The Lorentz - invariant deformation of the Whitham system for the non-linear Klein-Gordon equation.

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Abstract

We consider the deformation of the Whitham system for the non-linear Klein-Gordon equation having the Lorentz-invariant form. Using the Lagrangian formalism of the initial system we obtain the first non-trivial correction to the Whitham system in the Lorentz-invariant approach.

1 Introduction.

We will consider the deformation of the Whitham system for the non-linear Klein-Gordon equation

\[ \varphi_{tt} - \varphi_{xx} + V' (\varphi) = 0 \]  

(1.1)

The Whitham method is connected with the slow modulations of the periodic or quasiperiodic m-phase solutions of the nonlinear systems

\[ F^i (\varphi, \varphi_t, \varphi_x, \ldots) = 0, \quad i = 1, \ldots, n, \quad \varphi = (\varphi^1, \ldots, \varphi^n) \]  

(1.2)

which are represented usually in the form

\[ \varphi^i (x, t) = \Phi^i (k(U) x + \omega(U) t + \theta_0, U) \]  

(1.3)

In these notations the functions k(U) and \( \omega(U) \) play the role of the "wave numbers" and "frequencies" of m-phase solutions and \( \theta_0 \) are the initial phase shifts.

The functions \( \Phi^i (\theta) \) satisfy the system

\[ F^i (\Phi, \omega^a \Phi_a, k^\beta \Phi_\beta, \ldots) \equiv 0, \quad i = 1, \ldots, n \]  

(1.4)

and we choose (in a smooth way) some function \( \Phi (\theta, U) \) for every U as having "zero initial phase shifts". The full set of m-phase solutions of (1.2) can then be represented in the form (1.3).
For the \( m \)-phase solutions of (1.2) we have then \( k(U) = (k^1(U), \ldots, k^m(U)), \omega(U) = (\omega^1(U), \ldots, \omega^m(U)), \theta_0 = (\theta^1, \ldots, \theta^m) \), where \( U = (U^1, \ldots, U^N) \) are the parameters of the solution. We require also that all the functions \( \Phi_i(\theta, U) \) are \( 2\pi \)-periodic with respect to every \( \theta^\alpha, \alpha = 1, \ldots, m \).

For one-phase solutions of (1.1) we have obviously

\[
(\omega^2 - k^2)\Phi_{\theta\theta} + V'(\Phi) = 0 \tag{1.5}
\]

where \( \Phi(\theta, \omega^2 - k^2) \) is \( 2\pi \)-periodic function of \( \theta \).

In Whitham approach (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]) the parameters \( U \) become the slow functions of \( x \) and \( t \):

\[
U^\alpha = U^\alpha(X, T),
\]

where \( X = \epsilon x, T = \epsilon t (\epsilon \to 0) \).

More precisely (see [4]), we try to find the asymptotic solutions

\[
\varphi^i(\theta, X, T) = \sum_{k \geq 0} \Psi_{(k)}(S(X, T) \frac{\theta_0}{\epsilon} + \theta_0, X, T) \epsilon^k \tag{1.6}
\]

(where all \( \Psi_{(k)} \) are \( 2\pi \)-periodic in \( \theta \)) which satisfy the system (1.2), i.e.

\[
F^i(\varphi, \epsilon \varphi_T, \epsilon \varphi_X, \ldots) = 0 \quad i = 1, \ldots, n
\]

The function \( S(X, T) = (S^1(X, T), \ldots, S^m(X, T)) \) is the ”modulated phase” of the solution (1.6) and the function \( \Psi_{(0)}(\theta, X, T) \) belongs at every \( X \) and \( T \) to the family of \( m \)-phase solutions of (1.2). We have then

\[
\Psi_{(0)}(\theta, X, T) = \Phi(\theta + \theta_0(X, T), U(X, T)) \tag{1.7}
\]

and

\[
S^\alpha(X, T) = \omega^\alpha(U), \quad S^\alpha_X(X, T) = k^\alpha(U)
\]

as follows from the substitution of (1.6) in the system (1.2).

The functions \( \Psi_{(k)}(\theta, X, T) \) are defined from the linear systems

\[
\hat{L}^i_{j[U,\theta_0]}(X, T) \Psi_{(k)}^j(\theta, X, T) = f_{(k)}^i(\theta, X, T) \tag{1.8}
\]

where \( \hat{L}^i_{j[U,\theta_0]}(X, T) \) is a linear operator given by the linearizing of the system (1.4) on the solution (1.6). The resolvability conditions of the systems (1.8) can be written as the orthogonality conditions of the functions \( f_{(k)}(\theta, X, T) \) to all the ”left eigen vectors” (the eigen vectors of adjoint operator) \( \kappa_{[\theta+\theta_0(X, T)]}^{[q]}(\theta, \theta_0(X, T)) \) of the operator \( \hat{L}^i_{j[U,\theta_0]}(X, T) \) corresponding to zero eigen-values. The resolvability conditions of (1.8) for \( k = 1 \) together with

\[
k^\alpha_T = \omega^\alpha_X
\]

give the Whitham system for \( m \)-phase solutions of (1.2) playing the central role in the slow modulations approach.
We have for the equation (1.1)

\[ \hat{L}_{[k,\omega,\theta_0]} = (\omega^2 - k^2) \frac{\partial^2}{\partial \theta^2} + V''(\Phi(\theta + \theta_0,k,\omega)) \]

and the function \( \Phi(\theta + \theta_0,k,\omega) \) plays the role of both "left" and "right" eigen-vector corresponding to zero eigen-value.

The function \( f_{(1)}(\theta,X,T) \) is defined as

\[ f_{(1)} = -S_{TT} \Phi_\theta - 2S_T \Phi_{\theta T} + S_{XX} \Phi_\theta + 2S_X \Phi_{\theta X} \quad (1.9) \]

and the Whitham system can be written as

\[ \left( \omega \int_0^{2\pi} \Phi_\theta^2 \frac{d\theta}{2\pi} \right)_T = \left( k \int_0^{2\pi} \Phi_\theta^2 \frac{d\theta}{2\pi} \right)_X \quad (1.10) \]

\[ k_T = \omega_X \]

The Whitham system is a so-called system of Hydrodynamic Type, which can be written in the form

\[ A^\nu_{\mu}(U) U^\mu_T = B^\nu_{\mu}(U) U^\mu_X \quad (1.11) \]

with some matrices \( A(U) \) and \( B(U) \). In generic case the system (1.11) can be resolved w.r.t. the time derivatives of \( U \) and written in the evolution form

\[ U^\mu_T = V^\nu_{\mu}(U) U^\mu_X \quad , \quad \nu = 1, \ldots, N \quad (1.12) \]

(112)

The Lagrangian properties of the Whitham system were investigated by Whitham (3) who suggested also the method of "averaging" of the Lagrangian function to get the Lagrangian function for the Whitham system.

Another important procedure is the procedure of "averaging" of local Hamiltonian structures suggested by B.A. Dubrovin and S.P. Novikov (14, 22, 23). The Dubrovin - Novikov procedure gives a field-theoretical Hamiltonian structure of Hydrodynamic Type for the system (1.12) with a Hamiltonian function having the hydrodynamic form \( H = \int h(U)dX \). Dubrovin - Novikov bracket for the system (1.12) has the form

\[ \{ U^\nu(X), U^\mu(Y) \} = g^\nu\mu(U) \delta'(X - Y) + b^\nu\mu(U) U_\lambda^X \delta(X - Y) \quad (1.13) \]

which is called also the local Poisson bracket of Hydrodynamic Type.

The Hamiltonian properties of the systems (1.12) are strongly correlated with their integrability properties. Thus it was proved by S.P. Tsarev (24) that all the diagonalizable systems (1.12) having Dubrovin - Novikov Hamiltonian structure can in fact be integrated (S.P. Novikov conjecture). Actually the same is true also for the diagonalizable systems
having more general weakly-nonlocal Mokhov-Ferapontov or Ferapontov Hamiltonian structures \([25, 26, 27, 28, 29, 30, 31]\). Let us mention here that the generalization of Dubrovin - Novikov procedure for the weakly-nonlocal case was suggested in \([32]\).

The theory of the systems and hierarchies \((1.12)\) having the bi-Hamiltonian formalism plays the basic role in the theory of Frobenius manifolds \((33, 34, 35, 36, 38)\) giving the classification of of Topological Quantum Fields Theories based on WDVV-equation. Let us say here that Dubrovin - Novikov procedure gives also the possibility of construction of bi-Hamiltonian formalism of the Whitham system \((1.12)\) if the initial system \((1.2)\) has a local field-theoretical bi-Hamiltonian formalism. As a corollary of this fact, the Whitham hierarchies corresponding to the most of the integrable systems (like KdV or NLS) give in fact the important examples of Frobenius manifolds.

The higher corrections to Topological Quantum Field theories require the deformations \((37, 39, 41)\) of the Hydrodynamic Type hierarchies \((1.12)\) having the form

\[
U_\nu^{\mu} = V_\mu^{\nu}(U) U_\chi^\mu + \sum_{k \geq 2} v_\nu^{(k)}(U, U_X, \ldots, U_{kX}) \epsilon^{k-1} \tag{1.14}
\]

where all \(v_\nu^{(k)}\) are smooth functions polynomial in the derivatives \(U_X, \ldots, U_{kX}\) and having degree \(k\) according to the following gradation rule:

1) All the functions \(f(U)\) have degree 0;
2) The derivatives \(U_\nu^{\mu}X\) have degree \(k\);
3) The degree of the product of two functions having certain degrees is equal to the sum of their degrees.

The deformation \((1.14)\) of the system \((1.12)\) implies also the deformation of the corresponding (bi-)Hamiltonian structures \((1.13)\)

\[
\{U_\nu^{\mu}(X), U_\mu^{\nu}(Y)\} = \{U_\nu^{\mu}(X), U_\mu^{\nu}(Y)\}_0 +
\]

\[
+ \sum_{k \geq 2} \sum_{s=0}^k B^{\nu\mu}_{(k)s}(U, U_X, \ldots, U_{(k-s)X}) \delta^{(s)}(X - Y) \tag{1.15}
\]

where all \(B^{\nu\mu}_{(k)s}\) are polynomial w.r.t. derivatives \(U_X, \ldots, U_{(k-s)X}\) and have degree \((k-s)\).

The deformation \((1.14)\) of the hierarchy of Hydrodynamic Type with the deformation \((1.15)\) of the corresponding bi-Hamiltonian structure plays the basic role in the procedure of deformation of Frobenius manifold \((37, 39, 41)\). Let us say here that the general theory of deformations of integrable hierarchies of Hydrodynamic Type as well as the bi-Hamiltonian structures of Dubrovin - Novikov type is being actively studied by now. Let us also point out here the recent papers where the important results in this area were obtained \([10, 32, 33, 41, 43]\). Let us call the deformation \((1.14)\) of any system of Hydrodynamic Type the Dubrovin - Zhang deformation.

As was first pointed out in \([3]\) the higher corrections in Whitham method satisfy the more complicated equations including "dispersive terms” and the Whitham system \((1.11)\)
should in fact contain also the higher derivatives ("dispersion") being considered in the next orders of \( \epsilon \).\(^1\)

In [46] the general procedure of deformation of Hyperbolic Whitham systems based on the "renormalization" of parameters was suggested. The procedure suggested in [46] gives the deformation of the Whitham system (1.12) having the Dubrovin - Zhang form (1.14). This method can be considered from our point of view as the solution of the first part of the problem set by B.A. Dubrovin and connected with the deformations of Frobenius manifolds. Namely, B.A. Dubrovin problem requires the construction of the deformation of the Whitham system (1.12) in Dubrovin - Zhang form and the corresponding bracket (1.13) also having Dubrovin - Zhang form (1.15). The solution of Dubrovin problem for the case of bi-Hamiltonian integrable systems gives in fact the deformation of Frobenius manifolds defined by the corresponding Whitham hierarchies which are bi-Hamiltonian according to Dubrovin - Novikov procedure. Let us say here also that in [47] the Lagrangian properties of the deformations of Whitham systems suggested in [46] were investigated. However, the generalization of Dubrovin - Novikov procedure giving the deformations of the "averaged" brackets (1.13) was not considered yet.

2 Lorentz-invariant deformation of the Whitham system.

In this paper we will consider the deformation of the Whitham system for the equation (1.1) having Lorentz-invariant form. Like in [46, 47] we will use here the "renormalization" of parameters of one-phase solutions of (1.1) after the construction of the solution (1.6). Namely, we introduce the "renormalized" modulated phase

\[
S(X, T, \epsilon) = \sum_{k \geq 0} S(\kappa)(X, T) \epsilon^k
\]

which is the "physically observable" quantity and make then the "re-expansion" of the series (1.6) using the higher derivatives of \( S(X, T, \epsilon) \) instead of the parameter \( \epsilon \). The function \( \Phi \left( \frac{S(X, T, \epsilon)}{\epsilon} + \theta, S_X(X, T, \epsilon), S_T(X, T, \epsilon) \right) \) (2.1) will play now the role of the main approximation in the "renormalized" expansion while all the higher terms will contain the higher derivatives of the function \( S(X, T, \epsilon) \). The terms of the new expansion will now be \( \epsilon \)-dependent and constructed according to the new "gradation rule" put on the derivatives of \( S(X, T, \epsilon) \).

In this situation we can omit in fact the "unobservable" parameter \( \epsilon \) and use just the functionals of the "renormalized" modulated phase \( S(X, T) = S(X, T, \epsilon) \). Following [46, 47] let us then omit the parameter \( \epsilon \) in all calculations (or put formally \( \epsilon = 1 \)),

\(^1\)Also the multi-phase Whitham method was first considered in [5, 6, 7].
however, we will keep the notations \( X \) and \( T \) for the spatial and time variables just to emphasize that the parameters of one-phase solution are the slow functions of \( x \) and \( t \). Like in [46, 47] we define now the "renormalization rule" which determines the renormalized phase \( S(X, T, \varepsilon) \). Namely, we look now for the solution of (1.1) having the form

\[
\varphi(\theta, X, T) = \Phi(S(X, T) + \theta, S_X, S_T) + \sum_{k \geq 1} \tilde{\Psi}(k)(S(X, T) + \theta, X, T)
\]

where the functions \( \tilde{\Psi}(k) \) are the local functionals of \( S(X, T) \) and its derivatives having gradation degree \( k \) (defined below). All the functions \( \tilde{\Psi}(k)(\theta, X, T) \) are defined from the linear systems

\[
\hat{L}_{[S_X, S_T]} \tilde{\Psi}(k)(\theta, X, T) = \tilde{f}(k)(\theta, X, T)
\]

where

\[
\hat{L}_{[S_X, S_T]} = (S_T^2 - S_X^2) \frac{\partial^2}{\partial \theta^2} + V''(\Phi(\theta, S_X, S_T))
\]

is the linearization of the equation (1.5) on the function \( \Phi(\theta, S_X, S_T) \) and \( \tilde{f}(k)(\theta, X, T) \) is the discrepancy having gradation degree \( k \). The system (2.3) defines the function \( \tilde{\Psi}(k)(\theta, X, T) \) modulo the function \( c(X, T) \Phi_\theta(\theta, S_X, S_T) \) which belongs to the kernel of the operator \( \hat{L}_{[S_X, S_T]} \) at every \( X \) and \( T \). We require now that the function \( S(X, T) \) is normalized in the "optimal" way, such that

\[
\int_0^{2\pi} \Phi_\theta(\theta, S_X, S_T) \tilde{\Psi}(k)(\theta, X, T) \frac{d\theta}{2\pi} = 0
\]

for all \( X \) and \( T \).

The condition (2.4) defines now uniquely the function \( \tilde{\Psi}(k)(\theta, X, T) \) satisfying the system (2.3).

Let us speak now about the "gradation rule" we are going to use here.

Let us say first that the gradation used in [46, 47] and giving the deformation of the Whitham system in Dubrovin - Zhang form was defined by the \( X \)-derivatives of the parameters \( U \) of \( m \)-phase solutions (1.3). Namely, in [46, 47] the following gradation rule was used:

1) The functions \( f(\omega, k) = f(S_T, S_X) \) have degree 0;
2) The derivatives \( \omega_{kX}, k_{kX} \) have degree \( k \);
3) The degree of the product of two functions having certain degrees is equal to the sum of their degrees.

According to this gradation rule the higher \( T \)-derivatives of the parameters \( (\omega, k) \) do not have certain degrees and it is required just that the functions \( \omega_{kT}, k_{kT} \) can be
represented as the (infinite) series of terms having degree $\geq k$. The expression of $(k_T, \omega_T)$ in terms of $X$ derivatives of the functions $(k, \omega)$

$$
\begin{align*}
    k_T &= \omega_X \\
    \omega_T &= \sum_{k \geq 1} \sigma_{(k)}(k,\omega,k_X,\omega_X,\ldots)
\end{align*}
$$

plays then the role of the deformation of the Whitham system having Dubrovin-Zhang form. All the functions $\sigma_{(k)}(k,\omega,k_X,\omega_X,\ldots)$ have degree $k$ and are defined from the compatibility conditions

$$
\int_0^{2\pi} \Phi_\theta(\theta, S_X, S_T) f_{(k)}(\theta, X, T) \frac{d\theta}{2\pi} = 0
$$

for the system (2.3) in the $k$-th order. The system (2.5) plays then the role of the "full Whitham system" including all the orders of asymptotic expansion.

In this paper, however, we have to use some Lorentz-invariant gradation rule to provide the Lorentz-invariant deformation of the Whitham system (1.10).

Let us introduce now two differential operators

$$
\hat{\xi}_1 = S_T \frac{\partial}{\partial T} - S_X \frac{\partial}{\partial X}, \quad \hat{\xi}_2 = S_X \frac{\partial}{\partial T} - S_T \frac{\partial}{\partial X}
$$

Easy to see that both $\hat{\xi}_1$ and $\hat{\xi}_2$ are Lorentz-invariant. Let us use notation $\mu = S_T^2 - S_X^2$. We have

$$
\hat{\xi}_1 S = S_T^2 - S_X^2 = \mu, \quad \hat{\xi}_2 S = 0
$$

We put by definition $\text{deg } f(\mu) = 0$ for any smooth function $f(\mu)$. Easy to see then that the expressions

$$
\hat{\xi}_1 \mu = 2S_T^2 S_{TT} + 2S_X^2 S_{XX} - 4S_T S_X S_{XT}
$$

and

$$
\hat{\xi}_2 \mu = 2S_T S_X (S_{TT} + S_{XX}) - 2(S_T^2 + S_X^2) S_{XT}
$$

are both Lorentz-invariant functions. We then define the general function having degree 1 in the form

$$
f[S] = a(\mu) \hat{\xi}_1 \mu + b(\mu) \hat{\xi}_2 \mu
$$

Let us note that the function $\nu = S_{TT} - S_{XX}$ does not have certain degree in this approach and will be represented as the infinite sum of the terms having degree $\geq 1$.

Let us define now the functions with higher degrees using the operator $\hat{\xi}_1$. Namely, we introduce the family $\mathcal{M}$ of functions with certain gradation degrees using the following rule:
1) All the smooth functions $f(\mu)$ belong to $\mathcal{M}$ and have degree 0;
2) The functions $Q = \xi_1\mu$, $P = \xi_2\mu$ belong to $\mathcal{M}$ and $\deg Q = \deg P = 1$.
3) If the functions $F$ and $G$ belong to $\mathcal{M}$ then $FG$ belongs to $\mathcal{M}$ and

$$\deg (FG) = \deg F + \deg G$$

4) If the function $F$ belongs to $\mathcal{M}$ then the function $\hat{\xi}_1 F$ also belongs to $\mathcal{M}$ and

$$\deg (\hat{\xi}_1 F) = \deg F + 1$$

Easy to see that all the functions from the family $\mathcal{M}$ are Lorentz-invariant.

The function $\nu = S_{TT} - S_{XX}$ does not belong to the family $\mathcal{M}$ and we do not prescribe any certain degree to this function. Instead, we put now the conditions

$$S_{TT} - S_{XX} = \sum_{k \geq 1} \nu(k)$$  \hspace{1cm} (2.7)

where every $\nu(k)$ is represented as a sum functions belonging to $\mathcal{M}$ and having degree $k$.

The system (2.7) will now play the role of the deformation of the Whitham system in the Lorentz-invariant representation. Every function $\nu(k)$, $k \geq 1$ is defined now from the compatibility conditions (2.6) of the system (2.3) in the $k$-th order.

The equation

$$S_{TT} - S_{XX} = \nu(1)$$

coincides with the Whitham system (1.10) and we have

$$\nu(1) = - \left[ \left( \frac{\omega}{\partial T} - k \frac{\partial}{\partial X} \right) W(\mu) \right] \big/ W(\mu) = - \left( \hat{\xi}_1 W(\mu) \right) / W(\mu)$$  \hspace{1cm} (2.8)

where

$$W(\mu) = \int_{0}^{2\pi} \Phi_0^2(\theta, \mu) \frac{d\theta}{2\pi}$$

Let us come back to the expansion (2.2). We will assume now that every function $\tilde{\Psi}(k)(\theta, X, T)$ is represented by a sum of functions which belong to the family $\mathcal{M}$ (at every $\theta$) and have degree $k$. For the function

$$\phi(\theta, X, T) = \varphi(\theta - S(X, T), X, T) = \Phi(\theta, X, T) + \sum_{k \geq 1} \tilde{\Psi}(k)(\theta, X, T)$$  \hspace{1cm} (2.9)

we have the equation
\[(S_T^2 - S_X^2) \phi_{\theta\theta} + V'(\phi) + 2 \left( S_T \frac{\partial}{\partial T} - S_X \frac{\partial}{\partial X} \right) \phi + (S_{TT} - S_{XX}) \phi + \phi_{TT} - \phi_{XX} = 0 \]

It’s not difficult to check the relation

\[
\frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X^2} = -\hat{\xi}_1^+ \frac{1}{\mu} \hat{\xi}_1 + \hat{\xi}_2^+ \frac{1}{\mu} \hat{\xi}_2 =
\]

\[
= \frac{1}{\mu} \left( \hat{\xi}_1^2 - \hat{\xi}_2^2 \right) - \frac{1}{\mu^2} \left( \hat{\xi}_1 \mu \right) \hat{\xi}_1 + \frac{1}{\mu^2} \left( \hat{\xi}_2 \mu \right) \hat{\xi}_2 + \frac{1}{\mu} \left( S_{TT} - S_{XX} \right) \hat{\xi}_1
\]

where

\[
\hat{\xi}_1^+ = -\frac{\partial}{\partial T} S_T + \frac{\partial}{\partial X} S_X \quad , \quad \hat{\xi}_2^+ = -\hat{\xi}_2
\]

We have then

\[
\mu \phi_{\theta\theta} + V'(\phi) + 2 \hat{\xi}_1 \phi_{\theta} + (S_{TT} - S_{XX}) \phi_{\theta} +
\]

\[
+ \frac{1}{\mu} \left( \hat{\xi}_1^2 - \hat{\xi}_2^2 \right) \phi - \frac{1}{\mu^2} \left( \hat{\xi}_1 \mu \right) \left( \hat{\xi}_1 \phi \right) + \frac{1}{\mu^2} \left( \hat{\xi}_2 \mu \right) \left( \hat{\xi}_2 \phi \right) + \frac{1}{\mu} \left( S_{TT} - S_{XX} \right) \left( \hat{\xi}_1 \phi \right) = 0
\]

Using the relation (2.7) we can write then in the \( k \)-th order:

\[
\mu \tilde{\Psi}_{(k)\theta} + V''(\Phi) \tilde{\Psi}_{(k)} = -\sum_{l=1}^{k} \nu_{(l)} \left( \tilde{\Psi}_{(k-l)\theta} + \frac{1}{\mu} \hat{\xi}_1 \tilde{\Psi}_{(k-l-1)} \right) - \frac{1}{\mu} \left( \hat{\xi}_1^2 \tilde{\Psi}_{(k-2)} \right) +
\]

\[
+ \frac{1}{\mu^2} \left( \hat{\xi}_1 \mu \right) \left( \hat{\xi}_1 \tilde{\Psi}_{(k-2)} \right) - 2 \hat{\xi}_1 \tilde{\Psi}_{(k-1)\theta} + \frac{1}{\mu} \left( \hat{\xi}_2 \phi \right)^{[k]} - \frac{1}{\mu^2} \left( \hat{\xi}_2 \mu \right) \left( \hat{\xi}_2 \phi \right)^{[k-1]} - V'_{(k)} \quad (2.10)
\]

(where \( \tilde{\Psi}_{(q)} \equiv 0 \) for \( q < 0 \), and we put also \( \tilde{\Psi}_{(0)} \equiv \Phi(\theta, S_X, S_T) \)).

The expressions \( (\hat{\xi}_2 \phi)^{[k]} \) and \( (\hat{\xi}_2 \phi)^{[k-1]} \) represent here the terms of order \( k \) and \( k-1 \) of the gradated expansions of \( \hat{\xi}_2 \phi \) and \( \xi_2 \phi \) respectively and \( V'_{(k)} \) is a sum of functions having degree \( k \) given by the expansion of \( V' \) (except \( V''(\Phi)\tilde{\Psi}_{(k)} \)).

To use the equation (2.10) we have to define now the action of the operator \( \hat{\xi}_2 \) on the family \( \mathcal{M} \). In general, the expressions \( \hat{\xi}_2 F, F \in \mathcal{M} \) do not have certain degrees according to our gradation rule, so we have to represent these expressions as the infinite series of terms having certain degrees. We will assume naturally that the expansion of \( \hat{\xi}_2 F \) will always start with a term of degree \( k + 1 \) if \( \text{deg} F = k \). Let us prove now the following Lemma:

**Lemma 2.1.**

The relation (2.7) defines uniquely the gradated expansion of any expression \( (\hat{\xi}_2 F) \) where \( F \) is a function having certain degree.
Proof.

We have by definition that the functions \((\hat{\xi}_1\mu)\) and \((\hat{\xi}_2\mu)\) both belong to the family \(\mathcal{M}\) and have degree 1.

It’s not difficult to check the relation

\[
\left[\hat{\xi}_2, \hat{\xi}_1\right] = \hat{\xi}_2 \hat{\xi}_1 - \hat{\xi}_1 \hat{\xi}_2 = \frac{1}{\mu} (\hat{\xi}_2\mu) \hat{\xi}_1 - \frac{1}{\mu} (\hat{\xi}_1\mu) \hat{\xi}_2 + (S_{TT} - S_{XX}) \hat{\xi}_2 = \\
= \frac{1}{\mu} (\hat{\xi}_2\mu) \hat{\xi}_1 - \frac{1}{\mu} (\hat{\xi}_1\mu) \hat{\xi}_2 + \left(\sum_{k \geq 1} \nu(k)\right) \hat{\xi}_2
\]

We have then

\[
\hat{\xi}_2 \hat{\xi}_1 \mu = \hat{\xi}_1 \hat{\xi}_2 \mu + \sum_{k \geq 1} \left(\hat{\xi}_2\mu\right) \nu(k)
\]

which gives the gradated representation of the function \(\hat{\xi}_2 \hat{\xi}_1 \mu\).

Let us consider now the function \(\hat{\xi}_2 \hat{\xi}_2 \mu = \hat{\xi}_2^2 \mu\). By direct calculation the following relation can be proved

\[
\hat{\xi}_2^2 \mu = \hat{\xi}_1^2 \mu - 2\mu \hat{\xi}_1 (S_{TT} - S_{XX}) - \frac{2}{\mu} \left(\hat{\xi}_1\mu\right)^2 + \frac{2}{\mu} \left(\hat{\xi}_2\mu\right)^2 + \\
+ 3 \left(\hat{\xi}_1\mu\right) (S_{TT} - S_{XX}) - 2\mu (S_{TT} - S_{XX})^2
\]

Writing again

\[
\hat{\xi}_2^2 \mu = \hat{\xi}_1^2 \mu - \frac{2}{\mu} \left(\hat{\xi}_1\mu\right)^2 + \frac{2}{\mu} \left(\hat{\xi}_2\mu\right)^2 - 2\mu \sum_{k \geq 1} \hat{\xi}_1 \nu(k) + \\
+ 3 \left(\hat{\xi}_1\mu\right) \sum_{k \geq 1} \nu(k) - 2\mu \left(\sum_{k \geq 1} \nu(k)\right)^2
\]

we get the gradated representation of \(\hat{\xi}_2^2 \mu\).

All the other (monomial) expressions \(\hat{\xi}_2 F\) with \(\deg F = k\) can be represented in one of the following two forms

1) \(\xi_2 F = \hat{\xi}_2 \hat{\xi}_1 F'\), \(\deg F' = k - 1\);
2) \(\xi_2 F = \xi_2 F_1 F_2\), \(\deg F_1 < k\), \(\deg F_2 < k\).

We have then

\[
\hat{\xi}_2 \hat{\xi}_1 F' = \hat{\xi}_1 \hat{\xi}_2 F' + \frac{1}{\mu} (\hat{\xi}_2\mu) \hat{\xi}_1 F' - \frac{1}{\mu} (\hat{\xi}_1\mu) \hat{\xi}_2 F' + \left(\sum_{k \geq 1} \nu(k)\right) \hat{\xi}_2 F'
\]

\[
\hat{\xi}_2 F_1 F_2 = F_1 \hat{\xi}_2 F_2 + F_2 \hat{\xi}_2 F_1
\]
so we can use the induction with respect to $\deg F$.

According to Lemma 2.1 it’s natural to consider now the functions $f(\mu)$ ($f$ any smooth), $A = \hat{\xi}_1 \mu$, $B = \hat{\xi}_2 \mu$, $A_{(k)} = \hat{\xi}_1^k A$, $B_{(k)} = \hat{\xi}_1^k B$, as the ”generators” for the graded expansions of the solutions (2.2). We have then according to our rule: $\deg f(\mu) = 0$, $\deg A = \deg B = 1$, $\deg A_{(k)} = \deg B_{(k)} = k + 1$.

We will need also another technical Lemma:

**Lemma 2.2.**

For any function $F$ of degree $k$ the terms $\hat{\xi}_2 F^{[k+1]}$, $\ldots$, $\hat{\xi}_2 F^{[k+s]}$, ($s \geq 1$) of the graded expansion of $\hat{\xi}_2 F$ are defined by the terms $\nu_{(1)}$, $\ldots$, $\nu_{(s)}$ of the system (2.7).

Proof.

Easy to see that this statement is true for $\hat{\xi}_2 \hat{\xi}_1 \mu$ ($F = \hat{\xi}_1 \mu$) and $\hat{\xi}_2^2 \mu$ ($F = \hat{\xi}_2 \mu$). Using then the same induction with respect to $\deg F$ and the relations (2.14)-(2.15) we get the statement of the Lemma.

Lemma 2.2 is proved.

Using Lemma 2.2 we can prove now the following important Lemma:

**Lemma 2.3.**

The solvability condition for the system (2.10) in the $k$-th order defines uniquely the term $\nu_{(k)}$ of the system (2.7) provided that the terms $\nu_{(1)}$, $\ldots$, $\nu_{(k-1)}$ and corrections $\tilde{\Psi}_{(1)}$, $\ldots$, $\tilde{\Psi}_{(k-1)}$ are known.

Proof.

Indeed, we have for $k \geq 2$

$$\frac{1}{\mu} \left( \hat{\xi}_2 \phi \right)^{[k]} = \sum_{l=0}^{k-2} \sum_{s=1}^{k-l-1} \left( \hat{\xi}_2 \left( \tilde{\Psi}_{(l)} \right)^{[l+s]} \right)^{[k]}$$

where $s \leq k - 1$, $k - l - s \leq k - 1$.

In the same way

$$\left( \hat{\xi}_2 \phi \right)^{[k-1]} = \sum_{l=0}^{k-2} \left( \hat{\xi}_2 \tilde{\Psi}_{(l)} \right)^{[k-1]}$$

where $k - l - 1 \leq k - 1$.

All the functions $\tilde{\Psi}_{(0)}$, $\ldots$, $\tilde{\Psi}_{(k-2)}$ depend on $\nu_{(1)}$, $\ldots$, $\nu_{(k-2)}$ (and $(k, \omega)$). Using Lemma 2.2 we can claim then that both the expressions above are defined by the terms $\nu_{(1)}$, $\ldots$, $\nu_{(k-1)}$. 

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Looking at the other terms in the right-hand part of (2.10) we can see that they all depend on \( \nu(1), \ldots, \nu(k-1), \tilde{\Psi}(0), \ldots, \tilde{\Psi}(k-1) \) except one term \( \nu(k) \tilde{\Psi}(0) \). We have then that the orthogonality condition (2.6) in the order \( k \) gives the relation

\[
\nu(k) \int_{0}^{2\pi} \frac{\Phi^2}{2\pi} d\theta = G(k, \omega, \nu(1), \ldots, \nu(k-1))
\]

where \( G(k) \) is some smooth functional of \((k, \omega, \nu(1), \ldots, \nu(k-1))\).

Lemma 2.3 is proved.

Let us use now the natural choice of the functions \( \Phi(\theta, \mu) \) determined by the requirement that \( \Phi(\theta, \mu) \) has a local minimum at the point \( \theta = 0 \). Easy to see that \( \Phi(\theta, \mu) \) is a symmetric function of \( \theta \): \( \Phi(-\theta, \mu) = \Phi(\theta, \mu) \) in this case. The orthogonality condition (2.6) for \( k = 1 \) gives the formula (2.8) for the function \( \nu(1) \) and we obtain the Whitham system as the main term of the system (2.7). The function \( \tilde{f}(1) \) given by

\[
\tilde{f}(1) = -\nu(1)\Phi_\theta - 2S_T\Phi_\theta T + 2S_X\Phi_\theta X
\]

is anti-symmetric in \( \theta \): \( \tilde{f}(1)(-\theta, X, T) = -\tilde{f}(1)(\theta, X, T) \). Let us formulate now the following Lemma about the solutions \( \tilde{\Psi}(k) \) of the system (2.3) proved in [47]:

**Lemma 2.4.**

1) For a smooth periodic anti-symmetric function \( \tilde{f}(k)(\theta) \) satisfying the orthogonality conditions (2.6) the solution \( \tilde{\Psi}(k)(\theta) \) satisfying the normalization conditions (2.4) is a smooth periodic anti-symmetric function:

\[
\tilde{\Psi}(k)(-\theta) = -\tilde{\Psi}(k)(\theta).
\]

2) For a smooth periodic symmetric function \( \tilde{f}(k)(\theta) \) the solution \( \tilde{\Psi}(k)(\theta) \) satisfying the normalization conditions (2.4) is a smooth periodic symmetric function:

\[
\tilde{\Psi}(k)(-\theta) = \tilde{\Psi}(k)(\theta).
\]

We can claim then that the function \( \tilde{\Psi}(1)(\theta, X, T) \) is a periodic anti-symmetric function of \( \theta \): \( \tilde{\Psi}(1)(-\theta, X, T) = -\tilde{\Psi}(1)(\theta, X, T) \).

Easy to see now that the discrepancy function \( \tilde{f}(2) \) can be represented in the form

\[
\tilde{f}(2) = -\nu(2)\Phi_\theta(\theta, \mu) + \tilde{f}'(2)(\theta, X, T)
\]

where \( \tilde{f}'(2) \) is symmetric in \( \theta \): \( \tilde{f}'(2)(-\theta, X, T) = \tilde{f}'(2)(\theta, X, T) \).

Using the orthogonality conditions (2.6) we obtain then: \( \nu(2) \equiv 0 \) for the second term of the system (2.7). We have then \( \tilde{f}(2) = \tilde{f}'(2) \) and \( \tilde{\Psi}(2) \) is then a symmetric function of \( \theta \): \( \tilde{\Psi}(2)(-\theta, X, T) = \tilde{\Psi}(2)(\theta, X, T) \). Using the induction we obtain the following Lemma:

**Lemma 2.5.**

Under the choice of the functions \( \Phi(\theta, \mu) \) given above the following statements are true:

\(^2\)For another normalization of the functions \( \Psi(k) \) the analogous property was pointed out in [5].
1) All the even terms $\nu_{(2l)}(k, \omega, \ldots)$ in the deformation of Whitham system (2.7) are identically zero: $\nu_{(2l)} \equiv 0$;

2) All the odd corrections $\tilde{\Psi}_{(2l+1)}(\theta, X, T)$, $l \geq 0$ in (2.2) are anti-symmetric in $\theta$: $\tilde{\Psi}_{(2l+1)}(-\theta) = -\tilde{\Psi}_{(2l+1)}(\theta)$;

3) All the even corrections $\tilde{\Psi}_{(2l)}(\theta, X, T)$, $l \geq 1$ in (2.2) are symmetric in $\theta$: $\tilde{\Psi}_{(2l)}(-\theta) = \tilde{\Psi}_{(2l)}(\theta)$.

Thus we can rewrite the system (2.7) in the form

$$S_{TT} - S_{XX} = \sum_{l \geq 0} \nu_{(2l+1)}$$

where every $\nu_{(2l+1)}$ is a sum of functions belonging to the family $\mathcal{M}$ and having degree $2l + 1$.

Let us point out also that the system (2.16) inherits the momentum and the energy conservation laws of the system (1.1) which are given by the restriction of the corresponding laws to the solutions (2.2) and then integration with respect to $\theta$. The corresponding relations are given then by the (infinite) series depending on $k$, $\omega$ and their derivatives which can be written according to the gradation rule. Thus we can write

$$\langle \varphi_T \varphi_X \rangle_T = \langle \frac{\varphi_T^2}{2} + \frac{\varphi_X^2}{2} - V(\varphi) \rangle_X$$

(momentum conservation)

$$\langle \frac{\varphi_T^2}{2} + \frac{\varphi_X^2}{2} + V(\varphi) \rangle_T = \langle \varphi_T \varphi_X \rangle_X$$

(energy conservation).

We assume here that the function $\varphi(\theta, X, T)$ is given by the expression (2.2) and the notation $\langle \ldots \rangle$ is used here for the ”averaging” procedure given by the integration w.r.t. $\theta$: $\langle \ldots \rangle \equiv \int_0^{2\pi} \ldots d\theta / 2\pi$.

3 Lagrangian formalism and the deformation procedure.

Let us use now the Lagrangian formalism of the initial system (1.1) to obtain the first non-trivial correction $\nu_{(3)}$ in the system (2.16). We will use here the scheme suggested in [47] for the Lagrangian systems. It is well known that the equation (1.1) can be represented in the Lagrangian form

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3The analogous Lemma is true also for the deformation of the Whitham system having Dubrovin - Zhang form ([47]).

4The dispersion corrections arising here are in fact rather different from those considered in [5] because of the ”total renormalization” of parameters used in our approach.
\[ \frac{\delta}{\delta \varphi(x,t)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ -\frac{1}{2} \varphi_t^2 + \frac{1}{2} \varphi_x^2 + V(\varphi) \right] dx \, dt = 0 \]

We have to add the variable \( \theta \) and introduce the action functional

\[ \Sigma[\varphi] = \int \int \int \left[ -\frac{1}{2} \varphi_T^2 + \frac{1}{2} \varphi_X^2 + V(\varphi) \right] \frac{d\theta}{2\pi} \, dX \, dT \quad (3.1) \]

In terms of the function \( \phi(\theta, X, T) \) we can write then

\[
\Sigma = \int \int \int_{0}^{2\pi} \left[ -\frac{1}{2} S_T^2 (\phi_\theta)^2 + \frac{1}{2} S_X^2 (\phi_\theta)^2 + V(\phi) \right] \frac{d\theta}{2\pi} \, dX \, dT + \\
+ \int \int \int_{0}^{2\pi} \left[ - S_T \phi_\theta \phi_T + S_X \phi_\theta \phi_X \right] \frac{d\theta}{2\pi} \, dX \, dT + \\
+ \int \int \int_{0}^{2\pi} \frac{1}{2} \left[ - (\phi_T)^2 + (\phi_X)^2 \right] \frac{d\theta}{2\pi} \, dX \, dT \quad (3.2)
\]

The system (2.16) is equivalent to the equation

\[ \frac{\delta \Sigma}{\delta S(X,T)} = 0 \]

where the function \( \phi(\theta, X, T) \) is given by the relation (2.9).

For the determination of \( \nu_{(3)} \) we need to write just the part \( \Sigma' = \Sigma_{(0)} + \Sigma_{(2)} \) of the action (3.2), where

\[ \Sigma_{(0)} = \int \int \int_{0}^{2\pi} \left( -\frac{1}{2} \mu \phi_\theta^2 + V(\Phi) \right) \frac{d\theta}{2\pi} \, dX \, dT \]

and

\[ \Sigma_{(2)} = \frac{1}{2} \int \int \int_{0}^{2\pi} \left( -\mu \tilde{\Psi}_{(1) \theta}^2 + V''(\Phi) \tilde{\Psi}_{(1)}^2 \right) \frac{d\theta}{2\pi} \, dX \, dT + \\
+ \int \int \int_{0}^{2\pi} \left( 2 S_T \Phi_\theta T - 2 S_X \Phi_\theta X \right) \tilde{\Psi}_{(1)} \frac{d\theta}{2\pi} \, dX \, dT + \\
+ \frac{1}{2} \int \int \int_{0}^{2\pi} \left( -\Phi_T^2 + \Phi_X^2 \right) \frac{d\theta}{2\pi} \, dX \, dT \]

or, equivalently:

\[ \Sigma_{(2)} = \int \int \frac{1}{2\mu} \langle \Phi_{\mu}^2 \rangle \left( \hat{\xi}_{2\mu} \right)^2 \, dX \, dT + \\
+ \int \int \left[ -\frac{1}{2} \mu \langle Z^2(\theta, \mu) \rangle + \frac{1}{2} \langle V''(\Phi) Z^2(\theta, \mu) \rangle + 2 \langle \Phi_\theta Z(\theta, \mu) \rangle - \frac{1}{2\mu} \langle \Phi_{\mu}^2 \rangle \right] \left( \hat{\xi}_{1\mu} \right)^2 \, dX \, dT \]
where the function $Z(\theta, \mu)$ is defined from the representation of $\tilde{\Psi}_1$ in the form

$$\tilde{\Psi}_1(\theta, X, T) = Z(\theta, \mu) \left( \hat{\xi}_1 \mu \right)$$

after the resolving of the system (2.3) for $k = 1$ with the normalization conditions (2.4).

(Let us note that $\tilde{f}_1 = -\nu(1) \Phi_{\theta} - 2\hat{\xi}_1 \Phi_{\theta} = \left( \frac{W'(\mu)}{W(\mu)} \Phi_{\theta} - 2\Phi_{\theta \mu} \right) \left( \hat{\xi}_1 \mu \right)$ for $k = 1$.)

It can be shown after some calculations that

$$\frac{\delta \Sigma(2)}{\delta S(X, T)} = \left[ - \left( S_{TT} - S_{XX} + \hat{\xi}_1 \right) \left( \hat{\xi}_2 \mu \right)^2 \frac{d}{d\mu} + \hat{O}_2 \left( \hat{\xi}_2 \mu \right) \right] \frac{1}{\mu} \langle \Phi^2 \rangle +$$

$$+ \left[ -2 \left( S_{TT} - S_{XX} + \hat{\xi}_1 \right) \left( \hat{\xi}_1 \mu \right)^2 \frac{d}{d\mu} + 2\hat{O}_1 \left( \hat{\xi}_1 \mu \right) \right] \times$$

$$\times \left[ -\frac{1}{2} \mu \langle Z^2(\theta, \mu) \rangle + \frac{1}{2} \langle V''(\Phi) Z^2(\theta, \mu) \rangle + 2 \langle \Phi_{\theta \mu} Z(\theta, \mu) \rangle - \frac{1}{2\mu} \langle \Phi^2 \rangle \right]$$

where the operators $\hat{O}_1$ and $\hat{O}_2$ are given by the expressions

$$\hat{O}_1 = 2\hat{\xi}_1^2 + \left( 4(S_{TT} - S_{XX}) - \frac{1}{\mu} \left( \hat{\xi}_1 \mu \right) \right) \hat{\xi}_1 + \frac{1}{\mu} \left( \hat{\xi}_2 \mu \right) \hat{\xi}_2 -$$

$$- \frac{1}{\mu^2} \left( \hat{\xi}_1 \mu \right)^2 + \frac{1}{\mu^2} \left( \hat{\xi}_2 \mu \right)^2 + \frac{2}{\mu} \left( \hat{\xi}_1 \mu \right) (S_{TT} - S_{XX})$$

$$\hat{O}_2 = 2\hat{\xi}_1 \hat{\xi}_2 - \frac{1}{\mu} \left( \hat{\xi}_2 \mu \right) \hat{\xi}_1 + \frac{1}{\mu} \left( \hat{\xi}_1 \mu \right) \hat{\xi}_2 + 2(S_{TT} - S_{XX}) \hat{\xi}_2$$

To obtain the function $\nu(3)$ we need in fact just the main term (of degree 3) of the gradated expansion of this expression. To get this term we have to change first of all the expression $S_{TT} - S_{XX}$ by it’s main terms $\nu(1)$ in the gradated expansion. Besides that, we have to take the main terms in the gradated expansions of the expressions $\hat{\xi}_2 \hat{\xi}_1 \mu$ and $\hat{\xi}_2 \hat{\xi}_2 \mu$ arising in the expression above. Easy to see that these terms can be just extracted from the formulae (2.12) and (2.13) and are equal to

$$\hat{\xi}_1 \hat{\xi}_2 \mu + \nu(1) \hat{\xi}_2 \mu$$

and

$$\hat{\xi}_1^2 \mu - \frac{2}{\mu} \left( \hat{\xi}_1 \mu \right)^2 + \frac{2}{\mu} \left( \hat{\xi}_2 \mu \right)^2 - 2\mu \hat{\xi}_1 \nu(1) + 3 \left( \hat{\xi}_1 \mu \right) \nu(1) - 2\mu \nu(1)^2$$

respectively.

The expression $\delta \Sigma(0)/\delta S(X, T)$ is equal to
\[(S_{TT} - S_{XX})W(\mu) + \hat{\xi}_1 W(\mu)\]

so we have

\[
\left( \frac{\delta \Sigma_{(0)}}{\delta S(X,T)} \right)^{[3]} = \nu_{(3)} W(\mu)
\]

(and the Whitham system in the order 1).

Finally we have for \(\nu_{(3)}\):

\[
\nu_{(3)} = -\frac{1}{W(\mu)} \left( \frac{\delta \Sigma_{(2)}}{\delta S(X,T)} \right)^{[3]}
\]

The work was partially supported by the grant of President of Russian Federation (MD-8906.2006.2) and Russian Science Support Foundation.

**References**

[1] G. Whitham, A general approach to linear and non-linear dispersive waves using a Lagrangian, *J. Fluid Mech.* 22 (1965), 273-283.

[2] G. Whitham, Non-linear dispersive waves, *Proc. Royal Soc. London Ser. A* 139 (1965), 283-291.

[3] G. Whitham, Linear and Nonlinear Waves. Wiley, New York (1974).

[4] Luke J.C., A perturbation method for nonlinear dispersive wave problems, *Proc. Roy. Soc. London Ser. A*, 292, No. 1430, 403-412 (1966).

[5] M.J. Ablowitz, D.J. Benney., The evolution of multi-phase modes for nonlinear dispersive waves, *Stud. Appl. Math.* 49 (1970), 225-238.

[6] M.J. Ablowitz., Applications of slowly varying nonlinear dispersive wave theories, *Stud. Appl. Math.* 50 (1971), 329-344.

[7] M.J. Ablowitz., Approximate methods for obtaining multi-phase modes in nonlinear dispersive wave problems, *Stud. Appl. Math.* 51 (1972), 17-55.

[8] W.D. Hayes., Group velocity and non-linear dispersive wave propagation, *Proc. Royal Soc. London Ser. A* 332 (1973), 199-221.

[9] A.V. Gurevich, L.P. Pitaevskii., Decay of initial discontinuity in the Korteweg - de Vries equation, *JETP Letters* 17 (1973), 193-195.

[10] A.V. Gurevich, L.P. Pitaevskii., Nonstationary structure of a collisionless shock waves, *Sov. Phys. JETP* 38 (1974), 291-297.
[11] Flaschka H., Forest M.G., McLaughlin D.W., Multiphase averaging and the inverse spectral solution of the Korteweg - de Vries equation, Comm. Pure Appl. Math., - 1980.- Vol. 33, no. 6, 739-784.

[12] S.Yu. Dobrokhotov and V.P.Maslov., Konechnozonnye pochti periodicheskie reshenniya v WKB prizlizhenii., Itogi Nauki, ser. Matem. 1980, T. 15, 3-94, (in Russian), Translation: S.Yu. Dobrokhotov and V.P.Maslov., Finite-Gap Almost Periodic Solutions in the WKB Approximation. J. Soviet. Math., 1980, V. 15, 1433-1487.

[13] V.E. Zakharov, S.V. Manakov, S.P. Novikov, L.P. Pitaevskii, Teoriya solitonov. Metod obratnoi zadachi. Nauka, Moscow 1980. (ed. S. P. Novikov) (in Russian), Translation: S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, and V.E. Zakharov., Theory of solitons. The inverse scattering method., Plemun, New York 1984.

[14] B.A.Dubrovin and S.P.Novikov., Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov - Whitham averaging method, Soviet Math. Dokl., Vol. 27, (1983) No. 3, 665-669.

[15] P.D. Lax, C.D. Levermore., The small dispersion limit for the Korteweg - de Vries equation I, II, and III. Comm. Pure Appl. Math., 36 (1983), 253-290, 571-593, 809-830.

[16] S.P. Novikov., The geometry of conservative systems of hydrodynamic type. The method of averaging for field-theoretical systems., Russian Math. Surveys. 40 : 4 (1985), 85-98.

[17] V.V. Avilov, S.P. Novikov., Evolution of the Whitham zone in KdV theory, Soviet Phys. Dokl. 32 (1987), 366-368.

[18] A.V. Gurevich, L.P. Pitaevskii., Averaged description of waves in the Korteweg - de Vries - Burgers equation, Soviet Phys. JETP 66 (1987), 490-495.

[19] V.V. Avilov, I.M. Krichever, S.P. Novikov., Evolution of the Whitham zone in the Korteweg - de Vries theory, Soviet Phys. Dokl. 32 (1987), 564-566.

[20] I.M. Krichever., ”The averaging method for two-dimensional ”integrable” equations”, Functional Anal. Appl. 22 (1988), 200-213.

[21] R. Haberman., ”The modulated phase shift for weakly dissipated nonlinear oscillatory waves of the Korteweg-de Vries type”, Stud. Appl. Math. 78 (1988), no. 1, 73–90.

[22] B.A. Dubrovin and S.P. Novikov., Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, Russian Math. Survey, 44 : 6 (1989), 35-124.

[23] B.A.Dubrovin and S.P.Novikov., Hydrodynamics of soliton lattices, Sov. Sci. Rev. C, Math. Phys., 1993, V.9. part 4. P. 1-136.
[24] S.P. Tsarev., On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, *Soviet Math. Dokl.*, **31**: 3 (1985), 488-491.

[25] O.I. Mokhov and E.V. Ferapontov., Nonlocal Hamiltonian operators of hydrodynamic type associated with constant curvature metrics, *Russian Math. Surveys*, **45**:3 (1990), 218-219.

[26] E.V. Ferapontov., Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type, *Functional Anal. and Its Applications*, Vol. 25, No. 3 (1991), 195-204.

[27] E.V. Ferapontov., Dirac reduction of the Hamiltonian operator $\delta^{ij}\frac{\partial}{\partial x^i}$ to a submanifold of the Euclidean space with flat normal connection, *Functional Anal. and Its Applications*, Vol. 26, No. 4 (1992), 298-300.

[28] E.V. Ferapontov., Nonlocal matrix Hamiltonian operators. Differential geometry and applications, *Theor. and Math. Phys.*, Vol. 91, No. 3 (1992), 642-649.

[29] E.V. Ferapontov., Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, *Amer. Math. Soc. Transl.*, (2), 170 (1995), 33-58.

[30] M.V. Pavlov., Elliptic coordinates and multi-Hamiltonian structures of systems of hydrodynamic type., *Russian Acad. Sci. Dokl. Math.* **59**: 3 (1995), 374-377.

[31] A.Ya. Maltsev, S.P. Novikov. On the local systems hamiltonian in the weakly nonlocal Poisson brackets., *Physica D* **156** (2001), 53-80.

[32] A.Ya. Maltsev. ”The averaging of non-local Hamiltonian structures in Whitham’s method”, *Intern. Journ. of Math. and Math. Sci.*, **30**:7 (2002) 399-434.

[33] B.A. Dubrovin., ”Integrable systems in topological field theory”, *Nucl. Phys.*, **B379** (1992), 627-689.

[34] B.A. Dubrovin., Integrable Systems and Classification of 2-dimensional Topological Field Theories, ArXiv: [hep-th/9209040](http://arxiv.org/abs/hep-th/9209040)

[35] B.A. Dubrovin., Geometry of 2d topological field theories, ArXiv: [hep-th/9407018](http://arxiv.org/abs/hep-th/9407018)

[36] B.A. Dubrovin., ”Flat pencils of metrics and Frobenius manifolds”, ArXiv: [math.DG/9803106](http://arxiv.org/abs/math.DG/9803106) In: Proceedings of 1997 Taniguchi Symposium ”Integrable Systems and Algebraic Geometry”, editors M.-H. Saito, Y. Shimizu and K. Ueno, 47-72. World Scientific, 1998.

[37] B.A. Dubrovin, Y. Zhang., Bihamiltonian Hierarchies in 2D Topological Field Theory At One-Loop Approximation, *Commun. Math. Phys.* **198** (1998), 311-361.

[38] B.A. Dubrovin., ”Geometry and analytic theory of Frobenius manifolds”, ArXiv: [math.AG/9807034](http://arxiv.org/abs/math.AG/9807034)
[39] B.A.Dubrovin, Y.Zhang., Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants., ArXiv: math.DG/0108160

[40] P. Lorenzoni., Deformations of bihamiltonian structures of hydrodynamic type, *J. Geom. Phys.* **44** (2002), 331-371.

[41] B.A.Dubrovin, Y.Zhang., Virasoro Symmetries of the Extended Toda Hierarchy, ArXiv: math.DG/0308152

[42] Si-Qi Liu, Youjin Zhang., Deformations of Semisimple Bihamiltonian Structures of Hydrodynamic Type, ArXiv: math.DG/0405146

[43] Si-Qi Liu, Youjin Zhang., On the Quasitriviality of Deformations of Bihamiltonian Structures of Hydrodynamic Type, ArXiv: math.DG/0406626

[44] Boris Dubrovin, Si-Qi Liu, Youjin Zhang., ”On Hamiltonian perturbations of hyperbolic systems of conservation laws”, ArXiv: math.DG/0410027

[45] Boris Dubrovin, Youjin Zhang, Dafeng Zuo., ”Extended affine Weyl groups and Frobenius manifolds – II”, ArXiv: math.DG/0502365

[46] A.Ya. Maltsev., ”Whitham systems and deformations”, *Journ. Math. Phys.* **47**, (2006).

[47] A.Ya. Maltsev., ”The deformations of Whitham systems and Lagrangian formalism”, ArXiv: nlin.SI/0601050