A SINGLE EXPONENTIAL BOUND FOR THE REDUNDANT VERTEX THEOREM ON SURFACES

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ABSTRACT. Let $s_1, t_1, \ldots, s_k, t_k$ be vertices in a graph $G$ embedded on a surface $\Sigma$ of genus $g$. A vertex $v$ of $G$ is “redundant” if there exist $k$ vertex disjoint paths linking $s_i$ and $t_i$ ($1 \leq i \leq k$) in $G$ if and only if such paths also exist in $G - v$. Robertson and Seymour proved in Graph Minors VII that if $v$ is “far” from the vertices $s_i$ and $t_j$ and $v$ is surrounded in a planar part of $\Sigma$ by $l(g, k)$ disjoint cycles, then $v$ is redundant. Unfortunately, their proof of the existence of $l(g, k)$ is not constructive. In this paper, we give an explicit single exponential bound in $g$ and $k$.

1. INTRODUCTION

In their graph minors series of papers, Robertson and Seymour obtained some major results: finite graphs are well-quasi-ordered [RS04] for the minor relation, the $k$-disjoint path problem is polynomial [RS95]. And some notions introduced such as face-width [RS94] and some intermediate results such as the structure Theorem for graph excluding a $K_t$ minor [RS03] also proved to be of major importance. Unfortunately, these papers are hard to read and some constant are only given in existential statements. This has the unfortunate effect that no algorithms given in [RS95] is explicit.

Some parts such as the generalised Kuratowski Theorem for surfaces have been rewritten and are now well understood to the point that they appear in textbooks [Die05]. But some parts such as the structure Theorem for graphs excluding a $K_t$ minor [KW11] of the unique Linkage Theorem [KW10] took much longer to be worked on. Unfortunately, until recently, no explicit constant was known because of a result given in [RSS8] whose first proofs were only existential and which is needed in [KW10, KW11]: the redundant vertex Theorem on surfaces. In [KW10], the author write “At the moment, we believe that we also have a much shorter proof of […] the aspects of RSS8 which we use.” But they seem to have never published their proof.

As already stated, the first proof RSS8 is only existential. Later, Seymour and Johnson announced a new still existential proof but never published it. Using ideas of this new proof, Huynh [Huy09] obtained a new existential proof. This proof is still existential but only because it lacks a topological argument which the author and some co-authors recently filled [CHR13]. Although their bound is the first explicit one, it in tower of exponential in $k$ and the $g$. In the meantime, Adler et col. [AKK+11] proved that $2^{k-2} \leq l(0, k) \leq \text{cst}^k$.

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In this paper, we use the same approach as in \cite{Huy09,CHR13} but with a more careful analysis, we prove that \( l(g,k) \leq \text{cst}^{g+k} \) nested cycles are enough to ensure that the central vertex is redundant.

2. Statement of the Theorem

A \( k \)-pattern in a graph \( G \) is a collection \( \Pi := \{\{s_i, t_i\} : 1 \leq i \leq k\} \) of \( k \) pairwise disjoint subsets of \( V(G) \), where each set in \( \Pi \) has size one or two (i.e. \( s_i \) may be equal to \( t_i \)). The vertex set of \( \Pi \) is the set \( V(\Pi) := \cup \Pi \). A \( \Pi \)-linkage in \( G \) is a collection \( \mathcal{L} := \{L_1, \ldots, L_k\} \) of pairwise disjoint paths of \( G \) where each \( L_i \) has ends \( s_i \) and \( t_i \). A vertex \( v \) of \( G \) is redundant (with respect to \( \Pi \)), if \( G \) has a \( \Pi \)-linkage if and only if \( G - v \) does. In the following, we identify \( \mathcal{L} \) with the underlying graph \( \cup \mathcal{L} \). Note that allowing a singleton \( \{s\} \) in \( \Pi \) may seem strange because there is a \( \Pi \)-linkage in \( G \) if and only if there is a \( (\Pi - s) \)-linkage in \( G - s \) but we allow them for technical reasons which will become clear later on.

A surface is a connected compact 2-manifold possibly with boundaries. Oriented surfaces can be obtained by adding “handles” to the sphere, and non-orientable surfaces, by adding “crosscaps” to the sphere. The Euler genus \( g(\Sigma) \) of a surface \( \Sigma \) (or just genus) is twice the number of handles if \( \Sigma \) is orientable, and is the number of crosscaps otherwise. We denote the boundary of \( \Sigma \) by \( \partial(\Sigma) \). A curve in \( \Sigma \) is a continuous function \( \gamma : [0, 1] \to \Sigma \) (we identify \( \gamma \) with its image), and the ends of \( \gamma \) are the points \( \gamma(0) \) and \( \gamma(1) \). A path in \( \Sigma \) is an injective curve, and a \( \partial(\Sigma) \)-path is a path whose ends lie in \( \partial(\Sigma) \). A path is contractible if it bounds a disc, and two paths \( \mu \) and \( \nu \) in \( \Sigma \) are homotopic if \( \mu \) can be continuously distorted into \( \nu \). A surface with boundary \( \Omega \) with a closed disc \( \Delta(\Omega) \subseteq \Omega \) is disk with \( s \) strips if \( \Omega \setminus \Delta(\Omega) \) has \( s \) components called strips which are homeomorphic to \([0, 1]\times]0,1[\]. The ends of a strips are the components of the closure of \( S \) minus \( S \), and its sides are its subsets homeomorphic to \([0]\times]0,1[\) and \([1]\times]0,1[\).

A \( k \)-pattern in a surface with boundary \( \Sigma \) is a collection \( \Pi = \{\{s_i, t_i\} : 1 \leq i \leq k\} \) of \( k \) pairwise disjoint subsets of \( \partial(\Sigma) \), each of size one or two. A topological \( \Pi \)-linkage is a collection \( \Gamma := \{\gamma_1, \ldots, \gamma_k\} \) of disjoint \( \partial(\Sigma) \)-paths in \( \Sigma \) where each \( \gamma_i \) has ends \( s_i \) and \( t_i \). If \( \Sigma \) contains a \( \Pi \)-linkage, we say that \( \Pi \) is topologically feasible. When considering a disc with strip \( \Omega \), we further forbid vertices in \( V(\Pi) \) to be incident with strip (i.e. meet the closure of a strip). As for \( \Pi \)-linkage in a graph, we abuse notation and indentify \( \Gamma \) with a the corresponding subset of \( \Sigma \). Moreover, if \( G \) is embedded in \( \Sigma \) and \( \Gamma \) subset of \( \Sigma \) is a subgraph of \( G \), we also see \( \Gamma \) as a \( \Pi \)-linkage in \( G \).

Let \( \Pi \) be a \( k \)-pattern in a graph \( G \) embedded in a surface \( \Sigma \). A \( t \)-dartboard in \( G \) (with respect to \( \Pi \)) is subgraph of \( G \) whose components are \( v \) and cycles \( C_1, \ldots, C_t \) such that

i. each \( C_i \) bounds a disc \( \Delta_i \) in \( \Sigma \);
ii. \( v \in \Delta_1 \subset \ldots \Delta_t \);
iii. \( V(\Pi) \) is disjoint from the interior of \( \Delta_t \).

The vertex \( v \) is the centre of the dartboard.

Our main Theorem is the following:

\textbf{Theorem 1.} Let \( l(g,k) = (20k/9) \cdot (3e^{10/3e})^{3(g-1)+2k} \). The centre of any \( l(g,k) \)-dartboard with respect to a \( k \)-pattern \( \Pi \) in a graph \( G \) embedded on a surface \( \Sigma \) of genus \( g \) is redundant.
3. From the general case to reduced instances

Our proof has two main steps. In this Section, we prove the first part in which we reduce the problem to a so called “reduced instance”.

Let $L$ be a $\Pi$ linkage in a graph $G$ embedded in a surface $\Sigma$ for some $k$-pattern $\Pi$. Let $C$ be a $t$-dartboard with respect to $\Pi$. A subpath $P = v_0v_1 \ldots v_p$ of $L$ such that there exists $0 \leq i < t - p$ with the property that each $v_i$ belongs to $V(C_{i+j})$ ($0 \leq j \leq p$) is increasing (from $i$ to $i+p$), and it is decreasing if the reverse path is increasing. A valley is a subpath $v_{-p} \ldots v_p$ of $C$ with both $v_{-p} \ldots v_0$ and $v_p \ldots v_0$ decreasing from $t$ to $t-p$. A bad valley is a valley whose vertices $v_{-p}$ and $v_p$ belong to the same end of a strip of $\Omega$.

Let $L$ be a $\Pi$ linkage in a graph $G$ embedded in a disc with strips $\Omega$ for some $k$-pattern $\Pi$ in $G$. We say that $(\Omega,G,\Pi,L)$ is a reduced $t$-instance if

i. $\Pi$ is a $k$-pattern in $\Omega$;
ii. $bd(\Omega)$ is a subgraph of $G$;
iii. $G - E(L)$ is a $t$-dartboard $C$ such that $\Delta_t = \Delta(\Omega)$;
iv. $V(G) = V(C) = V(L)$;
v. the components of $L \cap \Delta(\Omega)$ are valleys which are not bad.

We prove Theorem 1 as a corollary of the following theorem.

**Theorem 2.** Let $f(s,k) = (20k/9) \cdot (3e^{10/(3e)})^s$. If $(\Omega,G,\Pi,L)$ is a reduced $f(s,k)$-instance on a disc with $s$ strips, then the centre of the dartboard $G - E(L)$ is redundant.

We now prove Theorem 1 assuming Theorem 2.

**Theorem 1.** Let $l(g,k) = (20k/9) \cdot (3e^{10/(3e)})^{3(g-1)+2k}$. The centre of any $l(g,k)$-dartboard with respect to a $k$-pattern $\Pi$ in a graph $G$ embedded on a surface $\Sigma$ of genus $g$ is redundant.

**Proof.** Let $f$ be the function in the statement of Theorem 2. We claim that $l(g,k) = f(3(g-1) + 2k,k)$ satisfy the conditions of the Theorem. Obviously, if $G$ contains no $\Pi$-linkage, then removing the centre of the dartboard will not change anything. So we can assume that $G$ contains a $\Pi$-linkage $C$. Assume for a contradiction that the theorem does not hold for some $g$ and $k$. Let us choose a counter example with $|V(G)| + |E(G)|$ minimum. Let $C$ be a $t$-dartboard with respect to $\Pi$ for $t = l(g,k)$, and let $v$ and $C_1, \ldots, C_t$ and $v$ be its components in order.

**Claim 1.** $C = G - E(L)$ and $V(G) = V(C) = V(L)$.

**Subproof.** We can delete all edges not in $E(C) \cup E(L)$, and contract all edges in $E(C) \cap E(L)$. If $x$ belongs to the symmetric difference $V(L) \Delta V(C)$, then we can contract any edge $xy$ incident with $x$ onto $y$. All cases yield smaller counter-examples.

Note that in this step we may end up identifying $s_i$ and $t_i$. This is the main reason why we allow $s_i = t_i$ in our $k$-patterns.

**Claim 2.** The first and last edge of every $L \in L$ are contained in $\Delta_t$.

**Proof.** If not, we can move the vertex $x \in V(\Pi)$ to the other end of the faulty edge and remove the edge resulting in a smaller counter-example.
Claim 3. No edge $e$ of $L \cap \Delta_t$ has both ends on the same cycle $C_i$.

Subproof. If not, let $C'_i$ be the cycle of $C_i \cup \{e\}$ which contains $e$ and which bounds a disc containing $v$. Replacing $C_i$ with $C'_i$ and removing all the edges in $E(C_i) \setminus E(C'_i)$ yields a smaller counter example. \hfill \square

Claim 4. Every component of $L \cap \Delta_t$ is a valley.

Subproof. If not, there exist a subpath $P = v_p \ldots v_0$ of some $L \in \mathcal{L}$ such that $v_p \ldots v_0$ and $v_0$ increases from $i$ to $i + p$. Such a path is a hill from $i$. Among all hills choose one from a minimum $i$. Let $Q$ be the subpath of $C_i$ such that $P \cup Q$ bounds a disc $\Delta_P$ which does not contain $v$. No $L' \in \mathcal{L}$ crosses $Q$. Indeed, since $L$ contains disjoint paths, any such $L'$ would then contain a hill from $i - 1$ which contradict the choice of $i$. We can thus replace $P$ in $L$ by $Q$ and remove all the edges in $P$ to obtain a smaller counter example.

So far we focused only on edge inside $\Delta_t$. Let us study edges outside $\Delta_t$. Every such edge $xy$ together with the radius $vx$ and $vy$ define a closed curve $\mu_{xy}$ and any two such curves are disjoint except from the base point $v$. An edge $e$ is non contractible if $\mu_e$ is and two edges $e$ and $f$ are homotopic if $\mu_e$ and $\mu_f$ are.

Claim 5. For every edge $e$ outside $\Delta_t$ such that $\mu_e$ bounds a disc $\Delta_e$, there exists a vertex of $V(\Pi)$ in the interior of $\Delta_e$.

Subproof. If not, let $\Delta'_i = \Delta_i \cup \Delta_t$ and $C'_i$ be the boundary of $\Delta'_i$. Replacing $C_i$ with $C'_i$ and removing the edges in $E(C_i) \setminus E(C'_i)$ yields a smaller counter-example. \hfill \square

Note that if $\mu_{xy}$ bounds a disc $\Delta_{xy}$ and $\mu_{x'y'}$ bound a disc $\Delta_{x'y'}$, then either $\Delta_{xy}$ and $\Delta_{x'y'}$ are disjoint (except from $v$) or one contains the other. Let $E_c$ contain the edges such that the corresponding discs are maximal.

Claim 6. The set $E_c$ contains at most $2k$ edges.

Subproof. This follows from the fact that, because of Claim 5, each such maximal disc must contain at least one vertex of $V(\Pi)$. \hfill \square

Claim 7. There are at most $3(g - 1)$ homotopy classes of non contractible edges.

Subproof. Let $D$ contain one curve $\mu_e$ from each homotopy class of non contractible edge. The curves in $D$ are the edges of a loopless simple graph embedded on $\Sigma$ with one vertex $v$. Let $H$ be such a graph maximal with the property that $D \subseteq E(H)$. Every face of $H$ is then a disc which is bounded by exactly 3 edges. Thus, $3|F(H)| = 2|E(H)|$. When combining this equality and Euler’s formula (i.e. $|V(H)| + |F(H)| = |E(H)| + 2 - g$), we obtain that $3 + 2|E(H)| \geq 3|E(H)| + 6 - 3g$, and thus $|D| \leq |E(H)| \leq 3g - 3$. \hfill \square

For each homotopy class $\mathcal{E}$ of non contractible edges, we can choose a strip in $\Sigma$ whose sides belong to $\mathcal{E}$ and which contains $\mathcal{E}$ and no other edge of $G$. For each edge $e \in E_c$ if we remove from $\Delta_e \setminus \Delta_t$ a small disc around a vertex of $V(\Pi)$, we also obtain a strip. In this way, we can embed $G$ on a first disc with at most $3(g - 1) + |E_c|$ strips $\Omega_1$.

Unfortunately, although we know that at least $|E_c|$ vertices of $V(\Pi)$ are not incident with strips of $\Omega_1$, some other vertex $v \in V(\Pi)$ may be. But then, because of Claim 2 there is a face $F$ of $G$ contained in some strip $S$ such that $v$ is incident with no strip of $\Omega_1 \setminus F$. Removing $F$ from $\Omega_1$ “splits” $S$ in two, and after at most
2k − |E_c| such splitting, we obtain a disc with at most $3(g - 1) + 2k$ strips $\Omega$ with
$V(\Pi)$ being incident no strip of $\Omega$. Thus $(\Omega, G, \Pi, L)$ is almost a reduced $t$-instance.
The only remaining problem is that there could be bad valleys.

**Claim 8.** The instance $(\Omega, G, \Pi, L)$ is a reduced $t$-instance.

*Subproof.* Suppose that $P = v_{-p} \ldots v_p$ is a valley and $v_{-p}$ and $v_p$ both belong to a
same end of a strip $S$. By following $P$ along $L$ in both directions, we obtain a path
$P' = v_{-p-2} \ldots v_{p+2}$ such that $v_{p+2}v_{p+1}$ and $v_{p+2}v_{p+1}$ are edge from $C_{t-1}$ to $C_t$
and $v_{p-1}v_p$ and $v_{p+1}v_p$ cross $S$ in the same direction. The path $P'$ looks like a
hill as defined in Claim 4 except that the hill is so high that it “traverses the sky”
through $S$. We thus define mountains from $i$ as a subpath $v_{-p-1-i} \ldots v_{p+1+i}$ of $L$
such that
- $v_{p-1-i} \ldots v_{p-1}$ and $v_{p+1+i} \ldots v_{p+1}$ increases from $t-i$ to $t$,
- and $v_{p-1}v_p$ and $v_{p+1}v_p$ cross a strip $S$;
- $v_{-p} \ldots v_p$ is a bad hill.

What we just proved is that if there is a bad hill, then there exists a mountain.
As in the proof of Claim 4 we can easily shortcut a mountain from some minimal $i$
and obtain a smaller counter-example.

We have now finished our cleaning process and we can apply Theorem 2. Indeed,
$(\Omega, G, \Pi, L)$ is a reduced $f(3(g - 1) + 2k, k)$-instance on a disc with at most
$(3(g - 1) + 2k)$-strips. The centre $v$ is thus redundant.

4. The Proof for Reduced Instances

The strategy to prove Theorem 2 is the following. Let $(\Omega, G, \Pi, L)$ be a reduced
$t$-instance with $t$ large. We first find a topological $\Pi$-linkage $\Gamma$ which crosses the
strip “few” times (using Theorem 4). The idea is to try to realise this topological
linkage in $G$. To do so, two cases arise. If $L$ crosses all the strips “enough” time,
then we explicitly realise $\Gamma$ in $G$. Otherwise, we can cut “small” strips and reduce
to a $k + “few”$ disjoint path problem on a disc with less strips.

Our tool to find a good topological linkage is the following Theorem of Geelen et co-
GHR13.

**Theorem 4.** Let $\Sigma$ be a surface with boundary, $\Pi$ be a topologically feasible $k$-
pattern in $\Sigma$, and $P$ be a non-separating $\text{bd}(\Sigma)$-path in $\Sigma$ whose ends are disjoint
from $V(\Pi)$. There exists a $\Pi$-linkage $\Gamma$ in $\Sigma$ such that each path $\gamma \in \Gamma$ intersects $P$
at most twice.

We also need the two following easy Lemmas.

**Lemma 1.** Let $\Pi$ be a $k$-pattern in a graph $G$ embedded on a disc $\Delta$. Let $L$ be a
$\Pi$-linkage in $G$. If $(\Delta, G, \Pi, L)$ is a reduced $t$-instance for $t \geq k$, then $v$ is redundant.

*Proof.* The proof is by induction on $k$. A *border path* in $L$ is an $L \in L$ such that one
of the component of $\Delta \setminus L$ contains no path in $L$. Any such a path which meets $C_{t-1}$
can always be rerouted to a subpath of $C_t$ linking its ends. Let thus assume that no
border path in $L$ meets $C_{t-1}$. Since all $\Pi$-linkage in a disc have at least one border
path, then $L' = L \cap \Delta_{t-1}$ is a $\Pi'$ linkage in $G' = G \cap \Delta_{t-1}$ for some $k'$-pattern
$\Pi'$ with $k' < k$. Note that $(\Delta_{t-1}, G', \Pi', L')$ is a reduced instance and $t - 1 > k'$.
There thus exists a $\Pi'$-linkage $L''$ which avoids $v$ in $G'$. But then $L'' \cup (L \setminus \Delta_{t-1})$ is
a $\Pi$-linkage in $G$ which avoids $v$. 

The following Lemma can be proved in a very similar way. We thus leave the proof to the reader.

**Lemma 2.** Let $G = P_k \times C_n$ be a cylinder embedded on a disc $\Delta$ such that $\text{bd}(\Delta)$ is a cycle $C$ of $G$, and let $\Pi$ be a $k$-pattern in $G$ with $V(\Pi) \subseteq V(C)$ (and thus $n \geq 2k$). If $\Pi$ is topologically feasible, then $G$ contains a $\Pi$-linkage.

Let $(\Omega, G, \Pi, \mathcal{L})$ be a reduced instance. The size $|S|$ of a strip $S$ of $\Omega$ is the number $p$ of components of $S \cap \mathcal{L}$. We say that $\mathcal{L}$ crosses $S$ $p$ times.

**Lemma 3.** Let $(\Omega, G, \Pi, \mathcal{L})$ be a reduced $t$-instance on a disc with $s$ strips for $t \geq 2k^3$. If for each $1 \leq i \leq s$, $|S_i| \geq 3k^3i + 1$, then the centre $v_{\text{centre}}$ of the dartboard $G \setminus \mathcal{L}$ is redundant.

**Proof.** We prove the Lemma by induction on the lexicographic order on $(s,k)$. Suppose that the set $\mathcal{L}_0$ of the singletons of $\mathcal{L}$ is nonempty. Remove $E(C_t)$ from $G$. As long as some degree 2 vertex $u$, contract an edge incident with $u$ and then remove from $\Omega$ the faces incident with $\text{bd}(\Delta_t)$. We then obtain reduced $(t-1)$-instance for a $(<k)$-pattern $\Pi'$ in a graph $G'$ embedded on a disc with $s$ strips. Since $(k-1)3^s < k^3s - 1$, then $v_{\text{centre}}$ is redundant. There thus exists a $\Pi'$ linkage $\mathcal{L}'$ in $G'$. But the $\mathcal{L}' \cup (\mathcal{L} \setminus \Delta_{t-1}) \cup \mathcal{L}_0$ is a $\Pi$-linkage in $G$ which avoids $v_{\text{centre}}$.

For $1 \leq i \leq s$, let $\Omega_i = \Omega \setminus (S_1 \cup \cdots \cup S_i)$. Note that $\Omega_0 = \Omega$ and $\Omega_s = \Delta_t$. Because $\mathcal{L}$ contains no bad valley, on each end of $S_i$, the middle $k^3i + 1$ paths have to either “go over” the $k^3i$ paths on their left or on their right. They thus contains subpaths going from $C_{t-k^3i+1}$ through $S_i$ and then down to $C_{t-k^3i+1}$. Let $\mathcal{P}_i$ contain these $k^3i + 1$ subpaths, and let $\mathcal{Q}_i$ contain the middle $2k^3i - 1$ paths of $\mathcal{P}_i$ (i.e. $\mathcal{Q}_i$ leaves $|k^3i - 1/2|$ paths of $\mathcal{P}_i$ on one side and $|k^3i - 1/2| + 1$ ones on the other side).

**Claim 9.** There exists a topological linkage $\Gamma$ such that for $0 \leq i \leq s$, $\Gamma$ crosses $S_i$ at most $2k^3i^{-1}$. Moreover, we can suppose that in each strip $S_i$, $\Gamma \cap S_i$ is a subgraph of $\mathcal{Q}_i \cap S_i$.

**Subproof.** We prove by induction on $0 \leq i \leq s$ that there exists a topological $\Pi$-linkage $\Gamma_i$ such that for $1 \leq j \leq i$, $\Gamma_j$ crosses each strip $S_j$ at most $2k^3j^{-1}$. For $i = 0$, we set $\Gamma_0 = \mathcal{L}_0$. So suppose that $\Gamma_{i-1}$ has been defined for $1 \leq i \leq s$. By construction, $\Gamma_{i-1} \cap \Omega_{i-1}$ is a $\Pi_{i-1}$-linkage for some $w$-pattern $\Pi_{i-1}$ such that $w \leq k + 2k^3i^{-1} + 2k^3i^{-1} + \cdots + 2k^3(i-1)^{-1} = k^3i^{-1}$. By Theorem 4 there exists a $\Pi_{i-1}$-linkage $\Gamma'_i$ which crosses $S_i$ at most $2w \leq 2k^3i^{-1}$ times, and since $2k^3i^{-1} = |\mathcal{Q}_i|$, we can suppose that $\Gamma'_i \cap S_i$ is a subgraph of $\mathcal{Q}_i \cap S_i$. But then $\Gamma_i = \Gamma'_i \cup (\Gamma_{i-1} \setminus \Omega_{i-1})$ is a $\Pi$-linkage in $\Omega$ which satisfies the induction conditions for $i$. The linkage $\Gamma = \Gamma_s$ satisfies the conditions of the Claim. \qed

In the remaining of this proof, we show that we can indeed realise in $G$ the topological linkage $\Gamma$ given by the previous Claim. To do so, we first fix an orientation of $\Delta_t$ so that we can order elements of $\text{bd}(\Delta_t)$ from left to right. As in the proof of Claim 9 for $0 \leq i \leq s$, $\Gamma_i := \Gamma \cap \Omega_i$ is a $\Pi_i$-linkage for some $(\leq k^3)$-pattern $\Pi_i$. A boundary segment $l$ of $\Omega_i$ is a component of $\text{bd}(\Delta(\Omega_i)) \cap \text{bd}(\Omega_i)$. Let $x_0, \ldots, x_p$ be the vertices of $V(\Pi_i)$ in a boundary segment $\alpha$ of $\Omega$ in order. An $\alpha$-pyramid is a set of paths $\mathcal{M}_l = \{M_0, \ldots, M_p\}$ such that for $0 \leq m \leq p/2$, $M_m$ and $M_{p-m}$ respectively link $x_m$ and $x_{p-m}$ and $C_{t-m}$. 


Claim 10. Let $\alpha_1$ and $\alpha_2$ be the two boundary paths of $\Omega_{s-1}$. There exists in $G \cap \Delta_i$ disjoint $\alpha_i$-pyramids which do not meet $P_s$.

Subproof. We prove by induction on $0 \leq i < s$ that there exists a set $M_i$ of disjoint path such that for each boundary segment $\alpha$ of $\Omega_i$, $M_i$ contains an $\alpha$-pyramid, and $M_i$ is disjoint from the sets $P_j$ for $i < j \leq s$.

The existence of $M_0$ follows from the fact that $L$ is a $\Pi$ linkage and $\Pi$ contains no singleton. Indeed, since $\Pi$ contains no singleton, no path $L \in L \cap \Delta_i$ with an end in $V(\Pi)$, let $\alpha$ be a boundary segment of $\Omega_0$, and let $x_0, \ldots, x_p$ be the vertices of $V(\Pi) \cap \alpha$ in order. The path $L_i$ leaving $x_i$ either has to go “over” the $i$ path leaving $x_0, \ldots, x_{i-1}$ or it has to go “over” the $p - i$ path leaving $x_{i+1}, \ldots, x_p$. It thus as to meet $C_{i-\min(i,p-1)}$. We can thus replace $L_i$ by a subpath so that $M_0$ satisfies the required property. Note that $L$ may produce a subpath for each of its ends but these two subpaths do no meet because $L$ has to “go over” the paths in some $P_r$.

Suppose now that $M_{i-1}$ exists for $1 \leq i < s$. The boundary segments of $\Omega_i$ are precisely boundary segments of $\Omega_{i-1}$ which are not incident with $S_i$ and the two unions of the ends of $S_i$ with the boundary segment of $\Omega_{i-1}$ to which they are incident to. For a boundary segment $\alpha$ of $\Omega_i$ which is also a boundary segment of $\Omega_{i-1}$, we put in $M_i$ the paths of the $\alpha$-pyramid in $M_{i-1}$. So let $\alpha_{\text{left}} \cup \beta \cup \alpha_{\text{right}}$ be a boundary segment of $\Omega_i$ in which $\beta$ is an end of $S_i$ and $\alpha_{\text{left}}$ and $\alpha_{\text{right}}$ are boundary segments of $\Omega_{i-1}$ incident with $\beta$ which are respectively on the left and on the right of $\beta$. Let $x_0, \ldots, x_p-1, y_0, \ldots, y_{q-1}, z_0, \ldots, z_{r-1}$ be the vertices of $\Pi_i$ on $\alpha_{\text{left}} \cup \beta \cup \alpha_{\text{right}}$ in order with all $x_j \in \alpha_{\text{left}}, y_j \in \beta$ and $z_j \in \alpha_{\text{right}}$. Let $\tilde{\mathcal{P}}$ contain the paths of $P_i \setminus S_i$ with an end in $\beta$. Let $\mathcal{P}_{\text{middle}}$ contains the paths of $\tilde{\mathcal{P}} \cap (Q_i \setminus S_i)$, and $P_{\text{left}}$ and $P_{\text{right}}$ contain respectively the paths on the left and on the right of $\mathcal{P}_{\text{middle}}$. One of $P_{\text{left}}$ and $P_{\text{right}}$ contains $\lceil k3^{i-1}/2 \rceil$ paths and the other contains $\lfloor k3^{i-1}/2 \rfloor + 1$ paths. So both contain at least $\lfloor k3^{i-1}/2 \rfloor$ paths.

Let $F_0, \ldots, F_{p-1}$ be the paths of the $\alpha_{\text{left}}$-pyramid in $M_i$ taken from left to right, and let $H_0, \ldots, H_{r-1}$ be the paths of the $\alpha_{\text{right}}$-pyramid in $M_i$ taken from left to right. Let $\tilde{\mathcal{M}}$ be the set of paths which contains

i. the paths $F_j$ for $0 \leq j \leq (p-1)/2$,

ii. the paths obtained by following $F_{p-1-j}$, then going right along $C_{t-j}$ to the $j^\text{th}$ path of $P_{\text{left}}$ on the right and then along this path down to $C_{t-k3^i+1}$ for $0 \leq j < (p-1)/2$,

iii. the paths in $P_{\text{middle}}$ with $y_j$ as an end ($0 \leq j < q$),

iv. the paths obtained by following $H_{r-1-j}$, then going left along $C_{t-j}$ to the $j^\text{th}$ path of $P_{\text{right}}$ on the left and then along this path down to $C_{t-k3^i+1}$ for $0 \leq j < (p-1)/2$,

v. the path $H_{r-1-j}$ for $0 \leq j \leq (r-1)/2$.

Note that these paths are well defined because, $\Pi_{i-1}$ is a $k3^{i-1}$-pattern, which implies that $p + r \leq k3^{i-1}$, and thus we respectively only send $\lfloor p/2 \rfloor \leq |P_{\text{left}}|$ paths in $P_{\text{left}}$ and $\lceil r/2 \rceil \leq |P_{\text{right}}|$ paths in $P_{\text{right}}$. Now, $\tilde{\mathcal{M}}$ contains $p + q + r \leq k3^i$ paths, and since in the case $\text{iii}$ and $\text{iv}$ the path reach $C_{t-k3^i+1}$, we can thus obtain a $\alpha_{\text{left}} \cup \beta \cup \alpha_{\text{right}}$-pyramid by shortening paths in $\tilde{\mathcal{M}}$. We put the paths of this pyramid in $M_i$. Since the paths in $M_{i-1}$ were disjoint from the paths in $P_j$ for
We can then try to apply Lemma 3. If we can, then we know that 2\(|k3^s/2|\) exists. Indeed, as already noted, we can suppose that we cannot directly apply there exists \(i\). Let \(P_{\beta_i}\) be the paths of \(P_s \setminus S_s\) with an end in \(\beta_i\) \((i = 1, 2)\). We extend the paths in the pyramids by sending, for \(i = 1, 2\),

- the path on the left of the \(\alpha_i\)-pyramid to the left and use at most \([k3^s/2]\) paths on the right of \(P_{\beta_{i-1}}\), to reach \(C_{t-k3^s+1}\);
- the path on the right of the \(\alpha_i\)-pyramid to the right and use at most \([k3^s/2]\) paths on the left of \(P_{\beta_i}\), to reach \(C_{t-k3^s+1}\).

Note that there rerouting are possible because there are \([k3^s/2]\) or \([k3^s/2]+1\geq[k3^s/2]\) paths free to accommodate them. In the end, we have found a set \(M\) of disjoint path in \(G \cap \Delta_t\) linking the vertices of \(V(\Pi_s)\) to \(C_{t-k3^s+1}\). Since \(\Pi_s\) is a \(k3^s\)-pattern, Lemma 3 implies that there is a \(\Pi_s\)-linkage \(L'\) in \(P \cup \{C_1 \cup \cdots \cup C_{t-k3^s+1}\}\). But then \(L' \cup (L \setminus \Delta_t)\) is a \(\Pi\)-linkage in \(G\) which avoids \(v_{\text{centre}}\). \(\square\)

We now finish the proof of the Theorem.

**Theorem 2.** Let \(f(s, k) = (20k/9) \cdot (3e^{10/(3\epsilon)})^s\). If \((\Omega, G, \Pi, L)\) is a reduced \(f(s, k)\)-instance on a disc with \(s\) strips, then the centre of the dartboard \(G - E(L)\) is redundant.

**Proof.** Let us order the strips \(S_1, \ldots, S_s\) by increasing size, and let \(|S_i|\) be the size of \(S_i\). If we can apply Lemma 3 then we know that we only need \(2k3^s \leq f(s, k)\) cycles to realise \(\Pi\). If not, then there exists a strip \(i\) such that \(|S_i| \leq 3k3^s\). Let \(i_1\) be the maximum such \(i\). Then \(L \cap \Omega_{i_1}\) is a \(\Pi'\)-linkage for some \(k + |S_i| + \cdots + |S_{i_1}|\) pattern \(\Pi'\). We know that \(|S_{i_1}| \leq |S_i| \leq 3k3^{s_1}\). So \(\Pi'\) is a \((\leq k + 3k_13^{s_1})\) pattern in \(\Omega_{i_1}\).

We can then try to apply Lemma 3. If we can, then we know that \(2(k + 3k_13^{s_1})^{3s_2-i_1}\) cycles are enough to realise \(\Pi'\) in \(G \cap \Omega_{i_1}\), and thus to realise \(\Pi\) in \(G\). If not, then there exists \(i_2 > 0\) such that \(|S_{i_1+i_2}| \leq 3(k + 3k_13^{s_1})^{3s_2-i_1}\), and we can iterate.

More formally, let \(\xi = 10/3\). We recursively define \(i_1 > 0\) to be the maximum index such that \(|S_{i_1+i_2+\cdots+i_{i-1}}| \leq k \cdot \xi^{i} \cdot i_1 \cdot i_2 \cdots i_{i-1} 3^{s_1+i_2+\cdots+i_{i-1}}\). We can suppose that \(i_1\) exists. Indeed, as already noted, we can suppose that we cannot directly apply Lemma 3 so there exists \(i\) such that \(|S_i| \leq 3k3^s\). But then surely \(|S_i| \leq k \cdot \xi \cdot i \cdot 3^s\). So suppose that \(i_1\) exist and that \(i_{i_1+1}\) does not. Then \(L \cap \Omega_{i_1+i_2+\cdots+i_{i-1}}\) is \(\Pi'\)-pattern for some \(k'\)-pattern \(\Pi'\). Let us now bound \(k'\).

\[
k' = k + (|S_1| + \cdots + |S_{i_1}|) + (|S_{i_1+1}| + \cdots + |S_{i_1+i_2}|) + \ldots
+ (|S_{i_1+i_2+\cdots+i_{i-1}+1}| + \cdots + |S_{i_1+i_2+\cdots+i_{i-1}}|)
\leq k + i_1|S_{i_1}| + i_2|S_{i_1+i_2}| + \cdots + i_{i_1}|S_{i_1+i_2+\cdots+i_{i-1}}|
\leq k + k \cdot \xi^1 \cdot i_1 \cdot 3^{s_1}
\quad + k \cdot \xi^2 \cdot i_1 \cdot i_2 \cdot 3^{s_2+i_2} + \ldots
\quad + k \cdot \xi^i \cdot i_1 \cdot i_2 \cdots i_{i-1} \cdot 3^{s_1+i_2+\cdots+i_{i-1}}
\leq k \cdot \xi^i \cdot i_1 \cdot i_2 \cdots i_{i} \cdot 3^{s_1+i_2+\cdots+i_{i-1}} \left(1 + \frac{1}{\xi \cdot i_{i} \cdot 3^{s_1}}\right)
\]
We can thus apply Lemma 3. Let the central vertex $v$ realise $\Gamma$ is bounded by
\[
\leq k \cdot \xi \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l \cdot 3^{i_1+i_2} + \cdots + 1 \cdot \xi \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + 1
\]
\[
\leq k \cdot \xi \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l \cdot 3^{i_1+i_2} + \cdots + 1 \cdot \frac{(3\xi)^{l+1} - 1}{3\xi - 1}
\]
\[
\leq k \cdot \frac{1}{3} (3\xi)^{l+1} - 1 \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l
\]
\[
|S_{i_1+i_2+i_3}| > k \cdot \xi \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l
\]
\[
> \xi + 3\xi - 1 \cdot \left( \frac{k}{3} \cdot \frac{(3\xi)^{l+1} - 1}{3\xi - 1} \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l \right)3^j
\]
\[
> \frac{(3\xi)^{l+1} - 1}{3} k' 3^j
\]
\[
> \frac{3\xi - 1}{3} k' 3^j = 3k' 3^j
\]

But then, because $i_{i+1}$ does not exist, for every $j > 0$,
\[
|S_{i_1+i_2+i_3}| > k \cdot \xi \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l
\]
\[
> \xi + 3\xi - 1 \cdot \left( \frac{k}{3} \cdot \frac{(3\xi)^{l+1} - 1}{3\xi - 1} \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l \right)3^j
\]
\[
> \frac{(3\xi)^{l+1} - 1}{3} k' 3^j
\]
\[
> \frac{3\xi - 1}{3} k' 3^j = 3k' 3^j
\]

We can thus apply Lemma 3. Let $s' = s - (i_1 + \cdots + i_l)$. The number of cycles need to realise $\Gamma$ is bounded by
\[
C = 2k' 3^{s'} < 2k \cdot \frac{1}{3l} (3\xi)^{l+1} - 1 \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l
\]
\[
\leq 2k \cdot \frac{1}{3l} (3\xi)^{l+1} - 1 \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^{i_1+i_2} + \cdots + i_l
\]
\[
\leq 2k \cdot \frac{3\xi - 1}{3\xi - 1} \cdot \xi \cdot i_1 \cdot i_2 \cdots i_l \cdot 3^s
\]
\[
\leq \frac{20}{9} k \left( \frac{i_1 + \cdots + i_l}{l} \right)^l 3^s \quad \text{by convexity}
\]
\[
\leq \frac{20}{9} k \left( \frac{s}{l} \right)^l 3^s \quad \text{because } i_1 + \cdots + i_l \leq s
\]

But $\left( \frac{s}{l} \right)^l$ is maximum for $l = \xi s/e$. So $C \leq (20k/9) \cdot e^{10s/(3e)} \cdot 3^s = f(s, k)$. The central vertex $v_{centre}$ is thus redundant as claimed. \qed

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