Orbital and epicyclic frequencies in massive scalar-tensor theory with self-interaction

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Abstract Observations of the electromagnetic signals originating from an accretion discs in the close vicinity of a compact object can be used to test spacetime in the strong field regime. More specifically, the so-called quasi-periodic oscillations, observed in the X-ray light curves of some pulsars, might carry features of the underlying theory of gravity making them a promising tool for testing the modifications of Einstein’s theory. Although there are different ways of explaining the quasi-periodic oscillations, one thing most of the models have in common is that they incorporate in certain way the radius of the innermost stable circular orbit, and the orbital and the epicyclic frequencies of particles moving around the compact object. In this paper we study the aforementioned quantities in the context of massive scalar-tensor theory including self-interaction of the scalar field. Unlike the massless scalar field case, the neutron stars in these theories can have large deviations from pure general relativity for values of the free parameters that are in agreement with the observations. Thus the deviations in the orbital and epicyclic frequencies can reach large values.

Keywords Modified gravity · Scalar-tensor theory · Quasi-periodic oscillations · Neutron stars

Ones of the most natural and well-motivated modified theories of gravity are the scalar-tensor theories (STTs) of gravity, in which gravity is mediated by the spacetime metric tensor and a dynamical scalar field. In the past decades the massless STT with coupling function of the form \( \alpha = \beta \varphi \) (where \( \varphi \) is the scalar field and \( \beta \) is a free parameter) gained strong interest. The reason is that unlike the Brans-Dicke scalar-tensor theory for example, in the weak field regime its predictions coincide with GR but in the strong field regime significant deviations could be observed. It was studied in the context of neutron stars for the first time in Damour and Esposito-Farese (1993), predicting the well-known effect of spontaneous scalarization, later generalized to slow and rapid rotation (Damour and Esposito-Farèse 1996; Sotani 2012; Pani and Berti 2014; Doneva et al. 2013). However, in the recent years the astrophysical observations of binary pulsars set tight constraints on the allowed values for the free parameter \( \beta \) in the theory (Antoniadis et al. 2013; Demorest et al. 2010), which makes the scalarized neutron stars in massless STTs almost indistinguishable from GR in the static case, leaving freedom for larger deviation only when rapid rotation is considered (Doneva et al. 2013). Recently it was shown in Ramazanoğlu and Pretorius (2016), Popchev (2015), Yazadjiev et al. (2016) that the picture changes significantly if a potential with a massive scalar-field term is added to the Lagrangian of the theory. The mass term effectively suppresses the scalar field beyond its Compton wavelength which leads to a significant increases of the interval of allowed values for the parameter \( \beta \). The studies showed that the presence of the mass term suppresses the spontaneous scalarization, but still significant deviations from GR...
can be observed for values of the parameters in agreement with the observations. Recently those studies were extended in Staykov et al. (2018), Popchev et al. (2018) by adding a self-interaction quartic term in the potential which additionally suppresses the scalarization of the neutron star. Thus the effect is similar to the presence of mass therm, even though some qualitative differences can be observed (Staykov et al. 2018).

The recent detections of gravitational waves (Abbott et al. 2018), even more the binary neutron star merger (Abbott et al. 2017) with its multi-messenger detection, pointed once again towards the importance of the proper study of modified theories of gravity in both gravitational wave and in the electromagnetic spectrum. One important, but still not well understood phenomena in the electromagnetic spectrum, are the so-called quasi-periodic oscillations (QPO) observed in the X-ray light curves of some pulsars. The QPOs are Hz to kHz oscillations in the X-ray flux of compact object (neutron stars and black hole candidates). The kHz QPOs are supposed to originate from the inner edge of the accretion disk, which means that they may turn out to be excellent probes for the strong field regime in the vicinity of a compact object (Stella and Vietri 1999; Maselli et al. 2015).

In most accretion models (Abramowicz et al. 2010), the innermost stable circular orbit (ISCO) is the inner edge of the accretion disc. On the other hand, ISCO may turn out to be important for the compact object mergers due to the fact that after ISCO, the two bodies should start falling rapidly to each other. The QPO origin is not well understood and different models explaining the QPO exist that are based on different mechanisms (see van der Klis 2006 for a comprehensive review). In general there are two main classes of models based on different mechanisms behind the QPOs. The first one is based on orbital and epicyclic motion of matter around the central object (for example Miller et al. 1998; Stella and Vietri 1999; Stella 2001; Abramowicz et al. 2004; Pappas 2012; Motta et al. 2014; Pappas and Sotiriou 2015; Maselli et al. 2015; Staykov et al. 2015), and the second one is based on oscillations and instabilities in an accretion disc around the compact object (for example Rezzolla et al. 2003a,b; Montero et al. 2004; Fragile et al. 2016; de Avellard et al. 2018). In one way or another, though, most QPO models incorporate the orbital and the epicyclic frequency of a particle on a circular orbit as well as the radius of ISCO. QPO frequencies in different alternative theories of gravity were examined in DeDeo and Psaltis (2004), Doneva et al. (2014), Vincent (2014), Maselli et al. (2015), Staykov et al. (2015), Maselli et al. (2017). We will extend the results in Doneva et al. (2014) by adding a massive scalar field with self-interaction that can significantly alter not only the quantitative deviations from GR compared to the massless scalar field case, but also the qualitative differences. Similar to previous studies, instead of employing a specific QPO model, in the present paper we will focus on the behavior of these parameters in massive scalar-tensor theories. This will help us estimate up to what extend the observed QPO frequencies deviate from pure GR and whether these deviations are observationally relevant.

This paper is structured as follow. In Sect. 1 we present the mathematical basics. In the first part of that section we present the field equations for constructing the neutron star model and the corresponding boundary conditions one has to employ. In Sect. 2 we present the general scheme for deriving the orbital and the epicyclic frequencies of a particle moving on a circular orbit in stationary and axisymmetric spacetimes, as well as the conditions for determining the radius of ISCO. In Sect. 3 we present and discuss the numerical results for massive scalar-tensor theories with self-interacting scalar field. The paper ends with a Conclusion.

### 1 The background solution

For simplicity, the mathematical part of this paper is in the more convenient Einstein frame, but all presented results in the following section are in the physical Jordan frame.

The general form of the Einstein frame STT action is

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R_g - 2 \hat{g}_{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_{\text{matter}}(A^2(\phi) \hat{g}_{\mu\nu}, \chi),
\]

where \( R_g \) is the Ricci scalar curvature with respect to the Einstein frame metric \( \hat{g}_{\mu\nu} \). In the Einstein frame, the scalar-tensor theories are specified by the function \( A(\phi) \), which controls the conformal transformation between the metric in the Einstein and the Jordan frames, and the scalar-field potential \( V(\phi) \). As one can see, \( A(\phi) \) appears in the action of the matter leading to a direct coupling between the matter and the scalar field. This is true only in the Einstein frame, though, while in the physical Jordan frame no direct coupling is present and thus the weak equivalence principle is satisfied.

The Jordan and the Einstein frame metrics, \( g_{\mu\nu} \) and \( \hat{g}^*_{\mu\nu} \) respectively, are connected via a conformal transformation \( g_{\mu\nu} = A^2(\phi) \hat{g}^*_{\mu\nu} \) and the relations between the scalar field in the two frames is given by \( \Phi = A^{-2}(\phi) \), where \( \Phi \) is the Jordan frame scalar field. The energy-momentum tensor transformation between both frames is given by the relation \( T^*_{\mu\nu} = A^2(\phi) T_{\mu\nu} \), where \( T^*_{\mu\nu} \) and \( T_{\mu\nu} \) are the Einstein and the Jordan frame ones, respectively. A detailed discussion about the two frames and the transformation between both of them in the context of neutron star physics can be found for example in Doneva et al. (2013), Yazadjiev et al. (2016).

We will adopt the following conformal factor and potential (Yazadjiev et al. 2016; Staykov et al. 2018)

\[
A(\phi) = e^{\frac{1}{2} \sqrt{\phi^2}}, \quad V(\phi) = 2m^2\phi^2 + \lambda \phi^4.
\]
This potential is the simplest one leading to a massive self-interacting scalar field. The first term in the potential $V(\phi)$ is the standard massive term, considered in previous studies of massive STT (Popchev 2015; Ramazanoğlu and Pretorius 2016; Yazadjiev et al. 2016; Doneva and Yazadjiev 2016) while the second term describes the self-interaction of the scalar field (Staykov et al. 2018; Popchev et al. 2018).

We will employ the slow rotation approximation in first order of the angular velocity $\Omega$. This approximation is suitable for our purposes, because it allows us to study with good accuracy models rotating with frequency up to about a few hundred Hz, which covers the majority of the observed pulsars. In addition we consider stationary and axisymmetric spacetime as well as stationary and axisymmetric scalar field and matter configurations. The Einstein frame spacetime metric in this case can be written in the following form

\begin{equation}
\begin{aligned}
ds^2 &= -e^{2\phi}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \\
&\quad - 2\omega(r, \theta)r^2 \sin^2 \theta d\phi dt.
\end{aligned}
\end{equation}

Within this approximation the metric function $\omega$ is of linear order of $\Omega$ while the rotational corrections to the other metric functions, the scalar field, the fluid energy density and the pressure are of order $O(\Omega^2)$.

The dimensionally reduced Einstein frame field equations in slow rotation approximation, derived from the action (1) and using the metric (3), and the equation for hydrostatic equilibrium are the following

\begin{equation}
\begin{aligned}
\frac{1}{r^2} \frac{d}{dr} \left[ r(1 - e^{-2\Lambda}) \right] &= 8\pi GA^4(\phi)\rho + e^{-2\Lambda}\left( \frac{d\phi}{dr} \right)^2 + \frac{1}{2} V(\phi), \\
\frac{2}{r} e^{-2\Lambda} \frac{d\phi}{dr} - \frac{1}{r^2}(1 - e^{-2\Lambda}) &= 8\pi GA^4(\phi)\rho + e^{-2\Lambda}\left( \frac{d\phi}{dr} \right)^2 - \frac{1}{2} V(\phi), \\
\frac{d^2\phi}{dr^2} + \left( \frac{d\phi}{dr} - \frac{dA}{dr} + \frac{2}{r} \right) \frac{d\phi}{dr} &= \frac{4\pi GA(\phi)A^4(\phi)(\rho - 3\phi)e^{2\Lambda}}{4} + \frac{1}{4} \frac{dV(\phi)}{d\phi} e^{2\Lambda}, \\
\frac{e^{\phi - A}}{r^4} \frac{d}{dr}\left[ e^{-(\phi + A)} r^4 \partial_r \bar{\omega} \right] + \frac{1}{r^2 \sin^3 \theta} \partial_{\theta} \left[ \sin^3 \theta \partial_{\theta} \bar{\omega} \right] &= 16\pi GA^4(\phi)(\rho + p) \bar{\omega}, \\
\frac{dp}{dr} &= -(\rho + p) \left( \frac{d\phi}{dr} + \alpha(\phi) \frac{d\phi}{dr} \right),
\end{aligned}
\end{equation}

where the function $\bar{\omega}$ is defined as $\bar{\omega} = \Omega - \omega$, and the coupling function $\alpha(\phi)$ is given by $\alpha(\phi) = \frac{2ln A(\phi)}{d\phi/d\psi}$. The hydrodynamical quantities that enter in the above field equations are the Jordan frame pressure $p$ and energy density $\rho$. They are related to the Einstein frame ones, $\rho_e$ and $p_e$, in the following way $\rho_e = A^4(\phi)\rho$.

The above system of equations (4), supplemented with the equation of state for the stellar matter and the appropriate boundary conditions, describes the interior and the exterior spacetime of a neutron star in massive scalar-tensor theories, and it is used for deriving the background solutions used in this study. The exterior space-time of a neutron star is described by the system (4), by setting $\rho = p = 0$.

The regularity of the metric functions and the scalar field at the center of the star leads to the following boundary conditions $A(0) = 0$, and $\frac{d\phi}{dr}(0) = 0$. The energy density at the center is $\rho(0) = \rho_c$, where $\rho_c$ is a constant input parameter. From the requirement for asymptotic flatness at infinity we have $\lim_{r \to \infty} \phi(r) = 0$, $\lim_{r \to \infty} \psi(r) = 0$ (see e.g. Yazadjiev et al. 2014). The coordinate radius $r_S$ of the star in the Einstein frame is determined by the standard condition $\rho(r_S) = 0$. The physical Jordan frame radius $R_S$ on the other hand is related to $r_S$ in the following way $R_S = A(\phi(r_S))r_S$.

As one can see, in the slow rotation approximation the field equation for $\bar{\omega}$ is decoupled from the rest of the field equations in (4). We can further simplify it by separating the angular and the radial dependence (see e.g. Yazadjiev et al. 2016) reaching the following equation for the radial part of $\bar{\omega}$:

\begin{equation}
\frac{e^{\phi - A}}{r^4} \frac{d}{dr} \left[ e^{-(\phi + A)} r^4 \frac{d\bar{\omega}}{dr} \right] = 16\pi GA^4(\phi)(\rho + p) \bar{\omega}.
\end{equation}

The natural boundary condition for $\bar{\omega}$ to ensure its regularity at the center of the star is $\frac{d\bar{\omega}}{dr}(0) = 0$, and at infinity we require that $\lim_{r \to \infty} \bar{\omega} = \Omega$.

In the numerical calculations we are using the dimensionally reduced parameters $m_\psi \to m_\psi R_0$ and $A \to A R_0^2$, where $R_0 = 1.47664$ km is one half of the solar gravitational radius.

## 2 ISCO, orbital and epicyclic frequencies

Here, we will briefly present the basic steps in the derivation of the equations for the radius of ISCO, the radial and vertical epicyclic frequencies and for the orbital frequency (Ryan 1995; Shibata and Sasaki 1998; Pappas and Apostolatos 2012; Maselli et al. 2015). For simplicity, the equations will be given using the general form of the metric describing rotating compact objects in the Jordan frame

\begin{equation}
\begin{aligned}
ds^2 &= g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + 2g_{t\phi}dt d\phi + g_{\phi\phi}d\phi^2,
\end{aligned}
\end{equation}

where all the metric functions depend only on the coordinates $r$ and $\theta$. The connection to the Einstein frame metric,
where the background solutions are calculated, is discussed in the previous section.

Massive particles in gravitational field move on timelike geodesics of the Jordan frame metric (6). The stationary and axial Killing symmetries of the metric, generated by the Killing vectors \( \frac{\partial}{\partial \theta} \) and \( \frac{\partial}{\partial \phi} \), give rise to two constants of motion, namely \( E = -u_t \) which corresponds to the energy per unit mass and \( L = u_\phi \) which corresponds to the angular momentum per unit mass. \( u^\mu = \dot{x}^\mu / d\tau \) is the four-velocity of the particle. One can easily show that the conservation laws can be rewritten in the form

\[
\frac{dt}{d\tau} = \frac{E g_{\phi\phi} + L g_{t\phi}}{g^2},
\]

\[
\frac{d\phi}{d\tau} = -\frac{E g_{\phi\phi} + L g_{t\phi}}{g^2},
\]

where \( g^2 = g_{\phi\phi}^2 - g_{tt} g_{\phi\phi} \) is defined for simplicity. The normalization condition for the four-velocity \( g^\mu u_\mu u_\nu = -1 \), give us

\[
g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + E^2 U(r, \theta) = -1,
\]

where we have defined

\[
U(r, \theta) = g_{\phi\phi} + 2l g_{\phi\phi} + l^2 g_{tt},
\]

and \( l = L/E \) is the proper angular momentum.

In the equatorial plane, \( \theta = \pi/2 \), the problem reduces to one dimensional problem with an effective equation of motion

\[
\dot{r}^2 = V(r),
\]

and an effective potential

\[
V(r) = g_{rr}^{-1} \left[ -1 - E^2 U \left( r, \theta = \frac{\pi}{2} \right) \right].
\]

For given \( E \) and \( L \) of the particle, the stable circular orbit with a radius \( r_c \) is determined by the conditions \( V(r_c) = 0 = V'(r_c) \) and \( V''(r_c) > 0 \), where the derivative with respect to \( r \) is denoted with a prime. The radius of ISCO is located at the point where the second derivative of the potential vanishes - \( V''(r_c) = 0 \). The angular velocity \( \Omega_p \) of a particle moving on a circular orbit in the equatorial plane can be found from the geodesic equation written in the form

\[
\frac{d}{d\tau} \left( g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) = \frac{1}{2} \partial_\mu g_{\sigma\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau}.
\]

For the radial coordinate this equation translates into

\[
\partial_r g_{tt} \left( \frac{dt}{d\tau} \right)^2 + 2 \partial_r g_{t\phi} \frac{dt}{d\tau} \frac{d\phi}{d\tau} + \partial_r g_{\phi\phi} \left( \frac{d\phi}{d\tau} \right)^2 = 0,
\]

from which, by taking into account the definition for the angular velocity \( \Omega_p = \frac{d\phi}{dt} = \frac{df}{d\tau} \), we obtain

\[
\Omega_p = \frac{d\phi}{dt} = -\frac{\partial_r g_{t\phi} \pm \sqrt{(\partial_r g_{t\phi})^2 - \partial_r g_{tt} \partial_r g_{\phi\phi}}}{\partial_r g_{\phi\phi}}.
\]

The positive sign in the above equation corresponds to prograde orbits and the negative sign to retrograde ones. In the present work, we are considering only prograde orbits.

To derive the epicyclic frequencies one should investigate small perturbations in radial and in vertical direction of a stable orbit. The radial and the vertical perturbations we write in the form

\[
r(t) = r_c + \delta r(t), \quad \theta(t) = \frac{\pi}{2} + \delta \theta(t),
\]

where \( \delta r(t) \) and \( \delta \theta(t) \) are the perturbations to the stable circular orbit with coordinate radius \( r_c \) in the equatorial plane (\( \theta = \pi/2 \)). The perturbations could be written explicitly in the form \( \delta r(t) \sim e^{2\pi i \nu_r t} \) and \( \delta \theta(t) \sim e^{2\pi i \nu_\theta t} \). By substituting (16) into Eq. (9) one can obtain the expressions for the radial and the vertical epicyclic frequencies:

\[
v_r^2 = \frac{(g_{tt} + \Omega_p g_{t\phi})^2}{2(2\pi)^2 g_{rr}} \delta_r^2 U \left( r_c, \frac{\pi}{2} \right)
\]

\[
v_\theta^2 = \frac{(g_{tt} + \Omega_p g_{t\phi})^2}{2(2\pi)^2 g_{\theta\theta}} \delta_{\theta}^2 U \left( r_c, \frac{\pi}{2} \right).
\]

For static neutron stars the orbital frequency and the vertical epicyclic frequency coincide, i.e. \( \nu_\theta = \nu_r \). At ISCO the radial epicyclic frequency is equal to zero, and for smaller radii it is negative, which shows a radial instability for orbits with a radius smaller than ISCO.

## 3 Numerical results

The observational constraint on the parameter \( \beta \) in the massless scalar field case is \( \beta > -4.5 \) (Demorest et al. 2010; Antoniadis et al. 2013). As we have already commented, the picture changes considerably in the case of nonzero scalar field mass. If the mass of the scalar field is in the range

\[10^{-16} \text{ eV} \lesssim m_\phi \lesssim 10^{-9} \text{ eV},\]

which roughly corresponds to \( 10^{-6} \lesssim m_\phi \lesssim 10 \) in our dimensionless units, no constrains can be placed on the parameter \( \beta \) from the binary pulsar observations. Thus, we are left with the constraint \( 3 \lesssim -\beta \lesssim 10^3 \), which is significantly wider compared to the massless case. As long as the self-interaction constant \( \lambda \) is concerned, the only constraint that we have is that it should be positive in order for the potential to be positive, and we restrict ourself to the values
used in Staykov et al. (2018). The motivation behind these constraints have been thoroughly discussed in Ramazanoğlu and Pretorius (2016), Yazadjiev et al. (2016), Staykov et al. (2018) and we refer the reader to these papers for more details.

In this study we used one of the popular realistic EOS with maximal mass and typical radii in agreement with the observations (Lattimer 2012; Özel 2013), namely the SLy EOS (Douchin and Haensel 2001) and more precisely their piecewise polytropic approximation (Read et al. 2009). As a matter of fact the calculations were performed for a second modern EOS, the APR4 EOS (Akmal et al. 1998), and we have obtained qualitatively similar results. Since the system of reduced field equations (4) has a three parameter ($\beta$, $m_\phi$ and $\lambda$) family of solutions, we have chosen to present results only for the SLy EOS in order to avoid unnecessary complication of the presented graphs.

We have studied both static models with $f = 0$ Hz and models rotating with frequency $f = 160$ Hz that fall into the validity of the slow rotation approximation, where $f$ is the rotational frequency of the star ($f = \Omega^2 \pi$). However, for such values of the frequency the effect of rotation on the studied parameters, except for the difference between the orbital frequency and the vertical epicyclic frequencies, is very small. That is why in most of the cases we will present only the static case results while the effect of the rotation will only be discussed mainly concerning the difference between the orbital frequency and the vertical epicyclic frequency.

### 3.1 Massive scalar-tensor theory

We start our study with the simpler case of STT with a massive scalar term and no self interaction. Since we have a three parameter ($\beta$, $m_\phi$ and $\lambda$) family of solutions, this will help the presentation and the understanding of the results with self-interaction in the next subsection.

In Fig. 1 we study the massless STT but for values of the parameter $\beta$ that are outside the observational limit $\beta > -4.5$ in the massless scalar field case. The reason is that typically the differences between the neutron star solutions in GR and in massive STT increase with the decrease of the mass of the scalar field $m_\phi$. Thus, the STT with $m_\phi = 0$ represent the maximum deviation from Einstein’s theory of gravity one can have in massive STT for a fixed $\beta$. In the top panel of Fig. 1 we plot the radius of ISCO as a function of the mass for massless STT with different values for the parameter $\beta$ (in different patterns and colours). Bottom panel: The orbital frequency as a function of the mass for massless STT. The same notations as in the top panel are used.

![Fig. 1](image-url)  
Top panel: The radius of ISCO as a function of the mass for massless STT with different values for the parameter $\beta$ (in different patterns and colours). Bottom panel: The orbital frequency as a function of the mass for massless STT. The same notations as in the top panel are used.

Fig. 1 Top panel: The radius of ISCO as a function of the mass for massless STT with different values for the parameter $\beta$ (in different patterns and colours). Bottom panel: The orbital frequency as a function of the mass for massless STT. The same notations as in the top panel are used.

In the top panel of Fig. 1 we plot the radius of ISCO as a function of the mass of the star. When ISCO is inside the star, i.e. all circular orbits in the exterior space are stable, we plot the radius of the star instead. The transitions between the two regimes forms a cusp. It is interesting to point out that with the decrease of the parameter $\beta$, the mass, at which ISCO is equal to the radius of the star, increases significantly compared to the GR one. In the same time the slope of the part of the graph, with ISCO bigger than the radius of the star, became steeper with the decrease of $\beta$. For high values of $\beta$ close to the observational limit of $-4.5$, the mass at which ISCO is equal to the neutron star radius, slightly decreases compared to the GR one, while for smaller $\beta$ it starts to rapidly increase. In the bottom panel of Fig. 1, we plot the orbital frequency at ISCO as a function of the mass. If ISCO is inside the star and all circular orbits above the stellar surface are stable, we calculate the orbital frequency at the surface of the star. Thus, similar to the top panel a cusp on the graphs appear. As one can see large deviations from GR are observed only for $\beta$ much smaller compared to the observational constraint in agreement with previous studies (DeDeo and Psaltis 2004; Doneva et al. 2014). This shows once again that the inclusion of a mass of the scalar field can lead to very interesting observational consequences compared to the massless case.
In Fig. 2 we go one step further and plot the ISCO and the orbital frequency for models in STT with nonzero scalar field mass. The results are calculated for \( \beta = -10 \) and using some representative values for the \( m_\phi \). As one can see, for large scalar field mass, ISCO becomes equal to the stellar radius for models with masses lower than the GR one, which allows for quite significant deviations from GR both in ISCO and in the orbital frequency. In addition, for larger \( m_\phi \) the radius of ISCO is larger and \( \nu_r \) is smaller than the GR case while the situation changes for small or zero \( m_\phi \). Thus, inclusion of a scalar field mass leads to qualitative differences from the massless case. For the presented in the figure values for the parameters, the radius of ISCO increases with more than 60%, compared to GR, and the orbital frequency decreases with more than 40% compared to GR. We find these models very interesting because they can have significant deviations from GR and allow for direct comparison with GR. There are models with ISCO in the same mass interval in both GR and the modified theory which have significant deviations between each other. This is not the case for the massless or the pure massive theory in which direct comparisons between models with the same mass as in GR are not possible.

### 3.2 Massive scalar-tensor theory with self-interaction

In this section we proceed to adding the final ingredient, i.e. presenting the results in the case of massive STT with self-interaction. We will concentrate on two values for the parameter \( \beta \), namely \( \beta = -6 \) and \( \beta = -10 \), and several different representative combinations of values for the scalar field mass and for the self-interaction constant. In Fig. 3 we plot the radius of ISCO as a function of the neutron star mass. In the top panel we plot models with \( \beta = -6 \), and in the bottom \( \beta = -10 \). As one can see, with the increase of the mass of the scalar field or the self-interaction constant \( \lambda \), the results converge to the GR ones. The deviations from GR in the top panel are quite small for the chosen values of the parameters (less than 6%), and in the bottom panel, due to the smaller value for \( \beta \), the deviations are more significant (up to 40%). This result is expected when compared to the behavior of the equilibrium solution properties (Staykov et al. 2018). However, it is interesting to point out that for a fixed nonzero value of \( \lambda \) the qualitative behavior can change. For example for \( \lambda = 0.1 \), one can clearly see in the bottom panel (\( \beta = -10 \)) that the deviations with respect to GR are larger for larger values of \( m_\phi \) contrary to the \( \lambda = 0 \) case in Fig. 2. Moreover, as the results in Staykov et al. (2018) show, the deviations in the neutron star mass and radius from the GR values decrease more or less monotonically with the increase of \( \lambda \) and \( m_\phi \). One of the reasons for this behavior is that the neutron star masses for which the radius of ISCO becomes equal to the neutron star radius can differ a lot for models with different \( \lambda \) and \( m_\phi \).

In Fig. 4 we plot the orbital frequency as a function of the neutron star mass. The point at which ISCO is equal to the neutron star radius is marked by a cusp on the graphs. As one can see, for lower neutron star masses where ISCO is inside the star and all of the orbits outside the star are stable, the frequency is higher compared to the GR one. For more massive models where ISCO is outside the star, however, the frequency is generally lower compared to the GR one. As one can expect by the results for the ISCO presented in Fig. 3, for beta \( \beta = -6 \) the frequency increases with the increase of the mass of the filed, as well as with the increase of the coupling constant \( \lambda \). The maximal deviation is about 4% in this case. For \( \beta = -10 \)
the models with higher mass of the scalar field show larger deviations (up to about 35%). In this case as well the results converge to GR with the increase of $\lambda$.

In Fig. 3 we present the maximal radial epicyclic frequency as a function of the mass. If the radial frequency does not have a maximum outside of the star, we plot the frequency at the stellar surface. In this case as well the frequency is always lower compared to the GR one (with maximal deviations roughly 10% for $\beta = -6$ and 50% for $\beta = -10$). One can see that $\nu_{r \text{max}}$ has similar qualitative behavior to the rest of the quantities presented in this section.

All of the previous results were in the nonrotating case. As we commented, the reason is the small effect the slow rotation has on the radius of ISCO, the orbital frequency and the radial epicyclic frequency. The radius of the ISCO, for example, shrinks with only about 4% for a frequency $f = 160$ Hz. The deviations in the orbital frequency and the radial epicyclic frequency are of the same magnitude. The only quantity that should be calculated in the rotating case is the nodal precession frequency defined as the difference between the orbital frequency and the vertical epicyclic frequency $\nu_n = \nu_p - \nu_\theta$. In the nonrotating limit $\nu_p = \nu_\theta$ and thus $\nu_n = 0$ which makes $\nu_n$ strongly dependent on the rotational rate of the star.

In Fig. 6 we plot the nodal precession frequency as a function of the neutron star mass for the same combinations of $\beta$ and $m_\phi$ as in the previous section. If all circular orbits are stable and ISCO is inside the star, the nodal precession frequency at the surface of the star is plotted while in the case when ISCO is above the stellar surface, that happens for larger neutron star masses, $\nu_n$ at ISCO is shown. Both regimes are clearly separated by a cusp in the graphs. Due to the fact that the nodal precession frequency is zero in the
nonrotating case, we have performed the calculations of $\nu_n$ for a nonzero rotational rate, more precisely $f = 160$ Hz for which the slow rotation approximation is still satisfied. In both panels one can see that if $\nu_n$ is calculated on the surface of the star, the frequencies in GR are larger than for STT. Contrary, if the nodal precession frequency is calculated at ISCO, the GR values are lower than the STT cases. In the bottom panel (for $\beta = -10$), one can see the non-monotonic behavior of the deviation from GR with the increase of the parameters $\lambda$ and $m_\phi$, which we have already discussed above. The deviations in the top panel are below 2% for all combinations of the parameters that are plotted while in the bottom panel they are of the order of 10%.

In the end of the discussion of the numerical results an additional comment concerning the figures should be made. For some of the examined models, see (Yazadjiev et al. 2016) and (Staykov et al. 2018), the scalarized branch can have the maximum mass between the bifurcation points, which as it is known, marks the transition point between stable and unstable models. In the figures above we did not mark in a different way the unstable part of the scalarized branch (if it is present) and one should have in mind this when studying the figures.

### 4 Conclusion

The quasi-periodic oscillations observed in the X-ray flux of some pulsars are still not well explained phenomenon which source is supposed to originate from the vicinity of the compact object. Although there are multiple models explaining the QPOs, the majority of those models incorporate in one way or another the orbital and the epicyclic frequency of a particle moving in a circular orbit around the central object. Due to the fact the above mentioned frequencies are
based on geodesic motion in orbits close to ISCO, they are closely related to the space-time geometry around the compact object. In this context, examining those frequencies in modified gravity can give us a clue about the possibility to test and restrict gravitational theories using electromagnetic signals originating from the vicinity of the compact object.

In this paper we studied the radius of ISCO, the orbital and the epicyclic frequencies of a particle moving on a circular orbit around neutron stars in massive STT and massive STT with self-interacting scalar field. In both cases, significant deviations from GR are allowed for values of the parameters in correlation with observations. Our results show that the radius of ISCO is always bigger than the corresponding one in GR, and the orbital and the epicyclic frequencies are always lower compared to the GR ones. For the examined set of values for the free parameters in the theories we found that the maximal deviation from GR for ISCO in STT with self-interaction is up to 40%. The orbital frequency is lower than the GR one with about 35%, and the maximal radial frequency decreases from GR with about 50%. We should note that we have chosen moderate values of the parameters of the theory and larger deviations are in general possible.

We conducted our numerical study in the so-called slow rotation approximation, which covers the rotation rates of most of the observed pulsars. In this approximation for rotational frequency $f = 160$ Hz (the highest one we have studied) the radius of ISCO decreases with about 4% compared to the static case for GR and the modified theories with all combinations of parameters. The deviations from the static case for the rest of the studied quantities are of the same magnitude. This is considerably smaller than the possible deviations due to modifications of Einstein’s theory which means that even the results in the static case can be used to put constraints on the parameters of the theory.

It is worth mentioning that for the case with lower value of $\beta$ we studied ($\beta = -10$) we observed a non-monotonic behavior in the deviation from GR with some of the parameters. This is a direct result from the fact that the mass of the models, at which ISCO is equal to the neutron star radius, changes with the parameters in the theory in non-monotonic manner.

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