Queueing Subject To Action-Dependent Server Performance: Utilization Rate Reduction

Michael Lin Nuno C. Martins Richard J. La

Abstract—We consider a discrete-time system comprising a first-come-first-served queue, a non-preemptive server, and a stationary non-work-conserving scheduler. New tasks enter the queue according to a Bernoulli process with a pre-specified arrival rate. At each instant, the server is either busy working on a task or is available. When the server is available, the scheduler either assigns a new task to the server or allows it to remain available (to rest). In addition to the aforementioned availability state, we assume that the server has an integer-valued activity state. The activity state is non-decreasing during work periods, and is non-increasing otherwise. In a typical application of our framework, the server performance (understood as task completion probability) worsens as the activity state increases. In this article, we build on and transcend recent stabilizability results obtained for the same framework. Specifically, we establish methods to design scheduling policies that not only stabilize the queue but also reduce the utilization rate—understood as the infinite-horizon time-averaged portion of time the server is working. This article has a main theorem leading to two key results: (i) We put forth a tractable method to determine, using a finite-dimensional linear program (LP), the infimum of all utilization rates that can be achieved by scheduling policies that are stabilizing, for a given arrival rate. (ii) We propose a design method, also based on finite-dimensional LPs, to obtain stabilizing scheduling policies that can attain an utilization rate arbitrarily close to the aforementioned infimum. We also establish structural and distributional convergence properties, which are used throughout the article, and are significant in their own right.

I. INTRODUCTION

In this article, we adopt the discrete-time framework proposed in [1], in which a scheduler governs when tasks waiting in a first-come-first-served queue are assigned to a server. The server is non-preemptive, and has an internal state comprising two components: (i) the availability state and (ii) activity state. The former indicates whether the server is busy or available, and the latter takes values in a finite set \{1, \ldots, n_s\} that accounts for the intensity of the effort put in by the server. The activity state depends on current and previous scheduling decisions, and it is useful for modelling performance-influencing factors, such as the state of charge of the batteries of an energy harvesting module that powers one or more components of the server. As a rule, the activity state may increase while the server is busy and, otherwise, decrease gradually while the server is available (or resting).

In our framework, which follows [1], an instantaneous service rate function ascribes to each possible activity state a probability that the server can complete a task in one time-step. According to our assumption of non-preemption, once the server becomes busy working on a task, it becomes available again only when the task is completed. When the server is available, the scheduler decides, based on the activity state and the size of the queue, whether to assign a new task to the server. Although our results remain valid for any instantaneous service rate function, in many applications it is decreasing, which causes the server performance (understood as task completion probability) to worsen as the activity state increases. The vital trade-off the scheduler faces, in this case, is whether to assign a new task when the server is available or allow it to remain available (rest) to possibly ameliorate the activity state as a way to improve future performance.

A. Problem Statements and Comparison to [1]

Besides introducing and justifying in detail the formulation adopted here, in [1] the authors characterize the supremum of all arrival rates for which there is a scheduler that can stabilize the queue. The analysis in [1] also shows that such a supremum can be computed by a finite search, and identifies simple stabilizing scheduler structures, such as those with a threshold-type configuration.

In this article, we build on the analysis in [1] to design schedulers that not only guarantee stability but also lessen the utilization rate, which we will define precisely later on and can be interpreted as the proportion of time in which the server is working. Specifically, throughout this article, we will investigate and provide solutions to the following two problems.

Problem 1. Given a server and a stabilizable arrival rate\(^1\), determine a tractable method to compute the infimum of all utilization rates that can be achieved by a stabilizing scheduling policy. Such a fundamental limit is important to determine how effective any given stabilizing policy is in terms of the utilization rate.

Problem 2. Given a server and a stabilizable arrival rate, determine a tractable method to design stabilizing scheduling policies whose utilization rate is arbitrarily close to the fundamental limit.

B. Overview of Main Results and Technical Approach

In [1] Theorem 1 states our main result, from which we obtain Corollaries 1 and 2 that constitute our solutions to Problems 1 and 2, respectively. The following are key consequences of these corollaries. (i) According to Corollary 1, the infimum utilization rate (alluded to in Problem 1) can

---

1A given arrival rate is deemed stabilizable when there is a scheduling policy for which the queue is stable in the sense specified in [1] and that will be precisely defined also in this article later on.
be computed by solving a finite-dimensional linear program (LP). (ii) If the arrival rate is stabilizable by the server, then Corollary 2 guarantees that, for each positive gap $\delta$, there is a stabilizing scheduling policy whose utilization rate exceeds the infimum (characterized by Corollary 1) by at most $\delta$. Notably, such a scheduling policy can be obtained from a solution of a suitably-specified finite-dimensional LP.

Our technical approach builds on the concepts and techniques introduced in [1]. In particular, we use an appropriately-constructed auxiliary finite-state controlled Markov chain (denoted in [1] as reduced process) to obtain the above-mentioned LP-based solution methods.

This article is mathematically more intricate than [1], which is unsurprising considering that it tackles not only stabilization but also regulation of the utilization rate. Among the new concepts and techniques put forth to prove Theorem 1 the distributional convergence results of [5], and the potential-like method used to establish them, are of singular importance—they are also original and relevant in their own right.

C. Related Literature

As mentioned earlier, to the best of our knowledge, our work is the first to study the problem of lessening the utilization rate of a server whose performance is time-varying and dependent on an internal state that reflects its activity history. For this reason, there are no other results to which we can directly compare our findings.

An earlier study that examined a system that closely resembles ours is that of Savla and Frazzoli [2]. They studied the problem of designing a maximally stabilizing task release control policy, using a differential system model. Under an assumption that the service time function is convex, they derived bounds on the maximum throughput achievable by any admissible policy for a fixed task workload distribution. In addition, they showed the existence of a maximally stabilizing threshold policy when the tasks have the same workload. Finally, they demonstrated that the maximum achievable throughput increases when the workload is not deterministic. However, they did not consider the problem of minimizing utilization rate in their study.

In addition to the aforementioned study, there are a few research fields that share a key aspect of our problem, which is to design a scheduling policy to optimize the performance with respect to one objective, subject to one or more constraints. For instance, wireless energy transfer has emerged as a potential solution to powering small devices that have low-capacity batteries or cannot be easily recharged, e.g., Internet-of-Things (IoTs) devices [3], [4]. Since the devices need to collect sufficient energy before they can transmit and the transmission rate is a function of transmit power, a transmitter has to decide (i) when to harvest energy and (ii) when to transmit and at what rate. For example, the studies reported in [5], [6], [7] examined the problem of maximizing throughput in wireless networks in which communication devices are powered by hybrid access points via wireless energy transfer. In a related study, Shan et al. [8] studied the problem of minimizing the total transmission delay or completion time of a given set of packets.

Integrated production scheduling and (preventive) maintenance planning in manufacturing, where machines can fail with time-varying rates, shares similar issues as scheduling devices powered by wireless energy transfer [9], [10], [11]. In more traditional approaches, the problems of production scheduling and maintenance scheduling are considered separately, and equipment failures are treated as random events that need to be coped with. When the machine failure probability, or rate, is time-varying and depends on the length of time (age) elapsed since the last (preventive) maintenance, the overall production efficiency can be improved by jointly considering both problems. For instance, the authors of [11] formulated the problem using an MDP model with the state consisting of the system’s age (since the last preventive maintenance) and the inventory level, and investigated the structural properties of optimal policies.

Another area that shares a similar objective is the maximum hand-offs control or sparse control [12], [13], [14], [15], [16]. The goal of the maximum hands-off control is to design a control signal that maximizes the time at which the control signal is equal to zero and inactive. For instance, the authors of [12] showed that, under the normality condition, the optimal solution sets of a maximum hands-off control problem and an associated $L^1$-optimal control problem coincide. Moreover, they proposed a self-triggered feedback control algorithm for infinite-horizon problems, which leads to a control signal with a provable sparsity rate, while achieving practical stability of the system. In another study [13], Chatterjee et al. provided both necessary conditions and sufficient conditions for maximum hands-off control problem. Ikeda and Nagahara [14] considered a linear time-invariant system and showed that, if the system is controllable and the dynamics matrix is nonsingular, the optimal value of the optimal control problem for the maximum hands-off control is continuous and convex in the initial condition.

Finally, another research problem, which garnered much attention in wireless sensor networks and is somewhat related to the maximum hands-off control, is duty-cycle scheduling of sensors. A common objective for the problem is to minimize the total energy consumption subject to performance constraints on delivery reliability and delays [12]. The authors of [15] proposed using a reinforcement learning-based control mechanism for inferring the states of neighboring sensors in order to minimize the active periods. In another study, Vigorito et al. studied the problem of achieving energy neutral operation (i.e., keep the battery charge at a sufficient level) while maximizing the awake times [19]. In order to design a good control policy, they formulated the problem as an optimal tracking problem, more precisely a linear quadratic tracking problem, with the aim of keeping the battery level around some target value.

D. Paper Structure

This article has five sections. After the introduction, in [11] we describe the technical framework, including the controlled Markov chain that models the server. In [11], we also introduce a relevant auxiliary reduced process, define key quantities
and maps that quantify the utilization rate, characterize key policy sets, specify the notion of stability used throughout the article, and establish certain preliminary results. Our main theorem and key results are stated in [III] while [IV] and [V] present continuity and distributional convergence properties, respectively, that are required in the proof of our main theorem. We defer the most intricate proofs, some of which also require additional auxiliary results, to the appendices at the end of the article. The main body of the article ends with brief conclusions in [VI].

II. TECHNICAL FRAMEWORK AND KEY DEFINITIONS

This section starts with a synopsis of the discrete-time framework put forth thoroughly in [I]. Henceforth, we replicate from [I] what is strictly necessary to make this article self-contained. In this section, we also introduce the concepts, sets, operators and notation that are required to formalize and later on solve Problems 1 and 2.

Remark 1. According to the approach in [I], each discrete-time $k$ represents a continuous-time interval, or epoch, whose duration can be made arbitrarily small. Considering that this representation is detailed in [II], here we proceed directly to the description of the resulting discrete-time framework and we refer to each epoch $k$ simply as time (instant) $k$, with $k \in \mathbb{N} := \{0, 1, 2, \ldots \}$.

A. Stochastic Discrete-Time Framework

As in [I], we consider that the server is represented by the MDP $Y := \{Y_k \in Y : k \in \mathbb{N}\}$. The state of the server at time $k$ is $Y_k := (S_k, W_k)$, whose components are the activity state $S_k$ and the availability state $W_k$ taking values in $\mathbb{S} := \{1, \ldots, n_s\}$ and $\mathbb{W} := \{A, B\}$, respectively. Here, $W_k = A$ indicates that the server is available at time $k$, while $W_k = B$ signals that the server is busy. Consequently, the state-space of the server is represented as

$$\mathbb{Y} := \mathbb{S} \times \mathbb{W}. \tag{1}$$

The MDP $X := \{X_k \in X : k \in \mathbb{N}\}$ represents the overall system comprising the server $Y$ and the queue length. More specifically, the state of the system is $X_k := (Y_k, Q_k)$, where $Q_k$ is the length of the queue at time $k$, and the state-space of $X$ is:

$$X := \mathbb{S} \times \left( (\mathbb{W} \times \mathbb{N}) \setminus (B, 0) \right). \tag{2}$$

Notice that $X$ excludes the impossible case in which the server would be busy working with an empty queue.

The action of the scheduler at time $k$ is represented by $A_k$, which takes values in the set $\mathbb{A} := \{\mathcal{R}, \mathcal{W}\}$. The scheduler directs the server to work at time $k$ when $A_k = \mathcal{W}$ and instructs the server to rest when $A_k = \mathcal{R}$. Since the server is non-preemptive, once it is busy working on a task it is not allowed to rest until the task is completed and it becomes available again. This constraint and the fact that no new tasks can be assigned when the queue is empty, lead to the following set of admissible actions for each possible state $x = (s, w, q)$ in $X$:

$$a_x := \begin{cases} \{\mathcal{R}\} & \text{if } q = 0, \text{ (impose ‘rest’ when queue is empty)} \\ \{\mathcal{W}\} & \text{if } q > 0 \text{ and } w = \mathcal{B}, \text{ (non-preemptive server)} \\ \mathbb{A} & \text{otherwise.} \end{cases} \tag{3}$$

We assume that tasks arrive according to a Bernoulli process $\{B_k; k \in \mathbb{N}\}$. The arrival rate is denoted by $\lambda := P(B_k = 1)$.

1) Activity-Dependent Server Performance: In our formulation, the efficiency or performance of the server is modeled with the help of an instantaneous service rate function $\mu : \mathbb{S} \to (0, 1)$. More specifically, if the server works on a task at time $k$, the probability that it completes the task before time $k + 1$ is $\mu(S_k)$. This holds irrespective of whether the task is newly assigned or inherited as ongoing work. Thus, $\mu$ quantifies the effect of the activity state on the performance of the server. The results presented throughout this article are valid for any choice of $\mu$ with codomain $(0, 1)$.

2) Dynamics of the activity state: We assume that (i) $S_{k+1}$ is equal to either $S_k$ or $S_k + 1$ when $A_k$ is $\mathcal{W}$ and (ii) $S_{k+1}$ is either $S_k$ or $S_k - 1$ if $A_k$ is $\mathcal{R}$. The state-transition probabilities for $S_k$ are specified below for every $s$ and $s'$ in $\mathbb{S}$:

$$P_{S_{k+1}|S_k, A_k}(s' | s, \mathcal{W}) = \begin{cases} \rho_{s,s+1} & \text{if } s' = s + 1 \\ 1 - \rho_{s,s+1} & \text{if } s' = s \\ 0 & \text{otherwise} \end{cases} \tag{4a}$$

$$P_{S_{k+1}|S_k, A_k}(s' | s, \mathcal{R}) = \begin{cases} \rho_{s,s-1} & \text{if } s' = s - 1 \\ 1 - \rho_{s,s-1} & \text{if } s' = s \\ 0 & \text{otherwise} \end{cases} \tag{4b}$$

where the parameters $\rho_{s,s'}$ quantify the likelihood that the activity state will transition to a greater or lesser value, depending on whether the action is $\mathcal{W}$ or $\mathcal{R}$, respectively. Here, we assume that $\{\rho_{s,s+1} : 1 \leq s < n_s\}$ and $\{\rho_{s,s-1} : 1 < s \leq n_s\}$ take values in $(0, 1)$. We also adopt the convention that $\rho_{1,0} = \rho_{n_s,n_s+1} = 0$.

3) Transition probabilities for $X_k$: We consider that $S_{k+1}$ is independent of $(\mathcal{W}_{k+1}, Q_{k+1})$ when conditioned on $(X_k, A_k)$. Under this assumption, the transition probabilities for $X_k$ can be written as follows:

$$P_{X_{k+1}|X_k, A_k}(x' | x, a) = P_{S_{k+1}|X_k, A_k}(s' | x, a) \times P_{W_{k+1}, Q_{k+1}|X_k, A_k}(w', q' | x, a) \tag{5}$$

for every $x, x'$ in $X$ and $a$ in $\mathbb{A}$.

We assume that, at each time $k$, the events that (i) there is a new task arrival and (ii) a task being serviced is completed are independent when conditioned on $X_k$ and $\{A_k = \mathcal{W}\}$.
Hence, the transition probability $P_{W_{k+1},Q_{k+1}|X_{k},A_{k}}$ in (5) is given by the following:

$$P_{W_{k+1},Q_{k+1}|X_{k},A_{k}}(w', q' | x, W)$$

(6a)

$$= \begin{cases} 
\mu(s) \lambda & \text{if } w' = A \text{ and } q' = q, \\
\mu(s) (1 - \lambda) & \text{if } w' = A \text{ and } q' = q - 1, \\
(1 - \mu(s)) \lambda & \text{if } w' = B \text{ and } q' = q + 1, \\
(1 - \mu(s))(1 - \lambda) & \text{if } w' = B \text{ and } q' = q, \\
0 & \text{otherwise},
\end{cases}$$

$$P_{W_{k+1},Q_{k+1}|X_{k},A_{k}}(w', q' | x, R)$$

(6b)

$$= \begin{cases} 
\lambda & \text{if } w' = A \text{ and } q' = q + 1, \\
1 - \lambda & \text{if } w' = A \text{ and } q' = q, \\
0 & \text{otherwise}.
\end{cases}$$

**Definition 1. (MDP $X$)** The MDP with input $A_k$ and state $X_k$, which at this point is completely defined, is denoted by $X$.

Table I summarizes the notation for MDP $X$.

| $W$ | set of activity states $\{1, \ldots, u\}$ |
| $A_{k} := (\{A, B\}$ | server availability ($A = \text{available}, B = \text{busy}$) |
| $\lambda$ | server state at time $k$ (takes values in $W$) |
| $\lambda$ | server state at time $k$ (takes values in $Y$) |
| $\lambda$ | server state component $S \times W$ |
| $\lambda$ | state space formed by $S \times (W \times N \setminus (B, 0))$ |
| $\lambda$ | system state at time $k$ (takes values in $X$) |
| $\lambda$ | possible actions ($R = \text{rest}, W = \text{work}$) |
| $\lambda$ | set of actions admissible at a given state $x$ in $X$ |
| $\lambda$ | action chosen at time $k$. |
| $\lambda$ | probability mass function |

**Definition 4. (System stability, stabilizability and $\Theta_{S}(\lambda)$).** For a given policy $\theta$ in $\Theta_R$, the system $X^\theta$ is stable if it satisfies the following properties:

(i) The number of transient states is finite and, hence, there is at least one recurrent communicating class.

(ii) All recurrent communicating classes are positive recurrent.

An arrival rate $\lambda$ is said to be stabilizable when there is a policy $\theta$ in $\Theta_R$ for which $X^\theta$ is stable. We also define $\Theta_{S}(\lambda)$ to be the set of randomized policies in $\Theta_R$ that stabilize the system for a stabilizable arrival rate $\lambda$.

Before we proceed, let us point out a useful fact under any stabilizing policy $\theta$ in $\Theta_S(\lambda)$.

**Lemma 1.** A stable system $X^\theta$ has a unique positive recurrent communicating class, which is aperiodic. Therefore, there is a unique stationary probability mass function (PMF) for $X^\theta$.

**Definition 5.** Given an arrival rate $\lambda > 0$ and a stabilizing policy $\theta$ in $\Theta_S(\lambda)$, we denote the unique stationary PMF and positive recurrent communicating class of $X^\theta$ by $\pi^\theta(x) = (\pi^\theta(x) : x \in \mathbb{X})$ and $\mathbb{C}_\theta$, respectively.

**B. Utilization Rate: Definition and Infimum**

Subsequently, we proceed to define the concepts and maps required to formalize the analysis and computation of the utilization rate, and its infimum alluded to in the statements of Problems 1 and 2.

**Definition 6. (Utilization rate function)** The function that determines the utilization rate in terms of a given stabilizable arrival rate $\lambda$ and a stabilizing policy $\theta$, is defined as:

$$U(\lambda, \theta) := \sum_{x \in \mathbb{X}} \pi^\theta(x) \theta(x), \quad \lambda \in (0, \lambda^*), \quad \theta \in \Theta_S(\lambda)$$

The utilization rate quantifies the probability that the server is working, in the stationary limit. Notably, $U(\lambda, \theta)$, computed for $X$ with arrival rate $\lambda$ and stabilized by $\theta$, coincides with the probability limit of the utilization rate, as defined for instance in [20] (with $U = \{0, 1\}$), when the averaging horizon tends to infinity. Using our notation, the aforesaid probability limit can be stated as follows:

$$\text{plim}_{N \to \infty} \frac{\sum_{k=0}^{N} I_{A_k = W}}{N + 1} = U(\lambda, \theta), \quad \lambda \in (0, \lambda^*), \quad \theta \in \Theta_S(\lambda)$$

where $I_{A_k = W}$ is 1 when $A_k = W$ and 0 otherwise. Hence, the utilization rate can also be viewed as the proportion of time in which the server is working, in the infinite time-horizon limit.

**Definition 7.** The infimum utilization rate for a given stabilizable arrival rate $\lambda$ is defined as:

$$U^*(\lambda) := \inf_{\theta \in \Theta_S(\lambda)} U(\lambda, \theta), \quad \lambda \in (0, \lambda^*).$$
C. Auxiliary MDP $\mathcal{Y}$

We proceed with describing an underlying controlled Markov chain whose state takes values in $\mathcal{Y}$ and approximates the server state of $X$ under a subclass of policies in $\Theta_R$, subject to an assumption that the queue always has a task to service whenever the server becomes available. We endow it with a reward function, which is the utilization of the server, and denote this auxiliary MDP by $\mathcal{Y}$ and its state at time $k$ by $\mathcal{Y}_k := (\mathcal{S}_k, W_k)$ in order to emphasize that it takes values in $\mathcal{Y}$. The action chosen at time $k$ is denoted by $\mathcal{A}_k$. We use the overline to denote the auxiliary MDP and any other variables associated with it, in order to distinguish them from those of the server state in $X$.

Under certain conditions, which we will identify later on, we can determine important properties of $X$ by analysing $\mathcal{Y}$. Notably, we will use the fact that $\mathcal{Y}$ is finite to compute $\pi^*$ via a finite-dimensional LP, and also to simplify the proofs of our main results.

As the queue size is no longer a component of the state of $\mathcal{Y}$, we eliminate the dependence of the admissible action sets on $q$, which was explicitly specified in (3) for MDP $X$, while still ensuring that the server is non-preemptive. More specifically, the set of admissible actions at each element $\mathcal{Y} := (\mathcal{S}, \mathcal{W})$ of $\mathcal{Y}$ is given by

$$\overline{\mathcal{A}} := \begin{cases} \mathcal{W} & \text{if } \mathcal{W} = B, \text{ (non-preemptive server)} \\ \mathcal{A} & \text{if } \mathcal{W} = A. \end{cases} \quad (7)$$

Consequently, for any given realization of the current state $\mathcal{Y}_k = (\mathcal{S}_k, \mathcal{W}_k)$, $\overline{\mathcal{A}}_k$ is required to take values in $\overline{\mathcal{A}}_{\mathcal{W}_k}$. We define the transition probabilities that specify $\mathcal{Y}$, as follows:

$$P_{\mathcal{Y}_{k+1}|\mathcal{Y}_k, \mathcal{A}_k}(\mathcal{Y}' | \mathcal{Y}, \mathcal{A}) := P_{S_{k+1}|S_k, A_k}(\mathcal{S}' | \mathcal{S}, \mathcal{A}) \times P_{W_{k+1}|W_k, A_k}(\mathcal{W}' | \mathcal{W}, \mathcal{A}), \quad (8)$$

where $\mathcal{Y}$ and $\mathcal{Y}'$ are in $\mathcal{Y}$, and $\mathcal{A}$ is in $\overline{\mathcal{A}}_{\mathcal{W}}$. The right-hand terms of (8) are defined, in connection with $X$, as follows:

$$P_{S_{k+1}|S_k, A_k}(\mathcal{S}' | \mathcal{S}, \mathcal{A}) := P_{X_{k+1}|X_k, A_k}(\mathcal{S}' | \mathcal{S}, \mathcal{A}), \quad (9)$$

$$P_{W_{k+1}|W_k, A_k}(\mathcal{W}' | \mathcal{W}, \mathcal{A}) := \begin{cases} \mu(\mathcal{S}) & \text{if } \mathcal{W}' = A \\ 1 - \mu(\mathcal{S}) & \text{if } \mathcal{W}' = B \end{cases} \quad (10a)$$

$$P_{W_{k+1}|W_k, A_k}(\mathcal{W}' | \mathcal{W}, \mathcal{A}) := \begin{cases} 1 & \text{if } \mathcal{W}' = A \\ 0 & \text{if } \mathcal{W}' = B \end{cases} \quad (10b)$$

From (10a) and (10b), we can deduce the following equality valid for all $q \geq 1$,

$$P_{\mathcal{W}_{k+1}|\mathcal{Y}_k, \mathcal{A}_k}(\mathcal{W}' | \mathcal{Y}, \mathcal{W}) = \sum_{q'=0}^\infty P_{W_{k+1}, Q_{k+1}|X_k, A_k}(\mathcal{W}', q' | (\mathcal{Y}, q), \mathcal{W}), \quad (11)$$

which holds for any $\mathcal{W}'$ in $\mathcal{W}$ and $\mathcal{Y}$ in $\mathcal{Y}$. Notice that the right-hand side (RHS) of (11) does not change when we vary $q$ across the positive integers. From this, in conjunction with (5), (3) and (9), we also have, for all $q \geq 1$,

$$P_{\mathcal{Y}_{k+1}|\mathcal{Y}_k, \mathcal{A}_k}(\mathcal{Y}' | \mathcal{Y}, \mathcal{W}) = \sum_{q'=0}^\infty P_{X_{k+1}|X_k, A_k}(\mathcal{Y}', q' | (\mathcal{Y}, q), \mathcal{W}). \quad (12)$$

The equality in (12) indicates that $P_{\mathcal{Y}_{k+1}|\mathcal{Y}_k, \mathcal{A}_k}$ also characterizes the transition probabilities of the server state $Y_k = (S_k, W_k)$ in $X$ when the current queue size is positive. This is consistent with our earlier viewpoint that $\mathcal{Y}$ behaves as the server state in $X$ when the queue is nonempty.

D. Stationary Policies and Stationary PMFs of $\mathcal{Y}$

Analogously to the MDP $X$, we only consider stationary randomized policies for $\mathcal{Y}$, which are defined below.

**Definition 8** ($\Phi_R$). We restrict our attention to stationary randomized policies acting on $\mathcal{Y}$, which are specified by a mapping $\phi : \mathcal{Y} \to [0, 1]$, as follows:

$$P_{\mathcal{X}_k|\mathcal{Y}_k, \ldots, \mathcal{Y}_0}(W | \mathcal{Y}_k, \ldots, \mathcal{Y}_0) = \phi(\mathcal{Y}_k)$$

$$P_{\mathcal{X}_k|\mathcal{Y}_k, \ldots, \mathcal{Y}_0}(R | \mathcal{Y}_k, \ldots, \mathcal{Y}_0) = 1 - \phi(\mathcal{Y}_k)$$

for every $k$ in $\mathbb{N}$ and $\mathcal{Y}_k, \ldots, \mathcal{Y}_0$ in $\mathcal{Y}$. We define $\Phi_R$ as the set of all stationary randomized policies for $\mathcal{Y}$ that satisfy (7).

Henceforth, we use $\mathcal{Y}$ to denote the auxiliary MDP $\mathcal{Y}$ under a policy $\phi$ in $\Phi_R$.

Following the approach in (1), we restrict our analysis to the subset $\Phi^+_R$ of $\Phi_R$ defined as follows:

$$\Phi^+_R := \{ \phi \in \Phi_R \mid \phi(1, A) > 0 \}$$

The main benefit of focusing on policies in $\Phi^+_R$, as stated in (1 Corollary 1), is that $\mathcal{Y}$ has a unique stationary PMF (denoted $\pi^*$) for every $\phi$ in $\Phi^+_R$. Specifically, that strategies in $\Phi^+_R$ rule out the case in which $(1, A)$ is an absorbing state, guarantees the uniqueness of the stationary PMF. Furthermore, from (1 Lemmas 2 and 4) we conclude that restricting to $\Phi^+_R$ any search that seeks to determine bounds or fundamental limits with respect to stabilizing policies incurs no loss of generality.

E. Service Rate of $\mathcal{Y}$ and Précis of Stabilizability Results

We start by defining the service rate of $\mathcal{Y}$ for a given policy $\phi$ in $\Phi^+_R$:

$$\mathcal{R} := \sum_{\mathcal{Y} \in \mathcal{Y}} \mu(\mathcal{S}) \phi(\mathcal{Y}) \pi^*(\mathcal{S}).$$

The maximal service rate $\mathcal{R}$ for $\mathcal{Y}$ is defined below.

$$\mathcal{R} := \sup_{\phi \in \Phi^+_R} \mathcal{R}.$$
threshold policy $\phi_\tau$, among the finitely many defined in [1 (6)], maximizes $\pi^\tau_{\phi_\tau}$.

**Definition 9.** We define the map $\mathcal{X} : \Phi^+_R \rightarrow \Theta_R$ as follows:

$$\mathcal{X}(\phi) := \vartheta^\phi, \quad \phi \in \Phi^+_R,$$

where

$$\vartheta^\phi(x) := \begin{cases} \phi(y) & \text{if } q > 0 \\ 0 & \text{otherwise} \end{cases}, \quad x \in X \quad (13)$$

It follows from its definition that $\mathcal{X}$ yields a policy for $X$ that acts as the given $\phi$ in $\Phi^+_R$ when the queue is not empty and imposes rest otherwise.

**Convention:** We reserve $\tau$, without a superscript, to denote a design parameter. Namely, it is a desired service rate that will be imposed as a constraint in the definition of the following policy sets.

**Definition 10. (Policy sets $\Phi^+_R(\tau)$ and $\Phi^+_R(\tau)$)** Given $\tau$ in $(0, \tau^*)$, we define the following policy sets:

$$\Phi^+_R(\tau) := \{ \phi \in \Phi^+_R | \tau^\phi = \tau \}$$

$$\Phi^+_R(\tau) := \{ \phi \in \Phi^+_R | \tau^\phi = \tau \}, \quad \epsilon \in [0, 1]$$

where $\Phi^+_R$ is defined as

$$\Phi^+_R := \{ \phi \in \Phi_R | \phi(1, A) \geq \epsilon \}, \quad \epsilon \in [0, 1]$$

We also define the following class of policies generated from $\Phi^+_R(\tau)$ and $\Phi^+_R(\tau)$ through $\mathcal{X}$:

$$\mathcal{X} \Phi^+_R(\tau) := \{ \mathcal{X}(\phi) : \phi \in \Phi^+_R(\tau), \tau \in (0, \tau^*) \}, \quad \epsilon \in (0, 1]$$

$$\mathcal{X} \Phi^+_R(\tau) := \{ \mathcal{X}(\phi) : \phi \in \Phi^+_R(\tau), \tau \in (0, \tau^*) \}$$

The following proposition establishes important stabilization properties for the policies in $\mathcal{X} \Phi^+_R(\tau)$.

**Proposition 1.** Let the arrival rate $\lambda$ in $(0, \tau^*)$ be given. If $\tau$ is in $(\lambda, \tau^*)$, then $X^0$ is stable, irreducible and aperiodic for any $\theta$ in $\mathcal{X} \Phi^+_R(\tau)$.

**Proof.** Stability of $X^0$ can be established using the same method adopted in [1] to prove [1, Theorem 3.2], which uses [1, Lemma 8] to establish a contradiction when $X^0$ is assumed not stable. That $X^0$ is irreducible follows from the fact that, under any policy $\theta$ in $\mathcal{X} \Phi^+_R(\tau)$, all states of $X^0$ communicate with $(1, A, 0)$. That the probability of transitioning away from $(1, A, 0)$ is less than one implies that the chain is aperiodic.

An immediate consequence of Proposition 1 is that $\{ \mathcal{X}(\phi) : \phi \in \Phi^+_R(\tau) \}$ is a nonempty subset of $\Theta_S(\lambda)$ when $\lambda < \tau \leq \tau^*$. This implies that, as far as stabilizability is concerned, there is no loss of generality in restricting our analysis to policies with the structure in $\Phi^+_R$. More interestingly, from Theorem 11 which will be stated and proved later on in Section III we can conclude that restricting our methods for solving Problem 2 to policies of the form $\Phi^+_R$ also incurs no loss of generality.

The following projection map will be important going forward.

**Definition 11. (Policy projection map $\mathcal{Y}$)** Given $\lambda$ in $(0, \tau^*)$, we define a mapping $\mathcal{Y} : \Theta_S(\lambda) \rightarrow \Phi^+_R$, where

$$\mathcal{Y}(\theta) := \varphi^\theta, \theta \in \Theta_S(\lambda)$$

with

$$\varphi^\theta(y) := \frac{\sum_{q \in \mathbb{N}} \theta(q, q) \pi^\theta(q, y)}{\sum_{q \in \mathbb{N}} \pi^\theta(q, y)}, \quad y \in \mathbb{Y},$$

where $\mathbb{Y} := \{ q \in \mathbb{N} | (\mathbb{Y}, q) \in X \}, \mathbb{Y} \in \mathbb{Y}$.

Notice that although the map $\mathcal{Y}$ depends on $\lambda$, for simplicity of notation, we chose not to denote this dependence explicitly. It is worthwhile to note that the map $\mathcal{Y}$, for a given $\lambda$ less than $\tau^*$, allows us to establish the following remark comparing the service rate notions for $X$ and $\mathbb{Y}$.

**Remark 2.** Given $\lambda$ in $(0, \tau^*)$ and $\tau$ in $(\lambda, \tau^*)$, our analysis in [1] implies that the following hold:

$$\lambda \overset{(i)}{=} \nu^\theta \overset{(ii)}{=} \mathcal{Y}(\theta) \leq \tau^*, \quad \theta \in \Theta_S(\lambda) \quad (14a)$$

$$\lambda \overset{(iii)}{=} \nu^\phi \overset{(i)}{<} \tau \leq \tau^*, \phi \in \Phi^+_R(\tau) \quad (14b)$$

where $\nu^\theta := \sum_{x \in X} \pi^\theta(x \theta(x) \mu(x)$ is the service rate of $X^0$. Notably, (i) and (ii) follow from [1, Lemma 4]. Using a similar argument, (iii) follows from the fact that $\mathcal{X}(\phi)$ is stabilizing, as guaranteed by Proposition 1 when $\tau$ is in $(\lambda, \tau^*)$.

F. Utilization Rate of $\mathbb{Y}$ and Computation via LP

We now proceed to defining the utilization rate of $\mathbb{Y}^\tau$ for a given $\phi$ in $\Phi^+_R$. Subsequently, we will define and propose a linear programming approach to computing the infimum of the utilization rates attainable by any policy for $\mathbb{Y}$ subject to a given service rate.

**Definition 12.** Given a policy $\phi$ in $\Phi^+_R$, the following function determines the utilization rate of $\mathbb{Y}^\tau$:

$$\mathcal{U}(\phi) := \sum_{\mathbb{Y} \in \mathbb{Y}} \pi^\phi(y) \mathcal{Y}(\phi)$$

**Definition 13. (Infimum utilization rate $\mathcal{U}^+_R$ and $\mathcal{U}^+_{\mathbb{Y}}$)** The infimum utilization rate of $\mathbb{Y}$ for a given service rate $\tau$ is defined as

$$\mathcal{U}^+_R(\tau) := \inf_{\phi \in \Phi^+_R(\tau)} \sum_{\mathbb{Y} \in \mathbb{Y}} \pi^\phi(y) \mathcal{Y}(\phi), \quad \epsilon \in (0, 1]$$

We also define the following approximate infimum utilization rates:

$$\tilde{\mathcal{U}}^+_R(\tau) := \inf_{\phi \in \Phi^+_R(\tau)} \sum_{\mathbb{Y} \in \mathbb{Y}} \pi^\phi(y) \mathcal{Y}(\phi), \quad \epsilon \in (0, 1]$$

Notice that the infimum that determines $\tilde{\mathcal{U}}^+_R$ and $\mathcal{U}^+_{\mathbb{Y}}$ is well-defined because there is a unique stationary PMF $\pi^\phi$ for each policy $\phi$ in $\Phi^+_R$.

**Remark 3.** Notice that since $\Phi^+_R(\tau) = \bigcup_{\epsilon \in (0, 1]} \Phi^+_R(\tau)$, we conclude that the following holds:

$$\tilde{\mathcal{U}}^+_R(\tau) = \lim_{\epsilon \rightarrow 0^+} \mathcal{U}^+_R(\tau) \quad (15)$$
We now proceed to outlining efficient ways to compute \( \mathcal{Z}_R^+ \), which is relevant because, as Corollary 1 indicates in (111) we can use it to compute \( \mathcal{B}^*(\lambda) \) when \( \lambda < \nu^* \). Hence, below we follow the approach in [21 Chapter 4] to construct approximate versions of \( \mathcal{Z}_R^+ \) that are computable using a finite-dimensional LP. Subsequently, we will obtain the policies in \( \Phi_R^+ \) corresponding to solutions of the LP, as is done in [21 Chapter 4]. The policies obtained in this way will form a set for each \( \epsilon \in (0, 1) \) that will be useful later on.

**Definition 14. (\( \epsilon \)-LP utilization rate \( \mathcal{Z}_L^\epsilon(\nu) \))** Let \( \epsilon \) be a given constant in \([0, 1]\) and \( \nu \) be a pre-selected service rate in \([0, \nu^*]\). The \( \epsilon \)-LP utilization rate \( \mathcal{Z}_L^\epsilon(\nu) \) is defined as:

\[
\mathcal{Z}_L^\epsilon(\nu) := \min_{\ell \in \mathcal{L}} \sum_{Y \in \mathcal{Y}} \ell_{Y,W} \quad \text{s.t. (16b)-(16e)}
\]

where the minimization is carried out over the following set:

\[
\mathcal{L} := \prod_{\pi \in \mathcal{T}_R, Y \in \mathcal{Y}} \{\ell_{Y,W} \geq 0\}
\]

Every solution is subject to the following constraints and is compactly represented as \( \ell := \prod_{\pi \in \mathcal{T}_R, Y \in \mathcal{Y}} \{\ell_{Y,W}\} \):

\[
(1 - \epsilon)\ell(1, A, W) \geq \epsilon \ell(1, A, R) \quad \text{(16b)}
\]

\[
\sum_{\{Y \in \mathcal{Y} \mid W \in \mathcal{W}_R\}} \mu(\pi)\ell_{Y,W} = \nu \quad \text{(16c)}
\]

\[
\sum_{Y \in \mathcal{Y}} \sum_{\pi \in \mathcal{T}_R} \ell_{Y,\pi} = 1 \quad \text{(16d)}
\]

and the equality below guarantees that every solution will be consistent with \( \nu \):

\[
\sum_{Y \in \mathcal{Y}} \sum_{\pi \in \mathcal{T}_R} \ell_{Y,\pi} R_{Y+1,Y,\pi,\pi}(\nu, \nu') = \sum_{\pi \in \mathcal{T}_R} \ell_{Y,\pi}, \quad Y \in \mathcal{Y} \quad \text{(16e)}
\]

**Definition 15. (Solution set \( \mathcal{L}^\epsilon(\nu) \))** For each \( \epsilon \in [0, 1] \) and \( \nu \) in \((0, \nu^*)\), we use \( \mathcal{L}^\epsilon(\nu) \) to represent the set of solutions of the LP specified by (16). We adopt the convention that \( \mathcal{L}^\epsilon(\nu) \) is empty if and only if the LP is not feasible.

**G. LP-based Policy Sets**

For each solution \( \ell \in \mathcal{L}^\epsilon(\nu) \) we can obtain a corresponding policy \( \phi_\ell \) in \( \Phi_R^+ \) for \( \mathcal{Y} \) as follows:

\[
\phi_\ell(Y) := \begin{cases} 
\ell_{Y,W} & \text{if} \ R \in \mathcal{T}_R \text{ and } \ell_{Y,R} > 0, \ Y \in \mathcal{Y} \\
1 & \text{otherwise}, \ Y \in \mathcal{Y}
\end{cases}
\]

**Remark 4.** Subject to the definition in (17), the constraint (16b) is equivalent to \( \phi_\ell(1, A) \geq \epsilon \), which holds for every solution \( \ell \in \mathcal{L}^\epsilon(\nu) \).
Proof. It follows immediately from [21] Theorem 4.3] that (18a) holds and \( \Phi_L^δ(\tau) \) dominates \( \Phi_R^δ(\tau) \). Furthermore, Proposition 5 from [IV] implies that the following limit holds:

\[
\lim_{\epsilon \to 0^+} \mathcal{W}_L^\epsilon(\tau) = \mathcal{W}_L^0(\tau) \tag{19}
\]

That (18b) holds is a consequence of (15), (18a) and (19).

Before proceeding to describe our main results, we define the following class of policies \( X \) that can be generated from solutions of the LP (16):

\( \mathcal{X} \Phi_L^\epsilon(\tau) := \{ \mathcal{X}(\phi) : \phi \in \Phi_L^\epsilon(\tau), \tau \in (0, \tau^*) \}, \epsilon \in (0, 1] \)

See Fig. 1 for a representation of the relationships among most of the policy sets for the MDPs \( X \) and \( Y \).

III. MAIN RESULTS

This section starts with Theorem 1 which is our main result. Subsequently, we state Corollaries 1 and 2 that undergird our methods to tackle Problems 1 and 2, respectively.

Theorem 1. Let an arrival rate \( \lambda \) in \( (0, \tau^*) \) be given. For each positive gap \( \delta \), there exist a service rate \( \tau^{\delta, \lambda} \) in \( (\lambda, \tau^*) \) and \( e^{\delta, \lambda} \) in \( (0, 1] \) such that \( \Phi_L^{e^{\delta, \lambda}}(\tau^{\delta, \lambda}) \) is nonempty and the following inequality holds:

\[
\mathcal{W}(\lambda, \theta) \leq \mathcal{W}_R^\delta(\lambda) + \delta, \quad \theta \in \mathcal{X} \Phi_L^{e^{\delta, \lambda}}(\tau^{\delta, \lambda}) \tag{20}
\]

Remarks 5 and 6 will expound the importance of Theorem 1 and its two corollaries. Our proof of the theorem given below relies on the continuity properties and distributional convergence results established in [IV] and [V] respectively.

Proof. Since it follows from Theorem 2 in [IV] that \( \mathcal{W}_L^0 \) is continuous and non-decreasing, we know that there is \( \tau^\dagger \) in \( (\lambda, \tau^*) \) such that the following inequality holds:

\[
\mathcal{W}_L^0(\tau^\dagger) \leq \mathcal{W}_L^0(\lambda) + \frac{1}{2} \delta \tag{21}
\]

Since \( \lambda \) is stabilizable, a stabilizing policy \( \theta \in \Theta_S(\lambda) \) exists. By [11] Lemma 2, \( \mathcal{W}(\theta) \) has non-zero probability to choose to work at state \( (1, A) \) and \( \mathcal{W}(\theta) = \lambda \) by (13a). Therefore, \( \Phi_R^\epsilon(\lambda) \) is nonempty for some positive \( \epsilon^\dagger \). From Proposition 5 in [IV] we can select \( \epsilon^\dagger \) in \( (0, \epsilon^\dagger] \) such that the following holds:

\[
\mathcal{W}_L^\epsilon(\tau^\dagger) \leq \mathcal{W}_L^0(\tau^\dagger) + \frac{1}{2} \delta, \quad \epsilon \in (0, \epsilon^\dagger] \tag{22}
\]

From Proposition 6 in [IV] we know that we can choose \( e^{\delta, \lambda} \) in \( (0, \epsilon^\dagger] \) such that the following holds:

\[
\mathcal{W}_L^{e^{\delta, \lambda}}(\tau^\dagger) \leq \mathcal{W}_L^{e^{\delta, \lambda}}(\tau^\dagger), \quad \tau^\dagger \in (\lambda, \tau^\dagger) \tag{23}
\]

In [IV] we develop in sequence several results that ultimately lead to Theorem 3 which establishes an important distributional convergence result that takes hold when \( \tau \) in \( (\lambda, \tau^\dagger) \) is selected as close as needed to \( \lambda \). Using Corollary 5 stated also in [IV] which follows immediately from Theorem 3 we conclude that, based on our choice of \( e^{\delta, \lambda} \) above, we can select \( \tau^{\delta, \lambda} \) in \( (\lambda, \tau^\dagger) \) such that the following inequality holds:

\[
\mathcal{W}(\lambda, \mathcal{X}(\phi)) \leq \mathcal{W}(\phi) + \frac{1}{2} \delta, \quad \phi \in \Phi_L^{e^{\delta, \lambda}}(\tau^{\delta, \lambda}) \tag{24}
\]

Hence, using our choices for \( e^{\delta, \lambda} \) and \( \tau^{\delta, \lambda} \), we infer from (21-24) that the following inequality holds:

\[
\mathcal{W}(\lambda, \mathcal{X}(\phi)) \leq \mathcal{W}_L^0(\lambda) + \delta, \quad \phi \in \Phi_L^{e^{\delta, \lambda}}(\tau^{\delta, \lambda}) \tag{25}
\]

which, together with (18b), leads to (20).

We proceed with stating a proposition that provides an utilization-rate counterpart for (ii) in (14a) and whose proof we omit because it follows immediately from [11] Lemmas 3 and 4.

Proposition 4. Given \( \lambda \) in \( (0, \tau^*) \), the following equality holds for any \( \theta \) in \( \Theta_S(\lambda) \):

\[
\mathcal{W}(\mathcal{W}(\theta)) = \mathcal{W}(\lambda, \theta) \tag{26}
\]

Corollary 1. The following equality holds:

\[
\mathcal{W}(\lambda, \tau^\dagger) = \mathcal{W}_R^\dagger(\lambda), \quad \lambda \in (0, \tau^*) \tag{27}
\]

Proof. It ensues from Proposition 3 and (i)-(ii) in (14a) that the following holds for any \( \lambda \) in \( (0, \tau^*) \):

\[
\mathcal{W}(\lambda, \theta) = \mathcal{W}(\mathcal{W}(\theta)) \geq \mathcal{W}_R^\dagger(\lambda), \quad \theta \in \Theta_S(\lambda) \tag{28}
\]

Since the inequality above holds for any \( \theta \) in \( \Theta_S(\lambda) \), the following inequality is satisfied for any \( \lambda \) in \( (0, \tau^*) \):

\[
\mathcal{W}(\lambda) \geq \mathcal{W}_R^\dagger(\lambda) \tag{29}
\]

We conclude the proof by remarking that (29) and Theorem 1 imply (27).

Remark 5 (Solution of Problem 1). Corollary 1 is significant because, in conjunction with (18b), it indicates that \( \mathcal{W}(\lambda) \) can be computed using the finite dimensional LP (16) for \( \epsilon = 0 \) and \( \tau = \lambda \).

Section IV-A discusses a numerical example and a graphical method to determine \( \mathcal{W}_L^0(\tau) \) for all values of \( \tau \) in \( [0, \tau^*] \). The graphical method leverages the analysis in Appendix A which establishes that \( \mathcal{W}_L^0(\tau) \) is non-decreasing and convex.

The following corollary follows directly from Theorem 1 and Corollary 1.

Corollary 2. Let an arrival rate \( \lambda \) in \( (0, \tau^*) \) be given. For each positive gap \( \delta \) there exist a service rate \( \tau^{\delta, \lambda} \) in \( (\lambda, \tau^*) \) and \( e^{\delta, \lambda} \) in \( (0, 1] \) such that \( \Phi_L^{e^{\delta, \lambda}}(\tau^{\delta, \lambda}) \) is nonempty and the following inequality holds:

\[
\mathcal{W}(\lambda, \mathcal{X}(\phi)) \leq \mathcal{W}_R^\dagger(\lambda) + \delta, \quad \phi \in \mathcal{X} \Phi_L^{e^{\delta, \lambda}}(\tau^{\delta, \lambda}) \tag{30}
\]

Remark 6 (Solution to Problem 2). While, as explained in Remark 5 \( \mathcal{W}(\lambda) \) can be computed effectively for any stabilizable \( \lambda \), Corollary 2 ascertains that we can address Problem 2. Specifically, given a stabilizable \( \lambda \) and any positive gap \( \delta \), Corollary 2 guarantees that we can find \( \tau \) and \( \epsilon \) such that any policy \( \theta \) in \( \mathcal{X} \Phi_L^\epsilon(\tau) \) is not only stabilizing but the utilization rate of \( \mathcal{X} \theta \) does not exceed \( \mathcal{W}(\lambda) + \delta \). The proof of Theorem 1 outlines a method for selecting such \( \tau \) and \( \epsilon \). This is a significant result because any solution of the LP (16) can be used to obtain a policy in \( \mathcal{X} \Phi_L^\epsilon(\tau) \).
IV. CONTINUITY AND MONOTONICITY OF $\mathcal{U}_L^\epsilon$

We proceed with establishing three properties of $\mathcal{U}_L^\epsilon$ that are needed in the proof of our main results in [II].

The following proposition establishes that when, for a given $\boldsymbol{\tau}$ in $(0, \tau^\epsilon)$, $\mathcal{H}^\epsilon(\boldsymbol{\tau})$ is viewed as a function of $\epsilon$, it is right continuous at $0$.

**Proposition 5.** Let $\boldsymbol{\tau}$ in $(0, \tau^\epsilon)$ be given. For any positive $\delta$, there is $\epsilon$ such that $\mathcal{U}_L^\epsilon(\boldsymbol{\tau}) \leq \mathcal{U}_L^\epsilon(\boldsymbol{\tau}) + \delta$.

**Proof.** The statement of the proposition is false if and only if there exists some $\boldsymbol{\tau}$ in $(0, \tau^\epsilon)$ for which $d := \lim_{\epsilon \to 0^+} \mathcal{U}_L^\epsilon(\boldsymbol{\tau}) - \mathcal{U}_L^\epsilon(\boldsymbol{\tau}) > 0$. We proceed to proving the proposition by contradiction by showing that the inequality above does not hold. Take $\epsilon$ positive such that $d := \mathcal{U}_L^\epsilon(\boldsymbol{\tau}) - \mathcal{U}_L^\epsilon(\boldsymbol{\tau})$ is in $[\underline{d}, \overline{d}]$. Select $\ell^\epsilon$ and $\ell^0$ in $\mathbb{L}^\epsilon(\boldsymbol{\tau})$ and $\mathbb{L}^0(\boldsymbol{\tau})$, respectively. Define $\ell^{\epsilon \nu} := \frac{1}{\nu}(\ell^\epsilon + 2\ell^0)$, which satisfies (16a)-(16c). Given that $\epsilon$ is positive, $\ell^{\epsilon \nu}$ will also satisfy (16b) for some positive $\epsilon^*$, which implies that $\mathcal{U}_L^\epsilon(\boldsymbol{\tau}) - \mathcal{U}_L^\epsilon(\boldsymbol{\tau}) \leq \frac{1}{\nu}d < \overline{d}$. \hfill $\square$

The following proposition establishes a useful monotonicity property in terms of $\boldsymbol{\tau}$.

**Proposition 6.** Let $\overline{\boldsymbol{\tau}}$ and $\overline{\boldsymbol{\tau}}$ in $(0, \tau^\epsilon)$ be given with $\overline{\boldsymbol{\tau}} \leq \overline{\boldsymbol{\tau}}$. There exists a positive $\epsilon^*$ such that the following holds:

$$\mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}}) \leq \mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}}), \quad \overline{\boldsymbol{\tau}} \in (\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}), \quad \epsilon \in (0, \epsilon^*)$$

**Proof.** From (16a), (16c), and the fact that $\min_{s \in S} \mu(s)$ is positive, we get

$$\mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}}) \leq \frac{1}{\min_{s \in S} \mu(s)} \overline{\boldsymbol{\tau}}, \quad \overline{\boldsymbol{\tau}} \in (0, \tau^\epsilon), \quad \epsilon \in (0, 1]$$

We can find $\overline{\boldsymbol{\tau}}$ in $(0, \tau^\epsilon)$ such that the following inequality holds:

$$\frac{1}{\min_{s \in S} \mu(s)} \overline{\boldsymbol{\tau}} \leq \mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}})$$

Since $\overline{\boldsymbol{\tau}}$ is stabilizable, a stabilizing policy $\theta \in \Theta_S(\overline{\boldsymbol{\tau}})$ exists. By (II) Lemma 2, $\phi^\theta := \mu(\theta)$ has non-zero probability to choose to work at state $(1, A)$ and $\mathcal{U}_L^\phi = \overline{\boldsymbol{\tau}}$ by (14a). We construct an $\ell^{\epsilon \nu}$ by the following definitions:

$$\ell_{\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}}^{\epsilon \nu} := \overline{\boldsymbol{\tau}} \phi^\theta(\overline{\boldsymbol{\tau}})$$

$$\ell_{\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}}}^{\epsilon \nu} := \overline{\boldsymbol{\tau}} \phi^\theta(1 - \phi^\theta(\overline{\boldsymbol{\tau}})) \quad \overline{\boldsymbol{\tau}} \in \overline{\boldsymbol{\tau}}$$

It is clear that $\ell^{\epsilon \nu}$ satisfies (16b) with some positive $\epsilon^*$ since $\phi^\theta(1, A) > 0$ and all other constraints in (16). Therefore, $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$ is nonempty for some positive $\epsilon^*$. Consequently, we can further invoke Proposition 2 to infer that $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$ and $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$ are nonempty for every $\epsilon \in (0, \epsilon^*)$. Now, let $\epsilon$ be an arbitrary selection in $(0, \epsilon^*)$ and $\ell^{\epsilon \nu}$ and $\ell^{\epsilon \nu}$ be elements of $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$ and $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$, respectively. From (16c) we conclude that, for any $\boldsymbol{\tau}$ in $(\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\tau}})$, $\ell^{\epsilon \nu} := (\mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}}))^{(\overline{\boldsymbol{\tau}})} + (\mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}}))^{(\overline{\boldsymbol{\tau}})} \overline{\boldsymbol{\tau}} - \overline{\boldsymbol{\tau}} \overline{\boldsymbol{\tau}}$ satisfies (16b)-(16c). From (16a) and the definition of $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$ and $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$, we use $\ell^{\epsilon \nu}$ in $\mathbb{L}^\epsilon(\overline{\boldsymbol{\tau}})$ and obtain the following inequality:

Furthermore, from (30) and (31), the following inequalities hold, which completes the proof:

$$\mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}}) \leq \frac{p - \overline{\boldsymbol{\tau}} - \mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}})}{p - \overline{\boldsymbol{\tau}}} + \frac{p - \overline{\boldsymbol{\tau}} - \mathcal{U}_L^\epsilon(\overline{\boldsymbol{\tau}})}{p - \overline{\boldsymbol{\tau}}}$$

The following theorem establishes important structural properties for $\mathcal{U}_L^0$. We provide a proof of the theorem in Appendix A.

**Theorem 2.** The 0-LP utilization rate function $\mathcal{U}_L^0 : [0, \tau^\epsilon] \to [0, 1]$ is non-decreasing, piecewise affine and convex.

A. A Graphical Method and Numerical Example

We proceed to describe a method to obtain $U_L^\phi(\phi^\theta)$ graphically. The main idea is to use our proof for Theorem 2 (Appendix A) to establish the following three-step method:

(Step 1) Compute $\mathcal{U}_L^\phi(\phi^\theta)$ and $\mathcal{U}_L^\phi(\phi^\theta)$ for all $\phi^\theta$ in $\{1, \ldots, n_s + 1\}$.

(Step 2) Identify the convex hull of the set $\{(\mathcal{U}_L^\phi, \mathcal{U}_L^\phi) : \phi^\theta \in \{1, \ldots, n_s + 1\}\}$.

(Step 3) Determine $\mathcal{U}_L^\phi(\phi^\theta) : [0, \tau^\epsilon] \to [0, 1]$ as the lower boundary of the convex hull.

We will use the following example to illustrate our method, and to motivate the observations at the end of this section.

**Example 1.** Consider that the system is characterized by $n_s = 5$ and the following transition probabilities for $S_\nu$, which approximate the differential equation that describes the server state evolution in [2]:

$$\rho_{s,s+1} := \frac{s + s - 1}{5}, \quad \rho_{s,s-1} := \frac{s + s - 1}{5}$$

The service rate function $\mu(1, \ldots, \mu(5))$ is set to be $0.01, 0.5, 0.2, 0.5, 0.05$.

We now proceed to apply the graphical method to our example. The following table lists the results obtained from step 1.

| $\tau$ | 1 | 2 | 3 | 4 | 5 | 6 |
|--------|---|---|---|---|---|---|
| $\mathcal{U}_L^\phi$ | 0.0000 | 0.0347 | 0.1993 | 0.1947 | 0.3000 | 0.0500 |
| $\mathcal{U}_L^\phi(\phi^\theta)$ | 0.0000 | 0.2383 | 0.4309 | 0.6316 | 0.8571 | 1.0000 |

**TABLE II**

Results of step 1 applied to Example 1.

The pairs $(\mathcal{U}_L^\phi, \mathcal{U}_L^\phi(\phi^\theta))$, for $\phi^\theta$ in $\{1, \ldots, n_s + 1\}$, are the centers of the dark-red circles in the following figure, and the shaded area is their convex hull, whose construction is step 2. Finally, as described in step 3 and represented in the figure, the lower boundary of the convex hull is $\mathcal{U}_L^\phi(\phi^\theta) : [0, \tau^\epsilon] \to [0, 1]$. 

Our analysis for this example also leads to the following observations. (i) As established by \[1\] (8) and Theorems 1 and 2], \(v_n\) is maximized by a threshold policy. For our example, \(v_5\) is 0.3, which is achieved for \(\phi_5\) when \(\tau = 5\). (ii) The corner points of \(\Lambda_0(v)\) are among the pairs obtained in step 1. (iii) As our example illustrates, \(\Lambda_0^\tau\) is not necessarily monotonic with respect to \(\tau\).

V. KEY DISTRIBUTIONAL CONVERGENCE RESULTS: A POTENTIAL-LIKE APPROACH

We start with the following lemma that is applicable for any time-homogeneous finite Markov chain. It establishes the existence of a potential-like function that will be useful later on. The proof of the lemma is deferred to Appendix C.

**Lemma 2.** Let a time-homogeneous Markov chain \(M := \{M_k : k \in \mathbb{N}\}\) taking values in a finite set \(M\) and a reward function \(R : M \times M \to \mathbb{R}_+\) be given. If \(M\) has a unique recurrent communicating class, there exists a map \(\mathcal{H} : M \to \mathbb{R}_+, \) which we designate as potential-like, for which the following holds for every \(m \in M\):

\[
\begin{align*}
\mathbb{E} \left[ R(M_{k+1}, M_k) \mid M_k = m \right] &= \mathbb{E} \left[ \mathcal{H}(M_{k+1}) - \mathcal{H}(M_k) \mid M_k = m \right] + r_{avg},
\end{align*}
\]

where the average reward \(r_{avg}\) can be computed using the stationary PMF \(\phi^M : M \to [0, 1]\) of \(M\) as

\[
r_{avg} := \sum_{m \in M} \mathbb{E} \left[ R(M_{k+1}, M_k) \mid M_k = m \right] \phi^M(m).
\]

The following lemma is the first step towards proving Theorem 3 which is the main result of this section.

**Lemma 3.** Let \(\lambda \in (0, \tau')\) and \(\epsilon \in (0, 1)\) be given. If \(\Phi^\mu_\lambda(\lambda)\) is nonempty, there is a positive constant \(\beta_\lambda, \epsilon\) such that the following inequality holds for every \(v \in (\lambda, \tau')\):

\[
\sum_{s \in S} \pi^\mu(s, A, 0) \leq \frac{\lambda - \lambda}{\beta_\lambda, \epsilon}, \quad \theta = \mathcal{X} \Phi^\mu_\lambda(v)
\]

Before we proceed with the proof of Lemma 3 we note that one should expect it to be somewhat involved because it needs to ascertain that the inequality in (33) holds (uniformly) for all policies in \(\mathcal{X} \Phi^\mu_\lambda(v)\). We decided to include the proof below, as opposed to deferring it to an appendix, because we find it to involve an instructive use of a potential-like function guaranteed by Lemma 2 to exist.

**Proof.** Select \(v \in (\lambda, \tau')\), and let \(\phi\) be any policy in \(\Phi^\mu_\lambda(v)\), which we know from Proposition 2 is nonempty, and set \(\theta = \mathcal{X} \Phi^\mu_\lambda(v)\). Recall that \(X^\mu\) is stable by Proposition 1. In our proof we will make use of Lemma 2 by selecting \(M = \mathcal{Y}^\mu\) and \(\mathcal{R}(y', y) = \mu(s)\) for all \(y'\) and \(y\) in \(\mathcal{Y}\), where we recall that \(y := (s, w)\). We define \(s^* = \arg\max_{s \in S} \mathcal{H}(s, A)\), where \(\mathcal{H}\) is the potential-like map obtained from Lemma 2 for the aforementioned choices of \(M\) and \(\mathcal{R}\).

The following hitting time will be central in our proof:

\[
T^\mu_\lambda := \min \{ k \geq 1 \mid X^\mu_{k} = (s^*, A, 0), \ X_0 = x \},
\]

where we adopt the convention that \(T^\mu_\lambda\) is infinite if \(X^\mu_{k} = (s^*, A, 0)\) never occurs for \(k \geq 1\). We will also use the following lower bound:

\[
T^\mu_\lambda := \min \{ k \geq 1 \mid y'(X^\mu_{k}) \leq v^*, \ X_0 = x \},
\]

where \(y'(x) := q + \mathcal{H}(y)\) and \(v^* := \mathcal{H}(s^*, A)\). Here, we also adopt the convention that \(T^\mu_\lambda\) is infinite if \(y'(X^\mu_{k}) \leq v^*\) never occurs for \(k \geq 1\). Notice that since \(y'(s^*, A, 0) = v^*\), we have \(T^\mu_\lambda \leq T^\mu_\lambda, x \in X\).

Subsequently, we use \(T^\mu_\lambda, T^\mu_\lambda, s^*, A, 0\) and \(\lambda\) to obtain a lower bound for \(E[T^\mu_{(s^*, A, 0)}]\) - the return time of \((s^*, A, 0)\) - which will ultimately lead to the proof of (33).

As we argue subsequently, the following lower bound for \(E[T^\mu_{(s^*, A, 0)}]\), which we will derive later in this proof, leads to (33) almost immediately:

\[
E[T^\mu_{(s^*, A, 0)}] \geq \frac{1}{v/\lambda - 1}
\]

We start by using the law of total probability to conclude that the following inequality holds:

\[
\begin{align*}
E[T^\mu_{(s^*, A, 0)}] &\geq (1 + E[T^\mu_{(s^*, A, 0)}])P^\mu_{(s^*, A, 1)}((s^*, A, 1)|(s^*, A, 0)) \\
&\geq (1 + E[T^\mu_{(s^*, A, 0)}])P^\mu_{(s^*, A, 1)}((s^*, A, 1)|(s^*, A, 0)) \\
&\geq (1 + E[T^\mu_{(s^*, A, 0)}])\min((1 - \rho_{s^*, s', -1}) + \frac{v/\lambda - 1}{1 - \rho_{s^*, s', -1}})
\end{align*}
\]

According to (22) (Theorem, p. 227), (33) implies that:

\[
\pi^\mu(s^*, A, 0) \leq \frac{\lambda - \lambda}{1 - \rho_{s^*, s', -1}}
\]

At this point we intend to use the following inequality to relate \(\pi^\mu(s^*, A, 0)\) with \(\sum_{s \in S} \pi^\mu_A(s, A, 0)\) :

\[
\pi^\mu(s^*, A, 0) \geq \sum_{s \in S} \pi^\mu(s, A, 0)P(X^\mu_{k+2n_s} = (s^*, A, 0)|X^\mu_k = (s, A, 0))
\]

Recall from Proposition 1 that \(X^\mu\) is irreducible. Moreover, we can show that there is positive \(\beta_{\lambda, \epsilon}\) satisfying

\[
P(X^\mu_{k+2n_s} = (s', A, 0)|X^\mu_k = (s, A, 0)) \geq \beta_{\lambda, \epsilon}
\]

for all \(\theta \in \mathcal{X} \Phi^\mu_\lambda(v)\). For example, one can verify

\[
\beta_{\lambda, \epsilon} = \epsilon(1 - \lambda)^{2n_s} \min_{i \in S} (1 - \mu(i))^{2n_s} \min_{j \in S} \mu(j)
\]

satisfies the inequality in (37). The proof of (33) follows from (36) and (37) with \(\beta_{\lambda, \epsilon} := (1 - \rho_{s^*, s', -1})\beta_{\lambda, \epsilon}\).
Proof of (34): We now proceed to proving that (34) holds. We start with the following equalities that hold for any \( x \) satisfying \( V(x) > v^* \), which implies \( q > 0 \):

\[
E[Q_k^\theta - qX_{k-1}^\theta = x] = \lambda - \Phi(s, w)\mu(s) \tag{39}
\]

\[
E[H(X_k^\theta) - H(y)|X_{k-1}^\theta = x] = E[H(Y_k^\theta) - H(y)|Y_{k-1}^\theta = y] \tag{40}
\]

\[
(\psi) = \Phi(s, w)\mu(s) - \mu'(\bar{\tau})
\]

In proving (39) and (40), we used the fact that if \( V(x) > v^* \) holds, \( q \geq 1 \), which, since \( \theta = H(\phi) \), implies that the policy \( \phi \) is applied. In addition, we used Lemma 2 to establish (i), where we used the fact that, for our choices of \( M \) and \( R \), \( r_{avg} \) is \( \bar{\tau} \). Hence, by adding the terms of (39) and (40) we conclude that the following holds when \( x \) is such that \( V(x) > v^* \) holds:

\[
E[V(X_k^\theta) - V(x)|X_{k-1}^\theta = x] = \lambda - \bar{\tau}. \tag{41}
\]

Because \( T_{Y_k^\theta}^{s^*,A,1} \geq k \) implies \( V(X_k^\theta) > v^* \), from (41) we obtain

\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ (V(X_k^\theta) - V(x)|X_{k-1}^\theta = x) \mid X_0 = (s^*, A, 1) \right] = 1 \tag{42}
\]

where \( T_{Y_k^\theta}^{s^*,A,1} \leq k \) is 1 when \( T_{Y_k^\theta}^{s^*,A,1} > k \) holds, and is 0 otherwise.

By the definition of \( V(x) = q + H(y) \), conditional on \( \{X_0 = (s^*, A, 1)\} \), for every \( K \) in \( N \),

\[
\sum_{k=1}^{N} \left( \frac{1}{k} \right) (V(X_k^\theta) - V(x)|X_{k-1}^\theta = x) \mid X_0 = (s^*, A, 1) \leq \min \{K, T_{Y_k^\theta}^{s^*,A,1} \} + \max y \in Y H(y) - v^* \]

Since \( X_k^\theta \) is positive recurrent, \( \mathbb{E} \left[ T_{Y_k^\theta}^{s^*,A,1} \right] < \infty \). Therefore, the dominated convergence theorem ([22], pp.179-180) allows us to interchange the order of the summation and the expectation in (42). After the interchange, we have

\[
\mathbb{E} \left[ \sum_{k=1}^{\infty} (V(X_k^\theta) - V(x)|X_{k-1}^\theta = x) \mid X_0 = (s^*, A, 1) \right] = \mathbb{E} \left[ V(X_0^\theta) - V(x) \mid X_0 = (s^*, A, 1) \right] \leq \min \{K, T_{Y_k^\theta}^{s^*,A,1} \} + \max y \in Y H(y) - v^*
\]

Theorem 3. Let \( \lambda \in (0, \bar{\tau}) \) and \( \epsilon \in (0, 1) \) be given. Suppose \( \Phi_\theta(\lambda) \) is non-empty, and let \( \beta_{\lambda, \epsilon} \) be a positive constant satisfying Lemma 2. Then, there is a positive constant \( \eta \), such that the following inequality holds for all \( \bar{\tau} \in (\lambda, \bar{\tau})^* \):

\[
\sum_{y \in Y} \left| \mathcal{P}_\theta(y) - \sum_{q \geq 0} \mathcal{P}_\theta(y, q) \right| \leq \frac{\beta_{\lambda, \epsilon} + \eta}{\beta_{\lambda, \epsilon}} (\bar{\tau} - \lambda)^2 + \frac{3}{\beta_{\lambda, \epsilon}} (\bar{\tau} - \lambda)
\]

In Appendix [1], we provide a proof for Theorem 3, where the existence of \( \beta_{\lambda, \epsilon} \) follows from Lemma 3. As was the case with Lemma 2, but even more so, the proof of Theorem 3 is rather involved because the inequality in (43) must hold uniformly for all \( \phi \) in \( \Phi_\theta(\bar{\tau}) \). The following corollary is an immediate consequence of Theorem 3.

Corollary 3. Let \( \lambda \in (0, \bar{\tau}) \) and \( \epsilon \in (0, 1) \) be given. If \( \Phi_\theta(\lambda) \) is non-empty, there is a positive constant \( \eta_{\lambda, \epsilon} \) such that the following inequality holds for all \( \bar{\tau} \in (\lambda, \bar{\tau})^* \):

\[
\left| \mathcal{P}_\theta(\phi) - \mathcal{P}_\theta(\bar{\tau}, \bar{\tau} \phi) \right| \leq \frac{\beta_{\lambda, \epsilon} + \eta_{\lambda, \epsilon}}{\beta_{\lambda, \epsilon}} (\bar{\tau} - \lambda)^2 + \frac{3}{\beta_{\lambda, \epsilon}} (\bar{\tau} - \lambda)
\]

We remind the reader that \( \mathcal{P}_\theta(\phi) \) implicitly depends on \( \lambda \) via the stationary PMF \( \mathcal{P}_\theta^{\lambda} \) in Definition [2].

VI. CONCLUSIONS

We put forth a methodology to design policies that schedule tasks from a queue to a server whose performance depends on the scheduling history. Our approach builds on and generalizes previous work that sought to design stabilizing non-preemptive policies. This article introduces methods to design non-preemptive policies that are not only stabilizing but also lessen the so-called utilization rate, which accounts for the proportion of time the server is working. Given a rate of arrival of tasks at the queue, our two main results yield a tractable method to compute the infimum of the utilization rates that are attainable by all stabilizable non-preemptive policies, and characterize subsets of conveniently-structured policies whose utilization rate is arbitrarily close to the infimum.

APPENDIX

A. Structural Results for \( \mathcal{P}_\theta(\bar{\tau}) \) and Proof of Theorem 2

Define \( \Phi^+ \) to be the set of policies in \( \Phi_\theta(\bar{\tau}) \) which are deterministic except for at most at one state where the policy randomizes between two admissible actions. In other words,

\[
\Phi^+ := \left\{ \phi \in \Phi_\theta \mid \text{there is } S_\phi \subset S \text{ such that (i) } |S \setminus S_\phi| \leq 1 \right\}
\]

for each \( \phi \) in \( \Phi_\theta(\bar{\tau}) \), let \( \mathcal{P}_\theta(\bar{\tau}) \) be the set of stationary PMFs of \( \mathcal{P}_\theta(\phi) \). The proof of [21] Theorem 4.4] tells us that, given
a non-empty solution set $\mathbb{L}_0^0(\bar{\gamma})$ for LP (16), there exist an optimal occupation measure $f^* \in \mathbb{L}_0^0(\bar{\gamma})$, a policy $\phi^* \in \Phi^1$, and a stationary PMF $\pi^* \in \Pi^1(\phi^*)$ such that the following equalities hold:

$$f^*_{\bar{\gamma}, R} + f^*_{\bar{\gamma}, W} = \pi^*(\bar{\gamma})$$

$$f^*_{\bar{\gamma}, W} = \pi^*(\bar{\gamma})\phi^*(\bar{\gamma}), \ \bar{\gamma} \in \bar{\gamma}$$

Hence, we can rewrite $W_0^0(\bar{\gamma})$ as

$$W_0^0(\bar{\gamma}) = \min_{\pi \in \Pi(\phi), \phi \in \Phi^1} \sum_{\bar{\gamma} \in \bar{\gamma}} \pi(\bar{\gamma})\phi(\bar{\gamma}) \text{ s.t.} \sum_{\bar{\gamma} \in \bar{\gamma}} \pi(\bar{\gamma})\phi(\bar{\gamma}) = \bar{\gamma}$$

(44)

We shall further divide $\Phi^1$ into three subsets where the probabilities to choose to work at the state $(1, A)$ are one, between zero and one, or zero and consider the LP (44) on each of the subsets in Lemmas [4] through [6]. Before we proceed with the proof, we restate the definition of threshold policies from [1].

We define a threshold policy $\phi_T$ as

$$\phi_T(s, w) := \begin{cases} 
0 & \text{if } s \geq \tau \text{ and } w = A, \\
1 & \text{otherwise}.
\end{cases}$$

**Lemma 4.** For every $\phi \in \Phi^1$ with $\phi(1, A) = 1$, there exist $\tau_1, \tau_2 \in \mathbb{S} \cup \{n_\phi + 1\}$ and $\alpha \in [0, 1]$ such that

$$\pi^\phi = (1 - \alpha)\pi^\tau_1 + \alpha\pi^\tau_2,$$

$$\bar{\gamma}(\phi) = (1 - \alpha)\bar{\gamma}(\phi_{\tau_1}) + \alpha\bar{\gamma}(\phi_{\tau_2}).$$

**Proof.** We define the mapping $T : \Phi_R \rightarrow \mathbb{S} \cup \{0\}$, where

$$T(\phi) := \max\{\bar{\gamma} \in \mathbb{S} \mid \phi(\bar{\gamma}, A) = 1\}, \ \phi \in \Phi_R.$$ We assume that $T(\phi) = 0$ if the set on the RHS is empty. We first observe that $T(\phi) \geq 1$ since $\phi(1, A) = 1$ and the only positive recurrent communicating class is $\{\bar{\gamma} \in \bar{\gamma} \mid \bar{\gamma} \geq T(\phi)\}$. Second, consider the following policy $\phi'$:

$$\phi'(\bar{\gamma}) = \begin{cases} 
\phi(\bar{\gamma}) & \text{if } \bar{\gamma} \geq T(\phi) \\
1 & \text{otherwise}
\end{cases}$$

It is clear that $\phi'$ has the same service rate and utilization rate as $\phi$ because both policies have the same positive recurrent communicating class and the policies inside the class are identical.

Recall that there is only one state, say $s'$, where $\phi$ randomizes between two actions. Thus, if $s' < T(\phi)$, $\phi'$ is just a threshold policy $\phi_T(\phi') + 1$. On the other hand, if $s' > T(\phi)$, $\phi'$ is of the following form:

$$\phi'(\bar{\gamma}) = \begin{cases} 
\gamma & \text{if } \bar{\gamma} = A \text{ and } \bar{\gamma} = s'
\end{cases}$$

(45)

Suppose that $\tau_1 = T(\phi) + 1$ and $\tau_2 = s' + 1$. We rewrite $\gamma$ in (45) as

$$\gamma = \frac{\alpha \cdot \pi^\tau_2 (\tau_2 - 1, A)}{\alpha \cdot \pi^\tau_2 (\tau_2 - 1, A) + (1 - \alpha)\pi^\tau_1 (\tau_2 - 1, A)}$$

(46)

for some $\alpha \in (0, 1)$. Note that, for every $\gamma \in (0, 1)$, we can find an appropriate $\alpha \in (0, 1)$ that satisfies (46) because $\pi^\tau_1 (\tau_2 - 1, A) > 0$ and $\pi^\tau_2 (\tau_2 - 1, A) > 0$ from the fact that $T(\phi) < s'$.

By solving the global balance equations for $\bar{\gamma}$ under the policy $\phi'$, we get the following stationary PMF. Its derivation is provided in Appendix [B] for every $\bar{\gamma} \in \bar{\gamma}$.

$$\pi^\phi(\bar{\gamma}) = (1 - \alpha)\pi^\tau_1 (\bar{\gamma}) + \alpha \cdot \pi^\tau_2 (\bar{\gamma})$$

(47)

The service rate can be obtained using the stationary PMF.

$$\pi^\phi = \sum_{\bar{\gamma} \in \bar{\gamma}} \mu(\bar{\gamma})\pi^\phi(\bar{\gamma})\phi(\bar{\gamma})$$

Substituting the RHS of (47) for $\pi^\phi(\bar{\gamma})$, we obtain

$$\pi^\phi = \sum_{\bar{\gamma} \in \bar{\gamma}} \left( \mu(\bar{\gamma})\alpha \cdot \pi^\tau_2 (\bar{\gamma}) + (1 - \alpha)\pi^\tau_1 (\bar{\gamma}) \right)\phi(\bar{\gamma})$$

(48)

Using the definition of $\phi'$ in (45) and the fact that $T(\phi) < s'$, we get

$$\phi'(\bar{\gamma}) = \begin{cases} 
\gamma & \text{if } \bar{\gamma} = A \text{ and } \bar{\gamma} = s'
\end{cases}$$

$$\phi_T(\phi')(\bar{\gamma}) = 1$$

First, using the expression in (46) for $\gamma$ in the first term, we get

$$\pi^\phi(\bar{\gamma}) = (1 - \alpha)\pi^\tau_1 (\bar{\gamma}) + \alpha \cdot \pi^\tau_2 (\bar{\gamma})$$

(49)

Second, we conclude $\pi^\phi(\bar{\gamma})\phi_{\tau_1}(\bar{\gamma}) = \pi^\phi(\bar{\gamma})\phi_{\tau_2}(\bar{\gamma})$ for all $\bar{\gamma} \in \bar{\gamma} \setminus \{(\tau_2 - 1, A)\}$ by considering the following three cases: (i) If $\bar{\gamma} \geq \tau_2$ and $w = A$, we have $\phi_{\tau_1}(\bar{\gamma}, w) = \phi_{\tau_2}(\bar{\gamma}, w) = 0$ from the definition of $\phi_{\tau_1}$ and $\phi_{\tau_2}$. (ii) If $\bar{\gamma} < \tau_2 - 1$, then $\pi^\phi(\bar{\gamma}) = 0$ (because $\bar{\gamma}$ is transient). (iii) If $\bar{\gamma} = B$, then $\phi_{\tau_1}(\bar{\gamma}, w) = \phi_{\tau_2}(\bar{\gamma}, w) = 1$. As a result, we get

$$\pi^\phi(\bar{\gamma}) = \sum_{\bar{\gamma} \in \bar{\gamma} \setminus \{(\tau_2 - 1, A)\}} \mu(\bar{\gamma})\left( (1 - \alpha)\pi^\tau_1 (\bar{\gamma}) \phi_{\tau_1}(\bar{\gamma}) + \alpha \pi^\tau_2 (\bar{\gamma}) \phi_{\tau_2}(\bar{\gamma}) \right)$$

(50)

Summing (49) and (50), we get

$$\pi^\phi = \pi^\phi' = \sum_{\bar{\gamma} \in \bar{\gamma}} \mu(\bar{\gamma})\left( (1 - \alpha)\pi^\tau_1 (\bar{\gamma}) \phi_{\tau_1}(\bar{\gamma}) + \alpha \pi^\tau_2 (\bar{\gamma}) \phi_{\tau_2}(\bar{\gamma}) \right)$$

(51)

Following similar steps, we can show $\bar{\gamma}(\phi) = (1 - \alpha)\bar{\gamma}(\phi_{\tau_1}) + \alpha\bar{\gamma}(\phi_{\tau_2})$. Finally, we include $\alpha$ at zero and one for the
Lemma statement to consider the case where \( \phi \) is a deterministic policy without the randomization.

**Lemma 5.** For every \( \phi \in \Phi^\dagger \) with \( \phi(1, A) \in (0, 1) \), there exist \( \tau_2 \in S \cup \{n_s + 1\} \) and \( \beta \in [0, 1] \) such that

\[
\phi(1, A) = \beta \phi(1, A) + \nu(1, A),
\]

\[
\phi = \beta \phi(1, A) + \nu,
\]

\[
\alpha(\phi) = \beta \alpha(\phi) + \nu(\phi).
\]

**Proof.** Because \( \phi \) randomizes between two actions only at state \((1, A)\), \( \phi \) is deterministic at all other states. There are two cases to consider: (i) \( T(\phi) > 0 \) and \( T(\phi) = 0 \). In the first case, \( \phi \) has the same service rate and utilization rate as the threshold policy \( \phi_{T(\phi) + 1} \). In the second case,

\[
\phi(\gamma) = \begin{cases} 
\gamma & \text{if } \gamma = (1, A), \\
1 & \text{if } \gamma = B, \\
0 & \text{otherwise}.
\end{cases}
\]

The rest of the proof is identical to that of Lemma 4 after replacing (a) \( \phi_{T(\phi)} \) with \( \phi_{T(\phi) + 1} \) and (b) \( \phi_{T(\phi) + 1} \) with \( \phi \), which is a policy that always rests with \( \phi_{T(\phi) + 1} = \nu(\phi) = 0 \).

Before we state the final lemma, note that, when \( \phi(1, A) = 0 \), the process \( \gamma \) could have two possible recurrent communicating classes. For such a policy \( \phi \), the utilization rate \( \nu \) is not well defined. Hence, for a policy \( \phi \) with \( \phi(1, A) = 0 \), we define a set of pairs consisting of a service rate and a utilization rate.

\[
SU(\phi) = \left\{ \left( \sum_{\gamma \in \gamma} \nu(\gamma) \phi(\gamma), \sum_{\gamma \in \gamma} \phi(\gamma) \phi(\gamma) \right) : \pi \in \Pi(\phi) \right\}
\]

**Lemma 6.** For every \( \phi \in \Phi^\dagger \) with \( \phi(1, A) = 0 \), there exist \( \tau_1, \tau_2 \in S \cup \{n_s + 1\} \) and \( \alpha \in [0, 1] \) such that

\[
SU(\phi) = \left\{ \left( (1 - \alpha)\phi(1, A) + \alpha \phi(1, A), \phi(1, A) \right) : \beta \in [0, 1] \right\}
\]

**Proof.** If \( T(\phi) = 0 \) which implies that the policy always rests, it is clear that \((1, A)\) is an absorbing state and the service rate and the utilization rate are both zero. If \( T(\phi) > 0 \), we can represent \( \phi \) as

\[
\phi(\gamma) = \begin{cases} 
0 & \text{if } \gamma = (1, A), \\
\phi(\gamma) & \text{otherwise},
\end{cases}
\]

where \( \phi' = \phi_{T(\phi) + 1} \) as in (45). The MC now has two positive recurrent communicating classes, and the stationary PMF can be any convex combination of stationary PMFs of \( \phi \) and \( \phi_1 \). This is also true for utilization rate and service rate.

**Proof of Theorem 2.** By Lemmas 4 through 6, the utilization rate and the service rate pair for every policy in \( \Phi^\dagger \) can be written as a convex combination of rate pairs of at most two threshold policies and \((0, 0)\). Hence, the optimization problem \( (44) \) can be transformed into the following optimization problem over two variables \( \alpha, \beta \in [0, 1] \) for convex combination and two thresholds \( \tau_1, \tau_2 \in S \cup \{n_s + 1\} \):

\[
\min_{\alpha, \beta \in [0, 1]} \beta(1 - \alpha)\phi(\tau_1) + \alpha\phi(\tau_2)
\]

\[
s.t. \quad \beta(1 - \alpha)\phi(\tau_1) + \alpha\phi(\tau_2) = \tau
\]

It is clear from this argument that \( \{ \phi(\gamma) : \gamma \in [0, \gamma] \} \) forms the lower boundary of the convex hull of \( \{ (0, 0) \} \), \( \{ (\phi_{\tau_1}, \phi(\gamma)) : \gamma \in S \cup \{n_s + 1\} \} \). Because there are a finite number of rate pairs of threshold policies, the lower bound of this convex hull is non-decreasing, piece-wise affine and convex for \( \gamma \in [0, \gamma] \).

**B. Derivation of Stationary PMF in (47)**

In order to prove that (47) is the correct stationary PMF, it suffices to show that the specified PMF satisfies the following global balance equations:

\[
\pi^{\gamma} \phi(\gamma) = \sum_{\gamma' \in \gamma} \pi^{\gamma'}(\gamma') \pi^{\gamma', \gamma} \quad \text{for all } \gamma \in \gamma,
\]

where \( \pi^{\gamma} \phi(\gamma) \) is the one-step transition matrix of \( \gamma \). To this end, we shall demonstrate that the RHS of (47) is equal to the RHS of (52).

First, we break the RHS of (52) into two terms.

\[
\sum_{\gamma \in \gamma} \pi^{\gamma} \phi(\gamma) = \pi^{\gamma} \left( \tau_2 - 1, A \right) \pi^{\gamma \tau_2} + \sum_{\gamma \in \gamma \setminus \{ \tau_2 - 1, A \}} \pi^{\gamma} \phi(\gamma)
\]

We then rewrite each term on the RHS: from (47) and (45), we have

\[
(53) = \left( \alpha \cdot \pi^{\gamma_1 \tau_1} (\tau - 2, A) + (1 - \alpha) \pi^{\gamma_1 \tau_1} (\tau - 2, A) \right)
\]

\[
\times \left( (1 - \gamma) \pi^{\gamma_2 \tau_2} (\tau_2 - 1, A) + \gamma \pi^{\gamma_2 \tau_2} (\tau_2 - 1, A) \right)
\]

Substituting the expression in (46) for \( \gamma \),

\[
(53) = \left( \alpha \cdot \pi^{\gamma_1 \tau_1} (\tau - 2, A) \pi^{\gamma_2 \tau_2} (\tau_2 - 1, A) \right)
\]

\[
+ \left( 1 - \alpha \right) \pi^{\gamma_1 \tau_1} (\tau - 2, A) \pi^{\gamma_2 \tau_2} (\tau_2 - 1, A)
\]

Second, from (47)

\[
(54) = \sum_{\gamma \in \gamma \setminus \{ \tau_2 - 1, A \}} \left( \alpha \cdot \pi^{\gamma_2 \tau_2} (\gamma) + (1 - \alpha) \pi^{\gamma_2 \tau_2} (\gamma) \right)
\]

From (45), for all \( \gamma = (\gamma_1, \gamma_2) \in \gamma \setminus \{ \tau_2 - 1, A \} \), we have \( \phi(\gamma) = \phi_\gamma (\gamma_2) \) and \( \pi^{\gamma_2 \tau_2} (\gamma) = \pi^{\gamma_2 \tau_2} (\gamma) \). Moreover, because \( \phi_{\tau_2} \) is a deterministic policy with a threshold on the activity state of the server, \( \pi^{\gamma_2 \tau_2} (\gamma) = 0 \) for all \( \gamma = (\gamma_1, \gamma_2) \) with \( \gamma < \tau_2 - 1 \). Hence, for all \( \gamma \in \gamma \setminus \{ \tau_2 - 1, A \} \) with \( \pi^{\gamma_2 \tau_2} (\gamma) > 0 \), together with the assumption \( \tau_1 \leq \tau_2 \), we have

\[
\phi_\gamma (\gamma) = \phi_\gamma (\gamma) = \begin{cases} 
0 & \text{if } \gamma \geq \tau_2 \text{ and } \gamma_2 = A \\
1 & \text{if } \gamma = B
\end{cases}
\]
and, consequently, \( \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j+1} = \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j+1} \). Therefore,

\[
\sum_{\mathbf{Y} \in \mathcal{Y}\setminus \{( r_{-1}, A) \}} \left( \alpha \cdot \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j} \right) \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j+1} + (1 - \alpha) \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j} \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j+1}.
\]

(56)

Substituting the new expressions in (55) and (56) for (53) and (54), respectively, we obtain

\[
\sum_{\mathbf{Y}} \mathbf{P}_{\mathbf{Y}}(\mathbf{Y}) \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j} = \alpha \cdot \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j} + (1 - \alpha) \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j},
\]

where the equality follows from the fact that \( \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j+1} \) and \( \mathbf{P}_{\mathbf{Y}, \mathbf{Y}}^{j+2} \) are the stationary PMFs of \( \mathbf{Y}_{\mathbf{Y}}^{j+2} \) and \( \mathbf{Y}_{\mathbf{Y}}^{j+2} \), respectively.

C. Proof of Lemma 2

We shall first construct a temporary function \( f \) that will be used to construct a potential function satisfying all conditions in the lemma. For the simplicity of exposition, suppose that the states in \( \mathbf{M} \) are ordered in some arbitrary fashion and let \( n^* \) be some state belonging to the recurrent communicating class. First, assign \( f(n^*) = 0 \). Next, for each \( m \in \mathbf{M} \setminus \{ n^* \} =: \mathbf{M}^- \) (with \( n_M^* := |\mathbf{M}^-| \)), we rewrite the constraints in (32) as

\[
f(m) - E[f(M_{k+1}) | M_k = m] = f(m) - \sum_{\mathbf{m} \in \mathbf{M}} f(\mathbf{m}) P_{M_{k+1}|M_k}(\mathbf{m} | m)
\]

\[
= (1 - P_{M_{k+1}|M_k}(m | m)) f(m) - \sum_{\mathbf{m} \in \mathbf{M}^-} f(\mathbf{m}) P_{M_{k+1}|M_k}(\mathbf{m} | m)
\]

\[
r_{avg} - E[R(M_{k+1}, M_k) | M_k = m] =: \xi_m
\]

These constraints can be put in a matrix form as follows:

\[
\begin{bmatrix} \mathbf{B} \end{bmatrix} \mathbf{f} = \xi
\]

(58)

where \( \mathbf{f} = (f(m) : m \in \mathbf{M}^-) \) and \( \xi = (\xi_m : m \in \mathbf{M}^-) \) are \( n_M^* \)-dimensional column vectors, and \( \mathbf{B} \) is an \( n_M^- \times n_M^- \) matrix whose elements are given by

\[
B_j,l = \begin{cases} 1 - P_{M_{k+1}|M_k}(j | j) & \text{if } j = l \\ P_{M_{k+1}|M_k}(l | j) & \text{if } j \neq l \end{cases}, \quad j, l \in \mathbf{M}^-.
\]

To complete the proof, we need the following lemma.

Lemma 7. The matrix \( \mathbf{B} \) is weakly chained diagonally dominant.

Proof. First, the matrix is weakly diagonally dominant because

\[
|B_{j,l}| = 1 - P_{M_{k+1}|M_k}(j | j) = \sum_{l \in \mathbf{M}^- \setminus \{j\}} P_{M_{k+1}|M_k}(l | j)
\]

\[
\geq \sum_{l \in \mathbf{M}^- \setminus \{j\}} P_{M_{k+1}|M_k}(l | j) = \sum_{l \in \mathbf{M}^- \setminus \{j\}} |B_{j,l}|.
\]

Second, for any state \( j \in \mathbf{M}^- \) with \( P_{M_{k+1}|M_k}(n^* | j) > 0 \), the \( j \)th row of \( \mathbf{B} \) is strictly diagonally dominant (SDD) because

\[
|B_{j,j}| > \sum_{l \in \mathbf{M}^- \setminus \{j\}} |B_{j,l}|.
\]

Finally, note that the \( j \)th row of \( \mathbf{B} \), \( j \in \mathbf{M}^- \), is not SDD only if \( P_{M_{k+1}|M_k}(n^* | j) = 0 \). Suppose that there exists a row of \( \mathbf{B} \), say the \( l \)th row, which is not SDD. Then, since \( M \) has only one recurrent communicating class, there exist (i) some \( l \in \mathbf{M}^- \) such that the \( l \)-th row of \( \mathbf{S} \) is SDD and (ii) a path from state \( l \) to state \( l^+ \) in the directed graph associated with matrix \( \mathbf{B} \). This proves that the matrix \( \mathbf{B} \) is weakly chained diagonally dominant.

Since weakly chained diagonally dominant matrix is non-singular [23], there is a unique solution to the set of linear equations in (58), which is then assigned to the temporary function \( f(m) \), \( m \in \mathbf{M}^- \). Recall that, by construction, the function \( f \) satisfies the condition (32) at all states \( m \in \mathbf{M}^- \). We now prove that the condition (32) holds at state \( n^* \) as well.

First, we can show that the following equality holds:

\[
r_{avg} = \sum_{m \in \mathbf{M}} \theta^M(m) \left( \mathbb{E} [R(M_{k+1}, M_k) | M_k = m] - \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = m] \right),
\]

where \( \theta^M \) is the stationary PMF of \( M \). From the definition of stationary PMF,

\[
\sum_{m \in \mathbf{M}} \theta^M(m) \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = m] = \sum_{m \in \mathbf{M}} \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = m].
\]

Therefore, we get

\[
\sum_{m \in \mathbf{M}} \theta^M(m) \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = m] = 0,
\]

and the equality in (59) follows from the definition of \( r_{avg} \).

Rewriting (59) using the equality in (57), we obtain

\[
r_{avg} = \theta^M(n^*) \left( \mathbb{E} [R(M_{k+1}, M_k) | M_k = n^*] - \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = n^*] \right) + \sum_{m \in \mathbf{M}^-} \theta^M(m) r_{avg}.
\]

Moving the second term on the RHS to the LHS, we obtain

\[
\left( 1 - \sum_{m \in \mathbf{M}^-} \theta^M(m) \right) r_{avg} = \theta^M(n^*) \mathbb{E} [R(M_{k+1}, M_k) | M_k = n^*] - \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = n^*].
\]

Thus, we have the desired equality \( \mathbb{E} [R(M_{k+1}, M_k) | M_k = n^*] - \mathbb{E} [f(M_{k+1}) - f(M_k) | M_k = n^*] = r_{avg} \) because \( \theta^M(n^*) > 0 \) from the assumption that \( n^* \) belongs to the unique positive recurrent communicating class.

Finally, we define the nonnegative potential-like function \( \mathcal{H} : \mathbf{M} \to \mathbb{R}_+ \), where \( \mathcal{H}(m) = f(m) - \min_{m' \in \mathbf{M}} f(m') \) for
all \( m \in \mathbb{M} \). From its construction, the function \( \mathcal{K} \) is non-negative and satisfies all the constraints in the lemma.

**D. Auxiliary Results and Proof of Theorem 3**

We make use of the following lemmas (Lemmas 5 through 10) to complete the proof of the theorem. Let \( \mathbf{P}^\phi \) and \( \pi^\phi \) be the one-step transition matrix and the stationary PMF (given as a row vector), respectively, of \( \mathbf{Y}^\phi \). Recall that we defined \( \Phi_R^\phi \) to be the set of \( \phi \in \Phi_R \) such that \( \phi(1, A) \geq \epsilon \) and that, for any \( \varpi \) in \((\lambda, \varpi')\), the set \( \Phi_R^\phi(\varpi) \) is nonempty by Proposition 2.

**Lemma 8.** There exists a positive constant \( \eta_e \) such that, for any distribution \( \pi \) over \( \varpi \), we have

\[
\sum_{\phi=1}^{\infty} \|\pi(\mathbf{P}^\phi)^r - \pi^\phi\|_1 \leq \eta_e \quad \text{for all } \phi \in \Phi_R^\phi.
\]

**Proof.** First, we can find positive \( \alpha_e \) such that, for all \( \phi' \in \Phi_R^\phi \),

\[
P(\mathbf{Y}_{k+2n_s} = (n_s, B) \mid \mathbf{Y}_k = \varpi) \geq \alpha_e, \quad \varpi \in \varpi.
\]

One can verify that, for example,

\[
\alpha_e = \epsilon(1 - \mu(n_s))^{2n_s} n_s^{-1}(1 - \mu(s))\rho_{s+1, s}\rho_{s, s+1}
\]

satisfies the inequality in (60).

Next, we follow an analysis that is similar to the proof of Theorem 4.16 of [24]. We define a function \( h: \mathbb{R}^{2n_s \times 2n_s} \rightarrow \mathbb{R}_+ \) with

\[
h(\mathbf{P}) = \frac{1}{2} \max_{i, j, \ell} \sum_{k=1}^{2n_s} |P_{i, \ell} - P_{j, \ell}|,
\]

where \( P_{i, \ell} \) is the element in the \( i \)th row and \( \ell \)th column of matrix \( \mathbf{P} \).

Note that since \( \phi \in \Phi_R^\phi \), \( \alpha_e \) every element in the column of \( (\mathbf{P}^\phi)^{2n_s} \) corresponding to \((n_s, B)\) is lower-bounded by some positive \( \alpha_e \). Thus, equation (4.6) of [24] tells us

\[
h((\mathbf{P}^\phi)^{2n_s}) \leq 1 - \alpha_e.
\]

Proceeding with the proof, for every \( r \geq 2n_s \) and \( \kappa = \lfloor r/2n_s \rfloor \),

\[
h((\mathbf{P}^\phi)^r) = h((\mathbf{P}^\phi)^{r - 2n_s}) (h((\mathbf{P}^\phi)^{2n_s}))^\kappa
\]

\[
\leq h((\mathbf{P}^\phi)^{r - 2n_s}) (h((\mathbf{P}^\phi)^{2n_s}))^\kappa
\]

\[
\leq (1 - \alpha_e)^\kappa \leq (1 - \alpha_e)^{r - 1} = K_e \sigma_e^r,
\]

where \( K_e = (1 - \alpha_e)^{-1} \) and \( \sigma_e = (1 - \alpha_e)^{1/2n_s} \). The first inequality follows from Lemma 4.3 of [24], which states \( h(\mathbf{P})h(\mathbf{P}^l) \leq h(\mathbf{P})h(\mathbf{P}^l) \) for any two stochastic matrices \( \mathbf{P} \) and \( \mathbf{P}^l \). The second inequality follows from the observation \( h(\mathbf{P}) \leq 1 \) for any stochastic matrix \( \mathbf{P} \), and \((1 - \alpha_e) < 1\) leads to the final inequality. Combining with Lemma 4.3 of [24] and the fact that the sum of all elements of \( \pi - \pi^\phi \) equals zero, we know that, for every \( r \geq 2n_s \),

\[
\|\pi(\mathbf{P}^\phi)^r - \pi^\phi\|_1 = \|\pi(\mathbf{P}^\phi)^r - \pi^\phi(\mathbf{P}^\phi)^r\|_1
\]

\[
\leq h((\mathbf{P}^\phi)^r) \|\pi - \pi^\phi\|_1 \leq 2K_e \sigma_e^r.
\]

Hence, the inequality in (61) yields the following bound:

\[
\sum_{r=1}^{\infty} \|\pi(\mathbf{P}^\phi)^r - \pi^\phi\|_1
\]

\[
= \sum_{r=1}^{2n_s} \|\pi(\mathbf{P}^\phi)^r - \pi^\phi\|_1 + \sum_{r=2n_s+1}^{\infty} \|\pi(\mathbf{P}^\phi)^r - \pi^\phi\|_1
\]

\[
\leq 4n_s + \sum_{r=2n_s+1}^{\infty} 2K_e \sigma_e^r = 4n_s + \frac{2K_e \sigma_e^{2n_s+1}}{1 - \sigma_e} =: \eta_e
\]

Define \( g^\phi(\phi) \) to be a row vector representing the distribution of server state under the stationary PMF \( \pi^\phi(\phi) \) of \( X^\phi(\phi) \), which is given by

\[
g^\phi(\phi)(\varpi) = \sum_{\varpi \in \varpi} \pi^\phi(\phi)(\varpi, \varpi), \quad \varpi \in \varpi.
\]

**Lemma 9.** Fix \( \varpi \) in \((\lambda, \varpi')\) and \( \epsilon \in (0, 1) \), and let \( \beta_{\lambda, \epsilon} \) be a positive constant satisfying Lemma 7. Then, the following bound holds for every \( \phi \in \Phi_R^\phi(\varpi) \) and \( r \in \mathbb{N} \):

\[
\|g^\phi(\phi) - \mathbf{P}^\phi\|_1 \leq 2r(\varpi - \lambda) / \beta_{\lambda, \epsilon}
\]

**Proof.** Let \( \mathbf{P}^\phi \) be the one-step transition matrix of \( \mathbf{Y} \) under the policy \( \phi_0 \), which always chooses \( R \) when the server is available. We denote the row of \( \mathbf{P}^\phi \) (resp. \( \mathbf{P}^\phi_0 \)) corresponding to the server state \( \varpi = (s, w) \in \mathbb{Y} \) by \( \mathbf{P}^\phi_0 \) (resp. \( \mathbf{P}^\phi_0 \)).

Since \( g^\phi(\phi) \) remains the same after one step transition, using the equality in (62), we can rewrite \( g^\phi(\phi) \) as

\[
g^\phi(\phi) = \sum_{s \in \mathbb{S}} [\pi^\phi(\phi)(s, A, 0) \mathbf{P}^\phi_0] + \sum_{s \in \mathbb{S}} \left( \sum_{q=1}^{\infty} \pi^\phi(\phi)(s, w, q) \mathbf{P}^\phi_0 \right)
\]

\[
= \sum_{s \in \mathbb{S}} [\pi^\phi(\phi)(s, A, 0) \mathbf{P}^\phi_0] + \sum_{s \in \mathbb{S}} \left( \sum_{q=1}^{\infty} \pi^\phi(\phi)(s, w, q) \mathbf{P}^\phi_0 \right)
\]

\[
= \sum_{s \in \mathbb{S}} [\pi^\phi(\phi)(s, A, 0) \mathbf{P}^\phi_0] + \sum_{s \in \mathbb{S}} \left( \sum_{q=1}^{\infty} \pi^\phi(\phi)(s, w, q) \mathbf{P}^\phi_0 \right).
\]

Applying (63) iteratively, we obtain

\[
g^\phi(\phi) = g^\phi(\phi) + \mathbf{P}^\phi(\mathbf{P}^\phi)^{r-1}, \quad r \in \mathbb{N}.
\]

Subtracting the first term on the RHS of (63) from both sides and taking the norm,

\[
\|g^\phi(\phi) - g^\phi(\phi)\|_1 = \|\gamma^\phi r (\mathbf{P}^\phi)^{r-1}\|_1
\]

\[
\leq \|\gamma^\phi\|_1 \sum_{r=1}^{\infty} (\mathbf{P}^\phi)^{r-1} = r \|\gamma^\phi\|_1
\]

15
where the last equality is a consequence of $\|P\|_{\infty} = 1$ for a stochastic matrix $P$. Substituting the expression for $\gamma^\phi$ and using the inequality $\|P^\phi - P^\gamma\|_1 \leq 2$ for all $y \in Y$, we get

$$r \|\gamma^\phi\|_1 \leq 2r \left( \sum_{s \in S} \pi^\phi(s, A, 0) \right).$$

Thus, we get

$$\left\| \phi^\mathcal{X}(\phi) - \phi^\mathcal{X}(\phi) (P^\phi)^r \right\|_1 \leq 2r \left( \sum_{s \in S} \pi^\phi(s, A, 0) \right) \leq \frac{r(\| - \lambda)}{\beta_{\lambda, c}}.$$  

The last inequality holds because $\beta_{\lambda, c}$ satisfies Lemma 3. □

**Lemma 10.** Fix $\pi$ in $(\lambda, \pi^\phi)$ and $\epsilon$ in $(0, 1)$, and let $\beta_{\lambda, c}$ and $\eta_\epsilon$ be positive constants satisfying Lemmas 3 and 9, respectively. Then, the following inequality holds for every $N \in \mathbb{N}$:

$$\left\| \phi_{\pi}(\phi) - \pi^\phi \right\|_1 \leq \frac{\eta_\epsilon}{N} \sum_{r=1}^{N-1} \left\| \phi_{\pi}(\phi) - \phi_{\pi}(\phi) (P^\pi)^r \right\|_1 \leq \frac{(N + 1)(\| - \lambda)}{\beta_{\lambda, c}} + \frac{\eta_\epsilon}{N},$$

where the last inequality utilizes $\sum_{r=1}^{N-1} r = N(N + 1)/2$. □

1) Proof of Theorem 3 We have

$$\sum_{y \in Y} \pi^\phi(y) - \sum_{q > 0} \pi^\phi(q, y, q) \leq \left\| \pi^\phi - \phi_{\pi}(\phi) \right\|_1 + \sum_{y \in Y} \left\| \phi_{\pi}(\phi) - \sum_{q > 0} \pi^\phi(q, y, q) \right\|_1 = \left\| \pi^\phi - \phi_{\pi}(\phi) \right\|_1 \sum_{s \in S} \pi^\phi(s, A, 0) \leq \frac{\eta_\epsilon}{N} \sum_{r=1}^{N-1} \left\| \phi_{\pi}(\phi) - \phi_{\pi}(\phi) (P^\pi)^r \right\|_1 \leq \frac{(N + 1)(\| - \lambda)}{\beta_{\lambda, c}} + \frac{\eta_\epsilon}{N},$$

where the final inequality follows from Lemmas 3 and 10. By selecting $N = \left\lceil \frac{\eta_\epsilon}{(\| - \lambda)^2} \right\rceil$, we obtain the inequality in (43):

$$\sum_{y \in Y} \pi^\phi(y) - \sum_{q > 0} \pi^\phi(q, y, q) \leq (\| - \lambda)^{\frac{1}{2}} + \left( \frac{\eta_\epsilon}{(\| - \lambda)^2} + 1 + 1 \right)(\| - \lambda) + \frac{\eta_\epsilon}{(\| - \lambda)^2} \leq \frac{\beta_{\lambda, c}}{\beta_{\lambda, c}} + \frac{3}{\beta_{\lambda, c}}(\| - \lambda)$$

References

[1] M. Lin, R. J. La, and N. C. Martins, “Stabilizing a queue subject to action-dependent server performance,” Concurrently submitted to the IEEE Transactions on Automatic Control (available at arXiv:1903.00135), 2020.

[2] K. Savla and E. Frazzoli, “A dynamical queue approach to intelligent task management for human operators,” Proceedings of the IEEE, vol. 100, pp. 672–686, March 2012.

[3] S. Bi, Y. Zeng, and R. Zhang, “Wireless powered communication networks: An overview,” IEEE Wireless Communications, vol. 23, pp. 10–18, April 2016.

[4] D. Niyato, D. I. Kim, M. Maso, and Z. Han, “Wireless powered communication networks: Research directions and technological approaches,” IEEE Wireless Communications, vol. 24, pp. 88–97, December 2017.

[5] Y. L. Che, L. Duan, and R. Zhang, “Spatial throughput maximization of wireless powered communication networks,” IEEE Journal on Selected Areas in Communications, vol. 33, pp. 1534–1548, August 2015.

[6] H. Ju and R. Zhang, “Throughput maximization in wireless powered communication networks,” IEEE Transactions on Wireless Communications, vol. 13, pp. 418–428, January 2014.

[7] G. Yang, C. K. Ho, R. Zhang, and Y. L. Guan, “Throughput maximization for massive MIMO systems powered by wireless energy transfer,” IEEE Journal on Selected Areas in Communications, vol. 33, pp. 1640–1650, August 2015.

[8] F. Shan, J. Luo, W. Wu, and X. Shen, “Optimal wireless power transfer scheduling for delay minimization,” in Proceedings of the IEEE INFOCOM, 2016.

[9] C. R. Cassady and E. Kutanoglu, “Integrating preventive maintenance planning and production scheduling for a single machine,” IEEE Transactions on Reliability, vol. 54, pp. 304–309, June 2005.

[10] N. M. Majid, M. Alouai-Selsolii, Abdelmoula, and Mofahid, “International journal of production,” An Integrated Production and Maintenance Planning Model with Time Windows and Shortage Cost, vol. 49, pp. 2265–2283. 2011.

[11] X. Yao, X. Xie, M. C. Fu, and S. I. Marcus, “Optimal joint preventive maintenance and production policies,” Naval Research Logistics, vol. 52, pp. 668–681, October 2005.

[12] M. Nagahara, D. E. Quevedo, and D. Nesci, “Maximum hands-off control: A paradigm of control effort minimization,” IEEE Transactions on Automatic Control, vol. 61, pp. 735–747, March 2016.

[13] D. Chatterjee, M. Nagahara, D. E. Quevedo, and K. M. Rao, “Characterization of maximum hands-off control,” Systems & Control Letters, vol. 94, pp. 31–36, 2016.

[14] T. Ikeda, M. Nagahara, and K. Kashima, “Consensus by maximum hands-off distributed control with sampled-data state observation,” in Proceedings of the IEEE Conference on Decision and Control (CDC), pp. 962–968, 2016.

[15] T. Ikeda and M. Nagahara, “Value function in maximum hands-off control for linear systems,” Automatica, vol. 64, pp. 190–195, 2016.

[16] T. Ikeda and K. Kashima, “Sparse optimal feedback control for continuous-time systems,” in Proceedings of the 18th European Control Conference (ECC), pp. 3728–3733, 2019.

[17] C. M. Vigorito, D. Ganesan, and A. G. Barto, “Adaptive control of duty cycling in energy-harvesting wireless sensor networks,” in Proceedings of the IEEE Global Telecommunications Conference, 2008.

[18] Z. Liu and I. Elhanany, “Rl-mac: A qos-aware reinforcement learning based mac protocol for wireless sensor networks,” in Proceedings of the IEEE Conference on Networking, Sensing and Control, 2006.

[19] C. M. Vigorito, D. Ganesan, and A. G. Barto, “Adaptive control of duty cycling in energy-harvesting wireless sensor networks,” in Proceedings of the IEEE Communications Society Conference on Sensor, Mesh and Ad Hoc Communications and Networks, 2007.

[20] V. Jog, R. J. La, and N. C. Martins, “Channels, learning, queuing and remote estimation systems with a utilization-dependent component,” ArXiV, 2019.

[21] E. Altman, Constrained Markov decision processes. Chapman & Hall/CRC, 1999.

[22] G. Grimmett and D. Stirzaker, Probability and Random Processes, third ed. Oxford, 2001.

[23] P. Azimzadeh and P. A. Forsyth, “Weakly chained matrices, policy iteration, and impulse control,” SIAM Journal on Numerical Analysis, vol. 54, no. 3, pp. 1341–1364, 2016.

[24] E. Seneta, Non-negative matrices and Markov chains. Springer Science & Business Media, 2006.