POWER CORRECTIONS AND LANDAU SINGULARITY *

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ABSTRACT

In the dispersive approach of Dokshitzer, Marchesini and Webber, standard power-behaved contributions of infrared origin are described with the notion of an infrared regular QCD coupling. I argue that their framework suggests the existence of non-standard contributions, arising from short distances (hence unrelated to renormalons and the operator product expansion), which appear in the process of removing the Landau singularity of the perturbative coupling. A natural definition of an infrared finite perturbative coupling is suggested within the dispersive method. Implications for the tau hadronic width and the lattice determination of the gluon condensate, where $\mathcal{O}(1/Q^2)$ contributions can be generated, are pointed out.

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1. Introduction

The study of power corrections in QCD has been the subject of active investigations in recent years. Their importance for a precise determination of $\alpha_s$ has been recognized, and various techniques (renormalons [1], finite gluon mass [2-6], dispersive approach [7]) have been devised to cope with situations where the standard operator product expansion (OPE) does not apply. Standard power-behaved contributions in QCD arise from non-perturbative effects at low scale, reflecting the non-trivial vacuum structure. In this paper, I concentrate on the dispersive approach [7], based on the notion of an infrared (IR) regular [8] (see also [9]) QCD coupling, where a non-perturbative contribution to the coupling, essentially restricted to low scales, parametrizes the power corrections. I point out that within this framework, it is very natural to expect the existence of new type of power contributions of ultraviolet (UV) origin, hence not controlled by the OPE, related to the removal of the IR Landau singularity presumably present in the perturbative part of the coupling. After a brief review of the dispersive approach (section 2), a simple “dispersive” method [10-13] (see also [14]) of removing the Landau singularity is suggested in section 3 as a convenient definition of a “regularized” perturbative coupling; the full IR regular coupling is then obtained as the sum of the “regularized perturbative coupling” and of the “genuine” non-perturbative piece. The former differs from the perturbative coupling by power corrections which are computed in section 4 and 5 using Borel transform techniques. In sharp contrast to the genuine non-perturbative part of the coupling, these corrections are not restricted to low energy, and can thus induce “perturbative” power contributions of ultraviolet origin in Euclidean observables, considered in section 6.1. The “non-perturbative” part of the power corrections, induced by the corresponding piece of the coupling, is discussed in section 6.2, and the framework of [7] is extended in a straightforward way by relaxing the assumption that this piece is confined to the IR region. Section 7 deals with Minkowskian observables. As a sample of applications, I discuss briefly in section 8 inclusive $\tau$-decay, and the gluon condensate on the lattice. A critical assessment and concluding remarks are given in section 9. Some more technical issues are developed in two appendices. In particular, in Appendix A the stability [12] against higher order corrections of the value of the IR fixed point of the “dispersively regularized” perturbative coupling is proved within some restrictions.

2. Dispersive approach to power corrections

Consider the contribution to an Euclidean (quark dominated) observable arising from dressed single gluon virtual exchange, which takes the generic form (after
subtraction of the Born term):

\[ D(Q^2) = \int_0^\infty \frac{dk^2}{k^2} \alpha_s(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \]  

(2.1)

The IR regularity of the coupling is implemented through a dispersion relation:

\[ \alpha_s(k^2) = - \int_0^\infty \frac{d\mu^2}{\mu^2 + k^2} \rho(\mu^2) \]

\[ \equiv k^2 \int_0^\infty \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{\text{eff}}(\mu^2) \]  

(2.2a)

\[ \alpha_s(k^2) = - \int_0^\infty \frac{d\mu^2}{\mu^2 + k^2} \rho(\mu^2) \]

\[ \equiv k^2 \int_0^\infty \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{\text{eff}}(\mu^2) \]  

(2.2b)

where \( \rho(\mu^2) = - \frac{1}{2\pi i} \text{Disc} \{ \alpha_s(-\mu^2) \} \equiv - \frac{1}{2\pi i} \{ \alpha_s[-(\mu^2 + i\epsilon)] - \alpha_s[-(\mu^2 - i\epsilon)] \} \) is the time like “spectral density”, and the “effective coupling” \( \alpha_{\text{eff}}(\mu^2) \) is defined by:

\[ \frac{d\alpha_{\text{eff}}}{d \ln \mu^2} = \rho(\mu^2) \]  

(2.3)

i.e.

\[ \alpha_{\text{eff}}(\mu^2) = - \int_{\mu^2}^{\infty} \frac{d\mu'^2}{\mu'^2} \rho(\mu'^2) \]  

(2.4)

For small \( \alpha_s \), \( \alpha_{\text{eff}}(\mu^2) = \alpha_s(\mu^2) + O(\alpha_s^3) \). Eq.(2.2) guarantees the absence of Landau singularity in the whole first sheet of the complex \( k^2 \) plane. The coupling \( \alpha_s(k^2) \) might be understood as a universal “physical” QCD coupling (not to be confused with e.g. the MS coupling), an analogue of the Gell-Mann - Low QED effective charge, hopefully defined through an extension to QCD of the QED “dressed skeleton expansion” [15,16]: such a program, which would give a firm field theoretical basis to the “naive non-abelization” procedure [17,6,18] familiar in renormalons calculations, has been initiated in [19] (a different ansatz is however suggested in [7]). In the “large \( \beta_0 \)” limit of QCD, as implemented through the “naive non-abelization” procedure, \( \alpha_s(k^2) \) then coincides with the V-scheme coupling [15] (but differs [19] from it at finite \( \beta_0 \)).

Inserting eq.(2.2) into eq.(2.1) one gets:

\[ D(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \rho(\mu^2) \left[ \mathcal{F} \left( \frac{\mu^2}{Q^2} \right) - \mathcal{F}(0) \right] \]  

(2.5a)

\[ \equiv \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \mathcal{F} \left( \frac{\mu^2}{Q^2} \right) \]  

(2.5b)

where \( \mathcal{F} \) is the “characteristic function” [7]:

\[ \mathcal{F} \left( \frac{\mu^2}{Q^2} \right) = \int_0^\infty \frac{dk^2}{k^2 + \mu^2} \varphi \left( \frac{k^2}{Q^2} \right) \]  

(2.6)
\[ F \equiv -\frac{dF}{d\ln \mu^2}. \]

Eq. (2.6) shows that the “characteristic function” is just the \( O(\alpha_s) \) Feynman diagram computed with a finite gluon mass \( \mu^2 \) [4,6], and that the Feynman diagram kernel \( \varphi \left( \frac{k^2}{Q^2} \right) \) (also called “distribution function” in [18]) is proportional to the time-like discontinuity of \( F \).

The authors of [7] moreover suggest that \( \alpha_s(k^2) \) comprises both a “perturbative” and a “non-perturbative” part:

\[ \alpha_s(k^2) = \alpha_s^{PT}(k^2) + \delta \alpha_s^{NP}(k^2) \]  

and similarly:

\[ \alpha_{\text{eff}}(\mu^2) = \alpha_{\text{eff}}^{PT}(\mu^2) + \delta \alpha_{\text{eff}}^{NP}(\mu^2) \]

(\text{the meaning of the quotes “” is clarified below}) where it is assumed that each term in eq.(2.7) satisfy separately a similar dispersion relation to eq.(2.2), e.g.:

\[ \delta \alpha_s^{NP}(k^2) = k^2 \int_0^\infty \frac{d\mu^2}{(\mu^2 + k^2)^2} \delta \alpha_{\text{eff}}^{NP}(\mu^2) \]

Furthermore, \( \delta \alpha_s^{NP} \) generates “non-perturbative” power corrections:

\[ \delta D_{NP}(Q^2) = \int_0^\infty \frac{dk^2}{k^2} \delta \alpha_s^{NP}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \]

\[ = \int_0^\infty \frac{d\mu^2}{\mu^2} \delta \alpha_{\text{eff}}^{NP}(\mu^2) \hat{F} \left( \frac{\mu^2}{Q^2} \right) \]

The crucial further assumption of [7] is that the “non-perturbative” contribution \( \delta \alpha_s^{NP}(k^2) \), which reflects confinement physics, is essentially restricted to the IR domain (in accordance with the OPE ideology of [20]), in order to comply with the requirement that no power correction inconsistent with the ones expected from the OPE arises from the UV behavior of \( \delta \alpha_s^{NP}(k^2) \).

In fact, the precise meaning of “\( \alpha_s^{PT}(k^2) \)” was left open in [7]. It is one of the main purpose of this paper to fill up this gap. Let us first note that both “\( \alpha_s^{PT}(k^2) \)” and “\( \alpha_{\text{eff}}^{PT}(\mu^2) \)” should be IR regular: the former from the very assumption it satisfies the dispersion relation, and thus cannot have any Landau singularity, the latter because any singularity at finite \( \mu^2 \) (the dispersive variable) will make the dispersion relation and its output “\( \alpha_s^{PT}(k^2) \)” (hence \( \delta \alpha_s^{NP}(k^2) \)) ill-defined. It follows that none of them can be given by such a simple form as (e.g.) the one-loop coupling. Nevertheless a simple and attractive ansatz exists. I shall assume that \( \alpha_s^{PT}(k^2) \) (\textit{defined by a Borel sum}), see eq.(4.1) below) has no non-trivial IR fixed point, but instead develops a Landau singularity on the space-like axis. Thus \( \alpha_s^{PT}(k^2) \) cannot satisfy the dispersion relation eq.(2.2), and the Landau singularity has to be removed by hand. This means one should understand “\( \alpha_s^{PT} \)” in eq.(2.7) as being:

\[ \alpha_{s,\text{reg}}^{PT} = \alpha_s^{PT} + \delta \alpha_s^{PT} \]
which differs from the (Borel-summed) $\alpha_s^{PT}$ by “perturbative” power corrections $\delta\alpha_s^{PT}$ which remove the singularity. Upon insertion into eq.(2.1), $\alpha_{s,\text{reg}}$ generates a “regularized perturbation theory” [21,22] piece of $D(Q^2)$:

$$D_{\text{reg}}^{PT}(Q^2) \equiv \int_0^\infty \frac{dk^2}{k^2} \alpha_{s,\text{reg}}^{PT}(k^2) \varphi \left( \frac{k^2}{Q^2} \right)$$

(2.12)

which, as we shall see in section 6, contributes new, “perturbative” type of power corrections. One can therefore write $\alpha_s$ as:

$$\alpha_s = \alpha_s^{PT} + \delta\alpha_s$$

(2.13)

where the total modification $\delta\alpha_s$ comprises both a “perturbative” and a “non-perturbative” part:

$$\delta\alpha_s = \delta\alpha_s^{PT} + \delta\alpha_s^{NP}$$

(2.14)

We shall see that, contrary to $\delta\alpha_s^{NP}$, $\delta\alpha_s^{PT}$ is in general not restricted to low $k^2$, and thus $\delta\alpha_s \simeq \delta\alpha_s^{PT} \gg \delta\alpha_s^{NP}$ at large $k^2$. Within these assumptions (which shall be relaxed in section 6.2), the determination of $\alpha_{s,\text{reg}}^{PT}$ and $\delta\alpha_s^{PT}$ becomes a physical question, rather than a matter of convention involved in the split eq.(2.14) between different components. In the time-like region, eq.(2.14) is paralleled by:

$$\alpha_{\text{eff}}(\mu^2) = \alpha_{\text{eff}}^{PT}(\mu^2) + \delta\alpha_{\text{eff}}^{PT}(\mu^2) + \delta\alpha_{\text{eff}}^{NP}(\mu^2)$$

(2.15)

with:

$$\alpha_{s,\text{reg}}^{PT}(k^2) = k^2 \int_0^\infty \frac{d\mu^2}{(\mu^2 + k^2)^2} \left[ \alpha_{\text{eff}}^{PT}(\mu^2) + \delta\alpha_{\text{eff}}^{PT}(\mu^2) \right]$$

(2.16)

In the next section, I investigate the simplest choice for $\delta\alpha_s^{PT}$, namely $\delta\alpha_{\text{eff}}^{PT} \equiv 0$, which implies:

$$D_{\text{reg}}^{PT}(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}^{PT}(\mu^2) \hat{F}(\frac{\mu^2}{Q^2})$$

(2.17)

3. Dispersive regularization

A simple ansatz for $\alpha_{s,\text{reg}}^{PT}$ is illustrated by the following example, which contains the essential ingredients of the general argument (and is also relevant to “large $\beta_0$” QCD). Consider the “minimal regularization” of the one loop coupling, obtained by just removing the Landau pole at $k^2 = \Lambda^2$, i.e. define:

$$\alpha_{s,\text{reg}}^{PT}(k^2) \equiv \frac{1}{\beta_0 \ln(k^2/\Lambda^2)} - \frac{1}{\beta_0 \frac{k^2}{\Lambda^2}} - 1$$

$$\equiv \alpha_s^{PT}(k^2) + \delta\alpha_s^{PT}(k^2)$$

(3.1)
(β₀ > 0). The resulting α_{s,reg}^{PT}(k^2) is analytic in the complex k^2 plane, with a cut on the negative k^2 axis (the time-like region in our notation), and satisfies the dispersion relation eq.(2.2). The simple, but crucial observation which makes the above model interesting is that the corresponding time-like discontinuity is entirely given by that of the one loop coupling:

$$\rho_{reg}(\mu^2) = \rho_{PT}(\mu^2) \equiv -\frac{1}{\beta_0} \frac{1}{\ln^2 \frac{\mu^2}{\Lambda^2} + \pi^2} \quad (3.2)$$

One notices that ρ_{PT}(μ²) is continuous, finite and negative in the whole range 0 < μ² < ∞ of the dispersive variable, and vanishes both for μ² → ∞ and μ² → 0. It follows the corresponding α_{eff,reg}^{PT} coincides with α_{eff}^{PT} (i.e. δα_{eff}^{PT} ≡ 0) and is obtained by substituting ρ with ρ_{PT} in eq.(2.4), which gives:

$$\alpha_{eff}^{PT}(\mu^2) = \frac{1}{\pi \beta_0} \left[ \frac{\pi}{2} - \arctan\left(\frac{1}{\pi \ln \frac{\mu^2}{\Lambda^2}}\right) \right] \quad (3.3a)$$

≡ \frac{1}{\pi \beta_0} \arctan\left(\frac{\pi}{\ln \frac{\mu^2}{\Lambda^2}}\right) \quad (μ^2 > \Lambda^2) \quad (3.3b)$$

≡ \frac{1}{\pi \beta_0} \arctan\left(\frac{\pi}{\ln \frac{\mu^2}{\Lambda^2}}\right) + \frac{1}{\beta_0} \quad (0 < μ^2 < \Lambda^2)$$

(this coupling was introduced previously in [6] (see also [13]) with a somewhat different motivation). The “effective coupling” α_{eff}^{PT}(μ²) is IR finite and satisfies the RG equation:

$$\frac{d\alpha_{eff}^{PT}}{d \ln \mu^2} = -\frac{1}{\pi \beta_0} \sin^2(\pi \beta_0 \alpha_{eff}^{PT}) \quad (3.4)$$

It therefore increases from 0 to the IR fixed point value 1/β₀ as μ² is decreased from ∞ to 0. The corresponding dispersively generated α_{s,reg}^{PT}(k^2) differs from the one-loop coupling α_{s}^{PT}(k^2) by power corrections (see eq.(3.1)), and approaches also 1/β₀ as k² → 0. There is thus non-commutativity of resummation (of e.g. the series obtained when α_{eff}^{PT}(μ²) is expanded in powers α_{s}^{PT}(k²)) and integration under the dispersive integral eq.(2.2b), reflecting the non-trivial IR fixed point of α_{eff}^{PT}(μ²): this is an example of the general phenomenon discussed in [23,24].

The features observed for the one loop coupling, namely, negative definite ρ_{PT}(μ²), vanishing of ρ_{PT}(μ² = 0), and IR finitness of α_{eff}^{PT}(μ²) are likely to remain true at any number of loops. Indeed, it seems reasonable to assume that the only singularity of α_{s}^{PT}(k²) on the first sheet of the cut complex k² plane is the space-like Landau singularity, and in particular that α_{s}^{PT}(k²), hence its discontinuity ρ_{PT}(μ²), remain finite on the time-like axis. If there is no time-like singularity (and no non-perturbative thresholds are expected in the perturbative part of the coupling), the discontinuity should be continuous for 0 < μ² < ∞ and cannot change sign without going through
real values of $\alpha_s^{PT}$. But real values are in general not compatible with the constant \(i\pi\) imaginary part acquired by $\ln k^2$ upon analytic continuation from the space-like to the time-like region. For instance, assume $\alpha_s^{PT}$ satisfies the 2-loop RG equation:

$$
\frac{d\alpha_s^{PT}}{d\ln k^2} = -\beta_0 (\alpha_s^{PT})^2 - \beta_1 (\alpha_s^{PT})^3
$$

whose solution is:

$$
\beta_0 \ln \frac{k^2}{\Lambda^2} = \frac{1}{a_s} + \lambda \ln a_s
$$

with $a_s = \alpha_s/(1 + \lambda\alpha_s)$ and $\lambda = \beta_1/\beta_0$. One finds, upon going to the time-like region, setting $\ln k^2 = \ln \mu^2 + i\pi$ ($\mu^2 > 0$), $a_s = |a_s| \exp i\theta$ and taking the imaginary part:

$$
\pi \beta_0 = \sin \theta |a_s| - \lambda \theta
$$

where I assumed $\theta > 0$ (the sign is appropriate for RG trajectories in the domain of attraction of the trivial UV fixed point). It is clear that $\sin \theta$ can vanish (hence $\text{Im } \alpha_s$ can change sign) at finite $\mu^2$ only in the special cases where $n\lambda = -\beta_0$ ($n$ positive integer), which are excluded anyway since $\lambda > 0$. Assuming the discontinuity indeed does not change sign, asymptotic freedom fix it to be negative. Furthermore, if one assumes that $\alpha_s^{PT}(k^2)$ approaches the trivial IR fixed point for $k^2 \to 0$, i.e. that $\alpha_s^{PT}(k^2 = 0) = 0$, then $\alpha_{eff}^{PT}(\mu^2)$ is IR finite. Indeed one gets (see eq.(2.4)):

$$
\alpha_{eff}^{PT}(\mu^2) = - \int_{\mu^2}^{\infty} d\mu^2 \rho_{PT}(\mu^2) (3.8a)
$$

$$
= \int_{0}^{\mu^2} \frac{d\mu^2}{\mu^2} \rho_{PT}(\mu^2) + \alpha_{eff,IR}^{PT} (3.8b)
$$

where:

$$
\alpha_{eff,IR}^{PT} \equiv \alpha_{eff}^{PT}(\mu^2 = 0) = - \int_{0}^{\infty} \frac{d\mu^2}{\mu^2} \rho_{PT}(\mu^2) (3.9)
$$

and the integrals converge at $\mu^2 = 0$ since $\rho_{PT}$ vanishes there. It is therefore natural to define $\alpha_s^{PT,reg}$ through the dispersion relation:

$$
\alpha_s^{PT,reg}(k^2) = - \int_{0}^{\infty} \frac{d\mu^2}{\mu^2 + k^2} \rho_{PT}(\mu^2) (3.10a)
$$

$$
= k^2 \int_{0}^{\infty} \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{eff}^{PT}(\mu^2) (3.10b)
$$

i.e. take $\delta\alpha_{eff}^{PT} \equiv 0$ in eq.(2.15). Although this suggestion is new in the present context, I realized while writing this article that the resulting “dispersive regularization” of the Landau singularity has actually been proposed [10,11] almost 40 years ago in QED, and has been revived recently in QCD [12,13]. Since $\alpha_{eff}^{PT}(\mu^2)$ has a non-trivial
IR fixed-point, $\alpha_{s,\text{reg}}^{PT}(k^2)$ differs [23,24] from $\alpha_{s}^{PT}(k^2)$ by an infinite set of “dispersively generated” power corrections $\delta\alpha_{s}^{PT}(k^2)$, of perturbative origin, which remove the Landau singularity. It also follows from eq.(3.9)-(3.10) that $\alpha_{s,\text{reg}}^{PT}(k^2 = 0) = \alpha_{\text{eff}}^{PT}_{\text{IR}}$.

For a more general example then the one-loop coupling, assume the Landau singularity is a cut starting in the space-like region at $k^2 = \Lambda^2$, and there is no further singularity in the first sheet of the complex $k^2$ plane. One can then write the dispersion relation:

$$\alpha_{s}^{PT}(k^2) = -\int_{-\Lambda^2}^{\infty} \frac{d\mu^2}{\mu^2 + k^2} \rho_{PT}(\mu^2)$$

(3.11)

Splitting-off the space-like discontinuity from $-\Lambda^2$ to 0 one immediately gets eq.(2.11) with $\alpha_{s,\text{reg}}^{PT}$ as in eq.(3.10a) and:

$$\delta\alpha_{s}^{PT}(k^2) = \int_{-\Lambda^2}^{0} \frac{d\mu^2}{\mu^2 + k^2} \rho_{PT}(\mu^2)$$

(3.12)

whose large $k^2$ expansion is:

$$\delta\alpha_{s}^{PT}(k^2) = -\sum_{n=1}^{\infty} b_{n}^{PT} \left(-\frac{\Lambda^2}{k^2}\right)^n$$

(3.13)

where the $b_{n}^{PT}$'s are real numbers:

$$b_{n}^{PT} = \int_{-\Lambda^2}^{0} \frac{d\mu^2}{\mu^2} \left(\frac{\mu^2}{\Lambda^2}\right)^n \rho_{PT}(\mu^2)$$

(3.14)

Furthermore, if one assumes that $\alpha_{s}^{PT}(k^2 = 0)$ vanishes, we have (setting $k^2 = 0$ in eq.(3.12)):

$$\alpha_{s,\text{reg}}^{PT}(k^2 = 0) = \delta\alpha_{s}^{PT}(k^2 = 0) = \int_{-\Lambda^2}^{0} \frac{d\mu^2}{\mu^2} \rho_{PT}(\mu^2) \equiv b_{0}^{PT}$$

$$= -\int_{0}^{\infty} \frac{d\mu^2}{\mu^2} \rho_{PT}(\mu^2) \equiv \alpha_{\text{eff}}^{PT}_{\text{IR}}$$

(3.15)

4. Borel transform techniques

A more specific method to obtain the “perturbative power corrections” $\delta\alpha_{s}^{PT}(k^2)$ makes use of the “RS invariant Borel transform” [25-27], i.e. I shall assume that $\alpha_{s}^{PT}$ is given by:

$$\alpha_{s}^{PT}(k^2) = \int_{0}^{\infty} dz \exp \left(-z\beta_{0} \ln \frac{k^2}{\Lambda^2}\right) \tilde{\alpha}_{s}(z) \quad (k^2 > \Lambda^2)$$

(4.1)

where it is convenient to choose $\Lambda$ as the Landau singularity of $\alpha_{s}^{PT}$. The “RS invariant Borel transform” $\tilde{\alpha}_{s}(z)$ is simply related to the ordinary transform (and
coincides with the latter in the “t Hooft scheme” (eq.(3.5)) if β₁ = 0); for the one-loop coupling, ˜αₛ(z) ≡ 1. In this section, I assume ˜αₛ(z) has no IR renormalons singularities, so that eq.(4.1) defines unambiguously αₛ^{PT}(k²) for k² > Λ². Taking the time-like discontinuity of eq.(4.1), one gets:

\[ \rho_{PT}(\mu^2) = \int_0^\infty dz \exp \left( -z\beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \tilde{\rho}(z) \quad (\mu^2 > \Lambda^2) \]  

(4.2)

with:

\[ \tilde{\rho}(z) = -\frac{1}{\pi} \sin(\pi \beta_0 z) \tilde{\alpha}_s(z) \]  

(4.3)

Hence, using eq.(3.8a):

\[ \alpha_{eff}^{PT}(\mu^2) = \int_0^\infty dz \exp \left( -z\beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \tilde{\alpha}_{eff}(z) \quad (\mu^2 > \Lambda^2) \]  

(4.4)

where ˜α_{eff}(z) is the RS invariant Borel transform of α_{eff}^{PT}(μ²):

\[ \tilde{\alpha}_{eff}(z) = -\frac{\tilde{\rho}(z)}{z\beta_0} = \frac{\sin(\pi \beta_0 z)}{\pi \beta_0 z} \tilde{\alpha}_s(z) \]  

(4.5)

Eq.(4.5) is just the relation between the “modified Borel transforms” of absorptive and dispersive parts first obtained in [27]. The oscillations of the sin(πβ₀z) factor in eq.(4.5) account for the absence of Landau singularity of α_{eff}^{PT}(μ²) (despite its presence in α_s^{PT}(k²)). For instance in the case of the one-loop coupling where ˜α_s(z) ≡ 1, eq.(4.4) and (4.5) reproduce eq.(3.3a). Note the alternative ansatz ˜α_{eff}(z) ≡ 1, i.e. assuming α_{eff}^{PT}(μ²) itself is the one-loop coupling (hence singular at μ² = Λ²), cannot give a consistent answer upon insertion into the dispersion relation (and would imply renormalons in ˜α_s(z)!)}. Furthermore, the assumption that α_{eff}^{PT}(μ²) is IR regular explains [23,24] that the right hand side of eq.(3.10b) may differ from its Borel sum α_s^{PT} by power corrections, and also that ˜α_s(z) has no IR renormalons generated by the low energy part of the dispersive integral in eq.(3.10b) (which would reflect the ambiguity of integrating over an IR singular α_{eff}^{PT}).

An alternative way to derive eq.(4.5) starts from eq.(3.10b), where one freely replaces α_{eff}^{PT}(μ²) by its Borel representation eq.(4.4) inside the dispersive integral (although this is not justified for μ² < Λ²!). Interchanging the order of the z and μ² integrations yields α_s^{PT}(k²) (not α_s^{PT,reg}(k²)!) as in eq.(4.1), with:

\[ \tilde{\alpha}_s(z) = \tilde{\alpha}_{eff}(z) \int_0^\infty \frac{d\mu^2}{(\mu^2 + k^2)^2} \exp \left( -z\beta_0 \ln \frac{\mu^2}{k^2} \right) \]  

(4.6)
To derive the power corrections in $\delta \alpha_{PT}$ it is convenient (although not absolutely necessary, see Appendix A2) to first split the dispersive integral eq.(3.10b) at $\mu^2 = k^2$:

$$\alpha_{s,reg}^{PT}(k^2) = k^2 \int_0^{k^2} \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{eff}^{PT}(\mu^2) + k^2 \int_{k^2}^{\infty} \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{eff}^{PT}(\mu^2) \quad (4.7)$$

The second integral contributes only to the Borel sum eq.(4.1), and not to the power corrections, since one can use the Borel representation of $\alpha_{eff}^{PT}$ (eq.(4.4)) there (taking $k^2 > \Lambda^2$). On the other hand, expanding the dispersive kernel in the first integral in inverse powers of $k^2$ one gets:

$$k^2 \int_0^{k^2} \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{eff}^{PT}(\mu^2) = \sum_{n=1}^{\infty} (-1)^{n+1} I_n(k^2) \quad (4.8)$$

with

$$I_n(k^2) \equiv \int_0^{k^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{k^2} \right)^n \alpha_{eff}^{PT}(\mu^2) \quad (4.9)$$

The $I_n(k^2)$'s are standard IR renormalons integrals. If the coupling $\alpha_{eff}^{PT}(\mu^2)$ has a non-trivial IR fixed point, they differ [23,24] from their corresponding Borel sums $I_n^{PT}(k^2)$ by a power correction. Putting:

$$I_n^{PT}(k^2) = \int_0^{\infty} dz \exp \left( -z \beta_0 \ln \frac{k^2}{\Lambda^2} \right) \tilde{I}_n(z) \quad (k^2 > \Lambda^2) \quad (4.10)$$

one gets:

$$\tilde{I}_n(z) = \tilde{\alpha}_{eff}(z) \int_0^{k^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{k^2} \right)^n \exp \left( -z \beta_0 \ln \frac{\mu^2}{k^2} \right)$$

$$= \tilde{\alpha}_{eff}(z) \frac{1}{1 - \frac{z}{z_n}} \quad (z_n = \frac{n}{\beta_0}) \quad (4.11)$$

whereas:

$$I_n(k^2) = I_n^{PT}(k^2) + b_n^{PT} \left( \frac{\Lambda^2}{k^2} \right)^n \quad (4.12)$$

with [23]:

$$b_n^{PT} = I_n - I_n^{PT} \quad (4.13)$$

where:

$$I_n \equiv I_n(k^2 = \Lambda^2) = \int_0^{\Lambda^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\Lambda^2} \right)^n \alpha_{eff}^{PT}(\mu^2) \quad (4.14)$$

and:

$$I_n^{PT} \equiv I_n^{PT}(k^2 = \Lambda^2) = \int_0^{\infty} dz \tilde{\alpha}_{eff}(z) \frac{1}{1 - \frac{z}{z_n}} \quad (4.15)$$
To derive these results (see also section 7.1) one splits the integral in eq.(4.9) at \( \mu^2 = \Lambda^2 \). The low energy integral is just \( I_n \left( \frac{\Lambda^2}{k^2} \right)^n \), whereas in the high energy integral, one can use the Borel representation eq.(4.4) of \( \alpha_{eff}^{PT} \) to get \( I_n^{PT}(k^2) - I_n^{PT} \left( \frac{\Lambda^2}{k^2} \right)^n \). Adding the two pieces yields eq.(4.12). Note that, provided \( \tilde{\alpha}_s(z) \) has no renormalons, so does \( \tilde{I}_n(z) \), since the zeroes of \( \tilde{\alpha}_{eff}(z) \) (eq.(4.5)) sit precisely at the would-be renormalons positions when \( n \) is an integer. Consequently, the power correction in eq.(4.12) and the constants \( b_n^{PT} \) are real and unambiguous, but nevertheless \( I_n(k^2) \) differs from its well defined Borel sum \( I_n^{PT(k^2)} \): this is an example of the phenomenon pointed out in [24]. Since the power corrections in \( \alpha_{s,reg}^{PT}(k^2) \) are given by those in the \( I_n(k^2) \)'s, one recovers eq.(3.13) from eq.(4.8).

It is instructive to rederive the result eq.(3.1) for the regularized one-loop coupling with the above method. In this case, not only \( I_n^{PT} \), but also the constants \( I_n \) and \( b_n^{PT} \) can be computed from the Borel transform \( \tilde{\alpha}_{eff}(z) = \frac{\sin(\pi \beta_0 z)}{\pi \beta_0 z} \), since \( \alpha_{s}^{PT}(k^2) \) satisfies for \( 0 < k^2 < \Lambda^2 \) the \( z < 0 \) Borel representation:

\[
\alpha_{s}^{PT}(k^2) = - \int_{-\infty}^{0} dz \exp \left( -z \beta_0 \ln \frac{k^2}{\Lambda^2} \right) \tilde{\alpha}_s(z) \quad (0 < k^2 < \Lambda^2) \quad (4.16)
\]

with \( \tilde{\alpha}_s(z) \equiv 1 \). Taking the time-like discontinuity of eq.(4.16), one finds:

\[
\rho_{PT}(\mu^2) = - \int_{-\infty}^{0} dz \exp \left( -z \beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \tilde{\rho}(z) \quad (0 < \mu^2 < \Lambda^2) \quad (4.17)
\]

with \( \tilde{\rho}(z) = -\frac{1}{\pi} \sin(\pi \beta_0 z) \). From eq.(3.8b) one deduces:

\[
\alpha_{eff}^{PT}(\mu^2) = \alpha_{eff|IR}^{PT} - \int_{-\infty}^{0} dz \exp \left( -z \beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \tilde{\alpha}_{eff}(z) \quad (4.18)
\]

\[
(0 < \mu^2 < \Lambda^2)
\]

Note that eq.(4.4) and (4.18) imply (for \( \mu^2 = \Lambda^2 \)):

\[
\alpha_{eff|IR}^{PT} = \int_{-\infty}^{\infty} dz \tilde{\alpha}_{eff}(z) = \frac{1}{\beta_0} \quad (4.19)
\]

Inserting eq.(4.18) into eq.(4.14) , one gets:

\[
I_n = \alpha_{eff|IR}^{PT} - \bar{I}_n^{PT} \quad (4.20)
\]

with

\[
\bar{I}_n^{PT} = \int_{-\infty}^{0} dz \tilde{\alpha}_{eff}(z) \frac{1}{1 - \frac{z}{s_n}} \quad (4.21)
\]

hence (eq.(4.13)):

\[
b_n^{PT} = \alpha_{eff|IR}^{PT} - (I_n^{PT} + \bar{I}_n^{PT}) \quad (4.22)
\]
which, upon substitution into eq.(3.13) yields:

$$\delta \alpha_s^{PT}(k^2) = \alpha_{eff,IR}^{PT} \frac{\Lambda^2}{1 + \frac{\Lambda^2}{k^2}} + \sum_{n=1}^{\infty} \left( I_n^{PT} + \bar{I}_n^{PT} \right) \left( - \frac{\Lambda^2}{k^2} \right)^n$$

(4.23)

But:

$$I_n^{PT} + \bar{I}_n^{PT} = \int_{-\infty}^{\infty} \frac{dz}{\pi} \frac{\sin(\pi \beta_0 z)}{\pi \beta_0 z} \frac{1}{1 - \frac{z}{z_n}} = \frac{1}{\beta_0} (1 - (-1)^n) = \begin{cases} 0 & (n \text{ even}) \\ \frac{2}{\beta_0} & (n \text{ odd}) \end{cases}$$

(4.24)

which gives, since $\alpha_{eff,IR}^{PT} = \frac{1}{\beta_0}$ (the latter value is actually universal [12] and holds beyond one loop within some assumptions, see Appendix A):

$$b_n^{PT} = (-1)^n \frac{1}{\beta_0}$$

(4.25)

A similar method could be applied to the two-loop coupling, where [25]:

$$\tilde{\alpha}_s(z) = \exp(\lambda z \ln \lambda z) \frac{\exp(-\lambda z)}{\Gamma(1 + \lambda z)}$$

(4.26)

Here one should take into account the fact that $\tilde{\alpha}_s(z)$ is complex for $z < 0$, since the Landau singularity is a cut rather than a pole in this case.

Infrared fixed point case: No “regularization” is needed if one assumes that the Borel-summed $\alpha_s^{PT}(k^2)$ satisfies by itself the dispersion relation eq.(2.2). A simple example of such a coupling, relevant to the “small $\beta_0$” limit of QCD [28], is provided by the two loop beta function (eq.(3.5)) with $\frac{2}{\beta_0} < 0$, which has an IR fixed point $\alpha_s^{PT} |_{IR} = -\frac{\beta_0}{\beta_1}$. As noted by Uraltsev [29], for large $\beta_0$, more precisely if $\beta_1 < 0$ but $\frac{2}{\beta_0} + \beta_0 > 0$, there are complex Landau singularities, which move to the second sheet when $\beta_0$ is decreased and $\frac{2}{\beta_0} + \beta_0 < 0$. In the latter case ( barring further finite singularities ) $\alpha_{s,reg}^{PT} \equiv \alpha_s^{PT}$ and all power corrections vanish ( i.e. $I_n = I_n^{PT}$).

5. Renormalons in $\tilde{\alpha}_s(z)$

Up to now, I assumed that $\tilde{\alpha}_s(z)$ has no renormalons. In reality, this is likely not to be the case, since $\alpha_s$ is a physical coupling analogous to the Gell-Mann Low effective charge in QED, which probably does have renormalons. The general idea of constructing the regularized perturbative coupling through a dispersion relation from the discontinuity of the perturbative coupling still applies, but there are conceptual

aIf however $\tilde{\alpha}_s(z)$ has renormalons (see next section), $\alpha_s^{PT}$ (defined by a Borel sum) can of course never satisfy eq.(2.2).
as well as technical complications, since in this case even the latter cannot be defined by a Borel sum as in eq.(4.2) without additional prescription. I shall limit myself to the following simple example: assume the “physical” QCD coupling $\bar{\alpha}_s^{PT}(k^2)$ (I use a superscript “bar” to avoid confusion with the “t’Hooft coupling” below) is given by the standard IR renormalon integral:

$$\bar{\alpha}_s^{PT}(k^2) = \int_0^{k^2} \frac{dk^2}{k^2} \left( \frac{k^2}{k^2} \right)^n \alpha_s^{PT}(k^2)$$  \hspace{1cm} (5.1)

where $\alpha_s^{PT}(k^2)$ is the “t’Hooft coupling” of eq.(3.5). It has been shown in [23] that $\bar{\alpha}_s^{PT}(k^2)$ satisfies the (ordinary) Borel representation with respect to $\alpha \equiv \alpha_s^{PT}(k^2)$ :

$$\bar{\alpha}_s^{PT}(k^2) = \int_0^{\infty} dz \exp \left( -\frac{z}{\alpha} \right) \frac{\exp \left( -\frac{\bar{\alpha}_s^{PT}(z)}{\beta_0} \right)}{1 - \frac{z}{\beta_0}}$$  \hspace{1cm} (5.2)

with $\delta = \frac{\bar{\alpha}_s^{PT}(z)}{\beta_0}$. Note that the standard Borel transform singularity at $z = z_n$ is a cut [30] if $\beta_1 \neq 0$. It follows from [27] that the corresponding modified, “RS invariant” Borel transform $\tilde{\alpha}_s(z)$ has a simple pole singularity at $z = z_n$. For instance, if $\beta_1 = 0$ (in which case the ordinary and modified Borel transforms coincide), one has $\alpha_s(z) = \frac{1}{1-z_n}$, hence (eq.(4.5)) $\tilde{\alpha}_s(z) = \frac{\sin(\pi \beta_0 z)}{\pi \beta_0} \frac{1}{1-z_n}$, which coincides with the one-loop $\tilde{I}_n(z)$ (see eq.(4.11)). The latter fact is not accidental, as one can show [18] (see section 7) that the time-like discontinuity of $\bar{\alpha}_s^{PT}(k^2)$ in eq.(5.1) (properly extended (section 7) to the complex $k^2$ plane) is given by the analogous integral (which implies a peculiar definition of the Borel sum in eq.(5.2) to be consistent with eq.(5.1)) :

$$\bar{\rho}_{reg}(\mu^2) = \int_0^{\mu^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\mu^2} \right)^n \rho_{PT}(\mu^2)$$  \hspace{1cm} (5.3)

hence:

$$\bar{\alpha}_{eff,reg}(\mu^2) = \int_0^{\mu^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\mu^2} \right)^n \alpha_{eff}^{PT}(\mu^2)$$  \hspace{1cm} (5.4)

(where $\rho_{PT}$ and $\alpha_{eff}^{PT}$ are the corresponding quantities for $\alpha_s^{PT}$). Eq.(5.4) shows that $\bar{\alpha}_{eff,reg}(\mu^2)$ coincides with $I_n(k^2 = \mu^2)$ (eq.(4.9)), and differs from the corresponding Borel sum $\bar{\alpha}_{eff}^{PT}(\mu^2) = \int_0^{\infty} dz \exp \left( -z \beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \bar{\tilde{\alpha}}_{eff}(z)$ by a power correction, as we have seen in section 4. One can also show (section 7) that the $\bar{\alpha}_{s,reg}(k^2)$ following from the dispersive regularization procedure:

$$\bar{\alpha}_{s,reg}(k^2) = -\int_0^{\infty} \frac{d\mu^2}{\mu^2 + k^2} \bar{\rho}_{reg}(\mu^2) = k^2 \int_0^{\infty} \frac{d\mu^2}{(\mu^2 + k^2)^2} \bar{\alpha}_{eff,reg}(\mu^2)$$  \hspace{1cm} (5.5)
is simply given by the similar formula:

\[ \tilde{\alpha}_{s,\text{reg}}^\text{PT}(k^2) = \int_0^{k^2} \frac{d k'^2}{k'^2} \left( \frac{k'^2}{k^2} \right)^n \alpha_{s,\text{reg}}^\text{PT}(k^2) \] (5.6)

(which displays a certain “self-consistency” of the procedure).

6. “Perturbative” and “non-perturbative” power corrections

Upon insertion of eq.(2.13), the representation eq.(2.1) for the Euclidean observable \( D(Q^2) \) can be split, as we have seen, into a “regularized perturbation theory” [21,22] and a “genuine non-perturbative” [7] piece:

\[ \begin{align*}
D(Q^2) &= \int_0^{\infty} \frac{d k^2}{k^2} \tilde{\alpha}_{s,\text{reg}}^\text{PT}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) + \int_0^{\infty} \frac{d k^2}{k^2} \delta \alpha_s^{NP}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \\
&\equiv D_{\text{reg}}^{\text{PT}}(Q^2) + \delta D_{NP}(Q^2)
\end{align*} \] (6.1)

Each of these contributions, which I consider in turn, generates power corrections.

6.1. “Perturbative” power corrections

The “regularized perturbation theory” piece \( D_{\text{reg}}^{\text{PT}}(Q^2) \) may be decomposed, following eq.(2.11), as:

\[ D_{\text{reg}}^{\text{PT}}(Q^2) = D_{\text{PT}}^{\text{PT}}(Q^2) + \delta D_{\text{PT}}^{\text{PT}}(Q^2) \] (6.2)

with:

\[ D_{\text{PT}}^{\text{PT}}(Q^2) = \int_0^{\infty} \frac{d k^2}{k^2} \alpha_{s}^{\text{PT}}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \] (6.3)

and:

\[ \delta D_{\text{PT}}^{\text{PT}}(Q^2) = \int_0^{\infty} \frac{d k^2}{k^2} \delta \alpha_{s}^{\text{PT}}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \] (6.4)

Provided \( \alpha_{s}^{\text{PT}}(k^2) \) has no non-trivial IR fixed point, and satisfies for small enough \( k^2 \) the \( z < 0 \) Borel representation eq.(4.16), \( D_{\text{PT}}^{\text{PT}}(Q^2) \) can be identified [23] to the perturbation theory Borel sum:

\[ D_{\text{PT}}^{\text{PT}}(Q^2) = \int_0^{\infty} dz \exp \left( -z \beta_0 \ln \frac{Q^2}{\Lambda^2} \right) \tilde{D}(z) \quad (Q^2 > \Lambda^2) \] (6.5a)

with:

\[ \tilde{D}(z) = \bar{\alpha}_s(z) \int_0^{\infty} \frac{d k^2}{k^2} \varphi \left( \frac{k^2}{Q^2} \right) \exp \left( -z \beta_0 \ln \frac{k^2}{Q^2} \right) \] (6.5b)

Eq.(6.5) is obtained in practice by freely using the \( k^2 > \Lambda^2 \) Borel representation of \( \alpha_{s}^{\text{PT}}(k^2) \) (eq.(4.1)) in eq.(6.3), and permutting the \( k^2 \) and \( z \) integrations. (This
procedure can be justified \[23\] by splitting the integral in eq.(6.3) at \(k^2 = Q^2\). The \(k^2 > Q^2\) piece poses no problem. The \(k^2 < Q^2\) piece is defined by analytic continuation from the low \(Q^2\) region, where one can use the \(z < 0\) Borel representation eq.(4.16) of the coupling to derive a \(z < 0\) Borel representation.

\(D_{\text{reg}}^{PT}(Q^2)\) differs from the Borel sum \(D_{\text{PT}}(Q^2)\) by “perturbative” power corrections \(\delta D_{\text{PT}}(Q^2)\). These corrections are expected, the presence of the Landau singularity of \(\alpha_s^{PT}(k^2)\) in the integration range making \(D_{\text{PT}}\) ill-defined (an ambiguity reflected in the usual way through the presence of IR renormalons at \(z = z_n > 0\) in \(\tilde{D}(z)\)), whereas \(D_{\text{reg}}^{PT}\) is unambiguous. However, the important point is that \(\delta \alpha_s^{PT}(k^2)\) is only moderately suppressed at high \(k^2\), i.e. most likely (eq.(3.13)) \(\delta \alpha_s^{PT}(k^2) = O(\Lambda^2/k^2)\) for \(k^2 \gg \Lambda^2\). Consequently, \(\delta D_{\text{PT}}(Q^2)\) will also get (apart from the ambiguous IR contributions which must be present to cancell the IR renormalons ambiguities in \(D_{\text{PT}}\) additional \emph{unambiguous} contributions originating from the UV region, hence unrelated to IR renormalons and the OPE, e.g.:

\[
\int_0^\infty \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \approx - \sum_{p=1}^{\infty} A_p b_p^{PT} \left( - \frac{\Lambda^2}{Q^2} \right)^p
\]

(6.6)

where:

\[
A_p = \int_0^\infty \frac{dk^2}{k^2} \varphi \left( \frac{k^2}{Q^2} \right) \left( \frac{Q^2}{k^2} \right)^p
\]

(6.7)

is a number, and I used the asymptotic expansion eq.(3.13) of \(\delta \alpha_s^{PT}(k^2)\) (throughout this section, I also assume \(\alpha_s^{PT}\) has no renormalons, so that the \(b_i^{PT}\)’s themselves are real and unambiguous).

Let us derive the result for \(\delta D_{\text{PT}}\) in the typical (cf. eq.(5.1)) case where:

\[
D(Q^2) = \int_0^{Q^2} \frac{dk^2}{k^2} \alpha_s(k^2) \left( \frac{k^2}{Q^2} \right)^n
\]

(6.8)

i)Assume first \(n\) is an integer. One proceeds by disentangling [30-32] long from short distances and split \[18\] the integral in eq.(6.8) at the arbitrary IR scale \(\Lambda_i^2 = c \Lambda^2\) \((c > 1)\):

\[
\delta D_{\text{PT}}(Q^2) = \int_0^{\Lambda_i^2} n \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \left( \frac{k^2}{Q^2} \right)^n + \int_{\Lambda_i^2}^{Q^2} n \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \left( \frac{k^2}{Q^2} \right)^n
\]

\[
\quad \equiv \delta D_{\text{ld}}^{PT} + \delta D_{\text{sd}}^{PT}
\]

(6.9)

The “long distance” part \(\delta D_{\text{ld}}^{PT}\) contributes the (ambiguous) power correction:

\[
\delta D_{\text{ld}}^{PT}(Q^2) = K_n^{PT}(\Lambda_i^2) \left( \frac{\Lambda_i^2}{Q^2} \right)^n
\]

(6.10)
with

\[ K_n^{PT}(\Lambda_t^2) = \int_0^{\Lambda_t^2} n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \delta \alpha_s^{PT}(k^2) \]  

(6.11)

This “IR” power correction cancels the \( z_n = n/\beta_0 \) IR renormalon ambiguity present in \( D_{PT} \), and is best combined with the similar contribution to \( D_{PT} \) from the same integration range to yield the unambiguous \( O((\Lambda^2/Q^2)^n) \) power correction:

\[ \int_0^{\Lambda_t^2} n \frac{dk^2}{k^2} \alpha_s^{PT}(k^2) \left( \frac{k^2}{Q^2} \right)^n = \left( \frac{\Lambda^2}{Q^2} \right)^n \int_0^{\Lambda_t^2} \frac{dk^2}{k^2} \alpha_s^{PT}(k^2) \left( \frac{k^2}{\Lambda^2} \right)^n \]  

(6.12)

On the other hand, the “short distance” part yields:

\[ \delta D_{sd}^{PT}(Q^2) = -\sum_{p>n} \frac{nb_p^{PT}}{n-p} \left[ -\frac{\Lambda^2}{Q^2} \right]^p - \left( nb_n^{PT} \ln \frac{Q^2}{\Lambda^2} + \text{const} \right) \left( -\frac{\Lambda^2}{Q^2} \right)^n \]  

(6.13)

Eq.(6.13) may be easily obtained by substituting eq.(3.13) into the second integral in eq.(6.9). I give a more general derivation, where it is not necessary to assume that the expansion eq.(3.13) is valid down to \( k^2 = \Lambda_t^2 \). It is convenient to separate the first \( n \) terms of the asymptotic expansion and define:

\[ \delta \alpha_s^{PT}(k^2) \equiv -\sum_{p=1}^{n} b_p^{PT} \left( -\frac{\Lambda^2}{k^2} \right)^p + \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \]  

(6.14)

Then:

\[ \delta D_{sd}^{PT}(Q^2) = -\sum_{p=1}^{n} b_p^{PT} \left( -\frac{\Lambda^2}{Q^2} \right)^p \int_{\Lambda_t^2}^{Q^2} \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^{n-p} + \int_{\Lambda_t^2}^{Q^2} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n \]  

(6.15)

The second integral in eq.(6.15) is now UV convergent, and can be expressed as:

\[ \int_{\Lambda_t^2}^{Q^2} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n = \int_{\Lambda_t^2}^{\infty} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n \]

\[ - \int_{Q^2}^{\infty} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n \]  

(6.16)

But:

\[ \int_{\Lambda_t^2}^{\infty} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n = \left( \frac{\Lambda^2}{Q^2} \right)^n \int_{\Lambda_t^2}^{\infty} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{\Lambda^2} \right)^n \]

\[ = \text{const} \left( \frac{\Lambda^2}{Q^2} \right)^n \]  

(6.17)

whereas, using eq.(3.13):

\[ \int_{Q^2}^{\infty} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n = \sum_{p>n} \frac{nb_p^{PT}}{n-p} \left( -\frac{\Lambda^2}{Q^2} \right)^p \]  

(6.18)
On the other hand, the first term in eq.(6.15) gives:

\[- \sum_{p=1}^{n} b^{PT}_p \left( -\frac{\Lambda^2}{Q^2} \right)^p \int_{\Lambda_i^2}^{Q^2} \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^{n-p} = - \sum_{p<n} \frac{nb^{PT}_p}{n-p} \left( -\frac{\Lambda^2}{Q^2} \right)^p \]

\[- \left( nb^{PT}_n \ln \frac{Q^2}{\Lambda^2} + \text{const} \right) \left( -\frac{\Lambda^2}{Q^2} \right)^n \] (6.19)

Eq.[(6.15)-(6.19)] then yields eq.(6.13). Combining eq.(6.10) and (6.13), one ends up with:

\[\delta D_{PT}(Q^2) = - \sum_{p \neq n} \frac{nb^{PT}_p}{n-p} \left( -\frac{\Lambda^2}{Q^2} \right)^p - n \left( b^{PT}_n \ln \frac{Q^2}{\Lambda^2} + D^{PT}_n \right) \left( -\frac{\Lambda^2}{Q^2} \right)^n \] (6.20)

where the constant \( D^{PT}_n \) is independant of the arbitrary IR scale \( \Lambda_I \), but complex and ambiguous - thus cancelling the IR renormalon ambiguity arising from a simple pole at \( z = z_n \) in \( \tilde{D}(z) \), and cannot be computed straightforwardly (apart from its imaginary part, see Appendix A) from the asymptotic expansion eq.(3.13).

Furthermore, \( \delta D_{PT}(Q^2) \) for a general observable as in eq.(2.1) may be easily obtained from eq.(6.6) and (6.20), splitting the integral in eq.(6.1) at \( k^2 = Q^2 \), and expanding the kernel for \( k^2 \leq Q^2 \):

\[\varphi(k^2/Q^2) = \sum_{n=1}^{\infty} n c_n \left( \frac{k^2}{Q^2} \right)^n \] (6.21)

(where I assumed for simplicity absence of logarithmic terms, see point ii) below). One gets:

\[\delta D_{PT}(Q^2) = - \sum_{n} b^{PT}_n d_n \left( -\frac{\Lambda^2}{Q^2} \right)^n - \sum_{n} n c_n \left( b^{PT}_n \ln \frac{Q^2}{\Lambda^2} + D^{PT}_n \right) \left( -\frac{\Lambda^2}{Q^2} \right)^n \] (6.22)

where

\[d_n = A_n + \sum_{p \neq n} \frac{pc_p}{p-n} \] (6.23)

depends only on the \( \varphi \) kernel.

ii) I also quote the analogous result for the log-enhanced kernel where \( n \) is integer, but:

\[D(Q^2) = \int_{0}^{Q^2} \frac{dk^2}{k^2} \alpha_s(k^2) \left( \frac{k^2}{Q^2} \right)^n \ln \frac{Q^2}{k^2} \] (6.24)

With similar methods, one gets:

\[\delta D_{PT}(Q^2) = - \sum_{p \neq n} \frac{nb^{PT}_p}{(n-p)^2} \left( -\frac{\Lambda^2}{Q^2} \right)^p - n \left( \frac{1}{2} b^{PT}_n \ln^2 \frac{Q^2}{\Lambda^2} + E^{PT}_n \ln \frac{Q^2}{\Lambda^2} + F^{PT}_n \right) \left( -\frac{\Lambda^2}{Q^2} \right)^n \] (6.25)

where \( E^{PT}_n \) and \( F^{PT}_n \) are complex and ambiguous (they cancell the IR renormalon ambiguity arising from a double pole in \( \tilde{D}(z) \)).
Non-integer $n$: in such a case it is no more necessary to introduce an intermediate scale $\Lambda_i$, and log-enhanced terms are absent. Specifically, suppose $0 < n < 1$. Then, with $D(Q^2)$ as in eq.(6.8):

$$\delta D_{PT}(Q^2) = \int_0^\infty n \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \left( \frac{k^2}{Q^2} \right)^n - \int_{Q^2}^\infty n \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \left( \frac{k^2}{Q^2} \right)^n$$

$$\equiv K_n^{PT} \left( \frac{\Lambda^2}{Q^2} \right)^n - \sum_{p=1}^{\infty} nb_{PT}^p \left( - \frac{\Lambda^2}{Q^2} \right)^p$$  \hspace{1cm} (6.26)

where $K_n^{PT}$ is again a complex, ambiguous constant corresponding to a simple IR renormalon pole at $z = z_n$:

$$K_n^{PT} = \int_0^\infty n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \delta \alpha_s^{PT}(k^2) \equiv K_n^{PT}(\Lambda^2_i = \infty)$$  \hspace{1cm} (6.27)

As an application of the above results, one can derive the “perturbative” power corrections generated by the regularized one loop coupling eq.(3.1). Using:

$$\delta \alpha_s^{PT}(k^2)_{\text{one-loop}} = - \frac{1}{\beta_0} \frac{\Lambda^2}{k^2}$$  \hspace{1cm} (6.28)

one finds they are given by the Beneke-Braun formula [6]:

$$\delta D_{PT}(Q^2)_{\text{one-loop}} = - \frac{1}{\beta_0} \int_0^\infty \frac{dk^2}{k^2} \frac{\Lambda^2}{k^2} \frac{\Lambda^2}{1 - \frac{\Lambda^2}{k^2}} \varphi \left( \frac{k^2}{Q^2} \right)$$

$$\equiv - \frac{1}{\beta_0} \left[ \mathcal{F}(\frac{\Lambda^2}{Q^2}) - \mathcal{F}(0) \right]$$  \hspace{1cm} (6.29)

since eq.(2.6) implies:

$$\mathcal{F}(\frac{\mu^2}{Q^2}) - \mathcal{F}(0) = \int_0^\infty \frac{dk^2}{k^2} \frac{\mu^2}{1 + \frac{\mu^2}{k^2}} \varphi \left( \frac{k^2}{Q^2} \right)$$  \hspace{1cm} (6.30)

i.e. the once subtracted dispersion relation eq.(6.30) for the characteristic function may be seen as a peculiar case of the formula eq.(6.4) for $\delta D_{PT}(Q^2)$, with the substitutions: $\Lambda^2 \rightarrow -\mu^2$ and $\delta \alpha_s^{PT}(k^2) \rightarrow - \frac{\mu^2}{1 + \frac{\mu^2}{k^2}}$ (which imply $b_{PT}^p \rightarrow (-1)^{p+1}$ in eq.(3.13)).

If $\varphi$ behaves as in eq.(6.21), this observation implies (eq.(6.22)) the small $\mu^2$ behavior:

$$\mathcal{F}(\frac{\mu^2}{Q^2}) - \mathcal{F}(0) = \sum_n d_n \left( - \frac{\mu^2}{Q^2} \right)^n + \sum_n n c_n \left( \ln \frac{Q^2}{\mu^2} + \tilde{d}_n \right) \left( - \frac{\mu^2}{Q^2} \right)^n$$  \hspace{1cm} (6.31)

where the constant $\tilde{d}_n$ can be computed explicitly. If $0 < n < 1$, one gets instead from eq.(6.26) (for $D(Q^2)$ as in eq.(6.8)):

$$\mathcal{F}(\frac{\mu^2}{Q^2}) - \mathcal{F}(0) = - \frac{\pi n}{\sin(\pi n)} \left( \frac{\mu^2}{Q^2} \right)^n + \sum_{p=1}^{\infty} \frac{n}{n-p} \left( - \frac{\mu^2}{Q^2} \right)^p$$  \hspace{1cm} (6.32)
where I used the identity:

\[
\int_0^\infty \frac{n \, dk^2}{k^2} \left( \frac{k^2}{\mu^2} \right)^n \frac{\mu^2}{\mu^2 + k^2} \equiv \frac{\pi n}{\sin(\pi n)}
\]  

(6.33)

It is interesting to note that any \( O\left((k^2/Q^2)^n\right) \) term in the low energy expansion of the kernel \( \varphi\left(\frac{k^2}{Q^2}\right) \) is in one-to-one correspondence, for \( n \) integer, with a non-analytic term \( \left(\frac{\mu^2}{Q^2}\right)^n \ln \frac{Q^2}{\mu^2} \) in \( F\left(\frac{\mu^2}{Q^2}\right) \), which explains the connection [4,6] between non-analytic terms in the characteristic function and IR renormalons. However these non-analytic terms are not quite of IR origin, since they arise from the analogue in eq.(6.30) of the \( \delta D_{PT}^{sd} \) piece of eq.(6.9), and not from (the analogue of) \( \delta D_{PT}^{ld} \); they correspond to an \( UV \) enhancement (letting \( Q^2 \to \infty \) in eq.(6.9)) rather than to an IR one. More generally, the leading log terms in \( F\left(\frac{\mu^2}{Q^2}\right) \) and \( \delta D_{PT}(Q^2) \), and in particular the analytic terms if there are no log (which implies, barring cases where \( b_n^{PT} = 0 \) (see below), the vanishing of the corresponding coefficient \( c_n \)) are unambiguous and of short distance origin (see eq.(6.22), (6.25) and (6.31)). Comparing eq.(6.22) and (6.31) show they are simply related by a \( b_n^{PT} \) factor. On the other hand, the sub-leading log terms (in particular the constant terms associated to a log) are ambiguous and of (partially) IR origin. It is actually not possible, without reintroducing an arbitrary IR cut-off \( \Lambda_I \), to disentangle unambiguously terms of IR and UV origin within the sub-leading log terms of eq.(6.22) and (6.31) (the exception to the previous statement is the case \( n \neq \) integer, where no IR cut-off \( \Lambda_I \) needs to be introduced, see eq.(6.26)). Thus, for \( n \) integer, non-analytic terms are “related to”, but do not really arise from, long distances (this is also apparent from the fact that their coefficient is proportionnal (eq.(6.22)) to the product \( c_n b_n^{PT} \) of a long distance \( \times \) a short distance parameter).

The previous remarks suggest a simple generalization of eq.(6.29) to an arbitrary coupling: assume the leading term is in eq.(6.22) is an \( O\left((\Lambda^2/Q^2)^n\right) \) power correction entirely of short distance origin, i.e. that \( c_i = 0 \) for \( i < n \) and \( \varphi \) is \( O\left((k^2/Q^2)^n\right) \) at small \( k^2 \). Then:

\[
\delta D_{PT}(Q^2) \simeq -b_n^{PT} d_n \left( -\frac{\Lambda^2}{Q^2} \right)^n
\]

\[
Q^2 \gg \Lambda^2
\]

(6.34)

while eq.(6.31) shows that the leading small \( \mu^2 \) behavior of \( F\left(\frac{\mu^2}{Q^2}\right) \) is analytic and given by:

\[
F\left(\frac{\mu^2}{Q^2}\right) - F(0) \simeq d_n \left( -\frac{\mu^2}{Q^2} \right)^n
\]

\[
\mu^2 \ll Q^2
\]

(6.35)

Eq.(6.34)-(6.35) agree with eq.(6.29) in the one loop case, where (eq.(4.25))

\[
b_n^{PT} = (-1)^n \frac{1}{\beta_0}
\]

19
In the complementary case where a peculiar $b_n^{PT}$ vanishes while $c_n \neq 0$, it is not necessary any more to introduce an IR cut-off $\Lambda_I$, since all integrals in eq.(6.15) are separately IR convergent. Setting $\Lambda_I = 0$ (with $D(Q^2)$ as in eq.(6.8)) one gets:

$$\delta D_{PT}(Q^2) = - \sum_{n=1}^{\infty} \frac{n b_n^{PT}}{n - p} \left( - \frac{\Lambda^2}{Q^2} \right)^p + \int_0^\infty n \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) \right]_{(n)} \left( \frac{k^2}{Q^2} \right)^n$$

(6.36)

where the last integral is an IR $O((1/Q^2)^n)$ power correction (with all other contributions arising from short distances). Note that a non-analytic term is still present in $F(Q^2)$, and is actually crucial to reproduce the IR power correction in $\delta D_{PT}(Q^2)$ (see section 7, where a more general derivation of these results, which relies directly on the representation eq.(2.17) and does not assume the dispersion relation eq.(6.30), is given).

6.2. Non-perturbative power corrections

In [7], a condition of sufficiently fast UV damping (i.e. of an exponential or at least of a rather high power suppression at large $k^2$) was imposed on the “non-perturbative” modification $\delta \alpha_s^{NP}(k^2)$. The assumption that $\delta \alpha_s^{NP}(k^2)$ is essentially restricted to low $k^2$ was motivated by the ideology of “soft confinement” of [20], who put forward the idea of gluon condensation as an essentially IR phenomenon, which could be described entirely within the OPE. However there is no fundamental reason, as is by now widely appreciated, that all power contributions should be of IR origin, and in fact the dispersive framework strongly suggests the existence of power contributions arising from short distances, as exemplified in section 6.1 through the simplest dispersive regularization procedure. For the sake of generality, I shall therefore relax this assumption. The split eq.(2.14) of $\delta \alpha_s$ into “perturbative” and “non-perturbative” components now becomes a matter of convention, since one can no more argue as in section 2 that $\delta \alpha_s \simeq \delta \alpha_s^{PT}$ at large $k^2$. However still assume that $\delta \alpha_s^{NP}(\mu^2)$ itself is exponentially suppressed at large $\mu^2$, in order to be able to expand under the integral in eq.(2.9): this is a rather strong restriction, but represents a straightforward extension of the framework of [7]. Then eq.(2.9) yields the asymptotic expansion:

$$\delta \alpha_s^{NP}(k^2) = - \sum_{n=1}^{\infty} b_n^{NP} \left( - \frac{\Lambda^2}{k^2} \right)^n$$

(6.37)

where the integer moments:

$$b_n^{NP} = \int_0^\infty n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\Lambda^2} \right)^n \delta \alpha_{eff}^{NP}(\mu^2)$$

(6.38)

are no more required to vanish as in [7]. It follows that in eq.(2.14):

$$\delta \alpha_s(k^2) = - \sum_{n=1}^{\infty} b_n \left( - \frac{\Lambda^2}{k^2} \right)^n$$

(6.39)
with:

\[ b_n \equiv b_{n}^{PT} + b_{n}^{NP} \]  

(6.40)

It is clear that all results of section 6.1 also apply to \( \delta D_{NP}(Q^2) \) (eq.(2.10a)) or to the total power contribution:

\[ \delta D(Q^2) \equiv \int_{0}^{\infty} \frac{d k^2}{k^2} \delta \alpha_s(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \]  

(6.41)

if one substitutes \( \delta \alpha_s^{PT}(k^2) \) with \( \delta \alpha_s^{NP}(k^2) \) or \( \delta \alpha_s(k^2) \), and \( b_{n}^{PT} \) with \( b_{n}^{NP} \) or \( b_{n} \) (of course for \( \delta \alpha_s^{NP} \) all subleading logs constants are also unambiguous). In particular, the conclusion of section 6.1 that it is not possible in general to disentangle in an unambiguous way for \( n \) integer the power corrections of IR origin from those which arise from short distances is also valid here.

The present general framework is still compatible with the assumption [7,20] that \( \delta \alpha_s^{NP}(k^2) \) (or even the total \( \delta \alpha_s(k^2) \)) is restricted to low \( k^2 \), but this question has to be decided by fitting the data, rather than imposed a priori. For instance, if one assumes that \( \delta \alpha_s^{NP}(k^2) \equiv \left[ \delta \alpha_s^{NP}(k^2) \right]_{(n)} \), i.e. that \( b_{p}^{NP} = 0 \) for \( 1 \leq p \leq n \), then one obtains immediately for the observable of eq.(6.8) (from the analogue of eq.(6.36), see also eq.(6.26)) :

\[ \delta D_{NP}(Q^2) = \int_{0}^{\infty} n \frac{d k^2}{k^2} \delta \alpha_s^{NP}(k^2) \left( \frac{k^2}{Q^2} \right)^n - \int_{Q^2}^{\infty} n \frac{d k^2}{k^2} \delta \alpha_s^{NP}(k^2) \left( \frac{k^2}{Q^2} \right)^n \]

\[ \equiv K_{n}^{NP} \left( \frac{\Lambda^2}{Q^2} \right)^n - \sum_{p>2} n b_{p}^{NP} \left( \frac{\Lambda^2}{Q^2} \right)^p \]  

(6.42)

where:

\[ K_{n}^{NP} = \int_{0}^{\infty} n \frac{d k^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \delta \alpha_s^{NP}(k^2) \]  

(6.43)

i.e. in this example the leading power correction is indeed of IR origin and controlled by the OPE. Imposing similarly that \( \delta \alpha_s^{NP}(k^2) \) be exponentially suppressed, i.e. that \( b_{n}^{NP} = 0 \) for all \( n \)'s, leads to the large \( Q^2 \) result (for the general observable of eq.(6.1), assuming eq.(6.21)):

\[ \delta D_{NP}(Q^2) = \sum_{n=1}^{\infty} c_n K_{n}^{NP} \left( \frac{\Lambda^2}{Q^2} \right)^n \]  

(6.44)

All non-perturbative power corrections, arising essentially from the infrared, are then in one to one correspondence with a term \( c_n \) in the low energy expansion of the Feynman diagram kernel, hence [30-32] with a related operator in the OPE. Alternatively, one could impose that the total \( \delta \alpha_s(k^2) \) be exponentially suppressed, by requiring that \( b_{n}^{NP} = -b_{n}^{PT} \) for all \( n \)'s (given the \( b_{n}^{PT} \)'s, a theorem [33] guarantees there is an infinity of solutions \( \delta \alpha_{eff}(\mu^2) \) for the resulting moment problem following from eq.(6.38);
for instance (see [34] for a related suggestion), these constraints may be fulfilled by expressing $\delta\alpha_s^{NP}(k^2)$ as an (eventually infinite) sum of time-like poles. Nevertheless, all such restrictions have no fundamental basis, barring the (arbitrary) requirement that only those power corrections to the Borel sum which are controlled by the OPE should be present.

7. Minkowskian observables

For Euclidean observables, the alternative representation in term of the characteristic function (eq.(2.5 b), (2.10 b) and (2.17)), although technically convenient, is not really indispensable. The situation is different for Minkowskian observables $R(Q^2)$ (such as cross sections or inclusive decay rates), for which the representation eq.(2.1) is not in general available. In such cases the characteristic function $\mathcal{F}_R(\frac{\mu^2}{Q^2})$ is usually given by two distinct pieces, for instance:

$$\mathcal{F}_R(\frac{\mu^2}{Q^2}) = \begin{cases} \mathcal{F}(-)(\frac{\mu^2}{Q^2}) & \mu^2 < Q^2 \\ \mathcal{F}(+)(\frac{\mu^2}{Q^2}) & \mu^2 > Q^2 \end{cases}$$  \hspace{1cm} (7.1)

(where $\mathcal{F}(-)$ is the sum of a real and a virtual contribution, while $\mathcal{F}(+)$ contains only the virtual contribution), and thus cannot satisfy the dispersion relation eq.(2.6). Then $R$ and $\delta R_{NP}$ are given directly by eq.(2.5 b) (respectively (2.10 b)) (with $D \to R$ and $\mathcal{F} \to \mathcal{F}_R$), i.e.:

$$R(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{eff}(\mu^2) \mathcal{F}_R(\frac{\mu^2}{Q^2})$$  \hspace{1cm} (7.2)

and similarly:

$$R_{reg}^{PT}(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{eff}^{PT}(\mu^2) \mathcal{F}_R(\frac{\mu^2}{Q^2})$$  \hspace{1cm} (7.3)

where $\alpha_{eff}^{PT}$ is obtained from the discontinuity of the (Borel summed) $\alpha_s^{PT}$ as explained in section 4 (I assume $\tilde{\alpha}_s(z)$ has no renormalons), and is IR finite.

Let us derive eq.(7.2)-(7.3) in the peculiar case where $R(Q^2)$ is related to the time-like discontinuity of an Euclidean observable $D(Q^2)$ which satisfies the dispersion relation:

$$D(Q^2) = Q^2 \int_0^\infty \frac{dQ'^2}{(Q'^2 + Q^2)^2} \ R(Q'^2)$$  \hspace{1cm} (7.4)

Then:

$$\frac{dR}{d\ln Q^2} = \rho_D(Q^2)$$  \hspace{1cm} (7.5)

where $\rho_D(Q^2) \equiv -\frac{1}{2\pi i} \text{Disc}\{D(-Q^2)\}$ ($Q^2 > 0$) is the time-like “spectral density” of $D(Q^2)$. On the other hand, if $D(Q^2)$ satisfies the representation eq.(2.1), the corresponding $\rho_D(Q^2)$ is given in term of the spectral density $\rho(\mu^2)$ of $\alpha_s(k^2)$ by:

$$\rho_D(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \rho(\mu^2) \varphi\left(\frac{\mu^2}{Q^2}\right)$$  \hspace{1cm} (7.6)
Eq. (7.6) follows [18] by performing the change of variable $x = k^2/Q^2$ in eq. (2.1), and performing the analytic continuation to the time-like region with the new integrand:

$$D(Q^2) = \int_0^\infty \frac{dx}{x} \alpha_s(xQ^2) \varphi(x)$$  \hspace{1cm} (7.7)

The corresponding $R(Q^2)$ from eq. (7.5) is then given by [18]:

$$R(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{e,f}(\mu^2) \varphi\left(\frac{\mu^2}{Q^2}\right)$$  \hspace{1cm} (7.8)

A similar argument applied to $D_{reg}^{PT}(Q^2)$ (eq. (2.12)) yields (since $Disc\{\alpha_{s,reg}^{PT}(-\mu^2)\} \equiv Disc\{\alpha_{s,reg}^{PT}(-\mu^2)\}$):

$$\rho_{D,reg}^{PT}(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \rho^{PT}(\mu^2) \varphi\left(\frac{\mu^2}{Q^2}\right)$$  \hspace{1cm} (7.9)

and:

$$R_{reg}^{PT}(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{e,f}^{PT}(\mu^2) \varphi\left(\frac{\mu^2}{Q^2}\right)$$  \hspace{1cm} (7.10)

where $\rho_{D,reg}^{PT}(Q^2) \equiv -\frac{1}{2\pi i} Disc\{D_{reg}^{PT}(-Q^2)\}$ and:

$$\frac{dR_{reg}^{PT}}{d\ln Q^2} = \rho_{D,reg}^{PT}(Q^2)$$  \hspace{1cm} (7.11)

Eq. (7.8) and (7.10) suggest that [18]:

$$\tilde{F}_R(\frac{\mu^2}{Q^2}) \equiv \varphi\left(\frac{\mu^2}{Q^2}\right)$$  \hspace{1cm} (7.12)

I complete the argument and show that: i) $R_{reg}^{PT}(Q^2)$ in eq. (7.10) has the correct perturbative expansion and Borel transform expected from eq. (7.4) and ii) $D_{reg}^{PT}(Q^2)$ in eq. (2.12) or (2.17) is related to $R_{reg}^{PT}(Q^2)$ by the dispersion relation eq. (7.4).

i) Eq. (7.4) implies [27] the relation between the Borel transforms of absorptive and dispersive parts:

$$\tilde{R}(z) = \frac{\sin(\pi \beta_0 z)}{\pi \beta_0 z} \tilde{D}(z)$$  \hspace{1cm} (7.13)

This relation is indeed satisfied since the Borel sum corresponding to eq. (7.10) is:

$$R_{PT}(Q^2) = \int_0^\infty dz \exp\left(-z\beta_0 \ln\frac{Q^2}{\Lambda^2}\right) \tilde{R}(z)$$  \hspace{1cm} (7.14)

with (an analogue of eq. (6.5b)):

$$\tilde{R}(z) = \tilde{\alpha}_{e,f}(z) \int_0^\infty \frac{d\mu^2}{\mu^2} \varphi\left(\frac{\mu^2}{Q^2}\right) \exp\left(-z\beta_0 \ln\frac{\mu^2}{Q^2}\right)$$  \hspace{1cm} (7.15)
Eq. (7.14)-(7.15) are obtained [23] by freely substituting $\alpha_{\text{eff}}^{PT}$ by its Borel representation eq.(4.4) into eq.(7.10), and permutting the orders of integration. In this case however, because $\alpha_{\text{eff}}^{PT}$ has a non-trivial IR fixed point, $R_{\text{reg}}^{PT}$ differs [23,24], as we shall see below, from its Borel sum $R_{\text{PT}}^{PT}$ (in sharp contrast with $D_{\text{PT}}^{PT}$ in eq.(6.3)). Eq.(7.15), together with eq.(4.6) and (6.5b), reproduce eq.(7.13).

ii) Substituting $R$ in the dispersion relation eq.(7.4) with $R_{\text{reg}}^{PT}$ (eq.(7.10)), one finds, after permutting the orders of integration:

$$D_{\text{reg}}^{PT}(Q^2) \equiv Q^2 \int_0^{\infty} \frac{dQ'^2}{(Q'^2 + Q^2)^2} R_{\text{reg}}^{PT}(Q^2)$$

$$= \int_0^{\infty} \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}^{PT}(\mu^2) \dot{F}(\frac{\mu^2}{Q^2})$$

with:

$$\dot{F}(\frac{\mu^2}{Q^2}) \equiv Q^2 \int_0^{\infty} \frac{dQ'^2}{(Q'^2 + Q^2)^2} \varphi\left(\frac{\mu^2}{Q^2}\right)$$

$$= \mu^2 \int_0^{\infty} \frac{dk^2}{(k^2 + \mu^2)^2} \varphi\left(\frac{k^2}{Q^2}\right)$$

where the change of variable $\frac{\mu^2}{Q^2} = \frac{k^2}{Q^2}$ has been performed in the second step. Eq.(7.17) indeed agrees with the expected relation for $\dot{F}$ obtained by taking the $\mu^2$-derivative of eq.(2.6), and shows that eq.(7.16) reproduces eq.(2.17), hence eq.(2.12).

Finally, let us justify eq.(A.1) of Appendix A, i.e. the statement that:

$$\text{Disc}\{D_{\text{reg}}^{PT}(-Q^2)\} = \text{Disc}\{D_{\text{PT}}^{PT}(-Q^2)\} \quad (Q^2 > 0)$$

This result follows immediately by applying to $D_{\text{PT}}^{PT}(Q^2)$ in eq.(6.3) the same argument [18] which leads to eq.(7.9), and reflects the basic feature of the dispersive regularization procedure that $\text{Disc}\{\alpha_{\text{s,reg}}^{PT}(-\mu^2)\} \equiv \text{Disc}\{\alpha_{\text{s}}^{PT}(-\mu^2)\}$. However now there is a caveat, since $\alpha_{\text{s}}^{PT}(k^2)$ does not satisfy the dispersion relation eq.(2.2). In particular, it could have complex singularities in the $k^2$ plane (in addition to the standard space-like Landau singularity), which would make eq.(7.7) (with $\alpha_s \rightarrow \alpha_{\text{s}}^{PT}$) meaningless at complex $Q^2$, and obstruct the analytic continuation to the time-like region. The procedure of [18] seems however to be safe if one assumes the absence of such singularities (as is the case for the one-loop coupling).

7.1. Perturbative power corrections

From the results of section 6, we know that $D_{\text{reg}}^{PT}$ (eq.(2.17)) differs from its Borel sum $D_{\text{PT}}$. It is therefore natural to expect that also here $R_{\text{reg}}^{PT}$ differs from the corresponding Borel sum $R_{\text{PT}}$ (eq.7.14) by “perturbative” power corrections $\delta R_{\text{PT}}$. 24
To determine them, one can proceed as in section 4 and Appendix A, and split the integral in eq.(7.3) at $\mu^2 = \Lambda^2$:

$$R_{\text{reg}}^{PT}(Q^2) = \int_0^{\Lambda^2} \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}^{PT}(\mu^2) \tilde{\mathcal{F}}_R(\frac{\mu^2}{Q^2}) + \int_{\Lambda^2}^{\infty} \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}^{PT}(\mu^2) \tilde{\mathcal{F}}_R(\frac{\mu^2}{Q^2})$$

$$\equiv R_{\text{reg},<}^{PT}(Q^2) + R_{>\text{reg}}^{PT}(Q^2)$$

(7.18)

Using the Borel representation of $\alpha_{\text{eff}}^{PT}$ (eq.(4.4)), $R_{>\text{reg}}^{PT}$ can be written as:

$$R_{>\text{reg}}^{PT}(Q^2) = R_{PT}(Q^2) - R_{<\text{reg}}^{PT}(Q^2)$$

(7.19)

where (for $Q^2 > \Lambda^2$):

$$R_{<\text{reg}}^{PT}(Q^2) \equiv \int_0^{\infty} dz \tilde{\alpha}_{\text{eff}}(z) \left[ \int_0^{\Lambda^2} \frac{d\mu^2}{\mu^2} \tilde{\mathcal{F}}_R(\frac{\mu^2}{Q^2}) \exp \left(-z\beta_0 \ln \frac{\mu^2}{\Lambda^2}\right) \right]$$

(7.20)

Thus:

$$R_{\text{reg}}^{PT}(Q^2) = R_{PT}(Q^2) + \delta R_{PT}(Q^2)$$

(7.21)

with:

$$\delta R_{PT}(Q^2) = R_{\text{reg},<}^{PT}(Q^2) - R_{<\text{reg}}^{PT}(Q^2)$$

(7.22)

The “perturbative” power corrections are then obtained by taking the low $\mu^2$ expansion of $\tilde{\mathcal{F}}_R(\frac{\mu^2}{Q^2})$ inside the corresponding integrals of finite support $[0, \Lambda^2]$ in eq.(7.18) and (7.20) (note that, since $\mu^2 < \Lambda^2 < Q^2$, $\delta R_{PT}(Q^2)$ depends only on the “low energy piece” $\mathcal{F}_{(-)}$ of $\mathcal{F}_R$). For instance, an analytic term $n \left(\frac{\mu^2}{Q^2}\right)^n$ (with $n > 0$ integer) in the low-$\mu^2$ expansion of $\tilde{\mathcal{F}}_R(\frac{\mu^2}{Q^2})$ contributes a power correction $b_n^{PT} \left(\frac{\Lambda^2}{Q^2}\right)^n$ (with $b_n^{PT}$ given in eq.(4.13) ). The same result holds if $n \neq \text{integer}$, in particular in the phenomenologically important case $n = 1/2$ (“1/Q power corrections” [35]). In the one-loop coupling case, the result eq.(4.25) for $b_n^{PT}$ (which contains the necessary ambiguous imaginary part when $n \neq \text{integer}$) agrees with the Beneke-Braun formula eq.(6.29) (with $\mathcal{F} \rightarrow \mathcal{F}_{(-)}$, see a general derivation in Appendix B ). This result is remarkable, since it does not rely on a dispersion relation for $\mathcal{F}_{(-)}$: in particular, for $n$ integer, such an analytic term may be a “subtraction” term, unrelated to the discontinuity of $\mathcal{F}_{(-)}$ (Appendix B). Similarly, a non-analytic term $n \left(\frac{\mu^2}{Q^2}\right)^n (c_n \ln \frac{Q^2}{\mu^2} + \tilde{d}_n)$ in $\tilde{\mathcal{F}}_R (n \text{ integer})$ contributes a (log-enhanced) power correction:

$$c_n \left(\frac{\Lambda^2}{Q^2}\right)^n \left(\tilde{b}_n^{PT} \ln \frac{Q^2}{\Lambda^2} + \tilde{d}_n^{PT} \right) + \tilde{d}_n^{PT} \left(\frac{\Lambda^2}{Q^2}\right)^n$$

(7.23)

with $\tilde{b}_n^{PT} = J_n - J_n^{PT}$, where:

$$J_n = \int_0^{\Lambda^2} n \frac{d\mu^2}{\mu^2} \left(\frac{\mu^2}{\Lambda^2}\right)^n \ln \frac{\Lambda^2}{\mu^2} \alpha_{\text{eff}}^{PT}(\mu^2)$$

(7.24)
and:

\[ J_n^{PT} = \int_0^\infty dz \, \tilde{\alpha}_{eff}(z) \left[ \int_0^\Lambda^2 \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\Lambda^2} \right)^n \ln \frac{\Lambda^2}{\mu^2} \exp \left( -z\beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \right] = \frac{1}{n} \int_0^\infty dz \, \tilde{\alpha}_{eff}(z) \left( 1 - \frac{1}{z_n} \right)^2 \]  

(7.25)

is the Borel sum corresponding to \( J_n \). The same techniques applied to the representation eq.(2.17) of an Euclidean quantity \( D(Q^2) \) reproduce the results of section 6.1, with explicit expressions similar to eq.(7.24),(7.25) for the coefficients of the subleading log terms. Note that \( J_n^{PT} \), hence \( \tilde{b}_n^{PT} \), are ambiguous, due to the presence of an IR renormalon (a simple pole) at \( z = z_n \) in the Borel transform, the simple zero in \( \tilde{\alpha}_{eff}(z) \) only partially cancelling the double pole in the integrand of eq.(7.25). This is another example of the relation [4,6] between non-analytic terms in the characteristic function and IR renormalons. This relation too can only be understood if \( \alpha_{eff}^{PT}(\mu^2) \) has a non-trivial IR fixed point: otherwise, if one assumes e.g. \( \alpha_{eff}^{PT}(\mu^2) \) is given by the one-loop coupling (i.e. \( \tilde{\alpha}_{eff}(z) \equiv 1 \)), one could associate IR renormalons even to analytic terms in the low \( \mu^2 \) expansion of \( F_R(\frac{\mu^2}{Q^2}) \)!

On the other hand, the coefficients \( b_n^{PT} \) of the leading-log parts (and in particular of the analytic parts if there are no accompanying log) are unambiguous for \( n \) integer (eq.(4.13)) if \( \tilde{\alpha}(z) \) has no renormalon, which suggests (in agreement with the analysis of section 6.1) they should be associated to short-distances: this point is tricky, since all power corrections formally originate (see eq.(7.22)) from integration over low \( \mu^2 \), and shows it is misleading to use the \( \alpha_{eff} \) representations eq.(2.5b) or (7.2) to separate long from short distances, at the difference (section 6) of the \( \alpha_s \)-representation eq.(2.1).

The above interpretation is reinforced by the following observation: the time-like discontinuity of the leading log contributions to \( \delta D_{PT} \), which arise (as discussed in section 6.1) from short distances, are related to the corresponding leading log contributions to \( \frac{d\delta R_{PT}}{d\ln Q^2} \) by eq.(7.5). For instance, in the case of the Euclidean quantity \( D(Q^2) \) of eq.(6.8) with \( n \) integer, one gets for the associated (through eq.(7.12)) time-like observable: \( \delta R_{PT}(Q^2) = b_n^{PT} \left( \frac{\Lambda^2}{Q^2} \right)^n \) (which is entirely short-distance!), whereas the corresponding leading log contribution to \( \delta D_{PT} \) is (see eq.(6.20)):

\[ -nb_n^{PT} \ln \frac{Q^2}{\Lambda^2} \left( -\frac{\Lambda^2}{Q^2} \right)^n \]

whose time-like discontinuity is indeed related to \( \delta R_{PT} \) by eq.(7.5). Similarly, for the quantity of eq.(6.24), one gets \( \delta R_{PT}(Q^2)|_{leading \, \log} = b_n^{PT} \left( \frac{\Lambda^2}{Q^2} \right)^n \ln \frac{Q^2}{\Lambda^2} \), whereas the corresponding leading log contribution to \( \delta D_{PT} \) is (see eq.(6.25)):

\[ -\frac{1}{2}nb_n^{PT} \ln^2 \frac{Q^2}{\Lambda^2} \left( -\frac{\Lambda^2}{Q^2} \right)^n \]

whose time-like discontinuity is again related to \( \frac{d\delta R_{PT}}{d\ln Q^2} \) by eq.(7.5). The basic reason for these relations is as follows. The dispersion relation in eq.(7.16) implies the “inverse” relation [18,36]:

\[ R_{reg}^{PT}(Q^2) = \frac{1}{2\pi i} \oint_{Q^2=\pm Q^2} \frac{dQ^2}{Q^2} \frac{d\delta R_{reg}}{d\ln Q^2} \]

(7.26)
But it is clear that, at least formally, the Borel sum $R_{PT}$ is also related to the Borel sum $D_{PT}$ by eq.(7.26) and $R_{PT}$ may actually be obtained by substituting $D_{reg}^{PT}$ with $D_{PT}$ of eq.(6.5a) into eq.(7.26), and permutting (as usual) the order of integrations! It follows the same statement is true for $\delta R_{PT}$ and $\delta D_{PT}$, i.e. we have:

$$\delta R_{PT}(Q^2) = \frac{1}{2\pi i} \oint_{|Q'^2|=Q^2} \frac{dQ'^2}{Q'^2} \delta D_{PT}(Q'^2)$$  \hspace{1cm} (7.27)

which accounts for the above mentioned relations. This argument remains formal as long as no precise definition of the (ambiguous) Borel sums $D_{PT}$ and $R_{PT}$ (hence of $\delta D_{PT}$ and $\delta R_{PT}$) is given: the principal part prescription is adequate here, since the previous argument is valid with it.

7.2. Non-perturbative power corrections

They are contained in (cf. eq.(2.10b)):

$$\delta R_{NP}(Q^2) = \int_0^\infty \frac{d\mu^2}{\mu^2} \delta \alpha_{eff}^{NP}(\mu^2) \tilde{F}_R(\frac{\mu^2}{Q^2})$$  \hspace{1cm} (7.28)

and are obtained [7] by taking the low $\mu^2$ expansion of $\tilde{F}_R(\frac{\mu^2}{Q^2})$ inside the integral (since $\delta \alpha_{eff}^{NP}(\mu^2)$ is assumed to be exponentially suppressed at large $\mu^2$). For instance, a non-analytic term $n \left(\frac{\mu^2}{Q^2}\right)^n (c_n \ln \frac{Q^2}{\mu^2} + \bar{d}_n)$ in $\tilde{F}_R$ ($n$ integer) contributes a log-enhanced power correction:

$$c_n \left(\frac{\Lambda^2}{Q^2}\right)^n \left(b_n^{NP} \ln \frac{Q^2}{\Lambda^2} + \bar{b}_n^{NP}\right) + \bar{d}_n b_n^{NP} \left(\frac{\Lambda^2}{Q^2}\right)^n$$  \hspace{1cm} (7.29)

where $b_n^{NP}$ is given in eq.(6.38), and:

$$\bar{b}_n^{NP} = \int_0^\infty \frac{d\mu^2}{\mu^2} \left(\frac{\mu^2}{\Lambda^2}\right)^n \ln \frac{\Lambda^2}{\mu^2} \delta \alpha_{eff}^{NP}(\mu^2)$$  \hspace{1cm} (7.30)

Note that eq.(7.29) has exactly the same structure as the corresponding contribution to $\delta R_{PT}(Q^2)$ (eq.(7.23)) with the substitutions $b_n^{PT} \rightarrow b_n^{NP}$ and $\bar{b}_n^{PT} \rightarrow \bar{b}_n^{NP}$! Again, the leading log terms terms with a coefficient $b_n^{NP}$ (an analytic, integer moment) should be associated to short distances, while the sub-leading log terms, with a coefficient $\bar{b}_n^{NP}$ (a non-analytic moment), are partly long distance. One sees once more it is not possible, for $n$ integer, to disentangle unambiguously these two type of contributions (which get mixed once one changes the scale $\Lambda$ inside the log in eq.(7.29)),

\[\text{This is a different prescription that the one which leads to eq.(A.1) and to a vanishing timelike discontinuity of } \delta D_{PT}; \text{ of course, given any prescription for } D_{PT}, \text{ one can always define the corresponding } R_{PT} \text{ by requiring that it is related to } D_{PT} \text{ by eq.(7.26), but in general this would result in different prescriptions for the Borel sums } D_{PT} \text{ and } R_{PT}.\]
unless $c_n = 0$ or $b_{n}^{NP} = 0$. The exception is again the case $n \neq \text{integer}$, where there are no logs of UV origin.

The non-perturbative power corrections in $R(Q^2)$ may also be derived from those in the associated Euclidean quantity $D(Q^2)$, since the dispersion relation eq.(7.4) and its inverse eq.(7.26) hold between $\delta D_{NP}(Q^2)$ and $\delta R_{NP}(Q^2)$ (note it is always possible to reconstruct a $D(Q^2)$ corresponding to a given $R(Q^2)$ using the relation eq.(7.12), so the method below is general). In particular we have (eq.(7.5)):

$$\frac{d(\delta R_{NP})}{d \ln Q^2} = -\frac{1}{2\pi i} Disc\{\delta D_{NP}(-Q^2)\} \quad (Q^2 > 0) \quad (7.31)$$

Consider for instance $D(Q^2)$ in eq.(6.24) with $n$ integer, which is the Euclidean quantity associated to:

$$R(Q^2) = \int_{0}^{Q^2} n \frac{d\mu^2}{\mu^2} \alpha_{eff}(\mu^2) \left(\frac{\mu^2}{Q^2}\right)^n \ln \frac{Q^2}{\mu^2} \quad (7.32)$$

and assume [7] $\delta \alpha_s^{NP}$ is restricted to low $k^2$, so that all $b_{p}^{NP}$'s vanish for $p$ integer. Then one gets for $Q^2 \gg \Lambda^2$:

$$\delta D_{NP}(Q^2) \simeq K_n^{NP} \left(\frac{\Lambda^2}{Q^2}\right)^n \ln \frac{Q^2}{\Lambda^2} + \text{const} \quad (7.33)$$

with $K_n^{NP}$ given in eq.(6.43), whereas (eq.(7.29)):

$$\delta R_{NP}(Q^2) \simeq \bar{b}_n^{NP} \left(\frac{\Lambda^2}{Q^2}\right)^n \quad (7.34)$$

Eq.(7.31) then implies:

$$K_n^{NP} = -(-1)^n n \bar{b}_n^{NP} \quad (7.35)$$

i.e.:

$$\int_{0}^{\infty} \frac{dk^2}{k^2} \left(\frac{k^2}{\Lambda^2}\right)^n \delta \alpha_s^{NP}(k^2) = (-1)^n n \int_{0}^{\infty} \frac{d\mu^2}{\mu^2} \left(\frac{\mu^2}{\Lambda^2}\right)^n \ln \frac{\mu^2}{\Lambda^2} \delta \alpha_{eff}(\mu^2) \quad (n = \text{integer}) \quad (7.36)$$

Similarly, if $n \neq \text{integer}$ and:

$$R(Q^2) = \int_{0}^{Q^2} n \frac{d\mu^2}{\mu^2} \alpha_{eff}(\mu^2) \left(\frac{\mu^2}{Q^2}\right)^n \quad (7.37)$$

the associated Euclidean quantity is as in eq.(6.8), and one gets (eq.(6.42)) (again assuming the $b_{p}^{NP}$'s vanish for integer $p$):

$$\delta D_{NP}(Q^2) \simeq K_n^{NP} \left(\frac{\Lambda^2}{Q^2}\right)^n \quad (7.38)$$
while:
\[ \delta R_{NP}(Q^2) \simeq b_n^{NP} \left( \frac{\Lambda^2}{Q^2} \right)^n \] (7.39)

In this case eq.(7.31) implies:
\[ K_n^{NP} = \frac{\pi n}{\sin(\pi n)} b_n^{NP} \quad (n \neq \text{integer}) \] (7.40)

(which is the exact replica of eq.(A.16) of Appendix A!), i.e.:
\[ \int_0^\infty \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \delta \alpha_s^{NP}(k^2) = \frac{\pi n}{\sin(\pi n)} \int_0^\infty \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\Lambda^2} \right)^n \delta \alpha_{e\text{ff}}^{NP}(\mu^2) \quad (n \neq \text{integer}) \] (7.41)

(eq.(7.41) is valid for \(0 < n < 1\) even if \(\delta \alpha_s^{NP}(k^2) = \mathcal{O}(\Lambda^2/k^2)\) at large \(k^2\) since the integral on the left hand side of eq.(7.41) is still UV convergent in this case).

Alternatively, eq.(7.35) and (7.40) may be derived by comparing the two expressions for \(\delta D_{NP}(Q^2)\) obtained from the equivalent representations eq.(2.10a) and (2.10b), choosing \(D(Q^2)\) as in eq.(6.8). Eq.(2.10a) yields eq.(7.38). On the other hand, using eq.(6.31) and (6.32) for \(n\) integer and \(0 < n < 1\) respectively, one gets from eq.(2.10b):
\[ \delta D_{NP}(Q^2) \simeq -(-1)^n n \, b_n^{NP} \left( \frac{\Lambda^2}{Q^2} \right)^n \] (7.42)

if \(n = \text{integer}\), and:
\[ \delta D_{NP}(Q^2) \simeq \frac{\pi n}{\sin(\pi n)} b_n^{NP} \left( \frac{\Lambda^2}{Q^2} \right)^n \] (7.43)

if \(0 < n < 1\), which reproduce eq.(7.35) and (7.40) upon comparison with eq.(7.38). The relations eq.(7.36) and (7.41) may be useful, since the moments are treated in [7] as fit parameters which constrain the shape of \(\delta \alpha_{e\text{ff}}^{NP}(\mu^2)\), hence \(\delta \alpha_s^{NP}(k^2)\), and it may be easier to find a fit for the latter quantity than for the former (which must be a complicated oscillating function to satisfy the constraint that its integers moments \(b_n^{NP}\) (eq.(6.38)) vanish).

8. Applications

8.1. Hadronic width of the \(\tau\) lepton

It is usually expressed in term of the quantity \(R_\tau\), itself related to the total \(e^+e^-\) annihilation cross-section into hadrons \(R_{e^+e^-}\) and to the Adler \(D_{e^+e^-}\) function by:
\[ R_\tau(m_\tau^2) = 2 \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left( 1 - \frac{s}{m_\tau^2} \right)^2 \left( 1 + 2 \frac{s}{m_\tau^2} \right) R_{e^+e^-}(s) \]
\[ = \frac{1}{2\pi i} \int_{|s|=m_\tau^2} \frac{ds}{s} \left( 1 - \frac{s}{m_\tau^2} \right)^3 \left( 1 + \frac{s}{m_\tau^2} \right) D_{e^+e^-}(s) \] (8.1)
In the dressed single gluon exchange approximation one has (with the parton model normalization removed):

\[ R_{e^+e^-}(Q^2) = 1 + \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \mathcal{F}_{e^+e^-}(\frac{\mu^2}{Q^2}) \]  

(8.2)

where \( \mathcal{F}_{e^+e^-} \) has been computed in [6,7,18]. Eq.(8.1) and (8.2) imply:

\[ R_\tau(m_\tau^2) = 1 + \int_0^\infty \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \mathcal{F}_\tau(\frac{\mu^2}{m_\tau^2}) \]  

(8.3)

with [7,18]:

\[ \mathcal{F}_\tau(\frac{\mu^2}{m_\tau^2}) = 2 \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(1 + 2 \frac{s}{m_\tau^2}\right) \mathcal{F}_{e^+e^-}(\frac{\mu^2}{s}) \]  

(8.4)

In particular, one finds [7] in the small \( \mu^2 \) limit:

\[ \mathcal{F}_\tau(\frac{\mu^2}{m_\tau^2}) - \mathcal{F}_\tau(0) \simeq -d_1^\tau \frac{\mu^2}{m_\tau^2} + ... \]  

(8.5)

with:

\[ d_1^\tau = \frac{16}{3\pi}(4 - 3\zeta(3)) \]  

(8.6)

which implies a leading \( 1/m_\tau^2 \) power correction\(^c\) of UV origin:

\[ \delta R_\tau(m_\tau^2) \simeq b_1 d_1^\tau \frac{\Lambda^2}{m_\tau^2} \]  

(8.7)

According to the discussion of section 7, this term originates directly from the integral on the circle (eq.(8.1)) of a corresponding leading UV \( \mathcal{O}(1/Q^2) \) term in \( \delta D_{e^+e^-}(Q^2) \) (such a term is present [6] in the large \( \beta_0 \) limit in \( \delta D_{e^+e^-}^{\text{PT}}(Q^2) \)). For a numerical estimate, assume:

\[ b_1 \simeq b_1^{\text{PT}}|\text{one - loop} = -\frac{1}{\beta_0} \]

and take: \( \Lambda = \Lambda_V = 2.3\Lambda_{\overline{\text{MS}}} \) (this choice of parameters corresponds to the large \( \beta_0 \) estimate [6] for \( \delta R_\tau^{\text{PT}} \)). Then one gets (for 3 flavors), assuming \( \alpha_s^{\text{MS}}(m_\tau^2) = 0.32 \):

\[ \delta R_\tau(m_\tau^2) \simeq -0.063 \]

\(^c\)While this paper was in writing, I learned about the article [37], where the presence of \( 1/m_\tau^2 \) terms arising from “dispersive regularization” of the coupling is also pointed out. The implementation of this idea is however very different from the present one: there it is applied directly to the “effective charge” defined by the \( D_{e^+e^-} \) function itself. I thank A. Kataev for bringing this reference to my attention.
which represents a sizable correction with respect to the (principal-value) Borel sum estimate [6] (still in the large $\beta_0$ limit): $R_\tau(m_2^2) - 1 \simeq 0.227$, or to the experimental value [38] $R_\tau(m_2^2) - 1 \simeq 0.20$.

Note also that a corresponding $1/Q^2$ power correction is absent from $R_{\tau^+\tau^-}(Q^2)$ (for which the leading power correction (of UV origin) is only $\mathcal{O}(1/Q^4)$ [7]).

8.2. Gluon condensate on the lattice

Power corrections of UV origin may be relevant for the lattice determination of the gluon condensate, since they are likely to affect the so-called “perturbative tail”. To see this, consider the following model [39] for the lattice plaquette $W(\alpha)$:

$$Q^4 \times W(\alpha) = \int_0^{Q^2} dk^2k^2\alpha_s(k^2/Q^2, \alpha)$$

where $Q$ is the UV cutoff (of the order of the inverse lattice size) and $\alpha$ the bare coupling constant. Eq.(8.8) is a peculiar case of eq.(6.8) for $n = 2$; with the renormalized coupling $\alpha_s$ given by eq.(2.13), one expects the short distance expansion of $W(\alpha)$ to involve $\mathcal{O}(1/Q^2)$ contributions which can screen the “physical”, “genuine non-perturbative” gluon condensate $\mathcal{O}(1/Q^4)$ contribution. Indeed one gets:

$$W(\alpha) = W_{PT}(\alpha) + \delta W(\alpha)$$

where $Q^4 \times W_{PT}(\alpha)$ is the Borel sum which defines the quartically divergent “perturbative tail” of eq.(8.8), while (see eq.(6.20) and Appendix A2):

$$Q^4 \times \delta W(\alpha) = b_1\Lambda^2Q^2 - (b_2 \ln \frac{Q^2}{\Lambda^2} + \tilde{D}_2 \pm i\pi b_2^{PT})\Lambda^4 + \mathcal{O}(1/Q^2)$$

is the subdominant “power correction” term, which involves still quadratic and logarithmic divergences. Eq.(8.10) suggests that, after the ambiguous $\pm i\pi b_2^{PT}$ imaginary part has been absorbed into the Borel integral , the first detected correction to the resulting (principal value regularized) Borel sum of the perturbative tail may be the $b_1\Lambda^2Q^2$ quadratically divergent term (such a term may even have been detected in a preliminary analysis [40] of the results of [39], where 8 orders of the perturbative expansion of $W_{PT}(\alpha)$ have been computed). Furthermore, the UV finite “gluon condensate” is buried into the constant $\tilde{D}_2$, which contains both “perturbative” and “non-perturbative” contributions from $\delta\alpha_s^{PT}$ and $\delta\alpha_s^{NP}$ respectively (I have set $\tilde{D}_2 \triangleq D_2^{PT} + D_2^{NP}$). It appears however impossible to fix this constant independently of the (arbitrary) choice of the scale $\Lambda$ inside the log divergence, and to separate contributions of IR and UV origin in $\tilde{D}_2$. The condition $b_2 = 0$ (i.e. the absence of such a divergence) thus appears as a minimal requirement for an unambiguous definition of the condensate as a quantity of genuine IR origin. The situation
here appears even more severe then in the case $b_2 = 0$, where the definition of the condensate depends on the (partially arbitrary) \cite{41} definition of the “regularized sum” (through principal value prescription or else) \cite{21,22} of perturbation theory, and its extraction from lattice data already faces serious difficulties \cite{42}.

9. Conclusion

In this paper, I have given arguments to support the existence, within the dispersive approach \cite{7}, of power corrections, some of them of short distance origin (hence unrelated to - thus not inconsistent with - the OPE), which appear naturally when the Landau singularity in the running coupling is removed. For an euclidean observable $D(Q^2)$, the situation can be summarized as follows: introducing an IR cut-off at $k^2 = \Lambda_I^2$ as in section 6, one can split eq.(2.1) into a “long distance” and a “short distance” part:

\[ D(Q^2) = \int_{0}^{\Lambda_I^2} \frac{dk^2}{k^2} \alpha_s(k^2) \varphi \left( \frac{k^2}{Q^2} \right) + \int_{\Lambda_I^2}^{\infty} \frac{dk^2}{k^2} \alpha_s^{PT}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) + \int_{\Lambda_I^2}^{\infty} \frac{dk^2}{k^2} \delta \alpha_s(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \]

which represents an example of OPE “a la SVZ” \cite{30-32}. In the short distance part, I have further split the IR regular coupling $\alpha_s$ into a “perturbative” and a “power correction” piece (eq.(2.13)). The long distance part yields, for large $Q^2$, power corrections, which one can parametrize \cite{8} with the IR regular coupling, and are consistent with the OPE. The integral over the perturbative coupling in the short distance part represents a form of “regularized perturbation theory” \cite{21,22} (choosing the IR cut-off $\Lambda_I$ above the Landau singularity $\Lambda$ of $\alpha_s^{PT}$). The last integral in eq.(9.1) yield at large $Q^2$ the new power corrections of short distance origin discussed in this paper, unrelated to the OPE. Equivalently, eq.(9.1) can be rewritten as:

\[ D(Q^2) = \int_{0}^{\infty} \frac{dk^2}{k^2} \alpha_s^{PT}(k^2) \varphi \left( \frac{k^2}{Q^2} \right) + \int_{0}^{\Lambda_I^2} \frac{dk^2}{k^2} \delta \alpha_s(k^2) \varphi \left( \frac{k^2}{Q^2} \right) + \int_{\Lambda_I^2}^{\infty} \frac{dk^2}{k^2} \delta \alpha_s(k^2) \varphi \left( \frac{k^2}{Q^2} \right) \]

which shows that the (ambiguous) Borel sum of perturbation theory (the first integral on the right hand side of eq.(9.2)) receive two types of power corrections at large $Q^2$: the long distances ones (the second integral), which correspond to the standard OPE “condensates” contribution \cite{20} (and contain both an ambiguous, “perturbative” component coming from the $\delta \alpha_s^{PT}$ piece of $\delta \alpha_s$ and a “genuine non-perturbative” component from the $\delta \alpha_s^{NP}$ piece); and those arising from short distances (the last integral). If the short distance power corrections are neglected \cite{8} (i.e. if one assumes that $\delta \alpha_s(k^2)$ is essentially a low $k^2$ modification and decreases sufficiently fast at large $k^2$), one recovers the standard view (see e.g. \cite{6}, \cite{22}) that the first correction to the Borel sum is given by the OPE.

Since the $\delta \alpha_s^{PT}$ piece (which eliminates the Landau singularity) is however a priori not restricted to low $k^2$, it induces “perturbative power corrections” $\delta D_{PT}$ which arise
both from long distances (where they remove the IR renormalons ambiguities of the Borel sum) and from short distances. They stand on the same level as radiative corrections, and are best looked upon as part of the “correct” resummation prescription of perturbation theory, yielding a “regularized perturbation theory” [21,22] which differs from Borel summation (even barring IR renormalons problems). There is no contradiction with the OPE, which does not require that all power contributions be of long distance origin.

The occurrence of “non-OPE”, “short-distance” power contributions is even more conspicuous from the “\(\alpha_{\text{eff}}\) representation” eq.(2.17). Performing an analogue split at \(\mu^2 = \Lambda^2\) as in section 7.1, one gets:

\[
D(Q^2) \simeq \int_0^{\Lambda^2} \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}(\mu^2) \hat{\mathcal{F}}(\mu^2/Q^2) + \int_{\Lambda^2}^{\infty} \frac{d\mu^2}{\mu^2} \alpha_{\text{eff}}^{PT}(\mu^2) \hat{\mathcal{F}}(\mu^2/Q^2) \tag{9.3}
\]

where \(\alpha_{\text{eff}} \equiv \alpha_{\text{eff}}^{PT} + \delta\alpha_{\text{eff}}^{NP}\) (both contributions are assumed to be IR regular), and the \(\delta\alpha_{\text{eff}}^{NP}\) piece has been neglected in the high \(\mu^2\) integral, since I assume it is exponentially suppressed there. Again, the low \(\mu^2\) integral generates power contributions at large \(Q^2\), which one can parametrize [8] with low \(\mu^2\) moments of the total IR regular effective coupling \(\alpha_{\text{eff}}\) (after expanding \(\hat{\mathcal{F}}\)). But it is clear that any term in the low \(\mu^2\) expansion of \(\hat{\mathcal{F}}\) (either analytic or non-analytic) can a priori contribute a power correction, whereas only the non-analytic terms are related (section 6) to OPE and long distances [4,6,7]. It seems artificial to eliminate the analytic contributions, which are associated to short distances [4,6], and require the first few analytic low \(\mu^2\) moments of \(\alpha_{\text{eff}}\) to vanish; furthermore, they certainly cannot vanish if, as is likely, \(\alpha_{\text{eff}}\) remains positive at low scales (this requirement looks more plausible, as argued below, if one postulate it [7] for the \(\delta\alpha_{\text{eff}}^{NP}\) piece only)!

I should stress that the short distance “perturbative power corrections” here discussed should not be confused with the effective \(1/Q^2\) power correction which represents the estimate [32,43,44] of the effect of UV renormalons on the remainder of the Borel sum when the perturbative expansion is truncated at its minimum term: even if the Borel summation is performed exactly [36] (using, say, a principal part prescription) the short distance \(1/Q^2\) power correction in \(\delta D_{PT}\) still remains (moreover they are also present in observables with an UV cut-off at \(Q^2\), such as \(D(Q^2)\) in eq.(6.8), or the lattice plaquette (section 8.2), which do not have any UV renormalons!).

The occurrence of power corrections to the Borel sum arising from “IR regularized” couplings has been noted before in [6] (the resulting resummation for Minkowskian observables has also been considered in [18]). The point of view put forward here however departs from the one in [6], which disfavor the use of IR regular couplings such as \(\alpha_{\text{PT,reg}}\) on the ground of the assumption [6,22] that the leading power correction to the (Borel-summmed) perturbation theory should be given by the OPE: as argued in this paper, the framework of [7], although not in contradiction with the OPE,

\(d\)This condition cannot be imposed [33] on all of them, since they are defined on a finite interval.
does suggest the opposite assumption as a natural alternative, since the dispersive regularization [10-13] of the coupling generates “for free” (eq.(2.17)) the “minimal” power corrections necessary to remove the Landau singularity. Furthermore, there is then no a priori reason that the genuine “non-perturbative” modification $\delta \alpha_s^{NP}(k^2)$ be itself restricted to low $k^2$, i.e. that the $b_n^{NP}$'s (eq.(6.38)) vanish. The split eq.(2.14) of $\delta \alpha_s$ into a “perturbative” and a “non-perturbative” component then becomes to some extent a matter of convention, and such is the split eq.(6.40) of the total $b_n$.

If one still insists on implementing the notion of “low energy modification”, two natural options appear. The first one assumes it should concern only the $\delta \alpha_s^{NP}$ part of the coupling; this choice leads to the picture of [7], where $b_n^{NP} = 0$ and the “genuine” non-perturbative power corrections are always consistent with the OPE, but where “perturbative” power corrections from $\delta \alpha_s^{PT}$ foreign to the OPE could remain. The proper definition of $\delta \alpha_s^{PT}$ becomes a physical question, rather then a matter of convention, since $\delta \alpha_s \simeq \delta \alpha_s^{PT}$ at large $k^2$ in this case. The alternative (more artificial in my opinion) is to have the total $\delta \alpha_s$ restricted to low $k^2$, which would make the present framework consistent with the standard view as explained above. This option requires that $b_n^{NP} = -b_n^{PT}$, so either the condition [7] $b_n^{NP} = 0$ has to be relaxed, or a proper redefinition of “$\delta \alpha_s^{PT}$” and “$\delta \alpha_s^{NP}$” has to be found (by reshuffling part of $\delta \alpha_s^{NP}$ into the new “$\delta \alpha_s^{PT}$”), such that “$b_n^{NP}” = “b_n^{PT}” = 0. Whether this can be achieved in a unique way at all is not clear. The point of view adopted here is that such questions should be decided by the data, rather then imposed a priori, and the $b_n$’s considered as free parameters.

An important qualitative difference between the “perturbative” and the “non-perturbative” power corrections, which could help defining a “correct” splitting of $\delta \alpha_s$, concerns the notion of “mismatch”[20] between radiative and power corrections. In many processes, it has been (apparently successfully) postulated [20] that the formally leading $O(\alpha_s)$ radiative corrections are actually numerically negligible (for $Q^2$ low enough, but still high enough to have convergence of the short distance expansion in inverse powers of $Q^2$) compared to the genuine “non-perturbative” power corrections. This “mismatch” is likely to be absent for the “perturbative” power corrections here considered, which are presumably of the same size as the radiative corrections. For instance, in the case of the regularized one-loop coupling eq.(3.1), one finds that for $k^2 > 2\Lambda^2$ the “radiative term” $\alpha_s^{PT}(k^2) = \frac{1}{\beta_0 \ln \frac{k^2}{\Lambda^2}}$ is of comparable size to the first “perturbative” power correction $\frac{1}{\beta_0} \frac{\Lambda^2}{k^2}$! Note also that “mismatch” between radiative and “non-perturbative” power corrections then requires also $\delta R_{PT}(Q^2) \ll \delta R_{NP}(Q^2)$ for $Q^2 \gg \Lambda^2$ (even if $\delta \alpha_s^{PT}(k^2) \gg \delta \alpha_s^{NP}(k^2)$ at large $k^2$), which gives a rough justification to the simultaneous neglect of radiative and “perturbative” power corrections implicitly performed in standard QCD sum rules analysis. It would be interesting to investigate the constraints on $\delta \alpha_s^{NP}$ necessary to implement the “mismatch”.

How unique is the present proposal? The choice of Borel summation to define the
“unregularized” sum of perturbation theory is as a rather natural one [45]. However, there is still considerable freedom in the definition of $\alpha_{s,\text{reg}}$ and $\delta \alpha_{s}^{PT}$. For instance, as an alternative to eq.(3.10), one might consider shopping-off the time-like discontinuity of the perturbative coupling at $\mu^2 = c^2 \Lambda^2$ instead of $\mu^2 = 0$:

$$\alpha_{s,\text{reg}}^{PT}(k^2) = - \int_{c^2 \Lambda^2}^{\infty} \frac{d\mu^2}{\mu^2 + k^2} \rho^{PT}(\mu^2)$$  \hspace{1cm} (9.4)

or even define $\alpha_{s,\text{reg}}^{PT}(k^2) \equiv \alpha_{s}^{PT}(k^2)$ (eq.(A.21)). The choice $\delta \alpha_{\text{eff}}^{PT} \equiv 0$ in eq.(2.16) appears still a very natural one, since one only disturbs in a minimal way the information contained in perturbation theory. Of course, as long as it is only a question of definition, it matters little what one calls “perturbative” and “non-perturbative”; but the distinction becomes a meaningful one once one starts postulating specific physical properties (such as “low-energy restriction” or “mismatch”) for any of these pieces.

Another possible limitation of the present proposal is the condition of exponential suppression of $\delta \alpha_{\text{eff}}(\mu^2)$: this excludes such familiar model as the “Richardson-like” type of IR finite coupling:

$$\alpha_{s}(k^2) = \frac{1}{\beta_0 \ln (c^2 + k^2/\Lambda^2)}$$  \hspace{1cm} (9.5)

($c^2 \geq 1$) where the large $k^2$ expansion of $\delta \alpha_{s}$ contains logs. Moreover, the possibility that $\alpha_{s}^{PT}$ might itself be afflicted by renormalons ambiguities is worrysome, since it makes the whole scheme more cumbersome, and with a less definite starting point. Nevertheless, given the generic possibility of power corrections unrelated to the OPE arising from the idea of a universal IR regular QCD coupling, it would be worthwhile to develop practical ways to get phenomenological evidence for presence or absence of such terms.

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Appendices

A Some results on the dispersively regularized coupling

Assuming $\alpha_s^{PT}(k^2)$ has no complex singularities, it was shown in section 7 that in the time-like region ($Q^2 > 0$) $\text{Disc}\{D_{\text{reg}}^{PT}(-Q^2)\} = \text{Disc}\{D_{\text{PT}}(-Q^2)\}$, which reflects the basic assumption of the dispersive regularization method that $\text{Disc}\{\alpha_s^{PT}(\mu^2)\} \equiv \text{Disc}\{\alpha_s^{PT}(-\mu^2)\}$. This observation, which can equivalently be expressed as:

$$\text{Disc}\{\delta D_{PT}(-Q^2)\} \equiv 0 \quad (Q^2 > 0) \quad (A.1)$$

has two interesting consequences, when applied to the observable of eq.(6.8): i) it leads to a relation between the $b_n^{PT}$'s and the IR renormalons residues in $\tilde{D}(z)$, and ii) it implies the universality of the one-loop value of the IR fixed point:

$$\alpha_s^{PT}(k^2 = 0) = \alpha_{\text{eff}|IR}^{PT} = \frac{1}{\beta_0} \quad (A.2)$$

which is valid for a general coupling, provided $\tilde{\alpha}_s(z)$ has no renormalons, and $\alpha_s^{PT}(k^2)$ has no complex singularities and vanishes for $k^2 \to 0$.

A1 Universality of $\alpha_{\text{eff}|IR}^{PT}$

Let us first consider point ii). If $0 < n < 1$, eq.(A.1) determines the phase of $K_n^{PT}$ (eq.(6.27)) as:

$$K_n^{PT} = (-1)^n K_n \quad (A.3)$$

with $K_n$ real, in order to give in eq.(6.26) a contribution:

$$\delta D_{PT}(Q^2) \supset K_n \left(-\frac{\Lambda^2}{Q^2}\right)^n \quad (A.4)$$

unambiguous in the time-like region ($Q^2 < 0$). Since the imaginary part of $K_n^{PT}$ must cancel the one in $D_{PT}(Q^2)$ generated by the IR renormalon, one deduces that in the space-like region ($Q^2 > 0$):

$$\frac{1}{\pi} \text{Im} D_{PT}(Q^2) = K_n \frac{\sin(\pi n)}{\pi} \left(\frac{\Lambda^2}{Q^2}\right)^n \quad (0 < n < 1) \quad (A.5)$$

On the other hand, $\frac{1}{\pi} \text{Im} D_{PT}(Q^2)$ is related (for any $n$) to the IR renormalon residue $K_n$ of the ordinary Borel transform $D(z)$ (in e.g. the scheme where the inverse $\beta$ function has only two terms) by [22,23]:

$$\frac{1}{\pi} \text{Im} D_{PT}(Q^2) = \left[K_n \frac{1}{\Gamma(1+\delta)} \right] \left(\frac{\tilde{\Lambda}^2}{Q^2}\right)^n \quad (A.6)$$
where, for $z \to z_n$ (assuming $\tilde{\alpha}_s(z)$ has no renormalon):

$$D(z) \simeq \frac{\bar{K}_n}{(1 - \frac{z}{z_n})^{1+\delta}} \quad (A.7)$$

with $[30] \; \delta = \frac{\delta_0}{\beta_0} z_n$, and $\bar{\Lambda}/\Lambda$ is an $n$-independant constant. Eq.(A.7) follows e.g. from the relation [27] between ordinary and modified Borel transforms singularities and eq.(6.5b), which yields the $RS$ invariant Borel transform:

$$\tilde{D}(z) = \tilde{\alpha}_s(z) \frac{1}{1 - \frac{z}{z_n}} \quad (A.8)$$

hence, for $z \to z_n$ (if $\tilde{\alpha}_s(z)$ has no renormalon):

$$\tilde{D}(z) \simeq \frac{\bar{K}_n}{1 - \frac{z}{z_n}} \quad (A.9)$$

with:

$$\bar{K}_n = \tilde{\alpha}_s(z_n) \quad (A.10)$$

Comparing eq.(A.5) and (A.6), and letting $n \to 0$, one gets:

$$K_0 = \frac{\bar{K}_0}{\beta_0} = \frac{1}{\beta_0} \quad (A.11)$$

where the last step follows also from [27] which implies, for $n \to 0$ (hence $z_n \to 0$), that $\bar{K}_0 = \bar{K}_0$, hence:

$$\bar{K}_0 = \tilde{\alpha}_s(z = 0) = 1 \quad (A.12)$$

On the other hand, eq.(6.27) and (A.3) imply:

$$K_0 = K_0^{PT} = \left[ \int_0^\infty n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \delta \alpha_s^{PT}(k^2) \right]_{n \to 0} = \alpha_{eff|IR}^{PT} \quad (A.13)$$

where the last equality is a consequence of the assumption that $\alpha_s^{PT}(k^2 = 0) = 0$, since then $\delta \alpha_s^{PT}(k^2 = 0) = \alpha_{eff|IR}^{PT}$ (eq.(3.15)), and of the observation that the integral in eq.(A.13) is dominated for $n \to 0$ by the $k^2 \to 0$ region (where it is IR divergent). Eq.(A.11) and (A.13) prove eq.(A.2). This argument explains and extends to all orders the stability of $\alpha_{eff|IR}^{PT}$ with respect to higher order corrections pointed out in [12] (where a general result has also been announced). Note it is crucial that $\tilde{\alpha}_s(z)$ has no renormalons, and $\alpha_s^{PT}(k^2)$ no complex singularities. Otherwise, one could start from an arbitrary $\alpha_{eff}(\mu^2)$, with an arbitrary $\alpha_{eff|IR}^{PT}$, and reconstruct $\alpha_{s,reg}^{PT}(k^2)$ via the dispersion relation eq.(3.10b). But the resulting $\tilde{\alpha}_s(z)$ would in general have renormalons (from eq.(4.6)), and even if this is avoided by having $\tilde{\alpha}_{eff}(z)$ vanish at $z = z_n$, it is not guaranteed the resulting $\alpha_s^{PT}(k^2)$ from eq.(4.1) has no complex singularities,
or vanishes at \( k^2 = 0 \)!. These conditions are in particular satisfied if \( \alpha_s^{PT}(k^2) \) is the sum of an arbitrary finite number of two-loop ’t Hooft couplings (eq.(3.5)).

**A2 Relation between \( b_n^{PT} \) and the IR renormalons residues**

To prove point i), assume first \( n \) is an integer. Then eq.(A.1) requires similarly that the ambiguous imaginary part of the constant \( D_n^{PT} \) in eq.(6.20) is given by \( \pm i \pi b_n^{PT} \left( -\frac{\Lambda^2}{Q^2} \right)^n \), so that it can be merged with the log to give a contribution:

\[
\delta D_{PT}(Q^2) \supset -n b_n^{PT} \ln \left( -\frac{Q^2}{\Lambda^2} \right) \left( -\frac{\Lambda^2}{Q^2} \right)^n \quad (A.14)
\]

unambiguous in the time-like region \( (Q^2 < 0) \). Since the imaginary part of \( D_n^{PT} \) must cancell the one in \( D_{PT}(Q^2) \) generated by the IR renormalon, one deduces that in the space-like region \( (Q^2 > 0) \):

\[
\frac{1}{\pi} \text{Im} D_{PT}(Q^2) = n b_n^{PT} \left( -\frac{\Lambda^2}{Q^2} \right)^n \quad (A.15)
\]

which relates \( b_n^{PT} \) to the \( z = z_n \) IR renormalon residue of the standard integral eq.(6.8) through eq.(A.6) (eq.(A.15) also suggests the IR renormalon residue should vanish if \( b_n^{PT} = 0 \), so that there should be no discontinuity neither in the space-like nor in the time-like region). Furthermore, eq.(A.15) is also valid for \( n \neq \text{integer} \) (defining a complex \( b_n^{PT} \) either through eq.(3.14) or eq.(4.13)). This statement follows immediately from eq.(A.3) and (A.5) and the relation:

\[
K_n^{PT} = \frac{\pi n}{\sin(\pi n)} \ b_n^{PT} \quad (0 < n < 1) \quad (A.16)
\]

To prove the latter, one can either use eq.(3.12) into eq.(6.27) and compare to eq.(3.14) with the help of the identity eq.(6.33), or start from the relation:

\[
\delta \alpha_s^{PT}(k^2) = \alpha_{\text{reg},<}^{PT}(k^2) - \alpha_<^{PT}(k^2) \quad (A.17)
\]

with:

\[
\alpha_{\text{reg},<}^{PT}(k^2) \equiv k^2 \int_0^{\Lambda^2} \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{\text{eff}}^{PT}(\mu^2) \quad (A.18)
\]

and:

\[
\alpha_<^{PT}(k^2) \equiv \int_0^\infty dz \ \tilde{\alpha}_{\text{eff}}(z) \left[ k^2 \int_0^{\Lambda^2} \frac{d\mu^2}{(\mu^2 + k^2)^2} \exp \left( -z\beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \right] \quad (k^2 > \Lambda^2) \quad (A.19)
\]

To derive eq.(A.17), one splits the dispersive integral eq.(3.10b) at \( \mu^2 = \Lambda^2 \), and write:

\[
\alpha_s^{PT}(k^2) = \alpha_{\text{reg},<}^{PT}(k^2) + \alpha_<^{PT}(k^2) \quad (A.20)
\]
where:
\[
\alpha^{PT}(k^2) \equiv k^2 \int_{\Lambda^2}^{\infty} \frac{d\mu^2}{(\mu^2 + k^2)^2} \alpha_{eff}^{PT}(\mu^2)
\]  
(A.21)
and uses the Borel representation eq.(4.4) of \(\alpha_{eff}^{PT}(\mu^2)\) to get:
\[
\alpha^{PT}(k^2) = \alpha^{PT}_s(k^2) - \alpha^{PT}_< (k^2)
\]  
(A.22)
Assume now \(0 < n < 1\). Upon insertion of eq.(A.17), eq.(6.27) becomes:
\[
K^{PT}_n = \int_{0}^{\infty} n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \alpha^{PT}_{reg,<}(k^2) - \int_{0}^{\infty} n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \alpha^{PT}(k^2)
\]  
(A.23)
But, permutting the \(k^2\) and \(\mu^2\) integrations, one gets:
\[
\int_{0}^{\infty} n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \alpha^{PT}_{reg,<}(k^2) = \int_{0}^{\Lambda^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\Lambda^2} \right)^n \alpha_{eff}^{PT}(\mu^2) \left[ \mu^2 \int_{0}^{\infty} \frac{dk^2}{(\mu^2 + k^2)^2} \left( \frac{k^2}{\mu^2} \right)^n \right]
\]  
(A.24)
whereas:
\[
\int_{0}^{\infty} n \frac{dk^2}{k^2} \left( \frac{k^2}{\Lambda^2} \right)^n \alpha^{PT}_<(k^2) = \int_{0}^{\infty} dz \tilde{\alpha}_{eff}(z) \int_{0}^{\Lambda^2} n \frac{d\mu^2}{\mu^2} \left( \frac{\mu^2}{\Lambda^2} \right)^n \exp \left( -z\beta_0 \ln \frac{\mu^2}{\Lambda^2} \right) \times \left[ \mu^2 \int_{0}^{\infty} \frac{dk^2}{(\mu^2 + k^2)^2} \left( \frac{k^2}{\mu^2} \right)^n \right]
\]  
(A.25)
which, together with eq.(4.13), proves eq.(A.16). In deriving eq.(A.25), I have freely used the Borel representation (eq.(A.19)) of \(\alpha^{PT}_s(k^2)\) inside the integral on the left hand side of eq.(A.25) down to \(k^2 = 0\) (although it is valid, similarly to eq.(4.1), only for \(k^2 > \Lambda^2\)), and permuted the order of \(k^2\) and \(z\) integrations. This procedure is similar to the one which gives the “correct” result eq.(6.5) for the integral eq.(6.3) over \(\alpha^{PT}_s(k^2)\), and its justification [23] is essentially the same: namely \(\alpha^{PT}_<(k^2)\) also contains a Landau singularity (which cancels (eq.(A.22)) the one in \(\alpha^{PT}_s(k^2)\), since \(\alpha^{PT}_<(k^2)\) is IR regular).
Note that, comparing eq.(A.15) and (A.6), and letting \(n \to 0\), one gets:
\[
b^{PT}_0 = \frac{\tilde{K}_0}{\beta_0} = \frac{1}{\beta_0}
\]  
(A.26)
But assuming \(\alpha^{PT}_s(k^2 = 0)\) vanishes implies (eq.(3.15)) \(b^{PT}_0 = \alpha^{PT}_{effIR}\) (this relation can also be derived from eq.(4.13) ), which gives an alternative proof of eq.(A.2).
B Expressions for power corrections in term of $\delta \alpha_s$

Although a Minkowskian quantity $R(Q^2)$ cannot be parametrized directly in term of $\alpha_s$ through a real integral representation similar to eq.(2.1), it may be interesting to point out that simple expressions do exist for the power corrections $\delta R_{PT}$ and $\delta R_{NP}$ themselves in term of $\delta \alpha_s^{PT}$ and $\delta \alpha_s^{NP}$ respectively. In the former case, they lead to an alternative proof of the Beneke-Braun formula [6].

B1 Perturbative power corrections

The result follows from a comparison of the similar expressions eq.(7.22) and (A.17) for $\delta R_{PT}$ and $\delta \alpha_s^{PT}$. As noted in section 7, for $Q^2 > \Lambda^2$ $\delta R_{PT}(Q^2)$ depends only on $F(\frac{\mu^2}{Q^2})$, the “low energy” characteristic function (eq.(7.1)). I shall assume the latter satisfies a (subtracted) dispersion relation. In general subtractions terms are present, since there is no constraint on the high $\mu^2$ behavior of $F(-)$. 

i) Consider first the case of one subtraction at $\mu^2 = 0$: then $F(-)$ is given by (see eq.(6.30)):

$$F(-)(\frac{\mu^2}{Q^2}) - F(-)(0) = \int_0^\infty \frac{dk^2}{k^2} \frac{-\mu^2}{1 + \frac{\mu^2}{k^2}} \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

(B1)

Taking the $\mu^2$ derivative and inserting the result into the expressions eq.(7.18) and (7.20) for $R_{PT,reg}^{PT}$ and $R_{<}^{PT}$ one gets immediately, after permutting the integrals and using eq.(A.18):

$$R_{PT,reg}^{PT}(Q^2) = \int_0^\infty \frac{dk^2}{k^2} \alpha_{PT}^{PT}(k^2) \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

(B.2)

whereas, using eq.(A.19) down to $k^2 = 0$ (this is again justified because $\alpha_s^{PT}(k^2)$ contains a Landau singularity) one finds:

$$R_{<}^{PT}(Q^2) = \int_0^\infty \frac{dk^2}{k^2} \alpha_{<}^{PT}(k^2) \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

(B.3)

Subtracting eq.(B.3) from eq.(B.2) one ends up with:

$$\delta R_{PT}(Q^2) = \int_0^\infty \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

(B.4)

ii) Assume next two subtractions. Then:

$$F(-)(\frac{\mu^2}{Q^2}) - F(-)(0) = a_0 \frac{\mu^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left( \frac{-\mu^2}{1 + \frac{\mu^2}{k^2}} + \frac{\mu^2}{1 + \frac{\mu^2}{k^2}} \right) \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

(B.5)

where:

$$a_0 \equiv F'(-)(0)$$
(Eq.(B.1) and (B.5) are analogues of eq.(2.6); the discontinuity \( \varphi(\frac{k^2}{Q^2}) \) can however no more be interpreted as a Feynman diagram kernel, and neither should the subtraction terms necessarily be identified to the real contribution, and the dispersive integral to the virtual one). Following the same steps as in i), one gets:

\[
R_{\text{reg,}\,<}^{PT}(Q^2) = -a_0 I_1 \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \alpha_{\text{reg,}\,<}^{PT}(k^2) - I_1 \frac{\Lambda^2 k^2}{k^2 + Q^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.6}
\]

where \( I_1 \) is defined in eq.(4.14). Similarly, one finds:

\[
R_{\,<}^{PT}(Q^2) = -a_0 I_1^{PT} \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \alpha_{\,<}^{PT}(k^2) - I_1^{PT} \frac{\Lambda^2 k^2}{k^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.7}
\]

where \( I_1^{PT} \) is defined in eq.(4.15). One deduces:

\[
\delta R_{PT}(Q^2) = -a_0 b_1^{PT} \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) - b_1^{PT} \frac{\Lambda^2 k^2}{k^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.8}
\]

with \( b_1^{PT} \) as in eq.(4.13). The subtraction term in the integrand insures convergence of the integral at infinity since \( \delta \alpha_s^{PT}(k^2) \simeq b_1^{PT} \frac{\Lambda^4}{k^2} + O\left(\frac{\Lambda^4}{k^2}\right) \) (eq.(3.13)). Generalization to an arbitrary number of subtractions is clear from eq.(B.4) and (B.8), which are analogues of eq.(6.4) for Minkowskian observables. Note that non-analytic terms in the large \( Q^2 \) expansion of \( \delta R_{PT}(Q^2) \) and renormalons come only from the dispersive integral in eq.(B.5), and are in one-to-one correspondence with terms in the low energy expansion of \( \varphi(\frac{k^2}{Q^2}) \).

iii) Extension to the general case where subtractions away from \( \mu^2 = 0 \) are required proceeds along similar lines. Assuming e.g. the second subtraction is at \( \mu^2 = Q^2 \):

\[
F(\frac{\mu^2}{Q^2}) - F(0) = a_1 \frac{\mu^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \frac{-\mu^2}{k^2 + Q^2} + \frac{\mu^2 k^2}{(k^2 + Q^2)^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.9}
\]

where:

\[
a_1 \equiv F(1)
\]

one finds:

\[
R_{\text{reg,}\,<}^{PT}(Q^2) = -a_1 I_1 \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \alpha_{\text{reg,}\,<}^{PT}(k^2) - I_1 \frac{\Lambda^2 k^2}{k^2 + Q^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.10}
\]

and:

\[
R_{\,<}^{PT}(Q^2) = -a_1 I_1^{PT} \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \alpha_{\,<}^{PT}(k^2) - I_1^{PT} \frac{\Lambda^2 k^2}{k^2 + Q^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.11}
\]

Hence:

\[
\delta R_{PT}(Q^2) = -a_1 b_1^{PT} \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) - b_1^{PT} \frac{\Lambda^2 k^2}{k^2 + Q^2} \right] \varphi(\frac{k^2}{Q^2}) \tag{B.12}
\]
The subtraction term in the integrand is easily managed, since eq.(B.12) can be put in the form:

$$\delta R_{PT}(Q^2) = -B_1 b_1^{PT} \Lambda^2 Q^2 + \int_0^{Q^2} \frac{dk^2}{k^2} \delta \alpha_s^{PT}(k^2) \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

$$+ \int_{Q^2}^{\infty} \frac{dk^2}{k^2} \left[ \delta \alpha_s^{PT}(k^2) - b_1^{PT} \Lambda^2 \right] \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

(B.13)

where:

$$B_1 = a_1 + \int_0^{Q^2} \frac{dk^2}{k^2} \frac{k^2 Q^2}{(k^2 + Q^2)^2} \varphi(-) \left( \frac{k^2}{Q^2} \right) + \int_{Q^2}^{\infty} \frac{dk^2}{k^2} \left[ \frac{k^2 Q^2}{(k^2 + Q^2)^2} - \frac{Q^2}{k^2} \right] \varphi(-) \left( \frac{k^2}{Q^2} \right)$$

is a number.

In the one-loop coupling case, eq.(B.4), (B.8) and (B.12) reproduce again the Beneke-Braun result eq.(6.29), provided one understands $F$ as being the analytic continuation of $F$ to the $\mu^2/Q^2 < 0$ region:

$$\delta R_{PT}(Q^2) = -\frac{1}{\beta_0} \left[ F(-) \left( -\frac{\Lambda^2}{Q^2} \right) - F(-)(0) \right]$$

(B.14)

Eq.(B.14) follows from the observation (which parallels a similar one in section 6) that the dispersion relations for $F(-)(\mu^2/Q^2) - F(-)(0)$ (eq.(B.1), (B.5) and (B.9)) may be seen as peculiar cases of the formulas for $\delta R_{PT}(Q^2)$ (eq.(B.4), (B.8) and (B.12) respectively), with the substitutions: $\Lambda^2 \rightarrow -\mu^2$ and $\delta \alpha_s^{PT}(k^2) \rightarrow \frac{k^2}{1+\frac{\mu^2}{k^2}}$. Note also the results of this section are easily extended to the rather general case where $F(-)(\mu^2/Q^2)$ can be written as the sum of a function which satisfies a dispersion relation (hence has no complex singularities) and a function analytic around the origin - a generalized “subtraction term” (but which may have complex singularities at finite distance from the origin).

### B2 Non perturbative power corrections

Formulas which allow an explicit connection of $\delta R_{NP}$ with $\delta \alpha_s^{NP}$ also exist, similar to those of Appendix B1 for $\delta R_{PT}$ (but valid only at large $Q^2$). Indeed, starting from the general expression:

$$\delta R_{NP}(Q^2) = \int_0^{Q^2} \frac{d\mu^2}{\mu^2} \delta \alpha_s^{NP}(\mu^2) \hat{F}(-)(\frac{\mu^2}{Q^2}) + \int_{Q^2}^{\infty} \frac{d\mu^2}{\mu^2} \delta \alpha_s^{NP}(\mu^2) \hat{F}(+)(\frac{\mu^2}{Q^2})$$

(B.15)

the assumed exponential decrease of $\delta \alpha_s^{NP}(\mu^2)$ allows to write, up to exponentially small corrections at large $Q^2$:

$$\delta R_{NP}(Q^2) \simeq \int_0^{Q^2} \frac{d\mu^2}{\mu^2} \delta \alpha_s^{NP}(\mu^2) \hat{F}(-)(\frac{\mu^2}{Q^2})$$

$$\simeq \int_0^{\infty} \frac{d\mu^2}{\mu^2} \delta \alpha_s^{NP}(\mu^2) \hat{F}(-)(\frac{\mu^2}{Q^2})$$

(B.16)
Assume now \( \mathcal{F}_{(-)}(\frac{\mu^2}{Q^2}) \) satisfies the twice subtracted dispersion relation at zero \( \mu^2 \) eq.(B.5). Then one finds after substitution in eq.(B.16), and using eq.(2.9):

\[
\delta R_{NP}(Q^2) \simeq -a_0 b_1^{NP} \frac{\Lambda^2}{Q^2} + \int_0^\infty \frac{dk^2}{k^2} \left[ \delta \alpha_s^{NP}(k^2) - b_1^{NP} \frac{\Lambda^2}{k^2} \right] \varphi(-) \left( \frac{k^2}{Q^2} \right)
\]  

(B.17)

Eq.(B.17) has exactly the same form as eq.(B.8) for \( \delta R_{PT}(Q^2) \), with the substitutions \( \delta \alpha_s^{PT} \rightarrow \delta \alpha_s^{NP} \), and \( b_1^{PT} \rightarrow b_1^{NP} \). A result similar to eq.(B.12), with the same substitutions, also holds if one starts from the dispersion relation eq.(B.9) with a subtraction at non-zero \( \mu^2 \), and of course the analogue of eq.(B.4), i.e.:

\[
\delta R_{NP}(Q^2) \simeq \int_0^\infty \frac{dk^2}{k^2} \delta \alpha_s^{NP}(k^2) \varphi(-) \left( \frac{k^2}{Q^2} \right)
\]  

(B.18)

holds if only one subtraction at \( \mu^2 = 0 \) is necessary (eq.(B.1)). It follows that similar formulas are also valid for the total power correction \( \delta R = \delta R_{PT} + \delta R_{NP} \), with the substitutions \( \delta \alpha_s^{PT} \rightarrow \delta \alpha_s = \delta \alpha_s^{PT} + \delta \alpha_s^{NP} \) and \( b_1^{PT} \rightarrow b_1 = b_1^{PT} + b_1^{NP} \).

If one now requires \( \delta \alpha_s^{NP} \) (resp. \( \delta \alpha_s \)) be restricted to low \( k^2 \), i.e. that \( b_n^{NP} = 0 \) (resp. \( b_n = 0 \)), then the subtraction terms in eq.(B.17) vanish, since they are proportional to \( b_1^{NP} \) (resp. \( b_1 \)) and one ends up with eq.(B.18) (or a similar result for \( \delta \bar{R} \)) whatever the subtractions needed for \( \mathcal{F}_{(-)} \). Comparing eq.(B.16) and (B.18), one gets the identity (valid only if \( \delta \alpha_s^{NP} \) is restricted to low \( k^2 \)):

\[
\int_0^\infty \frac{dk^2}{k^2} \delta \alpha_s^{NP}(k^2) \varphi(-) \left( \frac{k^2}{Q^2} \right) = \int_0^\infty \frac{d\mu^2}{\mu^2} \delta \alpha_s^{NP}(\mu^2) \dot{\mathcal{F}}_{(-)}(\frac{\mu^2}{Q^2})
\]  

(B.19)

from which eq.(7.36) and (7.41) may also be derived.
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