INTERMEDIATE VALUE PROPERTY FOR THE ASSOUAD DIMENSION OF MEASURES

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Abstract. Hare, Mendivil, and Zuberman have recently shown that if $X \subset \mathbb{R}$ is compact and of non-zero Assouad dimension $\dim_A X$, then for all $s > \dim_A X$, $X$ supports measures with Assouad dimension $s$. We generalize this result to arbitrary complete metric spaces.

1. Introduction and notation

The Assouad dimension has its origins in metric geometry and embedding problems, see e.g. \cite{10, 7}. More recently, it has been investigated thoroughly in fractal geometry. For a few of the many recent advances, see \cite{12, 4, 8, 5, 2, 3}. The Assouad dimension of a metric space $X = (X, d)$, denoted $\dim_A X$, is defined to be the infimum of those $\alpha \geq 0$ such that the following uniform bound holds for all $x \in X$, $0 < r < R < \infty$:

$$N(x, R, r) \leq \frac{C(R)}{r^\alpha}.$$

Here $N(x, R, r)$ denotes the minimal number of closed $r$-balls needed to cover the closed $R$-ball $B(x, R) = \{y \in X : d(x, y) \leq R\}$ and $0 < C < \infty$ is a constant independent of $x$, $r$, and $R$. If there is no $\alpha \in [0, \infty]$ fulfilling the requirement (1.1), we assign $\dim_A X$ the value $\infty$.

There is a close analogue of this concept for measures. Suppose that $\mu$ is a (Borel regular outer-) measure on $X$. For simplicity, we always assume that our measures are fully supported, that is $\text{spt } \mu = X$. The Assouad dimension of $\mu$, denoted by $\dim_A \mu$, is the infimum of those $\alpha \geq 0$ such that for all $x \in X$, $0 < r < R < \infty$,

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{C(R)}{r^\alpha},$$

where again $C < \infty$ is a constant depending only on $\alpha$. If (1.2) is not fulfilled for any $\alpha < \infty$, we define $\dim_A \mu = \infty$. This concept has appeared in the literature also under the names upper regularity dimension and upper Assouad dimension and measures satisfying the condition (1.2) have been called $\alpha$-homogeneous.

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It is well known and easy to see that \( \dim_A X < \infty \) if and only if \( X \) is geometrically doubling in the following sense: There is a constant \( C < \infty \) such that 
\[
N(x, 2r, r) \leq C
\]
for all \( x \in X \), \( 0 < r < \infty \). Likewise, \( \dim_A \mu < \infty \) if and only if 
\[
\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq C,
\]
for some constant \( C < \infty \), and for all \( x \in X \) and \( 0 < r < \infty \).

The investigation of the relation between the Assouad dimension of sets and that of measures supported on them was pioneered by Vol’berg and Konyagin \[13, 14\], who proved that for a compact metric space \( X \) with finite Assouad dimension,
\[
\dim_A X = \inf \{ \dim_A \mu : \text{spt} \mu \subset X \}. \tag{1.3}
\]
Note that \( \dim_A X \) is an obvious lower bound for \( \dim_A \mu \), whenever \( \mu \) is supported on \( X \). Luukkainen and Saksman \[11\] later extended \((1.3)\) for complete metric spaces. These results, however, left open the question whether all values \( s > \dim_A X \) are attainable for \( \dim_A \mu \). In other words, is it possible that the Assouad dimension of measures omits some values \( s > \dim_A X \)? Plausibly not, but an example of Hare, Mendivil, and Zuberman \[6, Proposition 3.8\] of an infinite compact set \( X \subset \mathbb{R} \) such that
\[
\{ \dim_A \mu : \text{spt} \mu \subset X \} = \{0, \infty\}
\]
suggests that this could be a subtle issue. An analogous intermediate value problem for the Assouad dimension of subsets has been considered in \[15, 11\].

Quite recently, Hare, Mendivil, and Zuberman \[6\], have shown that if \( X \subset \mathbb{R} \) is compact and \( \dim_A X > 0 \), then \( \dim_A \mu \) indeed attains all values \( s > \dim_A X \) when \( \mu \) ranges over all measures supported on \( X \). In this note, we prove

**Theorem 1.1.** If \( X \) is a complete metric space with \( 0 < \dim_A X < \infty \), then for all \( s > \dim_A X \), there is a measure \( \mu \) on \( X \) such that \( \dim_A \mu = s \).

Theorem \[1.4\] extends the result of Hare, Mendivil, and Zuberman to arbitrary complete metric spaces and thus settles the problem raised in \[6, Remark 4.3\]. We stress that there is no imposed upper bound on \( R \) in the definitions \((1.1)\) and \((1.2)\) and our result thus makes perfect sense also for unbounded \( X \).

Our starting point to prove Theorem \[1.1\] is similar to that in \[6\] in that we both rely on the construction of generalized cubes in \[9\], see Theorem \[2.1\] below. Given \( s > \dim_A X \), the result in \[6\] is obtained by a specific construction yielding a measure with \( \dim_A \mu = s \). Our approach is different in that we consider a family of measures \( \mu_p \) and relying on the continuity of the map \( p \mapsto \dim_A \mu_p \) verify that \( \dim_A \mu_p = s \) for some value of \( p \).
A result analogous to Theorem 1.1 for the lower Assouad dimension follows by essentially the same proof. We discuss this issue briefly in the last section of the paper.

2. Toolbox: Generalized cubes

We begin by restating [9, Theorem 2.1], which is our main tool. It provides us an analogue of the M-adic cubes in Euclidean spaces. These “generalized nested cubes” will be used throughout these notes.

**Theorem 2.1.** Let \( X \neq \emptyset \) be a metric space such that \( N(x, 2r, r) < \infty \) for all \( x \in X, r > 0 \). For each \( 0 < \delta < 1 \), there exists at most countable collections \( Q_k, k \in \mathbb{Z} \), of Borel sets having the following properties:

(i) \( X = \bigcup_{Q \in Q_k} Q \) for every \( k \in \mathbb{Z} \),

(ii) If \( Q \in Q_k, Q' \in Q_{k'}, k' \geq k \) and \( Q \cap Q' \neq \emptyset \), then \( Q' \subset Q \).

(iii) For every \( Q \in Q_k \), there is a distinguished point \( x_Q \in X \) so that \( B(x_Q, \frac{1}{C \delta^k}) \subset Q \subset B(x_Q, C \delta^k) \),

where \( 0 < C < \infty \) is a constant that depends only on \( \delta \).

(iv) There exists a point \( x_0 \in \bigcap_{k \in \mathbb{Z}} \{ x_Q : Q \in Q_k \} \).

(v) \( \{ x_Q : Q \in Q_k \} \subset \{ x_Q : Q \in Q_{k+1} \} \) for all \( k \in \mathbb{Z} \).

We make some remarks and introduce some further notation. We first note that, even if \( \delta \) is fixed, the families \( Q_k \) are in no way unique. However, for the sake of simplicity, when we refer to Theorem 2.1 with a given value \( 0 < \delta < 1 \), we always assume the families \( Q_k \) have been fixed. We will call the elements of \( \bigcup_{k \in \mathbb{Z}} Q_k \) cubes and refer to the point \( x_Q \) as the center of the cube \( Q \). The distinguished point \( x_0 \) may be considered the “origin” of the space \( X \). We also denote by \( Q_0 \) the unique element of \( Q_0 \) that contains \( x_0 \). This is the “unit cube” of \( X \).

If \( Q \in Q_k, Q' \in Q_{k+m}, m \in \mathbb{N}, \) and \( Q' \subset Q \), we denote \( Q' \prec_m Q \). We also abbreviate \( \prec_1 \) to \( \prec \). If \( Q' \prec Q \), we say that \( Q \) is the parent of \( Q' \), and that \( Q' \) is a child of \( Q \). More generally, we will refer to cubes \( Q' \prec_m Q, m \in \mathbb{N}, \) as offspring of \( Q \). It is convenient to think about the child-cube \( Q' \prec Q \) with \( x_{Q'} = x_Q \) as a distinguished central subcube of \( Q \). If \( Q' \prec Q \) is not a central subcube, we say that \( Q \) is a boundary cube. Let

\[ M_Q = |\{ Q' : Q' \prec Q \}| \]

stand for the number of children of \( Q \). If \( \dim_A X < \infty \), then \( M_Q \) is bounded and we denote

\[ M := \max_{n \in \mathbb{Z}, Q \in Q_n} M_Q. \]

To demonstrate Theorem 2.1 and our later considerations, it is helpful to keep in mind the following simple example.
Example 2.2. Let $\delta = \frac{1}{3}$, $X = \mathbb{R}$ and let $Q_k$ consist of the triadic half-open intervals

$$Q_k = \{ \left[ \left( j - \frac{1}{2} \right)3^{-k}, \left( j + \frac{1}{2} \right)3^{-k} \right] : j \in \mathbb{Z} \}.$$ 

Now $x_Q$ is literally the center point of each interval $Q$ and we may, for instance, fix $x_0 = 0$ so that $Q_0 = \left[ -\frac{1}{2}, \frac{1}{2} \right]$. Moreover, $M_Q = 3$ for all $Q \in Q_k$, $k \in \mathbb{Z}$ and thus $M = 3$.

Our next lemma provides us doubling measures that respect the construction of the generalized cubes. For its proof, see [9, Theorem 3.1 and Remark 5.1 (2)].

Lemma 2.3. Let $0 < \delta < \frac{1}{3}$ and let $0 < p < \frac{1}{M}$. Then there is a measure $\mu_p$ on $X$ such that $\dim_A \mu_p < \infty$ and so that for all $Q' \prec Q$ it holds that

$$\mu_p(Q') = \begin{cases} p \mu_p(Q), & \text{if } Q' \text{ is a boundary cube,} \\ (1 - (M_Q - 1)p) \mu_p(Q), & \text{if } Q' \text{ is the central subcube of } Q . \end{cases} \quad (2.1)$$

Moreover, if $t > \dim_A \mu$, then $\dim_A \mu_p < t$ for some choice of $\delta$ and $p$.

Remark 2.4. (1) The condition (2.1) implies that (having the nested cubes construction fixed), the measures $\mu_p$ are uniquely defined up to a multiplicative constant. If we further require that $\mu_p(Q_0) = 1$, the measure $\mu_p$ is thus completely determined by this condition.

(2) It is instructive to think about the measures $\mu_p$ in the setting of the Example 2.2 where they are uniquely defined for all $0 < p < \frac{1}{2}$ via (2.1) and the condition $\mu_p(-\frac{1}{2}, \frac{1}{2}] = 1$. In this case, the measure of each triadic interval is split among its triadic child-intervals in such a way that the central child inherits $(1 - 2p)$ times the mass of its parent and the rest is split equally between the two boundary children. It is an easy exercise to verify that $\dim_A \mu_p = -\log_3 p$, for all $0 < p \leq \frac{1}{3}$. Thus, $\dim_A \mu_p$ varies continuously in $p$ and attains all values $\geq 1 = \dim_A \mathbb{R}$. Our main result, Theorem 1.1 and its proof, reflect this phenomenon in the more abstract setting of Lemma 2.3.

Before starting with the proofs, we introduce our last bit of notation. For each $t > 0$, we denote by $n_t$ the smallest integer such that $\delta^{n_t} \leq t$, that is,

$$n_t = \lceil \log t / \log \delta \rceil .$$

This notation allows us to switch to a logarithmic scale in the ratio $R/r$. Namely, if $N = n_r - n_R$, then

$$\frac{1}{C} \delta^{-N} \leq \frac{R}{r} \leq C \delta^{-N} . \quad (2.2)$$

Here, and in what follows, we denote by $C$ a positive and finite constant that only depends on $\delta$ and $M$ and whose precise value is of no importance.

To conclude this section, we provide the following variant of [6, Lemma 3.5].
Lemma 2.5. Suppose that $\dim_A X > 0$. Then, for a suitably small $\beta > 0$ there are arbitrarily large $N \in \mathbb{N}$ and cubes

$$Q^N \prec Q^{N-1} \prec \ldots \prec Q^1 \prec Q^0$$

(2.3)

such that at least $\beta N$ of the cubes $Q^1, \ldots, Q^N$ are boundary cubes.

Proof. Suppose that $\beta > 0$ does not satisfy the requirement of the lemma. We prove the lemma by deriving a lower bound for $\beta$. To that end, we fix $N_0$ so large that any cube sequence as in (2.3) contains at most $\beta N$ boundary cubes. Then, for some $N_0 \in \mathbb{N}$, all $k \in \mathbb{Z}$, all $Q \in Q_k$, and all $N \geq N_0$, it holds that

$$|\{Q' : Q' \prec_N Q\}| \leq \binom{N}{\lfloor \beta N \rfloor} M^{\beta N} \quad \text{(2.4)}$$

To see this, label the offspring of $Q$ as follows. Denote by $Q^1_1$ the central child-cube of $Q$ and by $Q^2_1, \ldots, Q^M_1$ the boundary children. For each $Q^i_1$, denote by $Q^i_2, \ldots, Q^i_M$ the boundary children. Continuing this labelling for $N$ generations of offspring of $Q$, each $Q' \prec_N Q$ gets labelled with a sequence $i \in \{1, \ldots, M\}^N$. But, for each $Q^i_1 \prec_N Q$, $i = i_1 \ldots i_N$, only at most $\beta N$ of the symbols $i_k$ differ from 1, and for each such $i_k$, there are at most $M$ admissible values. Whence the bound (2.4).

Let $x \in X$, $0 < r < R < \infty$ and consider a cube $Q \in Q_{nR}$ with $Q \cap B(x, R) \neq \emptyset$. There are at most $C$ such cubes. Switching to the logarithmic scale $N = n_r - n_R$ and recalling (2.4), we observe that the ball $B(x, R)$ may be covered by

$$C \left( \frac{N}{\lfloor \beta N \rfloor} \right) M^{\beta N}$$

(2.5)

cubes $Q' \in Q_{n_r}$. Using trivial bounds for factorials, e.g.

$$n^n e^{1-n} \leq n! \leq (n + 1)^{n+1} e^{-n},$$

and taking logarithms in base $\delta$, we note that

$$\binom{N}{\lfloor \beta N \rfloor} \leq C \delta^{-\kappa(\beta)N},$$

where $\kappa(\beta) \to 0$ as $\beta \to 0$. Thus, (2.5) is bounded from above by

$$C \delta^{-N(\kappa(\beta) - \beta \log_3 M)}.$$

Since each $Q' \in Q_{n_r}$ may be covered by $C$ balls of radius $r$, and because $\frac{R}{r} \geq C \delta^{-N}$, recall (2.2), it follows that

$$N(x, R, r) \leq C \left( \frac{R}{r} \right)^{\kappa(\beta) - \beta \log_3 M}.$$

This upper bound is valid irrespective of $x$, $r$ and $R$ and thus

$$\dim_A X \leq \kappa(\beta) - \beta \log_3 M,$$

yielding the required lower bound for $\beta$. \qed
3. Proof of Theorem 1.1

In the remainder of the paper, we prove the following two lemmas. Theorem 1.1 follows readily from these and the last claim of Lemma 2.3. We fix $0 < \delta < \frac{1}{2}$ and consider the families $Q_k$ provided by Theorem 2.1 along with the measures $\mu_p$ provided by Lemma 2.3.

**Lemma 3.1.** The map $p \mapsto \dim_A \mu_{\delta,p}$ is continuous on the interval $]0, \frac{1}{M}[.$

**Lemma 3.2.** If $\dim_A X > 0$, then $\lim_{p \downarrow 0} \dim_A \mu_p = \infty.$

**Proof of Lemma 3.1.** For all $x \in X$, $t > 0$, and $x \in Q \in Q_{n_1}$, we have

$$\frac{1}{C} \mu_p(Q) \leq \mu_p(B(x, t)) \leq C \mu_p(Q). \quad (3.1)$$

More precisely, the constant $C$ may be taken to be uniform in $p$, if $p$ is bounded away from 0, which we can of course assume. To verify this is an easy exercise using (2.1), or see [9, proof of Theorem 3.1].

Consider $x \in X$ and $0 < r < R < \infty$. We switch to the logarithmic scale $N = n_r - n_R$. Let $Q_R \in Q_{n_R}, Q_r \in Q_{n_r}$ be the unique cubes containing $x$. Then $Q_r \prec N Q_R$ and by virtue of (3.1),

$$\frac{1}{C} \frac{\mu_p(Q_R)}{\mu_p(Q_r)} \leq \frac{\mu_p(B(x, R))}{\mu_p(B(x, r))} \leq C \frac{\mu_p(Q_R)}{\mu_p(Q_r)}. \quad (3.2)$$

Consider the parental line of cubes from $Q_R$ to $Q_r$ by denoting $Q^{N} = Q_r, Q^{N-1}, \ldots, Q^{1} < Q^0 = Q_R$. Let $\mathcal{B} = \{i \in \{1, \ldots, N\} : Q^i$ is a boundary cube$\}$ and $\mathcal{C} = \{1, \ldots, N\} \setminus \mathcal{B}$. Using (2.1), it follows that

$$\frac{\mu_p(Q_r)}{\mu_p(Q_R)} = p^{\lvert \mathcal{B} \rvert} \prod_{n \in \mathcal{C}} (1 + (1 - M_{Q^{n-1}})p).$$

Thus, given $0 < p < p' < \frac{1}{M}$, we have

$$\left(\frac{1 - (M - 1)p'}{1 - (M - 1)p}\right)^N \leq \frac{\mu_p(Q_R)/\mu_p(Q_r)}{\mu_{p'}(Q_R)/\mu_{p'}(Q_r)} \leq \left(\frac{p'}{p}\right)^N.$$

Recalling (2.2) and (3.2) and expressing these upper and lower bounds as

$$\left(\frac{p'}{p}\right)^N = \delta^{N(\log_3 p' - \log_3 p)},$$

$$\left(\frac{1 - (M - 1)p'}{1 - (M - 1)p}\right)^N = \delta^{N(\log_3 (1 - (M - 1)p') - \log_3 (1 - (M - 1)p))},$$

we note that if one of the measures $\mu_p, \mu_{p'}$ satisfies the condition (1.2) with the exponent $\alpha$, then the other satisfies it with the exponent $\alpha + \varepsilon$, where

$$\varepsilon = \max \{\log_3 p - \log_3 p', \log_3 (1 - (M - 1)p') - \log_3 (1 - (M - 1)p)\}.$$

Thus, we observe a quantitative modulus of continuity for $p \mapsto \dim_A \mu_p$. $\varepsilon$
Proof of Lemma 3.2. We estimate \( \dim A \mu_p \) from below. Let \( \beta > 0 \) be the constant from Lemma 2.5. Then, there are arbitrarily long offspring chains

\[
Q^N \prec Q^{N-1} \prec \ldots \prec Q^1 \prec Q^0
\]

with at least \( \beta N \) of the cubes \( Q^1, \ldots, Q^N \) boundary cubes. Whence, using (2.1),

\[
\mu_p(Q^N) \leq p^\beta N \mu_p(Q^0).
\]

Let \( x = x_{Q^N} \) and pick \( n \in \mathbb{Z} \) such that \( Q^0 \in Q_n \). Then, for \( r = \frac{1}{C} \delta^n + N, R = C\delta^n \), we have \( B(x, r) \subset Q^N \) and \( Q^1 \subset B(x, R) \). Whence

\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq \frac{\mu(Q^0)}{\mu(Q^N)} \geq p^{-\beta N} = \delta^{-N(\beta \log_\delta p)} \geq C \left( \frac{R}{r} \right)^{\beta \log_\delta p}.
\]

Since \( N \), and thus the ratio \( R/r \), may be arbitrarily large, this shows that

\[
\dim A \mu_p \geq \beta \log_\delta p \to \infty,
\]
as \( p \downarrow 0 \). \( \square \)

4. Attainable values for the lower Assouad dimension

In this final section, we discuss the following “dual” result for Theorem 1.1.

Theorem 4.1. If \( X \) is a complete metric space with \( \dim A X < \infty \), then for all \( 0 < s < \dim A X \), there is a measure \( \mu \) with support \( X \) such that \( \dim A \mu = s \).

Here \( \dim A X \) is the lower Assouad dimension of \( X \) defined as the supremum of the exponents \( \alpha \), for which there is a constant \( C < \infty \) such that

\[
N(x, R, r) \geq C \left( \frac{R}{r} \right)^{\alpha},
\]
for all \( x \in X \) and \( 0 < r < R < \text{diam}(X) \). Analogously, \( \underline{\dim} A \mu \) is the supremum of those \( \alpha \), for which

\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \geq C \left( \frac{R}{r} \right)^{\alpha}
\]
holds irrespective of \( x \in X \), \( 0 < r < R < \text{diam}(X) \).

For a compact set of reals, this result is also due to Hare, Mendivil, and Zuber-
man, see [6, Theorem 4.1]. In its full generality, Theorem 4.1 may be proved using ideas very similar to our proof of Theorem 1.1. Thus, we only sketch the idea and leave the details for the interested reader.

To begin with, we observe that \( p \mapsto \underline{\dim} A \mu_p \) is continuous. This follows by the very same proof as Lemma 3.1. Likewise, switching to a chain of central cubes in the proof of Lemma 3.2 gives \( \lim_{p \downarrow 0} \underline{\dim} A \mu_p = 0 \). So if we knew that for \( s < \dim A X \), we had \( \underline{\dim} A \mu_p > s \) for a suitably chosen \( \delta \) and \( p \), the claim would follow just as in the case of Theorem 1.1. This holds in some nice cases such as the Example 2.2, but unfortunately it does not hold in general if there is variation in the numbers \( M_Q \).
However, the argument can still be saved by considering mass distributions more general than those defined by (2.1). Given \( L \in \mathbb{N} \), let \( 0 < p < \frac{1}{M} \) and let \( \eta = (\eta_1, \ldots, \eta_L) \) be a probability vector, where all the weights \( \eta_i \) are non-zero. For each cube \( Q \in Q_k \) (\( \delta^k < C \text{diam}(X) \)), we consider \( L \) distinguished central subcubes \( Q(1), \ldots, Q(L) \prec Q \) whose distance to the complement of \( Q \) is comparable to \( \delta^k \). Note that this is possible, provided \( \delta \) is small enough depending on \( L \) and \( \dim A X \), see e.g. [8, Proof of Theorem 3.2]. Again, each boundary cube \( Q' \prec Q \) inherits \( p \) times the mass of its parent and the rest is distributed among the central child-cubes \( Q(1), \ldots, Q(L) \) in the proportion of the weights \( \eta_1, \ldots, \eta_L \). Denote the resulting measure by \( \mu_{p, \eta} \). Note that the measures \( \mu_p \) considered earlier correspond to the case \( L = 1 \) and the trivial probability vector \( \eta = (1) \).

The proof of Lemma 3.1 still works in this setting and implies that

\[
(p, \eta) \mapsto \dim \mu_{p, \eta} \text{ is continuous.} \tag{4.1}
\]

Moreover, if e.g. \( p \to 0 \) and \( \eta \to (1,0,\ldots,0) \), then it is very easy to see, by following a cube chain \( Q^N \prec Q^{N-1} \prec \cdots \prec Q^1 \prec Q^0 \), where \( Q^{i+1} = Q^i(1) \) for each \( i = 0, \ldots, N-1 \), that

\[
\dim \mu_{p, \eta} \to 0. \tag{4.2}
\]

Finally, a result of Käenmäki and Lehrbäck [8] implies that given \( s < \dim A X \), it is possible to choose \( \delta, L, p, \) and \( \eta \) so that

\[
\dim \mu_{p, \eta} > s. \tag{4.3}
\]

In fact, for (4.3), it is enough to consider the uniform probability vector \( \eta = (\frac{1}{L}, \frac{1}{L}, \ldots, \frac{1}{L}) \) for a suitably chosen \( \delta > 0 \) and \( L \in \mathbb{N} \), see [8, Theorem 3.4]. Theorem 4.1 now follows by combining (4.1)–(4.3).

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