SPLITTINGS AND SYMBOLIC POWERS OF SQUARE-FREE MONOMIAL IDEALS

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ABSTRACT. We study the symbolic powers of square-free monomial ideals via symbolic Rees algebras and methods in prime characteristic. In particular, we prove that the symbolic Rees algebra and the symbolic associated graded algebra are split with respect to a morphism which resembles the Frobenius map and that exists in all characteristics. Using these methods, we recover a result by Hoa and Trung which states that the normalized \(a\)-invariants and the Castelnuovo-Mumford regularity of the symbolic powers converge. In addition, we give a sufficient condition for the equality of the ordinary and symbolic powers of this family of ideals, and relate it to Conforti-Cornuéjols conjecture. Finally, we interpret this condition in the context of linear optimization.

1. Introduction

Symbolic powers of ideals have been studied intensely over the last two decades (see [DDSG+18] for a recent survey). Particular attention has been given to square-free monomial ideals, as in this setting methods from combinatorics, convex geometry, and linear optimization can be utilized to investigate properties of symbolic powers. For instance, the Cohen-Macaulay property of all symbolic powers of a square-free monomial ideal is characterized in terms of the combinatoric structure of its underlying simplicial complex [TT11, TT12, Var11]. In addition, the symbolic Rees algebra of monomial ideals is Noetherian [HHT07]; we note that this phenomenon does not hold for an arbitrary ideal in a polynomial ring [Rob85].

In this article, we propose a technique to deal with questions about symbolic powers of square-free monomial ideals. Specifically, we study the symbolic Rees algebras of a square-free monomial ideal via methods in prime characteristic. This combination, to the best of our knowledge, has not been previously used in combinatorial commutative algebra. In order for our results to hold over fields of arbitrary characteristic, we consider the map that raises every monomial to a power and resembles the Frobenius map. Considering this map, we obtain that the symbolic Rees algebra and the symbolic associated graded algebra are split in this general context (see Theorem 4.3). In particular, these symbolic algebras are \(F\)-pure in prime characteristic (see Corollary 4.4).

Motivated by the behavior of the Castelnuovo-Mumford regularity for powers of ideals [CHT99, Kod00], Herzog, Hoa, and Trung [HT02] asked whether the limit

\[
\lim_{n \to \infty} \frac{\operatorname{reg}(R/I^{(n)})}{n}
\]

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exists for every homogeneous ideal $I$ in a polynomial ring $R$. It is known that the function $\text{reg}(R/I(n))$ is bounded by a linear function on $n$ if $I$ is a monomial ideal. This follows because the regularity of a monomial ideal is bounded by the degree of the least common multiple of the generators [BH95, HT98].

Herzog and Hoa showed that the limit above exists for square-free monomial ideals. In fact, they showed a stronger version for the $a$-invariants [HT10, Theorems 4.7 and 4.9]. As a first consequence of our methods, we recover this result, providing an alternative proof.

**Theorem A** (see Theorem 3.5 and Corollary 3.6). Let $I$ be a square-free monomial ideal. Then,

\[
\lim_{n \to \infty} \frac{a_i(R/I(n))}{n}
\]

exists for every $0 \leq i \leq \dim R/I$. In particular,

\[
\lim_{n \to \infty} \frac{\text{reg}(R/I(n))}{n}
\]

exists.

The function $\text{reg}(R/I(n))$ is in fact a linear quasi-polynomial if $I$ is a monomial ideal [HT02]. In addition, as a consequence of our previous result, we recover properties of this quasi-polynomial in Corollary 3.8. Along the way of proving Theorem A, we showed that

\[a_i(R/I(n)) \geq ma_i(R/I(\lceil \frac{n}{m} \rceil))\]

for every $n, m \in \mathbb{N}$. In addition, we showed that

\[\text{depth}(R/I(n)) \leq \text{depth}(R/I(\lceil \frac{n}{m} \rceil))\]

for every $n, m \in \mathbb{N}$ (see Theorem 3.4). Similar results were previously obtained for the Stanley depth of symbolic powers [SF17].

Conforti and Cornuéjols [CC90] made a conjecture in the context of linear optimization. This conjecture was translated as a characterization of the set of square-free monomials ideals whose symbolic and ordinary powers are equal [GRV09, GVV07]. The following definition is needed to state this conjecture.

**Definition B.** A square-free monomial ideal $I$ of height $c$ is König if there exists a regular sequence of monomials in $I$ of length $c$. The ideal $I$ is said to have the packing property if every ideal obtained from $I$ by setting any number of variables equal to 0 or 1 is König.

The Conforti-Cornuéjols conjecture can be stated as follows.

**Conjecture C** ([CC90]). A square-free monomial ideal $I$ is packed if and only if $I^n = I(n)$ for every $n \in \mathbb{N}$.

We point out that one direction of the conjecture is already know. Explicitly, if $I^n = I(n)$ for every $n \in \mathbb{N}$, then $I$ is packed.

Motivated by this conjecture Hà asked\(^1\) if there exists a number $N$, in terms of $I$, such that if $I^n = I(n)$ for $n \leq N$, then the equality holds for every $n \in \mathbb{N}$. If the answer to this question is $\text{ht}(I)$, then Conforti-Cornuéjols. This is expected as a similar property is known for integral closure of ordinary powers. Specifically, it suffices to verify the equality up to the analytic spread of $I$ minus one [Sin07] (see also [RRV03]). In our next main result, we answer Hà’s question.

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**Theorem D** (see Theorem 4.8). Let $I$ be a square-free monomial ideal and $\mu(I)$ its minimal number of generators. Then, $I^n = I^{(n)}$ for every $n \leq \left\lceil \frac{\mu(I)}{2} \right\rceil$ if and only if $I^n = I^{(n)}$ for every $n \in \mathbb{N}$.

The previous result gives a finite algorithm to verify Conjecture C for a specific monomial ideal. We refer to the work of Gitler, Valencia and Villarreal [GVV07, Remark 3.5] for a different algorithm to verify Conjecture C. In Example 4.9 we show that the bound given by $N$ in Theorem D is sharp.

In Section 5, we recall the ideas from linear optimization that originally gave rise to Conjecture C. In particular, we translate Theorem D to this context in Theorem 5.2, showing that the Max-Flow-Min-Cut property of clutters can be verified with a finite process (see [BT82, Corollary 2.3] for a related result).

## 2. Notation

In this section we set up the notation used throughout the entire manuscript. We assume $R = k[x_1, \ldots, x_d]$ is a standard graded polynomial ring over the field $k$ and $\mathfrak{m} = (x_1, \ldots, x_d)$. The ideal $I \subseteq R$ is assumed to be a monomial ideal.

For a fixed $m \in \mathbb{Z}_{>0}$ we set $R^{1/m} = k[x_1^{1/m}, \ldots, x_d^{1/m}]$. We denote by $I^{1/m}$ the ideal of $R^{1/m}$ generated by $\{f^{1/m} \mid f \in I \text{ monomial}\}$.

We note that $R$ and $R^{1/m}$ are isomorphic as rings. Then, the category of $R$-modules is naturally equivalent to the category of $R^{1/m}$-modules. For an $R$-module, $M$, we denote by $M^{1/m}$ the corresponding $R^{1/m}$-module. Given that $R \subseteq R^{1/m}$, $M^{1/m}$ obtains a structure of $R$-module via restriction of scalars. In addition, $I$ corresponds to $I^{1/m}$ under this isomorphism. We consider the analogous notation for $A^{1/m} = k[x_1^{1/m}, \ldots, x_d^{1/m}, t^{1/m}]$. We often refer to the containment $R \subseteq R^{1/m}$ and $A \subseteq A^{1/m}$.

For a vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Q}_{\geq 0}^d$ we denote by $x^\mathbf{a}$ the monomial $x_1^{a_1} \cdots x_d^{a_d}$.

**Definition 2.1.** Given $n \in \mathbb{N}$, we denote by $I^{(n)}$ the $n$th-symbolic power of $I$:

$$I^{(n)} = \bigcap_{p \in \text{Min}(R/I)} I^n R_p \cap R.$$

We now consider algebras associated to ordinary and symbolic powers of ideals.

**Definition 2.2.** We consider the following graded algebras.

(i) The Rees algebra of $I$: $\mathcal{R}(I) = R[It] = \oplus_{n \in \mathbb{N}} I^{n} t^{n} \subseteq A$.

(ii) The associated graded algebra of $I$: $\text{gr}_I(R) = \oplus_{n \in \mathbb{N}} I^{n}/I^{n+1}$.

(iii) The symbolic Rees algebra of $I$: $\mathcal{R}^s(I) = \oplus_{n \in \mathbb{N}} I^{(n)} t^{n} \subseteq A$.

(iv) The symbolic associated graded algebra of $I$: $\text{gr}_I^s(R) = \oplus_{n \in \mathbb{N}} I^{(n)}/I^{(n+1)}$.

## 3. Castelnuovo-Mumford Regularity and $a$-invariants

In this section we study the graded structure of the symbolic powers. In particular, we prove Theorem A. The techniques here are inspired by methods in prime characteristic used to bound $a$-invariants of $F$-pure graded rings [HR76, DSNnB18]. In particular, the results proved in this section are motivated by the fact that the symbolic Rees and associated graded algebra are $F$-pure for every prime (see Corollary 4.4).

If $M$ is a graded $R$-module, we denote by

$$a_i(M) = \max \{j \mid H^i_m(M)_j \neq 0\}$$

for $i = 0, 1, \ldots$.
the $i$-th $a$-invariant of $M$ for $0 \leq i \leq \dim M$. The Castelnuovo-Mumford regularity of $M$ is defined as
\[ \reg(M) = \max\{a_i(M) + i\}. \]

We now introduce the notation necessary to formalize the idea of $m$-roots.

**Remark 3.1.** Let $I \subseteq R$ be a square-free monomial and $Q_1, \ldots, Q_s$ its minimal primes. We have
\[ I^{(m)} = Q_1^{n} \cap \ldots \cap Q_s^{n} \subseteq (Q_1^{nm})^{1/m} \cap \ldots \cap (Q_s^{nm})^{1/m} = (Q_1^{nm} \cap \ldots \cap Q_s^{nm})^{1/m} = (I^{(nm)})^{1/m}. \]

We now define a splitting from $R^{1/m}$ to $R$ which is inspired by the trace map in prime characteristic. In fact, these maps are the same if $k = \mathbb{F}_p$ and $m = p$.

**Definition 3.2.** For $m \in \mathbb{Z}_{>0}$, we define the $R$-homomorphism $\Phi^R_m : R^{1/m} \to R$ induced by
\[ \Phi^R_m(x^{a/m}) = \begin{cases} x^{a/m} & a \equiv 0 \pmod{m}; \\ 0 & \text{otherwise}. \end{cases} \]

Where $0 = (0, \ldots, 0) \in \mathbb{N}^d$. We note that $\Phi^R_m$ restricted to $R$ is the identity. Then, $R$ is isomorphic to a direct summand of $R^{1/m}$. We also consider the analogous map $\Phi^A_m : A^{1/m} \to A$. We now show that our splitting is compatible with symbolic powers.

**Lemma 3.3.** Let $I$ be a square-free monomial ideal. Then,
\[ \Phi^R_m((I^{(nm+j)})^{1/m}) \subseteq I^{(n+1)} \]
for every $n$, $m \in \mathbb{N}$ and $1 \leq j \leq m$.

**Proof.** We first prove our claim when $I$ is a prime monomial ideal $Q = (x_{i_1}, \ldots, x_{i_d})$ for some $1 \leq i_1 < \cdots < i_d \leq d$. In this case, we have that $(Q^{(nm+j)})^{1/m}$ is generated as a $k$-vector space by
\[ \{(x^a)^{1/m} \mid a_{i_1} + \ldots + a_{i_d} \geq nm + j\} \]
Let $(x^a)^{1/m} \in (Q^{(nm+j)})^{1/m}$ such that $\Phi^R_m((x^a)^{1/m}) \neq 0$, then $a/m \in \mathbb{N}^d$ and $a_{i_1} + \ldots + a_{i_d} \geq nm + j$. Set $b = (b_1, \ldots, b_d) = a/m$, we have $b_1 + \ldots + b_d \geq n + \frac{j}{m}$, and hence $\Phi^R_m((x^a)^{1/m}) \in Q^{(n+1)}$ as desired. Now, let $I$ be an arbitrary square-free monomial ideal and let $Q_1, \ldots, Q_s$ be its minimal primes, then $(I^{(nm+j)})^{1/m} = (Q_1^{(nm+j)})^{1/m} \cap \ldots \cap (Q_s^{(nm+j)})^{1/m}$. Therefore,
\[ \Phi^R_m((I^{(nm+j)})^{1/m}) \subseteq \Phi^R_m((Q_1^{(nm+j)})^{1/m}) \cap \ldots \cap \Phi^R_m((Q_s^{(nm+j)})^{1/m}) \subseteq Q_1^{(n+1)} \cap \ldots \cap Q_s^{(n+1)} = I^{(n+1)}, \]

hence the result follows. \qed

As a consequence of the previous result we obtain the following relations on depths and $a$-invariants of symbolic powers; these relations are key ingredients in the proof of Theorem A.

**Theorem 3.4.** Let $I$ be a square-free monomial ideal and $n$, $m \in \mathbb{N}$. Then
\begin{enumerate}
\item $\depth(R/I^{(n)}) \leq \depth(R/I^{((\frac{n}{m})^i)})$.
\item $a_i(R/I^{(n)}) \geq ma_i(R/I^{((\frac{n}{m})^i)})$ for every $0 \leq i \leq \dim R/I$.
\end{enumerate}
Proof. By Lemma 3.3 the natural map \( \iota: R/I^{(n+1)} \to R^{1/m}/(I^{nm+j})^{1/m} \) splits for every \( n, m \in \mathbb{N} \) and \( 1 \leq j \leq m \) with the splitting \( \Phi^n_m \) as in Lemma 3.3. Therefore, the module \( H^n_m(R/I^{(n+1)}) \) is a direct summand of \( H^n_m(R^{1/m}/(I^{nm+j})^{1/m}) \) for every \( 1 \leq i \leq \dim R/I \). We note that

\[
H^n_m(R^{1/m}/(I^{nm+j})^{1/m}) = (H^n_m(R/I^{nm+j}))^{1/m}.
\]

We conclude that \( H^n_m(R/I^{nm+j}) = 0 \) implies that \( H^n_m(R^{1/m}/(I^{nm+j})^{1/m}) = 0 \), and hence \( H^n_m(R/I^{(n+1)}) = 0 \). Therefore,

\[
\text{depth}(R/I^{(n+1)}) \geq \text{depth}(R/(I^{nm+j})),
\]

which proves the first part.

From Equation (3.1), we have

\[
a_i(R/I^{(n+1)}) \leq a_i(R^{1/m}/(I^{nm+j})^{1/m}) = \frac{1}{m} a_i(R/I^{nm+j}),
\]

and the second part follows. \( \square \)

As a consequence of Theorem 3.4 we recover the following limits for \( a \)-invariants of symbolic powers.

**Theorem 3.5 ([HT10, Theorem 4.7]).** Let \( I \) be a square-free monomial ideal. Then,

\[
\lim_{n \to \infty} \frac{a_i(R/I^{(n)})}{n}
\]

exists for every \( 0 \leq i \leq \dim R/I \).

Proof. Fix \( i \). The sequence \( \{\frac{\text{reg}(R/I^{(n)})}{n}\}_{n \in \mathbb{N}} \) has an upper bound [HHT07, 3.3], then so does \( \{\frac{a_i(R/I^{(n)})}{n}\}_{n \in \mathbb{N}} \). Set \( \alpha_n = \frac{a_i(R/I^{(n)})}{n} \) for every \( n \in \mathbb{N} \) and \( \alpha = \sup\{\alpha_n\} \).

If \( \alpha = -\infty \), we have that \( \alpha_n = -\infty \) for every \( n \) and the claim follows. We now assume that \( \alpha \neq -\infty \) and show that \( \lim \alpha_n = \alpha \). We note that Theorem 3.4(2) implies

\[
\alpha_n \geq \frac{m}{n} \left\lfloor \frac{n}{m} \right\rfloor \alpha_{\left\lfloor \frac{n}{m} \right\rfloor} \quad \text{for every } n, m \in \mathbb{N}.
\]

Fix \( \epsilon \in \mathbb{R}_{>0} \) and let \( t \in \mathbb{N} \) such that \( \alpha - \alpha_t < \epsilon \). It suffices to show \( \alpha_n > \alpha_t \) for every \( n \geq t^2 \) as this implies that \( \alpha - \alpha_n < \epsilon \). Let \( n = tq + r \) for some \( q, r \in \mathbb{N} \) and \( r \in [0, t-1] \), then \( n \geq t^2 \), we obtain \( q \geq t \) and then, \( 0 < t - r \leq q - r < q + 1 \), and so \( \left\lfloor \frac{n}{q+1} \right\rfloor = t \). Applying Inequality (3.2) with \( m = q + 1 \) we have \( \alpha_n \geq \alpha_t + \frac{t}{t+q+1} \alpha_t > \alpha_t \), which finishes the proof. \( \square \)

As a corollary we obtain that the related limit for the Castelnuovo-Mumford regularity of symbolic powers exists.

**Corollary 3.6 ([HT10, Theorem 4.9]).** Let \( I \) be a square-free monomial ideal. Then,

\[
\lim_{n \to \infty} \frac{\text{reg}(R/I^{(n)})}{n} = \max \left\{ \lim_{n \to \infty} \frac{a_i(R/I^{(n)})}{n} \right\}.
\]

Proof. The result follows from Theorem 3.5 and the following equalities

\[
\lim_{n \to \infty} \frac{\text{reg}(R/I^{(n)})}{n} = \lim_{n \to \infty} \max \left\{ \alpha_i(R/I^{(n)}) + i \right\} = \lim_{n \to \infty} \max \left\{ \frac{a_i(R/I^{(n)})}{n} \right\} = \max \left\{ \lim_{n \to \infty} \frac{a_i(R/I^{(n)})}{n} \right\}.
\]
Remark 3.7. If $I$ is a square-free monomial ideal, then
\[
\lim_{n \to \infty} \frac{\text{reg}(I^n)}{n} = \lim_{n \to \infty} \frac{\text{reg}(R/I^n) + 1}{n} = \lim_{n \to \infty} \frac{\text{reg}(R/I^n)}{n}.
\]
Let $\alpha(I^n)$ denote the smallest degree of a nonzero element of $I^n$. The Waldschmidt constant is defined by
\[
\hat{\alpha}(I) = \lim_{n \to \infty} \frac{\alpha(I^n)}{n}.
\]
Then,
\[
\hat{\alpha}(I) \leq \lim_{n \to \infty} \frac{\text{reg}(I^n)}{n} = \lim_{n \to \infty} \frac{\text{reg}(R/I^n)}{n}.
\]
We recall that $R^s(I)$ is a Noetherian algebra [HHT07, 3.2]; therefore, $\text{reg}(I^n)$ agrees with a linear quasi-polynomial $\gamma(n) + \theta(n)$ for $n \gg 0$. As a consequence of Theorem 3.4 we obtain that the leading coefficient of this quasi-polynomial is constant for square-free monomial ideals and an bound for $\theta(n)$.

Corollary 3.8. [HT10, Theorem 4.9] Let $I$ be a square-free monomial ideal. Then, $\gamma(n)$ is equal to a constant $\gamma$ for $n \gg 0$ and $\theta(n) \leq \dim(R/I) + 1$.

Proof. From Corollary 3.6 it follows that $\gamma(n)$ must be equal to $\gamma := \sup\{a_i(R/I^n)\}$ for $n \gg 0$. For the second claim, we observe that
\[
\gamma(n) + \theta(n) = \text{reg}(R/I^n) + 1 = \max\{a_i(R/I^n) + i\} + 1 \leq \gamma(n) + \dim(R/I) + 1.
\]
for $n \gg 0$. We conclude that $\theta(n) \leq \dim(R/I) + 1$ for every $n \in \mathbb{N}$.

The previous results, together with results for matroids [MT17] and low dimension [HT16], motivated Minh and Trung [MT17] to ask the following question.

Question 3.9 ([MT17]). Let $I$ be a square-free monomial ideal. Is $\text{reg}(R/I^n)$ a linear polynomial for $n \gg 0$?

It is a classical result that for any homogeneous ideal $I$, $\text{reg}(I^n)$ is a linear function $b(I)n + c(I)$ for $n \gg 0$ (see [CHT99, Kod00]). In general, not much is known about the invariant $c(I)$ besides the fact that it is non-negative [TW05, 3.3]. Corollary 3.8 provides an upper bound for $c(I)$ for a wide family of square-free monomial ideals.

Corollary 3.10. Let $I$ be a square-free monomial ideal such that $I^n = I^{(n)}$ for every $n \gg 0$, then $c(I) \leq \dim(R/I) + 1$. In particular, this holds for bipartite edge ideals.

Proof. The result follows by the assumption and Corollary 3.8. The case of edge ideals follows because they satisfy $I^n = I^{(n)}$ for every $n \gg 0$ [SVV94, Theorem 5.9].

4. ASSOCIATED GRADED ALGEBRAS AND EQUALITY OF SYMBOLIC AND ORDINARY POWERS

In this section, we study the graded algebras defined in Definition 2.2. This is in order to prove Theorem D. Our strategy is the following. We first show that the symbolic Rees and associated graded algebras split from its rings of $m$-roots in Theorem 4.3. Then, in Theorem 4.5, we characterize the equality of symbolic and ordinary powers in terms of this splitting. Finally, we use this characterization to prove Theorem D. We start with introducing rings of $m$-roots for the Rees and associated graded algebras.
Notation 4.1. We set
\[ \mathcal{R}(I)^{1/m} := \oplus_{n \in \mathbb{N}} I^{n/m} t^{n/m} \subseteq A^{1/m} \]
and
\[ \mathcal{R}^s(I)^{1/m} := \oplus_{n \in \mathbb{N}} (I^{(n)})^{1/m} t^{n/m} \subseteq A^{1/m}. \]
We consider the ideals
\[ \mathcal{J}(I) = \oplus_{n \in \mathbb{N}} I^{n+1} t^n \subseteq \mathcal{R}(I) \]
and
\[ \mathcal{J}^s(I) = \oplus_{n \in \mathbb{N}} I^{(n+1)} t^n \subseteq \mathcal{R}^s(I). \]
A classical result states that \( \text{gr}_I(R) \) is reduced if and only if \( I^n = I^{(n)} \) for every \( n \in \mathbb{N} \) [HHTZ08, Corollary 1.6]. As a consequence of this result, and its proof, one has that \( \mathcal{J}^s(I) \) is a radical ideal. We set,
\[ \text{gr}_I^s(R)^{1/m} = (\mathcal{R}^s(I)/\mathcal{J}^s(I))^{1/m}. \]

Remark 4.2.
1. By Remark 3.1 we have the inclusion
\[ \mathcal{R}^s(I)^{1/m} \subseteq \oplus_{n \in \mathbb{N}} (I^{(n)})^{1/m} t^{n/m} = \mathcal{R}^s(I)^{1/m}. \]
2. Since \( I \subseteq R \) is monomial, we have that \( I^n \subseteq (I^{nm})^{1/m} \). Then,
\[ \mathcal{R}(I) = \oplus_{n \in \mathbb{N}} I^{n+1} t^n \subseteq \oplus_{n \in \mathbb{N}} I^{nm+1} t^n = \mathcal{R}(I)^{1/m}. \]

Theorem 4.3. Let \( I \) be a square-free monomial ideal. Then, the maps induced by the observation in Remark 3.1
\[ \mathcal{R}^s(I) \rightarrow \mathcal{R}^s(I)^{1/m} \quad \text{and} \quad \text{gr}_I^s(R) \rightarrow \text{gr}_I^s(R)^{1/m} \]
split for every \( m \in \mathbb{N} \).

Proof. Fix \( m \in \mathbb{N} \) and let \( \Phi_m^R \) be the splitting in Definition 3.2. We define
\[ \varphi : \mathcal{R}^s(I)^{1/m} \rightarrow \mathcal{R}^s(I) \]
to be the homogeneous morphism of \( \mathcal{R}^s(I) \)-modules induced by \( \varphi(r^{1/m} t^{n/m}) = \Phi_m^R(r^{1/m}) t^{n/m} \)
if \( m \) divides \( n \), and \( \varphi(r^{1/m} t^{n/m}) = 0 \) otherwise. The map \( \varphi \) is well-defined because
\[ \Phi_m^R((I^{((n+1)m)})^{1/m}) \subseteq I^{(n+1)} \]
for every \( n \in \mathbb{N} \) by Lemma 3.3, and it is \( \mathcal{R}^s(I) \)-linear since \( \Phi_m^R \) is \( R \)-linear. If \( r \in I^{(n)} \subseteq (I^{nm})^{1/m} \), then \( \varphi(r t^{nm/m}) = \Phi_m^R(r) t^n = r t^n \) because \( \Phi_m^R \) is a splitting. We conclude that \( \varphi \)
is also a splitting.
Consider the ideal \( \mathcal{J} = \mathcal{J}^s(I) \) as in Notation 4.1. By Lemma 3.3 we obtain
\[ \varphi((I^{(nm+1)})^{1/m} t^{n+1} t^n) \subseteq I^{(n+1)} t^n \]
for every \( n \in \mathbb{N} \). Therefore, \( \varphi(\mathcal{J}^{1/m}) \subseteq \mathcal{J} \), i.e., \( \mathcal{J} \) is compatible with \( \varphi \). This induces a splitting \( \overline{\varphi} : (\mathcal{R}^s(I)/\mathcal{J})^{1/m} \rightarrow \mathcal{R}^s(I)/\mathcal{J} \). The conclusion follows.

Theorem 4.3 has the following consequence if the field \( k \) has positive characteristic.

Corollary 4.4. Let \( I \) be a square-free monomial ideal. If \( k \) is a perfect field of prime characteristic \( p \), then \( \mathcal{R}^s(I) \) and \( \text{gr}_I^s(R) \) are \( F \)-pure.
Proof. We note that the rings involved are $F$-finite. Then, they are $F$-pure if and only if they are $F$-split [HR74, Corollary 5.3]. Since $k$ is perfect, we have that $R^{1/p}$ and $R^{s(I)}/R$ correspond to the rings of $p$-roots of $R$ and $R^s(I)$ respectively. In addition, $(J^s(I))^{1/p}$ is the ideal of $R^s(I)^{1/p}$ that corresponds to $J^s(I)$. Since $J^s(I)$ is radical, $gr_I^s(R)$ is a reduced ring. Then, $gr_I^s(R)^{1/p}$ corresponds to the ring of $p$-roots of $gr_I^s(R)$. Then, the result follows from Theorem 4.3 with $m = p$. \qed

The following Theorem provides necessary and sufficient conditions for the equality of ordinary and symbolic powers of square-free monomial ideals.

**Theorem 4.5.** Let $I$ be a square-free monomial ideal. Then, the following are equivalent.

1. $\Phi_m^R((I^{nm+1})^{1/m}) \subseteq I^{n+1}$ for every $m, n \in \mathbb{N}$.
2. $\Phi_m^R((I^{nm+1})^{1/m}) \subseteq I^{n+1}$ for some $m \in \mathbb{N}_{>1}$ and every $n \in \mathbb{N}$.
3. $I^n = I^{(n)}$ for every $n \in \mathbb{N}$.

Proof. Clearly (1) implies (2). We now assume (2) and prove (3). We first prove that $J(I)$ is a radical ideal. Let $f \in I^n$ be a monomial such that $ft^n \in \sqrt{J(I)}$ then there exists $e \in \mathbb{N}$ such that $(ft^n)^m^e \in J(I)$. We observe that $ft^e \in I^{nm^e+1}$. We note that the assumption in (2) implies $\Phi_m^R((I^{nm^e+1})^{1/m}) \subseteq I^{nm^e+1}$. Therefore, $f^{m^e-1}t^{nm^e-1} = \Phi_m^A((I^{nm^e+1})^{1/m}) \subseteq \Phi_m^A((I^{nm^e+1})^{1/m}) \subseteq \Phi_m^A((I^{nm^e+1})^{1/m}) t^{nm^e-1} \subseteq I^{nm^e+1} t^{nm^e-1} \subseteq J(I)$.

A decreasing induction on $e$ shows $ft^n \in J(I)$. Then, $gr_I(R) = R/I \cap J(I)$ is reduced. As a consequence, $I^n = I^{(n)}$ for every $n \in \mathbb{N}$ [HHT08, Corollary 1.6].

Now, we assume that (3). By Lemma 3.3, we have $\Phi_m^R((I^{nm+1})^{1/m}) \subseteq I^{n+1}$ for every $n, m \in \mathbb{N}$, therefore (1) follows. \qed

We now state an open problem given by Hà at the BIRS-CMO workshop on *Ordinary and Symbolic Powers of Ideals* during the summer of 2017 at Casa Matemática Oaxaca.

**Problem 4.6 (Hà).** Let $I$ be a square-free monomial ideal. Find a number $N \in \mathbb{N}$, in terms of $I$, such that $I^n = I^{(n)}$ for every $n \leq N$ implies $I^n = I^{(n)}$ for every $n \in \mathbb{N}$.

This problem strongly related to Conforti-Cornuéjols (Conjecture C). In fact, if $N = \lambda(I)$ satisfies the conclusion of Problem 4.6, then Conjecture C follows [DDSG+18, Remark 4.19]. Hà also asked for an optimal value for Problem 4.6, then Conjecture C follows [DDSG+18, Remark 4.19]. Hà also asked for an optimal value for Problem 4.6. In Example 4.9, we prove that our bound is sharp.

If one assumes Conjecture C, then $N = \lceil \frac{d+1}{2} \rceil$ would work [HM10, Remark 4.8]. As a consequence of our methods, we solve Problem 4.6 by giving $N$ in terms of the number of generators of $I$ in Theorem 4.8. For the proof of this result, we need the following well-known lemma. We include its proof for the sake of completeness.

We denote by $\mu(I)$ the minimal number of generators of $I$. If $I$ is generated by the monomials $x_{\alpha_1}, \ldots, x_{\alpha_n}$, we denote by $I^{[m]}$ the ideal generated by $x_{\alpha_1^{m_1}}, \ldots, x_{\alpha_n^{m_n}}$.

**Lemma 4.7.** Let $I$ be a monomial ideal. If $r \geq \mu(I)(m-1) + 1$, then $I^r = I^{r-m}I^{[m]}$.

Proof. Let $u = \mu(I)$ and $x_{\alpha_1}, \ldots, x_{\alpha_u}$ a minimal set of generators of $I$. Let $\alpha_1, \ldots, \alpha_u$ be natural numbers such that $\alpha_1 + \ldots + \alpha_u = r$, then by assumption there must exist $\alpha_i$ such that $\alpha_i \geq m$. Therefore,

$$x_{\alpha_1^{\alpha_1}} \ldots x_{\alpha_u^{\alpha_u}} \in I^{r-m}I^{[m]}.$$
This shows that $I' \subseteq I^{−m}I^{[m]}$. To obtain the other containment, we observe that $I^{[m]} \subseteq I^m$.

**Theorem 4.8.** Let $I$ be a square-free monomial ideal. If $I^n = I^{(n)}$ for every $n \leq \lceil \frac{\mu(I)}{2} \rceil$, then $I^n = I^{(n)}$ for every $n \in \mathbb{N}$.

**Proof.** We show that

$$\Phi_2^R((I^{2n+1})^{1/2}) \subseteq I^{n+1}$$

for every $n \in \mathbb{N}$ and then the result follows by Theorem 4.5, (1) $\Rightarrow$ (3). By assumption and Lemma 3.3 this inclusion holds for $n < \lceil \frac{\mu(I)}{2} \rceil$, as for these values $I^{(n+1)} = I^{n+1}$. We fix $n \geq \lceil \frac{\mu(I)}{2} \rceil$. Then,

$$2n + 1 \geq \mu(I) + 1$$

and hence $(I^{2n+1})^{1/2} = (I^{2(n-1)+1})^{1/2}I$ by Lemma 4.7. Therefore, by induction

$$\Phi_2^R((I^{2n+1})^{1/2}) = \Phi_2^R((I^{2(n-1)+1})^{1/2}I) = \Phi_2^R((I^{2(n-1)+1})^{1/2})I \subseteq I^nI = I^{n+1},$$

finishing the proof. □

We point out that Theorem 4.8 relates to an open problem stated by Francisco, Hā, and Mermin [FHM13, Problem 5.14(a)]. The following example shows that the number given in Theorem 4.8 is sharp.

**Example 4.9.** Let $R = k[x_1, \ldots, x_{2t-1}]$ for some $t \geq 2$, and let

$$I = (x_1x_2, x_2x_3, \ldots, x_{2t-2}x_{2t-1}, x_{2t-1}x_1).$$

Then $I^n = I^{(n)}$ for every $n < t = \lceil \frac{\mu(I)}{2} \rceil$, whereas $I' \neq I^{(t)}$ [LTar, Corollary 4.5].

**Corollary 4.10.** Let $I$ be a square-free monomial ideal. Then, $I^n = I^{(n)}$ for every $n \in \mathbb{N}$ if and only if $x_1 \cdots x_dI^{2n+1} \subseteq (I^{n+1})^{[2]}$ for every $n \leq \frac{\mu(I)}{2}$.

**Proof.** Let $m \in \mathbb{N}$. Since $R^{1/m}$ and $R$ are regular of the same dimension, we have that $	ext{Hom}_R(R^{1/m}, R) \cong R^{1/m}$ as $R^{1/m}$-modules [Fed83, Lemma 1.6 (1)]. Furthermore, standard computations show that the map $\Psi_m : R^{1/m} \to R$ induced by

$$\Psi_m(x^{a/m}) = \begin{cases} x^{(a-(m-1)1)/m} & \text{if } a \equiv -1 \pmod{m}; \\ 0 & \text{otherwise,} \end{cases}$$

where $1 = (1, \ldots, 1) \in \mathbb{N}^d$, is a generator of $\text{Hom}_R(R^{1/m}, R)$ as $R^{1/m}$-module (this is a standard computation in prime characteristic $p$ when $m = p$). We now focus on the case $m = 2$. We stress that we are not making any assumption on the characteristic of the field.

We note that $x_1 \cdots x_dI^{2n+1} \subseteq (I^{n+1})^{[2]}$ if and only if

$$(x_1 \cdots x_d)^{1/2}(I^{2n+1})^{1/2} \subseteq (I^{n+1}) R^{1/2},$$

because $R^{1/2}$ is isomorphic to $R$ and $I$ is monomial. In addition, $(x_1 \cdots x_d)^{1/2}(I^{2n+1})^{1/2} \subseteq I^{n+1}R^{1/2}$ is equivalent to $\Psi_2((x_1 \cdots x_d)^{1/2}(I^{2n+1})^{1/2}) \subseteq I^{n+1}$ for every $n \in \mathbb{N}$ [Fed83, Lemma 1.6 (2)]. Since $\Phi_2^R(−) = \Psi_2((x_1 \cdots x_d)^{1/2} − )$, the result follows by Theorem 4.5. □

We note that Theorem 4.8 and Corollary 4.10 give an algorithm to verify Conjecture C for a specific ideal.
5. Applications to Linear Optimization

In this brief section we translate our result to the context of linear programming. For more on this topic, we refer to [HT18].

A clutter $\mathcal{C} = (V, E)$ is a collection of subsets $E$ of $V = \{v_1, \ldots, v_n\}$ such that every two elements of $E$ are incomparable with respect to inclusion. We denote by $M := M(\mathcal{C})$ the $n \times m$ matrix with entries equal to 0 or 1, such that its columns are the incidence vectors of the sets in $E$. Given $c \in \mathbb{Z}_{\geq 0}^n$, by the Strong Duality Theorem we have the following equality of dual linear programs.

\[
\min\{c \cdot x \mid x \in \mathbb{R}_{\geq 0}^n, Mx \geq 1\} = \max\{1 \cdot y \mid y \in \mathbb{R}_{\geq 0}^m, M^Ty \leq c\},
\]

where $1 = (1, 1, \ldots, 1) \in \mathbb{N}^d$. We say that $\mathcal{C}$ packs for $c$ if Equation (5.1) has optimal solutions $x \in \mathbb{Z}_{\geq 0}^n$ and $y \in \mathbb{Z}_{\geq 0}^m$. The clutter $\mathcal{C}$ satisfies the Max-Flow-Min-Cut (MFMC) property if it packs for every $c \in \mathbb{Z}_{\geq 0}^n$.

We now recall a lemma that allows us to rephrase Theorem 4.8 in this context.

Lemma 5.1 ([FHM13, Lemma 5.9]). Set

\[
\gamma(c) := \min\{c \cdot x \mid x \in \mathbb{Z}_{\geq 0}^n, Mx \geq 1\}, \quad \text{and} \quad \sigma(c) := \max\{1 \cdot y \mid y \in \mathbb{Z}_{\geq 0}^m, M^Ty \leq c\}.
\]

Then,

(i) $\mathbf{v}^c \in \mathcal{I}(t)$ if and only if $t \leq \gamma(c)$.

(ii) $\mathbf{v}^c \in \mathcal{I}'$ if and only if $t \leq \sigma(c)$.

We are now ready to present the main theorem of this section which is related to previous results in integer programming [BT82, Corollary 2.3].

Theorem 5.2. The clutter $\mathcal{C}$ packs for every $c \leq \lfloor \frac{m}{2} \rfloor 1$ if and only if $\mathcal{C}$ satisfies the MFMC property.

Proof. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ denote the column vectors of $A$. We denote by $I$ the ideal of the polynomial ring $R = k[v_1, \ldots, v_n]$ generated by the monomials $(v^{b_1}, \ldots, v^{b_m})$.

From the definitions, the MFMC property implies the packing property. We focus on the other direction. We assume $\mathcal{C}$ packs for every $c \leq \lfloor \frac{m}{2} \rfloor 1$. Let $\mathbf{v}^c$ be a minimal monomial generator of $\mathcal{I}(t)$ for some $1 \leq t \leq \lfloor \frac{m}{2} \rfloor$, then it is clear that it divides $\mathbf{v}^{t1}$. Moreover, by Lemma 5.1, we have $\gamma(c) = t$. This is because reducing a component of $c$ by 1, reduces the optimal solution $\gamma(c)$ by at most 1. Therefore, by assumption we have $\sigma(c) = \gamma(c) = t$ and hence $\mathbf{v}^c \in \mathcal{I}'$ by Lemma 5.1. We conclude that $\mathcal{I}(t) = \mathcal{I}'$ for $t = 1, \ldots, \lfloor \frac{m}{2} \rfloor = \lfloor \frac{m(t)}{2} \rfloor$. By Theorem 4.8 it follows that $\mathcal{I}(t) = \mathcal{I}$ for every $t \geq 1$. Now, let $c \in \mathbb{Z}_{\geq 0}^n$ arbitrary. Then $\mathbf{v}^c \in \mathcal{I}^{\gamma(c)} = \mathcal{I}^{\gamma(c)}$. We have $\sigma(c) \geq \gamma(c)$ by Lemma 5.1. \hfill \Box

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