ON HYPERBOLIC SETS OF POLYNOMIALS

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Abstract. Let $f$ be an infinitely-renormalizable quadratic polynomial and $J_\infty$ be the intersection of forward orbits of "small" Julia sets of its simple renormalizations. We prove that $J_\infty$ contains no hyperbolic sets.

1. Introduction

Let $f$ be a rational function of degree at least 2 considered as a dynamical systems $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}}$. An $f$-invariant compact set $X \subset \hat{\mathbb{C}}$ is said to be hyperbolic if $f: X \to X$ is uniformly expanding, i.e., for some $C > 0$ and $\lambda > 1$, $|D(f^m)(x)| \geq C\lambda^m$ for all $x \in X$ and all $m \geq 0$ (here $D$ stands for the spherical derivative and $f^m$ is $m$-iterate of $f$). In particular, any repelling periodic orbit of $f$ is a hyperbolic set. The closure of all repelling periodic orbits of $f$ is the Julia set $J(f)$ of $f$. Hyperbolic sets of $f$ are contained in $J(f)$. Apart of repelling periodic orbits, $f$ admits plenty of infinite (Cantor) hyperbolic sets [26]. Attracting periodic orbits (if any) along with their basins are contained in the complement $\hat{\mathbb{C}} \setminus J(f)$ (which is called the Fatou set of $f$). See e.g. [3] for an introduction to complex dynamics and [27] for a recent survey.

If $J(f)$ is a hyperbolic set by itself, i.e., $f: J(f) \to J(f)$ is uniformly expanding, then $f$ is called a hyperbolic rational map. Equivalently, the Fatou set $\hat{\mathbb{C}} \setminus J(f)$ is not empty and consists of basins of attractions of attracting periodic orbits. Hyperbolic rational maps are analogous to Axiom A diffeomorphisms and their dynamics has been intensively studied and very well understood. The famous 'Density of Hyperbolicity Conjecture (DHC)' in holomorphic dynamics - sometimes also called the Fatou conjecture - asserts that any rational map (polynomial) can be approximated by hyperbolic rational maps (polynomials) of the same degree.

In what follows $f$ (unless mentioned explicitly) is a quadratic polynomial $f_c(z) = z^2 + c$. The DHC (as well as a more general
MLC: Mandelbrot set Locally Connected) is widely open for the quadratic family \( f_c \), too (DHC for \( f_c \) as strongly believed accumulates in itself the essence of the general DHC). After a breakthrough work of Yoccoz [8] on the MLC, the only obstacle for proving DHC for quadratic polynomials are so-called infinitely-renormalizable ones, see [24].

Somewhat informally, a quadratic polynomial \( f_c \) with connected Julia set is called renormalizable if, for some topological disks \( U, V \) around the critical point 0 of \( f_c \) and for some \( p \geq 2 \) (called period of the renormalization), the restriction \( f_c^p : U \to V \) is conjugate to another quadratic polynomial \( f_{c'} \) with connected Julia set (see [6] for exact definitions and the theory of polynomial-like mappings). The map \( F := f_c^p : U \to V \) is then a renormalization of \( f_c \) and the set \( K(F) = \{ z \in U : F^n(z) \in U \text{ for all } n \geq 1 \} \) is a "small" (filled in) Julia set of \( f_c \). If \( f_{c'} \) is renormalizable by itself, then \( f_c \) is called twice renormalizable, etc. If \( f_c \) admits infinitely many renormalizations, it is called infinitely-renormalizable. Recall that the renormalization \( F \) is simple if any two sets \( f^i(K(F)), f^j(K(F)) \), \( 0 \leq i < j \leq p − 1 \), are either disjoint or intersect each other at a unique point which does not separate either of them.

To state our main result - which is Theorem 1.1 - let \( f(z) = z^2 + c \) be infinitely renormalizable. Let \( 1 = p_0 < p_1 < ... < p_n < ... \) be the sequence of consecutive periods of simple renormalizations of \( f \) and \( J_n \) denotes the "small" Julia set of the \( n \)-renormalization (where \( J_0 = J(f) \)). Then \( p_{n+1}/p_n \) is an integer, \( f^{p_n}(J_n) = J_n \), for any \( n \), and \( \{J_n\}_{n=1}^\infty \) is a strictly decreasing sequence of continua without interior, all containing 0. Let

\[
J_\infty = \cap_{n \geq 0} \cup_{j=0}^{p_n-1} f^j(J_n)
\]

be the intersection of orbits of the "small" Julia sets. \( J_\infty \) is a compact \( f \)-invariant set which contains the omega-limit set \( \omega(0) \) of 0. Each component of \( J_\infty \) is wandering, in particular, \( J_\infty \) contains no periodic orbits of \( f \). Note that a hyperbolic set in \( J_\infty \) (if existed) could not be repelling, that is any forward orbit of a point sufficiently close to this set must be in the set itself, since otherwise shadowing periodic orbits must be in \( J_\infty \).

**Theorem 1.1.** \( J_\infty \) contains no hyperbolic sets.

The conclusion of Theorem 1.1 would obviously hold provided

(1.1) \( J_\infty \) is totally disconnected.
(1.1) is true indeed for many classes of maps (including real ones) where it follows from ‘complex bounds’ [29] (meaning roughly that the sequence of renormalizations is compact) [15], [7], [21], [11], [12], [13]. See also [9], [10]. However, (1.1) breaks down in general: see [22], [28] for the existence of such maps and [16], [17], [18] (see also [5]) for explicit combinatorial conditions on \( f_c \) for (1.1) to fail. Yoccoz [31] posed a problem which seems to be equivalent to the following: find a necessary and sufficient condition on the combinatorics of \( f_c \) for (1.1) to hold. At present, the gap between known sufficient and necessary conditions is still very big.

Another well-known open problem is to give necessary and sufficient conditions so that the Julia set \( J(f) \) is locally-connected. For example, if (1.1) does not hold then \( J(f) \) is not locally-connected. Theorem 1.1 implies

**Theorem 1.2.** Let \( f(z) = z^2 + c \) and \( f \) has no irrational indifferent periodic orbits. Then \( J(f) \) is locally-connected at every point of any hyperbolic set \( X \) of \( f \). In particular, there are at least one and at most finitely many external rays landing at each \( x \in X \).

**Remark 1.1.** The case that \( f \) does have an irrational cycle seems to be open and requires a separate consideration, see [4] though. Note also that Theorem 1.2 removes the only restriction in Proposition 2.11 of [1] for degree 2 polynomials without irrational cycles.

Theorem 1.2 has been known for the following quadratic maps \( f \). If \( f \) has an attracting cycle, then \( f \) is hyperbolic and the whole \( J(f) \) is locally-connected. The same conclusion holds if \( f \) has a parabolic cycle [9]. The first part of Yoccoz’s result (see e.g., [8]) says that \( J(f) \) is locally-connected if \( f \) has no indifferent irrational cycles and at most finitely many times renormalizable. This allows us to reduce the proof of Theorem 1.2 to the case of \( f \) as in Theorem 1.1, hence, by the latter, to the case when \( X \) is disjoint from \( J_\infty \) in which case it is well-known that Yoccoz puzzle pieces shrink to each point of \( X \) [22], [19]. This shows that \( J(f) \) is locally connected at points of \( X \). The last claim follows then from [14], see also [30] and [19].

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2. Preliminaries

Here we collect, for further references, necessary notations and general facts which are either well-known [24], [23] or follow readily
from the known ones. Let \( f(z) = z^2 + c \) be infinitely renormalizable. We keep the notations of the Introduction.

(A). Let \( G \) be the Green function of the basin of infinity \( A(\infty) = \{ z | f^n(z) \to \infty, n \to \infty \} \) of \( f \) with the standard normalization at infinity \( G(z) = \ln|z| + O(1/|z|) \). The external ray \( R_t \) of argument \( t \in S^1 = \mathbb{R}/\mathbb{Z} \) is a gradient line to the level sets of \( G \) that has the (asymptotic) argument \( t \) at \( \infty \). \( G(z) \) is called the (Green) level of \( z \in A(\infty) \) and the unique \( t \) such that \( z \in R_t \) is called the (external) argument (or angle) of \( z \). A point \( z \in J(f) \) is accessible if there is an external ray \( R_t \) which lands at \( i.e. \), converges to \( z \). Then \( t \) is called an (external) argument (angle) of \( z \).

Let \( \sigma : S^1 \to S^1 \) be the doubling map \( \sigma(t) = 2t (mod 1) \). Then \( f(R_t) = R_{\sigma(t)} \).

(B). Given a small Julia set \( J_n \) containing 0, sets \( f^j(J_n) \) \( 0 \leq j < p_n \) are called small Julia sets of level \( n \). Each \( f^j(J_n) \) contains \( p_{n+1}/p_n \geq 2 \) small Julia sets of level \( n+1 \). We have \( J_n = -J_n \). Since all renormalizations are simple, for \( j \neq 0 \), the symmetric companion \( -f^j(J_n) \) of \( f^j(J_n) \) can intersect the orbit \( \text{orb}(J_n) = \bigcup_{j=0}^{p_n-1} f^j(J_n) \) of \( J_n \) only at a single point which is periodic. On the other hand, since only finitely many external rays converge to each periodic point of \( f \), the set \( J_\infty \) contains no periodic points. In particular, each component \( K \) of \( J_\infty \) is wandering, i.e., \( f^i(K) \cap f^j(K) = \emptyset \) for all \( 0 \leq i < j < \infty \). All this implies that \( \{ x, -x \} \subset J_\infty \) if and only if \( x \in K_0 := \cap_{n=1}^\infty J_n \).

Given \( x \in J_\infty \), for every \( n \), let \( j_n(x) \) be the unique \( j \in \{ 0, 1, \ldots, p_n-1 \} \) such that \( x \in f^j(x)(J_n) \). Let \( J_{x,n} = f^{j_n(x)}(J_n) \) be a small Julia set of level \( n \) containing \( x \) and \( K_x = \cap_{n \geq 0} J_{x,n} \), a component of \( J_\infty \) containing \( x \).

In particular, \( K_0 = \cap_{n \geq 0} J_n \) is the component of \( J_\infty \) containing 0 and \( K_c = \cap_{n=1}^\infty f(J_n) \), the component containing \( c \).

The map \( f : K_x \to K_f(x) \) is two-to-one if \( x \in K_0 \) and one-to-one otherwise. Moreover, for every \( y \in J_\infty \), \( f^{-1}(y) \cap J_\infty \) consists of two points if \( y \in K_c \) and consists of a single point otherwise.

(C). Given \( n \geq 0 \), the map \( f^{p_n} : f(J_n) \to f(J_n) \) has two fixed points: the separating fixed point \( \alpha_n \) (that is, \( f(J_n) \setminus \{ \alpha_n \} \) has at least two components) and the non-separating \( \beta_n \) (so that \( f(J_n) \setminus \beta_n \) has a single component).

For every \( n > 0 \), there are two rays \( R_{t_n} \) and \( R_{\tilde{t}_n} \) \( 0 < t_n < \tilde{t}_n < 1 \) to the non-separating fixed point \( \beta_n \in f(J_n) \) of \( f^{p_n} \) such that the component \( \Omega_n \) of \( C \setminus (R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n) \) which does not contain 0 has two characteristic properties:
The rays \( R \) we denote \( U \). Introduce an (unbounded) domain \( Y \) to \( \Omega \). The lengths of the arcs \( S_{n,c} = \{ t : R_t \subset \Omega \} \) between \( t_n \) and \( t_n^\prime \).

Denote \( t_n' = t_n + \frac{t_n - t_n^\prime}{2p_n}, \quad \tilde{t}_n = \frac{t_n - t_n^\prime}{2p_n}. \)

The rays \( R_{t_n}, \ R_{\tilde{t}_n} \) land at a common point \( \beta_n' \in f^{-p_n}(\beta_n) \cap \Omega_n \). Introduce an (unbounded) domain \( U_n \) with the boundary to be two curves \( R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n \) and \( R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n' \). Then \( c \in U_n \) and \( f^{-n} : U_n \to \Omega_n \) is a two-to-one branched covering. Also, \( f(J_n) = \{ z : f^{kp_n}(z) \in \overline{U}_n, k = 0, 1, \ldots \} \).

We denote \( s_n = [t_n, t_n'] \cup [\tilde{t}_n, \tilde{t}_n^\prime] \).

**D1**. Given a compact set \( Y \subset J(f) \) denote by \( \prod (Y) \) (or simply \( Y \), if the map is fixed) the set of arguments of the external rays which have their limit sets in \( Y \). It follows from (C) that \( \tilde{K}_c = \bigcap_{n=1}^\infty \{ [t_n, t_n'] \cup [\tilde{t}_n, \tilde{t}_n] \} \), i.e., it is either a single-point set or a two-point set. Consider the latter case, i.e., \( \tilde{K}_c = \{ \tau_1, \tau_2 \} \). Let \( S_c \) be the shortest arc in \( S^1 \) with the end points \( \tau_1, \tau_2 \). It follows from (C):

1. \( S_c \) contains \( c \) and contains no the forward orbit of \( \beta_n \),
2. for every \( 1 \leq j < p_n \) consider arguments (angles) of the the external rays which land at \( f^{-j}(\beta_n) \). The angles split \( S^1 \) into finitely many arcs. Then the length of any such arc is bigger than the length of the arc \( S_{n,c} = \{ t : R_t \subset \Omega \} \) between \( t_n \) and \( t_n^\prime \).
3. (unlinking) for each positive \( j \neq k \), one of the two arcs \( S^1 \setminus \{ \sigma^k(\tau_1), \sigma^k(\tau_2) \} \) contains both points \( \sigma^j(\tau_1), \sigma^j(\tau_2) \).

Now, since \( \tilde{K}_c \) contains at most two angles, \( \tilde{K}_c \) contains at most two different accessible points.

**D2**. Given \( \nu \in [0,1) \) there exists a unique minimal rotation set \( \Lambda_\nu \subset S^1 \) of \( \sigma : S^1 \to S^1 \) with the rotation number \( \nu \) \cite{2}. Recall that a closed subset \( \Lambda \) of \( S^1 \) is a rotation set of \( \sigma \) with the rotation number \( \nu \) if \( \sigma(\Lambda) \subset \Lambda \) and \( \sigma : \Lambda \to \Lambda \) extends to a map of \( S^1 \) which lifts to an orientation preserving continuous map \( F : \mathbb{R} \to \mathbb{R} \) with \( F - \text{Id} \) to be \( 1 \)-periodic and the fractional part of the rotation number of \( F \) to be equal to \( \nu \). Then \( \nu \) is irrational if and only if \( \Lambda_\nu \) is infinite, in this case there is a unique closed semi-circle containing \( \Lambda_\nu \) so that the end points of this semi-circle belong to \( \Lambda_\nu \).
3. Accessibility

We define a telescope following essentially [25]. Given \( x \in J(f) \), \( r > 0 \), \( \delta > 0 \), \( k \in \mathbb{N} \) and \( \kappa \in (0, 1) \), an \((r, \kappa, \delta, k)\)-telescope at \( x \in J \) is collections of times \( 0 = n_0 < n_1 < \ldots < n_k = n \) and disks \( B_l = B(f^{n_l}(x), r), l = 0, 1, \ldots, k \) such that, for every \( l > 0 \): (i) \( l/n_l > \kappa \), (ii) there is a univalent branch \( g_{n_l} : B(f^{n_l}(x), 2r) \to C \) of \( f^{-n_l} \) so that \( g_{n_l}(f^{n_l}(x)) = x \) and, for \( l = 1, \ldots, k, d(f^{n_l-1} \circ g_{n_l}(B_l), \partial B_l) > \delta \) (clearly, here \( f^{n_l-1} \circ g_{n_l} \) is a branch of \( f^{-n_l} \) that maps \( f^{n_l}(x) \) to \( f^{n_l-1}(x) \)). The trace of the telescope is a collection of sets \( B_{l,0} = g_{n_l}(B_l), l = 0, 1, \ldots, k \). We have: \( B_{k,0} \subset B_{k-1,0} \subset \ldots \subset B_{1,0} \subset B_{0,0} = B(x, r) \). By the first point of intersection of a ray \( R_t \), or an arc of \( R_t \), with a set \( E \) we mean a point of \( R_t \cap E \) with the minimal level (if it exists).

**Theorem 3.1.** [25] Given \( r > 0 \), \( \kappa \in (0, 1) \), \( \delta > 0 \) and \( C > 0 \) there exist \( M > 0 \), \( \tilde{l}, \bar{k} \in \mathbb{N} \) and \( K > 1 \) such that for every \((r, \kappa, \delta, k)\)-telescope the following hold. Let \( k > \bar{k} \). Let \( u_0 = u \) be any point at the boundary of \( B_k \) such that \( G(u) \geq C \). Then there are indexes \( 1 \leq l_1 < l_2 < \ldots < l_j = k \) such that \( l_1 < \tilde{l}, l_{i+1} < Kl_i, i = 1, \ldots, j-1 \) as follows. Let \( u_k = g_{n_k}(u) \in \partial B_{k,0} \) and let \( \gamma_k \) be an infinite arc of an external ray through \( u_k \) between the pint \( u_k \) and \( \infty \). Let \( u_{k,k} = u_k \) and, for \( l = 1, \ldots, k-1 \), let \( u_{k,l} \) be the first point of intersection of \( \gamma_k \) with \( \partial B_{l,0} \). Then, for \( i = 1, \ldots, j \),

\[
G(u_{k,l_i}) > M2^{-n_l_i}.
\]

Applying Theorem 3.1 as in [25] we obtain the following statement. See also [1] for a direct proof of part (1).

**Corollary 3.1.** Let \( X \) be a hyperbolic set for \( f \). (1) To every point \( x \in X \) one can assign a non-empty set \( A_x \subset S^1 \) such that for every \( t \in A_x \), the external ray \( R_t \) lands at \( x \). (2) The set \( A = \{(x, t) : x \in X, t \in A_x \} \) is closed in \( C \times S^1 \). (3) Moreover, for each \( \mu > 0 \) there is \( C(\mu) > 0 \) such that for all \( x \in X \) and all \( t \in A_x \), the first intersection of \( R_t \) with \( \partial B(x, \mu) \) has the level at least \( C(\mu) \).

**Proof.** As \( f : X \to X \) is expanding, there exist \( \lambda > 1, \rho > 0 \) and \( j_0 \) such that, for every \( x \in X \) and every \( k > 0 \), there exists a (univalent) branch \( g_{k,x} : B(f^{k_j}(x), \rho) \to C \) of \( f^{-k_j} \) with \( g_{k,x}(f^{k_j}(x)) = x \) and \( |g'_{k,x}(y)/g_{k,x}(z)| < 2 \) for \( y, z \in B(f^{k_j}(x), \rho) \). Moreover, \( |g'_{k,x}(z)| < \lambda^{-k} \) for all \( k > 0 \) and \( x \in X \). Therefore, there are \( r > 0 \), \( \delta > 0 \) and \( \kappa = 1/j_0 \) such that for any \( k > 1 \), every point \( x \in X \) admits an \((r, \kappa, \delta, k)\)-telescope with \( n_k = k_j_0 \). In fact, \( n_i = i j_0 \) for \( i = 0, 1, \ldots, k \). Let \( B_{k,0}(x) \subset B_{k-1,0}(x) \subset \cdots \subset B_{1,0}(x) \subset B_{0,0}(x) \)
be the corresponding trace. Define $C_0 = \inf_{y \in J(\ell)} \max_{z \in B(x, r)} G(z)$. It is easy to see that $C_0 > 0$. For each $x \in X$ we choose a point $u(x) \in \partial B(x, r)$ such that $G(u(x)) \geq C_0$. By Theorem 3.1 there are $\tilde{l}$ and $\tilde{k}$ such that for each $k > \tilde{k}$ and each $x \in X$ the following hold. There are $1 \leq l_{1,k}(x) < l_{2,k}(x) < \cdots < l_{j_k, k}(x) = k$ such that $l_{i,k}(x) < \tilde{l}$, $l_{i+1,k}(x) < K l_{i,k}(x)$, $i = 1, \ldots, j_k - 1$. Let $\gamma_k(x)$ be an arc of an external ray between the point $u_k(x) = g_{k, x}(u(f^{k0_j}(x)))$ and $\infty$. Let $u_{k, l}(x)$ be the first intersection of $\gamma_k(x)$ with $\partial B_{l, 0}(x)$. Then, for $i = 1, \ldots, j_k - 1$,

$$G(u_{k, l_{i,k}}(x)) > M 2^{-l_{i,k}(x)j_0}.$$  

For all $i = 1, \ldots, j_k - 1$,

$$i \leq l_{i,k}(x) < K^i \tilde{l}.$$ 

Denote by $t_k(x)$ the argument of an external ray that contains the arc $\gamma_k(x)$. It is enough to prove

**Lemma 3.2.** (i) If $(x_m)_{m>0} \subset X$, $x_m \to y$ and $t_{k_m}(x_m) \to \tau$ for some $k_m \to \infty$, then $R_\tau$ lands at $y$. (ii) Moreover, for each $\mu > 0$ there is $C(\mu) > 0$ such that for all pairs $(y, \tau)$ like this the first intersection of $R_\tau$ with $\partial B(y, \mu)$ has the level at least $C(\mu)$.

Indeed, assuming this lemma, we can define $A_x$ as a (non-empty) set of angles $t$ so that there are $x_m \in X$ and $k_m \to \infty$ with $x_m \to x$ and $t_{k_m}(x_j) \to t$. By (3.1) and (3.2) and since the level of the point $u_k(x)$ tends to zero as $k \to \infty$ uniformly in $x \in X$, we get that the ray $R_t$ indeed lands at the point $x$. It is then an elementary exercise to check that the set $A$ is closed. Claim (ii) implies obviously the part (3).

So, let’s prove Lemma 3.2 Let $(y, t)$ be as in the lemma. Pick any $\mu \in (0, r)$. Fix $l_0$ so that $\lambda^{-l_0} r < \mu/2$ and let

$$C(\mu) = \frac{M}{2^{k^0_l} l_j}.$$ 

There is $m_0$ such that for all $m > m_0$ and all $l > l_0$, $B_{l_0, 0}(x_m) \subset B(y, \mu)$. Then for every $m > m_0$,

$$G(u_{k_m, l_{0,m}}(x_m)) > C(\mu).$$ 

Hence, for every $m > m_0$ the first intersection of $\gamma_{k_m}(x_m)$ with $B(y, \mu)$ has level at least $C(\mu)$. It follows, for any $0 < C < C(\mu)$, the sequence of arcs of the rays $R_{t_{k_m}(x_m)}$ between the levels $C$ and $C(\mu)$ do not exit $B(y, \mu)$ for all $m > m_0$. As $C > 0$ can be chosen arbitrary small, Lemma 3.2 follows. □
4. A COMBINATORIAL PROPERTY

Let $f$ be an infinitely-renormalizable quadratic polynomial. First, we prove the following combinatorial fact (a maximality property) about $f$.

Let $\omega(t)$ be the omega-limit set of $t \in S^1$ under $\sigma : t \mapsto 2t(mod1)$ and $\sigma(E) = \cup_{t \in E} w(t)$.

**Lemma 4.1.** (i) $\sigma^{-1}(\tilde{K}_c) \subset \omega(\tilde{K}_c)$

(ii) $\tilde{J}_\infty = \omega(\tilde{K}_c)$

**Proof.** (a) $\sigma : \tilde{J}_\infty \to \tilde{J}_\infty$ has no periodic points. On the other hand, for each $t \in S^1$, $\sigma$ maps $\omega(t)$ onto itself. Hence, for each $t \in \tilde{J}_\infty$, the expanding map $\sigma : \omega(t) \to \omega(t)$ is not injective. It follows from (E), that then $\omega(t)\cap \tilde{K}_c \neq \emptyset$ for all $t \in \tilde{J}_\infty$.

(b) If $\tilde{K}_c$ consists of a single angle $\tau_c$, then (a) implies immediately that $\sigma^{-1}(\tau_c) \subset \omega(\tau_c)$ and we are done in this case.

(c) It remains to deal with a two-point set $\tilde{K}_c = \{\tau_1, \tau_2\}$. Let us assume the contrary, i.e., $\sigma^{-1}(\tau_1) \cup \sigma^{-1}(\tau_2)$ is not a subset of $\omega(\tilde{K}_c) = \omega(\tau_1) \cup \omega(\tau_2)$. Hence, by (a) and by the assumption, either $\sigma^{-1}(\tau_1)$ or $\sigma^{-1}(\tau_2)$ is a subset of each $\omega(\tau_i)$, $i = 1, 2$. Let, say, $\sigma^{-1}(\tau_1) \subset \omega(\tau_1) \cap \omega(\tau_2)$. By the assumption, there is $\tau \in \sigma^{-1}(\tau_2)$ such that $\tau \notin \omega(\{\tau_1, \tau_2\})$. Let $L$ be a semi-circles $S^1 \setminus \sigma^{-1}(\tau_1)$ that contains $\tau$. We claim that it is enough to show that for each $p_n$ and some $j_n > 0$ the arc $L$ contains $\sigma^{j_n p_n - 1}(\tau_1)$. Indeed, assume this is the case. Then, by (D1)3, Sect. 2, $L$ must contain one of the arcs $S^1 \setminus \{\sigma^{j_n p_n - 1}(\tau_1), \sigma^{j_n p_n - 1}(\tau_2)\}$ and, by (D1)2, the lengths of all such arcs are uniformly away from 0. Hence, there is a subsequence $n_i \to \infty$ such that the sequences $\sigma^{j_n p_n - 1}(\tau_1)$ and $\sigma^{j_n p_n - 1}(\tau_2)$ converge to points $a_1$ and $a_2$ respectively, where $a_1 \neq a_2$ and $a_1, a_2 \in \overline{L}$. On the other hand, $a_1, a_2 \in \tilde{K}_0$. But, from the assumption, $\tilde{K}_0 \cap L = \sigma^{-1}(\tau_1)$. Therefore, $\{a_1, a_2\} = \sigma^{-1}(\tau_1)$. But then $\sigma^{j_n p_n}(\tau_1)$ and $\sigma^{j_n p_n}(\tau_2)$ converge to the same point $\tau_1$ which is a contradiction with (D1)$2_\infty$.

(d) To show that, for each $n$, the arc $L$ contains a point $\sigma^{j_n p_n - 1}(\tau_1)$, for some $j_n > 0$, let us assume the contrary. So, we fix $n > 0$ and assume that $\{\sigma^{j p_n - 1}(\tau_1) : j > 0\} \subset L' := S^1 \setminus L$. Following the notations of (C), let $\Omega_n^0 = f^{-1}(\Omega_n)$. Then $J_n = \{z : f^{j_n p_n}(z) \in \Omega_n^0, n = 0, 1, \cdots\}$ and $\Omega_n^0$ is bounded by two bi-infinite curves and two angular (open) "arcs at infinity" which are two components $S_{0,0}, S'_{0,0}$ of $\sigma^{-1}(S_{n,0})$. Now, define $\epsilon(t) = 0$ if $t \in S_{0,0}$ and $\epsilon(t) = 1$ if $t \in S'_{0,0}$. To every $t \in \tilde{J}_n$ such that $\sigma^{j_n p_n}(t) \notin \partial(S_{0,0} \cup S'_{0,0})$ for all
In particular, \( k > 0 \), we assign a point in \( S^1 \) as follows:

\[
\theta(t) = \sum_{j=0}^{\infty} \frac{\epsilon(\sigma^j p_n(t))}{2^{j+1}}.
\]

In particular, \( \theta_0 := \theta(\tau_1/2) \) and \( \theta_1 := \theta(\tau_1/2 + 1/2) = \theta_0 + 1/2 \) are well-defined. By the assumption, for all \( k > 0 \) the points \( \sigma^k(\theta_0) \) belong to the same semi-circle \( C_{\theta_0} \) of \( S^1 \) with the end points \( \theta_0 \) and \( \theta_1 \). It follows that the set \( \omega(\theta_0) \subset C_{\theta_0} \) is a rotation set of \( \sigma \). Let \( E \subset \omega(\theta_0) \) be such that \( \sigma : E \rightarrow E \) is minimal. As \( \tilde{J}_n \) contains no periodic points of \( \sigma^{p_n} \), \( E \) contains no periodic points of \( \sigma \), too. Hence, \( E \) is infinite. By [2], see also (D2), for every closed infinite minimal rotation set of \( \sigma \) there is a unique closed semi-circle containing it and in this case the end points of the semi-circle belong to the set. Thus \( \theta_0, \theta_1 \in E \). It follows that there are two sequences \( j_i, j_i' \rightarrow \infty \) so that \( \sigma^{j_i p_n}(\tau_1/2) \rightarrow \tau_1/2 \) and \( \sigma^{j_i' p_n}(\tau_1/2) \rightarrow \tau_1/2 + 1/2 \) inside of the same semi-circle bounded by \( \tau_1/2, \tau_1/2 + 1/2 \). But then \( \sigma^{j_i p_n}(\tau_1) \) and \( \sigma^{j_i' p_n}(\tau_1) \) both tend to \( \tau_1 \) from different sides, in a contradiction with (D1)1\( _\infty \). This completes the proof of part (i) of the statement. Let us prove (ii). Obviously, \( \omega(\tilde{K}_c) \subset \tilde{J}_\infty \). Let’s show the opposite. Following (B), for each \( t \in \tilde{J}_\infty \), either \( t \in \omega(\tilde{K}_c) \) or \( \sigma^k(t) \in \sigma^{-1}(\tilde{K}_c) \), for some \( k \geq 0 \). By (i), \( \sigma^{-1}(\tilde{K}_c) \subset \omega(\tilde{K}_c) \), hence, \( t \in \omega(\tilde{K}_c) \) in any case.

5. Proof of Theorem 1.1

1. Assume the contrary and let \( X \subset J_\infty \) be a compact \( f \)-invariant hyperbolic set. In particular, Corollary 3.1 applies.
2. Replacing \( X \) by its subset if necessary we can assume that \( f : X \rightarrow X \) is a minimal map.
3. \( 0 \notin X \), hence, \( c \notin X \), too.
4. As \( J_\infty \) contains no cycles, \( X \) is an infinite set. If we assume that \( f : X \rightarrow X \) is one-to-one, then \( f : X \rightarrow X \) is an expanding homeomorphism of a compact set, therefore, \( X \) is finite, a contradiction.
5. Thus, \( f : X \rightarrow X \) is not one-to-one. Then, by (E), \( X_c := X \cap K_c \neq \emptyset \). On the other hand, by step 3, \( c \notin X_c \).
6. By (D), \( K_c \) consists of at most two arguments. As any point of \( X \) is accessible, the set \( K_c \) is also non-empty and \( X_c \) consists of either one or two different points. Let \( x_1 \in X_c \) and \( x_2 \in J(f) \) be any other point. Given \( r > 0 \) and \( n \), let \( W_n(x_1, r) \) be a component of \( B(x_1, r) \cap \Omega_n \) (see (C), Sect. 2 where \( \Omega_n \) is defined) that contains the point \( x_1 \). We use repeatedly the following
Claim 1. Given \( \hat{r} > 0 \) and \( \hat{C} > 0 \), there is \( \hat{n} \in \mathbb{N} \) as follows. For \( k = 1, 2 \), suppose that, for some angles \( \hat{t}_k \), the ray \( R_{i_k} \) lands at \( x_k \) and let \( z_k \) be the first intersection of \( R_{i_k} \) with \( \partial B(x_k, \hat{r}/2) \).

Assume: (a) \( G(z_k) > \hat{C} \) for \( k = 1, 2 \), (b) \( |x_1 - x_2| < \hat{r}/2 \), (c) one of the following holds: (i) \( t_n - \hat{t}_n \to 0 \) as \( n \to \infty \), or (ii) \( \hat{t}_1, \hat{t}_2 \) belong to a single component of \( s_n = [t_n, t'_n] \cup [\hat{t}_n, \hat{t}_n] \). Then \( x_2 \in W_n(x_1, \hat{r}) \) for each \( n > \hat{n} \).

Indeed, the length of each component of \( s_n \) goes to zero as \( n \to \infty \). Hence, as \( \hat{r} \) and \( \hat{C} \) are constants and \( n \) is big enough, condition (c) implies that a curve which consists of an arc of \( R_{i_k} \) from \( x_1 \) to \( z_1 \), then the shortest arc of the equipotential containing \( z_1 \) from \( z_1 \) to the first intersection \( u_2 \) with \( R_{i_3} \) and then back along \( R_{i_2} \) from \( u_2 \) to \( x_2 \) belongs to \( \Omega_n \) and \( B(x_1, \hat{r}) \) simultaneously.

7. Fix \( r > 0 \) small enough. Let \( a \in X_c \). By (E) and Steps 2, 3 and 5, \( f^{-1}(a) \) consists of two points which are both in \( X \). Let \( a_{-1} \) be one of them. Consider its (well-defined and unique) backward orbits in \( X \): \( a_{-m}, m = 1, \ldots \). Let \( a' \) be a limit point of the sequences \( a_{-p_n} \), i.e., \( a' = \lim_{i \to \infty} a_{-p_{n_i}} \). As \( a' \) belongs to \( K_c \) and \( X \) at the same time, hence, \( a' \in X_c \).

Claim 2. For all \( i \) large enough, \( a_{-p_{n_i}} \in W_{n_i}(a', r) \).

Indeed, by Corollary 3.1 there is a subsequence \( (n'_i) \) of \( (n_i) \) and a converging sequence \( t_i \in A_{a_{-p_{n'_i}}} \) such that \( t := \lim_{i \to \infty} t_i \) and \( t \in A_{a'} \). We have: \( t_i \in s_{n_i} \) for all \( i \). Then Claim 1 of Step 6 applies for each \( i \) big enough, with \( \hat{r} = r, \hat{C} = C(r/2), x_1 = a', x_2 = a_{-p_{n'_i}}, \hat{t}_1 = t, \hat{t}_2 = t_i \) and \( z_1, z_2 \) defined by this data as in Claim 1. Indeed, (a) holds by By Corollary 3.1(3) and (b) is obvious. Moreover, if (c) breaks down, since \( t_i \to t \), then \( t_i \) and \( t \) must lie at the same component of \( s_{n'_i} \).

By the conclusion of Claim 1, \( a_{-p_{n'_i}} \in W_{n'_i}(a', r) \) for each \( i \) big enough. Finally, as \( A_{a'} \) consists of at most two points (therefore, the sequence \( (n_i) \) has at most two limit points), Claim 2 follows.

8. Consider the case \( X_c = \{ a \} \). Let \( f^{-1}(a) = \{ a_{-1}, a_{-1}^* \} \). As \( X_c = \{ a \} \), there is a subsequence \( (n_i) \) such that back ward images \( a_{-p_{n_i}}, a_{-p_{n_i}}^* \) of \( a_{-1}, a_{-1}^* \) respectively tend to the same point \( a \). By Claim 2 Step 7, for each \( i \) large, \( a_{-p_{n_i}}, a_{-p_{n_i}}^* \in W_{n_i}(a, r) \). Therefore, the following two sets (which are preimages of \( W_{n_i}(a, r) \) by \( f^{p_{n_i}} \)): \( V_{n_i} := g_{p_{n_i}, a_{-(p_{n_i})}}(W_{n_i}(a, r)) \) and \( V_{n_i}^* := g_{p_{n_i}, a_{-(p_{n_i})}}^*(W_{n_i}(a, r)) \), are disjoint with their closures (because preimages of \( B(a, r) \) along points of \( X \) shrink exponentially) and both are contained in \( W_{n_i}(a, r) \).
Fix such \( n = n_i \). Then we get that, for every \( j > 0 \), \( 2^j \) preimages of \( a \in J_{c,n} \) by the map \( f^{2^j n} : J_{c,n} \to J_{c,n} \) are contained in the (disjoint) closures of \( V_n \) and \( V_n^* \). As the set of all those preimages are dense in \( J_{c,n} \), we get a contradiction with the fact that \( J_{c,n} \) is a continuum.

9. Consider the remaining case \( X_c = \{ a, b \} \), \( a \neq b \). Note that then each point \( a \) and \( b \) is accessible by a single ray \( R_{t(a)} \) and \( R_{t(b)} \) respectively. Hence, any point \( u \) from the grand orbits of \( a \) and \( b \) is a landing point of precisely one ray \( R_{t_u} \). Let us prove that \( f^{-1}(X_c) \subset X \). Let \( f(w) = x \in X_c \). By Lemma 4.1 one can find \( y \in X_c \) and \( m_i \to \infty \) such that \( \sigma^{m_i}(t_y) \to t_w \) and \( f^{m_i}(y) \) tends to some point \( \tilde{w} \in X \). By Corollary 3.1, \( t_w \in A_{\tilde{w}} \). But \( \sigma(t_w) = t_x \), hence, \( f(\tilde{w}) = x \) and \( A_{\tilde{w}} = \{ t_w \} \). Thus \( w = \tilde{w} \in X \).

10. We have just proved that \( \{ a_{-1}, a_{-1}^* \} = f^{-1}(a) \subset X \) and \( \{ b_{-1}, b_{-1}^* \} = f^{-1}(b) \subset X \). Now, we repeat the consideration as in Step 8. The sequences \( a_{-(p_n)}, a_{-(p_n)}^*, b_{-(p_n)}, b_{-(p_n)}^* \) must have all their limit points in \( X_c \). As \( r > 0 \) is small enough, \( \overline{B(a,r)} \cap \overline{B(b,r)} = \emptyset \). By Claim 2 of Step 7, for each \( n \) large, all 4 points \( a_{-(p_n)}, a_{-(p_n)}^*, b_{-(p_n)}, b_{-(p_n)}^* \) are in \( W_n(a,r) \cup W_n(b,r) \). Fixing \( n \) large, for each disk \( B(x,r) \), \( x \in \{ a, b \} \), there are two univalent branches of \( f^{-p_n} \) which are defined in \( B(x,r) \) such that they map \( W_n(a,r) \cup W_n(b,r) \) inside \( W_n(a,r) \cup W_n(b,r) \). Hence, for every \( j > 0 \), \( 4^j \) preimages of \( X_c \in J_{c,n} \) by the map \( f^{2^j n} : J_{c,n} \to J_{c,n} \) are contained in the (disjoint) closures of \( W_n(a,r) \) and \( W_n(b,r) \). As the set of all those preimages are dense in \( J_{c,n} \), we again get a contradiction with the fact that \( J_{c,n} \) is a continuum.

**Remark 5.1.** The combinatorial property for quadratic polynomials of Lemma 4.1 implies that if \( X \subset J_\infty \) is a minimal hyperbolic set then \( f^{-1}(X) \cap J_\infty = X \) provided \( f \) is quadratic, and this leads to a contradiction. Therefore, a small modification of the proof gives us the following claim for infinitely-renormalizable unicritical polynomial \( f(z) = z^d + c \) with any \( d \geq 2 \): \( J_\infty \) contains no hyperbolic sets \( X \) such that \( f : X \to X \) is minimal and \( f^{-1}(X) \cap J_\infty = X \).

6. Final remarks

A hyperbolic set of a rational map always carries an invariant measure with a positive Lyapunov exponent. Conjecturally, \( J_\infty \) as in Theorem 1.1 never carries such a measure. We cannot prove this conjecture in the full generality so far, but we can easily prove at least that \( F := f|_{J_\infty} \) is not "chaotic". Namely,
1. Every $F$-invariant probability measure $\mu$ has zero entropy, $h_\mu(F) = 0$.

2. Topological entropy of $F$ is zero, $h_{\text{top}}(F) = 0$.

Proving it we can assume $\mu$ is ergodic due to Ergodic Decomposition Theorem, see e.g. [26, Theorem 2.8.11a]. Start by observing that, for every $n$ and $0 \leq j < p_n$, $\mu(f^j(J_n)) = 1/p_n$, hence, $\mu$ has no atoms and $\mu(K) = 0$ for every component $K$ of $J_\infty$. Therefore, if $J'_\infty = J_\infty \setminus \cup_{n \in \mathbb{Z}} f^n(K_0)$ where $K_0$ is the component of $J_\infty$ containing 0 (see (B) of Sect. [2]), then $F : J'_\infty \to J'_\infty$ is an automorphism. Suppose to the contrary that $h_\mu(F) > 0$. Then 1. follows from Rokhlin entropy formula, [26, Theorem 2.9.7], saying that $h_\mu(F) = \int \log \text{Jac}_\mu(F) d\mu$. Here $\text{Jac}_\mu$ is Jacobian with respect to $\mu$, equal to 1 $\mu$-a.e., since $\mu$ must be supported on $J'_\infty \subset J_\infty$ where $F$ is an automorphism. A condition to be verified to apply Rokhlin formula is the existence of a one-sided countable generator of bounded entropy, proved to exist by Mañé, see e.g. [26, Lemma 11.3.2] and inclusion [26, (11.4.8)], due to positive Lyapunov exponent $\chi_\mu(F) := \int \log |F'| d\mu \geq \frac{1}{2} h_\mu(F) > 0$ (Ruelle’s inequality). Thus $h_\mu(F) > 0$ has led to a contradiction.

2. follows from 1. by variational principle $h_{\text{top}}(F) = \sup_\mu h_\mu(F)$. Compare item 4 in Section [5]. Here finiteness of $X$ is replaced by zero entropy.

The same proof with obvious modifications holds for $f(z) = z^d + c$, $d \geq 2$.

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