Toward a Complete Analysis of the Global Structure of Kerr-Newman Spacetime

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Abstract

An attempt is made to supplement Carter’s partial investigation of the global structure of Kerr-Newman spacetime on the symmetry axis. Namely, the global structures of $\theta = \text{const.}$ timelike submanifolds of Kerr-Newman metric starting from the symmetry axis all the way down to the equatorial plane are studied by introducing a new time coordinate slightly different from the usual Boyer-Lindquist time coordinate. It turns out that the maximal analytic extension of $\theta = \theta_0 \ (0 \leq \theta_0 < \pi/2)$ submanifolds is the same as that of the symmetry axis first studied by Carter whereas $\theta = \pi/2$ equatorial plane has the topology identical to that of the Reissner-Nordstrom spacetime. General applicability of this method to Kerr-Newman-type black hole solutions in other gravity theories is discussed as well.

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I. Introduction

Generally speaking, a complete analysis of the global structure of full, 4-dim. spacetime solution of Einstein equations is a formidable task. In certain special cases, however, there are intrinsically singled-out timelike 2-dim. submanifold, “T” of the full, 4-dim. manifold, “M” which can be simply analyzed. For instance, in the Schwarzschild or Reissner-Nordstrom (RN) solution\(^1\), the spacetime has topology of \(M = R^2 \times S^2\) because of the spherical symmetry and we can think of each point of the timelike 2-dim. submanifold \(T = R^2\) as representing a two-sphere \(S^2\) whose radius is the \(r\)-value at that point. In these two spherically-symmetric cases, the global structures of the submanifolds \(T\) exactly mirror those of the full 4-dim. manifold \(M\). In the case of Kerr-Newman solution\(^2\), however, this is not the case. There are incomplete, inextendable geodesics in the Kerr-Newman spacetime which do not lie in any totally geodesic, timelike 2-dim. submanifold \(T\). Despite this fact, by adopting the new coordinates introduced by Boyer-Lindquist\(^3\) (which can be thought of as the generalization of the Schwarzschild coordinates), Carter\(^3\) was able to demonstrate the global structure of a timelike 2-dim. submanifold of the Kerr-Newman solution. And the 2-dim. submanifold is the \((t, r)\)-plane in Boyer-Lindquist coordinates which represent the “symmetry axis” \((\theta = 0)\) of the Kerr-Newman spacetime. Then by applying the methods of Finkelstein\(^4\) and Kruskal\(^5\), he obtained the maximal analytic extension (depicted in the Carter-Penrose diagram\(^1\)) of the geometry on the symmetry axis. He then went on to conjecture that the analytic extension of the full, 4-dim. Kerr-Newman manifold would have basically the same topology except that the “ring singularity” at \(\Sigma = r^2 + a^2 \cos^2 \theta = 0\)\(^1\) of the full spacetime cannot be represented in his 2-dim. picture, i.e., in the extended conformal diagram of the symmetry axis\(^3\). In the present work, we would like to supplement Carter’s partial investigation of the global structure of Kerr-Newman spacetime. Namely, we consider the geometry of the “\(\theta = \text{const.}\)” timelike submanifolds of Kerr-Newman spacetime including both the “symmetry axis” \((\theta = 0)\) and the “equatorial plane” \((\theta = \pi/2)\) and examine their global structures. Since the geometry of \(\theta = \theta_0\) (with \(0 < \theta_0 \leq \pi/2\)) submanifolds is essentially 3-dim. whereas that of the symmetry axis \((\theta_0 = 0)\) is \textit{effectively} 2-dim. (since it
has a degeneracy along $\phi$ direction), things get more involved compared to the case of the symmetry axis discussed by Carter. Therefore in order to make the analysis tractable, we shall introduce a “new time coordinate” $\tilde{t}$ slightly different from the usual Boyer-Lindquist$^3$ time coordinate $t$ based on the philosophy that the global structure remains unaffected under coordinate changes. From there one can then apply the same methods of Finkelstein and Kruskal to obtain the maximal analytic extensions of the conformal diagrams of the $\theta = \theta_0$ submanifolds. Then one can readily see that the extended conformal diagrams turn out to be identical to the one obtained by Carter for the cases $0 \leq \theta_0 < \pi/2$ as has been anticipated. For the case $\theta_0 = \pi/2$, i.e., the equatorial plane, however, one can see that the conformal diagram turns out to be that of the RN spacetime$^{1,3}$ explicitly showing the existence of the ring singularity $\Sigma = 0$ (which of course becomes $r = 0$ on $\theta_0 = \pi/2$ equatorial plane) owned by the full, 4-dim. Kerr-Newman manifold. As stated earlier, the peculiarity with the global structure of the symmetry axis is that the 2-dim. metric of the symmetry axis is everywhere analytic and non-singular and hence the ring singularity $\Sigma = 0$ is absent there in its extended conformal diagram although it does exist in full, 4-dim. manifold. As a result, on the symmetry axis, one can pass through the ring singularity (or more precisely, through $r = 0$) and extend to “negative” values of $r$. This possibility causes some trouble since in this region containing the ring singularity, there may exist closed timelike curves which lead to the causality violation as pointed out by Carter$^3$. As we shall see shortly, the maximally extended conformal diagram of the $\theta = \theta_0$ (with $0 < \theta_0 < \pi/2$) submanifolds of Kerr-Newman spacetime remain the same, i.e., still take the same structure as that of the symmetry axis. The extended conformal diagram of the equatorial plane, on the other hand, takes exactly the same structure as that of the RN solution. As a result, it does have the ring singularity at $\Sigma = 0$ (or more precisely at $r = 0$ since $\theta_0 = \pi/2$) and one cannot, on the equatorial plane, extend to negative values of $r$. In short, the result of our analysis of the global structure of $\theta = \theta_0$ (with $0 < \theta_0 \leq \pi/2$) submanifolds supplements that of the global structure of the symmetry axis studied by Carter to bring us a clearer overview of the global structure of the full Kerr-Newman spacetime. And particularly, our study of the
global topology of the equatorial plane confirms the existence of the ring singularity and supports the general belief that the spacetime produced by physically realistic collapse of even nonspherical bodies would be qualitatively similar to the spherical case, i.e., the RN geometry.

II. Global structure of $\theta = \text{const.}$ submanifolds of Kerr-Newman geometry

Now consider the stationary, axisymmetric Kerr-Newman solution of the Einstein-Maxwell equations. The Kerr-Newman metric solution is given in Boyer-Lindquist coordinates as

$$ds^2 = -\left[\frac{\Delta - a^2 \sin^2 \theta}{\Sigma}\right]dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma}dtd\phi + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma}\right] \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$$  \hspace{1cm} (1)

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta \equiv r^2 - 2Mr + a^2 + e^2$ with $M$ being the mass, $a$ being the angular momentum per unit mass and $e$ being the total $U(1)$ charge of the hole. We are now particularly interested in the $\theta = \text{const.}$ timelike surfaces as submanifolds of this Kerr-Newman spacetime. Namely, consider the $\theta = \theta_0$ (0 $\leq \theta_0 \leq \pi/2$) timelike submanifolds of Kerr-Newman spacetime with the metric

$$ds^2 = -\left[\frac{\Delta - a^2 \sin^2 \theta_0}{\Sigma}\right]dt^2 - \frac{2a \sin^2 \theta_0 (r^2 + a^2 - \Delta)}{\Sigma}dtd\phi + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta_0}{\Sigma}\right] \sin^2 \theta_0 d\phi^2 + \frac{\Sigma}{\Delta}dr^2$$  \hspace{1cm} (2)

where $\Sigma = r^2 + a^2 \cos^2 \theta_0$ now. These $\theta = \theta_0$ surfaces have metrics which are literally 3-dim. in structure and possess an off-diagonal component in an intricate way. Thus in order to make the study of global structure of $\theta = \theta_0$ submanifolds tractable, here we consider a coordinate transformation which is defined only on the $\theta = \text{const.}$ timelike submanifolds. Namely consider a transformation from the Boyer-Lindquist time coordinate “$t$” to a new time coordinate “$\tilde{t}$” given by

$$\tilde{t} = t - (a \sin^2 \theta_0) \phi$$  \hspace{1cm} (3)

with other spatial coordinates ($r$, $\phi$) remaining unchanged. Note that the new time coordinate $\tilde{t}$ is different from the old one $t$ only for “rotating” case ($a \neq 0$) and even then only for $\theta_0 \neq 0$. In terms of this “new” time coordinate $\tilde{t}$, the metric of $\theta = \theta_0$ submanifolds becomes
\[ ds^2 = -\frac{\Delta}{\Sigma} d\tilde{t}^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma \sin^2 \theta_0 \left( -\frac{a}{\Sigma} d\tilde{t} + d\phi \right)^2 \]  

\[ = -N^2(r) d\tilde{t}^2 + h_{rr}(r) dr^2 + h_{\phi\phi}(r) [N^\phi(r) d\tilde{t} + d\phi]^2. \]

Remarkably, the metric now takes on the structure of simple “diagonal” ADM’s (2 + 1) space-plus-time split form with the lapse, shift functions and the spatial metric components being given respectively by

\[ N^2(r) = \frac{\Delta}{\Sigma}, \quad N^\phi(r) = -\frac{a}{\Sigma}, \]

\[ h_{rr}(r) = N^{-2}(r), \quad h_{\phi\phi}(r) = \Sigma \sin^2 \theta_0, \quad h_{r\phi}(r) = h_{\phi r}(r) = 0. \]

In terms of this new time coordinate \( \tilde{t} \), therefore, it becomes clearer that the \( \theta = \theta_0 \) submanifolds of Kerr-Newman spacetime have the topology of \( R^2 \times S^1 \) (which was not so transparent in the original Boyer-Lindquist time coordinate \( t \)) and hence becomes better-suited for the study of global structure. That is to say, the global structure of the submanifolds \( T = R^2 \) would mirror that of the full \( \theta = \theta_0 \) submanifolds since each point of \( T = R^2 \) can be thought of as representing \( S^1 \). We would like to add a comment here. Of course it is true that it is the manifold itself that has topology, not the metric. Therefore, regardless of the metrics one chooses, they all describe the same manifold with a single topology. However, the point we would like to make here is that the metric given in new time coordinate \( \tilde{t} \) (eq.(4)) demonstrates more clearly that the \( \theta = \theta_0 \) submanifolds it describes has the topology of \( R^2 \times S^1 \) than the metric in Boyer-Lindquist time coordinate \( t \) (eq.(2)) does.

Firstly, consider the \( \theta_0 = 0, \pi \) timelike submanifolds representing the “symmetry axis” of the Kerr-Newman spacetime with the metric being given by

\[ ds^2 = -(\frac{\Delta}{r^2 + a^2}) dt^2 + (\frac{\Delta}{r^2 + a^2})^{-1} dr^2. \]

Note that this metric of the symmetry axis is effectively 2-dim. since it is degenerate along the \( \phi \) direction. And this diagonal, 2-dim. structure of the metric of the symmetry axis allowed a complete analysis of its global structure as had been carried out by Carter\(^3\).

Secondly, consider the \( \theta_0 = \pi/2 \) surface which represents the “equatorial plane” of Kerr-
Newman spacetime. The metric of this submanifold is obtained in the new time coordinate \( \tilde{t} \) by setting \( \theta_0 = \pi/2 \) in eq.(4)

\[
ds^2 = -N^2(r)d\tilde{t}^2 + N^{-2}(r)dr^2 + r^2[N^\phi(r)d\tilde{t} + d\phi]^2
\]

with the lapse \( N(r) \) and the shift \( N^\phi(r) \) in above (2 + 1)-split form being given by

\[
N^2(r) = \Delta r^2 = \left[1 - \frac{2M}{r} + \left(\frac{a^2 + e^2}{r^2}\right)\right],
\]

\[
N^\phi(r) = -\frac{a}{r^2}.
\]

Finally, note that the metric of the \( \theta = \theta_0 \) \((0 < \theta_0 < \pi/2)\) submanifolds given in eq.(4) are everywhere non-singular including \( r = 0 \) and possess exactly the same causal structure (except for the appearance of ergoregion) as that of the symmetry axis \( (\theta = 0) \). Therefore, the maximal analytic extension of the \( \theta = \theta_0 \) \((0 < \theta_0 < \pi/2)\) submanifolds representing their global structure is essentially the same as that of the symmetry axis first studied by Carter\(^3\). The metric of the equatorial plane \( (\theta_0 = \pi/2) \) given in eq.(6), however, possesses a curvature singularity at \( r = 0 \) as expected (since it is the “ring singularity”, \( r = 0, \theta_0 = \pi/2 \)) whereas it exhibits almost the same causal structure (again except for the presence of the ergoregion) as that of the symmetry axis. As a result, the maximal analytic extension of the equatorial plane is identical to that of the RN spacetime\(^1,3\). Detailed analysis of the maximal analytic extension of the \( \theta = \text{const.} \) submanifolds including the transformations to the Kruskal-type coordinates in which the metric can be cast into the form

\[
ds^2 = \Omega^2(r)(dT^2 + dX^2) + \Sigma \sin^2 \theta_0(N^\phi(r)d\tilde{t} + d\phi)^2
\]

with \( (T, X) \) being the Kruskal-type coordinates and \( \Omega(r) \) being the associated conformal factor and the exposition of Carter-Penrose conformal diagrams will be reported in a separate publication.

And what makes this type of concrete analysis of the global structure possible is the fact that in the new time coordinate \( \tilde{t} \) given in eq.(3), it becomes more apparent that the \( \theta = \text{const.} \) submanifolds of Kerr-Newman spacetime with the metric being given by eq.(4) or (6) has the
topology of $R^2 \times S^1$ which was not so manifest in the old, Boyer-Lindquist time coordinate $t$. Thus it seems now natural to ask the physical meaning of the new time coordinate $\tilde{t}$. To get a quick answer to this question, we go back and look at the coordinate transformation law given in eq.(3) relating the two time coordinates $t$ and $\tilde{t}$. Namely, taking the dual of the transformation law $\delta \tilde{t} = \delta t - (a \sin^2 \theta_0) \delta \phi$, we get

\[
\left( \frac{\partial}{\partial \tilde{t}} \right)^\mu = \left( \frac{\partial}{\partial t} \right)^\mu - \frac{1}{(a \sin^2 \theta_0)} \left( \frac{\partial}{\partial \phi} \right)^\mu,
\]

or

\[
\tilde{\xi}^\mu = \xi^\mu - \frac{1}{(a \sin^2 \theta_0)} \psi^\mu \tag{8}
\]

where $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial \phi)^\mu$ denote Killing fields corresponding to the time-translational and the rotational isometries of the Kerr-Newman black hole spacetime respectively and $\tilde{\xi}^\mu = (\partial/\partial \tilde{t})^\mu$ denotes the Killing field associated with the isometry under the new time translation. Now this expression for the new time translational Killing field $\tilde{\xi}^\mu$ implies that in “new” time coordinate $\tilde{t}$, the time translational generator is given by the linear combination of the old time translational generator and the rotational generator. In plain English, this means that in “new” time coordinate, the action of new time translation consists of the action of old time translation and the action of rotation in opposite direction to $a$, i.e., to the rotation direction of the hole. Thus the new time coordinate $\tilde{t}$ can be interpreted as the coordinate, say, of a frame which rotates around the axis of the spinning Kerr-Newman black hole in opposite direction to that of the hole. Further, by considering the angular velocity, the angular momentum per unit mass (which will be defined concretely later on) and the surface gravity at the event horizon of the hole both in the original Boyer-Lindquist time coordinate $t$ and in the new time coordinate $\tilde{t}$ and then comparing them, one can explore the relative physical meaning between $t$ and $\tilde{t}$ in a more comprehensive manner. Thus in the following we shall do this. As stated above, since the Kerr-Newman spacetime is a stationary, axisymmetric solution, it possesses two Killing fields $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial \phi)^\mu$ associated with the time-translational and rotational isometries, respectively. And it is their linear combination, $\chi^\mu = \xi^\mu + \Omega H \psi^\mu$ which is normal to the Killing horizon of the rotating
Kerr-Newman solution. In addition, normally this is the defining equation of the angular velocity of the event horizon, $\Omega_H^6$. Thus from this equation, we first determine the location of the event horizon and next its angular velocity. Since the Killing horizon is defined to be a surface on which the Killing field $\chi^\mu$ becomes null, in order to find the event horizon, we look for zeros of $\chi^\mu \chi_\mu = 0$. A straightforward calculation shows that the Killing field $\chi^\mu$ becomes null at points where $\Delta = r^2 - 2Mr + a^2 + e^2 = 0$ both in the Boyer-Lindquist time coordinate, $t$ and in the new time coordinate, $\tilde{t}$. Thus we have regular inner and outer horizons at $r^\pm = M \pm \sqrt{M^2 - a^2 - e^2}$ and $r = r_+$ is the event horizon provided $M^2 \geq a^2 + e^2$.

Now we are in a position to compute the angular velocity at this event horizon. The angular velocity of the event horizon is given by

$$
\Omega_H = \left. \frac{d\phi}{dt} \right|_{r^+} = -\frac{g_{t\phi}}{g_{\phi\phi}} \left. \right|_{r^+} = \frac{a}{r_{+}^2 + a^2},
$$

(9)

$$
\tilde{\Omega}_H = \left. \frac{d\phi}{d\tilde{t}} \right|_{r^+} = -\frac{g_{t\phi}}{g_{\phi\phi}} \left. \right|_{r^+} = \frac{a}{r_{+}^2 + a^2 \cos^2 \theta_0} \quad (0 \leq \theta_0 \leq \pi/2)
$$

as measured in the Boyer-Lindquist time coordinate $t$ and in new time coordinate $\tilde{t}$, respectively.

Next, notice that in the Boyer-Lindquist coordinates (irrespective of choosing $t$ or $\tilde{t}$ as its time coordinate), the metric component $g_{\phi\phi}$ represents $(\text{proper distance from the axis of rotation})^2$. This suggests that the angular velocity we computed above may be written as

$$
\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{(\text{angular momentum per unit mass})}{(\text{proper distance from the axis of rotation})^2}
$$

and as a result the “angular momentum per unit mass” at some point from the axis of rotation may be identified with

$$
J = g_{\phi\phi} \Omega = -g_{t\phi}.
$$

(10)

(As we have seen, the quantity so defined as above admits clear interpretation of the angular momentum per unit mass at some point from the axis of rotation particularly for $\theta =$const. submanifolds. But generally, it should be distinguished from the total angular momentum of the entire Kerr-Newman spacetime measured in the asymptotic region,
\( \hat{J} = (16\pi)^{-1} \int_S \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \psi^\beta \) in the notation convention of ref. 6 with \( S \) being a large sphere in the asymptotic region and \( \psi^\mu = (\partial/\partial \phi)^\mu \) being the rotational Killing field introduced earlier. Thus in these definitions, angular momentum per unit mass \( J \) may change under coordinate transformations although the total angular momentum \( \hat{J} \) remains coordinate-independent. Thus at the horizon \( r = r_+ \), the angular momentum per unit mass in the usual Boyer-Lindquist time \( t \) and in the new time coordinate \( \tilde{t} \) are given respectively by

\[
J_H = | g_{t\phi}(r_+) | = a \sin^2 \theta_0 \left( \frac{r_+^2 + a^2}{r_+^2 + a^2 \cos^2 \theta_0} \right),
\]

\[
\hat{J}_H = | g_{t\phi}(r_+) | = a \sin^2 \theta_0 \quad (0 \leq \theta_0 \leq \pi/2).
\]

Finally, we turn to the computation of the surface gravity of the \( \theta = \text{const.} \) submanifolds of Kerr-Newman black hole. In physical terms, the surface gravity \( \kappa \) is the force that must be exerted to hold a unit test mass at the horizon and it is given in a simple formula as

\[
\kappa^2 = -\frac{1}{2} (\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu)
\]

where \( \chi^\mu \) is as given earlier and the evaluation on the horizon is understood. And since there are two regular Killing horizons at \( r = r_+ \) and \( r = r_- \), we define surface gravities at each of the two horizons correspondingly. Again a straightforward calculation yields

\[
\kappa_\pm = \frac{(r_\pm - r_\mp)}{2(r_\pm^2 + a^2)},
\]

\[
\tilde{\kappa}_\pm = \frac{(r_\pm - r_\mp)}{2(r_\pm^2 + a^2 \cos^2 \theta_0)} \quad (0 \leq \theta_0 \leq \pi/2)
\]

in Boyer-Lindquist time coordinate \( t \) and in new time coordinate \( \tilde{t} \) respectively and where we redefined \( \tilde{\kappa}_\pm \rightarrow [\sin^2 \theta_0 (1 + \sin^2 \theta_0)/2]^{1/2} \kappa_\pm \) for \( 0 < \theta_0 \leq \pi/2 \), but not for \( \theta_0 = 0 \). Thus from eqs. (9), (11) and (13), we can relate quantities with “tilde” in new time coordinate \( \tilde{t} \) and those in Boyer-Lindquist time coordinate \( t \) as

\[
\hat{J}_H = \left( \frac{r_+^2 + a^2 \cos^2 \theta_0}{r_+^2 + a^2} \right) J_H < J_H,
\]

\[
\hat{\Omega}_H = \left( \frac{r_+^2 + a^2}{r_+^2 + a^2 \cos^2 \theta_0} \right) \Omega_H > \Omega_H,
\]

\[
\tilde{\kappa}_\pm = \left( \frac{r_\pm^2 + a^2}{r_\pm^2 + a^2 \cos^2 \theta_0} \right) \kappa_\pm > \kappa_\pm.
\]
Note that in Boyer-Lindquist time coordinate \( t \), the angular velocity \( \Omega_{H} \) and the surface gravity \( \kappa_{\pm} \) at the horizon are independent of the polar angle \( \theta = \theta_{0} \) \( (0 \leq \theta_{0} \leq \pi/2) \), i.e., they remain the same for any value of \( \theta_{0} \). In contrast, in the new time coordinate \( \tilde{t} \), both \( \tilde{\Omega}_{H} \) and \( \tilde{\kappa}_{\pm} \) do have dependence on the polar angle \( \theta_{0} \) in such a way that they increase with \( \theta_{0} \), i.e., they get minimized at the symmetry axis \( (\theta_{0} = 0) \) whereas get maximized on the equatorial plane \( (\theta_{0} = \pi/2) \). In addition, from eq.(14), note that generally \( \tilde{\Omega}_{H} > \Omega_{H} \), \( \tilde{J}_{H} < J_{H} \) and \( \tilde{\kappa}_{\pm} > \kappa_{\pm} \) (where the inequalities hold for \( \theta_{0} \neq 0 \) and up to the redefinition of \( \kappa \)). These results indicate that in the “new” time coordinate \( \tilde{t} \), the \( \theta = \text{const.} \) submanifolds of the Kerr-Newman black hole has greater angular velocity yet smaller angular momentum per unit mass and greater surface gravity than they do in the usual Boyer-Lindquist time coordinate \( t \). Particularly here, “possessing greater angular velocity while smaller angular momentum per unit mass” may first look erroneous. But if one really looks into the details, one can realize that it is no nonsense since it arises from “same coordinate distance but different proper distances” from the axis of rotation to the horizon in the two time coordinates \( t \) and \( \tilde{t} \). Now from the greater angular velocity, we are led to the conclusion that the new time coordinate \( \tilde{t} \) defined by the eq.(3) appears to be the time coordinate, say, of a frame which rotates around the axis of the spinning Kerr-Newman black hole in opposite direction to that of the hole with an angular velocity that increases with the polar angle \( \theta_{0} \). Then the smaller angular momentum per unit mass and greater surface gravity can be attributed to the fact that as the angular momentum per unit mass decreases when transforming from the Boyer-Lindquist to new time coordinate, the surface gravity is expected to increase due to the effect of centrifugal force. And this conclusion agrees with our earlier quick interpretation of the “new” time coordinate \( \tilde{t} \).

IV. Application to Kerr-Newman-type solutions in other gravity theories

In this section, we shall illustrate that the same type of transformation, \( \tilde{t} = t - (a \sin^{2} \theta_{0}) \phi \) from the Boyer-Lindquist time coordinate \( t \) to the new time coordinate \( \tilde{t} \) allows us to explore global structure of \( \theta = \text{const.} \) submanifolds of Kerr-Newman-type solutions found in other gravity theories, most notably low energy effective string theories. Thus before we begin, it
seems worth mentioning the connection between black hole physics and string theory. In the study of string theory, much of the recent attention has been focussed on the construction of classical solutions such as solitonlike solutions including black hole solutions. The motivation for such study is the following. In order to study non-perturbative string theory, one must include, in addition to the standard Fock space states, the soliton states in the spectrum. In particular, some of the underlying symmetries of string theory may become manifest only after including these solitonic states in the spectrum. As a result, several black hole solutions have been found as stable extended solitonlike states in low energy string theory which can be thought of as the string theory analogue of important black hole solutions of Einstein gravity.

In the present work, we are particularly interested in rotating, charged Kerr-Newman-type black hole solutions containing dilaton and axion corrections in low energy string theory. These solutions were obtained first by Sen and then in a more general form (including the Newman-Unti-Tamburino (NUT) parameter) by Gal’tsov and Kechkin. As it is the case with the Kerr-Newman solution in Einstein-Maxwell theory, obtaining these rotating, charged black hole solutions in low energy string theory by directly solving classical field equations is highly involved and impractical. They were, therefore, obtained by a method for generating new solutions from the known ones which can be thought of as a generalization of Ehlers-Harrison transformations.

First, we begin with the rotating, charged black hole solution in low energy effective heterotic string theory found by Sen. Sen considered the low energy effective theory of heterotic string described by the action in 4-dim.

\[
S = \int d^4 x \sqrt{g} e^{-\Phi} (R + \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{8} F_{\mu\nu} F^{\mu\nu})
\]

where \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the field strength of the Maxwell field \( A_{\mu} \), \( \Phi \) is the dilaton field and

\[
H_{\mu\nu\lambda} = \partial_{\mu} B_{\nu\lambda} + \partial_{\nu} B_{\lambda\mu} + \partial_{\lambda} B_{\mu\nu} - [\omega_3(A)]_{\mu\nu\lambda}
\]

with \( B_{\mu\nu} \) and \([\omega_3(A)]_{\mu\nu\lambda}\) being the antisymmetric tensor gauge field and the gauge Chern-Simons term respectively (the Lorentz Chern-Simons term has not been included in the
definition of $H_{\mu \nu \lambda}$). And the action above, which is written in string frame with $g_{\mu \nu}$, can be transformed to the usual Einstein conformal frame with $\hat{g}_{\mu \nu}$ (in which the black hole solution will be given later on) via the conformal transformation

$$\hat{g}_{\mu \nu} = e^{-\Phi} g_{\mu \nu}.$$  

Then, in order to obtain a rotating, charged black hole solution in this low energy effective string theory, Sen employed the “twisting procedure” that generates inequivalent classical solutions starting from a given classical solution of string theory. In particular, using the method for generating charged black hole solution from a charge-neutral solution, Sen constructed the rotating, charged black hole solution by starting from a rotating, uncharged black hole solution, i.e., the Kerr solution. In Einstein conformal frame, Sen’s metric solution can be cast, in terms of Boyer-Lindquist coordinates, to

$$ds^2 = -\left[\frac{\Delta - a^2 \sin^2 \theta}{\Sigma_s}\right] dt^2 - \frac{2a \sin^2 \theta ([r(r + r_-) + a^2] - \Delta)}{\Sigma_s} dtd\phi$$

$$+ \frac{[r(r + r_-) + a^2]^2 - \Delta a^2 \sin^2 \theta}{\Sigma_s} \sin^2 \theta d\phi^2 + \frac{\Sigma_s}{\Delta} dr^2 + \Sigma_s d\theta^2$$

where

$$\Sigma_s = r(r + r_-) + a^2 \cos^2 \theta,$$

$$\Delta = r^2 - 2Mr + a^2, \quad r_- = 2M \sinh^2 \left(\frac{\alpha}{2}\right).$$

with $\alpha$ being an arbitrary number. This solution is a 3-parameter family of rotating, charged black hole solution with the mass $M_s$, charge $Q_s$, angular momentum $J_s$ and magnetic dipole moment $\mu_s$ being given respectively by

$$M_s = \frac{M}{2} (1 + \cosh \alpha), \quad Q_s = \frac{M}{\sqrt{2}} \sinh \alpha,$$

$$J_s = \frac{Ma}{2} (1 + \cosh \alpha), \quad \mu_s = \frac{Ma}{\sqrt{2}} \sinh \alpha$$

and with horizons being located at

$$r_{\pm} = M_s - \frac{Q_s^2}{2M_s} \pm [M_s^2 (1 - Q_s^2/(2M_s^2))^2 - \frac{J_s^2}{M_s^2}]^{1/2}.$$
Now, just as we did for the Kerr-Newman solution in Einstein-Maxwell theory, here we consider the \( \theta = \theta_0 \) (\( 0 \leq \theta_0 \leq \pi/2 \)) timelike submanifolds of Sen’s solution above and next perform the same type of transformation from the Boyer-Lindquist time coordinate \( t \) to the new time coordinate \( \tilde{t} \) as the one given earlier

\[
\tilde{t} = t - (a \sin^2 \theta_0) \phi
\]

with other spatial coordinates remaining unchanged. Then in terms of this new time coordinate \( \tilde{t} \), the metric of \( \theta = \theta_0 \) submanifolds again take on the structure of simple, diagonal, ADM’s \((2 + 1)\) space-plus-time split form

\[
ds^2 = -N^2(r)d\tilde{t}^2 + h_{rr}(r)dr^2 + h_{\phi\phi}(r)[N^\phi(r)d\tilde{t} + d\phi]^2
\]

where the lapse, shift functions and the spatial metric components are given respectively by

\[
N^2(r) = \frac{\Delta}{\Sigma_s}, \quad N^\phi(r) = -\frac{a}{\Sigma_s}, \quad (17)
\]

\[
h_{rr}(r) = N^{-2}(r), \quad h_{\phi\phi}(r) = \Sigma_s \sin^2 \theta_0, \quad h_{r\phi}(r) = h_{\phi r}(r) = 0.
\]

Next, we turn to the rotating, charged black hole solution in dilaton-axion gravity found by Gal’tsov and Kechkin\(^{10}\). These authors also considered the low energy effective theory of heterotic string represented by the action in 4-dim.

\[
S = \int d^4x \sqrt{\hat{g}}(\hat{R} + 2\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} e^{4\Phi} \partial_\mu A \partial^\mu A - e^{-2\Phi} F_{\mu\nu} F^{\mu\nu} - AF_{\mu\nu} \tilde{F}^{\mu\nu})
\]

where \( \tilde{F}^{\mu\nu} = \frac{1}{2} e^{\mu\alpha\beta} F_{\alpha\beta} \) denotes the dual of \( F_{\mu\nu} \). Compared to the action taken by Sen, this action above is written in Einstein conformal frame with metric \( \hat{g}_{\mu\nu} \) and the field strength of the Kalb-Ramond field, \( H_{\mu\nu\lambda} \) that appeared in Sen’s action is now transformed into the Peccei-Quinn axion field \( A \) using the dilaton-axion dual symmetry \( SL(2, R) \). Since the axion field appears explicitly in this format of the theory, the low energy heterotic string theory considered by Gal’tsov and Kechkin is usually called, Einstein-Maxwell-Dilaton-Axion (EMDA) theory. Then, in order to construct a rotating, charged black hole solution in this EMDA theory, they noted that there are two important symmetries in bosonic sector of
the low energy heterotic string theory which can be used to generate new classical solutions. One of them is target space duality $0(d, d + p)$, which is valid for EMDA system with $p$ abelian gauge fields whenever variables are independent of $d$-spacetime coordinates. This is the symmetry that Sen employed to construct his solution and this solution generation method is called, “twisting procedure”. The second symmetry is a dilaton-axion (or electric-magnetic) duality $SL(2, R)$, which arises in the 4-dim. case for which the field streng of the Kalb-Ramond field, $H_{\mu\nu\lambda}$ can be transformed into the axion field $A$ as mentioned earlier. It says that a pair $(\Phi, A)$ parametrizes the $SL(2, R)/SO(2)$ coset. Gal’tsov and Kechkin took one step further and attempted combining these two symmetries within a larger group for the case, $p = 1, d = 1$ in 4-dim. And the combined symmetry group turned out to be larger than the product of the two symmetries. Moreover, its nontrivial part generalizes Ehlers-Harrison transformations known in the Einstein-Maxwell theory. The group also contains scale and gauge transformations. Utilizing these two symmetries, then, they built a simple way to construct rotating, charged black hole solutions in EMDA theory from known solutions to vacuum Einstein theory such as Kerr solution. Thus, in a sense, the solution generation method of Gal’tsov and Kechkin can be thought of as an extension of twisting procedure by Sen and their solution (particularly the electrically- charged solution without magnetic charge and NUT parameter) is given, in Boyer-Lindquist coordinates, by

$$\begin{align*}
 ds^2 &= -\left[\frac{\Delta_d - a^2 \sin^2 \theta}{\Sigma_d}\right] dt^2 + \frac{2a \sin^2 \theta \left[\left(r(r + r_\perp) + a^2\right) - \Delta_d\right]}{\Sigma_d} dt d\phi \\
 &\quad + \left[\frac{[r(r + r_\perp) + a^2]^2 - \Delta_d a^2 \sin^2 \theta}{\Sigma_d}\right] \sin^2 \theta d\phi^2 + \frac{\Sigma_d}{\Delta_d} dr^2 + \Sigma_d d\theta^2 \\
 &\quad \text{(19)}
\end{align*}$$

where

$$\Sigma_d = r(r + r_\perp) + a^2 \cos^2 \theta,$$

$$\Delta_d = r(r + r_\perp) - 2Mr + a^2, \quad r_\perp = \frac{Q^2}{M}$$

and the coordinate “$r$” here corresponds to $r_0$ in the definition of Gal’tsov and Kechkin. This rotating, charged black hole solution is characterized by the mass $M$, the electric charge $Q$, and the angular momentum per unit mass $a$. Its horizons are located at
\[ r_{\pm} = M + \frac{Q^2}{2M} \pm [M^2(1 - \frac{Q^2}{2M^2})^2 - a^2]^{1/2} \]

and the boundary of the ergosphere, i.e., the static limit is located at

\[ r_{\pm} = M + \frac{Q^2}{2M} \pm [M^2(1 - \frac{Q^2}{2M^2})^2 - a^2 \cos^2 \theta]^{1/2}. \]

Now, as before, here we consider the \( \theta = \theta_0 \) \((0 \leq \theta_0 \leq \pi/2)\) timelike submanifolds of this rotating, charged black hole solution in EMDA theory and next perform the transformation from the Boyer-Lindquist time \( t \) to a new time coordinate \( \tilde{t} \)

\[ \tilde{t} = t - (a \sin^2 \theta_0)\phi \]

as we did before. One can see that, again, in this new time coordinate \( \tilde{t} \), the metric of \( \theta = \theta_0 \) submanifolds take on the structure of simple, diagonal ADM’s \((2 + 1)\) space-plus-time split form

\[ ds^2 = -N^2(r)d\tilde{t}^2 + h_{rr}(r)dr^2 + h_{\phi\phi}(r)\left[N^{\phi}(r)d\tilde{t} + d\phi\right]^2 \]

where the lapse, shift functions and the spatial metric components are given respectively by

\[
N^2(r) = \frac{\Delta_d}{\Sigma_d}, \quad N^{\phi}(r) = -\frac{a}{\Sigma_d}, \quad (20)
\]

\[
h_{rr}(r) = N^{-2}(r), \quad h_{\phi\phi}(r) = \Sigma_d \sin^2 \theta_0, \quad h_{r\phi}(r) = h_{\phi r}(r) = 0.
\]

Note first that in all the three cases we have considered thus far, i.e., the Kerr-Newman solution in Einstein-Maxwell theory, the rotating, charged black hole solution by Sen and by Gal’tsov and Kechkin in low energy effective theory of heterotic string, the metric of \( \theta = \text{const.} \) timelike submanifolds can be cast to a simple, diagonal ADM’s \((2 + 1)\) split form by a “single”, common transformation law for time coordinate, \( \tilde{t} = t - (a \sin^2 \theta_0)\phi \). This fact seems to suggests that the transformation law from the Boyer-Lindquist time coordinate to the new time coordinate we found, or equivalently the selection of new time coordinate \( \tilde{t} \) was not accidental, after all. Namely, the coordinate transformation to the frame which “rotates around the axis of the spinning hole in opposite direction to that of the hole with an
angular velocity that increases with the polar angle $\theta_0$ seems to have remarkable physical significance with general applicability. Secondly note that, again in terms of this new time coordinate $\tilde{t}$, it becomes clearer that the $\theta = \theta_0$ submanifolds of rotating, charged black hole spacetime solutions in low energy effective string theory have the topology of $R^2 \times S^1$. The global structures of the submanifolds $T = R^2$, then, would mirror those of the full $\theta = \theta_0$ submanifolds since each point of $T = R^2$ can be thought of as representing $S^1$. As we have observed in the Kerr-Newman spacetime case, now we can conclude that the global structure (upon the maximal analytic extension) of the $\theta = \theta_0$ ($0 \leq \theta_0 < \pi/2$) submanifolds is essentially the same as that of the symmetry axis first studied by Carter and the global structure of the equatorial plane ($\theta_0 = \pi/2$) is identical to that of the RN spacetime.

IV. Discussions

We now conclude with some comments worth mentioning. First, it is interesting to note that the metric of $\theta =$const. submanifolds of Kerr-Newman spacetime expressed in the “new” time coordinate $\tilde{t}$ as given in eq.(4) or (6) takes precisely the same structure as that of 3-dim. anti-de Sitter ($AdS_3$) black hole solution discovered recently by Banados, Teitelboim and Zanelli (BTZ). This indicates, among other things, that the choice of coordinates in which BTZ adopted their metric solution ansatz in ref.7 corresponds to the “new” time coordinates $\tilde{t}$ we have introduced in the present work, not the usual Boyer-Lindquist time coordinate. Indeed it has been demonstrated in detail by this author that by applying the same “complex coordinate transformation scheme” as the one Newman et al. employed in “deriving” Kerr solution from Schwarzschild solution and Kerr-Newman solution from RN solution, one can likewise derive rotating BTZ black hole solution from the nonrotating BTZ black hole solution. And in doing so the underlying spirit is that a 3-dim. geometry can be thought of as the $\theta = \pi/2$-slice of the 4-dim. geometry in which one introduces the null tetrad of basis vectors. Now, much as the Kerr-Newman solution “derived” in this manner naturally comes in the Boyer-Lindquist coordinates (via the coordinate transformation from the Kerr coordinates, as is well-known), the rotating BTZ solution derived in the same manner (i.e., by Newman’s complex coordinate transformation method) comes in Boyer-
Lindquist-type coordinates as well. Then next by performing a transformation to the "new" time coordinate, \( \tilde{t} = t - a\phi \) (i.e., \( \theta_0 = \pi/2 \)- case of the general transformation given in eq.(3)), the derived rotating BTZ black hole solution finally can be put in the form originally constructed by BTZ. A merit of this one-to-one correspondence between the \( \theta = \text{const.} \) submanifolds of Kerr-Newman black hole and the rotating BTZ solution is that now one can carry out the maximal analytic extension of the \( \theta = \text{const.} \) submanifolds of Kerr-Newman spacetime following a similar procedure taken for the complete study of the global structure of rotating BTZ black hole solution which was possible since it has been done in the new time coordinate in which the metric takes much simpler form and exhibits more clearly that the spacetime has the topology of \( R^2 \times S^1 \).

Next, we would like to point out the complementary roles played by the two alternative time coordinates \( t \) and \( \tilde{t} \). The Boyer-Lindquist time coordinate \( t \) is the usual Killing time coordinate with which one can obtain the rotating hole’s characteristics such as the angular velocity of the event horizon or the surface gravity as measured by an outside observer who is "static" with respect to, say, a distant star. The "new" time coordinate \( \tilde{t} \), on the other hand, is a kind of unusual in that it can be identified with the time coordinate of a non-static frame which rotates around the axis of the Kerr-Newman black hole in opposite direction to that of the hole with an angular velocity that increases with the polar angle. This new time coordinate, however, is particularly advantageous in exploring the global structure of the \( \theta = \text{const.} \) submanifolds of Kerr-Newman black hole since it allows one to transform to Kruskal-type coordinates and hence eventually allows one to draw the Carter-Penrose diagrams much more easily than the case when one employs the usual Boyer-Lindquist time coordinate.

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