BEHAVIOR OF TORSION FUNCTIONS OF SPACELIKE CURVES IN LORENTZ-MINKOWSKI SPACE

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Abstract. In this paper, we introduce the pseudo-torsion functions along spacelike curves whose curvature vector field has isolated lightlike points in Lorentz-Minkowski 3-space, and prove the fundamental theorem. Moreover, we analyze the behavior of the torsion function at such points. As a corollary, we obtain a necessary and sufficient condition for real analytic spacelike curves to be planar.

1. Introduction

We denote by $L^3$ the Lorentz-Minkowski 3-space equipped with the Lorentzian inner product $\langle \cdot, \cdot \rangle$. Consider a spacelike regular curve $\gamma : I \rightarrow L^3$, where $I$ is an open interval. Without loss of generality, we may assume that $\gamma$ is parametrized by arclength, that is, $\langle \gamma'(s), \gamma'(s) \rangle = 1$ holds for all $s \in I$, where the prime means $d/ds$. Then, we call $\kappa(s) := \gamma''(s)$ the curvature vector field along $\gamma(s)$. Although the tangent vector $\gamma'(s)$ is spacelike, $\kappa(s)$ may be spacelike, timelike, or lightlike. We list the known results concerning curves in $L^3$ depending on the causal characters of $\kappa(s)$ as follows:

- In the case that $\kappa$ is not lightlike on $I$, the curve $\gamma(s)$ is said to be a spacelike Frenet curve. The curvature and torsion functions can be defined as in the case of Euclidean 3-space. Then, the Frenet-Serret type formula is obtained, which yields the fundamental theorem for such spacelike curves [19, 14] (cf. §2). However, if $\kappa$ admits a lightlike point, such a procedure cannot be proceeded.

- In the case that $\kappa(s)$ is lightlike for each $s \in I$, although the Frenet frame cannot be defined in a similar manner, one can define another frame which satisfies the Frenet-Serret type formula. The fundamental theorem for such spacelike curves also holds. For more details, see [19, 10, 14] (cf. §22).

- With respect to timelike or lightlike curves, for a timelike curve $\gamma : I \rightarrow L^3$, one can consider the curvature vector field $\kappa$ in a similar way. Since $\kappa$ is always spacelike, the curvature and torsion functions can be defined in the usual way [19, 14]. Also, for a non-degenerate lightlike curve, a suitable frame can be defined [2, 7, 5, 11, 14] (see also [15, 19, 15, 1]). In [6], another treatment of the curvatures of curves in semi-Euclidean space is given. Global properties of spacelike curves in $L^3$ are investigated in [12].

Hence, we may say that the previous studies have not clarified the structure of spacelike curves whose curvature vector field has isolated lightlike points.
In this paper, we deal with spacelike curves whose curvature vector field $\kappa(s)$ has isolated lightlike points. We introduce a torsion-like invariant called the pseudo-torsion function, which yields the fundamental theorem for such spacelike curves (Theorem 3.3, Corollary 3.6). An application of such the fundamental theorem for mixed type surfaces in $L^3$ can be found in [5].

This paper is organized as follows. In Section 2, we review the several formulas of vectors and curves in $L^3$. In Section 3, we deal with spacelike curves having points where curvature vector field $\kappa(s)$ is lightlike. At such points, the torsion function is unbounded, in general. We obtain the coefficient of the divergent term of the torsion function (Theorem 3.3) for spacelike curves of type $L_k$ (cf. Definition 2.2). As a corollary, we obtain a necessary and sufficient condition for real analytic spacelike curves with non-zero curvature vector to be planar (Corollary 3.11).

2. Preliminaries

We denote by $L^3$ the Lorentz-Minkowski 3-space with the standard Lorentz metric $(\cdot,\cdot)$. Namely,

$$(x,x)=x^2+y^2-z^2$$

holds for each $x=(x,y,z)\in L^3$, where $x^T$ stands for the transpose of the column vector $x$. A vector $x\in L^3$ is called spacelike if $(x,x)>0$ or $x=0$. Similarly, if $(x,x)<0$ (resp. $(x,x)=0$), $x$ is called timelike (resp. lightlike). For $x\in L^3$, we set $|x|:=\sqrt{(x,x)}$.

For vectors $v, w\in L^3$, the vector product $v\times w$ is given by $v\times w:=Zv\times\epsilon w$, where $\times\epsilon$ means the standard cross product of the Euclidean 3-space $R^3$, and we set $Z:=\text{diag}(1,1,-1)$. Then, it holds that

$$\det(u,v,w)=(u,v\times w),$$

$$u\times(v\times w)=(u,v)w-(u,w)v,$$

$$\langle v\times w, v\times w \rangle = -\langle v,v \rangle \langle w,w \rangle + \langle v,w \rangle^2$$

for $u, v, w\in L^3$. In particular, $v\times w$ is orthogonal to $v$ and $w$. To calculate the vector product of a lightlike vector, the following formula is useful.

**Fact 2.1 ([5] Lemma 4.3)**. Let $v\in L^3$ be a spacelike vector. Take a lightlike vector $w\in L^3$ such that $\langle v, w \rangle = 0$. Then, either $v\times w = |v||w|$ or $v\times w = -|v||w|$ holds.

The isometry group of $L^3$ is described as the semidirect product $\text{Isom}(L^3) = O(2,1)\ltimes L^3$, where $O(2,1)$ is the Lorentz group which consists of square matrices $A$ of order 3 such that $A^TZA=Z$. We set $SO(2,1)$ and $SO^+(2,1)$ as

$$SO(2,1):=\{A\in O(2,1)\mid \det A=1\},$$

$$SO^+(2,1):=\{A=(a_{ij})\in SO(2,1)\mid a_{33}>0\},$$

respectively. Orientation-preserving isometries form the subgroup $SO(2,1)\ltimes L^3$ of the isometry group $\text{Isom}(L^3)$.

2.1. Spacelike curves in $L^3$. Let $I$ be an open interval. A regular curve $\gamma: I \rightarrow L^3$ is called spacelike if each tangent vector is spacelike. By a coordinate change, we may assume that $\gamma$ is parametrized by arclength. That is, $e(s):=\gamma'(s)$ gives the spacelike tangent vector field of unit length, where the prime means $d/ds$. As in the introduction, we call $\kappa(s):=\gamma''(s)$ the curvature vector field along $\gamma(s)$. If $\kappa(s)$ is nowhere zero, then $\gamma(s)$ is said to be of non-zero curvature vector.

In a general parametrization $\gamma=\gamma(t)$, the curvature vector field is written as

$$\kappa(t) = \frac{\gamma'(t) \times (\gamma'(t) \times \gamma''(t))}{|\gamma'(t)|^3}$$
where the dot means $d/dt$. Here, we used (2.2). So we have the following:

Let $\gamma(t)$ be a spacelike curve in $\mathbb{L}^3$, which may not be of unit speed. Then, $\gamma(t)$ has non-zero curvature vector if and only if $\dot{\gamma}(t)$ and $\dot{\gamma}(t)$ are linearly independent.

For a spacelike curve with non-zero curvature vector, we prepare several terminologies as follows:

**Definition 2.2.** Let $\gamma : I \to \mathbb{L}^3$ be a spacelike curve with non-zero curvature vector.

- A point $s_0 \in I$ is said to be a curvature-lightlike point, if $\kappa(s_0)$ is a lightlike vector. If there exists an open neighborhood $J$ of a curvature-lightlike point $s_0$ such that $J \setminus \{s_0\}$ consists of non-curvature-lightlike points, then $s_0$ is called isolated.
- The curve $\gamma(s)$ is said to be a spacelike Frenet curve, if it has no curvature-lightlike points. In particular, if $\kappa(s)$ is a spacelike (resp. timelike) vector field along $\gamma(s)$, then $\gamma(s)$ is said to be of type $S$ (resp. type $T$).
- If $\kappa(s)$ is a lightlike vector field along $\gamma(s)$, then $\gamma(s)$ is called a spacelike curve of type $L$.
- The function $\theta(s)$ defined by
  $$\theta(s) := \langle \kappa(s), \kappa(s) \rangle$$

is called the causal curvature function. Then, $\gamma(s)$ is of type $S$ (resp. type $T$, type $L$) if and only if

$$\theta(s) > 0 \quad (\text{resp. } \theta(s) < 0, \ \theta(s) \equiv 0)$$

holds on $I$.

- Letting $k \in \mathbb{Z}$ be a positive integer, $\gamma(s)$ is said to be of type $L_k$ at $s_0 \in I$, if

$$\theta(s_0) = \cdots = \theta^{(k-1)}(s_0) = 0, \ \theta^{(k)}(s_0) \neq 0.$$

Let us remark that, if $\gamma(s)$ is of type $L_k$ at $s_0 \in I$, then $s_0$ is an isolated curvature-lightlike point. Moreover, if $\gamma : I \to \mathbb{L}^3$ is real analytic, then the type must be either $S$, $T$, $L$ or $L_k$ (cf. Corollary 3.11).

In the following, we review the fundamental properties of spacelike Frenet curves or spacelike curves of type $L$, as explained in [14].

### 2.2. Spacelike Frenet curves.

Let $\gamma : I \to \mathbb{L}^3$ be a spacelike Frenet curve parametrized by arclength. Denote by $e(s) = \gamma'(s)$ (resp. $\kappa(s) = \gamma''(s)$) the unit tangent vector field (resp. the curvature vector field) along $\gamma(s)$.

The curvature function is defined as $\kappa(s) := \sqrt{\theta(s)}$ (resp. $|\kappa(s)|$). We set the sign $\sigma_\gamma \in \{+1,-1\}$ as $\sigma_\gamma = -1$ (resp. $+1$) if $\gamma$ is of type $S$ (resp. type $T$). Then,

$$n(s) := \frac{1}{\kappa(s)} \kappa(s), \quad b(s) := \sigma_\gamma e(s) \times n(s)$$

are respectively called the principal normal vector field, and the binormal vector field, which satisfy $\det(e, n, b) = 1$. The torsion function is defined as $\tau(s) := \langle n'(s), b(s) \rangle$. The system of the equations, $e' = \kappa n$, $n' = \sigma_\gamma (\kappa e + \tau b)$, and $b' = \sigma_\gamma \tau n$, is called the Frenet-Serret type formula. Then, it holds that

$$\det(\gamma', \gamma'', \gamma''') = -\theta \tau.$$

(2.4)
2.3. Spacelike curves of type $L$. Let $\gamma : I \to \mathbb{L}^3$ be a unit speed spacelike curve of type $L$. By definition, $\kappa(s)$ is a lightlike vector field. Then, there exists a lightlike vector field $\beta(s)$ such that

$$(e(s), \beta(s)) = 0, \quad (\kappa(s), \beta(s)) = 1.$$ 

Such a vector field $\beta(s)$ is uniquely determined. We call $\beta(s)$ the pseudo-binormal vector field. Then, $\mu(s) := -(\kappa(s), \beta(s))$ is called the pseudo-torsion function.\footnote{In \cite{11}, $-\mu(s)$ is called the pseudo-torsion function. However, to maintain the consistency with the notations for spacelike curves of type $L_h$, we use the minus sign. We also remark that the terminology ‘pseudo-binormal vector field’ is not used in \cite{11}.} 

The system of the equations, $e' = \kappa$, $\kappa' = -\mu \kappa$, and $\beta' = -e + \mu \beta$, is called the Frenet-Serret type formula. In terms of the frame $F := (e, \kappa, \beta)$, the formula is rewritten as

$$F' = F \begin{pmatrix} 0 & 0 & -1 \\ 1 & -\mu & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

3. Spacelike curves whose curvature vector field has isolated lightlike points

In this section, we investigate spacelike curves having isolated curvature-lightlike points.

3.1. Pseudo-torsion function and Frenet-Serret type formula. Let $\gamma : I \to \mathbb{L}^3$ be a unit speed spacelike curve with non-zero curvature vector. Denote by $e(s) = \gamma'(s)$ (resp. $\kappa(s) = \gamma''(s)$) the unit tangent vector field (resp. the curvature vector field) along $\gamma(s)$. The causal curvature function is given by $\theta(s) = (\kappa(s), \kappa(s))$.

Let $s_0 \in I$ be a curvature-lightlike point, namely, $\kappa(s_0)$ is a lightlike vector. By Fact 2.1,

$$e(s_0) \times \kappa(s_0) = \varepsilon \kappa(s_0)$$

holds for some $\varepsilon \in \{+1, -1\}$. We call $\varepsilon$ the sign. When we emphasize $\gamma(s)$ and $s_0$, we also denote by $\varepsilon = \text{sgn}(\gamma, s_0)$.

Lemma 3.1. Let $\gamma(s)$ be a spacelike curve with non-zero curvature vector, and $s_0 \in I$ be an isolated curvature-lightlike point. Then, there exists a non-vanishing vector field $\xi(s)$ along $\gamma(s)$ defined on a neighborhood of $s_0 \in I$ such that

$$e(s) \times \kappa(s) = \varepsilon \kappa(s) + \theta(s) \beta(s)$$

holds. (We call such a vector field $\beta(s)$ the pseudo-binormal vector field along $\gamma(s)$.)

Proof. Without loss of generality, we may assume that $\gamma(s)$ is parametrized by arclength. Since $(e(s), e(s)) = 1$, there exist $\alpha_1(s), \alpha_2(s)$ such that

$$(3.1) \quad e(s) = \begin{pmatrix} \cosh(\alpha_1(s)) \cos(\alpha_2(s)) \\ \cosh(\alpha_1(s)) \sin(\alpha_2(s)) \\ \sinh(\alpha_1(s)) \end{pmatrix}$$

holds. Then, $\kappa(s) = e'(s)$ is given by

$$(3.2) \quad \kappa(s) = \begin{pmatrix} \alpha_1' \sin \alpha_1 \cos \alpha_2 - \alpha_2' \cosh \alpha_1 \sin \alpha_2 \\ \alpha_1' \sin \alpha_1 \sin \alpha_2 + \alpha_2' \cosh \alpha_1 \cos \alpha_2 \\ \alpha_1' \cosh \alpha_1 \end{pmatrix}.$$

The causal curvature function $\theta(s)$ is calculated as

$$\theta(s) = (\alpha_2' \cos \alpha_1 - \alpha_1')(\alpha_2' \cosh \alpha_1 + \alpha_1').$$
In the case of $\epsilon = 1$, (3.2) and (3.3) yield

$$e(s) \times \kappa(s) = \kappa(s) + (\alpha'_1 + \alpha'_2 \cosh \alpha_1)\xi(s),$$

where

$$\xi(s) := \begin{pmatrix} \sinh \alpha_1 \cos \alpha_2 - \sin \alpha_2 \\ \sinh \alpha_1 \sin \alpha_2 + \cos \alpha_2 \\ \cosh \alpha_1 \end{pmatrix}.$$  

Since $\xi(s) \neq 0$, we have $\alpha'_1 + \alpha'_2 \cosh \alpha_1 = 0$ at $s_0$. We remark that $\alpha'_1 - \alpha'_2 \cosh \alpha_1 \neq 0$ holds by the assumption of non-zero curvature vector $\kappa(s) \neq 0$. Then, (3.3) yields that $e(s) \times \kappa(s) = \kappa(s) + \theta(s)\beta(s)$, where we set

$$\beta(s) := \frac{1}{\alpha'_1(s) - \alpha'_2(s) \cosh \alpha_1(s)} \xi(s),$$

and hence, we obtain the desired result. A similar argument can be applied in the case of $\epsilon = -1$.

**Proposition 3.2.** The pseudo-binormal vector field $\beta(s)$ is a lightlike vector field satisfying

$$\langle e(s), \beta(s) \rangle = 0, \quad \langle \kappa(s), \beta(s) \rangle = 1,$$

Moreover, setting $\mu(s) := -\langle \kappa'(s), \beta(s) \rangle$, we have

$$\kappa' = -\theta e - \mu \kappa + \left(\mu \theta + \frac{1}{2} \theta'\right) \beta, \quad \beta' = -e + \mu \beta.$$

**Proof.** Since

$$\beta(s) = -\frac{1}{\theta(s)} (e(s) \times \kappa(s) - \kappa(s))$$

holds for $s \neq s_0$, we can verify that $\beta(s)$ is a lightlike vector for each $s \neq s_0$. By the continuity, we have that $\beta(s)$ is a lightlike vector field. Similarly, (3.3) holds. Since $\det(e, \kappa, \beta) = \langle e \times \kappa, \beta \rangle = \langle e - \theta \beta, \beta \rangle = \epsilon$, we obtain (3.5). With respect to (4.2), set $\kappa' = P e + Q \kappa + R \beta$. Taking the inner products of $e, \kappa, \beta$, we have $P = -\theta, Q = -\mu, R = \mu \theta + \theta'/2$, where we used (3.4). Similarly, we have $\beta' = -e + \mu \beta$, and hence (4.5) holds. \qed

In the matrix form,

$$F' = F \begin{pmatrix} 0 & -\theta & -1 \\ 1 & -\mu & 0 \\ 0 & \mu \theta + \frac{1}{2} \theta' & \mu \end{pmatrix}$$

holds, where we set $F := (e, \kappa, \beta)$. We call $\mu(s)$ the pseudo-torsion function. If we emphasize the sign $\epsilon = \text{sgn}(\gamma, s_0)$, we write $\mu(s) = \mu_\epsilon(s)$.

**Example 3.3 ([13] Remark 2.2).** We set $\gamma : (\infty, -1) \rightarrow L^3$ as

$$\gamma(s) := \begin{pmatrix} \cos s + s \sin s \\ s \sqrt{s^2 - 1 - \log (s + \sqrt{s^2 - 1})} / 2 \\ \sin s - s \cos s \\ (s^2 - 2s^2 + 2s^2 + 2) \end{pmatrix}.$$

The causal curvature function $\theta(s)$ can be calculated as $\theta(s) = (s^4 - s^2 - 1)/(s^2 - 1)$. Then, we may check that $\theta(s_0) = 0$ and $\theta'(s_0) \neq 0$ hold at $s_0 = -\sqrt{1 + \sqrt{3}/\sqrt{2}} \mp 1.272$. Hence, $\gamma(s)$ is of type $L_1$ at $s_0$. The pseudo-torsion function $\mu(s)$ can be calculated as

$$\mu(s) = \frac{\sqrt{s^2 - 1} \left(s^6 - 2s^4 - 2s^2 + 2 - s (s^4 - 2s^2 + 2) \right)}{(s^2 - 1) (s^4 - s^2 - 1)}.$$
Although the denominator of $\mu(s)$ has zero at $s = s_0$, we can check that $\mu(s)$ can be analytically extended across $s = s_0$, and $\lim_{s \to s_0} \mu(s) = \sqrt{5/2} - 11/(2\sqrt{2})$ holds.

### 3.2. Fundamental theorem.

Let $\gamma(s)$ be a spacelike curve with non-zero curvature vector, $s_0 \in I$ be a curvature-lightlike point, and $\epsilon = \text{sgn}(\gamma, s_0) \in \{+1, -1\}$ be the sign. Take an isometry $T$ of $\mathbb{L}^3$, and set

$$\hat{\gamma}(s) := T \circ \gamma(s).$$

Then, the causal curvature function $\hat{\theta}(s)$ of $\hat{\gamma}(s)$ coincides with that of $\gamma(s)$. Hence, $s_0 \in I$ is also a curvature-lightlike point for $\hat{\gamma}(s)$.

**Lemma 3.4.** Let $\epsilon = \text{sgn}(\gamma, s_0) \in \{+1, -1\}$ be the sign of $\gamma(s)$ at $s_0$. If $T$ is orientation preserving (resp. orientation reversing), we have $\hat{\epsilon} = \epsilon$. (resp. $\hat{\epsilon} = -\epsilon$).

Moreover, the pseudo-torsion $\hat{\mu}(s)$ of $\hat{\gamma}(s)$ coincides with that $\mu(s)$ of $\gamma(s)$.

We omit the proof since this lemma can be verified directly by using the following formula:

$$(Tv) \times (Tw) = (\det T)(v \times w)$$

holds for $v, w \in \mathbb{L}^3$, $T \in O(2,1)$.

By a parallel translation, we may assume that $\gamma(s_0) = 0$. We would like to find an orientation preserving isometry $T \in SO(2,1)$ so that the frame

$$F := (e, \kappa, \beta)$$

has simplified form at $s_0 \in I$. First, we can find $T_1 \in SO^+(2,1)$ such that $T_1 e(s_0) = (1,0,0)^T$. Since $\kappa(s_0), \beta(s_0)$ are lightlike and perpendicular to $e(s_0)$, we have

$$T_1 \kappa(s_0) = a(0,1,\pm 1)^T, \quad T_1 \beta(s_0) = \frac{1}{2a}(0,1,\mp 1)^T,$$

where $a \in \mathbb{R}$ is a non-zero constant. Hence, there exists $T_2 \in SO^+(2,1)$ such that $T_2 T_1 F$ at $s_0 \in I$ is given by either $E_+, E_-, E'_+$ or $E'_-$, where

$$E_{\sigma_1} := \begin{pmatrix}
1 & 0 & 0 \\
0 & \sigma_1/\sqrt{2} & \sigma_1/\sqrt{2} \\
0 & -1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}, \quad E_{\sigma_2}' := \begin{pmatrix}
1 & 0 & 0 \\
0 & -\sigma_2/\sqrt{2} & -\sigma_2/\sqrt{2} \\
0 & 1/\sqrt{2} & -1/\sqrt{2}
\end{pmatrix},$$

and $\sigma_1, \sigma_2 \in \{+1, -1\}$. If we set $S \in SO(2,1)$ as $S := \text{diag}(1,-1,-1)$, then $E_{\pm} = SE_{\pm}'$ hold. Hence, setting $T := ST_2 T_1 \in SO(2,1)$, we have the following:

Let $\gamma(s)$ be a spacelike curve with non-zero curvature vector, $s_0 \in I$ be a curvature-lightlike point, and $\epsilon = \text{sgn}(\gamma, s_0)$ be the sign. Then there exists $T \in SO(2,1)$ such that $T F(s_0) = E_{\epsilon}$.

By the existence and uniqueness of the solution to the ODE with the initial condition $F(s_0) = E_+$ or $F(s_0) = E_-$, we obtain the fundamental theorem of spacelike curves having isolated curvature-lightlike points:

**Theorem 3.5.** Let $I$ be an open interval, and fix $s_0 \in I$. Also let $\theta(s), \mu(s)$ be two smooth functions defined on $I$ so that the zero of $\theta(s)$ is $s_0$ only. For each choice of the sign $\epsilon \in \{+1, -1\}$, there exists a unit speed spacelike curve $\gamma(s) : I \to \mathbb{L}^3$ with non-zero curvature vector such that

(a) the causal curvature and pseudo-torsion functions coincide with $\theta(s)$ and $\mu(s)$, respectively.

(b) the sign $\text{sgn}(\gamma, s_0)$ coincides with $\epsilon$.

Moreover, such a curve is unique up to orientation preserving isometries of $\mathbb{L}^3$.

Since the steps to follow are similar as in Euclidean space (e.g. see Theorem 5.2), we omit the proof.
Lemma 3.7. Let \( \gamma(s) : I \to \mathbb{L}^3 \) be a unit speed spacelike curve with non-zero curvature vector, and \( s_0 \in I \) be an isolated curvature-lightlike point. Then the torsion function \( \tau(s) \) of \( \gamma(s) \) is given by

\[
\tau(s) = -\mu(s) - \frac{1}{2} \theta'(s).
\]

Proof. Substituting \( \gamma' = e, \gamma'' = k, \gamma''' = k' \) into \( \det(\gamma', \gamma'', \gamma''') = -\theta \tau \) as in (2.4), and applying (3.6), we may verify the desired identity.

Theorem 3.8. For a positive integer \( k \), let \( \gamma(s) : I \to \mathbb{L}^3 \) be a spacelike curve of type \( L_k \) at \( s_0 \in I \) parametrized by arclength. Then

\[
\lim_{s \to s_0} (s - s_0) \tau(s) = -\frac{1}{2} k
\]

holds. In particular, \( \tau(s) \) is unbounded at \( s_0 \).

Proof. By the definition of type \( L_k \) in Definition 2.2, the division lemma [4] yields that there exists a function \( \hat{\theta}(s) \) such that

\[
\theta(s) = (s - s_0)^k \hat{\theta}(s) \quad (\hat{\theta}(s_0) \neq 0)
\]

holds. Together with Lemma 3.7, we have

\[
\tau = -\mu - \frac{1}{2} \left( k \frac{s - s_0}{s - s_0} + \frac{\hat{\theta}'}{\hat{\theta}} \right).
\]

Hence, \( (s - s_0) \tau(s) \to -k/2 \) holds as \( s \to s_0 \). 

We remark that, for a planar curve having a cusp singularity, the behavior of curvature function at the cusp is investigated [16].

Example 3.9. Let \( \gamma(s) \) be the spacelike curve of type \( L_1 \) given in Example 3.2. The torsion function \( \tau(s) \) can be calculated as

\[
\tau(s) = \frac{-s^6 + 2s^4 + 2s^2 - 2}{\sqrt{s^2 - 1} (s^4 - s^2 - 1)}
\]

which is unbounded at \( s_0 = -\sqrt{1 + \sqrt{5}/\sqrt{2}} \). We can check that \( \lim_{s \to s_0} (s - s_0) \tau(s) = -1/2 \), which verifies Theorem 3.8.

\[\text{\textsuperscript{2}The division lemma is also called Hadamard's lemma, cf. [3] Lemma 3.4}.\]
As a corollary of Theorem 3.8, we obtain the planarity condition for analytic spacelike curves (Corollary 3.11). For spacelike curves of type $S$, $T$ or $L$, the planarity condition is given as follows:

**Fact 3.10** ([14, Theorem 2.3], [13, Corollary 3.2], [4, Remark 7]). Let $\gamma : I \to \mathbb{L}^3$ be a spacelike curve.

- Assume that $\gamma(s)$ is a Frenet curve. Then, $\gamma(s)$ is included in an affine plane if and only if its torsion function is identically zero.
- Assume that $\gamma(s)$ is of type $L$. Then, $\gamma(s)$ is a planar curve included in a lightlike plane.

So, it is natural to ask the planarity condition for general spacelike curves. In analytic case, we have the following:

**Corollary 3.11.** A real analytic spacelike planar curve with non-zero curvature vector must be either a spacelike Frenet curve with vanishing torsion or a spacelike curve of type $L$.

**Proof.** Let $\gamma : I \to \mathbb{L}^3$ be a real analytic spacelike curve with non-zero curvature vector. By the analyticity, $\gamma(s)$ must be of type $S$, $T$, $L$ or $L_k$. By Fact 3.10, it suffices to show that if $\gamma(s)$ is of type $L_k$, then $\gamma(s)$ never be planar. Assume that $\gamma(s)$ is of type $L_k$ at $s_0 \in I$, and that $\gamma(s)$ is planar. Since $\gamma(s) (s \neq s_0)$ is a Frenet curve which is included in an affine plane, the torsion $\tau(s)$ of $\gamma(s) (s \neq s_0)$ is identically zero. However, by Theorem 3.8 $\tau(s)$ is unbounded at $s = s_0$, which is a contradiction. Hence, any spacelike curve of type $L_k$ never be planar.

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