Self-Dual Connections and the Equations of Fundamental Fields in a Weyl–Cartan Space

V. V. Kassandrov* and J. A. Rizcallahb

aInstitute of Gravitation and Cosmology, Peoples’ Friendship University of Russia, Moscow, 117198 Russia
bSchool of Education, Lebanese University, Beirut, Lebanon
*e-mail: vkassan@sci.pfu.edu.ru

Abstract—Spaces with a Weyl-type connection and torsion of a special type that are determined by the structure of the differentiability conditions in the algebra of complex quaternions are considered. These conditions are consistent only if the curvature of the connection is self-dual. The Maxwell and Yang–Mills fields associated with the irreducible components of the connection also turn out to be self-dual, so that the corresponding equations are fulfilled on the solutions of the generating system. Using the twistor structure of the latter, its general solution is obtained. The singular locus has a string-like (particle-like) structure generating the self-consistent algebraic dynamics of the string system.

DOI: 10.1134/S1063779618010203

The most general geometrodynamical approach to constructing the fundamental dynamics of fields/particles in practice proves to be unproductive due both to the absence of clear criteria for selecting the very geometry and the arbitrary choice of a Lagrangian. On the other hand, building the theory on some exceptional algebraic structure, primarily of the quaternion type, we can hope to uniquely determine the space-time geometry induced by it and to derive the equations of fundamental quaternion fields from the intrinsic properties of the original algebra alone.

Such an approach, proposed in [1, 2] (for more recent work, see [3]), is based on the statement of differentiability condition for functions of an algebraic ($\mathbb{A}$-) variable; these generalize the Cauchy–Riemann conditions, known in complex analysis, to the case of an associative but noncommutative algebra $\mathbb{A}$. Such conditions are stated in a componentless form invariant with respect to multiplication ($\cdot$) in the algebra $\mathbb{A}$

$$dF = \Phi dZ \Psi,$$  

(1)

where $F = F(Z) \in \mathbb{A}$ is an $\mathbb{A}$-valued function of variable $Z \in \mathbb{A}$; $\Phi = \Phi(Z), \Psi = \Psi(Z)$ are auxiliary $\mathbb{A}$-valued functions (“semiderivatives” of the main function $F(Z)$); and $dF$ is the linear part of the increment (differential) of $F(Z)$ corresponding to the increment of the argument $dZ$.

In previous works, the algebra of complex quaternions (biquaternions) $\mathbb{B}$, which is isomorphic to the full algebra of $2 \times 2$ matrices over $\mathbb{C}$, was used as a space-time algebra $\mathbb{A}$. However, the $4\mathbb{C}$-coordinate space of $\mathbb{B}$ was reduced to the subspace $\mathbf{M}$ of Hermitian matrices $Z \mapsto X = X^\dagger$ with the Minkowski metric, so that the entire construction turned out to be Lorentz-invariant. The physical fields associated with differentiable $\mathbb{B}$-functions remain $\mathbb{C}$-valued. In addition, the main class of solutions of (1) corresponds [2] to the case of the equality $\Psi(Z) = F(Z)$ (or $\Phi(Z) = F(Z)$); therefore, the $\mathbb{B}$-differentiability conditions, which play the role of primary field equations, finally take the form

$$dF = \Phi dXF.$$  

(2)

In the nontrivial case of $\mathbb{B}$-functions (matrices) with $\det F = 0$, we decompose (2) by columns to obtain the so-called generating system of equations (GSE)

$$d\xi = \Phi dX \xi$$  

(3)

for the 2-spinor field $\xi = \{\xi_a(X)\}$ and the 4-vector field $\Phi = \{\Phi_{AA}(X)\}$. Here, $A, B, \ldots, A', B' \ldots = 0, 1$, and the sign of $\mathbb{B}$-multiplication (matrix multiplication) is omitted.

It is precisely the system of equations (3), which is purely algebraic in nature that was used to construct algebrodynamics, that is, the dynamics of fields and
the corresponding singularities, being particle-like formations. We give below a number of the most important properties and consequences of system (3), referring to the authors’ works for the proofs (see references in [3]).

By eliminating the field $\Phi(X)$, we reduce the GSE to the system of equations of shear-free null congruences (SFCs) $\xi^A \partial_{AA} \xi_C = 0$. Similar to the SFC equations, we derive the general solution of the original GSE as

$$\Pi^{(C)}(\xi^A, \tau_A) = \Pi^{(C)}(\xi^A, X_{\xi^A} \xi^A) = 0, C = 1, 2,$$  \hspace{1cm} (4)

where $\Pi^{(C)}$ is a pair of arbitrary (holomorphic) functions of the twistor argument with 4 complex (spinor) components related by the twistor incidence relation $\tau_A = X_{\xi^A} \xi^A$. At each point $X \in \mathbf{M}$, solving system (4) with respect to $\xi^A$, we arrive at the 2-spinor field $\xi(X)$ (generally multivalued), each continuous branch of which satisfies both fundamental relativistic equations: the complex eikonal equation $\partial_{AA} \xi_C \partial_{AA} \xi^C = 0$ (for each spinor component) and the wave equation $\Box G = 0$ (for the quotient of components $G = \xi_1/\xi_0$).

Note that (4) is an invariant generalization of the so-called Kerr theorem for describing SFCs.

As for $\Phi(X)$, it is essentially a gauge field, since the GSE is form-invariant under transformations of the form

$$\xi \mapsto \alpha \xi, \Phi_{AA} \mapsto \Phi_{AA} - \partial_{AA} \alpha,$$  \hspace{1cm} (5)

where the gauge parameter $\alpha = \alpha(\xi, X\xi)$ depends on $X$ only through the components of the transformed twistor $W = (\xi, X\xi)$ (the so-called weak gauge invariance [4]).

Moreover, the compatibility conditions of the overdetermined GSE (3) $d\xi = 0 = R^C_{\xi^C} R = (\Phi dX \Phi) \wedge dX$ imply the self-duality of the curvature 2-form of the effective connection $\Omega = : = \Phi dX \Phi$ on the GSE solutions (the so-called weak self-duality [4]). Against the background of a space with the Minkowski metric, the connection $\Omega$ already possesses nonmetricity (of the Weyl type) and torsion of a specific form [4]. The very GSE equations (in form (2)) can be considered as the equations of covariantly constant fields in the corresponding space with the Weyl–Cartan connection [2] $dF = \Omega F$. Note that covariantly constant fields in Weyl–Cartan affine connection spaces of different types can be generally used for interesting geometric interpretations of electromagnetism [5].

Because of the self-duality of the curvature of the $GL(2, \mathbb{C})$ connection $\Omega$ on the GSE solutions, the equations of source-free gauge fields are fulfilled: the Maxwell equations for the trace part of the curvature and the $SL(2, \mathbb{C})$ Yang–Mills equations for the traceless one [6]. The associated electromagnetic and Yang–Mills fields are singular at the points

$$P_\tau = \det \left| \frac{d\Pi^C}{d\xi^A} \right| = 0,$$  \hspace{1cm} (6)

which correspond to the multiple roots of Eq. (4) for the spinor $\xi^A$. If the corresponding singular subsets are bounded in the three-dimensional space, these can be considered as particle-like formations, which even possess some properties of quantum particles. For example, an electric charge is a multiple of the minimum (elementary) charge of a Kerr–Newman singular ring. Solutions with electrically neutral singularities have also been found.

From systems (4) and (6) it follows that such formations have generally the nature of closed strings. Thus, the system of algebraic equations (4), (6), which follows from the GSE, determines the nontrivial algebraic dynamics of a system of closed strings on $\mathbf{M}$.

In a particular case, such a system degenerates into a locus of singular points on a single worldline, in the spirit of the well-known Wheeler–Feynman concept of the one-electron Universe. Such collective algebraic dynamics has been studied in detail in [7]. At least in the case of an arbitrary polynomial worldline, it turns out to be conservative: a set of Lorentz-invariant conservation laws is fulfilled. There are also other universal physically interesting properties of the dynamics of the roots of the original algebraic system, including their (asymptotic) merging and clustering. The more complicated general case of string algebraic dynamics is yet to be studied.

REFERENCES

1. V. V. Kassandrov, *Algebraic Structure of Space–Time and Algebroadynamics* (Moscow: izd. Universiteta Druzhby Narodov, 1992) [in Russian].
2. V. V. Kassandrov, Grav. Cosmol. 1, 216 (1995), V. V. Kassandrov, Acta Applic. Math. 50, 197 (1998).
3. V. V. Kassandrov, Phys. Atom. Nucl. 72, 813 (2009), V. V. Kassandrov, in *Space–Time Structure. Algebra and Geometry*, Ed. by D. G. Pavlov, Gh. Atanasiu, and V. Balan (Moscow: Lilia Print, 2007), p. 441.
4. V. V. Kassandrov and J. A. Rizcallah, Grav. Cosmol. 22, 230 (2016), V. V. Kassandrov and J. A. Rizcallah, arXiv: gr-qc/0012109.
5. V. V. Kassandrov and J. A. Rizcallah, Gen. Rel. Grav. 46, 1772 (2014), V. V. Kassandrov and J. A. Rizcallah, Grav. Cosmol. 21, 273 (2015).
6. V. V. Kassandrov and J. A. Rizcallah, Int. J. Geom. Methods Mod. Phys. 14, 1750031 (2017), arXiv:1612.06718 [gr-qc].
7. V. V. Kassandrov, I. Sh. Khasanov, and N. V. Markova, J. Phys. A: Math. Theor. 48, 395204 (2015).

Translated by N. Berestova

PHYSICS OF PARTICLES AND NUCLEI Vol. 49 No. 1 2018