The elliptic scattering theory of the 1/2-XYZ and higher order Deformed Virasoro Algebras

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Abstract

Bound state excitations of the spin 1/2-XYZ model are considered inside the Bethe Ansatz framework by exploiting the equivalent Non-Linear Integral Equations. Of course, these bound states go to the sine-Gordon breathers in the suitable limit and therefore the scattering factors between them are explicitly computed by inspecting the corresponding Non-Linear Integral Equations. As a consequence, abstracting from the physical model the Zamolodchikov-Faddeev algebra of two \( n \)-th elliptic breathers defines a tower of \( n \)-order Deformed Virasoro Algebras, reproducing the \( n = 1 \) case the usual well-known algebra of Shiraishi-Kubo-Awata-Odake \[\text{(1)}\].

To our friend Daniel, ad memoriam

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1 Prelude

At this stage, integrable one-dimensional quantum spin chains are a widely studied subject in theoretical physics. In fact, they are interesting models in themselves as integrable (qualitative) description of interactions between microscopic magnets in solid state physics phenomena and, besides, they may be thought of as (lattice) regularisations of two dimensional quantum field theories (cf. for instance [2], [3]): in this respect, they opened the road of the application of techniques, like the Algebraic Bethe Ansatz, which would be otherwise considerably difficult if applied directly to the infinite degrees of freedom case. Finally, recent and surprising findings (see [4] and the subsequent development) showed that hamiltonians of some integrable spin chains coincide with the mixing matrices of gauge invariant operators of $\mathcal{N} = 4$ super Yang-Mills theories, opening a new way to test the AdS/CFT correspondence.

Given this growing interest in the subject, we started by studying the most general one dimensional spin 1/2 chain, the so-called 1/2-XYZ, which describes spin 1/2 degrees of freedom on a lattice with anisotropic nearest-neighbour interaction. In the infinite volume limit and with finite lattice spacing (thermodynamic limit), a suitable scaling of the elliptic nome allowed the authors of [2] to obtain an integrable quantum field theory in the two dimensional plane, the sine-Gordon model. Therefore, it is reasonable to think of the 1/2-XYZ chain (in the thermodynamic limit) as an elliptic deformation of the sine-Gordon field theory. Moreover, in a recent paper [5] we proved that the same conclusion holds if we also allow the lattice spacing to go to zero in such a way to keep finite the lattice length on a circumference (cylinder geometry). In fact, starting from the Algebraic Bethe Ansatz equations we wrote down the equivalent Non-Linear Integral Equations (NLIEs) describing the anti-ferromagnetic ground state, its first excitations, – i.e. spin waves propagating along the chain –, and bound states thereof, which turn out to be labelled by a positive integer $n$ [5]. These excitations become in the sine-Gordon limit the solitons/antisolitons and breathers respectively. Therefore, we may think of them as elliptic deformed counterparts of solitons/antisolitons and breathers, and definitely borrow these names from the field theory parlance. By inspecting the NLIE in the thermodynamic limit, we started to compute scattering factors between excitations [5]. Our aim was to reveal, within a clear and unique framework, the elliptic scattering factors which had been already proposed [6, 7, 8], without however transparent connection to physical models. In this respect, the scattering matrix between excitations in the repulsive regime – elliptic solitons and antisolitons – was computed and showed to coincide with Zamolodchikov’s S-matrix [6]. The elliptic nome of such a matrix was found shifted with respect to the one appearing in the parameters defining the XYZ chain, thereby proving the so-called Smirnov’s conjecture. On the other hand, in the attractive regime the scattering factor between two first ($n = 1$) elliptic breathers was also computed [5] and showed to coincide on one side with the Lukyanov-Mussardo-Penati S-factor [7, 8], on the other side with the elliptic structure function of
the Deformed Virasoro Algebra \[1\].

In this paper we will continue the calculation of S-matrices by computing all the possible scattering factors between two \(n\) and \(m\) elliptic breathers, with \(n, m \geq 1\). As expected, these factors are elliptic deformations of the corresponding ones between sine-Gordon breathers \[9\]. From the mathematical point of view they may be used, when \(n = m\), to define an order \(n\) Deformed Virasoro Algebra (DVA). For the well-known DVA (order \(n = 1\)) \[1\] is obtained by quantizing the Frenkel-Reshetikhin Poisson bracket \[10\] and all the other higher order ones \((n > 1)\) may be thought of in the same manner.

2 An outlook on the theoretical background

The spin 1/2-XYZ model with periodic boundary conditions is a spin chain with hamiltonian, written in terms of Pauli matrices \(\sigma^x,y,z\),

\[ H = \frac{1}{2} \sum_{n=1}^{N} \big( J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z \big). \tag{2.1} \]

Here \(N\) is the number of lattice sites and because of the periodicity the site \(N + 1\) is identified with the site 1. The three (real) coupling constants \(J_x, J_y, J_z\) may be reparametrised (up to an overall multiplicative constant) in terms of elliptic functions with nome \(\exp\left(-\pi \frac{K'}{K}\right)\):

\[ J_x = 1 + k \sin^2 2\eta, \quad J_y = 1 - k \sin^2 2\eta, \quad J_z = \cosh 2\eta, \tag{2.2} \]

where \(\eta\) is real. As for the notations used in this paper we address the reader, for instance, to the Section 2 of \[5\]. As in \[5\], we constrain our analysis to the disordered regime, i.e.

\[ 0 < \exp\left(-\pi \frac{K'}{K}\right) < 1, \quad 0 < \eta < K, \tag{2.3} \]

which are equivalent to \(k > 0, K > 0, K' > 0\) or \(|J_z| < J_y < J_x\). We also distinguish between the repulsive regime \(0 < \eta < \frac{K}{2}\), and the attractive regime \(\frac{K}{2} < \eta < K\).

The spin 1/2-XYZ chain enjoys many interesting limit cases. If \(K' \to \infty\), which implies \(K \to \frac{\pi}{2}\), we have \(J_x \to J_y \to 1, J_z \to \cos 2\eta\), so that we recover the XXZ chain in the massless anti-ferromagnetic regime. If \(K \to \infty (K' \to \frac{\pi}{2})\), we discover that \(J_y \to J_z \to \frac{1}{\cosh^2 2\eta}, J_x \to 1 + \tanh^2 2\eta\), which means that we obtain the XXZ chain in the anti-ferromagnetic massive regime. As mentioned in the prelude, particular attention should be payed to the double scaling limit of site number and coupling constant

\[ N \to \infty, \quad K' = \frac{8\eta}{\pi} \log N + \text{finite terms}, \tag{2.4} \]
on the cylinder of constant circumference \( R \) and lattice spacing \( \Delta = \frac{R}{N} \to 0 \). In this case we obtain the sine-Gordon model on a cylinder of circumference \( R \) with coupling constant \( \beta^2 = 1 - \frac{2n}{\pi} \), i.e. with Lagrangian \( \mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{m_0^2}{8\pi^2 \beta} (\cos \sqrt{8\pi \beta} \phi - 1) \). Eventually, if even \( R \) goes to infinity, we gain the plane geometry, where the famous results of [2] apply.

The spectrum of the spin 1/2-XYZ model was found in [11] by means of the Algebraic Bethe Ansatz. In particular, the Bethe equations read

\[
\left[ \frac{H(i\alpha_j + \eta)\Theta(i\alpha_j + \eta)}{H(i\alpha_j - \eta)\Theta(i\alpha_j - \eta)} \right]^N = -e^{-4\pi i \frac{\eta}{\mathcal{K}}} \prod_{k=1}^{M} \frac{H(i\alpha_j - i\alpha_k + 2\eta)\Theta(i\alpha_j - i\alpha_k + 2\eta)}{H(i\alpha_j - i\alpha_k - 2\eta)\Theta(i\alpha_j - i\alpha_k - 2\eta)},
\]

where \( j = 1, \ldots, M \) and \( \nu \) is an integer. These equations (2.5) are valid when

\[
m_1 \eta = 2m_2 \mathcal{K},
\]

where \( m_1 \) and \( m_2 \) are integers, and when \( 2M = N, \mod m_1 \). The corresponding eigenvalues of the transfer matrix \( \Lambda_N(\alpha) \) may be written in terms of the Bethe roots as

\[
\frac{\Lambda_N(\alpha)}{\Theta(0)} = e^{2\pi i \frac{\eta}{\mathcal{K}}} H(i\alpha + \eta)^N \Theta(i\alpha + \eta)^N \prod_{j=1}^{M} \frac{H(i\alpha_j - i\alpha + 2\eta)\Theta(i\alpha_j - i\alpha + 2\eta)}{H(i\alpha_j - i\alpha)\Theta(i\alpha_j - i\alpha)} +
\]

\[
+ e^{-2\pi i \frac{\eta}{\mathcal{K}}} H(i\alpha - \eta)^N \Theta(i\alpha - \eta)^N \prod_{j=1}^{M} \frac{H(i\alpha_j - i\alpha_j + 2\eta)\Theta(i\alpha - i\alpha_j + 2\eta)}{H(i\alpha - i\alpha_j)\Theta(i\alpha - i\alpha_j)}.
\]

While writing (2.5, 2.7) we borrowed the notations from [11]: in particular, all the elliptic functions have nome \( \exp \left( -\pi \frac{\mathcal{K}'}{\mathcal{K}} \right) \).

### 3 The Non-Linear Integral Equation

In [5] we described the vacuum and the first excitations of the spin 1/2-XYZ chain upon transforming the Bethe equations into an equivalent NLIE, one for each state. In Algebraic Bethe Ansatz language the vacuum is given by \( \nu = 0 \) and by filling the real interval \( -\frac{\mathcal{K}'}{2\mathcal{K}}, \frac{\mathcal{K}'}{2\mathcal{K}} \) with Bethe roots. Excitations over the vacuum are expressed by \( N_h \) (real) holes, \( N_c \) close (complex) pairs, \( N_w \) wide (complex) pairs and \( N_{sc} \) (complex) self-conjugated roots, the vacuum obviously corresponding to \( N_h = N_c = N_w = N_{sc} = 0 \) (cf. [5] for details). In fact, any of these states may be described in a compact way by means of the following NLIE for the unknown function \( \tilde{Z}_N \)

\[
\tilde{Z}_N(\theta) = N \tilde{F}(\theta) + \sum_{k=1}^{N_h} \tilde{\chi}(\theta - h_k) - \sum_{k=1}^{N_c} \tilde{\chi}(\theta - c_k) - \sum_{k=1}^{N_w} \tilde{\chi}(\theta - w_k) -
\]

\[
- \sum_{k=1}^{N_{sc}} \tilde{\chi}(\theta - s_k) + 2 \int_{-\frac{\mathcal{K}'}{2\mathcal{K}}}^{\frac{\mathcal{K}'}{2\mathcal{K}}} d\eta \tilde{G}(\theta - \eta) \Im \ln \left[ 1 + e^{i \tilde{Z}_N(\eta + i\theta)} \right],
\]

(3.1)
where we assume for granted from [5] the definitions of the functions of the renormalised rapidity \( \theta \), \( \tilde{Z}_N(\theta) = Z_N\left(\frac{\theta}{K}\right) \) (similarly for \( F \) and \( \chi \)) and \( \tilde{G}(\theta) = \frac{\theta}{K} G\left(\frac{\theta}{K}\right) \). Nonetheless, we recall for the lazy reader the explicit formulæ for the forcing term, the kernel and the scattering phase

\[
\tilde{F}(\theta) = \sum_{n=-\infty}^{+\infty} \frac{1}{n} \frac{i \sinh \frac{2n(K-\eta)\pi}{K} - \left(1 - \frac{n}{K}\right) \cos n\pi \sinh \frac{2nK\pi}{K}}{\sinh \frac{2nK\pi}{K} + \sinh \frac{2n(K-2)\pi}{K}} e^{-2i\pi n \theta \frac{K}{K'}}, \tag{3.2}
\]

\[
\tilde{G}(\theta) = \frac{\eta}{KK'} \sum_{n=-\infty}^{+\infty} \frac{\sinh \frac{K}{2} \left(2n(K-2\eta)\pi\right)}{2 \sinh \frac{2n(K-2\eta)\pi}{K'} \cosh \frac{2n\eta\pi}{K'}} e^{-2i\pi n \theta \frac{K}{K'}}, \tag{3.3}
\]

\[
\tilde{\chi}(\theta) = \int_0^\theta dx 2\pi \tilde{G}(x) = \sum_{n=-\infty}^{+\infty} \frac{\sinh \frac{2n(K-2\eta)\pi}{K} - \left(1 - \frac{n}{K}\right) \cos n\pi \sinh \frac{2nK\pi}{K}}{2 \sinh \frac{2n(K-2\eta)\pi}{K'} \cosh \frac{2n\eta\pi}{K'}} e^{-2i\pi n \theta \frac{K}{K'}} n, \quad \lvert \text{Im}\theta \rvert < 2 \frac{K^2}{\eta} - 2K. \tag{3.4}
\]

Besides, we required reference to the so-called second determination of \( \tilde{\chi} \), \( \tilde{\chi}_{II} \), in that wide roots lie, by definition, outside the strip \( \lvert \text{Im}\theta \rvert < \min\{2K, 2K^2/\eta - 2K\} \), where formula (3.4) rigorously holds. Actually, here we only need the second determination in the attractive regime \( \frac{K}{2} < \eta < K \)

\[
\tilde{\chi}_{II}(\theta) = \tilde{\chi}(\theta) - \tilde{\chi}(\theta - 2i(K^2/\eta - K) \text{sgnIm}\theta), \tag{3.5}
\]

or better, spitting the upper from the lower band, we may equivalently introduce the two functions

\[
\tilde{\chi}_{II}^{(\epsilon)}(\theta) = \tilde{\chi}_{II}(\theta) \quad \text{when} \quad \epsilon \text{Im}\theta > 2K^2/\eta - 2K, \quad \epsilon = \pm 1. \tag{3.6}
\]

It is convenient to re-express (3.5) in terms of the multiplicative independent variables

\[
x = \exp\left(-\frac{2i\pi \eta \theta}{KK'}\right), \quad q = \exp\left(-\frac{4\pi K}{K'}\right), \quad p = \exp\left(-\frac{4\pi \eta}{K'}\right), \tag{3.7}
\]

which naturally lead to the definitions (\( |a| < 1 \))

\[
(x; a) = \prod_{s=0}^{\infty} (1 - xa^s), \quad \Theta_a(x) = (x; a)(ax^{-1}; a)(a; a). \tag{3.8}
\]

The latter satisfies the multiplicative properties

\[
\Theta_a(ax) = \Theta_a(x^{-1}) = -x^{-1} \Theta_a(x). \tag{3.9}
\]

Thanks to the definitions (3.8) we may write (3.6) explicitly

\[
\tilde{\chi}_{II}^{(\epsilon)}(\theta) = i \ln \left[ \Theta_{p^2}\left(xq^{\frac{c+1}{2}}p^{\frac{c-1}{2}}\right) \Theta_{p^2}\left(x^{-1}q^{\frac{1-c}{2}}p^{-\frac{1+c}{2}}\right) \right]^\epsilon. \tag{3.10}
\]
4 The bound states, i.e. the elliptic breathers

In the attractive regime $\frac{K}{2} < \eta < K$ a soliton and an antisoliton may form a bound state, which may be named – in analogy with the sine-Gordon theory – elliptic breather. It is represented as Bethe root configuration by adding to the Fermi-Dirac sea of the vacuum a string of complex (conjugated) roots. After remembering [2] that in the thermodynamic limit $N \to \infty$ the breather $B_n(\theta)$ with (real) rapidity $\theta$ can exist only if the integer $n$ satisfies

$$\frac{n}{n+1} K < \eta,$$

we must consider separately the two cases of $n$ even and odd: the breather $B_{2k+2}(\theta)$ is described by the string ($\theta$ real)

$$\theta \pm iK \pm iK2h \left(1 - \frac{K}{\eta}\right), \quad h = 0, 1, \ldots, k,$$  

while the breather $B_{2k+1}(\theta)$ by the string

$$\theta + i\frac{K^2}{\eta}, \quad \theta \pm iK \pm iK(2h-1) \left(1 - \frac{K}{\eta}\right), \quad h = 1, \ldots, k.$$

As a consequence of the constraint (4.1) the previous strings contain wide roots only, i.e. they are entirely in the second analyticity region of $\tilde{Z}_N(\theta)$. An explanation of this fact will be given in a forthcoming paper [12] and will justify the constraint (4.1) itself.

We are now ready to study the collision two breathers of the XYZ spectrum. Besides the clear interest as an example of non-relativistic factorised (integrable) theory, this scattering acquires a peculiar rôle in a mathematical perspective, since we showed in a previous publication [5] that in the case of two low-lying ($n = 1$) breathers it furnishes, – as Zamolodchikov-Faddeev algebra –, the Deformed Virasoro Algebra (DVA) of [1]. In order to obtain the asymptotic states, we shall consider the thermodynamic ($N \to \infty$) limit of the chain.

5 Scattering factors between two elliptic breathers

When the number of lattice sites $N$ goes to infinity, the NLIE undergoes a significant simplification in that the convolution term can be neglected with respect to the other terms [3, 13, 14]. Hence, when $N \to \infty$ the NLIE (3.1), specified for the configuration containing an elliptic breather $B_{2k'+2}(\theta_1)$ and another breather $B_{2k+2}(\theta_2)$, takes on the
simplified form

\[ \tilde{Z}_N(\theta) = N \tilde{F}(\theta) - \sum_{h'=0}^{k'} \sum_{\epsilon'=\pm 1} \tilde{\chi}_{II}^{(\epsilon')} \left( \theta - \theta_1 + i\epsilon' K + i\epsilon' K 2h' \left( 1 - \frac{K}{\eta} \right) \right) - \sum_{h=0}^{k} \sum_{\epsilon=\pm 1} \tilde{\chi}_{II}^{(\epsilon)} \left( \theta - \theta_2 + i\epsilon K + i\epsilon K 2h \left( 1 - \frac{K}{\eta} \right) \right) \].

(5.1)

Now, we need to compute \( \tilde{Z}_N(\theta) \) on each root of the \( n \) even string

\[ \theta_1(\epsilon', h') = \theta_1 + i\epsilon' K + i\epsilon' K 2h' \left( 1 - \frac{K}{\eta} \right), \quad \epsilon' = \pm 1, \quad h' = 0, 1, \ldots, k' \],

(5.2)

and sum up over all the roots (i.e. over both \( \epsilon' \) and \( h' \)). From the definition of counting function \( \tilde{Z}_N(\theta) \) we know (see discussion in Section 6 of [5]) that the sum of the scattering phases (i.e. the terms involving \( \tilde{\chi}_{II} \), but not the forcing term) will equalise exactly the scattering phase \( i \ln S_{2k'+2}(\theta_1), \theta_1 = \theta_1 - \theta_2 \), between the elliptic breathers \( B_{2k+2}(\theta_1) \) and \( B_{2k+2}(\theta_2) \). In other words, we can interpret on a physical ground \( S_{2k'+2}(\theta_1) \) as the scattering amplitude. Technically, we shall pay attention to the positions of the roots (5.2) in the second analyticity strip of \( \tilde{\chi} \), i.e. \( \epsilon' \Im \theta_1(\epsilon', h') > 2K^2/\eta - 2K \). Therefore, we consider the second determination of the functions \( \tilde{\chi}_{II}^{(\epsilon)}(\theta) \) in (5.1): this entails that we must apply again (3.5), now with new input \( \tilde{\chi}_{II}^{(\epsilon')}(\theta) \) in the r.h.s.. We obtain the new functions \( \tilde{\chi}_{II}^{(\epsilon', \epsilon)}(\theta)_{II} \)

\[ \tilde{\chi}_{II}^{(\epsilon', \epsilon)}(\theta)_{II} = i \ln \left[ -xq^{\frac{\epsilon + \epsilon'}{2}} \frac{\Theta_{p^2}(x^{-1}q^{1-\frac{\epsilon + \epsilon'}{2}})}{\Theta_{p^2}(x^{-1}q^{1-\frac{\epsilon + \epsilon'}{2}})} \frac{\Theta_{p^2}(xq^{2p})}{\Theta_{p^2}(xq^{2p})} \right] \],

(5.3)

or explicitly

\[ \tilde{\chi}_{II}^{(+, +)}(\theta)_{II} = i \ln \left[ -xq^{\frac{\epsilon + \epsilon'}{2}} \frac{\Theta_{p^2}(x^{-1}q^{1-\frac{\epsilon + \epsilon'}{2}})}{\Theta_{p^2}(x^{-1}q^{1-\frac{\epsilon + \epsilon'}{2}})} \frac{\Theta_{p^2}(xq^{2p})}{\Theta_{p^2}(xq^{2p})} \right] \],

(5.4)

\[ \tilde{\chi}_{II}^{(-, -)}(\theta)_{II} = i \ln \left[ -xq^{-1} \frac{\Theta_{p^2}(xp)\Theta_{p^2}(x^{-1}q^2)}{\Theta_{p^2}(xp)\Theta_{p^2}(x^{-1}q^2)} \right] \],

(5.5)

\[ \tilde{\chi}_{II}^{(-, +)}(\theta)_{II} = \tilde{\chi}_{II}^{(+, -)}(\theta)_{II} = i \ln \left[ -x \frac{\Theta_{p^2}(xp)\Theta_{p^2}(x^{-1}q)}{\Theta_{p^2}(xp)\Theta_{p^2}(x^{-1}q)} \right] \].

(5.6)

Hence, the scattering factor to compute reads

\[ S_{2k'+2k+2}(\theta_{12}) = \exp \left[ i \sum_{h'=0}^{k'} \sum_{h=0}^{k} \sum_{\epsilon'=\pm 1} \sum_{\epsilon=\pm 1} \tilde{\chi}_{II}^{(\epsilon', \epsilon)} \left( \theta_{12} + i(\epsilon' + \epsilon)K + i(\epsilon'h' + \epsilon h)2K \left( 1 - \frac{K}{\eta} \right) \right) \right]_{II} \],

(5.7)

and we can use the explicit expression (5.3) to write down

\[ S_{2k'+2k+2}(\theta_{12}) = \prod_{h'=0}^{k'} \prod_{h=0}^{k} \prod_{\epsilon'=\pm 1} \prod_{\epsilon=\pm 1} \left[ -x \frac{\Theta_{p^2}(xp^{-1}q^{1-f}q^{-1-f}p)}{\Theta_{p^2}(xp^{-1}q^{1+f}q^{-1-f}p)} \right] \],

(5.8)
with the shorthand \( f = \frac{\epsilon + \epsilon'}{2} + \epsilon h + \epsilon' h' \) and where, with a little abuse of notation, we defined again \( x \) as
\[
x = \exp \left( -\frac{2i\pi \eta \theta_{12}}{KK'} \right).
\]

This expression may be rearranged as
\[
S_{2k'+2,2k+2}(\theta_{12}) = \prod_{h' = 0}^{k'} \prod_{h = 0}^{k} \left[ p^{f-1} q^{1-f} \frac{\Theta_{p^2}(xp^{1-f}q^{f-1})\Theta_{p^2}(xp^{-f-1}q^{f+1}p)}{\Theta_{p^2}(xp^{-f-1}q^{f+1})\Theta_{p^2}(xp^{1-f}q^{f-1})} \right],
\]
and, after performing the product over \( \epsilon, \epsilon' \), as
\[
S_{2k'+2,2k+2}(\theta_{12}) = \prod_{h'=-k'}^{k'} \prod_{h=-k}^{k} \left[ p^{-1} q \frac{\Theta_{p^2}(xp^{1-h-h'}q^{h+h'-1})\Theta_{p^2}(xp^{-h-h'-1}q^{h'+1})}{\Theta_{p^2}(xp^{-h-h'-1}q^{h+h'})\Theta_{p^2}(xp^{1-h-h'}q^{h+h'-1})} \right].
\]

Now, the product on \( h' \) gives
\[
S_{2k'+2,2k+2}(\theta_{12}) = \left( \frac{q}{p} \right)^{(2k'+2)(2k+2)} \prod_{h=-k-1}^{k} \frac{\Theta_{p^2}(xp^{k-h}q^{-k'+h})\Theta_{p^2}(xp^{k-h-2}q^{-k'+h+2})\Theta_{p^2}(xp^{k-h-1}q^{-k'+h+1})}{\Theta_{p^2}(xp^{k-h-1}q^{-k'+h})\Theta_{p^2}(xp^{k-h-2}q^{-k'+h+2})\Theta_{p^2}(xp^{k-h}q^{-k'+h+1})} = \left( \frac{q}{p} \right)^{(2k'+2)(2k+2)} \prod_{h=-k}^{k} \frac{\Theta_{p^2}(xp^{k-h-q^{-k-k'}})\Theta_{p^2}(xp^{k-h-q^{-k-k'}}-2)\Theta_{p^2}(xp^{k-h-q^{-k-k'}+2})}{\Theta_{p^2}(xp^{k-h-q^{-k-k'}})\Theta_{p^2}(xp^{k-h-q^{-k-k'}}+2)\Theta_{p^2}(xp^{k-h-q^{-k-k'}}-2)}
\]
\[
\prod_{h=-k}^{k} \frac{\Theta_{p^2}(xp^{k-h-q^{-k-k'}})\Theta_{p^2}(xp^{k-h-q^{-k-k'}}+2)\Theta_{p^2}(xp^{k-h-q^{-k-k'}}-2)}{\Theta_{p^2}(xp^{k-h-q^{-k-k'}})\Theta_{p^2}(xp^{k-h-q^{-k-k'}}+2)\Theta_{p^2}(xp^{k-h-q^{-k-k'}}-2)}.
\]

Eventually, we can reach the final form after performing algebraic manipulations in order to highlight the exchange symmetry \( x \rightarrow x^{-1} \) and with the positions \( 2k' + 2 = n, 2k + 2 = m \)
\[
S_{n,m}(\theta_{12}) = x^{n} \Theta_{p^2}(x^{-1}q^{\frac{m+n}{2}}p^{\frac{m+n}{2}})\Theta_{p^2}(xp^{1-n}q^{-\frac{m+n}{2}})\Theta_{p^2}(x^{-1}q^{\frac{m+n}{2}}p^{-\frac{m+n}{2}}) \Theta_{p^2}(xp^{-1}q^{-\frac{m+n}{2}})\Theta_{p^2}(x^{-1}p^{\frac{m+n}{2}})
\]
\[
\prod_{l=1}^{m+n-1} \left[ x^{2} \Theta_{p^2}(x^{-1}q^{l}p^{-l})\Theta_{p^2}(xp^{-1}q^{l}) \Theta_{p^2}(xp^{-1}p^{-l})\Theta_{p^2}(xp^{1-l}q^{l}) \right].
\]

Albeit this formula has been obtained for both \( n \) and \( m \) even, it is easy but lengthy, by following the same path, to show that indeed formula (5.13) gives the scattering factor between the general \( n \)-th and the \( m \)-th breather. However, two precautions should be observed when dealing with (5.13). The first one is that the product over \( l \) is present only if \( m \geq 2 \); the second one concerns that this product runs over the integers if \( n \)
and $m$ have the same parity, whereas over the half-integers if $n$ and $m$ have different parities. We end this part by noticing two simple properties of (5.13)

$$S_{n,m}(\theta)S_{n,m}(-\theta) = 1, \quad S_{n,m}(\theta) = S_{m,n}(\theta). \quad (5.14)$$

**Remark 1:** The scattering factors between the elliptic breathers, (5.13), are not independent. More precisely, they satisfy the decomposition

$$S_{n+m,n'+m'} \left[ \theta - \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (n-m-n'+m') \right] =$$

$$= S_{n,m'} \left[ \theta - \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (n+m+n'+m') \right] \cdot$$

$$\cdot S_{m,m'} \left[ \theta - \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (-n-m+n'+m') \right] \cdot$$

$$\cdot S_{n,n'} \left[ \theta - \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (n+m-n'-m') \right] \cdot$$

$$\cdot S_{m,n} \left[ \theta - \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (-n-m-n'-m') \right]. \quad (5.15)$$

In particular, this links the general scattering amplitude $S_{n,n}$ to that of the fundamental scalar particle $S_{1,1}$ (or, in mathematical language, to the DVA structure function):

$$S_{n,n}(\theta) = \prod_{l=1-n}^{n-1} S_{1,1} \left[ \theta - \left( \frac{2iK}{2} - \frac{2iK^2}{\eta} \right) l \right]. \quad (5.16)$$

**Remark 2:** Let $B_n(\theta)$ be the Zamolodchikov-Faddeev generator of the $n$-th breather with rapidity $\theta$ and satisfy the exchange algebra

$$B_n(\theta_1)B_m(\theta_2) = S_{n,m}(\theta_1-\theta_2)B_m(\theta_2)B_n(\theta_1). \quad (5.17)$$

The relation (5.15) is consistent with the following connection between the creation operator $B_{n+m}(\theta)$ and the operators $B_n(\theta)$ and $B_m(\theta)$:

$$B_{n+m} \left[ \theta + \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (m-n) \right] = B_m \left[ \theta + \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (n+m) \right] \cdot$$

$$\cdot B_n \left[ \theta - \left( \frac{iK}{2} - \frac{iK^2}{2\eta} \right) (n+m) \right]. \quad (5.18)$$

Consecutive applications of this property allows us to write the generic breather generator $B_n$ as a product of fundamental scalar generators $B_1$. This fact becomes important as soon as we realise [5] that $B_1(\theta)$ coincide with the mode generator of DVA, $T(z)$, which enjoys a particular structure [1]. Hopefully, this should clarify the physical rôle played by DVA. Finalising the parallelism, (5.18) is the elliptic deformation of the analogous relation in the sine-Gordon theory (see footnote 9 in the seminal paper [9]).
5.1 Recovering the trigonometric limit

Let the trigonometric limit be defined by \( K' \to \infty \) or, equivalently, by \( K \to \pi/2 \). In this respect a key limit is

\[
\Theta_{p^2}(x^{-1}q^l p^{-l}) \to \frac{1}{i} \tanh \left[ \frac{\theta}{2} + \frac{i \pi b_l}{2} \right], \tag{5.19}
\]

where we have introduced the quantity \( b \) defined by \( \eta \), and then, thanks to the nature of that limit, connected to the sine-Gordon coupling constant (see section 2) as follows:

\[
\eta = \frac{\pi}{2(b + 1)}; \quad b = \frac{\beta^2}{1 - \beta^2}. \tag{5.20}
\]

Application of the limit (5.19) to the scattering factor (5.13) gives for us

\[
S_{n,m}(\theta) \to \frac{\tanh \left[ \frac{\theta}{2} + \frac{i \pi b}{4}(n-m) \right] \tanh \left[ \frac{\theta}{2} + \frac{i \pi b}{4}(m+n) \right]}{\tanh \left[ \frac{\theta}{2} - \frac{i \pi b}{4}(n-m) \right] \tanh \left[ \frac{\theta}{2} - \frac{i \pi b}{4}(m+n) \right]} \prod_{l=1+\frac{b_n}{2}}^{m+n-1} \tanh^2 \left( \frac{\theta}{2} + \frac{i \pi b_l}{2} \right) \prod_{l=1+\frac{b_n}{2}}^{m+n-1} \tanh^2 \left( \frac{\theta}{2} - \frac{i \pi b_l}{2} \right).
\]

\[
= \frac{\sinh \theta + i \sin \frac{\pi b}{2}(n-m)}{\sinh \theta - i \sin \frac{\pi b}{2}(m+n)} \cdot \frac{\sinh \theta + i \sin \frac{\pi b}{2}(m+n)}{\sinh \theta - i \sin \frac{\pi b}{2}(n-m)} \cdot \prod_{l=1}^{m-1} \frac{\sin^2 \left[ i \theta_{0} - \frac{\pi b}{4}(n-m+2l) \right] \cos^2 \left[ i \theta_{0} + \frac{\pi b}{4}(n-m+2l) \right]}{\sin^2 \left[ -i \theta_{0} - \frac{\pi b}{4}(n-m+2l) \right] \cos^2 \left[ -i \theta_{0} + \frac{\pi b}{4}(n-m+2l) \right]}. \tag{5.21}
\]

As expected, this limiting formula coincides with the scattering amplitude between the \( n \)-th and the \( m \)-th breather in the sine-Gordon model (i.e. formula (4.19) of [9]).

6 Higher Order Deformed Virasoro Algebras

Continuing the identification between the fundamental scalar generator \( B_1(\theta) \) and the DVA generator \( T(z) \), we can now define higher order Deformed Virasoro Algebras. The order \( n \) deformed Virasoro algebra is the central extension of the Zamolodchikov-Faddeev algebra 2 of two breathers creators \( B_n \), i.e.

\[
B_n(\theta_1)B_n(\theta_2) = S_{n,n}(\theta_1 - \theta_2)B_n(\theta_2)B_n(\theta_1), \tag{6.1}
\]

where

\[
S_{n,n}(\theta) = -x^2 \frac{\Theta_{p^2}(x^{-1}q^np^{-n})\Theta_{p^2}(xp^{-1}q_n)}{\Theta_{p^2}(xp^np-n)\Theta_{p^2}(x^{-1}p^nnq)} \prod_{l=1}^{n-1} \left[ x^2 \frac{\Theta_{p^2}^2(x^{-1}q^l p^{-l})\Theta_{p^2}^2(xp^{-1}q^l)}{\Theta_{p^2}^2(xq^l p^{-l})\Theta_{p^2}^2(x^{-1}p^l q^l)} \right]. \tag{6.2}
\]

When \( n = 1 \), the algebra (6.1) coincides with the Deformed Virasoro Algebra \( Vir_{p,q} \) without central extension. In order to clarify the relation between the general (6.1)

2Some authors prefer to define the Zamolodchikov-Faddeev algebra with the central extension: in this case it would simply coincide with a higher order deformed Virasoro algebra.
(or, which is the same, the corresponding higher order deformed Virasoro algebra) and the particular \( \mathcal{V}_{p,q} \), we consider the semiclassical limit of (6.1) around the particular values \( q = p^h \) (or \( K = \eta h \), \( h = 1, 2 \), characterised by the special property that at them \( S_{n,n}(\theta) = 1 \). From the algebraic perspective, this means that we can define a Poisson bracket around each point for which \( q = p^h \). In this respect, the order \( n \) Zamolodchikov-Faddeev algebra (generated by \( B_n \)) turns out to be a quantisation of such Poisson structure. Explicitly, if we measure the distance through \( \beta \) such that \( q_1 - \beta^2 = p^h \), these Poisson brackets are defined by

\[ \{ B_n(\theta_1), B_n(\theta_2) \} = \frac{\partial}{\partial \beta} S_{n,n}(\theta_1 - \theta_2)|_{\beta=0} B_n(\theta_1) B_n(\theta_2) \tag{6.3} \]

where

\[
\frac{\partial}{\partial \beta} S_{n,n}(\theta_{12})|_{\beta=0} = C(n, h) \ln p \left[ \frac{x^{-1}}{1-x^{-1}} - \frac{x}{1-x} + \right. \\
+ \sum_{m=0}^{\infty} \left( \frac{-2x^{-1}p^{2m}}{1-x^{-1}p^{2m}} + \frac{2x^{-1}p^{2m+1}}{1-x^{-1}p^{2m+1}} + \frac{2xp^{2m}}{1-xp^{2m}} - \frac{2xp^{2m+1}}{1-xp^{2m+1}} \right) ,
\]

and besides

\[
C(n, h) = \sum_{l=1-n}^{n-1} (-1)^{(1-h)l(n-|l|)} \frac{h}{2} (-1)^{h+1} \Rightarrow C(n, 1) = \frac{n^2}{2}, \quad C(n, 2) = \frac{(-1)^n - 1}{2} . \tag{6.5}
\]

Interpreting this relation in terms of modes (see for instance the discussion in Section 5 of [15]), we remark that the Poisson structure around \( q = p^h \) is proportional to the Frenkel-Reshetikhin Poisson bracket [10], which defines the classical Deformed Virasoro Algebra, without central extension. Even this conclusion has led us to consider the breather algebras as higher order versions of the fundamental DVA\( ^3 \).

7 Conclusive outlook

In this paper we have continued the study of scattering factors between bound states of the spin 1/2-XYZ model, commenced in [5] and restricted there to the simplest one. Following the method exposed in [5], any scattering factor has been extracted from the infrared limit of the NLIE describing the pairwise collision. In the end, we have found the complete set of scattering factors between bound states (elliptic breathers) of the spin 1/2-XYZ chain (in the attractive regime). This physical information has allowed us to re-interpret the braiding relations between the corresponding scattering operators as a consistent mathematical definition of a family of Deformed Virasoro Algebras,

\(^3\)Notice that it is ‘fundamental’ even in a physical sense as it describes the scattering of the fundamental scalar particle.
parameterized by a positive integer $n$. Of course, at $n = 1$ the usual Deformed Virasoro Algebra \cite{1} is recovered and $n$ ranges from this value to a maximum depending on the chain parameters. And all the members of this family have been proved to reproduce, in the classical limit, the Poisson bracket structure discovered by Frenkel and Reshetikhin \cite{10} (and defining the classical Deformed Virasoro Algebra). For these reasons, these algebras are all eligible to the name of higher order Deformed Virasoro Algebras and generalise the seminal quantisation by Shiraishi-Kubo-Awata-Odake \cite{1}. Besides, the search for a family of DVAs seems to be a quite intriguing problem, though we cannot guaranty that we have found the amplest family neither the only one. Nevertheless, these results should shed light on the peculiar rôle of DVA in field and lattice theory.

In an incoming paper \cite{12} these results and others will be used to highlight the non-relativistic nature of the thermodynamic theory and the link between bound state energies and the poles of the Zamolodchikov $Z_4$ S-matrix \cite{6}.

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**References**

1. J. Shiraishi, H. Kubo, H. Awata, S. Odake, *A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions*, Lett. Math. Phys. **38** (1996), 33 and [q-alg/9507034](http://arxiv.org/abs/q-alg/9507034).

2. A. Luther, *Eigenvalue spectrum of interacting massive fermions in one dimension*, Phys. Rev. **B14**, No. 5 (1976), 2153; J.D. Johnson, S. Krinski, B.M. McCoy, *Vertical-Arrow correlation length in the eight-vertex model and the low-lying excitations of the XYZ Hamiltonian*, Phys. Rev. **A8** (1973), 2526.

3. C. Destri, H.J. de Vega, *Unified approach to Thermodynamic Bethe Ansatz and finite size corrections for lattice models and field theories*, Nucl. Phys. **B438** (1995) 413 and [hep-th/9407117](http://arxiv.org/abs/hep-th/9407117).

4. J.A. Minahan, K. Zarembo, *The Bethe Ansatz for N=4 super Yang-Mills*, JHEP**03** (2003), 013 and [hep-th/0212208](http://arxiv.org/abs/hep-th/0212208).
[5] D. Fioravanti, M. Rossi, From finite geometry exact quantities to (elliptic) scattering amplitudes for spin chains: the 1/2-XYZ, JHEP08(2005), 010 and hep-th/0504122.

[6] A.B. Zamolodchikov, Z₄-Symmetric factorised S-matrix in two space-time dimensions, Commun. Math. Phys. 69 (1979), 165;

[7] S. Lukyanov, A note on the Deformed Virasoro Algebra, Phys.Lett. B367 (1996) 121 and hep-th/9509037;

[8] G. Mussardo, S. Penati, A quantum field theory with infinite resonance states. Nucl. Phys. B567 (2000), 454 and hep-th/9907039;

[9] A.B. Zamolodchikov, Al.B. Zamolodchikov, Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models, Ann. Phys. 120 (1979), 253;

[10] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and W-algebras, Commun. Math. Phys. 178 (1996), 237 and q-alg/9505025;

[11] L.A. Takhtadjan, L.D. Faddeev, The quantum method of the inverse problem and the Heisenberg XYZ model, Russ. Math. Surveys 34, 5 (1979), 11;

[12] D. Fioravanti, M. Rossi, Bound states of the XYZ model and the Z₄ Zamolodchikov S-matrix, in preparation;

[13] D. Fioravanti, A. Mariottini, E. Quattrini, F. Ravanini, Excited state Destri-de Vega equation for sine-Gordon and restricted sine-Gordon models, Phys. Lett. B390 (1997), 243 and hep-th/9608091;

C. Destri, H.J. de Vega, Non linear integral equation and excited–states scaling functions in the sine-Gordon model, Nucl. Phys. B504 (1997), 621 and hep-th/9701107.

[14] G. Feverati, F. Ravanini, G. Takacs, Nonlinear Integral Equation and Finite Volume Spectrum of Sine-Gordon Theory, Nucl. Phys. B540 (1999), 543 and hep-th/9805117.

[15] J. Avan, L. Frappat, M. Rossi, P. Sorba, Poisson structures on the center of the elliptic algebra $\mathcal{A}_{q,p}(\hat{sl}(2)_{c})$, Phys. Lett. A235 (1997), 323 and q-alg/9705012.