The PSPACE-hardness of understanding neural circuits

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Abstract

In neuroscience, an important aspect of understanding the function of a neural circuit is to determine which, if any, of the neurons in the circuit are vital for the biological behavior governed by the neural circuit: i.e., which sets of neurons, when deactivated, lead to an elimination of the behavior being studied. Typically, one is interested in finding the smallest such sets. A similar problem is to determine whether a given small set of neurons may be enough for the behavior to be displayed, even if all other neurons in the circuit are deactivated. Such a subset of neurons form what is called a degenerate circuit for the behavior being studied.

Recent advances in experimental techniques have provided researchers with tools to activate and deactivate subsets of neurons with a very high resolution, even in living animals. The data collected from such experiments may be of the following form: when a given subset of neurons is deactivated, is the behavior under study observed?

This setting leads to the algorithmic question of determining the minimal vital or degenerate sets of neurons, when one is given as input a description of the neural circuit. The algorithmic problem entails both figuring out which subsets of neurons should be perturbed (activated/deactivated), and then using the data from those perturbations to determine the minimal vital or degenerate sets. Given the large number of possible perturbations, and the recurrent nature of neural circuits, the possibility of a combinatorial explosion in such an approach has been recognized in the biology and the neuroscience literature, e.g. in a paper of Koch (Science, 337 (6094), pp. 531–532) and more recently in a paper of Kumar et al. (Trends in Neurosciences 36 (10), pp. 579–586). In a recent paper, Ramaswamy (biorxiv, 2019) took a step towards formulating the question in terms of computational complexity theory and established NP-hardness for some of these problems.

In this paper, we prove that the problems of finding minimal or minimum-size degenerate sets, and of finding the set of vital neurons, of a neural circuit given as input, are in fact PSPACE-hard. Further, the hardness results hold even when all the neurons in the neural circuit are threshold neurons with weights coming from a fixed, constant size set and have a bounded number of connections. More importantly, we prove our hardness results by showing that a simpler problem, that of simulating such neural circuits, is itself PSPACE-hard.

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1 Introduction

1.1 Background

In neuroscience, an important aspect of understanding the functioning of a neural circuit in a biological system is determining which neurons in the neural circuit are critical for the functioning of the system. For example, Flood et al. [3] showed that when the activity of a single specific pair of neurons in Drosophila is suppressed, certain feeding behaviors of the organism are eliminated. For the same organism, Bohra et al. [1] identified a similar small set of specific neurons whose inactivation leads to the elimination of the organism’s aversive response to bitter taste. The term vital set has been proposed for such sets of neurons [14]. A related aspect is that of identifying subsets of neurons such that as long as neurons in such a subset remain active, the inactivation of any other neurons outside the subset does not eliminate the behavior. The term degeneracy has been proposed to describe such phenomena (both in the setting of neuroscience, as well as in the setting of other biological systems) [2].

In agreement with the terminology, the term degenerate circuit was proposed in [14] to describe such subsets of neurons. Typically, with respect to a given behavior, one would be interested in determining the smallest or minimal vital or degenerate subsets of neurons, or the sizes of such sets.

Recent advances in experimental technology, especially optogenetics, have allowed researchers to achieve precise selective activation and deactivation of specific subsets of neurons, even those of live animals, and to record changes in the behavior of such neurons as a result of such perturbations (see, e.g. [4, 12, 18] for some recent advances related to these techniques). Thus, while studying vital or degenerate neurons for a given behavior, a researcher may be able to collect data of the form: when a particular subset $S$ of neurons is deactivated, is the behavior being studied still displayed?

This setting leads to an algorithmic question: given a description of the neural circuit and of the behavior under study (which will typically be encoded as the eventual activation of some output neurons), determine the properties of the vital or degenerate sets of neurons for that behavior. A particular algorithm for this problem may follow the above strategy of reading out the behavior of the neural circuit in response to selective activation and deactivation of specific neurons.

Given the large number of perturbations possible, it is not surprising, however, that the spectre of combinatorial explosion does cloud this strategy: we briefly mention three works in this direction here. Koch [7] highlighted combinatorial explosion as being a roadblock in understanding the behavior of general biological systems with heterogeneous components. In 2013, Vlachos et al. [17] proposed a “prediction and identification challenge”, where they invited readers to determine the functionalities of synthetic neural circuits using a set of allowed observations. Kumar et al. [9] noted the possibility of combinatorial explosion specifically in the setting of perturbative experiments in neuroscience by “selective modulation”, especially pointing to the recurrent nature of neural circuits as a possible source of difficulty. We remark here that by its nature, a neural circuit is recurrent in the sense that the activation state of a neuron at a given time can depend upon its own state at a previous time, and it is also in this aspect that models of neural circuits differ qualitatively from the usual circuit models in computational complexity theory.

In a recent paper, Ramaswamy [14] took a step towards studying this algorithmic difficulty in the context of computational theory. Starting with formal notions of vital sets and degenerate sets in the context of models of neural circuits, he formalized the following problems (here, following Ramaswamy’s notation, “$k$-vital set” denotes a vital set of size exactly $k$; we defer formal definition to section 2). Given as an input a description of the neural circuit, determine

1. whether there is degenerate circuit of size $k$.
2. whether there is a minimal degenerate circuit of size $k$.
3. a minimum size degenerate circuit.
The main result of Ramaswamy’s paper [14] is that each of these problems is NP-hard.

1.2 Our contributions

We show that problems 1-4 above are in fact PSPACE-hard. We also show that it is PSPACE-hard to find \((c \cdot \log n)\)-approximate minimum degenerate circuit, for any constant \(c\) and \(n\) is size of neural circuit. In fact, the PSPACE hardness for these problems turns out to be an immediate corollary of our main result, which establishes the PSPACE hardness for a superficially much simpler problem: that of simulating a neural circuit.

More specifically, we use a standard, synchronous, discrete-time model of neural circuits where each neuron is a vertex in a directed graph (which is not necessarily acyclic, in order to account for the recurrent nature of neural circuits). At each time \(t\), every neuron computes a Boolean function of the activation states at time \(t-1\) of those neurons from which it receives an incoming edge (in other words, we have a uniform conduction delay of 1). We further restrict these Boolean function to be threshold functions with coefficients and threshold potential coming from a fixed, constant size set of small integers. Such neurons are referred to as threshold neurons in the literature. Further, the neural circuit has a specified input neuron \(I\) with no incoming edges, and a specified output neuron \(O\) with no outgoing edges. The Neural-Circuit-Simulation problem asks: Given a neural circuit \(C\) as above as input, and given the initial state where at time \(t = 0\), neuron \(I\) is stimulated (i.e., set to 1), and all other neurons in \(C\) are not stimulated (i.e., set to 0), is there a future time \(t \geq 0\) when \(O\) becomes stimulated (i.e., set to 1)? Our main result then says:

**Theorem 1.1.** The problem Neural-Circuit-Simulation is PSPACE-complete.

As stated above, this result holds even when each neuron is restricted to be a threshold neuron with constant coefficients and threshold potential. Further, it also holds when each neuron in the input circuit is constrained to have at most 6 connections. We also emphasize that the recurrent nature of the neural circuits is the main feature underlying the result.

A more formal description of the result and the required technical definitions can be found in section 2. The proof of the main theorem appears in section 3 (as the proof of Theorem 3.7). The PSPACE-hardness of problems 1-4 in Ramaswamy’s list, claimed above, is an easy consequence of the PSPACE-hardness of the Neural-Circuit-Simulation problem. For completeness, we provide the proofs at the end of section 3.

1.3 Related work

We conclude the introduction with a brief description of some related work. Various models have been proposed to model the computational aspects of biological neurons; we refer to the book chapter by Koch, Mo and Softky [8] for a concise comparison of various such models. An important class among these is of models based on thresholds, where each neuron computes either a linear or polynomial threshold function at each time. Low-degree polynomial threshold neurons (known as sigma-pi neurons in the literature) were proposed to improve upon linear threshold neurons (also known as McCulloch–Pitts neurons) with respect to the modeling of “dendritic trees”: we refer to the survey of Mel [13, section 4.4.4] for further discussion of the motivation behind different threshold models.

A particular linear threshold model that has been extensively studied is the spiking neuron model [10, 11]. This is a continuous time model where each neurons performs a weighted integration (over time) of
“spikes” of activation that it observes from other neurons it is connected to, and fires only at those times when the integral crosses a threshold. Maas [10] showed that this model can be immensely powerful: for a given $d$, there is a fixed spiking neural network $N(d)$, such that for any Turing machine $M$ with $d$ tapes, there is a suitable rational assignment of weights that each neuron in $N(d)$ uses to weigh its different neurons, such that with the corresponding weights, $N(d)$ can simulate $M$ in the sense that it can encode the output of $M$ in the (continuous time) timing of its spike activity (we refer to [10] for further details). Note, however, that this is in sharp contrast to the model studied in our paper: here, the conduction delays between neurons are uniform and fixed to be 1, and further, the weights used by the neurons are drawn from a fixed, constant size set. The spiking neuron model has also been the subject of some recent work dealing with questions of asynchronous computation [5, 6].

Finally, we refer to the paper by Schmitt [15] for theoretical comparisons between the power of various models of neurons.

2 Preliminaries

**Definition 2.1 (Neural circuits).** A neural circuit is a directed graph $G = (V, E)$, with each vertex $v \in V$ equipped with a Boolean function $f_v : \{0, 1\}^{\mathcal{P}(v)} \to \{0, 1\}$. Here, $\mathcal{P}(v)$ denotes the set of those vertices $u$ in $V$ for which the directed edge $(u, v)$ is present in $E$. Following standard convention we also refer to the vertices as neurons. Each neural circuit has two specially designated neurons: the input neuron $I$ and the output neuron $O$. The state of a neural circuit at time $t$ is a specification of a bit $v(t) \in \{0, 1\}$ for each neuron $v$ in the circuit. A neuron $v$ is said to be stimulated at time $t$ if $v(t) = 1$, and non-stimulated otherwise.

We now proceed to define the dynamics of a neural circuit. At time $t = 0$, we set the state of the circuit such that $I(0) = 1$ for the input neuron $I$ and $v(0) = 0$ for all other neurons $v$. For $t \geq 0$, the state of the circuit at time $t + 1$ is obtained by each vertex $v$ evaluating its function $f_v$ based on inputs at time $t$. Formally, given $v \in V$, suppose $p(v) = \{u_1, u_2, \ldots, u_k\}$. Then we have

$$v(t + 1) = f_v(u_1(t), u_2(t), \ldots, u_k(t)).$$

(1)

In other words, the neural circuits we consider have a conduction delay of one unit time on every edge: the stimulation state of any neuron $v$ at a given time instant $t$ is available to all other neurons that have $v$ as a parent at time $t + 1$. We will therefore often write the update equation (eq. (1)) in the following abbreviated form:

$$v \leftarrow f_v(u_1, u_2, \ldots, u_k).$$

(2)

**Definition 2.2 (Non-trivial neural circuits).** We say that a neural circuit $G = (V, E)$ is non-trivial if, starting from the configuration $\sigma_0$ at time $t = 0$ in which only the input node $I$ is stimulated (i.e., $\sigma_0(I) = 1$ and $\sigma_0(v) = 0$ for all $v \neq I$), there is a time $t > 0$ at which the output node is stimulated, i.e., $O(t) = 1$.

**Definition 2.3 (Threshold Function).** A Boolean function $f : \{0, 1\}^k \to \{0, 1\}$ is said to be a threshold function if there are (possibly negative) integers $b, w_1, w_2, \ldots, w_k$ such that

$$f(x_1, x_2, \ldots, x_k) = \left\lfloor \sum_{i=1}^{k} w_i x_i \geq b \right\rfloor := \begin{cases} 1 & \text{when } \sum_{i=1}^{k} w_i x_i \geq b. \\ 0 & \text{when } \sum_{i=1}^{k} w_i x_i < b. \end{cases}$$

The parameter $b$ is referred to as the threshold potential of $f$, while the parameters $w_1, w_2, \ldots, w_k$ are referred to as its weights. By a slight abuse of terminology, we will use the phrase “$f$ has weights of absolute value at most $W$” to signify that $|b|, |w_1|, \ldots, |w_k| \leq W$.  

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Threshold neural circuits. In any realistic model of neural circuits, the Boolean functions that each neuron is allowed to compute at a given time step must be suitably constrained. In this paper, we constrain the neurons in our neural circuits to be compute only threshold function. In our computational complexity results, we will further restrict the neurons appearing in our reduction to only draw their weights from a fixed, constant sized set.

A neural circuit is said to be a threshold neural circuit if, for every neuron \( v \) in the neural circuit, the corresponding function \( f_v \) is a threshold function. We note here the well known fact that well known Boolean AND and OR functions can be represented by threshold functions with small weights. In particular, we have \( x_1 \lor x_2 = [x_1 + x_2 \geq 1] \), and \( x_1 \land x_2 = [x_1 + x_2 \geq 2] \).

We briefly remark on two aspects of the neural circuit model considered in this paper. The first is that all conduction delays are uniformly set to 1. This is in contrast, e.g., to the spiking neuron model where the conduction delays can vary over edges. Since our goal in this paper is to establish complexity theoretic hardness results for algorithmic problems on neural circuits, the use of a simplified model of conduction delay only serves to strengthen our results: we show that these algorithmic problems remain hard even for simplified model of neural circuits that we consider.

The second aspect is regarding comparisons with the usual Boolean circuit model in computational complexity theory. As discussed in the introduction, the hardness results for neural circuit arise as consequence of these circuits being recurrent (the directed graph underlying a neural circuit is not required to be acyclic). This is the main point of departure from the usual Boolean circuit model, and accounts for the significantly higher computational complexity of problems addressing neural circuit models (e.g., the simulation problem for Boolean circuits is trivially in \( P \), in contrast to the \( \text{PSPACE} \)-hardness result for the simulation of neural circuits proven here).

2.1 The problems

Finally, we give a formal description of the algorithmic problems studied in this paper.

**Problem 1 (Neural-Circuit-Simulation).** \( \text{INPUT:} \) A threshold neural circuit \( G = (V, E) \) with input node \( I \) and output node \( O \) along with the threshold functions \( f_v \) at the neurons \( v \in V \). The weights of the threshold functions are integers, and are drawn from a constant size set fixed in advance.  
\( \text{OUTPUT:} \)

- **YES:** if \( G \) is non-trivial. That is, if, with the stimulation state in which only \( I \) is stimulated (and none of the other nodes are stimulated), the neural circuit can reach a state in which the node \( O \) is stimulated.
- **NO:** otherwise.

Before describing the other problems, we recall the formalization of the notions of vital sets and degenerate circuits due to Ramaswamy [14]. We first formalize the notion of deactivating or silencing a subset of neurons in a neural circuit.

**Definition 2.4 (Silencing of a subset of neurons of a neural circuit).** Given a neural circuit \( G = (V, E) \) and a subset of neurons \( S \), silencing of the set \( S \) means that for all \( t \geq 0 \), and for all neurons \( v \in S \), \( \sigma_t(v) = 0 \), i.e., neurons in \( S \) never stimulate.

We now recast Ramaswamy’s definitions in our terms. Recall that a non-trivial neural circuit (Definition 2.2) is one in which starting from the initial condition in which only the input neuron \( I \) is stimulated, there is a future time \( t \) at which the output neuron \( O \) gets stimulated.

**Definition 2.5 (Degenerate Circuit [14]).** Given a neural circuit \( G = (V, E) \), a set \( N \subseteq V \) of neurons is said to constitute a degenerate circuit for \( G \) if the circuit obtained by silencing the neurons in \( V \setminus N \) is non-trivial if and only if \( G \) is non-trivial. We assume that the input neuron \( I \) and the output neuron \( O \) are always contained in any degenerate circuit. A minimal degenerate circuit for \( G \) is a degenerate circuit \( N \) of \( G \) such that no proper subset \( N' \) of \( N \) forms a degenerate circuit of \( G \). A minimum degenerate circuit for \( G \) is a degenerate circuit of \( G \) of minimum size.
A degenerate circuit gives us the notion of a sub-circuit of a non-trivial neural circuit $C$ that is capable of showing the same behavior as $C$. Note also that given a neural circuit $C$ there always exists at least one degenerate circuit $C$, which is $C$ itself.

Note also that given a neural circuit, finding a minimum degenerate circuit is equivalent to finding a sub-circuit of the smallest size which is capable of showing the same behavior as the original circuit.

**Definition 2.6 (Vital Set [14]).** Given a neural circuit $G = (V, E)$, a set of neurons $S \subseteq V \setminus \{I, O\}$ is said to be a vital set of neurons if it has a non-empty intersection with every degenerate circuit of the neural circuit $G$. A vital set of size $k$ is called a $k$-vital set. A minimal vital set $S$ is a vital set of $G$ such that no proper subset $S'$ of $S$ is a vital set of $G$. A minimum vital set is a vital set of minimum size.

Note that it follows from the definition that if $S$ is a nonempty vital set of a non-trivial neural circuit, then silencing the nodes in $S$ will ensure that starting from the initial condition in which only the input node $I$ is set to 1, the output node $O$ will never stimulate. We can now list the computational problems formalized by Ramaswamy [14], as described informally in the introduction.

**Problem 2 ($k$-Degenerate-Circuit).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$, an output neuron $O \in V$ and a positive integer $k$ ($k \geq 2$).

OUTPUT: A degenerate circuit $N$ of $G$ of size $k$.

**Problem 3 (Minimal-Degenerate-Circuit).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$, an output neuron $O \in V$.

OUTPUT: A minimal degenerate circuit $N$ of $G$.

**Problem 4 (Minimum-Degenerate-Circuit).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$ and an output neuron $O \in V$.

OUTPUT: A minimum degenerate circuit $N$ of $G$.

**Problem 5 ($k$-Vital-Set).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$ and an output neuron $O \in V$ and a number $k$.

OUTPUT: A $k$-vital set $N$ of $G$.

**Problem 6 (1-Vital-Set).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$ and an output neuron $O \in V$.

OUTPUT: The set of neurons $N \subseteq V$, such that for every $v \in N$, the set $\{v\}$ is a 1-vital set of $G$, and such that for every $v \notin N$, the set $\{v\}$ is not a 1-vital set of $G$.

**Problem 7 ($k$-Degenerate-Circuit-Decision).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$ and an output neuron $O \in V$.

OUTPUT:

YES: if there exists a degenerate circuit of $G$ of size $k$.
NO: otherwise.

**Problem 8 (Minimal-Degenerate-Circuit-Decision).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$, an output neuron $O \in V$.

OUTPUT:

YES: if there exists a minimal degenerate circuit of $G$ of size at least 3.
NO: otherwise.

**Problem 9 (Minimum-Degenerate-Circuit-Decision).** INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$ and an output neuron $O \in V$.

OUTPUT:

YES: if the size of any minimum degenerate circuit of $G$ is at least 3.
NO: otherwise.
Problem 10 (1-Vital-Sets-Decision). INPUT: A neural circuit $G = (V, E)$ with an input neuron $I \in V$ and an output neuron $O \in V$.

OUTPUT:
- YES: if the size of the set of 1-vital sets of $G$ is non-empty, i.e., if there is a vertex $v \in V \setminus \{I, O\}$ such that every degenerate circuit of $G$ includes $v$.
- NO: otherwise.

Finally, we record the following standard result, which will provide the source problem for our PSPACE-hardness reductions.

Theorem 2.1 (True Quantified Boolean Formula (TQBF) is PSPACE-complete). The following problem is PSPACE-complete.

INPUT: A fully quantified Boolean formula

$$\exists x_n \forall x_{n-1} \exists x_{n-2} \ldots \exists x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_n)$$

where $\phi(x_1, x_2, \ldots, x_n)$ is a 3-CNF formula in $n$ variables such that each variable $x_i$ appears at most 4 times in $\phi$. We also assume that $n$ is odd, and that the quantifiers alternate strictly: the $i^{th}$ quantification is $\exists x_i$ if $i$ is odd, and $\forall x_i$ if $i$ is even. We further assume that $\phi(x_1, x_2, \ldots, x_n) = 0$ when $x_1 = x_2 = \ldots = x_n = 0$.

OUTPUT: YES: if the input quantified Boolean formula is true, NO otherwise.

We briefly mention the standard methods by which the constraints assumed for the TQBF instance can be enforced. The constraint that each variable appears a bounded number of times in $\phi$ can be enforced in the same manner as that for 3-SAT [16]. To ensure that $\phi$ evaluates to false when all variables are set to 0, we replace $\phi$ by $\phi' := \phi \land (\overline{x_{n+1}} \lor x_{n+1} \lor \overline{x_{n+1}})$, where $x_{n+1}$, $x_{n+2}$ and $x_{n+3}$ are fresh variables not appearing in $\phi$, and introduce existential quantification over these fresh variables. The resulting TQBF instance is true if and only if the original instance was true (note that the position at which the existential quantifiers on $x_{n+1}$, $x_{n+2}$ and $x_{n+3}$ are introduced does not matter for this deduction). Finally, to ensure that $n$ is odd and that the quantifiers alternate strictly, we add extra dummy variables (which do not appear in $\phi'$) along with the requisite quantifiers so as to enforce strict alternation.

3 Computational hardness of simulating neural circuits

In this section, we prove our main result: the PSPACE-hardness of the problem NEURAL-CIRCUIT-SIMULATION. We start with a description of a simple counter gadget that will be useful in the reduction.

3.1 The counter threshold neural circuit

In this subsection, we will give a construction of a bounded-degree threshold counter neural circuit which satisfies the following requirements: given a positive integer $n$, we require a gadget with an input neuron $I$ and a set of $n$ specified neurons such that when the neural circuit is started in the initial state where $I = 1$ and all other neurons are set to 0, the activation states of the $n$ specified neurons go through, in sequence, each of the binary integers from 0 to $2^n - 1$, possibly after a warm-up time known in advance. In Corollary 3.6, we show that the gadget given here has this property.

Note also that all the neurons in the constructed counter gadget compute threshold functions, and further, the weights of these threshold functions are integers coming from the set $\{0, 1, -1, -2\}$. Further the maximum degree of any neuron in the construction is at most 6 (see fig. 1).

Construction of the threshold counter neural circuit counting 0 to $2^n - 1$ (see fig. 1) The counter gadget has an input neuron $I$, and $n$ variable neurons $x_{1,0}, x_{2,0}, \ldots, x_{n,0}$. For $2 \leq i \leq n - 1$, we
further introduce neurons $y'_i$, $y_i$, $a_i$ and $b_i$. Finally, we introduce two sets of auxiliary neurons: we first have the neurons $x'_{1,0}$, $x'_{2,0}$, $x_{1,1}$, $x_{1,2}, \ldots, x_{1,2n}$, and also the auxiliary neurons $x_{i,1}, x_{i,2}, \ldots, x_{i,2n+4-2i}$, for each integer $i$ satisfying $2 \leq i \leq n$.

The connections in the circuit depend on the stimulation condition of neurons. If stimulation of a neuron $v$ at any time $t > 0$ depends on the stimulation state of neuron $u$ at time $t - 1$, then there is an edge $(u, v)$.

Now, we describe the initial condition for the counter neural circuit. Thereafter, we see the stimulation conditions of neurons, upon satisfaction of which neurons in the counter neural circuit stimulate at any time $t > 0$.

**Initial condition:** At time $t = 0$, the neuron $I$ is stimulated (i.e., $I(0) = 1$) and all other neurons in the circuit are set to be non-stimulated (i.e., their states are set to 0).

**Stimulation conditions:** The neuron $I$ stimulates only at time $t = 0$: we have $I(0) = 1$ and $I(t) = 0$ for all $t \geq 1$. The neuron $x_{i,0}$ stimulates at time $t$ if either $I$ or $x'_{i,0}$ is stimulated at time $t - 1$. Formally,

$$x_{i,0} \leftarrow I \lor x'_{i,0}. \quad (3)$$

$x'_{i,0}$ stimulates at time $t$ if $x_{i,0}$ is stimulated at time $t - 1$:

$$x'_{i,0} \leftarrow x_{i,0}. \quad (4)$$

$x'_{2,0}$ stimulates at time $t$ if either $I$ is stimulated or $x_{2,0}$ is not stimulated at time $t - 1$:

$$x'_{2,0} \leftarrow I \lor \overline{x_{2,0}}. \quad (5)$$

$x_{2,0}$ stimulates at time $t$ if $x'_{2,0}$ is stimulated at time $t - 1$:

$$x_{2,0} \leftarrow x'_{2,0}. \quad (6)$$

$y'_2$ stimulates at time $t$ if $x_{2,0}$ is stimulated at time $t - 1$:

$$y'_2 \leftarrow x_{2,0}. \quad (7)$$

For $2 < i < n$, $y'_i$ stimulates at time $t$ if both $x_{i,0}$ and $y_{i-1}$ are stimulated at time $t - 1$:

$$y'_i \leftarrow x_{i,0} \land y_{i-1}. \quad (8)$$

For $1 < i < n$, $y_i$ stimulates at time $t$ if $y'_i$ is stimulated at time $t - 1$, $a_i$ stimulates at time $t$ if $y_i$ is stimulated and $x_{i+1,0}$ is not stimulated at time $t - 1$, and $b_i$ stimulates at time $t$ if $y_i$ is not stimulated and $x_{i+1,0}$ is stimulated at time $t - 1$:

$$y_i \leftarrow y'_i, \quad (9)$$

$$a_i \leftarrow y_i \land \overline{x_{i+1,0}}, \quad \text{and} \quad (10)$$

$$b_i \leftarrow \overline{y_i} \land x_{i+1,0}. \quad (11)$$

For $2 < i \leq n$, $x_{i,0}$ stimulates at time $t$ if at time $t - 1$ either $a_{i-1}$ or $b_{i-1}$ is stimulated:

$$x_{i,0} \leftarrow a_{i-1} \lor b_{i-1}. \quad (12)$$

For $1 \leq j \leq 2n$, $x_{i,j}$ stimulates at time $t$ if $x_{i,j-1}$ is stimulated at time $t - 1$:

$$x_{i,j} \leftarrow x_{i,j-1}. \quad (13)$$

Finally, for $2 \leq i \leq n$ and $1 \leq j \leq 2n + 4 - 2i$, $x_{i,j}$ stimulates at time $t$ if $x_{i,j-1}$ is stimulated at time $t - 1$:

$$x_{i,j} \leftarrow x_{i,j-1}. \quad (14)$$

We start with the following simple observation.
Observation 3.1. The neuron $x_1$ stimulates for the first time at time 1, and thereafter, it changes its state at every time step. Formally, we have

$$ x_{1,0}(t) = \begin{cases} 0 & \text{when } t \geq 0 \text{ is even}, \\ 1 & \text{when } t \geq 1 \text{ is odd}. \end{cases} \quad (15) $$

Proof. From eq. (3), $x_{1,0}$ stimulates at time $t = 1$, as the initial conditions give $I(0) = 1$ and $x'_{1,0}(0) = 0$. Thereafter, we have $I(t) = 0$ for all $t \geq 1$, i.e., at any time $t > 1$, $x_{1,0}(t) = x'_{1,0}(t-1)$. From eq. (4), at any time $t' > 0$, $x_{1,0}(t') = x_{1,0}(t' - 1)$. These imply that at any time $t > 1$, $x_{1,0}(t) = x_{1,0}(t - 2)$. We say above that $x_{1,0}(1) = 1$, and also know that $x_{1,0}(0) = 0$ (due to the initial conditions). Thus, for any even time $t$, $x_{1,0}(t) = 0$ and for any odd time $t$, $x_{1,0}(t) = 1$.

The following lemma builds upon the above observation to fully characterize the stimulation profile of the key neurons in the gadget.

Lemma 3.2. For $2 \leq i \leq n$, the neuron $x_{i,0}$ is stimulated for the first time at time $t = 2^{i-1} + 2i - 4$, and thereafter, changes its state after every $2^{i-1}$ time-steps. Formally, we have

$$ x_{i,0}(t) = \begin{cases} 0 & 0 \leq t \leq 2^{i-1} - 2i + 5, \\ 1 & 2 \cdot 2^{i-1} - 2i - 4 \leq t \leq (l + 1) \cdot 2^{i-1} - 2i - 5; \ l \geq 1 \ \text{odd}, \\ 0 & 2 \cdot 2^{i-1} - 2i - 4 \leq t \leq (l + 1) \cdot 2^{i-1} - 2i - 5; \ l \geq 2 \ \text{even}. \end{cases} \quad (16) $$

Similarly, for $2 \leq i < n$, the neuron $y_i$ is stimulated for the first time at time $2^i + 2i - 4$, remains stimulated at the next time step i.e. at time $2^i + 2i - 3$, and thereafter changes its state. It then repeats this behavior with a period of $2^i$. Formally, we have

$$ y_i(t) = \begin{cases} 1 & 1 \cdot 2^i + 2i - 4 \leq t \leq l \cdot 2^i + 2i - 3 \text{ for some integer } l \geq 1, \\ 0 & \text{otherwise}. \end{cases} \quad (17) $$

Proof. We prove the lemma by induction on the value of $i$. We first prove the base case for the first part of lemma (eq. (16)) and then for the second part of lemma (eq. (17)).

**Base Case for eq. (16)** ($i = 2$): From eq. (5), $x_{2,0}'$ stimulates for the first time at time $t = 1$, as the initial conditions give $I(0) = 1$ and $x_{2,0}(0) = 0$. Thereafter, we have $I(t) = 0$ for all $t \geq 1$, i.e., at any time $t' \geq 2$, $x_{2,0}'(t') = 1 - x_{2,0}(t' - 1)$. From eq. (6), for $t > 0$, $x_{2,0}(t) = x_{2,0}'(t - 1)$. This implies that $x_{2,0}$ stimulates for the first time at time 2. This proves the first item of eq. (16) for the base case. For $t > 2$, $x_{2,0}(t) = x_{2,0}'(t - 1) = 1 - x_{2,0}(t - 2)$. Since $x_{2,0}(1) = 0$ and $x_{2,0}(2) = 1$, it therefore follows that $x_{2,0}(t) = 1$ when $2k \leq t \leq 2k + 1$ for odd $k \geq 1$, and $x_{2,0}(t) = 0$ when $2k \leq t \leq 2k + 1$ for even $k \geq 2$. This proves the second and third item of eq. (16) for the base case.

**Base Case for eq. (17)** ($i = 2$): From the initial conditions, we have $y_2(0) = y_2'(0) = 0$. Using eq. (9), we therefore get $y_2(1) = 0$. From eqs. (7) and (9), we have, for $t \geq 2$, $y_2(t) = x_{2,0}(t - 2)$. From the above base case analysis of eq. (16), $x_{2,0}(t - 2) = 1$ if and only if $2k \leq t - 2 \leq 2k + 1$ for odd $k \geq 1$, i.e., if $4\ell \leq t < 4\ell + 1$ for some integer $\ell \geq 1$ (here, we put $k = 2\ell - 1$). This proves the base case for eq. (17).

We now proceed with the induction. Our induction hypothesis is that eqs. (16) and (17) are true for all $2 \leq i \leq k - 1$, for some $3 \leq k \leq n$. We prove inductively then that they are true also for $i = k$. We start with eq. (16).

From eq. (12) and the initial conditions, we get that $x_{k,0}(0) = x_{k,0}(1) = 0$. For any time $t \geq 2$, we have, from eqs. (10) to (12):}

$$ x_{k,0}(t) = a_{k-1}(t - 1) \lor b_{k-1}(t - 1) $$

$$ = (y_{k-1}(t - 2) \land x_{k,0}(t - 2)) \lor (y_{k-1}(t - 2) \land x_{k,0}(t - 2)) $$

$$ = y_{k-1}(t - 2) \lor x_{k,0}(t - 2) $$

To complete the proof, we first record the following consequence of the induction hypothesis.
Consequence 3.3. If \( x_{k,0}(l \cdot 2^{k-1} + 2k - 6) = x_{k,0}(l \cdot 2^{k-1} + 2k - 5) = a \) for some integer \( l \geq 1 \) and \( a \in \{0, 1\} \) then \( x_{k,0}(t) = 1 - a \), whenever \( t \) satisfies \( l \cdot 2^{k-1} + 2k - 4 \leq t \leq (l + 1) \cdot 2^{k-1} + 2k - 5 \).

Proof. From the induction hypothesis, we have
\[
y_{k-1}(t) = 1 \text{ if and only if } s \cdot 2^{k-1} + 2k - 6 \leq t \leq s \cdot 2^{k-1} + 2k - 5 \text{ for some integer } s \geq 1. \tag{18}
\]

If \( x_{k,0}(l \cdot 2^{k-1} + 2k - 6) = x_{k,0}(l \cdot 2^{k-1} + 2k - 5) = a \) for some integer \( l \geq 1 \) then using eqs. (18) and (17), we get \( x_{k,0}(l \cdot 2^{k-1} + 2k - 4) = x_{k,0}(l \cdot 2^{k-1} + 2k - 3) = 1 - a \). Since \( y_{k-1}(t) = 0 \) when \( l \cdot 2^{k-1} + 2k - 4 \leq t \leq (l + 1) \cdot 2^{k-1} + 2k - 7 \) (from eq. (18)), we see from eq. (17), that \( x_{k,0} \) remains in state 1 – \( a \) till time \( (l + 1) \cdot 2^{k-1} + 2k - 5 \).

Now, since \( y_{k-1} \) stimulates for the first time at time \( 2^{k-1} + 2k - 6 \), we see from eq. (17) that \( x_{k,0} \) stimulates for the first time at time \( 2^{k-1} + 2k - 4 \), i.e., \( x_{k,0}(2^{k-1} + 2k - 6) = x_{k,0}(2^{k-1} + 2k - 5) = 0 \). Now, iteratively using Consequence 3.3 proves eq. (16) for \( i = k \).

We now prove eq. (17) for \( i = k \) for \( 3 \leq k < n \). From the initial conditions and eq. (9), we have \( y_{k}(0) = y_{k}(1) = 0 \). For \( t \geq 2 \), from eqs. (8) and (9):
\[
y_{k}(t) = y_{k}(t - 1) = y_{k-1}(t - 2) \wedge x_{k,0}(t - 2). \tag{19}
\]

Now, from the induction hypothesis, we have
\[
y_{k-1}(t - 2) = 1 \text{ if and only if } l' \cdot 2^{k-1} + 2k - 4 \leq t \leq (l' + 1) \cdot 2^{k-1} + 2k - 3 \text{ for some integer } l' \geq 1. \tag{20}
\]

Further, since we have already established that the induction hypothesis implies eq. (16) for \( i = k \), we also have (assuming the induction hypothesis) that
\[
x_{k,0}(t - 2) = 1 \text{ if and only if } l' \cdot 2^{k-1} + 2k - 2 \leq t \leq (l' + 1) \cdot 2^{k-1} + 2k - 3 \text{ for some odd } l' \geq 1. \tag{21}
\]

Thus, for \( t \geq 2 \), \( y_{i}(t) = 1 \) if and only if \( t \) satisfies the conditions in both eqs. (20) and (21). A direct calculation (using \( k \geq 3 \)) then shows that this forces \( l = l' + 1 \). We then get that
\[
y_{k}(t) = 1 \text{ if and only if } (l' + 1) \cdot 2^{k-1} + 2k - 4 \leq t \leq (l' + 1) \cdot 2^{k-1} + 2k - 3 \text{ for some odd } l' \geq 1.
\]

The last condition can be written as \( \alpha \cdot 2^{k} + 2k - 4 \leq t \leq \alpha \cdot 2^{k} + 2k - 3 \), where \( \alpha = (l' + 1)/2 \geq 1 \) is an integer.” This completes the induction. \( \square \)

We now record three corollaries of the above lemma.

Corollary 3.4. The neuron \( x_{1,2n} \) stimulates for the first time at time \( 2n + 1 \), and thereafter changes its state at every time step. Formally,
\[
x_{1,2n}(t) = \begin{cases} 0 & 0 \leq t \leq 2n, \\ 1 & t = k + 2n; k \geq 1 \text{ odd}, \\ 0 & t = k + 2n; k \geq 2 \text{ even}. \end{cases}
\]

Proof. From eq. (13), we have \( x_{1,2n}(t) = x_{1,0}(t - 2n) \), for all \( t \geq 2n \), and from the same equation and initial conditions, we have \( x_{1,2n}(t) = 0 \) for \( 0 \leq t \leq 2n \). Together with Observation 3.1, this implies the claim of the corollary. \( \square \)
We now prove our main theorem (Theorem 1.1 in the introduction). We begin with a more formal restatement.

Theorem 3.7. If \( m \) is a positive integer, then the problem is \( \text{PSPACE-hard} \) if and only if for some odd positive integer \( l \), \( l \cdot 2^l - 1 \leq t \leq (l + 1) \cdot 2^l - 1 \). Also, from Corollary 3.4, for \( t' \geq 0 \), \( x_{i,2n}(t' + 2n) = 1 \) if and only if the \( i \)-th significant bit in the binary representation of \( t' \) is 1, and for all \( 2 \leq i \leq n \), \( x_{i,2n}(t' + 2n) = 1 \) if and only if the \( i \)-th least significant bit in the binary representation of \( t' \) is 1. This establishes the claim.

3.2 The reduction

We now prove our main theorem (Theorem 1.1 in the introduction). We begin with a more formal restatement.

Theorem 3.7. The Neural-Circuit-Simulation problem is PSPACE-hard, even when restricted to neural circuits of maximum degree at most 6, and such that every neuron computes a threshold function with weights of absolute value at most 2.

Proof. We reduce the TQBF problem to the bounded degree Neural-Circuit-Simulation problem. In particular, corresponding to any given TQBF instance, we construct a threshold neural circuit (having maximum degree 6) with an input and output neuron such that starting with the initial condition in which only the input neuron is stimulated at time \( t = 0 \), the output neuron stimulates at some time \( t > 0 \) if and only if the quantified Boolean formula is true.

Let the given QBF formula be \( \exists x_n \forall x_{n-1} \ldots \exists x_3 \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_n) \), where \( \phi(x_1, x_2, \ldots, x_n) \) is a 3-CNF formula with \( m \) clauses. As stated in Theorem 2.1, we can assume that each variable \( x_i \) occurs at most 4 times in \( \phi \), and further that the number \( n \) of quantifications is odd, with all the odd-indexed variables quantified with a \( \exists \) quantifier and all the even-indexed variables quantified with a \( \forall \) quantifier. From Theorem 2.1, we also assume that \( \phi \) is false for \( x_1 = x_2 = \ldots = x_n = 0 \). Our goal now is to create (in time polynomial in the size of \( \phi \)) a threshold neural circuit \( C_\phi \) with an input neuron \( I \) and an output neuron \( O \) such that when \( C_\phi \) is started in the initial state in which only \( I \) is stimulated at time \( t = 0 \), the output neuron \( O \) stimulates at some future time \( t > 0 \) if and only if \( \exists x_n \forall x_{n-1} \ldots \exists x_3 \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_n) \) is true.

Construction of the threshold neural circuit corresponding to a QBF (see Fig. 2): We first describe the nodes in \( C_\phi \). Of course, \( C_\phi \) contains an input neuron \( I \) and an output neuron \( O \). The first important component of our construction is an \( n \)-variable counter neural circuit (as discussed in
The input neuron $I$ of $C_\phi$ is identified with the input neuron of the counter neural circuit. Further, the neuron $x_{1,2n}$ in the counter neural circuit is relabelled as $x_1$ and for all $2 \leq i \leq n$, the node $x_{i,2n+4+i}$ in the counter neural circuit is relabelled as $x_i$. As we will see later, these relabelled neurons $x_i$, $1 \leq i \leq n$, will correspond to the variables $x_i$ of the same name appearing in $\phi$.

In addition to the nodes in the counter neural circuit, we have several auxiliary nodes. First, for each $1 < i \leq n$, we introduce $m + i - 1$ neurons $z_{i,1}, \ldots, z_{i,m+i-1}$. For odd $1 \leq i \leq n$, we introduce a neuron $s_{i,0}$ while for even $1 < i < n$, we introduce two neurons $s'_{i,0}, s_{i,0}$ and $s_{i,1}$. For odd $1 < i \leq n$, we introduce two neuron $p_i$ and $q_i$. Further, for each clause $c_i$, where $1 \leq i \leq m$, in $\phi$, we introduce a clause neuron $c_i$, of the same name. We also introduce, for each $1 < i \leq n, i - 1$ auxiliary clause neurons $d_{i,1}, \ldots, d_{i,i-1}$.

The connections in the circuit depends on the stimulation condition of neurons. If stimulation of a neuron $u$ at any time $t > 0$ depends on the stimulation state of neuron $u$ at time $t - 1$, then there is an edge $(u, v)$. Now, we describe the initial condition for the neural circuit $C_\phi$. Thereafter, we see the stimulation conditions of neurons, upon satisfaction of which neurons in the neural circuit stimulate at any time $t > 0$.

**Initial Condition:** At time $t = 0$, the neuron $I$ is stimulated and all other neurons are set to be non-stimulated.

**Stimulation Condition:** The stimulation conditions of the neurons appearing as part of the counter neural circuit remain the same as those discussed earlier. We now describe the stimulation conditions of the other nodes.

For $1 < i \leq n$, $z_{i,1}$ stimulates at time $t$ if $x_i$ is stimulated at time $t - 1$. Formally,

$$z_{i,1} \leftarrow x_i. \quad (22)$$

For even $1 < i < n$ and $1 < j \leq m + i - 1$, $z_{i,j}$ stimulates at time $t$ if $z_{i,j-1}$ stimulates at time $t - 1$. Formally,

$$z_{i,j} \leftarrow z_{i,j-1}. \quad (23)$$

For odd $1 < i \leq n$ and $1 < j \leq m + i - 2$, $z_{i,j}$ stimulates at time $t$ if $z_{i,j-1}$ stimulates at time $t - 1$. Formally,

$$z_{i,j} \leftarrow z_{i,j-1}. \quad (24)$$

For odd $1 < i \leq n$, $p_i$ stimulates at time $t$ if at time $t - 1$, $z_{i,m+i-3}$ is stimulated and $z_{i,m+i-2}$ is not stimulated, and $q_i$ stimulates at time $t$ if at time $t - 1$, $z_{i,m+i-3}$ is not stimulated and $z_{i,m+i-2}$ is stimulated. Formally,

$$p_i \leftarrow z_{i,m+i-3} \land \overline{z_{i,m+i-2}}. \quad (25)$$

$$q_i \leftarrow \overline{z_{i,m+i-3}} \land z_{i,m+i-2}. \quad (26)$$

For odd $1 < i \leq n$, $z_{i,m+i-1}$ stimulates at time $t$ if, at time $t - 1$, at least one of $s'_{i-1,0}$ or $s_{i-1,0}$ is stimulated and neither $p_i$ nor $q_i$ is stimulated. Formally,

$$z_{i,m+i-1} \leftarrow (s'_{i-1,0} \lor s_{i-1,0}) \land \overline{p_i} \land \overline{q_i}. \quad (27)$$

We note here a couple of simple consequences of eqs. (22) to (26), and the initial conditions.

**Observation 3.8.** For any $t \geq 0$, for even $1 < i < n$ and $1 \leq j \leq m + i - 1$, $z_{i,j}(t + j) = x_i(t)$, and $z_{i,j}(t) = 0$ for $t \leq j$.

**Observation 3.9.** For any $t \geq 0$, for odd $1 < i \leq n$ and $1 \leq j \leq m + i - 2$, $z_{i,j}(t + j) = x_i(t)$, and $z_{i,j}(t) = 0$ for $t \leq j$. 


Using the above observations, we get the following claim, which will help us in analyzing the update in eq. (27).

**Claim 3.10.** For any $t \geq 0$ and for odd $1 < i \leq n$,

$$
\overline{p}_i(t) \land \overline{q}_i(t) = \begin{cases} 
0 & \text{if } t = l \cdot 2^{i-1} + 2n + m + i - 2 \text{ for some integer } l \geq 1 \\
1 & \text{otherwise}.
\end{cases} \quad (28)
$$

**Proof.** The initial condition implies that $p_i(0) = q_i(0) = 0$, so we concentrate on the case $t \geq 1$. Using eqs. (25) and (26) and Observation 3.9, we then have

$$p_i(t) \lor q_i(t) = (z_{i,m+i-3}(t-1) \land \overline{z}_{i,m+i-2}(t-1)) \lor (z_{i,m+i-3}(t-1) \land z_{i,m+i-2}(t-1))$$

$$= z_{i,m+i-3}(t-1) \lor z_{i,m+i-2}(t-1). \quad (29)$$

When $t \leq m+i-2$, Observation 3.9 implies that the right hand side of eq. (29) is 0. On the other hand, when $t = t' + m + i - 1$ for some $t' \geq 0$, we use Observation 3.9 along with eq. (29) to get

$$p_i(t) \lor q_i(t) = x_i(t' + 1) \oplus x_i(t').$$

Now, recalling that the neuron $x_i$ is just a relabelled version of the neuron $x_{i,2n+4-2l}$ in the counter circuit (since $i > 1$), we see from Corollary 3.5 that $x_i(t' + 1) \oplus x_i(t') = 1$ if and only if $t' = l \cdot 2^{i-1} + 2n - 1$, for some integer $l \geq 1$. We thus get that $p_i(t) \lor q_i(t) = 1$ if and only if $t = l \cdot 2^{i-1} + 2n + m + i - 2$ for some integer $l \geq 1$. This proves the claim, since $p_i(t) \land q_i(t) = p_i(t) \lor q_i(t)$.

\[\square\]

We now continue with our description of the stimulation conditions. The clause neuron $c_i$ stimulates at time $t$ if the stimulation states of the variable neurons $x_j$ corresponding to the variables appearing in $c_i$ give a satisfying assignment for the clause $c_i$ at time $t - 1$. For $2 < i \leq m$, $d_{i,1}$ stimulates at time $t$ if $c_i$ is stimulated at time $t - 1$, while $d_{i,2,1}$ stimulates at time $t$ if both $c_1$ and $c_2$ are stimulated at time $t - 1$.

$$d_{i,1} \leftarrow c_1 \land c_2, \quad (30)$$

$$d_{i,1} \leftarrow c_i, \text{ when } i > 2. \quad (31)$$

For $3 < i \leq m$ and $1 < j < i - 1$, $d_{ij}$ stimulates at time $t$ if $d_{i,j-1}$ is stimulated at time $t - 1$.

$$d_{i,j} \leftarrow d_{i,j-1}. \quad (32)$$

For $2 < i \leq m$, $d_{i,i-1}$ stimulates at time $t$ if both $d_{i-1,i-2}$ and $d_{i,i-2}$ are stimulated at time $t - 1$.

$$d_{i,i-1} \leftarrow d_{i-1,i-2} \land d_{i,i-2}. \quad (33)$$

The neuron $s_{1,0}$ stimulates at time $t$ if $d_{m,m-1}$ stimulates at time $t - 1$.

$$s_{1,0} \leftarrow d_{m,m-1}. \quad (34)$$

Before proceeding, we record here a simple consequence of eqs. (30) to (34) and the initial conditions.

**Observation 3.11.** For any $t \geq 0$, we have $s_{1,0}(t + m) = 1$ if and only if $c_i(t) = 1$ for all $1 \leq i \leq m$. Also, $s_{1,0}(t) = 0$ for all $t \leq m$.

For even $1 < i < n$, $s_{i,0}'$ stimulates at time $t$ if $s_{i-1,0}$ is stimulated and $z_{i,m+i-1}$ is not stimulated at time $t - 1$. Formally,

$$s_{i,0}' \leftarrow s_{i-1,0} \land \overline{z}_{i,m+i-1}. \quad (35)$$
For even \( 1 < i < n \), \( s_{i,0} \) stimulates at time \( t \) if at least one of \( s_{i,0}' \) and \( z_{i+1,m+i} \) is stimulated and neither of \( p_{i+1} \) and \( q_{i+1} \) is stimulated at time \( t - 1 \). Formally,
\[
s_{i,0} \leftarrow (s_{i,0}' \lor z_{i+1,m+i}) \land \overline{p_{i+1}} \land \overline{q_{i+1}}. \tag{36}
\]
For even \( 1 < i < n \), \( s_{i,1} \) stimulates at time \( t \) if both \( s_{i-1,0} \) and \( z_{i,m+i-1} \) are stimulated at time \( t - 1 \). Formally,
\[
s_{i,1} \leftarrow s_{i-1,0} \land z_{i,m+i-1}. \tag{37}
\]
For odd \( 1 < i \leq n \), \( s_{i,0} \) stimulates at time \( t \) if both \( s_{i-1,0} \) and \( s_{i-1,1} \) are stimulated at time \( t - 1 \). Formally,
\[
s_{i,0} \leftarrow s_{i-1,0} \land s_{i-1,1}. \tag{38}
\]
Finally, \( O \) stimulates at time \( t \) if \( s_{n,0} \) stimulates at time \( t - 1 \).
\[
O \leftarrow s_{n,0}. \tag{39}
\]

Note that the size of \( C_{\phi} \) is \( \text{poly}(n) \), and the above description of \( C_{\phi} \) can be constructed in time \( \text{poly}(n) \) given \( \phi \) as input.

Observe that all neurons in the above circuits have small degree and have only threshold gates. We record this formally in the following observation.

**Observation 3.12.** The update function of all neurons in \( C_{\phi} \) as constructed above are threshold update functions with weights of absolute value at most 2. Further each neuron has degree at most 6.

**Proof.** Except for the neurons \( z_{i,m+i-1} \) for odd \( 1 < i \leq n \), \( s_{i,0} \) for even \( 1 < i < n \), and the clause neurons \( c_i \) for \( 1 \leq i \leq m \), all other neurons in fig. 2 either copy the stimulation state of some neuron (e.g., in eq. (22), \( z_{2,1} \) copies the stimulation state of \( x_2 \)) or compute conjunctions or disjunctions of stimulation states of two neurons. All of these are easily seen to be threshold functions with weights of absolute values at most 2. The clause neurons \( c_i \) compute a conjunction of three other neurons (possibly negated), and this also is a threshold function with weights coming from the same set. We now consider the remaining neurons \( (z_{i,m+i-1} \) for odd \( 1 < i \leq n \) and \( s_{i,0} \) for even \( 1 < i < n \)).

We begin by noting that the right hand side of the update equation (27) of the neurons \( z_{i,m+i-1} \) for odd \( 1 < i \leq n \) is 1 if and only if the threshold function \( [s_{i-1,0}' + s_{i-1,0} - 2p_i - 2q_i \geq 1] \) evaluates to 1 (this is because when the variables are restricted to take values in the Boolean domain \( \{0, 1\} \), the latter happens if and only if at least one of \( s_{i-1,0}' \) and \( s_{i-1,0} \) evaluates to 1, and neither of \( p_i \) and \( q_i \) do). Similarly for the neurons \( s_{i,0} \) for even \( 1 < i < n \), we observe that the right hand side of the update equation (36) is 1 if and only if the threshold function \( [s_{i,0}' + z_{i+1,m+i} - 2p_{i+1} - 2q_{i+1} \geq 1] \) evaluates to 1 (this is because when the variables are restricted to take values in the Boolean domain \( \{0, 1\} \), the latter happens if and only if at least one of \( s_{i,0}' \) and \( z_{i+1,m+i} \) evaluates to 1, and neither of \( p_{i+1} \) and \( q_{i+1} \) do).

We now check that \( C_{\phi} \) has bounded degree. Indeed, the degree of a node \( x_i, 1 \leq i \leq n \), is two more than the number of clauses in which the corresponding variable \( x_i \) occurs, so that their degree in the graph is at most 6 (since we started with a \( \phi \) in which each variable occurs at most 4 times). From the description above (see also fig. 2) we can check that all other nodes have degree at most 6 (the degree of 6 is also achieved by neurons \( x_{i,0} \) for odd \( 1 < i \leq n \), in the counter gadget and \( s_{i,0} \), for even \( 1 < i < n \)).

We now begin the analysis of the reduction with the following two claims.

**Claim 3.13.** For any even \( 1 < i < n \), \( s_{i,0}(t) = z_{i+1,m+i}(t) = 0 \) if \( t = l \cdot 2^l + 2n + m + i \) for some integer \( l \geq 1 \).

**Proof.** From Claim 3.10, we see that \( p_{i+1}(t-1) \land q_{i+1}(t-1) = 0 \) when \( t = l \cdot 2^l + 2n + m + i \) for some integer \( l \geq 1 \). The claim then follows from eqs. (27) and (36).
Lemma 3.16. by the assumption on the TQBF instance, $2 = x_1^i + 2n + m + i - 1$ is such that $s_i,0(t) = z_{i+1,m+i}(t) = 1$. Then we have $s_i(0(t'') = z_{i+1,m+i}(t'')$ for all $t'$ satisfying $t' \leq (l + 1) \cdot 2^l + 2n + m + i - 1$.

Proof. From Claim 3.10, we see that $\overline{p}_{i+1}(t'') \land \overline{q}_{i+1}(t'') = 0$ only when $t'' = t'' \cdot 2^l + 2n + m + i - 1$ for some integer $t'' \geq 1$. It follows that for all $t''$ such that $l \cdot 2^l + 2n + m + i \leq t'' \leq (l + 1) \cdot 2^l + 2n + m + i - 2$ for some integer $l \geq 0$, we have $\overline{p}_{i+1}(t'') \land \overline{q}_{i+1}(t'') = 1$. Thus, when $l \cdot 2^l + 2n + m + i \leq t'' \leq (l + 1) \cdot 2^l + 2n + m + i - 2$, the update equations in eqs. (27) and (36) simplify to

$$s_i,0(t'') \cdot 1 = s_i,0(t') \land z_{i+1,m+i}(t''),$$

and

$$z_{i+1,m+i}(t'') = s_i,0(t'').$$

It then follows that if $t$ is such that $l \cdot 2^l + 2n + m + i \leq t \leq (l + 1) \cdot 2^l + 2n + m + i - 2$ and $s_i,0(t) = z_{i+1,m+i}(t) = 1$, then it is also the case that $s_i,0(t + 1) = z_{i+1,m+i}(t + 1) = 1$. The claim then follows immediately.

Next we record an observation based on the properties of the counter neural circuit proved in Corollary 3.6.

Claim 3.15. For any $t \geq 0$, $s_i,0(t + m + 1) = 1$ if and only if $a(t) = x_1(t) = x_2(t) \ldots x_n(t)$ is a satisfying assignment for $\phi$. In particular, for any $t' \leq 2n + m + 1$, $s_i,0(t') = 0$. Further, for any $t' \geq 2n + m + 1$, $s_i,0(t') = 1$ if and only if $t' = 2n+m+1 + \sum_{i=1}^n a_i 2^{i-1} \mod 2^n$ for some satisfying assignment $a_1, a_2, \ldots, a_n$ of $\phi$.

Proof. From Observation 3.11, we see that the neuron $s_i,0$ stimulates at some time $t' \geq m + 1$ if and only if all the clause neurons $c_i$ stimulate at the time $t' - m$. The latter in turn, happens if and only if the stimulation state of the neurons $x_1, x_2, \ldots, x_n$ forms a satisfying assignment for $\phi$ at time $t' - m - 1$. This proves the first part of the claim.

For the second part, Corollaries 3.4 and 3.5 imply that for all $1 \leq i \leq n$, and for $t \leq 2n$, we have $x_i(t) = 0$ (recall that $x_i$ is identical to the neuron $x_i,2n$ of the counter gadget considered in Corollaries 3.4 and 3.5, while for $2 \leq i \leq n$, $x_i$ is identical to the neuron $x_i,2n+2-2l$ of the same gadget). On the other hand, from our assumption on the TQBF instance, $\phi(x_1, x_2, \ldots, x_n)$ is false when $x_1 = x_2 = \ldots = x_n = 0$. The first part proved above thus implies that for any $t'$ such that $m + 1 \leq t' \leq 2n + m + 1$, we have $s_i,0(t') = 0$. When $0 \leq t' \leq m$, we have $s_i,0(t') = 0$ from Observation 3.11. These prove the second part of our claim.

For the third part, Corollaries 3.4 and 3.5 imply that for $a \neq 0$, $a(t) = a_1 a_2 \ldots a_n$ if and only if $t = 2n + \sum_{i=1}^n a_i 2^{i-1} \mod 2^n$. This part therefore follows from the already proved first part (since, by the assumption on the TQBF instance, $0$ is not a satisfying assignment of $\phi$).

Finally, we state the following main claim regarding the behavior of the neural circuit $C_{\phi}$.

Lemma 3.16. The times at which the nodes in $C_{\phi}$ labeled $s_i', s_i,1$ (for even $1 < i < n$) and $s_i,0$ (for $1 \leq i \leq n$) stimulate are characterized as follows:

1. For any $t \geq 2n$, and odd $1 \leq i \leq n$, we have $s_i,0(t + m + i) = 1$ if and only if the formula

$$\forall x_1 \exists x_2 \exists x_3 \ldots \exists x_{i-1} \phi(x_1, x_2, \ldots, x_{i-1}, a_1, a_{i+1}, \ldots, a_n)$$

is true when $a_j = x_j(t)$ for all $j \geq i$. Further, $s_i,0(t') = 0$ for all $t' < 2n + m + i$.

2. For any $t \geq 2n$, and even $1 \leq i \leq n$, we have $s_i,1(t + m + i) = 1$ if and only if $x_i(t) = 1$ and the formula

$$\forall x_1 \exists x_2 \exists x_3 \ldots \exists x_{i-1} \phi(x_1, x_2, \ldots, x_{i-1}, a_1, a_{i+1}, \ldots, a_n)$$

is true when $a_j = x_j(t)$ for all $j \geq i - 1$. Further, $s_i,1(t') = 0$ for all $t' < 2n + m + i$. 

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3. For any $t \geq 2n$, and even $1 \leq i \leq n$, we have $s'_{i,0}(t + m + i) = 1$ if and only if $x_i(t) = 0$ and the formula

$$\forall x_{i-2} \exists x_{i-3} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{i-2}, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

is true when $a_j = x_j(t)$ for all $j \geq 1 - i$. Further, $s'_{i,0}(t') = 0$ for all $t' < 2n + m + i$.

4. For any $t \geq 2n + 1$, and even $1 \leq i \leq n$, we have $s_{i,0}(t + m + i) = 1$ if and only if there exists $t'$ satisfying $2n + \left\lceil (t - 2n)/2^k \right\rceil \cdot 2^k - 1 \leq t' < t$ such that $x_i(t') = 0$ and the formula

$$\forall x_{i-2} \exists x_{i-3} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{i-2}, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

is true when $a_j = x_j(t')$ for all $j \geq 1 - i$. Further, $s_{i,0}(t) = 0$ for all $t < 2n + m + i + 1$.

*Proof.* We prove the lemma by induction on the value of $i$. In the base case, $i = 1$, item 1 follows immediately from Claim 3.15, which characterizes the times $t' \geq 0$ at which $s_{i,0}(t') = 1$, and items 2 to 4 are vacuously true as $i$ is odd. Thus, the base case is established.

For the induction, we suppose that for some $2 \leq k \leq n$, items 1 to 4 of the lemma are true for all $i$ satisfying $1 \leq i \leq k - 1$, and show that this implies that they remain true for $i = k$ as well. We divide the inductive step into two cases, based on whether $k$ is even or odd.

**Case 1:** $k > 1$ is odd. In this case items 2 to 4 are vacuously true, as $i = k$ is odd. Thus, only item 1 remains to be proved. From eq. (38), we know that for any $t' \geq 1$, $s_{i,0}(t') = 1$ if and only if $s_{k-1,0}(t' - 1) = 1$. From the induction hypothesis, item 2 of the lemma is true for $i = k - 1$, so that we have $s_{k-1,0}(t' - 1) = 0$ when $1 \leq t' < 2n + m + k$ (and, also $s_{k,0}(0) = 0$ according to the initialization conditions). Thus, we get $s_{k,0}(t') = 0$ when $0 \leq t' < 2n + m + k$, which establishes the second part of item 1. We now proceed to prove the first part of item 1.

For any $t \geq 2n$, we have (again from eq. (38)) that $s_{k,0}(t + m + k) = 1$ if and only if $s_{k-1,0}(t + m + k - 1) = 1$. From the induction hypothesis (specifically, items 2 and 4 of the lemma) applied with $i = k - 1$, the latter holds if and only if the following two conditions are satisfied:

(a) $x_{k-1}(t) = 1$ and the formula

$$\forall x_{k-3} \exists x_{k-4} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{k-3}, a_{k-2}, a_{k-1}, a_k, \ldots, a_n)$$

is true when $a_j = x_j(t)$ for all $j \geq k - 2$.

(b) there exists a $t'$ satisfying $2n + \left\lceil (t - 2n)/2^{k-1} \right\rceil \cdot 2^{k-1} \leq t' < t$ such that $x_{k-1}(t') = 0$ and the formula

$$\forall x_{k-3} \exists x_{k-4} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{k-3}, a_{k-2}', a_{k-1}', a_k', \ldots, a_n')$$

is true when $a_j' = x_j(t')$ for all $j \geq k - 2$.

Now, from Corollary 3.6, we know that for all $t'' \geq 2n$, the $n$-bit binary integer $x_n(t'')x_{n-1}(t'') \ldots x_1(t'')$ is exactly the remainder obtained on dividing $t'' - 2n$ by $2^n$. Since $\left\lceil (t' - 2n)/2^{k-1} \right\rceil = \left\lceil (t - 2n)/2^{k-1} \right\rceil$, this implies that we have $x_j(t) = x_j(t')$ for all $j \geq k$, so that, we have $a_j = a_j'$ for all $j \geq k$. Thus, we conclude that for $t \geq 2n$, we have $s_{k,0}(t + m + k) = 1$ if and only if the following condition (equivalent to the conjunction of the conditions in items a and b above) holds: there exist $a_{k-2}, a_{k-2}' \in \{0, 1\}$ such that both the formulas

$$\forall x_{k-3} \exists x_{k-4} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{k-3}, a_{k-2}, 1, a_k, \ldots, a_n)$$

and

$$\forall x_{k-3} \exists x_{k-4} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{k-3}, a_{k-2}', 0, a_k, \ldots, a_n)$$
are true, when \( a_j = x_j(t) \) for all \( j \geq k \). But this latter condition is equivalent to the condition that the formula

\[
\forall x_{k-1} \exists x_{k-2} \forall x_{k-3} \exists x_{k-4} \ldots \forall x_2 \exists x_1 \, \phi(x_1, x_2, \ldots, x_{k-3}, x_{k-2}, x_{k-1}, a_k, \ldots, a_n)
\]

is true. This establishes the claim in item 1 of the lemma for \( i = k \).

**Case 2**: \( 2 \leq k < n \) is even. In this case, item 1 of the lemma is vacuously true, as \( i = k \) is even. Thus only items 2 to 4 remain to be proved. We first consider item 2.

From eq. (37), we know that for any \( t' \geq 1 \), \( s_{k,1}(t') = 0 \) if \( s_{k-1,0}(t' - 1) = 0 \) (and also that \( s_{k,1}(0) = 0 \), which is enforced by the initial condition). From the induction hypothesis, item 1 of the lemma is true for \( i = k - 1 \), so that we have \( s_{k-1,0}(t' - 1) = 0 \) when \( 1 \leq t' < 2n + m + k \). Thus, we get \( s_{k,1}(t') = 0 \) when \( 0 \leq t' < 2n + m + k \), which establishes the second part of item 2. We now proceed to prove the first part of item 2.

For any \( t' \geq 2n \), we have (again from eq. (37)) that \( s_{k,1}(t + m + k) = 1 \) if and only if \( s_{k-1,0}(t + m + k - 1) = z_{k,m+k-1}(t + m + k - 1) = 1 \). From Observation 3.8, \( z_{k,m+k-1}(t + m + k - 1) = 1 \) if and only if \( x_k(t) = 1 \). On the other hand, from the induction hypothesis (specifically, item 1 of the lemma) applied with \( i = k - 1 \), \( s_{k-1,0}(t + m + k - 1) = 1 \) if and only if the formula

\[
\forall x_{k-2} \exists x_{k-3} \ldots \forall x_2 \exists x_1 \, \phi(x_1, x_2, \ldots, x_{k-3}, a_{k-1}, a_k, a_{k+1}, \ldots, a_n)
\]

is true when \( a_j = x_j(t) \) for all \( j \geq k - 1 \). Together, these two observations finish the proof of item 2 for \( i = k \).

We now proceed to prove item 3 for \( i = k \). We note that the argument is virtually identical on the one already given for item 2, but we include the details for completeness. From eq. (35), we know that for any \( t' \geq 1 \), \( s_{k,0}'(t') = 0 \) if \( s_{k-1,0}(t' - 1) = 0 \) (and also that \( s_{k,0}'(0) = 0 \), which is enforced by the initial condition). From the induction hypothesis, item 1 of the lemma is true for \( i = k - 1 \), so that we have \( s_{k-1,0}(t' - 1) = 0 \) when \( 1 \leq t' < 2n + m + k \). Thus, we get \( s_{k,0}'(t') = 0 \) when \( 0 \leq t' < 2n + m + k \), which establishes the second part of item 3. We now proceed to prove the first part of item 3.

For any \( t' \geq 2n \), we have (again from eq. (35)) that \( s_{k,0}'(t + m + k) = 1 \) if and only if \( s_{k-1,0}(t + m + k - 1) = 1 \) and \( z_{k,m+k-1}(t + m + k - 1) = 0 \). From Observation 3.8, \( z_{k,m+k-1}(t + m + k - 1) = 0 \) if and only if \( x_k(t) = 0 \). On the other hand, from the induction hypothesis (specifically, item 1 of the lemma) applied with \( i = k - 1 \), \( s_{k-1,0}(t + m + k - 1) = 1 \) if and only if the formula

\[
\forall x_{k-2} \exists x_{k-3} \ldots \forall x_2 \exists x_1 \, \phi(x_1, x_2, \ldots, x_{k-3}, a_{k-1}, a_k, a_{k+1}, \ldots, a_n)
\]

is true when \( a_j = x_j(t) \) for all \( j \geq k - 1 \). Together, these two observations finish the proof of item 3 for \( i = k \).

We now proceed to prove item 4 for \( i = k \). Suppose, for the sake of contradiction, that there exists a time \( t' \) where \( 0 \leq t' < 2n + m + k + 1 \), such that \( s_{k,0}(t') = 1 \). Without loss of generality, choose \( t' \) to be the smallest such time. Note that \( t' > 1 \), as \( s_{k,0}(0) = 0 \) is enforced by the initial conditions, while \( s_{k,0}(1) = 0 \) (using eq. (36)) because both \( s_{k,0}'(0) = z_{k+1,m+k}(0) = 0 \) (as enforced by the initial condition). On the other hand, we also have \( s_{k,0}(t' - 1) = s_{k,0}(t' - 2) = 0 \), using the choice of \( t' \) as the smallest time for which \( s_{k,0}(t') = 1 \), and by the observation above that \( t' > 1 \). Further, as we have already established item 3 of the lemma for \( i = k \) using the induction hypothesis, we can apply the second part of that item to deduce that \( s_{k,0}'(t' - 1) = s_{k,0}'(t' - 2) = 0 \) (since \( t' - 2 < t' - 1 < 2n + m + k \)). But then, eq. (27) implies that \( z_{k+1,m+k}(t' - 1) = 0 \), which further implies \( s_{k,0}(t') = 0 \) (from eq. (36) as \( z_{k+1,m+k}(t' - 1) = s_{k,0}'(t' - 1) = 0 \)). The latter is a contradiction to the assumption that \( s_{k,0}(t') = 1 \). We therefore must have \( s_{k,0}(t') = 0 \) for all \( t' < 2n + m + k + 1 \). This establishes the second part of item 4. We now proceed to prove the first part of item 4. We begin by recording a few observations.

Define \( t_k = \lceil (t - 2n)/2k \rceil \geq 0 \), and \( t_k := 2n + \lceil (t - 2n)/2k \rceil \cdot 2k = 2n + t_k \cdot 2k \). Thus, if \( t' \) is as in the statement of item 4, we have

\[
t_k = t_k \cdot 2k + 2n \leq t' \leq t - 1 \leq (t_k + 1) \cdot 2k + 2n - 2.
\]
From Claim 3.10, we see that for any \( t'' \geq 2n \), we have \( p_{k+1}(t'' + m + k) \wedge q_{k+1}(t'' + m + k) = 1 \) whenever \( t'' \) satisfies
\[
l' \cdot 2^k + 2n \leq t'' \leq (l' + 1) \cdot 2^k + 2n - 2 \text{ for some integer } l' \geq 0.
\]

In view of eq. (40), this implies that
\[
\frac{p_{k+1}(t'' + m + k) \wedge q_{k+1}(t'' + m + k)}{t_k} = 1 \text{ if } t_k \leq t'' < t.
\] (41)

We now prove the forward direction of the equivalence claimed in the first part of item 4 of the lemma. Suppose therefore that there exists a \( t' \) satisfying \( 2n + \lceil (t - 2n)/2^k \rceil \cdot 2^k = t_k \leq t' < t \), for some \( t \geq 2n + 1 \), such that \( x_k(t') = 0 \) and the formula
\[
\forall x_{k-2} \exists x_{k-1} \ldots \forall x_2 \exists x_1 \varphi(x_1, x_2, \ldots, x_{k-2}, a_{k-1}, a_k, a_{k+1}, \ldots, a_n)
\]
is true when \( a_j = x_j(t') \) for all \( j \geq k - 1 \). Since we have already established item 3 of the lemma for \( i = k \) (assuming the induction hypothesis), our assumption implies that \( s'_{k,0}(t' + m + k) = 1 \). The update expressions in eqs. (27) and (36), along with the observation in eq. (41), then imply that
\[
z_{k+1, m+k}(t' + m + k + 1) = s_{k,0}(t' + m + k + 1) = 1.
\] (42)

Since \( t' \) satisfies eq. (40), we see therefore that the time \( t' + m + k + 1 \) satisfies the hypothesis of Claim 3.14, with the integer \( l \) in the claim set to \( \ell_k \). From Claim 3.14, we then get that \( s_{k,0}(t'') = 1 \) for all \( t'' \) satisfying
\[
t' + m + k + 1 \leq t'' \leq (\ell_k + 1) \cdot 2^k + 2n + m + k - 1.
\]

In particular, this implies that \( s_{k,0}(t + m + k) = 1 \), since, as observed in eq. (40), \( t + m + k \leq (\ell_k + 1) \cdot 2^k + 2n + m + k - 1 \). This proves the forward direction of the equivalence claimed in the first part of item 4.

We now consider the other direction of the equivalence. Suppose therefore that \( s_{k,0}(t + m + k) = 1 \) for some \( t \geq 2n + 1 \). We first note that this implies \( t \neq t_k \). This is because when \( t = t_k = 2n + \ell_k \cdot 2^k \), we must have \( \ell_k \geq 1 \) (since \( t \geq 2n + 1 \)). But then, Claim 3.13 would imply that \( s_{k,0}(t + m + k) = s_{k,0}(\ell_k \cdot 2^k + 2n + m + k) = 0 \). Thus, we must have \( t \neq t_k \).

If now, let \( t_1 \) be the smallest integer satisfying both \( s_{k,0}(t_1 + m + k) = 1 \) and \( t_k < t_1 \leq t \) (such a \( t_1 \) exists as \( t \) satisfies both these conditions). We claim that for such a \( t_1 \), we must have \( s'_{k,0}(t_1 + m + k - 1) = 1 \). We prove this by dividing the argument into two cases.

**Case (a) \( (t_1 = t_k + 1) \):** In this case \( s_{k,0}(t_k + m + k + 1) = 0 \). Note that if we establish that \( z_{k+1,m+k}(t_k + m + k) = 0 \) it will therefore follow from eq. (36) that \( s'_{k,0}(t_k + m + k) = s'_{k,0}(t_1 + m + k - 1) = 1 \).

Now, we note that when \( \ell_k \geq 1 \), the claim \( z_{k+1,m+k}(t_k + m + k) = z_{k+1,m+k}(\ell_k \cdot 2^k + 2n + m + k) = 0 \) is directly implied by Claim 3.13. For the remaining case when \( \ell_k = 0 \) so that \( t_k = 2n \), we observe that we have \( s'_{k,0}(2n + m + k - 1) = 0 \) (from the already established second part of item 3 of the induction hypothesis for the case \( i = k \)) and also \( s_{k,0}(2n + m + k - 1) = 0 \) (from the already established second part of item 4 of the induction hypothesis for the case \( i = k \)). Together with eq. (27), these imply again that \( z_{k+1,m+k}(2n + m + k) = 0 \). Thus, we always have \( z_{k+1,m+k}(t_k + m + k) = 0 \), and as observed above, this implies that if \( s_{k,0}(t_k + m + k + 1) = 1 \) then \( s'_{k,0}(t_k + m + k) = s'_{k,0}(t_1 + m + k - 1) = 1 \).

**Case (b) \( (t_k + 2 \leq t_k \leq t) \):** In this case \( s_{k,0}(t_k + m + k) = 1 \), and \( s_{k,0}(t_k + m + k - 1) = s_{k,0}(t_1 + m + k - 2) = 0 \), by the choice of \( t_k \). Since \( s_{k,0}(t_k + m + k) = 1 \), eq. (36) implies that either \( s'_{k,0}(t_k + m + k - 1) = 1 \) or \( z_{k+1,m+k}(t_k + m + k - 1) = 1 \). We want to show that \( s'_{k,0}(t_k + m + k - 1) = 1 \). Suppose, for the sake of contradiction, that \( s'_{k,0}(t_k + m + k - 1) = 0 \), so that we must have \( z_{k+1,m+k}(t_k + m + k - 1) = 1 \). Equation (27) then implies that \( s'_{k,0}(t_k + m + k - 2) = 1 \) as we have \( s_{k,0}(t_k + m + k - 2) = 0 \). Since \( t_k \leq t_k - 2 < t \), eq. (41) implies that \( p_{k+1}(t_k + m + k - 2) \wedge q_{k+1}(t_k + m + k - 2) = 1 \). In conjunction with eq. (36), \( s'_{k,0}(t_k + m + k - 2) = 1 \) would then imply that \( s_{k,0}(t_k + m + k - 1) = 1 \). This, however, is a contradiction, since \( s_{k,0}(t_k + m + k - 1) = 0 \) by the choice of \( t_k \). Thus we must have \( s'_{k,0}(t_k + m + k - 1) = 1 \).
In all the cases, we therefore have \( s'_{k,0}(t_1 + m + k - 1) = 1 \). Let \( t' = t_1 - 1 \). Then, \( s'_{k,0}(t' + m + k) = 1 \), and we also have \( 2n \leq t_k \leq t' < t \). By item 3 of the lemma for \( i = k \) (which we have already established above using the induction hypothesis), this implies that \( x_k(t') = 0 \) and

\[
\forall x_{k-2} \exists x_{k-3} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{k-2}, a_{k-1}, a_k, a_{k+1}, \ldots, a_n)
\]

is true when \( a_j = x_j(t') \) for all \( j \geq k - 1 \). Together, these imply the second direction of the equivalence claimed in first part of item 4 of the lemma, for the case \( i = k \). This completes the induction. \( \square \)

Lemma 3.16 immediately implies the following.

**Lemma 3.17.** The output neuron \( O \) stimulates at some time \( t \geq 0 \) if and only if the quantified Boolean formula

\[
\exists x_n \forall x_{n-1} \exists x_{n-2} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{n-1}, x_n)
\]

is true.

**Proof.** Note that the output neuron \( O \) stimulates at some time \( t \) if and only \( s_{n,0} \) stimulates at time \( t - 1 \) (see eq. (39)). Recall that \( n \) is odd. Thus, from item 1 of Lemma 3.16 (applied with \( i = n \)), \( s_{n,0} \) stimulates at some time \( t - 1 \) if and only if there exists \( a_n \in \{0, 1\} \) such that the quantified Boolean formula

\[
\forall x_{n-1} \exists x_{n-2} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{n-1}, a_n)
\]

is true. But this is equivalent to the statement that the quantified Boolean formula

\[
\exists x_n \forall x_{n-1} \exists x_{n-2} \ldots \forall x_2 \exists x_1 \phi(x_1, x_2, \ldots, x_{n-1}, a_n)
\]

is true. This completes the proof. \( \square \)

Finally, we note that the above lemma proves Theorem 3.7. \( \square \)

We have shown that \textsc{Neural-Circuit-Simulation} is \textsc{PSPACE}-hard. Finally we observe that \textsc{Neural-Circuit-Simulation} problem is also in \textsc{PSPACE}.

**Proposition 3.18.** \textsc{Neural-Circuit-Simulation} is in \textsc{PSPACE}.

**Proof.** Suppose that we are given a neural circuit \( G = (V, E) \) with input neuron \( I \) and output neuron \( O \). Starting with the initial condition in which only \( I \) is stimulated, we simulate \( G \) for \( 2^{|V|} + 1 \) units of time. If \( O \) does not stimulate before this time we return \textsc{NO}, otherwise we return \textsc{YES}.

The total number of possible states of the neural circuit \( G \) is \( 2^{|V|} \) (as every neuron has only two possible stimulation states). Further, given a stimulation state of the neural circuit at a given time, the stimulation state of the circuit at the next time step is completely specified by the update functions. Thus, if \( O \) does not become stimulated by time \( 2^{|V|} + 1 \), the simulation must have entered a loop, thus ensuring that \( O \) will never enter a stimulated state even if the simulation were continued indefinitely. This shows that the above algorithm returns \textsc{YES} if and only if \( G \) is non-trivial.

Finally, we observe that the space needed to run the above algorithm is \( O(\text{poly}(|V|)) \) (corresponding to the space needed to store the states of the neurons and the time counter). \( \square \)
4 Corollaries of the hardness of Neural-Circuit-Simulation

We now prove the hardness of problems 1-4 in Ramaswamy’s list, quoted in the introduction. We begin with the following simple observation which follows easily from Theorem 3.7.

Proposition 4.1. For \( k = 2 \), \( k \)-Degenerate-Circuit-Decision is \( \text{PSPACE} \)-hard.

Proof. We reduce the TQBF problem to \( k \)-Degenerate-Circuit-Decision for \( k = 2 \). Given a TQBF instance \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi(x_1, x_2, \ldots, x_n) \) (as in Theorem 2.1), we construct the same neural circuit \( C_\phi \) as in the proof of Theorem 3.7 (fig. 2). From Lemma 3.17, \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if \( C_\phi \) is non-trivial. We now show that \( C_\phi \) is non-trivial if and only if there is no degenerate circuit of \( C_\phi \) of size 2.

From eq. (39), \( O \) cannot stimulate if we silence the neuron \( s_{n,0} \). This implies that if \( C_\phi \) is non-trivial, then any degenerate circuit of \( C_\phi \) must contain the neuron \( s_{n,0} \). Recall from Definition 2.5 that the input neuron \( I \) and the output neuron \( O \) are both always contained in any degenerate circuit. Thus, we see that if \( C_\phi \) is non-trivial then any degenerate circuit of \( C_\phi \) must contain the neurons \( I, O \) and \( s_{n,0} \), and therefore must have size at least 3. On the other hand, in the case where \( C_\phi \) is non-trivial, the set \( \{I, O\} \) is a degenerate circuit of \( C_\phi \) of size 2 (this is because in the remaining circuit, the neuron \( s_{n,0} \) is silenced, and hence, as observed above, \( O \) can never stimulate).

Thus, we see that \( C_\phi \) has a degenerate circuit of size 2 if and only if \( C_\phi \) is not non-trivial. By construction, \( C_\phi \) is non-trivial if and only if \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true. Altogether, this gives a polynomial-time reduction from the TQBF problem to \( k \)-Degenerate-Circuit-Decision for \( k = 2 \). This proves for \( k = 2 \), \( k \)-Degenerate-Circuit-Decision is \( \text{PSPACE} \)-hard. \( \square \)

Proposition 4.2. Minimal-Degenerate-Circuit-Decision is \( \text{PSPACE} \)-hard.

Proof. We reduce the TQBF problem to Minimal-Degenerate-Circuit-Decision. Given a TQBF instance \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi(x_1, x_2, \ldots, x_n) \) (as defined in Theorem 2.1), we construct the neural circuit \( C_\phi \) (as in fig. 2). We show that \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if every minimal degenerate circuit of \( C_\phi \) of size at least 3.

As we have seen in the proof of Proposition 4.1, \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if \( C_\phi \) does not have a degenerate circuit of size 2. On the other hand, \( C_\phi \) always has a degenerate circuit of size greater than 2 (namely, \( C_\phi \) itself). Also, every degenerate circuit is of size at least 2, as it must contain the neurons \( I \) and \( O \) (cf. Definition 2.5). Thus, we see that \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if any minimal degenerate circuit of \( C_\phi \) is of size at least 3. \( \square \)

Proposition 4.3. Minimum-Degenerate-Circuit-Decision is \( \text{PSPACE} \)-hard.

Proof. This proof is identical to the proof of Proposition 4.2. Again, we reduce the TQBF problem to Minimum-Degenerate-Circuit-Decision. Given a TQBF instance \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) (as defined in Theorem 2.1), we construct the neural circuit \( C_\phi \) (as in fig. 2), and show that the instance is true if and only if a minimum degenerate circuit of \( C_\phi \) of size 2 does not exist.

Arguing exactly as in the proof of Proposition 4.2, we see that \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if any degenerate circuit (in particular any minimum degenerate circuit) of \( C_\phi \) is of size at least 3. This gives a polynomial time reduction from the TQBF problem to Minimum-Degenerate-Circuit-Decision, and hence shows that Minimum-Degenerate-Circuit-Decision is \( \text{PSPACE} \)-hard. \( \square \)

We also record here a hardness result for an approximation version of the problem. For \( \alpha > 1 \), we say that a degenerate circuit \( C \) of a given neural circuit \( G \) is a \( \alpha \)-approximate minimum degenerate circuit if the size of \( C \) is at most \( \alpha s \), where \( s \) is the size of a minimum degenerate circuit of \( G \).
Proposition 4.4. Fix any integer \( c \geq 1 \). It is PSPACE-hard to find a \((c \cdot \lceil \log N \rceil)\)-approximate minimum degenerate circuit of an input neural circuit of size \( N \).

Proof. We give a polynomial-time Turing reduction from TQBF to the \((c \cdot \lceil \log N \rceil)\)-approximate minimum degenerate circuit problem. Given a TQBF instance \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi(x_1, x_2, \ldots, x_n) \) (as defined in Theorem 2.1), we construct the neural circuit \( C_\phi \) (as in fig. 2). Note that the size of \( C_\phi \) is \( N \leq p(n) \) for some fixed polynomial \( p \).

Now, we begin by finding a \((c \cdot \lceil \log N \rceil)\)-approximate minimum degenerate circuit \( B \) of \( C_\phi \). If the size of \( B \) is more than \( 2 \cdot c \cdot \lceil \log N \rceil \), then we answer YES (i.e., \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true). This is correct since in this case, the size of any minimum degenerate circuit of \( C_\phi \) must be at least 3, so that, as argued in the proof of Proposition 4.2, \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) must be true.

If the size of \( B \) is at most \( 2 \cdot c \cdot \lceil \log N \rceil \), we simulate \( B \) for time \( 2^{|B|} + 1 \leq 2^2 c \cdot \log N + 1 \leq \text{poly}(n) \) starting from the initial condition in which only the input neuron is stimulated and answer YES if the output neuron stimulates during the simulation and NO otherwise. The correctness is then guaranteed by the facts that \( B \) is a degenerate circuit of \( C_\phi \), and that \( C_\phi \) is non-trivial if and only if \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true. The sufficiency of the time of simulation follows as in the proof of Proposition 3.18. \( \square \)

Proposition 4.5. 1-Vital-Sets-Decision is PSPACE-hard.

Proof. We reduce the TQBF problem to 1-Vital-Sets-Decision. Given a TQBF instance \( \phi \) (as defined in theorem 2.1), we construct the neural circuit \( C_\phi \) (as in fig. 2). We now show that \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if the set of 1-vital sets of \( C_\phi \) is non-empty.

The set of 1-vital sets contains those neurons (except \( I \) and \( O \)) which belong to all the degenerate circuits. As we have seen in the proof of Proposition 4.1, \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if all the degenerate circuits of \( C_\phi \) contain the neurons \( I, O, \) and \( s_{n,0} \). On the other hand, if \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is false, then \( C_\phi \) is not non-trivial so that \{\( I, O \)\} is a degenerate circuit for \( C_\phi \). Thus, we see that \( \exists x_n \forall x_{n-1} \ldots \exists x_1 \phi \) is true if and only if the set of 1-vital sets of \( C_\phi \) is non-empty. This gives a polynomial-time reduction from the TQBF problem to 1-Vital-Sets-Decision, and hence shows that 1-Vital-Sets-Decision is PSPACE-hard. \( \square \)

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Figure 1: Threshold counter neural circuit
Figure 2: Threshold neural circuit for n-variable QBF