A formula to estimate the Sobolev embedding constant on domains with minimally smooth boundary

Kazuaki Tanaka\textsuperscript{1*}, Kouta Sekine\textsuperscript{2}, Makoto Mizuguchi\textsuperscript{1}, Shin’ichi Oishi\textsuperscript{2,3}

\textsuperscript{1}Graduate School of Fundamental Science and Engineering, Waseda University,
\textsuperscript{2}Faculty of Science and Engineering, Waseda University,
\textsuperscript{3}CREST, JST

Abstract. In this paper, we proposed a formula to estimate an upper bound of the Sobolev type embedding constant from $W^{1,q}(\Omega)$ to $L^p(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3, \cdots$) with minimally smooth boundary. An estimation of the embedding constant can be derived by calculating concrete values of the following two constants. The first one is the classical Sobolev embedding constant on $\mathbb{R}^n$. The second one is the extension constant $A_q(\Omega)$ that satisfies $\|\nabla(Eu)\|_{L^q(\mathbb{R}^n)} \leq A_q(\Omega)\left(\|\nabla u\|_{L^q(\Omega)} + \sigma^{1/q}\|u\|_{L^q(\Omega)}\right)$, $\forall u \in W^{1,q}(\Omega)$ for given $\sigma > 0$ and an extension operator $E : W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^n)$. The main contribution of this paper is to propose a formula giving a concrete value of $A_q(\Omega)$ for an extension operator on a domain $\Omega$ with minimally smooth boundary. We also presented some examples of estimating the embedding constant for certain fixed domains.

Key words: embedding constant; extension operator; Sobolev inequality

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 2, 3, \cdots$) be a domain with minimally smooth boundary whose definition will be introduced in Definition 4. Let $p \in (n/(n-1), \infty)$ and set $q = np/(n+p)$. We are concerned with a concrete estimation of the embedding constant $C_p(\Omega)$ from $W^{1,q}(\Omega)$ to $L^p(\Omega)$, i.e., $C_p(\Omega)$ satisfies

$$\|u\|_{L^p(\Omega)} \leq C_p(\Omega)\|u\|_{W^{1,q}(\Omega)}, \quad \forall u \in W^{1,q}(\Omega),$$

where the norm $\|\cdot\|_{W^{1,q}(\Omega)}$ denotes the $\sigma$-weighted $W^{1,q}$ norm defined as $\|\cdot\|_{W^{1,q}(\Omega)}^q := \|\nabla \cdot\|_{L^q(\Omega)} + \sigma^{1/q}\|u\|_{L^q(\Omega)}$ for given $\sigma > 0$.

It has been well known that Sobolev type embedding theorems are important in studies on partial differential equations. There have been a lot of works on such theorems and their applications, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The classical Sobolev embedding theorem has been well known. Moreover, the best constant in the classical Sobolev inequality on $\mathbb{R}^n$ was independently shown by Aubin [1] and Talenti [17] in 1976 (see Theorem A.1). Since all elements $u \in W_0^{k,q}(\Omega)$ can be regarded as elements of $W^{k,q}(\mathbb{R}^n)$ by zero extension outside $\Omega$,\textsuperscript{2} the embedding constants satisfying (1) with the restriction $u \in W_0^{k,q}(\Omega)$ also can be estimated for a general domain $\Omega \subset \mathbb{R}^n$ by calculating the classical embedding constant on $\mathbb{R}^n$. Removing the restriction, however, such a simple extension cannot be constructed. The embedding constants without the restriction are concretely estimated if one can construct a linear and bounded operator $E$ from $W^{1,q}(\Omega)$ to $W^{1,q}(\mathbb{R}^n)$ such that $(Eu)(x) = u(x), \forall x \in \Omega$ and can derive a concrete value of the extension constant $A_q(\Omega)$ satisfying

$$\|\nabla(Eu)\|_{L^q(\mathbb{R}^n)} \leq A_q(\Omega)\left(\|\nabla u\|_{L^q(\Omega)} + \sigma^{1/q}\|u\|_{L^q(\Omega)}\right), \quad \forall u \in W^{1,q}(\Omega).$$

\textsuperscript{*}E-mail addresses: *imahazimari@fuji.waseda.jp

\textsuperscript{1}The paper [1] had been written in French. Aubin’s works containing [1] are summarized in [2] in English.

\textsuperscript{2}The space $W_0^{k,q}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ commonly defined.
Such an operator $E$ is called an extension operator. There have been some construction methods of extension operators. For example, the reflection method originates from Whitney [18] and Hestenes [10], whose summary can be found in, e.g., [19, 20]. The Calderón extension theorem originates from [6], which is summarized in, e.g., [19]. Stein [14] has shown that extension operators can be constructed on domains with minimally smooth boundary.

The main contribution of this paper is to propose a formula giving a concrete value of $A_q(\Omega)$ for the extension operator constructed by Stein’s method. Stein first constructed an extension operator on the special Lipschitz domain and then expanded this to that on domains with minimally smooth boundary. In his method, the regularized distance plays an important role, which is a $C^\infty$ function approximating the distance from a given closed set $S \subset \mathbb{R}^n$ to any point in its complement $S^c$. After the appearance of Stein’s construction method, the regularized distance was generalized to a one-parameter family of smooth functions by Fraenkel [21]. We will construct extension operators using Stein’s method with the generalized regularized distance to derive the embedding constant.

This paper consists of the following sections. In Section 2, we propose formulae for estimating the extension constant and the embedding constant. In Section 3, we show a proof of the main theorem proposed in Section 2. In Section 4, we present results of estimating the embedding constant for two fixed domains.

2 Formula for estimating the embedding constant

In this section, we propose a formula for estimating the embedding constant $C_p(\Omega)$. In Subsection 2.1, some notations and definitions are introduced. In Subsection 2.2, we propose a formula giving the concrete value of the extension constant. A concrete value of the extension constant and the best constant in the classical Sobolev inequality [1, 17] enable us to estimate the embedding constant.

2.1 Preparation

Throughout this paper, the following notations are used:

- $\mathbb{N} = \{1, 2, 3, \cdots \}$ and $\mathbb{N}_0 = \{0, 1, 2, \cdots \}$;
- $B(x, r)$ is an open ball whose center is $x$ and whose radius is $r \geq 0$;
- for any point $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, define $|x| := (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$;
- for any set $S \subset \mathbb{R}^n$, $S^c$ is its complementary set and $\overline{S}$ is its closure set;
- for any set $S \subset \mathbb{R}^n$ and any $\varepsilon > 0$, define $S^{\varepsilon} := \{x \in \mathbb{R}^n : B(x, \varepsilon) \subset S\}$;
- for any point $x \in \mathbb{R}^n$ and any set $S \subset \mathbb{R}^n$, define $\text{dist}(x, S) := \inf\{|x - y| : y \in S\}$;
- for any function $f$, $\text{supp} f$ denotes the support of $f$;
- for any function $f$ over $\mathbb{R}$, $f'$ denotes the ordinary derivative of $f$;
- for any function $f$ over $\mathbb{R}^n$ ($n = 2, 3, \cdots$), $\partial_{x_i} f$ denotes the partial derivative of $f$ with respect to the $i$-th component $x_i$ of $x$.

Some functional spaces are also introduced as follows: Let $L^p(\Omega)$ ($1 \leq p < \infty$) be the functional space of $p$-th power Lebesgue integrable functions over $\Omega$. Let $W^{k,p}(\Omega)$ ($k \in \mathbb{N}$, $1 \leq p < \infty$) be the $k$-th order $L^p$ Sobolev space on $\Omega$; in particular, we denote $H^k(\Omega) := W^{k,2}(\Omega)$.

We then introduce some definitions as follows:

*3 The special Lipschitz domain will be defined in Definition 3.
*4 The generalized regularized distance will be defined in Definition 2.
**Definition 1 (Mollifier).** A nonnegative function $\rho \in C^\infty (\mathbb{R}^n)$ is said to be a mollifier if

1. $\rho (x) = 0$ if $|x| \geq 1$;
2. $\int_{\mathbb{R}^n} \rho (x) \, dx = 1$.

For example, the function

$$\rho (x) := \begin{cases} c \exp \left( \frac{-1}{1-|x|^2} \right), & |x| < 1, \\ 0, & |x| \geq 1 \end{cases} \quad (3)$$

becomes a mollifier, where $c$ is chosen so that $\int_{\mathbb{R}^n} \rho (x) \, dx = 1$.

**Definition 2 (Regularized distance [21]).** Let $S$ be an arbitrary closed set in $\mathbb{R}^n$. Then, for given any $\xi \in (0, 1)$, there exists a function $RD_{S, \xi} : S^c \to \mathbb{R}$ such that, for all $x \in S^c$,

1. $(1 + \xi)^{-2} \text{dist} (x, S) \leq RD_{S, \xi} (x) \leq (1 - \xi)^{-2} \text{dist} (x, S)$
2. $RD_{S, \xi}$ is in $C^\infty (S^c)$ and satisfies $\left| \frac{\partial^\alpha}{\partial x^\alpha} RD_{S, \xi} (x) \right| \leq P_\alpha (\xi \text{dist} (x, S))^{1-|\alpha|}$.

Here, $P_\alpha$ is not depend on $S$ but only on a multi-index $\alpha \in \mathbb{N}_0^n$. The function $RD_{S, \xi}$ is called regularized distance from $S$.

**Definition 3 (Special Lipschitz domain [14]).** Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ ($n = 2, 3, \cdots$) be a function satisfying the Lipschitz condition

$$|\phi (x) - \phi (y)| \leq M |x - y|, \quad \forall x, y \in \mathbb{R}^{n-1}$$

for some $M > 0$. Then, $\Omega$ is called a special Lipschitz domain if it is written as $\Omega := \{(x', x_n) \in \mathbb{R}^n : x_n > \phi (x')\}$ with $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$.

The positive number $M$ in Definition 3 is called the Lipschitz constant of $\Omega$. Generalizing the special Lipschitz domain, the domain with minimally smooth boundary is defined as follows:

**Definition 4 (Domain with minimally smooth boundary [14]).** An open set $\Omega \subset \mathbb{R}^n$ ($n = 2, 3, \cdots$) is said to be a domain with minimally smooth boundary if there exist $\varepsilon > 0$, $N \in \mathbb{N}$, $M > 0$, and a sequence $\{U_i\}_{i \in \mathbb{N}}$ of open subsets of $\mathbb{R}^n$ such that

1. for any $x \in \partial \Omega$, $B (x, \varepsilon) \subset U_i$ holds for some $i \in \mathbb{N}$;
2. no point in $\mathbb{R}^n$ belongs to more than $N$ of the $U_i$;
3. for any $i \in \mathbb{N}$, there exists a special Lipschitz domain $\Omega_i$, whose Lipschitz bound is not more than $M$, such that $U_i \cap \Omega = U_i \cap \Omega_i$.

The positive number $M$ in Definition 4 is called the Lipschitz constant of $\Omega$, and $N$ in Definition 4 is called the overlap number of $\Omega$. To avoid confusion, $M$ and $N$ are sometimes denoted by $M_\Omega$ and $N_\Omega$, respectively.

*5From Lemma B. 2, the existence of the regularized distance is guaranteed. One can also find a formula giving the concrete value of $P_\alpha$ in the lemma.
2.2 The main theorem

The following theorem is the main theorem that gives a formula to estimate the extension constant.

**Theorem 1.** Let \( n \in \{2, 3, \ldots \} \) and let \( p \in [1, \infty) \). Suppose that \( \Omega \subset \mathbb{R}^n \) is a domain with minimally smooth boundary defined in Definition 4. Then, for any \( k \in \mathbb{N}_0 \), there is a linear and bounded operator \( E \) from \( W^{k,p}(\Omega) \) to \( W^{k,p}(\mathbb{R}^n) \) such that \( (Eu)(x) = u(x) \), \( \forall x \in \Omega \).

Let \( \gamma \) be any given positive number. Then,

\[
\| \nabla (Eu) \|_{L^p(\mathbb{R}^n)} \leq A_p(\Omega) \left( \| \nabla u \|_{L^p(\Omega)} + \gamma \| u \|_{L^p(\Omega)} \right), \quad \forall u \in W^{1,p}(\Omega)
\]

holds for

\[
A_p(\Omega) = \begin{cases} 
NA' + 1, & R \leq \gamma, \\
b_\varepsilon (6NA + NA' + 3) n^{1/p}/\gamma, & R > \gamma,
\end{cases}
\]

where \( R := b_\varepsilon (6NA + NA' + 3) n^{1/p}/(NA' + 1) \). Here, the constants appearing in (5) are determined as follows: Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a function satisfying the property

\[
\int_1^\infty \psi(t) \, dt = 1, \quad \int_1^\infty t^m \psi(t) \, dt = 0, \quad \forall m \in \mathbb{N}.^6
\]

Let \( A_0 \) be a constant satisfying \( |\psi(t)| \leq A_0/t^2 \), \( \forall t \geq 1 \). Let \( A_1 \) be a constant satisfying \( |\psi(t)| \leq A_1/t^3 \), \( \forall t \geq 1 \). Let \( N \) and \( M \) be the overlap number and the Lipschitz constant of \( \Omega \), respectively. Let \{\( U_i \}_{i \in \mathbb{N}} \) be the sequence as in Definition 4. Let \( \varepsilon \) be a positive number satisfying that \( \{x \in \mathbb{R}^n : B(x, 3\varepsilon/4) \subset U_i \} \) are not empty for all \( i \in \mathbb{N} \) and if \( \text{dist}(x, \partial \Omega) \leq \varepsilon/2 \) then \( x \in U_i^{3\varepsilon/4} \) holds for some \( i \in \mathbb{N} \). Let \( \rho \) be the mollifier defined in (3); let \( b_\varepsilon := \frac{4}{\varepsilon} \int_{\mathbb{R}^n} |\partial_x \rho(x)| \, dx \).

Set \( P = \int_{\mathbb{R}^n} |(n-1) \rho_* (|x|) + |x| \rho'_* (|x|)| (1 - |x|)^{-1} \, dx \), where \( \rho_* : \mathbb{R} \to \mathbb{R} \) is the function, s.t., \( \rho_* (|x|) = \rho(x), \quad x \in \mathbb{R}^n \). For any \( \tau > 0 \) and any \( \xi \in (0, 1) \), put

\[
Q := Q_{\Omega, \tau, \xi, p} := \frac{p(1+\tau)(1+\xi)^2}{(p+1)\tau^{1+1/p} (1-\xi)^2} \sqrt{1+M^2}
\]

and

\[
B := B_{\Omega, \xi, \tau} := A_1P(1+\xi)^2 (1+\tau) \sqrt{1+M^2}.
\]

Using the above notations, \( A \) and \( A' \) in (5) are determined by

\[
A := \{(A_0Q)^p + 1\}^{1/p}
\]

and

\[
A' := \max \left\{ 2^{p-1} (A_0Q)^p + 1, \left[(n-1) 2^{p-1} (BQ)^p + \{(A_0 + B) Q\}^p + 1\right]^{1/p} \right\},
\]

respectively.

Let us remark the following fact: the value \( A_p(\Omega) \) derived by Theorem 1 monotonically decreases with decreasing \( \xi \in (0, 1) \). Moreover, \( A_p(\Omega) \to A_p(\Omega) \big|_{\xi=0} (\xi \downarrow 0) \) holds. Therefore, \( A_p(\Omega) \big|_{\xi=0} + \delta \) with any positive number \( \delta \) becomes an upper bound of the extension constant, while the range of \( \xi \) is \( (0, 1) \).

\(^6\)A concrete example of \( \psi \) can be seen in (24).
A concrete value of the extension constant derived by Theorem 1 and the best constant in the classical Sobolev inequality \([1, 17]\)^7 enable us to estimate a concrete value of the embedding constant as follows:

**Corollary 1.** Let \(n \in \{2, 3, \ldots\}\) and \(\sigma > 0\). Let \(p \in (n/(n-1), \infty)\) and set \(q = np/(n+p)\). Suppose that \(\Omega \subset \mathbb{R}^n\) is a domain with minimally smooth boundary defined in Definition 4. Then,

\[
\|u\|_{L^p(\Omega)} \leq C_p(\Omega) \|u\|_{W^{1,q}(\Omega)}, \quad \forall u \in W^{1,q}(\Omega)
\]

holds for

\[
C_p(\Omega) = 2^{\frac{q-1}{q}} T_p A_q(\Omega) .^8
\]

Here, \(A_q(\Omega)\) is the extension constant derived by Theorem 1 with \(\gamma = \sigma^{1/q}\); the constant \(T_p\) is defined by

\[
T_p = \pi^{-\frac{1}{2}} n^{-\frac{q}{2}} \left(\frac{q-1}{n-q}\right)^{\frac{1}{2}} \left\{\frac{\Gamma(1+\frac{n}{q}) \Gamma(n)}{\Gamma\left(\frac{n}{q}\right) \Gamma(1+n-\frac{n}{q})}\right\}^{\frac{1}{2}}
\]

with the Gamma function \(\Gamma\).

This Corollary follows from the following inequality:

\[
\|u\|_{L^p(\Omega)} \leq \|Eu\|_{L^p(\mathbb{R}^n)} \\
\leq T_p \|\nabla Eu\|_{L^q(\mathbb{R}^n)} \\
\leq T_p A_q(\Omega) \left(\|\nabla u\|_{L^q(\Omega)} + \sigma^{1/q} \|u\|_{L^q(\Omega)}\right) \\
\leq 2^{\frac{q-1}{q}} T_p A_q(\Omega) \|u\|_{W^{1,q}(\Omega)}, \forall u \in W^{1,q}(\Omega).
\]

### 3 Proof of the main theorem

In this section, a proof of Theorem 1 is provided. In Subsection 3.1, we explain how to construct the extension operators for domains with minimally smooth boundary. In Subsection 3.2, we show a proof of Theorem 1.

#### 3.1 Construction of extension operator

We describe Stein’s construction method of extension operators [14]. Stein first constructed an extension operator on a special Lipschitz domain. Then, he expanded this to that on a domain with minimally smooth boundary.

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^7 The best constant in the classical Sobolev inequality shown by Aubin and Talenti [1, 17] can be seen in Theorem A.1.

^8 The constant \(C_p(\Omega)\) such that \(\|u\|_{L^p(\Omega)} \leq C_p(\Omega) \|u\|_{H^1(\Omega)}, \forall u \in H^1(\Omega)\), is also important [12, 13]. We can easily obtain a formula giving a concrete value of \(C_p(\Omega)\) using Hölder’s inequality with additional assumptions for \(\Omega\) and \(p\) (see Corollary C.1).
Extension operator on a special Lipschitz domain*9

Let \( \Omega' \subset \mathbb{R}^n \) (\( n = 2, 3, \ldots \)) be a special Lipschitz domain; \( \Omega' \) is written as the form \( \Omega' := \{(x', x_n) \in \mathbb{R}^n : x_n > \phi (x') \} \), \( x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) with a Lipschitz continuous function \( \phi : \mathbb{R}^{n-1} \to \mathbb{R} \) whose Lipschitz constant is \( M_{\Omega'} \). Since \( (1 + M_{\Omega'}^2)^{-1/2} \text{dist} (x, (\overline{\Omega'})^c) \geq \phi (x') - x_n, \forall (x', x_n) \in (\overline{\Omega'})^c \), it follows that

\[
C_{\Omega', \xi} \text{RD}_{\Omega', \xi} (x', x_n) \geq \phi (x') - x_n, \forall (x', x_n) \in (\overline{\Omega'})^c
\]

for any \( \xi > 0 \), where \( C_{\Omega', \xi} := (1 + \xi)^2 \sqrt{1 + M_{\Omega'}^2} \). We denote \( g_{\Omega', \tau, \xi}^* := (1 + \tau) C_{\Omega', \xi} \text{RD}_{\Omega', \xi} \) for any \( \tau > 0 \), which ensures that \( g_{\Omega', \tau, \xi}^* (x', x_n) \geq (1 + \tau)(\phi (x') - x_n) \). For any \( k \in \mathbb{N}_0 \) and any \( p \in [1, \infty) \), the following operator \( E_{\Omega', \tau, \xi} \) becomes extension operator from \( W^{k,p} (\Omega') \) to \( W^{k,p} (\mathbb{R}^n) \):

\[
(E_{\Omega', \tau, \xi} u) (x', x_n) := \begin{cases} 
 u (x', x_n), & \forall (x', x_n) \in \overline{\Omega}, \\
 \int_1^{\infty} u (x', x_n + t g_{\Omega', \tau, \xi}^* (x', x_n)) \psi (t) \, dt, & \forall (x', x_n) \in (\overline{\Omega'})^c,
\end{cases}
\]

where \( \psi : \mathbb{R} \to \mathbb{R} \) is a function satisfying the property (6).

Extension operator on a domain with minimally smooth boundary

Let \( \Omega \) be a domain with minimally smooth boundary. Let \( \{U_i\}_{i \in \mathbb{N}} \) be the sequence as in Definition 4. Let \( \varepsilon \) be a positive number satisfying that \( U_i^{2\varepsilon} \) are not empty for all \( i \in \mathbb{N} \) and if \( \text{dist} (x, \partial \Omega) \leq \varepsilon /2 \) then \( x \in U_i^{\varepsilon} \) holds for some \( i \in \mathbb{N} \).\(^{10} \) Let \( \rho \) be a mollifier, and put \( \rho_\varepsilon (x) := \varepsilon^{-n} \rho (xe^{-1}) \). Let \( \chi_i \) be the characteristic function of \( U_i^{\varepsilon} \), and put \( \lambda_i^\varepsilon (x) := (\chi_i * \rho_{4\varepsilon}^i (x)) \). Put

\[
U_0 = \left\{ x \in \mathbb{R}^n : \text{dist} (x, \Omega) < \frac{1}{4} \varepsilon \right\},
\]

\[
U_+ = \left\{ x \in \mathbb{R}^n : \text{dist} (x, \partial \Omega) < \frac{3}{4} \varepsilon \right\},
\]

and

\[
U_- = \left\{ x \in \Omega : \text{dist} (x, \partial \Omega) > \frac{1}{4} \varepsilon \right\}.
\]

Let \( \chi_0, \chi_+, \) and \( \chi_- \) be the corresponding characteristic functions of \( U_0, U_+, \) and \( U_- \), respectively. Let \( \lambda_0^\varepsilon := \chi_0 * \rho_{4\varepsilon}^i, \lambda_+^\varepsilon := \chi_+ * \rho_{4\varepsilon}^i, \) and \( \lambda_-^\varepsilon := \chi_- * \rho_{4\varepsilon}^i \). Put

\[
\Lambda_+^\varepsilon := \frac{\lambda_+^\varepsilon}{\lambda_+^\varepsilon + \lambda_-^\varepsilon} \quad \text{and} \quad \Lambda_-^\varepsilon := \frac{\lambda_-^\varepsilon}{\lambda_+^\varepsilon + \lambda_-^\varepsilon}.
\]

\(^9\)The extension operator on a special Lipschitz domain presented here is a little general one. Namely, Stein set \( \tau \) and \( \xi \) in concrete values because he focused on just proving the existence of the extension operators in his original theory [14]. The selections of \( \tau \) and \( \xi \) are important for accuracy of the corresponding embedding constant.

\(^{10}\)In Stein’s original method [14], the assumption for \( \varepsilon > 0 \) is just to be small enough. However, since bounds for the derivatives of \( \lambda_i^\varepsilon \) increase with decreasing \( \varepsilon \), a small \( \varepsilon \) makes the corresponding extension constant large. Due to this, we should select the value of \( \varepsilon \) with taking this property in consideration. Concrete selections of \( \varepsilon \) for fixed \( \Omega \) can be seen in Subsection 4.2.
To each $U_i$ there corresponds a special Lipschitz domain $\Omega_i$ as in Definition 4. Let $E^i_{\Omega_i, \tau, \xi}$ be the extension operator for $W^{k,p}(\Omega_i)$. For any $k \in N_0$ and any $p \in [1, \infty)$, the following operator $E^i_{\Omega_i, \tau, \xi}$ becomes extension operator from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^n)$:

$$
(E_{\Omega, \tau, \xi, \varepsilon} u) (x) := \Lambda^+_\varepsilon (x) \left( \frac{\sum_{i=1}^{\infty} \lambda_i^\varepsilon (x) E^i_{\Omega_i, \tau, \xi} (\lambda_i^\varepsilon u) (x)}{\sum_{i=1}^{\infty} \lambda_i^\varepsilon (x)^2} \right) + \Lambda^- \varepsilon (x) u (x)
$$

for all $x \in \mathbb{R}^n$.

Here, one can observe that

- $\text{supp} \lambda_i^\varepsilon \subset U_i$, and $\lambda_i^\varepsilon (x) = 1$ if $x \in U_i^{\frac{1}{2}\varepsilon}$;
- if $x \in \text{supp} \Lambda^+_\varepsilon$, then $\sum_{i \in \mathbb{N}} \lambda_i^\varepsilon (x) \geq 1$;
- bounds of the derivatives of $\lambda_i^\varepsilon$ are independent on $i$ but depend only on the $L^1$ norm of the corresponding derivatives of $\rho^+_\varepsilon$;
- $\lambda_0^\varepsilon (x) = 1$ if $x \in \overline{\Omega}$;
- $\lambda_i^\varepsilon (x) = 1$ if $\text{dist} (x, \partial \Omega) \leq \varepsilon/2$;
- $\lambda_i^\varepsilon (x) = 1$ if $x \in \Omega$ and $\text{dist} (x, \partial \Omega) \geq \varepsilon/2$;
- the supports of $\lambda_0^\varepsilon$, $\lambda_i^\varepsilon$, and $\lambda_i^\varepsilon$ are contained in the $\varepsilon/2$-neighborhood of $\Omega$, in the $\varepsilon$-neighborhood of $\partial \Omega$, and in $\Omega$, respectively;
- the functions $\lambda_0^\varepsilon$, $\lambda_i^\varepsilon$, and $\lambda_i^\varepsilon$ are bounded in $\mathbb{R}^n$, and all their partial derivatives are also bounded;
- all the derivatives of $\Lambda_i^\varepsilon$ and $\Lambda_i^\varepsilon$ are bounded on $\mathbb{R}^n$;
- $\Lambda_i^\varepsilon + \Lambda_i^\varepsilon$ is 1 on $\overline{\Omega}$ and is 0 outside the $\varepsilon/2$-neighborhood of $\Omega$.

### 3.2 Proof of Theorem 1

In the following, we describe a proof of Theorem 1. Let $\Omega' \subset \mathbb{R}^n (n = 2, 3, \cdots)$ be a special Lipschitz domain; $\Omega'$ is written as the form $\Omega' := \{(x', x_n) \in \mathbb{R}^n : x_n > \phi (x')\}$, $x' = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$ with a Lipschitz continuous function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ whose Lipschitz constant is $M_{\Omega'}$. Let $E_{\Omega', \tau, \xi}$ be the extension operator defined in (9). At the beginning, we estimate the constants $A_{p, \tau, \xi} (\Omega')$ and $A'_{p, \tau, \xi} (\Omega')$ satisfying

$$
\| E_{\Omega', \tau, \xi} u \|_{L^p (\mathbb{R}^n)} \leq A_{p, \tau, \xi} (\Omega') \| u \|_{L^p (\Omega')} , \quad u \in H^1 (\Omega')
$$

and

$$
\| \nabla (E_{\Omega', \tau, \xi} u) \|_{L^p (\mathbb{R}^n)} \leq A'_{p, \tau, \xi} (\Omega') \| \nabla u \|_{L^p (\Omega')} , \quad u \in H^1 (\Omega'),
$$

respectively. Since $C^\infty (\Omega')$ is dense in $H^1 (\Omega')$, it suffices to estimate $A_{p, \tau, \xi} (\Omega')$ and $A'_{p, \tau, \xi} (\Omega')$ for $u \in C^\infty (\Omega')$. Hereafter, we write $u_y = \partial_y u$, $g^* = g^*_{\Omega', \tau, \xi}$, $g^*_y = \partial_y g^*$, and $E = E_{\Omega', \tau, \xi}$ for simplicity.

**The first step: estimating $A_{p, \tau, \xi} (\Omega')$**

If $y < \phi (x)$ with $y \in \mathbb{R}$ and $x = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$, it follows that

$$
| (Eu) (x, y) | = \left| \int_1^\infty u (x, y + tg^* (x, y)) \psi (t) dt \right|
$$
By changing the variable integration as \( t = \frac{r^2}{t^2} \), we have

\[
\int_{1}^{\infty} \left| u \left( x, y + tg^* (x, y) \right) \right| \frac{dt}{t^2},
\]

(13)

Setting \( z := y - \phi (x) \), we have \( g^* (x, y) \geq (1 + \tau) (\phi (x) - y) = (1 + \tau) |z| \). We also have \( \phi (x) - y \geq \text{dist} \left( (x, y), \overline{\Omega} \right) \) for all \( x \in \mathbb{R}^{n-1} \) and \( y \in \mathbb{R} \). Since \( \text{dist} \left( (x, y), \overline{\Omega} \right) \geq (1 - \xi)^2 \text{RD}_{\Omega', \xi} (x, y) \) holds from Definition 2, it follows that

\[
|z| = \phi (x) - y \geq \text{dist} \left( (x, y), \overline{\Omega} \right)
\geq (1 - \xi)^2 \text{RD}_{\Omega', \xi} (x, y)
= (1 - \xi)^2 (1 + \tau)^{-1} C_{\Omega', \xi}^{-1} g^* (x, y).
\]

(14)

Now, recall that \( g^* = (1 + \tau) C_{\Omega', \xi} \text{RD}_{\Omega', \xi} \). From (14), we obtain \( g^* (x, y) \leq a |z| \), where \( a := a_{\Omega', \tau, \xi} := (1 + \tau)(1 + \xi)^2(1 - \xi)^{-2} \sqrt{1 + M_{\Omega'}^2} \). Putting \( s = z + tg^* (x, y) \), it follows from (13) that

\[
|(Eu) (x, y)| \leq A_0 \int_{1}^{\infty} \left| u \left( x, y + tg^* (x, y) \right) \right| \frac{dt}{t^2}
= A_0 g^* (x, y) \int_{z + g^* (x, y)}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| (s - z)^{-2} ds
\leq A_0 a |z| \int_{\tau |z|}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| (s - z)^{-2} ds
\leq A_0 a |z| \int_{\tau |z|}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| s^{-2} ds.
\]

By changing the variable integration as \( \tau (y - \phi (x)) = \tau z = w \), we have

\[
\int_{-\infty}^{\phi (x)} \left| (Eu) (x, y) \right|^p dy
\leq \left( \frac{aA_0}{\tau} \right)^p \int_{-\infty}^{\phi (x)} \left( \tau |z| \int_{\tau |z|}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| s^{-2} ds \right)^p dy, \quad z = y - \phi (x)
= \left( \frac{aA_0}{\tau^{1+1/p}} \right)^p \int_{-\infty}^{0} \left( |w| \int_{|w|}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| s^{-2} ds \right)^p dw
= \left( \frac{aA_0}{\tau^{1+1/p}} \right)^p \int_{0}^{\infty} \left( \int_{|w|}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| s^{-2} ds \right)^p |w|^{(p+1)-1} dw.
\]

Hardy’s inequality \(^{11} \) gives

\[
\int_{-\infty}^{\phi (x)} \left| (Eu) (x, y) \right|^p dy \leq \left( \frac{pA_0a}{(p+1)\tau^{1+1/p}} \right)^p \int_{0}^{\infty} \left( \int_{|w|}^{\infty} \left| u \left( x, s + \phi (x) \right) \right| s^{-1} \right)^p s^p ds
= \left( \frac{pA_0a}{(p+1)\tau^{1+1/p}} \right)^p \int_{0}^{\infty} \left| u \left( x, s + \phi (x) \right) \right|^p ds
= \left( \frac{pA_0a}{(p+1)\tau^{1+1/p}} \right)^p \int_{\phi (x)}^{\infty} \left| u \left( x, y \right) \right|^p dy.
\]

(15)

Moreover, from the definition of the extension operator (9), we have

\[
\int_{\phi (x)}^{\infty} \left| (Eu) (x, y) \right|^p dy = \int_{\phi (x)}^{\infty} \left| u \left( x, y \right) \right|^p dy.
\]

\(^{11}\)Hardy’s inequality can be found in Lemma B. 1.
From (15) and (16), it follows that
\[
\left( \int_{-\infty}^{\infty} |(Eu)(x,y)|^p \, dy \right)^{1/p} \leq \{(A_0Q)^p + 1\}^{1/p} \left( \int_{\phi(x)}^{\infty} |u(x,y)|^p \, dy \right)^{1/p},
\]
where \( Q \equiv Q_{\Omega, \tau, \xi, p} := p a_{\Omega, \tau, \xi} / \{(p + 1)^{1+1/p}\}. \) Integrating the both side of (17) by \( x \), we find that (11) holds for
\[ A_{p, \tau, \xi} (\Omega') = \{(A_0Q)^p + 1\}^{1/p}. \]

**The second step: estimating** \( A'_{p, \tau, \xi} (\Omega') \)

Definition 2 and Lemma B.2 ensure that \( |g^*_x (x,y)| \leq B/A_1 \) for \( j \in \{1, 2, \cdots, n\} \), where \( B \) appeared in the assumption of Theorem 1. If \( y < \phi (x) \) with \( y \in \mathbb{R} \) and \( x = (x_1, x_2, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \), it follows that
\[
\partial_y (Eu)(x,y) = \partial_y \int_1^{\infty} u(x,y + tg^* (x,y)) \psi(t) \, dt
\]
\[
= \int_1^{\infty} u_y (x,y + tg^* (x,y)) (1 + tg^*_y (x,y)) \psi(t) \, dt
\]
\[
= \int_1^{\infty} u_y (x,y + tg^* (x,y)) \psi(t) \, dt
\]
\[
+ g^*_y (x,y) \int_1^{\infty} u_y (x,y + tg^* (x,y)) t \psi(t) \, dt.
\]

Therefore, we have
\[
|\partial_y (Eu)(x,y)|
\]
\[
\leq \left| \int_1^{\infty} u_y (x,y + tg^* (x,y)) \psi(t) \, dt \right|
\]
\[
+ \left| g^*_y (x,y) \right| \left| \int_1^{\infty} u_y (x,y + tg^* (x,y)) t^3 \psi(t) \, dt \right|
\]
\[
\leq (A_0 + B) \int_1^{\infty} |u_y (x,y + tg^* (x,y))| \, dt \cdot \frac{dt}{t^2}, \quad y < \phi (x).
\]

From the similar discussion in the first step, we have
\[
\int_{-\infty}^{\infty} |\partial_y (Eu)(x,y)|^p \, dy \leq \{(A_0 + B)Q\}^p + 1 \int_{\phi(x)}^{\infty} |u_y (x,y)|^p \, dy.
\]

On the other hand, for \( j \in \{1, 2, \cdots, n-1\} \) and \( y < \phi (x) \), it follows that
\[
\partial_{x_j} (Eu)(x,y)
\]
\[
= \partial_{x_j} \int_1^{\infty} u(x,y + tg^* (x,y)) \psi(t) \, dt
\]
\[
= \int_1^{\infty} \left\{ u_{x_j} (x,y + tg^* (x,y)) + u_y (x,y + tg^* (x,y)) tg^*_{x_j} (x,y) \right\} \psi(t) \, dt
\]
\[
= \int_1^{\infty} u_{x_j} (x,y + tg^* (x,y)) \psi(t) \, dt
\]
\[
+ g^*_{x_j} (x,y) \int_1^{\infty} u_y (x,y + tg^* (x,y)) t \psi(t) \, dt.
\]
Therefore, we have

\[
\left| \partial_{x_j} (Eu) (x, y) \right|
\leq \left| \int_1^\infty u_{x_j} (x, y + tg^* (x, y)) \psi (t) \, dt \right|
+ \left| g_{x_j}^* (x, y) \right| \left| \int_1^\infty u_y (x, y + tg^* (x, y)) t \psi (t) \, dt \right|
\leq A_0 \int_1^\infty \left| u_{x_j} (x, y + tg^* (x, y)) \right| \frac{dt}{t^2}
+ B \int_1^\infty \left| u_y (x, y + tg^* (x, y)) \right| \frac{dt}{t^2}, \quad y < \phi (x).
\]

Since \((s + t)^p \leq 2^{p-1} (s^p + t^p)\) holds for \(s, t > 0\) and \(p > 1\), it follows from the similar discussion in (15) that

\[
\int_{-\infty}^{\phi (x)} \left| \partial_{x_j} (Eu) (x, y) \right|^p \, dy
\leq 2^{p-1} \int_{-\infty}^{\phi (x)} \left| A_0 a |z| \int_{\tau |z|}^{\infty} \left| u_{x_j} (x, s + \phi (x)) \right| s^{-2} \, ds \right|^p \, dy
+ 2^{p-1} \int_{-\infty}^{\phi (x)} \left| Ba |z| \int_{\tau |z|}^{\infty} \left| u_y (x, s + \phi (x)) \right| s^{-2} \, ds \right|^p \, dy
\leq 2^{p-1} (A_0 Q)^p \int_{\phi (x)}^{\infty} \left| u_{x_j} (x, y) \right|^p \, dy + 2^{p-1} (BQ)^p \int_{\phi (x)}^{\infty} \left| u_y (x, y) \right|^p \, dy.
\]

Therefore, it follows that

\[
\int_{-\infty}^{\infty} \left| \partial_{x_j} (Eu) (x, y) \right|^p \, dy
\leq \left\{ 2^{p-1} (A_0 Q)^p + 1 \right\} \int_{\phi (x)}^{\infty} \left| u_{x_j} (x, y) \right|^p \, dy + 2^{p-1} (BQ)^p \int_{\phi (x)}^{\infty} \left| u_y (x, y) \right|^p \, dy
\quad (19)
\]

for \(j \in \{1, 2, \cdots, n - 1\}\). From (18) and (19), we have

\[
\sum_{j=1}^{n} \int_{-\infty}^{\infty} \left| \partial_{x_j} (Eu) (x, y) \right|^p \, dy
= \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} \left| \partial_{x_j} (Eu) (x, y) \right|^p \, dy + \int_{-\infty}^{\infty} \left| \partial_y (Eu) (x, y) \right|^p \, dy
\leq \left\{ 2^{p-1} (A_0 Q)^p + 1 \right\} \sum_{j=1}^{n-1} \int_{\phi (x)}^{\infty} \left| u_{x_j} (x, y) \right|^p \, dy
+ (n - 1) 2^{p-1} (BQ)^p \int_{\phi (x)}^{\infty} \left| u_y (x, y) \right|^p \, dy
+ \left\{ (A_0 + B) Q \right\}^p + 1 \int_{\phi (x)}^{\infty} \left| u_y (x, y) \right|^p \, dy
\]

\[
= \left\{ 2^{p-1} (A_0 Q)^p + 1 \right\} \sum_{j=1}^{n-1} \int_{\phi (x)}^{\infty} \left| u_{x_j} (x, y) \right|^p \, dy.
\]
Recall that \( \Omega \subset i \mathcal{W} \) operator for 

The third step: estimating the extension constant \( A_p (\Omega) \)

This step concludes the proof of Theorem 1. Let us introduce the following lemma:

**Lemma 1.** Let \( S \subset \mathbb{R}^n \) and let \( p \in [1, \infty) \). Let \( \{ a_i (x) \}_{i \in \mathbb{N}} \subset L^p (S) \) satisfy that at most \( N \) of \( a_i (x) \) are not zero for each \( x \).

\[
\left( \int_S \left| \sum_{i \in \mathbb{N}} a_i (x) \right|^p \ dx \right)^{\frac{1}{p}} \leq N^{1 - \frac{1}{p}} \left( \sum_{i \in \mathbb{N}} \int_S |a_i (x)|^p \ dx \right)^{\frac{1}{p}}.
\]

This lemma follows from the inequality

\[
\left| \sum_{i \in \mathbb{N}} a_i (x) \right|^p \leq N^{p-1} \sum_{i \in \mathbb{N}} |a_i (x)|^p.
\]

Recall that \( \Omega \subset \mathbb{R}^n \) \( (n = 2, 3, \cdots) \) is a domain with minimally smooth boundary. To each \( U_i \) there corresponds a special Lipschitz domain \( \Omega_i \) as in Definition 4. Let \( E_{i \lambda_i, \tau, \xi} \) be the extension operator for \( W^{k,p} (\Omega_i) \). Then, it follows that \( A = \sup_{i \in \mathbb{N}} A_{p, \tau, \xi} (\Omega_i) \) and \( A' = \sup_{i \in \mathbb{N}} A'_{p, \tau, \xi} (\Omega_i) \). Let \( E_{\Omega, \tau, \xi, \varepsilon} \) be the extension operator defined in (10). Note that

\[
|\partial_{x_j} \lambda_0^\varepsilon| \leq \int_{\mathbb{R}^n} |\partial_{x_j} \rho_{\frac{1}{4\varepsilon}} (x)| \ dx = \frac{4}{\varepsilon} \int_{\mathbb{R}^n} |\partial_{x_j} \rho (x)| \ dx = b_{\varepsilon}; \quad (20)
\]

this bound is not depend on the index \( i \). Likewise, \( |\partial_{x_j} \lambda_0^\varepsilon| \), \( |\partial_{x_j} \lambda_+^\varepsilon| \), and \( |\partial_{x_j} \lambda_-^\varepsilon| \) are bounded by \( b_{\varepsilon} \). The following inequality holds:

\[
|\partial_{x_j} \Lambda_+^\varepsilon| = \left| (\partial_{x_j} \lambda_0^\varepsilon) \frac{\lambda_+^\varepsilon}{\lambda_+^\varepsilon + \lambda_-^\varepsilon} + \lambda_0^\varepsilon \left( \partial_{x_j} \lambda_+^\varepsilon \right) \left( \lambda_+^\varepsilon + \lambda_-^\varepsilon \right) - \lambda_+^\varepsilon \left( \partial_{x_j} \lambda_+^\varepsilon + \partial_{x_j} \lambda_-^\varepsilon \right) \right|
\leq 3b_{\varepsilon} =: b_+;
\]

\( |\partial_{x_j} \Lambda_-^\varepsilon| \) is also bounded by \( 3b_{\varepsilon} =: b_- \). Moreover, recall that if \( x \in \text{supp} \Lambda_+^\varepsilon \), then \( \sum_{i \in \mathbb{N}} \lambda_i^\varepsilon (x) \geq 1 \).

Let \( u \in H^1 (\Omega) \) and let \( U^* := \bigcup U_i^{\xi_i/2} \). In the following proof, we simply denote \( \sum_{i \in \mathbb{N}} \) by \( \sum \), \( E_{\Omega, \tau, \xi, \varepsilon} \) by \( E \), \( E_{\Omega_i, \tau, \xi} \) by \( E_i \), \( \lambda_0^\varepsilon \) by \( \lambda_i \), \( \Lambda_+^\varepsilon \) by \( \Lambda_+ \), and \( \Lambda_-^\varepsilon \) by \( \Lambda_- \). It follows that

\[
\| \nabla (Eu) \|_{L^p (\mathbb{R}^n)} \leq \left( \sum_j \int_{\mathbb{R}^n} |\partial_{x_j} (Eu)|^p \ dx \right)^{1/p}
\]
\[
\leq \left( \sum_j \int_{\mathbb{R}^n} \left| \partial_{x_j} \Lambda_+ \right| \left( \frac{\sum \lambda_i E^i (\lambda_i u)}{\sum \lambda_i^2} \right)^p \, dx \right)^{1/p} \\
+ \left( \sum_j \int_{\mathbb{R}^n} \left| \Lambda_+ (\circ) \right|^p \, dx \right)^{1/p} \\
+ \left( \sum_j \int_{\mathbb{R}^n} \left| (\partial_{x_j} \Lambda_-) u \right|^p \, dx \right)^{1/p} + \left( \sum_j \int_{\mathbb{R}^n} \left| \Lambda_- (\partial_{x_j} u) \right|^p \, dx \right)^{1/p},
\]

(21)

where
\[
\circ := \frac{\left( \partial_{x_j} \sum \lambda_i E^i (\lambda_i u) \right) \left( \sum \lambda_i^2 \right) - \left( \sum \lambda_i E^i (\lambda_i u) \right) \left( \partial_{x_j} \sum \lambda_i^2 \right)}{\left( \sum \lambda_i^2 \right)^2}.
\]

From Lemma 1, the first term of (21) is evaluated as
\[
\leq b_+ \left( \sum_j \int_{U^*} \left| \frac{\sum \lambda_i E^i (\lambda_i u)}{\sum \lambda_i^2} \right|^p \, dx \right)^{1/p} \\
\leq b_+ n^{1/p} \left( \int_{U^*} \left| \sum \lambda_i E^i (\lambda_i u) \right|^p \, dx \right)^{1/p} \\
\leq b_+ N^{1-1/p} n^{1/p} \left( \sum_j \int_{U^*_i} \left| E^i (\lambda_i u) \right|^p \, dx \right)^{1/p} \\
\leq b_+ N^{1-1/p} A n^{1/p} \left( \sum_j \int_{\Omega} \left| \lambda_i u \right|^p \, dx \right)^{1/p} \\
\leq b_+ N A n^{1/p} \left( \int_{\Omega} |u|^p \, dx \right)^{1/p}.
\]

The second term of (21) is evaluated as
\[
\left( \sum_j \int_{\mathbb{R}^n} \left| \Lambda_+ (\circ) \right|^p \, dx \right)^{1/p} \\
\leq \left( \sum_j \int_{U^*} \left| \frac{\partial_{x_j} \sum \lambda_i E^i (\lambda_i u)}{\sum \lambda_i^2} \right|^p \, dx \right)^{1/p} + \left( \sum_j \int_{U^*} \left| \frac{\sum \lambda_i E^i (\lambda_i u)}{\sum \lambda_i^2} \left( \partial_{x_j} \sum \lambda_i^2 \right) \right|^p \, dx \right)^{1/p}.
\]

(22)

The first term of (22) is evaluated as
\[
\left( \sum_j \int_{U^*} \left| \frac{\partial_{x_j} \sum \lambda_i E^i (\lambda_i u)}{\sum \lambda_i^2} \right|^p \, dx \right)^{1/p}
\]

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The second term of (23) is evaluated as
\[
\left( \sum_j \int_{U^*} \left| \sum_i \left( \partial_{x_j} \lambda_i \right) E^i (\lambda_i u) + \sum_i \lambda_i \left( \partial_{x_j} E^i (\lambda_i u) \right) \right|^p \, dx \right)^{1/p}
\]

\[
\leq \left( \sum_j \int_{U^*} \left| \sum_i \left( \partial_{x_j} \lambda_i \right) E^i (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[
+ \left( \sum_j \int_{U^*} \sum_i \lambda_i \left( \partial_{x_j} E^i (\lambda_i u) \right) \right|^p \, dx \right)^{1/p}
\]

The first term of (23) is evaluated as
\[
\left( \sum_j \int_{U^*} \left| \sum_i \left( \partial_{x_j} \lambda_i \right) E^i (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[
\leq \left( \sum_j \int_{U^*} \left| \sum_i \left( \partial_{x_j} \lambda_i \right) E^i (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[
\leq \left( \sum_j \int_{U^*} \left| \sum_i \left( \partial_{x_j} \lambda_i \right) E^i (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[
\leq N^{1-1/p} \left( \sum_j \sum_i \int_{U_i} b_{x_i}^p \left| E^i (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

The second term of (23) is evaluated as
\[
\left( \sum_j \int_{U^*} \sum_i \lambda_i \left( \partial_{x_j} E^i (\lambda_i u) \right) \right|^p \, dx \right)^{1/p}
\]

\[
\leq \left( \sum_j \int_{U^*} \sum_i \left| \partial_{x_j} E^i (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[
\leq N^{1-1/p} \left( \sum_i \sum_j \int_{U^*} \left| \partial_{x_j} (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[
\leq N^{1-1/p} A' \left( \sum_i \sum_j \int_{U_i} \left| \partial_{x_j} (\lambda_i u) \right|^p \, dx \right)^{1/p}
\]

\[\text{valid for } \lambda \in L^1 \cap L^p, \quad b_{x_i} \in L^1 \cap L^p, \quad A' \text{ constant}\]
From the above evaluations, we have

\[
N^{1/p}A^t \left( \sum_n \sum_j \int_\Omega \left| (\partial_{x_j} \lambda_i) u + \lambda_i (\partial_{x_j} u) \right|^p \, dx \right)^{1/p} \\
\leq N^{1/p}A^t \left( \sum_n \int_\Omega \sum_j \left| (\partial_{x_j} \lambda_i) u \right|^p \, dx \right)^{1/p} \\
+ N^{1/p}A^t \left( \sum_n \int_\Omega \sum_j \left| \lambda_i (\partial_{x_j} u) \right|^p \, dx \right)^{1/p} \\
\leq N\lambda^i \left\{ \left( \sum_n b^p \int_\Omega \left| u \right|^p \, dx \right)^{1/p} + \left( \int_\Omega \sum_j \left| \partial_{x_j} u \right|^p \, dx \right)^{1/p} \right\} \\
\leq N\lambda^i \left\{ b_n^{1/p} \left( \int_\Omega \left| u \right|^p \, dx \right)^{1/p} + \left( \int_\Omega \sum_j \left| \partial_{x_j} u \right|^p \, dx \right)^{1/p} \right\}.
\]

The second term of (22) is evaluated as

\[
\left( \sum_j \int_{U^*} \left| \frac{\sum \lambda_i E^i (\lambda_i u)}{(\sum \lambda^2_i)^2} \right|^p \, dx \right)^{1/p} \\
\leq \left( \sum_j \int_{U^*} \left| \frac{\sum \lambda_i E^i (\lambda_i u)}{(\sum \lambda^2_i)^2} \right|^p \, dx \right)^{1/p} \\
\leq \left( \sum_j \int_{U^*} \left| \frac{\sum \lambda_i E^i (\lambda_i u)}{(\sum \lambda^2_i)^2} \right|^p \, dx \right)^{1/p} \\
\leq 2b_n \left( \sum_j \int_{U^*} \left| \sum \lambda_i E^i (\lambda_i u) \right|^p \, dx \right)^{1/p} \\
\leq 2b_n N^{1/p}n^{1/p} \left( \sum \int_{U^*} \left| E^i (\lambda_i u) \right|^p \, dx \right)^{1/p} \\
\leq 2b_n N^{1/p}n^{1/p} A^{1/p} \left( \sum \int_{\Omega} \left| \lambda_i u \right|^p \, dx \right)^{1/p} \\
\leq 2b_n N A^{1/p} \left( \int_\Omega \left| u \right|^p \, dx \right)^{1/p}.
\]

From the above evaluations, we have

\[
\| \nabla (Eu) \|_{L^p(\mathbb{R}^n)} \\
\leq b_+ N A^{1/p} \left( \int_\Omega \left| u \right|^p \, dx \right)^{1/p} + N b_n A^{1/p} \left( \int_\Omega \left| u \right|^p \, dx \right)^{1/p} \\
+ N\lambda^i \left\{ b_n^{1/p} \left( \int_\Omega \left| u \right|^p \, dx \right)^{1/p} + \left( \int_\Omega \sum_j \left| \partial_{x_j} u \right|^p \, dx \right)^{1/p} \right\}.
\]
+ 2N\varepsilon A n^{1/p} \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} + b_- n^{1/p} \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} \\
+ \left( \int_{\Omega} \sum_j |\partial_{x_j} u|^p \, dx \right)^{1/p} \\
= (NA' + 1) \left( \int_{\Omega} \sum_j |\partial_{x_j} u|^p \, dx \right)^{1/p} \\
+ (b_+ NA + \varepsilon NA + 2b_+ NA + b_+ NA' + b_-) n^{1/p} \left( \int_{\Omega} |u|^p \, dx \right)^{1/p} \\
= (NA' + 1) \|\nabla u\|_{L^p(\Omega)} + b_\varepsilon (6NA + NA' + 3) n^{1/p} \|u\|_{L^p(\Omega)}.

Hence, the inequality (4) holds for

\begin{align*}
A_p(\Omega) &= \begin{cases} 
(NA' + 1), & R \leq \gamma, \\
\varepsilon (6NA + NA' + 3) n^{1/p}/\gamma, & R > \gamma,
\end{cases}
\end{align*}

where \( R := \varepsilon (6NA + NA' + 3) n^{1/p}/(NA' + 1). \)

\section{Estimation of the embedding constant}

In this section, we present some examples of estimating the embedding constant \( C_p(\Omega) \) defined in (1) using Theorem 1 and Corollary 1. In order to apply Theorem 1, there are a number of constants to be determined. We first describe how to determine them. Throughout this section, we set \( \rho \) as the mollifier defined in (3) and set \( \sigma = 1. \)

\subsection{Determining the constants with verified numerical computation}

The constants \( A_0, A_1, P, \) and \( \int_{\mathbb{R}^n} \left| \partial_{x_j} \rho(x) \right| \, dx \) in Theorem 1 were calculated by verified numerical computation. All computation errors, i.e., rounding errors and truncated errors, were strictly estimated; therefore, the accuracy of all results is mathematically guaranteed. All computations were carried out on a Windows 7, Intel(R) Core(TM) i7 860 CPU 2.80 GHz with 16.0 GB RAM by using MATLAB 2012b with INTLAB version 8, a toolbox for verified numerical computations [22]. Recall that

\begin{align*}
A_0 &= \sup \left\{ |t^2 \psi(t)| : t \geq 1 \right\}, \\
A_1 &= \sup \left\{ |t^3 \psi(t)| : t \geq 1 \right\}, \\
P &= \int_{\mathbb{R}^n} \left\{ (n - 1) \rho_\varepsilon (|x|) + |x| \rho_\varepsilon (|x|) \right\} (1 - |x|)^{-1} \, dx,
\end{align*}

and

\( b_\varepsilon = \frac{4}{\varepsilon} \int_{\mathbb{R}^n} \left| \partial_{x_j} \rho(x) \right| \, dx, \)

where \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) is a function satisfying (6). For example, the function

\[ \psi(t) := \frac{e^{i\omega(t-1)}}{\pi t} \text{Im} \left( e^{-\omega(t-1)i} \right), \quad \omega = C_\omega e^{-i\pi/4} = \frac{C_\omega}{\sqrt{2}} (1 - i) \]

satisfies the property (6) for any \( C_\omega > 0; \) a simple proof can be seen in, e.g., [7, 14]. Using the function \( \psi \) in (24) with \( C_\omega = 4.83, \) we derived the following estimation results:

\[ A_0 \in [12.8860, 12.8861]; \]
\[ A_1 \in [12.9325, 12.9326]; \]
\[ P \in [7.45592, 7.50131]; \]
\[ \int_{\mathbb{R}^2} |\partial_{x_j} \rho(x)| \, dx \in [1.86412, 1.92770]. \]

Here, we set \( n = 2 \) in calculating \( P \) and \( \int_{\mathbb{R}^2} |\partial_{x_j} \rho(x)| \, dx \). In the cases of \( n = 3, 4, \cdots \), we can also derive their estimations by verified numerical computation.

### 4.2 Estimation results for fixed domains

In the following, we present estimation results for two fixed domains \( \Omega \) with minimally smooth boundary.

**Example A**

Let \( \Omega \subset \mathbb{R}^2 \) be the domain as in Fig. 1 (a). We set \( \{U_i\}_{i \in \mathbb{N}} \) as follows: we first define the two sets among \( U_i \)'s as in Fig. 1 (b); then, \( U_i \)'s \((i = 1, 2, \cdots, 8)\) were obtained by symmetry reflections; finally, we defined the other \( U_i \)'s \((i = 9, 10, 11, \cdots)\) as empty sets. In this case, we chose \( M = 1, N = 2, \) and \( \varepsilon = 0.25 \). One can find in Fig. 1 (c) that these constants satisfy the required conditions mentioned in Theorem 1.

Figure 2 (a) shows the relationship between \( \tau \) and \( A_q(\Omega) \) in the cases of \( p = 4, 6, \) and 8; recall that \( q = 2p/(2+p) \). One can observe that \( A_q(\Omega) \) first decreases with increasing \( \tau \), then it reaches a minimum point, and thereafter it monotonically increases with increasing \( \tau \). The relationship between \( p \) and the constant \( \tau \) minimizing \( A_q(\Omega) \) can be seen in Fig. 2 (b). For example, in the cases of \( p = 4, 6, \) and 8, each \( A_q(\Omega) \) is minimized at the points \( \tau \approx 8.12, 5.83, \) and 5.06, respectively.

Figure 2 (c) shows the relationship between \( p \) and \( C_p(\Omega) \); we chose \( \tau \) which makes \( A_q(\Omega) \) (and \( C_p(\Omega) \)) as small as possible. Recall that all results in Fig. 2 were mathematically guaranteed with verified numerical computation.

**Example B**

Let \( \Omega \subset \mathbb{R}^2 \) be the domain as in Fig. 3 (a), of which boundary is composed of five semicircles and a straight line. We set \( \{U_i\}_{i \in \mathbb{N}} \) as follows: we first set \( U_i \)'s \((i = 1, 2, \cdots, 6)\) as in Fig. 3 (b)–(d); then, we got the other \( U_i \)'s \((i = 7, 8, \cdots, 10)\) by symmetrical reflection; the other
Figure 2: (a): The relationship between $\tau$ and $A_q(\Omega)$ with $p = 4, 6, \text{ and } 8$. (b): between $p$ and $\tau$ minimizing $A_q(\Omega)$. (c): between $p$ and $C_p(\Omega)$. (c).

$U_i$’s ($i = 11, 12, \cdots$) were defined as empty sets. In this case, we chose $M = 1$, $N = 2$, and $\varepsilon = \frac{2\sin(\pi/8)}{\sin(\pi/8) + 1}$. The selection of $\varepsilon$ depends on the smallest semicircle that composes the boundary of $\Omega$. One can find in Fig. 3 (e) that $\varepsilon = \frac{2\sin(\pi/8)}{\sin(\pi/8) + 1}$ satisfies the required condition in Theorem 1. The graphs of $A_q(\Omega)$, $\tau$ minimizing $A_q(\Omega)$, and $C_p(\Omega)$ are also displayed in Fig. 4.

Figure 3: (a): The domain $\Omega$ of Example B. (b)–(d): the open set $U_i$ ($i = 1, 2, \cdots, 10$); they describe $U_1$ to $U_6$, and the other $U_i$ ($i = 7, 8, \cdots, 10$) can be obtained by symmetrical reflection. (e): this shows how to determine $\varepsilon$.

5 Conclusion

We proposed a formula giving a concrete value of the embedding constant $C_p(\Omega)$ from $W^{1,q}(\Omega)$ to $L^p(\Omega)$ defined in (1) by calculating the extension constant $A_q(\Omega)$ defined in (2) for the extension operator constructed by Stein’s method [14]. Here, $\Omega$ is only assumed to be a domain with minimally smooth boundary defined in Definition 4. The formulae proposed in this paper
are summarized in Theorem 1 and Corollary 1. The main contribution of this paper is providing
Theorem 1, which enables us to calculate the extension constant $A_q(\Omega)$.

In addition, we presented some estimation results of the embedding constants. All estimation
results are mathematically guaranteed with verified numerical computation, while the derived
constants may not be sharp because of some over-estimations.

A  The best constant in the classical Sobolev inequality

The following theorem gives the best constant in the classical Sobolev inequality.

**Theorem A.1** (T. Aubin [1] and G. Talenti [17]). Let $u$ be any function in $H^1(\mathbb{R}^n)$ ($n = 2, 3, \cdots$). Moreover, let $q$ be any real number such that $1 < q < n$, and set $p = nq/(n-q)$. Then,

$$
\|u\|_{L^p(\mathbb{R}^n)} \leq T_p \|\nabla u\|_{L^q(\mathbb{R}^n)}
$$

holds for

$$
T_p = \pi^{-\frac{1}{2}} n^{-\frac{1}{4}} \left( \frac{q-1}{n-q} \right)^{1-\frac{1}{q}} \left\{ \frac{\Gamma \left( 1 + \frac{n}{q} \right) \Gamma (n)}{\Gamma \left( \frac{n}{q} \right) \Gamma \left( 1 + n - \frac{n}{q} \right)} \right\}^{\frac{1}{n}}
$$

with the Gamma function $\Gamma$.

B  Lemmas for proving Theorem 1

The following two lemmas were used in the proof of Theorem 1.

**Lemma B.1** (G.H. Hardy, et al. [23]). Let $p \in \mathbb{N}$ and let $r > 0$. Suppose that a function
$f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) \geq 0$, $\forall x \in \mathbb{R}$. Then, it follows that

$$
\left( \int_0^\infty \left( \int_0^x f(y) \, dy \right)^p \, dx \right)^{1/p} \leq \frac{p}{r} \left( \int_0^\infty (yf(y))^p \, dy \right)^{1/p},
$$

and

$$
\left( \int_0^\infty \left( \int_x^\infty f(y) \, dy \right)^p \, dx \right)^{1/p} \leq \frac{p}{r} \left( \int_x^\infty (yf(y))^p \, dy \right)^{1/p}.
$$

**Lemma B.2** (L.E. Fraenkel [21]). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying Lipschitz continuous condition, i.e., for some constant $M > 0$, it follows that

$$
|f(x) - f(y)| \leq M |x-y|, \quad \forall x, y \in \mathbb{R}^n.
$$
Suppose that there is an open set $G \subset \mathbb{R}^n$, s.t., $f(x) > 0, \forall x \in G$. Then, for given any $\varepsilon \in (0,1)$, there is a function $g \in C^\infty(G)$ such that

$$(1 + \varepsilon)^{-2} f(x) \leq g(x) \leq (1 - \varepsilon)^{-2} f(x), \quad \forall x \in G$$

and

$$\left| \frac{\partial^{\alpha}}{\partial x^{\alpha}} g(x) \right| \leq P_\alpha M^\alpha \{\varepsilon f(x)\}^{1 - |\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \geq 1. \quad (25)$$

Here, $P_\alpha$ is the constant determined as follows: Let $\rho$ be the mollifier defined in (9). Let $\rho_* : \mathbb{R} \to \mathbb{R}$ be the function, s.t., $\rho_* (|x|) = \rho (x), \ x \in \mathbb{R}^n$. The multi-index $\alpha$ is written as $\alpha = \beta + \gamma$ for $\beta, \gamma \in \mathbb{N}_0^n$ with $|\gamma| = 1$. Then, the inequality (25) holds for

$$P_\alpha = \int_{\mathbb{R}^n} \frac{|\frac{\partial^{\beta}}{\partial x^{\beta}} \rho_1(y)| (1 + |y|)^{|\gamma|}}{(1 - |y|)} dy, \quad (26)$$

where $\rho_1(y) := (n - 1) \rho_* (|y|) + |y| \rho_* (|y|)$.

### C The embedding constant from $H^1(\Omega)$ to $L^p(\Omega)$

Corollary C.1, which comes from Theorem 1 and Theorem A.1, gives a concrete estimation of the embedding constant from $H^1(\Omega)$ to $L^p(\Omega)$ under the suitable assumptions for $\Omega$ and $p$.

**Corollary C.1.** Let $n \in \{2, 3, \cdots \}$ and let $\sigma > 0$. Let $p \in (n/(n - 1), 2n/(n - 2))$ if $n \geq 3$ and let $p \in (n/(n - 1), \infty)$ if $n = 2$. Set $q = np/(n + p)$. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with minimally smooth boundary defined in Definition 4. Then,

$$\|u\|_{L^p(\Omega)} \leq C'_p(\Omega) \|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

holds for

$$C'_p(\Omega) = 2^{1/2} |\Omega|^{1/n - q} T_p A_q(\Omega).$$

Here, $A_q(\Omega)$ is the extension constant derived by Theorem 1 with $\gamma = \sigma^{1/2}$; the constant $T_p$ is defined by

$$T_p = \pi^{-\frac{1}{2}n - \frac{1}{2}} \left(\frac{q - 1}{n - q}\right)^{\frac{1}{2}} \left\{ \frac{\Gamma \left(1 + \frac{n}{q}\right) \Gamma(n)}{\Gamma \left(\frac{n}{q}\right) \Gamma \left(1 + n - \frac{n}{q}\right)} \right\}^{\frac{1}{2}}$$

with the Gamma function $\Gamma$.

**Proof.** Let $u \in H^1(\Omega)$. From the same discussion in (8), it follows that

$$\|u\|_{L^p(\Omega)} \leq T_p A_q(\Omega) \left(\|\nabla u\|_{L^q(\Omega)} + \sigma^{1/2} \|u\|_{L^q(\Omega)}\right). \quad (27)$$

Since $q \in (1, 2)$ holds when $p \in (n/(n - 1), 2n(n - 2))$, Hölder’s inequality gives

$$\|\nabla u\|_{L^q(\Omega)}^q \leq \left(\int_{\Omega} |\nabla u(x)|^{q \frac{2}{q}} dx \right)^{\frac{q}{2}} \left(\int_{\Omega} |1|^{\frac{2}{2-q}} dx \right)^{\frac{2-q}{q}}$$

$$= |\Omega|^{\frac{q-2}{2-q}} \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{q}{2}},$$
where $|\Omega|$ is the measure of $\Omega$. Therefore, it follows that

$$\|\nabla u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{p}} \|\nabla u\|_{L^2(\Omega)}.$$  \hfill (28)

In the same manner, we have

$$\|u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{p}} \|u\|_{L^2(\Omega)}.$$  \hfill (29)

From (27), (28), and (29), it follows that

$$\|u\|_{L^p(\Omega)} \leq |\Omega|^{\frac{2-q}{pq}} T_p A_q(\Omega) \left( \|\nabla u\|_{L^2(\Omega)} + \sigma^{1/2} \|u\|_{L^2(\Omega)} \right)$$

$$\leq 2^{1/2} |\Omega|^{\frac{2-q}{pq}} T_p A_q(\Omega) \|u\|_{H^1(\Omega)}.$$  \hfill $\square$

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