Arithmetic cohomology over finite fields and special values of $\zeta$-functions

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Summary. We construct cohomology groups with compact support $H^i_c(X_{ar}, \mathbb{Z}(n))$ for separated schemes of finite type over a finite field, which generalize Lichtenbaum’s Weil-etale cohomology groups for smooth and projective schemes. In particular, if Tate’s conjecture holds, and rational and numerical equivalence agree up to torsion, then the groups $H^i_c(X_{ar}, \mathbb{Z}(n))$ are finitely generated, form an integral model of $l$-adic cohomology with compact support, and admit a formula for the special values of the $\zeta$-function of $X$.

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1 Introduction

In [19], Lichtenbaum introduced the Weil-etale topology in order to produce finitely generated cohomology groups for varieties over finite fields which are related to special values of zeta-functions. In [6], we calculated the precise relationship between Weil-etale cohomology groups and etale cohomology groups. In particular, if one assumes Tate’s conjecture on the bijectivity of the cycle map and Beilinson’s conjecture that rational and numerical equivalence agree up to torsion, then for smooth and projective varieties, the Weil-etale cohomology groups of the motivic complex have all properties expected by Lichtenbaum, and allow a new interpretation of results of Kahn [15]. However, for non-smooth or non-proper schemes, the Weil-etale cohomology groups are not finitely generated in general. In this paper, we use ideas of Voevodsky to construct a modified version $H^i_c(X_{ar}, \mathbb{Z}(n))$ of Weil-etale cohomology which we call arithmetic cohomology (with compact support). Arithmetic cohomology groups are expected to be finitely generated, and related to special values of $\zeta$-functions for every separated scheme of finite type over a finite field.

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To construct arithmetic cohomology groups, we first define an intermediate Grothendieck topology, called eh-topology, which is generated by etale covers and abstract blow-ups. The eh-topology bears the same relationship to etale cohomology as Voevodsky’s cdh-topology to the Nisnevich topology. The first advantage of the eh-topology is that cohomology groups with compact support $H^i_{\text{eh}}(X_{\text{eh}}, F)$ can be defined independently of the choice of a compactification. An important tool to calculate eh-cohomology groups is the following

**Theorem 1.1**

a) For a separated scheme of finite type $X$ over a field of characteristic not dividing $m$, $H^i(X_{\text{et}}, \mu^\otimes m) \cong H^i(X_{\text{eh}}, \mu^\otimes m)$.

b) Under resolution of singularities for schemes up to dimension $d$, the etale- and eh-hypercohomology of the motivic complex agree for every smooth scheme $X$ of dimension at most $d$.

The eh-cohomology groups can be used to give a generalization of Tate’s conjecture.

**Conjecture 1.2**

For every separated scheme of finite type $X$ over $\mathbb{F}_q$, and every $n \in \mathbb{Z}$, the cycle map from eh-motivic cohomology to the Galois fixed part of $l$-adic cohomology of $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$,

$$H^i_c(X_{\text{eh}}, \mathbb{Z}(n)) \otimes \mathbb{Q}_l \to H^i_c(\bar{X}_{\text{et}}, \mathbb{Q}_l(n))^{\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)}$$

is an isomorphism, and the Galois-module $H^i_c(\bar{X}_{\text{et}}, \mathbb{Q}_l(n))$ is semi-simple at the eigenvalue $1$.

A homological version has been considered by Jannsen [13]. If resolution of singularities and the aforementioned conjectures of Tate and Beilinson (involving only smooth and projective schemes) hold, then Conjecture 1.2 holds.

Arithmetic cohomology (with compact support) for separated schemes of finite type over the finite field $\mathbb{F}_q$ is defined by applying Lichtenbaum’s idea [19] of replacing the Galois group $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ by the Weil-group $G$ to eh-cohomology. More precisely, a Weil-eh-sheaf on $X$ is an eh-sheaf on $\bar{X}$ together with an action of $G$, and Weil-eh cohomology (with compact support) $H^i_c(X_{\text{Wh}}, F)$ of the sheaf $F$ is defined as the cohomology of the complex $R\Gamma(G, R\Gamma_c(\bar{X}_{\text{eh}}, F))$. All results on the relationship between Weil-etale cohomology and etale cohomology proved in [6] carry over to the present situation.

The groups $H^i_c(X_{\text{ar}}, \mathbb{Z}(n))$ are defined as the Weil-eh cohomology groups with compact support of the motivic complex $\mathbb{Z}(n)$ of Suslin-Voevodsky (we set $\mathbb{Z}(n) = \text{colim}_{p|n} \mu^\otimes n[-1]$ for $n < 0$). Arithmetic cohomology groups are expected to satisfy the following

**Conjecture 1.3**

For all $n \in \mathbb{Z}$ and schemes $X$ separated and of finite type over $\mathbb{F}_q$, the groups $H^i_c(X_{\text{ar}}, \mathbb{Z}(n))$ are finitely generated, vanish for almost all $i$, and form an integral model for $l$-adic cohomology with compact support for all $l \neq p$,

$$H^i_c(X_{\text{ar}}, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \cong H^i_c(X_{\text{et}}, \mathbb{Z}_l(n)).$$
For \( l = p \), we get a new theory which agrees with logarithmic de Rham-Witt cohomology for smooth and proper schemes. Arithmetic cohomology groups should be related to special values of zeta-functions in the following way [19]:

**Conjecture 1.4** The weighted alternating sum of the ranks equals the order of the zeta-function

\[
\rho_n := \sum_i (-1)^i \cdot \operatorname{rank} H^i_c(\mathcal{X}_{ar}, \mathbb{Z}(n)) = \operatorname{ord}_{s=n} \zeta(\mathcal{X}, s),
\]

and for \( s \mapsto n \),

\[
\zeta(\mathcal{X}, s) \sim \pm (1 - q^{n-s})^{\rho_n} \cdot \chi(H_*^c(\mathcal{X}_{ar}, \mathbb{Z}(n)), e) \cdot q^{\chi(n)}.
\]

Here \( e \in H^1((\mathbb{F}_q)_{ar}, \mathbb{Z}(0)) \cong \mathbb{Z} \) is a generator, \( \chi(H_*^c(\mathcal{X}_{ar}, \mathbb{Z}(n)), e) \) is the Euler-characteristic of the complex

\[
\cdots \rightarrow H^{i-1}_c(\mathcal{X}_{ar}, \mathbb{Z}(n)) \xrightarrow{\cup e} H_i^c(\mathcal{X}_{ar}, \mathbb{Z}(n)) \xrightarrow{\cup e} H^{i+1}_c(\mathcal{X}_{ar}, \mathbb{Z}(n)) \rightarrow \cdots,
\]

and \( \chi(n) = \sum_{0 \leq i \leq n} (-1)^{i+j}(n-i) \cdot \dim H^j_e(\mathcal{X}_{et}, \Omega^j) \) is a generalization of the correcting factor of Milne [21].

We show several implications between the above conjectures, and the following

**Theorem 1.5** Conjectures 1.2, 1.3 and 1.4 hold in the following instances:

a) For every \( n \), if \( \mathcal{X} \) is a curve.

b) For \( n \leq 0 \), if \( \mathcal{X} \) is a scheme of dimension at most \( d \), and resolution of singularities for schemes of dimension at most \( d \) exists.

c) For every \( \mathcal{X} \) and every \( n \), if the conjectures of Tate and Beilinson hold, and resolution of singularities exists.

Finally, we give an example showing that essentially every statement given above is incorrect if one uses the etale topology.

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## 2 The etale h-topology

We introduce a Grothendieck topology which is finer than the etale topology, and has several advantages over the etale topology. For instance, there are
well-defined cohomology groups with compact support, and there is a long exact sequence of cohomology groups for blow-ups. On the other hand, for locally constant torsion coefficients, one gets the same cohomology groups as for the etale topology.

We use the term Grothendieck topology on a subcategory $C$ of the category of schemes in the sense of [1, II 1]. In particular, any morphism $Y \to X$ that can be dominated by a covering $U \to X$ is itself a covering. If we are given for every object $X$ of $C$ a class of morphisms with target $X$, then the intersection of all topologies containing these morphisms is again a topology, called the Grothendieck topology generated by these morphisms [1, II 1.1.6].

**Definition 2.1** The etale h-topology (or short eh-topology) on a suitable subcategory of the category of schemes is the Grothendieck topology generated by the following coverings:

1) etale coverings
2) abstract blow-ups: Assume we have a cartesian square

\[
\begin{array}{ccc}
Z' & \xrightarrow{i'} & X' \\
\downarrow f' & & \downarrow f \\
Z & \xrightarrow{i} & X,
\end{array}
\]

where $f$ is proper, $i$ is a closed embedding, and $f$ induces an isomorphism $X' - Z' \sim X - Z$. Then $(X' \xrightarrow{f'} X, Z \xrightarrow{i} X)$ is a covering.

The definition is motivated by Voevodsky’s cdh-topology, which is generated by Nisnevich covers and abstract blow-ups. For singular schemes, the cdh-topology has better properties than the Nisnevich topology, and similarly the eh-topology has better properties than the etale topology.

**Example.** Every scheme $X$ is covered by its irreducible components, $X^\text{red} \to X$ is a covering, and for every blow up $X'$ of $X$ with center $Z$, $(X' \to X, Z \to X)$ is a covering.

**Remark.** It is tempting to use the h-topology of Voevodsky instead of the eh-topology, in order to use alterations of de Jong instead of resolution of singularities. However, if one does so, then one loses the mod $p$ information. Similarly, Voevodsky’s method to define motivic cohomology groups as Ext-groups in the derived category of mixed motives does not give well-behaved cohomology groups for the etale topology (the resulting cohomology groups will be $p$-divisible). Even with rational coefficients we do not know how to prove the analog of Proposition 4.2 for an alteration, and hence we cannot prove Theorem 4.3 for finer topologies than the eh-topology.

We recall the following facts from [23]:

**Lemma 2.2** a) Every proper morphism $p : X' \to X$, such that for every point $x \in X$ there is a point in $p^{-1}(x)$ with the same residue field, is an eh-covering.
b) Every abstract blow-up \((X' \to X, Z \to X)\) has a refinement \((\tilde{X} \to X, Z \to X)\) such that every irreducible component of \(\tilde{X}\) has dimension not larger than the dimension of \(X\).

**Proof.** a) \[23\] Lemma 5.7

b) We can replace \(X'\) by the disjoint union of its irreducible components. We claim that removing all irreducible components of dimension larger than the dimension of \(X\) from \(X'\) gives a refinement \((\tilde{X} \to X' \to X, Z \to X)\).

Indeed, let \(T\) be one of the irreducible components, and let \(\eta\) be the generic point of \(T\). If \(\eta\) maps to \(X - Z\) under \(f\), then \(T\) is birational to a component to \(X\), hence \(\dim T \leq \dim X\). If \(\eta\) maps to \(Z\), then the map \(T \to X\) factors through \(Z\), and we can remove \(T\), because by a) the resulting scheme is still a covering. \(\Box\)

A covering as in a) is called proper eh-covering. If \(X\) is integral, then a proper eh-covering which is an isomorphism over a neighborhood of the generic point of \(X\) is called a proper birational eh-covering. Every proper eh-cover of an integral scheme \(X\) admits a proper birational refinement by the argument in \[23\] Lemma 5.7.

**Proposition 2.3** Every eh-cover of \(X\) has a refinement of the form

\[
\{U_i \to X' \to X\}_{i \in I},
\]

where \(\{U_i \to X'\}_{i \in I}\) is an etale cover, and \(X' \to X\) is a proper eh-cover.

**Proof.** \[23\] Prop. 5.9 \(\Box\)

We now fix a perfect field \(k\), let \(\text{Sch}/k\) be the category of separated schemes of finite type over \(k\), and \(\text{Sm}/k\) the full subcategory of smooth schemes. For \(d \in \mathbb{N} \cup \infty\), we denote by \(\text{Sch}^d/k\) and \(\text{Sm}^d/k\) the full subcategory consisting of schemes of Krull dimension at most \(d\) of \(\text{Sch}/k\) and \(\text{Sm}/k\), respectively. By Lemma \[23\] b), we can consider the eh-topology on \(\text{Sch}^d/k\). We write \((\text{Sch}^d/k)_{\text{eh}}\) for the category \(\text{Sch}^d/k\) equipped with the eh-topology, and \((\text{Sch}^d/k)_{\text{et}}\) for the topos of eh-sheaves on \(\text{Sch}^d/k\). Similarly, \((\text{Sm}^d/k)_{\text{et}}\) is the category of smooth schemes equipped with the etale topology, and \((\text{Sm}^d/k)_{\text{et}}\) is the topos of etale sheaves on \(\text{Sm}^d/k\).

**Definition 2.4** For \(d \in \mathbb{N} \cup \infty\) we denote by \(R(d)\) the strong form of resolution of singularities for varieties up to dimension \(d\), i.e. the following two conditions:

- For every integral separated scheme \(X \in \text{Sch}^d/k\), there is a proper, birational map \(f: Y \to X\) with \(Y \in \text{Sm}/k\).
- For every smooth scheme \(X \in \text{Sm}^d/k\) and every proper birational map \(f: Y \to X\), there is a sequence of blow-ups along smooth centers \(X_n \to X_{n-1} \to \cdots \to X_1 \to X\) such that the composition \(X_n \to X\) factors through \(f\).
If char $k = 0$, $R(\infty)$ holds by Hironaka’s theorem. By Abhyankar, $R(2)$ is known in general, and $R(3)$ is known for algebraically closed fields of characteristic $p > 5$. Condition $R(d)$ implies that every scheme in $\text{Sch}^d/k$ is locally smooth for the eh-topology.

Let $\rho_d : (\text{Sch}^d/k)_{\text{eh}} \to (\text{Sm}^d/k)_{\text{et}}$ be the canonical maps of sites, and for $a \geq b$ let $\iota : (\text{Sch}^a/k)_{\text{eh}} \to (\text{Sch}^b/k)_{\text{eh}}$ and $\sigma : (\text{Sm}^a/k)_{\text{eh}} \to (\text{Sm}^b/k)_{\text{eh}}$ be the canonical morphism of topoi induced by the restriction map.

**Lemma 2.5** Assume that $R(d)$ holds.

a) The functor $\rho_d$ induces a morphism of topoi $\rho : (\text{Sch}^d/k)_{\text{eh}} \to (\text{Sm}^d/k)_{\text{et}}$.

b) For every $a \geq b$, there is a commutative diagram

$$
\begin{array}{ccc}
(\text{Sch}^a/k)_{\text{eh}} & \xleftarrow{\iota^*} & (\text{Sm}^a/k)_{\text{et}} \\
\downarrow \iota & & \downarrow \sigma_* \\
(\text{Sch}^b/k)_{\text{eh}} & \xleftarrow{\iota^*} & (\text{Sm}^b/k)_{\text{et}}
\end{array}
$$

Proof. a) The only point which needs explanation is the left exactness of $\rho^*_d$. On a smooth scheme $S$, the presheaf pull-back $\rho^*_d \mathcal{F}(S)$ agrees with $\mathcal{F}(S)$. Since the system of coverings occurring in the definition of the sheafification functor can be assumed to be filtered by [20, III 2.2 a)], and since coverings by smooth schemes are cofinal in the system of all eh-covers by $R(d)$, the statements follows [20, App. A, Prop. 7]

b) By the same argument as in a), it suffices to know the value of the presheaf pull-back on smooth schemes of dimension at most $a$ in order to calculate $\rho^*_b$ and $\rho^*_a$. But on such a scheme, $\rho^*_b \sigma_* \mathcal{F} = \mathcal{F} = \iota_* \rho^*_a \mathcal{F}$. $\square$

We do not know that the presheaf pull-back $\rho^*_d$ is left exact. This is because equalizers do not exist in the category of smooth schemes, so that the colimit system defining the presheaf pull-back is not filtered, see [3, Rem. 3.7].

By resolution of singularities, every proper birational map to a smooth scheme can be refined by blow-ups along smooth centers. Since a blow-up $X' \to X$ along a smooth center satisfies the hypothesis of Lemma 2.2 a), Proposition 2.3 gives

**Corollary 2.6** Let $X \in \text{Sm}^d/k$ and assume that condition $R(d)$ holds. Then every eh-cover of $X$ has a refinement of the form

$$\{U_i \to X' \to X\},$$

where $\{U_i \to X'\}$ is an etale cover, and $X' \to X$ is a composition of blow-ups along smooth centers.

The following lemma will be applied several times.
Lemma 2.7 (Devisage Lemma) Let $P(X)$ be a property for schemes in $\text{Sch}^d/k$. Assume the following:

i) Condition $R(d)$.

ii) If $Z \subseteq X$ is a closed subscheme of $X$ with open complement $U$, and if $P(\cdot)$ holds for two of the three schemes $X, U$ and $Z$, then it also holds for the third.

iii) $P(X)$ holds for all smooth and projective $X \in \text{Sm}^d/k$.

Then $P(X)$ holds for all $X \in \text{Sch}^d/k$.

Proof. We proceed by induction on the dimension of $X$. Given $X$, we can assume by noetherian induction that $P(Z)$ holds for all closed subschemes $Z$ of $X$, and using property ii) we can reduce to the case that $X$ is integral. By Chow’s lemma, there is a projective scheme $X_1$ and an open subscheme $U_1$ of $X_1$ isomorphic to an open subscheme of $X$. Let $X_2$ be a desingularization of $X_1$, then there is an open subscheme $U_2$ of $X_2$ isomorphic to an open subscheme of $X$. By condition iii) we have $P(X_2)$, and by condition ii) this implies $P(U_2)$ and then $P(X)$.

3 Cohomology for the eh-topology

The usual argument with generators [20, III Lemma 1.3] shows that the categories $(\text{Sch}^d/k)^\sim_{\text{eh}}$ have enough injectives. Given an eh-sheaf $F \in (\text{Sch}^d/k)^\sim_{\text{eh}}$ and $X \in \text{Sch}^d/k$, the cohomology groups $H^i(X_{\text{eh}}, F)$ are defined as the derived functors of the global section functor $F \to \Gamma(X_{\text{eh}}, F)$. For $X \in \text{Sch}/k$, we let $\mathbb{Z}(X)$ be the free presheaf $U \mapsto \mathbb{Z}[\text{Hom}_{\text{Sch}}(U, X)]$ represented by the scheme $X$, and let $\mathbb{Z}_{\text{eh}}(X)$ be its associated eh-sheaf. Sheafification is necessary, because the eh-topology is not subcanonical.

Lemma 3.1 For every $a \geq b$, the canonical morphism of topoi $\iota^*: (\text{Sch}^a/k)^\sim_{\text{eh}} \to (\text{Sch}^b/k)^\sim_{\text{eh}}$ induces an isomorphism of cohomology groups. Moreover, $H^i(X_{\text{eh}}, F) = \text{Ext}^i_{(\text{Sch}^a/k)^\sim_{\text{eh}}}(\mathbb{Z}_{\text{eh}}(X), F)$ for every eh-sheaf $F \in (\text{Sch}^d/k)^\sim_{\text{eh}}$ of abelian groups and $X \in \text{Sch}^d/k$.

Proof. The first statement is proved as in [20, III Prop. 3.1]: Clearly $\iota_*^*$ is exact, and it suffices to show that $F \cong \iota_*^*F$. This is clear on the presheaf level, by definition of the presheaf pull-back, and $\iota_*^*F$ is a sheaf on $(\text{Sch}^d/k)_{\text{eh}}$ by Lemma 2.2 b). The second statement follows because $\text{Hom}_{(\text{Sch}^a/k)^\sim_{\text{eh}}}(\mathbb{Z}_{\text{eh}}(X), F) \cong F(X)$ for every eh-sheaf $F \in (\text{Sch}^d/k)^\sim_{\text{eh}}$ of abelian groups and $X \in \text{Sch}^d/k$. \qed
Proposition 3.2 Every abstract blow-up square \( \square \), gives rise to a long exact sequence of cohomology groups:

\[
\cdots \to H^i(X_{eh}, \mathcal{F}) \xrightarrow{i^*} H^i(Z_{eh}, \mathcal{F}) \oplus H^i(X_{eh}', \mathcal{F}) \xrightarrow{f^*} H^i(Z_{eh}', \mathcal{F}) \to \cdots.
\]

(2)

Proof. By Lemma 3.1, it suffices to show that there is a short exact sequence of eh-sheaves

\[
0 \to Z_{eh}(Z') \xrightarrow{i^*} Z_{eh}(Z) \oplus Z_{eh}(X') \xrightarrow{i^*} Z_{eh}(X) \to 0.
\]

(3)

Exactness on the clear on the presheaf level because \( i^* \) is injective. To check exactness in the middle, let \( x \in Z_{eh}(Z)(U) \oplus Z_{eh}(X')(U) \) be a section of the middle term over a scheme \( U \). By going to an eh-cover, we can assume that \( U \) is integral and that \( x = (\sum_i n_i \alpha_i, \sum_j m_j \beta_j) \) is represented by a linear combination of pairwise different morphisms \( \alpha_i : U \to Z \) and pairwise different morphisms \( \beta_j : U \to X' \) such that \( \sum_i n_i \alpha_i \circ i = \sum_j m_j f \circ \beta_j \in Z(X)(U) \). If \( \beta_j \) maps the generic point of \( U \) to \( Z' \), then \( \beta_j \) factors through \( Z' \), hence changing \( x \) by an element in the image of \( Z(Z')(U) \), we can assume that every summand \( \beta_j \) of \( x \) sends the generic point of \( U \) to \( X' - Z' \). We claim that this implies that \( x = 0 \), because no two \( \alpha_i \) or \( \beta_j \) can become equal in \( Z(X)(U) \). This is clear for the \( \alpha_i \) since \( i \) is injective and the \( \beta_j \) don’t have image in \( Z \). On the other hand, if \( \beta_1 \) and \( \beta_2 \) are two maps which map the generic point of \( U \) to \( X' - Z' \), and which become equal when composed with \( f \), then because \( f : X' - Z' \to X - Z \) is an isomorphism, \( \beta_1 \) and \( \beta_2 \) agree on the generic point of \( U \), hence are equal.

To show exactness on the right, let \( f \in Z(X)(U) \) be a morphism and consider the eh-covering \( U \times_X Z, U \times_X X' \) of \( U \). Restricting \( f \) to this covering, we get two maps \( f_Z : U \times_X Z \to U \to X \) and \( f_{X'} : U \times_X X' \to U \to X \), which clearly lie in the image of \( Z(Z)(U \times_X Z) \) and \( Z(X')(U \times_X X') \), respectively.

Remark. The presheaf analog of (3) is not exact in the middle, as stated in 29 Lemma 12.1. Take for example \( U = \text{Spec} \, k[x, y]/(xy) \), \( X \) the affine plane and \( X' \) be the blow-up of \( X \) at the origin. Take \( g \) to be the map \( U \to X' \) which embeds the line \( x = 0 \) into the exceptional divisor \( Z' \), and maps the line \( y = 0 \) to any line of \( X' \) intersecting the exceptional divisor in the image of the origin \( (0, 0) \in U \). If \( \tau \) is the reflection of \( U \) sending \( y \) to \( -y \), then \( g - g \circ \tau \) becomes zero when composed with the projection \( X' \to X \), but does not factor through \( Z' \).

We now come to the definition of cohomology with compact support.

Definition 3.3 Let \( U \in \text{Sch}^d/k \) and let \( j : U \to X \) be a compactification of \( U \), i.e. a proper scheme over \( k \) containing \( U \) as a dense open subscheme. Let \( i : Z \to X \) be the closed complement of \( U \) with the reduced subscheme structure. For an eh-sheaf \( \mathcal{F} \in (\text{Sch}^d/k)_{eh} \), let \( \mathcal{F} \to \mathcal{I} \) be a resolution by
injective eh-sheaves. We define the eh-cohomology with compact support of $U$ to be
\[ R\Gamma_c(U_{eh}, \mathcal{F}) = \text{cone}(I(X) \to I(Z))[-1]. \]

**Lemma 3.4** The above definition is independent of the choice of $X$.

**Proof.** Given two compactifications $X$ and $X'$, we can by the usual argument assume that there is a map $f : X' \to X$ which is the identity on $U$. Let $Z' = X' - U \to X'$ and $Z = X - U \to X$ be the closed complements of $U$ in $X'$ and $X$, respectively, with the reduced subscheme structure. Since $Z' \cong Z \times_X X'$ as topological spaces, and since the eh-cohomology of $Z'$ and $(Z')_{\text{red}}$ agree, we can assume that $Z' \cong Z \times_X X'$, so that $Z', X', Z$ and $X$ form a blow-up square (1). Consider the diagram
\[ I \cdot (X) \xrightarrow{i^*} I \cdot (Z) \xrightarrow{\text{cone}(i^*)} \]
\[ f^* \downarrow \quad f'^* \downarrow \quad \downarrow \]
\[ I \cdot (X') \xrightarrow{i'^*} I \cdot (Z') \xrightarrow{\text{cone}(i'^*)}. \]

Applying $\text{Hom}(-, I^*)$ to the exact sequence (3), we see that the right vertical map is an isomorphism. \qed

As usual, eh-cohomology groups with compact support are contravariant for proper maps and covariant for open embeddings. For a closed embedding $Z \subseteq X$ with open complement $U$ there is a long exact sequence
\[ \cdots \to H^j_c(U_{eh}, \mathcal{F}) \to H^j_c(X_{eh}, \mathcal{F}) \to H^j_c(Z_{eh}, \mathcal{F}) \to \cdots . \]

**Remark.** Another approach to define cohomology with compact support is to let the $Z_{eh}^c(X)$ be the eh-sheaf associated to the presheaf which sends an irreducible scheme $V$ to the free abelian group on closed subschemes $Z \subseteq V \times X$ such that the projection $Z \to V$ is an open embedding. If $X$ is proper, then $Z_{eh}^c(X) = Z_{eh}(X)$, because then $Z \cong V$ can be identified with the graph of a morphism $V \to X$. Hence if one defines cohomology with compact support of the sheaf $\mathcal{F}$ as $H^j_c(X_{eh}, \mathcal{F}) = \text{Ext}^j_{eh}(Z_c(X), \mathcal{F})$, then this agrees with Definition 3.3 by Lemma 3.1. For an open subscheme $U \subseteq X$ with closed complement $Z$, one can prove as in [3, Prop. 3.8] that there is a short exact sequence of eh-sheaves (this fails for the etale topology)
\[ 0 \to Z^c_{eh}(Z) \to Z^c_{eh}(X) \to Z^c_{eh}(U) \to 0, \]

hence the two definitions agree in general in view of (1).

**Lemma 3.5** Let $\mathcal{F} \in (\text{Sch}/k)_{eh}$ be a constructible sheaf of abelian groups. In the blow-up square (1), let $g$ be the composition $Z' \to X$. Then there is an exact triangle in the derived category of etale sheaves on $X$,
\( \mathcal{F} \to Rf_*\mathcal{F} \oplus i_*\mathcal{F} \to Rg_*\mathcal{F} \to \mathcal{F}[1]. \)

In particular, \( \mathcal{F} \) is a sheaf for the \( \text{eh} \)-topology, and there is a long exact sequence \( \mathbb{H} \) for the etale cohomology of \( \mathcal{F} \).

**Proof.** Since \( \mathcal{F} \) is constructible, \( \mathcal{F} \) is torsion and \( i^*\mathcal{F}, f^*\mathcal{F}, \) and \( g^*\mathcal{F} \) are the restrictions of \( \mathcal{F} \) to \( X, X', \) and \( Z' \), respectively. It suffices to show that there is a short exact sequence of etale sheaves

\[
0 \to \mathcal{F} \to f_*\mathcal{F} \oplus i_*\mathcal{F} \to g_*\mathcal{F} \to 0,
\]

and isomorphisms \( R^sf_*\mathcal{F} \cong R^sg_*\mathcal{F} \) for \( s > 0 \). If \( x \) is a geometric point over \( X - Z \), then \( (R^sf_*\mathcal{F})_x = (i_*\mathcal{F})_x = 0 \) for all \( s \geq 0 \). Since \( f \) is an isomorphism in a neighborhood of \( x \), \( (R^sf_*\mathcal{F})_x = 0 \) for \( s > 0 \), and the stalk at \( x \) of the sequence becomes the isomorphism \( (\mathcal{F})_x \sim (i_*\mathcal{F})_x \). For \( x \) a geometric point over \( Z \), there are isomorphisms \( (\mathcal{F})_x \sim (i_*\mathcal{F})_x \) and \( (R^sf_*\mathcal{F})_x \sim (R^sg_*\mathcal{F})_x \) by the proper base-change theorem.

Finally, if \( \mathcal{F} \) satisfies the sheaf property for a class of morphisms, then it also satisfied the sheaf property for the Grothendieck topology generated by it. \( \square \)

If the abstract blow-up \( f : X' \to X \) is finite, then in the Lemma it suffices to assume that \( \mathcal{F} \) is locally constructible. Indeed, in this case \( R^sf_*\mathcal{F} = R^sg_*\mathcal{F} = 0 \) for \( s > 0 \), and the proper base-change theorem is not needed.

Consider the canonical morphism of topoi \( \tau : (\text{Sch}/k)_{\text{et}} \to (\text{Sch}/k)_{\text{eh}} \).

**Theorem 3.6** Let \( \mathcal{F} \in (\text{Sch}/k)_{\text{et}} \) be a constructible sheaf. Then \( R^s\tau_*\mathcal{F} = 0 \) for \( s > 0 \). In particular, for every \( X \in \text{Sch}/k \),

\[
H^s(X_{\text{et}}, \mu_m^{\otimes n}) \cong H^s(X_{\text{eh}}, \mu_m^{\otimes n}).
\]

**Proof.** By the Lemma, \( \mathcal{F} \) is a sheaf for the \( \text{eh} \)-topology, and we identify \( \tau^*\mathcal{F} \) with \( \mathcal{F} \). Let \( C' \) be the cone of the canonical map of complexes of etale sheaves \( \gamma : \mathcal{F} \to R\gamma_*\mathcal{F} \). It suffices to show that \( H^i(X_{\text{et}}, C') = 0 \) for every scheme \( X \) in \( \text{Sch}/k \). Assume we have a non-zero element \( 0 \neq u \in H^i(X_{\text{et}}, C') \), and that \( X \) is a scheme of smallest dimension admitting such an element. Given an abstract blow-up diagram \( \mathbb{D} \), then according to Propositions 3.2 and Lemma 3.5 there is a map of long exact sequences

\[
\begin{align*}
H^i(X_{\text{et}}, \mathcal{F}) &\xrightarrow{\tau_X} H^i(Z_{\text{et}}, \mathcal{F}) \oplus H^i(X'_{\text{et}}, \mathcal{F}) \xrightarrow{\tau_{X'}} H^i(Z'_{\text{et}}, \mathcal{F}) \\
&\downarrow \quad \downarrow \quad \downarrow \\
H^i(X_{\text{eh}}, \mathcal{F}) &\xrightarrow{\tau_Z} H^i(Z_{\text{eh}}, \mathcal{F}) \oplus H^i(X'_{\text{eh}}, \mathcal{F}) \xrightarrow{\tau_{Z'}} H^i(Z'_{\text{eh}}, \mathcal{F})
\end{align*}
\]

If \( X' \) is an irreducible component of \( X \) and \( Z \) the union of the remaining components, and if \( \tau_X \) is not an isomorphism, then either \( \tau_{X'} \) or \( \tau_Z \) is not an
isomorphism, because \( \tau_{2^s} \) is an isomorphism by minimality of the dimension of \( X \). Hence we can by induction on the number of irreducible components of \( X \) assume that \( X \) is integral.

Since \( \tau^*C = 0 \), there is an eh-covering of \( X \) such that \( C \) is quasi-isomorphic to zero when restricted to this covering. We can by Proposition \[23\] assume that the covering is a composition of an etale cover \( \{U_i \to X'\} \), and a proper eh-cover \( X' \to X \). Replace \( f : X' \to X \) by a proper birational refinement, and let \( Z \) be the closed subscheme of \( X \) where \( f \) is not an isomorphism. Then we get a diagram as above, and by minimality of \( X \), \( \tau_2 \) and \( \tau_{2^s} \) are isomorphisms, hence \( \tau_{X'} \) cannot be an isomorphism for all \( i \), and thus \( C|_{X'} \) is not quasi-isomorphic to zero. But then it is also not quasi-isomorphic to zero on the etale cover \( \{U_i \to X'\} \), a contradiction. \( \square \)

4 Motivic, Hodge and de Rham cohomology

Consider the restriction of the motivic complex \( \mathbb{Z}(n) \in K^-(\text{Sm}^d/k) \), \( n \geq 0 \), of Suslin-Voevodsky \[23\] Def. 3.1], a bounded above complex of etale sheaves of abelian groups on \( \text{Sm}^d/k \). Abusing notation, we simply write \( \mathbb{Z}(n) \) for the extension \( \rho_d^*\mathbb{Z}(n) \in K^-(\text{Sch}^d/k) \). Under \( R(a) \), the cohomology of \( \rho_d^*\mathbb{Z}(n) \) does not depend on \( d \) as long as \( d \leq a \) by Lemmas \[2.5\] b) and \[5.1\]. For negative \( n \), we set \( \mathbb{Z}(n) = \colim_{\rho_d|\mathbb{Z}} \mu_{\mathbb{Z}}^{\otimes n}[-1] \).

For an abelian group \( A \), we write \( A(n) \) for the complex \( A \otimes \mathbb{Z}(n) \). If \( A \) is torsion free, then \( H^i(X_{eh}, A(n)) \cong H^i(X_{eh}, \mathbb{Z}(n)) \otimes A \).

**Lemma 4.1** Under \( R(d) \), there are quasi-isomorphism \( \mathbb{Z}(0) \cong \mathbb{Z}, \mathbb{Z}(1) \cong \mathbb{G}_m[-1] \) and \( \mathbb{Z}/m(n) \cong \mu_{m}^{\otimes n} \) for all \( n \in \mathbb{Z} \) and char \( k /m \) in \( D^-(\text{Sch}^d/k) \).

In particular, \( H^i(X_{eh}, \mathbb{Z}/m(n)) \cong H^i(X_{et}, \mu_{m}^{\otimes n}) \).

**Proof.** This follows by exactness of \( \rho_d^* \) from the corresponding statement for smooth schemes \[23\] Lemma 3.2], because \( \rho_d^*\mathbb{Z} = \mathbb{Z}, \rho_d^*\mathbb{G}_m = \mathbb{G}_m \) and \( \rho_d^*\mu_{m}^{\otimes n} = \mu_{m}^{\otimes n} \). The final statement follows with Theorem \[3.6\]. \( \square \)

**Proposition 4.2** Let \( n \in \mathbb{Z}, X \in \text{Sm}/k \), \( X' \) the blow-up of \( X \) along the smooth center \( Z \), and \( Z' = Z \times_X X' \). Then there is a long exact sequence

\[
\cdots \to H^i(X_{et}, \mathbb{Z}(n)) \to H^i(Z_{et}, \mathbb{Z}(n)) \oplus H^i(X'_{et}, \mathbb{Z}(n)) \to H^i(Z'_{et}, \mathbb{Z}(n)) \to \cdots.
\]

**Proof.** It suffices to prove exactness rationally, with mod \( p \) and with mod \( m \)-coefficients for \( p /m \). For mod \( m \)-coefficients, we have \( \mathbb{Z}/m(n) \cong \mu_{m}^{\otimes n} \), hence the claim follows from Lemma \[3.5\]. The result with mod \( p \)-coefficients is \[ IV \S 1] because of \( \mathbb{Z}/p^{r}(n) \cong W_r \Omega^{r}_{X, \log} \[8\]. Finally, rationally motivic cohomology and etale motivic cohomology agree \[ IV \ Prop. 3.6\], hence the result with rational coefficients is \[18\] Lemma IV 3.12]. \( \square \)
**Theorem 4.3** If $n \in \mathbb{Z}$, $X \in \text{Sm}^d/k$ and condition $R(d)$ holds, then

$$H^i(X_{et}, \mathbb{Z}(n)) \cong H^i(X_{et}, \mathbb{Z}(n)).$$

**Proof.** Let $C$ be the cone of the canonical map of complexes of etale sheaves $\mathbb{Z}(n) \to R(\rho_d)_* \mathbb{Z}(n)$. It suffices to show that $H^i(X_{et}, C') = 0$ for every scheme $X \in \text{Sm}^d/k$. Assume we have a non-zero element $0 \neq u \in H^i(X_{et}, C')$, and that $X$ is a scheme of smallest dimension admitting such an element. Since $\rho_d^* C' = 0$, there is an eh-covering of $X$ such that $C'$ is quasi-isomorphic to zero when restricted to this covering. By Corollary 4.4 we can assume that the covering has a composition of an etale cover $\{U_i \to X'\}$, and a composition of blow-ups along smooth centers $X' \to X$. Given a blow-up $X'$ of $X$ along the smooth center $Z$, we can find by Propositions 4.5 and 4.6 a map of long exact sequences

$$\begin{align*}
H^i(X_{et}, \mathbb{Z}(n)) &\to H^i(Z_{et}, \mathbb{Z}(n)) \oplus H^i(X'_{et}, \mathbb{Z}(n)) \to H^i(Z_{et}', \mathbb{Z}(n)) \\
\tau_X \downarrow &\quad \tau_Z \downarrow \tau_{X'} \downarrow \\
H^i(X_{et}, \mathbb{Z}(n)) &\to H^i(Z_{et}, \mathbb{Z}(n)) \oplus H^i(X'_{et}, \mathbb{Z}(n)) \to H^i(Z_{et}', \mathbb{Z}(n)).
\end{align*}$$

By minimality of $X$, $\tau_Z$ and $\tau_{Z'}$ are isomorphisms, and we conclude that $u|_{X'}$ is non-zero. In particular, $C'|_{X'}$ is not quasi-isomorphic to zero. But then it is also not quasi-isomorphic to zero on the etale cover $\{U_i \to X'\}$, a contradiction. \hfill \Box

**Corollary 4.4** Assume $R(d)$, let $n \in \mathbb{Z}$ and $X \in \text{Sch}^d/k$. If $c_{d_l}(k) < \infty$ for all primes $l$ dividing $m$, then the groups $H^i(Z_{et}, \mathbb{Z}/m(n))$ are finitely generated.

**Proof.** By Lemma 27 we can assume that $X$ is smooth and projective. Write $m = m' \cdot p^r$ with $p \nmid m'$. Finite generation of $H^i(X_{et}, \mu^\otimes m)$ is well-known. On the other hand, by Theorem 4.5 and 4.6, $H^i(Z_{et}, \mathbb{Z}/p^r(n)) \cong \mathbb{H}^i(Z_{et}, \mathbb{Z}/p^r(n)) \cong H^{i-n}(X_{et}, \mu^\otimes p^r)$. The latter group is finitely generated by [10] Prop. 4.18. \hfill \Box

**Proposition 4.5** Under resolution of singularities, the rational eh-motivic cohomology groups $H^i(X_{et}, \mathbb{Q}(n))$ agree with Voevodsky’s rational motivic cohomology groups $\text{Hom}_{DM} - (M(X), \mathbb{Q}(n)[i])$.

**Proof.** By [23] Thm. 1.5], Voevodsky’s motivic cohomology groups agree with the Nisnevich cohomology groups of the motivic complex for smooth $X$, $\text{Hom}_{DM} - (M(X), \mathbb{Z}(n)[i]) = H^i(X_{Nis}, \mathbb{Z}(n))$. Using the fact that rationally Nisnevich and etale cohomology agree, this implies the claim for smooth $X$: \hfill \Box
The definition we use allows us to give better bounds on resolution of singularities required.

\[ H^i(X_{\text{Nis}}, \mathbb{Q}(n)) \xrightarrow{\sim} \text{Hom}_{DM^-(M(X), \mathbb{Q}(n)[i])} \]

\[ \approx \]

\[ H^i(X_{\text{et}}, \mathbb{Q}(n)) \xrightarrow{\sim} H^i(X_{\text{eh}}, \mathbb{Q}(n)). \]

For arbitrary \( X \), we can proceed by induction on the dimension of \( X \) from the smooth case, using the blow-up sequences for both theories.

**Proposition 4.6** Assume \( R(d + r) \), let \( n \in \mathbb{Z} \) and \( X \in \text{Sch}^d/k. \)

a) (Projective bundle formula) There are canonical isomorphisms

\[ \bigoplus_{j=0}^{r} H^{i-2j}_c(X_{\text{eh}}, \mathbb{Z}(n-j)) \cong H^i_c(\mathbb{P}^r_X, \mathbb{Z}(n)). \]

b) (Affine bundle formula) There are canonical isomorphisms

\[ H^i_c((\mathbb{A}^r_X)_{\text{eh}}, \mathbb{Z}(n)) \cong H^{i-2r}_c(X_{\text{eh}}, \mathbb{Z}(n-r)). \]

**Proof.** a) Comparing the long exact sequences for cohomology with compact support \( \Xi \), we can by Lemma 2.7 assume that \( X \) is smooth and projective. In this case, we can consider etale cohomology instead of eh-cohomology by Theorem 4.3. The isomorphism is given by \( \sum p^*_r \cup \xi^j \), where \( p_r : \mathbb{P}^r_X \to X \) is the projection, and \( \xi \in H^2((\mathbb{P}^r_X)_{\text{et}}, \mathbb{Z}(1)) \cong \text{Pic} \mathbb{P}^r_X \) is the class of \( \mathcal{O}(1) \).

With mod \( m \)-coefficients, \( p \) \( \mathbb{Z}/m \), this map is an isomorphism by [20 Prop. 10.1] because \( \mathbb{Z}/m(n)_{\text{et}} \cong \mu_m^\otimes n \) for every \( n \in \mathbb{Z} \). For \( p^r \)-coefficients, the map is an isomorphism by [9 I Thm. 2.2.11] because \( \mathbb{Z}/p^r(n)_{\text{et}} \cong W_r \Omega^p_{X, \log} \). Rationally, the map is an isomorphism by [23 Thms. 1.5, 4.5], because motivic cohomology and etale motivic cohomology agree.

b) Again we first reduce to the case that \( X \) is smooth and projective. In this case, by (4), the section \( \mathbb{P}^{r-1} \to \mathbb{P}^r_X \) gives a long exact sequence

\[ \cdots \to H^i_c((\mathbb{A}^r_X)_{\text{eh}}, \mathbb{Z}(n)) \to H^i_c((\mathbb{P}^r_X)_{\text{eh}}, \mathbb{Z}(n)) \xrightarrow{\partial^r} H^i_c((\mathbb{P}^{r-1}_X)_{\text{eh}}, \mathbb{Z}(n)) \to \cdots. \]

(6)

It is easy to see that \( 0^* \xi = \xi \), so that \( 0^*(p^*_r \cup \xi^j) = p^*_r \cup \xi^j \) for \( 0 \leq j \leq r-1 \). Consequently, \( 0^* \) is surjective, and ker \( 0^* \cong H^{i-2r}_c(X_{\text{eh}}, \mathbb{Z}(n-r)). \)

**Remark.** Since \( H^i((\mathbb{P}^1_X)_{\text{et}}, \mathbb{Z}/m(0)) \cong H^i(k_{\text{et}}, \mathbb{Z}/m) \oplus H^i(k_{\text{et}}, \mu_m^\otimes -1) \), the projective bundle formula cannot hold in general without modifying the definition of motivic cohomology for negative \( n \).

**Remark.** In view of Proposition 4.6(b) and in analogy with the formula for the \( \zeta \)-function, it would be natural to define negative eh-cohomology as

\[ H^i_c(X_{\text{eh}}, \mathbb{Z}(n)) = H^{i-2n}_c((\mathbb{A}^r_X)_{\text{eh}}, \mathbb{Z}). \]

The definition we use allows us to give better bounds on resolution of singularities required.
4.1 Hodge and de Rham cohomology

Consider the sheaf of differentials $\Omega^n$ on $\text{Sm}^d/k$ and its pull-back $\rho^*\Omega^n$ to $(\text{Sch}^d/k)_{\text{eh}}$. Note that $\rho^*\Omega^n$ does not agree with the sheaf of differentials for non-smooth schemes. In this section we study the Hodge cohomology groups $H^i(X_{\text{eh}}, \rho^*\Omega^n)$ and de Rham cohomology groups $H^i_{\text{DR}}(X_{\text{eh}}) = H^i(X_{\text{eh}}, \rho^*\Omega^\cdot)$ for the eh-topology. We will need Hodge cohomology groups with compact support in the formula for $\zeta$-values below. On the other hand, the eh-version of de Rham cohomology generalizes Hartshorne’s de Rham cohomology [12] for fields of characteristic 0. We start with the following analog of Theorem 4.3.

**Theorem 4.7** If $X \in \text{Sm}^d/k$ and condition $R(d)$ holds, then

\[
H^i(X, \Omega^n_X) \cong H^i(X_{\text{eh}}, \rho^*\Omega^n);
\]
\[
H^i_{\text{DR}}(X) \cong H^i_{\text{DR}}(X_{\text{eh}}).
\]

**Proof.** Because $\Omega^n$ is quasi-coherent, $H^i(X, \Omega^n) \xrightarrow{\sim} H^i(X_{\text{et}}, \Omega^n)$. Now the proof works exactly as the proof of Theorem 4.3 using the Hodge cohomology analog of Proposition 4.2: If $X'$ is the blow-up of the smooth scheme $X$ along the smooth center $Z$, then there is a long exact sequence [9, IV Thm. 1.2.1]

\[
\cdots \to H^i(X, \Omega^n) \to H^i(Z, \Omega^n) \oplus H^i(X', \Omega^n) \to H^i(Z', \Omega^n) \to \cdots.
\]

The statement for de Rham-cohomology follows from the spectral sequence from Hodge to de Rham cohomology. \qed

**Corollary 4.8** Assume $R(d)$ and let $X \in \text{Sch}^d/k$. Then the $k$-vector spaces $H^i_{\text{eh}}(X_{\text{eh}}, \rho^*\Omega^n)$ and $H^i_{\text{DR}}(X_{\text{eh}})$ are finite dimensional.

**Proof.** This follows from the smooth, projective case by the same argument as Corollary 4.4. \qed

**Proposition 4.9** Assume $R(d + r)$ and let $X \in \text{Sch}^d/k$.

a) (Projective bundle formula) There are canonical isomorphisms

\[
\bigoplus_{j=0}^{r} H^i_{\text{eh}}(X_{\text{eh}}, \rho^*\Omega^{n-j}) \cong H^i_{\text{eh}}(\mathbb{P}^r_X)_{\text{eh}}, \rho^*\Omega^n).
\]

b) (Affine bundle formula) There are canonical isomorphisms

\[
H^i_{\text{eh}}((\mathbb{A}^r_X)_{\text{eh}}, \rho^*\Omega^n) \cong H^i_{\text{eh}}(X_{\text{eh}}, \rho^*\Omega^{n-r})).
\]
Proof. a) By the usual method we reduce to the smooth and proper case. In this case, the isomorphism is given by \( \sum p_r^* \xi^j \), where \( p_r : \mathbb{P}^r_X \to X \) is the projection and \( \xi \) the image of \( O(1) \) under the map \( \text{Pic} \mathbb{P}^r_X \cong H^1((\mathbb{P}^r_X)_{\text{et}}, \mathbb{Q}_m) \xrightarrow{d \log} H^1((\mathbb{P}^r_X)_{\text{et}}, \Omega^1) \), see [11, §6.3].

b) Follows formally from a) as in Proposition 4.6.
\[ \blacksquare \]

Recall the definition of algebraic de Rham cohomology of Hartshorne. Given a scheme \( X \in \text{Sch}/k \), we can use a Čech-covering argument and assume that there exists a closed immersion of \( X \) into a smooth scheme \( W \in \text{Sm}/k \). Then the de Rham cohomology of \( X \) is defined as the hypercohomology of the formal completion of the de Rham complex \( \Omega^\cdot W \) of \( W \) along \( X \) [12, II §1]

\[ H^i_{\text{DR}}(X) = H^q(\hat{W}, \hat{\Omega}^\cdot W). \]

If \( k \) is of characteristic 0, then this is independent of the choice of \( W \) by [12, Thm. 1.4].

**Theorem 4.10** If \( X \) is a scheme of characteristic zero, then

\[ H^i_{\text{DR}}(X_{\text{ch}}) \cong H^i_{\text{DR}}(X). \]

**Proof.** We proceed by induction on the dimension of \( X \). By [2] and [12, Prop. 4.1], both sides admit a Mayer-Vietoris sequence for a closed cover. Hence by induction on the number of irreducible components we can reduce to the case that \( X \) is integral. If \( X \) is smooth, then both sides agree with the usual de Rham-cohomology of \( X \) by Theorem 4.7. In general, by resolution of singularities, we can find a blow-up square [1] with \( X' \) smooth. Now we can compare the long exact sequence [2] to the corresponding long exact sequence for de Rham cohomology [12, Theorem 4.4] to complete the proof. \[ \blacksquare \]

### 5 Arithmetic cohomology

From now on we fix a finite field \( \mathbb{F}_q \), and denote by \( R(d) \) the existence of resolution of singularities for schemes over the algebraic closure of \( \mathbb{F}_q \). A detailed version of the following discussion can be found in [10].

Let \( G \subseteq \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \) be the Weil group, i.e. the free abelian group of rank 1 generated by the Frobenius endomorphism \( \varphi \).

**Definition 5.1** A Weil-\( \text{ch-sheaf} \) \( F \) on \( \text{Sch}/\mathbb{F}_q \) is an \( \text{ch-sheaf} \) on \( \text{Sch}/\overline{\mathbb{F}}_q \) together with an action over the Frobenius endomorphism, \( \psi : F \to \varphi_\ast F \), i.e. a compatible family of isomorphisms \( \psi_S : F(S) \to F(S \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q) \) for every scheme \( S/\overline{\mathbb{F}}_q \). We write \( (\text{Sch}/\mathbb{F}_q)_{\text{Wh}} \) for the topos of Weil-\( \text{ch-sheaves} \) on \( \text{Sch}/\mathbb{F}_q \).
Every eh-sheaf $F$ of $\text{Sch}/\mathbb{F}_q$ gives rise to a Weil-eh-sheaf on $\text{Sch}/\mathbb{F}_q$ by pulling back $F$ along $\text{Spec} \bar{\mathbb{F}}_q \to \text{Spec} \mathbb{F}_q$, and restricting the resulting Galois action to the Weil group. Conversely, there is a push-forward map from Weil-eh-sheaves on $\text{Sch}/\mathbb{F}_q$ to eh-sheaves on $\text{Sch}/\mathbb{F}_q$, giving a morphism of topoi $\gamma : (\text{Sch}/\mathbb{F}_q)_{\text{Wh}} \to (\text{Sch}/\mathbb{F}_q)_{\text{eh}}$.

For a Weil-eh-sheaf $F$, we define Weil-eh-cohomology $H^i(X_{\text{Wh}}, F)$ as the derived functor of $F \mapsto F(\bar{X})^G$. Similarly, we define Weil-eh-cohomology with compact support $H^i_c(X_{\text{Wh}}, F)$ of the sheaf $F$ as the cohomology of the complex $R\Gamma_c(F(\bar{X})_{\text{eh}}, F)$.

Let $e \in H^1((\mathbb{F}_q)_{\text{Wh}}, \mathbb{Z}) \cong \mathbb{Z}$ be a generator. Since $e^2 \in H^2((\mathbb{F}_q)_{\text{Wh}}, \mathbb{Z}) = 0$, the sequence

$$\cdots \to H^{i-1}_e(X_{\text{Wh}}, F) \xrightarrow{\cup e} H^i_e(X_{\text{Wh}}, F) \xrightarrow{\cup e} H^{i+1}_e(X_{\text{Wh}}, F) \to \cdots$$

is a complex. The results of [6] carry over to the present situation:

**Theorem 5.2** Let $X \in \text{Sch}/\mathbb{F}_q$ and let $F$ be a complex of eh-sheaves.

a) There are long exact sequences

$$\to H^i_e(X_{\text{eh}}, F) \to H^i(X_{\text{Wh}}, F) \to H^{i-1}_e(X_{\text{eh}}, F) \otimes \mathbb{Q} \to H^{i+1}_e(X_{\text{eh}}, F) \to \cdots$$

b) If the cohomology sheaves of $F$ are torsion, then

$$H^i_e(X_{\text{eh}}, F) \cong H^i(X_{\text{Wh}}, F)$$

c) If the cohomology sheaves of $F$ are uniquely divisible, then

$$H^i_e(X_{\text{Wh}}, F) \cong H^i_e(X_{\text{eh}}, F) \oplus H^{i-1}_e(X_{\text{eh}}, F),$$

and cup product with $e$ is given by the matrix $(0 0) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In particular, the sequence (7) is exact.

Since $G$ has cohomological dimension 1, the Leray spectral sequence for composition of functors breaks up into short exact sequences

$$0 \to H^{i-1}_e(\bar{X}_{\text{eh}}, F)_G \to H^i_e(X_{\text{Wh}}, F) \to H^i_e(\bar{X}_{\text{eh}}, F)_G \to 0. \quad (8)$$

Because the Leray spectral sequence is multiplicative, this implies that we have a commutative diagram

$$\begin{array}{ccc}
H^i_e(X_{\text{Wh}}, F) & \xrightarrow{\text{surj}} & H^i_e(\bar{X}_{\text{eh}}, F)_G \\
\cup e & & \downarrow \\
H^{i+1}_e(X_{\text{Wh}}, F) & \xleftarrow{\text{inj}} & H^i_e(\bar{X}_{\text{eh}}, F)_G,
\end{array}$$

where the right vertical map is induced by the identity map.

**Corollary 5.3** The Galois-modules $H^i_e(\bar{X}_{\text{eh}}, \mathbb{Q}(n))$ are semi-simple at the eigenvalue $1$. 
Proof. Semi-simplicity at the eigenvalue 1 is equivalent to the canonical map\
\[ H^i_c(X_{eh}, \mathbb{Q}(n))^G \to H^i_c(X_{eh}, \mathbb{Q}(n)) \]
being an isomorphism. Using (9) and (8) we get the diagram
\[
\begin{array}{cccccccc}
0 & \to & \text{im } \alpha_{i-1} & \overset{\text{im}(- \cup e)}{\to} & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & H^{i-1}_c(X_{eh}, \mathbb{Q}(n))^G & \to & \ker(- \cup e) & \to & \ker \alpha_i & \to 0.
\end{array}
\]
By Theorem 5.2 c), ker(- \cup e) = im(- \cup e), and the corollary follows.  

Definition 5.4 For \( n \in \mathbb{Z} \), we define arithmetic cohomology with compact support as Weil-etale-cohomology of the motivic complex, 
\[ R\Gamma_c(X_{ar}, \mathbb{Z}(n)) = R\Gamma(X_{et}, \mathbb{Z}(n)). \]

Arithmetic cohomology with coefficients in \( A \) is defined as the Weil-etale-cohomology of the complex \( \mathbb{Z}(n) \otimes A \).

Recall that the Weil-etale-cohomology groups \( H^i_W(X, \mathbb{Z}(n)) \) of [6] are defined as the cohomology groups of the complex \( R\Gamma(G, R\Gamma(X_{et}, \mathbb{Z}(n))). \) In order to apply the results of [6], we record the following consequence of Theorem 4.3:

Corollary 5.5 If \( X \) is smooth and projective, then under \( R(\dim X) \) we have 
\[ H^i_c(X_{eh}, \mathbb{Z}(n)) \cong H^i_c(X_{ar}, \mathbb{Z}(n)). \]

Lemma 5.6 If \( A \) is torsion free, then there are isomorphisms
\[ H^i_c(X_{eh}, \mathbb{Z}(n))^G \otimes A \cong H^i_c(X_{eh}, A(n))^G, \]
\[ H^i_c(X_{eh}, \mathbb{Z}(n)) \otimes A \cong H^i_c(X_{eh}, A(n)) \]

Proof. By torsion freeness we have \( H^i_c(X_{eh}, \mathbb{Z}(n)) \otimes A \cong H^i_c(X_{eh}, A(n)). \) The map on coinvariants is an isomorphism, because tensor product and coinvariants commute. For invariants, we compare the sequence (8) with \( \mathbb{Z}(n) \) and \( A(n) \)-coefficients:
\[
\begin{array}{cccccccc}
H^{i-1}_c(X_{eh}, \mathbb{Z}(n))^G \otimes A & \to & H^i_c(X_{eh}, A(n))^G & \to & H^i_c(X_{eh}, \mathbb{Z}(n))^G \otimes A \\
\downarrow & & \downarrow & & \downarrow \\
H^{i-1}_c(X_{eh}, A(n))^G & \to & H^i_c(X_{eh}, A(n))^G & \to & H^i_c(X_{eh}, \mathbb{Z}(n))^G.
\end{array}
\]

The following corollary is an immediate consequence of Proposition 4.6.
Corollary 5.7 Assume \( R(d + r) \), let \( X \in \text{Sch}^d/F_q \) and \( n \in \mathbb{Z} \).

a) (Projective bundle formula) There are canonical isomorphisms

\[
\bigoplus_{j=0}^{r} H^{i-2j}_c(X_{\text{ar}}, \mathbb{Z}(n - j)) \cong H^{i}_c((\mathbb{P}_X)_{\text{ar}}, \mathbb{Z}(n)).
\]

b) (Affine bundle formula) There are canonical isomorphisms

\[
H^{i}_c((\mathbb{A}^{r} \times X)_{\text{ar}}, \mathbb{Z}(n)) \cong H^{i-2r}_c(X_{\text{ar}}, \mathbb{Z}(n - r)).
\]

Proposition 5.8 Let \( X \in \text{Sch}^d/F_q \) and assume \( R(d) \).

a) \( H^{i}_c(X_{\text{ar}}, \mathbb{Z}(n)) = 0 \) for \( i > \max\{2d + 1, n + d + 1\} \).

b) If \( r \) is the number of irreducible components of \( X \) of maximal dimension \( d \), then there is a canonical surjection \( H^{2d+1}_c(X_{\text{ar}}, \mathbb{Z}(d)) \to \mathbb{Z}^r \).

Proof. a) This follows from the smooth projective case \cite{6} with the argument of Lemma \cite{2}.

b) Let \( X = Y \cup \bigcup_i X_i \), where \( Y \) is the union of irreducible components of dimension smaller than \( d \) and the \( X_i \) are the irreducible components of dimension \( d \). Then by \cite{2}, a) and induction on the number of irreducible components, we get a surjection \( H^{2d+1}_c(X_{\text{ar}}, \mathbb{Z}(d)) \to \bigoplus H^{2d+1}_c(X_i_{\text{ar}}, \mathbb{Z}(d)) \), hence we can assume that \( X \) is irreducible. If \( W \) is a compactification of \( X \), then by \cite{10} and a) we get a surjection \( H^{2d+1}_c(X_{\text{ar}}, \mathbb{Z}(d)) \to H^{2d+1}_c(W_{\text{ar}}, \mathbb{Z}(d)) \), and this is compatible with maps between compactifications. Finally, by \( R(d) \) we can assume that there is a blow-up square \cite{10} with \( W' \) smooth and projective. Again by a) and \cite{2} the map \( H^{2d+1}(W_{\text{ar}}, \mathbb{Z}(d)) \to H^{2d+1}(W'_{\text{ar}}, \mathbb{Z}(d)) \) is surjective, and by \cite{10} and Corollary \cite{5} the latter group is \( \mathbb{Z} \). \( \square \)

6 Tate’s conjecture, finite generation, and \( l \)-adic cohomology

In order not to confuse \( l \)-adic cohomology (\( l \neq p \)) with cohomology with \( \mathbb{Z}(n) \otimes \mathbb{Z}_l \)-coefficients, we write \( l \)-adic cohomology as

\[
H^i(X_{\text{et}}, \hat{\mathbb{Z}}_l(n)) := \lim_r H^i(X_{\text{et}}, \mu_r^{\otimes n});
\]

\[
H^i(X_{\text{et}}, \hat{\mathbb{Q}}_l(n)) := H^i(X_{\text{et}}, \hat{\mathbb{Z}}_l(n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l;
\]

and similarly for cohomology with compact support. These groups agree with the continuous cohomology groups of Jannsen \cite{14}. By Theorem \cite{5} we could define \( l \)-adic cohomology with the eh-topology. Assume \( R(\dim X) \) and consider the following natural map

\[
H^i_c(X_{\text{ar}}, \mathbb{Z}(n)) \to H^i_c(X_{\text{ar}}, \mathbb{Z}/l^n(n)) \cong H^i_c(X_{\text{et}}, \mu_r^{\otimes n}).
\]

(10)
The isomorphisms come from Lemma 4.1 and Theorem 5.2 b).

In the following conjectures, let $X \in \text{Sch}/\mathbb{F}_q$ and $n \in \mathbb{Z}$.

**Conjecture K**(X,n): The groups $H^i_c(X_{\text{ar}},\mathbb{Z}(n))$ form an integral model for $l$-adic cohomology with compact support for all $l \neq p$, i.e. the limit of the maps 
\[ (10) \] induces an isomorphism
\[ H^i_c(X_{\text{ar}},\mathbb{Z}_l(n)) \overset{\sim}{\longrightarrow} H^i_c(X_{\text{et}},\hat{\mathbb{Z}}_l(n)). \]

**Conjecture L**(X,n): For all $i$, the groups $H^i_c(X_{\text{ar}},\mathbb{Z}(n))$ are finitely generated abelian groups.

**Proposition 6.1** Assume $R(d)$.

a) Conjecture $L(X,n)$ for $X \in \text{Sch}^d/\mathbb{F}_q$ implies $K(X,n)$, the finiteness of $H^i_c(X_{\text{ar}},\mathbb{Z}(n))$ for $i \notin [2n, n+d+1]$, and the vanishing for $i \notin [0, 2d+1]$.

b) Conjecture $L(X,n)$ for all smooth and projective $X \in \text{Sch}^d/\mathbb{F}_q$ implies $L(X,n)$ for all $X \in \text{Sch}^d/\mathbb{F}_q$. The same statement is true for $K(X,n)$.

**Proof.** a) The finite generation of $H^i_c(X_{\text{ar}},\mathbb{Z}(n))$ implies by the long exact coefficient sequence that
\[ H^i_c(X_{\text{ar}},\mathbb{Z}_l(n)) \cong \lim_{\to} H^i_c(X_{\text{ar}},\mathbb{Z}(n))/l^r \cong \lim_{\to} H^i_c(X_{\text{ar}},\mathbb{Z}/l^r(n)). \]

The latter group is isomorphic to $H^i_c(X_{\text{et}},\hat{\mathbb{Z}}_l(n))$ as in (10). The $l$-adic cohomology groups $H^i_c(X_{\text{et}},\hat{\mathbb{Z}}_l(n))$ are finite or zero outside the given intervals by [17, Thm. 3].

b) This follows easily with Lemma 2.7. □

For a smooth and projective variety $X$, let $CH^n(X)$ and $A^n_{\text{num}}(X)$ be the free abelian group on closed integral subschemes of $X$ of codimension $n$ modulo rational and numerical equivalence, respectively.

**Conjecture (Tate/Beilinson)** For all smooth and projective varieties $X/\mathbb{F}_q$ and all $n \in \mathbb{Z}^{\geq 0}$, rational and numerical equivalence for algebraic cycles of codimension $n$ on $X$ agree up to torsion, and the order of the pole of the zeta function $\zeta(X,s)$ at $s = n$ is equal to the rank of $A^n_{\text{num}}(X)$:
\[ \dim CH^n(X) \otimes \mathbb{Q} = \dim A^n_{\text{num}}(X) \otimes \mathbb{Q} = \text{ord}_{s=n} \zeta(X,s). \]

By Tate [23, Thm. 2.9], this implies that for all smooth and projective $X$, the cycle map
\[ CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}(\bar{X}_{\text{et}},\hat{\mathbb{Q}}_l(n))^\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \]
is an isomorphism, and that $H^{2n}(\bar{X}_{\text{et}},\hat{\mathbb{Q}}_l(n))$ is semi-simple at eigenvalue 1.
Theorem 6.2 If $R(d)$ and the Tate-Beilinson conjecture hold, then $L(X, n)$ holds for all $X \in \text{Sch}^d/F_q$ and all $n$.

Proof. In view of Corollary 5.5, we get $L(X, n)$ for all smooth and projective varieties in $\text{Sch}^d/F_q$ from [21 Thm. 8.4]. The general case follows by Proposition 6.1 b).

The following conjecture can be thought of as the dual of the generalized Tate conjecture of Jannsen [13 Conj. 12.4, 12.6] for arbitrary $X \in \text{Sch}/F_q$ and $n \in \mathbb{Z}$.

Conjecture $J(X, n)$ For all $l \neq p$, the canonical map

$$H^i_c(X_{\text{et}}, \hat{Q}_l(n)) \to H^i_c(X_{\text{et}}, \hat{Q}_l(n))^{\text{Gal}(\overline{F}_l/F_q)}$$

is an isomorphism, and the Galois-module $H^i_c(X_{\text{et}}, \hat{Q}_l(n))$ is semi-simple at the eigenvalue 1.

We have $CH^n(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{2n}(X_{\text{et}}, \mathbb{Q}(n))$ because rationally Zariski and etale motivic cohomology agree. Hence for smooth and projective $X$, $J(X, n)$ specializes to Tate’s conjecture on the bijectivity of the cycle map under resolution of singularities, because then $H^{2n}(X_{\text{et}}, \mathbb{Q}(n)) \xrightarrow{\sim} H^{2n}(X_{\text{et}}, \mathbb{Q}(n))$ by Theorem 4.3. By Proposition 8.2 a) below, $J(X, n)$ is wrong if one uses the etale topology.

Lemma 6.3 We have $\dim H^i_c(X_{\text{et}}, \hat{Q}_l(n))^G = \dim H^i_c(X_{\text{et}}, \hat{Q}_l(n))_G$, and there is a short exact sequence of finite dimensional $\mathbb{Q}_l$-vector spaces

$$0 \to H^{i-1}_c(X_{\text{et}}, \hat{Q}_l(n))^G \to H^i_c(X_{\text{et}}, \hat{Q}_l(n)) \to H^i_c(X_{\text{et}}, \hat{Q}_l(n))_G \to 0.$$

Proof. The first statement follows from the exact sequence of finite dimensional $\mathbb{Q}_l$-vector spaces:

$$0 \to H^i_c(X_{\text{et}}, \hat{Q}_l(n))^G \to H^i_c(X_{\text{et}}, \hat{Q}_l(n)) \xrightarrow{1-\varphi} H^i_c(X_{\text{et}}, \hat{Q}_l(n)) \to H^i_c(X_{\text{et}}, \hat{Q}_l(n))_G \to 0.$$

Taking the inverse limit over $r$ of the short exact sequences of finite groups

$$0 \to H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n})^G \to H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n}) \xrightarrow{1-\varphi} H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n}) \to H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n})_G \to 0,$$

and comparing with the kernel and cokernel of $1-\varphi$ on $H^i_c(X_{\text{et}}, \hat{Q}_l(n))$, we get $\lim (H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n}))^G \otimes \mathbb{Q}_l \cong H^i_c(X_{\text{et}}, \hat{Q}_l(n))^G$, and $\lim (H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n})) \otimes \mathbb{Q}_l \cong H^i_c(X_{\text{et}}, \hat{Q}_l(n))^G$. Taking the inverse limit of

$$0 \to H^{i-1}_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n})_G \to H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n}) \to H^i_c(X_{\text{et}}, \mathbb{Q}_l^{\otimes n})^G \to 0,$$

the Lemma follows. □
Theorem 6.4 Assume $R(\dim X)$. Then $K(X, n) \Leftrightarrow J(X, n)$, and $J(X, n)$ for all smooth and projective $X \in \text{Sch}^d/F_q$ implies $J(X, n)$ for all $X \in \text{Sch}^d/F_q$.

The second statement reproves Theorem 12.7 of Jannsen [13].

Proof. Assuming $K(X, n)$, Theorem 5.2(c), and multiplicity of the cycle map imply that cup product with the image of $e$ under the canonical map $\mathbb{Z} \cong H^1((\mathbb{F}_q)_{ar}, \mathbb{Z}(0)) \rightarrow \mathbb{Q}_l \cong H^1((\mathbb{F}_q)_{et}, \mathbb{Q}_l(0))$ gives a long exact sequence of $l$-adic cohomology groups $H_c^i(X_{et}, \mathbb{Q}_l(n))$. Using the short exact sequences of Lemma 6.3, the argument of Corollary 5.8 shows semi-simplicity of $H_c^i(X_{et}, \mathbb{Q}_l(n))$ at the eigenvalue $1$ for all $i$.

We now prove by induction on $i$ that the map (11) is an isomorphism. By Corollary 5.8 and semi-simplicity, we have a commutative diagram

$$
\begin{array}{ccc}
H^{i-1}_c(X_{et}, \mathbb{Q}_l(n)) & \longrightarrow & H^{i-1}_c(X_{et}, \mathbb{Q}_l(n))_G \\
\downarrow & & \downarrow \\
H^{i-1}_c(X_{et}, \hat{\mathbb{Q}}_l(n))_G & \longrightarrow & H^{i-1}_c(X_{et}, \hat{\mathbb{Q}}_l(n))_G.
\end{array}
$$

Since $H^{i-1}_c(X_{et}, \mathbb{Q}_l(n)) \cong H^{i-1}_c(X_{et}, \hat{\mathbb{Q}}_l(n))_{\Gal(\mathbb{F}_q/\mathbb{F}_p)}$, and higher Galois cohomology is torsion, a Hochschild-Serre spectral sequence argument shows that the upper left map is an isomorphism. Hence the induction hypothesis implies that the right hand map is an isomorphism. By [8] and Lemma 6.8, we have a commutative diagram with exact rows

$$
\begin{array}{ccc}
H^{i-1}_c(X_{et}, \mathbb{Q}_l(n))_G & \longrightarrow & H^{i}_c(X_{ar}, \mathbb{Q}_l(n))_G \\
\downarrow & & \downarrow \\
H^{i-1}_c(X_{et}, \hat{\mathbb{Q}}_l(n))_G & \longrightarrow & H^{i}_c(X_{et}, \hat{\mathbb{Q}}_l(n))_G.
\end{array}
$$

Because the left hand map is an isomorphism, so is the right hand map, which concludes the induction step.

To prove the converse implication, note that by Lemma 11, it suffices to show $K(X, n)$ after tensoring with $\mathbb{Q}_l$. By hypothesis, the left hand vertical map, hence the right hand vertical map in (12) is an isomorphism for all $i$, and we conclude that the middle vertical map in (13) is an isomorphism. The final statement follows from Proposition 6.1.

\[ \square \]

7 Values of zeta-functions

We give a conjectural formula for values of zeta functions in terms of arithmetic cohomology, inspired by Lichtenbaum [19]. Fix $X \in \text{Sch}/F_q$, let $\zeta(X, s)$ be the zeta function of $X$, $\chi(H^*_c(X_{ar}, \mathbb{Z}(n), e)$ the Euler-characteristic of the complex
\[ \cdots \to H_{c}^{-1}(X_{ar}, \mathbb{Z}(n)) \xrightarrow{\cup e} H_{c}^{1}(X_{ar}, \mathbb{Z}(n)) \xrightarrow{\cup e} H_{c}^{1+1}(X_{ar}, \mathbb{Z}(n)) \to \cdots, \]

and

\[ \chi(n) = \sum_{0 \leq i \leq n} (-1)^{i+1}(n - i) \cdot \dim H_{c}^{i}(X_{eh}, \rho^{*}O^{i}). \]

**Conjecture Z(X,n)** The alternating sum \( \sum_{i}(-1)^{i} \cdot \text{rank } H_{c}^{i}(X_{ar}, \mathbb{Z}(n)) \) is zero, the weighted alternating sum of the ranks equals the order of the zeta-function

\[ \sum_{i}(-1)^{i} \cdot \text{rank } H_{c}^{i}(X_{ar}, \mathbb{Z}(n)) = \text{ord}_{s=n} \zeta(X, s) =: \rho_{n}, \]

and for \( s \mapsto n \),

\[ \zeta(X, s) \sim \pm (1 - q^{n-s})^{\rho_{n}} \cdot \chi(H_{c}^{*}(X_{ar}, \mathbb{Z}(n)), e) \cdot q^{\chi(n)}. \quad (14) \]

Note that the finiteness of the sum, and the finiteness of the sum defining the Euler characteristic is part of the conjecture.

**Theorem 7.1** Under \( R(d) \), \( L(X, n) \) for all \( X \in \text{Sch}^{d}/\mathbb{F}_{q} \) implies \( Z(X, n) \) for all \( X \in \text{Sch}^{d}/\mathbb{F}_{q} \).

**Proof.** By \( L(X, n) \) and Theorem 5.2 c), the alternating sum of the ranks of the groups \( H_{c}^{i}(X_{ar}, \mathbb{Z}(n)) \) equals

\[ \sum_{i}(-1)^{i} (\text{rank } H_{c}^{i}(X_{eh}, \mathbb{Z}(n)) + \text{rank } H_{c}^{i-1}(X_{eh}, \mathbb{Z}(n))) = 0. \]

From Lemma 6.3 and Proposition 6.1 a), we get

\[ \sum_{i}(-1)^{i} \cdot \text{rank } H_{c}^{i}(X_{ar}, \mathbb{Z}(n)) \]

\[ = \sum_{i}(-1)^{i} \cdot (\dim H_{c}^{i-1}(\bar{X}_{et}, \hat{\mathbb{Q}}(n))_{G} + \dim H_{c}^{i}(\bar{X}_{et}, \hat{\mathbb{Q}}(n))_{G}) \]

\[ = -\sum_{i}(-1)^{i} \dim H_{c}^{i}(\bar{X}_{et}, \hat{\mathbb{Q}}(n))_{G} = -\sum_{i}(-1)^{i} \dim H_{c}^{i}(\bar{X}_{et}, \hat{\mathbb{Q}}(n))_{\varphi = q^{n}} \]

By Grothendieck's formula for \( \zeta(X, s) \), the latter agrees with \( \text{ord}_{s=n} \zeta(X, s) \).

For smooth and projective varieties, the formula (14) holds by [6, Thm. 9.1] and Corollary 5.2. For arbitrary \( X \), note that by \( L(X, n) \) and Theorem 5.2 c), the cohomology groups of the complex \( (H^{*}(X, \mathbb{Z}(n)), e) \) are finite. By the argument of Lemma 2.7, it suffices to show the following: If \( U \subseteq X \) is an open subscheme with complement \( Z \), and the formula (14) holds for two of the three schemes \( Z, X \) and \( U \), then it also holds for the third. Consider the double complex
Remark. We cannot use de Jong’s Theorem on alterations to prove Theorem \ref{thm:main}. Taking horizontal cohomology, we see that the double complex is exact.

Taking vertical cohomology, we get the $E_1$-terms of a spectral sequence whose $E_1$-terms are finite, which converges to zero, and which has only finitely many differentials, i.e. $E_r = E_\infty$ for $r >> 0$. An inspection shows that the equality

$$\chi(H^*_c(X, \mathbb{Z}(n)), \epsilon) = \chi(H^*_c(Z, \mathbb{Z}(n)), \epsilon) \cdot \chi(H^*_c(U, \mathbb{Z}(n)), \epsilon)$$

is equivalent to the product of the orders of the $E_1$-terms on a anti-diagonal being equal for two adjacent anti-diagonals, i.e. $\prod_i |E^{i-1}_1| = \prod_i |E^{i+1}_{1-r-1}|$.

But it is easy to see that this property is preserved under differentials, i.e.

$\prod_i |E^{i-1}_r| = \prod_i |E^{i+1}_{r+1}|$ if and only if $\prod_i |E^{i+1}_r| = \prod_i |E^{i-1}_{r+1}|$.

Now the claim follows because the spectral sequence converges to zero, and $E_r = E_\infty$ for $r >> 0$, hence both sides equal one for $r >> 0$.

For the $\rho$-part, it is easy to see that for fixed $i$, $\sum_j (-1)^j \dim H^j(X, \rho^j \Omega^1)$ is compatible with the localization sequences\ref{locseq}, thus in view of Theorem \ref{thm:main} we can deduce the result from the smooth and projective case\ref{thm:main} Thm. 0.1. \qed

Remark. We cannot use de Jong’s Theorem on alterations to prove Theorem \ref{thm:main} because it is not clear how the formula $Z(X, n)$ behaves under finite etale Galois extensions. Also, it does not suffice to assume $K(X, n)$ instead of $L(X, n)$. Indeed, one can construct a diagram of the form\ref{diag}, with torsion vertical cohomology groups, where all vertical cohomology groups for two of the three complexes are zero, but the vertical cohomology of the third complex is not finitely generated (because if one considers the spectral sequence to the double complex\ref{diag}, one can have a differential $d_3$ that is non-trivial for infinitely many primes).

8 Examples

**Theorem 8.1**

a) If $R(d)$ holds, then $L(X, n)$ holds for all $X \in \text{Sch}^d/F_q$ and $n \leq 0$.

b) If $\dim X \leq 1$, then $L(X, n)$ holds for all $n \in \mathbb{Z}$.

c) Let $X$ be a surface for which every irreducible component is birationally equivalent to a surface satisfying Tate’s conjecture. Then $L(X, n)$ and $Z(X, n)$ holds for $n \leq 1$. 

\textit{Proof.} a) For \(X\) smooth and projective this is \cite[Prop. 9.2]{ref} and Corollary \ref{cor}. The general case follows with Lemma \ref{lemma}.

b) The statement is easy for zero-dimensional schemes, so it suffices by Lemma \ref{lemma} to show the statement for a smooth and projective curve. But then the result is the combination of \cite[Prop. 9.4]{ref}, \cite[Thm. 8.4]{ref} and Corollary \ref{cor}.

c) By the curve case and Lemma \ref{lemma} we can assume that \(X\) is smooth and projective. In this case, we apply \cite[Thm. 9.3]{ref} for \(n = 1\), and a) for \(n \leq 0\). \(\square\)

**Example 1.** (Zero-dimensional schemes) Since arithmetic cohomology groups and zeta functions are invariant under nilpotent extensions and compatible with coproducts, it suffices to consider the case \(X = \text{Spec} \mathbb{F}_q^r\). The zeta-function is

\[
\zeta(\mathbb{F}_q^r, s) = \zeta(\mathbb{F}_q^r, rs) = \frac{1}{1 - q^{-rs}}.
\]

Let \(w_{rn} = |\mathbb{Q}/\mathbb{Z}(n)^{\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q^r)}| = q^{rn} - 1\) if \(n \neq 0\), and \(w_0 = 1\). Then

\[
H^i_c((\mathbb{F}_q^r)_{\text{ar}}, \mathbb{Z}(n)) = \begin{cases} 
\mathbb{Z} & n = 0, i = 0, 1; \\
\mathbb{Z}/w_{rn} & n \neq 0, i = 1; \\
0 & \text{otherwise.}
\end{cases}
\]

(16)

This is clear for \(n = 0\). For \(n \neq 0\), we use that \(H^i_c((\mathbb{F}_q^r)_{\text{ar}}, \mathbb{Z}(n)) = H^{i-1}((\mathbb{F}_q^r)_{\text{et}}, \mathbb{Q}/\mathbb{Z}(n))\), because cohomology with rational coefficients vanishes. For the \(p\)-part, one checks easily that \(\chi(n) = rn\) for \(n \geq 0\) and \(\chi(n) = 0\) for \(n < 0\). For \(n \neq 0\), formula (14) becomes the identity \(1 - q^{-rn} = \pm (1 - q^{n-s})^0 w_{n^{-1}}^{-1} q^\chi(n)\). For \(n = 0\), we have \(\chi(H^*_c((\mathbb{F}_q^r)_{\text{ar}}, \mathbb{Z}(0)), e) = \frac{1}{r}\), because the map \(H^0_c((\mathbb{F}_q^r)_{\text{ar}}, \mathbb{Z}) \rightarrow H^1_c((\mathbb{F}_q^r)_{\text{ar}}, \mathbb{Z})\) is an injection with cokernel \(\mathbb{Z}/r\). Indeed, it is induced by the map \((\mathbb{Z}/r)^G \rightarrow (\mathbb{Z}/r)^G\), and the former is generated by \((1, \ldots, 1)\), whereas the latter is generated by the class of \((1, 0, \ldots, 0)\).

Formula (14) holds because \(\lim_{s \rightarrow 0} \frac{1 - e^{s(q^{-1})}}{1 - q^{-1}} = r\).

**Example 2.** Let \(C = \mathbb{P}^1/0 \sim 1\) be the node, and \(\mathbb{P}^1\) be its normalization. Consider the blow-up square

\[
\begin{array}{c}
\text{Spec } \mathbb{F}_q \amalg \text{Spec } \mathbb{F}_q \xrightarrow{\iota} \mathbb{P}^1 \\
\downarrow \iota' \quad \downarrow \\
\text{Spec } \mathbb{F}_q \rightarrow C
\end{array}
\]

The corresponding long exact sequence \ref{seq} breaks up into short exact sequences

\[
0 \rightarrow H^{i-1}_c((\mathbb{F}_p)_{\text{et}}, \mathbb{Z}(n)) \rightarrow H^i_c(C_{\text{et}}, \mathbb{Z}(n)) \rightarrow H^i_c((\mathbb{P}^1)_{\text{et}}, \mathbb{Z}(n)) \rightarrow 0,
\]

(18)

because both maps \(\iota^*\) and \(\iota'^*\) have image the diagonal of \(H^i_c((\mathbb{F}_p)_{\text{et}}, \mathbb{Z}(n)) \oplus H^i_c((\mathbb{F}_p)_{\text{et}}, \mathbb{Z}(n))\). Using the projective bundle formula and Example 1, we get
Arithmetic cohomology over finite fields and special values of \( \zeta \)-functions

\[
\text{rank } H^i_c(C_{\text{ar}}, \mathbb{Z}(n)) = \begin{cases} 
2 & (n, i) = (0, 1); \\
1 & (n, i) = (0, 0), (0, 2), (1, 2), (1, 3); \\
0 & \text{otherwise.}
\end{cases}
\]

For the weighted alternating sum of the ranks we get \( \rho_1 = -1 \), and \( \rho_n = 0 \) for \( n \neq 0 \). For the precise value of the zeta-function, one calculates from (18), Example 1 and the projective bundle formula

\[
|H^i_c(C_{\text{ar}}, \mathbb{Z}(n))_{\text{tor}}| = \begin{cases} 
w_n & i = 1, 2; \\
w_{n-1} & i = 3; \\
0 & \text{otherwise.}
\end{cases}
\]

This gives \( \chi(H^*_c(C_{\text{ar}}, \mathbb{Z}(n)), e) = w_{n-1}^{-1} \). The same argument shows that \( \chi(n) = n - 1 \) for \( n > 0 \) and \( \chi(n) = 0 \) for \( n \leq 0 \), hence for \( s \mapsto n \)

\[
\zeta(C, s) = \frac{1}{1 - q^{-s}} = \pm (1 - q^{n-s})^{\rho_n} \cdot q^{\chi(n)} \cdot w_{n-1}^{-1}.
\]

**Example 3.** We give an example which shows that all conjectures above are wrong if stated for the etale topology instead of the eh-topology (we received help from C. Weibel in constructing this example). Let \( X \) be a normal surface over \( \mathbb{F}_q \) of Reid with one singular point \( P \), such that the blow-up \( X' \) of \( X \) at \( P \) is smooth and has a node \( C \) as the exceptional divisor see [25, 6.6].

**Proposition 8.2** a) We have

\[
H^i(X_{\text{et}}, \mathbb{Q}) = \begin{cases} 
\mathbb{Q} & i = 0; \\
0 & i > 0;
\end{cases}
\]

\[
H^i(X_{\text{eh}}, \mathbb{Q}) = \begin{cases} 
\mathbb{Q} & i = 0, 2; \\
0 & i > 0.
\end{cases}
\]

b) Let \( i : P \to X, i' : C \to X' \) be the closed embeddings. Then \( H^2_c(X, \text{cone}(\mathbb{Q} \to i_*\mathbb{Q})) \cong 0 \) and \( H^2_c(X', \text{cone}(\mathbb{Q} \to i'_*\mathbb{Q})) \cong \mathbb{Q} \), i.e. etale cohomology with compact support of \( X - P \) depends on the compactification.

c) We have \( \mathbb{Q}/\mathbb{Z} \subseteq H^3_W(X, \mathbb{Z}) \). In particular, Weil-etale motivic cohomology groups are not finitely generated in general, even for proper schemes.

d) We have

\[
H^i(X_{\text{et}}, \hat{\mathbb{Q}}) = \begin{cases} 
\hat{\mathbb{Q}} & i \leq 3; \\
0 & i > 3.
\end{cases}
\]

In particular, Conjecture \( K(X, 0) \) is wrong if we use etale cohomology instead of eh-cohomology.

e) We have
strictly local ring of \(X\) and \(H\) the analog of Theorems 5.2 b) and 3.6, we get.

By a) and the analog of Theorem 5.2 c) the extreme terms vanish, and using geometric point \(x\) cohomology is torsion. Hence \(H_{\text{et}}\) we have \(\operatorname{ord}_s\Xi_s = 0\) and since \(\operatorname{ord}_s\Xi_s = 0\) and the analog of (19).

Proof. a) Let \(g : \text{Spec } F \rightarrow X\) be the generic point of \(X\). Since \(X\) is normal, we have \(g_*\underline{\mathbb{Q}} \cong \mathbb{Q}\), and \(R^pg_*\underline{\mathbb{Q}} = 0\) for \(s > 0\), because the stalk of \(R^pg_*\underline{\mathbb{Q}}\) at a geometric point \(x \in X\) is \(H^s(K_{\text{et}}, \underline{\mathbb{Q}}_X)\), where \(K\) is the field of fractions of the strictly local ring of \(X\) at \(x\). The latter group is zero because higher Galois cohomology is torsion. Hence \(H^i(X_{\text{et}}, \underline{\mathbb{Q}}) \cong H^i(F_{\text{et}}, \underline{\mathbb{Q}}) = 0\) for \(i > 0\).

To calculate the \(\text{eh}\)-cohomology of \(\mathbb{Q}\), we apply the long exact sequence (3), using \(H^i(X_{\text{eh}}, \underline{\mathbb{Q}}) = H^i(P_{\text{eh}}, \underline{\mathbb{Q}}) = 0\) for \(i > 0\), and the analog of (19).

b) This follows by a) from the long exact sequence (1) and \(H^1(\mathbb{F}_q, \underline{\mathbb{Q}}) = 0\) and \(H^1(\mathbb{C}, \underline{\mathbb{Q}}) \cong \mathbb{Q}\), (19).

c) The long exact coefficient sequence for Weil-\(\text{et}\) cohomology gives

\[
\cdots \rightarrow H^2_{\text{et}}(X, \mathbb{Q}) \rightarrow H^2_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \rightarrow H^3_{\text{et}}(X, \mathbb{Z}) \rightarrow H^3_{\text{et}}(X, \mathbb{Q}) \rightarrow \cdots. \tag{20}
\]

By a) and the analog of Theorem 5.2 c) the extreme terms vanish, and using the analog of Theorems (20) b) and (10) we get

\[
H^2(X_{\text{eh}}, \mathbb{Q}/\mathbb{Z}) \cong H^2(X_{\text{et}}, \mathbb{Q}/\mathbb{Z}) \cong H^3_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \cong H^3_{\text{et}}(X, \mathbb{Z}).
\]

But by a), the former group contains \(\mathbb{Q}/\mathbb{Z} = H^2(X_{\text{eh}}, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z}\). (This phenomenon does not occur for the Weil-\(\text{eh}\)-topology, because the extreme terms in (20) do not vanish.)

d) This can be calculated using blow-up sequences by Lemma (5).

e) Counting points, we get for the zeta-function the formula

\[
\zeta(X, s) \cdot \zeta(C, s) \cong \zeta(X', s) \cdot \zeta(P, s),
\]

and since \(\operatorname{ord}_s=0\zeta(\mathbb{F}_q, s) = -1\), \(\operatorname{ord}_s=0\zeta(X', s) = -1\), and \(\operatorname{ord}_s=0\zeta(C, s) = \operatorname{ord}_s=0\zeta(A^1, s) = 0\), we have \(\operatorname{ord}_s=0\zeta(X, s) = -2\). But in view of \(H^1_{\text{et}}(X, \mathbb{Q}) \cong H^1(X_{\text{et}}, \mathbb{Q}) \otimes H^{-1}(X_{\text{et}}, \mathbb{Q})\) we have \(-1\) for the weighted alternating sum of Weil-\(\text{et}\)-cohomology groups, and \(-2\) for the weighted alternating sum of arithmetic cohomology groups.

\[\square\]

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