Extended BPH Renormalization of Cutoff Scalar Field Theories

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Abstract

We show that general cutoff scalar field theories in four dimensions are perturbatively renormalizable through the use of diagrammatic techniques and an adapted BPH renormalization method. Weinberg’s convergence theorem is used to show that operators in the Lagrangian with dimension greater than four, which are divided by powers of the cutoff, produce perturbatively only local divergences in the two-, three-, and four-point correlation functions. We also show that the renormalized Green’s functions are the same as in ordinary $\Phi^4$ theory up to corrections suppressed by inverse powers of the cutoff. These conclusions are consistent with those of existing proofs based on the renormalization group.

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I. Introduction

The BPH (Bogoliubov-Parasiuk-Hepp) renormalization procedure\(^{(1)}\) is the oldest method by which quantum field theories are renormalized. Although this technique is well known, it lacks a clear connection with the general understanding given by the renormalization group, which in more recent times has given a more physical understanding to renormalization. Our purpose in this paper is to give a natural extension of the BPH approach in order to show that general cutoff field theories may be understood also by this more classic method.

Classic proofs of renormalization employing the BPH method are based on the use of a recursive algorithm to consistently remove all the divergences in the calculation of Greens functions\(^{(2)(3)}\). Weinberg’s convergence theorem is the most important ingredient\(^{(7)}\). It guarantees that at each order in perturbation theory the new divergences are local in the external momenta, and therefore may be removed perturbatively from all Greens functions by redefining the couplings of (a finite number of) local operators in the Lagrangian. The types of theories amenable to this treatment are only those containing interactions with couplings of non-negative dimension.

The main focus here is to deal with more general interactions, those operators which have dimension greater than four in the Lagrangian and which have couplings with inverse powers of the cutoff \(\Lambda\) (e.g. \(\frac{1}{\Lambda^2} \Phi^6\), \(\frac{1}{\Lambda^4} \Phi^5 \partial^2 \Phi\)). These operators, together with the inverse cutoff factors, present problems to the classic proofs of renormalization, as Weinberg’s convergence theorem may not be used directly to show that the divergences are local. This will be explained further below. The technique offered in this paper to circumvent this problem depends on rearranging the Feynman integrals so that divergences of graphs containing these higher dimensional vertices are shown to be local (hence primitive).

Consider the example of a scalar Lagrangian containing the operator \(\frac{1}{\Lambda^2} \Phi^6\), with \(\Lambda\) a momentum cutoff. At tree level such an interaction obviously vanishes as \(\Lambda\) becomes large. Although a \(\Phi^6\) term increases the degree of divergences in Feynman diagrams, powers of the cutoff coming from the above vertex suppress the final divergence in the loop integrations. Naively, such an interaction is power counting renormalizable, although one needs to be more precise in justifying this. The fault with naive power counting arguments is that the “mixing” of the cutoff between the increased number of divergences (a result from insertions of six-point vertices) with the vertex suppressions of \(\frac{1}{\Lambda^2}\) makes the use of Weinberg’s theorem invalid.

Power counting may alternatively be justified in the framework of the Wilson type renormalization group (RG)\(^{(4)}\). Indeed, these higher dimensional operators in cutoff Lagrangians have been understood for some time in terms of Wilson type RG flows. The crux of this method is that the scaling procedure
naturally distinguishes between operators of two types, relevant (also marginal) and irrelevant; and the RG flow equations dictate that the only effect of the irrelevant operators is to modify the physics with order \( \frac{1}{\Lambda^2} \) corrections. Elegant proofs of renormalization have been put forth using only the RG equations, with little mention of Feynman diagrams and without using Weinberg’s theorem at all.\(^{(5)}\) The general “effective” theory can then be shown to be perturbatively renormalizable, essentially by dimensional analysis. Contrasted with RG, the recursive BPH method is graphical in nature and uses combinatorical arguments to handle the potentially divergent Feynman diagrams.

The fact that the higher dimensional operators do give local divergences is not as straightforward (as it is in RG) in terms of diagrammatic perturbation theory, and will be elucidated in what follows. In this work we use the more traditional techniques of BPH and Weinberg’s convergence theorem to show that general cutoff scalar field theories are perturbatively renormalizable.

II. Renormalization Method

**Standard Theories**

The perturbative renormalization course must accomplish two tasks. First, the Green’s functions must be made well-defined, and second the procedure must be outlined order by order as to carry out the removal of divergences. This problem, at least for standard power counting renormalizable theories like \( \Phi^4 \) theory in four dimensions, was solved many years ago. The solution amounts to removing all divergences associated with any set of internal loop momenta. The Bogoliubov recursive formula is the mathematical statement:

\[
\bar{R}(G) = U(G) + \sum_{\{\gamma_1, \ldots, \gamma_n\} \in P(G)} U\left(\frac{G}{\gamma_1, \ldots, \gamma_n}\right) \prod_{i=1}^{n} \left(-T_{\gamma_i} \bar{R}(\gamma_i)\right), \quad \gamma_i \cap \gamma_j = \emptyset \quad (1)
\]

The set \( \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) ranges over all partitions of 1PIR non-overlapping subgraphs of \( G \). \( T_{\gamma} \) is a subtraction operator - Taylor expansion of degree \( \delta(\gamma) \) in \( \gamma \)'s external momenta. \( U(\gamma) \) is defined to be the value of the Feynman diagram, and \( \bar{R}(\gamma) \) is the value of \( U(\gamma) \) with all of its subdivergences removed (in a recursive manner). We follow the notation of standard texts.\(^{(2)}\)\(^{(3)}\)

In short, the expression (1) tells us to perform appropriate Taylor expansions in the external momenta of all the subgraphs. Weinberg’s theorem guarantees that the graph’s divergent behavior is local and contained in the finite order Taylor expansion. The coefficients of the expansion order by order in perturbation theory then are absorbed into the definition of the coupling constants and correspond to the removal of all the possible local divergences in the graphs that contribute to a Green’s function.
General Theories

More general theories contain higher dimensional interactions; we take the scalar Lagrangian (in Euclidean space) to be

\[ L = \frac{1}{2} \partial \Phi \partial \Phi + \frac{1}{2} m^2 \Phi^2 - \frac{\lambda}{4!} \Phi^4 + \sum_i \frac{\lambda_i}{\Lambda_i^{2n-4}} O_i^{(2n)} + L_{ct}, \]  

(2)

where \( O_i^{(2n)} \) are all operators \( i \) with canonical dimension \( 2n \) described below. For simplicity we only consider Lagrangians even under \( \Phi \rightarrow -\Phi \). This is a matter of convenience, as there are fewer renormalization parameters (by odd symmetry no 1- or 3-point counterterms), but our method is easily extended. The vertices \((\partial^{2n})(\Phi^{2m})\) with \( n + m > 2 \) span all of the possible higher dimensional Poincare-invariant operators in the Lagrangian.

The four dimensional theory is regulated by using a momentum cutoff, and powers of this cutoff \( \Lambda \) are used to give the proper dimensions for the higher dimensional terms in the theory. The couplings for the higher dimensional operators may also contain explicit “soft dimensional” logarithms of the cutoff \((\text{e.g., } \lambda_1 \log (\frac{\Lambda^m}{\Lambda^2} \Phi^6))\). The following discussion is most transparent if we explicitly exhibit the counterterm Lagrangian:

\[ L_{ct} = \frac{1}{2} (Z_\Phi - 1) \partial \Phi \partial \Phi + \frac{1}{2} (Z_m - 1) m^2 \Phi^2 - (Z_\lambda - 1) \frac{\lambda}{4!} \Phi^4, \]  

(3)

where \( Z_\Phi, Z_m, \) and \( Z_\lambda \) are the counterterms adjusted perturbatively to renormalize the 2- and 4-point functions.

The topology of a graph \( G \) containing \( v \) vertices of the type \( \Phi^4 \), \( v_{2m}^{(2n)} \) vertices of the type \((\partial^{2n})(\Phi^{2m})\) with \( n + m > 2 \) (counted with derivatives and fields in any order), \( I \) internal lines, and \( E \) external lines gives the topological relations:

\[ 4v + \sum_{m+n>2} (2m) v_{2m}^{(2n)} = 2I + E, \]  

(4)

\[ L = I - v + 1 - \sum_{m+n>2} v_{2m}^{(2n)} \quad L \equiv \text{number of loops} \]  

(5)

We have separated the marginal operator \( \Phi^4 \) from the irrelevant ones by counting \( v \) and \( v_{2m}^{(2n)} \) separately. The graph \( G \) then has a superficial degree of divergence, \( \delta(G) \).
\[
\delta(G) = 4 - E + \sum_{G's \text{ vertices}} (2m + 2n - 4)\psi^{(2n)}_{2m} \tag{6}
\]

The same graph, though, comes with suppression factors of \(1/\Lambda\) raised to some power. These factors enter through the dimensions of the higher terms in the cutoff Lagrangian (e.g., \(1/\Lambda^{2m+2n} (\partial^{2n})^2 \Phi^{2m}, 2m+2n > 4\)). When the inverse powers of the cutoff are considered within the power counting we see that the superficial degree of divergence is lowered by an amount \(\sum (2m + 2n - 4)\psi^{(2n)}_{2m}\). Effectively the power counting for the graph \(G\) is modified to \(\delta'(G) = 4 - E\), although \(\delta\) is the measure of the actual divergences from the loop integrations. We see, at least naively, that these theories are power counting renormalizable, although we must first prove that the divergences are local. Considering the use of Weinberg’s theorem, we must carefully distinguish between \(\delta\) and \(\delta'\). This is essentially the crux of this paper.

**Weinberg’s Theorem**

Weinberg’s convergence theorem establishes that if a graph \(G\) has \(\delta(G) < 0\), and all of its subgraphs \(\Gamma\) have \(\delta(\Gamma) < 0\), then \(G\) is a well-behaved finite expression. For primitively divergent graphs \(G\), one has \(\delta(G) \geq 0\), and all of its subgraphs \(\Gamma\) have \(\delta(\Gamma) < 0\). Indirectly we ascertain that the divergence in \(G\) is a polynomial of degree \(\delta(G)\) in its external momenta.

Simply replacing \(\delta\) with \(\delta'\) in the use of Weinberg’s theorem is not allowed since the divergence from loop integrations is really measured by \(\delta\). Take for example a graph \(G\) with \(\delta'(G) \geq 0\) and all of whose subgraphs \(\Gamma\) satisfy \(\delta'(\Gamma) < 0\), but with \(\delta(\Gamma) \geq 0\) for some \(\Gamma\). The usual argument of differentiating and integrating the graph with respect to its external momenta in order to show that the net divergence is local does not apply. Even though all subgraphs have \(\delta' < 0\), some of them may still possess loop divergences \((\delta \geq 0)\). Thus after differentiating \(G\) with respect to its external momenta a number \(\delta'(G) + 1\) times we are not guaranteed a finite integral. Then by integrating the differentiated graph with respect to the external momenta we may not be sure that the divergence is polynomial.

In other words, \(\delta\), rather than \(\delta'\), power counting is pertinent to the convergence theorem. When all the vertices of a graph correspond to relevant/marginal operators (\(\Phi^4\) vertices only in the case of a \(Z_2\)-symmetric 4-dimensional scalar theory), the superficial degrees of divergence \(\delta\) and \(\delta'\) are equal. In this case the naive power counting is justified by the use of the convergence theorem since the cutoff enters calculations only through divergent integrals. We only face a problem using Weinberg’s theorem with graphs containing the higher dimensional vertices, when \(\delta(G)\) and \(\delta'(G)\) are not equal.
In order to clarify the formalities of the proof, given in section III, a summary of the method is presented below. In renormalizing the theory using the BPH method (as opposed to scaling arguments), the inverse cutoff factors from the vertices are distinguished from the loop structure of a Feynman diagram, as:

\[ U(G) = \left( \frac{1}{\Lambda^{\delta - \delta'}} \right) \cdot \int (\prod_{j=1}^{L} d^{d}l_j) I(l_1, \ldots, l_L; p_1, \ldots, p_E) \]

\[ \delta(G) - \delta'(G) = \sum_{\text{G's vertices}} (2m + 2n + 4) v_{2m}^{(2n)} \]  

(7)

We then re-organize the loop structure and its \( \delta' \) subtractions into many integrals – each one of which has at worst a primitive divergence in view of Weinberg's theorem. By keeping careful record of the cutoff factors the divergences are shown to be local using \( \delta' \). Thus the process involves rearranging Feynman integrals, the use of Weinberg’s theorem to show that the divergences are local, and then counting powers of the cutoff. This procedure ultimately shows that using Weinberg’s theorem initially with \( \delta' \) counting is justified.

We start by noting that in order to renormalize a divergent diagram we have to perform all necessary subtractions to the original graph \( G \) to render its sub-integrations finite. This is done with \( \delta \) power counting and includes all the counterterms pertaining to the higher dimensional vertices. Weinberg’s theorem tells us that the net divergence in the loop integrations of the graph \( G \) plus its lower order counterterms is a polynomial in \( G \)'s external momenta, and that in general a truly primitive graph with a superficial degree of divergence \( \delta \) will then require counterterms from all operators of dimension \( \delta \) or less.

The greater than four dimensional counterterms are by dimensional analysis, however, divided by the appropriate power of \( \Lambda \) (i.e. \( \frac{f(\ln \Lambda/m)}{\Lambda^{2m+2n-4}} \) for a \( \lambda_i(\partial^{2n})(\Phi^{2m}) \) vertex) - they are also suppressed. These counterterms are added to the bare Lagrangian (or likewise perturbatively to the graphs), but then subtracted back out. The subtracted terms generate sets of new graphs with fewer loops and simpler topological structure, which may be analyzed through the same procedure. As we will see, only the 2- and 4-point counterterms need to be explicitly added perturbatively to the Greens functions.

A very simple example is illustrated in figure 1. Figure 1(a) shows a divergent one loop contribution to the 6-point function which, for the sake of argument, could be embedded into some larger graph. By adding and subtracting the divergent piece, illustrated graphically in figure 1(b), we re-organize the original diagram into a finite integral plus a tree-level interaction. The tree-level interaction (suppressed polynomial
subtraction) was not only added to the original graph, but added and subtracted. Weinberg’s theorem forces the expression in figure 1(a) to split into two pieces:

\[ \frac{1}{\Lambda^2} \left[ \delta(G) = 0 \text{ dimensional (finite) function} \right] + \frac{1}{\Lambda^2} \left[ \text{polynomial divergence of degree } \delta(G) = 0 \right] \]

(8)

Here the cutoff factor from the $\frac{1}{\Lambda^2} \Phi^6$ vertex has been separated from the Feynman integrals.

We continue this procedure until the general graph $G$ and its lower order counterterms according to $\delta' = 4 - E$ power counting are transformed into sets of graphs $G_j$ plus counterterms according to $\delta$ power counting. Each set now has all subdivergences in loop integrations removed; the net divergent behavior of $G$ is local in view of the convergence theorem. In the notation used in this paper, we have that $\bar{R}'(G) = \sum \bar{R}(G_j)$; each $\bar{R}(G_j)$ contains no divergent sub-integrations and has only a net primitive divergence, so $\bar{R}'(G)$ (the graph with all of its lower order counterterms according to $\delta'$ counting) also has at worst a local divergence.

This inductive method relies upon adding and subtracting many counterterms to the original graph. Each of which vanish at least as fast as $1/\Lambda^2$ as $\Lambda \to \infty$. This procedure re-organizes the Feynman integrals so that the divergent structure of the perturbative Greens functions may be dissected through using Weinberg’s theorem. In this manner the usual renormalization tools may be used to justify $\delta' = 4 - E$ power counting. By keeping track of the cutoff factors, the $O(\frac{1}{\Lambda^2})$ bounds on the renormalized Green’s functions are also found. (By further bookkeeping on the number of loops in the perturbative expansion, logarithmic corrections to the bounds may in principle be found. We shall not do so here.)

The conclusions of this analysis are summarized below:

1. A general diagram $G$ which has all of its subdivergences removed according to $\delta'$ power counting has at worst a divergence which is a polynomial in the external momenta of degree $\delta'(G)$. This is analogous to Weinberg’s theorem but with the modified power counting.

2. A renormalized graph containing at least one of the higher-dimensional vertices is at most proportional to $E^2/\Lambda^2$ (times logs) and vanishes as $\Lambda \to \infty$. This is with the physical scale $E$ ($p_i^2 < \Lambda^2$) of the Green’s functions fixed below $\Lambda$.

III. Proof of Renormalization

The proof will be broken up into three steps and will follow the BPH method in organizing the renormal-
ization. In section (a) we deal with the simplest graphical structures - namely disjoint renormalization parts which do not contain subdivergences. We re-organize the subtractions as described previously and show that only the two- and four-point Greens functions need renormalization. This part most clearly demonstrates the adding and subtracting procedure, allowing us to effectively use Weinberg’s theorem but with δ’ power counting. In section (b) the subtractions are organized that render graphs finite, but only for those which do not contain nested subdivergences (more explicitly, only graphs whose one particle irreducible, 1PIR, subgraphs do not themselves contain further subdivergent integrations). The point here is to build the recursion formula and to illustrate once again how the adding and subtracting process is used to show that the divergences are local. In section (c) we deal with general Feynman diagrams. In this section the $R'$ operation is utilized on the subgraphs to successively remove the subdivergences according to δ'. The rearrangement is used to show how $R'(G)$ of a general graph G breaks into a specific form, a sum of primitively divergent integrals, $\sum \bar{R}(G_i)$, thus justifying that the only counterterms necessary are those by δ’ counting. In this manner, the Bogoliubov recursion formula will be derived but with δ’ subtractions.

**Disjoint Subgraphs**

To this extent, consider first the subgraph $\Gamma$ below in figure 2(a). $\Gamma$ is made up of two disjoint parts $\gamma_1$ and $\gamma_2$ such that $\Gamma = \gamma_1 \cup \gamma_2$ and $\gamma_1 \cap \gamma_2 = \emptyset$. Both $\gamma_1$ and $\gamma_2$ themselves have no subdivergences, hence they have at worst primitive divergences of degrees $\delta(\gamma_1)$ and $\delta(\gamma_2)$. Following Dyson’s prescription the divergence from the loop integrations in $\Gamma$ is removed by replacing:

$$U(\Gamma) \to (1 - T_{\gamma_1})(1 - T_{\gamma_2})U(\Gamma) \quad (9)$$

The operation $T_\gamma$ is an order $\delta(\gamma_1)$ Taylor expansion in the external momenta of $\gamma_1$, and corresponds to including the appropriate counterterms in the bare Lagrangian.

More general subgraphs $\Gamma$ have many disjoint components, and may be broken into several disconnected 1PIR components $\gamma_i$ so that $\Gamma = \cup_{i=1}^n \gamma_i$, $\gamma_i \cap \gamma_j = \emptyset$. Assume for now that all of the $\gamma_i$ have no subdivergences (i.e., $U(\gamma_i) = \bar{R}(\gamma_i)$), so that they all have at worst primitive divergences. All divergences coming from the loop integrations are eliminated by replacing (as in figure 2b):

$$U(\Gamma) \to \prod_{i=1}^n (1 - T_{\gamma_i})U(\Gamma) \equiv R(\Gamma) \quad (10)$$
When the subgraph contains higher dimensional vertices, this expression is excessively oversubtracted. $T_\gamma$ is a Taylor expansion of degree $\delta(\gamma)$ not $\delta'(\gamma)$, with some $\delta(\gamma_i) > \delta'(\gamma_i)$. This means that all counterterms, including the ones which would renormalize the irrelevant operators, have been added to make all sub-integrations finite. This is clearly unnecessary by dimensional analysis, in view of the overall factor $1/\Lambda^{2m+2n-4}$ in front of these higher dimensional vertices $(\partial^{2n})\Phi^{2m}$ ($m + n > 2$). Some divergences are “eaten up” by the vertex suppressions. To re-iterate this point, the most general counterterm to these higher dimensional operators by dimensional analysis has the form $f(\ln \Lambda^m)/\Lambda^{2m+2n-4}$, and vanishes as $\Lambda \to \infty$.

The only subtractions we want to make are according to $\delta'$ power counting. With this in mind, the necessary change in the above prescription is then to use $\delta'$:

$$U(\Gamma) \to \prod_{i=1}^n (1 - T'_{\gamma_i}) U(\Gamma) \equiv R'(\Gamma)$$

This primed expression means that in subtracting the subgraph $\Gamma$ only the bare four-point and two-point parameters need adjustment (since $\delta' = 4 - E$), or likewise only the counterterms to $Z_m$, $Z_\lambda$, and $Z_\Phi$ need to be adjusted. Now we show that this expression truly yields a finite result.

Following the discussion in the introduction let’s add and subtract the unnecessary renormalizations and rewrite $R'(\Gamma)$ in (11) as:

$$R'(\Gamma) = \prod_{i=1}^n (1 - T'_{\gamma_i}) U(\Gamma) = \prod_{i=1}^n \left\{ 1 - T_{\gamma_i} + T_{\gamma_i} - T'_{\gamma_i} \right\} U(\Gamma)$$

$$= \sum_{\text{partitions } \gamma_j \in A} \prod_{\gamma_i \in A} (1 - T_{\gamma_i}) \prod_{\beta_k \in B} (T_{\beta_k} - T'_{\beta_k}) U(\Gamma)$$

The sum extends over the ways in which the IPIR parts of $\Gamma$ may be partitioned into two sets $A = \cup \gamma_j$ and $B = \cup \beta_k$ such that $\Gamma = A \cup B$. More explicitly (12) may be expanded,

$$R'(\Gamma) = \prod_{i=1}^n (1 - T_{\gamma_i}) U(\Gamma) + \sum_{j=1}^n \left\{ \prod_{i \neq j} (1 - T_{\gamma_i}) \right\} \left\{ (T_{\gamma_j} - T'_{\gamma_j}) U(\Gamma) \right\} + \ldots$$

The $(1 - T_\gamma)$ subtraction eliminates the loop divergence coming from the integration in $U(\gamma)$, and individually the operation $(T_\beta - T'_{\beta}) U(\Gamma)$ in (13) replaces the 1PIR renormalization part $\beta$ in $\Gamma$ with a polynomial in its external momenta. The $(T_\beta - T'_{\beta})$ operation when acting on the primitively divergent 1PIR graph $U(\beta)$ in fact always results in a polynomial with terms divided by powers of at least $\Lambda^2$. 

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\textbf{T - T’ Operation}

The fact that $T - T'$ is proportional to at least $1/\Lambda^2$ is found by carefully separating the inverse cutoff factors that arise from the vertices in $\beta$ from its loop momentum structure. The vertices in $\beta$ contribute the cutoff factor:

$$f_\beta \left( \frac{\ln \Lambda}{m} \right) \frac{1}{\Lambda^{\delta(\beta) - \delta'(\beta)}}$$

(14)

$$\delta(\beta) - \delta'(\beta) = \sum_{\beta \text{'s vertices}} (2m - 2n - 4)e_{2m}^{(2n)}$$

to the value of the renormalization part $\beta$ (where $f_\beta$ is a product of possible logarithmic factors in the tree-level vertices), and the operation $(T_\beta - T'_\beta)U(\Gamma)$ on the primitively divergent loop integral above is a polynomial of the form:

$$g_1 \left( \frac{\ln \Lambda}{m} \right)p^{\delta(\beta)} + g_2 \left( \frac{\ln \Lambda}{m} \right)\Lambda^2 p^{\delta(\beta) - 2} + \ldots + g_k \left( \frac{\ln \Lambda}{m} \right)\Lambda^{\delta(\beta) - \delta'(\beta) - 2}p^{\delta'(\beta) + 2}$$

(15)

Recall that $T$ and $T'$ are Taylor expansions of order $\delta$ and $\delta'$ respectively, and the coefficients $g_k$ are logarithmic factors of the cutoff which arise from the divergence of the 1PIR renormalization part $\beta$.

The value of $(T_\beta - T'_\beta)U(\Gamma)$ then has the form (combining the vertex factors and the above polynomial subtractions),

$$f_\beta \left( \frac{\ln \Lambda}{m} \right) \left( g_1 \left( \frac{\ln \Lambda}{m} \right)p^{\delta(\beta)} + g_2 \left( \frac{\ln \Lambda}{m} \right)\Lambda^2 p^{\delta(\beta) - 2} + \ldots + g_k \left( \frac{\ln \Lambda}{m} \right)\Lambda^{\delta(\beta) - \delta'(\beta) - 2}p^{\delta'(\beta) + 2} \right) U(\Gamma)$$

(16)

$$\equiv \sum_{j=1} U'_j(\beta)U(\Gamma)$$

(17)

The form in (17) is written in a manner to show how the $(T_\beta - T'_\beta)$ operation effectively reduces the renormalization part $\beta$ into several tree level interactions, $\sum U'_j(\beta)$, given by the polynomial above in (16). The sum of terms in (16) (the polynomial times the vertex factors) contain only terms monomial in $\beta$’s external momenta divided by powers of the cutoff, which is similar to replacing the subgraph $\beta$ under this operation with a sum of higher dimensional suppressed vertices. In general we have products of the $T - T'$ operations acting on a set $\{\beta_k\}$, as in (12); then each of the disjoint $\beta_k$ is replaced with several tree-level vertices - all of them divided by at least $\Lambda^2$. Lastly it is important to note that $T - T'$ acting on \textit{any} primitively divergent set of graphs $\tilde{R}(G)$ is a polynomial whose terms are divided by at least $\Lambda^2$.  

10
The effect of changing $(1 - T'_{\beta}) \to (1 - T_{\beta}) + (T_{\beta} - T'_{\beta})$ allows the divergent behavior to be understood in view of Weinberg’s theorem. Take one of the terms in the expansion given in (12), where we denote one of the partitions of $\Gamma$ into two sets $\Gamma_1$ and $\Gamma_2$ by the index $p$:

$$R'(\Gamma_p) = \prod_j (1 - T_{\gamma_j}) \left\{ \prod_k (T_{\beta_k} - T'_{\beta_k}) U(\Gamma) \right\}$$

(18)

where,

$$\Gamma_p = \{\gamma_j\} \cup \{\beta_k\} = \Gamma_1 \cup \Gamma_2$$

Since all of the elements $\gamma_j$ and $\beta_k$ are disjoint from one another, all the subtractions commute and we are left with two functions multiplying each other. We write (18) as the product below in order to illustrate this point:

$$R'(\Gamma_p) = \left( \prod_j (1 - T_{\gamma_j}) U(\Gamma_1) \right) \cdot U'(\Gamma_2)$$

(19)

where,

$$U'(\Gamma_2) = \prod_k (T_{\beta_k} - T'_{\beta_k}) U(\Gamma_2)$$

$$= \prod_k \left( \sum_{i_k = 1}^\infty U'_{i_k}(\beta_k) \right)$$

(20)

The original set of disjoint renormalization parts of $\Gamma$ has been partitioned into two new sets $\Gamma_1 = \{\gamma_k\}$ and $\Gamma_2 = \{\beta_j\}$. All of the elements $\beta$ in $\{\beta_k\}$ have been replaced with tree level interactions, $\sum_{i_k} U'_{i_k}(\beta_k)$, in view of the previous discussion, and the remaining parts $\Gamma_1 = \{\gamma_k\}$ have their divergences in the loop integrations completely removed according to Dyson’s prescription (i.e., subtractions according to $\delta$ power counting).

Note that $U'(\Gamma_2)$, written out in (20), is a sum of many terms found by factoring the product of the polynomials:

$$U'(\Gamma_2) = \prod_k \left( \sum_{i_k = 1}^\infty U'_{i_k}(\beta_k) \right) = \sum_{\text{all sets } \{i_1, i_2, \ldots\}} \prod_k U'_{i_k}(\beta_k)$$

(21)

This means that out of every polynomial $(T_{\beta_k} - T'_{\beta_k}) U(\beta_k)$ for all $\beta_k \in \Gamma_2$ we take one term $i_k$, then we sum over all the distinct ways of doing this.
The expression (18) is thus a finite function \( (\delta(\Gamma) - \delta'(\Gamma) = 0) \) or a finite function divided by powers of the cutoff since for each \( \Gamma_p \):

\[
R'(\Gamma_p) = U'(\Gamma_2) \cdot \frac{f_{\Gamma_1}}{\Lambda^{\delta(\Gamma_1)-\delta'(\Gamma_1)}} \cdot [\delta(\Gamma_1) \text{ dimensional finite function}]
\]

(22)

The factors \( \Lambda^{\delta(\Gamma_1)-\delta'(\Gamma_1)} \) and \( f_{\Gamma_1}(\ln \frac{\Lambda}{m}) \) come from the vertices (since the couplings may have explicit logarithms) in \( \Gamma_1 \) and even further suppress the polynomial terms in \( U'(\Gamma_2) \). The only case in which the expansion in (12) does not lead to terms (18) divided by at least \( \Lambda^2 \) is when there are no high-dimensional vertices in \( \Gamma \), \( \delta(\Gamma) = \delta'(\Gamma) \). This is when the usual BPH renormalization procedure is regained.

Consider the subgraph in Figure 3(a). The subtraction according to \( \delta' \) in this example splits into two pieces (figure 3b):

\[
(1 - T'_{\gamma_1})(1 - T'_{\gamma_2})U(\Gamma) = \frac{1}{\Lambda^2} \left( \delta(\Gamma) = 0 \text{ dim. finite function} \right)
+ \left( \frac{\ln(\frac{\Lambda}{m})}{\Lambda^2} \right) \left( \delta(\gamma_1) = 0 \text{ dim. finite function} \right)
\]

(23)

In the figure, the dashed and solid boxes represent the \( T - T' \) and \( T \) operations on the two renormalization parts \( \gamma_1 \) and \( \gamma_2 \).

Summing over all the ways in which the original \( \Gamma = \{\gamma_j\} \) may be split into two sets \( \Gamma_1 = \{\gamma_k\} \) and \( \Gamma_2 = \{\beta_j\} \) generates the terms in (12), each one of which has the form above in (22). If \( \Gamma \) contains a high-dimensional vertex, than each element in the expansion is divided by at least \( \Lambda^2 \).

To summarize so far, we see that the prescription according to \( \delta' \) counting (a modified Dyson’s prescription),

\[
U(\Gamma) \rightarrow \prod_{i=1}^{n} (1 - T'_{\gamma_i})U(\Gamma)
\]

leads to finite expressions (as \( \Lambda \rightarrow \infty \)). In the case that the subgraph \( \Gamma \) contains one of the higher dimensional vertices these results are divided by powers of the cutoff. Note also that if a subgraph \( \gamma \) satisfies \( \delta(\gamma) = \delta'(\gamma) \), which means no irrelevant operators in \( \gamma \), then no substitutions take place \( (T_\gamma - T'_{\gamma})U(\Gamma) = 0 \). We regain the conventional BPH prescription using \( \delta \) power counting on graphs containing no higher dimensional vertices.
Now we consider Feynman diagrams which contain many subgraphs but no nested subdivergences. According to $\delta'$ power counting the overall divergence from a particular subgraph $\Gamma$ in $G$ is removed by subtracting it from the graph as follows:

$$\prod_{\gamma \in \Gamma} (1 - T'_{\gamma}) U(G) = U(G) + \sum_{\beta \in P(\Gamma)} U\left( \frac{G}{\{\beta_1, \beta_2, \ldots, \beta_n\}} \right) \prod_k (-T'_{\beta_k}) U(\beta_k)$$

(25)

where $\Gamma = \cup \gamma_i$ and $P(\Gamma)$ extend over the partitions of $\Gamma$ into sets $\{\beta_k\}$, not counting the null set. Next, the $\delta'$ divergences from all of $G$’s subgraphs are removed. The subtractions from all of $G$’s subgraphs are summed with the condition that we count the equivalent ones only once. This amounts to summing only over the partitions of $G$ (this set generates all the partitions of every subgraph only once):  

$$\bar{R}'(G) = U(G) + \sum_{\beta \in P(G)} U\left( \frac{G}{\{\beta_1, \beta_2, \ldots, \beta_n\}} \right) \prod_k (-T'_{\beta_k}) U(\beta_k)$$

(26)

For the moment assume that the 1PIR renormalization parts $\beta_k$ do not themselves contain loop subdivergences. Equation (26) is then the expression for the graph $G$ with all of its subdivergences removed according to $\delta'$ counting. The usual BPH arguments use Weinberg’s theorem to deduce that the overall divergence of (26) is a local polynomial. However, certain sub-integrals in the above may still be divergent – just divided by powers of the cutoff since we are counting with $\delta'$. Thus we are not yet in a position to say that $\bar{R}'(G)$ has a local divergence. By re-organizing the above form we intend to show that it truly does have a local divergence in view of the convergence theorem.

Split the subtraction in equation (26) and we have:

$$\bar{R}'(G) = U(G) + \sum_{\beta \in P(G)} U\left( \frac{G}{\{\beta_1, \beta_2, \ldots, \beta_n\}} \right) \prod_k (-T'_{\beta_k}) U(\beta_k)$$

$$= \sum_{\Gamma_j} U'(\Gamma_j) \left\{ U\left( \frac{G}{\Gamma_j} \right) + \sum_{\{\gamma_1, \ldots, \gamma_n\} \in P(\Gamma_j)} U\left( \frac{G}{\{\Gamma_j, \gamma_1, \ldots, \gamma_n\}} \right) \prod_{i=1}^{a} (-T'_{\gamma_i} U(\gamma_i)) \right\}$$

(27)

$$U'(\Gamma_j) \equiv \prod_{i=1}^{n} (T'_{\beta_i} - T'_{\beta_i} U(\beta_i), \quad \Gamma_j = \cup_{i=1}^{n} \beta_i$$

(The sum over $\Gamma_j$ is over all partitions of $G$ ($\Gamma_1 = \emptyset, \Gamma_2 = \{\gamma_1\}, \{\gamma_1, \gamma_2\}, \ldots, \Gamma_n = \{\gamma\}$) including G, and the next sum extends over all the partitions of $\frac{G}{\Gamma_j}$ not including $\frac{G}{\Gamma_j}$.)
The factorization expressed in (27) is rather direct and follows from the structure of G - that all of its 1PIR components \( \gamma_i \) in every subgraph do not themselves contain further subdivergences (according to \( \delta \)). \( U(\gamma_i) \) is at most primitively divergent for all \( \gamma_i \). In more general cases we need to first remove the subdivergences in \( \gamma_i \) recursively, namely with the \( \bar{R}'(\gamma_i) \) operation.

Equation (27) expresses the re-organization of a Feynman diagram G together with its \( \delta' = 4 - E \) counterterms into a set of new diagrams with subtractions given by \( \delta \) counting. The sum over the partitions \( P(\frac{G}{\gamma_j}) \) together with the subtraction \( T \) renders all subintegrations in \( \frac{G}{\gamma_j} \) completely finite. Individually each set denoted by \( \frac{G}{\gamma_j} \) contains only true primitive momentum divergences in view of Weinberg's theorem.

The grouping of the subtractions in equation (27) should be contrasted with that of (26). Previously the sub-integrations in G may be divergent, just divided by hard powers of the cutoff. But by adding and subtracting the extra terms we arrive at Feynman integrals which have completely convergent subgraphs, but also divided by powers of the cutoff. Weinberg's theorem may now be used to justify that the overall divergence in \( \bar{R}'(G) \) is local since every subgraph \( \gamma \) of \( U'(\Gamma_k)U(\frac{G}{\gamma_k}) \) together with its corresponding counterterms has effectively \( \delta(\gamma) < 0 \).

Take as an example a graph with three 1PIR parts \( (\gamma_1, \gamma_2, \gamma_3) \), where each may be at most primitively divergent. In one term of the above expansion the \( T - T' \) operates on the part \( \gamma_3 \); the \( U'(\{\gamma_3\}) \) is thus a polynomial in \( \gamma_3 \)'s external momenta as discussed in the first part of the paper. The graph \( U(\frac{G}{\gamma_3})U'(\{\gamma_3\}) \) contains three divergences, and they have all been subtracted out by the operation:

\[
\sum_{\Gamma \in P(\frac{G}{\gamma_3})} U(\frac{G}{\Gamma, \gamma_3}) \prod_{\beta \in \Gamma} (-T_\beta) U(\beta)
\]

\[
= (-T_{\gamma_1} - T_{\gamma_2} + T_{\gamma_1} T_{\gamma_2}) U'(\gamma_3) U(\{\gamma_1, \gamma_2\})
\]

By Weinberg's theorem the net divergence in the four diagrams, including \( U'(\gamma_3)U(\frac{G}{\gamma_3}) \), must be a polynomial in G's momenta. The finite part of the integral is divided by whatever factors of the cutoff are present, those in the piece \( U'(\gamma_3) \) and the vertices in \( U(\frac{G}{\gamma_3}) \), and is at least \( \Lambda^2 \) since \( T - T' \) produces suppressions proportional to \( 1/\Lambda^2 \).

In more detail let's inspect the re-organized expression in equation (27), which we write in a more revealing notation:

\[
\bar{R}'(G) = \sum_{G_k} U(G_k) + \sum_{\{\gamma_1, \ldots, \gamma_n\} \in P(G_k)} U(\frac{G}{\{\gamma_1, \ldots, \gamma_n\}}) \prod_{i=1}^n (-T_{\gamma_i} U(\gamma_i))
\]

(28)
where
\[
U(G_k) = \left( \prod_{i=1}^{a} (T_{\beta_i} - T'_{\beta_i}) \right) U\left( \frac{G}{\{\beta_1, \ldots, \beta_n\}} \right), \quad \text{and} \quad \{\beta_1, \ldots, \beta_n\} \in P(G)
\]

Each \(G_i\) above is a graph containing many 'new' vertex insertions and has the same topological structure as \(G\) except for being at least one loop less. The \(T - T'\) operation has replaced particular loops in the original graph \(G\) with polynomials in the loop’s external momenta. Each polynomial leads to a number of terms divided by at least \(\Lambda^2\), as described previously in section III(a). These new graphs should then further be denoted by all possible combinations of the new vertices (the terms in the suppressed polynomial) inserted from the \(T - T'\) acting on different renormalization parts. Considering Weinberg’s theorem, we may write (29) as:

\[
\bar{R}'(G) = \bar{R}(G) + \sum \bar{R}(G_i)
\]

a sum of primitively divergent graphs.

The convergence theorem tells us indirectly that each \(\bar{R}(G_i)\) has at worst a divergence which is polynomial. The simplest piece in (29) is found by taking \(\Gamma_k = \{\emptyset\}\):

\[
\bar{R}(G) = U(G) + \sum_{P(G_k)} U\left( \frac{G}{\{\gamma_1, \ldots, \gamma_n\}} \right) \prod_{i=1}^{n} (-T_{\gamma_i} U(\gamma_i))
\]

\[
= \frac{f(\ln \frac{\Lambda}{m})}{\Lambda^{\delta(G) - \delta'(G)}} \left( \text{divergent polynomial of degree } \delta(G) + \text{finite function of } \dim \delta(G) \right)
\]

The overall cutoff factor coming from the higher vertices in \(G\) has been extracted from the original structure of the graph. The form in the brackets is given by the locality of the divergence, since all necessary counterterms to render \(G\)’s subintegrations finite have been included. After subtracting out the local divergence in \(\bar{R}(G)\) (of overall degree \(\delta'\)), this entire quantity must be proportional to the hard vertex cutoff factors in \(G\), times the remaining terms in the polynomial and the finite function. This is at most \(1/\Lambda^2\) since \(\delta(G) - \delta'(G) \geq 2\) for a graph \(G\) containing at least one higher dimensional vertex. The function \(f(\ln \frac{\Lambda}{m})\) comes from the possible logarithmic factors in the vertices and gives only a small correction to the hard powers of the suppression.

The remaining terms in (29) are more complicated owing to the \(T - T'\) operation. For example, suppose \((T - T')\) acting on a subgraph \(\gamma\), \((T - T')U(\gamma)\), leads to three terms: \(\frac{m^4}{\Lambda^2} \ln \frac{\Lambda}{m}, \frac{m^2}{\Lambda^4} \ln \frac{\Lambda}{m}, \text{ and } \frac{m^4}{\Lambda^2} \ln \frac{\Lambda}{m}\). Each one may be thought of as a vertex in the diagram \(G_j\). The last has an internal momentum flowing
through and leads to a higher degree of divergence in the overall integration in $G_j$ than the first two terms. The second term however gives a result an order in $\Lambda^2$ higher than if we had inserted $\frac{m^4}{e^2} \ln \Lambda \Lambda$ in its place.

As a result, the structure of $\tilde{R}(G_j)$ will break up into primitive divergences of varying degrees. The matter is complicated when we consider that there are many of these polynomials in any reduced diagram, all from the many $T - T'$ operations acting on the disjoint renormalization parts (in the $U'(\Gamma_k)$ found in equation (29) for example). This is illustrated in figure 4 where the dashed boxes around the subgraphs represent the $T - T'$ operation. Notationally each $G_j$ and its counterterms actually represent many diagrams as mentioned before, labeled by choosing one term out of each polynomial from the $T - T'$ operations in (29) factors into a sum, as in the previous section of the paper (eqns. (20)-(21)). Define $U'_n(\beta_i)$ to be the $n^{th}$ term in the polynomial $(T\beta_i - T'\beta_i)U(\beta_i)$. Then,

$$\prod_{i=1}^a (T\beta_i - T'\beta_i)U(\beta_i) = \prod_{i=1}^n \left( \sum_{i=1}^{N_i} U'_n(\beta_i) \right)$$

Summing over the sets $\{n_1, \ldots, n_a\}$ takes us over all combinations of making a product out of ‘$a$’ terms, one term $n_i$ from each of the polynomials. The number $N$ ($N = n_1 n_2 \cdots n_a$) of distinct combinations is in general quite large, and depends on how many terms $n_i$ there are in each of the $(T\beta_i - T'\beta_i)$.

The major difference here from section III(a) is that the terms $\prod_{i=1}^a U'_n(\beta_i)$ are in fact embedded in the graph $G_k$ in (29), and hence contribute to the Feynman integrals. Recall that the form of $U'_n(\beta_i)$ depends on the renormalization part $\beta_i$, being one monomial out of the following,

$$\frac{f_{\beta_i}(ln\Lambda m)}{\Lambda^{\delta(\beta_i) - \delta'(\beta_i)} } \left( g_1 (ln\Lambda m) p^{\delta(\beta_i)} + g_2 (ln\Lambda m) \Lambda^2 p^{\delta(\beta_i) - 2} + \ldots + g_k (ln\Lambda m) \Lambda^{\delta(\beta_i) - \delta'(\beta_i) - 2} p^{\delta'(\beta_i) + 2} \right)$$

All the pieces in (33) are divided by at least $\Lambda^2$, but carry different dimensions of momentum, mass, and the cutoff. '$P$' refers generally to the momentum flowing through the vertex, either internal loop momentum or off an external leg. Each of the distinct combinations in (32) thus gives a different internal loop momentum structure and contributes differently to the overall divergence.

Explicitly, the general $\tilde{R}'(G_k)$ must break into $N = n_1 n_2 \cdots n_a$ terms as follows:
\[ R'(G_k) = \frac{h_1 (ln \frac{\Delta}{m})}{\Lambda \sum_{2m+2n-4}^{2m+2n} v_{2m}^{(2n)}} \left( \text{divergent polynomial of degree } \delta(G) + \text{ finite function of dim. } \delta(G) \right) \]
\[ + \frac{h_2 (ln \frac{\Delta}{m})}{\Lambda \sum_{2m+2n-4}^{2m+2n} v_{2m}^{(2n)}} \Lambda^2 \left( \text{divergent polynomial of degree } \delta(G) - 2 + \text{ finite function of dim. } \delta(G) - 2 \right) \]
\[ + \frac{h_3 (ln \frac{\Delta}{m})}{\Lambda \sum_{2m+2n-4}^{2m+2n} v_{2m}^{(2n)}} m^2 \left( \text{divergent polynomial of degree } \delta(G) - 2 + \text{ finite function of dim. } \delta(G) - 2 \right) \]
\[ + \ldots + \frac{h_n (ln \frac{\Delta}{m})}{\Lambda^{\delta(G) - \delta'(G)}} \prod_{i=1}^{a} \Lambda^{\delta(\beta_i) - \delta'(\beta_i) - 2} \left( \text{divergent polynomial + finite function} \right) \]

(34)

(The \( h_i \) represent logarithmic factors - those originally present in the vertices of \( G \) times those from the coefficients of the \( T - T' \) polynomial.)

Lastly, the cutoff factors in front of the expressions above are understood as follows. The number of suppression factors coming from all of the vertices in \( G \),

\[ \delta(G) - \delta'(G) = \sum_{\text{vertices in } G} (2m + 2n - 4) v_{2m}^{(2n)} \]

must be greater than or equal to the number coming from any portion of the graph \( G \),

\[ \sum_{i=1}^{a} \left( \delta(\beta_i) - \delta'(\beta_i) \right) = \sum_{\text{vertices in all } \beta_i} (2m + 2n - 4) v_{2m}^{(2n)} \]

(35)

So

\[ \delta(G) - \delta'(G) \geq \sum_{i=1}^{a} \left( \delta(\beta_i) - \delta'(\beta_i) \right) \]

and thus every term in \( \bar{R}(G_k) \) in (34) has at least the suppression \( 1/\Lambda^2 \) in front since \( a \geq 1 \) for graphs containing at least one higher dimensional vertex. All the finite functions in (34) are then divided by at least \( \Lambda^2 \).

Overall each \( \bar{R}(G_j) \) has a divergence that is local and of degree \( \delta'(G) = 4 - E \). It also has finite parts suppressed by factors of at least \( 1/\Lambda^2 \). The original expression for the renormalized graph, \( R'(G) \), is a sum of the terms above in (34). Hence \( \bar{R}'(G) \) also has an overall primitive divergence of degree \( \delta'(G) = 4 - E \) which is local. The remainder is divided by at least \( \Lambda^2 \):
\[ R'(G) = (1 - T'_G) \bar{R}'(G) \leq \frac{E^2}{\Lambda^2} \text{ (times lns)} \] (38)

In actual practice none of this is necessary for we just have to remove the divergences according to \( \delta' \) power counting. However, first they must be shown to be local in the external momenta - thus the reason for the above analysis.

(c) General Graphs

Lastly, we consider the most general Feynman diagrams - those that contain nested divergences. One proceeds recursively in subtracting out all subdivergences in a general graph by replacing the lower order subgraphs \( U(\gamma) \) with \( \bar{R}'(\gamma) \), their value with all of their own subdivergences (according to \( \delta' \)) removed.

Recall that the \( \bar{R}'(\gamma) \) above is written as a set of graphs \( \gamma_k \) (in equation (30) for example) with all of the necessary counterterms to render their sub-integrations finite. These graphs \( \gamma \) may be subgraphs of some larger graph \( G \), so to continue the outlined procedure we rewrite the net subtraction to remove the overall divergence in \( \bar{R}'(\gamma) \) as:

\[ (-T'_\gamma) \bar{R}'(\gamma) = \sum_k (-T_{\gamma_k} + T_{\gamma_k} - T'_{\gamma_k}) \bar{R}(\gamma_k) \] (39)

The sum over \( k \) symbolizes the different ways in which the original graph \( \gamma \) has been partitioned into new graphs, which depends on the various ways we collect terms from the polynomials \( T - T' \) acting on \( \gamma \)'s subgraphs. This can be done inductively for more complicated graphs. The explicit recipe for rewriting the subtraction in (39) is presented in Appendix A, which holds inductively in the recursive procedure to higher perturbative order. In section III(b) the rewriting (or adding and subtracting procedure) was demonstrated on the simpler graphs.

By replacing the lower order subgraphs with their renormalized counterparts we arrive at the recursion formula similar to the one (27):

\[ \bar{R}'(G) = U(G) + \sum_{\{\gamma_1, \ldots, \gamma_n\} \in P(G), \gamma_i \cap \gamma_j = \emptyset} U(\frac{G}{\gamma_1, \ldots, \gamma_n}) \prod_{i=1}^{n_{\gamma_i}} \left( \sum_{a_i} (-T_{\gamma_i,a_i} + T_{\gamma_i,a_i} - T'_{\gamma_i,a_i}) \bar{R}(\gamma_i,a_i) \right) \] (40)

Equation (40) is the original Bogoliubov recursion formula, but with \( \delta' \) subtractions.
Next, the recursion relation in equation (40) may be iterated in order to renormalize the graph and generate all sets of new graphs, as before. The iteration must be handled as previously – factoring the new graphs into sets which have no subdivergences, so that the convergence theorem may be applied. That this is possible is due to the fact that \( T - T' \) acting on any primitive divergence is a polynomial with terms divided by at least \( \Lambda^2 \). The example given in figure 5 illustrates this procedure for a more complicated graph, which is written out in detail in order to demonstrate how \( \delta' \) subtractions may be 'converted' into sets of graphs renormalized with subtractions according to \( \delta \). Note that in the example the five loop 'cage' subdiagram possesses only a local divergence, so we do not worry about subtractions on its own subgraphs.

In this example we explicitly see how the original graph \( G \) breaks into seven sets of integrals which may only possess primitive divergences.

The equation above in (40) may be expressed as:

\[
\tilde{R}'(G) = \sum_{G_k} \left( U(G_k) + \sum_{\{\gamma_1, \ldots, \gamma_n\} \in P(G_k), \gamma_i \cap \gamma_j = \emptyset} U(\frac{G_k}{\{\gamma_1, \ldots, \gamma_n\}}) \prod_{i=1}^{n} (-T_i \tilde{R}(\gamma_i)) \right)
\]

The graphs \( G_k \) are denoted by all topological ways in which the subgraphs in \( G \) are acted upon by the \( T - T' \) operation, and further by the various terms in each of their corresponding polynomial. The many \( T - T' \) that contribute to the definition of \( G_k \) may be disjoint or nested. Correspondingly, each \( G_k \) and its subtractions may be viewed as a graph similar to \( G \) but with certain renormalized subgraphs \( \tilde{R}(\Gamma) \) of \( G \) replaced with various (monomial) higher dimensional vertices.

Since each of the terms in the sum over \( G_k \) in (41) has no subdivergences, it must have at most a polynomial divergence in its loop integrations. Counting powers of the cutoff tells us that the net divergence in the graph \( G_k \) is measured by \( \delta' = 4 - E \), \( E \) being the number of external legs, with a part remaining that is divided by at least \( \Lambda^2 \). Explicitly \( \tilde{R}'(G) \) in (41) takes on the form:

\[
\tilde{R}'(G) = \frac{h_1(ln \frac{\Lambda}{\mu})}{\Lambda} \left( \text{divergent polynomial of degree } \delta(G) + \text{finite function of dim. } \delta(G) \right)
+ \frac{h_2(ln \frac{\Lambda}{\mu})}{\Lambda} \left( \text{divergent polynomial of degree } \delta(G) - 2 + \text{finite function of dim. } \delta(G) - 2 \right)
+ \ldots + \frac{h_n(ln \frac{\Lambda}{\mu})}{\Lambda^2} \left( \text{divergent polynomial of order } 4-E+2 + \text{finite function of dimension } (4-E+2) \right)
\]

The subtraction that removes the overall divergence in \( \tilde{R}'(G) \) according to \( \delta' \) leaves:

\[
R'(G) = (1 - T'_G) \tilde{R}'(G) \leq \frac{E^2}{\Lambda^2} \text{ (times lns)}
\]
Considering the form above in equation (41) or more explicitly in (42), we may add and subtract the polynomial pieces and write (43) also as:

$$\left(-T' G\right) \bar{R}'(G) = \sum_k \left(-T_{G_k} + T_{G_k} - T'_{G_k}\right) \bar{R}(G_k) \tag{44}$$

$\delta'$ naive power counting is justified inductively through the use of Weinberg's theorem. The form in (44) is exactly of the type we had previously in (39), and in section III(b) it was shown explicitly on the simpler cases. The fact that $T - T'$ acting on a primitive divergence is a suppressed polynomial is responsible for the inductive proof, since at each perturbative order only new primitive divergences are encountered. These divergences are either renormalized or 'shrunk' to a suppressed vertex.

We may simply reabsorb the divergences in the two- and four-point Greens functions order by order in perturbation theory. In actual practice only the iterative procedure using $\delta'$ counting is necessary. The outlined roundabout procedure of adding and subtracting polynomials to the original graph is there to establish the perturbative effect of the irrelevant operators - that they produce local divergences and contributions to the renormalized Greens functions suppressed by at least $\Lambda^2$.

IV. Conclusions

We have presented a method to understand the effects of irrelevant operators in perturbative calculations. By adding and subtracting terms to a graph plus its two- and four-point counterterms, thus rearranging the Feynman integrals, we have effectively extended the BPH method to the case of power counting with $\delta'$. The renormalization of the general scalar theory then follows by organizing the subtractions in the standard manner of BPH. The bounds on the graphs which contain the higher dimensional vertices follow by keeping careful record of the cutoff factors.

These operators $\frac{1}{\Lambda^{2n}} G^{(2n)}_i$ ($2n > 2$) produce only local divergences in the two and four-point Greens functions and corrections to a four dimensional field theory that are of order $E^2/\Lambda^2$. Our contribution here is by showing how this may be understood through a natural generalization of the BPH approach. Extensions to scalar theories in $d \neq 4$ dimensions, or to theories not symmetric under $\Phi \to -\Phi$ is straightforward.

Acknowledgements

It is a pleasure to thank Eric D'Hoker for helpful conversations and guidance during the stages of this work.
Appendix A

Here we explicitly show how the general graphs with $\delta'$ subtractions explicitly break into renormalized (according to $\delta$ power counting) ones, that:

\[ \tilde{R}'(G) = \sum_i \tilde{R}(G_i) \quad (B1) \]

We start by using the explicit solution to the Bogoliubov recursion formula in (1). Denote a forest $F$ of a graph $G$ to be a finite set of 1PIR subgraphs $\gamma_i$ of $G$ such that either:

\[ \gamma_i \subset \gamma_j, \quad \gamma_j \supset \gamma_i, \quad \text{or} \quad \gamma_i \cap \gamma_j = \emptyset \quad (B2) \]

Normal forests $F_n(G)$ of $G$ are defined not to contain the entire graph $G$ and full forests do. Then the explicit solution to the Bogoliubov recursion formula is\(^{(2)}\):

\[ \tilde{R}'(G) = \sum_{U \in F_n(G)} \prod_{\gamma \in U} \left(-T'_{\gamma}\right)U(G) \quad (B3) \]

and

\[ R'(G) = \sum_{U \in F(G)} \prod_{\gamma \in U} \left(-T'_{\gamma}\right)U(G) \quad (B4) \]

In (B4) $T'_{\gamma}$ is defined to be zero if $\delta'(\gamma) < 0$, and we consider all forests of $G$. The subtractions are to be performed inside to out in accord with the nested nature of the elements in the forest. Next we add and subtract the additional subtractions so that:

\[ \tilde{R}'(G) = \sum_{U \in F_n(G)} \prod_{\gamma \in U} \left(-T'_{\gamma} + T_{\gamma} - T_{\gamma}\right)U(G) \quad (B5) \]

In order to prove the factorization we need to show that (B5) splits into a sum over graphs $G_i$, defined below, together with the full subtractions:
\[
\bar{R}'(G) = \sum_i \sum_{U \in F_n(G_i)} \prod_{\gamma \in U} (-T_\gamma) U(G_i)
\]

We start by expanding the expression (B5) while respecting the order of the nested differentiations. This generates terms labeled by how we may act \(T - T'\) on different sets \(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}\) of 1PIR subgraphs of \(G\). The different sets are conveniently labeled by the forests of \(G\), so in fact the expansion extends over all possible forests.

In order to continue we define \(F_S(G)\) to be a forest of \(G\) not including the elements \(S = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}\). Then the sum (B5) breaks into:

\[
\bar{R}'(G) = \sum_{S \in F(G)} \prod_{\beta \in S} \left( T_\beta - T'_\beta \right) \sum_{U \in F_S(G)} \prod_{\gamma \in U} (-T_\gamma) U(G)
\] (B6)

Recall that all subtractions are to performed from the most nested on out. Let’s switch the order of the subtractions to follow this rule. Define the maximal elements in \(S\) to be the set \(S_m = \{\gamma^1, \gamma^2, \ldots, \gamma^{m_S}\}\) such that all elements of \(S\) are contained in them and for all \(i\) and \(j\) that \(\gamma^i \cap \gamma^j = \emptyset\). Then (B6) is:

\[
\bar{R}'(G) = \sum_{S \in F(G)} \sum_{U \in F(G/S_m)} \prod_{\gamma \in U} (-T_\gamma) \left( \prod_{\beta \in S} \left( T_\beta - T'_\beta \right) \sum_{V \in F_S(S_m)} \prod_{\alpha \in V} (-T_\alpha) \right) U(G)
\] (B7)

The new graph \(U(G_i)\), where \(i\) denotes a forest \(S\) of \(G\) and individual terms in all of the subtractions \(T_\beta - T'_\beta\), is:

\[
U(G_i) = \prod_{\beta \in S} \left( T_\beta - T'_\beta \right) \sum_{V \in F_S(S_m)} \prod_{\alpha \in V} (-T_\alpha) U(G)
\] (B8)

The subtraction operators \(T - T'\), by virtue of the forest formula, must operate on only primitive divergences since all subdivergences have been removed by subtractions \(T\). The 1PI elements of the maximal set \(S_m\) have been shrunk to points, and the forests of \(G_i\) are exactly the forests of \(G/S_m\), \(F(G_i) = F(S_m)\). Furthermore, the power counting is not altered when multiple operations \(T - T'\) are nested. For two graphs \(\alpha \subset \beta\), the effective superficial degree of divergence for the reduced graph \(U(\beta_i) = U(\frac{\beta}{\alpha})(T_\alpha - T'_\alpha)\) term \(i\bar{R}(\alpha)\) satisfies
\( \delta(\beta) \geq \delta(\beta_i) \). This is a consequence of the fact that the \( T - T' \) operation always results in a polynomial whose terms are suppressed by hard powers of the cutoff.

As an example consider the simplest case, when \( S_m = S \). All elements \( \{\gamma_1, \ldots, \gamma_n\} \) in \( S \) are maximal (i.e. \( \gamma_i \not\subset \gamma_j \)), and expression (B8) splits into:

\[
U(G_i) = U\left(\frac{G}{S}\right) \prod_{\beta \in S} \left( T_\beta - T'_\beta \right) \sum_{V \in F(\beta)} \prod_{\alpha \in V} (-T_\alpha)U(\beta) \\
= U\left(\frac{G}{S}\right) \prod_{\beta \in S} (T_\beta - T'_\beta) \bar{R}(\beta)
\]

All subdivergences in the 1PI parts of \( S \) are eliminated, and the parts themselves are replaced with higher-dimensional vertices.

Further examples are slightly more complicated, as some of the \( T - T \) operations are nested. Topologically, however, all subdivergences are removed by the \( T \) operation in the sum in (B6). The forest \( S \) is divided into disjoint maximal elements \( \gamma^i \) which may be further sub-divided into their maximal sub-elements. Further sub-divisions cease when the set is reached which is itself maximal. Then the \( T - T' \) operations on this maximal set are polynomials divided by powers of the cutoff. Working backwards from the most nested subgraphs, the \( T - T' \) operations always act on subgraphs which are fully subtracted, hence lead to polynomials suppressed by at least \( \Lambda^2 \).

We have thus arrived at (B1). The sum over forests of \( G \) and individual terms in the \( T - T' \) operations, denote the possible new graphs \( G_i \).
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Figure 1  Divergent structure of simple graph. Here $T'$ on $U(G)$ is zero.

a. $U(G) = \begin{array}{c}
\includegraphics{figure1a.png}
\end{array}$

b. $U(G) = \begin{array}{c}
\includegraphics{figure1b.png}
\end{array}$

Figure 2  Standard renormalization of a subgraph.

a. $\begin{array}{c}
\includegraphics{figure2a.png}
\end{array}$

b. $\begin{array}{c}
\includegraphics{figure2b.png}
\end{array}$
Figure 3  Simple re–organization of subtractions into two renormalized subgraphs.

\[ \chi_i - \chi_j = T - T' \]

Figure 4  The T–T' operation leads to several monomial insertions.

\[ \sum_u \]

\[ = T - T' \]
Figure 5  Re-organization of T' subtractions (a) into seven renormalized sets (b).

\[ T' = \square \]

(a)

(b)

Set 1
Figure 5 (cont.)

Set 2

\[
+ \sum_i \left( \begin{array}{ccc}
\text{Set 3} & & \\
\text{Set 4} & & \\
\text{Set 5} & & \\
\end{array} \right)
\]

\[
\begin{array}{ccc}
\text{Set 6} & & \\
\text{Set 7} & & \\
\end{array}
\]

\[
= T
\]

\[
= T - T'
\]