O \), where \( D^+_q \) is a closed disk in \( \mathbb{R}^2 \) centered at \( q \) such that \( D^+_q \cap X = \{q\} \) and \( [-1 - \frac{1}{n}, n] \subset I^+_q \subset \mathbb{R} \). Similarly, let \( W^-_q = D^-_q \times I^-_q \ (q \in Z^-) \) be a compact product neighborhood of \( \{q\} \times [n, 1 - \frac{1}{n}] \) in \( O \), where \( D^-_q \) is a closed disk in \( \mathbb{R}^2 \) centered at \( q \) such that \( D^-_q \cap X = \{q\} \) and \( [n, 1 - \frac{1}{n}] \subset I^-_q \subset \mathbb{R} \). We choose the sets \( D^+_q \ (q \in Z^+) \) and \( D^-_q \ (q \in Z^-) \) so as to be pairwise disjoint.

4. Inside each \( W^+_q \ (q \in Z^+) \) or \( W^-_q \ (q \in Z^-) \), \( f \) is conjugate to the standard Morse function which admits a standard canceling pair of critical points, the one which has a smaller \( z \)-coordinate is of index 1 and the other is of index 2.
(5) The lines \( \{q\} \times [-1 - \frac{1}{n}, \infty) \) \( (q \in \mathbb{Z}^+) \) and \( \{q\} \times (-\infty, 1 - \frac{1}{n}] \) \( (q \in \mathbb{Z}^-) \)
are transverse to the level sets of \( f \) everywhere except at critical
points.

(6) \( f(x, y, z) = z \) outside the union of \( W_q^+ \)'s \( (q \in \mathbb{Z}^+) \) and \( W_q^- \)'s \( (q \in \mathbb{Z}^-) \).

Then, the family of the level sets of \( f \) defines a singular foliation on \( O \). The
singularities are the critical points of \( f \). We see that each level set contains at
most one critical point. We can also observe that for each \( z \in \mathbb{R} \) the endset
pair \((\mathcal{E}_z, \mathcal{E}_z^*)\) of the level set \( f^{-1}(z) \) is identified with: \( (\mathcal{E} - \mathbb{Z}, \mathcal{E}^*) \times \{z\} \)
if \( |z| \leq 1 \), \( (\mathcal{E}^+ - \mathbb{Z}^+, \mathcal{E}^*^+) \times \{z\} \) if \( z > 1 \), and \( (\mathcal{E}^- - \mathbb{Z}^-, \mathcal{E}^*^-) \times \{z\} \) if \( z < -1 \).

By the self-similarity property, all of these are homeomorphic to \((\mathcal{E} - \mathbb{Z}, \mathcal{E}^*)\).

As the next step, we define

\[
C = \bigcup_{q_n \in \mathbb{Z}^+} \{q_n\} \times \left[-1 - \frac{1}{n}, \infty\right) \cup \bigcup_{q_n \in \mathbb{Z}^-} \{q_n\} \times \left(-\infty, 1 - \frac{1}{n}\right]
\]

and

\[
\Omega = O - C.
\]

Then, by Propositions 3.1 and 3.2 we see that \( \Omega \) is diffeomorphic to \( \mathbb{R}^3 \). Each
level set \( L_z = f^{-1}(z) \cap \Omega \) of \( f|_{\Omega} \) is obtained from \( f^{-1}(z) \) by deleting the
points of intersection with \( C \). Since all the critical points of \( f \) are removed
by this deletion, every \( L_z \) is now a non-singular smooth surface. Let \( \mathcal{G} \) be the
foliation on \( \Omega \) thus obtained. Here, observe that if the point of intersection
of \( f^{-1}(z) \) and \( \{q\} \times I \) \( (q \in \mathbb{Z}) \) and \( I \) is either \([-1 - \frac{1}{n}, \infty) \) or \((-\infty, 1 - \frac{1}{n}]\)
not a critical point, then the deletion yields one puncture (or, one planar end) on \( f^{-1}(z) \), while if the point of intersection is a critical point, then the
deletion yields two punctures (or, two planar ends). Now, let \( Z_z \) be the set
of all ends of \( L_z \) newly produced by these deletions. Then, the endset pair
of \( L_z \) is expressed as \((\mathcal{E}_z \cup Z_z, \mathcal{E}_z^*)\), where \((\mathcal{E}_z, \mathcal{E}_z^*)\) is the endset pair of \( f^{-1}(z) \).

Since, as remarked above, each \( f^{-1}(z) \) contains at most one critical point,
it follows from the property (2) of \( f \) that \( Z_z \) is identified with: \( Z \) if \( |z| \leq 1 \),
the union of \( Z^+ \) and \( F_z \) if \( z > 1 \), the union of \( Z^- \) and \( F_z \) if \( z < -1 \), where \( F_z \)
was a (possibly empty) finite subset of \( \mathbb{R}^2 - X \). (Supplementary explanation:
If \( z \geq 2 \), \( f^{-1}(z) \) does not intersect \( \{q\} \times (-\infty, 1 - \frac{1}{n}] \) for any \( q \in \mathbb{Z}^- \). So,
in this case, \( F_z \) is either a singleton or empty depending on whether there
exists a critical point on \( f^{-1}(z) \cap \{q\} \times [-1 - \frac{1}{n}, \infty) \) for some \( q \in \mathbb{Z}^+ \). If
\( 1 < z < 2 \), we see that \( f^{-1}(z) \) intersects \( \{q\} \times (-\infty, 1 - \frac{1}{n}] \) for at most
finitely many \( q \in \mathbb{Z}^- \).) Therefore, \((\mathcal{E}_z \cup Z_z, \mathcal{E}_z^*, Z_z, \infty)\), is identified with:
\((\mathcal{E}, \mathcal{E}^*, Z, \infty) \times \{z\} \) if \( |z| \leq 1 \), \((\mathcal{E}^+ \cup F_z, \mathcal{E}^*^+, Z^+ \cup F_z, \infty) \times \{z\} \) if \( z > 1 \), and
\((\mathcal{E}^- \cup F_z, \mathcal{E}^*^-, Z^- \cup F_z, \infty) \times \{z\} \) if \( z < -1 \).
Lemma 4.1. If $F$ is a finite subset of $\mathbb{R}^2 - X$, then there is a homeomorphism $h : \mathcal{E} \cup F \to \mathcal{E}$ such that $h$ is the identity on $\mathcal{E}^*$ and that $h(Z \cup F) = Z$.

Proof. Let $F = \{x_1, \cdots, x_r\}$. Take a point $p$ in $\mathcal{E}^*$ and any sequence $\{p_i\}_{i=1}^{\infty}$ in $Z$ converging to $p$. We define a bijection $h : \mathcal{E} \cup F \to \mathcal{E}$ by: $h(x_k) = p_k$ for $k = 1, \cdots, r$, $h(p_i) = p_{r+i}$ for $i \geq 1$, and $h$ is the identity otherwise. Then, the continuity of $h$ easily follows. □

By this lemma and the self-similarity property, the 4-tuple $(\mathcal{E}_z \cup z, \mathcal{E}_z^*, Z_z, \infty)$ for the leaf $L_z$ is homeomorphic to $(\mathcal{E}, \mathcal{E}^*, Z, \infty)$ for every $z \in \mathbb{R}$. Hence, we can conclude that all the leaves $L_z$ of $\Omega$ is diffeomorphic to $\Sigma$.

As the final step, we will transform the foliation $(\Omega, \mathcal{G})$ so that the resulting foliation satisfies the property (P). To do so, take an arbitrary point $(x_0, y_0)$ from $\mathbb{R}^2 - X$ and put $V = \{(x, y, z) \in \mathbb{R}^3 \mid (x-x_0)^2 + (y-y_0)^2|z| < 1\}$. Next, choose any diffeomorphism $\Psi : \mathbb{R}^3 \to V$ which preserves leafwise the vertical foliation $dx = dy = 0$. Then, we can see that the foliation $\Psi(\mathcal{G})$ on $\Psi(\Omega)$ satisfies the property (P) in §2, while $\Psi(\Omega)$ is unbounded in the direction of $z$. Therefore, by Proposition 2.5, we obtain a complete closed uni-leaf foliation on $\mathbb{B}^3$ with leaves diffeomorphic to $\Sigma$. This completes the proof of Theorem 1.2. □

Examples of surfaces with the self-similarity property. In the case of planar surfaces, $\mathcal{E}^*$ and $Z$ are empty, hence, to check the self-similarity, we have only to show that $\mathcal{E}_1 \cup_{e_1} \mathcal{E}_2$ is homeomorphic to $\mathcal{E}$ for some $e \in \mathcal{E}$, where $(\mathcal{E}_i, e_i)$, $i = 1, 2$, are copies of $(\mathcal{E}, e)$. The following surfaces satisfy such a property: $\mathbb{R}^2$, $\mathbb{R}^2$ minus a discrete closed infinite set, $\mathbb{R}^2$ minus a Cantor set, and $S^2$ minus a Cantor set.

In the case of nonplanar surfaces, there are also many examples. Here, we give one family of surfaces $\Sigma(r)$, $r \in \mathbb{N}$ (Example (2) in §1 is $\Sigma(1)$). The endset pair $(\mathcal{E}, \mathcal{E}^*)$ of $\Sigma(r)$ is described as follows:

\[
\mathcal{E} = \{e, e_{i_1}, e_{i_1i_2}, \cdots, e_{i_1i_2\cdots i_r} \mid i_k \in \mathbb{N}, 1 \leq k \leq r \},
\]

\[
\mathcal{E}^* = \{e, e_{i_1}, e_{i_1i_2}, \cdots, e_{i_1i_2\cdots i_{r-1}} \mid i_k \in \mathbb{N}, 1 \leq k \leq r - 1 \},
\]

\[
Z = \{e_{i_1i_2\cdots i_r} \mid i_k \in \mathbb{N}, 1 \leq k \leq r \}.
\]

Let $\mathcal{E}(\ell)$ denote the $\ell$-th derived set of $\mathcal{E}$. Then, for $1 \leq \ell \leq r - 1$,

\[
\mathcal{E}(\ell) = \{e, e_{i_1}, e_{i_1i_2}, \cdots, e_{i_1i_2\cdots i_{r-\ell}} \mid i_k \in \mathbb{N}, 1 \leq k \leq r - \ell \},
\]

and $\mathcal{E}(r) = \{e\}$. For each $1 \leq k \leq r$, $e_{i_1i_2\cdots i_k}$ converges to $e_{i_1i_2\cdots i_{k-1}}$ as $i_k \to \infty$ while $i_1, i_2, \cdots, i_{k-1}$ being fixed, and $e_{i_1}$ converges to $e$ as $i_1 \to \infty$.
Now, put
\[ E^+ = \{ e, e_{i_1}, e_{i_1i_2}, \ldots, e_{i_1i_2\cdots i_r} \mid i_1 \text{ is even and } i_2, \ldots, i_r \text{ are arbitrary} \}, \]
\[ E^- = \{ e, e_{i_1}, e_{i_1i_2}, \ldots, e_{i_1i_2\cdots i_r} \mid i_1 \text{ is odd and } i_2, \ldots, i_r \text{ are arbitrary} \}, \]
\[ E^{++} = E^+ \cap E^*, \ E^{--} = E^- \cap E^*, \ Z^+ = E^+ \cap Z, \ Z^- = E^- \cap Z, \ e^+ = e^- = e, \]
and \( h = id : E^+ \vee e^+ = e^- E^- \rightarrow E. \) Then, the 4-tuple \((E, E^*, Z, e)\) of the surface \( \Sigma(r) \) satisfies the self-similarity.

**Figure 2.** \( \Sigma(2) \)

**Question.** List up all the open orientable surfaces whose endsets satisfy the self-similarity property.

**Question.** Can a surface which does not satisfy the self-similarity property be realized as a leaf of a uni-leaf foliation on \( \mathbb{B}^3 \)?
5. Higher dimensional leaves

In this section we consider the case of higher dimensional leaves. We give two results. The first one is the following.

**Theorem 5.1.** Let $M$ be a simply connected open $n$-manifold, $n \geq 3$, with a smooth foliation $\mathcal{F}$ by leaves diffeomorphic to $\mathbb{R}^{n-1}$. Then, $\mathcal{F}$ is smoothly conjugate to a complete closed (and necessarily, uni-leaf) foliation on $\mathbb{B}^n$.

This result is kindly taught by the referee to the authors.

**Proof.** Let $M$ and $\mathcal{F}$ be as in the hypothesis of the theorem. Then, it follows from the deep result of Palmeira [14] that there are an open set $\Omega$ of $\mathbb{R}^n$ diffeomorphic to $\mathbb{R}^n$ and a diffeomorphism $h : M \to \Omega$ such that $h(\mathcal{F})$ coincides with the restriction to $\Omega$ of the foliation of $\mathbb{R}^n$ by horizontal hyperplanes. Here, we may assume that $\Omega$ satisfies the property (1) of Proposition 2.5. In fact, if $\text{pr}_n(\Omega)$ is bounded, it suffices to replace $\Omega$ with $(\text{id}_{\mathbb{R}^{n-1}} \times \varphi)(\Omega)$, where $\varphi$ is an arbitrary diffeomorphism from the open interval $(\inf \text{pr}_n(\Omega), \sup \text{pr}_n(\Omega))$ to $\mathbb{R}$. Clearly, $h(\mathcal{F})$ satisfies the property (2) of Proposition 2.5. Thus, by Proposition 2.5, there exists a diffeomorphism $\Phi$ from $\Omega$ to $\mathbb{B}^n$ such that $\Phi(h(\mathcal{F}))$ is a complete closed foliation on $\mathbb{B}^n$. □

Let $\mathbb{B}^n$ denote the closed unit $n$-ball, and $\text{pr}_i$ the projection from a product space to its $i$-th factor. The next result in this section is the following.

**Theorem 5.2.** Let $n \geq 3$. Suppose that $F$ is a connected compact $(n-1)$-dimensional smooth submanifold of $\mathbb{B}^{n-1} \times \mathbb{R}$ such that $F \cap (\partial \mathbb{B}^{n-1} \times \mathbb{R}) = \partial \mathbb{B}^{n-1} \times \{0\} = \partial F$ and that $F$ is transverse to $\partial \mathbb{B}^{n-1} \times \mathbb{R}$ at $\partial F$. Let $E$ be a closed subset of $F$ satisfying that

1. $F - E$ is connected,
2. $E$ contains $\partial F$, and that
3. there exists a neighborhood $U$ of $E$ in $F$ such that $\text{pr}_1 : \mathbb{B}^{n-1} \times \mathbb{R} \to \mathbb{B}^{n-1}$ maps $U$ diffeomorphically to $\text{pr}_1(U)$ and that $\text{pr}_1^{-1}(\text{pr}_1(U)) \cap F = U$.

Then, there is a codimension 1 complete closed smooth foliation of $\mathbb{B}^n$ with a leaf diffeomorphic to $F - E$.

**Proof.** The proof is essentially the same as the one in the surface case. Let $n$, $F$ and $E$ be as above. We take a Morse function $f : (\mathbb{B}^{n-1} - \text{pr}_1(E)) \times \mathbb{R} \to \mathbb{R}$ so that

1. $f = \text{pr}_2$ on $(\mathbb{B}^{n-1} - \text{pr}_1(E)) \times [(\infty, -1] \cup [1, \infty)]$, and that
2. $0$ is a regular value of $f$ with $f^{-1}(0) = F - E$. 

Next, we take an exhausting sequence \( \{K_i\} \) of codimension 0 compact submanifolds in \( \mathbb{B}^{n-1} - \text{pr}_1(E) \), and a family of injective smooth paths \( c_p : [0, \infty) \to (\mathbb{B}^{n-1} - \text{pr}_1(E)) \times \mathbb{R}, p \in \text{Crit}(f) \), satisfying the same six conditions with the ones in §3. Then, \( M = (\mathbb{B}^{n-1} - \text{pr}_1(E)) \times \mathbb{R} - \bigcup_{p \in \text{Crit}(f)} c_p([0, \infty)) \) is diffeomorphic to \( (\mathbb{B}^{n-1} - \text{pr}_1(E)) \times \mathbb{R} \), and \( \Omega = M \cup (\mathbb{B}^{n-1} \times (2, \infty)) \) is diffeomorphic to \( \mathbb{B}^n \). Finally, by exactly the same argument given in §3 we complete the proof of Theorem 5.2.

\[ \square \]

**Remark.** For example, we may take as \( E - \partial F \) the Whitehead continuum, the Menger sponge, and so on.

6. **Higher codimensions**

**Proposition 6.1.** Let \( q \) and \( q' \) be positive integers such that \( 1 \leq q < q' \). Given a connected \( p \)-dimensional manifold \( L \), if there is a codimension \( q \) complete closed smooth foliation on \( \mathbb{B}^{p+q} \) with a leaf diffeomorphic to \( L \), then, there is a codimension \( q' \) complete closed smooth foliation on \( \mathbb{B}^{p+q'} \) with a leaf diffeomorphic to \( L \).

**Proof.** Suppose \( F \) is a codimension \( q \) complete closed smooth foliation on \( \mathbb{B}^{p+q} \) with a leaf diffeomorphic to \( L \). Then the foliation on \( \mathbb{B}^{p+q} \times \mathbb{B}^{q'-q} \) defined by \( F \times \{z\} \ (F \in F, z \in \mathbb{B}^{q'-q}) \) as leaves is a codimension \( q' \) complete closed smooth foliation and has a leaf diffeomorphic to \( L \). Since \( \mathbb{B}^{p+q} \times \mathbb{B}^{q'-q} \) is diffeomorphic to \( \mathbb{B}^{p+q'} \) by a quasi-isometric diffeomorphism, the conclusion follows.

Combining Proposition 6.1 with Theorem 1.1 and Theorem 5.2 we obtain

**Theorem 6.2.** Let \( L \) be \( \Sigma \) in Theorem 1.1 or \( F - E \) in Theorem 5.2, and let \( p = \dim L \). Then, for any positive integer \( q \), there is a codimension \( q \) complete closed smooth foliations on the open unit ball \( \mathbb{B}^{p+q} \) having \( L \) as a leaf.

Similarly, by Proposition 6.1 and Theorem 1.2 we have

**Theorem 6.3.** Let \( L \) be \( \Sigma \) in Theorem 1.2. Then, for any positive integer \( q \), there is a codimension \( q \) complete closed smooth uni-leaf foliation on the open unit ball \( \mathbb{B}^{3+q} \) having \( L \) as a leaf.

**Acknowledgements.** The authors would like to thank the referee for many invaluable suggestions, which have made the manuscript much more readable. They also thank Ryoji Kasagawa and Atsushi Sato for their interest in our work and many helpful comments.
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