GLOBAL $C^\infty$ IRREGULARITY OF THE \linebreak \linebreak \bar\partial–NEUMANN PROBLEM FOR WORM DOMAINS

MICHAE L CHRIST

University of California, Los Angeles

0. Introduction.

Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain with $C^\infty$ boundary, equipped with the standard Hermitian metric inherited from $\mathbb{C}^n$. The $\bar\partial$–Neumann problem for $(p,q)$ forms in $\Omega$ is the boundary value problem

$$\begin{cases} 
\Box u = f & \text{in } \Omega \\
u \mathcal{D} \rho = 0 & \text{on } \partial \Omega \\
\bar\partial u \mathcal{D} \rho = 0 & \text{on } \partial \Omega 
\end{cases}$$

where $\rho$ is a defining function for $\Omega$, $\Box = \bar\partial \bar\partial^* + \bar\partial^* \bar\partial$, $u, f$ are $(p,q)$ forms, and $\mathcal{D}$ denotes the interior product of forms. Under the stated hypotheses on $\Omega$, this problem is uniquely solvable for every $f \in L^2(\Omega)$. The Neumann operator $N$, mapping $f$ to the solution $u$, is continuous on $L^2(\Omega)$. The Bergman projection $B$ is the orthogonal projection of $L^2(\Omega)$ onto the closed subspace of $L^2$ holomorphic functions on $\Omega$, and is related to $N$ by $B = I - \bar\partial^* N \bar\partial$.

$N$ and $B$ are $C^\infty$ pseudolocal if $\Omega$ is strictly pseudoconvex, or more generally, is of finite type [Ca1]. Both preserve $C^\infty(\Omega)$ under certain weaker hypotheses [BS2],[Ca2]. For any pseudoconvex, smoothly bounded $\Omega$ and any finite exponent $s$, there exists a strictly positive weight $w \in C^\infty(\Omega)$ such that the Neumann operator and Bergman projection with respect to the Hilbert space $L^2(\Omega, w(x)dx)$ map the Sobolev space $H^t(\Omega)$ boundedly to $H^t(\Omega)$, for all $0 \leq t \leq s$ [K1]. It has remained an open question whether $N$ and $B$, defined with respect to the standard metric, preserve $C^\infty(\Omega)$ without further hypotheses on $\Omega$. An affirmative answer would have significant consequences [BL].

\begin{minipage}{\textwidth}
\footnotesize
Research supported by National Science Foundation grant DMS-9306833. I am indebted to D. Barrett, J. J. Kohn, P. Matheos and J. McNeal for stimulating conversations and useful comments on the exposition.
\end{minipage}
Theorem. There exist pseudoconvex, smoothly bounded domains \( \Omega \in \mathbb{C}^2 \) for which the Neumann operator on \((0,1)\) forms and Bergman projection fail to preserve \( C^\infty(\overline{\Omega}) \).

Examples are the worm domains, originally introduced by Diederich and Fornæss [DF] for another purpose\(^1\) but long considered likely candidates for \( N \) and \( B \) to fail to be globally regular in \( C^\infty \).

The proof depends on the observation of Barrett [B] that for each worm domain \( W \), for all sufficiently large \( s \), \( N \) and \( B \) fail to map \( H^s(W) \) boundedly to \( H^s(W) \). We establish for each worm domain an \( \textit{a priori} \) inequality of the form \( \|Nf\|_{H^s} \leq C_s \|f\|_{H^s} \), valid for all \( f \in C^\infty(W) \) such that \( Nf \in C^\infty(W) \), for a sequence of exponents \( s \) tending to \( \infty \). If \( N \) were to preserve \( C^\infty(W) \), then since it is a bounded linear operator on \( L^2(W) \) and since \( C^\infty(W) \) is dense in \( H^s(W) \), it would follow that \( N \) maps \( H^s(W) \) boundedly to itself, for a sequence of values of \( s \) tending to \( \infty \), a contradiction. More accurately, our inequality is valid only for certain subspaces of \( L^2(W) \) preserved by \( \Box \), but this still suffices to contradict the theorem of Barrett.

An analogous counterexample in the real analytic context is already known [Ch1]: there exists a bounded, pseudoconvex domain \( \Omega \subset \mathbb{C}^2 \) having real analytic boundary, such that the Szegö projection fails to preserve \( C^\omega(\partial\Omega) \). That result and its proof are however not closely related to the \( C^\infty \) case.

\( §§1 \) through 3 review material on worm domains and the \( \bar{\partial} \)-Neumann problem, and present some routine but tedious reductions. \( §4 \) formalizes a class of two-dimensional problems subsuming those to which the reductions lead. The analysis of those problems is contained in \( §§5 \) and 6.

1. Reduction to the Boundary.

The \( \bar{\partial} \)-Neumann problem is a boundary value problem for an elliptic partial differential equation, and as such is amenable to treatment by the method of reduction to a pseudodifferential equation on the boundary. This reduction has been carried out in detail for domains in \( \mathbb{C}^2 \) by Chang, Nagel and Stein [CNS]. We review here certain of their computations and direct consequences thereof.

Assume \( \Omega \subset \mathbb{C}^2 \) to be a smoothly bounded domain. The equation \( \Box u = f \) on \( \Omega \) for \((0,1)\) forms in \( C^\infty(\overline{\Omega}) \) is equivalent to an equation \( \Box^+ v = g \) on \( \partial\Omega \), where \( v, g \) are sections of a certain complex line bundle\(^2\) \( \mathcal{B}^{0,1} \). Let \( \rho \) be a smooth defining function for \( \Omega \) and define \( \bar{\omega}_2 = \bar{\partial}\rho \), and \( \bar{\omega}_1 = (\partial\rho/\partial z_2) \bar{dz}_1 - (\partial\rho/\partial z_1) \bar{dz}_2 \).

\( v \) is related to \( u \) by \( u = Pv + Gf \), where \( P \), \( G \) are respectively Poisson and Green operators for the elliptic system \( \Box u = f \) with Dirichlet boundary conditions. In particular, if \( f \in C^\infty(\overline{\Omega}) \), then \( u \in C^\infty(\overline{\Omega}) \) if and only if the same holds for \( v \). More precisely, \( G \) maps

---

\(^1\)Some but not all worm domains have nontrivial Nebenhülle[FS, p. 111], whereas all worm domains are counterexamples to global regularity.

\(^2\)\( \mathcal{B}^{0,1} \) is defined to be the quotient of the restriction to \( \partial\Omega \) of \( \mathcal{T}^{0,1} \mathbb{C}^2 \), modulo the span of \( \bar{\partial}\rho \). Sections of \( \mathcal{B}^{0,1} \) may be identified with scalar-valued functions times \( \bar{\omega}_1 \), hence with scalar-valued functions.
in order to conclude that $u \in H^s$ it suffices to know that $v \in H^{s-1/2}$.

On the other hand, $g = (\bar{\partial}Gf \, \bar{\partial}q)$, restricted to $\partial \Omega$. If $f \in H^s$ then $\bar{\partial}Gf \in H^{s+1}(\Omega)$, so its restriction to the boundary belongs to $H^{s+1/2}(\partial \Omega)$. Thus in order to show that $N$ preserves $H^s(\Omega)$ it suffices to show that if $\Box^+ v \in H^{s+1/2}(\partial \Omega)$, then $v \in H^{s-1/2}(\partial \Omega)$, assuming always that $s > 1/2$.

On $\partial \Omega$ a Cauchy-Riemann operator is the complex vector field $\bar{L} = (\partial_{\bar{z}_1} \rho) \bar{\partial}_{\bar{z}_2} - (\partial_{\bar{z}_2} \rho) \partial_{\bar{z}_1}$. Define $L$ to be the complex conjugate of $\bar{L}$. The characteristic variety $^3$ of $L$ is a real line bundle $\Gamma$. Assuming $\Omega$ to be pseudoconvex and the set of strictly pseudoconvex points to be dense in $\partial \Omega$, $\Gamma$ splits smoothly and uniquely as $\Gamma^+ \cup \Gamma^-$, where each fiber of $\Gamma^\pm$ is a single ray, and where $\Gamma^+$ is distinguished from $\Gamma^-$ by the requirement that the principal symbol of $[\bar{L}, L]$ is nonpositive on $\Gamma^+$, modulo terms spanned by the symbols of the real and imaginary parts of $\bar{L}$ and a term of order 0. Equivalently, the principal symbol of $[\bar{L}, \bar{L}^*]$ is nonnegative on $\Gamma^+$, modulo the same kinds of error terms.

$\Box^+$ is a classical pseudodifferential operator of order $+1$. Its principal symbol vanishes everywhere on $\Gamma^+$ but nowhere else. Microlocally in a conic neighborhood of $\Gamma^+$, $\Box^+$ takes the form

$$\Box^+ = Q\bar{L}L + F_1 \bar{L} + F_2 L + F_3$$

where $Q$ is an elliptic pseudodifferential operator of order $-1$, and each $F_j$ is a pseudodifferential operator of order less than or equal to $-1$. Since $\Box^+$ is elliptic except on $\Gamma^+$, for any pseudodifferential operator $G$ of order zero whose symbol vanishes identically in some neighborhood of $\Gamma^+$, one has for all $u \in C^\infty$ and all $N < \infty$

$$(1.1) \quad \|Gu\|_{H^{t+1}(\partial \Omega)} \leq C \|\Box^+ u\|_{H^t(\partial \Omega)} + C_N \|u\|_{H^{-N}(\partial \Omega)}.$$

Let $A$ be an elliptic pseudodifferential operator of order $+1$ such that $A \circ Q$ equals the identity on $L^2(\partial \Omega)$, modulo an operator smoothing of infinite order. Composing on the left with $A$, the equation $\Box^+ v = g$ may be rewritten as $Lv = \tilde{g}$ microlocally in a conic neighborhood of $\Gamma^+$, where

$$(1.2) \quad L = \bar{L}L + B_1 \bar{L} + B_2 L + B_3,$$

$\|\tilde{g}\|_{H^t} \leq C \|g\|_{H^{t+1}} + C_N \|v\|_{H^{-N}}$ for any finite $N$, and each $B_j$ is an operator of order less than or equal to zero. Therefore in order to show that the Neumann operator satisfies an a priori inequality of the form $\|Nf\|_{H^t(\Omega)} \leq C \|f\|_{H^t(\Omega)}$ for all $f \in C^\infty(\overline{\Omega})$ such that $Nf \in C^\infty(\overline{\Omega})$, it suffices to establish an priori inequality for all $v \in C^\infty(\partial \Omega)$ of the form

$$(1.3) \quad \|v\|_{H^t} \leq C \|Lv\|_{H^t} + C \|v\|_{H^{t'}} + C \|Qv\|_{H^{t+2}}$$

where $t = s - 1/2$, for some $t' < t$ and some pseudodifferential operator $\tilde{Q}$ of order zero whose symbol vanishes identically in some neighborhood of $\Gamma^+$.

\textsuperscript{3}By the characteristic variety of a pseudodifferential operator we mean the conic subset of the cotangent bundle on which its principal symbol vanishes.
2. Worm Domains.

A worm domain in $\mathbb{C}^2$ is an open set of the form

$$W = \{ z : |z_1 + e^{i \log |z_2|^2}|^2 < 1 - \phi(\log |z_2|^2) \}$$

where the function $\phi$ vanishes identically on some interval $[-r, r]$ of positive length, and is constructed [DF] so as to guarantee that $W$ will be pseudoconvex with $C^\infty$ boundary, and will be strictly pseudoconvex at every boundary point except those on the exceptional annulus $A \subset \partial W$ defined as

$$A = \{ z : z_1 = 0 \text{ and } |\log |z_2|^2| \leq r \}.$$

The circle group acts as a group of automorphisms of $W$ by $z \mapsto R_\theta z = (z_1, e^{i \theta} z_2)$. It acts on functions by $R_\theta f(z) = f(R_\theta z)$, and on $(0, 1)$ forms by $R_\theta(f_1 dz_1 + f_2 dz_2) = (R_\theta f_1)dz_1 + (R_\theta f_2)e^{-i \theta} dz_2$. The Hilbert space $L^2(0, 1; W)$ of square integrable $(0, k)$ forms decomposes as the orthogonal direct sum $\bigoplus_{j \in \mathbb{Z}} H^k_j$ where $H^k_j$ is the set of all $(0, k)$ forms $f$ satisfying $R_\theta f \equiv e^{ij \theta} f$. $\partial$ is an unbounded linear operator from $H^k_j$ to $H^{k+1}_j$, $B$ maps $H^0_j$ to itself, and the Neumann operator $N$ maps $H^1_j$ to $H^1_{j+1}$ boundedly, for each $j$.

For each $k$ and each $s \geq 0$, the Sobolev space $H^s(W)$ likewise decomposes as an orthogonal direct sum of subspaces $H^s_j$. It is known that for any smoothly bounded, pseudoconvex domain $\Omega \subset \mathbb{C}^2$, for any exponent $s \geq 0$, if $N$ maps $H^s(\Omega)$ boundedly to itself, then $B$ also maps $H^s(\Omega)$ boundedly to itself [BS1], where $H^s$ denotes the first instance a space of one forms, and in the second, a space of functions. Because $N, B$ preserve the summands $H_j$, the same proof shows\(^4\) that for any fixed $j$, if $N$ maps the space $H^s_j$ of $(0, 1)$ forms boundedly to itself, then $B$ maps the space $H^s_j$ of functions boundedly to itself. Barrett [B] that for each worm domain, for all sufficiently large $s$, for all $j$, $B$ fails to map $H^s_j$ boundedly to itself. Therefore in order to prove that $N$, acting on $(0, 1)$ forms, fails to preserve $C^\infty(W)$, it suffices to establish the following result for a single index $j$.

**Proposition 1.** For each worm domain there exists a discrete subset $S \subset \mathbb{R}^+$ such that for each $s \notin S$ and each $j \in \mathbb{Z}$ there exists $C_{s,j} < \infty$ such that for every $(0,1)$ form $u \in H^1_j \cap C^\infty(W)$ such that $Nu \in C^\infty(W)$,

$$\|Nu\|_{H^s(W)} \leq C_{s,j} \|u\|_{H^s(W)}.$$  \hspace{1cm} (2.1)

The defining function $\rho = 1 - \phi(\log |z_2|^2) - |z_1 + e^{i \log |z_2|^2}|^2$ for $W$ is invariant under $R_\theta$, as is the $(0, 1)$ form $\tilde{\omega}_2$ defined above. $\tilde{\omega}_1$ satisfies $R_\alpha \tilde{\omega}_1 = \exp(-i \alpha) \tilde{\omega}_1$ for all $\alpha$, but it may also be made invariant by multiplying it by the function $(z_1, re^{i \theta}) \mapsto e^{i \theta}$, which is smooth in a neighborhood of $\overline{W}$. We work henceforth with this modified $\tilde{\omega}_1$.

\(^4\)This follows from the argument of Boas and Straube [BS1] because all elements of their proof may be chosen to be invariant under the automorphisms $R_\theta$.
\(\Box^+\) commutes with \(R_\theta\) for all \(\theta\). Indeed, \(\Box^+ v = \bar{\partial}Pv + \bar{\partial}\rho\) [CNS]. \(\Box\) commutes with \(R_\theta\), hence so must \(P\). \(\bar{\partial}\) commutes with \(R_\theta\), and the Hermitian metric on \(\mathbb{C}^2\) and \(\bar{\partial}\rho\) are likewise \(R_\theta\)-invariant. Thus all ingredients in the above expression for \(\Box^+\) are invariant, hence so is \(\Box^+\) itself.

Identify square integrable sections of \(\mathcal{B}^{0,1}\) with scalar-valued \(L^2\) functions as above, and decompose \(L^2(\partial\mathcal{W}) = \oplus \mathcal{H}_j(\partial\mathcal{W})\) where \(\mathcal{H}_j\) is the subspace of those functions satisfying \(R_\theta f \equiv e^{ij\theta} f\). Then \(\Box^+\) maps \(\mathcal{H}_j(\partial\mathcal{W}) \cap C^\infty\) to \(\mathcal{H}_j(\partial\mathcal{W})\). We have seen in §1 that Proposition 1 would be a consequence of the validity of (1.3) for all \(v \in C^\infty(\partial\mathcal{W}) \cap \mathcal{H}_j\), for \(t = s - 1/2\).

Fix \(j\) and assume henceforth that \(u\) belongs to \(\mathcal{H}_j(\mathcal{W})\) and to \(C^\infty\). Then the associated boundary function \(v\) belongs to \(\mathcal{H}_j \cap C^\infty(\partial\mathcal{W})\). Henceforth we work exclusively on the boundary, and simplify notation by writing simply \(\mathcal{H}_j\) rather than \(\mathcal{H}_j(\partial\mathcal{W})\).

Note that \(L, L\) take \(\mathcal{H}_j \cap C^\infty\) to \(\mathcal{H}_{j+1}\) and to \(\mathcal{H}_{j-1}\), respectively. The operator \(A\) introduced after (1.1) may be constructed to be \(R_\theta\)-invariant, for both \(\Box^+\) and \(\bar{\partial}\) are invariant while \(\check{L}, L\) are automorphic of certain degrees, so that averaging the equation \(\Box^+ = Q\check{L}L + F_1L + F_2L + F_3\) with respect to \(R_\theta d\theta\) produces an invariant \(Q\) and \(F_3\), and operators \(F_1, F_2\) automatic of the appropriate degrees. This \((\check{L}L + B_1\check{L} + B_2L + B_3)v \in H^t\) microlocally near \(\Gamma^+\), where \(B_1, B_2, B_3\) map \(\mathcal{H}_j\) to \(\mathcal{H}_i\) for \(i = j - 1, j + 1, j\) respectively.

Since \(W\) is strictly pseudoconvex at all points not in \(\mathcal{A}\), it follows as in [K2] that on the complement of any neighborhood of \(\mathcal{A}\), the \(H^{s+1}\) norm of \(v\) is majorized by \(C\|\Box^+ v\|_{H^{s+1}(\partial\mathcal{W})} + C\|v\|_{H^{-N}}\), hence by \(C\|Lv\|_{H^s} + C\|v\|_{H^{-N}} + C\|\check{Q}v\|_{H^{s+2}}\) with \(\check{Q}\) as in (1.3). This estimate is one derivative stronger than that which we seek. In particular, it now suffices to control the \(H^s\) norm of \(v\) in an arbitrarily small neighborhood \(U\) of \(\mathcal{A}\), and to do so microlocally near \(\Gamma^+\).

Fix a \(C^\infty\) cutoff function \(\varphi\) supported in a small neighborhood of \(\mathcal{A}\) but identically equal to 1 in a smaller neighborhood, and fix an open set \(\mathcal{V}\) disjoint from a neighborhood of \(\mathcal{A}\) such that \(\nabla \varphi\) is supported in \(\mathcal{V}\). By Leibniz’s rule and the pseudolocality of pseudodifferential operators, the \(H^s\) norm of \(\mathcal{L}(\varphi v - v)\) is majorized by \(C\|v\|_{H^{s+1}(\mathcal{V})} + C\|v\|_{H^{-N}}\) for any \(N < \infty\). Thus by replacing \(v\) with \(\varphi v\) we may reduce matters to the case where \(v\) is supported in an arbitrarily small neighborhood \(W\) of \(\mathcal{A}\). Therefore it suffices to prove the existence of some \(W\) and an exponent \(s'\ < s\) such that (1.3) holds (with \(t\) replaced by \(s\)) for all \(v \in C^\infty_0\) supported in \(W\).

In a neighborhood of \(\mathcal{A}\) in \(\partial\mathcal{W}\) introduce coordinates \((x, \theta, t)\) where

\[
\begin{align*}
z_2 &= e^{x + it\theta}, \\
z_1 &= e^{2ix}(e^{it}(1 - \phi(2x)) - 1)
\end{align*}
\]

with \(2|x| < r + \delta\) and \(|t| < \delta\) for some small \(\delta > 0\). In these coordinates \(\mathcal{A} = \{t = 0, \ |x| \leq r/2\}\). Setting

\[
\gamma(x, t) = 2\left[e^{-it} - 1 + \phi(2x) - i\phi'(2x)\right]/\left[1 - \phi(2x)\right],
\]

the vector field \(\bar{L} = \partial_x + i\partial_\theta + \gamma\partial_t\) annihilates both \(z_1\) and \(z_2\). Hence it differs from what was previously denoted as \(\bar{L}\) by multiplication on the left by a nonvanishing factor, which
may be verified to have the form $b(x, t) \exp(i\theta)$. For $|x| \leq r/2$, $\phi(x) \equiv 0$ and consequently $\gamma(x, t) \equiv 2(e^{-it} - 1)$. Therefore

\begin{equation}
\bar{L} = \partial_x + i\partial_\theta + it\alpha(t)\partial_t \quad \text{ where } |x| \leq r/2, \text{ with } \text{Re } \alpha(0) \neq 0.
\end{equation}

The representation $\mathcal{L} = \bar{L} \bar{L} + B_1 \bar{L} + B_2 L + B_3$ in terms of the new $\bar{L}$ remains valid with modified coefficients $B_i$ that now preserve each $\mathcal{H}_j$, once the operator formerly denoted by $\mathcal{L}$ is multiplied by $|b|^{-2}$.

There exists a unique $C^\infty$ real-valued function $\mu$, independent of $\theta$, such that $[\bar{L}, L] = i\mu(x, t)\partial_t + i\nu(x, t) \text{Re } \bar{L}$ for unique real-valued, $C^\infty$ coefficients $\mu, \nu$. The pseudoconvexity of $\partial W$ means that $\mu$ does not change sign. Replacing $t$ by $-t$ if necessary, we may assume that $\mu \geq 0$.

Fix an integer $k$. We identify functions of $(x, t) \in \mathbb{R}^2$ with elements of $\mathcal{H}_k$ via the correspondence $u(x, t) \mapsto u(x, t)e^{ik\theta}$; $\partial_\theta$ then becomes multiplication by $ik$. For the remainder of the paper we work in $\mathbb{R}^2$. Define

$$D = \{(x, t) : |x| \leq r/2 \text{ and } t = 0\}.$$ 

By incorporating $ik$ into the $B_j$ we may further rewrite $\mathcal{L}$, when restricted to $\mathcal{H}_k$, as

$$\mathcal{L}_0 = \bar{\ell} \ell + B_1 \bar{\ell} + B_2 \ell + B_3$$

where $\bar{\ell}$ is a complex vector field in $\mathbb{R}^2$ which, for $|x| \leq r/2$, takes the form

$$\bar{\ell} = \partial_x + it\alpha(t)\partial_t.$$ 

Here $\ell$ denotes the conjugate of $\bar{\ell}$, each $B_j$ is a pseudodifferential operator of order $\leq 0$ in $\mathbb{R}^2$, and $\text{Re } \alpha(0) \neq 0$. Note that the commutator of the real and imaginary parts of $\bar{\ell}$ is not forced to vanish identically, because $\alpha$ is not real-valued. We have $[\bar{\ell}, \ell] = i\mu(x, t)\partial_t + i\nu(x, t) \text{Re } \bar{\ell}$ with the same coefficients as in (2.3).

Letting $(\xi, \tau)$ be Fourier variables dual to $(x, t)$, define $\tilde{\Gamma} = \{(x, t, \xi, \tau) : (x, t) \in D \text{ and } \xi = 0\}$. $\mathcal{L}_0$ is not elliptic at points of $D$, but is elliptic at most points in its complement; $\tilde{\Gamma}$ is the intersection of the characteristic variety of $\mathcal{L}_0$ with $\{(x, t, \xi, \tau) : (x, t) \in D\}$. Decompose $\tilde{\Gamma} = \tilde{\Gamma}^+ \cup \tilde{\Gamma}^-$ where $\tilde{\Gamma}^+ = \tilde{\Gamma} \cap \{\tau > 0\}$. Thus the principal symbol of $i\mu\partial_t$, namely $-\mu \tau$, is nonpositive in a conic neighborhood of $\tilde{\Gamma}^+$.

\footnote{Boas and Straube [BS2] have shown the $\bar{\partial}$–Neumann problem to be globally $C^\infty$ hypoelliptic whenever there exists a real vector field on the boundary that is transverse to the complex tangent space and has a certain favorable commutation property. If $\text{Re } \alpha(0)$ were to vanish then $\partial_\tau$ would be such a vector field. Thus nonvanishing of $\text{Re } \alpha(0)$ is for our purpose an essential feature of worm domains.}
3. Two Pseudodifferential Manipulations.

By an operator we will always mean a classical pseudodifferential operator, that is, one whose symbol admits a full asymptotic expansion in homogeneous terms of integral degrees. \( \sigma_j(T) \) denotes the \( j \)-th order symbol of \( T \) (in the Kohn-Nirenberg calculus), always with respect to the fixed coordinate system \((x, t, \xi, \tau)\). Henceforth we work under the convention that \( A, B, E \) denote operators whose orders are less than or equal to \(0, 0, -1\) respectively, whose meanings are permitted to change freely from one occurrence to the next, even within the same line. \( A \) denotes always an operator having the additional property that \( \sigma_0(A)(x, t, \xi, \tau) \equiv 0 \) for all \((x, t) \in \mathcal{D}\). Any operator of type \( E \) may be regarded as one of type \( A \). Two operators are said to agree microlocally in some conic open set if the full symbol of their difference vanishes identically there.

Any operator \( B \) may be written in the form \( B = \beta(x) + E \circ \ell + A \) microlocally in a conic neighborhood of \( \bar{\Gamma}^+ \), where \( \beta \) denotes both a \( C^\infty \) function and the operator defined by multiplication by that function. Indeed, \( \sigma_0(B)(x, 0, \tau) \) depends only on \((x, \mathrm{sgn}(\tau))\) and we define \( \beta(x) \) to be this quantity for \( \tau > 0 \). Then where \( \tau > 0 \) and \(|x| \leq r/2\), \( \sigma_0(B)(x, 0, \xi, \tau) \) is divisible by \( \xi = -i \sigma_1(\ell)(x, 0, \xi, \tau) \). \( \sigma_{-1}(E)(x, 0, \xi, \tau) \) is then uniquely determined for such \( x, \tau \) by the equation \( \sigma_0(B) = \beta(x) + \sigma_1(\ell) \cdot \sigma_{-1}(E) \). Define \( E \) to be any operator of order \(-1\) whose principal symbol satisfies this equation when restricted to \((x, t) \in \mathcal{D}\) and to a conic neighborhood of \( \bar{\Gamma}^+ \). Then simply define \( A = B - \beta - E \circ \ell \).

Writing \( B_1 = \beta_1(x) + E_1 \circ \ell + A_1 \) and similarly \( B_2 = \beta_2(x) + E_2 \circ \ell + A_2 \), and expressing \( E\ell \ell = E\ell \ell \) plus an operator of order \( \leq 0 \), we obtain \( \mathcal{L}_0 = (I + E)\ell \ell + (\beta_1 + A)\ell + (\beta_2 + A)\ell + B \). Composing both sides with a parametrix for \( I + E \) and modifying the definition of \( \mathcal{L}_0 \) to include this factor, we have \( \mathcal{L}_0 = \ell \ell + (\beta_1 + A)\ell + (\beta_2 + A)\ell + B \). Writing finally \( B = \beta_3(x) + E \circ \ell + A \) results in

\[
(3.1) \quad \mathcal{L}_0 = \ell \ell + (\beta_1 + A)\ell + (\beta_2 + A)\ell + (\beta_3 + A)
\]

where the \( \beta_j \) are \( C^\infty \) functions depending only on \( x \).

We next reduce the question of a priori \( H^s \) inequalities to \( L^2 \), simultaneously for all \( s \). Fix an operator \( Q \) of order \( 0 \) that is elliptic in some conic neighborhood of \( \bar{\Gamma}^- \), whose symbol vanishes identically in some conic neighborhood of \( \bar{\Gamma}^+ \). Fix an exponent \( s > 0 \), for which we seek an a priori inequality for all \( \|v\| \in C^\infty \) of the form

\[
(3.2) \quad \|v\|_{H^s} \leq C\|\mathcal{L}_0 v\|_{H^s} + C\|v\|_{H^{s'}} + C\|Qv\|_{H^{s+2}}
\]

for some \( s' < s \). Having such an inequality for a sequence of exponents \( s \) tending to \(+\infty\) would imply Proposition 1 by the preceding discussion. In particular, the \( H^{s+2} \) norm of \( Qv \) is already under control by virtue of (1.1), while the \( H^0 \) norm of \( v \) is harmless because the Neumann operator is bounded on \( L^2 \), and \( \|v\|_{H^{s'}} \leq \varepsilon \|v\|_{H^s} + C_{\varepsilon, N} \|v\|_{H^{-N}} \) for any \( \varepsilon > 0 \) and \( N < \infty \).

Fix a \( C^\infty \), strictly positive function \( m = m(\xi, \tau) \), homogeneous of degree 1 for large \(|(\xi, \tau)|\) and identically equal to \((1 + \tau^2)^{1/2} \) in a conic neighborhood of \( \{\xi = 0\} \). Define \( \Lambda^s \) to be the Fourier multiplier operator on \( \mathbb{R}^2 \) with symbol \( m(\xi, \tau)^s \).
Substituting \( v = \Lambda^{-s}u \) and \( g = \Lambda^{-s}f \), estimation of the \( H^s \) norm of \( v \), modulo a lower order norm, in terms of that of \( g \) is equivalent to estimation of the \( H^0 \) norm of \( u \) in terms of that of \( f \), modulo a negative order norm of \( u \). The equation \( \mathcal{L}_0 v = g \) becomes \( \Lambda^{-s}\mathcal{L}_0\Lambda^s u = f \). Write \( \Lambda^{-s}\ell\Lambda^s = \Lambda^{-s}\ell\Lambda^s \cong \Lambda^{-s}\ell\Lambda^s \) and similarly for other terms, and note that \( \Lambda^{-s}(\beta_j + A)\Lambda^s = \beta_j + A \) with the same function \( \beta_j \).

\( \Lambda^{\pm s} \) commutes with \( \partial_x \) and with \( \partial_t \), so \( \Lambda^{-s}(\partial_x + i\alpha t\partial_t)\Lambda^s = \partial_x + \Lambda^{-s}t\Lambda^s \cong \Lambda^{-s}\alpha\Lambda^s\partial_t \).

Applying the Fourier transform gives immediately \( \Lambda^{-s}[t,\Lambda^s]\partial_t = -s + E \), microlocally in some conic neighborhood of \( \tilde{\Gamma}^+ \), modulo operators smoothing there of infinite order. Since \( \Lambda^{-s}[\alpha,\Lambda^s]\partial_t \) is of order 0,

\[
\Lambda^{-s}t\Lambda^s \cong \Lambda^{-s}\alpha\Lambda^s\partial_t = \alpha\partial_t + \Lambda^{-s}[t,\Lambda^s]\alpha\partial_t + t \circ \Lambda^{-s}[\alpha,\Lambda^s] \circ \partial_t + E
= \alpha \circ (t\partial_t - s) + tB + E,
\]

microlocally near \( \tilde{\Gamma}^+ \). Thus microlocally near \( \tilde{\Gamma}^+ \), \( \Lambda^{-s}\mathcal{L}_0\Lambda^s = \mathcal{L}_s \) becomes

\[
\mathcal{L}_s = \bar{\ell}_s\ell_s + (\beta_1 + A)\bar{\ell}_s + (\beta_2 + A)\ell_s + (\beta_3 + A)
\]

where \( \bar{\ell}_s, \ell_s \) are first-order differential operators differing from \( \bar{\ell}, \ell \) respectively by terms of order zero, and taking the forms \( \bar{\ell}_s = \partial_x + i\alpha(t\partial_t - s), \ell_s = \partial_x - i\alpha(t\partial_t - s) \) for \( |x| \leq r/2 \), where \( \alpha \) depends only on \( t \) and the \( \beta_i \) only on \( x \).

To see that \( \bar{\ell}_s \) does take the form claimed for \( |x| \leq r/2 \), express \( \bar{\ell} = \partial_x + i\alpha t\partial_t \) modulo terms \( \gamma(x,t)\partial_x \) and \( \gamma(x,t)\partial_t \) where \( \gamma \equiv 0 \) for \( |x| \leq r/2 \). Then \( \Lambda^{-s}[\gamma(x,t)\partial_x,\Lambda^s] \) and \( \Lambda^{-s}[\gamma(x,t)\partial_t,\Lambda^s] \) are operators of the type \( A \), since they have nonpositive orders and their symbols of order zero vanish identically for \( |x| < r/2 \). \( \square \)

4. A Two-Dimensional Problem And Preliminary Inequalities.

The remainder of the paper consists of a self-contained analysis of a special class of pseudodifferential equations in a real two-dimensional region. We begin by describing the equations in question and fixing notation, which in some respects differs from that of preceding sections.

Fix an interval \( I = [-r, r] \subset \mathbb{R} \). Denote by \((x,t) \in \mathbb{R}^2\) coordinates in a neighborhood \( U \) of \( \mathcal{D} = I \times \{0\} \). The interval \( \mathcal{D} \) corresponds to the degenerate annulus embedded in the boundary of the worm domain, and will be the focus of attention. The convention concerning the symbols \( A, B, E \) introduced at the outset of §3 remains in force.

Consider a one parameter family of pseudodifferential operators of the form

\[
(4.1) \quad \mathcal{L}_s = \bar{L}L + (\beta_1(x) + A)\bar{L} + (\beta_2(x) + A)L + (\beta_3(x) + A)
\]

where the \( \beta_j \) are \( C^\infty \) functions. Suppose that \( \bar{L}, L \) are first-order differential operators depending on the real parameter \( s \), and that \(-L\) is the formal adjoint of \( \bar{L} \), modulo an
operator of order zero. Suppose that where $|x| \leq r$, they take the special forms\(^6\)

$$\bar{L} = \partial_x + ia(x)(t\partial_t + s) + O(t^2)\partial_t$$

$$L = \partial_x - i\bar{a}(x)(t\partial_t + s) + O(t^2)\partial_t.$$  

Here $O(t^2)$ denotes multiplication by a smooth function divisible by $t^2$ on the region $U$. $a$ and the coefficients $\beta_j$ are assumed independent of $s$, but $A$ and the terms $O(t^2)\partial_t$ are permitted to depend on $s$.

Assume that

\begin{equation}
Re a(x) \neq 0 \quad \text{for all } x \in I,
\end{equation}

and that there exist smooth real-valued coefficients $\mu, \nu$ such that $[\bar{L}, L] = i\mu(x, t)\partial_t + i\nu(x, t)\text{Re} \bar{L}$, satisfying

\begin{equation}
\mu \geq 0 \quad \text{at every point of } U.
\end{equation}

Because $\bar{L}^* = -L$ modulo a term of order zero, $L$ has the same real part as $\bar{L}$. A change of variables of the form $(x, t) \mapsto (x, h(x, t))$, with $h(x, 0) \equiv 0$ where $|x| \leq r$, therefore reduces matters to the case where the real parts of both $\bar{L}$ and $L$ are everywhere parallel to $\partial_x$, and $\bar{L} = \partial_x + i\bar{a}(x, t)(t\partial_t + s) + O(t^2)\partial_t$ on $I \times \mathbb{R}$, with $\bar{a}$ real-valued and nonvanishing. Rewrite $\bar{a}(x, t) = a(x) + O(t)$, and incorporate the contribution of $O(t)$ into the various terms $O(t^2)\partial_t$ and $A$. (4.3) is invariant under diffeomorphism and hence $\mu$ cannot change sign, so the coefficient of $t$ in the Taylor expansion of $\mu(x, t)$ about $t = 0$ must vanish identically, for $|x| \leq r$. This forces $\partial_x a(x) \equiv 0$ there. Thus

\begin{equation}
\bar{L} = \partial_x + ia(t\partial_t + s) + O(t^2)\partial_t
\end{equation}

for $|x| \leq r$, where $a$ is a nonzero real constant. Moreover, $\partial_x$ may be expressed in $U$ as a nonvanishing scalar multiple of $\bar{L} + L$, modulo an operator of order zero. From now on we work in these new coordinates.

Define $\Gamma = \{(x, t, \xi, \tau) : (x, t) \in D \text{ and } \xi = 0\}$. Decompose $\Gamma = \Gamma^+ \cup \Gamma^-$ where $\Gamma^+ = \{\tau > 0\} \cap \Gamma$. Then by (4.3), in some conic neighborhood of $\Gamma^+$ the principal symbol of $[\bar{L}, L]$ equals a nonpositive symbol, modulo terms in the span of the symbols of $\bar{L}, L$ and a term of order zero.

The symbol $\| \cdot \|$, with no subscript, denotes the norm in $L^2(U)$, while $\| \cdot \|_t$ denotes any fixed norm for the Sobolev space $H^t$ of functions having $t$ derivatives in $L^2$ and supported in $U$. The goal of the remainder of the paper is the following a priori estimate.

\(^6\)No assumption is now made on the vanishing or nonvanishing of the coefficient of $\partial_t$ in $\bar{L}$ where $|x| > r$, but the strict pseudoconvexity of $W$ outside the exceptional annulus was used to reduce Proposition 1 to Proposition 2 below.
Proposition 2. Let \( \{L_s\} \) be a family of operators of the form (4.1) satisfying all of the hypotheses introduced above. Then there exist a discrete exceptional set \( S \subset [0, \infty) \) such that for any \( s \not\in S \) and any pseudodifferential operator \( Q \) of order 0 whose principal symbol is nonzero in some conic neighborhood of \( \Gamma^- \), there exist \( C < \infty \) and a neighborhood \( W \) of \( D \) such that for every \( C^\infty \) function \( u \) supported in \( W \\
\|u\| + \|\bar{L}u\| + \|Lu\| \leq C \cdot (\|L_su\| + \|u\|_{-1} + \|Qu\|_1) \\
\) 

The key conclusions are that there is no loss of derivatives in estimating \( u \) in terms of \( L_su \), and that this holds for a sequence of values of \( s \) tending to \( +\infty \). The assumption that \( u \in C^\infty \) is essential. All hypotheses of \( \S 4 \) are satisfied by the family of operators \( L_s \) derived in \( \S 2 \) and \( \S 3 \). Proposition 2 thus implies the validity of (3.2), and hence of Proposition 1, which in turn implies our Theorem.

Our first preliminary estimate is a standard one valid for all \( s \in \mathbb{R} \).

Lemma 1. For each exponent \( s \) and each \( Q \) there exists \( C < \infty \) such that
\[
\|\partial_x u\| \leq C\|u\| + C\|L_su\| + C\|Qu\|_1
\]
for every \( u \in C^\infty_0(U) \).

Proof. For \( (x,t) \in D, \sigma_2(L_s)(x,t,\xi,\tau) = 0 \) if and only if \( (x,t,\xi,\tau) \in \Gamma \). Therefore the characteristic variety of \( L_s \) in \( T^*W \) is contained in an arbitrarily small conic neighborhood of \( \Gamma \) as \( \delta \to 0 \). Consequently there exists an operator \( \tilde{Q} \) of order zero such that firstly, \( T^*W \) is contained in the union of the two regions where \( \tilde{Q} \) is elliptic and \( \tau > 0 \), and secondly, the symbol of \( \tilde{Q} \) is supported in the union of the two regions where \( L_s \) is elliptic, and where \( Q \) is elliptic.

Since \( L_s \) is elliptic outside a small conic neighborhood of \( \Gamma \), the \( H^2 \) norm of \( u \) is majorized away from \( \Gamma \) by \( \|L_su\| + \|u\|_{-1} \), while in a conic neighborhood of \( \Gamma^- \) the \( H^1 \) norm of \( u \) is majorized by \( \|Qu\|_1 + \|u\|_{-1} \).

Write \( \langle f, g \rangle = \int_U f\bar{g} \, dx \, dt \). By Gårding’s inequality and the fact that \( i\mu \cdot i\tau \leq 0 \) in the support of the symbol of \( \tilde{Q} \),
\[
- \Re \langle \bar{L}Lu, u \rangle \geq c\|Lu\|^2 + c\|\bar{Lu}\|^2 - C\|Lu\| \cdot \|u\| - C\|L\| \cdot \|u\| - C\|u\|^2 - C\|\tilde{Q}u\|^2.
\]

The second condition imposed on \( \tilde{Q} \) ensures that
\[
\|\tilde{Q}u\|_1 \leq C\|L_su\| + C\|Qu\|_1 + C\|u\|_{-1}.
\]

Estimating \( \langle (L_s - \bar{L}L)u, u \rangle \) by Cauchy-Schwarz thus leads to
\[
\|\bar{L}u\| + \|Lu\| \leq C\|L_su\| + C\|u\| + C\|Qu\|_1.
\]

But \( \partial_x \) may be expressed as a linear combination of \( \bar{L} \) and of \( L \) modulo an operator of order 0. \qed
Lemma 2. There exists $C < \infty$ such that for every $f \in C^1(R)$ and every $\varepsilon > 0$,

$$\|f\|_{L^2[\varepsilon, 2\varepsilon]} \leq C\|f\|_{L^2[-2\varepsilon, -\varepsilon]} + C\varepsilon\|\partial_x f\|_{L^2(R)}.$$ 

Likewise

$$|f(0) - f(-\varepsilon)| \leq C\varepsilon^{1/2}\|\partial_x f\|_{L^2}.$$

The conclusions are invariant under translation, and the lemma will be invoked in that more general form.

Proof. For each $x \in (\varepsilon, 2\varepsilon)$, $|f(x) - f(x - 3\varepsilon)| \leq \int_{-2\varepsilon}^{2\varepsilon} |\partial_x f(y)| \, dy$ and both conclusions follow from the triangle and Cauchy-Schwarz inequalities. \qed

To simplify notation define

$$\mathfrak{B} = \|L_s u\| + \|u\|_{-1} + \|Qu\|_{1}.$$ 

Let $\delta > 0$ be a small constant to be chosen in §6, and assume $u$ to be supported in

$$W \subset \{(x, t) : |t| < \delta, \ |x| < r + \delta\}.$$ 

Applying Lemma 2 to the function $x \mapsto u(x, t)$ for each $t$ and applying Lemma 1 gives the following estimate, under the hypotheses of Lemma 1.

Lemma 3.

$$\|\partial_x u\| + \|u\| \leq C\|u\|_{L^2(I \times (-\delta, \delta))} + C\mathfrak{B}.$$ 

5. Limiting Operators and Mellin Transform.

Let $a$ be a nonvanishing $C^\infty$, real-valued function. For $\zeta \in \mathbb{C}$ define the ordinary differential operator

$$H_\zeta = (\partial_x + i\zeta a(x))(\partial_x - i\zeta a(x)) + \beta_1(x)(\partial_x + i\zeta a(x)) + \beta_2(x)(\partial_x - i\zeta a(x)) + \beta_3(x),$$

acting on functions of $x \in I$. Only the case of constant $a$ will be needed in this paper, but the general case arises in another problem and hence merits discussion.

Definition 1. $\mathfrak{S}$ is defined to be the set of all $\zeta \in \mathbb{C}$ such that there exists a solution $g$ of $H_\zeta g \equiv 0$ on $I$, satisfying $g(-r) = g(r) = 0$.

For any complex number $w$ we write $\langle w \rangle = (1 + |w|^2)^{1/2}$. 

11
Lemma 4. \( \mathcal{G} \) is a discrete subset of \( \mathbb{C} \), and for any compact subset \( K \) of \([0, \infty)\), the set of all \( \zeta \in \mathcal{G} \) having real part in \( K \) is finite. For each \( s \) such that \( \mathcal{G} \cap (s + i\mathbb{R}) = \emptyset \) there exists \( C < \infty \) such that for all \( \zeta \in s + i\mathbb{R} \), for all \( f, \varphi, \psi \in C^\infty(I) \) satisfying \( H_\zeta f = \varphi + \partial_x \psi \), one has

\[
\|f\|_{L^2(I)} + \langle \zeta \rangle^{-1} \|\partial_x f\|_{L^2(I)} \\
\leq C\langle \zeta \rangle^{-1/2} (|f(-r)| + |f(r)|) + C\langle \zeta \rangle^{-2} \|\varphi\|_{L^2(I)} + C\langle \zeta \rangle^{-1} \|\psi\|_{L^2(I)}.
\]

Proof. Throughout this proof, all norms without subscripts denote \( L^2 \) norms. The selfadjoint part of \( \tilde{H}_\zeta \), applied to \( f \), equals \( (\partial_x - \gamma a(x))(\partial_x + \gamma a(x))f \), modulo \( O(\gamma \|f\| + \|\partial_x f\|) \), in the \( L^2(I) \) norm. Therefore for \( s \) in any fixed compact subset of \( \mathbb{R} \) and any \( \zeta = s + i\gamma \in s + i\mathbb{R} \), for any \( f \) vanishing at both endpoints of \( I \),

\[-\Re \langle H_\zeta f, f \rangle \geq \|\partial_x f\|^2 + \gamma^2 \int_I |f|^2 |a|^2 - O(\gamma \|f\|^2 + \|f\| \cdot \|\partial_x f\|).
\]

The coefficient \( a \) vanishes nowhere, while

\[|\langle H_\zeta f, f \rangle| = |\langle \varphi + \partial_x \psi, f \rangle| \leq \|f\| \cdot \|\varphi\| + \|\partial_x f\| \cdot \|\psi\|.
\]

Combining the last two inequalities and invoking the Cauchy-Schwarz inequality and small constant – large constant trick, one obtains

\[
\gamma^2 \|f\| + |\gamma| \cdot \|\partial_x f\| \leq C\|\varphi\| + C|\gamma| \cdot \|\psi\|
\]

for all sufficiently large \( |\gamma| \), under the additional hypothesis that \( f \) vanishes at both endpoints of \( I \).

There exists a unique solution \( \phi_\zeta \) of \( H_\zeta \phi_\zeta = 0 \) on \( I \), satisfying \( \phi_\zeta(-r) = 0 \), \( \partial_x \phi_\zeta(-r) = 1 \). Then \( \phi_\zeta(r) \) is an entire holomorphic function of \( \zeta \), and \( \zeta \in \mathcal{G} \Leftrightarrow \phi_\zeta(r) = 0 \). We have seen that \( \zeta \notin \mathcal{G} \) provided that the imaginary part of \( \zeta \) is sufficiently large, when the real part stays in a bounded set. Thus \( \phi_\zeta(r) \) is nonconstant, so has discrete zeros.\(^7\)

To prove (5.1) let \( \zeta = s + i\gamma \) and \( f \in C^2(I) \) be given, and decompose \( f = g + h \) where \( H_\zeta g \equiv 0 \) and \( h \) vanishes at the endpoints of \( I \). The hypothesis \( \mathcal{G} \cap (s + i\mathbb{R}) = \emptyset \) means that the Dirichlet nullspace of \( H_\zeta \) is \( \{0\} \), so by elementary reasoning we conclude that for each \( \gamma \) there exists \( C < \infty \) such that \( \|h\| + \|\partial_x h\| \leq C\|\varphi\| + \|\psi\| \), since \( H_\zeta h = H_\zeta f = \varphi + \partial_x \psi \). Moreover, since \( H_\zeta \) depends continuously on \( \zeta \), \( C \) may be taken to be independent of \( \zeta \) in any compact subset of \( \mathbb{C}\setminus\mathcal{G} \). When \( |\gamma| \) is sufficiently large, on the other hand, (5.2) implies \( \|h\| + \langle \zeta \rangle^{-1} \|\partial_x h\| \leq C\langle \zeta \rangle^{-2} \|\varphi\| + C\langle \zeta \rangle^{-1} \|\psi\| \). Thus the component \( h \) of \( f \) satisfies (5.1).

\(^7\)An alternative method of proof would be to combine (5.2) with general results from the perturbation theory of linear operators [Ka], utilizing again the holomorphic dependence of \( H_\zeta \) on \( \zeta \).
We have $H_\zeta g = 0$, so that clearly $\|g\|$ and $\|\partial_x g\|$ are majorized by $C|f(r)| + C|f(-r)|$, uniformly for $\zeta$ in any compact set disjoint from $\mathcal{S}$. Assuming henceforth that $|\gamma|$ is large, the equation gives the inequality

$$\|\partial^2_x g\| \leq C\gamma^2\|g\| + C|\gamma| \cdot \|\partial_x g\|.$$ Integrating by parts as in the proof of (5.2) yields

$$(5.3) \quad \|\partial_x g\|^2 + \gamma^2\|g\|^2 \leq C|g(-r)\partial_x g(-r)| + C|g(r)\partial_x g(r)|.$$ To control the right hand side we use the bound

$$|\partial_x g(-r)| + |\partial_x g(r)| \leq C|\gamma|^{1/2}\|\partial_x g\| + C|\gamma|^{-1/2}\|\partial^2_x g\|.$$ Indeed, setting $v = \partial_x g$, for any $r' \in [r - |\gamma|^{-1}, r]$

$$|v(r) - v(r')| \leq C \int_{r'}^r |\partial_x v| \leq C|\gamma|^{-1/2}\|\partial_x v\|_{L^2}.$$ Then

$$|v(r)| \leq |\gamma| \int_{r-|\gamma|^{-1}}^{r'} |v(r) - v(r')| \, dr' + |\gamma| \int_{r-|\gamma|^{-1}}^{r} |v(r')| \, dr'$$ and the desired bound follows by Cauchy-Schwarz.

Putting this into (5.3), introducing a parameter $\lambda \in \mathbb{R}^+$ and applying Cauchy-Schwarz yields

$$\gamma^2\|g\|^2 + \|\partial_x g\|^2$$

$$\leq C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^{-1}\|\partial^2_x g\|^2$$

$$\leq C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^{-1}(\gamma^4\|g\|^2 + \gamma^2\|\partial_x g\|^2)$$

$$\leq C\lambda|g(-r)|^2 + C\lambda|g(r)|^2 + C\lambda^{-1}|\gamma| \cdot \|\partial_x g\|^2 + C\lambda^{-1}|\gamma|^{-3}\|g\|^2.$$ Choose $\lambda$ to be a large constant times $|\gamma|$. Then the last two terms on the right-hand side may be absorbed into the left, leaving

$$\gamma^2\|g\|^2 + \|\partial_x g\|^2 \leq C|\gamma| \cdot |g(-r)|^2 + C|\gamma| \cdot |g(r)|^2.$$ Since $g = f$ at the endpoints of $I$, this is the desired inequality for $g$. Adding it to that for $h$ concludes the proof.
Definition 2.

\[ S = \{ s \in [0, \infty) : \text{there exists } \gamma \in \mathbb{R} \text{ such that } s - \frac{1}{2} + i\gamma \in \mathbb{C} \} . \]

Lemma 4 guarantees that \( S \) is discrete.

Specialize now to the case where \( a(x) \equiv a \), the real constant in (4.4). Define

\[
L_s = (\partial_x + ia(t\partial_t + s)) \circ (\partial_x - ia(t\partial_t + s)) + \beta_1(x)(\partial_x + ia(t\partial_t + s)) + \beta_2(x)(\partial_x - ia(t\partial_t + s)) + \beta_3(x).
\]

Expanding the last term in the expression \( L_s = L_s + (L_s - L_s) \) gives

\[
L_s u = \Phi + \partial_x \Psi
\]

where

\[
\Phi = L_s u + (t\partial_t)^2Au + t\partial_tAu + Au
\]

\[
\Psi = t\partial_tAu + Au
\]

To reach (5.4) we may for instance express \( t\partial_t \circ O(t^2)\partial_t \) as \((t\partial_t)^2A + t\partial_tA + A\), since multiplication by \( t \) is an operator of the type \( A \). Likewise \([A,t\partial_t] = t[A,\partial_t] + [A,t]\partial_t\) is an operator of type \( A \), because \( \sigma_{-1}([A,t]) = c\partial_t\sigma_0(A) \) vanishes identically for \((x,t) \in D\) since \( \sigma_0(A) \) itself vanishes there.

The partial Mellin transform of \( f \) with respect to the \( t \) variable is defined to be

\[
\hat{f}(x,\gamma) = \int_0^\infty f(x,t)t^{-i\gamma}t^{-1}dt,
\]

provided that the integral converges. If \( f(x,\cdot) \in C^\infty[0,\infty) \) has bounded support for each \( x \), then the integral defining \( \hat{f}(x,\gamma) \) converges absolutely whenever \( \gamma \) has strictly positive imaginary part, and \( \hat{f}(x,\gamma) \) extends to a meromorphic function of \( \gamma \in \mathbb{C} \), whose only possible poles are at \( \gamma = 0, -i, -2i, \ldots \). Clearly

\[
(t\partial_t f)(x,\gamma) = i\gamma \hat{f}(x,\gamma)
\]

for all such \( f \). Consequently

\[
(L_s u)(x,\gamma) = H_{s+i\gamma} \hat{u}(x,\gamma) \quad \text{for all } \gamma \in \mathbb{C} \setminus \{0, -i, -2i, \ldots \}.
\]

The Mellin inversion and Plancherel formulas read

\[
f(x,t) = c \int_\mathbb{R} \hat{f}(x,\gamma) t^{i\gamma}d\gamma, \quad \int_0^\infty |f(x,t)|^2 t^{-1}dt = c' \int_\mathbb{R} |\hat{f}(x,\gamma)|^2 d\gamma.
\]

It follows directly from the definitions that \((t^{1/2}f)(x,\gamma) = \hat{f}(x,\gamma + \frac{i}{2})\) for all \( \gamma \in \mathbb{R} \). Thus the Plancherel identity may be rewritten as

\[
\int_0^\infty |f(x,t)|^2 dt = \int_0^\infty |t^{1/2}f(x,t)|^2 t^{-1}dt = c' \int_\mathbb{R} |\hat{f}(x,\gamma + \frac{i}{2})|^2 d\gamma.
\]
6. Proof of the Main Estimate.
We may now estimate \( u \) in terms of \( \mathcal{L}_s u \). To begin,

\[
\int_I \int_{[0,\delta]} |u(x,t)|^2 \, dx \, dt = c' \int_I \int_{\mathbb{R}} |\hat{u}(x,\gamma+i\frac{\delta}{2})|^2 \, d\gamma \, dx.
\]

Assume that \( s \notin S \), and write \( \zeta = s - \frac{1}{2} + i\gamma \). With \( \Phi, \Psi \) defined as in (5.4), \( H_\zeta \hat{u}(x,\gamma+i\frac{1}{2}) = \hat{\Phi}(x,\gamma+i\frac{1}{2}) + \partial_x \hat{\Psi}(x,\gamma+i\frac{1}{2}) \). Applying Lemma 4 on \( I \) yields for each \( \gamma \in \mathbb{R} \)

\[
(6.1) \quad \int_I \hat{u}(x,\gamma+i\frac{1}{2}) \, dx \leq C \int_I |\Phi(x,\gamma+i\frac{1}{2})|^4 \, dx + C \int_I |\Psi(x,\gamma+i\frac{1}{2})|^2 \, dx
\]
\[
+ C|\hat{u}(-r,\gamma+i\frac{1}{2})|^2 + C|\hat{u}(r,\gamma+i\frac{1}{2})|^2.
\]

**Lemma 5.** Assume \( W \subset \{(x,t) : |t| < \delta, |x| < r + \delta\} \). Then there exists \( C < \infty \) such that

\[
\int_I \int_{\mathbb{R}} |\Phi(x,\gamma+i\frac{1}{2})|^2 \, d\gamma \, dx \leq C \delta^2 \|u\|^2 + C\mathfrak{B}^2
\]

and

\[
\int_I \int_{\mathbb{R}} |\Psi(x,\gamma+i\frac{1}{2})|^2 \, d\gamma \, dx \leq C \delta^2 \|\partial_x u\|^2 + C\mathfrak{B}^2.
\]

Granting the lemma, we conclude from (6.1) that

\[
\|u\|_{L^2(I \times [0,\delta])}^2 \leq C \delta^2 \|u\|^2 + C \delta^2 \|\partial_x u\|^2 + C \mathfrak{B}^2 + C \int_{\mathbb{R}} |u(r,t)|^2 \, dt + C \int_{\mathbb{R}} |u(-r,t)|^2 \, dt.
\]

But by Lemma 2 and the assumption that \( u(x,t) \equiv 0 \) for \( |x| > r + \delta \), this last term is dominated by \( C \delta \|\partial_x u\|^2 \). Thus

\[
\|u\|_{L^2(I \times [0,\delta])}^2 \leq C \delta^2 \|u\|^2 + C \delta \|\partial_x u\|^2 + C\mathfrak{B}^2.
\]

All the same reasoning applies on the region \( I \times (-\delta,0] \), after the change of variables \( t \mapsto -t \). Thus

\[
\|u\|_{L^2(I \times (-\delta,0])} \leq C \delta^{1/2} (\|u\| + \|\partial_x u\|) + C\mathfrak{B}.
\]

Combining this with Lemma 3 gives

\[
\|u\| + \|\partial_x u\| \leq C \delta^{1/2} (\|u\| + \|\partial_x u\|) + C\mathfrak{B},
\]

so choosing \( \delta \) to be sufficiently small gives \( \|u\| \leq C\mathfrak{B} \), concluding the proof. \( \square \)
Proof of Lemma 5. The principal term in the double integral of the lemma for $\Phi$ is of course the contribution of $L_su$:

$$
\int_I \int_{\mathbb{R}} \left| (L_su)^*(x, \gamma + \frac{i}{2}) \right|^2 \langle \gamma \rangle^{-4} d\gamma dx
\leq \int_I \int_{\mathbb{R}} \left| (L_su)^*(x, \gamma + \frac{i}{2}) \right|^2 d\gamma dx
\leq C \|L_su\|^2,
$$
as desired.

Any operator $A$ of order $\leq 0$ satisfying $\sigma_0(A)(x, t, \xi, \tau) \equiv 0$ for $(x, t) \in D$ satisfies

$$
\|Au\|^2 \leq C \delta^2 \|u\|^2 + C \|u\|_{-1}^2
$$
for all $u$ supported in $W$, as $\delta \to 0$. A typical term of $\Phi$ resulting from $L_su - L_su$ is $(t\partial_t)^2 Au$. Its contribution to the first double integral in Lemma 5 is

$$
\int_I \int_{\mathbb{R}} \left| ((t\partial_t)^2 Au)^*(x, \gamma + \frac{i}{2}) \right|^2 \langle \gamma \rangle^{-2} d\gamma dx
\leq C \int_I \int_{\mathbb{R}} \left| (Au)^*(x, \gamma + \frac{i}{2}) \right|^2 d\gamma dx
= C \int_I \int_{[0, \delta]} |Au(x, t)|^2 dx dt
\leq C \delta^2 \|u\|^2 + C \|u\|_{-1}^2.
$$
A typical constituent of $\Psi$ is the term $t\partial_t Au$. Its contribution is dominated by

$$
C \int_I \int_{\mathbb{R}} \left| (t\partial_t Au)^*(x, \gamma + \frac{i}{2}) \right|^2 \langle \gamma \rangle^{-2} d\gamma dx
\leq C \int_I \int_{\mathbb{R}} \left| (Au)^*(x, \gamma + \frac{i}{2}) \right|^2 d\gamma dx
$$
and the remainder of the calculation is as above. □

Comments. The author can advance no reason why the method of reduction to the boundary should be essential to this analysis. Working directly on $\overline{W}$ might well result in a shorter proof. On the other hand, the analysis in §§4-6 applies with minor modification to a broader class of equations unconnected with the $\bar{\partial}$–Neumann problem.

Some refinements of this analysis and related observations will appear in [Ch2].

References

[B] D. Barrett, *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. 168 (1992), 1-10.

[BL] S. Bell and E. Ligocka, *A simplification and extension of Fefferman’s theorems on biholomorphic mappings*, Invent. Math. 57 (1980), 283-289.

[BS1] H. Boas and E. Straube, *Equivalence of regularity for the Bergman projection and the $\bar{\partial}$–Neumann operator*, Manuscripta Math. 67 (1990), 25-33.
[BS2] _____, Sobolev estimates for the $\bar{\partial}$–Neumann operator on domains in $\mathbb{C}^n$ admitting a defining function that is plurisubharmonic on the boundary, Math. Zeitschrift 206 (1991), 81-88.

[Ca1] D. Catlin, Subelliptic estimates for the $\bar{\partial}$–Neumann problem on pseudoconvex domains, Annals of Math. 126 (1987), 131-191.

[Ca2] _____, Global regularity of the $\bar{\partial}$–Neumann problem, Proc. Symp. Pure Math. 41 (1984), 39-49.

[CNS] D.-C. Chang, A. Nagel and E. M. Stein, Estimates for the $\bar{\partial}$–Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$, Acta Math. 169 (1992), 153-228.

[Ch1] M. Christ, The Szegö projection need not preserve global analyticity, Annals of Math. (to appear).

[Ch2] _____, Global $C^\infty$ irregularity for mildly degenerate elliptic operators, preprint.

[DF] K. Diederich and J. E. Fornæss, Pseudoconvex domains: an example with nontrivial Nebenhülle, Math. Ann. 225 (1977), 275-292.

[FS] J. E. Fornæss and B. Stensønes, Lectures On Counterexamples In Several Complex Variables, Mathematical Notes 33, Princeton University Press, Princeton, NJ, 1987.

[Ka] T. Kato, Perturbation Theory For Linear Operators, Springer-Verlag, New York, 1966.

[K1] J. J. Kohn, Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds, Trans. AMS 181 (1973), 273-292.

[K2] _____, Estimates for $\bar{\partial}_b$ on pseudoconvex CR manifolds, Proc. Symp. Pure Math. 43 (1985), 207-217.

Department of Mathematics, UCLA, Los Angeles, CA. 90095-1555
E-mail address: christ @ math.ucla.edu