What’s Decidable about Syntax-Guided Synthesis?

Benjamin Caulfield¹, Markus N. Rabe¹, Sanjit A. Seshia¹, and Stavros Tripakis¹,²

¹University of California, Berkeley
²Aalto University

Abstract. Syntax-guided synthesis (SyGuS) is a recently proposed framework for program synthesis problems. The SyGuS problem is to find an expression or program generated by a given grammar that meets a correctness specification. Correctness specifications are given as formulas in suitable logical theories, typically amongst those studied in satisfiability modulo theories (SMT). In this work, we analyze the decidability of the SyGuS problem for different classes of grammars and correctness specifications. We prove that the SyGuS problem is undecidable for the theory of equality with uninterpreted functions (EUF). We identify a fragment of EUF, which we call regular-EUF, for which the SyGuS problem is decidable. We prove that this restricted problem is EXPTIME-complete and that the sets of solution expressions are precisely the regular tree languages. For theories that admit a unique, finite domain, we give a general algorithm to solve the SyGuS problem on tree grammars. Finite-domain theories include the bit-vector theory without concatenation. We prove SyGuS undecidable for a very simple bit-vector theory with concatenation, both for context-free grammars and for tree grammars. Finally, we give some additional results for linear arithmetic and bit-vector arithmetic along with a discussion of the implication of these results.

1 Introduction

Program synthesis is an area concerned with the automatic generation of a program from a high-level specification of correctness. The specification may either be total, e.g., in the form of a simple but unoptimized program, or partial, e.g., in the form of a logical formula or even a collection of test cases. Regardless, one can typically come up with a suitable logic in which to formally capture the class of specifications. Traditionally, program synthesis has been viewed as a deductive process, wherein a program is derived from the constructive proof of the theorem that states that for all inputs, there exists an output, such that the desired correctness specification holds [20], with no assumptions made about the syntactic form of the program. However, over the past decade, there has been a successful trend in synthesis in which, in addition to the correctness specification, one also supplies a hypothesis about the syntactic form of the desired program. Such a hypothesis can take many forms: partial programs with “holes” [21,22], component libraries [16,14], protocol scenarios [23], etc. Moreover, the synthesis of verification artifacts, such as invariants [7], also makes use of “templates” constraining their syntactic structure. The intuition is that such syntactic restrictions on the form of
the program reduce the search space for the synthesis algorithms, and thus speed up the overall synthesis or verification process.

Syntax-guided synthesis (SyGuS) [2] is a recently-proposed formalism that captures this trend as a new class of problems. More precisely, a SyGuS problem comprises a logical specification $\varphi$ in a suitable logical theory $T$ that references one or more typed function symbols $f$ that must be synthesized, along with one or more formal languages $L$ of expressions of the same type as $f$, with the goal of finding expressions $e \in L$ such that when $f$ is replaced by $e$ in $\varphi$, the resulting formula is valid in $T$. The formal language $L$ is typically given in the form of a grammar $G$. Since the SyGuS definition was proposed about three years ago, it has been adopted by several groups as a unifying formalism for a class of synthesis efforts, with a standardized language (Synth-LIB) and an associated annual competition. However, the theoretical study of SyGuS is still in its infancy. Specifically, to our knowledge, there are no published results about the decidability or complexity of syntax-guided synthesis for specific logics and grammars. In this paper, we present a theoretical analysis of the syntax-guided synthesis problem. We analyze the decidability of the SyGuS problem for different classes of grammars and logics. For grammars, we consider arbitrary context-free grammars, tree grammars, and grammars specific to linear real arithmetic and linear integer arithmetic. For logics, we consider the major theories studied in satisfiability modulo theories (SMT) [5], including equality and uninterpreted functions (EUF), finite-precision bit-vectors (BV), and arrays—extensional or otherwise (AR), as well as theories with finite domains (FD). Our major results are as follows:

- For EUF, we show that the SyGuS problem is undecidable over tree grammars. These results extend straightforwardly for the theory of arrays. (See Section 3.)
- We present a fragment of EUF, called regular-EUF, for which the SyGuS problem is EXPTIME-complete given regular tree grammars. We prove that the sets of solution to regular-EUF problems are in one-to-one correspondence with the regular tree languages. (See Section 4.)
- For arbitrary theories with finite domains (FD) defined in Section 5, we show that the SyGuS problem is decidable for tree grammars, but undecidable for arbitrary context-free grammars.
- For BV, we show (perhaps surprisingly) that the SyGuS problem is undecidable for the classes of context-free grammars and tree grammars. (See Section 6.)

See Table 1 for a summary of our main results. In addition, we also consider certain restricted grammars specific to the theory of linear arithmetic over the reals and integers.

| Theory \ Grammar Class | Regular Tree | Context-free |
|-------------------------|--------------|--------------|
| Finite-Domain           | D            | U            |
| Bit-Vectors             | U            | U            |
| Arrays                  | U            | U            |
| EUF                     | U            | U            |
| Regular-EUF             | D            | ?            |

Table 1. Summary of main results, organized by background theories and classes of grammars. “U” denotes an undecidable SyGuS class, “D” denotes a decidable class, and “?” indicates that the decidability is currently unknown.
order substitution replaces not only each function symbol (or Terms and Substitutions

We review some key definitions and results used in the rest of the paper.

2 Preliminaries

We follow the book by Baader and Nipkow \[3\]. A signature (or ranked alphabet) \( \Sigma \) consists of a set of function symbols with an associated arity, a non-negative number indicating the number of arguments. For example \( \Sigma = \{ f : 2, a : 0, b : 0 \} \) consists of binary function symbol \( f \) and constants \( a \) and \( b \). For any arity \( n \geq 0 \), we let \( \Sigma^{(n)} \) denote the set of function symbols with arity \( n \) (the \( n \)-ary symbols). We will refer to the 0-ary function symbols as constants.

For any signature \( \Sigma \) and set of variables \( X \) such that \( \Sigma \cap X = \emptyset \), we define the set \( T(\Sigma, X) \) of \( \Sigma \)-terms over \( X \) inductively as the smallest set satisfying:

- \( \Sigma^{(0)}, X \subseteq T(\Sigma, X) \)
- For all \( n \geq 1 \), all \( f \in \Sigma^{(n)} \), and all \( t_1, \ldots, t_n \in T(\Sigma, X) \), we have \( f(t_1, \ldots, t_n) \in T(\Sigma, X) \).

We define the set of ground terms of \( \Sigma \) to be the set \( T(\Sigma, \emptyset) \) (or short \( T(\Sigma) \)). We define the subterms of a term recursively as \( \text{Subterms}(g(s_1, \ldots, s_k)) = \{ g(s_1, \ldots, s_k) \} \cup \bigcup_{s \in S} \text{Subterms}(s) \), which we lift to sets of terms, \( \text{Subterms}(S) = \bigcup_{s \in S} \text{Subterms}(s) \).

We say that a set \( S \) of terms is subterm-closed if \( \text{Subterms}(S) = S \).

For a set \( y_1, \ldots, y_k \) of variables (or constants) and terms \( t_1, \ldots, t_k, s \), the term \( s\{t_1/y_1, \ldots, t_k/y_k\} \) is formed by replacing each instance of each \( y_i \) in \( s \) with \( t_i \). We call \( \sigma := \{ t_1/y_1, \ldots, t_k/y_k \} \) a substitution. Substitutions extend in the natural way to formulae, by applying the substitution to each term in the formula.

We extend substitution to function symbols with arity \( > 0 \), where it is also called second-order substitution. For a function symbol \( f \) of arity \( k \), a signature \( \Sigma \), and a fresh set of variables \( \{ x_1, \ldots, x_k \} \), a substitution to \( f \) in \( \Sigma \) is a term \( w \in T(\Sigma, \{ x_1, \ldots, x_k \}) \). Given a term \( s \in T(\Sigma \cup f) \), the term \( s\{w/f\} \) is formed by replacing each occurrence of any term \( f(s_1, \ldots, s_k) \) in \( s \) with \( w(s_1/x_1, \ldots, s_k/x_k) \) (sometimes written \( w(s_1, \ldots, s_k) \)). We say that \( x_1, \ldots, x_k \) are the bound variables of \( f \). Intuitively, second-order substitution replaces not only \( f \) by \( w \), but also replaces the arguments \( s_1, \ldots, s_k \) of each function application \( f(s_1, \ldots, s_k) \) by the bound variables.

A context \( B \) is a term in \( T(\Sigma, \{ x \}) \) with a single occurrence of \( x \). For \( s \in T(\Sigma) \), we write \( B[s] \) for \( B\{s/x\} \).

Logical Theories A first-order model \( M \) in \( \Sigma \), also called \( \Sigma \)-model, is a pair consisting of a set \( \text{dom}(M) \) called its domain and a mapping \((-)^M \). The mapping assigns to each function symbol \( f \in \Sigma^F \) with arity \( n \geq 0 \) a total function \( f^M : \text{dom}(M)^n \to \text{dom}(M) \), and to each relation \( R \in \Sigma^R \) of arity \( n \) a set \( R^M \subseteq \text{dom}(M)^n \).

A formula is a boolean combination of relations over terms. The mapping induced by a model \( M \) defines a natural mapping of formulas \( \varphi \in L(\Sigma) \) to truth values, written \( M \models \varphi \) (we also say \( M \) satisfies \( \varphi \)). For some set \( \Phi \) of first-order formulas, we say (LRA and LIA), as well as bit-vectors (BV) where the grammars generate arbitrary but well-formed expressions in those theories and discuss the decidability of the problem in Section 2. The paper concludes in Section 8 with a discussion of the results, their implications, and directions for future work.
A theory $\mathcal{T} \subseteq L(\Sigma^F \cup \Sigma^R)$ is a set of formulas. We say $M$ is a model of $\mathcal{T}$ if $M \models \mathcal{T}$, and use $\text{Mod}(\mathcal{T})$ to denote the set of models of $\mathcal{T}$. A first-order formula $\varphi$ is valid in $\mathcal{T}$ if for all $M \in \text{Mod}(\mathcal{T})$, $M \models \varphi$. A theory is complete if for all formulas $\varphi \in L(\Sigma)$ either $\varphi$ or $\neg \varphi$ is valid.

Given a set of ground equations $E \subseteq \mathcal{F}(\Sigma) \times \mathcal{F}(\Sigma)$ and terms $s, t \in \mathcal{F}(\Sigma)$, we say that $s \rightarrow_t E t$ if there exists an $(l, r)$ in $E$ and a context $C$ such that $C[l] = s$ and $C[r] = t$. For example, if $E := \{a = g(b)\}$, then $h(a) \rightarrow_E h(g(b))$. Let $=_{E}$ be the symmetric and transitive closure of $\rightarrow_E$. We will sometimes write $E \models s = t$ instead of $s =_{E} t$. We will use $[s]_E$ to represent the set $\{t \mid s =_{E} t\}$. Birkhoff’s Theorem states that for any ranked alphabet $\Sigma$, set $E \subseteq \mathcal{F}(\Sigma) \times \mathcal{F}(\Sigma)$ and $s, t \in \mathcal{F}(\Sigma)$, $E \models s = t$ if and only if for every model $M$ in $\Sigma$ such that $M \models \bigwedge_{e \in E} e$ it holds $M \models s = t$.

In this work, we consider the common quantifier-free background theories of SMT solving: propositional logic (SAT), bit-vectors (BV), difference logic (DL), linear real arithmetic (LRA), linear integer (Presburger) arithmetic (LIA), the theory of arrays (AR), and the theory of uninterpreted functions with equality (EUF). For detailed definitions of these theories, see [5,4].

For the theory of EUF it is common to introduce the If-Then-Else operator (ITE) as syntactic sugar [6,5,4]. We follow this tradition and allow EUF formulas to contain terms of the form $\text{ITE}(\varphi, t_1, t_2)$, where $\varphi$ is a formula, and $t_1$ and $t_2$ are terms. To desugar EUF formulas we introduce an additional constant $c_{\text{ite}}$ and add two constraints $\varphi \rightarrow (c_{\text{ite}} = t_1)$ and $\neg \varphi \rightarrow (c_{\text{ite}} = t_2)$ for each ITE term $\text{ITE}(\varphi, t_1, t_2)$. As we will see in Section 3 the presence syntactic sugar such as the ITE operator in the grammar of SyGuS problems may have a surprising effect on the decidability of the SyGuS problem.

**Grammars and Automata** A context-free grammar (CFG) is a tuple $G = (N, T, S, R)$ consisting of a finite set $N$ of nonterminal symbols with a distinguished start symbol $S \in N$, a finite set $T$ of terminal symbols, and a finite set $R$ of production rules, which are tuples of the form $(N, (N \cup T)^*)$. Production rules indicate the allowed replacements of non-terminals by sequences over nonterminals and terminals. The language, $L(G)$, generated by a context-free grammar is the set of all sequences that contain only terminal symbols that can be derived from the start symbol using the production rules.

Tree grammars are a more restrictive class of grammars. They are defined relative to a ranked alphabet $\Sigma$. A regular tree grammar $G = (N, S, \Sigma, R)$ consists of a set $N$ of non-terminals, a start symbol $S \in N$, a ranked alphabet $\Sigma$, and a set $R$ of production rules. Production rules are of the form $A \rightarrow g(t_1, t_2, ..., t_k)$, where $A \in N$, $g$ is in $\Sigma$ and has arity $k$, and each $t_i$ is in $N \cup T_{\Sigma}$. For a given tree-grammar $G$ we write $L(G)$ for the set of trees produced by $G$. The regular tree languages are the languages produced by some regular tree grammar. Any regular tree grammar can be converted to a CFG by simply treating the right-hand side of any production as a string, rather than a tree. Thus, the undecidability results for SyGuS given regular tree grammars extend to undecidability results for SyGuS given CFGs.

Let $\Sigma$ be a signature of a background theory $\mathcal{T}$. We define a tree grammar $G = (N, S, \Sigma, P)$ to be $\mathcal{T}$-compatible (or $\Sigma$-compatible) if $\Sigma \subseteq \Sigma^F \cup \Sigma^R$ and the arities for all symbols in $\Sigma$ match those in $\Sigma$.

A deterministic bottom-up (or rational) tree automaton $A$ is a tuple $(Q, \Sigma, \delta, Q_F)$. Here, $Q$ is a set of states, $Q_F \subseteq Q$, and $\Sigma$ is a ranked alphabet. The function $\delta$ maps a symbol $g \in \Sigma^{(k)}$ and states $q_1, \ldots, q_k$, to a new state $q'$, for all $k$. If no such $q'$ exists,
δ(g, q₁, . . . , qₖ) is undefined. We can inductively extend δ to terms, where for all g ∈ Σ(ₖ) and all s₁, . . . , sₖ ∈ T(Σ), we set δ(g(s₁, . . . , sₖ)) := δ(g, δ(s₁), . . . , δ(sₖ)).

The language accepted by A is the set L(A) := \{ s ∈ T(Σ), δ(s) ∈ Q_F \}. There exist transformations between regular tree grammars and rational tree automata [8], and we will sometimes also define SyGuS problems in terms of rational tree automata rather than a regular tree grammars.

Syntax-Guided Synthesis We follow the definition of SyGuS given by Alur et al. [2], but we focus on the case to find a replacement for a single designated function symbol f with a candidate expression (the program), which is generated by a given grammar G. Let T be a background theory over signature Σ, and let G be a class of grammars. Given a function symbol f with arity k, a formula ϕ over the signature Σ∪{f}, and a grammar G ∈ G of terms in T(Σ, {x₁, . . . , xₖ}), the SyGuS problem is to find a term w ∈ L(G) such that the formula ϕ{w/f} is valid or to determine the absence of such a term. We represent the SyGuS problem as the tuple (ϕ, T, G, f).

The variables x₁, . . . , xₖ that may occur in the generated term w stand for the k arguments of f. For each function application of f the higher-order substitution ϕ{w/f} then replaces x₁, . . . , xₖ by the arguments of the function application.

Note that the original definition of SyGuS allows for universally quantified variables, while our definition above admits no variables. This is equivalent as universally quantified variables can be replaced with fresh constants without affecting validity.

Example 1. Consider the following example SyGuS problem in linear integer arithmetic. Let the type of the function to synthesize f be int × int → int and let the specification be given by the logical formula

ϕ₁ : ∀x, y f(x, y) = f(y, x) ∧ f(x, y) ≥ x.

We can restrict the set of expressions f(x, y) to be expressions generated by the grammar below:

```
Term := x | y | Const | ITE(Cond, Term, Term)
Cond := Term ≤ Term | Cond ∧ Cond | ¬Cond | (Cond)
```

It is easy to see that a function computing the maximum over x and y, such as ITE(x ≤ y, y, x), is a solution to the SyGuS problem. There are, however, other solutions, such as ITE(7 ≤ y ∧ 7 ≤ x, ITE(x ≤ y, y, x), 10). The function computing the sum of x and y would satisfy the specification, but cannot be constructed in the grammar.

3 SyGuS-EUF is Undecidable

We use SyGuS-EUF to denote the class of SyGuS problems (ϕ, EUF, G, f) where G is a grammar generating expressions that are syntactically well-formed expressions in EUF for f. In this section, we prove that SyGuS-EUF is undecidable. The proof of undecidability is a reduction from the simultaneous rigid E-unification problem (SREU) [11]. We say that a set E := \{ e₁, . . . , eₙ \} of equations between terms in T(Σ, V) together with an equation e* between terms in T(Σ, V) forms a rigid expression, denoted E ⊨ e*. A solution to E ⊨ e* is a substitution σ, such that e*σ and eᵢσ are ground
for each \( e_i \in E \) and \( E \sigma \vdash e^* \sigma \). Given a set \( S \) of rigid equations, the SREU problem is to find a substitution \( \sigma \) that is a solution to each rigid equation in \( S \), and is known to be undecidable \([11]\).

**Reducing SREU to SyGuS-EUF.** We start the reduction with constructing a boolean expression \( \Phi_S \) for a given set of rigid equations \( S \) over alphabet \( \Sigma \) and variables \( V := \{ x_1, \ldots, x_m \} \). Let each \( r_i \in S \) be \( e_{i,1}, \ldots, e_{i,l_i} \vdash e^*_i \), where \( e_{i,1}, \ldots, e_{i,l_i} \), and \( e^*_i \) are equations between terms in \( T(\Sigma, V) \). We associate with each rigid expression \( r_i \in S \) a boolean expression \( \psi_i := ( \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_f \land \bigwedge_{k \neq j} a_k \neq a_j ) \rightarrow e^*_i \sigma_f \), where \( \sigma_f \) is the substitution \( \{ f(a_1)/x_1, \ldots, f(a_m)/x_m \} \). The symbol \( f \) is a unary function symbol to be synthesized and \( a_1, \ldots, a_m \) are fresh constants \( (a_i \notin \Sigma \text{ for all } i) \). We set \( \Phi_S := \bigwedge_i \psi_i \).

Next we give the grammar \( G_S \), which generates the terms that may replace \( f \) in \( \Phi_S \). We define \( G_S \) to have the starting nonterminal \( A_1 \) and the following rules:

\[
\begin{align*}
A_1 & \to \ITE(x = a_1, S', A_2) \\
A_2 & \to \ITE(x = a_2, S', A_3) \\
\ldots \\
A_{m-1} & \to \ITE(x = a_{m-1}, S', A_{m-1}) \\
A_m & \to \ITE(x = a_m, S', \bot)
\end{align*}
\]

where \( \bot \) is a fresh constant (\( \bot \notin \Sigma \) and \( \bot \neq a_i \) for all \( i \)). Additionally, for each \( g \in \Sigma \) we add a rule \( S' \rightarrow g(S', \ldots, S') \), where the number of argument terms of \( g \) matches its arity.

**Lemma 1.** The SREU problem \( S \) has a solution if and only if the SyGuS-EUF problem \( \rho_S := (\Phi_S, \Sigma, G_S, \bot) \) has a solution over the ranked alphabet \( \Sigma \).

**Proof.** The main idea behind this proof is that each \( f(a_i) \) in \( \Phi_S \) represents the variable \( x_i \) in \( S \). Any replacement to \( f \) found in \( G_S \) corresponds to a substitution on all variables \( x_i \) in \( S \) that grounds the equations in the SREU problem.

\( \vdash \): Let \( \sigma_u := \{ u_1/x_1, \ldots, u_m/x_m \} \) be a solution to \( S \), where each \( u_i \) is a ground term in \( T(\Sigma) \). We consider the term \( w(x) := \ITE(x = a_1, u_1, \ITE(x = a_2, u_2, \ldots, \ITE(x = a_m, u_m, \bot)) \ldots , \) which is in the language of the grammar \( G_S \). To show that \( \Phi_S \{ w/f \} \) is valid, it suffices to show that for each model \( M \) of \( \Sigma \cup \{ a_1, \ldots, a_m \} \cup V \) and for each \( \psi_i \) we have \( M \models \psi_i \{ w/f \} \). If \( M \models [ \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_f \land \bigwedge_{k \neq j} a_k \neq a_j ] \{ w/f \} \), then \( M \models \psi_i \{ w/f \} \) holds trivially. We handle the remaining case below, giving justifications to the right of each new equation.

1. Assume \( M \models [ \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_f \land \bigwedge_{k \neq j} a_k \neq a_j ] \{ w/f \} \)
2. \( M \models \bigwedge_{k \neq j} a_k \neq a_j \)
3. For each \( j \): \( M \models w(a_j) = u_j \)
4. For each \( j \): \( M \models (e_{i,j} \sigma_f) \{ w/f \} \leftrightarrow e_{i,j} \sigma_u \)
5. \( M \models \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_u \)
6. \( M \models \{ e_{i,j} | j = 1, \ldots, l_i \} \sigma_u \lor e^*_i \sigma_u \)
7. \( \{ e_{i,j} | j = 1, \ldots, m \} \sigma_u \lor e^*_i \sigma_u \) (def. SyGuS)
8. \( M \models e^* \sigma_u \)
9. \( M \models (e^* \sigma_f) \{ w/f \} \)

(6,7, Birkhoff’s Thm.)
Therefore, \( M \models \Phi_S \) and we get that \( w \) is a solution to the SyGuS problem \( \rho_S \).

\[
\because: \quad \text{Let } w(x) \text{ and } \sigma_u \text{ be defined as before and assume that } w \text{ is a solution to the SyGuS problem } \rho_S. \text{ Each } u_i \text{ in } w \text{ is ground, since the nonterminal } S' \text{ in } G_S \text{ can only produce ground terms. Chose any } r_i \in S. \text{ We will show for every model } M \text{ on } \Sigma \cup V, \text{ that if } M \models \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_u \text{ then } M \models e_i^* \sigma_u. \text{ By Birkhoff’s theorem, this implies } e_{i,1} \sigma_u, \ldots, e_{i,l_i} \sigma_u \vdash e_i^* \sigma_u.
\]

1. Assume \( M \models \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_u \)
2. Let \( \hat{M} \) be a model over \( \Sigma \cup V \cup \{a_1, \ldots, a_m\} \) such that \( \hat{M} \upharpoonright \Sigma \cup V = M \) and \( \hat{M} \) assigns each \( a_i \) to a distinct new element not in \( \text{dom}(M) \).
3. \( \hat{M} \models w(a_j) = u_j \) (2)
4. For each \( j \): \( \hat{M} \models (e_{i,j} \sigma_f)\{w/f\} \leftrightarrow e_{i,j} \sigma_u \) (3)
5. \( \hat{M} \models \bigwedge_{j=1}^{l_i} e_{i,j} \sigma_u \) (1,2)
6. \( \hat{M} \models \bigwedge_{j=1}^{l_i} (e_{i,j} \sigma_f)\{w/f\} \) (4,5)
7. \( \hat{M} \models \psi_i\{w/f\} \) (w is a SyGuS solution)
8. \( \hat{M} \models (e_i^* \sigma_f)\{w/f\} \) (6,7)
9. \( \hat{M} \models e_i^* \sigma_u \) (3,8)
10. \( M \models e_i^* \sigma_u \) (2,9)

Thus \( e_{i,1} \sigma_u, \ldots, e_{i,l_i} \sigma_u \vdash e_i^* \sigma_u \) and \( \sigma_u \) is a solution to \( S \). \( \Box \)

**Theorem 1.** The SyGuS-EUF problem is undecidable.

**Remark on EUF without ITE.** A key step in the proof of Lemma 1 is the use of ITE statements to allow a single expression \( w \) to encode instantiations of multiple different variables. As discussed in Section 2, ITE statements are commonly part of EUF, but some definitions of EUF do not allow for ITE statements [19]. While this syntactic sugar has no effect on the complexity of the validity of EUF formulas, the undecidability of SyGuS-EUF may depend on the availability of ITE operators. It remains open whether there exist alternative proofs of undecidability that do not rely on ITE statements.

We use SyGuS-Arrays to denote the class of SyGuS problems \( (\phi, \text{Arrays}, G, f) \), where Arrays is the theory of arrays [5], and \( G \) is a grammar such that \( L(G) \) are syntactically well-formed expressions in Arrays for \( f \). There is a standard construction for representing uninterpreted functions as read-only arrays [5]. Therefore, the undecidability of SyGuS-Arrays follows from the undecidability of SyGuS-EUF, as we state below.

**Corollary 1.** The SyGuS-Arrays problem is undecidable.

### 4 Regular SyGuS-EUF

This section describes a fragment of EUF, which we call regular-EUF, for which the SyGuS problem is decidable.

**Definition 1.** We call \( (\phi, \text{EUF}, G, f) \) a regular SyGuS-EUF problem if \( G \) contains no ITE expressions and \( \phi \) is a regular-EUF formula as defined below.

A regular-EUF formula is a formula \( \phi := \bigwedge_{i} \psi_i \) over some ranked alphabet \( \Sigma \), where each \( \psi_i \) satisfies the following conditions:
1. It is a disjunction of equations or the negation of equations.
2. It does not contain any ITE expressions.
3. It contains at most one occurrence of $f$ per equation.
4. It satisfies one of the following cases:
   - Case 1: The symbol $f$ only occurs in positive equations.
   - Case 2: The symbol $f$ occurs in exactly one negative equation, and nowhere else.

We define any disjunction $\psi$ that satisfies the above conditions as regular. We will refer to a regular $\psi$ as case-1 or case-2, depending on which of the above cases is satisfied. Note that every regular-EUF formula is in conjunctive normal form.

We will show that for every regular $\psi_i$, we can construct a regular tree automaton $A_{\psi_i}$ accepting precisely the solutions to the SyGuS-EUF problem on $\psi_i$. The set of solutions to $\phi$ then becomes $L(G) \cap \bigcap_i L(A_{\psi_i})$, where $G$ is the grammar of possible replacements. The grammar $G$ can be represented as a deterministic bottom-up tree automaton $A_G$ whose size is exponential in $|G| \geq 8$. The product-automaton construction can be used to determine if $L(G) \cap \bigcap_i L(A_{\psi_i})$ is non-empty, which would imply that a solution exists to the corresponding SyGuS problem. This construction takes $O(|A_G| \cdot \prod_i |A_{\psi_i}|)$ time and space. Note that this is at most exponential even when some of the automata have size exponential in $|\phi|$ or $|G|$.

The connection between sets of ground equations and regular tree languages was first observed by Kozen [17], who showed that a language $L$ is regular if and only if there exist a set $E$ of ground equations and collection $S$ of ground terms such that $L = \bigcup_{s \in S} [s]_E$. The following, very similar theorem shows that a certain set of equivalence classes of a ground equational theory can be represented by a regular tree automaton.

**Theorem 2.** Let $E$ be a set of ground equations over the alphabet $\Sigma$, and let $C$ be a subterm-closed set of terms such that every term in $E$ is in $C$. There exists a regular tree automaton without accepting states $A_{E,C} := (Q, \Sigma, \delta)$ such that a state in $Q$ represents an equivalence class of a term in $C$. More formally, this means that for all terms $s, t \in T(\Sigma)$ such that there exist terms $s', t' \in C$ so that $s =_E s'$ and $t =_E t'$, it holds that $s =_E t$ if and only if $\delta(s) = \delta(t)$.

**Proof.** Let $Q := \{q_s \mid s \in C\}$. For each term $g(s_1, \ldots, s_k) \in C$, for $g \in \Sigma_k$, let $\delta(g.q_{s_1}, \ldots, q_{s_k}) = q_g(s_1, \ldots, s_k)$.

We define the function $merge(q, q')$ to operate on $A_{E,C}$ as follows: First, remove $q'$ from $q$. For all $q_1, \ldots, q_{n-1}, q_{n+1}, \ldots, q_k, q''$ and $g$ such that $\delta(q, q_1, \ldots, q_{n-1}, q', q_{n+1}, \ldots, q_k) = q''$, add $\delta(q, q_1, \ldots, q_{n-1}, q, q_{n+1}, \ldots, q_k) = q'''$ to $\delta$. If there already exists some $q'''$ such that $\delta(q, q_1, \ldots, q_{n-1}, q', q_{n+1}, \ldots, q_k) = q'''$, then $merge(q'', q''')$.

Now for each $s = t$ in $E$, call $merge(q_s, q_t)$. A simple inductive argument will show that the resulting automaton is $A_{E,C}$. \hfill \square

Let $\psi := e_1 \lor e_2 \lor \ldots \lor e_{k-1} \lor \neg e_k \lor \ldots \lor \neg e_{k+r}$ be a regular formula. Let $P := \{e_1, \ldots, e_{k-1}\}$ and $N := \{e_k, \ldots, e_{k+r}\}$. We can rewrite the formula $\psi$ to the normal form $\psi := (\bigwedge_{e \in N} e) \rightarrow (\bigvee_{e \in P} e)$. Solving the SyGuS problem for $\psi_i$ then becomes a problem of finding a $w$ such that $N \{w/f\} \vdash e \{w/f\}$ for some $e \in P$. The technique to form the automaton $A_{\psi_i}$ that represents the solutions to $\psi_i$ depends on whether $\psi_i$ is case-1 or case-2.
Assume that \( \psi \) is case-1 and chose some \( s = t \in P \). Assume \( f \) is not in \( s = t \). If \( N' \vdash s = t \), then \( \psi_i \) is trivially solvable. If \( N \not\vdash s = t \), then \( s = t \) can be removed from \( \psi_i \) to yield an equally solvable formula. Now assume \( f \) is in \( s = t \). Without loss of generality, there is a context \( B \) and a set of terms \( s_1, \ldots, s_{arity(f)} \) such that \( s = B[f(s_1, \ldots, s_{arity(f)})] \). Let \( C := \text{Subterms}(N) \cup \text{Subterms} \{ s_1, \ldots, s_{arity(f)} \} \) and let \( A_{P,C} := (Q, \Sigma, \delta) \) the automaton defined in the proof of theorem 2. For each \( q \in Q \), there is a ground term \( u_q \) such that \( \delta(u_q) = q \). Let \( Q' \subseteq Q \) be the set of states \( q \) such that \( \delta(B[u_q]) = \delta(t) \). By theorem 2 \( P \vdash B[u_q] = t \) if and only if \( q \in Q' \). Therefore, for any replacement, \( w \), of \( f \), \( P \vdash (s = t) \{ w/f \} \) if and only if \( \delta(w(s_1, \ldots, s_{arity(f)})) \in Q' \).

Let \( A_{s=t} := (Q, \Sigma \cup \{ x_1, \ldots, x_{arity(f)} \}, \delta', Q') \) be a tree automaton with accepting states \( Q' \). For each \( x_i \), let \( \delta'(x_i) := \delta(s_i) \). For all \( u \in T(\Sigma) \), let \( \delta'(u) := \delta(u) \). A simple inductive argument will show that for any replacement \( w \) of \( f \), \( \delta(w(s_1, \ldots, s_{arity(f)})) = \delta'(w(x_1, \ldots, x_{arity(f)})) \). Thus, \( L(A_{s=t}) \) defines the precise set of terms \( w \) such that \( P \vdash (s = t) \{ w/f \} \).

The set of solutions to \( \psi \) can be given by the automaton \( A_{\psi} \), whose language is \( \bigcup_{s \in E} L(A_{s=t}) \). This can be found in time and space exponential in \( |N| \) using the product construction for tree automata [8].

![Fig. 1. The automaton \( A_1 \) accepting the solutions to \( \psi_1 \) in example 2.](image)

**Example 2.** Let \( \psi := (g(a) = b \land g(b) = a) \rightarrow f(a) = g(g(b)) \). Note that this is a case-1 regular EUF clause. If we set \( E := \{ g(a) = b, g(b) = a \} \) and \( C := \{ a, b, g(a), g(b), g(g(b)) \} \), then \( A := A_{E,C} \) is the automaton from figure 1 (excluding the accepting state and \( x \) transition). Since the argument of \( f \) in \( f(a) = g(g(b)) \) is \( a \) and \( A \) parses \( a \) to state-1, a transition from \( x \) to state-1 is added to \( A \). Since \( g(g(b)) \) parses to state-2 in \( A \), state-2 is set as an accepting state in \( A \). So \( A \) accepts the replacements \( w \) to \( f \) such that \( \{ w/f \} \) is valid.

Assume \( \psi \) is case-2 and let \( s = t \) be the equation in \( N \) that contains \( f \). Without loss of generality, there is a context \( B \) and a set of terms \( s_1, \ldots, s_{arity(f)} \) such that \( s = B[f(s_1, \ldots, s_{arity(f)})] \). Let \( N' := N \setminus \{ s = t \} \), and let \( C := \text{Subterms}(N' \cup P) \cup \text{Subterms} \{ t, s_1, \ldots, s_{arity(f)} \} \). Choose some \( u = u' \in C \). If \( N' \vdash u = u' \), then every replacement to \( f \) is a solution. So assume \( N' \not\vdash u = u' \). Let \( w \) be a replacement to \( f \) such that \( s' := s \{ w/f \} = B[w(s_1, \ldots, s_{arity(f)})] \) and \( N' \{ w/f \} \vdash u = u' \). Let \( s' := s \{ w/f \} \). Assume \( s' \) is not \( N' \)-equivalent to any term in \( C \), let \( C' := C \cup \text{Subterms} \{ s' \} \) and let \( A_{N',C'} := (Q, \Sigma, \delta) \). We know \( \delta(s') \) has no outgoing edges: if it did, \( s' \) would be \( N' \)-equivalent to some term in \( C \). By construction, \( A_{N' \cup \{ s' = t \}, C'} \) is equivalent to calling \( \text{merge}(\delta(s'), \delta(t)) \) on \( A_{N',C'} \). Since \( \delta(s') \) has no
Example 3. Let \( \psi := (g(h(g(a))) = a \land f(g(a)) = a) \rightarrow h(g(a)) = a \). Note that this is a case-2 regular-EUF clause. If we set \( E := \{g(h(g(a))) = a\} \) and \( C := \{a, g(a), h(g(a)), g(h(g(a)))\} \), then \( A := A_{E,C} \) is the automaton from the left side of figure 4 (excluding the accepting state and \( x \) transition). Since the argument of \( f \) in \( f(g(a)) = a \) is \( a \) and \( A \) parses \( a \) to state-2, a transition from \( x \) to state-2 is added to \( A \). If we choose a replacement \( w \) such that \( w(g(a)) \) parses to state-3 in \( A \), then applying the equation \( w(g(a)) = a \) merges state-3 with state-1. This, in turn, forces a merge between the new state and state-2, yielding the automaton on the right side of figure 4. This automaton parses \( h(g(a)) \) and \( a \) to the same state, so state-3 is an accepting state. This does not occur if \( w(g(a)) \) parses to state-1 or state-2 in \( A \), so they are not accepting states. So \( A \) accepts the replacements \( w \) to \( f \) such that \( \psi\{w/f\} \) is valid.

We can summarize the above construction in the following lemma.

**Lemma 2.** The regular SyGuS-EUF problem is in EXPTIME.

The relationship between regular tree languages and the regular SyGuS-EUF problem is quite deep. Using the following lemma and the above constructions, we can see that a tree language is regular if and only if it is the set of solutions to a regular SyGuS-EUF problem.

**Lemma 3.** Let \( A := (Q, \Sigma \cup \{x_1, \ldots, x_k\}, \delta, Q_F) \) be a tree automaton. There exists a regular disjunctive formula \( \psi_A \) such that \( L(A) \) is the set of solutions to \( \psi_A \).
Proof. Let $T_Q$ be a subterm-closed set of terms such that for each state $q \in Q$, there is a term $u_q$ such that $\delta(u_q) = q$. Without loss of generality, assume that each $u_q \in T_Q$ is a subterm of some term in $L(A)$. Let $\sigma := \{x_i \mapsto c_i \mid i \in \{0, \ldots, k\}\}$ for some new constants $c_1, \ldots, c_k$. Let $N_Q := \{g(u_q, \sigma, \ldots, u_q, \sigma) = u_q \sigma \mid r \geq 0, g \in \Sigma_r, q_1, \ldots, q_r, q' \in Q, \delta(q, q_1, \ldots, q_r) = q'\}$ and $P_Q := \{f(c_1, \ldots, c_k) = u_q \mid q \in Q_F\}$. Finally set $\psi_A := (\bigwedge_{c \in N_Q} e) \rightarrow (\bigvee_{e \in P_Q} e)$. Using the construction from theorem 2 it is easy to check that the set of solutions to $\psi$ are precisely $L(A)$. \hfill \Box

We can also use the above lemma to show that regular SyGuS-EUF is EXPTIME-complete, as we will see below.

**Lemma 4.** The regular SyGuS-EUF problem is EXPTIME-hard.

**Proof.** We reduce from the EXPTIME-complete problem of determining whether a set of regular tree automata have languages with a non-empty intersection [24]. Let $A_1, \ldots, A_k$ be a set of regular tree automata over some alphabet $\Sigma$. For each automaton $A_i$, construct the formula $\psi_{A_i}$ as described in lemma 3. Let $\phi := \bigwedge_i \psi_{A_i}$. Let $f$ be a nullary function symbol to be synthesized, and let $G$ be a grammar such that $L(G) := T(\Sigma)$. The solutions to the regular SyGuS-EUF problem $(\phi, \Sigma, G, f)$ are the members of the set $\bigcap_i L(A_i)$. Therefore, $(\phi, \Sigma, G, f)$ has a solution if and only if $\bigcap_i L(A_i)$ is non-empty. \hfill \Box

Using the above lemma and lemma 2 we can conclude the following theorem.

**Theorem 3.** The regular SyGuS-EUF problem is EXPTIME-complete.

In concluding this section, we remark that the case-1 and case-2 restrictions on regular clauses are necessary. For lack of space, we exclude the details; the appendix contains an example elaborating on this point.

## 5 Finite-Domain Theories

In addition to the “standard” theories, we also consider a family of theories that we term **finite-domain (FD) theories**. Formally, an FD theory is a complete theory that admits one domain (up to isomorphism), and whose only domain is finite. For example, consider group axioms with a constant $a$ and the statements $\forall x : (x = 0) \lor (x = a) \lor (x = a \cdot a)$ and $a \cdot a \cdot a = 0$. This is an FD theory, since, up to isomorphism, the only model of this theory is the integers with addition modulo 3. Also Boolean logic and the theory of fixed-length bit-vectors without concatenation are FD theories. Bit-vector theories with (unrestricted) concatenation allow us to construct arbitrarily many distinct constants and are thus not FD theories.

In this section we give a general algorithm for any complete finite-domain theory for which validity is decidable. Let $\mathcal{T}$ be a such a theory and let $M$ be a model of $\mathcal{T}$ with a finite domain $dom(M)$. Assume without loss of generality that for every element $c \in dom(M)$ there is a constant $f$ in $M$ such that $f^M = c$.

We consider a SyGuS problem with a correctness specification $\varphi$ in theory $\mathcal{T}$, a function symbol $f$ to synthesize, and a tree grammar $G = (N, S, F, P)$ generating the set of candidate expressions. Let $a := a_1, \ldots, a_r$ be the constants occurring in $\varphi$. The expression $e$ generated by $G$ to replace $f$ can be seen as a function mapping $a$ to an
| Iteration# | $E_S$ | $E_A$ | $E_B$ |
|------------|------|------|------|
| 1          | none | $x$  | $y$  |
| 2          | $x \oplus y$ | $\neg y$ | none |
| 3          | $\neg y \oplus y \equiv \top$ | none | $(x \oplus y) \oplus \neg y \equiv \neg x$
|            |      |      | $(x \oplus y) \oplus x \equiv y$ |
| 4          | $\neg y \oplus \neg x \equiv \top$ | none | $(\neg y \oplus x) \oplus \neg x \equiv \top$
|            |      |      | $(\neg y \oplus x) \oplus x \equiv y$ |
| 5          | none | none | none |

Table 2. This table shows the expressions added to the sets $E_S$, $E_A$, and $E_B$ when we apply the algorithm to the SyGuS problem in Example 4. For readability, we simplify the expressions, indicated by the symbol \( \equiv \). Expressions that are syntactically new, but do not represent a new function are struck out. When no new function is added, “none” is written in the cell.

Element in $\text{dom}(M)$. If the domain of $M$ is finite there are only finitely many candidate functions, but it can be non-trivial to determine which functions can be generated by $G$. In the following, we describe an algorithm that iteratively determines the set of functions that can be generated by each non-terminal in the grammar $G$.

For each $V \in N$, we maintain a set $E_V$ of expressions $e$. In each iteration and for each production rule $V \rightarrow f(t_1, \ldots, t_k)$ for $V$ in $G$, we consider the expressions $f(t_1^*, \ldots, t_k^*)$ where $t_i^* := t_i$ if $t_i$ is an expression (i.e. $t_i \in T_F$) and $t_i^* \in E_{V'}$ if $t_i$ is a non-terminal $V'$. Given such an expression $e$, we compute the function table, that is the result of $e\{c/a\}$ for each $c \in \text{dom}(M)^r$, compare it to the function table of the expressions currently in $E_V$. Our assumption of decidability of the validity problem for $T$ guarantees that this operation is decidable. If $e$ represents a new function, we add it to the set $E_V$.

The algorithm terminates, after an iteration in which no set $E_V$ changed. As there are only finitely many functions from $\text{dom}(M)^r$ to $\text{dom}(M)$ and the sets $E_V$ grow monotonously, the algorithm eventually terminates. To determine the answer to the SyGuS problem, we then check whether there is an expression $e$ in $E_S$, for which $\varphi\{e/f\}$ is valid.

**Theorem 4.** Let $T$ be a complete theory for which validity is decidable and which has a finite-domain model $M$. The SyGuS problem for $T$ and $T$-compatible tree grammars is decidable.

**Example 4.** Consider the SyGuS problem over boolean expressions with the specification $\varphi = x \oplus f$, where $\oplus$ denotes the XOR operation and $f$ is the function symbol to synthesize from the following tree grammar (we use infix operators for readability):

$$S \rightarrow (A \oplus B)$$
$$A \rightarrow \neg B \mid x$$
$$B \rightarrow (S \oplus A) \mid y$$
The grammar generates boolean functions of variables $x$ and $y$ and the updates to $E_A$, $E_B$, and $E_S$ during each iteration of the proposed algorithm are given in Table 2.

The next step in the algorithm is to determine if any of the three expressions $E_S := \{ x \oplus y, \neg y \oplus y, \neg y \oplus \neg x \}$ make the formula $\varphi\{e/f\}$ valid, which is not the case.

6 Bit-Vectors

In this section, we show that the SyGuS problem for the theory of bit-vectors is undecidable - even when we restrict the problem to tree grammars. The proof makes use of the fact that we can construct (bit-)strings with the concatenation operation and can compare arbitrarily large strings with the equality operation. This enables us to encode the problem of determining if the languages of CFGs with no $\varepsilon$-transitions have non-empty intersection, which is undecidable [15].

**Theorem 5.** The SyGuS problem for the theory of bit-vectors is undecidable for both the class of context-free grammars and the class of BV-compatible tree grammars.

**Proof.** We start with the proof for the class of context-free grammars. Given two context-free grammars $G_1 = (N_1, S_1, T_1, R_1)$ and $G_2 = (N_2, S_2, T_2, R_2)$, we define a SyGuS problem with a single context-free grammar $G = (N, S, T, R)$ that has a solution iff the intersection of $G_1$ and $G_2$ is not empty. The proof idea is to express the intersection of the two grammars as the equality between two expressions, each generated by one of the grammars. The new grammar thus starts with the following production rule:

$$S \rightarrow S_1 = S_2$$

We then have to translate the grammars $G_1$ and $G_2$ into grammars $G_1'$ and $G_2'$ that produce expressions in the bit vector theory instead of arbitrary strings over their alphabets. There is a string produced by both $G_1$ and $G_2$ if and only if the constructed grammars $G_1'$ and $G_2'$ can produce a pair of equal expressions. We achieve this by encoding each letter as a bit string of the fixed length $1 + \log_2 |T_1 \cup T_2|$, and by intercalating concatenation operators ($\oplus$) in the production rules: We encode each production rule $(N, P)$ with $P = p_1 p_2 \ldots p_n$ as $(N, P')$ with $P' = p'_1 \oplus p'_2 \oplus \ldots \oplus p'_n$, where $p'_i = p_i$ if $p_i \in N$, and otherwise $p'_i$ are the fixed-length encodings of the terminal symbols. We then define $N = S \cup N'_1 \cup N'_2$, $T = \{0, 1, \oplus, =\}$, and $R = R'_1 \cup R'_2 \cup \{(S, S_1' = S_2')\}$. The correctness constraint $\varphi$ of our SyGuS problem then only states $\varphi := \neg f$, where $f : \mathbb{B}$ is the function symbol, a constant, to synthesize. As each character in the alphabets of the context-free grammars was encoded using bit vectors the same length, the comparison of the bit vectors is equivalent to the comparison between the strings of characters of the grammars $G_1$ and $G_2$ and the SyGuS problem has a solution if and only if the intersection of the languages of the context-free grammars $G_1$ and $G_2$ is empty.

Note that the context-free grammar $G$ can also be interpreted as a $BV$-compatible tree grammar, where $BV$ is the theory of bit-vectors. Although it is efficiently decidable whether two tree grammars produce a common tree, the expressions produced by the tree-interpretation of $G_1$ and $G_2$ will be equivalent as long as their leaves are equivalent. Thus, the equality of the expression trees in the interpretation of the bit vector theory still coincides with the intersection of the given context-free grammars $G_1$ and $G_2$.

$\Box$
We only used the concatenation operation of the bit-vector theory for the proof. That is, SyGuS is even undecidable for fragments of the theory of bit-vectors for which basic decision problems are easier than the general class; for example, the theory of fixed-sized bit-vectors with extraction and composition [9] for which satisfiability of conjunctions of atomic constraints is polynomial-time solvable unlike the general case which is NP-hard.

**Remark 1.** This proof only relies on the comparison of arbitrarily large values in the underlying logical theory. It may thus be possible to extend the proof to other theories involving numbers, such as LIA, LRA, and difference logic. The problem here is that these proofs tend to depend on syntactical sugar. Consider the case of LIA. If the signature allows us to use arbitrary integer constants, such as 42, it is simple to translate the proof above into a proof of undecidability of SyGuS for LIA and CFGs. For the standard signature of LIA, however, which just includes the integer constants 0 and 1 (larger integers can then be expressed as the repeated addition of the constant 1) the proof scheme above does not apply.

## 7 Other Background Theories

In this section, we remark on the decidability for some other classes of SyGuS problems. These results are straightforward, but the classes do occur in practice, and so they are worth mentioning.

**Linear real arithmetic (LRA) with arbitrary affine expressions.** Consider the family of SyGuS problems where:

i) the specification \( \varphi \) is a Boolean combination of linear constraints over real-valued variables \( \vec{x} := x_1, x_2, \ldots, x_n \) and applications of the function \( f \) to be synthesized. For simplicity, we assume a single function \( f \) of arity \( n \); the arguments below generalize.

ii) The grammar \( G \) is the one generating *arbitrary affine expressions* over \( \vec{x} \) to replace applications of \( f \). Thus, the application \( f(\vec{t}) \), where \( \vec{t} := t_1, t_2, \ldots, t_n \) is a vector of LRA terms, is replaced by an expression of the form \( a_0 + \sum_{i=1}^{n} a_it_i \).

Thus, for a fixed set of variables \( \vec{x} \) there is a fixed grammar for all formulas \( \varphi \).

This case commonly arises in invariant synthesis when the invariant is hypothesized to be an affine constraint over terms in a program. In this case, the solution of the SyGuS problem reduces to solving the \( \exists \forall \) SMT problem

\[
\exists a_0, a_1, \ldots, a_n. \forall x_1x_2\ldots x_n. \left( \varphi[f(\vec{t})/a_0 + \sum_{i=1}^{n} a_it_i]\right)
\]

which reduces to a formula with first-order quantification over real variables. Since the theory of linear real arithmetic admits quantifier elimination, the problem is solvable using any of a number of quantifier elimination techniques, including classic methods such as Fourier-Motzkin elimination [10] and the method of Ferrante and Rackhoff [13], as well as more recent methods for solving exists-forall SMT problems (e.g., [12]).
This decidability result continues to hold for grammars that generate bounded-depth conditional affine expressions. However, the case of unbounded-depth conditional affine expressions is still open, to our knowledge.

A similar reduction, for the case of affine expressions, can be performed for linear arithmetic over the integers (LIA), requiring quantifier elimination for Presburger arithmetic. Thus, this case is also decidable.

**Finite-precision bit-vector arithmetic (BV) with arbitrary bit-vector functions.** Consider the family of SyGuS problems where:

i) the specification \( \phi \) is an arbitrary formula in the quantifier-free theory of finite-precision bit-vector arithmetic [5,4] over a collection of \( k \) bit-vector variables whose cumulative bit-width is \( w \). Let \( f \) be a bit-vector function to be synthesized with output bit-width \( m \).

ii) The grammar \( G \) is the one generating arbitrary bit-vector expressions over these \( k \) variables, using all the operators defined in the theory. In other words, \( G \) imposes no major syntactic restriction on the form of the bit-vector function \( f \).

Thus, for a fixed set of bit-vector variables there is a fixed grammar for all formulas \( \phi \). This class of SyGuS problems has been studied as the synthesis of “bitvector programs” (in applications such as code optimization and program deobfuscation) from components (bit-vector operators and constants) [16,14]. It is easy to see that this class is decidable. A simplistic (but not very efficient) way to solve it is to enumerate all \( 2^{m2^w} \) possible semantically-distinct bitvector functions over the \( k \) variables and check, via an SMT query, whether each, when substituted for \( f \) will make the resulting formula valid.

### 8 Discussion

In this paper, we have presented a first theoretical analysis of the SyGuS problem, focusing on its decidability for various combinations of logical theories and grammars. The main results of the paper are summarized in Table 11 augmented by the decidability of the simple but common SyGuS classes described in Section 7. We conclude with a few remarks about the results, connections between them, and their relevance in practice.

Consider the theory of finite-precision bit-vector arithmetic (BV). We have seen in Section 7 that the SyGuS problem is decidable when the logical formula is an arbitrary BV formula and the grammar allows the function to be replaced by any bit-vector function over the constants in the formula. However, we have also seen that the SyGuS problem is undecidable when an arbitrary context-free grammar can be used to restrict the space of bit-vector functions to be synthesized (see Section 6). These results may seem to contradict the intuition (stated in Section 1) that syntax guidance restricts the search space for synthesis and thus makes the problem easier to solve. We thus have to be careful which classes of grammars we pick to restrict SyGuS problems.

For future work, it would be good to study the LIA and LRA background theories in more detail. In particular, we would like to determine if these theories are decidable when grammars are provided, and whether the use of conditionals without bounding expression tree depth affects the decidability. Further, for SyGuS classes that are decidable, it would be useful to perform a more fine-grained characterization of problem complexity, especially with regard to special classes of grammars.
References

1. R. Alur, M. Martin, M. Raghothaman, C. Stergiou, S. Tripakis, and A. Udupa. Synthesizing Finite-state Protocols from Scenarios and Requirements. In *Proceedings of Haifa Verification Conference (HVC)*, volume 8855 of *LNCS*. Springer, 2014.

2. Rajeev Alur, Rastislav Bodik, Garvit Juniwal, Milo M. K. Martin, Mukund Raghothaman, Sanjit A. Seshia, Rishabh Singh, Armando Solar-Lezama, Emina Torlak, and Abhishek Udupa. Syntax-guided synthesis. In *Proceedings of Formal Methods in Computer-Aided Design (FM CAD)*, pages 1–17, October 2013.

3. Franz Baader and Tobias Nipkow. *Term rewriting and all that*. Cambridge university press, 1999.

4. Clark Barrett, Pascal Fontaine, and Cesare Tinelli. The Satisfiability Modulo Theories Library (SMT-LIB). [www.SMT-LIB.org](http://www.SMT-LIB.org), 2016.

5. Clark Barrett, Roberto Sebastiani, Sanjit A. Seshia, and Cesare Tinelli. Satisfiability modulo theories. In Armin Biere, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability*, volume 4, chapter 8. IOS Press, 2009.

6. Randal E Bryant, Steven German, and Miroslav N Velev. Exploiting positive equality in a logic of equality with uninterpreted functions. In *Proceedings of Computer Aided Verification (CAV)*, pages 470–482. Springer, 1999.

7. Michael Colón, Sriram Sankaranarayanan, and Henny Sipma. Linear invariant generation using non-linear constraint solving. In *Proceedings of Computer Aided Verification (CAV)*, pages 420–432, 2003.

8. H. Comon, M. Dauchet, R. Gilleron, C. Löding, F. Jacquemard, D. Lugiez, S. Thierry, and M. Tommasi. Tree automata techniques and applications. Available on: [http://www.grappa.univ-lille3.fr/tata](http://www.grappa.univ-lille3.fr/tata) 2007. release October, 12th 2007.

9. David Cyrluk, Oliver Möller, and Harald Rueß. An efficient decision procedure for the theory of fixed-sized bit-vectors. In *Proceedings of Computer Aided Verification (CAV)*, pages 60–71. Springer, 1997.

10. G. B. Dantzig and B. C. Eaves. Fourier-Motzkin elimination and its dual. *Journal of Combinatorial Theory A*, 14:288–297, 1973.

11. Anatoli Degtyarev and Andrei Voronkov. The undecidability of simultaneous rigid e-unification. *Theoretical Computer Science*, 166(1):291–300, 1996.

12. Bruno Dutertre. Solving exists/forall problems with Yices. In *Proceedings of the International Workshop on Satisfiability Modulo Theories (SMT)*, 2015.

13. Jeanne Ferrante and Charles Rackoff. A decision procedure for the first order theory of real addition with order. *SIAM Journal of Computing*, 4(1):69–76, 1975.

14. Sumit Gulwani, Susmit Jha, Ashish Tiwari, and Ramarathnam Venkatesan. Synthesis of loop-free programs. *SIGPLAN Notices*, 46:62–73, June 2011.

15. John E Hopcroft and Jeffrey D Ullman. *Introduction to automata theory, languages, and computation*. Pearson Education India, 1979.

16. Susmit Jha, Sumit Gulwani, Sanjit A. Seshia, and Ashish Tiwari. Oracle-guided component-based program synthesis. In *Proceedings of the 32Nd ACM/IEEE International Conference on Software Engineering (ICSE)*, pages 215–224, 2010.

17. Dexter Kozen. Complexity of finitely presented algebras. In *Proceedings of the ninth annual ACM symposium on Theory of computing*, pages 164–177. ACM, 1977.

18. Dexter Kozen. On the myhill-nerode theorem for trees. *Bull. Europ. Assoc. Theor. Comput. Sci.*, 47:170–173, 1992.

19. Daniel Kroening and Ofer Strichman. *Equality Logic and Uninterpreted Functions*, pages 59–80. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

20. Z. Manna and R. Waldinger. A deductive approach to program synthesis. *ACM TOPLAS*, 2(1):90–121, 1980.
21. Armando Solar-Lezama, Rodric Rabbah, Rastislav Bodík, and Kemal Ebcioğlu. Programming by sketching for bit-streaming programs. In Proceedings of the 2005 ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI), pages 281–294, 2005.

22. Armando Solar-Lezama, Liviu Tancau, Rastislav Bodík, Sanjit A. Seshia, and Vijay Saraswat. Combinatorial sketching for finite programs. In ASPLOS, pages 404–415, 2006.

23. Abhishek Udupa, Arun Raghavan, Jyotirmoy V. Deshmukh, Sela Mador-Haim, Milo M.K. Martin, and Rajeev Alur. TRANSIT: Specifying protocols with concolic snippets. In Proceedings of the 34th ACM SIGPLAN conference on Programming Language Design and Implementation, pages 287–296, 2013.

24. Margus Veanes. On computational complexity of basic decision problems of finite tree automata. Technical report, UPMAIL Technical Report 133, Uppsala University, Computing Science Department, 1997.
A Omitted details from Sec. 4

At the end of Sec. 4 we remarked that the case-1 and case-2 restrictions on regular clauses are necessary. The following example gives a clause that includes one positive and one negative equation in which an \( f \) appears. The set of solutions to the corresponding SyGuS-EUF problem is not a regular tree language. More specifically:

Let \( \Sigma := \{ g : 1, g' : 1, h : 1, a : 0, b : 0, c : 0 \} \) be a ranked alphabet, and let \( f \) be a unary function symbol to be synthesized. Let \( N := \{ f(a) = b, g(a) = a, g'(a) = a, h(a) = \text{true}, h(b) = c, g(c) = c, g'(c) = c \} \) and \( \phi := (\bigwedge_{e \in P} e) \rightarrow f(b) = c \). Define \( G \) to be the tree grammar with start symbol \( S \) and the following rules:

\[
S \rightarrow g(S) \mid g'(S) \mid h(A) \\
A \rightarrow g(A) \mid g'(A) \mid h(A) \mid x.
\]

We will show that the set of solutions to the regular SyGuS-EUF problem \((\phi, \Sigma, G, f)\) is not a regular tree language.

Let \( w(x) \in L(G) \) be a replacement to \( f \) and \( E' := E\{w/f\} \). By the rules of \( G \), there must be a context \( B \) and a term \( t(x) \) over the alphabet \( \{ g : 1, g' : 1 \} \) such that \( w(x) = B[h(t(x))] \). We can see that \( b = E' w(a) = E' B[h(t(a))] = E' B[h(a)] = E' B[b] \). Also, \( w(b) = E' c \iff h(t(b)) = E' c \iff t(b) = E' b \). The terms \( t(x) \) such that \( t(b) = b \) are precisely those of the form \( B[B[...B[x]...] ...] \). Therefore, the set of solutions to the above regular SyGuS-EUF is \( L := \{ B[h(B[B[...B[x]...] ...])] \mid B \text{ is any context over } \{ g : 1, g' : 1 \} \} \).

We now use the Myhill-Nerode theorem for regular tree languages [18], stated below:

**Theorem 6 (Myhill-Nerode theorem for regular tree languages [8]).** Given a tree language \( L \) over ranked alphabet \( \Sigma \), we define \( s \equiv_L t \) if \( C[s] \iff C[t] \) for each context \( C \) and terms \( s \) and \( t \) over \( \Sigma \). The following are equivalent:

1. \( L \) is regular
2. \( \equiv_L \) has finitely many equivalence classes
3. \( L \) is accepted by a rational tree automaton.

Using this theorem, it is easy to check that \( L \) is not regular.