Abstract. Canonical Polyadic Decomposition (CPD) of a third-order tensor is a minimal decomposition into a sum of rank-1 tensors. We find conditions for the uniqueness of individual rank-1 tensors in CPD and present an algorithm to recover them. We called the algorithm “algebraic” because it relies only on standard linear algebra: we compute Kronecker products, the null space of a matrix, and eigen/singular value decomposition. Our new conditions for uniqueness and the working assumptions for the algorithm are more relaxed than, for instance, the famous Kruskal bound. We address both the case where all tensor dimensions are strictly smaller than the rank and the case where a dimension is at least equal to the rank. In the latter case it is known that if \(2 \leq I \leq J \leq K\), then the CPD of a generic \(I \times J \times K\) tensor of rank \(R \leq K\) is unique if and only if \(R \leq (I - 1)(J - 1)\). An existing algebraic algorithm (based on simultaneous diagonalization of a set of matrices) computes CPD under the more restrictive constraint \(R(R - 1) \leq I(I - 1)J(J - 1)/2\) and optimization based algorithms fail to compute the CPD in a reasonable amount of time in the low dimensional case \(I = 3, J = 7, K = R = 12\). By comparison, in our approach it takes less than 1 sec to compute CPD of a generic \(3 \times 7 \times 12\) tensor of rank 12. We show for \(R \leq 24\) and conjecture for \(R > 25\) that our algorithm can recover the rank-1 tensors in CPD up to \(R \leq (I - 1)(J - 1)\).

Key words. canonical polyadic decomposition, Candecomp/Parafac decomposition, CP decomposition, tensor, uniqueness of CPD, uni-mode uniqueness

AMS subject classifications. 15A69, 15A23

1. Introduction.

1.1. Basic terminology. Let \(\mathcal{T} \in \mathbb{R}^{I \times J \times K}\) denote a third-order tensor with entries \(t_{ijk}\). By definition, \(\mathcal{T}\) is rank-1 if it equals the outer product of three nonzero vectors \(a \in \mathbb{R}^I, b \in \mathbb{R}^J,\) and \(c \in \mathbb{R}^K\): \(\mathcal{T} = a \circ b \circ c\), which means that \(t_{ijk} = a_i b_j c_k\) for all values of indices.

A Polyadic Decomposition of \(\mathcal{T}\) expresses \(\mathcal{T}\) as a sum of rank-1 terms:

\[
\mathcal{T} = \sum_{r=1}^{R} a_r \circ b_r \circ c_r,
\]

or

\[
t_{ijk} = \sum_{r=1}^{R} a_{ir} b_{jr} c_{kr}
\]

(1.1)

where \(a_r = [a_{1r} \ldots a_{Ir}]^T \in \mathbb{R}^I, b_r = [b_{1r} \ldots b_{Jr}]^T \in \mathbb{R}^J, c_r = [c_{1r} \ldots c_{Kr}]^T \in \mathbb{R}^K, 1 \leq r \leq R\). If the number \(R\) of rank-1 terms in (1.1) is minimal, then (1.1) is called the Canonical Polyadic Decomposition (CPD) of \(\mathcal{T}\) and \(R\) is called the rank of \(\mathcal{T}\) (denoted by \(r_T\)).
It is clear that in (1.1) the rank-1 terms can be arbitrarily permuted and that vectors within the same rank-1 term can be arbitrarily scaled provided the overall rank-1 term remains the same. The CPD of a tensor is unique when it is only subject to these trivial indeterminacies.

We write (1.1) as $T = [A, B, C]_R$, where the matrices $A := [a_1 \ldots a_R] \in \mathbb{R}^{I \times R}$, $B := [b_1 \ldots b_R] \in \mathbb{R}^{J \times R}$ and $C := [c_1 \ldots c_R] \in \mathbb{R}^{K \times R}$ are called the first, second and third factor matrix of $T$, respectively.

It may happen that the CPD of a tensor $T$ is not unique but all CPDs of $T$ share the same factor matrix in some mode. In this case we say that this factor matrix of $T$ is unique. It is well known that if two or more rank-1 terms in the CPD of $T$ have collinear vectors in some mode, then CPD is not unique. Nevertheless, the factor matrix in the same mode can still be unique.

Our presentation is in terms of real-valued tensors and real-valued factor matrices for notational convenience. Complex variants are easily obtained by taking into account complex conjugations.

We conclude this subsection with some remarks on terminology. The CPD was introduced by F.L. Hitchcock in [16] and was later referred to as Canonical Decomposition (Candecomp) [1], Parallel Factor Model (Parafac) [11, 13], and Topographic Components Model [20]. Uniqueness of one factor matrix is called uni-mode uniqueness in [10, 23]. Uniqueness of the CPD is often called essential uniqueness in Engineering papers and specific identifiability in Algebraic Geometry papers.

1.2. Problem statement. In this paper we deal with the following problems:

P1: Is a given PD $T = [A, B, C]_R$ canonical ($r_T = R$)?

P2: Is the third (resp. first or second) factor matrix in a given CPD $T = [A, B, C]_R$ unique?

P3: Is a given CPD $T = [A, B, C]_R$ unique?

P4: Assume that a CPD $T = [A, B, C]_R$ is unique but unknown. Is it possible to compute CPD algebraically? That is, is it possible to recover the CPD from $T$ by means of conventional linear algebra (basically by taking the orthogonal complement of a subspace and computing generalized eigenvalue decomposition (GEVD))? Problems P1–P3 are important for applications in the sense that, thanks to its uniqueness, CPD is currently becoming a standard tool for signal separation and data analysis, with concrete applications in telecommunication, array processing, machine learning, etc. [3, 4, 17].

Problem P4 is important from a computational point view in the following sense. In practice, the tensor $T$ is known up to an unknown noise tensor $N$ and one deals with a perturbed version of (1.1):

$\hat{T} = T + N = [A, B, C]_R + N$.

The factor matrices of $T$ are approximated by a solution of the optimization problem

$$\min \|\hat{T} - [A, B, C]_R\|,$$  \text{s.t.}  $A \in \mathbb{R}^{I \times R}$, $B \in \mathbb{R}^{J \times R}$, $C \in \mathbb{R}^{K \times R}$,

where $\| \cdot \|$ denotes a suitable norm [21]. In Example 2.6 we demonstrate that in some cases, even if the decomposition is exact, the optimization approach to compute the CPD may require many initializations, although the solution can be computed algebraically. Thus, the algebraic solution may serve as a good initialization.
It is known that the tensor rank is NP-hard over \( \mathbb{R} \) (and \( \mathbb{C} \)) \cite{14, 15}. There is a number of results obtained in Algebraic Geometry (see \cite{19} and the references therein) which allow to compute or estimate rank of a particular tensor even if none of its factor matrices is unique. Another kind of results related to \( P_1 - P_3 \) concerns the generic uniqueness of CPD (also known as generic identifiability), i.e. the case where one answers \( P_1 - P_3 \) for a generic tensor (in this case the answers are given in terms of tensor’s rank and dimensions). In this paper we do not study generic uniqueness and we always consider \( P_1 \) and \( P_2 \) together, that is, we present conditions that guarantee both that the decomposition is canonical and that one of the factor matrices is unique. In short, we proceed as follows. First, we derive conditions on \( A \), \( B \), and \( C \) that guarantee the tensor rank and uniqueness of a single matrix. Second, we show that under additional constraints on the factor matrices the overall CPD is unique. Third, we impose further constraints, to guarantee that the CPD can be computed algebraically and present the algorithm. That is, in this paper

answer to \( P_4 \) \( \Rightarrow \) answer to \( P_3 \) \( \Rightarrow \) answer to \( P_1 - P_2 \).

There are several papers on \( P_1 - P_2, P_4 \) and many on \( P_3 \). We refer the readers to \cite{6, 7} and \cite{8} and the references therein for recent and an overview of early results on uniqueness and algebraic algorithms, respectively. In this paper we further improve methods from \cite{6–8} and obtain new answers to \( P_1 - P_4 \).

2. Main results and organization of the paper.

2.1. Basic notation and conventions. Throughout the paper \( C^k_n \) denotes the binomial coefficient,

\[
C^k_n = \begin{cases} 
\frac{n!}{k!(n-k)!}, & \text{if } k \leq n, \\
0, & \text{if } k > n;
\end{cases}
\]

\( r_A \), range(\( A \)), and ker(\( A \)) denote the rank, the range, and the null space of a matrix \( A \), respectively; \( k_A \) (the \( k \)-rank of \( A \)) is the largest number such that every subset of \( k_A \) columns of the matrix \( A \) is linearly independent;

\( P_{\{l_1, \ldots, l_k\}} \) denotes the set of all permutations of the set \( \{l_1, \ldots, l_k\} \).

We follow the convention that if some of \( l_1, \ldots, l_k \) coincide, then the set \( P_{\{l_1, \ldots, l_k\}} \) contains identical elements, yielding card \( P_{\{l_1, \ldots, l_k\}} = k! \). For instance, \( P_{\{1,1,1\}} \) consists of the six identical entries (1, 1, 1).

Let \( S^{m+l}(\mathbb{R}^{K^{m+l}}) \) denote a subspace of \( \mathbb{R}^{K^{m+l}} \) that consists of vectorized versions of \( K \times \cdots \times K \) symmetric tensors of order \( m+l \), yielding \( \dim S^{m+l}(\mathbb{R}^{K^{m+l}}) = C^{m+l}_{K^{m+l}} \).

To simplify the presentation and w.l.o.g. we will assume throughout the paper that the third dimension \( K \) of the tensor \( T = [A, B, C ]_R \) coincides with \( r_C \), yielding \( r_C = K \leq R \). This can always be achieved in a “dimensionality reduction” step (see, for instance, \cite[Subsection 1.4]{8}).

2.2. Main identity. Let \( T = (t_{ijk})_{i,j,k=1}^{I,J,K} \) have a PD \( T = [A, B, C ]_R \). The following identity

\[
R_{m,l}(T) := \Phi_{m,l}(A, B)S^{m+l}(C)^T, \quad m \geq 1, \quad l \geq 0,
\]

\(2.1\),
in which

\[ \mathbf{R}_{m,l}(\mathcal{T}) \in \mathbb{R}^{(IJ)^{m+l} \times K^{m+l}} \] is constructed from the tensor \( \mathcal{T} \),

\[ \Phi_{m,l}(\mathbf{A}, \mathbf{B}) \in \mathbb{R}^{(IJ)^{m+l} \times M(m,l,R)} \] is constructed from the matrices \( \mathbf{A} \) and \( \mathbf{B} \),

\[ \mathbf{S}_{m+l}(\mathbf{C}) \in \mathbb{R}^{K^{m+l} \times M(m,l,R)} \] is constructed from the matrix \( \mathbf{C} \),

\[ M(m,l,R) = C_R^{m-l}C_{m+l-1}^{m-l} + C_R^{m-l}C_{m+l-1}^{m-l} + \cdots + C_R^{m+l}C_{m+l-1}^{m+l-l} \] (2.2)

(see Definitions 2.1–2.3 below) is the heart of our derivation. The proof of (2.1) as well as equivalent constructions of the matrices \( \mathbf{R}_{m,l}(\mathcal{T}) \) and \( \Phi_{m,l}(\mathbf{A}, \mathbf{B}) \) are given in Section 3.

The results on algorithm obtained in this paper admit the following interpretation. PD (1.1) can be considered as a system of \( IJK \) equations (the number of entries of \( \mathcal{T} \)) with \( (I + J + K)R \) unknowns (the entries of \( \mathbf{A}, \mathbf{B} \), and \( \mathbf{C} \)). In this paper we find new systems of equations with only \( KR \) unknowns (the entries of \( \mathbf{C} \)). Then we show that the new system can be easily solved by reduction to a generalized eigenvalue problem. When the matrix \( \mathbf{C} \) is known, the remaining factor matrices \( \mathbf{A} \) and \( \mathbf{B} \) can be recovered from (1.1) as in [8].

In the particular case \( m = 1 \) and \( l = 0 \), the matrix \( \mathbf{R}_{1,0}(\mathcal{T}) \in \mathbb{R}^{I \times K} \) is called the matrix unfolding of \( \mathcal{T} \) and identity (2.1) takes the form

\[ \mathbf{R}_{1,0}(\mathcal{T}) = [\mathbf{a}_1 \otimes \mathbf{b}_1 \ldots \mathbf{a}_R \otimes \mathbf{b}_R] \mathbf{C}^T =: (\mathbf{A} \otimes \mathbf{B}) \mathbf{C}^T = (\Phi_{1,0}(\mathbf{A}, \mathbf{B}) \mathbf{S}_1(\mathbf{C})^T), \] (2.3)

in which “\( \otimes \)” denotes the Khatri-Rao product or column-wise Kronecker product.

For \( m \geq 2 \) and \( l = 0 \), identity (2.1) was used implicitly in [6, 7] \( \mod 2 \) and [5] \( \mod 2 \) to obtain new results on the uniqueness of the CPD and algebraic computation, respectively. A variant of identity (2.1) with \( m \geq 2 \) and \( l = 0 \) was used explicitly in [8] to obtain new results on algebraic computation (in [8, eq. (1.16)] the matrix \( \mathcal{R}_m(\mathbf{C}) \) coincides with \( \mathbf{S}_{m+0}(\mathbf{C}) \) and the matrices \( \mathbf{R}_m(\mathcal{T}) \) and \( \mathbf{C}_m(\mathbf{A}) \otimes \mathbf{C}_m(\mathbf{B}) \) are obtained from \( \mathbf{R}_{m,0}(\mathcal{T}) \) and \( \Phi_{m,0}(\mathbf{A}, \mathbf{B}) \), respectively, by deleting the zero rows and by replacing multiple rows by a single row; the link between notations used in [8] and [5] is explained in [8, Subsection 1.6]). In [6, 7] we presented several generalizations of the famous Kruskal theorem [18]

\[ k_A + k_B + k_C \geq 2R + 2, \text{ then } r_T = R, \text{ and the CPD } \mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R \text{ is unique.} \] (2.4)

It is one of the contributions of [8] that condition (2.4) not only implies the uniqueness of the CPD but also guarantees that the CPD can be found algebraically.

In this paper we use (2.1) for \( m \geq 2 \) and \( l \geq 0 \), i.e., we also consider \( l > 0 \). We show that under certain conditions (more relaxed than the conditions in [8]) it is possible to recover the matrix \( \mathbf{C} \) from the subspace ker\((\mathbf{R}_{m,l}(\mathcal{T})) \cap S^{m+l}(\mathbb{R}^{K^{m+l}})\). In contrast to [6, 7] and [8] we do not use compound matrices (at least not explicitly). In the remaining part of this subsection we explain how the matrices in (2.1) are constructed.

One can easily check that any number from \( \{1, \ldots, I^{m+l}J^{m+l}\} \) can be uniquely represented as \((i - 1)J^{m+l} + j\) and that any number from \( \{1, \ldots, K^{m+l}\} \) can be
uniquely represented as \( \hat{k} \), where

\[
\hat{i} := (i_1 - 1)I^{m+l-1} + (i_2 - 1)I^{m+l-2} + \cdots + (i_{m+l-1} - 1)I + i_{m+l},
\]

\[
\hat{j} := (j_1 - 1)J^{m+l-1} + (j_2 - 1)J^{m+l-2} + \cdots + (J_{m+l-1} - 1)J + J_{m+l},
\]

\[
\hat{k} := (k_1 - 1)K^{m+l-1} + (k_2 - 1)K^{m+l-2} + \cdots + (K_{m+l-1} - 1)K + K_{m+l},
\]

\[
i_1, \ldots, i_{m+l} \in \{1, \ldots, I\}, \quad j_1, \ldots, j_{m+l} \in \{1, \ldots, J\}, \quad k_1, \ldots, k_{m+l} \in \{1, \ldots, K\}.
\]

**Definition 2.1.** Let \( T \in \mathbb{R}^{l \times J \times K} \). The \( I^{m+l}J^{m+l}K^{m+l} \)-by-\( M^{m+l} \) matrix whose \((\hat{i} - 1)J^{m+l} + \hat{j}, \hat{k})\)th entry is

\[
\frac{1}{m!(m+l)!} \sum_{(s_1, \ldots, s_{m+l}) \in P(i_1, \ldots, s_{m+l})} \det \left[ \begin{array}{ccc} t_{i_1j_1s_1} & \cdots & t_{i_1j_ms_m} \\ \vdots & & \vdots \\ t_{i_ms_1} & \cdots & t_{i_ms_m} \end{array} \right] \cdot t_{i_{m+1}j_{m+1}s_{m+1}} \cdots t_{i_{m+l}j_{m+l}s_{m+l}}
\]

is denoted by \( R_{m,l}(T) \).

Columns of the matrices \( \Phi_{m,l}(A, B) \) and \( S_{m+l}(C) \) are indexed by \((m+l)\)-tuples \((r_1, \ldots, r_{m+l})\) such that

\[
1 \leq r_1 \leq r_2 \leq \cdots \leq r_{m+l} \leq R
\]

and the set \( \{r_1, \ldots, r_{m+l}\} \) contains at least \( m \) distinct elements.

It is easy to show that there exist \( M(m, l, R) \) (see eq. (2.2)) \((m+l)\)-tuples which satisfy condition (2.9), yielding that matrices \( \Phi_{m,l}(A, B) \) and \( S_{m+l}(C) \) have \( M(m, l, R) \) columns. We follow the convention that \((m+l)\)-tuples (2.9) are ordered lexicographically: the \((m+l)\)-tuple \((r'_1, \ldots, r'_{m+l})\) is preceding the \((m+l)\)-tuple \((r''_1, \ldots, r''_{m+l})\) if and only if either \( r'_1 \neq r''_1 \) or there exists \( k \in \{1, \ldots, m + l - 1\} \) such that \( r'_k = r''_k \) and \( r'_k < r''_{k+1} \).

**Definition 2.2.** Let \( A \in \mathbb{R}^{I \times R} \), \( B \in \mathbb{R}^{J \times R} \). The \( I^{m+l}J^{m+l}K^{m+l} \)-by-\( M(m, l, R) \) matrix whose \(((\hat{i} - 1)J^{m+l} + \hat{j}, (r_1, \ldots, r_{m+l}))\)th entry is

\[
\frac{1}{(m!)^2} \sum_{(s_1, \ldots, s_{m+l}) \in P(r_1, \ldots, r_{m+l})} \det \left[ \begin{array}{ccc} a_{i_1s_1} & \cdots & a_{i_1s_m} \\ \vdots & & \vdots \\ a_{i_ms_1} & \cdots & a_{i_ms_m} \end{array} \right] \cdot \det \left[ \begin{array}{ccc} b_{j_1s_1} & \cdots & b_{j_1s_m} \\ \vdots & & \vdots \\ b_{j_ms_1} & \cdots & b_{j_ms_m} \end{array} \right] \cdot a_{i_{m+1}s_{m+1}} \cdots a_{i_{m+l}s_{m+l}} \cdot b_{j_{m+1}s_{m+1}} \cdots b_{j_{m+l}s_{m+l}}
\]

is denoted by \( \Phi_{m,l}(A, B) \).

An alternative columnwise description of \( \Phi_{m,l}(A, B) \) are given at the end of Section 3.

**Definition 2.3.** Let \( C \in \mathbb{R}^{K \times R} \). The \( K^{m+l} \)-by-\( M(m, l, R) \) matrix whose \((r_1, \ldots, r_{m+l})\)th column is

\[
\frac{1}{(m+l)!} \sum_{(s_1, \ldots, s_{m+l}) \in P(r_1, \ldots, r_{m+l})} c_{s_1} \otimes \cdots \otimes c_{s_{m+l}}
\]

is denoted by \( S_{m+l}(C) \).

Our overall derivation generalizes ideas from [8] \((l = 0)\) to \( l \geq 0 \). In this paper, we do not formally prove that the new conditions for algebraic computation \((l \geq 1)\) are
are vectorized rank-1 matrices. In other words, if the third factor matrix of the matrix has full column rank, yielding found from to \([5, 8]\) for an algebraic algorithm. We have the following result.

\[
\ker(R_{m,t}(T)) = \ker(\Phi_{m,t}(A, B)S_{m+l}(C)^T) .
\]

Let \(U_m\) be a matrix such that its columns form a basis for \(\text{range}(S_{m+l}(C)^T)\). We require that

\[
\text{the matrix } \Phi_{m,t}(A, B)U_m \text{ has full column rank and } k_C = K.
\]

(We say that a matrix has full column rank if its columns are linearly independent, implying that it cannot have more columns than rows.) It is clear that by assumption (2.11) we can “cancel” \(\Phi_{m,t}(A, B)\) in (2.10):

\[
\ker(R_{m,t}(T)) = \ker(S_{m+l}(C)^T).
\]

In the paper we prove that assumption (2.12) allows us to recover \(C\) from \(\ker(S_{m+l}(C)^T)\) (and hence from \(T\)) up to column permutation and scaling by means of GEVD. Simulations indicate that for certain values of dimensions and rank, (2.11) does not hold for \(l = 0\) but does hold for some \(l \geq 1\). We suggest two reasons for this finding. First, if \(l = 0\), then the matrix \(S_{m+0}(C)\) has full column rank [8], yielding that the matrix \(U_m\) is square and nonsingular (for instance, the identity). Thus, by (2.11), the matrix \(\Phi_{m,0}(A, B)\) must have full column rank. If \(l \geq 1\), then the matrix \(U_m\) has more rows than columns, yielding that \(\Phi_{m,l}(A, B)U_m\) has fewer columns than \(\Phi_{m,l}(A, B)\). Second, the columns of \(\Phi_{m,l}(A, B)\) (see end of Section 3) are linear combinations of vectors of the form \(a_{i_1} \otimes \cdots \otimes a_{i_m} \otimes b_{j_1} \otimes \cdots \otimes b_{j_m}\). It is known that for matrices with such Kronecker structured columns one can expect that linear dependencies of columns will disappear or become “weak” if \(l\) is large enough. (For instance, if \(x_1, x_2, x_3\) are linearly independent vectors in \(\mathbb{R}^3\) and \(A = [x_1 \ x_2 \ x_3 \ x_1 + x_2 x_1 + x_3 x_2 + x_1 x_2 + x_3 x_1 + x_2 x_3]\), then \(A \in \mathbb{R}^{3 \times 7}\), \(r_A = 3, k_A = 2\), but \(A \odot A \in \mathbb{R}^{8 \times 7}\), \(r_{A \odot A} = k_{A \odot A} = 6\), and even \(A \odot A \odot A \in \mathbb{R}^{27 \times 7}\), \(r_{A \odot A \odot A} = k_{A \odot A \odot A} = 7\). Hence, by increasing the value \(l\), (2.11) is expected to be relaxed.

\section{2.3. At least one factor matrix of \(T\) has full column rank.} In this subsection we consider PD \(T = (t_{ijk})_{i,j,k=1}^{I,J,K} = [A, B, C]_R\) and assume that the matrix \(C\) has full column rank, yielding \(r_C = K = R\). This case is well studied. We refer the reader to [6, Subsection 1.2.3] for an overview of known results on uniqueness and to [5, 8] for an algebraic algorithm.

Note that, if the matrix \(C\) is known, then the matrices \(A\) and \(B\) can be easily found from \(R_{1,0}(T)C^{-1} = A \odot B\) (see (2.3)) using the fact that the columns of \(A \odot B\) are vectorized rank-1 matrices. In other words, if \(r_C = R\), then the uniqueness of the third factor matrix of \(T\) implies the uniqueness of the overall CPD.

We have the following result.

\textbf{Theorem 2.4.} Let \(T = (t_{ijk})_{i,j,k=1}^{I,J,K} = [A, B, C]_R\), \(r_C = K = R, l \geq 0\), and let the matrix \(R_{2,l}(T)\) be defined as in Definition 2.1. Assume that

\[
\dim \left( \ker(R_{2,l}(T)) \cap S^{2+l}(\mathbb{R}^{K^{2+l}}) \right) = R.
\]

(2.13)
Then

1. \( r_T = R \) and the CPD of \( T \) is unique; and
2. the CPD of \( T \) can be found algebraically.

Note that by (2.1), Definition 2.3, and the assumption \( r_C = K = R \),

\[
\ker(R_{2,l}(T)) \bigcap S^{2+l}(\mathbb{R}^{K^{2+l}}) = \ker(\Phi_{2,l}(A, B)S_{2+l}(C)^T) \bigcap S^{2+l}(\mathbb{R}^{K^{2+l}}) \supseteq \\
\ker(S_{2+l}(C)^T) \bigcap S^{2+l}(\mathbb{R}^{K^{2+l}}) \ni x \otimes \cdots \otimes x, \quad x \text{ is a column of } C^{-T}.
\]

Thus, condition (2.13) in Theorem 2.4 means that we require the subspace to have the minimal possible dimension.

The procedure that constitutes the proof of Theorem 2.4(2) is summarized in Algorithm 1. From Definition 2.1 it follows that the rows of the matrix \( R_{2,l}(T) \) are vectorized versions of \( K \times \cdots \times K \) symmetric tensors of order \( 2 + l \). Thus, in step 2, we find the vectors \( w_1, \ldots, w_R \) that form a basis of the orthogonal complement to \( \text{range}(R_{2,l}(T)^T) \) in the space \( S^{2+l}(\mathbb{R}^{K^{2+l}}) \). It can be shown (see Lemma 4.7) that assumptions \( r_C = R \) and (2.13) guarantee that

\[
\ker(R_{2,l}(T)) \bigcap S^{2+l}(\mathbb{R}^{K^{2+l}}) = \text{range} \left( C^{-T} \otimes \cdots \otimes C^{-T} \right). \quad (2.14)
\]

In steps 4–5 we recover \( C^{-T} \) from \( W \) using (2.14) as follows. By (2.14), there exists a unique nonsingular \( R \times R \) matrix \( M \) such that

\[
W = \left( C^{-T} \otimes \cdots \otimes C^{-T} \right) M^T. \quad (2.15)
\]

In step 4, we construct the tensor \( W \) whose vectorized frontal slices are the vectors \( w_1, \ldots, w_R \). Reshaping both sides of (2.15) we obtain the CPD \( W = [C^{-T}, C^{-T} \otimes \cdots \otimes C^{-T}, M]_R \). In step 5, we find the CPD by means of the GEVD using the fact that all factor matrices of \( W \) have full column rank.

Theorem 2.4 and Algorithm 1 extend results from [5, 8] (\( l = 0 \)) to \( l \geq 0 \). It was shown in [5] that the CPD of an \( I \times J \times R \) generic tensor of rank \( R \) is unique and can be computed by Algorithm 1 with \( l = 0 \) if \( R(R - 1) \leq I(I - 1)J(J - 1)/2 \). On the other hand, it is known [2, Propositions 5.2, 5.4], [22, Theorem 2.7] that for \( \min(I, J) \geq 3 \) generic uniqueness holds if and only if \( R \leq (I - 1)(J - 1) \). One can easily check that if \( \min(I, J) \geq 3 \), then the bound \( R \leq (I - 1)(J - 1) \) is more relaxed than the bound \( R(R - 1) \leq I(I - 1)J(J - 1)/2 \). The following example demonstrates that the CPD can also be computed by Algorithm 1 for \( R \leq (I - 1)(J - 1) \leq 24 \).

**Example 2.5.** We consider \( I \times J \times (I - 1)(J - 1) \) tensors generated as a sum of \( R = (I - 1)(J - 1) \) random rank-1 tensors (i.e., the tensors were generated by a PD \([A, B, C]_R \) in which the entries of \( A, B, \text{ and } C \) are independently drawn from the standard normal distribution \( N(0, 1) \)). We assume that condition (2.13) holds for some \( l \geq 0 \) and we run Algorithm 1 for \( l = 0, 1, \ldots, \) until condition (2.13) is met. We test all cases \( I \times J \times (I - 1)(J - 1) \) such that \( I \geq 3, J \geq 3, \) and \( (I - 1)(J - 1) \leq 24 \). The timings \( t_1 \) and \( t_2 \) are averaged over 100 random tensors. The results are shown in Table 2.1.

Simulations show that (2.13) indeed holds for some \( l \leq 2 \) and that \( l \) is constant for tensors of the same dimensions and rank. By comparison, the algebraic algorithm from [5, 8] is limited to the cases where \( l = 0 \). To get insight into algorithm complexity we included the computational time. We implemented Algorithm 1 in MATLAB.
2014a (the implementation was not optimized), and we did experiments on a computer with Intel Core 2 Quad CPU Q9650 3.00 GHz × 4 and 8GB memory running Ubuntu 12.04.5 LTS. We omit the implementation details because of space limitations. However, it is worth noting that the computational cost of steps 3–6 is negligible compared to the cost of steps 1–2 in which we construct (mean computation time 12.04.5 LTS. We omit the implementation details because of space limitations. How-

very difficult to find the CPD by means of numerical optimization. We restarted

it for at most 500 iterations. In 4 cases the residual \( \| \mathcal{T} - [\mathbf{A}_{est}, \mathbf{B}_{est}, \mathbf{C}_{est}]_{12} \|/\| \mathcal{T} \| \) was of the order of 0.0001 but in all cases the estimated factor matrices were far from the true matrices.

**Algorithm 1** (Computation of CPD, \( K = R \) (see Theorem 2.4(ii)))

**Input:** \( \mathcal{T} \in \mathbb{R}^{I \times J \times K} \) and \( I \geq 0 \) with the property that there exist \( \mathbf{A} \in \mathbb{R}^{I \times R} \), \( \mathbf{B} \in \mathbb{R}^{J \times R} \), and \( \mathbf{C} \in \mathbb{R}^{K \times R} \) such that \( R \geq 2 \), \( \mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_{R} \), \( r_{C} = R \), and (2.13) holds.

**Output:** Matrices \( \mathbf{A} \in \mathbb{R}^{I \times R} \), \( \mathbf{B} \in \mathbb{R}^{J \times R} \) and \( \mathbf{C} \in \mathbb{R}^{K \times R} \) such that \( \mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_{R} \)

1: Construct the \( I^{2+1} \times J^{2+1} \times K^{2+1} \) matrix \( \mathbf{R}_{2,1}(\mathcal{T}) \) by Definition 2.1.
2: Find \( \mathbf{w}_{1}, \ldots, \mathbf{w}_{R} \) that form a basis of \( \ker(\mathbf{R}_{2,1}(\mathcal{T})) \cap S^{2+1}(\mathbb{R}^{R^{2+1}}) \)
3: \( \mathbf{W} \leftarrow [\mathbf{w}_{1} \ldots \mathbf{w}_{R}] \)
4: Reshape the \( R^{2+1} \times R \) matrix \( \mathbf{W} \) into an \( R \times R^{1+1} \times R \) tensor \( \mathcal{W} \)
5: Compute the CPD
   \[ \mathcal{W} = [\mathbf{C}^{-T}, \mathbf{C}^{-T} \odot \cdots \odot \mathbf{C}^{-T}, \mathbf{M}]_{R} \quad (\mathbf{M} \text{ is a by-product}) \quad (\text{GEVD}) \]
6: Find the columns of \( \mathbf{A} \) and \( \mathbf{B} \) from the equation \( \mathbf{A} \odot \mathbf{B} = \mathbf{R}_{1,0}(\mathcal{T}) \mathbf{C}^{-T} \)

2.4. No factor matrix of \( \mathcal{T} \) is required to have full column rank. In this subsection we consider the PD \( \mathcal{T} = (t_{ijk})_{i,j,k=1}^{I,J,K} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_{R} \) and extend results of the previous subsection to the case \( r_{C} = K \leq R \).

2.4.1. Results on uniqueness of one factor matrix and overall CPD. We have two results on uniqueness of the third factor matrix.
TABLE 2.1

Values of parameter \( l \) and computational cost of Algorithm 1 for \( I \times J \times (I-1)(J-1) \) tensors of rank \( R = (I-1)(J-1) \leq 25 \) (see Example 2.5 for the meaning of \( C_{R+1}^{2+1} \), \( t_1 \), and \( t_2 \)).

| \( I \times J \times (I-1)(J-1) \) | \( l \) | \( C_{R+1}^{2+1} \) | \( t_1 \) (sec) | \( t_2 \) (sec) |
|---|---|---|---|---|
| 3 \times 3 \times 4 | 0 | 10 | 0.012 | 0.008 |
| 3 \times 4 \times 6 | 0 | 21 | 0.022 | 0.013 |
| 3 \times 5 \times 8 | 0 | 36 | 0.038 | 0.013 |
| 3 \times 6 \times 10 | 0 | 55 | 0.060 | 0.014 |
| 3 \times 7 \times 12 | 1 | 364 | 0.368 | 0.035 |
| 3 \times 8 \times 14 | 1 | 560 | 0.725 | 0.071 |
| 3 \times 9 \times 16 | 1 | 816 | 1.342 | 0.156 |
| 3 \times 10 \times 18 | 1 | 1140 | 2.333 | 0.284 |
| 3 \times 11 \times 20 | 1 | 1540 | 4.259 | 0.773 |
| 3 \times 12 \times 22 | 1 | 2024 | 6.119 | 0.970 |
| 3 \times 13 \times 24 | 1 | 2600 | 9.386 | 1.698 |
| 4 \times 4 \times 9 | 0 | 45 | 0.047 | 0.013 |
| 4 \times 5 \times 12 | 1 | 364 | 0.367 | 0.034 |
| 4 \times 6 \times 15 | 1 | 680 | 0.988 | 0.108 |
| 4 \times 7 \times 18 | 2 | 5985 | 22.375 | 8.566 |
| 4 \times 8 \times 21 | 2 | 10626 | 56.758 | 36.272 |
| 4 \times 9 \times 24 | 2 | 17550 | 150.261 | 210.018 |
| 5 \times 5 \times 16 | 1 | 816 | 1.321 | 0.152 |
| 5 \times 6 \times 20 | 2 | 8855 | 41.213 | 22.903 |
| 5 \times 7 \times 24 | 2 | 17550 | 139.622 | 212.346 |

THEOREM 2.7. Let \( T = (t_{ijk})_{i,j,k=1}^{l,l,l} = [A, B, C]_R \), \( r_C = K \leq R \), \( m = R - K + 2 \), and \( l_1, \ldots, l_m \) be nonnegative integers. Let also

the matrices \( \Phi_{1,l_1}(A, B) \), \( \ldots, \Phi_{m,l_m}(A, B) \) be defined as in Definition 2.2 and

the matrices \( S_{1+l_1}(C) \), \( \ldots, S_{m+l_m}(C) \) be defined as in Definition 2.3.

Let \( U_1, \ldots, U_m \) be matrices such that their columns form bases for \( \text{range}(S_{1+l_1}(C)^T) \), \( \ldots, \text{range}(S_{m+l_m}(C)^T) \), respectively. Assume that

(i) \( k_C \geq 1 \); and

(ii) \( A \odot B \) has full column rank; and

(iii) the matrices \( \Phi_{1,l_1}(A, B)U_1, \ldots, \Phi_{m,l_m}(A, B)U_m \) have full column rank.

Then \( r_T = R \) and the third factor matrix of \( T \) is unique.

According to the following theorem the number of matrices in (iii) in Theorem 2.7 can be reduced to one if \( R \leq \min(k_A, k_B) + K - 1 \).

THEOREM 2.8. Let \( T = (t_{ijk})_{i,j,k=1}^{l,l,l} = [A, B, C]_R \), \( r_C = K \leq R \), \( m = R - K + 2 \), and \( l \geq 0 \). Let also the matrices \( \Phi_{m,l}(A, B) \) and \( S_{m+l}(C) \) be defined as in Definition 2.2 and Definition 2.3, respectively. Let \( U_m \) be a matrix such that its columns form a basis for \( \text{range}(S_{m+l}(C)^T) \). Assume that

(i) \( k_C \geq 1 \); and

(ii) \( A \odot B \) has full column rank; and

(iii) \( \min(k_A, k_B) \geq m - 1 \); and

(iv) the matrix \( \Phi_{m,l}(A, B)U_m \) has full column rank.

Then \( r_T = R \) and the third factor matrix of \( T \) is unique.
The assumptions in Theorems 2.7 and 2.8 complement each other as follows: in Theorem 2.7 we do not require that the matrices \( \Phi_k \) and in Theorem 2.8 we do not require that the condition \( \min(k_{A}, k_{B}) \geq m - 1 \) holds and in Theorem 2.8 we do not require that the matrices \( \Phi_{k,l} (A, B) U_k \), \( 1 \leq k \leq m - 1 \) have full column rank.

It was shown in [7, Proposition 1.20] that if \( T \) has two PDs \( T = [A, B, C]_R \) and \( T = [A, B, C]_R \) that share the factor matrix \( C \) and if the condition

\[
\max(\min(k_{A}, k_{B} - 1), \min(k_{A} - 1, k_{B})) + k_{C} \geq R + 1
\]  

(2.16)

holds, then both PDs consist of the same rank-one terms. Thus, combining Theorems 2.7–2.8 with [7, Proposition 1.20] we obtain the following result on uniqueness of the overall CPD.

**Theorem 2.9.** Let the assumptions in Theorem 2.7 or Theorem 2.8 hold and let \( \tau_{T} = R \) and the CPD of tensor \( T \) is unique.

**2.4.2. Algebraic algorithm for CPD.** We have the following result on algebraic computation.

**Theorem 2.10.** Let \( T = (t_{ijk})_{i,j,k=1}^{l,m,k} = [A, B, C]_R \), \( \tau_{C} = K \leq R \), \( m = R - K + 2 \), and \( l \geq 0 \). Let also the matrices \( \Phi_{m,l}(A, B) \) and \( S_{m+l}(C) \) be defined as in Definition 2.2 and Definition 2.3, respectively. Let \( U_m \) be a matrix such that its columns form a basis for \( \text{range}(S_{m+l}(C)^T) \). Assume that

(i) \( k_{C} = K \); and
(ii) \( A \odot B \) has full column rank; and
(iii) the matrix \( \Phi_{m,l}(A, B) U_m \) has full column rank.

Then \( \tau_{T} = R \), the CPD of \( T \) is unique and can be found algebraically.

In Section 4 we explain that the assumptions in Theorem 2.10 are more restrictive than the assumptions in Theorem 2.9. This proves the statement on rank and uniqueness in Theorem 2.10. To prove the statement on algebraic computation we will explain in Section 4 that Theorem 2.10 can be reformulated as follows.

**Theorem 2.11.** Let \( T = (t_{ijk})_{i,j,k=1}^{l,m,k} = [A, B, C]_R \), \( \tau_{C} = K \leq R \), \( m = R - K + 2 \), and \( l \geq 0 \). Let also the matrix \( R_{m,l}(T) \) be defined as in Definition 2.1. Assume that

(i) \( k_{C} = K \); and
(ii) \( A \odot B \) has full column rank; and
(iii) \( \dim(\ker(R_{m,l}(T)) \cap S^{m+l}(R^{K^{m+l}})) = C_{R}^{K-1} \).

Then \( \tau_{T} = R \), the CPD of \( T \) is unique and can be found algebraically.

Note that if \( k_{C} = K \), then by (2.1) and Lemma 4.6 (i) below,

\[
\dim(\ker(R_{m,l}(T)) \cap S^{m+l}(R^{K^{m+l}})) = \dim(\ker(\Phi_{m,l}(A, B) S_{m+l}(C)^T) \cap S^{m+l}(R^{K^{m+l}})) \geq C_{R}^{K-1}.
\]

(2.17)

Thus, assumption (iii) of Theorem 2.11 means that we require the subspace to have the minimal possible dimension. That is, we suppose that the factor matrices \( A, B, \) and \( C \) are such that the multiplication by \( \Phi_{m,l}(A, B) \) in (2.1) does not increase the overlap between \( \ker(S_{m+l}(C)^T) \) and \( S^{m+l}(R^{K^{m+l}}) \). In other words, we suppose that the multiplication by \( \Phi_{m,l}(A, B) \) does not cause additional vectorized \( K \times \cdots \times K \) symmetric tensors of order \( m + l \) to be part of the null space of \( R_{m,l}(T) \). This is key to the derivation. By the assumption, as we will explain further in this section,
the only vectorized symmetric tensors in the null space of \( R_{m,l}(T) \) admit a direct connection with the factor matrix \( C \), from which \( C \) may be retrieved. On the other hand, the null space of \( R_{m,l}(T) \) can obviously be computed from the given tensor \( T \).

The algebraic procedure based on Theorem 2.11 consists of three phases and is summarized in Algorithm 2. In the first phase we find the \( K \times C prefix_mat{K}{-1} \) matrix \( F \) such that

\[
\text{every column of } F \text{ is orthogonal to exactly } K - 1 \text{ columns of } C \quad (2.18)
\]

\[
\text{any vector that is orthogonal to exactly } K - 1 \text{ columns of } C \text{ is proportional to a column of } F. \quad (2.19)
\]

Since \( k_C = K \) any \( K - 1 \) columns of \( C \) define a unique column of \( F \) (up to scaling). Thus, (2.18)–(2.19) define the matrix \( F \) up to column permutation and scaling. A special representation of \( F \) (called \( B(C) \)) was studied in [8]. It was shown in [8] that the matrix \( F \) can be considered as an unconventional variant of the inverse of \( C \):

\[
\text{every column of } C \text{ is orthogonal to exactly } C^{{K-2}}_{R-1} \text{ columns of } F, \quad (2.20)
\]

\[
\text{any vector that is orthogonal to exactly } C^{{K-2}}_{R-1} \text{ columns of } F \text{ is proportional to a column of } C. \quad (2.21)
\]

(Note that, since \( k_C = K \), multiplication by the Moore–Penrose pseudo-inverse \( C^\dagger \) yields \( CC^\dagger = I_K \). In contrast, for \( F \) we consider the product \( FC \).) It can be shown (see Lemma 4.7) that under the assumptions in Theorems 2.10–2.11:

\[
k_F \geq 2, \quad \text{the matrix } F^{(m+l-1)} \text{ has full column rank and} \quad (2.22)
\]

\[
\ker(R_{m,l}(T)) \cap S^{m+l}([\mathbb{R}^{K^{m+l}}]) = \text{range } (F^{(m+l)}) \quad (2.23)
\]

where

\[
F^{(m+l-1)} := \underbrace{F \odot \cdots \odot F}_{m+l-1}, \quad F^{(m+l)} := \underbrace{F \odot \cdots \odot F}_{m+l} \quad (2.24)
\]

If \( K = R \) (as in Subsection 2.3), then \( m = R - K + 2 = 2 \), (2.23) coincides with (2.14) (\( F \) coincides with \( C^{-T} \) up to column permutation and scaling), and the first phase of Algorithm 2 coincides with steps 1–5 of Algorithm 1. For \( K < R \) (implying \( m > 2 \)) we work as follows. From Definition 2.1 it follows that the rows of the matrix \( R_{m,l}(T) \) are vectorized versions of \( K \times K \) symmetric tensors of order \( m+l \). Thus, in step 2, we find the vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_{C^{K-1}_R} \) that form a basis of the orthogonal complement to \( \text{range}(R_{m,l}(T)^T) \) in the space \( S^{m+l}([\mathbb{R}^{K^{m+l}}]) \) (the existence of such a basis follows from assumption (iii) of Theorem 2.11). By (2.23), there exists a unique nonsingular \( C^{K-1}_R \times C^{K-1}_R \) matrix \( M \) such that

\[
W = F^{(m+l)}M^T. \quad (2.25)
\]

In step 4, we construct the tensor \( W \) whose vectorized frontal slices are the vectors \( \mathbf{w}_1, \ldots, \mathbf{w}_{C^{K-1}_R} \). Reshaping both sides of (2.25) we obtain the CPD \( W = [F, F^{(m+l-1)}], \) \( M \) in which the matrices \( F^{(m+l-1)} \) and \( M \) have full column rank and \( k_F \geq 2 \). By [12], the CPD of \( W \) can be computed by means of GEVD.

In the second and third phase we use \( F \) to find \( A, B, C \). There are two ways to do this. The first way is to find \( C \) from \( F \) by (2.20)–(2.21) and then to recover \( A \).
Algorithm 2 (Computation of CPD, $K \leq R$ (see Theorem 2.11))

Input: $T \in \mathbb{R}^{I \times J \times K}$ and $l \geq 0$ with the property that there exist $A \in \mathbb{R}^{I \times R}$, $B \in \mathbb{R}^{J \times R}$, and $C \in \mathbb{R}^{K \times R}$ such that $T = [A, B, C]_R$ and assumptions (i)–(iii) in Theorem 2.11 hold.

Output: Matrices $A \in \mathbb{R}^{I \times R}$, $B \in \mathbb{R}^{J \times R}$ and $C \in \mathbb{R}^{K \times R}$ such that $T = [A, B, C]_R$

Phase 1: Find the matrix $F \in \mathbb{R}^{K \times C^{-1}}_R$ such that $F$ coincides with $B(C)$ up to (unknown) column permutation and scaling

1: Construct the $I^{m+l}J^{m+l} \times K^{m+l}$ matrix $R_{m,l}(T)$ by Definition 2.1.
2: Find $w_1, \ldots, w_{C^{-1}}$ that form a basis of ker($R_{m,l}(T)$) $\cap S^{m+l}(\mathbb{R}^{K^{m+l}})$
3: $W \leftarrow [w_1 \ldots w_{C^{-1}}]$
4: Reshape the $K^{m+l} \times C^{-1}$ matrix $W$ into an $K \times K^{m+l-1} \times C^{-1}$ tensor $W$
5: Compute the CPD $W = [F, F \odot \ldots \odot F, M]_{C^{-1}}$ ($M$ is a by-product) (GEVD)

Phase 2 and Phase 3 (can be taken verbatim from [8, Algorithms 1,2])

Example 2.12. Table 2.2 contains some examples of CPDs which can be computed by Algorithm 2 and cannot be computed by algorithms from [8]. The tensors were generated by a PD $[A, B, C]_R$ in which the entries of $A$, $B$, and $C$ are independently drawn from the standard normal distribution $N(0, 1)$. The timings $t_1$ and $t_2$ are averaged over 100 random tensors for each set of dimensions $I \times J \times K$ and rank $R$.

In our (suboptimal) implementation in steps 1–2 we construct a positive semi-definite $C^{m+l} \times C^{m+l} \times C^{m+l}$ matrix (mean computation time $t_1$) and compute its smallest $C^{K^{-1}}$ eigenvalues (mean computation time $t_2$). The computational cost of steps 3–5 is negligible compared to the cost of steps 1–2.

Uniqueness of the CPDs follows from Theorem 2.11. By comparison, the results of [7] guarantee uniqueness only for rows $1$–$4$ (see [7, Table 3.1]).
3. Derivation of the main identity. Let $\mathcal{T} = (t_{ijk})_{i,j,k=1}^{I,J,K} = [A, B, C]_R$. In this section we establish a link between the matrix $R_{m,l}(\mathcal{T})$ defined in subsection 2.2 and the factor matrices $A$, $B$, and $C$. We show that the matrix $R_{m,l}(\mathcal{T})$ is obtained from $\mathcal{T}$ by taking the following steps: 1) taking the $(m+l)$th Kronecker power of $\mathcal{T}$; 2) making two partial skew-symmetrizations and one partial symmetrization of the result; 3) reshaping the result into an $I^{m+l}J^{m+l}K^{m+l}$ matrix. The main identity is obtained by applying steps 1)-3) to the both sides of (1.1).

3.1. Step 1: Kronecker product power of $\mathcal{T}$. The Kronecker product square of $\mathcal{T}$, $\mathcal{T}^{(2)} := \mathcal{T} \otimes \mathcal{T}$, is an $I \times J \times K$ block-tensor whose $(i, j, k)$th block is the $I \times J \times K$ tensor $t_{ijk} \mathcal{T}$. Equivalently, $\mathcal{T}^{(2)}$ is an $I^2 \times J^2 \times K^2$ tensor whose $(\tilde{i}, \tilde{j}, \tilde{k}) := (i_1 - 1)I + i_2, (j_1 - 1)J + j_2, (k_1 - 1)K + k_2)$th entry is

$$t_{ijk}^{(2)} = t_{i_1j_1k_1}t_{i_2j_2k_2}.$$

Similarly, the $(l+m)$th Kronecker product of $\mathcal{T}$,

$$\mathcal{T}^{(m+l)} := \mathcal{T} \otimes \cdots \otimes \mathcal{T},$$

is an $I^{m+l} \times J^{m+l} \times K^{m+l}$ tensor whose $(\tilde{i}, \tilde{j}, \tilde{k})$th entry is

$$t_{ijk}^{(m+l)} = t_{i_1j_1k_1}t_{i_2j_2k_2} \cdots t_{i_{m+l}j_{m+l}k_{m+l}},$$

where $\tilde{i}$, $\tilde{j}$, and $\tilde{k}$ are defined in (2.5), (2.6), and (2.7), respectively. One can easily check that if $\mathcal{T} = \sum_{r=1}^R a_r \circ b_r \circ c_r$, then

$$\mathcal{T}^{(m+l)} = \sum_{r_1, \ldots, r_{m+l}=1}^R (a_{r_1} \otimes \cdots \otimes a_{r_{m+l}}) \circ (b_{r_1} \otimes \cdots \otimes b_{r_{m+l}}) \circ (c_{r_1} \otimes \cdots \otimes c_{r_{m+l}}).$$

3.2. Step 2: two partial skew-symmetrizations and one partial symmetrization of a reshaped version of $\mathcal{T}^{(m+l)}$. Recall that a higher-order tensor is said to be symmetric (resp. skew-symmetric) with respect to a given group of indices or partially symmetric (resp. skew-symmetric) if its coordinates do not alter by an arbitrary permutation of these indices (resp. if the sign changes with every interchange of two arbitrary indices in the group).

Let us recall the operations of (complete) symmetrization and skew-symmetrization. With a general $k$th-order $L \times \cdots \times L$ tensor $\mathcal{N}$ one can associate its symmetric part $S^k(\mathcal{N})$ and skew-symmetric part $\Lambda^k(\mathcal{N})$ as follows. By construction, $S^k(\mathcal{N})$ is a tensor whose entry with indices $l_1, \ldots, l_k$ is equal to

$$\frac{1}{k!} \sum_{(p_1, \ldots, p_k) \in P_{l_1, \ldots, l_k}} n_{p_1 \cdots p_k}. \tag{3.2}$$

That is, to get $S^k(\mathcal{N})$ we should take the average of $k!$ tensors obtained from $\mathcal{N}$ by all possible permutations of the indices. Similarly, $\Lambda^k(\mathcal{N})$ is a tensor whose entry with indices $l_1, \ldots, l_k$ is equal to

$$\begin{cases} \frac{1}{k!} \sum_{(p_1, \ldots, p_k) \in P_{l_1, \ldots, l_k}} \text{sgn}(p_1, \ldots, p_k)n_{p_1 \cdots p_k}, & \text{if } l_1, \ldots, l_k \text{ are distinct}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$
where
\[
\text{sgn}(p_1, \ldots, p_k) \text{ denotes the signature of the permutation } (p_1, \ldots, p_k).
\]
The definition of \( \Lambda^k(N) \) differs from that of \( S^k(N) \) in that the signatures of the permutations are taken into account and that the entries of \( \Lambda^k(N) \) with repeated indices are necessarily zeros. One can easily check that if permutations are taken into account and that the entries of \( \Lambda^k(N) \) with repeated indices are necessarily zeros.

\[
\Lambda^k(d_1 \circ \cdots \circ d_k) = \frac{1}{k!} \sum_{(p_1, \ldots, p_k) \in P_{(1, \ldots, k)}} \sigma(p_1, \ldots, p_k) d_{p_1} \circ \cdots \circ d_{p_k},
\]
\[
S^k(d_1 \circ \cdots \circ d_k) = \frac{1}{k!} \sum_{(p_1, \ldots, p_k) \in P_{(1, \ldots, k)}} d_{p_1} \circ \cdots \circ d_{p_k}.
\]

Partial (skew-)symmetrization is a (skew-)symmetrization with respect to a given group of indices. Instead of presenting the formal definitions we illustrate both notions for an \( M \times L \times L \) tensor \( N = (n_{m,1,l_1,l_2})_{m,l_1,l_2=1}^{M,L,L} \). Partial symmetrization with respect to the group of indices \( \{2,3\} \) maps the tensor \( N \) to a tensor that we denote by \((I_M \circ S^2)N\), whose entry with indices \((m, l_1, l_2)\) is equal to
\[
\sum_{(p_1, p_2) \in P_{(1,1,2)}} n_{ml_1l_2} = n_{ml_1l_2} + n_{ml_2l_1}.
\]

Similarly, by \((I_M \circ \Lambda^2)N\) we denote the tensor whose entry with indices \((m, l_1, l_2)\) is equal to
\[
\begin{cases} 
\text{sgn}(p_1, p_2)n_{ml_1l_2} = n_{ml_1l_2} - n_{ml_2l_1}, & \text{if } l_1 \neq l_2, \\
0, & \text{if } l_1 = l_2.
\end{cases}
\]

If \( N = d_1 \circ d_2 \circ d_3 \in \mathbb{R}^{M \times L \times L} \), then
\[
(I_M \circ S^2)(d_1 \circ d_2 \circ d_3) = d_1 \circ S^2(d_2 \circ d_3) = d_1 \circ d_2 \circ d_3 + d_1 \circ d_3 \circ d_2,
\]
\[
(I_M \circ \Lambda^2)(d_1 \circ d_2 \circ d_3) = d_1 \circ \Lambda^2(d_2 \circ d_3) = d_1 \circ d_2 \circ d_3 - d_1 \circ d_3 \circ d_2.
\]

Thus, operations \((I_M \circ S^2)\) and \((I_M \circ \Lambda^2)\) symmetrize and skew-symmetrize the horizontal slices of \( N \).

Let us reshape the tensor \( T^{(m+1)} \) into an \( I \times I \times J \times \cdots \times J \times K \times \cdots \times K \) (each letter is repeated \( m+1 \) times) tensor \( \tilde{T}^{(m+1)} \) as:
\[
\tilde{T}^{(m+1)} = \sum_{r_1, \ldots, r_{m+1}=1}^R (a_{r_1} \circ \cdots \circ a_{r_{m+1}}) \circ (b_{r_1} \circ \cdots \circ b_{r_{m+1}}) \circ (c_{r_1} \circ \cdots \circ c_{r_{m+1}}).
\]

Then the entries of \( \tilde{T}^{(m+1)} \) are given by
\[
\tilde{T}^{(m+1)}_{i_1 \ldots i_{m+1} j_1 \ldots j_{m+1} k_1 \ldots k_{m+1}} = t_{i_1 j_1 k_1} t_{i_2 j_2 k_2} \cdots t_{i_{m+1} j_{m+1} k_{m+1}}.
\]

From (3.1) and (3.9) it follows that \( \tilde{T}^{(m+1)} \) is just a higher-order representation of \( T^{(m+1)} \).
A new tensor \( \tilde{T}^{(m+l)}_{\text{AS}} \) is obtained from \( \tilde{T}^{(m+l)} \) by applying two partial skew-symmetrizations and one partial symmetrization as follows:

\[
\tilde{T}^{(m+l)}_{\text{AS}} := \left[ (\Lambda^m \circ I_1 \circ \cdots \circ I_l) \circ (\Lambda^m \circ I_1 \circ \cdots \circ I_j) \circ S^{m+l} \right] \tilde{T}^{(m+l)}. \tag{3.10}
\]

To obtain \( \tilde{T}^{(m+l)}_{\text{AS}} \) we first skew-symmetrize \( \tilde{T}^{(m+l)} \) with respect to the group of indices \( \{1, \ldots, m\} \) (the first \( m \) “I” dimensions), then we skew-symmetrize the result with respect to the group of indices \( \{m + l + 1, \ldots, 2m + l\} \) (the first \( m \) “J” dimensions), and, finally, we symmetrize the result with respect to the group of indices \( \{2m + 2l + 1, \ldots, 3m + 3l\} \) (all “K” dimensions). From (3.2), (3.3), and (3.9), it follows that the \((i_1, \ldots, i_{m+l}, j_1, \ldots, j_{m+l}, k_1, \ldots, k_{m+l})\)th entry of the tensor \( \tilde{T}^{(m+l)}_{\text{AS}} \) is equal to zero if some index is repeated in \( i_1, \ldots, i_m \) or \( j_1, \ldots, j_m \) and is equal to

\[
\frac{1}{(m+l)!} \sum_{(s_1, \ldots, s_{m+l}) \in P_{1^{s_1}, \ldots, s_{m+l}}} \left[ \frac{1}{m!} \sum_{(q_1, \ldots, q_m) \in P_{1^{q_1}, \ldots, q_m}} \text{sgn}(q_1, \ldots, q_m) \times \left[ \frac{1}{m!} \sum_{(p_1, \ldots, p_m) \in P_{1^{p_1}, \ldots, p_m}} \text{sgn}(p_1, \ldots, p_m) \prod_{i=1}^{m+l} \text{det} \begin{pmatrix} t_{i_1 q_1 s_m} & \cdots & t_{i_1 q_m s_m} \\ \vdots & \ddots & \vdots \\ t_{i_{m+l} q_1 s_m} & \cdots & t_{i_{m+l} q_m s_m} \end{pmatrix} \right] \right] =
\frac{1}{m!} \sum_{(s_1, \ldots, s_{m+l}) \in P_{1^{s_1}, \ldots, s_{m+l}}} \left[ \frac{1}{m!} \sum_{(q_1, \ldots, q_m) \in P_{1^{q_1}, \ldots, q_m}} \text{sgn}(q_1, \ldots, q_m) \times \left[ \frac{1}{m!} \sum_{(p_1, \ldots, p_m) \in P_{1^{p_1}, \ldots, p_m}} \text{sgn}(p_1, \ldots, p_m) \prod_{i=1}^{m+l} \text{det} \begin{pmatrix} t_{i_1 j_1 s_m} & \cdots & t_{i_1 j_m s_m} \\ \vdots & \ddots & \vdots \\ t_{i_{m+l} j_1 s_m} & \cdots & t_{i_{m+l} j_m s_m} \end{pmatrix} \right] \right]
\]

otherwise (we used twice the Leibniz formula for the determinant). Thus, by (2.8), the tensor \( \tilde{T}^{(m+l)}_{\text{AS}} \) and the matrix \( R_{m,l}(T) \) have the same entries (in step 3 it will be shown that \( R_{m,l}(T) \) is a matrix unfolding of \( \tilde{T}^{(m+l)}_{\text{AS}} \)).

Let us apply partial skew-symmetrizations and partial symmetrization to the right-hand side of (3.8); from (3.8), (3.10), (see also (3.6)–(3.7) for the properties of the outer product) it follows that

\[
\tilde{T}^{(m+l)}_{\text{AS}} = \sum_{r_1, \ldots, r_{m+l} = 1}^{R} \left[ (\Lambda^m(a_{r_1} \circ \cdots \circ a_{r_m}) \circ a_{r_{m+1}} \circ \cdots \circ a_{r_{m+l}}) \circ S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}}) \right] = \sum_{r_1, \ldots, r_{m+l} = 1}^{R} \tilde{T}^{A,B}_{r_1, \ldots, r_{m+l}} \circ S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}}),
\tag{3.11}
\]

where the expressions \( \Lambda^m(a_{r_1} \circ \cdots \circ a_{r_m}) \) and \( \Lambda^m(b_{r_1} \circ \cdots \circ b_{r_m}) \) are defined in (3.5), the expression \( S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}}) \) is defined in (3.4), and, by definition,

\[
\tilde{T}^{A,B}_{r_1, \ldots, r_{m+l}} := \Lambda^m(a_{r_1} \circ \cdots \circ a_{r_m}) a_{r_{m+1}} \circ \cdots \circ a_{r_{m+l}} \circ S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}}) b_{r_{m+1}} \circ \cdots \circ b_{r_{m+l}}
\]
(recall that the vectors \(a_r\) and \(b_r\) are columns of the matrices \(A\) and \(B\), respectively).

Note that by construction, \(S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}})\) is a completely symmetric tensor (that is, the expression \(S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}})\) does not change after any permutation of the vectors \(c_{r_1}, \ldots, c_{r_{m+l}}\)). Taking this fact into account we can group the summands in (3.11) as follows

\[
\hat{T}^{(m+l)}_{\text{AAS}} = \sum_{1 \leq r_1 \leq \cdots \leq r_{m+l} \leq R} \left( \sum_{(p_1, \ldots, p_{m+l})} F_{p_1, \ldots, p_{m+l}}^{AB} \right) \circ S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}}). \tag{3.12}
\]

### 3.3. Step 3: Reshaping (unfolding) of \(\hat{T}^{(m+l)}_{\text{AAS}}\) into the matrix \(R_{m,l}(T)\)

We define the matricization operation \(\text{Matr} : \mathbb{R}^{I \times J \times J \times J \times K \times K} \rightarrow \mathbb{R}^{I \times J \times J \times J \times K \times K_{m+l}}\) as follows: the \((i_1, \ldots, i_{m+l}, j_1, \ldots, j_{m+l}, k_1, \ldots, k_{m+l})\)th entry of a tensor is mapped to the \((i-1)J^{m+l} + j, k)\)th entry of a matrix, where \(i, j, k\) are defined in (2.5), (2.6), and (2.7), respectively. One can easily verify that

\[
\text{Matr}(a_{i_1} \circ \cdots \circ a_{i_{m+l}} \circ b_{j_1} \circ \cdots \circ b_{j_{m+l}} \circ c_{k_1} \circ \cdots \circ c_{k_{m+l}}) = \\
[a_{i_1} \otimes \cdots \otimes a_{i_{m+l}} \otimes b_{j_1} \otimes \cdots \otimes b_{j_{m+l}} \otimes c_{k_1} \otimes \cdots \otimes c_{k_{m+l}}]^T
\]

and that \(R_{m,l}(T) = \text{Matr}(\hat{T}^{(m+l)}_{\text{AAS}})\).

What is left to show is that the matricization of the right-hand side of (3.12) coincides with the matrix \(\Phi_{m,l}(A, B)S_{m+l}(C)^T\). In the sequel, when no confusion is possible, we will use \(S^k\) and \(\Lambda^k\) to denote “symmetrization” and “skew-symmetrization” of vector representations of a certain tensor: if \(d_1, \ldots, d_k \in \mathbb{R}^L\), then the vectors \(S^k(d_1 \otimes \cdots \otimes d_k)\) and \(\Lambda^k(d_1 \otimes \cdots \otimes d_k)\) are computed in the same way as in (3.4)–(3.5) but with “\(\circ\)” replaced by “\(\otimes\)”:  

\[
S^k(d_1 \otimes \cdots \otimes d_k) = \frac{1}{k!} \sum_{(p_1, \ldots, p_k) \in P_{1, \ldots, k}} d_{p_1} \otimes \cdots \otimes d_{p_k},
\]

\[
\Lambda^k(d_1 \otimes \cdots \otimes d_k) = \frac{1}{k!} \sum_{(p_1, \ldots, p_k) \in P_{1, \ldots, k}} \sigma(p_1, \ldots, p_k)d_{p_1} \otimes \cdots \otimes d_{p_k}.
\]

Hence, by (3.11), (3.12), and (3.13)

\[
R_{m,l}(T) = \text{Matr}(\hat{T}^{(m+l)}_{\text{AAS}}) = \\
\sum_{1 \leq r_1 \leq \cdots \leq r_{m+l} \leq R} \left( \sum_{(s_1, \ldots, s_{m+l})} f_{s_1, \ldots, s_{m+l}}^{AB} \right) S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}})^T = \tag{3.15}
\]

\[
\sum_{1 \leq r_1 \leq \cdots \leq r_{m+l} \leq R} \phi(A, B)_{r_1, \ldots, r_{m+l}} S^{m+l}(c_{r_1} \circ \cdots \circ c_{r_{m+l}})^T,
\]

where

\[
f_{s_1, \ldots, s_{m+l}}^{AB} := \Lambda^m(a_{s_1} \otimes \cdots \otimes a_{s_m}) \otimes a_{s_{m+1}} \otimes \cdots \otimes a_{s_{m+l}} \otimes \\
\Lambda^m(b_{s_1} \otimes \cdots \otimes b_{s_m}) \otimes b_{s_{m+1}} \otimes \cdots \otimes b_{s_{m+l}},
\]

\[
\phi(A, B)_{r_1, \ldots, r_{m+l}} := \sum_{(s_1, \ldots, s_{m+l})} f_{s_1, \ldots, s_{m+l}}^{AB}.
\]
We show that $\phi(A,B)_{r_1,\ldots,r_{m+1}}$ is the zero vector if the set $\{r_1,\ldots,r_{m+1}\}$ has fewer than $m$ distinct elements and that $\phi(A,B)_{r_1,\ldots,r_{m+1}}$ is a column of $\Phi_{m,l}(A,B)$ otherwise. From (3.14) and the Leibniz formula for the determinant it follows that the entries of the vector $\Lambda^k(d_1 \otimes \cdots \otimes d_k)$ are all possible $k \times k$ minors of the matrix

$$D := \{|d_1| \ldots |d_k|\}$$

divided by $k!$. In particular, if some of the vectors $d_i$ coincide, then $\Lambda^k(d_1 \otimes \cdots \otimes d_k)$ is the zero vector. Hence, the vector $f^A_{s_1,\ldots,s_{m+1}}$ has entries

$$\frac{1}{(m!)^2} \det \begin{pmatrix} a_{i_1,s_1} & \cdots & a_{i_1,s_m} \\ \vdots & \ddots & \vdots \\ a_{i_m,s_1} & \cdots & a_{i_m,s_m} \end{pmatrix} \cdot \det \begin{pmatrix} b_{j_1,s_1} & \cdots & b_{j_1,s_m} \\ \vdots & \ddots & \vdots \\ b_{j_m,s_1} & \cdots & b_{j_m,s_m} \end{pmatrix}$$

where $i_1,\ldots,i_{m+1} \in \{1,\ldots,l\}$ and $j_1,\ldots,j_{m+1} \in \{1,\ldots,J\}$. In particular, if the set $\{r_1,\ldots,r_{m+1}\}$ has fewer than $m$ distinct elements, then $f^A_{s_1,\ldots,s_{m+1}}$ are zero vectors for all $(s_1,\ldots,s_{m+1}) \in P_{\{r_1,\ldots,r_{m+1}\}}$, yielding that $\phi(A,B)_{r_1,\ldots,r_{m+1}}$ is the zero vector.

Hence, by Definition 2.2, the matrix $\Phi_{m,l}(A,B)$ has columns $\phi(A,B)_{r_1,\ldots,r_{m+1}}$, where $(r_1,\ldots,r_{m+1})$ satisfies (2.9). Thus, (3.15) coincides with (2.1).

4. Proofs related to Subsections 2.3 and 2.4. In this section we 1) prove Theorems 2.7 and 2.8; 2) show that the assumptions in Theorem 2.10 are more restrictive than the assumptions in Theorem 2.9, which implies the statement on uniqueness in Theorem 2.10; 3) prove that assumption (iii) in Theorem 2.10 is equivalent to assumption (iii) in Theorem 2.11; 4) prove statements (2.22)–(2.23); 5) prove Theorem 2.4.

4.1. Proofs of Theorems 2.7 and 2.8. In the sequel, $\omega(\lambda_1,\ldots,\lambda_R)^T$ denotes the number of nonzero entries of $[\lambda_1 \ldots \lambda_R]^T$. The following condition $(W_m)$ was introduced in [6, 7] in terms of $m$-th compound matrices. In this paper we will use the following (equivalent) definition of $(W_m)$.

**Definition 4.1.** We say that condition $(W_m)$ holds for the triplet of matrices $(A,B,C) \in \mathbb{R}^{I \times R} \times \mathbb{R}^{J \times R} \times \mathbb{R}^{K \times R}$ if $\omega(\lambda_1,\ldots,\lambda_R) \leq m - 1$ whenever

$$\text{rank}(AD\text{diag}(\lambda_1,\ldots,\lambda_R)B^T) \leq m - 1 \quad \text{and} \quad [\lambda_1 \ldots \lambda_R]^T \in \text{range}(C^T), \quad (4.1)$$

where $\text{diag}(\lambda_1,\ldots,\lambda_R)$ denotes a square diagonal matrix with $\lambda_1,\ldots,\lambda_R$ on the main diagonal.

Since the rank of the product $AD\text{diag}(\lambda_1,\ldots,\lambda_R)B^T$ does not exceed the rank of the factors and $\text{rank}(\text{diag}(\lambda_1,\ldots,\lambda_R)) = \omega(\lambda_1,\ldots,\lambda_R)$, we always have the implication

$$\omega(\lambda_1,\ldots,\lambda_R) \leq m - 1 \quad \Rightarrow \quad \text{rank}(AD\text{diag}(\lambda_1,\ldots,\lambda_R)B^T) \leq m - 1. \quad (4.2)$$

By Definition 4.1, condition $(W_m)$ holds for the triplet $(A,B,C)$ if and only if the opposite implication in (4.2) holds for all $[\lambda_1 \ldots \lambda_R]^T \in \text{range}(C^T) \subset \mathbb{R}^R$.

The following results on rank and uniqueness of one factor matrix have been obtained in [6].

**Proposition 4.2.** (see [6, Proposition 4.9]) Let $T = (t_{ijk})_{i,j,k=1} = |A,B,C|_R$, $r_C = K \leq R$. Assume that

(i) $k_C \geq 1$;
(ii) $A \odot B$ has full column rank;
(iii) conditions $(W_m)\ldots,(W_1)$ hold for the triplet of matrices $(A,B,C)$.
Thus, (4.5) takes the form
\[ r_T = R \text{ and the third factor matrix of } \mathcal{T} \text{ is unique.} \]

**Proposition 4.3.** (see [6, Corollary 4.10]) Let \( \mathcal{T} = (t_{ijk})_{i,j,k=1}^{I,J,K} = [A, B, C]_R, \)
\( r_C = K \leq R. \) Assume that
(i) \( k_C \geq 1; \)
(ii) \( A \odot B \) has full column rank;
(iii) \( \min(k_A, k_B) \geq m - 1; \)
(iv) condition \((W_m)\) holds for the triplet of matrices \((A, B, C)\).

Then \( r_T = R \) and the third factor matrix of \( \mathcal{T} \) is unique.

One can easily notice the similarity between the assumptions in Theorems 2.7–2.8 and the assumptions in Propositions 4.2–4.3. The proofs of Theorems 2.7–2.8 follow from Propositions 4.2–4.3 and the following lemma.

**Lemma 4.4.** Let \( A \in \mathbb{R}^{I \times R}, B \in \mathbb{R}^{J \times R}, \) and \( C \in \mathbb{R}^{K \times R}, \) \( r_C = K \leq R, \) \( k \leq m = R - K + 2, \) and let \( l \) be a nonnegative integer. Let also the matrix \( \Phi_{k,l}(A, B) \) be defined as in Definition 2.2, the matrix \( S_{k+l}(C) \) be defined as in Definition 2.3, and \( U \) be a matrix such that its columns form a basis for \( \text{range}(S_{k+l}(C)^T). \) Assume that

the matrix \( \Phi_{k,l}(A, B)U \) has full column rank. \hspace{1cm} (4.3)

Then condition \((W_k)\) holds for the triplet of matrices \((A, B, C)\).

**Proof.** Let (4.1) hold for \( m = k. \) We need to show that \( \omega(\lambda_1, \ldots, \lambda_R) \leq k - 1. \) Since \( [\lambda_1 \ldots \lambda_R]^T \in \text{range}(C^T) \) and \( r_C = K, \) there exists a unique vector \( x \in \mathbb{R}^K \) such that \( [\lambda_1 \ldots \lambda_R] = x^TC. \) Hence, we need to show that \( x \) is orthogonal to at least \( R - k + 1 \) columns of \( C. \)

By (4.1), there exist \( \tilde{A} \in \mathbb{R}^{I \times R} \) and \( \tilde{B} \in \mathbb{R}^{J \times R} \) such that

\[ A \text{Diag}(\lambda_1, \ldots, \lambda_R)B^T = \tilde{A} \tilde{B}^T \]

and \( \max(r_{\tilde{A}}, r_{\tilde{B}}) \leq k - 1. \) Since \( ab^T \lambda = a \circ b \circ \lambda, \) we can consider (4.4) as an equality of two PDs of an \( I \times J \times 1 \) tensor

\[ \sum_{r=1}^{R} a_r \circ b_r \circ \lambda_r = \sum_{r=1}^{R} \tilde{a}_r \circ \tilde{b}_r \circ 1. \]

Hence, by (2.1),

\[ \Phi_{k,l}(A, B)S_{k+l}(x^TC)^T = \Phi_{k,l}(A, B)S_{k+l}([\lambda_1 \ldots \lambda_R]^T) = \Phi_{k,l}(\tilde{A}, \tilde{B})S_{k+l}([1 \ldots 1]^T). \] \hspace{1cm} (4.5)

Since \( \max(r_{\tilde{A}}, r_{\tilde{B}}) \leq k - 1, \) it follows from Definition 2.2 that \( \Phi_{k,l}(\tilde{A}, \tilde{B}) \) is the zero matrix (cf. explanation at the end of Section 3). Besides, it easily follows from Definition 2.3 that

\[ S^T_{k+l}(C)(x \odot \cdots \odot x) = S^T_{k+l}(x^TC). \]

Thus, (4.5) takes the form

\[ \Phi_{k,l}(A, B)S^T_{k+l}(C)(x \odot \cdots \odot x) = \Phi_{k,l}(A, B)S^T_{k+l}(x^TC) = 0. \]
Hence, by (4.3), the vector $x \otimes \cdots \otimes x$ is orthogonal to the range of $S_{k+l}(C)$. In particular,
\[
(x \otimes \cdots \otimes x)^T \sum_{(x_1, \ldots, x_{k+l}) \in P(m, \ldots, m)} c_{x_1} \otimes \cdots \otimes c_{x_{m+l}} = (x^T c_{r_1}) \cdots (x^T c_{r_{k-1}})(x^T c_{r_k})^{l+1} = 0
\]
for all $(k+l)$-tuples $(r_1, \ldots, r_k)$ such that $1 \leq r_1 < \cdots < r_k \leq R$, yielding that $x$ is orthogonal to at least $R - k + 1$ columns of $C$. \qed

4.2. Proof of statement on rank and uniqueness in Theorem 2.10. In Lemma 4.5 below we prove that $\min(k_A, k_B) \geq m$. It is clear that condition $\min(k_A, k_B) \geq m$ and assumption (i) in Theorem 2.10 imply assumption (iii) in Theorem 2.8 and condition (2.16). Hence, by Theorem 2.9, $r_T = R$ and the CPD of tensor $T$ is unique.

**Lemma 4.5.** Let assumptions (i) and (iii) in Theorem 2.10 hold. Then $\min(k_A, k_B) \geq m$.

**Proof.** Assume to the contrary that $k_A < m$ or $k_B < m$. W.l.o.g. we assume that the first $m$ columns of $A$ are linearly dependent. We will get a contradiction with assumption (iii) by constructing a nonzero vector $f \in \text{range}(S_{m+l}(C)^T)$ such that $\Phi_{m,l}(A, B)f = 0$. Since $k_2 = K$, there exists $x \in \mathbb{R}^K$ such that
\[
x^T c_1 \neq 0, \ldots, x^T c_m \neq 0, \quad x^T c_{m+1} = \cdots = x^T c_R = 0.
\]
We set $f = S_{m+l}(C)^T(x \otimes \cdots \otimes x)$ and we index the entries of $f$ by $(m+l)$-tuples as in (2.9). One can easily show that $f$ has entries $(x^T c_{r_1}) \cdots (x^T c_{r_{m+l}})$. Hence, by (4.6),
\[
(x^T c_{r_1}) \cdots (x^T c_{r_{m+l}}) = 0, \text{ if } \{r_1, \ldots, r_{m+l}\} \setminus \{1, \ldots, m\} \neq \emptyset,
\]
\[
(x^T c_{r_1}) \cdots (x^T c_{r_{m+l}}) \neq 0, \text{ if } \{r_1, \ldots, r_{m+l}\} \setminus \{1, \ldots, m\} = \emptyset.
\]
On the other hand, by Definition 2.2 and the assumption of linear dependence of the vectors $a_1, \ldots, a_m$, the columns of $\Phi_{m,1}(A, B)$ indexed by the $(m+l)$-tuples (2.9) such that $\{r_1, \ldots, r_{m+l}\} \setminus \{1, \ldots, m\} = \emptyset$ are zero. Hence, $\Phi_{m,1}(A, B)f = 0$. \qed

4.3. Properties of the matrix $S_{m+l}(C)^T$. The following auxiliary Lemma will be used in Subsections 4.4 and 4.5. Since the proof is rather long and technical, it is included in the supplementary materials.

**Lemma 4.6.** Let $C \in \mathbb{R}^{K \times R}$, $k_C = K$, $m = R - K + 2$, $l \geq 0$, let $F$ satisfy (2.18)-(2.19), and let $F^{(m+l)}$ be defined by (2.24). Then
\begin{enumerate}
\item[(i)] $\dim(\ker(S_{m+l}(C)^T) \cap S_{m+l}(\mathbb{R}^{K^{m+l}})) = C_K^{-1}$;
\item[(ii)] $\ker(S_{m+l}(C)^T) \cap S_{m+l}(\mathbb{R}^{K^{m+l}}) = \text{range}(F^{(m+l)})$;
\item[(iii)] $\text{range}(S_{m+l}(C)^T) = S_{m+l}(C)^T(S_{m+l}(\mathbb{R}^{K^{m+l}}))$;
\item[(iv)] $\dim(\text{range}(S_{m+l}(C)^T)) = C_{K+m+l-1}^{-1} - C_K^{-1}$.  
\end{enumerate}

4.4. Proof of equivalence of Theorems 2.10 and 2.11. We prove that assumption (iii) in Theorem 2.10 is equivalent to assumption (iii) in Theorem 2.11. By (2.17), it is sufficient to prove that
\[
\dim\left(\ker(R_{m,l}(T)) \cap S_{m+l}(\mathbb{R}^{K^{m+l}})\right) \geq C_K^{-1} + 1 \iff \text{the matrix } \Phi_{m,l}(A, B)U_m \text{ does not have full column rank}.  
\]
To prove (4.7) we will use the following result for \( X := \Phi_{m,l}(A, B) \), \( Y := S_{m+l}(C)^T \), and \( E := S^{m+l}(\mathbb{R}^{K^{m+l}}) \): if \( E \) is a subspace and \( X \) and \( Y \) are matrices such that \( XY \) is defined, then

\[
\dim (\ker(XY) \cap E) \geq \dim (\ker(Y) \cap E) + 1 \iff \\
\text{there exists a nonzero vector } f \in E \setminus \ker(Y) \text{ such that } XYf = 0. 
\]

(4.8)

We have

\[
\dim \left( \ker(R_{m,l}(T)) \right) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) \geq C_R^{K-1} + 1 \overset{(2.1)}{\iff} \\
\left\{ \begin{array}{l}
\dim \left( \ker(\Phi_{m,l}(A, B)S_{m+l}(C)^T) \right) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) \geq C_R^{K-1} + 1 = \\
\dim \left( \ker(S_{m+l}(C)^T) \right) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) + 1 \\
\text{there exists a nonzero vector } f \in S^{m+l}(\mathbb{R}^{K^{m+l}}) \setminus \ker(S_{m+l}(C)^T) \\
\text{such that } \Phi_{m,l}(A, B)S_{m+l}(C)^Tf = 0 \\
\end{array} \right. \\
\overset{(4.8)}{\iff} \\
\text{the matrix } \Phi_{m,l}(A, B)U_m \text{ does not have full column rank,}
\]

where the equality in the second statement holds by Lemma 4.6 (i) and the last equivalence follows from \( \text{range}(U_m) = \text{range}(S_{m+l}(C)^T) \).

4.5. Proof of the statement on algebraic computation in Theorem 2.10.

The overall procedure that constitutes the proof of the statement on algebraic computation is summarized in Algorithm 2 and explained in Subsection 2.4.2. In this subsection we prove statements (2.22)–(2.23).

**Lemma 4.7.** Let assumptions (i) and (iii) in Theorem 2.11 hold and let \( F \) satisfy (2.18)–(2.19). Then (2.22)–(2.23) hold.

**Proof.** The implication \( k_C = K \Rightarrow (2.22) \) was proved in [8, Proposition 1.10]. In Subsection 4.4 we proved that assumption (iii) in Theorem 2.10 holds. By (2.1), Theorem 2.10 (iii), and Lemma 4.6 we have

\[
\ker(R_{m,l}(T)) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) = \ker(\Phi_{m,l}(A, B)S_{m+l}(C)^T) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) = \\
\ker(S_{m+l}(C)^T) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) = \text{range } \left( F^{(m+l)} \right),
\]

which completes the proof of (2.23). \( \square \)

4.6. Proof of Theorem 2.4. We check the assumptions in Theorem 2.11 for \( m = 2 \). Assumption (i) holds since \( r_C = K = R \) implies \( k_C = K \) and assumption (iii) coincides with (2.13). To prove assumption (ii) we assume to the contrary that \( (A \odot B)[\lambda_1 \ldots \lambda_R]^T = 0 \). Then \( \text{rank}(A\text{Diag}(\lambda_1, \ldots, \lambda_R)B^T) = 0 \). In Subsection 4.4 we explained that assumption (iii) in Theorem 2.10 also holds. Hence, by Lemma 4.4, condition (W2) holds for the triplet \( (A, B, C) \). Hence, at most one of the values \( \lambda_1, \ldots, \lambda_R \) is not zero. If such a \( \lambda_r \) exists, then \( a_r = 0 \) or \( b_r = 0 \) yielding that \( \min(k_A, k_B) = 0 \). On the other hand, by Lemma 4.5, \( \min(k_A, k_B) \geq 2 \), which is a contradiction. Hence, \( \lambda_1 = \cdots = \lambda_R = 0 \).

5. Discussion. A number of conditions (called \( (K_m) \), \( (C_m) \), \( (U_m) \), and \( (W_m) \)) for uniqueness of CPD of a specific tensor have been proposed in [6, 7]. It was shown that each subsequent condition in \( (K_m), \ldots, (W_m) \) is more general than the preceding one, but harder to use. Verification of conditions \( (K_m) \) and \( (C_m) \) reduces to the
computation of matrix rank. In contrast, conditions \((U_m)\) and \((W_m)\) are not easy to check for a specific tensor but hold automatically for generic tensors of certain dimensions and rank [9].

In this paper we have proposed new sufficient conditions for uniqueness that can be verified by the computation of matrix rank, are more relaxed than \((K_m)\) and \((C_m)\), but cannot be more relaxed than \((W_m)\). Nevertheless, examples illustrate that in many cases the new conditions may be considered as an “easy to check analogue” of \((U_2) \iff (W_2))\) and \((W_m)\).

We have also proposed an algorithm to compute the factor matrices. The algorithm relies only on standard linear algebra, and has as input the tensor \(T\), the tensor rank \(R\), and a nonnegative integer parameter \(l\). The algorithm basically reduces the problem to the construction of a \(C_{K+m+l-1}^{m+l} \times C_{K+m+l-1}^{m+l}\) matrix \(Q\), the computation of its \(C_{K-1}^{R-1}\)-dimensional null space, and the GEVD of a \(C_{K-1}^{R-1} \times C_{K-1}^{R-1}\) matrix pencil, where \(m = R - K + 2\). For \(l = 0\), Algorithms 1 and 2 coincide with algorithms from [5] and [8], respectively. Our derivation is different from the derivations in [5] and [8] but has the same structure: from the CPD \(T = [A, B, C]\) we derive a set of equations that depend only on \(C\); we find \(C\) from the new system by means of GEVD, and then recover \(A\) and \(B\) from \(T\) and \(C\).

It is interesting to note that the new algorithm (with \(l = 1\)) computes the CPD of a generic \(3 \times 7 \times 12\) tensor of rank 12 in less than 1 second while optimization-based algorithms (we checked a Gauss-Newton dogleg trust region method) fail to find the solution in a reasonable amount of time.

We have demonstrated that our algorithm (with \(l \leq 2\)) can find the CPD of a generic \(I \times J \times K\) tensor of rank \(R\) if \(R \leq K \leq (I - 1)(J - 1)\) and \(R \leq 24\). We conjecture that the algorithm (possibly with \(l \geq 3\)) can also find the CPD for \(R \geq 25\). (It is known that the CPD of a generic tensor is not unique if \(R > (I - 1)(J - 1)\).) In that case the \(C_{K+m+l-1}^{m+l} \times C_{K+m+l-1}^{m+l}\) matrix \(Q\) becomes large and the computation, as it is proposed in the paper, becomes infeasible. Since the null space of \(Q\) is just \(R\)-dimensional the approach may possibly be scaled by using iterative methods to compute the null space.

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S.1. Two auxiliary lemmas. For easy referencing, we present a combinatorial identity and a simple generalization of the rank-nullity theorem (both results are no doubt known, but we do not know a reference to them).

Lemma S.1.1. Let $m, l, R$ be nonnegative integers. Then

$$C^0_{l+1} C_{R}^{m+l} + C^1_{l+1} C_{R}^{m+l-1} + \cdots + C^l_{l+1} C_{R}^{m+1} + C_{l+1}^{l+1} C_{R}^{m-1} = C_{R+l+1}^{m+l},$$

where we follow the convention that $C_{R}^{m+l-p} = 0$ if $m+l-p > R$.

Proof. The identity can be obtained by equating the coefficients of $x^{m+l}$ in $(1 + x)^{l+1} (1 + x)^R$ and $(1 + x)^{R+l+1}$. \( \Box \)

Lemma S.1.2. Let $X$ be a matrix and $E$ a subspace such that $\text{range}(X^T) \subseteq E$. Then

(i) $\dim(\ker(X) \cap E) + \dim(\ker(X)) = \dim E$;

(ii) $\text{range}(X) = \ker(X)$,

where the subspace $\ker(X)$ denotes the image of $E$ under $X$.

Proof. Let $P$ be a matrix whose columns form a basis for the subspace $E$. Then $\dim(\ker(X) \cap E) = \dim(\ker(XP))$, $X(E) = \text{range}(XP)$, and the matrix $XP$ has $\dim E$ columns. Hence, by the rank-nullity theorem,

$$\dim(\ker(X) \cap E) + \dim(\ker(X)) = \dim(\ker(XP)) + \dim(XP) = \dim E.$$  

Since, $\text{range}(X^T) \subseteq \text{range}(P)$, it follows that $\text{range}(X) = \text{range}(XP) = X(E)$. \( \Box \)

S.2. A lower bound on the rank of the matrix $S_{m+l}(C)$. We need some additional notation. Let $m \geq 2$, $l \geq 0$, $p \geq 0$ and $m \leq m+l-p \leq R$. With an $(m+l-p)$-tuple

$$(i_1, \ldots, i_{m+l-p})$$

such that $1 \leq i_1 < \cdots < i_{m+l-p} \leq R$ we associate the set

$$E_{i_1, \ldots, i_{m+l-p}} := \{(i_1, \ldots, i_{m+l-p}, q_1, \ldots, q_p) : 1 \leq q_1 \leq \cdots \leq q_p \leq 2 + l - p \}.$$  

In other words, the set $E_{i_1, \ldots, i_{m+l-p}}$ consists of the $(m+l)$-tuples that are obtained by merging the $(m+l-p)$-tuple $(i_1, \ldots, i_{m+l-p})$ with $p$-combinations with repetitions of the set $\{i_1, \ldots, i_{m+l-p}\}$. It is clear that for a fixed $p$ there exist $C_{R}^{m+l-p}$ sets $E_{i_1, \ldots, i_{m+l-p}}$ and each set $E_{i_1, \ldots, i_{m+l-p}}$ contains $C_{2+l-p+p+1}^{p} = C_{l+1}^{p}$ $(m+l)$-tuples. Let $E$ be the union of all sets $E_{i_1, \ldots, i_{m+l-p}}$. Then, by Lemma S.1.1 the set $E$ contains exactly

$$C^0_{l+1} C_{R}^{m+l} + C^1_{l+1} C_{R}^{m+l-1} + \cdots + C^l_{l+1} C_{R}^{m+1} + C_{l+1}^{l+1} C_{R}^{m-1} = C_{R+l+1}^{m+l},$$

$(m+l)$-tuples (we follow the convention that $C_{R}^{m+l-p} := 0$ if $m+l-p > R$). Since, by construction, each $(m+l)$-tuple of $E_{i_1, \ldots, i_{m+l-p}}$ contains exactly $m+l-p \geq m$ distinct
elements, it follows that each \((m+l)\)-tuple of \(E\) contains at least \(m\) distinct elements.
Let \(S_{m+l}(C)\) denote the \(K^m \times (C_{m+l}^m - C_{m-1}^m)\) matrix with columns \((2.10)\), where \((r_1, \ldots, r_{m+l}) \in E\). Then \(S_{m+l}(C)\) is a submatrix of \(S_{m+l}(C)\). We have the following lemma.

**Lemma S.2.1.** Let \(C \in \mathbb{R}^{K \times R}\), \(k_C = K\), and \(m = R - K + 2\). Then the matrix \(S_{m+l}(C)\) has full column rank. In particular, \(r_s_{m+l}(C) \geq C_{R+l+1} - C_{R-1}^m\).

**Proof.** Suppose that there exists \(f \in \mathbb{R}^{C_{R+l+1} - C_{R-1}^m}\) such that \(S_{m+l}(C)f = 0\). We show that \(f = 0\). We assume that the entries of \(f\) are indexed by \((m+l)\)-tuples \((r_1, \ldots, r_{m+l}) \in E\), that is, in \(S_{m+l}(C)f = 0\) the column of \(S_{m+l}(C)\) associated with the \((m+l)\)-tuple \((i_1, \ldots, i_{m+l-p}, i_{q_1}, \ldots, i_{q_p})\) is multiplied by \(f_{i_1, \ldots, i_{m+l-p}, i_{q_1}, \ldots, i_{q_p}}\).

To show that all entries \(f_{i_1, \ldots, i_{m+l-p}, i_{q_1}, \ldots, i_{q_p}}\) are zero we proceed by induction on \(p = l, l-1, \ldots, \max(0, m + l - R):\) in the \(p\)th step we assume that the identities

\[
f_{i_1, \ldots, i_{m+l-p}, i_{q_1}, \ldots, i_{q_p}} = 0 \quad 1 \leq i_1 < \cdots < i_{m+l-p} \leq R, \quad 1 \leq q_1 \leq \cdots \leq q_p \leq 2 + l - p
\]

hold for \(\bar{p} = l, l-1, \ldots, p-1\) and prove that the identities hold for \(\bar{p} = p\).

(i) Induction hypothesis: \(p = l\). We show that

\[
f_{i_1, \ldots, i_{m+l-p}, i_{q_1}, \ldots, i_{q_p}} = 0 \quad \text{for} \quad 1 \leq i_1 < \cdots < i_m \leq R, \quad 1 \leq q_1 \leq \cdots \leq q_l \leq 2.
\]

We give the proof for the case \(i_1 = 1, \ldots, i_m = m\), the other cases follow similarly. Thus, we show that \(f_{1, \ldots, m, q_1, \ldots, q_l} = 0\) for \(1 \leq q_1 \leq \cdots \leq q_l \leq 2\).

Since \(k_C = K\), the square matrix \([c_1, c_2, c_{m+1}, \ldots, c_R]\) is nonsingular. Let \(u_1\) and \(u_2\) denote the first and the second column of \([c_1, c_2, c_{m+1}, \ldots, c_R]^T\), respectively. Then

\[
\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} [c_1, c_2, c_{m+1}, \ldots, c_R] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.
\]

(S.2.1)

Let \(x = t_1 u_1 + t_2 u_2\). Then the vector \(x^{(m+l)} := x \otimes \cdots \otimes x\) is orthogonal to the columns of the matrix \(S_{m+l}(C)\) indexed by the \((m+l)\)-tuples \((r_1, \ldots, r_{m+l}) \in E \setminus E_{1, \ldots, m}\). Indeed, if \(\{r_1, \ldots, r_{m+l}\} \setminus \{1, \ldots, m\} \neq \emptyset\), then by \((2.10)\) and \((S.2.1)\),

\[
x^{(m+l)}^T \frac{1}{(m+l)!} \sum_{\{s_1, \ldots, s_{m+l}\} \in P_{(r_1, \ldots, r_{m+l})}} c_{s_1} \otimes \cdots \otimes c_{s_{m+l}} = (x^T c_{r_1}) \cdots (x^T c_{r_{m+l}}) = 0.
\]

(S.2.2)

Hence

\[
0 = x^{(m+l)}^T S_{m+l}(C)f = \sum_{\{r_1, \ldots, r_{m+l}\} \in E_{1, \ldots, m}} (x^T c_{r_1}) \cdots (x^T c_{r_{m+l}}) f_{r_1, \ldots, r_{m+l}} = \sum_{1 \leq q_1 \leq \cdots \leq q_l \leq 2} (x^T c_{1}) \cdots (x^T c_{m}) (x^T c_{q_1}) \cdots (x^T c_{q_l}) f_{1, \ldots, m, q_1, \ldots, q_l} = \quad \text{(S.2.3)}
\]

\[
(x^T c_{1}) \cdots (x^T c_{m}) \sum_{1 \leq q_1 \leq \cdots \leq q_l \leq 2} (x^T c_{q_1}) \cdots (x^T c_{q_l}) f_{1, \ldots, m, q_1, \ldots, q_l}.
\]

Since \(k_C = K\), at most one of the vectors \(u_1\) and \(u_2\) can be orthogonal to any of the vectors \(c_3, \ldots, c_m\). Hence,

\[
(x^T c_{1}) \cdots (x^T c_{m}) = t_1 t_2 (t_1 u_1^T c_3 + t_2 u_2^T c_3) \cdots (t_1 u_1^T c_m + t_2 u_2^T c_m) \neq 0
\]
for generic \( t_1, t_2 \in \mathbb{R} \). Hence, by (S.2.3),

\[
\sum_{1 \leq q_1 \leq \cdots \leq q_l \leq 2} (x^T c_{q_1}) \cdots (x^T c_{q_l}) f_{1, \ldots, m, q_1, \ldots, q_l} = 0 \quad (S.2.4)
\]

for generic \( t_1, t_2 \in \mathbb{R} \). By construction of \( x \), the \( l + 1 \) products \((x^T c_{q_1}) \cdots (x^T c_{q_l})\), \( 1 \leq q_1 \leq \cdots \leq q_l \leq 2 \), coincide with the monomials \( t_1^{q_1} t_2^{q_2} t_1^{l-1} t_2^{q_2} \). Thus, identity (S.2.4) expresses the fact that a polynomial in \( t_1 \) and \( t_2 \) with coefficients \( f_{1, \ldots, m, q_1, \ldots, q_l} \) vanishes for generic \( t_1, t_2 \in \mathbb{R} \). It is well known that this is possible only if the polynomial is identically zero, yielding that \( f_{1, \ldots, m, q_1, \ldots, q_l} = 0 \) for \( 1 \leq q_1 \leq \cdots \leq q_l \leq 2 \).

(ii) Inductive step. We show that

\[
f_{i_1, \ldots, i_{m+l-p}, i_1, \ldots, i_p} = 0 \quad \text{for} \quad 1 \leq i_1 < \cdots < i_{m+l-p} \leq R, \quad 1 \leq q_1 \leq \cdots \leq q_p \leq 2 + l - p
\]
or, equivalently, that \( f_{i_1, \ldots, i_{m+l-p}, i_1, \ldots, i_p} = 0 \) for

\[
(i_1, \ldots, i_{m+l-p}, i_1, \ldots, i_p, q_1, \ldots, q_p) \in \bigcup_{1 \leq i_1 < \cdots < i_{m+l-p} \leq R} E_{i_1, \ldots, i_{m+l-p}}.
\]

We give the proof for the case \( i_1 = 1, \ldots, i_{m+l-p} = m + l - p \), the other cases follow similarly. Thus, we show that \( f_{1, \ldots, m+l-p, q_1, \ldots, q_p} = 0 \) for \( 1 \leq q_1 \leq \cdots \leq q_p \leq 2 + l - p \). The derivation is very similar to that of the induction hypothesis.

Since \( k_C = K \), the \( K \times K \) matrix \([c_1 \ldots c_{2+l-p} c_{m+l-p+1} \ldots c_R]\) is nonsingular.

Let \( u_1, \ldots, u_{2+l-p} \) denote the first \( 2 + l - p \) columns of \([c_1 \ldots c_{2+l-p} c_{m+l-p+1} \ldots c_R]^{-T}\).

Then

\[
\begin{bmatrix}
  u_1^T \\
  \vdots \\
  u_{2+l-p}^T
\end{bmatrix}
\begin{bmatrix}
  c_1 & \ldots & c_{2+l-p} & c_{m+l-p+1} & \ldots & c_R
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 1 & 0 & \ldots & 0
\end{bmatrix}. \quad (S.2.5)
\]

Let \( x = t_1 u_1 + \cdots + t_{2+l-p} u_{2+l-p} \). Let

\[
E_{\tilde{p}>p} := \bigcup_{\tilde{p}<p} \bigcup_{1 \leq i_1 < \cdots < i_{m+l-p} \leq R} E_{i_1, \ldots, i_{m+l-p}}
\]

and let the sets \( E_{\tilde{p}>p} \) and \( E_{\tilde{p}=p} \) be defined similarly. Then \( E = E_{\tilde{p}>p} \cup E_{\tilde{p}>p} \cup E_{\tilde{p}=p} \).

Then, by (2.10), (S.2.2), and (S.2.5), the vector \( x^{(m+l)} := x \otimes \cdots \otimes x \) is orthogonal to the columns of the matrix \( S_{m+l}(C) \) indexed by the \( (m+l) \)-tuples

\[
(r_1, \ldots, r_{m+l}) \in E_{\tilde{p}>p} \cup (E_{\tilde{p}=p} \setminus E_{1, \ldots, m+l-p}).
\]

Hence, similarly to (S.2.3) we obtain

\[
0 = x^{(m+l)} S_{m+l}(C) f = \sum_{(r_1, \ldots, r_{m+l}) \in E} (x^T c_{r_1}) \cdots (x^T c_{r_{m+l}}) f_{r_1, \ldots, r_{m+l}} = \\
\sum_{(r_1, \ldots, r_{m+l}) \in E_{\tilde{p}>p} \cup E_{1, \ldots, m+l-p}} (x^T c_{r_1}) \cdots (x^T c_{r_{m+l}}) f_{r_1, \ldots, r_{m+l}}.
\]
Since, by the induction assumption, \( f_{r_1,...,r_{m+1}} = 0 \) for \((r_1,\ldots,r_{m+l})\in E_{l-p}\), we have

\[
0 = \sum_{(r_1,\ldots,r_{m+1})\in E_{l-1+m+m+\ldots}} (x^T c_{r_1}) \cdots (x^T c_{r_{m+1}}) f_{r_1,\ldots,r_{m+1}} = \\
\sum_{1\leq q_1\leq \cdots \leq q_p \leq 2+l-p} \left( x^T c_1 \right) \cdots \left( x^T c_{m+1-p} \right) \left( x^T c_{q_1} \right) \cdots \left( x^T c_{q_p} \right) f_{1,\ldots,m+l-p,q_1,\ldots,q_p} = \\
\left( x^T c_1 \right) \cdots \left( x^T c_{m+1-2} \right) \sum_{1\leq q_1\leq \cdots \leq q_p \leq 2+l-p} \left( x^T c_{q_1} \right) \cdots \left( x^T c_{q_p} \right) f_{1,\ldots,m+l-p,q_1,\ldots,q_p}.
\]

(S.2.6)

Since \(k_{C} = K\), at most \(1+l-p\) of the vectors \(u_1,\ldots,u_{2+l-p}\) can be orthogonal to any of the vectors \(c_{3+l-p},\ldots,c_{m+l-p}\). Hence,

\[
(x^T c_1) \cdots (x^T c_{m+l-p}) = \\
t_1 \cdots t_{2+l-p} (t_1 u_1^T c_{3+l-p} + \cdots + t_{2+l-p} u_{2+l-p}^T c_{3+l-p}) \cdots \\
(t_1 u_1^T c_{3+l-p} + \cdots + t_{2+l-p} u_{2+l-p}^T c_{m+l-p}) \neq 0
\]

(S.2.7)

for generic \(t_1,\ldots,t_{2+l-p} \in \mathbb{R}\). Hence, by (S.2.6),

\[
\sum_{1\leq q_1\leq \cdots \leq q_p \leq 2+l-p} (x^T c_{q_1}) \cdots (x^T c_{q_p}) f_{1,\ldots,m+l-p,q_1,\ldots,q_p} = 0
\]

(S.2.8)

for generic \(t_1,\ldots,t_{2+l-p} \in \mathbb{R}\). By construction of \(x\), the \(C_{l+1}^{p}\) products \((x^T c_{q_1}) \cdots (x^T c_{q_p})\), \(1\leq q_1 \leq \cdots \leq q_p \leq 2+l-p\), coincide with the monomials \(\{t_1^{q_1} \cdots t_{2+l-p}^{q_p}\}_{\alpha_1+\cdots+\alpha_{2+l-p}=p}\). Thus, identity (S.2.8) expresses the fact that a polynomial in \(t_1,\ldots,t_{2+l-p}\) with coefficients \(f_{1,\ldots,m+l-p,q_1,\ldots,q_p}\) vanishes for generic \(t_1,\ldots,t_{2+l-p} \in \mathbb{R}\). It is well known that this is possible only if the polynomial is identically zero, yielding that \(f_{1,\ldots,m+l-p,q_1,\ldots,q_p} = 0\) for \(1 \leq q_1 \leq \cdots \leq q_p \leq 2+l-p\). \(\blacksquare\)

**S.3. A lower bound on the dimension of \(\text{ker}(S_{m+l}(C)^T) \cap S^{m+l}(\mathbb{R}^{K^{m+l}})\).**

In this subsection we prove the following result.

**Lemma S.3.1.** Let \(C \in \mathbb{R}^{K \times R}\), \(k_{C} = K\), \(m = R - K + 2\), \(l \geq 0\), let \(F\) satisfy (2.19)–(2.20), and let \(F^{(m+l)}\) be defined by (2.25). Then

(i) The matrix \(F^{(m+l)}\) has full column rank, that is \(r_{F^{(m+l)}} = C_{R}^{K-1}\);

(ii) \(\text{ker}(S_{m+l}(C)^T) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) \supset \text{range}(F^{(m+l)})\).

In particular, \(\dim \left( \text{ker}(S_{m+l}(C)^T) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) \right) \geq C_{R}^{K-1}\).

**Proof.** Statement (i) was proved in \(\Box\) Proposition 1.10.]

Let \(f\) be a column of the matrix \(F\). Then \(f^{(m+l)}\) is a column of \(F^{(m+l)}\). It is clear that \(f^{(m+l)} \in S^{m+l}(\mathbb{R}^{K^{m+l}})\). To prove (ii) we need to show that \(S_{m+l}(C)^T f^{(m+l)} = 0\).

By Definition 2.3, the \((r_1,\ldots,r_{m+l})\)th entry of the vector \(S_{m+l}(C)^T f^{(m+l)}\) is

\[
\left( \frac{1}{(m+l)!} \sum_{(s_1,\ldots,s_{m+l}) \in P_{(r_1,\ldots,r_{m+l})}} c_{s_1} \otimes \cdots \otimes c_{s_{m+l}} \right)^T f^{(m+l)} = \sum_{(s_1,\ldots,s_{m+l}) \in P_{(r_1,\ldots,r_{m+l})}} (c_{s_1}^T f) \cdots (c_{s_{m+l}}^T f) = (c_{r_1}^T f) \cdots (c_{r_{m+l}}^T f).
\]
Since the vector $f$ is orthogonal to exactly $K - 1$ columns of $C$, the fact that at least $m$ indices of $r_1, \ldots, r_{m+l}$ are distinct, and $m + l \geq R - K + 2 + l > R - (K - 1)$ it follows that $(c^T_{r_1}f) \cdots (c^T_{r_{m+l}}f) = 0$, which completes the proof of (ii). □

S.4. Proof of Lemma 4.6. We set $X = S_{m+l}(C)^T$ and $E = S^{m+l}(\mathbb{R}^{K^{m+l}})$ in Lemma S.1.2. Then statement (iii) follows from Lemma S.1.2 (ii). Besides, by Lemma S.1.2 (i),

$$\dim \left( \ker(S_{m+l}(C)^T) \cap S^{m+l}(\mathbb{R}^{K^{m+l}}) \right) + \text{range}(S_{m+l}(C)^T) = C^{K-1}_R.$$  \hfill (S.4.1)

Statements (i), (ii), and (iv) follow from (S.4.1), Lemmas S.2.1–S.3.1 and the identity

$C^{K-1}_R = C^{m-1}_R$.

REFERENCES

[1] I. Domanov and L. De Lathauwer. Canonical polyadic decomposition of third-order tensors: reduction to generalized eigenvalue decomposition. *SIAM J. Matrix Anal. Appl.*, 35(2):636–660, 2014.