The massive modular Hamiltonian

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Abstract

We compute the vacuum local modular Hamiltonian associated with a space ball region in the free scalar massive Quantum Field Theory. We give an explicit expression on the one particle Hilbert space in terms of the massive Legendre differential operator and the Green integral for the Helmholtz operator. The quadratic form of the massive modular Hamiltonian is expressed in terms of an integral of the energy density with parabolic distribution and of a Yukawa potential, that here appears intrinsically. We then get the formula for the local entropy of a Klein-Gordon wave packet. This gives the vacuum relative entropy of a coherent state on the double cone von Neumann algebras associated with the free scalar QFT. Among other points, we provide the passivity characterisation of the modular Hamiltonian within the standard subspace set up.

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1 Introduction

In this paper we provide the formula for the vacuum modular Hamiltonian associated with the free massive scalar quantum field in a bounded spacetime region (double cone), an open problem in Quantum Field Theory that is standing up since the Tomita-Takesaki modular theory uncovered in the 70’s the intrinsic modular evolution associated with a normal state of a von Neumann algebra, see [35].

The Bisognano-Wichmann theorem [6] showed the geometric nature of the modular flow for the wedge region case, in a general framework. The bounded region problem has then been subject of work by several researchers over the years, see e.g. [8, 13, 15, 33, 34, 37]. The underlying picture in our approach is to consider the massive set up as a deformation of the massless one. Our analysis is valid in arbitrary spacetime dimension \(d + 1\), with \(d > 1\).

**Background.** Let \(\mathcal{M}\) be a von Neumann algebra and \(\varphi\) a faithful normal state on \(\mathcal{M}\). As is well known, the Tomita-Takesaki modular theory provides us with a canonical one-parameter group of \(\mathcal{M}\) associated with \(\varphi\), the modular group \(\sigma^\varphi\). Thus the quantum system \(\mathcal{M}\) is equipped with an intrinsic evolution \(\sigma^\varphi\), that is characterised by the KMS thermal equilibrium condition [35]. In the GNS representation, the modular group is implemented by a unitary one-parameter group \(\Delta^\varphi\), whose generator \(\log \Delta^\varphi\) is the modular Hamiltonian associated with \(\varphi\) (see [26] for a discussion on the matter).

In Quantum Field Theory, for each spacetime region \(O\), we have the von Neumann algebra \(\mathcal{A}(O)\) of observables localised in \(O\). By the Reeh-Schlieder theorem, the vacuum vector is cyclic and separating for \(\mathcal{A}(O)\) if both \(O\) and its causal complement have non empty interiors, see [18]. Thus the restriction of vacuum state \(\varphi\) to \(\mathcal{A}(O)\) is faithful and the modular Hamiltonian associated with \(\mathcal{A}(O)\) and \(\varphi\) gives us a local Hamiltonian \(\log \Delta_O\).

The problem of computing \(\log \Delta_O\) is then natural. Among other motivations, the modular Hamiltonian is related to Araki’s relative entropy [2], that is recently playing an important role in Quantum Field Theory, also in relation with quantum entropy/energy inequalities; an account of this wide research topic goes beyond the purpose of this introduction (see [3, 10, 24, 28, 40] and refs. therein).

As said, if \(W\) is a wedge region, there is an important model independent result [6]: \(\Delta_W^{\times 2\pi}\) is identified with the \(2\pi\)-rescaled boost unitary transformations leaving \(W\) globally invariant, a result rich of consequences, see [18]. For a different spacetime region \(O\), the understanding of the local modular structure is definitely more problematic because, in general, \(\Delta_O\) has no geometric description due to the lack of enough spacetime symmetries. A basic issue concerns the case where \(O\) is a double cone, the causal envelop of a time-zero ball. In the free massless QFT, the geometric description of the double cone vacuum modular group was derived in [19]. The general conformal case was later analysed solely in terms of the local von Neumann algebras [17][7]. Among other results, we mention here the forward light cone case in the free massless QFT [8] and the approximate local estimates in [16].

The modular theory has a version for standard subspaces, see [21, 32, 22]. Let \(\mathcal{H}\) be a complex Hilbert space and \(H \subset \mathcal{H}\) a standard subspace, i.e. \(H\) is a real linear, closed subspace of \(\mathcal{H}\) such that \(H \cap iH' = \{0\}\) and \(\mathcal{H} + i\mathcal{H} = \mathcal{H}\), with \(H'\) the symplectic complement of \(H\). The modular operator \(\Delta_H\) associated with \(H\) is a canonical positive, non-singular selfadjoint operator on \(\mathcal{H}\) associated with \(H\) that satisfies

\[
\Delta_H^s H = H, \quad s \in \mathbb{R}.
\]
In a free scalar QFT, the vacuum modular (unitary) group associated with the von Neumann algebra of a region $O$ is the second quantisation of a unitary one-parameter group in the one-particle Hilbert space, indeed of the modular group of the local standard subspace $H(O)$. We shall henceforth denote by $\Delta_O$ the modular operator $\Delta_{H(O)}$ on the one particle Hilbert space. Also, we set $H(B) = H(O)$, $\Delta_B = \Delta_O$ if $O$ is the double cone with basis the unit space ball centred at origin $B$. Hence, in this paper

**Modular Hamiltonian of $B = \log \Delta_B$.**

**Massless Hamiltonian.** In the massless case, $\Delta_B^{is}$ is associated with a one parameter group of conformal transformations that globally preserve $O$ [19]. Nonetheless, the description of $\log \Delta_B$ has not been worked out so far. This will be our starting point.

In terms of the wave Cauchy data, we shall see that the local massless modular Hamiltonian is given by

$$\log \Delta_B = -2\pi i \begin{bmatrix} 0 & \frac{1}{2}(1 - r^2) \nabla^2 - r \partial_r - D \\ \frac{1}{2}(1 - r^2) & 0 \end{bmatrix},$$

(1)

with $B$ the unit space ball and $D = (d - 1)/2$ the scaling dimension of the free scalar field. Namely

$$\log \Delta_B = -2\pi i \begin{bmatrix} 0 & M \\ L & 0 \end{bmatrix},$$

(2)

with

$$M = \text{Multiplication operator by } \frac{1}{2}(1 - r^2),$$

(3)

$$L = \text{Legendre operator } \frac{1}{2}(1 - r^2) \nabla^2 - r \partial_r - D.$$

(4)

As we shall see, the right hand side of (1) gives indeed an essentially selfadjoint operator on the one particle Hilbert space on the smooth, compactly supported function domain.

We shall derive the following formula for the massless modular Hamiltonian of $B$ in terms of the classical stress-energy tensor $T$:

$$-(\Phi, \log \Delta_B \Phi) = 2\pi \int_{x_0 = 0} \frac{1 - r^2}{2} (T_{00})_\Phi(x) dx + \pi D \int_{x_0 = 0} \Phi^2 dx,$$

(5)

with $\Phi$ a real wave with smooth, compactly supported Cauchy data.

The right hand side of (5) is similar to a formula for the modular Hamiltonian sketched by Casini, Huerta and Myers [9, (2.23)] in terms of the QFT stress-energy tensor, when the Cauchy data are supported in $B$.

**Massive Hamiltonian.** The above formula (5) gives a first hint for the structure of the massive modular Hamiltonian: one would be tempted to replace the massless stress energy tensor $T$ with the massive stress energy tensor $T^{(m)}$. This would amount to replace in (1) the Legendre operator with its massive perturbation $\frac{1}{2}(1 - r^2)(\nabla^2 - m^2) - r \partial_r - D$.

This guess leads only to a part of the formula. Here we are at a delicate point. The quadratic form of $K_m$ restricts to the quadratic form of $\log \Delta_B$ on $C^\infty_0 (B)$. As we shall
see, in the massive case, on $C^\infty_0(B)$, we have \[ \frac{1}{2\pi} \log \Delta_B = K^B_m \]

\[
K^B_m = \begin{bmatrix}
0 & M \\
L_m & 0
\end{bmatrix},
\]

with \[
L_m = \frac{1}{2} (1 - r^2)(\nabla^2 - m^2) - r \partial_r - D - \frac{1}{2} m^2 G^B_m.
\]

here $G^B_m$ is the operator associated with the quadratic form of the restriction of the inverse Helmholtz operator $H_m$

\[
H_m = -\nabla^2 + m^2.
\]

$G^B_m$ is given by a Green function integral; in particular, in $d = 3$ space dimensions,

\[
G^B_m f(x) = \frac{1}{4\pi} \int_B \frac{e^{-m|x-y|}}{|x-y|} f(y) dy.
\]

Then $\log \Delta_B$ is determined by complex linearity on a dense set of the Hilbert space which is a domain of essential selfadjointness. One important point is that $K^B_m$ is a compact perturbation of $K^B_0$.

Due to the presence of the term proportional to $G^B_m$, $i \log \Delta_B$ does not act locally on $B$. Yet it acts locally on a large subspace of waves, those with Cauchy data $f, g$ supported in $B$ with $f$ is the propagator $f = (\nabla^2 - m^2) h$ of a compactly supported function $h$.

One remarkable point here is the intrinsic appearance of the Yukawa potential $e^{-mr}/4\pi r$, a fact that deserves further physical insight [41].

The quadratic form $-(\Phi, \log \Delta_B \Phi)$ on $H(B)$ is the entropy $S_\Phi$ and is expressed in terms of the energy density of $\Phi$ as we shall now explain.

The measure of information. One main consequence of our analysis is a formula for the entropy density carried by a wave packet. Let’s explain the framework.

Suppose $\Phi$ is a scalar wave packet, a solution of the wave or Klein-Gordon equation $(\Box + m^2) \Phi = 0$. At given time, we can measure the signal contained say in space ball $B$. The quantity $S_\Phi$ that represents the mean information stored by $\Phi$ in $B$ at that time is called the entropy of $S_\Phi$ with respect of $B$ and has been introduced in [24, 25, 11], although different space regions (wedges) were there considered.

With $\mathcal{H}$ a complex Hilbert space and $H \subset \mathcal{H}$ a factorial standard subspace, one defines the entropy of a vector $k \in \mathcal{H}$ with respect to $H$ as

\[
S_k = \Im(k, P_H i \log \Delta_H k).
\]

Here, $P_H : H + H' \to H$, $P_H : h + h' \mapsto h$, is the cutting projection associated with $H$, $\Delta_H$ is the modular operator associated with $H$ and $H'$ is the symplectic complement of $H$.

Motivated by Quantum Field Theory, one equips the waves’ real linear space with a complex Hilbert space structure, where the imaginary part of the scalar product is given by the time independent symplectic form

\[
\Im(\Phi, \Psi) = \frac{1}{2} \int_{x_0 = t} (\Psi \partial_0 \Phi - \Phi \partial_0 \Psi) dx.
\]
Then one considers the local net of standard subspaces associated of the resulting \( \mathcal{H}: H(B) \) is the closure of the real linear space of waves with Cauchy data supported in \( B \). So one defines \( S_\Phi \) as the entropy of the vector \( \Phi \) with respect to \( H(B) \). We have

\[
S_\Phi = \pi \int_B (\Psi \Phi' - \Phi \Psi') dx ,
\]

(time zero integral) with \( \Psi = i \log \Delta_B \Phi \) and the prime denotes the time derivative and \( \Delta_{H(B)} = \Delta_B \).

One then have to compute (9). In [25.11], this computation has been worked out in the case of a Klein-Gordon wave for a half-space region (whose causal envelop is a wedge), a case motivated by the a study of the Quantum Null Energy Condition inequality.

From the point of view of information theory, it is however natural to consider the case \( B \) is a bounded region. Note that the computation in [25.11] relies on the explicit knowledge of the modular Hamiltonian \( \log \Delta_W \).

Now, \( P_{H(B)} \) acts by cutting the Cauchy data [11], so we have at our disposal all the ingredients to compute the local entropy \( S_\Phi(R) \) of the wave packet \( \Phi \) in the (causal envelop of) the radius \( R \) space ball \( B_R(\bar{x}) \) around the space point \( \bar{x} \), at time \( t \). We shall see that \( S_\Phi(R) \) is the sum of three terms:

\[
S_\Phi(R) = \pi \int_{B_R(\bar{x})} \frac{R^2 - r^2}{R} (T_{00}^{(m)}(t, x))_\Phi dx
+ \pi \frac{d - 1}{2R} \int_{B_R(\bar{x})} \Phi^2(t, x) dx
+ \pi \frac{m^2}{R} \int_{B_R(\bar{x}) \times B_R(\bar{x})} G_m(x - y)\Phi(t, x)\Phi(t, y) dxdy
\]

with \( r = |x - \bar{x}| \) and \( T_{00}^{(m)} \) the energy density of \( \Phi \):

\[
T_{00}^{(m)} = \frac{1}{2} (\frac{1}{2} |(\partial_0 \Phi)|^2 + |\nabla x \Phi|^2 + m^2 \Phi^2).
\]

The first term is the main one. The last term is a sort of boundary effect, it does not survive when the region \( B_R(\bar{x}) \) approaches a half-space. We however expect further physical insight to be spelled out in this regard.

We note here the appearance of the parabolic distribution \( \frac{1}{2}(1 - r^2) \) in stress-energy term of the formula for the modular Hamiltonian. We wonder about possible deep roots for this, somehow similarly with the appearance of Wigner semicircular distribution [39] in the Free Probability framework [38]. The parabolic distribution in three dimensional space is a higher-dimensional generalisation of the Wigner semicircular distribution and is related to the marginal distribution of a spherical distribution.

**Content of this paper.** Our paper is organised as follows. First, we collect functional analytic results. Among these, we have a standard subspace version of the Pusz and Woronowicz [31] complete passivity characterisation, up to a proportionality constant, of the modular Hamiltonian. This reflects the second principle of thermodynamics. By our results, our local Hamiltonian is completely passive in our standard subspace sense.
Then we describe the massless modular group in the wave set up, so we get an explicit description of the massless Hamiltonian. Next, we proceed to the massive case by a series of deformation arguments. We then provide our local entropy formula for a Klein-Gordon wave packet and discuss some of the implications in Quantum Field Theory.

Our general references for Operator Algebras and Quantum Field Theory are [14, 18, 35].

2 Abstract set up

We begin with abstract, functional analytical results.

2.1 Stability of skew-selfadjointness

Let $H$ be a real Hilbert space and $T : D(T) \subset H \to H$ a (real) linear operator with $D(T)$ dense in $H$.

We shall say that $T$ is skew-Hermitian if $T \subset -T^*$ and skew-selfadjoint if $T = -T^*$.

Let $H_C$ be the usual complexification of $H$, so $H_C = H \oplus iH$ with the usual complex structure. Then $T$ extends to a complex linear operator $T_C$ on $H_C$ and one can apply most results for complex linear operators.

In particular, $T$ is the infinitesimal generator of a one-parameter group of orthogonal operators on $H$ iff $T$ is skew-Hermitian and $\text{ran}(K \pm 1) = H$.

Recall that $T$ is a Fredholm operator if $T$ is closed and $\dim \ker(T), \text{codim ran}(T) < \infty$; then the index is defined by

$$\text{ind}(T) = \dim \ker(T) - \text{codim ran}(T).$$

We shall use the following stability result [20, Chapter IV]: if $P : H \to H$ is a linear operator:

$T$ Fredholm, $P$ compact $\implies T + P$ Fredholm & $\text{ind}(T + P) = \text{ind}(T)$.

Let $H'$ be a Hilbert space equivalent to $H$, namely $H$ and $H'$ are the same linear space with equivalent norms. Clearly, we may regard $T$ also as an operator on $H'$.

**Proposition 2.1.** Let $H, H'$ be as above, $K$ a skew-selfadjoint operator on $H$ and $P : H \to H$ a compact linear operator.

If $K + P$ is skew-Hermitian on $H'$, then $K + P$ is skew-selfadjoint on $H'$.

**Proof.** By considering the complexification $K_C$, we may assume that our operators act on a complex Hilbert space and are complex linear.

As $K$ is skew-selfadjoint on $H$, $K \pm 1$ is invertible. As $P$ is compact,

$$\text{ind}(K + P \pm 1) = \text{ind}(K \pm 1) = 0.$$

Clearly the index as operator on $H$ and on $H'$ is the same. Now, $K + P$ is Hermitian as operator on $H'$, so $\ker(K + P \pm 1) = \{0\}$. Therefore $\text{ran}(K + P \pm 1) = H$. We conclude that $K + P \pm 1$ is invertible, thus $K + P$ is skew-selfadjoint on $H'$. \qed
2.2 Real linear invariant subspaces

We now characterise the closed real linear subspaces of a complex Hilbert space $\mathcal{H}$ that are invariant for a one parameter unitary group.

**Proposition 2.2.** Let $\mathcal{H}$ be a Hilbert space, $H \subset \mathcal{H}$ a closed, real linear subspace and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ a selfadjoint operator. With $V(s) = e^{isA}$, $s \in \mathbb{R}$, $K = iA$ and $C$ (resp. $B$) the real algebra of complex, continuous functions $g$ on $\mathbb{R}$ vanishing at $\infty$ (resp. complex, bounded Borel functions on $\mathbb{R}$) such that $g(-t) = \bar{g}(t)$, the following are equivalent:

(i) $V(s)H = H$, $s \in \mathbb{R}$,

(ii) $g(A)H \subset H$, $g \in C$,

(iii) $g(A)H \subset H$, $g \in B$,

(iv) $(K^2 - 1)^{-1}H \subset H$ and $K(K^2 - 1)^{-1}H \subset H$,

(v) $(K \pm 1)^{-1}H \subset H$,

(vi) $K|_{H}$ is skew-selfadjoint on $H$, namely $D(K) \cap H$ is dense in $H$, $K(D(K) \cap H) \subset H$ and $K : (D(K) \cap H) \subset H \to H$ is skew-selfadjoint.

**Proof.** (i) $\Rightarrow$ (v): Let $h$ be a real $L^1$-function on $\mathbb{R}$ and $g = \hat{h}$ its Fourier transform. Then

$$g(A) = \int_{\mathbb{R}} h(s)V(s)ds.$$  

If (i) holds true, then $g(A)H \subset H$ for such $g$. In particular (v) holds, similarly as in (10).

(v) $\Rightarrow$ (iv) follows by the identities

$$2K(K^2 - 1)^{-1} = (K + 1)^{-1} + (K - 1)^{-1}, \quad 2(K^2 - 1)^{-1} = (K + 1)^{-1} - (K - 1)^{-1}.$$  

(iv) $\Rightarrow$ (ii): The complexification $\mathcal{C}_C$ of $\mathcal{C}$ is the $C^*$-algebra $C$ of continuous functions on $\mathbb{R}$ vanishing at infinity, namely every $F \in C$ is written uniquely as

$$F = f + ig, \quad f, g \in \mathcal{C}, \quad f(t) = \frac{1}{2}[F(t) + \bar{F}(-t)], \quad g(t) = \frac{1}{2i}[F(t) - \bar{F}(-t)].$$

Note that $\mathcal{C}$ contains the real algebra $\mathcal{C}_0$ of functions of the form $f(t) = p((t^2 + 1)^{-1}, it(t^2 + 1)^{-1})$ with $p$ a two-variables polynomial with real coefficients and zero constant coefficient. Note that $\mathcal{C}$ and $\mathcal{C}_0$ are closed with respect to complex conjugation.

Since $\mathcal{C}_0$ separates the points of $\mathbb{R}$, given any $f \in \mathcal{C}$, there exists a sequence $F_n = f_n + ig_n$ with $f_n, g_n \in \mathcal{C}_0$ such that $F_n \to f$ uniformly on $\mathbb{R}$, thus $f_n \to f$ uniformly on $\mathbb{R}$. We conclude that $\mathcal{C}_0$ is norm dense in $\mathcal{C}$. Therefore we have (ii).

(ii) $\Rightarrow$ (iii): Let $g \in \mathcal{B}$. Given $h \in H$, $k \in H^\perp$ (real orthogonal), by Lusin’s theorem there exists a bounded sequence of continuous functions $g_n$ such that $g_n \to g$ almost everywhere w.r.t. the spectral measure of $A$ associated with $h, k$. By replacing $g_n(t)$ with $g_n(t) + \bar{g}_n(-t)$, we may assume that $g_n \in \mathcal{C}$. By Lebesgue’s dominated convergence theorem, we have $(k, g_n(A)h) \to (k, g(A)h)$. As $\Re(k, g_n(A)h) = 0$, we have $\Re(k, g(A)h) = 0$, that implies $g(A)H \subset H$ because $h, k$ can be arbitrarily chosen.
(vi) ⇒ (v): $K|_H$ generates a one-parameter group $V_H$ of orthogonal operators on $H$, therefore

$$(1 ± K)^{-1}H = -\int_0^\infty e^{-s}V_H(±s)ds H \subset H . \quad (10)$$

Clearly (i) ⇒ (vi), (iii) ⇒ (ii) and (iii) ⇒ (i). □

If $A$ and $H$ satisfy the condition is the above proposition, we shall say that $H$ is $iA$-invariant.

Let $\mathcal{H}$ be a complex Hilbert space. A standard subspace $H$ of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ with

$\bar{H} + iH = \mathcal{H}, \quad H \cap iH = \{0\} . \quad (10)$

With $H$ a standard subspace of $\mathcal{H}$, let $K_H : D(K_H) \subset H \to H$ be a real linear, densely defined operator. Then we can extend $K_H$ to a complex linear operator $K : D(K) \subset \mathcal{H} \to \mathcal{H}$, with dense domain $D(K) = D(K_H) + iD(K_H)$, defined by

$K(h + ik) = K_Hh + iK_Hk , \quad h, k \in D(K_H) . \quad (10)$

We shall say that $K_H$ is $\partial$-symplectic if the real quadratic form $\beta(h, K_Hk)$, $h, k \in D(K_H)$, is symmetric, with $\beta(h, k) = \Re(h, k)$. Clearly, the infinitesimal generator of a one-parameter continuous group of symplectic maps is $\partial$-symplectic.

**Corollary 2.3.** With $K_H$ and $K$ as above, $K$ is skew-hermitian on $\mathcal{H}$ iff $K_H$ is skew-Hermitian on $H$ w.r.t. the real part of the scalar product and $\partial$-symplectic on $D(K_H)$.

In this case, if $K_H$ is essentially skew-selfadjoint on $H$, $K$ is essentially skew-selfadjoint on $\mathcal{H}$ and the skew-selfadjoint closure $\tilde{K}$ of $K$ leaves $H$ invariant as in Prop. 2.2

**Proof.** Note that $(h, K_Hk) = -(K_Hh, k)$ for all $h, k \in D(K_H)$ iff $K_H$ is skew-Hermitian and $\partial$-symplectic on $D(K_H)$. In this case, we now check that $K$ is skew-hermitian on $D(K)$. Indeed,

$$(h + ik, K(h + ik)) = (h + ik, K_Hh + iK_Hk) = (h, K_Hh) + (k, K_Hk) − i(k, K_Hh) + i(h, K_Hk)$$

$$= i(k, K_Hh) − i(h, K_Hk) = −i(K_Hk, h) + i(K_Hh, k) = −(K(h + ik), h + ik) .$$

Concerning the last part of the statement, assume that $K_H$ is essentially skew-selfadjoint on $H$ and $\partial$-symplectic on $D(K_H)$. Then, the range of $K_H \pm 1$ is dense in $H$, which implies that the range of $K \pm 1$ is dense in $\mathcal{H}$, so $K$ is essentially skew-selfadjoint and $\tilde{K}$ is skew-selfadjoint. Since $K$ extends $K_H$, we have that $(K \pm 1)^{-1}$ extends $(K_H \pm 1)^{-1}$, therefore

$$(K \pm 1)^{-1}\text{ran}(K_H) = (K_H \pm 1)^{-1}\text{ran}(K_H) = D(K_H) \subset H .$$

As $\text{ran}(K_H)$ is dense in $H$, we then have $(K \pm 1)^{-1}H \subset H$, thus the corollary follows by Prop. 2.2. □

Note now that, if $T_1, T_2$ are bounded linear operator on a Hilbert space $H$ with $\text{ker}(T_1) = \text{ker}(T_1) = \{0\}$ and $\text{R}(T_1) \cap \text{R}(T_2)$ dense, then

$$T_1 − T_2 = T_1(T_2^{-1} − T_1^{-1})T_2$$
on a dense domain, so
\[ T_2^{-1} - T_1^{-1} \text{ compact } \implies T_1 - T_2 \text{ compact} . \] (11)

Suppose then that \( H \) is a real Hilbert space with two embeddings \( H \subset \mathcal{H}_1, H \subset \mathcal{H}_2 \) in the complex Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \), isometric with respect to the real part of the scalar product, and \( A_k : D(A_k) \subset \mathcal{H}_k \to \mathcal{H}_k \) a selfadjoint operator such that \( H \) left invariant by \( K_k = -i_k A_k \) as in Prop. 2.2 with \( i_k \) the complex structure of \( \mathcal{H}_k \). We have:

**Proposition 2.4.** Let \( A_1, A_2 \) be as above, with \( D(A_1) \cap D(A_2) \) dense. Then
\[
i_1 A_1 |_H - i_2 A_2 |_H \text{ compact } \implies g(A_1) |_H - g(A_2) |_H \text{ compact}
\]
for every continuous complex function \( g \in \mathcal{C} \) (see Prop. 2.2).

**Proof.** By the identity (11), we have
\[
i_1 A_1 |_H - i_2 A_2 |_H \text{ compact } \implies i_1 (A_1 \pm i_1)^{-1} |_H - i_2 (A_2 \pm i_2)^{-1} |_H \text{ compact} ;
\]
therefore, if \( i_1 A_1 |_H - i_2 A_2 |_H \) is compact, then
\[
p(i_1 (A_1 + i_1)^{-1}, i_1 (A_1 - i_1)^{-1}) |_H - p(i_2 (A_2 + i_2)^{-1}, i_2 (A_2 - i_2)^{-1}) |_H
\]
is compact, for every two-variable polynomial \( p \) such that \( p(-x, -y) = p(y, x) \).

The proposition then follows by uniformly approximating \( g(\lambda) \) by \( p(i(\lambda + i)^{-1}, i(\lambda - i)^{-1}) \) on \( \mathbb{R} \), with \( p \) a polynomial as above, similarly as in the proof of Prop. 2.2. \( \square \)

### 2.3 Orthogonal and cutting projections

Let \( H \subset \mathcal{H} \) be a standard subspace of the complex Hilbert space \( \mathcal{H} \) and \( \Delta_H, J_H \) be the modular operator and conjugation of \( H \), namely
\[ S_H = J_H \Delta_H \]
is the polar decomposition of the antilinear involution \( S_H : h + ik \mapsto h - ik, h, k \in H \), *(Tomita’s operator)*. Then
\[ \Delta_H^s H = H, \quad J_H H = H', \]
s \( \in \mathbb{R} \), where \( H' = iH^\perp \) the symplectic complement of \( H \), with \( \perp \) denoting the orthogonal w.r.t. the real part of the scalar product. \( \Delta_H^s \) and \( \log \Delta_H \) are called the modular unitary group and the modular Hamiltonian of \( H \).

In this section, and in the following one, we assume \( H \) to be factorial, i.e. \( H \cap H' = \{0\} \), that is \( H + H' \) is dense in \( \mathcal{H} \); equivalently 1 is not an eigenvalue of \( \Delta_H \).

Associated with \( H \), there are three other standard subspaces: the real orthogonal \( H^\perp \), the symplectic complement \( H' \), and \( iH \). We shall consider the three associated real linear projections
\[
E_H : H + H^\perp \to H, \quad h + h^\perp \mapsto h,
\]
\[
P_H : H + H' \to H, \quad h + h' \mapsto h,
\]
\[
Q_H : H + iH \to H, \quad h + ik \mapsto h.
\]
Note that $E_H, P_H, Q_H$ are closed, densely defined real linear operators, $E_H$ is bounded and $P_H$ is the cutting projection [11]. We have

$$E_H = (1 + \Delta_H)^{-1} + J_H \Delta_H^{1/2} (1 + \Delta_H)^{-1}, \quad (12)$$

$$P_H = (1 - \Delta_H)^{-1} + J_H \Delta_H^{1/2} (1 - \Delta_H)^{-1}, \quad (13)$$

$$Q_H = \frac{1}{2} (1 + S_H); \quad (14)$$

more precisely, (13) means that $P_H$ is the closure of the operator $a(\Delta_H) + b(\Delta_H) J_H$ on $D(a(\Delta_H)) \cap D(b(\Delta_H))$, with

$$a(\lambda) = (1 - \lambda)^{-1}, \quad b(\lambda) = \lambda^{1/2} (1 - \lambda)^{-1}, \quad \lambda \in (0, +\infty) \setminus \{1\}.$$

Formulas (12) and (13) were obtained respectively in [15] (see also [30]) and in [11]; formula (14) is straightforward. We then have

$$E_H = 2Q_H (1 + \Delta_H)^{-1}, \quad P_H = 2Q_H (1 - \Delta_H)^{-1},$$

hence

$$P_H = E_H (1 + \Delta_H) (1 - \Delta_H)^{-1} = E_H \coth \left( \frac{1}{2} \log \Delta_H \right) \quad (15)$$

(denoting by the same symbol an operator and its closure).

**Proposition 2.5.** $P_H i$ restricts to a skew-selfadjoint operator on $H$ and

$$P_H i|_H = i \coth \left( \frac{1}{2} \log \Delta_H \right)|_H. \quad (16)$$

**Proof.** $H$ is $i \tanh(\frac{1}{2} \log \Delta_H)$-invariant by Prop. 2.2 because the hyperbolic tangent is a bounded, odd function; so $i \tanh(\frac{1}{2} \log \Delta_H)|_H$ is bounded, skew-selfadjoint on $H$. Therefore, its inverse $-i \coth(\frac{1}{2} \log \Delta_H)|_H$ is a skew-selfadjoint operator on $H$. Namely, $H$ is $i \coth(\frac{1}{2} \log \Delta_H)$-invariant. Equation (15) then gives (16) by restriction. □

We end this subsection by noting the following relations

$$P_H^* = -i P_H i; \quad (17)$$

$$P_H i \log \Delta_H = i \log \Delta_H P_H. \quad (18)$$

The first relation is proved in [11], the second one is valid on $D(P_H i \log \Delta_H) \cap D(i \log \Delta_H P_H)$ because $H$ is $i \log \Delta_H$-invariant.

### 2.4 Entropy and quadratic forms

Recall the entropy of a vector $k$ in a Hilbert space $\mathcal{H}$ with respect to a factorial standard subspace $H$ is defined by

$$S_k = \Im(k, P_H i \log \Delta_H k) = \Re(k, P_H^* i \log \Delta_H k), \quad (19)$$

where $P_H$ is the cutting projection $P_H : H + H^* \to H$. Formula (19) is to be understood in the sense of quadratic forms, namely $S_k = S(k, k)$ as we now define.
Let $D = D(\sqrt{\log \Delta_H} F)$, with $F$ the spectral projection of $\Delta_H$ relative to the interval $(0, 1)$. Then $D$ is a dense linear subspace of $\mathcal{H}$. With $h, k \in D$, we set

$$S(h, k) = \Re(h, P^*_H \log \Delta_H k) = \Im(h, P_H i \log \Delta_H k). \quad (20)$$

More precisely, following the discussion in [11], let $E(\lambda)$ be the spectral family of $\Delta_H$, the right hand side of (20) is defined by on $D$ by

$$\Im(h, P_H i \log \Delta_H k) = \int_0^{+\infty} a(\lambda) \log \lambda d(h, E(\lambda)k) - \int_0^{+\infty} b(\lambda) \log \lambda d(h, J_H E(\lambda)k). \quad (21)$$

Note that $b(\lambda) \log \lambda$ is a bounded function, so the right integral is always finite. Moreover, $a(\lambda) \log \lambda$ is bounded on $(1, +\infty)$ and positive on $(0, 1)$. So the above formula is well defined by the spectral theorem, provided $h, k \in D$. In particular,

$$S_k < \infty \iff k \in D,$$

[11] Prop. 2.4.

Set $D_0 = D(a(\Delta_H) \log \Delta_H \cap D(b(\Delta_H) \log \Delta_H) \subset D(\log \Delta_H P_H) \cap D$.

**Lemma 2.6.** $P^*_H \log \Delta_H$ is essentially selfadjoint on $D_0$ and $P^*_H \log \Delta_H|_{D_0} = \log \Delta_H P_H|_{D_0}$.

**Proof.** By equations (17), we have

$$(P^*_H \log \Delta_H)^* h = \log \Delta_H P_H h = -i \log \Delta_H P_H h = -i P_H i \log \Delta_H h = P^*_H \log \Delta_H h$$

for all $h \in D_0$, namely $(P^*_H \log \Delta_H)^*|_{D_0} = P^*_H \log \Delta_H|_{D_0} = \log \Delta_H P_H|_{D_0}$, so it suffices to show that the closure $\log \Delta_H P_H$ of $\log \Delta_H P_H|_{D_0}$ is selfadjoint.

With $0 < \varepsilon < 1$, let $E_\varepsilon$ be the spectral projection of $\log \Delta_H$ relative to the set $(-\varepsilon^{-1}, -\varepsilon) \cup (\varepsilon, \varepsilon^{-1})$. Then $E_\varepsilon (H + H') \subset H + H'$ by using Prop. [2.2] moreover $E_\varepsilon (H + H') \subset D_0$. So $E_\varepsilon \mathcal{H} \subset D(\log \Delta_H P_H)$ because $\log \Delta_H P_H E_\varepsilon$ is bounded by formula (13) and $E_\varepsilon$ commutes with $\log \Delta_H P_H$, thus also with $(\log \Delta_H P_H)^*$. We conclude that $(\log \Delta_H P_H)^* E_\varepsilon = \log \Delta_H P_H E_\varepsilon$ and the lemma follows since $E_\varepsilon \nearrow 1$ as $\varepsilon \searrow 0$. \hfill $\square$

**Proposition 2.7.** $S$ is a real linear, closed, symmetric, positive quadratic form on $D$.

**Proof.** $S$ is real linear and positive. By Lemma 2.6, $S$ is also symmetric on $D$. Thus $S$ closable on $D_0$, being associated with the closable, real linear operator $P^*_H \log \Delta_H$ by eq. (20), cf. the proof of [20] Thm. 1.27. Indeed, $S$ is closed on $D$ because $D_0$ is a core for $\log \Delta_H P_H$, so a form core for $S$. \hfill $\square$

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2.5 Standard subspaces and passivity

We provide here a standard subspace version of Pusz-Woronowicz’s complete passivity characterisation of the modular Hamiltonian in the $C^*$-algebraic setting [31].

Let $\mathcal{H}$ be a complex Hilbert space and $H$ standard subspace. In this section, $A$ is a selfadjoint linear operator on $\mathcal{H}$ such that $e^{isA}H = H$, $s \in \mathbb{R}$, namely $H$ is $iA$-invariant as in Prop. 2.2.

We shall say that $A$ is active/passive with respect to $H$ if
\[ \pm(\xi, A\xi) \geq 0, \quad \xi \in D(A) \cap H. \]
$A$ is n-active/passive w.r.t. $H$ if the generator of $e^{itA} \otimes e^{itA} \cdots \otimes e^{itA}$ is active/passive with respect to the $n$-fold tensor product $H \otimes H \otimes \cdots \otimes H$, (closed real linear span of monomials $h_1 \otimes h_2 \otimes \cdots \otimes h_n$, $h_i \in H$, cf. [27]). $A$ is completely active/passive if $A$ is $n$-active/passive for all $n \in \mathbb{N}$.

Note that $A$ is active/passive iff
\[ \pm(\xi, A\xi) \geq 0, \quad \xi \in \mathcal{D}, \]
with $\mathcal{D} \subset D(A) \cap H$ a real linear space such that the closure of $\mathcal{D}$ in the graph topology of $A$ is equal to $D(A) \cap H$.

Since $e^{itA}$ leaves $H$ globally invariant, $e^{itA}$ commutes with $\Delta_H^{i\theta}$ and $J_H$. We have
\[ J_H \Delta_H J_H = \Delta_H^{-1}, \quad J_H A J_H = -A. \tag{22} \]

Proposition 2.8. $\log \Delta_H$ is completely passive w.r.t. $H$.

Proof. $\log \Delta_H$ is passive w.r.t. $H$, see [24]. Hence it is completely passive because the modular unitary group of $H \otimes H \otimes \cdots \otimes H$ is $\Delta_H^{i\theta} \otimes \Delta_H^{i\theta} \cdots \otimes \Delta_H^{i\theta}$, see [27].

Let $\mathcal{D}_{an} \subset \mathcal{H}$ be the subspace of vectors with bounded spectrum with respect to both $A$ and $\log \Delta$.

Lemma 2.9. If $A$ is passive w.r.t. $H$, then $A \log \Delta_H$ is a positive selfadjoint operator on $\mathcal{H}$.

Proof. Denote by $S_H$ the Tomita operator of $\mathcal{H}$. As $S_H$ commutes with $e^{itA}$, $\Delta_H^{i\theta}$ and $J_H$, we have $S_H \mathcal{D}_{an} = \mathcal{D}_{an}$. Let $\xi \in \mathcal{D}_{an}$, so $(1 + S_H)\xi \in H$. Then, by passivity,
\[ 0 \geq ((1 + S_H)\xi, A(1 + S_H)\xi), \tag{23} \]
thus
\[ 0 \geq ((1 + S_H)i\xi, A(1 + S_H)i\xi) = ((1 - S_H)\xi, A(1 - S_H)\xi). \tag{24} \]
Summing up (23) and (24) we get
\[ 0 \geq (\xi, A\xi) + (S_H\xi, AS_H\xi) = (\xi, A\xi) + (J_H \Delta_H^{1/2} \xi, AJ_H \Delta_H^{1/2} \xi) = (\xi, A\xi) - (J_H \Delta_H^{1/2} \xi, J_H A \Delta_H^{1/2} \xi) = (\xi, A\xi) - (A \Delta_H^{1/2} \xi, A \Delta_H^{1/2} \xi) = (\xi, A(1 - \Delta_H)\xi), \]
thus $A(1 - \Delta_H) \leq 0$ because $\mathcal{D}_{an}$ is a core for $A(1 - \Delta_H)$. But $A(1 - \Delta_H) \leq 0$ is equivalent to $A \log \Delta_H \geq 0$. \qed
We shall say that a standard subspace is abelian if $\Delta_H = 1$, see \cite{[22]}. 

**Theorem 2.10.** $A$ is completely active with respect to $H$ iff $\log \Delta_H = \lambda A$ for some $\lambda \leq 0$.

**Proof.** Assume that $A$ is completely active with respect to $H$. Let $\Lambda \subset \mathbb{R}^2$ be the joint spectrum of $A$ and $\log \Delta_H$. By Lemma 2.9, $\Lambda$ is contained in the region $Q = \{(a, b) \in \mathbb{R}^2 : ab \geq 0\}$. By (22) we have $-\Lambda = \Lambda$ and by complete passivity we have $\Lambda + \Lambda + \cdots + \Lambda \subset Q$ (finite sum).

Let $C_1, C_2$ be two different points in $\Lambda$. Then $n_1C_1 + n_2C_2 \in Q$ for all integers $n_1, n_2 \in \mathbb{Z}$. Thus $\Lambda \subset Z$. If $Z$ is vertical, then $\log \Delta_H = 0$, namely $H$ is abelian, so $\log \Delta_H = \lambda A$ with $\lambda = 0$. If $Z$ is not vertical then $A = \lambda \log \Delta_H$ with $\lambda > 0$.

For the converse, it remains to show that $A$ is completely active if $\Delta_H = 1$, namely if $H$ is abelian. In this case, the scalar product of $\mathcal{H}$ is real on $\mathcal{H}$, and $iH$ is the real orthogonal complement of $H$. As $A$ maps $H$ into $iH$, we have $\Re(h, Ah) = 0$ for $h \in D(A) \cap H$, thus $(h, Ah) = 0$ because $A$ is selfadjoint. So $A$ is active, thus completely active by repeating this argument for $H \otimes H \cdots \otimes H$. \(\square\)

### 3 Preliminaries on the waves’ space

Let $\mathcal{S}$ denote the real linear space of smooth, compactly supported real functions on $\mathbb{R}^d$. We shall always assume $d \geq 2$ unless otherwise specified.

As is known, if $f, g \in \mathcal{S}$, there is a unique smooth real function $\Phi$ on $\mathbb{R}^{d+1}$ which is a solution $\Phi$ of the Klein-Gordon equation with mass $m \geq 0$ \((\Box + m^2)\Phi = 0\) (a wave packet or, briefly, a wave) with Cauchy data $\Phi|_{x_0=0} = f$, $\partial_0 \Phi|_{x_0=0} = g$. We set $\Phi = w_m(f, g)$ and denote by $\mathcal{T}_m$ the real linear space of these $\Phi$’s; we will often use the identification $\mathcal{S}^2 \longleftrightarrow \mathcal{T}_m$, \((f, g) \longleftrightarrow \Phi|_{x_0=0} = f, \partial_0 \Phi|_{x_0=0} = g\). We set $\mathcal{S}^2 \longleftrightarrow \mathcal{T}_m$, \((f, g) \longleftrightarrow \Phi|_{x_0=0} = f, \partial_0 \Phi|_{x_0=0} = g\). \(\Phi = w_m(f, g)\) and denote by $\mathcal{T}_m$ the real linear space of these $\Phi$’s; we will often use the identification \(\Phi = \Phi(w_m(f, g))\) and denote by $\mathcal{T}_m$ the real linear space of these $\Phi$’s; we will often use the identification

\[ (\Box + m^2)\Phi = 0 \]

The one particle Hilbert space is $\mathcal{H}_m = L^2(\mathcal{H}_m, \delta(p^2 - m^2))$ with $\mathcal{H}_m$ the mass $m$ hyperboloid. We denote by $(\cdot, \cdot)_m$ the scalar product of $\mathcal{H}_m$. The Fourier transform of a $\Phi \in \mathcal{T}_m$ is a distribution of the form $\hat{\Phi}(p) = \delta(p^2 - m^2) F(p)$ with a compactly supported smooth function $F : \mathcal{H}_m \cup (-\mathcal{H}_m) \rightarrow \mathbb{C}$, so that $\mathcal{T}_m$ real linearly embeds into $\mathcal{H}_m$ by $\Phi \mapsto (2\pi)^d/2 F|_{\mathcal{H}_m}$. We may thus consider $\mathcal{T}_m$ as a dense subset of $\mathcal{H}_m$.

Consider the symplectic form on $\mathcal{T}_m$

\[ \beta(\Phi, \Psi) = \frac{1}{2} \int_{x_0=0} (\Psi \partial_0 \Phi - \Phi \partial_0 \Psi) dx. \]

This is the imaginary part of the restriction of the scalar product of $\mathcal{H}_m$ to $\mathcal{T}_m$:

\[ \Im((\langle f_1, g_1 \rangle, \langle f_2, g_2 \rangle)_m = \beta((\langle f_1, g_1 \rangle, \langle f_2, g_2 \rangle)) .\]
In particular, \( \Im \langle \cdot, \cdot \rangle \) is therefore actually independent of \( m \). We denote by \( H^s_m \) the real Hilbert space of real valued tempered distributions \( f \in S'(\mathbb{R}^d) \) such that

\[
||f||^2_s = ||f||^2_{s;m} = \int_{\mathbb{R}^d} (|p|^2 + m^2)^s |\hat{f}(p)|^2 dp < +\infty, \quad s \in \mathbb{R}.
\]

(27)

It is clear that \( S \) is dense in \( H^{\pm 1/2}_m \) and that \( \mu_m : H^{1/2}_m \to H^{-1/2}_m \) with

\[
\hat{\mu_m}f(p) = \sqrt{|p|^2 + m^2} \hat{f}(p).
\]

(28)

is a unitary operator.

Then

\[
i_m = \begin{bmatrix} 0 & \mu_m^{-1} \\ -\mu_m & 0 \end{bmatrix},
\]

(29)

namely \( i_m(f, g) = \langle \mu_m^{-1}g, -\mu_m f \rangle \), is a unitary operator \( i_m \) on \( H_m = H^{1/2}_m \oplus H^{-1/2}_m \).

As \( i_m^2 = -1 \), the unitary \( i_m \) defines a complex structure (multiplication by the imaginary unit) on \( H_m \) that becomes a complex Hilbert space with scalar product

\[
(\Phi, \Psi) = (\Phi, \Psi)_m = \beta(\Phi, i_m \Psi) + i\beta(\Phi, \Psi).
\]

In terms of the Cauchy data, the scalar product is given by

\[
(\langle f, g \rangle, \langle h, k \rangle)_m = \frac{1}{2} \left( (f, \mu_m h) + (g, \mu_m^{-1} k) + i[(h, g) - (f, k)] \right), \quad \langle f, g \rangle, \langle h, k \rangle \in H^{1/2}_m \oplus H^{-1/2}_m,
\]

(where \( (\cdot, \cdot) \) is the standard \( L^2 \) scalar product). Furthermore the map

\[
\langle f, g \rangle \in H_m \mapsto w_m(f, g) \in \mathcal{H}_m
\]

is isometric and complex linear, so it extends to a unitary operator. We will then identify these spaces; namely the one particle Hilbert space is

\[
\mathcal{H}_m = \text{real Hilbert space } H_m \text{ with complex structure given by } i_m.
\]

Note that, when \( m = 1 \), \( H^{\pm 1/2}_m \) is a Sobolev space with usual definition. For \( m > m' > 0 \), the norms \( ||\cdot||_{\pm 1/2,m} \) are equivalent on the same linear space:

\[
||f||_{-1/2,m} \leq ||f||_{-1/2,m'} \leq \sqrt{\frac{m}{m'}} ||f||_{-1/2,m} \quad \text{(30)}
\]

\[
||f||_{1/2,m'} \leq ||f||_{1/2,m} \leq \sqrt{\frac{m}{m'}} ||f||_{1/2,m'} \quad \text{(31)}
\]

\( H^{1/2}_m \) and \( H^{-1/2}_m \) are naturally a dual pair under the bilinear form

\[
\langle f, g \rangle \in H^{1/2}_m \times H^{-1/2}_m \mapsto \int_{\mathbb{R}^d} \bar{f} \hat{g}.
\]

(32)

Let \( Z \) be an open, non-empty subset of \( \mathbb{R}^d \) and denote by \( Z' \) the interior of its complement. We shall denote by \( H_m(Z) \) the closed, real linear subspace of \( \mathcal{H}_m \)

\[
H_m(Z) = \{ \Phi \in \mathcal{T}_m(Z) \}^-,
\]
where \( \mathcal{T}_m(Z) = \{ \Phi = w_m(f, g) \in \mathcal{T}_m : \text{supp}(f), \text{supp}(g) \subset Z \} \).

If \( Z \) and \( Z' \) are non-empty, then \( H_m(Z) \) is a standard subspace of \( \mathcal{H}_m \), the one associated with \( Z \). We shall be mainly interested in the case \( Z \) or \( Z' \) is the unit ball \( B \) of \( \mathbb{R}^d \). Clearly, setting
\[
H_m^{\pm 1/2}(Z) = \text{closure of } C_0^\infty(Z) \text{ in } H_m^{\pm 1/2},
\]
we have
\[
H_m(Z) = H_m^{1/2}(Z) \oplus H_m^{-1/2}(Z).
\]

Here and in the following, \( C_0^\infty(Z) \) denotes the space of real \( C^\infty \) function on \( \mathbb{R}^d \) with compact support in \( Z \). By duality \([1]\), see also \([13, 21]\), we have

\[
H_m(B)' = H_m(B').
\]

**Lemma 3.1.** (\([15]\) Prop. A.2) \( H_m^{\pm 1/2}(B) \) and \( H_0^{\pm 1/2}(B) \) are the same linear space with equivalent norms, \( m \geq 0 \).

**Proof.** Note first that \( H_0^{-1/2}(B) \subset H_m^{-1/2}(B) \), indeed \( H_0^{-1/2}(B) \) has a bounded embedding into \( H_m^{-1/2}(B) \). Let’s show the reverse inclusion, namely
\[
f \in H_m^{-1/2}(B) \implies f \in H_0^{-1/2}(B).
\]
As \( f \) is compactly supported, \( \hat{f} \) is analytic; let \( C > 0 \) be such that \( |\hat{f}(p)|^2 \leq C \) if \( |p| \leq 1 \). Then we have
\[
\int_{|p| < 1} \frac{|\hat{f}(p)|^2}{|p|} dp \leq C \int_{|p| < 1} \frac{dp}{|p|} < \infty
\]
\((d \geq 2)\). On the other hand,
\[
\int_{|p| \geq 1} \frac{|\hat{f}(p)|^2}{|p|} dp < \infty
\]
because \( f \in H_m^{-1/2}(B) \); therefore \( f \in H_0^{-1/2}(B) \). The norms on \( H_0^{-1/2}(B) \) and \( H_m^{-1/2}(B) \) are then equivalent by the open mapping theorem, as can be checked directly too.

Similarly, we have \( H_m^{1/2}(B) \subset H_0^{1/2}(B) \). For the converse inclusion, let \( f \in H_0^{1/2}(B) \) and \( C > 0 \) be such that \( |p||\hat{f}(p)|^2 \leq C \) if \( |p| \leq 1 \).
\[
\frac{1}{\sqrt{1 + m^2}} \int_{|p| < 1} |\hat{f}(p)|^2 |p|^2 + m^2 dp \leq \int_{|p| < 1} \frac{|\hat{f}(p)|^2 |p|}{|p|} dp \leq C \int_{|p| < 1} \frac{dp}{|p|} < \infty.
\]
\( \square \)

## 4 Massless Hamiltonian

We start with the analysis of the modular Hamiltonian in the massless case for two reasons. Firstly, the formula for the massless Hamiltonian has not been worked out so far and it already shows new and interesting aspects. Secondly, the massive Hamiltonian will be derived by a deformation of the massless one.
4.1 The modular group, \( m = 0 \)

In the following, \( O \) will denote the double cone on the Minkowski spacetime \( \mathbb{R}^{d+1} \) with base the open unit ball \( B \) centred at the origin in the time zero hyperplane \( \mathbb{R}^d \). As above, \( H_0(O) = H_0(B) \) is the standard subspace in the massless, scalar one-particle Hilbert space \( \mathcal{H}_0 \). We assume \( d \geq 2 \).

The modular group \( \Delta_B^{is} \) associated with \( H_0(B) \) has been computed in [19] in terms of the action on the field or, equivalently, on the spacetime test-functions. \( \Delta_B^{is} \) is associated with a one parameter group of conformal transformation that preserve \( O \), namely

\[
(u, v) \mapsto (Z(u, s), Z(v, s)),
\]

where \( Z \) is given by

\[
Z(z, s) = \frac{g(z, s)}{f(z, s)}
\]

with

\[
f(z, s) = \frac{(1 + z) + e^{-s}(1 - z)}{2}, \quad g(z, s) = \frac{(1 + z) - e^{-s}(1 - z)}{2}.
\]

Here we need to compute \( \Delta_B^{is} \) it in terms of waves. Let \( \Phi \) be a wave and set

\[
(V(s)\Phi)(u, v) = \gamma(u, v; s)\Phi(Z(u, s), Z(v, s)),
\]

with

\[
u = x_0 + r, \quad v = x_0 - r, \quad r = |x| \equiv \sqrt{x_1^2 + \cdots + x_d^2}
\]

and we omit the remaining spherical coordinates as the action is trivial on them.

The cocycle \( \gamma \) given by

\[
\gamma(u, v; s) = F(u, s)F(-v, -s), \quad F(z, s) \equiv f^{-D}(z, s),
\]

with

\[
D = \frac{d - 1}{2};
\]

\( D \) is the scaling dimension. We have:

**Theorem 4.1.** The modular group of \( H_0(B) \) is given by

\[
\Delta_B^{-is} = V(2\pi s).
\]

**Proof.** That \( \Delta_B^{-is/2\pi} \) is associated with the flow \( Z \) [34] and the cocycle \( \gamma \) follows rather directly from the test function formula [19], but for the determination of the value of the constant \( D \). Now, as in [19], \( H_0(B) \) is equivalent to \( H_0(W) \), with \( W \) a wedge region, by a unitary operator that is obtained by composing unitaries associated with translations and ray inversion map. The cocycle is associated with the Jacobian of the ray inversion map. Thus, by computation similar to those in [19], the cocycle is of the form \( f^{-D} \) for some constant \( D > 0 \).

However, it will follows from Lemma 4.9 that \( D = (d - 1)/2 \) is the only value compatible with the complex linearity of the generator of \( V \). Of course, \( D \) can be determined by direct calculations too. \( \square \)
4.2 The generator of the modular group, \( m = 0 \)

We now compute the \( K_0 = \frac{d}{ds} V(s) \big|_{s=0} \), the generator of \( V \) in Theorem 4.1, that is proportional to the modular Hamiltonian.

With the above notations, denoting by a prime the derivative with respect to \( s \)-parameter, we have (Identity 7.2)

\[
Z(z, 0) = z, \quad Z'(z, 0) = \frac{1 - z^2}{2}
\]

and

\[
\gamma(u, v; 0) = 1, \quad \gamma'(u, v; 0) = -\frac{D}{2}(u + v) = -Dx_0.
\]

Therefore

\[
(V(s)\Phi)(u, v)' = \gamma'(u, v; s)\Phi(Z(u, s), Z(v, s))
+ \gamma(u, v; s)\left( \partial_u \Phi(Z(u, s), Z(v, s))Z'(u, s) + \partial_v \Phi(Z(u, s), Z(v, s))Z'(v, s) \right).
\]

We then have:

**Proposition 4.2.** We have

\[
(K_0\Phi)(u, v) = -\frac{D}{2}(u + v)\Phi - \frac{1}{2}u^2\partial_u\Phi - \frac{1}{2}v^2\partial_v\Phi + \frac{1}{2}\partial_u\Phi + \frac{1}{2}\partial_v\Phi.
\]

*In terms of the \( x_0, r \) coordinates*

\[
(K_0\Phi)(x_0, r) = \frac{1}{2}(1 - (x_0^2 + r^2))\partial_0\Phi - x_0r\partial_r\Phi - D x_0\Phi. \tag{37}
\]

**Proof.** We compute:

\[
(K_0\Phi)(u, v) = (V(s)\Phi)(u, v)' \big|_{s=0}
= \gamma'(u, v; 0)\Phi(u, v) + \partial_u\Phi(u, v)Z'(u, 0) + \partial_v\Phi(u, v)Z'(v, 0)
= -\frac{D}{2}(u + v)\Phi(u, v) + \frac{1}{2}\partial_u\Phi(u, v)(1 - u^2) + \frac{1}{2}\partial_v\Phi(u, v)(1 - v^2)
= -\frac{D}{2}(u + v)\Phi - \frac{1}{2}u^2\partial_u\Phi - \frac{1}{2}v^2\partial_v\Phi + \frac{1}{2}\partial_u\Phi + \frac{1}{2}\partial_v\Phi.
\]

Now

\[
\partial_u = \frac{1}{2}(\partial_0 + \partial_r), \quad \partial_v = \frac{1}{2}(\partial_0 - \partial_r),
\]

so

\[
(K_0\Phi)(x_0, r) = -\frac{1}{4}\left((x_0 + r)^2(\partial_0 + \partial_r)\Phi + (x_0 - r)^2(\partial_0 - \partial_r)\Phi\right) - Dx_0\Phi + \frac{1}{2}\partial_0\Phi
= -\frac{1}{2}(x_0^2 + r^2)\partial_0\Phi - x_0r\partial_r\Phi - Dx_0\Phi + \frac{1}{2}\partial_0\Phi
= \frac{1}{2}(1 - (x_0^2 + r^2))\partial_0\Phi - x_0r\partial_r\Phi - D x_0\Phi. \tag{38}
\]

□
Corollary 4.3. We have

\[
(K_0 \Phi)|_{x_0=0} = \frac{1}{2} (1 - r^2) \partial_0 \Phi|_{x_0=0},
\]

(39)

\[
(\partial_0 K_0 \Phi)|_{x_0=0} = \frac{1}{2} (1 - r^2) \nabla^2 \Phi - r \partial_r \Phi - D \Phi|_{x_0=0}.
\]

(40)

Proof. The first equality follows immediately from (37). Again from (37) we have

\[
(\partial_0 K_0 \Phi)|_{x_0=0} = \frac{1}{2} (1 - r^2) \partial_0 \Phi - r \partial_r \Phi - D \Phi|_{x_0=0}
\]

thus (40) holds.

The above corollary translates into the following:

Proposition 4.4. We have

\[
K_0 : w_0(f, g) \mapsto w_0 \left( \frac{1}{2} (1 - r^2) g, \frac{1}{2} (1 - r^2) \nabla^2 f - r \partial_r f - Df \right).
\]

In other words, \( K_0 \) is the operator on \( S^2 \) given by the \( 2 \times 2 \) matrix

\[
K_0 = \begin{bmatrix}
0 & \frac{1}{2} (1 - r^2) \\
\frac{1}{2} (1 - r^2) \nabla^2 - r \partial_r - D & 0
\end{bmatrix}.
\]

(41)

Now the symplectic form \( \beta \) on \( T_0 \) (26) is given, in terms of Cauchy data, as

\[
\beta(\Phi, \Psi) = \frac{1}{2} \int_{\mathbb{R}^d} (f_2 g_1 - f_1 g_2) dx.
\]

(42)

with \( \Phi = w_0(f_1, g_1), \Psi = w_0(f_2, g_2) \).

Thus

\[
2 \beta(\Phi, K_0 \Phi) = \int_{x_0=0} (K_0 \Phi) \partial_0 \Phi - \Phi \partial_0 (K_0 \Phi) dx
\]

\[
= \int_{\mathbb{R}^d} \frac{1}{2} (1 - r^2) g^2 dx - \int_{\mathbb{R}^d} \frac{1}{2} (1 - r^2) f \nabla^2 f dx + \int_{\mathbb{R}^d} r f \partial_r f dx + D \int_{\mathbb{R}^d} f^2 dx.
\]

(43)

So, taking into account Identity (7.1), we have

\[
2 \beta(\Phi, K_0 \Phi) = \frac{1}{2} \int_{\mathbb{R}^d} (1 - r^2) \left( g^2 + \nabla f^2 \right) dx + D \int_{\mathbb{R}^d} f^2 dx,
\]

namely

\[
\beta(\Phi, K_0 \Phi) = \frac{1}{4} \int_{x_0=0} (1 - r^2) \left( (\partial_0 \Phi)^2 + |\nabla \Phi|^2 \right) dx + \frac{D}{2} \int_{x_0=0} \Phi^2 dx.
\]

(44)

So we have obtained the following proposition.

Proposition 4.5. With \( A_0 = -\iota_0 K_0 \), we have

\[
(\Phi, A_0 \Phi) = \beta(\Phi, K_0 \Phi) = \frac{1}{2} \int_{x_0=0} (1 - r^2) (T_{00}) \Phi dx + \frac{D}{2} \int_{x_0=0} \Phi^2 dx
\]

(45)

with \( \langle T_{00} \rangle \Phi = \frac{1}{4} ((\partial_0 \Phi)^2 + |\nabla \Phi|^2) \) the energy density given by classical stress-energy tensor \( T \).

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Proof. Since $A_0$ is selfadjoint, $(\Phi, A_0 \Phi)$ is real, so

$$(\Phi, A_0 \Phi) = \Im(\Phi, A_0 \Phi) = \Im(\Phi, i_0 A_0 \Phi) = \Im(\Phi, K_0 \Phi) = \beta(\Phi, K_0 \Phi),$$

so the proposition follows from (44). □

Since

$$\beta(K_0 \Phi, \Psi) + \beta(\Phi, K_0 \Psi) = 0,$$

by the polarisation identity we then get

$$\beta(\Phi, K_0 \Psi) = \frac{1}{2} \left( \beta(\Phi + \Psi, K_0 (\Phi + \Psi)) - \beta(\Phi - \Psi, K_0 (\Phi - \Psi)) \right)$$

$$= \frac{1}{2} \int_{x_0=0} (1 - r^2) (T_{00})_{\Phi, \Psi} dx + \frac{D}{2} \int_{x_0=0} \Phi \Psi dx \quad (46)$$

4.3 $K_0$ acting on $H_0$

Set $H_0(B) = H_0^{1/2}(B) \oplus H_0^{-1/2}(B)$ for the standard subspace associated with $B$ as before. Let

$$M_0 : D(M_0) \subset H_0^{-1/2} \rightarrow H_0^{1/2}, \quad L_0 : D(L_0) \subset H_0^{1/2} \rightarrow H_0^{-1/2}$$

be the closures of the operators

$$M_0 = \frac{1}{2} (1 - r^2)$$

$$L_0 = \frac{1}{2} (1 - r^2) \nabla^2 - r \partial_r - D$$

on $S$. ($M_0$ is the multiplication operator by $\frac{1}{2} (1 - r^2)$). We shall indeed see that the both operator $M_0$ and $L_0$ on $S$ are closable as $M_0^* \subset L_0$ on $S$.

Denote by $K_0 = -\frac{\pi}{2\pi} \log \Delta_B$ the modular Hamiltonian relative to $B$ ($m = 0$).

Lemma 4.6. $S^2$ is a core for $K_0$ (as real linear operator).

Proof. By the geometrical action of the modular group $\Delta_B^{is}$, and duality, one sees that $C_0^\infty(B)^2 + C_0^\infty(B')^2$ is a dense, real linear subspace of $H_0$, globally $\Delta_B^{is}$-invariant, contained in the domain of the generator $K_0$; thus it is a core for $K_0$. So $S^2$ is a core too being a larger subspace still contained in $D(K_0)$. □

So we have:

Theorem 4.7. The massless Hamiltonian $\log \Delta_B$ is given by $-2 \pi A_0 = \log \Delta_B$ with

$$K_0 = \begin{bmatrix} 0 & M_0 \\ L_0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & M_0 \\ -M_0^* & 0 \end{bmatrix},$$

and $A_0 = -i_0 K_0$. 

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Proof. By Lemma 4.6, we have the first equality. As $A_0$ is selfadjoint, we have $K_0^* = -K_0$, thus the second equality holds. □

Next corollary will be generalised in Section 5.5.

Corollary 4.8. $A_0$ is completely active with respect to $H_0(B)$.

Proof. Immediate by Theorem 2.10. □

Corollary 4.9. We have $K_0 = -i_0 K_0 i_0$. Thus

$$\mu_0 M_0 \mu_0 = M_0^* = -L_0.$$ 

Proof. Immediate by the complex linearity of $K_0$. □

Note, in particular, that the constant $D$ in the expression of the operator $L_0$ is fixed by the above corollary.

Corollary 4.10. $K_0^B = K_0|_{H_0(B)}$ is skew-selfadjoint on $H_0(B)$.

Proof. Since $H_0(B)$ is globally invariant for $e^{sK_0}$, we may apply Prop. 4.11. □

Remark 4.11. By Cor. 4.10, we have indirectly solved a Dirichlet problem for the degenerate elliptic operators $L_0 M_0 - 1$ and $M_0 L_0 - 1$, see [4]. In the next section we shall see a more general degenerate Dirichlet problem for an integro-differential operator.

5 Massive Hamiltonian

Let $M_m$ be the real linear operator

$$M_m : D(M_m) \subset H_m^{-1/2} \to H_m^{1/2}, \quad M_m f = \frac{1 - r^2}{2} f,$$ 

with domain $D(M_m) = \{f \in H_m^{-1/2} : M_m f \in H_m^{1/2}\}$ and $M_m^*$ the adjoint of $M$ with respect to the real Hilbert space scalar products

$$M_m^* : D(M_m^*) \subset H_m^{1/2} \to H_m^{-1/2}.$$ 

Note that $S$ is a core for $M_m$; indeed the transpose of $M_m|S$ under the duality (32) is $M_m$. While the formal expression $M_m$ for $M$ does not depend on $m$, the operator $M_m^*$ will have a different expression depending on $m$.

Let also $L_m : D(L_m) \subset H_m^{1/2} \to H_m^{-1/2}$ be the closure of the real linear operator on $S$

$$L_m = \frac{1}{2}(1 - r^2)(\nabla^2 - m^2) - r \partial_r - D + \frac{1}{2} m^2 (\nabla^2 - m^2)^{-1}$$ (48)

The actions of $L_m$ and $M_m$ on $S$ are naturally defined in Fourier transform.
5.1 The operator $\tilde{K}_m$

As a first step towards the formula for the massive Hamiltonian, we are going to single out a natural deformation operator $\tilde{K}_m$ of the massless skew-adjoint massless Hamiltonian $K_0$. As we shall see, the quadratic form associated with $\tilde{K}_m$ will restrict to the quadratic form of the massive skew-adjoint Hamiltonian on the ball $B$.

**Proposition 5.1.** $\mu_m M_m \mu_m = -L_m$ on the Schwartz space $S(\mathbb{R}^d)$.

**Proof.** For simplicity, denote by $\hat{\mu}$ the function $\hat{\mu}(p) = \sqrt{|p|^2 + m^2}$. We have $\nabla_p \hat{\mu}_m = \frac{p}{\hat{\mu}_m}$, so

$$\nabla^2 \hat{\mu}_m = \nabla_p \cdot \frac{P}{\hat{\mu}_m} = \sum_k \frac{\mu_m - p_k \hat{\mu}_m}{\hat{\mu}_m^2} = \frac{1}{\hat{\mu}_m} \sum_k \frac{\mu_m^2 - p_k^2}{\hat{\mu}_m^2} = \frac{1}{\hat{\mu}_m} (d - \frac{|p|^2}{\mu_m^2}),$$

$$\nabla^2 \hat{\mu}_m^2 = d - 1.$$

With $f \in S(\mathbb{R}^d)$, we have

$$\frac{1}{2}(\mu_m(1 - r^2)\mu_m f) = \frac{1}{2} \hat{\mu}_m (1 + \nabla^2_p) (\hat{\mu}_m \hat{f})$$

$$= \frac{1}{2} \hat{\mu}_m^2 + \frac{1}{2} \hat{\mu}_m \nabla_p (\hat{\mu}_m \hat{f})$$

$$= \frac{1}{2} \hat{\mu}_m^2 (1 + \nabla^2_p) \hat{f} + p \cdot \nabla_p \hat{f} + \frac{1}{2} (d - \frac{|p|^2}{\mu_m^2}) \hat{f}$$

$$= \frac{1}{2} \hat{\mu}_m^2 (1 + \nabla^2_p) \hat{f} + p \cdot \nabla_p \hat{f} + D \hat{f} + \frac{1}{2} \frac{m^2}{\hat{\mu}_m^2} \hat{f}$$

$$= \frac{1}{2} |p|^2 (1 + \nabla^2_p) \hat{f} + p \cdot \nabla_p \hat{f} + \frac{1}{2} m^2 (1 + \nabla^2_p) \hat{f} + \frac{1}{2} \frac{m^2}{\hat{\mu}_m^2} \hat{f}$$

$$= (\nabla^2_p + \frac{1}{2} |p|^2 (1 - r^2) + \frac{1}{2} \frac{m^2}{\mu_m^2} f) \nabla^2 \hat{\mu}_m^2 = (-L_m f)^\dagger$$

as desired. \qed

**Lemma 5.2.** With $M_m$ and $M_m^*$ as above, we have $\mu_m^* M_m^* = M_m \mu_m$, namely $M \mu_m$ is a selfadjoint operator on $H_m^1$. Here $\mu_m : H_m^1 \to H_m^{-1/2}$ is the unitary operator given by (28).

**Proof.** $g \in H_m^{1/2}$ belongs to $D((M_m \mu_m)^*)$ iff there exists $g' \in H_m^{1/2}$ such that

$$(g, M_m \mu_m f)_{1/2} = (g', f)_{1/2} \forall f \in D(M_m \mu_m).$$

Now denote by $h$ the function $h(x) = \frac{1}{2} (1 - r^2)$; we have

$$(g, M_m \mu_m f)_{1/2} = \int (\mu_m g) M_m \mu_m f dx = \int (\mu_m g) h(\mu_m f) dx = \int h(\mu_m g)(\mu_m f) dx$$

thus $g \in D((M_m \mu_m)^*)$ iff $h \mu_m g \in H_m^{1/2}$, namely iff $g \in D(M_m \mu_m)$, and then we have $(M_m \mu_m)^* g = M_m \mu_m g$. \qed
As $\mu_m$ is unitary, we have the formula
\[
M_m^* = \mu_m M_m \mu_m.
\]
Let $\tilde{K}_m : D(\tilde{K}_m) \subset H_m \rightarrow H_m$ be the real linear operator
\[
\tilde{K}_m = \begin{bmatrix} 0 & M_m \\ -M_m^* & 0 \end{bmatrix}.
\]
The domain of $\tilde{K}_m$ is given by $D(\tilde{K}_m) = D(M_m^*) \oplus D(M_m) \subset H_{m^{1/2}} \oplus H_{m^{-1/2}}$.

We then have:

**Proposition 5.3.**
\[
\tilde{K}_m = \begin{bmatrix} 0 & M_m \\ -M_m^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mu_m M_m \mu_m \\ -\mu_m M_m \mu_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & M_m \\ -L_m & 0 \end{bmatrix}.
\]

**Proof.** By the above discussion, we have the first equality and the second equality holds on $S(\mathbb{R}^d)^2$. So it remains to show that $S(\mathbb{R}^d)$ is a core for $\mu_m M_m \mu_m$, namely that $\mu_m S(\mathbb{R}^d)$ is a core for $M_m$, which is true because $\mu_m S(\mathbb{R}^d) = S(\mathbb{R}^d)$ as is manifest in Fourier transform.

**Remark 5.4.** It is useful to compare with the corresponding structure in the case of the half-space $x_1 > 0$, whose casual envelop is the wedge region $W_1$. In this case, the modular group is given by the rescaled pure Lorentz transformations in the $x_1$-direction. A computation analogous to the one given in Sect. 4 then gives
\[
2\pi \log \Delta_{W_1} = \iota_m \left[ \begin{array}{cc} 0 & 1 \\ x_1(\nabla^2 - m^2) + \partial_1 & 0 \end{array} \right].
\]
Note that the upper-right entry in the above matrix is independent of the mass $m \geq 0$.

### 5.2 The operator $K^B_m$

Recall that $H_m(B) = H_m^{1/2}(B) \oplus H_m^{-1/2}(B)$ . Let $E_m$ be the orthogonal projection $\mathcal{H}_m \rightarrow H_m(B)$. Thus
\[
E_m = \begin{bmatrix} E_m^+ & 0 \\ 0 & E_m^- \end{bmatrix},
\]
with $E_m^\pm$ the orthogonal projection $H_m^{\pm1/2} \rightarrow H_m^{\pm1/2}(B)$.

We define $K^B_m$ as the operator on $H_m(B)$ given by the compression of $\tilde{K}_m$ on $H_m(B)$, namely by restricting the quadratic form of $\tilde{K}_m$ to $H_m(B)$:
\[
K^B_m = E_m \tilde{K}_m | H_m(B);
\]
thus $K^B_m : D(K^B_m) \subset H_m(B) \rightarrow H_m(B)$ with $D(K^B_m) = D(\tilde{K}_m) \cap H_m(B)$.

As the only non local term in $\tilde{K}_m$ is $\frac{1}{2} m^2 \mu_m^2$, we have on $H_m(B)$$K^B_m = \begin{bmatrix} \frac{1}{2}(1 - r^2)(\nabla^2 - m^2) - r \partial_r - D & \frac{1}{2}(1 - r^2) \\ \frac{1}{2}(1 - r^2) & 0 \end{bmatrix} - \frac{1}{2} m^2 \begin{bmatrix} 0 & G^B_m \\ G^B_m & 0 \end{bmatrix}.
\]

(49)
with
\[ G_m^B = E_m^\mu m^{-2}. \]
That is to say
\[ K_m^B = \begin{bmatrix} 0 & M_m \\ I_m^B & 0 \end{bmatrix} \text{ on } H_m(B) \] (50)
with \( I_m^B \) the sum of the local, massive Legendre operator \( \frac{1}{2}(1-r^2)(\nabla^2 - m^2) - r \partial_r - D \) and the operator \( -\frac{1}{2}m^2 G_m^B \), that we further discuss in Sect. 5.3.

Note that
\[ (\Phi, K_m^B \Psi) = (\Phi, K_m \Psi), \quad \Phi \in H_m, \Psi \in D(K_m^B), \]
thus
\[ (\Phi, K_m^B \Psi) = -(K_m^B \Phi, \Psi), \quad \Phi, \Psi \in D(K_m^B), \]
namely \( K_m^B \) is skew-Hermitian on its domain.

**Proposition 5.5.** \( K_m^B \) is skew-selfadjoint on \( H_m(B), \ m \geq 0. \)

**Proof.** \( K_0^B \) is skew-selfadjoint on \( H_0(B) \) by Corollary [4.10] Moreover
\[ K_m^B = K_0^B - P_m - V_m^B \] (51)
with
\[ P_m = m^2 \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2) & 0 \end{bmatrix}, \quad V_m^B = \frac{1}{2}m^2 \begin{bmatrix} 0 & 0 \\ G_m^B & 0 \end{bmatrix}, \] (52)
and \( P_m, V_m^B \) are compact operators \( H_m(B) \to H_m(B) \) by Proposition [5.6].

We now apply Proposition [2.7]. Since \( K_0^B \) is skew-selfadjoint in \( H_0(B) \) and \( K_m^B \) is skew-Hermitian in \( H_m(B) \), it follows that \( K_m^B \) is skew-selfadjoint in \( H_m(B). \)

\[ \square \]

### 5.3 More on the operator \( G_m^B \)

With \( m > 0 \), we have considered the operator
\[ G_m^B = E_m^\mu m^{-2} \cdot (-\nabla^2 + m^2)^{-1}|_{H_m^{1/2}(B)} : H_m^{1/2}(B) \to H_m^{-1/2}(B). \] (53)

The operator \( V_m^B \) in (52) acts on \( H_m(B) \) and is a term in the expression for \( K_m^B \).

Let \( f, g \in C_0^\infty(B) \subset H_m^{1/2}(B) \) and \( \Lambda \) be the embedding \( H_m^{1/2}(B) \hookrightarrow H_m^{-3/2}(B) \). We have
\[ (g, G_m^B f)_{-1/2} = (g, \mu^{-2}_m f)_{-1/2} = (\Lambda g, \Lambda f)_{-3/2} = (g, \Lambda^* \Lambda f)_{-1/2} \]
thus \( G_m^B = \Lambda^* \Lambda \) as operator \( H_m^{-1/2}(B) \to H_m^{-1/2}(B) \), in particular it is positive selfadjoint.

In the following proposition, the multiplication operator by \( \frac{1}{2}(1-r^2) \) acting on different spaces is denoted by the same symbol \( M \).

**Proposition 5.6.** \( M_m : H_m^{1/2}(B) \to H_m^{-1/2}(B) \) and \( G_m^B : H_m^{1/2}(B) \to H_m^{-1/2}(B) \) are compact operators. Therefore, both the operators \( P_m \) and \( V_m^B \) given by (52) are compact on \( H_m(B). \)
Proof. First we show that the multiplication operator \( \frac{1}{2}(1 - r^2) : H^{-1/2}_m(B) \rightarrow H^{-1/2}_m(B) \) is bounded. Let \( h \) be a real, smooth, compactly supported function on \( \mathbb{R}^d \) that agrees with \( \frac{1}{2}(1 - r^2) \) on \( B \). Then \( \frac{1}{2}(1 - r^2)f = hf \) if \( h \in H^{-1/2}_m(B) \). As the multiplication by \( h \) is a bounded operator \( H^{s}_m \rightarrow H^{s}_m [36] \), we see that \( \frac{1}{2}(1 - r^2) : H^{-1/2}_m(B) \rightarrow H^{-1/2}_m(B) \) is bounded for every \( s \in \mathbb{R} \), in particular with \( s = -1/2 \).

Now, we have a factorization

\[
H^{-1/2}_m(B) \xrightarrow{N_m} H^{-1/2}_m(B) \xrightarrow{\frac{1}{2}(1-r^2)} H^{-1/2}_m(B)
\]

where the upright arrow operator is bounded and the vertical embedding is compact by Rellich’s theorem.

Similarly, \( G_m^B \) factors as:

\[
H^{1/2}_m(B) \xrightarrow{G_m^B} H^{1/2}_m(B) \xrightarrow{\Lambda^*\Lambda} H^{-1/2}_m(B)
\]

where the both the embeddings \( H^{1/2}_m(B) \hookrightarrow H^{-1/2}_m(B) \) and \( \Lambda \) are compact by Rellich’s theorem, thus also \( \Lambda^*\Lambda \) is compact. \( \square \)

As \( \mu_m^{-2} = - (\nabla^2 - m^2)^{-1} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \), we have

\[
(\mu_m^{-2}f)(x) = \int_{\mathbb{R}^d} G_m(x - y) f(y) dy , \quad f \in S ,
\]

with \( G_m \) the Green function for the Helmholtz operator

\[
G_m(x) = \frac{1}{(2\pi)^{d/2}} \left( \frac{m}{|r|} \right)^{d/2-1} K_{d/2-1}(mr) , \quad (54)
\]

where \( K_\alpha \) is a modified Bessel function of the second kind. We may denote \( \mu_m^{-2} \) also by \( G_m \).

Now, if \( \Phi_k = w_m(f_k, g_k) \in T_m(B) \), we have

\[
(\Phi_1, V_m^B \Phi_2) = (\Phi_1, V_m \Phi_2)
\]

with

\[
V_m = \begin{bmatrix} 0 & 0 \\ G_m & 0 \end{bmatrix} , \quad (55)
\]

therefore

\[
\Re(\Phi_1, i_m V_m^B \Phi_2) = -\Im(\Phi_1, V_m \Phi_2) = (f_1, G_m f_2)_{L^2} = \int_{B \times B} G_m(x - y) f_1(x) f_2(y) dx dy . \quad (56)
\]
5.4 The operators $K_m$ and $A_m = -i m K_m$

We now define the complex linear operator $K_m$ on $\mathcal{H}_m$ by extending $K^B_m$ by complex linearity, namely

$$K_m(\Phi + i m \Psi) = K^B_m \Phi + i m K^B_m \Psi, \ \Phi, \Psi \in D(K^B_m),$$

so

$$D(K_m) = D(K^B_m) + i m D(K^B_m).$$

We shall see that $A_m = i m K_m$ is proportional to the mass $m$ local modular Hamiltonian $\log \Delta_B$.

**Proposition 5.7.** $A_m$ is essentially selfadjoint on $D(K_m)$. Moreover,

$$e^{isA_m} H_m(B) = H_m(B), \quad s \in \mathbb{R}.$$ 

**Proof.** Immediate by Corollary 2.3. \hfill \square

We now give an explicit formula for the quadratic form on $H_m(B)$ associated with $A_m$, in terms of the stress-energy tensor and a Green integral operator with Yukawa kernel.

With $A_m = -i m K_m$, note first that

$$(\Phi, A_m \Phi) = \Im i(\Phi, A_m \Phi) = \Im (\Phi, i m A_m \Phi) = \Im (\Phi, K_m \Phi) = \beta(\Phi, K_m \Phi) = \beta(\Phi, \tilde{K}_m \Phi), \quad \text{(57)}$$

if $\Phi \in H_m(B)$ is in the domain of $A_m$.

**Theorem 5.8.** If $\Phi \in T_m(B)$, we have

$$\langle \Phi, A_m \Phi \rangle = \frac{1}{2} \int_B (1 - r^2) \langle T^{(m)}_{00} \rangle \Phi^2 d\mathbf{x} + \frac{D}{2} \int_B \Phi^2 d\mathbf{x} + \frac{1}{2} m^2 \int_{B \times B} G_m(\mathbf{x} - \mathbf{y}) \Phi(\mathbf{x}) \Phi(\mathbf{y}) d\mathbf{x} \quad \text{(58)}$$

($t = 0$ integrals) with $\langle T^{(m)}_{00} \rangle = \frac{1}{2} ((\partial_0 \Phi)^2 + |\nabla \Phi|^2 + m^2 \Phi^2)$ the energy density given by the classical, massive stress-energy tensor $T^{(m)}$.

**Proof.** Let $\Phi_m = w_m(f, g) \in T_m(B)$. Since $K^B_m = K^B_0 - P_m - V^B_m$ and the symplectic form $\beta$ is mass independent, by the identities (57) we have

$$(\Phi_m, A_m \Phi_m) = \beta(\Phi_m, \tilde{K}_m \Phi_m) = \beta(\Phi_0, K_0 \Phi_0) - \beta(\Phi_m, P_m \Phi_m) - \beta(\Phi_m, V_m \Phi_m),$$

so the theorem follows by the $m = 0$ case, Prop. 4.5 eq. (56) and the identity

$$-\beta(\Phi_m, P_m \Phi_m) = \frac{m^2}{2} \int_B (1 - r^2) \Phi^2_m \ d\mathbf{x} \ .$$

\hfill \square
5.5 The massive modular local Hamiltonian

This section contains our main result Theorem 5.13.

Denote by $P_m$ the cutting projection on $\mathcal{H}_m$ relative to $H_m(B)$.

**Proposition 5.9.** $P_m$ is given by the matrix

$$P_m = \begin{bmatrix} P_+ & 0 \\ 0 & P_- \end{bmatrix},$$

with $P_\pm : D(P_\pm) \subset H^{1/2}_m \to H^{1/2}_m$ the operator of multiplication by the characteristic function $\chi_B$ of $B$ in $H^{1/2}_m$.

**Proof.** By duality (33), $H_m(B)' = H_m(B')$, therefore $D(P_m) = H_m(B) + H_m(B')$.

By Prop. 2.5, $P_{m^\dagger}|_{H_m(B)}$ is a real linear, densely defined operator on $H_m(B)$. Clearly $P_m$ acts as in (59) on vectors in $H_m(B) + H_m(B')$ given by smooth functions, hence on all $D(P_m)$ as $P_m$ is a closed operator [11].

By Prop. 2.5, $P_{m^\dagger}|_{H_m(B)}$ is a real linear, densely defined operator on $H_m(B)$.

**Proposition 5.10.** $(P_{m^\dagger} - P_0)_{H_m(B)}$ is a compact operator on $H_m(B)$, $m \geq 0$.

**Proof.** By (59), we have

$$P_{m^\dagger} = \begin{bmatrix} 0 & P_+ \mu_m^{-1} \\ -P_- \mu_m & 0 \end{bmatrix}.$$

As $P_{m^\dagger}$ is skew-selfadjoint, it suffices to show that $P_- \mu_m - P_- \mu_0 : H^{1/2}_m(B) \to H^{1/2}_m(B)$ is compact. Namely, that

$$f \in H^{1/2}_m(B) \mapsto (\mu_m - \mu_0)f|_B \in H^{-1/2}_m(B)$$

is compact. Now, in Fourier transform,

$$\left((\mu_m - \mu_0)f\right)(p) = \left(\sqrt{|p|^2 + m^2} - \sqrt{|p|^2}\right)\hat{f}(p) = \frac{m^2}{\sqrt{|p|^2 + m^2} + \sqrt{|p|^2}}\hat{f}(p)$$

so $(\mu_m - \mu_0) : H^{1/2}_m \to H^{1/2}_m$ is a bounded operator. We have the following diagram

$$\begin{array}{ccc}
H^{1/2}_m & \xrightarrow{P_- \mu_m - P_- \mu_0} & H^{-1/2}_m(B) \\
\downarrow \mu_m - \mu_0 & & \downarrow \iota_2 \\
H^{1/2}_m & \xrightarrow{\iota_1} & H^0_m = L^2(\mathbb{R}^d) \xrightarrow{\chi_B} H^0_m(B)
\end{array}$$

where $\chi_B$ is the multiplication by the characteristic function of $B$ in $L^2(\mathbb{R}^d)$, i.e. the orthogonal projection $L^2(\mathbb{R}^d) \to L^2(B)$, and $\iota_1$, $\iota_2$ are natural embeddings. Therefore $P_- \mu_m - P_- \mu_0 : H^{1/2}_m \to H^{-1/2}_m$ is compact because all the operators in the diagram are bounded and the embedding $\iota_2 : H^0_m(B) \hookrightarrow H^{-1/2}_m(B)$ is compact by Rellich’s theorem.

\[\square\]
Lemma 5.11. The point spectrum of $\Delta_B = \Delta_{B,m}$ is empty.

Proof. It was proved by Figliolini and Guido [15 Prop. 3.5] that 1 is not eigenvalue of $(\Delta_B + 1)(\Delta_B - 1)^{-1}$, and the same argument in that proof shows that no $\lambda \in \mathbb{R}$ can be an eigenvalue, implying the same for $\Delta_B$. □

We mention that Lemma 5.11 implies that the promotion of $\Delta_B$ on the Bose Fock space has no point spectrum but 1 with multiplicity one. Then the vacuum modular group of acts ergodically on the second quantisation von Neumann algebra $A_m(B)$ the Bose Fock space over $\mathcal{H}_m$. Therefore, $A_m(B)$ is a factor of type III$_1$ [12], see [1] [15].

The massive modular Hamiltonian can be characterised as “minimal” perturbation of the massless modular Hamiltonian in the sense of Proposition 5.12 or Corollary 5.14.

Proposition 5.12. $K_m = \pi_m A_m$ is the unique skew-selfadjoint operator $K = \pi A$ on $\mathcal{H}_m$, leaving $H_m(B)$ invariant (as in Prop. 2.2), such that

$$i_m f_\lambda(A)|_{H_m(B)} - i_0 f_\lambda(A_0)|_{H_m(B)}$$

is a compact operator on $H_m(B)$ for all $\lambda > 0$, with $f_\lambda(x) = e^{-\lambda x^2} \tanh(\pi x)$.

Proof. $K_m^B$ is a compact perturbation of $K_0^B$ by Proposition 5.6 and this implies that

$$\pi_m f_\lambda(A_m)|_{H_m(B)} - \pi_0 f_\lambda(A_0)|_{H_m(B)}$$

is also compact by Prop. 2.4, any $\lambda > 0$.

On the other hand, assume now that the operator (62) is compact on $H_m(B)$. Then

$$T \equiv i_m f_\lambda(A)|_{H_m(B)} - i_m f_\lambda(A_m)|_{H_m(B)}$$

is also a compact, by taking the side by side difference of equations (62) and (63).

Since $H_m(B)$ is $K$ and $K_m$-invariant, both $K$ and $K_m$ commute with $\Delta_{B,m}$. Thus $\Delta_{B,m}^i |_{H_m(B)}$ commutes with the compact operator $T$, so with $T^* T$.

Thus, the finite-dimensional eigen-spaces of $T^* T$ are left invariant by $\Delta_{B,m}^i$, so the point spectrum of $\Delta_{B,m}$ would not be empty if $T \neq 0$. By Lemma 5.11 we must then have $T = 0$. Therefore, for all $\lambda > 0$, we have $f_\lambda(A) = f_\lambda(A_m)$ on $H_m(B)$, thus on $\mathcal{H}_m$ because $H_m(B) + i m H_m(B)$ is dense and $f_\lambda$ is bounded. Since $e^{-\lambda x^2} \searrow 1$ as $\lambda \searrow 0$, we conclude that $\tanh(\pi A) = \tanh(\pi A_m)$, so $A = A_m$ because the hyperbolic tangent is one-to-one. □

Theorem 5.13. The modular Hamiltonian $\log \Delta_B$ associated with the unit ball $B$ in the free scalar, mass $m$ quantum field theory is given by

$$-2\pi A_m = \log \Delta_{B,m}.$$ 

Thus, on $H_m(B)$, we have

$$\log \Delta_{B,m} = 2\pi i_m \begin{bmatrix} 0 & \frac{1}{2} (1 - r^2) (\nabla^2 - m^2) - r \partial_r - D - \frac{1}{2} m^2 G_mB_m^B \frac{1}{2} (1 - r^2) \\ \frac{1}{2} (1 - r^2) (\nabla^2 - m^2) - r \partial_r - D - \frac{1}{2} m^2 G_mB_m^B \frac{1}{2} (1 - r^2) \end{bmatrix} \tag{64}$$

where $G_mB_m^B$ is the operator (53). $G_mB_m^B$ is given by the Helmholtz Green integral operator, with Yukawa potential if $d = 3$, see (56).
Proof. Due to Prop. 5.12 it suffices to show that

$$imf\left(\frac{1}{2}\log \Delta_{B,m}\right)\bigg|_{H_m(B)} - i_0f\left(\frac{1}{2}\log \Delta_{B,0}\right)\bigg|_{H_0(B)}$$

is compact (where $H_0(B)$ and $H_m(B)$ are identified), all $\lambda > 0$.

Now, formula (16) for the cutting projection and Prop. 5.10 entails that

$$im\coth\left(\frac{1}{2}\log \Delta_{B,m}\right)\bigg|_{H_m(B)} - i_0\coth\left(\frac{1}{2}\log \Delta_{B,0}\right)\bigg|_{H_0(B)}$$

is densely defined and compact, thus

$$im\tanh\left(\frac{1}{2}\log \Delta_{B,m}\right)\bigg|_{H_m(B)} - i_0\tanh\left(\frac{1}{2}\log \Delta_{B,0}\right)\bigg|_{H_0(B)}$$

is compact by equation (11). Therefore also the operator (65) is compact by applying Prop. 2.4, because

$$e^{-\lambda x^2}\tanh(x/2) = g\left(\tanh(x/2)\right)$$

with $g(x) = ix\exp\left(-\frac{\lambda}{2\pi} \log^2 \frac{1-x}{1+x}\right)$.

Corollary 5.14. $A_m = -\frac{1}{2}\log \Delta_{B,m}$ is the unique selfadjoint operator $A$ on $H_m$, leaving $H_m(B)$ invariant, such that $(imA - i_0A_0)|_{H_m(B)}$ is densely defined and compact on $H_m(B)$.

Proof. $K^B_m$ is a compact perturbation of $K^B_0$ by Prop. 5.6. Conversely, assume that $K|_{H_m(B)}$ is a compact perturbation of $K^B_0$ as above, with $K = imA$. Then $(imf\lambda|_{H_m(B)} - i_0f\lambda(A_0)|_{H_m(B)})$ is compact by Prop. 2.4. So $K = K_m$ by Prop. 5.13.

It would be interesting to have a direct proof of the following corollary, that manifests the thermodynamical nature of the modular Hamiltonian, cf. [31].

Corollary 5.15. $A_m$ is completely active with respect to $H_m(B)$.

Proof. Immediate by Theorem 2.10.

Let’s now consider balls of different radius. If $F$ is a function on $\mathbb{R}^n$ and $\lambda > 0$, we set $F_\lambda(x) = F(\lambda x)$. If $\Phi \in T_m$, we have

$$\left((\Box + \lambda^2 m^2)\Phi_\lambda\right)(x) = \lambda^2\left((\Box)\Phi(\lambda x) + m^2\Phi^2(\lambda x)\right) = 0,$$

thus $\Phi_\lambda \in T_{\lambda m}$ and we have a real linear map bijection $\delta_\lambda : T_m \to T_{\lambda m}$

$$\delta_\lambda : \Phi \in T_m \to \lambda^D\Phi_\lambda \in T_{\lambda m}.$$ (66)

In terms of the Cauchy data, we have $\delta_\lambda : w_m(f, g) \to w_{\lambda m}(\lambda^D f_\lambda, \lambda^{D+1}g_\lambda)$.

If $\Phi, \Psi \in T_m(B)$

$$\beta(\delta_\lambda\Phi, \delta_\lambda\Psi) = \frac{\lambda^d}{2}\int_{x=0} \left(\Psi(\lambda x)(\partial_0\Phi)(\lambda x) - \Phi(\lambda x)(\partial_0\Psi)(\lambda x)\right)dx = \beta(\Phi, \Psi),$$

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namely $\delta_\lambda$ preserves the symplectic form $\beta$. One can see that $\delta_\lambda$ intertwines the complex structure on $\mathcal{H}_m$ and $\mathcal{H}_{\lambda m}$, so $\delta_\lambda$ is a unitary

$$\delta_\lambda : \mathcal{H}_m \to \mathcal{H}_{\lambda m}.$$ 

Let $B_R$ the ball in $\mathbb{R}^d$ with center at the origin and radius $R > 0$. With $H_m(B_R)$ the standard subspace of $\mathcal{H}_m$ associated with $B_R$, we have

$$\delta_\lambda : H_m(B_R) \to H_{\lambda m}(B_{\lambda^{-1} R}).$$

With $\Delta_{R,m}$ the modular operator of $H_m(B_R)$, we then have $\Delta_{\lambda^{-1} R, \lambda m} = \delta_\lambda \Delta_{R,m} \delta_\lambda^{-1}$, so

$$\log \Delta_{m,R} = \delta_R \log \Delta_{R,m} \delta_R^{-1}.$$  

(67)

Now, $G_m$ is given by (54), so we have

$$G_{\lambda m}(x) = \frac{1}{(2\pi)^{d/2}} \left( \frac{\lambda m}{r} \right)^{d/2 - 1} \cdot K_{d/2 - 1}(m \lambda r) = \lambda^{(d-1)/2} G_m(\lambda x).$$

In particular, if $d = 3$, we have $K_{1/2}(r) = \left( \frac{\pi}{2r} \right)^{1/2} e^{-r}$ and

$$G_m(x) = \frac{1}{4\pi} \frac{e^{-mr}}{r}$$

see e.g. [36, Chapter 3]. $G_m$ is the Yukawa potential.

The above discussion yields the following formula for the matrix elements of the modular Hamiltonian.

**Theorem 5.16.** With $\log \Delta_{R,m}$ the mass $m$ modular Hamiltonian for the radius $R$ ball $B_R$, we have

$$-\Re(\Phi, \log \Delta_{R,m} \Psi) =$$

$$2\pi \int_{B_R} \frac{R^2 - r^2}{2R} (T_0^{(m)})_{\Phi, \Psi} d\mathbf{x} + 2\pi \frac{d - 1}{R} \int_{B_R} \Phi \Psi d\mathbf{x} + m^2 \frac{\pi}{R} \int_{B_R \times B_R} G_m(\mathbf{x} - \mathbf{y}) \Phi(\mathbf{y}) \Psi(\mathbf{x}) d\mathbf{x} d\mathbf{y},$$

where $\Phi, \Psi \in \mathcal{T}_m(B_R)$.

**Proof.** Immediate by the above discussion. \(\square\)

**Remark 5.17.** We make a comment on the locality of the massive Hamiltonian. The skew-selfadjoint modular Hamiltonian on $H_m(B)$ is proportional to $K_m^B$ and the only non-local term in $K_m^B$ is $V_m$. Thus $K_m^B$ acts locally on $(f, g)$ if $G_m^B$ acts locally on $f$.

Now, $\nabla^2 - m^2$ is a local operator, thus $G_m^B$ acts locally on $(\nabla^2 - m^2)|_{C^\infty_0(B)}$. Call $T$ the closure of the operator $-\nabla^2 + m^2|_{C^\infty_0(B)}$ as an operator on $L^2(B) \to L^2(B)$). Then $T$ is a closed, positive Hermitian operator, with lower bound $m^2$, and $T$ is local on its domain. The range $R(T)$ of $T$ is closed in $L^2(B)$, thus $R(T) = \ker(T^*)^\perp$.

Now $T^* : D(T^*) \subset L^2(B) \to L^2(B)$ is the operator $-\nabla^2 + m^2$ with domain $D(T^*) = \{ f \in L^2(B) : (-\nabla^2 + m^2)f \in L^2(B) \}$. 

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With \( \{ \varphi_k \} \) a basis for ker\( (T^*) \), we conclude that \( G^B_m \) is local on the orthogonal complement of \( \varphi_1, \varphi_2, \ldots \) in \( L^2(B) \). That is, if \( \langle f, g \rangle \in D(K^B_m) \cap H_m(\tilde{B}) \), with \( \tilde{B} \) a closed ball contained in \( B \), we have
\[
K^B_m \langle f, g \rangle \in H_m(\tilde{B}) \iff \int_B f(x) \varphi_k(x) dx = 0, \quad \forall k,
\]
(massive case) namely iff \( f \) has zero mean on \( B \) with respect to every \( \varphi_k \).

6 Entropy density of a wave packet

With \( \mathcal{H}_m \) our wave Hilbert space, the following proposition gives an explicit formula for the action of the cutting projection \( P_m \) on waves.

**Proposition 6.1.** Let \( \Phi \in \mathcal{T}_m \) and set \( \Psi = K_m \Phi \). Then \( \Psi = w_m(f, g) \in H_m(B) + H_m(B') \) and \( P_m \Psi = w_m(\chi_B f, \chi_B g) \), with \( \chi_B \) the characteristic function of \( B \).

**Proof.** As \( \Phi \) belongs to \( D(\log \Delta_B) = D(K_m) \), it follows that \( K_m \Phi \in D(P_m) \), thus \( P_m \Psi = w_m(\chi_B f, \chi_B g) \) by Prop. 5.9.

We can now compute the local entropy of a wave packet \( \Phi \).

**Proposition 6.2.** Let \( \Phi \in \mathcal{T}_m \). The entropy \( S_\Phi \) of \( \Phi \) with respect to \( H_m(B) \) is given by
\[
S_\Phi = \pi \int_B (\Psi \Phi' - \Phi \Psi') dx,
\]
with \( \Psi = K_m \Phi \) and the prime denotes the time derivative.

**Proof.** The proof now follows from Proposition 6.1 in analogy with the the proof given in \[11\] for the wedge region case.

**Theorem 6.3.** The entropy \( S_\Phi \) of the Klein-Gordon wave \( \Phi \in \mathcal{T}_m \) in the unit ball \( B \), i.e. with respect to \( H_m(B) \), is given by
\[
S_\Phi = 2\pi \int_B \frac{1 - r^2}{2} T^{(m)}_{00} \phi dx + \pi D \int_B \Phi^2 dx + \pi m^2 \int_{B \times B} G_m(x - y) \Phi(y) \Phi(x) dx dy.
\]
In particular, \( S_\Phi \) is finite if \( \Phi \in \mathcal{T}_m \).

**Proof.** With \( \Phi, \Psi \in \mathcal{T}_m \), define the real linear quadratic form
\[
q_B(\Phi, \Psi) = 2\pi \int_B \frac{1 - r^2}{2} T^{(m)}_{00} \phi dx + \pi D \int_B \Phi \Psi dx + \pi m^2 \int_{B \times B} G_m(x - y) \Phi(y) \Psi(x) dx dy.
\]
Clearly, \( q_B(\Phi, \Phi) < \infty \) if \( \Phi \in \mathcal{T}_m \). We want to show that
\[
S(\Phi, \Phi) = q_B(\Phi, \Phi), \quad \Phi \in \mathcal{T}_m,
\]
Figure 1: The spherical massless wave $\Phi(r) = \frac{\sin(r-t)}{r}$, $d = 3$, at time $t = 0$ and $t = 1$ ($\Phi$ is not everywhere defined). In this example, the entropy densities per area $S(r)$ are close at different times.

where $S$ is the entropy form with respect to $H_m(B)$. As $S$ is a closed by Prop. 2.7 it suffices to show that $q_B$ is closable on $T_m$ and that (70) holds on a form core for $S$.

Now, $q_B$ is given by

$$q_B(\Phi, \Psi) = \Re(\Phi, P_m^* A_m P_m \Psi), \quad \Phi, \Psi \in T_m,$$

similarly as in [11, Thm. 3.5]. The real linear operator $P_m^* A_m P_m$ is Hermitian, thus closable. As $q_B$ is positive, $q_B$ is closable, cf. the proof of [20, Thm. 1.27].

On the other hand, (70) holds if $\Phi \in T_m(B)$ by Theorem 5.8. Then it holds if $\Phi \in D(\log \Delta_B)$ by the same argument, using Prop. 6.2. As $D(\log \Delta_B)$ is a form core for $S$ (see Sect. 2.4), we conclude that $S = q_B$ on $T_m$. \hfill \Box

Denote by $B_R(\bar{x})$ the radius $R$ space ball around the point $\bar{x} \in \mathbb{R}^d$.

**Corollary 6.4.** The entropy $S_\Phi(R) = S_\Phi(R, t, \bar{x})$ of the wave packet $\Phi \in T_m$ in the space region $B_R(\bar{x})$ at time $t$ is given by

$$S_\Phi(R) = \pi \int_{B_R(\bar{x})} \frac{R^2 - r^2}{R} \langle \Phi(0_0) \rangle \Phi d\mathbf{x}$$

$$+ \frac{\pi D}{R} \int_{B_R(\bar{x})} \Phi^2 d\mathbf{x} + \frac{\pi m^2}{R} \int_{x, y \in B_R(\bar{x})} G_m(x - y) \Phi(y) \Phi(x) d\mathbf{x} d\mathbf{y},$$

with $r = |x - \bar{x}|$, $(x_0 = t$ integral).

**Proof.** In view of Theorem 5.16 and the formula for the cutting projection in Proposition 6.1 the theorem follows, in the time zero case, by the entropy formula (19). By translation covariance, we get the formula at arbitrary time. \hfill \Box

For large $R$, $S_\Phi(R)/R$ gets proportional to the total local energy $E = \int_{B_R(\bar{x})} \langle 0_0 \rangle \Phi(t, x) d\mathbf{x}$:

$$\frac{S_\Phi(R)}{R} \sim \pi E,$$
as \( R \to \infty \) like in [24, (41)], see also [28].

It follows that, at fixed time \( t \), the local entropy \( S_{\Phi}(R, x) = S_{\Phi}(R, t, x) \) has an expansion

\[
S_{\Phi}(R, x) = \pi \left( \left\langle T_{00} \right\rangle_{\Phi}(t, x) + D\Phi^2(t, x) \right) V_d R^{d-1} + \ldots
\]

as \( R \to 0 \); here \( A_{d-1}(R) = 2\pi^{d/2} R^{d-1}/\Gamma(d/2) \) is the area of the \((d-1)\)-dimensional sphere \( \partial B_R \) and \( V_d = A_{d-1}(1)/d \) is the volume of \( B \). So the energy density of the wave packet \( \Phi \) around a point gets proportional to the area of the sphere boundary of \( B_R \), as expected by the holographic area theorems for the entropy, in black hole and other contexts, see [5].

Note that the leading coefficient of entropy density \( S_{\Phi}(R, x)/A_{d-1}(R) \) is the sum of two terms, that are proportional one to the energy \( \left\langle T_{00} \right\rangle_{\Phi}(x) \) and the other to the square of the height \( \Phi^2(x) \) of the wave packet at the point \( x \). In other words, it depends quadratically on the height and on the space and time slopes of \( \Phi \).

As \( R \to \infty \), we then have the asymptotic bound for the local entropy

\[
S_{\Phi}(R) \leq \pi E R
\]

with \( E \) the total energy in the ball \( B_R \), namely the Bekenstein bound holds, see [28] and references therein.

6.1 Quantum Field Theory

By the analysis in [25, 11], we have an immediate corollary in Quantum Field Theory concerning the local vacuum relative entropy of a coherent state.

**Corollary 6.5.** Let \( A_m(O_R) \) be the von Neumann algebra associated with the double cone \( O_R \) (the causal envelope of \( B_R \)) by the free, neutral quantum field theory with mass \( m \geq 0 \). The relative entropy \( S(\varphi_\Phi|\varphi) \) (see [2]) between the vacuum state \( \varphi \) and the coherent state \( \varphi_\Phi \) associated with the one-particle wave \( \Phi \in H_m \) is given by \( S_{\Phi}(R) \) by Corollary 6.4 (with \( B_R \) centred at the origin).

**Proof.** As shown in [11], \( S(\varphi_\Phi|\varphi) \) is equal to the entropy of the vector \( \Phi \in H_m \) with respect to the standard subspace \( H_m(O_R) \). So Corollary 6.4 applies. \( \square \)

7 Appendix. Elementary relations

For the reader’s convenience, we collect a couple of elementary identities that are used in the text. Note first that \( \partial_r \) is the partial derivative in the direction \( \vec{r} = (\frac{x_1}{r}, \ldots, \frac{x_d}{r}) \), thus

\[
r \partial_r f = \vec{x} \cdot \nabla f,
\]

for any \( f \in S \).
Identity 7.1. Let $f \in S$. We have
\[
\int_{\mathbb{R}^d} \frac{1}{2} (1 - r^2) |\nabla f|^2 \, dx = - \int_{\mathbb{R}^d} \frac{1}{2} (1 - r^2) f \nabla^2 f \, dx + \int_{\mathbb{R}^d} r f \partial_r f \, dx.
\]

Proof. The identity follows immediately by the following two relations:
\[
- \int_{\mathbb{R}^d} \frac{1}{2} (1 - r^2) f \nabla^2 f \, dx = \int_{\mathbb{R}^d} \frac{1}{2} (1 - r^2) |\nabla f|^2 \, dx + \frac{d}{2} \int_{\mathbb{R}^d} f^2 \, dx, \quad (72)
\]
\[
\int_{\mathbb{R}^d} r f \partial_r f \, dx = - \frac{d}{2} \int_{\mathbb{R}^d} f^2 \, dx. \quad (73)
\]
Concerning the second relation, by (71) we have
\[
\int_{\mathbb{R}^d} r f \partial_r f \, dx = \sum_k \int_{\mathbb{R}^d} x_k f \partial_k f \, dx = \frac{1}{2} \sum_k \int_{\mathbb{R}^d} x_k \partial_k (f^2) \, dx
\]
\[
= - \frac{1}{2} \sum_k \int_{\mathbb{R}^d} f^2 \, dx = - \frac{d}{2} \int_{\mathbb{R}^d} f^2 \, dx.
\]
Then the first relation follows by
\[
\int_{\mathbb{R}^d} (1 - r^2) f \nabla^2 f \, dx = - \sum_k \int_{\mathbb{R}^d} \partial_k ((1 - r^2) f) \partial_k f \, dx
\]
\[
= 2 \sum_k \int_{\mathbb{R}^d} x_k f \partial_k f \, dx - \sum_k \int_{\mathbb{R}^d} (1 - r^2) (\partial_k f)^2 \, dx
\]
\[
= -d \int_{\mathbb{R}^d} f^2 \, dx - \int_{\mathbb{R}^d} (1 - r^2) |\nabla f|^2 \, dx.
\]

Identity 7.2. With $Z$ and $\gamma$ as in Section 3, we have
\[
Z'(z, s)_{s=0} = (1 - z^2)/2,
\]
\[
\gamma'(u, v; 0) = -\frac{D}{2} (u + v) = -D x_0.
\]

Proof. Denoting by a prime the derivative with respect to the $s$-parameter, we have
\[
f(z, 0) = 1, \quad f'(z, 0) = -\frac{e^{-s}(1 - z)}{2} \big|_{s=0} = \frac{1 - z}{2},
\]
\[
g(z, 0) = z, \quad g'(z, s) = \frac{e^{-s}(1 - z)}{2} \big|_{s=0} = \frac{1 - z}{2}.
\]
Since $Z = g/f$, we get
\[
Z'(z, s)_{s=0} = (1 - z^2)/2.
\]
We have
\[ \gamma'(u, v; s)|_{s=0} = (F(u, s)F(-v, -s))'|_{s=0} = F'(u, s)|_{s=0} - F'(-v, -s)|_{s=0}. \]

Since
\[ F'(z, s)|_{s=0} = -d(f(z, s))^{-D-1}f'(z, s)|_{s=0} = \frac{D}{2}(1 - z), \]
we have
\[ \gamma'(u, v; 0) = -\frac{D}{2}(u + v) = -D x_0. \]

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