CHARACTERISTIC FUNCTIONS OF $p$-ADIC INTEGRAL OPERATORS

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To Jasper Stokman on his 50th birthday with admiration

Abstract. Let $P \in \mathbb{Q}_p[x,y]$, $s \in \mathbb{C}$ with sufficiently large real part, and consider the integral operator

$$(A_{P,s}f)(y) := \frac{1}{1 - p^{-1}} \int_{\mathbb{Z}_p} |P(x,y)|^s f(x) dx$$

on $L^2(\mathbb{Z}_p)$. We show that if $P$ is homogeneous of degree $d$ then for each character $\chi$ of $\mathbb{Z}_p^\times$ the characteristic function $\det(1 - uA_{P,s,\chi})$ of the restriction $A_{P,s,\chi}$ of $A_{P,s}$ to the eigenspace $L^2(\mathbb{Z}_p)_{\chi}$ is the $q$-Wronskian of a set of solutions of a (possibly confluent) $q$-hypergeometric equation, where $q = p^{-1-ds}$. In particular, the nonzero eigenvalues of $A_{P,s,\chi}$ are the reciprocals of the zeros of such $q$-Wronskian.

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1. Introduction

Let $P \in \mathbb{Q}_p[x,y]$ be a $p$-adic polynomial and $s \in \mathbb{C}$. We consider the operator

\[(A_{P,s}f)(y) = \frac{1}{1 - p^{-1}} \int_{\mathbb{Z}_p} |P(x,y)|^s f(x) \, dx\]

on the Hilbert space $H := L^2(\mathbb{Z}_p)$ when $\text{Re}(s)$ is not too negative. In this case $A_{P,s}$ is a well-defined Hilbert-Schmidt (in particular, compact) operator. Therefore, its spectrum consists of eigenvalues $\lambda_i \neq 0$ (if present) and also 0, which may or may not be an eigenvalue. Moreover, if $|P(x,y)| = |P(y,x)|$ and $s \in \mathbb{R}$ then $A_{P,s}$ is self-adjoint, hence $\lambda_i \in \mathbb{R}$ and we have a spectral decomposition $H = \text{Ker}A_{P,s} \oplus \bigoplus_i H(\lambda_i)$, where $H(\lambda_i)$ is the $\lambda_i$-eigenspace of $A_{P,s}$ ([11]). The goal of this paper is to compute $\lambda_i$.

We mostly focus on the case when $P$ is homogeneous of degree $d$. Then $A_{P,s}$ commutes with the group of units $\mathbb{Z}_p^\times$, hence preserves each eigenspace $H_\chi \subset H$ of $\mathbb{Z}_p^\times$ corresponding to a character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$. Denote the restriction of $A_{P,s}$ to $H_\chi$ by $A_{P,s,\chi}$. If $\text{Re}(s)$ is sufficiently large then $A_{P,s,\chi}$ is trace class with exponential decay of eigenvalues, so we can define its characteristic function – the Fredholm determinant

\[h_{P,s,\chi}(u) := \det(1 - u A_{P,s,\chi}) = \prod_i (1 - u \lambda_i)^{\text{mult}(\lambda_i)},\]

where $\text{mult}(\lambda_i)$ is the algebraic multiplicity of $\lambda_i$. Then $h_{P,s,\chi}(u)$ is an entire function of Hadamard order 0 whose zeros are the reciprocals of $\lambda_i$.

Our main result is the following theorem. Let $P(x,y) := x^d - y^l Q(x,y)$, where $Q(0) = 1$. Let $k := k_0 + \delta_{1,\chi}$, where $k_0$ is the order of the pole at 0 of the 2-variable zeta-function $Z(Q, \chi, s, z)$ defined by (2.2). Define the $q$-Wronski matrix of a collection of functions $f_1, ..., f_k$ of a complex variable $u$ to be the $k$-by-$k$ matrix with entries

\[W(f_1, ..., f_k)(u)_{ji} := f_i(q^{j-i}u).\]

Finally, let $\Phi_i$ be the solutions of the $q$-hypergeometric equation defined by (4.6), and

\[f_i(u) := (u; q)_{\infty} \Phi_i(u),\]

where

\[(u; q)_{\infty} := \prod_{n=0}^{\infty} (1 - uq^n);\]

then $u^{-\nu_i} f_i(u)$ are entire functions taking value 1 at the origin.

**Theorem 1.1.** The characteristic function $h_{P,s,\chi}$ is a limit of functions of the form

\[h(u) = \frac{(\beta u)^{-\sum_{i=1}^k \nu_i}}{\prod_{1 \leq i < j \leq k} (q^{\nu_j} - q^{\nu_i})} \det W(f_1, ..., f_k)(\beta u),\]

as $\nu_1, ..., \nu_k \to +\infty$ and $\beta \to \infty$ if $k_0 > 0$, where $q := p^{-1-ds}$. 
We note that such a limit can itself be written as a Wronskian of a set \( f_1, \ldots, f_k \) of solutions of a confluent \( q \)-hypergeometric difference equation. Also note that the order of this difference equation is typically bigger than \( k \), so that \( f_1, \ldots, f_k \) don’t span the space of its meromorphic solutions over the field of \( q \)-elliptic functions, so that the \( q \)-Wronskian does not factor explicitly. As a result, the eigenvalues of \( A_{P,s} \) are typically given by transcendental functions of \( p^s \), in contrast with Igusa’s theorem \([I]\) that integrals 
\[
\int_{\mathbb{Z}_p} |P(x)|^s |dx|
\]
for a \( p \)-adic polynomial \( P \) are always rational functions of \( p^s \). We show, however, that for special values of parameters these eigenvalues may be algebraic.

Theorem 1.1 is proved in Section 4. Our proof of Theorem 1.1 provides a method of computing the function \( h_{P,s,\chi} \). This method is based on realizing the operator \( A_{P,s,\chi} \) as a first order \( q \)-difference operator on a space of analytic functions.

By the same method, analogous results can be obtained over a general non-archimedian local field \( F \), and for more general integral operators in which the character \( w \mapsto |w|^s \) is replaced by any multiplicative character of \( F \). Further, they can be extended to integral operators whose kernels are of the form 
\[
\prod_{i=1}^m \chi_i(P_i(x,y))
\]
where \( P_i \) are homogeneous polynomials and \( \chi_i \) are multiplicative characters, and to linear combinations of such kernels. Since these extensions are straightforward, we will not discuss them in detail.

**Remark 1.2.** One of our motivations for writing this paper was the desire to understand the eigenvalues of the \( p \)-adic Hecke operators defined in \([BK]\), in particular the operators \( T_x \) discussed in \([K]\), Theorem 2, which are the Hecke operators of \([BK]\) for \( \mathbb{P}^1 \) with 4 parabolic points and \( G = PGL_2 \). More precisely, in this paper we study what happens for \( p \)-adic integral operators “generically”, which helps recognize, by contrast, the special features of the situation of \([K],[BK]\). Namely, as explained in \([K]\), Subsection 2.4, this situation should belong to the realm of “\( p \)-adic integrable systems” (a notion which hasn’t yet been precisely defined), and as a result the eigenvalues of \( T_x \) are algebraic numbers\footnote{This was checked in 2007 by M. Vlasenko in examples \([V]\) but the general proof has not been written down and will be given in our forthcoming joint paper with E. Frenkel.} in contrast with the present paper, where the eigenvalues are (almost certainly) transcendental.

The organization of the paper is as follows. In Section 2 we discuss zeta-functions of univariate \( p \)-adic polynomials, and define a 2-variable zeta-function of such a polynomial, which is a rational function of variables \( s \) and \( z \) depending on a character of \( \mathbb{Z}_p^\times \). In Section 3 we realize the operator \( A_{P,s,\chi} \) for homogeneous \( P \) as a first order \( q \)-difference operator on the space of analytic functions on a disk written in terms of the 2-variable zeta-function of the univariate polynomial corresponding to \( P \). This allows us to write the equation for an eigenvector of \( A_{P,s} \) as a first order \( q \)-difference equation. In Section 4 we recall the definition
and properties of \(q\)-hypergeometric functions and then use them to solve the \(q\)-difference equation for an eigenvector of \(A_{p,s,\chi}\). This allows us to express the characteristic function \(h_{p,s,\chi}\) as a \(q\)-Wronskian of a set of entire solutions of a \(q\)-hypergeometric equation, thereby proving Theorem 1.1. In Section 5 we give examples of explicit computation of \(h_{p,s,\chi}\) for specific classes of homogeneous polynomials \(P\). Finally, in Section 6 we consider the non-homogeneous case. We first show that the operator \(A_{p,s}\) is trace class for sufficiently large \(\text{Re}(s)\), and then consider an example of a non-homogeneous \(P\) and show that in this example the problem of finding eigenvalues of \(A_{p,s}\) leads to more complicated functional equations, namely Mahler’s recursions (AS, BCCD).

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2. Zeta-functions

Let \(\chi\) be a character of \(Z_p^\times\). Let \(Q \in \mathbb{Q}_p[x]\) be a polynomial with constant term 1 of degree \(r\). Let \(s \in \mathbb{C}\) with \(\text{Re}(s) > -\frac{1}{r}\) if \(r \geq 1\). Define the zeta-function

\[
\zeta(Q, \chi, s) := \frac{1}{1 - p^{-1}} \int_{\mathbb{Z}_p^\times} |Q(x)|^s \chi(x) |dx|,
\]

where \(|dx|\) is the additive Lebesgue measure in which the volume of \(\mathbb{Z}_p\) is 1.

For example, if \(|Q(x)| = a\) when \(|x| = 1\) then \(\zeta(Q, \chi, s) = a^s \delta_{1,\chi}\). The function \(\zeta(Q, \chi, s)\) is a special case of the Igusa zeta-function of \(Q\) and therefore is rational in \(p\) by the result of [I].

Write \(Q_m\) for the polynomial \(Q_m(x) := Q(x^p - m)\). Then

\[
\frac{1}{1 - p^{-1}} \int_{\mathbb{Z}_p^\times} |Q(x)|^s \chi(x^p^m) |dx| = p^m \zeta(Q_m, \chi, s).
\]

Let \(c\) be the leading coefficient of \(Q\). If \(m \ll 0\) then \(|Q_m(x)| = 1\) when \(|x| = 1\), so \(\zeta(Q_m, \chi, s) = \delta_{1,\chi}\). Likewise, if \(m \gg 0\), we have \(|Q_m(x)| = |c| p^m r^s\) when \(|x| = 1\), so \(\zeta(Q_m, \chi, s) = |c|^s p^{mrs} \delta_{1,\chi}\). So if \(|c| = 1\) and \(m\) is far enough from zero then

\[
(2.1) \quad 
\zeta(Q_m, \chi, s) = p^{\max(m,0)rs} \delta_{1,\chi}.
\]

Example 2.1. Assume that \(Q \in \mathbb{Z}_p[x]\) and its reduction \(\overline{Q}\) to \(\mathbb{F}_p[x]\) also has degree exactly \(r\) and no roots in \(\mathbb{F}_p\) (so \(r \geq 2\)). Then \(|c| = 1\), so \(|Q(x)| = 1\) when \(|x| \leq 1\) and \(|Q(x)| = |x|^r\) when \(|x| > 1\). So

\[
\zeta(Q_m, 1, s) = p^{\max(m,0)rs} \delta_{1,\chi},
\]

for all \(m\).
**Example 2.2.** Let $Q(x) = 1 - x$, so $r = 1$. If $|x| = 1$ then $|1 - p^{-m}x| = 1$ if $m < 0$ and $|1 - p^{-m}x| = p^m$ if $m > 0$. Thus formula (2.1) holds for $m \neq 0$. For $m = 0$, we compute:

$$(1 - p^{-1})\zeta(1 - x, \chi, s) = \int_{\mathbb{Z}_p^\times} |1 - x| \chi(x) |dx| =$$

$$\sum_{k=2}^{p-1} \int_{k+p\mathbb{Z}_p} \chi(x) |dx| + \sum_{n \geq 1} p^{-ns} \sum_{k=1}^{p-1} \int_{1+kp^n+p^{n+1}\mathbb{Z}_p} \chi(x) |dx|.$$  

Thus for $\chi = 1$ we get

$$\zeta(1 - x, 1, s) = 1 - \frac{p^{-1}(1 - p^{-s})}{(1 - p^{-1})(1 - p^{-s-1})}.$$  

If $\chi \neq 1$ then let $\ell$ be the smallest integer such that $\chi$ is trivial on $1 + p^\ell \mathbb{Z}_p$. Then we get

$$\zeta(1 - x, \chi, s) = -\frac{p^{-1}(1 - p^{-\ell(s+1)+1})}{(1 - p^{-1})(1 - p^{-s-1})}.$$  

So for all $\chi$ we get

$$\zeta(1 - x, \chi, s) = \delta_{1,\chi} - \frac{p^{-1}(1 - p^{-\ell(s+1)+1})}{(1 - p^{-1})(1 - p^{-s-1})},$$

where for $\chi = 1$ we set $\ell = 0$.

Define the **2-variable zeta function** of $Q, \chi$ by

$$(2.2) \quad Z(Q, \chi, s, z) := \sum_{m \in \mathbb{Z}} (\zeta(Q_m, \chi, s) - \delta_{1,\chi}) z^m.$$  

It is easy to see that this series is finite in the negative direction and converges for $|z| < \min(1, p^{-\text{Re}(s)})$ to a rational function of $p^s, z$ which has the form

$$(2.3) \quad Z(Q, \chi, s, z) = Z_0(Q, \chi, s, z) + \delta_{1,\chi} \left( \frac{|z|^s}{1 - p^s z} - \frac{1}{1 - z} \right),$$

where $Z_0 \in \mathbb{C}(p^s)[z, z^{-1}]$ is a Laurent polynomial. For instance, in Example [2.1](#) we have $Z_0 = 0$, while in Example [2.2](#) $Z_0 = -\frac{p^{-1}(1 - p^{-l(s+1)+1})}{(1 - p^{-1})(1 - p^{-s-1})}$ is independent of $z$. However, it is clear that in general $Z_0$ may contain any integer power of $z$.

### 3. Realization of $A_{P,s,\chi}$ on analytic functions

Let $P \in \mathbb{Q}_p[x, y]$ be a homogeneous polynomial of degree $d$. Without loss of generality we may (and will) assume that

$$P(x, y) = x^{d-l} y^l Q\left(\frac{x}{y}\right),$$

where $Q$ is a polynomial with constant term 1, of some degree $r \leq l$. Let $s \in \mathbb{C}$, and consider the operator $A_{P,s,\chi}$ given by (1.1) acting on $H_\chi$. We are interested in nonzero eigenvalues of $A_{P,s,\chi}$.
Recall that for two linear endomorphisms $B, C$ of a vector space, nonzero eigenvalues of $BC$ and $CB$ are the same. Thus we may assume without loss of generality that $l = d$.

The space $H_\chi$ can be identified with the space $\ell_2$ of sequences $\{f_n\}$ such that $\sum_{n \geq 0} |f_n|^2 < \infty$ by the assignment $f_n = p^{-\frac{n}{2}} f(p^n)$, $n \geq 0$.

In terms of this presentation, we have

$$(A_{P,s,\chi} f)(y) = \frac{|y|^d ds}{1 - p^{-1}} \sum_{n \geq 0} p^{\frac{n}{2}} f_n \int_{|x| = p^{-n}} |Q(\frac{x}{y})|^s \chi(x p^{-n}) |dx|.$$  

Therefore, replacing $\frac{y}{x}$ with $x$ in the integral, we obtain

$$(A_{P,s,\chi} f)_m = \frac{p^{\frac{m}{2}}}{1 - p^{-1}} \sum_{n \geq 0} p^{\frac{n}{2}} f_n \int_{|x| = p^{m-n}} |Q(x)|^s \chi(x p^{m-n}) |dx|.$$  

So we get that

$$(A_{P,s,\chi} f)_m = \sum_{n \geq 0} a_{mn} f_n$$

where $a_{mn} := p^{\frac{m+n}{2} - dms} \zeta(Q_{m-n}, \chi, s)$.

**Example 3.1.** Assume that we are in the situation of Example 2.1. Then

$$(A_{P,s,\chi} f) = \sum_{n \geq 0} a_{mn} f_n$$

Let us represent the sequence $f_n$ by the analytic function

$$F(z) = \sum_{n \geq 0} f_n p^{\frac{n}{2}} z^n$$

in the disc $|z| < p^{-\frac{1}{2}}$. This identifies $H_\chi$ with the space $\mathcal{H}$ of such analytic functions whose boundary values are in $L^2$, with the norm being the usual $L^2$ norm on the boundary. Multiplying (3.1) by $p^{\frac{n}{2}} z^n$ and adding up over $m \geq 0$, we get

$$(A_{P,s,\chi} F)(z) = \sum_{m \geq 0} \sum_{n \geq 0} p^{\frac{m+n}{2} - dms} \zeta(Q_{m-n}, \chi, s) z^{m-n} f_n z^n =$$

$$\delta_{1,\chi} \frac{F(p^{-1})}{1 - p^{-d} z^2} + \sum_{m \geq 0} \sum_{n \geq 0} p^{\frac{m+n}{2} - dms} (\zeta(Q_{m-n}, \chi, s) - \delta_{1,\chi}) z^{m-n} f_n z^n =$$

$$\delta_{1,\chi} \frac{F(p^{-1})}{1 - p^{-d} z^2} + (Z(Q, \chi, s, p^{-d} z) F(p^{-1-d} z^2))_+$$

where $G_+$ is the regular part of a Laurent series $G$.

Thus we obtain the following proposition

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\[Note that Proposition 3.2 continues to hold when \( d \) is not necessarily an integer. Indeed, the function \(|y|^d |Q(x/y)|\) makes sense for non-integer \( d \).\]
Proposition 3.2. If \( l = d \) then the operator \( A_{P,s} \) on \( H_\chi \) is equivalent to the operator on \( \mathcal{H} \) given by the formula

\[
(A_{P,s,\chi} F)(z) = \delta_{1,\chi} \left( \frac{F(p^{-1}) - F(p^{-1-ds}z)}{1 - p^{-ds}z} + \frac{|c|^s F(p^{-1-ds}z)}{1 - p^{-(d-r)s}z} \right) + (Z_0(Q, \chi, s, p^{-ds}z) F(p^{-1-ds}z))^+. \]

Conjugating this by \( |x|^{d-l} \), we obtain a similar formula for general \( l \in [r, d] \):

\[
(A_{P,s,\chi} F)(z) = \delta_{1,\chi} \left( \frac{F(p^{-1-(d-l)s}) - F(p^{-1-ds}z)}{1 - p^{-(d-l)s}z} + \frac{|c|^s F(p^{-1-ds}z)}{1 - p^{-(l-r)s}z} \right) + (Z_0(Q, \chi, s, p^{-ls}z) F(p^{-1-ds}z))^+. \]

This immediately implies

Corollary 3.3. If \( \text{Re}(s) > -\frac{1}{\max(2(d-l), 2(l-r), d)} \) then the operator \( A_{P,s,\chi} \) is well defined and trace class, with exponential decay of eigenvalues. Moreover, if \( \chi \neq 1 \) then this is so for \( \text{Re}(s) > -\frac{1}{2} \).

So we will assume from now on that \( \text{Re}(s) \) varies in this range.

4. \( q \)-HYPERGEOMETRIC FUNCTIONS AND PROOF OF THEOREM

4.1. Basic hypergeometric series. The general basic hypergeometric series is defined by the formula

\[
\phi_{r-1}(a_1, ..., a_i; b_1, ..., b_{r-1}; q, u) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_i; q)_n}{(b_1; q)_n \cdots (b_{r-1}; q)_n(q; q)_n} (-1)^n q^{\frac{n(n-1)}{2}} u^n,
\]

where \( \ell \leq r \) and \( (a, q)_n \) is the \( q \)-Pochhammer symbol:

\[
(a, q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),
\]

see [GR]. We assume that \( |q| < 1 \) and \( b_j \) are not equal to a nonpositive integer power of \( q \). Then this series converges for \( |u| < 1 \) if \( r = \ell \) and for all \( u \) if \( r > \ell \).

There is also Bailey’s variant of the basic hypergeometric series

\[
\phi_{r-1}(a_1, ..., a_i; b_1, ..., b_{r-1}; q, u) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_i; q)_n}{(b_1; q)_n \cdots (b_{r-1}; q)_n(q; q)_n} u^n,
\]

which converges in the disc \( |u| < 1 \) for all \( \ell, r \). The two versions agree if \( r = \ell \), and \( \phi_{r-1} \) can be obtained from \( \phi_{m-1} \) with \( m = \max(\ell, r) \) by specializing some parameters to 0. Also, \( \phi_{r-1} \) can be obtained from \( \phi_{\ell-1} \) by sending \( a_{\ell+1}, ..., a_r \) to \( \infty \).
4.2. The \( q \)-hypergeometric difference equation. It is easy to see that \( \ell\varphi_{r-1} \) satisfies the difference equation

\[
(4.1) \quad u(1 - a_1T)\ldots(1 - a_\ell T)\Phi(u) = (1 - b_1q^{-1}T)\ldots(1 - b_{r-1}q^{-1}T)(1 - T)\Phi(u),
\]

where \((T\Phi)(u) := \Phi(qu)\). This equation is called the \( q \)-hypergeometric difference equation.

We will mostly use the special case \( r = \ell \), when \( \phi = \varphi \) and

\[
\ell\phi_{\ell-1}(a_1, \ldots, a_\ell; b_1, \ldots, b_{\ell-1}; q, u) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n\ldots(a_\ell; q)_n}{(b_1; q)_n\ldots(b_{\ell-1}; q)_n(q; q)_n} u^n.
\]

Equation (4.1) immediately implies that \( \ell\phi_{\ell-1} \) is meromorphic in the whole complex plane with at most simple poles at the points \( 1, q^{-1}, q^{-2}, \ldots \), present for generic parameter values.

Example 4.1. The Heine hypergeometric function \( {}_2\phi_1(a, b; c; q, u) \) satisfies the equation

\[
(4.2) \quad u(1 - aT)(1 - bT)\Phi(u) = (1 - cT)(1 - T)\Phi(u).
\]

One may also consider a more general \( q \)-hypergeometric equation

\[
(4.3) \quad \Phi_{\ell,r,i}(a_1, \ldots, a_\ell; b_1, \ldots, b_r; q, u) := u^i \ell\varphi_{r-1}(\frac{a_1}{b_1}, \ldots, \frac{a_\ell}{b_\ell}, \ldots, \frac{a_i}{b_i}, \ldots; q, u),
\]

where the \( i \)-th term in the \( b \)-list is omitted. These solutions form a basis of solutions if and only if \( r \geq \ell \) (or, equivalently, the equation is regular at \( 0 \)).

4.3. Confluent limits. The \( q \)-hypergeometric difference equation has many interesting limits when various parameters go to \( 0 \) and \( \infty \) in various coordinated ways, which are called confluent limits. Under these limits, the \( q \)-hypergeometric equation turns into confluent \( q \)-hypergeometric equations which may be irregular at \( 0 \) and \( \infty \), and \( q \)-hypergeometric functions turn into confluent \( q \)-hypergeometric functions. For example, the \( q \)-hypergeometric functions \( \ell\varphi_{r-1} \) and \( \ell\phi_{r-1} \) are confluent unless \( \ell = r \).

A basic example of a confluent \( q \)-hypergeometric function is the entire function

\[
(4.4) \quad J(a, q, u) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} u^n}{(a; q)_n(q; q)_n} = \varphi_1(0; a; q, u).
\]

which will arise in the next section. This function satisfies the \( q \)-Bessel difference equation

\[
-uT\Phi(u) = (1 - aq^{-1}T)(1 - T)\Phi(u)
\]
and expresses in terms of the Hahn-Exton $q$-Bessel function
\[ J_\nu(x; q) := x^{\nu} \frac{(q^{\nu+1}; q)_\infty}{(q, q)_\infty} J(q^{\nu+1}, q, qx^2) \]
also known as the third Jackson $q$-Bessel function, see [KSw, KS, GR].

4.4. The spectrum of a first order difference operator and proof of Theorem 1.1. Let $\mathcal{H}$ be the Hilbert space of analytic functions in the disk $|z| < 1$ with $L^2$ boundary values, with norm
\[ \|F\|^2 = \frac{1}{2\pi} \int_{|z|=1} |F(z)|^2 d\theta. \]
For a meromorphic function $G$ at a point $a \in \mathbb{C}$, let $\text{pr}(G, a)$ be the principal part of $G$ at $a$ (a polynomial in $\frac{1}{z-a}$ with zero constant term). Let $q \in \mathbb{C}$, $|q| < 1$. Let $R(z) \in \mathbb{C}(z)$ be a nonzero rational function equipped with a subset $\{z_1, ..., z_k\}$ of its poles in the disk $|z| < |q|^{-1}$ that we call marked, which includes all its poles in the smaller disk $|z| \leq 1$.

Define an operator on $\mathcal{H}$ by the formula
\[ (B_R F)(z) = R(z) F(qz) - \sum_{j=1}^k \text{pr}(R(z) F(qz), z_j). \]
It is easy to see that this operator is well defined and trace class. The goal of this subsection is to compute the characteristic function $h_R(u) = \det(1 - uB_R)$ of $B_R$. This will imply Theorem 1.1 as the operator $A_{P,s,\chi}$ is a special case of $B_R$.

We start with the case when $z_1, ..., z_k$ are simple poles and $z_i \neq 0$ for all $i$. Let $b_i = z_i^{-1}$. Then
\[ R(z) = \beta \frac{(1 - a_1 z)(1 - a_\ell z)}{(1 - b_1 z)(1 - b_\ell z)} \]
for some $\ell \geq k$, $\beta \neq 0$. Let $b_j^{-1} R_j$ be the residues of $R$ at the poles $z_j$, $j \in [1, k]$; so we have
\[ R_j = -\beta \frac{(1 - a_1 b_j^{-1})(1 - a_\ell b_j^{-1})}{(1 - b_1 b_j^{-1})(1 - b_\ell b_j^{-1})}, \]
where the $j$-th factor in the denominator is omitted. Now the operator $B_R$ takes the form
\[ (B_R F)(z) = \beta \frac{(1 - a_1 z)(1 - a_\ell z)}{(1 - b_1 z)(1 - b_\ell z)} F(qz) + \sum_{j=1}^k R_j F(qb_j^{-1}) \frac{1}{1 - b_j z}. \]
So the eigenvalue equation for $B_R$ looks like
\[ \lambda F(z) = \beta \frac{(1 - a_1 z)(1 - a_\ell z)}{(1 - b_1 z)(1 - b_\ell z)} F(qz) + \sum_{j=1}^k C_j \frac{1}{1 - b_j z}. \]
where $C_j := R_j F(qb_j^{-1})$.

Consider the difference equation

$$F(z) = u \frac{(1-a_1 z)(1-a_2 z) \ldots (1-a_\ell z)}{(1-b_1 z) \ldots (1-b_r z)} F(qz) + \frac{1}{1-b_j z}.$$ 

For small $u$ it has a unique power series solution, obtained by iterating the equation:

$$F_j(z, u) = \frac{1}{1-b_j z} + u \frac{(1-a_1 z)(1-a_2 z) \ldots (1-a_\ell z)}{(1-b_1 z) \ldots (1-b_r z)} \frac{1}{1-b_j z} + \ldots = \frac{1}{1-b_j z} \cdot \epsilon_{+1} \varphi_r(a_1 z, \ldots, a_\ell z, q; b_1 z, \ldots, qb_j z, \ldots, b_r z; q, u).$$

So for the eigenvector we have

$$F(z, \lambda^{-1}) = \sum_{j=1}^k \lambda^{-1} C_j F_j(z, \beta \lambda^{-1}) =$$

(4.5) $$\sum_{j=1}^k \frac{\lambda^{-1} C_j}{1-b_j z} \cdot \epsilon_{+1} \varphi_r(a_1 z, \ldots, a_\ell z, q; b_1 z, \ldots, qb_j z, \ldots, b_r z; q, \beta \lambda^{-1}).$$

So we get

$$C_i = R_i F(qb_i^{-1}) =$$

$$\sum_{j=1}^k \frac{\lambda^{-1} R_i C_j}{1-qb_i^{-1}b_j} \cdot \epsilon_{+1} \varphi_r(\frac{a_1}{b_i}, \ldots, \frac{a_\ell}{b_i}, q; \frac{b_1}{b_i}, \ldots, \frac{b_r}{b_i}, q, \beta \lambda^{-1}).$$

This can be written as

$$\sum_{j=1}^k C_j M_{ji} (\beta \lambda^{-1}) = 0,$$

where

$$M_{ii}(u) := \epsilon \varphi_{r-1}(\frac{a_1}{b_i}, \ldots, \frac{a_\ell}{b_i}, b_i; q, u),$$

and for $j \neq i$

$$M_{ji}(u) := \frac{\epsilon \varphi_{r-1}(\frac{a_1}{b_i}, \ldots, \frac{a_\ell}{b_i}, \frac{b_j}{b_i}, \frac{b_i}{b_j}; q, u) - \epsilon \varphi_{r-1}(\frac{a_1}{b_i}, \ldots, \frac{a_\ell}{b_i}, \frac{b_i}{b_j}; q, qu)}{1 - \frac{b_j}{b_i}},$$

where in both cases the $i$-th term in the $b$-list is omitted. Hence

$$M_{ji}(u) = u^{\nu_i} \frac{\prod_{m \neq j} (1-b_m T)}{\prod_{m \neq i} (1-\frac{b_m}{b_i})} \Phi_i(u),$$

where $m$ varies in $[1, k]$ and

(4.6) $$\Phi_i(u) := \Phi_{\ell,r,i}(a_1, \ldots, a_\ell; b_1, \ldots, b_k, q^{-1}b_{k+1}, \ldots, q^{-1}b_r; q, u)$$
are the solutions of the \( q \)-hypergeometric difference equation defined by formula (4.3) (with shifted parameters). Thus we have a \( k \)-by-\( k \) matrix \( M(u) \), and the eigenvalues of \( B_R \) are solutions of the equation

\[
\det M(\beta \lambda^{-1}) = 0.
\]

The matrix \( M(u) \) expresses in terms of the \( q \)-Wronski matrix of the functions \( \Phi_1, \ldots, \Phi_k \). Indeed, the \( q \)-Wronski matrix

\[
W(u) = W(\Phi_1, \ldots, \Phi_k)
\]

has entries

\[
W_{n_i}(u) = T_n^1 - \prod_{j \neq i}(1 - b_i T_j).
\]

But by the Newton interpolation formula we have

\[
T_n^1 = \sum_{j=1}^k b_j^{1-n} \prod_{m \neq j}(1 - b_m T) \prod_{m \neq j}(1 - b_m b_j),
\]

Thus, setting \( D := \text{diag}(\prod_{m \neq i}(1 - b_m)) \) and \( N(u) := \text{diag}(u^{-\nu}) \), we have

\[
M(u) = V^{-1}DW(u)N(u)D^{-1},
\]

where \( V \) is the Vandermonde matrix with entries

\[
V_{n_j} = b_j^{1-n}.
\]

In particular, we find that the eigenvalues of \( B_R \) are solutions of the equation

\[
\det W(\beta \lambda^{-1}) = 0.
\]

If \( k = r = \ell \), the matrix \( W(u) \) is the \( q \)-Wronski matrix of a basis of solutions of the \( q \)-hypergeometric difference equation, so its determinant factorizes, resulting in the eigenvalues being of the form \( \prod_{i<j} a_i u^{\nu_i} q^n \), \( n \geq 0 \). This is not surprising since in this case the matrix of the operator \( B_R \) is triangular, so the eigenvalues are just the diagonal entries.

Now to obtain the characteristic function of \( B_R \) we need to eliminate the poles of the function \( \det W(\beta \lambda^{-1}) \) (i.e., “clear denominators”). Since \( q \)-hypergeometric functions have first order poles at 1, \( q^{-1} \), \( q^{-2} \) etc., this is achieved by replacing \( \Phi_i(u) \) by

\[
f_i(u) = (u; q)_\infty \Phi_i(u).
\]

The functions \( f_i(u) \) satisfy a simple modification of the \( q \)-hypergeometric equation, and we have

**Proposition 4.2.**

\[
\det(1 - u \beta^{-1} B_R) = u^{-\sum_{i=1}^k \nu_i} \prod_{i<j}(q^{\nu_i} - q^{\nu_j}) \det W(f_1, \ldots, f_k)(u).
\]

**Proof.** Since \( \det V = \prod_{i<j}(q^{\nu_i} - q^{\nu_j}) \), the two sides of (4.7) are entire functions of Hadamard order 0 with the same zeros, both having value 1 at the origin. The Proposition follows. \( \square \)
Remark 4.3. This method also gives explicit $q$-hypergeometric formulas for eigenvectors of the operator $B_R$, which are given by (4.5), where $C := (C_1, \ldots, C_k)$ is a null row-vector of the matrix $V^{-1}DW(\beta\lambda^{-1})$. Moreover, if the operator $B_R$ is self-adjoint, the orthogonality relations for the eigenvectors yield interesting identities with $q$-hypergeometric functions.

Now, if the simple poles $z_j = b_j^{-1}$, $j \in [1, k]$ of $R$ are allowed collide and produce multiple poles, including one at the origin (which would give a fully general function $R(z)$) then the characteristic function of $B_R$ is given by a confluent limit of the formula of Proposition 4.2. But by Proposition 3.2, the operator $A_{P, s, \chi}$ is a special case of $B_R$, with $k = k_0 + \delta_1\chi$, where $k_0$ is the order of pole of $Z(Q, \chi, s, z)$ at the origin, i.e., is obtained from the generic situation by sending $\nu_1, \ldots, \nu_k_0$ to $+\infty$ and $\beta \to \infty$ if $k_0 > 0$. Therefore Proposition 4.2 and Proposition 3.2 imply Theorem 1.1.

Remark 4.4. The same results with obvious changes extend over any non-archimedian local field $F$ instead of $\mathbb{Q}_p$. Namely, $\mathbb{Z}_p$ should be replaced by the ring of integers $\mathcal{O}_F$, $p \in \mathbb{Z}_p$ by a uniformizer $\pi \in \mathcal{O}_F$ and $p \in \mathbb{C}$ by the order of the residue field $\mathcal{O}_F/\pi\mathcal{O}_F$ of $F$. Furthermore, the character $w \mapsto |w|^s$ in the integral can be replaced with any multiplicative character $w \mapsto \eta(w) := |w|^s\eta_0(w)$, where $\eta_0$ is a character of $\mathcal{O}_F^\times$, and it is easy to see that the corresponding function $Z_0(P, \chi, s, \eta_0, z)$ is still a Laurent polynomial of $z$. Finally, we can further extend this theory to kernels of the form $\prod_{i=1}^n \eta_i(P(x, y))$, and in fact to any sufficiently nice homogeneous kernels, as well as linear combinations of such kernels. Since these extensions are straightforward, we will not discuss them in detail.

4.5. The case when $R$ is a Laurent polynomial. To illustrate the confluent limit of Proposition 4.2, let us consider the case when $R$ is a Laurent polynomial:

$$R(z) = z^{-k}(1 - a_1z)\ldots(1 - a_\ell z), \ k \geq 1, \ell \geq k.$$ 

This is an important special case, as it occurs for $A_{P, s, \chi}$ with $\chi \neq 1$. The simplest interesting example $k = 1, \ell = 2$ is worked out even more explicitly in Section 5.5.

For simplicity assume that $a_i$ are distinct; the general case is similar. Then we have

$$(B_RF)(z) = (R(z)F(qz))_+.$$ 

To find the spectrum of $B_R$, let us introduce a parameter $t$ and let

$$R_t(z) = (-t)^{-k}b_1\ldots b_k(1 - a_1z)\ldots(1 - a_\ell z)\left(1 - t^{-1}b_1z\right)\ldots\left(1 - t^{-1}b_kz\right),$$

for some distinct $b_i \neq 0$. Then $R_t(z) \to R(z)$ as $t \to 0$, so $\lim_{t \to 0} B_{R_t} \to B_R$. Thus we can find the characteristic function of $B_R$ by finding the characteristic function of $B_{R_t}$ and taking the limit $t \to 0$. 

In the limit $t \to 0$, the difference equation for the functions $\Phi_i$ takes the form

$$ (1 - a_1 T) \cdots (1 - a_\ell T) \Psi = u^{-1} T^k \Psi, \tag{4.8} $$

which can be written as

$$ (-1)^{\ell} a_1 \cdots a_\ell (1 - a_1^{-1} T^{-1}) \cdots (1 - a_\ell^{-1} T^{-1}) \Psi = q^\ell u^{-1} T^{k-\ell} \Psi. \tag{4.9} $$

This equation is regular at $\infty$, so it has a basis of (confluent) $q$-hypergeometric solutions with power behavior at infinity, $\Psi_1, \ldots, \Psi_\ell$, such that $\Psi_i(u) \in u^{\mu_i} \mathbb{C}[[u^{-1}]]$, where $\mu_i := -\log_q a_i$. Namely,

$$ \Psi_i(u) = u^{\mu_i} \Psi_i^0(u), $$

and

$$ \Psi_i^0(u) = e^{-k \phi_{\ell-1}} \left( 0, \ldots, 0; \frac{a_1}{a_1}, \ldots, \frac{a_\ell}{a_\ell}; q, \frac{(-1)^{\ell} q^\ell a_1^{-\ell-k}}{a_1 \cdots a_\ell u} \right). $$

is an entire function in $u^{-1}$ equal to 1 at $\infty$.

Let $\theta(u) = \theta(u, q) := (q; q)_\infty (-u; q)_\infty (-qu^{-1}; q)_\infty$ be the Jacobi theta function. We have $\theta(qu) = u^{-1} \theta(u)$. Thus conjugating (4.8) by $\theta(u)^{\frac{1}{k}}$, we get for every solution $\Psi$ of this equation:

$$ u(1 - a_1 u^{\frac{1}{k}} T) \cdots (1 - a_\ell u^{\frac{1}{k}} T) \theta^{\frac{1}{k}} \Psi = (u^{\frac{1}{k}} T)^k \theta^{\frac{1}{k}} \Psi, $$

i.e.,

$$ (1 - a_1 u^{\frac{1}{k}} T) \cdots (1 - a_\ell u^{\frac{1}{k}} T) \Psi_s = q^{\frac{1}{k}} T^k \Psi_s, $$

where $\Psi_s := \theta^{\frac{1}{k}} \Psi$ (using appropriate branches of the function $z^{\frac{1}{k}}$). This equation has $k$ independent solutions with power behavior near zero (in terms of the variable $u^{\frac{1}{k}}$), $E_1, \ldots, E_k$, which can be found using the power series method. Namely,

$$ E_m(u) = u^{\frac{2\pi i m}{k}} E(e^{\frac{2\pi i m}{k}} u^{\frac{1}{k}}) $$

where $E$ is an entire function with $E(0) = 1$.

We have

$$ E(v) = \sum_{j=1}^\ell \xi_j(v) \theta(v^k)^{\frac{1}{k}} \Psi_j(v^k), $$

where $\xi_j(q^{\frac{1}{k}} v) = \xi_j(v)$. Also the functions

$$ \tilde{\xi}_j(v) := v^{k \mu_j} \theta(v^k)^{\frac{1}{k}} \xi_j(v^k) $$

must be meromorphic in $v$ away from the origin. We have

$$ \tilde{\xi}_j(q^{\frac{1}{k}} v) = q^{\mu_j} v^{-1} \tilde{\xi}_j(v). $$

Thus

$$ \tilde{\xi}_j(v) = \theta(q^{-\mu_j} v, q^\frac{1}{k}) \eta_j(v), $$
where $\eta_j$ is an elliptic function with period $q^\frac{1}{r}$. We conclude that

$$E(v) = \sum_{j=1}^{\ell} \eta_j(v)\theta(a_jv, q^\frac{1}{r})\Psi_j^0(v^k).$$

**Lemma 4.5.** The elliptic functions $\eta_j$ are constant.

**Proof.** Assume the contrary. Let $p$ be a pole of $\eta_l$ for some $l$, and assume that its order $r$ is maximal among all $\eta_j$ at $p$. Let $\lim_{v \to p}(v-p)^r\eta_j(v) = c_j$; so $c_j \neq 0$. Thus

$$(v-p)^rE(vq^\frac{1}{r}) = q^{-\frac{n(n-1)}{2r}}v^{-n}\sum_{j=1}^{\ell}(v-p)^r\eta_j(v)a_j^{-n}\theta(a_jv, q^\frac{1}{r})\Psi_j^0(q^n v^k).$$

So setting $v = p$, we obtain (since $E$ is entire):

$$\sum_{j=1}^{\ell} d_j a_j^{-n}\Psi_j^0(q^n p^k) = 0,$$

$d_j := c_j\theta(a_j, q^\frac{1}{r})$. Consider these equations for $n = -N, -N+1, \ldots, -N+\ell-1$ as a linear system with respect to the unknowns $x_j := d_j a_j^{-N}$. The matrix of this system has entries $\alpha_{ij} = a_j^{-i+1}\Psi_j^0(q^{-i+1} p^k)$. When $N \to +\infty$, this matrix approaches a Vandermonde matrix (as $\Psi_j^0(\infty) = 1$). Thus it is nondegenerate for $N \gg 0$, which yields that $d_j = 0$, i.e., $c_j\theta(a_j, q^\frac{1}{r}) = 0$ for all $j$. We conclude that $\theta(a_j, q^\frac{1}{r}) = 0$, so the only pole of $\eta_l$ on $\mathbb{C}^\times / q^\infty$ is $a_l^{-1}$. Thus $r \geq 2$ (as there are no elliptic functions with just one simple pole). We also see that for all $j$ with $\theta(a_j, q^\frac{1}{r}) \neq 0$ we have $c_j = 0$, i.e., the order of the pole of $\eta_j$ at $p$ is $\leq r-1$. So for such $j$ set $c'_j := \lim_{v \to p}(v-p)^{r-1}\eta_j(v)$. Then, evaluating the identity

$$(v-p)^{r-1}E(vq^\frac{1}{r}) = q^{-\frac{n(n-1)}{2r}}v^{-n}\sum_{j=1}^{\ell}(v-p)^{r-1}\eta_j(v)a_j^{-n}\theta(a_jv, q^\frac{1}{r})\Psi_j^0(q^n v^k)$$

at $p$, we obtain

$$\sum_{j: c_j \neq 0} c_j a_j^{-n+1}\theta(a_j, q^\frac{1}{r})\Psi_j^0(q^n v^k) + \sum_{j: c_j = 0} c'_j a_j^{-n}\theta(a_j, q^\frac{1}{r})\Psi_j^0(q^n v^k) = 0,$$

and the same argument with the Vandermonde matrix gives $c_l\theta'(a_l, q^\frac{1}{r}) = 0$, hence $c_l = 0$, a contradiction. $\square$

Thus, setting $\zeta := e^{\frac{2\pi i}{r}}$ we obtain

**Proposition 4.6.**

$$E(v) = \sum_{j=1}^{\ell} \eta_j \theta(a_jv, q^\frac{1}{r})\Psi_j^0(v^k),$$
\( \eta_j \in \mathbb{C} \), and the constants \( \eta_j \) are uniquely determined by the condition that \( E \) is holomorphic at 0 and \( E(0) = 1 \).

Proof. We just have to justify uniqueness, which follows since the difference of two such expressions will have an admissible power asymptotics at 0 for a solution of the corresponding difference equation. \( \square \)

Remark 4.7. The constants \( \eta_j \) admit product formulas which can be derived by taking the confluent limit of the connection matrix of the \( q \)-hypergeometric equation. We will not give these formulas here except in the simplest example of the \( q \)-Bessel equation, see Subsection 5.5.

Recall that \( \det(\zeta(m(n-1))_{1 \leq m,n \leq k} \mathbb{R}) \leq \frac{(k-1)(k-2)}{2} \frac{1}{k^2} \). So we arrive at the following result.

Proposition 4.8. We have

\[
\det(1 - u B_R) = i^{-\frac{(k-1)(k-2)}{2}} k^{-\frac{1}{2}} \det(\zeta(m(n-1))T^{n-1}E(\zeta_i u^\frac{1}{k}))_{1 \leq m,n \leq k},
\]

where \( E \) is the entire function given by the formula of Proposition 4.6.

Note that while the entries of this matrix are functions of \( u^\frac{1}{k} \), the determinant is a function of \( u \).

5. Examples

In this section we consider a number of examples of computation of the characteristic function \( h_{P,s,\chi} \). In all of these examples, the parameter \( k \) (the size of the Wronski matrix) equals 1, so the characteristic function turns out to be (confluent) \( q \)-hypergeometric. While these examples are all instances of Theorem 1.1, to make the discussion more concrete, we do all computations explicitly from scratch and then compare the answers to existing literature on \( q \)-special functions.

5.1. The situation of Example 2.1. Our first example is the setting of Example 2.1. A typical polynomial \( P \) of this kind is \( P(x,y) = y^{d-r}(x^2 + y^2)^\frac{r}{2} \) for even \( r \), where \( p \) has remainder 3 modulo 4. In this case \( Z_0 = 0 \), so \( A_{P,s,\chi} = 0 \) when \( \chi \neq 1 \), so it remains to consider the case \( \chi = 1 \). Then, by Proposition 3.2, we have

\[
(A_{P,s,1}F)(z) = \frac{F(p^{1})}{1 - pqz} + \frac{(p^s - 1)pqzF(qz)}{(1 - pqz)(1 - p^{rs+1}qz)}.
\]

So the eigenvalue equation has the form

\[
\lambda F(z) = \frac{F(p^{1})}{1 - pqz} + \frac{(p^s - 1)pqzF(qz)}{(1 - pqz)(1 - p^{rs+1}qz)}.
\]

If \( r = 0 \) then \( P = y^d \), so \( A_{P,s,1} \) has rank 1 with the only nonzero eigenvalue \( \lambda = \frac{1}{1-q} \). The same thing happens if \( s = 0 \). So let us assume that \( r > 0, s \neq 0 \). Then we have

\[
\lambda F(0) = F(p^{1}).
\]
This implies that $\lambda \neq 0$. Now, if $F(p^{-1}) = 0$ then there are no nonzero solutions (looking at the order of $F$ at zero), so $F(p^{-1}) \neq 0$. Then we may set $F(0) = 1$, so that $\lambda = F(p^{-1})$. So the difference equation for the eigenvector looks like

$$F(z) = \frac{1}{1 - pqz} + \lambda^{-1} \frac{(p^r - 1)pqzF(qz)}{(1 - pqz)(1 - prs + 1qz)}.$$ 

Iterating this, we get the expression for $F$:

$$F(z) = \frac{1}{1 - pqz} + \lambda^{-1} \frac{(p^r - 1)pqz}{(1 - pqz)(1 - pq^2z)(1 - prs + 1qz)} + \frac{(p^r - 1)^2p^2q^3z^2}{(1 - pqz)(1 - pq^2z)(1 - pq^3z)(1 - prs + 1qz)} + \ldots$$ 

So we have

$$\lambda = F(p^{-1}) = \frac{1}{1 - q} + \lambda^{-1} \frac{(p^r - 1)q}{(1 - q)(1 - q^2)(1 - prs + 1qz)} + \frac{(p^r - 1)^2q^3}{(1 - q)(1 - q^2)(1 - q^3)(1 - prs + 1qz)} + \ldots$$ 

Multiplying the numerators and denominators on the right hand side of this equation by $1 - p^r$ and multiplying both sides by $\lambda^{-1}$, we obtain

**Proposition 5.1.** We have

$$h_{P,s,1}(u) = J(p^r, q, (1 - p^r)u),$$

where $J(a, q, u) = \psi_1(0, a; q, u)$ is the entire function given by the formula (4.4). In particular, the eigenvalues of the operator $A = A_{P,s,1}$ are the solutions of the equation

$$J(p^r, q, (1 - p^r)\lambda^{-1}) = 0$$

and are simple.

Thus the characteristic function of $A_{P,s,1}$ expresses via the Hahn-Exton $q$-Bessel function with parameter

$$\nu = -1 - \frac{rs}{1 + ds}.$$

**Remark 5.2.** We obtain the following formula for the eigenvector $F(z)$ with eigenvalue $\lambda$:

$$F(z) = \frac{1}{1 - pqz} \cdot \psi_2(0, q; pq^2z, apqz; q, (1 - a)pqz\lambda^{-1}).$$

where $a := p^r$. So setting $p^2q = b$ and replacing $z$ with $p^{-\frac{1}{2}}z$, we get that the functions

$$F_i(z) := \frac{1}{1 - bz} \cdot \psi_2(0, q; bqz, abz; q, b^2q^{-1}zu_i),$$

for $u_i$ roots of the equation $J(a, q, u) = 0$, are orthogonal on the unit circle $|z| = 1$. 

Remark 5.3. This calculation implies that the unbounded self-adjoint operator 
$$(1-p^s) A_{P,s,1}^{-1}$$ has a limit $B_0$ as $s \to 0$, which has 1-dimensional kernel. Moreover, the inverse $\hat{A}_0$ of $B_0$ on the orthogonal complement of $\text{Ker}B_0$ is trace class and has characteristic function

$$\det(1-u\hat{A}_0) = \lim_{a \to 1} (a-1) \frac{J(a,q,u)}{u} = \sum_{n=0}^{\infty} (-1)^n q^{(n+1)\frac{a}{2}} u^n.$$ 

Consider now the case $l = \frac{r+d}{2}$ and $s > -\frac{1}{d}$. Then the operator $A = A_{P,s,1}$ has matrix

$$a_{mn} = q^{\frac{m+n}{2}} q^{\frac{m+n-\min(m,n)}{2}},$$

where $q := p^{-1} q^{-1} = q^{\nu+1}$. Note that this operator is trace class whenever $|q| < 1$, $|q| < |q|^{-1}$ and self-adjoint if in addition $q, q \in \mathbb{R}$.

Proposition 5.4. If $0 < q, q < 1$ then the operator $A$ is positive.

Proof. It suffices to show that for any $0 < q < 1$, the matrix with entries

$$b_{mn} := q^{-\min(m,n)}, 0 \leq m, n \leq N$$

is positive definite. By Sylvester’s criterion, for this it suffices to show that its determinant $D_N$ is positive (as we know that 0 is not an eigenvalue). Subtracting from the first row of this matrix $q$ times the second row, we get

$$D_N = q^{-N} (1-q) D_{N-1},$$

with $D_0 = 1$. So $D_N = q^{-N(N+1)/2} (1-q)^N > 0$. □

Corollary 5.5. If $|q| < 1, |q| < |q|^{-1}$ then the zeros of the function $(q, q) \infty J(q, q, u)$ are all simple. Moreover, for $0 < q < 1$, they are real if in addition $0 < q \leq q^{-1}$, and nonnegative if $0 < q \leq 1$.

Proof. For $q = 1$ the result follows from Remark 5.3 so assume $q \neq 1$. Then the first statement follows from Proposition 5.4 (interpolated to non-integer values of $d$). For the second statement it suffices to assume that $q \neq q^{-1}$. So the operator $A$ is a self-adjoint trace class operator. Hence its eigenvalues $\lambda_i$ are real, and by Proposition 5.4 they are positive if $q < 1$. But $(1-q)\lambda_i^{-1} = u_i$, where $u_i$ are the zeros of $J(q, q, u)$. This implies the second statement. □

Remark 5.6. 1. We note that Corollary 5.3 is not new and is given just for illustration purposes. Namely, the second statement of Corollary 5.3 is proved in [KS] using a different method. Also a proof similar to ours is given in [SS].

2. The assumptions of Corollary 5.3 cannot be dropped. For example, if $q = q^{-m+1}$ for $m \geq 0$ (i.e., $\nu = -m$) then $(q, q) \infty J(q, q, u)$ has a zero of order $m$ at $u = 0$ (so a multiple zero for $m \geq 2$). This also shows that when $q, q \in (0, 1)$ and $q$ is close to $q^{-m+1}$ then there have to be zeros of $(q, q) \infty J(q, q, u)$ with argument close to $2\pi k/m$ or $2\pi (k+1)/m$ (depending on the sign of $1 - q^{m-1} q$). So for $m \geq 2$ they cannot be all real.
A more complete analysis of the zeros of the function $J$ is carried out in \cite{AM1}. In particular, it is shown there that for $q, q \in (0, 1)$ the number of non-real zeros is always finite.

5.2. A rank 1 perturbation. The operator $A$ from Subsection 5.1 for $r = d$ is a rank 1 perturbation of the inverse to a second order difference operator. Namely, consider the self-adjoint trace class operator $\tilde{A}$ with kernel $|P(x,y)|^{s-1} \frac{1}{1-p^{-ds}}$, i.e., $\tilde{A} = \frac{1}{1-pq}(A - \frac{1}{1-p}\Pi)$, where $\Pi$ is the projector to constant functions. This operator has a limit at $s = 0$ where the kernel becomes $\frac{1}{d} \log |P(x,y)|$. As before, $\tilde{A}$ vanishes on $H_\chi$ with $\chi \neq 1$, so let us consider it on $H_1 \approx \ell^2$. Then the matrix of $\tilde{A}$ in the standard basis is given by

$\tilde{a}_{mn} = p^{\frac{m+n}{2}} p^{-\min(m,n)} ds - 1$.

There is an obvious eigenvector $(1, 0, ...)$ with eigenvalue 0 (the indicator function of $\mathbb{Z} \times p$), so let’s consider the action of $\tilde{A}$ on its orthogonal complement, i.e., on the space $\ell^2_0$ of square summable sequences $\{f_n\}$ such that $f_0 = 0$. This space is identified with $H$ by the assignment $F(z) = \sum_{n \geq 1} p^z f_n z^{n-1}$. In terms of this realization we have

$(\tilde{A} F)(z) = \frac{qzF(qz) - p^{-1} F(p^{-1})}{(1-z)(1-pqz)}$.

Thus the eigenvalue equation has the form

$\lambda F(z) = \frac{qzF(qz) - p^{-1} F(p^{-1})}{(1-z)(1-pqz)}$.

If $\lambda = 0$ then we get $qzF(qz) = p^{-1} F(p^{-1})$, i.e. $F = 0$. Thus $\lambda \neq 0$. Now, if $F(p^{-1}) = 0$ then there are no nonzero solutions (looking at the order of $F$ at zero), so $F(p^{-1}) \neq 0$. Then we may set $F(0) = 1$, so that

$\lambda = -p^{-1} F(p^{-1})$.

Thus we have

$F(z) = \frac{1}{(1-z)(1-pqz)} + \lambda^{-1} \frac{qz}{(1-z)(1-pqz)} F(qz)$.

Iterating this, we get the expression for $F$:

$F(z) = \frac{1}{(1-z)(1-pqz)} + \lambda^{-1} \frac{qz}{(1-z)(1-qz)(1-pqz)(1-pq^2z)} + \lambda^{-2} \frac{q^3 z^2}{(1-z)(1-qz)(1-q^2z)(1-pqz)(1-pq^2z)(1-pq^3z)} + ...$

So setting $z = p^{-1}$, we have

$1 + \lambda^{-1} \frac{p^{-1}}{(1-p^{-1})(1-q)} + \lambda^{-2} \frac{qp^{-2}}{(1-p^{-1})(1-p^{-1}q)(1-q)(1-q^2)}$. 

Thus we get

**Proposition 5.7.** The characteristic function of $\tilde{A}$ is given by the formula

$$\det(1 - u\tilde{A}) = J(p^{-1}, q, -p^{-1}u),$$

so the eigenvalues of $\tilde{A}$ are the solutions of the equation

$$(5.1) \quad J(p^{-1}, q, -p^{-1}A^{-1}) = 0$$

and are simple.

Note that the Hahn-Exton parameter $\nu$ now equals $-\frac{ds}{ds+1}$.

Consider now the inverse operator $\tilde{A}^{-1}$. Let $F = \tilde{A}^{-1}G$. Then we have

$$G(z) = \frac{qzF(qz) - p^{-1}F(p^{-1})}{(1 - z)(1 - pqz)}.$$

Thus $G(0) = -p^{-1}F(p^{-1})$. So we have

$$(\tilde{A}^{-1}G)(z) = (1 - q^{-1}z)(z^{-1} - p)G(q^{-1}z) - z^{-1}G(0) = ((1 - q^{-1}z)(z^{-1} - p)G(q^{-1}z))_+.$$ 

This shows that $\tilde{A}^{-1}$ is an unbounded self-adjoint operator, which is a self-adjoint extension of the symmetric operator $L = L_\nu$ given by

$$(LG)(z) := ((1 - q^{-1}z)(z^{-1} - p)G(q^{-1}z))_+$$

with initial domain being the space of polynomials $\mathbb{C}[z]$ with norm

$$\|G\|^2 = \frac{1}{2\pi} \int_{|z|=\frac{1}{2}} |G(z)|^2 d\theta.$$

Writing $G(z)$ as $\sum_{n \geq 0} p^{n}g_{n}z^{n}$, we obtain an expression for $L$ as a difference operator on the space of sequences whose matrix in the standard basis is a Jacobi matrix:

$$(Lg)_{n} = q^{-n-1}(p^{\frac{1}{2}}g_{n+1} - (1 + pq)g_{n} + p^{\frac{1}{2}}qg_{n-1}),$$

where we agree that $g_{-1} = 0$. It follows that the self-adjoint extension of $L$ given by $\tilde{A}^{-1}$ has discrete spectrum, and its eigenvalues are the solutions $u_i$ of the equation

$$J(p^{-1}, q, -p^{-1}u) = 0.$$

This analysis is carried out in [SS]. Let us explain its results in more detail. Consider the homogeneous difference equation $Lg = 0$. Its basic solutions (generically) are $g_{n} = \beta^{n}$ where $\beta$ is a root of the characteristic equation

$$p^{\frac{1}{2}}\beta^{2} - (1 + pq)\beta + p^{\frac{1}{2}}q = 0.$$ 

The roots of this equation are $\beta_{1} = p^{\frac{1}{2}}$ and $\beta_{2} = p^{\frac{1}{2}}q = p^{\frac{1}{2} - ds}$. So in the range $-\frac{1}{d} < s \leq -\frac{1}{2d}$ (i.e., $\nu \geq 1$) we have $p^{-\frac{1}{2} - ds} \geq 1$, so the equation $Lg = 0$ has a unique, up to scaling, solution in $\ell_{2}$ (namely, $p^{-\frac{1}{2}}$). This implies (after some
work, done in [SS]) that the operator $L$ is essentially self-adjoint on $C[z]$. On the other hand, if $s > -\frac{1}{2q}$ (i.e., $|\nu| < 1$) then $p^{-\frac{1}{2}ds} < 1$, so both basic solutions are in $\ell_2$. This implies (again after some work done in [SS]) that the operator $L$ is not essentially self-adjoint but rather has von Neumann deficiency indices $(1, 1)$. So $L$ has a 1-parameter family of self-adjoint extensions parametrized by $(\alpha_1, \alpha_2) \in \mathbb{RP}^1$ obtained by adding the function $\psi_{\alpha_1,\alpha_2} := \frac{\alpha_1}{1-z} + \frac{\alpha_2}{1-pqz}$ to the initial domain of $L$, which makes it essentially self-adjoint (if $s = 0$, i.e., $q = p^{-1}$, we should add $\psi_{\alpha_1,\alpha_2} = \frac{\alpha_1}{1-z} + \frac{\alpha_2}{(1-2z)^2}$). Moreover, the self-adjoint extension of $L$ given by $\tilde{A}^{-1}$ corresponds to the point $(1, 0) \in \mathbb{RP}^1$. Finally, if $-\frac{1}{2q} < s \leq 0$ (i.e., $0 \leq \nu < 1$) then $L$ is positive and $\tilde{A}^{-1}$ is its Friedrichs extension, as it has the smallest domain, see [L]. This comes from the fact that the decay of Taylor coefficients of the function $\psi_{\alpha_1,\alpha_2}$ is fastest if and only if $\alpha_2 = 0$.

For example, in the case $s = 0$, i.e., $q = p^{-1}$, we get the difference operator $(Lg)_n = p^n(p^{\frac{1}{2}}g_{n+1} - 2x_n + p^{\frac{1}{2}}g_{n-1})$, i.e. $L = DD^*$ where $(Dg)_n = p^{n-1/2}(g_n - g_{n-1})$.

The eigenvalues of the Friedrichs extension of $L$ are, therefore, solutions of the equation $J(p^{-1}, p^{-1}, -p^{-1}u) = 0$. This is shown in [KRS].

**Remark 5.8.** A more complete account of the spectral theory of difference operators (also called $q$-Sturm-Liouville theory) can be found in the book [AM2].

**Remark 5.9.** We obtain the following formula for the eigenvector $F(z)$ with eigenvalue $\lambda$:

$$F(z) = \frac{1}{(1-z)(1-pqz)} \cdot 2\phi_2(0, q; pq^2z, pqz; q, -z\lambda^{-1}).$$

Thus the eigenvectors are

$$F_i(z) = \frac{1}{(1-z)(1-pqz)} \cdot 2\phi_2(0, q; pq^2z, pqz; q, pzu_i),$$

where $u_i$ are the roots of the equation $J(p^{-1}, q, u) = 0$.

The fact that the inverse of $\tilde{A}$ is a difference operator $L$ implies that we also have

$$F_i(z) = \sum_{n=0}^{\infty} J(p^{-1}, q, q^{n+1}u_i)z^n.$$ 

Thus we obtain the orthogonality relations

$$\sum_{n \geq 0} p^{-n}J(p^{-1}, q, q^{n+1}u_i)J(p^{-1}, q, q^{n+1}u_j) = 0, \ i \neq j$$

which are the Hahn-Exton $q$-analogs of the Fourier-Bessel orthogonality relations with $p^{-1} = q^{n+1}$ (see [KSw], p.2).
5.3. **Continuous limit.** We can also consider the “archimedian” limit of the operator $\tilde{A}|_{H_1}$ from the previous subsection as $p \to 1$ (this does not have a $p$-adic interpretation but makes sense analytically). For this pick $a > 0$ and think of $q^{\frac{a}{2}} f_n$ as values of some function $f$ at $q^{\frac{-a}{2}}$. Then the limit $p \to 1$ will be a continuous limit which transforms Riemann sums into integrals over $[0, 1]$. The condition $\sum_n |f_n|^2 < \infty$ translates into the condition that

$$\int_0^1 |f(x)|^2 dx < \infty,$$

i.e., $f \in L^2[0, 1]$. Also the operator $\tilde{A}$ after appropriate renormalization converges to

$$(5.2) \quad (\tilde{A}_\infty(a)f)(x) = \frac{1}{a} \int_0^1 (\max(x, y)^{-a} - 1)(xy)^{-\frac{a}{2}} f(y) dy.$$

Note that the operator $\tilde{A}_\infty(a)$ is unitarily equivalent to $\tilde{A}_\infty(1)$ via the change of variable $x \to x^a$. Since $q \to 1$ as $p \to 1$ (for fixed $s$), and since the $q$-Bessel function $J_\nu(x; q)$ degenerates in this limit to the classical Bessel function $J_\nu(x)$, the eigenvalues of $\tilde{A}_\infty(a)$ are proportional to inverse squared zeros of $J_\nu$, where $\nu = -\frac{ds}{ds+1}$.

Furthermore, $\tilde{A}_\infty(a)$ is inverse to a second order differential operator. Namely, consider the differential operator

$$L_\nu := -\partial^2 + \frac{\nu^2 - \frac{1}{4}}{x^2}.$$

Assume that $\nu > -1$ and consider the action of $L_\nu$ on the space of functions $g$ on $[0, 1]$ such that $gx^{-\nu-\frac{1}{2}}$ is a polynomial and $g(1) = 0$. This defines an essentially self-adjoint operator. Its eigenvalues and eigenvectors are found in the standard way (for $\nu = 0$ this is the classical problem of vibrations of a circular membrane, i.e., the Dirichlet problem for the Laplacian on the disk). Namely, consider the solution of the equation $L_\nu \psi = \psi$ which behaves as $x^{\nu+\frac{1}{2}}$ at 0. This solution is $x^{\frac{\nu}{2}} J_\nu(x)$. Thus the solution of $L_\nu \psi = \lambda^2 \psi$ with such behavior at 0, up to scaling, is $x^{\frac{\nu}{2}} J_\nu(\lambda x)$. So the eigenvalues of $L_\nu$ are squared zeros of the Bessel function $J_\nu(x)$.

On the other hand, let us compute the Green function of $L_\nu$. This is also standard. We need to find the solution of the equation $L_\nu^2 g(x, y) = \delta(x - y)$ with the above boundary conditions. So $g(x, y) = b(y)(1 - y^{-2\nu})x^{\nu+\frac{1}{2}}$ when $x < y$ and $g(x, y) = b(y)(x^{\nu+\frac{1}{2}} - x^{-\nu+\frac{1}{2}})$ when $x > y$ with gluing condition $b(y) = -\frac{y^{\nu+\frac{1}{2}}}{2\nu}$. Thus

$$g(x, y) = \frac{1}{2\nu} (\max(x, y)^{-2\nu} - 1)(xy)^{\nu+\frac{1}{2}}.$$

So we have

$$\tilde{A}_\infty(2\nu)^{-1} = L_\nu.$$
where \( \nu = -\frac{ds}{ds+1} \). This gives another way to see that eigenvalues of \( L_\nu \) are squared zeros of the Bessel function \( J_\nu(x) \).

**Remark 5.10.** The differential operator \( L_\nu \) is the continuous limit of the difference operator \( L_\nu \) considered in the previous subsection.

### 5.4. The case when \( Z_0 \) is a constant

Assume now that \( Q \in \mathbb{Z}_p[x] \) and the reduction \( \overline{Q} \) of \( Q \) modulo \( p \) also has degree \( d \). In this case the function \( Z_0(Q, \chi, s, z) \) defined by (2.2), (2.3) is constant as a function of \( z \), i.e.,

\[
Z_0(Q, \chi, s, z) = \zeta(Q, \chi, s) - \delta_{1, \chi}.
\]

The case \( \chi \neq 1 \) is trivial, so we now consider the case \( \chi = 1 \). For brevity we will denote \( \zeta(Q, 1, s) - 1 \) by \( \beta = \beta(s) \). The case when \( \beta(s) = 0 \) has already been discussed in Subsection 5.1, so we assume that \( \beta(s) \neq 0 \). For instance, if \( Q \) has \( m \) zeros in \( \mathbb{F}_p \) all of which are simple then by the computation in Example 2.2 we have

\[
\beta(s) = -m \frac{p^{-1}(1 - p^{-s})}{(1 - p^{-1})(1 - p^{-s-1})}.
\]

We have

\[
R(z) = \frac{1}{1 - z} - \frac{1}{1 - p q z} + \beta(1 - a z)(1 - a^{-1} p q z)
\]

for \( a \) found from the quadratic equation

\[
\beta(1 - a)(1 - a^{-1} p q) = 1 - p q.
\]

Thus eigenvectors of \( A_{P, s, 1} \) are solutions of the \( q \)-difference equation

\[
\lambda F(z) = \beta(1 - a z)(1 - a^{-1} p q z) \frac{F(z)}{(1 - z)(1 - p q z)} + F(p^{-1}) \frac{1}{1 - p q z}.
\]

Setting \( z = 0 \), we have

\[
(\lambda - \beta) F(0) = F(p^{-1}).
\]

First consider the case when \( F(p^{-1}) \neq 0 \). Then \( F(0) \neq 0 \), so setting \( F(0) = 1 \), we get \( \lambda - \beta = F(p^{-1}) \). Thus, using that \( \lambda \neq \beta \), we have

\[
F(z) = \frac{1 - \beta \lambda^{-1}}{1 - p q z} + \beta \lambda^{-1} \frac{(1 - a z)(1 - a^{-1} p q z)}{(1 - z)(1 - p q z)} F(q z).
\]

Hence, iterating, we obtain

\[
F(z) = \frac{1 - \beta \lambda^{-1}}{1 - p q z} + \beta \lambda^{-1} \frac{(1 - a z)(1 - a^{-1} p q z)}{(1 - z)(1 - p q z)} \frac{1 - \beta \lambda^{-1}}{1 - p q^2 z} + \beta^2 \lambda^{-2} \frac{(1 - a z)(1 - a q z)(1 - a^{-1} p q z)}{(1 - z)(1 - q z)(1 - p q z)} \frac{1 - \beta \lambda^{-1}}{1 - p q^3 z} + \ldots
\]
Substituting $z = p^{-1}$, we have

$$1 = \lambda^{-1} \frac{1}{1 - q} + \beta \lambda^{-2} \frac{(1 - ap^{-1})(1 - a^{-1}q)}{(1 - p^{-1})(1 - q)} \frac{1}{1 - q^2} + \beta^2 \lambda^{-3} \frac{(1 - ap^{-1})(1 - ap^{-1}q)(1 - a^{-1}q)(1 - a^{-1}q^2)}{(1 - p^{-1})(1 - p^{-1}q)(1 - q)(1 - q^2)} \frac{1}{1 - q^3} + ...$$

Since $\beta = \frac{1 - pq}{(1 - a)(1 - a^{-1}pq)} = -\frac{1 - p^{-1}q^{-1}}{(1 - a^{-1})(1 - ap^{-1}q^{-1})}$, this can be written as

$$2 \phi_1(ap^{-1}q^{-1}, a^{-1}; p^{-1}q^{-1}; q, \beta \lambda^{-1} a^{-1}) = 0.$$  

So we see that the eigenvalues of $A$ are the solutions of the equation (5.3).

It remains to consider the case $F(p^{-1}) = 0$. In this case we get

$$\lambda F(z) = \frac{\beta(1 - az)(1 - a^{-1}pqz)}{(1 - z)(1 - pqz)} F(qz).$$

Setting $z = p^{-1}q^{k-1}$, $k \geq 1$, we see that $F(p^{-1}q^{k-1}) = 0$ implies $F(p^{-1}q^k) = 0$ unless $ap^{-1}q^{k-1} = 1$ or $a^{-1}q^k = 1$. If $F(p^{-1}q^k) = 0$ for all $k \geq 0$ then $F = 0$, so we see that we must have $a = pq^{-k+1}$ or $a = q^k$ for some $k \geq 1$. In this case

$$\beta = \frac{1 - pq}{(1 - q^k)(1 - pq^{-k+1})},$$

and we have eigenvalues $\beta q^j$, $j \geq 0$, whose eigenvectors have the form $z^j + O(z^{j+1})$, in addition to $k$ eigenvalues which solve equation (5.3) (as the $q$-hypergeometric function on the left hand side of this equation specializes to a polynomial of degree $k$).

So in this case the spectrum consists of two different parts, finite and infinite. The infinite part is fairly trivial ($\beta q^j$, $j \geq 0$), while the finite part is more interesting and consists of the numbers $\beta \gamma_i^{-1}$, where $\gamma_1, ..., \gamma_k$ are the algebraic functions of $q$ which are zeros of the $k$-th little $q$-Jacobi polynomial $p_k(q^{-1}; x; p^{-1}q^{-2}, 1; q)$ ([KoS]). If $s \in \mathbb{Q}$, these are algebraic numbers.

We note that as $k \rightarrow \infty$ (i.e., $\beta \rightarrow 0$), we find ourselves in the situation of Proposition 5.1 for $r = d$. This recovers [KSw], Proposition A1, stating that the little $q$-Jacobi polynomials can be degenerated to the $q$-Bessel function $J(a; q, u)$.

These two cases ($F(p^{-1}) \neq 0$ and $F(p^{-1}) = 0$) can be nicely unified as follows. Recall that the function $2 \phi_1$ for generic parameters has at most simple poles at $1, q^{-1}, q^{-2}, ...$ and no other singularities. Thus the function

$$\widetilde{2 \phi_1}(a, b, c, q, u) := 2 \phi_1(a, b, c, q, u)(u; q)_\infty$$

is entire (it is obtained from $2 \phi_1$ by “clearing denominators”).

**Proposition 5.11.** The characteristic function of $A_{P,s,1}$ is

$$h_{P,s,1}(u) = \tilde{2 \phi_1}(ap^{-1}q^{-1}, a^{-1}; p^{-1}q^{-1}; q, \beta u).$$
Thus the eigenvalues of $A_{P,s,1}$ are the solutions of the equation
\[ 2\tilde{\phi}_1(ap^{-1}q^{-1},a^{-1},p^{-1}q^{-1};q,\beta\lambda^{-1}) = 0. \]

**Example 5.12.** Let $P(x,y) = x - y$. Then $d = m = 1$ and
\[ \beta = \frac{1 - pq}{(1 - p)(1 - q)}. \]

So the equation for $a$ takes the form
\[ (1 - a)(1 - ap^{-1}pq) = (1 - p)(1 - q), \]
which gives $a = p$ or $a = q$. We have
\[ 2\phi_1(q^{-1},p^{-1};p^{-1}q^{-1};q,u) = 1 - \frac{1 - p}{1 - pq}u. \]

(On the first little $q$-Jacobi polynomial $p_1$). Thus for the eigenvalues we get the equation
\[ \left( 1 - \frac{1 - p}{1 - pq} \beta\lambda^{-1} \right) (\beta\lambda^{-1};q)_\infty. \]

So the eigenvalues are
\[ (5.4) \quad \lambda_0 = \frac{1}{1 - q}, \quad \lambda_{k+1} = \frac{1 - pq}{(1 - p)(1 - q)} q^k, \quad k \geq 0. \]

In fact, this is easy to see in a different way: the operator $A$ is the convolution with the function $\frac{|x|_s}{1 - p}$ on $\mathbb{Z}_p$, so its eigenvalues are the values of the Fourier transform of this function, which are easily shown to be exactly the list (5.4).

5.5. **The case $\chi \neq 1$ with $Z_0$ having a first order pole at 0 and $\infty$.** Consider now the case $\chi \neq 1$ with $Z_0$ having the form $z^{-1}(1 - az)(1 - bz)$. This happens, for instance, in the case
\[ P(x,y) = x^d - p^{-1}xyP_*(x,y) + y^d, \]
where $P_*(x,y) \in \mathbb{Z}_p[x,y]$ is a homogeneous polynomial of degree $d - 2$ whose coefficients of $x^{d-2}$ and $y^{d-2}$ have norm 1. In this case
\[ Q(x) = 1 - p^{-1}xQ_*(x) + x^d, \]
where $Q_*(x) = P_*(x,1)$. Then we have
\[ Z_0(P,\chi,s,z) = \zeta(1 - x,\chi,s)(p^{-1}q z^{-1} + z^{-1}) + p^s\zeta(Q_*,\chi,s). \]

Let
\[ \gamma = \gamma(s) := \frac{p^s\zeta(Q_*,\chi,s)}{\zeta(1 - x,\chi,s)} \]
and let
\[ z^{-1} + \gamma + p^{-1}q^{-1}z = z^{-1}(1 - az)(1 - bz). \]

Then after rescaling $z$ the eigenvalue equation for $A_{P,s,\chi}$ takes the form
\[ (5.5) \quad \lambda F(z) = (z^{-1}(1 - az)(1 - bz)F(qz))_. \]
So let us solve (5.5) for general $a, b$.

We can rewrite (5.5) in the form

$$\lambda F(z) = z^{-1}(1 - az)(1 - bz)F(qz) - z^{-1}F(0).$$

To solve (5.6), consider the deformed equation

$$\lambda F(z) = -\frac{t^{-1}(1 - az)(1 - bz)F(qz)}{1 - t^{-1}z} + \frac{t^{-1}(1 - at)(1 - bt)F(qt)}{1 - t^{-1}z}$$

which degenerates to (5.5) as $t \to 0$, and let us solve (5.7). We have

$$(1 - at)(1 - bt)F(qt) = (1 + 
\lambda F(z)).$$

Assume first that $F(qt) \neq 0$. Then $F(0) \neq 0$, and setting $F(0) = 1$, we get

$$F(z) = -\lambda^{-1}t^{-1}(1 - az)(1 - bz)\frac{1 + \lambda^{-1}t^{-1}}{1 - t^{-1}z}F(qz) + \frac{1 + \lambda^{-1}t^{-1}}{1 - t^{-1}z}.$$

Iterating this, we obtain

$$F(z) = \frac{1 + \lambda^{-1}t^{-1}}{1 - t^{-1}z} - \lambda^{-1}t^{-1}\frac{1}{1 - t^{-1}z} + \frac{1 + \lambda^{-1}t^{-1}}{1 - t^{-1}z}$$

Thus, setting $z = qt$, we have

$$\frac{1 + t\lambda}{(1 - at)(1 - bt)} = F(qt) = \frac{1 + \lambda^{-1}t^{-1}}{1 - q} - \lambda^{-1}t^{-1}\frac{1}{1 - q}$$

$$+ \lambda^{-2}t^{-2}\frac{(1 - aqt)(1 - aq^2t)(1 - bqt)(1 - bq^2t) + \lambda^{-1}t^{-1}}{1 - q + \lambda^{-1}t^{-1}}$$

So we get that the eigenvalues $\lambda$ are the solutions of the equation

$$2\phi_1(at, bt; 0; -\lambda^{-1}t^{-1}) = 0.$$
Now let us send $c$ to 0 and use that
\[
\lim_{c \to 0} 2\phi_1(\alpha, \frac{c}{c}; \gamma; q, cz) = \phi_1(\alpha; \gamma; q, \beta z)
\]
(see [GR]). This yields
\[
2\phi_1(at, bt; 0; q, -ut^{-1}) = \frac{(bt; q)_{\infty} \theta(au; q)}{(\frac{b}{a}; q)_{\infty} \theta(au, q)} \phi_1(au; \frac{a}{b}; q, -\frac{q^2}{bu}) + \frac{(at; q)_{\infty} \theta(bu; q)}{(\frac{a}{b}; q)_{\infty} \theta(bu, q)} \phi_1(bt; \frac{b}{a}; q, -\frac{q^2}{au}).
\]
Thus the equation $2\phi_1(at, bt; 0; q, -ut^{-1}) = 0$ yields
\[
\frac{(bt; q)_{\infty} \theta(au; q)}{(\frac{b}{a}; q)_{\infty}} \phi_1(au; \frac{a}{b}; q, -\frac{q^2}{bu}) + \frac{(at; q)_{\infty} \theta(bu; q)}{(\frac{a}{b}; q)_{\infty}} \phi_1(bt; \frac{b}{a}; q, -\frac{q^2}{au}) = 0
\]
Now sending $t$ to 0, we get

**Proposition 5.13.** The eigenvalues of $A_{P,s,\lambda}$ are the solutions of the equation
\[
E(a, b; q, \lambda^{-1}) = 0,
\]
where
\[
E(a, b; q, u) := \frac{\theta(au; q)J(\frac{aq}{b}; q, -\frac{q^2}{bu})}{(\frac{b}{a}; q)_{\infty}(q; q)_{\infty}} + \frac{\theta(bu; q)J(\frac{bq}{a}; q, -\frac{q^2}{au})}{(\frac{a}{b}; q)_{\infty}(q; q)_{\infty}}.
\]
Indeed, the case $F(qt) = 0$ is irrelevant for us since then we must have $\lambda = t^{-1}q^m$ for some $m$, and these eigenvalues, if present, go to $\infty$ as $t \to 0$, since $m$ has to remain bounded.

As before, by growth considerations Proposition 5.13 implies that
\[
\det(1 - uA_{P,s,\lambda}) = Cu^m E(a, b; q; u)
\]
for some $C \neq 0, m \in \mathbb{Z}$. In particular, this means that $E(a, b; q; u)$ is meromorphic at 0. Note also that the two summands in $E$ form a basis in the space of solutions of the $q$-Bessel difference equation transformed by $u \mapsto u^{-1}$ over the field of elliptic functions:
\[
abqu^2 K(q^2 u) - (1 + (a + b)u)K(qu) + K(u) = 0.
\]
This equation is irregular at 0, but it has a single, up to scaling, solution $\tilde{E}(u)$ meromorphic at 0, which can be found by the power series method. Moreover, it is easy to see that $\tilde{E}(u)$ is in fact holomorphic at 0 and does not vanish there (as equation (5.7) degenerates to the equation $K(qu) = K(u)$ near $u = 0$). So up to scaling we must have $\tilde{E}(u) = E(u)$, hence $m = 0$. Thus we get

**Corollary 5.14.** The function $E(a, b; q, u)$ extends holomorphically to the origin with $E(a, b; q, 0) \neq 0$, and
\[
\det(1 - uA_{P,s,\lambda}) = \frac{E(a, b; q, u)}{E(a, b; q, 0)}.
\]
In fact, it is shown in [Mo], Theorem 1 that $E(a, b; q, 0) = 1$. Thus one has 
\[ \det(1 - uA_{P,s,E}) = E(a, b; q, u) \].

6. The non-homogeneous case

6.1. The trace class property. Now let $P \in \mathbb{Q}_p[x, y]$ be any polynomial (not necessarily homogeneous). Let us first show that if $\text{Re}(s)$ is sufficiently large then the operator $A_{P,s}$ is trace class. First recall the following well known lemma.

**Lemma 6.1.** Let $A$ be a bounded operator on a Hilbert space $H$ and let $A_k \to A$ be a sequence of bounded operators on $H$ of ranks $\leq r_k$, $k \geq 0$, such that $r_k - 1 \leq 2r_k$. If $M := \sum_{k \geq 1} r_k \| A_{k-1} - A \| < \infty$ then $A$ is trace class.

**Proof.** Recall that for any bounded operator $B$ of rank $r$ we have $\text{Tr}(B) \leq r \| B \|$.

Now, we have 
\[ A = \lim_{k \to \infty} A_k = A_0 + (A_1 - A_0) + \ldots + (A_k - A_{k-1}) + \ldots, \]
an absolutely convergent series with respect to the operator norm. So given a finite rank operator $C$, we have
\[ \text{Tr}(AC) = \text{Tr}(A_0C) + \sum_{k \geq 1} \text{Tr}((A_k - A_{k-1})C). \]
Now, the rank of $A_k - A_{k-1}$ is at most $r_k + r_{k-1} \leq 2r_k$, while its norm is at most $\| A_{k-1} - A \| + \| A_k - A \|$. Thus
\[ |\text{Tr}(AC)| \leq (\| A_0 \| + \sum_{k \geq 1} 2r_k (\| A_{k-1} - A \| + \| A_k - A \|)) \| C \|. \]
Hence
\[ \text{Tr}|A| \leq \| A_0 \| + \sum_{k \geq 1} 2r_k (\| A_{k-1} - A \| + \| A_k - A \|) \leq \| A_0 \| + 4M. \]

Now let $\gamma_P$ be the log-canonical threshold of $P$ ([Mu]), i.e.,
\[ \gamma_P = \sup(\gamma : \text{vol}(|P(x, y)| < t) = O(t^\gamma), t \to 0). \]

**Proposition 6.2.** If $\text{Re}(s) > 1 - \frac{1}{2}\gamma_P$ then $A_{P,s}$ is trace class.

**Proof.** Let $\text{Re}(s) = \rho > 0$. Let $A_k$ be the integral operator on $\mathbb{Z}_p$ with kernel $D_k(x, y) := \frac{1}{1 - p^{-\rho}} \max(p^{-k}, |P(x, y)|)^{s}$. It is clear that $\| A - A_k \|$ is dominated by its Hilbert-Schmidt norm, hence
\[ \| A - A_k \| \leq Cp^{-k(\rho + \frac{1}{2}\gamma_P)}. \]
for large $k$. On the other hand, we have 
\[ P(x, y) - P(x', y') = P_1(x, y)(x - x') + P_2(x', y)(y - y') \]
for some polynomials $P_1, P_2$, so
\[ |P(x, y) - P(x', y')| < p^\ell \max(|x - x'|, |y - y'|) \]
for some $\ell$. This implies that if $|P(x, y)| = p^{-k}$ and $|x' - x|, |y' - y| \leq p^{-k-\ell}$ then $|P(x', y')| = |P(x, y)|$. Thus if $|x' - x|, |y' - y| \leq p^{-k-\ell}$ then $D_k(x, y) = D_k(x', y')$. We conclude that the rank of $A_k$ is $\leq p^{k+\ell}$. Thus by Lemma 6.1 $A$ is trace class if $\rho > 1 - \frac{1}{2} \gamma_P$. 

For example, if $P$ has zeros in $\mathbb{Z}_p^2$ but they are all smooth points of the curve $P(x, y) = 0$ then $\gamma_P = 1$ and $A_{P, s}$ is trace class whenever $\text{Re}(s) > \frac{1}{2}$.

6.2. An example. Proposition 6.2 allows us to define the characteristic function $h_{P, s}(u)$ for a general polynomial $P$. Computing this function, however, seems difficult in general, as we no longer have a decomposition into the sectors $H_\chi$, and even when we do, the answer goes beyond the theory of $q$-hypergeometric functions.

To demonstrate what may happen in the non-homogeneous case, consider the polynomial
\[ P(x, y) = x^{2d} + y^2, \]
where $d > 1$ and $p = 4k + 3$. We have $|P(x, y)| = \max(|x|^d, |y|)$. Thus the corresponding operator $A_{P, s}$ still commutes with $\mathbb{Z}_p^\times$ and $A_{P, s, \chi} = 0$ if $\chi \neq 1$, while the matrix of $A = A_{P, s, 1}$ is
\[ a_{mn} = p^{-\frac{m+n}{2} - 2s \min(m, dn)}. \]

For simplicity assume that $s \geq 0$. In terms of analytic functions $F(z)$ we get
\[ (AF)(z) = \sum_{n \geq 0} \sum_{m > dn} z^m p^{-2dn s - \frac{m}{2}} f_n + \sum_{n \geq 0} \sum_{m \leq dn} z^m p^{-2ms - \frac{m}{2}} f_n = \frac{(1 - p^{-2s})z}{(1 - z)(1 - p^{-2sz})} F(p^{2ds - 1} z^d) + \frac{F(p^{-1})}{1 - p^{-2sz}}. \]

So the eigenvalue equation looks like
\[ \lambda F(z) = \frac{(1 - p^{-2s})z}{(1 - z)(1 - p^{-2sz})} F(p^{2ds - 1} z^d) + \frac{F(p^{-1})}{1 - p^{-2sz}}. \]
Thus $\lambda F(0) = F(p^{-1})$. If $F(p^{-1}) = 0$ then $F = 0$, so $F(p^{-1}) \neq 0$, hence $F(0) \neq 0$, $\lambda \neq 0$. So we may set $F(0) = 1$, which gives $\lambda = F(p^{-1})$. Hence we get
\[ F(z) = \frac{1}{1 - p^{-2sz} z} + \lambda^{-1} \frac{(1 - p^{-2s})z}{(1 - z)(1 - p^{-2sz})} F(p^{2ds - 1} z^d). \]

Let $s > -\frac{1}{2d}$ and $q = p^{-\frac{2s+1}{d-1}}$. Then, replacing $z$ by $p^{-1}q^{-d}z$, we have
\[ F(p^{-1}q^{-d}z) = \frac{1}{1 - q^{-1}z} + \lambda^{-1} \frac{(p^{-1}q^{-d} - q^{-1})z}{(1 - p^{-1}q^{-d}z)(1 - q^{-1}z)} F(p^{-1}q^{-d}z). \]
So setting $G(z) := F(p^{-1}q^{-d}z)$ and $u = \lambda^{-1}$, we get

\begin{equation}
G(z) = \frac{1}{1 - q^{-1}z} + u \frac{(p^{-1}q^{-d} - q^{-1})z}{(1 - p^{-1}q^{-d}z)(1 - q^{-1}z)} G(z^d).
\end{equation}

This shows that $G$ is a $d$-Mahler series (of degree 2) in the sense of \[BCCD\], which is a generalization of the notion of $d$-automatic and $d$-regular series (see \[AS\]).

Iterating (6.1), for any $u$ we get a solution

\[
G(z, u) := \frac{1}{1 - q^{-1}z} + u \frac{(p^{-1}q^{-d} - q^{-1})z}{(1 - p^{-1}q^{-d}z)(1 - q^{-1}z)(1 - q^{-1}z^d)} + \frac{(p^{-1}q^{-d} - q^{-1})^2 z^d}{(1 - p^{-1}q^{-d}z)(1 - p^{-1}q^{-d}z^d)(1 - q^{-1}z)(1 - q^{-1}z^d)(1 - q^{-1}z^d)^2} + \ldots;
\]

Namely, it is clear that this series converges (uniformly on every compact set of points $(z, u)$) in the region $|z| < \min(|q|, |pq^d|)$. Now set $z = q^d$, which belongs to this region. So setting $\beta := u(q^{-1} - p^{-1}q^{-d})$, we obtain

\[
1 = \frac{\beta q}{(1 - p^{-1}q^{-d})(1 - q^{-d-1})} - \frac{\beta^2 q^{d+1}}{(1 - p^{-1}q^{-d})(1 - p^{-1})(1 - q^{d-1})(1 - q^{d-1})} + \frac{\beta^3 q^{2d+1}}{(1 - p^{-1}q^{-d})(1 - p^{-1})(1 - p^{-1}q^{d-1})(1 - q^{d-1})(1 - q^{d-1})} + \ldots.
\]

Define the “automatic Pochhammer symbol”

\[
[a; q]_{d,n} = (1 - aq)(1 - aq^d)(1 - aq^{d^2})\ldots(1 - aq^{d^{n-1}}).
\]

Let

\[
K(a; q, u) = \sum_{n=0}^{\infty} (-1)^n q^{\frac{d^n-1}{d-1}} u^n [a; q]_{d,n}[q^{-1}; q^d]_{d,n}
\]

(an entire function). We obtain

**Proposition 6.3.** We have

\[
\det(1 - uA_{P,s,1}) = K(p^{-1}q^{-d}; q, (q^{-1} - p^{-1}q^{-d})u).
\]

Thus the eigenvalues $A_{P,s,1}$ are the solutions of the equation

\[
K(p^{-1}q^{-d}; q, (q^{-1} - p^{-1}q^{-d})\lambda^{-1}) = 0.
\]

**Proof.** Indeed, the right hand side is an entire function of order 0 since its Taylor coefficients are rapidly decaying. \qed
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