Twistor property of GKZ-hypergeometric systems

Takuro Mochizuki

Abstract

We study the mixed twistor $\mathcal{D}$-modules associated to meromorphic functions. In particular, we describe their push-forward and specialization under some situations. We apply the results to study the twistor property of a type of GKZ-hypergeometric systems, and to study their specializations. As a result, we obtain some isomorphisms of mixed TEP-structures in the local mirror symmetry.

Keywords: mixed twistor $\mathcal{D}$-module, generalized Hodge structure, polarization, GKZ-hypergeometric system, local mirror symmetry.

MSC2010: 14F10 32C38 32S35

1 Introduction

1.1 GKZ-hypergeometric systems and related $\mathcal{D}$-modules

Let $T^n$ denote the $n$-dimensional complex algebraic torus, i.e., $T^n = \{(t_1, \ldots, t_n) \mid t_i \in \mathbb{C}^*\}$. For any $a \in \mathbb{Z}^n$, we set $t^a := \prod_{i=1}^n t_i^{a_i}$. For any finite subset $\mathcal{A}$ in $\mathbb{Z}^n$, we consider the algebraic function $F_{\mathcal{A}} = \sum_{a \in \mathcal{A}} \alpha_a t^a$ on $T^n \times \mathbb{C}^A$, where $(\alpha_a \mid a \in \mathcal{A})$ is the standard coordinate system on $\mathbb{C}^A$. Let $L(F_{\mathcal{A}})$ denote the line bundle $\mathcal{O}_{T^n \times \mathbb{C}^A}$ with the flat connection $d + dF_{\mathcal{A}}$. It gives an algebraic holonomic $\mathcal{D}$-module on $T^n \times \mathbb{C}^A$, denoted by $L(F_{\mathcal{A}})$. Let $\pi_{\mathcal{A}} : T^n \times \mathbb{C}^A \rightarrow \mathbb{C}^A$ denote the projection. We obtain the holonomic algebraic $\mathcal{D}$-modules $\pi_{\mathcal{A}}^0 L(F_{\mathcal{A}}) (\ast = \ast, !)$ on $\mathbb{C}^A$.

We also have their reduction with respect to a natural $T^n$-action. Set $G_{\mathcal{A}} := \sum \alpha_a$ on $(\mathbb{C}^*)^A$, and let $L(G_{\mathcal{A}})$ denote the algebraic flat bundle $(\mathcal{O}_{(\mathbb{C}^*)^A}, d + dG_{\mathcal{A}})$. We have the $T^n$-action on $\mathbb{C}^A$ given by $(s_1, \ldots, s_n)(\alpha_a) = (s^{-a} \alpha_a)$. Let $S_{\mathcal{A}}$ denote the quotient space $(\mathbb{C}^*)^A/T^n$, and let $\pi_{\mathcal{A}}' : (\mathbb{C}^*)^A \rightarrow S_{\mathcal{A}}$ denote the projection. We obtain the $\mathcal{D}$-modules $\pi_{\mathcal{A}}^0 L(G_{\mathcal{A}}) (\ast = \ast, !)$. We have the open subset $S_{\mathcal{A}}^{reg}$ of $S_{\mathcal{A}}$ determined by a regularity condition on which $\pi_{\mathcal{A}}^0 L(G_{\mathcal{A}})$ are locally free $\mathcal{O}_{S_{\mathcal{A}}^{reg}}$-modules. The flat bundle $\pi_{\mathcal{A}}^0 L(G_{\mathcal{A}})|_{S_{\mathcal{A}}^{reg}}$ is particularly interesting.

Such $\mathcal{D}$-modules are isomorphic to a type of (reduced) GKZ-hypergeometric systems. (See [1], [14]. See also [39], [53].) One of the main purposes in this paper is to study the natural mixed twistor structure and some related structures on the $\mathcal{D}$-modules. The author hopes that our study will be useful to understand the generalized Hodge theoretic property of GKZ-hypergeometric systems.

Remark 1.1 Take a tuple of complex numbers $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$. We have the algebraic line bundle with a flat connection $L(F_{\mathcal{A}}, \beta) = (\mathcal{O}_{T^n \times \mathbb{C}^A}, d + dF_{\mathcal{A}} + \sum \beta_i dt_i/t_i)$, and we obtain the $\mathcal{D}$-modules $\pi_{\mathcal{A}}^0 L(F_{\mathcal{A}}, \beta)$ on $\mathbb{C}^A$, and their reduction on $S_{\mathcal{A}}$. Although we can study the mixed twistor structures on the $\mathcal{D}$-modules with the same method, we shall not discuss them in this paper.

1.2 TEP-structures associated to Landau-Ginzburg models

The GKZ-hypergeometric systems have been intensively studied from various viewpoints. One of the interesting is that they are closely related with some important objects in the mirror symmetry such as Frobenius manifolds, or more roughly TEP structures.

Let us consider the case where $\mathcal{A}$ is the set of the primitive vectors of the one dimensional cones in a fan of a smooth weak Fano projective toric variety $X$. Then, the family of Laurent polynomials $G_{\mathcal{A}}$ is the Landau-Ginzburg model corresponding to $X$. In this case, we have $\pi_{\mathcal{A}}^0 L(G_{\mathcal{A}}) = \pi_1^0 L(G_{\mathcal{A}})$, and the $\mathcal{D}$-module $\pi_{\mathcal{A}}^0 L(G_{\mathcal{A}})$
underlies the Frobenius manifold, or more roughly the TEP-structure associated to the Landau-Ginzburg model. (See \[8, 9, 10, 20, 40, 43,\] etc.)

Here, recall that a TE-structure on a complex manifold \(M\) in the sense of \[13\] is a locally free \(\mathcal{O}_{\mathbb{C}^4 \times M}\)-module \(V\) with a meromorphic flat connection \(\nabla : V \rightarrow V \otimes \mathcal{O}_{\mathbb{C}^4 \times M}(\log(\{0\} \times M))\). If it is equipped with a perfect pairing \(P : V \otimes j^*V \rightarrow \lambda^n \mathcal{O}_{\mathbb{C}^4 \times M}\) such that \(j^*P(a \otimes j^*b) = (-1)^n P(b \otimes j^*a)\) for \(n \in \mathbb{Z}\), then \((V, \nabla, P)\) is called a TEP-structure or more precisely TEP-structure on \(M\). Here, \(j : \mathbb{C} \times M \rightarrow \mathbb{C} \times M\) is given by \(j(\lambda, Q) = (-\lambda, Q)\). We shall often omit to denote \(\nabla\), i.e., \((V, \nabla, P)\) is denoted by \((V, P)\).

In the study on the mirror symmetry for weak Fano toric manifolds, with the notation of \[40\], T. Reichelt and C. Sevenheck obtained a TEP-structure \((V, P)\) on \(S_{\text{reg}}^A\) such that \(V_{\{1\} \times S_{\text{reg}}^A}\) is equal to \(\pi^0_{\bar{A}} L(G_A)|_{S_{\text{reg}}^A}\), and they proved a generalized Hodge theoretic property of \((V, P)\) in \[40\] Theorem 5.3] partially based on a work of C. Sabbah [42]. (See also the work of A. Douai and Sabbah [8, 9] on the algebraic construction of Frobenius manifolds, the work of H. Iritani [20] for the analytic construction of TEP-structure associated to Landau-Ginzburg models and the work of H. Fan [41] for the \(L^2\)-theoretic constructions.)

For more general \(A\), Reichelt and Sevenheck [41] studied the TE-structures and TEP-structures related to the GKZ-hypergeometric systems \(\pi^0_{\bar{A}} L(F_A)\) and \(\pi^0_{\bar{A}} L(G_A)\). They are motivated by the mirror symmetry for complete intersections in weak Fano toric manifolds. Let \(A\) be a fan corresponding to the toric manifold obtained as the direct sum \(\bigoplus_{i=1}^r L_i\), where \(L_i\) are nef line bundles on a weak Fano toric manifold \(X\). Inspired by the work of E. Mann and T. Mignon [31], Reichelt and Sevenheck constructed some TEP-structures on \(S_{\text{reg}}^A\). Namely, with the notation in [41] \(\{0\}\), they constructed a morphism of TE-structures \((\text{id} \times \phi)^* \tilde{\phi} : (\text{id} \times \phi)^* (\tilde{\mathcal{N}}_{\bar{A}}^{0,0}) \rightarrow (\text{id} \times \phi)^* (\tilde{\mathcal{M}}_{\bar{A}}^{0,0})\), whose restriction to \(\{1\} \times S_{\text{reg}}^A\) is equal to the morphism of \(D\)-modules \(\pi^0_{\bar{A}} L(G_A)|_{S_{\text{reg}}^A} \rightarrow \pi^0_{\bar{A}} L(G_A)|_{S_{\text{reg}}^A}\). The image \(\text{Im}(\text{id} \times \phi)^* \tilde{\phi}\) with a natural pairing is a TEP-structure. According to [31] and [41], roughly saying, \((\text{id} \times \phi)^* \tilde{\phi}\) is isomorphic to the reduced quantum \(D\)-module of the zero set of a general section of \(\bigoplus L_i\) on \(X\). (See [31] and [41] for a more precise statement.)

The TE-structures \((\text{id} \times \phi)^* (\tilde{\mathcal{N}}_{\bar{A}}^{0,0})\) and \((\text{id} \times \phi)^* (\tilde{\mathcal{M}}_{\bar{A}}^{0,0})\) and the TEP-structure \(\text{Im}(\text{id} \times \phi)^* \tilde{\phi}\) are constructed in a rather algebraic way. Pursuing the Hodge theoretic aspect of the mirror symmetry (see [25]), Reichelt-Sevenheck also constructed some TE-structures and a TEP-structure in a Hodge theoretic way, which are conjectured to equal \((\text{id} \times \phi)^* (\tilde{\mathcal{N}}_{\bar{A}}^{0,0})\), \((\text{id} \times \phi)^* (\tilde{\mathcal{M}}_{\bar{A}}^{0,0})\) and \((\text{id} \times \phi)^* \tilde{\phi}\) [41] Conjecture 6.13.

We shall give another Hodge theoretic construction of TE-structures and TEP-structures associated to the Landau-Ginzburg models, on the basis of the theory of mixed twistor \(D\)-modules (see [7]). In particular, when \(A\) is obtained from a fan corresponding to \(\bigoplus L_i\) as above, with the notation in [7.1] we shall construct a morphism of TE-structures \(\mathfrak{M}_{\bar{A}}^{\text{reg}} \rightarrow \mathfrak{M}_{\bar{A}}^{\text{reg}}\) on \(S_{\text{reg}}^A\), which underlies a morphism of mixed twistor \(D\)-modules. We also construct a TEP-structure \(\mathfrak{M}^{\text{reg}}_{\bar{A}}\) on \(S_{\text{reg}}^A\) such that (i) it underlies a pure twistor \(D\)-module, (ii) it is isomorphic to the image of \(\mathfrak{M}^{\text{reg}}_{\bar{A}}\) on \(S_{\text{reg}}^A\). We shall prove the following which exhibits a generalized Hodge theoretic property of the TE-structures and the TEP-structure in [41].

**Theorem 1.2 (Theorem 7.2)** We have the following commutative diagram:

\[
\begin{array}{ccc}
(id \times \phi)^* (\tilde{\mathcal{N}}_{\bar{A}}^{0,0}) & \xrightarrow{\cong} & \lambda^{-r} \mathfrak{M}^{\text{reg}}_{\bar{A}} \\
\downarrow & & \downarrow \\
(id \times \phi)^* (\tilde{\mathcal{M}}_{\bar{A}}^{0,0}) & \xrightarrow{\cong} & \lambda^{-r} \mathfrak{M}^{\text{reg}}_{\bar{A}}
\end{array}
\]  

As a result, \(\text{Im}(\text{id} \times \phi)^* \tilde{\phi}\) is isomorphic to \(\lambda^{-r} \mathfrak{M}^{\text{reg}}_{\bar{A}}\).

We shall also study the relation between the Hodge theoretic construction due to Reichelt-Sevenheck and our construction (14) and Proposition 6.67.

### 1.3 Specializations and an application to local mirror correspondence

Another main purpose in this paper is to study the specialization of the twistor structure and the polarization of GKZ-hypergeometric systems. When \(A := A \cup \{a_0\} \in \mathbb{Z}^n\), it is significant to clarify the relation among the mixed twistor \(D\)-modules over \(\pi^0_{\bar{A}} L(G_A)\) \((* = *, !)\) and \(\pi^0_{\bar{A}} L(G_A)\) \((* = *, !)\). It is motivated by the study of the local mirror symmetry [5, 20, 27, 28, 29].
For simplicity, let $X$ be a smooth weak Fano toric surface. Let $\mathcal{A}(X)$ be the set of the primitive vectors of the one dimensional cones in a fan corresponding to $X$. We have the algebraic function $\tilde{G}_{\mathcal{A}(X)} := G_{\mathcal{A}(X)} + 1$ on $(\mathbb{C}^*)^{\mathcal{A}(X)}$. In the local mirror symmetry, we are interested in the comparison of some objects associated to $X$ and $\tilde{G}_{\mathcal{A}(X)}$. Indeed, according to T.-M. Chiang, A. Klemm, S.-T. Yau and E. Zaslow [3], a generating function of the genus 0 local Gromov-Witten invariants of $X$ is a solution of the GKZ-hypergeometric systems given on $S_{\mathcal{A}(X)} = (\mathbb{C}^*)^{\mathcal{A}(X)}/T_2$ related to $\tilde{G}_{\mathcal{A}(X)}$, where the local Gromov-Witten invariants are roughly Gromov-Witten invariants for the canonical bundle $K_X$ of $X$. (See [5] for a more precise statement. See also [3], [27] and [50].)

For the mirror symmetry, according to A. Givental [16], (see also [20], [40]), there exists an isomorphism of the Frobenius manifolds associated to any weak Fano toric manifold and the corresponding Landau-Ginzburg model. In their study on local mirror symmetry [27, 28, 29], Y. Konishi and S. Minabe pursued an analogue of the isomorphism. For any $\mathcal{A}(X)$, their construction in §6.4.3). The pairings on the graded pieces are also described in terms of $(\mathcal{A}(X), Q)$ on $S_{\mathcal{A}(X)}^{\text{reg}}$, which is closely related with the GKZ-hypergeometric system studied in [5]. It seems natural to expect that a mixed Frobenius manifold should be constructed from the variation of mixed Hodge structure, and that there should exist an isomorphism between the mixed Frobenius manifolds.

In this paper, we realize the expectation in a rough level. We shall not study mixed Frobenius manifolds. Instead, we consider more rough objects which we call mixed TEP-structures. A mixed TE-structure on a complex manifold $M$ consists of the following:

- A TE-structure $\mathcal{V}$ on $M$.
- An increasing filtration $\mathcal{W} = (\mathcal{W}_m(\mathcal{V}) | m \in \mathbb{Z})$ on $\mathcal{V}$ such that (i) $\mathcal{W}_m = 0$ ($m << 0$) and $\mathcal{W}_m = \mathcal{V}$ ($m >> 0$), (ii) $\text{Gr}_m^\mathcal{W}(\mathcal{V})$ are locally free $\mathcal{O}_{\mathbb{C}^1 \times M}$-modules, (iii) $\mathcal{W}_m$ are preserved by the connection of $\mathcal{V}$.

If each $\text{Gr}_m^\mathcal{W}(\mathcal{V})$ is equipped with a non-degenerate pairing $P_m : \text{Gr}_m^\mathcal{W}(\mathcal{V}) \otimes j^* \text{Gr}_m^\mathcal{W}(\mathcal{V}) \to \lambda^{-m} \mathcal{O}_{\mathbb{C}^1 \times M}$ such that $(\text{Gr}_m^\mathcal{W}, P_m)$ is a TEP-structure, then $(\mathcal{V}, \mathcal{W}, \{P_m\})$ is called a mixed TEP-structure. For example, a graded polarized variation of mixed Hodge structure naturally induces a mixed TEP-structure by the Rees construction.

On the local $A$-side, we have the mixed TE-structure $\mathcal{V}(\mathcal{A}(X))$ associated to the above variation of the mixed Hodge structure induced by the Rees construction. It is also described as a mixed TE-structure mentioned in [124] given as follows. We set $\mathcal{A}(K_X) := \{(a, 1) \in \mathbb{Z}^3 | a \in \mathcal{A}(X)\} \cup \{(0, 0, 1)\}$. It is the set of the primitive vectors in the one dimensional cones in a fan corresponding to $K_X$ if we naturally regard $K_X$ as a toric manifold. We naturally have $S_{\mathcal{A}(X)} = S_{\mathcal{A}(K_X)}$ and $S_{\mathcal{A}(X)}^{\text{reg}} = S_{\mathcal{A}(K_X)}^{\text{reg}}$. After the appropriate shift of the weight filtration, $\mathcal{V}(\mathcal{A}(X))$ is isomorphic to the mixed TE-structure $(S_{\mathcal{A}(K_X)}^{\text{reg}}, \mathcal{W})$ over the $\mathcal{D}$-module $\mathcal{D}_{\mathcal{A}(K_X)}^{0, L}(G_{\mathcal{A}(K_X)})$. Here, $\mathcal{D}_{\mathcal{A}(K_X)}^{0, \mathcal{A}(K_X), +}$ is the $TE$-structure mentioned in [124] and $\mathcal{W}$ is the filtration induced by the weight filtration of the mixed twistor $\mathcal{D}$-module. We have the pairings $P_{\mathcal{A}(K_X), +, k}$ ($k \in \mathbb{Z}$) on the graded pieces $\text{Gr}_k^\mathcal{W} \mathcal{D}_{\mathcal{A}(K_X), +}$ associated to $(0, 0, -1) \in \mathbb{Z}^3$ (see [6.3]). The pairings on the graded pieces are also described in terms of $(T_{Q_2}^2, \mathcal{G}_{\mathcal{A}(X), Q})$ ($Q \in S_{\mathcal{A}(X)}^{\text{reg}}$) (see [6.4.3]).

According to the general theory of toric manifolds, we have a natural isomorphism $H^2(X, \mathbb{C}) \otimes \mathbb{C}^* \simeq S_{\mathcal{A}(X)}$. So, we have the universal covering map $\chi : H^2(X, \mathbb{C}) \to S_{\mathcal{A}(X)}$ induced by the exponential map.
Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.

Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.

Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.

Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.

Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.

Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.

Theorem 1.3 (Theorem 7.43 for a more precise statement.) We have appropriate non-empty open subsets $U_1 \subset \mathcal{U}_X$ and $U_2 \subset \chi^{-1}(S^\text{reg}_{\mathcal{A}(X)})$, a holomorphic isomorphism $\varphi : U_1 \simeq U_2$, and an isomorphism of the mixed TEP-structures

$$(\text{QDM}(X, K_X), \tilde{W}, \{P_m\}) \simeq \varphi^* \chi^* (\mathcal{G}^\text{reg}_{\mathcal{A}(K_X)}^\text{reg}, \tilde{W}, \{P_{\mathcal{A}(K_X), \ast, m}\})).$$

The objects and the isomorphisms are equivariant with respect to the actions of $2\pi\sqrt{-1}H^2(X, \mathbb{Z})$.
whose poles are contained in \( D \). We need more preliminaries to obtain the relation between polarizations on the graded pieces (Proposition 5.3.9).

We also need the compatibility of various standard functors for \( D \)-modules and flat bundles, which are explained in the appendix sections \( A \), \( B \) and \( C \).

Acknowledgement. This study grows out of my attempt to understand the works of H. Iritani [19, 20], Y. Konishi, S. Minabe [27, 28, 29] and T. Reichelt, C. Sevenheck [40, 41]. I am grateful to them for discussions. I thank G. Wilkin for his interest to this study and his kindness. I am grateful to M.-H. Saito for his kindness and support. I thank A. Ishii, Y. Tsuchimoto, T. Xue and K. Vilonen for their kindness.

This work is partially supported by the Grant-in-Aid for Scientific Research (C) (No. 22540078), the Grant-in-Aid for Scientific Research (A) (No. 22244003) and the Grant-in-Aid for Scientific Research (S) (No. 24224001), Japan Society for the Promotion of Science. This research was partially completed while the author was visiting the Institute for Mathematical Sciences, National University of Singapore in 2014.

2 \( D \)-modules associated to meromorphic functions

2.1 Purity condition

2.1.1 \( D \)-modules associated to meromorphic functions

Let \( X \) be a complex manifold with a hypersurface \( D \). Let \( \mathcal{O}_X(*D) \) be the sheaf of meromorphic functions on \( X \) whose poles are contained in \( D \). Let \( D \) denote the duality functor on the category of holonomic \( D \)-modules on \( X \). For any coherent \( D_X \)-module \( M \), we set \( M(*D) := \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} M \) and \( M(!D) := D(\mathcal{D}(M)(*D)) \). The restriction of \( M(*D) \) to \( X \setminus D \) is naturally isomorphic to \( M|_{X \setminus D} \). We have the canonical morphism \( M(!D) \to M(*D) \) whose restriction to \( X \setminus D \) is the identity.

Let \( f \) be a meromorphic function on \( (X, D) \), i.e., a section of \( \mathcal{O}_X(*D) \). Let \( (f)_0 \) and \( (f)_{\infty} \) denote the effective divisors obtained as the zeroes and the poles of \( f \). The supports of the divisors are denoted by \( |(f)_0| \) and \( |(f)_{\infty}| \).

We obtain the meromorphic flat bundle \( L_0(f, D) := \mathcal{O}_X(*D)v \) with \( \nabla v = \nabla f \). We naturally regard it as a \( \mathcal{D}_X \)-module. We set \( L_0(f, D) := L_0(f,D)(!D) = D(L_0(-f,D)) \). The image of the canonical morphism \( L_0(f, D) \to L_0(f,D) \) is independent of the choice of \( D \) such that \( |(f)_{\infty}| \subset D \), and denoted by \( L(f) \). We have natural isomorphisms \( L_0(f, D) \simeq L_0(f, D) \) for \( \ast = !, \ast \). Indeed, \( L(f) \to L_0(f, D) \) naturally induces \( L(f)(!D) \to L_0(f,D) \) which is clearly an isomorphism. By using the duality, we obtain \( L_0(f, D) \to L(f)(!D) \).

When \( D = |(f)_{\infty}| \), we set \( L_0(f) := L_0(f, |(f)_{\infty}|) \).

2.1.2 Purity

We continue to use the notation in [2.1.1]. We introduce a condition.

Definition 2.1 We say that a meromorphic function \( f \) on \( X \) is pure at \( P \in X \), if the canonical morphism \( L_0(f) \to L_0(f) \) is an isomorphism on a neighbourhood of \( P \). We say that \( f \) is pure if it is pure at any point of \( X \).

Because \( D(L_0(f)) \simeq L_0(-f) \) and \( D(L_0(f)) \simeq L_0(-f) \), we have the following easy lemmas.

Lemma 2.2 Let \( f \) be a meromorphic function on \( X \).

- \( f \) is pure if and only if \(-f \) is pure.
- \( f \) is pure if and only if the morphisms \( L_0(f) \to L_0(f) \) and \( L_0(-f) \to L_0(f) \) are epimorphisms.
Let $\varphi : Y \rightarrow X$ be a proper morphism of complex manifolds such that $\varphi$ induces an isomorphism $Y \setminus \varphi^{-1}(D) \simeq X \setminus D$.

**Lemma 2.3** Let $f$ be a meromorphic function on $(X, D)$. Suppose that $|(|f|)_{\infty}| = |f|_{\infty}$ and that $|f|_{\infty}$ is pure. Then, $f$ is also pure, and it satisfies $|f|_{\infty} = D$.

**Proof** We set $f_Y := \varphi^*(f)$ and $D_Y := \varphi^{-1}(D)$. Because $D = \varphi(|f_Y|) \subset |f| \subset D$, we have $|f|_{\infty} = D$. We have $\varphi^*(L_s(f_Y, D_Y)) = 0 (i \neq 0)$, and $\varphi^0(L_s(f_Y, D_Y)) \simeq \varphi^0(L_s(f_Y, D_Y))$. Hence, we have $\varphi^0(L_Y(f_Y, D_Y)) \simeq L_Y(f, D_Y)$. By the duality, we have $\varphi^0(L_Y(f_Y, D_Y)) \simeq L_Y(f, D_Y)$. Then, from the purity of $f_Y$, we obtain that $L_Y(f, D) \rightarrow L_Y(f, D)$ is an isomorphism.

### 2.1.3 Vanishing of cohomology for pure functions

The purity condition sometimes implies the vanishing of the cohomology. We mention a typical case. Let $X$ be a complex manifold with a hypersurface $D$. Let $f$ be a meromorphic function on $(X, D)$ such that (i) $|f|_{\infty}$ is pure, (ii) $f$ is pure. Let $F : X \rightarrow S$ be a proper morphism of complex manifolds.

**Proposition 2.4** Suppose that $R^kF_*((\Omega^1_X \otimes O(D))^0) = 0$ for $(i, j) \in \mathbb{Z}_+^2$ such that $i + j > \dim X$. Then, we have $F^kL(f) = 0 (k \neq 0)$.

**Proof** Because $DL(f) \simeq L(-f)$ and because $-f$ is also pure, it is enough to prove that $F^kL(f) = 0$ for $k > 0$. We have only to prove the claim locally around any point of $S$. We set $\omega_X := \Omega^1_X$ and $\omega_S := \Omega^1_S$. Recall that $F_1L(f)$ is obtained as $RF_*\left(\omega_X^{\text{dim} X} \otimes F^{-1}(D_S \otimes \omega_S) \right) \otimes F_1 L(f)$. We have the standard free resolution $D_X \otimes \Omega^1_X$ of the right $D_X$-modules $\omega_X$. By the assumption, if $i + j > 0$, we have

$$R^jF_*\left(\left[\Omega^1_X \otimes F^{-1}D_S \otimes \omega_S \right] \otimes F_1 L(f) \right) \simeq R^jF_*\left(\Omega^1_X \otimes L(f) \right) \otimes F^{-1}D_S \otimes \omega_S = 0.$$  

Hence, we have the desired vanishing $F^kL(f) = 0$ for $k > 0$.

**Example 2.5** The condition of Proposition 2.4 is satisfied if $F$ is factorized into the composite of morphisms of complex manifolds $X \xrightarrow{\rho} X' \xrightarrow{\rho'} S$ such that (i) $D' = \rho(D)$ is also a hypersurface of $X'$, (ii) $\rho$ induces $X \setminus D \simeq X' \setminus D'$ (ii) $O_{X'}(D')$ is relatively ample with respect to $F'$.

### 2.2 $D$-modules associated to non-degenerate meromorphic functions

#### 2.2.1 A non-degeneracy condition

Let $X$ be a complex manifold with a simple normal crossing hypersurface $D$. Let $f$ be a meromorphic function on $(X, D)$. In this paper, we shall often use the following non-degeneracy condition.

**Definition 2.6** $f$ is called non-degenerate along $D$ if the following holds for a small neighbourhood $N$ of $|f|_{\infty}$.

- $(f)_0 \cap N$ is reduced and non-singular.
- $N \cap (|(f)|_{\infty} \cup D)$ is normal crossing.

Let $D = \bigcup_{i \in A} D_i$ and $|f|_{\infty} = \bigcup_{i \in A} D_i$ be the irreducible decompositions. For any non-empty subset $I \subset A$, we set $D_I := \bigcap_{i \in I} D_i$ and $D_I^\prime := D_i \setminus \bigcup_{j \notin I} D_j$. If $I = \emptyset$, we set $D_\emptyset = X$. Then, the second condition can be reworded that $D_I^\prime$ is transversal with $|(f)|_{\infty}$ for any $I \subset A$ with $I \cap A_f \neq \emptyset$.

**Remark 2.7** Suppose that a meromorphic function $f$ on $(X, D)$ is non-degenerate along $D$. Let $D'$ be a hypersurface of $X$ such that $|(f)|_{\infty} \subset D' \subset D$. Then, $f$ is non-degenerate along $D'$. But, the converse does not hold in general. Namely, even if a meromorphic function $f$ on $(X, D')$ is non-degenerate along $D'$, it is not necessarily non-degenerate along $D$. For example, set $X = \mathbb{C}^2$, $f = (z_1 - z_2)/z_2$, $D' = \{z_2 = 0\}$ and $D = \{z_1 = 0\} \cup \{z_2 = 0\}$.  

6
We reword the condition in terms of local coordinate systems. We set $D_f^\circ := \bigcup_{i \in \Lambda_f} D_i$, where $\Lambda_f := \Lambda \setminus \Lambda_I$. We have $D = |(f)\infty| \cup D_f^\circ$. Let $Q \in |(f)\infty|$. We take a holomorphic coordinate neighbourhood $(X_Q; z_1, \ldots, z_n)$ around $Q$ such that

$$|f\infty|\cap X_Q = \bigcup_{i=1}^{\ell_1} \{z_i = 0\}, \quad D_f^\circ \cap X_Q = \bigcup_{i=\ell_1+1}^{\ell_1+\ell_2} \{z_i = 0\}.$$  

Let $k_i$ denote the pole order of $f$ along $\{z_i = 0\}$. Then, we have an expression $f = f_0 \prod_{i=1}^{\ell_1} z_i^{-k_i}$ where $f_0$ is holomorphic. Let $I \subset \Lambda_f$ be determined by $Q \in D_I^\circ$. Note $I \cap \Lambda_f \neq \emptyset$. If $f$ is non-degenerate along $D$, then the divisor $(f|_{(D_I^\circ)})$ is reduced and non-singular. Conversely, if the above holds for any $Q \in |(f)\infty|$, then $f$ is non-degenerate.

**Remark 2.8** See Definition 6.3 and Lemma 6.9 for the relation between the non-degeneracy condition in §6.5 and the condition in Definition 2.6. We postpone to discuss the exact relation between the (cohomologically) tameness condition for algebraic functions (see §6.3) and the conditions in Definitions 2.1 and 2.6.

### 2.2.2 Convenient coordinate systems

Suppose that a section $f$ of $O_X(*D)$ is non-degenerate along $D$. We have a holomorphic coordinate system $(X_Q; z_1, \ldots, z_n)$ around $Q \in |(f)\infty|$ satisfying the following conditions.

- $|(f)\infty|\cap X_Q = \bigcup_{i=1}^{\ell_1} \{z_i = 0\}$ and $D_f^\circ \cap X_Q = \bigcup_{i=\ell_1+1}^{\ell_1+\ell_2} \{z_i = 0\}$.

- If $Q \in |(f)\infty|$, we have $f_1|_{X_Q} = z_0 \prod_{i=1}^{\ell_1} z_i^{-k_i}$ for some $k \in \mathbb{Z}_{>0}^{\ell_1}$.

- If $Q \notin |(f)\infty|$, we have $f_1|_{X_Q} = \prod_{i=1}^{\ell_1} z_i^{-k_i}$ for some $k \in \mathbb{Z}_{>0}^{\ell_1}$.

In this paper, such a coordinate system is called a convenient coordinate system.

### 2.2.3 Families of non-degenerate functions

Let $X \to S$ be a smooth morphism of complex manifolds. Let $D$ be a simple normal crossing hypersurface in $X$ with the irreducible decomposition $D = \bigcup_{i \in \Lambda} D_i$. Suppose that the induced morphisms $D_f^\circ \to S$ are smooth for any $I \subset \Lambda$. For any $s \in S$, let $(X_s, D_s)$ denote the fibers of $(X, D)$ over $s \in S$. In such a situation, we sometimes consider a condition which is stronger than the non-degeneracy along $D$.

**Definition 2.9** Let $f$ be a meromorphic function on $(X, D)$ which is non-degenerate along $D$. If moreover $f|_{X_s}$ is non-degenerate along $D_s$ for any $s \in S$, we say that $f$ is non-degenerate along $D$ over $S$.

### 2.2.4 Purity in the non-degenerate case

Let $X$ be a complex manifold with a normal crossing hypersurface $D$. Let $f$ be a meromorphic function on $(X, D)$.

**Lemma 2.10** Suppose that $D = |(f)\infty|$ and that $f$ is non-degenerate along $D$. Then, $f$ is pure.

**Proof** We have only to check the claim locally around any point of $|(f)\infty|$. We use a convenient coordinate system as in §2.2.2. We have a natural isomorphism $L_i(f)(*D) \simeq L_s(f)(*D)$. Hence, for a large $N$, $u := \prod_{i=1}^{\ell_1} z_i^{-k_i} N$ is a section of $L_i(f)$.

We have a natural morphism $\varphi : D_X \to L_i(f)$ given by $P \mapsto Pu$. Let us check that the composite of $\varphi$ and $L_i(f) \to L_s(f)$ is an epimorphism. It is enough to observe that the image of $u$ in $L_s(f)$ generates $L_s(f)$. For $m \in \mathbb{Z}^{m_1}$, let $z^m = \prod_{i=1}^{\ell_1} z_i^{m_i}$. If $Q \in |(f)\infty|$, note $\partial_{z_0}(z^m v) = z^{m-k} v$. If $Q \notin |(f)\infty|$, note $z_1\partial_{z_1}(z^m v) = -k_i z^{m-k} v + m_i z^m v$. Then, the claim easily follows.

Hence, the natural morphism $\kappa : L_i(f) \to L_s(f)$ is an epimorphism. As remarked in Lemma 2.2, because the dual of $\kappa$ is the epimorphism $L_i(-f) \to L_s(-f)$, we obtain that $\kappa$ is a monomorphism.
2.2.5 Expressions of the D-modules associated to non-degenerate functions

Let $V_D\mathcal{D}_X$ denote the sheaf of subalgebras of $\mathcal{D}_X$ generated by $\Theta_X(\log D)$ over $\mathcal{O}_X$. Let $f$ be a meromorphic function on $(X, D)$ which is non-degenerate along $D$. Then, $L(f)$ and $L(f)(D) = L(f)\otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$ are naturally $V_D\mathcal{D}_X$-modules.

**Lemma 2.11** We naturally have $L_!(f, D) \simeq \mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f)$ and $L_*(f, D) \simeq \mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f)(D)$.

**Proof** Because $(\mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f))(\ast D) \simeq (\mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f)(D))(\ast D) \simeq L(f)(\ast D)$, we have natural morphisms:

$$D_X \otimes_{V_D\mathcal{D}_X} L(f)(D) \xrightarrow{\alpha_1} L_*(f, D)$$

$$L_!(f, D) \xrightarrow{\alpha_2} D_X \otimes_{V_D\mathcal{D}_X} L(f).$$

It is enough to prove that the morphisms are isomorphisms locally around any point of $D$. We use a convenient coordinate system in $(\mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f))(\ast D)$.

Let $v$ be a frame of $L(f)$ over $\mathcal{O}_X(\ast f \infty)$ such that $\nabla v = v df$. Because $a_1 : L_!(f, D) \longrightarrow L(f)$ is an epimorphism, we can locally take a section $v'$ of $L_!(f, D)$ which is mapped to $v$ via $a_1$. We consider the submodule $V_D\mathcal{D}_X v' \subset L_!(f, D)$. It is coherent over $V_D\mathcal{D}_X$. Note that $L(f)$ is coherent over $V_D\mathcal{D}_X$. Because $\text{Ker} a_1 \cap V_D\mathcal{D}_X v' = \text{Ker}(a_1 |_{V_D\mathcal{D}_X v'})$, it is coherent over $V_D\mathcal{D}_X$. In particular, it is locally finitely generated over $V_D\mathcal{D}_X$. Take a generator $f_1, \ldots, f_m$ of $\text{Ker} a_1 \cap V_D\mathcal{D}_X v'$. The supports of $f_j$ are contained in $D$. We can take a large $N$ such that $\prod_{i=1}^{\ell_1+\ell_2} z_i^{N} f_j = 0$ in $L_!(f, D)$, because $L_!(f, D)(\ast D) = L(f)(\ast D)$. Then, it is easy to see that $\prod_{i=1}^{\ell_1+\ell_2} z_i^{N} (V_D\mathcal{D}_X v' \cap \text{Ker}(a_1)) = 0$. We set $u := \prod_{i=1}^{\ell_1+\ell_2} z_i^{N} v'$, and then we have $V_D\mathcal{D}_X v'' \cap \text{Ker}(a_1) = 0$.

The morphism $V_D\mathcal{D}_X v'' \longrightarrow L(f)$ induced by $a_1$ gives an isomorphism $V_D\mathcal{D}_X v'' \longrightarrow \mathcal{O}_X(\ast f \infty) \prod_{i=1}^{\ell_1+\ell_2} z_i^{N} v$ of $V_D\mathcal{D}_X$-modules. In particular, $V_D\mathcal{D}_X v''$ is naturally an $\mathcal{O}_X(\ast f \infty)$-module.

We set $v := \prod_{i=1}^{\ell_1+\ell_2} z_i^{N} v'$. We have $z_i \partial_i v = -k_i u^{(3)} \prod_{i=1}^{\ell_1+\ell_2} z_i^{N} f$ for $i = 1, \ldots, \ell_1$, and $z_i \partial_i v = N v^{(3)}$ for $i = \ell_1 + 1, \ldots, \ell_1 + \ell_2$. Take $0 \leq p_1, \ldots, p_{\ell_1+\ell_2} \leq N$. We set $p := (p_1, \ldots, p_{\ell_1+\ell_2})$. We set $v_p := \prod_{i=1}^{\ell_1+\ell_2} \partial_i^{p_i} v$. Note that $a_1(v_p) = C_p \prod_{i=1}^{\ell_1+\ell_2} z_i^{N-p_i} v$ for a non-zero constant $C_p$. We consider the following morphism induced by $a_1$:

$$V_D\mathcal{D}_X \cdot \prod_{i=1}^{\ell_1+\ell_2} \partial_i^{p_i} v \longrightarrow L(f)$$

Let us observe that $(\ref{2.2.2})$ is a monomorphism. If $p = (0, \ldots, 0)$, it has already been observed. If $p_i > 0$, set $p_i' := p_j (j \neq i)$ and $p_i := p_i - 1$. We have $\partial_i v_p = p_i' v_p$. Let $s \in \text{Ker}(a_1) \cap V_D\mathcal{D}_X v_p$. We have $s = \partial_s s'$ for some $s' \in V_D\mathcal{D}_X v_p$. We have $0 = a_1(z_i s) = a_1(z_i \partial_i s') = 0$. We have $z_i \partial_i s' = 0$ in $V_D\mathcal{D}_X v_p \simeq \mathcal{O}_X(\ast f \infty) \prod_{j=1}^{N-p_j} v$.

But, because $N - p_j > 0$, it is easy to see that if a section $s'$ of $\mathcal{O}_X(\ast f \infty)$ satisfies $z_i \partial_i s' = 0$ then $s' = 0$. Hence, we obtain $s = 0$. In particular, the induced morphism $V_D\mathcal{D}_X v_{(N, \ldots, N)} \longrightarrow L(f)$ is a monomorphism. Because $a_1(v_{(N, \ldots, N)})$ is $v$ multiplied by a non-zero constant, $V_D\mathcal{D}_X v_{(N, \ldots, N)} \longrightarrow L(f)$ is also an epimorphism, i.e., an isomorphism.

Hence, we have a $V_D\mathcal{D}_X$-homomorphism $a_2 : L(f) \longrightarrow L_!(f, D)$ such that $a_1 \circ a_2 = \text{id}$. It induces a $\mathcal{D}_X$-homomorphism $\beta_2 : \mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f) \longrightarrow L_!(f, D)$. We set $g := \beta_2 \circ a_2$ which is an endomorphism of $L_!(f, D)$.

For the dual, $g$ induces the identity on $D_!(f, D)$. Hence, we obtain that $g$ is the identity. In particular, $(\ref{3.3})$ is an epimorphism and $\beta_2$ is an isomorphism. Because $\mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f)$ is generated by $v$, $\beta_2$ is an epimorphism.

Hence, $a_2$ and $\beta_2$ are isomorphisms.

For any non-negative integer $m$, we consider the $\mathcal{O}_X(\ast f \infty)$-homomorphism $\gamma_m : L(f)((m+1)D) \longrightarrow \mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f)(D)$ given by

$$\gamma_m \left( \prod_{i=1}^{\ell_1+\ell_2} z_i^{-1} \prod_{i=1}^{\ell_1+\ell_2} z_i^{-m-1} v \right) = (-1)^{\ell_2 m(m-1)} \prod_{i=1}^{\ell_1+\ell_2} \partial_i^{m} \otimes \prod_{i=1}^{\ell_1+\ell_2} z_i^{-1} v$$

Let $\iota_m : L(f)(mD) \longrightarrow L(f)((m+1)D)$ be the natural inclusion. Then, we have $a_1 \circ \gamma_m \circ \iota_m = a_1 \circ \gamma_{m-1}$. We obtain a $\mathcal{O}_X(\ast f \infty)$-homomorphism $\gamma : L(f)(\ast D) \longrightarrow \mathcal{D}_X \otimes_{V_D\mathcal{D}_X} L(f)(D)$. By the construction, $\gamma$ is an epimorphism, and $a_1 \circ \gamma$ is the identity. Then, we obtain that $a_1$ is an isomorphism.
2.2.6 De Rham complexes

We give some complexes which are quasi-isomorphic to the de Rham complexes of $L_*(f, D)$ when $f$ is non-degenerate along $D$. (See [10], [46], [57] for the case $|(f)_0| \cap |(f)_\infty| = \emptyset$.) We set $d_X := \dim X$.

We have the natural complex of right $D_X$-modules $\Omega^*_X \otimes_{\mathcal{O}_X} D_X[d_X]$ which is a right $D_X$-free resolution of $\Omega_X := \Omega^*_X$. We have the subcomplex $\Omega^*_X(\log D)(-D) \otimes_{\mathcal{O}_X} V_D D_X[d_X]$, which is a right $V_D D_X$-free resolution of $\Omega_X$. Indeed, the natural morphism $\Omega_X \otimes V_D D_X \to \Omega_X$ induces a quasi-isomorphism $\Omega^*_X(\log D)(-D) \otimes_{\mathcal{O}_X} V_D D_X[d_X] \simeq \ker$ of complexes of right $V_D D_X$-modules.

Suppose that $f$ is non-degenerate along $D$. Because $L_*(f, D) \simeq D_X \otimes_{V_D D_X} L(f)(D)$ according to Lemma 2.11, we have the following natural isomorphisms:

$$\Omega_X \otimes_{D_X} L_*(f, D) \simeq \Omega_X \otimes_{V_D D_X} L(f)(D) \simeq \Omega^*_X(\log D)(-D) \otimes L(f)(D)[d_X] \simeq \big(\Omega^*_X(\log D)(*(f)_\infty), d + df\big)[d_X]$$

(5)

Because $L_l(f, D) \simeq D_X \otimes_{V_D D_X} L(f)$ according to Lemma 2.11, we have the following isomorphisms:

$$\Omega_X \otimes_{D_X} L_l(f, D) \simeq \Omega_X \otimes_{V_D D_X} L(f) \simeq \Omega^*_X(\log D)(-D) \otimes L(f)[d_X] \simeq \big(\Omega^*_X(\log D)(-(f)_\infty), d + df\big)[d_X]$$

(6)

Coherent expression Let $H$ be any divisor in $X$. Let us consider the complex

$$\Omega^l_{X-1}(\log D)(H) \xrightarrow{a_1} \Omega^l_X(\log D)(H + (f)_\infty) \xrightarrow{a_2} \Omega^l_{X+1}(\log D)(H + 2(f)_\infty),$$

where $a_i$ are induced by the multiplication of $df$.

Lemma 2.12 If $f$ is non-degenerate along $D$, we have $\Im(a_1) = \ker a_2$ on a neighbourhood of $|(f)_\infty|$.

Proof We have only to check the claim locally around any point $Q$ of $|f| \cap |(f)_\infty|$. Let us consider the case $Q \in |(f)_0| \cap |(f)_\infty|$. Then, $f = z_n \prod_{i=1}^{\ell_1} z_i$. We have the following local section of $\Omega^1_X(\log D)$:

$$\tau := df \cdot \prod_{i=1}^{\ell_1} z_i^{k_i} = dz_n - \sum_{i=1}^{\ell_1} k_i z_n \frac{dz_i}{z_i}$$

We have $\tau|Q \neq 0$. We consider the following on a neighbourhood of $Q$:

$$\Omega^l_{X-1}(\log D) \xrightarrow{b_1} \Omega^l_X(\log D) \xrightarrow{b_2} \Omega^l_{X+1}(\log D)$$

Here, $b_i$ are induced by the multiplication of $\tau$. Because $\tau|Q \neq 0$, we have $\Im(b_1) = \ker(b_2)$. Then, in the case $Q \notin |(f)_0|$, the claim of the lemma follows. The case $Q \notin |(f)_0|$ can be argued similarly, and it is well known.

We set $\Omega^l_X(\log D, f) := \Omega^l_X(\log D)(\ell(f)_\infty)$. Together with the differential $d + df$, we obtain a complex $\big(\Omega^l_X(\log D, f), d + df\big)$. We also have $\big(\Omega^l_X(\log D, f)(-D), d + df\big)$. We obtain the following from the previous lemma.

Lemma 2.13 The following natural morphisms are quasi-isomorphisms:

$$(\Omega^l_X(\log D, f), d + df) \longrightarrow \big(\Omega^l_X(\log D)(*(f)_\infty), d + df\big)$$

$$(\Omega^l_X(\log D, f)(-D), d + df) \longrightarrow \big(\Omega^l_X(\log D)(-(f)_\infty), d + df\big)$$

Hence, we have the following quasi-isomorphisms:

$$\Omega_X \otimes_{D_X} L_*(f, D) \simeq \big(\Omega^l_X(\log D, f), d + df\big)[d_X]$$

$$\Omega_X \otimes_{D_X} L_l(f, D) \simeq \big(\Omega^l_X(\log D, f)(-D), d + df\big)[d_X]$$

9
Kontsevich complexes Let \( \Omega_{X,f,D}^k \) denote the kernel of the following morphism induced by the multiplication of \( df \):

\[
\Omega_{X}^k(\log D) \xrightarrow{df} \frac{\Omega_{X}^{k+1}(\log D)((f)_\infty)}{\Omega_{X}^{k+1}(\log D)}.
\]

Lemma 2.14 \( \Omega_{X,f,D}^k \) are locally free.

**Proof** We have \( \Omega_{X,f,D}^k = \Omega_{X}^k(\log D) \) outside \( |(f)_\infty| \). Locally around any point of \( (f)_\infty \), we have local decompositions \( \Omega_{X}^k(\log D) = A_k \oplus B_k \) such that the multiplication of \( df \) induces \( B_k \simeq A_{k+1}(\log D)_\infty \). Hence, we have \( \Omega_{X,f,D}^k = A_k \oplus B_k(-f)_\infty \), and the claim of the lemma follows.

The multiplication of \( df \) induces \( df : \Omega_{X,f,D}^k \to \Omega_{X,f,D}^{k+1} \). By the commutativity \( [d,df] = 0 \), the exterior derivative induces \( d : \Omega_{X,f,D}^k \to \Omega_{X,f,D}^{k+1} \). Hence, we obtain the complex \( (\Omega_{X,f,D}^k,d+df) \). We also obtain the complex \( (\Omega_{X,f,D}^k(-D),d+df) \).

Lemma 2.15 The following natural morphisms are quasi-isomorphisms:

\[
(\Omega_{X,f,D}^k,d+df) \to (\Omega_{X}^k(\log D),d+df)
\]

\[
(\Omega_{X,f,D}^k(-D),d+df) \to (\Omega_{X}^k(\log D)(-D),d+df)
\]

2.2.7 The push-forward by a projection

Let us consider the case \( (X,D) = (X_0,D_0) \times S \). Suppose that \( X_0 \) is compact. Let \( f \) be a meromorphic function on \( (X,D) \). Let \( \pi : X \to S \) denote the projection. Let \( n := \dim X_0 \).

Lemma 2.16 Suppose that \( f \) is non-degenerate along \( D \). We have the following natural isomorphisms:

\[
\pi_+(L_*(f,D)) \simeq \mathbb{R}^\pi_+ (\Omega_{X/S}^{\bullet+n}(\log D)(\ast(f)_\infty),d+df)
\]

\[
(7)
\]

\[
\pi_+(L_!(f,D)) \simeq \mathbb{R}^\pi_+ (\Omega_{X/S}^{\bullet+n}(\log D)-D)(\ast(f)_\infty),d+df)
\]

\[
(8)
\]

**Proof** Let us consider \( (7) \). By Lemma 2.11, we have \( L_*(f,D) \simeq D_X \otimes_{V_D} D(f)(D) \). We have \( D_X = D_{X_0} \otimes D_S \) and \( V_D D_X = V_{D_0} D_{X_0} \otimes D_S \). Let \( p : X \times X_0 \to X_0 \) be the projection. We obtain

\[
\pi_+(L_*(f,D)) \simeq \mathbb{R}^\pi_+ \left( p^* \bigl( \Omega_{X}^0(\log D)_\infty \bigl) \otimes \mathcal{O}(D) \right)
\]

\[
\simeq \mathbb{R}^\pi_+ \left( \Omega_{X/S}^{\bullet+n}(\log D)(-D) \otimes \mathcal{O}_X(L_!(f,D)) \right)
\]

\[
(9)
\]

It implies \( (7) \). Similarly, we obtain \( (8) \) from the expression \( L_!(f,D) \simeq D_X \otimes_{V_D} \mathcal{O}_X(L(f)(-D)) \) in Lemma 2.11 as in \( (6) \).

Suppose moreover that \( f \) is non-degenerate along \( D \) over \( S \). (See Definition 2.13 for this stronger condition.) As in \( (2.2.6) \), we can naturally define the complexes \( (\Omega_{X/S}^{\bullet+n}(\log D),d+df) \) and \( (\Omega_{X/S,f,D}^{\bullet+n},d+df) \) in the relative setting.

Lemma 2.17 If \( f \) is non-degenerate along \( D \) over \( S \), then we have the following natural isomorphisms:

\[
\pi_+(L_*(f,D)) \simeq \mathbb{R}^\pi_+ (\Omega_{X/S}^{\bullet+n}(\log D),d+df)
\]

\[
\simeq \mathbb{R}^\pi_+ (\Omega_{X/S,f,D}^{\bullet+n},d+df)
\]

\[
(10)
\]

\[
\pi_+(L_!(f,D)) \simeq \mathbb{R}^\pi_+ (\Omega_{X/S}^{\bullet+n}(\log D)(-D),d+df)
\]

\[
\simeq \mathbb{R}^\pi_+ (\Omega_{X/S,f,D}^{\bullet+n}(-D),d+df)
\]

\[
(11)
\]

**Proof** As in \( (2.2.6) \), we obtain the isomorphisms from Lemma 2.16.

Corollary 2.18 Suppose that \( f \) is non-degenerate along \( D \) over \( S \). Then, \( \pi_+(L_*(f,D)) \) are flat bundles on \( S \), i.e., locally free \( \mathcal{O}_S \)-modules with an integrable connection.

**Proof** The right hand sides of \( (10) \) and \( (11) \) are \( \mathcal{O}_S \)-coherent. Then, the claim of this corollary follows from a well known result, i.e., if a \( \mathcal{D}_X \)-module is coherent of \( \mathcal{O}_X \), then it is a flat bundle.
2.3 Some functions satisfying the purity condition

We give some examples of meromorphic functions which are not necessarily non-degenerate but satisfy the purity condition. Let \( X \) denote a complex manifold with a normal crossing hypersurface \( D \).

2.3.1 Basic cases

We set \( X^{(1)} := X \times \mathbb{P}^1 \) and \( D^{(1)} := (D \times \mathbb{P}^1) \cup \{ \infty \} \times X \). Take \( f, g \in \mathcal{O}_X(*)D \), and we consider a meromorphic function \( F := \tau f + g \) on \((X^{(1)}, D^{(1)})\). We give some sufficient conditions for \( F \) to be pure on an open subset in \( X^{(1)} \).

Lemma 2.19 Suppose the following.

- \( g \) is non-degenerate along \( D \).
- \( |(f)_0| \cap |(f)_{\infty}| = \emptyset \), and \( |(f)_0| \subset |(g)_{\infty}| \). In particular, \((f)_0 \subset D \).
- \( D = |(f)_{\infty}| \cup |(g)_{\infty}| \).

Then, \( F \) is pure on \( X^{(1)} \).

Proof Let us consider the \( \mathcal{D}_{X^{(1)}}(*D^{(1)}) \)-module \( \mathcal{V} = \mathcal{O}_{X^{(1)}}(*D^{(1)})v \) with \( \nabla v = vdF \). It is enough to prove that \( \mathcal{V}[vD^{(1)}] \to \mathcal{V} \) is an epimorphism around any point of \((P, \tau) \in X^{(1)} \). We take a holomorphic coordinate system \((x_1, \ldots, x_n) \) around \( P \) such that \( D = \bigcup_{i=1}^n \{ x_i = 0 \} \).

Let us consider the case \( \tau \neq \infty \). For a large \( N \), we have \( v_1 = \prod_{i=1}^\ell x_i^N v \in \mathcal{V}(\mathcal{D}D^{(1)}) (\ast = \ast, !) \). It is enough to prove that \( v_1 \) generates \( \mathcal{V} \) as a \( \mathcal{D}_{X^{(1)}} \)-module. For any \( m \in \mathbb{Z}^\ell \), we have

\[
\partial_i(x^m v) = (x_i^{-1} m_i + \partial_i g + \tau \partial_i f)x^m v.
\]

We also have \( \partial_\tau v = f v \). For \( i = 1, \ldots, \ell \), we obtain

\[
x_i \partial_i(x^m v) - (f^{-1} x_i \partial_i f) - \tau \partial_\tau(x^m v) = x^m (x_i \partial_i g + m_i) v.
\]

(12)

For \( i = \ell + 1, \ldots, n \), we have

\[
\partial_i(x^m v) - (f^{-1} \partial_i f) \tau \partial_\tau(x^m v) = x^m \partial_i(g) v.
\]

(13)

By using (12) and (13), we obtain \( \mathcal{O}_{X^{(1)}}(*g_{\infty}) v_1 \) is contained in \( \mathcal{D}_{X^{(1)}} v_1 \). Then, by using \( \partial_\tau v = f v \), we obtain \( \mathcal{V} \) is contained in \( \mathcal{D}_{X^{(1)}} v_1 \) around \((P, \tau) \).

Let us consider the case \( \tau = \infty \). Let \( \kappa := \tau^{-1} \). It is enough to see that \( v_1 = \kappa^N \prod_{i=1}^\ell x_i^N v \) generates \( \mathcal{V} \) around \((P, \infty) \). We have \( \kappa \partial_\kappa v = -\tau \partial_\tau v = -\kappa^{-1} f v \). As in the case \( \tau \neq \infty \), we obtain \( \mathcal{O}_{X^{(1)}}(*g_{\infty}) v_1 \) is contained in \( \mathcal{D}_{X^{(1)}} v_1 \) by using (12) and (13). Then, by using \( \kappa \partial_\kappa v = -\kappa^{-1} f v \), we obtain that \( \mathcal{V} \) is contained in \( \mathcal{D}_{X^{(1)}} v_1 \).

The following lemma is easy to see.

Lemma 2.20 Suppose the following:

- \( g = 0 \).
- \( D = |(f)_{\infty}| \).
- \((f)_0 \) is smooth and reduced, and \( D \cup |(f)_0| \) is normal crossing.

Then, \( F = \tau f \) is non-degenerate on \(|\tau \neq 0| \times X \). In particular, \( F \) is pure on \(|\tau \neq 0| \times X \).
2.3.2 A variant

Let $X$ be a complex manifold with a simple normal crossing hypersurface $D$. We set $X^{(2)} := \mathbb{P}_1 \times X$. Let $D^{(2)}$ denote the union of $\{ \infty \times \mathbb{P}_1 \times X$ and $\mathbb{P}_1^{(2)} \times \{ 0, \infty \} \times X$ and $\mathbb{P}_1^{(2)} \times \{ 0, \infty \} \times X$. Let $h$ be a meromorphic function on $(X, D)$ such that (i) $D = (h)_\infty$, (ii) $h$ is non-degenerate along $D$, (iii) $(h)_0$ is reduced and non-singular on $X$. We have the meromorphic function $F = t^{-1} \tau + th$ on $(X^{(2)}, D^{(2)})$.

We set $D_0 := \{ t = 0 \} \times X$ and $D_2 := \{ t = \infty \} \times X \cup (\mathbb{P}_1^{(2)} \times D)$ in $\mathbb{P}_1 \times X$. We take a projective birational morphism $\varphi : Y \to \mathbb{P}_1 \times X$ such that (i) $g := \varphi^*(th)$ is non-degenerate, (ii) $D_Y := \varphi^{-1}(D_1 \cup D_2)$ is normal crossing, (iii) $Y \setminus \varphi^{-1}(D_1) \simeq (\mathbb{P}_1 \times X) \setminus D_1$. We set $f := \varphi^*(t^{-1})$.

**Lemma 2.21** $F = \tau f + g$ is pure on $\tilde{Y} := \mathbb{P}_1 \times Y$.

**Proof** Because $D_2 \setminus D_1 \subset \{(th)_\infty\}$, we have $\varphi^{-1}(D_2 \setminus D_1) \subset [(g)_\infty]$. We clearly have $\varphi^{-1}(D_1) = [(f)_\infty]$. Hence, we have $D_Y = [(f)_\infty] \cup [(g)_\infty]$. We also have $\varphi^{-1}(D_1) = [(f)_\infty]$ and $[(f)_\infty] \subset [(g)_\infty]$. Then, we obtain the claim of Lemma 2.21 from Lemma 2.19.

**Lemma 2.22** $F$ is pure on $\mathbb{P}_1 \times \mathbb{P}_1 \times X$.

**Proof** Let $\tilde{\varphi} : \tilde{Y} \to X^{(2)}$ be the induced morphism. Let $\tilde{D}_Y$ be the union of $\{ \infty \} \times Y$ and $\mathbb{P}_1 \times D_Y$ in $\tilde{Y}$. By the previous lemma, $\tilde{F} := \tau f + g$ is pure on $\tilde{Y}$. We also have $\tilde{D}_Y = [(F)_\infty]$. Hence, the natural morphism $L_t(F, D_Y) \to L_* (\tilde{F}, \tilde{D}_Y)$ is an isomorphism. Because $L_*(F, D_1 \cup D_2) \simeq \tilde{\varphi}_* L_*(\tilde{F}, \tilde{D}_Y)$, we obtain that $L_t(F, D_1 \cup D_2) \to L_* (F, D_1 \cup D_2)$ is an isomorphism, i.e., $F$ is pure on $X^{(2)}$.

2.4 Push-forward

2.4.1 A statement

Let $X$ be a complex manifold with a simple normal crossing hypersurface $D$. We set $Y := X \times \mathbb{P}_1$ and $D^{(0)} := (X \times \{ \infty \}) \cup (D \times \mathbb{P}_1)$ and $D^{(1)} := D^{(0)} \cup (X \times \{ 0 \})$. Let $f$ and $g$ be meromorphic functions on $(X, D)$. We assume the following.

- The divisor $(f)_0 \cap (X \setminus D)$ is reduced and non-singular.

We set $Z_f := [(f)_0]$. Let $[z_0 : z_1]$ be a homogeneous coordinate system of $\mathbb{P}_1$, and we set $t := z_0 / z_1$. We obtain a meromorphic function $tf$ on $Y$. The pull back of $g$ by the projection $Y \to X$ is also denoted by $g$. We set $F := tf + g$.

We have the $D$-modules $L_* (F, D^{(0)}_Y)$ and $L_* (F, D^{(1)}_Y)$ on $Y$. We have the natural exact sequence:

$$0 \to L_* (F, D^{(0)}_Y) \to L_* (F, D^{(1)}_D) \to L_* (F, D^{(1)}_Y) \to 0 \quad (14)$$

Let $\pi : Y \to X$ be the projection. We prove the following proposition in 2.4.3.

**Proposition 2.23** We have $\pi^+_i (M) = 0$ (i ≠ 0) for

$$M = L_* (F, D^{(0)}_Y), \quad L_* (F, D^{(1)}_D), \quad L_* (F, D^{(1)}_Y) / L_* (F, D^{(0)}_Y).$$

We have the following isomorphisms:

$$\begin{align*}
\pi^0_* L_* (F, D^{(0)}_Y) &\simeq L_* (g, D)(\ast Z_f) / L_* (g, D) \simeq \text{Ker} \left( L_* (g)(\ast Z_f)(\ast D) \to L_* (g, D) \right) \quad (15) \\
\pi^0_* L_* (F, D^{(1)}_Y) &\simeq L_* (g)(\ast Z_f)(\ast D) \quad (16) \\
\pi^0_* L_* (F, D^{(1)}_D) / L_* (F, D^{(0)}_Y) &\simeq L_* (g, D) \quad (17)
\end{align*}$$

The push-forward of $\text{Ker} (14)$ is isomorphic to the following standard exact sequence

$$0 \to \text{Ker} \left( L_* (g)(\ast Z_f)(\ast D) \to L_* (g, D) \right) \to L_* (g)(\ast Z_f)(\ast D) \to L_* (g, D) \to 0$$
By the duality, we obtain the following.

**Corollary 2.24** We have $\pi_i^+ M = 0 \ (i \neq 0)$ for

$$M = L_i(F, D_Y^{(0)}), \ L_i(F, D_Y^{(1)}), \ Ker\{ L_i(F, D_Y^{(1)}) \to L_i(F, D_Y^{(0)}) \}.$$  

We have the following isomorphisms:

$$\pi_+^0 (L_i(F, D_Y^{(0)})) \simeq Ker\{ L_i(g, D)(!Z_f) \to L_i(g, D) \} \simeq Cok\{ L_i(g, D) \to L_i(g)(!Z_f)(!D) \}$$

$$\pi_+^0 (L_i(F, D_Y^{(1)})) \simeq L_i(g)(!Z_f)(!D)$$

$$\pi_+^0 (Ker\{ L_i(F, D_Y^{(1)}) \to L_i(F, D_Y^{(0)}) \}) \simeq L_i(g, D)$$

The push-forward of

$$0 \to Ker\{ L_i(F, D_Y^{(1)}) \to L_i(F, D_Y^{(0)}) \} \to L_i(F, D_Y^{(1)}) \to L_i(F, D_Y^{(0)}) \to 0$$

is isomorphic to the standard exact sequence:

$$0 \to L_i(g, D) \to L_i(g)(!Z_f)(!D) \to Cok\{ L_i(g, D) \to L_i(g)(!Z_f)(!D) \} \to 0$$

\[\Box\]

**Remark 2.25** Note that the restriction of the morphism $L_i(F, D_Y^{(0)}) \to L_i(F, D_Y^{(0)})$ to $\mathbb{P}^1 \times (X \setminus D)$ is an isomorphism. Indeed, the restriction of $F$ to $\mathbb{P}^1 \times (X \setminus D)$ is non-degenerate along $\{\infty\} \times (X \setminus D)$. In particular, the restriction of the morphism $\pi_0^0 L_i(F, D_Y^{(0)}) \to \pi_0^0 L_i(F, D_Y^{(0)})$ to $X \setminus D$ is an isomorphism. Indeed, $\pi_0^0 L_i(F, D_Y^{(0)})|_{X \setminus D}$ are isomorphic to $i_*(\mathcal{O}_{Z_f \setminus D}, d + dg')$, where $i : Z_f \setminus D \to X \setminus D$ is the inclusion, and $g'$ is the restriction of $g$ to $Z_f \setminus D$.

\[\Box\]

Let us consider the case that $f$ is moreover non-degenerate along $D$. In this case, $Z_f$ is smooth, and $Z_f \cup D$ is normal crossing. Let $\iota : Z_f \to X$ denote the inclusion. We set $D_{Z_f} := D \cap Z_f$. We set $g_0 := g_{|Z_f}$. We have the $\mathcal{D}$-modules $L_*(g_0, D_{Z_f})$ on $Z_f$.

**Corollary 2.26** If $f$ is moreover non-degenerate along $D$, we have the following isomorphisms:

$$\pi^0_+ L_* (F, D_Y^{(0)}) \simeq L_*(g, D)(!Z_f) / L_*(g, D) \simeq \iota_+ L_*(g_0, D_{Z_f})$$

$$\pi^0_+ L_* (F, D_Y^{(0)}) \simeq Ker\{ L_*(g, D)(!Z_f) \to L_*(g, D) \} \simeq \iota_+ L_*(g_0, D_{Z_f})$$

The image of $\pi^0_+ L_1(F, D_Y^{(0)}) \to \pi^0_+ L_*(F, D_Y^{(0)})$ is naturally isomorphic to $\iota_+ L_0(g_0)$.

\[\Box\]

**Proof** The first isomorphism in [20] is given in Proposition 2.23. If $Z_f \cup D$ is normal crossing, we clearly have the second isomorphism in [20]. We obtain the isomorphisms in [21] by the duality. According to Remark 2.25, the restriction of the morphism $\pi_0^0 L_1(F, D_Y^{(0)}) \to \pi_0^0 L_*(F, D_Y^{(0)})$ to $X \setminus D$ is an isomorphism. Hence, the image of $\pi^0_+ L_1(F, D_Y^{(0)}) \to \pi^0_+ L_*(F, D_Y^{(0)})$ is identified with the image of a non-zero morphism $\iota_+ L_0(g_0, D_{Z_f}) \to \iota_+ L_0(g_0, D_{Z_f})$, which is isomorphic to $\iota_+ L_0(g_0)$.

\[\Box\]
2.4.2 Extensions

We give a general preliminary which is a variant of Beilinson’s construction [4]. Let \( h \) be any meromorphic function on a complex manifold \( Z \). We set \( D_h := \{ |h|_0 \} \cup \{ |h|_\infty \} \). Let \( \mathcal{O}_Z(*D_h)[s] := \mathcal{O}_Z(*D_h) \otimes_{\mathbb{C}} \mathbb{C}[s] \). For any pair of integers \((a,b)\) with \( a < b \), we consider the meromorphic flat bundle

\[
I^{a,b}_h := s^a \mathcal{O}_Z(*D_h)[s]/s^b \mathcal{O}_Z(*D_h)[s]
\]

with the connection \( \nabla \) determined by \( \nabla s^j = s^{j+1} dh/h \). For any holonomic \( \mathcal{D}_Z \)-module \( M \), we set \( \Pi^{a,b}_h M := I^{a,b}_h \otimes_{\mathcal{O}_Z} M \).

We return to the situation in \[\ref{2.4.1}\. We prove the following proposition in \[\ref{2.4.4}\. Proposition 2.27 Suppose that \((f)_0 \cap X \setminus D \) is smooth and reduced. Then, we have \( \pi_+^t (\Pi^{a,b}_t L_*(F)(*t)) = 0 \) \((*= *, !)\) for \( j \neq 0 \), and we have the following natural isomorphisms:

\[
\pi_+^t (\Pi^{a,b}_t L_*(F)(*t)) \simeq \left( \Pi^{a,b}_{-i} L_*(f) \right) (!(!f_0))(*D)
\]

\[
\pi_0^t (\Pi^{a,b}_t L_1(F))(*t) \simeq \left( \Pi^{a,b}_{-i} L_1(f) \right)(!*(!f_0))(*D)
\]

2.4.3 Proof of Proposition 2.23

Let us obtain the first isomorphism in \[\ref{2.4.1}\. We set \( \mathcal{O}_X(*D)[t] := \mathcal{O}_X(*D) \otimes_{\mathbb{C}} \mathbb{C}[t] \). Let \( R\pi_* \) denote the ordinary push-forward of sheaves by \( \pi \). We have \( R^j \pi_*(L_*(F)) = 0 \) for \( j > 0 \), and \( \pi_*(L_*(F)) \simeq \mathcal{O}_X(*D)[t] \) as an \( \mathcal{O}_X \)-module. Hence, \( \pi_+ L_*(F) \) is represented by the following complex:

\[
\mathcal{O}_X(*D)[t] \xrightarrow{\partial^+_t} \mathcal{O}_X(*D)[t]
\]

Here, the second term sits in the degree 0. The action of vector fields \( V \) on \( X \) is given by \( V(t^i) = t^i(V(f)+V(g)) \).

It is easy to see that the morphism \( \mathcal{O}_X(*D)[t] \xrightarrow{\partial^+_t} \mathcal{O}_X(*D)[t] \) is a monomorphism. Let us look at the cokernel. We consider the following morphism of \( \mathcal{O}_X(*D) \)-modules:

\[
\mathcal{O}_X(*D)[t] \rightarrow L_*(g,D),(Z_f), \quad t^i \mapsto (-1)^i f^{i-1}
\]

Here, we use the natural identification \( L_*(g,D),(Z_f) = \mathcal{O}_X(*D),(Z_f) \) as \( \mathcal{O}_X \)-module. It is a morphism of \( \mathcal{D}_X \)-modules. Indeed, for a holomorphic vector field \( V \) on \( X \), we have

\[
V(t^i) = t^{i+1} V(f) + t^i V(g), \quad V(f^{i-1}(1-i)! \d i f^{i-1} V(f) + V(g)f^{i-1}(1-i)! \d i f^{i-1} = 0.
\]

For \( i > 0 \), we have \( \partial_t t^i = f t^{i-1} + f t^i \) which is mapped to \( i(-1)^i (i-1)! f^{i-1} + f(-1)^i i! f^{i-1} = 0 \). We also have \( \partial_t t^0 + f t^0 = f \) which is mapped to 1. Hence, we obtain the morphism of the \( \mathcal{D} \)-modules:

\[
\text{Cok} \left( \mathcal{O}_X(*D)[t] \xrightarrow{\partial^+_t} \mathcal{O}_X(*D)[t] \right) \rightarrow L_*(g,D),(Z_f) / L_*(g,D)
\]

(22)

We consider filtrations \( \mathcal{F}_j(\mathcal{O}_X(*D)[t]) = \bigoplus_{i \leq j} \mathcal{O}_X(*D)[t^i] \) and \( \mathcal{F}_j L_*(g,D),(Z_f) = L_*(g,D)((j+1)Z_f) \). It is easy to see that the induced morphism on the graded modules is an isomorphism. Hence, (22) is an isomorphism. Thus, we obtain the first half of \([15]\. Let us prove \([17]\. Let \( \mathcal{O}_X(*D)[t, t^{-1}] := \mathcal{O}_X(*D) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \). Let \( \iota : X \times \{0\} \rightarrow X \times \mathbb{P}^1 \) denote the inclusion. We identify \( L_*(F, D^{(1)}_\iota)/L_*(F, D^{(0)}_\iota) \) with the \( \mathcal{O}_Y \)-module

\[
\mathcal{M}_1 := \iota_* \left( \mathcal{O}_X(*D)[t, t^{-1}] / \mathcal{O}_X(*D)[t] \right) v
\]

and the connection \( \nabla \) given by \( \nabla v = vd(F) \). Then, \( \pi_+ \left( L_*(F, D^{(1)}_\iota)/L_*(F, D^{(0)}_\iota) \right) \) is represented by the complex \( \mathcal{M}_1 \xrightarrow{\partial^+_t} \mathcal{M}_1 \). Here, the second term sits in the degree 0. The kernel and the cokernel are denoted by \( \text{Ker}_1 \).
and Cok$_1$, respectively. For the identification $\mathcal{M}_1 = \bigoplus_{j=1}^{\infty} \mathcal{O}_X(*D)t^{-j}v$, we set $\mathcal{F}_n := \bigoplus_{j=n}^{\infty} \mathcal{O}_X(*D)t^{-j}v$. We have $\partial_t + f : \mathcal{F}_n \rightarrow \mathcal{F}_{n+1}$, which induces an isomorphism $\text{Gr}_n^{\mathcal{F}} \simeq \text{Gr}_{n+1}^{\mathcal{F}}$ for $n \geq 1$. Hence, it is easy to check $\text{Ker}_{n} = 0$, and that the natural $\mathcal{O}_X$-morphism $L_{\ast}(g, D) \rightarrow \mathcal{O}_X(*D)t^{-1}v = \mathcal{F}_1$ induces an isomorphism $L_{\ast}(g, D) \rightarrow \text{Cok}_1$ which is compatible with the flat connection.

Let us observe (14). We consider $\mathcal{M}_2 = \mathcal{O}_X(*D)[t,t^{-1}]v$ with $\nabla v = vdF$. As before, $\pi_{\ast}(L_{\ast}(F, D_{\ast}))$ is represented by $\mathcal{M}_2 \rightarrow \mathcal{M}_2$, where the second term sits in the degree 0. The kernel and the cokernel are denoted by Ker$_2$ and Cok$_2$, respectively. We set $\mathcal{M}_0 := \mathcal{O}_X(*D)[t]v$. We have the natural exact sequence $0 \rightarrow \mathcal{M}_0 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_1 \rightarrow 0$. Because $\mathcal{M}_i \rightarrow \mathcal{M}_i$ $(i = 0, 1)$ are monomorphisms, we obtain that Ker$_2 = 0$. We also have the following exact sequence of $D$-modules:

$$0 \rightarrow L_{\ast}(g, D)(*Z_{\ast})/L_{\ast}(g, D) \rightarrow \text{Cok}_2 \rightarrow L_{\ast}(g, D) \rightarrow 0$$

It implies that Cok$_2(*Z_{\ast}) \simeq L_{\ast}(g, D)(*Z_{\ast})$. Hence, we have a uniquely induced morphism $L_{\ast}(g, D)(*Z_{\ast}) \rightarrow \text{Cok}_2$. Because Cok$_2(*D) \simeq \text{Cok}_2$, we have a uniquely induced morphism

$$\rho : L_{\ast}(g)(!*Z_{\ast})(*D) \simeq L_{\ast}(g, D)(*Z_{\ast})(!*D) \rightarrow \text{Cok}_2.$$

Let us prove that the morphism is an isomorphism. Around any point of $Z_{\ast} \setminus D$, we have $\psi_{\ast}(\text{Cok}_2) \simeq \psi_{\ast}(L_{\ast}(g))$ and $\phi_{\ast}(\text{Cok}_2) \simeq \phi_{\ast}(L_{\ast}(g)(*Z_{\ast})/L_{\ast}(g))$. By a direct computation, we can check that the natural morphism $\psi_{\ast}(\text{Cok}_2) \rightarrow \phi_{\ast}(\text{Cok}_2)$ is non-zero. It implies that $\psi_{\ast}(\text{Cok}_2) \rightarrow \phi_{\ast}(\text{Cok}_2)$ is an isomorphism. Hence, $\rho$ is an isomorphism on $X \setminus D$, and then it follows that $\rho$ is an isomorphism on $X$. We also obtain the second isomorphism in (15), and that the push-forward of (18) is isomorphic to (19).

2.4.4 Proof of Proposition 2.27

Let us prove the claim in the case $\ast = \ast$. We have the following exact sequence:

$$0 \rightarrow \Pi_{\ast}^{-1,b}L_{\ast}(F)(*t) \rightarrow \Pi_{\ast}^{b}L_{\ast}(F)(*t) \rightarrow L_{\ast}(F)(*t)s^{\ast} \rightarrow 0$$

Then, we obtain $\pi_{\ast}^{0}(\Pi_{\ast}^{b}L_{\ast}(F)(*t)) = 0 (j \neq 0)$ by applying Proposition 2.25 and an easy induction. We also have the exact sequence:

$$0 \rightarrow \pi_{\ast}^{0}(\Pi_{\ast}^{-1,b}L_{\ast}(F)(*t)) \rightarrow \pi_{\ast}^{0}(\Pi_{\ast}^{b}L_{\ast}(F)(*t)) \rightarrow L_{\ast}(g)(*(f)_{0})(*D) \rightarrow 0$$

By an easy induction, we obtain that the following natural morphisms are isomorphisms:

$$\pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(*t)) \rightarrow \pi_{\ast}^{0}(\Pi_{\ast}^{b}L_{\ast}(F)(*t))((*D)) \rightarrow \pi_{\ast}(\Pi_{\ast}^{-1,b}L_{\ast}(F)(*t))((*(f)_{0})(*D)).$$

Hence, it is enough to obtain an isomorphism $\pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(*t)) \simeq \Pi_{\ast}^{b}L_{\ast}(g)$ on $X \setminus (D \cup Z_{\ast})$.

We have the following representative of $\pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(*t))$:

$$\bigoplus_{j=a}^{b-1} \mathcal{O}_X(*D)[t]s^{j} \rightarrow \bigoplus_{j=a}^{b-1} \mathcal{O}_X(*D)[t]s^{j}$$

Here, the morphism $\kappa$ is given by $\partial_t + f + st^{-1}$. The action of any vector field $V \in \Theta_X$ is given by $V(s^{j}) = (V(g)+tV(f))s^{j}$. The kernel of $\kappa$ is clearly 0. The natural inclusion of $\bigoplus_{j=a}^{b-1} L_{\ast}(g)t^{-1}s^{j}$ into $\bigoplus_{j=a}^{b-1} L_{\ast}(g)[t]t^{-1}s^{j}$ induces an isomorphism of $\bigoplus_{j=a}^{b-1} L_{\ast}(g)t^{-1}s^{j}$ and the cokernel of $\kappa$. We have $V(s^{j}t^{-1}) = V(g)s^{j}t^{-1} + V(f)s^{j}$. Because $V(s^{j}) = fs^{j} + s^{j+1}t^{-1}$, we have $V(f)s^{j} = gV(f)s^{j} + s^{j+1}t^{-1}$ on $X \setminus (Z_{\ast} \cup D)$. Thus, we obtain the claim in the case $\ast = \ast$.

Let us consider the case $\ast = \ast$. We can deduce it from the claim in the case $\ast = \ast$ by using the duality. But, we give a more direct argument which would be instructive for our study later. As in the case of $\ast = \ast$, we have $\pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(tt)) = 0 (j \neq 0)$ and the following natural isomorphisms:

$$\pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(tt)) \rightarrow \pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(tt))((D)) \rightarrow \pi_{\ast}(\Pi_{\ast}^{b}L_{\ast}(F)(tt))((Z_{\ast})(D)).$$

15
Hence, it is enough to obtain an isomorphism \( \pi^0_+ (\Pi^{a,b}_t L_t(F)(t)) \simeq \Pi^{a,b}_{t-1} L_t(g) \) on \( X \setminus (D \cup Z_f) \).

We have the following representative of \( \pi^0_+ (\Pi^{a,b}_t L_t(F)(t)) \) on \( X \setminus (Z_f \cup D) \):

\[
\bigoplus_{j=0}^{b-1} \mathcal{O}_X[t]s^j \xrightarrow{\kappa} \bigoplus_{j=0}^{b-1} \mathcal{O}_X[t]s^j
\]

The morphism \( \kappa \) and the action of \( V \in \Theta_X \) are given by the same formula. The natural inclusion of \( \bigoplus_{j=0}^{b-1} L_t(g)s^j \) into \( \bigoplus_{j=0}^{b-1} L_t(g)[t]s^j \) induces an isomorphism of \( \bigoplus_{j=0}^{b-1} L_t(g)[t]s^j \) and the cokernel of \( \kappa \). On \( X \setminus (Z_f \cup D) \), we have \( V(s^j) = V(g)s^0 + V(f)ts^1 \) and \( \kappa(s^j+1) = f^0s^0 + s^1 \). Hence, we have \( V(f^0s^0) = V(g)(f^0s^0) + fV(f^{-1})f^0s^1 \).

Thus, we obtain the claim in the case \( \star = ! \).

**Remark 2.28** Let us look at the restriction of the natural morphism \( \pi^0_+ \Pi^{a,b}_t L_t(F)(t) \to \pi^0_+ \Pi^{a,b}_t L_t(F)(*t) \) to \( X \setminus (D \cup Z_f) \). Because \( \kappa(s^j) = f^0s^0 + s^1t \), we have \( fs^1 = -t^{-1}s^1 \) in \( \pi^0_+ \Pi^{a,b}_t L_t(F)(*t) \).

Hence, under the above isomorphisms, it is identified with \( \Pi^a_t L(g) \to \Pi^a_t L(g) \) induced by the multiplication of \(-s\).}

### 2.4.5 A consequence

We continue to use the notation in 2.4.1. For simplicity, we consider the case \( g = 0 \). Motivated by the descriptions of some hypergeometric systems in \[4\] and \[50\], we give a remark on another description of \( \pi^0_+ (L_t, D^0 Y(t)) \). We set \( Y_0 := X \times \mathbb{P}^1_t \times \mathbb{P}^1_s \). Let \( p_i \) denote the projection of \( Y_0 \) onto the \( i \)-th component. We set \( D_0 := \mathbb{P}^1_t(D) \cup \mathbb{P}^1_s(\{0, \infty\}) \cup \mathbb{P}^1_s(\{0, \infty\}) \). We regard \( t \) and \( s \) as meromorphic functions on \((Y_0, D_0)\). Let us consider \( L_*(t(f+s), D_0) \).

Let \( \pi_{ts} : X \times \mathbb{P}^1_t \times \mathbb{P}^1_s \to X, \pi_t : X \times \mathbb{P}^1_t \to X \) and \( \pi_s : X \times \mathbb{P}^1_s \to X \) denote the projections. Set \( D_0 = (X \setminus \{0, \infty\}) \cup (D \times \mathbb{P}^1_t) \), and \( D_0 = (X \setminus \{0, \infty\}) \cup (D \times \mathbb{P}^1_s) \).

**Proposition 2.29** We have the following natural isomorphisms:

\[
\pi_{ts+} \mathcal{O}_{X \times \mathbb{P}^1_t} (t(f+s), D_0) \simeq \pi_{ts+} \mathcal{O}_X (t(f+s), D_0) \simeq \pi_{ts+} \mathcal{O}_X (t(f+s), D_0) \simeq \mathcal{O}_X (t(f) e)(D_0).
\]

**Proof** Let \( p_{12} \) and \( p_{13} \) denote the projection of \( Y_0 \) onto \( X \times \mathbb{P}^1_t \) and \( X \times \mathbb{P}^1_s \), respectively. By Proposition 2.28, we have

\[
p_{13+} L_* (t(f+s), D_0) \simeq \mathcal{O}_{X \times \mathbb{P}^1_t} (t(f+s), D_0) \simeq \mathcal{O}_X (t(f+s), D_0) = 0 \quad (i \neq 0).
\]

We also have the following by Proposition 2.28

\[
p_{12+} L_* (t(f+s), D_0) \simeq L_* (t(f+s), D_0) = 0 \quad (i \neq 0).
\]

Then, the claim of the proposition follows.

### 2.4.6 Complement for a non-resonant case

We give a remark on a non-resonant case. Take \( \alpha \in \mathbb{C} \setminus \mathbb{Z} \). We consider the line bundle \( L_{\alpha,Y}(t) := \mathcal{O}_Y (D^0_Y(t)) e \), with the flat connection \( \nabla e = e(d(f)+\alpha df)/t \), where \( e \) is a global frame. Similarly, we consider the line bundle \( L_{\alpha,X} := \mathcal{O}_X (D \cup \{0\}) v \) with the connection \( \nabla v = v (-\alpha df / f) \). The following is essentially contained in Theorem 1.5 of 2.

**Proposition 3.1** We have a natural isomorphism \( \pi^0_+ L_{\alpha,Y}(t) \simeq L_{\alpha,X} \).

**Proof** We use the notation in 2.4.1. Indeed, \( \pi^0_+ L_{\alpha,Y}(t) \) is represented by the complex

\[
\mathcal{O}_X (D)[t, t^{-1}] e \xrightarrow{\alpha} \mathcal{O}_X (D)[t, t^{-1}] e.
\]

Here, the second term sits in the degree 0, and \( a(g) = (t \alpha_0 + \alpha + t f) g \). The action of a holomorphic vector field \( V \) on \( t e \) is given as \( V(t^e) = t^{-1} (Vf) e \). We define an \( \mathcal{O}_X (D) \)-homomorphism \( \Phi : \mathcal{O}_X (D)[t, t^{-1}] e \to \mathcal{O}_X (D) v \) by \( \Phi (f e) = \Gamma(-\alpha + 1) \Gamma(-\alpha + j - 1) f^{-j} v \). Then, we can check that \( \Phi \) is compatible with the connections, and \( \Phi \circ a = 0 \). It induces a morphism of \( \mathcal{D} \)-modules \( \pi^0_+ L_{\alpha,Y}(t) \to L_{\alpha,X} \). We can check that the induced morphism is an isomorphism.
2.5 Specialization

2.5.1 Statement

Let $X$ be a complex manifold with a simple normal crossing hypersurface $D$. Let us consider meromorphic functions $f$ and $g$ on $(X, D)$. We set $X^{(1)} := X \times \mathbb{C}_\tau$ and $D^{(1)} := D \times \mathbb{C}_\tau$. We have the meromorphic function $\tau f + g$ on $(X^{(1)}, D^{(1)})$ and the associated $D$-modules $M_{\tau f + g} := L_*(\tau f + g, D^{(1)})$ on $X^{(1)}$ for $\tau = *, !$. Let $K_{\tau f + g}$ and $C_{\tau f + g}$ denote the kernel and the cokernel of $M_{\tau f + g} \rightarrow M_{\tau f + g}(\tau)$ for $\tau = *, !$. Let $i_1 : X \rightarrow X^{(1)}$ be given by $i_1(Q) = (Q, 0)$. We shall prove the following proposition in [2.5.2, 2.5.4]

Proposition 2.31 If $|f|_0 \cap |(f)_\infty| = \emptyset$, we have the following:

\[ C_{\tau f + g} \simeq i_0 + L_* (g, D), \quad K_{\tau f + g} \simeq i_0 + L_* (g, D)(|f|_\infty). \]

\[ K_1_{\tau f + g} \simeq i_0 + L_1 (g, D), \quad C_{\tau f + g} \simeq i_0 + L_1 (g, D)(|f|_\infty). \]

2.5.2 The case $g = 0$ and $D = |(f)_\infty|$

First, we consider the case $g = 0$ and $D = |(f)_\infty|$. In this case, $\tau f$ is non-degenerate along $D^{(1)}$. Hence, we have $M_{\tau f, 0} = M_{\tau f, 0}$ which we denote by $M_f$. We also have $K_{\tau f, 0} \simeq K_{\tau f, 0}$ and $C_{\tau f, 0} \simeq C_{\tau f, 0}$. We have a global section $v$ of $M_f$ such that $M_f = O_{X^{(1)}}(\tau D^{(1)})(v)$ with $\nabla v = v d(\tau f)$. We described the $V$-filtration $U_0(M_f)$ of $M_f$ along $\tau = 0$ in [27], which we recall here. We use the convention that $\tau_0 + \alpha$ is locally nilpotent on $U_\alpha / U_{\alpha, \tau_0}$. We describe the filtration locally around any point of $(Q, 0) \in D \times \{0\}$. Suppose $Q \not\in |(f)_\infty|$. We have $Gr^r_0(M_f(\tau)) = 0$ unless $j \in \mathbb{Z}$, and $U_j(M_f(\tau)) = \tau^{-j} M_f$ for $j \in \mathbb{Z}$. Suppose $Q \in |(f)_\infty|$. We take a holomorphic coordinate system $(z_1, \ldots, z_n)$ around $Q$ such that $D = \{z_i = 0\}$ and $f = z^k$ for some $k \in \mathbb{Z}_{>0}$. For $0 \leq \alpha < 1$, we set $p = [\alpha k] := ([\alpha k_1], \ldots, [\alpha k_n])$, where $[a] := \max\{n \in \mathbb{Z} | n \leq a\}$. Let $\delta = (1, \ldots, 1) \in \mathbb{Z}_{>0}^\ell$. Let $\pi : X^{(1)} \rightarrow X$ denote the projection. We may naturally regard $\pi^* D_X$ as a sheaf of subalgebras in $D_{X^{(1)}}$. Locally around $(Q, 0)$, we have

\[ U_\alpha(M_f(\tau)) = \pi^* D_X(O_{X^{(1)}}(x - \delta^p v) = \pi^* D_X \left( \sum_{j=0}^{\infty} O_{X^{(1)}}(x - \delta^p (\tau f)^j v) \right) \]

in $O_{X^{(1)}}(\tau D^{(1)})v$. We have $U_1(M(\tau)) = \tau^{-1} U_0 M$. We obtain

\[ U_{<0} M + \tau \partial_\tau U_0 M = \pi^* D_X \left( \sum_{j=1}^{\infty} O_{X^{(1)}}(x - \delta^j (\tau f)^j v) \right). \]

Hence, $U_0 M / (U_{<0} M + \tau \partial_\tau U_0 M) \simeq \tau_0 \left( D_X(O_X(D)) \right) \simeq \tau_0 O_X(D)$. It implies $C_{\tau f, 0} \simeq \tau_0 O_X(D)$ if $D = |(f)_\infty|$. By using the duality, we obtain $K_{\tau f, 0} \simeq \tau_0 O_X(D)$ if $D = |(f)_\infty|$. We consider the case $|g|_0 \cap |(g)_\infty| = \emptyset$. We define $M_0 := L(\tau f)$. Set $M := M_{\tau f, g}$. We put $D_0 := |(f)_\infty| \cup |(g)_\infty|$. We have the hypersurface $D_1 \subset D$ such that $D_0 \cup D_1 = D$ and $\text{codim}_X(D_0 \cap D_1) \geq 2$. We set $D_2 := |(g)_\infty| \cup D_1$. We have $M \simeq M_0 \otimes L_*(g, D_2^{(1)})$. We naturally have $M(\tau) \simeq M_0(\tau) \otimes L_*(g, D_2^{(1)})$. We shall observe that $M(\tau) \simeq M_0(\tau) \otimes L_*(g, D_2^{(1)})$.

We have the $V$-filtration $U_\tau (M_0(\tau))$ of $M_0(\tau)$. Set $U_\alpha (M(\tau)) := U_\alpha (M_0(\tau)) \otimes L_*(g, D_2^{(1)})$ for any $\alpha \in \mathbb{R}$.

Lemma 2.32 $U_\tau (M(\tau))$ is the $V$-filtration of $M(\tau)$.

**Proof** By the construction, $\tau \partial_\tau + \alpha$ is locally nilpotent on $U_\alpha M(\tau)/U_{<\alpha} M(\tau)$. Let us prove that $U_\alpha M$ is $VD_{X^{(1)}}$-coherent. If $Q \not\in |(f)_\infty|$, the claim is clear. Let us consider the case $Q \in |(f)_\infty|$. We take a holomorphic
coordinate system \((x_1, \ldots, x_n)\) around \(Q\) such that \(D = \bigcup_{i=1}^f \{x_i = 0\}\) and \(g = x^{-a}\) and \(f = f_1 x^{-b}\), where \(f_1\) is nowhere vanishing. Here \(a, b \in \mathbb{Z}_{\geq 0}\). Put \(p := [a b]\. Let \(\delta = (1, \ldots, 1) \in \mathbb{Z}^f\).

We use the identifications \(M_0 = \mathcal{O}_X^{\mathbb{N}}((s(f))_1^\infty) v \) with \(\nabla v = v d(\tau f)\), and \(L_*(g, D_1^{(1)}) = \mathcal{O}_X((s D_1^{(1)}) e\) with \(\nabla e = e dg\). It is enough to prove that \(U_\alpha(M(\mathcal{M}_s^r)) \otimes L_*(g, D_2^{(1)})\) is generated by \(x^{-\delta - p} v \otimes e\) over \(\mathcal{V}_X(\mathcal{M}_s^r)\).

Set \(s(a) := \{i \mid a_i \neq 0\}\) and \(s(b) := \{i \mid b_i \neq 0\}\). For \(i \in s(a) \cup s(b)\), and for any \(q \in \mathbb{Z}_{\geq 0}\), we have
\[\partial_i x_i (x^{-\delta - p} v \otimes e) = x^{-\delta - p} q (-pi - qi - ai x^{-a} - bi \tau f_1 x^{-b} + \tau x_i x^{-b} \partial_i f_1) v \otimes e.\]

Because \(\tau \partial_i (v \otimes e) = f_1 \tau x^{-b} (v \otimes e)\), we have
\[\partial_i x_i (x^{-\delta - p} v \otimes e) + (bi - \tau f_1 x_i \partial_i f_1) \tau \partial_i (x^{-\delta - p} v \otimes e) = x^{-\delta - p} q (-pi - qi - ai x^{-a} - bi \tau f_1 x^{-b} + \tau x_i x^{-b} \partial_i f_1) v \otimes e.\]
(24) By using (24), we obtain that \(x^{-\delta - p} v \otimes e \in \mathcal{M}_s^r(\mathcal{V}_X(\mathcal{M}_s^r)(v \otimes e)\). Let \(\delta_1 := (1, \ldots, 1) \in \mathbb{Z}^{s(b)}.\) For any \(Q \in \pi^* \mathcal{D}_X\) and for any \(h \in \mathcal{O}_X((s D_1^{(1)})\), we have \(\partial_i (Q(x^{-\delta_1} v) \otimes he) = \partial_i Q(x^{-\delta_1} v) \otimes he + Q(x^{-\delta_1} v) \otimes \partial_i (he).\)

Hence, we can easily deduce that \(U_\alpha(M) \subset \mathcal{V}_X(\mathcal{M}_s^r)x^{-\delta - p} (v \otimes e)\). Thus, we obtain the lemma.

We set \(U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)}) : = U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\) for any \(\alpha \in \mathbb{R}\).

Lemma 2.33 \(U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\) is the \(V\)-filtration of \(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)}\).

**Proof** For \(\alpha < 1\), we have \(U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)}) = U_\alpha(M_0) \otimes L_*(g, D_2^{(1)})\), which is coherent over \(\mathcal{V}_X(\mathcal{M}_s^r)\). For \(\alpha \geq 1\), we have \(U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)}) = \sum_{\beta + \gamma < \alpha} \partial_\gamma u_\beta (M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\). We obtain that \(U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\) are coherent over \(\mathcal{V}_X(\mathcal{M}_s^r)\) for any \(\alpha\). We have \(\tau \partial + \alpha \) are nilpotent on \(U_\alpha U_{\leq \alpha}\).

Hence, \(U_\alpha(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\) is the \(V\)-filtration of \(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\).

Because the induced morphism
\[\partial \mathcal{M}_s^r_0 (\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})/U_{\leq 0} (\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)}) \rightarrow U_1 (\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})/U_{< 1} (\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\]
(25) is an isomorphism, we have \(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)}) \simeq \mathcal{M}_s^r(\mathcal{M}_s^r)\). We obtain \(C_\alpha g (\gamma_0) \simeq M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\) \(\simeq \mathcal{M}_s^r(\mathcal{M}_s^r)\). We also obtain \(K_\alpha g \simeq M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\). Under the assumption \(|\gamma_0| \cap |\gamma_0| = \emptyset\), it is naturally isomorphic to \(M_0(\mathcal{M}_s^r) \otimes L_*(g, D_2^{(1)})\).

Thus, we obtain the claims for \(K_\alpha g \) and \(C_\alpha g \) in the case \(|\gamma_0| \cap |\gamma_0| = \emptyset\). By using the duality, we also obtain the claims for \(C_\alpha g \) and \(K_\alpha g \) in this case.

2.54 The general case

Let us consider the general case. We take a projective algebro morphism of complex manifolds \(G : X' \rightarrow X\) such that (i) \(D' := G^{-1}(D)\) is a simple normal crossing hypersurface, (ii) \(X \cup D' \simeq X \setminus D\), (iii) \(|g'|_0 \cap |g'|_0 = \emptyset\), where \(f' := G^*(f)\). We set \(g' := G^*(g)\). We have \(f'_0 = G^*((f)_0)\) and \(f'_0 = G^*((f))_0\). In particular, we have \(|(f')_0| \cap |(f')_0| = \emptyset\). We have \(C_\alpha g \simeq L_*(g', D')\) and \(K_\alpha g \simeq L_*(g', D')\).

The induced morphism \(X^{(1)} \rightarrow X^{(1)}\) is also denoted by \(G\). We have \(G^0_+(\mathcal{M}_s^{f, g'}) \simeq M_{s, f, g}\) and \(G^0_+(\mathcal{M}_s^{f, g'}) = 0\) for \(i \neq 0\). We obtain \(G_+^{\mathcal{M}_s^{f, g'}}(\mathcal{M}_s^{f, g'}) \simeq M_{s, f, g}\) and \(G_+^{\mathcal{M}_s^{f, g'}}(\mathcal{M}_s^{f, g'}) = 0\) for \(i \neq 0\). We have \(G_+^{\mathcal{M}_s^{f, g'}}(L_*(g', D')) \simeq L_*(g', D')\) and \(G_+^{\mathcal{M}_s^{f, g'}}(L_*(g', D')) = 0\) for \(i \neq 0\). We also have \(G_+^{\mathcal{M}_s^{f, g'}}(L_*(g', D')) \simeq L_*(g', D')\) and \(G_+^{\mathcal{M}_s^{f, g'}}(L_*(g', D')) = 0\) for \(i \neq 0\). Let \(I\) be the image of \(\mathcal{M}_s^{f, g'}(\mathcal{M}_s^{f, g'}) \rightarrow \mathcal{M}_s^{f, g'}(\mathcal{M}_s^{f, g'})\). Because \(G_+^{\mathcal{M}_s^{f, g'}}(\mathcal{M}_s^{f, g'}) \simeq C_\alpha g \) for \(i \neq 0\), we obtain that \(G_+^{\mathcal{M}_s^{f, g'}}(\mathcal{M}_s^{f, g'}) = 0\) for \(i \neq 0\). Hence, we obtain \(G_+^{\mathcal{M}_s^{f, g'}}(\mathcal{M}_s^{f, g'}) \simeq C_\alpha g \). Thus, we obtain the claims for \(C_\alpha g \) and \(K_\alpha g \) by duality. By using the duality, we obtain the claims for \(C_\alpha g \) and \(K_\alpha g \). Thus, the proof of Proposition 2.31 is finished.
2.6 Nearby cycle functor and Push-forward

2.6.1 Beilinson’s functors and variants

Let $h$ be a meromorphic function on a complex manifold $Y$. Let $M$ be a holonomic $\mathcal{D}_Y$-module. Suppose $|(h)_0| \cap |(h)_\infty| = \emptyset$. We recall the functors of Beilinson [1]:

$$\psi^{(a)}_h(M) := \lim \operatorname{Cok}(\Pi^{\alpha,b}_h M((h)_0) \to \Pi^{\alpha,b}_h M((h)_0))$$

$$\Xi^{(a)}_h(M) := \lim \operatorname{Cok}(\Pi^{\alpha-1,b}_h M((h)_0) \to \Pi^{\alpha,b}_h M((h)_0)).$$

On any relatively compact subset $K$ in $Y$, if $b$ is sufficiently large, the cokernel of $\Pi^{\alpha,b}_h M((h)_0) \to \Pi^{\alpha,b}_h M((h)_0)$ is isomorphic to $\psi^{(a)}_h(M)$ and the kernel is isomorphic to $\psi^{(b)}_h(M)$. On $K$, if $N$ is sufficiently large, $\Xi^{(a)}_h(M)$ is isomorphic to the cokernel of $\Pi^{\alpha+1,a+N}_h M((h)_0) \to \Pi^{\alpha,a+N}_h M((h)_0)$ and the kernel of $\Pi^{\alpha-N,a+1}_h M((h)_0) \to \Pi^{\alpha-N,a}_h M((h)_0)$. (See the argument in [26] 4.1.4, for example.)

In the case $|(h)_0| \cap |(h)_\infty| \neq \emptyset$ and $M = \mathcal{O}_Y(D)$ for a normal crossing hypersurface $D \subset Y$, we also consider a variant. For simplicity, we assume that (i) $(h)_\infty$ is smooth and reduced, (ii) $D = |(h)_0|$, (iii) $D \cup |(h)_\infty|$ is normal crossing. We set $Z := |(h)_0|$.

**Lemma 2.34** We have a canonical morphism $\Pi^{\alpha+1,b+1}_h \mathcal{O}_Y \to \Pi^{\alpha,b}_h \mathcal{O}_Y(\!\!|D\!\!)$ such that the composite with $\Pi^{\alpha,b}_h \mathcal{O}_Y(\!\!|Z\!\!) \to \Pi^{\alpha,b}_h \mathcal{O}_Y$ is the canonical one.

**Proof** Let $K$ and $C$ denote the kernel and the cokernel of the morphism $\Pi^{\alpha+1,b+1}_h \mathcal{O}_Y(\!\!|Z\!\!) \to \Pi^{\alpha,b+1}_h \mathcal{O}_Y$. We set $D_Z := D \cap Z$. Let $\iota : Z \to Y$ denote the inclusion. By a direct computation, we can check $C \simeq \iota_+ \mathcal{O}_Z(\!\!|D_Z\!\!)s^a$ and $K \simeq \iota_* \mathcal{O}_Z(\!\!|D_Z\!\!)s^b$. We have the induced monomorphism $\Pi^{\alpha+1,b+1}_h \mathcal{O}_X(\!\!|Z\!\!/K) \to \Pi^{\alpha,b+1}_h \mathcal{O}_X$.

Let us consider the canonical morphism $\Pi^{\alpha+1,b+1}_h \mathcal{O}_Y \to \Pi^{\alpha,b+1}_h \mathcal{O}_Y$. The composite with $\Pi^{\alpha,b+1}_h \mathcal{O}_Y \to C$ is 0. Hence, we have an induced morphism $\Pi^{\alpha+1,b+1}_h \mathcal{O}_Y \to \Pi^{\alpha,b+1}_h \mathcal{O}_Y(\!\!|K\!\!)$. Hence, we obtain $\Pi^{\alpha+1,b+1}_h \mathcal{O}_Y \to \Pi^{\alpha,b}_h \mathcal{O}_Y(\!\!|Z\!\!)$ such that the composite with $\Pi^{\alpha,b}_h \mathcal{O}_Y(\!\!|Z\!\!) \to \Pi^{\alpha,b}_h \mathcal{O}_Y$ is the canonical one.

Let us consider the following induced morphism

$$\rho_N : \Pi^{\alpha+1,a+N}_h \mathcal{O}_Y((h)_\infty)(\!\!|D\!\!) \to \Pi^{\alpha,a+N}_h \mathcal{O}_Y((h)_\infty)((h)_0). \quad (26)$$

We set $\Xi^{(a)}_h(\mathcal{O}_X(\!\!|D\!\!)) := \lim \operatorname{Cok}(\rho_N)$. 

**Lemma 2.35** For any relatively compact subset $K \subset X$, there exists $N_0$ such that $\operatorname{Cok}(\psi^{(a)}_N)_K \to \operatorname{Cok}(\psi^{(a)}_N)_K$ is an isomorphism if $N_0 \leq N_1 \leq N_2$.

**Proof** As mentioned above, the claim is well known outside of $Z$. Hence, the induced morphism

$$\operatorname{Cok}(\rho_{N_2})_K((h)_\infty) \to \operatorname{Cok}(\rho_{N_1})_K((h)_\infty)$$

is an isomorphism. Take $Q \in Z \cap K$, and take a holomorphic local coordinate neighbourhood $(Y_Q; x_1, \ldots, x_n)$ of $Y$ around $Q$ such that $h_1|_{Y_Q} = x_n^{-1} \prod_{i=1}^t x_i^{k_i}$, where $k_i \in \mathbb{Z}_{>0}$. It is enough to prove that $\phi^{(0)}_{x_n} \operatorname{Cok}(\rho_{N_2})_K \to \phi^{(0)}_{x_n} \operatorname{Cok}(\rho_{N_1})_K$ is an isomorphism. Set $h_1 := \prod_{i=1}^t x_i^{k_i}$ on $Z_Q := Y_Q \cap Z$. Let $\iota_1 : Z_Q \to Y_Q$ be the inclusion. We have the following commutative diagram:

$$\phi^{(0)}_{x_n} \Pi^{\alpha+1,a+N}_h \mathcal{O}_Y((h)_\infty) \to \phi^{(0)}_{x_n} \Pi^{\alpha,a+N}_h \mathcal{O}_Y((h)_\infty)((h)_0)$$

$$\downarrow \simeq$$

$$\iota_{1+\Pi^{\alpha+1,a+N}} \mathcal{O}_{Z_Q}((h)_1) \to \iota_{1+\Pi^{\alpha,a+N}} \mathcal{O}_{Z_Q}$$

Hence, $\phi^{(0)}_{x_n} \operatorname{Cok}(\rho_{N_2})$ is identified with $\iota_{1+\mathcal{O}_{Z_Q}}(\Pi^{\alpha,a+N}_{h_1} \mathcal{O}_{Z_Q}((h)_1) \to \Pi^{\alpha,a+N}_{h_1} \mathcal{O}_{Z_Q})$. Then, the claim is reduced to the above standard case. 

19
2.6.2 The push-forward of the nearby cycle sheaf

Let $X$ be a complex manifold with a normal crossing hypersurface $D$. Let $f$ and $g$ be meromorphic functions on $(X, D)$. We set $Y := \mathbb{P}^1 \times X$ and $D^{(1)} := (\mathbb{P}^1 \times D) \cup \{\infty\} \times X$. Let $\pi : Y \to X$ be the projection. We consider $F := \tau f + g$ on $(Y, D^{(1)})$. We suppose the following.

- $F$ is pure on $\{\tau \neq 0\} \times X$.
- $g$ is pure on $X \setminus |(f)_{\infty}|$.
- $D = |(f)_{\infty}| \cup |(g)_{\infty}|$, $|(f)_{0}| \subset D$ and $|(f)_{0}| \cap |(f)_{\infty}| = \emptyset$.

Note that we have $\pi^0_+ \psi^\tau_{(a)} L_*(F) = \pi^0_+ \psi^\tau_{(a)} L_*(F)$ and $\Xi_{f^{-1}}^a(g) \simeq \Xi_{f^{-1}}^a L_*(g)$ by the assumptions.

**Proposition 2.36** We have a natural isomorphism $\pi^0_+ \psi^\tau_{(a)} L_*(F) \simeq \Xi_{f^{-1}}^a L_*(g)$.

**Proof** By the assumption of the purity of $F$ on $\{\tau \neq 0\} \times X$, we have $\Pi_{\tau}^{a,N} L_*(F)(\tau) = \Pi_{\tau}^{a,N} L_*(F)(\tau)$. Let $\mathcal{I}$ be the image of $\Pi^a_{\tau} L_*(F)(\tau) \to \Pi^a_{\tau} L_*(F)(\tau)$. The support of the kernel $K^a_{\tau}$ and the cokernel $C^a_{\tau}$ are contained in $\tau = 0$. Hence, we have $\pi^0_+ K^a_{\tau} = 0$ and $\pi^0_+ C^a_{\tau} = 0$ for $i \neq 0$. By Proposition 2.27, we have $\pi^0_+ \Pi^{a,N} L_*(F)(\tau) = 0$ for $\tau \neq 0$ and $\tau_1, \tau_2 \in \{\ast, 1\}$. We can easily deduce that $\pi^0_+ \mathcal{I} = 0$ unless $i = 0$. Hence, we have the following exact sequence

$$0 \to \pi^0_+ K^a_{\tau} \to \pi^0_+ \Pi^a_{\tau} L_*(F)(\tau) \to \pi^0_+ \Pi^a_{\tau} L_*(F)(\tau) \to 0$$

If $|a - b|$ is sufficiently large, we have $C^a_{\tau} \simeq \psi^\tau_{(a)} L_*(F)$. Hence, we have

$$\pi^0_+ \psi^\tau_{(a)} L_*(F) \simeq \text{Cok} \left( \pi^0_+ \Pi^a_{\tau} L_*(F)(\tau) \to \pi^0_+ \Pi^a_{\tau} L_*(F)(\tau) \right)$$

By Proposition 2.27, we have the following natural isomorphisms:

$$\pi^0_+ \Pi^{a,N} L_*(F)(\tau) \simeq \Pi_{f^{-1}}^{a,N} L_*(g)(\tau)$$

By $|(g)_{\infty}| \cup |(f)_{\infty}| = D$ and the purity of $g$ on $X \setminus |(f)_{\infty}|$, we have

$$\Pi_{f^{-1}}^{a,N} L_*(g)(\tau) \simeq \Pi_{f^{-1}}^{a,N} L_*(g)(\tau), \quad \Pi_{f^{-1}}^{a,N} L_*(g)(\tau) \simeq \Pi_{f^{-1}}^{a,N} L_*(g)(\tau).$$

As in Remark 2.28, the morphism $\pi^0_+ \Pi^{a,N} L_*(F)(\tau) \to \pi^0_+ \Pi^{a,N} L_*(F)(\tau)$ is identified with

$$\Pi_{f^{-1}}^{a,N+1} L_*(g)(\tau) \to \Pi_{f^{-1}}^{a,N} L_*(g)(\tau).$$

Hence, we obtain the desired isomorphism.

Let $\iota_0 : X \to \mathbb{P}^1 \times X$ be the inclusion given by $\iota_0(P) = (0, P)$.

**Corollary 2.37** Under the assumption, we have a natural isomorphism $\psi^\tau_{(a)} L_*(F) \simeq \iota_0 \Xi_{f^{-1}}^a L_*(g)$.

We have the canonical nilpotent map $N$ on $\psi^\tau_{(a)} L_*(F)$. The following proposition is clear by the above description.

**Proposition 2.38** Under the same assumption, we have $\text{Ker} N \simeq \iota_0 L_*(g, D)$ and $\text{Cok} N \simeq \iota_0 L_*(g, D)$. The canonical morphism $\text{Ker} N \to \text{Cok} N$ is identified with the canonical morphism $L_*(g, D) \to L_*(g, D)$.

**Remark 2.39** Note that $\text{Ker} N$ and $\text{Cok} N$ are isomorphic to the kernel and the cokernel of $L_*(F)(\tau) \to L_*(F)(\tau)$. The isomorphisms in Proposition 2.31 and Proposition 2.38 are consistent.
2.6.3 A variant

Let us give a similar statement in a slightly different situation. We continue to use the notation in \(2.34, 2.35\).

Suppose that (i) \(g = 0\), (ii) \(D = \{(f)\}_{\mathbb{R}}\), (iii) \((f)_{\mathbb{R}}\) is non-singular and reduced, and \(|(f)_{\mathbb{R}}| \cup D\) is normal crossing. Note that \(F = \tau f\) is non-degenerate on \(\{\tau \neq 0\} \times X\). We set \(Z := \{(f)_{\mathbb{R}}\}\), and let \(\iota_Z : Z \hookrightarrow X\) denote the inclusion.

**Proposition 2.40** We have a natural isomorphism \(\pi_0^0 \psi_\tau^{(a)} L_s(F) \simeq \Xi_{f,0}^{(a)} \mathcal{O}_X\).

**Proof** As in the proof of Proposition 2.36, we have

\[
\pi_0^0 \psi_\tau^{(a)} L_s(F) \simeq \text{Cok} \left( \pi_+^0 \Pi^a_{\tau} \Pi^a_{\tau} + N L_!(\tau) \xrightarrow{\kappa_N} \pi_+^0 \Pi^a_{\tau} \Pi^a_{\tau} + N L_!(F)(\tau) \right).
\]

Recall that we have the following isomorphisms:

\[
\pi_+^0 \Pi^a_{\tau} \Pi^a_{\tau} + N L_!(\tau) \simeq \Pi^a_{\tau} \Pi^a_{\tau} + N \mathcal{O}_X(\tau)(D)
\] (27)

\[
\pi_+^0 \Pi^a_{\tau} \Pi^a_{\tau} + N L_!(F)(\tau) \simeq \Pi^a_{\tau} \Pi^a_{\tau} + N \mathcal{O}_X(Z)(D)
\] (28)

Then, we obtain the claim of Proposition 2.40 from the following lemma.

**Lemma 2.41** Under the isomorphisms (27) and (28), the image of \(\kappa_N\) is equal to the image \(\rho_N\) in (26) if \(N\) is sufficiently large.

**Proof** We can compare the restriction of \(\rho_N\) and \(\kappa_N\) to \(X \setminus (Z \cup D)\) as in Remark 2.28. Take any \(N_1 \geq N\). We have the morphisms \(\kappa_{N_1}\) and \(\rho_{N_1}\). The morphisms \(\kappa_N\) and \(\rho_N\) are induced by \(\kappa_{N_1}\) and \(\rho_{N_1}\). Because \((\kappa_{N_1} - \rho_{N_1})|_{X \setminus (D \cup Z)} = 0\), the difference \(\kappa_N - \rho_N\) factors through \(\iota_Z + \mathcal{O}_Z(S D Z)\), where \(D_Z := Z \cap D\). Hence, we obtain that \(\kappa_N - \rho_N = 0\). Thus, we finish the proof of Lemma 2.41 and Proposition 2.40.

**Corollary 2.42** If \(Z \cap D = \emptyset\), we have \(\pi_0^0 \psi_\tau^{(0)} L_s(tf) \simeq \Xi_{f,0}^{(0)} \mathcal{O}_X\).

Let \(Q\) be a point of \(Z \cap D\). We take a convenient coordinate system \((x_1, \ldots, x_n)\) with \(f = x_n \prod_{i=1}^\ell x_i^{-k_i}\) around \(Q\). We have the following morphism by applying \(\phi_{x_n}^{(0)}\) to the canonical morphism:

\[
\phi_{x_n}^{(0)} \Pi^{a+1,b+1}_f \mathcal{O}_X(Z)(D) \longrightarrow \phi_{x_n}^{(0)} \Pi^{a,b}_f \mathcal{O}_X(Z)(D)
\] (29)

We set \(h := \prod_{i=1}^\ell x_i^{-1}\). Then, (29) is identified with

\[
\iota_Z + \Pi^{a+1,b+1}_h \mathcal{O}_Z(D) \longrightarrow \iota_Z + \Pi^{a,b}_h \mathcal{O}_Z(D).
\]

Hence, we have natural isomorphism \(\phi_{x_n}^{(0)}(\text{Cok}) \simeq \iota_Z + \psi_\tau^{(a)}(\mathcal{O}_Z)\).

3 Mixed twistor \(\mathcal{D}\)-modules associated to meromorphic functions

3.1 Mixed twistor \(\mathcal{D}\)-modules

We recall some operations for \(\mathcal{R}\)-modules and twistor \(\mathcal{D}\)-modules. See [33, 36] and [42] for more details.

3.1.1 \(\mathcal{R}\)-modules

Let \(X\) be a complex manifold. We set \(X := \mathbb{C}_\lambda \times X\). Let \(p_X : X \longrightarrow X\) denote the projection. Let \(\mathcal{R}_X \subset \mathcal{D}_X\) be the sheaf of subalgebras generated by \(\lambda p_X^* \Theta_X\) over \(\mathcal{O}_X\). We set \(d_X := \text{dim } X\). The pull back \(p_X^* \Theta_X\) is also denoted by \(\Omega_X\). For any left \(\mathcal{R}_X\)-module \(\mathcal{M}\), \(\lambda^{-d_X} \Theta_X \otimes \mathcal{O}_X \mathcal{M}\) is naturally a right \(\mathcal{R}_X\)-module, by which the category of left \(\mathcal{R}_X\)-modules and the category of right \(\mathcal{R}_X\)-modules are equivalent. In this paper, \(\mathcal{R}_X\)-modules mean...
left $\mathcal{R}_X$-modules. Let $D^b_c(\mathcal{R}_X)$ denote the derived category of cohomologically bounded coherent complexes of $\mathcal{R}_X$-modules.

An $\mathcal{R}_X$-module is equivalent to an $O_X$-module $\mathcal{M}$ with a meromorphic relative flat connection $D^j : \mathcal{M} \to \Omega^1_{\mathcal{X}/\mathbb{C}_\lambda} \otimes \mathcal{M}$ where $\lambda_0 := \{0\} \times X$. The operator $\mathbb{D} := \lambda D^j$ is also often used, and called a family of flat $\lambda$-connections.

The easiest example of $\mathcal{R}_X$-module is the line bundle $O_X$ with the meromorphic relative flat connection $D^j$ determined by $\mathbb{D}(1) = 0$. It is just denoted by $O_X$.

Let $\mathcal{M}_i (i = 1, 2)$ be $\mathcal{R}_X$-modules. Then, $\mathcal{M}_1 \otimes_O \mathcal{M}_2$ and $\mathcal{M}_1 \oplus \mathcal{M}_2$ are naturally $\mathcal{R}_X$-modules. The tensor product $\mathcal{M}_1 \otimes_O \mathcal{M}_2$ is denoted just by $\mathcal{M}_1 \otimes \mathcal{M}_2$, if there is no risk of confusion.

We define the duality functor $D_X : D^b_c(\mathcal{R}_X) \to D^b_c(\mathcal{R}_X)$ by

$$D_X(\mathcal{M}) := R\mathbb{H}om_{\mathcal{R}_X}(\mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1})[\dim X].$$

(We will review more details in [C2].) Note that $D_XO_X$ is naturally isomorphic to $\lambda^{d\lambda}O_X = O_X(-d_x\lambda_0)$. If $\mathcal{M}$ is an $\mathcal{R}_X$-module underlying a mixed twistor $\mathcal{D}$-module, then $H^j(D_X\mathcal{M}) = 0$ unless $j = 0$. In that case, we identify $D_X\mathcal{M}$ and $H^0D_X\mathcal{M}$.

Let $f : X \to Y$ be any morphism of complex manifolds. We set

$$\mathcal{R}_{Y \leftarrow X} := \lambda^{-d\lambda}O_X \otimes f^{-1}O_Y \otimes ^{-1}(\mathcal{R}_Y \otimes \lambda^{d\lambda}O_Y).$$

It is naturally a left $f^{-1}\mathcal{R}_Y$-module and a right $\mathcal{R}_X$-module. For any object $\mathcal{M}$ in $D^b_c(\mathcal{R}_X)$, we set

$$f_!(\mathcal{M}) := Rf_!(\mathcal{R}_{Y \leftarrow X} \otimes ^L_{\mathcal{R}_X} \mathcal{M})$$

in the derived category of $\mathcal{R}_Y$-modules. (We shall review more details on the push-forward in [C2].) The $j$-th cohomology sheaves of $f_!(\mathcal{M})$ are denoted by $f_j!(\mathcal{M})$. If $\mathcal{M}$ is good relative to $f$, i.e., for any compact subset $K \subset Y$, there exists good filtration of $\mathcal{M}_{f^{-1}(K)}$. Moreover, suppose that the support of $\mathcal{M}$ is proper over $f$. Then, $f_!\mathcal{M}$ is an object in $D^b_c(\mathcal{R}_Y)$, and we have a natural isomorphism $f_!D_X\mathcal{M} \simeq D_Yf_!\mathcal{M}$.

Let $j : \mathbb{C}_\lambda \to \mathbb{C}_\lambda$ be given by $j(\lambda) = -\lambda$. The induced morphism $j \times \text{id}_\mathcal{X} : \mathcal{X} \to \mathcal{X}$ is also denoted by $j$. For any $\mathcal{R}_X$-module $\mathcal{M}$, we naturally regard $j^*\mathcal{M}$ as an $\mathcal{R}_X$-module by the natural isomorphism $j^*\mathcal{R}_X \simeq \mathcal{R}_X$.

Let $H$ be a hypersurface in $X$. We set $H := \mathbb{C}_\lambda \times H$. For any $\mathcal{R}_X$-module $\mathcal{M}$, we set $\mathcal{M}(\star H) := \mathcal{M} \otimes O_X(\star H)$. In particular, $\mathcal{R}_X(\star H) := \mathcal{R}_X \otimes O_X(\star H)$. We may naturally consider $\mathcal{R}_X(\star H)$-modules. For any $\mathcal{M} \in D^b_c(\mathcal{R}_X(\star H))$, we set

$$D_{X(\star H)}(\mathcal{M}) := R\mathbb{H}om_{\mathcal{R}_X(\star H)}(\mathcal{M}, \mathcal{R}_X(\star H) \otimes \Omega_X^{-1})[\dim X].$$

Let $f : X \to Y$ be a proper morphism of complex manifolds. Let $H_Y$ be a hypersurface in $Y$, and we set $H_X := f^{-1}(H_Y)$. For any $\mathcal{M} \in D^b_c(\mathcal{R}_X(\star H_X))$, we have

$$f_!(\mathcal{M}) := Rf_!(\mathcal{R}_{Y \leftarrow X} \otimes ^L_{\mathcal{R}_X} \mathcal{M}) \simeq Rf_!(\mathcal{R}_{Y \leftarrow X}(\star H_X) \otimes ^L_{\mathcal{R}_X(\star H_X)} \mathcal{M})$$

in the derived category of $\mathcal{R}_Y(\star H_Y)$-modules. If $f$ induces an isomorphism $X \setminus H_X \simeq Y \setminus H_Y$, we have $f_1^!\mathcal{M} = 0$ unless $j = 0$. We shall identify $f_!\mathcal{M}$ and $f_1^!\mathcal{M}$ in that case.

### 3.1.2 $\mathcal{R}$-triples

We set $S := \{\lambda \in \mathbb{C} | |\lambda| = 1\}$. Let $\mathcal{D}b_{S \times X/S}$ denote the sheaf of distributions on $S \times X$ which are continuous with respect to $S$. (See [H2].) For local sections $P \in \mathcal{R}_X|S \times X$ and $Q \in \mathcal{D}b_{S \times X/X}$, the local section $P \bullet Q := PQ$ of $\mathcal{D}b_{S \times X/S}$ is naturally defined. Let $\sigma : S \times X \to S \times X$ be given by $\sigma(\lambda, x) = (-\lambda, x)$. For local sections $\sigma^*P \in \sigma^*\mathcal{R}_X|S \times X$ and $Q \in \mathcal{D}b_{S \times X/X}$, the local section $\sigma^*(P) \bullet Q := \sigma^*(P)Q$ is defined. Thus, the sheaf $\mathcal{D}b_{S \times X/X}$ is naturally $(\mathcal{R}_X|S \times X, \sigma^*\mathcal{R}_X|S \times X)$-module.

Let $\mathcal{M}_i (i = 1, 2)$ be $\mathcal{R}_X$-modules. A sesqui-linear pairing of $\mathcal{M}_1$ and $\mathcal{M}_2$ is a $(\mathcal{R}_X|S \times X, \sigma^*\mathcal{R}_X|S \times X)$-homomorphism $C : \mathcal{M}_1|S \times X \otimes \sigma^*\mathcal{M}_2|S \times X \to \mathcal{D}b_{S \times X/S}$. Such a tuple $T = (\mathcal{M}_1, \mathcal{M}_2, C)$ is called an $\mathcal{R}_X$-triple.
Let $T_i = (M_{1i}, M_{2i}, C_i)$ be $R_X$-triples. A morphism of $R_X$-triples $\varphi : T_1 \rightarrow T_2$ is a pair of $R_X$-homomorphisms $\varphi_1 : M_{21} \rightarrow M_{11}$ and $\varphi_2 : M_{22} \rightarrow M_{22}$ such that $C(\varphi_1(m_2), \sigma^*m_1) = C(m_2, \sigma^*\varphi_2(m_1))$. For any morphism, we set $\text{Ker}(\varphi) := (\text{Cok}(\varphi_1), \text{Ker}(\varphi_2), C_{\text{Ker}\varphi})$, where $C_{\text{Ker}\varphi}$ denotes the naturally induced sesqui-linear pairing. Similarly, we set

$$\text{Cok}(\varphi) := (\text{Ker}(\varphi_1), \text{Cok}(\varphi_2), C_{\text{Cok}\varphi}), \quad \text{Im}(\varphi) := (\text{Im}(\varphi_1), \text{Im}(\varphi_2), C_{\text{Im}\varphi}).$$

For an $R_X$-triple $T = (M_1, M_2, C)$, we set $T^* := (M_2, M_1, C^*)$, where $C^*(m_2, \sigma^*m_1) := \sigma^*C(m_1, \sigma^*m_2)$. For a morphism $\varphi = (\varphi_1, \varphi_2) : T_1 \rightarrow T_2$ of $R_X$-triples, we set $\varphi^* = (\varphi_2, \varphi_1)$.

We set $U_X(a, b) := (\lambda^0O_X, \lambda^0O_X, C_0)$, where $C_0(f, \sigma g) := f \sigma g$. For any $R$-triple $T := (M_1, M_2, C)$, we define $T \otimes U_X(a, b) := (\lambda^aM_1, \lambda^bM_2, C)$, where $C$ is induced by the natural identification $(\lambda^aM_1)_{S \times X} = (\lambda^bM_2)_{S \times X}$. Particularly, $U_X(-n, n)$ is denoted by $T(n)$, called the $n$-th Tate object, and $T \otimes T(n)$ is called the $n$-th Tate twist of $T$. We use the identification $T(n) \simeq T(−n)^*$ given by the morphism $((-1)^n, (−1)^n)$.

A morphism $S : T \rightarrow T^* \otimes T(−n)$ such that $S^* = (−1)^nS$ is called a sesqui-linear duality of weight $n$. For example, we have $U_X(p, q) \rightarrow U_X(p, q)^* \otimes T(−(p − q))$ given by $((-1)^p, (−1)^q)$.

Let $f : X \rightarrow Y$ be any morphism of complex manifolds. Let $T$ be an $R_X$-triple. We have a naturally defined push-forward $f_!$ in the derived category of $R_Y$-triples. See [42]. It induces a functor from the derived category of $R_X$-triples to derived category of $R_Y$-triples. The $j$-th cohomology $R_Y$-triple of $f_!T$ is denoted by $f_!^jT$. For $T = (M_1, M_2, C)$, the $R$-triple $f_!^jT$ consists of $(f_!^jM_1, f_!^jM_2, f_!^jC)$.

Let $H$ be a hypersurface. The sheaf $\mathcal{D}B_{S \times X/S}(\ast H)$ is naturally $(R_X(\ast H)_{S \times X}, \ast^*R_X(\ast H)_{S \times X})$-modules. For $R_X(\ast H)$-modules $M_i (i = 1, 2)$, a sesqui-linear pairing of $M_1$ and $M_2$ is a $(R_X(\ast H)_{S \times X}, \ast^*R_X(\ast H)_{S \times X})$-homomorphism $C : M_1_{S \times X} \times \ast^*M_2_{S \times X} \rightarrow \mathcal{D}B_{S \times X/S}(\ast H)$. For a given $R_X$-triple $T = (M_1, M_2, C)$, we obtain an $R_X(\ast H)$-triple $T(\ast H) := (M_1(\ast H), M_2(\ast H), C(\ast H))$, where $C(\ast H)$ is a naturally induced sesqui-linear pairing of $M_1(\ast H)$ and $M_2(\ast H)$.

### 3.1.3 Mixed twistor $D$-modules

A pure twistor $D_X$-module of weight $n$ is an $R_X$-triple $T = (M_1, M_2, C)$ consisting some conditions. For example, we impose that $M_1$ are flat over $O_{C_{\lambda}}$ and that the characteristic variety of $M_i$ are contained in the product of $C_{\lambda}$ and Lagrangian varieties of $T^*X$. A polarization of $T$ is a sesqui-linear duality $S : T \rightarrow T^* \otimes T(−n)$ satisfying some conditions. The precise conditions are given based on the strategy due to M. Saito [48]. We refer the detail to [42] [45] and [32] [33]. A pure twistor $D$-module which admit a polarization is called a polarizable pure twistor $D$-module. In this paper, we consider only polarizable pure twistor $D$-modules, and hence we often omit “polarizable”.

A mixed twistor $D$-module is an $R$-triple $T$ with a finite increasing filtration $W$ by integers satisfying some conditions. For example, we impose that $G^{\mathcal{D}B}_W(T)$ are polarizable pure twistor $D$-modules of weight $n$. The precise conditions are given based on the strategy due to M. Saito [52]. We refer the detail to [36].

We recall some important properties of mixed twistor $D$-modules. Let $MTM(X)$ denote the category of mixed twistor $D$-modules on $X$.

**Proposition 3.1** The category $MTM(X)$ is abelian. More concretely, we have the following.

- Let $\varphi : (T_1, W_1) \rightarrow (T_2, W_2)$ be a morphism in $MTM(X)$. Then, the $R$-triples $\text{Ker}(\varphi)$ and $\text{Im}(\varphi)$ with the induced filtrations are mixed twistor $D$-modules. Moreover, $\varphi$ is strict with respect to the filtration $W$, and $\text{Im}(\varphi)$ with the induced filtration is a mixed twistor $D$-module.

Moreover, the category of polarizable pure twistor $D$-modules of weight $w$ is abelian and semisimple.

**Proposition 3.2** Let $f : X \rightarrow Y$ be a projective morphism of complex manifolds.

- Let $T$ be a polarizable pure twistor $D_X$-module of weight $w$. Then, $f_!^jT$ are polarizable pure twistor $D_Y$-modules of weight $w + j$.

- Let $(T, W)$ be a mixed twistor $D$-module on $X$. Let $W_k f_!^jT$ be the image of $f_!^jW_{k−j}T \rightarrow f_!^jT$. Then, $(f_!^jT, W)$ are mixed twistor $D$-modules.
Lemma 3.4 Let $\mathcal{T}$ be a mixed twistor $\mathcal{D}$-module on $X$. Let $(\mathcal{M}_1, \mathcal{M}_2, C)$ be the underlying $\mathcal{R}_X$-triple.

- $D_X\mathcal{M}_i \simeq \mathcal{H}^0(D_X\mathcal{M}_i)$.
- We have the naturally induced sesquilinear pairing $D\mathcal{C}$ of $D_X\mathcal{M}_1$ and $D_X\mathcal{M}_2$, and the $\mathcal{R}_X$-triple $D\mathcal{T} = (D\mathcal{M}_1, D\mathcal{M}_2, D\mathcal{C})$ with the induced filtration is a mixed twistor structure.
- For any morphism $\phi : X \to Y$, we have a natural isomorphism $Df_!^\dagger D\mathcal{T} \simeq f_!^{-1}D\mathcal{T}$.

3.1.4 Underlying $D$-modules

Let $(\mathcal{T}, W)$ be a mixed twistor $\mathcal{D}$-module on $X$. The $\mathcal{R}$-triple $\mathcal{T}$ is described as $(\mathcal{M}_1, \mathcal{M}_2, C)$. Let $\iota_1 : X \to X$ be given by $\iota_1(P) = (1, P)$. We obtain the $\mathcal{D}$-module $\Xi_{DR}(\mathcal{T}) := \iota_1^{-1}(\mathcal{M}_2/\lambda - 1\mathcal{M}_2)$ which we call the $\mathcal{D}$-module underlying $\mathcal{T}$. It is naturally equipped with the filtration $W$. For any morphism of mixed twistor $\mathcal{T}$-modules $\varphi : (\mathcal{T}(1), W) \to (\mathcal{T}(2), W)$, we have the induced morphism of filtered $\mathcal{D}$-modules $\Xi_{DR}(\varphi) : \Xi_{DR}(\mathcal{T}(1), W) \to \Xi_{DR}(\mathcal{T}(2), W)$.

The following lemma is easy to see.

Lemma 3.4 Let $\varphi : (\mathcal{T}(1), W) \to (\mathcal{T}(2), W)$ be any morphism of mixed twistor $\mathcal{T}$-modules.

- We have $\text{Ker}\, \Xi_{DR}(\varphi) = \Xi_{DR}(\text{Ker}(\varphi))$, $\text{Im}\, \Xi_{DR}(\varphi) = \Xi_{DR}(\text{Im}(\varphi))$, and $\text{Cok}\, \Xi_{DR}(\varphi) = \Xi_{DR}(\text{Cok}(\varphi))$.
- $\varphi$ is an epimorphism (resp. monomorphism) if and only if $\Xi_{DR}(\varphi)$ is an epimorphism (resp. monomorphism).

The following lemma is easy to see by construction of the functors.

Lemma 3.5 Let $(\mathcal{T}, W)$ be a mixed twistor $\mathcal{D}$-module on $X$.

- Let $f : X \to Y$ be a projective morphism of complex manifolds. Then, we have $f_!^\dagger \Xi_{DR}(\mathcal{T}) \simeq \Xi_{DR}(f_!^\dagger \mathcal{T})$. The induced filtrations $W$ are also equal.
- We have a natural isomorphism $D_X\Xi_{DR}(\mathcal{T}) \simeq \Xi_{DR}D\mathcal{T}$. The induced filtrations $W$ are also equal.

3.1.5 Integrable case

Let $\mathcal{D}_X$ be the sheaf of holomorphic differential operators on $X$. Let $\mathcal{R}_X \subset D_X$ be the sheaf of subalgebras generated by $\mathcal{R}_X$ and $\lambda^2 \partial_\lambda$. The correspondence $\mathcal{M} \to \lambda^{-\dim X}\Omega_X \otimes \mathcal{M}$ also induces an equivalence between the category of left $\mathcal{R}_X$-modules and the category of right $\mathcal{R}_X$-modules. An $\mathcal{R}_X$-module is equivalent to an $\mathcal{O}_X$-module with a meromorphic flat connection $\nabla : \mathcal{M} \to \Omega^1_X(\log\lambda^0) \otimes \mathcal{O}(\lambda^0) \otimes \mathcal{M}$.

Let $\mathcal{M}_i (i = 1, 2)$ be $\mathcal{R}_X$-modules. Let $\theta$ be the polar coordinate of $S$. We have $\partial_\theta = \sqrt{-1}(\lambda \partial_\lambda - \nabla_\theta)$. It naturally acts on $\mathcal{M}_i$. A sesquilinear pairing of $\mathcal{M}_1$ and $\mathcal{M}_2$ is a $(\mathcal{R}_X|_{S \times X}, \sigma^*\mathcal{R}_X|_{S \times X})$-homomorphism $C : \mathcal{M}_1|_{S \times X} \times \sigma^*\mathcal{M}_2|_{S \times X} \to \mathcal{D}_S|_{S \times X}/S$ which is compatible with the action of $\partial_\theta$. Such a tuple $(\mathcal{M}_1, \mathcal{M}_2, C)$ is called an $\mathcal{R}_X$-triple or integrable $\mathcal{R}_X$-triple. For any morphism of complex manifolds $f : X \to Y$, the $\mathcal{R}_X$-triples $f_!^\dagger \mathcal{T}$ are naturally integrable if $\mathcal{T}$ is integrable.

A mixed twistor $\mathcal{D}$-module $(\mathcal{T}, W)$ is called integrable if the $\mathcal{R}$-triple $\mathcal{T}$ is integrable and the filtration $W$ is compatible with the action of $\lambda^2 \partial_\lambda$. Propositions 3.1 and 3.2 are naturally extended to the integrable case.

Let $(\mathcal{T}, W)$ be an integrable mixed twisted $\mathcal{D}_X$-module. Take an injective resolution $\mathcal{I}$ of $\mathcal{R}_X \otimes \Omega^1_X(\dim X)$ as a left $(\mathcal{R}_X, \mathcal{R}_X)$-module. Then, $D_X\mathcal{M}_i \simeq \text{RHom}_{\mathcal{R}_X}(\mathcal{M}, \mathcal{I})$ are naturally $\mathcal{R}_X$-modules. The action of $\lambda^2 \partial_\lambda$ is given by $(\lambda^2 \partial_\lambda f)(m) = \lambda^2 \partial_\lambda (f(m)) - f(\lambda^2 \partial_\lambda m)$. It is shown that $D\mathcal{C}$ is compatible with the action of $\partial_\theta$. (See [30].) Hence, $D(\mathcal{T}, W)$ is also naturally integrable.

We shall give some complements on the duality and the push-forward of $\mathcal{R}$-modules in [30].
3.2 Integripable mixed twistor $\mathcal{D}$-modules associated to non-degenerate functions

3.2.1 Pure twistor $\mathcal{D}$-modules associated to meromorphic functions

Let $X$ be a complex manifold with a hypersurface $D$. We set $\mathcal{X} := \mathbb{C}_x \times X$ and $\mathcal{D} := \mathbb{C}_x \times D$. Let $f$ be a meromorphic function on $(X, D)$. We have a wild harmonic bundle $E(f)$ on $(X, D)$. It consists of a Higgs bundle $(\mathcal{O}_{\mathcal{X} \setminus D}, e, df)$ with a pluri-harmonic metric $h$ determined by $h(e, e) = 1$, where $e$ denotes a global frame of the line bundle. Recall that we have the associated polarizable pure twistor $\mathcal{D}$-module $(\mathcal{L}(f), \mathcal{L}(f), C(f))$ of weight $0$ on $X$, where $\mathcal{L}(f)$ is an $\mathcal{O}_X$-module, and $C(f)$ is a sesqui-linear pairing $\mathcal{L}(f)|_{\mathcal{X} \setminus D} \times \mathcal{L}(f)|_{\mathcal{X} \setminus D} \to \mathcal{D}\mathcal{O}_{\mathcal{X} \setminus D}$. Here $S := \{ |x| = 1 \} \subset \mathbb{C}_x$. A polarization is given by $(id, id)$.

Let $d_X := \dim X$. We have the polarizable pure twistor $\mathcal{D}$-module $T(f) := (\lambda^{dx} \mathcal{L}(f), \mathcal{L}(f), C(f))$ of weight $d_X$. The natural polarization is given by $((-1)^{dx}, (-1)^{dx})$.

The restriction $\mathcal{L}(f)|_{\mathcal{X} \setminus D}$ is equipped with a global frame $v$ such that $\mathcal{L}(f)|_{\mathcal{X} \setminus D}$ is isomorphic to $\mathcal{O}_{\mathcal{X} \setminus D}v$ with the family of flat $\lambda$-connections $\mathcal{D}$ determined by $\mathcal{D}v = vdf$. We have

$$C(f)(v, \sigma v) = \exp(-\lambda \bar{\tau} + \lambda^{-1} f).$$

In general, it is not easy to describe $\mathcal{L}(f)$ explicitly. The following lemma is clear by the construction of $\mathcal{L}(f)$ in [33], and it will be used implicitly.

Lemma 3.6 We naturally have $\mathcal{L}(f)(\ast D) \simeq \mathcal{O}_X(\ast D)v$, and the natural morphism $\mathcal{L}(f) \to \mathcal{L}(f)(\ast D)$ is a monomorphism.

Lemma 3.7 If $|\langle f \rangle_0 \cap D = \emptyset$ and $|\langle f \rangle_\infty| = D$, then $\mathcal{L}(f)$ is naturally isomorphic to $\mathcal{O}_X(\ast D)v$.

Proof If $D$ is normal crossing, the claim follows from the construction of $\mathcal{L}(f)$ in [33]. Although we are mainly interested in the case where $D$ is normal crossing, we give a proof in the general case. We take a projective morphism $\varphi : \mathcal{X} \to X$ of complex manifolds such that (i) $D' := \varphi^{-1}(D)$ is normal crossing, (ii) $\varphi$ induces an isomorphism $\mathcal{X} \setminus D' \simeq X \setminus D$. We set $f' := \varphi^*(f)$. We have $\mathcal{L}(f') = \mathcal{O}_{\mathcal{X}'}(\ast D'v)$ with a frame $v'$ such that $\mathcal{D}v' = v'df'$. Because $\mathcal{L}(f')\ast D' = \mathcal{L}(f')$, we obtain $\varphi_!(\mathcal{L}(f')) \simeq \varphi_!(\mathcal{L}(f')(\ast D))$. We also have that $\mathcal{L}(f)$ is a direct summand of $\varphi_!(\mathcal{L}(f'))$ and $\varphi_!(\mathcal{L}(f'))|_{\mathcal{X} \setminus D} = \mathcal{L}(f)|_{\mathcal{X} \setminus D}$. Then, we can deduce the claim of the lemma.

Proposition 3.8 Suppose that (i) $D$ is normal crossing, (ii) $f$ is non-degenerate along $D$, (iii) $D = |\langle f \rangle_\infty|$. Then, $\mathcal{L}(f)$ is naturally isomorphic to $\mathcal{O}_X(\ast D)v$.

Proof We have only to check the claim locally around any point of $|\langle f \rangle_0 \cap |\langle f \rangle_\infty|$. Let $(z_1, \ldots, z_n)$ be a convenient coordinate system with $f = z_n \prod_{i=1}^k z_i^{-k_i}$ for some $k_i > 0$. Note that $\prod_{i=1}^k z_i^{N}v$ is a section of $\mathcal{L}(f)$. We have $\partial_n v = v \prod_{i=1}^k z_i^{-k_i}$. Then, it is easy to deduce that $\mathcal{L}(f) \supset \mathcal{R}_X v = \mathcal{O}_X(\ast D)v$.

3.2.2 Mixed twistor $\mathcal{D}$-modules associated to meromorphic functions

For $\ast = \ast, !$, we have the $\mathcal{R}_X$-modules $\mathcal{L}(f)(\ast D)$ obtained as the localization $\mathcal{L}(f)$. They are denoted by $\mathcal{L}_\ast(f, D)$ in this paper. If $D = |\langle f \rangle_\infty|$, they are also denoted by $\mathcal{L}(f)$. We have the natural morphisms of $\mathcal{R}_X$-modules $\mathcal{L}_\ast(f, D) \to \mathcal{L}(f) \to \mathcal{L}_\ast(f, D)$. It induces $\mathcal{L}_\ast(f, D)(\ast D) \simeq \mathcal{L}(f)(\ast D)$.

For $\ast = \ast, !$, we have the mixed twistor $\mathcal{D}$-modules $\mathcal{T}(f)[\ast D]$ obtained as the localizations of $\mathcal{T}(f)$. They are denoted by $\mathcal{T}_\ast(f, D)$ in this paper. If $D = |\langle f \rangle_\infty|$, they are also denoted by $\mathcal{T}_\ast(f)$. The underlying $\mathcal{R}_X$-triples of $\mathcal{T}_\ast(f, D)$ and $\mathcal{T}_\ast(f, D)$ are $(\lambda^{dx} \mathcal{L}_\ast(f, D), \mathcal{L}_\ast(f, D), C(f)(\ast D))$ and $(\lambda^{dx} \mathcal{L}_\ast(f, D), \mathcal{L}_\ast(f, D), C(f)(\ast D))$, respectively. The weight filtrations of $\mathcal{T}_\ast(f, D)$ are denoted by $W$. We have natural morphisms of mixed twistor $\mathcal{D}$-modules $\mathcal{T}_\ast(f, D) \to \mathcal{T}_\ast(f, D) \to \mathcal{T}_\ast(f, D)$.

The meromorphic family of flat connections $\mathcal{D}^f$ on $\mathcal{L}(f)(\ast D)$ is naturally extended to a meromorphic flat connection $\nabla$ given by $\nabla v = vdf(\lambda^{-1} f)$. Let $C(f)(\ast D)$ be the sesqui-linear pairing of $\mathcal{L}(f)(\ast D)$ and $\mathcal{L}(f)(\ast D)$ induced by $C(f)$. Due to (33), $C(f)(\ast D)$ is compatible with the actions of $\lambda^2 \partial_x$. 

Lemma 3.9 $\mathcal{T}_\ast(f, D)$ are integrable mixed twistor $\mathcal{D}$-modules, and $\mathcal{T}(f, D)$ is an integrable pure twistor $\mathcal{D}$-module. The morphisms $\mathcal{T}_\ast(f, D) \to \mathcal{T}(f, D) \to \mathcal{T}_\ast(f, D)$ are integrable.
Proof By using [36] Lemma 3.2.4, we obtain that the underlying $\mathcal{R}$-triples of $\mathcal{T}_*(f, D)$ are integrable. Then, we obtain that $\mathcal{T}(f, D)$ is an integrable pure twistor $\mathcal{D}$-module. By using [36] Lemma 7.1.35, we obtain that the filtrations $W$ of $\mathcal{T}_*(f, D)$ are also integrable. Hence, we obtain that $\mathcal{T}_*(f, D)$ are integrable mixed twistor $\mathcal{D}$-modules. By using [36] Lemma 7.1.37, we obtain the integrability of the morphisms.

Lemma 3.10 The $\mathcal{D}$-modules $\Xi_{DR}(\mathcal{T}_*(f, D))$ and $\Xi_{DR}(\mathcal{T}(f))$ are naturally identified with $L_*(f, D)$ and $L(f)$. The morphisms $\Xi_{DR}(\mathcal{T}_*(f, D)) \rightarrow \Xi_{DR}(\mathcal{T}(f)) \rightarrow \Xi_{DR}(\mathcal{T}_*(f, D))$ are naturally identified with $L_*(f, D) \rightarrow L(f) \rightarrow L_*(f, D)$.

Proof Let $\mathcal{M}$ be an $\mathcal{R}_X$-module underlying $\mathcal{T}_*(f, D)$ ($* = *!$) or $\mathcal{T}(f, D)$. Let $g$ be a holomorphic function on an open subset $U \subset X$. Because $\mathcal{T}_*(f, D)$ are integrable, the KMS-spectrum of $\mathcal{M}|_U$ along $g$ are contained in $\mathbb{R} \times \{0\}$. Hence, the specialization of the $V$-filtration along $g$ at $\lambda = 1$ is equal to the $V$-filtration of $\tau_1^{-1}(\mathcal{M}/(\lambda - 1)\mathcal{M})$, where $\tau_1 : X \rightarrow \mathcal{X}$ given by $\tau_1(P) = (1, P)$. Then, the claim is clear.

In the proof of Lemma 3.10, the KMS-spectrum of the $\mathcal{R}_X$-modules are contained in $\mathbb{Q} \times \{0\}$, indeed. We can directly check it by standard computations.

Lemma 3.11 Suppose that (i) $D = |(f)_\infty|$, (ii) $f$ is pure at $P \in X$. Then, the canonical morphisms $\mathcal{T}_*(f, D) \rightarrow \mathcal{T}(f) \rightarrow \mathcal{T}_*(f, D)$ are isomorphisms on a neighbourhood of $P$.

Proof It follows from that the morphisms of the underlying $\mathcal{D}$-modules are isomorphisms.

Corollary 3.12 Suppose that (i) $D$ is normal crossing, (ii) $D = |(f)_\infty|$, (iii) $f$ is non-degenerate. Then the canonical morphisms $\mathcal{T}_*(f) \rightarrow \mathcal{T}(f) \rightarrow \mathcal{T}_*(f)$ are isomorphisms. In particular, we have isomorphisms $\mathcal{L}_*(f, D) \simeq \mathcal{O}_X(*D) v$.

3.2.3 Expression of $\tilde{\mathcal{R}}$-modules in non-degenerate cases

Suppose that $D$ is normal crossing. Let us describe $\mathcal{R}_X$-modules $\mathcal{L}_*(f, D)$ in the case that $f$ is non-degenerate along $D$. We do not assume $D = |(f)_\infty|$. Let $V_D \mathcal{R}_X$ denote the sheaf of subalgebras in $\mathcal{R}_X$ generated by $\lambda \cdot p^*_X \mathcal{O}_X(\log D)$ over $\mathcal{O}_X$. Let $V_D \tilde{\mathcal{R}}_X$ denote the sheaf of subalgebras in $\tilde{\mathcal{R}}_X$ generated by $V_D \mathcal{R}_X$ and $\lambda^2 \partial_X$. We have the $V_D \tilde{\mathcal{R}}_X$-modules $\mathcal{L}(f)$ and $\mathcal{L}(f)(D) = \mathcal{L}(f) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$.

Proposition 3.13 We have natural isomorphisms of $\tilde{\mathcal{R}}_X$-modules

$$\mathcal{L}_*(f, D) \simeq \mathcal{R}_X \otimes_{V_D \mathcal{R}_X} \mathcal{L}(f)(D), \quad \mathcal{L}_!(f, D) \simeq \mathcal{R}_X \otimes_{V_D \mathcal{R}_X} \mathcal{L}(f).$$

Proof We have natural isomorphisms of $\tilde{\mathcal{R}}_X(*D)$-modules

$$\mathcal{L}_*(f, D)(*D) \simeq (\mathcal{R}_X \otimes_{V_D \mathcal{R}_X} \mathcal{L}(f)(D))(*D), \quad \mathcal{L}_!(f, D)(*D) \simeq (\mathcal{R}_X \otimes_{V_D \mathcal{R}_X} \mathcal{L}(f))(*D).$$

Note that if we are given an $\mathcal{R}_X$-endomorphism $g$ of $\mathcal{L}_*(f, D)$ such that $g|_{\mathcal{O}_X(X \setminus D)} = 0$, then we have $g = 0$. Hence, it is enough to prove that there exist isomorphisms as $\mathcal{R}_X$-modules.

The claim is clear outside $|(f)_0| \cap |(f)_\infty|$ by the construction of $\mathcal{L}_*(f, D)$ in [36] §5.3. We set

$$\mathcal{L}_*(f, D) := \mathcal{R}_X \otimes_{V_D \mathcal{R}_X} \mathcal{L}(f)(D), \quad \mathcal{L}^!(f, D) := \mathcal{R}_X \otimes_{V_D \mathcal{R}_X} \mathcal{L}(f).$$

Let $Q \in |(f)_0| \cap |(f)_\infty|$. We take a convenient coordinate system $(z_1, \ldots, z_n)$ with $D = \bigcup_{i=1}^{\ell_1+\ell_2} \{z_i = 0\}$ and $f = z_n \prod_{i=1}^{\ell_1} z_i^{-k_i}$. We set $X_1 := \{(z_1, \ldots, z_{\ell_1}, z_n)\}$ and $X_2 := \{(z_{\ell_1+1}, \ldots, z_{n-1})\}$. We set $f_1 := z_n \prod_{i=1}^{\ell_1} z_i^{-k_i}$ and $D_1 := \bigcup_{i=1}^{\ell_1} \{z_i = 0\}$ and $D_2 := \bigcup_{i=\ell_1+1}^{\ell_1+\ell_2} \{z_i = 0\}$. We have $\mathcal{L}_*(f_1, D_1) = \mathcal{L}(f_1) = \mathcal{L}_*(f_1, D_1)$. In particular, we have $\mathcal{L}_*(f_1, D_1)[*z_2] = \mathcal{L}_*(f_1, D_1)$ for $j = 1, \ldots, \ell_1$. We also have $\mathcal{L}_*(f_2, D_2) \simeq \mathcal{L}_*(f_2, D_2)$. Hence, $\mathcal{L}_*(f_2, D_2)[*z_2] \simeq \mathcal{L}_*(f_2, D_2)[*z_2]$ for $i = \ell_1 + 1, \ldots, \ell_1 + \ell_2$. We have $\mathcal{L}_!(f, D) \simeq \mathcal{L}_!(f_1, D_1) \otimes \mathcal{L}_!(f_2, D_2)$. Then, it is easy to check that $\mathcal{L}_!(f, D)[*z_2] \simeq \mathcal{L}_!(f, D)$ for $i = \ell_1 + 1, \ldots, \ell_1 + \ell_2$. We obtain the desired isomorphism by the characterization in [36] Proposition 5.3.1].

26
3.2.4 De Rham complexes in non-degenerate cases

Let us consider the case that \((X, D) = S \times (X_0, D_0)\), where \(X_0\) is a complex manifold with a normal crossing hypersurface \(D_0\). Let \(f\) be a meromorphic function on \((X, D)\) which is non-degenerate along \(D\). Let \(\pi : X \to S\) denote the projection. We have the mixed twistor \(\mathcal{D}\)-modules \(\pi_!^\dagger \mathcal{T}_*(f, D)\) on \(S\). Let us describe the underlying \(\mathcal{R}_S\)-modules \(\pi_!^\dagger \mathcal{L}_*(f, D)\), by assuming that \(X_0\) is projective.

Set \(n := \dim X_0\). Let \(p_\lambda : X \to X\) denote the projection. For any \(\mathcal{R}_X\)-module \(M\), we set

\[
\text{DR}^n_{X/S} M := \lambda^{-m-n} p_\lambda^*(\Omega_{X/S}^{n+n}) \otimes \mathcal{O}_X M.
\]

Then, we obtain the complex \(\text{DR}^n_{X/S} M\). We naturally have \(\pi_!^\dagger M \simeq \mathbb{R}^\dagger \pi_* \text{DR}_X^\dagger M\).

Suppose \(|(f)\infty| = D\). We have \(\mathcal{L}_*(f, D) = \mathcal{L}(f)\) which is isomorphic to \(\mathcal{O}_X(*D)v\) with \(Dv = vd\). We set \(\mathring{\Omega}_{X/S} := \lambda^{-f} p_\lambda^* \Omega_{X/S}^f\). Then, we immediately obtain the following natural isomorphism:

\[
\text{DR}^n_{X/S} \mathcal{L}(f) \simeq (\mathring{\Omega}_{X/S}^{n+n}(D), d + \lambda^{-1} df)
\]

Let us consider the case that \(D\) is not necessarily \(|(f)\infty|\). We set \(\mathring{\Omega}_{X/S} := \lambda^{-f} p_\lambda^* \Omega_{X/S}^f\). We obtain the complexes \((\mathring{\Omega}_{X/S}(D), d + \lambda^{-1} df)\) and \((\mathring{\Omega}_{X/S}(D)(-D), d + \lambda^{-1} df)\).

**Lemma 3.14** We have the following natural quasi-isomorphisms:

\[
\text{DR}^n_{X/S} \mathcal{L}_*(f, D) \simeq (\mathring{\Omega}_{X/S}^{n+n}(D)(\ell(f)\infty), d + \lambda^{-1} df)
\]

\[
\text{DR}^n_{X/S} \mathcal{L}_*(f, D) \simeq (\mathring{\Omega}_{X/S}^{n+n}(D)(-D)(\ell(f)\infty), d + \lambda^{-1} df)
\]

Hence, \(\pi_!^\dagger \mathcal{L}_*(f, D)\) are expressed as the push-forward of the right hand side of \((31)\) and \((32)\).

**Proof** It follows from the expression in Proposition 3.13.

Suppose moreover that \(f\) is non-degenerate along \(D\) over \(S\). We use the notation in \((22.7)\). We set \(\mathring{\Omega}_{X/S}(D, f) := \mathring{\Omega}_{X/S}(D)(\ell(f)\infty)\), and we obtain the following complexes:

\[
(\mathring{\Omega}_{X/S}(D), d + \lambda^{-1} df), \quad (\mathring{\Omega}_{X/S}(D)(-D), d + \lambda^{-1} df).
\]

Similarly, we set \(\mathring{\Omega}_{X/S,f,D} := \lambda^{-f} p_\lambda^* \Omega_{X/S,f,D}^f\), and we obtain the following complexes:

\[
(\mathring{\Omega}_{X/S,f,D}(D), d + \lambda^{-1} df), \quad (\mathring{\Omega}_{X/S,f,D}(D)(-D), d + \lambda^{-1} df).
\]

As in the case of \(\mathcal{D}\)-modules in \((22.6)\) and \((22.7)\), we have the following lemma.

**Lemma 3.15** Suppose that \(f\) is non-degenerate along \(D\) over \(S\). Then, we have the following natural quasi-isomorphisms:

\[
\text{DR}^n_{X/S} \mathcal{L}_*(f, D) \simeq (\mathring{\Omega}_{X/S}^{n+n}(D), d + \lambda^{-1} df) \simeq (\mathring{\Omega}_{X/S,f,D}^{n+n}, d + \lambda^{-1} df)
\]

\[
\text{DR}^n_{X/S} \mathcal{L}_*(f, D) \simeq (\mathring{\Omega}_{X/S}^{n+n}(D)(-D), d + \lambda^{-1} df) \simeq (\mathring{\Omega}_{X/S,f,D}^{n+n}(-D), d + \lambda^{-1} df)
\]

Hence, \(\pi_!^\dagger \mathcal{L}_*(f, D)\) are expressed as the push-forward of the complexes in \((33)\) and \((34)\).

We also remark the following lemma.

**Lemma 3.16** Suppose that \(f\) is non-degenerate along \(D\) over \(S\). Then, \(\pi_!^\dagger \mathcal{L}_*(f, D)\) are locally free \(\mathcal{O}_{C_X \times S}\)-modules.
\textbf{Proof} The specialization at $\lambda = 1$ are locally free $\mathcal{O}_S$-modules by Corollary 3.17. So, by the general property of mixed twistor $D$-modules, we obtain that $\pi_1^!\mathcal{L}_*(f, D)$ are locally free $\mathcal{O}_{C \times S}$-modules.

\textbf{Corollary 3.17} If $f$ is non-degenerate along $D$ over $S$, the $\mathcal{O}_{C \times S}$-modules

$$\mathbb{R}^i\pi_* (\Omega_{X/S}^*(\log D)(\ast f)_\infty, d + \lambda^{-1} df) \simeq \mathbb{R}^i\pi_* (\Omega_{X/S}^*(\log D, f), d + \lambda^{-1} df)$$

are locally free. The $\mathcal{O}_{C \times S}$-modules

$$\mathbb{R}^i\pi_* (\Omega_{X/S}^*(\log D)(\ast f)_\infty, d + \lambda^{-1} df) \simeq \mathbb{R}^i\pi_* (\Omega_{X/S}^*(\log D, f), d + \lambda^{-1} df)$$

are also locally free.

### 3.3 The real structure of the localization of some mixed twistor $D$-modules

Let $X$ be a complex manifold. For a mixed twistor $D$-module $\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)$ on $X$, we have

$$j^* \mathcal{T} = (j^* \mathcal{M}_1, j^* \mathcal{M}_2, j^* C), \quad DT = (D\mathcal{M}_1, D\mathcal{M}_2, DC).$$

We set $\tilde{\gamma}^* \mathcal{T} := (j^* D\mathcal{M}_2, j^* D\mathcal{M}_1, j^* D C^*)$. We will naturally identify $j^* D\mathcal{M}_1$ and $D j^* \mathcal{M}_1$. Recall that a real structure of $\mathcal{T}$ is an isomorphism $\kappa : \tilde{\gamma}^* \mathcal{T} \simeq \mathcal{T}$ satisfying $\tilde{\gamma}^* \kappa \circ \kappa = \text{id}$.

#### 3.3.1 Basic case

Let $d_X := \dim X$. We have the isomorphism $\nu_X : D\mathcal{O}_X \simeq \mathcal{O}_X \lambda^{d_X}$, whose specialization at $\{\lambda_0\} \times X$ ($\lambda_0 \neq 0$) is equal to the morphism in \([A.2.1]\). We have the natural identification $j^* \mathcal{O}_X = \mathcal{O}_X$ given by the pull back of functions. We have $D\nu_X = (-1)^{d_X} \nu_X$. As shown in \([36]\), the isomorphism

$$(\nu_X^{-1}, (-1)^{d_X} \nu_X) : \tilde{\gamma}^* \mathcal{U}_X(d_X, 0) \simeq \mathcal{U}_X(d_X, 0)$$

gives a real structure.

Let $Y$ be a smooth hypersurface of $X$. We set $d_Y := \dim Y = d_X - 1$. Let $\iota : Y \rightarrow X$ be the inclusion. We have the following natural morphisms of integrable mixed twistor $D$-modules:

$$\mathcal{U}_X(d_X, 0)[\ast Y] \rightarrow \iota_* \mathcal{U}_Y(d_Y, 0) \otimes T(-1)$$

(35)

$$\iota_! \mathcal{U}_Y(d_Y, 0) \rightarrow \mathcal{U}_X(d_X, 0)[\ast Y]$$

(36)

The morphisms are induced by $\mathcal{R}_X$-homomorphisms $\mathcal{O}_X[\ast Y] \rightarrow \iota_! \mathcal{O}_Y \lambda^{-1}$ and $\iota_! \mathcal{O}_Y \rightarrow \mathcal{O}_X[\ast Y]$. Locally, if $X$ is equipped with a holomorphic coordinate system $(x_1, \ldots, x_n)$ such that $Y = \{x_1 = 0\}$, the morphisms are given by $\lambda x_1^{-1} \mapsto \iota_! (dx_1/\lambda)^{-1}$ and $\iota_! (dx_1/\lambda)^{-1} \mapsto -\partial_1(1)$.

\textbf{Proposition 3.18} The natural morphisms (35) and (36) are compatible with the real structures.

\textbf{Proof} Let us look at the compatibility of the morphisms in (35) which is the commutativity of the following diagrams:

\begin{align*}
\mathcal{O}_X[\ast Y] & \xleftarrow{\nu_X} D(\mathcal{O}_X[\ast Y]) & \mathcal{O}_X[\ast Y] & \xleftarrow{(-1)^{d_Y} \nu_X} D(\mathcal{O}_X[\ast Y] \lambda^{d_X}) \\
\iota_! \mathcal{O}_Y \lambda^{d_Y + 1} & \xleftarrow{-\nu_Y} D(\iota_! \mathcal{O}_Y \lambda^{-1}) & \iota_! \mathcal{O}_Y \lambda^{-1} & \xleftarrow{(-1)^{d_Y + 1} \nu_Y} D(\iota_! \mathcal{O}_Y \lambda^{d_Y + 1})
\end{align*}

To check the commutativity, it is enough to compare the specialization along $\{\lambda_0\} \times X$ for any $\lambda_0 \neq 0$. Then, it is reduced to Proposition [A.3] We can check the compatibility of (36) similarly.

28
3.3.2 Mixed twistor $D$-modules associated to holomorphic functions

Let $F$ be any holomorphic function on $X$. We have the $\mathcal{R}_X$-module $\mathcal{L}(F)$ given by $\mathcal{O}_X e$ with $\mathcal{D}e = e dF$, where $e$ is a global frame. Let $\mathcal{L}(F)^{\dagger} = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}(F), \mathcal{O}_X)$ is naturally isomorphic to $\mathcal{O}_X e^{\dagger}$ with $\mathcal{D}e^{\dagger} = e^{\dagger}(-dF)$, where $e^{\dagger}$ is the dual frame of $e$. Hence, we have the isomorphism $j^* \mathcal{L}(F)^{\dagger} \simeq \mathcal{L}(F)$ given by $j^* e^{\dagger} \leftrightarrow e$.

As in the case of $\mathcal{O}_X$, we have a natural isomorphism $\nu : j^* \mathcal{D}(F) \simeq \lambda^{d_X} j^* \mathcal{L}(F)^{\dagger} \simeq \lambda^{d_X} \mathcal{L}(F)$. Hence, $\mathcal{T}(F) = (\lambda^{d_X} \mathcal{L}(F), \mathcal{L}(F), C_F)$ is equipped with a real structure given by $(\nu^{-1}, (1)^{d_X} \nu)$.

Let $F_Y$ denote the restriction of $F$ to $Y$. We have the mixed twistor $D$-module $\mathcal{T}(F_Y)$ on $Y$ with real structure on $Y$. It is given as $\mathcal{T}(F_Y) = (\lambda^{d_Y} \mathcal{L}(F_Y), \mathcal{L}(F_Y), C_{F_Y})$. We have natural isomorphisms $\mathcal{L}(F)[*Y] \simeq \mathcal{L}(F) \otimes \mathcal{O}_X[*Y] \ (\ast = \ast, !)$ and $\iota_1 \mathcal{L}(F_Y) \simeq \mathcal{L}(F) \otimes \iota_1 \mathcal{O}_Y$. We obtain a morphism $\mathcal{L}(F)[*Y] \to \iota_1 \mathcal{L}(F_Y) \lambda^{-1}$ from the morphism $\mathcal{O}_X[*Y] \to \iota_1 \mathcal{O}_Y \lambda^{-1}$ by taking the tensor product with $\mathcal{L}(F)$. We also obtain $\iota_1 \mathcal{L}(F_Y) \to \mathcal{L}(F)[!Y]$ from $\iota_1 \mathcal{O}_Y \to \mathcal{O}_X[!Y]$. As in the basic case [3.3.1] we have the morphisms of the mixed twistor $D$-modules:

$$\mathcal{T}(F)[*Y] \to \iota_1 \mathcal{T}(F_Y) \otimes (-1) \quad (37)$$

$$\iota_1 \mathcal{T}(F_Y) \to \mathcal{T}(F)[!Y] \quad (38)$$

The following proposition is proved by the argument for Proposition 3.18.

**Proposition 3.19** The morphisms (37) and (38) are compatible with the real structures.

3.4 Push-forward

We use the notation in §2.4.1. We have the twistor $D$-modules $\mathcal{T}_*(F, D^{(1)}_Y) \ (\ast = \ast, !)$.

**Proposition 3.20** For $\mathcal{T} = \mathcal{T}_*(F, D^{(1)}_Y), \mathcal{T}_*(F, D^{(0)}_Y), \mathcal{T}_*(F, D^{(1)}_Y)/\mathcal{T}_*(F, D^{(0)}_Y)$, we have $\pi_*^0(\mathcal{T}) = 0 \ (i \neq 0)$. We also have the following natural isomorphisms of the integrable mixed twistor $D$-modules with real structure:

$$\pi_*^0 \mathcal{T}_*(F, D^{(1)}_Y) \simeq \left( \mathcal{T}_*(g, D) \otimes \mathcal{T}(-1) \right)[!(f)_0][*D] \quad (39)$$

$$\pi_*^0 \left( \mathcal{T}_*(F, D^{(1)}_Y)/\mathcal{T}_*(F, D^{(0)}_Y) \right) \simeq \mathcal{T}_*(g, D) \otimes \mathcal{T}(-1)[*D] \quad (40)$$

$$\pi_*^0 \left( \mathcal{T}_*(F, D^{(0)}_Y) \right) \simeq \ker \left( \left( \mathcal{T}_*(g, D) \otimes \mathcal{T}(-1) \right)[!(f)_0][*D] \to \mathcal{T}_*(g, D) \otimes \mathcal{T}(-1)[*D] \right) \quad (41)$$

**Proof** We have the natural morphisms integrable mixed twistor $D$-modules:

$$\pi_*^0 \mathcal{T}_*(F, D^{(1)}_Y) \to \pi_*^0 \mathcal{T}_*(F, D^{(1)}_Y)[*D] \leftarrow \left( \pi_*^0 \mathcal{T}_*(F, D^{(1)}_Y)[!(f)_0][*D] \right)[*D]. \quad (42)$$

According to Proposition 2.23, the induced morphisms of the underlying $D$-modules are isomorphisms. Hence, we obtain that the morphisms in (42) are isomorphisms. We have

$$\pi_*^0 \mathcal{T}_*(F, D^{(1)}_Y) \ast(f)_0)(*D) \simeq \pi_*^0 \left( \mathcal{T}_*(F, D^{(1)}_Y)/\mathcal{T}_*(F, D^{(0)}_Y) \right) \ast(f)_0)(*D).$$

Hence, we have

$$\pi_*^0 \mathcal{T}_*(F, D^{(1)}_Y) \simeq \left( \pi_*^0 \left( \mathcal{T}_*(F, D^{(1)}_Y)/\mathcal{T}_*(F, D^{(0)}_Y) \right) \ast(f)_0 \right)[*D].$$

Let us describe the underlying $\mathcal{R}_X(*D)$-modules $\pi_*^0 \left( \mathcal{T}_*(F, D^{(1)}_Y)/\mathcal{T}_*(F, D^{(0)}_Y) \right)(*D)$. The $\mathcal{R}$-module

$$\pi_*^0 \left( \mathcal{L}_*(F, D^{(1)}_Y)/\mathcal{L}_*(F, D^{(0)}_Y) \right)(*D)$$

is expressed by $\lambda^{-1} \mathcal{L}_*(g, D)(*D) dt/t$ which is naturally isomorphic to $\lambda^{-1} \mathcal{L}_*(g, D)(*D)$. Let $\iota_0 : X \to Y$ be given by $\iota_0(Q) = (Q, 0)$. We have

$$\text{Ker} \left( \mathcal{L}_*(F, D^{(1)}_Y) \to \mathcal{L}_*(F, D^{(0)}_Y) \right)(*D \times \mathbb{P}^1) \simeq \iota_0 \ast \mathcal{L}_*(g, D)(*D)[\partial_t][\partial_t]$$
By a direct computation, we have

\[ \lambda^{dv} \mathcal{L}_*(g, D)(*D)[\partial_t][\partial_t] \rightarrow \lambda^{dv-1} \mathcal{L}_*(g, D)(*D)[\partial_t][\partial_t] \]

Here, the first term sits in the degree \(-1\). The cokernel is isomorphic to \(\lambda^{dv} \mathcal{L}_*(g, D)(*D) = \lambda^{dv+1} \mathcal{L}_*(g, D)(*D)\).

By a direct computation, we have \(\langle C(\lambda^{-1} dt \cdot \partial_t \otimes 1, \sigma^*(\lambda^{-1} dt/t), \chi) \rangle = 2\pi \sqrt{-1}\). Hence, we have

\[ \pi_1^0 \mathcal{T}_*(F, D_Y^{(1)}) \simeq (\lambda^{dv+1} \mathcal{L}_*(g, D), \lambda^{-1} \mathcal{L}_*(g, D), C_0)[!(f)_0][*D] = \big( \mathcal{T}_*(g, D) \otimes \mathcal{T}(-1) \big)[!(f)_0][*D]. \]

The isomorphism is clearly integrable. For the comparison of the real structure, it is enough to check the case \[3.19\]. It is enough to check it on \(X \setminus D\). Then, it is reduced to Proposition \[3.19\]. Thus, we obtain Proposition \[3.20\].

**Corollary 3.21** For \(\mathcal{T} = \mathcal{T}_*(F, D_Y^{(1)}), \mathcal{T}_*(F, D_Y^{(0)}), \mathcal{Ker}(\mathcal{T}_*(F, D_Y^{(1)}) \rightarrow \mathcal{T}_*(F, D_Y^{(0)}))\), we have \(\pi_1^0(\mathcal{T}) = 0\) (\(i \neq 0\)). We also have the following natural isomorphisms of the integrable mixed twistor \(\mathcal{D}\)-modules with real structure:

\[ \pi_1^0 \mathcal{T}_*(F, D_Y^{(1)}) \simeq \mathcal{T}_*(g, D)[*f_0][!D] \]

\[ \pi_1^0 \big( \mathcal{Ker}(\mathcal{T}_*(F, D_Y^{(1)}) \rightarrow \mathcal{T}_*(F, D_Y^{(0)})) \big) \simeq \mathcal{T}_*(g, D)[!D] \]

\[ \pi_1^0 \mathcal{T}_*(F, D_Y^{(0)}) \simeq \text{Cok} \big( \mathcal{T}_*(g, D)[!D] \rightarrow \mathcal{T}_*(g, D)[*f_0][!D] \big) \]

**Proof** Because \(\mathcal{T}_*(F, D_Y^{(0)}) \ast \simeq \mathcal{T}_*(F, D_Y^{(0)}) \otimes \mathcal{T}(\dim Y)\), the claim follows from Proposition \[3.20\].

Let us consider the case that \(f\) is moreover non-degenerate along \(D\). In this case \(Z_f\) is smooth, and \(Z_f \cup D\) is normal crossing. As in \[2.4.1\], let \(\iota : Z_f \rightarrow X\) denote the inclusion, and we set \(D_{Z_f} := D \cap Z_f\) and \(g_0 := g|_{Z_f}\).

We have the integrable mixed twistor \(\mathcal{D}\)-modules \(\mathcal{T}_*(g_0, D_{Z_f})\) with real structure on \(Z_f\). By using Proposition \[3.19\], we obtain the following corollary.

**Corollary 3.22** If \(f\) is moreover non-degenerate along \(D\), we have the following isomorphisms of the integrable mixed twistor \(\mathcal{D}\)-modules with real structure:

\[ \pi_1^0 \mathcal{T}_*(F, D_Y^{(0)}) \simeq \iota_! \mathcal{T}_*(g_0, D_{Z_f}) \otimes \mathcal{T}(-1) \]

\[ \pi_1^0 \mathcal{T}_*(F, D_Y^{(0)}) \simeq \iota_! \mathcal{T}_*(g_0, D_{Z_f}) \otimes \mathcal{T}(-1) \]

The image of the morphism \(\pi_1^0 \mathcal{T}_*(F, D_Y^{(0)}) \rightarrow \pi_1^0 \mathcal{T}_*(F, D_Y^{(0)})\) is \(\iota_! \mathcal{T}_{\min}(g_0) \otimes \mathcal{T}(-1)\) under the isomorphisms.

### 3.5 Specialization

Let \(X\) be a complex manifold with a simple normal crossing hypersurface \(D\). We set \(X^{(1)} := X \times \mathbb{C}_\tau\) and \(D^{(1)} := D \times \mathbb{C}_\tau\). Let \(f\) and \(g\) be meromorphic functions on \((X, D)\). We have the meromorphic function \(F = \tau f + g\) on \((X^{(1)}, D^{(1)})\). We have the associated integrable mixed twistor \(\mathcal{D}\)-modules \(\mathcal{T}_*(F, D)\) (\(* = *, !\)) with real structure on \(X^{(1)}\). Let \(\mathcal{K}_{*, f, g}\) and \(\mathcal{C}_{*, f, g}\) denote the kernel and the cokernel of \(\mathcal{T}_*(F, D)[!]\tau \rightarrow \mathcal{T}_*(F, D)[*\tau]\).

**Proposition 3.23** If \(|!(f)_0| \cap |!(f)_\infty| = \emptyset\), we have the following isomorphism of the integrable mixed twistor \(\mathcal{D}\)-modules with real structure:

\[ \mathcal{C}_{*, f, g} \simeq \mathcal{T}_*(g, D) \otimes \mathcal{T}(-1), \quad \mathcal{K}_{*, f, g} \simeq \mathcal{T}_*(g, D)[!*f]\]

\[ \mathcal{K}_{!, f, g} \simeq \mathcal{T}_*(g, D), \quad \mathcal{C}_{!, f, g} \simeq \mathcal{T}_*(g, D)[*f] \otimes \mathcal{T}(-1). \]
Proof First, let us consider the case $g = 0$ and $|(f)\infty| = D$. In this case, $F = \tau f$ is non-degenerate along $D^{(1)}$. We have $T,(F, D) = T(F)$. The underlying $R_{\chi}(1)$-module $L(F)$ is $O_{\chi}(1)(*D^{(1)})w$ with $w = vD$. The $V$-filtration of $L(F)(*\tau)$ is computed in [37]. (We impose the condition $\tau d + \alpha \lambda$ is nilpotent on $U_{\alpha}/U_{<\alpha}$.) Let $Q$ be any point of $X$. If $Q \notin D$, we have $U_j(L(F)(*\tau)) = \tau^{-j}L(F)$ around $(Q, 0)$. Let $Q \in D$. We take a convenient coordinate point $(z_1, \ldots, z_n)$ such that $f = \prod_{i=1}^{t} z_i^{-k_i}$. Let $\pi : X^{(1)} \rightarrow X$ denote the projection. We naturally regard $\pi^*R_{\chi}$ as the sheaf of subalgebras in $R_{\chi}(1)$. For $0 \leq \alpha < 1$, we set $p = [\alpha k]$. We have

$$U_{\alpha}(L(F)(*\tau)) = \pi^*R_{\chi}(O_{\chi}(1)x^{-\delta - p}F) = \pi^*R_{\chi}(\sum_{j=0}^{\infty} O_{\chi}(1)x^{-\delta - p}F^j)$$

in $O_{\chi}(1)(*D^{(1)})w$. We have $U_1(L(F)(*\tau)) = \tau^{-1}U_0(L(F)(*\tau))$. Then, we have

$$U_{<0}(L(F)) + \tau \partial_u U_0(L(F)) = \sum_{j=1}^{\infty} O_{\chi}(1)x^{-\delta - p}F^j$$

Hence, we have

$$\frac{U_0(L(F))}{U_{<0}(L(F)) + \tau \partial_u U_0(L(F))} \simeq \iota_*O_X[*D]$$

Here $\iota : C_{\lambda} \times X \times \{0\} \rightarrow X^{(1)}$. It implies that

$$\text{Cok}(L(F)[\iota]) \rightarrow L(F)[\iota^\tau] \simeq \iota_+(\lambda^{-1}O_X[*D])$$

By the duality, we obtain that

$$\text{Ker}(L(F)[\iota]) \rightarrow L(F)[\iota^\tau] \simeq \iota_+(O_X[D])$$

Hence, we have

$$\text{Cok}(T(F)[\iota]) \rightarrow T(F)[\iota^\tau] \simeq \iota_+(U_X(dX, 0)[*D] \otimes T(-1))$$

$$\text{Ker}(T(F)[\iota]) \rightarrow T(F)[\iota^\tau] \simeq \iota_+(U_X(dX, 0)[*D])$$

Let us consider the case that $|(g)\infty| \cap |(g)\infty| = \emptyset$. We put $D_0 := |(f)\infty| \cup |(g)\infty|$. We have the hypersurface $D_1 \subset D$ such that $D_0 \cup D_1 = D$ and $\text{codim}_X(D_0 \cap D_1) \geq 2$. We set $D_2 := |(g)\infty| \cup D_1$.

Lemma 3.24 We have $L(\tau f)[*\tau] \otimes L_{*\tau}(g, D_2^{(1)}) \simeq L_{*\tau}(F, D_2^{(1)}[*\tau])$ for $*, \tau \in \{*, !\}$.

Proof We have $L(\tau f)[*\tau] \otimes L_{*\tau}(g, D_2^{(1)})[*\tau] \simeq L_{*\tau}(F, D_2^{(1)}[*\tau])$. We set

$$U_{\alpha}(L(\tau f)[*\tau] \otimes L_{*\tau}(g, D_2^{(1)})) := U_{\alpha}(L(\tau f)[*\tau]) \otimes L_{*\tau}(g, D_2^{(1)}).$$

We have that (i) $tU_{\alpha} \subset U_{\alpha-1}$ and $\delta U_{\alpha} \subset U_{\alpha+1}$, (ii) $t\partial \lambda + \alpha$ is nilpotent on $U_{\alpha}/U_{\alpha}$, (iii) $U_{\alpha}/U_{<\alpha}$ is strict. We have $tU_{\alpha} = U_{\alpha-1}$ for $\alpha < 1$, and $\delta U \rightarrow U_{\alpha+1}$ if $* = !$, we have that $tU_{\alpha} = U_{\alpha-1}$. If $* = !$, we have that $\delta U \rightarrow U_{\alpha+1}$ is an isomorphism. So, it is enough to prove that $U_{\alpha} (\alpha < 1)$ are $R_X$-coherent. We can prove it by the argument in the proof of Proposition 2.31.

We obtain the following:

$$\text{Cok}(L_{\tau}(F)[\iota]) \rightarrow L_{\tau}(F)[*\tau] \simeq \text{Cok}(L(\tau f)[\iota]) \rightarrow L(\tau f)[*\tau] \otimes L_{\tau}(g, D_2^{(1)}) \simeq \iota_+\left(\lambda^{-1}O_X[*\tau] \otimes L_{\tau}(g, D_2^{(1)}) \otimes \iota_+(\lambda^{-1}L_{\tau}(g, D)[*\tau])\right) (45)$$

31
Lemma 3.26 Suppose that
\[ \nabla \] the claims in this case.
\[ \kappa \]
\[ \lambda \]
\[ 3.6.2 \] Good filtration
\[ \text{Note that, because } |(g)_0| \cap |(g)_\infty| = \emptyset, \text{ we obtain the isomorphisms } a_i \text{ from Proposition } 3.13. \text{ Then, we obtain the claims in this case.} \]

We can reduce the general case to the case \[ |(g)_0| \cap |(g)_\infty| = \emptyset \] as in the proof of Proposition 2.34. For the comparison of the real structure, it is enough to check it on \( X \setminus D \). Then, the claim is reduced to Proposition 3.19.

3.6 \( \mathbb{C}^*\)-homogeneous \( \mathcal{R} \)-modules

3.6.1 Homogeneity
Suppose that a complex manifold \( X \) is equipped with a \( \mathbb{C}^* \)-action, i.e., a morphism \( \mu : \mathbb{C}^* \times X \to X \) satisfying
\[ \mu(a_1a_2, x) = \mu(a_1, \mu(a_2, x)) \text{ and } \mu(1, x) = x. \]
We consider the action of \( \mathbb{C}^* \) on \( \mathcal{O}_X \) given by the multiplication. Let \( \mathcal{O}_X \) be a \( \mathbb{C}^* \)-module.

Let \( \mathcal{M} \) be an \( \mathcal{R}_X \)-module. It is equivalent to an \( \mathcal{O}_{\mathbb{C}^* \times X} \)-module \( \mathcal{M} \) with a meromorphic flat connection \( \nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_{\mathbb{C}^* \times X} \otimes \mathcal{O}_X(\mathcal{A}^0), \text{ where } \mathcal{A}^0 := \{0\} \times X. \) We have the \( \mathcal{O}_{\mathbb{C}^* \times X} \)-module \( \tilde{\mu}^* \mathcal{M} \) on \( \mathbb{C}^* \times X \) equipped with the meromorphic flat connection \( \tilde{\mu}^* \nabla \). We can easily check that
\[ (\tilde{\mu}^* \nabla)(\tilde{\mu}^* \mathcal{M}) \subset \tilde{\mu}^* \mathcal{M} \otimes \Omega^1_{\mathbb{C}^* \times X} (\log(\mathbb{C}^* \times \mathcal{A}^0)) \otimes \mathcal{O}^*(\mathbb{C}^* \times \mathcal{A}^0). \]
Hence, \( \tilde{\mu}^* \mathcal{M} \) is naturally an \( \mathcal{R}_{\mathbb{C}^* \times X} \)-module.

Let \( p : \mathbb{C}^* \times X \to X \) denote the projection. Let \( p_1 : \mathbb{C}^* \times \mathbb{C}^* \times X \to X \) be given by
\[ p_1(a_1, a_2, x) = x, \quad p_2(a_1, a_2, x) = \tilde{\mu}(a_2, x), \quad p_3(a_1, a_2, x) = \tilde{\mu}(a_1a_2, x). \]
We have the morphisms \( p_{23, 1} \) id \( \times \tilde{\mu}, \mu \times \mu \times \text{id} : \mathbb{C}^* \times \mathbb{C}^* \times X \to \mathbb{C}^* \times X \) given by
\[ p_{23}(a_1, a_2, x) = (a_2, x), \quad \text{(id} \times \tilde{\mu})(a_1, a_2, x) = (a_1, \tilde{\mu}(a_2, x)), \quad (\mu \times \text{id})(a_1, a_2, x) = (a_1a_2, x). \]

Definition 3.25 An \( \mathcal{R}_X \)-module \( \mathcal{M} \) is called \( \mathbb{C}^* \)-homogeneous if we have an isomorphism of \( \mathcal{R}_{\mathbb{C}^* \times X} \)-modules
\( \kappa : \mu^* \mathcal{M} \simeq \tilde{\mu}^* \mathcal{M} \) satisfying the cocycle condition
\( (\text{id} \times \tilde{\mu})^* \kappa \circ p_{23, 1} = (\mu \times \text{id})^* \kappa. \)

The restriction of \( \tilde{\mu} \) to \( \mathbb{C}^* \times X \) is a free action, and the quotient space is \( X \). The condition in Definition 3.25 implies that there exists a \( \mathcal{D} \)-module \( M \) such that the restriction \( \mathcal{M}|_{\mathbb{C}^* \times X} \) is isomorphic to the pull back of \( M \) by \( \mathbb{C}^*_\lambda \times X \to X \). Indeed, \( M \) is given as the specialization of \( M \) to \( \{1\} \times X \), i.e., \( M = \Xi_{\text{DR}}(\mathcal{M}) := \iota^{-1}_1(\mathcal{M} / (\lambda - 1) \mathcal{M}), \) where \( \iota_1 : X \to X \) is given by \( \iota_1(x) = (1, x). \)

Lemma 3.26 Suppose that \( \mathcal{M} \) is a locally free \( \mathcal{O}_X \)-module, for simplicity. Then, the torus action is uniquely determined by the connection \( \nabla \).

Proof Let \( \mathfrak{g} \) be the holomorphic fundamental vector field of the action \( \mu \) on \( X \), i.e., \( \mathfrak{g} = T_{(1, Q)} \mu_*(\partial/\partial a) \) for any \( Q \in X \). The holomorphic fundamental vector field of \( \tilde{\mu} \) on \( X \) is given by \( \lambda \partial \mu + \mathfrak{g} \). The \( \mathbb{C}^* \)-action on \( \mathcal{M} \) induces a differential operator \( L : \mathcal{M} \to \mathcal{M} \) such that \( L(f) = fL(s) + (\lambda \partial \mu + \mathfrak{g})f \cdot s \). On \( \mathbb{C}^*_\lambda \times X \), we can easily check that \( L(s) = \nabla_{\lambda \partial \mu + \mathfrak{g}}(s) \). The equality holds on \( \mathbb{C}^*_\lambda \times X \). Because the \( \mathbb{C}^* \)-action is determined by \( L \), it is uniquely determined by the connection \( \nabla \).

3.6.2 Good filtration
Let us consider the case where the action \( \mu \) is trivial, i.e., \( \mu(a, x) = x \) for any \( (a, x) \in \mathbb{C}^* \times X \). The action \( \tilde{\mu} \) is just the multiplication on \( \mathbb{C}^*_\lambda \). Suppose that \( \mathcal{M} \) is strict, i.e., the multiplication of \( \lambda - \lambda_0 \) is a monomorphism for any \( \lambda_0 \in \mathbb{C} \). We also assume that \( \mathcal{M} \) underlies a regular singular mixed twistor \( \mathcal{D} \)-module.

We set \( M := \Xi_{\text{DR}}(\mathcal{M}) \). Let us recall that \( M \) is equipped with the good filtration \( F_* \) such that \( M \) is isomorphic to the analytification of the Rees module \( R(M, F_*) \). The filtration \( F \) is obtained as follows. Let \( p_\lambda : \mathbb{C}^*_\lambda \times X \to X \) be the projection.
Lemma 3.27 We naturally have $p^*_\lambda (M)(\ast \lambda) = \mathcal{M}(\ast \lambda)$.

Proof We have a natural isomorphism $p^*_\lambda (M)(\ast \lambda)|_{\mathbb{C}^\times \times X} = \mathcal{M}(\ast \lambda)|_{\mathbb{C}^\times \times X}$. Because both $p^*_\lambda (M)(\ast \lambda)$ and $\mathcal{M}(\ast \lambda)$ are regular singular, the isomorphism is extended on $\mathbb{C} \times X$.

For any section $s$ of $M$, the number $i(s) := \min \{i \mid \lambda^i p^*_\lambda (s) \in \mathcal{M}\}$ exists because $\mathcal{M}$ is a coherent $R_X$-module. It determines the filtration $F$ on $M$, i.e., $F_j(M) = \{s \in M \mid i(s) \leq j\}$. We have $F_j \mathcal{D}_X \cdot F_k(M) \subset F_{j+k}(M)$, where $F_j \mathcal{D}_X$ denote the sheaf of differential operators whose orders are less than $j$.

We set $R(M, F) = \sum F_j M \lambda^j$. By taking the analytification with respect to $\lambda$, we obtain an $\mathcal{R}_X$-module $\mathcal{R}(M, F)$.

Lemma 3.28 We have a natural isomorphism $\mathcal{R}(M, F)(\ast \lambda) \simeq \mathcal{M}(\ast \lambda)$. By the construction, we have $\mathcal{R}(M, F) \subset \mathcal{M}$. Let $\mathcal{P}_\lambda : \mathbb{P}^1 X \rightarrow X$ be the projection. Let $M'$ be the sheaf on $\mathbb{P}^1 \times X$ such that (i) the restriction to $\mathbb{P}^1 \times \{0\} \times X$ is equal to the restriction of $\mathcal{P}_\lambda(M)(\ast \infty)$, (ii) the restriction to $\mathbb{C} \times X$ is equal to $\mathcal{M}$. Because $\mathcal{M}$ is $\mathbb{C}^*$-equivariant, $M'$ is also $\mathbb{C}^*$-equivariant. It is easy to observe that $R\mathcal{P}_\lambda M' = 0$ for $i > 0$, and that the natural morphism $\mathcal{P}_\lambda M' \rightarrow \mathcal{M}_{[1]} \times X$ is an epimorphism. For any local section $s$ of $\mathcal{M}_{[0]} \times X$, we take a section $\tilde{s}$ of $\mathcal{P}_\lambda M'$ which is mapped to $s$.

Let $\omega : X \rightarrow X$ be given by $\omega (P) = (0, P)$. Because $Gr^F (M) \simeq \omega^{-1} (\mathcal{M}/\mathcal{M})$, the Sym$^\ast \Theta_X$-module $Gr^F (M)$ is coherent. Hence, $F$ is a good filtration.

We recall the following full faithfulness of the functor from the category of good filtered $\mathcal{D}$-modules to $\mathcal{R}$-modules.

Lemma 3.29 Let $\mathcal{M}_i$ $(i = 1, 2)$ be coherent $\mathcal{D}_X$-modules with a good filtration $F^{(i)}$ on $X$. We have the natural correspondence between morphisms of $\mathcal{R}_X$-modules $\mathcal{R}(\mathcal{M}_1, F^{(1)}) \rightarrow \mathcal{R}(\mathcal{M}_2, F^{(2)})$ and morphisms of filtered $\mathcal{D}$-modules $(\mathcal{M}_1, F^{(1)}) \rightarrow (\mathcal{M}_2, F^{(2)})$.

Proof A morphism of filtered $\mathcal{D}$-modules naturally induce a morphism of the associated Rees modules which are compatible with the connections. Let $f : \mathcal{R}(\mathcal{M}_1, F^{(1)}) \rightarrow \mathcal{R}(\mathcal{M}_2, F^{(2)})$ be a morphism of $\mathcal{R}$-modules. By taking the specialization to $\lambda = 1$, we obtain a morphism $f_1 : \mathcal{M}_1 \rightarrow \mathcal{M}_2$. Let us prove that it preserves the filtrations.

As in the proof of Lemma 3.28, we extend $\mathcal{R}(\mathcal{M}_i, F^{(i)})$ to $\mathcal{O}_{\mathbb{P}^1 \times X} \{\ast \infty\} \times X$)-modules which are also denoted by $\mathcal{R}(\mathcal{M}_i, F^{(i)})$. We have the morphism $\tilde{f} : \mathcal{R}(\mathcal{M}_1, F^{(1)})(\ast \lambda) \rightarrow \mathcal{R}(\mathcal{M}_2, F^{(2)})(\ast \lambda)$ induced by $f$. Let $\mathcal{P}_\lambda : \mathbb{P}^1 X \rightarrow X$ be the projection. We have the induced morphism $\mathcal{P}_{\lambda_\ast} (\tilde{f}) : \mathcal{P}_{\lambda_\ast} (\mathcal{R}(\mathcal{M}_1, F^{(1)})(\ast \lambda)) \rightarrow \mathcal{P}_{\lambda_\ast} (\mathcal{R}(\mathcal{M}_2, F^{(2)})(\ast \lambda))$. The sheaf $\mathcal{P}_{\lambda_\ast} (\mathcal{R}(\mathcal{M}_i, F^{(i)})(\ast \lambda))$ is isomorphic to $\mathcal{M}_i(\ast \lambda^{-1})$. Because $\mathcal{P}_{\lambda_\ast} (f(\lambda s)) = \lambda \mathcal{P}_{\lambda_\ast} (\tilde{f}(s))$, and because $\mathcal{P}_{\lambda_\ast} (\tilde{f})$ is compatible with the action of $\lambda \partial_\lambda$, we obtain that

$$\mathcal{P}_{\lambda_\ast} (\tilde{f}) \left( \sum_{j=-N}^N s_j \lambda^j \right) = \sum_{j=-N}^N f_1 (s_j) \lambda^j$$

Hence, $\tilde{f}$ comes from a morphism of the Rees modules compatible with the grading. Then, we obtain that $f_1$ preserves the filtrations.

3.6.3 $\mathcal{R}$-modules associated to homogeneous meromorphic functions

Let $X$ be a complex manifold with a $\mathbb{C}^*$-action $\mu$ as above. For any $a \in \mathbb{C}^*$, the morphism $\mu (a, \bullet) : X \rightarrow X$ is denoted by $\mu_a$. Let $D$ be a hypersurface of $X$ such that $\mu_a (D) = D$ for any $a \in \mathbb{C}^*$. Let $f$ be a meromorphic function on $(X, D)$ such that $\mu_a^* (f) = a f$ for any $a \in \mathbb{C}^*$.

Proposition 3.30 The $\mathcal{R}$-modules $\mathcal{L}_* (f, D)$ are $\mathbb{C}^*$-homogeneous.
Proof First, let us consider the $\tilde{R}_X(*D)$-module $Q(f)$ induced by $O_X(*D)$ with the meromorphic flat connection $d + d(\lambda^{-1} f)$. We set $X_1 := C^{*} \times X$ and $D_1 := C^{*} \times D$. We set $f_1 := p^*(f)$. We naturally have $p^*O_X(*D_1) \simeq O_{X_1}(\lambda D_1) \simeq \tilde{\mu}^*O_X(*D)$. Because $\tilde{\mu}^*(\lambda^{-1} f) = p^*(\lambda^{-1} f) = \lambda^{-1} f_1$, the pull back of the connections are also equal. Hence, we have $p^*Q(f) \simeq \tilde{\mu}^*Q(f)$. Similarly, we can check the cocycle condition for $Q(f)$ as in Definition 3.23. It is enough to check that we have natural isomorphisms $p^*L_\ast(f) \simeq \tilde{\mu}^*L_\ast(f)$ which induce the above isomorphism $p^*Q(f) \simeq \tilde{\mu}^*Q(f)$.

Take $(a_0, \lambda_0, Q_0) \in X_1 = C^{*} \times C^{*} \times X$. Let us check that there exists a unique isomorphism $p^*L_\ast(f) \simeq \tilde{\mu}^*L_\ast(f)$ around $(a_0, \lambda_0, Q_0)$ which induces $p^*Q(f) \simeq \tilde{\mu}^*Q(f)$. Let $g$ be a holomorphic function defined on a neighbourhood of $(a_0, \lambda_0, Q_0)$ such that $g^{-1}(0) = D$. We set $g_1 := g \circ \mu$ defined on a neighbourhood of $(a_0, Q_0)$. We have $p^*L_\ast(f)[*g_1] = p^*L_\ast(f)$ on a neighbourhood of $(\lambda_0, a_0, Q_0)$. It is enough to prove that $\tilde{\mu}^*L_\ast(f)[*g_1] = \tilde{\mu}^*L_\ast(f)$ on a neighbourhood of $(\lambda_0, a_0, Q_0)$.

We set $Q_1 := \mu(a_0, Q_0)$. We take a small neighbourhood $U = U_{\lambda_0} \times U_{Q_1} \subset X$ of $(\lambda_0, Q_1)$ such that $g$ is defined on $U_{Q_1}$. Let $t_g : U_{Q_1} \rightarrow U_{Q_1} \times C$ be the graph. We may assume that we have the $V$-filtration of $t_g L_\ast(f)$ on $U \times \mathcal{C}_t$. In the case $* = \ast$, the induced morphism $t : Gr^V_0 \rightarrow Gr^V_{-1}$ is an isomorphism. In the case $* = !$, the induced morphism $\delta_t : Gr^V_{-1} \rightarrow Gr^V_0$ is an isomorphism.

Let $U' \subset \mathcal{X}_t$ be a small neighbourhood of $(a_0, \lambda_0, Q_0)$ such that $\mu(U') \subset U$. Let $\tilde{\mu}_1 : U' \times \mathcal{C}_t \rightarrow U \times \mathcal{C}_t$ be the induced morphism. We naturally have $t_{g_1} L_\ast(f) = \bigoplus_{j=0}^\infty \partial_{t_{g_1} t_{\tilde{\mu}_1}} L_\ast(f)$ and $\tilde{\mu}_1^* L_\ast(f) = \bigoplus_{j=0}^\infty \partial_{t_{g_1} t_{\tilde{\mu}_1}} \tilde{\mu}_1^* L_\ast(f)$. The natural isomorphisms $\tilde{\mu}_1^* L_\ast(f) \simeq \mu_1^* L_\ast(f)$ induce isomorphisms $\tilde{\mu}_1^* t_{g_1} L_\ast(f) \simeq \mu_1^* t_{g_1} L_\ast(f)$.

We set $V_\ast(t_{g_1} \mu_1^* L_\ast(f)) := \mu_1^* V_\ast(t_{g_1} L_\ast(f))$. By the construction, $V_\ast(t_{g_1} \mu_1^* L_\ast(f))$ are $\mathcal{V} \mathcal{R}_{X_1 \times \mathcal{C}_t} U' \times \mathcal{C}_t$. Because $V_\ast(t_{g_1} L_\ast(f))$ are coherent over $V_\mathcal{R}_{X_1 \times \mathcal{C}_t}$, we obtain that $V_\ast(t_{g_1} \mu_1^* L_\ast(f))$ are pseudo-coherent over $O_{X_1 \times \mathcal{C}_t}$ and finitely generated over $\mathcal{V} \mathcal{R}_{X_1 \times \mathcal{C}_t}$ and hence coherent over $\mathcal{V} \mathcal{R}_{X_1 \times \mathcal{C}_t}$. Then, we obtain that $V_\ast(t_{g_1} \mu_1^* L_\ast(f))$ is a $V$-filtration of $t_{g_1} \mu_1^* L_\ast(f)$ along $t$. In the case $* = \ast$, the morphism $t : Gr^V_0 \rightarrow Gr^V_{-1}$ is an isomorphism. In the case $* = !$, the morphism $\delta_t : Gr^V_{-1} \rightarrow Gr^V_0$ is an isomorphism. Hence, we have $\mu_1^* L_\ast(f) \simeq Q(f)[*g_1] \simeq p^* L_\ast(f)[*g_1]$. Thus, the proof of Proposition 3.30 is finished.

4 Graded sesqui-linear dualities

4.1 Sesqui-linear dualities and graded sesqui-linear dualities

Let $X$ be a complex manifold. Let $(\mathcal{T}, W)$ be a mixed twistor $\mathcal{D}$-module on $X$.

- A sesqui-linear duality of weight $w$ on $(\mathcal{T}, W)$ is a morphism $S : (\mathcal{T}, W) \rightarrow (\mathcal{T}, W)^* \otimes \mathcal{T}(-w)$ such that $S^* = (-1)^w S$.

- A graded sesqui-linear duality on $(\mathcal{T}, W)$ is a tuple of sesqui-linear dualities $S_w$ ($w \in \mathbb{Z}$) of weight $w$ on $Gr^W_w \mathcal{T}$.

- A graded sesqui-linear duality $(S_w \mid w \in \mathbb{Z})$ on $(\mathcal{T}, W)$ is called a graded polarization if each $S_w$ is a polarization of $Gr^W_w \mathcal{T}$.

Remark 4.1 The two notions are the same if $(\mathcal{T}, W)$ is pure. But, in general, they are not directly related. A sesqui-linear duality $S$ of weight $w$ on a mixed twistor $\mathcal{D}$-module $(\mathcal{T}, W)$ induces just morphisms $Gr^W_w (\mathcal{T}) \rightarrow (Gr^W_{-\ell+2w} (\mathcal{T}))^* \otimes \mathcal{T}(-w)$.

If $(\mathcal{T}, W)$ is pure of weight $w$, a sesqui-linear duality of weight $w$ is called just a sesqui-linear duality.

4.2 Induced graded sesqui-linear dualities

4.2.1 Pure case

Let $X$ be a complex manifold. Let $\mathcal{T}$ be a pure twistor $\mathcal{D}$-module of weight $w$ on $X$. Let $D$ be an effective divisor of $X$. Recall that we have the mixed twistor $\mathcal{D}$-modules $\mathcal{T}[\ast D]$ ($\ast = *, !$) obtained as the localizations. Note that a polarization $\mathcal{S}$ of $\mathcal{T}$ induces a graded polarization $\mathcal{S}[\ast D]$ of $\mathcal{T}[\ast D]$ as explained in 3.6. By the same construction, from a sesqui-linear duality $S$ of $\mathcal{T}$, we obtain a graded sesqui-linear duality $S[\ast D] = (S[\ast D])_j \mid j \in \mathbb{Z}$.
Let us recall the local construction, for which we can take a holomorphic function $f$ such that $D = (f)_0$. (The graded sesqui-linear duality is eventually independent from the choice of $f$. See [36].)

Let $\mathcal{N} : \psi_f^{(a)}(\mathcal{T}) \to \psi_f^{(a-1)}(\mathcal{T})$ be the canonical morphism. Recall that the weight filtration of $\psi_f^{(a)}(\mathcal{T})$ is the shift of the monodromy weight filtration of $\mathcal{N}$, i.e., $W(\mathcal{N})_j \psi_f^{(a)}(\mathcal{T}) = W_{w+1-2a+j} \psi_f^{(a)}(\mathcal{T})$. In particular, the induced morphisms

$$\text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T}) \xrightarrow{\mathcal{N}_j} \text{Gr}_w^{W} \psi_f^{(a-j)}(\mathcal{T}) = \text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T}) \otimes \mathcal{T}(-j)$$

are isomorphisms for $j \geq 0$. The primitive part $P \text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T}) (j \geq 0)$ is defined to be the kernel of $\mathcal{N}^j : \text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T}) \to \text{Gr}_w^{W} \psi_f^{(a-j)}(\mathcal{T})$. We formally set $P \text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T}) = 0$ for $j < 0$, in this paper.

We have natural isomorphisms:

$$\text{Gr}_w^{W} \mathcal{T}[f] \simeq \begin{cases} T & (j < 0) \\ \text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T}) & (j = 0) \\ P \text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T}) & (j > 0) \end{cases}$$

A sesqui-linear duality $\mathcal{S}[(f)_0]_{w+1+\ell}$ on $P \text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T}) (\ell \geq 0)$ is induced by the composite of the following morphisms:

$$\text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T}) \xrightarrow{a_1} \text{Gr}_w^{W} \psi_f^{(-\ell)}(\mathcal{T}) = \text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T}) \otimes \mathcal{T}(-\ell) \xrightarrow{a_2} \text{Gr}_w^{W} \psi_f^{(-\ell)}(\mathcal{T}) \otimes \mathcal{T}(-w-\ell) \xrightarrow{a_3} \left(\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})\right)^\ast \otimes \mathcal{T}(-w-\ell) \xrightarrow{a_4} \left(\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})\right)^\ast \otimes \mathcal{T}(-w-1-\ell)$$

(47)

Here, $a_1$ is induced by $(-\mathcal{N})^\ell$, $a_2$ is induced by $\mathcal{S}$, and $a_4 (i = 3, 4, 5)$ are the isomorphisms in [36].

For $j < 0$, let $P' \text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T})$ denote the image of $P \text{Gr}_w^{W} \psi_f^{(a-j)}(\mathcal{T}) \to \text{Gr}_w^{W} \psi_f^{(a-j)}(\mathcal{T})$. We have the following natural isomorphisms:

$$\text{Gr}_w^{W} \mathcal{T}[f] \simeq \begin{cases} T & (j > 0) \\ \text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})^\ast & (j = 0) \\ P' \text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T}) (j < 0) \end{cases}$$

Because $P' \text{Gr}_w^{W} \psi_f^{(a)}(\mathcal{T}) \simeq P \text{Gr}_w^{W} \psi_f^{(a-j)}(\mathcal{T})$, the pure twistor $\mathcal{D}$-modules $P \text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T}) (\ell \geq 0)$ are equipped with the induced sesqui-linear dualities $\mathcal{S}[(f)_0]_{w+1-\ell}$. They are induced by the composite of the following:

$$\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T}) \xrightarrow{b_1} \text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})^\ast \otimes \mathcal{T}(-w) \xrightarrow{b_2} \text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T})^\ast \otimes \mathcal{T}(-w+1) = \left(\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})\right)^\ast \otimes \mathcal{T}(-w+1) \xrightarrow{b_3} \left(\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})\right)^\ast \otimes \mathcal{T}(-w+1+\ell)$$

(48)

Here, $b_1$ is induced by $\psi_f^{(1)}(\mathcal{S})$, $b_3$ is the inverse of the induced morphism of $(-1)^\ell(\mathcal{N}^{\ell})^\ast$, and $b_i (i = 2, 4)$ are the natural isomorphisms. It is also induced by the composite of the following:

$$\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T}) \xrightarrow{c_1} \text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T}) \otimes \mathcal{T}(\ell) \xrightarrow{c_2} \text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})^\ast \otimes \mathcal{T}(-w+\ell) = \text{Gr}_w^{W} \psi_f^{(0)}(\mathcal{T})^\ast \otimes \mathcal{T}(-w+1+\ell) \xrightarrow{c_3} \left(\text{Gr}_w^{W} \psi_f^{(1)}(\mathcal{T})\right)^\ast \otimes \mathcal{T}(-w+1+\ell)$$

(49)

Here, $c_1$ is the inverse of the induced morphism of $\mathcal{N}^{\ell}$, $c_2$ is induced by $\mathcal{S}$, and $c_3$ is the natural morphism.

The graded sesqui-linear dualities $\mathcal{S}[D]$ and $\mathcal{S}[\ast D]$ induce graded sesqui-linear dualities on the kernel and the cokernel of the morphism $\mathcal{T}[D] \to \mathcal{T}[\ast D]$.

Remark 4.2 If $\mathcal{S}$ is a polarization, then $\mathcal{S}[\ast D]$ are graded polarizations.
4.2.2 Mixed case

The construction was also generalized in the mixed case [36]. Let \((\mathcal{T}, W)\) be a mixed twistor \(\mathcal{D}\)-module on \(X\). Let \(D\) be an effective divisor of \(X\). A graded sesqui-linear duality \(\mathcal{S} = (\mathcal{S}_w \mid w \in \mathbb{Z})\) of \((\mathcal{T}, W)\) induces graded sesqui-linear dualities \(\mathcal{S}[\ast D]\) of the mixed twistor \(\mathcal{D}\)-modules \((\mathcal{T}[\ast D], W)\). We recall the local construction. If we are given a holomorphic function \(f\) such that \(D = (f)_{\geq 0}\), then \(\psi_f^{(a)}(\mathcal{T})\) is equipped with the two filtrations. One is the filtration \(L\) induced by the weight filtration of \(\mathcal{T}\). The other is the relative monodromy weight filtration \(W\) of \(\mathcal{N}\) with respect to \(L\), which is equal to the weight filtration of the mixed twistor \(\mathcal{D}\)-module \(\psi_f^{(a)}(\mathcal{T})\). The induced filtration \(L\) on \(\text{Gr}^W \psi_f^{(a)}(\mathcal{T})\) has a canonical splitting due to Kashiwara:

\[
\text{Gr}^W \psi_f^{(a)}(\mathcal{T}) = \bigoplus_w \text{Gr}^L_w \text{Gr}^W \psi_f^{(a)}(\mathcal{T}) = \bigoplus_w \text{Gr}^W \psi_f^{(a)}(\mathcal{T})
\]

Hence, the sesqui-linear dualities \(\mathcal{S}_w\) on \(\text{Gr}^L_w(\mathcal{T})\) \((w \in \mathbb{Z})\) induce sesqui-linear dualities of \(\text{Gr}^W_j \psi_f^{(a)}(\mathcal{T})\) \((j \in \mathbb{Z})\). We have the decomposition:

\[
\text{Gr}^L_k(\mathcal{T}[\ast D]) = A_{1,k} \oplus A_{2,k}
\]

Here, \(A_{1,k}\) is the sum of the direct summands of \(\text{Gr}^L_k(\mathcal{T})\) whose strict supports are not contained in \(D\), and the support of \(A_{2,k}\) is contained in \(D\). As shown in [36], \(A_{2,k}\) is naturally isomorphic to a subobject in \(\text{Gr}^k \psi_f^{(0)}(\mathcal{T})\). Hence, it is equipped with the induced sesqui-linear duality, which is the \(k\)-th entry of \(\mathcal{S}[\ast D]\). It \(\mathcal{S}\) is a graded polarization, then \(\mathcal{S}[\ast D]\) are graded polarizations.

4.3 Push-forward

4.3.1 A condition

We introduce a condition on the push-forward of mixed twistor \(\mathcal{D}\)-modules equipped with a graded sesqui-linear duality. Let \(F: X \to Y\) be a projective morphism of complex manifolds. Let \((\mathcal{I}, W)\) be a mixed twistor \(\mathcal{D}\)-module on \(X\) with a graded sesqui-linear duality \(\mathcal{S} = (\mathcal{S}_j \mid j \in \mathbb{Z})\). Recall that we have the induced complex

\[
F_{i+1}^0 \mathcal{G}_j^W \mathcal{I} \xrightarrow{a_{j+1}^{i+1}} F_i^0 \mathcal{G}_j^W \mathcal{I} \xrightarrow{a_j^i} F_{i+1}^1 \mathcal{G}_j^W \mathcal{I},
\]

and \(\ker a_j^i / \text{im} a_{j+1}^{i+1}\) is naturally isomorphic to \(\text{Gr}_{j+1}^W F_i^0 \mathcal{I}\). Here, \(a_j^i\) are induced by the extensions \(0 \to \text{Gr}_{j-1}^W \to W_j/W_{j-2} \to \text{Gr}_{j}^W \to 0\). We set \(\alpha_j := a_{j+1}^{-1}\) and \(\beta_j := a_j^i\). We have

\[
F_{i+1}^0 \mathcal{G}_j^W \mathcal{I} \xrightarrow{\alpha_j} F_i^0 \mathcal{G}_j^W \mathcal{I} \xrightarrow{\beta_j} F_{i+1}^1 \mathcal{G}_j^W \mathcal{I}.
\]

As the Hermitian adjoint, we have the following:

\[
\left( F_{i+1}^0 \mathcal{G}_j^W \mathcal{I} \right)^* \xleftarrow{\alpha_j^*} \left( F_i^0 \mathcal{G}_j^W \mathcal{I} \right)^* \xrightarrow{\beta_j^*} \left( F_{i+1}^1 \mathcal{G}_j^W \mathcal{I} \right)^*.
\]

We also have the induced isomorphism \(F_i^0 \mathcal{S}_j : F_i^0 \mathcal{G}_j^W \mathcal{I} \xrightarrow{\cong} (F_i^0 \mathcal{G}_j^W \mathcal{I})^* \otimes \mathcal{T}(-j)\).

Lemma 4.3 Let \(\mathcal{I}_j\) denote the image of \(\ker \beta_j \cap \ker (\alpha_j^* \circ F_i^0 \mathcal{S}_j) \to \ker \beta_j / \text{im} \alpha_j\). Then, the morphism

\[
\ker \beta_j \cap \ker (\alpha_j^* \circ F_i^0 \mathcal{S}_j) \to \left( \ker \beta_j \cap \ker (\alpha_j^* \circ F_i^0 \mathcal{S}_j) \right)^* \otimes \mathcal{T}(-j)
\]

induced by \(F_i^0 \mathcal{S}_j\) is factorized as follows:

\[
\ker \beta_j \cap \ker (\alpha_j^* \circ F_i^0 \mathcal{S}_j) \to \mathcal{I}_j \xrightarrow{\iota_j} \mathcal{I}_j \otimes \mathcal{T}(-j) \to \left( \ker \beta_j \cap \ker (\alpha_j^* \circ F_i^0 \mathcal{S}_j) \right)^* \otimes \mathcal{T}(-j)
\]

Namely, we have an induced sesqui-linear duality \(\nu_j\) of weight \(j\) on \(\mathcal{I}_j\).
Definition 4.4 We say that Condition \( \mathcal{S} \) is satisfied for the morphism \( F : X \to Y \) and the mixed twistor \( \mathcal{D} \)-module \( (\mathfrak{A}, \mathfrak{W}) \) with the graded sesqui-linear duality \( \mathcal{S} \) if the following holds:

(A) The morphisms \( \ker \beta_j \cap \ker (\alpha_j^* \circ F_j^0 \mathcal{S}_j) \to \ker \beta_j / \im \alpha_j \) are epimorphisms for any \( j \).

Remark 4.5 If \( F_j^0 \mathcal{S}_j \) is a polarization of \( F_j^0 \mathfrak{A}_j \mathfrak{W}_j \), then \( \ker \beta_j \cap \ker (\alpha_j^* \circ F_j^0 \mathcal{S}_j) \to \ker \beta_j / \im \alpha_j \) is an isomorphism.

According to Lemma 4.3 if the condition (A) is satisfied, we have the induced sesqui-linear duality \([F_j^0 \mathcal{S}_j]\) of weight \( j \) on \( \mathfrak{A}_j \mathfrak{W}_j \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
F_j^0 \mathfrak{A}_j \mathfrak{W}_j & \xrightarrow{F_j^0 \mathcal{S}_j} & (F_j^0 \mathfrak{A}_j \mathfrak{W}_j)^* \otimes \mathcal{T}(-j) \\
\uparrow & & \downarrow \\
\ker \beta_j \cap \ker (\alpha_j^* \circ F_j^0 \mathcal{S}_j) & \longrightarrow & \left( \ker \beta_j \cap \ker (\alpha_j^* \circ F_j^0 \mathcal{S}_j) \right)^* \otimes \mathcal{T}(-j) \\
\downarrow & & \uparrow \\
\mathfrak{A}_j \mathfrak{W}_j & \xrightarrow{[F_j^0 \mathcal{S}_j]} & (\mathfrak{A}_j \mathfrak{W}_j)^* \otimes \mathcal{T}(-j)
\end{array}
\]

The tuple \( ([F_j^0 \mathcal{S}_j] | j \in \mathbb{Z}) \) is denoted by \( [F_j^0 \mathcal{S}] \).

4.3.2 Statements

Let \( F : X \to Y \) be a projective morphism of complex manifolds. Let \( \mathcal{T} \) be a pure twistor \( \mathcal{D} \)-module of weight \( w \) on \( X \) with a sesqui-linear duality \( \mathcal{S}_X \). Let \( D_Y \) be an effective divisor of \( Y \). We set \( D_X := F^* D_Y \). As explained in \([4.3.1]\), we have the mixed twistor \( \mathcal{D} \)-modules \( \mathcal{T}[\ast D_X] \) with the induced graded sesqui-linear duality \( \mathcal{S}_X[\ast D_X] = (\mathcal{S}_X[\ast D_X]_m | m \in \mathbb{Z}) \). We shall prove the following theorem in \([4.3.3]\).

Theorem 4.6 Condition (A) is satisfied for the projective morphism \( F \) and the mixed twistor \( \mathcal{D} \)-module \( \mathcal{T}[\ast D_X] \) with the graded sesqui-linear duality \( \mathcal{S}_X[\ast D_X] \).

The pure twistor \( \mathcal{D} \)-module \( F_0^0 \mathcal{T} \) of weight \( w \) is equipped with the induced sesqui-linear duality \( \mathcal{S}_Y := F_0^0 \mathcal{S}_X \). Moreover, it induces a graded sesqui-linear duality \( \mathcal{S}_Y[\ast D_Y] \) of \( F_0^0 (\mathcal{T})[\ast D_Y] \cong F_0^0 (\mathcal{T}[\ast D_X]) \). We shall prove the following theorem in \([4.3.3]\).

Theorem 4.7 We have \( [F_0^0 \mathcal{S}_X[\ast D_X]] = \mathcal{S}_Y[\ast D_Y] \).

Corollary 4.8 Suppose that \( \mathcal{S} \) is a polarization, and that \( F_i^0 \mathcal{T}[\ast D_X] = 0 \) \((i \neq 0)\). Then, \( [F_i^0 \mathcal{S}_X[\ast D_X]] \) are graded polarizations.

Proof The assumptions imply that \( \mathcal{S}_Y \) is a polarization. Then, the claim follows from \( [F_i^0 \mathcal{S}_X[\ast D_X]] = \mathcal{S}_Y[\ast D_Y] \).

Let \( \mathcal{K} \) and \( \mathcal{C} \) denote the kernel and the cokernel of the natural morphism \( \mathcal{T}[\ast D_X] \to \mathcal{T}[\ast D_X] \). They are naturally equipped with the induced graded sesqui-linear dualities \( \mathcal{S}_K = (\mathcal{S}_{K,j} | j \in \mathbb{Z}) \) and \( \mathcal{S}_C = (\mathcal{S}_{C,j} | j \in \mathbb{Z}) \).

Corollary 4.9 Suppose that \( F_i^0 \mathcal{T}[\ast D_X] = F_i^0 \mathcal{K} = F_i^0 \mathcal{C} = 0 \) \((i \neq 0)\). Then, Condition (A) is satisfied for the morphism \( F \) and the mixed twistor \( \mathcal{D} \)-modules \( \mathcal{K} \) (resp. \( \mathcal{C} \)) with \( \mathcal{S}_K \) (resp. \( \mathcal{S}_C \)). Moreover, we have \( [F_i^0 \mathcal{S}_{K,j}] = \mathcal{S}_Y[\ast D_Y]_j \) for \( j < w \) and \( [F_i^0 \mathcal{S}_{C,j}] = \mathcal{S}_Y[\ast D_Y]_j \) for \( j > w \). Under the assumptions, if \( \mathcal{S} \) is a polarization, \( [F_i^0 \mathcal{S}_K] \) and \( [F_i^0 \mathcal{S}_C] \) are graded polarizations.
Proof Under the assumption of the corollary, we also have \( F_i^T = 0 \) \((i \neq 0)\). Then, the claims immediately follow from Theorem 4.10.

For the proof of the theorems we give an argument in the case \( * = * \), and the other case is similar. So, to simplify the description, we denote \( S_X \circ D_X \) and \( S_Y \circ D_Y \) by \( S_{X,j} \) and \( S_{Y,j} \) in the following proof. Because it is enough to consider the issue locally on \( Y \), we may assume to have a holomorphic function \( g_Y \) such that \( D_Y = (g_Y)_0 \). The pull back \( g_Y \circ F \) is denoted by \( g \). We shall use the notation in \([4.3.1]\) with \( \Sigma = T[\ast D_X] \).

### 4.3.3 Preliminary

We have a natural isomorphism \( F_{1+1}^2 Gr_m^W \psi_g^{(0)}(T) \simeq \psi_g^{(0)} F_1^T \) of mixed twistor \( D \)-modules. By the spectral sequence for \( \psi_g^{(0)}(T) \) with the weight filtration, we have the complex

\[
F_{1+1}^2 Gr_m^W \psi_g^{(0)}(T) \xrightarrow{\xi_j} F_1^2 Gr_m^W \psi_g^{(0)}(T) \xrightarrow{\eta_j} F_1^1 Gr_{m-1}^W \psi_g^{(0)}(T),
\]

and \( \text{Ker} \eta_j/\text{Im} \xi_j \) is naturally isomorphic to \( Gr_{m}^W \psi_g^{(0)}(T) \simeq Gr_{m}^W \psi_g^{(0)} F_1^T \).

For \( \ell \geq 0 \), let \( P_{\ell} Gr_{w+1+j}^W \psi_g^{(0)}(T) \) denote the image of \( P Gr_{w+1+j}^W \psi_g^{(0)}(T) \rightarrow Gr_{w+1+j}^W \psi_g^{(0)}(T) \). Note \( P_{\ell} Gr_{w+1+j}^W \psi_g^{(0)}(T) = 0 \) if \( j + 2\ell < 0 \). We have the primitive decomposition

\[
Gr_{w+1+j}^W \psi_g^{(0)}(T) = \bigoplus_{\ell \geq 0} P_{\ell} Gr_{w+1+j}^W \psi_g^{(0)}(T).
\]

**Lemma 4.10** For \( j \geq 0 \), the morphism \( F_1^0 P Gr_{w+1+j}^W \psi_g^{(0)}(T) \rightarrow F_1^1 Gr_{w+1+j}^W \psi_g^{(0)}(T) \) factors through

\[
F_1^1 P_0 Gr_{w+1+j}^W \psi_g^{(0)}(T) \oplus F_1^1 P_1 Gr_{w+1+j}^W \psi_g^{(0)}(T).
\]

**Proof** Note that \( \xi_j \) and \( \eta_j \) are compatible with the canonical morphisms

\[
\mathcal{N}^T : F_1^1 Gr_m^W \psi_g^{(0)}(T) \rightarrow F_1^1 Gr_m^W \psi_g^{(-\ell)} T = F_1^1 Gr_{m-\ell}^W \psi_g^{(0)} T \otimes T(-\ell).
\]

For \( j \geq 0 \), the morphisms \( Gr_{w+1+j}^W \psi_g^{(0)}(T) \rightarrow Gr_{w+1+j}^W \psi_g^{(-j)}(T) = Gr_{w+1-j}^W \psi_g^{(0)}(T) \otimes T(-j) \) are isomorphisms. Then, the claim easily follows.

The restriction of \( \eta_{w+1+j} \) to \( F_1^0 P Gr_{w+1+j}^W \psi_g^{(0)}(T) \) induces the following morphisms:

\[
\eta_{k,w+1+j} : F_1^0 P Gr_{w+1+j}^W \psi_g^{(0)}(T) \rightarrow F_1^1 P_k Gr_{w+1+j}^W \psi_g^{(0)}(T), \quad (k = 0, 1).
\]

Note that \( Gr_{w+1}^W \text{Cok}(\psi_g^{(1)}(T) \rightarrow \psi_g^{(0)}(T)) \simeq P Gr_{w+1}^W \psi_g^{(0)} T \). By using the spectral sequence for the cokernel, we obtain the following complex

\[
F_{1-1}^1 P Gr_{w+1+j}^W \psi_g^{(0)}(T) \xrightarrow{\kappa_{1-j}} F_1^0 P Gr_{w+1+j}^W \psi_g^{(0)}(T) \xrightarrow{\kappa_{2-j}} F_1^1 P Gr_{w+1+j}^W \psi_g^{(0)}(T).
\]

We have \( \kappa_{2,w+1+j} = \eta_{0,w+1+j} \) by construction.

**Lemma 4.11** We have the following commutative diagram up to signature:

\[
\begin{array}{ccc}
F_1^1 P Gr_{w+1+j}^W \psi_g^{(0)}(T) & \xrightarrow{\eta_{w+1+j}} & F_1^1 P_1 Gr_{w+1+j}^W \psi_g^{(0)}(T) \\
\downarrow \simeq & & \downarrow \simeq \\
\left( F_1^0 P Gr_{w+1+j}^W \psi_g^{(0)}(T)^* \otimes T(-w-j-1) \right) & \xrightarrow{\kappa_{w+1+j}} & \left( F_1^1 P Gr_{w+1+j+2}^W \psi_g^{(0)}(T)^* \otimes T(-w-j-1) \right)
\end{array}
\]

The vertical arrows are induced by the induced sesqui-linear dualities of \( P Gr_{w+k}^W \psi_g^{(0)}(T) \) \((k > 0)\).
Proof Note that we have the following diagram which is commutative up to signs.

\[
\begin{array}{c}
F_w^0 \text{Gr}^W_{w+1+j} \psi_g^{(0)}(T) \\ \downarrow \eta_{w+1+j} \\
F_w^1 \text{Gr}^W_{w+1-j} \psi_g^{(0)}(T) \otimes T(-j) \\ \downarrow \eta_{w-1-j} \\
F_w^1 \text{Gr}^W_{w-j} \psi_g^{(0)}(T) \otimes T(-j) \\
\end{array}
\]

\[
\begin{array}{c}
F_w^0 \text{Gr}^W_{w+1-j} \psi_g^{(0)}(T^*) \otimes T(-w-j) \\ \downarrow a_1 \\
(F_w^0 \text{Gr}^W_{w+1+j} \psi_g^{(0)}(T))^* \otimes T(-w-j-1) \\
\end{array}
\]

Here, \(a_i\) are morphisms induced by the natural isomorphisms \(\psi_g^{(0)}(T^*) \simeq \psi_g^{(1)}(T)^* \simeq \psi_g^{(0)}(T) \otimes T(-1)\), and \(\eta_{w+1-j}'\) denotes \(\eta_{w+1-j}\) for \(T^*\). Let \(\mu : F_w^1 \text{Gr}^W_{w-j} \psi_g^{(0)}(T) \rightarrow (F_w^1 \text{Gr}^W_{w+j+2} \psi_g^{(0)}(T)) \otimes T(-w-j-1)\) denote the composite of the right vertical arrows. The restriction of \(\mu\) to \(F_w^1 P_0 \text{Gr}^W_{w+j} \psi_g^{(0)}(T)\) is 0, and the restriction of \(\mu\) to \(F_w^1 P_1 \text{Gr}^W_{w+j} \psi_g^{(0)}(T)\) induces an isomorphism

\[
F_w^1 P_1 \text{Gr}^W_{w+j} \psi_g^{(0)}(T) \xrightarrow{\approx} (F_w^1 P \text{Gr}^W_{w+j+2} \psi_g^{(0)}(T))^* \otimes T(-w-j-1)
\]

Thus, we obtain (55).

We have the natural morphism

\[
\text{Ker}\left(F_w^0 P \text{Gr}^W_{w+1+j} \psi_g^{(0)}(T) \xrightarrow{\eta_{w+1+j}} F_w^1 \text{Gr}^W_{w+j} \psi_g^{(0)}(T)\right) \rightarrow P \text{Gr}^W_{w+1+j} F_w^0 \psi_g^{(0)}(T).
\]

Lemma 4.12 The morphism (56) is an epimorphism.

Proof Let \(T = (M_1, M_2, C)\). Let \(W\) denote the filtration of \(\psi_g^{(0)} M_2\) associated to the weight filtration of \(\psi_g^{(0)} T\). It is enough to prove that

\[
\text{Ker}\left(F_w^0 P \text{Gr}^W_{w+1+j} \psi_g^{(0)} M_2 \rightarrow F_w^1 \text{Gr}^W_{w+j} \psi_g^{(0)} M_2\right) \rightarrow P \text{Gr}^W_{w+1+j} F_w^0 \psi_g^{(0)} M_2
\]

is an epimorphism. Let \(f_1\) be a section of \(\text{Ker}\left(F_w^0 \text{Gr}^W_{w+1+j} \psi_g^{(0)} M_2 \rightarrow F_w^1 \text{Gr}^W_{w+j} \psi_g^{(0)} M_2\right)\) such that \(\mathcal{N} \cdot 1 \cdot f_1 = \partial f_2\) for \(f_2 \in F_w^{-1} \text{Gr}^W_{w-j} \psi_g^{(0)} M_2 \lambda^{+1}\). We have \(f'_2 \in F_w^{-1} \text{Gr}^W_{w+1+j+1} \psi_g^{(0)} M_2\) such that \(\mathcal{N} \cdot 1 \cdot f'_2 = f_2\). Then, \(f_1 - \partial f'_2\) is a section of \(\text{Ker}\left(F_w^0 P \text{Gr}^W_{w+1+j} \psi_g^{(0)} M_2 \rightarrow F_w^1 \text{Gr}^W_{w+j} \psi_g^{(0)} M_2\right)\). Hence, (56) is an epimorphism.

Lemma 4.13 Under the identification \(\text{Gr}^W_{w+1}(T[sg]) \simeq P \text{Gr}^W_{w+1} \psi_g^{(0)} (T)\), the kernel of

\[
F_w^0 P \text{Gr}^W_{w+1} \psi_g^{(0)} T \xrightarrow{\eta_{w+1}} F_w^1 \text{Gr}^W_{w} \psi_g^{(0)} T
\]

is contained in \(\text{Ker} \beta_{w+1}\).

Proof Recall that \(\phi_g^{(0)} T\) is equal to the image of \(\psi_g^{(1)} T \rightarrow \psi_g^{(0)} T\). Let \(C\) denote the cokernel. Let \(C_1 \subset \psi_g^{(0)} T\) denote the inverse image of \(P \text{Gr}^W_{w+1} \psi_g^{(0)} T \subset C\) by the projection \(\psi_g^{(1)} T \rightarrow C\). The extension \(0 \rightarrow \phi_g^{(0)} T \rightarrow C_1 \rightarrow P \text{Gr}^W_{w+1} \psi_g^{(0)} T \rightarrow 0\) induces the following morphism:

\[
F_w^0 P \text{Gr}^W_{w+1} \psi_g^{(1)} T \rightarrow F_w^1 \psi_g^{(0)} T.
\]
Because $\phi_g^{(0)}$ is an exact functor, the kernel of $\beta_{w+1}$ is equal to the kernel of $\phi_g^{(0)}$. Moreover, the kernel of $\phi_g^{(0)}$ is equal to the kernel of the following induced morphism

$$F^0_1 P G_{w+1}^W \psi_g^{(1)} \mathcal{T} \xrightarrow{c_1} G_{w+1}^W F^1_1 \phi_g^{(0)}(\mathcal{T}).$$

We have the following complex associated to $\phi_g^{(0)}(\mathcal{T})$ with the weight filtration, and $G_{w+1}^W F^1_1 \phi_g^{(0)}(\mathcal{T})$ is $\text{Ker} c_3/\text{Im} c_2$:

$$F^0_1 G_{w+1}^W \phi_g^{(0)}(\mathcal{T}) \xrightarrow{c_2} F^1_1 G_w^W \phi_g^{(0)}(\mathcal{T}) \xrightarrow{c_3} F^2_1 G_{w-1}^W \phi_g^{(0)}(\mathcal{T}).$$

We have $G_w^W \psi_g^{(0)}(\mathcal{T}) = G_w^W \phi_g^{(0)}(\mathcal{T})$. The image of $\eta_{w+1}$ is contained in $\text{Ker} c_3$, and $c_1$ is the composite of

$$F^0_1 P G_{w+1}^W \psi_g^{(1)} \mathcal{T} \xrightarrow{\eta_{w+1}} \text{Ker} c_3 \xrightarrow{\text{Ker} c_3/\text{Im} c_2}.$$

Then, the claim of the lemma follows.

4.3.4 Proof of Theorem 4.6 and Theorem 4.7

For $j > w$, we have natural isomorphisms $G_j^w F^i_1 \mathcal{T}[\ast D_X] \simeq G_j^w F^i_1 \psi_g \mathcal{T}$. For $j > w$, the morphism $\kappa_{ij}$ is identified with $\alpha_j$. For $j > w + 1$, the morphism $\kappa_{ij}$ in (53) is identified with $\beta_j$. We also have Lemma 4.13

Hence, by Lemma 4.11, we have

$$\text{Ker} \left( F^0_1 P G_{w+1}^W \psi_g^{(0)}(\mathcal{T}) \rightarrow F^1_1 G_{w+1}^W \psi_g^{(0)}(\mathcal{T}) \right) \simeq \text{Ker} \beta_j \cap \text{Ker} (\alpha_j^* \circ F^0_1 S_{X,j}).$$

Hence, Lemma 4.12 implies Theorem 4.6

Let us consider the morphism

$$F^0_1 G_{w+1+j}^W \psi_g^{(0)}(\mathcal{T}) \xrightarrow{F^0_1 \psi_g^{(0)} S_X \circ (-\mathcal{N})^j} \left( F^0_1 G_{w+1+j}^W \psi_g^{(0)}(\mathcal{T}) \right)^{\ast} \otimes \mathcal{T}(-w - j - 1)$$

It induces a sesqui-linear duality for $\text{Ker} \eta_{w+1+j} \subset F^0_1 G_{w+1+j}^W \psi_g^{(0)}(\mathcal{T})$:

$$\text{Ker} \eta_{w+1+j} \rightarrow \left( \text{Ker} \eta_{w+1+j} \right)^{\ast} \otimes \mathcal{T}(-w - j - 1)$$

It is factorized as follows:

$$\text{Ker} \eta_{w+1+j} \rightarrow G_{w+1+j}^W F^0_1 \psi_g^{(0)}(\mathcal{T}) \xrightarrow{b} G_{w+1+j}^W F^0_1 \psi_g^{(0)}(\mathcal{T}^{\ast})^{\ast} \otimes \mathcal{T}(-w - j - 1) \rightarrow \left( \text{Ker} \eta_{w+1+j} \right)^{\ast} \otimes \mathcal{T}(-w - j - 1) \quad (61)$$

By construction, the restriction of $b$ to $P G_{w+1+j}^W F^0_1 \psi_g^{(0)}(\mathcal{T}) \simeq G_{w+1+j}^W F^0_1 \mathcal{T}[\ast D_X] = G_{w+1+j}^W F^0_1 \mathcal{T}$ is the induced sesqui-linear duality $\mathcal{S}_{Y,w+1+j}$. Then, the claim of Theorem 4.7 follows.

4.4 Basic examples of induced sesqui-linear dualities

Let $X$ be a complex manifold. Set $d := \dim X$. We have the pure twistor $\mathcal{D}$-module $\mathcal{U}_X(\mathcal{D},0) = (\mathcal{O}_X \lambda^d, \mathcal{O}_X, C_0)$. Here, $C_0$ is given by $C_0(s_1, \sigma^* s_2) = s_1 \cdot \sigma^*(s_2)$. The canonical polarization $\mathcal{U}_X(\mathcal{D},0) \rightarrow \mathcal{U}_X(\mathcal{D},0)^{\ast} \otimes \mathcal{T}(-d)$ is given by $(\lambda^d, 1) \mapsto ((-1)^d \cdot 1 \cdot \lambda^d, (-1)^d \cdot \lambda^d \cdot \lambda^{-d})$.

Let $D = \sum k_i D_i$ be an effective divisor of $X$ such that $\bigcup D_i$ is normal crossing. We describe the induced graded polarization on $\mathcal{U}_X(\mathcal{D},0)[\ast D]$. 

40
4.4.1 The simplest case

Let us consider the case that $D = (t)_0$ for a coordinate function $t$. We will not distinguish $D$ and $|(t)_0|$. We describe the induced graded polarizations of $U_X(0,d)|[*t]$ ($* = *,!$). It is enough to describe the induced polarizations on $\psi^0_tU_X(d,0) \simeq U_X(d,0)|[*t]/U_X(0,d)$ and $\psi^1_tU_X(d,0) \simeq \text{Ker}(U_X(d,0)[!t] \longrightarrow U_X(d,0))$.

**Lemma 4.14** Let $\iota : D \longrightarrow X$ be the inclusion as above. The natural isomorphisms

$$\psi^0_tU_X(d,0) \simeq \iota_1U_D(d-1,0) \otimes T(-1), \quad \psi^1_tU_X(d,0) \simeq \iota_1U_D(d-1,0)$$

are compatible with the polarizations.

**Proof** The natural isomorphism $\psi_{t \cdot \delta}U_X(d,0) \otimes U_D(-1,0) \simeq U_D(d-1,0)$ is compatible with the polarizations. Then, the claim follows from [36 Proposition 4.3.2].

We give a more explicit description of the polarizations.

**Lemma 4.15** The induced polarization $\psi^1_tU_X(d,0) \longrightarrow (\psi^1_tU_X(d,0))^* \otimes T(-d+1)$ is given by

$$([t^{-1}\lambda^{d'}], \{0_1\}) \longmapsto ((-1)^d[0_1]\lambda^{d-1}, (-1)^d[t^{-1}\lambda]).$$

The induced polarization $\psi^0_tU_X(d,0) \longrightarrow ([\psi^0_tU_X(d,0)]^* \otimes T(-d-1)$ is given by

$$([0_1][\lambda^{d'}, [t^{-1}]) \longmapsto ((-1)^d+1[t^{-1}][\lambda^{d+1}, (-1)^d+1[0_1]^{-1}\lambda].$$

**Proof** Recall that we have a natural isomorphism of $\psi^{1}_tU_X(d,0) \simeq \iota_1\psi_{t \cdot \delta}U_X(d,0) \otimes U_X(-1,0)$ studied in [36 Proposition 4.3.1]. In this case, the isomorphisms of the $\mathcal{R}$-modules are given as follows:

$$\psi^0_t(O_X) \simeq \iota_1\psi_{t \cdot \delta}(O_X)\lambda^{-1}, \quad [t^{-1}] \longmapsto (dt/\lambda)^{-1}\lambda^{-1}$$

$$\psi^1_t(O_X) \simeq \iota_1\psi_{t \cdot \delta}(O_X), \quad [0_1] \longmapsto -(dt/\lambda)^{-1}.$$  

Then, according to [36 Proposition 4.3.2], the induced polarization of $\psi^1_tU_X(d,0)$ is given by (62). Similarly, the isomorphism $\psi^0_tU_X(d,0) = U_X(d,0)[*t]/U_X(0,d) \simeq \iota_1\psi_{t \cdot \delta}U_X(d,0) \simeq U_X(0,-1)$ is given by

$$([-0_1][\lambda^{d'}, [t^{-1}]) \longmapsto (dt/\lambda)^{-1}\lambda^{d'}, (dt/\lambda)^{-1}\lambda^{-1}).$$

Then, we obtain the claim for $\psi^0_tU_X(d,0)$.

We have an isomorphism of mixed twistor $\mathcal{D}$-modules

$$U_X(d,0)[*t]/U_X(0,d) \simeq \text{Ker}(U_X(d,0)[!t] \longrightarrow U_X(d,0)) \otimes T(-1)$$

given by $([0_1][\lambda^{d'}, [t^{-1}]) \longmapsto ([t^{-1}][\lambda^{d+1}, [0_1]^{-1}\lambda^{-1}].$ It is compatible with the polarizations (62) and (63).

4.4.2 Normal crossing case

Let us consider the case that $X$ is equipped with a holomorphic coordinate $(x_1, \ldots, x_d)$ such that $D = \sum_{i=1}^m k_iD_i$ for some $(k_1, \ldots, k_m) \in \mathbb{Z}_{>0}^m$, where $D_i = \{x_i = 0\}$. We have the decomposition

$$\text{Gr}^W(U_X(d,0)[*D]) = \bigoplus_{j \geq d} \text{Gr}^W_j(U_X(d,0)[*D]).$$

For $J \subset \{1, \ldots, m\}$, we set $D_J := \bigcap_{i \in J} D_i$. Let $\iota_J : D_J \longrightarrow X$ denote the inclusion. We have

$$\text{Gr}^W_j(U_X(d,0)[*D_J]) \simeq \bigoplus_{|J|=j} \iota_JU_D(d-j,0) \otimes T(-j).$$  

(64)

Note that $\iota_JU_D(d-j,0) \otimes T(-j)$ is equipped with the natural polarization $((-1)^d, (-1)^d)$.
Proposition 4.16 Under the natural isomorphism [64], the induced polarization on $i_j \mathcal{U}_d (d-j,0) \otimes T(-j)$ is equal to $\left((-1)^d \prod_{i \in J} k_i^{-1}, (-1)^d \prod_{i \in J} k_i^{-1}\right)$. 

Proof It is enough to consider the induced polarization on $\mathcal{G}_m^W \mathcal{U}_X (d,0)$. For $a = (a_i) \in \mathbb{Z}^m$, we set $\psi(a) := \psi_{X_1}^{(a_1)} \circ \cdots \circ \psi_{X_m}^{(a_m)}$ and $\psi(a) := \psi_{X_1}^{(a_1)} \circ \cdots \circ \psi_{X_m}^{(a_m)}$. Let $0 = (0, \ldots, 0) \in \mathbb{Z}^m$ and $1 = (1, \ldots, 1) \in \mathbb{Z}^m$. We set $f := \prod_{i=1}^m x_i^{k_i}$. We have $\psi_f^{(a)} \mathcal{U}(d,0) = (\psi_f^{(a)} \mathcal{O}_X)^d, \psi_{\mathcal{O}_X}, \psi_f^{(a)} \mathcal{C})$. We consider $\phi(a) \psi_f^{(a)} \mathcal{U}_X (d,0)$.

Recall that $\psi_f^{(a)} (\mathcal{O}_X)$ is isomorphic to $\text{Cok} \left( \Pi_{f,\infty} \mathcal{O}_X ![f] \rightarrow \Pi_{f,\infty} \mathcal{O}_X[*f] \right)$. Then, $\phi(0) \psi_f^{(a)} (\mathcal{O}_X)$ is isomorphic to

$$\text{Cok} \left( \Pi_{0,\infty}^\infty \psi(1) \mathcal{O}_X \rightarrow \Pi_{0,\infty}^\infty \psi(0) \mathcal{O}_X \right).$$

The morphism is induced by $(\lambda s)^j \prod_{i=1}^m \delta_i \mapsto (\lambda s)^{j+m} \prod_{i=1}^m k_i x_i^{-1}$. Hence, we have the following isomorphism:

$$\phi(0) \psi_f^{(a)} (\mathcal{O}_X) \simeq \bigoplus_{0 \leq j \leq m-1} \psi(0)(\mathcal{O}_X)(\lambda s)^j \tag{65}$$

Recall that $\psi_f^{(a)} (\mathcal{O}_X)$ is also isomorphic to $\text{Ker} \left( \Pi_{f,\infty} \mathcal{O}_X ![f] \rightarrow \Pi_{f,\infty} \mathcal{O}_X[*f] \right)$, and that $\phi(0) \psi_f^{(a)} (\mathcal{O}_X)$ is isomorphic to

$$\text{Ker} \left( \Pi_{0,\infty} \mathcal{O}_X \rightarrow \Pi_{0,\infty} \psi(0) \mathcal{O}_X \right).$$

We obtain the following isomorphism:

$$\phi(0) \psi_f^{(1)} (\mathcal{O}_X) \simeq \bigoplus_{-m+1 \leq j \leq 0} \psi(1)(\mathcal{O}_X)(\lambda s)^j.$$

Hence, $\phi(0) \psi_f^{(0)} \mathcal{U}_X (d,0)$ is naturally isomorphic to $\bigoplus_{j=0}^{m-1} \psi^0(\mathcal{U}_X (d,0) \otimes T(j))$, where $\psi(0) \mathcal{U}_X (d,0)$ is

$$\left(\psi(1) \mathcal{O}_X \lambda^d, \psi(0) \mathcal{O}_X, \psi(0) \mathcal{C}\right) \simeq \bigoplus_{j=0}^{m-1} \mathcal{U}_d (d-m,0) \otimes T(-m).$$

The canonical morphism $\mathcal{N}_j : \phi(0) \psi_f^{(a)} \mathcal{U}_X (d,0) \rightarrow \phi(0) \psi_f^{(a-j)} \mathcal{U}_X (d,0)$ is given as follows; the second component is given as $b(\lambda s)^i \mapsto b(\lambda s)^j$ if $i \leq a - j + m - 1$, and $b(\lambda s)^i \mapsto 0$ otherwise. The first component is given similarly.

Let us describe the morphism $\phi(0) \psi_f^{(0)} \mathcal{U}_X (d,0) \rightarrow \phi(0) \psi_f^{(0)} \mathcal{U}_X (d,0)^* \otimes T(-d-1)$ induced by the polarization of $\mathcal{U}_X (d,0)$. We have the isomorphism $\psi_f^{(0)} (\mathcal{O}_X \lambda^d) \simeq \psi_f^{(1)} (\mathcal{O}_X \lambda^d) \lambda^{-1}$ induced by $(\lambda s)^j \mapsto -(\lambda s)^{j+1} \lambda^{-1}$. It induces an identification $\phi(0) \psi_f^{(0)} (\mathcal{O}_X \lambda^d) \simeq \phi(0) \psi_f^{(1)} (\mathcal{O}_X \lambda^d) \lambda^{-1}$. Note that the isomorphism

$$\text{Cok} \left( \Pi_{0,\infty} \mathcal{O}_X \rightarrow \Pi_{0,\infty} \mathcal{O}_X \right) \simeq \text{Ker} \left( \Pi_{0,\infty} \mathcal{O}_X \rightarrow \Pi_{0,\infty} \mathcal{O}_X \right)$$

induces the isomorphism

$$\bigoplus_{1 \leq j \leq m} \psi(0)(\mathcal{O}_X)(\lambda s)^j \simeq \text{Cok} \left( \Pi_{0,\infty} \psi(1) \mathcal{O}_X \rightarrow \Pi_{0,\infty} \psi(0) \mathcal{O}_X \right) \simeq \text{Ker} \left( \Pi_{0,\infty} \psi(1) \mathcal{O}_X \rightarrow \Pi_{0,\infty} \psi(0) \mathcal{O}_X \right)$$

$$\cup \bigoplus_{-m+1 \leq j \leq 0} \psi(1)(\mathcal{O}_X)(\lambda s)^j \tag{66}$$

which is given by $(\lambda s)^{p+m} \prod_{i=1}^m (k_i x_i^{-1}) \mapsto (\lambda s)^p \prod_{i=1}^m \delta_i$. In all, the induced morphism

$$\phi(0) \psi_f^{(0)} (\mathcal{S}) : \bigoplus_{0 \leq j \leq m-1} \psi(0)(\mathcal{O}_X)(\lambda s)^j \rightarrow \bigoplus_{-m+1 \leq j \leq 0} \psi(1)(\mathcal{O}_X)(\lambda s)^j \lambda^{-1}$$

is given by $(\lambda s)^{p+m} \prod_{i=1}^m (k_i x_i^{-1}) \mapsto (-1)^{d+1} \lambda^{-1} (\lambda s)^{p+1} \prod_{i=1}^m \delta_i$. Hence, the second component of the induced polarization of $\mathcal{G}_m^W \phi(0) \psi_f^{(a)} \mathcal{U}(d,0)$ is given by the isomorphisms $\prod_{i=1}^m (k_i x_i^{-1}) \mapsto (-1)^{d+m} \lambda^{-m} \prod_{i=1}^m \delta_i$. The first component is obtained in the same way.

The isomorphism $\psi(0) \mathcal{O}_X \simeq \iota \mathcal{O}_{D(m)} \lambda^{-m}$ is given by $\prod_{i=1}^m (x_i^{-1} \lambda) \mapsto \prod_{i=1}^m (dx_i / \lambda)^{-1}$, and the isomorphism $\psi(1) \mathcal{O}_X \simeq \iota \mathcal{O}_{D(m)} \lambda^{-m}$ is given by $\prod_{i=1}^m (\lambda x_i) \mapsto \prod_{i=1}^m (dx_i / \lambda)^{-1}$. Then, the claim of Proposition 4.16 follows.
4.5 Nearby cycle functors and maximal functors

We give a relation between the induced graded sesqui-linear dualities on the nearby cycle sheaves and the maximal sheaves. Although we do not use the results directly in this paper, the argument will be given in\[5\].

Let $\mathcal{T}$ be a pure twistor $\mathcal{D}$-module of weight $w$ on $X$. Let $f$ be a holomorphic function on $X$. We have the following exact sequences of mixed twistor $\mathcal{D}$-modules:

$$
0 \rightarrow \psi_j^{(a+1)} \mathcal{T} \rightarrow \Xi_j^{(a)} \mathcal{T} \rightarrow \mathcal{T}^{(a)[*f]} \rightarrow 0
$$

$$
0 \rightarrow \mathcal{T}^{(a)[*f]} \rightarrow \Xi_j^{(a)} \mathcal{T} \rightarrow \psi_j^{(a)} \mathcal{T} \rightarrow 0
$$

Hence, we have the following isomorphisms:

$$
\text{Gr}^W_{w-2a+j} \psi_j^{(a+1)} \mathcal{T} \cong \text{Gr}^W_{w-2a+j} \Xi_j^{(a)} \mathcal{T} \quad (j < 0)
$$

$$
\text{Gr}^W_{w-2a+j} \Xi_j^{(a)} \mathcal{T} \cong \text{Gr}^W_{w-2a+j} \psi_j^{(a)} \mathcal{T} \quad (j > 0)
$$

The following exact sequences have the unique splittings:

$$
0 \rightarrow \text{Gr}^W_{w-2a} \psi_j^{(a+1)} \mathcal{T} \rightarrow \text{Gr}^W_{w-2a} \Xi_j^{(a)} \mathcal{T} \rightarrow \mathcal{T}^{(a)} \rightarrow 0
$$

$$
0 \rightarrow \mathcal{T}^{(a)} \rightarrow \text{Gr}^W_{w-2a} \Xi_j^{(a)} \mathcal{T} \rightarrow \text{Gr}^W_{w-2a} \psi_j^{(a)} \mathcal{T} \rightarrow 0
$$

In other words, we have $\text{Gr}^W_{w-2a} \Xi_j^{(a)} \mathcal{T} = \mathcal{T}^{(a)} \oplus \text{Gr}^W_{w-2a} \psi_j^{(a)} \mathcal{T}$, and $\text{Gr}^W_{w-2a} \psi_j^{(a)} \mathcal{T} = \text{Gr}^W_{w-2a} \psi_j^{(a+1)} \mathcal{T}$ in $\text{Gr}^W_{w-2a} \Xi_j^{(a)} \mathcal{T}$. The isomorphism $\text{Gr}^W_{w-2a} \psi_j^{(a+1)} \mathcal{T} \cong \text{Gr}^W_{w-2a} \psi_j^{(a)} \mathcal{T}$ is induced by the canonical morphism $\mathcal{N} : \psi_j^{(a+1)} \mathcal{T} \rightarrow \psi_j^{(a)} \mathcal{T}$.

Let $\mathcal{N} : \Xi_j^{(0)} \mathcal{T} \rightarrow \Xi_j^{(0)} \mathcal{T} \otimes \mathcal{T}(-1)$ be the canonical morphism. Let $W(\mathcal{N})$ denote the monodromy weight filtration of $\mathcal{N}$. For $j \geq 0$, let $P \text{Gr}^W_{j}(\mathcal{N}) \Xi_j^{(0)} \mathcal{T}$ denote the primitive part, i.e., the kernel of $\mathcal{N}^{j+1} : \text{Gr}^W_{j}(\mathcal{N}) \Xi_j^{(0)} \mathcal{T} \rightarrow \text{Gr}^W_{j+1}(\mathcal{N}) \Xi_j^{(0)} \mathcal{T} \otimes \mathcal{T}(-j-1)$.

**Lemma 4.17** We have $W(\mathcal{N})_j = W_{w+j}$ ($j \in \mathbb{Z}$), and

$$
P \text{Gr}^W_{j}(\mathcal{N}) \Xi_j^{(0)} \mathcal{T} \simeq \begin{cases} 
\mathcal{T} & (j = 0) \\
P \text{Gr}^W_{w+j} \psi_j^{(0)} \mathcal{T} & (j > 0)
\end{cases} \quad (67)
$$

**Proof** It is enough to consider the case $w = 0$. Because the morphism $\mathcal{N} : \Xi_j^{(0)} \mathcal{T} \rightarrow \Xi_j^{(0)} \mathcal{T} \otimes \mathcal{T}(-1)$ induces $W_j \Xi_j^{(0)} \mathcal{T} \rightarrow (W_{j-2} \Xi_j^{(0)} \mathcal{T}) \otimes \mathcal{T}(-1)$. Let us observe that the induced morphism $\text{Gr}^W_{j}(\mathcal{N}) \Xi_j^{(0)} \mathcal{T} \rightarrow \text{Gr}^W_{j+1}(\mathcal{N}) \Xi_j^{(0)} \mathcal{T}$ is an isomorphism for $j > 0$. Note that $\mathcal{N} : \Xi_j^{(0)} \mathcal{T} \rightarrow \Xi_j^{(-1)} \mathcal{T}$ factors through $\Xi_j^{(0)} \mathcal{T} \rightarrow \psi_j^{(0)} \mathcal{T} \rightarrow \Xi_j^{(-1)} \mathcal{T}$.

Hence, for $j > 0$, we have the following commutative diagram:

\begin{align*}
\text{Gr}^W_j \Xi_j^{(0)} \mathcal{T} & \overset{\mathcal{N}^{j}}{\rightarrow} \text{Gr}^W_j \Xi_j^{(-1)} \mathcal{T} \\
\cong & \Downarrow \\
\text{Gr}^W_j \psi_j^{(0)} \mathcal{T} & \overset{\mathcal{N}^{j-1}}{\rightarrow} \text{Gr}^W_j \psi_j^{(-1)} \mathcal{T}
\end{align*}

Hence, we have $W = W(\mathcal{N})$ on $\Xi_j^{(0)} \mathcal{T}$. We also obtain (67) for $j > 0$. We have the following commutative diagram:

\begin{align*}
\text{Gr}_2^W \Xi_j^{(0)} \mathcal{T} & \longrightarrow \text{Gr}_0^W \Xi_j^{(0)} \mathcal{T} \otimes \mathcal{T}(-1) = \mathcal{T} \oplus \text{Gr}_0^W \psi_j^{(0)} \mathcal{T} \\
& \cong \\
\text{Gr}_2^W \psi_j^{(0)} \mathcal{T} & \longrightarrow \text{Gr}_0^W \psi_j^{(0)} \mathcal{T} = \text{Gr}_0^W \psi_j^{(1)} \mathcal{T} \\
& \cong \\
\text{Gr}_2^W \psi_j^{(1)} \mathcal{T}
\end{align*}

43
Then, we obtain \([07]\) for \(j = 0\).

Let \(S\) be a Hermitian sesqui-linear duality of \(T\). For \(j \geq 0\), let \(S_f^{(0)}\) denote the composite of the following morphisms:

\[
\Xi_f^{(0)}(T) \xrightarrow{(-)^{N_j}} \Xi_f^{(-j)}(T) = \Xi_f^{(0)}(T) \otimes T(-j) \rightarrow \Xi_f^{(0)}(T^*) \otimes T(-j - w) \approx \Xi_f^{(0)}(T)^* \otimes T(-j - w)
\]

The middle morphism is induced by \(S\). Let \(S_f^{(0)}\) denote the composite of the following morphisms:

\[
\psi_f^{(0)}(T) \xrightarrow{(-)^{N_j}} \psi_f^{(-j)}(T) = \psi_f^{(0)}(T) \otimes T(-j) \rightarrow \psi_f^{(0)}(T^*) \otimes T(-w - j) \approx \psi_f^{(1)}(T)^* \otimes T(-w - j)
\]

\[
\Xi_f^{(0)}(T) \xrightarrow{S_f^{(0)}} \Xi_f^{(0)}(T)^* \otimes T(-w - j - 1)\]

**Proposition 4.18** For \(j \geq 0\), the following diagram is commutative:

\[
\Xi_f^{(0)}(T) \xrightarrow{S_f^{(0)}} \Xi_f^{(0)}(T)^* \otimes T(-w - j - 1)
\]

Here, the vertical morphisms are natural ones.

**Proof** We can check the claim by a direct computation. We remark that, in \([08]\), the isomorphism \(T(1)^* \approx T(-1)\) is given by \((-1, -1)\).

**Corollary 4.19** Suppose that \(S\) is a polarization. For \(j \geq 1\), the morphism \(S_f^{(0)}\) induces a polarization of \(PGr_{w+j}^{(1)}(T)\). It is equal to the polarization of \(PGr_{w+j}^{(1)}(T)\) under the natural isomorphism. In particular, \(\Xi_f^{(0)}(T)\) is equipped with an induced graded polarization. 

## 5 Comparisons of polarizations

### 5.1 A specialization

#### 5.1.1 Statements

Let \(X\) be a complex manifold with a normal crossing hypersurface \(D\). Set \(d := \dim X\). Let \(f, g \in O_X(*D)\). We consider a meromorphic function \(F := \tau f + g\) on \(X \times \mathbb{P}^1_+\), and the associated mixed twistor \(D\)-modules \(T(F)[\tau]\). They are equipped with natural real structure.

**Assumption 5.1** We assume the following.

- \(|(f)_0| \cap |(f)_\infty| = \emptyset\), and \(|(f)_0| \subset |(g)_\infty|\). We also have \(D = |(f)_\infty| \cup |(g)_\infty|\).
- \(F\) is pure on \(X \times \{\tau \neq 0\}\), and \(g\) is pure on \(X \setminus |(f)_\infty|\).

For example, the assumptions are satisfied in the cases in Lemma [2.19] and Lemma [2.21].

We set \(D^{(1)} := (D \times \mathbb{P}^1) \cup \{\infty\} \times X\). Recall that \(T(F)\) denotes the image of \(T_0(F, D^{(1)}) \rightarrow T_0(F, D^{(1)})\), which is a pure twistor \(D\)-module of weight \(d + 1\). We have \(T_0(F, D^{(1)})[\star \tau] = T_0(F)[\star \tau]\) for \(\star = *, 1\) by the assumption.

Let \(\iota : X \rightarrow X \times \mathbb{P}^1_+\) be given by \(\iota(x) = (x, 0)\). According to Proposition [5.23], for the morphism \(\varphi : T(F)[\tau] \rightarrow T(F)[\star \tau]\), we have the following isomorphisms:

\[
\text{Cok}(\varphi) \approx \iota_! T_0(g, D) \otimes T(-1), \quad \text{Ker}(\varphi) \approx \iota_! T_0(g, D)
\]
The polarization $S_F = ((-1)^{d+1}, (-1)^{d+1})$ of $T(F)$ induces graded polarizations of $T(F)[\star \tau]$, and they induce graded polarizations of $\text{Cok}(\varphi)$ and $\text{Ker}(\varphi)$. We denote them by $S_{\text{Cok}(\varphi)} = (S_{\text{Cok}(\varphi), w} \mid w \in \mathbb{Z})$ and $S_{\text{Ker}(\varphi)} = (S_{\text{Ker}(\varphi), w} \mid w \in \mathbb{Z})$. Note, by applying $\psi^{(0)}_\tau$, we have the following exact sequence:

$$0 \longrightarrow \text{Ker}(\varphi) \longrightarrow \psi^{(1)}_\tau T(F) \longrightarrow \psi^{(0)}_\tau T(F) \longrightarrow \text{Cok}(\varphi) \longrightarrow 0$$

For the weight filtrations of $\text{Cok}(\varphi)$ and $\text{Ker}(\varphi)$, we have

\begin{align*}
\text{Gr}^W_{d+1+j} \text{Cok}(\varphi) &\simeq P \text{Gr}^W_{d+1+j} \psi^{(0)}_\tau (T(F)) \ (j > 0), \\
\text{Gr}^W_{d+1+j} \text{Ker}(\varphi) &\simeq P \text{Gr}^W_{d+1+j} \psi^{(1)}_\tau (T(F)) \ (j > 0),
\end{align*}

The isomorphisms are compatible with the induced polarizations.

We also have other polarizations. Let $T(g)$ denote the image of $T((g, D) \longrightarrow T_s(g, D))$, which is a pure twistor $D$-module of weight $d$. It is equipped with the natural polarization $S_\varphi = ((-1)^d, (-1)^d)$. Because $T(g)[\star (f)_\infty] = T_s(g, D)$, we have the induced graded polarizations $\iota_1 S_g[\star (f)_\infty]$ of $\iota_1 T_s(g, D) \otimes T(-1)$, and $\iota_1 S_g[\star (f)_\infty]$ of $\iota_1 T(g, D)$.

**Proposition 5.2** Under the isomorphisms $\text{Cok}(\varphi) \simeq \iota_1 T_s(g, D) \otimes T(-1)$ and $\text{Ker}(\varphi) \simeq \iota_1 T(g, D)$, we have $S_{\text{Cok}(\varphi)} = \iota_1 S_g[\star (f)_\infty]$ and $S_{\text{Ker}(\varphi)} = \iota_1 S_g[\star (f)_\infty]$.

**5.1.2 Some consequences**

Before going to the proof of Proposition 5.2 we give a consequence. Let $X, D, f$ and $g$ be as in Prop. 5.1.1. Let $\rho : X \longrightarrow Y$ be a projective morphism of complex manifolds. We assume that $R^i \rho_* \Omega_X^j = 0$ for any $i > 0$ and $j = 0, \ldots, \dim X$. Set $\rho_1 := \rho \times \text{id}_{\mathbb{P}^1} : X \times \mathbb{P}^1 \longrightarrow Y \times \mathbb{P}^1$. Note $\rho_1^* T(F) = 0$ unless $i \neq 0$ by Proposition 2.4. Let $\iota_1 Y : Y \longrightarrow Y \times \mathbb{P}^1$ be the inclusion induced by $\{0\} \longrightarrow \mathbb{P}^1$.

**Corollary 5.3** Suppose that $\rho^* T_s(g, D) = 0 (i \neq 0)$.

- **Condition (A)** is satisfied for the morphism $\rho$ and the mixed twistor $D$-module $T_s(g, D)$ with the graded polarization $S_g[\star (f)_\infty]$. The induced graded sesqui-linear duality $[\rho^0_1 S_g[\star (f)_\infty]]$ on $\rho^0_1 T_s(g, D)$ is a graded polarization.

- Let $\mathcal{K}$ and $\mathcal{C}$ denote the kernel and the cokernel of $\rho^0_1 T_s(g, D) \simeq K_Y$ and $\iota_1 \rho^0_1 T_s(g, D) \otimes T(-1) \simeq C_Y$. We have $\iota_1 [\rho^0_1 S_g[\star (f)_\infty]] = S_{\mathcal{K}_Y}$ and $\iota_1 [\rho^0_1 S_g[\star (f)_\infty]] = S_{\mathcal{C}_Y}$. Here, $S_{\mathcal{C}_Y}$ and $S_{\mathcal{K}_Y}$ are the graded polarizations induced by the graded polarization $\rho^0_1 S_F$ of $\rho^0_1 T(F)$.

**Proof** It follows from Corollary 4.9 and Proposition 5.2.

Let $\kappa : X' \longrightarrow X$ be a projective morphism of complex manifolds such that (i) $D' = \kappa^{-1}(D)$ is normal crossing, (ii) $\kappa$ induces $X' \setminus D' \simeq X \setminus D$, (iii) Assumption 5.4 is satisfied for $f' = \kappa^*(f)$ and $g' = \kappa^*(g)$. We have $\kappa^* T_s(g', D') = 0 (i \neq 0)$, and we have natural isomorphisms $\kappa^0_1 T_s(g', D') \simeq T_s(g, D)$. We set $\rho' := \rho \circ \kappa$. We have natural isomorphisms $\rho'^0_1 T_s(g', D') \simeq \rho^0_1 T_s(g, D)$.

**Corollary 5.4**

- **Condition (A)** holds for $\kappa$ and $T_s(g', D')$ with $S_{g'}[\star (f')_\infty]$. We have $[\kappa^0_1 S_{g'}[\star (f')_\infty]] = S_g[\star (f)_\infty]$ under the natural isomorphisms.

- **Condition (A)** holds for $\rho'$ and $T_s(g', D')$ with $S_{g'}[\star (f')_\infty]$. We have $[\rho'^0_1 S_{g'}[\star (f')_\infty]] = [\rho^0_1 S_{g'}[\star (f)_\infty]]$ under the natural isomorphisms.
5.1.3 Preliminary

Let us return to the proof of Proposition [5.2]. We shall give an argument for the polarizations on \( \text{Cok}(\varphi) \cong t_1 \mathcal{T}(g, D) \otimes T(-1) \). The argument for the other can be given in a similar way. The comparison of the polarizations on \( \text{Gr}_W^{d+2} \text{Cok}(\varphi) \cong \text{Gr}_W^{d+2}(t_1 \mathcal{T}(g, D) \otimes T(-1)) \) is given as in [1.4.1]. Let us study the other parts. We use the identification \( \text{Gr}_W^{d+2+j} \text{Cok}(\varphi) = P \text{Gr}_W^{d+2+j}(\psi^{(0)} T) \) for \( j \geq 0 \). In the proof, \( t_1 \mathcal{S}_g[*(f)_{\infty}] \) are denoted by \( \mathcal{S}_{g,j} \).

Let \( \mathcal{L}(F) \) denote the \( \mathcal{R}_X \pi^! \)-module underlying \( T(F) \). Let \( \mathcal{L}(g) \) denote the \( \mathcal{R}_X \)-module underlying \( T(g) \). By the assumption, we have \( \mathcal{L}(F)[*\tau] = \mathcal{L}(F, D^{(1)})[*\tau] = \mathcal{L}_s(F, D^{(1)})[*\tau] \) and \( \mathcal{L}_s(g, D) = \mathcal{L}(g)[*(f)_{\infty}] \).

Let \( \pi : X \times \mathbb{P}^1 \to X \) be the projection. Take a sufficiently large \( N \). We have

\[
\pi_i^0(\Pi^{\infty,N}_\tau \mathcal{L}(F)[*\tau])(*D) = \text{Cok} \left( \bigoplus_{i=0}^{N-1} \pi_* \mathcal{L}(F)(\lambda)^i \mapsto \bigoplus_{i=0}^{N-1} \pi_* \mathcal{L}(F)(\lambda)^i \lambda^{-1} d\tau / \tau \right)(*D)
\]

Here, \( \kappa \) is given by \( d\tau \cdot (\partial \tau + \lambda^{-1} f + s / \tau) \). Hence, \( \pi_i^0 \Pi^{\infty,N}_\tau \mathcal{L}(F)[*\tau] \) has the frame \( (\lambda)^j \lambda^{-1} d\tau / \lambda \) \((a \leq j \leq N - 1)\). We have the isomorphism of \( \mathcal{R} \)-modules \( \pi_i^0(\Pi^{\infty,N}_\tau \mathcal{L}(F)[*\tau])(*D) \cong \lambda^{-1} \Pi^{0,N}_{\tau-1}\mathcal{L}(g)(*D) \) given by \( \lambda^{-1}(\lambda)^j \lambda^{-1} d\tau \to \lambda^{-1}(\lambda)^j \).

\[
\pi_i(\Pi^{\infty,N}_\tau \mathcal{L}(F)[!\tau])(*D) = \text{Cok} \left( \bigoplus_{i=0}^{N-1} \pi_* \mathcal{L}(F)(\lambda)^i \mapsto \bigoplus_{i=0}^{N-1} \pi_* \mathcal{L}(F)(\lambda)^i \lambda^{-1} d\tau \right)(*D)
\]

Here, \( \kappa \) is given in the same way. Hence, \( \pi_i^0 \Pi^{\infty,N}_\tau \mathcal{L}(F)[!\tau](*D) \) has the frame \( (\lambda)^j f d\tau / \lambda \) \((a \leq j \leq N - 1)\). We have the isomorphism \( \pi_i^0(\Pi^{\infty,N}_\tau \mathcal{L}(F)[!\tau])(*D) \cong \Pi^{0,N}_{\tau-1}\mathcal{L}(g)(*D) \) given by \( (\lambda)^j f d\tau / \lambda \to (\lambda)^j \).

**Lemma 5.5** We have the following commutative diagram:

\[
\begin{array}{ccc}
\pi_i^0(\Pi^{\infty,N}_\tau \mathcal{L}(F)[!\tau])(*D) & \longrightarrow & \pi_i^0(\Pi^{\infty,N}_\tau \mathcal{L}(F)[*\tau])(*D) \\
\cong & \downarrow & \cong \\
\Pi^{\infty,N}_{\tau-1}\mathcal{L}(g)(*D) & \longrightarrow & \Pi^{\infty,N}_{\tau-1}\mathcal{L}(g)(*D) \lambda^{-1}
\end{array}
\]

Here, the vertical isomorphisms are given as above, the upper horizontal arrow is the natural one, and the lower horizontal arrow is given by \( (\lambda)^j \to (-\lambda)(\lambda)^{j+1} \).

**Proof** We have \( f(\lambda)^j d\tau / \lambda = -\lambda(\lambda)^j \lambda^{-1} d\tau / \tau \) in \( \pi_i^0(\Pi^{\infty,N}_\tau \mathcal{L}(F)[*\tau])(*D) \). Then, we can check the claim directly.

Because \( |(f)_0| \subset |(g)_{\infty}| \) and \( |(f)|_{\infty} \cup |(g)|_{\infty} = D \), we have

\[
\pi_i^0 \psi^{(a)}(\mathcal{L}F) = \pi_i^0 \text{Cok} \left( \Pi^{\infty,N}_\tau \mathcal{L}(F)[!\tau] \longrightarrow \Pi^{\infty,N}_\tau \mathcal{L}(F)[*\tau] \right) \\
\cong \text{Cok} \left( \Pi^{\infty,N}_{\tau+1}\mathcal{L}(g) \lambda^{-1} |D| \longrightarrow \Pi^{\infty,N}_{\tau-1}\mathcal{L}(g) \lambda^{-1} |D| \right) \cong \mathcal{L}(g) \lambda^{-1}
\]

In particular, \( \pi_i^0 \psi^{(0)}(\mathcal{L}F) \cong \mathcal{L}(g) \lambda^{-1} \). We also have

\[
\pi_i^0 \psi^{(a)}(\mathcal{L}F) \cong \text{Cok} \left( \Pi^{\infty,N}_{\tau+1}\mathcal{L}(g)[!D] \longrightarrow \Pi^{\infty,N}_{\tau-1}\mathcal{L}(g)[*D] \right) \cong \mathcal{L}(g). \]

In particular, \( \pi_i^0 \psi^{(1)}(\mathcal{L}g) \lambda^{d+1} = \mathcal{L}(g) \lambda^{d+1} \). The isomorphism \( \Xi_i^{(1)}(\mathcal{L}g) \lambda^{-1} \cong \Xi_i^{(0)}(\mathcal{L}g) \) is induced by \( \lambda^{-1}(\lambda)^j \to - (\lambda)^j \).

**Remark 5.6** Note that the isomorphisms are not necessarily integrable, i.e., the actions of \( \lambda \partial \lambda \) are not compatible, in general. We also remark that it is not clear whether \( \pi_i^0(\psi^{(0)} T(F) \cong \mathcal{L}(g) \otimes T(-1) \) as an \( \mathcal{R} \)-triple. We avoid to consider it.
We have a natural morphism

\[ \psi_\tau^0(T(F) \to \text{Cok}(\psi_\tau^1(T(F) \to \psi_\tau^0(T(F)) \simeq \text{Cok} \left( T(F)[\pi] \to T(F)[\star \tau] \right) \simeq \epsilon_1 T_\tau(g,D) \otimes T(-1). \]

It induces \( \pi_1^0 \psi_\tau^0(T(F) \to T_\tau(g,D) \otimes T(-1). \) The morphisms of the underlying \( R \)-modules are identified with the following natural morphisms:

\[ \pi_1^0 \psi_\tau^1(L(F)\lambda^{d+1}) \simeq \Xi_{j-1}^0(L(g)\lambda^{d+1}) \leftarrow \text{Ker} \left( \Xi_{j-1}^0(L(g)\lambda^{d+1}) \to \Xi_{j-1}^{(-1)}(L(g)\lambda^{d+1}) \right) \simeq L_{\lambda}(g,D)\lambda^{d+1} \]

\[ \pi_1^0 \psi_\tau^0(L(F)) \simeq \Xi_{j-1}^0(L(g)\lambda^{-1}) \to \text{Cok} \left( \Xi_{j-1}^{(-1)}(L(g)\lambda^{d+1}) \to \Xi_{j-1}^{(0)}(L(g)\lambda^{-1}) \right) \simeq L_\lambda(g,D)\lambda^{-1} \]

Hence, the morphisms of the underlying \( R \)-modules of \( \text{Gr}^W_{d+2+j} \pi_1^0 \psi_\tau^0(T(F) \simeq \text{Gr}^W_{d+2+j}(T_\tau(g,D) \otimes T(-1)) \simeq P \text{Gr}^W_{d+2+j}(\psi_{\tau-1}(T(g) \otimes T(-1)) \text{ are induced by the following natural morphisms:} \]

\[ \pi_1^0 \psi_\tau^1(L(F)\lambda^{d+1}) \simeq \Xi_{j-1}^0(L(g)\lambda^{d+1}) \leftarrow \psi_\tau^{(1)}(L(F)) \lambda^{d+1} \]

\[ \pi_1^0 \psi_\tau^0(L(F)) \simeq \Xi_{j-1}^0(L(g)\lambda^{-1}) \to \psi_{\tau-1}(L(g)\lambda^{-1}) \]

### 5.1.4 Proof of Proposition 5.2

The polarization \( S_{\text{Cok},d+2+j} P \text{Gr}^W_{d+2+j} \psi_\tau^0(T(F) \text{ is induced by the composite of the following morphisms:} \)

\[ \psi_\tau^0(T(F) \xrightarrow{(-N)^j} \psi_\tau^{(-j)}(T(F)) = \psi_\tau^{(-j)}(T(F)) \otimes T(-j) \]

\[ \xrightarrow{S_{\text{Cok}} \psi_\tau^{(0)}(T(F)^* \otimes T(-j-d-1) \rightarrow \psi_\tau^{(1)}(T(F)^* \otimes T(-j-d-1) \simeq \psi_\tau^{(0)}(T(F)^* \otimes T(-j-d-2) \quad (71) \]

The polarization \( S_{\text{g},d+2+j} P \text{Gr}^W_{d+2+j} T_\tau(g,D) \otimes T(-1) \text{ is induced by the composite of the following morphisms:} \)

\[ \psi_{\tau-1}(T(g) \otimes T(-1)) \xrightarrow{(-N)^{j-1}} \psi_{\tau-1}(T(g) \otimes T(-1)) \otimes T(-j+1) \]

\[ \xrightarrow{S_{\text{g}} \psi_{\tau-1}(T(g)^* \otimes T(-1)) \otimes T(-j-d-1) \simeq \psi_{\tau-1}(T(g)^* \otimes T(-j-d-2) \quad (72) \]

Recall that morphism of \( R \)-triples are given as pairs of the morphisms of the underlying \( R \)-modules. It is enough to compare the morphisms of the underlying \( R \)-modules for \( S_{\text{Cok},d+j+2} \) and \( S_{\text{g},d+j+2}. \)

Let us look at the second component of (71):

\[ \psi_\tau^0(L(F) \xrightarrow{(-N)^j} \psi_\tau^0(L(F)(\lambda))^{-(j-1)} \xrightarrow{(-1)^{d+1}} \psi_\tau^0(L(F)\lambda^{d+1})(\lambda)^{-j-1} \xrightarrow{-1} \psi_\tau^0(L(F)\lambda^{d+1})(\lambda)^{-j-d-2} \quad (73) \]

We apply \( \pi_1^0 \) to (73). By the computation in 5.1.3 we obtain

\[ \Xi_{j-1}^0(L(g)\lambda^{-1}) \xrightarrow{(-N)^j} \Xi_{j-1}^0(L(g)\lambda^{-1})(\lambda)^{-j} \xrightarrow{(-1)^{d+1}} \Xi_{j-1}^0(L(g)\lambda^{-1})(\lambda)^{-j-d-1} \]

\[ \simeq \Xi_{j-1}^0(L(g)\lambda^{-1})(\lambda)^{-j-d-1} \quad (74) \]

Note that the composite

\[ \Xi_{j-1}^0(L(g)\lambda^{-1})(\lambda)^{-j-d-1} \simeq \Xi_{j-1}^0(L(g)\lambda^{-1})(\lambda)^{-j-d-2} \simeq \Xi_{j-1}^0(L(g)\lambda^{-1})(\lambda)^{-j-d-2} \]

47
is the natural isomorphism, i.e., it is induced by $\lambda^{-1}(\lambda s)^j \mapsto (\lambda s)^{-1}$. Hence, the induced morphism
\[ \Xi_t^0(\mathcal{L}(g)\lambda^{-1}) \longrightarrow \Xi_t^0(\mathcal{L}(g)\lambda^{d+1})(\lambda s)^{-d-j-2} \]
is given by the composite of the following.
\[ (\mathcal{L}(g)\lambda^{-1}) \xrightarrow{(\mathcal{L}(g)\lambda^{-1})} (\mathcal{L}(g)\lambda^{d+1})(\lambda s)^{-j} \xrightarrow{(\mathcal{L}(g)\lambda^{d+1})(\lambda s)^{-j}} (\mathcal{L}(g)\lambda^{d+1})(\lambda s)^{-j-d-2} \]

Then, as in (4.5) we can check that the induced diagram is commutative:
\[ \begin{array}{c}
\Xi_{t-1}^0(\mathcal{L}(g)\lambda^{-1}) \\
\downarrow \\
\Xi_{t-1}^0(\mathcal{L}(g)\lambda^{d+1})(\lambda s)^{-d-2-j}
\end{array} \xrightarrow{\psi_{i-1}^0(\mathcal{L}(g)\lambda^{-1})} \begin{array}{c}
\Xi_{t-1}^0(\mathcal{L}(g)\lambda^{d+1})(\lambda s)^{-d-2-j}
\end{array} \]

Hence, we obtain that the second components of $\mathcal{S}_{\text{COK} d+1+j}$ and $\mathcal{S}_{g*,d+1+j}$ are equal. We can compare the first component similarly. Thus, the proof of Proposition 5.8 is finished.

**Remark 5.7** Because $\mathcal{S}$ is a polarization, it is enough to check the commutativity of (75) up to the signature, which can make the argument easier.

### 5.2 A push-forward

#### 5.2.1 Statement

Let $X$ be a complex manifold with a normal crossing hypersurface $D$. Set $d := \dim X$.

**Assumption 5.8** Let $f$ be a meromorphic function on $(X, D)$ such that (i) $(f)_\infty = D$, (ii) $f$ is non-degenerate along $D$, (iii) $(f)_0$ is smooth.

We set $Z := |(f)_0|$. We have the meromorphic function $tf$ on $X \times \mathbb{P}^1$. We have the associated mixed twistor $\mathcal{D}$-modules $\mathcal{T}(tf)[\ast t]$ and $\mathcal{T}(tf)$. Let $\pi : X \times \mathbb{P}^1 \longrightarrow X$ be the projection. According to Proposition 3.20 and Corollary 5.21 we have $\pi^i_0(\mathcal{T}(tf))[\ast t] = 0$ for $i \neq 0$, and we have the isomorphisms
\[ \pi^0_0(\mathcal{T}(tf))[\ast t] \cong \mathcal{U}_X(d, 0)[\ast Z][\ast D] \cong \mathcal{T}(-1), \quad \pi^0_0(\mathcal{T}(tf))[\ast t] \cong \mathcal{U}_X(d, 0)[\ast Z][\ast D]. \]

Recall that $\mathcal{T}(tf)$ is the image of $\mathcal{T}(tf) \longrightarrow \mathcal{T}(tf)$. We have $\mathcal{T}(tf) = (\mathcal{L}(tf)\lambda^{d+1}, \mathcal{L}(tf), C)$. The polarization is given by $\mathcal{S}_{tf} = ((-1)^{d+1}, (-1)^{d+1})$. It induces graded polarizations $\mathcal{S}_{tf*}[\ast t]$ on $\mathcal{T}(tf)$ (resp. $\mathcal{T}(tf)$) on $\mathcal{T}(tf)$ (resp. $\mathcal{T}(tf)[\ast t]$). The polarization $\mathcal{S}_0 = ((-1)^{d+1}, (-1)^{d+1})$ of $\mathcal{U}_X(d, 0)$ induces graded polarizations $\mathcal{S}_0[\ast (f_0)][\ast (f)_\infty]$ on $\mathcal{U}_X(d, 0)[\ast Z][\ast D]$, and $\mathcal{S}_0[\ast (f)_0][\ast (f)_\infty]$ on $\mathcal{U}_X(d, 0)[\ast Z][\ast D] \otimes \mathcal{T}(-1)$.

**Proposition 5.9**

- $\pi^i_0(\mathcal{G}^W T(tf)[\ast t]) = 0$ unless $i \neq 0$. In particular, Condition (A) holds for the projection $\pi$ and the mixed twistor $\mathcal{D}$-modules $\mathcal{T}(tf)[\ast t]$ with $\mathcal{S}_{tf*}[\ast t]$, and the mixed twistor $\mathcal{D}$-modules $\mathcal{T}(tf)$ with $\mathcal{S}_{tf*}$.

- The isomorphisms $\pi^0_0(\mathcal{T}(tf))[\ast t] \cong \mathcal{U}_X(d, 0) \otimes \mathcal{T}(-1)[\ast Z][\ast D]$ and $\pi^0_0(\mathcal{T}(tf))[\ast t] \cong \mathcal{U}_X(d, 0)[\ast Z][\ast D]$ are compatible with the graded polarizations.

Before going to the proof of Proposition 5.9 we give a consequence. Let $\rho : X \longrightarrow Y$ be any projective morphism of complex manifolds. We suppose $\rho^i_0(\mathcal{U}_X(d, 0)[\ast Z][\ast D]) = 0$ for $i \neq 0$. Note that it implies $\rho^i_0(\mathcal{U}_X(d, 0)[\ast Z][\ast D]) = 0$ for $i \neq 0$.

**Corollary 5.10** Condition (A) holds for $\rho$ and $\mathcal{U}_X(d, 0)[\ast Z][\ast D]$ with $\mathcal{S}_0[\ast (f)_0][\ast (f)_\infty]$. We have
\[ [\rho^0_0(\mathcal{S}_0[\ast (f)_0][\ast (f)_\infty])] = [(\rho \circ \pi^0_0(\mathcal{S}_{tf*}[\ast (t)_0])]. \]

Condition (A) holds for $\rho$ and $\mathcal{U}_X(d, 0)[\ast Z][\ast D]$ with $\mathcal{S}_0[\ast (f)_0][\ast (f)_\infty]$. We have
\[ [\rho^0_0(\mathcal{S}_0[\ast (f)_0][\ast (f)_\infty])] = [(\rho \circ \pi^0_0(\mathcal{S}_{tf*}[\ast (t)_0])]. \]
Proof According to Corollary 5.3 and Lemma 2.21, Condition (A) holds for \( \rho \circ \pi \) and \( T(tf)[st] \) with \( \mathcal{S}_f[st] \). Then, the claim follows from Proposition 5.9.

Let us return to the proof of Proposition 5.9. We shall prove the first claim of the proposition in 5.2.2–5.2.10. The second claim can be proved similarly. For the comparison of polarizations, it is enough to check the coincidence up to signatures thanks to the positivity of polarizations. This remark could simplify the following argument although we shall also give arguments for signatures.

5.2.2 The push-forward

Let us look at the isomorphisms (70) more closely. First, we see the isomorphisms of the underlying \( \mathcal{R} \)-modules.

Outside of \( D \), we have

\[
\pi_1^0 \mathcal{L}(f)[st] \simeq \operatorname{Cok}(\mathcal{O}_X[t] \xrightarrow{\varphi} \mathcal{O}_X[t] \cdot (dt/t \lambda)), \quad \varphi(h t^j) = (jht^{-1} + h f t^j/\lambda) dt.
\]

It has a global frame \( dt/t \lambda \). Outside of \( D \cup Z \), we have \( \partial_x(dt/t) = (\partial_x f) dt/\lambda = \varphi(f^{-1} \partial_x f) \), and hence \( \nabla[dt/t] = 0 \). We have already known \( \pi_1^0 \mathcal{L}(f)[st] \simeq \lambda^{-1} \mathcal{O}_X[|Z|][sD] \). Hence, we have

\[
\pi_1^0 (\mathcal{L}(f)[st]) \simeq \lambda^{-1} \mathcal{O}_X[|Z|][sD] \cdot [dt/t]
\]

Outside of \( D \), we have

\[
\pi_1^0 (\mathcal{L}(f)[st]) \simeq \operatorname{Cok}(\mathcal{O}_X[t] \xrightarrow{\varphi} \mathcal{O}_X[t] \cdot (dt/\lambda))
\]

Here, \( \varphi \) is given in the same way. We have the generator \([dt/\lambda] \). We have

\[
\partial_\lambda(dt/\lambda) = (\partial_\lambda f) dt + f dt/\lambda \cdot (\partial_\lambda f) t = \partial_\lambda(t(\partial_\lambda f)) dt \equiv 0
\]

Hence, \([fdt/\lambda] \) is a flat section. We have

\[
\pi_1^0 \mathcal{L}(f)[st] \simeq \mathcal{O}_X[*Z][|D|] \cdot [fdt/\lambda]
\]

We remark \( \partial_\lambda[fdt/\lambda] = 0 \). We also remark \( fdt/\lambda = -\partial_\lambda \otimes dt/\lambda \) in \( \operatorname{Cok}(\mathcal{O}_X \times \mathbb{P}^1[|t|] \rightarrow \mathcal{O}_X \times \mathbb{P}^1[|t|] dt/\lambda) \).

Remark 5.11 The isomorphisms are integrable, i.e., they are compatible with the actions of \( \lambda^2 \partial_\lambda \).

Let us compute \( \pi_1 C([fdt/\lambda], \sigma^*(dt/\lambda)) \). We take a \( C^\infty \)-function \( \rho \) on \( \mathbb{R} \) such that \( \rho(t) = 1 \) \( (t \leq 1/2) \) and \( \rho(t) = 0 \) \( (t \geq 1) \). We set \( \chi(t) = \rho(t|t|^2) \). Note that the support of \( C(\partial_\lambda, t^{-1}) \) is contained in \( \{ t = 0 \} \). We have

\[

\pi_1\mathcal{L}([fdt/\lambda], \sigma^*(dt/\lambda)) = \pi_1 C(-\partial_\lambda \otimes dt/\lambda, \sigma^*(dt/\lambda)) = \frac{1}{2\pi \sqrt{-1}} \int C(-\partial_\lambda, \sigma^*(1/t)) \chi dt d\sigma
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int T^{-1} \partial_\lambda \chi dt d\sigma = - \int_0^\infty \rho'(s) ds = 1 \quad (77)
\]

Hence, the above isomorphisms of \( \mathcal{R} \)-modules induce the isomorphisms of mixed twistor \( D \)-modules.

5.2.3 Comparison of the pure part

The support of \( T(t)[st]/T_\ast(t) \) is \( X \times \{ t = 0 \} \). The restriction of \( T(t)[st]/T_\ast(t) \) to \( (X \setminus D) \times \{ t = 0 \} \) is pure of weight \( d + 2 \). As remarked in 5.1.11 it is equipped with the induced polarization given by

\[
(\partial_\lambda \lambda^{d+1}, t^{-1}) \leftrightarrow ((-1)^{d+1} \lambda^{d+2}, (-1)^d \partial_\lambda \lambda^{-1}).
\]

On \( X \setminus D \), \( \pi_1 T(t)[st]/T_\ast(t) \) is pure of weight \( d + 2 \), and the induced polarization is given by

\[
([fdt/\lambda] \lambda^{d+1}, [t^{-1} dt/\lambda]) \leftrightarrow ((-1)^{d+1} [t^{-1} dt/\lambda] \lambda^{d+2}, (-1)^d [fdt/\lambda] \lambda^{-1})
\]

Note that \([fdt/\lambda] = -[\partial_\lambda dt/\lambda] \). Hence, the isomorphism

\[
\pi_1 (T(t)[st]/T_\ast(t))|_{X \setminus D} \simeq U_X(d, 0) \otimes T(-1)|_{X \setminus D}
\]

is compatible with the polarizations. It also implies the isomorphism

\[
\pi_1 (T(t)[st])|_{X \setminus (D \cup Z)} \simeq U_X(d, 0) \otimes T(-1)|_{X \setminus (D \cup Z)}
\]

is compatible with the polarizations.
5.2.4 The associated graded objects

The following lemma implies the first claim of Proposition 5.9.

**Lemma 5.12** We have \( \pi_i^W \text{Gr}^W \mathcal{T}(tf)[*t] (i \neq 0) \) and \( \pi_i^W \text{Gr}^W \mathcal{T}_*(tf) (i \neq 0) \). In particular, we have

\[
\text{Gr}^W \pi_i^W \mathcal{T}_*(tf) \simeq \pi_i^W \text{Gr}^W \mathcal{T}_*(tf), \quad \text{Gr}^W \pi_i^W \mathcal{T}(tf)[*t] \simeq \pi_i^W \text{Gr}^W \mathcal{T}(tf)[*t].
\]

**Proof** We give an argument for the case \(* = *\). The argument for the other case is similar. We have \( \text{Gr}^W \mathcal{T}(tf)[*t] = \text{Gr}^W \mathcal{T}_*(tf) = 0 \) for \( j < d + 1 \), and \( \text{Gr}^W_{d+1} \mathcal{T}(tf)[*t] = \text{Gr}^W_{d+1} \mathcal{T}_*(tf) = \mathcal{T}(tf) \). The support of \( \text{Gr}^W_j \mathcal{T}(tf)[*t] \) and \( \text{Gr}^W_j \mathcal{T}_*(tf) \) are contained in \( X \times \{ t = 0 \} \). Hence, it is enough to prove \( \pi_i^W \mathcal{T}(tf) = 0 \) for \( i \neq 0 \).

Let \( A_1 \) and \( A_2 \) denote the kernel of \( \mathcal{T}(tf) \rightarrow \mathcal{T}(tf) \) and the cokernel of \( \mathcal{T}(tf) \rightarrow \mathcal{T}_*(tf) \). The support of \( A_1 \) are contained in \( \{ t = 0 \} \). Hence, we obtain \( \pi_i^W \mathcal{T}(tf) = 0 (i \neq 0) \), and we have the following exact sequences:

\[
0 \rightarrow \pi_i^W A_1 \rightarrow \pi_i^W \mathcal{T}(tf) \rightarrow \pi_i^W \mathcal{T}_*(tf) \rightarrow 0
\]

Then, we obtain the claim of Lemma 5.12.

By Lemma 5.12, it is enough to compare the induced polarizations on \( \text{Gr}^W_j \pi_i^W \mathcal{T}(tf)[*t] \simeq \pi_i^W \text{Gr}^W_j (\mathcal{T}(tf)[*t]) \).

5.2.5 Comparison on the push-forward of the pure part

We have

\[
\pi_i^W \mathcal{T}(tf) = \text{Im}(\pi_i^W \mathcal{T}(tf)[!t] \stackrel{a}{\longrightarrow} \pi_i^W \mathcal{T}(tf)[*t]) = \text{Im}(\mathcal{U}_X(d,0)[*Z][!D] \rightarrow \mathcal{U}_X(d,0)[!Z][*D])
\]

The morphism \( a \) is given by \([dt/\lambda] \mapsto [dt/\lambda] \). If \( f = x \) is a coordinate function, then

\[
[dt/\lambda] = x^{-1}[xdt/\lambda] = \partial_x[dt/t\lambda]
\]

Hence, \( \pi_i^W \mathcal{T}(tf) \) is the image of

\[
\mathcal{U}_X(d,0)[*Z][!D] \rightarrow \text{Ker}\left(\mathcal{U}_X(d,0)[!Z] \rightarrow \mathcal{U}_X(d,0)\right)[*D] \otimes \mathbf{T}(-1)
\]

It is isomorphic to \( \mathcal{U}_X(d,0)[*Z]/\mathcal{U}_X(d,0) \simeq \iota_Z|_Z|\mathcal{U}_X(d-1,0) \otimes \mathbf{T}(-1) \), where \( \iota_Z : Z \rightarrow X \) denotes the inclusion.

We have the two polarizations on \( \pi_i^W \mathcal{T}(tf) \). One is induced by the polarization \((-1)^{d+1}, (-1)^{d+1}\) of \( \mathcal{T}(tf) \), for which we have

\[
([dt/\lambda] \lambda^{d+1}, [dt/\lambda]) \leftrightarrow (-1)^{d+1}[dt/\lambda] \lambda^{d+1}, (-1)^{d+1}[dt/\lambda]). \quad (78)
\]

The other is induced by the isomorphism \( \pi_i^W \mathcal{T}(tf) \simeq \mathcal{U}_X(d,0)[*Z]/\mathcal{U}_X(d,0) \) and the polarization of \( \mathcal{U}_X(d,0) \).

**Lemma 5.13** They are the same.

**Proof** We identify \( \mathcal{U}_X(d,0) \) with \( \mathcal{O}_X[dt/(t\lambda)]A^d, \mathcal{O}_X[fdt/\lambda], C \), where \( C \) is the induced pairing. The polarization is given by \( ([dt/(t\lambda)]A^d, [fdt/\lambda]) \leftrightarrow ((-1)^d[fdt/\lambda]A^d, (-1)^d[dt/(t\lambda)]) \). We may assume that \( f = x \) for a coordinate function. According to the computation in 4.4.1, the second polarization is given by

\[
\partial_x[dt/(t\lambda)]A^d \leftrightarrow (-1)^{d+1}x^{1}[xdt/\lambda]A^{d+1}
\]

\[
x^{-1}[xdt/\lambda] \leftrightarrow (-1)^{d+1}\partial_x[dt/(t\lambda)]A^{1-1}
\]

By comparing with (78), we obtain that the second one is equal to the first one.
5.2.6 A variant of the maximal functor

To describe the underlying $\mathcal{R}$-modules of $\pi_1^0(\mathcal{L}(tf))$, we introduce a functor for $\mathcal{R}$-modules. (See 5.2.1 for the $\mathcal{D}$-module version.) We have the smooth $\mathcal{R}_X(f)$-module $\Pi_{f-1}^{a,b} \mathcal{O}_X$ given as follows:

$$
\Pi_{f-1}^{a,b} (\mathcal{O}_X \lambda^m) := \bigoplus_{j=a}^{b-1} \mathcal{O}_X ((Z \cup D)) \lambda^m \langle \lambda s \rangle^j
$$

The flat $\lambda$-connection is given by $\mathbb{D}(\lambda s)^i = -(\lambda s)^{i+1} df/f$. As in the case of $\mathcal{D}$-modules, we have the canonical morphism:

$$
\Pi_{f-1}^{a+1,b+1} (\mathcal{O}_X \lambda^m)[*Z][!!D] \longrightarrow \Pi_{f-1}^{a,b} (\mathcal{O}_X \lambda^m)[!!Z][*D]
$$

(79)

We define $\Xi_{f-1}^{(a)}(\mathcal{O}_X \lambda^m)$ as the cokernel of (79).

We shall prove the following lemma in 5.2.7.

Lemma 5.14 We have a natural isomorphism:

$$
\pi_1^0 (\Pi_{f-1}^{a,N} \mathcal{L}(tf)[*t]) \simeq \Pi_{f-1}^{a,N} (\mathcal{O}_X \lambda^{-1})[*Z][!!D]
$$

(80)

We also have a natural isomorphism:

$$
\pi_1^0 (\Pi_{f-1}^{a,N} \mathcal{L}(tf)[*t]) \simeq \Pi_{f-1}^{a,N} (\mathcal{O}_X)[*Z][!!D]
$$

(81)

We shall prove the following lemma in 5.2.8.

Lemma 5.15 We have natural isomorphisms

$$
\pi_1^0 \psi_1^{(0)} (\mathcal{L}(tf)) \simeq \Xi_{f-1}^{(0)} (\mathcal{O}_X \lambda^{-1}), \quad \pi_1^0 \psi_1^{(1)} (\mathcal{L}(tf)) \simeq \Xi_{f-1}^{(0)} (\mathcal{O}_X)
$$

such that the following diagram is commutative:

$$
\begin{array}{ccc}
\pi_1^0 \psi_1^{(0)} (\mathcal{L}(tf))(\lambda s) & \longrightarrow & \Xi_{f-1}^{(0)} (\mathcal{O}_X \lambda^{-1})(\lambda s) \\
\alpha_1 \downarrow & & \alpha_2 \downarrow \\
\pi_1^0 \psi_1^{(1)} (\mathcal{L}(tf)) & \longrightarrow & \Xi_{f-1}^{(0)} (\mathcal{O}_X)
\end{array}
$$

(82)

Here, $\alpha_1$ is the natural one, and $\alpha_2$ is given by $\lambda^{-1}(\lambda s)^{i+1} \longmapsto -(\lambda s)^i$.

5.2.7 The construction of isomorphisms (80) and (81)

Take a sufficiently large $N$. We have $\pi_1^0\Pi_{f-1}^{a,N} \mathcal{L}(tf)[*t] = 0 (i \neq 0)$. Outside of $D$, we have

$$
\pi_1^0 \Pi_{f-1}^{a,N} \mathcal{L}(tf)[*t] \simeq \text{Cok} \left( \bigoplus_{i=a}^{N-1} \mathcal{O}_X [t](\lambda s)^i \xrightarrow{\varphi} \bigoplus_{i=a}^{N-1} \mathcal{O}_X [t](\lambda s)^i (dt/\lambda t) \right).
$$

$$
\varphi (tf^i g(\lambda s)^i) = (tf^i g^i (\lambda s)^i) = (tf^i g^i (\lambda s)^i) dt.
$$

Outside of $D \cup Z$, we have the frame $(\lambda s)^i [dt/\lambda t] (a \leq i < N)$. For any local holomorphic coordinate system $(x_1, \ldots, x_n)$, we have $\partial_j (\lambda s)^i [dt/\lambda t] = (\lambda s)^i [dt/\lambda t]$. We have

$$
\varphi ((\lambda s)^i (\partial_j f)) = (\lambda s)^i (\partial_j f) (f/\lambda) dt + (\lambda s)^{i+1} (\partial_j f) (dt/\lambda t) \equiv 0.
$$

If $f = x$ is a coordinate function, we obtain

$$
x \partial_x ((\lambda s)^i \cdot dt/\lambda t) = x (\lambda s)^i \cdot dt/\lambda \equiv - (\lambda s)^{i+1} \cdot dt/\lambda t.
$$

51
Hence, \( \pi_1^0 \Pi_{i,N}^a \mathcal{L}(tf)[st](Z \cup D) \) is isomorphic to \( \Pi_{i-1}^{N} \mathcal{O}_X \lambda^{-1} \) by \( (\lambda s)^i [dt/\lambda t] \mapsto (\lambda s)^i \lambda^{-1} \). We define the isomorphism by \( (\lambda s)^i [dt/\lambda t] \mapsto (\lambda s)^i \lambda^{-1} \).

We have \( \pi_1^0 \Pi_{i,N}^a \mathcal{L}(tf)[!t] = 0 \) for \( i \neq 0 \). Outside of \( D \), we have

\[
\pi_1^0 \Pi_{i,N}^a \mathcal{L}(tf)[!t] \simeq \text{Cok} \left( \bigoplus_{i=0}^{N-1} \mathcal{O}_X [t] (\lambda s)^i \to \bigoplus_{i=0}^{N-1} \mathcal{O}_X [t] (\lambda s)^i dt/\lambda \right)
\]

Outside of \( Z \cup D \), we have the frame \( (\lambda s)^i dt/\lambda (a \leq i < N) \).

Let us consider the sections \( (\lambda s)^i [f dt/\lambda] \). If \( f = x \) is a coordinate function, we have

\[
\lambda \partial_x (\lambda s)^i \cdot x dt/\lambda = (\lambda s)^i \cdot \partial_t (\lambda s)^i \cdot x \cdot dt/\lambda
\]

Because \( \partial_t (\lambda s)^i dt = (\lambda s)^i dt + (xt/\lambda) dt(\lambda s)^i \), we have

\[
\lambda x \partial_x (\lambda s)^i [dt/\lambda] = -(\lambda s)^i [dt/\lambda].
\]

We define the isomorphism by the correspondence \( (\lambda s)^i [dt/\lambda] \mapsto (\lambda s)^i \). Thus, we obtain Lemma 5.14.

### 5.2.8 Proof of Lemma 5.15

The morphism \( \pi_0^0 \Pi_{i,N}^a \mathcal{L}(tf)[!t] \to \pi_1^0 \Pi_{i,N}^a \mathcal{L}(tf)[st] \) is given by \( (\lambda s)^i [dt/\lambda] \mapsto (\lambda s)^i [dt/\lambda] \). We can identify it with \( \Phi : \mathcal{O}_X [!Z][!!D] \to \Pi_{i-1}^{N} \mathcal{O}_X \lambda^{-1} [!!Z][!!D] \).

Outside of \( D \cup Z \), it is given by

\[
(\lambda s)^i [f dt/\lambda] \mapsto (\lambda s)^i [f dt/\lambda] = -(\lambda s)^i [dt/\lambda t]
\]

Hence, \( \Phi \) is induced by \( \Pi_{i-1}^{N} \mathcal{O}_X \to \Pi_{i-1}^{N} \mathcal{O}_X \lambda^{-1} \) given by \( (\lambda s)^i \mapsto - (\lambda s)^i \lambda^{-1} \). By the identification \( s^i \mapsto - s^i \), it is identified with \( \Pi_{i-1}^{N+1} \mathcal{O}_X \lambda^{-1} [!!Z][!!D] \to \Pi_{i-1}^{N} \mathcal{O}_X \lambda^{-1} [!!Z][!!D] \) and \( \Pi_{i-1}^{N} \mathcal{O}_X [!!Z][!!D] \to \Pi_{i-1}^{N} \mathcal{O}_X [!!Z][!!D] \). We obtain the following:

\[
\pi_0^0 \psi_1^{(0)} (\mathcal{L}(tf)) \simeq \text{Cok} \left( \Pi_{i-1}^{N+1} \mathcal{O}_X \lambda^{-1} [!!Z][!!D] \to \Pi_{i-1}^{N} \mathcal{O}_X \lambda^{-1} [!!Z][!!D] \right) = \Xi_{i-1}^{(0)} (\mathcal{O}_X \lambda^{-1})
\]

\[
\pi_0^0 \psi_1^{(1)} (\mathcal{L}(tf)) \simeq \text{Cok} \left( \Pi_{i-1}^{N+1} \mathcal{O}_X [!!Z][!!D] \to \Pi_{i-1}^{N} \mathcal{O}_X [!!Z][!!D] \right) = \Xi_{i-1}^{(1)} (\mathcal{O}_X)
\]

We have the commutativity of \( \Xi_2 \) by the construction. Thus, the proof of Lemma 5.14 is finished.

### 5.2.9 Morphisms of the underlying \( \mathcal{R} \)-modules

The polarization of \( \text{Gr}_{d+1+j}^W \mathcal{T}(tf)[st] \simeq P \text{Gr}_{d+1+j}^W \psi_1^{(0)} (\mathcal{T}(tf)) (j > 0) \) is induced by the composite of the following morphism:

\[
\psi_1^{(0)} (\mathcal{T}(tf)) \longrightarrow \psi_1^{(-j)} (\mathcal{T}(tf)) = \psi_1^{(0)} (\mathcal{T}(tf)) \otimes \mathcal{T}(-j)
\]

\[
\simeq \psi_1^{(0)} (\mathcal{T}(tf)^*) \otimes \mathcal{T}(-d - 1 - j) \simeq \psi_1^{(1)} (\mathcal{T}(tf)^*) \otimes \mathcal{T}(-d - 1 - j) \simeq \psi_1^{(0)} (\mathcal{T}(tf)^*) \otimes \mathcal{T}(-d - 2 - j)
\]

(83)

The morphism of the second \( \mathcal{R} \)-modules is given by the composite of the following morphisms:

\[
\psi_1^{(0)} (\mathcal{L}(tf)) \simeq \psi_1^{(-j)} (\mathcal{L}(tf)) = \psi_1^{(0)} (\mathcal{L}(tf)) (\lambda s)^{-j} \xrightarrow{(-1)^{d+1}} \psi_1^{(0)} (\mathcal{L}(tf) \lambda^{d+1}) (\lambda s)^{-j-d-1}
\]

\[
\xrightarrow{(-1)} \psi_1^{(1)} (\mathcal{L}(tf) \lambda^{d+1}) (\lambda s)^{-j-d-2}
\]

(84)
We have the following commutative diagram

\[
\begin{array}{ccc}
\pi^0_1 \psi^0_t (L(tf)) & \xrightarrow{(-N)^j} & \pi^0_1 \psi^0_t (L(tf)(\lambda_s)^{-j}) \\
\downarrow & & \downarrow \\
\Xi^0_{f-1}(\mathcal{O}_X \lambda^{-1}) & \xrightarrow{(-N)^j} & \Xi^0_{f-1}(\mathcal{O}_X \lambda^{-1})(\lambda_s)^{-j} \\
\end{array}
\]

Recall the commutativity of (82). Then, we also have the following commutative diagram

\[
\begin{array}{ccc}
\pi^0_1 \psi^0_t (L(tf)(\lambda_s)^{-j}) & \xrightarrow{(-1)^{d+2}} & \pi^0_1 \psi^0_t (L(tf)(\lambda^{d+1})(\lambda_s)^{-j-1-d}) \\
\downarrow & & \downarrow \\
\Xi^0_{f-1}(\mathcal{O}_X \lambda^{-1})(\lambda_s)^{-j} & \xrightarrow{(-1)^{d+1}} & \Xi^0_{f-1}(\mathcal{O}_X \lambda^{d+1})(\lambda_s)^{-j-1-d} \\
\end{array}
\]

Hence, after applying \( \pi^0_1 \), we can identify (84) with the composite of the following:

\[
\Xi^0_{f-1}(\mathcal{O}_X \lambda^{-1}) \xrightarrow{(-N)^j} \Xi^0_{f-1}(\mathcal{O}_X \lambda^{-1})(\lambda_s)^{-j} \xrightarrow{(-1)^{d+1}} \Xi^0_{f-1}(\mathcal{O}_X \lambda^{d+1})(\lambda_s)^{-j-1-d} \quad (85)
\]

We have a similar description of the morphism of the first \( \mathcal{R} \)-modules for (84).

**5.2.10 Comparison of the polarization on the other parts**

Let us compare the induced polarizations on

\[
\phi_{x_0}^0 \pi^0_1 P \mathcal{G}_{d+3+j}^W \psi^0_t (T(tf)) \simeq \mathcal{G}_{d+3+j}^W \left( \mathcal{U}_X (d, 0) \otimes T(-1)[*D] \right) \quad (j \geq 0).
\]

Outside of \( Z \), the comparison can be done in a way similar to that in [5.1]. On \( Z \setminus D \), we have already given the comparison in Lemma 5.13. So, it is enough to give the argument for the direct summands of \( \pi^0_1 P \mathcal{G}^W \psi^0_t T \) whose supports are contained in \( Z \cap D \). We have only to consider the issue locally around any point of \( Z \cap D \). We may assume to have a holomorphic coordinate \((x_0, x_1, \ldots, x_{d-1})\) such that \( f = x_0 \prod_{i=1}^{m} x_i^{-k_i} \) for some \((k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 0}^m\). We set \( x^k := \prod_{i=1}^{m} x_i^{k_i} \). We compare the graded polarizations on

\[
\phi_{x_0}^0 \pi^0_1 P \mathcal{G}_{d+3+j}^W \psi^0_t (T(tf)) \simeq \psi_{x_0}^0 \psi^0_t \left( \mathcal{U}_X (d, 0) \otimes T(-1)[*D] \right) \\
\simeq \mathcal{G}_{d+3+j}^W \left( \psi_{x_0}^1 (\mathcal{U}_X (d, 0) \otimes T(-1))[*D] \right) \quad (86)
\]

Because \( \Xi^0_{f-1} \mathcal{O}_X = \text{Cok} \left( \Pi_{f-1}^{1,N+1} \mathcal{O}_X[*D] \rightarrow \Pi_{f-1}^{0,N} \mathcal{O}_X[*D] \right) \), we have

\[
\phi_{x_0}^0 \Xi^0_{f-1} \mathcal{O}_X = \text{Cok} \left( \Pi_{x_0}^{1,N+1} \psi_{x_0}^0 \mathcal{O}_X[*D] \rightarrow \Pi_{x_0}^{0,N} \psi_{x_0}^1 \mathcal{O}_X[*D] \right).
\]

Naturally, it is isomorphic to \( \psi_{x_0}^0 \psi_{x_0}^1 \mathcal{O}_X \). We have the isomorphism

\[
(\lambda_s)^i x_0^{-1} \mathcal{O}_X \longrightarrow -\partial_{x_0} \otimes (\lambda_s)^i.
\]

Then, we have \( \phi_{x_0}^0 \Xi^0_{f-1} \mathcal{O}_X \simeq \psi_{x_0}^1 \psi_{x_0}^0 \mathcal{O}_X \).

**Lemma 5.16** The morphisms of the underlying \( \mathcal{R} \)-modules of

\[
\phi_{x_0}^0 \pi^0_1 P \mathcal{G}_{d+3+j}^W \psi^0_t (T(tf)) \simeq \mathcal{G}_{d+3+j}^W \left( \psi_{x_0}^1 (\mathcal{U}_X (d, 0) \otimes T(-1))[*D] \right) \\
\simeq \mathcal{G}_{d+3+j}^W \left( \psi_{x_0}^0 \psi_{x_0}^1 (\mathcal{U}_X (d, 0) \otimes T(-1))[*D] \right) \quad (87)
\]

53
are equal to the morphisms induced by the composite of the above morphisms:

\[ \phi_{x_0}^{(0)} \pi_t^{(1)} (\mathcal{L}(t) \lambda^{d+1}) \simeq \phi_{x_0}^{(0)} (O_X \lambda^{d+1}) \simeq \psi_{x_0}^{(1)} (O_X \lambda^{d+1}) \]

\[ \phi_{x_0}^{(0)} \pi_t^{(1)} (\mathcal{L}(t)) \simeq \phi_{x_0}^{(0)} (O_X \lambda^{-1}) \simeq \psi_{x_0}^{(1)} (O_X \lambda^{-1}) \]

**Proof** Note that we have \( \psi_t^{(0)} \mathcal{T}(t) \rightarrow \mathcal{T}(t)[st] / \mathcal{T}(t) \), which induces

\[ \pi_t^{(0)} \psi_t^{(1)} (\mathcal{T}(t)) \rightarrow \pi_t^{(0)} \left( \mathcal{T}(t)[st] / \mathcal{T}(t) \right) \simeq \frac{U_X(d,0) \otimes T(-1)[|Z|][*D]}{t_z U_Z(d-1,0) \otimes T(-1)} \]

We have the following morphisms of the underlying \( \mathcal{R} \)-modules:

\[ \pi_t^{(0)} \psi_t^{(1)} (\mathcal{L}(t) \lambda^{d+1}) \simeq \Xi_{\mathcal{L}(t)} \lambda^{d+1} \simeq \text{Ker} \left( O_X \lambda^{d+1} [Z] \rightarrow t_z O_Z \lambda^{d+1} \right) \]

\[ \pi_t^{(0)} \psi_t^{(1)} (\mathcal{L}(t)) \simeq \Xi_{\mathcal{L}(t)} \lambda^{-1} \simeq \text{Cok} \left( t_z O_Z \lambda^{-1} \rightarrow O_X \lambda^{-1} [Z] \right) \]

Note that \( \psi_t^{(0)} \mathcal{T}(t) \rightarrow \mathcal{T}(t)[st] / \mathcal{T}(t) \) is identified with \( \psi_t^{(1)} \mathcal{T}(t) \rightarrow \text{Cok} \left( \Xi_{\mathcal{L}(t)} \lambda^{-1} \rightarrow \Xi_{\mathcal{L}(t)} \lambda^{-1} \right) \). Hence, \( b_2 \) is identified with \( \Xi_{\mathcal{L}(t)} \lambda^{-1} \rightarrow \Xi_{\mathcal{L}(t)} \lambda^{-1} \). We have a similar description for \( b_1 \). Then, the claim is clear.

We have \( A : \psi_x^{(0)} \psi_x^{(1)} (O_X \lambda^{-1}) \rightarrow \psi_x^{(1)} \psi_x^{(0)} (O_X \lambda^{d+1}) \lambda^{-d-2} \) such that the following is commutative:

\[
\begin{array}{ccc}
\phi_{x_0}^{(0)} \Xi_{\mathcal{L}(t)} \lambda^{-1} & \xrightarrow{(-1)^{d+1}} & \phi_{x_0}^{(0)} \Xi_{\mathcal{L}(t)} \lambda^{d+1} \\
\downarrow & & \downarrow \\
\psi_x^{(1)} \psi_x^{(0)} (O_X \lambda^{-1}) & \xrightarrow{A} & \psi_x^{(1)} \psi_x^{(1)} (O_X \lambda^{d+1}) \lambda^{-d-2}
\end{array}
\]

The morphism \( A \) is obtained as the following morphism, induced by the multiplication of \((-1)^{d+1}\) and the correspondence \((\lambda s)^{d+1} x_0^{-1} \leftarrow - \mathcal{D}_{x_0} \otimes (\lambda s)^t \), i.e., induced by the correspondence \((\lambda s)^{d+1} x_0^{-1} \leftarrow (-1)^d \mathcal{D}_{x_0} \otimes (\lambda s)^t \):

\[
\text{Cok} \left( \Pi_{x_0}^{\infty, \psi_x^{(1)} (O)} [D] \rightarrow \Pi_{x_0}^{\infty, \psi_x^{(1)} (O)} [sD] \right) \simeq \text{Cok} \left( \Pi_{x_0}^{\infty, \psi_x^{(0)} (O)} [D] \rightarrow \Pi_{x_0}^{\infty, \psi_x^{(0)} (O)} [sD] \right)
\]

The polarization of \( \mathcal{U}_X(d,0) \otimes T(-1) \) is given by

\[
(\lambda^{d+1}, \lambda^{-1}) \leftarrow (-1)^{d+1} \lambda^{-1} \lambda^{d+2}, (-1)^{d+1} \lambda^{d+1} \lambda^{-d-2})
\]

The polarization of \( \mathcal{T} = \psi_x^{(1)} \mathcal{U}_X(d,0) \otimes T(-1) \) is induced by

\[
(x_0^{-1} \lambda^{d+1}, \mathcal{D}_{x_0} \lambda^{-1}) \leftarrow ((-1)^{d+1} \mathcal{D}_{x_0} \lambda^{-1} \lambda^{d+1}, (-1)^{d+1} x_0^{-1} \lambda^{d+1} \lambda^{-d-1})
\]

(88)

Let us look at the induced morphism \( \psi_x^{(0)} (\mathcal{T}) \rightarrow \psi_x^{(0)} (\mathcal{T}^*) \otimes \mathcal{T}(-d-2) \), which is given by

\[
\psi_x^{(0)} (\mathcal{T}) \simeq \psi_x^{(0)} (\mathcal{T}^*) \otimes \mathcal{T}(-d-1) \simeq \psi_x^{(1)} (\mathcal{T})^* \otimes \mathcal{T}(-d-1)
\]

\[
\simeq \left( \psi_x^{(1)} (\mathcal{T}) \otimes (\mathcal{T}(-1))^* \otimes \mathcal{T}(-d-1) \simeq \psi_x^{(0)} (\mathcal{T})^* \otimes \mathcal{T}(-d-2)
\]

(89)

The second component is given by

\[
\psi_x^{(0)} (\psi_x^{(0)} (O_X \lambda^{-1}) \simeq \psi_x^{(0)} (\psi_x^{(0)} (O_X \lambda^{d+1})) \lambda^{-d-2}
\]

Here, \( a_1 \) is induced by (88). The composite is equal to \( A \). We obtain that, the polarizations of (88) have the same morphism for the second underlying \( \mathcal{R} \)-modules. We can similarly compare the morphisms for the first underlying \( \mathcal{R} \)-modules. Thus, the proof of Proposition 5.3 is finished.
6 Mixed twistor $\mathcal{D}$-modules and GKZ-hypergeometric systems

6.1 GKZ-hypergeometric systems

6.1.1 The associated toric varieties

We consider a finite subset $A := \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$, where $a_i = (a_{ij} | j = 1, \ldots, n)$. We assume that $A$ generates $\mathbb{Z}^n$. Formally, we set $a_0 := (0, \ldots, 0)$.

We set $T^n := \text{Spec} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. The associated complex manifold is also denoted by $T^n$ if there is no risk of confusion. We consider a morphism $\psi_A : T^n \to \mathbb{P}^m$ given by $\psi_A(t_1, \ldots, t_n) = [1 : t^{a_1} : \ldots : t^{a_m}]$, where $t^{a_i} := \prod_{j=1}^n t_j^{a_{ij}}$. Let $X_A$ denote the closure of $\psi_A(T^n)$ in $\mathbb{P}^m$.

For any subset $B \subset \mathbb{R}^n$, let $\text{Conv}(B)$ denote the convex hull of $B \subset \mathbb{R}^n$. Note that 0 is an interior point of $\text{Conv}(A \cup \{0\})$ if and only if 0 is an interior point of $\text{Conv}(A)$.

For any face $\sigma$ of $\text{Conv}(A \cup \{0\})$, we put $I_{\sigma} := \sigma \cap (A \cup \{0\})$ and $J_{\sigma} := (A \cup \{0\}) \setminus I_{\sigma}$. We have the subspace $\mathbb{P}_\sigma := \{[z_0 : z_1 : \cdots : z_m] | z_j = 0 (a_j \in J_{\sigma})\}$.

We set $P^*_\sigma := \{[z_0 : \ldots : z_m] \in \mathbb{P}_\sigma | z_i \neq 0 (i \in I_{\sigma}), z_i = 0 (i \in J_{\sigma})\}$. Recall the following.

**Proposition 6.1 (Proposition 1.9 [15])** For any face $\sigma$ of $\text{Conv}(A \cup \{0\})$, we have the non-empty intersection $X_A = X_A \cap P^*_\sigma$, and it is a $T^n$-orbit of $[z_0 : \cdots : z_m]$, where $z_i^\sigma = 1 (i \in I_{\sigma})$ or $z_i^\sigma = 0 (i \in J_{\sigma})$. We have the decomposition into the orbits $X_A = \bigcup_{\sigma} X^\sigma_A$ where $\sigma$ runs through the set of faces of $\text{Conv}(A \cup \{0\})$.

Recall that we have a desingularization of $X_A$ in the category of toric varieties. (See [6] for example.) Namely, there exist a smooth fan $\Sigma$ and a toric birational morphism $\varphi_\Sigma : X_\Sigma \to X_A$, where $X_\Sigma$ denotes the smooth toric variety associated to $\Sigma$. In that situation, we set $D_\Sigma := \bigcup_{\tau \in \Sigma(1)} D_{\tau}$, where $\Sigma(1)$ denotes the set of 1-dimensional cones of $\Sigma$, and $D_{\tau} \subset X_\Sigma$ denotes the hypersurface corresponding to $\tau$. We have $T^n = X_\Sigma \setminus D_{\Sigma}$.

6.1.2 $\mathcal{D}$-modules associated to families of Laurent polynomials

Suppose that we are given an algebraic map $\gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1))$. It determines a family of Laurent polynomials $\psi_A \times \text{id}_S)^*(s/\psi_A)$ on $T^n \times S$ denoted by $F_\gamma$. We obtain the algebraic $\mathcal{D}$-module $L_{T^n \times S}(F_\gamma)$ on $T^n \times S$ given as the line bundle $\mathcal{O}_{T^n \times S} \cdot e$ with $\nabla e = e \cdot \nabla F_\gamma$, where $e$ denotes a global frame. We obtain the following algebraic $\mathcal{D}$-modules on $\mathbb{P}^m \times S$:

$$L_*(A, S, \gamma) := (\psi_A \times \text{id}_S)^*, L_{T^n \times S}(F_\gamma) \quad (*) = !,*$$

The image of $L_!(A, S, \gamma) \to L_*(A, S, \gamma)$ is denoted by $L_{\text{min}}(A, S, \gamma)$, which is the minimal extension of $L_{T^n \times S}(F_\gamma)$ on $\mathbb{P}^m \times S$ via $\psi_A \times \text{id}_S$. Let $\pi : \mathbb{P}^m \times S \to S$ denote the projection. We obtain the algebraic $\mathcal{D}$-modules $\pi_+ L_*(A, S, \gamma) \quad (*) = !,*$ and $\pi_+ L_{\text{min}}(A, S, \gamma)$.

It is convenient for us to take a toric desingularization $\varphi_\Sigma : X_\Sigma \to X_A$ when we work in the complex analytic setting or when we are interested in the twistor property of the $\mathcal{D}$-modules. We set $(X_{\Sigma,S}, D_{\Sigma,S}) := (X_\Sigma, D_\Sigma) \times S$.

Let $\varphi_\Sigma$ denote $X_\Sigma \hookrightarrow \mathbb{P}^m \times S$ which is the composite of $\varphi_\Sigma \times \text{id}_S$ and the inclusion $X_A \times S \hookrightarrow \mathbb{P}^m \times S$.

We have a meromorphic function $F_{\gamma, \Sigma} := \varphi_\Sigma^*(s/\psi_A)$ on $(X_{\Sigma,S}, D_{\Sigma,S}) \times S$. We obtain the $\mathcal{D}_{X_{\Sigma,S}}$-modules $L_*(F_{\gamma, \Sigma}, D_{\Sigma,S}) \quad (*) = !,*$ on $X_{\Sigma,S}$. Then, we have $\varphi_{\Sigma,S}^* L_*(F_{\gamma, \Sigma}, D_{\Sigma,S}) = 0$ for $i \neq 0$. In the algebraic case, we have $\varphi_{\Sigma,S}^* L_*(F_{\gamma, \Sigma}, D_{\Sigma,S}) \cong L_*(A, S, \gamma)$. In the complex analytic setting, we adopt it as the definition, i.e., $L_*(A, S, \gamma) := \varphi_{\Sigma,S}^* L_*(F_{\gamma, \Sigma}, D_{\Sigma,S})$. Let $\pi_\Sigma : X_\Sigma \times S \to S$ denote the projection. Then, we have the following natural isomorphisms for $\pi_\Sigma = !,*$:

$$\pi_{\Sigma,+} L_*(F_{\gamma, \Sigma}, D_{\Sigma,S}) \cong \pi_{\Sigma,*} L_*(A, S, \gamma)$$

(90)

If we are given another desingularization $\varphi_{\Sigma'} : X_{\Sigma'} \to X_A$, we can find a smooth fan $\Sigma''$ and toric morphisms $\psi_1 : X_{\Sigma''} \to X_{\Sigma}$ and $\psi_2 : X_{\Sigma''} \to X_{\Sigma'}$ such that $\varphi_{\Sigma'} \circ \psi_1 = \varphi_{\Sigma'} \circ \psi_2 =: \varphi_{\Sigma''}$. We have natural isomorphisms
(ψ₁ × idₘ) + L₁(F₁,Σ₁, D₁, S₁) ≅ (ψ₁ × idₘ) + L,m(F₁,Σ₁, D₁, S₁) ≅ L₁(F₁,Σ₁, D₁, S₁). We have similar isomorphisms for L₁(F₁,Σ₁, D₁, S₁). Hence, we have the following natural isomorphisms on Pₘ × S:

\[ \pi^i_+L_*(F_γ,Σ,S,γ) ≅ \pi^0_+L_*(F_γ,Σ, D,S) \approx \pi^i_+L_*(F_γ,Σ, D,S) \approx \pi^i_+L_*(F_γ,Σ, D,S) \]

In this sense, the D-modules L₁(A, S, γ) are independent of the choice of a resolution. We also have the following natural isomorphisms:

\[ \pi^i_+L_*(F_γ,Σ,S,γ) ≅ \pi^i_+L_*(F_γ,Σ, D,S) ≅ \pi^i_+L_*(F_γ,Σ, D,S) ≅ \pi^i_+L_*(A, S, γ) \quad (91) \]

**Remark 6.2** Although it is convenient for us to take a toric desingularization Xₐ of Xₐ, we can also use any toric completion of Tₙ for the study of πₙ⁺Lₙ(A, S, γ). Let Xₐ be any n-dimensional smooth toric variety. We fix an inclusion Tₙ ⊂ Xₐ. The family of Laurent polynomials F_γ gives a meromorphic function F_γ,Xₐ on (Xₐ, S, D). Then, we naturally have πₙ⁺Lₙ(F_γ,Xₐ, D) ≅ πₙ⁺Lₙ(F_γ,Σ, D) ≅ πₙ⁺Lₙ(A, S, γ).

Let γ₁ : S → H₀(Pₘ, O(1)) (i = 1, 2) be holomorphic maps. We give some conditions under which we have π₁⁺L₁(A, S, γ₁) ≅ π₁⁺L₁(A, S, γ₂). We have the Tₙ-action on Pₘ given by \((t₁, \ldots, tₙ)[z_0 : \ldots : z_m] = [t^a₁z_0 : \ldots : t^mz_m]\). For b ∈ Tₙ, let \(m_b\) denote the induced action on Pₘ, and let \(m^b\) denote the induced action on H₀(Pₘ, O(1)).

**Lemma 6.3** Suppose that there exists a holomorphic map \(k : S → Tₙ\) such that γ₁ = m⁺γ₂. Then, we have the natural isomorphisms π⁺L₁(A, S, γ₁) ≅ L₁(A, S, γ₂). If γ₁ and k are algebraic, then we have the isomorphisms of algebraic D-modules.

**Proof** We have the morphism \(m_κ : Pₘ × S → Pₘ × S\) induced by κ and the identity of S. We have an isomorphism \(m⁺L₁(A, S, γ₁) ≅ L₁(A, S, γ₂)\). Applying π⁺, we obtain the desired isomorphisms.

The following lemma is clear by construction.

**Lemma 6.4** Let \(ρ : S → C\) be a holomorphic map such that γ₁ = γ₂ + ρ · z₀. Then, we have the isomorphisms of the analytic D-modules L₁(A, S, γ₁) ≅ L₁(A, S, γ₂). In particular, we have the isomorphisms of the analytic D-modules π⁺L₁(A, S, γ₁) ≅ π⁺L₁(A, S, γ₂).

### 6.1.3 Non-degenerate families of sections

Let \(p₁ : Pₘ × S → S\) be the projection. Let \(γ : S → H₀(Pₘ, O_{Pₘ}(1))\) be any holomorphic map. We have the expression γ(x) = \(\sum_{i=0}^m γ_i(x)z_i\). It determines a family of Laurent polynomials F_γ(x, t) = \(\sum_{a∈σ} γ_i(x)t^a\). For any face σ of Conv(σ \cup \{0\}), we consider the associated family of Laurent polynomials F_γ,σ(x, t) = \(\sum_{a∈σ} γ_i(x)t^a\). We can regard it as a function on Tₙ × S. Let df,σ denote the exterior derivative of F_γ,σ, which is a section of Ω₁_{Tₙ × S}. We use the following non-degeneracy condition which is a minor generalization of the classical non-degeneracy condition \(30\) (see also \(11, 32\)) in the sense that we also consider the derivatives in the S-direction.

**Definition 6.5** γ is called non-degenerate at ∞ for Xₐ if we have (F_γ,σ)⁻¹(0) ∩ (df,σ)⁻¹(0) = \(∅\) for any face σ of Conv(σ \cup \{0\}) such that 0 \(∉\) σ.

The condition is equivalent to that the zero-divisors of F_γ,σ are smooth and reduced for any face σ of Conv(σ \cup \{0\}) such that 0 \(∉\) σ. We remark the following.

**Lemma 6.6** If σ \(⊆\) Conv(σ \cup \{0\}), we have (F_γ,σ)⁻¹(0) ∩ (df,σ)⁻¹(0) = \(∅\) if and only if (df,σ)⁻¹(0) = \(∅\).

**Proof** Under the assumption, there exists \(β = (β₁, \ldots, βₙ) \in (Qₙ)^σ\) such that \(β \cdot aₖ = 1\) for any \(aₖ \in I_σ\). Because \(\sum_{j=1}^n β_jt_j\partial_j\) \(aₖ\) = \(β \cdot aₖ \cdot t^a = t^a\), we have F_γ,σ⁻¹(0) ⊃ (df,σ)⁻¹(0). Hence, the condition F_γ,σ⁻¹(0) ∩ (df,σ)⁻¹(0) = \(∅\) is equivalent to that (df,σ)⁻¹(0) = \(∅\).

Let us reword the condition. We have the section s₀ of the line bundle \(p₁O_{Pₘ}(1)\) such that s₀|{x} = γ(x) ∈ H₀(Pₘ, O_{Pₘ}(1)) for any x ∈ S. We set \(H_γ := s₀⁻¹(0) \subset Pₘ × S\). We also put \(H_∞ := \{z₀ = 0\} \subset Pₘ\).
Lemma 6.7 Let $\gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1))$ be a morphism. For any face $\sigma$ of Conv($A \cup \{0\}$), the following conditions are equivalent.

- $(F_{\gamma, \sigma})^{-1}(0) \cap (dF_{\gamma, \sigma})^{-1}(0) = \emptyset$.
- There exists an open neighbourhood $U_1$ of $X^*_{\mathcal{A}} \times S$ in $\mathbb{P}^m \times S$ such that (i) $H_\gamma \cap U_1$ is smooth (ii) $H_\gamma$ is transversal with $X^*_{\mathcal{A}} \times S$.

Proof Let $Q \in X^*_{\mathcal{A}}$ be the point $[z_0^0 : \cdots : z_m^m]$ as in [6.11]. By using the $T^n$-action on $X^*_{\mathcal{A}}$, we consider the map $\psi : T^n \times S \to X^*_{\mathcal{A}} \times S$ given by $\psi(t) = t \cdot Q$. Take any $i_0 \in I_\sigma$. We have

$$\psi^*(s_\gamma)/\psi^*(z_{i_0}) = \sum_{a_i \in I_\sigma} \gamma_i(x)(t^{a_i} - a_{i_0}) = : G_{\gamma, \sigma}(x, t).$$

Then, the second condition holds if and only if (i) $G_{\gamma, \sigma}$ is not constantly 0, (ii) the divisor $(G_{\gamma, \sigma}) = (F_{\gamma, \sigma})$ is smooth. It is equivalent to that $(dF_{\gamma, \sigma})^{-1}(0) \cap (dF_{\gamma, \sigma})^{-1}(0) = \emptyset$.

Corollary 6.8 $\gamma$ is non-degenerate at $\infty$ for $X_\mathcal{A}$ if and only if the following holds:

- There exists a neighbourhood $U$ of $H_\infty \times S$ in $\mathbb{P}^m \times S$ such that $H_\gamma \cap U$ is a smooth hypersurface in $U$.
- If $\sigma$ is a face of Conv($A \cup \{0\}$) such that $0 \notin \sigma$, then $H_\gamma$ is transversal with $X^*_{\mathcal{A}} \times S$.

The first condition in this corollary is trivial if the image of $\gamma$ does not contain $0 \in H^0(\mathbb{P}^m, \mathcal{O}(1))$. The conditions particularly imply that $\gamma(U) \notin \mathbb{C} \cdot z_0$ for any open subset $U \subset S$.

Let $\varphi_\Sigma : X_\Sigma \to X_\mathcal{A}$ be any toric desingularization. Let $\varphi_\Sigma : X_\Sigma \to \mathbb{P}^m$ be the composite of $\varphi_\Sigma$ and the inclusion $X_\mathcal{A} \to \mathbb{P}^m$.

Lemma 6.9 Let $\gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1))$ be a holomorphic map which is non-degenerate at $\infty$ for $X_\mathcal{A}$. Then, $F_{\gamma, \Sigma}$ is non-degenerate along $D_\Sigma \times S$. We also have $|F_{\gamma, \Sigma}(\infty)| = \varphi_{\Sigma}^{-1}(H_\infty) \times S$.

Proof Let $T_\rho$ be a $T^n$-orbit in $X_\Sigma$ contained in $\varphi_{\Sigma}^{-1}(H_\infty)$. Note that $\varphi_{\Sigma}(T_\rho)$ is contained in $X_\Sigma \cap H_\infty$, and the morphism $T_\rho \to \varphi_{\Sigma}(T_\rho)$ is smooth. Let $\varphi_{\Sigma, S} := \varphi_{\Sigma} \times \text{id}_S$. The restriction of $\varphi_{\Sigma, S}(s_\gamma)$ to $T_\rho \times S$ is not 0, and the 0-divisor of $\varphi_{\Sigma, S}(s_\gamma)|_{T_\rho \times S}$ is smooth. Then, the claims of the lemma are easy to see.

Any element $s \in H^0(\mathbb{P}^m, \mathcal{O}(1))$ determines a morphism $\gamma_s$ from the one-point set to $H^0(\mathbb{P}^m, \mathcal{O}(1))$.

Definition 6.10 We say that $s$ is non-degenerate at $\infty$ for $X_\mathcal{A}$ if $\gamma_s$ is non-degenerate at $\infty$ for $X_\mathcal{A}$ in the sense of Definition 6.5.

It is standard that there exist non-empty Zariski open subsets $U \subset H^0(\mathbb{P}^m, \mathcal{O}(1))$ such that any $s \in U$ is non-degenerate at $\infty$ for $X_\mathcal{A}$. Let $U^*_\mathcal{A}$ be the union of such open subsets.

6.1.4 Basic examples of non-degenerate family of sections

The following lemma is clear.

Lemma 6.11 A holomorphic map $\gamma : S \to U^*_\mathcal{A}$ is non-degenerate at $\infty$ for $X_\mathcal{A}$.

Note that the converse is not necessarily true. The following lemma is clear.

Lemma 6.12 Let $\gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1))$ be a holomorphic map. Let $\Phi_{\gamma} : H_\gamma \to \mathbb{P}^m$ be the induced morphism. Suppose that $H_\gamma$ is a smooth hypersurface, and that any critical value of $\Phi_{\gamma}$ is not contained in $X_\mathcal{A} \cap H_\infty$. Then, $\gamma$ is non-degenerate at $\infty$ for $X_\mathcal{A}$.
Example 6.13 We set \( W := \{ \sum_{i=1}^{n} \alpha_i z_i \} \subset H^0(\mathbb{P}^m, \mathcal{O}(1)) \). Then, the inclusion \( \iota : W \to H^0(\mathbb{P}^m, \mathcal{O}(1)) \) is non-degenerate at \( x \) for any \( x \).

Indeed, we set \( H_i := (\mathbb{C}^{m+1} \setminus \{(0, \ldots, 0)\}) \times W := \{ (z_0, \ldots, z_m; \alpha_0, \ldots, \alpha_m) \in \mathbb{C}^{m+1} \mid \sum_{i=1}^{m} \alpha_i z_i = 0 \} \). Then, \( \tilde{H}_i \) is clearly smooth. Because \( H_i \) is the quotient of \( \tilde{H}_i \) by the natural free \( \mathbb{C}^* \)-action, \( H_i \) is also smooth. Let \( H_i \to \mathbb{C}^{m+1} \setminus \{(0, \ldots, 0)\} \) be the projection. Then, the critical values are \( \{ (z_0, 0, \ldots, 0) \mid z_0 \in \mathbb{C}^* \} \). Hence, the set of the critical values of \( \Phi_x : H_i \to \mathbb{P}^m \) is \( \{ [1 : 0 : \cdots : 0] \} \). Then, by Lemma 6.12, we obtain that \( \iota \) is non-degenerate.

Set \( W^* := \{ \sum_{i=1}^{m} \alpha_i z_i \mid \alpha_i \in \mathbb{C}^* \} \). We have the action of \( T^* \) on \( W^* \) by \( t \cdot (\alpha_1, \ldots, \alpha_m) = (t^{\alpha_1} \alpha_1, \ldots, t^{\alpha_m} \alpha_m) \). The action is free because \( A \) generates \( \mathbb{Z}^n \). Let \( q : W^* \to W^*/T^* \) be the projection.

Lemma 6.14 Let \( \gamma : S \to W^* \) be any holomorphic map such that \( q \circ \gamma \) is submersive. Then, \( \gamma \) is non-degenerate at \( x \) for any \( x \).

Proof Let \( \sigma \) be any face of \( \text{Conv}(A \cup \{0\}) \) such that \( 0 \notin \sigma \). Take any \( (P, x) \in H \subset \mathbb{P}^m \times S \) such that \( P \in X_A \).

We have \( P \in H(X, x) \). Let \( \Delta_\epsilon := \{ z \in \mathbb{C} \mid |z| < \epsilon \} \) for \( \epsilon > 0 \). For any holomorphic map \( g : \Delta_\epsilon \to W^* \), we have the family of hyperplanes \( H_{g} \subset \mathbb{P}^m \times \Delta_\epsilon \), and we obtain the induced map \( \Phi_x : H_{g} \to \mathbb{P}^m \). Because \( P \neq \{ 1 : 0 : \cdots : 0 \} \), we can find a holomorphic map \( g_0 : \Delta_\epsilon \to W^* \) such that \( g_0(0) = \gamma(x) \), \( g_0 \) is a morphism, and \( \Phi_x : H_{g_0} \to \mathbb{P}^m \) is an isomorphism.

Example 6.15 Let \( r : W^*/T^* \to W^* \) be any section of \( q \). Then, the induced morphism \( r : W^*/T^* \to H^0(\mathbb{P}^m, \mathcal{O}(1)) \) is non-degenerate.

6.1.5 D-modules associated to non-degenerate families of sections

Let \( \gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1)) \) be any holomorphic map. Suppose that \( \gamma \) is non-degenerate at \( x \) for any \( x \).

We have the \( D \)-modules \( L_x(A, S, \gamma) \) on \( \mathbb{P}^m \times S \). We set \( H_{\infty} := H_{\infty} \times S \).

Lemma 6.16 We naturally have \( L_x(A, S, \gamma)(*H_{\infty,S}) \simeq L_x(A, S, \gamma)(* = *, !) \).

Proof For a desingularization \( \varphi : \Sigma \to \Sigma \), we obtain the \( D \)-modules \( L_x(F_{\gamma, \Sigma}, D_{\Sigma,S}, \gamma) \). Because \( F_{\gamma, \Sigma} \) is non-degenerate along \( D_{\Sigma,S} \), we have

\[
L_x(A, S, \gamma)(*H_{\infty,S}) \simeq \varphi_{\Sigma,S}^{-1} L_x(F_{\gamma, \Sigma}, D_{\Sigma,S}, \gamma) \simeq L_x(A, S, \gamma).
\]

Thus, we are done.

Let \( K_{A,S,\gamma} \) and \( C_{A,S,\gamma} \) denote the kernel and the cokernel of \( L_t(A, S, \gamma) \to L_t(A, S, \gamma) \).

Corollary 6.17 We have \( M(*H_{\infty,S}) \simeq M \) for the \( D \)-modules \( M = K_{A,S,\gamma}, C_{A,S,\gamma}, L_{\infty}(A, S, \gamma) \).

Proposition 6.18 We have \( \pi^1_M = 0 \) \((i \neq 0)\) for the \( D \)-modules \( M = K_{A,S,\gamma}, C_{A,S,\gamma}, L_{\infty}(A, S, \gamma) \).

Proof Let us prove \( \pi^1_M L_x(A, S, \gamma) = 0 \) \((i \neq 0)\). By using the duality, it is enough to prove that \( \pi^1_M L_x(A, S, \gamma) = 0 \) \((i > 0)\) and for \(* = *, !\). By Lemma 6.16, we have \( R^j \pi_* (L_x(A, S, \gamma) \otimes \Omega^0_{E,S}) = 0 \) for any \( j > 0 \) and any \( i \). By using the expression

\[
\pi^1_M L_x(A, S, \gamma) \simeq \Omega^{j + \pi + \gamma}_{E,S} \otimes L_x(A, S, \gamma),
\]

we obtain the claim.

Let us prove the vanishings for \( K_{A,S,\gamma}, C_{A,S,\gamma} \) and \( L_{\infty}(A, S, \gamma) \). By Corollary 6.17, we obtain \( \pi^1_M K_{A,S,\gamma} = 0, \pi^1_M C_{A,S,\gamma} = 0, \pi^1_M L_{\infty}(A, S, \gamma) = 0 \) for \( i > 0 \). Let \( \gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1)) \) denote the composite of \( \gamma \) and the multiplication of \(-1 \) on \( H^0(\mathbb{P}^m, \mathcal{O}(1)) \). We naturally have \( DK_{A,S,\gamma} \simeq C_{A,S,\gamma}, D\mathcal{C}_{A,S,\gamma} \simeq K_{A,S,\gamma} \), and \( DL_{\infty}(A, S, \gamma) \simeq L_{\infty}(A, S, \gamma) \). Then, we obtain \( \pi^1_M K_{A,S,\gamma} = 0, \pi^1_M C_{A,S,\gamma} = 0, \pi^1_M L_{\infty}(A, S, \gamma) = 0 \) for \( i < 0 \).
Corollary 6.19 The following are exact sequences:

\[ 0 \to \pi_+^0 K_{A,S,\gamma} \to \pi_+^0 L_i(A, S, \gamma) \to \pi_+^0 L_{\text{min}}(A, S, \gamma) \to 0 \]
\[ 0 \to \pi_+^0 L_{\text{min}}(A, S, \gamma) \overset{\pi_+^0}{\to} L_{s}(A, S, \gamma) \to \pi_+^0 C_{A,S,\gamma} \to 0 \]

In particular, \( \pi_+^0 L_{\text{min}}(A, S, \gamma) \) is isomorphic to the image of \( \pi_+^0 L_i(A, S, \gamma) \to \pi_+^0 L_{s}(A, S, \gamma) \).

Corollary 6.20 For any desingularization \( \varphi_{\Sigma} : X_{\Sigma} \to X_A \), we have \( \pi_+^{i} L_{s}(F_{\gamma, \Sigma}, D_{\Sigma,S}) = 0 \) for \( i \neq 0 \).

Corollary 6.21 Suppose that \( \gamma \) satisfies the stronger condition that \( \gamma(S) \subset U_{A}^{\text{reg}} \). Then, the \( D \)-modules \( \pi_+^0 L_{s}(A, S, \gamma) \), \( \pi_+^0 L_{\text{min}}(A, S, \gamma) \), \( \pi_+^0 K_{A,S,\gamma} \) and \( \pi_+^0 C_{A,S,\gamma} \) are locally free \( \mathcal{O}_S \)-modules with a flat connection.

**Proof** We obtain the claims for \( \pi_+^0 L_{s}(A, S, \gamma) \) from Corollary 2.18 and the isomorphisms (90). Then, the claims for the others follows.

Suppose moreover that 0 is an interior point of \( \text{Conv}(A) \). In this case, we have \( L_{i}(A, S, \gamma) \simeq L_{\text{min}}(A, S, \gamma) \simeq L_{s}(A, S, \gamma) \), which is denoted by \( L(A, S, \gamma) \). We have \( \pi_+^0 L_{s}(A, S, \gamma) = 0 \) for \( i \neq 0 \). For a desingularization \( \varphi_{\Sigma} : X_{\Sigma} \to X_A \), we have the \( D \)-module \( L(F_{\gamma, \Sigma}) = L_{i}(F_{\gamma, \Sigma}, D_{\Sigma,S}) = L_{s}(F_{\gamma, \Sigma}, D_{\Sigma,S}) \). We have the following corollary as a special case.

Corollary 6.22 If 0 is an interior point of \( \text{Conv}(A) \), we have \( \pi_+^{i} L(F_{\gamma, \Sigma}) = 0 \) for \( i \neq 0 \). We naturally have \( \pi_+^0 L(F_{\gamma, \Sigma}) \simeq \pi_+^0 L(A, S, \gamma) \).

Example 6.23 The \( D \)-modules associated to the non-degenerate section in Example 6.13 are called the GKZ-hypergeometric systems. The \( D \)-modules associated to the non-degenerate sections in Example 6.13 are called the reduced GKZ-hypergeometric systems. Note that it is independent of the choice of a section, according to Lemma 6.3.

### 6.1.6 De Rham complexes

Let \( \gamma : S \to H^0(\mathbb{P}^m, \mathcal{O}(1)) \) be a holomorphic map which is non-degenerate at \( \infty \) for \( X_A \). Take a toric desingularization \( \varphi_{\Sigma} : X_{\Sigma} \to X_A \).

**Proposition 6.24** If 0 is contained in the interior part of \( \text{Conv}(A) \), we have the following natural isomorphism for \( L(F_{\gamma, \Sigma}) = L_{i}(F_{\gamma, \Sigma}, D_{\Sigma,S}) = L_{s}(F_{\gamma, \Sigma}, D_{\Sigma,S}) \):

\[ \pi_+^0 L_{s}(F_{\gamma, \Sigma}, D_{\Sigma,S}) \simeq \mathbb{R}^n \pi_+^{*}(\Omega_{X_{\Sigma,S}/S}^{*}(\ast D_{\Sigma,S}), d + dF_{\gamma, \Sigma}) \]  
\[ (\ast) \]

If \( \gamma \) is algebraic, it means \( \pi_+^0 L_{s}(A, S, \gamma) \simeq \mathbb{R}^n \pi_+^{*}(\Omega_{X_{\Sigma,S}/S}^{*}(\ast D_{\Sigma,S}), d + dF_{\gamma, \Sigma}) \).

If 0 is a boundary point of \( \text{Conv}(A \cup \{0\}) \), we have the following isomorphisms:

\[ \pi_+^0 L_{s}(F_{\gamma, \Sigma}, D_{\Sigma,S}) \simeq \mathbb{R}^n \pi_+^{*}(\Omega_{X_{\Sigma,S}/S}^{*}(\log D_{\Sigma,S})(\ast (F_{\gamma, \Sigma})_{\infty}), d + dF_{\gamma, \Sigma}) \]  
\[ (\ast) \]

**Proof** We immediately \((\ast)\) from \( L(F_{\gamma, \Sigma}, D_{\Sigma,S}) \simeq (O_{X_{\Sigma,S}}(\ast D_{\Sigma,S}), d + dF_{\gamma, \Sigma}) \). By applying the arguments in \(2.2.6\) and \(2.2.7\) we obtain \((\ast)\) and \((\ast)\).

Let us rewrite \((\ast)\) and \((\ast)\) in terms of \( X_A \). We set \( D_A := X_A \cap \left( \bigcup_{i=0}^{m}(z_i = 0) \right) \) and \( D_{A,\infty} := X_A \cap H_{\infty} \).

Let \( \Omega_{X_A}^{*}(\log D_A) \) denote the sheaf of logarithmic \( i \)-forms \((X_A, D_A)\) as in \([3]\). Let \( \Omega_{X_A,D_A}^{*}(\ast) \) denote the sheaf of \( i \)-forms on \( X_A \) whose restrictions to each stratum of \( D_A \) is 0 as in \([7]\). Let \( \Omega_{X_A,S/S}^{*}(\log D_{A,S}) \) and \( \Omega_{(X_A,D_A)\times S/S}^{*}(\ast) \) denote the pull back of \( \Omega_{X_A}^{*}(\log D_A) \) and \( \Omega_{(X_A,D_A)}^{*}(\ast) \) by the projection \( X_A \times S \to X_A \), respectively.

59
Proposition 6.25 We have the following isomorphisms:

\[ \pi^0_+ L_*(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega^{*}_{X_A \times S/S}(\log D_{A,S})(*D_{A,\infty,S}), d + dF_\gamma \right) \]

\[ \pi^0_+ L_1(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega^{*}_{(X_A, D_A) \times S/S}(\ast D_{A,\infty,S}), d + dF_\gamma \right) \]

Proof According to [3] Lemma 6.1, we have

\[ R\varphi_* \Omega^{i}_{X_\Sigma} (\log D_{\Sigma}) \simeq \Omega^{i}_{X_A} (\log D_A). \]

For a smooth toric variety \( X_\Sigma \), we have \( \Omega^{i}_{X_\Sigma, D_\Sigma} \simeq \Omega^{i}_{X_\Sigma} (\log D_{\Sigma})(-D_{\Sigma}). \) According to [7] Proposition 1.8, we have

\[ R\varphi_* \Omega^{i}_{X_\Sigma} (\log D_{\Sigma})(-D_{\Sigma}) \simeq \Omega^{i}_{X_A, D_A}. \]

Then, we obtain (95) and (96) from (93) and (94).

In the algebraic setting, we set \( X^\text{aff}_A := X_A \setminus H_\infty \) and \( D^\text{aff}_A := D_A \setminus H_\infty \). We obtain the following similarly.

Proposition 6.26 We have the following isomorphisms:

\[ \pi^0_+ L_*(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega^{*}_{X^\text{aff}_A \times S/S}(\log D_{A,\text{aff},S}), d + dF_\gamma \right) \]

\[ \pi^0_+ L_1(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega^{*}_{(X^\text{aff}_A, D_{A,\text{aff}}) \times S/S}(\ast D_{A,\text{aff},S}), d + dF_\gamma \right) \]

6.2 Mixed twistor \( \mathcal{D} \)-modules

6.2.1 Mixed twistor \( \mathcal{D} \)-modules associated to families of Laurent polynomials

We use the notation in [46.1]. Let \( \gamma : S \rightarrow H^0(\mathbb{P}^m, \mathcal{O}(1)) \) be a holomorphic map. We take a toric desingularization \( \varphi : X_\Sigma \rightarrow X_A \). Let \( d_S := \text{dim } S \). We have the integrable mixed twistor \( \mathcal{D} \)-modules \( \mathcal{T}_*(F_\gamma, \Sigma, D_\Sigma, S) \) \((* = *, !)\) with real structure on \( X_\Sigma \times S:\)

\[ \mathcal{T}_*(F_\gamma, \Sigma, D_\Sigma, S) = \left( \lambda^{n+d_S} \mathcal{L}_*(F_\gamma, \Sigma, D_\Sigma, S), \mathcal{L}_*(F_\gamma, \Sigma, D_\Sigma, S), \mathcal{C}_*(F_\gamma, \Sigma, D_\Sigma, S) \right). \]

\[ \mathcal{T}_!(F_\gamma, \Sigma, D_\Sigma, S) = \left( \lambda^{n+d_S} \mathcal{L}_!(F_\gamma, \Sigma, D_\Sigma, S), \mathcal{L}_!(F_\gamma, \Sigma, D_\Sigma, S), \mathcal{C}_!(F_\gamma, \Sigma, D_\Sigma, S) \right). \]

The weight filtration is denoted by \( W \). The image of \( \mathcal{T}_!(A, S, \gamma) \rightarrow \mathcal{T}_*(A, S, \gamma) \) is denoted by \( \mathcal{T}_{\text{min}}(A, S, \gamma) \).

On \( \mathbb{P}^m \times S \), we obtain the \( \mathcal{R}_{\mathbb{P}^m \times S} \)-modules \( \mathcal{L}_*(A, S, \gamma) := \varphi^0_{\Sigma, S+} \mathcal{L}_*(F_\gamma, \Sigma, D_\Sigma, S) \) and the integrable mixed twistor \( \mathcal{D} \)-modules with induced real structure \( \mathcal{T}_*(A, S, \gamma) := \mathcal{T}^0_{\Sigma, S+} + \mathcal{T}_*(F_\gamma, \Sigma, D_\Sigma, S) \):

\[ \mathcal{T}_*(A, S, \gamma) = \left( \lambda^{n+d_S} \mathcal{L}_*(A, S, \gamma), \mathcal{L}_*(A, S, \gamma), \mathcal{C}_*(A, S, \gamma) \right) \]

\[ \mathcal{T}_!(A, S, \gamma) = \left( \lambda^{n+d_S} \mathcal{L}_!(A, S, \gamma), \mathcal{L}_!(A, S, \gamma), \mathcal{C}_!(A, S, \gamma) \right) \]

Here, \( \mathcal{C}_*(A, S, \gamma) \) \((* = *, !)\) are obtained as the push-forward of \( \mathcal{C}_!(F_\gamma, \Sigma, D_\Sigma, S) \). They are twistor enhancement of \( \mathcal{L}_*(A, S, \gamma) \). As in the case of \( \mathcal{D} \)-modules ([65.1], 22), they are independent of the choice of a toric desingularization. We also obtain integrable mixed twistor \( \mathcal{D} \)-modules with real structure \( \pi^!_*(\mathcal{T}_!(A, S, \gamma)) \) \((* = *, !)\) on \( S \) which are naturally isomorphic to \( \pi^!_{\Sigma, 1}(\mathcal{T}_*(F_\gamma, \Sigma, D_\Sigma, S)). \)

Remark 6.27 Let \( X_{\Sigma_1} \) be any \( n \)-dimensional smooth toric variety with a fixed inclusion \( \mathbb{T}^n \subset X_{\Sigma_1} \). We have the meromorphic function \( F_{\gamma, \Sigma_1} \) on \( X_{\Sigma_1, D_{\Sigma_1}} \) associated to the family of Laurent polynomials \( F_\gamma \). We have the associated integrable mixed twistor \( \mathcal{D} \)-modules \( \mathcal{T}_*(F_{\gamma, \Sigma_1}, D_{\Sigma_1, S}) \). As in the case of \( \mathcal{D} \)-modules, we naturally have \( \pi^!_{\Sigma_1, 1}(\mathcal{T}_*(F_{\gamma, \Sigma_1}, D_{\Sigma_1, S})) \simeq \pi^!_1(\mathcal{T}_*(A, S, \gamma)). \)
6.2.2 Non-degenerate family of sections

Let $\gamma : S \rightarrow H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ be a holomorphic map which is non-degenerate at $\infty$ for $X_A$.

**Proposition 6.28** We naturally have $L_\ast(A, S, \gamma)(\ast H_\infty, S) \simeq L_\ast(A, S, \gamma)$.

**Proof** We obtain the claim by using the argument in Lemma 6.10 together with Corollary 3.12 and Proposition 6.28.

Let $K_{A, S, \gamma}$ and $\mathcal{C}_{A, S, \gamma}$ denote the kernel and the cokernel of $T_\ast(A, S, \gamma) \rightarrow T_\ast(A, S, \gamma)$.

**Corollary 6.29** Let $\mathcal{M}$ be the $R_{\mathbb{P}^m \times S}$-modules underlying $K_{A, S, \gamma}$, $\mathcal{C}_{A, S, \gamma}$ or $T_{\text{min}}(A, S, \gamma)$. Then, we have $\mathcal{M}(\ast H_\infty, S) \simeq \mathcal{M}$.

We obtain the following from Proposition 6.18.

**Proposition 6.30** We have $\pi_+^0 T = 0$ $(i \neq 0)$ for the integrable mixed twistor $D$-modules for $T = T_\ast(A, S, \gamma)$ $(\ast = \ast, \star) T_{\text{min}}(A, S, \gamma)$, $K_{A, S, \gamma}$, and $\mathcal{C}_{A, S, \gamma}$.

We obtain the following corollary.

**Corollary 6.31** We have the following exact sequences of integrable mixed twistor $D$-modules on $S$:

\[
0 \rightarrow \pi_+^0 K_{A, S, \gamma} \rightarrow \pi_+^0 T_\ast(A, S, \gamma) \rightarrow \pi_+^0 T_{\text{min}}(A, S, \gamma) \rightarrow 0
\]

\[
0 \rightarrow \pi_+^0 T_{\text{min}}(A, S, \gamma) \rightarrow \pi_+^0 T_\ast(A, S, \gamma) \rightarrow \pi_+^0 \mathcal{C}_{A, S, \gamma} \rightarrow 0
\]

As a result, the image of the morphism $\pi_+^0 T_\ast(A, S, \gamma) \rightarrow \pi_+^0 T_{\ast}(A, S, \gamma)$ is naturally isomorphic to the integrable pure twistor $D$-module $\pi_+^0 T_{\ast}(A, S, \gamma)$. Moreover, we have

\[
\text{Gr}_{n+d_S} W \pi_+^0 T_\ast(A, S, \gamma) \simeq \pi_+^0 T_{\ast}(A, S, \gamma) \simeq \text{Gr}_{n+d_S} W \pi_+^0 T_\ast(A, S, \gamma).
\]

**Corollary 6.32** For any desingularization $\varphi : X_\Sigma \rightarrow X_A$, we have $\pi_{\Sigma}^i (T_\ast(F, \Sigma, D_{\Sigma, S})) = 0$ $(i \neq 0)$.

6.2.3 Descriptions in terms of the de Rham complexes

Let $\gamma : S \rightarrow H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$ be a morphism which is non-degenerate at $\infty$ for $X_A$. We give descriptions of the $R$-modules $\pi_0^0 L_\ast(A, S, \gamma)$. We use the notation in 4.3.2.4 and 6.1.6.

**Proposition 6.33** Take a toric desingularization $\gamma : X_\Sigma \rightarrow X_A$. If 0 is an interior point of $\text{Conv}(A \cup \{0\})$, we have

\[
\pi_{\Sigma}^0 (L_\ast(F, \Sigma)) \simeq \mathbb{R}^n \pi_\Sigma^0 (\Omega^\ast_{X_{\Sigma}/S}(\ast D_{\Sigma, S}), d + \lambda^{-1} dF_{\gamma, \Sigma}).
\]

If moreover $\gamma$ is algebraic, it means $\pi_{\Sigma}^0 (L_\ast(F, \Sigma)) \simeq \mathbb{R}^n \pi_\Sigma^0 (\Omega^\ast_{X_{\Sigma}/S}(\ast D_{\Sigma, S}), d + \lambda^{-1} dF_{\gamma})$.

If 0 is a boundary point of $\text{Conv}(A \cup \{0\})$, we have the following natural isomorphisms:

\[
\pi_{\Sigma}^0 L_\ast(F, \Sigma, D_{\Sigma, S}) \simeq \mathbb{R}^n \pi_\Sigma^0 (\Omega^\ast_{X_{\Sigma}/S}(\log D_{\Sigma, S}) (* F_{\gamma, \Sigma}), d + \lambda^{-1} dF_{\gamma, \Sigma}) \tag{99}
\]

\[
\pi_{\Sigma}^0 L_\ast(F, \Sigma, D_{\Sigma, S}) \simeq \mathbb{R}^n \pi_\Sigma^0 (\Omega^\ast_{X_{\Sigma}/S}(\log D_{\Sigma, S})(\ast D_{\Sigma, S}), d + \lambda^{-1} dF_{\gamma, \Sigma}) \tag{100}
\]

**Proof** They are the special cases of the isomorphisms given in 4.3.2.4.

Let $q : \mathbb{C} \times X_A \times S \rightarrow X_A$ be the projection. Let $\Omega^\ast_{X_{A}, S}(\log D_{A, S})$ denote $\lambda^{-1} q^* \Omega^\ast_{X_{A}}(\log D_{A})$. Let $\Omega^\ast_{X_{A}, D_{A}}$ denote $\lambda^{-1} q^* \Omega^\ast_{X_{A}, D_{A}}$. [End of text]
Proposition 6.34 We have the following natural isomorphisms:

\[
\pi_0^! \Lambda_*(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega_{X_A \times S/S}^*(\log D_{A,S})(*D_{A,\infty,S}), d + \lambda^{-1}dF_{\gamma} \right) \tag{101}
\]

\[
\pi_0^! \Lambda_1(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega_{(X_A, D_A) \times S/S}^*(\log D_{A,S}), d + \lambda^{-1}dF_{\gamma} \right) \tag{102}
\]

In the algebraic setting, we have the following isomorphisms:

\[
\pi_0^! \Lambda_*(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega_{X_A^{aff} \times S/S}^*(\log D_{A,S}^{aff}), d + \lambda^{-1}dF_{\gamma} \right) \tag{103}
\]

\[
\pi_0^! \Lambda_1(A, S, \gamma) \simeq \mathbb{R}^n \pi_* \left( \Omega_{(X_A^{aff}, D_A^{aff}) \times S/S}^*(d + \lambda^{-1}dF_{\gamma}) \right) \tag{104}
\]

Proof We obtain the isomorphisms from Proposition 6.33 with the isomorphisms (101) and (102).

Corollary 6.35 The image of the natural morphism

\[
\mathbb{R}^n \pi_* \left( \Omega_{(X_A^{aff}, D_A^{aff}) \times S/S}^*(d + \lambda^{-1}dF_{\gamma}) \right) \longrightarrow \mathbb{R}^n \pi_* \left( \Omega_{X_A^{aff} \times S/S}^*(\log D_{A,S}^{aff}), d + \lambda^{-1}dF_{\gamma} \right) \tag{105}
\]

underlies a pure twistor \(D\)-module.

Let \(U_{A}^{reg} \subset H^0(\mathbb{P}^m, \mathcal{O}(1))\) be the open subset as in 6.13. Suppose that the image of \(\gamma\) is contained in \(U_{A}^{reg}\). In this case, \(\pi_0^! \left( \Lambda_*(A, S, \gamma) \right)\) are locally free \(\mathcal{O}_{C \times S}\)-modules. Let \(\pi_1^! \left( \Lambda_*(A, S, \gamma) \right)^\vee\) be the dual as \(\mathcal{O}_{C \times S}\)-modules, which are naturally equipped with the meromorphic connection. They are naturally isomorphic to \(\lambda^{-dS} D_{\pi_1^!}^0 \left( \Lambda_*(A, S, \gamma) \right)\) up to signatures. The real structure of \(\pi_1^! \mathcal{T}(A, S, \gamma)\) gives

\[
\pi_0^! \Lambda_1(A, S, \gamma) \simeq j^* D_{\pi_1^!}^0 \left( \lambda^{n+dS} \Lambda_*(A, S, \gamma) \right) \simeq \lambda^{-n-dS} j^* D_{\pi_2^!}^0 \Lambda_*(A, S, \gamma) \simeq \lambda^{-n} \left( j^* \pi_0^! \Lambda_*(A, S, \gamma) \right)^\vee
\]

Corollary 6.36 \(\pi_0^! \Lambda_1(A, S, \gamma)\) is naturally identified with \(\lambda^{-n} j^* \mathbb{R}^n \pi_* \left( \Omega_{X_A^{aff} \times S/S}^*(\log D_{A,S}^{aff}), d + \lambda^{-1}dF_{\gamma} \right)^\vee\). The image of the natural morphism

\[
\lambda^{-n} j^* \mathbb{R}^n \pi_* \left( \Omega_{X_A^{aff} \times S/S}^*(\log D_{A,S}^{aff}), d + \lambda^{-1}dF_{\gamma} \right)^\vee \longrightarrow \mathbb{R}^n \pi_* \left( \Omega_{X_A^{aff} \times S/S}^*(\log D_{A,S}^{aff}), d + \lambda^{-1}dF_{\gamma} \right)
\]

underlies an integrable pure twistor \(D\)-module.

6.2.4 Graded polarizations

Let \(A = \{a_1, \ldots, a_m\}\) and \(a_0 := (0, \ldots, 0)\). Let \(\gamma : S \longrightarrow H^0(\mathbb{P}^m, \mathcal{O}(1))\) be a morphism which is non-degenerate at \(\infty\) for \(X_A\). We obtain the family of Laurent polynomials

\[
F_{\gamma}(x, t) = \sum_{i=0}^{m} \gamma_i(x) t^{a_i}
\]

Take \(b \in \mathbb{Z}^n\) such that \(0\) is contained in the interior part of \(\text{Conv}(A \cup \{b\})\). It gives a monomial \(t^b\). We shall observe that the mixed twistor \(D\)-modules \(\pi_0^! \mathcal{T}(A, S, \gamma)\) are equipped with graded sesqui-linear dualities depending on the choice of \(b\).

Let \(\varphi_{\Sigma} : X_{\Sigma} \longrightarrow X_A\) be a toric desingularization. We have the meromorphic function \(t_{\Sigma}^b\) on \((X_{\Sigma}, D_{\Sigma})\) induced by \(t^b\). We also have the meromorphic functions \(F_{\gamma, \Sigma}\) and \(t_{\Sigma, S}^b\) on \((X_{\Sigma}, S, D_{\Sigma}, S)\) induced by \(F_{\gamma}\) and \(t^b\).

Lemma 6.37 Suppose that \(|(t^b_{\Sigma})_0| \cap |(t^b_{\Sigma})_\infty| = \emptyset\). Then, the other conditions in Lemma 2.4 are satisfied for \(g = F_{\gamma, \Sigma}\) and \(f = t_{\Sigma, S}^b\).
Proof By the non-degeneracy assumption on $\gamma, F_{\gamma, \Sigma}$ is non-degenerate along $D_{\Sigma, S}$. We clearly have $|t_{\Sigma, S}^m| \cup |(t_{\Sigma, S}^\infty)| \subset D_{\Sigma, S}$. It is enough to prove that $D_{\Sigma, S} = |t_{\Sigma, S}^m| \cup |(F_{\gamma, \Sigma})|$.

We set $a_{m+1} := b$ and $A := A \cup \{a_{m+1}\}$. We set $\tilde{S} := S \times \mathbb{C}_\tau$. We can naturally regard $H^0(\mathbb{P}^{m+1}, O(1)) = H^0(\mathbb{P}^{m}, O(1)) \times (\mathbb{C} \cdot z_{m+1})$. We consider the morphism $\tilde{\gamma} := \gamma \times id : \tilde{S} \rightarrow H^0(\mathbb{P}^{m+1}, O(1))$. It is enough to prove that $F_{\tilde{\gamma}}(x, \tau, t) = F_{\gamma}(x, t) + \tau t_{\Sigma, S}^\infty$ is non-degenerate at $\infty$ for $X_A$. Let $\sigma$ be any face of $\text{Conv}(A)$. If $b \not\in \sigma$, then $\sigma$ is a face of $\text{Conv}(A \cup \{0\})$ such that $0 \not\in \sigma$. Hence, we have $(dF_{\gamma, \sigma})^{-1}(0) = (dF_{\gamma, \sigma})^{-1}(0) = 0$ by applying Lemma 6.37 to $A$. If $b \in \sigma$, then $\partial_t F_{\gamma, \sigma} = \sigma^*_{\Sigma, S}$ is nowhere vanishing. Thus, we obtain the claim of the lemma.

We have the canonical polarization $((-1)^{n+d_\Sigma}((-1)^{n+d_\Sigma})$ on the pure twistor $D$-module $T_{(F_{\gamma, \Sigma}, D_{\Sigma, S})}$ of weight $n + d_\Sigma$ on $X_{\Sigma, S}$, which we denote by $S_{\gamma, \Sigma}$. By Lemma 6.37 we have

$$T_{*}(F_{\gamma, \Sigma}, D_{\Sigma, S}) = T_{*(F_{\gamma, \Sigma}, D_{\Sigma, S})}[(\sigma_{\Sigma, S})^\infty].$$

As explained in [4.2.3], we have the induced graded polarization $S_{\gamma, \Sigma}[\sigma_{\Sigma, S}]^\infty$ on $T_{*(F_{\gamma, \Sigma}, D_{\Sigma, S})}$, which we denote by $S_{\gamma, \Sigma, b}$. By Corollary 5.4 Condition (A) holds for the projection $\pi_{\Sigma, \gamma}$ and the mixed twistor $D$-modules $T_{*(F_{\gamma, \Sigma}, D_{\Sigma, S})}$ with $S_{\gamma, \Sigma, b}$, and we obtain the graded polarization $[\pi_{\Sigma, \gamma}]^*_{\gamma, \Sigma, b}$ on $T_{*(F_{\gamma, \Sigma}, D_{\Sigma, S})} \cong T_{*(A, S, \gamma)}$. We obtain the following lemma from Corollary 5.4.

Lemma 6.38 The graded polarizations $[\pi_{\Sigma, \gamma}]^*_{\gamma, \Sigma, b}$ on $\pi_{\Sigma, \gamma} T_{*(A, S, \gamma)}$ are independent of the choice of a toric desingularization $\varphi_{\Sigma}$. We denote them by $S_{\gamma}(A, S, \gamma, b)$.

If $0$ is contained in the interior part of $\text{Conv}(A)$, then $\pi_{\Sigma, \gamma} T_{*(A, S, \gamma)}$ is pure, and $S_{\gamma}(A, S, \gamma, b)$ is equal to $\pi_{\Sigma, \gamma}^* S_{\gamma}$ for any toric desingularization $\varphi_{\Sigma} : X_A \rightarrow X_A$.

6.2.5 The smooth part and the induced graded pairings

Recall that we have the open subset $U_{\gamma, \Sigma}^{\text{reg}}$ in $H^0(\mathbb{P}^{m}, O_{\mathbb{P}^{m}}(1))$. (See 6.13) For any $\gamma : S \rightarrow H^0(\mathbb{P}^{m}, O_{\mathbb{P}^{m}}(1))$, we set $S_{\gamma}^{\text{reg}} := \gamma^{-1}(U_{\gamma, \Sigma}^{\text{reg}})$.

Proposition 6.39 Let $A$ be the $R_{\Sigma}$-modules underlying $\pi_{\gamma} T$ for $T = T_{*(A, S, \gamma)} (\gamma = *, !, \min, K_{A, \Sigma, \gamma}$ and $C_{A, S, \gamma}$). Then, $M_{\gamma}^{\text{reg}}$ is a locally free $O_{C_{A} \times S_{\gamma}}$-module. In particular, the mixed twistor $D$-module $\pi_{\gamma} T_{\gamma}^{\text{reg}}$ comes from a graded polarizable variation of mixed twistor structure on $S_{\gamma}^{\text{reg}}$. It is admissible along $S \setminus S_{\gamma}^{\text{reg}}$.

Proof It follows from Corollary 6.24 and the general property of mixed twistor $D$-modules.

Corollary 6.40 The restriction of $[99]$ and $[100]$ to $S_{\gamma}^{\text{reg}}$ are locally free $O_{C_{A} \times S_{\gamma}}$-modules. Equivalently, the restriction of $[101], [102], [103]$ and $[104]$ to $S_{\gamma}^{\text{reg}}$ are locally free $O_{C_{A} \times S_{\gamma}}$-modules. The restriction of the image of $[105]$ to $S_{\gamma}^{\text{reg}}$ is also an $O_{C_{A} \times S_{\gamma}}$-module.

Let $\gamma : S \rightarrow H^0(\mathbb{P}^{m}, O_{\mathbb{P}^{m}}(1))$ be a morphism which is non-degenerate at $\infty$ for $X_A$. Suppose that (i) $S_{\gamma}^{\text{reg}} \neq \emptyset$, (ii) we are given a hypersurface $Y$ which contains $S \setminus S_{\gamma}^{\text{reg}}$. Take $b \in \mathbb{Z}^n$ such that $0$ is an interior point of $\text{Conv}(A \cup \{b\})$. We have the mixed twistor $D$-modules $\pi_{\gamma} T_{\gamma}^{\text{reg}}$ with the induced graded polarization $S_{\gamma}(A, S, \gamma, b)$ on $S$. They are also equipped with the natural real structure. We set

$$\mathcal{V}_{\gamma}(A, S, \gamma) := \pi_{\gamma}^* L_{\Sigma}(A, S, \gamma)(\gamma Y).$$

We obtain the filtration $\mathcal{V}_{\gamma}(A, S, \gamma)$ and the graded pairing $P_{\gamma}(A, S, \gamma, b) = (P_{\gamma}(A, S, \gamma, b)_k | k \in \mathbb{Z})$ by the procedure in [3.2.3]. We explain this in this situation. The $R$-modules $\pi_{\gamma}^* L_{\Sigma}(A, S, \gamma)$ are equipped with the filtration $W$ underlying the weight filtration of $\pi_{\gamma}^* T_{\gamma}(A, S, \gamma)$. We set

$$\mathcal{V}_{\gamma}(A, S, \gamma) := \left( W_{k+d_\Sigma} \pi_{\gamma}^* L_{\Sigma}(A, S, \gamma) \right)(\gamma Y).$$

The polarization and the real structure of $\text{Gr}_{k+d_\Sigma} W_{k+d_\Sigma} \pi_{\gamma}^* T_{\gamma}(A, S, \gamma)$ induce a pairing $P_{\gamma}(A, S, \gamma, k)$ of weight $k$ on $\text{Gr}_{k} W_{k+d_\Sigma} \mathcal{V}_{\gamma}(A, S, \gamma)$. Thus, we obtain a graded pairing $P_{\gamma}(A, S, \gamma) = (P_{\gamma}(A, S, \gamma, k) | k \in \mathbb{Z})$ on $(\mathcal{V}_{\gamma}(A, S, \gamma), \mathcal{W})$. In this way, we obtain mixed TEP structures $(\mathcal{V}_{\gamma}(A, S, \gamma), \mathcal{W}, P_{\gamma}(A, S, \gamma))$.
6.2.6 Algebraicity

Let $Z$ be any smooth complex quasi-projective variety. An integrable mixed twistor $\mathcal{D}$-module $(\mathcal{T}, W)$ on $Z$ is called algebraic if the following holds:

- Let $\overline{Z}$ be a smooth projective manifold with an open immersion $Z \rightarrow \overline{Z}$. Then, there exists an integrable mixed twistor $\mathcal{D}$-module $\mathcal{T}'$ on $\overline{Z}$ such that $\mathcal{T}'_{\overline{Z}} = \mathcal{T}$.

We also have the notion of the algebraicity of the underlying $\overline{\mathcal{R}}$-modules of integrable mixed twistor $\mathcal{D}$-modules. Let $\mathcal{D}_{\mathcal{A} \times Z}$ denote the sheaf of algebraic linear differential operators on $\mathcal{C}_\lambda \times Z$. Let $\Theta^a_Z$ denote the algebraic tangent sheaf of $Z$. Let $p : \mathcal{C}_\lambda \times Z \rightarrow Z$ denote the projection. Let $\overline{\mathcal{R}}^a_Z \subset \mathcal{D}_{\mathcal{C}_\lambda \times Z}$ denote the sheaf of subalgebras generated by $\lambda \cdot p^* \Theta_Z$ and $\lambda^2 \partial_\lambda$ over $\mathcal{O}_{\mathcal{C}_\lambda \times Z}$. We say that an $\overline{\mathcal{R}}_Z$-module $\mathcal{M}$ is algebraic if there exists an $\overline{\mathcal{R}}^a_Z$-module $\mathcal{M}^a$ such that $\mathcal{M}$ is isomorphic to the analytification of $\mathcal{M}^a$.

We naturally regard $H^0(\mathbb{P}^m, \mathcal{O}(1))$ as a quasi-projective variety. Let $\gamma : S \rightarrow H^0(\mathbb{P}^m, \mathcal{O}(1))$ be an algebraic morphism of smooth quasi-projective varieties.

**Proposition 6.41**

- The integrable mixed twistor $\mathcal{D}$-modules $\mathcal{T}_*(\mathcal{A}, S, \gamma)$ and $\pi_1^* \mathcal{T}_*(\mathcal{A}, S, \gamma)$ are algebraic. The underlying $\overline{\mathcal{R}}$-modules of $\mathcal{T}_*(\mathcal{A}, S, \gamma)$ and $\pi_1^* \mathcal{T}_*(\mathcal{A}, S, \gamma)$ are algebraic.

- For any toric desingularization $\varphi_\Sigma : X_\Sigma \rightarrow X_\mathcal{A}$, the integrable mixed twistor $\mathcal{D}$-modules $\mathcal{T}_*(F_{\gamma, S}, D_{\Sigma, S})$ are algebraic. The underlying $\overline{\mathcal{R}}$-modules of $\mathcal{T}_*(F_{\gamma, S}, D_{\Sigma, S})$ are algebraic.

**Proof** The claims for $\mathcal{T}_*(F_{\gamma, S}, D_{\Sigma, S})$ is clear. Then, we obtain the claims for the others by construction.

6.3 Description as a specialization

6.3.1 $\mathcal{D}$-modules

Let $\mathcal{A} = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n$ be a finite subset which generates $\mathbb{Z}^n$. We take $a_{m+1} \in \mathbb{Z}^n$ such that $0$ is contained in $\text{Conv}(\mathcal{A})$, where $\mathcal{A} = \mathcal{A} \cup \{a_{m+1}\}$. Let us compare the $\mathcal{D}$-modules associated to $\mathcal{A}$ and $\mathcal{A}$.

We identify $\mathbb{P}^m$ with the subspace of $\mathbb{P}^{m+1}$ determined by $z_{m+1} = 0$. We naturally regard $H^0(\mathbb{P}^{m+1}, \mathcal{O}(1)) = H^0(\mathbb{P}^m, \mathcal{O}(1)) \times \mathbb{C} : z_{m+1}$.

Let $\gamma : S \rightarrow H^0(\mathbb{P}^m, \mathcal{O}(1))$ be a morphism which is non-degenerate at $\infty$ for $X_\mathcal{A}$. We set $\overline{S} := S \times \mathbb{C}_\tau$. We have the induced map $\overline{\gamma} = \gamma \times \text{id} : \overline{S} \rightarrow H^0(\mathbb{P}^{m+1}, \mathcal{O}(1))$ given by $\overline{\gamma}(x, \tau) = \gamma(x) + \tau z_{m+1}$. Let $\pi : \mathbb{P}^m \times S \rightarrow S$ and $\overline{\pi} : \mathbb{P}^{m+1} \times \overline{S} \rightarrow \overline{S}$ denote the projections.

**Proposition 6.42**

- We have $L_* (\overline{\mathcal{A}}, \overline{S}, \overline{\gamma}) \simeq L_!(\overline{\mathcal{A}}, \overline{S}, \overline{\gamma}) \simeq L_{\text{min}}(\overline{\mathcal{A}}, \overline{S}, \overline{\gamma})$. We also have $\overline{\pi}_i^* L_* (\overline{\mathcal{A}}, \overline{S}, \overline{\gamma}) = 0$ for $i \neq 0$.

- Let $\iota : S \rightarrow \overline{S}$ be the inclusion induced by $\{0\} \rightarrow \mathbb{C}$. The kernel and the cokernel of the morphism

$$\overline{\pi}_i^0 L_{\text{min}}(\overline{\mathcal{A}}, \overline{S}, \overline{\gamma})(!\tau) \rightarrow \overline{\pi}_i^0 L_{\text{min}}(\overline{\mathcal{A}}, \overline{S}, \overline{\gamma})(*\tau)$$

are naturally isomorphic to $\iota_+ \overline{\pi}_i^0 L_*(\mathcal{A}, S, \gamma)$ and $\iota_+ \pi_i^0 L_*(\mathcal{A}, S, \gamma)$, respectively.

**Proof** Let $\varphi_\Sigma : X_\Sigma \rightarrow X_\mathcal{A}$ be a toric desingularization. We may assume to have a toric morphism $\varphi_{\Sigma, \mathcal{A}} : X_\Sigma \rightarrow X_\mathcal{A}$ which is a desingularization. For the meromorphic function $f = t^{m+1}$ on $X_\Sigma$, we may assume $|f|_0 \cap |f|_\infty = \emptyset$.

We have $F_{\overline{\Sigma}, \overline{\gamma}} = F_{S, \gamma} + \tau t^{m+1}$. We can check that the assumption in Lemma 2.19 is satisfied for $F_{S, \gamma}$ and $t^{m+1}$. Hence, we have the purity of $F_{\overline{\Sigma}, \overline{\gamma}}$ on $(X_\Sigma, \overline{\mathcal{S}}, D_{\Sigma, \mathcal{S}})$. Let $\overline{\pi}_\Sigma : X_\Sigma \rightarrow \overline{\mathcal{S}}$ be the projection. By using Proposition 2.4, we obtain $\overline{\pi}_\Sigma^* L(F_{\overline{\Sigma}, \overline{\gamma}}, D_{\Sigma, \mathcal{S}}) = 0$ for $i \neq 0$. Thus, we obtain the first claim of Proposition 6.42.
Let \( \iota_1 : X_{2,s} \rightarrow X_{2,s} \) denote the inclusion induced by \( \{0\} \rightarrow C \). According to Proposition 2.31, the kernel and the cokernel of the morphism \( L(F_{\tilde{S}, \gamma}, D_{2,s})(t) \rightarrow L(F_{\tilde{S}, \gamma}, D_{2,s})(s) \) are naturally isomorphic to \( \iota_{1+L}(F_{S, \gamma}, D_{2,s}) \) and \( \iota_{1+L}(F_{S, \gamma}, D_{2,s}) \), respectively.

Let \( \iota_2 : X_2 \times S \rightarrow S \) be the projection. By Proposition 4.18, we have \( \pi_i^0 L_*(F_{S, \gamma}, D_{2,s}) = 0 \) for \( i \neq 0 \). We also have the vanishing in the first claim. Then, we obtain the second claim of Proposition 6.42.

We have the canonical nilpotent map \( N \) on \( \psi_+ \pi_i^0 L_*(\tilde{A}, \tilde{S}, \gamma) \simeq \psi_+ \pi_i^0 L_*(\tilde{A}, \tilde{S}, \gamma) \) induced by \( t \partial_t \).

**Corollary 6.43** We have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ker } N & \rightarrow & \text{Cok } N \\
\simeq & & \simeq \\
\pi_i^0 L_*(A, S, \gamma) & \rightarrow & \pi_i^0 L_*(A, S, \gamma).
\end{array}
\]

The lower horizontal arrow is the natural morphism.

**Proof** It follows from Proposition 2.38.

### 6.3.2 Graded polarized mixed twistor \( D \)-modules

Let us show the relations between the mixed twistor \( D \)-modules associated \((A, S, \gamma)\) and \((\tilde{A}, \tilde{S}, \tilde{\gamma})\), although we have already used it in 6.2.4 implicitly.

**Proposition 6.44**

- We have \( T_*(\tilde{A}, \tilde{S}, \tilde{\gamma}) \simeq T_*(\tilde{A}, \tilde{S}, \tilde{\gamma}) \simeq T_{\min}(\tilde{A}, \tilde{S}, \tilde{\gamma}) \), and \( \pi_i^0 T_{\min}(\tilde{A}, \tilde{S}, \tilde{\gamma}) = 0 \) for \( i \neq 0 \).

- Let \( K(\tilde{A}, \tilde{S}, \tilde{\gamma}) \) and \( C(\tilde{A}, \tilde{S}, \tilde{\gamma}) \) denote the kernel and the cokernel of the following morphism of the mixed twistor \( D \)-modules:

\[
\pi_i^0 T_*(\tilde{A}, \tilde{S}, \tilde{\gamma})[\tau] \rightarrow \pi_i^0 T_*(\tilde{A}, \tilde{S}, \tilde{\gamma})[\tau].
\]

Then, we have the following natural isomorphisms:

\[
K(\tilde{A}, \tilde{S}, \tilde{\gamma}) \simeq \iota_1 \pi_i^0 T_*(A, S, \gamma), \quad C(\tilde{A}, \tilde{S}, \tilde{\gamma}) \simeq \iota_1 \pi_i^0 T_*(A, S, \gamma) \otimes T(-1).
\]

The isomorphisms are compatible with the real structures.

**Proof** We obtain the first claim from Proposition 6.42. As in the proof of Proposition 6.42 by using the vanishing of the cohomology in Proposition 6.40 and the first claim of this proposition, we obtain the isomorphisms from Proposition 6.24. We can easily compare the real structures by using Proposition 3.18.

On the other hand, we have the graded polarizations \( S_*(A, S, \gamma, a_{m+1}) \) on \( \pi_i^0 T_*(A, S, \gamma) \) as explained in 6.2.4. They induce graded sesqui-linear dualities on \( \iota_1 \pi_i^0 T_*(A, S, \gamma) \) and \( \iota_1 \pi_i^0 T_*(A, S, \gamma) \otimes T(-1) \). On the other hand, as explained in 6.2.4, the pure twistor \( D \)-module \( \pi_i^0 T_{\min}(\tilde{A}, \tilde{S}, \tilde{\gamma}) \) is equipped with the induced polarization. It induces the graded polarizations \( S_K \) and \( S_C \) of \( K(\tilde{A}, \tilde{S}, \tilde{\gamma}) \) and \( C(\tilde{A}, \tilde{S}, \tilde{\gamma}) \), as explained in 6.2.4.

**Proposition 6.45** The isomorphisms in Proposition 6.42 are compatible with the graded polarizations.

**Proof** We use the notation in the proof of Proposition 6.42. Let \( K_1 \) and \( C_1 \) be the kernel and the cokernel of the morphism of the mixed twistor \( D \)-modules \( T_*(F_{\gamma, \Sigma}, D_{2,s}) \rightarrow T_*(F_{\gamma, \Sigma}, D_{2,s}) \). According to Proposition 6.2, the isomorphisms \( K_1 \simeq \iota_1 T_*(F_{\gamma, \Sigma}, D_{2,s}) \) and \( C_1 \simeq \iota_1 T_*(F_{\gamma, \Sigma}, D_{2,s}) \) are compatible with the graded sesqui-linear dualities. Then, the claims of the propositions follow from Corollary 4.9 and the construction of the graded polarizations.

Let \( N : \tilde{\psi}_{\tau, -\delta} \pi_i^0 T_*(\tilde{A}, \tilde{S}, \tilde{\gamma}) \otimes U(-1, 0) \rightarrow \tilde{\psi}_{\tau, -\delta} \pi_i^0 T_*(\tilde{A}, \tilde{S}, \tilde{\gamma}) \otimes U(0, -1) \) be the morphism given by the pair \((-t \partial_t, -t \partial_t)\).
Corollary 6.46 We have the following commutative diagram:
\[
\begin{align*}
\text{Ker}(\mathcal{N}) & \longrightarrow \text{Cok}(\mathcal{N}) \otimes T(1) \\
\simeq & \quad \simeq \\
\pi^0_\tau \mathfrak{T}(\mathcal{A}, S, \gamma) & \longrightarrow \pi^0_\tau \mathfrak{T}_s(\mathcal{A}, S, \gamma)
\end{align*}
\] (106)

The horizontal arrows are the natural morphisms.

Proof Because \( \iota \cdot \text{Cok}(\mathcal{N}) \) and \( \iota \cdot \text{Ker}(\mathcal{N}) \) are naturally isomorphic to \( \mathcal{C}(\tilde{\mathcal{A}}, \tilde{S}, \tilde{\gamma}) \) and \( \mathcal{K}(\tilde{\mathcal{A}}, \tilde{S}, \tilde{\gamma}) \), we obtain the vertical isomorphisms in (106) from Proposition 6.45. The commutativity of the diagram follows from Corollary 6.46.

6.3.3 The induced mixed TEP-structures

We set \( \mathcal{V} := \pi^0_\tau \mathcal{L}(\tilde{\mathcal{A}}, \tilde{S}, \tilde{\gamma})(\ast \tau) \). Let \( \lambda \mathcal{N} : \psi_{\tau,-\delta}(\mathcal{V}) \longrightarrow \psi_{\tau,-\delta}(\mathcal{V}) \) be the morphism induced by \( \lambda \tau \partial_\tau \). We restate the result in [B.3] in this situation.

Assumption 6.47 \( \mathcal{V} \) is regular singular along \( \tau = 0 \). There exists an open subset \( \tilde{B} \) in \( \tilde{S} \) such that \( \mathcal{V}|_{\tilde{B}} \) is a locally free \( \mathcal{O}_{\mathcal{C}_\lambda \times \tilde{B}}(\ast \tau) \)-module.

We set \( B := \iota^{-1}(\tilde{B}) \) which is an open subset in \( S \). Under the assumption \( \text{Cok}(\lambda \mathcal{N})|_B \) is a locally free \( \mathcal{O}_{\mathcal{C}_\lambda \times B} \)-module. We also have the pairing \( P\mathring{\mathcal{A}}_{k} \) of weight \( n \) on \( \mathcal{V}|_B \) induced by the real structure and the polarization of \( \pi^0_\tau \mathcal{L}(\tilde{\mathcal{A}}, \tilde{S}, \tilde{\gamma}) \). By the procedure in [B.1.3] we have the weight filtration \( \mathcal{W}(1) \) and the graded pairing \( (P^{(1)} \mid k \in \mathbb{Z}) \) on \( \text{Cok}(\lambda \mathcal{N})|_B \) and \( \text{Ker}(\lambda \mathcal{N})|_B \).

Note that \( \text{Cok}(\lambda \mathcal{N}) \) is the underlying \( \mathcal{R} \)-module of \( \text{Cok}(\mathcal{N}) \otimes T(1) \simeq \pi^0_\tau \mathcal{L}_s(\mathcal{A}, S, \gamma) \), i.e., \( \text{Cok}(\lambda \mathcal{N}) \) is isomorphic to \( \pi^0_\tau \mathcal{L}_s(\mathcal{A}, S, \gamma) \). As explained in [6.2.5] we obtain the weight filtration \( \mathcal{W}(2) \) and the graded pairing \( (P^{(2)} \mid k \in \mathbb{Z}) \) on \( \text{Cok}(\lambda \mathcal{N})|_B \) from the real structure and the graded polarization of \( \pi^0_\tau \mathcal{L}(\mathcal{A}, S, \gamma) \). Similarly, \( \text{Ker}(\lambda \mathcal{N}) \) is the underlying \( \mathcal{R} \)-module of \( \text{Ker}(\mathcal{N}) \simeq \pi^0_\tau \mathfrak{T}(\mathcal{A}, S, \gamma) \), i.e., \( \text{Ker}(\lambda \mathcal{N}) \) is isomorphic to \( \pi^0_\tau \mathfrak{T}(\mathcal{A}, S, \gamma) \). As explained in [6.2.5] we have the filtration \( \mathcal{W}(2) \) and the graded pairing \( (P^{(2)} \mid k \in \mathbb{Z}) \) on \( \text{Ker}(\lambda \mathcal{N}) \).

Proposition 6.48 On \( \text{Cok}(\lambda \mathcal{N}) \) and \( \text{Ker}(\lambda \mathcal{N}) \), we have \( \mathcal{W}(1) = \mathcal{W}(2) \) and \( P^{(1)}_{\mathring{\mathcal{A}}_k} = P^{(2)}_{\mathring{\mathcal{A}}_k} \) for any \( k \in \mathbb{Z} \).

Proof It follows from Proposition [B.6] and the isomorphisms \( \text{Cok}(\mathcal{N}) \otimes T(1) \simeq \pi^0_\tau \mathfrak{T}_s(\mathcal{A}, S, \gamma) \) and \( \text{Ker}(\mathcal{N}) \simeq \pi^0_\tau \mathfrak{T}(\mathcal{A}, S, \gamma) \) in Corollary 6.46.

6.4 A regular singular case

6.4.1 The \( \mathcal{D} \)-modules

Let \( \mathcal{A} = \{ a_1, \ldots, a_m \} \) be as in [6.1.1] Suppose that there exists \( \alpha \in (\mathbb{Z}^n)^\vee \) such that \( \alpha(a_i) = 1 \) for any \( a_i \in \mathcal{A} \). Then, it is well known that the associated GKZ-systems are regular singular. In [3], [27] and [56], the \( \mathcal{D} \)-modules are described in terms of the Gauss-Manin connection for the relative cohomology groups of some families. We review it in the language of \( \mathcal{D} \)-modules, which fits to the theory of mixed twistor \( \mathcal{D} \)-modules.

With an appropriate choice of a frame of \( \mathbb{Z}^n \), we may assume that \( a_i = (b_i, 1) \in \mathbb{Z}^{n-1} \times \mathbb{Z} \) for each \( a_i \in \mathcal{A} \). Let \( \mathcal{B} \subset \mathbb{Z}^{n-1} \) denote the image of \( \mathcal{A} \) via the projection of \( \mathbb{Z}^{n-1} \times \mathbb{Z} \longrightarrow \mathbb{Z}^{n-1} \). In other words, we have \( \mathcal{A} = \{ (b, 1) \mid b \in \mathcal{B} \} \). For simplicity, we impose the following in this subsection.

Assumption 6.49 We assume that \( \mathcal{B} \) generates \( \mathbb{Z}^{n-1} \), and \( 0 \) is contained in the interior part of \( \text{Conv}(\mathcal{B}) \).

Let \( X_\mathcal{B} \) denote the closure of the image of the morphism \( \psi_\mathcal{B} : T^{n-1} \longrightarrow \mathbb{P}^m \) as in [6.1.1]. We set \( W := \left\{ \sum_{i=1}^{m} \alpha_i z_i \mid \alpha_i \in \mathbb{C} \right\} \subset H^0(\mathbb{P}^m, \mathcal{O}(1)) = \left\{ \sum_{i=0}^{m} \alpha_i z_i \right\} \). We use the notation in [6.1.3].
Lemma 6.50 Let $\gamma : S \rightarrow W$ be a morphism of complex manifolds. The following conditions are equivalent.

- $\gamma$ is non-degenerate at $\infty$ for $X_A$.
- If $\gamma = \sum_{i=1}^{m} \gamma_i z_i$, the family of Laurent polynomials $F_{\gamma,B} = \sum_{i=1}^{m} \gamma_i t^{b_i} \text{ is Conv}(B)$-regular in the sense that $F_{\gamma,B,\sigma}^{-1}(0) \cap (dF_{\gamma,B,\sigma})^{-1}(0) = \emptyset$ for any face $\sigma$ of Conv($B$).

Proof We have the family of Laurent polynomials $F_{\gamma} = \sum \gamma_i t^{a_i}$ associated to $A$ and $\gamma$. For any face $\sigma$ of $B$, we have the face $\sigma(A)$ of $A$ given by $\sigma(A) := \{ (1, c) \mid c \in \sigma \}$. We have $F_{\gamma,\sigma(A)} = t_n F_{\gamma,B,\sigma}$ and $dF_{\gamma,\sigma(A)} = t_n dF_{\gamma,B,\sigma} + dt_n F_{\gamma,B,\sigma}$. We have $F_{\gamma,B,\sigma}^{-1}(0) = F_{\gamma,\sigma(A)}^{-1}(0)$ and $dF_{\gamma,\sigma(A)}^{-1}(0) = F_{\gamma,B,\sigma}^{-1}(0) \cap (dF_{\gamma,B,\sigma})^{-1}(0)$. Then, the claim is clear.

Let $\gamma : S \rightarrow W$ be any holomorphic map. Let us study the $D$-modules $L_s(A,S,\gamma)$ and $\pi_+^* L_s(A,s,\gamma)$. It is convenient to consider a toric compactification which is different from that in [6.1.]. We set $X_B^0 := X_B \times \mathbb{P}^1$ which is naturally a toric variety. We take a toric desingularization $\phi$. We take a toric desingularization $\phi$ which is naturally a toric variety. We take a toric desingularization $\phi$ which is naturally a toric variety.

Lemma 6.50 We have natural isomorphisms $\pi_+^* L_s^0(B,S,\gamma) \simeq \pi_+^* L_s(A,S,\gamma)$.

Proof We take a toric desingularization $\phi : X_{\Sigma_1} \rightarrow X_A$. We may assume that we have a toric morphism $\rho : X_{\Sigma_1} \rightarrow X_{\Sigma_0}^0$ which is birational. We have natural isomorphisms $\rho_* L_+(F_{\gamma,\Sigma_1},D_{\Sigma_1},s) \simeq L_+(F_{\gamma,\Sigma_0},D_{\Sigma_0},s)$.

They induce the desired isomorphisms.

The family of Laurent polynomials $F_{\gamma}$ is described as $F_{\gamma} = t_n f_{\gamma}$, $f_{\gamma} := \sum_{i=1}^{m} \gamma_i t^{b_i}$.

Let $Z_{f_{\gamma}} \subset X_{\Sigma_0}$ denote the zero set of $f_{\gamma}$. Suppose that $\gamma$ is non-degenerate at $\infty$ for $X_A$. As remarked in Lemma 6.50, the family of Laurent polynomials $f_{\gamma}$ is Conv($B$)-regular. According to Proposition 2.23 we have

$$\pi_0^0 L_s(A,S,\gamma) \simeq \pi_0^0 L_+(F_{\gamma,\Sigma_0},D_{\Sigma_0},s) \simeq \pi_0^0 (\mathcal{O}_{X_{\Sigma_0},s}(*D_{\Sigma_0},s)).$$ (107)

$$\pi_0^0 L_+(A,S,\gamma) \simeq \pi_0^0 L_+(F_{\gamma,\Sigma_0},D_{\Sigma_0},s) \simeq \pi_0^0 (\mathcal{O}_{X_{\Sigma_0},s}(*D_{\Sigma_0},s)).$$ (108)

We have $\pi_0^0 (\mathcal{O}_{X_{\Sigma_0},s}(*Z_{f_{\gamma}},(*D_{\Sigma_0},s))) = 0$ and $\pi_0^0 (\mathcal{O}_{X_{\Sigma_0},s}(*Z_{f_{\gamma}})(*D_{\Sigma_0},s))) = 0$ for $i \neq 0$, which follow from $\pi_0^0 L_+(A,S,\gamma) = 0$ for $i \neq 0$.

If moreover the image of $\gamma$ is contained in $W \cap U_{\text{reg}}$, the $D_S$-modules (107) and (108) are flat bundles on $S$. The fiber of (107) over $s \in S$ is the relative cohomology group $H^{n-1}(T^{n-1},Z_{f_{\gamma}(s)})$ with $\mathbb{C}$-coefficient, where $Z_{f_{\gamma}(s)} := Z_{f_{\gamma}(s)} \cap T^{n-1}$. The fiber of (108) over $s \in S$ is $H^{n-1}(X_{\Sigma_0} \setminus Z_{f_{\gamma}(s)},D_{\Sigma_0} \setminus Z_{f_{\gamma}(s)})$.

Remark 6.52 Under the identification $X_{\Sigma_0} = \mathbb{P}^1_s \times X_{\Sigma_0}$, as remarked in [2.4.5] (107) is also naturally isomorphic to $\pi_0^0 (\mathcal{O}_{X_{\Sigma_0},s}(*Z_{f_{\gamma}(s)})(*D_{\Sigma_0},s)))$. 


6.4.2 The mixed twistor $D$-modules

We continue to use the notation in (6.4.1). Suppose that $\gamma$ is non-degenerate at $\infty$ for $X_\mathcal{A}$. We have the mixed twistor $D$-modules $\mathcal{T}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)$ $(\ast = +, 1)$ on $X_{\Sigma_0, s}$. We obtain the mixed twistor $D$-modules on $\mathbb{P}^m \times \mathbb{P}^1 \times S$:

$$
\mathcal{T}_*(B, S, \gamma) := \mathcal{T}_{\Sigma_0, S} \mathcal{T}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)
$$

They are independent of the choice of a toric desingularization $\varphi_{\Sigma_0} : X_{\Sigma_0} \to X_B$. We have the natural isomorphisms $\pi_{\Sigma_0}^* \mathcal{T}_*(B, S, \gamma) \simeq \pi_{\Sigma_0}^* \mathcal{T}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s) \ast = +, 1)$. As in the case of $D$-modules, we have the following.

Lemma 6.53 We have natural isomorphisms $\pi_{\Sigma_0}^* \mathcal{T}_*(B, S, \gamma) \simeq \pi_{\Sigma_0}^* \mathcal{T}_*(A, S, \gamma)$ for $\ast = +, 1$.

Lemma 6.54 The $\mathcal{R}$-modules $\mathcal{L}_*(B, S, \gamma)$ and $\mathcal{L}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)$ are $\mathbb{C}^*$-homogeneous. (See (3.6.6) for the notion of homogeneity.)

Proof We consider the $\mathbb{C}^*$-action on $T^n$ given by $a \cdot (t_1, \ldots, t_{n-1}, t_n) = (a t_1, \ldots, t_{n-1}, a t_n)$. It induces a $\mathbb{C}^*$-action on $X_{\Sigma_0, s}$. We have $a f_{\gamma, \Sigma_0} = a f_{\gamma, \Sigma_0}$. By Proposition 3.30 the $\mathcal{R}$-modules $\mathcal{L}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)$ are $\mathbb{C}^*$-homogeneous. Then, we obtain that $\mathcal{L}_*(B, S, \gamma)$ are also $\mathbb{C}^*$-homogeneous.

Let $q_S : \mathbb{P}^m \times \mathbb{P}^1 \times S \to \mathbb{P}^m \times S$ and $q_{\Sigma_0} : X_{\Sigma_0, s} \to X_{\Sigma_0, s}$ be the projections.

Corollary 6.55 The $\mathcal{R}$-modules $q_S^* \mathcal{L}_*(B, S, \gamma)$ and $q_{\Sigma_0}^* \mathcal{L}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)$ are $\mathbb{C}^*$-homogeneous for the trivial $\mathbb{C}^*$-actions on $\mathbb{P}^m \times S$ and $X_{\Sigma_0, s}$.

Proof Because $q_S$ and $q_{\Sigma_0}$ are $\mathbb{C}^*$-equivariant, the first claim follows from Lemma 6.54. The second follows from Lemma 3.28.

Corollary 6.56 The $\mathcal{R}$-modules $\pi_{\Sigma_0}^0 \mathcal{L}_*(B, S, \gamma)$ and $\pi_{\Sigma_0}^0 \mathcal{L}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)$ are $\mathbb{C}^*$-homogeneous for the trivial $\mathbb{C}^*$-actions on $S$. In particular, the underlying $D$-modules are equipped with the good filtrations such that the analytification of the $\mathcal{R}$-modules are isomorphic to $q_S^0 \mathcal{L}_*(B, S, \gamma)$ and $q_{\Sigma_0}^0 \mathcal{L}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s)$.

Proposition 6.57 Suppose that $\gamma : S \to W$ is non-degenerate at $\infty$ for $X_\mathcal{A}$. Then, we have the following natural isomorphisms of the integrable mixed twistor $D$-modules compatible with the real structure:

$$
\pi_{\Sigma_0}^0 \mathcal{T}_*(A, S, \gamma) \simeq \pi_{\Sigma_0}^0 \mathcal{T}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s) \cap (\mathcal{U}_{X_{\Sigma_0, s}}(n + d_{\Sigma_0, s}) \ast [D_{\Sigma_0, s}])
$$

Thus, we obtain the following isomorphisms:

$$
\pi_{\Sigma_0}^0 \mathcal{T}_*(A, S, \gamma) \simeq \pi_{\Sigma_0}^0 \mathcal{T}_*(F_\gamma, \Sigma_0, D_{\Sigma_0}, s) \cap (\mathcal{U}_{X_{\Sigma_0, s}}(n + d_{\Sigma_0, s}) \ast [D_{\Sigma_0, s}])
$$

Thus, we obtain the claim of the proposition.

Let $M$ be the pure Hodge module on $X_{\Sigma_0, S}$ of weight dim $S + n - 1$, corresponding to the constant sheaf. Let $M(-1)$ be the $(-1)$-th Tate twist of $M$. We obtain the mixed Hodge module $\pi_{\Sigma_0}^0 M(-1) \ast [D_{\Sigma_0, S}]$ which induces the mixed twistor $D$-module in the right hand side of (109). We also have the mixed Hodge module $\pi_{\Sigma_0}^0 M \ast [D_{\Sigma_0, S}]$. By Lemma 3.28 we obtain the following.

68
Corollary 6.58 The Hodge filtrations of $\pi^0_{\Sigma_0}[M(-1)[Z_{f_s}] + D_{\Sigma_0, s}]$ and $\pi^0_{\Sigma_0}[M[Z_{f_s}] + D_{\Sigma_0, s}]$ are equal to the good filtrations in Corollary 6.56.

Let $\iota: Z_{f_s} \rightarrow X_{\Sigma_0, s}$ be the inclusion of the complex manifolds. We have the following natural morphisms of integrable mixed twistor $D$-modules on $S$:

$$\pi^0_{\Sigma_0}[U_{X_{\Sigma_0, s}}(n + dS - 1, 0)[Z_{f_s}] + D_{\Sigma_0, s}] \xrightarrow{a_1} \pi^0_{\Sigma_0}[U_{Z_{f_s}}(n + dS - 1, -1)]$$

$$\pi^0_{\Sigma_0}[U_{X_{\Sigma_0, s}}(n + dS, -1)[Z_{f_s}] + D_{\Sigma_0, s}]$$

(113)

Proposition 6.59 We have $\pi^0_{\Sigma_0}[U_{Z_{f_s}}(n + dS, -1)] = \text{Im} a_1 \oplus \text{Ker} a_2$. We also have

$$\text{Gr}^W_{n + dS} \left( \pi^0_{\Sigma_0}[U_{X_{\Sigma_0, s}}(n + dS - 1, 0)[Z_{f_s}] + D_{\Sigma_0, s}] \right) \simeq \text{Im} a_1$$

$$\simeq \text{Gr}^W_{n + dS} \left( \pi^0_{\Sigma_0}[U_{X_{\Sigma_0, s}}(n + dS, -1)[Z_{f_s}] + D_{\Sigma_0, s}] \right)$$

(114)

Proof It follows from Corollary 6.31.

6.4.3 Graded polarizations and induced pairings

Let $b = (0, -1) \in \mathbb{Z}^{n-1} \times \mathbb{Z}$. Note that $0$ is contained in the interior part of $\text{Conv}(A \cup \{b\})$.

Let $\gamma : S \rightarrow W$ be a morphism which is non-degenerate at infinity for $X_A$. As explained in 6.2.4, we obtain graded polarizations $S_\gamma(A, S, \gamma, b)$ of $\pi^0_{\Sigma_0}T_\gamma(A, S, \gamma)$. Suppose moreover that the image of $\gamma$ is contained in $W \cap U_{\Sigma_0}$. Then, $V_\gamma^\prime(B, S, \gamma) := \pi^0_{\Sigma_0}L_\gamma(F_{\gamma, \Sigma_0}, D_{\Sigma_0})$ is locally free $O_{C_\lambda \times S}$-modules. Let $W$ denote the filtration of $V_\gamma^\prime(B, S, \gamma)$ underlying the weight filtration of $\pi^0_{\Sigma_0}T_\gamma(F_{\gamma, \Sigma_0}, D_{\Sigma_0})$. We set $\widetilde{W}_kV_\gamma^\prime(B, S, \gamma) := W_{dS + k}V_\gamma^\prime(B, S, \gamma)$. Then, the polarization and the real structure of $\text{Gr}^W_{dS + k}\pi^0_{\Sigma_0}T_\gamma(F_{\gamma, \Sigma_0}, D_{\Sigma_0})$ induce a pairing $\widetilde{P}_k$ of weight $k$ on $\text{Gr}^W_{dS + k}V_\gamma^\prime(B, S, \gamma)$, as explained in 6.2.5. Let us give a description of the pairings $\widetilde{P}_k$ in the case $*=*$. We have the following isomorphism:

$$V_\gamma^\prime(B, S, \gamma) \simeq \pi^0_{\Sigma_0}[O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s}]$$

(115)

Note that $O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s}$ is equipped with the filtration $W$ which underlies the weight filtration of $U_{X_{\Sigma_0, s}}(n + dS, -1)[Z_{f_s}] + D_{\Sigma_0, s}]$. By using the spectral sequence, we have the following complex

$$\pi^1_{\Sigma_0}\text{Gr}^{W}_{dS + n + k + 1} \left( \lambda^{-1}O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s} \right) \xrightarrow{a_k}$$

$$\pi^0_{\Sigma_0}\text{Gr}^{W}_{dS + n + k} \left( \lambda^{-1}O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s} \right) \xrightarrow{a_k}$$

$$\pi^1_{\Sigma_0}\text{Gr}^{W}_{dS + n + k - 1} \left( \lambda^{-1}O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s} \right)$$

(116)

and the cohomology is isomorphic to $\text{Gr}^W_{n + k}V_\gamma^\prime(B, S, \gamma)$. The real structure and the polarization of

$$\pi^0_{\Sigma_0}\text{Gr}^{W}_{dS + n + k} U_{X_{\Sigma_0, s}}(n + dS, -1)[Z_{f_s}] + D_{\Sigma_0, s}]$$

induces a pairing $\widetilde{P}_{n+k}$ of weight $n+k$ on $\pi^0_{\Sigma_0}\text{Gr}^{W}_{dS + n + k} \left( \lambda^{-1}O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s} \right)$. The pairing $\widetilde{P}_{n+k}$ induces the following isomorphism:

$$\Phi_{\widetilde{P}_{n+k}} : \pi^0_{\Sigma_0}\text{Gr}^{W}_{dS + n + k} \left( \lambda^{-1}O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s} \right) \simeq \lambda^{-n-k}\pi^0_{\Sigma_0}\text{Gr}^{W}_{dS + n + k} \left( \lambda^{-1}O_{C_\lambda \times X_{\Sigma_0, s}}[Z_{f_s}] + D_{\Sigma_0, s} \right)$$

\[\Box\]
As the dual of (116), we have the following:

\[ \pi_{\Sigma_0}^1 \mathcal{G}r_{d_{S_0} + n + k - 1}^W \left( \lambda^{-1} \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \right) \xrightarrow{\beta^\vee} \pi_{\Sigma_0}^0 \mathcal{G}r_{d_{S_0} + n + k}^W \left( \lambda^{-1} \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \right) \xrightarrow{\alpha^\vee} \pi_{\Sigma_0}^{-1} \mathcal{G}r_{d_{S_0} + n + k + 1}^W \left( \lambda^{-1} \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \right) \]  \quad (117)

We obtain the subsheaf \( \ker \beta \cap \ker (\alpha \circ \Phi_{\tilde{P}_{n+k}}) \) in \( \mathcal{P}_{n+k} \). Condition (A) for \( \pi_{\Sigma_0} \) and \( \mathcal{U}_{X_{\Sigma_0}} \) is satisfied with the graded polarization. According to Proposition 6.60, it is enough to consider the case where \( \pi_{\Sigma_0} \cap \mathcal{U}_{X_{\Sigma_0}} = 0 \). Let \( \mathcal{F} \) be the Hodge filtration on \( \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \boxtimes \mathcal{O}_{|S_d|} \). We have the following natural isomorphism:

\[ \mathcal{G}r_{n+k}^W \left( \lambda^{-1} \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \right) \cong \bigoplus_{I \in \Lambda(k)} I^{|I|} \left( \lambda^{-k} \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \right) \]

Let \( F^* \) be the Hodge filtration on \( H^*(D_{\Sigma_0}, I) \) which is a decreasing filtration. We set \( F^j H^m(D_{\Sigma_0}, I) := F^{-j} H^m(D_{\Sigma_0}, I) \). Let \( R_F H^m(D_{\Sigma_0}, I) \) be the Rees module:

\[ R_F H^m(D_{\Sigma_0}, I) := \sum F^j H^m(D_{\Sigma_0}, I) \lambda^{-j} = \sum F^j H^m(D_{\Sigma_0}, I) \lambda^j \]

The associated \( \mathcal{O}_{X_{\Sigma_0}} \)-module is also denoted by the same notation. We have the following natural isomorphism:

\[ \pi_{\Sigma_0}^1 \mathcal{G}r_{n+k}^W \left( \lambda^{-1} \mathcal{O}_{X_{\Sigma_0}} \boxtimes \mathcal{O}_{|Z_f|} \right) \cong \bigoplus_{I \in \Lambda(k)} I^{|I|} \left( \lambda^{-k} R_F H^{n-k+i}(D_{\Sigma_0}, I) \right) \bigoplus \bigoplus_{I \in \Lambda(k)} I^{|I|} \left( \lambda^{-k-1} R_F H^{n-2-k+i}(Z_f, I) \right) \]  \quad (119)

We obtain the complex (110) by applying the Rees construction to the following exact sequence of pure Hodge structures:

\[ \bigoplus_{I \in \Lambda(k)} H^{n-k}(D_{\Sigma_0}, I) \otimes \mathbb{Q}(-k) \bigoplus_{I \in \Lambda(k+1)} H^{n-k-2}(Z_f, I) \otimes \mathbb{Q}(-k-1) \to \bigoplus_{I \in \Lambda(k-1)} H^{n-k}(D_{\Sigma_0}, I) \otimes \mathbb{Q}(-k) \bigoplus_{I \in \Lambda(k)} H^{n-2-k}(Z_f, I) \otimes \mathbb{Q}(-k-1) \to \bigoplus_{I \in \Lambda(k-2)} H^{n-k+2}(D_{\Sigma_0}, I) \otimes \mathbb{Q}(-k+1) \bigoplus_{I \in \Lambda(k)} H^{n-k}(Z_f, I) \otimes \mathbb{Q}(-k) \]  \quad (120)
Here, $Q(i)$ denote the $i$-th Tate object in the category of Hodge structures. We can obtain the complex \(120\) by using the theory of mixed Hodge modules \(52\), or more explicitly by using the theory of mixed Hodge complexes (see \(38\)).

We have the ordinary Poincaré pairing on \(P_I\) on \(H^{n-|I|}(D_{S_0,I})\) and \(P_{Z,f,I}\) on \(H^{n-|I|-2}(Z_{f,I})\), i.e.,

\[
P_I(a,b) = \int_{D_{S_0,I}} ab, \quad P_{Z,f,I}(a,b) = \int_{Z_{f,I}} ab.
\]

For \(i \in \Lambda\), let \(p_i(f)\) be the pole order of \(f\) along \(D_i\). For any integer \(\ell\), let \(\epsilon(\ell) := (-1)^{(\ell-1)/2}\). We set

\[
\tilde{P}_I := \epsilon(n-|I|) \cdot \prod_{i \in I} p_i(f)^{-1} \cdot P_I, \quad \tilde{P}_{Z,f,I} := \epsilon(n-2-|I|) \cdot \prod_{i \in I} p_i(f)^{-1} \cdot P_{Z,f,I}.
\]

We have the pairing of weight \(n-|I|\) on \(R_F H^{n-|I|}(D_{S_0,I})\) induced by \(\tilde{P}_I\), and the pairing of weight \(n-|I|-2\) on \(R_F H^{n-|I|-2}(Z_{f,I})\) induced by \(\tilde{P}_{Z,f,I}\). The induced pairings are also denoted by \(\tilde{P}_I\) and \(\tilde{P}_{Z,f,I}\) respectively.

**Proposition 6.61** We have \(\tilde{P}_{n+k} = \bigoplus_{I \in \Lambda(k-1)} \tilde{P}_I \oplus \bigoplus_{I \in \Lambda(k)} \tilde{P}_{Z,f,I}\).

**Proof** It follows from Proposition \(4.16\) and Proposition \(3.10\).

### 6.5 Mixed TEP-structures on reduced GKZ-hypergeometric systems

#### 6.5.1 Preliminary

As in \(6.4.1\) we consider \(A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n\) which generates \(\mathbb{Z}^n\). We set \(N_A := \mathbb{Z}^n\) and \(M_A := \mathbb{Z}^m\). Let \(e_1, \ldots, e_m\) be the standard basis of \(M_A\). We have the surjective morphism \(\Xi_A : M_A \twoheadrightarrow N_A\). Let \(L_A := \ker \Xi_A\). We obtain the exact sequence

\[
0 \longrightarrow L_A \overset{\Theta_A}{\longrightarrow} M_A \overset{\Xi_A}{\longrightarrow} N_A \longrightarrow 0.
\]

By taking the dual, we obtain the exact sequence \(0 \longrightarrow N_A^\vee \overset{\Xi_A^\vee}{\longrightarrow} M_A^\vee \overset{\Theta_A^\vee}{\longrightarrow} L_A^\vee \longrightarrow 0\). For a finitely generated free abelian group \(A\), we set \(A_{C^*} := C^* \otimes_{\mathbb{Z}} A\). We can naturally regard it as a complex algebraic variety or a complex manifold. Particularly, we set \(S_A := L_A^\vee\). We have the natural surjection \(\Theta_A^\vee : M_A^\vee_{C^*} \twoheadrightarrow S_A\).

We have the action \(\rho_1\) of \(N_{A,C^*}\) on \(M_A^\vee\), induced by \(-\phi\). We also have the natural action \(\rho_0\) of \(N_{A,C^*}\) on itself by the multiplication. We can naturally regard \(\Theta_A : M_A^\vee_{C^*} \twoheadrightarrow S_A\) as the quotient of the projection \(N_{A,C^*} \times M_A^\vee_{C^*} \twoheadrightarrow M_A^\vee_{C^*}\) via the above actions of \(N_{A,C^*}\).

The identifications \(M_A = \mathbb{Z}^m\) and \(N_A = \mathbb{Z}^n\) induce the coordinate systems \((w_1, \ldots, w_m)\) on \(M_A^\vee_{C^*}\) and \((t_1, \ldots, t_n)\) on \(N_{A,C^*}\). We set \(G_A := \sum_{i=1}^m w_i\) on \(M_A^\vee_{C^*}\). The algebraic function \(F_A = \sum_{i=1}^m w_i t_i\) on \(N_{A,C^*}\) is \(N_{A,C^*}\)-invariant, and \(G_A\) is the descent of \(F_A\). Any splitting \(\gamma_A : L_A^\vee \twoheadrightarrow M_A^\vee\) of \(\Theta_A^\vee\) induces an algebraic morphism \(\gamma_A : L_A^\vee_{C^*} \twoheadrightarrow M_A^\vee_{C^*}\), and we obtain \(L_A^\vee_{C^*} \times M_A^\vee_{C^*} \twoheadrightarrow M_A^\vee_{C^*}\). The splitting \(\gamma_A\) also gives an isomorphism \(M_A^\vee \simeq N_A^\vee \times L_A^\vee\), and hence \(M_A^\vee_{C^*} \simeq N_{A,C^*} \times L_{A,C^*}\). The pull back of \(F_A\) by \(id \times \gamma_A\) is equal to \(G_A\) under the identification.

#### 6.5.2 Mixed twistor \(D\)-modules and the induced mixed TEP-structures

We naturally regard \(M_A^\vee_{C^*} = (C^*)^m = \{ \sum_{i=1}^m a_i z_i \mid a_i \neq 0 \}\), and \(N_{A,C^*} = T^n\) in \(6.1.1\). Let \(\gamma_A : L_A^\vee_{C^*} \twoheadrightarrow M_A^\vee_{C^*}\) be the splitting as above. Note that it is non-degenerate at \(\infty\) for \(X_A\), as remarked in Example \(6.14\). We obtain the following integrable mixed twistor \(D\)-modules with real structure on \(S_A\):

\[
\mathcal{T}_{A,*} := \pi_A^* T_\gamma (A,S_A,\gamma_A) \quad (* = \ast, !).
\]

The underlying \(D\)-modules are the reduced GKZ-hypergeometric systems. As in Lemma \(6.3\), \(\mathcal{T}_{A,*}\) are independent of the choice of a splitting \(\gamma_A\).
We set $T_{A,\min} := \pi^0_{\min}(A, S_A, \gamma_A)$. By Corollary \[6.31\] it is isomorphic to the image of the natural morphism $T_{A,!} \to T_{A,*}$, and $Gr^m W T_{A,*} = Gr^m W T_{A,!} = Gr^m W T_{A,\min} = T_{A,\min}$. If $0$ is contained in the interior part of $\text{Conv}(A)$, we have $T_{A,!} = T_{A,*} = T_{A,\min}$.

We set $L_{A,*} := \mathcal{L}_*(A, S_A, \gamma_A)$ and $\nu_{A,*} := \pi^0_1 L_{A,*}$. The $\mathcal{R}$-modules $\nu_{A,*}$ underlie $T_{A,*}$. More precisely, $T_{A,*}$ is expressed as a pair of $\mathcal{R}$-modules $\lambda^m \nu_{A,*}$ and $\nu_{A,*}$ with the induced sesqui-linear pairing, and $T_{A,!*}$ is expressed as a pair of $\mathcal{R}$-modules $\lambda^m \nu_{A,*}$ and $\nu_{A,*}$ with the induced sesqui-linear pairing. Let $W$ denote the filtration of $\nu_{A,*}$ underlying the weight filtration of $T_{A,*}$. We set $\nu_k \nu_{A,*} := \nu_{k+n-m} \nu_{A,*}$. Note dim $S_A = m - n$.

Let $L_{A,*}$ be the image of $L_{A,!} \to L_{A,*}$, and set $\nu_{A,\min} := \pi^0_1 L_{A,\min}$. It is naturally isomorphic to the image of $\nu_{A,!*} \to \nu_{A,*}$ by Corollary \[6.31\]. The $\mathcal{R}$-module $\nu_{A,\min}$ underlies $T_{A,\min}$. We have $Gr^m W \nu_{A,*} \simeq Gr^m W \nu_{A,*} \simeq \nu_{A,\min}$.

If $0$ is an interior point of $\text{Conv}(A)$, then $T_{A,*} = T_{A,\min} = T_{A,!*}$ is pure of weight $m$, and it is equipped with the canonical polarization. As explained in Proposition \[6.2.3\] even if $0$ is not a point of $\text{Conv}(A)$, we have the graded polarizations $S_{A,*}, b$ of $T_{A,*}$ depending on the choice of $b \in \mathbb{Z}^n$ such that $0$ is an interior point of $\text{Conv}(A \cup \{b\})$. The weight $m$-part of $S_{A,*}, b$ are independent of $b$.

Let $S_{A,*} := \mathbb{Z}^{-1}_\mathcal{R}(U_{\mathcal{R}})$. The restriction $\nu_{A,*}|_{S_{A,*}^\mathcal{R}}$ are locally free $O_{C_A \times S_{A,*}^\mathcal{R}}$-modules. Take a hypersurface $Y$ such that $Y \subset S_A \setminus S_{A,*}^\mathcal{R}$. The real structure and the graded polarization of $S_{A,*}$ induce a graded pairing $P_{A,*}, b$ of $(\nu_{A,*}(\ast Y), W)$. In this way, we obtain mixed TEP-structures $(\nu_{A,*}(\ast Y), W, P_{A,*}, b)$.

Remark 6.62 Let $\mathbb{C}[M_A]$ denote the group ring of $M_A$ over $\mathbb{C}$. We may naturally regard $M'_{A,C,*}$ as the algebraic variety $\text{Spec} \mathbb{C}[M_A]$. Then, $T_{A,*}$ is algebraic in the sense of \[6.2.6\].

Remark 6.63 If $0$ is contained in the interior part of $\text{Conv}(A)$, we shall often omit the subscripts $*, !$ because $T_{A,*} = T_{A!*} = T_{A,\min}$.

6.5.3 Homogeneity

Let $e^1, \ldots, e^m$ denote the dual basis of $M'_A$. We have the morphism $Z \to M'_A$ given by $1 \mapsto v := \sum_{i=1}^m e^i$. It induces a $\mathbb{C}^*$-action on $M'_{A,C,*}$. By the composition $Z \to M'_A \to L'_A$, we obtain a $\mathbb{C}^*$-action on $S_A$. The map $\Theta'_A : M'_{A,C,*} \to S_A$ is $\mathbb{C}^*$-equivariant. For the $\mathbb{C}^*$-action, we have $a^* G_A = a \cdot G_A$ for any $a \in \mathbb{C}^*$.

We consider the action of $\mathbb{C}^*$ on $C_A$ given by the multiplication. For the diagonal $\mathbb{C}^*$-action on $C_A \times M'_{A,C,*}$, we have $a^*(\lambda^{-1} G_A) = \lambda^{-1} G_A$ for any $a \in \mathbb{C}^*$.

Lemma 6.64 The $\mathcal{R}$-modules $\nu_{A,*}$ on $S_A$ are $\mathbb{C}^*$-homogeneous in the sense of \[3.6.1\].

Proof Take any toric desingularization $\varphi_\Sigma : X_\Sigma \to X_A$. We have the $\mathbb{C}^*$-action on $X_\Sigma \times S_A$ which is the extension of $T^\Sigma \times S_A \simeq M'_{A,C,*}$. For the action, we have $a^*(\lambda^{-1} F_{\gamma_A, \Sigma}) = \lambda^{-1} F_{\gamma_A, \Sigma}$.

We set $Y := X_\Sigma \times S_A$ and $D_Y := D_\Sigma \times S_A$. By Proposition \[3.39\] the $\mathcal{R}$-modules $\mathcal{L}_*(F_{\gamma_A, \Sigma}, D_Y)$ are $\mathbb{C}^*$-homogeneous. Then, $\nu_{A,*} \simeq \pi^0_1 \mathcal{L}_*(F_{\gamma_A, \Sigma}, D_Y)$ are also $\mathbb{C}^*$-homogeneous. It is easy to check that the $\mathbb{C}^*$-actions are independent of the choice of $\varphi_\Sigma$.

6.5.4 Variation of Hodge structure

Let us consider the special case $\Theta'_A(v) = 0$. Note that the condition $\Theta'_A(v) = 0$ is equivalent to the standard criterion for the G-KZ-system to be regular singular. Indeed, we have $\Theta'_A(v) = 0$ if and only if there exists $\alpha \in N'_A$ such that $\alpha(\Xi_A(e_i)) = 1$ ($i = 1, \ldots, m$). If $\Theta'_A(v) = 0$, we have $\alpha \in N'_A$ such that $\Xi'_A(\alpha)(v)$, and hence we have $(\alpha, \Xi_A(e_i)) = (\Xi_A(\alpha), e_i) = (v, e_i) = 1$. Conversely, suppose that there exists $\alpha \in N'_A$ such that $\alpha(\Xi_A(e_i)) = 1$. Because $\Xi'_A(\alpha), e_i = 1$, we have $\Xi'_A(\alpha) = v$. As a result, we obtain the following.

Lemma 6.65 If $\Theta'_A(v) = 0$, the mixed twistor $\mathcal{D}$-modules $T_{A,*}$ are regular singular.

Let $M_{A,*}$ be the $\mathcal{D}$-module on $S_A$ underlying $T_{A,*}$, i.e., $M_{A,*} := \nu_{A,*}/(\lambda - 1) \nu_{A,*}$. Because the $\mathbb{C}^*$-action on $S_A$ is trivial, we have the good filtration $F$ on $M_{A,*}$ such that the analytification of the Rees module $R(M_{A,*}, F)$ is isomorphic to $\nu_{A,*}$, as remarked in Lemma \[3.28\].
Proposition 6.66 \( T_{A,+}|^{S_{A}^{\text{reg}}} \) comes from a graded polarizable variation of mixed Hodge structure on \( S_{A}^{\text{reg}} \). For any algebraic embedding \( S_{A}^{\text{reg}} \subset Z \), the graded polarizable variation of mixed Hodge structure is admissible along \( Z \setminus S_{A}^{\text{reg}} \).

**Proof** As remarked in Proposition 6.39 \( T_{A,+}|^{S_{A}^{\text{reg}}} \) comes from a graded polarizable variation of mixed twistor structure. The sesqui-linear pairings of the underlying \( R \)-modules are preserved by the action of \( S^{1} = \{ \lambda \in \mathbb{C}^{*} \mid |\lambda| = 1 \} \). Hence, the patching of the \( R_{X} \)-module \( \lambda^{m} V_{A} \) and \( \sigma^{n} V_{A} \) is compatible with the \( \mathbb{C}^{*} \)-action. By the correspondence of variations of mixed Hodge structure and variation of mixed twistor structure with \( \mathbb{C}^{*} \)-action, we obtain that the variation of mixed twistor structure comes from a variation of mixed Hodge structure.

By the algebraicity in Proposition 6.41 and the general property of mixed twistor \( D \)-modules, the variations of mixed twistor structure are admissible along \( Z \setminus S_{A}^{\text{reg}} \). Hence, we easily obtain the admissibility of the variation of mixed Hodge structure.

6.5.5 Appendix: Comparison with the construction of Reichelt-Sevenheck

As in [D.2] we set \( V := H^{0}(\mathbb{P}^{m}, \mathcal{O}(1)) \). Let \( Z \subset \mathbb{P}^{m} \times V \) be the 0-set of the universal section of \( \mathcal{O}_{\mathbb{P}^{m}}(1) \otimes \mathcal{O}_{V} \). We decompose \( V = V_{1} \times V_{2} \), where \( V_{1} := \{ (\alpha_{2}, 0) \mid \alpha_{2} \in \mathbb{C} \} \) and \( V_{2} := \{ \sum_{i=1}^{m} \alpha_{i} z_{i} \mid \alpha_{i} \in \mathbb{C} \} \). We identify \( V_{2} = M_{\lambda}^{\text{reg}} \otimes \mathbb{C} \).

We have the splitting \( \gamma_{A} : S_{A} = L_{\lambda}^{\text{reg}} \otimes \mathbb{C}^{*} \rightarrow M_{\lambda}^{\text{reg}} \otimes \mathbb{C}^{*} \subset V_{2} \). We obtain \( \text{id} \times \gamma_{A} : V_{1} \times S_{A} \rightarrow V_{1} \times V_{2} \). Let \( Z_{A} \) be the fiber product of \( Z \) and \( V_{1} \times S_{A} \) over \( V_{1} \times V_{2} \). We have the naturally induced morphisms \( q_{1} : Z_{A} \rightarrow \mathbb{P}^{m} \) and \( q_{2} : Z_{A} \rightarrow V_{1} \times S_{A} \).

Recall that we have the morphism \( \psi_{A} : T^{n} \rightarrow \mathbb{P}^{m} \) induced by \( A \). Let \( U := \psi_{A}(T^{n}) \). We set \( Z_{A,U} := Z_{A} \times_{\mathbb{P}^{m}} U \). Let \( \iota_{Z_{A,U}} : Z_{A,U} \rightarrow \mathbb{P}^{m} \times (V_{1} \times S_{A}) \) be the inclusion.

Let us consider the pure Hodge module \( (\mathcal{O}_{Z_{A,U}}, F) \) and the mixed Hodge modules \( t_{Z_{A,U}}(\mathcal{O}_{Z_{A,U}}, F) \) as in [D.2.3]. Let \( M^{IC}(Z_{A,U}) \) denote the image of the morphism \( t_{Z_{A,U}}(\mathcal{O}_{Z_{A,U}}, F) \rightarrow t_{Z_{A,U}}(\mathcal{O}_{Z_{A,U}}, F) \). Let \( \pi_{V_{1} \times S_{A}} : \mathbb{P}^{m} \times (V_{1} \times S_{A}) \rightarrow V_{1} \times S_{A} \) be the projection. Then, we obtain the mixed Hodge modules \( \pi_{V_{1} \times S_{A}}^{*}(\mathcal{O}_{Z_{A,U}}, F) \) and \( \pi_{V_{1} \times S_{A}}^{*}(M^{IC}(Z_{A,U})) \). By applying the procedure in [D.1.1] we obtain the following \( \mathcal{R}_{S_{A}} \)-modules:

\[
\begin{align*}
G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(\mathcal{O}_{Z_{A,U}}, F)), & \quad G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(M^{IC}(Z_{A,U}))).
\end{align*}
\]

**Proposition 6.67** We have isomorphisms of \( \mathcal{R}_{X} \)-modules:

\[
\lambda \cdot V_{A} \simeq G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(\mathcal{O}_{Z_{A,U}}, F))
\]

We also have the following commutative diagram:

\[
\begin{array}{ccc}
\lambda \cdot V_{A!} & \simeq & G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(\mathcal{O}_{Z_{A,U}}, F)) \\
\downarrow & & \downarrow \\
\lambda \cdot V_{A} & \simeq & G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(\mathcal{O}_{Z_{A,U}}, F))
\end{array}
\]

**Proof** We can obtain the claim from Proposition D.6 by using the non-characteristic pull back. We can also prove it directly by the argument in the proof of Proposition D.6.

**Corollary 6.68** We naturally have \( \lambda \cdot V_{A,\min} \simeq G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(M^{IC}(Z_{A,U}))) \).

**Proof** Reichelt-Sevenheck proved that \( G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(M^{IC}(Z_{A,U}))) \) is naturally isomorphic to the image of \( G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(\pi_{V_{1} \times S_{A}}^{*}(M^{IC}(Z_{A,U}))) \rightarrow G_{0} \mathcal{F}_{L_{S_{A}}^{\text{loc}}}(V_{A!} \rightarrow V_{A}, \text{Corollary 6.31}) \). As mentioned in [6.5.2] \( V_{A,\min} \) is naturally isomorphic to the image of \( V_{A!} \rightarrow V_{A}, \) by Corollary 6.31. Hence, we have the desired isomorphism.
7 Quantum \( \mathcal{D} \)-modules

7.1 Some mixed twistor \( \mathcal{D} \)-modules in mirror symmetry

7.1.1 Toric data

Let \( X \) be an \( n \)-dimensional smooth weak Fano toric variety. Let \( \Sigma \) be a fan of \( X \). Let \( \Sigma(1) = \{ \rho_1, \ldots, \rho_m \} \) denote the set of the 1-dimensional cones in \( \Sigma \). Let \( [\rho_i] \in \mathbb{Z}^n \) be the primitive generator of \( \rho_i \cap \mathbb{Z}^n \).

Let \( K_X \) denote the canonical bundle of \( X \). Let \( L_j \) (\( j = 1, \ldots, r \)) be nef line bundles on \( X \) such that \((K_X \otimes \bigotimes_{j=1}^r L_j)^n\) is nef. We may assume that \( L_j = \mathcal{O}(\sum_{i=1}^m \beta_{ji} D_i) \) (\( j = 1, \ldots, r \)) for some \( \beta_{ji} \in \mathbb{Z}_{\geq 0} \), where \( D_i \) are the hypersurfaces corresponding to the cones \( \rho_i \). Let \( \mathbb{Z}^{n+r} = \mathbb{Z}^n \oplus \mathbb{Z}^r \). Let \( n_1, \ldots, n_r \) denote the standard basis of \( \mathbb{Z}^r \). We set \( a_i = \mathbb{Z}^n \oplus \mathbb{Z}^r \) as follows:

\[
a_i := \begin{cases}
[r_i] + \sum_{j=1}^r \beta_{ji} n_j & (i = 1, \ldots, m), \\
n_{i-m} & (i = m + 1, \ldots, m + r).
\end{cases}
\]

We put \( A := \{a_1, \ldots, a_{r+m}\} \). We also set \( a_{r+m+1} := -\sum_{i} n_j \) and \( \tilde{A} := A \cup \{a_{r+m+1}\} \). Note that 0 is contained in the interior part of the convex hull of \( \text{Conv}(A) \). (We shall use \( \tilde{A} \) in \[7.2\].)

7.1.2 \( \tilde{\mathcal{R}} \)-modules with filtration and graded pairing associated to \( A \)

Applying the construction in \[6.3\] we obtain the integrable mixed twistor \( \mathcal{D} \)-modules \( \mathcal{T}_{A,*}(\ast = \ast, 1) \) with real structure on \( S_A \). They are equipped with the graded polarization \( S_{A,*}a_{r+m+1} \). Let \( \mathcal{V}_{A,*} \) denote their underlying \( \tilde{\mathcal{R}} \)-modules. They are \( \mathbb{C}^* \)-homogeneous as in \[6.3.3\]. By the procedure in \[8.2.2\] we obtain the filtration \( \tilde{W} \) and the graded pairing \( (P_{A,*}A_k | k \in \mathbb{Z}) \) on \( V_{A,*}(S^r_{\mathbb{Z}}) \).

We apply the same construction to \( \mathcal{T}_{A,*} \otimes T(n + r) \). Let \( \mathfrak{M}_{A,*} \) be the underlying \( \tilde{\mathcal{R}} \)-modules. They are \( \mathbb{C}^* \)-homogeneous. The restriction \( \mathfrak{M}_{A,*}(S^r_{\mathbb{Z}}) \) is equipped with the filtration \( \tilde{W} \) and the graded pairings \( \mathcal{P}_{A,*} := (\mathcal{P}_{A,*}A_k | k \in \mathbb{Z}) \). Thus, we obtain mixed TEP-structures \( (\mathcal{P}_{A,*}, \tilde{W}, \mathcal{P}_{A,*}) \) on \( S^r_{\mathbb{Z}} \).

By construction, we have \( \mathfrak{M}_{A,*} = \lambda^{n+r} \mathcal{V}_{A,*} \) and \( \tilde{W}_k \mathfrak{M}_{A,*} = \lambda^{n+r} \tilde{W}_{k+2(n+r)} \mathcal{V}_{A,*} \). We have \( \mathfrak{M}_{k\tilde{W}} \mathfrak{M}_{A,*} = \lambda^{n+r} \mathfrak{M}_{k\tilde{W}k+2(n+r)} \mathcal{V}_{A,*} \). The graded pairings on \( \mathfrak{M}_{k\tilde{W}} \mathfrak{M}_{A,*}(\ast \lambda) = \mathfrak{M}_{k\tilde{W}k+2(n+r)} \mathcal{V}_{A,*}(\ast \lambda) \) are equal.

7.1.3 Comparison

Let \( [z_0 : \cdots : z_{m+r}] \) be the homogeneous coordinate system on \( \mathbb{P}^{m+r} \). Let \( W^* := \{\sum_{i=1}^{m+r} \alpha_i z_i \} \subset H^0(\mathbb{P}^{m+r}, \mathcal{O}(1)) = M_A^0 \otimes \mathbb{C}^* \). The splitting \( S_A = L_A^0 \otimes \mathcal{O} \to M_A^0 \otimes \mathbb{C}^* \) induces a morphism \( \varphi : S^r_{\mathbb{Z}} \to W^* \). We set \( W^r_{\mathbb{Z}} := W^* \cap U^r_{\mathbb{Z}} \). We have \( \varphi(S^r_{\mathbb{Z}}) \subset W^r_{\mathbb{Z}} \). There exists the involution \( \omega' \) on \( M_A^0 \otimes \mathbb{C}^* = (\mathbb{C}^*)^{m+r} \) given by \( \omega'(z_1, \ldots, z_{m+r}) = (z_1, \ldots, z_m, -z_{m+1}, \ldots, -z_{m+r}) \). It induces an involution \( \omega \) on \( S_A \).

In \[11\] \( \S 6 \), by enhancing GKZ-hypergeometric systems, Reicheit and Sevenheck constructed \( \tilde{\mathcal{R}} \)-modules \( \tilde{\mathcal{N}}_{A}^{\mathbb{P}^{m+r}} \) and \( \tilde{\mathcal{M}}_{A}^{-(\mathbb{P}^{m+r})} \) on \( W^* \) with a morphism of \( \tilde{\mathcal{R}} \)-modules \( \tilde{\varphi} : \tilde{\mathcal{N}}_{A}^{\mathbb{P}^{m+r}} \to \tilde{\mathcal{M}}_{A}^{-(\mathbb{P}^{m+r})} \).

Remark 7.1 We remark that “\( \mathcal{R}_{\mathcal{L}_X} \)-module” in \[11\] means “\( \tilde{\mathcal{R}} \)-module” in this paper.

The restrictions \( \tilde{\mathcal{N}}_{A}^{\mathbb{P}^{m+r}} \) and \( \tilde{\mathcal{M}}_{A}^{-(\mathbb{P}^{m+r})} \) are smooth, i.e., locally free \( \mathcal{O}_{\mathcal{L}_X \times \mathbb{P}^{m+r}} \)-modules. We take the pull back of them as \( \mathcal{O} \)-modules by the morphism \( \text{id} \times \varphi : \mathcal{L}_X \times S^r_{\mathbb{Z}} \to \mathcal{L}_X \times W^r_{\mathbb{Z}} \). Then, the pull back \( (\text{id} \times \varphi)^* \) and \( (\text{id} \times \varphi)^* \) are equipped with the induced meromorphic flat connection. Mann-Mignon \[31\] and Reichelt-Sevenheck \[41\] proved that \( \mathcal{Q} \mathcal{M}_{A}^{\mathbb{P}^{m+r}} := \omega^* (\text{id} \times \varphi)^* \) plays an important role in the mirror symmetry for the quantum \( \mathcal{D} \)-modules of complete intersections in the weak Fano toric variety \( X \). It is one of our main purpose in this paper to complement on their study by giving a twistor description of the quantum \( \mathcal{D} \)-modules. Namely, to understand \[41\] Conjecture 6.13, we prove the following.
Theorem 7.2 We have the following commutative diagram:

\[
\begin{array}{ccc}
(id \times g) \cdot (\tilde{\mathcal{M}}_{A}^{(-r,0,0)}) & \xrightarrow{\cong} & \lambda^{-r} \mathcal{O}_{A_{I}}|_{S_{A}^{\text{reg}}} \\
(id \times g) \cdot \tilde{\phi} & \downarrow & \\
(id \times g) \cdot (\mathcal{M}_{A}^{(-r,0,0)}) & \xrightarrow{\cong} & \lambda^{-r} \mathcal{O}_{A_{I}}|_{S_{A}^{\text{reg}}}
\end{array}
\]

(121)

Here, the right vertical arrow is the canonical morphism. As a result, we have \( \omega^{*}(\lambda^{-r} \mathcal{O}_{A_{I}}|_{S_{A}^{\text{reg}}}) \).

The proof of this theorem will be given in [4.1.4, 4.1.8].

Corollary 7.3 \((id \times g)^{*}(\tilde{\mathcal{M}}_{A}^{(0,0,0)})\) and \((id \times g)^{*}(\mathcal{M}_{A}^{(-r,0,0)})\) underlie integrable mixed twistor \(\mathcal{D}\)-modules, and \(\omega^{*}(\mathcal{O}_{A_{I}}|_{S_{A}^{\text{reg}}})\) underlies an integrable pure twistor \(\mathcal{D}\)-module.

With the notation in the last part of [6.5.5] we obtain the following from Corollary 6.68.

Corollary 7.4 \(\text{Im } \tilde{\phi} \) is isomorphic to \(\lambda^{-1}G_{0}^{\mathcal{L}_{\text{loc}}(\varphi_{1} \times S_{A_{I}}, \mathcal{M}^{IC}(\mathcal{Z}_{\mathcal{A}_{I}}))} \).

Corollary 7.5 Let \(r = 1\) and \(\mathcal{L}_{1} = K_{\mathcal{X}}\). Then, \(\text{Im } \tilde{\phi}\) is described as in Proposition 6.59 in terms of polarized variation of Hodge structure associated to the zero set of the family of Laurent polynomials associated to the set \(\mathcal{B}\) which is related with \(\mathcal{A}\) as in §0.4.3.

7.1.4 Preliminary (1)

Let us recall some constructions and commutative diagrams in [11]. Set \(V := H^{0}(\mathbb{P}^{m+r}, \mathcal{O}(1))\), and let \(V^{\vee}\) be the dual space. Set \(A^{1} := \{a_{0}, a_{1}, \ldots, a_{m+r}\} \subseteq \mathbb{Z} \times \mathbb{Z}^{m+r}\), where \(a_{0} = (1,0) \in \mathbb{Z} \times \mathbb{Z}^{m+r}\) and \(a_{1} = (1, a_{i}) \in \mathbb{Z} \times \mathbb{Z}^{m+r}\). We have the \(\mathcal{D}_{V}\)-modules \(\mathcal{M}_{A_{1}}^{(-r,0,0)}\) and \(\mathcal{M}_{A_{1}}^{(-2r,0,0)}\) associated to the GKZ-hypergeometric systems as in [11] §2.2. We shall use the following lemma later.

Lemma 7.6 Any endomorphism of \(\mathcal{M}_{A_{1}}^{(-2r,0,0)}\) is the multiplication of a complex number.

Proof We regard \(\mathbb{P}^{m+r}\) as the quotient space of \(V^{\vee}\setminus \{0\}\) by the natural \(\mathcal{C}^{*}\)-action. We have the coordinate system \((z_{0}, z_{1}, \ldots, z_{m+r})\) on \(V^{\vee}\) corresponding to the homogeneous coordinate system \((z_{0}, z_{1}, \ldots, z_{m+r})\). Set \(T^{n+r} := \{(t_{0}, t_{1}, \ldots, t_{n+r}) \mid t_{i} \in \mathbb{C}^{*}\}\) and \(T^{n+r+1} := \{(t_{0}, t_{1}, \ldots, t_{n+r+1}) \mid t_{i} \in \mathbb{C}^{*}\}\). We have the morphism \(h: T^{n+r+1} \longrightarrow V^{\vee}\) given by \(h(t_{0}, t_{1}, \ldots, t_{n+r+1}) = (t_{0}, t_{1}^{a_{1}}, \ldots, t_{n+r}^{a_{m+r}})\). Let \(\mathcal{L}_{V}\) denote the Fourier transform from \(\mathcal{D}_{V}\)-modules to \(\mathcal{D}_{V}\)-modules. According to [11] Lemma 2.10, Proposition 5.1, we have a natural isomorphism \(\mathcal{L}_{V}\) isomorphic to \(\mathcal{M}^{(-2r,0,0)}\). Then, the claim is clear.

We decompose \(V = V_{1} \times W\), where \(V_{1} = \{a_{0}z_{0} \mid a_{0} \in \mathbb{C}\}\) and \(W = \{\sum_{i=1}^{m+r} a_{i}z_{i} \mid a_{i} \in \mathbb{C}\}\). We have the family of the Laurent polynomials \(\Phi_{A} = \sum_{i=1}^{m+r} a_{i}t^{a_{i}}\) on \(T^{n} \times W\). We have the algebraic morphism \(\Phi_{A}: T^{n+r} \times W \longrightarrow \mathbb{C} \times W = V_{1} \times W\) induced by \(-\Phi_{A}\) and \(\text{id}_{W}\). By using the Radon transforms, Reichelt-Sevenheck constructed the following commutative diagram of \(\mathcal{D}_{V}\)-modules [11] Theorem 2.11, Corollary 5.5]:

\[
\begin{array}{ccc}
\mathcal{M}_{A_{1}}^{(-2r,0,0)} & \longrightarrow & \mathcal{H}^{0}(\Phi_{A}\mathcal{O}_{T^{n+r} \times W}) \\
\downarrow & & \downarrow \\
\mathcal{M}_{A_{1}}^{(-r,0,0)} & \longleftarrow & \mathcal{H}^{0}(\Phi_{A}\mathcal{O}_{T^{n+r} \times W})
\end{array}
\]

(122)

Let \(\mathcal{L}_{W}\) denote the Fourier-Laplace transform over \(W\), which gives a functor from the category of \(\mathcal{D}_{V}\)-modules to the category of \(\mathcal{D}_{V_{1}}\times W\)-modules. (See [11]) Let \(\tau\) be the coordinate system of \(V_{1}\) obtained as the dual of \(a_{0}\). Let \(\mathcal{L}_{W}^{\text{loc}}\) be the localized Fourier-Laplace transform, i.e., \(\mathcal{L}_{W}^{\text{loc}}(M) := \mathcal{L}_{W}(M)|_{C_{\tau}}\) for any \(\mathcal{D}_{V}\)-modules \(M\).
We have the following commutative diagrams on $\mathbb{C}^*_x \times W$ induced by \([122]\), and according to \([41]\) Proposition 3.3, Corollary 5.5], the horizontal arrows are isomorphisms:

\[
\begin{align*}
\text{FL}_{W}^{\text{loc}} \mathcal{M}_{A}^{-\lfloor 2r \frac{d}{2} \rfloor} & \xrightarrow{\cong} \text{FL}_{W}^{\text{loc}} \mathcal{H}^{0} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W} \\
\text{FL}_{W}^{\text{loc}} \mathcal{M}_{A}^{-\lfloor r \frac{d}{2} \rfloor} & \xrightarrow{\cong} \text{FL}_{W}^{\text{loc}} \mathcal{H}^{0} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W}
\end{align*}
\]

(123)

We also have \(\text{FL}_{W}^{\text{loc}} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W} \simeq \text{FL}_{W}^{\text{loc}} \mathcal{H}^{0} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W}\).

### 7.1.5 Preliminary (2)

Set \(\lambda = \tau^{-1}\). Let \(i_{W} : W \rightarrow H^{0}(\mathbb{P}^{m+r}, \mathcal{O}(1))\) be the inclusion. We naturally regard \(\pi_{1}^{0} \mathcal{L}_{i}(A, W, i_{W})|_{\mathbb{C}^*_x \times W}\) as \(\mathcal{D}_{\mathbb{C}^*_x \times W}\)-modules.

**Lemma 7.7** We have the following commutative diagram:

\[
\begin{align*}
\text{FL}_{W}^{\text{loc}} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W} & \xrightarrow{\cong} \pi_{1}^{0} \mathcal{L}_{i}(A, W, i_{W})|_{\mathbb{C}^*_x \times W} \\
\text{FL}_{W}^{\text{loc}} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W} & \xrightarrow{\cong} \pi_{1}^{0} \mathcal{L}_{i}(A, W, i_{W})|_{\mathbb{C}^*_x \times W}
\end{align*}
\]

(124)

**Proof** Let \(\mathbb{P}_{V_{i}}\) be the projective completion of \(V_{i}\). We take a smooth complex algebraic variety \(Y\) with an open immersion \(\iota : T^{n+r} \times W \subset Y\) and a projective morphism \(q : Y \rightarrow \mathbb{P}_{V_{i}} \times W\) such that (i) \(q \circ \iota = \Phi_{A}\), (ii) \(D_{Y} := Y \setminus (T^{n+r} \times W)\) is normal crossing. The composite of \(q\) and the projection \(\mathbb{P}_{V_{i}} \times W \rightarrow \mathbb{P}_{V_{i}}\) naturally gives a meromorphic function \(F_{Y}\) on \((Y, D_{Y})\). We set \(F_{Y} := -F_{Y_{1}}\). Because \([F_{Y}]_{0} \cap [F_{Y}]_{\infty} = 0\), \(F_{Y}\) is non-degenerate along \(D_{Y}\). We set \(Y^{(1)} := \mathbb{C}^{*}_{x} \times Y\) and \(D^{(1)} := \mathbb{C}^{*}_{x} \times D_{Y}\). Set \(D_{Y}^{(0)} := [F_{Y}]_{\infty}\).

We have the \(\mathcal{D}\)-modules \(L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y})\) on \(Y^{(1)}\). The image \(L(\lambda^{-1}F_{Y})\) of the canonical morphism \(L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y}) \rightarrow L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y})\) is given by \(\mathcal{O}_{Y^{(1)}}(\ast D^{(1)}_{Y})\) with \(d + d(\lambda^{-1}F_{Y})\).

We naturally have \(L(\lambda^{-1}F_{Y}) \simeq (\text{id}_{\mathbb{C}^{*}_{x}} \times q)^{*} L(-\lambda^{-1}\alpha_{0})\). We also have \(L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y}) = \mathcal{O}_{Y^{(1)}}(\ast D^{(1)}_{Y}) \otimes L(\lambda^{-1}F_{Y})\) by Lemma \([2.41]\). Then, we naturally have

\[
(\text{id}_{\mathbb{C}^{*}_{x}} \times q)^{*} L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y}) \simeq (\text{id}_{\mathbb{C}^{*}_{x}} \times q)^{*} L(\lambda^{-1}\alpha_{0}).
\]

(125)

Let \(\pi : \mathbb{C}^{*}_{x} \times V_{i} \times W \rightarrow \mathbb{C}^{*}_{x} \times W\) be the projection. We obtain the following from \([122]\):

\[
\text{FL}_{W}^{\text{loc}} \Phi_{A}^{\text{loc}} \mathcal{O}_{T^{n+r} \times W} \simeq \pi_{1}^{*} + (\text{id}_{\mathbb{C}^{*}_{x}} \times q)^{*} L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y})
\]

(126)

We take a desingularization \(\varphi_{\Sigma_{1}} : X_{\Sigma_{1}} \rightarrow X_{A}\). We have the open immersion \(\iota_{1} : T^{n+r} \subset X_{\Sigma_{1}}\). We may assume to have a projective morphism \(q_{1} : Y \rightarrow X_{\Sigma_{1}} \times W\) such that \(q_{1} \circ \iota = \iota_{1} \times i_{W}\).

We naturally regard \(L_{s}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W})\) and \(L_{\min}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W})\) as \(\mathcal{D}\)-modules on \(\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W\). Note that \(F_{i_{W}, \Sigma_{1}}\) is non-degenerate along \(D_{\Sigma_{1}, W}\). We have the decomposition \(D_{\Sigma_{1}, W} = D^{1}_{\Sigma_{1}, W} \cup D^{2}_{\Sigma_{1}, W}\) such that \(D^{1}_{\Sigma_{1}, W} = [F_{i_{W}, \Sigma_{1}]}_{\infty}\). We naturally have \(L_{\min}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W}) = \mathcal{O}_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W}(\ast (\mathbb{C}^{*}_{x} \times D^{1}_{\Sigma_{1}, W}))\) with the flat connection \(d + d(\lambda^{-1}F_{i_{W}, \Sigma_{1}})\). By Proposition \([3.13]\) and Lemma \([2.11]\) we naturally have

\[
L_{s}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W})|_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W} = L(F_{i_{W}, \Sigma_{1}})|_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W} \otimes \mathcal{O}_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W}(\ast (\mathbb{C}^{*}_{x} \times D_{\Sigma_{1}, W})).
\]

We naturally have \(L(\lambda^{-1}F_{Y}) = (\text{id}_{\mathbb{C}^{*}_{x}} \times q_{1})^{*} L(F_{i_{W}, \Sigma_{1}})|_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W}\), and hence

\[
(\text{id}_{\mathbb{C}^{*}_{x}} \times q_{1})^{*} L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y}) \simeq L_{s}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W})|_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W}.
\]

We have \(\pi_{1} \circ (\text{id}_{\mathbb{C}^{*}_{x}} \times q) = (\text{id}_{\mathbb{C}^{*}_{x}} \times \pi_{\Sigma_{1}}) \circ (\text{id}_{\mathbb{C}^{*}_{x}} \times q_{1})\). Hence, we naturally have

\[
\mathcal{D}_{\Sigma_{1}} \mathcal{D}_{\Sigma_{1}} L_{s}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W})|_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W} \simeq (\text{id}_{\mathbb{C}^{*}_{x}} \times \pi_{\Sigma_{1}})^{*} \left( L_{s}(F_{i_{W}, \Sigma_{1}}, D_{\Sigma_{1}, W})|_{\mathbb{C}^{*}_{x} \times X_{\Sigma_{1}} \times W} \right)
\]

\[
\simeq \pi_{1}^{*} \circ (\text{id}_{\mathbb{C}^{*}_{x}} \times q_{1})^{*} L_{s}(\lambda^{-1}F_{Y}, D^{(1)}_{Y}).
\]

(127)

Then, we obtain the claim of Lemma \([127]\).
7.1.6 Meromorphic extension on $\mathbb{C}_\lambda \times W$

By the construction, $\mathcal{L}_{\lambda, W}^\ast$ is naturally extended to $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules, which we denote by $\mathcal{L}_{\lambda, W}^\ast$. We use the notation $\mathcal{L}_{\lambda, W}^\ast = \mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$. By the construction, $\mathcal{L}_{\lambda, W}^\ast$ is also naturally extended to $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules, which are equal to $\mathcal{L}_{\lambda, W}^\ast$. The commutative diagrams are naturally extended to the commutative diagrams of $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules, and we obtain the following:

\[
\begin{array}{ccc}
\mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) & \xrightarrow{\mathcal{L}_{\lambda, W}^\ast} & \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \\
\end{array}
\]

7.1.7 Duality

For any complex manifold $B$, let $\mathcal{D}(\ast \lambda)$ denote the duality functor for $\mathcal{D}_{\mathbb{C}_\lambda \times B}(\ast \lambda)$-modules.

**Lemma 7.8** We have the following commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
\mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) & \xleftarrow{\sim} & \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \\
\end{array}
\]

**Proof** Let us consider the left square. By using the compatibility of the duality and the functors used in the construction of $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules, we can observe that there exists the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) & \xleftarrow{\sim} & \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \\
\end{array}
\]

Here, $\mathcal{D}(\ast \lambda)$ denotes the ordinary duality functor for the $\mathcal{D}$-modules. It induces the left square in $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules. Then, we obtain the commutativity of the right square in $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules by using the compatibility of the duality and the functors used in $\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)$-modules.

We give some remarks on the choices of the duality isomorphisms. The following lemma follows from Lemma 7.6.

**Lemma 7.9** Let $\ell_i : j^\ast \mathcal{D}(\ast \lambda) \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \xrightarrow{\ell_i} j^\ast \mathcal{D}(\mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)))$ be the isomorphisms induced by isomorphisms $\ell_i : \mathcal{D}(\ast \lambda) \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \xrightarrow{\ell_i} \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda))$. Then, there exists $\beta \in \mathbb{C}^\ast$ such that $\ell_1 = \beta \ell_2$.

In $\mathbb{R}$, Rechelt-Sevenheck uses the isomorphism $j^\ast \mathcal{D}(\ast \lambda) \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \xrightarrow{\ell_i} j^\ast \mathcal{D}(\mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)))$ induced by the duality isomorphism $\mathcal{D}(\ast \lambda) \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)) \xrightarrow{\ell_i} \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda))$ which is constructed in terms of Koszul complexes. As remarked in Lemma 7.6, it is equal to $k_1$ up to the multiplication of a non-zero constant.

Let $\mathcal{D}$ denote the duality functor for $\mathcal{R}$-modules. We also have the duality functor $\mathcal{D}$ given for $\mathcal{R}$-modules. (See $\mathcal{C}$) By construction, we have $\mathcal{D}(\ast \lambda) \mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda))$ is equal to $\mathcal{D}(\mathcal{L}_{\lambda, W}^\ast(\mathcal{D}_{\mathbb{C}_\lambda \times W}(\ast \lambda)))$. 

77
Lemma 7.10 We have the following commutative diagram:

\[
\begin{array}{ccc}
D^{(\star \lambda)}(\pi^0_L^\star(A,W,\omega))(\star \lambda) & \xrightarrow{\sim} & D^{(\star \lambda)}(\pi^0_L^\star(A,W,\omega))(\star \lambda) \\
\downarrow k_3 & & \downarrow k_3 \\
\pi^0_L(A,W,\omega)(\star \lambda) & \xrightarrow{=} & \pi^0_L(A,W,\omega)(\star \lambda)
\end{array}
\]

Here, the right vertical arrow is induced by \(j^*D^{(\star \lambda)}\pi^0_L(A,W,\omega)\) \(\sim \pi^0_L(A,W,\omega)\), and the upper horizontal arrow is given in Proposition C.10.

**Proof** It follows from Proposition C.21.

7.1.8 End of the proof of Theorem 7.2

By the construction \[\ref{7.4} \] \([6], \ref{7.5}\) \(T_{\Sigma_1}^\circ\) is an \(\mathcal{R}\)-submodule of \(\mathcal{M}_{A|C_A^\times W_{\omega}}\), and \(\mathcal{e}_{\circ\Sigma_1}\) is an \(\mathcal{R}\)-submodule of \(\mathcal{M}_{A|C_A^\times W_{\omega}}\).

Lemma 7.11 The restriction of \(c_2 \circ c_1\) to \(C_A \times W_{\omega}\) induces an isomorphism

\[
*(\mathcal{M}_{A|C_A \times W_{\omega}}(\star \lambda)) \simeq \lambda^n \pi^0_L(A,W,\omega)(\star \lambda).
\]

**Proof** Take a toric desingularization \(\phi_{\Sigma_1}: X_{\Sigma_1} \to X_A\) as in the proof of Lemma 6.7. We have the decomposition \(D_{\Sigma_1} = D_{\Sigma_1}^1 \cup D_{\Sigma_1}^2\) as above. Let \(\mathcal{M}_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\log D_{\Sigma_1}^1, W_{\omega})\) be as in \([6, 2.3]\) We have the following natural isomorphisms:

\[
\mathcal{M}_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\log D_{\Sigma_1}^1, W_{\omega}) \simeq R^{n+r} \pi_*(\Omega_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\log D_{\Sigma_1}^1, W_{\omega}))(\star \lambda) \\
\simeq \pi^0_L(A,W,\omega)(\star \lambda).
\]

The composite of the isomorphisms is equal to \(c_2\). By considering the push-forward of \(\phi_{\Sigma_1}\) for the middle term, by the isomorphism in \([3]\), we obtain the following isomorphisms:

\[
\mathcal{M}_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\log D_{\Sigma_1}^2, W_{\omega}) \simeq R^{n+r} \pi_*(\Omega_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\log D_{\Sigma_1}^2, W_{\omega}))(\star \lambda) \\
\simeq \pi^0_L(A,W,\omega)(\star \lambda).
\]

By Proposition \([6, 3.3]\) we have \(\pi^0_L(A,W,\omega)(\star \lambda) = R^{n+r} \pi_*(\Omega_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\log D_{A,W}(\omega))\).

According to \([6, 3.5, 3.20]\), the restriction of \(c_2 \circ c_1\) induces an isomorphism of \(\mathcal{M}_{X_{\Sigma_1} \times W_{\omega}(\omega)}(\star \lambda)\) with \(\lambda^n \pi^0_L(A,W,\omega)(\star \lambda)\) for the middle term. Then, the claim of Lemma 7.11 follows.

Lemma 7.12 The restriction of \(b_2 \circ b_1\) induces the following isomorphism:

\[
\mathcal{N}_{\circ\Sigma_1}(\star \lambda) := \lambda^n \pi^0_L(A,W,\omega)(\star \lambda)
\]

**Proof** For any locally free \(\mathcal{O}_{C_A \times W_{\omega}(\star \lambda)}\)-module \(\mathcal{V}\) with a flat connection \(\nabla\), let \(\mathcal{V}^\star\) denote the dual as a meromorphic flat bundle, i.e., \(\mathcal{V}^\star = \mathcal{H}om\mathcal{O}_{C_A \times W_{\omega}(\star \lambda)}(\mathcal{V}, \mathcal{O}_{C_A \times W_{\omega}(\star \lambda)})\) with the induced flat connection. We have the following diagram:

\[
\begin{array}{cccc}
D^{(\star \lambda)}(\mathcal{M}_{A|C_A \times W_{\omega}}(\star \lambda)) & \xrightarrow{\sim} & D^{(\star \lambda)}(\pi^0_L(A,W,\omega)(\star \lambda)) & \xrightarrow{\sim} & D^{(\star \lambda)}(\pi^0_L(A,W,\omega)(\star \lambda)) \\
\downarrow d_1 & & \downarrow d_2 & & \downarrow d_3 \\
(\mathcal{M}_{A|C_A \times W_{\omega}}(\star \lambda))^\star & \xrightarrow{\sim} & (\pi^0_L(A,W,\omega)(\star \lambda))^\star & \xrightarrow{\sim} & (\pi^0_L(A,W,\omega)(\star \lambda))^\star
\end{array}
\]
Here, $d_i$ ($i = 1, 2$) are the isomorphisms given for flat bundles, and $d_3$ is the isomorphism given for smooth $\mathcal{R}$-modules. The left square is clearly commutative. The right square is commutative up to signatures by Lemma \ref{lem:7.11}.

We have the following isomorphisms:

$$\lambda^{m+r} \pi_1^0 \mathcal{L}_i(A, W, i_W)_{|C_\lambda \times W^{reg}} \simeq j^* D \pi_1^0 \mathcal{L}_s(A, W, i_W)_{|C_\lambda \times W^{reg}}$$

$$\simeq j^* \text{Hom}_{C_\lambda \times W^{reg}} \left( \pi_1^0 \mathcal{L}_s(A, W, i_W)_{|C_\lambda \times W^{reg}}, \lambda^{m-n} \mathcal{O}_{C_\lambda \times W^{reg}} \right)$$  (132)

Hence, $d_3$ and the restriction of the morphism $k_3$ induces the following isomorphism:

$$\lambda^{m+r} \pi_1^0 \mathcal{L}_i(A, W, i_W)_{|C_\lambda \times W^{reg}}, \lambda^{n-r} \mathcal{O}_{C_\lambda \times W^{reg}} \simeq \lambda^{m-n} \pi_1^0 \mathcal{L}_i(A, W, i_W)_{|C_\lambda \times W^{reg}}$$

Applying the construction in §7.2, we obtain the following commutative diagram on $W^{reg}$:

$$\text{By Lemma } 7.11 \text{ and Lemma } 7.12, \text{ we obtain the following commutative diagram on } W^{reg}:$$

$$\begin{array}{ccc}
\ast \hat{N}_{\mathcal{A}|C_\lambda \times W^{reg}} & \xrightarrow{\simeq} & \lambda^{-r} \left( \lambda^{m+r} \pi_1^0 \mathcal{L}_i(A, W, i_W) \right) \\
\downarrow & & \downarrow \\
\ast \hat{M}_{\mathcal{A}|C_\lambda \times W^{reg}} & \xrightarrow{\simeq} & \lambda^{-r} \left( \lambda^{m+r} \pi_1^0 \mathcal{L}_s(A, W, i_W) \right)
\end{array}$$

By taking the pull back by id $\times \varphi$, we obtain the desired commutative diagram (121). Thus, the proof of Theorem 7.2 is finished.

**7.2 Description as a specialization**

This subsection is the continuation of §7.1 and it is the preparation for Theorem 7.43.

**7.2.1 The $\tilde{\mathcal{R}}$-module with pairing associated to $\tilde{A}$**

Applying the construction in §6.5, we obtain the integrable pure twistor $\mathcal{D}$-module $\mathcal{T}_{\tilde{A}}$ on $S_{\tilde{A}}$. Let $\mathcal{V}_{\tilde{A}}$ denote their underlying $\tilde{\mathcal{R}}$-module which is $\mathbb{C}^*$-homogeneous. We have the filtration $\tilde{W}$ and the graded pairing $P_{\tilde{A}}$ induced by the real structure and the graded polarization on $\mathcal{V}_{\tilde{A}}$. In this case, $\tilde{W}$ is pure of weight $n + r$.

We apply the same construction to $\mathcal{T}_{\tilde{A}} \otimes T(n + r)$. We obtain the underlying $\tilde{\mathcal{R}}$-module $\mathcal{W}_{\tilde{A}}$ on $S_{\tilde{A}}$ which is $\mathbb{C}^*$-homogeneous. The restriction $\mathcal{W}_{\tilde{A}|S_{\tilde{A}}^{reg}}$ is equipped with the filtration $\tilde{W}$ which is pure of weight $-n - r$, and the pairing $\mathcal{P}_{\tilde{A}}$ of weight $-n - r$. In other words, we obtain a TEP($n + r$)-structure ($\mathcal{W}_{\tilde{A}|S_{\tilde{A}}^{reg}}, \mathcal{P}_{\tilde{A}}$). By construction, we have $\mathcal{W}_{\tilde{A}} = \lambda^{n+r} \mathcal{V}_{\tilde{A}}$, and $P_{\tilde{A}} = \mathcal{P}_{\tilde{A}}$ on $\mathcal{V}_{\tilde{A}}(*\lambda) = \mathcal{V}_{\tilde{A}}(*\lambda)$.

**7.2.2 Relation of $\mathcal{W}_{\tilde{A}}$ and $\mathcal{W}_{\tilde{A}}$**

We set $\tilde{S}_{\tilde{A}} := S_{\tilde{A}} \times \mathbb{C}_\gamma$. Applying the construction in §6.3, we obtain the integrable pure twistor $\mathcal{D}$-module $\mathcal{T}(\tilde{A}, \tilde{S}_{\tilde{A}}, \tilde{\gamma}_{\tilde{A}})$ and the underlying $\tilde{\mathcal{R}}$-module $\mathcal{L}(\tilde{A}, \tilde{S}_{\tilde{A}}, \tilde{\gamma}_{\tilde{A}})$ on $\mathbb{P}^{m+r+1} \times \tilde{S}_{\tilde{A}}$. Let $\tilde{\pi} : \mathbb{P}^{m+r+1} \times \tilde{S}_{\tilde{A}} \rightarrow \tilde{S}_{\tilde{A}}$ be the projection. We set $\mathcal{W}_{\tilde{A}} := \lambda^{n+r} \pi_1^0 \mathcal{L}(\tilde{A}, \tilde{S}_{\tilde{A}}, \tilde{\gamma}_{\tilde{A}})(*\tau)$. It is equipped with the pairing $\mathcal{P}_{\tilde{A}}$ of weight $-n - r$ as in the case of §7.2.4.

**Lemma 7.13** We can naturally identify $S_{\tilde{A}}$ with $S_{\tilde{A}} \times \mathbb{C}_\tau^\times \subset \tilde{S}_{\tilde{A}}$ under which $\mathcal{W}_{\tilde{A}|S_{\tilde{A}}} = \mathcal{W}_{\tilde{A}}$. In other words, $\mathcal{W}_{\tilde{A}}$ is a meromorphic extension of $\mathcal{W}_{\tilde{A}}$ on $\tilde{S}_{\tilde{A}}$. We also have the $\mathbb{C}^*$-action on $\tilde{S}_{\tilde{A}}$ for which the inclusions $S_{\tilde{A}} \simeq S_{\tilde{A}} \times \{0\} \subset \tilde{S}_{\tilde{A}}$ and $S_{\tilde{A}} \rightarrow \tilde{S}_{\tilde{A}}$ are equivariant.
We may assume that the splitting $\tilde{\kappa}_\mathcal{A} : L^\vee_{\mathcal{A}} \to M^\vee_{\mathcal{A}}$ is given by the direct sum of $\kappa_{\mathcal{A}} : L^\vee_{\mathcal{A}} \to M^\vee_{\mathcal{A}}$ and $\mathcal{Z}\mathcal{Y} \to \mathcal{Z}\mathcal{E}^{m+r+1}_{m+r+1}$. Then, we have $F_{\gamma_{\mathcal{A}}}(x, \tau, t) = F_{\gamma_{\mathcal{A}}}(x, t) + \tau t^{m+r+1}$. Then, by the construction of $\tilde{\gamma}_{\mathcal{A}}$, we have $\tilde{\gamma}_{\mathcal{A}}|_{\mathcal{S}_{\mathcal{A}}} = \gamma_{\mathcal{A}}$. We obtain $\mathcal{T}(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}_{\mathcal{A}}, \tilde{\gamma}_{\mathcal{A}})|_{\mathcal{P}m \times \mathcal{S}_{\mathcal{A}}} = \mathcal{T}(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}_{\mathcal{A}}, \tilde{\gamma}_{\mathcal{A}})$, and hence $\tilde{\mathcal{P}}_{\mathcal{A}|S_{\mathcal{A}}} = \tilde{\mathcal{P}}_{\mathcal{A}}$. The claim on the $\mathcal{C}^*$-action is also easy to see.

Recall the following due to Reichelt-Sevenheck who proved it in a more general situation.

**Proposition 7.14** ([30]) $\tilde{\mathcal{P}}_{\mathcal{A}}$ is regular along $\tau$.

**Lemma 7.15** We consider the morphism $\lambda N : \psi_{\tau,-\delta}\tilde{\mathcal{P}}_{\mathcal{A}} \to \psi_{\tau,-\delta}\tilde{\mathcal{P}}_{\mathcal{A}}$, where $N$ is induced by $\tau \partial_{\tau}$.

- The kernel and the cokernel of $\lambda N$ are isomorphic to $\mathcal{P}_{\mathcal{A}|1}$ and $\mathcal{P}_{\mathcal{A}|s}$, respectively.

- Let $W(N)$ denote the weight filtration of $\lambda N$ on $\psi_{\tau,-\delta}\tilde{\mathcal{P}}_{\mathcal{A}}$. The induced filtrations on $\text{Ker}(\lambda N)$ and $\text{Cok}(\lambda N)$ are also denoted by $W(N)$. Then,

$$W(N)_k(\text{Cok}(\lambda N)) \simeq \tilde{W}_{k-n-r}(\mathcal{P}_{\mathcal{A}|s})$$

$$W(N)_k(\text{Ker}(\lambda N)) \simeq \tilde{W}_{k-n-r}(\mathcal{P}_{\mathcal{A}|1})$$

**Proof** The first follows from the isomorphisms in Corollary 6.40 and the constructions of $\mathcal{P}_{\mathcal{A}}$ and $\tilde{\mathcal{P}}_{\mathcal{A}}$. The second also follows from the isomorphisms in Corollary 6.40, the comparison of the filtrations $W(N)$ and $W$ on $\psi_{\tau,-\delta}(\mathcal{T})$, and the constructions of the filtrations $\tilde{W}$.

Let $\widetilde{B}$ be any open set in $\tilde{\mathcal{S}}_{\mathcal{A}}$ such that $\tilde{\mathcal{P}}_{\mathcal{A}}$ is locally free $\mathcal{O}_{\mathcal{C}_A \times \mathcal{B}}$-module, and regular singular along $\tau = 0$. The polarization and the real structure of $\pi_0^0(\mathcal{T}(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}_{\mathcal{A}}, \tilde{\gamma}_{\mathcal{A}}))$ induce a pairing $\mathcal{P}_{\mathcal{A}}$ of weight $-n-r$ on $\tilde{\mathcal{P}}_{\mathcal{A}|\mathcal{B}}$. We clearly have $\tilde{\mathcal{P}}_{\mathcal{A}|\mathcal{B}|\{\tau = 0\}} = \mathcal{P}_{\mathcal{A}|\mathcal{B}|\{\tau = 0\}}$. We obtain the induced graded pairing $\text{sp}_\tau(\mathcal{P}_{\mathcal{A}})$ on $(\text{Cok}(\lambda N), \tilde{W})$ by the procedure in [B.3]. We also have $\text{sp}_\tau(\tilde{\mathcal{P}}_{\mathcal{A}})$ on $(\text{Ker}(\lambda N), \tilde{W})$.

**Lemma 7.16** We have $\text{sp}_\tau(\mathcal{P}_{\mathcal{A}}) = \mathcal{P}_{\mathcal{A}|s}$ under the isomorphism $(\text{Cok}(\lambda N), \tilde{W}) \simeq (\mathcal{P}_{\mathcal{A}|s}, \tilde{W})$. We also have $\text{sp}_\tau(\tilde{\mathcal{P}}_{\mathcal{A}}) = \mathcal{P}_{\mathcal{A}|1}$ under the isomorphism $(\text{Ker}(\lambda N), \tilde{W}) \simeq (\mathcal{P}_{\mathcal{A}|1}, \tilde{W})$.

**Proof** It follows from Proposition 6.48.
We have the natural morphisms:

\[
\begin{array}{ccc}
L_X^\vee & \longrightarrow & L_A^\vee \\
\simeq & \downarrow & \simeq \\
H^2(X, \mathbb{Z}) & \longrightarrow & H^2(Y, \mathbb{Z}) \\
\end{array}
\]

We take a frame \(\eta_1, \ldots, \eta_\ell\) of \(H^2(X, \mathbb{Z})\) such that each \(\eta_i\) is the first Chern class of a nef line bundle. Let \(\eta_{\ell+1} \in H^2(Y, \mathbb{Z})\) be the first Chern class of the tautological line bundle \(O(1)\) of \(Y\) over \(X\). By the assumption on the line bundles \(L_j\), \(\eta_{\ell+1}\) is also a nef class. The tuple \(\eta_1, \ldots, \eta_{\ell+1}\) gives a frame of \(H^2(Y, \mathbb{Z})\). Let \(\xi_1 \in L_A^\vee\) be the elements corresponding to \(\eta_i\). We have \(\xi_{\ell+1} = \xi\).

The tuple \(\xi_1, \ldots, \xi_\ell\) gives a frame of \(L_A^\vee\). It gives a coordinate system \((x_1, \ldots, x_\ell)\) on \(S_A\) with which \(S_A \simeq (\mathbb{C}^*)^\ell\). Similarly, the tuple \(\xi_1, \ldots, \xi_{\ell+1}\) gives a frame of \(L_A^\vee\). It gives a coordinate system \((x_1, \ldots, x_{\ell+1})\) on \(S_A\) with which \(S_A \simeq (\mathbb{C}^*)^{\ell+1}\). We have \(\tau = x_{\ell+1}\) under the identification in Lemma 7.13. Recall the following due to Iritani.

**Lemma 7.17** \((20)\) There exists \(\epsilon > 0\) such that the following holds:

- \(B_A = \{(x_1, \ldots, x_{\ell+1}) \mid 0 < |x_i| < \epsilon \ (i = 1, \ldots, \ell + 1)\}\) is contained in \(S_A^{\text{reg}}\).

- \(B_A = \{(x_1, \ldots, x_\ell) \mid 0 < |x_i| < \epsilon \ (i = 1, \ldots, \ell)\}\) is contained in \(S_A^{\text{reg}}\).

**Proof** The first claim is proved in \(20\) Appendix A1. Let \(\sigma\) be a face of \(\text{Conv}(A \cup \{0\})\) such that \(0\) is not contained in \(\sigma\). Then, \(\sigma\) is a face of \(\text{Conv}(A)\). We have the equality \(F_{\gamma_A, \sigma} = F_{\gamma_A, \sigma}\), and \(F_{\gamma_A, \sigma}\) does not contain \(x_{\ell+1}\). Hence, the second claim follows.

We consider the following subset \(\overline{B}_A\) in \(\overline{S}_A\):

\[
\overline{B}_A = \{(x_1, \ldots, x_{\ell+1}) \mid 0 < |x_i| < \epsilon \ (i = 1, \ldots, \ell), \ |x_{\ell+1}| < \epsilon\}
\]

The restriction of \(\overline{\mathfrak{M}}_{A|B_A}\) is a locally free \(\mathcal{O}_{\mathbb{C}^* \times \overline{B}_A(*x_{\ell+1})}\)-module, and it is regular along \(x_{\ell+1}\).

### 7.2.4 Coverings

We set \(\mathcal{S}_A := \mathbb{C} \otimes L_A^\vee\). The exponential map \(\mathbb{C} \rightarrow \mathbb{C}^*\) induces the covering map \(\chi_A : \mathcal{S}_A \rightarrow S_A\). We obtain an \(\mathfrak{R}\)-module \(\chi_A^* \mathfrak{M}_A\), equipped with the filtration \(\chi_A^* \tilde{W}\) on \(\mathcal{S}_A\). The restriction to \(\mathcal{S}_A^{\text{reg}} := \chi_A^{-1}(S_A^{\text{reg}})\) are equipped with graded pairings \(\chi_A^* \mathfrak{P}_A\).

We set \(\tilde{\mathcal{S}}_A := \mathbb{C} \otimes L_A^\vee\). By the decomposition \(L_A^\vee = L_A^\vee \oplus \mathcal{Z} Y\) in the proof of Lemma 7.13, we naturally have \(\tilde{\mathcal{S}}_A = \mathcal{S}_A \times \mathbb{C}\). The map \(\chi_A\) and the identity on \(\mathbb{C}\) induces \(\tilde{\chi}_A : \tilde{\mathcal{S}}_A \rightarrow \tilde{S}_A\).

Take a frame \(\xi_1, \ldots, \xi_\ell\) of \(L_A^\vee\) and an open subset \(\overline{B}_A\) as in 7.2.3. The restriction of \(\tilde{\chi}_A^* \tilde{\mathfrak{M}}_A\) to \(\tilde{\chi}_A^{-1}(\overline{B}_A)\) is equipped with the pairing \(\tilde{\chi}_A^* \tilde{\mathfrak{P}}_A\) of weight \(-n - r\).

**Lemma 7.18** The restriction of the mixed TEP-structure \(\chi_A^* (\mathfrak{M}_{A|S_A^{\text{reg}}} \tilde{W}, \mathfrak{P}_{A*})\) to \(\tilde{\chi}_A^{-1}(B_A)\) \((*=*,!\) are obtained from \(\tilde{\chi}_A^* (\tilde{\mathfrak{M}}_A, \tilde{\mathfrak{P}}_A)_{|\tilde{B}_A}\) by the procedure in \(7.1.3\).

### 7.2.5 Logarithmic extension and endomorphisms of \(\mathfrak{M}_A\)

This is the continuation of 7.2.3. Let us recall the explicit description of the logarithmic extension of \(\mathfrak{M}_A\) due to Reichelt and Sevenheck in \(40\). See \(40\) for more details. We also give an easy remark on uniqueness of automorphisms, although it is also essentially implied in \(40\).

Let \(\overline{B}_A\) be as in Lemma 7.17. We set

\[
\overline{B}_A := \{(x_1, \ldots, x_{\ell+1}) \mid |x_i| < \epsilon \ (i = 1, \ldots, \ell + 1)\}.
\]

First, the following holds.
Proposition 7.19 (§3.2 of [40]) There exists a locally free $\mathcal{O}_{\mathcal{C}_X \times B_A}$-module $\Omega_{\mathfrak{A}}$ with an isomorphism

$$\Omega_{\mathfrak{A}|\mathcal{C}_X \times B_A} \cong \mathfrak{A}|B_A$$

such that the following holds:

- The meromorphic flat connection $\nabla$ of $\mathfrak{A}|B_A$ gives a meromorphic flat connection $\nabla^\Omega_{\mathfrak{A}}$ of $\Omega_{\mathfrak{A}}$ which is logarithmic along $\{x_i = 0\}$ $(i = 1, \ldots, \ell + 1)$.

- The residues $\text{Res}_{x_i}(\nabla^\Omega_{\mathfrak{A}})|_{\mathcal{C}_X \times \{0\}}$ are nilpotent.

Such $\Omega_{\mathfrak{A}}$ is unique up to canonical isomorphisms.

They considered the residual connection $\nabla^E$ on $E := \mathfrak{A}|_{\mathcal{C}_X \times \{0\}}$. They give a quite explicit description of the meromorphic flat bundle $(E, \nabla^E)$. Let $N_i : E \longrightarrow \lambda^{-1}E$ $(i = 1, \ldots, \ell + 1)$ denote $\text{Res}_{x_i}(\nabla^\Omega_{\mathfrak{A}})|_{\mathcal{C}_X \times \{0\}}$. We have the fundamental vector fields $\tilde{\nu} = \sum_{i=1}^{\ell+1} k_i x_i \partial_{x_i}$ of the $C^*$-action on $\mathfrak{A}$ (see §6.5.3). Note that for the $C^*$-homogeneity of $\mathfrak{A}$, the fundamental vector field of the $C^*$-action on $\mathcal{C}_X \times \mathfrak{A}$ is $\lambda \partial_{\lambda} + \tilde{\nu}$. Let us consider the $C^*$-homomorphism $L : E \longrightarrow E$ given by $\nabla^E(\lambda \partial_{\lambda}) + \sum_{i=1}^{\ell+1} k_i N_i$. Because $[L, N_i] = 0$, we have $[L, \lambda N_i] = \lambda N_i$. For the following proposition, and more detailed description in terms of the cohomology group of $Y$, see Lemma 3.8 and its proof in [40].

Proposition 7.20 (§3.2 of [40]) There exists a frame $u_1, \ldots, u_{\text{rank } E}$ for which we have the following:

- $L(u_j) = c_j u_j$ for $c_j \in \mathbb{Z}$.

- We have $c_1 < c_i$ ($i \neq 1$).

- We have $N_i u_j = \sum \alpha_{k_j} u_k$ for some $\alpha_{k_j} \in \mathbb{Z}$. In particular, $\text{Sym} \left( \bigoplus_{i=1}^{\ell+1} \mathbb{C} N_i \right)$ acts on $F := \bigoplus_{i=1}^{\text{rank } E} \mathbb{C} u_i$.

- The map $\text{Sym} \left( \bigoplus_{i=1}^{\ell+1} \mathbb{C} N_i \right) \longrightarrow F$ given by $P(N_1, \ldots, N_{\ell+1}) \longrightarrow P(N_1, \ldots, N_{\ell+1})$ is surjective.

By the first property in Proposition 7.20, we have the $C^*$-action on $E$ which induces $L$. As a result, we have a vector space $H$ with an increasing filtration $F$ indexed by integers such that the $C^*$-equivariant bundle $E$ is isomorphic to the analytification of the Rees module of $(H, F)$. Let $d_0 := \min \{ d \mid \text{Gr}^F_d(H) \neq \emptyset \}$. The second property implies that $\dim F_{d_0}(H) = 1$. The third and the fourth properties imply that $F_{d_0}(H)$ generates $H$ over the induced actions of the residues. The following is also essentially implied in the proof of [40, Lemma 3.8].

Corollary 7.21 For any endomorphism $\varphi$ of the $\mathfrak{A}$-module $\mathfrak{A}|\mathfrak{B}$, there exists a complex number $\alpha$ such that $\varphi = \alpha \text{id}$.

Proof: Because $\Omega_{\mathfrak{A}^{\mathfrak{A}}}$ is the Deligne extension of $\mathfrak{A}|\mathfrak{A}^{\mathfrak{A}}$ to $\mathfrak{C}_X \times \mathfrak{B}$, the morphism $\varphi$ is extended to an endomorphism of the $O$-module $\mathfrak{A}$ compatible with $\nabla^\Omega_{\mathfrak{A}}$. Hence, it induces an endomorphism $\varphi^E$ of $(E, \nabla^E)$ compatible with the actions of the residues. Because the $C^*$-action is determined by $\nabla^E$ and the residues, $\varphi^E$ is $C^*$-equivariant. Hence, it induces an endomorphism $\varphi^H$ of $(H, F)$ compatible with the induced actions of residues. Because $F_{d_0}$ generates $H$ over the actions of the residues, $\varphi^H$ is uniquely determined by the restriction $\varphi^H|_{F_{d_0}}$. Because $\dim F_{d_0} = 1$, we have $\alpha \in \mathbb{C}$ such that $\varphi^H = \lambda \varphi^H$. If $\alpha = 0$, then $\varphi = 0$. Hence, we have $\varphi = \alpha \text{id}$.

Corollary 7.22 Let $U \subset S_A$ be any connected open subset such that $B_A \subset U$. For any endomorphism $\varphi$ of the $\mathfrak{A}$-module $\mathfrak{A}|_{S_A}$, there exists a complex number $\alpha$ such that $\varphi = \alpha \text{id}$.

Corollary 7.23 Let $P' : \mathfrak{A}|_{S_A^{\text{reg}}} \otimes j^* \mathfrak{A}|_{S_A^{\text{reg}}^*} \longrightarrow \lambda^n r \mathcal{O}_{\mathcal{C}_X \times S_A^{\text{reg}}}$ be a morphism of $\mathfrak{A}$-modules. Then, there exists a complex number $\alpha$ such that $P' = \alpha \mathfrak{A}$. 

82
7.2.6 Logarithmic extension and endomorphisms of $\mathfrak{U}_A$

We can easily deduce the same property for $\mathfrak{U}_{A^*}$ ($\ast = *, !$). Let $B_A$ be as in Lemma 7.17. We set

$$\mathfrak{B}_A := \{(x_1, \ldots, x_\ell) \mid |x_i| < \epsilon \ (i = 1, \ldots, \ell)\}. $$

We regard $\mathfrak{B}_A$ as $\{(x_1, \ldots, x_{\ell + 1}) \in \mathfrak{B}_A \mid x_{\ell + 1} = 0\}$. We obtain the vector bundle $\Omega\mathfrak{B}_A$. It is equipped with the endomorphism $\lambda \text{Res}_{x_{\ell + 1}}(\nabla\Omega\mathfrak{B}_A)$, and the induced connection.

**Lemma 7.24** The conjugacy classes of $\lambda \text{Res}_{x_{\ell + 1}}(\nabla\Omega\mathfrak{B}_A)$ are independent of the choice of $(\lambda, P) \in \mathbb{C}_\lambda \times \mathfrak{B}_A$.

**Proof** By the construction, $\mathfrak{U}_A$ comes from a harmonic bundle. It is easy to see that $\mathfrak{U}_A$ is equal to the prolongation of the family of $\lambda$-flat bundles $\mathcal{P}_0(\mathfrak{U}_A|B_A)$ associated to tame harmonic bundles, as remarked in [10]. Then, the claim follows from a general theory of harmonic bundles [32].

Let $\Omega\mathfrak{B}_A$ be the cokernel of $\lambda \text{Res}_{x_{\ell + 1}}(\nabla\Omega)$. It is a locally free $\mathcal{O}_{\mathbb{C}_\lambda \times \mathfrak{B}_A}$-module. The following is clear by the relation of $\mathfrak{U}_{A^*}$ and $\mathfrak{U}_A$.

**Lemma 7.25** We have natural isomorphisms of $\mathcal{R}$-modules $\Omega\mathfrak{B}_A|\mathbb{C}_\lambda \times B_A \simeq \mathfrak{U}_{A^*}|\mathbb{C}_\lambda \times B_A$.

Set $E_A := \Omega\mathfrak{B}_A|\mathbb{C}_\lambda \times \{0\}$. It is equipped with the induced connection $\nabla^{E_A}$. We obtain an explicit description of $(E_A, \nabla^{E_A})$ from the description of $(E, \nabla^E)$ in [10]. Let $N_i : E_A \to \lambda^{-1}E_A$ $(i = 1, \ldots, \ell)$ denote $\text{Res}_{x_{\ell + 1}}(\nabla^{E_A})|\mathbb{C}_\lambda \times \{0\}$. We have the fundamental vector fields $\mathfrak{p} = \sum_{i=1}^{\ell} k_i x_i \partial_{x_i}$ of the $\mathbb{C}^*$-action on $S_A$ (see (7.5.20)). Note that for the $\mathbb{C}^*$-homogeneity of $\mathfrak{U}_{A^*}$, the fundamental vector field of the $\mathbb{C}^*$-action on $\mathbb{C}_\lambda \times S_A$ is $\lambda \partial_\lambda + \mathfrak{p}$. Let us consider the $\mathbb{C}$-homomorphism $L : E_A \to E_A$ given by $\nabla^{E}(\lambda \partial_\lambda) + \sum_{i=1}^{\ell} k_i N_i$. Because $[L, N_i] = 0$, we have $[L, \lambda N_i] = \lambda N_i$. The following is a direct consequence of Proposition 7.20.

**Proposition 7.26** There exists a frame $u_1, \ldots, u_{\text{rank}E_A}$ for which we have the following:

- $L(u_j) = c_j u_j$ for $c_j \in \mathbb{Z}$.
- We have $c_1 < c_i$ $(i \neq 1)$.
- We have $N_i u_j = \sum \beta_{kj}^i u_k$ for $\beta_{kj}^i \in \mathbb{Z}$. In particular, $\text{Sym} \left( \bigoplus_{i=1}^{\ell} \mathbb{C} N_i \right)$ acts on $F_A := \bigoplus_{i=1}^{\text{rank}E_A} \mathbb{C} u_i$.
- The map $\text{Sym} \left( \bigoplus_{i=1}^{\ell} \mathbb{C} N_i \right) \to F_A$ given by $P(N_1, \ldots, N_\ell) \mapsto P(N_1, \ldots, N_\ell) u_1$ is surjective.

We omit a more detailed description in terms of the cohomology group of $X$.

By the first property in Proposition 7.26, we have the $\mathbb{C}^*$-action on $E_A$ which induces $L$. As a result, we have a vector space $H_A$ with an increasing filtration $F$ indexed by integers such that the $\mathbb{C}^*$-equivariant bundle $E_A$ is isomorphic to the analytification of the Rees module of $(H_A, F)$. Let $d_0 := \min \{ d \mid \text{Gr}^d_F(H) \neq 0 \}$. The second property implies that $\dim F_{d_0}(H_A) = 1$. The third and the fourth properties imply that $F_{d_0}(H_A)$ generates $H_A$ over the induced actions of the residues.

**Corollary 7.27** Let $U \subset S_A$ be any connected open subset such that $B_A \subset U$. For any endomorphism $\varphi$ of the $\mathcal{R}$-module $\mathfrak{U}_{A^*}|\mathbb{C}_\lambda \times U$ $(\ast = *, !)$, there exists a complex number $\alpha$ such that $\varphi = \alpha \text{id}$.

**Proof** We obtain the claim for $\mathfrak{U}_{A^*}$ by the argument in the proof of Corollary 7.21 and Corollary 7.22. Because $\mathfrak{U}_{A^*} \simeq \lambda^{2n-m+r} j^* D\mathfrak{U}_{A^*}$, we obtain the claim for $\mathfrak{U}_A$.

We have an isomorphism $\Psi : j^* (\mathfrak{U}_{A^*}|\mathbb{C}_\lambda \times U)^\vee \simeq \lambda^{-n-r} \mathfrak{U}_{A^*}|\mathbb{C}_\lambda \times U$ induced by the duality isomorphism $\mathfrak{U}_{A^*} \simeq \lambda^{2n-m+r} j^* D\mathfrak{U}_{A^*}$.

**Corollary 7.28** Let $U \subset S_{\mathfrak{U}}^{\text{res}}$ be any open subset such that $B_A \subset U$. For any morphism of $\mathcal{R}$-modules $\Psi' : j^* (\mathfrak{U}_{A^*}|\mathbb{C}_\lambda \times U)^\vee \to \lambda^{-n-r} \mathfrak{U}_{A^*}|\mathbb{C}_\lambda \times U$, there exists a complex number $\alpha$ such that $\Psi' = \alpha \Psi$. 

83
7.3 Preliminary on the A-side

We recall some basic matters on the quantum products of toric varieties. For example, see [19, 20, 27, 28, 10, 11] for more details.

7.3.1 The cohomology ring of projective bundles

Let $X$ be a complex manifold. For simplicity, we assume $H^{2i+1}(X, \mathbb{Z}) = 0$ for any $i$. We shall omit the coefficient of the cohomology group if the coefficient is $\mathbb{C}$. Let $E$ be a locally free $O_X$-module of rank $r$. Let $E^\vee$ denote the dual. Let $Y := \mathbb{P}(E \otimes O_X) = \text{Proj}(\text{Sym}^*(E \otimes O_X))$ denote the projective completion of $E^\vee$. We have the natural inclusions $i_0 : X \cong \mathbb{P}(O) \rightarrow Y$ and $i_\infty : H_\infty := \mathbb{P}(E) \rightarrow Y$. We have $i_0(X) \subset E^\vee$ and $Y = E^\vee \cup H_\infty$.

Let $\pi : Y \rightarrow X$ denote the projection. Let $\pi^*$ and $i_0^*$ denote the pull back of the cohomology. We have $H^*(Y) = \text{Ker} i_0^* \oplus \text{Im} \pi^*$. Let $i_{0*} : H^i(X) \rightarrow H^{i+2r}(Y)$ and $\pi_* : H^i(Y) \rightarrow H^{i-2r}(X)$ denote the Gysin map. We have $H^*(Y) = \text{Im} i_{0*} \oplus \text{Ker} \pi_*$. Let $\mathcal{O}_Y(1)$ denote the tautological line bundle of $Y$ over $X$. We set $\gamma := c_1(\mathcal{O}_Y(1)) \in H^2(Y)$. We have $H^*(Y) = \bigoplus_{j=0}^r \gamma^j \cdot \pi^*(H^*(X))$. Note that $\gamma$ is the cohomology class representing $H_\infty$. In particular, we have $i_0^* \gamma = 0$. Hence, $\text{Ker} i_0^* = \bigoplus_{j=1}^r \pi^* H^j(X) \gamma^j$. Because $c_{r+1}(E^\vee \otimes O_Y) = 0$, we have

$$\gamma \cdot \sum_{j=0}^r \gamma^{r-j} \pi^* c_j(E^\vee) = 0. \quad (133)$$

Let $N : H^*(Y) \rightarrow H^*(Y)$ be determined by $N(\sigma) = \gamma \sigma$. By $(133)$, we have $\text{Im} N = \bigoplus_{i=1}^r \gamma^i \cdot \pi^*(H^j(X))$, and $\text{Cok} N \cong \text{Im} \pi^*$.

**Lemma 7.29** We have $\text{Im} i_{0*} = \text{Ker} N$.

**Proof** Because $H_\infty \cap i_0(X) = 0$, we have $\text{Im} i_{0*} \subset \text{Ker} N$. By comparison of the dimension, we obtain $\text{Im} i_{0*} = \text{Ker} N$.]

In particular, $\text{Ker}(N) \cap H^{2r}(Y) = i_{0*} H^0(X, \mathbb{C})$ is one dimensional, and that $\sum_{j=0}^r \gamma^{r-j} \pi^* c_j(E^\vee)$ is a base. By considering the restriction to a ball in $X$, we obtain that $i_{0*}(1) = \sum_{j=0}^r \gamma^{r-j} \pi^* c_j(E^\vee)$. In particular, we have $i_{0*} \sigma = \pi^* \sigma \sum_{j=0}^r \gamma^{r-j} \pi^* c_j(E^\vee)$.

7.3.2 Quantum products of $Y$ in the weak Fano toric case

Suppose that (i) $X$ is a smooth weak Fano toric variety, (ii) $E$ is the direct sum of holomorphic line bundles $L_i$ ($i = 1, \ldots, r$), (iii) $L_i$ ($i = 1, \ldots, r$) and $(\bigotimes_{i=1}^r L_i \otimes K_X)^\vee$ are nef. Then, $Y$ is also a weak Fano toric manifold.

Let $\text{Eff}^+(Y) \subset H_2(Y, \mathbb{Z})$ denote the subset of the homology classes of non-empty algebraic curves. We set $\text{Eff}(Y) := \text{Eff}^+(Y) \cup \{0\}$. We take a homogeneous base $\phi_1, \ldots, \phi_X(1)$ of $H^*(Y, \mathbb{C})$. Let $\phi^1, \ldots, \phi^X$ be the dual base of $H^*(Y)$ with respect to the Poincaré pairing.

For $d \in \text{Eff}(Y)$, let $Y_{0,3,d}$ denote the moduli stack of stable maps $f : C \rightarrow Y$ where $C$ denotes a 3-pointed pre-stable curve with genus $0$, and $f$ denotes a morphism such that the homology class of $f(C)$ is $d$. Let $p : C_{0,3,d} \rightarrow Y_{0,3,d}$ be the universal curve, and let $\text{ev}_i : C_{0,3,d} \rightarrow X$ be the evaluation map at the $i$-th marked point. Let $[Y_{0,3,d}]_{\text{vir}}$ denote the virtual fundamental class of $Y_{0,3,d}$. For $\alpha_i \in H^* (Y)$ ($i = 1, 2, 3$), we obtain the number $(\alpha_1, \alpha_2, \alpha_3)_{0,3,d} := \int [Y_{0,3,d}]_{\text{vir}} \prod_{i=1}^3 \text{ev}_i^* \alpha_i$ called the Gromov-Witten invariants of $Y$.

According to a result of Iritani [19] (see also [20] and [40]), there exists an open subset $U \subset H^2(Y, \mathbb{C})$ such that the following holds:

- $U$ is of the form $\{u \in H^2(Y, \mathbb{C}) \mid \text{Re}(u, d) < -M, \forall d \in \text{Eff}^+(Y)\}$ for some $M > 0$.
- For any $u \in U$, the quantum product $\bullet_u$ on $H^*(Y, \mathbb{C})$ is convergent:

\[ u \bullet Y = \sum_{d \in \text{Eff}(Y)} \sum_{i=1}^{\chi(Y)} \langle \alpha, \beta, \phi_i \rangle_{0,3,d}^Y \phi^i \cdot e^{\langle u, d \rangle} \]
Lemma 7.30 For a large $M' > 0$, we have
\[ \pi^* \{ \sigma \in H^2(X, \mathbb{C}) \mid \text{Re}(\sigma, d) < -M', \forall d \in \text{Eff}^*(X) \} \times \{ c \cdot \gamma | c \in \mathbb{C}, \text{Re}(c) < -M' \} \subset U. \]

Proof Any algebraic curve $C$ is homologous to a curve $C_0 \cup \bigcup_{j=1}^{r} C_j \cup \bigcup_{i=1}^{l} F_i$ where $C_0$ is contained in $i_0(X)$, $C_j$ are contained in $\mathbb{P}(L_j) \subset Y$, and $F_i$ $(i = 1, \ldots, l)$ are contained in fibers of $\pi : Y \rightarrow X$. We have
\[
\sum_{j=0}^{r} \text{Re}(\sigma + c_\gamma , [C_j]) + \sum_{i=1}^{l} \text{Re}(\sigma + c_\gamma , [F_i]) \leq \sum_{j=0}^{r} \text{Re}(\sigma, C_j) + \sum_{i=1}^{l} \text{Re}(c, \gamma, [F_i])
\]
Then, the claim of the lemma is clear.

7.3.3 Degenerated quantum products on $H^*(Y)$ and $H^*(X)$

Let $\text{Eff}(Y, \gamma)$ denote the set of $[C] \in \text{Eff}(Y)$ such that $\langle \gamma , [C] \rangle = 0$. We set $\text{Eff}^*(Y, \gamma) := \text{Eff}(Y, \gamma) \cap \text{Eff}^*(Y)$. Let $\sigma \in U_X$. As in [28], by taking the limit of $\bullet_{(\sigma, c)}$ for $\text{Re}(c) \rightarrow -\infty$, we obtain the following product $\bullet_{\sigma}$ on $H^*(X)$:
\[
\alpha \bullet_{\sigma} \beta := \sum_{d \in \text{Eff}(Y, \gamma)} \sum_{i=1}^{\chi(Y)} \langle \alpha, \beta \rangle_{0,3,d} \cdot \phi^i \cdot e^{(\sigma, \pi, d)}
\]

As in [7, 3.1] let $N : H^*(Y) \rightarrow H^*(Y)$ be the endomorphism given by the cup product of $\gamma$, i.e., $N(\beta) := \gamma \cup \beta$ for $\beta \in H^*(Y)$. Note that $\gamma \bullet_{\sigma} \beta = \beta \bullet_{\sigma} \gamma = \beta \cup \gamma$ for any $\beta \in H^*(Y)$, which follows from the divisor axiom. Hence, $\text{Im}(N)$ and $\text{Ker}(N)$ are the ideal of the algebra $(H^*(Y), \bullet_{\sigma})$. By the natural isomorphism of vector spaces $H^*(X) \cong \text{Cok}(N)$, we obtain the induced product $\bullet_{\sigma}^{\text{Cok}}$ on $H^*(X)$. The algebra $(H^*(X), \bullet_{\sigma}^{\text{Cok}})$ is denoted by $Q_\sigma H^*(X, \text{Cok}^\vee)$ in this paper. The multiplication $\text{Cok}(N) \times \text{Ker}(N) \rightarrow \text{Ker}(N)$ and the natural isomorphism $\text{Ker}(N) \cong H^*(X)$ induces a structure of $Q_\sigma H^*(X, \text{Cok}^\vee)$-module on $H^*(X)$. The $Q_\sigma H^*(X, \text{Cok}^\vee)$-module is denoted by $K_\sigma H^*(X, \text{Cok}^\vee)$. The multiplication of $b_1 \in Q_\sigma H^*(X, \text{Cok}^\vee)$ and $b_2 \in K_\sigma H^*(X, \text{Cok}^\vee)$ is denoted by $b_1 \bullet_{K_\sigma} b_2$.

7.3.4 Filtration and the graded pairings

Let $W(N)$ be the weight filtration of the nilpotent map $N$ on $H^*(Y)$. For any $\sigma \in U_X$, by shifting the filtrations of Konishi-Minabe in [28], we set
\[
\mathbb{W}_{k} Q_\sigma H^*(X, \text{Cok}^\vee) := \text{Im} \left( W(N)_{k+n+r} H^*(Y) \rightarrow Q_\sigma H^*(X, \text{Cok}^\vee) \right).
\]

Here, $Q_\sigma H^*(X, \text{Cok}^\vee)$ is identified with $\text{Cok}(N)$. Note that
\[
\text{Gr}_{k-n-r}^{\mathbb{W}} Q_\sigma H^*(X, \text{Cok}^\vee) \simeq \begin{cases} P \text{Gr}^{W(N)}_k H^*(Y) & (k \geq 0) \\ 0 & (k < 0) \end{cases}
\]

Here, $P \text{Gr}^{W(N)}_k H^*(Y)$ is the primitive part, i.e., the kernel of $N^{k+1} : \text{Gr}^{W(N)}_k H^*(Y) \rightarrow \text{Gr}^{W(N)}_{k-2} H^*(Y)$. The Poincaré pairing $(\cdot, \cdot)_Y$ on $H^*(Y)$ and the nilpotent map $N$ induces the following symmetric pairing on $\text{Gr}^{\mathbb{W}}_{k-n-r} Q_\sigma H^*(X, \text{Cok}^\vee)$:
\[
P^{\text{Cok}^\vee}_{k-n-r}(a, b) := (a, N^k b)_Y
\]

It is easy and standard to see that the pairings are non-degenerate.

We also have the filtration $W$ on $K_\sigma H^*(X, \text{Cok}^\vee)$ given as follows:
\[
\mathbb{W}_{k} K_\sigma H^*(X, \text{Cok}^\vee) := K_\sigma H^*(X, \text{Cok}^\vee) \cap W(N)_{k+n+r} H^*(Y).
\]

85
Here $K_\sigma H^*(X, E^\vee)$ is identified with $\text{Ker}(N)$. We have
\[
\text{Gr}_{k-n-r}^W K_\sigma H^*(X, E^\vee) \simeq \begin{cases} 
  P_k^\prime \text{Gr}_{k}^W(N) H^*(Y) & (k \leq 0), \\
  0 & (k > 0).
\end{cases}
\]

Here, $P_k^\prime \text{Gr}_{k}^W(N) H^*(Y)$ is the image of $P \text{Gr}_{k}^W(N) H^*(Y)$ by $N^{-k}$. Hence, we have the induced pairings $(\cdot, \cdot)_{k-n-r}$ on $\text{Gr}_{k-n-r}^W K_\sigma H^*(X, E^\vee)$.

Remark 7.31 Consider the filtration $L_{-n-r-2} := 0 \subset L_{-n-r-1} := \text{Im} N \subset L_{-n-r} := H^*(Y)$. We have the relative monodromy filtration $W(N; L)$ of $N$ with respect to $L$. See [22, 55]. It induces the filtration on $\text{Cok}(N)$, which is exactly the above filtration. Similarly, consider the filtration $L_{-n-r-1} := 0 \subset L_{-n-r} = \text{Ker} N \subset L_{-n-r+1} := H^*(Y)$. Then, according to [22] and [55], we have the relative monodromy filtration $W(N; L')$ of $N$ with respect to $L'$. It induces the filtration on $\text{Ker}(N)$ which is exactly the above filtration. See [52] for more thorough on relative weight filtrations. See also [36] for a summary.

7.3.5 Appendix

Let us recall the relation between the degenerated quantum products and local Gromov-Witten invariants in some special cases. We essentially follow [28].

Let $d \in \text{Eff}^*(X)$. Let $X_{0,3,d}$ be the moduli stack of stable maps $f : C \to X$ where $C$ denotes a 3-pointed pre-stable curve with genus 0, and $f$ denotes a morphism such that the homology class of $f(C)$ is $d$. Let $p : C_{0,3,d} \to X_{0,3,d}$ be the universal curve, and let $\mu : C_{0,3,d} \to X$ be the universal map. Let us consider the following concavity condition for $d$:

(B) $H^0(C, f^*E^\vee) = 0$ for any $(C, f) \in X_{0,3,d}$.

Recall that $E$ is the direct sum of the nef line bundles $L_i$ $(i = 1, \ldots, r)$. We recall the following standard lemma.

Lemma 7.32 Suppose that $\langle c_1(L_i^\vee), d \rangle < 0$ for any $i$. Then, Condition (B) holds for $d$.

Proof Take $d \in \text{Eff}^*(X)$ and $(C, f) \in X_{0,3,d}$. Let $C = \bigcup C_a$ be the irreducible decomposition. Because $L_i$ is nef, the degree of $f^*(L_i^\vee)|_{C_a}$ is non-negative for any $a$. By the assumption $\langle c_1(L_i^\vee, d) \rangle < 0$, we have an irreducible component $C_a$ of $C$ such that the degree of $f^*(L_i^\vee)|_{C_a}$ is strictly negative. We also remark that $C$ is connected. Then, we have $H^0(C, f^*L_i) = 0$, i.e., Condition (B) holds for $d$.

If Condition (B) holds for $d$, we obtain the locally free sheaf $R^1p_*\mu^*E^\vee$ on $X_{0,3,d}$. Let $[X_{0,3,d}]^\text{vir}$ denote the virtual fundamental class of $X_{0,3,d}$. Let $ev_i : X_{0,3,d} \to X$ be the evaluation map at the $i$-th marked point. For any vector bundle $E$, let $e(E)$ denote the Euler class. Then, for any $\alpha_i \in H^*(X)$ $(i = 1, 2, 3)$, we set
\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,d}^{X, E^\vee, \text{vir}} := \int_{[X_{0,3,d}]^\text{vir}} \prod_{i=1,2,3} ev_i^*(\alpha_i) \cdot e(Rp_*\mu^*E^\vee).
\]

They are called $e^{-1}$-twisted Gromov-Witten invariants or local Gromov-Witten invariants.

For any $\alpha, \beta \in H^*(X)$, let $\alpha \cup \beta$ denote the ordinary cup product. We take a homogeneous base $\rho_1, \ldots, \rho_t$ of $H^*(X)$. Let $\rho^1, \ldots, \rho^t$ denote the dual base of $H^*(X)$ with respect to the Poincaré pairing of $X$.

Proposition 7.33 Suppose one of the following holds:

Case 1. $X$ is Fano, and $L_i$ $(i = 1, \ldots, r)$ are ample. Note that Condition (B) holds for any $d \in \text{Eff}^*(X)$.
Case 2. $X$ is a weak Fano surface, $r = 1$, and $\mathcal{L}_1 = K_X^{-1}$. Note that Condition (B) holds for any $d \in \text{Eff}^*(X)$ with $\langle c_1(K_X), d \rangle \neq 0$.

Then, we have the following for $\sigma \in \mathcal{U}_X$ and for $\alpha, \beta \in Q_\sigma H^*(X, \mathcal{E}^\vee)$:

$$\alpha \bullet_\sigma^\mathcal{E} \beta = \begin{cases} \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(Case 1)} \\ \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X), (c_1(K_X), d) \neq 0} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(Case 2)} \end{cases}$$

(134)

We also have the following for $\sigma \in \mathcal{U}_X$ and for $\alpha \in Q_\sigma H^*(X, \mathcal{E}^\vee)$, and $\beta \in K_\sigma H^*(X, \mathcal{E}^\vee)$:

$$\alpha \bullet_{K, \sigma}^\mathcal{E} \beta = \begin{cases} \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(Case 1)} \\ \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X), (c_1(K_X), d) \neq 0} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(Case 2)} \end{cases}$$

(135)

Proof For the proof of the proposition, we study the degenerated quantum products $\bullet_\sigma$ on $H^*(X)$. Any element $\alpha \in H^*(Y)$ has the expression $\alpha = \sum_{j=0}^\ell \pi^* \alpha_j \cdot \gamma^j$ where $\alpha_j \in H^*(X)$.

Lemma 7.34 Let $\sigma \in \mathcal{U}_X$. Suppose Case 1 or Case 2. For $\alpha = \sum_{j=0}^\ell \pi^* (\alpha_j) \gamma^j$ and $\beta = \sum_{j=0}^\ell \pi^* (\beta_j) \gamma^j$, we have the following:

$$\begin{align*} \alpha \bullet_\sigma \beta &= \begin{cases} \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(Case 1)} \\ \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X), (c_1(K_X), d) \neq 0} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(Case 2)} \end{cases} \\ &= \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(135)} \end{align*}$$

Before giving a proof of Lemma 7.34, we deduce Proposition 7.33 from Lemma 7.34. We give an argument in Case 1. The other case can be given similarly. Because $i_0^* i_{0*}(\rho^j) = e(\mathcal{E}^\vee) \rho^j$, we obtain the following for $\alpha, \beta \in H^*(X)$ from (135):

$$\begin{align*} \alpha \bullet^\mathcal{E} \beta &= \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) e(\mathcal{E}^\vee) \rho_i \\ &= \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, \beta, e(\mathcal{E}^\vee) \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \quad \text{(137)} \end{align*}$$

Thus, we obtain (134). We identify $\text{Ker}(N) = i_0_\sigma H^*(X)$. We have $i_{0*}(\beta) = e(\mathcal{E}^\vee) \beta + \sum_{j=1}^\ell c_j \gamma^j$ for some $c_j \in H^*(X)$. Hence we have the following for $\alpha \in H^*(X)$ and $\beta \in H^*(X)$:

$$\begin{align*} i_{0*}(\alpha \bullet^\mathcal{E} \beta) &= \pi^*(\alpha) \bullet^\mathcal{E} \pi^*(\beta) = \pi^*(\alpha) \cup i_{0*} \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, e(\mathcal{E}^\vee) \beta, e(\mathcal{E}^\vee) \beta \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) i_{0*} \rho_i \\ &= i_{0*} \left( \alpha \cup \beta + \sum_{d \in \text{Eff}^*(X)} \sum_{i=1}^\ell \langle \alpha, e(\mathcal{E}^\vee) \beta, e(\mathcal{E}^\vee) \beta \rangle_{X,\mathcal{E}^\vee,\mathcal{E}^\vee, -1}^{X,\mathcal{E}^\vee, -1} e(\sigma, d) \rho_i \right) \quad \text{(138)} \end{align*}$$
Thus, we obtain \(135\). It remains to prove Lemma 7.34.

Let us prove Lemma 7.34 in Case 1. For any algebraic curve \(C\) in \(Y\), let \([C] \in H_2(Y)\) denote the homology class of \(C\).

Lemma 7.35 Let \(C\) be a non-empty algebraic curve in \(Y\). Then, we have \(\langle \gamma, [C] \rangle = 0\) if and only if \(C \subset i_0(X)\).

Proof Because \(\mathcal{O}_Y(1) = \mathcal{O}(H_\infty)\), the second condition implies the first. Let us prove that the first condition implies the second. We may assume that \(C\) is irreducible. If \(C \subset H_\infty\), we clearly have \(\langle \gamma, [C] \rangle > 0\). Hence, we have \(C \not\subset H_\infty\). If \(C \cap H_\infty \neq \emptyset\), we have \(\langle \gamma, [C] \rangle > 0\). Hence, we obtain that \(C \cap H_\infty = \emptyset\). Let \(\varphi : \tilde{C} \to C \subset E^Y\) be the normalization. Let \(q : E^Y \to X\) be the projection. Note that \(q(C)\) is not a point. We obtain the morphism \(q \circ \varphi : \tilde{C} \to X\) and a section \(s\) of \((q \circ \varphi)^*E^Y\). Because \((q \circ \varphi)^*L_i\) are ample, we have \(s = 0\). It means \(C \subset i_0(X)\). Thus, we obtain Lemma 7.35.

Lemma 7.36 For \(d \in \text{Eff}^i(Y, \gamma)\) and for \(i > 0\), we have \(\langle \alpha_1, \alpha_2, \gamma^i \alpha_3 \rangle_{0,3,d}^Y = 0\).

Proof The support of the cohomology class \(\gamma^i \alpha_3\) is contained in \(H_\infty\). For \((C, f) \in Y_{0,3,d}\), we have \(f(C) \subset i_0(X)\). Then, the claim of Lemma 7.36 is clear.

By Lemma 7.35, we have \(\text{Eff}^*(Y, \gamma) = \text{Eff}^*(X)\). By Lemma 7.36 for \(d \in \text{Eff}^*(Y, \gamma)\) and for \(\alpha_i = \sum_{j=0}^r \pi^* \alpha_{i,j} \gamma^j\), we have

\[
\langle \alpha_1, \alpha_2, \gamma_3 \rangle_{0,3,d}^Y = \langle \pi^* \alpha_{1,0}, \pi^* \alpha_{2,0}, \pi^* \alpha_{3,0} \rangle_{0,3,d}^Y = \int_{[X_{0,3,d}]^{vir}} \prod_{i=1}^{ev}_{i}^* \langle \alpha_{i,0} \rangle \cdot c(R^1 p_* \mu^* E^Y)
\]

\[
= \langle \alpha_{1,0}, \alpha_{2,0}, \alpha_{3,0} \rangle_{0,3,d}^X, E^Y, e^{-1} \tag{139}
\]

We set \(\phi_{ij} := \rho_i \gamma^j (1 \leq i \leq \ell, 0 \leq j \leq r)\) which give a frame of \(H^*(Y)\). Let \(\phi_{ij} (1 \leq i \leq \ell, 0 \leq j \leq r)\) be the dual base with respect to the Poincaré pairing of \(Y\). Note that \(\phi_{ii} = \delta_{ii}(\rho_i^t)\). Indeed, it is enough to check \(\langle \phi_{ii}, \alpha_i(\rho_i^t) \rangle = 0 (\ell > 0), \langle \phi_{kk}, \alpha_k(\rho_k^t) \rangle = 0 (k \neq i)\) and \(\langle \phi_{ks}, \alpha_s(\rho_s^t) \rangle = 1\), which can be checked by direct computations. Then, we obtain the equality \(139\) in Case 1.

Let us prove \(136\) in Case 2. It is exactly the case given in \(28\). We just revisit it in a slightly different way. Let \(d \in \text{Eff}^*(Y, \gamma)\). Note that the virtual dimension of \(Y_{0,3,d}\) is 3. Indeed, because we have \(K_Y = \mathcal{O}_Y(2)\), the virtual dimension of \(Y_{0,n,d}\) is \(n-3 + \dim Y - \langle c_1(K_Y)\rangle\), \(d = n\).

Lemma 7.37 For \(\alpha_i = \alpha_{i,0} + \alpha_{i,1} \gamma (\alpha_{ij} \in H^*(X), i = 1, 2, 3)\), we have

\[
\langle \alpha_1, \alpha_2, \gamma_3 \rangle_{0,3,d}^Y = \langle \alpha_{1,0}, \alpha_{2,0}, \alpha_{3,0} \rangle_{0,3,d}^Y
\]

Proof By the dimension reason, we have \(\langle \alpha_{1,0}, \alpha_{2,0}, \alpha_{3,0} \rangle_{0,3,d}^Y = 0\) unless the cohomological degree of \(\alpha_i\) are 2. Moreover, by the divisor axiom, we have \(\langle \alpha_{1,2}, \alpha_{3,0} \rangle_{0,3,d}^Y = 0\) if one of \(\alpha_i\) is \(\gamma\). Then, the claim of Lemma 7.37 follows.

Lemma 7.38 Suppose that \(d \in \text{Eff}^*(Y, \gamma)\) satisfies \(\langle \pi^* c_1(K_X) \rangle, d = 0\). Then, we have \(\langle \alpha_1, \alpha_2, \gamma_3 \rangle_{0,3,d}^Y = 0\) for any \(\alpha_i \in H^*(Y)\).

Proof It is enough to consider the case where \(\alpha_i \in \pi^* H^2(X)\). Let us observe that the induced morphism \(Y_{0,3,d} \to X_{0,3,d}\) is smooth and the fibers are \(\mathbb{P}^1\). Take \((C, g) \in X_{0,3,d}\). Because \(K_X^d\) is nef, and because we have \(\langle c_1(K_X^d), [g(C)] \rangle = 0\), the restriction of \(g^* K_X\) to each irreducible component of \(C\) is of degree 0. In particular, we have \(H^1(C, g^* K_X) = 0\), and \(\dim H^0(C, g^* K_X) = 1\). Let \(C_{0,3,d}\) denote the universal curve over \(X_{0,3,d}\). Let \(\mu : C_{0,3,d} \to X\) be the universal morphism. Let \(p : C_{0,3,d} \to X_{0,3,d}\) denote the projection. Then, \(p_* \mu^*(\mathcal{O} \otimes K_X^d)\) is a locally free sheaf, and the projectivization is naturally isomorphic to \(Y_{0,3,d}\). Then, it is easy to see that \(\langle Y_{0,3,d} \rangle^{vir}\) is the pull back of \(\langle X_{0,3,d} \rangle^{vir}\) via the natural morphism \(q : Y_{0,3,d} \to X_{0,3,d}\). Because \(q_* (\langle Y_{0,3,d} \rangle^{vir}) = 0\), we obtain \(\langle \alpha_1, \alpha_2, \gamma_3 \rangle_{0,3,d}^Y = 0\). Thus, we obtain Lemma 7.38.
Lemma 7.39 Let \( d \in \text{Eff}(Y, \gamma) \). If \( (\pi^* c_1(K_X), d) < 0 \), then we have \( f(C) \subset i_0(X) \) for any \((C, f) \in Y_{0,3,d} \). In particular, \( d \) comes from \( \text{Eff}^*(X) \).

Proof Let \((C, f) \in Y_{0,3,d} \). We have \( f(C) = \sum m_i [C_i] \) \((m_i > 0)\) for algebraic curves in \( Y \). Note that \( C \) is connected. Suppose \( f(C) \cap H_\infty \neq \emptyset \). If \( f(C) \not\subset H_\infty \), we have \(< \gamma_i, [C_i] > 0 \) for some \( i_0 \) and hence \(< \gamma, [f(C)] > 0 \) which contradicts with the assumption. If \( f(C) \subset H_\infty \), we have \(< \gamma, f(C) > = < c_1(K_X), d > 0 \) which also contradicts with the assumption. Hence, we have \( f(C) \cap H_\infty = \emptyset \). Then, \( f \) is equivalent to a morphism \( g : C \to X \) and a section \( s \) of \( g^*(K_X) \). But, because \(< c_1(K_X), d > < 0 \), we have \( s = 0 \). Thus, we obtain Lemma 7.39.

We can deduce the equality (139) in Case 2 from Lemma 7.34, Lemma 7.38 and Lemma 7.39 as in Case 1. Thus, Lemma 7.34 and Proposition 7.33 are proved.

7.4 Associated quantum \( \mathcal{D} \)-modules

We continue to use the notation in (7.3). We recall some basic matters on the quantum \( \mathcal{D} \)-modules of toric varieties. Again, see [19], [20], [27], [28], [40], [41] for more details.

7.4.1 Quantum \( \mathcal{D} \)-modules associated to degenerated quantum products on \( H^*(X) \)

We set \( \text{QDM}(X, \mathcal{E}^r) := \mathcal{O}_{\mathbb{C}^r} \otimes \mathcal{O}_{\mathcal{U}_X} \otimes H^*(X) \). As in [28], we have the meromorphic connection \( \nabla^{\mathcal{E}^r} \) on \( \text{QDM}(X, \mathcal{E}^r) \) which is a variant of the connections of Dubrovin and Givental. It is given as follows. Let \( \mu_X \) be the grading operator on \( H^*(X) \) defined by \( \mu(a) := ka \) for \( a \in H^{2k}(X) \). We can naturally identify \( H^2(X) \otimes \mathcal{O}_{\mathcal{U}_X} \) with the tangent sheaf of \( \mathcal{U}_X \), and so can regard \( \xi \in H^2(X) \) as a vector field on \( \mathcal{U}_X \). We can naturally regard \( b \in H^*(X) \) as a section of \( \text{QDM}(X, \mathcal{E}^r) \). Then, \( \nabla^{\mathcal{E}^r} \) is determined by the following on \( \mathbb{C}^r \times \mathcal{U}_X \):

\[
(\nabla^{\mathcal{E}^r}_\xi b)_{(\lambda, \sigma)} = -\lambda^{-1} \xi \delta_{\sigma} b, \quad (\nabla^{\mathcal{E}^r}_{\lambda \partial_{\sigma}} b)_{(\lambda, \sigma)} = \lambda^{-1} E \delta_{\sigma} b + \mu_X(b), \quad ((\lambda, \sigma) \in \mathbb{C}^r \times \mathcal{U}_X).
\]

The filtrations \( \widetilde{W} \) of \( \text{QDM}(X, \mathcal{E}^r) \) \((\sigma \in \mathcal{U}_X)\) give filtration \( \widetilde{W} \) on \( \text{QDM}(X, \mathcal{E}^r) \). It is preserved by the connection \( \nabla^{\mathcal{E}^r} \). We have the induced meromorphic connection on \( \text{Gr}^{\mathcal{E}^r}_k \text{QDM}(X, \mathcal{E}^r) \), denoted by the same notation.

We regard \( b_i \in \text{Gr}^{\mathcal{E}^r}_k \text{QDM}(X, \mathcal{E}^r) \) \((i = 1, 2)\) as sections of \( \text{Gr}^{\mathcal{E}^r}_k \text{QDM}(X, \mathcal{E}^r) \). We set

\[
\mathcal{P}^{\mathcal{E}^r}_k(b_1, b_2) := \lambda^{-k} \mathcal{P}^{\mathcal{E}^r}_k(b_1, b_2).
\]

Thus, we obtain a morphism \( \mathcal{P}^{\mathcal{E}^r}_k : \text{Gr}^{\mathcal{E}^r}_k \text{QDM}(X, \mathcal{E}^r) \otimes j^* \text{Gr}^{\mathcal{E}^r}_k \text{QDM}(X, \mathcal{E}^r) \to \lambda^{-k} \mathcal{O}_{\mathbb{C}^r \times \mathcal{U}_X} \).

We will give a proof of the following lemma by using the description as a specialization of the quantum \( \mathcal{D} \)-module of \( Y \) in (7.3.3).

Lemma 7.40 \( \nabla^{\mathcal{E}^r} \) is flat, and \( \mathcal{P}^{\mathcal{E}^r}_k \) are pairings of weight \( k \) on \((\text{Gr}^{\mathcal{E}^r}_k \text{QDM}(X, \mathcal{E}^r), \nabla^{\mathcal{E}^r}) \). In other words, \((\text{QDM}(X, \mathcal{E}^r), \widetilde{W}, \{\mathcal{P}^{\mathcal{E}^r}_k\})\) is a mixed TEP-structure.

We also set \( \mathcal{K}(X, \mathcal{E}^r) := \mathcal{O}_{\mathbb{C}^r} \otimes H^*(X) \). The flat connection \( \nabla^{\mathcal{E}^r} \) on \( \mathcal{K}(X, \mathcal{E}^r) \) is equipped with the meromorphic flat connection \( \nabla^{\mathcal{E}^r} \) that satisfies the following:

\[
(\nabla^{\mathcal{E}^r}_\xi b)_{(\lambda, \sigma)} = -\lambda^{-1} \xi \delta_{\sigma} b, \quad (\nabla^{\mathcal{E}^r}_{\lambda \partial_{\sigma}} b)_{(\lambda, \sigma)} = \lambda^{-1} E \delta_{\sigma} b + \mu_X(b) + rb, \quad ((\lambda, \sigma) \in \mathbb{C}^r \times \mathcal{U}_X)
\]

Here, \( b \in H^*(X) \) and \( \xi \in H^2(X) \). The filtration \( \widetilde{W} \) is preserved by \( \nabla^{\mathcal{E}^r} \). We obtain a morphism \( \mathcal{P}^{\mathcal{E}^r}_k : \text{Gr}^{\mathcal{E}^r}_k \mathcal{K}(X, \mathcal{E}^r) \times j^* \text{Gr}^{\mathcal{E}^r}_k \mathcal{K}(X, \mathcal{E}^r) \to \lambda^{-k} \mathcal{O}_{\mathbb{C}^r \times \mathcal{U}_X} \) by the formula:

\[
\mathcal{P}^{\mathcal{E}^r}_k(b_1, b_2) = (-1)^{k-n-r} \lambda^{-k} \mathcal{P}^{\mathcal{E}^r}_k(b_1, b_2)
\]

We will give a proof of the following lemma by using the description as a specialization of the quantum \( \mathcal{D} \)-module of \( Y \) in (7.3.3).

Lemma 7.41 \( \nabla^{\mathcal{E}^r} \) is flat, and \( \mathcal{P}^{\mathcal{E}^r}_k \) are pairings of weight \( k \) on \((\text{Gr}^{\mathcal{E}^r}_k \mathcal{K}(X, \mathcal{E}^r), \nabla^{\mathcal{E}^r}) \). In other words, \((\mathcal{K}(X, \mathcal{E}^r), \widetilde{W}, \{\mathcal{P}^{\mathcal{E}^r}_k\})\) is a mixed TEP-structure.
7.4.2 Quantum $\mathcal{D}$-module of $Y$

We recall the quantum $\mathcal{D}$-module associated to the quantum products of $Y$. We set $\mathcal{U}_γ := \mathcal{U}_γ/2\pi \sqrt{-1}T\mathbb{Z}_γ$. We set $\mathcal{U}_γ^1 := \mathcal{U}_γ \times \mathcal{U}_γ$ which is the quotient of $\mathcal{U}_γ$ by the action of $2\pi \sqrt{-1}T\mathbb{Z}_γ$. We embed $\mathcal{U}_γ^1 \to \mathbb{C}^*$ induced by $c \mapsto \exp(c)$. The quantum products $\bullet_{\sigma,e}$ depends only on $(\sigma,e')$. So, we denote them by $\bullet_{\sigma,e'}$.

We set $\text{QDM}(Y) := \mathcal{O}_{\mathcal{C}_X \times \mathcal{U}_γ^1} \otimes H^*(Y)$. Recall that we have the meromorphic flat connection $\nabla^G$ on $\text{QDM}(Y)$ due to Dubrovin and Givental. Let $\mu_Y$ be the grading operator on $H^*(Y)$, given by $\mu_Y(b) := kb$ for $b \in H^{2k}(Y)$. We naturally regard $\xi \in \mathbb{H}^2(Y)$ as a vector field on $\mathcal{U}_γ$, which induces a vector field on $\mathcal{U}_γ^1$. We also naturally regard $b \in H^*(Y)$ as a section of $\text{QDM}(Y)$. Then, $\nabla^G$ is determined by the following on $\mathcal{C}_X \times \mathcal{U}_γ^1$:

$$\begin{align*}
\left(\nabla^G_{\xi} b\right)_{(\lambda,\sigma,t)} &= -\lambda^{-1} \xi \bullet_{(\sigma,t)} b, \\
\left(\nabla^G_{\lambda \partial b}\right)_{(\lambda,\sigma,t)} &= \lambda^{-1} E \bullet_{(\sigma,t)} b + \mu_Y(b), \\
&\quad ((\lambda,\sigma,t) \in \mathbb{C}^* \times \mathcal{U}_γ^1).
\end{align*}$$

It is equipped with the pairing $\mathcal{P}_Y$ of weight $-n - r$ induced by the Poincaré pairing:

$$\mathcal{P}_Y(b_1,j \ast b_2) := \lambda^{n+r} \left(b_1, b_2\right)$$

It is well known that $\mathcal{P}_Y$ is $\nabla^G$-flat. It is clearly equivariant with respect to the action of $2\pi \sqrt{-1}H^2(X,\mathbb{Z})$.

7.4.3 Logarithmic extension and the specialization

Put $\mathcal{U}_γ^1 := \mathcal{U}_γ \cup \{0\}$ which is a neighbourhood of 0 in $\mathbb{C}$. We set $\overline{\mathcal{U}_γ^1} := \mathcal{U}_X \times \mathcal{U}_γ^1$. We set $\overline{\text{QDM}(Y)} := \mathcal{O}_{\mathcal{C}_X \times \overline{\mathcal{U}_γ^1}} \otimes H^*(Y)$. We naturally regard $\overline{\text{QDM}(Y)}_{\mu_{\overline{\mathcal{U}_γ^1}}} = \overline{\text{QDM}(Y)}$. The meromorphic flat connection $\nabla^G$ of $\overline{\text{QDM}(Y)}$ naturally gives a meromorphic flat connection on $\overline{\text{QDM}(Y)}$ which is logarithmic along $t = 0$. The pairing $\mathcal{P}_Y$ is naturally extended to the following morphism which is also denoted by $\mathcal{P}_Y$:

$$\overline{\text{QDM}(Y)} \otimes j^* \overline{\text{QDM}(Y)} \to \lambda^{n+r} \mathcal{O}_{\mathcal{C}_X \times \overline{\mathcal{U}_γ^1}}.$$ 

We set $\overline{\text{QDM}(Y)}^{sp} := \mathcal{O}_{\mathcal{C}_X \times \mathcal{U}_X} \otimes H^*(Y)$. We naturally have $\text{QDM}(Y) \to \overline{\text{QDM}(Y)}_{\mu_{\mathcal{U}_X}} = \overline{\text{QDM}(Y)}$. The residue $\text{Res}_t(\nabla^G) : \lambda \overline{\text{QDM}(Y)}^{sp} \to \overline{\text{QDM}(Y)}^{sp}$ is given by the multiplication of $-\lambda^{-1} \gamma$. The specialization of the connection $\nabla^G$ is induced by the degenerated quantum products:

$$\begin{align*}
\left(\nabla^G_{\xi} b\right)_{(\lambda,\sigma)} &= -\lambda^{-1} \xi \bullet_{(\sigma)} b, \\
\left(\nabla^G_{\lambda \partial b}\right)_{(\lambda,\sigma)} &= \lambda^{-1} E \bullet_{(\sigma)} b + \mu_Y(b), \\
&\quad ((\lambda,\sigma) \in \mathcal{C}_X \times \mathcal{U}_X)
\end{align*}$$

It is equipped with the pairing $\mathcal{P}_Y$ of weight $-n - r$ given by the formula (140).

By the construction $\psi_{-\varepsilon}(\overline{\text{QDM}(Y)}(st))$ is naturally isomorphic to $\overline{\text{QDM}(Y)}^{sp}$ with $\nabla^G^{sp}$. By the construction, we have the following isomorphisms which are compatible with the meromorphic connections:

$$\text{Cok}(\lambda \text{Res}_t(\nabla^G)) \simeq \text{QDM}(X, E^\vee), \quad \text{Ker}(\lambda \text{Res}_t(\nabla^G)) \simeq K(X, E^\vee).$$

By the construction, we have the following which implies Lemma 7.40 and Lemma 7.41.

**Lemma 7.42** The weight filtration $\widehat{W}$ on $\text{QDM}(X, E^\vee)$ and the pairings $\mathcal{P}_Y^{E^\vee}$ on $\text{Gr}^W_k \text{QDM}(X, E^\vee)$ are constructed from $(\text{QDM}(Y), \nabla^G, \mathcal{P}_Y)$ by the procedure in B.1.3.

7.5 Local mirror isomorphism

Recall that the isomorphism $L_X^\vee \simeq H^2(X,\mathbb{Z})$ is induced by $M_X^\vee : \to H^2(X,\mathbb{Z})$ given by $\sum_{i=1}^m a_i e_i^\vee \to \sum a_i |D_i|$, where $e_i^\vee (i = 1,\ldots,m)$ is the dual frame of the standard basis $e_1,\ldots,e_m$.

Let $\eta_i,\ldots,\eta_l$ be a frame of $H^2(X,\mathbb{Z})$ such that the first Chern class of $K_X^\vee \otimes \bigotimes_{i=1}^l L_i^\vee$ is contained in the cone generated by $\eta_i (i = 1,\ldots,l)$. We can check the existence of such a frame in an elementary way. We have a frame $\xi_1,\ldots,\xi_l$ of $L_X^\vee$ corresponding to $\eta_1,\ldots,\eta_l$ by the above isomorphism. For $M \in \mathbb{R}$, we set

$$B_M(\eta) := \left\{ \sum_{i=1}^l a_i \eta_i \in H^2(X,\mathbb{C}) \mid \text{Re}(a_i) < -M (i = 1,\ldots,l) \right\},$$
\[ B_M'(\xi) := \left\{ \sum_{i=1}^{\ell} \beta_i \xi_i \in \mathfrak{S}_A = L_X^\vee \otimes \mathbb{C} \mid \text{Re}(\beta_i) < -M \ (i = 1, \ldots, \ell) \right\}. \]

We obtain the following theorem as a corollary of the theorem of Givental, according to which we have isomorphisms between the mixed TEP-structures associated to local A-models and local B-models in \([27, 28]\). It is also related with \([41, \text{Conjecture 6.14}]\). We use the notation in \([7.2]\) and \([7.4]\).

**Theorem 7.43** We have the following:

1. An open subset \( U_1 \subset \mathcal{U}_X \subset H^2(X, \mathbb{C}) \); It contains \( B_{M_1}(\eta) \) for a positive number \( M_1 \); It is preserved by the natural action of \( 2\pi\sqrt{-1}H^2(X, \mathbb{Z}) \).

2. An open subset \( U_2 \subset \mathcal{S}^{\text{reg}}_A \); It contains \( B_{M_2}(\xi) \) for a positive number \( M_2 \); It is preserved by the natural action of \( 2\pi\sqrt{-1}\mathbb{Z}L_X^\vee \).

3. A holomorphic isomorphism \( \varphi : U_1 \simeq U_2 \); It preserves the Euler vector fields.

4. Isomorphism of mixed TEP-structures

\[ \Phi : (\text{QDM}(X, E^\vee), \tilde{W}, \{\alpha P^E_k | k \in \mathbb{Z}\})_{|U_1} \simeq \varphi^* \chi^*_A(\mathfrak{W}_A, \tilde{W}, \{\mathfrak{P}_{A, *, k} | k \in \mathbb{Z}\})_{|U_2} \]

\[ \Phi_1 : (K(X, \mathcal{E}^\vee), \tilde{W}, \{\alpha P^E_k | k \in \mathbb{Z}\})_{|U_1} \simeq \varphi^* \chi^*_A(\mathfrak{W}_A, \tilde{W}, \{\mathfrak{P}_{A, *, k} | k \in \mathbb{Z}\})_{|U_2} \]

**Remark 7.44** If the Euler vector fields are 0, then the \( \tilde{R} \)-modules are \( \mathbb{C}^* \)-homogeneous with respect to the trivial action on the base space. So, they are equipped with the Hodge filtration. Because the Hodge filtration can be recovered from the connection, the isomorphisms in Theorem 7.43 also preserve the Hodge filtrations.
Remark 7.45 If \( r = 1 \) and \( L_1 = K_X \), the mixed TEP-structure is expressed as in [6.4] in terms of the variation of Hodge structure associated to \( B \). Here, \( B \) is related to \( A \) as in [6.4.1].

Remark 7.46 We set \( U'_1 := U_1/(2\pi \sqrt{-1}H^2(X,Z)) \) and \( U'_2 := U_2/(2\pi \sqrt{-1}L^2_X) \). We have the induced isomorphism \( \varphi' : U'_1 \simeq U'_2 \). We may regard \( U'_1 \subset H^2(X,Z) \otimes \mathbb{C}^* \) and \( U'_2 \subset L^2_X \otimes \mathbb{C}^* \). They are naturally extended to neighbourhoods \( \mathcal{U}'_1 \subset H^2(X) \otimes \mathbb{C} \) and \( \mathcal{U}'_2 \subset L^2_X \otimes \mathbb{C} \). At this moment, it is not clear to the author if we could also deduce the comparison of the weight filtrations from [21].

Remark 7.47 In [11], it is announced that a result in [21] implies the comparison of the TE-structures in Theorem 7.44. At this moment, it is not clear to the author if we could also deduce the comparison of the weight filtrations from [21].

A Duality of meromorphic flat bundles

A.1 Isomorphism

Let \( X \) be a complex manifold with a hypersurface \( H \). Let \( d_X := \dim X \). Let \( M \) be a reflexive meromorphic flat bundle on \((X,H)\), i.e., \( M \) is a \( \mathcal{D} \)-module and that \( M \) is a coherent and reflexive \( \mathcal{O}_X(*)H \)-module. We set \( M' := \mathcal{H}om_{\mathcal{O}_X(*)H}(M, \mathcal{O}_X(*)H) \) which is naturally a reflexive meromorphic flat connection. In this section, let \( \mathcal{D} \) denote the dual category of the \( \mathcal{D}_X \)-local system. As well known, \( \mathcal{D}_{X(*)H}(M) := \mathcal{D}(M)(*)H \) is isomorphic to \( M' \) as a meromorphic flat connection. Let us recall the construction of the isomorphism \( \nu_M : \mathcal{D}_{X(*)H}(M) \xrightarrow{\simeq} M' \) given in [36].

Let \( \Theta_X \) denote the tangent sheaf of \( X \). We set \( \Theta_X^p := \bigwedge^p \Theta_X \) for \( p \geq 0 \). We have the Spencer resolution \( \mathcal{D}_X \otimes \Theta_X^* \) of \( \mathcal{O}_X \) by a free left \( \mathcal{D} \)-modules. We set \( \Omega_X := \Omega_X^d \). The dual line bundle is denoted by \( \Omega_X^{-1} \). We have the following natural identification:

\[
\mathcal{D}_{X(*)H}(M) \simeq \mathcal{H}om_{\mathcal{D}_X(*)H}(\mathcal{D}_X \otimes \Theta_X^* \otimes M, \mathcal{D}_X \otimes \Omega_X^{-1}(*)H)[d_X].
\]

The degree 0 term in the right hand side is \( \mathcal{H}om_{\mathcal{D}_X(*)H}(\mathcal{D}_X \otimes \Omega_X^{-1} \otimes M, \mathcal{D}_X \otimes \Omega_X^{-1}) \simeq \mathcal{D}_X \otimes M' \). The isomorphism \( \nu_M \) is induced by the canonical isomorphism \( \mathcal{D}_{X(*)H}(M) \simeq \mathcal{D}_X \otimes \Theta^* \otimes M' \) whose degree 0-term is the identity.

We have another description of the isomorphism \( \nu_M \) in the case \( M \) has no poles. Let \( L_{M'} \) be the sheaf of flat section of \( M' \) which is \( \mathbb{C} \)-local system. We have a natural isomorphism \( \text{DR} \mathcal{D}_{X(*)H}(M) = R\mathcal{H}om_{\mathcal{D}_X}(M, \mathcal{O}_X[d_X]) \simeq L_{M'}[d_X] \). We have \( \text{DR}(\nu_M) = (-1)^{d_X} \text{id} \) which characterize \( \nu_M \).

Under the natural identification \( \mathcal{D}_{X(*)H} \circ \mathcal{D}_{X(*)H}(M) = M \), we have the induced morphism \( \mathcal{D}_{X(*)H} \nu_M : \mathcal{D}_{X(*)H}(M) \simeq M' \). We shall use the following lemma implicitly. (See [36].)

Lemma A.1 We have \( \mathcal{D}_{X(*)H} \nu_M = (-1)^{d_X} \nu_M \).

A.1.1 A commutative diagram

For any morphism of meromorphic flat bundles \( f : M_1 \rightarrow M_2 \) on \((X,H)\), the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{D}_{X(*)H}M_2 & \xrightarrow{\mathcal{D}_{X(*)H}f} & \mathcal{D}_{X(*)H}M_1 \\
\nu_{M_2} \downarrow & & \nu_{M_1} \downarrow \\
M_2' & \xrightarrow{f'} & M_1'
\end{array}
\]

Let \( M_i \) be reflexive meromorphic flat connections on \((X,H)\). Let \( \rho : M_1 \rightarrow \mathcal{D}_{X(*)H}M_2 \) be a morphism of \( \mathcal{D} \)-modules. We have the induced morphisms \( \nu_{M_2} \circ \rho : M_1 \rightarrow M_2' \) and \( \nu_{M_1} \circ \mathcal{D}_{X(*)H}\rho : M_2 \rightarrow M_1' \). We shall use the following lemma in Proposition 7.2.
Lemma A.2 We have $\nu_{M_i} \circ D_{X(\ast H)}\rho = (-1)^{d_X}(\nu_{M_2} \circ \rho)^\vee$.

Proof We have the following commutative diagram:

$$
\begin{array}{ccc}
D_{X(\ast H)}(M_1) & \xrightarrow{\nu_{M_1}} & D_{X(\ast H)}(M_2) \\
\downarrow & & \downarrow \\
M_1^\vee & \xleftarrow{(\nu_{M_2} \circ \rho)^\vee} & M_2
\end{array}
$$

Here, the right vertical arrow is the composite of $D_{X(\ast H)}(M_2) \xrightarrow{\nu_{M_2}^\vee} (M_2^\vee)^\vee \simeq M_2$. Note that $\nu_{M_1} \circ D_{X(\ast H)}\rho$ precisely means the composite of $M_2 \simeq D_{X(\ast H)}(D_{X(\ast H)}M_2) \xrightarrow{D_{X(\ast H)}\rho} D_{X(\ast H)}M_1 \simeq M_1^\vee$. We remark that the composite of the following natural isomorphisms is the multiplication of $(-1)^{d_X}$.

$$
M_2 \longrightarrow (M_2^\vee)^\vee \longrightarrow D_{X(\ast H)}(M_2^\vee) \xrightarrow{D_{X(\ast H)}\nu_{M_2}} D_{X(\ast H)}(D_{X(\ast H)}M_2) \longrightarrow M_2
$$

Then, the claim of Lemma A.2 follows.

A.2 Duality and localizations of flat bundles

This subsection is a preliminary for §A.3 and §3.3

A.2.1 Statement

Let $X$ be a complex manifold with a smooth hypersurface $Y$. Set $d_X := \dim X$. Let $\iota : Y \longrightarrow X$ be the inclusion. Let $M$ be a flat bundle on $X$, i.e., a locally free $\mathcal{O}_X$-module with a flat connection. The pull back of $M$ to $Y$ is denoted by $M_Y$. For any $D_X$-module $N$, we set $N(\ast Y) := N \otimes \mathcal{O}_X(\ast Y)$ and $N(!Y) := D_{X}(D_{X}N(\ast Y))$.

We have the natural morphisms $\rho_{M,1} : M(\ast Y) \longrightarrow \iota_1 M_Y$ and $\rho_{M,2} : \iota_1 M_Y \longrightarrow M(!Y)$. Locally, they are given as follows. Let $(x_1, \ldots, x_n)$ be a holomorphic local coordinate system of $X$ such that $Y = \{x_1 = 0\}$. We have $\iota_1 M_Y \simeq \bigoplus_{n=0}^{\infty} \iota_1 M_Y(dx_1)^{-1} \partial_{x_1}^n$. Then, we have $\rho_{M,1}(f : x_1^{-1}) = \iota_1(f_Y(dx_1)^{-1})$ for any section $f$ of $M$. And, we have $\rho_{M,2}(g_Y(dx_1)^{-1}) = -\partial_{x_1}(\tilde{g})$ in $M(!Y)$ for any section $g$ of $M_Y$, where $\tilde{g}$ is a section of $M$ satisfying $\tilde{g}|_Y = g$ and $\partial_{x_1}(\tilde{g}) = 0$ in $M$. They are independent of the choice of coordinate systems, and we have the global morphisms.

The isomorphism $\nu_M : DM \simeq M^\vee$ induces $\nu_{M_Y} : D(M(\ast Y)) \simeq M^\vee(!Y)$ and $\nu_{M_Y} : D(M(!Y)) \simeq M^\vee(\ast Y)$. We have $\nu_{M_Y} : DM_Y \longrightarrow M_Y^\vee$ determined by a similar condition. The composite $D_{\iota_1} M_Y \simeq \iota_1 DM_Y \simeq \iota_1 M^\vee_Y$ is denoted by $\iota_1 \nu_{M_Y}$. We shall prove the following proposition in §A.2.2 and §A.2.4

Proposition A.3 We have $\nu_{M_Y} \circ D\rho_{M,1} = -\rho_{M_Y} \circ \iota_1 \nu_{M_Y}$ and $\iota_1 \nu_{M_Y} \circ D\rho_{M,2} = \rho_{M_Y} \circ \nu_{M_Y}$.

Clearly, it is enough to prove the case $M = \mathcal{O}_X$ for which we naturally identify $M^\vee = \mathcal{O}_X$. We shall $\rho_{M,i}$ by $\rho_i$, and $\nu_{M,i}$ by $\nu_i$ and $\nu_{M,Y}$ by $\nu_Y$ and $\nu_Y$, respectively.

We set $d_Y := \dim Y = d_X - 1$. Because $D\nu_X = (-1)^{d_X} \nu_X$ and $D\nu_Y = (-1)^{d_Y} \nu_Y$, the two equalities are equivalent. Hence, it is enough to prove $\iota_1 \nu_Y$.

We have only to prove the claim locally around any point of $Y$. We may assume that $X = \mathbb{C}_t \times Y$. We have natural isomorphisms $\mathcal{O}_X \simeq \mathcal{O}_{\mathbb{C}_t} \otimes \mathcal{O}_Y$ and $D\mathcal{O}_X \simeq D\mathcal{O}_{\mathbb{C}_t} \otimes D\mathcal{O}_Y$. The equality $\iota_1 \nu_Y \circ D\rho_2 = \rho_1 \circ \nu_X$ means the commutativity of the following diagram:

$$
\begin{array}{ccc}
D(\mathcal{O}_{\mathbb{C}_t}(\ast t)) \otimes D\mathcal{O}_Y & \xrightarrow{\nu_{C_t}(\ast t) \otimes \nu_Y} & \mathcal{O}_{\mathbb{C}_t}(\ast t) \otimes \mathcal{O}_Y \\
\downarrow D\rho_2 \otimes \text{id} & & \downarrow \rho_1 \otimes \text{id} \\
D(\iota_0 \mathcal{O}_{\{0\}}) \otimes D\mathcal{O}_Y & \xrightarrow{\iota_0 \nu_{\mathcal{O}_{\{0\}}} \otimes \nu_Y} & \iota_0 \mathcal{O}_{\{0\}} \otimes \mathcal{O}_Y
\end{array}
$$

Hence, it is enough to consider the case $\dim X = 1$, i.e., $X = \mathbb{C}_t$ and $Y = \{0\}$, which we assume in the following.
A.2.2 Commutativity of duality and push-forward

First, let us describe the isomorphism $D_{t+}O_Y \simeq \iota _+DO_Y$. We naturally have $\iota _+O_Y = \bigoplus _{j=0}^{\infty } \iota _*\mathbb{C} \cdot \partial ^j/(dt)^{-1}$. We have the exact sequence $0 \to D_X \xrightarrow{\kappa _0} D_X \xrightarrow{\kappa _1} \iota _+O_Y \to 0$, where $\kappa _0(P) = P \cdot t$ and $\kappa _1(1) = \iota _*(dt)^{-1}$. It induces the exact sequence:

$$0 \to \text{Hom}_{D_X}(D_X, D_X \otimes \Omega _X^{-1}) \xrightarrow{\kappa _1^t} \text{Hom}_{D_X}(D_X, D_X \otimes \Omega _X^{-1}) \xrightarrow{\kappa _0^t} D(\iota _+O_Y) \to 0$$

Here, $\kappa _0^t(g)(P) = g(Pt)$. Under the natural identifications $\text{Hom}_{D_X}(D_X, D_X \otimes \Omega _X^{-1}) \simeq D_X \otimes \Omega _X^{-1}$, we have $\kappa _0^t(Q) = tQ$.

We naturally have $\iota _+DO_Y = \bigoplus _{j=0}^{\infty } \iota _*\text{Hom}(\mathbb{C}, \mathbb{C}) \cdot \partial ^j/(dt)^{-1}$.

Lemma A.4 The isomorphism $D(\iota _+O_Y) \simeq \iota _+D(O_Y)$ is given by $\kappa _1^t((dt)^{-1}) \mapsto -\iota _*(\text{id} \cdot (dt)^{-1})$.

Proof Note that the following diagram is commutative, due to M. Saito [51]:

$$\begin{array}{ccc}
DR(D_{t+}O_Y) & \xrightarrow{\sim} & DR(\iota _+DO_Y) \\
\downarrow & & \downarrow \\
D_{t+}DRO_Y & \xrightarrow{\sim} & \iota _+DRO_Y
\end{array}$$

Here, the vertical arrows are the exchange of the de Rham functors with the others, and the horizontal arrows are the exchange of the duality functor and the push-forward. We describe the induced morphism $DR(\iota _+O_Y) \simeq \iota _*DO_Y$. We set $\omega _X^{\text{top}} = \mathcal{D}b_X^0[2]$ and $\omega _Y^{\text{top}} = CY$. Then, it is described as the composite of the following:

$$R\text{Hom}_{D_X}(\iota _+O_Y, \omega _X^{\text{top}})[1] \xrightarrow{\alpha } R\text{Hom}_{\mathbb{C}}(\iota _*DO_Y, \omega _X^{\text{top}}) \xrightarrow{b} R\text{Hom}_{\mathbb{C}}(\iota _*DO_Y, \omega _Y^{\text{top}}) \xrightarrow{\varepsilon } \text{Hom}_{\mathbb{C}}(\iota _*DO_Y, \iota _*\omega _Y^{\text{top}}) = \iota _*\text{Hom}(\mathbb{C}, \mathbb{C}). \quad (141)$$

Let us describe the morphisms more explicitly. We have $\omega _X^{\text{top}} = \mathcal{D}b_X^0[1] = (\mathcal{D}b_X^0 \xrightarrow{\partial } \mathcal{D}b_X^1)$, where $\mathcal{D}b_X^0$ sits in the degree $-1$. We have

$$R\text{Hom}_{D_X}(\iota _+O_Y, \omega _X^{\text{top}})[1] \simeq \text{Hom}_{D_X}(\iota _+O_Y, \mathcal{D}b_X^0[1]) \simeq \mathcal{D}b_{X,Y}^{0,0}[1],$$

where $\mathcal{D}b_{X,Y}$ denotes the sheaf of $(0, 1)$-currents on $X$ whose supports are contained in $Y$. The last isomorphism is given by $\Psi \mapsto \Psi (\iota _*(dt)^{-1})$. We have the section $-\overline{\partial }/(t-1)$ of $\mathcal{D}b_{X,Y}^{0,1}$. Recall that, for a complex manifold $Z$ with $\dim Z = d_Z$, we identify the total complex of $\Omega _Z^{\bullet }[d_Z] \otimes \mathcal{D}b_{Z}^{0,\bullet }[d_Z]$ with $\mathcal{D}b_{Z}^{0,\bullet }[2d_Z]$ by the correspondence $\eta ^{d_Z+i} \otimes \overline{\partial }^{d_Z+i} \mapsto (\partial _i \eta ^{d_Z+i} \otimes \overline{\partial }^{d_Z+i})$. We have

$$a(-\overline{\partial }/(1/t))((\iota _*\partial ^j/(dt)^{-1})dt) = -\partial ^j/(dt)(\iota _*\overline{\partial }/(1/t)), \quad a(-\overline{\partial }/(1/t))((\iota _*\partial ^j/(dt)^{-1})dt) = \partial ^j/(dt)^{-1}$$

Hence, we have $(b \circ a)(-\overline{\partial }/(t-1))(\iota _*(dt)^{-1}) = \overline{\partial }/(dt/t)$. Note that $c$ is induced by the trace morphism $\iota _*\omega _Y^{\text{top}} \to \omega _X^{\text{top}}$, for which we have $\iota _*(1) \mapsto \overline{\partial }/(t-1)$. Hence, we have $(c \circ b \circ a)(-\overline{\partial }/(t-1)) = \iota _*1$.

We have the quasi-isomorphism:

$$\text{Hom}(\iota _+O_Y, \mathcal{D}b_X^{0,\bullet }) \simeq DR(\iota _+O_Y) \simeq \left(\text{Hom}_{D_X}(D_X, O) \xrightarrow{\iota } \text{Hom}_{D_X}(D_X, O)\right) \quad (142)$$

The complexes (142) are naturally quasi-isomorphic to the total complex associated to the following double complex:

$$\begin{array}{c}
\text{Hom}_{D_X}(D_X, \mathcal{D}b_X^{0,0}) \xrightarrow{\iota } \text{Hom}_{D_X}(D_X, \mathcal{D}b_X^{0,0}) \\
\downarrow \overline{\partial } \downarrow \overline{\partial } \\
\text{Hom}_{D_X}(D_X, \mathcal{D}b_X^{0,1}) \xrightarrow{\iota } \text{Hom}_{D_X}(D_X, \mathcal{D}b_X^{0,1}) \xrightarrow{\iota } \text{Hom}_{D_X}(D_X, \mathcal{D}b_X^{0,1})
\end{array} \begin{pmatrix}
(0, -1) & \rightarrow & (1, -1) \\
\downarrow & & \downarrow \\
(0, 0) & \rightarrow & (1, 0)
\end{pmatrix}$$

94
Here, the right square in the parenthesis indicates the degrees of the double complex. Then, we can easily deduce $\partial(t^{-1}) = 1$ in $\mathbb{C}$.

Hence, the isomorphism $\text{DR} \iota + \mathcal{D} \mathcal{O}_Y \simeq \text{DR} \mathcal{D}(\iota + \mathcal{O}_Y)$ is identified with $\iota_* \text{Hom} \left( \mathbb{C}, \mathcal{C} \right) \rightarrow \text{Cok}(\mathcal{O} \xrightarrow{\iota} \mathcal{O})$ for which $\iota_* \text{id}$ is mapped to the image of $-1$ in $\text{Cok}(\mathcal{O} \xrightarrow{\iota} \mathcal{O})$. Then, the claim of Lemma A.4 follows.

The isomorphism $\mathcal{D} \mathcal{O}_Y \rightarrow \mathcal{O}_Y$ is given by $\text{id} \mapsto -1$. We have the induced isomorphism $\iota + \nu_Y : \mathcal{D}(\iota + \mathcal{O}_Y) \simeq \iota + \mathcal{O}_Y$. The following lemma is an immediate corollary of Lemma A.4.

**Lemma A.5** We have $\iota + \nu_Y \left( \kappa_1^1 ((dt)^{-1}) \right) = -\kappa_1(1)$.

**A.2.3 A description of $\nu_X$**

Let us describe the isomorphism $\nu_X : \mathcal{D} \mathcal{O}_X \simeq \mathcal{O}_X$. We use the following natural free resolution of $\mathcal{O}_X$:

$$\mathcal{O}_X \simeq (\mathcal{D}_X \xrightarrow{\partial} \mathcal{D}_X)$$

It induces the following free resolution of $\mathcal{D} \mathcal{O}_X$:

$$\mathcal{D} \mathcal{O}_X \simeq (\mathcal{D}_X \otimes \Omega_X^{-1} \xrightarrow{\partial} \mathcal{D}_X \otimes \Omega_X^{-1})$$

We have the morphism of $\mathcal{D}_X$-modules $\mu : \mathcal{D}_X \rightarrow \mathcal{D}_X \otimes \Omega_X^{-1}$ given by $a_j \partial^j \rightarrow (-1)^j \partial^j a_j \otimes (dt)^{-1}$. We have the following commutative diagram:

$$\begin{array}{c}
\mathcal{D}_X \otimes \Omega_X^{-1} \xrightarrow{\partial} \mathcal{D}_X \otimes \Omega_X^{-1} \\
-\mu^{-1} \downarrow \quad \mu^{-1} \downarrow \\
\mathcal{D}_X \xrightarrow{\partial} \mathcal{D}_X
\end{array}$$

(143)

It induces $\mathcal{D} \mathcal{O}_X \rightarrow \mathcal{O}_X$. We can check that the induced morphism $\text{DR} \mathcal{O}_X \simeq \text{DR} \mathcal{D} \mathcal{O}_X$ is given by $\mathbb{C} \simeq \text{Hom}(\mathbb{C}, \mathbb{C})$, $1 \mapsto -\text{id}$. Hence, it is $\nu_X$.

**A.2.4 End of the proof of Proposition A.3**

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{D}_X & \xrightarrow{-\partial} & \mathcal{D}_X \\
\text{id} & \downarrow & \downarrow \\
\mathcal{D}_X & \xrightarrow{-\partial t} & \mathcal{D}_X \mathcal{O}_X \xrightarrow{\varphi_1} \mathcal{O}_X (t)
\end{array}$$

(144)

Here $\varphi_1(1) = t^{-1}$. We also have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{D}_X & \xrightarrow{t} & \mathcal{D}_X \\
-\text{id} & \downarrow & \downarrow \\
\mathcal{D}_X & \xrightarrow{t \partial} & \mathcal{D}_X \mathcal{O}_Y \xrightarrow{\varphi_2} \mathcal{O}_X (t)
\end{array}$$

(145)

95
Here $\varphi_2(1) = 1$. As the dual of (145), we obtain the following:

$$
D(t+O_X) \xrightarrow{\varphi^1} D_X \otimes \Omega_X^1 \xrightarrow{-\partial_t} D_X \otimes \Omega_X^1
$$

(146)

We have the following commutative diagram:

$$
\begin{array}{c}
D_X \otimes \Omega_X^1 \xrightarrow{(\partial_t \lambda)} D_X \\
\downarrow \mu^{-1} \hspace{1cm} \downarrow \nu^{-1} \hspace{1cm} \downarrow \varphi^1 \\
D_X \xrightarrow{-(\lambda \partial_t)} D_X \xrightarrow{-\varphi^1} O_X
\end{array}
$$

Because $\varphi_1(1) = t^{-1}$, we have $D\rho_2 \circ \nu^{-1}_X(t^{-1}) = -\kappa_1'(t^{-1})$. We also have $\rho_1(t^{-1}) = \kappa_1(1)$. Then, we obtain $D\rho_2 \circ \nu^{-1}_X = (t+O_X)^{-1} \circ \rho_1$ by Lemma A.3. Thus, the proof of Proposition A.3 is finished.

### A.3 Duality and nearby cycle functors

We shall generalize Proposition A.3. This subsection is a preliminary for (B.3).

#### A.3.1 Some induced isomorphisms

Let $t$ be a function on $X$ such that the zero divisor $(t)_0$ is smooth and reduced, i.e., the zero set $H$ of $t$ is smooth and reduced as a subscheme. Let $M$ be a regular singular meromorphic flat bundle on $(X,H)$.

We use the natural identification $(\Pi^{a,b}_t M)^\vee \simeq \Pi_t^{-b+1,-a+1} M^\vee$. (See [36, S2.3] where the identification is explained for $R$-modules. The identification for flat bundles is obtained as the specialization at $\lambda = 1$.) We have the isomorphisms $\nu_{\Pi^{a,b}_t M}: (D_X(*) \Pi^{a,b}_t M) \simeq \Pi_t^{b+1,-a+1} M^\vee$, which induce

$$
\nu_{\Pi^{a,b}_t M,*}: D_X(\Pi^{a,b}_t M) \simeq \Pi_t^{b+1,-a+1} M^\vee, \quad \nu_{\Pi^{a,b}_t M,!}: D_X(\Pi_t^{a,b} M) \simeq \Pi_t^{b+1,-a+1} M^\vee.
$$

We have the following isomorphism for a sufficiently large $N$:

$$
\psi^{(a)}_t(M) \simeq \text{Ker}(\Pi_t^{b,N,a}(M) \rightarrow \Pi_t^{a,N,a}(M)) \simeq \text{Cok}(\Pi_t^{a+N}(M) \rightarrow \Pi_t^{a+N}(M)).
$$

Hence, we have the following canonical isomorphisms:

$$
\nu_{M,\psi^{(a)}_t}: D_H \psi^{(a)}_t(M) \simeq \psi^{(-a+1)}(M^\vee)
$$

We have the $V$-filtration $U$ of $M$ along $t$. Here, we adopt the condition that $\partial_t + \alpha$ is nilpotent on $\psi_{t,\alpha}(M) := U_\alpha(M)/U_{<\alpha}(M)$. The natural perfect pairing $M \otimes M^\vee \rightarrow O_X(*)$ naturally induces a perfect pairing of $O_X$-modules $V^{-1}(M) \otimes V^{-1}(M^\vee) \rightarrow O_X$. It induces a perfect pairing $\psi_{t,-1}(M) \times \psi_{t,-1}(M^\vee) \rightarrow O_X$ of flat bundles. It gives the following natural identification:

$$
\psi_{t,-1}(M)^\vee \simeq \psi_{t,-1}(M^\vee).
$$

(147)

We have the canonical isomorphisms $\xi^{(0)}_t : \psi^{(0)}_t(M) \simeq t+\psi_{t,-1}(M)$ and $\xi^{(1)}_t : \psi^{(1)}_t(M) \simeq t+\psi_{t,-1}(M)$. The construction of the isomorphisms is explained in the case of $R$-modules in [36, S4.3]. The construction for $D$-modules is obtained as the specialization to $\lambda = 1$. 

96
A.3.2 Some commutative diagrams

**Proposition A.6** The following diagram is commutative:

\[
D_X \psi_t^{(1)}(M) \xrightarrow{\nu_{M,\nu_t^{(1)}}} \psi_t^{(0)}(M^\vee) \xrightarrow{\xi_t^{(1)}} D_X(1) \xrightarrow{\xi_t^{(0)}} \psi_t^{(0)}(M^\vee) \]

Here, the lower horizontal arrow is induced as the composite of the natural isomorphisms \( D_X \psi_t^{(1)}(M) \simeq t_+ D_H \psi_t^{(0)}(M) \simeq t_+ \psi_t^{(1)}(M^\vee) \simeq t_+ \psi_t^{(0)}(M^\vee) \).

The following diagram is also commutative:

\[
D_X \psi_t^{(0)}(M) \xrightarrow{\nu_{M,\nu_t^{(0)}}} \psi_t^{(1)}(M^\vee) \xrightarrow{\xi_t^{(0)}} D_X(0) \xrightarrow{\xi_t^{(1)}} \psi_t^{(1)}(M^\vee) \]

Here, the lower horizontal arrow is induced as the composite of the natural isomorphisms \( D_X \psi_t^{(0)}(M) \simeq t_+ D_H \psi_t^{(0)}(M) \simeq t_+ \psi_t^{(0)}(M^\vee) \simeq t_+ \psi_t^{(0)}(M^\vee) \).

**Proof** Let \( V \) be any regular singular meromorphic flat bundle on \((X,H)\). We set \( V_t := V(tH) \). Let \( K_V \) and \( C_V \) denote the kernel and the cokernel of \( V_t \to V_w \). We have the following natural commutative diagram:

\[
0 \to D_X C_V \xrightarrow{\nu_{V,C}} D_X V_t \xrightarrow{\nu_V} D_X V_w \xrightarrow{\nu_{V,K}} D_X K_V \to 0
\]

Let \( K_{0V} \) and \( C_{0V} \) denote the kernel and the cokernel of the morphism \( \psi_{t,-1}(V) \to \psi_{t,-1}(V) \). We have natural isomorphisms \( \xi_{C,V} : C_V \simeq t_+ C_{0V} \) and \( \xi_{K,V} : K_V \simeq t_+ K_{0V} \). The construction of the isomorphisms is explained in [36, §4.3.2] for \( R \)-modules. The construction for \( D \)-modules is obtained as the specialization at \( \lambda = 1 \). Under the natural identification \( \psi_{t,-1}(V)^\vee = \psi_{t,-1}(V^\vee) \), we have \( C_{0V}^\vee = K_{0V^\vee} \) and \( C_{0V} = K_{0V^\vee}^\vee \).

**Lemma A.7** The following diagram is commutative:

\[
D_X K_V \xrightarrow{\nu_{V,K}} C_V \xrightarrow{\xi_{C,V}} D_X(0) \xrightarrow{\xi_{K,V}} \psi_t^{(1)}(M^\vee) \]

The lower horizontal arrow is the composite of \( D_X(t_+ K_{0V}) \simeq t_+ D_H K_{0V} \simeq t_+ C_{0V}^\vee \simeq t_+ C_{0V^\vee} \).

The following diagram is commutative:

\[
D_X C_V \xrightarrow{-\nu_{V,C}} K_V \xrightarrow{\xi_{K,V}} D_X(0) \xrightarrow{\xi_{C,V}} \psi_t^{(0)}(M^\vee) \]

The lower horizontal arrow is the composite of \( D_X(t_+ C_{0V}) \simeq t_+ D_H C_{0V} \simeq t_+ C_{0V^\vee} \simeq t_+ K_{0V^\vee} \).

**Proof** It is enough to consider the case that the monodromy of \( V \) along the loop around \( t = 0 \) is unipotent. Moreover, we may assume that the logarithm of the monodromy is a Jordan block. Then, we have a flat subbundle \( L \subset V \) of rank one such that \( K_L \simeq K_V \) and \( K_{0L} \simeq K_{0V} \). We also have \( V^\vee \to L^\vee \) which induces \( C_{0V^\vee} \simeq C_{0L^\vee} \) and \( C_V \simeq C_L \). Then, we obtain [150] from the claim for \( L \), which follows from Proposition A.3. We obtain [151] similarly.

We have \( K_{0H^{-1},t^{-1}M} \simeq \psi_{t,-1}(M) \) and \( C_{0H^{p+1},t^{-1}M} \simeq \psi_{t,-1}(M^\vee) \). Hence, we obtain the commutativity of [148] from Lemma A.7. We obtain the commutativity of [149] similarly.
A.3.3 Specialization of pairings

Let $M_i$ be regular singular meromorphic flat bundles on $(X, H)$. Let $P : M_1 \otimes M_2 \to \mathcal{O}_X(*H)$ be a morphism of $\mathcal{D}$-modules. We have the naturally induced pairing $\psi_{t,-1}(P) : \psi_{t,-1}(M_1) \otimes \psi_{t,-1}(M_2) \to \mathcal{O}_H$.

The pairing $P$ and the isomorphism $\nu_{M_2}$ induce a morphism $\Psi_P : M_1 \simeq D_{X(*H)}M_2$. For $a = 0, 1$, we have the induced morphisms

$$\psi^a(M_1) \to \psi^a(D_X M_2) \simeq D_X \psi^{(a+1)}(M_2).$$

By the natural isomorphisms $\iota_\psi \psi_{t,-1}(M_1) \simeq \psi^a(M_1)$ and the isomorphism $\nu_{\psi_{t,-1}(M_2)}$, we obtain

$$\iota_\psi \psi_{t,-1}(M_1) \simeq D_X \iota_\psi \psi_{t,-1}(M_2) \simeq \iota_\psi D_H \psi_{t,-1}(M_2) \simeq \iota_\psi \psi_{t,-1}(M_2)^\vee.$$

It comes from a morphism $\rho_a : \psi_{t,-1}(M_1) \simeq \psi_{t,-1}(M_2)^\vee$, which is equivalent to a pairing

$$P_a : \psi_{t,-1}(M_1) \times \psi_{t,-1}(M_2) \to \mathcal{O}_H.$$

**Corollary A.8** We have $P_0 = \psi_{t,-1}(P)$ and $P_1 = -\psi_{t,-1}(P)$.

**Proof** Let $\mu : \psi_{t,-1}(M_1) \to \psi_{t,-1}(M_2)^\vee$ be the morphism induced by $\psi_{t,-1}(P)$. The push-forward $\iota_\psi \mu : \iota_\psi \psi_{t,-1}(M_1) \to \iota_\psi \psi_{t,-1}(M_2)^\vee$ is the composite of the following morphisms:

$$\iota_\psi \psi_{t,-1}(M_1) \simeq \psi^a(M_1) \to \psi^a(M_2^\vee) \simeq \iota_\psi \psi_{t,-1}(M_2^\vee)$$

By the construction, $\iota_\psi \rho_a$ is composite of the following morphisms:

$$\iota_\psi \psi_{t,-1}(M_1) \simeq \psi^a(M_1) \to \psi^a(M_2^\vee) \simeq \psi^a(D_X M_2) \simeq D_X \psi^{(a+1)}(M_2) \simeq D_X \iota_\psi \psi_{t,-1}(M_2) \simeq \iota_\psi D_X \psi_{t,-1}(M_2) \simeq \iota_\psi \psi_{t,-1}(M_2^\vee)$$

(152)

By Proposition[A.6], we have $\iota_\psi \mu = \iota_\psi \rho_0$ and $\iota_\psi \mu = -\iota_\psi \rho_1$.

A.4 Push-forward

This subsection is a preliminary for §B.4.

A.4.1 Statement

Let $X$ be a complex manifold. Let $M_i$ be flat bundles on $X$. Let $P : M_1 \otimes M_2 \to \mathcal{O}_X$ be a morphism of $\mathcal{D}$-modules. By the isomorphism $\nu_{M_2} : D_X M_2 \simeq M_2\pi$, $P$ is equivalent to a morphism $\varphi : M_1 \to D_X M_2$.

Let $F : X \to S$ be a smooth proper morphism of complex manifolds. We have the flat bundles $F_+^i M_i$ on $S$. We obtain the following morphism:

$$F_+^i M_1 \to F_+^i D_X M_2 \to D_S F_{+}^{-j} M_2$$

The composite is denoted by $F_+^i \varphi$. By the isomorphism $\nu_{F_{+}^{-j} M_2} : D_S F_{+}^{-j} M_2 \simeq F_{+}^{-j} M_2^\vee$, we obtain a flat morphism

$$P(F_+^i \varphi) : F_+^i M_1 \otimes F_{+}^{-j} M_2 \to \mathcal{O}_S.$$

We give a more direct expression of the pairing $P(F_+^i \varphi)$. For each $Q \in S$, we set $X_Q := F^{-1}(Q)$, and let $M_{i,Q}$ denote the restriction of the flat bundles $M_i$ to $X_Q$. Let $d := \dim X - \dim S$. Let $H^{d+j}_{\text{DR}}(X_Q, M_{i,Q})$ denote the de Rham cohomology of the flat bundle $M_{i,Q}$. The fiber of $F_+^i(M_i)_{|Q}$ is naturally identified with $H^{d+j}_{\text{DR}}(X_Q, M_{i,Q})$. The pairing $P$ naturally induces the following pairing $P_{F,j,Q}$:

$$H^{d+j}_{\text{DR}}(X_Q, M_{1,Q}) \times H^{d+j}_{\text{DR}}(X_Q, M_{2,Q}) \to H^{2d}_{\text{DR}}(X_Q, \mathbb{C}) \to \mathbb{C}$$

Here, $H_{\text{DR}}^{2d}(X_Q, \mathbb{C}) \to \mathbb{C}$ is induced by $(2\pi \sqrt{-1})^{-d} \int_{X_Q}$. The family $\{P_{F,j,Q} | Q \in S\}$ gives a flat morphism $P_{F,j} : F_+^i M_1 \otimes F_{+}^{-j} M_2 \to \mathcal{O}_S$. We shall prove the following proposition in §A.4.2 A.4.3

**Proposition A.9** We have $P(F_+^i \varphi) = \epsilon(d)(-1)^{j}P_{F,j}$, where $\epsilon(d) = (-1)^{d(d-1)/2}$.  

98
A.4.2 Preliminary

We have only to prove the claim locally around any point of $S$. So, we may assume that $S$ is a multi-disc. Let $t$ be a holomorphic function on $S$ such that the zero divisor $(t)_0$ is smooth and reduced. Let $S_0 := \| (t)_0 \|$. The pull back $t \circ F$ is also denoted by $t$. Set $X_0 := \{ Q \in X \mid t(Q) = 0 \}$. The induced morphism $X_0 \to S_0$ is denoted by $F_0$. We naturally identify the restriction of the flat bundles $M_i$ to $X_0$ with $\psi_{t,-1}(M_i)$. We also identify the restriction of $F^j_+ M_i$ to $S_0$ with $\psi_{t,-1} F^j_+ (M_i)$, which is also naturally isomorphic to $F^j_0 \psi_{t,-1} (M_i)$.

We obtain the following pairings as the restriction of $P$, $P(F^j_+ \varphi)$ and $P_{F,j}$:

$$
\psi_{t,-1} P : \psi_{t,-1}(M_1) \otimes \psi_{t,-1}(M_2) \to O_{S_0}
$$

$$
\psi_{t,-1} P(F^j_+ \varphi) : \psi_{t,-1} F^j_+(M_1) \otimes \psi_{t,-1} F^{-j}_+(M_2) \to O_{S_0}
$$

$$
\psi_{t,-1} P_{F,j} : \psi_{t,-1} F^j_+(M_1) \otimes \psi_{t,-1} F^{-j}_+(M_2) \to O_{S_0}
$$

By the construction, we have $(\psi_{t,-1} P)_{F_0,j} = \psi_{t,-1} P_{F,j}$.

Let $\varphi_0 : \psi_{t,-1} M_1 \to D_{X_0} \psi_{t,-1} M_2$ be the morphism corresponding to $\psi_{t,-1} P$.

**Lemma A.10** We have $\psi_{t,-1} P(F^j_+ \varphi) = P(F^j_+ \varphi_0)$. As a consequence, to prove Proposition A.9 it is enough to consider the case where $S$ is a point.

**Proof** Let $\iota_{X_0} : X_0 \to X$ be the inclusion. By Proposition A.6, the morphism $\iota_{X_0} \varphi_0 : \iota_{X_0} \psi_{t,-1} M_1 \to \iota_{X_0} D_{X_0} \psi_{t,-1} M_2$ is identified with the composite of the following morphism:

$$
\psi_t(0)(M_1) \xrightarrow{\psi_t(0)} \psi_t(0)(D_X M_2) \simeq D_X \psi_t(1)(M_2)
$$

(153)

We have the morphism $\rho_j : \psi_{t,-1} F^j_+ M_1 \to D_{S_0} \psi_{t,-1} F^j_+ M_2$ corresponding to $\psi_{t,-1} P(F^j_+ \varphi)$. Let $\iota_{S_0} : S_0 \to S$ be the inclusion. By Proposition A.6, $\iota_{S_0} \rho_j$ is identified with

$$
\psi_t(0) F^j_+ M_1 \xrightarrow{\psi_t(0) \rho_j} \psi_t(0) D_S F^j_+ M_2 \simeq D_S \psi_t(1) F^{-j}_+ M_2.
$$

(154)

Note that the following diagram of the natural isomorphisms is commutative, which can be checked easily:

$$
\begin{array}{ccc}
D_S \psi_t(1) F^{-j}_+ M_2 & \xrightarrow{\simeq} & \psi_t(0) D_S F^{-j}_+ M_2 \\
\simeq & & \simeq \\
D_S F^{-j}_+ \psi_t(1) M_2 & \xrightarrow{\simeq} & F^{-j}_+ D_X \psi_t(1) M_2 \\
\end{array}
$$

Then, the morphism (154) is identified with the push-forward of (153) by $F$. Hence, we have $\rho_j = F^j_+ \varphi_0$, which implies the first claim of the lemma. The second claim follows from the first claim and the equality $(\psi_{t,-1} P)_{F_0,j} = \psi_{t,-1} P_{F,j}$.

A.4.3 The case where $S$ is a point

Let $d := \text{dim } X$. Let $L_i$ be the sheaf of flat sections of $M_i$. Let $L_i^\vee$ denote the local system obtained as the dual of $L_i$. The pairing $P$ induces $\langle \cdot, \cdot \rangle : L_1 \times L_2 \to \mathbb{C}_X$, which induces $v : L_1 \to L_2^\vee$. We have the natural isomorphism $\text{DR } M_i \simeq L_i[d]$ and $\text{DR } D M_2 \simeq R\text{Hom}_{\mathcal{D}X} (M_2, \mathcal{O}_X[d]) \simeq L_2[d]$. The morphism $\varphi$ induces $(-1)^d v : L_1[d] \to L_2^\vee[d]$.

Let us describe the induced morphism $L_1[d] \to R\text{Hom}_{\mathcal{D}X} (\mathcal{D} M_2, \mathcal{D} \mathcal{O}[d])$. A section $e$ of $L_1$ naturally gives a section of $(L_1[d])^{-d}$, which is denoted by $e'$. We have the section $(-1)^d v(e')$ of $\text{Hom}_{\mathcal{D}X} (M_2, \mathcal{O}[d])^{-d}$. The image of $(-1)^d v(e')$ in $\text{Hom}_{\mathcal{D}X} (\mathcal{D} M_2, \mathcal{D} \mathcal{O}[d])^{-d}$ is denoted by $F_e$. Then, we have

$$
F_e(\eta^{d+j} m) = (-1)^d \eta^{d+j} \cdot (-1)^d v(e')(m) = (-1)^{d+j} \eta^{d+j} \langle e, m \rangle'.
$$

Here, $\eta^{d+j}$ is a section of $(\mathcal{O}[d])^{d+j}$, $m$ is a section of $M_2$, and $(e, m)'$ denotes the section of $(\mathcal{O}[d])^{-d}$ corresponding to $(e, m)$.
We have the natural quasi isomorphisms $\text{DR}O[d] \simeq \text{Tot}(\Omega^* \otimes \mathfrak{D} \mathcal{B}^{\mathcal{O}}[d]) \simeq \mathfrak{D} \mathcal{B}^{\mathcal{O}}[2d]$, where the latter is given by $\eta^{d+j} \otimes \bar{\eta}^{d+i} \mapsto \epsilon(d)(-1)^{dj} \eta^{d+j} \wedge \bar{\eta}^{d+i}$. Let $\bar{F}_e$ denote the image of $F_e$ in $\text{Hom}_{\mathcal{C}_X}(\text{DR}M_2, \mathfrak{D} \mathcal{B}^{\mathcal{O}}[2d])^{-d}$. It is given by

$$\bar{F}_e(\eta^{d+j}m) = \epsilon(d)(-1)^{dj}\eta^{d+j}(e, m)$$

which is naturally regarded as a section of $(\mathfrak{D} \mathcal{B}^{\mathcal{O}}[2d])^{-d}$. The morphism $L_1[d] \rightarrow \text{Hom}_{\mathcal{C}_X}(\text{DR}M_2, \mathfrak{D} \mathcal{B}^{\mathcal{O}}[2d])^{-d}$ is extended to the morphism $\text{DR}M_1 \rightarrow \text{Hom}_{\mathcal{C}_X}(\text{DR}M_2, \mathfrak{D} \mathcal{B}^{\mathcal{O}}[2d])^{-d}$ induced by the multiplication $\text{DR}M_1 \times \text{DR}M_2 \rightarrow \mathfrak{D} \mathcal{B}^{\mathcal{O}}[2d]$ given by $(x^{d+k}e, y^{d+j}m) \mapsto \epsilon(d)(-1)^{kd}x^{d+k}\eta^{d+j}(e, m)$. Then, we obtain the claim of Proposition A.9 in the case where $S$ is a point. By Lemma A.10, the proof of Proposition A.9 is finished.

### B Pairings and their functoriality

#### B.1 Mixed TEP-structures and their functoriality

**B.1.1 Pairing of weight $w$ and graded pairing**

Let $X$ be a complex manifold with a hypersurface $H$. Set $\mathcal{X} := \mathbb{C}_\lambda \times X$. For any $\mathcal{O}_X$-module $N$, let $N(*H) := N \otimes \mathcal{O}_X(*C_\lambda \times H)$. An $\mathcal{R}_X(*H)$-module $\mathcal{V}$ is called smooth if (i) $\mathcal{V}|_{C_\lambda \times (X \setminus H)}$ is locally free as an $\mathcal{O}_X$-module, (ii) $\mathcal{V}$ is $\mathcal{O}_X(*H)$-coherent. For a smooth $\mathcal{R}_X(*H)$-module $\mathcal{V}$, a pairing of weight $w$ on $\mathcal{V}$ is defined to be an $\mathcal{R}_X$-homomorphism $P: \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{O}_X(*H)$ such that (i) $j^*P = (-1)^wP \circ \chi$, (ii) $P$ is non-degenerate, i.e., the induced morphism $\mathcal{V} \rightarrow \text{Hom}_{\mathcal{O}_X}(*H)(j^*\mathcal{V}, \mathcal{O}_X(*H))$ is an isomorphism. Such a pair $(\mathcal{V}, P)$ is a TEP-$(-w)$-structure in the sense of [19]. Note that $P$ naturally induces $P(*\lambda): \mathcal{V}(\lambda) \otimes j^*\mathcal{V}(\lambda) \rightarrow \mathcal{O}_X(*H)(\lambda)$, and $P$ is the restriction of $P(*\lambda)$ to $\mathcal{V} \otimes j^*\mathcal{V}$. We also obtain pairings $P(a)$ of weight $w + 2a$ on $\lambda^{-a}\mathcal{V}$ as the restriction of $P(*\lambda)$. We have $P(a)(\lambda^{-a}v_1, j^*(\lambda^{-a}v_2)) = (-1)^w\lambda^{-2a}P(v_1, j^*v_2)$.

Suppose that a smooth $\mathcal{R}_X(*H)$-module $\mathcal{V}$ is equipped with an exhaustive increasing filtration $\mathcal{W} \subset \mathcal{R}_X(*H)$-submodules indexed by integers such that $\text{Gr}_w(\mathcal{V})$ are also smooth $\mathcal{R}_X(*H)$-modules. A graded pairing of $(\mathcal{V}, \mathcal{W})$ is defined to be a tuple of pairings $P_w$ ($w \in \mathbb{Z}$) of weight $w$ on $\text{Gr}_w^c(\mathcal{V})$. Such a filtered smooth $\mathcal{R}_X(*H)$-module $\mathcal{V}$ with a graded pairing $(P_w, w \in \mathbb{Z})$ is called a mixed TEP-structure. For any integer $a$, we set $\mathcal{W}_{k+2a}(\lambda^{-a}\mathcal{V}) := \lambda^{-a}\mathcal{W}_{k}\mathcal{V}$. We naturally have $\text{Gr}_w^c(\lambda^{-a}\mathcal{V}) = \lambda^{-a}\text{Gr}_w^c(\mathcal{V})$. The pairing $P_k$ on $\text{Gr}_k^c(\mathcal{V})$ induces $P_{k+2a}$ on $\text{Gr}_{k+2a}(\lambda^{-a}\mathcal{V})$. Thus, we obtain another mixed TEP-structure which consists of a filtered $\mathcal{R}_X(*H)$-module $(\lambda^{-a}\mathcal{V}, \mathcal{W})$ and a graded pairing $(P_w, w \in \mathbb{Z})$.

A mixed TEP-structure $(\mathcal{V}, \mathcal{W}, \mathcal{P})$ is called integrable if (i) $\mathcal{V}$ is an $\mathcal{R}_X(*H)$-module, (ii) $\mathcal{W}$ is a filtration by $\mathcal{R}_X(*H)$-submodules, (iii) $P_w$ are also $\mathcal{R}_X(*H)$-homomorphisms. In that case, $(\mathcal{V}, \mathcal{W}, \mathcal{P})$ is also called a mixed TEP-structure.

**B.1.2 Specialization of pairings**

Suppose that $Y$ is a hypersurface of $X$ given as $t = 0$ for a holomorphic function $t$ such that $dt$ is nowhere vanishing. We introduce a procedure of pairings in the context of $\mathcal{R}_X(*Y)$-modules, which we call specialization. The procedure can also work in the non-integrable case.

We set $\mathcal{X} := \mathbb{C}_\lambda \times X$ and $\mathcal{Y} := \mathbb{C}_\lambda \times Y$. Let $p_\lambda: \mathcal{X} \rightarrow X$ be the projection. Let $V\mathcal{R}_X \subset \mathcal{R}_X$ denote the sheaf of subalgebras generated by $\lambda p_\lambda \mathcal{O}_X(\log Y)$ and $\lambda^2 \partial_\lambda$ over $\mathcal{O}_X$.

Let $\mathcal{V}$ be a smooth $\mathcal{R}_X(*Y)$-module. We assume the following:

**B0** $\mathcal{V}$ is regular along $t$, i.e., there exists a filtration $V_*(\mathcal{V}) = (V_a(\mathcal{V}) | a \in \mathbb{R})$ by $\mathcal{R}_X$-submodules such that (i) $V_a(\mathcal{V})$ are locally free $\mathcal{O}_X$-submodules and $\text{Gr}_a^c(\mathcal{V})$ are locally free $\mathcal{O}_Y$-modules, (ii) $\lambda p_\lambda V_a \subset V_{a+1}$ and $tV_a = V_{a-1}(\mathcal{V})$, (iii) $\lambda a t + \lambda a$ are nilpotent on $\text{Gr}_a^c(\mathcal{V})$.

We obtain $\mathcal{R}_Y$-modules $\text{Gr}_a^c(\mathcal{V})$ which are smooth. We are particularly interested in $\text{Gr}_1^c(\mathcal{V})$. It is also denoted as $\psi_{t, \cdot}(\mathcal{V})$. We also assume the following:

**B1** The conjugacy classes of $(\lambda \mathcal{N})|_{(\lambda, W)} : \psi_{t, -\cdot}(\mathcal{V})(\lambda, Q) \rightarrow \psi_{t, -\cdot}(\mathcal{V})(\lambda, Q)$ are independent of the choice of $(\lambda, Q) \in \mathcal{Y}$. Here, $N: \psi_{t, -\cdot}(\mathcal{V}) \rightarrow \psi_{t, -\cdot}(\mathcal{V}) \lambda^{-1}$ denote the morphism induced by the multiplication of $t\partial_1$. 
Let $P$ be a pairing of weight $w$ on $V$. We have the induced morphism $P : V_{-1}V \otimes j^*V_{-1}V \rightarrow \lambda^{-w}O_X$. Then, we have the naturally induced pairing of weight $w$:

$$\psi_{t,-\delta}(P) : \psi_{t,-\delta}(V) \otimes j^*\psi_{t,-\delta}(V) \rightarrow \lambda^{-w}O_Y$$

Let $W(N)$ denote the monodromy weight filtration of $\lambda N$ on $\psi_{t,-\delta}(V)$. By the assumption, $W(N)$ is a filtration by subbundles. We have the induced non-degenerate pairings:

$$\psi_{t,-\delta}(P)_k : Gr^W_{-k}(\psi_{t,-\delta}(V)) \times j^*Gr^W_{-k}(\psi_{t,-\delta}(V)) \rightarrow \lambda^{-w}O_Y$$

For $k \geq 0$, let $PGr^W_k \psi_{t,-\delta}(V)$ denote the primitive part, i.e., the kernel of $N^{k+1} : Gr^W_k \psi_{t,-\delta}(V) \rightarrow \lambda^{-k-1}Gr^W_{-k-2} \psi_{t,-\delta}(V)$. We have the induced pairings:

$$sp_t(P)_{w+k} := \psi_{t,-\delta}(P) \circ (N^k \times 1) : PGr^W_{-k}(\psi_{t,-\delta}(V)) \times j^*PGr^W_{-k}(\psi_{t,-\delta}(V)) \rightarrow \lambda^{-w-k}O_Y$$

By using $\psi_{t,-\delta}(P) \circ (id \times j^*N) = -\psi_{t,-\delta}(P) \circ (N \times id)$, we can easily check that $sp_t(P)_{w+k}$ is a pairing of weight $w + k$ on $PGr^W_{-k}(\psi_{t,-\delta}(V))$.

Let $k \leq 0$. Let $P' Gr^W_k \psi_{t,-\delta}(V) \subset Gr^W_k \psi_{t,-\delta}(V)$ denote the image of $\lambda^{-k}PGr^W_{-k} \psi_{t,-\delta}(V)$ by $N^{-k}$. We have the isomorphism

$$id \times j^*N^{-k} : P' Gr^W_k \psi_{t,-\delta}(V) \times j^*(\lambda^{-k}PGr^W_{-k} \psi_{t,-\delta}(V)) \cong P' Gr^W_k \psi_{t,-\delta}(V) \times j^*P' Gr^W_k \psi_{t,-\delta}(V).$$

We obtain the induced pairing of weight $w + k$.

$$sp_t(P)_{w+k} := \psi_{t,-\delta}(P) \circ (id \times j^*N^{-k})^{-1} : P' Gr^W_k \psi_{t,-\delta}(V) \times j^*P' Gr^W_k \psi_{t,-\delta}(V) \rightarrow \lambda^{-w-k}O_Y$$

It is also induced by $(-1)^k \psi_{t,-\delta}(P)_{-k} \circ (N^k \times id)^{-1}$.

**B.1.3 Filtration and graded pairings**

We continue to use the notation in [B.1.2]. Let $W(N)$ denote the filtration on $\text{Cok}(\lambda N)$ induced by the filtration $W(N)$ on $\psi_{t,-\delta}(V)$. It is equipped with the flat connection. We set

$$\tilde{W}_k \text{Cok}(\lambda N) := W(N)_{k-w} \text{Cok}(\lambda N).$$

In other words, $\tilde{W}_k \text{Cok}(\lambda N)$ is the image of $W(N)_{k-w} \psi_{t,-\delta}(V) \rightarrow \text{Cok}(\lambda N)$. We have

$$Gr^W_{k} \text{Cok}(\lambda N) \simeq \begin{cases} PGr^W_{k-w} \psi_{t,-\delta}(V) & (k \geq w) \\ 0 & (k < w) \end{cases}$$

Hence, we have the pairing of weight $k$ on $Gr^W_{k} \text{Cok}(\lambda N)$ induced by $sp_t(P)_k$. The induced pairing is also denoted by $sp_t(P)_k$. The tuple $(sp_t(P)_k \mid k \in \mathbb{Z})$ is denoted by $sp_t(P)$. Thus, we obtain a mixed TEP-structure $(\text{Cok}(\lambda N),\tilde{W},sp_t(P))$.

Similarly, we set $\tilde{W}_k \text{Ker}(\lambda N) := W(N)_{k-w} \cap \text{Ker}(\lambda N)$. We have

$$Gr^W_{k} \text{Ker}(\lambda N) \simeq \begin{cases} P'Gr^W_{k-w} \psi_{t,-\delta}(V) & (k \leq w) \\ 0 & (k > w) \end{cases}$$

Hence, we have the pairing $sp_t(P)_k$ of weight $k$ on $Gr^W_{k} \text{Ker}(\lambda N)$. The induced pairing is also denoted by $sp_t(P)_k$. Thus, we obtain a mixed TEP-structure $(\text{Ker}(\lambda N),\tilde{W},sp_t(P))$, where $sp_t(P) := (sp_t(P)_k \mid k \in \mathbb{Z})$. 

101
B.1.4 Dependence on defining functions

Let $\tau$ be a holomorphic function on $X$ such that (i) $Y = \{ \tau = 0 \}$, (ii) $d\tau$ is nowhere vanishing. The $V$-filtrations for $t$ and $\tau$ are the same, and we naturally have $\psi_{t,-\delta}(V) = \psi_{\tau,-\delta}(V)$ as $O_Y$-modules. Let $N_t$ and $N_\tau$ denote the morphisms $\psi_{t,-\delta}(V) \to \lambda^{-1}\psi_{t,-\delta}(V)$ induced by $t\partial_t$ and $\tau\partial_\tau$. We remark the following standard lemma.

Lemma B.1 We have $N_t = N_\tau$ and $\psi_{t,-\delta}(P) = \psi_{\tau,-\delta}(P)$. As a result, we have $\text{Cok}(N_t) = \text{Cok}(N_\tau)$ and $\text{Ker}(N_t) = \text{Ker}(N_\tau)$. On $\text{Cok}(N_t) = \text{Cok}(N_\tau)$ and $\text{Ker}(N_t) = \text{Ker}(N_\tau)$, the induced connections are also equal. Moreover, the induced filtrations and the graded pairings are also equal.

In other words, the induced mixed TEP-structures are well defined in the sense that they are independent of the choice of a defining function of $Y$. Note that the induced connections are not the same on $\psi_{t,-\delta}(V) = \psi_{\tau,-\delta}(V)$, in general.

B.2 Pairings associated with real structure and graded sesqui-linear duality

B.2.1 Compatibility of sesqui-linear duality and real structure

Let $X$ be a complex manifold. Let $T = (M', M'', C)$ be a pure twistor $D$-module of weight $w$ on $X$. Let $S : T \to T^*(-w)$ be a sesqui-linear duality. Let $\kappa : \tilde{\gamma}^*T \to T$ be a real structure. We say that $S$ and $\kappa$ are compatible if the following diagram is commutative.

\[\begin{array}{ccc}
\tilde{\gamma}^*T & \longrightarrow & \tilde{\gamma}^*(T^*)(-w) \\
\kappa & \downarrow & \kappa' \uparrow \\
T & \longrightarrow & T^*(-w)
\end{array}\]

Recall that $S$ is expressed as a pair of morphisms $M' \xleftarrow{S'} M''\lambda^w$ and $M' \xrightarrow{S''} M'\lambda^{-w}$ satisfying $S' = S''$, and that $\kappa$ is expressed as a pair of morphisms $D_X j^*M'' \xleftarrow{\kappa'} M'$ and $D_X j^*M' \xrightarrow{\kappa'} M''$ satisfying $j^*D_X \kappa'' \circ \kappa' = \text{id}$ and $\kappa'' \circ j^*D_X \kappa' = \text{id}$. Here, $D_X$ denotes the duality functor for $R_X$-modules. Note that $\tilde{\gamma}^*(T(-w)) \simeq \tilde{\gamma}^*(T)(-w)$ is given by $((-1)^w, (1)^w)$. Then, the commutativity of the above diagram means the commutativity of the following:

\[\begin{array}{ccc}
j^*D_X (M') & \xrightarrow{((-1)^w, j^*D_X S'')} & j^*D_X (M'')\lambda^{-w} \\
\kappa'' & \downarrow & \kappa' \uparrow \\
M'' & \xrightarrow{S''} & M'\lambda^{-w}
\end{array}\]

Because $\kappa'' = (j^*D_X \kappa')^{-1}$, the commutativity means $\kappa' \circ S'' = (-1)^w j^*D_X (\kappa' \circ S'')$.

Let $(T, W)$ be a mixed twistor $D$-module with a real structure $\kappa$ and a graded sesqui-linear duality $(S_w | w \in \mathbb{Z})$. We say that the real structure and the graded sesqui-linear duality are compatible if the induced real structure and the sesqui-linear duality on $\text{Gr}_w(T)$ are compatible for any $w$.

B.2.2 The associated TEP-structure in the pure case

Let $T = (M', M'', C)$ be an integrable pure twistor $D$-module of weight $w$ on a complex manifold $X$ with a real structure $\kappa$ and a polarization $S$ which are compatible and integrable. We have the following isomorphisms of $R_X$-modules:

\[\begin{array}{ccc}
M'' & \xrightarrow{S''} & M'\lambda^{-w} \\
\kappa' & \downarrow & \lambda^{-w} j^*D_X M''
\end{array}\]

Let $H$ be a hypersurface of $X$. Suppose that $M'(\ast H)$ and $M''(\ast H)$ are smooth $R_X(\ast H)$-modules. We have a natural isomorphism

\[\nu_{M''} : (D_X M'')(\ast H) \simeq \lambda^{d_X} \cdot M''(\ast H)^\vee\]
whose restriction to \(\{\lambda_0\} \times X\) (\(\lambda_0 \neq 0\)) is given by the morphism in \([A.1]\). We consider the pairing \(P\) induced by \((-1)^{d_X} \nu_{M'}\) and \(\kappa' \circ S''\):

\[
P : \mathcal{M}'(\ast H) \times j^* \mathcal{M}''(\ast H) \longrightarrow \lambda^{d_X-w} \mathcal{O}_X(\ast H)
\]

**Proposition B.2** We have \(j^* P \circ \text{exchange} = (-1)^{d_X-w} P\). In other words, \(P\) is a pairing of weight \(w - d_X\) on \(\mathcal{M}''(\ast H)\). In other words, \((\mathcal{M}''(\ast H), P)\) is a TEP-structure \((\mathcal{M}'', \mathcal{M}'', \mathcal{M}''(\ast H), \{\mathcal{M}''(\ast H)\})\).

**Proof** The claim of this proposition follows from the relation \(\kappa' \circ S'' = (-1)^w j^* D(\kappa' \circ S'')\) and Lemma \([A.2]\).

For any \(n \in \mathbb{Z}\), the pure twistor \(D\)-module \(\mathcal{T} \otimes \mathcal{T}(-n)\) of weight \(w + 2n\) naturally equipped with the polarization and the real structure which are compatible. The polarization is given by \((-\mathcal{M}'(\ast H), -\mathcal{M}'(\ast H))\). The real structure is given by \((-\mathcal{M}'(\ast H), -\mathcal{M}'(\ast H))\). Hence, the induced pairing is \(P(n)\) which was introduced in \([B.1.1]\).

**Example B.3** Let \(\mathcal{T}(\mathcal{F}) = (\lambda^{d_X} \mathcal{L}(\mathcal{F}), \mathcal{L}(\mathcal{F}), C)\) be the integrable pure twistor \(D\)-module of weight \(d_X\) associated to a holomorphic function \(F\) on \(X\), where \(d_X := \dim X\). The polarization \(S\) is given by \(((-1)^{d_X}, (-1)^{d_X})\). The real structure is given by \((-\mathcal{M}'(\ast H), -\mathcal{M}'(\ast H))\). Hence, the induced pairing is just the multiplication.

**B.2.3** The associated mixed TEP-structure

Let \((\mathcal{T}, W)\) be a mixed twistor \(D\)-module on \(X\). Let \((\mathcal{M}', \mathcal{M}'', C)\) be the underlying \(\mathcal{R}_X\)-triple of \(\mathcal{T}\). We often pick \(\mathcal{M}'\), and we say that \(\mathcal{M}'\) is the underlying \(\mathcal{R}_X\)-module of \(\mathcal{T}\). We have the filtration \(W\) on \(\mathcal{M}'\) such that \(W_j \mathcal{M}'\) are the underlying \(\mathcal{R}\)-modules of \(W_j \mathcal{T}\), which we call the filtration of \(\mathcal{M}'\) underlying the weight filtration of \((\mathcal{T}, W)\). In the following, we shall often use the shifted filtration \(\mathcal{W}\) on \(\mathcal{M}''\) given by \(\mathcal{W}_k(\mathcal{M}'') := W_{k+d_X} \mathcal{M}'\).

Let \((\mathcal{T}, W)\) be an integrable mixed twistor \(D\)-module with real structure and a graded sesqui-linear duality which are compatible and integrable. Let \(H\) be a hypersurface of \(X\). Let \(\mathcal{M}''\) be the underlying \(\mathcal{R}_X\)-module. Suppose that \(\mathcal{M}''(\ast H)\) is a smooth \(\mathcal{R}_X(\ast H)\)-module. Let \(W\) be the filtration of \(\mathcal{M}''\) underlying the weight filtration of \(\mathcal{T}\). We set \(\mathcal{W}_k(\mathcal{M}''(\ast H)) := W_{k+d_X} \mathcal{M}'\). Each \(\mathcal{G}_{k} \mathcal{W}(\mathcal{M}''(\ast H))\) is equipped with the pairing \(P_k\) of weight \(k\) induced by the real structure and the sesqui-linear duality. In this way, we obtain a mixed TEP-structure \((\mathcal{M}'(\ast H), \mathcal{W}, \{P_k\})\).

**Remark B.4** The underlying \(\mathcal{R}\)-module of \(\mathcal{T} \otimes \mathcal{T}(-n)\) is \(\mathcal{M}''(\ast H)\). The filtration \(\mathcal{W}(\mathcal{M}''(\ast H))\) induced by the mixed twistor structure \((\mathcal{T}, W) \otimes \mathcal{T}(-n)\) is given by \(\mathcal{W}_k(\mathcal{M}''(\ast H)) = \lambda^{2k} \mathcal{W}_k(\mathcal{M}''(\ast H))\), which is equal to the shift of the filtration in \([B.1.1]\). The graded pairing on \((\mathcal{M}''(\ast H), \mathcal{W})\) induced by the real structure and the graded sesqui-linear duality on \((\mathcal{T}, W) \otimes \mathcal{T}(-n)\) is equal to that induced by the graded pairing on \((\mathcal{M}''(\ast H), \mathcal{W})\) by the procedure in \([B.1.1]\).

**B.3** Comparison of the specializations

**B.3.1** Statement

Let \(X\) be a complex manifold with a smooth hypersurface \(Y\). Suppose that \(Y\) is given as \(\{t = 0\}\) for a holomorphic function \(t\) such that \(dt\) is nowhere vanishing. Let \(\mathcal{T} = (\mathcal{M}_1, \mathcal{M}_2, C)\) be an integrable pure twistor \(D\)-module of weight \(w\) on \(X\) with a sesqui-linear duality \(S\) and a real structure \(\kappa\) which are compatible and integrable.

**Assumption B.5** We assume that \(\mathcal{M}_1(\ast Y)\) are smooth \(\mathcal{R}_X(\ast Y)\)-modules, and regular along \(Y\). (See the condition \((B0)\) in \([B.1.2]\) for the regularity.)

Set \(d := \dim X\). We have the pairing \(P\) of weight \(w - d\) on \(\mathcal{M}_2(\ast Y)\). We have the endomorphism \(\lambda N : \psi_{t, -\delta}(\mathcal{M}_2) \to \psi_{t, -\delta}(\mathcal{M}_2)\), where \(N\) is the induced by \(t\partial_t\). Note that the condition \((B1)\) is satisfied because
\( \mathcal{M}_2 \) comes from a pure twistor \( D \)-module. By the procedure in \([B.1.3]\) we obtain filtrations \( \widetilde{W}^{[1]} \) and graded pairings \( (P_k^{[1]} \mid k \in \mathbb{Z}) := sp_\nu (P) \) on \( \text{Cok}(\lambda N) \) and \( \text{Ker}(\lambda N) \).

Let \( \mathcal{N} : \widetilde{\psi}_{t,-\delta}(T) \otimes \mathcal{U}(-1,0) \to \widetilde{\psi}_{t,-\delta}(T) \otimes \mathcal{U}(0,1) \) be given by \((-N,-N)\), where \( N : \psi_{t,-\delta}(\mathcal{M}_1) \to \lambda^{-1}\psi_{t,-\delta}(\mathcal{M}_1) \) are induced by \( i\partial_t \). According to the isomorphism in Proposition 4.3.1 of \([36]\), we have the following commutative diagram:

\[
\begin{array}{ccc}
\psi^{(1)}(T) & \xrightarrow{\cong} & \psi^{(0)}(T) \\
\downarrow & & \downarrow \\
\iota \widetilde{\psi}_{t,-\delta}(T) \otimes \mathcal{U}(-1,0) & \xrightarrow{\mathcal{N}} & \iota \widetilde{\psi}_{t,-\delta}(T) \otimes \mathcal{U}(0,1)
\end{array}
\]

Here, the upper horizontal arrow is the canonical morphism. Hence, we have the following natural isomorphisms:

\[
\iota \text{Cok}(\mathcal{N}) \cong \text{Cok}(\psi^{(1)}(T) \to \psi^{(0)}(T)) \cong \text{Cok}(\mathcal{T}[!t] \to \mathcal{T}[*t])
\]

\[
\iota \text{Ker}(\mathcal{N}) \cong \text{Ker}(\psi^{(1)}(T) \to \psi^{(0)}(T)) \cong \text{Ker}(\mathcal{T}[!t] \to \mathcal{T}[*t])
\]

So, \( \text{Cok}(\mathcal{N}) \otimes \mathcal{T}(1) \) and \( \text{Ker}(\mathcal{N}) \) are equipped with the real structure and the graded sesqui-linear duality which are compatible and integrable. The underlying \( \mathcal{R}_Y \)-modules of \( \text{Cok}(\mathcal{N}) \otimes \mathcal{T}(1) \) and \( \text{Ker}(\mathcal{N}) \) are \( \text{Cok}(\lambda N) \) and \( \text{Ker}(\lambda N) \), respectively. Applying the procedure in \([B.2.3]\) we obtain filtrations \( \widetilde{W}^{[2]} \) and graded pairings \( (P_k^{[2]} \mid k \in \mathbb{Z}) \) on \( \text{Cok}(\lambda N) \) and \( \text{Ker}(\lambda N) \).

**Proposition B.6** We have \( \widetilde{W}^{[1]} = \widetilde{W}^{[2]} \) and \( P_k^{[1]} = P_k^{[2]} \) for any \( k \in \mathbb{Z} \). In other words, the induced mixed TEP-structures are the same.

For the monodromy weight filtration \( W(N) \) of \( \mathcal{N} \) on \( \widetilde{\psi}_{t,-\delta}(T) \), we have \( W(N) \iota \widetilde{\psi}_{t,-\delta}(T) = W_j \psi_{t,-\delta}(T) \). Then, we can check the equality for the filtrations easily. We shall compare the pairings in \([B.3.2][B.3.3]\).

### B.3.2 Preliminary

Let \( \mathcal{V} \) be a smooth \( \mathcal{R}_X(*Y) \)-module which is regular along \( t = 0 \). We have a natural isomorphism \( \nu_{\mathcal{V}} : D(\mathcal{V})(*Y) \cong \lambda^d \cdot \mathcal{V}^\vee \) whose specialization at \( \{\lambda_0\} \times X \) \( \lambda_0 \neq 0 \) is equal to the morphism in \([A.1]\). By the procedure in \([A.3.1]\) it induces the following isomorphisms of \( \mathcal{R}_X \)-modules for \( a = 0,1 \):

\[
\nu_{\mathcal{V},\psi^a_t} : D_X \psi^a_t(\mathcal{V}) \cong \psi^{(a+1)}(\mathcal{V}) \lambda^d
\]

**Lemma B.7** The following diagram is commutative:

\[
\begin{array}{ccc}
D_X \psi^{(1)}_t(\mathcal{V}) & \xrightarrow{\nu_{\mathcal{V},\psi^{(1)}_t}} & \psi^{(0)}_t(\mathcal{V}) \lambda^d \\
\cong & & \cong \\
D_X \iota \psi_{t,-\delta}(\mathcal{V}) & \xrightarrow{\cong} & \iota \psi_{t,-\delta}(\mathcal{V}) \lambda^{d-1}
\end{array}
\]

Here, the vertical arrows are the isomorphisms in \( \S 4.3 \) of \([36]\), and the lower horizontal arrow is induced as the composite of the isomorphisms \( D_X (\iota \psi_{t,-\delta}(\mathcal{V})) \cong \iota \psi_{t,-\delta}(\mathcal{V}) \lambda^{-1} \cong \iota \psi_{t,-\delta}(\mathcal{V}) \lambda^{-1} \).

The following diagram is commutative:

\[
\begin{array}{ccc}
D_X (\psi^{(0)}_t(\mathcal{V})) & \xrightarrow{\nu_{\mathcal{V},\psi^{(0)}_t}} & \psi^{(1)}_t(\mathcal{V}) \lambda^d \\
\cong & & \cong \\
D_X (\iota \psi_{t,-\delta}(\mathcal{V}) \lambda^{-1}) & \xrightarrow{\cong} & \iota \psi_{t,-\delta}(\mathcal{V}) \lambda^d
\end{array}
\]

The vertical arrows and the lower horizontal arrows are given as in the diagram \( (1.55) \).

104
Proof. We have only to prove the commutativity of the diagrams after taking the specializations to $\{\lambda_0\} \times X$ for any generic $\lambda_0 \neq 0$, which follows from Proposition [4.6].

Let $V_i$ ($i = 1, 2$) be smooth $\mathcal{R}_X(*Y)$-modules which are regular along $t = 0$. Let $P' : V_1 \otimes j^*V_2 \longrightarrow \lambda^{-m}\mathcal{O}_X(*Y)$ be a morphism of $\mathcal{R}_X(*Y)$-modules. We have the induced morphism of $\mathcal{R}_Y$-modules

$$\psi_{t,-\delta}(P') : \psi_{t,-\delta}(V_1) \otimes j^*\psi_{t,-\delta}(V_2) \longrightarrow \lambda^{-m}\mathcal{O}_Y.$$ 

The pairing $P'$ and the isomorphism $(-1)^d\nu_{\psi_{t,-\delta}(V_2)}$ induce an $\mathcal{R}_X$-homomorphism $V_1 \longrightarrow \lambda^{-m-d}(j^*D_XV_2)(*Y)$. It induces the following morphisms for $a = 0, 1$:

$$\psi^{(a)}_t(V_1) \longrightarrow \lambda^{-m-d}D_X\psi^{(1-a)}_t(V_2).$$

By the isomorphisms $\lambda^{-1+a}\cdot t\psi_{t,-\delta}(V_1) \simeq \psi^{(a)}_t(V_1)$ and $\lambda^{-a}t\psi_{t,-\delta}(V_2) \simeq \psi^{(1-a)}_t(V_2)$, we obtain the following:

$$t\psi_{t,-\delta}(V_1) \longrightarrow \lambda^{-m-d+1}j^*D_X(t\psi_{t,-\delta}(V_2)) \simeq \lambda^{-m-d+1}t\psi_{t,-\delta}(V_2) \quad (157)$$

The morphism $\psi^{(a)}_t(V_1)$ and $(-1)^d\nu_{\psi_{t,-\delta}(V_2)}$ induce a pairing $P' : \psi_{t,-\delta}(V_1) \times j^*\psi_{t,-\delta}(V_2) \longrightarrow \lambda^{-m}\mathcal{O}_Y$. We obtain the following lemma from Lemma [B.7] as in the case of Corollary [A.8]. We remark the twist of the signatures, i.e., we use $(-1)^d\nu^\lambda$ and $(-1)^d\nu_{\psi_{t,-\delta}(V_2)}$.

**Lemma B.8** We have $\psi_{t,-\delta}(P') = P'_1$ and $\psi_{t,-\delta}(P') = -P'_0$.

### B.3.3 Proof of Proposition [B.6]

For $a = 0, 1$, from the morphisms $\psi^{(a)}_t(S) : \psi^{(a)}_t(T) \longrightarrow \psi^{(a)}_t(T)^* \otimes T(-w - 1 + 2a)$ and $\psi^{(a)}_t(\tilde{S}) : \tilde{\psi}^{(a)}_t(T) \simeq \psi^{(a)}_t(T)$, we have the following $\mathcal{R}_X$-homomorphisms:

$$\psi^{(a)}_t(M_2) \overset{\delta^{(a)}}{\longrightarrow} \lambda^{-w-1+2a}\psi^{(1-a)}_t(M_1) \overset{\delta^{(a)}}{\longrightarrow} \lambda^{-w-1+2a}j^*D_X\psi^{(a)}_t(M_2) \quad (158)$$

The composite $b^{(a)}_2 \circ \delta^{(a)}_1$ comes from an isomorphism $\psi_{t,-\delta}(M_2)\lambda^{-1} \simeq \lambda^{-w-1}j^*D_X(\psi_{t,-\delta}(M_2)\lambda^{-1})$, in the case $a = 0$, or $\psi_{t,-\delta}(M_2) \simeq \lambda^{-w+1}j^*D_X(\psi_{t,-\delta}(M_2))$ in the case $a = 1$. Together with $(-1)^d\nu\psi_{t,-\delta}(M_2)$, we obtain a pairing $P_a : \psi_{t,-\delta}(M_2) \times j^*\psi_{t,-\delta}(M_2) \longrightarrow \lambda^d\mathcal{O}_Y$.

**Lemma B.9** We have $\psi_{t,-\delta}(P) = \tilde{P}_0 = \tilde{P}_1$.

**Proof.** According to §4.3 of [36], we have the following commutative diagram:

$$\begin{array}{ccc}
\psi^{(0)}_t(T) & \overset{\psi^{(0)}_t(S)}{\longrightarrow} & \psi^{(0)}_t(T)^* \otimes T(-w - 1) \\
\simeq \downarrow & & \simeq \downarrow \\
\iota_t\psi_{t,-\delta}(T) \otimes \mathcal{U}(0, -1) & \longrightarrow & (\iota_t\psi_{t,-\delta}(T) \otimes \mathcal{U}(0, -1))^* \otimes T(-w - 1)
\end{array} \quad (159)$$

The lower horizontal arrow is induced by $\psi_{t,-\delta}(S)$. It means that the following diagram is commutative:

$$\begin{array}{ccc}
\psi^{(0)}_t(M_2) & \overset{\lambda^{(0)}}{\longrightarrow} & \psi^{(1)}_t(M_1)\lambda^{-w-1} \\
\simeq \downarrow & & \simeq \downarrow \\
\iota_t\lambda^{-1}\psi_{t,-\delta}(M_2) & \overset{\psi_{t,-\delta}(S')}{\longrightarrow} & \iota_t\psi_{t,-\delta}(M_1)\lambda^{-w-1}
\end{array} \quad (160)$$

We have the pairing $P' : M_1(*Y) \times j^*M_2(*Y) \longrightarrow \lambda^d\mathcal{O}_X(*Y)$ induced by $\kappa' : M_1 \longrightarrow j^*DM_2$ and $(-1)^d\nu_{M_2(*Y)}$. The morphism $b^{(0)}_2$ comes from $\psi_{t,-\delta}(M_1) \longrightarrow j^*D(\psi_{t,-\delta}(M_2)\lambda^{-1})$. Together with the
morphism \((-1)^{d-1}\nu_{\psi_t,-\delta}(M_2)\), we obtain a pairing \(P'_t: \psi_{t,-\delta}(M_1) \times j^*\psi_{t,-\delta}(M_2) \rightarrow \lambda^d\mathcal{O}_Y\). We obtain 
\[\psi_{t,-\delta}(P) = P_0\] from the equality \(\psi_{t,-\delta}(P') = P'_0\) in Lemma [B.8] and the commutativity (160).

We also have the following commutative diagram from (161):

\[
\begin{array}{ccc}
\psi_t^{(1)}(M_2) & \xrightarrow{b^{(1)}} & \psi_t^{(0)}(M_1) \lambda^{-w+1} \\
\downarrow & & \downarrow \\
\iota_t \psi_{t,-\delta}(M_2) & \xrightarrow{-\psi_{t,-\delta}(S')} & \iota_t \psi_{t,-\delta}(M_1) \lambda^{-w}
\end{array}
\]

The morphism \(b^{(1)}_t\) comes from a morphism \(\lambda^{-1}\psi_{t,-\delta}(M_1) \rightarrow j^*D(\psi_{t,-\delta}(M_2))\). It induces \(P'_t: \psi_{t,-\delta}(M_1) \times j^*\psi_{t,-\delta}(M_2) \rightarrow \lambda^d\mathcal{O}_Y\). From the equality \(\psi_{t,-\delta}(P') = -P'_0\) and the commutativity of the diagram (161), we obtain \(\psi_{t,-\delta}(P) = \overline{P}_1\). Thus, the proof of Lemma [B.9] is finished.

The graded sesqui-linear duality on \(Gr^{W}_{\nu_0+1+k} Cok(N)\) is induced by \(\psi_t^{(0)}(S) \circ (-N)^k\). Hence, by Lemma [B.9] we obtain \(P[1] = P[2]\) on \(Cok(N)\). The graded sesqui-linear duality on \(Gr^{W}_{\nu_0-1-k} Ker(N)\) is induced by \(\psi_t^{(1)}(S) \circ N^k\). Hence, by Lemma [B.9] we obtain \(P[1] = P[2]\) on \(Ker(N)\). Thus the proof of Proposition [B.3] is finished.

**B.4 Push-forward**

Let \(X\) be a complex manifold. We set \(d_X := \dim X\). Let \(\mathcal{T} = (M_1, M_2, C)\) be an integrable pure twistor \(\mathcal{D}\)-module of weight \(w\) on \(X\) with a sesqui-linear duality \(\mathcal{S}\) and a real structure \(\kappa\) which are compatible and integrable. Suppose that \(\mathcal{T}\) is smooth, i.e., \(M_i\) are locally free \(\mathcal{O}_X\)-modules. We have the associated pairing \(P(S, \kappa)\) on \(M_2\) of weight \(w - d_X\).

Let \(F: X \rightarrow Y\) be a smooth projective morphism. We set \(d_Y := \dim Y\). We have the pure twistor \(\mathcal{D}\)-modules \(F^*_t\mathcal{T} = (F^*_t\mathcal{M}_1, F^*_t\mathcal{M}_2, F^*_t\mathcal{C})\). They are equipped with the induced real structure \(\kappa_t\). We also have the induced morphisms \(S_t: F^*_t\mathcal{T} \rightarrow (F^*_t\mathcal{T})^* \otimes \mathcal{T}(-w)\). Note that \(F^*_t\mathcal{M}_i\) are locally free \(\mathcal{O}_Y\)-modules. As in [B.2.2] we have the associated morphism

\[P(F_t\mathcal{S}, F_t\kappa)_i: F^*_t\mathcal{M}_2 \otimes F^*_t\mathcal{M}_2 \rightarrow \lambda^{-w+d_X}\mathcal{O}_Y\]

induced by the composite of the following morphisms:

\[F^*_t\mathcal{M}_2 \xrightarrow{F^*_t\mathcal{S}} \lambda^{-w}F^*_t\mathcal{M}_1 \xrightarrow{F^*_t\kappa} \lambda^{-w}F^*_t\mathcal{M}_2 \simeq \lambda^{-w}j^*D F^*_t\mathcal{M}_2 \simeq \lambda^{-w+d_X}j^*(F^*_t\mathcal{M}_2)^* \]

The last isomorphism is given by \((-1)^{d_X}\nu_{F^*_t\mathcal{M}_2}\). We give an expression of \(P(F_t\mathcal{S}, F_t\kappa)_i\) in terms of \(P\).

Let \(Q \in Y\) be any point of \(Y\). We set \(X_Q := F^{-1}(Q)\). Let \(F_Q: X_Q \rightarrow \{Q\}\) be the restriction of \(F\). Let \(\mathcal{T}_Q := (M_1, M_2, C_Q)\) be the restriction of \(\mathcal{T}\) to \(X_Q\) as smooth \(\mathcal{R}\)-triple. Let \(F^*_t\mathcal{T}_Q\) be the restriction of \(F^*_t\mathcal{T}\) to \(Q\) as smooth \(\mathcal{R}\)-triple. We naturally have \(F^*_Q(F^*_t\mathcal{T}_Q) = F^*_t\mathcal{T}_Q\).

We set \(\overline{\Omega}^1_X := \lambda^{-1}p_\lambda^*\Omega_X^1\) and \(\overline{\Omega}^1_Q := \lambda^p\overline{\Omega}^1_X\). We have the de Rham complex \(\overline{\Omega}^*_X \otimes (M_2\mathcal{O}_X)\) for \(M_2\mathcal{O}_X\). We have

\[F^*_Q\mathcal{M}_2 = R^{d+2}F_*(\overline{\Omega}^*_{\mathcal{X}_Q} \otimes (M_2\mathcal{O}_X)).\]

Set \(d := d_X - d_Y = \dim X_Q\). Let \(\Omega^{p,q}_{X,Q}\) denote the sheaf of \(C^\infty(0,q)\)-forms on \(X_Q\). Let \(\Omega^{p,q}_{\mathcal{X}_X} := p_\lambda^*(\Omega_{\mathcal{X}_X}^p)\otimes p_\lambda^{-1}\mathcal{O}_{X,Q}\). The pairing \(P\) pairing the following:

\[\left(\lambda^{-k}\Omega_{\mathcal{X}_X}^{k,p}\mathcal{O}_{X,Q/C} \otimes M_2\right) \otimes j^*\left(\lambda^{-(d-k)}\Omega_{\mathcal{X}_X}^{d-k,d-p}\mathcal{O}_{X,Q/C} \otimes M_2\right) \rightarrow \lambda^{-d-w+d_X}\Omega_{\mathcal{X}_X}^{d}\]

given by \((\xi^{k,p,m_1}, \xi^{d-k,d-p,m_2}) \mapsto \xi^{k,p}\xi^{d-k,d-p}(m_1, m_2)\). Note that \(-d - w + d_X = -w + d_Y\). It induces

\[P_{t,Q}: R^{d+2}F_*(\overline{\Omega}^*_{\mathcal{X}_X} (M_2\mathcal{O}_X)) \otimes j^*R^{d-i}F_*(\overline{\Omega}^*_{\mathcal{X}_X} (M_2\mathcal{O}_X)) \rightarrow \lambda^{-w+d_X}\mathcal{O}_X\]

By varying \(Q\), we obtain \(P_{F,t}: F^*_t\mathcal{M}_2 \otimes F^*_t\mathcal{M}_2 \rightarrow \lambda^{-w+d_X}\mathcal{O}_Y\).
Proposition B.10 We have \( P(F_1S, F_1\kappa)_i = \epsilon(d)(-1)^d P_{F_i} \).

\[ \text{Proof} \quad \text{It is enough to compare the specializations along } \{ \lambda \} \times X \text{ for } \lambda \neq 0, \text{which follows from Proposition A.9} \]

\[ \Box \]

C Some functoriality of \( \tilde{\mathcal{R}} \)-modules

C.1 Preliminary

C.1.1 Some sheaves

For any complex manifold \( X \), let \( \mathcal{X} := \mathbb{C}_\lambda \times X \). Let \( p_\lambda : \mathcal{X} \rightarrow X \) be the projection. Set \( d_\lambda := \text{dim} \mathcal{X} \). Let \( \mathcal{O}_{\mathcal{X}} \) denote the sheaf of functions on \( \mathcal{X} \) which are locally constant in the \( \mathcal{X} \)-direction, and holomorphic in the \( \lambda \)-direction.

Let \( \Omega^1_\mathcal{X} \) denote the sheaf of holomorphic 1-forms on \( \mathcal{X} \). Let \( \Omega^1_\mathcal{X} \oplus \lambda^{-1}p_\lambda^*\Omega^1_\mathcal{X} \) and \( \Omega^p_\mathcal{X} := \bigwedge^p \Omega^1_\mathcal{X} \). In particular, we set \( \Omega^1_\mathcal{X} := \Omega^1_X \). As in the case of \( D \)-modules, for any left \( \mathcal{R}_X \)-module \( N \), we naturally obtain a right \( \mathcal{R}_X \)-module \( \Omega^1_\mathcal{X} \otimes N \). Conversely, for any right \( \mathcal{R}_X \)-module \( N^r \), we naturally obtain a left \( \mathcal{R}_X \)-module \( \Omega^1_\mathcal{X} \otimes N^r \).

Let \( \Theta_X \) be the sheaf of holomorphic vector fields on \( \mathcal{X} \). We set \( \Omega^1_X := \lambda p_\lambda^*\Theta_X \). We set \( \Theta_X^\perp := \bigwedge^p \Theta_X \). We obtain the Spencer resolution \( \mathcal{R}_X \otimes \Theta_X \) of \( \mathcal{O}_X \) by locally free left \( \mathcal{R}_X \)-modules. (See also Lemma C.1 below.) We have the decomposition:

\[ \mathcal{O}_X = \bigoplus_{j=0}^\infty (\partial_\lambda \lambda^j) \otimes \lambda^j \mathcal{N}. \]

\[ \text{We denote it by } \nabla \text{ which is explicitly described as follows.} \]

- Let \( P \) and \( m \) denote local sections of \( \mathcal{R}_X \) and \( \mathcal{N} \), respectively. For any \( j \geq 0 \), we have local sections \( P_{ij} \) (\( i = 0, \ldots, j \)) of \( \mathcal{R}_X \) such that \( P_{ij}(\partial_\lambda \lambda^j) = \sum_{i=0}^j \partial_\lambda P_{ij} \lambda^j \) in \( \mathcal{R}_X \). Then, \( P \nabla(\partial_\lambda \lambda^j) = \sum_{i=0}^j \partial_\lambda \lambda^j \).

- We also have \( \partial_\lambda \lambda^j \nabla(\partial_\lambda \lambda^j) = \partial_\lambda^{j+1} \lambda^j \nabla = \lambda^j \partial_\lambda^{j+1} \lambda + j(j-1) \partial_\lambda^{j+1} \lambda^j m \).

We have the inclusion \( \iota : \mathcal{N} \rightarrow \mathcal{N}(\lambda^2 \partial_\lambda) \) given by \( \iota (m) = 1 \otimes m \). It is an \( \mathcal{R}_X \)-homomorphism. We have the decomposition:

\[ \mathcal{N}(\lambda^2 \partial_\lambda) = \bigoplus_{j=0}^\infty (\partial_\lambda \lambda^j) \nabla \iota (\mathcal{N}) \]

\[ \text{(163)} \]

We also have the \( \mathcal{R}_X \)-action \( \nabla \) on \( \mathcal{N}(\lambda^2 \partial_\lambda) \) given as follows.

- For any \( P \) and \( m \) as above, we set \( P \nabla(\partial_\lambda \lambda^j) = \partial_\lambda \lambda^j P \nabla \).

If moreover \( \mathcal{N} \) is an \( \tilde{\mathcal{R}}_X \)-module, the \( \mathcal{R}_X \)-action \( \nabla \) is extended to the \( \tilde{\mathcal{R}}_X \)-action \( \nabla \) on \( \mathcal{N}(\lambda^2 \partial_\lambda) \) given as follows:

\[ \partial_\lambda \lambda^j \nabla(\partial_\lambda \lambda^j) = \partial_\lambda \lambda^j \partial_\lambda \lambda^2 \lambda^j m - \partial_\lambda^{j+1} \lambda^{j+2} m. \]

It is easy to check that \( \mathcal{N}(\lambda^2 \partial_\lambda) \) is an \( \tilde{\mathcal{R}}_X \)-module. (See also Lemma C.1 below.) We have the decomposition:

\[ \mathcal{N}(\lambda^2 \partial_\lambda) = \bigoplus_{j=0}^\infty (\partial_\lambda \lambda^j) \nabla \iota (\mathcal{N}) \]

\[ \text{(164)} \]
For any $\ell \in \mathbb{Z}$, we set $\lambda^\ell \mathcal{N}(\lambda^2 \partial_\lambda) := \bigoplus_{j=0}^{\infty} \partial_\lambda^j \otimes \lambda^{2j+\ell} \mathcal{N}$. It is equipped with the natural $\tilde{R}$-actions $\triangledown$ and $\bigtriangledown$, obtained as above. We naturally have $\lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda) = \lambda^2 \triangledown \mathcal{N}(\lambda^2 \partial_\lambda) = \lambda^2 \bigtriangledown \mathcal{N}(\lambda^2 \partial_\lambda)$. We have the well defined morphisms of sheaves $\partial_\lambda \triangledown : \lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda) \to \mathcal{N}(\lambda^2 \partial_\lambda)$ and $\partial_\lambda \bigtriangledown : \lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda) \to \mathcal{N}(\lambda^2 \partial_\lambda)$.

It is easy to check the following lemma.

**Lemma C.1** Let $\mathcal{N}$ be an $\tilde{R}_X$-module.

- For any local section $m$ of $\mathcal{N}$, we have the following relation
  $$(\partial_\lambda \lambda^2) \triangledown (m) + (\partial_\lambda \lambda^2) \bigtriangledown (m) = \mu (\partial_\lambda \lambda^2 m).$$

- For the above isomorphism $\Phi : \tilde{R}_X \otimes_{\tilde{R}_X} \mathcal{N} \simeq \mathcal{N}(\lambda^2 \partial_\lambda)$, we have
  $$\partial_\lambda \lambda^2 \bigtriangledown (\partial_\lambda \lambda^2) \otimes m) = \Phi (\partial_\lambda \lambda^2 \bigtriangledown (-\partial_\lambda \lambda^2) \times m) + \Phi (\partial_\lambda \lambda^2 \bigtriangledown \partial_\lambda \lambda^2 \otimes \partial_\lambda m).$$

For any $P \in \tilde{R}_X$, we have the following:

$$P \bigtriangledown (\partial_\lambda \lambda^2 \otimes m) = \Phi (\partial_\lambda \lambda^2 \bigtriangledown (P m)).$$

- $\partial_\lambda \triangledown$ gives a morphism $(\lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda), \triangledown) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \triangledown)$, and $\partial_\lambda \bigtriangledown$ gives a morphism $(\lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown)$.

By the lemma, for any $\tilde{R}_X$-module $\mathcal{N}$, we obtain the following exact sequence:

$$0 \to (\lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda), \triangledown) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \to \mathcal{N} \to 0$$

Here, $a_1$ is the projection onto the 0-th component with respect to the decomposition [163]. We also have the following exact sequence:

$$0 \to (\lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \to \mathcal{N} \to 0$$

Here, $a_2$ is the projection onto the 0-th component with respect to the decomposition [163]. We obtain the following $\tilde{R}_X$-resolutions of $\mathcal{N}$:

$$S^\triangledown_\lambda := \left( \left( \lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda), \triangledown \right) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \right)$$

$$S^\bigtriangledown_\lambda := \left( \left( \lambda^2 \mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown \right) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \right)$$

**Example C.2** Let us consider the case $\mathcal{N} = \tilde{R}_X$. We consider the $\tilde{R}_X$-action given by the left multiplication. We have the isomorphism $\tilde{R}_X(\lambda^2 \partial_\lambda) \simeq \tilde{R}_X$ given by $\sum \partial_\lambda^j \otimes \lambda^{2j+1} \to \sum \partial_\lambda^j \partial_\lambda^{2j+1}$. Under the isomorphism, $\partial_\lambda \lambda^2 \triangledown$ is the left multiplication of $\partial_\lambda \lambda^2$, and $\partial_\lambda \lambda^2 \bigtriangledown$ is the right multiplication of $-\lambda^2 \partial_\lambda$. Hence, we naturally have $\tilde{R}_X \otimes \tilde{R}_X^\times \simeq S^\triangledown_\lambda (\tilde{R}_X \otimes \tilde{R}_X^\times)$.

**C.1.3 Exchange**

We define $\Psi : \mathcal{N}(\lambda^2 \partial_\lambda) \to \mathcal{N}(\lambda^2 \partial_\lambda)$ by $\Phi((\partial_\lambda \lambda^2) \triangledown \mu (m)) := (\partial_\lambda \lambda^2) \bigtriangledown \mu (m)$. By the construction, $\Psi$ gives an isomorphism $(\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown)$.

**Lemma C.3** $\Psi$ also gives an isomorphism $\Psi : (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown) \to (\mathcal{N}(\lambda^2 \partial_\lambda), \bigtriangledown)$.

**Proof** It is enough to check that $\Psi((\partial_\lambda \lambda^2) \bigtriangledown \mu (m)) = (\partial_\lambda \lambda^2) \bigtriangledown \Psi ((\partial_\lambda \lambda^2) \bigtriangledown \mu (m))$ due to the decomposition [163]. By the commutativity of $\partial_\lambda \lambda^2$ and $\partial_\lambda \lambda^2 \bigtriangledown$, we have only to check $\Psi ((\partial_\lambda \lambda^2) \bigtriangledown \mu (m)) = (\partial_\lambda \lambda^2) \bigtriangledown \mu (m)$.

We have the induced isomorphisms $\Psi : S^\bigtriangledown_\lambda (\mathcal{N}) \to S^\bigtriangledown_\lambda (\mathcal{N})$ and $\Psi : S^\bigtriangledown_\lambda (\mathcal{N}) \to S^\bigtriangledown_\lambda (\mathcal{N})$.

**Example C.4** If $\mathcal{N} = \tilde{R}_X$, under the isomorphism $\tilde{R}_X \simeq \tilde{R}_X(\lambda^2 \partial_\lambda)$, the isomorphism $\Psi : \tilde{R}_X \to \tilde{R}_X$ is given by $\Psi(\sum \partial_\lambda m_j) = \sum m_j (-\partial_\lambda)^j$. 

108
C.1.4 Bi-modules

Let $\mathcal{O}_\mathcal{C}_\lambda(\lambda^2 \partial_\lambda)$ denote the sheaf of subalgebras in $\overline{\mathcal{R}}_X$ generated by $\lambda^2 \partial_\lambda$ over $\mathcal{O}_\mathcal{C}_\lambda$. Let $B$ be an $\mathcal{O}_\mathcal{C}_\lambda$-module. An action of $\lambda^2 \partial_\lambda$ on $B$ is a morphism of sheaves $\rho(\lambda^2 \partial_\lambda) : B \to B$ such that $\rho(\lambda^2 \partial_\lambda)(fP) = f \rho(\lambda^2 \partial_\lambda)(P) + (\lambda^2 \partial_\lambda f) P$ for any local sections $f \in \mathcal{O}_\mathcal{C}$ and $P \in B$. If $B$ is a sheaf of algebras over $\mathcal{O}_\mathcal{C}$, we moreover impose the Leibniz rule $\rho(\lambda^2 \partial_\lambda)(PQ) = \rho(\lambda^2 \partial_\lambda)(P)Q + P \rho(\lambda^2 \partial_\lambda)(Q)$. If we are given a sheaf of algebras $\mathcal{B}$ over $\mathcal{O}_\mathcal{C}$ equipped with an action of $\lambda^2 \partial_\lambda$, the sheaf $\mathcal{B} \otimes \mathcal{O}_\mathcal{C}_\lambda(\lambda^2 \partial_\lambda)$ is naturally a sheaf of algebras. The multiplication is given by $(P \otimes (\lambda^2 \partial_\lambda))^k \cdot (Q \otimes (\lambda^2 \partial_\lambda))^j = \sum_{k=0}^j (k : j) P \rho(\lambda^2 \partial_\lambda)^j(Q) \otimes (\lambda^2 \partial_\lambda)^{k+j-j}$, where $(k : j)$ denote the binomial coefficients.

We can naturally regard $\mathcal{R}_X$ as a left $\mathcal{O}_\mathcal{C}_\lambda(\lambda^2 \partial_\lambda)$-module. The action of $\lambda^2 \partial_\lambda$ on $\mathcal{R}_X$ is given by $\rho(\lambda^2 \partial_\lambda)(P) = [\lambda^2 \partial_\lambda, P]$ in $\overline{\mathcal{R}}_X$ which satisfies the Leibniz rule $\rho(\lambda^2 \partial_\lambda)(PQ) = \rho(\lambda^2 \partial_\lambda)(P)Q + P \rho(\lambda^2 \partial_\lambda)(Q)$. The sheaf $\mathcal{R}_X \otimes \mathcal{O}_\mathcal{C}_\lambda(\lambda^2 \partial_\lambda)$ is naturally a sheaf of algebras. We have the isomorphism $\mathcal{R}_X \otimes \mathcal{O}_\mathcal{C}_\lambda \mathcal{O}_\mathcal{C}_\lambda(\lambda^2 \partial_\lambda) \simeq \overline{\mathcal{R}}_X$ given by $\sum P \otimes (\lambda^2 \partial_\lambda)^j \mapsto \sum P_j(\lambda^2 \partial_\lambda)^j$.

Because $\mathcal{O}_\mathcal{C}_\lambda$ is the center of $\mathcal{R}_X$, we have the naturally defined sheaf of algebras $\mathcal{R}_X \otimes \mathcal{O}_\mathcal{C}_\lambda \mathcal{R}_X$ on $\mathcal{X}$. It is naturally an $\mathcal{O}_\mathcal{C}_\lambda(\lambda^2 \partial_\lambda)$-module. The action of $\lambda^2 \partial_\lambda$ is given by $\rho_1(\lambda^2 \partial_\lambda)(P \otimes Q) = \rho(\lambda^2 \partial_\lambda)(P) \otimes Q + P \otimes \rho(\lambda^2 \partial_\lambda)(Q)$.

Let $\mathcal{N}$ be an $\mathcal{A}_X$-module. Let $\ell$ (resp. $r$) denote the $\mathcal{R}_X$-action or $\overline{\mathcal{R}}_X$-action on $\mathcal{N}$ induced by $k_1$ (resp. $k_2$). We set $\mathcal{N}(\lambda^2 \partial_\lambda) := \bigoplus_{\ell=0}^{\infty} \partial_\lambda^\ell \otimes \lambda^{2\ell} \mathcal{N}$. The $\mathcal{R}_X$-action $\ell$ on $\mathcal{N}$ and the construction $\nabla$ give an $\overline{\mathcal{R}}_X$-action $\ell \nabla$ on $\mathcal{N}(\lambda^2 \partial_\lambda)$. The induced multiplication of $g \in \mathcal{R}_X$ and $c \in \mathcal{N}(\lambda^2 \partial_\lambda)$ is denoted by $g(\ell \nabla) c$. We obtain an $\mathcal{R}_X$-action by taking the restriction, which is also denoted by $\ell \nabla$. For $a \in \mathcal{R}_X$, let $\ell \nabla(a)$ denote the endomorphism of the sheaves $\mathcal{N}(\lambda^2 \partial_\lambda)$ given by $m \mapsto a(\ell \nabla)m$. We use the notation $\ell \nabla, r \nabla$ and $r \nabla$ similarly.

Lemma C.5 We have $r \nabla(a_1) \circ \ell \nabla(a_2) = \ell \nabla(a_2) \circ r \nabla(a_1)$ for any $a_1, a_2 \in \overline{\mathcal{R}}_X$.

Proof By using the explicit construction of $\nabla$ and $\nabla$, we can observe that $r \nabla(a_1) \circ \ell \nabla(a_2) = \ell \nabla(a_2) \circ r \nabla(a_1)$ if $a_1, a_2 \in \mathcal{R}_X$. By the assumption, we have $\ell \nabla(\partial_\lambda^2) = \ell \nabla(\partial_\lambda^2)$ and $r \nabla(\partial_\lambda^2) = r \nabla(\partial_\lambda^2)$. By using Lemma C.1, we obtain $r \nabla(a_1) \circ \ell \nabla(a_2) = \ell \nabla(a_2) \circ r \nabla(a_1)$ if either of $a_1$ or $a_2$ is $\partial_\lambda^2$.

The following lemma is clear by construction.

Lemma C.6 We have $r \nabla(a_1) \circ \ell \nabla(a_2) = \ell \nabla(a_2) \circ r \nabla(a_1)$ and $r \nabla(a_1) \circ \ell \nabla(a_2) = \ell \nabla(a_2) \circ r \nabla(a_1)$ for any $a_1, a_2 \in \mathcal{R}_X$.

As mentioned in the proof of Lemma C.5, we have $r \nabla(\partial_\lambda^2) = \ell \nabla(\partial_\lambda^2)$ and $r \nabla(\partial_\lambda^2) = \ell \nabla(\partial_\lambda^2)$, the morphism $\Psi$ in [C.1.3] is independent of the choices of $\ell$ and $r$. It exchanges $(r \nabla, \ell \nabla)$ and $(\ell \nabla, r \nabla)$.

Example C.7 We consider $\mathcal{N} := \mathcal{R}_X \otimes \overline{\mathcal{R}}_X^{-1}$ as an $\mathcal{A}_X$-module. Here, the $\mathcal{R}_X$-actions $\ell$ and $r$ are induced by the left and the right multiplications. The action of $\lambda^2 \partial_\lambda$ is given by $\lambda^2 \partial_\lambda(P) = [\lambda^2 \partial_\lambda, P]$ in $\overline{\mathcal{R}}_X$.

We have the isomorphism $\mathcal{N}(\lambda^2 \partial_\lambda, \ell \nabla) \simeq \mathcal{R}_X \otimes \mathcal{R}_X (\mathcal{N}, \ell) = \mathcal{R}_X \otimes \overline{\mathcal{R}}_X^{-1} \simeq \mathcal{R}_X \otimes \overline{\mathcal{R}}_X^{-1} (d\lambda)^{-1}$. Under the isomorphism, $\ell \nabla$ is equal to the left multiplications, and $r \nabla$ is equal to the $\mathcal{R}_X$-action induced by the right multiplication.

We have the isomorphism $\mathcal{N}(\lambda^2 \partial_\lambda, r \nabla) \simeq \mathcal{R}_X \otimes \mathcal{R}_X (\mathcal{N}, r) = \mathcal{R}_X \otimes \overline{\mathcal{R}}_X^{-1} \simeq \mathcal{R}_X \otimes \overline{\mathcal{R}}_X^{-1} (d\lambda)^{-1}$. Under the isomorphism, $r \nabla$ is equal to the $\mathcal{R}_X$-action induced by the right multiplication, and $\ell \nabla$ is equal to the left multiplication.

C.1.5 Functoriality for inner homomorphisms

Let $\mathcal{N}_1$ be an $\mathcal{R}_X$-module. Let $\mathcal{N}_2$ be an $\mathcal{A}_X$-module. Let $\mathcal{H}_{\text{Hom}}(\mathcal{N}_1, \mathcal{N}_2^\ell)$ be the sheaf of $\mathcal{R}_X$-homomorphisms from $\mathcal{N}_1$ to $(\mathcal{N}_2^\ell, \ell)$. It is equipped with the $\mathcal{R}_X$-action induced by $r$. It is naturally extended to an $\overline{\mathcal{R}}_X$-action,
where the action of $\lambda^2 \partial_\lambda$ is given by $\lambda^2 \partial_\lambda(f(m)) = \lambda^2 \partial_\lambda(f(m)) - f(\lambda^2 \partial_\lambda m)$. We have the $\tilde{R}_X$-actions $\nabla$ and $\nabla^\flat$ on $\Hom_{R_X}(N_1,N_2^\flat(\lambda^2 \partial_\lambda))$.

Let $\Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat)$ denote the sheaf of $R_X$-homomorphisms $N_1 \rightarrow (N_2(\lambda^2 \partial_\lambda),\xi \nabla)$, where $\xi \nabla$ is denoted by $N_2(\lambda^2 \partial_\lambda)$. By Lemma C.9 we have the $\tilde{R}_X$-action on $\Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat)$ induced by $r \nabla$ on $N_2(\lambda^2 \partial_\lambda)$. The induced $\tilde{R}_X$-action is also denoted by $r \nabla$. By Lemma C.10 we have the $R_X$-action on $\Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat)$ induced by $r \nabla$ on $N_2(\lambda^2 \partial_\lambda)$. It is extended to an $\tilde{R}_X$-action. The action of $\lambda^2 \partial_\lambda$ is given by $(\lambda^2 \partial_\lambda g)(m) = \lambda^2 \partial_\lambda (\xi \nabla)(g(m)) - g(\lambda^2 \partial_\lambda m)$. The induced $\tilde{R}_X$-action is denoted by $r \nabla$.

We define a morphism of sheaves $F : \Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat) \rightarrow \Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat)$ given as follows. For any local section $\sum \partial_\lambda^i \otimes \lambda^2 g_i$ of $\Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)$ and any local section $m$ of $N_1$, we set

$$F\left(\sum \partial_\lambda^i \otimes \lambda^2 g_i\right)(m) := \sum \partial_\lambda^i \otimes \lambda^2 g_i(m).$$

If $N_1$ is $R_X$-coherent, then $F$ is an isomorphism. We can check the following lemma by a direct computation.

**Lemma C.8** For local sections $a \in \tilde{R}_X$ and $b \in \Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)$, we have $F(a \nabla b) = a(r \nabla)F(b)$ and $F(a \nabla b) = a(r \nabla)F(b)$.

We have the following natural isomorphism by the decomposition (169):

$$\Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat) \simeq \Hom_{R_X}(N_1(\lambda^2 \partial_\lambda),N_2(\lambda^2 \partial_\lambda)^\flat)$$

**Lemma C.9** We have the following commutative diagram:

$$\begin{array}{ccc}
\Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat) & \xrightarrow{b} & \Hom_{R_X}(N_1(\lambda^2 \partial_\lambda)^\flat, N_2(\lambda^2 \partial_\lambda)^\flat) \\
a_1 & & a_2 \\
\Hom_{R_X}(\lambda^2 N_1,N_2(\lambda^2 \partial_\lambda)^\flat) & \xrightarrow{b} & \Hom_{R_X}(\lambda^2 N_1(\lambda^2 \partial_\lambda)^\flat, N_2(\lambda^2 \partial_\lambda)^\flat)
\end{array}$$

The horizontal arrows $b$ are (166). The morphism $a_2$ is given by $\partial_\lambda \nabla$ on $N_1(\lambda^2 \partial_\lambda)$. The morphism $a_1$ is determined by $a_1(g)(m) = -\partial_\lambda (\xi \nabla)(g(m)) + g(\partial_\lambda m)$ for any $g \in \Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat)$ and $m \in \lambda^2 N_1$.

**Proof** For any $g \in \Hom_{R_X}(N_1,N_2(\lambda^2 \partial_\lambda)^\flat)$ and $m = \lambda^2 m_1 \in \lambda^2 N_1$, we have

$$(a_2 \circ b)(g)(\iota(\lambda^2 m_1)) = b(g)\left(\iota(\lambda^2) \nabla (\iota(m_1))\right) = b(g)\left(-\iota(\partial_\lambda \lambda^2) \nabla(\iota(\lambda^2 m_1)) + \iota(\partial_\lambda \lambda^2 m_1)\right)$$

$$= (-\partial_\lambda \lambda^2)(\xi \nabla)(g(m_1)) + g(\partial_\lambda \lambda^2 m_1) = -\partial_\lambda (\xi \nabla)(g(m)) + g(\partial_\lambda m).$$

Thus, we obtain the claim of the lemma.

**Corollary C.10** We have the following natural isomorphism of $\tilde{R}_X$-complexes:

$$S_X^\flat \Hom_{R_X}(N_1,N_2^\flat) \simeq \Hom_{R_X}(S_X^\flat N_1,N_2^\flat(\lambda^2 \partial_\lambda)^\flat) \boxtimes [1]$$

$$S_X^\flat \Hom_{R_X}(N_1,N_2^\flat) \simeq \Hom_{R_X}(S_X^\flat N_1,N_2^\flat(\lambda^2 \partial_\lambda)^\flat) \boxtimes [1]$$

**Proof** We obtain (168) from Lemma C.8 and Lemma C.9. We obtain (169) by exchanging $\nabla$ and $\nabla^\flat$.

### C.2 Push-forward

#### C.2.1 Push-forward of $R$-modules and $\tilde{R}$-modules

We recall the push-forward of $R$-modules and $\tilde{R}$-modules. See [242] for more details. Let $X$ be any complex manifold. Let $C^\infty X$ denote the sheaf of $C^\infty$-functions $F$ on $X$ such that $\overline{\partial}_X F = 0$. Let $\Omega^{\bullet,q}_X$ denote the sheaf of $C^\infty$ $(0,q)$-forms on $X$. Let $C^\infty_X$ denote the sheaf of $C^\infty$-functions on $X$. We define

$$\tilde{C}^\infty_X := \bigoplus_{p+q=j} p^{-1}_X \Omega^{\bullet,q}_X \otimes p^{-1}_X C^\infty_X(\overline{\partial} X \otimes_{\mathcal{O}_X} C^\infty_X).$$
With the exterior derivative in the $X$-direction, we obtain the complexes $(\Omega^\cdot_X, d)$ and $(C_X^\cdot, d)$. The natural inclusion gives a quasi-isomorphism $(\Omega^\cdot_X, d) \rightarrow (C_X^\cdot, d)$.

Let $f : X \rightarrow Y$ be any morphism of complex manifolds. The induced morphism $X \rightarrow Y$ is also denoted by $f$. We set $R_{X,Y,f} := R_X \otimes_{O_Y} f^{-1}R_Y$. It is naturally a sheaf of algebras. It is also naturally an $O_{C_X}(\lambda^2\partial_X)$-module, and the action of $\lambda^2\partial_X$ satisfies the Leibniz rule. Hence, we obtain the sheaf of algebras $A_{X,Y,f} := R_{X,Y,f} \otimes_{O_Y} O_Y(\lambda^2\partial_Y)$ as in the case of $A_X$. We have a natural inclusion $\tilde{R}_X \rightarrow A_{X,Y,f}$ and $f^{-1}\tilde{R}_Y \rightarrow A_{X,Y,f}$. We set $R_{Y,-X} := \pi_X \otimes_{O_X} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1})$. It is naturally a left $R_{X,Y,f}$-module. We have the natural action of $\lambda^2\partial_X$ on $f^{-1}(\tilde{R}_Y)$ given by $\lambda^2\partial_X(Q) := [\lambda^2\partial_X(Q) in f^{-1}(\tilde{R}_Y)]$. It induces an action of $\lambda^2\partial_X$ on $R_{Y,-X}$, with which $R_{Y,-X}$ is an $A_{X,Y,f}$-module. We have a canonical locally free $R_{X,Y,f}$-resolution $(\Omega^\cdot_X[dX] \otimes_{O_X} R_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1})$ of $R_{Y,-X}$. It is also an $A_{X,Y,f}$-resolution. We have the following natural quasi-isomorphism of $A_{X,Y,f}$-complexes:

$$ (\Omega^\cdot_X[dX] \otimes_{O_X} R_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) \rightarrow (C_X^\cdot, d) \otimes_{O_X} R_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) $$

Let $M^\cdot$ be any $R_X$-complex. We obtain the following $R_Y$-complex:

$$ f_1M^\cdot := f_1 \left( (\Omega^\cdot_X[dX] \otimes_{O_X} R_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) \right) \otimes_{O_X} M^\cdot $$

It induces a functor $D^b(R_X) \rightarrow D^b(R_Y)$. We denote the right hand side of (170) by $Rf_!(\tilde{R}_{Y,-X} \otimes_{\tilde{R}_X} M^\cdot)$. If $M^\cdot$ is cohomologically $R_X$-coherent and good relative to $f$, then $f_1M^\cdot$ is also cohomologically $R_X$-coherent. If $M^\cdot$ is an $\tilde{R}_X$-complex, $f_1M^\cdot$ is naturally an $\tilde{R}_Y$-complex. Hence, (170) gives a functor $D^b(\tilde{R}_X) \rightarrow D^b(\tilde{R}_Y)$.

C.2.2 Another expression of the push-forward of $\tilde{R}$-modules

We put

$$ \tilde{C}_X^\cdot := \bigoplus_{p+q=j} p_X^{-1}T_X^{0,q} \otimes p_X^{-1}C_X^\cdot \left( \Omega^p_X \otimes_{O_X} C_X^\cdot \right). $$

With the exterior derivative $d$, we obtain the complexes $(\tilde{\Omega}^\cdot_X, d)$ and $(\tilde{C}_X^\cdot, d)$, and we have the natural quasi-

isomorphism $(\Omega^\cdot_X, d) \rightarrow (\tilde{C}_X^\cdot, d)$.

Let $f : X \rightarrow Y$ be a morphism of complex manifolds. We set $\tilde{R}_{X,Y,f} := \tilde{R}_X \otimes_{O_Y} f^{-1}\tilde{R}_Y$. We set $\tilde{R}_{Y,-X} := \tilde{R}_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1})$. It is naturally an $\tilde{R}_{X,Y,f}$-module. We have a canonical locally free $R_{X,Y,f}$-resolution $(\tilde{\Omega}^\cdot_X[dX + 1] \otimes_{O_X} \tilde{R}_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1})$ of $\tilde{R}_{Y,-X}$. We have the following natural quasi-

isomorphism of $\tilde{R}_X \otimes f^{-1}\tilde{R}_Y$-complexes:

$$ (\tilde{\Omega}^\cdot_X[dX + 1] \otimes_{O_X} \tilde{R}_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) \rightarrow (\tilde{C}_X^\cdot, d) \otimes_{O_X} \tilde{R}_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) $$

Let $M^\cdot$ be any $\tilde{R}_X$-complex. We obtain the following $\tilde{R}_Y$-complex:

$$ \tilde{f}_1M^\cdot := \tilde{f}_1 \left( (\tilde{\Omega}^\cdot_X[dX + 1] \otimes_{O_X} \tilde{R}_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) \right) \otimes_{O_X} \tilde{M}^\cdot $$

It induces a functor $D^b(\tilde{R}_X) \rightarrow D^b(\tilde{R}_Y)$. We denote the right hand side of (171) by $R\tilde{f}_!(\tilde{R}_{Y,-X} \otimes_{\tilde{R}_X} \tilde{M}^\cdot)$.

Lemma C.11 We have the following natural isomorphism:

$$ (\tilde{C}_X^\cdot, d) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) \otimes_{O_X} \tilde{M}^\cdot \cong S^\cdot_X \left( (\tilde{C}_X^\cdot, d) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \Omega_Y^{-1}) \right) \otimes_{O_X} \tilde{M}^\cdot $$

111
Let $\ell$ and $r$ denote the $R_Y$-action (resp. $\tilde{R}_Y$-action) on $R_Y \otimes \Omega_Y^{-1}$ (resp. $\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1}$) induced by the left and right multiplications. Let $\ell \nabla$ (resp. $r \nabla$) denote the $\tilde{R}_Y$-action on $(R_Y \otimes \tilde{\Omega}_Y^{-1})(\lambda^2 \partial_\lambda)$ induced by $\ell$ and $\nabla$ (resp. $r$ and $\nabla$). We have the isomorphism $\Omega_Y^{-1} \simeq \lambda^{-2} \tilde{\Omega}_Y^{-1}$ given by $\tau \mapsto \tau(d\lambda)^{-1}$. As mentioned in Example C.7, they induce the following isomorphism:

$$
(\tilde{R}_Y \otimes (\lambda^{-2} \tilde{\Omega}_Y^{-1}), \ell, r) \simeq \left((R_Y \otimes \Omega_Y^{-1})(\lambda^2 \partial_\lambda), \ell \nabla, r \nabla\right).
$$

(173)

We obtain (172) from (173).

**Corollary C.12** We have the natural isomorphism $\tilde{f}_! \mathcal{M}^* \simeq S_X^r f_! \mathcal{M}^*$ of $\tilde{R}_Y$-complexes. In particular, we have a natural isomorphism $f_! \mathcal{M}^* \to f_! \mathcal{M}^*$ in the derived category of $\tilde{R}_Y$-modules.

**C.2.3 Trace morphisms**

Let $\mathfrak{D}b_X/c_\lambda$ be the sheaf of distributions $F$ on $X$ such that $\partial_\lambda F = 0$. We set

$$
\mathfrak{D}b_X := \bigoplus_{p+q=j} p^{-1}_\lambda \Omega_X^0 \otimes p^{-1}_\lambda c_\lambda \left(\tilde{\Omega}_X^0 \otimes \mathcal{O}_X \mathfrak{D}b_X/c_\lambda\right).
$$

With the exterior derivative $d$ in the $X$-direction, we obtain the complex $(\mathfrak{D}b_X, d)$. We have the natural quasi-isomorphism $(\mathfrak{D}b_X, d) \to (\mathfrak{D}b_X, d)$. We also set

$$
\mathfrak{D}b^j_X := \bigoplus_{p+q=j} p^{-1}_\lambda \Omega_X^0 \otimes p^{-1}_\lambda c_\lambda \left(\tilde{\Omega}_X^0 \otimes \mathcal{O}_X \mathfrak{D}b_X/c_\lambda\right).
$$

With the exterior derivative, we obtain the complex $(\mathfrak{D}b^j_X, d)$. The natural inclusion $(\mathfrak{D}b_X, d) \to (\mathfrak{D}b^j_X, d)$ is a quasi-isomorphism.

Let $f : X \to Y$ be a proper morphism. Recall that we have the trace morphism $tr_f : \lambda^{d_X} f_! \mathcal{O}_X[d_X] \to \lambda^{d_Y} \mathcal{O}_Y[d_Y]$ in the derived category of $\tilde{R}_Y$-modules:

$$
\lambda^{d_X} f_! \mathcal{O}_X[d_X] \simeq f_! \left(\lambda^{d_X} \mathfrak{D}b_X^* [2d_X] \otimes f^{-1}_* \mathcal{O}_Y f^{-1}(R_Y \otimes \tilde{\Omega}_Y^{-1})\right) \simeq f_! \left(\lambda^{d_X} \mathfrak{D}b_X^* [2d_X] \otimes \mathcal{O}_Y (R_Y \otimes \tilde{\Omega}_Y^{-1})\right) \xrightarrow{a} \lambda^{d_Y} \mathfrak{D}b^*_Y [2d_Y] \otimes \mathcal{O}_Y (R_Y \otimes \tilde{\Omega}_Y^{-1}) \simeq \lambda^{d_Y} \mathcal{O}_Y[d_Y].
$$

(174)

Here, $a$ is given by the integration of currents along $f$ multiplied by $(2\pi i) -d_X+d_Y$. We also have the trace morphism $\tilde{tr}_f : \lambda^{d_X} \tilde{f}_! \mathcal{O}_X[d_X] \to \lambda^{d_Y} \mathcal{O}_Y[d_Y]$:

$$
\lambda^{d_X} \tilde{f}_! \mathcal{O}_X[d_X] \simeq f_! \left(\lambda^{d_X} \mathfrak{D}b^*_X [2d_X + 1] \otimes f^{-1}_* \mathcal{O}_Y f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1})\right) \simeq f_! \left(\lambda^{d_X} \mathfrak{D}b^*_X [2d_X + 1] \otimes f^{-1}_* \mathcal{O}_Y (\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1})\right) \xrightarrow{b} \lambda^{d_Y} \mathfrak{D}b^*_Y [2d_Y + 1] \otimes \mathcal{O}_Y (\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1}) \simeq \lambda^{d_Y} \mathcal{O}_Y[d_Y]
$$

(175)

Here, $b$ is given by the integration of currents along $f$ multiplied by $(2\pi i) -d_X+d_Y$. The following is clear by the construction of the morphisms.

**Lemma C.13** $\tilde{tr}_f$ is equal to the composite of $\lambda^{d_X} \tilde{f}_! \mathcal{O}_X[d_X] \to \lambda^{d_X} \tilde{f}_! \mathcal{O}_X[d_X]$ and $tr_f$.

**Proof** As in Lemma C.11 we can identify the morphism (175) with that obtained from (174) by the functor $S_X^r$. Hence, the claim is clear.

112
C.2.4 Complement

Let \( \mathcal{N} \) be any \( \widetilde{\mathcal{R}}_X \)-module. We have the morphism of sheaves

\[
\left( \mathcal{O}_X^\bullet [d_X] \otimes_{f^{-1} \mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}_X}) \right) \otimes_{\mathcal{O}_X} \left( \mathcal{N}(\lambda^2 \partial_\lambda), \nabla \right) \rightarrow \left( \mathcal{O}_X^\bullet [d_X] \otimes_{f^{-1} \mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}_X}) \right) \otimes_{\mathcal{O}_X} \mathcal{N} \right) (\lambda^2 \partial_\lambda)
\]

(176)

by \( g \otimes (\partial_\lambda \otimes \lambda^{2j}) \mapsto \partial_\lambda \otimes (g \otimes \lambda^{2j}) \). By construction, the following holds.

Lemma C.14 The morphism (176) is an \( \widetilde{\mathcal{R}}_X \)-homomorphism with respect to the natural \( f^{-1} \mathcal{R}_Y \)-action on the left hand side, and the \( f^{-1} \mathcal{R}_Y \)-action \( \nabla \) on the right hand side. The morphism (176) is compatible with the actions \((\partial_\lambda \lambda^2) \nabla \) on both sides.

Corollary C.15 The morphism (176) induces an isomorphism \( f_!(\mathcal{M}^\bullet (\lambda^2 \partial_\lambda), \nabla) \rightarrow (f_!(\mathcal{M}^\bullet))(\lambda^2 \partial_\lambda, \nabla) \). It also induces an isomorphism \( f_! S_X^0(\mathcal{M}) \cong S_X f_!(\mathcal{M}) \).

C.3 Duality

C.3.1 Duality of \( \mathcal{R} \)-modules and \( \widetilde{\mathcal{R}} \)-modules

Let \( X \) be a complex manifold with a hypersurface \( H \). Set \( d_X := \dim X \). We naturally regard \( \lambda^{d_X} \mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1} \) as an \( \mathcal{A}_X(\ast \text{mod}) \)-module. We take an \( \mathcal{A}_X(\ast \text{mod}) \)-injective resolution \( \mathcal{G}^\bullet_0 \) of \( \lambda^{d_X} \mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1} \). For any \( \mathcal{R}_X(\ast \text{mod}) \)-module \( \mathcal{N} \), let \( \mathcal{H}om_{\mathcal{R}_X(\ast \text{mod})}(\mathcal{N}, \mathcal{G}^\bullet_0) \) denote the sheaf of \( \mathcal{R}_X(\ast \text{mod}) \)-homomorphisms \( \mathcal{N} \rightarrow (\mathcal{G}^\bullet_0, \ell) \). It is naturally an \( \mathcal{R}_X(\ast \text{mod}) \)-module by \( r \). If \( \mathcal{N} \) is an \( \mathcal{R}_X(\ast \text{mod}) \)-module, it is naturally an \( \widetilde{\mathcal{R}}_X(\ast \text{mod}) \)-module. The action of \( \lambda^2 \partial_\lambda \) is given by \( (\lambda^2 \partial_\lambda f'(m) = \lambda^2 \partial_\lambda (f(m)) - f(\lambda^2 \partial_\lambda m) \).

For any \( \mathcal{R}_X(\ast \text{mod}) \)-complex \( \mathcal{M}^\bullet \), we obtain the following \( \mathcal{R}_X(\ast \text{mod}) \)-complex:

\[
D_X(\ast \text{mod}) \mathcal{M}^\bullet := \mathcal{H}om_{\mathcal{R}_X(\ast \text{mod})}(\mathcal{M}^\bullet, \mathcal{G}^\bullet_0)
\]

(177)

If \( \mathcal{M}^\bullet \) is an \( \mathcal{R}_X(\ast \text{mod}) \)-complex, then \( D_X(\ast \text{mod}) \mathcal{M}^\bullet \) is naturally an \( \widetilde{\mathcal{R}}_X(\ast \text{mod}) \)-complex. We shall often denote the right hand side of (177) by \( R\mathcal{H}om_{\mathcal{R}_X(\ast \text{mod})}(\mathcal{M}^\bullet, \lambda^{d_X} \mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1}) \) if there is no risk of confusion.

C.3.2 Another expression of the duality of \( \widetilde{\mathcal{R}} \)-modules

We give another expression of the duality functor for \( \widetilde{\mathcal{R}} \)-modules. We naturally regard \( \widetilde{\mathcal{R}}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1} \) as \( \mathcal{R}_X(\ast H) \otimes \mathcal{R}_X(\ast H) \)-module. Let \( \ell \) (resp. \( r \)) denote the \( \mathcal{R}_X(\ast \text{mod}) \)-actions induced by the left multiplication (resp. the right multiplication). We take an injective \( \mathcal{R}_X(\ast H) \otimes \mathcal{R}_X(\ast H) \)-resolution \( \mathcal{G}^\bullet_1 \) of \( \lambda^{d_X} \mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1} \). For any \( \mathcal{R}_X(\ast \text{mod}) \)-module \( \mathcal{N} \), let \( \mathcal{H}om_{\mathcal{R}_X(\ast \text{mod})}(\mathcal{N}, \mathcal{G}^\bullet_1) \) denote the sheaf of \( \mathcal{R}_X(\ast \text{mod}) \)-homomorphisms \( \mathcal{N} \rightarrow (\mathcal{G}^\bullet_1, \ell) \). It is naturally an \( \mathcal{R}_X(\ast \text{mod}) \)-module induced by the \( \mathcal{R}_X(\ast \text{mod}) \)-action \( r \).

For any \( \mathcal{R}_X(\ast \text{mod}) \)-complex \( \mathcal{M}^\bullet \), we obtain the following \( \mathcal{R}_X(\ast \text{mod}) \)-complex:

\[
\tilde{D}_X(\ast \text{mod}) \mathcal{M}^\bullet := \mathcal{H}om_{\mathcal{R}_X(\ast \text{mod})}(\mathcal{M}^\bullet, \mathcal{G}^\bullet_1)
\]

(178)

We shall denote the right hand side of (178) by \( R\mathcal{H}om_{\mathcal{R}_X(\ast \text{mod})}(\mathcal{M}^\bullet, \mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1}) \) if there is no risk of confusion.

We have the \( \mathcal{R}_X(\ast \text{mod}) \)-action \( \ell \nabla \) on \( (\mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1})(\lambda^2 \partial_\lambda) \) induced by the \( \mathcal{R}_X(\ast \text{mod}) \)-action \( \ell \) and the construction \( \nabla \). We also have the \( \mathcal{R}_X(\ast \text{mod}) \)-action \( r \nabla \) on \( (\mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1})(\lambda^2 \partial_\lambda) \) induced by the \( \mathcal{R}(\ast \text{mod}) \)-action \( r \) and the construction \( \nabla \). Similarly, we have the \( \mathcal{R}_X(\ast \text{mod}) \)-actions \( \ell \nabla \) and \( r \nabla \) on \( \mathcal{G}^\bullet_0(\lambda^2 \partial_\lambda) \). As in (177), we have the isomorphism

\[
\left( (\mathcal{R}_X(\ast H) \otimes \overline{\mathcal{O}_X}^{-1})(\lambda^2 \partial_\lambda), \ell \nabla, r \nabla \right) \cong \left( \mathcal{R}_X(\ast H) \otimes (\lambda^{-2} \overline{\mathcal{O}_X}^{-1}), \ell, r \right).
\]

(179)
Hence, we may assume to have a quasi-isomorphism of \( \mathcal{R}_X(\ast H) \otimes \mathcal{R}_X(\ast H) \)-complexes
\[
\lambda^2 G_0^\ast (\lambda^2 \delta_\lambda)[1] \to G_1^\ast.
\] (180)

From Corollary \([9,10]\) and \([180]\), we have the following morphisms of \( \mathcal{R}_X(\ast H) \)-complexes:
\[
S_X^\ast \text{Hom}_{\mathcal{R}_X(\ast H)}(M^\ast, G_0^\ast) \simeq \text{Hom}_{\mathcal{R}_X(\ast H)}(S_X^\ast M^\ast, \lambda^2 G_0^\ast (\lambda^2 \delta_\lambda)^{\vee}[1]) \to \text{Hom}_{\mathcal{R}_X(\ast H)}(S_X^\ast M^\ast, G_1^\ast)
\] (181)

We also have the following natural quasi-isomorphisms:
\[
S_X^\ast \text{Hom}_{\mathcal{R}_X(\ast H)}(M^\ast, G_0^\ast) \to \text{Hom}_{\mathcal{R}_X(\ast H)}(M^\ast, G_0^\ast)
\] (182)
\[
\text{Hom}_{\mathcal{R}_X(\ast H)}(M^\ast, G_1^\ast) \to \text{Hom}_{\mathcal{R}_X(\ast H)}(S_X^\ast M^\ast, G_1^\ast)
\] (183)

**Proposition C.16** If \( M^\ast \) is cohomologically \( \mathcal{R}_X(\ast H) \)-coherent, then \([181]\) is a quasi-isomorphism. Together with \([182]\) and \([183]\), we obtain the following isomorphisms in the derived category of cohomologically \( \mathcal{R}_X(\ast H) \)-coherent complexes:
\[
D_X M^\ast \xrightarrow{\sim} D_X S_X^\ast M^\ast \xrightarrow{\sim} S_X^\ast D_X M^\ast \xrightarrow{\sim} D_X M^\ast
\]

**Proof** For simplicity of the description, we omit to denote \((\ast H)\) in the proof. It is enough to prove the claim for an \( \mathcal{R}_X \)-module \( M \) which is coherent over \( \mathcal{R}_X \). We begin with the following lemma.

**Lemma C.17** Let \( 0 \to N_1 \to N_2 \to N_3 \to 0 \) be an exact sequence of coherent \( \mathcal{R}_X \)-modules. Let \( I \) be any injective \( \mathcal{R}_X \)-module. Then, the following is exact.
\[
0 \to \text{Hom}_{\mathcal{R}_X}(N_3(\lambda^2 \delta_\lambda), I(\lambda^2 \delta_\lambda)) \to \text{Hom}_{\mathcal{R}_X}(N_2(\lambda^2 \delta_\lambda), I(\lambda^2 \delta_\lambda)) \to \text{Hom}_{\mathcal{R}_X}(N_1(\lambda^2 \delta_\lambda), I(\lambda^2 \delta_\lambda)) \to 0
\] (184)

**Proof** The sequence \([184]\) is rewritten as follows:
\[
0 \to \text{Hom}_{\mathcal{R}_X}(N_3, I(\lambda^2 \delta_\lambda)) \to \text{Hom}_{\mathcal{R}_X}(N_2, I(\lambda^2 \delta_\lambda)) \xrightarrow{\beta} \text{Hom}_{\mathcal{R}_X}(N_1, I(\lambda^2 \delta_\lambda)) \to 0
\] (185)

It is enough to prove that \( \beta \) is an epimorphism. For any non-negative integer \( L \), we set \( F_L I(\lambda^2 \delta_\lambda) := \bigoplus_{j=0}^{\infty} \partial_\lambda^j \otimes \lambda^{2j} I \). They are naturally an \( \mathcal{A}_X \)-submodules of \( I(\lambda^2 \delta_\lambda) \). Let \( F_L \text{Hom}_{\mathcal{R}_X}(N_i, I(\lambda^2 \delta_\lambda)) \) be the image of \( \text{Hom}_{\mathcal{R}_X}(N_i, I(\lambda^2 \delta_\lambda)) \). Because \( N_i \) is \( \mathcal{R}_X \)-coherent, the filtration \( F \) is exhaustive on \( \text{Hom}_{\mathcal{R}_X}(N_i, I(\lambda^2 \delta_\lambda)) \).

By construction, we have \( \text{Gr}_F^L \text{Hom}_{\mathcal{R}_X}(N_i, I(\lambda^2 \delta_\lambda)) \simeq \text{Hom}_{\mathcal{R}_X}(N_i, \lambda^{2j} I) \). Because \( I \) is \( \mathcal{R} \)-injective, we have the surjectivity of \( \text{Gr}_F^L \text{Hom}_{\mathcal{R}_X}(N_2, I(\lambda^2 \delta_\lambda)) \to \text{Gr}_F^L \text{Hom}_{\mathcal{R}_X}(N_1, I(\lambda^2 \delta_\lambda)) \). Then, we obtain the desired surjectivity.

The claim of Proposition \([C.16]\) is reduced to the following lemma.

**Lemma C.18** For any coherent \( \mathcal{R}_X \)-module \( N \), the natural morphism
\[
\text{Hom}_{\mathcal{R}_X}(N(\lambda^2 \delta_\lambda), \lambda^2 G_0^\ast (\lambda^2 \delta_\lambda)^{\vee}[1]) \to \text{Hom}_{\mathcal{R}_X}(N(\lambda^2 \delta_\lambda), G_1^\ast)
\] (186)
is a quasi-isomorphism.

**Proof** The morphism \([186]\) is a quasi-isomorphism if \( N = \mathcal{R}_X \). It is enough to prove that \([186]\) is a quasi-isomorphism locally around any point of \( X \). We may assume to have an \( \mathcal{R}_X \)-free resolution \( P^\ast \) of \( N \). We have the following commutative diagram:
\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{R}_X}(N(\lambda^2 \delta_\lambda), \lambda^2 G_0^\ast (\lambda^2 \delta_\lambda)^{\vee}[1]) & \xrightarrow{\alpha_3} & \text{Hom}_{\mathcal{R}_X}(N(\lambda^2 \delta_\lambda), G_1^\ast)
\\ \downarrow{\alpha_2} & & \downarrow{\alpha_3}
\\ \text{Tot} \text{Hom}_{\mathcal{R}_X}(P^\ast(\lambda^2 \delta_\lambda), \lambda^2 G_0^\ast (\lambda^2 \delta_\lambda)^{\vee}[1]) & \xrightarrow{\alpha_4} & \text{Tot} \text{Hom}_{\mathcal{R}_X}(P^\ast(\lambda^2 \delta_\lambda), G_1^\ast)
\end{array}
\]

Here, \( \text{Tot} C^\ast \) denote the total complex of a double complex \( C^\ast \). As remarked above, \( \alpha_4 \) is a quasi-isomorphism. Because \( (G_1^\ast, \ell) \) is \( \mathcal{R}_X \)-injective, \( \alpha_3 \) is a quasi-isomorphism. As for \( \alpha_2 \), note that \( (\lambda^2 G_0^\ast, \ell) \) is \( \mathcal{R}_X \)-injective. We have the isomorphism exchanging \( \vee \) and \( \wedge \) as in \([C.13]\). Hence, \( \alpha_2 \) is a quasi-isomorphism by Lemma \([C.17]\). Then, we obtain that \( \alpha_1 \) is a quasi-isomorphism. Thus we obtain Lemma \([C.18]\) and hence Proposition \([C.16]\).
C.3.3 Smooth case
Let \(\mathcal{M}\) be a smooth \(\tilde{\mathcal{R}}_X(\ast H)\)-module. Let \(\mathcal{M}'\) be the smooth \(\mathcal{R}_X(\ast H)\)-module \(\text{Hom}_{\mathcal{O}_X(\ast H)}(\mathcal{M}, \mathcal{O}_X(\ast H))\) with the induced meromorphic connection. As remarked in [36], we have a natural isomorphism \(\tilde{D}_{X(\ast H)}\mathcal{M} \simeq \lambda^{dx}\mathcal{M}'\) in the derived category. Together with Proposition C.16, we obtain \(\tilde{D}_{X(\ast H)}\mathcal{M} \simeq \lambda^{dx}\mathcal{M}'\).

We also have the isomorphism \(\tilde{D}_{X(\ast H)}\mathcal{M} \simeq \lambda^{dx}\mathcal{M}'\) given in a more direct and standard way, by using the Spencer resolution \(\tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^\ast\) of \(\mathcal{O}_X(\ast H)\):

\[
\text{RHom}_{\tilde{\mathcal{R}}_X(\ast H)}(\mathcal{M}, \lambda^{dx}\tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^{-1})[d_X + 1] \simeq \\
\text{Hom}_{\tilde{\mathcal{R}}_X(\ast H)}(\tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^0 \otimes \mathcal{M}, \lambda^{dx}\tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^{-1})[d_X + 1] \simeq \\
\tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^0 \otimes \lambda^{dx}\mathcal{M}' \simeq \lambda^{dx}\mathcal{M}'
\]

Lemma C.19 The isomorphisms \(\tilde{D}_{X(\ast H)}\mathcal{M} \simeq \lambda^{dx}\mathcal{M}'\) are equal, up to signatures.

Proof We have the natural quasi-isomorphism \(S_X^0((\mathcal{R}_X(\ast H) \otimes \mathcal{O}_X(\ast H)) \otimes \mathcal{O}_X(\mathcal{M}) \rightarrow S_X^0(\mathcal{M})\). We have the following natural isomorphisms:

\[
S_X^0((\mathcal{R}_X(\ast H) \otimes \mathcal{O}_X(\ast H)) \otimes \mathcal{O}_X(\mathcal{M}) \simeq S_X^0((\mathcal{R}_X(\ast H) \otimes \mathcal{O}_X(\ast H)) \otimes \mathcal{O}_X(\mathcal{M}) \simeq (\mathcal{R}_X(\ast H) \otimes \mathcal{O}_X(\ast H)) \otimes \mathcal{O}_X(\mathcal{M})
\]

Then, the isomorphism in Proposition C.16 is expressed as follows:

\[
\text{Hom}_{\tilde{\mathcal{R}}_X(\ast H)}(\tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^0 \otimes \mathcal{M}, \tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^{-1})[d_X + 1] \simeq \\
\text{hom}(\tilde{\mathcal{R}}_X(\ast H)(S_X^0((\mathcal{R}_X(\ast H) \otimes \mathcal{O}_X(\ast H)) \otimes \mathcal{O}_X(\mathcal{M}), \tilde{\mathcal{R}}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^{-1})[d_X + 1] \simeq \\
S_X^0(\text{Hom}_{\tilde{\mathcal{R}}_X(\ast H)}(\mathcal{R}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^0 \otimes \mathcal{M}, \mathcal{R}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^{-1})[d_X] \simeq \\
\text{Hom}_{\tilde{\mathcal{R}}_X(\ast H)}(\mathcal{R}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^0 \otimes \mathcal{M}, \mathcal{R}_X(\ast H) \otimes \tilde{\mathcal{O}}_X^{-1})[d_X]
\]

Then, the claim is clear by the constructions of the isomorphisms \(\tilde{D}_{X(\ast H)}\mathcal{M} \simeq \lambda^{dx}\mathcal{M}'\).

C.4 Compatibility of push-forward and duality
C.4.1 Compatibility for \(\mathcal{R}\)-modules

Let us recall the compatibility of the push-forward and the duality for \(\mathcal{R}\)-modules studied in [36]. Let \(Y\) be a complex manifold.

We set \(\mathcal{M}_{Y0} := (\mathcal{R}_Y \otimes \tilde{\mathcal{P}}_Y^{-1}) \circ \mathcal{O}_Y(\mathcal{R}_Y \otimes \tilde{\mathcal{P}}_Y^{-1})\), where \(\circ \mathcal{O}_Y\) means that we consider the actions \(\ell \circ \mathcal{O}_Y\) for the tensor product over \(\mathcal{O}_Y\). It is equipped with the \(\mathcal{R}_Y\)-actions \(r_i (i = 1, 2)\) induced by the action \(r\) on the \(i\)-th factor. We also have the \(\mathcal{R}_Y\)-action \(\ell \otimes \mathcal{O}_Y\) induced by the actions \(\ell\). We have the action of \(\lambda^2\partial_\lambda\) on \(\mathcal{M}_{Y0}\) induced by the actions on \(\mathcal{R}_Y \otimes \tilde{\mathcal{P}}_Y^{-1}\) with the Leibniz rule. Each \(\mathcal{R}_Y\)-action with the action of \(\lambda^2\partial_\lambda\) gives an \(\mathcal{R}_Y\)-action.

We set \(\mathcal{M}_{Y1} := (\mathcal{R}_Y \otimes \tilde{\mathcal{P}}_Y^{-1}) \circ (\mathcal{R}_Y \otimes \tilde{\mathcal{P}}_Y^{-1})\), where \(\circ \mathcal{O}_Y\) means that we consider the action \(r\) on the first factor and the action \(\ell\) on the second factor for the tensor product over \(\mathcal{O}_Y\). It is equipped with the \(\mathcal{R}_Y\)-action \(\ell_1\) induced by the action \(\ell\) on the first factor, and the \(\mathcal{R}_Y\)-action \(\ell_2\) induced by the action \(r\) on the second factor. We also have the \(\mathcal{R}_Y\)-action \(r \otimes \mathcal{O}_Y\) induced by the action \(r\) on the first factor and the action \(\ell\) on the second factor. As in the case of \(\mathcal{M}_{Y0}\), we have the action of \(\lambda^2\partial_\lambda\) on \(\mathcal{M}_{Y1}\) with which each \(\mathcal{R}_Y\)-action is extended to an \(\mathcal{R}_Y\)-action.

Recall that we have a unique isomorphism of sheaves \(\Psi_1 : \mathcal{M}_{Y1} \rightarrow \mathcal{M}_{Y0}\) such that (i) \(\Psi_1 \circ r_2 = r_2 \circ \Psi_1, \Psi_1 \circ \ell_1 = (\ell \otimes \ell) \circ \Psi_1\) and \(\Psi_1 \circ (r \otimes \ell) = r_1 \circ \Psi_1\), (ii) \(\Psi_1\) induces the identity on \(\tilde{\mathcal{P}}_Y^{-1} \otimes \tilde{\mathcal{P}}_Y^{-1}\). Here, \(\Psi_1 \circ r_2 = r_2 \circ \Psi_1\) means \(\Psi_1(r_2(P)m) = r_2(P)\Psi_1(m)\) for any local sections \(P \in \mathcal{R}_X\) and \(m \in \mathcal{M}_{Y1}\), and the meaning of \(\Psi_1 \circ \ell_1 = (\ell \otimes \ell) \circ \Psi_1\) and \(\Psi_1 \circ (r \otimes \ell) = r_1 \circ \Psi_1\) are similar. The uniqueness is clear by the conditions (i) and (ii). Locally, \(\Psi_1\) is given as follows. For any local frame \(\tau\) of \(\tilde{\mathcal{P}}_Y^{-1}\), we have \(\Psi_1((Q \otimes \tau) \circ m) = (\ell \otimes \ell)(Q)((1 \otimes \tau) \circ m)\) for \(Q \in \mathcal{R}_X\) and \(m \in \mathcal{R}_X \otimes \tilde{\mathcal{P}}_Y^{-1}\). It is compatible with the actions of \(\lambda^2\partial_\lambda\).
Let $f : X \rightarrow Y$ be a proper morphism. We obtain the morphism

$$\lambda^d f_!(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}))|d_X| \rightarrow \lambda^d f_! \mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}|d_Y| \tag{189}$$

in the derived category of $\mathcal{A}_Y$-modules as the composite of the following morphisms (see [C.1.4] for $\mathcal{A}_Y$):

$$\lambda^d f_!(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}))|d_X| \zeta \lambda^d f_! \left(\left(\left(\mathcal{O}_{\mathcal{Y}}[2d_x] \otimes_{\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})\right) \otimes_{\mathcal{O}_Y} \left(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})\right)\right) \right) \otimes_{\mathcal{O}_Y} \left(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}\right) \zeta \lambda^d f_!(\mathcal{O}_X|d_X|) \otimes_{\mathcal{O}_Y} \left(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}\right) \rightarrow \lambda^d f_!(\mathcal{O}_X|d_X|) \otimes_{\mathcal{O}_Y} \left(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}\right)|d_Y| \tag{190}$$

Here, $\alpha_1$ is induced by $\Psi_1^{-1}$, and $\alpha_2$ is induced by the trace morphism (see [C.2.3]).

Let $\mathcal{M}$ be a coherent $\mathcal{R}_X$-module which is good relative to $f$. Then, we have the natural morphism

$$f_! D_X \mathcal{M} \rightarrow D_Y f_! \mathcal{M} \tag{191}$$

obtained as the composite of the following morphisms:

$$f_! D_X \mathcal{M} \simeq f_! Rf_! \mathcal{R}_{\mathcal{O}_X} \left( \mathcal{M}, \lambda^d \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})|d_X| \right) \rightarrow f_! Rf_! Rf_{-1} \mathcal{R}_Y \left( \mathcal{R}_{\mathcal{O}_X} \otimes_{\mathcal{O}_Y} \mathcal{M}, \mathcal{R}_{\mathcal{O}_X} \otimes_{\mathcal{O}_Y} \lambda^d f_! \left( \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})\right)\right)|d_X| \rightarrow Rf_! Rf_{-1} \mathcal{R}_Y \left( f_! \mathcal{M}, \lambda^d f_! \left( \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})\right)\right)|d_X| \rightarrow Rf_! Rf_{-1} \mathcal{R}_Y \left( f_! \mathcal{M}, \lambda^d \left( \mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1} \right)|d_Y| \right) \simeq D_Y f_! \mathcal{M} \tag{192}$$

If $\mathcal{M}$ is also an $\widetilde{\mathcal{R}}_X$-module, then [191] is a morphism in the derived category of $\widetilde{\mathcal{R}}_Y$-modules.

### C.4.2 Compatibility for $\widetilde{\mathcal{R}}$-modules

We set $\widetilde{\mathcal{M}}_{Y^0} : = \left( \widetilde{\mathcal{R}}_Y \otimes \overline{\mathcal{O}}_Y^{-1}\right)^{\otimes_{\mathcal{O}_Y} (\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})}$. The tensor product is taken as in the case of $\mathcal{M}_{Y^0}$. We have the $\widetilde{\mathcal{R}}_X$-actions $r_i$ ($i = 1, 2$) and $\ell \otimes \ell$ on $\widetilde{\mathcal{M}}_{Y^0}$ as in the case of $\mathcal{M}_{Y^0}$. We set $\widetilde{\mathcal{M}}_{Y^1} : = \left( \widetilde{\mathcal{R}}_Y \otimes \overline{\mathcal{O}}_Y^{-1}\right)^{\otimes_{\mathcal{O}_Y} (\mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1})}$, where the tensor product is taken as in the case of $\mathcal{M}_{Y^1}$. We have the $\widetilde{\mathcal{R}}_X$-actions $r_2, \ell_1$ and $\ell \otimes \ell$ on $\widetilde{\mathcal{M}}_{Y^1}$ as in the case of $\mathcal{M}_{Y^1}$.

We have a unique isomorphism of sheaves $\widetilde{\Psi}_1 : \widetilde{\mathcal{M}}_{Y^1} \rightarrow \widetilde{\mathcal{M}}_{Y^0}$ such that (i) $\widetilde{\Psi}_1 \circ r_2 = r_2 \circ \widetilde{\Psi}_1$, $\widetilde{\Psi}_1 \circ \ell_1 = (\ell \otimes \ell) \circ \widetilde{\Psi}_1$, (ii) $\widetilde{\Psi}_1$ induces the identity on $\overline{\mathcal{O}}_X^{-1} \otimes \overline{\mathcal{O}}_X^{-1}$. For any local frame $\tau$ of $\overline{\mathcal{O}}_Y^{-1}$, we have $\widetilde{\Psi}_1 ((Q \otimes \tau) \otimes m) = (\ell \otimes \ell)(Q)((1 \otimes \tau) \otimes m)$.

**Lemma C.20** We have $\widetilde{\Psi}_1 \circ (r \otimes \ell) = r_1 \circ \widetilde{\Psi}_1$.

**Proof** It is enough to prove the equality locally around any point of $\mathcal{X} \setminus \mathcal{X}^0$. We may assume to have a frame $\tau$ of $\overline{\mathcal{O}}_Y^{-1}$ such that $r(v)(\tau^{-1}) = 0$ for any $v \in \Theta_X$. Let $Q \in \mathcal{R}_Y$ and $m \in \mathcal{R}_Y \otimes \overline{\mathcal{O}}_Y^{-1}$. For any $g \in \mathcal{O}_Y$, we have

$$r_1(g)\widetilde{\Psi}_1 ((Q \otimes \tau) \otimes m) = r_1(g)(\ell \otimes \ell)(Q)((1 \otimes \tau) \otimes m) = (\ell \otimes \ell)(Q)((g \otimes \tau) \otimes m) = (\ell \otimes \ell)(Q)((1 \otimes \tau) \otimes gm) = \widetilde{\Psi}_1 ((Q \otimes \tau) \otimes gm) = \widetilde{\Psi}_1 ((r \otimes \ell)(g)((Q \otimes \tau) \otimes m)). \tag{193}$$
For any $v \in \tilde{\Theta}_X$, we have
\[
\Psi \left( (r \otimes \ell)(v)((Q \otimes \tau) \otimes m) \right) = \Psi \left( (-Qv \otimes \tau) \otimes m \right) + \Psi(Q \otimes vm) = -(\ell \otimes \ell)(Qv)((1 \otimes \tau) \otimes m) + (\ell \otimes \ell)(Q)((1 \otimes \tau) \otimes vm) = (\ell \otimes \ell)(Q)r_1(v)((1 \otimes \tau) \otimes m) = r_1(v)\Psi \left( (Q \otimes \tau) \otimes m \right) \tag{194}
\]
Thus, we are done.

Let $f : X \rightarrow Y$ be a proper morphism as in \[\text{C.4.1}\]. We obtain the morphism
\[
\lambda^d \tilde{f}_1 \left( O_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} ) \right)[d_X + 1] \rightarrow \lambda^d \tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1}[d_Y + 1] \tag{195}
\]
in the derived category of $\tilde{R}_Y \otimes \tilde{\Omega}_Y$-modules as the composite of the following morphisms:
\[
\lambda^d \tilde{f}_1 \left( O_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} ) \right)[d_X + 1] \\
\simeq \lambda^d f_1 \left( (\tilde{\Omega}_X^\bullet \otimes_{\tilde{R}_X^\bullet}[2d_X + 2] \otimes \tilde{\Omega}_X) \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} ) \right) \otimes \tilde{\Omega}_X \left( O_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} ) \right) \\
\simeq \lambda^d f_1 \left( \tilde{\Omega}_X^\bullet \otimes_{\tilde{R}_X^\bullet}[2d_X + 2] \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} ) \right) \otimes \tilde{\Omega}_Y \left( \tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} \right) \\
\simeq \lambda^d f_1 \left( (O_X[d_X + 1]) \otimes \tilde{\Omega}_Y \left( \tilde{R}_X \otimes \tilde{\Omega}_X^{-1} \right) \right) \rightarrow \lambda^d \tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1}[d_Y + 1] \tag{196}
\]
Here, $\tilde{\alpha}_1$ is induced by $\Psi^{-1}$, and $\tilde{\alpha}_2$ is induced by the trace morphism (see \[\text{C.2.3}\]).

Let $\mathcal{M}$ be a coherent $\tilde{R}_X$-module which is good relative to $f$. Then, we have the natural isomorphism
\[
\tilde{f}_1 \tilde{D}_X \mathcal{M} \rightarrow \tilde{D}_Y \tilde{f}_1 \mathcal{M} \tag{197}
\]
in the derived category of $\tilde{R}_Y$-modules, obtained as the composite of the following morphisms:
\[
\tilde{f}_1 \tilde{D}_X \mathcal{M} \simeq Rf_1 R\text{Hom}_{\tilde{R}_X} \left( \mathcal{M}, \lambda^d \tilde{R}_X \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} )[d_X + 1] \right) \\
\rightarrow Rf_1 R\text{Hom}_{f^{-1}\tilde{R}_Y} \left( \tilde{R}_Y \otimes_{\tilde{\Omega}_Y^{-1}} \mathcal{M}, \tilde{R}_Y \otimes_{\tilde{\Omega}_Y^{-1}} \tilde{R}_Y \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} )[d_X + 1] \right) \\
\rightarrow R\text{Hom}_{\tilde{R}_Y} \left( \tilde{f}_1 \mathcal{M}, \lambda^d \tilde{R}_Y \otimes_{\tilde{\Omega}_Y^{-1}} \tilde{R}_Y \otimes_{f^{-1}O_Y} f^{-1}(\tilde{R}_Y \otimes \tilde{\Omega}_Y^{-1} )[d_X + 1] \right) \\
\rightarrow R\text{Hom}_{\tilde{R}_Y} \left( \tilde{f}_1 \mathcal{M}, \lambda^d \tilde{R}_Y \otimes_{\tilde{\Omega}_Y^{-1}} [d_Y + 1] \right) \simeq \tilde{D}_Y \tilde{f}_1 \mathcal{M}. \tag{198}
\]

### C.4.3 Comparison

Let $f : X \rightarrow Y$ be a proper morphism of complex manifolds as above. We shall prove the following proposition in \[\text{C.4.4} \text{C.4.5}\].

**Proposition C.21** Let $\mathcal{M}$ be a coherent $\tilde{R}_X$-module which is good relative to $f$. The following diagram is commutative:
\[
\begin{array}{ccc}
\tilde{f}_1 \tilde{D}_X \mathcal{M} & \simeq & \tilde{D}_Y \tilde{f}_1 \mathcal{M} \\
\simeq & & \simeq \\
\tilde{f}_1 D_X \mathcal{M} & \simeq & \tilde{D}_Y f_1 \mathcal{M}
\end{array} \tag{199}
\]
Here, the horizontal arrows are given in \[\text{C.4.1} \text{and C.4.2}\] and the vertical arrows are given by the isomorphisms in Corollary \[\text{C.12} \text{and Proposition C.16}\].

117
**C.4.4 Step 1**

We construct a natural transform \( \bar{f}_! \bar{D}_X \rightarrow \bar{D}_Y f_! \) modifying the constructions in [**C.4.1**] and [**C.4.2**].

We set \( \mathcal{M}_{Y_0} := (\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1})^\ell \otimes_{\mathcal{O}_Y} (\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1}) \). The tensor product is taken as in the case of \( \mathcal{M}_{Y_0} \). It is equipped with the \( \mathcal{R}_X \)-action \( r_2 \) induced by the \( \mathcal{R}_X \)-action \( r \) on the second factor. It is equipped with the \( \mathcal{R}_X \)-action \( \ell \otimes \ell \) induced by the \( \mathcal{R}_X \)-actions \( \ell \) on the first and second factors. It is equipped with the \( \mathcal{R}_X \)-action \( r_1 \) induced by the \( \mathcal{R}_X \)-action \( r \) on the first factor. Together with the action of \( \lambda^2 \partial_\lambda \) by \( \ell \otimes \ell \), it is extended to an \( \mathcal{R}_X \)-action which is also denoted by \( r_1 \).

We set \( \mathcal{M}_{Y_1} := (\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1})^\ell \otimes_{\mathcal{O}_Y} (\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1}) \). The tensor product is taken as in the case of \( \mathcal{M}_{Y_1} \). It is equipped with the \( \mathcal{R}_X \)-action \( r_2 \) induced by the \( \mathcal{R}_X \)-action \( r \) on the second factor. It is equipped with the \( \mathcal{R}_X \)-action \( r \otimes \ell \) induced by the \( \mathcal{R}_X \)-actions \( r \) and \( \ell \) on the first and second factors, respectively. It is equipped with the \( \mathcal{R}_X \)-action \( \ell_1 \) induced by the \( \mathcal{R}_X \)-action \( r \) on the first factor. Together with the action of \( \lambda^2 \partial_\lambda \) by \( r \otimes \ell \), it is extended to an \( \mathcal{R}_X \)-action which is also denoted by \( \ell_1 \).

We have the isomorphism \( \Psi'_1 : \mathcal{M}_{Y_1} \simeq \mathcal{M}'_{Y_0} \) determined by the conditions (i) \( \Psi'_1 \circ \ell_1 = (\ell \otimes \ell) \circ \Psi_1 \) and \( \Psi'_1 \circ r_2 = r_2 \circ \Psi_1 \), (ii) \( \Psi'_1 \) induces the identity on \( \mathcal{O}_Y \otimes \mathcal{O}_Y^{-1} \). It also satisfies the condition \( \Psi'_1 \circ (r \otimes \ell) = r_1 \circ \Psi'_1 \), which can be checked by the argument in Lemma [**C.20**].

Then, we obtain the morphism

\[
\lambda^d \bar{X} f_! \left( \mathcal{O}_X \otimes f^{-1} \mathcal{O}_y, f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1}) \right)[d_X + 1] \rightarrow \lambda^d \bar{Y} \mathcal{O}_y \otimes \mathcal{O}_Y^{-1}[d_Y + 1]
\]

in the derived category of \( \mathcal{R}_Y \otimes \mathcal{R}_Y \)-complexes, obtained as the composite of the following:

\[
\lambda^d \bar{X} f_! \left( \mathcal{O}_X \otimes f^{-1} \mathcal{O}_y, f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1}) \right)[d_X + 1] \\
\simeq \lambda^d \bar{X} f_! \left( \left( \mathcal{R}_X \otimes \mathcal{O}_Y^{-1} \right)[2d_X + 1] \otimes \mathcal{R}_X \right) \otimes \mathcal{R}_X \mathcal{O}_X \otimes f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1})[d_X + 1]
\]

\[
\simeq \lambda^d \bar{X} f_! \left( \left( \mathcal{R}_X \otimes \mathcal{O}_Y^{-1} \right)[2d_X + 1] \otimes f^{-1} \mathcal{M}_{Y_0} \right) \otimes \mathcal{R}_X \mathcal{O}_X \otimes f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1})[d_X + 1]
\]

\[
\simeq \lambda^d \bar{X} f_! \mathcal{O}_X[d_X + 1] \mathcal{O}_Y \otimes \mathcal{O}_Y^{-1}[d_Y + 1] \rightarrow \lambda^d \bar{Y} \mathcal{O}_y \otimes \mathcal{O}_Y^{-1}[d_Y + 1]
\]

Here \( a'_2 \) is induced by \( \Psi'_1 \), and \( a'_2 \) is induced by the trace morphism in [**C.2.3**]. Then, for any \( \mathcal{R}_X \)-module \( \mathcal{M} \) which is good relative to \( f \) and coherent over \( \mathcal{R}_X \), we obtain the morphism

\[
\bar{f}_! \bar{D}_X \mathcal{M} \rightarrow \bar{D}_Y f_! \mathcal{M}
\]

as the composite of the following morphisms:

\[
\bar{f}_! \bar{D}_X \mathcal{M} \rightarrow Rf_! R\mathcal{H}om_{\mathcal{R}_X} \left( \mathcal{M}, \lambda^d \mathcal{O}_X \otimes f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1})[d_X + 1] \right)
\]

\[
Rf_! R\mathcal{H}om_{\mathcal{R}_X} \left( \mathcal{M}, \lambda^d \mathcal{O}_X \otimes f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1})[d_X + 1] \right) \rightarrow R\mathcal{H}om_{\mathcal{R}_Y} \left( f_! \mathcal{M}, \lambda^d \mathcal{O}_Y \otimes \mathcal{O}_Y^{-1}[d_Y + 1] \right)
\]

\[
R\mathcal{H}om_{\mathcal{R}_Y} \left( f_! \mathcal{M}, \lambda^d \mathcal{O}_Y \otimes \mathcal{O}_Y^{-1}[d_Y + 1] \right) \simeq \bar{D}_Y f_! \mathcal{M}
\]

**Lemma C.22** The following diagram is commutative:

\[
\begin{array}{ccc}
\lambda^d \bar{X} f_! \left( \mathcal{O}_X \otimes f^{-1} \mathcal{O}_y, f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1}) \right)[d_X + 1] & \rightarrow & \lambda^d \bar{Y} \mathcal{O}_y \otimes \mathcal{O}_Y^{-1}[d_Y + 1] \\
\downarrow & & \downarrow c_2 \\
\lambda^d \bar{X} f_! \left( \mathcal{O}_X \otimes f^{-1} \mathcal{O}_y, f^{-1}(\mathcal{R}_Y \otimes \mathcal{O}_Y^{-1}) \right)[d_X + 1] & \rightarrow & \lambda^d \bar{Y} \mathcal{O}_y \otimes \mathcal{O}_Y^{-1}[d_Y + 1]
\end{array}
\]

Here, \( c_1 \) and \( c_2 \) are given by [**C.19**] and [**2.10**], and the left vertical arrow is given in Proposition [**C.16**].
**Proof** We can regard $\mathcal{M}_Y^r$ as an $A_Y$-module by $\ell \otimes \ell$ and $r_1$ with the action of $\lambda^2 \partial_\lambda$. We can also regard $\mathcal{M}_Y^0$ as an $A_Y$-module by $r \otimes \ell$ and $\ell_1$ with the action of $\lambda^2 \partial_\lambda$. As mentioned in Example C.7 we have the following isomorphisms:

$$(\mathcal{R}_Y \otimes \Omega_Y^{-1}(\lambda^2 \partial_\lambda), \ell \nabla, r \nabla) \simeq (\overline{\mathcal{R}}_Y \otimes \lambda^{-2} \Omega_Y^{-1}, \ell, r)$$

It induces the following isomorphisms:

$$\left(\mathcal{M}_Y^r(\lambda^2 \partial_\lambda), (\ell \otimes \ell) \nabla, r_1 \nabla\right) \simeq (\lambda^{-2} \overline{\mathcal{M}}_Y^r, \ell \otimes \ell, r_1) \quad (205)$$

$$\left(\mathcal{M}_Y^0(\lambda^2 \partial_\lambda), (r \otimes \ell) \nabla, \ell_1 \nabla\right) \simeq (\lambda^{-2} \overline{\mathcal{M}}_Y^0, r \otimes \ell, \ell_1) \quad (206)$$

The isomorphisms are compatible with the $\overline{\mathcal{R}}_X$-actions $r_2$. We have the following commutative diagram:

$$\xymatrix{ \mathcal{M}_Y^1(\lambda^2 \partial_\lambda) \ar[r] \ar[d] & \lambda^{-2} \overline{\mathcal{M}}_Y^1 \ar[d] \\
\mathcal{M}_Y^0(\lambda^2 \partial_\lambda) \ar[r] & \lambda^{-2} \overline{\mathcal{M}}_Y^0}
$$

Here, the vertical arrows are induced by $\Psi'_1$ and $\overline{\Psi}$, and the horizontal arrows are as in $\text{(205)}$ and $\text{(206)}$. Hence, the following diagram is commutative:

$$\lambda^d x f_1 \left(\mathfrak{Ob}_X/\mathfrak{C}_X[2dX + 2] \otimes f^{-1} \mathcal{M}_Y^0\right) \xrightarrow{\tilde{\alpha}} \lambda^d x f_1 \left(\mathfrak{Ob}_X/\mathfrak{C}_X[2dX + 2] \otimes f^{-1} \overline{\mathcal{M}}_Y^0\right) \quad (\text{205})$$

$$\lambda^d x \left(\mathfrak{Ob}_X[2dX + 1] \otimes f^{-1} \mathcal{M}_Y^0\right) \xrightarrow{\tilde{\alpha}'} \lambda^d x \left(\mathfrak{Ob}_X[2dX + 1] \otimes f^{-1} \overline{\mathcal{M}}_Y^0\right) \quad (\text{206})$$

We also have the compatibility of the trace morphism in Lemma C.13. Then, we obtain the claim of Lemma C.22 by construction of the morphisms.

The natural transform $\tilde{f}_1 \rightarrow f_1$ in Corollary C.12 induces $\tilde{D}_Y f_1 \rightarrow \tilde{D}_Y f_1$.

**Lemma C.23** The composite of $\tilde{f}_1 \rightarrow f_1$ and $\tilde{D}_Y f_1 \rightarrow \tilde{D}_Y f_1$ is equal to $\tilde{D}_Y f_1$.

**Proof** We obtain the claim by comparing the constructions (198) and (203) and by using Lemma C.22.

**C.4.5 Step 2**

We consider the following diagram:

$$\xymatrix{ \tilde{f}_1 \tilde{D}_X \ar[r]^{\alpha_1} \ar[d]_{\beta_1} & \tilde{f}_1 \tilde{D}_X S^X_\bullet \ar[r]^{\alpha_2} \ar[d]_{\beta_2} & f_1 D_X \ar[d]_{\beta_3} \\
\tilde{D}_Y f_1 \ar[r]^{\gamma_1} & \tilde{D}_Y f_1 S^X_\bullet \ar[r]^{\gamma_2} & D_Y f_1}
$$

(207)

The morphisms $\alpha_1$ and $\gamma_1$ are the natural ones. The morphisms $\beta_i$ ($i = 1, 2$) are given in C.4.4. The morphism $\beta_3$ is given in C.2.4. The morphism $\alpha_2$ is induced by $\tilde{f}_1 \rightarrow f_1$ in Corollary C.12 and $\tilde{D}_X S^X_\bullet \simeq S^X_\bullet D_X \rightarrow D_X$ in C.3.2. The morphism $\gamma_2$ is induced by $f_1 S^X_\bullet \simeq S^X_\bullet f_1$ in C.2.4 and $\tilde{D}_Y S^X_\bullet \simeq S^X_\bullet D_Y \rightarrow D_Y$ in C.3.2. The morphism $\beta_3$ is given in C.4.1. Clearly, the left square is commutative. For the proof of Proposition C.21 it is enough to prove the commutativity of the right square.

We have the following expression:

$$\tilde{f}_1 \tilde{D} S^X_\bullet M \simeq \lambda^d x R f_1 \left(\left(\mathfrak{Ob}_X[dX + 1] \otimes f^{-1}(\overline{\mathcal{R}}_Y \otimes \Omega_Y^{-1})\right) \otimes_{\mathcal{O}_X} R \mathcal{H}om_{\mathcal{R}_X} (S^X_\bullet M, \overline{\mathcal{R}}_X \otimes \Omega_X^{-1})\right)[dX + 1] \quad (208)$$

119
It is rewritten as follows:
\[
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X + 1] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} S^\vee_X \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \tag{209}
\]
We have the natural morphism from (209) to the following:
\[
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X + 1] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \cong \\
\lambda^{dx} Rf_! R\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{O}_X \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right)[d_X] \tag{210}
\]
Indeed, we have
\[
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X + 1] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} S^\vee_X \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \rightarrow \\
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X + 1] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \\
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \tag{211}
\]
We also have
\[
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X + 1] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} S^\vee_X \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \rightarrow \\
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} S^\vee_X \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \\
\lambda^{dx} Rf_! \left[ \left( \Omega_X^*[d_X] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \otimes_{\mathcal{O}_X} \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \mathcal{R}_X \otimes \Omega_X^{-1} \right)[d_X] \right] \tag{212}
\]
The composite of the morphisms (211) and (212) are equal. We have the following morphisms:
\[
Rf_! \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( \mathcal{M}, \lambda^{dx} \mathcal{O}_X[d_X] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \rightarrow \\
\mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( f_1 \mathcal{M}, f_1 \left( \lambda^{dx} \mathcal{O}_X[d_X] \otimes_{f^{-1}O_{Y'}} f^{-1}(\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \right) \\
\cong \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( f_1 \mathcal{M}, \lambda^{dx} f_1(\mathcal{O}_X)[d_X] \otimes_{\mathcal{O}_Y} (\widetilde{R}Y \otimes \widetilde{\Omega}^{-1}_{Y'}) \right) \\
\rightarrow \mathcal{R}\text{Hom}_{\mathcal{R}_X} \left( f_1 \mathcal{M}, \lambda^{dx} \mathcal{R}_Y \otimes \Omega_Y^{-1}[d_Y] \right) \rightarrow \mathcal{D} f_1 \mathcal{M} \tag{213}
\]
As the composite of the above morphisms, we obtain a morphism \( G : f_1 \bar{D}S^\vee_X \mathcal{M} \rightarrow \mathcal{D} f_1 \mathcal{M} \). By construction, we can check that both \( \beta_3 \circ \alpha_2 \) and \( \gamma_2 \circ \beta_2 \) are equal to \( G \). Indeed, \( \beta_3 \circ \alpha_2 \) is induced by (211), and \( \gamma_2 \circ \beta_2 \) is induced by (212). Thus, we obtain the commutativity of the right square of (207), and the proof of Proposition C.21 is finished.

D Comparison of some \( \mathcal{R} \)-modules

D.1 Preliminary

D.1.1 A construction

Let \( W_\tau = \mathbb{C}[t] \langle \partial_t \rangle \) be the Weyl algebra in the variable \( t \). We set \( W_\text{loc} := W_\tau \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{-1}] \). Let \( X \) be any smooth algebraic variety. Let \( p : \mathbb{C} \times X \rightarrow X \) be the projection. In the algebraic setting, we naturally identify \( p_*D_{\mathbb{C},X} \) with \( D_X \otimes W_\tau \). We naturally identify algebraic \( D_{\mathbb{C},X} \)-modules and algebraic \( D_X \otimes W_\tau \)-modules. We also naturally identify algebraic \( D_{\mathbb{C},X} \)-modules and algebraic \( D_X \otimes W_\text{loc} \)-modules.

We regard \( \mathbb{C}_r \) as the dual space of \( \mathbb{C}_t \) by the pairing \( (\tau, t) \mapsto \tau t \). Let \( M \) be an algebraic \( D_{\mathbb{C},X} \)-module. We have its partial Fourier-Laplace transform \( \text{FL}_X(M) \) on \( \mathbb{C}_r \times X \). As a transform from \( D_X \otimes W_\tau \)-modules to
$\mathcal{D}_X \otimes W_\tau$-modules, it is described as the quotient of $\partial_t - \tau : M[\tau] \to M[\tau]$. Here, the action of $\mathcal{D}_X \otimes W_\tau$ on $M[\tau]$ is given as follows. We have $v(m\tau^j) = (vm)\tau^j$ for a vector field $v$ on $X$, and $\partial_t(m\tau^j) = jm\tau^{j-1} - tm\tau^j$.

By taking the restriction, we have the localized partial Fourier transform $\text{FL}^\text{loc}_X(M)$ on $\mathbb{C}_*^\times X$. We set $\lambda := \tau^{-1}$. We may naturally regard $\text{FL}^\text{loc}_X(M)$ as a $\mathcal{D}_X \otimes W_\lambda$-module. Let us explicit it more explicitly. We have the action of $\mathcal{D}_{\mathbb{C}_*^\times X}$ on $M[\lambda, \lambda^{-1}]$ given as follows. We have $v(m\lambda^j) = (vm)\lambda^j$ for a vector field $v$ on $X$, and $\partial_\lambda(m\lambda^j) = tm\lambda^{j-2} + jm\lambda^{-1}$. Then, the $\mathcal{D}_{\mathbb{C}_*^\times X}$-module $\text{FL}^\text{loc}_X(M)$ is described as the quotient of

$$M[\lambda, \lambda^{-1}] \xrightarrow{\partial_\lambda - \lambda^{-1}} M[\lambda, \lambda^{-1}].$$

The natural inclusion $M \to M[\lambda, \lambda^{-1}]$ induces a $\mathcal{D}_X$-homomorphism $\text{loc} : M \to \text{FL}^\text{loc}_X(M)$.

Let $(M, F)$ be an algebraic filtered $\mathcal{D}_{\mathbb{C}_*^\times X}$-module, i.e., $M$ is an algebraic $\mathcal{D}_X$-module, and $F$ is a good filtration of $M$. Following [44], we set

$$G_0 \text{FL}^\text{loc}_X(M, F) := \sum_{j \in \mathbb{Z}} \lambda^j \text{loc}(F_j M) \subset \text{FL}^\text{loc}_X(M)$$

on $X$. It is naturally an algebraic $\mathcal{R}_X$-module. It is also equipped with the action of $\lambda^j \partial_\lambda$. Thus, we obtain an algebraic $\mathcal{R}_X$-module. Because $\partial F_j(M) \subset F_{j+1}(M)$, we have $\text{loc}(F_j M) \subset \lambda \text{loc}(F_{j+1} M)$, and hence $G_0 \text{FL}^\text{loc}_X(M, F) = \sum_{j \geq N} \lambda^j \text{loc}(F_j M)$ for any $N$.

We have the Rees module $R(M, F) = \sum F_j M\lambda^j$. We have the $\mathcal{R}_X$-homomorphism $R(M, F) \to \text{FL}^\text{loc}_X(M, F)$, and the image is $G_0 \text{FL}^\text{loc}_X(M, F)$.

The quotient of the following is denoted by $G_0' \text{FL}^\text{loc}_X(M, F)$:

$$\lambda R(M, F) \xrightarrow{\partial_\lambda - \lambda^{-1}} R(M, F)$$

It is naturally an algebraic $\mathcal{R}_X$-module.

**Lemma D.1** The image of the induced morphism $G_0' \text{FL}^\text{loc}_X(M, F) \to \text{FL}^\text{loc}_X(M)$ is $G_0 \text{FL}^\text{loc}_X(M, F)$. If the multiplication of $\lambda$ on $G_0' \text{FL}^\text{loc}_X(M, F)$ is injective, the natural morphism $G_0' \text{FL}^\text{loc}_X(M, F) \to G_0 \text{FL}^\text{loc}_X(M, F)$ is an isomorphism of $\mathcal{R}_X$-modules.

**Proof** The first claim is clear by the construction. The morphism $G_0' \text{FL}^\text{loc}_X(M, F) \to G_0 \text{FL}^\text{loc}_X(M, F)$ is an isomorphism after the localization with respect to $\lambda$. Then, the second claim follows.

We may interpret the construction $G_0' \text{FL}^\text{loc}_X(M, F)$ in terms of $\mathcal{R}$-modules. Let $p$ denote the projection of $\mathbb{C}_* \times X$ onto $X$, as above. We have the algebraic $\mathcal{R}_{\mathbb{C}_* \times X}$-module $\mathcal{L}(-t) = \left(O_{\mathbb{C}_* \times \mathbb{C}_* \times X}, d-d(\lambda^{-1} t)\right)$. Then, by definition, $G_0' \text{FL}^\text{loc}_X(M, F)$ is isomorphic to

$$\lambda \cdot p_0^0 \left(R(M, F) \otimes \mathcal{L}(-t)\right) \simeq \lambda \cdot \mathbb{R}^1 p_* \left(R(M, F) \otimes \mathcal{L}(-t) \xrightarrow{b} R(M, F) \otimes \mathcal{L}(-t) \otimes \left(\lambda^{-1} \Omega^1_{X \times \mathbb{C}_*} \right)\right)$$

Here, $b$ is induced by the meromorphic flat connection of $R(M, F) \otimes \mathcal{L}(-t)$. (See [3.2.4] for example.)

**Proposition D.2** Suppose that $(M, F)$ is a filtered $\mathcal{D}_{\mathbb{C}_* \times X}$-module underlying an algebraic mixed Hodge module. Then, the natural morphism $G_0' \text{FL}^\text{loc}_X(M, F) \to G_0 \text{FL}^\text{loc}_X(M, F)$ is an isomorphism.

**Proof** By the assumption, we have a mixed Hodge module on $\mathbb{P}_1^1 \times X$ whose underlying filtered $\mathcal{D}$-module $(M', F')$ on $\mathbb{P}_1^1 \times X$ satisfies $(M', F')|_{C \times X} = (M, F)$. As explained in [36] §12.5], the analytification of $R(M', F')$ underlies an integrable mixed twistor $\mathcal{D}$-module $(T, W)$ on $\mathbb{P}_1^1 \times X$. Let $\mathcal{V}(-t) = (\mathcal{L}(-t), \mathcal{L}(-t), C(-t))$ denote the $\mathcal{R}_{\mathbb{P}_1^1 \times X}$-triple as in [3.2.1]. As explained in [36] §11.3.2], we have a mixed twistor $\mathcal{D}$-module $((T, W) \otimes \mathcal{V}(-t))[\ast(t)\infty]$ on $\mathbb{P}_1^1 \times X$.

**Lemma D.3** The underlying $\mathcal{R}_{\mathbb{P}_1^1 \times X}$-module of $((T, W) \otimes \mathcal{V}(-t))[\ast(t)\infty]$ is the analytification of $R(M, F) \otimes \mathcal{L}(-t)$.

121
Proof The claim is clear on $C_i \times X$. Let $Q$ be any point of $X$. It is enough to check the claim locally around $(\infty, Q)$. Let $M$ and $\tilde{M}$ be the $R_{P_1|X} \times X$-modules underlying $T$ and $((T, W) \otimes V(-t))[s(t)_\infty]$, respectively. We have $\tilde{M}(s(t)_\infty) = M \otimes L(-t)$.

Let $\kappa := t^{-1}$. We consider $V_{t^{-1}} \subset R_{C_i \times X}$ which is generated by $\lambda t \Theta_{C_i \times X}(\log \kappa)$ over $O_{C_i \times C_i \times X}$. We have the $V$-filtration $V(M)$ of $M$ along $\kappa$. Then, $V_{t^{-1}}(M)$ is $V_{t^{-1}}$-coherent for any $c \in R$. Let $q : C_i \times X \to X$ be the projection. Note that $V_{t^{-1}}(M)$ are $q^* R_X$-coherent because $M$ is regular. We take sections $s_1, \ldots, s_m$ of $V_{t^{-1}}(M)$ which generates $V_{t^{-1}}(M)$ over $q^* R_X$. Let $v$ be the generator of $L(-t)$ such that $\nabla_v = vd(-\lambda^{-1}t)$.

Let $M_1$ be the $R_{C_i \times X}$-submodule of $M \otimes L(-t)$ generated by $s_i \otimes v$ ($i = 1, \ldots, m$). Let us observe $M_1 = \tilde{M} \otimes L(-t)$. It is enough to prove that $\kappa^{-n}(V_{t^{-1}} \otimes v) \subset M_1$ for any $n$. We have $V_{t^{-1}} \otimes v \subset M_1$. Suppose $\kappa^{-n}(V_{t^{-1}} \otimes v) \subset M_1$. Let $f$ be any section of $V_{t^{-1}}$. We have

$$\kappa \partial_t(\kappa^{-n}f \otimes v) = -nf \otimes v + \kappa^{-n}(\kappa \partial_t f) \otimes v - \kappa^{-n-1}f \otimes v.$$

We have sections $h_j$ of $q^* R_X$ such that $\kappa^{-n}(\kappa \partial_t f) \otimes v = \sum \kappa^{-n}h_j s_j \otimes v$. Hence, we obtain $\kappa^{-n-1}f \otimes v \in M_1$.

Suppose $c < 0$. For a sufficiently large $N$, $t^N s_i \otimes v$ ($i = 1, \ldots, m$) are sections of $\tilde{M}$. We also have that $t^N s_i$ generates $V_{-N} M$. Hence, we obtain that $t^N s_i \otimes v$ ($i = 1, \ldots, m$) generates $M \otimes L(-t)$ over $R_X$. It implies that $\tilde{M} = M \otimes L(-t)$.

Hence, the analytification of $p^0_1(R(M, F) \otimes L(-t))$ underlies a mixed twistor $D$-module. It implies that the multiplication of $\lambda$ on $p^1_1(R(M, F) \otimes L(-t))$ is injective. Then, the claim follows from the Lemma [D.1.1]

D.1.2 Comparison

Let $Y$ be a smooth projective variety. Let $X$ be any smooth algebraic variety. Let $\pi_1 : Y \times C_i \times X \to C_i \times X$ and $\pi_2 : Y \times C_i \times X \to X$ be the projections.

Let $(M, F)$ be a filtered $D$-module on $Y \times C_i \times X$ which underlies a mixed Hodge module. We obtain filtered $D$-modules $\pi_{1*}(M, F)$ on $C_i \times X$, which underlie mixed Hodge modules. Applying the construction in [D.1.1] we obtain $R_X$-modules $G_0 \text{FL}_X^{\text{loc}} \pi_{1*}(M, F)$.

Let $R(M, F)$ be the Rees module of $(M, F)$. It is naturally an algebraic $\tilde{R}_{Y \times C_i \times X}$-module. We have the $R_{Y \times C_i \times X}$-module $L(-t)$. We obtain an algebraic $\tilde{R}_{Y \times C_i \times X}$-module $R(M, F) \otimes L(-t)$. It underlies a mixed twistor $D$-module (Lemma [D.3.1]). We obtain $R_X$-modules $\pi_{2*}^0(R(M, F) \otimes L(-t))$.

Proposition D.4 We have natural isomorphisms $\pi^j_{2*} \pi_{1*}^0(R(M, F) \otimes L(-t)) \simeq G_0 \text{FL}_X^{\text{loc}} \pi^j_{1*}(M, F)$.

Proof Note that we have $\pi_{1*}^0(R(M, F)) \simeq R(\pi_{1*}^0(M, F))$ by the theory of mixed Hodge modules. Let $p_X : C_i \times X \to X$ denote the projection. We have $p_X \circ \pi_1 = \pi_2$. Let $L(-t)$ denote the $D$-module associated to $-t$ on $\mathbb{P}^1_X \times X$ as in [2.1.1]. Let us recall the following lemma.

Lemma D.5 For regular holonomic $D$-modules $M$ on $C_i \times X$, we have $p^j_{X*} M \otimes L(-t) = 0$ for $j \neq 0$.

Proof It is enough to consider the case $j = -1$. Let $s$ be any section of $M$ such that $\partial_t s = s = 0$, and let us check that $s = 0$. We have the $V$-filtration of $V_s M$ along $u = t^{-1}$. Suppose $s \neq 0$ around $(\infty, P) \in \mathbb{P}^1 \times X$. Because $M$ is regular along $u = t^{-1}$, we have $a_0 := \min\{a | s \in V_a M\}$. Because $\partial_t s \in V_{a_0}$, we also have $s \in V_{a_0}$ which contradicts the choice of $a_0$. Hence, we have $s = 0$ around $(\infty, P)$. Let $M_1 \subset M$ be the $D$-submodule generated by $s$. We have $\text{Supp}(M_1) \cap \{\infty\} \times X = \emptyset$. Let $P \in X$. By shrinking $X$ appropriately, we take an algebraic function $f$ on $X$ such that $f(P) = 0$ and $df$ is nowhere vanishing. We have the $V$-filtrations $V_f J$ of $M_1$ and $M_1 \otimes L(-t)$ along $f$. It is easy to see that $\text{Gr}^V_j(M \otimes L(-t)) \simeq \text{Gr}^V_j(M) \otimes L(-t)$. If $s$ is non-zero around $(t, P)$, there exists $b$ such that $s$ gives a non-zero section $[s]$ of $\text{Gr}^V_j(M)$ satisfying $\partial_t [s] - [s] = 0$. Hence, we can reduce the claim to the case $\dim X = 0$, where it can be checked easily. Thus, we obtain Lemma [D.5].

By the lemma, we have $p^0_{X*} \pi_{1*}^0(R(M, F) \otimes L(-t)) \simeq \pi_{2*}^0(R(M, F) \otimes L(-t))$. Then, the claim of Proposition [D.4] follows from Proposition [D.2].
D.2 \( \tilde{R} \)-modules associated to subvarieties of \( \mathbb{P}^m \)

D.2.1 Setting

We fix a homogeneous coordinate system \([z_0 : \cdots : z_m]\) of \( \mathbb{P}^m \). We set \( V := H^0(\mathbb{P}^m, \mathcal{O}(1)) \). We put \( V_1 := \{ \alpha_0 z_0, \alpha_0 \in \mathbb{C} \} \subset V \) and \( V_2 := \{ \sum_{i=1}^m \alpha_i z_i | \alpha_i \in \mathbb{C} \} \subset V \). We may regard \( \alpha_0 \) and \((\alpha_1, \ldots, \alpha_m)\) as coordinate systems of \( V_1 \) and \( V_2 \), respectively.

We set \( H_0 := \{ z_0 = 0 \} \). Let \( U \) be a smooth quasi-projective variety with an immersion \( \iota_U : U \to \mathbb{P}^m \setminus H_0 \). For simplicity, we assume that there exists a hypersurface \( H \subset \mathbb{P}^m \) for which \( \iota_U(U) \) is a closed subset of \( \mathbb{P}^m \setminus (H_0 \cup H) \). We do not assume that \( H \) is smooth.

We set \( V_2^* := V_2 \setminus \{ 0 \} \). We shall construct some \( \tilde{R} \)-modules on \( V_2^* \) associated to \( U \).

D.2.2 A construction

We take a smooth projective variety \( Y \) with an open immersion \( \iota_1 : U \to Y \) and a morphism \( \iota_2 : Y \to \mathbb{P}^m \) such that (i) \( D_Y := Y \setminus U \) is normal crossing, (ii) \( \iota_2 \circ \iota_1 = \iota_U \). We have \( D_Y = \iota_2(H \cup H_0) \). We set \( D_{Y,V_2^*} := D_Y \times V_2^* \). We set \( \iota_{2,V_2^*} := \iota_2 \times \text{id}_{V_2^*} : Y \times V_2^* \to \mathbb{P}^m \times V_2^* \).

We have the meromorphic function

\[
F_{Y,V_2^*} := \iota_{2,V_2^*}^* \left( \sum_{i=1}^m \alpha_i z_i / z_0 \right)
\]

on \((Y, D_Y) \times V_2^* \). We obtain the integrable mixed twistor \( \mathcal{D} \)-modules \( T_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \) \((*=\ast, !)\) and the underlying \( \tilde{R}_{Y \times V_2^*} \)-modules \( L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \). Let \( \pi_{Y,V_2^*} : Y \times V_2^* \to V_2^* \) be the projection. We obtain the integrable mixed twistor \( \mathcal{D} \)-modules \( \pi_{Y,V_2^*}^0 T_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \) and the underlying \( \tilde{R}_{V_2^*} \)-modules \( \pi_{Y,V_2^*}^0 L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \).

If \( Y' \) and \( \iota_1' \) are another choice, then we take \( Y'' \) with morphisms \( \iota''_1 : U \to Y'' \) \( a : Y'' \to Y' \) and \( b : Y'' \to Y \) such that \( a \circ \iota''_1 = \iota_1' \), \( b \circ \iota''_1 = \iota_1' \), \( \iota_2 \circ a = \iota_2 \circ b = \iota''_1 \). Then, we have natural commutative diagram of isomorphisms:

\[
\begin{array}{ccc}
\pi_{Y,V_2^*}^0 T_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) & \simeq & \pi_{Y',V_2^*}^0 T_s(F_{Y',V_2^*}^u, D_{Y',V_2^*}) \\
\downarrow & & \downarrow \\
\pi_{Y,V_2^*}^0 L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) & \simeq & \pi_{Y',V_2^*}^0 L_s(F_{Y',V_2^*}^u, D_{Y',V_2^*})
\end{array}
\]

In this sense, \( \pi_{Y,V_2^*}^0 L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \) and the underlying \( \tilde{R}_{V_2^*} \)-modules \( \pi_{Y,V_2^*}^0 L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \) are independent of the choice of \( Y \).

We shall observe that \( \pi_{Y,V_2^*}^0 L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \) are isomorphic to the \( \tilde{R}_{V_2^*} \)-modules given in [41].

D.2.3 Comparison with a construction of Reichelt-Sehebeck

Let \( Z \subset \mathbb{P}^m \times V \) be the 0-set of the universal section \( \sum \alpha_i z_i \) of \( \mathcal{O}_{\mathbb{P}^m}(1) \otimes \mathcal{O}_V \). The projections \( q_1 : Z \to \mathbb{P}^m \) and \( q_2 : Z \to V \) are smooth. We set \( Z_U := Z \times \mathbb{P}^m \). Let \( \iota_{Z_U} : Z_U \to \mathbb{P}^m \times V \). Let \( \pi_V : \mathbb{P}^m \times V \to V \) be the projection.

We have the variation of pure Hodge structure \( (\mathcal{O}_{Z_U}, F) \) with the canonical real structure, where the Hodge filtration \( F \) is given by \( F_0 = \mathcal{O}_{Z_U} \) and \( F_{-1} = 0 \). We have the associated pure Hodge module which is also denoted by \( (\mathcal{O}_{Z_U}, F) \). We obtain the mixed Hodge modules \( \iota_{Z,U}^*(\mathcal{O}_{Z_U}, F) \) \((*=\ast, !)\) on \( \mathbb{P}^m \times V \), and then \( \pi_{V,1}^0 \iota_{Z,U}^*(\mathcal{O}_{Z_U}, F) \) on \( V = V_1 \times V_2 \). By applying the procedure in [41,1.1.1] and by taking the restriction to \( V_2^* \subset V_2 \), we obtain the following \( \tilde{R}_{V_2^*} \)-modules:

\[
G_0 \mathcal{F} \mathcal{L} \mathcal{O}^\text{loc}_{V_2^*} \pi_{V,1}^0 \iota_{Z,U}^*(\mathcal{O}_{Z_U}, F)|_{V_2^*}
\]

Proposition D.6 We have isomorphisms of \( \tilde{R}_X \)-modules:

\[
\lambda \cdot \pi_{Y,V_2^*}^0 L_s(F_{Y,V_2^*}^u, D_{Y,V_2^*}) \simeq G_0 \mathcal{F} \mathcal{L} \mathcal{O}^\text{loc}_{V_2^*} \pi_{V,1}^0 \iota_{Z,U}^*(\mathcal{O}_{Z_U}, F)|_{V_2^*}
\]

132
We also have the following commutative diagram:

\[
\begin{array}{c}
\lambda \cdot \pi_{Y,V_2}^0 \mathcal{L}_i(F_{Y,V_2}^u, D_{Y,V_2}) \xrightarrow{=} G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}^0 t_{ZU}^!(O_{ZU}, F)|_{V_2}^* \\
\downarrow \\
\lambda \cdot \pi_{Y,V_2}^0 \mathcal{L}_s(F_{Y,V_2}^u, D_{Y,V_2}) \xrightarrow{=} G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}^0 t_{ZU}^*(O_{ZU}, F)|_{V_2}^*
\end{array}
\]

**Proof** We set \(Z_0 := Z \cap (\mathbb{P}^m \times V_1 \times V_2)\). Let \(\mathbb{P} V_1\) denote the projective completion of \(V_1\), i.e., \(\mathbb{P} V_1 = \mathbb{P}(V_1^* + \mathbb{C})\). Let \(Z_0\) be the closure of \(Z_0\) in \(\mathbb{P}^m \times \mathbb{P} V_1 \times V_2\). Let \(\iota_{Z_0} : Z_0 \xrightarrow{} \mathbb{P}^m \times \mathbb{P} V_1 \times V_2^*\) denote the natural inclusion.

By the construction of \(Z_0\), we have the following equality of meromorphic functions on \(Z_0\):

\[
\iota_{Z_0}^* \alpha_0 = -\iota_{Z_0}^* \left( \sum_{i=1}^m \alpha_i z_i / z_0 \right)
\]

Let \(Z_{0U} := Z_U \cap Z_0\), and let \(Z_{0U}\) denote the closure of \(Z_{0U}\) in \(Z_0\). Let \(q : Z_{0U} \xrightarrow{} \mathbb{P}^m \times V_2^*\) denote the naturally induced morphism. Note that \(Z_{0U}\) is naturally isomorphic to \(U \times V_2^*\), and that the restriction \(q_{Z_{0U}}\) is an immersion. We can take a smooth complex algebraic variety \(B\) with projective morphisms \(\varphi_1 : B \xrightarrow{} Z_{0U}\) and \(\varphi_2 : B \xrightarrow{} Y \times V_2^*\), and an open immersion \(j : U \times V_2^* \subset B\) such that (i) \(q \circ \varphi_1 = i_{Y,V_2} \circ \varphi_2\), (ii) \(\varphi_a \circ j (a = 1, 2)\) are the identity of \(U \times V_2^*\), (iii) \(D_B = B \setminus j(U \times V_2^*)\) is a normal crossing hypersurface. We set \(G := -\varphi_1^* \alpha_0 = \varphi_2^* F_Y^\natural\). We have the integrable mixed twistor \(\mathcal{D}\)-modules \(\mathcal{T}_G(G, D_B)\) on \(B\). Let \(\mathcal{L}_s(G, D_B)\) denote the underlying \(\mathcal{R}_B\)-modules. Then, we have the following natural isomorphisms and the commutative diagrams:

\[
\begin{array}{c}
\varphi_2^! \mathcal{L}_i(G, D_B) \xrightarrow{=} \mathcal{L}_i(F_{Y,V_2}^u, D_{Y,V_2}) \\
\downarrow \\
\varphi_2^! \mathcal{L}_s(G, D_B) \xrightarrow{=} \mathcal{L}_s(F_{Y,V_2}^u, D_{Y,V_2})
\end{array}
\]

We also have the following:

\[
\begin{array}{c}
\varphi_1^! \mathcal{L}_i(G, D_B) \xrightarrow{=} R(i_{ZU}^!(O_{ZU}, F)) \otimes \mathcal{L}(-\alpha_0) \\
\downarrow \\
\varphi_1^! \mathcal{L}_s(G, D_B) \xrightarrow{=} R(i_{ZU}^*(O_{ZU}, F)) \otimes \mathcal{L}(-\alpha_0)
\end{array}
\]

Then, the claim of the theorem follows from Proposition [D7].

Reichelt and Sevenheck considered the image \((M, F)\) of \(i_{ZU}^!(O_{ZU}, F) \xrightarrow{} i_{ZU}^*(O_{ZU}, F)\), that is the minimal extension of \(Z_U\) in \(\mathbb{P}^m \times V\). They proved that \(G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}^0 (M, F)|_{V_2}^*\) is isomorphic to the image of the natural morphism \(G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}^0 t_{ZU}^!(O_{ZU}, F)|_{V_2}^* \xrightarrow{=} G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}^0 t_{ZU}^*(O_{ZU}, F)|_{V_2}^*\).

**Corollary D.7** \(G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}(M, F)|_{V_2}^*\) is isomorphic to the image of the natural morphism

\[
\lambda \cdot \pi_{Y,V_2}^0 \mathcal{L}_i(F_{Y,V_2}^u, D_{Y,V_2}) \xrightarrow{=} \lambda \cdot \pi_{Y,V_2}^0 \mathcal{L}_s(F_{Y,V_2}^u, D_{Y,V_2}).
\]

In particular, \(G_0 \text{FL}_V^{\text{loc}} \pi_{V_1}(M, F)|_{V_2}^*\) underlying integrable pure twistor \(\mathcal{D}\)-module.

**Proof** The first claim follows from Proposition [D6], and the result of Reichelt and Sevenheck mentioned above. Because the morphism of the mixed twistor \(\mathcal{D}\)-modules \(\pi_{Y,V_2}^0 \mathcal{T}_i(F_{Y,V_2}^u, D_{Y,V_2}) \xrightarrow{=} \pi_{Y,V_2}^0 \mathcal{T}_s(F_{Y,V_2}^u, D_{Y,V_2})\) factors through the morphism of the pure twistor \(\mathcal{D}\)-modules

\[
\text{Gr}_W^{\text{dim} U + m} \pi_{Y,V_2}^0 \mathcal{T}_i(F_{Y,V_2}^u, D_{Y,V_2}) \xrightarrow{=} \text{Gr}_W^{\text{dim} U + m} \pi_{Y,V_2}^0 \mathcal{T}_s(F_{Y,V_2}^u, D_{Y,V_2}),
\]

the second claim follows.
References

[1] A. Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. 73, (1994), 269–290.

[2] A. Adolphson, S. Sperber, *A-hypergeometric systems that come from geometry*, Proc. Amer. Math. Soc. 140, (2012), 2033–2042.

[3] V. V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. 69, (1993), 349–409.

[4] A. Beilinson, *How to glue perverse sheaves*, in: *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, Lecture Notes in Math., 1289, Springer, Berlin, (1987), 42–51.

[5] T.-M. Chiang, A. Klemm, S.-T. Yau, E. Zaslow, *Local mirror symmetry: calculations and interpretations*, Adv. Theor. Math. Phys. 3, (1999), 495–565.

[6] D. A. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, 124, American Mathematical Society, Providence, RI, 2011.

[7] V. I. Danilov, *de Rham complex on toroidal variety*, Algebraic geometry (Chicago, IL, 1989), Lecture Notes in Math., 1479, Springer, Berlin, (1991), 26–38.

[8] A. Douai, C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. I.*, Ann. Inst. Fourier (Grenoble), 53, (2003), 1055–1116.

[9] A. Douai, C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures. II.*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, (2004), 1–18.

[10] H. Esnault, C. Sabbah, J.-D. Yu, (with an appendix by M. Saito), *E₁-degeneration of the irregular Hodge filtration*, arXiv:1302.4537.

[11] H. Fan, *Schrödinger equations, deformation theory and tt*-geometry*, arXiv:1107.1290.

[12] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131, Princeton University Press, Princeton, NJ, 1993.

[13] C. Hertling, *tt*-geometry, Frobenius manifolds, their connections, and the construction for singularities, J. Reine Angew. Math. 555, (2003), 77–161.

[14] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. 84, (1990), 255–271.

[15] I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Birkhäuser Boston Inc., Boston, MA, 1994.

[16] A. Givental, *A mirror theorem for toric complete intersections*, In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Progr. Math., 160, Birkhäuser Boston, Boston, MA, (1998) 141–175.

[17] C. Hertling, Y. Manin, *Unfoldings of meromorphic connections and a construction of Frobenius manifolds, in Frobenius manifolds*, Aspects Math., E36, Vieweg, Wiesbaden, (2004), 113–144.

[18] R. Hotta, K. Takeuchi, T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, 236, Birkhäuser Boston Inc., Boston, MA, 2008.

[19] H. Iritani, *Convergence of quantum cohomology by quantum Lefschetz*, J. Reine Angew. Math. 610, (2007), 29–69.

[20] H. Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. 222, (2009), 1016–1079.
[21] H. Iritani, E. Mann, T. Mignon, *Quantum Serre in terms of quantum D-modules*, arXiv:1412.4523

[22] M. Kashiwara, *A study of variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22**, (1986) 991–1024.

[23] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Springer-Verlag, Berlin, 1990

[24] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, **217**, American Mathematical Society, Providence, (2003).

[25] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry*, In *From Hodge theory to integrability and TQFT tt*-geometry* Proc. Sympos. Pure Math., **78**, Amer. Math. Soc., Providence, RI, (2008) 87–174.

[26] Y. Konishi and S. Minabe, *Local Gromov-Witten invariants of cubic surfaces via nef toric degeneration*, Ark. Mat. **47**, (2009), 345–360.

[27] Y. Konishi, S. Minabe, *Local B-model and mixed Hodge structure*, Adv. Theor. Math. Phys. **14**, (2010), 1089–1145.

[28] Y. Konishi, S. Minabe, *Mixed Frobenius Structure and Local A-model*, arXiv:1209.5550

[29] Y. Konishi, S. Minabe, *Local Quantum Cohomology and Mixed Frobenius Structure*, arXiv:1405.7476

[30] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32**, (1976), 1–31.

[31] E. Mann, T. Mignon, *Quantum D-modules for toric nef complete intersections*, arXiv:1112.1552

[32] T. Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules I, II*, Mem. AMS. **185**, (2007)

[33] T. Mochizuki, *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque **340**, Société Mathématique de France, Paris, 2011.

[34] T. Mochizuki, *Harmonic bundles and Toda lattices with opposite sign II*, Comm. Math. Phys. **328**, 1159–1198, DOI:10.1007/s00220-014-1994-0, the second part of arXiv:1301.1718

[35] T. Mochizuki, *Holonomic D-modules with Betti structure*, arXiv:1001.2336, to appear in *Mém. Soc. Math. France*

[36] T. Mochizuki, *Mixed twistor D-modules*, arXiv:1104.3366

[37] T. Mochizuki, *A twistor approach to the Kontsevich complexxes*, preprint

[38] C. Peters and J. Steenbrink, *Mixed Hodge structure*, Springer-Verlag, Berlin, 2008.

[39] T. Reichelt, *Laurent polynomials, GKZ-hypergeometric systems and mixed Hodge modules*, Compos. Math. **150**, (2014), 911–941

[40] T. Reichelt, C. Sevenheck, *Logarithmic Frobenius manifolds, hypergeometric systems and quantum D-modules*, arXiv:1010.2118, to appear in J. Alg. Geom.

[41] T. Reichelt, C. Sevenheck, *Non-affine Landau-Ginzburg models and intersection cohomology*, arXiv:1210.6527

[42] C. Sabbah, *Polarizable twistor D-modules* Astérisque, **300**, (2005)

[43] C. Sabbah, *Hypergeometric periods for a tame polynomial*, Port. Math. (N.S.) **63**, (2006), 173–226.

[44] C. Sabbah, *Fourier-Laplace transform of a variation of polarized complex Hodge structure*, J. Reine Angew. Math. **621**, (2008), 123–158
C. Sabbah, *Wild twistor D-modules*, in *Algebraic analysis and around*, Adv. Stud. Pure Math., 54, Math. Soc. Japan, Tokyo, (2009), 293–353.

C. Sabbah, J.-D. Yu. *On the irregular Hodge filtration of exponentially twisted mixed Hodge modules*, arXiv:1406.1339

K. Saito, A. Takahashi, *From primitive forms to Frobenius manifolds*, in *From Hodge theory to integrability and TQFT tt*-geometry*, 31–48, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

M. Saito, *Modules de Hodge polarisables*, Publ. RIMS., 24, (1988), 849–995.

M. Saito, *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble), 39, (1989), 27–72.

M. Saito, *Duality for vanishing cycle functors*, Publ. Res. Inst. Math. Sci. 25, (1989), 889–921.

M. Saito, *Induced D-modules and differential complexes*, Bull. Soc. Math. France 117, (1989), 361–387.

M. Saito, *Mixed Hodge modules*, Publ. RIMS., 26, (1990), 221–333.

M. Schulze and U. Walther, *Hypergeometric D-modules and twisted GaussManin systems*, J. Algebra 322, (2009), 3392–3409.

C. Simpson, *Mixed twistor structures*, math.AG/9705006.

J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure. I*, Invent. Math. 80, (1985), 489–542.

J. Stienstra, *Resonant hypergeometric systems and mirror symmetry*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, (1998), 412–452.

J.-D. Yu, *Irregular Hodge filtration on twisted de Rham cohomology*, Manuscripta Math. 144, (2014), 99–133

Address

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
takuro@kurims.kyoto-u.ac.jp