A CONJUGACY CLASS COUNTING IN TEICHM"ULLER
SPACE

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Abstract. Let $\gamma$ be a pseudo-Anosov homeomorphism and $X$ an element
of the Teichm"uller space of a genus $g$ surface. In this paper, we find asymptotics for the number of pseudo-Anosov homeomorphisms that are conjugate
to $\gamma$ and the axis of their action on Teichm"uller space intersects the ball of
radius $R$ centered at $X$, as $R$ tends to infinity.

1. Introduction

1.1. Statement of results. In [Mar04], Margulis obtained asymptotics for
the volume growth and orbit counting for balls of large radius, in the setting
of manifolds with negative curvature. Similar asymptotics were obtained in
[ABEM12] for the Teichm"uller space. To state their results, let $T_g$ be the Teich-
m"uller space of $S$, a surface of genus $g$, and denote the mapping class group
of $S$ by $\Gamma$. Given $X, Y \in T_g$, let $B(X,R)$ be the ball of radius $R$ centered
at $X$, where the distance here is measured with respect to the Teichm"uller
metric. Denoting the orbit of $Y$ under the action of $\Gamma$ by $\Gamma \cdot Y$, Theorem 1.2
of [ABEM12] gives

$$|\Gamma \cdot Y \cap B(X,R)| \sim \frac{\Lambda^2}{h \Vol(M_g)} e^{hR} \quad \text{as} \quad R \to \infty.$$ 

Here, $h = 6g - 6$ is the entropy of the Teichm"uller geodesic flow with respect
to Masur-Veech measure and $\Lambda$ is the Hubbard-Masur constant [ABEM12]
[Dum15]. The term $\Vol(M_g)$ is the normalized volume of the moduli space
$M_g$ as explained at the end of Section 2.2 of [ABEM12]. The cardinality of
a finite set $S$ is denoted by $|S|$ and $A(R)$ is said to be asymptotic to $B(R),
written $A(R) \sim B(R)$, if $A(R)/B(R) \to 1$ as $R \to \infty$. Theorem 1.3 of of the
same paper gives the following asymptotics for the volume of $B(X,R)$:

$$\Vol(B(X,R)) \sim \frac{\Lambda^2}{h} e^{hR} \quad \text{as} \quad R \to \infty.$$ 

Now fix $\gamma \in \Gamma$ to be a pseudo-Anosov homeomorphism and let $L_\gamma$ be the
axis of its action on Teichm"uller space, namely, the unique geodesic that is
kept fixed by this action. The cyclic group generated by $\gamma$, denoted by $\langle \gamma \rangle$, acts on $T_g$ properly discontinuously, hence we can form the quotient to be the
cylinder $C_\gamma = \langle \gamma \rangle \backslash T_g$. The elements of $C_\gamma$ are of the form $[Y] = \langle \gamma \rangle \cdot Y$ for
$Y \in T_g$. Since the action of $\gamma$ on $L_\gamma$ is by translation, the quotient $\tilde{\mathcal{C}}_\gamma = \langle \gamma \rangle \backslash \mathcal{C}$
is a closed geodesic in $\mathcal{C}_\gamma$. Define

$$B(\tilde{\mathcal{C}}_\gamma, R) = \{[Y] \in \mathcal{C}_\gamma : d([Y], \tilde{\mathcal{C}}_\gamma) \leq R\},$$

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where the distance in \( C \gamma \) is the one induced by the Teichmüller distance on its cover \( \mathcal{T}_g \). Define the \( \Gamma \)-orbit of \([X] \in C_g\) to be
\[
\Gamma \cdot [X] = \{[gX] : g \in \Gamma\}.
\]
The goal of this paper is to establish orbit counting and volume asymptotics, similar to the ones obtained in \([ABEM12]\), for \( B(\bar{L}_\gamma, R) \) instead of \( B(X, R) \).

To state these results, we need a few definitions. Define
\[
B_{\text{Ext}}(\bar{L}_\gamma) = \{\zeta \in \mathcal{MF} : \inf_{X \in \bar{L}_\gamma} \text{Ext}(\zeta, X) \leq 1\}.
\]

The action of \( \langle \gamma \rangle \) on \( B_{\text{Ext}}(\bar{L}_\gamma) \) is proper and discontinuous, given we remove the two endpoints of \( \bar{L}_\gamma \) from the set, hence we can form the quotient to be \( C_{\gamma, \text{Ext}} = \langle \gamma \rangle \backslash B_{\text{Ext}}(\bar{L}_\gamma) \).

The Thurston measure on \( \mathcal{MF} \) induces a measure on \( C_{\gamma, \text{Ext}} \), which we denote by \( \nu \) as well. The orbit counting asymptotics is given by:

**Theorem A.** Let \( \gamma \) and \( \bar{L}_\gamma \) be as above and \([X] \in C_g\). Then as \( R \to \infty \),
\[
|\Gamma \cdot [X] \cap B(\bar{L}_\gamma, R)| \sim \frac{\Lambda^2}{h \Vol(M_g)} \nu(C_{\gamma, \text{Ext}}) e^{hR}.
\]

Note that \( \Gamma \cdot [X] \cap B(\bar{L}_\gamma, R) \) is in one-to-one correspondence with the mapping class group translations \( g\bar{L}_\gamma \) of \( \bar{L}_\gamma \) that intersects \( B(X, R) \). This in turn is in one-to-one correspondence with the conjugates of \( \gamma \) whose axis intersects \( B(X, R) \). Therefore, we have the following:

**Corollary B.** Let \( \gamma \) be a pseudo-Anosov homeomorphism and \( X \in \mathcal{T}_g \). Then as \( R \to \infty \),
\[
|\{\gamma' \in \Gamma : \gamma' \text{ is conjugate to } \gamma \text{ and } \bar{L}_\gamma \cap B(X, R) \neq \emptyset\}| \sim \frac{\Lambda^2}{h \Vol(M_g)} \nu(C_{\gamma, \text{Ext}}) e^{hR}.
\]

The volume asymptotics is given by:

**Theorem C.** Let \( \gamma \) and \( \bar{L}_\gamma \) be as above. Then as \( R \to \infty \),
\[
\Vol(B(\bar{L}_\gamma, R)) \sim \frac{\Lambda^2}{h} \nu(C_{\gamma, \text{Ext}}) e^{hR}.
\]

1.2. **Remarks and the relation to other works.** If \( \Sigma \) is a surface of constant negative curvature \(-1\) and \( \Gamma \) its fundamental group, then Theorem 2.5 of \([EM93]\) gives
\[
|B(\bar{L}_\gamma, R) \cap \Gamma \cdot [x]| \sim \frac{\text{Length}(\bar{L}_\gamma)}{\text{Area}(\Sigma)} e^R \quad \text{as} \quad R \to \infty,
\]
where the terms in the above expression are defined in the same way as before.

A calculation in hyperbolic metric shows Theorem C in this setting, namely, as \( R \to \infty \),
\[
\text{Area}(B(\bar{L}_\gamma, R)) \sim \text{Length}(\bar{L}_\gamma) e^R.
\]

For \( M \) a compact manifold of (variable) negative curvature, we can define \( \Gamma, \gamma, \bar{L}_\gamma \) and \( L_\gamma \) similarly. The asymptotics for \( |\Gamma \cdot [x] \cap B(\bar{L}_\gamma, R)| \) can be obtained as a special case of common perpendicular counting. To explain this,
Lemma 5.3.) Since \( \gamma \subset B \) disjoint and they almost cover all of \( \gamma \), the perpendiculars from \( x' \) to \( L' \) of length less than \( R \), where such a perpendicular is defined as a locally geodesic path that starts from \( x' \) and arrives perpendicularly at \( L' \). It follows from Theorem 1 of [PP17] that for some constant \( c_\gamma > 0 \),
\[
|\Gamma \cdot [x] \cap B(L', R)| \sim c_\gamma e^{\delta R} \quad \text{as } R \to \infty.
\]
Here, \( \delta \) is the topological entropy of the geodesic flow on \( T^1 M \), the unit tangent bundle of \( M \). Moreover, under some additional conditions, an exponentially small error term is obtained for the above asymptotics. (See Theorem 3 of the same paper for the precise statement.)

As a final remark, let us mention that our methods for proving Theorem A are quite flexible. In particular, Theorem A can be proved for an arbitrary compact set \( K \subset C \) replacing \( L' \), and the proof is word for word the same, given we change an \( \epsilon \)-net of \( K \)–invariant to \( \epsilon \)-cover of \( K \). (See Section 1.3.) Volume asymptotics in this case can be obtained in the same way as we obtained Theorem C from Theorem A.

1.3. The outline of the proof. Theorem C proved at the end of Section 3.1 follows easily from Theorem A using an estimate obtained in Theorem 5.1 of [ABEM12]. Hence the main task is to prove Theorem A. Let \( \gamma \) and \( L \) be as in Section 1.1 and fix a point \( P \in T_\gamma \). By definition, \( \Gamma \cdot [P] \cap B(L', R) \) is in one to one correspondence with
\[
\langle \gamma \rangle \setminus (\Gamma \cdot P \cap B(L', R)) = \{ \langle \gamma \rangle \cdot X : X \in \Gamma \cdot P \cap B(L', R) \},
\]
where
\[
B(L', R) = \{ X \in T_\gamma : d(X, L') \leq R \}.
\]
Note that taking the quotient by \( \langle \gamma \rangle \) in (2) is justified since \( \Gamma \cdot P \cap B(L', R) \) is \( \langle \gamma \rangle \)–invariant.

Fix a point \( O \in L \) and for an \( \epsilon > 0 \), let \( O = X_0, X_1, \ldots, X_N = \gamma O \) be an \( \epsilon \)-net in \( [O, \gamma O] \), i.e., the geodesics connecting \( X_i \) to \( X_{i+1} \) are disjoint for \( 0 \leq i < N \) and \( \sup_{0 \leq i < N} d(X_i, X_{i+1}) < \epsilon \). Translating this net by the powers \( \gamma \), we get a \( \gamma \)-invariant \( \epsilon \)-net \( \ldots, X_{-1}, X_0, X_1, \ldots \) of \( L' \). Define the closest point map
\[
P : \mathcal{PMF} \to \mathbb{Z} \quad \text{by} \quad P[\zeta] = i \quad \text{if} \quad \text{Ext}(\zeta, X_i) = \inf_{j \in \mathbb{Z}} \text{Ext}(\zeta, X_j).
\]
Setting \( A_i = P^{-1}(i) \), we obtain a \( \gamma \)-invariant partition of \( \mathcal{PMF} \). Let \( S(X_i, A_i, R) \) be the sector of radius \( R \) centered at \( X_i \) and observing \( A_i \), namely, all the points \( Y \in T_\gamma \) such that \( d(X_i, Y) \leq R \) and the geodesic connecting \( X_i \) to \( Y \) hits the boundary at an element of \( A_i \). (see 5.1) for a precise definition.)

The main geometric idea of this paper is that \( S(X_i, A_i, R) \)'s are almost disjoint and they almost cover all of \( B(L', R) \). (See the discussion just before Lemma 5.3.) Since \( \gamma S(X_i, A_i, R) = S(X_{i+N}, A_{i+N}, R) \),
\[
\sum_{i=0}^{N-1} |\Gamma \cdot P \cap S(X_i, A_i, R)|
\]
gives a good approximation for $|\langle \gamma \rangle \setminus (\Gamma \cdot P \cap B(\mathcal{L}_\gamma, R))|$. The asymptotics of $|\Gamma \cdot X \cap S(X_i, \mathcal{A}_i, R)|$ as $R \to \infty$ is given by $\text{ABEM12}$. Summing up these asymptotics as the $\epsilon$–net $(X_i)$ in $\mathcal{L}_\gamma$ gets finer, namely $\epsilon \to 0$, we obtain the right hand side of (1).

To make these ideas work, we need to approximate each $A_i$ from inside and outside by open sets $U_i \subset A_i \subset V_i$ and squeeze the above sum between the corresponding sums for $U_i$ and $V_i$ replacing $A_i$. To prove that these lower and upper bounds both converge to the right-hand side of (1), we need the boundary of the partition $\{A_i\}$ to have measure zero. This is proved in Section 3 and the proof uses Theorem D (see Section 1.4). The only result from Section 3 that is used in the rest of the paper is Proposition 3.1. Section 4 is devoted to the statement and proof of Proposition 4.11, which is the main tool we use to compare extremal and Teichmüller lengths. In Section 5, we carry out the sector approximation scheme that we mentioned earlier. Both of the facts that $S(X_i, A_i, R)$’s are almost disjoint and that they almost cover $B(\mathcal{L}_\gamma, R)$ are applications of Proposition 4.11.

1.4. A formula for the derivative of extremal length. We end this introduction by stating a formula that we obtained in Section 3 in the course of proving Proposition 3.1. For $X \in \mathcal{T}_g$ and $\zeta \in \mathcal{MF}$, denote the extremal length of $\zeta$ in $X$ by $\text{Ext}(\zeta, X)$. Fixing $\zeta$, we can consider $E_\zeta = \text{Ext}(\zeta, \cdot)$ as a function from $\mathcal{T}_g$ to $\mathbb{R}$. This function is differentiable and its derivative at $X \in \mathcal{T}_g$, $d_X E_\zeta : T_X(\mathcal{T}_g) \to \mathbb{R}$, is given by the Gardiner’s formula [Gar84]

$$d_X E_\zeta(\mu) = 2\Re \int_X \mu \cdot V_X^{-1}(\zeta),$$

where $T_X(\mathcal{T}_g)$ is the tangent space to $\mathcal{T}_g$ at $X$, $\mu \in T_X(\mathcal{T}_g)$ is a Beltrami differential and the homeomorphism $V_X : Q(X) \to \mathcal{MF}$ is defined by sending a quadratic differential to its vertical measured foliation.

If we fix $X \in \mathcal{T}_g$ instead, we can define

$$E_X : \mathcal{MF} \to \mathbb{R} \quad \text{by} \quad E_X(\zeta) = \text{Ext}(\zeta, X).$$

In order to compute the derivative of $E_X$ we need a differential structure on $\mathcal{MF}$. In general, the manifold $\mathcal{MF}$ equipped with train-track charts is only piecewise linear. However, if $\zeta$ is generic, meaning that it does not have a leaf connecting any two singularities and all the singularities are simple, then $\mathcal{MF}$ is smooth at $\zeta$ (in a sense to be defined at the beginning of Section 3.2), hence the tangent space at this point to $\mathcal{MF}$, $T_\zeta \mathcal{MF}$, is defined. The derivative of $E_X$ at such a $\zeta$ is given by the following theorem: (for the precise statement see Theorem 3.4)

**Theorem D.** Fix $X \in \mathcal{T}_g$ and let $\zeta \in \mathcal{MF}$ be a generic measured foliation. Then $E_X$ is smooth at $\zeta$ and there is an $\eta \in T_\zeta \mathcal{MF}$ such that

$$d_\zeta E_X : T_\zeta \mathcal{MF} \to \mathbb{R} \quad \text{is given by} \quad d_\zeta E_X(\cdot) = \omega_{\text{Th}}(\eta, \cdot),$$

where $\omega_{\text{Th}}$ stands for the Thurston symplectic form (see Section 3.1 for a definition). Moreover, $\eta$ can be completely described in certain train track coordinates around $\zeta$. 
Notation. For a set $A$ and subsets $A_{\delta}$ indexed by $\delta \in (0, s)$ for some $s > 0$, we say $A_{\delta} \uparrow A$ as $\delta \downarrow 0$ if the following holds: $A_{\delta_2} \supseteq A_{\delta_1}$ for $\delta_2 < \delta_1$ and $\bigcup A_{\delta} = A$. Similarly, we say $B_{\delta} \downarrow B$ as $\delta \downarrow 0$ if $B_{\delta_2} \subseteq B_{\delta_1}$ for $\delta_2 < \delta_1$ and $\bigcap B_{\delta} = B$. Finally, for real numbers $a, b, c$ we write $a \preceq c$ if $|a - b| < c$.

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2. Background on Teichmüller space

Teichmüller space. Let $S$ be a compact surface of genus $g \geq 2$. We denote the Teichmüller space of $S$ by $\mathcal{T}_g$. This is the space of equivalent classes of orientation preserving homeomorphism $f: S \to X$, where $X$ is a Riemann surface and $f: S \to X$ is said to be equivalent to $g: S \to Y$ if there exists a biholomorphism $h: X \to Y$ such that $g$ is isotopic to $h \circ f$. We denote an element $[f: S \to X]$ of $\mathcal{T}_g$ by $X$ and keep the marking in the back of our mind. The mapping class group (or modular group) of $S$ is denoted by $\Gamma$. This is the group of orientation preserving homeomorphisms of $S$ up to isotopy. An element of mapping class group $[\gamma: S \to S]$ acts on $[f: S \to X] \in \mathcal{T}_g$ by change of marking, namely $[\gamma] [f] = [f \circ \gamma^{-1}]$. Taking the quotient of $\mathcal{T}_g$ by $\Gamma$ we obtain the moduli space $\mathcal{M}_g = \Gamma \backslash \mathcal{T}_g$.

Quadratic differentials. For a Riemann surface $X$, let $Q(X)$ be the space of holomorphic quadratic differentials (or quadratic differentials for short) on $X$. For a $\phi \in Q(X)$, define the norm of $\phi$ to be

$$|\phi| = \int_X |\phi(z)||dz|^2.$$ 

The union of $Q(X)$ for $X \in \mathcal{T}_g$ forms the space of quadratic differentials, denoted by $Q\mathcal{T}_g$. More precisely, $Q\mathcal{T}_g$ is the space of equivalent classes $[f: S \to (X, \phi)]$, where $f$ and $X$ are as before and $\phi \in Q(X)$. We denote $[f]$ by $(X, \phi)$ or just $\phi$. Sending $(X, \phi)$ to $X$ gives a projection map $\pi: Q\mathcal{T}_g \to \mathcal{T}_g$. The principal domain $\mathcal{P}(1, \ldots, 1) \subset Q\mathcal{T}_g$ is defined to be quadratic differentials with only simple zeros.

A flat chart for $(X, \phi)$ is a holomorphic chart $\varphi: U \subset \mathbb{C} \to X$ on which the pullback of $\phi$ is $dz^2$. The change of coordinates between two flat charts is of the form $z \to \pm z + c$. For $\phi \in Q\mathcal{T}_g$ and $A \in \text{SL}_2(\mathbb{R})$, $A\phi$ is defined as the unique element $\psi \in Q\mathcal{T}_g$ such that the change of marking $\phi \to A\psi$ is given by multiplication by $A$ on the corresponding flat charts. With this definition, Teichmüller geodesic flow is given by

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$ 

The cotangent space at $X \in \mathcal{T}_g$ is naturally identified with $Q(X)$, hence the norm on $Q(X)$ induces a Finsler norm on Teichmüller space. The resulting metric is called the Teichmüller metric, the distance between two points $X, Y \in \mathcal{T}_g$ is denoted by $d(X, Y)$ and the geodesic connecting $X$ to $Y$ is shown by
For $X \in T_g$ and $\zeta \in \mathcal{MF}$, there exists a unique $\phi \in Q(X)$ such that $\phi$ has $\zeta$ as its vertical measured foliation ([HM79]). For such $X$ and $\zeta$, define

$$[X, \zeta] = \{\pi(g_t\phi), \ 0 \leq t \leq \infty\}.$$  

**Extremal length.** Let $\mathcal{V}: QT_g \to \mathcal{MF}$ be the function that sends a quadratic differential to its vertical measured foliation. For $X \in T_g$, the restriction of $\mathcal{V}$ to $Q(X)$, $\mathcal{V}_X: Q(X) \to \mathcal{MF}$, is a homeomorphism, and the extremal length of $\zeta \in \mathcal{MF}$ at such an $X$ can be defined by

$$\text{Ext}(\zeta, X) = |\mathcal{V}_X^{-1}(\zeta)|.$$

The Busemann functions are defined by

$$\beta(\zeta, X) = \frac{1}{2} \log \text{Ext}(\zeta, X);$$
$$\beta(\zeta, X, Y) = \beta(\zeta, Y) - \beta(\zeta, X).$$

Note that $\beta(\zeta, X, Y)$ only depends on $[\zeta]$, hence it can be denoted by $\beta([\zeta], X, Y)$. Kerschhoff inequality states that

$$\frac{\text{Ext}(\zeta, Y)}{\text{Ext}(\zeta, X)} \leq e^{2d(X,Y)},$$

hence taking logarithms we obtain

$$\beta(\zeta, Y) \leq \beta(\zeta, X) + d(X, Y).$$

If we think of $\beta(\zeta, X)$ as the "length at infinity" of $[X, \zeta]$, the above can be thought of as the triangle inequality in $\triangle(\zeta, X, Y)$.

3. EQUIDISTANT MEASURED FOLIATIONS ARE NEGLEGIBLE

3.1. **Background on train tracks.** We define a train track to be an embedded 3–regular graph in $S$ such that its vertices are locally modeled on Figure 1. The vertices of this graph are called _switches_ and the edges are called _branches_ of the train track. In the same figure, $a$ is called an _incoming_ branch and $b, c$ are called _outgoing_ branches. A branch is said to be _large_ if it is the outgoing branch for both of its endpoints. A _splitting_ along the large branch $e$ is shown in Figure 2. A train track $\tau$ is said to be _complete_ if all the components of $S - \tau$ are cusped triangles.

![Figure 1. A switch](image)

A function $w$ from the set of branches of $\tau$ to $\mathbb{R}$ is called a _weight_ if we have $w(a) = w(b) + w(c)$ for every switch as in Figure 1. Let $W(\tau)$ be the set of all weights on $\tau$. A weight $w \in W(\tau)$ is said to be _positive_ (or a _measure_ on the train track $\tau$), denoted by $w > 0$, if $w(a) > 0$ for all the branches $a$ of $\tau$. Denote the set of all measures on $\tau$ by $W^+(\tau)$. 

Given $\mu \in W^+(\tau)$, we can foliate a rectangular neighborhood of $\tau$ according to $\mu$. Shrinking the components of the complement of this neighborhood, we get a measured foliation, denoted by $\mathcal{F}(\tau, \mu)$. A measured foliation $\zeta$ is said to be carried by $\tau$ if there exists a measure $\mu \in W^+(\tau)$ such that $\zeta = \mathcal{F}(\tau, \mu)$.

A measured foliation is called generic if it has only simple singularities and does not have a leaf connecting any two of its singularities. Let $\zeta \in \mathcal{MF}$ be generic and assume $\tau$ is a train track carrying $\zeta$, say, $\zeta = \mathcal{F}(\tau, \mu)$ for some $\mu \in W^+(\tau)$. Since $\zeta$ is generic, $\tau$ should be complete, hence $W^+(\tau)$ is of maximal dimension $6g - 6$ and

$$\varphi_\tau: W^+(\tau) \to \mathcal{MF} \text{ defined by } \mu_1 \mapsto \mathcal{F}(\tau, \mu_1)$$

gives a chart around $\zeta$. If the train track $\tau'$ carries $\zeta$ as well, then the change of coordinates $\varphi_\tau^{-1} \circ \varphi_{\tau'}$ is linear in a neighborhood of $\mu$. This gives $\mathcal{MF}$ a linear structure at such a $\zeta$. (see the explanation after Proposition 3.1.)

For a train track $\tau$, define the antisymmetric pairing $\omega_{Th}: W(\tau) \times W(\tau) \to \mathbb{R}$ by

$$\omega_{Th}(w_1, w_2) = \frac{1}{2} \sum_v \det \begin{pmatrix} w_1(b_v) & w_1(c_v) \\ w_2(b_v) & w_2(c_v) \end{pmatrix},$$

where the sum is over all the switches $v$ of $\tau$ and at each switch $v$, the incoming branch and outgoing branches are labeled by $a_v, b_v, c_v$ respectively, in such a way that $a_v b_v c_v$ is clockwise. Since $W(\tau)$ is a vector space, $T_\mu W^+(\tau)$ is naturally identified with $W(\tau)$ for every $\mu \in W^+(\tau)$, hence (3) gives an antisymmetric form on $W^+(\tau)$, denoted by $\omega_{Th}$ as well. It can be proved that $\omega_{Th}$ is invariant under the change of coordinates, hence it gives rise to an antisymmetric form on $\mathcal{MF}$, called the Thurston symplectic form.

Let $(X, \phi) \in QT_g$ and denote the zeros of $\phi$ on $X$ by $\Sigma$. A saddle triangulation (or triangulation for short) of $\phi$ is a triangulation of $X$ whose vertices belong to $\Sigma$ and the edges are straight lines in the flat metric induced by $\phi$. Fix such a triangulation $\Delta$ of $\phi$. For a triangle $ABC \in \Delta$, a comparison triangle is defined as a flat model of $ABC$, namely, this is a Euclidean triangle $A'B'C'$ together with a flat chart $\varphi: A'B'C' \to ABC$ that sends $A'$ to $A$, $B'$ to $B$ and $C'$ to $C$. (By triangle here, we mean the union of edges and the interior.) Note that the comparison triangle is unique up to translation and reflection from the origin.

For a triangulation $\Delta$, we can give the structure of a measured train track to the dual graph of $\Delta$ by defining the measure of an edge $e$, dual to the side $BC$ of a triangle $ABC \in \Delta$ to be $|\mathbb{R}(\overline{B'C'})|$ where $A'B'C'$ is the corresponding comparison triangle. Note that if $\mathcal{V}(\phi)$ is generic then $A'B'C'$ does not have a vertical side, hence the measure constructed above is indeed positive. The train track obtained in this way is called the train track adapted to $\Delta$. Observe that, as shown in Figure 2, a flip in the triangulation $\Delta$ corresponds to a splitting in the adapted train track and vice versa.

3.2. $E(X,Y)$ has measure zero. For given $X, Y \in \mathcal{T}_g$, define

$$E(X, Y) = \{ \zeta \in \mathcal{MF} : \text{Ext}_X(\zeta) = \text{Ext}_Y(\zeta) \}.$$

The goal of this section is to prove the following:
Figure 2. A splitting in the train track corresponds to a flip in the triangulation

**Proposition 3.1.** Let $X,Y \in T_g$ be distinct. Then $E(X,Y)$ is of Thurston measure zero.

A manifold $M$ with charts $\varphi_\alpha: U_\alpha \to V_\alpha \subset M$ is said to be smooth at $x \in M$ if the transition maps are smooth near $x$. More precisely, if for all indices $\alpha$ and $\beta$ such that $x \in V_\alpha \cap V_\beta$, $\varphi_{\alpha\beta} = \varphi_\beta^{-1} \circ \varphi_\alpha: U_{\alpha\beta} \to U_{\beta\alpha}$ is smooth on a neighbourhood of $\varphi_\alpha^{-1}x$ where $U_{\alpha\beta}$ is the domain of definition of $\varphi_{\alpha\beta}$. We define a manifold to be linear or analytic at a point in a similar way.

If $M$ is smooth at $x$, the tangent space to $M$ at $x$, denoted by $T_xM$, can be defined in the usual way. A function $f: M \to \mathbb{R}$ is said to be smooth (analytic, linear) at a smooth (analytic, linear) point $x \in M$, if it is smooth (analytic, linear) in a neighborhood of $\varphi_\alpha^{-1}(x)$ for a chart $\varphi_\alpha$ that covers $x$. If $f$ is smooth at $x$, the differential of $f$ at $x$, $d_xf: T_xM \to \mathbb{R}$, can be defined in the usual way.

For a given $X \in T_g$, define $N: Q(X) \to \mathbb{R}$ by $N(\phi) = |\phi|$. Since $Q(X)$ is a vector space, for every $\phi \in Q(X)$ we can identify the tangent space to $Q(X)$ at $\phi$, $T_\phi Q(X)$, with $Q(X)$. Define the following anti-symmetric pairing on $T_\phi Q(X)$:

$$\omega_\phi(\psi_1, \psi_2) = \frac{1}{4} \Im \left( \int_X \frac{\psi_1 \overline{\psi_2}}{|\phi|} \right).$$

Note that if $\phi \in P(1,\ldots,1)$, then $\omega_\phi$ is defined for all $\psi_1, \psi_2$, but it’s not necessarily so if $\phi$ has non-simple zeros.

**Lemma 3.2.** Given $X \in T_g$, let $\phi \in Q(X) \cap P(1,\ldots,1)$ and $\psi \in T_\phi Q(X) \simeq Q(X)$, then

$$d_\phi N(\psi) = 4\omega_\phi(i\phi, \psi).$$

**Proof.** The fact that $N$ is differentiable around $\phi$ follows from the proof of Theorem 5.3 of [Dum15]. With the notation introduced in that proof, we have

$$N(\phi) = N_0^i(\phi) + N_i^i(\phi),$$

and it is proved that both $N_0^i$ and $N_i^i$ are smooth in a neighborhood of $\phi$. The derivative of $N$ is computed in [Roy71] Lemma 1. □

**Construction.** Given $X \in T_g$, $\phi \in Q(X)$ and a triangulation $\Delta$ of $\phi$, let $(\tau, \mu)$ be the train track adapted to $\Delta$. For every $\psi \in Q(X)$, define $w(\Delta, \psi) \in$
$W(\tau)$ as follows: If the branch $e$ of the train track $\tau$ is dual to the side $BC$ of a triangle $ABC \in \Delta$, set
\[ w(\Delta, \psi)(e) = \frac{1}{2} \Re \int_B^C \frac{\psi}{\sqrt{\phi}}, \]
where the integral is taken over the side $BC$ of the triangle $ABC$, and the sign for $\sqrt{\phi}$ is chosen so that $\Re \int_B^C \sqrt{\phi} > 0$.

**Lemma 3.3.** Let $X \in T_g$ and assume $\phi \in Q(X)$ is such that $V(\phi)$ is generic. If $\Delta$ is a triangulation of $\phi$ and $\tau$ is its adapted train track, then $\varphi_{\tau}^{-1} \circ V$ is defined and smooth (even real analytic) in a neighbourhood of $\phi$ and its derivative at $\phi$, $D_\phi(\varphi_{\tau}^{-1} \circ V) \colon T_\phi Q(X) \to T_\mu W^+(\tau)$, is given by $\psi \mapsto w(\Delta, \psi)$.

**Proof.** If $\phi_1 \in Q(X)$ is near $\phi$, we can choose a triangulation of $\phi_1$, denoted by $\Delta(\phi_1)$, that is close to $\Delta = \Delta(\phi)$. Let $(\tau(\phi_1), \mu(\phi_1))$ be the measured train track adapted to $\phi_1$. Since $V(\phi)$ is generic, $\tau(\phi_1)$ is the same as $\tau = \tau(\phi)$ up to isotopy. This gives us a map
\[ W_\Delta \colon U \to W^+(\tau) \quad \text{defined by} \quad W_\Delta(\phi_1) = \mu(\phi_1), \]
where $U$ is a small neighbourhood of $\phi$. Let $A$ and $B$ be two of the zeros of $\phi$ such that $AB$ is the side of a triangle in $\Delta$ and assume $e$ is the branch of $\tau$ that is dual to $AB$. The zeros of $\phi_1$ vary as a complex analytic function of $\phi_1 \in Q(X)$, hence for $\phi_1$ close to $\phi$, $A_{\phi_1}$ and $B_{\phi_1}$ can be chosen such that
\[ W_\Delta(\phi_1)(e) = \int_{A_{\phi_1}}^{B_{\phi_1}} \Re \sqrt{\phi_1}. \]

$\int_{A_{\phi_1}}^{B_{\phi_1}} \Re \sqrt{\phi_1}$ is called a period function and its derivative is given by Douady-Hubbard formula to be $w(\Delta, \psi)$ (DH75). \hfill \square

Recall that for $X \in T_g$, $E_X \colon \mathcal{MF} \to \mathbb{R}$ is defined by $E_X(\zeta) = \text{Ext}(\zeta, X)$.

**Theorem 3.4.** Let $\zeta$ be a generic measured foliation and $X \in T_g$. Then $E_X$ is real analytic at $\zeta$ and its derivative, $d_\zeta E_X \colon T_\zeta \mathcal{MF} \to \mathbb{R}$, is given by
\[ d_\zeta E_X(\bullet) = \omega_{\text{Th}}(\eta, \bullet) \quad \text{(4)} \]
for some $\eta \in T_\zeta \mathcal{MF}$ that depends on $X$ and $\zeta$. Moreover, assuming that $\Delta$ is a triangulation of $\phi = V_X^{-1} \zeta$ and $(\tau, \mu)$ its adapted train track, then $\eta$ can be computed in the train track chart $\varphi_{\tau}$ to be $4w(\Delta, i\phi)$, namely,
\[ \eta = D_\mu \varphi_{\tau}(4w(\Delta, i\phi)) \quad \text{(5)} \]
where $D_\mu \varphi_{\tau} \colon W(\tau) \cong T_\mu W^+(\tau) \to T_\zeta \mathcal{MF}$ is the derivative of the chart $\varphi_{\tau} \colon W^+(\tau) \to \mathcal{MF}$ at $\mu$.

**Proof.** We start by taking derivatives of both sides of the identity
\[ N(\phi_1) = E_X(V(\phi_1)) \]
at $\phi$. Lemma 3.2 gives the derivative of the left hand side to be $\omega_{\phi}(4i\phi, \bullet)$. Using chain rule for the right hand side and the fact that $\omega_{\text{Th}}$ is the push-forward of $\omega_{\phi}$ by $V_X$ (Dum15 Theorem 5.8), we get
\[ d_\zeta E_X(\bullet) = \omega_{\text{Th}}(4D_\phi V_{\phi}(i\phi), \bullet). \]
This is (4) for \( \eta = 4D\nu_\phi(i\phi) \). To obtain \( D\nu_\phi(i\phi) \) in a train track chart, we should take derivative of both sides of

\[
\nu_X(\phi_1) = \varphi_\tau \circ W_\Delta(\phi_1),
\]

where \( W_\Delta \) is defined in the proof of Lemma 3.3 Chain rule and Lemma 3.3 then gives (5).

**Lemma 3.5.** Given \( X, Y \in T_g \) and a generic measured foliation \( \zeta \), let \( \eta_1, \eta_2 \in T_\zeta M\mathcal{F} \) be such that \( d\zeta E_X = \omega_{\text{Th}}(\eta_1, \cdot) \) and \( d\zeta E_Y = \omega_{\text{Th}}(\eta_2, \cdot) \). Then \( \eta_1 = \eta_2 \) implies \( X = Y \).

**Proof.** First let us describe \( w(\Delta, i\phi) \) for an arbitrary triangulation \( \Delta \) of a quadratic differential \( \phi \in QT_g \). Set \( (\tau, \mu) \) to be the train track adapted to \( \Delta \). Then by the definition of \( w(\cdot, \cdot) \), if a branch \( e \) of \( \tau \) is dual to the side \( BC \) of a triangle \( ABC \in \Delta \), then

\[
w(\Delta, i\phi)(e) = -\Im(B'C'),
\]

where the comparison triangle \( A'B'C' \) is chosen such that \( \Re(B'C') > 0 \). Note that \( \phi \) here is uniquely determined by the data \( (\tau, \mu, w(\Delta, i\phi)) \).

Now to prove the lemma, let \( \phi_1 \in Q(X) \) and \( \phi_2 \in Q(Y) \) be such that \( \nu(\phi_1) = \nu(\phi_2) = \zeta \). Choose an arbitrary triangulation \( \Delta_j \) for \( \phi_j \) and let \( (\tau_j, \mu_j) \) be its adapted train track \( (j = 1, 2) \). Since \( \zeta = \mathcal{F}(\tau_1, \mu_1) = \mathcal{F}(\tau_2, \mu_2) \), the two train tracks should have a common splitting \( (\tau, \mu) \) [PH92 Theorem 2.3.1]. Since every split along an edge corresponds to a flip in the dual triangulation, we obtain triangulations \( \Delta'_1, \Delta'_2 \) from \( \Delta_1, \Delta_2 \) such that they both have \( (\tau, \mu) \) as their adapted train track.

By Theorem 3.4, the derivative of \( \varphi_\tau : W^+(\tau) \rightarrow \mathcal{M}\mathcal{F} \) identifies \( \eta_j \) with \( 4w(\Delta_j, i\phi_j) \in T_\mu W^+(\tau) \) for \( j = 1, 2 \). Hence, the assumption \( \eta_1 = \eta_2 \) implies \( w(\Delta'_1, i\phi_1) = w(\Delta'_2, i\phi_2) \). However, as mentioned before, the data \( (\tau, \mu, w(\Delta, i\phi)) \) uniquely determines \( \phi \). This implies that \( \phi_1 = \phi_2 \), hence \( X = Y \).

**Proof of Proposition 3.1.** Let \( \mathcal{G} \) be the subset of \( \mathcal{M}\mathcal{F} \) consisting of generic measured foliations. Then \( \mathcal{G} \) has full measure, i.e., \( \nu(\mathcal{M}\mathcal{F} \setminus \mathcal{G}) = 0 \). Define

\[
f : \mathcal{M}\mathcal{F} \rightarrow \mathbb{R} \quad \text{by} \quad f(\zeta) = E_X(\zeta) - E_Y(\zeta),
\]

hence \( E(X, Y) = f^{-1}(0) \). Let \( \zeta \in E(X, Y) \cap \mathcal{G} \) be arbitrary, then according to Theorem 3.4 there exist \( \eta_1, \eta_2 \) such that

\[
d\zeta E_X(\cdot) = \omega_{\text{Th}}(\eta_1, \cdot) \quad \text{and} \quad d\zeta E_Y(\cdot) = \omega_{\text{Th}}(\eta_2, \cdot),
\]

hence the derivative \( d\zeta f : T_\zeta \mathcal{M}\mathcal{F} \rightarrow \mathbb{R} \) is given by

\[
d\zeta f(\cdot) = (\eta_1 - \eta_2, \cdot).
\]

Since \( X \neq Y \), by Lemma 3.5 we have \( \eta_1 - \eta_2 \neq 0 \). Non-degeneracy of the Thurston form ([PH92 Theorem 3.2.4]) then implies that \( d\zeta f \neq 0 \). Since \( f \) is smooth at \( \zeta \), \( f(\zeta) = 0 \) and \( d\zeta f \neq 0 \), \( f^{-1}(0) \) is locally a submanifold of codimension 1 around \( \zeta \). So there is a neighborhood \( U_\zeta \) of \( \zeta \) such that \( E(X, Y) \cap U_\zeta \) is of measure zero. Now, covering \( E(X, Y) \cap \mathcal{G} \) by countably many \( U_\zeta \)'s we obtain \( \nu(E(X, Y) \cap \mathcal{G}) = 0 \). Since \( \mathcal{G} \) has full measure, \( \nu(E(X, Y)) = 0 \).
4. Comparing Teichmüller and extremal lengths

4.1. Projection to a thick geodesic. In this section, we state a few general facts about the geodesics that lie completely in the thick part, where by a geodesic we always mean a bi-infinite geodesic in the Teichmüller space, unless otherwise stated. Denoting the covering map from Teichmüller space to the moduli space by \( \Pi: T_g \to M_g \), for a subset \( K \subset M_g \) we define \( \tilde{K} \) to be \( \Pi^{-1}(K) \).

**Definition 4.1.** Let \( K \subset M_g \) be compact. A Teichmüller geodesic \( G \) is said to be \( K \)-thick if \( G \subset \tilde{K} \).

Assume \( G \) is \( K \)-thick for some compact \( K \subset M_g \) and let \( X \in T_g \) and \( \zeta \in M \mathcal{F} \) be arbitrary. Define

\[
\text{proj}_G X = \{ Y \in G : d(X, Y) = d(X, G) \};
\]

\[
\text{proj}_G \zeta = \text{proj}_G \zeta = \{ Y \in G : \text{Ext}(\zeta, Y) = \text{Ext}(\zeta, G) \},
\]

where \( d(X, G) = \inf \{ d(X, Y) : Y \in G \} \) and \( \text{Ext}(\zeta, G) = \inf \{ \text{Ext}(\zeta, Y) : Y \in G \} \). Both \( \text{diam} (\text{proj}_G X) \) and \( \text{diam} (\text{proj}_G \zeta) \) are bounded by constants depending only on \( K \), where \( \text{diam} \) stands for the diameter of a set. The boundedness of \( \text{diam} (\text{proj}_G X) \) is a consequence of the contraction theorem of [Min96] and the boundedness of \( \text{diam} (\text{proj}_G \zeta) \) is also standard and follows, say, from Proposition 4.5.

There have been many analogies between the Teichmüller space, equipped with the Teichmüller metric, and a hyperbolic space. More specifically, we expect the Teichmüller metric to behave like a \( \delta \)-hyperbolic (Gromov hyperbolic) metric in the thick part. (see for example [Min96], [MM99], [Raf14].) The following is an instance of this phenomenon:

**Theorem 4.2.** ([Raf14] Theorem 8.1) Let \( K \subset M_g \) be compact and \( X, Y, Z \in T_g \). Then there are constants \( C \) and \( D \) only depending on \( K \) such that the following holds: If \( U, V \in [X, Y] \) are such that \( [U, V] \subset \tilde{K} \) and \( d(U, V) > C \), then for every \( W \in [U, V] \) we have

\[
\min \{ d(W, [Z, X]), d(W, [Z, Y]) \} < D.
\]

Note that the above theorem remains true if any number of the vertices of the triangle \( \triangle(X, Y, Z) \) belongs to the boundary of Teichmüller space. The following is a consequence of Theorem 4.2

**Proposition 4.3.** Let \( K \subset M_g \) be compact and \( G \) be a \( K \)-thick geodesic. Then there exists a constant \( C = C(K) \) such that for every \( X \in T_g \), \( Y \in G \) and \( H \in \text{proj}_G X \), the geodesic connecting \( X \) to \( Y \) passes through \( B(H, C) \), the ball of radius \( C \) centered at \( H \).

**Proof.** Let \( X, H, Y \) be as in the proposition. By Theorem 4.2 there exists \( C' = C'(K) \) such that for every \( Z \in [H, Y] \) there is \( W \in [X, H] \cup [X, Y] \) such that \( d(Z, W) < C' \). We claim that \( C = 3C' + 1 \) satisfies the statement. If \( d(H, Y) < 2C' + 1 \) then we are done, otherwise let \( Z \in [H, Y] \) be such that \( d(H, Z) = 2C' + 1 \). If the point \( W \) given by Theorem 4.2 lies in \( [X, H] \), then by triangle inequality in \( \triangle(W, H, Z) \) we obtain \( d(W, H) > C' + 1 \), hence

\[
d(X, Z) \leq d(X, W) + d(W, Z) < d(X, H) - 1,
\]

\[
d(X, Z) \leq d(X, W) + d(W, Z) < d(X, H) - 1,
\]
which contradicts the choice of $H$. This contradiction implies $W \in [X,Y]$. Triangle inequality in $\triangle(H,Z,W)$ then implies $d(H,W) < 3C' + 1$. □

**Corollary 4.4.** Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there is $C = C(\mathcal{K})$ such that for every $X \in T_g$, $H \in \text{proj}_g X$ and $Y \in \mathcal{G}$, we have

$$d(X,Y) \simeq_C d(X,H) + d(H,Y).$$

**Proof.** Let $\mathcal{G}$, $X$ and $H$ be as above and let $C = C(\mathcal{K})$ be the constant given by Proposition 4.3. If $[X,Y]$ intersects $B(H,C)$ at $Z$ then

$$d(X,Z) \simeq_C d(X,H) \text{ and } d(Z,Y) \simeq_C d(H,Y).$$

The statement follows from summing up these two estimates. □

Proposition 4.3 can be proved if $X \in T_g$ is replaced by a measured foliation $\zeta \in \mathcal{M}\mathcal{F}$. The proof parallels the one given above, only instead of the triangle inequality for triangles with a vertex at infinity, we should use Kerschoff inequality. (see the discussion at the end of Section 2.) Corollary 4.4 can be proved in this setting as well, hence we have the following:

**Proposition 4.5.** Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there exists a constant $C = C(\mathcal{K})$ such that for every $\zeta \in \mathcal{M}\mathcal{F}$, $H \in \text{proj}_g \zeta$ and $Y \in \mathcal{G}$, we have

$$\beta(\zeta,Y) \simeq_C \beta(\zeta,H) + d(H,Y).$$

Moreover, the geodesic connecting $\zeta$ to $Y$ passes through $B(H,C)$.

4.2. Busemann approximation. We make the following definition:

**Definition 4.6.** Let $\mathcal{K} \subset \mathcal{M}_g$ be compact. The geodesic $[X,Y]$ connecting two points $X,Y \in T_g$ is called $\mathcal{K}$–typical if it spends at least half of its time in $\tilde{\mathcal{K}}$.

We say that two geodesics $G_1: [0,a] \to T_g$ and $G_2: [0,b] \to T_g$, parametrized with respect to arc length, $D$–fellow travel, if $|a-b| < D$ and for all $0 \leq t \leq \min\{a,b\}$ we have

$$d(G_1(t),G_2(t)) < D.$$ 

**Theorem 4.7.** ([Raf14] Theorem 7.1) Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $C > 0$. Then there exists a constant $D = D(\mathcal{K},C)$ such that the following holds: for every $X,Y \in \tilde{\mathcal{K}}$ and $\tilde{X},\tilde{Y} \in T_g$ such that $d(X,\tilde{X})$ and $d(Y,\tilde{Y})$ are both less than $C$, the geodesics $[X,Y]$ and $[\tilde{X},\tilde{Y}]$ $D$–fellow travel.

Note that the conclusion of this theorem remains valid if $X = \tilde{X}$ belongs to the boundary of the Teichmüller space ([Raf14] Remark 7.2). In that case, we should allow $a = b = \infty$ in the definition of fellow traveling.

**Remark 4.8.** It is a consequence of this theorem that for every compact set $\mathcal{K} \subset \mathcal{M}_g$ and real number $C > 0$, there exists an enlargement $\mathcal{K}' \supset \mathcal{K}$, depending only $\mathcal{K}$ and $C$, such that if $X,Y,\tilde{X},\tilde{Y}$ are as in the theorem and $[X,Y]$ is $\mathcal{K}$–typical, then $[\tilde{X},\tilde{Y}]$ is $\mathcal{K}'$–typical.
Generically, two geodesic rays going to the same point in the boundary of Teichmüller space become exponentially close to each other. The precise statement is as follows:

**Theorem 4.9.** ([EMR21] Corollary ??) Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $C > 0$. Then there are positive numbers $\alpha = \alpha(\mathcal{K})$ and $D = D(\mathcal{K}, C)$ such that the following holds: If $X, Y \in \overline{\mathcal{K}}$ and $\zeta \in \mathcal{MF}$ are such that $\text{Ext}(\zeta, X) = \text{Ext}(\zeta, Y)$; $d(X, Y) < C$; and $Z_1 \in [X, \zeta)$ and $Z_2 \in [Y, \zeta)$ are such that $d(X, Z_1) = d(Y, Z_2) = T$, then

$$d(Z_1, Z_2) < D e^{-\alpha T}.$$  

We also need the following ([Min96] Corollary 4.1):

**Lemma 4.10.** Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then there exists a constant $C = C(\mathcal{K})$ such that for every $X, Y \in \mathcal{T}_g$ we have

$$\text{diam} (\text{proj}_G(X) \cup \text{proj}_G(Y)) < d(X, Y) + C.$$

The next proposition is the main tool that we use to relate the extremal and Teichmüller lengths:

**Proposition 4.11.** Let $\mathcal{K} \subset \mathcal{M}_g$ be compact and $\mathcal{G}$ be a $\mathcal{K}$–thick geodesic. Then for every $\epsilon > 0$ there exists $C = C(\mathcal{K}, \epsilon)$ such that the following holds: if $\zeta \in \mathcal{MF}$; $X, Y \in \mathcal{G}$; $Z \in [X, \zeta) \cap \overline{\mathcal{K}}$ and $H \in \text{proj}(Z, \mathcal{G})$ are such that the goedesic $[Z, H]$ is $\mathcal{K}$–typical and of length greater than $C$, then

$$d(Z, Y) - d(Z, X) \approx_\epsilon \beta(\zeta, Y) - \beta(\zeta, X) = \beta([\zeta], X, Y).$$

**Proof.** Let $H_\zeta \in \text{proj}_G \zeta$ and $C = C(\mathcal{K})$ be the constant given by Proposition 4.5, so there is $X' \in (\zeta, X]$ such that $d(X', H_\zeta) < C$. By Theorem 4.7 there is a constant $D$ depending on $\mathcal{K}$ and $\mathcal{G}$ such that $(\zeta, X')$ and $(\zeta, H_\zeta)$ $D$–fellow travel; as a result, we can find $Z_\zeta \in (\zeta, H_\zeta]$ with $d(Z, Z_\zeta) < D$. Since $H_\zeta \in \text{proj}_G \zeta$, by Lemma 4.10 there exists $D'$ depending on $D$ and $\mathcal{K}$ such that $d(H, H_\zeta) < D'$. Triangle inequality then implies $d(X', H) < D' + C$, hence $[Z, H]$ and $[Z, X']$ fellow travel, so $[Z, X']$ is $\mathcal{K}'$–typical for some enlargement $\mathcal{K}'$ of $\mathcal{K}$ (Remark 4.8) and since the constants $C, D, D'$ depend on $\mathcal{K}$, the enlargement $\mathcal{K}'$ only depends on $\mathcal{K}$ as well. Applying Proposition 4.5 once again gives $Y' \in ([\zeta], Y)$ with $d(Y', H_\zeta) < C$, hence $d(X', Y') < 2C$. Using Kerckhoff inequality we get

$$|\beta(\zeta, X', Y')| \leq d(X', Y') < 2C,$$

so by moving $Y'$ along $(\zeta, Y]$ by at most $2C$ we may obtain $Y''$ such that $\beta(\zeta, X') = \beta(\zeta, Y'')$. Let $Z$ be the point obtained by flowing $Y''$ along $[Y'', \zeta)$ by time $T = d(X', Z)$. By Theorem 4.9 there exists $T_0 = T_0(\mathcal{K}', C, \epsilon)$ such that if $T > T_0$ we have $d(Z, Z') < \epsilon$. Hence if $d(Z, H) > T_0 + D' + C$, we have

$$d(Z, Y) - d(Z, X) \approx_\epsilon d(Z', Y) - d(Z, X) = d(Y'', Y) - d(X', X)$$

$$= (\beta(\zeta, Y) - \beta(\zeta, Y'')) - (\beta(\zeta, X) - \beta(\zeta, X'))$$

$$= \beta(\zeta, Y) - \beta(\zeta, X).$$

$\square$
5. Proofs

5.1. Preliminary discussion. Fix a pseudo-Anosov homeomorphism $\gamma$ throughout this section and let $\mathcal{L}$ be the axis of its action on Teichmüller space and $L = \tau(\gamma)$ its translation length. Define $\mathcal{C}$ and $\bar{L}$ as in the introduction and denote the covering map from $\mathcal{T}$ to $\mathcal{C}$ by $\Pi$. To lighten the notation, we denote $L$ and $\bar{L}$ by $L$ and $\bar{L}$ respectively. Recall the following definitions:

$B(X, R) = \{ Y \in \mathcal{T} : d(X, Y) \leq R \}$;

$B(\mathcal{L}, R) = \{ Y \in \mathcal{T} : d(Y, \mathcal{L}) \leq R \}$;

$B_{\text{Ext}}(\mathcal{L}) = \{ \zeta \in \mathcal{M}(\mathcal{F}) : \text{Ext}(\zeta, \mathcal{L}) \leq 1 \}$;

and define the following:

$\text{Typ}(\mathcal{L}, K) = \{ Y \in \mathcal{T} : [\cdot, H] \text{ is } K\text{-typical for some } H \in \text{proj}_L Y \}$;

$\text{Typ}(X, K) = \{ Y \in \mathcal{T} : [X, Y] \text{ is } K\text{-typical} \}$.

Given an open subset $U$ of $\mathcal{PMF}$, let

$S(X, U, R) = \{ \pi(gt\phi) : \phi \in Q(X), [\mathcal{V}(\phi)] \in U \text{ and } 0 \leq t \leq R \}$;

$S_{\text{Ext}}(X, U) = \{ \zeta \in \mathcal{M}(\mathcal{F}) : \zeta \in U \text{ and } \text{Ext}(\zeta, X) \leq 1 \}$.

Let $X, P \in \mathcal{T}$ and assume $U$ is an open subset of $\mathcal{PMF}$. We need the following two facts from [ABEM12]:

• As $R \to \infty$,

$$|\Gamma \cdot P \cap S(X, U, R)| \sim \frac{\Lambda^2}{h \text{Vol}(\mathcal{M})} \nu(S_{\text{Ext}}(X, U)) e^{hR}. \tag{6}$$

• For every $\epsilon > 0$, there exists a compact set $K \subset \mathcal{M}$ depending on $X, P, \epsilon$ such that

$$\limsup_{R \to \infty} e^{-hR}|\Gamma \cdot P \cap S(X, U, R) \setminus \text{Typ}(X, K)| < \epsilon. \tag{7}$$

The first fact follows from [ABEM12] Theorem 2.9, since, with the notation of Proposition 2.1 of the same paper, we have

$$\int_U \lambda^{-\gamma}(q) ds_X(q) = \int_U \frac{d(\delta_X^+), \nu}{d\bar{s}} ds_X = (\delta_X^+)_* \nu(U) = \nu(S_{\text{Ext}}(X, U)),$$

where the first equality uses part (i) of the same proposition.

The second fact follows from [ABEM12] Theorem 2.7 since (again, with the paper’s notation) if $K' \subset \mathcal{PMF}$ is a compact subset of the principal domain of quadratic differentials with norm 1, then $K = \pi(K')$ is compact as well.

Let $[\gamma^\pm]$ be the set containing the two elements of $\mathcal{PMF}$ that are fixed by $\gamma$. By Theorem 6.9 of [MP89], the action of $\langle \gamma \rangle$ on $\mathcal{PMF} \setminus [\gamma^\pm]$ is a covering space action. Define

$$\mathcal{C}_{\text{Ext}} = \langle \gamma \rangle \setminus B_{\text{Ext}}(\mathcal{L}) \quad \text{and let} \quad \Pi_{\text{Ext}} : B_{\text{Ext}}(\mathcal{L}) \to \mathcal{C}_{\text{Ext}}$$

be the corresponding covering map. Since $\Gamma \cdot [P] \cap B(\mathcal{L}, R)$ is in one-to-one correspondence with $\langle \gamma \rangle \setminus (\Gamma \cdot P \cap B(\mathcal{L}, R))$, Theorem [A] is equivalent to the following:
Theorem 5.1. Let \( \gamma \in \mathcal{MCG} \) be pseudo-Anosov and \( \mathcal{L} = \mathcal{L}_\gamma \) its axis. Then for a given \( P \in \mathcal{T}_g \) we have

\[
|\langle \gamma \rangle \backslash (\Gamma \cdot P \cap B(\mathcal{L}, R))| \sim \frac{\Lambda^2}{h \text{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\text{Ext}}, \gamma) e^{hR},
\]

as \( R \to \infty \).

Theorem 5.1 is proved in Section 5.2. Assuming this theorem, we now give a proof of Theorem C.

Proof of Theorem C. Fix a point \( O \in \mathcal{L} \) and let \( Y \in \mathcal{T}_g \) be arbitrary. Pick a point \( H_Y \in \text{proj}_\mathcal{L}(Y) \) and choose \( k = k(Y) \in \mathbb{Z} \) in such a way that \( d(O, \gamma^k H_Y) < L \). Define

\[
h : \langle \gamma \rangle \backslash (\Gamma \cdot X \cap B(\mathcal{L}, R)) \to \Gamma \cdot X \cap B(O, R + L)
\]

by sending \( \langle \gamma \rangle \cdot (gX) \) to \( \gamma^k(gX) \) (there might be more than one option for \( H_Y gX \) and \( k(gX) \), then choose one.) Note that \( h \) is an injection, hence

\[
|\langle \gamma \rangle \backslash (\Gamma \cdot X \cap B(\mathcal{L}, R))| \leq |\Gamma \cdot X \cap B(O, R + L)|. \tag{8}
\]

By [ABEM12] Theorem 5.1, there exists a constant \( C \), only depending on the base point \( O \), such that for all \( X \in \mathcal{T}_g \) and \( R > 0 \),

\[
|\Gamma \cdot X \cap B(O, R)| < Ce^{hR}.
\]

This, combined with (8), implies that for \( C' = Ce^{hL} \) and every \( X \in \mathcal{T}_g \) we have

\[
|\langle \gamma \rangle \backslash (\Gamma \cdot X \cap B(\mathcal{L}, R))| < C'e^{hR}. \tag{9}
\]

Define the covering map

\[
\Pi_{\gamma, \Gamma} : \mathcal{C}_\gamma \to \mathcal{M}_g \quad \text{by} \quad \langle \gamma \rangle \cdot X \mapsto \Gamma \cdot X.
\]

Since \( \Pi_{\gamma} \) is a local diffeomorphism, we have

\[
\text{Vol}(B(\mathcal{L}, R)) = \int_{\mathcal{M}_g} |\Pi_{\gamma, \Gamma}^{-1}(X) \cap B(\mathcal{L}, R)| d\text{Vol}(X) = \int_{\mathcal{M}_g} |\langle \gamma \rangle \backslash (\Gamma \cdot X \cap B(\mathcal{L}, R))| d\text{Vol}(X).
\]

We multiply the left and right-hand side of this equation by \( e^{-hR} \) and take the limit as \( R \to \infty \). Since \( C' \) in (9) does not depend on \( X \), we can apply Lebesgue’s dominated convergence theorem to take the limit inside the integral. Theorem 5.1 then concludes the proof.

\( \square \)

5.2. Concluding the proof. The goal of this section is to give a proof of Theorem 5.1. Fix two points \( P \in \mathcal{T}_g \) and \( O \in \mathcal{L} \) for the rest of this section. To lighten the notation, we do not show the dependance of constants on \( \mathcal{L}, O, P \). So, for example, we write \( C = C(\mathcal{K}) \) instead of \( C = C(\mathcal{K}, \mathcal{L}, O, P) \). For \( n, m \in \mathbb{Z} \cup \{\pm \infty\} \), a sequence of points \( \mathcal{X} = (X_i)_{n \leq i \leq m} \subset \mathcal{L} \) is called a net if \( (X_i, X_{i+1})'s \) are disjoint for \( n \leq i < m \). For such a net \( \mathcal{X} \), \( \sup_{n \leq i \leq m} d(X_i, X_{i+1}) \) is called the mesh of \( \mathcal{X} \) and for an \( \epsilon > 0 \), a net with mesh less than \( \epsilon \) is called an \( \epsilon \)-net.
Let $\epsilon > 0$ and take an $\epsilon$-net $O = X_0, \ldots, X_N = \gamma O$ in $[O, \gamma O]$. Letting $X_{i+N} = \gamma X_i$, we obtain a $\gamma$-invariant $\epsilon$-net $\mathcal{X} = (\ldots, X_{-1}, X_0, X_1, \ldots)$ in $\mathcal{L}$, and every $\gamma$-invariant $\epsilon$-net containing $O$ is obtained in this way. From now on, by an $\epsilon$-net $\mathcal{X} = (X_i)_{i \in \mathbb{Z}}$ we always mean a $\gamma$-invariant $\epsilon$-net in $\mathcal{L}$ with $X_0 = O$, and we show the number $N$ such that $X_N = \gamma O$ by $N = N(\mathcal{X})$. For such an $\epsilon$-net $\mathcal{X}$, and for all $i \in \mathbb{Z}$, define

$$A_i(\mathcal{X}) = \{ \zeta \in \mathcal{MF} : \text{Ext}(\zeta, X_i) \leq 1 \text{ and } \beta(\zeta, X_i) = \inf_{j \in \mathbb{Z}} \beta(\zeta, X_j) \}.$$

**Lemma 5.2.** Let $\mathcal{X}$ be an $\epsilon$-net, then

$$\frac{1}{e^{(6g-6)}} \nu(\mathcal{C}_{\text{Ext}, \gamma}) \leq \sum_{i=0}^{N(\mathcal{X})-1} \nu(A_i(\mathcal{X})) \leq \nu(\mathcal{C}_{\text{Ext}, \gamma}).$$

*Proof.* Let $N = N(\mathcal{X})$. It follows from the definition that the interior of $A_i(\mathcal{X})$’s, denoted by $\hat{A}_i(\mathcal{X})$’s, are disjoint and $A_i(\mathcal{X})$’s are $\gamma$-equivariant, meaning that $\gamma A_i(\mathcal{X}) = A_{i+N}(\mathcal{X})$. These facts imply that $\Pi_{\text{Ext}, \gamma} : \hat{A}_i(\mathcal{X}) \to \mathcal{C}_{\text{Ext}, \gamma}$ is an injection for $i \in \mathbb{Z}$, and $\Pi_{\text{Ext}, \gamma}(A_i(\mathcal{X})) \subset C_{\text{Ext}, \gamma}$ are disjoint for $0 \leq i < N$. Thus we have

$$\sum_{i=0}^{N-1} \nu(A_i(\mathcal{X})) \leq \nu(\mathcal{C}_{\text{Ext}, \gamma}).$$

By the definition of $E(\cdot, \cdot)$, given at the beginning of Section 3.2, $A_i(\mathcal{X}) \cap A_j(\mathcal{X}) \subset E(X_i, X_j)$, hence by proposition 3.1

$$\partial A_i(\mathcal{X}) = \bigcup_{j \neq i} A_i(\mathcal{X}) \cap A_j(\mathcal{X}) \subset \bigcup_{j \neq i} E(X_i, X_j) \implies \nu(\partial A_i(\mathcal{X})) = 0.$$

From this, we get

$$\sum_{i=0}^{N-1} \nu(\hat{A}_i(\mathcal{X})) = \sum_{i=0}^{N-1} \nu(A_i(\mathcal{X})) \leq \nu(\mathcal{C}_{\text{Ext}, \gamma}).$$

This proves the right hand inequality in the statement of the lemma.

To prove the left hand inequality, let $\zeta \in B_{\text{Ext}}(\mathcal{L})$ be arbitrary and assume $H_\zeta \in \text{proj}_L \zeta$, so we have $\text{Ext}(\zeta, H_\zeta) \leq 1$. Since the mesh of $\mathcal{X}$ is less than $\epsilon$, if $X_{i_0}$ is the element of the net that is closest to $H_\zeta$, by triangle inequality (Kerckhoff inequality) in $\triangle(\zeta, H_\zeta, X_{i_0})$ we have

$$\beta(\zeta, X_{i_0}) \leq \beta(\zeta, H_\zeta) + \beta(H_\zeta, X_{i_0}) < \epsilon \implies \inf_{i \in \mathbb{Z}} \beta(\zeta, X_i) < \epsilon \implies \inf_{i \in \mathbb{Z}} \text{Ext}(\zeta, X_i) < e^{2\epsilon}.$$

If the latter infimum is attained at $i = i_1$ then we have $\zeta/\epsilon^i \in A_{i_1}$, hence $\zeta \in e^{i} A_{i_1}$. As a result,

$$B_{\text{Ext}}(\mathcal{L}) \subset \bigcup e^{i} A_i(\mathcal{X}) \implies C_{\text{Ext}, \gamma} \subset \bigcup \Pi_{\text{Ext}, \gamma}(e^{i} A_i(\mathcal{X})),$$

which implies

$$\frac{1}{e^{(6g-6)}} \nu(\mathcal{C}_{\text{Ext}, \gamma}) \leq \sum_{i=0}^{N-1} \nu(A_i(\mathcal{X})).$$

□
Let $\mathcal{X}$ be an $\epsilon$–net and for $\delta > 0$, define the following subsets of $\mathcal{MF}$:

$$\mathcal{U}_i^\delta(\mathcal{X}) = \{ \zeta \in \mathcal{MF} : \text{Ext}(\zeta, X_i) \leq 1 \text{ and } \beta(\zeta, X_i) < \inf_{j \neq i} \beta(\zeta, X_j) - \delta \};$$

$$\mathcal{V}_i^\delta(\mathcal{X}) = \{ \zeta \in \mathcal{MF} : \text{Ext}(\zeta, X_i) \leq 1 \text{ and } \beta(\zeta, X_i) < \inf_{j \neq i} \beta(\zeta, X_j) + \delta \}.$$

Note that $\mathcal{U}_i^\delta(\mathcal{X})$’s are open and $\gamma$–equivariant, and the same is true for $\mathcal{V}_i^\delta$’s. Moreover, $\mathcal{U}_i^\delta(\mathcal{X}) \subset \mathcal{A}_i(\mathcal{X}) \subset \mathcal{V}_i^\delta(\mathcal{X})$ and $\mathcal{U}_i^\delta(\mathcal{X}) \uparrow \mathcal{A}_i(\mathcal{X})$ as $\delta \downarrow 0$; also, $\mathcal{V}_i^\delta(\mathcal{X}) \downarrow \mathcal{A}_i(\mathcal{X})$ as $\delta \downarrow 0$. For a compact set $\mathcal{K} \subset \mathcal{M}_g$, define the following subsets of $\Gamma \cdot P$ associated to $\mathcal{U}_i^\delta(\mathcal{X})$ and $\mathcal{V}_i^\delta(\mathcal{X})$:

$$\mathcal{U}_i^\delta(\mathcal{X}, \mathcal{K}) = \Gamma \cdot P \cap S(X_i, [\mathcal{U}_i^\delta(\mathcal{X})]) \cap \text{Typ}(X_i, \mathcal{K});$$

$$\mathcal{V}_i^\delta(\mathcal{X}) = \Gamma \cdot P \cap S(X_i, [\mathcal{V}_i^\delta(\mathcal{X})]),$$

where for a subset $\mathcal{U} \subset \mathcal{MF}$, we denote $\{[\zeta] : \zeta \in \mathcal{U} \} \subset \mathcal{P}\mathcal{MF}$ by $[\mathcal{U}]$. As before, both $\mathcal{U}_i^\delta(\mathcal{X}, \mathcal{K})$’s and $\mathcal{V}_i^\delta(\mathcal{X})$’s are $\gamma$–equivariant. For the moment, let $N = N(\mathcal{X}), \mathcal{U}_i(R) = \mathcal{U}_i^\delta(\mathcal{X}, \mathcal{K}) \cap B(X_i, R)$ and $\mathcal{V}_i(R) = \mathcal{V}_i^\delta(\mathcal{X}) \cap B(X_i, R)$. As mentioned in 1.3, the idea of the proof is to show that for $\mathcal{K}$ large enough:

- For every $i \in \mathbb{Z}$, $\Pi_i : \mathcal{U}_i(R) \to \Gamma \cdot [P] \cap B(\mathcal{L}, R)$ is almost an injection and $\Pi_i(\mathcal{U}_i(R))$’s are more or less disjoint for $0 \leq i < N$ (Lemma 5.3).

Hence $\sum_{i=0}^{N-1} |\mathcal{U}_i(R)|$ gives a lower bound for $|\Gamma \cdot [P] \cap B(\mathcal{L}, R)|$.

- The union of $\Pi_i(\mathcal{V}_i(R+\epsilon))$’s for $0 \leq i < N$ cover almost all of $C(\mathcal{L}, R)$ (Lemma 5.4 + Lemma 5.5), hence $\sum_{i=0}^{N-1} |\mathcal{V}_i(R+\epsilon)|$ gives an upper bound for $|\Gamma \cdot [P] \cap B(\mathcal{L}, R)|$.

As $R \to \infty$, we can use (6) to count the points in each $\mathcal{U}_i(R)$ and $\mathcal{V}_i(R)$, and if $\epsilon, \delta$ are small enough ($\delta$ moves to 0 much faster than $\epsilon$), the upper and lower bounds obtained in this way are close to each other, and they approximate the right hand side of 5.1 from above and below.

**Lemma 5.3.** Let $\mathcal{X} = (X_i)_{i \in \mathbb{Z}}$ be an $\epsilon$–net. Then, for every compact $\mathcal{K} \subset \mathcal{M}_g$ and $\delta > 0$, there exists a constant $C = C(\epsilon, \mathcal{K}, \delta)$ such that for $i \neq j$,

$$\left(\mathcal{U}_i^\delta(\mathcal{X}, \mathcal{K}) \setminus B(X_i, C)\right) \cap \left(\mathcal{U}_j^\delta(\mathcal{X}, \mathcal{K}) \setminus B(X_j, C)\right) = \emptyset.$$

**Proof.** Fix the net $\mathcal{X}$ throughout the proof. Let $Y \in \mathcal{U}_i^\delta(\mathcal{X}, \mathcal{K})$ and $H \in \text{proj}_L Y$. We claim that there is a compact set $\mathcal{K}'$, only depending on $\mathcal{K}$, such that $[Y, H]$ is $\mathcal{K}'$–typical. Let $\zeta \in \mathcal{U}_i^\delta$ be such that $Y$ lies on $[X_i, \zeta]$. Note that, since the net’s mesh is less than $\epsilon$,

$$\beta(\zeta, X_i) = \inf_{k \in \mathbb{Z}} \beta(\zeta, X_k) \simeq_{\epsilon} \inf_{Y \in \mathcal{L}} \beta(\zeta, Y).$$

Because of the shape of $\beta(\zeta, X)$ as a function of $X \in \mathcal{L}$, described in Proposition 4.3, the latter infimum should be attained near $X_i$; namely, there exists $C_1 = C_1(\mathcal{K})$ such that if $H_{\zeta} \in \text{proj}_L \zeta$, then $d(X_i, H_{\zeta}) < C_1$. Thus, by Theorem 4.7 (\zeta, X_i) and (\zeta, H_{\zeta}) $D$–fellow travel for some $D = D(\mathcal{K}, C_1)$. As a result, if $\mathcal{Z}_{\zeta} \in (\zeta, H_{\zeta})$ is such that $d(Y, X_i) = d(\mathcal{Z}_{\zeta}, H_{\zeta})$ then $d(Y, \mathcal{Z}_{\zeta}) < D$. Since $H_{\zeta} = \text{proj}_L \mathcal{Z}_{\zeta}$, Lemma 4.10 implies $d(H_{\zeta}, H) < D_1$ for some $D_1 = D_1(\mathcal{K}, D)$. Triangle inequality then implies $d(X_i, H) < C_2 = C_1 + D_1$ and since $[Y, X_i]$ is $\mathcal{K}$–typical, Remark 4.8 implies that $[Y, H]$ is $\mathcal{K}'$–typical for an enlargement $\mathcal{K}' \supset \mathcal{K}$ that only depends on $\mathcal{K}, C_2$. 
This proves the claim.

If we apply Proposition 4.11 to $K'$ and $\delta/2$, we obtain $C_3 = C_3(K', \delta)$ such that if $Y$ and $H$ are as above, $d(Y, L) = d(Y, H) > C_3$ and $[Y, H]$ is $K'$-typical, then

$$d(Y, X_i) - d(Y, X_j) \approx_{\delta/2} \beta(\zeta, X_i) - \beta(\zeta, X_j).$$

Note that by triangle inequality in $\triangle(Y, H, X_i)$,

$$d(Y, H) \geq d(Y, X_i) - d(X_i, H) > d(Y, X_i) - C_2,$$

so for $C = C_2 + C_3$,

$$d(Y, X_i) > C \implies d(Y, H) > C_3.$$

As a result, if $Y \in U_0^0(\mathcal{X}, \mathcal{K}) \setminus B(X_i, C)$ we have

$$d(Y, X_i) - d(Y, X_j) \approx_{\delta/2} d(\zeta, X_i) - d(\zeta, X_j)
\implies d(Y, X_i) - d(Y, X_j) < -\delta/2 < 0
\implies d(Y, X_i) < d(Y, X_j).$$

The second implication is because $\zeta \in U_0^0(\mathcal{X})$ implies $d(\zeta, X_i) - d(\zeta, X_j) < -\delta$. If $Y \in U_0^0(\mathcal{K}) \setminus B(X_j, C)$ as well, then by changing the role of $X_i$ and $X_j$ in the argument above we obtain $d(Y, X_j) < d(Y, X_i)$. This contradiction proves that the intersection mentioned in the lemma is empty.

Proof of the lower bound for Theorem 5.1. For an $\epsilon$-net $\mathcal{X}$, a compact set $\mathcal{K} \subset \mathcal{M}_g$ and $\delta > 0$, let $C = C(\epsilon, \mathcal{K}, \delta)$ be the constant given by Lemma 5.3. For every $i \in \mathbb{Z}$, define

$$U_i^0(\mathcal{X}, \mathcal{K}; C, R) = U_i^0(\mathcal{X}, \mathcal{K}) \cap B(X_i, R) \setminus B(X_i, C).$$

We claim that

$$\Pi_\gamma : U_i^0(\mathcal{X}, \mathcal{K}; C, R) \to \Gamma \cdot [P] \cap B(\bar{L}, R)$$

is an injection. To prove this claim, assume $\Pi_\gamma(X) = \Pi_\gamma(Y)$ for $X, Y \in U_i^0(\mathcal{X}, \mathcal{K}; C, R)$. This implies that there exists $k \in \mathbb{Z}$ such that $\gamma^k X = Y$, so $Y$ belongs to the intersection of $U_i^0(\mathcal{X}, \mathcal{K}; C, R)$ with $\gamma^k U_i^0(\mathcal{X}, \mathcal{K}; C, R) = U_{i+kN}^0(\mathcal{X}, \mathcal{K}; C, R)$. By the definition of $C$, we have $k = 0$, hence $X = Y$. This proves the claim.

A similar argument implies that $\Pi_\gamma(U_i^0(\mathcal{X}, \mathcal{K}; C, R))$ are disjoint for $0 \leq i < N(\mathcal{X})$, so

$$\sum_{i=0}^{N(\mathcal{X})-1} |U_i^0(\mathcal{X}, \mathcal{K}; C, R)| \leq |\Gamma \cdot [P] \cap B(\bar{L}, R)|.$$

Multiplying both sides by $e^{-hR}$ and taking lim inf as $R \to \infty$, we get

$$\sum_{i=0}^{N(\mathcal{X})-1} s_i^0(\mathcal{X}, \mathcal{K}) \leq \liminf_{R \to \infty} e^{-hR} |\Gamma \cdot [P] \cap B(\bar{L}, R)|, \quad (10)$$

where $s_i^0(\mathcal{X}, \mathcal{K})$ is defined by

$$s_i^0(\mathcal{X}, \mathcal{K}) = \liminf_{R \to \infty} e^{-hR} |U_i^0(\mathcal{K}) \cap B(X_i, R)|$$. 
Note that (10) is valid for every \( \epsilon \)-net \( \mathcal{X} \), compact set \( \mathcal{K} \subset \mathcal{M}_g \) and \( \delta > 0 \). Now, fixing \( \mathcal{X} \) and \( \delta \), we let the compact sets \( (\mathcal{K}_n)_{n \in \mathbb{N}} \subset \mathcal{M}_g \) form an exhaustion of \( \mathcal{M}_g \). This means that \( \mathcal{K}_n \subset \mathcal{K}_{n+1} \) and \( \mathcal{M}_g = \bigcup_{n \in \mathbb{N}} \mathcal{K}_n \). Then, by (6) and (7),

\[
s_n^\delta(\mathcal{X}, \mathcal{K}_n) \uparrow \frac{\Lambda^2}{h \text{Vol}(\mathcal{M}_g)} \nu(\mathcal{U}_n^\delta(\mathcal{X})) \quad \text{as} \quad n \to \infty,
\]

so we get

\[
\frac{\Lambda^2}{h \text{Vol}(\mathcal{M}_g)} \sum_{i=0}^{N(\mathcal{X})-1} \nu(\mathcal{U}_i^\delta(\mathcal{X})) \leq \liminf_{R \to \infty} e^{-hR} |\Gamma \cdot [P] \cap B(\mathcal{L}, R)|.
\]

Now, keeping the net \( \mathcal{X} \) fixed in the above expression and letting \( \delta \downarrow 0 \), we have \( \mathcal{U}_i^\delta(\mathcal{X}) \uparrow \mathcal{A}_i(\mathcal{X}) \), hence \( \nu(\mathcal{U}_i^\delta(\mathcal{X})) \uparrow \nu(\mathcal{A}_i(\mathcal{X})) = \nu(\mathcal{A}_i(\mathcal{X})) \). Thus,

\[
\frac{\Lambda^2}{h \text{Vol}(\mathcal{M}_g)} \sum_{i=0}^{N(\mathcal{X})-1} \nu(\mathcal{A}_i(\mathcal{X})) \leq \liminf_{R \to \infty} e^{-hR} |\Gamma \cdot [P] \cap B(\mathcal{L}, R)|.
\]

Finally, using Lemma 5.2 and making the \( \epsilon \)-net \( \mathcal{X} \) finer, i.e., letting \( \epsilon \to 0 \), proves the lower bound

\[
\frac{\Lambda^2}{h \text{Vol}(\mathcal{M}_g)} \nu(\mathcal{C}_{\text{Ext, } \gamma}) \leq \liminf_{R \to \infty} e^{-hR} |\Gamma \cdot [P] \cap B(\mathcal{L}, R)|.
\]

\( \square \)

For the upper bound we first prove:

**Lemma 5.4.** Let \( \gamma \) and \( \mathcal{L} \) be as before. Then, for every \( \kappa > 0 \), there exists a compact set \( \mathcal{K} \subset \mathcal{M}_g \), only depending on \( \kappa \), such that

\[
\limsup_{R \to \infty} e^{-hR} |\langle \gamma \rangle \setminus (\Gamma \cdot [P \cap B(\mathcal{L}, R)] \setminus \text{Typ}(\mathcal{L}, \mathcal{K}))| < \kappa.
\]

**Proof.** Let

\[
h : \langle \gamma \rangle \setminus (\Gamma \cdot [P \cap B(\mathcal{L}, R)]) \to \Gamma \cdot [P \cap B(O, R + L)]
\]

be the map defined in the proof of Theorem C given after the statement of Theorem 5.1. By (7), there exists \( \mathcal{K}' \subset \mathcal{M}_g \), only depending on \( O \) and \( \kappa \), such that

\[
e^{-hR} |\Gamma \cdot [P \cap B(O, R + L)] \setminus \text{Typ}(O, \mathcal{K}')] < \kappa
\]

(11) for \( R \) large enough. Let \( [Y] \in \langle \gamma \rangle \setminus (\Gamma \cdot [P \cap B(\mathcal{L}, R)] \) and recall that \( h \) sends \( [Y] \) to \( \gamma^k Y \) such that \( d(O, \gamma^k H) < L \) for some \( H \in \text{proj}_{\mathcal{L}} Y \). Since \( d(O, \gamma^k H) < L \), by Remark 4.8 there exists an enlargement \( \mathcal{K} \supset \mathcal{K}', \) only depending on \( \mathcal{K}' \) and \( L \), such that \( [\gamma^k Y, O] \mathcal{K}' \)-typical implies \( [\gamma^k Y, \gamma^k H] \) is \( \mathcal{K} \)-typical. Thus, \( h \) sends

\[
\langle \gamma \rangle \setminus (\Gamma \cdot [P \cap B(\mathcal{L}, R)] \setminus \text{Typ}(\mathcal{L}, \mathcal{K})) \quad \text{to} \quad \Gamma \cdot [P \cap B(O, R + L)] \setminus \text{Typ}(O, \mathcal{K}').
\]

The injectivity of \( h \) and (11) then concludes the proof.

\( \square \)
Lemma 5.5. Let the axis $\mathcal{L}$ and the point $P \in T_g$ be as before, and assume $\mathcal{X} = (X_i)_{i \in \mathbb{Z}}$ is an $\epsilon$-net. Then, for every compact set $\mathcal{K} \subset \mathcal{M}_g$ and $\delta > 0$, there exists $C = C(\epsilon, \mathcal{K}, \delta)$ such that the following holds: if
\[
Y \in \Gamma \cdot P \cap B(\mathcal{L}, R) \cap \text{Typ}(\mathcal{L}, \mathcal{K}), \quad d(Y, \mathcal{L}) > C,
\]
and $i_0 \in \mathbb{Z}$ is such that $d(Y, X_{i_0}) = \inf_{i \in \mathbb{Z}} d(Y, X_i)$, then
\[
Y \in \mathcal{V}_{i_0}^{\delta}(\mathcal{X}) \cap B(X_{i_0}, R + \epsilon).
\]

Proof. Let
\[
Y \in \Gamma \cdot P \cap B(\mathcal{L}, R) \cap \text{Typ}(\mathcal{L}, \mathcal{K}).
\]
Note that $\inf_{i \in \mathbb{Z}} d(Y, X_i)$ is attained for some $i \in \mathbb{Z}$ because of the shape of $d(Y, X)$ as a function of $X \in \mathcal{L}$, described in Corollary 4.4. Let the geodesic from $X_{i_0}$ to $Y$ hit the boundary at $[\zeta]$, i.e., $Y \in [X_{i_0}, \zeta]$. By Proposition 4.11, there exists $C = C(\mathcal{K}, \delta)$ such that if $d(Y, \mathcal{L}) > C$, then for all $i \in \mathbb{Z}$ we have
\[
\beta(\zeta, X_i) - \beta(\zeta, X_{i_0}) \simeq_{\delta} d(Y, X_i) - d(Y, X_{i_0})
\]
which implies
\[
\implies \beta(\zeta, X_i) - \beta(\zeta, X_{i_0}) > d(Y, X_i) - d(Y, X_{i_0}) - \delta \geq -\delta
\]
implies
\[
\implies \beta(\zeta, X_{i_0}) < \beta(\zeta, X_i) + \delta
\]
implies
\[
\implies \zeta \in \mathcal{V}_{i_0}^{\delta}(\mathcal{X})
\]
Let $H \in \text{proj}_\mathcal{L} Y$ and note that since $Y \in B(\mathcal{L}, R)$, we have $d(Y, H) \leq R$. Choose $X_{i_1}$ to be the point of $\mathcal{X}$ that is closest to $H$. Since the mesh of $\mathcal{X}$ is less than $\epsilon$, we have $d(H, X_{i_1}) < \epsilon$, hence by triangle inequality $d(Y, X_{i_1}) < R + \epsilon$. So
\[
d(Y, X_{i_0}) = \inf_{j \in \mathbb{Z}} d(Y, X_j) < R + \epsilon.
\]
This proves the lemma.

Proof of the upper bound for Theorem 5.1. For an $\epsilon$-net $\mathcal{X}$, a compact set $\mathcal{K} \subset \mathcal{M}_g$ and $\delta > 0$, let $C = C(\epsilon, \mathcal{K}, \delta)$ be the constant given by Lemma 5.5 and for $R > C$, write
\[
\Gamma \cdot P \cap B(\mathcal{L}, R) = B_1(\mathcal{K}; C, R) \cup B_2(\mathcal{K}, R) \cup B_3(C),
\]
where
\[
B_1(\mathcal{K}; C, R) = \Gamma \cdot P \cap B(\mathcal{L}, R) \cap \text{Typ}(\mathcal{L}, \mathcal{K}) \setminus B(\mathcal{L}, C);
\]
\[
B_2(\mathcal{K}, R) = \Gamma \cdot P \cap B(\mathcal{L}, R) \setminus \text{Typ}(\mathcal{L}, \mathcal{K});
\]
\[
B_3(C) = \Gamma \cdot P \cap B(\mathcal{L}, C).
\]
Note that $B_1(\mathcal{K}, C, R) \setminus B_2(\mathcal{K}, R)$ is $\gamma$-invariant, hence we can form the quotient $\overline{B}_1(\mathcal{K}, C, R) = \langle \gamma \rangle \setminus B_1(\mathcal{K}, C, R)$ and $\overline{B}_2(\mathcal{K}, R) = \langle \gamma \rangle \setminus B_2(\mathcal{K}, R)$, $\overline{B}_3(C) = \langle \gamma \rangle \setminus B_3(C)$, and obtain
\[
|\langle \gamma \rangle \setminus (\Gamma \cdot P \cap B(\mathcal{L}, R))| \leq \overline{B}_1(\mathcal{K}, C, R) + \overline{B}_2(\mathcal{K}, R) + \overline{B}_3(C).
\]
Since $|\overline{B}_3(C)|$ is a constant only depending on $C$,
\[
\limsup_{R \to \infty} e^{-\gamma R} |\overline{B}_3(C)| \to 0.
\]
To control $|B_2(\mathcal{K}, R)|$, we define

$$
\kappa(\mathcal{K}) = \limsup_{R \to \infty} e^{-hR} |B_2(\mathcal{K}, R)|.
$$

To find an upper bound for $B_1(\mathcal{K}, C, R)$, note that by Lemma 5.5,

$$
B_1(\mathcal{K}, C, R) \subset \bigcup_{i \in \mathbb{Z}} (V^\delta_i(\mathcal{X}) \cap B(X_i, R + \epsilon))
$$

implies

$$
|B_1(\mathcal{K}, C, R)| \leq \sum_{i=0}^{N(\mathcal{X})-1} |V^\delta_i(\mathcal{X}) \cap B(X_i, R + \epsilon)|.
$$

Multiplying both sides of (12) by $e^{-hR}$ and taking the lim sup as $R \to \infty$ (while keeping $\mathcal{X}$, $\mathcal{K}$ and $\delta$ fixed), we obtain

$$
\limsup_{R \to \infty} e^{-hR} |\langle \gamma \rangle \setminus \Gamma \cdot P \cap B(\mathcal{L}, R)| \leq \sum_{i=0}^{N(\mathcal{X})-1} \lim_{R \to \infty} e^{-hR} |V^\delta_i \cap B(X_i, R + \epsilon)| + \kappa(\mathcal{K})
$$

$$
= \sum_{i=0}^{N(\mathcal{X})-1} \frac{\Lambda^2}{h \Vol(M_g)} e^{he} \nu(V^\delta_i(\mathcal{X})) + \kappa(\mathcal{K}),
$$

where the equality is by (6). Note that the above is valid for every $\epsilon$-net $\mathcal{X}$, compact set $\mathcal{K} \subset M_g$ and $\delta > 0$. Keeping $\mathcal{X}$ and $\delta$ fixed, we let $\mathcal{K}_n \uparrow M_g$ to be an exhaustion of $M_g$. By Lemma 5.4, $\kappa(\mathcal{K}_n) \downarrow 0$ as $n \to \infty$, thus

$$
\limsup_{R \to \infty} e^{-hR} |\langle \gamma \rangle \setminus \Gamma \cdot P \cap B(\mathcal{L}, R)| \leq \frac{\Lambda^2}{h \Vol(M_g)} e^{he} \sum_{i=0}^{N(\mathcal{X})-1} \nu(V^\delta_i(\mathcal{X})).
$$

As in the proof of the lower bound, we first let $\delta \downarrow 0$ and use $\bigcap_i V^\delta_i(\mathcal{X}) = \mathcal{A}_i(\mathcal{X})$, then let $\epsilon \downarrow 0$ and use Lemma 5.2 to obtain

$$
\limsup_{R \to \infty} e^{-hR} |\langle \gamma \rangle \setminus \Gamma \cdot P \cap B(\mathcal{L}, R)| \leq \frac{\Lambda^2}{h \Vol(M_g)} \nu(C_{\Ext, \gamma}).
$$

\[\square\]

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