Theory of damped Rabi oscillations at finite temperatures

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Abstract. Rabi oscillations in the presence of dissipation are theoretically studied at finite temperatures. The system that we consider is a quantum two-level system with both internal relaxations and external decays under an alternating field. The environment modeled by a set consisting of an infinite number of harmonic oscillators is employed in order to introduce temperature into the system. The equations of motion for the density matrix (Liouville equation) of such a quantum two-level system coupled to a Caldeira-Leggett-type environment are obtained with the Born-Markov approximation. Internal relaxations of the system are incorporated into the Liouville equations using the Redfield theory, resulting in Liouville equations with temperature-dependent relaxation rates. The exact solutions are obtained by using the Torrey method based on the Laplace transform. The solution for damped Rabi oscillations can be divided into elementary relaxation processes such as a simple decay and a damped oscillation. Since the coefficients of the solution involve physical relaxation rates corresponding to the processes, all the relaxation rates such as the dephasing rate are determined by simple calculations of the coefficients in these different relaxation processes.

1. Introduction

Coherent temporal oscillations of the population inversion in a two-level system driven by an external electromagnetic field are known as Rabi oscillations [1]. These oscillations produce quantum-mechanical superposition states, and Rabi oscillations are now widely used to manipulate the single-qubit operations required for quantum computation in various quantum two-level systems. In real systems, there are inherent dissipations that damp the Rabi oscillations, resulting in poor controllability of single-bit operations. Therefore, an urgent issue in this field is to find the origin of this dissipation. However, no technique has yet been developed for measuring damping coefficients in dynamical ranges, i.e., in-situ measurement, different from Spin echo [2] and Ramsey interference techniques.

Under these circumstances, we proposed a novel in-situ scheme for measuring all kinds of damping coefficients in a damped Rabi system at zero temperature [3, 4]. Here we extend our theory to finite temperature by introducing a Caldeira-Leggett-type environment [5]. In addition, temperature-dependent damping coefficients [6] are studied based on the Redfield theory [7, 8].
Two-level system

\[ H = \text{Hamiltonian for the system} \]

\[ \text{environment are given within the Born-Markov approximation as} \]

The two-level system is a part of a multi-level system. Internal relaxation rates are denoted by \( \gamma_0 \) and \( \gamma_1 \), and \( \Gamma_i \) indicates the outside decay rate from \( |i\rangle \). \( \Omega \) and \( \omega \) are the Rabi frequency and the driving frequency, respectively.

2. Redfield theory for damped Rabi oscillations

Let us consider damped Rabi oscillations in a two-level system at finite temperatures as shown in Fig. 1. Here \( \Gamma_0 (\Gamma_1) \) is the decay rate from the ground state \( |0\rangle \) with energy \( E_0 \) (an excited state \( |1\rangle \) with \( E_1 \) to the external system, and \( \gamma_0 (\gamma_1) \) is the rate of the internal relaxation from \( |0\rangle \) \( (|1\rangle) \) to \( |1\rangle \) \( (|0\rangle) \). The two-level system \( (n = 0, 1) \) in this manuscript is part of a multi-level system \( H^{(S)} = \sum_n |n\rangle E_n \langle n| \) \( (n = 0, 1, 2 \cdots) \) and the part other than the two-level system can be regarded as the external system.

The environment modeled by a set consisting of an infinite number of harmonic oscillators \( H^{(B)} = \sum_\alpha (\gamma_\alpha^2/2m_\alpha + m_\alpha \omega_\alpha^2 x_\alpha^2/2) \) coupled linearly to the system \( H^{(\text{int})} = \phi \sum_\alpha C_\alpha x_\alpha \) is employed in order to introduce temperature into the system. Here \( \phi (x_\alpha) \) indicates the degree of freedom of the system (the environment) and \( C_\alpha \) is a coupling constant between the system and the environment. The Hamiltonian for the whole system is written as \( H = H^{(S)} + H^{(B)} + H^{(\text{int})} + H^{(\Omega)} \). The first and second terms of the right-hand side describe the Hamiltonian for the system \( H^{(S)} \) and its bath \( H^{(B)} \). The third is their interaction Hamiltonian \( H^{(\text{int})} \). The fourth term of the right-hand side describes the Hamiltonian for the driving field \( H^{(\Omega)} \).

According to the Redfield theory, the Liouville equations for a system coupled to an environment are given within the Born-Markov approximation as [8]

\[
\dot{\rho}_{nm} = -i\omega_{nm}\rho_{nm} + \frac{[H^{(\Omega)}, \rho]_{nm}}{i\hbar} - \sum_{k,l} R_{nmkl}\rho_{kl}, \tag{1}
\]

where \( \rho_{nm} = \langle n|\rho|m\rangle \) and \( \omega_{nm} = (E_n - E_m)/\hbar \). The other terms are defined as

\[
R_{nmkl} = \delta_{\ell,m} \sum_\gamma \Gamma^{(+)}_{\gamma \gamma k} + \delta_{n,k} \sum_\gamma \Gamma^{(-)}_{\ell \gamma m} - \Gamma^{(+)}_{\ell mnk} - \Gamma^{(-)}_{\ell mnk}, \tag{2}
\]

\[
\Gamma^{(+)}_{\ell mnk} = \left( \frac{1}{\hbar} \right)^2 \int_0^\infty d\tau e^{-i\omega_{nk}\tau} Tr_B \left[ \tilde{H}^{(\text{int})}_{\ell m} (\tau) \tilde{H}^{(\text{int})}_{nk} (0) \rho^{(B)} \right], \tag{3}
\]

\[
\Gamma^{(-)}_{\ell mnk} = \left( \frac{1}{\hbar} \right)^2 \int_0^\infty d\tau e^{-i\omega_{kn}\tau} Tr_B \left[ \tilde{H}^{(\text{int})}_{\ell m} (0) \tilde{H}^{(\text{int})}_{nk} (\tau) \rho^{(B)} \right], \tag{4}
\]

\[
\tilde{H}^{(\text{int})}_{nm} (\tau) = \langle n|\tilde{H}^{(\text{int})}(\tau)|m\rangle, \tag{5}
\]

\[
\tilde{H}^{(\text{int})}(\tau) = e^{\tau H^{(\text{int})}} H^{(\text{int})} e^{-\tau H^{(\text{int})}}. \tag{6}
\]
To study damped Rabi oscillations starting with an initial condition \((\rho_{00} + \rho_{11} = 1)\), the multi-level system is divided into two parts to form a two-level system \((m, n = 0, 1)\) and the rest of the system is regarded as the external system. Suppose that the number of levels in the external system is sufficiently large. In this case, the probability density diffuses into the external system, i.e., \(\rho_{k\ell} (k, \ell \neq 0, 1) \ll 1\) during the typical decay time \(\tau_1 = \Gamma_1^{-1}\). This means that the probability density has little chance to return from the external system to the two-level system under this situation during the typical decay time. Thus, the third terms of the right-hand side in the Liouville equation (1) are approximated as \(\sum_{k, \ell=0,1} \Gamma_{nnk\ell}\rho_{k\ell}\). Furthermore, we make a secular approximation (rotation wave approximation) that only retains the terms \(\Gamma_{nnk\ell}\) with \(n - m = k - \ell\).

The equations of motion for the diagonal component of the density matrix is represented in the two-level system as \(\dot{\rho}_{nn} = \left[H^{(\Omega)}, \rho_{nn}\right]/\hbar - \sum_{k=0,1} R_{nnk\ell}\rho_{k\ell} (n = 0, 1)\), where

\[
R_{0000} = 2\text{Re}(\Gamma_{0110}^{(+)}) + 2 \sum_{\ell \neq 0,1} \text{Re}(\Gamma_{0\ell\ell0}^{(+)}) = \left(= \gamma_0 + \Gamma_0\right),
\]

\[
R_{1111} = 2\text{Re}(\Gamma_{1001}^{(+)}) + 2 \sum_{\ell \neq 0,1} \text{Re}(\Gamma_{1\ell\ell1}^{(+)}) = \left(= \gamma_0 + \Gamma_1\right),
\]

\(R_{1100} = -\gamma_0\) and \(R_{0011} = -\gamma_0\). Here the relations \(\Gamma_{k\ell} = \Gamma_{k\ell}^{(+)\ast}\) and \(R_{nnk\ell} = R_{nmk\ell}^{(+)}\) are used. Because the first term of the right-hand side in Eq. (7) (Eq. (8)) describes the transition from \(|0\rangle\) \((|1\rangle\) to \(|1\rangle\) \((|0\rangle\)), this term \(\gamma_0\) (\(\gamma_0\)) represents the internal relaxation rate. On the other hand, \(\Gamma_0\) (\(\Gamma_1\)) is the decay rate from \(|0\rangle\) \((|1\rangle\) to the external system since the second terms describes the transition to states in the external system. The equations of motion for the off-diagonal component are represented in the two-level system as \(\dot{\rho}_{nm} = -i\omega_{nm}\rho_{nm} + \left[H^{(\Omega)}, \rho_{nm}\right]/\hbar - \text{Re}(R_{nmmn})\rho_{nm} (n \neq m)\). For the frequency shift realized by coupling the external system and the environment, \(\omega_{nm}\) is renormalized as \(\bar{\omega}_{nm} = \omega_{nm} + \text{Im}(R_{nmmn})\). Similarly, the transverse relaxation rate \(\Gamma_1\) is given as \(\text{Re}(R_{n101}) = \text{Re}(R_{1010}) = \frac{1}{2}(\gamma_0 + \gamma_0) + \frac{1}{2}(\Gamma_0 + \Gamma_1) + \gamma_\varphi\), \((= \Gamma)\), where the pure dephasing rate is \(\gamma_\varphi = \text{Re}(\Gamma_{0000}^{(+)}) + \Gamma_{1111}^{(+)} - \Gamma_{0011}^{(+)} - \Gamma_{1100}^{(+)}\). Thus, the Liouville equations are now given in the two-level system as

\[
\dot{\rho}_{nm} = \left[H^{(\Omega)}, \rho_{nm}\right]/\hbar - \gamma_0\rho_{nm} - (1)^n\gamma_{01}\rho_{00} + (1)^n\gamma_{10}\rho_{11}, \quad (n = 0, 1),
\]

\[
\dot{\rho}_{nm} = -i\omega_{nm} + \left[H^{(\Omega)}, \rho_{nm}\right]/\hbar - \Gamma_1\rho_{nm}, \quad n \neq m, \quad (n, m = 0, 1),
\]

where \(\bar{\omega}_{nm}\) is rewritten as \(\omega_{nm}\). The driving field Hamiltonian \(H^{(\Omega)}\) is expressed by a matrix based on the eigen states \(|0\rangle\) and \(|1\rangle\) in the two-level system as

\[
H^{(\Omega)} = \begin{pmatrix}
0 & \frac{-\Omega_0}{\hbar} e^{-i\omega t} \\
\frac{-\Omega_0}{\hbar} e^{i\omega t} & 0
\end{pmatrix},
\]

where \(\Omega_0\) is proportional to the driving field amplitude. The Rabi oscillation becomes \(\Omega_0\) on resonance.

By introducing relations \(\hat{f} = \rho_{01} e^{-i\omega t} + \rho_{10} e^{i\omega t}, -i(\rho_{01} e^{-i\omega t} - \rho_{10} e^{i\omega t}), \rho_{11} - \rho_{00}, \rho_{00} + \rho_{11}\), in the way conventionally to rewrite the Liouville equations, Eqs. (9) and (10) are rewritten as

\[
\frac{d}{dt}\hat{f} = \hat{M}\hat{f},
\]

where

\[
\hat{M} = \frac{-1}{2}
\begin{pmatrix}
2\Gamma & 2\Delta & 0 & 0 \\
-2\Delta & 2\Gamma & 2\Omega_0 & 0 \\
0 & -2\Omega_0 & \Gamma_+ + 2\gamma_+ & \Gamma_- + 2\gamma_- \\
0 & 0 & \Gamma_- & \Gamma_+
\end{pmatrix},
\]

(13)
Here $\Gamma_{\pm} = \Gamma_{\perp} + \Gamma_{\parallel}, \gamma_{\pm} = \gamma_{\perp} \pm \gamma_{\parallel}$, and $\Delta = \omega_{\parallel} - \omega$. The Liouville equations (12) at $T = 0$ reduce to Eq. (9) in Ref. 3.

By using the spectral density function of a bath $J(\omega) = \sum_{\alpha} \lambda_{\alpha}^{2} \pi \delta(\omega_{\alpha} - \omega)/\hbar^{2}$, the real part of the coefficients in Eq. (3) can be represented as

$$\text{Re} \Gamma_{\ell m n k}^{(+)} = (a \sigma_x - b \sigma_z)_{\ell m}(a \sigma_x - b \sigma_z)_{n k} J(\omega_{n k}) \frac{e^{-\beta \hbar |\omega_{n k}|/2}}{\sinh(\beta \hbar |\omega_{n k}|/2)}, \tag{14}$$

where $\sigma_i (i = x, y, z)$ are the Pauli matrices. The others are $a = \langle 0 | \phi | 1 \rangle$, $b = ((1 | \phi | 1) - \langle 0 | \phi | 0 \rangle)/2$, and $\lambda_{\alpha} = C_{\alpha} \sqrt{\hbar/2m_{\alpha}\omega_{\alpha}}$. Thus, the internal relaxation rates are represented as $\gamma_{+} = C_{10} \coth(\beta \hbar |\omega_{01}|/2), \gamma_{-} = C_{10},$ and $\gamma_{\varphi} = C_{\varphi} T$. In an Ohmic bath, spectral density function $J(\omega)$ is given as $J(\omega) = \gamma_{\varphi} f(\omega/\omega_{c})$ where $\omega_{c}$ is the cut-off frequency and $\gamma$ is a frequency-independent coefficient. By using Eq. (14), we obtained $C_{10} = 4a^{2}J(|\omega_{10}|)$ and $C_{\varphi} = 8b^{2}\gamma_{f}(0)hk_{B}$ in the Ohmic bath. Here the outside decay rates are assumed to be $\Gamma_{i} = \Gamma_{i}^{(\beta)} e^{-\beta \Delta E_{i}} + \Gamma_{i}^{(\varphi)} (i = 0, 1)$, where the first and the second terms of the right-hand side are thermodynamic and quantum tunnelling decay rates. Here $\Delta E_{i}$ is the potential barrier.

3. Solution

The Liouville equation (12) can be analytically solved by using the Torrey method based on the Laplace transformation [3, 9]. As a result, the density matrix element $\rho_{11}$ is given as $\rho_{11} = \frac{1}{2}(A e^{\gamma_{11} t} + B \sin(\Omega_{R} t + \theta) e^{\gamma_{11} t})$. When there is no driving field, the density matrix element $\rho_{11}$ is given as $\rho_{11} = (1 - \gamma) e^{-\gamma_{R} t} + \gamma e^{\gamma_{R} t}$ [10]. Coefficients such as $A$ and $B$ are not expressed in this paper owing to their complicated forms. For a case where the decoherence time is sufficiently long in relation to the Rabi oscillation time $\tau_{R} = \frac{1}{\gamma_{11}}$, we have provided concrete expressions of the coefficient in a previous paper [3], where we studied damped Rabi oscillations at $T = 0$. It has been shown that the following relations were obtained from combinations of the decay constants $a, c, g$, and $h$ as $\gamma_{+} = g + h - 2a, \gamma_{\varphi} = a + 2c - \frac{3}{2}(g + h), \Gamma_{0} = g$, and $\Gamma_{1} = 2a - g$, where $\gamma_{10}$ in Eq. (28) of the previous paper is replaced by $\gamma_{+}$. We have proposed a scheme for measuring the rates of internal relaxation and external decay from these relations. This scheme can be extended to a finite temperature.

4. Summary

We have theoretically studied damped Rabi oscillations at finite temperatures, and have extended the Liouville equations at $T = 0$ to finite temperatures by using the Redfield theory. We have obtained a scheme for measuring the rates of internal relaxation and outside decay from the analytical solutions at finite temperatures.

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