A GENERALIZATION OF SCHMIDT NUMBER FOR MULTIPARTITE STATES

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Received Day Month Year
Revised Day Month Year

The Schmidt number is of crucial importance in characterizing the bipartite pure states. We explore and propose here a generalization of Schmidt number for states in multipartite systems. It is shown to be entanglement monotonic and valid for both pure and mixed states. In addition, the corresponding generalization of multipartite Schmidt coefficients is introduced. Our approach is applicable for systems with arbitrary number of parties and for arbitrary dimensions.

Keywords: Schmidt number; Schmidt coefficients; multipartite system; entanglement measure.

1. Introduction

Entanglement is shown to play a crucial role in quantum information processing and quantum computation. However, quantifying entanglement is not straightforward, and has become one of the most significant problems in this area. Several kinds of entanglement measures have been proposed for bipartite case. Yet there are no operational methods for multipartite states in general.

In contrast to the bipartite case, the situation is more involved in the multipartite case. There are many kinds of entanglement. For the simplest case, a three-qubit state can be either fully separable, biseparable, or genuinely entangled. An \(m\)-partite state might have many different kinds of cases: fully separable, 2-separable, 3-separable, \(\ldots\), \((m-2)\)-separable, and genuinely entangled, etc. On the other hand, the structure of the local rank of the multipartite case is a intricate one. The bipartite pure state \(|\psi\rangle\) is uniquely determined by its reductions but a tripartite pure state has three single-particle marginals of inequivalent rank. It is generally difficult to characterize different types of multipartite entanglement and
distinguish them from each other completely.

The Schmidt number is indispensable in characterization and quantification of entanglement associated with pure states, it can be used to characterize and quantify the degree of bipartite entanglement for pure state directly, such as entanglement formation, concurrence, etc. Then a natural idea in mind is that whether there is a corresponding quantity for the multipartite states. Unfortunately, the Schmidt decomposition is not valid for multipartite case. Only rare pure states in the multipartite case admit the Schmidt decomposition form. Where

\[ |\psi\rangle = \sum_{k=1}^{r} \lambda_k |e_k^{(1)}\rangle |e_k^{(2)}\rangle \otimes \cdots \otimes |e_k^{(m)}\rangle, \]

where \( r \leq \min\{N_1, N_2, \ldots, N_m\}\), \( N_i \) denotes the dimension of the \( i \)-th subsystem, \( \{|e_k^{(i)}\rangle\} \) is an orthonormal set of the \( i \)-th state space. For the simplest three-partite case, any three-qubit pure state can be written as \[ |\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\theta} |100\rangle + \lambda_2 |110\rangle + \lambda_3 |111\rangle, \]

where \( \lambda_i \geq 0\), \( \sum_i \lambda_i^2 = 1\), \( \theta \in [0, \pi] \), which is not a form of Schmidt decomposition. That is, there is no correspondence of Schmidt number for multipartite case in general. In Ref. [15], a generalized Schmidt number for a multipartite state \( |\psi\rangle \) is defined to be the minimal necessary number of summands in the representation of it as a sum of separable states. It is shown to be a multipartite entanglement measure but the scenario is not based on the Schmidt decomposition. In this paper, we try to extend the Schmidt number to multipartite systems according to the Schmidt decomposition of the subsystems. And as a closely related concept, the corresponding multipartite Schmidt coefficients will be discussed. As desired, we show that the generalized Schmidt number can be used to quantify entanglement for multipartite states as well. That is, we propose a new entanglement measure for multipartite states.

The rest of this paper is organized as follows. In Sec. 2, we review the origin Schmidt number and the Schmidt coefficients for the bipartite case. In Sec. 3, we give our method of extending the Schmidt number to the multipartite systems. We begin with the tripartite case and then discuss the generalized case. Both the pure states and the mixed states are considered. We show that our generalized Schmidt number is an entanglement monotone. Then, in Sec. 4, we construct the multipartite Schmidt coefficients based on our scenario of the multipartite Schmidt number. Sec. 5 lists two kinds of examples: the W state and the GHZ state. Finally, in Sec. 6, we draw the conclusion.

2. Preliminary

Throughout this paper, we only consider the finite-dimensional case since the Schmidt number for the continuous-variable system may be \( \infty \). Let \( |\psi\rangle = \sum_{i,j} d_{ij} |i_1\rangle |j_2\rangle \) be a pure state lives in \( \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \) and \( |\psi\rangle = \sum_k \lambda_k |e_k\rangle |f_k\rangle \) be
its Schmidt decomposition, where \{i_1\} and \{j_2\} are the standard computational bases of \(\mathbb{C}^{N_1}\) and \(\mathbb{C}^{N_2}\) respectively, \{e_k\} and \{f_k\} are orthonormal sets of \(\mathbb{C}^{N_1}\) and \(\mathbb{C}^{N_2}\) respectively. Then the Schmidt number of \(|\psi\rangle\) is defined by

\[
R_\psi = \text{rank}(\rho_1) = \text{rank}(\rho_2),
\]

where \(\rho_i\) denotes the reduced state of the \(i\)-th part. \(\lambda_k\)s are called the Schmidt coefficients of \(|\psi\rangle\). It is clear that \(R_\psi\) coincides with the rank of the coefficient matrix of \(|\psi\rangle\), i.e., \(R_\psi = \text{rank}(D)\), \(D = [d_{ij}]\); \(R_\psi\) is also the length of the Schmidt decomposition of \(|\psi\rangle\). For mixed state \(\rho\) acting on \(\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}\), the Schmidt number is defined by

\[
R_\rho = \inf_{D_2(\rho)} \max_{|\psi\rangle} R_\psi, \tag{3}
\]

where \(D_2(\rho) = \{p_i, |\psi_i\rangle : \rho = \sum p_i |\psi_i\rangle \langle \psi_i|\}\) is for all pure ensembles of the bipartite state \(\rho\). It is shown that \(R_\rho\) is entanglement monotonic.

3. The Schmidt number for multipartite case

Table 1. The Schmidt number of the three qubit pure state (\(r_i\) denotes the rank of \(\rho_i\), \('i - j' means \(\rho_{ij}\) is not a pure separable state, \('i \cdot j - k' means \(\rho_{ijk} = |\psi_{ijk}\rangle \langle \psi_{ijk}|\) is an entangled pure state, etc.)

| Type | Model | Reductions | \(R_\psi\) |
|------|-------|------------|-------------|
| 1\[23\] | 1 2 3 | \(r_1 = 1\) | 1 |
| 1\[23\] | 1 2 -3 | \(r_1 = 1, r_2 = r_3 = 2\) | 2 |
| 1\[23\] | 1 -2 3 | \(r_1 = 1, r_1 = r_2 = 2\) | 2 |
| 2\[13\] | 2 1 -3 | \(r_2 = 1, r_1 = r_3 = 2\) | 2 |
| GE figure (a) | \(r_1 = 2\), \(\rho_i\) is separable, \(i = 1, 2, 3\) | 3 |
| GE figure (a) | \(r_1 = 2\), \(\rho_i\) is entangled for some \(i\) | 4 |

We develop a method of extending the Schmidt number to multipartite case. We consider the three qubit case first. If \(|\psi\rangle\) is fully separable (i.e., 1\[23\] separable), then \(|\psi\rangle = |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle\). In such a case, the Schmidt number of \(|\psi\rangle\) can be defined to be 1. Suppose \(|\psi\rangle\) is bi-separable, without loss of generality, we assume that it is 1\[23\] separable and not fully separable, i.e., \(|\psi\rangle = |\psi_1\rangle |\psi_{23}\rangle\). Since \(|\psi_{23}\rangle\) is a bipartite entangled pure state, it has Schmidt number \(R_{\psi_{23}} = 2\), and then \(R_\psi\) can be viewed as \(R_{\psi_{23}}\), namely, \(R_\psi = 2\). If \(|\psi\rangle\) is genuinely entangled, then \(\text{rank}(\rho_1) = \text{rank}(\rho_{23}) = \text{rank}(\rho_2) = \text{rank}(\rho_3) = \text{rank}(\rho_{13}) = \text{rank}(\rho_{12}) = 2\), where \(\rho_\gamma\) denotes the reduction of the \(\gamma\)-subsystem(s). If \(\rho_{23}, \rho_{23}\) and \(\rho_{13}\) are separable, then \(R_{\rho_{12}} = R_{\rho_{13}} = R_{\rho_{23}} = 1\), we thus view \(R_\psi\) as \(\text{rank}(\rho_1) + R_{\rho_{23}} = \text{rank}(\rho_2) + R_{\rho_{13}} = \text{rank}(\rho_3) + R_{\rho_{12}} = 3\). If \(\rho_{23}, \rho_{12}, \rho_{13}\) is entangled, then \(\max\{R_{\rho_{12}}, R_{\rho_{23}}, R_{\rho_{13}}\} = 2\), we thus view \(R_\psi\) as \(\text{rank}(\rho_1) + 2 = 4\). That is, there are four types of entanglement indeed for the three qubit case (see Table II). (For \(i \in \{1, 2, 3, \ldots, m\}\), we denote by \(i\) the combination
consisting of all elements in \( \{1, 2, \ldots, m\} - \{i\} \), for instance, if \( m = 4, i = (2) \), then \( \bar{i} = (134) \).)

### Table 2. The Schmidt number of \(|\psi\rangle\) in the three-partite system.

| Type   | Reductions | \( R_\psi \) |
|--------|------------|--------------|
| 1[2]3  | \( r_1 = 1 \)  | 1            |
| 123    | \( r_1 = 1, r_2 = r_3 = R_{\psi_{23}} \) | \( R_{\psi_{23}} \) |
| 123    | \( r_3 = 1, r_1 = r_2 = R_{\psi_{12}} \) | \( R_{\psi_{12}} \) |
| 213    | \( r_2 = 1, r_1 = r_3 = R_{\psi_{13}} \) | \( R_{\psi_{13}} \) |
| GE     | \( r_i \geq 2, i = 1, 2, 3 \) | \( \max_i (r_i + R_{\rho_i}) \) |

### Table 3. The Schmidt number of \(|\psi\rangle\) in the four-partite system.

| Type   | Model | Local rank | \( R_\psi \) |
|--------|-------|------------|--------------|
| 1[2]34 | 1 2 3 4 | \( r_i = 1 \)  | 1            |
| 1234   | 1 2 -3 4 | \( r_1 = r_4 = 1 \) | \( R_{\psi_{234}} \) |
| 1234   | 1 -2 3 4 | \( r_3 = r_4 = 1 \) | \( R_{\psi_{124}} \) |
| 234    | 2 3 1 -4 | \( r_2 = r_3 = 1 \) | \( R_{\psi_{243}} \) |
| 234    | 2 3 1 -4 | \( r_2 = r_3 = 1 \) | \( R_{\psi_{243}} \) |
| 234    | 1 2 3 -4 | \( r_1 = r_3 = 1 \) | \( R_{\psi_{243}} \) |
| 234    | 1 2 3 -4 | \( r_1 = r_3 = 1 \) | \( R_{\psi_{243}} \) |
| 34     | 1 3 2 -4 | \( r_1 = r_3 = 1 \) | \( R_{\psi_{243}} \) |
| 1234   | 1 -2 3 4 | \( r_i = 1 \)  | \( R_{\psi_{1234}} \) |
| 1234   | 1 -2 3 4 | \( r_i \geq 2 \) | \( R_{\psi_{1234}} + R_{\psi_{243}} \) |
| 1234   | 1 -2 3 4 | \( r_i \geq 2 \) | \( R_{\psi_{1234}} + R_{\psi_{243}} \) |
| 1234   | 1 -2 3 4 | \( r_i \geq 2 \) | \( R_{\psi_{1234}} + R_{\psi_{243}} \) |
| GE     | figure (b) | \( r_i \geq 2 \) | \( \max_i (r_i + R_{\rho_i}) \) |

We now move to the \( N_1 \otimes N_2 \otimes N_3 \) case. We may assume that \( N_1 \leq N_2 \leq N_3 \). If \(|\psi\rangle\) is fully separable, then \(|\psi\rangle = |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle\) and thus the Schmidt number of \(|\psi\rangle\) can be considered to be 1. If \(|\psi\rangle\) is bi-separable, without loss of generality, we assume that it is \(1|2|3\) separable and not fully separable, i.e., \(|\psi\rangle = |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle\). Since \(|\psi_{23}\rangle\) is a bipartite entangled pure state, we let \( R_{\psi_{23}} = t, 2 \leq t \leq N_2 \), then \( R_\psi \) can be viewed as \( R_{\psi_{23}} \), that is, \( R_\psi = t \). If \(|\psi\rangle\) is genuinely entangled, then \( \text{rank}(\rho_1) = \text{rank}(\rho_{23}) = i \geq 2 \), \( \text{rank}(\rho_2) = \text{rank}(\rho_{13}) = j \geq 2 \) and \( \text{rank}(\rho_3) = \text{rank}(\rho_{12}) = k \geq 2 \). If \( \rho_{12}, \rho_{23} \) and \( \rho_{13} \) are separable, then \( R_{\rho_{12}} = R_{\rho_{13}} = R_{\rho_{23}} = 1 \), we thus view \( R_\psi \) as \( \max_i (\text{rank}(\rho_i) + R_{\rho_i}) \geq 3 \). If \( \rho_{23} \), or \( \rho_{12} \), or \( \rho_{13} \) is entangled, then \( \max\{R_{\rho_{12}}, R_{\rho_{23}}, R_{\rho_{13}}\} \geq 2 \), we thus view \( R_\psi \) as \( \max_i (\text{rank}(\rho_i) + R_{\rho_i}) \geq 4 \). That is, there are at most \( N_1 + N_3 \) types of entanglement for the \( N_1 \otimes N_2 \otimes N_3 \) case (see Table 2).
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Note. If $m > 3$, then there exist $|\psi\rangle$ and $|\phi\rangle$ in $m$-partite systems, such that $|\psi\rangle$ is $k$-separable, $|\phi\rangle$ is genuinely entangled but $R_\psi > R_\phi$.

A natural way of generalizing the Schmidt number to mixed states $\rho$ acting on $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \mathbb{C}^{N_3}$ can be defined by the convex roof structure

$$R_\rho := \inf_{D_k(\rho)} \max_{\psi_i} R_{\psi_i},$$

where the minimization is over all ensemble of $\rho$ (Hereafter, we denote by $D_k(\rho)$ the set of all ensembles of the $k$-partite state $\rho$). It is reasonable as this means that $\rho$ cannot be obtained by mixing pure states with Schmidt number lower than $R_\rho$ and that there exists an ensemble with Schmidt number at most $R_\rho$ to reach the state. It is clear that $R_\rho$ is an entanglement monotone since local rank of pure state is non-increasing under local operations and classical communication (LOCC).

Now we can establish the Schmidt number for the four-partite system as Table. Analogously, we can define the Schmidt number for mixed states via the convex roof structure as Eq. (4). This approach can be extended to $m$-partite case step by step for both pure and mixed states. That is, for $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \cdots \otimes \mathbb{C}^{N_m}$, the $R_\psi$ can be defined as the program discussed above: if $|\psi\rangle$ is not genuinely entangled and assume with on loss of generality that it is $12|34|5\cdots m$ separable (resp. $12|3|4|5\cdots m$ separable), then $R_\psi = R_{\psi_{12}} + R_{\psi_{34}} + R_{\psi_{5\cdots m}}$ (resp. $R_\psi = R_{\psi_{12}} + R_{\psi_{5\cdots m}}$); if $|\psi\rangle$ is genuinely entangled, then $R_\psi = \max(r_i + R_{\rho_i})$. For mixed state $\rho$ acting on $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \cdots \otimes \mathbb{C}^{N_m}$, it can be defined by the convex roof structure

$$R_\rho := \inf_{D_m(\rho)} \max_{\psi_i} R_{\psi_i},$$

where the minimization is over all ensemble of $\rho$. We now can conclude the following result.

Theorem: The Schmidt number defined as Eq. (5) is an entanglement monotone.

Similar to the bipartite case, a multipartite $\rho$ is fully separable iff $R_\rho = 1$. We now have established a complete hierarchy of Schmidt numbers that quantify the dimensions of the entanglement. The bipartite Schmidt number can be viewed as an entanglement measure since the Schmidt number fully reflects the dimensional of entanglement. From this point of view, the generalized Schmidt number can also be viewed as an entanglement measure for the multipartite case.

We illustrate the 'structure' of the Schmidt number for genuinely entanglement with the following figures. $r_i - R_{ij} - r_j'$ means $\rho_{ij}$ is a mixed state with Schmidt
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number $R_{ij}$. For the tripartite case, the Schmidt number

\[
\begin{array}{c}
\text{(a)}
\end{array}
\]

can be ‘explained’ by Figure (a): $R_\psi = \max_i \{r_i + R_{i\bar{i}}\}$. For the four-partite pure state $|\psi\rangle$, $R_\psi$ is determined by $r_i$ and $R_{i\bar{i}}$ while $R_{ij}$ is decided by the Schmidt number of the six bipartite reductions and the four single part reductions. It is clear that $R_\psi \leq \max_{i \neq j} \{r_i + r_j + R_{ij}\}$ (see the ‘relation’ among $r_i$, $r_j$ and $R_{ij}$ in Figure (b)).

**Note.** It is straightforward that the generalized Schmidt number is invariant under the invertible SLOCC (stochastic local operations and classical communication) since invertible SLOCC preserves the rank of the reduction\cite{19} (also see in Refs. 20\cite{20}, 21\cite{21}, 22\cite{22}).

4. The Schmidt coefficients for multipartite case

For the bipartite system, almost any entanglement measure or even any quantum correlation for pure states can be represented by its Schmidt coefficients\cite{4,13,23,24}. In this section, we discuss the Schmidt coefficients for the multipartite case based on the scenario of the multipartite Schmidt number. We begin with the tripartite case. Let $|\psi\rangle$ be a pure state in a $N_1 \otimes N_2 \otimes N_3$ system. If it is fully separable, the Schmidt coefficient is $\{1\}$. If it is $1|23$ separable, the Schmidt coefficients are defined as that of $|\psi_{23}\rangle$; similarly, we can define it for the types $1|23$ and $2|13$. If it is genuinely entangled, we assume without loss of generality that

\[ R_\psi = \text{rank}(\rho_{t_1}) + R_{\rho_{t_1}} + \text{rank}(\rho_{t_2}) + R_{\rho_{t_2}} + \cdots = \text{rank}(\rho_{t_k}) + R_{\rho_{t_k}}. \]

Let $\tilde{\rho}_{t_i} = \frac{1}{\sqrt{2}} \rho_{t_i}$, and let

\[ \sigma(\tilde{\rho}_{t_i}) = \{\lambda_p^{(i)}\}_{p=1}^{\text{rank}(\tilde{\rho}_{t_i})}, \]

where $\sigma(\cdot)$ denotes the set of eigenvalues of the described matrix. Let $|\psi_1^{(t_i)}\rangle$, $|\psi_2^{(t_i)}\rangle$, \ldots, $|\psi_r^{(t_i)}\rangle$ be elements in the pure state ensembles of $\rho_{t_i}$ such that $R_{\psi_j^{(t_i)}} = R_{\rho_{t_i}}$, $j = 1, 2, \ldots, r$. Assume that

\[ E(|\psi_j^{(t_i)}\rangle) = \max_j E(|\psi_j^{(t_i)}\rangle), \]

where $E(\cdot)$ denotes the expectation value. For the four-partite case, the Schmidt number $R_{ij}$ is decided by the Schmidt number of the six bipartite reductions and the four single part reductions. It is clear that $R_\psi \leq \max_{i \neq j} \{r_i + r_j + R_{ij}\}$ (see the ‘relation’ among $r_i$, $r_j$ and $R_{ij}$ in Figure (b)).

**Note.** It is straightforward that the generalized Schmidt number is invariant under the invertible SLOCC (stochastic local operations and classical communication) since invertible SLOCC preserves the rank of the reduction\cite{19} (also see in Refs. 20\cite{20}, 21\cite{21}, 22\cite{22}).
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For the three-partite case, we let \( S \) be an element in the pure state ensembles of \( \rho \) such that \( \lambda \) is defined to be the Schmidt coefficients of \( |\psi\rangle \) with \( \gamma = \frac{1}{\sqrt{2}} \). In such a case, we define the Schmidt coefficients to be 

\[
\sigma(\tilde{\rho}_i) \bigcup \{1/\sqrt{2}\}
\]

(6)

is defined to be the Schmidt coefficients of \( |\psi\rangle \); If \( E_i \) reaches a maximum for some \( i \) with \( R_{\rho_1} = 1 \),

\[
\sigma(\tilde{\rho}_i) \bigcup S_C^i
\]

(7)

is defined to be the Schmidt coefficients of \( |\psi\rangle \). The ratio \( \frac{1}{\sqrt{2}} \) here guarantees that the Schmidt coefficients are normalized, i.e., the sum of the squares is 1.

For the four-partite system, if it is not genuinely entangled, then it reduces to the three-partite case. For example, if \( |\psi\rangle \) is 1|234 separable, then \( |\psi\rangle = |\psi_1\rangle|\psi_{234}\). So we can define the Schmidt coefficients as that of \( |\psi_{234}\rangle \). If \( |\psi\rangle \) is 12|34 separable, then \( |\psi\rangle = |\psi_{12}\rangle|\psi_{34}\). In such a case, we define the Schmidt coefficients to be 

\[
\sigma(\tilde{\rho}_1) \bigcup \sigma(\tilde{\rho}_3)
\]

In these cases, the von Neumann entanglement entropy is clear. If it is genuinely entangled, we let \( |\phi^{(i)}_0\rangle \) be an element in the pure state ensembles of \( \rho_1 \) such that \( R_{\psi} = \text{rank}(\rho_1) + R_{\rho_1}, R_{\rho_2} = R_{\phi^{(i)}_0}, \) and \( E(|\phi^{(i)}_0\rangle) \) reaches the maximal over all elements \( |\phi^{(i)}\rangle \)'s in the ensembles of \( \rho_1 \) such that \( R_{\rho_1} = R_{\phi^{(i)}_0} \), where \( E(|\phi^{(i)}_0\rangle) \) is defined as 

\[
E(|\phi^{(i)}_0\rangle) := \max_j E_j(|\phi^{(i)}_0\rangle), 1 \leq j \leq 4.
\]

(8)

we let \( S_C(|\phi^{(i)}_0\rangle) = \{\gamma_k\} \) and \( S_C^i = \{\tilde{\gamma}_k\} \) with \( \tilde{\gamma}_k = \frac{\gamma_k}{\sqrt{2}} \). Then we call

\[
\sigma(\tilde{\rho}_0) \bigcup S_C^i
\]

(9)

the Schmidt coefficients of \( |\psi\rangle \).

**Note.** (i) \( \rho \) is fully separable iff the Schmidt coefficients is \( \{1\} \); (ii) the number of the Schmidt coefficients coincides with the Schmidt number; (iii) the Schmidt coefficients may not be unique.
5. Examples

We end our discussion with some examples. Two well known three qubit states are the W state and the GHZ state,

\[ W_3 = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle), \]

\[ GHZ_3 = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \]

For the state \( W_3 \), one can easily see that any coefficient matrix of any bipartite splitting is not of rank-one, so it is genuinely entangled. From Table 1, \( R_{W_3} = 4 \). The state \( GHZ_3 \) is also genuinely entangled. Table 1 indicates \( R_{GHZ_3} = 3 \). In Ref. 19, it is proved that \( W_3 \) and \( GHZ_3 \) are two types of genuinely entangled states under SLOCC classification, which meets our results. The Schmidt coefficients of \( W_3 \) are \( \{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{2}, \frac{1}{2} \} \). The Schmidt coefficients of \( GHZ_3 \) are \( \{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \} \). In addition, the entanglement dimensionality vector \( 27 \) of \( W_3 \), (2, 2, 2), coincides with that of \( GHZ_3 \). From this point of view the generalized Schmidt number provides a more strict classification of multipartite states than the scenario of entanglement dimensionality vector proposed in Ref. 27. In addition, it is worth noticing that the Schmidt number is different from the collectibility proposed in Ref. 28 since the collectibility of \( GHZ_3 \) is larger than that of \( W_3 \).

For the \( m \)-qubit W-state \( |W_m\rangle \) and the GHZ state

\[ W_m = \frac{1}{\sqrt{m}} (|0\cdots 01\rangle + |0\cdots 010\rangle + |1\cdots 00\rangle), \]

\[ GHZ_m = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes m} + |1\rangle^{\otimes m}), \]

one can check that \( R_{W_m} = 2(m-1) \) and \( R_{GHZ_m} = 3 \). It is worth mentioning here that all the reductions of the \( |W_m\rangle \) are genuinely entangled while all the reductions of the \( |GHZ_m\rangle \) are (fully) separable. Similarly, for the generalized GHZ state in the \( d^{\otimes m} \) system, the Schmidt number of \( |GHZ_m^{(d)}\rangle \) is \( d+1 \).

6. Conclusion

In this paper, the generalizations of the Schmidt number and the Schmidt coefficients for multipartite case are established from a mathematical point-of-view. We showed that the generalized Schmidt number is a well-defined entanglement measure since it is entanglement monotonic. Our results may shed new lights on the task of multipartite systems: the multipartite states can be classified via the generalized Schmidt number. We also hope that our mathematic scenario of the Schmidt number may induce some exact physical or operational meaning.

Going further, one can define the generalized entanglement formation in terms of the Schmidt coefficients. That is, if the Schmidt coefficients of \( |\psi\rangle \) are \( \{ \eta_i \} \), then we can define the generalized entanglement of formation by \( E(|\psi\rangle) := -\sum_i \eta_i^2 \log_2 \eta_i^2 \).
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It can be extended to mixed states via the convex roof structure. (Note that although the Schmidt coefficients may not be unique the generalized von Neumann entanglement entropy is unique.) The origin entanglement of formation for the bipartite case is an entanglement monotone, we conjecture that the generalized entanglement of formation is an entanglement monotone (the proof maybe a hard work due to the complex structure of both the multipartite states and the multipartite LOCC).

Acknowledgments

Y. Guo is supported by the National Natural Science Foundation of China under Grants No. 11301312 and 11171249, the Natural Science Foundation of Shanxi under Grant No. 2013021001-1 and 2012011001-2, and the Research start-up fund for Doctors of Shanxi Datong University under Grant No. 2011-B-01. H. Fan is supported by the ‘973’ program (Grant No. 2010CB922904).

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