On a Diophantine equation with five prime variables

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Abstract: Let \([x]\) denote the integral part of the real number \(x\), and \(N\) be a sufficiently large integer. In this paper, it is proved that, for \(1 < c < \frac{1121682}{5471123}, c \neq 2\), the Diophantine equation \(N = [p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c]\) is solvable in prime variables \(p_1, p_2, p_3, p_4, p_5\).

Keywords: Diophantine equation; prime number; exponential sum

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1 Introduction and main result

Let \([x]\) be the integral part of the real number \(x\). In 1933–1934, Segal [15, 16] first considered the Waring’s problem with non–integer degrees, who showed that for any sufficiently large integer \(N\) and \(c > 1\) being not an integer, there exists an integer \(k_0 = k_0(c) > 0\) such that the equation

\[N = [x_1^c] + [x_2^c] + \cdots + [x_k^c]\]

is solvable for \(k \geq k_0(c)\). Later, Segal’s bound for \(k_0(c)\) was improved by Deshouillers [4] and by Arkhilev and Zhitkov [1], respectively. Let \(G(c)\) be the least of the integers \(k_0(c)\) such that every sufficiently large integer \(N\) can be written as a sum of not more than \(k_0(c)\) numbers with the form \([n^c]\). In particular, Deshouillers [5] and Gritsenko [8] considered the case \(k = 2\) and gave \(G(c) = 2\) for \(1 < c < \frac{4}{3}\) and \(1 < c < \frac{55}{41}\), respectively.

In 1937, Vinogradov [19] solved asymptotic form of the ternary Goldbach problem. He proved that, for sufficiently large integer \(N\) satisfying \(N \equiv 1 \pmod{2}\), the following equation

\[N = p_1 + p_2 + p_3\]

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is solvable in primes $p_1, p_2, p_3$. As an analogue of the ternary Goldbach problem, in 1995, Laporta and Tolev [13] investigated the solvability of the following equation

$$N = [p_1^c] + [p_2^c] + [p_3^c]$$

in prime variables $p_1, p_2, p_3$. Define

$$R_s(N) = \sum_{N = [p_1^c] + [p_2^c] + \cdots + [p_s^c]} (\log p_1)(\log p_2)\cdots(\log p_s).$$

Laporta and Tolev [13] showed that the sum $R_3(N)$ has asymptotic formula for $1 < c < 17/16$ and gave

$$R_3(N) = \Gamma_3\left(1 + \frac{1}{c}\right)\Gamma\left(\frac{3}{c}\right)N^{3/c-1} + O\left(N^{3/c-1} \exp\left(-\frac{\log N}{3}\right)\right).$$

for any $0 < \delta < 1/3$. Later, Kumchev and Nedeva [12] improved the result of Laporta and Tolev [13], and enlarged the range of $c$ to $12/11$. Afterwards, Zhai and Cao [20] refined the result of Kumchev and Nedeva [12], who extended the range of $c$ to $258/235$. In 2018, Cai [3] enhanced the result of Zhai and Cao [20] and gave the upper bound of $c$ as $137/119$.

In 1938, Hua [10] proved that every sufficiently large integer $N$, which satisfies $N \equiv 5 \pmod{24}$, can be represented as five squares of primes, i.e.,

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2.$$

In this paper, as an analogue of Hua’s five square theorem, we shall investigate the solvability of the following Diophantine equation

$$N = [p_1^c] + [p_2^c] + [p_3^c] + [p_4^c] + [p_5^c]$$

in prime variables $p_1, p_2, p_3, p_4, p_5$, and devote to establish the following result.

**Theorem 1.1** Let $1 < c < \frac{11216182}{5471123}, c \neq 2$, and $N$ be a sufficiently large integer. Then we have

$$R_5(N) = \frac{\Gamma_5\left(1 + \frac{1}{c}\right)}{\Gamma\left(\frac{5}{c}\right)}N^{5/c-1} + O\left(N^{5/c-1} \exp\left(-\frac{\log N}{4}\right)\right),$$

where the implied constant in the $O$–term depends only on $c$.

**Notation.** Throughout this paper, we suppose that $1 < c < \frac{11216182}{5471123}, c \neq 2$. Let $p$, with or without subscripts, always denote a prime number; $\varepsilon$ always denote arbitrary small positive constant, which may not be the same at different occurrences. As usual,
we use \([x], \{x\} \) and \(\|x\|\) to denote the integral part of \(x\), the fractional part of \(x\) and the distance from \(x\) to the nearest integer, respectively. Also, we write \(e(x) = e^{2\pi ix}\); \(f(x) \ll g(x)\) means that \(f(x) = O(g(x))\); \(f(x) \asymp g(x)\) means that \(f(x) \ll g(x) \ll f(x)\).

We also define

\[
P = N^{1/c}, \quad \tau = P^{1-c-\varepsilon}, \quad S(\alpha) = \sum_{p \leq P} (\log p) e(\lfloor pf\rfloor \alpha),
\]

\[
S^*(\alpha) = \sum_{p \leq P} (\log p) e(p^c \alpha), \quad S(\alpha, X) = \sum_{X < p \leq 2X} (\log p) e(\lfloor p^c \rfloor \alpha),
\]

\[
S^*(\alpha, X) = \sum_{X < p \leq 2X} (\log p) e(p^c \alpha), \quad T(\alpha, X) = \sum_{X < n \leq 2X} e(\lfloor nc\rfloor \alpha).
\]

## 2 Preliminary Lemmas

In this section, we shall state some preliminary lemmas, which are required in the proof of Theorem 1.1.

**Lemma 2.1** Let \(f(x)\) be a real differentiable function in the interval \([a, b]\). If \(f'(x)\) is monotonic and satisfies \(|f'(x)| \leq \theta < 1\). Then we have

\[
\sum_{a < n \leq b} e^{2\pi if(n)} = \int_a^b e^{2\pi if(x)} \, dx + O(1).
\]

**Proof.** See Lemma 4.8 of Titchmarsh [17].

**Lemma 2.2** Let \(L, Q \geq 1\) and \(z_\ell\) be complex numbers. Then we have

\[
\left| \sum_{L < \ell \leq 2L} z_\ell \right|^2 \leq \left(2 + \frac{L}{Q}\right) \sum_{|q| < Q} \left(1 - \frac{|q|}{Q}\right) \sum_{L < \ell + q, \ell - q \leq 2L} z_{\ell + q} z_{\ell - q}.
\]

**Proof.** See Lemma 2 of Fouvry and Iwaniec [6].

**Lemma 2.3** Suppose that \(f(x) : [a, b] \to \mathbb{R}\) has continuous derivatives of arbitrary order on \([a, b]\), where \(1 \leq a < b \leq 2a\). Suppose further that

\[
|f^{(j)}(x)| \asymp \lambda_1 a^{1-j}, \quad j \geq 1, \quad x \in [a, b].
\]

Then for any exponential pair \((\kappa, \lambda)\), we have

\[
\sum_{a < n \leq b} e(f(n)) \ll \lambda_1^\kappa a^\lambda + \lambda_1^{-1}.
\]

**Proof.** See (3.3.4) of Graham and Kolesnik [7].
Lemma 2.4 Let $x$ be not an integer, $\alpha \in (0, 1)$, $H \geq 3$. Then we have

$$e(-\alpha \{x\}) = \sum_{|h| \leq H} c_h(\alpha)e(hx) + O\left(\min\left(1, \frac{1}{H\|x\|}\right)\right),$$

where

$$c_h(\alpha) = \frac{1 - e(-\alpha)}{2\pi i(h + \alpha)}.$$

Proof. See Lemma 12 of Buriev [2] or Lemma 3 of Kumchev and Nedeva [12]. □

Lemma 2.5 Suppose $Y > 1$, $\gamma > 0$, $c > 1$, $c \notin \mathbb{Z}$. Let $\mathcal{A}(Y; c, \gamma)$ denote the number of solutions of the inequality

$$|n_1^c + n_2^c - n_3^c - n_4^c| < \gamma, \quad Y < n_1, n_2, n_3, n_4 \leq 2Y,$

then

$$\mathcal{A}(Y; c, \gamma) \ll (\gamma Y^{4-c} + Y^2)Y^\varepsilon.$$

Proof. See Theorem 2 of Robert and Sargos [14]. □

Lemma 2.6 For $1 < c < 3$, $c \neq 2$, we have

$$\int_0^1 |S(\alpha)|^4 d\alpha \ll (P^{4-c} + P^2)P^\varepsilon.$$

Proof. By a splitting argument, it is sufficient to show that

$$\int_0^1 |S(\alpha, P/2)|^4 d\alpha \ll (P^{4-c} + P^2)P^\varepsilon.$$

Trivially, we have

$$\int_0^1 |S(\alpha, P/2)|^4 d\alpha = \sum_{P/2 < p_1, p_2, p_3, p_4 \leq P} (\log p_1) \cdots (\log p_4) \int_0^1 e\left(\left([p_1^c] + [p_2^c] - [p_3^c] - [p_4^c]\right)\alpha\right) d\alpha \ll (\log P)^4 \sum_{P/2 < n_1, n_2, n_3, n_4 \leq P} \frac{1}{[n_1^c] + [n_2^c] = [n_3^c] + [n_4^c]}.$$

On the other hand, if $[n_1^c] + [n_2^c] = [n_3^c] + [n_4^c]$, we can deduce that

$$|n_1^c + n_2^c - n_3^c - n_4^c| = |\{n_1^c\} + \{n_2^c\} - \{n_3^c\} - \{n_4^c\}| \leq 2.$$

By Lemma 2.5, we derive that

$$\int_0^1 |S(\alpha, P/2)|^4 d\alpha \ll (\log P)^4 \cdot \mathcal{A}(P/2; c, 2) \ll (P^{4-c} + P^2)P^\varepsilon,$$

which completes the proof of Lemma 2.6. □
Lemma 2.7 For \(1 < c < 3, c \neq 2\), we have
\[
\int_{-\tau}^{\tau} |S^*(\alpha)|^4 d\alpha \ll P^{4-c} \log^6 P.
\]

Proof. By a splitting argument, it is sufficient to show that
\[
\int_{-\tau}^{\tau} |S^*(\alpha, P/2)|^4 d\alpha \ll P^{4-c} \log^5 P.
\] (2.1)

We have
\[
\int_{-\tau}^{\tau} |S^*(\alpha, P/2)|^4 d\alpha
= \sum_{P/2 < p_1, p_2, p_3, p_4 \leq P} (\log p_1) \cdots (\log p_4) \int_{-\tau}^{\tau} e((p_1^c + p_2^c - p_3^c - p_4^c)\alpha) d\alpha
\ll \sum_{P/2 < p_1, p_2, p_3, p_4 \leq P} (\log p_1) \cdots (\log p_4) \min \left(\tau, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|}\right)
\ll \mathcal{U} \tau \log^4 P + \mathcal{V} \log^4 P,
\] (2.2)

where
\[
\mathcal{U} = \sum_{P/2 < n_1, n_2, n_3, n_4 \leq P} 1, \quad \mathcal{V} = \sum_{P/2 < n_1, n_2, n_3, n_4 \leq P} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|}.
\]

We have
\[
\mathcal{U} \ll \sum_{P/2 < n_1 \leq P} \sum_{P/2 < n_2 \leq P} \sum_{P/2 < n_3 \leq P} \sum_{P/2 < n_4 \leq P} \frac{1}{(n_1^c + n_2^c - n_3^c - n_4^c)^{1/c} \in \mathbb{P}^c}
\ll \sum_{P/2 < n_1, n_2, n_3 \leq P} \left(1 + \left(n_1^c + n_2^c - n_3^c + 1/\tau\right)^{1/2} - \left(n_1^c + n_2^c - n_3^c - 1/\tau\right)^{1/2}\right),
\]
and by the mean–value theorem
\[
\mathcal{U} \ll P^3 + \frac{1}{\tau} P^{4-c}.
\] (2.3)

Obviously, \(\mathcal{V} \leq \sum_{\ell} \mathcal{V}_\ell\), where
\[
\mathcal{V}_\ell = \sum_{P/2 < n_1, n_2, n_3, n_4 \leq P} \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|}.
\] (2.4)

and \(\ell\) takes the values \(\frac{2k}{\tau}\), \(k = 0, 1, 2, \ldots\), with \(\ell \ll P^c\). Then, we derive that
\[
\mathcal{V}_\ell \ll \frac{1}{\ell} \sum_{P/2 < n_1, n_2, n_3, n_4 \leq P} \frac{1}{(n_1^c + n_2^c + n_3^c + n_4^c)^{1/c} \in \mathbb{P}^c}
\]
\[
\ll \frac{1}{\ell} \sum_{P/2 < n_1, n_2, n_3, n_4 \leq P} \frac{1}{(n_1^c + n_2^c + n_3^c + n_4^c)^{1/c} \in \mathbb{P}^c}.
\]
For $\ell \geq \frac{1}{c}$ and $P/2 < n_1, n_2, n_3 \leq P$ with $n_1^c + n_2^c - n_3^c \asymp P^c$, it is easy to see that
\[
(n_1^c + n_2^c - n_3^c + 2\ell)^{1/c} - (n_1^c + n_2^c - n_3^c + \ell)^{1/c} > 1.
\]
Hence, by the mean–value theorem, we get
\[
\mathcal{V}_\ell \ll \frac{1}{\ell} \sum_{P/2 < n_1, n_2, n_3 \leq P} \left( (n_1^c + n_2^c - n_3^c + 2\ell)^{1/c} - (n_1^c + n_2^c - n_3^c + \ell)^{1/c} \right) \ll P^{4-c}. \quad (2.5)
\]
Combining (2.2)–(2.5), we obtain the desired estimate (2.1), which completes the proof of Lemma 2.7.

**Lemma 2.8** Let $3 < U < V < Z < X$ and suppose that $Z - \frac{1}{2} \in \mathbb{N}$, $X \geq 64Z^2U$, $Z \geq 4U^2$, $V^3 \geq 32X$. Assume further that $F(n)$ is a complex–valued function such that $|F(n)| \leq 1$. Then the sum
\[
\sum_{X < n \leq 2X} \Lambda(n) F(n)
\]
may be decomposed into $O(\log^{10} X)$ sums, each of which either of Type I:
\[
\sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} F(mk)
\]
with $K \gg Z$, where $a(m) \ll m^\varepsilon$, $MK \asymp X$, or of Type II:
\[
\sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} b(k) F(mk)
\]
with $U \ll M \ll V$, where $a(m) \ll m^\varepsilon$, $b(k) \ll k^\varepsilon$, $MK \asymp X$.

**Proof.** See Lemma 3 of Heath–Brown [9].

**Lemma 2.9** For any $\varepsilon > 0$, the pair $\left( \frac{32}{205} + \varepsilon, \frac{269}{410} + \varepsilon \right)$ is an exponential pair.

**Proof.** See the Corollary of Theorem 1 of Huxley [11].

**Lemma 2.10** For any real number $\theta$, there holds
\[
\min \left( 1, \frac{1}{H\|\theta\|} \right) = \sum_{h=-\infty}^{+\infty} a_h e(h\theta),
\]
where
\[
a_h \ll \min \left( \frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{h^2} \right).
\]
**Proof.** See p.245 of Heath–Brown [9].
Lemma 2.11 Let \(1 < c < \frac{11216182}{5471123}\), \(c \neq 2, P^\frac{5}{6} \ll X \ll P, H = X^{\frac{1414}{781589}}\) and \(c_h(\alpha)\) denote complex numbers such that \(|c_h(\alpha)| \ll (1 + |h|)^{-1}\). Then, for any \(\alpha \in (\tau, 1 - \tau)\), if \(M \ll X^{\frac{204417}{1563178}}\), we have

\[
S_I(\alpha) := \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} e((h + \alpha)(mk)^c) \ll X^{\frac{770175}{781589} \epsilon},
\]

where \(a(m) \ll m^\epsilon\) and \(MK \asymp X\).

Proof. Obviously, we have

\[
|S_I(\alpha)| \ll X^\epsilon \max_{|\xi| \in (\tau, H+1)} \sum_{M < m \leq 2M} \left| \sum_{K < k \leq 2K} e(\xi(mk)^c) \right|.
\]

Then we use Lemma 2.3 to estimate the inner sum over \(k\) in (2.6) with exponential pair \((\kappa, \lambda)\) and derive that

\[
S_I(\alpha) \ll X^\epsilon \max_{|\xi| \in (\tau, H+1)} \sum_{M < m \leq 2M} \left( |\xi|^\kappa X^\kappa K^{-1})^\kappa K^\lambda + \frac{K}{|\xi| X^\epsilon} \right)
\]

\[
 \ll X^\epsilon \max_{|\xi| \in (\tau, H+1)} \left( |\xi|^\kappa X^\kappa K^{-1})^\kappa K^\lambda + \frac{MK}{|\xi| X^\epsilon} \right)
\]

\[
 \ll X^\epsilon (H^\kappa X^\kappa + \lambda^{-1} K^\kappa + 1 - \epsilon - 1),
\]

From Lemma 2.9, by taking

\[
(\kappa, \lambda) = A^3BABABABABABABA \left( \frac{32}{205} + \frac{269}{410} + \epsilon \right)
\]

\[
= \left( \frac{13643}{643770} + \epsilon, \frac{580013}{643770} + \epsilon \right),
\]

we can see that, if \(M \ll X^{\frac{204417}{1563178}}\), then there holds

\[
S_I(\alpha) \ll X^{\frac{770175}{781589} \epsilon},
\]

which completes the proof of Lemma 2.11.

Lemma 2.12 Let \(1 < c < \frac{11216182}{5471123}\), \(c \neq 2, P^\frac{5}{6} \ll X \ll P, H = X^{\frac{1414}{781589}}\) and \(c_h(\alpha)\) denote complex numbers such that \(|c_h(\alpha)| \ll (1 + |h|)^{-1}\). Then, for any \(\alpha \in (\tau, 1 - \tau)\), if there holds \(X^{\frac{22828}{781589}} \ll M \ll X^{\frac{3998717}{10942246}}\), then we have

\[
S_{II}(\alpha) := \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} b(k) e((h + \alpha)(mk)^c) \ll X^{\frac{770175}{781589} \epsilon},
\]

where \(a(m) \ll m^\epsilon\), \(b(k) \ll k^\epsilon\) and \(MK \asymp X\).
Proof. Let $Q = X^{\frac{235}{233-AB} \log X}^{-1}$. From Lemma 2.2 and Cauchy’s inequality, we derive that

$$\left| S_{II}(\alpha) \right| \ll \sum_{|h| \leq H} |c_h(\alpha)| \sum_{K < k \leq 2K} b(k) \sum_{M < m \leq 2M} a(m)e((h + \alpha)(mk)^c)$$

$$\ll \sum_{|h| \leq H} |c_h(\alpha)| \left( \sum_{K < k \leq 2K} |b(k)| \right)^\frac{1}{2} \left( \sum_{K < k \leq 2K} \sum_{M < m \leq 2M} a(m)e((h + \alpha)(mk)^c) \right)^\frac{1}{2}$$

$$\ll K^{\frac{1}{2}+\varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left( \sum_{K < k \leq 2K} \frac{M}{Q} \sum_{0 \leq q < Q} \left( 1 - \frac{q}{Q} \right) \right) \times \sum_{M + q < m \leq 2M - q} a(m + q) a(m - q) e((h + \alpha)n^c \Delta_c(m, q)) \right)^\frac{1}{2}$$

$$\ll K^{\frac{1}{2}+\varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q < Q} \sum_{M < m \leq 2M} \sum_{K < k \leq 2K} e((h + \alpha)n^c \Delta_c(m, q)) \right)^\frac{1}{2}$$

(2.7)

where $\Delta_c(m, q) = (m + q)^c - (m - q)^c$. Thus, it is sufficient to estimate the sum

$$S_0 := \sum_{K < k \leq 2K} e((h + \alpha)n^c \Delta_c(m, q)).$$

By Lemma 2.3 with the exponential pair $(\kappa, \lambda) = AB(0, 1) = (\frac{1}{6}, \frac{2}{3})$, we have

$$S_0 \ll (|h + \alpha|X^{c-1}q)^\frac{1}{6}K^{\frac{2}{3}} + \frac{1}{|h + \alpha|X^{c-1}q}.$$

Putting the above estimate into (2.7), we obtain that

$$S_{II}(\alpha) \ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq q < Q} \sum_{M < m \leq 2M} \right)$$

$$\times \left( (|h + \alpha|X^{c-1}q)^\frac{1}{6}K^{\frac{2}{3}} + \frac{1}{|h + \alpha|X^{c-1}q} \right)^\frac{1}{2}$$

$$\ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{X^2}{Q} + \frac{X}{Q} \left( H^{\frac{1}{6}}X^{\frac{1}{2}(c-1)}MK^{\frac{2}{3}}Q^{\frac{2}{3}} + X^{1-c}M^{-1}Q^{\frac{1}{2}} \log Q \right) \right)^\frac{1}{2}$$

$$\ll X^{1+\varepsilon}Q^{-\frac{1}{2}} \sum_{|h| \leq H} |c_h(\alpha)| \ll X^{1+\varepsilon}Q^{-\frac{1}{2}} \sum_{|h| \leq H} \frac{1}{|h|} \ll X^{\frac{235}{233-AB}+\varepsilon},$$

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which completes the proof of Lemma 2.12.

Lemma 2.13 For $\alpha \in (\tau, 1-\tau)$, there holds

$$S(\alpha) \ll P^{\frac{770175}{471589}+\varepsilon}.$$  

Proof. First, we have

$$S(\alpha) = U(\alpha) + O(P^{1/2}),$$

where

$$U(\alpha) = \sum_{n \leq P} \Lambda(n)e([n^c]\alpha).$$

By a splitting argument, it is sufficient to prove that, for $P^{5/6} \ll X \ll P$ and $\alpha \in (\tau, 1-\tau)$, there holds

$$U^*(\alpha) := \sum_{X < n \leq 2X} \Lambda(n)e([n^c]\alpha) \ll X^{\frac{770175}{471589}+\varepsilon}.  \tag{2.8}$$

By Lemma 2.4 with $H = X^{\frac{11414}{781589}}$, we have

$$U^*(\alpha) = \sum_{X < n \leq 2X} \Lambda(n)e(n^c\alpha - \{n^c\} \alpha) = \sum_{X < n \leq 2X} \Lambda(n)e(n^c\alpha) - \{n^c\} \alpha)$$

$$= \sum_{X < n \leq 2X} \Lambda(n)e(n^c\alpha) \left( \sum_{|h| \leq H} c_h(\alpha)e(hn^c) + O\left( \min\left(1, \frac{1}{H||n^c||}\right) \right) \right)$$

$$= \sum_{|h| \leq H} c_h(\alpha) \sum_{X < n \leq 2X} \Lambda(n)e((h + \alpha)n^c) + O\left( \log X \cdot \sum_{X < n \leq 2X} \min\left(1, \frac{1}{H||n^c||}\right) \right).$$

By Lemma 2.10 and Lemma 2.3 with the exponential pair $(\kappa, \lambda) = AB(0, 1) = (\frac{1}{6}, \frac{2}{3})$, we derive that

$$\sum_{X < n \leq 2X} \min\left(1, \frac{1}{H||n^c||}\right)$$

$$= \sum_{X < n \leq 2X} \sum_{\ell = -\infty}^{+\infty} a_{\ell}e(\ell n^c) \ll \sum_{\ell = -\infty}^{+\infty} |a_{\ell}| \sum_{X < n \leq 2X} e(\ell n^c)$$

$$\ll \frac{X \log 2H}{H} + \sum_{1 \leq \ell \leq H} \frac{1}{\ell} \sum_{X < n \leq 2X} e(\ell n^c) + \sum_{\ell > H} \frac{H}{\ell^2} \sum_{X < n \leq 2X} e(\ell n^c)$$

$$\ll \frac{X \log 2H}{H} + \sum_{1 \leq \ell \leq H} \frac{1}{\ell} (X^{c-1} \ell)^{\frac{1}{6}} X^{\frac{1}{2}} + \frac{1}{\ell \ell^c-1}) + \sum_{\ell > H} \frac{H}{\ell^2} (X^{c-1} \ell)^{\frac{1}{6}} X^{\frac{1}{2}} + \frac{1}{\ell \ell^c-1})$$

$$\ll X^{\frac{770175}{471589}} \log X + H^{\frac{1}{3}} X^{\frac{c}{2}} + X^{1-c} \ll X^{\frac{770175}{471589}} \log X.  \tag{2.9}$$
Taking $U = X^{\log^{10} X}$, $V = X^{\log_{10}^{12} X}$, and $Z = [X^{\log_{10}^{18} X}]^2 \pm \frac{1}{2}$ in Lemma 2.8, it is easy to see that the sum
\[
\sum_{|h| \leq H} c_h(\alpha) \sum_{X < n \leq 2X} \Lambda(n)e((h + \alpha)n^c)
\]
can be represented as $O(\log^{10} X)$ sums, each of which either of Type I
\[
S_I(\alpha) = \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} e((h + \alpha)(mk)^c)
\]
with $K \gg Z, a(m) \ll m^\epsilon, MK \asymp X$, or of Type II
\[
S_{II}(\alpha) = \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq 2M} a(m) \sum_{K < k \leq 2K} b(k)e((h + \alpha)(mk)^c)
\]
with $U \ll M \ll V, a(m) \ll m^\epsilon, b(k) \ll k^\epsilon, MK \asymp X$. For the Type I sums, by noting the fact that $K \gg Z$ and $MK \asymp X$, we deduce that $M \ll X^{\log_{10}^{12} X}$. From Lemma 2.11, we have $S_I(\alpha) \ll X^{770175 + \epsilon}$. For the Type II sums, by Lemma 2.12, we have $S_{II}(\alpha) \ll X^{770175 + \epsilon}$. Therefore, we conclude that
\[
\sum_{|h| \leq H} c_h(\alpha) \sum_{X < n \leq 2X} \Lambda(n)e((h + \alpha)n^c) \ll X^{770175 + \epsilon}.
\]
(2.10)

From (2.8)–(2.10), we complete the proof of Lemma 2.13.

Lemma 2.14 For $\alpha \in (0, 1), c \notin \mathbb{Z}$, we have
\[
T(\alpha, X) \ll X^{\frac{c+1}{2}} \log X + \frac{1}{\alpha X^{c-1}}.
\]

Proof. Taking $H_1 = X^{\frac{2-c}{2}}$, and by Lemma 2.4, we deduce that
\[
T(\alpha, X) = \sum_{X < n \leq 2X} e((\alpha n^c - \{n^c\})\alpha)
= \sum_{X < n \leq 2X} e(\alpha n^c) \left( \sum_{|h| \leq H_1} c_h(\alpha)e(hn^c) + O\left( \min\left(1, \frac{1}{H_1\|n^c\|}\right) \right) \right)
= \sum_{|h| \leq H_1} c_h(\alpha) \sum_{X < n \leq 2X} e((h + \alpha)n^c) + O\left( \sum_{X < n \leq 2X} \min\left(1, \frac{1}{H_1\|n^c\|}\right) \right).
\]
(2.11)

From Lemma 2.10, we get
\[
\sum_{X < n \leq 2X} \min\left(1, \frac{1}{H_1\|n^c\|}\right) = \sum_{X < n \leq 2X} \sum_{k=\infty}^{\infty} a_k e(kn^c) \ll \sum_{k=-\infty}^{\infty} |a_k| \left| \sum_{X < n \leq 2X} e(kn^c) \right|.
\]
(2.12)
Then we shall use Lemma 2.3 with the exponential pair \((\kappa, \lambda) = AB(0, 1) = (\frac{1}{6}, \frac{2}{3})\) to estimate the sum over \(n\) on the right-hand side in (2.12), and derive that

\[
\sum_{X < n \leq 2X} \min \left(1, \frac{1}{H_1 \|n^c\|}\right) \ll \frac{X \log 2H_1}{H_1} + \sum_{1 \leq k \leq H_1} \frac{1}{k} \left| \sum_{X < n \leq 2X} e(kn^c) \right| + \sum_{k > H_1} \frac{H_1}{k^2} \left| \sum_{X < n \leq 2X} e(kn^c) \right|
\]

\[
\ll \frac{X \log 2H_1}{H_1} + \sum_{1 \leq k \leq H_1} \frac{1}{k} \left( (X^{c-1}k)^{\frac{2}{3}}X^{\frac{2}{3}} + \frac{1}{kX^{c-1}} \right)
\]

\[
+ \sum_{k > H_1} \frac{H_1}{k^2} \left( (X^{c-1}k)^{\frac{2}{3}}X^{\frac{2}{3}} + \frac{1}{kX^{c-1}} \right)
\]

\[
\ll X^{\frac{c+1}{4}} \log X + H_1^{\frac{2}{3}}X^{\frac{2c}{3} + \frac{1}{4}} + X^{1-c} \ll X^{\frac{c+1}{4}} \log X. \tag{2.13}
\]

Similarly, for the first term in (2.11), we have

\[
\sum_{|h| \leq H_1} c_h(\alpha) \sum_{X < n \leq 2X} e((h + \alpha)n^c)
\]

\[
= c_0(\alpha) \sum_{X < n \leq 2X} e(\alpha n^c) + \sum_{|h| \leq H_1} c_h(\alpha) \sum_{X < n \leq 2X} e((h + \alpha)n^c)
\]

\[
\ll \frac{1}{\alpha X^{c-1}} + \sum_{|h| \leq H_1} \frac{1}{h} \left( ((h + \alpha)X^{c-1})^{\frac{2}{3}}X^{\frac{2}{3}} + \frac{1}{(h + \alpha)X^{c-1}} \right)
\]

\[
\ll \frac{1}{\alpha X^{c-1}} + H_1^{\frac{2}{3}}X^{\frac{2c}{3} + \frac{1}{4}} + X^{1-c}
\]

\[
\ll \frac{1}{\alpha X^{c-1}} + X^{\frac{c+1}{4}} \log X. \tag{2.14}
\]

Combining (2.11)–(2.14), we complete the proof of Lemma 2.14. \(\blacksquare\)

### 3 Proof of Theorem 1.1

By the definition of \(S_5(N)\), it is easy to see that

\[
S_5(N) = \int_0^1 S^5(\alpha)e(-N\alpha)d\alpha = \int_{-\tau}^{1-\tau} S^5(\alpha)e(-N\alpha)d\alpha
\]

\[
= \int_{-\tau}^{\tau} S^5(\alpha)e(-N\alpha)d\alpha + \int_{\tau}^{1-\tau} S^5(\alpha)e(-N\alpha)d\alpha
\]

\[
= S_5^{(1)}(N) + S_5^{(2)}(N), \tag{3.1}
\]

say. In order to prove Theorem 1.1, we need the two following propositions, whose proofs will be given in the following two subsections.
Proposition 3.1 For $1 < c < \frac{11216182}{3471123}$, $c \neq 2$, there holds
\[
R_5^{(1)}(N) = \frac{\Gamma^5(1 + 1/c)}{\Gamma(5/c)} N^{5/c - 1} + O(N^{5/c - 1} \exp \left( \frac{-\log N}{4} \right)).
\]

Proposition 3.2 For $1 < c < \frac{11216182}{3471123}$, $c \neq 2$, there holds
\[
R_5^{(2)}(N) \ll N^{5/c - 1 - \varepsilon}.
\]

From Proposition 3.1 and Proposition 3.2, we obtain the result of Theorem 1.1.

3.1 Proof of Proposition 3.1

In this subsection, we shall concentrate on establishing Proposition 3.1. Define
\[
G(\alpha) = \sum_{m \leq N} \frac{1}{c} m^{\frac{1}{c} - 1} e(m\alpha),
\]
\[
\mathcal{H}_1(N) = \int_{-\tau}^{\tau} G^5(\alpha)e(-N\alpha)d\alpha,
\]
\[
\mathcal{H}(N) = \int_{\frac{-1}{2}}^{\frac{1}{2}} G^5(\alpha)e(-N\alpha)d\alpha.
\]

Then we can write
\[
R_5^{(1)}(N) = (R_5^{(1)}(N) - \mathcal{H}_1(N)) + (\mathcal{H}_1(N) - \mathcal{H}(N)) + \mathcal{H}(N). \quad (3.2)
\]

As is shown in Theorem 2.3 of Vaughan [18], we derive that
\[
\mathcal{H}(N) = \frac{\Gamma^5(1 + 1/c)}{\Gamma(5/c)} P^{5-c} + O(P^{4-c}). \quad (3.3)
\]

By Lemma 2.8 of Vaughan [18], we know that
\[
\mathcal{H}_1(N) - \mathcal{H}(N) \ll \int_{\frac{-1}{2}}^{\frac{1}{2}} |G(\alpha)|^5 d\alpha \ll \int_{\frac{-1}{2}}^{\frac{1}{2}} \alpha^{-\frac{5}{2}} d\alpha \ll \tau^{1-\frac{5}{2}} \ll P^{5-c-\nu} \quad (3.4)
\]
for some $\nu > 0$. Next, we consider the estimate of $|R_5^{(1)}(N) - \mathcal{H}_1(N)|$. We have
\[
R_5^{(1)}(N) - \mathcal{H}_1(N) \ll \int_{-\tau}^{\tau} |S^5(\alpha) - G^5(\alpha)| d\alpha
\]
\[
\ll \int_{-\tau}^{\tau} |S(\alpha) - G(\alpha)||S(\alpha)|^4 + |G(\alpha)|^4| d\alpha
\]
\[
\ll \sup |S(\alpha) - G(\alpha)| \times \left( \int_{-\tau}^{\tau} |S(\alpha)|^4 d\alpha + \int_{\frac{-1}{2}}^{\frac{1}{2}} |G(\alpha)|^4 d\alpha \right). \quad (3.5)
\]

From Lemma 2.8 of Vaughan [18], we know that
\[
G(\alpha) \ll \min (N^{\frac{1}{c}}, |\alpha|^{-\frac{1}{c}}).
\]

Therefore, there holds
\[
\int_{-\tau}^{\tau} |G(\alpha)|^4 \, d\alpha \ll \int_{0}^{\tau} \min \left( N^{1/2}, |\alpha|^{-1/2} \right)^4 \, d\alpha.
\]
\[
\ll \int_{0}^{\tau} N^2 \, d\alpha + \int_{\tau}^{1} \alpha^{-\frac{3}{2}} \, d\alpha \ll N^{2/3} - 1 \ll P^{1-\epsilon}.
\]  
(3.6)

For \(|\alpha| \leq \tau\), we have
\[
S(\alpha) = \sum_{p \leq P} (\log p)e(p^\epsilon \alpha) + O(\tau P) = S^*(\alpha) + O(\tau P).
\]  
(3.7)

Therefore, from Lemma 2.7, we obtain
\[
\int_{-\tau}^{\tau} |S(\alpha)|^4 \, d\alpha \ll \int_{-\tau}^{\tau} |S^*(\alpha)|^4 \, d\alpha + O(\tau^5 P^4) \ll P^{1-\epsilon} \log^6 P.
\]  
(3.8)

Finally, we consider the upper bound of \(|S(\alpha) - G(\alpha)|\) under the condition \(|\alpha| \leq \tau\).

Trivially, by (3.7), we have
\[
S(\alpha) = \sum_{n \leq P} \Lambda(n)e(n^\epsilon \alpha) + O(P^{1/2}) + O(\tau P)
\]
\[
= \sum_{n \leq P} \Lambda(n)e(n^\epsilon \alpha) + O(P^{1-\epsilon}).
\]  
(3.9)

From Lemma 2.1, we know that, for \(|\alpha| \leq \tau\) and \(u \geq 2\), there holds
\[
\sum_{1 < m \leq u} e(m\alpha) = \int_{1}^{u} e(t\alpha) \, dt + O(1).
\]

By partial summation and the above identity, we deduce that
\[
\sum_{n \leq P} \Lambda(n)e(n^\epsilon \alpha) = \int_{1}^{P} e(t^\epsilon \alpha) d \left( \sum_{n \leq t} \Lambda(n) \right) = \int_{1}^{P} e(t^\epsilon \alpha) \, dt + O(P \exp (- (\log P)^{1/3}))
\]
\[
= \int_{1}^{N} \frac{1}{c} u^\frac{1}{2} \log u \, du + O(P \exp (- (\log P)^{1/3}))
\]
\[
= \int_{1}^{N} \frac{1}{c} u^\frac{1}{2} \, d \left( \int_{1}^{u} e(t\alpha) \, dt \right) + O(P \exp (- (\log P)^{1/3}))
\]
\[
= \int_{1}^{N} \frac{1}{c} u^\frac{1}{2} \, d \left( \sum_{1 < m \leq u} e(m\alpha) + O(1) \right) + O(P \exp (- (\log P)^{1/3}))
\]
\[
= \sum_{m \leq N} \frac{1}{c} m^\frac{1}{2} \log m \, du + O(P \exp (- (\log P)^{1/3}))
\]
\[
= G(\alpha) + O(P \exp (- (\log P)^{1/3})).
\]  
(3.10)

From (3.9) and (3.10), we deduce that
\[
\sup_{|\alpha| \leq \tau} |S(\alpha) - G(\alpha)| \ll P \exp (- (\log P)^{1/3}).
\]  
(3.11)
inserting (3.6), (3.8) and (3.11) into (3.5), we get

$$\mathcal{R}_5^{(1)}(N) - \mathcal{K}_1(N) \ll P^{5-c} \exp \left(- \frac{1}{2} \log P\right).$$

By (3.2)–(3.4) and (3.12), we obtain the desired result of Proposition 3.1.

### 3.2 Proof of Proposition 3.2

In this subsection, we devote to prove Proposition 3.2. First, we have

$$S(\alpha) = \sum_{p \in P^{5/6}} (\log p) e\left([p^c] \alpha\right) + \sum_{p^{5/6} < p \in P} (\log p) e\left([p^c] \alpha\right).$$

By a splitting argument, (3.13) and Lemma 2.6, we deduce that

$$\mathcal{R}_5^{(2)}(N) \ll (\log P) \max_{p^{5/6} \leq X \leq P} \left| \int_{\tau}^{1-\tau} S^4(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| + P^{\frac{5}{2}} \int_{0}^{1} |S(\alpha)|^4 d\alpha$$

$$\ll (\log P) \max_{p^{5/6} \leq X \leq P} \left| \int_{\tau}^{1-\tau} S^4(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| + P^{\frac{5}{2} + \epsilon} (P^{4-c} + P^{2})$$

$$\ll (\log P) \max_{p^{5/6} \leq X \leq P} \left| \int_{\tau}^{1-\tau} S^4(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right| + P^{5-c-\epsilon}.$$  \hspace{1cm} (3.14)

For $P^{5/6} \ll X \ll P$, we have

$$\int_{\tau}^{1-\tau} S^4(\alpha) S(\alpha, X) e(-N\alpha) d\alpha$$

$$\leq \sum_{X < p \leq 2X} (\log p) \int_{\tau}^{1-\tau} S^4(\alpha) e\left([p^c] - N\alpha\right) d\alpha$$

$$\ll \sum_{X < p \leq 2X} (\log p) \left| \int_{\tau}^{1-\tau} S^4(\alpha) e\left([p^c] - N\alpha\right) d\alpha \right|.$$ 

By Cauchy’s inequality, we deduce that

$$\left| \int_{\tau}^{1-\tau} S^4(\alpha) S(\alpha, X) e(-N\alpha) d\alpha \right|$$

$$\ll X^{\frac{1}{2} + \epsilon} \left( \sum_{X < n \leq 2X} \left| \int_{\tau}^{1-\tau} S^4(\alpha) e\left([n^c] - N\alpha\right) d\alpha \right|^2 \right)^{\frac{1}{2}}$$

$$= X^{\frac{1}{2} + \epsilon} \left( \sum_{X < n \leq 2X} \int_{\tau}^{1-\tau} S^4(\alpha) e\left([n^c] - N\alpha\right) d\alpha \cdot \int_{\tau}^{1-\tau} S^4(\beta) e\left([n^c] - N\beta\right) d\beta \right)^{\frac{1}{2}}$$

$$= X^{\frac{1}{2} + \epsilon} \left( \int_{\tau}^{1-\tau} S^4(\beta) e(-N\beta) d\beta \int_{\tau}^{1-\tau} S^4(\alpha) T(\alpha - \beta, X) e(-N\alpha) d\alpha \right)^{\frac{1}{2}}$$
\[ X^{\frac{1}{2} + \varepsilon} \left( \int_{\tau}^{1-\tau} |S(\beta)|^4 \, d\beta \int_{\tau}^{1-\tau} |S(\alpha)|^4 |T(\alpha - \beta, X)| \, d\alpha \right)^{\frac{1}{2}}. \] (3.15)

For the inner integral in (3.15), we have
\[ \int_{\tau}^{1-\tau} |S(\alpha)|^4 |T(\alpha - \beta, X)| \, d\alpha \]
\[ \ll \left( \int_{(\tau,1-\tau) \cap \{ |\alpha - \beta| \leq X^{-c} \}} + \int_{(\tau,1-\tau) \cap \{ |\alpha - \beta| > X^{-c} \}} \right) |S^4(\alpha)T(\alpha - \beta, X)| \, d\alpha. \] (3.16)

For the first term on the right-hand side of (3.16), we use Lemma 2.13 and the trivial estimate \( T(\alpha - \beta, X) \ll X \) to deduce that
\[ \int_{(\tau,1-\tau) \cap \{ |\alpha - \beta| \leq X^{-c} \}} |S^4(\alpha)T(\alpha - \beta, X)| \, d\alpha \]
\[ \ll X \cdot \sup_{\alpha \in (\tau,1-\tau)} |S(\alpha)|^4 \times \int_{|\alpha - \beta| \leq X^{-c}} \, d\alpha \ll P^{3080700 \times \varepsilon} X^{1-c}. \] (3.17)

For the second term on the right-hand side of (3.16), by Lemma 2.13 and Lemma 2.14, we obtain
\[ \int_{(\tau,1-\tau) \cap \{ |\alpha - \beta| > X^{-c} \}} |S^4(\alpha)T(\alpha - \beta, X)| \, d\alpha \]
\[ \ll \int_{(\tau,1-\tau) \cap \{ |\alpha - \beta| > X^{-c} \}} |S(\alpha)|^4 \left( X^{\frac{1}{2} + \varepsilon} \log X + \frac{1}{|\alpha - \beta| X^{c-1}} \right) \, d\alpha \]
\[ \ll X^{\frac{1}{2} + \varepsilon} \times \int_{0}^{1} |S(\alpha)|^4 \, d\alpha + \sup_{\alpha \in (\tau,1-\tau)} |S(\alpha)|^4 \times \int_{|\alpha - \beta| > X^{-c}} \left\{ \frac{1}{|\alpha - \beta| X^{c-1}} \right\} \, d\alpha \]
\[ \ll X^{\frac{1}{2} + \varepsilon} \left( P^{1-c} + P^2 \right) P^\varepsilon + P^{3080700 \times \varepsilon} X^{1-c}. \] (3.18)

Combining (3.16) and (3.18), we conclude that
\[ \int_{\tau}^{1-\tau} |S^4(\alpha)T(\alpha - \beta, X)| \, d\alpha \ll X^{\frac{1}{2} + \varepsilon} \left( P^{1-c} + P^2 \right) P^\varepsilon + P^{3080700 \times \varepsilon} X^{1-c}. \] (3.19)

Inserting (3.19) into (3.15), we obtain
\[ \left| \int_{\tau}^{1-\tau} S^4(\alpha)S(\alpha, X)e(-N\alpha) \, d\alpha \right| \]
\[ \ll X^{\frac{1}{2} + \varepsilon} \left( X^{\frac{1}{2} + \varepsilon} \left( P^{1-c} + P^2 \right) P^\varepsilon + P^{3080700 \times \varepsilon} X^{1-c} \right) \left( P^{4-c} + P^2 \right) P^\varepsilon \]
\[ \ll X^{1 + \frac{11}{12}} \left( P^{4-c} + P^2 \right) P^\varepsilon + P^{\frac{1540350}{781589} \times \varepsilon} \left( P^{\frac{4}{3} - c} + P \right) X^{1 - \frac{1}{2}}. \]

For \( 1 < c < 2 \), we have
\[ \left| \int_{\tau}^{1-\tau} S^4(\alpha)S(\alpha, X)e(-N\alpha) \, d\alpha \right| \]
\[
\ll P^{c+11/14} \cdot P^{4-c+\varepsilon} + P^{\frac{1540350}{731123} + \varepsilon} \cdot P^{\frac{4-c}{2}} \cdot P^{\frac{5}{2}} \ll P^{5-c-\varepsilon}. \tag{3.20}
\]

For \(2 < c < \frac{11216182}{5471123}\), we have
\[
\left| \int_{-\tau}^{1-\tau} S^4(\alpha)S(\alpha, X)e(-N\alpha)d\alpha \right| \ll P^{c+11/14} \cdot P^{2+\varepsilon} + P^{\frac{1540350}{731123} + \varepsilon} \cdot P \cdot P^{\frac{5}{2}(1-\frac{c}{2})} \ll P^{5-c-\varepsilon}. \tag{3.21}
\]

From (3.14), (3.20) and (3.21), we deduce that
\[
\mathcal{R}^{(2)}_5(N) \ll P^{5-c-\varepsilon}
\]
provided that \(1 < c < \frac{11216182}{5471123}, c \neq 2\), which completes the proof of Proposition 3.2.

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