Monopole Loop Distribution and Confinement in SU(2) Lattice Gauge Theory

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Abstract

The abelian-projected monopole loop distribution is extracted from maximal abelian gauge simulations. The number of loops of a given length falls as a power nearly independent of lattice size. This power increases with $\beta = 4/g^2$, reaching five around $\beta = 2.85$, beyond which, it is shown, loops any finite fraction of the lattice size vanish in the infinite lattice limit.

A strong correlation has been established between confinement and abelian monopoles extracted in the maximal abelian gauge. The entire SU(2) string tension appears to be due to the monopole portion of the projected U(1) field\cite{1, 2}. Confinement appears to require the presence of large loops of monopole current of order the lattice size, possibly in a percolating cluster. As $\beta = 4/g^2$ is raised, the finite lattice theory undergoes a deconfining phase transition which is coincident with the disappearance of large monopole loops\cite{3}. This transition is interpreted as a finite-temperature phase transition, one which exists if one of the four lattice dimensions is kept finite as the others become infinite, but which disappears in the 4-d symmetric infinite lattice limit.

In the U(1) lattice gauge theory itself, monopoles have also been identified as the cause of the phase transition\cite{4}. However, in this theory, monopoles are interpreted as strong-coupling lattice artifacts which do not survive the continuum limit. The continuum limit is non-confining.

Assuming the presence of large monopole loops is a necessary condition for confinement, in order for the SU(2) theory to confine in the continuum limit it is necessary for some abelian monopoles to survive this limit, i.e. to exist as physical objects. Evidence has been presented of scaling of the monopole density which would make this so\cite{5}. However, it is not sufficient just to have some monopoles survive this limit; one needs large loops of a finite physical size to survive. Small loops of finite size on the lattice, which are by far the most abundant, will shrink to zero physical size as the lattice spacing goes to zero, becoming irrelevant. It is believed that to cause confinement, monopole loops must be at least as large as the relevant Wilson loops of nuclear size, and may need to span the entire space.

Consider a large but finite universe, represented as an $N^4$ lattice with lattice spacing $a$. The strong interactions should not care whether the universe is finite or infinite, so long as it is much larger than a hadron. One can then take the continuum limit as $a \to 0$, \ldots
$N \to \infty$, simultaneously, holding $Na$ constant at the universe size. It is then clear that the size of any object that is to remain of finite physical size in the continuum limit must also become infinitely large in lattice units, with linear dimension proportional to $N$, i.e., some finite fraction of the full lattice size. It would therefore appear that at the very least, monopole loops some finite fraction of the lattice size must survive the continuum limit if it is to be confining, and probably loops at least as large as the lattice itself. Recently, it was shown that if the plaquette is restricted to be greater than 0.5, the loop distribution function falls so fast that no monopole loops any finite fraction of the lattice size survive in the large lattice limit for any value of $\beta$ \cite{6}. Here it will be shown that for the standard Wilson action the same is true if $\beta > 2.85$.

By studying the monopole loop size distribution function, one can tell how quickly the probability of finding loops of increasing size decreases with loop size. This function turns out to have very little finite lattice size dependence for loops of length less than twice the lattice size. Thus one can fairly confidently extrapolate this function, which appears to be a simple power law, to the infinite lattice, from which the probability of finding loops of finite physical size can be determined. Due to the dependence of the power on $\beta$, it is shown below that the probability of finding loops whose length is a finite fraction of the lattice size vanishes in the large lattice limit for all $\beta > 2.85$.

Gauge configurations from the simulations were transformed to maximal abelian gauge using the adjoint field technique \cite{7}. Abelian monopole currents were then extracted using the DeGrand-Toussaint procedure \cite{8}. Sample sizes ranged from 500 configurations for the $20^4$ lattices (1500 at $\beta = 3.1$), to 5000 for the $8^4$ lattices, always after 1000 equilibration sweeps.

Define the loop distribution function, $p(l)$, as the probability, normalized per lattice site, of finding a monopole loop of length $l$ on a lattice of any size. The probability of finding a loop of length $N$ or larger on an $N^4$ lattice is given by $N^4 I(N)$ where $I$ is the integrated loop distribution function

$$I(M) = \int_M^{4N^4} p(l)dl,$$

(1)

where, since $N$ will be taken large, the discrete distribution has been replaced by a continuous one. The upper limit comes from the fact that each dual link can contain at most two abelian monopoles. To get confinement from the monopole mechanism, at least some finite fraction of lattices would have to contain loops of length order $N$ or larger. (Some would argue that loops of size $N^2$ or larger might be necessary due to loop crumpling). Conversely, if

$$\lim_{N \to \infty} N^4 I(N) = 0$$

(2)

then there will be no loops of length $N$ or larger on the $N^4$ lattice in the large lattice limit, and confinement from the monopole mechanism will not be possible.

In Fig. 1, $\log_{10} p(l)$ is plotted vs. $\log_{10}(l)$ for various $\beta$ and lattice sizes. The data are consistent with a power law, $p(l) \propto l^{-q}$, for loops up to around length $l = 1.5N$ (the size 4 loops, which fall well below the trend are excluded from the fits). This is consistent with the results of \cite{3} where the loop distribution was studied over a narrower $\beta$ range. The larger the lattice, the further the power law is valid before some deviation occurs at large $l$. Also note that the $12^4$ and $20^4$ data are virtually identical for loops up to
size 30 or so, and even the $8^4$ are hardly different. Linear fits were made for loop sizes in the range 6 to 1.1$N$ (only the $12^4$ fits are shown for clarity). For larger loop sizes, occasionally zero instances of a particular size was observed. Zeros cannot be plotted on the logarithmic scale, but if ignored the data will be biased upward. A moving average was used in this circumstance to properly account for the zero observations.

The deviations from linearity for large loops can be easily understood as a finite size effect coupled with the periodic boundary condition. For loops longer than about $1.5N$, there is a significant probability of reconnection through the boundary. This makes a would-be large loop terminate earlier than it would on an infinite lattice. Thus, on a finite lattice there must be a deficit of very large loops, and an excess of mid-size loops due to reconnection. At $\beta = 2.4$ and 2.5 this is especially apparent. The linear trend continues further for the $20^4$ lattice than for the $12^4$, which itself continues further than the $8^4$. At some point starting around $1.5N$, but not readily apparent until about $4N$, there is a bulge of excess probability, followed eventually by a steep drop that falls below the linear trendline. Often the data cuts off before it falls below, but the fact that no loops larger than those plotted occurred can be used to infer that the probability distribution must eventually fall below the trendline. For instance if one assumes that very large loops follow the trendline for the $12^4$ data at $\beta = 2.4$, then one can calculate that 3.5 instances of loops longer than 1250 lattice units should have been seen in the sample. The fact than none occurred implies that the data most likely does fall below the trend line for very large loops, and certainly could not remain significantly above it.

Because the power law trend continues further the larger the lattice and the deviations always occur for $l > N$, it seems quite reasonable to assume that on the infinite lattice one would have a pure power law. It is difficult to imagine what length other than the lattice size could set the scale for a change in behavior at extremely large loop sizes beyond those measured here. In addition, since the small loop data are nearly independent of lattice size, it would seem reasonable that the power, which is completely determined by the small loops, must also be essentially the same on the infinite lattice as observed here for the $12^4$ or $20^4$ lattices (some slight variations are visible on the $8^4$). Assuming this, one can easily predict the point at which condition (2) becomes satisfied, namely $q > 5$. For $q > 5$ the probability of having a monopole loop with length equal to any finite fraction of the lattice size $N$, or larger, vanishes in the large lattice limit, whereas for $q < 5$ the same becomes overwhelmingly likely. This is because for $q > 5$ not only is condition (2) satisfied, but also

$$\lim_{N \to \infty} N^4 I(N/M) = 0$$  \hspace{1cm} (3)$$

for any finite $M$.

The power $q$ is plotted as a function of $\beta$ in Fig. 2. A definite rising trend is observed, in sharp contradiction to the conclusion of ref. [9], where a constant value of $q$ was inferred from runs over a rather narrow $\beta$ range (2.3 to 2.5). It is seen that $q$ apparently passes 5 around $\beta = 2.85$. For any $\beta$ beyond this, including the continuum limit, $\beta \to \infty$, there will be no loops any finite fraction of the lattice size in the infinite lattice limit. A small residual finite size dependence in $q$ is possible, which could shift the infinite lattice critical $\beta$ to 2.9 or possibly 3.0, but it is hard to picture how this could prevent $q$ from ever reaching 5, which would be necessary to have large loops survive the continuum limit.
Error bars in Fig. 2 were obtained from the least squares fits. If data for different loop sizes were highly correlated, then these errors could be underestimated. As a test, correlation coefficients were computed between the numbers of each size loop occurring on a lattice, for samples of 1000 $12^4$ lattices at both $\beta = 2.6$ and 2.9. The degree of correlation was in all cases very small, with coefficients, $r$, averaging 0.02 (with a maximum of 0.04) at $\beta = 2.6$ and 0.003 at $\beta = 2.9$. These fall within the expected noise level of 0.03 for these sample sizes, and are certainly small enough to justify ignoring correlations in the fits. The autocorrelation in Monte Carlo time was also computed and found to fall below noise after two Monte Carlo sweeps.

One might also ask with what confidence one can reject the hypothesis $q \leq 5$ at $\beta = 3.0$ or 3.1. At $\beta = 3.1$ on the $20^4$ lattice, $q$ was found to be $7.65 \pm 0.32$. If one assumes $q = 5$ instead, then the number of loops of size 12 links and larger that should have occurred in the sample (based on the number of six-link loops) is 285, whereas the actual number was 35. The possibility of this occurring is, by raw Poisson statistics, less than $10^{-76}$. At $\beta = 3.0$ the same analysis rejects the hypothesis at the level $10^{-59}$. This shows that on these lattices it is virtually certain that $q$ exceeds 5, even if some latitude is allowed for the modest Monte Carlo time correlations.

It is interesting to note that the above conclusions are insensitive to the precise behavior of the distribution for $l > N$. If $p(l)$ follows a different power law with exponent $q'$ for $l > N$, $q'$ could be as small as unity and still condition (2) will hold. For instance, if $q = 5 + \epsilon$ and $q' = 1$ then

$$p(l) = \begin{cases} l^{-5-\epsilon} & , \ l \leq N \\ N^{-4-\epsilon}l^{-1} & , \ l > N \end{cases},$$

so

$$I(N) \propto N^{-4-\epsilon} \int_{N}^{8N^4} l^{-1} dl,$$

and $N^4I(N) \propto N^{-\epsilon} \ln(8N^3)$ which still vanishes as $N \to \infty$. Thus the presence of large enough loops to cause confinement is, paradoxically, controlled almost entirely by the distribution function for loops of length smaller than $N$, because this determines the base probability for the case $l = N$. This observation makes the above result far more robust, since the distribution for $l < N$ is seen to always follow a pure power law which is almost independent of lattice size. It was the distribution for $l > N$ which was somewhat less certain, but all that one needs to know about this part of the distribution is that it falls at least as fast as $l^{-1}$, not a very stringent requirement. As before, the loop size cutoff for the condition can be taken to be $N/M$ instead of $N$, where $M$ is some finite number. This further reduces the likelihood of the finite lattice size affecting the important small-loop part of the distribution.

On our lattices, as in many previous studies, confinement appears to occur at couplings for which loops that wrap completely around the periodic lattice are common, and the theory is deconfined when such loops are absent. This is illustrated in Fig. 3, where the probability that the largest loop has a non-zero winding number is plotted vs. $\beta$, along with the Polyakov loop. The rather sudden rise from zero in winding probability is coincident with the point of largest slope in the Polyakov loop, which signals deconfinement. For the $20^4$ lattice, the average Polyakov loop itself is not a very sensitive test of confinement, but moments of the Polyakov loop become decidedly non-Gaussian at
this point. For instance both the first (absolute value) and fourth moment are consistent with Gaussian for $\beta \leq 2.7$, but not for $\beta > 2.7$ where the distribution is not peaked at zero and the theory is deconfined. If loops in addition to the largest loop were also tested, the net winding probability would likely become very close to unity when deeply into the confining region. Loops which wind the lattice occur in pairs, with a Dirac sheet stretched between them. Once a winding pair exists, the loops can drift apart without increasing their length, stretching the Dirac sheet to cover an arbitrarily large section of lattice and incurring no additional energy penalty in doing so. Such a Dirac sheet could easily disrupt the values of Polyakov loops passing through it only once, resulting in a random, i.e. confined, Polyakov loop average. Very large non-winding loops would also have large Dirac sheets having the same effect. However, the winding configurations, having shorter loops, have a lower energy, so would appear first at the periodic lattice deconfinement transition.

Since large loops must disappear when $q$ hits 5, if one accepts the observation that large loops are associated with confinement, one would have to conclude that the infinite 4-d symmetric lattice must undergo a deconfining phase transition at a finite value of $\beta$ around 2.9, rather than at $\beta = \infty$ as is usually assumed. Of course, one could instead give up the link between abelian monopoles and confinement, despite the strong evidence in favor of a connection. Since large abelian monopole loops have been shown to be responsible for most if not all of the string tension, if the theory still confines when they are absent it will have a much smaller string tension. In this case the theory would confine in the continuum, but could be quite different in detail from the usual lattice theory in the crossover region of the Wilson action, since the latter would be highly contaminated with monopoles, seen here as essentially lattice artifacts.

Let us return to the possibility that the continuum non-abelian pure-gauge theory really doesn’t confine, an idea which has been suggested before [10, 11]. The renormalized coupling could have an infrared-stable fixed point, and stop increasing at some distance scale. The heavy-quark potential would be a logarithmically-modified Coulomb potential, and the gluons would be massless. How is one then to explain confinement in the real world? It may be that the real-world confinement that we see is a manifestation of chiral symmetry breaking [14, 12]. Even in a nonconfining theory, chiral symmetry can break spontaneously if the coupling is strong enough. Instantons will also likely play a role here. Once a chiral condensate is established, it can produce a confining force by expelling strong color fields; i.e. if the condensate abhors external color fields, hadrons will expel some condensate from their immediate vicinity, creating a region of higher vacuum energy proportional to their volume, a sort of chiral-expelled bag. Stretching of this bag would result in a linear potential. The energy density of the bag which would follow the movement of quarks may also be responsible for the quarks’ dynamical mass. This picture is supported by the observation that $\langle \bar{\psi} \psi \rangle$ is lowered in the neighborhood of a color source [3], indicating some expulsion of condensate. Similar ideas are contained in chiral quark models [14] where a polarized Dirac sea is responsible for the binding of the quarks in a baryon, and also in the instanton liquid model [15]. Both of these models are able to compute with fair accuracy a large number of low-energy properties of hadrons, and neither has an absolutely confining potential. Further details on this scenario and justifications are given in [1] and references therein.
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References

[1] A.S. Kronfeld, M.L. Laursen, G. Schierholz, and U.J. Weise, Phys. Lett. B 198 (1987) 516; T. Suzuki and I. Yotsuyanagi, Phys. Rev. D 42 (1990) 4257; T. Suzuki, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 176.

[2] S. Ejiri, S-i. Kitahara, Y. Matsubara, and T. Suzuki, Phys. Lett. B 343, (1995) 304; J.D. Stack, S.D. Neiman, and R.J. Wensley, Phys. Rev. D 50 (1994) 3399.

[3] V.G. Bornyakov, V.K. Mitrjushkin, and M. Müller-Preussker, Phys. Lett. B 284 (1992) 99; T. Suzuki et. al., Phys. Lett. B 347 (1995) 375.

[4] J.D. Stack and R. Wensley, Nucl. Phys. B 371 (1992) 597; Phys. Rev. Lett. 72 (1994) 21; W. Kerler, C. Rebbi, and A. Weber, Nucl. Phys. B (Proc. Suppl.) 47 (1996) 667.

[5] V.G. Bornyakov et. al., Phys. Lett. B, 261 (1991) 116.

[6] M. Grady, SUNY Fredonia report SUNY-FRE-98-01, hep-lat/9801016, (unpublished).

[7] K. Bernstein, G. Di Cecio, and R. W. Haymaker, Phys. Rev. D 55 (1997) 6730.

[8] T.A. DeGrand and T. Toussaint, Phys. Rev. D 22 (1980) 2478.

[9] A. Hart and M. Teper, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 497; Nucl. Phys. B (Proc. Suppl.) 63 (1998) 522.

[10] M. Grady, Z. Phys C, 39 (1988) 125.

[11] A. Patrascioiu, E. Seiler, and I.O. Stamatescu, Nuovo Cimento D 11 (1989) 1165; A. Patrascioiu, E. Seiler, V. Linke, and I.O. Stamatescu, Nuovo Cim. 104B (1989) 229.

[12] M. Grady, Nuovo Cim. 105A (1992) 1065.

[13] W. Feilmaier, M. Faber, and H. Markum, Phys. Rev. D, 39 (1989) 1409.

[14] Chr.V. Christov et. al., Prog. Nucl. Part. Phys. 37 (1996) 91, and references therein.

[15] E.V. Shuryak, Nuclear Phys. B 203 (1982) 93, 116, 140; 214 (1983) 237; D.I. Diakonov and V. Yu. Petrov, Nucl. Phys. B 245 (1984) 259, 272 (1986) 457, hep-lat/9810037.
Figure Captions

FIG. 1. Log-log plots of loop probability vs. loop length: (a) $\beta = 2.4$ (upper graph, right scale), $\beta = 2.5$ (left scale, shifted for clarity); (b) from upper to lower, data series are $\beta = 2.6, 2.7, 2.8, 2.9, 3.0, 3.1$; linear fits are to upper region of the $12^4$ data, as explained in text.

FIG. 2. Power, $q$, characterizing the decrease of loop probability with loop length, vs. $\beta$. Lines are drawn to guide the eye.

FIG. 3. Polyakov loop (filled symbols) and winding probability of largest loop (open symbols) vs. $\beta$. 
