Electronic transport properties of quasicrystals

S. Roche, G. Trambly de Laissardiere and D. Mayou

Laboratoire d’Etudes des Proprietes Electroniques des Solides-CNRS, Grenoble

We present a review of some results concerning electronic transport properties of quasicrystals. After a short introduction to the basic concepts of quasiperiodicity, we consider the experimental transport properties of electrical conductivity with particular focus on the effect of temperature, magnetic field and defects. Then, we present some heuristic approaches that tend to give a coherent view of different, and to some extent complementary, transport mechanisms in quasicrystals. Numerical results are also presented and in particular the evaluation of the linear response Kubo-Greenwood formula of conductivity in quasiperiodic systems in presence of disorder.

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I. INTRODUCTION

In 1984, D. Schechtman, I. Blech, D. Gratias and J.W. Cahn presented a new metastable phase of an AlMn binary alloy. The diffraction pattern was formed by intense Bragg peaks organized according to the icosahedral symmetry strictly forbidden from conventional crystallography. The underlying order was claimed to be described by the mathematical concept of quasiperiodicity.

The confirmation of a new state of matter has been an intense subject of controversy. In particular L. Pauling proposed alternative description of five-fold diffraction patterns based on icosahedral glasses formed by twins. However, the situation changed after the discovery of stable phases (icosahedral AlCuFe, AlPdMn, AlCuCo...) by Tsai et al. and the existence of quasiperiodic crystals (quasicrystals) is now well accepted. Furthermore, these materials have revealed a lot of unexpected physical properties.

The aim of this article is to review briefly the experimental results concerning electronic conductivity and then to present theoretical studies of electronic structure and electronic transport in these systems.

II. QUASIPERIODIC ORDER

A. Construction of the Fibonacci chain

To generate the Fibonacci chain, it is possible to use an inflation process starting from two incommensurate segments of respective lengths A and B. One adopts the rule $A \rightarrow AB$, and $B \rightarrow A$, which leads to the consecutive sequences $S_0 = A$, $S_1 = AB$, $S_2 = ABA$, $S_3 = ABAAB$, $S_4 = ABAABABA$, $S_5 = ABAABABAABAAB$ and so on. This inflation rule is related to the sequence of Fibonacci numbers given by

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1 \quad \text{with} \quad F_{n+1}/F_n \rightarrow \tau = (1 + \sqrt{5})/2 = 1.61803398...$$

In the infinite limit, the ratio of numbers of A and B of the semi-infinite chain $S_\infty = ABAABABAABAABAABAAB...$ is equal to the golden mean $\tau$.

The cut and project method is a geometrical algorithm to generate quasiperiodic structures in a D dimensional space starting from a periodic one in N dimensional space (with $N > D$). It can be related to the fact that every quasiperiodic function can be algebraically related to a periodic one in higher space. If one defines $f(x, y)$ a 2π-periodic function in $x$ and $y$ directions, Fourier decomposition of this function leads to $f(x, y) = \sum_{p,q} \alpha_{pq} \exp(2\pi p(x + qy))$. The quasiperiodic function $g(x) = f(x, y = x/\tau)$ in 1D space can be seen as a restriction to $y = x/\tau$ (line with irrational slope) of a periodic function in 2D space. We illustrate this cut and project method for the construction of the Fibonacci chain in figure figFCP where two main steps are distinguishable.

Firstly, one defines a window of width $B$ along a line $D$ of slope $p$. Next, the vertices of the square lattice in 2D that belong to the window, are projected perpendicularly on $D$. These projected points define a sequence of two distinct lengths A and B such that $A/B = p$. When $p$ is rational, the chain is periodic, whereas for irrational slope, it becomes quasiperiodic. The Fibonacci chain correspond to a slope $p = 1/\tau$ (where $\tau$ is the golden mean). The periodic approximant chains of Fibonacci are related to the sequence of rational numbers $p_n$, converging to the slope that determines the quasicrystal. For example, for the Fibonacci chain they are defined by $(1/1, 2/1, 3/2, 5/3, ..., p_n = F_{n-1}/F_n \rightarrow \tau)$. So, they do not share long range quasiperiodic order in a strict sense, but are close from the quasiperiodic chain as far as local order is concerned.
B. Quasiperiodic tilings

The possibility of quasiperiodic tilings of an euclidian space was known since the work of R. Penrose. On figure fig2 we represent a part of a Penrose tiling which is constructed from two rhombi of respective angles \( \left( \frac{2\pi}{5}, \frac{3\pi}{5} \right) \) and \( \left( \frac{\pi}{5}, \frac{4\pi}{5} \right) \) which are assembled according to local matching rules. The Penrose tiling is constrained to a long range five-fold orientational order, which corresponds to a diffraction pattern shown in figure figcdiff. An algebraic description of Penrose tiling is due to De Bruijn.

If we consider a tiling of a N-dimensional space \( \mathcal{E} \) a quasiperiodic tiling of a D-dimensional space \( \mathcal{E}_\parallel \) is described in a general method developed by M. Duneau and A. Katz. This method generalizes the example of figure figFCP, where \( \mathcal{E}_\parallel \) will contain the Fibonacci chain. The complementary space is called \( \mathcal{E}_\perp \), of \( (N-D) \) dimensions, and such that \( \mathcal{E}_\parallel + \mathcal{E}_\perp = \mathcal{E} \). In real space \( \mathcal{E}_\parallel \) the geometrical properties of quasiperiodic tilings can be summarized by the followings points:

- **Aperiodicity**: No discrete translation lets the tiling invariant.
- **Local Isomorphism (Conway theorem)**: Every finite partition of tiles of characteristic length \( L \) admits at least one identical replica in a distance of order \( 2L \). This frequent and regular reappearance of identical local environments reveals a mesoscopic homogeneity which gives an intermediate situation between periodic translational order and disordered tilings.
- **Self-similarity**: A tiling is said self-similar if it exists another one composed by smaller tiles (with a ratio of length of \( (1 + \sqrt{5})/2 = \tau \) for a Penrose tiling) which preserves all the vertices of the primary tiling. This property allows to define scale transformations from a tiling to an inflated or deflated one.

Physical quantities (like for example the charge density \( \rho(\mathbf{r}) \)) admits Fourier components with vector \( \mathbf{K} \) that can be indexed with integers. For example, 3D-icosahedral quasicrystals requires the use of a basis of six integers such that every \( \mathbf{K} \) vector of the reciprocal space is defined by

\[
\mathbf{K} = n_1 \mathbf{a}_1^\ast + n_2 \mathbf{a}_2^\ast + n_3 \mathbf{a}_3^\ast + n_4 \mathbf{a}_4^\ast + n_5 \mathbf{a}_5^\ast + n_6 \mathbf{a}_6^\ast
\]

where the \( \mathbf{a}_i^\ast \) are chosen according to the five-fold axis of an icosahedra, and every \( \mathbf{K} \) are decomposed in an orthonormal basis according to \( (h + \tau h', k + \tau k', l + \tau l') \) with \( h, h', k, k', l, l' \) some integers and \( \tau \) the golden mean.

C. Atomic cluster models

Once the quasiperiodic order was proposed as a candidate to explain the long-range special symmetry inherent to quasicrystals, the major work for crystallographers has been to develop structural models of atomic distribution. For icosahedral phases, atomic models for alloys such as \( \text{AlCuFe} \) and \( \text{AlPdMn} \) have been proposed, with an estimated accuracy of about 80 – 90% of the real structure.

We give on figures Jan1 and Jan2, an example of a structural model proposed by C. Janot for the icosahedral phase of \( \text{AlPdMn} \). In figure Jan1 the “elementary brick” known as the pseudo-Mackay icosahedron (P.M.I.) is used to construct the quasiperiodic tiling according to inflation rules (some additional atoms have to be added to fill entirely the quasiperiodic lattice).
III. EXPERIMENTAL TRANSPORT PROPERTIES OF QUASICRYSTALS

Here, we will focus on electronic transport properties as a function of temperature, magnetic field and defects concentration, and present essentially dc conductivity results.

A. Different classes of quasicrystals

Quasicrystals have been divided into two main families which are the icosahedral phases whose diffraction pattern have five-fold symmetry and which are quasiperiodic in the 3 directions of space, and decagonal phases which confine quasiperiodicity in a 2D plane, leaving one direction periodic. The second class gives the possibility to compare properties between quasiperiodic and periodic directions. For example, a strong conductivity anisotropy between quasiperiodic planes and periodic direction has been measured \(^{18}\). Yet in the following, we focus essentially on icosahedral phases that have the most striking properties.

B. Electronic conductivity of quasicrystals

Conductivity in icosahedral phases of high structural quality such as \(\text{AlCuFe}\) or \(\text{AlPdMn}\) alloys is very low, of the order of \(100 - 300 \Omega^{-1} \text{cm}^{-1}\) at zero temperature. Moreover, we illustrate on figure figicopar, one of the most unexpected transport properties of quasicrystals. Indeed, after annealing samples, with a consequent improvement of structural quality, the conductivity decreases \(^{19},^{20}\). This tendency is surprising since these phases are generally composed of good metals for which an increase of conductivity is expected with improvement of order. Consequently, the correlation between transport properties and quasiperiodic order has been an intense subject of discussion and controversy.

We also note that the curves \(\sigma(T)\) are nearly parallel. This point suggests to write \(\sigma(T) = \sigma_{4K} + \delta\sigma(T)\) with \(\sigma_{4K}\) a measure of conductivity at 4 Kelvin and \(\delta\sigma(T)\) the variation as a function of temperature which is nearly independent of the alloy (fig. figicopar). This is a general behavior of all the icosahedral quasicrystals of high structural quality and corresponds to an “inverse Mathiessen rule” \(^{20}\). Indeed, the Mathiessen rule \(\rho(T) = \rho_0 + \delta\rho(T)\) is characteristic of metallic alloys where \(\rho_0\) and \(\delta\rho(T)\) are respectively the resistivity due to static defects and scattering by phonons.

Generally, the phase diagram of quasicrystals in composition and temperature is very complex and the existence zone of quasiperiodic order is reduced. Around these particular zones of the phase diagram, some other periodic phases can be grown with a similar local order. These so-called approximants phases possess a common local order with quasicrystals and seem to have the same kind of physical properties. This is the case for alloys such as \((\text{AlCuFe}, \text{AlMnSi}, \text{AlPdMn},...\) which are associated with families of crystalline approximants phases like \(\alpha - \text{AlMnSi}, \alpha - \text{AlCuFe}, R - \text{AlCuFe},...\) whose smallest unit cell can have only \(\sim 130\) atoms. However the alloy \(\text{AlPdRe}\) is an exception \(^{21}\), since no approximant phases are known.

We show, in figure fig2, recent results where the conductivity of a small cubic approximant \(\alpha - \text{AlSiCuFe}\) (with a unit cell parameter of \(a = 12.33\) A) is compared to that of an icosahedral phase of \(\text{AlCuFe}\). The difference of the absolute conductivity is very small and in addition, the behavior of \(\delta\sigma(T) > 0\) is the same. In contrast, a crystalline “non-approximant” phase of similar stoichiometry (tetragonal \(\omega - \text{Al}_{7}\text{Cu}_{2}\text{Fe}\) phase) with cell parameters \(a = 6.34\) A and \(c = 14.87\) A, has a conductivity of the order of \(10^4 (\Omega \text{cm})^{-1}\) between 0 and 300K (with a metallic temperature dependence \(\delta\sigma(T) < 0\)).
Another interesting experimental result concerning approximants has been revealed by two successive approximants of the cubic phases of AlGaMgZr. A transition is observed from a metallic regime (for a cubic approximant 1/1 \((a = 14.2\text{\AA})\) \(\rho_{4K} = 58\mu\Omega\text{cm}\) and \(\rho(T) > 0\)) to a “quasiperiodic regime” for the approximant 2/1 (with parameter \(a = 23\text{\AA}\)) with a resistivity close to that of the icosahedral phase \(\rho_{4K} = 120\mu\Omega\text{cm}\) and \(\delta\rho(T) < 0\). This suggests that a minimal size of the unit cell is necessary to observe characteristic effects of quasiperiodic order.

The \(AlPdRe\) alloy is even more resistive with a behavior as a function of temperature different from the other quasicrystals. Typically at 4K, the resistivity of the alloy \(\rho(i - Al_{62.5}Pd_{22}Mn_{7.5}) \sim 10.000\ \mu\Omega\text{cm}\) whereas \(\rho(i - Al_{70.5}Pd_{21}Re_{8.5}) \sim 1.500.000\ \mu\Omega\text{cm}\) which is of the same order of that of doped semiconductors (note that Mn an Re belong to the same column of the Mendeelev table and Mn is in the 3d row whereas Re is in the 5d row).

So, the two icosahedral phases, defined by the same long-range order, with close atomic composition, have nevertheless a conductivity variation of several order of magnitude. The densities of states at the Fermi level, for alloys like \(AlCuFe, AlPdMn, ...\) are \(\sim 1/3\) of that of free electrons, whereas for \(AlPdRe\) it is \(\sim 1/10\) of that of free electrons. Consequently, the reduction of conductivity between \(i - AlPdMn\) and \(i - AlPdRe\) is not only due to a reduction of density of states, but depends also strongly on localization issues. In figure figRe1, we present the resistivity variation of \(AlPdRe\) as a function of temperature (from 24 to 25).

The resistivity at 4 Kelvin of the \(AlPdRe\) alloy is similar to that of resistive materials like doped semiconductors. So, it is legitimate to compare their respective behaviors as a function of temperature in order to characterize the possible underlying conduction mechanisms implied in \(AlPdRe\) at low temperature. The temperature dependence of \(\sigma(T) = 1/\rho(T)\) follows neither an exponential law of \(\exp(-E/k_BT)\) type which is characteristic of thermally activated processes, nor a \(\exp(-AT^{-\frac{1}{4}})\) law characteristic of variable range hopping mechanisms at low temperature. On the contrary, the conductivity of \(AlPdRe\) follows on a large range of temperature \(4 - 800K\), a power law \(\sigma(T) \sim T^\beta\) with \(1 < \beta < 1.2\) which remains unexplained.

As a comparison, we show the conductivity of the \(Al_{72}Ru\) alloy whose behavior is typical of thermally activated process of semiconductors (represented in fig. figPoon with \(\bullet\)). The behavior of \(\sigma(T)\) is described by \(\exp(-\Delta/k_BT)\) (with \(\Delta = 0.17eV\) is the gap width). The \(i - AlPdRe\) alloy is shown to follow \(\sigma(T) \sim T^\alpha\).

The figure figMIT shows the correlation between \(\sigma_{4K}\) and \(\sigma_{300K}\), which is an element to identify a metal-insulator transition. The position of \(AlPdRe\) on that plot is coherent with the special properties of this alloy at low temperature.

C. Quantum interferences in quasicrystals

At low temperature, quantum interference effects (QIE) have been clearly identified for the class of quasicrystals located on the metallic side, and over a large range of temperature \([0.3K - 100K]\) and magnetic field \([0 – 20]\ Tesla\). These phenomena of localization were expected in disordered systems of smaller resistivity. However, the dependence \(\delta\sigma(T)\) at low temperature and \(\delta\sigma(H)\) in magnetic field were analyzed convincingly by means of the theories of quantum interference effects (weak localization, electron-electron interaction...), even for the approximant cubic phase with small parameter \(a = 12.33\text{\AA}\) de AlCuFeS.

Concerning the \(i - AlPdRe\) phase, the description of \(\sigma(T, H)\) by means of QIE is however not possible. For these alloys, that seem to be on the insulating side of a metal-insulator transition, it would be in fact surprising to find for example weak localization effects.
D. Conclusion on experimental results

To conclude this experimental part, let us summarize the main points concerning electronic transport properties. Many quasicrystals and their approximant phases are typically composed by 60% to 70% of aluminum which is a very good metal, but they appear to be very bad conductors. In addition, they follow an “Inverse Mathiessen rule” as a function of temperature. We present in the figure figFCP, the resistivity as a function of temperature for typical metallic alloys and quasicrystals (from [33]).

These systems are close to a metal-insulator transition. On the metallic side, experiments reveal the importance of quantum interference effects and suggest that the local order on $15-30\AA$ determine the observed properties. On the insulating side, the power law $\sigma(T) \sim T^\alpha$ for the conductivity of $AlPdRe$ on a wide range of temperature is still unexplained.

To conclude, let us mention that other properties like diamagnetism and small number of carriers (deduced from Hall effect) are consistent with the tendency to localization observed in these systems. Also, optical conductivity measurements show that there is no Drude peak for icosahedral quasicrystals, which is again in contradiction with usual characteristic of metals [35].

IV. THEORY OF ELECTRONIC STRUCTURE OF QUASICRYSTALS

A. Localization and quasiperiodic order

1. Critical states and scaling of bands

In order to study the spectral properties of independent electrons in a quasiperiodic potential, one often uses a tight-binding model where the vertices $\mathcal{V}_T$ of a given tiling are chosen as atomic sites $|n\rangle$. The Hamiltonian then writes:

$$H = \sum_{<n,m> \in \mathcal{V}_T} |n\rangle \langle m| t_{nm} + \sum_n \varepsilon_n |n\rangle \langle n|$$

where quasiperiodicity is introduced either geometrically (respective positions of atomic sites), or through a modulation of energy sites or hopping term on a periodic lattice. One of the main results concerning electronic localization in Fibonacci chains is the power law behavior of the envelope of the wave function $|\psi_N| \sim N^{-\alpha}$, first studied by M. Kohmoto and B. Sutherland, and Ostlund et al [36], and referred to as critical states. Such a peculiar localization is shown in figure fig3 where an off-diagonal model is used ($\varepsilon_n = 0 \ \forall n$) with $t_B/t_A = \gamma$ the measure of quasiperiodicity, and $\alpha = \ln \gamma/\ln \tau$ exactly for $E = 0$ [37,38]. For 2D-Penrose lattices, the exponents are given typically by $3/8 < \alpha < 5/8$, depending on the eigenstates and the physical parameters [39].

The properties of the eigenstate $\psi_n$ have to be investigated through multifractal analysis [40]. This eigenstate is associated to a so-called 6-cycle, in relation to the transfer matrix properties. Indeed, for this particular energy, starting from the first 6 elementary matrix $P_{n=1,6}$ (with $n = F_n$ a Fibonacci number), the algebraic properties of Fibonacci numbers allow to define a renormalization scheme given by $P_{n+6} = P_n$ which constitute the 6-cycle. If $\psi_0$ and $\psi_1$ are taken as initial conditions, it can be shown that the fractal structure of $\psi_n$, shown in figure fig3, is determined by the renormalization:

$$P_n$$
\[ \psi_n = \gamma^{2p} \psi_0 \quad \text{when} \quad n = F_3 + F_6 + F_9 + \ldots + F_{3 \times (2p-1)} + F_{3 \times 2p} \]
\[ \psi_n = -\gamma^{2p+1} \psi_1 \quad \text{when} \quad n = 1 + F_2 + F_6 + \ldots + F_{3 \times 2p} + F_{3 \times (2p+1)} \]

with \( p \) a integer.

Concerning the spectral properties of quasiperiodic hamiltonians in 1 dimension, J. Bellissard et al. have proven rigorously the gap labelling theorem, which gives the value of integrated density of states IDoS with each gap of the electronic structure. For a Fibonacci chain, the heights of the plateaux of IDoS are given by \( j \tau/\tau + 1 \) \( \mod 1 \) where \( j \) is an integer (see figure figIDOS). C. Sire has determined the width and the position of gaps of periodic approximants of the Fibonacci chain in a perturbative regime.

In higher dimension, several numerical studies on discrete models (see for example studies on octagonal tiling or Penrose tiling) have complemented the description of critical states. It is important to note that localization aspects can be studied either directly in a pure quasiperiodic system, which becomes rapidly difficult in higher dimensions, or by investigating the properties of big approximants and focusing on the renormalization of properties from one approximant to another.

In figure figECP3, the localization of eigenstates is represented within a localized basis. The projection of the given state in the unit cell of a Penrose approximant allows to see three hierarchies of pentagona shift by an angle of \( \pi/5 \). These states can be investigated through multifractal analysis which lies in the evaluation of the moments of the probability distribution, associated to spectral measure. If \( \{|n\} \) is an orthonormal basis, one can write:

\[ \mu_q(\varepsilon_k) = \frac{\sum_{n=1}^{N} |\langle n|\Psi_k \rangle|^2 q}{(\sum_{n=1}^{N} |\langle n|\Psi_k \rangle|^2)^q} \sim N^{-(q-1)D_q} \]

where the exponents \( D_q \) are known as the multifractal dimensions of the spectrum.

Qualitatively, a critical state can be described as follows: Suppose that a given state \( \psi_L \) is mainly localized in a region of characteristic length \( L \). Then the Conway Theorem implies that a similar region must exists at a distance \( \leq 2L \). If \( L \) is sufficiently large, then both regions will be good candidate for a tunneling effect from let’s say the envelope of a given state is given by \( \psi_L \) to \( \psi_{2L} = z \psi_L \), where a damping factor \( z \) associated to the probability amplitude of the event is introduced. Then the case \( z = 0 \) corresponds to strictly localized states (Anderson localization), whereas \( |z| = 1 \) is the signature of extended states. For intermediate localization cases, like those related to fractal eigenstates, one writes:

\[ \psi_L \sim L^{-\ln|z|/\ln 2} \sim L^{-\alpha} \]

in a perfect quasicrystal \( |z| \neq 1 \) and will generally depend on the parameters of the considered model.

The nature of critical states can be also related to the scaling properties of electronic dispersion relation of approximants. Indeed, if we recall the argument of C. Sire, one can focus on the scaling properties of bandwidth of a series of approximants of a quasicrystal. If we consider an initial cube of length \( L \) in D dimensions with no restrictive condition on the atomic order (periodic, disordered, quasiperiodic...), then the spectrum of the infinite periodic system of unit cell \( L \) with \( L^D \) atoms will be composed by \( L^D \) bands. The typical bandwidth is then related to the overlap between two states \( \phi_1 \) and \( \phi_2 \) localized in adjacent blocks of length \( L \), and can be qualitatively linked with \( \Delta \varepsilon \sim |\langle \phi_1|H|\phi_2 \rangle| \).

For Bloch states with modulus \( |\phi(x)\rangle \sim 1/L^{D/2} \) and with \( t \) an average hopping amplitude from one site to another, then \( \Delta \varepsilon \sim t/L \) and the mean velocity writes \( v \sim \Delta \varepsilon/L^{-1} \sim t \) which is, as expected, independent of the arbitrary length \( L \) chosen. The same argument for a disordered cube involves the localization length \( \xi \) and the bandwidth is then \( \Delta \varepsilon \sim tL^{D-1} \exp(-L/\xi) \), which correspond when \( L \to \infty \) to a purely discrete spectrum. For a system that will be dominated by algebraic localization, i.e \( |\phi(x)\rangle \sim 1/L^{\alpha(x)} \) one finds a scaling behavior of bandwidths.
defined by an exponent $\beta > 1$ related to the distribution of $\alpha$. The mean group velocity as a function of the size $L$ is then

$$v \sim \frac{\Delta \varepsilon}{L^{\beta - 1}} \sim \frac{t}{L^{\beta - 1}}$$

which goes to zero when $L \to \infty$. The $\beta$ exponents have been studied for 1D and 2D systems and confirm this general argument.

2. Quantum diffusion in perfect quasiperiodic structures

Since the band velocity depends on the length scale, one expects non-ballistic propagation in a perfect quasiperiodic structure. The dynamic of an initially localized wave packet (such that $\psi_n(t = 0) = \delta_{n,n_0}$) is governed by the time dependent Schrödinger equation:

$$i\hbar \frac{d\psi_n}{dt} = -t(\psi_{n+1} - \psi_{n-1}) + \varepsilon_n \psi_n$$

H. Hiramoto and S. Abe have studied, in the diagonal model ($t_n = t \forall n$), the diffusion properties of such wave packets in Fibonacci chains and have shown that the mean square displacement was defined by an abnormal regime of propagation:

$$\sqrt{\langle (\Delta x)^2 \rangle} = \sqrt{\sum_n (n - n_0)^2 |\psi_n(t)|^2} \sim t^\gamma$$

where the $\gamma$ coefficients vary between the typical values of localized states ($\gamma = 0$ with $\sqrt{\langle (\Delta x)^2 \rangle}$ which remains finite when $t \to \infty$), and ballistic regimes $\gamma = 1$ for extended states. This abnormal regime of propagation of wave packets is specific to the correlations of a quasiperiodic potential.

The quantum diffusion properties are characterized by the whole distribution of moments associated to the spread of a given wave packet $\langle x^n \rangle(t) = t^{\sigma_q}$. Recently, F. Piéchon, using a perturbative renormalization group treatment, has investigated analytically the diffusion exponent of the wave packets in Fibonacci chains. In particular, he has shown that there exists a fraction of sites from which a quantum state will spread with exponents $\sigma_q = \ln \tilde{z} \ln \tau^{-3}$ whereas the complementary remaining sites are associated with $\sigma_q = \ln z \ln \tau^{-2}$ (where $\tilde{z}, z$ are the perturbative parameters related to the quasiperiodic potential). The main result of his study is the fact that diffusion exponents of wave packets are directly related to exponents of the spectral measure. We also note that for systems in higher dimension, G. Jona-Lasinio et al. have rigorously shown that a propagation of wave packets in hierarchical potential, through tunneling over arbitrary large scales is related to anomalous diffusion.
3. Landauer Resistance in quasiperiodic systems

By means of the formalism of transfer matrix, M. Kohmoto et al. have conjectured an algebraic increase of Landauer resistance. They have shown analytically, for energies corresponding to Q-cycles, that \( \rho_N \leq \rho_0 N^\alpha \). The authors extend heuristically their results to chaotic but bounded orbits, and rigorous mathematical approaches have demonstrated that for all \( E \) belonging to the spectrum \( \sigma(H) \) (singular continuous) but constituting a zero measure ensemble, the exact solutions of the Schrödinger equation \( H\psi = E\psi \), as well as the Landauer resistance were bounded by polynomials.

The works of Tsunetsumi have generalized these power law dependences of landauer resistance for two dimensional Penrose lattices. The Landauer conductance follows a general power law \( g(L) \sim L^\alpha \) whose typical exponents are given in figure Tsufig. The system studied consists of a finite part of a Penrose lattice which is connected to conducting leads of finite width M.

B. Electronic structure of quasicrystalline materials

To investigate electronic properties of more realistic materials, one can use ab-initio methods and study the general characteristics of electronic structure of approximant phases of quasicrystals. From ab-initio calculations, the main features can be summarized by the existence of a deep pseudo-gap at the Fermi level and by an overall structure composed of a high concentration of bands with small dispersion (see figure figSTB2) from ref. 57, 58.

The presence of a deep pseudo-gap at \( E_F \) is familiar to the alloys of Hume-Rothery type. This point has been experimentally confirmed by X-ray emission or absorption spectroscopy for icosahedral phases. The specific heat measures give also a weak TDoS at the Fermi level, of order \( \sim 1/3 \) of free electrons for icosahedral phases of AlCuFe, AlLiCu (see fig. figPseudga), and of \( \sim 1/10 \) for \( i-AlCuRu \) and \( i-AlPdRe \). The pseudo-gap is also observed experimentally for approximant phases.

1. Hume-Rothery mechanism and pseudo-gap at the Fermi level

Let us recall briefly the physical mechanism at the origin of pseudo-gap in metallic alloys. If we consider nearly free electrons, that is free electrons diffused by a weak pseudopotential \( V \), the hamiltonian \( \mathcal{H} \) of the system reads : \( \mathcal{H} = \mathcal{H}_0 + V \) with \( \mathcal{H}_0 = h^2 k^2 / 2m \) the kinetic energy of free electrons. Within the local approximation of pseudopotentials, \( V \) is given by :

\[
V(\mathbf{r}) = \sum_{\mathbf{K}} V_{\mathbf{K}} \exp(i\mathbf{K}.\mathbf{r})
\]

where \( \mathbf{K} \) is a vector of reciprocal space. The Fourier coefficient \( V_{\mathbf{K}} \) will couple the plane waves \( |\mathbf{k}\rangle \) and \( |\mathbf{k} - \mathbf{K}\rangle \). The kinetic energies of plane waves \( |\mathbf{k}\rangle \) and \( |\mathbf{k} - \mathbf{K}\rangle \) are close as soon as \( \mathbf{k} \) is close to the Bragg plane associated to \( \mathbf{K} \). The mixing between \( |\mathbf{k}\rangle \) and \( |\mathbf{k} - \mathbf{K}\rangle \) is strong near this Bragg plane. Furthermore, the stronger \( |V_{\mathbf{K}}|\), the stronger is this mixing.

Let us now consider the effect of one Bragg plane, associated to \( \mathbf{K} \), on a plane wave \( |\mathbf{k}\rangle \). As soon as \( \mathbf{k} \) is sufficiently close to the Bragg plane, we neglect the contribution of the other Bragg planes by considering only \( V_{\mathbf{K}} \) which couple states \( |\mathbf{k}\rangle \) and \( |\mathbf{k} - \mathbf{K}\rangle \) (two bands model), then the hamiltonian matrix is :
approximant suggests that localization properties of approximant s are similar to those of quasicrystals. This power law on a realistic model of state $|\Psi\rangle$ involves the pseudo-zone $A$, whereas for $i$-AlLiCu, the pseudo-zone $B$ is involved.

Soon after their discovery, quasicrystals and their approximant phases have been rapidly classified as Hume-Rothery alloys. In particular, J. Friedel and F. D'noyer have shown that the icosahedral phase of AlLiCu was stabilized by a strong interaction between a Fermi surface of electrons and a “pseudo Brillouin zone” (2). The latter was constructed from the Bragg planes corresponding to the brightest peaks of the diffraction pattern. In figure figZ, we present two examples of pseudo-Brillouin-zone of the icosahedral. The stabilization of $i$-AlCuFe involves the pseudo-zone $A$, whereas for $i$-AlLiCu, the pseudo-zone $B$ is involved.

2. Spiky structure of the density of states and localization

The spiky structure of the TDoS observed in figure figSTB2 corresponds to a high and inhomogeneous concentration of bands with small dispersion, whose first consequence as far as transport is concerned is a small velocity at Fermi energy $v(E_F) = (\partial E/\partial k)|_{E=E_F}$. It is interesting to note that even for simple models, where quasiperiodic order is introduced, for example through modulation of energy sites, the TDoS (see figPdg) also reveal a spiky structure (from[3]).

Recently, T. Fujiwara et al. have investigated the electronic properties of localization for 3D structural models of decagonal approximant phases $Al_{60}Cu_{30}Co_{14}$. Developing the eigenstates within a basis of spd orbitals, $\chi_{RL,l}(r) = (r|R,l | 0, 1, 2), \Phi_l(r) = \sum_{R,l=0,1,2} \alpha_{RL,l} \chi_{RL,l}(r)$, they have shown that the inverse participation ratio was described by a power law as a function of the approximant size. For a given Fermi level, they have evaluated:

$$P_\Psi = 1/\mu_2(\varepsilon(k)) = (\sum_{R,l} |\alpha_{RL,l}|^2)^2/\sum_{R,l} |\alpha_{RL,l}|^4$$

and shown that $P_\Psi \sim N^\nu$, for an average over states with energy close to $E_F$, and as a function of the number $N$ of atoms per unit cell. This quantity, which gives the number of sites which contribute significantly to the state $|\Psi\rangle$ for a given energy (see[2]), is hence defined by a scaling behavior of a state whose distribution of weight in a large unit cell is inhomogeneous (not uniformly delocalized). This power law on a realistic model of approximant suggests that localization properties of approximants are similar to those of quasicrystals.

These aspects of band structure allow to evaluate the behavior of Boltzmann conductivity in the relaxation time approximation RTA. The L.M.T.O. results obtained by T. Fujiwara et al. have confirmed that conductivity is very weak, and more concretely they get $\sigma_{DC} \sim 10 - 150(\Omega cm)^{-1}$ for the $\alpha$-AlMn(Si) phase (cubic lattice of Mackay icosahedra), which are anomalously low for metallic alloys. Their adjustable parameter is the scattering time $\tau$ for which they take standard values $\sim 10^{-14} - 10^{-15}$s. In addition, their study of the decagonal phase $d$ – AlCuCo agrees with the observed experimental anisotropy between quasiperiodic and periodic directions ($\sigma_{DC} \sim 3000(\Omega cm)^{-1}$) for the quasiperiodic one, whereas for periodic ($\sigma_{DC} \sim 15000(\Omega cm)^{-1}$) another analytical approach of a Bloch-Boltzmann type by S.E. Burkov et al. also confirms the general tendency of weak conductivity in quasicrystals.
Recently, we have proposed a model to analyze the role of atomic clusters (see section 2.3) in electronic structure of quasicrystalline materials\cite{68,69}. The model is a cluster of transition metal atoms in a metallic matrix. We analyzed the modifications of density of states due to multiple scattering effects. The essential result is that the spiky peaks of the TDoS observed from ab-initio calculations, and that seem to have been observed experimentally\cite{70}, could be the signature of states preferentially localized around structural cluster. We propose a generalization of the concept of virtual bound state to cluster virtual bound states.

We show in the figures figag1, different cluster virtual bound states according to the geometry of the structural entity (icosahedra, dodecahedra...\cite{68}). We note that, according to our results, the existence of spiky structure is favoured for icosahedral cluster for which lifetime of an electron will be longest. Also, figure figag2 shows how a cluster of clusters induce again more spiky structure in the density of states at the scale of a more complex cluster.

V. THEORETICAL APPROACHES TO ELECTRONIC TRANSPORT IN QUASICRYSTALS

A semi-classical Bloch-Boltzmann description of transport in quasicrystals seems unsufficient to take into account most of the aspects due to the special algebraic localization of these materials. Some specific transport mechanisms have been proposed to explain results like the inverse Matthiessen rule, or the temperature and defects influence on the conductivity, that need to go beyond a Bloch-Boltzmann analysis.

In particular, the possibility of two different unconventional transport mechanisms specific of these materials has been proposed\cite{20,57,71}. Transport could be dominated, for short relaxation times $\tau$ (defined by disorder) by hopping between "critical localized states", whereas for long time $\tau$ the regime could be dominated by non-ballistic propagation of wave packets between two scattering events. We present briefly these two mechanisms in the following.

A. Non-ballistic propagation between two collisions events

A possible interpretation of conductivity in quasicrystals\cite{20,71}, takes as a starting point the numerical results on anomalous diffusion of wave packets in perfect quasiperiodic structures\cite{72} (see section 4.1.1). Indeed for sufficiently long collision time $\tau$ (due to disorder), the propagation of wave packet follows typically a law such that $L(t) = At^\beta$ with $\beta < 1$ which defines a sub-ballistic regime with $A$, a constant which does not depend explicitly on time. It is then possible to estimate the conductivity by means of Einstein formula $\sigma = e^2 N(E_F) D(\tau)$. The diffusivity is given as a function of $\tau$ by $D(\tau) = L^2(\tau)/3\tau$ and $N(E_F)$ is the density of states at Fermi energy. The explicit dependence of conductivity with $\tau$ thus reads:

$$\sigma(\tau) \sim A\tau^{2\beta-1}$$

In consequence, provided that $\beta \leq 0.5$, conductivity will decrease with $\tau$, in other words with the diminution of defects. In the limit where $\beta \sim 0$ we will get $\sigma \sim 1/\tau$, which will be in agreement with the inverse Matthiessen rule that characterize quasicrystals\cite{72} (i.e a general form of the conductivity $\sigma(T) = \sigma_{4K} + \delta\sigma(T)$). Such values of exponents have been obtained in some regimes on octagonal tilings\cite{72}, but they correspond to strong quasiperiodic potential.

A similar argument is due to C. Sire\cite{72}, which states that for a given mean free path $l_{pm} = L$, the qualitative behavior of a quasicrystal and an approximant of unit cell $L$ will be the same. So, as for an approximant, we can write the mean velocity $v_F = v_F(L) \sim L^{-\alpha}$ with $L = v_F\tau$, the diffusivity then reads:

$$D(\tau) = L^2(\tau)/3\tau$$

$$\sigma(\tau) \sim A\tau^{2\beta-1}$$
\[ D(L) \sim v_F^2(L) \tau \sim L^{1-\alpha} \] which is equivalent to the first argument

J. Bellissard and H. Schulz-Baldes have shown recently, by a mathematical study of Kubo-Greenwood formula in a model of quasicrystals, that for long relaxation time, \( \tau \to \infty \), the transport law gives \( \sigma \sim \tau^{2\beta-1} \) where the \( \beta \) exponent is linked to anomalous quantum diffusion regime.

### B. Interband transition mechanisms and hopping transport

If one considers periodic hamiltonians whose eigenstates are inhomogeneously distributed in a large unit cell, an interband transition mechanism could allow a new kind of propagation modes. Indeed, the eigenstates of the perfect structure must have the generic Bloch form \( \Psi_{nk}(r) = u_{nk}(r)e^{ikr} \) with \( n \) a band index, \( k \) a wave vector and \( u_{nk} \) a periodic function defined in the unit cell (we suppose that the coefficients \( |u_{nk}(r)| \) have an unequal distribution of weight through the large unit cell, as we schematize in fig. figEC1. Then collision events can induce transitions from \( \Psi_{nk}(r) \to \Psi_{n'q}(r) \), which will correspond to a charge displacement in real space. This phenomenon is not taken into account in a Bloch-Boltzmann description.

The comparison between the contributions of this hopping mechanism and of the classical metallic conductivity (i.e. propagation of charge between two scattering events) could be made as follows. Suppose that \( \Delta \varepsilon \) is a typical bandwidth at a given scale \( L \) and \( \tau \) a finite lifetime induced by disorder. Then, within this framework of hopping mechanism, the dominant modes of transport will be given by the comparison between the hopping frequency \( \Delta \varepsilon/\hbar \) due to the quasiperiodic potential and the frequency \( 1/\tau \) induced by disorder. Consequently, the latter mechanism will dominate provided that \( 1/\tau > \Delta \varepsilon/\hbar \). This could be realised even for small disorder in quasicrystalline materials, given that band dispersion is very weak.

In this regime of interband transition, the hopping length will not depend anymore on collision time so that diffusivity will write \( D = L^2(\tau)/3\tau \sim 1/\tau \). This explains qualitatively the inverse Mathiessen rule, as well as the variation of conductivity with improvement of structural order \( \sigma \sim 1/\tau \). This interband mechanism has been also proposed by T. Fujiwara as a natural consequence of band structure and from numerical results on 2D Penrose lattices where random phasons defects are introduced in the lattices. We refer also to our studies of Landauer resistance of quasiperiodic chains and on quantum networks, and on some exact results on the role of phasons. C. Janot has also proposed a multiscale hopping mechanism in relation to hierarchical aspects of structural models.

### C. Quantum interference and correlated disorder

A first argument to justify the observation of quantum interference effects (weak localization, electron-electron interaction...) in quasicrystals is to assume that the general scaling theory of localization is applicable as soon as electrons are in a diffusive regime. The phase coherence is preserved only up to an inelastic length imposed by finite temperature or magnetic field.

From several experiments, an estimation of the elastic mean free path at zero temperature \( l_{pm} \sim 20\sim 30\text{Å} \) for \( AlCuFe, AlPdMn.. \) alloys has been proposed. Then the minimal Mott conductivity is \( \sigma_{\text{min}} \sim 200 \times (3/l_{pm}) \sim 20\sim 30\text{Ωcm}^{-1} \), with a ratio of \( \sigma_{\text{exp}}/\sigma_{\text{min}} \sim (100\sim 150)/(20\sim 30) \gg 1 \). This suggests that quantum interference do not dominate the conduction regime for these systems. But, if we now apply the same procedure...
for AlPdRe, assuming the same mean free path, \( l_{pm} \sim 20 - 30 \text{Å} \), we get \( \sigma_{exp}/\sigma_{min} \sim 1/(20 - 30) \ll 1 \). In conclusion the regime is now dominated by quantum interference at the origin of a metal-insulator transition. It could be that the quasiperiodic potential is stronger in AlPdRe, leading to a smaller conductance \( g < g_c \) for a cube with size equal to the mean free path (\( g_c \) is the universal critical conductance).

There is another way to conceive localization in quasicrystals, taking as a starting point the notion of correlated disorder. Indeed, one can expect that disorder in a quasicrystal preserves strong correlations between diffusive centers whereas classical theories of localization assume a completely random distribution of scatterers.

A first example in this context, is a calculation of quantum corrections, on the metallic side of an Anderson metal-insulator transition, for a disordered binary alloy with local and short range interaction between impurities. M.T. Béal-Monod and G. Forgac have evaluated the quantum corrections of the electronic conductivity, introducing conditional probability in the spatial distribution of diffusion centers, and have shown, for example, that a repulsive first neighbours interaction will decrease conductivity with regards to the uncorrelated case.

In the context of mesoscopic physics and phase coherent effects, M. B. Hastings et al. have analyzed the effects of real space symmetries of the diffusion potential. They have shown explicitly how symmetry constraints imposed on disorder, influence diagrammatic treatments.

One of us, has investigated the effect of some particular phason defects in Fibonacci chains and quantum networks Landauer resistance. These studies show how correlations of disorder induced by this kind of defects can lead to specific behavior of conductivity.

D. Kubo-Greenwood conductivity in quasiperiodic systems with disorder

From ab-initio calculations of band structure, it has been shown that Bloch-Boltzmann conductivity in pure approximants was weak, in good agreement with experiments. However, in order to study carefully the transport phenomena in quasicrystals, we need to go beyond the semi-classical approximations. In particular, the heuristic arguments given above suggest that the Bloch-Boltzmann picture of ballistic propagation between scattering events is not applicable.

Thus, a natural starting point for a study of transport is given by the Kubo-Greenwood linear response theory of transport coefficients, which makes no assumptions on the nature of states. At zero temperature and frequency one gets for a Fermi level \( E_F \):

\[
\sigma_{DC}(E_F) = \frac{2\pi e^2 \hbar}{\Omega} \text{Tr}\left[ \hat{V}_x \delta(E_F - \hat{H}) \hat{V}_x \delta(E_F - \hat{H}) \right]
\]

Recently, we have developed a numerical method based on the theory of orthogonal polynomials to evaluate directly this formula. We have investigated the relation between quantum diffusion of wave packets and conductivity at Fermi energy for a 3D quasiperiodic system. We used a diagonal tight-binding model where quasiperiodicity was introduced through a modulation of site energies. As propagation in pure quasiperiodic systems is not diffusive, we have also introduced a disordered potential which established a diffusive regime at long time of wave packets diffusion.

We clearly show that transport regime deviates from the prediction of a Bloch-Boltzmann approach which is applicable to metals in the limit of weak disorder. The Bloch-Boltzmann theory predicts that the conductivity
varies like $\sigma = \sigma_0 (V_0/V_{dis.})^2$. Thus, when the strength of the disorder potential is multiplied by $\sqrt{2}$, the conductivity must be divided by two. It is clear from figures figcond1 and figcond2 that this prediction is not verified. The conductivity is less sensitive to disorder than what Bloch-Boltzmann theory predicts. Another conclusion is also that for sufficiently strong quasiperiodic potential, the conductivity is nearly insensitive to disorder in the regions of pseudo-gaps. This result is quite interesting because the Fermi level of quasicrystalline materials has been shown to be located within a pseudo-gap region. It suggests that at these energies, the quasiperiodic potential acts like a strongly disordered potential and thus the conductivity is nearly independant to a small additional source of scattering.
VI. CONCLUSION

Experiments show that quasicrystals present remarkable electronic transport properties. In particular, they approach a metal-insulator transition when the quasiperiodic order is improved. Within the one electron scheme, the different theoretical approaches suggest an intermediate type of algebraic localization between purely extended states and exponentially localized states. Several schemes have been proposed to explain original transport properties, among which, non-ballistic quantum diffusion for long relaxation times $\tau$ or hopping mechanisms induced by interband transitions at shorter $\tau$. We note however that the role of electron-electron interactions has not been discussed up to now although they can play an important role close to a metal insulator transition. More precisely, the quasiperiodic potential tends to localize electrons and under these circumstances, the electron-electron interaction could thus lead to a Coulomb-gap (or pseudogap) even without disorder. Electron interactions could lead to changes in the band structure close to the Fermi level that cannot be described by the usual treatment of correlations used in ab-initio calculations (i.e the local density approximation (L.D.A.)). These changes can be sensitive to defects, scattering by phonons or variation of the Fermi Dirac distribution with temperature. This could deeply affect the transport properties of quasicrystals.

VII. ACKNOWLEDGMENTS

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Figure Captions:

Figure 1: Construction of the Fibonacci chain and an approximant one through cut and project algorithm.

Figure 2: Penrose tiling of a 2D space.

Figure 3: Schematic diffraction pattern of a Penrose tiling.

Figure 4: Successive atomic layers of a pseudo-Mackay icosahedron (PMI). 1: centered cube, 2: icosahedron, 3: icosidodecahedron, 4: PMI. The model of C. Janot distinguishes two PMI according to their atomic composition. A first family (PMI-A) is defined with 6 manganese atoms plus 6 paladium on the icosahedron, with the other sites occupied by aluminum, while a second family (PMI-T) has 20 paladium atoms among the 30 available sites on the icosidodecahedron, with aluminum atoms for the other sites.

Figure 5: Example of a plane section of the 3D structure, where the circles correspond to the equatorial sections of a PMI. A $\tau^3$ inflation is necessary to find a PMI of PMI.

Figure 6: Electronic conductivity for different quasicrystalline phases as a function of temperature and structural quality (courtesy of C. Berger).

Figure 7: Comparison of the $\sigma(T)$ between icosahedral phases and an approximant alloy of unit cell parameter $a = 12.33\,\text{Å}$.

Figure 8: Strong variation of the resistivity of AlPdRe as a function of temperature.

Figure 9: $\sigma(T)$ for $i-\text{Al}_{70}\text{Pd}_{20}\text{Re}_{10}$ and $\text{Al}_{2}\text{Ru}(\bullet)$.

Figure 10: $\sigma_{4K}$ as a function of $\sigma_{300K}$ for different icosahedral phases.

Figure 11: Magnetoconductivity $\Delta\sigma(H)/\sigma_0$ for $i-\text{AlCuFe}$ up to 20 Tesla.

Figure 12: General behaviors of conductivity for quasicrystals compared to metallic alloys (courtesy of C. Berger).

Figure 13: Fractal property of eigenstates localization (through $|\psi_n|$) in a pure Fibonacci chain.

Figure 14: Integrated density of states $IDoS(E)$ for the Fibonacci chain which appears to be a increasing continuous function of energy, not differentiable.

Figure 15: Projection of an eigenstate in the unit cell of a Penrose approximant (N=9349 sites). One can distinguish three hierarchical pentagonal entities formed by inhomogeneous distribution of amplitudes. The grey and black sites represent more than 90% of the total weight of the considered state.

Figure 16: Conductance $g(L)$ for $E = -3.4$ for different widths M in the semi-periodic direction.
Figure 17: Total densities of states (TDoS) of alloys with close composition: (A) approximant 1/1 $i-Al_{62.5}Cu_{25}Fe_{12.5}$ 128 atoms/unit cell, (B) non-approximant $\omega-Al_7Cu_2Fe$, 40 atoms/unit cell.

Figure 18: Dispersion curves $E(k)$ along the main directions of the reciprocal space.

Figure 19: Partial density of states measured by X-ray emission or absorption spectroscopy (from ref. 59). (a) pure Al, (b) $\omega-Al_7Cu_2Fe$, (c) rhombohedral approximant $Al_{62.5}Cu_{26.5}Fe_{11}$, (d) icosahedral phase $Al_{62}Cu_{25.5}Fe_{12.5}$.

Figure 20: Effect of Bragg diffraction on electronic bands. (a): $V_K = 0$, (b) $V_K \neq 0$.

Figure 21: Representation of a Pseudo-gap.
Figure 22: Examples of pseudo-Brillouin-zones of the icosahedral phase. A: 42 (30+12) facets (main pseudo-zone for \(AlCuFe, ALPdMn\); B: 60 facets (main pseudo-zone for \(AlCuLi\)). The arrows are issued from the peaks which together with all equivalent peaks (by the icosahedral symmetry) define the facets of the pseudo-zone.

Figure 23: TDoS of two cubic approximants with \(L=13\) et \(L=144\) sites per unit cell (courtesy of C. Sire).

Figure 24: Cluster virtual bound states. (b): Tetrahedron, (c): Cube, (d): Icosahedron, (e): Dodecahedron. (a): Friedel virtual bound state.

Figure 25: Cluster virtual bound states on an icosahedron (b) and on an icosahedron of icosahedron (c).

Figure 26: Schematic illustration of a critical state.

Figure 27: Schematic representation of an interband transition. \(D_g\) is the displacement of the gravity center of the wave packet.

Figure 28: Electronic conductivity as a function of Fermi energy for a quasiperiodic potential defined on a cubic lattice with nearest-neighbour hopping \(t = 1\). The potential is given by on-site energies \(\varepsilon_{ijk} = \varepsilon_i + \varepsilon_j + \varepsilon_k + V_{\text{dis.}}\), with \(\varepsilon_i = \pm V_{qp} = 0.7\) according to a Fibonacci sequence, and \(\varepsilon_{\text{dis.}}\) is randomly distributed between \([-V_{\text{dis.}}/2, V_{\text{dis.}}/2]\). The insert shows the average over sites \(j\) of \(D_j(t) = \frac{(r(t) - r_j)^2}{t}\) for a wave packet initially localized in \(j\). This shows the onset of a diffusiv regime for long time \((D_j(t) = \text{cste})\).

Figure 29: Electronic conductivity as a function of Fermi energy for the model defined on figure 29, but for a quasiperiodic potential, \(V_{qp} = 0.9\), and \(V_{\text{dis.}} = 2\).