Extending the Epoch of Reionization window with apt Foreground and Instrument modeling

RAUT DINESH

1 1079 Shukrawar Peth, Pune 411002, India

ABSTRACT

It is seen that foregrounds of 21cm Epoch of Reionization experiments, which are expected to have smooth spectral dependence, are dominant in a wedge shaped region of the Fourier space called as Foreground Wedge. A possible way forward to isolate the 21cm Epoch of Reionization (EoR) signal from the much larger foreground component is to focus on the remaining portion of the Fourier space called as the EoR window. There are in-fact three distinct regions in the Fourier space, (i) the Foreground wedge portion, (ii) the EoR window region which lies beyond Horizon wedge and (iii) the region in between the Horizon wedge and the Field of View wedge. This paper addresses two questions: (1) Whether the signal in between the two wedges is also a direct representation of the EoR brightness temperature fluctuations? and (2) How can we extract the cosmology information from this region considering that the foregrounds are much larger than the signal? The answer to the first question is yes, the visibilities are within a few percent, same as the EoR brightness temperature fluctuations for the cases considered. Secondly, the in between region is not foreground free, due to non-negligible sidelobes. But, one can possibly extract signal from this region if one models foregrounds and instruments accurately. As all the three components, the cosmological signal, the foregrounds and the thermal noise are calculated in the same space, the analysis suggested could be more straightforward.

Keywords: reionization, cosmology — large scale structure

1. INTRODUCTION

Epoch of Reionization is considered as one of the prime frontiers of observational cosmology. Evolution of the universe during this epoch can be studied through the redshifted 21 cm line signal originating from the neutral hydrogen (HI) in the intergalactic medium (IGM) (for reviews, see Fan et al. 2006; Furlanetto et al. 2006; Choudhury 2009; Pritchard & Loeb 2012). The observation of this cosmological signal is hindered due to the presence of much brighter astrophysical foregrounds (see, e.g., Furlanetto et al. 2006; Jelić et al. 2010) which are expected to dominate the HI signal by a factor of $10^4$-$10^5$ (Di Matteo et al. 2002; Oh & Mack 2003; Di Matteo et al. 2004; Ali et al. 2008). The foregrounds are expected to be smooth functions of frequency while the signal, as it traces HI distribution in the universe, is expected to oscillate rapidly with respect to the frequency. This principle has been used by experiment EDGES 1 (Bowman et al. 2018) in their analysis for extracting the global signal. To detect the fluctuating signal two different approaches has been proposed. One can take out the foreground signal by carefully modelling its frequency dependence (Santos et al. 2005; Bowman et al. 2006; Wang et al. 2006; Gleser et al. 2008; Liu et al. 2009a,b; Harker et al. 2009, 2010; Liu & Tegmark 2011; Petrovic & Oh 2011; Chapman et al. 2012; Bonaldi & Brown 2015; Ghosh et al. 2015). This approach has been called as Foreground Subtraction or Foreground Removal. It has been used to detect the 21cm signal by LOFAR (Low Frequency Array) 2 (Patil et al. 2017; Mertens et al. 2018; Morales et al. 2019; Gehlot et al. 2019). One can also measure the signal in the Fourier domain by focusing on the region where signal is expected to be much larger than the Foreground component (Datta et al. 2010; Vedantham et al. 2012; Morales et al. 2012; Trott et al. 2012; Parsons et al. 2012b; Pober et al. 2013; Hazelton et al. 2013; Liu et al. 2014a,b; Thya-garajan et al. 2015). The foregrounds, being smooth, tend to dominate in a wedge-shaped region in the $k_\parallel$-$k_\perp$ space, and hence the remaining portion of $k$-space ends up being nearly foreground free and is called as EoR window. Here $k_\parallel$ and $k_\perp$ are magnitudes of the Fourier modes in the directions parallel and perpendicular to the line of sight (LOS). This method, called as foreground avoidance, is already be-

---

1 https://www.haystack.mit.edu/ast/arrays/Edges/

2 http://www.lofar.org
ing used by the experiments like PAPER (Precision Array for Probing the Epoch of Reionization)\(^3\) (Ali et al. 2015) and MWA (The Murchison Widefield Array)\(^4\) (Paul et al. 2016; Beardsley et al. 2016).

The extent of Foreground Wedge or EoR window is set by the geometric delay corresponding to the Horizon. If one carefully considers the foregrounds in the \(k_{\perp}-k_{\parallel}\) space, the situation is more like one depicted in the Figure 1. There are three distinct regions, the wedge shaped region that lies below the line corresponding to the FoV wedge and which is dominated by foregrounds, the EoR window that lies above the line corresponding to the Horizon wedge and which is nearly Foreground free and the region in between the two lines. Both the above mentioned telescopes, MWA and PAPER, have a very large Field of View (FoV) and do not have a very large number of baselines. So MWA and PAPER focus on shorter baselines as against SKA which has much longer baselines. So MWA and PAPER automatically end up focusing on the blue region of the Figure 1 as \(k_{\perp}\) is proportional to the baseline distance. Telescope like SKA has a smallish FoV (About 4 deg at \(z = 9.0\)) which means that the FoV wedge occupies a smaller area in the \(k_{\perp}-k_{\parallel}\) space. As \(\Omega_{\text{fov}} \approx 4.8 \times 10^{-3}\)sr is relatively smaller\(^5\) and as the number of baselines are large, the green region of the Figure 1 that lies in between the two above mentioned lines is significant and can also yield good measurements of the signal provided the foregrounds are modelled properly and telescope calibration is done accurately.

The core idea that is being presented in this paper is that instead of subtracting the foregrounds in the real space they can be modelled and subtracted out in the Fourier space. The details of this procedure is given in the fifth section while the remaining sections give quantitative footing to this idea. It is expected that with careful modelling of foregrounds and telescope elements, one can achieve a good Signal to Noise ratio in the green region of the Figure 1 and utilize that region for measuring the signal. The author is also of the opinion that it would be almost impossible to build images and do foreground subtraction in the real space unless the foreground parameters are determined at accuracy of about 1 part in 10-100 thousand. This is because to build images one needs all the Fourier modes accurately measured. If the signal is not correctly measured in large and continuous portion of Fourier space (for example the Foreground Wedge region) then it would be impossible to reconstruct it by doing an inverse Fourier transform. For a faithful construction of the signal one would have to measure it accurately all over the Fourier space. And to measure it accurately in the Foreground wedge region, one would have to constrain foreground parameters to one part in 10-100 thousand or so as the signal is weaker by a factor of 10-100 thousand.

The paper is organized as follows: In section 2, I have discussed how to accurately get the visibilities corresponding to the EoR signal from the \(k\)-space brightness temperature fluctuations. In the third section, calculations of the second section are applied to a simulated cube obtained using 21cmFAST (Mesinger et al. 2011). In the fourth section, an account of the calculations of the thermal noise in \(k\)-space is given. One is expected to compare this noise with the signal. In the fifth section, the noise associated with Foreground modelling is considered. This can again compared to the signal and thermal noise estimates. Throughout this paper, I have plotted various quantities in the \(k_{\perp}-k_{\parallel}\) space. The cosmological parameters used for this study are \(\Omega_m = 0.308, \Omega_{\Lambda} = 1 - \Omega_m, h = 0.678, \Omega_b = 0.049, \sigma_8 = 0.84\) and \(n_s = 0.966\) (Planck Collaboration et al. 2016).

2. VISIBILITIES TO 21CM BRIGHTNESS TEMPERATURE FLUCTUATIONS

The fundamental equation of radio astronomy (Thompson et al. 2017; Taylor et al. 1999) that connects the intensity distribution as a function of \(\theta\) on sky to visibilities measured as a function baselines is given by,

\[
V(U, V) = \int_{\Omega} d^2\theta A_V(\theta) I_V(\theta)e^{-2\pi i U \cdot \theta} 
\]  

Here, \(v\) is frequency, \(\Omega\) is Field of View (FoV), \(U\) is baseline distance measured in units of wavelength \(\lambda\) and \(A_V(\theta)\) is the primary beam. As far as EoR is concerned, this equation

\(^{3}\) http://eor.berkeley.edu/
\(^{4}\) http://www.mwatelescope.org/
\(^{5}\) \(\Omega_{\text{fov}} = (\pi/4)(1.3/d_{\text{station}})^2\)
\(^{6}\) http://astronomers.skatelescope.org/documents/SKA-TEL-SKO-DD-001-1_BaselineDesign1.pdf
accomplishes Fourier Transforms with respect to two out of three co-ordinates. The two co-ordinates are distances perpendicular to the LOS or the Line of Sight, \( r_\perp \) and the remaining third co-ordinate is the distance along LOS, \( r_\parallel \). For small displacements along LOS, the frequency is linearly related to the LOS distances and so a Fourier Transform of visibilities with respect to the frequency (Morales & Hewitt 2004) gives a representation of Intensity in the Fourier or \( k \)-space.

\[
V(U, \eta) = \int_B dV(U, \nu) e^{-2\pi i \nu \eta} \\
= \int_{\Omega, B} d^2 \theta d\nu A_\nu(\theta) f_\nu(\theta) e^{-2\pi i (U \theta + \nu \eta)}
\]

The quantity \( |V(U, \eta)|^2 \) is directly related to the Power Spectrum of 21cm brightness temperature fluctuations with the components of wavevector \( k_\parallel \) (parallel to LOS) and \( k_\perp \) (perpendicular to LOS) are directly proportional to the parameter \( \eta \) and baselines \( d = U \times \lambda \), respectively. As shown in the appendix the calculations yields,

\[
V(k_\perp, k_\parallel) = \frac{1}{r_\parallel D_\parallel(z)} \int d\nu d^2 r_\perp I(r_\perp, r_\perp) \exp \left[ -i \left( r_\perp \cdot k_\perp + r_\parallel k_\parallel \right) \right] \\
\times \exp \left( -2\pi i \frac{r_\perp \cdot d}{D_\parallel(z) c} \frac{r_\parallel}{r_\parallel'} \right)
\]

\[
k_\perp = \frac{2\pi}{D_\parallel(z)} U \\
k_\parallel = \frac{2\pi}{r_\parallel} \eta \\
r_\parallel' = \frac{c(1+z)^2}{H(z)} v_{21}
\]

Above equation assumes a flat beam and is used for calculating the 21cm power spectrum. This is fine as the aim is to study the effect of the additional term, \( \exp \left( -2\pi i \frac{r_\perp \cdot d}{D_\parallel(z) c} \frac{r_\parallel}{r_\parallel'} \right) \), that is arising in the expression of power spectrum. For the case of computing power arising due to Foregrounds, I have assumed an Airy pattern beam (refer Figure 14).

As seen from the exact calculations, the expression for \( V(U, \eta) \) contains a extra factor of phase which I would call as \( h \)-function.

\[
h(r_\perp, r_\parallel) \equiv \exp \left( -2\pi i \frac{r_\perp \cdot d}{D_\parallel(z) c} \frac{r_\parallel}{r_\parallel'} \right)
\]

For short baselines \( |d| \approx 0 \) the phase is almost zero and so the \( h \)-function is unity. And hence the Fourier transformed visibilities are simply Fourier transformed intensity fluctuations whose square is directly proportional to the 21cm power spectrum. For a telescope like SKA1-Low the baselines range from a low of 35 metres, which is same as a station diameter, to a maximum of about 65km. As we have an additional factor of \( h \)-function in the expression, we have,

\[
V(k_\perp, k_\parallel) = \tilde{\delta}_{21}(k_\perp, k_\parallel) \otimes \tilde{h}(k_\perp, k_\parallel)
\]

That is Fourier transformed visibilities are proportional to the convolution of \( \tilde{\delta}_{21} \), 21cm brightness temperature fluctuations in \( k \)-space and \( \tilde{h} \), the Fourier transform of \( h \)-function for fixed \( d \). Note that theoretical 21cm power spectrum would correspond to \( P_{21}(k_\perp, k_\parallel) = |\tilde{\delta}_{21}(k_\perp, k_\parallel)|^2 \) while the one obtained in this approach would correspond to \( P(k_\perp, k_\parallel) = |V(k_\perp, k_\parallel)|^2 \) (Morales et al. 2019). From above equations one has,

\[
P(k_\perp, k_\parallel) \propto |\tilde{\delta}_{21}(k_\perp, k_\parallel) \otimes \tilde{h}(k_\perp, k_\parallel)|^2
\]

In this paper, it has been proposed that instead of trying to get the 21cm brightness temperature fluctuations by going to the real space, it could be worth exploring the option of doing the complete analysis in Fourier or \( k \)-space and \( \tilde{h} \) plane. The Fourier transformed visibilities can be directly related to the 21cm brightness temperature fluctuations in the Fourier space and Foregrounds and other sources of noise can also be analyzed in the \( k_\perp - k_\parallel \) plane. \( h \)-function is only a phase, summation of its Fourier transform gives zero imaginary part and unity for real part. The \( \tilde{h} \)-function can be plotted for some baselines. Note that baseline also represent \( k_\perp \) and for \( z = 90 \), that I consider in this paper, \( d = 300m \) roughly corresponds to \( k_\perp = 0.1Mpc^{-1} \). \( k_x \) and \( k_y \) are two components of \( k_\perp \). The real and imaginary parts of \( \tilde{h} \)-function for 3 representative baselines are plotted in Figures 2, 3 and 4. The plots are for \( d_x = d_y \). As expected, the real part is even under reflection while the imaginary part is odd under reflection. In the plots, \( k_y = 0 \). It is evident from Figure 3 that the \( \tilde{h} \)-function is almost like a delta function and so loss of power for the corresponding \( k \)-modes is small. As seen from Figure 4, the \( \tilde{h} \)-function is significant for non-zero arguments and this would cause a depreciation in the measured fluctuations and hence the power spectrum. The baselines components \( x \) and \( y \) can have different relationship in the baseline plane. If \( d_x \) and \( d_y \) are located in such a way that they subtend an angle of 15 degrees with respect to the positive \( x \)-axis then one gets Figure 5 For 30 degrees angle one gets Figure 6. Once we have obtained desired \( h \)-functions, the next step is to generate signal with the help of simulations and see how the convolution affects the power measured in terms of visibilities. The box-size used in this paper is 630Mpc as it corresponds to the approximate 4 degree FoV. The bandwidth used if 8MHz (\( \approx 160Mpc \)) and channel width is about 62.5kHz (128 channels in the bandwidth). Baseline range is from about 100m to 3km which also corresponds to \( k_\perp \) of about .03Mpc\(^{-1}\) to 1.0Mpc\(^{-1}\).
3. SIGNAL POWER SPECTRUM

To generate 21 cm brightness temperature cubes, I used publicly available package 21cmFAST \(^7\) (Mesinger et al. 2011). I have used box of size \((630\text{Mpc})^3\) with grid-size of \(256^3\). The cosmological parameters were as mentioned below while the reionization parameters were \(\zeta = 30\), \(M_{\min} = 4.9 \times 10^8\text{M}_\odot\), and \(R_{\text{HII,max}} = 50\text{Mpc}\). The redshift of computation was \(z = 9.0\). If one considers a slice along the frequency direction of width about 8MHz, it would correspond to LOS distance of about 160Mpc. As our observation slice is about one fourth of the observation width, one can average every four successive measurements along \(k_\perp\) and assign them to a mean \(k_\perp\)-bin. This also gives \(\Delta k_\parallel \approx 4\Delta k_\perp\), as required. The dimensionless 21 cm power spectrum is given as

\(^7\)https://github.com/andreimesinger/21cmFAST/
where measurements are unavailable. This is because LOS extent is about 160Mpc, about one fourth of the perpendicular extent which is about 630Mpc. The solid line corresponds to $\mu_{\min} = 0.95$ (approximately the Horizon wedge) while the dashed line corresponds to $\mu_{\min} = 0.25$ (approximately the Field of View wedge).

with $\theta_{\text{Hor}} = 90^\circ$.

As mentioned earlier, we would like to extract as much information as possible from the region in between: $\mu_{\text{Hor}} < \mu < \mu_{\text{FoV}}$. After obtaining the equations for $\mu_{\text{FoV}}$ and $\mu_{\text{Hor}}$, I discuss how to get visibilities corresponding to the 21 cm brightness temperature distribution. As mentioned earlier, they are convolution of 21 cm fluctuations in the $k$-space with the $h$-function. To see the effect of convolution, it is advisable to look at 21 cm fluctuations without any convolution. This is shown in Figure 7. To get the convolved power, $d_s$ and $d_i$ were divided into 20 logarithmically spaced $d$-bins along both $x$ and $y$ directions. Note that the $h$-function explicitly depends on the baseline $d$. The $h$-function and its Fourier transforms were computed for all these 200 bins separately. The $h$-function varies only little within each of these bins and so was used to perform the convolution with the 21 cm fluctuations obtained from 21cmFAST. The convolved cube was used to get convolved power and it plotted in the Figure 8. The difference between the true power and the power obtained after convolution is plotted in Figure 9. It can be seen that there is up-to 5-10% decrement in the power, especially for larger baselines. A similar conclusion for the case of PAPER telescope was obtained earlier by Parsons et al. (2012b), although the approach used was different. As one can see, the error rises for larger $k_{\perp}$. The er-
Figure 8. The 21 cm signal power spectrum, this time obtained by convolving with the $h$-function.

Figure 9. The difference between true power (Figure 7) and power obtained after convolution (Figure 8).

(15)

System temperature needed for estimation of power due to thermal noise is given by

$$T_{\text{sys}} = 60 \text{ K} \left( \frac{300 \text{ MHz}}{v_c} \right)^{2.55} + T_{\text{recr}},$$

where $v_c = 1420 M\nu z/(1 + z)$ is the observation frequency and $T_{\text{recr}} \approx 60 K$. As described in McQuinn et al. (2006); Parsons et al. (2012a) the thermal noise for each measured Fourier mode is given by,

$$\Delta^2_{\text{thermal}}(k) \approx \frac{x^2 \gamma k^3 \Omega}{2 \pi^2 |t| T_{\text{sys}}^2},$$

where $k = (k_\perp + k_\parallel)^{1/2}$ is absolute magnitude of the Fourier mode, $\Omega$ is the field of view of each interferometric element, $t$ is the integration time in seconds for the mode and $X, Y$ are cosmology dependent factors which are given as

$$X \approx 1.9 h^{-1} \left( \frac{c M\nu c}{\text{arcmin}} \right) \left( \frac{1 + z}{10} \right)^{0.2},$$

and

$$Y \approx 11.5 h^{-1} \left( \frac{c M\nu c}{\text{MHz}} \right) \left( \frac{1 + z}{10} \right)^{0.5} \left( \frac{\Omega_{m} h^2}{0.15} \right)^{-0.5}.$$  

(18)

Equation 16 is for a single $k$-mode sampled continuously for time $t$. As the Earth rotates, baselines rotate and the values of $k_\perp$ corresponding to the baselines also change as dictated by equations 4. The extent of time for which any mode can be sampled continuously is given by (Parsons et al. 2012a).

$$t_{\text{per-mode}} = t_{20} \left( \frac{\Omega_0}{\Omega} \right)^{1/2} \left( \frac{20}{|\mathbf{u}|} \right).$$

(19)

Here $\Omega_0$ is PAPER Field of View while $\Omega$ is the FoV of the concerned telescope, in this case SKA. $t_{20}$ is time of continuous sampling corresponding to a baseline of length $20 \times \lambda$. To get thermal noise for any mode for practical scenario one has to consider multiple factors. First one is that there would be binning involved in $k$-space and each $k$-bin can contain multiple samples, say $N_{k\text{-bin}}$. Each mode in a given bin would also be sampled multiple times in a day. This quantity is $N_t = t_{\text{per-day}} / t_{\text{per-mode}}$. Total samples for any $k$-bin would be $N_{k\text{-bin}} \times N_t$. Considering all these factors, one can arrive at following expression for thermal noise for each $k$-bin

$$\Delta^2_{\text{thermal}}(k) \approx \frac{x^2 \gamma k^3 \Omega}{2 \pi^2 2 t_{\text{per-mode}} t_{\text{days}} T_{\text{sys}}^2 \left( \frac{1}{(N_{\text{per-mode}})^{1/2}} \right)}.$$  

(20)

---

* http://astronomers.skatelescope.org/documents/SKA-TEL-SKO-DD-001-1_BaselineDesign1.pdf
where \( N_{\text{per-mode}} = N_{\text{k-bin}} \times N_{t} \) and \( t_{\text{days}} \) is the number of days telescope is observing. Note that a square root appears on \( N_{\text{per-mode}} \) as here the modes are added incoherently while there is no square root on \( t_{\text{per-mode}} \) as the mode is sampled coherently for this duration. Figure 10 plots this thermal noise in the cylindrical \( k \)-space for observing time of 6 hours each for 120 days. While, the Figure 11 plots the same quantity for a time period 5 times as big.

5. FOREGROUND NOISE

As mentioned in earlier sections, the main reason the EoR signal could be extracted is the fact that it oscillates rapidly as a function of frequency while the Foreground component is smooth as function of frequency. So to get the Foreground component, the idea is to smooth the total signal over fairly wide frequency interval (\( \geq 5 \text{MHz} \)) so that the fluctuating signal and the mode-mixing components would average out to a very small value. Then fit to the smoothed signal a physically motivated foreground model. In the case of statistical signal, the foreground fitting has to be done along each and every Line of Sight. That is, when one obtains image cube from smoothed visibilities, the telescope resolution would naturally yield a certain possible number of Line of sights through this cube. And the foreground model has to be fitted to each Line of Sight separately assuming that every LOS traces slightly different section of the sky (Jelić et al. 2010). Such a procedure would yield foreground component but with its parameters along each LOS having some uncertainties. Once we have a foreground model, it can be used to simulate visibilities and power corresponding to this model (Equation 1). Uncertainties in the parameters of the foreground model would give rise to Foreground Noise in the measurement of the signal power spectrum. There would be additional sources of errors due to instrumental mode mixing (Parsons et al. 2016; Mertens et al. 2018; Ghosh et al. 2018). The instrumental mode mixing would generate additional noise and that can be added in quadrature to the other two components of noise, thermal and foregrounds. Simulations of power generated in the Fourier space due to mode-mixing would require simulating the visibilities of the mode-mixing processes. A thorough treatment of the errors and power spectra of mode-mixing components on the lines presented in the above mentioned papers is beyond the scope of this paper. But, it could be taken up as future work. In this paper, this Foreground noise is simulated for some particular cases. We defer the detailed analysis for all the other cases for future work. This paper shows that this FG noise component is small enough to give a reasonable signal to noise ratio.

The Galactic Diffuse Synchrotron Emission (or GDSE) model considered in this paper is:

\[
T_{\text{GDSE}} = A_{\text{syn}} \left( \frac{\nu}{\nu_c} \right)^\beta
\]

(21)

\( A_{\text{syn}} \) is assumed to be 351K with 10% variation in its value across different Lines of Sights. \( \beta \), the spectral index, is assumed to be about 2.55 (running spectral component is also there with value of about 0.1) (Rogers & Bowman 2008). For the temperature distribution given by Equation 21, one can simulate the visibilities (Thompson et al. 2017; Taylor et al. 1999) and compute the power corresponding to these visibilities. For computing visibilities corresponding to the Foregrounds considered, entire sky was pixelized in 625 pixels. The pixelization of the sky is denser at the centre of the primary beam and coarser towards the horizon. The am-

**Figure 10.** Thermal noise in \( k_{\perp} - k_{\parallel} \) space for observing time of 720 hours (6 hours for 120 days).

**Figure 11.** Thermal noise in \( k_{\perp} - k_{\parallel} \) space for observing time of 3600 hours (6 hours for 600 days).
The amplitudes of GDSE for each pixel are chosen to be Gaussian variables with mean value of 351 K and standard deviation of 10% of the mean value. $\nu_c$ is 150 MHz. Once one has the mean realization, aim is to deviate around this mean realization and compute the noise in power arising from uncertainties in the Foreground parameters. As we know, the Fourier transformed visibilities are directly related to the power. One can now obtain multiple realizations (about 10) of the same and then look for standard deviation in the Foreground power obtained for different realizations. This would give rise to what I have been calling as Foreground Noise. Note that I have used extended Blackmann-Nuttall window (Thyagarajan et al. 2015) for simulating Foreground visibilities along the frequency axis. One can estimate the foreground noise if there is say 2% uncertainty in the estimation of GDSE amplitude along each LOS. This is shown in Figure 12. One can also estimate the Foreground noise if the $\beta$ parameter is uncertain by 10% along each LOS. This is shown in Figure 13.

As seen from the figures, the noise generated is comparable to the signal for the quoted uncertainties. One can also get a rough estimate of the Foreground noise for the case of GDSE amplitude uncertainty. If the primary beam falls to about 1% of its peak value for the first sidelobe. And if the error in the estimation of amplitude of GDSE is assumed to be about 1%. Then, for power spectrum of Foregrounds, this would correspond to a roughly $(0.01 \times 0.01)^2 = 10^{-8}$ amount of error and so this (noise in) power is seen to be comparable to the signal power. The Foreground noise is much larger in the wedge region as the most effective part of the primary beam is larger in that region. The beam for above analysis was assumed to be Airy pattern with first null occurring at about 4 degrees and the extent of the primary beam is all the way up to the Horizon. Note that this could be a pessimistic scenario as the beam would fall more rapidly as one observes the sky through higher order side-lobes. One can also check the effect of beam calibration errors on the measurement of power spectrum. The results are as depicted in Figure 15. Here foreground parameters are fixed while the beam is assumed to have some percentage variations about its mean value. The extents of variations are as depicted in Figure 14.
foreground model can be propagated through the instrument pipeline to simulate the visibilities and also uncertainties in them in the Fourier space. These uncertainties would act as foreground noise and can be added in quadrature to the thermal noise. Thus once one subtracts foreground visibilities, one gets signal visibilities with uncertainties. Writing technically:

$$V_{\text{Total}}(U, \eta) = V_{\text{Signal}}(U, \eta) + V_{\text{FG}}(U, \eta) + V_{\text{Other}}(U, \eta)$$

Here, $V_{\text{Other}}(U, \eta)$ includes other sources of visibilities like instrumental mode-mixing. Contribution of the this term to total visibilities could to be modelled. Thus its power contribution could be estimated and subtracted out. It would also carry noise and that should be added to other noise contributors. One would have to model the chromatic response of the instrument, (see e.g. Hazelton et al. 2013; Thiyagarajan et al. 2015) and also the calibrations and mis-direction errors because of point sources (see e.g. Datta et al. 2010; Morales et al. 2012). While the scintillation noise could be added to the other sources of noise (see e.g. Koopmans 2010; Vedantham & Koopmans 2016).

Note that in the above equation, the quantity on the left hand side has intrinsic standard deviation of the thermal noise. Now,

$$V_{\text{Signal}}(U, \eta) = V_{\text{Total}}(U, \eta) - V_{\text{FG}}(U, \eta) - V_{\text{Other}}(U, \eta)$$

Thus for the signal visibilities one gets

$$\sigma_{V_{\text{Signal}}}(U, \eta) = \sqrt{\sigma_{V_{\text{thermal}}}(U, \eta)^2 + \sigma_{V_{\text{FG}}}(U, \eta)^2 + \sigma_{V_{\text{Other}}}(U, \eta)^2}$$

Once one subtracts the foreground component this total deviation would be the one on signal and would yield total uncertainty in the signal. The Signal to Noise Ratio (SNR) for FG noise due to errors in the GDSE amplitude and thermal noise, added in quadrature, is plotted in Figure 16. As seen from the plot, there is considerable amount of region in between the two lines that can yield significant information about the signal from EoR.

There would be many sources that would contribute to the $\sigma_{V_{\text{Signal}}}(U, \eta)$ and they are not limited to parameters of the foreground. Even calibration errors in the primary beam or uncertainties in the positions of bright point sources would contribute to the $\sigma_{V_{\text{Signal}}}(U, \eta)$ and would add to total noise. Thus the SNR shown in Figure 16 is an estimate and as mentioned earlier it does not include noise generate due to instrumental mode mixing or other possible sources of errors. These extra sources of errors like ionosphere, direction dependent beams, bandpass, etc. pose significant challenges as they form excess contaminants and make the removal of the smooth foreground component very difficult. A careful analysis of all the effects would need much more effort and hence it has been

![Figure 15. Noise due to uncertainty in the determination of primary beam.](image-url)
deferred as future work. There one can address the question as to what sort of calibration requirements, both instrument and foreground, are needed so that the total noise remains controlled and below the signal strength.

APPENDIX

We have,

\[ V(U, \eta) = \int_{\Omega, B} d^2\theta d\nu A_v(\theta)I_v(\theta)e^{-2\pi i (U\theta + V\eta)} \]  

(25)

Now substituting as follows: \( D_c(z) \) is comoving distance to redshift \( z \):

\[ \theta = r_\perp / D_c(z) \]
\[ U = d/\lambda = d\nu/c \]

and writing \( V \) in terms of LOS distance \( r_\parallel \) (\( \nu_0 \) is the central frequency and \( r'_\parallel \) is derivative of LOS distance with respect to frequency),

\[ dv = dr_\parallel / r'_\parallel \]
\[ v = \nu_0 + \Delta v = \nu_0 + \Delta r_\parallel / r'_\parallel \]

yields,

\[ V(U, \eta) = \frac{1}{D_c(z)^2} \int d^2r_\perp dr_\parallel A(r_\perp, r_\parallel)I(r_\perp, r_\parallel) \]

\[ \times e^{-2\pi i (d\nu_0 + \Delta r_\parallel / r'_\parallel) / c} \]

As we use periodic boundary conditions in cosmology analysis, the origin can be shifted to the centre of the observation volume. Thus \( \Delta r_\parallel \) can be replaced by \( r_\parallel \). Considering this, dropping the phase term \( e^{-2\pi i \nu_0} \) (as an overall phase would not matter for squared visibilities) and identifying

\[ k_\perp = \frac{2\pi}{D_c(z)} d\nu_0 / c \]
\[ k_\parallel = \frac{2\pi}{r'_\parallel} \eta \]

one gets the desired equation,

\[ V(k_\perp, k_\parallel) = \frac{1}{r'_\parallel D_c(z)^2} \int dr_\parallel d^2r_\perp A(r_\perp, r_\parallel)I(r_\perp, r_\parallel) \]

\[ \times \exp \left[ -i \left( r_\perp \cdot k_\perp + r_\parallel k_\parallel \right) \right] \exp \left( -2\pi i \frac{r_\perp \cdot d\nu_0 / c}{D_c(z) r'_\parallel} \right) \]

REFERENCES

Ali, S. S., Bharadwaj, S., & Chengalur, J. N. 2008, MNRAS, 385, 2166, doi: 10.1111/j.1365-2966.2008.12984.x

Ali, Z. S., Parsons, A. R., Zheng, H., et al. 2015, ApJ, 809, 61, doi: 10.1088/0004-637X/809/1/61

Beardsley, A. P., Hazelton, B. J., Sullivan, I. S., et al. 2016, ApJ, 833, 102, doi: 10.3847/1538-4357/833/1/102

Bonaldi, A., & Brown, M. L. 2015, MNRAS, 447, 1973, doi: 10.1093/mnras/stu2601

Bowman, J. D., Morales, M. F., & Hewitt, J. N. 2006, ApJ, 638, 20, doi: 10.1086/498703

Bowman, J. D., Rogers, A. E. E., Monsalve, R. A., Mozdzen, T. J., & Mahesh, N. 2018, Nature, 555, 67, doi: 10.1038/nature25792

Chapman, E., Abdalla, F. B., Harker, G., et al. 2012, MNRAS, 423, 2518, doi: 10.1111/j.1365-2966.2012.21065.x

Choudhury, T. R. 2009, Current Science, 97, 841. https://arxiv.org/abs/0904.4596

Datta, A., Bowman, J. D., & Carilli, C. L. 2010, ApJ, 724, 526, doi: 10.1088/0004-637X/724/1/526

Di Matteo, T., Ciardi, B., & Miniati, F. 2004, MNRAS, 355, 1053, doi: 10.1111/j.1365-2966.2004.08443.x

Di Matteo, T., Perna, R., Abel, T., & Rees, M. J. 2002, ApJ, 564, 576, doi: 10.1086/324293

Fan, X., Carilli, C. L., & Keating, B. 2006, ARA&A, 44, 415, doi: 10.1146/annurev.astro.44.051905.092514
