The Simplified Spherical Harmonics Method For Radiative Heat Transfer

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Abstract. The Simplified $P_N$-Approximation was introduced some 50 years ago, as an intuitive three-dimensional extension to the one-dimensional slab $P_N$-formulation. The set of simplified- $P_N$ or $SP_N$ equations was formulated, such that they reduced to the standard $P_N$-approximation for a one-dimensional slab, resulting in $2(N+1)$ simultaneous first order differential equations, subject to a modified set of Marshak boundary conditions. In this article the method is developed further for radiative heat transfer applications by reducing the governing equations to a small set of simultaneous second-order, elliptic PDEs, similar to the simple $P_1$ equation. Each of the equations is subject to $P_1$-type boundary conditions. Several two-dimensional examples are presented, including scattering, strongly inhomogeneous temperatures and absorption coefficients.

1. Introduction
The radiative transfer equation (RTE) is an integro-differential equation in five independent variables (three in space and two in direction), which is exceedingly difficult to solve. Several approximate methods have been developed over time. Among these approximate methods the Spherical Harmonics Method (SHM), the Discrete Ordinates Method or the Finite Volume Method, and the Monte Carlo Method are presently used most frequently [1]. Both the Spherical Harmonics Method and DOM/FVM approximate the directional variation of the radiative intensity.

Like all methods, the SHM has a number of drawbacks. The lowest-order $P_N$-approximation, the $P_1$-approximation, has enjoyed great popularity because of its relative simplicity and compatibility with standard solution methods [1], but performs poorly in the optically thin limit and other nonisotropic radiative intensity fields, which to a great extent can be mitigated using the modified differential approximation approach [2, 3]. Mathematical complexity increases rapidly if higher-order $P_N$-approximations for multi-dimensional geometry are desired. Consequently, only a few and very limited multi-dimensional formulations of higher order have been presented [4, 5], as reviewed by Yang and Modest [6]. Recently, Modest and Yang [7] have shown how the $(N+1)^3$ first-order PDEs of the general three-dimensional $P_N$-approximation can be condensed into a set of $N(N+1)/2$ second-order, elliptic PDEs, which are compatible with, and can be embedded into, standard commercial codes. For two-dimensional problems the set can be reduced to $(N+1)^2/4$ elliptic PDEs. However, the set of equations is very complex, in particular if allowing for spatially varying absorption and anisotropic scattering, and includes many cross-derivatives, which makes it difficult to apply higher order $P_N$-methods to actual problems.

Facing these mathematical difficulties Gelbard [8] introduced the Simplified $P_N$-Approximation some
The general radiative transfer equation (RTE) is given by [1]

\[ \hat{s} \cdot \nabla I + I = (1 - \omega)I_b + \frac{\omega}{4\pi} \int_{4\pi} I(s') \Phi(\hat{s} \cdot \hat{s}') d\Omega', \]

(1)

where space coordinates have been nondimensionalized using the extinction coefficient, i.e., \( d\tau = \beta ds \) (as indicated by the subscript \( \tau \) in \( \nabla \)). Here \( I \) is radiative intensity, \( I_b \) is blackbody intensity (or Planck function), \( \omega = \sigma_s/\beta \) is the single scattering albedo and \( \sigma_s \) the scattering coefficient, and \( \Phi \) is the scattering phase function. Equation (1) requires the outgoing intensity to be specified everywhere along the surface of the enclosure.

In the one-dimensional \( P_N \)-approximation for a plane-parallel slab the radiative intensity is expanded into a series of Legendre polynomials \( P_l \)

\[ I(\tau, \mu) \approx \sum_{l=0}^{N} I_l(\tau)P_l(\mu), \]

(2)

where \( \mu = \cos \theta \) and \( \theta \) is the polar angle for the unit direction vector \( \hat{s} \) (see Fig. 1). Equation (2) is approximate because the series is truncated beyond \( l = N \), i.e., we assume \( I_l(\tau) = 0 \) for all \( l > N \). The scattering phase function for such a medium may also be expanded into Legendre polynomials, or [1]

\[ \Phi(\mu, \mu') = \sum_{m=0}^{M} A_m P_m(\mu')P_m(\mu), \]

(3)

where \( M \) is the order of approximation for the phase function. The general \( P_N \)-approximation for a one-dimensional anisotropically scattering slab may then be written as \( N + 1 \) simultaneous first-order partial
differential equations [1]

\[
\frac{k + 1}{2k+3} I'_{k+1}(\tau) + \frac{k}{2k-1} I'_{k-1}(\tau) + \left( 1 - \frac{\omega A_k}{2k+1} \right) I_k(\tau) = (1 - \omega) I_b(\tau) \delta_{0k}, \\
k = 0, 1, \ldots, N.
\] (4)

The Marshak boundary conditions for this set of equations, assuming diffuse surfaces, i.e., \( I_w = J_w/\pi \), where \( J_w \) is the radiosity at the wall, are given by [1]

\[
\sum_{i=0}^{N} I_i(0) \int_{0}^{1} P_i(\mu) P_{j-1}(\mu) \, d\mu = \frac{J_w}{\pi} \int_{0}^{1} P_{j-1}(\mu) \, d\mu, \quad i = 1, 2, \ldots, \frac{1}{2}(N+1); \quad (5a)
\]

\[
\sum_{i=0}^{N} I_i(\tau) \int_{-1}^{0} P_i(\mu) P_{j-1}(\mu) \, d\mu = \frac{J_w}{\pi} \int_{-1}^{0} P_{j-1}(\mu) \, d\mu, \quad i = 1, 2, \ldots, \frac{1}{2}(N+1). \quad (5b)
\]

While the developments of Larsen [9] and Pomraning [10] provide theoretical credentials to the \( SP_N \) method, they are rather tedious, and we will here only provide the intuitive development of Gelbard [8], as a basis for further development for radiative heat transfer applications. Depending on whether \( k \) is odd or even, Gelbard made the following substitutions in Eqs. (4) and (5):

\[
k \text{ odd} : \quad I_k(\tau) \rightarrow I_k(\tau_x, \tau_y, \tau_z), \quad I_k' = \frac{dI_k}{d\tau} \rightarrow \nabla_r \cdot I_k, \quad (6a)
\]

\[
k \text{ even} : \quad I_k(\tau) \rightarrow I_k(\tau_x, \tau_y, \tau_z), \quad I_k' = \frac{dI_k}{d\tau} \rightarrow \nabla_r I_k, \quad (6b)
\]

i.e., for every odd \( k \) the \( I_k \) becomes a vector and differentiation is replaced by the divergence operator, while even \( I_k \) remain scalars and their differentiation is replaced by the gradient operator. Substituting Eqs. (6) into Eq. (4) leads to

\[
k = 0, 2, \ldots, N - 1 \quad (\text{even}) : \quad \frac{k + 1}{2k+3} \nabla_r \cdot I_{k+1} + \frac{k}{2k-1} \nabla_r \cdot I_{k-1} + \alpha_k I_k = \alpha_k I_b \delta_{0k}, \quad (7a)
\]

\[
k = 1, 3, \ldots, N \quad (\text{odd}) : \quad \frac{k + 1}{2k+3} \nabla_r I_{k+1} + \frac{k}{2k-1} \nabla_r I_{k-1} + \alpha_k I_k = 0, \quad (7b)
\]

where

\[
\alpha_k = 1 - \frac{\omega A_k}{2k+1}. \quad (7c)
\]
Solving Eq. (7b) for $I_k$ and substituting the result into (7a) produces a set of simultaneous elliptic partial differential equations in the unknown scalars $I_k$ (k even):

$$k = 0, 2, \ldots, N - 1 \quad \text{(even)} :$$

$$\frac{(k + 1)(k + 2)}{(2k + 3)(2k + 5)} \nabla_\tau \left( \frac{1}{\alpha_{k+1}} \nabla_\tau I_{k+2} \right) + \frac{(k + 1)^2}{(2k + 3)(2k + 1)} \nabla_\tau \left( \frac{1}{\alpha_{k+1}} \nabla_\tau I_k \right)$$

$$+ \frac{k^2}{(2k - 1)(2k + 1)} \nabla_\tau \left( \frac{1}{\alpha_{k-1}} \nabla_\tau I_k \right) + \frac{k(k - 1)}{(2k - 1)(2k - 3)} \nabla_\tau \left( \frac{1}{\alpha_{k-1}} \nabla_\tau I_{k-2} \right) = \alpha_k (I_k - I_0 \delta_{0k}). \quad (8)$$

Similarly, sticking Eqs. (6) into the $P_N$ boundary conditions, Eqs. (5), gives us a consistent set of conditions for the $S_{PN}$-equations:

$$\sum_{k \text{ even}}^{N-1} I_k \int_0^1 P_k(\mu) P_{2i-1}(\mu) d\mu + \sum_{k \text{ odd}}^N \hat{n} \cdot I_k \int_0^1 P_k(\mu) P_{2i-1}(\mu) d\mu = J_w \int_0^1 P_{2i-1}(\mu) d\mu, \quad i = 1, 2, \ldots, \frac{1}{2}(N + 1), \quad (9)$$

or, with the definition of the Legendre polynomial half-moments $p_{m,i}^a$ given by

$$p_{m,i}^a = p_{2i-1}^a = \int_0^1 P_{m,i}^a(\bar{\mu}) P_{2i-1}(\bar{\mu}) d\bar{\mu}, \quad (10)$$

$$\sum_{k \text{ even}}^{N-1} p_{k,2i-1}^0 I_k + \sum_{k \text{ odd}}^N \hat{n} \cdot I_k = \frac{p_{0,2i-1}^0}{\pi} J_w, \quad i = 1, 2, \ldots, \frac{1}{2}(N + 1). \quad (11)$$

Again, eliminating the odd $I_k$ with Eq. (7b), this set of boundary conditions reduces to

$$\sum_{k \text{ even}}^{N-1} p_{k,2i-1}^0 I_k = \sum_{k \text{ odd}}^N \frac{k}{2k - 1} \hat{n} \cdot I_{k-1} + \frac{k + 1}{2k + 3} \hat{n} \cdot I_{k+1} + \frac{p_{0,2i-1}^0}{\pi} J_w, \quad i = 1, 2, \ldots, \frac{1}{2}(N + 1). \quad (12)$$

No direct formula for intensity is derived, but one may assume a series of the form

$$I(r, \hat{s}) = I_0(r) + I_1(r) \cdot \hat{s} + I_2(r) P_2^0(\hat{s}) + \ldots, \quad (13)$$

which is no longer a complete series of orthogonal functions and, therefore is not guaranteed to approach the exact answer in the limit. However, assuming this to be an orthogonal set, we can obtain incident radiation $G$ and radiative flux $q$ from their definitions as

$$G(r) = \int_{4\pi} I(r, \hat{s}) d\Omega = 4\pi I_0(r), \quad (14)$$

$$q(r) = \int_{4\pi} I(r, \hat{s}) \hat{s} d\Omega = \frac{4\pi}{3} I_1(r) = -\frac{4\pi}{3} \sum_{i=1}^{N-1} \hat{n}_i I_0 + \frac{2}{5} \hat{n} \cdot I_2. \quad (15)$$

While Eqs. (8) and (12) form a self-consistent set of $(N + 1)/2$ simultaneous elliptic partial differential equations and their boundary conditions, the problem can be further simplified by recognizing that the combination of variables

$$J_k = \frac{k + 1}{2k + 1} I_k + \frac{k + 2}{2k + 5} I_{k+2} \quad (16)$$
appears repeatedly in, both, governing equations and boundary conditions. Thus we may rewrite Eqs. (8)
\[ k = 0, 2, \ldots, N - 1 \text{ (even)} : \]
\[ \frac{k + 1}{2k + 3} \nabla_{r} \left( \frac{1}{\alpha_{k+1}} \nabla_{r} J_{k} \right) + \frac{k}{2k - 1} \nabla_{r} \left( \frac{1}{\alpha_{k-1}} \nabla_{r} J_{k-2} \right) = \alpha_{k} (I_{k} - I_{b} \delta_{0k}). \]  
(17)

A tabulation of the \( p_{n,j}^{\mu} \) has been given by Modest [12]; inspection shows that \( p_{n,j}^{0} = 0 \) if \( n + j = \) even, with the exception of \( n = j \), and boundary conditions (12) reduce to
\[ \frac{p_{2l-1,2i-1}^{0}}{\alpha_{2i-1}} \hat{\mathbf{n}} \cdot \nabla_{r} J_{2i-2} = \sum_{k=0}^{N} p_{2k,2i-1}^{0} I_{2k} - \frac{p_{0,2i-1}^{0}}{\pi} J_{w}, \quad i = 1, 2, \ldots, \frac{1}{2}(N + 1). \]  
(18)

The \( I_{k} \) on the right-hand sides may be eliminated by inverting Eq. (16), starting with \( k = N - 1 \) (and noting that \( I_{N+1} \equiv 0 \)). This results in individual partial differential equations for each \( I_{k} \), in which \( J_{l} \) (\( l \neq k \)) occur only as source terms without derivatives. Once the \( J_{k} \) have been determined, incident radiation and radiative flux are obtained from Eqs. (14) and (15) as
\[ G(r) = 4\pi \left[ J_{0}(r) - \frac{2}{3} J_{2}(r) + \frac{24}{55} J_{4}(r) + \ldots \right], \]  
(19)
\[ q(r) = -\frac{4\pi}{3\alpha_{1}} \nabla_{r} J_{0}(r). \]  
(20)

We will demonstrate this by looking in more detail at the \( SP_{1} \) and \( SP_{3} \) approximations (even orders, such as \( SP_{2} \), have also been formulated [13], but—based on the development shown here—appear to be as inappropriate as for the standard \( P_{N} \)-method).

2.1. \( SP_{1} \)-Approximation
With \( N = 1 \) we obtain a single equation and a single boundary condition from Eqs. (17) and (18), i.e.,

**Governing equation:**
\[ k = 0 : \quad \frac{1}{3} \nabla_{r} \left( \frac{1}{\alpha_{1}} \nabla_{r} J_{0} \right) = \alpha_{0} (I_{0} - I_{b}); \]  
(21)

**Boundary condition:**
\[ i = 1 : \quad \frac{p_{0,1}^{0}}{\alpha_{1}} \hat{\mathbf{n}} \cdot \nabla_{r} J_{0} = p_{0,1}^{0} (I_{0} - J_{w}/\pi). \]  
(22)

With \( p_{0,1}^{0} = \frac{1}{2} \) and \( p_{1,1}^{0} = \frac{1}{3}, \) and \( I_{0} = J_{0} \) from Eq. (16), we obtain
\[ \frac{1}{3} \nabla_{r} \left( \frac{1}{\alpha_{1}} \nabla_{r} J_{0} \right) = \alpha_{0} (J_{0} - I_{b}), \]  
(23)

with boundary condition
\[ \frac{1}{3\alpha_{1}} \hat{\mathbf{n}} \cdot \nabla_{r} J_{0} = \frac{1}{2} (J_{0} - J_{w}/\pi). \]  
(24)

Not surprisingly, using \( G = 4\pi I_{0} = 4\pi J_{0} \) shows that the \( SP_{1} \)-approximation is identical to the \( P_{1} \)-method as given, for example, by Modest [1].
2.2. \(SP_3\)-Approximation

Setting \(N = 3\) we get two simultaneous equations and two boundary conditions:

**Governing equations:**

\[
k = 0 : \quad \frac{1}{3} \nabla \cdot \left( \frac{1}{\alpha_1} \nabla \alpha_0 J_0 \right) = \alpha_0 (J_0 - I_b) = \alpha_0 \left( J_0 - \frac{2}{3} J_2 - I_b \right), \tag{25a}
\]

\[
k = 2 : \quad \frac{3}{7} \nabla \cdot \left( \frac{1}{\alpha_3} \nabla \alpha_2 J_2 \right) + \frac{2}{3} \nabla \cdot \left( \frac{1}{\alpha_1} \nabla \alpha_0 J_0 \right) = \alpha_2 J_2 = \frac{5}{3} \alpha_2 J_2, \tag{25b}
\]

or, subtracting \(2\times\text{Eq. (25)}\):

\[
k = 2 : \quad \frac{3}{7} \nabla \cdot \left( \frac{1}{\alpha_3} \nabla \alpha_2 J_2 \right) = \left( \frac{5}{3} \alpha_2 + \frac{4}{3} \alpha_0 \right) J_2 - 2 \alpha_0 (J_0 - I_b). \tag{25c}
\]

**Boundary conditions:**

\[
i = 1 : \quad \frac{p_{0,1}^{0}}{\alpha_1} \hat{n} \cdot \nabla \alpha_0 J_0 = p_{0,1}^{0} (I_0 - J_w) + p_{2,1}^{0} J_2, \tag{26a}
\]

\[
i = 2 : \quad \frac{p_{0,3}^{0}}{\alpha_3} \hat{n} \cdot \nabla \alpha_2 J_2 = p_{0,3}^{0} (I_0 - J_w) + p_{2,3}^{0} J_2. \tag{26b}
\]

With \(p_{2,1}^{0} = p_{2,3}^{0} = \frac{1}{8}, \ p_{0,3}^{0} = \frac{3}{7}, \ p_{0,3}^{0} = -\frac{1}{8}\), and eliminating the \(I_b\), the boundary conditions become

\[
i = 1 : \quad \frac{1}{3 \alpha_1} \hat{n} \cdot \nabla \alpha_0 J_0 = \frac{1}{2}(J_0 - \frac{2}{3} J_2 - J_w) + \frac{1}{3} \nabla \cdot J_2 = \frac{1}{2}(J_0 - J_w) - \frac{1}{3} J_2, \tag{26c}
\]

\[
i = 2 : \quad \frac{1}{7 \alpha_0} \hat{n} \cdot \nabla \alpha_2 J_2 = -\frac{1}{8}(J_0 - \frac{2}{3} J_2 - J_w) + \frac{5}{3} \nabla \cdot J_2 = -\frac{1}{8}(J_0 - J_w) + \frac{5}{3} J_2. \tag{26d}
\]

Unlike the regular \(P_3\)-approximation, \(SP_3\) has only two, and nearly separated, elliptic partial differential equations: Eqs. (25a) and (26c) for \(J_0\) and Eqs. (25c) and (26d) for \(J_2\), the only connection being the other \(J_6\) appearing in source terms.

3. Sample Calculations

Several one- and two-dimensional radiative heat transfer problems have been reported in [6, 7, 12], in particular a nonabsorbing, isotropically scattering medium confined in a rectangular enclosure with a heated strip was studied, as well as a two-dimensional rectangular enclosure with strongly varying temperatures and absorption coefficients. We will repeat some of the previous calculations here with the \(SP_N\) equations and compare them with regular \(P_N\) and Monte Carlo results. In all case cold and black bounding surfaces are chosen: this has no impact on solution complexity, but (i) makes presentation of results easier, and (ii) is a more severe test of the method, since reflecting boundaries cause directionally smoother intensities.

**Enclosure with varying properties** Although scattering poses no additional difficulty for the spherical harmonics method (as opposed to the DOM or FVM), we will limit ourselves to a nonscattering medium, in order to reduce the number of variables. The following nondimensional field will is studied for a square enclosure [12, 14]:

\[
I_b = 1 + 5r^2(2 - r^2), \tag{27a}
\]

\[
k = C_k \left[ 1 + 5(2 - r^2)^2 \right], \tag{27b}
\]

with

\[
r^2 = x^2 + y^2; \quad -1 \leq x \leq +1, \quad -1 \leq y \leq +1, \tag{27c}
\]
Incident radiation

![Figure 2. Incident radiation and radiative source for a square enclosure; optically thick case ($C_k = 1$).](image1)

![Figure 3. Incident radiation and radiative source for a square enclosure; optically intermediate case ($C_k = 0.1$).](image2)

i.e., the blackbody intensity (Planck function) is normalized with its minimum value (obtained at the center and the 4 corners), with a maximum value of $I_b = 6$ at a distance of $r = 1$ from the centerline. The absorption coefficient, normalized in terms of length units, has a maximum value of $\kappa = 16C_k$ at $x = y = 0$, and rapidly diminishes away from the center to a minimum value of $\kappa = C_k$ at the four corners. Thus the problem is radially one-dimensional, except for the conditions at the four perpendicular walls, which are assumed cold and black. The optical thickness of the square enclosure is $\tau_D = 18\sqrt{2}C_k$ along a diagonal, and $\tau_D = 23.5C_k$ along an $x = 0$ or $y = 0$ line, respectively. Here we investigate values of $C_k = 1$ (optically thick), $C_k = 0.1$ (optically intermediate), and $C_k = 0.01$ (optically thin conditions). Results for incident radiation $G = 4\pi I_b$ and divergence of the radiative flux $\nabla \cdot \mathbf{q}$ are shown in Figs. 2 through 4, comparing results from various orders of $P_N$ and $SP_N$ with those of a Monte Carlo simulation. All calculations were performed with a relatively fine $41 \times 41$ cell system to ensure adequate resolution of sharp gradients [especially in the case of the a zeroth-order photon Monte Carlo (PMC) method]. For the optically thick case, Fig. 2, all methods do very well, although $P_1$ has difficulty following the sharp peaks in both $G$ and $\nabla \cdot \mathbf{q}$, with $P_3/SP_3$ being almost exact, and $P_5/SP_5$ even closer to the PMC results. In the optically intermediate case, Fig. 3, both $P_1$ and $P_3$ have difficulties following the peaks and valleys of the incident radiation $G$, doing better with $\nabla \cdot \mathbf{q}$. Only $P_5$ follows the variations in $G$ well, and is essentially exact in the prediction of $\nabla \cdot \mathbf{q}$. It is seen that the $SP_N$ schemes actually outperform their corresponding $P_N$ counterparts. For the optically thin case, Fig. 4, $P_3$ and $P_5$ perform better than $P_1$, but it is clear that neither can follow the true variation in incident radiation. $SP_3$ and $SP_5$ while better than $P_1$ cannot follow the actual variation as well as their corresponding $P_N$. All methods predict $\nabla \cdot \mathbf{q}$ relatively well (i.e., the quantity of greatest importance, providing the radiative source term in the overall energy equation), but higher order methods perform only marginally better than $P_1$.

Figure 5 depicts radiative fluxes along a wall for the same 3 cases. It is seen that, for the optically thick case, $P_1$ incurs serious errors (10% at the center, 40% in the corners). This is not surprising, since it is well known that the $P_1$-approximation performs poorly for optically thick situations in the vicinity of temperature discontinuities [1]. $P_3$ and $P_5$, on the other hand, perform rather well. $SP_3$ and $SP_5$, while considerably better than $P_1$, cannot compete with their $P_N$ counterparts. This trend becomes more
obvious for optically intermediate and thin media. In the optically intermediate case \( P_1 \) does well near \( x = 0 \) (where the problem is close to 1D), but it fails miserably toward the corners. Again, both \( P_3 \) and \( P_5 \) perform extremely well, while \( SP_3 \) and \( SP_5 \), like \( P_1 \), are not able to follow the curvature in the heat flux profile. Finally, in the optically thin case, \( P_1 \), \( SP_3 \) and \( SP_5 \) essentially predict a constant value, while \( P_3 \) and \( P_5 \) perform better, but incur maximum errors of 10–15%.

**Enclosure with purely scattering medium** A black-walled square enclosure with walls of width \( L \) and a constant scattering coefficient \( \sigma_s = \text{const} \) with one heated wall has been considered by several investigators, primarily to investigate ray effects in the discrete ordinates (DOM) and finite volume methods (FVM). Modest and Yang[7] compared the performance of \( P_1 \), \( P_3 \) and their modified versions[3] to that of various levels of the FVM, finding that all methods perform miserably for this problem, the FVM being better at the heated wall, and the \( P_N \) methods having the advantage along the cold walls. Figures 6 and 7 show nondimensional incoming irradiation \( H^* = H/\sigma T^4 \) (which must always be positive), comparing various levels of \( P_N \) and \( SP_N \) and their modified versions. In the optically thin case, \( \tau_L = \sigma_s L = 0.1 \), (Fig. 6) both \( P_1 \) and \( P_3 \) perform poorly along all 3 surfaces, albeit in a different manner, leading to physically impossible negative irradiation along the hot bottom wall. \( SP_3 \) and \( SP_5 \) perform better than \( P_1 \), but it is observed that they display \( P_1 \)-type behavior, i.e., \( SP_3 \) is much closer to \( P_1 \) than to \( P_3 \). The modified versions all perform well for this wall-emission dominated configuration. Performance of \( P_N \) and \( SP_N \) becomes better with increasing optical thickness (Fig. 7), but the \( P_N \) vs. \( SP_N \) trend remains the same, i.e., while \( SP_3 \) and \( SP_5 \) show improvements over \( P_1 \), these improvements are fairly minor.

**4. Summary and Conclusions**

The Simplified Spherical Harmonics, or \( SP_N \) method was formulated for radiative heat transfer, resulting in \((N+1)/2\) simultaneously nearly decoupled, \( P_1 \)-like elliptic partial differential equations, subject to also nearly-decoupled Marshak boundary conditions. The \( SP_N \) methods provide potentially great savings over the standard \( P_N \) approach, which requires \( N(N+1)/2 \) strongly coupled PDEs with cross-derivatives. Judging from the results given here, it appears that the \( SP_3 \)-method constitutes a distinct improvement over \( P_1 \) for calculations of incident radiation \( G \) and flux divergences \( \mathbf{\nabla} \cdot \mathbf{q} \), but much less so as far as the

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**Figure 4.** Incident radiation and radiative source for a square enclosure; optically thick case \((C_k = 1)\).

**Figure 5.** Radiative flux along bottom wall \((y = -1)\).
Figure 6. Comparison of various $P_N$ and $SP_N$ for surface heat fluxes in a purely scattering square enclosure; optically thin case $\tau_L = 0.1$.

Figure 7. Comparison of various $P_N$ and $SP_N$ for surface heat fluxes in a purely scattering square enclosure; optically intermediate case $\tau_L = 1.0$. 
evaluation of wall fluxes is concerned. The $SP_3$-method requires a reasonable two simultaneous PDEs as opposed to the single PDE for $P_1$. While $SP_5$ is somewhat more accurate still, the increase in accuracy is relatively small, perhaps not justifying the additional numerical effort (3 simultaneous PDEs). More sample calculations will need to be carried out to assess the overall benefit of the $SP_N$ as opposed to more established RTE solution methods.

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