Andrade, Omori and Time-to-failure Laws from Thermal Noise in Material Rupture

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Using the simplest possible ingredients of a rupture model with thermal fluctuations, we provide an analytical theory of three ubiquitous empirical observations obtained in creep (constant applied stress) experiments: the initial Andrade-like and Omori-like $1/t$ decay of the rate of deformation and of fiber ruptures and the $1/(t_c - t)$ critical time-to-failure behavior of acoustic emissions just prior to the macroscopic rupture. The lifetime of the material is controlled by a thermally activated Arrhenius nucleation process, describing the cross-over between these two regimes. Our results give further credit to the idea proposed by Ciliberto et al. that the tiny thermal fluctuations may actually play an essential role in macroscopic deformation and rupture processes at room temperature. We discover a new re-entrant effect of the lifetime as a function of quenched disorder amplitude.

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Constant stress (so-called “creep”) experiments constitute a standard testing procedure in material sciences. The typical response to the sudden application of a constant stress is that the strain rate as well as the acoustic emission rate first jump to high values followed by slow universal power law decays, respectively called the Andrade law\textsuperscript{1} for the strain rate and the Omori law\textsuperscript{2} for the acoustic rate. Then, after a long decay whose duration may vary within extraordinary large bounds (see below), the rates rebound and accelerate (while the applied stress remains constant) by following a power law acceleration resulting in a finite-time singularity (the rupture of the sample). The two regimes of decelerating followed by accelerating rates and the lifetime of the structure are the result of a subtle interplay between the pre-existing micro-heterogeneity of the material and the self-organized evolving deformation and damage due to dislocation motion and/or micro-cracking. Up to now, there are no theory encompassing all these regimes. Here, we propose a simple mechanism that provides an explanation of all these observations, which is based on the recent proposal\textsuperscript{3, 4} that thermal noise is strongly renormalized by quenched heterogeneities. Based on the analysis of a simple fiber bundle rupture model, Refs.\textsuperscript{3, 4} showed that the average lifetime of the fiber-bundle takes an Arrhenius form with an effective temperature renormalized from the bare temperature $T$ to a value strongly amplified by the presence of the frozen disorder in the rupture thresholds $f_c(i)$, in agreement with experiments and numerical simulations. This result suggests that the usual assumption of neglecting the role of thermal fluctuations in material rupture processes at room temperature may actually be incorrect (see \textsuperscript{5} for early discussions): due to frozen heterogeneities, tiny thermal fluctuations can be amplified many times, thus actually controlling the time-dependent aspects of failure. Our purpose is to extend the analysis of this model by showing that it is able to reproduce all the empirical observations mentioned above in creep (constant applied stress) experiments. This shows that the simplest possible ingredients of a rupture model together with thermal fluctuations can render essentially all of the richness of creep experiments.

The democratic fiber-bundle model (DFBM) with thermal noise\textsuperscript{3, 4} can be seen as a mean field treatment of rupture. A macroscopic constant load $F = N f_0$ is applied at time $t = 0$ to an initially undamaged system made of a very large number $N$ of parallel elastic fibers (the results derived below are obtained in the thermodynamic limit $N \to \infty$). At all times, $F$ is shared democratically among all $(1 - \Phi(t))N$ surviving fibers, where $\Phi(t)$ is the fraction of broken fibers at time $t$. The externally applied force per surviving fiber is thus

$$f_a = \frac{f_0}{1 - \Phi(t)}.$$  \hspace{1cm} (1)

The strength of each fiber $i$ is characterized by a critical value $f_c(i)$ drawn for a distribution $P_d(f)$, centered on the mean equal to 1 and with variance $T_d$. Putting the mean strength to 1 sets the force scale. The heterogeneous strengths are constant and determined once for all, corresponding to a frozen disorder, which is “read” in a certain organized way as the rupture develops. Microscopic thermal fluctuations are taken into account by assuming that a fiber with load $f_a$ and threshold $f_c(i) > f_a$ has a non-zero probability $G(f_c(i) - f_a)$ to rupture per unit time governed by the rate with which a thermal fluctuation can activate a microscopic force $\Delta f_i \geq f_c(i) - f_a$ to pass the rupture threshold $f_c(i)$:

$$G(f_c(i) - f_a) = \frac{2}{\sqrt{2\pi T}} \text{erfc} \left( \frac{f_c(i) - f_a}{\sqrt{2T}} \right),$$

where $\text{erfc}(x)$ is the
complementary error function, \( T \) is the variance of the thermal force fluctuations \( \Delta f_i \) and \( \gamma \) is a microscopic constant rate fixing the time scale of the thermal activation process. This expression amounts to introducing a zero-mean normal distribution of thermal fluctuation forces \( \Delta f_i \) with variance \( \gamma_t \) and with correlation time proportional to \( 1/\gamma_t \).

We first follow §4 and introduce the distribution \( Q(f, t) \) of the rupture thresholds of the unbroken fibers at time \( t \). Obviously, \( Q(f, t) = 0 \) for \( f < f_a \), since all these fibers are already broken. Politi et al. [9] have shown that \( Q(f, t) \) can be approximated with a very high accuracy in the limit \( N \to \infty \) by the initial distribution \( P_d(f) \) of rupture strengths truncated at a lower value \( f_s(t) \),

\[
Q(f, t) = P_d(f), \quad \text{for} \ f > f_s(t) \tag{2}
\]

and 0 otherwise, where \( f_s(t) \) is determined by the self-consistent equation

\[
\dot{\Phi}(t) = \int_{-\infty}^{f_s(t)} df \ P_d(f) \tag{3}
\]

expressing that all fibers whose strengths are below \( f_s(t) \) have failed at some time before \( t \). This approximation for \( Q(f, t) \) with (3) amounts to view the time-dependent rupture process as a “front” propagating and “eating” the distribution \( P_d(f) \) from the weakest towards the strongest fibers. We also have by definition \( \Phi(t) = 1 - \int_{-\infty}^{f_s(t)} df \ Q(f, t) \). Taking the time derivative of \( \Phi(t) \) and replacing \( Q(f, t) \) by \(-Q(f, t) G(f - f_a)\) expressing that the rate of breaking is controlled by the thermally activated rupture process acting on each fiber independently, we get \( \dot{\Phi} = \int_{-\infty}^{f_s(t)} df \ Q(f, t) G(f - f_a) \). Putting (2) in this equation and taking for \( P_d(f) \) a normal distribution centered on 1 with variance \( T_d \) as in [3, 4] yields

\[
\dot{\Phi} = \frac{\gamma}{2} \int_{f_a}^{\infty} df \ \frac{1}{\sqrt{2\pi T}} \exp \left[ -\frac{(f - f_a)^2}{2T} \right] \text{erfc} \left( \frac{f - f_a}{\sqrt{2T}} \right) df. \tag{4}
\]

Making explicit \( P_d(f) \) in (3) gives

\[
\Phi = \frac{1}{2} \left[ \text{erf} \left( \frac{f_s - 1}{\sqrt{2T}} \right) + 1 \right], \quad f_s = 1 + \sqrt{2T} \text{erf} \left( 2\Phi - 1 \right), \tag{5}
\]

where \( y = \text{erf}(z) \) is the inverse function to the error function \( z = \text{erf}(y) \). Putting \( f_s \) in (4) gives

\[
\dot{\Phi} = R(\dot{\Phi}) \equiv \frac{\gamma}{2} \int_0^1 \text{erfc} \left[ L(\Phi, z) \right] dz, \tag{6}
\]

\[
L(\Phi, z) = \frac{1}{\sqrt{2T}} \left( 1 - \frac{f_0}{1 - \Phi} \right) + \mu \text{irf} (2z - 1), \tag{7}
\]

with \( \mu = \sqrt{T_d/T} \). The solution of equation (6) with (7) provides in principle all the information on the fraction \( \Phi(t) \) of broken fibers. This equation (7) is valid as long as the approximation (2) holds (see below).

The parameter \( \mu \) quantifies the relative importance of the thermal fluctuations compared with the quenched heterogeneities. The relevant regime for applications to macroscopic ruptures at room temperature is \( \mu > 1 \) and often \( \mu \gg 1 \), that is, thermal fluctuations are tiny contributions to the applied macroscopic mechanical forces. Indeed, assuming that the energy barrier to rupture a fiber corresponds to the Griffith energy \( g c^2 \), necessary for nucleating a crack of half-length \( c \) in the solid with surface energy \( g \), we obtain \( \mu \approx 1.5 - 4 \cdot 10^3 \) for \( c = 1 \) micron and \( \mu \approx 1.5 - 4 \) for \( c = 1 \) nanometer, using \( g = 10 - 50 \) erg/cm\(^2\) for most solids. Thus, even for the smallest microcracks, thermal fluctuations are very small in relative value.

It turns out that this regime \( \mu \geq 1 \) allows for a very convenient approximation of \( R(\dot{\Phi}) \) obtained by linearizing \( L(\Phi, z) \) with respect to \( z \). Then, the integral over \( z \) in (3) can be calculated explicitly to yield

\[
\dot{\Phi} = R(\dot{\Phi}) = \frac{\gamma T}{4\pi \mu D(\dot{\Phi}) U(\dot{\Phi})} \exp \left( -\frac{U(\dot{\Phi})}{T} \right), \tag{8}
\]

where

\[
U(\dot{\Phi}) = TL^2(\dot{\Phi}, \Phi) = \frac{1}{2} \left[ f_s(\Phi) - f_a(\Phi) \right]^2, \tag{9}
\]

and 
\[
D(\Phi^*) (1 - \Phi^*)^2 = \alpha, \quad \alpha = \frac{f_0}{\sqrt{2\pi T}} \tag{10}
\]

with \( \alpha \) a important physical parameter quantifying the strength of the disorder relative to the applied force. The explicit approximate solution of equation (10) is

\[
\Phi^*(\alpha) = \begin{cases} 
\frac{20 - \pi - 4(\pi - 4)\alpha}{24 - 2\pi + 8(\pi - 4)\alpha}, & \alpha < 3/2, \\
\frac{1}{2} \text{erfc} \left( \sqrt{\ln \alpha} \right) \alpha^{\sqrt{2\ln \alpha}} 1 + \alpha^{\sqrt{2\ln \alpha}}, & \alpha > 3/2. 
\end{cases} \tag{11}
\]
For example, this gives $\Phi^*(\alpha = 2) = 0.089$ compared with the exact value 0.092.

It follows from (5) that the time to reach some $\Phi$ is given by $\gamma T t = 4\pi\mu \int_0^\Phi D(z) U(z) e^{U(z)/T} dz$. For $0 < \Phi < \Phi^*(\alpha)$, due to the exponential factor $e^{U/T}$, the main contribution to the last integral comes from a small neighborhood of the upper integration limit. This yields

$$\gamma t \simeq 4\pi\mu \frac{D(\Phi)U(\Phi)}{A(\Phi)} e^{U(\Phi)/T}, \quad A(\Phi) = \frac{dU(\Phi)}{d\Phi}, \quad (12)$$

for $\Phi < \Phi^*$. This approximation is correct under the assumption that $e^{U(\Phi)/T}$ is rapidly increasing with $\Phi$, i.e., if $|A(\Phi)| \gg T$. The absolute value $|\ldots|$ stresses that this condition applies also for $\Phi > \Phi^*(\alpha)$. The same reasoning in this case gives a similar approximation

$$\gamma(t_\ast - t) \simeq 4\pi\mu \frac{D(\Phi)U(\Phi)}{|A(\Phi)|} e^{U(\Phi)/T}, \quad \Phi > \Phi^*; \quad (13)$$

where $t_\ast - t$ is the time to complete rupture. The condition $|A(\Phi)| \gg T$ shows that both relations $(12)$ and $(13)$ do not work in the vicinity of the minimum rate of fiber failures given by the solution of $(10)$, for which $A(\Phi^*) = 0$. For $0 < \Phi < \Phi^*$, combining $(5)$ and $(12)$, we obtain $U = T/t$ for $t < t^\ast$, where $t^\ast$ is defined by $\Phi(t^\ast) = \Phi^*$. This gives

$$U [\Phi(t)] = T \ln[\gamma(t_\ast - t)] \quad (\Phi^* < \Phi < \Phi_c), \quad (14)$$

valid for $\gamma t \gg 1$ (this condition simply means that the thermal fluctuations have had time to contribute several independent jolts). The constant of integration gives the ln $\gamma$ contribution determined from matching with the initial stage. Replacing the lhs of $(3)$ by $(14)$ and putting $f_\alpha \simeq f_0$ (for $\Phi$ small) gives, in view of equation $(5)$, the fraction rate

$$\phi \simeq \frac{1}{4\pi\mu t \ln t} \exp \left[ -\frac{1}{2Td} \left(1 - f_0 - \sqrt{2T\ln\gamma t}\right)^2 \right]. \quad (15)$$

Expression $(15)$ is one of our main results: the failure rate $\Phi$ of fibers decreases after application of the load proportionally to $1/t$, up to logarithm corrections. This $1/t$ decay lasts as long as $\Phi$ remains smaller than $\Phi^*$. This $1/t$ law is known in seismology as the Omori law. It is also ubiquitous in creep experiments with exponents that are often close to or smaller than our prediction $1$. For intermediate times such that $\gamma t < e^{1/2T}$, $\phi \sim \frac{1}{\ln t} e^{(1-f_0)\sqrt{2T\ln\gamma T}/T}$, which gives an apparent exponent $\sim 1/t^p$ with $p < 1$. For $\ln \gamma t \gg (1-f_0)^2/2T$, $p \to 1 + (T/Td)$ which is close to but slightly larger than 1. Numerical simulations confirm these predictions accurately. See for instance figure 2 of [5] which our theory explains quantitatively. Exact numerical integration and our analytical approximation coincide everywhere, excluding a time interval corresponding to a very small vicinity of the stationary point $\Phi^*$. Note that Andrade's law also derives from the deformation rate being proportional to $df_a(t)/dt = f_0\Phi/(1 - \Phi(t))^2 \propto \Phi$ as $\Phi(t)$ varies much more slowly than $\Phi$.

Let us now turn to the description of the second regime $\Phi(t) > \Phi^*$, relevant to obtain the failure rate up to global failure. Combining $(5)$ and $(13)$, we obtain the expression

$$U [\Phi(t)] = T \ln[\gamma(t_\ast - t)] \quad (\Phi^* < \Phi < \Phi_c), \quad (16)$$

which is analogous to $(14)$. The regime $\Phi^* < \Phi(t) < \Phi_c$ is strongly influenced by thermal fluctuations, so that the disorder term can be neglected to obtain, in view of $(10)$ and $(6)$, $\Phi_c - \Phi(t) = f_0 \sqrt{2T\ln\gamma(t_\ast - t)}$. Differentiating both sides of this expression with respect to $t$ yields the failure rate

$$\Phi(t) = C(t)/(t_\ast - t), \quad (17)$$

where $C(t) = f_0 T/[\epsilon(1-c)^2]$ with $c = \sqrt{2T\ln[\gamma(t_\ast - t)]}$. This is the second important result of our analysis (see also eq. (B11) in [5]), which shows that, for $\Phi > \Phi^*$, the failure rate accelerates towards a finite-time singularity approximately as $\sim 1/(t_\ast - t)$. Such a behavior has been documented extensively in experiments on rupture of heterogeneous material [6]. Our analysis provides a novel mechanism for the ubiquitous time-to-failure regime observed in heterogeneous material. Strong quenched heterogeneity has been shown to play an essential role in controlling the critical nature of the rupture process [7] and in the existence of a time-to-failure power law such as $(17)$. Here, we confirm that the heterogeneity is essential to renormalize the thermal fluctuations [6]. While the philosophy is similar, the mechanism is different and novel. As for the Omori law, the logarithmic corrections in $(17)$ may give an apparent exponent of the power law, slightly smaller than 1, as observed in experiments. Our numerical tests show that expression $(17)$ provides an approximation which coincide almost everywhere with the exact solution inside the interval $t^\ast < t < t_\ast$.

There is a simple physical interpretation of the transition between the two above mentioned rate behaviors $(15)$ and $(17)$. To explain the first (rate decaying) regime, consider the degenerate case of spontaneous fracture ($f_0 = 0, \Phi^* = 1$) for which $U(\Phi) = f_2/2$. As time increases, $f_2$ grows, the remaining fibers are stronger and the failure rate decays together with the rate of change of the energy barrier. The second regime can be qualitatively understood by taking the limit of zero disorder $(T_d = 0, \Phi^* = 0)$, leading to $U(\Phi) = (1 - f_2)^2/2$. The force $f_2$ per remaining fiber grows with time, the fibers break more and more easily and the failure rate grows to give the fracture in finite-time. In the intermediate case $0 < \Phi^* < \Phi_c$, due to the competition between the growth of $f_2$ and $f_a$, the two regimes co-exist. At early times, the
growth of \( f_a \) dominates giving the Omori and Andrade laws, followed by the growth of \( f_a \) in the second regime \( \Phi > \Phi^* \) giving the power law finite-time singularity.

Lastly, we turn to the behavior for \( \Phi \approx \Phi^* \), which turns out to provide the dominant contribution for the total time for rupture, as shown in [3, 4]. Indeed, the fiber bundle spends most of its time in the vicinity of the stationary point \( \Phi^* \), corresponding to the minimum failure rate. In this case, \( U \) can be expanded as

\[
U(\Phi) = U(\Phi^*) - B(\Phi^*)(\Phi - \Phi^*)^2, \quad B(\Phi) = -\frac{1}{2} \frac{d^2U(\Phi)}{d\Phi^2},
\]

and equation (8) becomes

\[
\Phi \simeq R(\Phi^*) \exp \left[ -\frac{B(\Phi^*)}{T} (\Phi - \Phi^*)^2 \right]. \tag{18}
\]

The solution of this equation is

\[
\text{erfi} \left( \frac{B(\Phi^*)}{T} (\Phi - \Phi^*) \right) = 2 \sqrt{\frac{B(\Phi^*)}{\pi T}} R(\Phi^*) (t - t^*), \tag{19}
\]

where \( \text{erfi}(z) = \frac{1}{2} \text{erf}(iz) \) is the imaginary error function. Using its asymptotics \( \text{erfi}(z) \sim \frac{1}{\sqrt{\pi}} z e^{z^2} \) for large \( z \) together with (19), equation (18) becomes

\[
\frac{d\Psi^2}{dt} \simeq \frac{T}{B(\Phi^*)} (t - t^*), \quad \Psi = \Phi - \Phi^*,
\]

whose solution yields the fracture rate

\[
\dot{\Phi} \approx \frac{\sqrt{T \text{sign}(t - t^*)}}{2|t - t^*| \sqrt{B(\Phi^*) \ln(\gamma|t - t^*|)}} \quad (\gamma|t - t^*| \gg 1).
\]

Expression (18) allows us additionally to calculate the total lifetime \( t_c \) of the fiber bundle:

\[
\gamma t_c = \frac{1}{R(\Phi^*)} \int_{-\infty}^{\infty} \exp \left[ -\frac{B(\Phi^*)}{T} (\Phi - \Phi^*)^2 \right] d\Phi. \tag{20}
\]

The calculation of this integral with the use of (8) gives

\[
\gamma t_c \simeq 4\pi \sqrt{\frac{\alpha}{T_d}} \frac{D(\Phi^*)U(\Phi^*)}{\sqrt{\text{B}(\Phi^*)}} \frac{T_d}{T} \exp \left[ -\frac{U(\Phi^*)}{T} \right]. \tag{21}
\]

This expression, together with (11), recovers the main result of [3, 4], while improving on the prefactors to the main Arrhenius-type dependence.

Using (9) and (11), \( U(\Phi^*) \) can be written explicitly

\[
U(\Phi^*) = \left[ \frac{\Phi_c - \Phi^*}{1 - \Phi^*} \pm \sqrt{\ln \left( \frac{\alpha}{(1 - \Phi^*)^2} \right)} \right]^2 \tag{22}
\]

where the sign + (resp. −) corresponds to the case \( \Phi^* > 1/2 \) (resp. \( \Phi^* < 1/2 \)). As shown in Fig. 1 \( U(\Phi^*) \) is a non-monotone function of \( T_d \). Due to the above mentioned competition between quenched disorder and the growth of the actual force \( f_a \), \( U(\Phi^*) \) decreases as long as \( T_d < T^*_d \) and then increases with increasing \( T_d \) beyond \( T^*_d \). The first regime \( T_d < T^*_d \) corresponds to the effect discovered in Refs. [3, 4] and mentioned above of the renormalization of thermal fluctuations by quenched disorder, and consequently of decreasing strength by increasing the disorder. Since a larger \( U(\Phi^*) \) corresponds to a large lifetime through (21), we uncover the new effect of a strengthening of the fiber system by increasing the disorder beyond a certain threshold. All our formulas have been checked by direct numerical integration with excellent agreements. We expect that extensions of the DFBM to non-mean field power law interactions [8] will not change our results qualitatively but may modify the Omori’s and time-to-failure exponents.

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