Ważewski Topological Principle 
and V-bounded Solutions of Nonlinear Systems

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Abstract

We use the Ważewski topological principle to establish a number of new sufficient conditions for the existence of proper (defined on the entire time axis) solutions of essentially nonlinear nonautonomous systems. The systems under consideration are characterized by the monotonicity property with respect to a certain auxiliary guiding function $W(t, x)$ depending on time and phase coordinates. Another auxiliary function $V(t, x)$, which is positively defined in the phase variables $x$ for any $t$, is used to estimate the deviation of the proper solutions from the origin.

1 Introduction

The goal of this paper is to lay down sufficient conditions under which the nonlinear nonautonomous system of ODEs

$$\dot{x} = f(t, x)$$

(1)

where $f : \Omega \rightarrow \mathbb{R}^n$ ($\Omega \subset \mathbb{R}^{1+n}$) has a solution $x(t)$ extendable on the entire time axis and possessing the property that a given positively definite (with respect to $x$-variables) function $V(t, x)$ is bounded along the graph of $x(t)$. We especially focus on getting estimates for the function $V(t, x(t))$. The main results are obtained by using the Ważewski topological principle [1, 2, 3, 4], and some of them generalize the results of V. M. Cheresiz [5].

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It should be noted that the Ważewski topological principle was successfully exploited for proving the existence of bounded solutions to some boundary value problems in [6] and to quasihomogeneous systems in [7, 8] (see also a discussion in [9]).

To apply the Ważewski principle, along with the function $V$ which can be naturally considered as an analogue of time-dependent norm, we use another auxiliary function $W(t, x)$. In general case, this function is a sign-changing one, but it must have positively definite derivative by virtue of the system (1) in the domain where $V \geq v_0$ for some constant $v_0 > 0$. We call $V$ and $W$ the estimating function and the guiding function respectively and we say that together they form the V–W-pair of the system. Note that the term "guiding function" we borrow from [10] (originally — "guiding potential"). Basically topological method of guiding functions, which was developed by M. A. Krasnosel’ski and A. I. Perov, is an effective tool for proving the existence of bounded solutions of essentially nonlinear systems too (see the bibliography in [10, 11]). But, except [9, 13], in all papers known to us only independent of time guiding functions were used.

In [5], the role of V–W-pair plays the square of Euclidean norm together with an indefinite nondegenerate quadratic form. It appears that in this case sufficient conditions for the existence of bounded solutions as well as the estimates of their norms coincide with those obtained by means of technique developed in [14, 15] for indefinitely monotone (not necessarily finite dimensional) systems.

We shall not mention here another interesting approaches in studying the existence problem of bounded solutions to nonlinear systems, because they have not been used in this paper. For the corresponding information the reader is referred to [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

This paper is organized as follows. Section 2 contains necessary definitions, in particular, the notion of V–W-pair is introduced and some additional conditions imposed on estimating and guiding functions are described. In section 3 we prove two main theorems concerning the existence and the uniqueness of V-bounded solution to a nonlinear nonautonomous system possessing V–W-pair. Finally, in section 4 we show how the results of section 3 can be applied in the case where the estimating and guiding functions are nonautonomous quadratic forms. In this connection it should be pointed out that guiding quadratic forms play an important role in the theory of linear dichotomous systems with (integrally) bounded coefficients [29, 30, 31].

2 The definition of V–W-pair and the main assumptions

Let $\Omega$ be a domain of $\mathbb{R}^{1+n} = \{t \in \mathbb{R} \times \{x \in \mathbb{R}^n\}$ such that the projection of $\Omega$ on the time axis $\{t \in \mathbb{R}\}$ covers all this axis, and let in the system (1) $f(\cdot) \in C(\Omega \mapsto \mathbb{R}^n)$. It
will be always assumed that each solution of the system has the uniqueness property.

**Definition 1.** A function $V(\cdot) \in C^1(\mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^+)$ of variables $t \in \mathbb{R}, x \in \mathbb{R}^n$ will be called the estimating function, if for any $t \in \mathbb{R}$ the function $V_t(\cdot) := V(t, \cdot) : \mathbb{R}^n \mapsto \mathbb{R}^+$ is positively definite, has a unique critical point, the origin, and satisfies the condition $\lim_{\|x\| \to \infty} V_t(x) = \infty$.

Note that, as is well known, for any $t \in \mathbb{R}$ and each $c > 0$ the set $V_t^{-1}([0,c]) := \{x \in \mathbb{R}^n : 0 \leq V_t(x) \leq c\}$ is compact, its boundary is a closed connected hypersurface $V_t^{-1}(c)$ surrounding the origin, and in addition, if $c_2 > c_1$, then the set $V_t^{-1}([0,c_1])$ is a proper subset of the set $V_t^{-1}([0,c_2])$.

**Definition 2.** A global solution $x(t), t \in I$ of the system (1) is said to be V-bounded if $\sup_{t \in I} V(t, x(t)) < \infty$.

For $U(\cdot) \in C^1(\Omega \mapsto \mathbb{R})$ we put

$$\dot{U}_f := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f.$$ 

**Definition 3.** For the system (1), a function $W(\cdot) \in C^1(\Omega \mapsto \mathbb{R})$ will be called the guiding function concordant with $V$ if for some $v_0 > 0$ such that $\Omega \cap V^{-1}([v_0, \infty)) \neq \emptyset$ there exist functions

$$a(\cdot) \in C(\Omega \cap V^{-1}([v_0, \infty)) \mapsto (0, \infty)), \quad G(\cdot) \in C([v_0, \infty) \mapsto (0, \infty)), \quad g(\cdot) \in C([v_0, \infty) \mapsto (0, \infty))$$

satisfying the inequalities

$$|V_f(t,x)| \leq a(t,x)G(V(t,x)) \quad \forall(t,x) \in \Omega \cap V^{-1}([v_0, \infty)),$$

$$W_f(t,x) \geq a(t,x)g(V(t,x)) \quad \forall(t,x) \in \Omega \cap V^{-1}([v_0, \infty)),$$

$$g(v) \geq g(v_0) > 0 \quad \forall v \geq v_0.$$ 

**Definition 4.** For the system (1), the estimating function $V$ and the concordant guiding function $W$ will be called the V–W-pair of this system.

Define

$$F(v) := \int_{v_0}^{v} \left( g(u)/G(u) \right) du.$$ 

On the half-line $v \geq v_0$, this function is monotonically increasing and has the inverse $F^{-1}(\cdot) : [0, \infty) \mapsto [v_0, \infty)$. 

3
Denote by $\Pi_t := \{t\} \times \mathbb{R}^n$ the "vertical" hyperplane in $\mathbb{R}^{1+n}$, and in so far suppose that the system (1) has $V$–$W$-pair which satisfies the following additional conditions:

(A): $\lim_{v \to \infty} F(v) = \infty$;

(B): $\int_{-\infty}^0 \alpha(s) \, ds = \int_0^\infty \alpha(s) \, ds = \infty$, where

$$\alpha(t) := \inf \{ a(t, x) : x \in \Omega_t, V_t(x) > v_0 \} ;$$

(C): there exist numbers $w^+, w^-(w^+ > w^-)$ such that

$$V^{-1}([0, v_0)) \subset W^{-1}([w^-, w^+)), \quad V^{-1}(v_0) \subset \Omega,$$

and in addition, for any $t \in \mathbb{R}$ the number $w^+$ belongs to the range of $W_t(\cdot) := W(t, \cdot) : \Omega_t \mapsto \mathbb{R}$ where $\Omega_t := \Pi_t \cap \Omega$.

(D): the domain $W$, which is defined as such a connected component of the set $W^{-1}(w^-, w^+)$ that contains $V^{-1}([0, v_0))$, has the property: for any sufficiently large by absolute value negative $t$ there exists a set $M_t \subset W_t \cup \partial W_t \cap W_t^{-1}(w^+)$, where $W_t := W \cap \Pi_t$, such that the set $M_t \cap \partial W_t \cap W_t^{-1}(w^+) \neq \emptyset$ is a retract of $\bigcup_{s \geq t} \partial W_s \cap W_s^{-1}(w^+)$, but is not a retract of $M_t$, and, besides,

$$\liminf_{t \to -\infty} \sup \{ V_t(x) : x \in M_t \} = \nu < \infty.$$

Remark 1. If $V^{-1}([0, v_0)) \subset \Omega$ and $W(V^{-1}(v_0)) \in [w^-, w^+]$, then one can redefine the guiding function in the domain $V^{-1}([0, v_0))$ in such a way that $V^{-1}([0, v_0)) \subset W^{-1}([w^-, w^+])$.

Remark 2. The condition (D) is fulfilled if for any negative sufficiently large by absolute value $t$ there exists a finite collection $\{M_{t,j}\}$ of compact manifolds with border such that: $\partial M_{t,j} \cap \partial M_{t,k} = \emptyset, j \neq k$; the interior of $M_{t,j}$ belongs to $W_t$; the set $\bigcup_{j \geq 1} \partial M_{t,j}$ is a retract of $\bigcup_{s \geq t} \partial W_s \cap W_s^{-1}(w^+)$ and

$$\liminf_{t \to -\infty} \max \{ V_t(x) : x \in \bigcup_{j \geq 1} M_{t,j} \} = \nu < \infty.$$

In fact, in this case, taking into account that any compact manifold can not be retracted to its border, it is sufficient to put $M_t := \bigcup_{j \geq 1} M_{t,j}$.

Remark 3. Since the set $M_t \cap W_t^{-1}(w^+)$ is not empty and in any point of this set the function $V$ takes values not less than $v_0$, we get the inequality $\nu \geq v_0$. 

In
3 The existence and the uniqueness of V-bounded solution

The lemma given below open the door to estimation of solutions of the system (1) by means of functions $V$ in the presence of V–W-pair.

**Lemma 1.** Suppose that the system (1) has V–W-pair satisfying the condition (A). Let this system has a global solution $x(t), t \in I \subseteq \mathbb{R}$, such that

$$W^* := \sup_{t \in J} W(t, x(t)) < \infty, \quad W_* := \inf_{t \in J} W(t, x(t)) > -\infty$$

$J := \{t \in I : V(t, x(t)) > v_0\}.$

Then for any $t_0 \in I$, in the case where $V(t_0, x(t_0)) \leq v_0$ and $J \neq \emptyset$, the following inequality holds true

$$V(t, x(t)) \leq F^{-1}(W^* - W_0) \quad \forall t \in I \cap [t_0, \infty) \tag{5}$$

where

$$W_0 = \inf \{W(t, x(t)) : t \in I, V(t, x(t)) = v_0\}.$$

If $V(t_0, x(t_0)) > v_0$, then in the case where $V(t, x(t)) > v_0$ for all $t \in I \cap [t_0, \infty)$, we have

$$V(t, x(t)) \leq F^{-1}(F(V(t_0, x(t_0))) + W^* - W(t_0, x(t_0))), \tag{6}$$

and otherwise

$$V(t, x(t)) \leq \max \{F^{-1}(F(V(t_0, x(t_0))) + W^* - W(t_0, x(t_0))), F^{-1}(W^* - W_0)\}, \tag{7}$$

where

$$W^0 = \sup \{W(t, x(t)) : t \in I, V(t, x(t)) = v_0\}.$$

In addition, if $[t_0, \theta] \subset I$ is such a segment that $(t_0, t_* \subset J$ and

$$V(t_0, x(t_0)) = V(t_0, x(t_0)) = v_0,$$

then

$$V(t, x(t)) \leq F^{-1}\left(\frac{1}{2}[W(t^*, x(t^*)) - W(t_0, x(t_0))]\right) \quad \forall t \in [t_*, t^*]. \tag{8}$$

If the condition (B) is fulfilled and $[t_0, \theta] \subset I$, where $\theta$ is determined by the equality

$$\int_{t_0}^{\theta} \alpha(s) ds = (W^* - W_*)/g(v_0), \tag{9}$$

then there exists $\tau \in [t_0, \theta]$ for which $V(\tau, x(\tau)) \leq v_0$. 

5
Proof. Let the condition (A) is fulfilled. Throughout this proof, put \( v(t) := V(t, x(t)) \). Then in view of (2), (3) we have

\[
\left| \frac{d}{dt} F(v(t)) \right| = \frac{g(v(t)) |\dot{v}(t)|}{G(v(t))} \leq \frac{d}{dt} W(t, x(t)) \quad \forall t \in J. \tag{10}
\]

If \( v(t_0) \leq v_0 \) and \( J \neq \emptyset \), then there exists an interval \((t_*, T) \subset J \cap [t_0, \infty)\) such that \( v(t_*) = v_0 \). Since \( F(v(t_*)) = 0 \), then, as a consequence of (10) and inequality \( W(t_*, x(t_*)) \geq W_0 \), we have

\[
F(v(t)) \leq W(t, x(t)) - W(t_*, x(t_*)) \leq W^* - W_0 \quad \forall t \in [t_*, T],
\]

and from this it follows that \( v(t) \leq F^{-1}(W^* - W_0) \) for all \( t \in [t_*, T] \). Taking into account that \( v_0 = F^{-1}(0) \leq F^{-1}(W^* - W_0) \) and the function \( F^{-1}(\cdot) \) is monotonically increasing, one ascertains that (5) is true for all \( t \in [t_0, \infty) \cap I \).

If now \( v(t_0) > v_0 \), then until \( v(t) > v_0 \) we have

\[
F(v(t)) - F(v(t_0)) \leq W(t, x(t)) - W(t_0, x(t_0)).
\]

In the case where \( v(t) > v_0 \) for all \( t \in [t_0, \infty) \cap I \), we obtain the inequality (6). Otherwise there exists the nearest to \( t_0 \) moment \( t_* > t_0 \) such that \( v(t_*) = 0 \). Then \( W(t_*, x(t_*)) \leq W^0 \), and on the segment \([t_0, t_*]\), we arrive at

\[
F(v(t)) - F(v(t_0)) \leq W^0 - W(t_0, x(t_0)) \Leftrightarrow v(t) \leq F^{-1}(F(v(t_0)) + W^0 - W(t_0, x(t_0))).
\]

Taking into account the estimate obtained above for \( v(t) \) on the segment \([t_*, T]\), one ascertains that the inequality (7) holds true.

Now let us estimate \( v(t) \) on \([t_*, t^*] \subset I \) under the condition that \((t_*, t^*) \subset J \) and \( v(t_*) = v(t^*) = v_0 \). Let \( \dot{t} \) be a point at which \( v(t) \) reaches its maximum on \([t_*, t^*]\). Then from the inequality (10) it follows that

\[
W(t^*, x(t^*)) - W(t_*, x(t_*)) \geq \int_{t_*}^{t^*} g(v(t)) |\dot{v}(t)| \frac{dt}{G(v(t))} + \int_{\dot{t}}^{t^*} g(v(t)) |\dot{v}(t)| \frac{dt}{G(v(t))} \geq
\]

\[
2F(v(\dot{t})) - F(v(t_*)) - F(v(t^*)) = 2F(v(\dot{t})) \geq 2F(v(t)) \quad \forall t \in [t_*, t^*],
\]

and we obtain the inequality (8).

Next, let the condition (B) is fulfilled and \([t_0, \theta] \subset I \). Let us prove that there exists a number \( \tau \) belonging to \([t_0, \theta]\) for which \( v(\tau) \leq v_0 \). Obviously, it is sufficient to consider the case where \( v(t_0) > v_0 \). If we suppose the contrary, i.e. that \( v(t) > v_0 \) for all \( t \in [t_0, \theta] \), then we can find such a small \( \epsilon > 0 \) that the inequality \( v(t) > v_0 \) and
thus the inequality $\frac{d}{dt} W(t, x(t)) \geq \alpha(t) g(v_0) > 0$ holds true for all $t \in [t_0, \theta + \epsilon] \subset J$.

From this in virtue of (4) we arrive at inequality

$$W^* \geq W(\theta + \epsilon, x(\theta + \epsilon)) > g(v_0) \int_{t_0}^{\theta} \alpha(s) \, ds + W_*$$

which contradicts the definition of $\theta$. Hence, there do exists a number $\tau \in [t_0, \theta]$ with the required property. \hfill \Box

**Remark 4.** If it is impossible to find $F^{-1}(\cdot)$ explicitly, then in order to obtain efficient estimates of solutions one can replace the function $F(v)$ by another appropriate strictly monotonic function $F_1(v)$, which satisfies the inequality $F_1(v) \leq F(v)$ for all $v > v_0$ and tends to infinity when $v \to \infty$.

Put

$$w^0(t) := \max \{ W_t(x) : x \in V_t^{-1}(v_0) \},$$

$$w_0(t) := \min \{ W_t(x) : x \in V_t^{-1}(v_0) \},$$

$$\omega_0 := \liminf_{t \to -\infty} w_0(t),$$

$$\tilde{\omega} := \liminf_{t \to -\infty} (\inf \{ W_t(x) : x \in M_t, V_t(x) \geq v_0 \}).$$

It is clear that the inequalities

$$w_- \leq \omega_0 \leq w_+, \quad w_- \leq \tilde{\omega} \leq w_+$$

holds true once the condition (D) is satisfied.

Now we are in position to prove the following statement.

**Theorem 1.** Assume that the system (1) has $V$–$W$-pair satisfying the conditions (A)-(D). Let there exists a number $V^*$ such that

$$V^* > \max \{ F^{-1}(F(v) + w_+ - w), F^{-1}(w_+ - \omega_0) \},$$

and the set $\text{cls} \left( V^{-1}([0, V^*]) \cap \mathcal{W} \right)$ (here $\text{cls}$ means the closure operation) belongs to the domain $\Omega$. Then the system (1) has a $V$-bounded solution $x_*(t)$, $t \in \mathbb{R}$, which satisfies the inequality

$$V(t, x(t)) \leq F^{-1} \left( \frac{1}{2} \left( \sup_{s \geq t} w^0(s) - \inf_{s \leq t} w_0(s) \right) \right) \leq F^{-1} \left( \frac{w_+ - w_-}{2} \right) =: v_*, \quad \forall t \in \mathbb{R}.$$
Proof. By the conditions (C) and (D) the set \( \partial \mathcal{W} \cap W^{-1}(w^+) \) does not intersect the set \( V^{-1}([0, v_0]) \). Then from the definition of guiding function it follows that the set \( \partial \mathcal{W} \cap W^{-1}(w^+) \) coincides with the set of exit points of integral curves of the system (1) from the domain \( \mathcal{W} \) and it consists of the strict exit points only.

By the condition (D) we can choose a sequence of moments \( t_j \to -\infty, \ j \to \infty \), and a sequence of sets \( \mathcal{M}_{t_j} \subset \mathcal{W}_{t_j} \cup [\partial \mathcal{W}_{t_j} \cap W^{-1}(w^+)] \) in such a way that

\[
\sup \{ V_{t_j}(x) : x \in \mathcal{M}_{t_j} \} \leq \nu + \delta, \quad \inf \{ V_{t_j}(x) : x \in \mathcal{M}_{t_j} \}, V_{t_j}(x) \geq v_0 \geq \tilde{\omega} - \delta, \\
V^* > \max \{ \inf F^{-1}(\nu + \delta + w^+) - \tilde{\omega} + \delta, \inf F^{-1}(w^+ - \omega_0) \} \geq \nu + \delta > v_0
\]

for sufficiently small \( \delta > 0 \) and for all \( j \), and the intersection of each \( \mathcal{M}_{t_j} \) with \( \partial \mathcal{W} \cap W^{-1}(w^+) \) be the retract for the set of exit points from \( \mathcal{W} \cap \{ (t, x) : t \geq t_j \} \) but there does not exist a retraction of \( \mathcal{M}_{t_j} \) on \( \mathcal{M}_{t_j} \cap \partial \mathcal{W} \cap W^{-1}(w^+) \). Then by Ważewski principle for any \( j \) there exists a point \( (t_j, x_{0j}) \in \mathcal{M}_{t_j} \) such that the nonextendable solution \( x_j(t), t \in I_j \), which satisfies the initial condition \( x_j(t_j) = x_{0j} \) has the property

\[
(t, x_j(t)) \in \mathcal{W} \quad \forall t \in [t_j, \infty) \cap I_j.
\]

Observe that \( V(t_j, x_{0j}) \leq \nu + \delta \), and thus by the lemma 1 setting \( I = [t_j, \infty) \cap I_j \), \( v_j(t) = V(t, x_j(t)) \) we obtain \( v_j(t) < V^*, \ t \in I \). Hence, taking into account the condition of the theorem we have

\[
(t, x_j(t)) \in \text{cls} \left( V^{-1}([0, V^*]) \cap \mathcal{W} \right) \subset \Omega, \quad t \in [t_j, \infty) \cap I_j.
\]

In view of this we conclude that \( [t_j, \infty) \subset I_j \).

Next, applying the lemma 1 again, we can find \( \tau_j \) for which \( v_j(\tau_j) \leq v_0 \), and from (B) it follows that \( \tau_j \to -\infty, \ j \to -\infty \). Besides, if there exists at least one \( t \geq \tau_j \) for which \( v_j(t) > v_0 \), then there exist moments \( t_*, t^* \) such that \( \tau_j \leq t_* < t, \ t < t_* \) and \( v_j(t_*) = v_j(t^*) = v_0 \), but \( v_j(t) > v_0 \) for \( t \in (t_*, t^*) \). Then in virtue of inequality (8), for any pair of such moments we have

\[
v_j(t) \leq F^{-1} \left( \frac{w^0(t_*) - w_0(t_*)}{2} \right) \quad \forall t \in [t_*, t^*],
\]

and thus \( v_j(t) \leq v_0 \) for all \( t \geq \tau_j \).

Now one can prove the existence of V-bounded solution \( x_s(t) \) by the known scheme (see, e.g., [5, 8, 10]. Namely, if we denote by \( x(t, t_0, x_0) \) the solution which for \( t = t_0 \) takes the value \( x_0 \), then setting \( \xi_j := x_j(0) \), we obtain the equalities

\[
x_j(t) = x(t, 0, x_j(0)) = x(t, 0, \xi_j), \quad t \in [t_j, \infty).
\]

Having selected from the sequence \( \xi_j \in \text{cls} \left( V_0^{-1}([0, v_*]) \cap \mathcal{W}_0 \right) \subset \Omega_0 \) a subsequence converging to \( x_s \), put \( x_s(t) := x(t, 0, x_s) \). Using the reductio ad absurdum reasoning
it is easy to show that on the maximal existence interval $I$ of this solution we have the inclusion

$$(t, x_*(t)) \in \text{cls}(V^{-1}([0, v_*]) \cap \mathcal{W}).$$

Therefore $I = \mathbb{R}$ and $V(t, x_*(t)) \leq v_*$ for all $t \in \mathbb{R}$. Finally, taking into account (13), we arrive at the inequality (12).

\[\square\]

Remark 5. As is easily seen from the proof of the Theorem 1, it is sufficient to require that the inequalities (2),(3) hold true on the set $\text{cls}(V^{-1}([v_0, V^*]) \cap \mathcal{W})$ only, and the inequalities (4) — for $v \in [v_0, V^*]$ only.

**Theorem 2.** Let $\tilde{\Omega}$ be a subdomain of the domain $\Omega$ and let

$$\Omega^* := \{(t, z) \in \mathbb{R} \times \mathbb{R}^n : z = x - y, \ (t, x) \in \tilde{\Omega}, \ (t, y) \in \tilde{\Omega}\}.$$

Suppose that there exist functions $U(\cdot) \in C^1(\Omega^* \mapsto \mathbb{R})$, $H(\cdot), h(\cdot) \in C(\mathbb{R}_+ \mapsto \mathbb{R}_+)$, $b(\cdot), \beta(\cdot) \in C(\mathbb{R} \mapsto (0, \infty))$ such that:

1) the function $h(\cdot)$ is nondecreasing, the function $H(\cdot)$ is strictly monotonically increasing, and in addition,

$$\limsup_{t \to \pm \infty} \frac{1}{b(t)} \left| \int_0^t \beta(t) h \circ H^{-1} \left( \frac{u}{b(t)} \right) \, dt \right| = \infty$$

for any $u > 0$;

2) for all $(t, x), (t, y) \in \tilde{\Omega}$, the following inequalities hold true

$$|U(t, x - y)| \leq b(t) H(V(t, x - y)),
\quad U_x'(t, x - y) + U_x'(t, x - y) \cdot (f(t, x) - f(t, y)) \geq \beta(t) h(V(t, x - y)),
$$

where $V(t, x)$ is an estimating function. Then the system (1) cannot have two different nonextendable solutions $x(t), y(t), t \in \mathbb{R}$, whose graphs lie in $\tilde{\Omega}$ and which have the property

$$\sup_{t \in \mathbb{R}} V(t, x(t) - y(t)) < \infty. \quad (14)$$

**Proof.** Suppose that the system (1) has a pair of solutions $x(t), y(t), t \in \mathbb{R}$ such that $(t, x(t)), (t, y(t)) \in \tilde{\Omega}$ for all $t \in \mathbb{R}$. Let us show that for these solutions the condition (14) fails.

Consider the functions $u(t) := U(t, x(t) - y(t)), \ v(t) := V(t, x(t) - y(t))$. By condition, the function $u(\cdot)$ does not decrease. Hence, there exist (either finite or infinite) limits $u_* = \lim_{t \to -\infty} u(t), \ u^* = \lim_{t \to \infty} u(t)$. If we suppose that $x(t) \neq y(t)$, then there exists $t_0$ such that $x(t_0) \neq y(t_0)$, from whence $v(t_0) > 0$ and $\dot{u}(t_0) > 0$. For this reason, $u^* > u(t_0) > u_*$. 

9
First suppose that $u_* \geq 0$. Then $u(t) > u(t_0) > 0$ for $t > t_0$. Since
\[ v(t) \geq H^{-1}(u(t_0)/b(t)), \]
then
\[ u(t) \geq u(t_0) + \int_{t_0}^{t} \beta(s) h \circ H^{-1} \left( \frac{u(t_0)}{b(s)} \right) \, ds \]
and
\[ H(v(t)) \geq \frac{1}{b(t)} \int_{t_0}^{t} \beta(s) h \circ H^{-1} \left( \frac{u(t_0)}{b(s)} \right) \, ds, \quad t \geq t_0. \]
Thus, $\limsup_{t \to \infty} v(t) = \infty$.

Now suppose that $u_* < 0$. Then there exists $t'$ such that $u(t') < 0$. Then $u(t) \leq u(t')$ for all $t < t'$ and $v(t) \geq H^{-1}(|u(t')|/b(t))$ for $t < t'$. Then
\[ u(t') - u(t) \geq \int_{t}^{t'} \beta(s) h \circ H^{-1} \left( \frac{|u(t')|}{b(s)} \right) \, ds, \quad t \leq t', \]
from whence, as above, we have $\limsup_{t \to \infty} v(t) = \infty$. \qed

4 Studying V-bounded solutions by means of pair of quadratic forms

Consider the case where the V–W-pair of the system (1) is a pair of quadratic forms
\[ V(t, x) = \langle B(t)x, x \rangle, \quad W(t, x) = \langle C(t)x, x \rangle, \quad (15) \]
where $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{R}^n$, $\{B(t)\}_{t \in \mathbb{R}}$ and $\{C(t)\}_{t \in \mathbb{R}}$ are families of symmetric nondegenerate operators in $\mathbb{R}^n$ smoothly depending on parameter $t$ and satisfying the conditions:

(a): for any $t \in \mathbb{R}$, the operator $B(t)$ is positively definite;
(b): for any $t \in \mathbb{R}$, there exist projectors $P_+(t), P_-(t)$ on corresponding invariant subspaces $\mathbb{L}_+(t), \mathbb{L}_-(t)$ of operator $C(t)$ such that the restriction of $C(t)$ on $\mathbb{L}_+(t)$ (on $\mathbb{L}_-(t)$) is a positively definite (negatively definite) operator.

Observe that since the subspaces $\mathbb{L}_+(t), \mathbb{L}_-(t)$ are mutually orthogonal then the projectors $P_+(t), P_-(t)$ are symmetric.

From $C(t)$-invariance of these subspaces it follows that $P_\pm(t)C(t) = C(t)P_\pm(t)$ and, as a consequence, we have the representation
\[ C(t) = (P_+(t) + P_-(t))C(t)(P_+(t) + P_-(t)) = P_+(t)C(t)P_+(t) + P_-(t)C(t)P_-(t). \]
Put

\[ C_+(t) := P_+(t)C(t)P_+(t), \quad C_-(t) = P_-(t)C(t)P_-(t) \]

(16)

Obviously, the kernel of the operator \( C_+(t) \) (operator \( C_-(t) \)) is the subspace \( \mathbb{L}_-(t) \) (subspace \( \mathbb{L}_+(t) \)), and the restriction of this operator on \( \mathbb{L}_+(t) \) (on \( \mathbb{L}_-(t) \)) is a positively definite (negatively definite) operator.

Let the right-hand side of the system (1) admits the representation

\[ f(t, x) = A(t, x)x + f_0(t) \]

where \( A(\cdot, \cdot) \in C(\Omega \mapsto \text{Hom} \mathbb{R}^n) \), \( f_0(t) := f(t, 0) \), and in addition,

(c): there exist numbers \( V^* > 0, w_- < 0, w^+ > 0 \) such that the domain \( \Omega \) contains the set

\[ V^{-1}([0, V^*]) \cap W^{-1}([w_-, w^+]) = \{(t, x) \in \mathbb{R}^{1+n} : \langle B(t)x, x \rangle \leq V^*, w_- \leq \langle C(t)x, x \rangle \leq w^+ \}. \]

The number \( v_0 \) in the definition of the guiding function must be chosen in such a way that the set inclusions from condition (C) hold true. Denote by \( \lambda = \lambda^+(t) \) and \( \lambda = \lambda_-(t) \), respectively, the maximal and the minimal characteristic values of the pencil \( C(t) - \lambda B(t) \). Since

\[ \lambda^+(t) = \max \{ \langle C(t)x, x \rangle : \langle B(t)x, x \rangle = 1 \}, \]
\[ \lambda^-(t) = \min \{ \langle C(t)x, x \rangle : \langle B(t)x, x \rangle = 1 \} \]

(see, e.g., [32]), then taking into account that the function \( W_t(x) \) has the unique critical point \( x = 0 \), we have

\[ w^0(t) := \max \{ \langle C(t)x, x \rangle : \langle B(t)x, x \rangle \leq v_0 \} = \lambda^+(t)v_0, \]
\[ w_0(t) := \min \{ \langle C(t)x, x \rangle : \langle B(t)x, x \rangle \leq v_0 \} = \lambda_-(t)v_0. \]

Hence, in order that the set inclusions from condition (C) hold true it is sufficient to assume that

(d): the inequalities

\[ \lambda_-(t)v_0 \geq w_-, \quad \lambda^+(t)v_0 \leq w^+ \quad \forall t \in \mathbb{R}. \]

are fulfilled

Now we impose a number of conditions on the mappings \( A(t, x) \) and \( f_0(t) \) to ensure the existence of \( V \)-bounded solution of the system (1) in virtue of the Theorem 1 and the Remark 5.
Let $\Lambda_V(t, x)$ be the maximal by absolute value characteristic value of the pencil

$$B(t)A(t, x) + A^*(t, x)B(t) + \dot{B}(t) - \lambda B(t)$$

(here $A^*$ is the operator conjugate with $A$), and let $\lambda_W(t, x)$ be the minimal characteristic value of the pencil

$$C(t)A(t, x) + A^*(t, x)C(t) + \dot{C}(t) - \lambda B(t).$$

Put

$$\varphi(t) := \sqrt{\langle B(t)f_0(t), f_0(t) \rangle}, \quad \psi(t) := \sqrt{\langle B^{-1}(t)C(t)f_0(t), C(t)f_0(t) \rangle}.$$

Then taking into account the inequalities

$$\left| \langle 2B(t)A(t, x) + \dot{B}(t) \rangle x, x \rangle \right| \leq |\Lambda_V(t, x)| \langle B(t)x, x \rangle,$$

$$\langle 2C(t)A(t, x) + \dot{C}(t) \rangle x, x \rangle \geq \lambda_W(t, x) \langle B(t)x, x \rangle,$$

$$\langle B(t)f_0(t), x \rangle \leq \varphi(t) \sqrt{\langle B(t)x, x \rangle},$$

$$\langle C(t)f_0(t), x \rangle \geq -\psi(t) \sqrt{\langle B(t)x, x \rangle}$$

we obtain

$$\left| \dot{V}_{f(t, x)}(t, x) \right| \leq |\Lambda_V(t, x)| V(t, x) + 2\varphi(t) \sqrt{V(t, x)},$$

$$\dot{W}_{f(t, x)}(t, x) \geq \lambda_W(t, x)V(t, x) - 2\psi(t) \sqrt{V(t, x)}.$$

Now impose on the system the following conditions:

(e): there exist positive constants $c_1, c_2, c_3, \sigma$ such that

$$c_2^2 < v_0, \quad \sigma \leq 1,$$

and in the domain $V^{-1}([v_0, V^*]) \cap W^{-1}([w_-, w_+])$ the inequalities

$$2\varphi(t) \leq c_1 |\Lambda_V(t, x)|, \quad 2\psi(t) \leq c_2 \lambda_W(t, x), \quad |\Lambda_V(t, x)| \leq c_3 (B(t)x, x)^\sigma \lambda_W(t, x);$$

holds true;

(f): the function $\alpha(t) := \inf \{\lambda_W(t, x) : x \in \Omega_t, \ V(t, x) > v_0 \}$ has the properties

$$\int_{-\infty}^0 \alpha(s) \, ds = \int_0^\infty \alpha(s) \, ds = \infty.$$
Put
\[ G(v) = c_3v^\sigma(v + c_1\sqrt{v}), \quad g(v) = v - c_2\sqrt{v}, \quad a(t,x) = \lambda_W(t,x), \]
and in order to satisfy the rest of conditions which guarantee the existence of \( V \)-bounded solution, define the family of ellipsoidal disks
\[ \mathcal{M}_t := \{t\} \times \{x \in \mathbb{L}_+(t) : \langle C(t)x,x \rangle \leq w^+ \}. \]

Now prove the following proposition.

**Lemma 2.** For any \( c > 0, t_0 \in \mathbb{R} \) there exists a retraction of the set \( W^{-1}(c) \) to the ellipsoid
\[ \{(t,x) \in \mathbb{R}^{1+n} : t = t_0, \quad x \in \mathbb{L}_+(t_0), \quad \langle C(t_0)x,x \rangle = c \}. \]

**Proof.** First we observe that for arbitrary \( t \in \mathbb{R} \) \( c > 0 \) there exists a retraction of \( \mathcal{N}_{t,c} := \{x \in \mathbb{R}^n : \langle C(t)x,x \rangle = c \} \) to the intersection of this set by the subspace \( \mathbb{L}_+(t) \).

In fact, one can define such a retraction by a mapping \( x \mapsto \theta(t,x)P_+(t)x \), provided that the scalar function \( \theta(t,x) \) is determined from condition \( \langle C_+(t)x,\theta(t,x)x \rangle = c \) for all \( x \in \mathcal{N}_{t,c} \). Since \( c > 0 \), then \( \mathcal{N}_{t,c} \cap \mathbb{L}_+(t) = \varnothing \), and hence, \( \langle C_+(t)x,x \rangle > 0 \) for all \( x \in \mathcal{N}_{t,c} \). Therefore
\[ \theta(t,x) = \sqrt{\frac{c}{\langle C_+(t)x,x \rangle}}. \]

Now it remains only to show that the set \( \{t_0\} \times \mathcal{N}_{t_0,c} = W^{-1}(c) \cap \Pi_{t_0} \) is a retract of \( W^{-1}(c) \). Introduce the operator \( S(t) := \sqrt{C^2(t)} = C_+(t) - C_-(t) \), where the operators \( C_\pm(t) \) are defined in (16). Then we get
\[ C(t) = S(t)(P_+(t) - P_-(t)) = (P_+(t) - P_-(t))S(t). \]

The quadratic form \( \langle C(t)x,x \rangle \) by means of the substitution \( x = \left[\sqrt{S(t)}\right]^{-1}y \) is reduced to \( \langle (P_+(t) - P_-(t))y,y \rangle \). Obviously, \( P_+(t) - P_-(t) \) is a symmetric orthogonal inversion operator:
\[ (P_+(t) - P_-(t))^* = P_+(t) - P_-(t), \quad (P_+(t) - P_-(t))^2 = E. \]

From the representation of projector via the Riesz formula (see, e.g., [29, c. 34]) it follows that the projectors \( P_\pm(t) \) smoothly depend on parameter. Therefore the mutually orthogonal subspaces \( \mathbb{L}_+(t) \) and \( \mathbb{L}_-(t) \) have constant dimensions \( n_+, n_- \) and define smooth curves \( \gamma_+, \gamma_- \) in Grassmannian manifolds \( G(n,n_+) \) and \( G(n,n_-) \) respectively. Since \( G(n,n_+) \) is a base space of a principal fiber bundle, namely,
\[ G(n, n_\perp) = O(n)/O(n_\perp) \times O(n_-), \] then there exists a smooth curve \( Q(t) \) in \( O(n) \), which is projected onto \( \gamma_+(t) \), the operator \( Q(t_0) \) being the identity element \( E \) of the group \( O(n) \). Obviously, \( \mathbb{L}_+(t) = Q(t)\mathbb{L}_+(t_0) \) and, as a consequence, \[ P_\pm(t) = Q(t)P_\pm(t_0)Q^{-1}(t). \]

From the above reasoning it follows that the change of variables
\[
x = \left[ \sqrt{S(t)} \right]^{-1} Q(t) \sqrt{S(t_0)} y
\]
reduces the quadratic form \( W(t, x) := \langle C(t)x, x \rangle \) to \( W(t_0, y) = \langle C(t_0)y, y \rangle \), and then the mapping
\[
\mathbb{R} \times \mathbb{R}^n \mapsto \{ t_0 \} \times \mathbb{R}^n : \quad (t, x) \mapsto \left( t_0, \sqrt{S(t)}Q^{-1}(t) \left[ \sqrt{S(t_0)} \right]^{-1} x \right)
\]
define a retraction of the set \( W^{-1}(c) \) to the set \( W^{-1}(c) \cap \Pi_{t_0} \). \(\square\)

Denote by \( \lambda = \lambda^\perp(t) \) the minimal characteristic value of the pencil \( P_+(t)[C(t) - \lambda B(t)]|_{L_+(t)} \).

Then
\[
\max\{ \langle B(t)x, x \rangle : x \in M_t \} = \frac{w^+}{\lambda^\perp(t)},
\]
and the condition (D) will be fulfilled once we suppose that \((g)\): the inequality
\[
\limsup_{t \to -\infty} \lambda^\perp(t) > 0
\]
holds true.

In this case we have
\[
\nu = \liminf_{t \to -\infty} \frac{w^+}{\lambda^\perp(t)}.
\]

From the above reasoning it follows that
\[
\min\{ \langle C(t)x, x \rangle : x \in M_t, \langle B(t)x, x \rangle \geq v_0 \} = \lambda^+(t) v_0.
\]

Hence,
\[
\tilde{\omega} = \liminf_{t \to -\infty} \lambda^+(t) v_0, \quad \omega_0 := \liminf_{t \to -\infty} \lambda^-(t) v_0.
\]

We have established the following result.
Theorem 3. Let the functions $V(t, x)$, $W(t, x)$ are defined by (15) and the system (1) satisfies the conditions (a)–(g). Put

$$F(v) := \frac{1}{c_3} \int_{v_0}^{v} \frac{u - c_2 \sqrt{u}}{u^{\sigma}(u + c_1 \sqrt{u})} \, du$$

and assume that the inequality (11) is valid where the numbers $\nu, \tilde{\omega}, \omega_0$ are defined by the formulae (17), (18), and also $\text{cls} \left( V^{-1}([0, V^*]) \cap W \right) \subset \Omega$. Then the system has a $V$-bounded solution $x(t)$ such that

$$V(t, x(t)) \leq F^{-1} \left( \frac{v_0}{2} \left[ \sup_{s \geq t} \lambda^+(s) - \inf_{s \leq t} \lambda^-(s) \right] \right) \quad \forall t \in \mathbb{R}.$$ 

To make the estimates of $V$-bounded solution more efficient let us utilize Remark 4 and estimate the $F(v)$ from below. If $\sigma < 1$, then for $u \geq c_2^2$ we have

$$\frac{u - c_2 \sqrt{u}}{u^{\sigma}(u + c_1 \sqrt{u})} = \frac{1 - c_2 u^{-1/2}}{u^{\sigma}(1 + c_1 u^{-1/2})} \geq \frac{\sqrt{v_0}}{\sqrt{v_0} + c_1} u^{-\sigma}(1 - (u/c_2^2)^{-1/2}) \geq \frac{\sqrt{v_0}}{\sqrt{v_0} + c_1} (u^{-\sigma} - c_2^{-1} u^{-1/2 - \sigma/2}).$$

Hence, in this case

$$F(v) \geq \frac{\sqrt{v_0}}{(1 - \sigma)(\sqrt{v_0} + c_1)c_3} \left[ v^{1-\sigma} - 2c_2^{1-\sigma} v^{(1-\sigma)/2} - v_0^{1-\sigma} + 2c_2^{1-\sigma} v_0^{(1-\sigma)/2} \right] =: F_1(v)$$

and

$$F_1^{-1}(z) = \left[ \frac{(1 - \sigma)(\sqrt{v_0} + c_1)c_3}{\sqrt{v_0}} \sqrt{\frac{v_0}{\sqrt{v_0} + c_1} z + \left( \frac{v_0^{(1-\sigma)/2} - c_2^{1-\sigma}}{c_2^{1-\sigma}} \right)^2} + c_2^{1-\sigma} \right]^{2/(1-\sigma)}.$$

If $\sigma = 1$, then

$$F(v) \geq \frac{\sqrt{v_0}}{(\sqrt{v_0} + c_1)c_3} \left[ \ln v - \ln v_0 + 2c_2 v^{-1/2} - 2c_2 v_0^{-1/2} \right].$$

In this case we put

$$F_1(v) := \frac{\sqrt{v_0}}{(\sqrt{v_0} + c_1)c_3} \left[ \ln v - \ln v_0 - 2c_2 v_0^{-1/2} \right].$$

Then

$$F_1^{-1}(z) = v_0 \exp \left( \frac{\sqrt{v_0} + c_1}{\sqrt{v_0}} z + 2c_2 v_0^{-1/2} \right).$$

Approaching the limit as $v_0 \to c_2^2$, we obtain the following proposition.
Theorem 4. Let the conditions of the theorem 3 hold true for $v_0 = (1 + \epsilon)c_2^2$ and for all sufficiently small $\epsilon > 0$. Then the system (1) has a solution $x(t)$ which is extendable on the entire real axis and which for any $t \in \mathbb{R}$ satisfies the inequality

$$\langle B(t)x(t), x(t) \rangle \leq \begin{cases} 
C_1 \sqrt{\sup_{s \geq t} \lambda^+(s) - \inf_{s \leq t} \lambda_-(s) + c_2^{1-\sigma}} & , \quad \sigma < 1, \\
(ce_2)^2 \exp \left[ C_2 \left( \sup_{s \geq t} \lambda^+(s) - \inf_{s \leq t} \lambda_-(s) \right) \right] & , \quad \sigma = 1,
\end{cases}$$

where $C_1 := \sqrt{(1 - \sigma)c_2}, C_2 := \frac{(c_2 + c_2^2)c_2^2}{2}$.

Lastly, we prove the following uniqueness theorem.

Theorem 5. Let for the system (1) the following conditions hold true:

1) there exists a domain $\hat{\Omega} \subseteq \Omega$ such that

$$f(t, x) - f(t, y) = \hat{A}(t, x, y)(x - y) \quad \forall (t, x, (t, y) \in \hat{\Omega},$$

where $\hat{A}(t, x, y) \in \text{Hom}(\mathbb{R}^n)$;

2) there exists a smooth family of operators $\{\hat{C}(t)\}_{t \in \mathbb{R}}$ in $\mathbb{R}^n$ and a function $\hat{\beta}(\cdot) \in C(\mathbb{R} \mapsto (0, \infty))$ such that the minimal characteristic value $\hat{\lambda}(t, x, y)$ of the pencil

$$\hat{C}(t)\hat{A}(t, x, y) + \hat{A}^*(t, x, y)\hat{C}(t) + \frac{d}{dt}\hat{C}(t) - \lambda B(t)$$

satisfies the inequality

$$\hat{\lambda}(t, x, y) \geq \hat{\beta}(t) \quad \forall (t, x, (t, y) \in \hat{\Omega};$$

3) the maximal by absolute value characteristic value $\hat{\Lambda}(t)$ of the pencil $\hat{C}(t) - \lambda B(t)$ satisfies the equality

$$\lim_{t \to \pm \infty} \sup \frac{1}{\hat{\Lambda}(t)} \left| \int_0^t \frac{\beta(s)}{\hat{\Lambda}(s)} ds \right| = \infty.$$

Then the system (1) has at most one solution $x(t)$ defined on the entire axis, with the graph belonging to $\hat{\Omega}$ and such that $\sup_{t \in \mathbb{R}} \langle B(t)x(t), x(t) \rangle < \infty$.

Proof. Having observed that the theorem’s conditions ensure the fulfillment of the inequalities

$$|\langle \hat{C}(t)(x - y), x - y \rangle| \leq \hat{\Lambda}(t) \langle B(t)(x - y), x - y \rangle,$$

$$\langle (2\hat{C}(t)\hat{A}(t, x, y) + \frac{d}{dt}\hat{C}(t))(x - y), x - y \rangle \geq \hat{\beta}(t)\langle B(t)(x - y), x - y \rangle,$$

it is sufficient to apply the theorem 2 for the case where $U(t, x) = \langle \hat{C}(t)x, x \rangle$, $V(t, x) = \langle B(t), x, x \rangle$. \qed
Conclusions.

The technique applied in this paper for studying the essentially nonlinear nonautonomous systems by means of a pair of auxiliary functions allows us to generalize a number of earlier known results concerning the questions of existence and uniqueness of bounded and proper solutions. In the case where the estimating function is a quadratic form with varying matrix the theorem 4 can be efficiently applied to establish asymptotic estimates of solutions for \( t \to \infty \).

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