How could the replica method improve accuracy of performance assessment of channel coding?

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Abstract. We explore the relation between the techniques of statistical mechanics and information theory for assessing the performance of channel coding. We base our study on a framework developed by Gallager in IEEE Trans. Inform. Theory 11, 3 (1965), where the minimum decoding error probability is upper-bounded by an average of a generalized Chernoff’s bound over a code ensemble. We show that the resulting bound in the framework can be directly assessed by the replica method, which has been developed in statistical mechanics of disordered systems, whereas in Gallager’s original methodology further replacement by another bound utilizing Jensen’s inequality is necessary. Our approach associates a seemingly ad hoc restriction with respect to an adjustable parameter for optimizing the bound with a phase transition between two replica symmetric solutions, and can improve the accuracy of performance assessments of general code ensembles including low density parity check codes, although its mathematical justification is still open.

1. Introduction
In the last few decades, much attention has been paid to the similarities between statistical mechanics and information theory. In general, inference or search problems that arise in research on communication, inference, learning, combinatorics and other information theory fields can be treated by regarding the system as a virtual spin system subject to disordered interactions [1, 2, 3]. In this way, problems in information theory have been successfully analyzed utilizing methods developed in statistical mechanics [4, 5, 6, 7, 8], and vice versa [9, 10].

This research trend has shown that the similarities between the two fields are not limited to the structure of problems but also apply to analysis techniques. However, because the development histories of the two frameworks have been relatively independent, there are still barriers which may hinder further expansion and deepening of this promising interdisciplinary research field. In order to overcome possible obstacles, it is of great importance to investigate the methodological relations between the two fields. This article is written under this motivation. More precisely, we explore the similarities and differences between the techniques of statistical mechanics and information theory in analyzing channel coding (or error correcting codes).

This article is organized as follows. In sections 2 and 3, we briefly review a standard framework of (classical) channel coding and a conventional methodology for assessing its performance, which was developed by Gallager in [11]. Sections 4 and 5 are the main parts of the current article. In section 4, we reconsider the channel coding problem by applying the replica method developed in statistical mechanics. Using the replica method makes it possible to avoid applying Jensen’s
inequality, which is required in the original methodology. This offers a novel interpretation of the origin of a superficially ad hoc restriction with respect to an adjustable parameter for tightening the upper-bound of the minimum decoding error probability that appears in the conventional approach. Applying the replica method does not change the assessed performance, though it can improve the accuracy of the performance assessment for general code ensembles, including low-density parity-check codes, as shown in section 5. The final section, section 6, is devoted to a summary and discussion.

2. Framework of channel coding
Consider a message \( m \in \{1, 2, \ldots, 2^K\} \) transmitted to a receiver through a classical noisy channel. For this purpose, \( m \) is, in general, mapped to a codeword of \( N \)-dimension \( x_m = (x_{m1}, x_{m2}, \ldots, x_{mN}) \in \{0, 1\}^N \) prior to the transmission. The mapping of \( m \rightarrow x_m \) (\( m = 1, 2, \ldots, 2^K \)) can equivalently be expressed as \( C = \{x_1, x_2, \ldots, x_{2^K}\} \) and is termed a channel coding or simply a (channel) code.

The receiver must infer the original message \( m \) from the received degraded codeword of \( N \)-dimension \( y = (y_1, y_2, \ldots, y_N) \). For simplicity, we assume a memoryless channel with the degradation process modeled by a conditional probability \( P(y|x_m) = \prod_{l=1}^N P(y_l|x_{ml}) \). We further assume that the message \( m \) is encoded by a method of source coding such that it is equally generated with a probability of \( 2^{-K} \), which is preferred for enhancing communication performance. Under these assumptions, Bayesian theory indicates that for a given code \( C \), the maximum likelihood (ML) decoding

\[
\hat{m}(y) = \arg\max_{s \in \{1, 2, \ldots, 2^K\}} \{P(y|x_s)\},
\]

(1)

minimizes the probability of decoding error

\[
P_e(C) = 2^{-K} \sum_{m, y} P(y|x_m)\Delta_{ML}(m, y),
\]

(2)

where \( \arg\max \{\cdots\} \) denotes the argument that maximizes \( \cdots \) and \( \Delta_{ML}(m, y) = 1 \) if the original message \( m \) is not correctly retrieved by equation (1) for a given \( y \) and \( \Delta_{ML}(m, y) = 0 \) otherwise. In the following, we address the problem of assessing how small a \( P_e(C) \) is achievable by selecting the optimal \( C \) among a given code ensemble.

3. Conventional scheme for analyzing channel coding
3.1. Generalized Chernoff’s bound
As \( \Delta_{ML}(m, y) \) depends on \( m \) and \( y \) in a highly nonlinear manner, direct evaluation of equation (2) is difficult. In order to avoid this difficulty, several techniques for upper-bounding this function have been developed in conventional information theory \[11, 12, 13\].

The inequality

\[
\Delta_{ML}(m, y) \leq \left( \sum_{s \neq m} \left( \frac{P(y|x_s)}{P(y|x_m)} \right)^{\mu} \right)^{\lambda},
\]

(3)

which holds for \( \forall \lambda > 0 \) and \( \forall \rho > 0 \), is key for this purpose. This is validated as follows. The right hand side is non-negative and therefore satisfies the inequality if \( \Delta_{ML}(m, y) = 0 \). If

\footnote{In the conventional argument of information theory, channel coding is examined independently of source coding without assuming a prior distribution of messages. However, we here assume that messages are uniformly distributed a priori as a result of source coding in order to emphasize the optimality of the maximum-likelihood decoding.}
\( \Delta_{\text{ML}}(m, y) = 1 \), there exists at least one message \( s \) for which \( P(y|x_s) \geq P(y|x_m) \). This means that for such a message, \( (P(y|x_s)/P(y|x_m))^\lambda > 1 \) holds in the summation of the right hand side of equation (3) since \( \lambda > 0 \) ensures the fraction is greater than unity and therefore the summation is greater than unity as all terms are non-negative. \( \rho > 0 \) also ensures the inequality is valid.

Substituting equation (3) into equation (2) yields a generalized Chernoff’s bound

\[
P_e(C) \leq 2^{-K} \sum_{m,y} P(y|x_m) \left( \sum_{s \neq m} \left( \frac{P(y|x_s)}{P(y|x_m)} \right)^\lambda \right)^\rho,
\]

which holds for \( \forall \lambda > 0 \) and \( \forall \rho > 0 \). This indicates that the accuracy of the upper-bound can be improved by minimizing the right hand side with respect to these parameters.

### 3.2. Ensemble average as an upper-bound for the minimum

Unfortunately, direct minimization of the right hand side of equation (4) is non-trivial due to the complicated dependence on \( C \). However, the expression can still be useful for assessing the minimum error probability among all possible codes, \( P_e = \min_{C \in \{ \text{all codes} \}} \{P_e(C)\} \), for classical channels.

For this purpose, we introduce an ensemble of all codes \( Q(C) = \prod_{s=1}^{2^K} Q(x_s) \), where \( Q(x) \) is an identical distribution for generating codewords \( x_1, x_2, \ldots, x_{2^K} \) independently. Averaging equation (4) with respect to \( Q(C) \) gives an upper-bound of \( P_e \) as

\[
P_e \leq \frac{P_e(C)}{Q(C)} \leq \sum_{C \in \{ \text{all codes} \}} Q(C) \left( 2^{-K} \sum_{m,y} P(y|x_m) \left( \sum_{s \neq m} \left( \frac{P(y|x_s)}{P(y|x_m)} \right)^\lambda \right)^\rho \right) = 2^{-K} \sum_{y} \left( \sum_{m=1}^{2^K} \sum_{x_m} Q(x_m)P(y|x_m)^{1-\lambda\rho} \right) \sum_{C \setminus x_m} \prod_{s \neq m} Q(x_s) \left( \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho,
\]

due to the fact that the minimum value over a given ensemble is always smaller than the average over the ensemble. Here, \( \overline{\cdot} \) represents the average over a code ensemble \( Q(C) \) and \( C \setminus x_m \) denotes a subset of \( C = \{x_1, x_2, \ldots, x_{2^K}\} \) from which only \( x_m \) is excluded.

### 3.3. Jensen’s inequality and random coding exponent

Equation (5) is still difficult to assess for large \( K \) because the right hand side involves the fractional moment of a sum of exponentially many terms \( \sum_{C \setminus x_m} \prod_{s \neq m} Q(x_s) \left( \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho \), the direct and rigorous evaluation of which requires an exponentially large computational cost even while the code ensemble is factorizable with respect to codewords. Jensen’s inequality

\[
\sum_{C \setminus x_m} \prod_{s \neq m} Q(x_s) \left( \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho \leq \left( \sum_{C \setminus x_m} \prod_{s \neq m} Q(x_s) \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho = (2^K - 1)^\rho \left( \sum_{x} Q(x)P(y|x)^\lambda \right)^\rho \leq 2^{\rho K} \left( \sum_{x} Q(x)P(y|x)^\lambda \right)^\rho,
\]

which holds for \( 0 < \rho \leq 1 \), is a standard technique of information theory to overcome this difficulty. Plugging this into equation (5), in conjunction with an additional restriction \( \rho \leq 1 \),
we obtain the expression

\[ P_e \leq P_e(Q) \leq 2^{\rho K} \sum_y \left( \sum_x Q(x) P(y|x)^{1-\rho \lambda} \right) \left( \sum_x Q(x) P(y|x)^{\lambda} \right)^\rho, \tag{7} \]

\((0 \leq \rho \leq 1),\) where \(2^{-K} \sum_{y=1}^2 \sum_{x_m} Q(x_m) P(y|x_m)^{1-\lambda \rho} = \sum_x Q(x) P(y|x)^{1-\lambda \rho}\) is utilized and the trivial case \(\rho = 0\) is included.

For any given \(0 \leq \rho \leq 1,\) the upper-bound of equation (7) is generally minimized by \(\lambda = 1/(1 + \rho),\) as assumed in Gallager’s paper [11]. The computational difficulty for assessing equation (7) is resolved for memoryless channels \(P(y|x) = \prod_{i=1}^N P(y|x_i)\) by assuming factorizable distributions \(Q(x) = \prod_{i=1}^N Q(x_i)\). This assumption naturally indicates that the upper-bound depends exponentially on the code length \(N\) as \(P_e \leq \exp[-N(-\rho R + E_0(\rho, Q))],\) where \(R = K/N\) and

\[ E_0(\rho, Q) = -\ln \left[ \sum_y \left( \sum_x Q(x) P(y|x)^{1+\rho} \right)^{1+\rho} \right], \tag{8} \]

are often termed the code rate and Gallager function, respectively. This means that if \(N\) is sufficiently large and the random coding exponent

\[ E_r(R) = \max_{0 \leq \rho \leq 1, Q} \{-\rho R \ln 2 + E_0(\rho, Q)\}, \tag{9} \]

is positive for a given \(R,\) there exists a code with a decoding error probability smaller than an arbitrary positive number. For a fixed \(Q(x),\) \(E_0(\rho, Q)\) is a convex upward function satisfying \(E_0(\rho = 0, Q) = 0\) and

\[ \frac{\partial}{\partial \rho} E_0(\rho, Q) \bigg|_{\rho=0} = \sum_{y,x} Q(x) P(y|x) \ln \frac{P(y|x)}{\sum_x Q(x) P(y|x)} \equiv \ln 2 \times I(Q), \tag{10} \]

where \(I(Q)\) represents the mutual information between \(x\) and \(y\) (in bits). This implies that the critical rate \(R_c\) below which \(E_r(R)\) becomes positive is given by \(\rho = 0\) as

\[ R_c = \max_Q \{I(Q)\}, \tag{11} \]

which agrees with the definition of the channel capacity [14].

As \(R\) is reduced from \(R_c,\) the value of \(\rho\) that optimizes the right hand side of equation (9) increases and reaches \(\rho = 1\) at a certain rate \(R_b.\) Below \(R_b,\) equation (9) is always optimized at the boundary \(\rho = 1.\) Figure 1 shows an example of \(E_r(R)\) for the binary symmetric channel (BSC), which is characterized by a crossover rate of \(0 \leq p \leq 1\) as \(P(1|0) = P(0|1) = p\) and \(P(1|1) = P(0|0) = 1 - p\) for binary alphabets \(x, y \in \{0, 1\}.)\n
\(E_r(R)\) characterizes an upper-bound of a typical decoding error probability of randomly constructed codes. However, surprisingly enough, it is known that for certain classes of channels, \(E_r(R)\) represents the performance of the best codes at the level of exponent for a relatively high code rate region \(R \geq R_a,\) which contains \(R = R_b,\) since \(E_r(R)\) agrees with the exponent of a lower bound of the best possible code [15]. This is far from trivial because the restriction \(\rho \leq 1,\) which governs \(E_r(R)\) of \(R \leq R_b,\) is introduced in an ad hoc manner when employing Jensen’s inequality in the above methodology.
Figure 1. Random coding exponent \( E_r(R) \) for BSC for a crossover rate \( p = 0.1 \). \( E_r(R) \) (solid curve) becomes positive for \( R < R_c \equiv 0.531 \). The functional form of \( E_r(R) \) for \( R < R_b \equiv 0.189 \) differs from that for \( R_b \leq R \leq R_c \). The broken curve represents the value of the upper-bound exponent that is maximized without the restriction \( \rho \leq 1 \).

4. Performance assessment by the replica method

4.1. Expanding the upper-bound for \( \rho = 1, 2, \ldots \)

In order to clarify the origin of the superficially artificial restriction \( \rho \leq 1 \), we evaluate the exponent without using Jensen’s inequality. For this purpose, we assess the right hand side of equation (5) analytically, continuing the expressions obtained for \( \rho = 1, 2, \ldots \) to \( \rho \in \mathbb{R} \). This is often termed the replica method \([16, 17]\).

For the current problem, the first step of the replica method is to evaluate the expression

\[
\sum_{\mathcal{C} \setminus \mathbb{x}^\rho \neq m} 2^K \prod_{s \neq m} Q(x_s) \left( \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho = \sum_{\{s^a\}_{a=1}^\rho \neq m} 2^K \prod_{\tau \neq m} \left( \sum_{x_{\tau}} Q(x_{\tau}) P(y|x_{\tau})^\lambda \sum_{a=1}^\rho \delta(s^a, \tau) \right) = \sum_{\{i_1, i_2, \ldots, i_{\rho}\}} W(i_1, i_2, \ldots, i_{\rho}) \prod_{t=1}^\rho \left( \sum_{x} Q(x) P(y|x)^\lambda \right)^{i_t}, \tag{12}
\]

analytically for \( \rho = 1, 2, \ldots \), where \( \delta(x, y) = 1 \) for \( x = y \) and vanishes otherwise, and \( W(i_1, i_2, \ldots, i_{\rho}) \) is the number of ways of partitioning \( \rho \) replica messages \( s^1, s^2, \ldots, s^\rho \) to \( i_1 \) states (out of \( \tau = 1, 2, \ldots, 2^K \) except for \( \tau = m \)) by one, to \( i_2 \) states by two, \ldots and to \( i_{\rho} \) states by \( \rho \). Obviously, \( W(i_1, i_2, \ldots, i_{\rho}) = 0 \) unless \( \sum_{t=1}^\rho i_t = \rho \). It is worth noting that the expression of the right hand side is valid only for \( \rho = 1, 2, \ldots \).
4.2. Saddle point assessment under the replica symmetric ansatz

Exactly evaluating equation (12) is difficult for large $K = NR$. However, in many systems, quantities of this kind scale exponentially with respect to $N$, which implies that the exponent characterizing the exponential dependence can be accurately evaluated by the “saddle point method” with respect to the partition of $\rho$, $(i_1, i_2, \ldots, i_\rho)$, under an appropriate assumption of the symmetry underlying the objective system in the limit of $N \rightarrow \infty$. The replica symmetry, for which equation (12) is invariant under any permutation of the replica indices $a = 1, 2, \ldots, \rho$, is critical for the current evaluation. This implies that it is natural to assume that, for large $N$, the final expression of equation (12) is dominated by a single term possessing the same symmetry, which yields the following two types of replica symmetric (RS) solutions:

- **RS1**: Dominated by $(i_1, i_2, \ldots, i_\rho) = (\rho, 0, \ldots, 0)$, giving

  \[
  \sum_{C, \mathbf{x}_m} 2^K \prod_{s \neq m} Q(x_s) \left( \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho \approx W(\rho, 0, \ldots, 0) \left( \sum_x Q(x) P(y|x)^\lambda \right)^\rho \\
  \approx 2^{N\rho R} \left( \sum_x Q(x) P(y|x)^\lambda \right)^\rho. \tag{13}
  \]

- **RS2**: Dominated by $(i_1, i_2, \ldots, i_\rho) = (0, 0, \ldots, 1)$, giving

  \[
  \sum_{C, \mathbf{x}_m} 2^K \prod_{s \neq m} Q(x_s) \left( \sum_{s \neq m} P(y|x_s)^\lambda \right)^\rho \approx W(0, 0, \ldots, 1) \left( \sum_x Q(x) P(y|x)^{\lambda \rho} \right)^1 \\
  \approx 2^{N R} \sum_x Q(x) P(y|x)^{\lambda \rho}. \tag{14}
  \]

Plugging these into the final expression of equation (5), in conjunction with $P(y|x) = \prod_{l=1}^N P(y|x_l)$ and $Q(x) = \prod_{l=1}^N Q(x_l)$, gives the exponents

\[
E_{RS1}(\rho, \lambda, Q, R) = -\rho R \ln 2 - \ln \left[ \sum_y \left( \sum_x Q(x) P(y|x)^{1-\lambda \rho} \right) \left( \sum_x Q(x) P(y|x)^\lambda \right)^\rho \right], \tag{15}
\]

and

\[
E_{RS2}(\rho, \lambda, Q, R) = -R \ln 2 - \ln \left[ \sum_y \left( \sum_x Q(x) P(y|x)^{1-\lambda \rho} \right) \left( \sum_x Q(x) P(y|x)^{\lambda \rho} \right) \right], \tag{16}
\]

where the suffixes RS1 and RS2 correspond to equations (13) and (14), respectively, as two candidates of the exponent $E(\rho, \lambda, Q, R)$ for upper-bounding the minimum decoding error probability as $P_e \leq \exp \left[ -N E(\rho, \lambda, Q, R) \right]$. 

4.3. Phase transition between RS solutions: origin of the restriction $\rho \leq 1$

Although we have so far assumed that $\rho$ is a natural number, both the functional forms of the saddle point solutions, (15) and (16), can be defined over $\rho \in \mathbb{R}$. Therefore, we analytically continue these expressions from $\rho = 1, 2, \ldots$ to $\rho \in \mathbb{R}$, and select the relevant solution for each set of $(\rho, \lambda, Q, R)$ in order to obtain the correct upper-bound exponent $E(\rho, \lambda, Q, R)$. This is the second step of the replica method.

For $\rho = 1, 2, \ldots$ and sufficiently large $N$, this can be carried out by selecting the solution of the lesser exponent value. Unfortunately, as yet a mathematically justified general guideline
for selection of the relevant solution for $\rho \leq 1$ has not been determined. Such a guideline is necessary for determining the channel capacity by assessment at $\rho = 0$. However, there is an empirical criterion for this purpose, which is indicated by the analysis of exactly solvable models [13]. In the current case, this means that for fixed $\lambda, Q$ and $R$ we should choose the solution for which the partial derivative with respect to $\rho$ at $\rho = 1$, $(\partial/\partial \rho) E_{R_1}(\rho, \lambda, Q, R)|_{\rho=1}$ or $(\partial/\partial \rho) E_{R_2}(\rho, \lambda, Q, R)|_{\rho=1}$, is lesser, as the relevant solution for $\rho \leq 1$. This criterion implies that $E_{R_1}(\rho, \lambda, Q, R)$ should be chosen to provide the tightest bound $E_{\text{replica}}(R) = \max_{0 \leq \rho, 0 \leq \lambda, Q} \{ E(\rho, \lambda, Q, R) \}$ for relatively large $R$, which yields the expression

$$E_{\text{replica}}(R) = \max_{0 \leq \rho, 0 \leq \lambda, Q} \{ E_{R_1}(\rho, \lambda, Q, R) \}$$

$$= \max_{0 \leq \rho, Q} \left\{ -\rho R \ln 2 - \ln \left[ \sum_y \left( \sum_x Q(y|x)P(y|x)^{1/\lambda} \right)^{1+\rho} \right] \right\}.$$  \hspace{1cm} (17)

As $R$ is reduced from $R = R_c$, below which equation (17) becomes positive, the value of $\rho$ that maximizes the right hand side of equation (17) increases from $\rho = 0$, keeping the relation $\lambda = 1/(1+\rho)$ at the maximum point. When $R$ reaches $R_b$, the optimal value of $\rho$ becomes unity and $\lambda = 1/2$, for which

$$\frac{\partial}{\partial \rho} E_{R_1}(\rho, \lambda, Q, R) \bigg|_{(\rho, \lambda, R) = (1, 1/2, R_b)} = \frac{\partial}{\partial \rho} E_{R_2}(\rho, \lambda, Q, R) \bigg|_{(\rho, \lambda, R) = (1, 1/2, R_b)} = 0.$$  \hspace{1cm} (18)

This implies that for $R < R_b$, $(\partial/\partial \rho) E_{R_2}(\rho, \lambda, Q, R)|_{\rho=1} < (\partial/\partial \rho) E_{R_1}(\rho, \lambda, Q, R)|_{\rho=1}$ holds when the condition for a maximum is satisfied. Therefore, we should not select $E_{R_1}(\rho, \lambda, Q, R)$, but rather $E_{R_2}(\rho, \lambda, Q, R)$ for assessing the tightest bound $E_{\text{replica}}(R) = \max_{0 \leq \rho, 0 \leq \lambda, Q} \{ E(\rho, \lambda, Q, R) \}$ for $R < R_b$, which yields

$$E_{\text{replica}}(R) = \max_{0 \leq \rho, 0 \leq \lambda, Q} \{ E_{R_2}(\rho, \lambda, Q, R) \}$$

$$= \max_{1 \leq \rho, 0 \leq \lambda, Q} \left\{ -R \ln 2 - \ln \left[ \sum_y \left( \sum_x Q(y|x)P(y|x)^{1-\lambda \rho} \right) \left( \sum_x Q(x)P(y|x)^{\lambda \rho} \right) \right] \right\}$$

$$= \max_{Q} \left\{ -R \ln 2 - \ln \left[ \sum_y \left( \sum_x Q(y|x)P(y|x)^{1/2} \right)^2 \right] \right\}.$$  \hspace{1cm} (19)

In the second line, any choice of $(\rho, \lambda)$ that satisfies $\lambda \rho = 1/2$ and $\rho \geq 1$ optimizes the exponent.

Although the style of the derivation seems somewhat different from that of the conventional approach, the exponents obtained by equations (17) and (19) are identical to those assessed using equation (13). Therefore, $E_{\text{replica}}(R) = E_{\text{rep}}(R)$ holds, implying that no improvement is gained by the replica method in the analysis of the ensemble of all codes.

Nevertheless, our approach is still useful for clarifying the origin of the seemingly artificial restriction $\rho \leq 1$ in the conventional scheme. The above analysis indicates that there is no such restriction as long as the upper-bound of equation (13) is directly evaluated. Instead, what is the most relevant is the breaking of the analyticity with respect to $\rho$ of the upper-bound exponent $E(\rho, \lambda, Q, R)$, which can be interpreted as a phase transition between the two types of replica symmetric solutions $E_{R_1}(\rho, \lambda, Q, R)$ and $E_{R_2}(\rho, \lambda, Q, R)$ in the terminology of physics. As a consequence, we have to appropriately switch the functional forms of the objective function in
order to correctly obtain the optimized exponent. However, this procedure, in practice, can be completely simulated by optimizing a single function in conjunction with introducing an additional restriction $\rho \leq 1$, which can be summarized by a conventional formula of the random coding exponent, namely equation (9).

Of course, it must be kept in mind that the mathematical validity of our methodology is still open while the known correct results are reproduced. Although applying the saddle point assessment is a major reason for the strengthening of mathematical rigor, the most significant issue in the current context is mathematical justification of the empirical criterion at $\rho = 1$ to select the appropriate solution for $\rho \leq 1$ when multiple saddle point solutions exist. Accumulated knowledge about error exponents of various codes in information theory [11, 19, 20, 21, 22] may be of assistance for solving this issue.

Although we have applied the replica method to an upper-bound following the conventional framework in order to clarify the relation to an information theory method, it can be utilized to directly assess the minimum possible decoding error probability. For a region of lower $R$, there still exists a gap between the lower- and upper-bounds of the error exponents of the best possible code. An analysis based on the replica method indicates that the lower-bound of the exponent, which corresponds to the upper-bound of the decoding error probability, agrees with the correct solution [23].

5. Analysis of low-density parity-check codes

5.1. Definition of an LDPC code ensemble

Although a novel interpretation is obtained, our approach does not update known results in the analysis of the ensemble of all codes. However, this is not the case in general; the replica method usually offers a smaller upper-bound than conventional schemes for general code ensembles. We will show this for an ensemble of low-density parity-check (LDPC) codes.

A $(k, j)$ LDPC code is defined by selecting $N - K$ parity checks composed of $k$ components, $x_{l_1} \oplus x_{l_2} \oplus \ldots \oplus x_{l_k} = 0$, out of $\binom{N}{k}$ combinations of indices for characterizing a binary codeword of length $N$, $x = (x_l) \in \{0, 1\}^N$, where $l_1, l_2, \ldots, l_k = 1, 2, \ldots, N$ and $\oplus$ denotes addition over the binary field. There are several ways to define an LDPC code ensemble. For analytical convenience, we here focus on an ensemble constructed by uniformly selecting $N - K$ ordered combinations of $k$ different indices $l_1, l_2, \ldots, l_k$, $\langle l_1 l_2 \ldots l_k \rangle$, for parity checks, so that each component index of codewords $l(=1, 2, \ldots, N)$ appears $j$ times in the total set of parity checks. A code $C$ constructed in this way is specified by a set of binary variables $c = \{c_{\langle l_1 l_2 \ldots l_k \rangle}\}$, where $c_{\langle l_1 l_2 \ldots l_k \rangle} = 1$ if the combination $\langle l_1 l_2 \ldots l_k \rangle$ is used for a parity check and $c_{\langle l_1 l_2 \ldots l_k \rangle} = 0$ otherwise.

For simplicity, we assume symmetric channels, where we can assume that the sent message $m$ is encoded into the null codeword $x = 0$. Under this assumption, the generalized Chernoff’s bound [4] for an LDPC code is expressed as

$$ P_e(C) \leq \sum_y P(y|0)^{1-\lambda \rho} \left( \sum_{x \neq 0} I(x|c) P(y|x)^{\lambda} \right)^{\rho}, \quad (20) $$

where

$$ I(x|c) = \prod_{\langle l_1 l_2 \ldots l_k \rangle} \left( 1 - c_{\langle l_1 l_2 \ldots l_k \rangle} + c_{\langle l_1 l_2 \ldots l_k \rangle} \delta(x_{l_1} \oplus x_{l_2} \oplus \ldots \oplus x_{l_k}, 0) \right), \quad (21) $$

returns unity if $x$ satisfies all the parity checks and vanishes otherwise, screening only codewords in the summation over $x$ in the right hand side of equation (20).
5.2. Performance assessment by the replica method

Unlike the random code ensemble explored in the previous section, a statistical dependence arises among codewords in an LDPC code. This yields atypically bad codes, the minimum distance of which is of the order of unity with a probability of algebraic dependence on \( N \). The contribution of such atypical codes causes the average of the decoding error probability over a naive LDPC code ensemble to decay algebraically with respect to \( N \), indicating that the error exponent vanishes even for a sufficiently small rate \( R \) [24]. However, we can reduce the fraction of the bad codes to as small as required by removing short cycles in the parity check dependence by utilizing certain feasible algorithms [25]. This implies that, in practice, the performance of the LDPC code ensembles can be characterized by analysis with respect to the typical codes utilizing the saddle point method as shown below [26].

In order to employ the replica method, we assess the average of the right hand side of equation (20) with respect to the LDPC code ensemble

\[
Q(c) = \frac{1}{\mathcal{N}(k,j)} \prod_{t=1}^{N} \delta \left( \sum_{(l_2,l_3,..,l_k)} c(l_2,l_3,..,l_k), j \right), \tag{22}
\]

where \( \mathcal{N}(k,j) = \sum_{c} \prod_{t=1}^{N} \delta \left( \sum_{(l_2,l_3,..,l_k)} c(l_2,l_3,..,l_k), j \right) \) stands for the number of \((k,j)\) LDPC codes. For \( \rho = 1, 2, \ldots \) and sufficiently large \( N \), evaluating this using the saddle method, substituting with \( P(y|x) = \prod_{t=1}^{N} P(y|x_t) \), gives an upper-bound for the average decoding error probability over an ensemble of typical LDPC codes from which atypically bad codes are expurgated as

\[
E_{LDPC}(\rho, \lambda, R) = - \text{Extr}_{\chi, \hat{\chi}} \left\{ \frac{\lambda^{k-1}}{k!} \sum_{b_1,b_2,..,b_k} \prod_{a=1}^{k} \chi(b_a) \prod_{t=1}^{k} \delta(b_1^a \oplus b_2^a \oplus \cdots \oplus b_k^a, 0) \\
+ \ln \left[ \sum_{y} P(y|0)^{1-\lambda^\rho} \left( \sum_{x} \hat{\chi}(x)^j \prod_{a=1}^{\rho} P(y|x_a)^{\lambda} \right) \right] \\
- \sum_{b} \hat{\chi}(b) \chi(b) - \left( \frac{j}{k} - j + j \ln \left[ \frac{(jN)^{1-1/k}}{((k-1)!)^{1/k}} \right] \right) \right\}, \tag{23}
\]

\( b = (b_1, b_2, \ldots, b^\rho) \in \{0, 1\}^\rho \) and \( x = (x^1, x^2, \ldots, x^\rho) \in \{0, 1\}^\rho \). \text{Extr}_X \) denotes the operation of extremization with respect to \( X \), which corresponds to the saddle point assessment of a certain complex integral and therefore does not necessarily mean maximization or minimization. An outline of the derivation is shown in Appendix A.

An RS solution which is relevant for \( 0 \leq \rho \leq 1 \) corresponding to RS1 in the previous section is obtained under the RS ansatz

\[
\chi(b) = q \int_{-1}^{+1} du \pi(u) \prod_{a=1}^{\rho} \left( \frac{1 + u(-1)^{b^a}}{2} \right), \tag{24}
\]

\[
\hat{\chi}(b) = \hat{q} \int_{-1}^{+1} d\hat{u} \hat{\pi}(\hat{u}) \prod_{a=1}^{\rho} \left( \frac{1 + \hat{u}(-1)^{b^a}}{2} \right), \tag{25}
\]

where \( q \) and \( \hat{q} \) are normalization variables that constrain the respective variational functions \( \pi(u) \) and \( \hat{\pi}(\hat{u}) \) to be distributions over \([-1, 1]\), making it possible to analytically continue the expression [23] from \( \rho = 1, 2, \ldots \) to \( \rho \in \mathbb{R} \). Carrying out partial extremization with respect to
\( q \) and \( \hat{q} \) yields an analytically continued RS upper-bound exponent

\[
E_{\text{RS}}^{\text{LDPC}}(\rho, \lambda, R) = -\text{Extr}_{\pi, \hat{\pi}} \left\{ \sum_{k} P(y|0)^{1-\lambda \rho} d\hat{u}_{0, \pi}(\hat{u}_{\mu}) \left( \sum_{x=0,1} \prod_{\nu=1}^{j} \left( \frac{1 + \hat{u}_{\mu}(-1)^{x}}{2} \right) P(y|x)^{\lambda} \right)^{\rho} \right\}
\]

where the functional extremization \( \text{Extr}_{\pi, \hat{\pi}} \{ \cdots \} \) can be performed numerically in a feasible time by Monte Carlo methods in practice \[27\].

5.3. Comparison of lower-bound estimates of error threshold
When the noise level is sufficiently small and the code length \( N \) is sufficiently large, there exists at least one \((k, j)\) LDPC code with a decoding error probability smaller than an arbitrary positive number. The maximum value of such noise levels is sometimes termed the error threshold.

Equation (26) can be utilized to assess a lower-bound of the error threshold. Table I shows the lower-bounds obtained by maximizing this equation with respect to \( \rho \geq 0 \) and \( \lambda \geq 0 \) for several sets of \((k, j)\). Estimates obtained by the conventional schemes utilizing Jensen’s inequality, which in the current case are determined by an upper-bound exponent

\[
E_{\text{Jensen}}^{\text{LDPC}}(\rho, \lambda, R) = -\text{Extr}_{u, \hat{u}} \left\{ \sum_{k} P(y|0)^{1-\lambda \rho} \left( \sum_{x=0,1} \prod_{\nu=1}^{j} \left( \frac{1 + \hat{u}_{\nu}(-1)^{x}}{2} \right) P(y|x)^{\lambda} \right)^{\rho} \right\}
\]

are also provided for comparison.

Table I indicates that, in general, the lower-bounds estimated by the replica method are not smaller than those of the conventional schemes. This implies that unlike the case of the ensemble of all codes, employing Jensen’s inequality can relax an upper-bound for general code ensembles and therefore there may be room for improvement in results obtained by conventional schemes based on this inequality.

6. Summary and discussion
In summary, we have explored the relation between statistical mechanics and information theory methods for assessing performance of channel coding, based on a framework developed by Gallager \[11\]. An average of a generalized Chernoff’s bound for probability of decoding error over a given code ensemble can be directly evaluated by the replica method of statistical mechanics, while Jensen’s inequality must be applied in a conventional information theory approach. The direct evaluation of the average associated a switch of two analytic functions in the random coding exponent known in information theory with a phase transition between two replica symmetric solutions obtained by the replica method. Better lower-bounds of the error threshold were obtained for ensembles of LDPC codes under the assumption that the replica method produces the correct results. This may motivate an improvement in the accuracy of performance assessment, refining the conventional methodologies.
| $R$ | $(j,k)$ | Jensen 1 | Jensen 2 | replica | Shannon |
|-----|--------|---------|---------|---------|---------|
| 1/2 | (3,6)  | 0.0678  | 0.0915  | 0.0998  | 0.109   |
| 2/5 | (3,5)  | 0.115   | 0.129   | 0.136   | 0.145   |
| 1/3 | (4,6)  | 0.1705  | 0.1709  | 0.173   | 0.174   |
| 1/3 | (2,3)  | 0       | 0.0670  | 0.0670  | 0.174   |
| 1/2 | (2,4)  | 0       | 0.0286  | 0.0286  | 0.109   |

Table 1. Lower-bound estimates of the error threshold of BSC. In columns “Jensen 1”, “Jensen 2” and “replica”, the estimates represent the critical crossover rates $p_c$, below which the maximized values of equation (26) or (27) are positive. In the evaluation, the exponents are maximized with respect to two parameters $\rho \geq 0$ and $\lambda \geq 0$ for “Jensen 2” and “replica” while a single parameter maximization with respect to $\rho \geq 0$, keeping $\lambda = 1/(1 + \rho)$, is performed for “Jensen 1”. “Shannon” represents the channel capacity for a given code rate $R$.

A characteristic feature of the methods developed in statistical mechanics is the employment of the saddle point assessment utilizing a certain symmetry underlying the objective system, which, in some cases, makes it possible to accurately analyze macroscopic properties of large systems even when there are statistical correlations or constraints among system components. Such approaches may also be useful for analyzing codes of quantum information, for which, in many cases, there arise non-trivial correlations among codewords for the purpose of dealing with noncommutativity of operators [28].

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Appendix A. Outline of derivation of equation (23)
Equation (23) is obtained by averaging the right hand side of equation (20) with respect to the $(k,j)$ LDPC code ensemble (22). For this assessment, we first evaluate the normalization constant $\mathcal{N}(k,j)$ utilizing the identity

$$
\delta \left( \sum_{l_2l_3...l_k} c_{l_2l_3...l_k}, j \right) = \frac{1}{2\pi i} \oint dZ_i Z_i^{-(j+1)} \prod_{l_2l_3...l_k} Z_i^{\sum c_{l_2l_3...l_k}} e^{\sum_{l_2l_3...l_k} c_{l_2l_3...l_k}(1 + Z_{l_1} Z_{l_2} \ldots Z_{l_k})},
$$

where $i = \sqrt{-1}$ and $\oint dZ$ denotes the contour integral along a closed curve surrounding the origin on the complex plane. Plugging this expression into $\mathcal{N}(k,j) = \sum c \prod_{l=1}^N \delta \left( \sum_{l_2l_3...l_k} c_{l_2l_3...l_k}, j \right)$ yields

$$
\mathcal{N}(k,j) = \frac{1}{(2\pi i)^N} \oint \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \prod_{l_2l_3...l_k} (1 + Z_{l_1} Z_{l_2} \ldots Z_{l_k}) \exp \left[ \sum_{l_2l_3...l_k} \ln (1 + Z_{l_1} Z_{l_2} \ldots Z_{l_k}) \right] \ldots + \text{higher order terms}
$$

$$
= \frac{1}{(2\pi i)^N} \oint \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \exp \left[ \sum_{l_2l_3...l_k} Z_{l_1} Z_{l_2} \ldots Z_{l_k} \ldots \right]
$$
where we have introduced the dummy variables

\[ N \approx (2\pi)^N \int \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \exp \left[ \sum_{(l_1,l_2,\ldots,l_k)} Z_{l_1} Z_{l_2} \cdots Z_{l_k} \right] \]

\[ N \approx (2\pi)^N \int \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \exp \left[ \frac{N^k}{k!} \left( \frac{1}{N} \sum_{l=1}^N Z_l \right)^k \right]. \quad (A.2) \]

Here, in the third to fifth lines we have omitted irrelevant higher order terms since they do not affect the following saddle point assessment. Inserting the identity \( 1 = N^{-1} \int d\bar{q}_0 \delta \left( \sum_{l=1}^N Z_l - Nq_0 \right) = (2\pi N)^{-1} \int d\bar{q}_0 \int_{-\infty}^{+\infty} d\bar{q}_0 \exp \left[ \bar{q}_0 \left( \sum_{l=1}^N Z_l - Nq_0 \right) \right] \) into this expression makes it possible to analytically integrate equation \( \text{(A.2)} \) with respect to \( Z_l \) \((l = 1, 2, \ldots, N)\). For large \( N \), the most dominant contribution to the resulting integral with respect to \( q_0 \) and \( \bar{q}_0 \) can be evaluated by the saddle point method as

\[
\frac{1}{N} \ln N(k,j) \approx \operatorname{Extr}_{q_0,\bar{q}_0} \left\{ \frac{N^{k-1}}{k!} \bar{q}_0^k - \bar{q}_0 q_0 + \ln \left( \frac{\bar{q}_0}{q_0} \right) \right\} = \frac{j}{k} - j + \ln \left[ \frac{(jN)^{j-j/k}}{(k-1)!} \right].
\]  \quad (A.3)

where the saddle point is given as \( q_0 = ((k-1)!)^{1/k} j^{1/k} N^{-1+1/k} \) and \( \bar{q}_0 = ((k-1)!)^{-1/k} (jN)^{1-1/k} \).

The average of the right hand side of equation \( \text{(20)} \) for \( \rho = 1, 2, \ldots \) can be evaluated in a similar manner. For this, we expand \( \left( \sum_{\mathbf{x}} \mathcal{I}^\rho(\mathbf{x}|\mathbf{c}) \mathcal{P}(\mathbf{y}|\mathbf{x})^\lambda \right)^\rho \) and take the average with respect to \( \mathbf{c} \), utilizing the LDPC code ensemble \( \text{(22)} \). For each fixed set of \( \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^\rho \), we obtain the expression

\[
\sum_{\mathbf{c}} \delta \left( \sum_{(l_1,l_2,\ldots,l_k)} c_{(l_1,l_2,\ldots,l_k)}, j \right) \prod_{a=1}^\rho \mathcal{I}(\mathbf{x}^a|\mathbf{c})
\]

\[
= \frac{1}{(2\pi)^N} \int \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \prod_{(l_1,l_2,\ldots,l_k)} \left( 1 + Z_{l_1} Z_{l_2} \cdots Z_{l_k} \prod_{a=1}^\rho \delta(x_{l_1}^a + x_{l_2}^a + \cdots + x_{l_k}^a, 0) \right)
\]

\[
\approx \frac{1}{(2\pi)^N} \int \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \exp \left[ \sum_{(l_1,l_2,\ldots,l_k)} Z_{l_1} Z_{l_2} \cdots Z_{l_k} \prod_{a=1}^\rho \delta(x_{l_1}^a + x_{l_2}^a + \cdots + x_{l_k}^a, 0) \right]
\]

\[
= \frac{1}{(2\pi)^N} \int \prod_{l=1}^N dZ_l Z_l^{-(j+1)} \times
\]

\[
\exp \left[ \sum_{b_1,b_2,\ldots,b_k} N^k \prod_{t=1}^k \left( \frac{1}{N} \sum_{l=1}^N Z_l \prod_{a=1}^\rho \delta(b_{l_1}^a + b_{l_2}^a + \cdots + b_{l_k}^a, 0) \right) \right], \quad (A.4)
\]

where we have introduced the dummy variables \( \mathbf{b}_t = (b_{l_1}^a, b_{l_2}^a, \ldots, b_{l_k}^a) \) \((t = 1, 2, \ldots, k)\) as

\[
\prod_{a=1}^\rho \delta(x_{l_1}^a + x_{l_2}^a + \cdots + x_{l_k}^a, 0) = \sum_{\mathbf{b}_1,\mathbf{b}_2,\ldots,\mathbf{b}_k} \left( \prod_{a=1}^\rho \prod_{t=1}^k \delta(x_{l_1}^a + b_{l_1}^a, b_{l_1}^a) \prod_{a=1}^\rho \delta(b_{l_1}^a + b_{l_2}^a + \cdots + b_{l_k}^a, 0) \right), \quad (A.5)
\]

in order to decouple \( x_{l_1}^a, x_{l_2}^a, \ldots, x_{l_k}^a \) of the left hand side. Inserting the identity

\[
1 = N^{-2^\rho} \int_\mathbf{b} \mathcal{D}(\mathbf{x}) \delta \left( \sum_{l=1}^N Z_l \prod_{a=1}^\rho \delta(x_{l_1}^a, b^a) - N \mathcal{X}(\mathbf{x}) \right)
\]
\[
\begin{align*}
&= \frac{1}{(2\pi N)^{2\rho}} \int \left( \prod_b d\chi(b) d\tilde{\chi}(b) \right) \exp \left[ \sum_b \tilde{\chi}(b) \left( \sum_{l=1}^{N} Z_l \prod_{a=1}^{\rho} \delta(x^a_l, b^a) - N \chi(b) \right) \right] \\
&= \frac{1}{(2\pi N)^{2\rho}} \int \left( \prod_b d\chi(b) d\tilde{\chi}(b) \right) \exp \left[ \sum_{l=1}^{N} Z_l \tilde{\chi}(x_l) - N \sum_b \tilde{\chi}(b) \chi(b) \right], \tag{A.6}
\end{align*}
\]

where \( x_l = (x^1_l, x^2_l, \ldots, x^\rho_l) \) \((l = 1, 2, \ldots, N)\), into equation (A.4) allows integration with respect to \( Z_l \) \((l = 1, 2, \ldots, N)\) to be performed analytically. The resulting expression enables us to take summations with respect to \( x_l \) \((l = 1, 2, \ldots, N)\) independently in assessing the average, which yields identical contributions for \( l = 1, 2, \ldots, N \) and leads to the saddle point evaluation of equation (23).

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