ERGODICITY OF NON-AUTONOMOUS DISCRETE SYSTEMS WITH NON-UNIFORM EXPANSION

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Abstract. We study the ergodicity of non-autonomous discrete dynamical systems with non-uniform expansion. As an application we get that any uniformly expanding finitely generated semigroup action of $C^{1+\alpha}$ local diffeomorphisms of a compact manifold is ergodic with respect to the Lebesgue measure. Moreover, we will also prove that every exact non-uniform expandable finitely generated semigroup action of conformal $C^{1+\alpha}$ local diffeomorphisms of a compact manifold is Lebesgue ergodic.

1. Ergodicity of finitely generated semigroup actions with non-uniform expansion

A local $C^r$-diffeomorphism $f : M \to M$ of a boundaryless compact differentiable manifold $M$ is said to be uniformly expanding if in some smooth metric $f$ stretches every tangent vector. To be precise, if for some choice of a Riemannian metric $\| \cdot \|$, there is $0 < \sigma < 1$ such that

$$\| Df(x)^{-1} \| < \sigma \quad \text{for all } x \in M.$$ 

In [23], Sullivan and Shub proved that every $C^{1+\alpha}$ uniformly expanding circle local diffeomorphism is ergodic with respect to Lebesgue measure. On the other hand, the regularity of this result cannot be improved. Indeed, Quas constructed in [20] a $C^1$ uniformly expanding map of the circle which preserves Lebesgue measure, but for which Lebesgue measure is non-ergodic. Although rather folklore is the extension to greater dimension of the Sullivan procedure, a rigorous proof that every $C^{1+\alpha}$ uniformly expanding local diffeomorphisms of $M$ is Lebesgue-ergodic can be easily deduced from [16, Theorem 1.1(c), pg. 167].

We will extend the usual definition of a uniformly expanding map to a semigroup $\Gamma$ finitely generated by local diffeomorphisms $f_1, \ldots, f_d$. Consider $\Omega = \{1, \ldots, d\}^\mathbb{N}$. For a given sequence $\omega = \omega_1\omega_2 \cdots \in \Omega$ we define the orbital branch corresponding to $\omega$ by

$$f^n_\omega = f_{\omega_n} \circ \cdots \circ f_{\omega_2} \circ f_{\omega_1} \quad \text{for all } n \geq 1.$$ 

We say that the action of $\Gamma$ on $M$ is uniformly expanding (along an orbital branch) if there exist $\omega \in \Omega$, $\lambda > 1$ and $C > 0$ such that for every $x \in M$,

$$\| Df^n_\omega(x)v \| \geq C\lambda^n \|v\| \quad \text{for all } v \in T_xM \text{ and } n \geq 1.$$ 

(1)

Finitely generated semigroup actions by uniformly expanding maps have been previously considered in [22]. Observe that (1) is more general an include semigroup non-necessarily generated by expanding maps. In order to extend the above result about the ergodicity of the Lebesgue measure for random uniformly expanding semigroup actions we need first some definitions.
A set \( A \subset M \) is \( \Gamma \)-invariant set if \( f(A) \subset A \) for all \( f \in \Gamma \). We say that the semigroup action of \( \Gamma \) on \( M \) is ergodic with respect to Lebesgue measure if \( m(A) \in \{0, 1\} \) for all \( \Gamma \)-invariant set \( A \) of \( M \) where \( m \) denotes the normalized Lebesgue measure of \( M \).

**Theorem A.** Every uniformly expanding finitely generated semigroup action of \( C^{1+\alpha} \) local diffeomorphisms of a compact manifold is ergodic with respect to Lebesgue measure.

The \( C^{1+\alpha} \)-regularity assumption behind the ergodicity theorems essentially related to the bounded distortion property which guarantees the preservation of density by the dynamics. There are many examples that show that \( C^{1} \)-regularity condition alone is not enough (see for instance \([6, 20]\)). For uniformly expanding actions of \( C^{2} \) endomorphisms the Theorem can be deduced from \([14, \text{Theorem 2.2}]\). We will get this theorem (for \( C^{1+\alpha} \) local diffeomorphisms), as a consequence of the following result which requires to introduce a generalization of uniformly expanding actions.

We say that the action of \( \Gamma \) is non-uniformly expanding (along an orbital branch) if there is \( \omega = \omega_1 \omega_2 \cdots \in \Omega \) such that for \( m \)-almost every \( x \in M \),

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df_{\omega_i}(f_{\omega_i}(x))^{-1}\| < 0. \tag{2}
\]

The action of \( \Gamma \) is said to be exact if for every open set \( B \) of \( M \) there are maps a sequence of maps \((g_n)_n\) in \( \Gamma \) such that

\[
M = \bigcup_{n \in \mathbb{N}} g_n(B) \quad \text{modulo a set of zero } m\text{-measure.}
\]

**Theorem B.** Every exact non-uniformly expanding finitely generated semigroup action of \( C^{1+\alpha} \) local diffeomorphisms of a compact manifold is ergodic with respect to Lebesgue measure.

It is clear that there are no uniformly expanding semigroup actions of diffeomorphisms. Indeed, by definition, there exist \( \omega \) and \( n \geq 1 \) large enough such that \( \|Df_{\omega_i}(x)^{-1}\| < 1 \) for all \( x \in M \). In other words, there exists an uniformly expanding map \( g \) in the semigroup \( \Gamma \) which forbids \( \Gamma \) to be a semigroup of diffeomorphisms. In fact, we will show that there are no non-uniformly expanding finitely generated semigroup actions of diffeomorphisms. Because of this, in \([10]\) the authors introduced a weak form of non-uniform expansion. Namely, they ask the existence of a constant \( a > 0 \) such that for \( m \)-almost every \( x \in M \) there is \( \omega \in \Omega \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df_{\omega_i}(f_{\omega_i}(x))^{-1}\| < -a. \tag{3}
\]

In this case we say that the action of \( \Gamma \) is non-uniformly strong expandable. They constructed a large class of examples of semigroup action of diffeomorphisms satisfying this non-uniform expansion. They proved the ergodicity of a finitely generated non-uniformly expanding action with a finite Markov partition. However, the existence of finite Markov partitions for finitely generated expanding actions seems to be crucial assumption, because they have
only finitely generated Markov partitions under even strong condition of conformality (see [17]).

In the recent paper [21], Theorem A, Rashid and Zamani claim that every non-uniformly strong expandable transitive finitely generated semigroup action of conformal local \( C^{1+a} \) diffeomorphisms is ergodic with respect to Lebesgue. However, the proof only works for group actions of diffeomorphisms (arguments in pg. 8, lines 3-4 in the proof of Theorem A cannot be correctly applied for forward invariant sets). Nevertheless, modifying slightly the assumptions replacing transitivity by exactness one can recover easily the result for semigroups. In fact, we will obtain this result assuming a weaker notion of non-uniformly expansion. Namely, we assume that the action of \( \Gamma \) is non-uniformly expandable, that is, for \( m \)-almost every \( x \in M \) there exists \( \omega \in \Omega \) such that (2) holds.

**Theorem C.** Every exact non-uniformly expandable finitely generated semigroup action of conformal \( C^{1+a} \) local diffeomorphisms of a compact manifold is ergodic with respect to Lebesgue measure.

Recall that a local diffeomorphism \( g \) is said to be conformal if there exists a function \( a : M \to \mathbb{R} \) such that for all \( x \in M \) we have that \( Dg(x) = a(x) \text{Isom}(x) \), where \( \text{Isom}(x) \) denotes an isometry of \( T_xM \). From the above result one obtains as a corollary the main result of [5] about the ergodicity of the expanding minimal semigroup actions of diffeomorphisms. A semigroup action generated by \( C^1 \)-diffeomorphisms \( f_1, \ldots, f_d \) of \( M \) is said to be expanding if for every \( x \in M \) there exists \( h \) in the inverse semigroup (the semigroup generated by inverse maps \( f_1^{-1}, \ldots, f_d^{-1} \)) such that \( \|Dh(x)^{-1}\| < 1 \). It is not difficult to see that if the semigroup action is expanding and minimal then action of the inverse semigroup is non-uniformly expandable and exact. Hence, by the above result one gets that the action is ergodic with respect to Lebesgue measure whether \( f_1, \ldots, f_d \) are conformal \( C^{1+a} \)-diffeomorphisms ([5, Thm. B]). We provide more details and new examples where Theorem C applies in the last section of this work.

Observe that conditions (1), (2) and (3) only require the existence of a sequence of functions satisfying the corresponding property. This is in fact because the above results are actually a consequence of an abstract theory in the context of non-autonomous discrete dynamical systems in compact metric spaces with non-uniform expansion. In the next section, we will develop this theory and in §3 we will provide the main results for non-autonomous discrete dynamical systems. After that in §4 we obtain as a consequence the above results.

2. Non-autonomous discrete dynamical systems with non-uniform expansion

A non-autonomous discrete dynamical system is a pair \( (M, f_{1,\omega}) \) where \( M \) is a compact metric space and \( f_{1,\omega} = (f_n)_{n \in \mathbb{N}} \) is a sequence of continuous maps from \( M \) to itself. As it is usual, for each \( k \in \mathbb{N} \), we denote by \( f_{k,\omega} \) the sequence of maps \( f_{k+n} : M \to M \) for \( n \in \mathbb{N} \) and

\[
    f^0_k = \text{id} \quad \text{and} \quad f^n_k = f_{k+n-1} \circ \cdots \circ f_{k+1} \circ f_k \quad n \in \mathbb{N}.
\]

Associated with this system we have a skew-product map \( F \) on \( M = \mathbb{N} \times M \) given by \( F(k, x) = (k + 1, f_k(x)) \). Observe that \( F^n(k, x) = (k + n, f^n_k(x)) \) for all \( n \geq 0 \).
We consider a Borel probability measure $m$ on $M$ which is non-singular for $f_{1,\infty}$, that is, both $m(f_n(A)) = 0$ and $m(f_n^{-1}(A)) = 0$ whenever $m(A) = 0$ for all $n \in \mathbb{N}$. We want to understand the long-term behavior of the fiberwise orbits of typical points in $M$ with respect to the measure $m$. To do this, we will study forward $f_{1,\infty}$-invariant sets, i.e., measurable sets $A$ so that $f_n(A) \subseteq A$ for all $n \in \mathbb{N}$. Namely, we will study the following definition:

**Definition 2.1.** We say that a measure $m$ is locally $f_{1,\infty}$-ergodic if for every forward $f_{1,\infty}$-invariant measurable set $A$ of $M$ with positive $m$-measure, there exists an open set $B$ of $M$ such that $m(B \setminus A) = 0$. If the measure of $B$ is uniformly bounded away from zero, we say that $m$ is locally strong $f_{1,\infty}$-ergodic. This means that there is $\varepsilon > 0$ such that for every forward $f_{1,\infty}$-invariant set $A$ of $M$ with positive $m$-measure there exists an open set $B$ with $m(B) > \varepsilon$ such that $m(B \setminus A) = 0$.

Firstly we give some basic properties of $m$ which will be useful later.

**Lemma 2.2.** The support of $m$ is a forward $f_{1,\infty}$-invariant set. Moreover, for any $r > 0$ there exists $b_1(r) > 0$ such that $m(B(x, r)) > b_1(r)$, for every $x \in \text{supp}(m)$.

*Proof.* Given $x \in \text{supp}(m)$. At he first, we claim that $f_n(x)$ also belongs to the support of $m$. By contradiction, assume that every small neighborhood of $f_n(x)$ has null $m$-measure. Since $m$ is non-singular and $f_n$ is a continuous map this implies that small neighborhoods of $x$ have also null $m$-measure. This contradicts $x \in \text{supp}(m)$. The second claim is straightforward. Assume again, by contradiction, that there exists $r > 0$ and a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{supp}(m)$ such that $m(B(x_n, r)) \to 0$ as $n \to \infty$. Since $\text{supp}(m)$ is a compact set, the sequence must accumulate at some point $z$ in the support of $m$. Then $m(B(z, r)) \leq \liminf_{n \to \infty} m(B(x_n, r)) = 0$ which contradicts $z \in \text{supp}(m)$.

In the sequel, we want to study the local ergodicity of non-autonomous systems. We will review the theory of hyperbolic preballs and hyperbolic times introduced by Alves [1] for autonomous systems and extended by Alves and Vilarinho [4] for random maps under assumptions of non-uniform expansions. This theory has been deeply studied and generalized in many works as [2, 3, 24]. We state it in the context of non-autonomous systems.

2.1. **Hyperbolic preballs:** Here, we give two sufficient conditions to get local ergodicity. This starts by introducing the notion of hyperbolic pre-balls.

**Definition 2.3.** Let $\delta > 0$ and $0 < \lambda < 1$. Given $n \geq 1$ and $(k, x) \in \mathcal{M}$, we say that a neighborhood $V^n_k(x)$ of $x$ in $M$ is a $(\delta, \lambda)$-hyperbolic preball of order $n$ of $f_{1,\infty}$ for the point $(k, x)$ if

1. the map $f_k^n : M \to M$ sends $V^n_k(x)$ homeomorphically onto the open ball $B(f_k^n(x), \delta)$ centered at the point $f_k^n(x)$ and of radius $\delta$,
2. for every $y, z \in V^n_k(x)$

$$d(f_k^n(y), f_k^n(z)) \leq \lambda^{n-i} d(f_k^n(y), f_k^n(z)) \quad \text{for } i = 0, \ldots, n - 1. \quad (4)$$

**Remark 2.4.** Notice that [1] and [2] can be extended to the closure of $V^n_k(x)$. 


In addition, we will need that the hyperbolic preballs have a good control of the distortion with respect to the measure \(m\). To be more clear, we give the following definition.

**Definition 2.5.** Let \(\delta > 0\) and \(0 < \lambda < 1\). We say that a point \((k, x) \in \mathcal{M}\) has an infinitely many \((\delta, \lambda)\)-hyperbolic preballs with bounded distortion if there exists a sequence of \((\delta, \lambda)\)-hyperbolic preballs \(V_k^n(x)\) of order \(n\) where \(n_i \to \infty\) and a constant \(K = K(\delta, \lambda, k) > 0\) such that for each \(i \in \mathbb{N}\),

\[
m(f_k^n(A)) m(f_k^n(B)) \leq K m(A) m(B) 
\]

for all pair of measurable sets \(A, B \subset V_k^n(x)\).

In what follows, we show local ergodicity under the assumption that almost every point has infinitely many hyperbolic preballs with bounded distortion. This assumption can be interpreted in two different ways. The first criterion will be used to get local ergodicity of non-uniform expanding non-autonomous systems. The second will be applied later for non-uniform expandable non-autonomous systems.

**2.1.1. First criterium: preballs with bounded distortion.** We assume the existence of a state \(\{k\} \times M\) in which almost every point has infinitely many hyperbolic preballs with bounded distortion.

**Proposition 2.6.** If there is \(k \in \mathbb{N}\) such that for \(m\)-almost every point \(x \in M\) there exist \(\delta = \delta(x) > 0\) and \(0 < \lambda = \lambda(x) < 1\) so that \((k, x)\) has infinitely many \((\delta, \lambda)\)-hyperbolic preballs with bounded distortion then \(m\) is locally \(f_{1,\infty}\)-ergodic.

**Proof.** Given \(\delta > 0\) and \(0 < \lambda < 1\), we define

\[
Z_{\delta, \lambda} = \{(k, x) \in \mathcal{M} : \text{the point } (k, x) \text{ has infinitely many } (\delta, \lambda)\text{-hyperbolic preballs}\}.
\]

Let \(Z\) be the union of \(Z_{\delta, \lambda}\) for \(0 < \delta\) and \(0 < \lambda < 1\). We denote by \(Z(k)\) the section of \(Z\) on \(\{k\} \times M\). That is, \(Z(k) = \{x \in M : (k, x) \in Z\}\). Since \(Z_{\delta, \lambda} \subset Z_{\delta', \lambda'}\) for any \(0 < \delta' \leq \delta\) and \(0 < \lambda \leq \lambda' < 1\) we can write

\[
Z \overset{\text{def}}{=} \bigcup_{0 < \delta} \bigcup_{0 < \lambda < 1} Z_{\delta, \lambda} = \bigcup_{n > 1} Z_{1/n, 1-1/n}.
\]

Let \(A\) be a forward \(f_{1,\infty}\)-invariant set of \(M\) with positive \(m\)-measure. Since the support of \(m\) is also forward \(f_{1,\infty}\)-invariant we can assume that \(A \subset \text{supp}(m)\). We need to show that there exists an open set \(B\) of \(M\) so that \(m(B \setminus A) = 0\). Notice that, by assumption, there exists \(k \in \mathbb{N}\) such that \(m(A \cap Z(k)) = m(A) > 0\). Thus, there exist \(\delta = \delta(A) > 0\) and \(0 < \lambda = \lambda(A) < 1\) such that \(\tilde{A} = A \cap Z_{\delta, \lambda}(k)\) has positive \(m\)-measure, where \(Z_{\delta, \lambda}(k) = \{x \in M : (k, x) \in Z_{\delta, \lambda}\}\). Moreover, since \(\tilde{A} \subset A\) and \(A\) is by assumption forward \(f_{1,\infty}\)-invariant then \(f_n(\tilde{A}) \subset A\) for all \(n \in \mathbb{N}\). Additionally, every point \((k, x)\) where \(x \in \tilde{A}\) has infinitely many \((\delta, \lambda)\)-hyperbolic preballs. The rest of the proof follows the argument of [4, Prop. 2.13] which is inspired by [2].

Let \(\gamma > 0\) be some small number. By the regularity of \(m\) and since \(m(\tilde{A}) > 0\), there is a compact set \(\tilde{A}_c \subset \tilde{A}\) and an open set \(\tilde{A}_o \supset \tilde{A}\) such that \(m(\tilde{A}_o \setminus \tilde{A}_c) \leq \gamma m(\tilde{A})\). Notice that, for any \(x \in \tilde{A}_c\) we have a \((\delta, \lambda)\)-hyperbolic preball \(V_k^n(x)\) of order \(n = n(x)\) contained in \(\tilde{A}_o\). Let \(W_k^n(x)\)
be the part of $V^i_k(x)$ which is sent homeomorphically by $f^m_k$ onto the open ball $B(f^m_k(x), \delta/4)$. By compactness, there are $x_1, \ldots, x_r \in \tilde{A}_c$ such that

$$\tilde{A}_c \subset W_1 \cup \cdots \cup W_r \quad \text{where} \quad W_i = W^i_k(x_i) \quad \text{and} \quad n_i = n(x_i) \quad \text{for} \quad i = 1, \ldots, r. \quad (6)$$

Assume that

$$\{n_1, \ldots, n_r\} = \{n_1^*, \ldots, n_r^*\} \quad \text{with} \quad n_1^* < \cdots < n_r^*.$$

Let $I_1$ be the maximal subset of $[1, \ldots, r]$ such that for each $i \in I_1$ both $n_i = n_i^*$ and $W_i \cap W_j = \emptyset$ for every $j \in I_1$ with $j \neq i$. Inductively we define $I_\ell$ for $\ell = 2, \ldots, s$ as follows: supposing that $I_1, \ldots, I_{\ell-1}$ have already been defined, let $I_\ell$ be a maximal set of $[1, \ldots, r]$ such that for each $i \in I_\ell$ both $n_i = n_i^*$ and $W_i \cap W_j = \emptyset$ for every $j \in I_1 \cup \cdots \cup I_\ell$ with $i \neq j$. Set $I = I_1 \cup \cdots \cup I_s$. By construction, we have that $W_i$ for $i \in I$ are pairwise disjoint sets.

We will prove that the family of set $V_i = V^i_k(x_i)$ for $i \in I$ covers $\tilde{A}_c$. Indeed, by construction, given any $W_j$ with $j = 1, \ldots, r$, there is some $i \in I$ with $n_i \leq n_j$ such that $W_j \cap W_i \neq \emptyset$. Taking images by $f^m_k$, we have $f^m_k(W_j) \cap B(f^m_k(x_i), \delta/4) \neq \emptyset$. Since $W_j$ is contained in the $(\delta, \lambda)$-hyperbolic preball $V_j$ of order $n_j$ and $n_i \leq n_j$, by definition of hyperbolic preballs,

$$\text{diam}(f^m_k(W_j)) \leq \lambda^{n_j-n_i} \text{diam}(f^m_k(W_i)) \leq \frac{\delta}{2}.$$ 

Hence $f^m_k(W_j) \subset B(f^m_k(x_i), \delta)$. This gives that $W_j \subset V_i$. Taking into account (5), we get that the family of sets $V_i$ for $i \in I$ covers $\tilde{A}_c$.

Observe that by the bounded distortion property (5) applied to $A = W_i$ and $B = V_i$ we get $K = K(\delta, \lambda, k) > 0$ such that

$$m(W_i) \geq K^{-1} \frac{m(B(f^m_k(x_i), \delta/4))}{m(B(f^m_k(x_i), \delta))} \frac{m(V_i)}{m(W_i)}.$$ 

According to Lemma 2.2, the measure of any ball centered at a point in the support of $m$ is lower comparable with its radius and thus we can find a constant $\tau = \tau(\delta, \lambda, k) > 0$ so that $m(W_i) \geq \tau m(V_i)$ for all $i \in I$. Hence

$$m(\bigcup_{i \in I} W_i) = \sum_{i \in I} m(W_i) \geq \tau \sum_{i \in I} m(V_i) \geq \tau m(\bigcup_{i \in I} V_i) \geq \tau m(\tilde{A}_c) \geq \frac{\tau}{2} m(\tilde{A}).$$

The last inequality is obtained from the fact that $m(\tilde{A}_c) > (1 - \gamma)m(\tilde{A})$ and choosing $\gamma > 0$ small enough which it is possible because the constant $\tau$ does not depend on $\gamma$. Now, we are going to prove the existence of $i \in I$ in such away that

$$\frac{m(W_i \setminus \tilde{A})}{m(W_i)} < \frac{2\gamma}{\tau}. \quad (7)$$

Indeed, otherwise we get the following contradiction.

$$\gamma m(\tilde{A}) \geq m(\tilde{A} \setminus \tilde{A}_c) \geq m(\bigcup_{i \in I} W_i \setminus \tilde{A}) \geq \frac{2\gamma}{\tau} m(\bigcup_{i \in I} W_i) > \gamma m(\tilde{A}).$$

Finally, we obtain the required open ball $B$. Since $f^m_k(\tilde{A}) \subset A$ and $f^m_k$ is injective on $W_i$, we have

$$m(f^m_k(W_i) \setminus \tilde{A}) \leq m(f^m_k(W_i) \setminus f^m_k(\tilde{A})) = m(f^m_k(W_i \setminus \tilde{A})).$$
By the distortion property, relation (7) and taking in mind that \( f_k^{n_i}(W_i) = B(f_k^{n_i}(x), \delta/4) \), we get
\[
\frac{m(B(f_k^{n_i}(x), \delta/4) \setminus A)}{m(B(f_k^{n_i}(x), \delta/4))} \leq \frac{m(f_k^{n_i}(W_i) \setminus \tilde{A})}{m(f_k^{n_i}(W_i))} \leq K \frac{m(W_i \setminus \tilde{A})}{m(W_i)} \leq \frac{2K\gamma}{\tau},
\]
which can obviously be made arbitrarily small, letting \( \gamma \to 0 \). From this, one easily deduces, taking an accumulation point of this balls, that there is a ball \( B \) of radius \( \delta/4 \) where the relative measure of \( A \) is one. This completes the proof.  

**Remark 2.7.** In the above proof, the radius of the obtained open ball depends only on \( \delta > 0 \) but it may be vary from an invariant set to another one. To get strong local \( f_{1,\infty} \)-ergodicity, we must ask that \( \delta > 0 \) and \( 0 < \lambda < 1 \), in the statement of the proposition, are uniform on \( x \). In other words, we need that \( m(Z_{\delta,\lambda}(k)) = 1 \), for some \( k \in \mathbb{N} \).

### 2.1.2. Second criterium: preballs with regularity.

Now, we assume that almost every point \( x \) has infinitely many hyperbolic preballs, but probably in different states \( |k| \times M \). This assumption is obviously weaker than the previous condition. To prove the local ergodicity, we also need to assume that the preballs have a good control of the regularity.

**Definition 2.8.** Let \( \delta > 0 \) and \( 0 < \lambda < 1 \). We say that a point \( (k, x) \in M \) has infinitely many regular \( (\delta, \lambda) \)-hyperbolic preballs if there exist a sequence of \( (\delta, \lambda) \)-hyperbolic preballs \( V_i = V_k^{n_i}(x) \) of order \( n_i \) where \( n_i \to \infty \) and a constant \( L = L(\delta, \lambda, k) > 0 \) such that
\[
m(B(x, R_i)) \leq L m(B(x, r_i)), \quad \text{for all } i \in \mathbb{N}
\]
where \( B(x, R_i) \) and \( B(x, r_i) \) are, respectively, the smallest ball around \( x \) containing \( V_i \) and the largest ball around \( x \) contained in \( V_i \).

The following proposition shows local ergodicity under the assumption of the existence of infinitely many regular preballs with bounded distortion. Here, we also need to assume that the metric measure space \( (M, d, m) \) satisfies the density point property. That is, for any measurable set \( A \) of \( M \),
\[
\lim_{r \to 0^+} \frac{m(A \cap B(x, r))}{m(B(x, r))} = 1, \quad \text{for } m\text{-almost all } x \in A.
\]

This property holds in any metric space for which Besicovitch’s Covering Theorem holds. In particular, it holds for any Borel probability measure in Euclidean spaces. Also, it is satisfied for any Borel probability measure in a Polish ultra-metric space and for the Cantor space \( 2^\mathbb{N} \) with the coin-tossing measure and the usual distance. In general metric spaces this is not necessarily the case [15]. As another relatively general mode of this property, one can refer to the weak locally doubling measure \( m \) (see [12, Thm. 3.4.3]) in the sense that
\[
\limsup_{r \to 0^+} \frac{m(B(x, 2r))}{m(B(x, r))} < \infty, \quad \text{for } m\text{-almost all } x \in M.
\]

**Proposition 2.9.** Assume that \( (M, d, m) \) satisfies the density point property. If for \( m\)-almost every point \( x \in M \) there are \( k = k(x) \in \mathbb{N} \), \( \delta = \delta(x) > 0 \) and \( 0 < \lambda = \lambda(x) < 1 \) so that \( (k, x) \) has infinitely many regular \( (\delta, \lambda) \)-hyperbolic preballs with bounded distortion then \( m \) is locally \( f_{1,\infty} \)-ergodic.
**Proof.** Let $A$ be a forward $f_{1,\infty}$-invariant set with positive $m$-measure. By the density point property $m$-almost every point in $A$ is a density point. That is, it satisfies (9). By the assumption we find a density point $x \in A$, $k = k(x) \in \mathbb{N}$, $\delta = \delta(x) > 0$ and $0 < \lambda = \lambda(x) < 1$ so that $(k, x)$ has a nested sequence of regular $(\delta, \lambda)$-preballs $V_i = V_k^{m_i}(x)$ of order $n_i \to \infty$ with bounded distortion. Let $z$ be an accumulation point of $f_k^{m_i}(x)$. Then taking a subsequence if it is necessary we have $B(z, \delta/2) \subset B(f_k^{m_i}(x), \delta)$ for all $i$ large enough. On the other hand, since $f_k^{m_i}(A) \subset A$ and $f_k^{m_i}$ is injective on $V_i$, we have

$$m(f_k^{m_i}(V_i) \setminus A) \leq m(f_k^{m_i}(V_i) \setminus f_k^{m_i}(A)) = m(f_k^{m_i}(V_i \setminus A)).$$

By the distortion property and the regularity of the preballs, having into account that $f_k^{m_i}(V_i) = B(f_k^{m_i}(x), \delta)$, it follows that for every $i$ large enough

$$\frac{m(B(z, \delta/2) \setminus A)}{m(M)} \leq \frac{m(f_k^{m_i}(V_i) \setminus A)}{m(f_k^{m_i}(V_i))} \leq K \frac{m(V_i \setminus A)}{m(V_i)} \leq K \frac{m(B(z, \delta) \setminus A)}{m(B(z, \delta))},$$

where $B(x, R_i)$ and $B(x, r_i)$ are, respectively, the smallest ball around $x$ containing $V_i$ and the largest ball around $x$ contained in $V_i$. Taking limit as $i \to \infty$, since $V_i$ is nested then $R_i \to 0$ and since $x$ is a density point of $A$, we get that $m(B(z, \delta/2) \setminus A) = 0$. This completes the proof of the proposition. \hfill \Box

**Remark 2.10.** To get strong local $f_{1,\infty}$-ergodicity it suffices to ask that $\delta > 0$ and $0 < \lambda < 1$ in the statement of the proposition are uniform on $x$.

**Remark 2.11.** Proof of Proposition 2.29 actually shows the following: if $x$ is a density point of a $f_{1,\infty}$-invariant set $A$ such that there is $k = k(x) \in \mathbb{N}$, $\delta = \delta(x) > 0$ and $0 < \lambda = \lambda(x) < 1$ then there is $z$ such that $m(B(z, \delta/2) \setminus A) = 0$.

### 2.2. Hyperbolic preballs with bounded distortion.

Here, we will study how we can get hyperbolic preballs with bounded distortion. First we need the following lemma.

**Lemma 2.12.** For each $n \in \mathbb{N}$, consider functions $\psi_n : M \to \mathbb{R}$ and assume that there exist $k \in \mathbb{N}$, $0 < \alpha \leq 1$, $\epsilon > 0$ and a constant $C_k = C_k(\epsilon, \alpha) > 0$ such that

$$|\psi_n(x) - \psi_n(y)| \leq C_k d(x, y)^\alpha, \quad \text{for all } x, y \in M \text{ with } d(x, y) < \epsilon \text{ and } n \geq k. \quad (10)$$

Then any $(\delta, \lambda)$-preball $V_k^m(x)$ of order $n$ for a point $(k, x) \in M$ with $0 < \delta \leq \epsilon$, $0 < \lambda < 1$ there is a constant $K = \exp(C_k \delta^\alpha (1 - \lambda^\alpha)^{-1}) > 0$ such that

$$K^{-1} \leq e_s \psi_{k(y)-S_n \psi(k,z)} \leq K, \quad \text{for all } y, z \in V_k^m(x)$$

where

$$S_n \psi = \sum_{i=0}^{n-1} \psi \circ F^i$$

denotes the $n$-th Birkhoff sum of a function $\psi : M \to \mathbb{R}$ given by $\psi(k, x) = \psi_k(x)$. 
Proof. For any pair of points $y, z \in V^n_k(x)$, by definition of $(\lambda, \delta)$-hyperbolic preball (see also Remark 2.4),

$$d(f_k^n(y), f_k^n(z)) \leq \lambda^{n-i}d(f_k^n(y), f_k^n(z)) \leq \lambda^{n-i}\delta \leq \epsilon \quad \text{for all } i = 0, \ldots, n - 1$$

and thus

$$|S_n\psi(k, y) - S_n\psi(k, z)| \leq \sum_{i=0}^{n-1} |\psi_{k+i}(f_k^i(y)) - \psi_{k+i}(f_k^i(z))|$$

$$\leq \sum_{i=0}^{n-1} C_k d(f_k^i(y), f_k^i(z))^\alpha \leq \sum_{i=0}^{n-1} C_k \lambda^{(n-i)\delta}\alpha.$$

It is then enough to take $K = \exp(\sum_{j=0}^{\infty} C_k \lambda^j \delta^\alpha) = \exp(C_k \delta^\alpha(1 - \lambda^\alpha)^{-1}) > 0$. \hfill $\Box$

In order to get the bounded distortion property we will need to suppose that the measure $m$ is $f_{1,\infty}$-conformal. That is, for each $n \in \mathbb{N}$ we have some function $\psi_n : M \to \mathbb{R}$ such that

$$m(f_n(A)) = \int_A e^{-\psi_n(x)} \ dm(x), \quad \text{for every measurable set } A \text{ so that } f_n|_A \text{ is injective}.$$ 

Surely, any absolutely continuous measure is conformal, by the definition. Also, there are several examples of conformal measures appearing in the literature (see [8], for a large class of examples).

In fact, the concept of $f_{1,\infty}$-conformal measure allows us to have varying Jacobians with respect to the dynamics in the sequence.

Proposition 2.13. Assume that $m$ is $f_{1,\infty}$-conformal as above and there exists $k \in \mathbb{N}$, $0 < \alpha \leq 1$, $\epsilon > 0$ such that the functions $(\psi_n)_n$ satisfy the locally Hölder condition [10]. Then any $(\delta, \lambda)$-preball of a point $(k, x) \in M$ with $0 < \delta \leq \epsilon$, $0 < \lambda < 1$ has bounded distortion, i.e., satisfies [5] with distortion constant $K = K(\delta, \lambda, C_k)$ uniform on $x$ and on the order of the preball.

Proof. We consider $0 < \delta \leq \epsilon$, $0 < \lambda < 1$ and a $(\delta, \lambda)$-hyperbolic preball $V^n_k(x)$ of order $n$ for a point $(k, x) \in M$. Let $A, B$ be a pair of measurable sets in $V^n_k(x)$. By the conformality of the measure, it is not hard to see that

$$m(f^n_k(A)) = \int_A e^{-S_n\psi(k,z)} \ dm(z) \leq \sup_{z \in A} e^{-S_n\psi(k,z)} m(A)$$

and

$$m(f^n_k(B)) = \int_B e^{-S_n\psi(k,y)} \ dm(y) \geq \inf_{y \in B} e^{-S_n\psi(k,z)} m(B)$$

where $S_n\psi$ denotes the $n$-th Birkhoff sum of a function $\psi : M \to \mathbb{R}$ given by $\psi(k, x) = \psi_k(x)$. From this and Lemma [2.12] one easily concludes the proposition. \hfill $\Box$
2.3. Hyperbolic times. Now, we will provide a sufficient condition to get a hyperbolic preball. In order to do this, we first need to restrict the class of non-autonomous discrete dynamical systems $f_{1,\infty} = (f_n)_n$ that we are considering.

We suppose that $f_n : M \to M$ for all $n \in \mathbb{N}$ are local homeomorphisms with uniform Lipschitz constant for the inverse branches. This means that there is a function $\varphi : M \to \mathbb{R}$ such that for each $(k, x) \in M$ there exists a neighborhood $V$ of $x$ so that $f_k : V \to f_k(V)$ is invertible and

$$d(y, z) \leq \varphi(k, x) d(f_k(y), f_k(z)), \quad \text{for all for every } y, z \in V.$$

**Definition 2.14.** Let $0 < \sigma < 1$. A positive integer $n \in \mathbb{N}$ is called $\sigma$-hyperbolic time of $f_{1,\infty}$ for the point $(k, x) \in M$ if

$$\prod_{i=n-\ell}^{n-1} \varphi(F^i(k, x)) \leq \sigma^\ell, \quad \text{for } \ell = 1, \ldots, n \text{ where } F^i(k, x) = (k + i, f_k^i(x)).$$

The following proposition shows that existence of hyperbolic times implies the existence of hyperbolic preballs.

**Proposition 2.15.** For any $\epsilon > 0$ there is $0 < \delta_k \leq \epsilon$ such that if $n \in \mathbb{N}$ is a $\sigma$-hyperbolic time of $f_{1,\infty}$ for a point $(k, x) \in M$ then $(k, x)$ has a $(\delta_k, \lambda)$-hyperbolic preball of order $n$ where $\lambda = \sigma$.

**Proof.** First of all we will set $\delta_k > 0$. To do this we fix $\epsilon > 0$. For each $k \in \mathbb{N}$, since $f^i_k$ is a local homeomorphism, for every $x \in M$ there is $0 < \delta_{k,x} \leq \epsilon$ such that $f^i_k$ sends a neighborhood $U(k, x)$ of $x$ homeomorphically onto an open ball of radius $\delta_{k,x}$ centered at $f^i_k(x)$ and satisfying

$$d(y, z) \leq \varphi(k, x) d(f_k(y), f_k(z)) \quad \text{for all } y, z \in U(k, x). \quad (11)$$

By compactness of $M$, we can choose a uniform radius $\delta_k > 0$. Otherwise we find a sequence of points $x_n \in M$ converging to a point $\bar{x}$ and with $\delta_{k,x_n} \to 0$. Hence, we obtain that $\delta_{k,\bar{x}}$ must be to zero obtaining a contradiction. Thus we get that $f_k : U(k, x) \to B(f_k(x), \delta_k)$ is a homeomorphism satisfying (11). Moreover, without loss of generality, using the order of $\mathbb{N}$, we can assume that $\delta_k \geq \delta_{k+1}$ for all $k \in \mathbb{N}$.

Now we will show the proposition by induction on $n$. Let $n = 1$ be a $\sigma$-hyperbolic time of a point $(k, x)$. This implies that $\varphi(k, x) \leq \sigma$. Let $V^1_k(x)$ be the neighborhood $U(k, x)$ of $x$ obtained above. Hence we have that $f_k$ sends homeomorphically $V^1_k(x)$ onto the open ball $B(f_k(x), \delta_k)$ and

$$d(y, z) \leq \varphi(k, x) d(f_k(y), f_k(z)) \leq \sigma d(f_k(y), f_k(z)), \quad \text{for all } y, z \in V^1_k(x).$$

Thus, $V^1_k(x)$ is a $(\delta_k, \sigma)$-hyperbolic preball of order $n = 1$ at the point $(k, x)$.

Now, assuming the proposition holds for $n$, we prove it for $n + 1$. Namely, we assume that if $n$ is a $\sigma$-hyperbolic time of a point $(k, x)$, there exists a $(\delta_k, \sigma)$-hyperbolic preball $V^n_k(x)$ and additionally it holds that

$$f^i_k(V^n_k(x)) \subset U(F^i(k, x)), \quad \text{for } i = 0, \ldots, n - 1.$$
Let \( n + 1 \) be a \( \sigma \)-hyperbolic time of a point \((k, x)\). Hence,

\[
\prod_{i=n+\ell}^{n-1} \varphi(F^i(F(k, x))) = \prod_{j=n+1-\ell}^{n} \varphi(F^j(k, x)) \leq \sigma^\ell, \quad \text{for } \ell = 1, \ldots, n
\]

and thus \( n \) is a \( \sigma \)-hyperbolic time of the point \( F(k, x) = (k + 1, f_k(x)) \). By induction, there exists a \((\delta_{k+1}, \sigma)\)-hyperbolic preball \( V \) of order \( n \) at the point \( F(k, x) \). This means that \( f_{k+1}^n \) sends homeomorphically \( V \) onto \( B(f_{k+1}^n(f_k(x)), \delta_{k+1}) \) and

\[
d(f_{k+1}^i(y), f_{k+1}^i(z)) \leq \sigma^{n-i}d(f_{k+1}^n(y), f_{k+1}^n(z)), \quad \text{for all } y, z \in V, i = 0, \ldots, n - 1. \tag{12}
\]

Notice that, in fact, \( V \subset B(f_k(x), \delta_k) \) since applying the above inequality for \( i = 0 \) and recalling that \( \delta_k \geq \delta_{k+1} \), we have that

\[
d(y, f_k(x)) \leq \sigma^n d(f_{k+1}^n(y), f_{k+1}^n(f_k(x))) \leq \sigma^n \delta_{k+1} < \delta_k.
\]

Therefore, there is a neighborhood \( V_{k+1}^{n+1}(x) \) of \( x \) which is sent homeomorphically by \( f_k \) onto \( V \). Moreover, \( V_{k+1}^{n+1}(x) \subset U(k, x) \). On the other hand, by the induction hypothesis, we have also that

\[
f_k^i(V_{k+1}^{n+1}(x)) = f_k^{i-1}(V) \subset U(F^{i-1}(F(k, x))) = U(F^i(x)) \quad \text{for } i = 1, \ldots, n.
\]

Now, we must show that for every \( y, z \in V_{k+1}^{n+1}(x) \) it holds that

\[
d(f_k^j(y), f_k^j(z)) \leq \sigma^{n+1-j}d(f_{k+1}^n(y), f_{k+1}^n(z)), \quad \text{for all } j = 0, \ldots, n. \tag{13}
\]

Applying (12) we obtain (13) for \( j = 1, \ldots, n \). Thus, it is enough to check it for \( j = 0 \). This follows applying recursively

\[
d(y, z) \leq \varphi(k, x)d(f_k(y), f_k(z)) \leq \cdots \leq \prod_{i=0}^{n} \varphi(F^i(k, x))d(f_{k+1}^n(y), f_{k+1}^n(z))
\]

for any \( y, z \in V_{k+1}^{n+1}(x) \). Since \( n + 1 \) is a \( \sigma \)-hyperbolic time of \( (k, x) \) we complete the proof. \( \square \)

2.4. **Expanding/expandable measures.** In this subsection, we study how to get hyperbolic times. We will continue assuming that \( f_{1,\infty} = (f_n) \) is a non-autonomous discrete system of local homeomorphisms \( f_n \) with uniform Lipschitz constant \( \varphi(n, x) = \varphi_n(x) \) for the inverse branches as in the previous section. Additionally we assume that

\[
\sup\{- \log \varphi(k, x) : x \in M, k \in \mathbb{N}\} < \infty.
\]

For \( k \in \mathbb{N} \) and \( a > 0 \), let \( M(k,a) \) be the set of points \( x \in M \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \varphi(F^i(x)) < -a.
\]

**Proposition 2.16.** If \( x \in M(k,a) \) then there is \( \sigma = \exp(-a/2) \) such that \( (k, x) \) has infinitely many \( \sigma \)-hyperbolic times.
Proof. For every $x \in M(k,a)$ and $N$ sufficiently large we have

$$\sum_{i=0}^{N-1} - \log \phi(F^i(k,x)) \geq Na.$$  

Taking $a_i = - \log \phi(F^i(k,x)) - a/2$, we have $a_0 + \cdots + a_{N-1} \geq aN/2$. By Pliss lemma (c.f. [4, Lemma 4.2]) with

$$c = a/2 \quad \text{and} \quad A = \sup \{- \log \phi(i,z) - a/2 : z \in M, i \in \mathbb{N}\} < \infty,$$

there are $t \geq \theta N$, $\theta = c/A$ and $1 \leq n_1 < \cdots < n_\ell \leq N$ such that

$$\sum_{i=n_j}^{n_{j-1}} a_i \geq 0, \quad \text{for } n = 0, \ldots, n_j - 1 \text{ and } j = 1, \ldots, t.$$  

Therefore,

$$\sum_{i=n_j}^{n_{j-1}} \log \phi(F^i(k,x)) \leq \frac{a}{2}(n_j - n), \quad \text{for } n = 0, \ldots, n_j - 1 \text{ and } j = 1, \ldots, t.$$  

By taking $0 < \sigma = \exp(-a/2) < 1$ and $\ell = n_j - n$, we get

$$\prod_{i=n_j-\ell}^{n_j-1} \phi(F^i(k,x)) \leq \sigma^\ell \quad \text{for } \ell = 0, \ldots, n_j - 1 \text{ and } j = 1, \ldots, t.$$  

This implies that $n_j$ for $j = 1, \ldots, t$ are $\sigma$-hyperbolic times of $F$ for $(k,x)$. Since $t \to \infty$ as $N \to \infty$ we obtain infinitely many hyperbolic times and complete the proof. \hfill \Box

Following [19], we say that a measure $m$ is $f_{1,\infty}$-expanding if there is $k \in \mathbb{N}$ so that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \phi(F^i(k,x)) < 0, \quad \text{for } m\text{-almost every } x \in M.$$  

Observe that equivalently, one can ask that the above limit holds at $(1,x)$, for $m$-almost every $x \in M$. If the limit is uniformly far away from zero, as in the above proposition, i.e., if there is $a > 0$ such that $m(M(1,a)) = 1$, we say that $m$ is strong $f_{1,\infty}$-expanding.

Similarly, we will say that $m$ is $f_{1,\infty}$-expansible if for $m$-almost every $x \in M$ there is $k = k(x) \in \mathbb{N}$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \phi(F^i(k,x)) < 0.$$  

In addition, if there is $a > 0$ uniform on $x$ so that for $m$-almost every $x \in M$ there is $k = k(x) \in \mathbb{N}$ such that $x \in M(k,a)$, then we say that $m$ is strong $f_{1,\infty}$-expansible.

As a consequence of the above proposition, we have the following:

**Corollary 2.17.** It holds that,
Hence, 
\[ n \alpha = V \theta n \]

for all \( f \theta \phi \) Lipschitz constant. 

2.5.1. Expanding and regular hyperbolic preballs can be obtained in this case. Consequently, by taking \( \sigma \) close enough to one, we get the following: 

\[ \lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta(F(k, x)) \leq \lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \varphi(F(k, x)) \]

\[ \leq \lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta(F^i(k, x)) + \frac{1}{2} \log \sigma. \]

Consequently, by taking \( \sigma = \sigma(x) \) close enough to one, we get the following:

2.5. **Locally geodesic metric spaces.** A metric space is said to be **locally geodesic** (or locally 1-quasiconvex) if each point has a neighborhood \( U \) such that for each pair of points \( x, y \in U \), there is a rectifiable curve \( \gamma \) joining \( x \) and \( y \) with length \( \ell(\gamma) = d(x, y) \). In this subsection, we assume that \((M, d)\) is locally geodesic and we show how expanding/expandable measures and regular hyperbolic preballs can be obtained in this case.

2.5.1. **Expanding/expandable measures.** In the two previous subsection, we assumed that the non-autonomous system \( f_{1,\infty} = (f_n)_{n} \), formed by local homeomorphisms \( f_n \) have uniform Lipschitz constant \( \varphi(n, x) = \varphi_n(x) \) for the inverse branches. That is, satisfying (2.3). An a priori weaker condition is to assume that maps \( f_n \) have pointwise Lipschitz constants \( \theta(n, x) = \varphi_n(x) > 0 \) for the inverse branches. That is, there is a positive bounded functions \( \theta_n : M \to \mathbb{R} \) such that for each \( x \in M \) it holds

\[ \theta_n(x)^{-1} = \lim inf_{y \to x} \frac{d(f_n(x), f_n(y))}{d(x, y)}. \]

According to [9, Cor. 2.4], any pointwise Lipschitz map on a locally geodesic metric space \((M, d)\) is uniformly Lipschitz. Moreover, by [9, Lemma 2.3], restricting \( f_n \) to a small neighborhood \( V \) of \( x \), one gets

\[ d(y, z) \leq \|\theta_n\|_{\infty, V} d(f_n(y), f_n(z)), \quad \text{for all } y, z \in V. \]

Thus we can take \( \varphi(n, x) = \|\theta_n\|_{\infty, V} \). In addition, if \( \theta : M \to \mathbb{R} \), given by \( \theta(n, x) = \varphi_n(x) \) is a continuous function (with the discrete topology in \( \mathbb{N} \)) or equivalently, \( \theta_n : M \to \mathbb{R} \) is a continuous map, for all \( n \in \mathbb{N} \) then one can get also an upper estimative. Indeed, since \( \varphi_n(x) < \sigma^{-1/2} \varphi_n(x) \), for all \( 0 < \sigma < 1 \), by the continuity, one can find a small neighborhood \( V = V(\sigma) \) of \( x \) such that

\[ \varphi(n, x) = \|\theta_n\|_{\infty, V} \leq \sigma^{-1/2} \varphi_n(x) = \sigma^{-1/2} \theta(n, x). \]

Hence,

\[ \lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta(F(k, x)) \leq \lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \varphi(F(k, x)) \]

\[ \leq \lim sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta(F^i(k, x)) + \frac{1}{2} \log \sigma. \]
Proposition 2.18. Given \( x \in M \), there is \( a = a(x) > 0 \) such that \( x \in M(k,a) \) if and only if

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta(F^i(k,x)) < 0.
\]

Moreover, \( m \) is (strong) \( F \)-expanding/expandable if and only if it holds

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta(F^i(k,x)) < -a
\]

under the corresponding quantification assumptions and \( a \geq 0 \).

2.5.2. Regular hyperbolic preballs. Next, we are going to show how we can get regular preballs. To do this, we need to impose some extra conditions on the metric measure space \((M,d,m)\) and also on the non-autonomous dynamical systems \(f_{1,\infty} = (f_n)_{n \in \mathbb{N}}\).

We will assume that the measure \( m \) is locally doubling (see [12], pg. 326]), i.e., there are \( \rho > 0 \) and \( L > 0 \) such that

\[
m(B(x,2r)) \leq L m(B(x,r))
\]

for each \( x \in M \) and each \( 0 < r \leq \rho \). Every locally doubling metric measure space satisfies the density point property.

Finally, we will impose that \( f_{1,\infty} = (f_n)_{n \in \mathbb{N}} \) is conformal in the sense that \( f_n \) is a conformal map for all \( n \in \mathbb{N} \). Namely, there is a function \( \phi_n : M \to \mathbb{R} \) such that for every \( x \in M \)

\[
\lim_{y \to x} \frac{d(f_n(x),f_n(y))}{d(x,y)} = e^{-\phi_n(x)}.
\]

Observe that, in this case

\[
\theta_n(x) = e^{\phi_n(x)}, \quad \text{for all } x \in M \text{ and } n \in \mathbb{N}.
\]

Proposition 2.19. Let \( f_{1,\infty} = (f_n)_{n \in \mathbb{N}} \) be a conformal non-autonomous discrete system on a locally geodesic compact metric measure space \((M,d,m)\), where \( m \) is locally doubling. If there are \( k \in \mathbb{N} \), \( 0 < e, 0 < \alpha \leq 1 \) and \( C_k = C_k(e,\alpha) > 0 \) such that

\[
|\phi_n(x) - \phi_n(y)| \leq C_k d(x,y)\alpha, \quad \text{for all } x, y \in M \text{ with } d(x,y) < e \text{ and } n \geq k,
\]

then any \((\delta,\lambda)\)-hyperbolic preball of order \( n \) of a point \((k,x) \in M\) with \( 0 < \delta \leq e, 0 < \lambda < 1 \) is regular, i.e., satisfies [5] with regularity constant \( L(e,\lambda,k) > 0 \), uniform on \( x \) and on the order of the pre-ball.

Proof. At the first, note that by compactness of \( M \), one can assume that any ball of radius less than \( e > 0 \) is contained in a geodesic neighborhood. On the other hand, it is not difficult to see that for every \((k,x) \in M \) and \( n \in \mathbb{N} \) it holds that

\[
\lim_{y \to x} \frac{d(f_n^k(x),f_n^k(y))}{d(x,y)} = e^{-S_n\phi(k,x)},
\]

where \( S_n\phi \) denotes the \( n \)-th Birkhoff sum of a function \( \phi : M \to \mathbb{R} \) given by \( \phi(k,x) = \phi_k(x) \).
Claim 2.20. For any $0 < \delta \leq \varepsilon$ and $0 < \lambda < 1$, there exists $K = \exp(C_k \delta^\alpha (1 - \lambda^\alpha)^{-1}) > 0$ such that for any $(\delta, \lambda)$-hyperbolic pre-ball $V^n_k(x)$ of order $n$ of a point $(k, x) \in M$, it holds

$$K^{-1} e^{-S_n\phi(k,x)} d(y, z) \leq d(f^n_k(y), f^n_k(z)) \leq Ke^{-S_n\phi(k,x)} d(y, z),$$

for all $y, z \in V^n_k(x)$.

Proof. By Lemma 2.12 and Hölder assumption of $\phi_n$, we find $K = \exp(C_k \delta^\alpha (1 - \lambda^\alpha)^{-1}) > 0$ such that

$$K^{-1} \leq e^{-S_n\phi(k,y)-S_n\phi(k,z)} \leq K, \quad \text{for all} \ y, z \in V^n_k(x).$$

In particular,

$$e^{-S_n\phi(k,y)} \leq Ke^{-S_n\phi(k,x)} \quad \text{and} \quad e^{S_n\phi(k,y)} \leq Ke^{S_n\phi(k,x)}, \quad \text{for all} \ y \in V^n_k(x).$$

This implies that the uniform norms $\|e^{-S_n\phi}\|_\infty$ and $\|e^{S_n\phi}\|_\infty$ in $V^n_k(x)$ are bounded by $Ke^{-S_n\phi(k,x)}$ and $Ke^{S_n\phi(k,x)}$ respectively. Let $y$ and $z$ be a pair of points in the closure of $V^n_k(x)$ and consider a geodesic $\gamma$ joints them, i.e., a rectifiable curve with length $\ell(\gamma) = d(y, z)$. According to [9, Lemma 2.3],

$$d(f^n_k(y), f^n_k(z)) \leq \|e^{-S_n\phi}\|_\infty \ell(\gamma) \leq Ke^{-S_n\phi(k,x)} d(y, z). \quad (14)$$

Notice that the inverse map of $f^n_k : V^n_k(x) \to B(f^n_k(x), \delta)$ is also conformal with pointwise Lipschitz constant given by the exponential of $S_n\phi(k, y)$. Hence, arguing similarly, as above, one has that

$$d(y, z) \leq Ke^{S_n\phi(k,x)} d(f^n_k(y), f^n_k(z)), \quad \text{for all} \ y, z \in V^n_k(x). \quad (15)$$

Putting together (14) and (15), we conclude the proof of the claim. \qed

Now, let $B(x, R)$ and $B(x, r)$ be, respectively, the smallest ball around $x$ containing $V^n_k(x)$ and the largest ball around $x$ contained in $V^n_k(x)$. Take $y$ and $z$ in the boundary of $V^n_k(x)$ so that $d(x, y) = R$ and $d(x, z) = r$. By the above claim

$$\delta K^{-1} e^{S_n\phi(k,x)} \leq r \leq Ke^{S_n\phi(k,x)} \delta. \quad (16)$$

In particular, the ratio of $r$ and $R$ do not depend on $n$. Equation (16) implies that $R \leq tr$, where $t = K^2 = \exp(2C_k \delta^\alpha (1 - \lambda^\alpha)^{-1})$. Since $m$ is locally doubling, being $\delta > 0$ small enough (this holds if $\varepsilon > 0$ is small) one gets that

$$\frac{m(B(x, R))}{m(B(x, r))} \leq \frac{m(B(x, tr))}{m(B(x, r))} \leq L < \infty$$

and this completes the proof. \qed

3. Main results on non-autonomous discrete dynamical systems

Now, we give the main results of the paper. In order to do this we summarize the assumptions that we need. We have a non-autonomous discrete system $f_{1,\infty}$ with $f_{1,\infty} = (f^n_{m})_{m \in \mathbb{N}}$ on a metric measurable space $(M, d, m)$ or equivalently a skew-product map

$$F : \mathbb{N} \times M \to \mathbb{N} \times M, \quad F(k, x) = (k + 1, f_k(x))$$

under the following assumptions:
(H1) **Hypothesis on metric space:** \((M, d)\) is a compact metric space.

(H2) **Hypothesis on the fiber maps:** \(f_n : M \to M\), for all \(n \in \mathbb{N}\), is a local homeomorphism with uniform Lipschitz constant for the inverse branches. That is, for every \(n \in \mathbb{N}\), there is a function \(\varphi_n : M \to \mathbb{R}\) such that for each \(x \in M\) there exists a neighborhood \(V\) of \(x\) in such away that \(f_n : V \to f_n(V)\) is invertible and

\[
d(y, z) \leq \varphi_n(x)d(f_n(y), f_n(z)), \quad \text{for all for every } y, z \in V.
\]

Additionally, we assume that \(\sup\{-\log \varphi_n(x) : x \in M, n \in \mathbb{N}\} < \infty\).

(H3) **Hypothesis on the measure:** \(m\) is a Borel probability on \(M\). We also assume that

i) \(m\) is \(f_{1,\infty}\)-non-singular, i.e., both \(m(f_n(A)) = 0\) and \(m(f_n^{-1}(A)) = 0\) whenever \(m(A) = 0\);

ii) \(m\) is locally Hölder \(f_{1,\infty}\)-conformal. That is, there are constants \(0 < \alpha \leq 1\), \(\varepsilon > 0\) and \(C_1 > 0\) such that for every \(n \in \mathbb{N}\) there is a map \(\psi_n : M \to \mathbb{R}\) so that

\[
m(f_n(A)) = \int_A e^{-\psi_n(x)} \, dm(x)
\]

for every measurable set \(A\) such that \(f_n|_A\) is injective and satisfying that

\[
|\psi_n(x) - \psi_n(y)| \leq C_1 d(x, y)^\alpha \quad \text{for all } x, y \in M \text{ with } d(x, y) < \varepsilon \text{ and } n \in \mathbb{N}.
\]

Recalling the notion of local ergodicity in Definition 2.1, we have the following main result.

**Theorem 3.1.** Let \(f_{1,\infty} = (f_n)_{n \in \mathbb{N}}\) be a non-autonomous discrete dynamical system on the metric measure space \((M, d, m)\) under the assumption (H1), (H2) and (H3). Suppose also that there is a \(a \geq 0\) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \varphi_{i+1}(f_i^1(x)) < -a \quad \text{for } m\text{-almost every } x \in M
\]

where \(f_n^0 = \text{id}\) and \(f_i^1 = f_i \circ \cdots \circ f_1\). Then the probability measure \(m\) is locally \(f_{1,\infty}\)-ergodic if \(a = 0\) and strong locally \(f_{1,\infty}\)-ergodic if \(a > 0\).

**Proof.** By assumption \(m\) is (strong) \(f_{1,\infty}\)-expanding for \(a = 0\) (resp. \(a > 0\)). According to Corollary 2.17 for \(m\)-almost every \(x \in M\) we have \(0 < \sigma = \sigma(x) < 1\) (resp. \(0 < \sigma < 1\) uniform on \(x\)) such that \((1, x)\) has infinitely many \(\sigma\)-hyperbolic times. By Propositions 2.15 and 2.13 there are \(0 < \delta_1 \leq \varepsilon\) and \(\lambda = \sigma\) such that \((1, x)\) has infinitely many \((\delta_1, \lambda)\)-hyperbolic preballs with bounded distortion. Finally by Proposition 4.2 (resp. Remark 2.7) we obtain that \(m\) is locally (strong) ergodic as we want to prove. \(\square\)

In order to state the second main result we need to impose slightly strong hypothesis on the measure metric space and the non-autonomous discrete dynamical system.

(H1*) **Hypothesis on metric space:** \((M, d)\) is a compact locally geodesic metric space.

(H2*) **Hypothesis on the fiber maps:** \(f_{1,\infty}\) is locally Hölder conformal. That is, there are constants \(0 < \alpha \leq 1\), \(\varepsilon > 0\) and \(C_1 > 0\) such that for each \(n \in \mathbb{N}\) there is a function
Hypothesis on the measure:

$$\phi_n : M \to \mathbb{R}$$ so that for every $$x \in M,$$

$$\lim_{y \to x} \frac{d(f_n(x), f_n(y))}{d(x, y)} = e^{-\phi_n(x)}$$

and

$$|\phi_n(x) - \phi_n(y)| \leq C_1 d(x, y)^a$$

for all $$x, y \in M$$ with $$d(x, y) < \varepsilon$$ and $$n \in \mathbb{N}.$$ Additionally, we assume that

$$\sup \{-\phi_n(x) : x \in M, n \in \mathbb{N}\} < \infty.$$ (H3*) Hypothesis on the measure: $$m$$ is a $$f_{1,\infty}$$-non-singular locally Hölder $$f_{1,\infty}$$-conformal Borel probability measure on $$M$$ as in [H3]. We also assume that $$m$$ is locally doubling, i.e., there are $$\rho > 0$$ and a constant $$L > 0$$ such that

$$m(B(x, 2r)) \leq L m(B(x, r))$$

for any ball $$B(x, r)$$ of radius $$0 < r \leq \rho$$ and $$x \in M.$$ Observe that by setting $$\theta_n(x) = e^{\phi_n(x)},$$ according to [2.5.1] we have that actually the maps $$f_n$$ are local homeomorphisms with uniform Lipschitz constant $$\varphi_n(x) = \|\theta_n\|_{\infty, V}$$ at a neighborhood $$V$$ of $$x.$$ Thus, hypothesis [H1*] [H3*] implies [H1] [H3].

**Theorem 3.2.** Let $$f_{1,\infty} = (f_n)_{n \in \mathbb{N}}$$ be a non-autonomous discrete dynamical system on the metric measure space $$(M, d, m)$$ under the assumption [H1*] [H2*] and [H3*]. Suppose also that there is $$a \geq 0$$ such that for $$m$$-almost every $$x \in M$$ there is $$k = k(x) \in \mathbb{N}$$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi_{k+i}(f_k^i(x)) < -a$$

where $$f_k^0 = \text{id}$$ and $$f_k^i = f_{k+i-1} \circ \cdots \circ f_k.$$ Then the probability measure $$m$$ is locally $$f_{1,\infty}$$-ergodic if $$a = 0$$ and strong locally $$f_{1,\infty}$$-ergodic if $$a > 0.$$

**Proof.** From Proposition [2.18] [17] implies that the measure $$m$$ is (strong) $$f_{1,\infty}$$-expandable if $$a = 0$$ (resp. if $$a > 0$$). Then, according to Corollary [2.17] for $$m$$-almost every $$x \in M$$ we have $$k = k(x) \in \mathbb{N}$$ and $$0 < \sigma = \sigma(x) < 1$$ (resp. $$0 < \sigma < 1$$ uniform on $$x$$) such that $$(k, x)$$ has infinitely many $$\sigma$$-hyperbolic times. By Propositions [2.15] [2.13] and [2.19], there are $$0 < \delta_k \leq \varepsilon$$ and $$\lambda = \sigma$$ such that $$(k, x)$$ has infinitely many regular $$(\delta_k, \lambda)$$-hyperbolic preballs with bounded distortion. Finally by Proposition [2.9] (resp. Remark [2.10]) we obtain that $$m$$ is locally (strong) ergodic. This completes the proof. \[\square\]

**Remark 3.3.** The assumptions [H1*] [H2] and [H3*] are satisfied if $$M$$ is a Riemannian compact manifold, $$m$$ is the normalized Lebesgue measure of $$M,$$ the fiber maps $$f_n : M \to M$$ are $$C^{1+\alpha}$$ local diffeomorphisms and the closure of $$f_{1,\infty} = (f_n)_{n \in \mathbb{N}}$$ is compact in the space of $$C^{1+\alpha}$$ local diffeomorphisms of $$M.$$ In this case, $$\theta_n(x) = \|Df_n(x)^{-1}\|$$ and $$\psi_n(x) = \log |\det Df_n(x)|.$$
where the fibers maps $f$ the form uniformly bounded. Thus, in order to satisfy also (H2*) we need to ask that $f \phi \Gamma$ a semigroup respectively. The compactness of the closure of $f_1, \ldots, f_d$ implies that $C_n, H_n$ and $\|\phi_n\|_\infty$ are, all of them, uniformly bounded. Thus, in order to satisfy also (H2) we need to ask that $f_\nu$ is conformal, i.e.,
\[ \|Df_\nu(x)^{-1}\| = \|Df_\nu(x)^{-1}\| \quad \text{for all } x \in M \text{ and } n \in \mathbb{N}. \]

4. Main results on semigroup actions

Let $(M, d, m)$ be a compact metric Borel probability space. We consider a skew-product of the form
\[ F : \Omega \times M \to \Omega \times M, \quad F(\omega, x) = (\sigma(\omega), f_\omega(x)). \]
where the fibers maps $f_\omega : M \to M$ are non-singular with respect $m$. We have in mind that $\sigma$ is the shift map on either $\Omega = \mathbb{N}$ or $\Omega = \{1, \ldots, d\}^\mathbb{N}$. In the first case we are modeling a non-autonomous dynamical systems $f_1, \ldots, f_d$ so that the fiber mas are locally constant. That is, $f_\omega = f_{\omega_1}$ if $\omega = (\omega_n)_{n \in \mathbb{N}}$ with $\omega_1 = i$. Now, we reinterpret in this setting some notions previously introduced for semigroup action or non-autonomous dynamical systems.

4.1. Ergodicity. We will say that $A \subset M$ is forward $F$-invariant set if $f_\omega(A) \subset A$ for all $\omega \in \Omega$. A forward $F$-invariant set $A$ with $m(A) > 0$ is called an ergodic component of $m$ with respect to $F$, if it does not admit any smaller forward $F$-invariant subset with positive $m$-measure. The measure $m$ is called $F$-ergodic if $M$ is an ergodic component. Equivalently, if $m(A) \in [0, 1]$ for all forward $F$-invariant measurable set $A$ of $M$. Finally, analogously to Definition 2.1 we define locally (strong) $F$-ergodicity in this context.

**Proposition 4.1.** If $m$ is locally strong $F$-ergodic then $m$ has finitely many ergodic components.

**Proof.** From the strong $F$-ergodicity we have $\epsilon > 0$ so that for any $F$-invariant set $A$ with positive measure there is an open ball $B$ of uniform fixed radius with $m(B) > \epsilon$ such that $m(B \setminus A) = 0$. Since $M$ is compact, there can be only finitely many disjoints $F$-invariant subsets with positive $m$-measure. Hence, we only have finitely many ergodic components of $m$. \qed

The formalism of the notion of exactness with respect to $m$, perviously defined (to the Lebesgue measure), in this context is the following. We say that $F$ is $m$-exact if for every open set $B$ of $M$, there are sequences $(u_k)_{k}$ and $(\omega_k)_{k}$ in $\mathbb{N}$ and $\Omega$ respectively such that
\[ M = \bigcup_{k \geq 1} f_{\omega_k}^u(B) \quad \text{modulo a set of zero } m\text{-measure}. \]

**Proposition 4.2.** If $F$ is $m$-exact and $m$ is locally $F$-ergodic then $m$ is $F$-ergodic.

**Proof.** Let $A$ be a forward $F$-invariant measurable set. By the local ergodicity of the measure $m$ we get an open set $B$ of $M$ such that $m(B \setminus A) = 0$. First observe the following.
Claim 4.3. For any function \( f \) we have that \( f(B) \setminus A \subset f(B \setminus A) \).

Proof. If \( x \in f(B) \setminus A \) then \( x = f(b) \notin A \) with \( b \in B \). Moreover, \( b \notin A \) since otherwise \( f(b) \in f(A) \subset A \). Thus, \( x \in f(B) \setminus A \) as required. \( \square \)

Now, using this claim and since \( F \) is \( m \)-exact and \( A \) is a forward \( F \)-invariant set, we get
\[
M \setminus A \subset \bigcup_{k \geq 1} f_{\omega_k}^n(B \setminus A)
\]
modulo a set of zero \( m \)-measure.

Since \( m \) is non-singular, we obtain that \( A \) has full \( m \)-measure and conclude the proof. \( \square \)

4.2. Proof of Theorems \( A \), \( B \) and \( C \) Let us consider a semigroup \( \Gamma \) finitely generated by \( C^{1+\alpha} \) local diffeomorphisms \( f_1, \ldots, f_d \) of a compact manifold \( M \). We consider the associated skew-product \( F \) as above. We will first deduce Theorem \( A \) from Theorem \( B \).

It is not difficult to see that the expansion assumption of Theorem \( A \) implies the non-uniform expansion assumption in Theorem \( B \). Thus, we only need to prove that \( F \) is exact (with respect to the Lebesgue measure \( m \)). This will be achieved in the following lemma:

Lemma 4.4. Assume that there exist \( \omega \in \Omega, C > 0 \) and \( \lambda > 1 \) such that
\[
||Df_\omega^n(x)|| \geq C\lambda^n||v|| \quad \text{for all } n \in \mathbb{N}, x \in M \text{ and } v \in T_xM.
\]

Then, given \( x \in M \) and \( \varepsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( M = f_\omega^n(B(x, \varepsilon)) \).

Proof. Assume by contradiction that \( M \neq f_\omega^n(B) \) for all \( n \in \mathbb{N} \) where \( B = B(p, \varepsilon) \) is the open ball of radius \( \varepsilon \) and centered at \( x \). Then, for each \( n \in \mathbb{N} \) we may a smooth curve \( \gamma_n \) joining \( f_\omega^n(p) \) to a point \( y_n \in M \setminus f_\omega^n(B) \) of length less than the diameter of the manifold. Since \( f_\omega^n \) is a local diffeomorphism, there is a unique curve \( \hat{\gamma}_n \) joining \( p \) to some point \( x \in M \setminus B \) such that \( f_\omega^n(\hat{\gamma}_n) = \gamma_n \). Hence the length of \( \gamma_n \) is
\[
\int ||\gamma_n'(t)|| dt = \int ||Df_\omega^n(\hat{\gamma}_n(t)) \cdot \gamma_n'(t)|| dt \geq C\lambda^n \int ||\gamma_n'(t)|| dt.
\]

But since length of \( \hat{\gamma}_n \) is larger than \( \varepsilon \) we arrive to a contradiction for \( n \) large enough. \( \square \)

Theorem \( B \) immediately follows from Remark \( 5.3 \), Theorem \( 5.1 \) and Proposition \( 4.2 \).

Similarly we will prove Theorem \( C \). First we need to prove that if \( \Gamma \) is non-uniformly expandable then the Lebesgue measure \( m \) is locally \( F \)-ergodic. To do this we proceed as in Theorem \( 5.2 \). Let \( A \subset M \) be a \( \Gamma \)-invariant set with \( 0 < m(A) < 1 \). Since \( \Gamma \) is non-uniformly expandable, we find a Lebesgue density point \( x \in A \) and a sequence \( \omega \in \Omega \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log ||Df_{\omega_{i+1}}(f_\omega^i(x))^{-1}|| < 0.
\]

Using Remark \( 5.3 \), we have that a non-autonomous dynamical system \( f_{1,\infty} = (f_n)_{n \in \mathbb{N}} \) where \( f_n = f_{\omega_n} \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \theta_{k+1}(f_1^i(x)) < 0.
\]
where \( \theta_n(x) = \|Df_n(x)^{-1}\| \) is the pointwise Lipschitz constant for the inverse branches of \( f_n \). According to Proposition 2.13 there is \( a = a(x) > 0 \) such that \( x \in M(1,a) \). Then, Proposition 2.16 implies that there is \( \sigma > 0 \) such that \((1,x)\) has infinitely many \( \sigma \)-hyperbolic times. By Propositions 2.15, 2.13 and 2.19 there are \( 0 < \delta_1 \leq \epsilon \) and \( \lambda = \sigma \) such that \((1,x)\) has infinitely many \((\delta_1, \lambda)\)-hyperbolic regular preballs with bounded distortion. Finally by Remark 2.11 we get that there is \( z \) such that \( m(B(z, \delta/2) \setminus A) = 0 \). This concludes that \( m \) is locally ergodic. Finally, since by assumption, also the action of \( \Gamma \) is exact, then Proposition 4.2 concludes that \( m \) is ergodic completing the proof of Theorem C.

4.3. Examples. We will show some new examples where our main result Theorem C applies. As we indicated in the introduction, [21, Thm. B] has a gap in its proof and only works for transitive group of diffeomorphisms. For semigroup action of local diffeomorphisms Theorem C requires that the action is exact instance transitive. From this theorem we cover the result in [5] on the ergodicity of the Lebesgue measure for expanding minimal conformal eomorphisms. But also Theorem C extends this result for semigroups of local diffeomorphisms as we will see below. First we introduce some definitions:

**Definition 4.5.** The action of a semigroup \( \Gamma \) of \( C^1 \) local diffeomorphisms of \( M \) is said to be backward expanding if there is for every \( x \in M \) there is \( h \in \Gamma \) such that \( \|Dh(x)^{-1}\| < 1 \).

Usually a semigroup action is said to be minimal if every orbit is dense. Since \( M \) is compact, this is equivalent to ask that the whole space can be covered by finitely many pre-images by elements of \( \Gamma \) of any open set. For this reason we introduce the following definition:

**Definition 4.6.** The action of a semigroup \( \Gamma \) of local diffeomorphisms of \( M \) is said to be backward minimal if for every open set \( U \subset M \) there are maps \( h_1, \ldots, h_n \) in \( \Gamma \) such that \( M = h_1(U) \cup \cdots \cup h_n(U) \).

Observe that if the action is backward minimal then it is also exact. Thus with the above definitions, the following result is a corollary of Theorem C.

**Corollary 4.7.** Every backward expanding and backward minimal semigroup action of conformal \( C^{1+\alpha} \) local diffeomorphisms of a compact manifold is ergodic with respect to Lebesgue measure.

**Proof.** We only need to note that if the action is backward expanding then also it is non-uniformly expandable. To do this, we first observe from the compactness of \( M \) and the \( C^1 \)-differentiability of the maps in \( \Gamma \) we get a finite open cover \( \{V_1, \ldots, V_m\} \) of \( M \) and maps \( h_1, \ldots, h_m \) in \( \Gamma \) such that \( \|Dh_i(x)^{-1}\| < \sigma < 1 \) for all \( x \in V_i \) for all \( i = 1, \ldots, m \). Thus, given any point \( x \in M \) we can construct a sequence \( (i_n)_{n \in \mathbb{N}} \) with \( i_n \in \{1, \ldots, m\} \) such that \( x \in V_{i_n} \) and \( h_{i_{n-1}, \ldots, h_{i_1}}(x) \in V_{i_n} \) for \( n \geq 2 \). Let \( k_n \) be the number of generators \( f_1, \ldots, f_k \) involved in the composition of \( h_{i_n} \). Observe that \( k_n \) only take finitely many values for all \( n \geq 1 \). In particular we have \( k \in \mathbb{N} \) such that \( k_n \leq k \) for all \( n \in \mathbb{N} \). Take \( \omega \in \Omega = \{1, \ldots, d\}^\mathbb{N} \) such that \( f^\omega_n = h_{i_n} \circ \cdots \circ h_{i_1} \) where \( \ell_n = k_1 + \cdots + k_n \) for all \( n \in \mathbb{N} \). Hence, by the conformality of the
generators of \( \Gamma \) we have
\[
\frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \log \| Df_{\omega_{i+1}}^j (f_{\omega_i}^j(x))^{-1} \| = -\frac{1}{\ell_n} \sum_{j=0}^{\ell_n} \log \| Dh_{ij+1} (h_{ij}(x)) \| \leq \frac{n}{\ell_n} \log \sigma \leq -\frac{1}{k} \log \sigma
\]

Now one only need to write \( \frac{1}{m} \sum_{i=0}^{m} a_i = \frac{\ell_m}{\ell_n} \sum_{i=0}^{\ell_m} a_i + \frac{1}{m} \sum_{i=\ell_m+1}^{\ell_n} a_i \) where \( \ell_m \leq n \leq \ell_{m+1} \).

Having into account that \( \ell_{m+1} - \ell_m \leq k \), \( \ell_m \leq km < kn \) and \( a_i = \log \| Df_{\omega_{i+1}}^i (f_{\omega_i}^i(x))^{-1} \| \) is uniformly bounded we conclude (3). \( \square \)

Theorem C can be also used to provide new examples of semigroup actions of diffeomorphisms which are not expanding as the following example show.

**Example 4.8.** Here, we give an example of a semigroup action which is exact and non-uniformly expanding, but not expanding. Consider the semigroup \( \Gamma \) generated by two \( C^{1+\alpha} \) diffeomorphisms \( f_0, f_1 \) on the unit interval \([0, 1]\) with the following properties:

1. \( f_0 \) and \( f_1 \) have both exactly two fixed points: \( f_0(0) = f_1(0) = 0 \) and \( f_0(1) = f_1(1) = 1 \);
2. \( Df_0(0) < 1, Df_0(1) = 1 \) and \( Df_1(0) > 1, Df_1(1) \leq 1 \);
3. \( \log Df_0(0)/ \log Df_1(0) \notin \mathbb{Q} \);
4. there are points \( 0 < a < c_1 < c_2 < b < 1 \) such that
   
   a. \( f_0([c_1, b]) \cup f_1([a, c_2]) \subseteq [a, b] \),
   b. \( Df_1(x) > 1 \) for all \( x \in [a, c_1] \) and \( Df_0(x) > 1 \) for all \( x \in [c_2, b] \),
   c. \( \min_{x \in [c_1, c_2]} \max \{ |Df_0(x), Df_1(x)| \} > 1 \).

Figure 1 shows a schematic graph of such diffeomorphisms. Since both generators have \( Df_1(1) \leq 1 \) the action of semigroup \( \Gamma \) is not backward expanding on \( M = [0, 1] \). We claim that the action of semigroup is non-uniformly expanding. More precisely, we show that for any \( x \neq 0, 1 \), there \( \omega = \omega(x) \in \Omega = \{0, 1\}^\mathbb{N} \) with
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df_{\omega_{i+1}}^i (f_{\omega_i}^i(x))^{-1} \| < 0. \quad (18)
\]

The conclusion consists of two parts, completely straightforward.

1. for any \( x \in (0, 1) \), there is a \( m = m(x) \) such that either \( f_0^m(x) \) or \( f_1^m(x) \) belongs to \([a, b]\);
2. for any \( x \in [a, b] \), there is a sequence \( \tilde{\omega} = (\tilde{\omega}_n)_{n \in \mathbb{N}} \in \Omega \) such that \( f_{\tilde{\omega}_0}^n(x) \in [a, b] \) and \( Df_{\tilde{\omega}_{n+1}}(f_{\tilde{\omega}_n}^n(x)) > 1 \) for any \( n \geq 0 \).

Now, for any \( x \in (0, 1) \), considering the concatenation \( \omega = \omega(x) \) of the words obtaining above we get that condition (15) holds along \( \omega \). To complete the proof we need to show that the action of \( \Gamma \) is exact. To do this, first we will observe that it is enough to prove that the orbit by the inverse semigroups, i.e., the semigroup generated by \( f_0^{-1} \) and \( f_1^{-1} \), of any point in \((0, 1)\) is dense in \( M = [0, 1] \). Indeed, the density of the backward orbit provides that for each open set \( U \) and point \( x \in (0, 1) \) we have a map \( h \in \Gamma \) such that \( x \in h(U) \). Since \((0, 1)\) is a Lindelöf space we can get a countable subcover and thus we get the action of \( \Gamma \) is exact. Now, the density of the backward orbit of any point \( x \in (0, 1) \) it follows by the non-resonant case in [15 Lem. 3]
which is our assumption that \( f_0 \) and \( f_1 \) has logarithmic rational independent derivatives at zero.

Finally, to conclude the paper, we will prove in the following proposition that there is no finitely generated semigroup action of diffeomorphisms in the assumptions of Theorem \[\text{B} \] as we claimed in the introduction.

**Proposition 4.9.** There are no non-uniformly expanding finitely generated semigroup actions of diffeomorphisms.

**Proof.** Suppose that \( \Gamma \) is a non-uniformly expanding finitely generated semigroup of diffeomorphisms. Hence, there exists \( \omega \in \Omega \) such that for \( m \)-almost every \( x \in M \) it holds

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df_{\omega^i}(x) \|^{-1} > 0.
\]

Since \( |T^{-1}|^{-1} \leq |\det T|^{1/s} \) for all linear operator \( T \) on a \( s \)-dimensional vector space, one has that

\[
\liminf_{n \to \infty} \frac{1}{n} \log |\det Df_{\omega}^n(x)| = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |\det Df_{\omega^i}(x)| > 0.
\]

Since \( f_i \) is a diffeomorphisms for all \( i = 1, \ldots, d \), changing variables we have that

\[
\int |\det Df_{\omega}^n(x)| \, dm(x) = 1.
\]
Hence, by Fatou-Lebesgue lemma since $|\det Df_{i}|$ is uniformly bounded for all $i = 1, \ldots, d$ and using the Jensen inequality we get that
\[
0 = \liminf_{n \to \infty} \frac{1}{n} \log \int |\det Df_{n}(x)| \, dm(x) \geq \liminf_{n \to \infty} \frac{1}{n} \int \log |\det Df_{n}(x)| \, dm(x) \geq \int \liminf_{n \to \infty} \log |\det Df_{n}(x)| \, dm(x) > 0.
\]
This provides a contradiction and concludes the proof of the proposition. \hfill \Box

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