An investigation on probabilistic relations in measurement

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Abstract. In the representational approach, measurement is defined on the basis of primitive (in the context) notion of empirical relation. To account for measurement uncertainty, it is quite natural to consider probabilistic (or, more generally, non deterministic) empirical relations. Here the notion of probabilistic relation is investigated. After a brief historical outline, we point out that probabilistic relations appear in everyday measurement situation. To demonstrate this, a simple illustrative experiment has been developed, whose results are presented and discussed.

1. Probabilistic empirical relation

In a formal approach, for defining measurement, it is convenient to refer it to a weaker notion. In the representational approach, measurement is defined on the basis of the concept of empirical relation, which is a primitive one in the context [5] notion of empirical relation. In most of the literature concerning such a theory, empirical relations are regarded as “deterministic”, in that they are associated to object according to a “yes-no” (“true-false”), classical logic [12]. To account for measurement uncertainty, it is instead quite natural to consider probabilistic (or, more generally, non deterministic) relations [7, 9, 11, 12]. This approach is more radical to the one provided by the early theory of errors [1], and was probably firstly conceived by Fechner himself [3]. The idea was fruitfully developed in psychophysics. Thurstone provided a (genial) representation of (inner) sensations in terms of probabilistic (or random) variables³. Link notes that Thurstone’s idea was already somewhat present, in nuce, in Fechner’s thought. Experimental work in psychophysics provided wide empirical support to the need, at least in that field, of conceiving empirical relations as non-deterministic [4-8]. We suggest that the idea of probabilistic relation can be applied to measurement in general, including physics and engineering. We have discussed elsewhere the theoretical aspects of this position. Here we present and discuss a (relatively) simple experiment, concerning voltage measurement, whose result can anyway be extended to situations where the resolution of the observation device interact with the fluctuations of the phenomenon under observation.

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² References are here listed in a chronological order, to provide a feeling of historical development of the subject.
³ We definitely prefer the term probabilistic to (the more common) random variable, since probability does not necessarily imply randomness [10].
2. Experimental set up
In order to provide some experimental data in support to the theoretical considerations, we have designed and performed an experiment in which two objects are compared to order them in respect of some property of interest. To obtain the input - output relationship, we need to have some control on one of the two. If one of them constitutes a fixed reference item, \( s \), the entity of the other, \( a \), can be varied either continuously or in steps smaller than the resolution of the observation device. Then we can evaluate the probability of observing the relations of equal, greater than, or less than, for small variations of the second object. We also measure the reference and the observed objects with an independent, high accuracy instrument in order to obtain additional information that will be used for displaying the results. The whole set-up is shown in Figure 1.

![Figure 1. Generic discrete comparison experiment.](image)

In principle it is possible to design such an experiment with any property, such as mass or voltage, given a proper comparator having a limited resolution. On the other hand there are practical implementation issues to be faced such as reference quantities stability and fine control on the variable item. In this regard voltage has been considered a convenient choice, in term of flexibility and ease of implementation. We have designed the voltage comparison experiment shown in Figure 2.

![Figure 2. Voltage comparison set-up.](image)

A stable fixed voltage source, \( V_s \), acts as reference object \( s \).
A second stable voltage source can be adjusted by acting on a voltage divider, to obtain the desired series of voltages states, \( V_a \).
The voltage comparator consists in a standard multiplexed data acquisition board, with a proper control software. Through the multiplexer the board connects an input channel at a time to the data conversion chain, so the measurement chain for the two voltages is the same and it performs as a comparator that is moved from one sample to the other.

The resolution of the comparator is the resolution of the data acquisition chain that includes a programmable gain amplifier to manage some flexibility in the input range. During the experiment the gain was set to obtain a ±10 V range and, since the analogue to digital converter has a 12 bits range, the overall resolution is set to 4.883 mV.

A higher accuracy bench voltmeter (Agilent 34401) is connected in parallel to the comparator, so that it is possible to measure the two voltage levels independently. During the experiment the reference voltage $V_s$ was stable in a ±0,1 mV range, while it was possible to control the variable voltage source $V_a$, down to 0,1 mV. The sampling frequency was set to 500 Hz, and 500 samples were acquired for each channel for a one second time history.

The software evaluates the difference $\Delta y = y_s - y_a$ for each couple of samples ad a logical operator evaluates if $\Delta y$ is equal, larger than or less than zero. The output of each sample is recorded obtaining a sequence of 500 logical output values was recorded.

Table 1. Comparator logical output values

| Objects | Readings | Comparator | Output |
|---------|----------|------------|--------|
| $a < s$ | $y_a < y_s$ | $\Delta y < 0$ | $-1$ |
| $a = s$ | $y_a = y_s$ | $\Delta y = 0$ | $0$ |
| $a > s$ | $y_a > y_s$ | $\Delta y > 0$ | $1$ |

So the overall set up, including the software data processing, realises a voltage comparator with known resolution and a logical (yes-no) output, for each of the investigated relations. For practical reasons such results have been conventionally codified with the values -1, 0 and +1.

Since the comparator is subjected to some noise, as the two voltage sources, even in presence at the input of two strictly equal voltages, the comparator output will present a distribution of logical values centred in zero but with considerable probability of lower/higher values.

3. Experimental procedure

After a one hour warm up of the voltage sources to reach a stable condition, the reference voltage $V_s$ is measured and the variable voltage source $V_a$ is set to a value lower than the reference with a nominal difference equivalent to more than 3 bits or 17 mV. A sequence of 500 samples is acquired and the comparison performed for each sample couple. In this situation, even in presence of some noise, the probability to obtain -1 at the output is almost one, due the large difference between the inputs.

Then the variable voltage source is increased in small steps with an increment in voltage that is less than the resolution of the comparator, and the test is repeated for all the values. The voltage sequence as measured at the input of the comparator by the bench voltmeter for all the 16 tests performed is presented in Figure 2.
It is worth noting that the amplitude of each step of $V_a$ is about half the resolution of the comparator, as shown by the dashed lines indicating a voltage variation corresponding to ± 1 bit across the reference level $V_s$.

The probability for each logical value at the output of the comparator is evaluated at each voltage step and it is possible to obtain a probability curve as a function of the $\Delta V$ at the comparator input, as presented in Figure 3.

In fact, the basic idea under this experiment, is quite general and it can be applied to a generic quantity, given a discrete comparator with a given resolution. In order to have a clearer view of this, it is possible to redraw the graph presented in Figure 3, considering a generic quantization process, instead of using specific voltage quantities. Scaling the voltage steps to the specific voltage resolution of the comparator that is used in this specific experiment we can find the graph presented in Figure 4.
4. Discussion

Probability distributions presented in Figure 4 show that even when at the input of the discrete comparator two quantities are almost equal, the probability to obtain an equal result by the comparator is less than one (green squares), and there are not negligible probabilities to obtain less than (blue circles) or greater than (red stars) results. This situation is related to the characteristics of the overall comparison chain, in particular the noise level and the resolution. The former is due to the contribution of both the quantities involved and the comparator, while the latter is typical of the comparator device. A higher signal to noise ratio and a smaller resolution would result in more steep changes in the distribution curves, and of course in a more accurate comparison, but even in this more favourable case the result will not be deterministic, since investigating the behaviour between two, even small, quantisation intervals all the three possible results could appear at the output, so the most appropriate description would be in any case a probability distribution.

5. References

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