Mathematical Analysis of A Tuberculosis-Lymphatic filariasis Co-infection Model.

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Abstract
An Ordinary Differential Equation(ODE) co-infection model of Tuberculosis-Lymphatic filariasis is proposed with 17 mutually disjoint compartments. We showed that the model is Mathematically and Epidemiologically well-posed and the disease-free equilibrium(DFE) of the co-infection model is locally asymptotically stable if $R_0 < 1$, it is unstable if $R_0 > 1$. The numerical results show that lymphatic filariasis infection increases susceptibility to tuberculosis infection. This is in agreement with literature that, persons with lowered immunity such as HIV, diabetes, immune disorder etc are at a higher risk of contacting infectious diseases. We also found that increasing the rate of diagnosis and treatment of active tuberculosis and symptomatic lymphatic filariasis cases, the incidence of co-infection in the community can be reduced, and that if resources are limited, efforts should be targeted at treating only the co-infected cases. Sensitivity analysis showed that increasing mosquito mortality rate, reducing the probability of mosquito infecting humans, reducing the probability of humans infecting mosquitoes, reducing mosquitoes recruitment rate by destroying mosquitoes breeding sites and reducing the number of times a mosquito bites human will bring down the reproduction number to less than unity.

Keywords: Tuberculosis, Lymphatic filariasis, Co-infection, Tuberculosis-Lymphatic filariasis.

1 Introduction

In recent times, we have witnessed a resurgence of deadly infectious diseases which were once thought to have been eradicated appearing in new geographical areas, where they were initially not present. Malaria, tuberculosis(TB), HIV/AIDS, Dengue fever, cholera, West Nile virus, lymphatic filariasis(LF), Chagas, Lassa fever e.t.c. and recently, the Corona virus disease, 2019 (COVID 19) are just a few diseases which continue to persist despite all efforts committed to getting them eradicated.

LF and TB, according to World Health Organisation, are endemic in India, Indonesia and Nigeria. There is an overlap of endemic regions between LF and TB which may lead to co-infection [1]. Moreover, microbiological studies have suggested that LF infection could affect the pathogenesis of TB in co-infection cases. Persons with lowered immunity such as HIV, diabetes, immune disorder etc are at a higher risk of contracting TB [2]. Filariasis is a group of human and animal infectious diseases caused by nematode parasites commonly called filariae [3]. The thread-like adult parasites live in vessels, tissues or body cavities of the vertebrate hosts. The female worms are viviparous and produce microscopic embryos called microfilariae [3]. The microfilariae circulate in the blood or migrate through the skin from where they are ingested by vectors during blood meal [3]. The worms have an estimated active reproductive
span of 4-6 years, producing millions of small immature microfilariae, which circulate in the peripheral blood [4]. They are transmitted from person to person by several species of mosquitoes [4]. The drug diethylcarbamazine (DEC) is used to kill the microfilariae in the blood stream. But in the case of elephantiasis, surgical procedure may be prescribed to reduce the swelling and relieve the obstruction. TB is an airborne transmitted disease caused by slowly-replicating bacterium. ([5], [4]). It affects the lungs (pulmonary TB) but can also affect other parts of the body (extra pulmonary TB). The disease is released into the air as droplets when those who have active TB cough or sneeze [6]. Individuals having latent TB cannot transmit TB [7]. Only 10% of the latent TB individuals actually develop active TB. Latent TB individuals can remain in the latent state for a long time, and may die without developing the disease [8]. The good news is that TB and LF can be treated.

TB and LF are widespread and serious public health problems in developing countries. While TB has continued to cause high mortality in humans, LF is a major cause of chronic morbidity in humans [4]. It is estimated that TB infects one-third of the world’s population resulting in 23 million deaths per year [1]. And over 120 million people are affected by lymphatic filariasis (LF) disease worldwide [4]. TB and LF are endemic in Nigeria, suggesting an overlap of endemic region between TB and LF that may lead to co-infection [1]. LF impedes the smooth functioning of the immune system [9] which can lower the immune defence system, making the individual to be highly susceptible to TB and other infectious diseases. Helminths co-infections with other diseases alone are thought to occur in over 800 million people and are especially prevalent among the global poor. ([10],[11], [12]). TB patients with helminth infections have severe pulmonary disease, which can interfere with diagnosis test for TB ([13], [14], [15]). Six high burden countries (HBC) account for 60% of global TB incidences. These countries are India, Indonesia, China, Nigeria, Pakistan and South Africa. According to WHO [16] TB mortality accounts for 1.4 million deaths and 0.4 million (additional) deaths due to HIV/AIDS and TB. There are about 600,000 new (incident) TB cases in Nigeria and 250,000 TB-related deaths (WHO, 2016). LF infect 120 million people globally with about 40 million disfigured and incapacitated [4]. India, Indonesia and Nigeria are the countries with the highest LF burden in the world.

Several mathematical models have been formulated to understand the transmission dynamics of TB and LF under various circumstances. [17] developed and analysed an LF disease transmission model to determine the impact of multi-interventions campaign via health education and sterile insect technique (SIT) on the spread of LF. [18] formulated and analysed a nonlinear differential equation model to study the effect of chemophylaxis on the exposed individuals. [19] developed a mathematical model to investigate the impact that vector genus specific dependent processes may have on overall LF transmission. [20] used lymphatic filariasis Simulation Model (LYMFASIM) to estimate the duration of Mass Drug Administration (MDA) required for elimination and residual levels of microfilaria (MI) and antigenaemia (Ag) prevalence reached after that duration in different transmission settings, varying from low to high. The result indicated that the duration of annual MDA increased with higher baseline endemicity and lower coverage. [21] used a model-based assessment to develop different plausible scale-up scenarios to reach global elimination and eradication of LF. They predicted the duration of MDA to reach local elimination for different transmission archetypes and estimated the required number of treatments and the implication of rapid scale-up. [22] used mathematical models to assess the feasibility and strategic value of including vector control in the GPELF initiative to achieve the global elimination of LF. [23] developed and analysed a mathematical model to quantify the potential effect that heterogeneous infection processes occurring in the major mosquito vector genera may have on parasite transmission and control. [24] formulated and analysed a deterministic differential equation model with two key control measures; quarantine and treatment. They assumed that no infection exists at the initial stage and that there is no vertical transmission in both human and mosquito population. [25] formulated a TB model with seasonality, [26] formulated and analysed a mathematical model for the transmission dynamics of TB with vaccination and screening of individuals to identify individuals with chronic TB cases to be placed on prompt TB treatment. [27] proposed and analysed an SEIR compartmental model to examine the population dynamics of TB with BCG vaccination as a control measure. [28] presented and analysed a new mathematical model for TB dynamics to study the effects of additional heterogeneities based on the level of awareness of TB within the population and active-case finding on the dynamics of the disease. [29] developed and analysed a mathematical model based on the level of exposure of individuals to mycobacterium TB. [30] formulated a co-infection model for malaria and rotavirus. [31] proposed and analysed a deterministic mathematical model for HIV/TB co-infection to examine the impact of HIV or TB infection with treatment of TB and management of HIV using Antiretroviral Therapy (ART). [32] proposed a system of nine nonlinear mathematical model to study the co-infection of both zika and malaria in a community where they are both
The pathogenesis of TB and LF, signs and symptoms, diagnosis and treatment as well as current epidemiological data are presented. Thereafter, an extensive review of TB and LF were carried out. Mathematical tools and techniques include: use of Lyapunov functionals, bifurcation analysis, numerical simulations and other methods used for the study of dynamical systems. ([37], [38], [39], [40], [41], [42], [43], [44], [45]). Based on the biology and natural history of TB and LF, we formulated seventeen novel deterministic mathematical co-infection models given below.

The total human and mosquito population at time $t$, denoted by $N_h(t)$ and $N_v(t)$ are divided into 17 mutually exclusive compartments. This is made up of 14 mutually-exclusive compartments for human population and 3 mutually-exclusive compartments for mosquito population, see Figure 1. The human population is made up of susceptible individuals $S_h(t)$, individual having latent TB $E_l(t)$, individuals having active TB $I_T(t)$, individuals treated of TB $T_l(t)$, individuals having latent LF $E_h(t)$, individuals having asymptomatic LF $A_h(t)$, individuals having symptomatic LF $I_h(t)$, dually infected individuals having latent LF and latent TB $E_{ht}(t)$, dually infected individuals having latent LF and active TB $E_{ht}(t)$, dually infected individuals with asymptomatic LF and latent TB $A_{ht}(t)$, dually infected individuals with asymptomatic LF and active TB $A_{ht}(t)$, dually infected individuals with symptomatic LF and latent TB $I_{ht}(t)$, dually infected individuals with symptomatic LF and active TB $I_{ht}(t)$ and individuals with treated LF $T_h(t)$. So that

$$N_h(t) = S_h(t) + E_l(t) + I_T(t) + T_l(t) + E_h(t) + A_h(t) + I_h(t) +$$

$$E_{ht}(t) + E_{ht}(t) + A_{ht}(t) + A_{ht}(t) + I_{ht}(t) + I_{ht}(t) + T_h(t)$$

Similarly, the mosquito population at time $t$ is made up of susceptible mosquito population, $S_v(t)$, exposed mosquito population, $E_v(t)$ and infected female mosquito population, $I_v(t)$, so that

$$N_v(t) = S_v(t) + E_v(t) + I_v(t)$$

Susceptible individuals acquire LF infection following effective contact with mosquito infected with LF at a rate given by

$$\lambda_v = \frac{\beta_v \sigma_v \sigma_h I_v}{\sigma_v N_v + \sigma_h N_h}$$

Susceptible mosquitoes can acquire LF when in contact with individuals infected with LF (both singly and dually infected) at a rate given by

$$\lambda_h = \frac{\beta_h \sigma_v \sigma_h \left( \eta_1 E_h + \eta_1 A_h + I_h \right) + \eta_v \left( \eta_2 E_{ht} + \eta_2 A_{ht} + I_{ht} \right) + \eta_v \omega_v \left( \eta_3 E_{ht} + \eta_3 A_{ht} + I_{ht} \right)}{\sigma_v N_v + \sigma_h N_h}$$
Susceptible individuals acquire TB following effective contact with an infected individual at a rate given by

$$\lambda_T = \frac{\beta_T (I_T + \theta_T (\eta_{T1} E_{hT} + \eta_{T2} A_{hT} + I_{hT}))}{N_h}$$ (2.5)

See Tables 1 and 2 for the definition of state variables and parameters in the model.
The model for TB-LF co-infection is given by the following deterministic system of nonlinear differential equations.

\[
\begin{align*}
\dot{S}_h &= \gamma_1 S_h - \lambda_h S_h + \mu_h S_h - \mu_v S_v \\
\dot{E}_h &= \gamma_2 E_h - \lambda_h E_h + \mu_h E_h - \mu_v E_v \\
\dot{A}_h &= \gamma_3 A_h - \lambda_h A_h + \mu_h A_h - \mu_v A_v \\
\dot{T}_h &= \gamma_4 T_h - \lambda_h T_h + \mu_h T_h - \mu_v T_v \\
\dot{I}_h &= \gamma_5 I_h - \lambda_h I_h + \mu_h I_h - \mu_v I_v \\
\dot{S}_v &= \mu_v S_v - \mu_v S_v \\
\dot{E}_v &= \mu_v E_v - \mu_v E_v \\
\dot{A}_v &= \mu_v A_v - \mu_v A_v \\
\dot{T}_v &= \mu_v T_v - \mu_v T_v \\
\dot{I}_v &= \mu_v I_v - \mu_v I_v \\
\end{align*}
\]

**Figure 1:** Model schematic diagram.
\[
\begin{align*}
\frac{dS_h}{dt} &= \lambda_h - \lambda_T S_h - \lambda_v S_h - \mu_h S_h, \\
\frac{dE_t}{dt} &= (1 - p_1)\lambda_T (S_h + \phi T + T_h) - \epsilon_1 \lambda_T E_t - \lambda_v E_t + \tau h_1 I_h - (\gamma_1 + \mu_h) E_t, \\
\frac{dI_T}{dt} &= p_1 \lambda_T (S_h + \phi T + T_h) + \epsilon_1 \lambda_T E_t + \gamma_1 E_t - \lambda_v I_T + \tau h_3 I_{hT} - (\tau T_1 + \delta_T + \mu_h) I_T, \\
\frac{dT_i}{dt} &= \tau T_1 I_T - \phi \lambda_T T_i - \lambda_v T_i - \mu_h T_i, \\
\frac{dE_h}{dt} &= \lambda_v (S_h + T_i) - \theta_1 \lambda_T E_h + \nu \lambda_v T_h + \tau T_2 E_{hT} - (k_1 + \mu_h) E_h, \\
\frac{dA_h}{dt} &= k_1 E_h + \tau T_3 A_{hT} - (k_2 + \mu_h) A_h - \theta_2 \lambda_T A_h, \\
\frac{dI_h}{dt} &= k_2 A_h + \tau T_4 I_{hT} - (\tau h_1 + \delta_L + \mu_h) I_h - \theta_3 \lambda_T I_h \\
\frac{dT_h}{dt} &= \tau h_1 I_h - \lambda_T T_h - \nu \lambda_v T_h - \mu_h T_h, \\
\frac{dE_{ht}}{dt} &= \lambda_v E_t + (1 - p_2)\theta_4 \lambda_T E_h - \epsilon_2 \lambda_T E_{ht} - (\gamma_2 + k_2 + \mu_h) E_{ht}, \\
\frac{dE_{ht}}{dt} &= \lambda_v I_T + p_2 \theta_1 \lambda_T E_h + \epsilon_2 \lambda_T E_{ht} + \gamma_2 E_{ht} - (\tau T_1 + k_4 + \delta_L + \mu_h) E_{hT}, \\
\frac{dA_{ht}}{dt} &= (1 - p_3)\theta_2 \lambda_T A_h - \epsilon_3 \lambda_T A_{ht} + k_3 E_{ht} - (\gamma_3 + \tau h_5 + \mu_h) A_{hT}, \\
\frac{dA_{ht}}{dt} &= p_3 \theta_2 \lambda_T A_h + \epsilon_3 \lambda_T A_{ht} + \gamma_3 A_{ht} + k_4 E_{ht} - (\tau T_3 + k_6 + \delta_T + \mu_h) A_{hT}, \\
\frac{dI_{ht}}{dt} &= (1 - p_4)\theta_3 \lambda_T I_h - \epsilon_4 \lambda_T I_{ht} + k_5 A_{ht} - (\gamma_4 + \tau h_2 + \delta_L + \mu_h) I_{ht}, \\
\frac{dI_{ht}}{dt} &= p_4 \theta_3 \lambda_T I_h + \epsilon_4 \lambda_T I_{ht} + k_6 A_{ht} + \gamma_4 I_{ht} - (\tau T_4 + \tau h_3 + \delta_L + \delta_T + \mu_h) I_{hT}, \\
\frac{dS_v}{dt} &= \Lambda_v - \lambda_h S_v - \mu_v S_v, \\
\frac{dE_v}{dt} &= \lambda_v S_v - k_v E_v - \mu_v E_v, \\
\frac{dI_v}{dt} &= k_v E_v - \mu_v I_v,
\end{align*}
\]
### Table 1: Description of state variables of model (2.6)

| Variable   | Description                                                                                   |
|------------|-----------------------------------------------------------------------------------------------|
| $S_h(t)$   | Population of susceptible individuals                                                          |
| $E_l(t)$   | Population of individuals with latent TB                                                      |
| $I_T(t)$   | Population of individuals with active TB                                                       |
| $T_t(t)$   | Population of individuals treated of TB                                                        |
| $E_h(t)$   | Population of individuals with latent LF                                                      |
| $A_h(t)$   | Population of individuals with asymptomatic LF                                                |
| $I_h(t)$   | Population of individuals with symptomatic LF                                                 |
| $T_h(t)$   | Population of individuals treated of LF                                                        |
| $E_{ht}(t)$| Population of dually infected individuals with latent LF and latent TB                         |
| $E_{hT}(t)$| Population of dually infected individuals with latent LF and active TB                         |
| $A_{ht}(t)$| Population of dually infected individuals with asymptomatic LF and latent TB                  |
| $A_{hT}(t)$| Population of dually infected individuals with asymptomatic LF and active TB                  |
| $I_{ht}(t)$| Population of dually infected individuals with symptomatic LF and latent TB                   |
| $I_{hT}(t)$| Population of dually infected individuals with symptomatic LF and active TB                   |
| $S_v(t)$   | Population of susceptible mosquitoes                                                          |
| $E_v(t)$   | Population of exposed mosquitoes.                                                              |
| $I_v(t)$   | Population of infectious mosquitoes.                                                           |

#### 2.1 Explanation of Terms of the Equation

The equations in (2.6) represent the derivatives of the different human and mosquito compartments with respect to time.

The first equation in (2.6) represent the derivatives of the susceptible human population, $S_h$. The first term on the right hand side is the recruitment term, $\Lambda_h$. The second represent the interaction between the susceptible humans, $S_h$ and humans infected with TB leading to new cases of latent TB in humans. These leave the susceptible human population, $S_h$ and move into the latent or exposed class for TB, $E_l$. The third term is the interaction between the susceptible humans, $S_h$ and the infectious mosquitoes, $I_v$ that lead to new cases of LF in humans. These leave the susceptible human population, $S_h$ and move into the latent or exposed class for LF, $E_h$. The last term is the death rate, $\mu_h$ for the susceptible human population, $S_h$.

The second equation in (2.6) represent the derivatives of the latent individuals that are exposed to TB. The first term represent the number of new cases of latent TB. The second represent latent TB individuals that developed active TB. These leave the latent class and move into the infectious class. The third represent the latent TB individuals who developed TB through exogenous reinfection. These also leave the latent class and move into the infectious class, $I_v$. The fourth term is the interaction between infectious mosquitoes and those with latent TB that produce new cases of co-infected individuals, $E_{ht}$ (latent LF and latent TB). These leave the latent TB class and moves into the latent LF and latent TB, $E_{ht}$ class. The fifth term represent the number of dually infected individuals with symptomatic LF and latent TB that was successfully treated of LF but still have latent TB. These move into the latent TB class. The sixth
Table 2: Description of parameters of model (2.6)

| Parameter | Description |
|-----------|-------------|
| $\mu_h, \mu_v$ | Natural death rates |
| $\Lambda_h, \Lambda_v$ | Recruitment rates |
| $\beta_T$ | Transmission rate of TB |
| $\beta_v$ | Transmission probability from mosquitoes to humans |
| $\beta_h$ | Transmission probability from humans to mosquitoes |
| $b_v$ | Number of bites on humans per unit time |
| $b_h$ | Number of bites per mosquito per unit time |
| $\sigma_v$ | Number of times a mosquito bites human per unit time |
| $\sigma_h$ | Maximum number of mosquito bites a human can receive per unit time |
| $\tau_{T1}, \ldots, \tau_{T4}$ | Treatment rates for TB |
| $\tau_{h1}, \tau_{h2}, \tau_{h3}$ | Treatment rates for LF |
| $\gamma_{T1}, \gamma_{T2}, \gamma_{T3}, \gamma_{T4}$ | TB progression rate |
| $\epsilon_{T1}, \epsilon_{T2}, \epsilon_{T3}$ | Exogeneous reinfection rate for TB |
| $k_{T1}, \ldots, k_{T6}$ | Progression rates for LF |
| $p_{T1}, \ldots, p_{T4}$ | Fraction of fast disease progression |
| $\nu$ | Reinflection rates for LF after treatment |
| $\theta_T$ | Increased infectiousness of dually-infected on TB transmission |
| $\eta_{T1}, \eta_{T2}$ | Reduced transmissibility of dually-infected on TB in comparison with infected with TB |
| $\eta_{T1}, \eta_{T2}, \eta_{T3}$ | Reduced transmissibility of LF by asymptomatic individuals in comparison with symptomatic individuals |
| $\eta_v$ | Mod. par. accounts for the increased infectioness of dually-infected|
| $\omega_v$ | Mod. par. accounts for the increased infectioness of dually-infected with active TB compared to those with latent TB |
| $\phi$ | Rate of TB reinfection after treatment |
| $k_v$ | Progression rates for mosquitoes |
| $\delta_L$ | Disease-induced death rate for individuals with LF |
| $\delta_T$ | Disease-induced death rate for individuals with active TB |

The third equation in (2.6) describe the rate of change of the individuals with active TB cases $I_{T}$. The first term represent the number of new cases of individual who progress faster to active TB. The second term are those who get active TB through exogenous reinfection. The third represent those who progress from the latent class to active TB. The fourth term is the interaction between infected mosquitoes and those with active TB that produce new cases of dually infected individual, $E_{hT}$ (latent LF and active TB). These leave the infected TB class and move into the latent LF and active TB class, $E_{hT}$. The fifth term represent the number of dually infected individuals with symptomatic LF and active TB who were successfully treated of LF but still have TB infection. The sixth term is the rate of treatment of active TB cases, $\tau_{T1}$. These leave the infected TB class and moves into the treated class. The seventh term is the disease-induced death of infected TB individuals and the last term is the death rate for the infected TB class.

The fourth equation in (2.6) represent the derivatives of the treated TB population. The first term represent those that have been treated of TB. The second term represent those that developed latent TB through exogenous reinfection. These leave the treated class for TB and move into the latent TB class. The third term on the right hand side is the interaction between infected mosquitoes and those that have been treated of TB leading to new cases of latent LF. These leave the TB treated class and move to the latent class, $E_h$. The last term is the death rate for the treated TB class.
The fifth equation in (2.6) represent the derivatives of the latent LF population. The first term represent the number of those that have latent LF. The second term represent the interaction between the individuals with active TB and those with latent LF disease that lead to new cases of dually infected individual, $E_{ht}$. These leave the latent LF class and move into the latent LF and latent TB class, $E_{ht}$. The third represent those who gets re-infected after successfully been treated of LF. The fourth represent those in the dually infected class $E_{ht}$ that have been treated of the TB disease but still have latent LF. The fifth term represent those that developed LF disease but does not show sign of the disease, they are referred to as asymptomatic LF disease cases. They leave the latent LF class and move to the asymptomatic LF class, $A_{ht}$. The last term represent the death rate for the latent LF class.

The sixth equation in (2.6) represent the derivatives of individuals with asymptomatic LF disease (those who have the disease but does not show symptoms of it). The first term represent those with asymptomatic LF disease. The second represent those that are dually infected with asymptomatic LF and active TB, but have been treated of TB. The third term are those who developed the symptoms of LF. These leave the asymptomatic class and move into the symptomatic class. The fourth is the death rate for the asymptomatic LF class. The last term is the interaction between the asymptomatic LF individuals and individuals with active TB cases, that lead to new cases of dually infected individuals, asymptomatic LF and latent TB individuals, $A_{ht}$. These leave the asymptomatic LF class and move into the $A_{ht}$ class.

The seventh equation in (2.6) represent the derivatives of the symptomatic LF population. The first term is the number of symptomatic LF cases. The second represent dually infected individual with symptomatic LF and active TB whose TB have been treated successfully but still has LF disease. The third term represent those that are undergoing treatment for LF, these leave the symptomatic LF class after successful treatment and move into the treated class $T_{ht}$. The fourth term is the disease-induced death. The fifth term is the natural mortality for the symptomatic LF population. The last is interaction between active TB cases and those with symptomatic LF cases, to produce new cases of dually infected individual with latent TB and symptomatic LF, $I_{ht}$. These leave the symptomatic LF class and moves into the dually infected class with latent TB and symptomatic LF, $I_{ht}$. The eighth term represent the number of new cases of the active TB individuals who were infected with LF disease. The second term represent the interaction of those previously treated of LF and infected mosquitoes to produce latent LF individual. These leave the treated LF class and move into the latent LF class, $A_{ht}$. The last term is the death rate for the treated LF population.

The ninth equation in (2.6) represent the derivatives of dually infected individual with latent TB and latent LF, $E_{ht}$. The first term represent the number of latent TB individuals that were infected with LF, $E_{ht}$. The second term represent the latent LF individual that were infected with TB to produce new cases of dually infected individual with latent TB and latent LF. The third term are the fractions of those latent LF cases who developed active TB. These leave the dually infected class of latent LT and latent TB, $E_{ht}$ and move into the dually infected class of latent LF and active TB, $E_{ht}$. The fourth term represent the individual who got infected through exogenous re-infection. These move into the dually infected class of latent LF and active TB, $E_{ht}$. The sixth term represent those that developed asymptomatic LF. These leave the latent LF and latent TB class, $E_{ht}$ and move into the asymptomatic LF and latent TB class, $A_{ht}$. The last term is the death rate for the latent LF and latent TB population.

The tenth equation in (2.6) represent the derivatives of the latent LF and active TB population, $E_{ht}$. The first term is the number of new cases of the active TB individuals who were infected with LF disease. The second represent the latent LF individuals who were infected with TB and progress to active TB, $E_{ht}$. The third term represent the number of latent LF and latent TB who gets TB through exogenous re-infection. The fourth represent the number of those who progress from $E_{ht}$ class with active TB. The fifth term represent those that have been treated of active TB, these leave the $E_{ht}$ class and move into the latent LF class, $E_{h}$. The sixth term represent the dually infected individual with latent LF and active TB that progress into the asymptomatic LF and active TB. The seventh term is the disease-induced death for the latent LF and active TB population, $E_{ht}$. The last term is the natural death rate for the latent LF and active TB population.
The eleventh equation in (2.6) represent the derivatives of asymptomatic LF and latent TB class. The first term the asymptomatic LF that developed latent TB. The second represent the asymptomatic LF that developed active TB. These leave the asymptomatic LF and latent TB class and move into the asymptomatic LF and active TB class. The third term represent the number of asymptomatic LF and latent TB individuals who developed active TB through exogenous re-infection. They leave the asymptomatic LF and latent TB class and move into the asymptomatic LF and active TB class. The fourth term represent the number of new cases of those with latent LF and latent TB who developed asymptomatic LF. The fifth term represent the number of those in the asymptomatic LF and latent TB class, \( A_{ht} \), who developed active TB. These leave the asymptomatic LF and latent TB class and move into the asymptomatic LF and active TB class, \( A_{ht} \). The sixth term represent the dually infected individuals with asymptomatic LF and latent TB who developed symptoms of LF. These leave the asymptomatic LF and latent TB class and move into the symptomatic LF and latent TB class, \( I_{ht} \). The last term is the death rate for the asymptomatic LF and latent TB class, \( \delta_{ht} \).

The twelfth equation in (2.6) represent the derivatives of the asymptomatic LF and active TB, \( A_{ht} \), population. The first term is the number of asymptomatic LF individuals that have active TB. The second term represent the dually infected individual with asymptomatic LF and latent TB who developed active TB through exogenous re-infection. The third term represent those in the asymptomatic LF and latent TB class, \( A_{ht} \), who developed active TB. The fourth term represent those in the latent LF and active TB class, \( E_{ht} \), who developed asymptomatic LF. The fifth term represent those who are undergoing treatment for active TB. These leave the asymptomatic LF and active TB class, \( A_{ht} \), and move into the asymptomatic LF and latent TB class, \( A_{ht} \). The sixth term represent those in the asymptomatic LF and active TB class, \( A_{ht} \), who developed symptoms of LF. These leave the asymptomatic LF and active TB class, \( A_{ht} \) and move into the symptomatic LF and active TB class, \( I_{ht} \). The seventh term is the rate of disease-induced death for those with active TB. The last term is the death rate for the symptomatic LF and active TB population.

The thirteenth equation in (2.6) represent the derivatives of the symptomatic LF and latent TB population, \( I_{ht} \). The first term is the number of asymptomatic LF individual that developed latent TB. The second term represent the symptomatic LF individuals that developed active TB. These leave the symptomatic LF and latent TB class, \( I_{ht} \) and move into the symptomatic LF and active TB class, \( I_{ht} \). The third term represent asymptomatic LF and latent TB individuals who developed active TB through exogenous re-infection. These leave the symptomatic LF and latent TB class, \( I_{ht} \) and move into the symptomatic LF and active TB class, \( I_{ht} \). The fourth term represent the dually infected individual in the asymptomatic LF and latent TB class, \( A_{ht} \), who developed symptoms of lymphatic filariasis. The fifth term represent the dually infected individual in the symptomatic LF and latent TB class, \( I_{ht} \), who developed active TB. These leave the symptomatic LF and latent TB class, \( A_{ht} \), and move into the symptomatic LF and active TB class, \( I_{ht} \). The sixth term represent dually infected individual in the symptomatic LF and latent TB class, \( I_{ht} \) and move into the symptomatic LF and active TB class, \( I_{ht} \). The seventh term represent the disease-induced death for LF infected individual. The last term is the death rate for the symptomatic LF and latent TB population.

The fourteenth equation in (2.6) represent the derivatives of the symptomatic LF and active TB population. The first term is the number of symptomatic LF individual that have active TB. The second term represent the dually infected individual with symptomatic LF and latent TB who developed active TB through exogenous re-infection. The third term represent symptomatic LF and latent TB individuals who developed active TB through exogenous re-infection. These leave the symptomatic LF and latent TB class, \( I_{ht} \) and move into the symptomatic LF and active TB class, \( I_{ht} \). The fourth term represent the dually infected individual in the asymptomatic LF and latent TB class, \( A_{ht} \), who developed active TB. These leave the symptomatic LF and latent TB class, \( A_{ht} \), and move into the symptomatic LF and active TB class, \( A_{ht} \). The fifth term represent symptomatic LF and latent TB individuals that developed active TB. These leave the symptomatic LF and latent TB class, \( A_{ht} \), and move into the symptomatic LF and active TB class, \( A_{ht} \). The sixth term represent those in the asymptomatic LF and active TB, class, \( A_{ht} \), who are undergoing treatment for LF. These leave the symptomatic LF and latent TB class, \( A_{ht} \) and move into the latent TB class, \( E_T \). The seventh term represent the disease-induced death for LF infected individual. The last term is the death rate for the symptomatic LF and active TB population.

The fifteenth equation in (2.6) represent the derivatives of the susceptible mosquito population, \( S_v \). The first term is the interaction between the susceptible mosquitoes, \( S_v \) and the infectious humans (\( E_h, A_h, I_h, E_{ht}, A_{ht}, I_{ht}, E_{ht}, A_{ht} \) and \( I_{ht} \)). The last term is the death rates for susceptible mosquitoes, \( S_v \).

The sixteenth equation in (2.6) represent the derivatives of the latent mosquito population, \( E_v \). The first term is the number of new cases of latently infected mosquitoes. The second term represent the number of latent mosquitoes that became infectious, these leave the latent class and move into the infectious class, \( I_v \). The last term is the death rate for the latent mosquitoes population.
The seventeenth equation in (2.6) represent the derivatives of the infectious class, $I_v$. The first term represent the number of infectious mosquitoes and the last term is the death rate for the infectious mosquitoes population.

### 3 Positivity of Solution of the Model

**Theorem 1:** The model (2.6) to be consistent with both human and mosquito population, all feasible solutions of the model will remain positive for all time $t > 0$.

**Proof:**
Consider the first equation of model (2.6), given below as

$$\frac{dS_h}{dt} = \Lambda_h - \lambda_T S_h - \lambda_v S_h - \mu_h S_h, \quad (3.1)$$

Without loss of generality, we can write equation (3.1) as

$$\frac{dS_h}{dt} \geq -(\lambda_T + \lambda_v + \mu_h)S_h, \quad (3.2)$$

$$\frac{dS_h}{S_h(t)} \geq -(\lambda_T + \lambda_v + \mu_h)dt \quad (3.3)$$

integrating (3.3) with respect to $t$ in $[0, t_1]$, yields

$$S_h(t_1) \geq S_h(0)e^{-\int_{0}^{t_1}(\lambda_T + \lambda_v + \mu_h)dt} > 0, \forall t > 0. \quad (3.4)$$

$$S_h(t) > 0, \forall t > 0.$$

Similarly, all state variables of the model (2.6) are positive, $\forall t > 0$.

**Theorem 2:** Let $(S_h, E_t, I_T, T_t, E_h, A_h, I_h, T_h, E_{ht}, E_{ht}, A_{ht}, A_{ht}, I_{ht}, I_{ht}, S_v, E_v, I_v)$ be trajectories of the system (2.6) with initial conditions and the biological feasible regions given by the set

$${\mathcal{D}}_1 = {\mathcal{D}}_{h1} \times {\mathcal{D}}_{v1} \text{ where } {\mathcal{D}}_{h1} = \{(S_h, E_t, I_T, T_t, E_h, A_h, I_h, T_h, E_{ht}, E_{ht}, A_{ht}, A_{ht}, I_{ht}, I_{ht}) \in \mathbb{R}_{+}^{14} : N_h \leq \frac{\Lambda_h}{\mu_h}\} \quad {\mathcal{D}}_{v1} = \{(S_v, E_v, I_v) \in \mathbb{R}_{+}^{3} : N_v \leq \frac{\Lambda_v}{\mu_v}\} \text{ is positively-invariant and attracts all the positive trajectories of (2.6)}$$

**Proof:**

Adding up the first 14 equations on the right hand side of (2.6), yields

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h - \delta_L I_h - \delta_T I_T - \delta_T E_{ht} - \delta_T A_{ht} - \delta_T I_{ht} - \delta_T I_{ht} - \delta_T I_{ht}. \quad (3.5)$$

From (3.5), it follows that $\frac{dN_h}{dt} \leq \Lambda_h - \mu_h N_h$. Hence, $\frac{dN_h}{dt} \leq 0$ if $N_h(t) \geq \frac{\Lambda_h}{\mu_h}$. we have that

$$N_h(t) \leq N_h(0)e^{-\mu_h t} + \frac{\Lambda_h}{\mu_h}(1 - e^{-\mu_h t}). \quad (3.6)$$
If $N_h(0) \leq \frac{\Lambda_h}{\mu_h}$, then $N_h(t) \leq \frac{\Lambda_h}{\mu_h}$ for all $t > 0$. Similarly, if $N_v(0) \leq \frac{\Lambda_v}{\mu_v}$, then $N_v(t) \leq \frac{\Lambda_v}{\mu_v}$ for all $t > 0$. Hence, the set $D_1$ is positively invariant.

Moreover, if $N_h(0) > \frac{\Lambda_h}{\mu_h}$ and $N_v(0) > \frac{\Lambda_v}{\mu_v}$, then either the orbits enters the domain $D_1$ in finite time or $N_h(t)$ asymptotically approaches $\frac{\Lambda_h}{\mu_h}$ as $t \to \infty$ and $N_v(t)$ asymptotically approaches $\frac{\Lambda_v}{\mu_v}$ as $t \to \infty$. Thus, the domain $D_1$ attracts all trajectories in $\mathbb{R}_{+}^{17}$. Since the domain $D_1$ is positively-invariant, it is enough to study the dynamics of the flows generated by the system (2.6) in $D_1$.

We conclude, therefore, that the model (2.6) is Mathematically and Epidemiologically well posed.

4 Local Asymptotic Stability (LAS) of the Disease-Free Equilibrium (DFE)

The DFE of the model (2.6) is given by

$$E_2 = (S_h^0, E_t, I_t, T_t, E_h^0, A_h^0, I_h^0, T_h^0, E_h, A_h, I_h, T_h, S_v^0, E_v^0, A_v^0, I_v, T_v) \quad (4.1)$$

$$= \left(\frac{\Lambda_h}{\mu_h}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{\Lambda_v}{\mu_v}, 0, 0\right). \quad (4.2)$$

Applying the method in [45], we investigate the LAS of the DFE:

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}. \quad (4.3)$$

And:

$$F_{11} = \begin{pmatrix} 0 & (1 - p_1)\beta_T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.4)$$

$$F_{12} = \begin{pmatrix} (1 - p_1)\beta_T \theta_T \eta_T & 0 & (1 - p_1)\beta_T \theta_T \eta_T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.5)$$

And:

$$F_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$  \quad (4.6)
\[
F_{22} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\eta_v G_2 & \eta_v \omega_v G_2 & \eta_v G_2 & \eta_v \omega_v G_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]  
(4.7)

Where \( G_1 = \frac{\beta_v \mu_v \lambda_v \sigma_v \sigma_h}{\sigma_v \lambda_v \mu_h + \sigma_h \lambda_h \mu_v} \) and \( G_2 = \frac{\beta_h \mu_h \lambda_v \sigma_v \sigma_h}{\sigma_v \lambda_v \mu_h + \sigma_h \lambda_h \mu_v} \)

\[
V = \begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}.
\]  
(4.8)

\[
V_{11} = \begin{pmatrix}
g_5 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_1 & g_6 & 0 & 0 & 0 & 0 \\
0 & 0 & g_1 & 0 & 0 & 0 \\
0 & 0 & -k_1 & g_2 & 0 & 0 \\
0 & 0 & 0 & -k_2 & g_3 & 0 \\
0 & 0 & 0 & 0 & 0 & g_7
\end{pmatrix}.
\]  
(4.9)

\[
V_{12} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -\tau h_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\tau h_3 & 0 & 0 & 0 & 0 \\
-\tau T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau T_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\tau T_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]  
(4.10)

\[
V_{21} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -\gamma_2 & g_8 \\
0 & 0 & 0 & 0 & -k_3 & 0 \\
0 & 0 & 0 & 0 & -k_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]  
(4.11)

\[
V_{22} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_9 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\gamma_3 & g_{10} & 0 & 0 & 0 & 0 & 0 \\
-k_5 & 0 & g_{11} & 0 & 0 & 0 & 0 \\
0 & -k_6 & -g_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -g_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -k_v & \mu_v
\end{pmatrix}.
\]  
(4.12)

where \( g_1 = k_1 + \mu_h, g_2 = k_2 + \mu_h, g_3 = \tau h_1 + \mu_h + \delta_L, g_4 = k_v + \mu_h, g_5 = \gamma_1 + \mu_h, g_6 = \tau T_1 + \mu_h + \delta_T, g_7 = \gamma_2 + \mu_h + k_3, g_8 = \tau T_2 + k_4 + \mu_h + \delta_L, g_9 = \gamma_3 + \mu_h + k_5 \)
\[ \rho(FV^{-1}) = R_0 = \max(R_L, R_T) \]

where

\[ R_L = \sqrt{\frac{\sigma^2 k_1 \sigma^2 k_2 \sigma^2 k_3 \sigma^2 k_4 (g_1 g_2 g_3 + k_1 (g_1 g_2 g_3 + k_2))}{(\lambda_1 \sigma + \lambda_2 \sigma + \lambda_3 \sigma)^2 g_1 g_2 g_3 g_4}} \]

and

\[ R_T = \frac{p_1 T_T}{\tau_2 + \gamma + \mu_1} + \frac{\gamma (1 - p_1) T_T}{(\gamma + \mu_2) (\tau_2 + \gamma + \mu_1)} \]

Using theorem 2 in [45], we claim the following.

**Theorem 3:** The DFE for the TB-LF co-infection model is locally asymptotically stable if \( R_0 < 1 \) and unstable if \( R_0 > 1 \).
Table 3: Baseline values and ranges of the parameters of the model 2.6.

| Parameters | Baseline values | Ranges | References |
|------------|----------------|--------|------------|
| $\mu_h$   | 0.02041 year$^{-1}$ | (0.0143, 0.03) | Assumed |
| $\mu_v$   | 0.0714 year$^{-1}$ | (0.0143, 0.03) | Assumed |
| $\Lambda_h$ | 3 768 410 year$^{-1}$ | (3,000,000, 4,000,000) | [46] |
| $\Lambda_v$ | 10000 year$^{-1}$ | (5,000, 15,000) | [46] |
| $\beta_T$ | Variable year$^{-1}$ | (1.55, 5) | [16] |
| $\beta_v$ | Variable year$^{-1}$ | (0, 1) | [16] |
| $\beta_h$ | Variable year$^{-1}$ | (0, 1) | [16] |
| $\sigma_v$ | 620.39 year$^{-1}$ | (600, 800) | [47] |
| $\sigma_h$ | 1168 year$^{-1}$ | (1000, 2000) | [47] |
| $\tau_{T1}$ | 1.5 year$^{-1}$ | (0, 2) | [6] |
| $\tau_{T2}$ | 1.5 year$^{-1}$ | (0, 2) | [6] |
| $\tau_{T3}$ | 1.5 year$^{-1}$ | (0, 2) | [6] |
| $\tau_{T4}$ | 1.5 year$^{-1}$ | (0, 2) | [6] |
| $\tau_{h1}$ | 1.5 year$^{-1}$ | (0, 2) | [47] |
| $\tau_{h2}$ | 1.5 year$^{-1}$ | (0, 2) | [47] |
| $\tau_{h3}$ | 1.5 year$^{-1}$ | (0, 2) | [47] |
| $\gamma_1$ | 0.5 year$^{-1}$ | [0, 1] | [48] |
| $\gamma_2$ | 0.7 year$^{-1}$ | [0, 1] | [48] |
| $\gamma_3$ | 0.7 year$^{-1}$ | [0, 1] | [48] |
| $\gamma_4$ | 0.7 year$^{-1}$ | [0, 1] | [48] |
| $\epsilon_1$ | 1.5 year$^{-1}$ | [1, 2.5] | [49] |
| $\epsilon_2$ | 1.8 year$^{-1}$ | [1, 2.5] | [49] |
| $\epsilon_3$ | 1.8 year$^{-1}$ | [1, 2.5] | [49] |
| $\epsilon_4$ | 1.8 year$^{-1}$ | [1, 2.5] | [49] |
| $k_1$ | 0.15 year$^{-1}$ | (0, 0.6) | Assumed |
| $k_2$ | 0.15 year$^{-1}$ | (0, 0.6) | Assumed |
| $k_3$ | 0.2 year$^{-1}$ | (0, 0.6) | Assumed |
| $k_4$ | 0.2 year$^{-1}$ | (0, 0.6) | Assumed |
| $k_5$ | 0.2 year$^{-1}$ | (0, 0.6) | Assumed |
| $k_6$ | 0.2 year$^{-1}$ | (0, 0.6) | Assumed |
| $\nu$ | 0.0001 year$^{-1}$ | (0, 1) | Assumed |
| $\eta_{T1}$ | 0.85 year$^{-1}$ | [0, 1] | Assumed |
| $\eta_{T2}$ | 0.85 year$^{-1}$ | [0, 1] | Assumed |
| $\eta_1$ | 0.3 year$^{-1}$ | [0.1, 0.8] | [50] |
| $\eta_2$ | 0.3 year$^{-1}$ | [0.1, 0.8] | [50] |
| $\eta_3$ | 0.3 year$^{-1}$ | [0.1, 0.8] | [50] |
| $\eta_0$ | 1.3 year$^{-1}$ | [1, 2] | [47] |

5 Sensitivity Analysis

Sensitivity analysis is commonly used to determine the robustness of model predictions to parameter values. It is used to discover parameters that have a high impact on the basic reproduction number $R_0$. According to ([53], [54], [55]), the normalized forward sensitivity index of a variable to a parameter is the ratio of the relative change in the variable to the relative change in the parameter.

**Definition:** The normalized forward sensitivity index of a variable $u$, that depends differentiably on a parameter, $p$, is defined as

$$
\frac{\Delta u}{\Delta p} = \frac{\partial u}{\partial p} \times \frac{p}{u}
$$
Here, we assess the impact of the parameters of the model on the basic reproduction number $R_0$, by computing the elasticity index.

For this analysis, we made use of values in Table 3.

**Table 4**: Elasticity indices of the basic reproduction number

| Parameters | Elasticity Index |
|------------|-----------------|
| $\mu$      | -1.45788        |
| $\sigma_v$ | +0.990603       |
| $\beta_v$  | +0.5            |
| $\beta_h$  | +0.5            |
| $\Lambda_v$| +0.490603       |
| $\Lambda_h$| -0.490603       |
| $k_v$      | +0.467277       |
| $k_2$      | -0.44009        |
| $\mu_h$    | +0.367504       |
| $\eta_{h1}$| +0.251683       |
| $k_3$      | +0.248288       |
| $\tau_{h1}$| -0.244939       |
| $k_1$      | +0.0598564      |
| $\sigma_h$ | +0.00939691     |
| $\eta_1$   | +0.0000285929   |
| $\delta_L$ | -0.000163293    |

$\gamma_p = \frac{\partial u}{\partial p} \times \frac{p}{u}$  \hspace{1cm} (5.1)
6 Numerical simulations

Here, we carried out numerical simulation. The ranges and baseline values of the parameters are listed in Table (3). We used demographic (and epidemiological) parameters relevant to Nigeria and from literature. In 2015, Nigeria population was estimated to be 184,635,279 [46].

Figure 2: New cases of (a) $E_{hT}$ (b) $A_{hT}$ and (c) $I_{hT}$ when $\theta_1$, $\theta_2$ and $\theta_3$ are varied.

Figure 3: New cases of (a) $E_t$ and (b) $I_T$ when $\theta_1$, $\theta_2$ and $\theta_3$ are varied.
Figure 4: New cases of (a) $E_T$ and (b) $I_T$ when $\theta_1$ is varied.

Figure 5: New cases of (a) $E_{hT}$ (b) $A_{hT}$ and (c) $I_{hT}$ when $\theta_1$ is varied.
Figure 6: New cases of (a) $E_t$ and (b) $I_T$ when $\theta_2$ is varied.

Figure 7: New cases of (a) $E_{hT}$ (b) $A_{hT}$ and (c) $I_{hT}$ when $\theta_2$ is varied.
Figure 8: New cases of (a) $E_t$ and (b) $I_T$ when $\theta_3$ is varied.

Figure 9: New cases of (a) $E_{hT}$ (b) $A_{hT}$ and (c) $I_{hT}$ when $\theta_3$ is varied.
Figure 10: New cases of (a) $E_a$ and (b) $I_T$ with $\tau_{h1}$, $\tau_{h2}$ and $\tau_{h3}$ varied.

Figure 11: New cases of (a) $E_{ht}$ (b) $A_{ht}$ and (c) $I_{ht}$ with $\tau_{h1}$, $\tau_{h2}$ and $\tau_{h3}$ varied.
Figure 12: New cases of (a) $E_T$ and (b) $I_T$ with $\tau_{t1}$ varied.

Figure 13: New cases of (a) $E_T$ and (b) $I_T$ with $\tau_{t2}$ varied.
Figure 14: New cases of (a) $E_t$ and (b) $I_T$ with $\tau_{h,3}$ varied.

Figure 15: New cases of (a) $E_t$ and (b) $I_T$, when those with symptomatic LF are treated and $\tau_{h,1}$ varied.
Figure 16: New cases of (a) $E_t$ and (b) $I_T$ when the two diseases are treated.

Figure 17: New cases of (a) $E_{hT}$ (b) $A_{hT}$ and (c) $I_{hT}$ when the two diseases are treated.
7 Results and Discussion

Table 4 provides the elasticity indices of $R_0$, to the 16 parameters. Looking at the elasticity indices in Table 4, the basic reproduction number is most sensitive to the death rate for mosquitoes, $\mu_v$, with an elasticity index of $-1.45788$. What this index means is that if we increase the death rate for mosquitoes by 1%, the basic reproduction number, $R_0$, will decrease by approximately 1.46%. The basic reproduction number is also sensitive to the following parameters: the number of times a mosquito bite humans, $\sigma_v$, with an elasticity index of $+0.990603$, rate of transmission of LF from mosquitoes to humans, $\beta_v$, with an elasticity index of 0.5, rate of transmission of LF from humans to mosquitoes, $\beta_h$, with an elasticity index of 0.5, recruitment rate for mosquitoes, $\Lambda_v$, with elasticity index of 0.490603, treatment rate for LF, $\tau_h$, with elasticity index of $-0.244939$ etc. The basic reproduction number is least sensitive to the disease induced death rate for humans, $\delta_L$, with an elasticity index of $-0.0000163293$.

Figures 2a, 2b, and 2c show the cumulative new cases of latent LF and active TB, asymptomatic LF and active TB, and symptomatic LF and active TB when $\theta_1$, $\theta_2$ and $\theta_3$ are varied from 1 to 15. $\theta_1$, $\theta_2$ and $\theta_3$ are modification parameters which account for the increased susceptibility of those with LF disease to TB. According to WHO (2018), LF impede the smooth functioning of the lymphatic system. The lymphatic system houses and maintain the human immune defence system. If the functions of the lymphatic system is affected because of the presence of filariasis, the immunity of the organism is compromised and this can open the door for TB and other infectious diseases. Figures 3a and 3b shows the cumulative new cases of latent and active TB when $\theta_1$, $\theta_2$ and $\theta_3$ are varied from 1 to 20. , indicating that those with LF disease are highly susceptible to TB and other infectious diseases. Similar trends was also observed in Figures 4a and 4b, Figures 5a, 5b and 5c when $\theta_1$ only was varied from 0 to 20. Figures 6a and 6b shows the cumulative new cases of latent TB $E_t$ and active TB, $I_T$ when the modification parameters $\theta_2$ was varied from 020. Figures 7a, 7b and 7c show the cumulative new cases of the co-infected individual with active TB cases. That is, latent LF and active TB $(E_{hT})$, asymptomatic if and active TB, and symptomatic LF and active TB $(I_{hT})$ when $\theta_2$ was varied from 0 – 20. Here, there was no significant change in the dynamic of the co-infected class. Compared to when $\theta_1$ was varied. The same trend was also observed in Figures 8a and 8b, 9a, 9b and 9c when $\theta_3$ was varied. Comparing Figures (3a and 3b), (5a and 5b) and (8a and 8b), it was found that those who have latent LF disease are highly susceptible to TB in the single and co-infected cases. Figure 10 show the cumulative new cases of latent TB $(E_t)$ and active TB $(I_T)$ when we applied LF-only treatment strategy. In this strategy, all the treatments for LF were adjusted from 0.2 – 0.9 i.e. $\tau_{h1} = \tau_{h2} = \tau_{h3} = 0.2$ and $\tau_{h1} = \tau_{h2} = \tau_{h3} = 0.9$. It was found that the cumulative new cases of latent TB $(E_t)$ and active TB $(I_T)$ dropped significantly. In the co-infected cases, there was a drop in the co-infected compartment containing symptomatic LF and active TB $(I_{hT})$, but in the other co-infected classes, containing active TB $(E_{hT})$ and $(A_{hT})$, there was no significant drop in the number of new co-infected cases. because only those with LF symptoms were treated (see Figures 11a, 11b and 11c). In Figures 12a and 12b, when the treatment for the latent LF infected individual was varied, we found that the cumulative new case of latent TB $(E_t)$ and active TB $(I_T)$ dropped significantly, but in Figure 13 and 14 there was no significant changes in the cumulative new cases of latent TB $(E_t)$ and active TB $(I_T)$ when the treatment $\tau_{h2}$ and $\tau_{h3}$ were varied respectively. In Figure 15, when all those with LF disease are treated and $\tau_{h1}$ varied from 0.2 – 0.9, there was a significant drop in the cumulative new cases of those with latent TB $(E_t)$ and active TB $(I_T)$. We found in Figure 16 a significant drop in the cumulative new cases of latent TB $(E_t)$ when the treatments rate was increased from 0 to 0.9. Similar trend was also observe in Figure 17.

8 Conclusion

We proposed and analysed 17 nonlinear Mathematical model to study how the endemicity of LF affect the population dynamics of TB. The mathematical model presented in this work provide mathematical and epidemiological insight into the transmission dynamics of TBLF co-infection. The numerical results presented in Figures 2, 3, 4, 5, 6, 7, 8 and 9 shows that LF infection increases susceptibility to TB infection. This is in agreement with [2] that, persons with lowered immunity such as HIV, diabetes, immune disorder etc are at a higher risk of contacting infectious diseases. The results presented in Figures 10 to 17 shows that increasing the rate of diagnosis and treatment of active TB
and symptomatic LF cases can reduce the incidence of co-infection in the community, and that if resources are limited, efforts should be targeted at treating only the co-infected cases or those with symptomatic LF cases only. The public health implications of the results in the sensitivity analysis is that increasing the death rate for mosquitoes, reducing the number of times a mosquito bites human (through the use of windows and door screens), reducing the rate of transmission of LF from mosquito to humans, reducing the rate of transmission of LF from human to mosquitoes, reducing the recruitment rate for mosquitoes (by destroying mosquitoes breeding sites using larvicide), reducing the progression rate for mosquitoes and increasing the treatment rate for LF, will bring down the basic reproduction number to less than one.

9 Abbreviation

ODE: Ordinary Differential Equation
DEC: Diethylcarbamazine
TB: Tuberculosis
LF: Lymphatic filariasis
WHO: World Health Organisation
SEIR: Susceptible Exposed Infectious Recovered
BCG: Bacille Calmette Guerin
HBC: High Burden Countries
LAS: Local Asymptotic Stability
DFE: Disease-free Equilibrium
ART: Anti-retro-viral Therapy

10 Declaration

10.1 Availability of Data and Material

All data used during this study are included in this article.

10.2 Competing Interest

The authors declared that they have no competing interests.

10.3 Funding

Not applicable

10.4 Author’s Contribution

EI, RA and IA formulated the model equations. EI wrote the introduction, Positivity of solutions, LAS of the DFE and Numerical simulations. RA and IA did the sensitivity analysis. The authors read and approved the manuscript for submission.
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