INTERACTION OF OSCILLATORY PACKETS OF WATER WAVES

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Abstract. For surface gravity water waves we give a detailed analysis of the interaction of two NLS described wave packets with different carrier waves. We separate the internal dynamics of each wave packet from the dynamics caused by the interaction and prove the validity of a formula for the envelope shift caused by the interaction of the wave packets.

1. Introduction. It is the purpose of this paper to bring together two kinds of recent approximation results, namely the justification of the NLS approximation for the water wave problem without surface tension in infinite and finite depth [10, 5] and the detailed description of the interaction of two oscillatory wave packets in NLS scaling for general dispersive wave systems [4]. This will allow us to give a detailed description of the interaction of two such wave packets for the water wave problem.

The 2D water wave problem without surface tension consists in finding the irrotational flow of an incompressible fluid in an infinitely long canal of finite or infinite depth with a free top surface under the influence of gravity. The bottom is assumed to be at infinity or to be impermeable. The pressure is assumed to be constant along the free top surface and particles at the top surface are assumed to stay at the top surface. In the Lagrangian formulation of the water wave problem which has been used in [5] for fixed time \( t \) the free surface of the fluid is written as

\[
\Gamma(t) = \{ (\tilde{X}_1(\alpha, t), \tilde{X}_2(\alpha, t)) = (\alpha + X_1(\alpha, t), X_2(\alpha, t)) : \alpha \in \mathbb{R} \}.
\]

It is a Jordan-curve which in case of finite depth has no intersection with the bottom \( \{(\alpha, -1) : \alpha \in \mathbb{R} \} \). Under the previous assumptions in this formulation the problem is completely determined by the evolution of the variables \( W = (Z_1, X_2, U_1) \), with \( U_1 = \partial_1 X_1 \) and \( Z_1 = K_0 X_1 \) where \( K_0 \) has the Fourier symbol \( \hat{K}_0(k) = -i \tanh(k) \). See Section 2 for more details.

For this problem we are interested in the evolution, respectively interaction, of two wave packets in NLS scaling, i.e., we consider initial conditions of the form

\[
W = \varepsilon A_{1,init}(\varepsilon x) e^{ik_1 x} \varphi(k_1) + \varepsilon A_{2,init}(\varepsilon x) e^{ik_2 x} \varphi(k_2) + \text{c.c.,}
\]

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with spatially localized amplitude functions $A_{j,\text{init}}$, basic spatial wave numbers $k_j$, with $k_2 > k_1 > 0$, $\varphi(k_j) \in \mathbb{C}^3$, and $0 < \varepsilon \ll 1$ a small perturbation parameter.

Figure 1. Two wave packets in NLS scaling

In [11] it has been proposed that in the limit of small $\varepsilon > 0$ the evolution of a single wave packet is described via the approximation

$$W = \varepsilon \Psi_{\text{NLS}} + \mathcal{O}(\varepsilon^2) = \varepsilon A(\varepsilon(x - ct), \varepsilon^2 t) e^{i(k_0 x - \omega_0 t)} \varphi(k_0) + \text{c.c.} + \mathcal{O}(\varepsilon^2),$$

with $c$ being the group velocity of the wave packet and $\omega_0 > 0$ being the basic temporal wave number associated to the basic spatial wave number $k_0 > 0$ via the dispersion relation of the water wave problem. The slow time scale is $T = \varepsilon^2 t \in \mathbb{R}$, the slow spatial scale is $X = \varepsilon(x - ct) \in \mathbb{R}$, and the complex-valued amplitude $A = A(X, T) \in \mathbb{C}$ solves in lowest order the NLS equation

$$\partial_T A = i \nu_1 \partial_X^2 A + i \nu_2 |A|^2,$$

with coefficients $\nu_j = \nu_j(k_0) \in \mathbb{R}$. An approximation theorem that the NLS equation makes correct predictions about the dynamics of the water wave problem without surface tension in infinite and finite depth has been shown only recently in [10, 5].

A straightforward generalization of the previous ansatz to two (or more) wave packets would be

$$W = \sum_{j=1,2} \varepsilon A(\varepsilon(x - c_j t), \varepsilon^2 t) e^{i(k_j x - \omega_j t)} \varphi(k_j) + \text{c.c.}.$$

However, such an ansatz does not lead to a consistent system of equations. The coupled system of NLS equations

$$\partial_T A_1(X_1, T) = i \nu_1(k_1) \partial_{X_1}^2 A_1(X_1, T) + i \nu_2(k_1) A_1(X_1, T) |A_1(X_1, T)|^2$$
$$+ i \nu_3(k_1, k_2) A_1(X_1, T) |A_2(X_2, T)|^2,$$

$$\partial_T A_2(X_2, T) = i \nu_1(k_2) \partial_{X_2}^2 A_2(X_2, T) + i \nu_2(k_2) A_2(X_2, T) |A_2(X_2, T)|^2$$
$$+ i \nu_4(k_1, k_2) A_2(X_2, T) |A_1(X_1, T)|^2,$$

depends via $X_2 = X_1 + \varepsilon^{-1}(c_1 - c_2) T$ in a singular way on the small perturbation parameter $0 < \varepsilon \ll 1$. This singular dependence has the simple consequence that pulses with different velocities, in other words, $c_1 \neq c_2$, do not interact in lowest order w.r.t. the small perturbation parameter $0 < \varepsilon \ll 1$, i.e., in lowest order the first wave packet behaves as if the second wave packet is not present, and vice versa. This has been first rigorously proved in [7] for a Klein-Gordon equation as original system. This property of non-interaction in lowest order has been discussed in [1] for various classes of original systems and general wave packets, and is also true for the water wave problem.

Our interest is in a more detailed description of the interaction. In [6, 9] formulas have been derived for the phase and envelope shift caused by the interaction of two pulses in soliton form for nonlinear wave equations. A first attempt to justify these formulas has been made in [2, 8], where the shift of the envelope has been estimated to be of order $\mathcal{O}(\varepsilon)$ for pulses in soliton form. This had improved estimates known from [7, 1] for the shifts of the envelopes from $\mathcal{O}(1)$ to $\mathcal{O}(\varepsilon)$. 
In [3] the existing results have been improved significantly. For the Klein-Gordon equation a separation of the internal and the interaction dynamics of general wave packets in NLS form up to high order has been made. The internal dynamics is described by (uncoupled) nonlinear and linear Schrödinger equations. The interaction dynamics is mainly described by phase shift functions $\Omega^{(1)}_j$ for the underlying carrier waves and pulse shift functions $\Psi^{(1)}_j$ for the envelopes. In [4] a framework for the detailed description of the interaction of wave packets in NLS scaling for general dispersive wave systems has been given. It is the purpose of this paper to explain that the framework developed in [4] applies to the water wave problem as well. In lowest order we have

$$\mathcal{W} = \varepsilon A^{(1)}_1 \left( \varepsilon (x - c_1 t + \varepsilon \Psi^{(1)}_1), \varepsilon^2 t \right) e^{i(k_1 x + \omega_1 t + \varepsilon \Omega^{(1)}_1)} \varphi(k_1)$$

$$+ \varepsilon A^{(1)}_2 \left( \varepsilon (x - c_2 t + \varepsilon \Psi^{(1)}_2), \varepsilon^2 t \right) e^{i(k_2 x + \omega_2 t + \varepsilon \Omega^{(1)}_2)} \varphi(k_2)$$

$$+ \mathcal{O}(\varepsilon^2) + \text{c.c.},$$

where the $A^{(1)}_j$ satisfy the decoupled NLS equations (1), with coefficients $\nu_j = \nu_j(k_m)$ for $j, m \in \{1, 2\}$, and describe the internal dynamics of the wave packets. The interaction dynamics is described by the phase shifts $\Omega^{(1)}_j$ and the envelope shifts $\Psi^{(1)}_j$ which satisfy phase shift formulas

$$\partial_t \Omega^{(1)}_1 = \gamma_{11} |A^{(1)}_2|^2,$$

$$\partial_t \Omega^{(1)}_2 = \gamma_{12} |A^{(1)}_1|^2,$$

and envelope shift formulas

$$\partial_t \Psi^{(1)}_1 = \gamma_{21} |A^{(1)}_2|^2,$$

$$\partial_t \Psi^{(1)}_2 = \gamma_{22} |A^{(1)}_1|^2,$$

with coefficients $\gamma_{ij} \in \mathbb{R}$ which depends on $k_1$, $k_2$, respectively. That the system really behaves as indicated by the displayed formulas comes from the form of the terms of $\mathcal{O}(\varepsilon^2)$ and higher. They have the same structure up to an order $\mathcal{O}(\varepsilon^{7/2})$. The internal dynamics is described by functions $A^{(n)}_j$ which satisfy decoupled linearized NLS equations. The interaction dynamics is described by the phase shifts $\Omega^{(1)}_j$, the envelope shifts $\Psi^{(1)}_j$, and higher order corrections $\Omega^{(2)}_j$. The ansatz is consistent to higher order since for $j = 1, 2$ and $m = 1, 2$ the variables depend in the following way on the coordinates

$$A^{(m)}_j = A^{(m)}_j \left( \varepsilon (x - c_j t + \varepsilon \Psi^{(1)}_j), \varepsilon^2 t \right),$$

$$\Omega^{(m)}_j = \Omega^{(m)}_j \left( \varepsilon (x - c_{3-j} t + \varepsilon \Psi^{(1)}_j), \varepsilon^2 t \right),$$

$$\Psi^{(1)}_j = \Psi^{(1)}_j \left( \varepsilon (x - c_{3-j} t), \varepsilon^2 t \right).$$

In addition to the system of differential equations, we get a system of algebraic equations and corresponding non-resonance conditions from the cancelation of mixed and higher order harmonics. It has been explained in [4] that the solutions of this system of amplitude equations exist globally in time.

The plan of the paper is as follows. After describing the water wave problem in more detail, we explain in Section 2 the NLS approximation results obtained for the water wave problem without surface tension in infinite and finite depth in [10, 5] for a single wave packet. We explain that all parts of the validity proofs except of the estimates for the residual transfer line for line if instead of one wave packet two wave packets are considered. In Section 3 we explain that the framework developed in [4] applies to the water wave problem, i.e., we give the detailed description of the interaction of the wave packets. We explain that in suitably chosen variables the internal dynamics and the interaction dynamics of the wave packets can be separated up to high order.

**Notation.** For $m_1, m_2 \in \mathbb{N}$, we define the weighted spaces $H^{m_2}(m_1) = \{ u \in L^2(\mathbb{R}) : \| u \|_{H^{m_2}(m_1)} < \infty \}$ with norm $\| u \|_{H^{m_2}(m_1)} = \| u \rho^{m_1} \|_{H^{m_2}(\mathbb{R})}$, where $\rho(x) = (1 + x^2)^{1/2}$, and
$H^{m_2}(\mathbb{R})$ is the Sobolev space of functions with weak derivatives up to order $m_2$ in $L^2(\mathbb{R})$. The Fourier transform is denoted by $\mathcal{F}$, so that $\hat{u}(k) := \mathcal{F}(u)(k) = \frac{1}{\sqrt{2\pi}} \int e^{-ikx}u(x)dx$ for $u \in L^2(\mathbb{R})$.

2. The NLS approximation for the water wave problem. Under the assumptions on the flow from the beginning of this paper the dynamics of the 2D water wave problem is completely determined by the evolution of the free surface $\Gamma(t)$ which is governed by

$$
\partial_t^2 X_1(1 + \partial_x X_1) + \partial_x X_2(1 + \partial_x^2 X_2) = 0, \quad (2)
$$

$$
\partial_t X_2 = \mathcal{K}(X) \partial_t X_1. \quad (3)
$$

The operator $\mathcal{K}(X)$ depends linearly on $U_1 = \partial_t X_1$, but nonlinearly on $X$. It is related to the Dirichlet-Neumann operator and its existence is a consequence of the incompressibility and irrotationality of the flow. It is defined by $\mathcal{K}(X)U_1 = \partial_{x_2}\phi|_{\Gamma(t)}$, where $\phi : \Omega(t) \to \mathbb{R}$ solves for fixed $t$ the boundary value problem

$$
\Delta \phi = 0, \quad \text{in} \quad \Omega(t),
$$

$$
\partial_{x_2} \phi = 0, \quad \text{for} \quad x_2 = -1,
$$

$$
\partial_x \phi = U_1, \quad \text{on} \quad \Gamma(t).
$$

The operator $\mathcal{K}(X)$ is of the form $\mathcal{K}(X) = \mathcal{K}_0 + \mathcal{S}_1(X)$, where $\mathcal{K}_0$ is the linear part of the operator $\mathcal{K}(X)$, and has the Fourier symbol $\hat{\mathcal{K}}_0(k) = -i \tanh(k)$. The nonlinear part $\mathcal{S}_1(X)$ has certain smoothing properties. In particular, $\mathcal{K}(X)$ (and as a consequence, $\mathcal{S}_1(X)$) depends analytically on $\partial_x X_1, X_2 \in Y_{\sigma,s}$ for any $\sigma > 0, s > 1$, where

$$
Y_{\sigma,s} = \{ f \in L^2(\mathbb{R}) : \| f \|_{Y_{\sigma,s}} = \left( \int (1 + k^2)^{\sigma} e^{2\pi |k|} |\mathcal{F}(f)(k)|^2 \, dk \right)^{1/2} < \infty \}.
$$

It is well known that functions in $Y_{\sigma,s}$ are analytic in a strip of width $2\sigma$ in the complex plane symmetric around the real axis. These spaces are used in [5] for the error estimates and the local existence and uniqueness of solutions with the help of the Cauchy-Kowalevskaya theorem.

Due to the behavior of the system at the wave number $k = 0$ the variable $X_1$ is unbounded in space and so it is advantageous to work with the variable $Z_1 = \mathcal{K}_0 X_1$. The system of equations for the water wave problem can be rewritten entirely in terms of the variables $\mathcal{W} = (Z_1, X_2, U_1)$, namely

$$
\partial_t \mathcal{W} = F_{\mathcal{W}}(\mathcal{W}) \quad (4)
$$

with

$$
F_{\mathcal{W}}(\mathcal{W}) = \begin{pmatrix}
\mathcal{K}_0 U_1 \\
-(1 - \mathcal{M}_2 Z_1 + (\partial_x X_2) \mathcal{K}_0 + (\partial_x X_2) \mathcal{S}_1(X))^{-1} (\partial_x Z_2)(1 + [\partial_t, \mathcal{S}_1(X)] U_1)
\end{pmatrix},
$$

where

$$
\mathcal{M}_2 = -\partial_\alpha (\mathcal{K}_0)^{-1}.
$$

$F_{\mathcal{W}}$ is an analytic mapping from $Y_{\sigma,s} \times Y_{\sigma,s} \times Y_{\sigma,s-1/2}$ into $Y_{\sigma,s-1/2} \times Y_{\sigma,s-1/2} \times Y_{\sigma,s-1}$.

In order to derive the NLS equation and to bring together [5] with [4] it is useful to diagonalize the linear part of (4). In Fourier space the linearization is given by

$$
\partial_t \begin{pmatrix}
\hat{Z}_1 \\
\hat{X}_2 \\
\hat{U}_1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & -i \tanh(k) \\
0 & 0 & -i \tanh(k) \\
0 & -ik & 0
\end{pmatrix}
\begin{pmatrix}
\hat{Z}_1 \\
\hat{X}_2 \\
\hat{U}_1
\end{pmatrix}.
$$

The eigenvalues of the matrix on the right hand side are given by $\lambda_j = i \omega_j$ for $j = 1, 2, 3$ with

$$
\omega_1(k) = 0, \quad \omega_2(k) = -\omega(k), \quad \omega_3(k) = \omega(k),
$$

respectively.
where
\[ \omega(k) = \text{sign}(k) \sqrt{k \tanh(k)}. \]

We write the original coordinates as a sum of the associated eigenvectors, i.e.,
\[ \left( \begin{array}{c} \hat{Z}_1 \\ \hat{X}_2 \\ \hat{U}_1 \end{array} \right) = \left( \begin{array}{ccc} 1 & \hat{s} & -\hat{s} \\ 0 & \hat{s} & -\hat{s} \\ 0 & 1 & 1 \end{array} \right) \left( \begin{array}{c} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{array} \right) = D(k) \left( \begin{array}{c} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{array} \right), \]
with \( \hat{s} = \hat{s}(k) = \sqrt{k^{-1} \tanh(k)} \). Due to the asymptotic behavior of \( \hat{s} \) it is easy to see that from \((Z_1, X_2, U_1) \in Y_{\sigma,s} \times Y_{\sigma,s} \times Y_{\sigma,s-1/2} \), it follows \((c_1, c_2, c_3) \in Y_{\sigma,s} \times Y_{\sigma,s-1/2} \times Y_{\sigma,s-1/2} \) and vice versa. The variables \( c = (c_1, c_2, c_3)^T \) satisfy
\[ \partial_t c = F(c) = D^{-1} F_W(Dc) \]
(5)

For the same reason, \( F_c \) is a smooth mapping from \( H^s \times H^{s-1/2} \times H^{s-1/2} \) into \( H^{s-1} \times H^{s-1} \times H^{s-1} \).

In order to derive the NLS equation we make the ansatz
\[ \left( \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right) = \sum_{\pm} \varepsilon A_{\pm 1}(\varepsilon t, \varepsilon^2 t) E_{\pm 1} \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + O(\varepsilon^2), \]
where \( E = e^{i(k_0 t - \omega_0 t)}, \quad \omega_0 = \omega(k_0), \) and \( \Lambda_j = A_{-j} \). We equate the coefficients of the \( \varepsilon^m E^j \) to zero and finally find that \( A_1 \) has to satisfy the NLS equation
\[ \partial_T A_1 = -i \frac{\omega''(k_0)}{2} \partial_X^2 A_1 + i \nu_2(k_0) A_1 |A_1|^2, \]
with a \( \nu_2(k_0) \in \mathbb{R} \).

The approximation result from [5] for the NLS approximation of the water wave problem in Lagrangian coordinates is then as follows.

**Theorem 2.1.** Fix \( s_A - 4 \geq s \geq 6 \), let \( \beta = 7/2 \), and let \( H^s = H^s \times H^s \times H^{s-1/2} \). For all \( C_A, C_0, T_0 > 0 \) there exist \( C_R, \varepsilon_0, T_1 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) the following is true. Let \( A \in \mathcal{C}([0, T_0], H^{s_A}) \) be a solution of (1) with
\[ \sup_{T \in [0, T_0]} \| A \|_{H^{s_A}} \leq C_A \]
and let \( W_{|t=0} = \varepsilon \Psi_{|t=0} + \varepsilon^\beta R_{|t=0} \in H^s \) with \( \| R_{|t=0} \|_{H^s} \leq C_0 \). Then there is a unique solution \( W = \varepsilon \Psi + \varepsilon^\beta R \in \mathcal{C}([0, T_1/\varepsilon^2], H^s) \) of (4) which satisfies
\[ \sup_{t \in [0, T_1/\varepsilon^2]} \| R(t) \|_{H^s} \leq C_R. \]

There seems to be a contradiction between the assumption that the solutions of the NLS equation are in some Sobolev space and the idea of solving the water wave problem (5) in spaces of analytic functions with the help of the Cauchy-Kowalevskaya theorem. However, the NLS approximation is strongly concentrated at integer multiples of the basic wave number \( k_0 \) and is of order \( O(\varepsilon^n) \) at \( k = nk_0 \). Hence, the Fourier transform decays as \( e^{-|\ln(\varepsilon)||k|} \) for \(|k| \to \infty \). Due to the Paley-Wiener theorem the NLS approximation is at least analytic in a strip of width \( O(1) \). The regularity of the solutions of the NLS equation determines the concentration of the Fourier modes around \( nk_0 \). Hence, a cut-off outside some \( O(1) \)-neighborhood of \( nk_0 \) will only change the approximation in higher order if the difference is measured in Sobolev spaces.

In order to explain that all parts of the proof except for the estimates for the residual transfer line for line to the case of two wave packets we recall the main ideas of the proof of Theorem 2.1. We consider an abstract evolutionary problem
\[ \partial_T v = \Lambda v + B(v, v) + \ldots, \]
with $\Lambda$ a linear and $B$ a symmetric bilinear operator. Suppose that $v$ is formally approximated by $\varepsilon\Psi$, i.e., that the residual
\[
\text{Res}(v) = -\partial_t v + \Lambda v + B(v,v) + \ldots
\]
is small for $v = \varepsilon\Psi$. For our choice of $\varepsilon\Psi$ in Section 3 we have
\[
\text{Res}(\varepsilon\Psi) = O(\varepsilon^\gamma)
\]
in some Sobolev space and with $\gamma = 11/2$. In order to prove Theorem 2.1 we have to estimate the error
\[
\varepsilon^\beta R = v - \varepsilon\Psi
\]
for all $t \in [0, T_1/\varepsilon^2]$ to be of order $O(\varepsilon^\beta)$ for a $\beta > 1$, i.e., we have to prove that $R$ is of order $O(1)$ for all $t \in [0, T_1/\varepsilon^2]$. The error $R$ satisfies
\[
\partial_t R = \Lambda R + 2\varepsilon^\alpha B(\Psi,R) + \varepsilon^\beta B(R,R) + \ldots + \varepsilon^{-\beta}\text{Res}(\varepsilon\Psi).
\]
In our case $\Lambda$ generates a uniformly bounded semigroup and so we are done aside from possible complicated functional analytic details, if a) $\alpha \geq 2$, b) $\beta > 2$ and c) $\varepsilon^{-\beta}\text{Res}(\varepsilon\Psi) = O(\varepsilon^2)$. The result then would follow by a rescaling of time, $T = \varepsilon^2 t$, and an application of Gronwall’s inequality. In our case, however, we have $\alpha = 1$. Since $\gamma = 11/2$ we can choose $\beta = 7/2$ and so the points b) and c) are satisfied easily. The difficulty is to control the term $2\varepsilon B(\Psi,R)$ in the linear evolution. In [5] this term has been eliminated by a near identity change of coordinates. Since the water wave problem is a quasilinear problem there is a loss of regularity associated with the transformation. For (5) there is a loss of regularity of half a derivative such that after the transformation the right hand side of (9) loses one derivative, i.e., we have a system of the form
\[
\partial_t \tilde{R} = \Lambda \tilde{R} + \varepsilon^\sigma G_3(\Psi,\tilde{\Psi},\tilde{R}) + O(\varepsilon^3)
\]
with $G$ a symmetric trilinear mapping and the right hand side a smooth mapping from $Y_{\sigma,s} \times Y_{\sigma,s-1/2} \times Y_{\sigma,s-1/2}$ into $Y_{\sigma,s-1} \times Y_{\sigma,s-3/2} \times Y_{\sigma,s-3/2}$. This loss of regularity of one derivative is compensated by making the weight via $\sigma$ time-dependent. The choice $\sigma(t) = \sigma_0 - \beta \varepsilon^2 t$ gives a dissipative term $-\varepsilon^2/\beta |k|$ and a smoothing of one derivative such that the solutions of (10) exist on an $O(1/\varepsilon^2)$ time scale in $Y_{\sigma(t),s} \times Y_{\sigma(t),s-1/2} \times Y_{\sigma(t),s-1/2}$.

The proof of [10] works very similar except that the term $2\varepsilon B(\Psi,R)$ is eliminated by a fully nonlinear transformation which is adapted to the case of infinite depth and avoids the use of analytic functions.

3. Interaction of NLS described wave packets. We already diagonalized the water wave problem in Fourier space, and so (5) with $\tilde{v} = \mathcal{F}c$ can be written as
\[
\partial_t \tilde{v}_j(l,t) = i\omega_j(l)\tilde{v}_j(l,t)
\]
\[+ \int_{\mathbb{R}} \sum_{j_1,j_2 \in \{1,2,3\}} s^j_{j_1,j_2}(l,l-l_2,l_2)\tilde{v}_{j_1}(l-l_2,t)\tilde{v}_{j_2}(l_2,t)dl_2 + \text{h.o.t.},
\]
for $j = 1,2,3$ which is exactly the starting point of the analysis in [4]. We refrain from rewriting [4] and restrict ourselves to the main ideas and to the formulation of the approximation result. We refer to [4] for the missing details. For the derivation of the extended modulation system, (11) is transferred into $x$-space. We denote the $n$-th component of the residual corresponding to the inverse Fourier transform of (11) by
\[
\text{Res}_n(V)(x,t) = -\partial_t V_n(x,t) + \mathcal{L}_n(\partial_x)V_n(x,t) + N_n((V_j)_{j \in \{1,2,3\}})(x,t).
\]
Plugging in the ansatz $V^{an} = (v_1^{an}(x, t), v_2^{an}(x, t), v_3^{an}(x, t))$ defined by

$$v_2^{an}(x, t) = \sum_{j=1, 2} \sum_{r=1}^{3} \varepsilon^r A_j^{(r)}(X_j, T)e^{iY_j} + c.c. + M_{mixed, 1}, \quad (13)$$

$$v_j^{an}(x, t) = M_{mixed, j}, \quad \text{for} \quad j \in \{1, 3\} \quad (14)$$

$$X_j = X + \varepsilon \omega'_1(k_j)t + \varepsilon^2 \Psi_1^{(1)}(X + \varepsilon \omega'_1(k_{3-j})t, T), \quad (15)$$

$$Y_j = k_jx + \omega_1(k_j)t + \sum_{l=1, 2} \varepsilon^l \Omega_1^{(l)}(X + \varepsilon \omega'_1(k_{3-j})t, T), \quad (16)$$

$$X = \varepsilon x, \quad T = \varepsilon^2 t, \quad (17)$$

into (12) leads to a number of conditions in order to make the residual as small as possible, in particular to Nonlinear Schrödinger equations for the $A_j^{(1)}$. The mixed and higher order harmonic terms $M_{mixed, j}$ satisfy

$$M_{mixed, j} = O(\varepsilon^2). \quad (18)$$

We set

$$\widetilde{Res}_j(V) = \sum_{j_1, j_2, j_3} \varepsilon^{j_1} \text{Res}_{j_1, j_2, j_3}^j(V)e^{i(j_2 Y_1 + j_3 Y_2)}$$

which is an implicit definition for the terms $\text{Res}_{j_1, j_2, j_3}^j(V)$.

- At $\varepsilon^3 e^{iY_1}$ we find

$$\text{Res}^1_{3, 1, 0} = t_{31} + \bar{t}_{32} A_1^{(1)}.$$  

The condition $t_{31}=0$ yields the NLS equation

$$\partial_2 A_1^{(1)}(X_1, T) = -i(\omega'_1(k_1)/2)\partial_1^2 A_1^{(1)}(X_1, T) + \gamma_1 |A_1^{(1)}(X_1, T)|^2 A_1^{(1)}(X_1, T), \quad (19)$$

with coefficient $\gamma_1 = \gamma_1(k_1) \in i\mathbb{R}$, and the condition $\bar{t}_{32}=0$ yields the phase shift formula

$$\Omega_1^{(1)}(X_2, T) = \gamma_{11} \int_{X_2}^{X_3} |A_2^{(1)}(\zeta, T)|^2 d\zeta, \quad (20)$$

so $\Omega_1^{(1)}$ is a real quantity and therefore a pure phase correction since $\gamma_{11} = \gamma_{11}(k_1, k_2) \in \mathbb{R}$.

- At $\varepsilon^4 e^{iY_1}$ we find the linear inhomogeneous evolution equation

$$\partial_2 A_1^{(2)}(X_1, T) = i(\omega''_1(k_1)/2)\partial_1^2 A_1^{(2)}(X_1, T) + t_{41}. \quad (21)$$

for $A_1^{(2)}$. Here, no coupling with terms involving $A_2^{(r)}$-variables occurs such that $A_1^{(2)}$ describes internal dynamics of a single pulse. Moreover, we find the envelope shift formula

$$\Psi_1^{(1)}(X_2, T) = \gamma_{21}(k_1, k_2) \int_{X_2}^{X_3} |A_2^{(1)}(\zeta, T)|^2 d\zeta, \quad (22)$$

with pre-factor $\gamma_{21}(k_1, k_2)$.

Finally we obtain a formula for the quantity $\Omega_1^{(2)}$ which can be interpreted as follows. Its real part is a second order correction to the phase shift, whereas its imaginary part gives a correction of the amplitude. We refrain from explicitly displaying the rather lengthy expression for the real part of $\Omega_1^{(2)}$ and only note that it is a pure integration of spatially localized terms. We refrain from explicitly displaying the expression for the imaginary part of $\Omega_1^{(2)}$ and only remark that $\text{Im}\Omega_1^{(2)}$ is spatially localized and so
the induced correction is small w.r.t. \( \varepsilon \) except during the collision of spatially localized wave packets. Since \( \text{Im}\Omega^{(2)} \) and \( \text{Im}\Omega^{(2)} \) are supposed to describe interaction dynamics we may assume that \( \Omega^{(2)} = \Omega^{(2)} = 0 \) initially. Moreover, due to the fact that \( \text{Im}\Omega^{(2)} \) and \( \text{Im}\Omega^{(2)} \) turn out to be spatially localized in the region of interaction, \( \text{Im}\Omega^{(2)} \) and \( \text{Im}\Omega^{(2)} \) play no role for the envelope shift.

- At \( \varepsilon^5e^{\beta t} \) we choose \( A^{(3)}_1 \) to satisfy

\[
\partial_t A^{(3)}_1(\mathbf{x}_1, T) + i(\omega'_1(k_1))\partial^2_t A^{(3)}_1(\mathbf{x}_1, T) + t_{51} = 0
\]

(23)

where \( t_{51} \) is at most linear in \( A^{(3)}_1 \) and \( A^{(3)}_1 \).

Then we have [4, Theorem 4.1]

**Theorem 3.1.** Let \( s \geq 2, m \geq 2, s_A \geq s + 10, l_1 \neq l_2, l_1, l_2 > 0, \) and let \( A_1^{(1)}|_{T=0} \).

\( A_1^{(1)}|_{T=0} \in H^{s_A}(m) \cap H^{s_A+2m}(0) \). Then for all \( T_0 > 0 \) there exist \( \varepsilon_0 > 0, C > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) we have

\[
\sup_{t \in [0, T_0]} \| \text{Res}(V^an) \|_{H^s(\mathbb{R})} \leq C\varepsilon^{11/2}.
\]

It is obvious that the \( H^s \) spaces can be replaced by the \( Y_{\gamma, s} \) spaces which were used in [5] in order to apply the Cauchy-Kowalevskaya theorem. Since the variables for which we derived the extended NLS approximation and the variables for which in [5] the Cauchy-Kowalevskaya theorem has been applied coincide Theorem 3.1 gives the necessary estimates for the residual terms occurring in the error equations associated to (5). Therefore, with the discussion from Section 2 we can conclude the following approximation theorem.

**Theorem 3.2.** Fix \( s_A - 4 \geq 6, \) let \( \beta = 7/2, \) and let \( \mathcal{H}^s = H^s \times H^s \times H^{s-1/2} \). For all \( C_A, C_0, T_0 > 0 \) there exist \( C_R, \varepsilon_0, T_1 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) the following is true. Let \( A_j^{(1)} \in C([0, T_0], H^{s_A}(2)) \) for \( j = 1, 2 \) be solutions of the NLS equation (19) with

\[
\sup_{T \in [0, T_0]} \| A_j^{(1)} \|_{H^{s_A}(2)} \leq C_A,
\]

and let \( W |_{t=0} = \varepsilon V^an |_{t=0} + \varepsilon^\beta R |_{t=0} \in \mathcal{H}^s \), with \( \| R |_{t=0} \|_{\mathcal{H}^s} \leq C_0 \). Then there is a unique solution \( W = \varepsilon V^an + \varepsilon^\beta R \in C([0, T_1/\varepsilon^2], \mathcal{H}^s) \) of (4) which satisfies

\[
\sup_{t \in [0, T_1/\varepsilon^2]} \| R(t) \|_{\mathcal{H}^s} \leq C_R.
\]

Thus Theorem 3.2 makes it rigorous that in suitable chosen coordinates it is possible to separate the internal dynamics of wave packets in NLS scaling from the dynamics caused by the interaction. The error of order \( O(\varepsilon^{7/2}) \) is much smaller than the change of the solution caused by the phase and envelope shift.

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