On Direct vs Indirect Data-Driven Predictive Control

Vishaal Krishnan  Fabio Pasqualetti

Abstract— In this work, we compare the direct and indirect approaches to data-driven predictive control of stochastic linear time-invariant systems. The distinction between the two approaches lies in the fact that the indirect approach involves identifying a lower dimensional model from data which is then used in a certainty-equivalent control design, while the direct approach avoids this intermediate step altogether. Working within an optimization-based framework, we find that the suboptimality gap measuring the control performance w.r.t. the optimal model-based control design vanishes with the size of the dataset only with the direct approach. The indirect approach has a higher rate of convergence, but its suboptimality gap does not vanish as the size of the dataset increases. This reveals the existence of two distinct regimes of performance as the size of the dataset of input-output behaviors is increased. We show that the indirect approach, by relying on the identification of a lower dimensional model, has lower variance and outperforms the direct approach for smaller datasets, while it incurs an asymptotic bias as a result of the process noise and a (possibly) incorrect assumption on the order of the identified model. The direct approach, however, does not incur an asymptotic bias, and outperforms the indirect approach for larger datasets. Ultimately, by revealing the existence of two non-asymptotic regimes for the performance of direct and indirect data-driven predictive control designs, our study suggests that neither approach is invariably superior and that the choice of design must, in practice, be informed by the available dataset.

Index Terms— Data-driven predictive control, direct and indirect data-driven control, system identification, generative and discriminative models.

I. INTRODUCTION

The interest in direct data-driven control is motivated by its promise to make the system identification step unnecessary for control design, and more optimistically by its bid to outperform the traditional system identification and model-based control design pipeline. While there has been a surge in literature devoted to developing techniques for direct data-driven control [1]–[5], the question of its success on the above fronts has largely remained unsettled. With this broad motivation, we seek to systematically compare the performance of the two paradigms in the predictive control setting for stochastic linear time-invariant systems.

We consider a discrete-time, stochastic linear time-invariant (LTI) system of the form:

\[ x_{t+1} = Ax_t + Bu_t + w_t, \quad y_t = Cx_t, \]

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{p \times n} \) are the system matrices, \( x_t \in \mathbb{R}^n \), \( u_t \in \mathbb{R}^m \) and \( y_t \in \mathbb{R}^p \) are the system state, control input and output at time \( t \in \mathbb{N} \), respectively, with \( w_t \in \mathbb{R}^n \) being the process noise generated by an i.i.d. process with distribution \( \mathcal{N}(0, \Omega_w) \). Furthermore, we assume that the pair \( (A, C) \) is observable and that \( (A, B) \) is controllable. The predictive control task is specified via the following optimization problem:

\[
\begin{align*}
\min_{u_0, \ldots, u_{T-1}} & \quad \mathbb{E} \left[ \sum_{t=0}^{T-1} \left( \alpha_t(u_t) + \beta_t(y_{t+1}) \right) \right] \\
\text{s.t.} & \quad y_t = Cx_t, \\
& \quad x_{t+1} = Ax_t + Bu_t + w_t, \\
& \quad x_0 = 0,
\end{align*}
\]

where the stage costs \( \alpha_t(u) = u^\top Q_t u \) and \( \beta_t(y) = (y - y_{ref})^\top R_t (y - y_{ref}) \), with \( Q_t > 0 \) and \( R_t > 0 \) for all \( t \in \{0, \ldots, T-1\} \) and \( y_{ref} \in \mathbb{R}^p \). The data-driven control problem studied in this paper is one of solving the control task (2) using a finite dataset of input-output behaviors \( \{x_{0:i}, u_{0:i}, y_{0:i}\} \) for System (1) with initial condition \( x_0 = 0 \):

\[ U = \left[ u^{(1)} \ldots u^{(N)} \right], \quad Y = \left[ y^{(1)} \ldots y^{(N)} \right], \]

where \( u^{(i)} = (u_{0:i}, \ldots, u_{T-1:i}) \) and \( y^{(i)} = (y_{0:i}, \ldots, y_{T:i}) \) for every \( i \in \{1, \ldots, N\} \). We assume that the inputs \( u_{0:i}^{(i)} \) for the control experiments are generated by an i.i.d. process with distribution \( \mathcal{N}(0, \Sigma_u) \).

Data-driven predictive control design essentially involves the mapping of the dataset of control experiments and the parameters of the control task onto a finite control sequence. Optimality of the control design depends on extracting system and task-relevant information from the dataset [6]–[8]. The presence of noise in the dataset introduces system and task-irrelevant information, and since both system and task are unknown, extracting relevant information from the dataset is key. If the information extraction step is not handled effectively, it results in suboptimality of the control design. Against this backdrop, the distinction between the direct and indirect approaches to data-driven control lie in the way the two approaches extract system and task-relevant information from the dataset. Yet, a complete understanding of the comparative advantages of the two approaches is lacking. Motivated by this need, we undertake in this paper a comparative study between the two approaches. We choose for comparison with the direct data-driven control design a certainty equivalent control design that utilizes a model.
identified from the dataset by ordinary least squares. We later discuss the implications of this choice for our results and conclusions, and emphasize that the qualitative insights we obtain on the comparative performances are much more general and not constrained by this choice.

**Contributions.** The primary contribution of this paper is a comparative study of the non-asymptotic performance of direct and indirect predictive control for stochastic LTI systems. Our investigation reveals the existence of two distinct non-asymptotic regimes for performance as the dataset size is increased, where one approach outperforms the other in each regime. More specifically, we find that the indirect approach, by relying on the identification of a low dimensional kernel representation, converges faster to its asymptotic performance (measured by its suboptimality gap) and outperforms the direct approach for smaller datasets. Conversely, the direct approach outperforms the indirect approach for larger datasets and achieves asymptotic performance with a vanishing suboptimality gap. Our technical contributions are as follows: We first formulate a unifying optimization-based framework that permits a comparative analysis of the direct and indirect approaches. We then present analytical results characterizing the asymptotic performance and sample complexity bounds for the two approaches, shedding light on their performances as the dataset size is increased, and their dependence on the system and noise parameters. Finally, we present results from numerical experiments validating our analysis and demonstrating the existence of the aforementioned non-asymptotic regimes in practice.

**Related work.** The control design problem considered in this paper, that of data-driven predictive control, has garnered considerable attention in recent years. Several techniques for design that combine learning with model predictive control have been proposed [9], [10]. In [11], the authors introduce the Data-Enabled Predictive Control (DeePC) method, which has then been applied to various settings [12]. In a recent work [13], the authors present experimental results comparing direct and indirect data-driven control in the presence of noise and nonlinearities in the underlying system. Furthermore, the authors present a framework to bridge the direct and indirect approaches based on regularization, which potentially allows for an efficient transition between the two approaches. Also, in [14] the authors compare the DeePC method with the Subspace Predictive Control (SPC) method, showing that the two methods are equivalent in the deterministic case, reasoning that the DeePC method implicitly estimates the same predictive model as SPC. Further, they investigate the comparative performance of the two methods in experiments. Also of particular relevance are works that have investigated the connection between persistency of excitation and data-driven control and system identification [15], [16], as are works that have investigated the sample complexity of LTI system identification [17], [18] and data-driven LQR design [19]. However, an analysis characterizing the comparative performance of the direct and indirect approaches and an understanding of the non-asymptotic case are still lacking.

The distinction between direct and indirect data-driven control parallels the distinction between discriminative and generative modeling approaches in machine learning classification, where a dataset is either used to train a classifier to simply learn (in a supervised setting) to classify input examples as in the discriminative case, or to learn a generative model for the classes as in the latter case. In this context, [20] is an early work comparing the performance of discriminative and generative classifiers as a function of dataset size. Several works [21], [22] have proposed interpolating schemes that result in hybrid discriminative/generative models to exploit the advantages of both classes of models, with reported success.

**II. DATA-DRIVEN CONTROL DESIGN**

In this section, we address the data-driven control design problem. To this end, we first take the view that the data-driven predictive control design problem is one of solving the control task (2) with a data-driven model of System (1). We then note that both the direct and indirect approaches can be studied within this framework by showing that even the direct approach to data-driven control design relies on an implicit model of the input-output behavior of the underlying system, even if such a model is not explicitly identified (as recently noted also in [14]). We then obtain a characterization of the suboptimality gap for the data-driven formulation of the control task (2), showing that it is controlled both above and below by the discrepancy between the model used in design and the true model. This allows for a comparison of the direct and indirect approaches to data-driven control, via the discrepancy between the implicit and true models in the case of the direct approach, and between the identified and true models for the indirect approach, respectively.

We begin by establishing a key property of the underlying control task (2) in the model-based setting, which will be useful in setting up the data-driven formulation. The output \( y = (y_1, \ldots, y_T) \) of System (1) (note that we have \( y_0 = 0 \) since \( x_0 = 0 \)) over the horizon \([1, \ldots, T]\) generated by control input \( u = (u_0, \ldots, u_{T-1}) \) is given by:

\[
y = Gy + G'w, \tag{3}
\]

where \( w = (w_0, \ldots, w_{T-1}) \) is the process noise, and:

\[
\begin{bmatrix}
CB & 0 & \ldots & 0 \\
CAB & CB & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
CA^{T-1}B & CA^{T-2}B & \ldots & CB
\end{bmatrix}
\]

is given by replacing \( B \) with \( I_n \) in the expression for \( G \) above. Now, with \( v = G'w \) in (3) and \( F(u, y) = \sum_{t=0}^{T-1} (\alpha_t(u_t) + \beta_t(y_{t+1})) \), we can express the control task (2) as follows:

\[
\min_{u \in \mathbb{R}^n} \mathbb{E}_v \{ F(u, y) \mid y = Gu + v \}. \tag{4}
\]

The following lemma establishes the certainty equivalence property for model-based predictive control of stochastic LTI systems with quadratic cost:
Lemma 2.1 (Certainty equivalent model-based predictive control) For \( u^* = \arg \min_{u \in \mathbb{R}^{m \times T}} F(u, G_u), \) we have \( E_v [F(u^*, G_u^* + v)] = \min_{u \in \mathbb{R}^{m \times T}} E_v [F(u, G_u + v)]. \)

We refer the reader to Appendix A for the proof. Lemma 2.1 establishes that given the input-output model \( G, \) certainty equivalence holds w.r.t. process noise. Therefore, the minimizer to the stochastic optimization problem \( (\ref{eq:stochastic_optimization}) \) can be equivalently obtained as the solution to the following deterministic optimization problem\(^1\):

\[
\min_{u \in \mathbb{R}^{m \times T}} F(Pu), \quad P = \begin{bmatrix} I & G \end{bmatrix}. \tag{5}
\]

This suggests the use of the above formulation \( (\ref{eq:stochastic_optimization}) \) as the control design procedure for the task \( (\ref{eq:task}) \). We now compute the minimizer \( u^* \) in \( (\ref{eq:stochastic_optimization}) \). With \( y_{\text{ref}} = I_T \otimes y_{\text{ref}}, \) we get:

\[ F(u, y) = u^T Q u + (y - y_{\text{ref}})^T R (y - y_{\text{ref}}), \]

where \( Q = \text{diag}(Q_0, \ldots, Q_{T-1}) \) and \( R = \text{diag}(R_0, \ldots, R_{T-1}) \). We further obtain:

\[ \nabla F(u, y) = 2 \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} u - 2 \begin{bmatrix} 0 \\ R y_{\text{ref}} \end{bmatrix}. \]

The minimizer \( u^* \) in \( (\ref{eq:stochastic_optimization}) \) satisfies \( P^T \nabla F(Pu^*) = 0 \) (first-order optimality condition). Substituting from the above, we get:

\[ P^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} Pu^* = P^T \begin{bmatrix} 0 \\ R y_{\text{ref}} \end{bmatrix}, \]

Simplifying the above, we get:

\[ u^* = (Q + G^T R G)^{-1} G^T R y_{\text{ref}}. \tag{6} \]

We note however, that the certainty equivalence property was established in Lemma 2.1 under the assumption of availability of the true input-output behavior model \( \hat{P} \). In the data-driven control setting, we do not have direct access to \( \hat{P} \) for control design by \( (\ref{eq:stochastic_optimization}) \), and either (i) an estimate of \( \hat{P} \) is obtained from noisy input-output behavior data \( U, Y \) (indirect data-driven control), or (ii) the data matrix \( \begin{bmatrix} U^T & Y^T \end{bmatrix}^T \) is itself used in place of the behavior model (direct data-driven control). In other words, data-driven control design involves the use of an estimate \( \hat{P} \) of the true behavior model \( P \). In indirect data-driven control design, such an estimate is explicitly obtained from data, whereas direct data-driven control design involves the use of an implicit estimate, as will be seen in the ensuing section. The data-driven control design problem is formulated by replacing \( \hat{P} \) for \( P \) in \( (\ref{eq:stochastic_optimization}) \):

\[
\min_{u \in \mathbb{R}^{m \times T}} F(\hat{P}u), \quad \hat{P} = \begin{bmatrix} I & \hat{G} \end{bmatrix}. \tag{7}
\]

Also, let \( \hat{u} = \arg \min_{u \in \mathbb{R}^{m \times T}} F(\hat{P}u) \). Following similar steps as in the computation of \( u^* \) in \( (\ref{eq:certainty_equivalent}) \), we get:

\[ \hat{u} = \left( Q + \hat{G}^T \hat{R} \hat{G} \right)^{-1} \hat{G}^T \hat{R} y_{\text{ref}}. \tag{8} \]

We note that the control input \( \hat{u} \) is not guaranteed to be optimal for the control task \( (\ref{eq:task}) \), owing to the mismatch between the estimate \( \hat{G} \) and the true model \( G \). The performance of \( \hat{u} \) is measured by its suboptimality gap given by:

\[ \text{Gap}(\hat{u}) = F(\hat{P}\hat{u}) - F(\hat{P}u^*). \tag{9} \]

This suboptimality gap serves as a metric for comparing the direct and indirect data-driven control design methodologies. We note that \( \text{Gap}(\hat{u}) \geq 0 \) for any \( \hat{u} \), and from the \( \mu \)-strong convexity of \( F(\hat{P}) \), it follows that \( \text{Gap}(\hat{u}) \geq \mu \|u - u^*\|^2/2 \). From the \( \nu \)-Lipschitz continuity of the gradient of \( F(\hat{P}) \), it follows that \( \text{Gap}(\hat{u}) \leq \nu \|u - u^*\|^2/2 \). Combining the above, we get:

\[ \mu/2 \|u - u^*\|^2 \leq \text{Gap}(\hat{u}) \leq \nu/2 \|u - u^*\|^2. \]

It follows from \( (\ref{eq:certainty_equivalent}) \) and \( (\ref{eq:suboptimality_gap}) \) that the error \( \|\hat{u} - u^*\| \) arises from the mismatch \( \Delta \) between \( \hat{G} \) and \( G \), and we see from the above that it also controls the suboptimality gap \( \text{Gap}(\hat{u}) \). This allows us to investigate the control performance measured by the suboptimality gap \( \text{Gap}(\hat{u}) \) via the mismatch \( \Delta \) between \( \hat{G} \) and \( G \).

A. Direct data-driven control

In direct data-driven control design, we would like to use the input-output behavior data matrix \( \begin{bmatrix} U^T & Y^T \end{bmatrix}^T \) directly in place of the model \( \hat{P} \) in \( (\ref{eq:certainty_equivalent}) \). The key idea here is to avoid identifying a model of input-output behaviors and to directly search for the optimal behavior for the task \( (\ref{eq:task}) \) within the span of observed behaviors contained in the data matrix \( \begin{bmatrix} U^T & Y^T \end{bmatrix}^T \). However, before obtaining the direct data-driven design formulation, we first note that the input-output behavior data satisfies:

\[ \begin{bmatrix} U \\ Y \end{bmatrix} = \mathcal{P} U + \begin{bmatrix} 0 \\ V \end{bmatrix}, \]

where \( V = [v^{(1)} \ldots v^{(N)}] \) is the matrix of noise realizations in the control experiments. We note that while the above relation reveals the underlying structure in the available input-output behavior data, we do not have access to \( \mathcal{P} \) and \( V \). The above relation reveals that as the number of experiments \( N \) increases, it may be possible to construct behaviors \( \begin{bmatrix} U^T & Y^T \end{bmatrix}^T z \) with \( z \in \mathbb{R}^N \), such that \( z \in \text{Ker}(U) \), but which nevertheless incur a low cost in design, i.e., \( F(\begin{bmatrix} U^T & Y^T \end{bmatrix}^T z) \) attains a low value. The corresponding control input would indeed be \( \hat{u} = Uz = 0 \), suggesting that the low cost can be attained without taking any control action. However, this is entirely misleading, as such behaviors are essentially constructed from the process noise components realized in the control experiments and contained in \( V \), and the corresponding control input \( \hat{u} = 0 \) may not actually incur a low suboptimality gap, as measured by \( \text{Gap}(\hat{u}) \). We therefore restrict our search within \( \text{Ker}^\perp(U) \),

\(^2\)where \( \mu = \min_{\lambda} \{Q + G^T R G\} \) and \( \nu = \max_{\lambda} \{Q + G^T R G\} \).

\(^3\)It is further possible to obtain a bound on \( \|\hat{u} - u^*\| \) as a function of \( \Delta \) and the system and task parameters. We do not, however, pursue a characterization of such a bound in the context of this paper.
the orthogonal complement of $\text{Ker}(U)$, to obtain the direct data-driven design formulation:
\[
\min_{z \in \mathbb{R}^N} F \left( \begin{bmatrix} U \\ Y \end{bmatrix} U^T U z \right).
\]
Now, for any $z' = U^T U z \in \text{Ker}^p(U)$, we have:
\[
U^T U z = \left( \begin{bmatrix} \mathcal{P} U + \begin{bmatrix} 0 \\ \mathcal{V} \end{bmatrix} \end{bmatrix} U^T U z = \left( \mathcal{P} U + \begin{bmatrix} 0 \\ \mathcal{V} \end{bmatrix} \right) z, \right.
\]
\[
\mathcal{P} + \begin{bmatrix} 0 \\ \mathcal{V} \end{bmatrix} \right) U z,
\]
where $u = U z \in \text{Col}(U)$. With $\hat{P}_{\text{direct}} = \mathcal{P} + \begin{bmatrix} 0 \\ \mathcal{V} \end{bmatrix}$, we can rewrite (10) as:
\[
\min_{u \in \text{Col}(U)} F \left( \hat{P}_{\text{direct}} u \right).
\]

The above is the sense in which the direct data-driven control design formulation employs implicitly an estimate $\hat{P}_{\text{direct}}$ of the true behavior model $\mathcal{P}$ as stated earlier, and can be connected to the general data-driven control design formulation (7). We now obtain the minimizer $\hat{u}_{\text{direct}}$ in (10) as:
\[
\hat{u}_{\text{direct}} = U \left( U^T Q U + Y_{\text{ref}} U^T \mathcal{R} Y_{\text{ref}} \right)^T Y_{\text{ref}} U^T.
\]
where $Y_{\text{ref}} = Y U^T U$. Furthermore, we note that the mismatch between $\hat{G}_{\text{direct}}$ and $\mathcal{G}$ is given by:
\[
\Delta_{\text{direct}} = \hat{G}_{\text{direct}} - \mathcal{G} = V U^T.
\]
We characterize the dependence of the implicit model error on the number of control experiments $N$ and the time horizon $T$ through the following theorem, for the Single-Input-Single-Output case ($p = m = 1$) for the sake of simplicity. We note that the result can be readily extended to the Multiple-Input-Multiple-Output case.

**Theorem 2.2 (Implicit model)** Let $p = m = 1$ and $N, T \in \mathbb{N}$ be such that the empirical covariance matrix $\Sigma_{\text{uu}} = UU^T / N$ is invertible. The implicit model error $\Delta_{\text{direct}}$, given by (12), satisfies:
\[
(i) \quad \mathbb{E} [\Delta_{\text{direct}}] = 0,
\]
\[
(ii) \quad \mathbb{P} \{ \| \Delta_{\text{direct}} \|_F \geq \epsilon \} \leq \frac{T^2}{N \epsilon^2} \frac{\sigma_{\text{min}} (\Sigma_{\text{uu}})}{\sigma_{\text{w}}^2},
\]
where $\sigma_{\text{min}} (\Sigma_{\text{uu}})$ is the smallest singular value of $\Sigma_{\text{uu}}, \sigma_{\text{w}}$ the variance of input $u$ and $\sigma_{\text{w}} = \sum_{i=0}^{T-1} CA^T \Omega_a A^T C^T$.

We refer the reader to Appendix [3] for the proof. Theorem 2.2 sheds light on both the asymptotic and non-asymptotic performance of the direct approach as the dataset size $N$ increases. It establishes that the direct approach does not incur an asymptotic bias, which converges to zero w.p. 1 at the rate $O \left( \frac{\epsilon}{\sqrt{N}} \right)$. The $T^2$ dependence has implications for the scalability of performance of the direct approach which, as we will also see from numerical experiments, deteriorates drastically with the control horizon length $T$.

### B. Indirect data-driven control

For the indirect approach, as stated earlier, we choose as the candidate a certainty-equivalent control design that utilizes a model identified from the dataset by ordinary least squares. To develop the indirect data-driven formulation, we first note that (5) can be rewritten as $[\mathcal{G} \ 1] \left[ \begin{bmatrix} u \\ y - v \end{bmatrix} \right] = 0$. Let $M_L \in \mathbb{R}^{pT \times pT}$ be a lower block-triangular block-Toeplitz matrix with first column block $[M_1 \ldots M_{L}]$ and row block $[M_1 0 \ldots 0]$, with $M_k \in \mathbb{R}^{p \times p}$ for all $p \in \{1, \ldots, L\}$. For $L \geq n + 1$, we note that there exists $M_k \in \mathbb{R}^{p \times p}$, $p \in \{1, \ldots, L\}$, with $M_1 = I_p$ such that the matrix $N_L = -M_L \mathcal{G} \in \mathbb{R}^{pT \times nT}$ is also lower block-triangular block-Toeplitz, with first column block $[N_1 \ldots N_{L}]$ and row block $[N_1 0 \ldots 0]$, with $N_k \in \mathbb{R}^{p \times m}$ for all $p \in \{1, \ldots, L\}$ and $N_L = 0$. We then get:
\[
[N_L \ M_L] \left[ \begin{bmatrix} u \\ y - v \end{bmatrix} \right] = \mathcal{G} \left[ \begin{bmatrix} u \\ y - v \end{bmatrix} \right] = 0.
\]
It can be readily seen that this corresponds to the delay operator representation [1], with $L \geq n + 1$:
\[
\sum_{\tau = 1}^{L} M_{\tau} (y_{t-\tau+1} - v_{t-\tau+1}) + \sum_{\tau = 1}^{L} N_{\tau} u_{t-\tau} = 0.
\]

Now since a pair $(M_L, N_L)$ satisfying $N_L = -M_L \mathcal{G}$ exists for any $L \geq n + 1$, we have $[N_L \ M_L] \mathcal{P} = [N_L \ M_L] \left[ \begin{bmatrix} I \\ 0 \end{bmatrix} \right] = 0$. Furthermore, since $M_1 = I_p$ we note that $M_L$ is invertible for any $L$. Therefore, $\mathcal{G} = -M_L^{-1} N_L$, and any such pair $(M_L, N_L)$ (for $L \geq n + 1$) satisfying the above allows for an equivalent representation of the input-output behavior model $\mathcal{P}$. In the data-driven control setting, however, the order $n$ of the underlying system is unknown, and parametric estimates $\hat{M}_1, \ldots, \hat{M}_L$ and $\hat{N}_1, \ldots, \hat{N}_L$, with $\hat{M}_1 = I_p$ are obtained for some choice $L \in \{1, \ldots, T\}$. This corresponds to the system identification problem, where the identified model is given by $\hat{G}_{id} = -\hat{M}_L^{-1} \hat{N}_L$, and the indirect data-driven control formulation is given by:
\[
\min_{u \in \mathbb{R}^{m \times T}} F (\hat{P}_{id} u), \quad \hat{P}_{id} = \left[ -\hat{M}_L^{-1} \hat{N}_L \right].
\]
In order to estimate parameters $\{M_k\}_{k=1}^{L}$ and $\{N_k\}_{k=1}^{L}$ for some choice $L$, we reorganize the dataset of input-output behaviors of length $T$ obtained from control experiments on System (1) into one of input-output behaviors of length $L$. For any length $T$ behavior $(u, y)$, we obtain $T - L + 1$ behaviors of length $L$ as the columns of $[H_L(u)^T \\ H_L(y)^T]^T$, where for any $z = (z_1, \ldots, z_T)$, $H_L(z)$ is the Hankel matrix of depth $L$. For $L \geq n + 1$, it follows from (13) that:
\[
[N_L \ldots N_1 \ M_L \ldots M_1] \left[ \begin{bmatrix} H_L(u) \\ H_L(y) - H_L(v) \end{bmatrix} \right] = 0.
\]

This follows from the fact that the observability matrix generated by the pair $(A, C)$ attains full column rank over a horizon of length $n$. 


We henceforth treat the SISO (Single-Input-Single-Output) case \((p = m = 1)\) for the sake of simplicity. We note that our results can be readily extended to the MIMO (Multiple-Input-Multiple-Output) case. For \(p = m = 1\), the parameters \(N_k, M_k\) above are scalars, and we let \(n = (n_L, \ldots, n_1)\) and \(m = (m_L, \ldots, m_1)\). Now, we note that the length-\(L\) behaviors do not all correspond to distinct control experiments starting from a zero initial state. Indeed, any length-\(L\) behavior of System (1) satisfies:

\[
\begin{pmatrix}
  u_{t+L-1} \\
  y_{t+L-1}
\end{pmatrix} = \begin{pmatrix}
  0 \\
  \mathcal{O}_L
\end{pmatrix} \tilde{G}_L \begin{pmatrix}
  A x_t \\
  u_{t+L-1}
\end{pmatrix} + \begin{pmatrix}
  0 \\
  \mathcal{V}_L
\end{pmatrix},
\]

where \(x_t = \begin{pmatrix}
  A^{t-1} B \\
  \vdots \\
  B
\end{pmatrix} u_{0:t-1}, \) and the matrices \(\mathcal{O}_L\) and \(\mathcal{G}_L\) are given by:

\[
\mathcal{O}_L = \begin{pmatrix}
  C & \mathcal{A} \\
  \mathcal{C}
\end{pmatrix}, \quad \mathcal{G}_L = \begin{pmatrix}
  C B & 0 & \cdots & 0 \\
  C A B & C B & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  C A^{L-1} B & C A^{L-2} B & \cdots & C B
\end{pmatrix},
\]

Assuming \((A, AB)\) is also a controllable pair, for \(L \geq n + 1\) we have:

\[
(n^\top \ m^\top) \begin{pmatrix}
  0 \\
  \mathcal{O}_L
\end{pmatrix} \mathcal{G}_L = 0.
\]

It follows that \(n + G_L^T m = 0\) and \(O_L^T m = 0\). For \(k \in \{1, \ldots, L-1\}\), we have \(n_k = -G_L^{(L-k+1)^T} m\). Furthermore, since \(O_L^T m = 0\), it follows for \(k = L\) that \(n_L + G_L^{(1)^T} m = n_L + B^T O_L^T m = n_L = 0\). For the problem of identifying the parameters \((n, m)\), we note that we do not have access to the matrices \(\mathcal{O}_L\) and \(\mathcal{G}_L\) above, nor to the (process) noise \(\nu\) from experiments. We thereby obtain a data-driven, ordinary least-squares formulation to solve (16):

\[
\min_{\tilde{n}, \tilde{m} \in \mathbb{R}^L} \left( n^\top \tilde{m} \right) \begin{pmatrix}
  1 \\
  N \sum_{t=1}^N H_L(u^{(k)}(t)) H_L(u^{(k)}(t))^\top \\
  H_L(u^{(k)}(t)) H_L(u^{(k)}(t))^\top
\end{pmatrix} \begin{pmatrix}
  \tilde{n} \\
  \tilde{m}
\end{pmatrix},
\] subject to \(\tilde{n}_1 = 1, \tilde{n}_L = 0\),

where \(\tilde{N} = N(T - L + 1)\). We now investigate the dependence of the identified parameter mismatch on the number of control experiments \(N\), the time horizon \(T\) and the model dimension \(L\). We first note that:

\[
E \left[ \frac{1}{N} \sum_{t=1}^N \begin{pmatrix}
  H_L(u^{(k)}(t)) & H_L(u^{(k)}(t))^\top \\
  H_L(u^{(k)}(t)) & H_L(u^{(k)}(t))^\top
\end{pmatrix} \right] = \begin{pmatrix}
  \Sigma_u & \Sigma_L \Sigma_u G_L^T + \mathcal{O}_L \Sigma_u O_L^T + \Sigma_v \\
  \Sigma_L \Sigma_u G_L^T + \mathcal{O}_L \Sigma_u O_L^T + \Sigma_v & \Sigma_L
\end{pmatrix}.
\]

Let \(\tilde{n} = E[\tilde{n}], \tilde{m} = E[\tilde{m}]\). It can be shown that since \((\tilde{n}, \tilde{m}) = E[(\tilde{n}, \tilde{m})]\), the pair \((\tilde{n}, \tilde{m})\) is the minimizer in (17) in expectation. For the unconstrained version of (17), in expectation we get:

\[
\Sigma_u \tilde{n} + \Sigma_L \tilde{m} = \lambda_{\min} \tilde{n}, \quad \tilde{G}_L \Sigma_u \tilde{n} + \tilde{G}_L \Sigma_L \tilde{m} + \mathcal{O}_L \Sigma_u O_L^T \tilde{m} + \Sigma_v \tilde{m} = \lambda_{\min} \tilde{m},
\]

where \(\lambda_{\min} \geq 0\) is the smallest eigenvalue of the covariance matrix above. It follows that:

\[
\lambda_{\min} \tilde{G}_L \tilde{n} + \mathcal{O}_L \Sigma_u O_L^T \tilde{m} + \Sigma_v \tilde{m} = \lambda_{\min} \tilde{m},
\]

Note that for \(\lambda_{\min} = 0\), we will need that \(O_L \Sigma_u O_L^T + \Sigma_v\) has a non-trivial kernel, which is precluded by the fact that \(\Sigma_v\) is positive definite and for \(L \leq n\), even \(O_L^T\) may have a non-trivial kernel. Thereby, for \(\lambda_{\min} > 0\), we have:

\[
\left( n^\top \ m^\top \right) \left( O_L \Sigma_u O_L^T + \Sigma_v \right)^{-1} \left( \begin{pmatrix}
  \Sigma_u \\
  \Sigma_v
\end{pmatrix} \tilde{n} - \lambda_{\min} \tilde{m} \right) = 0.
\]

Comparing the above to (17), we see that in the presence of process noise and/or for \(L < n\), we incur an asymptotic bias \(\|\mathcal{N} \tilde{n} - \tilde{m}\| > 0\), from the fact that \(\lambda_{\min} > 0\).

For the non-asymptotic case, we first note that the data matrices satisfy:

\[
\begin{pmatrix}
  H_L(u^{(k)}(t)) \\
  H_L(u^{(k)}(t))^\top
\end{pmatrix} = \begin{pmatrix}
  I \\
  \mathcal{O}_L
\end{pmatrix} H_L(u^{(k)}(t)) + \begin{pmatrix}
  0 \\
  \mathcal{O}_L
\end{pmatrix} A X_0 + \begin{pmatrix}
  0 \\
  \mathcal{O}_L
\end{pmatrix} H_L(u^{(k)}(t)),
\]

where \(X_0\) is the matrix containing the hidden initial states for the length-\(L\) behaviors in the data Hankel matrices. We see that the sample covariance matrix in (17) is given by:

\[
\frac{1}{\tilde{N}} \sum_{k=1}^N H_L(u^{(k)}(t)) H_L(u^{(k)}(t))^\top / \tilde{N} = \Sigma_{uL}, \quad \Sigma_{vL}.
\]

We now see that \(\sum_{k=1}^N H_L(u^{(k)}(t)) H_L(u^{(k)}(t))^\top / \tilde{N} = \Sigma_{uL}, \quad \Sigma_{vL} \)

where \(\Sigma_{uL}, \Sigma_{vL} \) are the covariance of the length-\(L\) inputs. Furthermore, we get \(H_L(u)(t) / \tilde{N} = \Sigma_{uL}, \quad \Sigma_{vL} \)

where \(\Sigma_{uL}, \Sigma_{vL} \) are the sample (cross) covariances between \(u, x_0\) and \(u, v\) respectively. A similar expression involving the sample (cross) covariances among \(u, x_0, v\) can be obtained for the block \(\sum_{k=1}^N H_L(u^{(k)}(t)) H_L(u^{(k)}(t))^\top / \tilde{N} \)

We note that the cross covariances vanish in expectation since \(u, x_0, v\) are independent. We further note that the matrices \(\Sigma_{uL}, \Sigma_{vL}, \Sigma_{uL}, \Sigma_{vL}, \Sigma_{uL}, \Sigma_{vL} \) are \(L\)-dimensional square matrices, and do not depend on \(T\). Following a similar analysis as in Theorem 22, we can infer that as the dataset size \(N\) increases, they converge to zero with high probability at a rate \(O\left( \frac{\tilde{T}^2}{N(T - L + 1)} \right) \), which is much faster than the rate \(O\left( \frac{T^2}{N} \right)\) we obtained for the direct data-driven case.

III. A COMPARISON OF DIRECT AND INDIRECT DATA-DRIVEN CONTROL

We now discuss the comparative performance of the direct and indirect approaches, drawing both from the analytical results of the previous section and results from numerical experiments (contained in Figures 12 and 3).

We see from Theorem 22 that the implicit model error in the direct approach vanishes asymptotically as \(N \to \infty\), and it therefore follows that the suboptimality gap vanishes asymptotically with the size of the dataset. However, we also see that the direct approach has high sample complexity, resulting in potentially large variance for finite values of \(N\), as clearly seen from the wider 95\% confidence bands for the direct approach in Figure 3.

From the discussion in the previous section and Figure 3 we see that the ordinary least squares-based indirect approach incurs an asymptotic bias (and consequently, a suboptimality gap) due to the process noise covariance and an incorrect assumption on the dimension of the underlying
system. While the asymptotic bias due to process noise can be partially mitigated by a total least squares-based approach [23] for high signal-to-noise ratio regimes, the bias due to an incorrect assumption on the model dimension will remain. Moreover, we note that a total least squares-based approach has pitfalls in low signal-to-noise ratio regimes in comparison to ordinary least squares, and will potentially have high variance (where ordinary least squares has a distinct advantage over the direct approach), and we believe that there exists no invariably superior candidate for the indirect approach. We note that the indirect approach has lower sample complexity than the direct approach owing to the intermediate step of identifying a lower $L$-dimensional model from the data. Furthermore, we observe from experiments that the indirect approach with a large $L \geq n + 1$ overfits noise in the data resulting in high variance (Figure [1]), and the direct approach outperforms the indirect approach in this case. We also see from Figure [1] that the indirect approach incurs a larger asymptotic bias with a lower value of $L \leq n$ but has lower variance for finite values of $N$.

Since the direct approach relies on an implicit $T$-dimensional model of the input-output behavior, we see that the indirect approach mitigates some of the hurdles faced by the (high dimensional) direct approach by explicitly identifying a lower $L$-dimensional model. Intuitively, we see that the implicit $T$-dimensional model in the direct approach effectively contains $O(T^2)$ parameters, while the identified $L$-dimensional model in the indirect approach contains $O(L)$ parameters. This explains the poor scaling behavior of the direct approach observed in Figure [2] where the performance of the direct approach deteriorates with increasing $T$, while the indirect approach remains unaffected (since $L$ is not a function of $T$). Furthermore, identifying a lower $L$-dimensional model not only has lower sample complexity, but also results in the expansion of the dataset available for identification. This can be seen from the fact that the dataset of $T$-dimensional behaviors contains $N$ samples, whereas by constructing $L$-dimensional Hankel matrices from $T$-dimensional behaviors essentially results in a dataset containing $N(T - L + 1)$ samples (it must however be noted that the $L$-dimensional behavior samples are not all i.i.d.).

IV. CONCLUDING REMARKS

Our study reveals the existence of two non-asymptotic regimes for the performance of direct and indirect data-driven predictive control designs, which precludes any conclusion that either of the two approaches is invariably superior. It also suggests that interpolating between the direct and indirect approaches may be useful in managing the underlying trade-offs effectively in practice, a direction that has recently been pursued in [13]. We note that for our comparison we chose a certainty-equivalent control design with ordinary least squares-based identification as the candidate for the indirect approach. While our results are, in principle, restricted by this particular choice, we believe that the qualitative insights we obtain on the comparative performances are much more general and not constrained by this choice. However, we do not comment on the capabilities of potentially more sophisticated techniques (for both direct and indirect data-driven control) in better managing the underlying tradeoffs, and a more fundamental analysis is likely necessary for a truly technique-agnostic comparison. Furthermore, a fine-grained analysis of the implications for data-driven control of the phenomena of measure concentration, data sparsity and scaling of signal-to-noise ratio may afford a finer characterization of the non-asymptotic regimes of performance and result in practically useful heuristics.

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APPENDIX

A. Proof of Lemma 2.7

By convexity of $F$ and Jensen’s inequality, we have:
\[
\mathbb{E}_\nu \left[ F(\bm{u}, \mathcal{G} \bm{u} + \nu) \right] \geq F(\bm{u}, \mathbb{E}_\nu [p \mathcal{G} \bm{u} + \nu]) \\
= F(\bm{u}, \mathcal{G} \bm{u}) \\
\geq \min_{\nu} F(\bm{u}, \mathcal{G} \bm{u}).
\]
Let $G(\bm{u}) = \mathbb{E}_\nu [p \mathcal{G} \bm{u} + \nu]$. For any $\bm{u}_1$ and $\bm{u}_2$, we have:
\[
G(\bm{u}_2) - G(\bm{u}_1) = \mathbb{E}_\nu \left[ F(\bm{u}_2, \mathcal{G} \bm{u}_2 + \nu) - F(\bm{u}_1, \mathcal{G} \bm{u}_1 + \nu) \right] \\
\geq \mathbb{E}_\nu \left[ \nabla^T F(\bm{u}_1, \mathcal{G} \bm{u}_1 + \nu) \delta \right] (\bm{u}_2 - \bm{u}_1) \\
= \mathbb{E}_\nu \left[ \nabla^T F(\bm{u}_1, \mathcal{G} \bm{u}_1 + \nu) \right] (\bm{u}_2 - \bm{u}_1).
\]
By the Dominated Convergence Theorem, we get:
\[
\mathbb{E}_\nu \left[ \nabla^T F(\bm{u}, \mathcal{G} \bm{u} + \nu) \right] = \nabla \mathbb{E}_\nu [F(\bm{u}, \mathcal{G} \bm{u} + \nu)] = \nabla G(\bm{u}).
\]
Substituting in the inequality above, we get:
\[
G(\bm{u}_2) - G(\bm{u}_1) \geq \nabla \mathbb{E}_\nu [F(\bm{u}, \mathcal{G} \bm{u} + \nu)] (\bm{u}_2 - \bm{u}_1).
\]
This establishes that the function $G$ is also convex. Furthermore, since $F$ is quadratic, it follows that $\nabla F$ is affine. Now, for $\bm{u}^* = \arg \min_{\nu} F(\bm{u}, \mathcal{G} \bm{u})$, we have $\nabla \mathbb{E}_\nu [F(\bm{u}, \mathcal{G} \bm{u}^* + \nu)] = \nabla F(\bm{u}^*, \mathcal{G} \bm{u}^*) = 0$ (by strict convexity of $F$ it also follows that $\nabla \mathbb{E}_\nu \mathcal{G}(\bm{u}) = 0$ only if $\bm{u} = \bm{u}^*$). Therefore, for any $\bm{u} \in \mathbb{R}^m$, we get that $G(\bm{u}) - G(\bm{u}^*) \geq \nabla \mathbb{E}_\nu [F(\bm{u}, \mathcal{G} \bm{u} + \nu)] (\bm{u} - \bm{u}^*) = 0$, which implies that $\bm{u}^*$ is the global minimizer of $G$. Thus, we have $\bm{u}^* = \arg \min_{\nu} \mathbb{E}_\nu [F(\bm{u}, \mathcal{G} \bm{u} + \nu)]$.

B. Proof of Theorem 2.2

The implicit model error $\Delta_{direct}$ is given by:
\[
\Delta_{direct} = VU^T = VU^T (UU^T)^{-1} \\
= \sum_{k=1}^{N} \mathcal{G} (\bm{u})^T (\sum_{k=1}^{N} \mathcal{G} (\bm{u}) (\bm{u})^T)^{-1} \\
= \sum_{k=1}^{N} \mathcal{G} \mathcal{G} (\bm{u}) (\bm{u})^T (\sum_{k=1}^{N} \mathcal{G} (\bm{u}) (\bm{u})^T)^{-1}.
\]
Since $\mathcal{G}$ and $\mathcal{G}^T$ are independent and $\mathbb{E}[\mathcal{G}] = 0$, we have:
\[
\mathbb{E} [\Delta_{direct}] = \sum_{k=1}^{N} \mathcal{G} \mathcal{G} (\bm{u}) (\bm{u})^T (\sum_{k=1}^{N} \mathcal{G} (\bm{u}) (\bm{u})^T)^{-1} \\
= 0.
\]
Let $\Sigma_{uu} = \sum_{k=1}^{N} \mathcal{G} (\bm{u}) (\bm{u})^T / N$ be the sample covariance of $\mathcal{G}$. Following the reasoning above, we also get that:
\[
\mathbb{E} \left[ \frac{1}{N} \mathcal{G} \mathcal{G} (\bm{u}) (\bm{u})^T \right] = 0.
\]
Now, we have:
\[
\| \Delta_{direct} \|_{F} \leq \frac{1}{N} \mathcal{G} \mathcal{G} (\bm{u}) (\bm{u})^T \leq \sigma_{\min} (\Sigma_{uu}) \| \frac{1}{N} VU^T \|_{F}.
\]
Furthermore, we have:
\[
\left\| \frac{1}{N} VU^T \right\|_{F} = \sqrt{\sum_{i,j=1}^{T} \left( \frac{1}{N} VU^T \right)_{ij}^2} \leq T \left\| \frac{1}{N} VU^T \right\|_{max},
\]
and it follows that:
\[
\| \Delta_{direct} \|_{F} \leq \frac{T}{\sigma_{\min} (\Sigma_{uu})} \left\| \frac{1}{N} VU^T \right\|_{max}.
\]
We now have:
\[
\mathbb{P} \left\{ \| \Delta_{direct} \|_{F} \geq \epsilon \right\} \leq \mathbb{P} \left\{ \frac{T}{\sigma_{\min} (\Sigma_{uu})} \left\| \frac{1}{N} VU^T \right\|_{max} \geq \epsilon \right\} \\
= \mathbb{P} \left\{ \left\| \frac{1}{N} VU^T \right\|_{max} \geq \frac{\epsilon \sigma_{\min} (\Sigma_{uu})}{T} \right\}.
\]
Applying Chebyshev’s inequality, we obtain:
\[
\mathbb{P} \left\{ \| \Delta_{direct} \|_{F} \geq \epsilon \right\} \leq \mathbb{P} \left\{ \left\| \frac{1}{N} VU^T \right\|_{max} \geq \sigma_{\min} (\Sigma_{uu}) \right\} \leq \frac{\epsilon^2}{\sigma_{\min} (\Sigma_{uu})^2}.
\]
We note that for all $i \in \{1, \ldots, T\}$, $\mathbb{P} \{ \| \Delta_{direct} \|_{F} \geq \epsilon \} \leq \frac{\epsilon^2}{\sigma_{\min} (\Sigma_{uu})^2}$.

Fig. 1. The figure shows the dependence of the suboptimality gap $\text{Gap}(\hat{u})$ defined in (9) on the dataset size $N$ for direct and indirect data-driven predictive control, for three different values $L = 2, 3, 4$ (dimension of the identified system). The underlying system matrices $A, B, C$ in (1) were randomly generated (entries are i.i.d. normal random samples) with $n = 3$ and $p = m = 1$ (Single Input Single Output case). The task horizon length was chosen to be $T = 5$, cost matrices $Q = I$, $R = I$ and $y_{\text{ref}} = 1$. Inputs in the control experiments are i.i.d. samples of the standard normal distribution and the process noise are i.i.d. samples of the normal distribution with zero mean and covariance $\Omega_w = 0.75I$. The plots show the (empirical) mean suboptimality gap from 50 trials (sets of control experiments for every $N$), along with the corresponding 95% confidence bands. With the direct approach as the reference, we observe that the convergence rate of the indirect approach deteriorates as $L$ increases, seen as an increase in the empirical mean suboptimality gap and a widening of the 95% confidence band, suggesting that the indirect approach is prone to overfitting the noise in the dataset at higher values of $L$.

Fig. 2. The figure shows the dependence of the suboptimality gap $\text{Gap}(\hat{u})$ defined in (9) on the dataset size $N$ for direct and indirect data-driven predictive control, for three different values $T = 4, 5, 6$ of the control horizon length. The experimental setup is otherwise as outlined in Figure 1 with $L = 2$ for the indirect approach. We observe that the sample complexity of the direct approach increases with the length $T$ of the control horizon, while the indirect approach remains unaffected by the control horizon length.

Fig. 3. The figure shows the dependence of the suboptimality gap $\text{Gap}(\hat{u})$ defined in (9) on the dataset size $N$ for direct and indirect data-driven predictive control, for three different values of the signal-to-noise ratio (control input to process noise) in the control experiments to generate the dataset. The experimental setup is otherwise as outlined in Figure 1 with length $T = 5$ of the control horizon and $L = 2$ for the indirect approach. We observe that the asymptotic suboptimality gap of the indirect approach increases as the signal-to-noise ratio decreases, and subsequently the direct approach outperforms the indirect approach at lower values of $N$, the dataset size.