Higher order correlations for fluctuations in the presence of fields

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Abstract

The higher order moments of the fluctuations for the thermodynamical systems in the presence of fields are investigated in the framework of a theoretical method. The method uses a generalized statistical ensemble consonant with the adequate expression for the internal energy. The applications refer to the case of a system in magnetoquasistatic field. In the case of linear magnetic media one finds that for the description of the magnetic induction fluctuations the Gaussian approximation is good enough. For nonlinear media the corresponding fluctuations are non-Gaussian, they having a non-null asymmetry. Additionally the respective fluctuations have characteristics of leptokurtic, mesokurtic and platykurtic type, depending on the value of the magnetic field strength comparatively with a scaling factor of the magnetization curve.

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I. INTRODUCTION

In our previous work [1] we have presented a phenomenological approach of the fluctuations for the generalized systems (i.e. thermodynamical systems in the presence of fields). In the respective work the fluctuations were evaluated only by second order numerical characteristics (correlations and moments). From a more general probabilistic perspective [2, 3, 4, 5, 6] the fluctuations in physical systems must be evaluated also by means of higher order numerical characteristics (higher order correlations and moments). In the present paper we aim to present an approach for the evaluation of the higher order correlations for the alluded generalized systems.

For our aim in Sec.II we will present the general considerations and relations regarding the approached problems. Next one in Sec.III we proceed to evaluations of higher order moments for linear magnetic systems (in the presence of magnetoquasistatic field), as well as for nonlinear magnetic media. The mentioned evaluations are finalized in some interesting relations able for comparisons with adequate experimental results.

II. THEORETICAL CONSIDERATIONS

A. Statistical ensembles for generalized systems

As in previous work [1] (and above reminded) we consider a generalized system described by the set of extensive parameters \((U, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m)\), where \(U\) denotes the generalized internal energy, \(X_i (i = 1, 2, \ldots, n)\) signify the usual extensive parameters (for systems in absence of fields), while \(Y_j (j = 1, 2, \ldots, m)\) mean the extensive parameters associated with the fields.

In the framework of fluctuation theory the thermodynamical quantities represent the mean (or expected) values of random variables. In the following we will note with \(\bar{A}\) the mean value of the random quantity \(A\). In such context we have \(U = E\), where \(E\) denotes the energy regarded as random variable.

We take the investigated system as a small part of a big, isolated ensemble. Generally speaking the probability density for the fluctuations of the quantities \(E, \{X_i\}_{i=1}^n\) and \(\{Y_j\}_{j=1}^m\)
are of the form

\[ w = Z^{-1} \exp \left( -\beta E - \sum_{i=1}^{n} \alpha_i X_i - \sum_{j=1}^{m} \gamma_j Y_j \right) \]  

(1)

where \( Z \) is given by the normalization condition of the probability.

\[ Z = \int \cdots \int \prod_{i=1}^{n} dX_i \exp (-\alpha_i X_i) \prod_{j=1}^{m} dY_j \exp (-\gamma_j Y_j) \int_\Gamma e^{-\beta E} d\Gamma \]  

(2)

d\( \Gamma \) represents the elementary volume in phase space.

The statistical integral \( Z \) is a function of the quantities \( \beta, \{\alpha_i\}_{i=1}^{n} \) and \( \{\gamma_j\}_{j=1}^{m} \). For the identification of the physical signification of the parameters \( \beta, \alpha_i \) and \( \gamma_j \) we note

\[ d \left( \ln Z + \beta E + \sum_{i=1}^{n} \alpha_i X_i + \sum_{j=1}^{m} \gamma_j Y_j \right) = \right] \]

\[ = \beta dE + \sum_{i=1}^{n} \alpha_i dX_i + \sum_{j=1}^{m} \gamma_j dY_j \]  

(3)

or

\[ d \left( \ln Z + \beta U + \sum_{i=1}^{n} \alpha_i X_i + \sum_{j=1}^{m} \gamma_j Y_j \right) = \right] \]

\[ = \beta dU + \sum_{i=1}^{n} \alpha_i dX_i + \sum_{j=1}^{m} \gamma_j dY_j \]  

(4)

where we take into account that in fact \( U = E \).

From the thermodynamics of the generalized systems it is known \([10, 11]\) that

\[ dU = T dS + \sum_{i=1}^{n} \xi_i dX_i + \sum_{j=1}^{m} \psi_j Y_j \]  

(5)

where \( S, T \) and \( \xi_i \) denote respectively the entropy, temperature and the field dependent intensive parameters, while \( \psi_j \) signify the supplementary parameters due to the presence of the fields.

From (4) and (5) one obtains:

\[ d \left( \ln Z + \beta U + \sum_{i=1}^{n} \alpha_i X_i + \sum_{j=1}^{m} \gamma_j Y_j \right) = \right] \]

\[ = \beta T dS + \sum_{i=1}^{n} \left( \alpha_i + \beta \tilde{\xi}_i \right) dX_i + \sum_{j=1}^{m} \left( \gamma_j + \beta \psi_j \right) d\overline{Y}_j \]  

(6)
One observes that in this relation the left hand term is an exact differential. This fact impose
the condition that the right hand term of the equation to be also an exact differential. From
the mentioned condition it directly results that we must have the following relations:

\[ \beta = \frac{1}{kT} \]  

(7)

\[ \alpha_i = -\beta \xi_i = -\frac{\xi_i}{kT} \]  

(8)

\[ \gamma_j = -\beta \psi_j = -\frac{\psi_j}{kT} \]  

(9)

In the above relations \( k \) denotes the Boltzmann’s constant.

We observe that the quantities \( \{ \alpha_i \}_{i=1}^n \) are functions of the field dependent intensive
parameters.

\section*{B. The evaluation of the higher order correlations and moments}

The mean values of the fluctuating quantities can be evaluated through the relations:

\[ U = \mathbb{E} = - \left[ \frac{\partial(\ln Z)}{\partial \beta} \right]_{\alpha_i,\gamma_j} , \quad i = 1, 2, \ldots, n \]

\[ j = 1, 2, \ldots, m \]  

(10)

\[ \bar{X}_i = - \left[ \frac{\partial(\ln Z)}{\partial \alpha_i} \right]_{\beta,\alpha_i,\gamma_j} , \quad l \neq i \]  

(11)

\[ \bar{Y}_j = - \left[ \frac{\partial(\ln Z)}{\partial \gamma_j} \right]_{\beta,\alpha_i,\gamma_l} , \quad l \neq j \]  

(12)

The expressions of the type \( \prod_i (\delta X_i)^{r_i} \prod_j (\delta Y_j)^{r_j} \) are called higher order correlations. By
using the statistical sum \( Z \) as above introduced for some of the respective correlations one
obtains the expressions:

\[ \delta X_a \delta X_b = \frac{\partial^2(\ln Z)}{\partial \alpha_a \partial \alpha_b} = - \frac{\partial \mathbb{X}_b}{\partial \alpha_a} = kT \frac{\partial \mathbb{X}_b}{\partial \xi_a} , \quad a, b = 1, 2, \ldots, n \]  

(13)

\[ \delta Y_a \delta Y_b = \frac{\partial^2(\ln Z)}{\partial \gamma_a \partial \gamma_b} = - \frac{\partial \mathbb{Y}_b}{\partial \gamma_a} = kT \frac{\partial \mathbb{Y}_b}{\partial \psi_a} , \quad a, b = 1, 2, \ldots, m \]  

(14)
\[ \delta X_a \delta Y_b = \frac{\partial^2 (\ln Z)}{\partial \alpha_a \partial \gamma_b} = -\frac{\partial Y_b}{\partial \alpha_a} = kT \frac{\partial Y_b}{\partial \xi_a}, \quad a = 1, 2, \ldots, n \]

\[ \delta X_a \delta X_b \delta X_c = -\frac{\partial^3 (\ln Z)}{\partial \alpha_a \partial \alpha_b \partial \alpha_c} = \frac{\partial^2 X_c}{\partial \alpha_a} \frac{\partial^2 X_c}{\partial \alpha_b} = k^2 T^2 \frac{\partial^2 X_c}{\partial \xi_a \partial \xi_b}, \quad a, b, c = 1, 2, \ldots, n \]

\[ \delta Y_a \delta Y_b \delta Y_c = -\frac{\partial^3 (\ln Z)}{\partial \gamma_a \partial \gamma_b \partial \gamma_c} = \frac{\partial^2 Y_c}{\partial \gamma_a} \frac{\partial^2 Y_c}{\partial \gamma_b} = k^2 T^2 \frac{\partial^2 Y_c}{\partial \psi_a \partial \psi_b}, \quad a, b, c = 1, 2, \ldots, m \]

\[ \delta X_a \delta Y_b \delta Y_c = -\frac{\partial^3 (\ln Z)}{\partial \alpha_a \partial \gamma_b \partial \gamma_c} = \frac{\partial^2 Y_c}{\partial \alpha_a} \frac{\partial^2 Y_c}{\partial \gamma_b} = k^2 T^2 \frac{\partial^2 Y_c}{\partial \xi_a \partial \psi_b}, \quad a = 1, 2, \ldots, n \]

\[ \delta Y_a \delta Y_b \delta Y_c = -\frac{\partial^3 (\ln Z)}{\partial \beta_a \partial \beta_b \partial \beta_c} = \frac{\partial^2 Y_c}{\partial \beta_a} \frac{\partial^2 Y_c}{\partial \beta_b} = k^2 T^2 \frac{\partial^2 Y_c}{\partial \psi_a \partial \psi_b}, \quad b, c = 1, 2, \ldots, m \]

The formulae for the correlations of orders higher than 3 are generally more complicated.

But the higher order moments can be obtained by means of the following recurrence formulae:

\[ (\delta X_a)^{n+1} = -\frac{\partial}{\partial \alpha_a} (\delta X_a)^n - n (\delta X_a)^{n-1} \frac{\partial X_a}{\partial \alpha_a} \]

\[ (\delta Y_a)^{n+1} = -\frac{\partial}{\partial \gamma_a} (\delta Y_a)^n - n (\delta Y_a)^{n-1} \frac{\partial Y_a}{\partial \gamma_a} \]

As examples we give here the expressions of the moments \((\delta X_a)^4\) and \((\delta Y_a)^4\).

\[ (\delta X_a)^4 = -\frac{\partial^3 X_a}{\partial \alpha_a^3} + 3 \left( \frac{\partial X_a}{\partial \alpha_a} \right)^2 = \]

\[ = \left( kT \frac{\partial}{\partial \xi_a} \right)^3 X_a + 3 \left( kT \frac{\partial X_a}{\partial \xi_a} \right)^2 \]

\[ (\delta Y_a)^4 = -\frac{\partial^3 Y_a}{\partial \gamma_a^3} + 3 \left( \frac{\partial Y_a}{\partial \gamma_a} \right)^2 = \]

\[ = \left( kT \frac{\partial}{\partial \psi_a} \right)^3 Y_a + 3 \left( kT \frac{\partial Y_a}{\partial \psi_a} \right)^2 \]

The fourth order moments are of interest for the evaluation of the so called excess coefficient:

\[ C_E = \frac{(\delta X_a)^4}{(\delta X_a^2)^2} - 3 \]

which is an indicator of the deviation from the gaussian distribution.
III. HIGHER ORDER MOMENTS FOR THE SYSTEMS SITUATED IN A MAGNETOQUASISTATIC FIELD

A. Linear magnetic media

Let us consider a uniformly magnetized continuous media, situated in a magnetoquasistatic field. The system is characterized by the extensive parameters \((U, V, N, B)\), where \(V, N\) and \(B\) denote respectively the volume, particle number and magnetic induction. In the case of linear magnetic systems the differential of the internal energy is given by \([10, 11]\)

\[
dU = TdS - \hat{p}dV + \hat{\zeta}dN + VH \cdot dB
\]  

(24)

For the sake of brevity we omitted the mean symbol from above the parameters \(V, N, B\), but we will take into account that in fact in relation (24) and in the expressions of the moments appear the mean values of the respective quantities.

The relations (13) and (16) show that the moments associated to the usual thermodynamic quantities, i.e. volume \(V\) or particle number \(N\), are functions of the parameters \(\hat{\xi}_i\), which depend on the fields constraints.

For example, in the case \(B = \text{const.}\) for the volume \(V\) one obtains:

\[
(\delta V)^2 = -kT \left( \frac{\partial V}{\partial \hat{p}} \right)_{T, \hat{\zeta}, B}
\]  

(25)

\[
(\delta V)^3 = k^2T^2 \left( \frac{\partial^2 V}{\partial \hat{p}^2} \right)_{T, \hat{\zeta}, B}
\]  

(26)

where \([10]\)

\[
\hat{p} (B = \text{const.}) = p_{B,N} = p - \frac{1}{2} H \cdot B - \frac{1}{2} H^2 \rho \frac{\partial \mu}{\partial \rho}
\]  

(27)

\(H\) signify the magnetic field strength, \(\mu\) is the magnetic permeability and \(\rho = \frac{N}{V}\).

By using the properties of the Jacobians the relation (25) can be transformed as follows:

\[
(\delta V)^2 = -kT \frac{\partial (V, \hat{\zeta}, T, B)}{\partial (\hat{p}, \hat{\zeta}, T, B)} = -kT \frac{\partial (V, \hat{\zeta}, T, B)}{\partial (\hat{p}, \hat{\zeta}, T, B)} \cdot \frac{\partial (V, N, T, B)}{\partial (\hat{p}, \hat{\zeta}, T, B)}
\]

\[
= -kT \frac{\left( \frac{\partial \hat{\zeta}}{\partial N} \right)_{T,V,B}}{\left( \frac{\partial \hat{p}}{\partial V} \right)_{T,N,B} \left( \frac{\partial \hat{\zeta}}{\partial N} \right)_{T,V,B} + \left( \frac{\partial \hat{\zeta}}{\partial V} \right)_{T,N,B}^2}
\]  

(28)
In the above relation we take into account the condition \(-\left(\frac{\partial \hat{\rho}}{\partial N}\right)_{T,V,B} = \left(\frac{\partial \hat{\varepsilon}}{\partial V}\right)_{T,N,B}\). Here is the place to point out that the result (28) is the same with the one obtained [1] within the Gaussian approximation.

Now let us evaluate the second and third order parameters of fluctuations for the magnetic induction \(B\). For simplicity we consider that volume and particle number are constant. By using the relations (14), (17) and (22) we find:

\[
\langle (\delta B)^2 \rangle = \frac{kT}{V} \left( \frac{\partial B}{\partial H} \right)_{T,V,N} = \frac{kT\mu}{V} \tag{29}
\]

\[
\langle (\delta B)^3 \rangle = \left( \frac{kT}{V} \right)^2 \left( \frac{\partial^2 B}{\partial H^2} \right)_{T,V,N} = \left( \frac{kT}{V} \right)^2 \frac{\partial \mu}{\partial H} = 0 \tag{30}
\]

The result (29) is identical with the one obtained [1] within the Gaussian approximation. We remark that in the case of linear magnetic media \(\langle (\delta B)^3 \rangle\) is null, because \(\mu\) is independent on \(H\). Additionally in this case the excess coefficient (23) is also null. These facts show that in the alluded case the Gaussian approximation is sufficient for a quantitative description of the fluctuations for \(B\).

**B. Nonlinear magnetic media**

In the case of nonlinear magnetic media \(\mu\) depends on \(H\). Therefore the evaluation of the moments of orders higher than 2 become necessary.

We approach such a case under the constraints when \(V = \text{const.} \) and \(N = \text{const.} \). Then for the internal energy \(U\) we have:

\[
\text{d}U = T\text{d}S + V \mathbf{H} \cdot \text{d}\mathbf{B}
\]

For the moments of 2, 3 and 4 order of \(B\) we obtain:

\[
\langle (\delta B)^2 \rangle = \frac{kT}{V} \left( \frac{\partial B}{\partial H} \right)_{T,V,N} \tag{31}
\]

\[
\langle (\delta B)^3 \rangle = \left( \frac{kT}{V} \right)^2 \left( \frac{\partial^2 B}{\partial H^2} \right)_{T,V,N} \tag{32}
\]
\[
(\delta B)^4 = \left(\frac{kT}{V}\right)^3 \left(\frac{\partial^3 B}{\partial H^3}\right)_{T,V,N} + 3 \left[\frac{kT}{V} \left(\frac{\partial B}{\partial H}\right)_{T,V,N}\right]^2 = \\
\left(\frac{kT}{V}\right)^3 \left(\frac{\partial^3 B}{\partial H^3}\right)_{T,V,N} + 3 \left[(\delta B)^2\right]^2
\]  

(33)

For finding the explicit expressions of \( (\delta B)^2 \), \((\delta B)^3\) and \((\delta B)^4\) it is necessary to know the expression of the function \( B = B(H) \). The most known such function is given by the Langevin equation:

\[
B = \mu_0 M_s \left( \coth \left( \frac{1}{a} \right) + \mu_0 H \right)
\]

(34)

where

\[
a = \frac{\mu_0 m H}{kT}
\]

(35)

\( M_s \) represents the saturation magnetization, \( \mu_0 \) is the vacuum permeability and \( m \) signify the magnetic moment of an individual molecule.

By means of some simple mathematical operations one finds:

\[
(\delta B)^2 = \frac{kT\mu_0}{V} \left[ \frac{\mu_0 m M_s}{kT} \left( \frac{1}{a^2} - \frac{1}{\sinh^2 a} \right) + 1 \right]
\]

(36)

\[
(\delta B)^3 = \frac{2\mu_0^3 m^2 M_s}{V^2} \left( \frac{\cosh a}{\sinh^3 a} - \frac{1}{a^3} \right)
\]

(37)

\[
(\delta B)^4 = \frac{2\mu_0^4 m^3 M_s}{V^3} \left( \frac{3}{a^4} + \frac{\sinh^2 a - 3\cosh^2 a}{\sinh^4 a} \right) + \\
+3 \left\{ \frac{kT\mu_0}{V} \left[ \frac{\mu_0 m M_s}{kT} \left( \frac{1}{a^2} - \frac{1}{\sinh^2 a} \right) + 1 \right]\right\}^2
\]

(38)

One of the most used dependence of \( B \) from \( H \) is given [12] by the function

\[
B = \mu_0 M_s \left[ 1 - \exp \left( -\frac{H^2}{2\sigma^2} \right) \right] + \mu_0 H
\]

(39)

where \( \sigma \) is a scaling factor.

The mentioned function imply a maximum for the differential permeability \( \mu_d = dB/dH \) at the value \( H = \sigma \).

By using the general formulas for the 2nd, 3rd and 4th order moments of the random variable \( B \) one obtains:

\[
(\delta B)^2 = \frac{\mu_0 kT}{V} \left\{ \frac{M_s H}{\sigma^2} \exp \left( -\frac{H^2}{2\sigma^2} \right) + 1 \right\}
\]

(40)
(\delta B)^3 = \left(\frac{kT}{V}\right)^2 \frac{\mu_0 M_s}{\sigma^2} \left(1 - \frac{H^2}{\sigma^2}\right) \exp\left(-\frac{H^2}{2\sigma^2}\right) \tag{41}

(\delta B)^4 = \left(\frac{kT}{V}\right)^3 \frac{\mu_0 M_s H}{\sigma^4} \left(\frac{H^2}{\sigma^2} - 3\right) \exp\left(-\frac{H^2}{2\sigma^2}\right) + 
+ 3 \left(\frac{\mu_0 kT}{V}\right)^2 \left\{\frac{M_s H}{\sigma^2} \exp\left(-\frac{H^2}{2\sigma^2}\right) + 1\right\}^2 \tag{42}

In the end we wish to note the following observations:

1. \((\delta B)^3\) change its sign in the point \(H = \sigma\), where the differential permeability \(\mu_d = \frac{dB}{dH}\) takes its maximal value [this means the inflection point of the function \(B = B(H)\)]. For \(H < \sigma\), the moment \((\delta B)^3\) is positive while for \(H > \sigma\) it is negative.

2. The fourth order moment \((\delta B)^4\) give informations about the deviation from the Gaussian approximation. This because it is implied in the so called excess coefficient \(C_E\).

In the here discussed case we have:

\[C_E = \frac{kT M_s H}{\mu_0 \sigma^1} \left(\frac{H^2}{\sigma^2} - 3\right) \exp\left(-\frac{H^2}{2\sigma^2}\right) \tag{43}\]

In probabilistic terminology [13] the distribution of a random variable is called leptokurtic, mesokurtic or platykurtic as the excess coefficient \(C_E\) satisfy the conditions \(C_E > 0, C_E = 0\) and \(C_E < 0\) respectively. Then by taking into account the expression (43) of \(C_E\) one can say that for the here studied situation the fluctuations of the magnetic induction \(B\) are leptokurtic, mesokurtic and platykurtic as the magnetic field strength \(H\) satisfy the conditions \(H > \sqrt{3}\sigma\), \(H = \sqrt{3}\sigma\) and \(H < \sqrt{3}\sigma\) respectively.

IV. SUMMARY AND CONCLUSIONS

1. We investigated the higher order moments of the fluctuations for complex thermodynamic systems (i.e. systems considered in the presence of fields). Our approach uses a generalized statistical ensemble. We considered the case when the energy \(E\), the usual extensive parameters \(\{X_i\}_{i=1}^n\) and the field parameters \(\{Y_j\}_{j=1}^m\) are fluctuating random variables. We find that the higher order moments of fluctuations depend on the field constraints.
2. The general results from the Sec.II were particularized for the case of a system situated in a magnetoquasistatic field. If the magnetic characteristics of such system are linear the third order moment of the magnetic induction and the excess coefficient are null. Therefore the description of the fluctuations for the magnetic induction can be done in the framework of Gaussian approximation.

3. For nonlinear magnetic media $\langle \delta B \rangle^3 \neq 0$ and consequently the fluctuations of $B$ deviate from the normal distribution. Additionally the respective deviation are characterized by the various value of the excess coefficient $C_E$ given by the formula (43). From the respective formula it results that the fluctuations of $B$ can be leptokurtic (for $H > \sqrt{3}\sigma$), mesokurtic (for $H = \sqrt{3}\sigma$) and respectively platykurtic (for $H < \sqrt{3}\sigma$) for the cases in which the function $B = B(H)$ is given by the relation (39).

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