THE PRIMITIVE IDEAL SPACE OF THE PARTIAL-ISOMETRIC CROSSED PRODUCT OF A SYSTEM BY A SINGLE AUTOMORPHISM

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Abstract. Let \((A, \alpha)\) be a system consisting of a \(C^*\)-algebra \(A\) and an automorphism \(\alpha\) of \(A\). We describe the primitive ideal space of the partial-isometric crossed product \(A \times_{\alpha}^{\text{piso}} \mathbb{N}\) of the system by using its realization as a full corner of a classical crossed product and applying some results of Williams and Echterhoff.

1. Introduction

Lindiarni and Raeburn in [8] introduced the partial-isometric crossed product of a dynamical system \((A, \Gamma^+, \alpha)\) in which \(\Gamma^+\) is the positive cone of a totally ordered abelian group \(\Gamma\) and \(\alpha\) is an action of \(\Gamma^+\) by endomorphisms of \(A\). Note that since the \(C^*\)-algebra \(A\) is not necessarily unital, we require that each endomorphism \(\alpha_s\) extends to a strictly continuous endomorphism \(\alpha_s\) of the multiplier algebra \(M(A)\). This for an endomorphism \(\alpha\) of \(A\) happens if and only if there exists an approximate identity \((a_\lambda)\) in \(A\) and a projection \(p \in M(A)\) such that \(\alpha(a_\lambda)\) converges strictly to \(p\) in \(M(A)\). We stress that if \(\alpha\) is extendible, then we may not have \(\alpha(1_{M(A)}) = 1_{M(A)}\). A covariant representation of the system \((A, \Gamma^+, \alpha)\) is defined for which the endomorphisms \(\alpha_s\) are implemented by partial isometries, and the associated partial-isometric crossed product \(A \times_{\alpha}^{\text{piso}} \Gamma^+\) of the system is a \(C^*\)-algebra generated by a universal covariant representation such that there is a bijection between covariant representations of the system and nondegenerate representations of \(A \times_{\alpha}^{\text{piso}} \Gamma^+\). This generalizes the covariant isometric representation theory that uses isometries to represent the semigroup of endomorphisms in a covariant representation of the system (see [3]). The authors of [8], in particular, studied the structure of the partial-isometric crossed product of the distinguished system \((B_{\Gamma^+}, \Gamma^+, \tau)\), where the action \(\tau\) of \(\Gamma^+\) on the subalgebra \(B_{\Gamma^+}\) of \(\ell^\infty(\Gamma^+)^*\) is given by the right translation. Later, in [4], the authors showed that \(A \times_{\alpha}^{\text{piso}} \Gamma^+\) is a full corner in a subalgebra of the \(C^*\)-algebra \(\mathcal{L}(\ell^2(\Gamma^+) \otimes A)\) of adjointable operators on the Hilbert \(A\)-module \(\ell^2(\Gamma^+) \otimes A \cong \ell^2(\Gamma^+, A)\). This realization led them to identify the kernel of the natural homomorphism \(q : A \times_{\alpha}^{\text{piso}} \Gamma^+ \to A \times_{\alpha}^{\text{iso}} \Gamma^+\) as a full corner of the compact operators \(\mathcal{K}(\ell^2(\mathbb{N}) \otimes A)\), when \(\Gamma^+ = \mathbb{N} := \mathbb{Z}^+\). So as an application, they recovered the Pimsner-Voiculescu exact sequence in [10]. Then in their subsequent work [5], they proved that for an extendible \(\alpha\)-invariant ideal \(I\) of \(A\) (see the definition in [1]), the partial-isometric crossed product \(I \times_{\alpha}^{\text{piso}} \Gamma^+\) sits naturally as an ideal in \(A \times_{\alpha}^{\text{piso}} \Gamma^+\) such that \((A \times_{\alpha}^{\text{piso}} \Gamma^+)/(I \times_{\alpha}^{\text{piso}} \Gamma^+) \cong A/I \times_{\alpha}^{\text{piso}} \Gamma^+\). This is actually a generalization of [2, Theorem 2.2]. They then combined these

2010 Mathematics Subject Classification. Primary 46L55.
Key words and phrases. \(C^*\)-algebra, automorphism, partial isometry, crossed product, primitive ideal.
results to show that the large commutative diagram of [8, Theorem 5.6] associated to the system \((B_\Gamma, \Gamma^+, \tau)\) is valid for any totally ordered abelian group, not only for subgroups of \(\mathbb{R}\). In particular, they use this large commutative diagram for \(\Gamma^+ = \mathbb{N}\) to describe the ideal structure of the algebra \(B_\mathbb{N} \times_{\pi}^\text{piso} \mathbb{N}\) explicitly.

Now here we consider a system \((A, \alpha)\) consisting of a C*-algebra \(A\) and an automorphism \(\alpha\) of \(A\). So we actually have an action of the positive cone \(\mathbb{N} = \mathbb{Z}^+\) of integers \(\mathbb{Z}\) by automorphisms of \(A\). In the present work, we want to study \(\text{Prim}(A \times_{\alpha}^\text{piso} \mathbb{N})\), the primitive ideal space of the partial-isometric crossed product \(A \times_{\alpha}^\text{piso} \mathbb{N}\) of the system. Since \(A \times_{\alpha}^\text{piso} \mathbb{N}\) is in fact a full corner of the classical crossed product \((B_\mathbb{Z} \otimes A) \times \mathbb{Z}\) (see [4, §5]), \(\text{Prim}(A \times_{\alpha}^\text{piso} \mathbb{N})\) is homeomorphic to \(\text{Prim}((B_\mathbb{Z} \otimes A) \times \mathbb{Z})\). Therefore it is enough to describe \(\text{Prim}((B_\mathbb{Z} \otimes A) \times \mathbb{Z})\). To do this, we apply the results on describing the primitive ideal space (ideal structure) of the classical crossed products from [12, 6]. So we consider the following two conditions:

1. when \(A\) is separable and abelian;
2. when \(A\) is separable and \(\mathbb{Z}\) acts on \(\text{Prim} A\) freely (see [2]).

For the first condition, by applying a theorem of Williams, \(\text{Prim}((B_\mathbb{Z} \otimes A) \times \mathbb{Z})\) is homeomorphic to a quotient space of \(\Omega(B_\mathbb{Z}) \times \Omega(A) \times \mathbb{T}\), where \(\Omega(B_\mathbb{Z})\) and \(\Omega(A)\) are the spectrums of the C*-algebras \(B_\mathbb{Z}\) and \(A\) respectively (recall that the dual \(\hat{\mathbb{Z}}\) is identified with \(\mathbb{T}\) via the map \(z \mapsto (\gamma_z : n \mapsto z^n)\)). By computing \(\Omega(B_\mathbb{Z})\), we parameterize the quotient space as a disjoint union, and then we precisely identify the open sets. For the second condition, we apply a result of Echterhoff which shows that \(\text{Prim}((B_\mathbb{Z} \otimes A) \times \mathbb{Z})\) is homeomorphic to the quasi-orbit space of \(\text{Prim}(B_\mathbb{Z} \otimes A) = B_\mathbb{Z} \times \text{Prim} A\) (see in [2] that this is a quotient space of \(\text{Prim}(B_\mathbb{Z} \otimes A)\)). Again by a similar argument to the first condition, we describe the quotient space and its topology precisely.

We begin with a preliminary section in which we recall the theory of the partial-isometric crossed products, and some discussions on the primitive ideal space of the classical crossed products briefly. In section 3 for a system \((A, \alpha)\) consisting of a C*-algebra \(A\) and an automorphism \(\alpha\) of \(A\), we apply the works of Williams and Echterhoff to describe \(\text{Prim}(A \times_{\alpha}^\text{piso} \mathbb{N})\) using the realization of \(A \times_{\alpha}^\text{piso} \mathbb{N}\) as a full corner of the classical crossed product \((B_\mathbb{Z} \otimes A) \times \mathbb{Z}\). As some examples, we compute the primitive ideal space of \(C(\mathbb{T}) \times_{\alpha}^\text{piso} \mathbb{N}\) where the action \(\alpha\) is given by rotation through the angle \(2\pi\theta\) with \(\theta\) rational and irrational. Moreover the description of the primitive ideal space of the Pimsner-Voiculescu Toeplitz algebra associated to the system \((A, \alpha)\) is completely obtained, as it is isomorphic to \(A \times_{\alpha}^\text{piso} \mathbb{N}\). We also discuss necessary and sufficient conditions under which \(A \times_{\alpha}^\text{piso} \mathbb{N}\) is GCR (postliminal or type I). Finally in the last section, we discuss the primitivity and simplicity of \(A \times_{\alpha}^\text{piso} \mathbb{N}\).

2. Preliminaries

2.1. The partial-isometric crossed product. A partial-isometric representation of \(\mathbb{N}\) on a Hilbert space \(H\) is a map \(V : \mathbb{N} \to B(H)\) such that each \(V_n := V(n)\) is a partial isometry, and \(V_{n+m} = V_n V_m\) for all \(n, m \in \mathbb{N}\).

A covariant partial-isometric representation of \((A, \alpha)\) on a Hilbert space \(H\) is a pair \((\pi, V)\) consisting of a nondegenerate representation \(\pi : A \to B(H)\) and a partial-isometric representation \(V : \mathbb{N} \to B(H)\) such that

\[
\pi(\alpha_n(a)) = V_n \pi(a) V_n^* \quad \text{and} \quad V_n^* V_n \pi(a) = \pi(a) V_n^* V_n
\]
for all $a \in A$ and $n \in \mathbb{N}$.

Note that every system $(A, \alpha)$ admits a nontrivial covariant partial-isometric representation [8, Example 4.6]: let $\pi$ be a nondegenerate representation of $A$ on $H$. Define $\Pi : A \to B(\ell^2(\mathbb{N}, H))$ by $(\Pi(a)\xi)(n) = \pi(\alpha_n(a))\xi(n)$. If

$$H := \text{span}\{\xi \in \ell^2(\mathbb{N}, H) : \xi(n) \in \pi(\alpha_n(1))H \text{ for all } n\},$$

then the representation $\Pi$ is nondegenerate on $H$. Now for every $m \in \mathbb{N}$, define $V_m$ on $H$ by $(V_m\xi)(n) = \xi(n + m)$. Then the pair $(\Pi|_H, V)$ is a partial-isometric covariant representation of $(A, \alpha)$ on $H$. One can see that if we take $\pi$ faithful, then $\Pi$ will be faithful as well, and $H = \ell^2(\mathbb{N}, H)$ whenever $\overline{\alpha}(1) = 1$ (e.g. when $\alpha$ is an automorphism).

**Definition 2.1.** A partial-isometric crossed product of $(A, \alpha)$ is a triple $(B, j_A, j_B)$ consisting of a $C^*$-algebra $B$, a nondegenerate homomorphism $i_A : A \to B$, and a partial-isometric representation $i_\mathbb{N} : \mathbb{N} \to M(B)$ such that:

(i) the pair $(j_A, j_\mathbb{N})$ is a covariant representation of $(A, \alpha)$ in $B$;

(ii) for every covariant partial-isometric representation $(\pi, V)$ of $(A, \alpha)$ on a Hilbert space $H$, there exists a nondegenerate representation $\pi \times V : B \to B(H)$ such that $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_\mathbb{N} = V$; and

(iii) the $C^*$-algebra $B$ is spanned by $\{i_\mathbb{N}(n)^*i_A(a)i_\mathbb{N}(m) : n, m \in \mathbb{N}, a \in A\}$.

By [8, Proposition 4.7], the partial-isometric crossed product of $(A, \alpha)$ always exists, and it is unique up to isomorphism. Thus we write the partial-isometric crossed product $B$ as $A \times^{\text{piso}} \mathbb{N}$.

We recall that by [8, Theorem 4.8], a covariant representation $(\pi, V)$ of $(A, \alpha)$ on $H$ induces a faithful representation $\pi \times V$ of $A \times^{\text{piso}} \mathbb{N}$ if and only if $\pi$ is faithful on the range of $(1 - V_n^*V_n)$ for every $n > 0$ (one can actually see that it is enough to verify that $\pi$ is faithful on the range of $(1 - V^*V)$, where $V := V_1$).

### 2.2. The primitive ideal space of crossed products associated to second countable locally compact transformation groups.

Let $\Gamma$ be a discrete group which acts on a topological space $X$. For every $x \in X$, the set $\Gamma \cdot x := \{s \cdot x : s \in \Gamma\}$ is called the $\Gamma$-orbit of $x$. The set $\Gamma_x := \{s \in \Gamma : s \cdot x = x\}$, which is a subgroup of $\Gamma$, is called the stability group of $x$. We say the $\Gamma$-action is free or $\Gamma$ acts on $X$ freely if $\Gamma_x = \{e\}$ for all $x \in X$. Consider a relation $\sim$ on $X$ such that for $x, y \in X, x \sim y$ if and only if $\Gamma \cdot x = \Gamma \cdot y$. One can see that this is an equivalence relation on $X$. The set of all equivalence classes equipped with the quotient topology is denoted by $\mathcal{O}(X)$ and called the quasi-orbit space, which is always a $T_0$-topological space. The equivalence class of each $x \in X$ is denoted by $\mathcal{O}(x)$ and called the quasi-orbit of $x$.

Now let $\Gamma$ be an abelian countable discrete group which acts on a second countable locally compact Hausdorff space $X$. So $(\Gamma, X)$ is a second countable locally compact transformation group with $\Gamma$ abelian. Then the associated dynamical system $(C_0(X), \Gamma, \tau)$ is separable with $\Gamma$ abelian, and so the primitive ideals of $C_0(X) \times_\tau \Gamma$ are known (see [12, Theorem 8.21]). Furthermore, the topology of $\text{Prim}(C_0(X) \times_\tau \Gamma)$ has been beautifully described [12, Theorem 8.39]. So here we want to recall the discussion on $\text{Prim}(C_0(X) \times_\tau \Gamma)$ in brief. See more in [12] that this is indeed a huge and deep discussion.
Let $N$ be a subgroup of $\Gamma$. If we restrict the action $\tau$ to $N$, then we obtain a dynamical system $(C_0(X), N, \tau|_N)$ with the associated crossed product $C_0(X) \rtimes_{\tau|_N} N$. Suppose that $X^\Gamma_N$ is the Green’s $(C_0(X) \otimes C_0(\Gamma/N)) \rtimes_{\tau \otimes \text{id}} \Gamma) - (C_0(X) \rtimes_{\tau|_N} N)$-imprimitivity bimodule whose structure can be found in [12] Theorem 4.22. If $(\pi, V)$ is a covariant representation of $(C_0(X), N, \tau|_N)$, then $\text{Ind}_N^\Gamma(\pi \times V)$ denotes the representation of $C_0(X) \rtimes_{\tau} \Gamma$ induced from the representation $\pi \times V$ of $C_0(X) \rtimes_{\tau|_N} N$ via $X^\Gamma_N$. Now for $x \in X$, let $\varepsilon_x : C_0(X) \to C \simeq \mathbb{B}(C)$ be the evaluation map at $x$ and $w$ a character of $\Gamma_x$. Then the pair $(\varepsilon_x, w)$ is a covariant representation of $(C_0(X), \Gamma_x, \tau|_{\Gamma_x})$ such that the associated representation $\varepsilon_x \times w$ of $C_0(X) \times \Gamma_x$ is irreducible, and hence by [12] Proposition 8.27, $\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times w)$ is an irreducible representation of $C_0(X) \rtimes_{\tau} \Gamma$. So ker $(\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times w))$ is a primitive ideal of $C_0(X) \rtimes_{\tau} \Gamma$. Note if a primitive ideal is obtained in this way, then we say it is induced from a stability group. In fact by [12] Theorem 8.21, all primitive ideals of $C_0(X) \rtimes_{\tau} \Gamma$ are induced from stability groups. Moreover since for every $w \in \hat{\Gamma}$ there is a $\gamma \in \hat{\Gamma}$ such that $w = \gamma|_{\Gamma_x}$, every primitive ideal of $C_0(X) \rtimes_{\tau} \Gamma$ is actually given by the kernel of an induced irreducible representation $\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x})$ correspondent to a pair $(x, \gamma)$ in $X \times \hat{\Gamma}$. To see the description of the topology of $\text{Prim}(C_0(X) \rtimes_{\tau} \Gamma)$, first note that if $(x, \gamma)$ and $(y, \mu)$ belong to $X \times \hat{\Gamma}$ such that $\Gamma \cdot x = \Gamma \cdot y$ (which implies that $\Gamma_x = \Gamma_y$) and $\gamma|_{\Gamma_x} = \mu|_{\Gamma_x}$, then by [12] Lemma 8.34,

$$\ker (\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x})) = \ker (\text{Ind}_{\Gamma_y}^\Gamma(\varepsilon_y \times \mu|_{\Gamma_y})).$$

So define a relation on $X \times \hat{\Gamma}$ such that $(x, \gamma) \sim (y, \mu)$ if

$$\Gamma \cdot x = \Gamma \cdot y \quad \text{and} \quad \gamma|_{\Gamma_x} = \mu|_{\Gamma_x}. \quad \text{(2.2)}$$

One can see that $\sim$ is an equivalence relation on $X \times \hat{\Gamma}$. Now consider the quotient space $X \times \hat{\Gamma}/\sim$ equipped with the quotient topology. Then we have:

**Theorem 2.2.** [12] Theorem 8.39] Let $(\Gamma, X)$ be a second countable locally compact transformation group with $\Gamma$ abelian. Then the map $\Phi : X \times \hat{\Gamma} \to \text{Prim}(C_0(X) \rtimes_{\tau} \Gamma)$ defined by

$$\Phi(x, \gamma) := \ker (\text{Ind}_{\Gamma_x}^\Gamma(\varepsilon_x \times \gamma|_{\Gamma_x}))$$

is a continuous and open surjection, and factors through a homeomorphism of $X \times \hat{\Gamma}/\sim$ onto $\text{Prim}(C_0(X) \rtimes_{\tau} \Gamma)$.

**Remark 2.3.** In the theorem above, note that $\text{Prim}(C_0(X) \rtimes_{\tau} \Gamma)$ is then a second countable space. This is because as it is mentioned in [12] Remark 8.40, the quotient map $q : X \times \hat{\Gamma} \to X \times \hat{\Gamma}/\sim$ is open. Moreover, $X$ and $\hat{\Gamma}$ both are second countable.

Theorem 2.2 can be applied to see that the primitive ideal space of the rational rotation algebra is homeomorphic to $T^2$. We skip it here and refer readers to [12] Example 8.45 for more details.

### 2.3. The primitive ideal space of crossed products by free actions.

Let $(A, \Gamma, \alpha)$ be a classical dynamical system with $\Gamma$ discrete. Then the system gives an action of $\Gamma$ on the spectrum $\hat{A}$ of $A$ by $s \cdot [\pi] := [\pi \circ \alpha_s^{-1}]$ for every $s \in \Gamma$ and $[\pi] \in \hat{A}$ (see [12] Lemma 2.8 and [11] Lemma 7.1]). This also induces an action of $\Gamma$ on $\text{Prim} A$ such that $s \cdot P := \alpha_s(P)$ for each $s \in \Gamma$ and $P \in \text{Prim} A$. 

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[Note: The text above is a natural representation of the given document, formatted for better readability.]
Recall that if \( \pi \) is a (nondegenerate) representation of \( A \) on \( H \) with \( \ker \pi = J \), then \( \text{Ind} \) \( \pi \) denotes the induced representation \( \tilde{\pi} \times U \) of \( A \times_\alpha \Gamma \) on \( \ell^2(\Gamma, H) \) associated to the covariant pair \((\tilde{\pi}, U)\) of \((A, \Gamma, \alpha)\) defined by
\[
(\tilde{\pi}(a)\xi)(s) = \pi(\alpha_s^{-1}(a))\xi(s) \quad \text{and} \quad (U_t\xi)(s) = \xi(t^{-1}s)
\]
for all every \( a \in A, \xi \in \ell^2(\Gamma, H), \) and \( s, t \in \Gamma \). Note that by \( \text{Ind} J \), we mean \( \ker(\text{Ind} \pi) \).

Now let \((A, \Gamma, \alpha)\) be a classical dynamical system in which \( A \) is separable and \( \Gamma \) is an amenable discrete countable group. If \( \Gamma \) acts on \( \text{Prim} A \) freely, then each primitive ideal \( \pi = P \) of \( A \) induces a primitive ideal of \( A \times_\alpha \Gamma \), namely \( \text{Ind} P = \ker(\text{Ind} \pi) \), and the description of \( \text{Prim}(A \times_\alpha \Gamma) \) is completely available:

**Theorem 2.4.** \([6, \text{Corollary 10.16}]\) Suppose in the system \((A, \Gamma, \alpha)\) that \( A \) is separable and \( \Gamma \) is an amenable discrete countable group. If \( \Gamma \) acts on \( \text{Prim} A \) freely, then the map
\[
\mathcal{O}(\text{Prim} A) \rightarrow \text{Prim}(A \times_\alpha \Gamma)
\]
\[
\mathcal{O}(P) \mapsto \text{Ind} P = \ker(\text{Ind} \pi)
\]
is a homeomorphism, where \( \pi \) is an irreducible representation of \( A \) with \( \ker \pi = P \). In particular, \( A \times_\alpha \Gamma \) is simple if and only if every \( \Gamma \)-orbit is dense in \( \text{Prim} A \).

We can apply the above Theorem to see that the irrational rotation algebras are simple. Readers can refer to \([6, \text{Example 10.18}]\) or \([12, \text{Example 8.46}]\) for more details.

### 3. The Primitive Ideal Space of \( A \times_\alpha^\text{piso} \mathbb{N} \) by Automorphic Action

First recall that if \( T \) is the isometry in \( B(\ell^2(\mathbb{N})) \) such that \( T(e_n) = e_{n+1} \) on the usual orthonormal basis \( \{e_n\}_{n=0}^\infty \) of \( \ell^2(\mathbb{N}) \), then we have
\[
\mathcal{K}(\ell^2(\mathbb{N})) = \overline{\text{span}}\{T_n(1 - TT^*)T_m^* : n, m \in \mathbb{N}\}.
\]

Now consider a system \((A, \alpha)\) consisting of a \( C^* \)-algebra \( A \) and an automorphism \( \alpha \) of \( A \). Let the triples \((A \times_\alpha^\text{piso} \mathbb{N}, j_A, v)\) and \((A \times_\alpha \mathbb{Z}, i_A, u)\) be the partial-isometric crossed product and the classical crossed product of the system respectively. Here our goal is to describe the primitive ideal space of \( A \times_\alpha^\text{piso} \mathbb{N} \) and its topology completely. See in \([4]\) that the kernel of the natural homomorphism \( q : (A \times_\alpha^\text{piso} \mathbb{N}, j_A, v) \rightarrow (A \times_\alpha \mathbb{Z}, i_A, u) \) given by \( q(\nu_n^*j_A(a)v_m) = u_n^*j_A(a)u_m \), is isomorphic to the algebra of compact operators \( \mathcal{K}(\ell^2(\mathbb{N})) \) \( \times \) \( A \). Therefore we have a short exact sequence
\[
0 \longrightarrow (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) \xrightarrow{\mu} A \times_\alpha^\text{piso} \mathbb{N} \xrightarrow{q} A \times_\alpha \mathbb{Z} \longrightarrow 0,
\]
where \( \mu(T_n(1 - TT^*)T_m^* \otimes a) = v_n^*j_A(a)(1 - v^*v)v_m \) for all \( a \in A \) and \( n, m \in \mathbb{N} \). So \( \text{Prim}(A \times_\alpha^\text{piso} \mathbb{N}) \) as a set, is given by the sets \( \text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) \) and \( \text{Prim}(A \times_\alpha \mathbb{Z}) \). With no condition on the system, we do not have much information about \( \text{Prim}(A \times_\alpha \mathbb{Z}) \) in general. However, by \([4, \text{Proposition 2.5}]\), we do know that \( \ker q \cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \) is an essential ideal of \( A \times_\alpha^\text{piso} \mathbb{N} \). Therefore \( \text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) \) which is homeomorphic to \( \text{Prim} A \), sits in \( \text{Prim}(A \times_\alpha^\text{piso} \mathbb{N}) \) as an open dense subset. We will identify this open dense subset, namely the primitive ideals \( \{\mathcal{I}_P : P \in \text{Prim} A\} \) of \( \text{Prim}(A \times_\alpha^\text{piso} \mathbb{N}) \) coming from \( \text{Prim} A \), shortly. Moreover see in \([4, \text{§5}]\) that \( A \times_\alpha^\text{piso} \mathbb{N} \) is a full corner of the classical crossed product \((B^\infty Z \otimes A) \times_{\beta \otimes \alpha^{-1}} Z, \) where \( B^\infty Z := \overline{\text{span}}\{1_n : n \in \mathbb{Z}\} \subset \ell^\infty(Z) \), and the action \( \beta \) of \( Z \) on \( B^\infty Z \) is given by translation such that \( \beta_m(1_n) = 1_{n+m} \) for all \( m, n \in \mathbb{Z} \). Thus \( \text{Prim}(A \times_\alpha^\text{piso} \mathbb{N}) \) is homeomorphic to \( \text{Prim}((B^\infty Z \otimes A) \times_{\beta \otimes \alpha^{-1}} Z) \), and
hence it suffices to describe $\text{Prim}((B_2 \otimes A) \times_{\beta_{\alpha^{-1}}} \mathbb{Z})$ and its topology. To do this, we will consider two conditions on the system that make us able to apply a theorem of Williams and a result by Echterhoff. We will also identify those primitive ideals of $A \times_{\alpha}^{\text{piso}} \mathbb{N}$ coming from $\text{Prim}(A \times_{\alpha} \mathbb{Z})$, which form a closed subset of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$.

But first, let us identify the primitive ideals $\mathcal{I}_P$.

**Proposition 3.1.** Let $\pi : A \to B(H)$ be a nonzero irreducible representation of $A$ with $P := \ker \pi$. If the pair $(\Pi, V)$ is defined as in [8] Example 4.6] (see [2], then the associated representation of $(A \times_{\alpha}^{\text{piso}} \mathbb{N}), j_A, v)$, which we denote by $(\Pi \times V, p)$, is irreducible on $\ell^2(\mathbb{N}, H)$, and does not vanish on $\ker q \simeq \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$.

**Proof.** To see that $(\Pi \times V, p)$ is irreducible, we show that every $\xi \in \ell^2(\mathbb{N}, H) \setminus \{0\}$ is a cyclic vector for $(\Pi \times V, p)$, that is $\ell^2(\mathbb{N}, H) = \text{span} \{(\Pi \times V, p)(x) : x \in (A \times_{\alpha}^{\text{piso}} \mathbb{N})\}$. We show that

\[
\mathcal{H} := \text{span}\{(\Pi \times V, p)(v_n^* j_A(a)(1 - v^* v)v_m)(\xi) : a \in A, n, m \in \mathbb{N}\}
\]

equals $\ell^2(\mathbb{N}, H)$ which is enough. By viewing $\ell^2(\mathbb{N}, H)$ as the Hilbert space $\ell^2(\mathbb{N}) \otimes H$, it suffices to see that each $e_n \otimes h$ belongs to $\mathcal{H}$, where $\{e_n\}_{n=0}^\infty$ is the usual orthonormal basis of $\ell^2(\mathbb{N})$ and $h \in H$. Since $\xi \neq 0$ in $\ell^2(\mathbb{N}, H)$, there is $m \in \mathbb{N}$ such that $\xi(m) \neq 0$ in $H$. But $\xi(m)$ is a cyclic vector for the representation $\pi : A \to B(H)$ as $\pi$ is irreducible. Thus we have $\overline{\text{span}}\{\pi(a)(\xi(m)) : a \in A\} = H$, and hence $\text{span}\{e_n \otimes (\pi(a)\xi(m)) : n \in \mathbb{N}, a \in A\}$ is dense in $\ell^2(\mathbb{N}) \otimes H \simeq \ell^2(\mathbb{N}, H)$. So we have to show that $\mathcal{H}$ contains each element $e_n \otimes (\pi(a)\xi(m))$. Calculation shows that

\[
e_n \otimes (\pi(a)\xi(m)) = (V_n^* \Pi(a)(1 - V^* V)V_m)(\xi) = (\Pi \times V, p)(v_n^* j_A(a)(1 - v^* v)v_m)(\xi),
\]

and therefore $e_n \otimes (\pi(a)\xi(m)) \in \mathcal{H}$ for every $a \in A$ and $n \in \mathbb{N}$. So we have $\mathcal{H} = \ell^2(\mathbb{N}, H)$.

To show that $(\Pi \times V, p)$ does not vanish on $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, first note that since $\pi$ is nonzero, $\pi(a)h \neq 0$ for some $a \in A$ and $h \in H$. Now if we take $(1 - TT^*) \otimes a \in \mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, then $((\Pi \times V, p)(\mu((1 - TT^*) \otimes a)) = (\Pi \times V, p)(j(a)(1 - v^* v)) \neq 0$. This is because for $(e_0 \otimes h) \in \ell^2(\mathbb{N}, H)$, we have

\[(\Pi \times V, p)(j_A(a)(1 - v^* v)(e_0 \otimes h)) = \Pi(a)(1 - V^* V)(e_0 \otimes h) = e_0 \otimes \pi(a)h,
\]

which is not zero in $\ell^2(\mathbb{N}, H)$ as $\pi(a)h \neq 0$. \hfill $\square$

**Remark 3.2.** The primitive ideals $\mathcal{I}_P$ are actually the kernels of the irreducible representations $(\Pi \times V, p)$ which form the open dense subset

\[\mathcal{U} := \{\mathcal{I} \in \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \nsubseteq \mathcal{I}\}\]

of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ homeomorphic to $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$. Now $\text{Prim}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)$ itself is homeomorphic to $\text{Prim}A$ via the (Rieffel) homeomorphism $P \mapsto \mathcal{K}(\ell^2(\mathbb{N})) \otimes P$. But $\mathcal{K}(\ell^2(\mathbb{N})) \otimes P$ is the kernel of the irreducible representation $(\text{id} \otimes \pi)$ of $\mathcal{K}(\ell^2(\mathbb{N})) \otimes A$, which indeed equals the restriction $(\Pi \times V, p)|_{\mathcal{K}(\ell^2(\mathbb{N})) \otimes A}$. Therefore we have

\[\mathcal{I}_P \cap (\mathcal{K}(\ell^2(\mathbb{N})) \otimes A) = \ker((\Pi \times V, p)|_{\mathcal{K}(\ell^2(\mathbb{N})) \otimes A}) = \ker(\text{id} \otimes \pi) = \mathcal{K}(\ell^2(\mathbb{N})) \otimes P.
\]

Consequently the map $P \mapsto \mathcal{I}_P$ is a homeomorphism of $\text{Prim}A$ onto the open dense subset $\mathcal{U}$ of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$. 


Now we want to describe the topology of $\text{Prim}((B_Z \otimes A) \times_{\partial \alpha^{-1}} Z) \simeq \text{Prim}(A \times_{piso} \mathbb{N})$ and identify the primitive ideals of $A \times_{piso} \mathbb{N}$ coming from $A \times_{\alpha} Z$ under the following two conditions:

1. when $A$ is separable and abelian, by applying a theorem of Williams, namely Theorem 2.2
2. when $A$ is separable and $Z$ acts on $\text{Prim} A$ freely, by applying Theorem 2.4

3.1. The topology of $\text{Prim}((B_Z \otimes A) \times_{\partial \alpha^{-1}} Z)$ when $A$ is separable and abelian.

Suppose that $A$ is separable and abelian. Then $(B_Z \otimes A) \times_{\partial \alpha^{-1}} Z$ is isomorphic to the crossed product $C_{\mathbb{N}}(\Omega(B_Z \otimes A)) \times_{\alpha} Z$ associated to the second countable locally compact transformation group $(Z, \Omega(B_Z \otimes A))$. Therefore by Theorem 2.2 $\text{Prim}((B_Z \otimes A) \times_{\partial \alpha^{-1}} Z)$ is homeomorphic to $\Omega(B_Z \otimes A) \times \mathbb{T}/ \sim$. But we want to describe $\Omega(B_Z \otimes A) \times \mathbb{T}/ \sim$ precisely. To do this, we need to analyze $\Omega(B_Z \otimes A)$, and since $\Omega(B_Z \otimes A) \simeq \Omega(B_Z) \times \Omega(A)$ (see [11, Theorem B.37] or [11, Theorem B.45]), we have to compute $\Omega(B_Z)$ first.

**Lemma 3.3.** Let $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ be the two-point compactification of $\mathbb{Z}$. Then $\Omega(B_Z)$ is homeomorphic to the open dense subset $\mathbb{Z} \cup \{\infty\}$.

**Proof.** First note that $B_Z$ exactly consists of those functions $f : \mathbb{Z} \to \mathbb{C}$ such that $\lim_{n \to -\infty} f(n) = 0$ and $\lim_{n \to \infty} f(n)$ exists. Thus the complex homomorphisms (irreducible representations) of $B_Z$ are given by the evaluation maps $\{\varepsilon_n : n \in \mathbb{Z}\}$, and the map $\varepsilon_\infty : B_Z \to \mathbb{C}$ defined by $\varepsilon_\infty(f) := \lim_{n \to \infty} f(n)$ for all $f \in B_Z$. So we have $\Omega(B_Z) = \{\varepsilon_n : n \in \mathbb{Z}\} \cup \{\varepsilon_\infty\}$. Note that the kernel of $\varepsilon_\infty$ is the ideal $C_0(\mathbb{Z}) = \text{span}\{1_n - 1_m : n < m \in \mathbb{Z}\}$ of $B_Z$. Now let $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ be the two-point compactification of $\mathbb{Z}$ which is homeomorphic to the subspace $X := \{-1\} \cup \{-1 + 1/(1 - n) : n \in \mathbb{Z}, n < 0\} \cup \{1 - 1/(1 + n) : n \in \mathbb{Z}, n \geq 0\} \cup \{1\}$ of $\mathbb{R}$. Then the map

\[ f \in B_Z \mapsto \tilde{f} \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}), \]

where

\[ \tilde{f}(r) := \begin{cases} 
\lim_{n \to \infty} f(n) & \text{if } r = \infty, \\
n(r) & \text{if } r \in \mathbb{Z}, \text{ and} \\
0 & \text{if } r = -\infty, 
\end{cases} \]

embeds $B_Z$ in $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})$ as the maximal ideal

\[ I := \{\tilde{f} \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}) : \tilde{f}(\infty) = 0\}. \]

Thus it follows that $\Omega(B_Z)$ is homeomorphic to $\tilde{I}$, and $\tilde{I}$ itself is homeomorphic to the open subset

\[ \{\pi \in C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^\wedge : \pi|_I \neq 0\} = \{\tilde{\varepsilon}_r : r \in (\mathbb{Z} \cup \{\infty\})\} \]

of $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^\wedge$ in which each $\tilde{\varepsilon}_r$ is an evaluation map. So by the homeomorphism between $C(\{-\infty\} \cup \mathbb{Z} \cup \{\infty\})^\wedge$ and $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, the open subset $\{\tilde{\varepsilon}_r : r \in (\mathbb{Z} \cup \{\infty\})\}$ is homeomorphic to the open (dense) subset $\mathbb{Z} \cup \{\infty\}$ of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ equipped with the relative topology. Therefore $\Omega(B_Z)$ is in fact
homeomorphic to $\mathbb{Z} \cup \{\infty\}$. One can see that $\mathbb{Z} \cup \{\infty\}$ is indeed a second countable locally compact Hausdorff space with

$$\mathcal{B} := \{\{n\} : n \in \mathbb{Z}\} \cup \{J_n : n \in \mathbb{Z}\}$$

as a countable basis for its topology, where $J_n := \{n, n+1, n+2, \ldots\} \cup \{\infty\}$ for every $n \in \mathbb{Z}$.

\[\] *Remark 3.4.* Before we continue, we need to mention that, if $A$ is a separable $C^*$-algebra (not necessarily abelian), then by [11, Theorem B.45] and using Lemma 3.3, $(C_0(\mathbb{Z}) \otimes A)$ and $(B_\mathbb{Z} \otimes A)$ are homeomorphic to $\mathbb{Z} \times \hat{A}$ and $(\mathbb{Z} \cup \{\infty\}) \times \hat{A}$ respectively. Also, $\text{Prim}(C_0(\mathbb{Z}) \otimes A)$ and $\text{Prim}(B_\mathbb{Z} \otimes A)$ are homeomorphic to $\mathbb{Z} \times \text{Prim} A$ and $(\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A$ respectively (note that these homeomorphisms are $\mathbb{Z}$-equivariant for the action of $\mathbb{Z}$). Since $C_0(\mathbb{Z}) \otimes A$ is an (essential) ideal of $B_\mathbb{Z} \otimes A$, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Z} \times \hat{A} & \begin{array}{c}
\text{id}
\end{array} & \rightarrow \mathbb{Z} \times \hat{A} \\
\downarrow \begin{array}{c}
i
\end{array} & & \downarrow \begin{array}{c}\iota
\end{array} \\
(\mathbb{Z} \cup \{\infty\}) \times \hat{A} & \begin{array}{c}
\Theta
\end{array} & \rightarrow (B_\mathbb{Z} \otimes A) \hat{\otimes} \text{Prim}(B_\mathbb{Z} \otimes A) \\
\end{array}
\begin{array}{c}
\downarrow \begin{array}{c}\iota
\end{array} & & \downarrow \begin{array}{c}\iota
\end{array} \\
(\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A & \begin{array}{c}
\text{id}
\end{array} & \rightarrow (\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A
\end{array}
$$

where $\Theta$ and $\hat{\Theta}$ are the canonical continuous, open surjections, and $\iota$ an $\iota$ are the canonical embedding maps. Now to see how $\mathbb{Z}$ acts on $(\mathbb{Z} \cup \{\infty\}) \times \hat{A}$ (and accordingly on $(\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A$), note that since the crossed products $(C_0(\mathbb{Z}) \otimes A) \times_{\beta \otimes \alpha^{-1}} \mathbb{Z}$ and $(C_0(\mathbb{Z}) \otimes A) \times_{\beta \otimes \text{id}} \mathbb{Z}$ are isomorphic (see [12, Lemma 7.4]), we have

$$n \cdot (m, [\pi]) = (m + n, [\pi]) \quad \text{and} \quad n \cdot (\infty, [\pi]) = (n + \infty, n \cdot [\pi]) = (\infty, [\pi \circ \alpha_n])$$

for all $n, m \in \mathbb{Z}$ and $[\pi] \in \hat{A}$. Accordingly

$$n \cdot (m, P) = (m + n, P) \quad \text{and} \quad n \cdot (\infty, P) = (\infty, \alpha_n^{-1} P)$$

for all $n, m \in \mathbb{Z}$ and $P \in \text{Prim} A$.

So when $A$ is separable and abelian, using Lemma 3.3, $\Omega(B_\mathbb{Z} \otimes A) = (\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$. Now to describe $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) / \sim$, note that by Remark 3.4, $\mathbb{Z}$ acts on $(\mathbb{Z} \cup \{\infty\}) \times \Omega(A)$ as follows:

$$n \cdot (m, \phi) = (m + n, \phi) \quad \text{and} \quad n \cdot (\infty, \phi) = (\infty, \phi \circ \alpha_n)$$

for all $n, m \in \mathbb{Z}$ and $\phi \in \Omega(A)$. Therefore, the stability group of each $(m, \phi)$ is $\{0\}$, and the stability group of each $(\infty, \phi)$ equals the stability group $\mathbb{Z}_\phi$ of $\phi$. Accordingly, the $\mathbb{Z}$-orbit of each $(m, \phi)$ is $\mathbb{Z} \times \{\phi\}$, and the $\mathbb{Z}$-orbit of $(\infty, \phi)$ is $\{\infty\} \times \mathbb{Z} \cdot \phi$, where $\mathbb{Z} \cdot \phi$ is the $\mathbb{Z}$-orbit of $\phi$. So for the pairs (or triples) $((m, \phi), z)$ and $((n, \psi), w)$ of $(\mathbb{Z} \times \Omega(A)) \times \mathbb{T}$, we have

$$((m, \phi), z) \sim ((n, \psi), w) \iff \begin{array}{l}
\mathbb{Z} \cdot (m, \phi) = \mathbb{Z} \cdot (n, \psi) \\
\mathbb{Z} \times \{\phi\} = \mathbb{Z} \times \{\psi\} \\
\mathbb{Z} \times \{\phi\} = \mathbb{Z} \times \{\psi\} \\
(\mathbb{Z} \cup \{\infty\}) \times \{\phi\} = (\mathbb{Z} \cup \{\infty\}) \times \{\psi\} \\
(\mathbb{Z} \cup \{\infty\}) \times \{\phi\} = (\mathbb{Z} \cup \{\infty\}) \times \{\psi\}.
\end{array}$$
The last equivalence follows from the fact that $\Omega(A)$ is Hausdorff. Therefore $((m, \phi), z)$ and $((n, \psi), w)$ are in the same equivalence class in $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ if and only if $\phi = \psi$, while $((m, \phi), z) \sim ((\infty, \psi), w)$ for every $\psi \in \Omega(A)$ and $w \in \mathbb{T}$, because

$$\mathbb{Z} \cdot (\infty, \psi) = \{\infty\} \times \mathbb{Z} \cdot \psi = \{\infty\} \times \overline{\mathbb{Z} \cdot \psi}.$$ 

Thus if $\phi \in \Omega(A)$, then all pairs $((m, \phi), z)$ for every $m \in \mathbb{Z}$ and $z \in \mathbb{T}$ are in the same equivalence class, which can be parameterized by $\phi \in \Omega(A)$. On the other hand, for the pairs $((\infty, \phi), z)$ and $((\infty, \psi), w)$, we have

$$((\infty, \phi), z) \sim ((\infty, \psi), w) \iff \mathbb{Z} \cdot (\infty, \phi) = \mathbb{Z} \cdot (\infty, \psi) \quad \text{and} \quad \gamma_z|_{\mathbb{Z} \cdot \phi} = \gamma_w|_{\mathbb{Z} \cdot \psi}.$$ 

Therefore

$$((\infty, \phi), z) \sim ((\infty, \psi), w) \iff \mathbb{Z} \cdot \phi = \mathbb{Z} \cdot \psi \quad \text{and} \quad \gamma_z|_{\mathbb{Z} \cdot \phi} = \gamma_w|_{\mathbb{Z} \cdot \psi},$$

which means if and only if the pairs $(\phi, z)$ and $(\psi, w)$ are in the same equivalence class in the quotient space $\Omega(A) \times \mathbb{T} / \sim$ homeomorphic to $\text{Prim}(A \times_{\alpha} \mathbb{N})$. Therefore $((\infty, \phi), z) \sim ((\infty, \psi), w)$ in $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ precisely when $(\phi, z) \sim (\psi, w)$ in $\Omega(A) \times \mathbb{T} / \sim$, and hence the class of each $((\infty, \phi), z)$ of the form $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ can be parameterized by the class of $(\phi, z)$ in $\Omega(A) \times \mathbb{T} / \sim$. So we can identify $((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} / \sim$ with the disjoint union

$$\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim).$$

Now we have:

**Theorem 3.5.** Let $(A, \alpha)$ be a system consisting of a separable abelian $C^*$-algebra $A$ and an automorphism $\alpha$ of $A$. Then $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ is homeomorphic to $\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$, equipped with the (quotient) topology in which the open sets are of the form

$$\{ U \subseteq \Omega(A) : U \text{ is open in } \Omega(A) \} \cup \{ U \cup W : U \text{ is a nonempty open subset of } \Omega(A), \text{ and } W \text{ is open in } (\Omega(A) \times \mathbb{T} / \sim) \}.$$ 

**Proof.** Since the quotient map $q : ((\mathbb{Z} \cup \{\infty\}) \times \Omega(A)) \times \mathbb{T} \rightarrow \Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$ is open, as well as $\bar{q} : \Omega(A) \times \mathbb{T} \rightarrow \Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$, for every $n \in \mathbb{Z}$, every open subset $O$ of $\Omega(A)$, and every open subset $V$ of $\mathbb{T}$, the forward image of open subsets $\{n\} \times O \times V$ and $J_n \times O \times V$ by $q$, forms a basis for the topology of $\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$, which is

$$\{ O \subseteq \Omega(A) : O \text{ is open in } \Omega(A) \} \cup \{ O \cup \bar{q}(O \times V) : O \text{ is a nonempty open subset of } \Omega(A), \text{ and } V \text{ is open in } \mathbb{T} \}.$$ 

As the open subsets $\bar{q}(O \times V)$ also form a basis for the quotient topology of $\Omega(A) \times \mathbb{T} / \sim$, we can see that each open subset of $\Omega(A) \sqcup (\Omega(A) \times \mathbb{T} / \sim)$ is either an open subset $U$ of $\Omega(A)$ or of the form $U \cup W$ for some nonempty open subset $U$ in $\Omega(A)$ and some open subset $W$ in $\Omega(A) \times \mathbb{T} / \sim$. \hfill $\Box$

**Remark 3.6.** Under the condition of Theorem 3.5, the primitive ideals of $\text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N})$ coming from $\text{Prim}(A \times_{\alpha} \mathbb{Z})$, which form the closed subset

$$\mathcal{F} := \{ J \in \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \subseteq J \},$$
are the kernels of the irreducible representations \((\mathrm{Ind}_Z^\mathbb{Z}(\phi \times G_\mathbb{Z})_0)\circ q\) corresponding to the equivalence classes of the pairs \((\phi, z)\) in \(\Omega(A) \times \mathbb{T}/\sim\) (again by using Theorem 2.2). Therefore if \(J_{[(\phi, z)]}\) denotes \(\ker(\mathrm{Ind}_Z^\mathbb{Z}(\phi \times G_\mathbb{Z})_0)\circ q\), then \(F = \{J_{[(\phi, z)]} : \phi \in \Omega(A), z \in \mathbb{T}\}\), and the map \([[(\phi, z)] \mapsto J_{[(\phi, z)]}\) is homeomorphism of \(\text{Prim}(A \times_\alpha \mathbb{Z}) \simeq \Omega(A) \times \mathbb{T}/\sim\) onto \(F\).

**Proposition 3.7.** Let \((A, \alpha)\) be a system consisting of a separable abelian \(C^*\)-algebra \(A\) and an automorphism \(\alpha\) of \(A\). Then \(A \times_\alpha^{\text{piso}} \mathbb{N}\) is GCR if and only if \(\mathbb{Z} \setminus \Omega(A)\) is a \(T_0\) space.

**Proof.** By [9, Theorem 5.6.2], \(A \times_\alpha^{\text{piso}} \mathbb{N}\) is GCR if and only if \(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q\) and \(A \times_\alpha \mathbb{Z} \simeq C_0(\Omega(A)) \times_\tau \mathbb{Z}\) are GCR. But since \(A\) is abelian, \(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A\) is automatically CCR, and hence it is GCR. Therefore \(A \times_\alpha^{\text{piso}} \mathbb{N}\) is GCR precisely when \(A \times_\alpha \mathbb{Z}\) is GCR. By [12, Theorem 8.43], \(A \times_\alpha \mathbb{Z}\) is GCR if and only if \(\mathbb{Z} \setminus \Omega(A)\) is a \(T_0\) space. \(\Box\)

**Proposition 3.8.** Let \((A, \alpha)\) be a system consisting of a separable abelian \(C^*\)-algebra \(A\) and an automorphism \(\alpha\) of \(A\). Then \(A \times_\alpha^{\text{piso}} \mathbb{N}\) is not CCR.

**Proof.** Note that \(A \times_\alpha^{\text{piso}} \mathbb{N}\) is CCR if and only if \((B_\mathbb{Z} \otimes \mathbb{A}) \times_{\beta \times \alpha^{-1}} \mathbb{Z} \simeq C_0(\Omega(B_\mathbb{Z} \otimes \mathbb{A})) \times_\tau \mathbb{Z}\) is CCR, because they are Morita equivalent (see [12, Proposition I.43]). Since for the \(\mathbb{Z}\)-orbit of a pair \((m, \phi)\), we have

\[
\mathbb{Z} \cdot (m, \phi) = \mathbb{Z} \times \{\phi\} = \mathbb{Z} \times \{\phi\} = (\mathbb{Z} \cup \{\infty\}) \times \{\phi\},
\]

it follows that \(\mathbb{Z}\)-orbit of \((m, \phi)\) is not closed in \(\Omega(B_\mathbb{Z} \otimes \mathbb{A}) = (\mathbb{Z} \cup \{\infty\}) \times \Omega(A)\).

Therefore by [12, Theorem 8.44], \(C_0(\Omega(B_\mathbb{Z} \otimes \mathbb{A})) \times_\tau \mathbb{Z}\) is not CCR, and hence \(A \times_\alpha^{\text{piso}} \mathbb{N}\) is not CCR. \(\Box\)

**Example 3.9.** (Pimsner-Voiculescu Toeplitz algebra) Suppose \(\mathcal{T}(A, \alpha)\) is the Pimsner-Voiculescu Toeplitz algebra associated to the system \((A, \alpha)\) (see [10]). It was shown in [4, §5] that \(\mathcal{T}(A, \alpha)\) is isomorphic to the partial-isometric crossed product \(A \times_\alpha^{\text{piso}} \mathbb{N}\) associated to the system \((A, \alpha^{-1})\). Therefore when \(A\) is abelian and separable, the description of \(\text{Prim}(\mathcal{T}(A, \alpha))\) follows completely from Theorem 3.5. In particular, for the trivial system \((\mathbb{C}, \text{id})\), \(\mathcal{T}(\mathbb{C}, \text{id})\) is the Toeplitz algebra \(\mathcal{T}(\mathbb{Z})\) of integers isomorphic to \(\mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N}\). So again by Theorem 3.5, \(\text{Prim}(\mathcal{T}(\mathbb{Z}))\) corresponds to the disjoint union \(\{0\} \sqcup \mathbb{T}\) in which every (nonempty) open set is of the form \(\{0\} \sqcup W\) for some open subset \(W\) of \(\mathbb{T}\). This description is known which coincides with the description of \(\text{Prim}(\mathcal{T}(\mathbb{Z}))\) obtained from the well-known short exact sequence \(0 \to \mathcal{K}(\ell^2(\mathbb{N})) \to \mathcal{T}(\mathbb{Z}) \to C(\mathbb{T}) \to 0\).

**Example 3.10.** Consider the system \((C(\mathbb{T}), \alpha)\) in which the action \(\alpha\) is given by rotation through the angle \(2\pi \theta\) with \(\theta\) rational. By using the discussion in [12, Example 8.46], \(\text{Prim}(C(\mathbb{T}) \times_\alpha^{\text{piso}} \mathbb{N})\) can be identified with the disjoint union

\[
\mathbb{T} \sqcup \mathbb{T}^2,
\]

in which by Theorem 3.5 each open set is given by

\[
\{U \subset \mathbb{T} : U \text{ is open in } \mathbb{T}\} \cup \{U \cup W : U \text{ is a nonempty open subset of } \mathbb{T}, \text{and } W \text{ is open in } \mathbb{T}^2\}.
\]
Moreover the orbit space \( \mathbb{Z} \setminus \mathbb{T} \) is homeomorphic to \( \mathbb{T} \), which is obviously \( T_0 \) (in fact Hausdorff). So it follows by Proposition 3.7 that \( C(\mathbb{T}) \times_\alpha^\text{piso} \mathbb{N} \) is GCR.

### 3.2. The topology of \( \text{Prim}((B_\mathbb{Z} \otimes A) \times_{B \otimes \alpha^{-1}} \mathbb{Z}) \) when \( A \) is separable and \( \mathbb{Z} \) acts on \( \text{Prim} A \) freely.

Consider a system \( (A, \alpha) \) in which \( A \) is separable, and \( \mathbb{Z} \) acts on \( \text{Prim} A \) freely. It follows that \( \mathbb{Z} \) acts on \( \text{Prim}(B_\mathbb{Z} \otimes A) \) freely too. This is because, firstly, by [11, Theorem B.45], \( \text{Prim}(B_\mathbb{Z} \otimes A) \) is homeomorphic to \( \text{Prim} B_\mathbb{Z} \times \text{Prim} A \), and hence it is homeomorphic to \( (\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A \). Then \( \mathbb{Z} \) acts on \( (\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A \) such that

\[
n \cdot (m, P) = (m + n, P) \quad \text{and} \quad n \cdot (\infty, P) = (\infty, \alpha_n^{-1}(P))
\]

for all \( n, m \in \mathbb{Z} \) and \( P \in \text{Prim} A \). Therefore the stability group of each \((\infty, P)\) equals the stability group \( \mathbb{Z}_P \) of \( P \), which is \( \{0\} \) as \( \mathbb{Z} \) acts on \( \text{Prim} A \) freely, and stability group of each \((m, P)\) is clearly \( \{0\} \). So in the separable system \( (B_\mathbb{Z} \otimes A, \mathbb{Z}, \beta \otimes \alpha^{-1}) \) (with \( \mathbb{Z} \) abelian), \( \mathbb{Z} \) acts on \( \text{Prim}(B_\mathbb{Z} \otimes A) \simeq (\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A \) freely. Therefore by Theorem 2.4, \( \text{Prim}(B_\mathbb{Z} \otimes A) \) is homeomorphic to the quasi-orbit space \( \text{O}(\text{Prim}(B_\mathbb{Z} \otimes A)) = \text{O}((\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A) \), which describes \( \text{Prim}(A \times_\alpha^\text{piso} \mathbb{N}) \) as well.

We want to describe the quotient topology of \( \text{O}((\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A) \) precisely, and identify the primitive ideals of \( A \times_\alpha^\text{piso} \mathbb{N} \) coming from \( \text{Prim}(A \times_\alpha \mathbb{Z}) \). We have

\[
\text{O}(m, P) = \text{O}(n, Q) \iff \widehat{Z} \cdot (m, P) = \widehat{Z} \cdot (n, Q)
\iff \widehat{Z} \times \{P\} = \widehat{Z} \times \{Q\}
\iff \widehat{Z} \times \{P\} = \widehat{Z} \times \{Q\}
\iff (\mathbb{Z} \cup \{\infty\}) \times \{P\} = (\mathbb{Z} \cup \{\infty\}) \times \{Q\}.
\]

Therefore \( \text{O}(m, P) = \text{O}(n, Q) \) if and only if \( \{P\} = \{Q\} \), and this happens precisely when \( P = Q \) by the definition of the hull-kernel (Jacobson) topology on \( \text{Prim} A \) (that is why the primitive ideal space of any C*-algebra is always \( T_0 \) [9, Theorem 5.4.7]). So all pairs \((m, P)\) for every \( m \in \mathbb{Z} \) have the same quasi-orbit which can be parameterized by \( P \in \text{Prim} A \), and since

\[
\widehat{Z} \cdot (\infty, Q) = \{\infty\} \times \widehat{Z} \cdot Q = \{\infty\} \times \widehat{Z} \cdot Q = \{\infty\} \times \widehat{Z} \cdot Q,
\]

\( \text{O}(m, P) \neq \text{O}(\infty, Q) \) for all \( m \in \mathbb{Z} \) and \( P, Q \in \text{Prim} A \). Moreover

\[
\text{O}(\infty, P) = \text{O}(\infty, Q) \iff \widehat{Z} \cdot (\infty, P) = \widehat{Z} \cdot (\infty, Q)
\iff \{\infty\} \times \widehat{Z} \cdot P = \{\infty\} \times \widehat{Z} \cdot Q.
\]

Thus \( \text{O}(\infty, P) = \text{O}(\infty, Q) \) if and only if \( \widehat{Z} \cdot P = \widehat{Z} \cdot Q \), which means if and only if \( P \) and \( Q \) have the same quasi-orbit \((\text{O}(P) = \text{O}(Q))\) in \( \text{O}(\text{Prim} A) \simeq \text{Prim}(A \times_\alpha \mathbb{Z}) \). So each quasi-orbit \( \text{O}(\infty, P) \) can be parameterized by the quasi-orbit \( \text{O}(P) \) in \( \text{O}(\text{Prim} A) \), and we can therefore identify \( \text{O}((\mathbb{Z} \cup \{\infty\}) \times \text{Prim} A) \) by the disjoint union

\[
\text{Prim} A \sqcup \text{O}(\text{Prim} A).
\]

Then we have:

**Theorem 3.11.** Let \((A, \alpha)\) be a system consisting of a separable C*-algebra \( A \) and an automorphism \( \alpha \) of \( A \). Suppose that \( \mathbb{Z} \) acts on \( \text{Prim} A \) freely. Then \( \text{Prim}(A \times_\alpha^\text{piso} \mathbb{N}) \) is
homeomorphic to \( \text{Prim} A \sqcup \mathcal{O}(\text{Prim} A) \), equipped with the (quotient) topology in which the open sets are of the form
\[
\{ U \subset \text{Prim} A : U \text{ is open in } \text{Prim} A \} \cup \\
\{ U \cup W : U \text{ is a nonempty open subset of } \text{Prim} A, \text{ and } W \text{ is open in } \mathcal{O}(\text{Prim} A) \}.
\]

**Proof.** Note that since by [12, Lemma 6.12], the quasi-orbit map \( q : \text{Prim}(B_Z \otimes A) \rightarrow \mathcal{O}(\text{Prim}(B_Z \otimes A)) \) is continuous and open, the proof follows from a similar argument to the proof of Theorem 3.11. So we skip it here. \( \square \)

**Remark 3.12.** Under the condition of Theorem 3.11, we want to identify the primitive ideals of \( \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) \) coming from \( \text{Prim}(A \times_{\alpha} \mathbb{Z}) \), which form the closed subset
\[
\mathcal{F} := \{ J \in \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) : \mathcal{K}(\ell^2(\mathbb{N})) \otimes A \simeq \ker q \subset J \}
\]
homeomorphic to \( \text{Prim}(A \times_{\alpha} \mathbb{Z}) \simeq \mathcal{O}(\text{Prim} A) \) (see Theorem 2.4). These ideals are actually the kernels of the irreducible representations \( (\text{Ind} \pi) \circ q = (\tilde{\pi} \times U) \circ q \) of \( A \times_{\alpha}^{\text{piso}} \mathbb{N} \), where \( \pi \) is an irreducible representation of \( A \) with \( \ker \pi = P \). But since the pair \( (\tilde{\pi}, U) \) is clearly a covariant partial-isometric representation of \( (A, \alpha) \), one can see that in fact, \( (\text{Ind} \pi) \circ q = \tilde{\pi} \times \text{piso} U \), where \( \tilde{\pi} \times \text{piso} U \) is the associated representation of \( A \times_{\alpha}^{\text{piso}} \mathbb{N} \) corresponding to the pair \( (\tilde{\pi}, U) \). Thus each element of \( \mathcal{F} \) is of the form \( \ker(\tilde{\pi} \times \text{piso} U) \) corresponding to the quasi-orbit \( \mathcal{O}(P) \), and therefore we denote \( \ker(\tilde{\pi} \times \text{piso} U) \) by \( \mathcal{O}(P) \). So the map \( \mathcal{O}(P) \rightarrow \mathcal{O}(P) \) is a homeomorphism of \( \mathcal{O}(\text{Prim} A) \) onto the closed subspace \( \mathcal{F} \) of \( \text{Prim}(A \times_{\alpha}^{\text{piso}} \mathbb{N}) \).

For the following remark, we need to recall that the primitive ideal space of any \( C^* \)-algebra \( A \) is locally compact [7, Corollary 3.3.8]. A locally compact space \( X \) (not necessarily Hausdorff) is called almost Hausdorff if each locally compact subspace \( U \) contains a relatively open nonempty Hausdorff subset (see [12, Definition 6.1.]). If a \( C^* \)-algebra is GCR, then it is almost Hausdorff (see the discussion on pages 171 and 172 of [12]). Finally if \( A \) is separable, then by applying [11, Theorem A.38] and [11, Proposition A.46], it follows that \( \text{Prim} A \) is second countable.

**Remark 3.13.** It follows from [13] that if \( (A, Z, \alpha) \) is a separable system in which \( Z \) acts on \( \hat{A} \) freely, then \( A \times_{\alpha} Z \) is GCR if and only if \( A \) is GCR and every \( Z \)-orbit in \( \hat{A} \) is discrete. But every \( Z \)-orbit in \( \hat{A} \) is discrete if and only if for each \( [\pi] \in \hat{A} \), the map \( Z \rightarrow Z \cdot [\pi] \) defined by \( n \mapsto n \cdot [\pi] = [\pi \circ \alpha_n^{-1}] \) is a homeomorphism, and this statement itself, by [12, Theorem 6.2 (Mackey-Glimm Dichotomy)], is equivalent to saying that the orbit space \( Z \backslash \hat{A} \) is \( T_0 \). Therefore we can rephrase the statement of [13] to say that if \( (A, Z, \alpha) \) is a separable system in which \( Z \) acts on \( \hat{A} \) freely, then \( A \times_{\alpha} Z \) is GCR if and only if \( A \) is GCR and the orbit space \( Z \backslash \hat{A} \) is \( T_0 \).

**Proposition 3.14.** Let \( (A, \alpha) \) be a system consisting of a separable \( C^* \)-algebra \( A \) and an automorphism \( \alpha \) of \( A \). Suppose that \( Z \) acts on \( \hat{A} \) freely. Then \( A \times_{\alpha}^{\text{piso}} \mathbb{N} \) is GCR if and only if \( A \) is GCR and the orbit space \( Z \backslash \hat{A} \) is \( T_0 \).

**Proof.** The proof follows from a similar argument to the proof of Proposition 3.7 and Remark 3.13. \( \square \)

**Example 3.15.** Consider the system \( (C(\mathbb{T}), \alpha) \) in which the action \( \alpha \) is given by rotation through the angle \( 2\pi \theta \) with \( \theta \) irrational. Then \( Z \) acts on \( \text{Prim}(C(\mathbb{T})) = C(\mathbb{T})^\sim = \mathbb{T} \)
freely (see [12, Example 8.45] or [6, Example 10.18]). Therefore by Theorem 3.11 Prim($C(T) \times_{\alpha}^{piso} N$) can be identified with the disjoint union $T \sqcup O(T)$. But the quasi-orbit space $O(T)$ contains only one point as each $Z$-orbit is dense in $T$ (see [12, Lemma 3.29]). Let us parameterize this only point by 0 (note that $O(T)$ is homeomorphic to the primitive ideal space of the irrational rotation algebra $A_\theta := C(T) \times_{\alpha} Z$ which is simple). So Prim($C(T) \times_{\alpha}^{piso} N$) is actually identified with
\[ T \sqcup \{0\}, \]
where each open set is given by
\[ \{U \subset T : U \text{ is open in } T\} \cup \{U \cup \{0\} : U \text{ is a nonempty open subset of } T\}. \]
Here we would like to mention that 0 in $T$ is the zero ideal if and only if Prim($T(A, \alpha)$) contains only one point as each $Z$-orbit is dense in $T$. Therefore by Theorem 3.11, $C(T) \times_{\alpha}^{piso} N$ is not GCR.

**Remark 3.16.** Recall that since the Pimsner-Voiculescu Toeplitz algebra $T(A, \alpha)$ is isomorphic to $A \times_{\alpha^{-1}}^{piso} N$ (see Example 3.9), if $A$ is separable and $Z$ acts on Prim $A$ freely, then the description of Prim($T(A, \alpha)$) is obtained completely from Theorem 3.11.

4. PRIMITIVITY AND SIMPLICITY OF $A \times_{\alpha}^{piso} N$

In this section, we want to discuss the primitivity and simplicity of $A \times_{\alpha}^{piso} N$. Recall that a $C^*$-algebra is called *primitive* if it has a faithful nonzero irreducible representation, and it is called *simple* if it has no nontrivial ideal.

**Theorem 4.1.** Let $(A, \alpha)$ be a system consisting of a $C^*$-algebra $A$ and an automorphism $\alpha$ of $A$. Then $A \times_{\alpha}^{piso} N$ is primitive if and only $A$ is primitive.

*Proof.* If $A \times_{\alpha}^{piso} N$ is primitive, it has a faithful nonzero irreducible representation $\rho : A \times_{\alpha}^{piso} N \to B(H)$. Then since the restriction of $\rho$ to the ideal $K(\ell^2(N)) \otimes A \simeq \ker q$ is nonzero, it gives an irreducible representation of $K(\ell^2(N)) \otimes A$ which is clearly faithful. So it follows that $K(\ell^2(N)) \otimes A$ is primitive, and therefore $A$ must be primitive as well.

Conversely, if $A$ is primitive, then it has a faithful nonzero irreducible representation $\pi$ on some Hilbert space $H$ ($P = \ker \pi = \{0\}$). We show that the associated irreducible representation $(\Pi \times V)\rho$ of $A \times_{\alpha}^{piso} N$ on $\ell^2(N, H)$ is faithful. By [8, Theorem 4.8], it is enough to see that if $\Pi(a)(1 - V^*V) = 0$, then $a = 0$. If $\Pi(a)(1 - V^*V) = 0$, then
\[ \Pi(a)(1 - V^*V)(e_0 \otimes h) = (e_0 \otimes \pi(a)h) = 0 \quad \text{for all } h \in H. \]
It follows that $\pi(a)h = 0$ for all $h \in H$, and therefore $\pi(a) = 0$. Since $\pi$ is faithful, we must have $a = 0$. This completes the proof. $\square$

**Remark 4.2.** Note that Theorem 4.1 simply means that in the homeomorphism $P \mapsto P$ mentioned in Remark 3.22, $P$ is the zero ideal if and only if $I_P$ is the zero ideal. This is because if $A \times_{\alpha}^{piso} N$ is primitive, then its zero ideal as one of its primitive ideals is of the form $I_P$ (coming from Prim $A$), as $K(\ell^2(N)) \otimes A \neq 0$. 

\[ \]
Finally it is not difficult to see that $A \times_{\alpha} \mathbb{N}$ is not simple. This is because as we see, it contains $K(\ell^2(\mathbb{N})) \otimes A$ as a nonzero ideal. Moreover if $K(\ell^2(\mathbb{N})) \otimes A = A \times_{\alpha} \mathbb{N}$, then $A \times_{\alpha} \mathbb{Z} \simeq (A \times_{\alpha} \mathbb{N})/(K(\ell^2(\mathbb{N})) \otimes A)$ must be the zero algebra. So it follows that $A = 0$, which is a contradiction as we have $A \neq 0$. Therefore $A \times_{\alpha} \mathbb{N}$ contains $K(\ell^2(\mathbb{N})) \otimes A$ as a proper nonzero ideal, and hence we have proved the following:

**Theorem 4.3.** Let $(A, \alpha)$ be a system consisting of a C*-algebra $A$ and an automorphism $\alpha$ of $A$. Then $A \times_{\alpha} \mathbb{N}$ is not simple.

**Acknowledgements.** This research is supported by Rachadapisek Sompote Fund for Postdoctoral Fellowship, Chulalongkorn University.

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