Spectral properties of a two dimensional photonic crystal with quasi-integrable geometry

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Abstract. In this paper we study the statistical properties of the allowed frequencies for electromagnetic waves propagating in two-dimensional photonic crystals with quasi-integrable geometry. We compute the level spacing, group velocity, and curvature distributions ($P(s)$, $P(v)$, and $P(c)$, respectively) and compare them with the corresponding random matrix theory predictions. Due to the quasi-integrability of the crystal we observe signatures of intermediate statistics in $P(s)$ and $P(c)$ for high refractive index contrasts.

1. Introduction

One of the most used tools to characterize quantum or wave systems, from the point of view of quantum chaos, is the distribution function $P(s)$ of the spacings $s_n = (E_n - E_{n-1})/\Delta$ of adjacent eigenenergies, where $\Delta$ is the mean level spacing. See for example [1]. For deterministic quantized Hamiltonian systems or wave cavities, such as billiards, it is well accepted that: (i) a $P(s)$ of the Poisson type,

$$P(s) = \exp(-s) \ ,$$

(1)
corresponds to integrable classical motion [2]; (ii) a $P(s)$ with a Wigner-Dyson form,

$$P(s) = \frac{\pi}{2} \exp\left(-\frac{\pi}{4} s^2\right) \ ,$$

(2)
is a signature of chaotic classical dynamics [3, 4] (this statement is known as the Bohigas-Giannoni-Schmidt conjecture); while (iii) if $P(s)$ has a form which is intermediate between Poisson and Wigner-Dyson, the dynamics of the classical counterpart should develop mixed chaos [5, 6].

With the use of billiard tables exhibiting the generic transition to chaos (billiards whose classical dynamics evolves from integrable to fully chaotic as a function of their geometrical parameters) it is possible to study the transition from Poisson to Wigner-Dyson in the shape of $P(s)$. As an example, recently, the study of $P(s)$ in the transition Poisson–Wigner-Dyson was performed by the use of a two-dimensional (2D) photonic crystal [7], where $s_n = (\omega_n^2 - \omega_{n-1}^2)/\Delta$ and $\omega^2$ are the eigenfrequencies of the propagating electromagnetic waves. Such photonic crystal was constructed as a periodic array of cylinders, with dielectric constant $\epsilon$, surrounded by air;
Figure 1. Unit cell scheme of the two-dimensional quasi-integrable photonic crystal considered in this paper. It consists of two infinitely long rectangular columns with dielectric constant $\epsilon_a$ surrounded by air, $\epsilon_b = 1$, embedded in a squared unit cell of linear size $b$.

in such a way that the unit cell of the crystal has the geometry of the Sinai billiard. That is, in the ray limit the dynamics of the system shows the generic transition to chaos as a function of $\epsilon$: When $\epsilon = 1$ the rays propagate in free space (integrable dynamics), while when $\epsilon \to \infty$ the rays hit hard cylinders (chaotic dynamics). So, $P(s)$ transits effectively from Poisson to Wigner-Dyson.

On the other hand, there is a family of dynamical billiards consisting of two or more integrable regions. Such billiards are known as quasi-integrable billiards. The dynamics of quasi-integrable billiards is not integrable neither chaotic, moreover, it can not be considered as a combination of both (as the mixed-chaotic dynamics proper of smooth billiards in the transition from integrability to chaos). Also, it has been observed [8, 9, 10, 11] that the quantum or wave counterparts of quasi-integrable systems obey the statistical properties of disordered systems having Anderson transition at the critical point. That is, quantized quasi-integrable systems show a $P(s)$ of the semi-Poisson type:

$$P(s) = 4s \exp(-2s) .$$

The predictions for $P(s)$ as given in Equations (2) and (3) correspond to systems preserving time-reversal symmetry, which is the case we explore in this paper.

In addition, we would like to stress that (i) most studies on photonic crystals are focus on the low-frequency part of the spectrum; and (ii) to date, not many studies on the spectral properties of quasi-integrable systems are available. Then, the purpose of this paper is to study the statistical properties of high-frequency electromagnetic waves propagating in photonic crystals with quasi-integrable geometry.

2. Model and Method

The 2D photonic crystal with quasi-integrable geometry we shall use has a unit cell that consists of two infinitely long rectangular columns having dielectric constant $\epsilon_a$ surrounded by air $\epsilon_b = 1$. See Figure 1. The periodic structure supports the propagation of two electromagnetic modes with either the electric ($E$-mode) or the magnetic ($H$-mode) field parallel to the columns. The wave equation for the monochromatic field $\vec{H}(\vec{r}, t) = \vec{H}(\vec{r}) \exp(-i\omega t)$ obtained from Maxwell’s equations is written as

$$\nabla \times \left[ \eta(\vec{r}) \nabla \times \vec{H}(\vec{r}) \right] = \left( \frac{\omega}{c} \right)^2 \vec{H}(\vec{r}) ,$$

where $\eta(\vec{r}) = 1/\epsilon(\vec{r}) = \sum_{\vec{G}} \eta(\vec{G}) \exp(i\vec{G} \cdot \vec{r})$ is the inverse dielectric function, which being periodic, is expanded in a Fourier series over the reciprocal lattice vectors $\vec{G}$. Since the system
is periodic, due to the Bloch theorem, $\hat{H}(\vec{r}) = \sum_{\vec{k}} \hat{h}_{\vec{k}}(\vec{G}) \exp \left( i \left( \vec{k} + \vec{G} \right) \cdot \vec{r} \right)$. Substitution of this expansion into Equation (4) leads to an infinite set of linear homogeneous equations for the coefficients $\hat{h}_{\vec{k}}$. This set of equations can be simplified in the long-wavelength limit, $k \to 0$, using the method proposed in [12]. Omitting details, we give the eigenvalue equation for the $E$-mode,

$$\det \left[ \frac{\omega^2}{c^2} \delta_{\vec{G},\vec{G}'} - G G' \eta \left( \vec{G} - \vec{G}' \right) \right] = 0, \quad \vec{G}, \vec{G}' \neq 0. \tag{5}$$

Equation (5) is a polynomial equation of infinite order with respect to $\omega^2$; its roots are $\omega^2_n(\vec{k})$.

Finally, the form factor of the structure of Figure 1 is given by

$$\eta \left( \vec{G} \right) = \frac{1}{\epsilon_a \epsilon_b} \left[ 1 - F \left( \vec{G} \right) \right] F \left( \vec{G} \right), \tag{6}$$

with

$$F \left( \vec{G} \right) = \frac{4}{G_x G_y} \left[ \sin \left( \frac{a}{2} G_x \right) \sin \left( \frac{h}{2} G_y \right) + \exp(-i\vec{G} \cdot \vec{r}) \sin \left( \frac{a'}{2} G_x \right) \sin \left( \frac{h'}{2} G_y \right) \right].$$

Here, $G_x$ and $G_y$ are the rectangular components of $\vec{G}$.

### 3. Results

Here we analyze the evolution of the spectral properties of the photonic crystal of Figure 1 as a function of $\epsilon_a$ (with $\epsilon_b$ fixed to 1). Without loss of generality, we choose $a = 0.54$, $a' = 0.25$, $h = 0.3$, $h' = 0.65$, and $\vec{r} = 0.375\hat{x} + 0.175\hat{y}$. All lengths are given in units of $b$. With this prescription for the geometrical parameters the filling fraction equals to 31.25%.

A detailed analysis of the frequency spectrum for the photonic crystal of Figure 1 (not shown here) reveals that due to finite matrix size effects only a fraction of the eigenfrequencies obtained from the the diagonalization of Equation (5) is useful, as in standard billiard problems. Also, as $\epsilon_a$ increases, the percentage of useful eigenfrequencies decreases. For example, for $\epsilon_a = 1.2$, 5, and 50, the percentage of useful eigenfrequencies is approximately 90%, 30%, and 25%, respectively. For this reason, below we diagonalize matrices of size $\sim 14,000$ and use for the statistics only the first 3,500 eigenfrequencies for all the values of $\epsilon_a$ considered.

In Figure 2 we show the level spacing distribution $P(s)$ for the quasi-integrable photonic crystal of Figure 1 with $\epsilon_a = 1.2$, 5, and 50. When $\epsilon_a = 1.2$, $P(s)$ is quite close to Equation (1), see Figure 2(a), since the system is still close to integrable. This result should be expected for any non-integrable photonic crystal, see for example [7]. In Figure 2(b), for $\epsilon_a = 5$, we already see a very well developed Wigner-Dyson shape for $P(s)$ even though the geometry of our photonic crystal does not develop chaos in the ray limit. We understand this result as the lack of resolution of the quasi-integrable geometry by the waves, due to the low refractive index contrast. Then, by further increasing the value of $\epsilon_a$, see Figure 2(c), we observe that $P(s)$ moves towards the semi-Poisson shape, as expected, since for high-refractive index contrasts the quasi-integrable geometry should be better realized by the waves moving inside our photonic crystal. So, we believe that by increasing further the value of $\epsilon_a$ and/or including higher frequencies in the statistical analysis, the level spacing distribution should become even closer to Equation (3).

To complement the spectral analysis of our system we also compute the group velocity distribution $P(v)$ and the curvature distribution $P(c)$. From random matrix theory it is well known that the group velocity distribution should be Gaussian [1],

$$P(v) = \frac{1}{\sqrt{2}} \exp \left( -\frac{v^2}{2} \right), \tag{7}$$
Figure 2. Level spacing distribution $P(s)$ for the quasi-integrable photonic crystal of Figure 1 with (a) $\epsilon_a = 1.2$, (b) $\epsilon_a = 5$, and (c) $\epsilon_a = 50$ (histograms). $\epsilon_b = 1$. Black, blue, and red lines correspond to Equations (1), (2), and (3), respectively.

while the curvature distribution decays as a power-law for large $c$ [13, 14],

$$P(c) = \frac{1}{2} \frac{1}{(1 + c^2)^{3/2}},$$

for systems showing the $P(s)$ given by Equation (2). The level velocities $v_n$ and the scaled curvatures $c_n$ are given, respectively, by

$$v_n = \sqrt{\beta(x_n)} \frac{\partial x_n}{\partial k}$$

and

$$c_n = \frac{\beta(x_n)}{\pi} \frac{\partial^2 x_n}{\partial k^2},$$

where $x_n = \omega_n^2/\Delta$ and $\beta(x) = \langle (\partial x_n/\partial k)^2 \rangle_x^{-1}$. The derivatives in Equation (9) were calculated as finite differences ($\partial x_n/\partial k \approx \Delta x_n/\Delta k$) with $\Delta k = 10^{-6}$.

Then, in Figures 3 and 4 we present plots of $P(v)$ and $P(c)$ for the quasi-integrable photonic crystal of Figure 1 with $\epsilon_a = 1.2$, 5, and 50. In Figure 3 we show that $P(v)$ evolves towards a Gaussian shape as $\epsilon_a$ increases, as predicted. However, we expected to observe a Gaussian shape already at $\epsilon_a = 5$, see Figure 3(b), for which $P(s)$ shows the Wigner-Dyson form. From Figure 4 we note that $P(c) \propto c^{-\alpha}$, for $c \rightarrow \infty$, with $\alpha > 3$. This behavior is typical of critical systems [15] and of quantized Hamiltonian systems in the transition from integrability to chaos. In the case $\epsilon_a = 5$ we expected to observe a clear behavior of the form $P(c) \propto c^{-3}$ which is only partially seen, see Figure 4(b).

4. Conclusions
In this work we study the statistical properties of the allowed frequencies for electromagnetic waves which propagate in two-dimensional photonic crystals with quasi-integrable geometry. We report the transition Poisson–Wigner-Dyson–semi-Poisson in the structure of the level spacing probability distribution $P(s)$ as a function of $\epsilon_a$ (partially confirmed by $P(v)$ and $P(c)$). The fact that $P(s)$ evolves towards the semi-Poisson shape for high refractive index contrasts emerge as a clear signature of the quasi-integrability of our photonic crystal.

Even though we showed results for $E$-modes only, we have verified that our conclusions include the corresponding $H$-modes too.

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Figure 3. Group velocity distribution $P(v)$ for the quasi-integrable photonic crystal of Figure 1 with (a) $\epsilon_a = 1.2$, (b) $\epsilon_a = 5$, and (c) $\epsilon_a = 50$ (histograms). $\epsilon_b = 1$. Blue lines correspond to Equation (7).

Figure 4. Curvature distribution $P(c)$ for the quasi-integrable photonic crystal of Figure 1 with (a) $\epsilon_a = 1.2$, (b) $\epsilon_a = 5$, and (c) $\epsilon_a = 50$ (histograms). $\epsilon_b = 1$. Blue lines, plotted to guide the eye, are proportional to $c^{-3}$; see Equation (8).

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