New exponential dispersion models for count data - properties and applications

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Abstract

In their fundamental paper on cubic variance functions (VFs), Letac and Mora (The Annals of Statistics, 1990) presented a systematic, rigorous and comprehensive study of natural exponential families (NEFs) on the real line, their characterization through their VFs and mean value parameterization. They presented a section that for some reason has been left unnoticed. This section deals with the construction of VFs associated with NEFs of counting distributions on the set of nonnegative integers and allows to find the corresponding generating measures. As EDMs are based on NEFs, we introduce in this paper three new classes of EDMs based on their results. For these classes, which are associated with simple VFs, we derive their mean value parameterization and their associated generating measures. We also prove that they have some desirable properties. Each of these classes is shown to be overdispersed and zero inflated in ascending order, making them as competitive statistical models for those in use in both, statistical and actuarial modeling. A numerical example of real data compares the performance of one class and demonstrates its superiority.

Keywords. Exponential dispersion model; natural exponential family; overdispersion; variance function; zero-inflated distribution

1 Introduction and background

Natural exponential families (NEFs) and exponential dispersion models (EDMs) on \( \mathbb{R} \) play an important role both in probability and statistical applications. Most of the frequently used distributions are indeed belonging to such models. However, a huge number of NEFs (or EDMs) have not been used in probabilistic or statistical modelling for two main reasons: they have not been revealed or do not have explicit functional forms (even not via power series expansions). This, despite the fact that they could have provided significant and new models useful in statistical applications. Indeed, the main purpose of this paper is to expose the statistical research community to various classes of such NEFs. A thorough discussion on this observation is presented in Bar-Lev and Kokonendji (2017).

One of the most forsaken reference representing the above situation is the fundamental paper of Letac and Mora (1990) (hereafter, LM, 1990) on NEFs which provides a thorough description and analytic properties of such families along with their mean value parameterization. In spite of the fact that their article received many citations, a major and important part of the article was somehow abandoned without being noticed. This part refers to the section dealing with the

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construction of NEFs of counting distributions on the set of nonnegative integers \( \mathbb{N}_0 \). These families are represented by either polynomial variance functions (VFs) or other nice forms. Moreover, in their Proposition 4.4 they explicitly present a formula that allows to compute, at least numerically, the counting measure \( \mu \) which generates the appropriate NEF in terms of its mean \( m \) (for further details see also Bar-Lev and Kokonendji, 2017). Such a formula requires (except for some limited special cases) numerical calculations of the \( n \)-th derivative of product of functions depending on the mean \( m \) which are needed for calculating the the mass of \( \mu \) at the point \( n \).

In our opinion, the reason why this formula as well as Proposition 4.4 of LM (1990) were not used is that in the eighties and nineties of the last century (when the LM, 1990, article was just published) there were no powerful mathematical programs that would allow the complex and cumbersome calculations of the mass of \( \mu \) on the nonnegative integers. Fortunately, nowadays, the situation has changed and existing powerful computing software are available and can be used to calculate the probabilities of the relevant NEFs. To our best knowledge, these families have not been considered since then, neither theoretically nor numerically. Our contribution is to fill this omission. A fact that will lead to exposure of numerous counting NEFs (as well as EDMs) that can serve as competitive statistical models for those in use (e.g., Poisson, Negative binomial) in both, statistical and actuarial modeling.

For this we need to present some preliminaries. As is well known, and as will seen in the sequel, EDMs are based on NEFs. Hence, we first need to present some basic properties of NEFs as VFs, statistical and actuarial modeling.

Let \( \mu \) be a positive Radon measure on \( \mathbb{R} \) with convex support \( C_\mu \). Consider the set

\[
D_\mu = \left\{ \theta \in \mathbb{R} : L_\mu(\theta) = \int e^{\theta x} \mu(dx) < \infty \right\},
\]

and assume that \( \Theta_\mu = \text{int } D_\mu \) is nonempty. Then, the NEF \( \mathcal{F} \) generated by \( \mu \) is defined by the set of probabilities

\[
\mathcal{F} = \mathcal{F}(\mu) = \{ F(\theta, \mu(dx)) = \exp \{ \theta x - k_\mu(\theta) \} \mu(dx) : \theta \in \Theta \},
\]

where \( k_\mu(\theta) = \ln L_\mu(\theta) \) is the cumulant transform of \( \mu \). \( k_\mu \) is strictly convex and real analytic on \( \Theta \) where \( k'(\theta) \) and \( k''(\theta), \theta \in \Theta \), are the respective mean and variance corresponding to \( F(\theta, \mu) \) and the open interval \( M = M_{\mathcal{F}} = k'(\Theta) \) is called the mean domain of \( \mathcal{F} \). For the sake of simplicity, when no ambiguity is caused, we shall henceforth suppress the dependence on \( \mu \) of all functionals involved.

Since the map \( \theta \mapsto k'(\theta) \) is one-to-one, its inverse function \( \psi : M \rightarrow \Theta \) is well defined. The variance corresponding to \( \mathcal{F} \) in terms of \( m \) is

\[
V(m) = 1/\psi'(m) = k''(\theta),
\]

and the map \( m \mapsto V(m) \) from \( M \) into \( \mathbb{R}^+ \) is called the VF of \( \mathcal{F} \). In fact, a VF of an NEF \( \mathcal{F} \) is a pair \((V, M)\) which uniquely determines an NEF within the class of NEFs (see Morris, 1982, and LM, 1990). It is important to emphasize that a VF is a transform, not of a particular distribution, but rather of a family \( \mathcal{F} \) in the sense that if two VFs \((V_1, M_1)\) and \((V_2, M_2)\) of two NEFs \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively, satisfy \( V_1 = V_2 \) on \( J = M_1 \cap M_2 \neq \emptyset \) then \( \mathcal{F}_1 = \mathcal{F}_2 \). This would imply that given a VF \((V, M)\) then the mean domain \( M \) is the largest open interval on which \( V \) is positive real analytic.

Now, for a given VF \((V, M)\) of an NEF \( \mathcal{F} \), let choose two primitives \( \psi(m) \) and \( \psi_1(m) \) of \( 1/V(m) \) and \( m/V(m) \), respectively. Then there exists a positive Radon measure \( \mu \) on \( \mathbb{R} \) such that

\[
\psi_1(m) = \int e^{\psi(m)x} \mu(dx), \quad k(\psi(m)) = \psi_1(m), m \in M,
\]
and
\[ \mathcal{F} = \{ F(m, \mu(dx)) = \exp\{ x\psi(m) - \psi_1(m) \} \mu(dx) : m \in M \}. \] (5)
The reparameterization of \( \mathcal{F} \) in (5) is called the mean value parameterization of \( \mathcal{F} \) (see LM, 1990, Proposition 2.3). Accordingly, an NEF has two natural presentations: one is parameterized by canonical parameter \( \theta \) and is given in (2) and the second by the mean parameter \( m \) and as given in (5). However, as far as statistical applications concern, the rather more important presentation is the mean value parameterization (as \( \theta \) is just an artificial parameter - the argument of the corresponding Laplace transform).

We now present the definitions of steep NEFs and EDMs:

- **Steep NEFs**: An NEF \( \mathcal{F} = \mathcal{F}(\mu) \) is called steep \( \iff \) its cumulant transform \( k_\mu(\theta) \) is essentially smooth convex function on \( D_\mu \) (defined in (1)) \( \iff M_F = int C_\mu \) (c.f., Barndorff-Nielsen, 1978; LM, 1990). We shall refer to this definition in the sequel.

- **EDMs**: Let \( \mathcal{F} = \mathcal{F}(\mu) \) be an NEF generated by \( \mu \) with VF \( (V, M) \) and Laplace and cumulant transforms \( L_\mu \) and \( k_\mu \), respectively. Denote \( \Lambda = \{ p \in \mathbb{R}^+ : L_p^\mu \text{ is a Laplace transform of some measure } \mu_p \} \), then \( \Lambda \) is nonempty due to convolution and is called the Jorgensen set (or the dispersion parameter space in the terminology of EDMs). It has been shown that \( \Lambda = \mathbb{R}^+ \iff \mu \) (and thus all members of \( \mathcal{F} \)) is infinitely divisible. If \( p \in \Lambda \), the NEF generated by \( \mu_p \) is the set probabilities of the form

\[ \mathcal{F}(\mu_p) = F(\theta, \mu_p(dx)) = \exp\{ \theta x - pk_{\mu_p}(\theta) \} \mu_p(dx) : \theta \in \Theta_{\mu_p} = \Theta_\mu \] (6)

and the set of NEFs

\[ \bigcup_{p \in \Lambda} \mathcal{F}(\mu_p) \]

was termed by Jorgensen (1987) the EDM corresponding to \( \mu \). In particular if \( \Lambda = \mathbb{R}^+ \) (i.e., \( \mu \) is infinitely divisible) then EDMs are used to describe the error component in generalized linear models. It is important to note that the VF corresponding to an EDM in (6) has the form

\[ (pV(m), pM), \]

where \( m \) is the mean corresponding to (6) and \( M \) is the mean corresponding to the NEF generated by \( \mu \). Obviously if \( M = \mathbb{R}, \mathbb{R}^+ \) or \( \mathbb{R}^- \) then so is \( pM \), the mean parameter space of the corresponding EDM.

Various types of VFs of NEFs have been presented and discussed in the literature (for a thorough survey see Bar-Lev and Kokonendji, 2017). Certainly, the most famous and practically used classes of VFs are polynomial VFs (up to degree 3) and the Tweedie class (c.f., Tweedie, 1984, Bar-Lev and Enis, 1986, and Jorgensen, 1987, 1997) having VFs of the form \( V(m) = \alpha m^\gamma \) where \( \alpha > 0, \gamma \in \mathbb{R}/(0, 1) \).

Nonetheless, it seems that another class of VFs is also very important and interesting. It is the class of VFs having a polynomial structure with degree \( r \geq 4 \), for which all of the respective cumulants and moments are also polynomials and can be simply derived. Indeed, in this respect of polynomial VFs, Bar-Lev (1987) (see also LM, 1990, Theorem 3.2) showed that any \( r \)-th degree polynomial of the form

\[ V(m) = \sum_{i=1}^{r} a_i m^i, m \in \mathbb{R}^+, r \in \mathbb{N}, a_i \geq 0, \sum_{i=1}^{r} a_i > 0, \] (8)
is a VF of an infinitely divisible NEF. The class of VFs of the form (3) is huge and contains, among many others, three out of the six NEFs having quadratic VFs characterized by Morris (1982) and all of the six NEFs having strictly cubic VFs characterized by LM (1990). However, for a polynomial with degree \( r \geq 4 \) (except for the case where \( a_1 > 0 \) and \( a_i = 0, i \geq 2 \)), no investigation has been carried out into which NEFs correspond to such VFs.

Recall that for a given VF \( V, \psi(m) \) and \( \psi_1(m) \) are primitives of \( dm/V(m) \) and \( mdm/V(m) \), respectively, i.e.,

\[
\theta = \psi(m) = \int \frac{dm}{V(m)} \tag{9}
\]

and

\[
k(\theta) = k(\theta(m)) = \psi_1(m) = \int \frac{mdm}{V(m)}. \tag{10}
\]

Accordingly, if \( V \) is of the general form (3) it is not possible to explicitly express \( \psi(m) \) and \( \psi_1(m) \), in which case the mean value parameterization (5) is useless for any practical consideration. If, however, for some special cases of the \( a_i \)'s coefficients, it can be calculated nicely and explicitly then so can be the corresponding likelihood function based on an appropriate random sample. This fact has a tremendous significance in statistical inference. Apropos, other references dealing with polynomial VFs not of the form (3) and present necessary and/or sufficient conditions for polynomials to be VFs of NEFs are Bar-Lev and Bshouty (1990); Bar-Lev, Bshouty and Enis (1991, 1992) and Bar-Lev, Bshouty, Enis and Ohayon (1992).

After this long introduction we arrive at the crux of the paper. LM (1990) have presented a subclass of (3) with the form

\[
(V,M) = \left( m \prod_{i=1}^{r} \left( 1 + \frac{m}{p_i} \right), \mathbb{R}^+ \right), \quad p_i > 0, i = 1, \ldots, r, r \in \mathbb{N}, \tag{11}
\]

and another subclass of the form

\[
V = \frac{m}{\prod_{i=1}^{r} \left( 1 - \frac{m}{p_i} \right)}, M = (0, \min(p_1, \ldots, p_r)), \quad p_i > 0, i = 1, \ldots, r, r \in \mathbb{N}. \tag{12}
\]

In their Proposition 4.4, LM (1990) proved, under mild conditions, that the NEFs corresponding to (11) and (12) are generated by counting measures on \( \mathbb{N}_0 \). But most importantly, they also derived expressions, using the Lagrange formula, the mass points \( \mu_n \) at \( n \in \mathbb{N}_0 \), where \( \mu(dx) = \sum_{n=0}^{\infty} \mu_n \delta_n(dx) \) and \( \delta_n \) is the Dirac mass at \( n \) (LM, 1990, Eq. 4.16). Such expressions were then used to derive the mass points \( \mu_n \) corresponding to VFs up to \( r = 2 \) (i.e., up to cubic VFs). The resulting discrete NEFs with cubic VFs that they found are

- the Abel NEF (also known as generalized Poisson; c.f., Consul, 1989, and Consul and Famoye, 2006);
- the Takács NEF (also known as the generalized negative binomial; c.f., Jain and Consul, 1971, and Consul and Famoye, 2006);
- The strict arcsine NEF;
- The large arcsine NEF.

All of the latter NEFs have explicit and closed forms for the expressions of their probabilities. Notwithstanding, for \( r \geq 3 \), Eq. 4.16 of LM (1990), which enables to derive expressions for the \( \mu_n \)'s, is quite intricate, cumbersome and involved with computing recursively derivatives of functions of
the two primitives $\psi(m)$ and $\psi_1(m)$. Consequently, for $r \geq 3$, the corresponding $\mu_n$’s in Eq. 4.16 of LM (1990), and thus also the NEF probabilities in $\text{(5)}$, cannot be presented neither in closed and explicit forms nor in terms of infinite sum (or some transcendental functions). They can be derived only through numerical calculations by either mathematical software (as MATHEMATICA, R or Python) or by writing appropriate computer programs. This explains our statement above that many NEFs (at least with polynomial VF structure and degree $r \geq 3$) have not been used for statistical modeling or applications for the mere fact that they have not been known before and thus not been considered and investigated. Therefore in this paper we intend to correct to a certain extent the ‘injustice’ caused to these discrete NEFs. Notice however an important point. When we refer to $\text{(5)}$ in a Bayesian framework and when $\text{(5)}$ serves as a prior distribution then the $\mu_n$’s calculation becomes superfluous and redundant when calculating the posterior distribution, as one can choose arbitrarily any two primitives $\psi(m)$ and $\psi_1(m)$. We shall further relate to this point in Section 3.

In particular, we present in the sequel three subclasses: two of the form $\text{(11)}$ and one of the form $\text{(12)}$. For convenience we shall call them classes though they are subsets of $\text{(11)}$ and $\text{(12)}$. These classes of VFs were chosen because of the relative simplicity of the calculations of $\psi(m)$ and $\psi_1(m)$ for which explicit expressions are available. The three VF classes are

$$V(m) = (1 + \frac{m}{p})^r, m(1 + \frac{m}{b})(1 + \frac{m}{p})^r, m/(1 - \frac{m}{p})^r,$$  

where $p > 0, b > 0, r \in \mathbb{N}_0$ and $m \in \mathbb{R}^+$. Later we will coin each class a name and discuss its properties. However, at this point, we will notice a very important fact. All three classes are of the form $\text{(7)}$ representing VFs of EDMs. Consequently, their corresponding probabilities belong to the realm of EDMs.

The paper is organized as follows. In Section 2 we will discuss further important aspects related to the practical implementation of Proposition 4.4 of LM, 1990, and also present three classes of VFs of the form $\text{(11)}$ and $\text{(12)}$ for which the associated NEFs are concentrated on $\mathbb{N}_0$. In fact, as we already noted, such classes form exponential dispersion models (EDMs) for count data (see item 2 of Section 2.2). When no confusion is caused we prefer at this stage the notion of NEFs rather than that of EDMs as most of the properties discussed in the sequel are inherent properties of NEFs. In Section 3 we present in three subsections each of the three classes presented in $\text{(13)}$. For each class we derive expressions for $\psi(m)$ and $\psi_1(m)$ which fulfills the premises of Proposition 4.4 of LM (1990). We then describe some of their properties. In particular it will be shown that the corresponding NEFs’ distributions are overdispersed and zero inflated in ascending order in $r$ (relative to the Poisson and negative binomial NEFs). A numerical example of real data, presented in Section 4, compares the performance of the first class in $\text{(13)}$ for $r = 1, ..., 5$ to other well used discrete distributions. This example demonstrates the superiority of the members of this class with $r = 4, 5$, vis-a-vis all other distributions. Section 5 is devoted to some concluding remarks.

2 Further aspects and analysis and presentation of the three classes

2.1 Further aspects and analysis

When considering NEFs, there are interesting and complex relationships between the generating measure $\mu$, the Laplace transform $L(\theta)$, the VF $(V, M)$ and the forms of $\psi(m)$ and $\psi_1(m)$ defined in $\text{(9)}$ and $\text{(10)}$, respectively. We demonstrate only four cases of those relationships:

1. $\mu$ is known but neither $L(\theta)$ or $(V, M)$ can be derived explicitly.
2. $\mu$ is known and so are $(V, M), \psi(m)$ and $\psi_1(m)$ but $L(\theta)$ is not and can be solved only via functional equation. A good example for the latter situation is the Kendall-Ressel NEF (c.f., Bar-Lev, Boukai and Landsman, 2016, and the references cited therein).

3. For most of the NEFs having a given polynomial VF structure it is impossible to find expressions for $\mu, L(\theta)$, and often not even for $\psi(m)$ and $\psi_1(m)$. However, such VFs can still be used to describe some probabilistic phenomena as described in Bar-Lev, Bshouty, Grünwald and Harremoës (2010) regarding Jeffreys and Shtarkov distributions (further details on this two distributions can be found in the latter reference). These two distributions play an important role in universal coding and minimum description length inference which are two central areas within the field of information theory. However, in some situations Shtarkov distributions exist while Jeffreys distributions do not. Indeed, the latter paper presented various classes of VFs of infinitely divisible NEFs (stemming from polynomial VFs) for which Shtarkov distributions exist while Jeffreys distributions do not (though none of their corresponding $\mu, L(\theta), \psi(m)$ or $\psi_1(m)$ could be explicitly expressed).

4. Consider a simple polynomial VF the form $(V, M) = (m(1 + m^r), \mathbb{R}^+), r \in \mathbb{N}_0$. This structure of polynomial VFs has been introduced by Hinde and Demétrio (1998) for overdispersed models and characterized by Kokonendji et al. (2007) to analyze overdispersed and zero-inflated count data. Further theoretical and data analysis of the HD class can be found in Kokonendji, Dossou-Gbéto and Demétrio (2004) and Kokonendji and Malouche (2008). Though the HD class is not a special case of (11), this class of VFs fulfill the premises of Proposition 4.4 of LM (1990). Thus, the corresponding NEFs distributions are supported on $\mathbb{N}_0$ and Proposition 4.4 of LM (1990) can be employed to compute the respective $\mu_n$'s. Nonetheless, for the HD class, although $\psi(m)$ in (10) can be nicely expressed as

$$\psi(m) = \int \frac{dm}{m(1 + m^r)} = \ln m - \frac{1}{r} \ln (m^r + 1) + a,$$

$\psi_1(m)$, however, has a very complex expression of the form

$$\psi_1(m) = \int \frac{dm}{1 + m^r} = m_2 F_1(1, \frac{1}{r}; 1 + \frac{1}{r}; -m^r) + b,$$

where

$$2 F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

is the generalized hypergeometric function of type 2 and 1, respectively, and $(d)_k$ is the Pochhammer symbol. Such an expression for $\psi_1(m)$ makes the corresponding mean value parameterization of the HD class unuseful for practical considerations.

The above four cases motivate the present paper to look for classes of VFs (namely, those in (13)) for which we can obtain relatively simple expressions for both $\psi(m)$ and $\psi_1(m)$ and for which Proposition 4.4 of LM (1990) is applicable. This is as opposed to the HD class of VFs which is too complicated for analysis (see (14)) when $r > 2$, and therefore is excluded from further consideration.

Now we present Proposition 4.4 of LM (1990), the results of which had been laid down for decades, but its practical significance is enormous. We present only the practical details of the proposition.

**Proposition 1** (Proposition 4.4 of LM, 1990). Let $\mathcal{F}$ be an NEF on $\mathbb{R}$ with VF $(V, M)$. Then $\mathcal{F}$ is concentrated on $\mathbb{N}_0$ such that $\mu_0 > 0$ and $\mu_1 > 0$ if and only if \( i) M = (0, b) \) for some $0 < b \leq \infty;
ii) There exists a real analytic function $\psi'_1$ on $M$ such that $\psi'_1(m) = m/V(m)$ on $M$ and such that $\psi'_1(0) = 1$.  

In this case, if $\psi_1$ is a primitive of $\psi'_1$ on $M$ and $G$ is real analytic on $M$ defined by $G(m) = m \exp(-\psi(m))$, $G(0) = 1$, 

and if

$$
\begin{align*}
\mu_0 &= \exp(\psi_1(m))|_{m=0}, \\
\mu_n &= \frac{1}{m} \left[ (d/dm)^{n-1} \exp(\psi_1(m)) \times \psi'_1(m) \times (G(m))^n \right] |_{m=0}, \ n \geq 1
\end{align*}
$$

(correcting the order of parentheses around $G$ in LM, 1990), then $\mu(dx) = \sum_{n=0}^{\infty} \mu_n \delta_n(dx)$ generates $\mathcal{F}$.

Remark 2 It can be readily concluded from the proof of the above proposition that the corresponding NEF $\mathcal{F}$ is concentrated on the set of nonnegative integers comes from the fact that $f(m) = V(m)/m$ is analytic around 0 with $f(0) = 1$.

2.2 Presentation of the three classes of VF's and some of their properties

As stated, our goal is to locate classes of VF's, subclasses of (11) and (12), for which we can derive explicit and relatively simple expressions both for $\psi(m)$ and $\psi_1(m)$. The three specific classes discussed here have already been presented in (13). Now, we will discuss and address each of them separately. The first two of these classes which belong to the realm of (11) are given by

$$(V, M) = \left( m(1 + \frac{m}{p}), \mathbb{R}^+ \right), \ p > 0, r = 0, 1, 2, 3, ...,$$  

where the special cases $r = 0$, $r = 1$ and $r = 2$ correspond, respectively, the Poisson, negative binomial and Abel (or generalized Poisson) NEF's. The class in (18) was called the ABM class and it was first presented by Awad, Bar-Lev and Makov (2016) in a Bayesian framework (further details regarding such a Bayesian framework for the ABM class can be found in Bar-Lev and Kokonendji, 2017).

The second class of VF's is a generalization of a class mentioned briefly in Bar-Lev and Kokonendji (2017) and is given by

$$(V, M) = \left( m(1 + \frac{m}{b})(1 + \frac{m}{p})^r, M = \mathbb{R}^+ \right), \ p > 0, b > 0, r \in \mathbb{N}_0.$$  

We term (19) as the LMS (Letac-Mora-Steep) class. Obviously when $b = p$ then the LMS class reduces to the ABM class. Accordingly, in the sequel we analyze the LMS class for $b \neq p$. Special cases of the (19) are the negative binomial NEF when $r = 0$ and the Takács NEF when $r = 1$ and $b = p/2$ (see LM, 1990, Table 2). The latter NEF is also known as the generalized negative binomial family (c.f., Devroye, 1992). Clearly if $b = p$ then the LMS class coincides with the ABM one. In fact, the LMS class contains the ABM one (except for the Poisson case), but a separate treatment is made for each class for several reasons. First, the ABM class has already been introduced earlier by Awad, Bar-Lev and Makov (2016). Secondly, the various calculations associated with the ABM class are easier and much simpler to perform, especially when they focus on the calculations of the $\mu_n$'s in (17). Finally, the simplicity of calculations makes the ABM class rather more attractive for applications and modeling (indeed, members of the ABM class are used in the numerical section as competitors to some well used distributions for count data modeling).
The third class of VFs, a subclass of (12), does not have a polynomial structure. It has the form

$$(V,M) = \left( \frac{m}{(1 - \frac{m}{p})^r}, M = (0,p) \right), \text{ where } p > 0, r \in \mathbb{N}_0,$$

and we call it as the LMNS (Letac-Mora-Non Steep) class.

Before we proceed to discuss each of the three classes separately in the subsections below, we will present a number of general comments regarding these classes (as well as any other classes too).

1. **Steepness**: The NEFs corresponding to the three classes of VFs are concentrated on $\mathbb{N}_0$, thus their convex support is $C = [0, \infty)$. Hence, the first two classes (ABM and LMS) belong to steep NEFs as their mean domain $M = \mathbb{R}^+$ coincides with $intC$. In contrast, the LMNS class is nonsteep as the corresponding mean domain $M = (0,p)$ is a proper subset of $(0, \infty)$.

2. **Infinitely divisibility and EDMs**: All of the three classes constitute infinitely divisible NEFs as they are subsets of (8) and thus the dispersion parameter space $\Lambda = \mathbb{R}_+$ (i.e., they are VFs for all $p \in \mathbb{R}_+$). Thus, as indicated above, they establish EDMs.

3. **The form of $\Theta$ and the determination of $\psi(m)$ and $\psi_1(m)$**: One needs to choose appropriately primitives $\psi(m)$ and $\psi_1(m)$ of $1/V(m)$ and $m/V(m)$, respectively, which satisfy the premises of Proposition 4.4 of LM (1990) and will allow the computation of the $\mu_n$’s in (17).

An appropriate primitive $\psi(m)$ fulfilling such premises is already available and determined by conditions (15) and (16). In order to choose a suitable primitive $\psi(m)$ we will first discuss the form of $\Theta = \psi(M)$. First we notice that set $\Theta$ is the image of $\mathbb{R}_+$ for the ABM and LMS classes and the image of $(0,p)$ of the LMNS class by the map $m \mapsto \theta = \psi(m)$. Thus, it has the form $(-\infty,q)$, for some $q \in \mathbb{R}$. Obviously, the calculation of the inverse function $m \mapsto \theta = \psi(m)$ cannot be done in an elementary way for $r > 2$ (and sometimes not also for $r = 2$). Let us also mention here a subtle point. Since LM (1990) have not chosen a particular constant of integration while computing a primitive $\psi_1$ of $\psi_1' = m/V(m)$, the generating $\mu$ of $F$, expressed by the Lagrange formula (see LM, 1990), is defined up to a multiplicative constant. Accordingly, several choices are possible for choosing $\psi_1(m)$:

i) The simplest way to choose $\psi_1(m)$ is to impose the condition by which $\lim_{\theta \to -\infty} k(\theta) = 0$, or equivalently, that $\lim_{m \to 0} \psi_1(m) = 0$ which we simply write as

$$\psi_1(0) = 0.$$

$$\text{(21)}$$

ii) If $\mu$ is bounded then one can impose conditions on $\mu$ to be a probability. The question arises, therefore, when $\mu$ is bounded. The following simple lemma (whose proof is presented, without any loss of generality, for the ABM class only) provides an answer.

**Lemma 3** The generating $\mu$ of the NEF $F$ is bounded iff $q \geq 0$.

**Proof.** $\iff$: If $q > 0$ then $k(0) < \infty$. Recall that $e^{k(0)}$ is the total mass of $\mu$. If $q = 0$, then for $\theta < 0$ we assume that $\lim_{\theta \to -\infty} k(\theta) = 0$ and write

$$k(\theta) = \int_{-\infty}^{\theta} k'(t)dt = \int_{0}^{\theta} k'(\psi(s))\psi'(s)ds = \int_{0}^{\Theta} \frac{s}{V(s)} ds,$$

$$= \int_{0}^{\Theta} \frac{ds}{(1 + \frac{1}{p})^r}.$$

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where in the last equality we used VF corresponding to the ABM class. Since \( \lim_{\theta \to 0} k'(\theta) = \infty \) we can claim that
\[
\lim_{\theta \to 0} k(\theta) = \int_0^\infty \frac{ds}{(1 + \frac{s}{r})^r} = \frac{p}{r-1}, r \geq 2.
\]
This shows that when \( q = 0 \), the total mass of \( \mu \) is \( e^{p/(r-1)} \). If \( \mu \) is normalized to make it a probability then \( \lim_{\theta \to -\infty} k(\theta) = 0 \) is no longer fulfilled after such a normalization.
\[\Rightarrow: \text{If } q < 0 \text{ the measure } \mu \text{ is unbounded since } 0 \text{ does not belong to the closure of } \Theta. \]

4. **Final Conditions on the choice of \( \psi \) and \( \psi_1 \):** in order to calculate the \( \mu_n \)'s in (14) we impose the following conditions, gathered from (15), (16) and (21), for determining the specific constants of integration when computing primitives \( \psi \) and \( \psi_1 \) fulfilling the premises of Proposition 4.4 of LM (1990). These conditions are
\[
\lim_{m \to 0} G(m) = \lim_{m \to 0} me^{-\psi(m)} = G(0) = 1, \psi'_1(0) = 1 \text{ and } \psi_1(0) = 0, \quad (22)
\]
where \( G(m) \) is defined in (10).

5. **Probability functions:** Once the primitives satisfying (22) are well determined, it is convenient to present the mean value parameterization of \( F \) in (15) by means of r.v.'s. Indeed let \( X \) be a r.v. whose distribution belongs to \( F \) in (15), then its probability function is
\[
P(X = n; m) = \mu_n \exp(\psi(m)n - \psi_1(m)), \ n = 0, 1, ..., m \in M. \quad (23)
\]

1. **Cumulants and moments:** As we have already mentioned, the cumulants (and thus also moments and central moments) of both classes ABM and LMS will also be polynomials. Their calculations are based on the following simple result (c.f., Bar-Lev, Bshouty, Enis, and Ohayon, 1992). Define an operator \( L \) acting on \( V \) by \( L(V) \equiv L_1(V) = VV' \) and \( L_n(V) = L(L_{n-1}(V)), n \in \mathbb{N}, \) with \( L_0(V) = V, \) then the \( r \)-th cumulant of \( F \), expressed in terms of \( m, \) is given by
\[
k_{r+2}(m) \equiv k^{(r+2)}(\mu(m)) = L_r(V(m)), \text{ for all } r = 0, 1, ..., \text{ and } m \in M, \quad (24)
\]
where \( k_j = k_j(m) \) stands for the \( j \)-th cumulant expressed in terms of \( m. \) Consequently, the skewness and kurtosis of \( F \) are easily obtained. The calculation of class LMNS cumulants is done using (24).

2. **The distribution of the minimal sufficient and complete statistic:** Let \( X \) be a r.v. having a distribution belonging to an NEF \( F \) with VF \((V, M)\) satisfying the premises of Proposition 1. If \((X_1, ..., X_n)\) are \( n \) independent replicas of \( X \) then \( Y_n = \sum_{i=1}^n X_i \) is a minimal sufficient and complete statistic for \( m \) (or \( \theta \)). In general, for NEFs, the distribution of \( Y_n \) is required for statistical inference, but it is not always easy to calculate. However, in the present case, the distribution of \( Y_n \) can be calculated in a manner similar to that of \( X \) by using the same conditions as in (22) and the corresponding \( \mu_n \)'s in (17), where the level of numerical complexity in calculating the distributions of \( X \) and \( Y_n \) is the same. A minor cultivation, however, is required as follows. If \((V, M)\), either of the form (11) or (12), is the VF corresponding to \( X \) with mean \( m \) then \( m^* = E(Y_n) = nm \) and \( V(Y_n) = nV(m) \), which by substituting \( m = m^*/n \) becomes \( V(Y_n) = nV(m^*/n) \). Accordingly, the VF corresponding to \( Y_n \) is \((V^*, M^*) = (nV(m^*/n), nM) \), where \( M^* = nM \) is \( \mathbb{R}^+ \) for the class in (11) and \((0, np)\) for the class in (12). The rest of the calculations of \( \psi^*, \psi_1^* \) and \( \mu_n^* \) for \((V^*, M^*)\) are completely similar to those for \( \psi, \psi_1 \) and \( \mu_n \) for \((V, M)\).
3. **Overdispersion**: Recall that in statistics, overdispersion is the presence of greater variability in a data set than would be expected based on a given statistical model. For instance, the Poisson NEF which is commonly used in practice to model count data (e.g., number of insurance claims; number of customers arriving into a queueing system). The theoretical mean and variance for the Poisson model are equal. On the other hand, in a large number of empirical data sets, the sample variance is considerably larger than the sample mean. Consequently, researchers have tried to model such data sets by families of distributions, such as the negative binomial and the generalized Poisson-Abel) distributions, for which the variance is larger than the mean. The statistical literature is full of articles on this subject, but we refrain from citing them for the sake of brevity. Now, back to our classes. Consider polynomial VF in (11) and denote by

$$V_r(m) = m \prod_{i=1}^{r} \left(1 + \frac{m}{p_i}\right), \quad r = 0, 1, ..., \text{with } \prod_{i=1}^{0} = 1,$$

and by $F_r$ be the NEF corresponding to $V_r$. Note again that $F_0, F_1$ and $F_2$ are, respectively, the Poisson, negative binomial and the Abel NEFs. Then trivially we have that the larger the $r$ the larger is $V_r$, i.e.,

$$m < V_1(m) < V_2(m) < ....$$

Firstly, the latter property indicates that all of the associated NEFs $F_i$ distributions are overdispersed with respect to the Poisson distribution, and secondly, there is an ascending order in $r$ of such an overdispersion. Similarly, this overdispersion property trivially holds also for the second class of LM (1990) given in (12). As the ABM, LMS and LMNS are subclasses of (12) or (11), they share the same overdispersion property. Moreover, one can simply realize that for any $r$ one has $V_{rABM} < V_{rLMS}$, i.e., the LMS class is more overdispersed than the ABM one.

3 Some analysis of the ABM, LMS and LMNS classes

In the following three subsections we will discuss the three classes in two aspects. One is to find explicit expressions for $\psi(m)$ and $\psi_1(m)$ that satisfy condition (22) and for which the Proposition 4.4 of LM (1990) is applicable. The second aspect is to show that the distributions of the relevant NEFs are zero-inflated with respect to the Poisson NEF and among themselves in an ascending order. Recall that a zero-inflated model is a statistical model based on a zero-inflated probability distribution, i.e. a distribution that allows for frequent zero-valued observations. In various insurance data the probability of the event of no claims during the insured period is rather large and the Poisson model does not fit. Various other models have been suggested in the realm of zero-inflated models in which the probability of zero is larger than the probability of nonzero. Such zero-inflated distributions are naturally overdispersed relative to the Poisson distribution. On this subject, too, the statistical literature is full of relevant articles, but we refrain from quoting them for reasons of brevity.

In each subsection we provide two propositions. One relates to the computations of $\psi(m), \psi_1(m)$ and $G(m)$ fulfilling condition (22); the second proposition relates to the zero-inflated property.

3.1 The ABM class

The ABM class has been first introduced by Awad, Bar-Lev and Makov (2016) and implemented for mortality projections in actuarial science. In this respect, the Lee-Carter model (Lee and Carter, 1992) and variants thereof (e.g., Renshaw and Haberman, 2006) is a largely acceptable method of
mortality forecasting. Awad, Bar-Lev and Makov (2016) have dealt with predicting mortality rates by embedding the Lee-Carter model within a Bayesian framework. They used the ABM class of counting distributions as alternatives to the Poisson counts of events (deaths) under the Lee-Carter modeling for mortality forecast and showed that members of the ABM class predicts better than the Poisson the mortality rates of elderly age people. This has been demonstrated for national data of the US, Ireland and Ukraine. Since the Bayesian approach was involved, it was not relevant there to calculate neither the constants of integration for the primitives \( \psi \) and \( \psi_1 \) (as determined by (22)) and nor the \( \mu_n \)’s in (17), as these constants and mass points are cancelled out while computing the appropriate posterior distribution (for further details see Bar-Lev and Kokonendji (2017)). They also did not demonstrate how the general expressions are obtained for \( \psi \) and \( \psi_1 \). Therefore, we will provide the appropriate proof.

**Proposition 4** Consider the ABM class given in (18) then the corresponding \( \psi(m) \), \( \psi_1(m) \) and \( G(m) \) fulfilling condition (22) have the forms

\[
\theta = \psi(m) = \log \frac{mp}{m+p} + \sum_{j=1}^{r-1} \frac{1}{j} \left[ \frac{p^j}{(m+p)^j} - 1 \right], \quad (25)
\]

\[
\psi_1(m) = \frac{p}{r-1} \left[ 1 - \left( \frac{p}{m+p} \right)^{r-1} \right], \quad r > 1, \quad (26)
\]

and

\[
G(m) = \frac{m+p}{p} \exp \left( \sum_{j=1}^{r-1} \frac{1}{j} \left[ \frac{p^j}{(m+p)^j} - 1 \right] \right). \quad (27)
\]

**Proof.** Let us begin with \( \psi(m) \). For the particular \( \mu \) mentioned Proposition 1 it must be such that \( \lim_{m \to 0} \psi(m) = -\infty \) and \( \lim_{m \to 0} G(m) = 1 \) (see (22)). As a consequence we write

\[
\psi(m) = c - \int_{m}^{\infty} \frac{dt}{t(1+t/p)^r} = c - \int_{1+\frac{m}{p}}^{\infty} \frac{dx}{(x-1)x^r}, \quad (28)
\]

where \( c \) is a constant to be determined by (22). Observe that the integral does exist since \( r > 0 \). Now, we split \( \frac{1}{(x-1)x^r} \) in partial fractions. Since 1 is a simple pole with residue 1, we express

\[
\frac{1}{(x-1)x^r} - \frac{1}{x-1} = -\frac{x^{r-1} + x^{r-2} + \ldots + x + 1}{x^r}
\]

or

\[
\frac{1}{(x-1)x^r} = \frac{1}{x-1} - \sum_{j=1}^{r} \frac{1}{x^j}.
\]

Hence

\[
\psi(m) = c - \left[ \log(x-1) - \log x + \sum_{j=1}^{r-1} \frac{1}{jx^j} \right]_{x=1+\frac{m}{p}}^{\infty}
\]

\[
= c + \log \frac{m}{m+p} + \sum_{j=1}^{r-1} \frac{p^j}{j(m+p)^j} \quad (29)
\]

and thus

\[
G(m) = (m+p) \exp \left( -c - \sum_{j=1}^{r-1} \frac{p^j}{j(m+p)^j} \right).
\]
By setting \( G(0) = 1 \) we obtain that

\[
c = \log p - \sum_{j=1}^{r-1} \frac{1}{j},
\]

and a result the primitive \( \psi(m) \) that fulfills the premises of Proposition 1 has the form \( 25 \) and \( G(m) \) has the form \( 27 \). Just note from \( 25 \) that it is not possible to express \( m \) as a function of \( \theta \), implying that the corresponding Laplace transform cannot be explicitly expressed as a function of \( \theta \). This is the situation that will prevail in the other classes of VFs under consideration.

Now, to find \( \psi_1(m) \) fulfilling \( 21 \), we write

\[
\psi_1(m) = \int \frac{dm}{(1 + \frac{m}{p})^r} = - \frac{p^r}{(r-1)(m+p)^{r-1}} + d,
\]

where the constant \( d \) is determined by the condition \( \psi_1(0) = 0 \), by which we obtain that \( d = \frac{p}{r-1} \). Consequently, the primitive \( \psi_1 \) to be used in Proposition 1 has the form \( 26 \).

Just note that by employing \( 25 \), \( 26 \) and \( 27 \) in \( 17 \), we find that for the ABM class

\[
\mu_0 = \mu_1 = 1.
\]

Now we go back to discussing the ABM class in the context of zero-inflated distributions. The probability mass at a point \( n, n = 0, 1, \ldots \), is given in \( 23 \) with \( \psi(m) \) and \( \psi_1(m) \) given in \( 25 \) and \( 26 \), respectively. Then, for \( F_r, r = 0, 1, \ldots \), the probability mass at 0 is

\[
P_r(0; p, m) = \mu_0 \exp \left( -\psi_1(r)(m) \right) = \exp \left( -\psi_1(r)(m) \right), r = 1, 2, \ldots,
\]

where \( \psi_1(r)(m) \) denote the \( \psi_1 \) function in \( 20 \) corresponding to \( F_r, r = 0, 1, \ldots \), and \( \mu_0 = 1 \) by \( 30 \).

Note that the probability at 0 of the Poisson NEF \( F_0 \) is \( e^{-m} \). We present the following proposition according to which the probability at 0 is an ascending function in \( r \), i.e., the larger the \( r \) the larger is the probability mass at zero. The meaning of this claim is that as \( r \) increases, \( F_r \) becomes more and more zero-inflated, a feature that enables the ABM class to serve as statistical models for zero-inflated data.

**Proposition 5** Consider the ABM class. Then with the notation above the ABM class satisfies

\[
e^{-m} < P_2(0; p, m) < P_3(0; p, m) < \ldots < P_r(0; p, m), r = 2, 3, \ldots
\]

**Proof.** By using \( 31 \) and \( 20 \) we have

\[
P_r(0; p, m) = \exp \left[ \frac{p}{r-1} \left( \frac{p}{m+p} \right)^{r-1} - 1 \right], r = 2, 3, \ldots,
\]

We first show \( P_r(0; p, m) > e^{-m} \) for all \( r \geq 2 \), i.e., that holds:

\[
e^{-m} < \exp \left[ \frac{p}{r-1} \left( \frac{p}{m+p} \right)^{r-1} - 1 \right], p > 0, m > 0
\]

or, equivalently, that

\[
(1 + x)^{r-1} \left[ 1 - (r-1)x \right] < 1, \text{ where } x = m/p > 0.
\]
Since $f(x) \equiv (1 + x)^{r-1} [1 - (r - 1)x]$ is strictly decreasing on $(0, \infty)$ with $f(0) = 1$, \(33\) follows. Now, we show that
\[
P_r(0; p, m) < P_{r+1}(0; p, m) \text{ for all } r \geq 2,
\]
or that
\[
\left( \frac{1}{1 + x} \right)^{r-1} - 1 < \left( \frac{1}{1 + x} \right)^r, x = m/p > 0.
\]
Multiplying both sides of the latter inequality by $(1 + x)^r$ and the simplifying, yields an inequality
\[
rx + 1 < \sum_{i=0}^{r} \binom{r}{i} x^i,
\]
which trivially holds. We have avoided showing that \(e^{-m} = P_0(0; p, m) < P_1(0; p, m)\), where \(P_1(0; p, m)\) is the probability at 0 of the negative binomial NEF for simplicity only. For showing this inequality one needs to compute \(\psi_1(m)\) for \(r = 1\) and the result is immediate. \(\blacksquare\)

### 3.2 The LMS class

**Proposition 6** Consider the LMS class given in [14] with \(b \neq p\). Then the corresponding \(\psi(m)\) and \(\psi_1(m)\) fulfilling condition \([22]\) have the forms
\[
\psi(m) = \ln \left( \frac{m}{b} \right) + (1 - \frac{b}{p})^{-r} \ln \left( \frac{m+b}{b} \right) \sum_{i=0}^{r-1} (-1)^{i+1} \binom{r}{i} \left( \frac{b}{p} \right)^i - \left( \frac{b}{p-b} \right)^r \ln \left( \frac{m}{b} + 1 \right) + b' \sum_{i=1}^{r-1} \frac{A_i}{(m+p)(p-b)} + c_0,
\]
where \(A_i = \sum_{j=i}^{r-1} (-1)^{j+r} \binom{r-j}{j} \left( \frac{b}{p} \right)^j\) and \(\sum_{i=1}^{0} = 0\),

and
\[
\psi_1(m) = b \left( \frac{p}{p-b} \right)^r \ln \left( \frac{m+b}{m+p} \right) + bp' \sum_{i=2}^{r} \frac{1}{(i-1)(p-b)r-i+1(m+p)^{r-i}} + d_0, \sum_{i=2}^{1} = 0,
\]
where
\[
c_0 = \log b - \left( 1 - \frac{p}{b} \right)^r \ln \left( \frac{b}{p} \right) \sum_{i=0}^{r-1} (-1)^{i+1} \binom{r}{i} \left( \frac{b}{p} \right)^i - b' \sum_{i=1}^{r-1} \frac{A_i}{i \cdot p' (p-b)^r} - 1
\]
and
\[
d_0 = -b \left( \frac{p}{p-b} \right)^r \ln \left( \frac{b}{p} \right) - bp' \sum_{i=2}^{r} \frac{1}{(i-1)(p-b)r-i+1p^{r-i}}.
\]

**Proof.** Bar-Lev and Kokonendji (2017) presented in their concluding section a special case of the LMS class with \(b = 1\) of the form
\[
V(m) = m(1 + m)(1 + \frac{m}{p})^r.
\]
However, they failed to correctly derive the corresponding expression for \(\psi(m)\) but correctly derived the expression for \(\psi_1(m)\). However, Accordingly, we will derive the correct expression for \(\psi(m)\) with \(b = 1\) and then use it to prove \([33]\) and then prove \([35]\). We first prove that for the special subclass \([38]\), the corresponding \(\psi(m)\) has the form

\[

\]
\[ \psi(m) = \ln m + (1 - p)^{-r} \ln (m + p) \sum_{i=0}^{r-1} (-1)^{i+1} \binom{r}{i} p^i - \left( \frac{p}{p-1} \right)^r \ln (m + 1) \]
\[ + \frac{1}{r} \sum_{i=1}^{r} \frac{A_i}{(m+p)^{i-1}} + c, \quad \text{where } A_i = \sum_{j=i}^{r-1} (-1)^{j+r} \binom{r-i}{j-i} p^j \text{ and } \sum_{i=1}^{0} = 0, \]

whereas an expression for the corresponding \( \psi_1(m) \) has the form (c.f., Bar-Lev and Kokonendji (2017))
\[ \psi_1(m) = \left( \frac{p}{p-1} \right)^r \ln \frac{m+1}{m+p} + p^r \sum_{i=2}^{r} \frac{1}{i-1} \left( \frac{i-1}{(i-1)(p-1) - (m+p)^{i-1}} \right) + d, \sum_{i=1}^{0} = 0. \]
and

$$\psi_1(m) = b\psi_{1,p/b}(m/b) + d,$$  \hfill (45)

where \(V(m)\) in (43) coincides up to a constant with (19). The proof of (34) and (45) is simple. Clearly,

$$\psi'(m)dm = \frac{dm}{V(m)} = \frac{dm}{bV_{p/b}(\frac{m}{b})} = \frac{dm'}{V_{p/b}(m')} = \psi'_{p/b}(m')dm'$$

which leads to

$$\psi(m) = \psi_{p/b}(m/b) + c,$$

and similarly to

$$\psi_1(m) = b\psi_{1,p/b}(m/b) + d.$$

Consequently, \(\psi\) and \(\psi_1\) corresponding to (19) are of the forms (34) and (35), respectively, with \(c_0\) and \(d_0\) being replaced by arbitrary constants \(c\) and \(d\). Now, by using (22) we simply find that \(c = c_0\) and \(d = d_0\) as given in (56) and (57), respectively. \(\blacksquare\)

A more convenient form of the \(\psi(m)\) and \(\psi_1(m)\) functions is

$$\psi(m) = \ln(m) + \sum_{i=1}^{r-1} c_i (m + p)^{-i} + c_r \ln(m + p) + c_{r+1} \ln(m + b) + c_0$$

$$\psi_1(m) = \sum_{i=1}^{r-1} d_i (m + p)^{-i} + d_r \ln(m + p) + d_{r+1} \ln(m + b) + d_0,$$

where the coefficients \(c_i, d_i, i = 1, \ldots, r\) are (using Newton’s binomial in the \(c_i\)’s),

$$c_i = \begin{cases} \frac{b^r}{(p-b)^r} \cdot \sum_{j=0}^{r-1} (-1)^{j+i} \binom{r}{j-i} \left(\frac{p}{p-b}\right)^j = \frac{p^i}{r} \left(1 - \left(\frac{p}{p-b}\right)^r \right)^{-i} & i = 1, \ldots, r-1; \\
\left(-\frac{p}{p-b}\right)^r \cdot \sum_{i=0}^{r-1} (-1)^{i+1} \binom{r}{i} \left(\frac{p}{p-b}\right)^i = \left(\frac{p}{p-b}\right)^r - 1 & i = r; \\
\end{cases}$$

$$d_i = \begin{cases} b^r \cdot \frac{p-b}{p-b} \cdot i^{-i} & i = 1, \ldots, r-1; \\
-b \cdot \frac{p-b}{p-b} \cdot i^r & i = r; \\
\left(b \cdot \frac{p-b}{p-b}\right) & i = r + 1. \\
\end{cases}$$

Integration constants \(c_0\) and \(d_0\) are easily obtained by

$$c_0 = - \left( \sum_{i=1}^{r-1} c_i (m + p)^{-i} + c_r \ln(m + p) + c_{r+1} \ln(m + b) \right) \bigg|_{m=0}$$

$$= - \sum_{i=1}^{r-1} c_i p^{-i} - c_r \ln p - c_{r+1} \ln b$$

$$d_0 = -Big \left( \sum_{i=1}^{r-1} d_i (m + p)^{-i} + d_r \ln(m + p) + d_{r+1} \ln(m + b) \right) \bigg|_{m=0}$$

$$= - \sum_{i=1}^{r-1} d_i p^{-i} - d_r \ln p - d_{r+1} \ln b$$
With respect to the zero-inflated property one also has a result similar to that of Proposition 1 for the ABM class. Therefore, we shall use notations similar to those used for the ABM class (just note that the case \( r = 0 \) corresponds to the negative binomial NEF). Accordingly, for \( r = 0, 1, \ldots \), we denote by \( F_i \) the NEF corresponding to \( V_i \), by \( \psi_r(m) \) and \( \psi_1(m) \) to denote the \( \psi \) and \( \psi_1 \), respectively, and by \( P_r(0; b, p, m) \) the probability mass at the point 0. as \( \mu_0 = e^{\psi_1(0)} = 1 \). Then

\[
P_r(0; b, p, m) = \mu_0 \exp (-\psi_1(m)) = \exp (-\psi_1(m))
\]

and by (35) and (37) it follows that

\[
P_r(0; b, p, m) = \exp \left\{ -\frac{p}{p-b} \ln \frac{p(m+b)}{b(m+p)} - p^r \sum_{i=2}^{r} \frac{1}{(i-1)(p-b)^{r-i+1}} \left[ \frac{1}{(m+p)^{i-1}} - \frac{1}{p^{i-1}} \right] \right\}.
\]

**Proposition 7** Consider the LMS class. Then with the notation above the LMS class satisfies

\[
P_0(0; b, p, m) < P_1(0; b, p, m) < \ldots < P_r(0; b, p, m), r = 1, 2, \ldots
\]

**Proof.** For simplicity we rewrite \( u_r = \log P_r(0; b, p, m) \) as follows

\[
u_r = -\left( \frac{p}{p-b} \right)^r \ln \frac{p(m+b)}{b(m+p)} - p^r \sum_{i=2}^{r} \frac{1}{(i-1)(p-b)^{r-i+1}} \left[ \frac{1}{(m+p)^{i-1}} - \frac{1}{p^{i-1}} \right]
\]

where

\[
A = \frac{p}{p-b} > 1, \quad B = \log \frac{b(m+b)}{p(m+p)} > 0, \quad c_i = \frac{(p-b)^i}{i} \left[ \frac{1}{p^i} - \frac{1}{(m+p)^i} \right] > 0.
\]

With these notations we have for all \( r \) the inequality

\[
u_{r+1} - u_r = A^r (A - 1) \left( B + \sum_{i=1}^{r-1} c_i \right) + A^{r+1} c_r > 0,
\]

and thus the desired result.

### 3.3 The LMNS class

The LMNS class is given by VFs of the form (20). Recall that the corresponding class of NEFs when \( r \geq 1 \) is non-steep with mean domain \((0, p)\), support \( \mathbb{N}_0 \) and convex support \([0, \infty)\). When \( r = 0 \) the corresponding VF is the Poisson one. Bryc and Ismail (2005) considered a special case \( V(m) = m/(1 - \frac{m}{p}) \) on the mean domain \((0, p)\) and compute explicitly a measure \( \mu \) such that \( \mathcal{F} = \mathcal{F}(\mu) \) is an NEF supported on \( \mathbb{N}_0 \).

We will act here in two ways. One way is similar to those used so far by computing primitives \( \psi \) and \( \psi_1 \) that fulfill condition (22). The second way is more direct by using the Lagrange formula, in much the same way as in the proof of Proposition 4.4 of LM (1990) and express the \( \mu_0 \)'s by means of Hermite polynomials. We start with the 'traditional' way and present two propositions.
Proposition 8 Consider the LMNS class given in (20) then the corresponding \( \psi(m) \), and \( \psi_1(m) \) fulfilling condition (22) have the forms

\[
\psi(m) = \ln m + \sum_{i=1}^{r} \frac{1}{i} \binom{r}{i} \left( -\frac{m}{p} \right)^i
\]

and

\[
\psi_1(m) = \frac{p}{r+1} \left[ 1 - (1 - \frac{m}{p})^{r+1} \right].
\]

Proof. Simple. One can readily verify that

\[
\theta = \psi(m) = \ln m + \sum_{i=1}^{r} \frac{1}{i} \binom{r}{i} \left( -\frac{m}{p} \right)^i + c
\]

and

\[
\psi_1(m) = -\frac{p}{r+1} (1 - \frac{m}{p})^{r+1} + d,
\]

which by employing the constraints in (22) we obtain

\[ c = 0 \text{ and } d = \frac{p}{r+1}, \]

and thus the desired result.

We now examine the zero-inflated property. We use similar notations as in the previous to cases and denote by \( P_r(X = n; p, m) = P_r(n; p, m) \) the probability at the point \( n \) of the NEF \( F_r \) associated with the VF \( V_r(m) = m/(1 - \frac{m}{p})^r, r = 0, 1, \ldots \)

Proposition 9 The LMNS class satisfies the zero-inflated property with an ascending order in \( r \) as follows

\[ e^{-m} < P_1(0; p, m) < \ldots < P_r(0; p, m), r = 1, 2, \ldots \]

Proof. As \( \mu_0 = 1 \) we have

\[
P_r(n; p, m) = \mu_n \exp \left[ \left( \ln m + \sum_{i=1}^{r} \frac{1}{i} \binom{r}{i} \left( -\frac{m}{p} \right)^i \right) n - \frac{p}{r+1} \left( 1 - (1 - \frac{m}{p})^{r+1} \right) \right],
\]

and thus

\[
P_r(0; p, m) = \exp \left[ -\frac{p}{r+1} \left( 1 - (1 - \frac{m}{p})^{r+1} \right) \right].
\]

Let us rewrite again \( u_r = \log P_r(0; p, m) \) as follows

\[
u_r = \left( \frac{1}{A^{r+1}} - 1 \right) < 0, \quad A = (1 - \frac{m}{p})^{-1} > 1.
\]

With this notation we have for \( r > 1 \):

\[
u_r - u_{r-1} = \frac{p(A - 1)}{(r+1)A^{r+1}} \left[ A^{r-1} - A^{r+1} r \right]
\]

\[
= \frac{p(A - 1)}{(r+1)A^{r+1}} \left[ A + A^2 + A^3 + \ldots + A^r - r \right] > 0.
\]

For the first inequality \( u_1 > -m \) we have just to write \( m + u_1 = \frac{m^2}{2p} > 0 \). This concludes the proof.
Remark 10 For the LMNS class we assumed that \( r \) is a natural number. However, all results obtained for this class are also correct for any real number \( r \geq 1 \) as the LMNS class of VFs can be shown to fulfill the premises of Proposition 4.4 of LM (1990). Consequently, the finite sum in (47) should be replaced by sum of entire series using the binomial series of Newton instead of the binomial formula of Pascal. Even the inequality \( (A^r - 1)/(A - 1) > r \) needed in the previous proposition, which is correct for \( A > 1 \) and a positive integer \( r \), is still correct when \( r \) is any positive number \( \geq 1 \) as

\[
\frac{A^r - 1}{A - 1} = \frac{r}{A - 1} \int_1^A t^{r-1} dt > r.
\]

However, we focus in this work only on classes for which we can obtain relatively simple expressions for both \( \psi(m) \) and \( \psi_1(m) \) in the form of finite sums and the like and not in sums of entire series. This is the reason why we have excluded the HD class (see (14)) from further consideration.

We now present a second way to compute the \( \mu_n \)'s for the NEFs corresponding to the LMNS class by means of Hermite polynomials, a way suggested to us by Gérard Letac (a personal communication).

**Proposition 11** Let \( F \) be the NEF corresponding to the VF \( m/(1 - m)^r \), \( 0 < m < p \). Then there exists a positive measure \( \nu \) on the set of positive integers \( \mathbb{N} \) such that \( F \) is generated by

\[
\mu = e^{\nu} = \delta_0 + \sum_{k=1}^{\infty} \frac{p^k}{k!} \nu^{*k},
\]

where \( \nu^{*k} \) is the \( k \)-th fold convolution of \( \nu \),

\[
\nu(n) = \frac{1}{n!n} \left[ \left( \frac{d}{dm} \right)^{n-1} e^{nP(m)} \right]_{m=0}
\]

and

\[
P(m) = -\sum_{k=1}^{\infty} \frac{(-r)_k}{k!k} m^k,
\]

where \( (-r)_k \) is the Pochhammer symbol

\[
(-r)_k = -r(-r+1)(-r+2)\cdots(-r+k-1).
\]

**Proof.** For simplicity, the proof is made for the special case \( p = 1 \) (as it similar for arbitrary \( p > 0 \)). For this case we have

\[
d\theta = \frac{dm}{V_F(m)} = (1 - m)^r \frac{dm}{m} = \frac{dm}{m} + \sum_{k=1}^{\infty} \frac{(-r)_k}{k!} m^{k-1} dm
\]

Thus \( \theta = \log m - P(m) \) which by denoting \( w = e^\theta \) we get \( m = we^{P(m)} \). Now apply the Lagrange formula which states that if \( h(w) = wg(h(w)) \) then

\[
h(w) = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[ \left( \frac{d}{dm} \right)^{n-1} (g(m))^n \right]_{m=0}.
\]

When applying this formula to \( m = h(w) = k'_u(\theta) \) and \( g(m) = e^{P(m)} \) we get

\[
k'_u(\theta) = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[ \left( \frac{d}{dm} \right)^{n-1} e^{nP(m)} \right]_{m=0}.
\]
Since $d\theta = dw/w$ we obtain

$$k_\mu(\theta) = \sum_{n=1}^{\infty} \frac{w^n}{n!n} \left( \frac{d}{dm} ight)^{n-1} e^{nP(m)} m_0 = \sum_{n=1}^{\infty} \nu(n)w^n,$$

and the remainder of the proof is standard. ■

**Example 12** For $r = 1$, $P(m) = m$ and $\nu(n) = n^{-2}/n!$.

**Example 13** For $r = 2$, $P(m) = 2m - m^2/2$ but the computation of

$$\left[ \left( \frac{d}{dm} \right)^{n-1} e^{n(2m-m^2/2)} \right]_{m=0}$$

is more delicate. For such a computation with use the formula for Hermite polynomials (see, Ranville, 1960, p.130) by which

$$e^{2xt-t^2} = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}.$$  

Setting $x = \sqrt{2n}$ and $t = \sqrt{n/2m}$ yield

$$e^{n(2m-m^2/2)} = \sum_{k=0}^{\infty} H_k(\sqrt{2n}) \left( \frac{n}{2} \right)^{k/2} \frac{m^k}{k!}.$$  

By employing the Taylor formula it follows that

$$\left[ \left( \frac{d}{dm} \right)^{n-1} e^{n(2m-m^2/2)} \right]_{m=0} = H_{n-1}(\sqrt{2n}) \left( \frac{n}{2} \right)^{(n-1)/2}$$

and thus

$$\nu(n) = \frac{1}{n!n} H_{n-1}(\sqrt{2n}) \left( \frac{n}{2} \right)^{(n-1)/2}.$$  

### 4 A numerical example

In this section we show that the ABM class is very well suited for fitting small counting data. Consider the well-known data sets of automobile insurance claims per policy over a fixed period of time that have been studied in Gossiaux and Lemaire (1981) for fitting Poisson and the negative Binomial distributions, Willmot (1987) for fitting the Poisson-inverse Gaussian distribution, and Gomez-Deniz and Calderin-Ojeda (2011) for fitting the Lindley distribution. The parameters of the fitted distributions are computed by maximum likelihood estimation.

We compare these four models with the ABM distributions in case of $r = 2, \ldots, 5$. Our parameters are estimated by moment fitting. The data set we chose contains just six counting values, $n_{i}^{\text{obs}}, i = 0, \ldots, 5$ is the number of policy holders that claimed $i$ times in the specified period. The first row in Table 1 lists these values. The other rows are the corresponding expected numbers from the fitting models.

Observe that the data show over-dispersion and zero-inflation: mean is 0.0865, variance is 0.1225, and $p_0 = 0.93$. In order to compare the fitting models we first compute four distance measures.

1. $l_2$ norm = $\sqrt{\sum_{k=0}^{5} \left( p_{k}^{\text{obs}} - p_{k}^{\text{exp}} \right)^2}$.  

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Table 1: Data set and 8 associated fitting models: Poisson (P), negative binomial (NB), Poisson-inverse Gaussian (PIG), Lindley (L), ABM $r = 2, \ldots, 5$.

| no. | data  | 0  | 1  | 2  | 3  | 4  | 5  |
|-----|-------|----|----|----|----|----|----|
| P   | 3719  | 3668.54 | 317.33 | 13.72 | 0.40 | 0.01 | 0.00 |
| NB  | 3719.22 | 3719.59 | 230.43 | 38.51 | 8.50 | 2.14 | 0.58 |
| PIG | 3718.58 | 234.54 | 34.86 | 8.32 | 2.45 | 0.80 | 0.00 |
| L   | 3676.17 | 302.46 | 20.08 | 1.21 | 0.07 | 0.00 | 0.00 |
| $r = 2$ | 3719.59 | 230.43 | 38.51 | 8.50 | 2.14 | 0.58 | 0.00 |
| $r = 3$ | 3719.13 | 231.48 | 37.94 | 8.41 | 2.16 | 0.61 | 0.00 |
| $r = 4$ | 3718.90 | 232.00 | 37.66 | 8.37 | 2.17 | 0.62 | 0.00 |
| $r = 5$ | 3718.77 | 232.32 | 37.50 | 8.34 | 2.18 | 0.63 | 0.00 |

Table 2: Distance measures of the fitting models.

|       | $l_2$  | TV     | RMSE  | Kullback-Leibler |
|-------|--------|--------|-------|------------------|
| P     | 0.02558 | 0.02133 | 41.77 | 0.01584          |
| NB    | 0.0008492 | 0.0009064 | 1.387 | 0.0002052        |
| PIG   | 0.001078 | 0.001021 | 1.760 | 0.0001987        |
| L     | 0.02116  | 0.01762 | 34.55 | 0.008742         |
| $r = 2$ | 0.0006238 | 0.0006803 | 1.019 | 0.0001659        |
| $r = 3$ | 0.0004432 | 0.0004192 | 0.7237 | 0.0001601        |
| $r = 4$ | 0.0004199 | 0.0003763 | 0.6858 | 0.0001589        |
| $r = 5$ | 0.0004339 | 0.0004475 | 0.7086 | 0.0001588        |

2. Total variation $= \frac{1}{2} \sum_{k=0}^{5} |p_{k}^{\text{obs}} - p_{k}^{\text{exp}}|$.

3. RMSE $= \sqrt{\frac{1}{6} \sum_{k=0}^{5} (n_{k}^{\text{obs}} - n_{k}^{\text{exp}})^2}$.

4. Kullback-Leibler divergence $= \sum_{k=0}^{5} p_{k}^{\text{obs}} \log \frac{p_{k}^{\text{obs}}}{p_{k}^{\text{exp}}}$.

Furthermore, we performed a Pearson’s chi-squared test,

$$\sum_{k=0}^{m} \frac{(b_{k}^{\text{obs}} - b_{k}^{\text{exp}})^2}{b_{k}^{\text{exp}}},$$

where $b_{k}^{\text{obs}}$ is the number of data in the $k$-th category. The categories are, in case of Poisson and Lindley $\{0\}, \{1\}, \{2\}, \{3, 4, 5\}$, and for all other distributions $\{0\}, \{1\}, \{2\}, \{3\}, \{4, 5\}$. Note that the highest category contains less than 5 expected counts. But we decided to do this because then the degrees of freedom become 2 (5 categories – 1 – number of estimated parameters), see also Willmot (1987). From the chi-square quantile we computed the p-value at an 0.05 significance level.

We may conclude that overall the ABM models, and more specifically the $r = 4$ and $r = 5$ models, perform the best. Although it can not be theoretically proven, it seems very intuitive to us that as $r$ increase, so does the p-value.
Table 3: Chi-squared tests of the fitting models.

|     | chi-squared | df | p-value |
|-----|-------------|----|---------|
| P   | 344.2       | 2  | ≈ 0     |
| NB  | 1.172       | 2  | 0.5565  |
| PIG | 0.5438      | 2  | 0.7619  |
| L   | 106.5       | 2  | ≈ 0     |
| r = 2 | 0.6412   | 2  | 0.7257  |
| r = 3 | 0.5451   | 2  | 0.7614  |
| r = 4 | 0.5099   | 2  | 0.7750  |
| r = 5 | 0.4929   | 2  | 0.7816  |

5 Concluding remarks

1. In this paper we have attempted in exposing ‘new’ EDMs of distribution supported on the set of nonnegative integers. Such EDMs can be represented only by their mean value parametrization whereas their respective generating measure can be computed via (17) by existing powerful mathematical software. The expressions obtained for the $\mu_n$’s will depend, of course, on the unknown parameters $p$ and/or $b$. Based on a random sample, the MLE is the sample mean whereas the parameters $p$ and $b$ can be estimated by the method of moments estimation. All that is said above depends, of course, on the ability to locate classes of VFs of the form (11) or (12) for which both $\psi$ and $\psi_1$ possess explicit and ‘nice expressions in terms of $m$. In such a case the likelihood function is well expressed, a fact that has a tremendous significance in statistics. Obviously, if such EDMs are used in a Bayesian framework there is no need to compute the $\mu_n$’s.

2. From the presentation of these three classes, it will be easy to see that more classes of the same type (i.e., subclasses of either (11) or (12)) can be constructed. However, we will suffice with presenting only the three classes.

3. The classes of EDMs introduced in this paper can be used, for example, as competitors and alternatives to the Poisson or negative binomial NEFs for modeling count data in various actuarial aspects and insurance claims. This has been indeed demonstrated in the numerical section. However, based on our experience in the insurance and actuarial industry, we have noticed that professionals are very concerned about using new (both discrete and continuous) distributions to estimate and evaluate various relevant parameters as the insurance risk factor. So in another paper of ours (Bar-Lev and Ridder, 2019) we considered, just for the sake of demonstration, the problem of computing the insurance risk factor

$$\ell(x) = P\left(\sum_{k=0}^{N} Y_k > x\right),$$

for large values of $x$, where $N$ is a r.v. counting the number of claims during a fixed period of time and the $Y_i$’s are the respective independent claim sizes. The conventional actuarial literature is full with models in which $N$ has either Poisson or negative binomial distributions whereas the $Y_i$’s have a common gamma or inverse Gaussian or even positive stable distribution. Bar-Lev and Ridder (2019) used ‘unconventional’ NEF distributions for $N$ by taking the Abel, strict arcsine and Takács NEFs (i.e., NEFs having cubic VFs characterized by LM, 1990). For data of a Swedish claims at a car insurance company they considered all combinations of the distributions of $N$ and the $Y_i$’s mentioned above and demonstrated that the best
fit for such data is obtained for the pair (arcsine, positive stable) with p-value equals .7460. All fit ranking after are, respectively, (arcsine, inverse Gaussian, p-value 0.4224), (Takács, gamma, p-value 0.4159), (Abel, positive stable, p-value 0.3089), (Takács, inverse Gaussian, p-value 0.2800), (Takács, positive stable, p-value 0.2701), (Abel, inverse Gaussian, p-value 0.2459) and (Abel, gamma, p-value 0.2101). As opposed to these, the worst fit has been obtained for pairs of the Poisson along with the gamma, inverse Gaussian and positive stable distributions with p-value less than .00001.

4. Consequently, we trust the ABM, LMS and LMNS classes (as well as other similar classes) are going to play a role of a ‘new generation’ of counting distributions and to have a ‘prosperous future’ in applications to actuarial science data as well as to other statistical data. Indeed, the present authors are now pursuing a project in which dozens of sets of count data from the statistical literature were collected. Such data were modeled by some conventional discrete distributions whereas we are attempting to fit either the ABM, LMS or LMNS distributions in order to test which of these models, conventional or ‘unconventional’ provide a better fit for these data.

5. One last remark. Researchers may avoid using the LMNS class as it is non-steep. However, another important class of NEFs having power VFs of the form \((V, M) = (\alpha m^\gamma, \mathbb{R}_+^\gamma), \alpha > 0, \gamma < 0\) (which belong to the Tweedie scale) is also non-steep. Indeed, for the latter class \(M = \mathbb{R}_+^\gamma\) whereas its convex support \(C = \mathbb{R}\). This class though is frequently used in various applications.

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