Schrödinger Sigma Models and Jordanian Twists

Io Kawaguchi\textsuperscript{1}, Takuya Matsumoto\textsuperscript{2} and Kentaroh Yoshida\textsuperscript{3}

\textsuperscript{*}Department of Physics, Kyoto University Kyoto 606-8502, Japan
\textsuperscript{†}School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

Abstract: We proceed to study the integrable structures of two-dimensional nonlinear sigma models defined on three-dimensional Schrödinger spacetimes. We show that anisotropic Lax pairs are equivalent with isotropic Lax pairs with flat conserved currents under non-local gauge transformations. Then a quite non-trivial realization of the undeformed Yangian symmetry $\mathcal{Y}({\mathfrak{sl}}(2))$ is revealed after an appropriate gauge fixing, which is determined by comparing the gauge transformation to a quantum Jordanian twist. As a result, an exotic symmetry found in [arXiv:1209.4147] may be interpreted as a Jordanian deformation of $\mathcal{Y}({\mathfrak{sl}}(2))$.

Keywords: Integrable Field Theory, Sigma Models, AdS-CFT Correspondence

ArXiv ePrint: 1305.6556[hep-th]

\textsuperscript{1}E-mail: io@gauge.scphys.kyoto-u.ac.jp
\textsuperscript{2}E-mail: tmatsumoto@usyd.edu.au
\textsuperscript{3}E-mail: kyoshida@gauge.scphys.kyoto-u.ac.jp
## Contents

1 Introduction 1  
2 Setup 3  
   2.1 Schrödinger spacetimes 3  
   2.2 Schrödinger sigma models 4  
   2.3 Anisotropic Lax pairs in the right description 5  
   2.4 Exotic symmetry 7  
3 Gauge transformations and Yangian $\mathcal{Y}(\text{sl}(2))$ 8  
   3.1 Non-local gauge transformations 9  
   3.2 The undeformed Yangian algebra 12  
   3.3 The relation to the exotic symmetry 14  
4 The gauge fixing from Jordanian twists 15  
   4.1 Quantum Jordanian twists 15  
   4.2 The Jordanian pull-back 18  
5 The left description revisited 19  
6 The geometric interpretation of twists 21  
7 Conclusion and Discussion 23  
A Twists and currents 24  
B The current algebra for $\mathcal{J}_\mu^{R\pm}$ 25  
C A different gauge fixing 32

## 1 Introduction

One of the most striking achievements in the study of the AdS/CFT correspondence [1–3] is the discovery of the integrable structure (For a comprehensive review, see [4]). On the string-theory side, it is closely related to the classical integrability of...
two-dimensional non-linear sigma models on symmetric cosets [5]. The symmetric cosets which can appear as consistent string backgrounds are classified in [6], including supercharges.

The next step is to consider integrable deformations of AdS spaces and spheres (For the works related to our issue, for example, see [7–11]). The deformed geometries are still represented by cosets [12] and are homogeneous, while the symmetric-space structure is lost. Thus the familiar way in symmetric spaces [13–18] is not applicable to discuss the integrable structure in the deformed cases any more. However, some new aspects appear due to the deformation and the integrable structure is enriched as in analogy with the deformations from the XXX models to the XXZ models.

For example, non-linear sigma models defined on squashed $S^3$ have a rich structure. The squashing breaks the isometry of round $S^3$, $SO(4) = SU(2)_L \times SU(2)_R$ to $SU(2)_L \times U(1)_R$. The sigma models have a couple of Yangians based on $SU(2)_L$ [19]. It is shown that the Yangians are preserved even after adding the Wess-Zumino term [20]. One may also consider an infinite-dimensional symmetry originated from $U(1)_R$. This is a quantum affine algebra at the classical level [21–23]. In the end, two different kinds of infinite-dimensional symmetries are realized simultaneously and hence the integrable structure should be called the “hybrid integrability”. The two algebras are gauge-equivalent [24].

It is also interesting to consider the classical integrable structure of the sigma models defined on three-dimensional Schrödinger spacetimes. The isometry is given by $SL(2, \mathbb{R})_L \times U(1)_R$. It has been shown that the sigma models possess a pair of $SL(2, \mathbb{R})_L$ Yangians and an affine extension of $q$-deformed Poincaré algebra (named an exotic symmetry) [25, 26]. The latter symmetry is based on $U(1)_R$. The mathematical background of the affine extension has not been elucidated.

In this paper we proceed to study the affine extension of $q$-deformed Poincaré algebra. Our aim is to unveil the mathematical background. We first construct a conserved current satisfying the flatness condition by performing a gauge transformation for the monodromy matrix based on $U(1)_R$. Then the BIZZ construction [15] leads to $SL(2, \mathbb{R})_R$ Yangian in a quite non-trivial representation. As a result, the gauge transformation can be identified as undoing the classical Jordanian twist. That is, the exotic symmetry found in [26] may be interpreted as the Jordanian twist of the Yangian.

This paper is organized as follows. In Section 2, we introduce the classical action of the Schrödinger sigma models and give a short review of the monodromy matrix based on $U(1)_R$. In Section 3, we construct a conserved current satisfying the flatness condition by performing a gauge transformation to the monodromy matrix. Then the
BIZZ construction leads to $SL(2, \mathbb{R})_R$ Yangian in a quite non-trivial representation. In Section 4, the mathematical formulation of quantum Jordanian twist is introduced. The relation to the gauge-fixing condition is explained from the point of view of the Jordanian twist. In Section 5, we argue the relation of the Jordanian twist to the Yangian generators based on $SL(2, \mathbb{R})_L$ through the left-right duality. In Section 6, we discuss the geometric interpretation of the classical Jordanian twist. Section 7 is devoted to conclusion and discussion.

The Appendix part might seem lengthy because messy computations, for example, of current algebras are explained in detail. We, however, believe that the calculation process would be valuable for the readers who are interested in the detailed computations. In Appendix A, we will show the detailed computations in performing a gauge transformation to the monodromy matrix and rewriting the conserved currents. Appendix B is devoted to illustrate the computation of the current algebra. In Appendix C, as a side note, we try to argue another flat conserved current, for the twist operation does not correspond to the Jordanian twist. Then the resulting algebra is a deformed Yangian which contains the third order terms even at the level-zero.

2 Setup

Let us first introduce the classical action of two-dimensional non-linear sigma models defined on three-dimensional Schrödinger spacetimes. Then we give a brief review of the Lax pairs and the monodromy matrices based on $U(1)_R$. The behavior of them under gauge transformations is explained. Finally we give a review on an infinite dimensional extension of the $q$-deformed Poincaré algebra, which is referred to as the exotic symmetry.

2.1 Schrödinger spacetimes

Schrödinger spacetimes in three dimensions are known as null-like deformations of AdS$_3$. The metric is given by [27–29]

$$ds^2 = L^2 \left[ -2e^{-2\rho}dudv + d\rho^2 - Ce^{-4\rho}dv^2 \right]. \quad (2.1)$$

The deformation is measured by a real constant parameter $C$. When $C = 0$, the metric (2.1) describes the AdS$_3$ space with the curvature radius $L$ and the isometry is $SO(2, 2) = SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$. When $C \neq 0$, $SL(2, \mathbb{R})_R$ is broken to $U(1)_R$ while $SL(2, \mathbb{R})_L$ is preserved. That is, the global symmetry that survives the deformation is $SL(2, \mathbb{R})_L \times U(1)_R$ in total. This symmetry yields two distinct pictures to describe the classical dynamics, as we will see later.
It is convenient to rewrite the metric (2.1) in terms of the left-invariant one-form,
\[ J = g^{-1}dg = -T^+ J^- - T^- J^+ + T^2 J^2 , \]

where \( g = e^{2\alpha T^+} e^{2\beta T^2} e^{2\gamma T^-} \in SL(2, \mathbb{R}) \), \( J^a = 2\mathrm{Tr}(T^a J) \).

The \( \mathfrak{sl}(2) \) generators \( T^a \) (\( a = \pm, 2 \)) satisfy the commutation relations \([T^a, T^b] = \varepsilon^{abc} T^c\), where the structure constants \( \varepsilon^{abc} \) are the totally anti-symmetric tensor normalized as \( \varepsilon^{-+2} = +1 \). The generators are normalized as \( \mathrm{Tr}(T^a T^b) = \frac{1}{2} \gamma^{ab} \) with the Killing metric \( \gamma^{ab} \). The \( \mathfrak{sl}(2) \) indices are raised and lowered by \( \gamma^{ab} \) and its inverse \( \gamma_{ab} \) respectively.

From here on, we work with the fundamental representation,
\[ T^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad T^- = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad T^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]

Note that \( T^2 \) is taken as the Cartan generator.

By using this one-form \( J \), the metric (2.1) can be rewritten into the following form,
\[ ds^2 = \frac{L^2}{2} \left[ \mathrm{Tr}(J^2) - 2C \mathrm{Tr}(T^- J)^2 \right] . \]

This expression enables us to regard the Schrödinger spacetimes as null-deformations of \( \text{AdS}_3 \). Now it is manifest that the metric (2.4) is invariant under the \( SL(2, \mathbb{R})_L \times U(1)_R \) transformations defined as
\[ g \mapsto e^{\beta_a T^a} g e^{-\alpha T^-} , \quad J \mapsto e^{\alpha T^-} J e^{-\alpha T^-} \]
with any \( \beta_a \) (\( a = \pm, 2 \)), \( \alpha \in \mathbb{R} \).

2.2 Schrödinger sigma models

Let us introduce here the classical action of two-dimensional non-linear sigma models defined on Schrödinger spacetimes in three dimensions. For simplicity, we shall refer to those as Schrödinger sigma models.

With the metric (2.4), the classical action and the Lagrangian is given by
\[ S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \ L[J] \quad \text{with} \quad L[J] = -\eta^{\mu\nu} \left[ \mathrm{Tr}(J_\mu J_\nu) - 2C \mathrm{Tr}(T^- J_\mu) \mathrm{Tr}(T^- J_\nu) \right] . \]

The base space is a two-dimensional Minkowski spacetime spanned by the coordinates \( x^\mu = (t, x) \) with the metric \( \eta^{\mu\nu} = \text{diag}(-1, +1) \). Then one can expand the one-form \( J \) by the world-sheet coordinates as \( J = J_\mu dx^\mu \). The rapidly dumping boundary condition
is taken so that the group-valued field \( g(x) \) approaches a constant element \( g_\infty \) at spatial
infinities\(^1\):

\[
g(x) \to g_\infty \quad \text{with} \quad x \to \pm \infty.
\]

That is, \( J_\mu \) vanishes very rapidly at spatial infinities. This setup is not appropriate
in considering some applications to string theory. However, it is suitable to study
infinite-dimensional symmetries generated by an infinite set of non-local charges in a
well-defined way. The Virasoro constraints are also not taken into account.

Taking a variation of the action (2.6) leads to the equations of motion,

\[
\partial^\mu J_\mu - 2C \text{Tr}(T^- J_\mu)[J^\mu, T^-] = 0,
\]

where it has already been rewritten by using the conservation law of the \( U(1)_R \) current,
\( \partial^\mu J^-_\mu = 0 \). When \( C = 0 \), the Schrödinger sigma models become \( SL(2, \mathbb{R}) \) principal
chiral models.

The Schrödinger sigma models are classically integrable \([25]\). The \( SL(2, \mathbb{R})_L \) and
\( U(1)_R \) symmetries give rise to two descriptions to describe the classical dynamics. One
is the left description based on \( SL(2, \mathbb{R})_L \). The other is the right description based
on \( U(1)_R \). For each of them, Lax pairs and monodromy matrices are constructed. Then
all Lax pairs lead to the identical equations of motion (2.8). The classical integrable
structure is similar to the hybrid one in the cases of squashed \( S^3 \) and warped \( AdS_3 \)
\([21–24]\).

### 2.3 Anisotropic Lax pairs in the right description

For later argument, let us introduce Lax pairs in the right description \([26]\). The expressions
of the Lax pairs are anisotropic to reflect the deformed target space. The anisotropic Lax pairs \( \mathcal{A}_\mu^R(x; \lambda_R) \) are given by\(^2\)

\[
\mathcal{A}_\mu^R(x; \lambda_R) = \frac{1}{1 - \lambda_R^2} \left\{ \begin{array}{c}
T^+ (J^-_\mu - \lambda_R \epsilon_{\mu\nu} J^-,\nu) \\
+ T^2 (-J^2_\mu - \lambda_R \epsilon_{\mu\nu} J^2,\nu \pm \sqrt{C} J^-_\mu) + \sqrt{C} \lambda_R^2 \epsilon_{\mu\nu} J^-,\nu
\end{array} \right\} + \lambda_R^2 \left( \pm \sqrt{C} \epsilon_{\mu\nu} J^2,\nu + CJ^-_\mu \right) \right). \tag{2.9}
\]

\(^1\)Precisely, the group-valued field depends on both time \( t \) and the spatial direction \( x \), namely
\( g = g(t, x) \). Hereafter the time dependence is suppressed like \( g = g(x) \) for simplicity.

\(^2\)We have changed the notation from \([26]\), precisely \( \mathcal{A}_\mu^R(x; \lambda_R)_{[\text{ours}]} = L_\mu^R(x; \lambda_R)_{[26]} \). As we
will see in Section 3, the anisotropy of the Lax pairs is superficial and turns out to be isotropic in essential.
Here $\lambda_{R\pm} \in \mathbb{C}$ are spectral parameters and the anti-symmetric tensor $\epsilon_{\mu\nu}$ is normalized as $\epsilon_{tx} = -\epsilon_{tx} = 1$. The equations of motion (2.8) and the flatness condition of the current $dJ + J \wedge J = 0$ are reproduced from the zero-curvature condition,

$$\left[ \partial_t - \mathcal{A}_t^{R\pm}(x; \lambda_{R\pm}), \partial_x - \mathcal{A}_x^{R\pm}(x; \lambda_{R\pm}) \right] = 0. \quad (2.10)$$

This ensures the integrability of the systems. Indeed, the monodromy matrices

$$U^{R\pm}(\lambda_{R\pm}) := \text{Pexp} \left[ \int_{-\infty}^{\infty} dx \mathcal{A}_x^{R\pm}(x; \lambda_{R\pm}) \right] \quad (2.11)$$

are conserved for any values of $\lambda_{R\pm} \in \mathbb{C}$, due to the condition (2.10);

$$\frac{d}{dt} U^{R\pm}(\lambda_{R\pm}) = 0. \quad (2.12)$$

This conservation law leads to an infinite number of conserved charges.

**General properties of Lax pairs and monodromy matrices.** In general, Lax pairs may be regarded as gauge fields and it is transformed under gauge transformations.

In the next section we will discuss a gauge transformation of $\mathcal{A}_\mu^{R\pm}(x; \lambda_{R\pm})$. Before that, it is worth seeing general transformation properties of Lax pairs under gauge transformations.

Let us denote an arbitrary Lax connection as $L_\mu(x; \lambda)$. Then the gauge transformation law is given by

$$[L_\mu(x; \lambda)]^{f(x)} := f(x)^{-1} L_\mu(x; \lambda) f(x) - f(x)^{-1} \partial_\mu f(x), \quad (2.13)$$

where $f(x)$ is an arbitrary $SL(2, \mathbb{R})$-valued function of the world-sheet. This transformation preserves the zero-curvature condition (2.10) because (2.13) can be rewritten with the covariant derivative as follows:

$$\partial_\mu - [L_\mu(x; \lambda)]^{f(x)} = f(x)^{-1} (\partial_\mu - L_\mu(x; \lambda)) f(x). \quad (2.14)$$

For any $SL(2, \mathbb{R})$-valued functions $g(x)$ and $h(x)$, the gauge transformation by the product $g(x)h(x)$ is equivalent to the product of the gauge transformations by $g(x)$ and $h(x)$:

$$[L_\mu(x; \lambda)]^{g(x)h(x)} = [[L_\mu(x; \lambda)]^{g(x)}]^{h(x)}. \quad (2.15)$$

Due to (2.14), the associated monodromy matrices are also transformed in the adjoint way,

$$[U(\lambda)]^{f} = f(-\infty)^{-1} U(\lambda) f(+\infty) \quad \text{with} \quad U(\lambda) := \text{Pexp} \left[ \int_{-\infty}^{\infty} dx \mathcal{L}_x(x; \lambda) \right].$$

The spatial boundary values $f(\pm \infty)$ are constant under the boundary condition (2.7) and hence the resulting monodromy matrix is still conserved.
2.4 Exotic symmetry

In the present case, due to the deformation, a part of the isometry of $\text{AdS}_3$, $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ is broken to $SL(2,\mathbb{R})_L \times U(1)_R$. Here we shall focus upon the breaking of $SL(2,\mathbb{R})_R$ to $U(1)_R$. The Noether current associated with $U(1)_R$ is given by

$$j_{\mu}^{-} = -J_{\mu}^{-}. \quad (2.16)$$

As the isometry of the target space, the $2, +$ components of $\mathfrak{sl}(2,\mathbb{R})$ are broken. However, as the symmetry of the sigma models, there still exist the conserved charges corresponding to these components. Those are realized in a non-local way and the conserved currents are given by [25, 26]

$$j_{\mu}^{R,2} = -e^{\sqrt{C}} \chi^{-}(J_{\mu}^{2} + \sqrt{C} \epsilon_{\mu\nu} J_{\nu}^{-}) ,$$
$$j_{\mu}^{R,+} = -e^{\sqrt{C}} \chi^{-}(J_{\mu}^{+} + \sqrt{C} \epsilon_{\mu\nu} J_{\nu}^{+} + CJ_{\mu}^{-}) . \quad (2.17)$$

Here $\chi^{-}(x)$ is the non-local field defined as

$$\chi^{-}(x) := -\frac{1}{2} \int_{-\infty}^{\infty} dy \, \epsilon(x-y) j_{\mu}^{R,-}(y) , \quad (2.18)$$

and $\epsilon(x)$ is the signature function defined as $\epsilon(x) := \theta(x) - \theta(-x)$ with the step function $\theta(x)$. The associated conserved charges are given by

$$Q^{R,a} = \int_{-\infty}^{\infty} dx \, \tilde{j}_{\mu}^{R,a}(x) \quad (a = \pm, 2) . \quad (2.19)$$

Note that the non-local descriptions are intrinsic to the sigma model realizations rather than the isometry.

In addition to (2.17), there is another set of conserved currents $\tilde{j}_{\mu}^{R,a}$ represented by

$$\tilde{j}_{\mu}^{-} = j_{\mu}^{-} = -J_{\mu}^{-} ,$$
$$\tilde{j}_{\mu}^{2} = -e^{\sqrt{C}} \chi^{-}(J_{\mu}^{2} - \sqrt{C} \epsilon_{\mu\nu} J_{\nu}^{-}) ,$$
$$\tilde{j}_{\mu}^{+} = -e^{\sqrt{C}} \chi^{-}(J_{\mu}^{+} - \sqrt{C} \epsilon_{\mu\nu} J_{\nu}^{+} + CJ_{\mu}^{-}) . \quad (2.20)$$

The currents $\tilde{j}_{\mu}^{R,a}$ are related to $j_{\mu}^{R,a}$ by flipping the signature of $\sqrt{C}$, where $j_{\mu}^{-}$ does not contain $\sqrt{C}$ and hence $\tilde{j}_{\mu}^{-}$ coincides with $j_{\mu}^{-}$. The associated charges

$$\tilde{Q}^{R,a} = \int_{-\infty}^{\infty} dx \, \tilde{j}_{\mu}^{R,a}(x) \quad (a = \pm, 2) \quad (2.21)$$

are also conserved.
The Poisson algebra of $Q^{R,a}$ is computed in [25] by imposing the canonical commutation relations among the dynamical variables $X(x) \in \{v(x), \rho(x), u(x)\}$ and the conjugate momenta,

$$\{X(x), \Pi_X(y)\}_P = \delta(x-y) \quad \text{with} \quad \Pi_X(y) = \frac{\partial L}{\partial \dot{X}}(y).$$

(2.22)

The dot means time-derivative $\dot{X} = \frac{\partial}{\partial t} X$. The resulting algebra turns out to be the classical analogue of $q$-deformed two-dimensional Poincaré algebra [30, 31];

$$\{Q^{R,2}, Q^{R,+}\}_P = Q^{R,+} \cosh(\xi Q^{R,-}), \quad \{Q^{R,2}, Q^{R,-}\}_P = -\frac{\sinh(\xi Q^{R,-})}{\xi},$$

$$\{Q^{R,-}, Q^{R,+}\}_P = Q^{R,2}.$$  

(2.23)

Note that the deformation parameter has been redefined from $C$ to $\xi$ as

$$\xi := \sqrt{C^2}. \quad (2.24)$$

This algebra is also referred as a non-standard $q$-deformation of $U(\mathfrak{sl}(2))$ with $q = e^\xi$ in comparison to the standard quantum deformation $U_q(\mathfrak{sl}(2))$ [32–34] (see also sec. 6.4 F in [35]). Since the relations in (2.23) are invariant under the sign flip $\xi \rightarrow -\xi$, another set of the charges $\tilde{Q}^{R,a}$ also generates a copy of the $q$-Poincaré algebra.

Notably, taking the mixed commutators $\{Q^{R,a}, \tilde{Q}^{R,b}\}_P$ into account, the whole algebra generates an infinite-dimensional symmetry. That is, the charges with tilde can be regarded as affine generators. We refer to this symmetry as exotic symmetry [26]. It is remarkable that the exotic symmetry has only semi-infinite levels as well as the Yangians. Even though the commutation relations among the charges are not completely determined yet, the following relation is enough for our later argument,

$$\{\tilde{Q}^{R,2}, Q^{R,2}\}_P = \cosh(\xi Q^{R,-})(\tilde{Q}^{R,2} - Q^{R,2}).$$

(2.25)

### 3 Gauge transformations and Yangian $\mathcal{Y}(\mathfrak{sl}(2))$

In this section we shall rewrite the exotic symmetry into the undeformed Yangian $\mathcal{Y}(\mathfrak{sl}(2))$ by performing a gauge transformation.

Let us first summarize some properties of the exotic symmetry. The list of them gives rise to an important hint to find out the gauge transformation. The properties are the following:

---

3 In the case of squashed spheres, the same technique leads to a quantum affine algebra $U_q(\hat{\mathfrak{sl}}(2))$ [23].
1. The charge algebra of the exotic symmetry exhibits the semi-infinite levels such as Yangians (For the concrete shape of root lattice, see [26]).

2. The spectral parameter relations in the left-right duality is the same as the ones in $SL(2)$ principal chiral models.

3. The classical $r/s$-matrices associated with the exotic symmetry suggest that the right description also belongs to the rational class$^4$.

These properties strongly suggest that the isotropic Lax pairs should exist in the right description.

If this observation is true, then there should be the gauge transformations between the anisotropic Lax pairs $A_{\mu}^{R\pm}(x; \lambda_{R\pm})$ in (2.9) and the isotropic ones. Then the isotropic Lax pairs are expected to take the following form

$$\mathcal{L}_{\mu}^{R\pm}(x; \lambda_{R\pm}) = \mathcal{J}_{\mu}^{R\pm} - \lambda_{R\pm} \epsilon_{\mu\nu} \mathcal{J}^{R\pm,\nu} \frac{1}{1 - \lambda_{R\pm}^2},$$

(3.1)

together with a conserved current $\mathcal{J}_{\mu}^{R\pm}(x)$ which satisfies the flatness condition.

We will construct concretely the isotropic Lax pairs with flat conserved currents from the anisotropic ones by performing non-local gauge transformations below. Then we present the current algebras for the flat conserved currents. The BIZZ procedure [15] is applicable to generate an infinite number of conserved charges. The resulting charge algebras are shown to be the undeformed Yangian $\mathcal{Y}(sl(2))$. The classical $r/s$-matrices for the isotropic Lax pairs are also computed by following the work [36]. Finally we discuss the relation between the exotic symmetry and the undeformed Yangian.

3.1 Non-local gauge transformations

The first we have to do is to find out a gauge transformation from the anisotropic Lax pair to the desired isotropic ones.

To get a hint, let us compare the asymptotic behavior around $\lambda_{R\pm} = \infty$. The isotropic Lax pairs $\mathcal{L}_{\mu}^{R\pm}(x; \lambda_{R\pm})$ are expected to vanish at $\lambda_{R\pm} = \infty$ from (3.1). On the other hand, the asymptotic behavior of the anisotropic Lax pairs $A_{\mu}^{R\pm}(x; \lambda_{R\pm})$ is fixed from the concrete expressions. It gives rise to the $\mathcal{O}(1)$ contributions at $\lambda_{R\pm} = \infty$ and the asymptotic forms are given by

$$A_{\mu}^{R\pm}(x; \infty) = \pm \sqrt{C} \left[ T^{\pm} \epsilon_{\mu\nu} J^{-,\nu} - T^{-} (\epsilon_{\mu\nu} J^{\pm,\nu} \pm \sqrt{C} J^{-}) \right].$$

(3.2)

$^4$Due to the presence of non-ultra local terms, this classification is not definite though.
At this point, one should notice that the Lax pairs that vanishes at \( \lambda_{R\pm} = \infty \) can be constructed from the original \( A_{\mu}^{R\pm}(x; \lambda_{R\pm}) \) by subtracting the remaining contribution (3.2). Then this subtraction is indeed realized as a gauge transformation, as we will see below.

The next is to consider the gauge transformation to realize the subtraction. To define the gauge transformation it is necessary to find out an appropriate gauge functions. Indeed, the gauge functions are given by the following non-local fields

\[
F^{\pm}(x) := \text{Pexp} \left[ \int_{-\infty}^{x} dy \, A_{\times}^{R\pm}(y, \infty) \right] K^{\pm},
\]

where \( K^{\pm} \) are arbitrary constant \( SL(2, \mathbb{R}) \) elements. Those are interpreted as formal solutions of the differential equation

\[
[\partial_{\mu} - A_{\mu}^{R\pm}(x; \infty)] F^{\pm}(x) = 0,
\]

and \( K^{\pm} \) corresponds to the overall constant factor of the solutions.

The gauge transformations of \( A_{\mu}^{R\pm}(x; \lambda_{R\pm}) \) by \( F^{\pm}(x) \) are performed as follows:

\[
[A_{\mu}^{R\pm}(x; \lambda_{R\pm})]^{F^{\pm}(x)} = F^{\pm}(x)^{-1} A_{\mu}^{R\pm}(x; \lambda_{R\pm}) F^{\pm}(x) - F^{\pm}(x)^{-1} \partial_{\mu} F^{\pm}(x)
\]

\[
= F^{\pm}(x)^{-1} \left( A_{\mu}^{R\pm}(x; \lambda_{R\pm}) - A_{\mu}^{R\pm}(x; \infty) \right) F^{\pm}(x). \tag{3.5}
\]

In the second equality the differential equation (3.4) has been used. Now it is obvious that \( [A_{\mu}^{R\pm}(x; \lambda_{R\pm})]^{F^{\pm}(x)} \) vanish at \( \lambda_{R\pm} \to \infty \). Furthermore, as a byproduct, it turns out that \( [A_{\mu}^{R\pm}(x; \lambda_{R\pm})]^{F^{\pm}(x)} \) take the desired isotropic forms,

\[
L_{\mu}^{R\pm}(x; \lambda_{R\pm}) := [A_{\mu}^{R\pm}(x; \lambda_{R\pm})]^{F^{\pm}(x)} = \frac{\mathcal{J}_{\mu}^{R\pm} - \lambda_{R\pm} \epsilon_{\mu\nu} \mathcal{J}_{\rho}^{R\pm, \nu}}{1 - \lambda_{R\pm}^2}. \tag{3.6}
\]

Here the currents \( \mathcal{J}_{\mu}^{R\pm} \) are defined as

\[
\mathcal{J}_{\mu}^{R\pm}(x) := -(gF^{\pm}(x))^{-1} \partial_{\mu}(gF^{\pm}(x)), \tag{3.7}
\]

which satisfy the flatness condition by definition. The conservation law can also be shown by direct computation.

Here we should be careful of the ambiguity of the constant factors \( K^{\pm} \) in (3.3). This ambiguity may be regarded as a kind of residual gauge freedom. The reason is as follows. If we take the gauge-fixing condition as

\[
L_{\mu}^{R\pm}(x; \lambda_{R\pm}) \to 0 \quad \text{as} \quad \lambda_{R\pm} \to \infty, \tag{3.8}
\]
then the gauge potential \( F^{\pm} \) are not completely fixed and then there is still ambiguity of the overall constant factor. This remaining freedom of the gauge transformation is identified with the choice of \( K^{\pm} \).

In fact, any choice of \( K^{\pm} \) works well as the isotropic Lax pairs by the construction. However, the expressions of the associated currents are sensitive to the choice. To see this, let us consider the right multiplication for \( K^{\pm} \) by a constant matrix element \( H \). Then the currents \( J_{\mu}^{R^{\pm}} \) are transformed in the adjoint way, namely

\[
J_{\mu}^{R^{\pm}} \rightarrow H^{-1} J_{\mu}^{R^{\pm}} H \quad \text{under} \quad K^{\pm} \rightarrow K^{\pm} H. \tag{3.9}
\]

This transformation triggers the mixing among the \( \mathfrak{sl}(2) \) components of \( J_{\mu}^{R^{\pm}} \). In particular, the resulting form of the current algebra (as well as the charge algebra) depends on the mixing. This fact implies the possibility that the exotic symmetry can be rewritten into some known algebra by a suitable gauge fixing of \( H \). This is the key observation in our argument.

Although we are going to discuss the current algebra hereafter, before that, we have to fix the remaining gauge freedom \( K^{\pm} \). In doing that, we would like to take the gauge that leads to the simplest form of the current algebra. How can we do that? What are hints to do that? The important clue is provided by the mathematical background on quantum Jordanian twists. The observation to find out how to fix the gauge is explained in the next section 4. Here we shall simply give the answer;

\[
K^{\pm} = \exp(\pm 2\xi T^2 Q_{-}^{R}) = \begin{pmatrix} e^{\pm \xi Q_{-}^{R}} & 0 \\ 0 & e^{\mp \xi Q_{-}^{R}} \end{pmatrix}. \tag{3.10}
\]

This gauge fixing leads to the undeformed Yangian, as we will see later. We also discuss an example of the different gauge-fixing in Appendix C.

The next is to show the current algebra under the choice (3.10). First of all, let us derive the expressions of the currents. The non-local fields \( \mathcal{F}^{\pm} \) are given by

\[
\mathcal{F}^{+}(x) = \exp[-2\xi T^{-} e^{-2\xi \chi^{-}(x)}(\chi^{2}(x) - \frac{1}{2}Q_{-}^{R,2})] \exp[+2\xi T^{2}(\chi^{-}(x) + \frac{1}{2}Q_{-}^{R})],
\]

\[
\mathcal{F}^{-}(x) = \exp[+2\xi T^{-} e^{+2\xi \chi^{-}(x)}(\tilde{\chi}^{2}(x) - \frac{1}{2}Q_{-}^{R,2})] \exp[-2\xi T^{2}(\chi^{-}(x) + \frac{1}{2}Q_{-}^{R})]. \tag{3.11}
\]

Then \( J_{\mu}^{R^{\pm}} \) are given by

\[
J_{\mu}^{R^{+},+}(x) = e^{+\xi Q_{-}^{-}} j_{\mu}^{R^{+,+}}(x) + \xi \epsilon_{\mu\nu} \partial^{\nu} \left[ e^{-2\xi (\chi^{-}(x) - \frac{1}{2}Q_{-}^{R})} (\chi^{2}(x) - \frac{1}{2}Q_{-}^{R,2}) \right],
\]

\[
J_{\mu}^{R^{+,+}}(x) = e^{-\xi Q_{-}^{-}} j_{\mu}^{R^{+,+}}(x) - \xi \epsilon_{\mu\nu} \partial^{\nu} \left[ e^{+2\xi (\chi^{-}(x) - \frac{1}{2}Q_{-}^{R})} (\tilde{\chi}^{2}(x) - \frac{1}{2}\tilde{Q}_{-}^{R,2}) \right],
\]

\[
J_{\mu}^{R^{-},2}(x) = -\epsilon_{\mu\nu} \partial^{\nu} \left[ e^{-2\xi \chi^{-}(x)} (\chi^{2}(x) - \frac{1}{2}Q_{-}^{R,2}) \right],
\]

\[
J_{\mu}^{R^{+},2}(x) = \epsilon_{\mu\nu} \partial^{\nu} \left[ e^{+2\xi \chi^{-}(x)} (\tilde{\chi}^{2}(x) - \frac{1}{2}\tilde{Q}_{-}^{R,2}) \right] - 11 -
\[ J^{R,-}_{\mu}(x) = -\epsilon_{\mu\nu} \partial^\nu \left[ e^{+2\xi^-}(x)\left(\chi^2 - \frac{1}{2}Q^{R,2}\right)\right], \]
\[ J^{R,+}_{\mu}(x) = +\frac{1}{2\xi} \epsilon_{\mu\nu} \partial^\nu \left[ e^{-2\xi}(x)\frac{1}{2}Q^{R,-}\right], \]
\[ J^{R,-}_{\mu}(x) = -\frac{1}{2\xi} \epsilon_{\mu\nu} \partial^\nu \left[ e^{+2\xi}(x)\frac{1}{2}Q^{R,-}\right]. \]

Here the non-local field \( \chi^- \) is given in (2.18). New non-local fields \( \chi^2(x) \) and \( \tilde{\chi}^2(x) \) are defined as, respectively,
\[
\chi^2(x) := -\frac{1}{2} \int_{-\infty}^{\infty} dy \epsilon(x-y) j^{R,2}(y), \quad \tilde{\chi}^2(x) := -\frac{1}{2} \int_{-\infty}^{\infty} dy \epsilon(x-y) \tilde{j}^{R,2}(y). \]

For the derivations of (3.11) and (3.12), see Appendix A.

The currents \( J^{R,\pm}_{\mu} \) are quite complicated, but the current algebra takes a simple form,
\[
\{ J^{R,\pm,a}_{t}(x), J^{R,\pm,b}_{t}(y) \}_P = \epsilon^{ab}_{\ c} J^{R,\pm,c}_{t}(x) \delta(x-y),
\{ J^{R,\pm,a}_{t}(x), J^{R,\pm,b}_{x}(y) \}_P = \epsilon^{ab}_{\ c} J^{R,\pm,c}_{x}(x) \delta(x-y) + \gamma^{ab}_{\ \varepsilon} \partial_\varepsilon \delta(x-y),
\{ J^{R,\pm,a}_{t}(x), J^{R,\pm,b}_{x}(y) \}_P = 0. \]

The detail of the computations is explained in Appendix B. Notably, this current algebra is exactly the same as the one in the usual \( SL(2,\mathbb{R}) \) principal chiral models.

### 3.2 The undeformed Yangian algebra

The next is to consider the charge algebra generated by the current \( J^{R,\pm}_{\mu} \).

First of all, let us derive a set of the conserved charges. The current \( J^{R,\pm}_{\mu} \) is non-local but satisfies the flatness condition by the definition (3.7). Thus one can generate an infinite number of conserved charges by applying the BIZZ construction [15]. The first few charges are inductively obtained as
\[
Y^{R,\pm}_{(0)} = \int_{-\infty}^{\infty} dx \ J^{R,\pm}_{t}(x),
\]
\[
Y^{R,\pm}_{(1)} = \frac{1}{4} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \epsilon(x-y) [J^{R,\pm}_{t}(x), J^{R,\pm}_{t}(y)] - \int_{-\infty}^{\infty} dx \ J^{R,\pm}_{x}(x),
\]
\[
Y^{R,\pm}_{(2)} = \frac{1}{12} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \ \epsilon(x-y) \epsilon(x-z) [J^{R,\pm}_{t}(x), J^{R,\pm}_{t}(y)] [J^{R,\pm}_{t}(x), J^{R,\pm}_{x}(y)] - \frac{1}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \epsilon(x-y) [J^{R,\pm}_{t}(x), J^{R,\pm}_{x}(y)] + \int_{-\infty}^{\infty} dx \ J^{R,\pm}_{t}(x). \]

In principle, one can derive the expression with an arbitrary order.
Then the current algebra (3.14) leads to the Poisson brackets of $Y_{(0)}^{R±,a}$ and $Y_{(1)}^{R±,a}$,
\[
\{Y_{(0)}^{R±,a}, Y_{(0)}^{R±,b}\}_P = \varepsilon^{abc} Y_{(0)}^{R±,c},
\{Y_{(1)}^{R±,a}, Y_{(0)}^{R±,b}\}_P = \varepsilon^{abc} Y_{(1)}^{R±,c},
\{Y_{(1)}^{R±,a}, Y_{(1)}^{R±,b}\}_P = \varepsilon^{abc} [\varepsilon Y_{(2)}^{R±,c} + \frac{1}{12} (Y_{(0)}^{R±})^2 Y_{(0)}^{R±,c} - Y_{(0)}^{R±,c}],
\]
up to a subtlety of non-ultra local terms\textsuperscript{5}. These are nothing but the defining relations of the Yangian algebra [32, 33]. Indeed, after some algebra, the Serre relations are also derived,
\[
\{\{Y_{(1)}^{+}, Y_{(1)}^{-}\}, Y_{(1)}^{2}\}_P = \frac{1}{4} (Y_{(1)}^{+} Y_{(0)}^{-} - Y_{(1)}^{+} Y_{(0)}^{-}) Y_{(0)}^{2}.
\]
Thus the set of the conserved charges $Y_{(n)}^{R±,a}$ ($n \geq 0$) generate the Yangian algebra $\mathcal{Y}(\mathfrak{sl}(2))$ in the sense of Drinfeld’s first realization [32, 33].

The charges can also be produced by expanding $U_{R±}(\lambda_{R±})$ around $\lambda_{R±} = \infty$ like
\[
[U_{R±}(\lambda_{R±})]^{F±} = F±(+\infty)^{-1} U_{R±}(\lambda_{R±}) F±(-\infty) = \exp \left[ \sum_{n=0}^{\infty} Y_{(n)}^{R±} \lambda_{R±}^{-n-1} \right].
\]
Note that the gauge transformation plays an important role in finding out the Yangian charges properly. If we naively expand the original $U_{R±}(\lambda_{R±})$ around $\lambda_{R±} = \infty$, then some messy expressions, which cannot be identified with the Yangian charges at a glance, would be obtained.

So far, the two sets of Yangians charges $Y_{(n)}^{R±,a}$ and $Y_{(n)}^{R±,a}$ have been constructed from $U_{R±}(\lambda_{R±})$ and $U_{R±}(\lambda_{R±})$ respectively. Note that $Y_{(n)}^{R±}$ does not coincide with $Y_{(n)}^{R±}$. This is contrast to the two sets of Yangians in the left description, where the currents contain the deformation parameter $\xi = \sqrt{C}/2$ while the dependence does not appear at the charge level\textsuperscript{6}. Summarizing, the two Yangians are degenerate in the left description while it is not the case in the right description.

Before closing this section, let us examine the $r/s$-matrices for $L_{x}^{R±}(x; \lambda_{R±})$ by following the work [36]. With the current algebra (3.14), the following bracket is evaluated as
\[
\{L_{x}^{R±}(x; \lambda_{R±}) \odot L_{x}^{R±}(y; \mu_{R±})\}_P
\]

\textsuperscript{5}In computing the algebra, we have followed a prescription utilized in [17].

\textsuperscript{6}For the level-zero charges, the terms proportional to $\xi$ are integrated out (see (5.2)). For the level-one charges, the integrated forms contain the $\xi$-dependent terms which are linear to the level-zero charges. Such terms can be removed by using the automorphism of Yangian. After all, there is no $\xi$-dependence.
\[ \begin{align*}
&= \left[ R^{\pm}_{\lambda R_{\pm}}(x; \lambda R_{\pm}) - s^{\pm}_{\lambda R_{\pm}}(x; \lambda R_{\pm}), \mathcal{L}^{R^{\pm}_{\lambda R_{\pm}}(x; \lambda R_{\pm}) \otimes 1} \delta(x - y) \\
&+ \left[ R^{\pm}_{\lambda R_{\pm}}(y; \mu R_{\pm}) + s^{\pm}_{\lambda R_{\pm}}(y; \mu R_{\pm}), 1 \otimes \mathcal{L}^{R^{\pm}_{\lambda R_{\pm}}(y; \mu R_{\pm})} \right] \right] \delta(x - y) \\
&- 2s^{\pm}_{\lambda R_{\pm}}(x; \lambda R_{\pm}) \partial \delta(x - y). \end{align*} \]

Here the \( r \)-matrices \( R^{\pm}_{\lambda R_{\pm}}(\mu R_{\pm}) \) and \( s \)-matrices \( s^{\pm}_{\lambda R_{\pm}}(\mu R_{\pm}) \) are given by, respectively,

\[
R^{\pm}_{\lambda R_{\pm}}(\mu R_{\pm}) = \frac{1}{2(\lambda R_{\pm} - \mu R_{\pm})} \left( \frac{\lambda R_{\pm}^2}{1 - \lambda R_{\pm}^2} + \frac{\mu R_{\pm}^2}{1 - \mu R_{\pm}^2} \right) \gamma_{ab} T^a \otimes T^b, \\
s^{\pm}_{\lambda R_{\pm}}(\mu R_{\pm}) = \frac{\lambda R_{\pm} + \mu R_{\pm}}{2(1 - \lambda R_{\pm}^2)(1 - \mu R_{\pm}^2)} \gamma_{ab} T^a \otimes T^b.
\]

These are exactly the same as those in the \( SL(2, \mathbb{R}) \) principal chiral models.

### 3.3 The relation to the exotic symmetry

It is of importance to elucidate the relation between the original exotic symmetry and the Yangian \( \mathcal{Y}(\mathfrak{sl}(2)) \) that is obtained after the gauge transformation. The charges at the level-zero are expressed in terms of the \( q \)-Poincaré generators \( Q^{R,a} \) as follows:

\[
\begin{align*}
\mathcal{Y}^{R+}_{(0)} &= e^{\xi Q^{R,-}} Q^{R,+} + \xi \left( e^{\xi Q^{R,-}} Q^{R,2} \right)^2, \\
\mathcal{Y}^{R,2}_{(0)} &= e^{\xi Q^{R,-}} Q^{R,2}, \\
\mathcal{Y}^{R,-}_{(0)} &= \frac{1 - e^{-2\xi Q^{R,-}}}{2\xi}.
\end{align*}
\]

These relations are obtained from the relations between \( J^{R+}_{\mu} \) and \( j^{R}_{\mu} \) in (3.12). Note that \( \mathcal{Y}^{R+}_{(0)} \) form the undeformed \( \mathcal{U}(\mathfrak{sl}(2)) \). In the mathematical language, the relations in (3.21) are regarded as a homomorphism from \( \mathcal{U}(\mathfrak{sl}(2)) \) to the \( q \)-deformed Poincaré algebra (2.23) at the classical level\(^7\). To express the level-one charge, we also need not only \( q \)-deformed Poincaré charges but also the non-trivial exotic charges \( \tilde{Q}^{R,2} \) and \( \tilde{Q}^{R,+} \) like

\[
\begin{align*}
\mathcal{Y}^{R+}_{(1)} &= \frac{1}{4} e^{\xi Q^{R,-}} \left[ \frac{1}{\xi} \{ \{ \tilde{Q}^{R,2}, Q^{R,+} \}_P, Q^{R,+} \}_P + \{ \tilde{Q}^{R,2}, Q^{R,+} \}_P Q^{R,2} \\
&- (\sinh(\xi Q^{R,-}) \tilde{Q}^{R,2} + \cosh(\xi Q^{R,-}) Q^{R,2} \right] Q^{R,2} \right], \\
\mathcal{Y}^{R,2}_{(1)} &= \frac{1}{4} \left( \{ \tilde{Q}^{R,2}, Q^{R,+} \}_P - \cosh(\xi Q^{R,-}) Q^{R,2} \right) + \frac{1}{4} Q^{R,2} (\tilde{Q}^{R,2} - Q^{R,2}), \\
\mathcal{Y}^{R,-}_{(1)} &= \frac{1}{4} e^{-\xi Q^{R,-}} (\tilde{Q}^{R,2} - Q^{R,2}).
\end{align*}
\]

\(^7\)We expect that an appropriate completion makes this map the isomorphism between \( \mathcal{U}(\mathfrak{sl}(2)) \) and the \( q \)-Poincaré algebra as the Poisson algebra.
The defining relations of the Yangian $\mathcal{Y}(\mathfrak{sl}(2))$ can be shown with the algebra of the exotic symmetry in (2.25). This is an alternative derivation of the Yangian $\mathcal{Y}(\mathfrak{sl}(2))$ without the current algebra of the flat conserved current. The relations in (3.21) and (3.22) are also regarded as a homomorphism from the Yangian $\mathcal{Y}(\mathfrak{sl}(2))$ to the exotic symmetry [26] as the classical Poisson algebra. It is an interesting issue to seek for the quantum-level expressions of the maps (3.21) and (3.22) with the Hopf algebraic structure.

Note that the $\xi \to 0$ limit of (3.21) and (3.22) is non-singular and the charges reduce to the Yangian algebra realized in the usual way in $SL(2, \mathbb{R})$ principal chiral models.

Now we have the two sets of Yangians $\mathcal{Y}^\pm(\mathfrak{sl}(2))$. For instance, the level-zero charges in $\mathcal{Y}^-(\mathfrak{sl}(2))$ are the same as those in (3.21), up to the replacement $(Q^R, \xi) \to (\tilde{Q}^R, -\xi)$. Either of the two Yangians is a subalgebra of the exotic symmetry through the maps (3.21) and (3.22). An interesting question is whether the direct sum of two Yangians $\mathcal{Y}^+(\mathfrak{sl}(2)) \oplus \mathcal{Y}^-(\mathfrak{sl}(2))$ is isomorphic to the exotic symmetry or not. For the moment, it would be difficult to answer it because the defining relations of the exotic symmetry has not completely been fixed yet. Presumably, one can get a clue by comparing them to the relations of the deformed Yangian $\mathcal{Y}_\xi(\mathfrak{sl}(2))$ [38] and its classical limit.

4 The gauge fixing from Jordanian twists

The gauge fixing of the constant part $K^\pm$ in (3.10) was given in an a priori way in the previous section\(^8\). The aim here is to explain how $K^\pm$ in (3.10) can be figured out from the mathematical background on quantum Jordanian twists. We first introduce the twisting procedure for the quantum case [32, 33, 37, 38]. Then we compare it to the gauge transformation as the classical counter-part and determine the explicit forms of $K^\pm$.

4.1 Quantum Jordanian twists

Let us recall the definition of the quasitriangular Hopf algebra introduced by Drinfeld [32, 33]. A triplet $(A, \Delta, R)$, consisting of a Hopf algebra $A$ with the coproduct $\Delta : A \to A \otimes A$ and an invertible element $R \in A \otimes A$, is called the quasitriangular Hopf algebra if $R$ satisfies the following properties:

$$\tilde{\Delta}(a)R = R\Delta(a) \quad \text{for any} \quad a \in A,$$

\(^8\)An example of the different gauge-fixing is discussed in Appendix C.
\[(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (4.1)\]

where \(\tilde{\Delta} := P_{12} \circ \Delta\) with the permutation operator \(P_{12} : A \otimes A \mapsto A \otimes A\). We denote \(\mathcal{R}_{12} = \mathcal{R} \otimes 1, \mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \mathcal{R}_{23} = 1 \otimes \mathcal{R}\) with \(\mathcal{R} = \sum_i a_i \otimes b_i\). As an important proposition, the universal R-matrix \(\mathcal{R}\) automatically satisfies the Yang-Baxter equation

\[\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}, \quad (4.2)\]

due to the properties (4.1). Thus quasitriangular Hopf algebra is the important structure behind quantum integrable models.

The quasitriangular Hopf algebra allows integrable deformations called twists [32, 33, 37]. Let us suppose the existence of an invertible element \(F \in A \otimes A\) satisfying the cocycle condition:

\[F_{12}(\Delta \otimes 1) = F_{23}(1 \otimes \Delta)F, \quad (4.3)\]

where \(F_{12} = F \otimes 1\) and \(F_{23} = 1 \otimes F\). With this twist operator, the twisted coproduct and the universal R-matrix can be introduced as

\[\Delta^{(F)}(a) := F\Delta(a)F^{-1}, \quad \mathcal{R}^{(F)} := \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1} \quad \text{with} \quad \mathcal{F}_{21} = P_{12} \circ \mathcal{F}. \quad (4.4)\]

Then the triplet \((A, \Delta^{(F)}, \mathcal{R}^{(F)})\) is also a quasitriangular Hopf algebra. As a result, the twisted R-matrix \(\mathcal{R}^{(F)}\) also satisfies the Yang-Baxter equation. The twist is referred as the Reshetikhin twist or the Jordanian twist if it is abelian \([F_{ij}, F_{kl}] = 0\) or non-abelian \([F_{ij}, F_{kl}] \neq 0\), respectively.

We are interested here in a deformation of Yangian \(A = \mathcal{Y}(sl(2))\) by the Jordanian twist operator [39]

\[F = \exp \left( \frac{1}{2} \hbar \otimes \ln \gamma \right) = 1 + \sum_{n=1}^{\infty} \frac{\xi^n}{n!} \prod_{l=0}^{n-1} (\hbar + 2l) \otimes f^n = 1 + \xi \hbar \otimes f + \frac{\xi^2}{2} \hbar (\hbar + 2) \otimes f^2 + \cdots, \quad (4.5)\]

where \(\gamma := 1 - 2\xi f\) and \(\xi \in \mathbb{C}\) is a deformation parameter. The generators \(e, h, f\) are the canonical basis of \(sl(2)\). There is a long history in studies of the Jordanian deformations in quantum integrable models. For the related works, for example, see [38–43]. Note that the twist becomes trivial \(F \to 1\) in \(\xi \to 0\) limit. The twist operator \(F\) in (4.5) satisfies the cocycle condition (4.3). the twisted universal R-matrix \(\mathcal{R}^{(F)}\) defined in (4.4) is also a solution of the Yang-Baxter equation (4.2).
To see the relation to our argument, let us consider the fundamental representation
\[ \rho : U(\mathfrak{sl}(2)) \to \text{End}(\mathbb{C}^2), \]
\[ \rho(e) = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(h) = e_{11} - e_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(f) = e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
where \( e_{ij} \) is a \( 2 \times 2 \) matrix of unity \((e_{ij})_{kl} = \delta_{ik}\delta_{jl}\). This is naturally lifted to the Yangian evaluation representation \( \rho_u : \mathcal{Y}(\mathfrak{sl}(2)) \to \text{End}(\mathbb{C}^2) \) with a spectral parameter \( u \in \mathbb{C} \).

For the Yang-Baxter equation with twisted R-matrix, one may consider the image of both sides by \((\rho_u \otimes \rho_v \otimes 1)\) \( \mathcal{R}_{12}^{(F)} \) with \( i = 1, 2 \). This is the RTT relation of twisted Yangian \( \mathcal{Y}_\xi(\mathfrak{sl}(2))^{\text{10}} \) [38]. The fundamental R-matrix \( \mathcal{R}_{12}^{(F)}(u) \in \text{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \) and T-operator \( T^{(F)}(u) \in \text{End}(\mathbb{C}^2 \otimes \mathcal{Y}_\xi(\mathfrak{sl}(2))[[u^{-1}]] \) are computed as
\[
\mathcal{R}_{12}^{(F)}(u) = (\rho \otimes \rho) \mathcal{F}_{21} (1 - P_{12}u^{-1}) \mathcal{F}^{-1} = (\rho \otimes \rho) \mathcal{F}_{21} \mathcal{F}^{-1} - P_{12}u^{-1} \\
= 1 + \xi e_{21} \otimes (e_{11} - e_{22}) - \xi (e_{11} - e_{22}) \otimes e_{21} + \xi^2 e_{21} \otimes e_{21} - e_{ij} \otimes e_{ji} u^{-1},
\]
\[
T^{(F)}(u) = (\rho \otimes 1) \mathcal{F}_{21} \cdot T(u) \cdot (\rho \otimes 1) \mathcal{F}^{-1} = \begin{pmatrix} 1 & 0 \\ \xi h & 1 \end{pmatrix} T(u) \begin{pmatrix} \gamma^{1/2} & 0 \\ 0 & \gamma^{-1/2} \end{pmatrix},
\]
where \( T(u) \) is the generating function of the undeformed Yangian \( T(u) \in \text{End}(\mathbb{C}^2 \otimes \mathcal{Y}(\mathfrak{sl}(2))[[u^{-1}]] \) and \( \gamma^{\pm 1/2} \) are interpreted as the formal power-series,
\[
\gamma^{\pm 1/2} = (1 - 2\xi \Gamma)^{\pm 1/2} = 1 + \sum_{n=1}^{\infty} \frac{\xi^n}{n!} \prod_{l=0}^{n-1} (2l + 1) \Gamma^n.
\]
The expression of \((\rho \otimes \rho) \mathcal{F}_{21} \mathcal{F}^{-1}\) in (4.7) was firstly described by Zakrzewski [43]. The universal form of it was obtained in [41]. The related quantum group is a non-standard quantization of \( U(\mathfrak{sl}(2)) \) given by [31], which is a \( q \)-deformed Poincaré algebra we have already encountered.

\[ \text{10} \] This algebra is often called the deformed Yangian in some literatures. On the other hand, the word, twisted Yangian, is sometimes utilized to refer to the remnant of Yangian at boundary.
Quantum \( \mathcal{Y}(\mathfrak{sl}(2)) \ni T(u) \xrightarrow{T(\mathcal{F})} T(\mathcal{F})(u) \in \mathcal{Y}(\mathfrak{sl}(2)) \)

v.s.

Classical \( \mathcal{Y}_{\text{cl}}(\mathfrak{sl}(2)) \ni \left[ U^R(\lambda_R) \right]^\mathcal{F} \xleftarrow{\mathcal{F}(x)} U^R(\lambda_R) \in \text{Exotic symm.} \)

**Figure 1.** The correspondence between quantum and classical twists. The twisted T-operator (upper right) is the quantum analogue of the anisotropic right monodromy (bottom right) on one hand. The gauge transformed isotropic monodromy (bottom left) is corresponding to the undeformed Yangian T-operator (upper left) on the other hand. The classical Jordanian twist (bottom middle) is working in the opposite direction in contrast to the quantum one (upper middle).

### 4.2 The Jordanian pull-back

We are now ready to compare the classical integrable structure of Schrödinger sigma models to quantum Jordanian twists.

In general, the T-operators and R-matrix in (4.6) in quantum theory correspond to the monodromy matrices and the classical r-matrix in classical theory. In the present case, the untwisted T-operator \( T(u) \) corresponds to the gauge transformed monodromy matrices \( \left[ U^R(\lambda_R) \right]^\mathcal{F} \) associated to the rational Lax pair \( \mathcal{L}_{\mu}^{\pm}(x; \lambda_{R_{\pm}}) \) in (3.6). On the other hand, the classical analogue of the twisted T-operator \( T(\mathcal{F})(u) \) corresponds to the monodromy matrices \( U^R(\lambda_R) \) obtained from the anisotropic right Lax pairs \( \mathcal{A}_{\mu}^{R_{\pm}}(x; \lambda_{R_{\pm}}) \) in (2.9). The relation of the quantities is depicted in Fig. 1.

Indeed, the non-local gauge transformation \( \mathcal{F}(x) \) in (3.3) is parallel to the classical analogue of the quantum Jordanian twist \( \mathcal{F} \) in (4.5), but the direction is opposite\(^{11}\). The gauge transformation corresponds to the inverse of the quantum Jordanian twist (see Fig. 1). Thus the gauge transformation should be called the classical Jordanian twist. The point is that the anisotropic Lax pairs \( \mathcal{A}_{\mu}^{R_{\pm}}(x; \lambda_{R_{\pm}}) \) are already Jordanian twisted at the classical level. The twist can be undone by performing the gauge transformation and hence the isotropic Lax pairs \( \mathcal{L}_{\mu}^{R_{\pm}}(x; \lambda_{R_{\pm}}) \) can be obtained. This is the mathematical background behind the exotic symmetry. That is, the exotic symmetry is identified with a twisted Yangian.

Let us compare the quantum T-operator \( T(u) \) to the monodromy matrix \( \left[ U^R(\lambda_R) \right]^\mathcal{F} \), where we take either of \( \lambda_{R_{\pm}} \) and write it as \( \lambda_R \). Then \( T(u) \) and \( \left[ U^R(\lambda_R) \right]^\mathcal{F} \) are given

---

\(^{11}\)In this section, we only discuss \( \mathcal{F}(x) := \mathcal{F}^+(x) \) and the argument for \( \mathcal{F}^-(x) \) is completely parallel.
by

\[
T(u) = \mathcal{F}_{21}^{-1} T(\mathcal{F})(u) \mathcal{F} = \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix} T(\mathcal{F})(u) \begin{pmatrix} \gamma^{-1/2} & 0 \\ 0 & \gamma^{1/2} \end{pmatrix},
\]

(4.9)

\[
[U_R(\lambda_R)]^F = \mathcal{F}(+\infty)^{-1} U_R(\lambda_R) \mathcal{F}(-\infty) = K^{-1} \begin{pmatrix} e^{\xi Q_{R^-}} & 0 \\ -\sqrt{2} \xi Q_{R^2} & e^{-\xi Q_{R^-}} \end{pmatrix} U_R(\lambda_R) K.
\]

(4.10)

Thus a canonical choice of the constant \(K^\pm\) in (3.3) turns out to be (3.10). The correspondence between (4.9) and (4.10) also tells us that the quantum twists \(\mathcal{F}_{21}\) and \(\mathcal{F}\) may be interpreted as the world-sheet boundary terms of the non-local gauge transformations \(\mathcal{F}(+\infty)\) and \(\mathcal{F}(-\infty)\), respectively.

5 The left description revisited

It is possible to describe the classical dynamics with the left description based on \(SL(2, \mathbb{R})_L\), though we have worked in the right description so far. Let us first recall the left description and then argue the Jordanian twist from the viewpoint of the left description through the left-right duality. In particular, the conserved currents associated with the two Yangians based on \(SL(2, \mathbb{R})_L\) can be rewritten in a simple form by using the twists \(\mathcal{F}^\pm\).

The Lax pairs in the left description based on \(SL(2, \mathbb{R})_L\) are given by [9, 19]

\[
\mathcal{L}_\mu^L(x; \lambda_{L^\pm}) = \mathcal{J}_\mu^L - \lambda_{L^\pm} \epsilon_{\mu\nu} \mathcal{J}^{L^\pm,\nu} \frac{1}{1 - \lambda_{L^\pm}^2} \]

(5.1)

with the improved \(SL(2, \mathbb{R})_L\) Noether current\(^{12}\),

\[
\mathcal{J}_\mu^L = \partial_\mu g \cdot g^{-1} - 2C \text{Tr}(T^- J_\mu) gT^- g^{-1} \pm \sqrt{C} \epsilon_{\mu\nu} \partial^\nu (gT^- g^{-1}).
\]

(5.2)

The last topological terms are required so that the current satisfies the flatness condition.

The right Lax pairs \(\mathcal{A}^{R^\pm}(x; \lambda_{R^\pm})\) in (2.9) are related to the left isotropic Lax pairs in (5.1) through a local gauge transformation by a group element \(g(x)\),

\[
[\mathcal{L}_\mu^L(x; \lambda_{L^\pm})]^g = \mathcal{A}_\mu^{R^\pm}(x; \lambda_{R^\pm}),
\]

(5.3)

\(^{12}\)We have changed the notations from the previous work [26], precisely \(\mathcal{J}_\mu^L_{\text{[ours]}} = j_\mu^{L^\pm}_{[26]}\) and \(\mathcal{L}_\mu^L(x; \lambda_{L^\pm})_{\text{[ours]}} = \mathcal{L}_\mu^L(x; \lambda_{L^\pm})_{[26]}\).
\[
\mathcal{L}_{\mu}^{L\pm}(x; \lambda_{L\pm}) \xrightarrow{g(x)} \mathcal{A}_{\mu}^{R\pm}(x; \lambda_{R\pm})
\]
\[
g_{\mathcal{F}\pm}g^{-1}(x) = g_{\pm}(x)
\]
\[
\mathcal{A}_{\mu}^{L\pm}(x; \lambda_{L\pm}) \xrightarrow{\text{local gauge trans.}} \mathcal{L}_{\mu}^{R\pm}(x; \lambda_{R\pm})
\]

**Figure 2.** The commutative diagram of the gauge transformations. The isotropic left/right Lax pairs \( \mathcal{L}_{\mu}^{L/R\pm}(x; \lambda_{L/R\pm}) \) (upper left/bottom right) and the left/right Lax pairs \( \mathcal{A}_{\mu}^{L/R\pm}(x; \lambda_{L/R\pm}) \) (bottom left/upper right) are related each other through the (non-)local gauge transformations (arrows), satisfying the commutative condition \( g_{\mathcal{F}\pm} = g_{\pm}g \). The sequence (upper left \( \rightarrow \) upper right \( \rightarrow \) bottom right) is given in (5.4).

under the parameter relations \( \lambda_{L\pm} = 1/\lambda_{R\pm} \). Then, taking account of the classical Jordanian twists (3.6) and the composite rule of the gauge transformations (2.15), one can find the sequence of the gauge transformations,

\[
[L_{\mu}^{L\pm}(x; \lambda_{L\pm})]^{g_{\mathcal{F}\pm}} = [A_{\mu}^{R\pm}(x; \lambda_{R\pm})]^{F_{\pm}} = L_{\mu}^{R\pm}(x; \lambda_{R\pm}).
\] (5.4)

The sequence of the dualities is illustrated in Fig. 2.

Now it is obvious from (5.4) that the left isotropic Lax pairs \( L_{\mu}^{L\pm}(x; \lambda_{L\pm}) \) are directly related to the right ones \( L_{\mu}^{R\pm}(x; \lambda_{R\pm}) \) through the gauge transformations by \( g_{\mathcal{F}\pm}(x) \). This implies that the left flat currents (5.2) should be written as a conjugation of the right flat current (3.7), namely

\[
J_{\mu}^{L\pm}(x) = \partial_{\mu}(g_{\mathcal{F}\pm}(x)) \cdot (g_{\mathcal{F}\pm}(x))^{-1}.
\] (5.5)

Indeed, the simple expressions in (5.5) agree with the currents in (5.2). It is now manifest that \( J_{\mu}^{L\pm} \) satisfy the flatness conditions. Note that the currents in (5.5) are obtained from the \( SL(2, \mathbb{R})_L \) current \( \partial_{\mu}g \cdot g^{-1} \) in \( SL(2, \mathbb{R}) \) principal chiral models by the replacement

\[
g \to g_{\pm} := g_{\mathcal{F}\pm}.
\] (5.6)

The dictionary of the left-right duality on some quantities is summarized below:

- Current: \( (g_{\pm})^{-1}J_{\mu}^{L\pm}g_{\pm} = -J_{\mu}^{R\pm} \),
- Lax pair: \( (g_{\pm})^{-1}(\partial_{\mu} - L_{\mu}^{L\pm}(x; \lambda_{L\pm}))g_{\pm} = \partial_{\mu} - L_{\mu}^{R\pm}(x; \lambda_{R\pm}) \),
- Spectral parameter: \( \lambda_{L\pm} = 1/\lambda_{R\pm} \),
- Monodromy matrix: \( g_{\pm}(+\infty)^{-1}U^{L\pm}(\lambda_{L\pm})g_{\pm}(-\infty) = [U^{R\pm}(\lambda_{R\pm})]^{F_{\pm}} \).
Note that the first relations for the currents are not the gauge transformations but identities. The currents are transformed in the same way as the Lax pairs like in (2.13). The others are associated with the gauge transformations by $g^\pm$.

Before closing this section, let us comment on other expressions of Lax pairs in the left description, $A^L_\mu(x; \lambda_{L\pm})$, which appears in Fig. 2. It is easy to derive the expressions,

$$A^L_\mu(x; \lambda_{L\pm}) = [L^L_\mu(x; \lambda_{L\pm})]^{-1}$$

where we have introduced the left non-local field $G^\pm$ so that the diagram in Fig. 2 is commutative;

$$G^\pm(x) := gF^\pm g^{-1}(x).$$

On the other hand, $A^R_\mu(x; \lambda_{R\pm})$ in (2.9) can be rewritten as

$$A^R_\mu(x; \lambda_{R\pm}) = [L^R_\mu(x; \lambda_{R\pm})]^g$$

The expressions in (5.7) are quite similar to those in (5.9). According to this similarity, one may expect the existence of the exotic symmetry even in the left description. It would be interesting to construct the corresponding generators concretely.

### 6 The geometric interpretation of twists

The isotropic Lax pairs for Schrödinger sigma models are obtained from the ones for $SL(2; \mathbb{R})$ principal chiral models by the formal replacement of the group element $g \rightarrow gF^\pm$, as we have seen in the previous section. Here we would like to consider the following questions. What happens to the target space by the replacement? How the Jordanian twists act on the Poincaré coordinates of the undeformed AdS3?

As in the case of squashed $S^3$ [24], the classical Lagrangian of Schrödinger sigma models (2.6) is written in a dipole-like form,

$$L = -\eta^{\mu\nu}\text{Tr}(J^L_\mu J^L_\nu)$$

with the left flat currents (5.5). It should be emphasized that both $J^L_+\text{ and } J^L_-$ are necessary to express the action, while the Lax pairs are constructed from either of them.
Thus, in order to construct Schrödinger spacetimes from AdS$_3$, it is necessary to use two kinds of the Jordanian twists $F^\pm$. The twists $gF^\pm$ take the values in $SL(2; \mathbb{R})$ and hence the new angle variables $(v^\pm, \rho^\pm, u^\pm)$ are formally introduced via the following relations,

$$g = e^{2v^+}T^+ e^{2\rho^+ T} e^{2u^-} \rightarrow g^\pm = gF^\pm = e^{2v^\pm T^+} e^{2\rho^\pm T^2} e^{2u^- T^-}. \quad (6.2)$$

We refer to the above coordinates as the dipole coordinates.

Plugging the definitions of $F^\pm$ in (3.11) into (6.2), the dipole coordinates are related to the usual coordinates of Schrödinger spacetimes,

$$u^\pm = [u \pm 2\xi \int_{-\infty}^{x} dy \left( \dot{\rho} - 2e^{-2\rho}(uv \pm \xi v') \right) \exp \left( \mp 4\xi \int_{y}^{x} dz e^{-2\rho} \dot{v} \right)] \exp \left( \mp 4\xi \int_{x}^{\infty} dy e^{-2\rho} \dot{v} \right),$$

$$v^\pm = v, \quad \rho^\pm = \rho \pm 2\xi \int_{x}^{\infty} dy e^{-2\rho} \dot{v}. \quad (6.3)$$

Here dot and prime mean time and spatial derivatives, respectively, on the world-sheet. In the $\xi \to 0$ limit (equivalently, $C \to 0$), the dipole coordinates $(v^\pm, \rho^\pm, u^\pm)$ reduce to the Poincaré coordinates $(v, \rho, u)$ of AdS$_3$.

The left currents are written in the dipole coordinates as follows:

$$J_{\mu}^{L,\pm} = \partial_{\mu}(gF^\pm) \cdot (gF^\pm)^{-1} = 2T^+ \partial_{\mu}v^\pm - 2v^\pm \partial_{\mu}\rho^\pm + 2e^{-2\rho^\pm}(v^\pm)^2 \partial_{\mu}u^\pm + 2T^2 \partial_{\mu}\rho^\pm - 2e^{-2\rho^\pm}v^\pm \partial_{\mu}u^\pm + 2T^- e^{-2\rho^\pm} \partial_{\mu}u^\pm. \quad (6.4)$$

Then the Lagrangian (6.1) can also be expressed in terms of the dipole coordinates as

$$L = -2\eta^{\mu\nu} \left[ -2e^{-2\rho} \partial_{\mu}u \partial_{\nu}v + \partial_{\mu}\rho \partial_{\nu}\rho - C^2 e^{-2\rho} \partial_{\mu}v \partial_{\nu}v \right]$$

$$= -2\eta^{\mu\nu} \left[ -(e^{-2\rho^+} \partial_{\mu}u^+ \partial_{\nu}v^- + e^{-2\rho^-} \partial_{\mu}u^- \partial_{\nu}v^+) + \partial_{\mu}\rho^+ \partial_{\nu}\rho^- \right]. \quad (6.5)$$

This expression is very impressive because it is quite similar to the classical action of $SL(2, \mathbb{R})$ principal chiral models. In the second line of (6.5), the dependence on the deformation parameter $C$ is encoded into the dipole coordinates. The expression in the dipole coordinates are very curious. It might be interpreted as principal chiral models over a kind of doubled geometry spanned by two distinct AdS$_3$ pieces. It would be interesting to elaborate the geometrical meaning of the dipole coordinates.

Recall that the $SL(2, \mathbb{R})_R$ symmetry in (3.21), which is realized in a nontrivial way after the gauge transformation, is not an isometry of target space but a symmetry of Schrödinger sigma models. This difference should be crucial. So, after the gauge transformation, the action (6.5) looks very close to principal chiral models, but still different from them.
7 Conclusion and Discussion

We have proceeded to study the affine extension of $q$-Poincare algebra in the Schrödinger sigma models. It has been shown that anisotropic Lax pairs are equivalent with isotropic Lax pairs with flat conserved currents under non-local gauge transformations. In this sense, the anisotropic Lax pairs are not anisotropic but isotropic in essential. Then a quite non-trivial realization of the undeformed Yangian symmetry $\mathcal{Y}(\mathfrak{sl}(2))$ has been revealed by comparing the gauge transformation to a quantum Jordanian twist. As a result, the exotic symmetry found in [26] may be interpreted as a Jordanian twist of $\mathcal{Y}(\mathfrak{sl}(2))$.

So far, we have discussed just three-dimensional Schrödinger spacetime (Sch$_3$), but it would be interesting to consider the embedding into the string-theory context. The present result may be directly applicable by considering a subspace like Sch$_3 \times S^1$. It is shown in [44] that the string sigma models correspond to the Jordanian deformations of $SL(2)$ spin chains$^{13}$.

Another interesting issue is to argue the relation between Jordanian twists and TsT transformations in string theory backgrounds. The TsT transformations may be reinterpreted as imposing twisted periodic boundary conditions [46]. The twist may be regarded as the abelian version of the Jordanian twist (the Reshetikhin twist) [37]. In fact, the Schrödinger spacetime can be realized by performing null Melvin twists which contain light-like T-dualities, while space-like T-dualities are assumed in [46]. Probably, it would be possible in general to show that the null Melvin twists [47–49] correspond to the Jordanian twists. We hope that we could report on the result in this direction in the near future.

Furthermore, the understanding of the Jordanian twist in the Schrödinger sigma models would be a key ingredient to consider a generalization from three dimensions to higher dimensions. In particular, it would enable us to study infinite-dimensional symmetries in higher dimensional Schrödinger spacetimes.

Acknowledgments

We would like to thank H. Itoyama, H. Kanno and S. Moriyama for useful discussions. The work of IK was supported by the Japan Society for the Promotion of Science (JSPS). TM also would like to thank A. Molev, K. Oshima and H. Yamane for valuable discussions and comments on the mathematical aspects.

$^{13}$The Jordanian deformations of the XXX models are originally discussed in [45].
Appendix

A Twists and currents

We shall derive here the explicit forms of $F^\pm(x)$ in (3.11) and the flat conserved currents in (3.12). The computations for $F^-(x)$ is completely parallel with that of $F^+(x)$ and hence we will concentrate on $F^+(x)$.

Let us start from the expression (3.3). The path-ordered factor can be rewritten as

$$P\exp\left[ \int_{-\infty}^{x} dy \ A_R^+(y; \infty) \right] = \exp\left[ 2\xi T^2 \chi^-(x) \right] \exp\left[ -2\xi T^- (\chi^2(x) - \frac{1}{2} Q^{R,2}) \right] \exp\left[ -\xi T^2 Q^{R,-} \right].$$  \hspace{1cm} (A.1)

Here we have used the following identity

$$P\exp\left[ \int_{b}^{a} dx \ (T^2 \psi^2(x) - T^- \psi^+(x) - T^+ \psi^-(x)) \right] = e^{\phi(a) T^2} P\exp\left[ -\int_{b}^{a} dx \ (T^- e^{\phi(x)} \psi^+(x) + T^+ e^{-\phi(x)} \psi^-(x)) \right] e^{-\phi(b) T^2},$$ \hspace{1cm} (A.2)

where $\psi^a(x)$ ($a = \pm, 2$) are arbitrary scalar functions and the potential $\phi(x)$ is introduced through $\partial_x \phi(x) = \psi^2(x)$. Note that $\psi^-(x) = 0$ in the present case. Thus we obtain the following form,

$$F^+(x) = P\exp\left[ \int_{-\infty}^{x} dy \ A_R^+(y; \infty) \right] \exp\left[ 2\xi T^2 Q^{R,-} \right] = \exp\left[ 2\xi T^2 \chi^-(x) \right] \exp\left[ -2\xi T^- (\chi^2(x) - \frac{1}{2} Q^{R,2}) \right] \exp\left[ +\xi T^2 Q^{R,-} \right].$$  \hspace{1cm} (A.3)

With the help of the relation

$$e^{h T^2} T^\pm e^{-h T^2} = e^{\pm h T^\pm},$$

the middle exponential factor with $T^-$ can be moved to the left-most. After that, the expression (3.11) has been derived.

The next is to derive (3.12). We start from (3.7). By using the relation (3.4), it is an easy task to rewrite $J_{\mu}^{R_+}(x)$ as follows:

$$J_{\mu}^{R_+}(x) = -(gF^+(x))^{-1} \partial_{\mu} (gF^+(x)) = F^+(x)^{-1} (-J_{\mu} - A_{\mu}^{R_+}(x; \infty)) F^+(x) = F^+(x)^{-1} \left( -T^- e^{-2\xi \chi^-} j_{\mu}^{R_+} + T^2 e^{-2\xi \chi^-} j_{\mu}^{R,-} - T^+ j_{\mu}^{R,-} \right) F^+(x).$$  \hspace{1cm} (A.4)
In the last equality, we have used the right conserved currents \( j^R \). It is convenient to express the twist as \( \mathcal{F}^+(x) = e^{A(x)T^-} e^{B(x)T^2} \) with
\[
A(x) = -2\xi e^{-2\xi\chi^-(x)} \left[ \chi^2(x) - \frac{1}{2} Q^{R,-} \right], \\
B(x) = +2\xi \left[ \chi^-(x) + \frac{1}{2} Q^{R,-} \right]. 
\]
(A.5)

By using the identities
\[
e^{hT^-} T^2 e^{-hT^-} = T^2 + hT^-, \\
e^{hT^-} T^+ e^{-hT^-} = T^+ + hT^2 + \frac{1}{2} h^2 T^-,
\]
the expression (A.4) is further rewritten as
\[
\mathcal{J}^R = -T^- e^{+B} \left[ e^{-2\xi\chi^-} (j^{R,+}_\mu + A j^{R,2}_\mu) + \frac{1}{2} A^2 j^{R,-}_\mu \right] \\
+ T^2 \left[ e^{-2\xi\chi^-} j^{R,2}_\mu + A j^{R,-}_\mu \right] - T^+ e^{-B} j^{R,-}_\mu. 
\]
(A.6)

Note that the non-local fields \( \chi^a \) are the potential of the conserved currents \( \epsilon_{\mu\nu} \partial^\nu \chi^a = -j^{R,a}_\mu \) (\( a = -, 2 \)). Therefore, by using the total derivative, each \( \mathfrak{sl}(2) \)-component can be written into the simple form (3.12),
\[
\mathcal{J}^R_{\mu, +} = e^{+B} \left[ e^{-2\xi\chi^-} (j^{R,+}_\mu + A j^{R,2}_\mu) + \frac{1}{2} A^2 j^{R,-}_\mu \right] = e^{B} \left[ j^{R,+}_\mu + \frac{1}{2} \epsilon_{\mu\nu} \partial^\nu (B A^2) \right], \\
\mathcal{J}^R_{\mu, -} = e^{-2\xi\chi^-} j^{R,2}_\mu + A j^{R,-}_\mu = \frac{1}{2\xi} \epsilon_{\mu\nu} \partial^\nu A, \\
\mathcal{J}^R_{\mu, +,-} = e^{-B} j^{R,-}_\mu = \frac{1}{2\xi} \epsilon_{\mu\nu} \partial^\nu e^{-B}. 
\]
(A.7)

B The current algebra for \( \mathcal{J}^R_{\mu, \pm} \)

Let us compute the current algebra for \( \mathcal{J}^R_{\mu, \pm} \). We are confined to \( \mathcal{J}^R_{\mu, +} \) below. The same argument is applicable to \( \mathcal{J}^R_{\mu, -} \) with the sign flip of \( \sqrt{C} = 2\xi \).

It is helpful to list the Poisson brackets between the components of \( J_\mu = g^{-1} \partial_\mu g \),
\[
\begin{align*}
\{ J^-_{\mu}(x), J^2(y) \}_p &= -J^-_{\mu}(x) \delta(x-y), \\
\{ J^-_{\mu}(x), J^+(y) \}_p &= -J^2_{\mu}(x) \delta(x-y), \\
\{ J^2(x), J^+(y) \}_p &= -(J^+_x + 8\xi J^-_x)(x) \delta(x-y), \\
\{ J^-_{\mu}(x), J^-_{\nu}(y) \}_p &= 0, \\
\{ J^-_{\mu}(x), J^2(y) \}_p &= -J^-_{\mu}(x) \delta(x-y), \\
\{ J^-_{\mu}(x), J^+(y) \}_p &= -J^2_{\mu}(x) \delta(x-y) - \partial_x \delta(x-y), \\
\{ J^2(x), J^-(y) \}_p &= J^-_{\mu}(x) \delta(x-y), \\
\{ J^2(x), J^2(y) \}_p &= \partial_x \delta(x-y), \\
\{ J^2(x), J^+(y) \}_p &= \partial_y \delta(x-y), \\
\end{align*}
\]
(B.1)
\begin{align*}
\{J^2_t(x), J^+_x(y)\}_p &= -J^+_x(x)\delta(x - y), \\
\{J^+_t(x), J^-_x(y)\}_p &= J^2_x(x)\delta(x - y) - \partial_x \delta(x - y), \\
\{J^+_t(x), J^2_x(y)\}_p &= (J^+_x + 4\xi^2 J^-_x)(x)\delta(x - y), \\
\{J^+_t(x), J^+_x(y)\}_p &= 4\xi^2 J^2_x(x)\delta(x - y) + 4\xi^2 \partial_x \delta(x - y).
\end{align*}

Note that the Poisson brackets between the spatial components are zero.

Let us first compute the current algebra for \(j^R_\mu(x)\). For this purpose, we need the Poisson brackets between \(\chi^-(x)\) and \(J^a_\mu(x)\). Since \(\chi^-(x)\) is written as

\[\chi^-(x) = \frac{1}{2} \int_{-\infty}^\infty dy \, \epsilon(x - y) J^-_t(y),\]

the Poisson brackets are given by

\[\{\chi^-(x), J^a_\mu(y)\}_p = \frac{1}{2} \int_{-\infty}^\infty dz \, \epsilon(x - z) \{J^-_t(z), J^a_\mu(y)\}_p.\]

The components of the bracket (B.3) are given by

\begin{align*}
\{\chi^-(x), J^-_t(y)\}_p &= 0, \quad \{\chi^-(x), J^-_x(y)\}_p = 0, \quad (B.4) \\
\{\chi^-(x), J^2_t(y)\}_p &= -\frac{1}{2} \epsilon(x - y) J^-_t(y), \\
\{\chi^-(x), J^+_t(y)\}_p &= -\frac{1}{2} \epsilon(x - y) J^2_t(y), \\
\{\chi^-(x), J^2_x(y)\}_p &= -\frac{1}{2} \epsilon(x - y) J^-_x(y), \\
\{\chi^-(x), J^+_x(y)\}_p &= -\frac{1}{2} \epsilon(x - y) J^2_x(y) - \delta(x - y).
\end{align*}

Note that we have a trivial Poisson bracket,

\[\{\chi^-(x), \chi^-(y)\}_p = 0.\quad (B.5)\]

With (B.1), (B.4) and (B.5), the current algebra for \(j^R_\mu\) is computed as

\begin{align*}
\{j^R_\mu^-(x), j^R_\mu^-(y)\}_p &= 0, \quad (B.6) \\
\{j^R_\mu^-(x), j^R_\mu^-(y)\}_p &= e^{2\xi x^-} j^R_\mu^-(x) \delta(x - y), \\
\{j^R_\mu^-(x), j^R_\mu^+(y)\}_p &= j^R_\mu^+(x) \delta(x - y),
\end{align*}
\[ \{ j^R_t(x), j^R_t(y) \}_p = -\xi(x-y) \left[ j^R_t(x) e^{2\xi} j^R_t(y) + e^{2\xi} j^R_t(x) j^R_t(y) \right] , \]
\[ \{ j^R_t(x), j^R_t(y) \}_p = e^{2\xi} j^R_t(x) \delta (x-y) \]
\[ \{ j^R_t(x), j^R_t(y) \}_p = -\xi(x-y) \left[ j^R_t(x) j^R_t(y) + e^{2\xi} j^R_t(x) j^R_t(y) \right] , \]
\[ \{ j^R_t(x), j^R_t(y) \}_p = 0 , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = e^{2\xi} j^R_x(x) \delta (x-y) , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = -e^{2\xi} j^R_x(x) \delta (x-y) , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = -4\xi e^{4\xi} j^R_x(x) \delta (x-y) + e^{4\xi} j^R(x) \partial \delta (x-y) \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = -\xi(x-y) \left[ j^R_x(x) e^{2\xi} j^R_x(y) + e^{2\xi} j^R_x(x) j^R_x(y) \right] , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = 0 , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = 2\xi e^{2\xi} j^R_x(x) \delta (x-y) , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = -\xi(x-y) \left[ j^R_x(x) e^{2\xi} j^R_x(y) + e^{2\xi} j^R_x(x) j^R_x(y) \right] , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = -\xi(x-y) \left[ j^R_x(x) e^{2\xi} j^R_x(y) + e^{2\xi} j^R_x(x) j^R_x(y) \right] , \]
\[ \{ j^R_x(x), j^R_x(y) \}_p = -\xi(x-y) \left[ j^R_x(x) e^{2\xi} j^R_x(y) + e^{2\xi} j^R_x(x) j^R_x(y) \right] . \]

The next task is to consider the Poisson brackets between \( e^{-2\xi(x)} \) and \( j^R_t(x) \).
which are evaluated as follows,

\[
\left\{ e^{-2\xi \chi^{-}}(x), j_{\mu}^{R,-}(y) \right\}_P = \xi e^{-2\xi \chi^{-}(x)} \int_{-\infty}^{\infty} dz \, \epsilon(x - z) \left\{ j_{\mu}^{R,-}(z), j_{\mu}^{R,-}(y) \right\}_P. \tag{B.7}
\]

The components of the bracket (B.7) are given by

\[
\begin{align*}
\left\{ e^{-2\xi \chi^{-}}(x), j_{t}^{R,-}(y) \right\}_P &= 0, & \left\{ e^{-2\xi \chi^{-}}(x), j_{x}^{R,-}(y) \right\}_P &= 0, \\
\left\{ e^{-2\xi \chi^{-}}(x), j_{t}^{R,2}(y) \right\}_P &= \xi \epsilon(x - y) e^{-2\xi \chi^{-}}(x) e^{2\xi \chi^{-}} j_{t}^{R,-}(y), \\
\left\{ e^{-2\xi \chi^{-}}(x), j_{x}^{R,2}(y) \right\}_P &= \xi \epsilon(x - y) e^{-2\xi \chi^{-}}(x) e^{2\xi \chi^{-}} j_{x}^{R,-}(y), \\
\left\{ e^{-2\xi \chi^{-}}(x), j_{x}^{R,+}(y) \right\}_P &= \xi \epsilon(x - y) e^{-2\xi \chi^{-}}(x) j_{x}^{R,2}(y) - 2\xi \delta(x - y).
\end{align*}
\tag{B.8}
\]

The brackets between \( \chi^{2}(x) \) and \( j_{\mu}^{R}(x) \) are also computed similarly,

\[
\left\{ \chi^{2}(x), j_{\mu}^{R,a}(y) \right\}_P = -\frac{1}{2} \int_{-\infty}^{\infty} dz \, \epsilon(x - z) \left\{ j_{\mu}^{R,a}(z), j_{\mu}^{R,a}(y) \right\}_P. \tag{B.9}
\]

The components of the bracket (B.9) are given by

\[
\begin{align*}
\left\{ \chi^{2}(x), j_{t}^{R,-}(y) \right\}_P &= \frac{1}{2} \epsilon(x - y) e^{2\xi \chi^{-}} j_{t}^{R,-}(y), \\
\left\{ \chi^{2}(x), j_{t}^{R,2}(y) \right\}_P &= -\frac{1}{2} \sinh \left( \xi Q_{R}^{-} \right) j_{t}^{R,2}(y) - \frac{\xi}{2} Q_{R}^{2} e^{2\xi \chi^{-}} j_{t}^{R,-}(y) \\
& \quad - \frac{1}{2} \epsilon(x - y) \left( e^{2\xi \chi^{-}} - e^{2\xi \chi^{-}(y)} \right) j_{t}^{R,2}(y) \\
& \quad - \xi \epsilon(x - y) \left( \chi^{2}(x) - \chi^{2}(y) \right) e^{2\xi \chi^{-}} j_{t}^{R,-}(y), \\
\left\{ \chi^{2}(x), j_{x}^{R,+}(y) \right\}_P &= -\frac{1}{2} \sinh \left( \xi Q_{R}^{-} \right) j_{x}^{R,+}(y) - \frac{1}{2} \epsilon(x - y) e^{2\xi \chi^{-}} j_{x}^{R,+}(y) \\
& \quad - \frac{\xi}{2} Q_{R}^{2} j_{x}^{R,2}(y) - \xi \epsilon(x - y) \left( \chi^{2}(x) - \chi^{2}(y) \right) j_{x}^{R,2}(y), \\
\left\{ \chi^{2}(x), j_{x}^{R,-}(y) \right\}_P &= \frac{1}{2} \epsilon(x - y) e^{2\xi \chi^{-}} j_{x}^{R,-}(y), \\
\left\{ \chi^{2}(x), j_{x}^{R,2}(y) \right\}_P &= e^{4\xi \chi^{-}(x)} \delta(x - y) - \frac{1}{2} \sinh \left( \xi Q_{R}^{-} \right) j_{x}^{R,2}(y) - \frac{\xi}{2} Q_{R}^{2} e^{2\xi \chi^{-}} j_{x}^{R,-}(y) \\
& \quad - \frac{1}{2} \epsilon(x - y) \left( e^{2\xi \chi^{-}} - e^{2\xi \chi^{-}(y)} \right) j_{x}^{R,2}(y) \\
& \quad - \xi \epsilon(x - y) \left( \chi^{2}(x) - \chi^{2}(y) \right) e^{2\xi \chi^{-}} j_{x}^{R,-}(y), \\
\left\{ \chi^{2}(x), j_{x}^{R,+}(y) \right\}_P &= -\frac{1}{2} \sinh \left( \xi Q_{R}^{-} \right) j_{x}^{R,+}(y) - \frac{1}{2} \epsilon(x - y) e^{2\xi \chi^{-}} j_{x}^{R,+}(y)
\end{align*}
\tag{B.10}
\]
Similarly, the Poisson bracket between $\chi^-(x)$ and $\chi^2(y)$ is given by

$$\left\{ e^{-2\xi\chi^-(x)}, \chi^2(y) \right\}_p = -\frac{1}{2} \sinh (\xi Q_{R^-}^-) e^{-2\xi\chi^-(x)}$$

$$-\frac{1}{2} \epsilon(x - y) \left[ 1 - e^{-2\xi\chi^-(x)} e^{2\xi\chi^-(y)} \right].$$

(B.11)

Similarly, the Poisson bracket between $\chi^2(x)$ and $\chi^2(y)$ can be evaluated as

$$\left\{ \chi^2(x), \chi^2(y) \right\}_p = \frac{1}{2} \sinh (\xi Q_{R^-}^-) \left( \chi^2(x) - \chi^2(y) \right)$$

$$+ \frac{1}{4} Q_{R^2} Q_{R^2} \left( e^{2\xi\chi^-(x)} - e^{2\xi\chi^-(y)} \right)$$

$$+ \frac{1}{2} \epsilon(x - y) \left( \chi^2(x) - \chi^2(y) \right) \left( e^{2\xi\chi^-(x)} - e^{2\xi\chi^-(y)} \right).$$

(B.12)

At this stage, let us introduce a flat and conserved current $I_\mu$ like

$$I_\mu^- = \epsilon_{\mu\nu} \partial^\nu \left( \frac{2}{\xi} e^{-2\xi\chi^-} \right), \quad I_\mu^2 = -\epsilon_{\mu\nu} \partial^\nu \left( e^{-2\xi\chi^-} \chi^2 \right),$$

$$I_\mu^+ = j_{R^+}^\mu + \xi \epsilon_{\mu\nu} \partial^\nu \left[ e^{-2\xi\chi^-} (\chi^2)^2 \right].$$

The flat and conserved current $J_{R^+}^\mu$ is related to $I_\mu$ through

$$J_{R^+}^{\mu,2} = e^{\xi Q_{R^+}^-} I_\mu^2, \quad J_{R^+}^{\mu,-} = I_\mu^- + \xi Q_{R^+} Q_{R^2} I_\mu^-,$$

$$J_{R^+}^{\mu,+} = e^{\xi Q_{R^+}^-} \left( I_\mu^+ + \xi Q_{R^2} I_\mu^2 + \frac{\xi^2}{2} (Q_{R^2})^2 I_\mu^- \right).$$

Using (B.6), (B.8), (B.10), (B.11) and (B.12), we can compute the current algebra for $I_\mu$. The Poisson brackets of the time components are given by

$$\left\{ I^-_t(x), I^-_t(y) \right\}_p = 0,$$

$$\left\{ I^-_t(x), I^2_t(y) \right\}_p = I^-_t(x) \delta(x - y) + \xi^2 Q_{(0)}^- I^-_t(x) I^2_t(y),$$

$$\left\{ I^-_t(x), I^+_t(y) \right\}_p = I^2_t(x) \delta(x - y) + \xi^2 Q_{(0)}^- I^-_t(x) I^+_t(y),$$

$$\left\{ I^2_t(x), I^2_t(y) \right\}_p = 0,$$

$$\left\{ I^2_t(x), I^+_t(y) \right\}_p = I^+_t(x) \delta(x - y) + \xi^2 Q_{(0)}^- I^-_t(x) I^+_t(y),$$

$$\left\{ I^+_t(x), I^+_t(y) \right\}_p = 0.\]
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^+_t(y) \}_p = \xi^2 Q^{-}_0 \left[ \mathcal{I}^+_t(x) \mathcal{I}^+_t(y) - \mathcal{I}^+_t(x) \mathcal{I}^+_t(y) \right]. \]

The Poisson brackets of the time and spatial components are given by
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^+_x(y) \}_p = 0, \tag{B.16} \]
\[ \{ \mathcal{I}^-_t(x), \mathcal{I}^2_x(y) \}_p = \mathcal{I}^-_x(x) \delta(x-y) + \xi^2 Q^{-}_0 \mathcal{I}^-_t(x) \mathcal{I}^-_x(y), \]
\[ \{ \mathcal{I}^-_t(x), \mathcal{I}^+_x(y) \}_p = \mathcal{I}^2_x(x) \delta(x-y) - \partial_x \delta(x-y) + \xi^2 Q^{-}_0 \mathcal{I}^-_t(x) \mathcal{I}^+_x(y), \]
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^-_x(y) \}_p = -\mathcal{I}^-_x(x) \delta(x-y) - \xi^2 Q^{-}_0 \mathcal{I}^-_t(x) \mathcal{I}^-_x(y), \]
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^2_x(y) \}_p = \partial_x \delta(x-y), \]
\[ \{ \mathcal{I}^2_t(x), \mathcal{I}^+_x(y) \}_p = \mathcal{I}^+_x(x) \delta(x-y) + \xi^2 Q^{-}_0 \mathcal{I}^+_t(x) \mathcal{I}^+_x(y), \]
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^-_x(y) \}_p = -\mathcal{I}^2_x(x) \delta(x-y) - \partial_x \delta(x-y) - \xi^2 Q^{-}_0 \mathcal{I}^+_t(x) \mathcal{I}^-_x(y), \]
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^-_x(y) \}_p = -\mathcal{I}^+_x(x) \delta(x-y) - \xi^2 Q^{-}_0 \mathcal{I}^+_t(x) \mathcal{I}^-_x(y), \]
\[ \{ \mathcal{I}^+_t(x), \mathcal{I}^2_x(y) \}_p = \xi^2 Q^{-}_0 \left[ \mathcal{I}^+_t(x) \mathcal{I}^+_x(y) - \mathcal{I}^+_t(x) \mathcal{I}^+_x(y) \right]. \]

The Poisson brackets of the spatial components are given by
\[ \{ \mathcal{I}^+_x(x), \mathcal{I}^+_x(y) \}_p = 0, \tag{B.17} \]
\[ \{ \mathcal{I}^-_x(x), \mathcal{I}^2_x(y) \}_p = \xi^2 Q^{-}_0 \mathcal{I}^-_x(x) \mathcal{I}^-_x(y), \]
\[ \{ \mathcal{I}^-_x(x), \mathcal{I}^+_x(y) \}_p = \xi^2 Q^{-}_0 \mathcal{I}^-_x(x) \mathcal{I}^2_x(y), \]
\[ \{ \mathcal{I}^+_x(x), \mathcal{I}^2_x(y) \}_p = 0, \]
\[ \{ \mathcal{I}^2_x(x), \mathcal{I}^+_x(y) \}_p = \xi^2 Q^{-}_0 \mathcal{I}^-_x(x) \mathcal{I}^+_x(y), \]
\[ \{ \mathcal{I}^+_x(x), \mathcal{I}^+_x(y) \}_p = \xi^2 Q^{-}_0 \left[ \mathcal{I}^2_x(x) \mathcal{I}^+_x(y) - \mathcal{I}^+_x(x) \mathcal{I}^2_x(y) \right]. \]

Here \( Q^{-}_0 \) is defined as
\[ Q^{-}_0 := \int_{-\infty}^{\infty} dx \mathcal{I}^-_t(x) = \frac{1}{\xi} \sinh \left( \xi Q^{R,-} \right). \tag{B.18} \]

Finally let us compute the current algebra for \( \mathcal{J}^{R+}_\mu \). For this purpose, we still need the Poisson brackets between \( Q^{R,a} (a = -, 2) \) and \( \mathcal{I}_\mu(x) \).
It is first necessary to compute the Poisson brackets between $Q^{R,a}$ ($a = -, 2$) and $j^R_\mu(x)$. Those can be obtained from the Poisson brackets (B.8) and (B.10) by taking the $x \to \infty$ limit, in which $\chi^\alpha(x)$ is replaced by $-\frac{1}{2}Q^{R,a}$. Thus the resulting brackets are given by

\[
\begin{align*}
\{ Q^{R,-}, j^{R,-}_\mu(x) \}_p &= 0, \\
\{ Q^{R,-}, j^{R,2}_\mu(x) \}_p &= e^{2\xi} \chi^{-} j^{R,-}_\mu(x), \\
\{ Q^{R,-}, j^{R,+}_\mu(x) \}_p &= j^{R,+}_\mu(x), \\
\{ Q^{R,2}, j^{R,-}_\mu(x) \}_p &= -e^{2\xi} \chi^{-} j^{R,-}_\mu(x), \\
\{ Q^{R,2}, j^{R,2}_\mu(x) \}_p &= -2e^{2\xi} \chi^{-} j^{R,2}_\mu(x) + \cosh(\xi Q^{R,-}) j^{R,2}_\mu(x), \\
\{ Q^{R,2}, j^{R,+}_\mu(x) \}_p &= -2e^{2\xi} \chi^{2} j^{R,2}_\mu(x) + \cosh(\xi Q^{R,-}) j^{R,+}_\mu(x).
\end{align*}
\]

In the same way, from (B.5), (B.11) and (B.12), we obtain the following Poisson brackets:

\[
\begin{align*}
\{ Q^{R,-}, \chi^{-}(x) \}_p &= 0, \\
\{ Q^{R,-}, \chi^{2}(x) \}_p &= 2\xi \left[ e^{2\xi} \chi^{-}(x) - \cosh(\xi Q^{R,-}) \right], \\
\{ Q^{R,2}, \chi^{-}(x) \}_p &= 2\xi \left[ -e^{2\xi} \chi^{-}(x) + \cosh(\xi Q^{R,-}) \right], \\
\{ Q^{R,2}, \chi^{2}(x) \}_p &= \frac{1}{2} \sinh(\xi Q^{R,-}) Q^{R,2} + \cosh(\xi Q^{R,-}) \chi^{2}(x) - e^{2\xi} \chi^{-}(x) \chi^{2}(x). 
\end{align*}
\]

These Poisson brackets lead to the Poisson brackets between $Q^{R,a}$ ($a = -, 2$) and $\mathcal{I}_\mu(x)$:

\[
\begin{align*}
\{ Q^{R,-}, \mathcal{I}_\mu^{-}(x) \}_p &= 0, \\
\{ Q^{R,-}, \mathcal{I}^2_\mu(x) \}_p &= \cosh(\xi Q^{R,-}) \mathcal{I}_\mu^{-}(x), \\
\{ Q^{R,-}, \mathcal{I}^+_\mu(x) \}_p &= \cosh(\xi Q^{R,-}) \mathcal{I}^2_\mu(x), \\
\{ Q^{R,2}, \mathcal{I}_\mu^{-}(x) \}_p &= -\cosh(\xi Q^{R,-}) \mathcal{I}_\mu^{-}(x), \\
\{ Q^{R,2}, \mathcal{I}^2_\mu(x) \}_p &= 0, \\
\{ Q^{R,2}, \mathcal{I}^+_\mu(x) \}_p &= \cosh(\xi Q^{R,-}) \mathcal{I}^+_\mu(x).
\end{align*}
\]
Since the current algebra of $I_\mu$ is given by (B.15), (B.16) and (B.17), with the brackets in (B.20), the current algebra of $J_{\mu}^{R+}$ is computed as

\[
\begin{align*}
\{ J_{t}^{R+,a}(x), J_{t}^{R+,b}(y) \} & = \varepsilon_{ab} \delta(x-y), \\
\{ J_{t}^{R+,a}(x), J_{x}^{R+,b}(y) \} & = \varepsilon_{ab} J_{x}^{R+,c}(x) \delta(x-y) + \gamma^{ab} \partial_{x} \delta(x-y), \\
\{ J_{x}^{R+,a}(x), J_{x}^{R+,b}(y) \} & = 0.
\end{align*}
\]  

(B.21)

At last, after long computations with messy expressions, we have obtained quite a simple algebra. Remarkably speaking, this current algebra is identical with the one for the usual $SL(2,\mathbb{R})$ principal chiral models, and it does not contain $C$ as well as non-local scalar functions.

C  A different gauge fixing

The choice of constant term in (3.3) affects the relations of currents algebra, as we have seen in Section 3.1. One might wonder what algebra is realized by a different gauge fixing. It sounds a fair question. As an example, let us replace the constant $K^+$ in (3.10) by $K^+ H$ with $H = e^{-\xi T^- Q^{R,-} e^{-\xi T^2 Q^{R,-}}}$. Then the twist operator is modified as

\[
\mathcal{F}^+(x) \to \mathcal{F}^+(x) H = e^{2\xi T^2 \chi^-(x)} e^{-2\xi T^- \chi^2(x)}.
\]  

(C.1)

Under this replacement, the currents are transformed in the adjoint way,

\[
J_{\mu}^{R+}(x) \to H^{-1} J_{\mu}^{R+}(x) H =: I_{\mu}(x),
\]  

(C.2)

where $I_{\mu}(x)$ is defined by (B.13). Since $I_{\mu}$ is also flat conserved current, an infinite number of conserved charges $Q_{(n \geq 0)}$ are generated by the BIZZ construction. For example, the first two charges are represented by

\[
Q_{(0)} = \int_{-\infty}^{\infty} dx I_t(x),
\]  

\[
Q_{(1)} = \frac{1}{4} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \epsilon(x-y) [I_t(x), I_t(y)] - \int_{-\infty}^{\infty} dx I_x(x).
\]  

(C.3)

The expression of the charges in terms of the current is identical, but the current algebra of $I_{\mu}$ is completely different from that of $J_{\mu}^{R+}$. Hence the algebra of $Q_{(n)}$ would not take the usual form of the Yangian $\mathcal{Y}(\mathfrak{sl}(2))$, though these would be isomorphic because the difference is just the gauge choice.
By using the current algebra of $I_{\mu}$, the Poisson brackets among $Q_{(n)}$ are evaluated. The Poisson brackets at the level-zero are given by

\[
\begin{align*}
\{Q_{(0)}^-, Q_{(0)}^2\}_P &= Q_{(0)}^- + \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(0)}^-, \\
\{Q_{(0)}^-, Q_{(0)}^+\}_P &= Q_{(0)}^+ + \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(0)}^+, \\
\{Q_{(0)}^2, Q_{(0)}^+\}_P &= Q_{(0)}^+ + \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(0)}^+ .
\end{align*}
\] (C.4)

Those at the level-one are

\[
\begin{align*}
\{Q_{(1)}^-, Q_{(1)}^2\}_P &= \{Q_{(0)}^2, Q_{(1)}^2\}_P = 0 , \\
\{Q_{(0)}^-, Q_{(1)}^2\}_P &= Q_{(1)}^- + \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(1)}^- , \\
\{Q_{(0)}^-, Q_{(1)}^+\}_P &= Q_{(1)}^+ + \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(1)}^+ , \\
\{Q_{(0)}^2, Q_{(1)}^-\}_P &= -Q_{(1)}^- - \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(1)}^- , \\
\{Q_{(0)}^2, Q_{(1)}^+\}_P &= Q_{(1)}^+ + \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(1)}^+ , \\
\{Q_{(1)}^+, Q_{(1)}^-\}_P &= -Q_{(1)}^- - \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(1)}^- , \\
\{Q_{(1)}^2, Q_{(1)}^-\}_P &= -Q_{(1)}^- - \xi^2 Q_{(0)}^- Q_{(0)}^- Q_{(1)}^- .
\end{align*}
\] (C.5)

At the level-two, the Poisson brackets are

\[
\begin{align*}
\{Q_{(1)}^-, Q_{(1)}^2\}_P &= Q_{(2)}^- + \frac{1}{12} \left[ -2 Q_{(0)}^+ Q_{(0)}^- + (Q_{(0)}^2)^2 \right] Q_{(1)}^- - Q_{(1)}^- + \xi^2 Q_{(0)}^- Q_{(1)}^- Q_{(1)}^- , \\
\{Q_{(1)}^-, Q_{(1)}^+\}_P &= Q_{(2)}^2 + \frac{1}{12} \left[ -2 Q_{(0)}^+ Q_{(0)}^- + (Q_{(0)}^2)^2 \right] Q_{(2)}^- - Q_{(2)}^- + \xi^2 Q_{(0)}^- Q_{(1)}^- Q_{(1)}^2 , \\
\{Q_{(1)}^2, Q_{(1)}^+\}_P &= Q_{(2)}^+ + \frac{1}{12} \left[ -2 Q_{(0)}^+ Q_{(0)}^- + (Q_{(0)}^2)^2 \right] Q_{(1)}^+ - Q_{(1)}^+ + \xi^2 Q_{(0)}^- Q_{(1)}^- Q_{(1)}^+ .
\end{align*}
\] (C.6)

Thus the resulting algebra is regarded as a one-parameter deformation of $\mathcal{Y}(\mathfrak{sl}(2))$. When $C = 0$, the defining relations of $\mathcal{Y}(\mathfrak{sl}(2))$ are reproduced.

It is also interesting to see the deformed Serre relations. There are five relations.

i) the first one,

\[
\{\{Q_{(1)}^-, Q_{(1)}^2\}_P, Q_{(1)}^-\}_P = \frac{1}{4} Q_{(0)}^- \left( Q_{(1)}^2 Q_{(1)}^- - Q_{(1)}^- Q_{(0)}^2 \right) + \xi^2 \left\{ Q_{(0)}^- Q_{(1)}^- Q_{(1)}^- , Q_{(1)}^- \right\}_P.
\]

ii) the second one,

\[
\{\{Q_{(1)}^-, Q_{(1)}^+\}_P, Q_{(1)}^-\}_P + \{\{Q_{(1)}^-, Q_{(1)}^2\}_P, Q_{(1)}^-\}_P
\]
\[ = \frac{1}{4} Q^2_0 \left( Q^+_1 Q^0_0 - Q^+_1 Q^0_0 \right) + \frac{1}{4} Q^2_0 \left( Q^-_1 Q^0_0 - Q^-_1 Q^0_0 \right) \]
\[ + \xi^2 \left\{ Q^-_0 Q^+_1 Q^2_0, Q^-_1 \right\}_p + \xi^2 \left\{ Q^-_0 Q^-_1 Q^2_1, Q^1_1 \right\}_p . \]

iii) the third one,
\[ \left\{ \left\{ Q^-_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p = \frac{1}{4} Q^2_0 \left( Q^-_1 Q^+_1 Q^2_0 - Q^-_1 Q^2_0 \right) + \xi^2 \left\{ Q^-_0 Q^-_1 Q^2_1, Q^1_1 \right\}_p . \]

iv) the fourth one,
\[ \left\{ \left\{ Q^-_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p + \left\{ \left\{ Q^2_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p \]
\[ = \frac{1}{4} Q^2_0 \left( Q^-_1 Q^+_1 Q^2_0 - Q^-_1 Q^2_0 \right) + \frac{1}{4} Q^2_0 \left( Q^-_1 Q^+_0 - Q^-_1 Q^2_0 \right) \]
\[ + \xi^2 \left\{ Q^-_0 Q^-_1 Q^2_1, Q^1_1 \right\}_p + \xi^2 \left\{ Q^-_0 Q^-_1 Q^2_1, Q^1_1 \right\}_p \]
\[ + \xi^2 Q^-_0 \left( \left\{ Q^-_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p - \left\{ \left\{ Q^2_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p . \]

v) the fifth one,
\[ \left\{ \left\{ Q^2_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p = \frac{1}{4} Q^2_0 \left( Q^2_1 Q^+_0 - Q^2_1 Q^2_0 \right) + \xi^2 \left\{ Q^-_0 Q^-_1, Q^+_1, Q^2_1 \right\}_p \]
\[ + \xi^2 Q^-_0 \left( \left\{ Q^-_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p - \left\{ \left\{ Q^2_1, Q^+_1 \right\}_p , Q^2_1 \right\}_p . \]

Thus all of the Serre relations are modified due to the deformation parameter \( \xi \).

Finally, it is worth showing how the deformed Yangian is embedded into the exotic symmetry [26]. The level-zero charges are embedded into the \( q \)-Poincare algebra as follows:
\[ Q^+_0 = Q^{R, +} + \frac{\xi}{2} \sinh(\xi Q^{R, -})(Q^{R, 2})^2, \]
\[ Q^2_0 = Q^{R, 2} \cosh(\xi Q^{R, -}), \]
\[ Q^-_0 = \frac{\sinh(\xi Q^{R, -})}{\xi}. \]  

(C.7)

The embedding of the level-one charges is more involved. The level-one charges are expressed in terms of the charges of the \( q \)-Poincare generators and \( \tilde{Q}^{R, 2} \) like
\[ Q^+_1 = \frac{1}{4} \left\{ Q^{R, +}, \left\{ Q^{R, +}, \tilde{Q}^{R, 2} \right\}_p \right\}_p - \frac{1}{4} \sinh(\xi Q^{R, -}) \tilde{Q}^{R, 2} Q^{R, +} - \frac{\xi}{8} (Q^{R, 2})^2 (\tilde{Q}^{R, 2} - Q^{R, 2}), \]
\[ Q^2_1 = \frac{1}{4} \left\{ \left\{ \tilde{Q}^{R, 2}, Q^{R, +} \right\}_p - \cosh(\xi Q^{R, -}) Q^{R, +} \right\}_p, \]
\[ Q^-_1 = \frac{1}{4} \left( \tilde{Q}^{R, 2} - Q^{R, 2} \right). \]  

(C.8)
Finally, $\tilde{Q}_{R,+}$ is contained in one of the level-two charges. For example, it appears in $Q_{(2)}$,

\[
Q_{(2)} = \frac{1}{8\xi}(\tilde{Q}_{R,+} - \cosh(\xi Q_{R,-})\{\tilde{Q}_{R,2}, Q_{R,+}\}_P - \frac{1}{3}\sinh^2(\xi Q_{R,-})Q_{R,+})
+ \frac{1}{8\xi}Q_{R,2}\sinh(\xi Q_{R,-})(Q_{R,2} + \tilde{Q}_{R,2} + \frac{4}{3}\sinh^2(\xi Q_{R,-})Q_{R,2}).
\]

(C.9)

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113]. [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428 (1998) 105 [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998) 253 [arXiv:hep-th/9802150].

[4] N. Beisert et al., “Review of AdS/CFT Integrability: An Overview,” arXiv:1012.3982 [hep-th].

[5] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69 (2004) 046002 [hep-th/0305116].

[6] K. Zarembo, “Strings on Semisymmetric Superspaces,” JHEP 1005 (2010) 002 [arXiv:1003.0465 [hep-th]].

[7] I. V. Cherednik, “Relativistically Invariant Quasiclassical Limits Of Integrable Two-Dimensional Quantum Models,” Theor. Math. Phys. 47 (1981) 422 [Teor. Mat. Fiz. 47 (1981) 225].

[8] L. D. Faddeev and N. Y. Reshetikhin, “Integrability of the principal chiral field model in (1+1)-dimension,” Annals Phys. 167 (1986) 227.

[9] J. Balog, P. Forgacs and L. Palla, “A Two-dimensional integrable axionic sigma model and T duality,” Phys. Lett. B 484 (2000) 367 [hep-th/0004180].

[10] D. Orlando, S. Reffert and L. I. Uruchurtu, “Classical integrability of the squashed three-sphere, warped AdS3 and Schrödinger spacetime via T-Duality,” J. Phys. A 44 (2011) 115401. [arXiv:1011.1771 [hep-th]].

[11] B. Basso and A. Rej, “On the integrability of two-dimensional models with $U(1) \times SU(N)$ symmetry,” arXiv:1207.0413 [hep-th].

[12] S. Schafer-Nameki, M. Yamazaki and K. Yoshida, “Coset Construction for Duals of Non-relativistic CFTs,” JHEP 0905 (2009) 038 [arXiv:0903.4245 [hep-th]].
[13] M. Lüscher, “Quantum nonlocal charges and absence of particle production in the two-dimensional nonlinear sigma model,” Nucl. Phys. B 135 (1978) 1.

[14] M. Lüscher and K. Pohlmeyer, “Scattering of massless lumps and nonlocal charges in the two-dimensional classical nonlinear sigma model,” Nucl. Phys. B 137 (1978) 46.

[15] E. Brezin, C. Itzykson, J. Zinn-Justin and J. B. Zuber, “Remarks about the existence of nonlocal charges in two-dimensional models,” Phys. Lett. B 82 (1979) 442.

[16] D. Bernard, “Hidden Yangians in 2-D massive current algebras,” Commun. Math. Phys. 137 (1991) 191.

[17] N. J. MacKay, “On the classical origins of Yangian symmetry in integrable field theory,” Phys. Lett. B 281 (1992) 90 [Erratum-ibid. B 308 (1993) 444].

[18] E. Abdalla, M. C. Abdalla and K. Rothe, “Non-perturbative methods in two-dimensional quantum field theory,” World Scientific, 1991.

[19] I. Kawaguchi and K. Yoshida, “Hidden Yangian symmetry in sigma model on squashed sphere,” JHEP 1011 (2010) 032. [arXiv:1008.0776 [hep-th]].

[20] I. Kawaguchi, D. Orlando and K. Yoshida, “Yangian symmetry in deformed WZNW models on squashed spheres,” Phys. Lett. B 701 (2011) 475. [arXiv:1104.0738 [hep-th]].

[21] I. Kawaguchi and K. Yoshida, “Hybrid classical integrability in squashed sigma models,” Phys. Lett. B 705 (2011) 251 [arXiv:1107.3662 [hep-th]].

[22] I. Kawaguchi and K. Yoshida, “Hybrid classical integrable structure of squashed sigma models: A short summary,” J. Phys. Conf. Ser. 343 (2012) 012055 [arXiv:1110.6748 [hep-th]].

[23] I. Kawaguchi, T. Matsumoto and K. Yoshida, “The classical origin of quantum affine algebra in squashed sigma models,” JHEP 1204 (2012) 115 [arXiv:1201.3058 [hep-th]].

[24] I. Kawaguchi, T. Matsumoto and K. Yoshida, “On the classical equivalence of monodromy matrices in squashed sigma model,” JHEP 1206 (2012) 082 [arXiv:1203.3400 [hep-th]].

[25] I. Kawaguchi and K. Yoshida, “Classical integrability of Schrodinger sigma models and q-deformed Poincare symmetry,” JHEP 1111 (2011) 094 [arXiv:1109.0872 [hep-th]].

[26] I. Kawaguchi and K. Yoshida, “Exotic symmetry and monodromy equivalence in Schrodinger sigma models,” JHEP 1302 (2013) 024 [arXiv:1209.4147 [hep-th]].

[27] D. Israel, C. Kounnas, D. Orlando and P. M. Petropoulos, “Electric/magnetic deformations of S**3 and AdS(3), and geometric cosets,” Fortsch. Phys. 53 (2005) 73 [hep-th/0405213].

[28] D. T. Son, “Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry,” Phys. Rev. D 78 (2008) 046003 [arXiv:0804.3972 [hep-th]].
[29] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” Phys. Rev. Lett. 101 (2008) 061601 [arXiv:0804.4053 [hep-th]].

[30] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, “Q deformation of Poincare algebra,” Phys. Lett. B 264 (1991) 331.

[31] Ch. Ohn, “A *-product on SL(2) and the corresponding nonstandard quantum-U(sl(2)),” Lett. Math. Phys. 25 (1992) 85.

[32] V. G. Drinfel’d, “Hopf algebras and the quantum Yang-Baxter equation,” Sov. Math. Dokl. 32 (1985) 254.

[33] “Quantum groups,” J. Sov. Math. 41 (1988) 898 [Zap. Nauchn. Semin. 155, 18 (1986)].

[34] M. Jimbo, “A q difference analog of U(g) and the Yang-Baxter equation,” Lett. Math. Phys. 10 (1985) 63.

[35] V. Chari and A. N. Pressley, “A Guide to Quantum Groups,” Cambridge University Press; 27/7/1995, ISBN-10: 0521558840, ISBN-13: 978-0521558846.

[36] J. M. Maillet, “New integrable canonical structures in two-dimensional models,” Nucl. Phys. B 269 (1986) 54.

[37] N. Reshetikhin, “Multiparameter quantum groups and twisted quasitriangular Hopf algebras,” Lett. Math. Phys. 20 (1990) 331.

[38] S. M. Khoroshkin, A. A. Stolin and V. N. Tolstoy, ”Deformation of Yangian Y(sl_2),” Comm. in Alg., 26 (1998) 1041 [arXiv:q-alg/9511005].

[39] M. Gerstenhaber, A. Giaquinto and S. D. Schack, “Quantum symmetry,” Quantum Groups (Lecture Notes in Mathematics Volume 1510) (P.P.Kulish, ed.), Springer Verlag, Berlin, 1992, pp. 9-46.

[40] P. P. Kulish, V. D. Lyakhovsky and A. I. Mudrov, “Extended jordanian twists for Lie algebras,” J. Math. Phys. 40 (1999) 4569 [math/9806014 [math.QA]].

[41] A. Ballesteros and F. J. Herranz, “Universal R-matrix for non-standard quantum sl(2, R),” J. of Phy. A 29 (1996) L311 [q-alg/9604008].

[42] P. P. Kulish and A. A. Stolin, “Deformed Yangians and Integrable Models,” Czech. J. Phys. 47 (1997) 1207, [arXiv:q-alg/9708024].

[43] S. Zakrzewski, “A Hopf star-algebra of polynomials on the quantum SL(2, R) for a ‘unitary’ R-matrix,” Lett. Math. Phys. 22 (1991) 287.

[44] T. Kameyama and K. Yoshida, “String theories on warped AdS backgrounds and integrable deformations of spin chains,” arXiv:1304.1286 [hep-th].

[45] A. Stolin and P. P. Kulish, “New rational solutions of Yang-Baxter equation and deformed Yangians,” Czech. J. Phys. 47 (1997) 123 [arXiv:q-alg/9608011].
[46] L. F. Alday, G. Arutyunov and S. Frolov, “Green-Schwarz strings in TsT-transformed backgrounds,” JHEP 0606 (2006) 018 [hep-th/0512253].

[47] C. P. Herzog, M. Rangamani and S. F. Ross, “Heating up Galilean holography,” JHEP 0811 (2008) 080 [arXiv:0807.1099 [hep-th]].

[48] J. Maldacena, D. Martelli and Y. Tachikawa, “Comments on string theory backgrounds with non-relativistic conformal symmetry,” JHEP 0810 (2008) 072 [arXiv:0807.1100 [hep-th]].

[49] A. Adams, K. Balasubramanian and J. McGreevy, “Hot Spacetimes for Cold Atoms,” JHEP 0811 (2008) 059 [arXiv:0807.1111 [hep-th]].