Book free 3-Uniform Hypergraphs

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Abstract

A k-book in a hypergraph consists of k Berge triangles sharing a common edge. In this paper we prove that the number of the hyperedges in a k-book-free 3-uniform hypergraph on n vertices is at most \( \frac{n^2}{8} (1 + o(1)) \).

1 Introduction

Let G be a graph. The vertex and the edge set of G are denoted by V(G) and E(G). If there are two triangles sitting on an edge in a graph, we call this a diamond. Whereas k triangles sitting on an edge is called a k-book, denoted by a \( B_k \). Similarly, let H be a hypergraph and the vertex and the edge set of H be denoted by V(H) and E(H). A hypergraph is called r-uniform if each member of E has size r. A hypergraph \( H = (V, E) \) is called linear if every two hyperedges have at most one vertex in common. A Berge cycle of length k, denoted by Berge-\( C_k \), is an alternating sequence of distinct vertices and distinct hyperedges of the form \( v_1, h_1, v_2, h_2, \ldots, v_k, h_k \) where \( v_i, v_{i+1} \in h_i \) for each \( i \in \{1, 2, \ldots, k-1\} \) and \( v_k v_1 \in h_k \). The hypergraph equivalent of k-books is defined similarly with k Berge triangles sharing a common edge. We say that this common edge is the base of the k-book.

The maximum number of edges in a triangle-free graph is one of the classical results in extremal graph theory and proved by Mantel in 1907 [13]. The extremal problem for diamond-free graphs follows from this. Given a graph G on n vertices and having \( \left\lceil \frac{n^2}{4} \right\rceil + 1 \) edges. Mantel showed that G contains a triangle. Rademacher (unpublished, and simplified later by Erdős in [6]) proved in the 1940s that the number of triangles in G is at least \( \left\lfloor \frac{n^2}{4} \right\rfloor \). Erdős conjectured in 1962 [7] that the size of the largest book in G is at least \( \frac{n}{6} \) and this was proved soon after by Edwards (unpublished, see also Khadžiivanov and Nikiforov [11] for an independent proof).
Theorem 1 (Edwards [4], Khadžiivanov and Nikiforov [11]). Every $n$-vertex graph with more than $\frac{n^2}{4}$ edges contains an edge that is in at least $\frac{n}{6}$ triangles.

Both Rademacher’s and Edwards’ results are sharp. In the former, the addition of an edge to one part in the complete balanced bipartite graph (note that in $G$ there is an edge contained in $\left\lfloor \frac{n}{2} \right\rfloor$ triangles) achieves the maximum. In the latter, every known extremal construction of $G$ has $\Omega(n^3)$ triangles. For more details on book-free graphs we refer the reader to the following articles [2], [13] and [16]. We look into the equivalent problem in the case of hypergraphs.

Given a family of hypergraphs $\mathcal{F}$, we say that a hypergraph $H$ is $\mathcal{F}$-free if for every $F \in \mathcal{F}$, the hypergraph $H$ does not contain a $F$ as a sub-hypergraph.

The systematic study of the Turán number of Berge cycles started with Lazebnik and Verstraëte [12], who studied the maximum number of hyperedges in an $r$-uniform hypergraph containing no Berge cycle of length less than five. Another result was the study of Berge triangles by Győri [8]. He proved that:

Theorem 2 (Győri [8]). The maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraphs on $n$ vertices is at most $\frac{n^2}{8}$.

It continued with the study of Berge five cycles by Bollobás and Győri [3]. In [9], Győri, Katona, and Lemons proved the following analog of the Erdős-Gallai Theorem [5] for Berge paths. For other results see [1, 10]. The particular case of determining the maximum number of the hyperedges of a triangle-free linear hypergraph on $n$ vertices is equivalent to the famous $(6,3)$-problem, which is a special case of a general problem of Brown, Erdős, and Sós. The following theorem of Ruzsa and Szemerédi plays important role in our paper:

Theorem 3 (Ruzsa and Szemerédi [15]). For any $\epsilon > 0$ there exists $n_0(\epsilon)$ such that if $n > n_0(\epsilon)$ then a Berge-triangle-free 3-uniform linear hypergraph on $n$ vertices has at most $\epsilon n^2$ hyperedges.

We continue the work on that and determine the maximum number of hyperedges for a $k$-book-free 3-uniform hypergraph. The main result is as follows:

Theorem 4. For a given $k \geq 2$ and $\epsilon > 0$ there exists $n_1(k, \epsilon)$ such that if $n > n_1(k, \epsilon)$ then a 3-uniform $B_k$-free hypergraph $H$ on $n$ vertices can have at most $\frac{n^2}{8} + \epsilon n^2$ edges.

The following example shows that this result is asymptotically sharp. Take a complete bipartite graph with color classes of size $\left\lfloor \frac{n}{4} \right\rfloor$ and $\left\lceil \frac{n}{4} \right\rceil$ respectively. Denote the vertices in each class with $x_i$ and $y_i$ respectively. Construct a graph by doubling each vertex and replacing each edge with two hyperedges as shown below (Figure 1). So essentially, we have replaced every graph edge with two hyperedges. The construction does not contain a $B_k$, as it does not contain a Berge triangle. With this, the number of hyperedges is $2 \times \frac{n^2}{16} = \frac{n^2}{8}$.

Figure 1: Replacing every edge $x_iy_i$ in the bipartite graph with two hyperedges $x_iy_jy_j'$ and $y_jy_j'x_i'$. 

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2 Proof of Theorem 4

Fix \( k \geq 2, \epsilon > 0 \) and set

\[
n_1(k, \epsilon) = \max \left\{ 18k + 12, n_0 \left( \frac{\epsilon}{6k^2 - 8k} \right) \right\}
\]

where \( n_0(.) \) is from Theorem 3. Suppose that \( n > n_1(k, \epsilon) \). Let \( H \) be a \( B_k \)-free 3-uniform hypergraph on \( n \) vertices. We are interested in the 2-shadow, i.e., let \( G \) be a graph with vertex set \( V(H) \) and

\[
E(G) = \{ ab \mid \{a, b\} \subset e \in E(H) \}\]

If an edge in \( G \) lies in more than one hyperedge in \( H \), we color it blue. Otherwise, we color it red. We define hypergraphs \( H_r \) and \( H_b \) in the following way. \( V(H_r) = V(H_b) = V(H) \), \( E(H_r) = \{ e \in E(H) \mid e \text{ contains two or three red edges of } G \} \) and \( E(H_b) = E(H) \setminus E(H_r) \). Note that each hyperedge in \( H_b \) contains two or three blue edges of \( G \).

**Claim 5.** The number of hyperedges in \( H_r \) is at most \( \frac{n^2}{8} \).

**Proof.** Denote the subgraph of \( G \) formed by the red colored edges by \( G_r \). Suppose that \( |E(G_r)| \geq \frac{n^2}{4} + 1 \). By Theorem 3 we have a book of size \( \frac{n}{6} \) in \( G_r \). Denote the vertices of the \( \frac{n}{6} \)-book in \( G_r \) with \( u, v \) and \( x_i, 1 \leq i \leq \frac{n}{6} \) respectively where \( uv \) is the base of the book. Denote the third vertex of the hyperedge containing edge \( uv \) by \( w \), set \( X = \{ x_i \mid 1 \leq i \leq \frac{n}{6} \} \) and for each \( x_i \in X \) denote the hyperedge containing \( ux_i \) and \( vx_i \) by \( ux_iy_i \) and \( vx_iz_i \) respectively.

Set \( E' := \emptyset \) and \( X' := \emptyset \). Go through the vertices of \( X \) and perform the following procedure for each of them. At the beginning of the process no vertex is marked.

If the current vertex \( x_i = w \) then mark it.

If \( x_i \) is unmarked then

- add \( x_i \) to \( X' \) and hyperedges \( ux_iy_i \) and \( vx_iz_i \) to \( E' \),
- if there exists \( j > i \) such that \( y_i = x_j \) then mark \( x_j \),
- if there exists \( \ell > i \) such that \( z_i = x_\ell \) then mark \( x_\ell \).

By definition of red edges and the procedure (i.e. it adds two new hyperedges to \( E' \) forming a Berge triangle with \( uvw \) at each step handling an unmarked vertex but at most one: when \( x_i = w \) the set of hyperedges \( E' \cup \{ uvw \} \) with vertex set \( X' \cup \{ u, v \} \) form a \( k' \)-book with base \( uvw \), where \( k' = |X'| \). Moreover at each step of the procedure whenever an unmarked vertex was added to \( X' \) then at most two more vertices became marked. Each unmarked vertex are in \( X' \) at the end of the procedure, therefore

\[
k' \geq \left| X \setminus \{ w \} \right| \geq \frac{n/6 - 1}{3}
\]

at the end of the procedure and it is at least \( k \) by the definition of \( n_1(k, \epsilon) \), but this is a contradiction.

Hence \( |E(G_r)| \leq \frac{n^2}{4} \) and

\[
|E(H_r)| \leq \frac{|E(G_r)|}{2} \leq \frac{n^2}{8}
\]

by the definition of red colored edges. \( \square \)
Note that edge \( uv \) for each of them. At the beginning of the process no vertex is marked.

**Proof.** Suppose that \( \{u, v\} \) is a pair of vertices which is contained in \( 2k - 1 \) hyperedges of \( H_b \). Note that edge \( uv \) is colored blue. Denote the third vertices of hyperedges containing \( u \) and \( v \) by \( x_1, \ldots, x_{2k-1} \) and set \( X = \{x_i \mid 1 \leq i \leq 2k - 1\} \). Observe that for each \( i \) at least one of \( ux_i \) and \( vx_i \) is colored blue.

Set \( E' := \emptyset \) and \( X' := \emptyset \). Go through the vertices of \( X \) and perform the following procedure for each of them. At the beginning of the process no vertex is marked.

If the current vertex \( x_i = x_{2k-1} \) and there is no marked vertex in \( X \) then do nothing.

Otherwise if \( x_i \) is unmarked then
- add \( x_i \) to \( X' \) and add \( ux_i v \) to \( E' \),
- if \( ux_i \) is colored blue denote a hyperedge containing it by \( ux_i y_i \) where \( y_i \neq v \) and add \( ux_i y_i \) to \( E' \),
- otherwise \( vx_i \) is colored blue, so denote a hyperedge containing it by \( vx_i y_i \) where \( y_i \neq u \) and add \( vx_i y_i \) to \( E' \),
- if there exists \( j > i \) such that \( y_i = x_j \) then mark \( x_j \).

If at the end of the procedure there is no marked vertex in \( X \) then set \( w = x_{2k-1} \) otherwise let \( w \) be an arbitrary marked vertex.

By definition of blue edges and the procedure (i.e. it adds two new hyperedges to \( E' \) forming a Berge triangle with \( uvw \) at each step handling an unmarked vertex but at most the last one) the set of hyperedges \( E' \cup \{uvw\} \) with vertex set \( X' \cup \{u, v\} \) form a \( k' \)-book with base \( uvw \) where \( k' = |X'| \).

Moreover if there is no marked vertex in \( X \) at the end of the process then \( X' = X \setminus \{x_{2k-1}\} \), otherwise at each step of the procedure whenever an unmarked vertex was added to \( X' \) than at most one more vertex became marked and each unmarked vertex are in \( X' \) at the end of the procedure. Therefore \( k' \geq k \), but it is a contradiction.

We now give an upper bound on the number of hyperedges in \( H_b \).

**Claim 7.** The number of hyperedges in \( H_b \) is at most \( en^2 \).

**Proof.** Take a hyperedge \( xyz \) in the sub-hypergraph \( H_b \). By Claim 6 there are at most \( 2k - 2 \) hyperedges of \( H_b \) containing each of the pairs of vertices \( xy, yz, \) and \( xz \). If we deleted all such hyperedges barring \( xyz \) we would delete at most \( 6k - 9 \) hyperedges. Therefore there is a linear 3-uniform subhypergraph \( H'_b \) of \( H_b \) with \( V(H'_b) = V(H_b) = V(H) \) and

\[
|E(H'_b)| \geq \frac{|E(H_b)|}{6k - 8}
\]

(i.e. a greedy algorithm can find an appropriate \( H'_b \)).

Consider a hyperedge \( e \) in \( H'_b \). Observe that \( H'_b \) is a \( B_k \)-free hypergraph, since it is a subhypergraph of \( H \), therefore the number of Berge triangles sitting on the edge \( e \) is at most \( k - 1 \). Apply the following greedy procedure until all the hyperedges are marked. In a step pick an unmarked hyperedge, mark it and delete an unmarked hyperedge of each Berge triangle containing the current hyperedge. Observe that this marked edge is not an edge of a triangle anymore. Define \( H''_b \) the
following way. Let $V(H''_b) = V(H'_b) = V(H)$ and $E(H''_b)$ contains the remaining hyperedges of $H'_b$. Observe that at most $k - 1$ edges were deleted in each step and marked edges were never deleted. Therefore

$$|E(H''_b)| \geq \frac{|E(H'_b)|}{k}.$$  

Moreover $H''_b$ is a Berge-triangle-free 3-uniform linear hypergraph therefore Theorem 3 can be applied with $\epsilon' = \frac{\epsilon}{6k^2 - 8k}$. We get that

$$\frac{|E(H_b)|}{6k^2 - 8k} \leq |E(H''_b)| \leq \frac{\epsilon n^2}{6k^2 - 8k}.$$  

\[ \square \]

**Proof of Theorem 4.** By definition $E(H)$ is a disjoint union of $E(H_r)$ and $E(H_b)$. By Claim 5 and Claim 7

$$|E(H)| \leq |E(H_r)| + |E(H_b)| \leq \frac{n^2}{8} + \epsilon n^2.$$  

\[ \square \]

3 Conclusions

Recall that both Turán numbers of triangle-free graph and $k$-book-free graphs on $n$ vertices are $\frac{n^2}{4}$, moreover Győri [8] proved that the maximum number of hyperedges in a Berge triangle-free 3-uniform hypergraphs on $n$ vertices is at most $\frac{n^2}{8}$. Given the similarities, we conjecture the following:

**Conjecture 1.** For a given $k \geq 2$ every 3-uniform $B_k$-free hypergraph $H$ on $n$ vertices ($n$ is large) has at most $\frac{n^2}{8}$ hyperedges.

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