Multi-time state mean-variance model in continuous time *

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Abstract: The objective of the continuous time mean-variance model is to minimize the variance (risk) of an investment portfolio with a given mean at terminal time. However, the investor can stop the investment plan at any time before the terminal time. To solve this problem, we consider minimizing the variances of the investment portfolio in the multi-time state. The advantage of this multi-time state mean-variance model is that we can minimize the risk of the investment portfolio within the investment period. To obtain the optimal strategy of the model, we introduce a sequence of Riccati equations, which are connected by a jump boundary condition. Based on this equations, we establish the relationship between the means and variances in the multi-time state mean-variance model. Furthermore, we use an example to verify that the variances of the multi-time state can affect the average of Maximum-Drawdown of the investment portfolio.

Keywords: mean-variance; multi-time state; stochastic control.

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OR/MS subject classification: Finance/portfolio; dynamic programming/optimal control.

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1 Introduction

To balance the return (mean) and risk (variance) in a single-period portfolio selection model, Markowitz (1952, 1959) proposed the mean-variance model. Since then, many related works focused on these topics. Under some mild assumptions, Merton (1972) solved the single-period problem analytically. Richardson (1989) studied a mean-variance model in which a single stock with a constant risk-free rate is introduced. Dynamic asset allocation in a mean-variance framework was studied by Bajouex-Besnainou and Portait (1998). Li and Ng (2000) embedded the discrete-time multi-period mean-variance problem within a multi-objective optimization framework and obtained an optimal strategy. By extending the embedding technique introduced in Li and Ng (2000) and applying the results from the stochastic LQ control in the continuous time case, Zhou and Li (2000) investigated an optima pair for the continuous-time mean-variance problem. Further results in the mean-variance problem include those with bankruptcy prohibition, transaction costs, and random parameters in an complete and incomplete markets (see Bielecki et al. (2005); Dai et al. (2010); Lim (2004); Lim and Zhou (2002); Xia (2005)).

The pre-committed strategies in the aforementioned multi-period and continuous time cases, differed from those of the single-period case. For further details, see (Kydland and Prescott, 1997). Basak and Chabakauri (2010) adopted a game theoretic approach to study the time inconsistency in the mean-variance model and Björk et al. (2014) studied the mean-variance problem with state dependent risk aversion.

In the financial market, for a given terminal time $T$, $Y^\pi(T)$ represents a portfolio asset with strategy $\pi(\cdot)$, while $\mathbb{E}[Y^\pi(T)]$ and $\text{Var}(Y^\pi(T)) = \mathbb{E}(Y^\pi(T) - \mathbb{E}[Y^\pi(T)])^2$ represent the mean and variance of $Y^\pi(T)$, respectively. In the classical mean-
variance model, we want to minimize the variance of the portfolio asset \( \text{Var}(Y^{\pi}(T)) \) for a given mean \( \mathbb{E}[Y^{\pi}(T)] = L \), where \( L \) is a constant. The investor can stop the investment plan at an uncertain horizon time \( \tau \) before the terminal time \( T \), where \( \tau \leq T \). Therefore, there are many related works on the mean-variance portfolio model with an uncertain horizon time. Martellini and Urošević (2006) considered static mean-variance analysis with an uncertain time horizon. Yi et al. (2008) studied the mean-variance model of a multi-period asset-liability management problem under uncertain exit time. Furthermore, see (Wu et al., 2011; Yao and Ma, 2010; Yu, 2013) for additional studies in this vein. However, in the literature of mean-variance model under uncertain or random exit time, we always suppose that the uncertain horizon time \( \tau \) satisfies a distribution (or a conditional distribution) and investigate the related mean-variance model at time \( \tau \).

However, in general, we do not know the information of \( \tau \) at initial time \( t_0 = 0 \). Given a probability space \((\Omega, \mathcal{F}, P)\), notice that for a given partition \( 0 = t_0 < t_1 < \cdots < t_N = T \) of interval \([0, T]\) and \( \omega \in \Omega \), there exists \( i \in \{0, 1, \cdots, N - 1\} \) such that \( \tau(\omega) \in [t_i, t_{i+1}] \). To reduce the variance of the portfolio asset \( Y^{\pi}(\cdot) \) at \( \tau \in (0, T] \), we consider minimizing the variances of the portfolio asset at multi-time state \( (Y^{\pi}(t_1), Y^{\pi}(t_2), \cdots, Y^{\pi}(t_N)) \) with constraint on means of multi-time state \( (Y^{\pi}(t_1), Y^{\pi}(t_2), \cdots, Y^{\pi}(t_N)) \). Therefore, we introduce the following multi-time state mean-variance model:

\[
J(\pi(\cdot)) = \sum_{i=1}^{N} \mathbb{E}(Y^{\pi}(t_i) - \mathbb{E}[Y^{\pi}(t_i)])^2, \tag{1.1}
\]

with constraint on the multi-time state mean,

\[
\mathbb{E}[Y^{\pi}(t_i)] = L_i, \quad i = 1, 2, \cdots, N. \tag{1.2}
\]

In this multi-time state mean-variance model, we can minimize the risk of the investment portfolio within the multi-time \((t_1, t_2, \cdots, t_N)\). It should be noted that the
multi-time state \((Y^\pi(t_1), Y^\pi(t_2), \cdots, Y^\pi(t_N))\) of the investment portfolio can affect the value of each other, and we cannot solve the multi-time state mean-variance model via one classical Riccati equation directly. To obtain the optimal strategy of the multi-time state mean-variance model, we introduce a sequence of Riccati equations, which are connected by a jump boundary condition (see equations (3.5) and (3.6)). Based on this sequence of Riccati equations, we investigate an optimal strategy (see Theorem 3.1) and establish the relationship between the means and variances of this multi-time state mean-variance model (see Lemma 3.1).

The Maximum-Drawdown of the asset \(Y^\pi(\cdot)\) is an important index to evaluate a strategy in the investment portfolio model, where the Maximum-Drawdown of the asset \(Y^\pi(\cdot)\) is defined in the interval \([0, h]\), \(h \leq T\), by

\[
\text{MD}^h_{Y^\pi} = \text{esssup} \{z \mid z = Y^\pi(t) - Y^\pi(s), \ 0 \leq t \leq s \leq h\}.
\]

Based on simulation results of the multi-time state mean-variance model (see subsection 4.2), we can see that the constrained condition (1.2) can affect the average of \(\text{MD}^h_{Y^\pi}\) of the portfolio asset \(Y^\pi(\cdot)\) (see Figure 3). The work is most closely related to the study of (Yang, 2018), in which the author established the necessary and sufficient conditions for stochastic differential systems with multi-time state cost functional.

The remainder of this paper is organized as follows. In Section 2, we formulate the multi-time state mean-variance model. Then, in Section 3, we investigate an optimal strategy and establish the relationship between multi-time state mean and variance for the proposed model. In Section 4, based on the main results of Section 3, we compare the multi-time state mean-variance model with classical mean-variance model. Finally, we conclude the paper in Section 5.
2 Multi-time state mean-variance model

Let $W$ be a $d$-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}(t)\}_{t \geq 0})$, where $\{\mathcal{F}(t)\}_{t \geq 0}$ is the $P$-augmentation of the natural filtration generated by $W$. We suppose the existence of one risk-free bond asset and $n$ risky stock assets that are traded in the market, where the bond satisfies the following equation:

$$
\begin{aligned}
&\begin{cases}
    \frac{dR_0(t)}{R_0(t)} = r(t) dt, & t > 0, \\
    R_0(0) = a_0 > 0,
\end{cases}
\end{aligned}
$$

and the $i$'th ($1 \leq i \leq n$) stock asset is described by

$$
\begin{aligned}
&\begin{cases}
    \frac{dR_i(t)}{R_i(t)} = b_i(t) dt + R_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t), & t > 0, \\
    R_i(0) = a_i > 0,
\end{cases}
\end{aligned}
$$

where $r(\cdot) \in \mathbb{R}$ is the risk-free return rate of the bond, $b(\cdot) = (b_1(\cdot), \cdots, b_n(\cdot)) \in \mathbb{R}^n$ is the expected return rate of the risky asset, and $\sigma(\cdot) = (\sigma_1(\cdot), \cdots, \sigma_n(\cdot))^T \in \mathbb{R}^{n \times d}$ is the corresponding volatility matrix. Given initial capital $x > 0$, $\gamma(\cdot) = (\gamma_1(\cdot), \cdots, \gamma_n(\cdot)) \in \mathbb{R}^n$, where $\gamma_i(\cdot) = b_i(\cdot) - r(\cdot)$, $1 \leq i \leq n$. The investor's wealth $Y^\pi(\cdot)$ satisfies

$$
\begin{aligned}
&\begin{cases}
    \frac{dY^\pi(t)}{Y^\pi(t)} = \left[ r(t)Y^\pi(t) + \gamma(t)\pi(t)^T \right] dt + \pi(t)\sigma(t)dW(t), \\
    Y^\pi(0) = y,
\end{cases}
\end{aligned}
$$

where $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot)) \in \mathbb{R}^n$ is the capital invested in the risky asset $R(\cdot) = (R_1(\cdot), \cdots, R_n(\cdot)) \in \mathbb{R}^n$ and $\pi_0(\cdot)$ is the capital invested in the bond. Thus, we have

$$
Y^\pi(\cdot) = \sum_{i=0}^{n} \pi_i(\cdot).
$$
In this study, we consider the following multi-time state mean-variance model:

\[
J_1(\pi(\cdot)) = \sum_{i=1}^{N} \mathbb{E}(Y^{\pi}(t_i) - \mathbb{E}[Y^{\pi}(t_i)])^2, \tag{2.2}
\]

with constraint on the multi-time state mean,

\[
\mathbb{E}[Y^{\pi}(t_i)] = L_i, \ i = 1, 2, \cdots, N, \tag{2.3}
\]

where \(0 = t_0 < t_1 < \cdots < t_N = T\). The set of admissible strategies \(\pi(\cdot)\) is defined as:

\[
\mathcal{A} = \left\{ \pi(\cdot) : \pi(\cdot) \in L^2_F[0,T;\mathbb{R}^n]\right\},
\]

where \(L^2_F[0,T;\mathbb{R}^n]\) is the set of all square integrable measurable \(\mathbb{R}^n\) valued \(\{\mathcal{F}_t\}_{t \geq 0}\) adaptive processes. If there exists a strategy \(\pi^*(\cdot) \in \mathcal{A}\) that yields the minimum value of the cost functional (2.2), then we say that the multi-time state mean-variance model (2.2) is solved.

We make the following assumptions to obtain the optimal strategy for the proposed model (2.2):

**H1:** \(r(\cdot), b(\cdot)\) and \(\sigma(\cdot)\) are bounded deterministic continuous functions.

**H2:** \(r(\cdot), \gamma(\cdot) > 0, \sigma(\cdot)\sigma(\cdot)^\top > \delta I\), where \(\delta > 0\) is a given constant and \(I\) is the identity matrix of \(\mathbb{S}^n\), \(\mathbb{S}^n\) is the set of symmetric matrices.

### 3 Optimal strategy

In this section, we investigate an optimal strategy \(\pi(\cdot)\) for the problem defined in (2.2), with a constraint on the multi-time state mean (2.3). Here, we describe how to construct an optimal strategy for (2.2) with constrained condition (2.3).

Similar to (Zhou and Li, 2000), we introduce the following multi-time state
mean-variance problem: minimizing the cost functional,

\[ J_2(\pi(\cdot)) = \sum_{i=1}^{N} \left( \frac{\mu_i}{2} \text{Var}(Y^\pi(t_i)) - \mathbb{E}[Y^\pi(t_i)] \right). \]  

To solve the cost functional (3.1), we employ the following model:

\[ J_3(\pi(\cdot)) = \sum_{i=1}^{N} \mathbb{E}\left[ \frac{\mu_i}{2} Y^\pi(t_i)^2 - \lambda_i Y^\pi(t_i) \right], \]  

where \( \mu_i > 0 \) and \( \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, N \). For given \( \mu_i > 0, i = 1, 2, \cdots, N \), we suppose \( \pi^*(\cdot) \) is an optimal strategy of cost functional (3.1). Based on Theorem 3.1 of (Zhou and Li, 2000), taking \( \lambda_i = 1 + \mu_i \mathbb{E}[Y^\pi(t_i)], i = 1, 2, \cdots, N \), we can show that \( \pi^*(\cdot) \) is an optimal strategy of cost functional (3.2). It should be noted that, we cannot solve the cost functional (3.2) by applying the embedding technique of (Zhou and Li, 2000) for the multi-time state mean-variance models via the classical Riccati equation directly. This is because the value \( Y^\pi(t_i) \) can affect \( Y^\pi(t_{i+1}) \), for \( i = 0, 1, \cdots, N - 1 \).

Denoting by

\[ \rho_i = \frac{\lambda_i}{\mu_i}, z^\pi_i(t) = Y^\pi(t) - \rho_i, t \leq t_i, i = 1, 2, \cdots, N, \]

\[ \beta(t) = \gamma(t)[\sigma(t)\sigma(t)^\top]^{-1}\gamma(t)^\top, t \leq T. \]

Thus, the cost functional (3.2) is equivalent to

\[ J_4(\pi(\cdot)) = \sum_{i=1}^{N} \mathbb{E}\left[ \frac{\mu_i}{2} z^\pi_i(t_i)^2 \right], \]  

where \( z^\pi_i(\cdot) \) satisfies

\[
\begin{cases}
    dz^\pi_i(t) = [r(t)z^\pi_i(t) + \gamma(t)\pi(t)^\top + \rho_i r(t)]dt + \pi(t)\sigma(t)dW(t), \\
    z^\pi_i(t_{i-1}) = Y^\pi(t_{i-1}) - \rho_i, t_{i-1} < t \leq t_i.
\end{cases}
\]  

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Now, we construct a new sequence of Riccati equations that are connected by a jump boundary condition, in which the jump boundary condition can offset the interaction of \( Y^\pi(t_{i+1}) \) and \( Y^\pi(t_i) \), for \( i = 0, 1, \cdots, N - 1 \). We first introduce a sequence of deterministic Riccati equations:

\[
\begin{cases}
    dP_i(t) = [\beta(t) - 2r(t)]P_i(t)dt, \\
    P_i(t_i) = \mu_i + P_{i+1}(t_i), \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \cdots, N,
\end{cases}
\]

and related equations,

\[
\begin{cases}
    dg_i(t) = [(\beta(t) - r(t))g_i(t) - \rho_i r(t) P_i(t)]dt, \\
    g_i(t_i) = g_{i+1}(t_i) + P_{i+1}(t_i)(\rho_i - \rho_{i+1}), \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \cdots, N,
\end{cases}
\]

where \( P_{N+1}(t_N) = 0, \ g_{N+1}(t_N) = 0, \ \rho_{N+1} = 0 \). Furthermore, by a simple calculation, we can obtain,

\[
\frac{g_i(t)}{P_i(t)} = \frac{g_i(t_i)}{P_i(t_i)} e^{-\int_{t_i}^{t} r(s)ds} + \rho_i (1 - e^{-\int_{t_i}^{t} r(s)ds}), \quad t_{i-1} \leq t \leq t_i, \quad i = 1, 2, \cdots, N,
\]

which is used to obtain the following results.

**Theorem 3.1.** Let Assumptions \( H_1 \) and \( H_2 \) hold, there exists an optimal strategy \( \pi^*(\cdot) \) for cost functional (3.3), where the optimal strategy \( \pi^*(\cdot) \) is given as follows:

\[
\pi^*(t) = \gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}[(\rho_i - \frac{g_i(t_i)}{P_i(t_i)})e^{-\int_{t_i}^{t} r(s)ds} - Y^\pi(t)], \quad t_{i-1} < t \leq t_i,
\]

where \( Y^\pi(t) = z^\pi(t) + \rho_i, \ t_{i-1} < t \leq t_i \) and \( i = 1, 2, \cdots, N \).

**Proof:** For any given \( i \in \{1, 2, \cdots, N\}, \ t_{i-1} < t \leq t_i \), applying Itô formula to
\[ z_i^\pi(t)^2 P_i(t) \text{ and } z_i^\pi(t)g_i(t), \text{ respectively, we have} \]

\[
\frac{1}{2} \left( 2z_i^\pi(t)P_i(t) \left[ r(t)z_i^\pi(t) + \gamma(t)\pi(t)^\top + \rho_i r(t) \right] + z_i^\pi(t)^2 \beta(t) - 2r(t) \right) P_i(t) \\
+ P_i(t)\pi(t)\sigma(t)\sigma(t)^\top \pi(t)^\top \right) dt + z_i^\pi(t)P_i(t)\pi(t)\sigma(t)dW(t) \\
= \frac{1}{2} \left( 2z_i^\pi(t)P_i(t) \left[ \gamma(t)\pi(t)^\top + \rho_i r(t) \right] + z_i^\pi(t)^2 \beta(t)P_i(t) \\
+ P_i(t)\pi(t)\sigma(t)\sigma(t)^\top \pi(t)^\top \right) dt + z_i^\pi(t)P_i(t)\pi(t)\sigma(t)dW(t)
\]

and

\[
dz_i^\pi(t)g_i(t) \\
= \left\{ g_i(t)\gamma(t)\pi(t)^\top + g_i(t)\rho_i r(t) + z_i^\pi(t)\beta(t)g_i(t) - \rho_i r(t)P_i(t) \right\} dt \\
+ g_i(t)\pi(t)\sigma(t)dW(t).
\]

We add the above two equations together and integrate from \( t_{i-1} \) to \( t_i \), it follows
that

\[
\mathbb{E} \left[ \frac{P_i(t_i) z_i^\mu(t_i)}{2} \frac{z_i^\mu(t_i)}{2} - \frac{P_i(t_{i-1}) z_i^\mu(t_{i-1})}{2} z_i^\mu(t_{i-1}) + z_i^\mu(t_i) g_i(t_i) - z_i^\mu(t_{i-1}) g_i(t_{i-1}) \right] \\
= \mathbb{E} \left[ \mu_i + \frac{P_{i+1}(t_i)}{2} z_i^\mu(t_i) - \frac{P_i(t_{i-1})}{2} z_i^\mu(t_{i-1}) + z_i^\mu(t_i) [g_{i+1}(t_i) + P_{i+1}(t_i)(\rho_i - \rho_{i+1})] - z_i^\mu(t_{i-1}) g_i(t_{i-1}) \right] \\
= \mathbb{E} \left[ \mu_i + \frac{P_{i+1}(t_i)}{2} z_i^\mu(t_i) - \frac{P_i(t_{i-1})}{2} z_i^\mu(t_{i-1}) + z_i^\mu(t_i) [g_{i+1}(t_i) + P_{i+1}(t_i)(\rho_i - \rho_{i+1})] - [z_i^\mu(t_{i-1}) + \rho_i - \rho_i] g_i(t_{i-1}) \right] \\
= \mathbb{E} \left[ \mu_i + \frac{P_{i+1}(t_i)}{2} z_i^\mu(t_i) - \frac{P_i(t_{i-1})}{2} z_i^\mu(t_{i-1}) + \frac{P_i(t_{i-1})}{2} z_i^\mu(t_{i-1}) - P_i(t_{i-1})(\rho_i - \rho_i) z_i^\mu(t_{i-1}) g_i(t_{i-1}) \right] \\
+ \frac{P_i(t_{i-1})}{2} z_i^\mu(t_{i-1})^2 - \frac{P_i(t_{i-1})}{2} z_i^\mu(t_{i-1})^2 - 2 \gamma(t) \pi(t) (z_i^\mu(t_i) P_i(t) + g_i(t)) \\
+ z_i^\mu(t_i) \beta(t) P_i(t) + 2 z_i^\mu(t_i) \beta(t) g_i(t) + 2 g_i(t) \rho_i r(t) \right] dt \\
= \mathbb{E} \left[ \int_{t_i}^{t_i} \left\{ \pi(t) + \gamma(t)(\sigma(t) \sigma(t)^\top) - 1 (z_i^\mu(t_i) + \frac{g_i(t)}{P_i(t)}) \sigma(t) P_i(t) \sigma(t)^\top \right. \\
\left. - \gamma(t) \pi(t) \sigma(t) \sigma(t)^\top - 1 \gamma(t) g_i(t) + 2 g_i(t) \rho_i r(t) \right] dt, \right.
\]

the third equality is derived by the following results,

\[
z_i^\mu(t_{i-1}) = Y^\pi(t_{i-1}) - \rho_i \\
= Y^\pi(t_{i-1}) - \rho_{i-1} + \rho_{i-1} - \rho_i \\
= z_i^\mu(t_{i-1}) + \rho_{i-1} - \rho_i,
\]
where \( z_i(t_0) = y_0, \rho_0 = 0. \)

Thus, we have

\[
\begin{align*}
\mathbb{E}\left[ \frac{\mu_i}{2} z_i^\pi(t_i)^2 - (\rho_{i-1} - \rho_i)^2 \frac{P_i(t_{i-1})}{2} - (\rho_{i-1} - \rho_i) g_i(t_{i-1}) \right] \\
+ \frac{P_{i+1}(t_i)}{2} z_i^\pi(t_i)^2 + P_{i+1}(t_i)(\rho_i - \rho_{i+1}) z_i^\pi(t_i) + z_i^\pi(t_i) g_{i+1}(t_i) \\
- \frac{P_i(t_{i-1})}{2} z_{i-1}^\pi(t_{i-1})^2 - P_i(t_{i-1})(\rho_{i-1} - \rho_i) z_{i-1}^\pi(t_{i-1}) - z_{i-1}^\pi(t_{i-1}) g_i(t_{i-1}) \right] \\
= \frac{1}{2} \mathbb{E} \int_{t_{i-1}}^{t_i} \left[ \pi(t) + \gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}(z_i^\pi(t) + \frac{g_i(t)}{P_i(t)})] \sigma(t) P_i(t) \sigma(t)^\top \\
[\pi(t) + \gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}(z_i^\pi(t) + \frac{g_i(t)}{P_i(t)})] \sigma(t) P_i(t) \sigma(t)^\top \\
- \gamma(t) P_i(t) \sigma(t) \sigma(t)^\top)^{-1} \gamma(t)^\top g_i(t)^2 + 2 g_i(t) r(t) \right] \, dt.
\end{align*}
\]

Adding \( i \) on both sides of equation (3.7) from 1 to \( N \), it follows that

\[
\begin{align*}
\sum_{i=1}^{N} \mathbb{E}\left[ \frac{\mu_i}{2} z_i^\pi(t_i)^2 - (\rho_{i-1} - \rho_i)^2 \frac{P_i(t_{i-1})}{2} - (\rho_{i-1} - \rho_i) g_i(t_{i-1}) \right] \\
+ \frac{P_{i+1}(t_i)}{2} z_i^\pi(t_i)^2 + P_{i+1}(t_i)(\rho_i - \rho_{i+1}) z_i^\pi(t_i) + z_i^\pi(t_i) g_{i+1}(t_i) \\
- \frac{P_i(t_{i-1})}{2} z_{i-1}^\pi(t_{i-1})^2 - P_i(t_{i-1})(\rho_{i-1} - \rho_i) z_{i-1}^\pi(t_{i-1}) - z_{i-1}^\pi(t_{i-1}) g_i(t_{i-1}) \right] \\
= \sum_{i=1}^{N} \mathbb{E}\left[ \frac{\mu_i}{2} z_i^\pi(t_i)^2 - (\rho_{i-1} - \rho_i)^2 \frac{P_i(t_{i-1})}{2} - (\rho_{i-1} - \rho_i) g_i(t_{i-1}) \right] \\
- \mathbb{E} \left[ \frac{P_i(t_{i-1})}{2} z_{i-1}^\pi(t_{i-1})^2 + P_i(t_{i-1})(\rho_{i-1} - \rho_i) z_{i-1}^\pi(t_{i-1}) + z_{i-1}^\pi(t_{i-1}) g_i(t_{i-1}) \right] \\
= \sum_{i=1}^{N} \frac{1}{2} \mathbb{E} \int_{t_{i-1}}^{t_i} \left[ \pi(t) + \gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}(z_i^\pi(t) + \frac{g_i(t)}{P_i(t)})] \sigma(t) P_i(t) \sigma(t)^\top \\
[\pi(t) + \gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}(z_i^\pi(t) + \frac{g_i(t)}{P_i(t)})] \sigma(t) P_i(t) \sigma(t)^\top \\
- \gamma(t) P_i(t) \sigma(t) \sigma(t)^\top)^{-1} \gamma(t)^\top g_i(t)^2 + 2 g_i(t) r(t) \right] \, dt.
\end{align*}
\]
This completes the proof. □

Based on the representation of $\mathbb{E}\left[ \sum_{i=1}^{N} \frac{\mu_i}{2} z_i^2(t_i) \right]$, we can obtain an optimal strategy $\pi^*(\cdot)$ for $J_4(\pi(\cdot))$, for $t \in (t_{i-1}, t_i)$, $i = 1, 2, \cdots, N$,

$$
\pi^*(t) = -\gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}(z^\pi(t) + \frac{g_i(t)}{P_i(t)})^\top 
$$

Note that

$$
g_i(t) = \frac{g_i(t_i)}{P_i(t_i)} e^{-\int_{t_{i-1}}^{t} r(s)ds} + \rho_i (1 - e^{-\int_{t_{i-1}}^{t} r(s)ds}), \quad t_{i-1} < t \leq t_i,
$$

where

$$
g_i(t_i) = \frac{g_{i+1}(t_i) + P_{i+1}(t_i)(\rho_i - \rho_{i+1})}{\mu_i + P_{i+1}(t_i)}, \quad i = 1, 2, \cdots, N,
$$

which leads to

$$
\pi^*(t) = \gamma(t)(\sigma(t)\sigma(t)^\top)^{-1}\left[ (\rho_i - \frac{g_i(t_i)}{P_i(t_i)})e^{-\int_{t_{i-1}}^{t} r(s)ds} - Y^*(t) \right], \quad t_{i-1} < t \leq t_i,
$$

where $Y^*(t) = z^\pi(t) + \rho_i$, $t_{i-1} < t \leq t_i$ and $i = 1, 2, \cdots, N$.

This completes the proof. □
Now, we consider the process of portfolio asset equation according to $\pi^*(\cdot)$,

\[
\begin{aligned}
\frac{dY^*(t)}{dt} &= [r(t)Y^*(t) + \gamma(t)\pi^*(t)]dr + \pi^*(t)\sigma(t)dW(t), \\
Y^*(0) &= y.
\end{aligned}
\] (3.10)

$\mathbb{E}[Y^*(\cdot)]$ and $\mathbb{E}[Y^*(\cdot)^2]$ satisfy the following linear ordinary differential equations:

\[
\begin{aligned}
\frac{d\mathbb{E}[Y^*(t)]}{dt} &= \left[r(t) - \beta(t)\mathbb{E}[Y^*(t)] + \left(\rho_i - \frac{g_i(t_i)}{P_i(t_i)}\right)e^{-\int_{t_i}^{t} r(s)ds}\beta(t)\right]dt, \\
Y^*(0) &= y, \ t_{i-1} < t \leq t_i, \ i = 1, 2, \cdots, N,
\end{aligned}
\] (3.11)

and

\[
\begin{aligned}
\frac{d\mathbb{E}[Y^*(t)^2]}{dt} &= \left[2r(t) - \beta(t)\mathbb{E}[Y^*(t)^2] + \left(\rho_i - \frac{g_i(t_i)}{P_i(t_i)}\right)^2e^{-\int_{t_i}^{t} 2r(s)ds}\beta(t)\right]dt, \\
Y^*(0)^2 &= y^2, \ t_{i-1} < t \leq t_i, \ i = 1, 2, \cdots, N.
\end{aligned}
\] (3.12)

In the following, we investigate the efficient frontier of the multi-time state mean-variance $\text{Var}(Y^*(t_i))$ and $\mathbb{E}[Y^*(t_i)]$.

**Lemma 3.1.** Let Assumptions $H_1$ and $H_2$ hold, the relationship of $\text{Var}(Y^*(t_i))$ and $\mathbb{E}[Y^*(t_i)]$ is given as follows:

\[
\text{Var}(Y^*(t_i)) = \text{Var}(Y^*(t_{i-1}))e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt} + \frac{\left(\mathbb{E}[Y^*(t_i)] - \mathbb{E}[Y^*(t_{i-1})]\right)e^{\int_{t_{i-1}}^{t_i} r(t)dt}}{e^{\int_{t_{i-1}}^{t_i} \beta(t)dt} - 1}.
\] (3.13)

where $i = 1, 2, \cdots, N$.

**Proof:** Combining equations (3.11) and (3.12), we have for $i = 1, 2, \cdots, N,$

\[
\mathbb{E}[Y^*(t_i)] = \mathbb{E}[Y^*(t_{i-1})]e^{\int_{t_{i-1}}^{t_i} [r(t) - \beta(t)]dt} + \left(\rho_i - \frac{g_i(t_i)}{P_i(t_i)}\right)(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}),
\] (3.14)
and

$$E[Y^*(t_i)^2] = E[Y^*(t_{i-1})^2]e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt} + (\rho_i - \frac{g_i(t_i)}{P(t_i)})^2\left(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}\right). \quad (3.15)$$

By equation (3.14), we have

$$\rho_i - \frac{g_i(t_i)}{P(t_i)} = \frac{E[Y^*(t_i)] - E[Y^*(t_{i-1})]e^{\int_{t_{i-1}}^{t_i} [r(t) - \beta(t)]dt}}{1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}}.$$ 

Plugging $\rho_i - \frac{g_i(t_i)}{P(t_i)}$ into equation (3.15), it follows that

$$E[Y^*(t_i)^2] = E[Y^*(t_{i-1})^2]e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt} + \left(E[Y^*(t_i)] - E[Y^*(t_{i-1})]e^{\int_{t_{i-1}}^{t_i} [r(t) - \beta(t)]dt}\right)^2 \left(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}\right),$$

and thus

$$\text{Var}(Y^*(t_i))(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt})$$

\begin{align*}
= & \left(E[Y^*(t_{i-1})^2] - [EY^*(t_{i-1})]^2\right)e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt}(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}) \\
+ & [EY^*(t_{i-1})]^2\left(e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt} - e^{\int_{t_{i-1}}^{t_i} [2r(t) - 2\beta(t)]dt}\right) \\
+ & \left(E[Y^*(t_i)] - E[Y^*(t_{i-1})]e^{\int_{t_{i-1}}^{t_i} [r(t) - \beta(t)]dt}\right)^2 + (e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt} - 1)[EY^*(t_i)]^2 \\
= & \left(E[Y^*(t_{i-1})^2] - [EY^*(t_{i-1})]^2\right)e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt}(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}) \\
+ & [EY^*(t_{i-1})]^2e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt} + [EY^*(t_i)]^2e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt} \\
- & 2E[Y^*(t_i)]E[Y^*(t_{i-1})]e^{\int_{t_{i-1}}^{t_i} [r(t) - \beta(t)]dt} \\
= & \left(E[Y^*(t_{i-1})^2] - [EY^*(t_{i-1})]^2\right)e^{\int_{t_{i-1}}^{t_i} [2r(t) - \beta(t)]dt}(1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}) \\
+ & e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}\left(E[Y^*(t_i)] - E[Y^*(t_{i-1})]e^{\int_{t_{i-1}}^{t_i} r(t)dt}\right)^2, \\
\end{align*}
which deduces that

\[ \text{Var}(Y^*(t_i)) = \text{Var}(Y^*(t_{i-1})) e^{\int_{t_{i-1}}^{t_i} [\beta(t)] dt} + \left( \frac{\mathbb{E}[Y^*(t_i)] - \mathbb{E}[Y^*(t_{i-1})] e^{\int_{t_{i-1}}^{t_i} [r(t)] dt}}{e^{\int_{t_{i-1}}^{t_i} [\beta(t)] dt} - 1} \right)^2. \]

This completes the proof. \( \Box \)

**Remark 3.1.** Specially, for \( i = 1 \), one obtains

\[ \text{Var}(Y^*(t_1)) = \frac{\left( \mathbb{E}[Y^*(t_1)] - y e^{\int_{t_0}^{t_1} [r(t)] dt} \right)^2}{e^{\int_{t_0}^{t_1} [\beta(t)] dt} - 1}, \]

which is the same as the efficient frontier in (Zhou and Li, 2000).

It should be noted that the optimal strategy \( \pi^*(\cdot) \) of cost functional (3.3) depends on the parameters \( \mu = (\mu_1, \cdots, \mu_N) \), \( \lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{R}^N \). We want to show that there exist \( \lambda \) and \( \mu \) such that the optimal strategy \( \pi^*(\cdot) \) of cost functional (3.3) is an optimal strategy of cost functional (3.2).

**Theorem 3.2.** Let Assumptions \( H_1, H_2 \) hold, and

\[ L_i - L_{i-1} e^{\int_{t_{i-1}}^{t_i} [r(t)] dt} > 0, \quad i = 1, 2, \cdots, N; \]

\[ [1 + P_{i+1}(t_i) \rho_{i+1} - g_{i+1}(t_i)] (1 - e^{-\int_{t_{i+1}}^{t_i} [\beta(t)] dt}) \]

\[ > [L_i - L_{i-1} e^{\int_{t_{i-1}}^{t_i} [r(t)] dt}] P_{i+1}(t_i), \quad i = 1, 2, \cdots, N - 1, \]

where \( L_0 = y \). There exists \( \lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_N^*) \), \( \mu = (\mu_1, \mu_2, \cdots, \mu_N) \in \mathbb{R}^N \) which are determined by

\[ \lambda_i^* = 1 + \mu_i \mathbb{E}[Y^*(t_i)], \quad \rho_i = \frac{\lambda_i^*}{\mu_i}, \quad i = 1, 2, \cdots, N, \]

such that the optimal strategy \( \pi^*(\cdot) \) of cost functional (3.3) is an optimal strategy of cost functional (3.2).
**Proof:** By Theorem 3.1, an optimal strategy of model (3.2) can be solved by (3.3), let

$$\lambda_i^* = 1 + \mu_i \mathbb{E}[Y^*(t_i)], \quad \rho_i = \frac{\lambda_i^*}{\mu_i}, \quad i = 1, 2, \cdots, N. \quad (3.18)$$

Note that $\mathbb{E}[Y^*(t_i)]$ depends on $\lambda_i^*$. To solve the parameters $\lambda_i^*, i = 1, 2, \cdots, N$, by equation (3.14), we first consider the case $i = N,$

$$\mathbb{E}[Y^*(t_N)] = \mathbb{E}[Y^*(t_{N-1})] e^{\int_{t_{N-1}}^{t_N} [r(t) - \beta(t)] dt} + \frac{\lambda_N^*}{\mu_N} \left[ 1 - e^{-\int_{t_{N-1}}^{t_N} \beta(t) dt} \right] \quad (3.19)$$

and

$$\lambda_N^* = 1 + \mu_N \mathbb{E}[Y^*(t_{N-1})] e^{\int_{t_{N-1}}^{t_N} [r(t) - \beta(t)] dt} + \lambda_N^* \left[ 1 - e^{-\int_{t_{N-1}}^{t_N} \beta(t) dt} \right].$$

Thus, we have

$$\lambda_N^* = e^{\int_{t_{N-1}}^{t_N} \beta(t) dt} + \mu_N \mathbb{E}[Y^*(t_{N-1})] e^{\int_{t_{N-1}}^{t_N} r(t) dt}. \quad (3.20)$$

Based on the representation of $\lambda_N^*$, by equation (3.19), we have

$$\mathbb{E}[Y^*(t_N)] = \frac{e^{\int_{t_{N-1}}^{t_N} \beta(t) dt} - 1}{\mu_N} + \mathbb{E}[Y^*(t_{N-1})] e^{\int_{t_{N-1}}^{t_N} r(t) dt},$$

which indicates that

$$\mu_N = \frac{e^{\int_{t_{N-1}}^{t_N} \beta(t) dt} - 1}{\mathbb{E}[Y^*(t_N)] - \mathbb{E}[Y^*(t_{N-1})] e^{\int_{t_{N-1}}^{t_N} r(t) dt}}. \quad (3.21)$$

Based on constrained condition (2.3) of $\mathbb{E}[Y^*(t_N)] = L_N$, $\mathbb{E}[Y^*(t_{N-1})] = L_{N-1}$ and condition (3.16), we can solve $\lambda_N^*$ and $\mu_N > 0$.

In the following, we consider the case $i = N - 1$. By equations (3.14) and (3.18), one obtains

$$\lambda_{N-1}^* = 1 + \mu_{N-1} \mathbb{E}[Y^*(t_{N-1})],$$
\[ \mathbb{E}[Y^*(t_{n-1})] - \mathbb{E}[Y^*(t_{n-2})]e^{\int_{t_{n-2}}^{t_{n-1}} r(t) - \beta(t) \, dt} \]

\[ = \left( \rho_{n-1} - \frac{g_{n-1}(t_{n-1})}{P_n(t_{n-1})} \right) \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right] \]

\[ = \left( \frac{\lambda_{n-1}^*}{\mu_{n-1}} - \frac{g_n(t_{n-1}) + P_n(t_{n-1})(\rho_{n-1} - \rho_n)}{\mu_{n-1} + P_n(t_{n-1})} \right) \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right] \]

\[ = \left( \frac{\lambda_{n-1}^* + P_n(t_{n-1}) \rho_n - g_n(t_{n-1})}{\mu_{n-1} + P_n(t_{n-1})} \right) \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right], \]

and thus

\[ \mathbb{E}[Y^*(t_{n-1})] - \mathbb{E}[Y^*(t_{n-2})]e^{\int_{t_{n-2}}^{t_{n-1}} r(t) - \beta(t) \, dt} \]

\[ = \frac{\lambda_{n-1}^* + P_n(t_{n-1}) \rho_n - g_n(t_{n-1})}{\mu_{n-1} + P_n(t_{n-1})} \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right]. \quad (3.22) \]

It follows that,

\[ \lambda_{n-1}^* = 1 + \mu_{n-1} \mathbb{E}[Y^*(t_{n-1})] \]

\[ = 1 + \frac{P_n(t_{n-1}) \rho_n - g_n(t_{n-1})}{u_{n-1} + P_n(t_{n-1})} \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right] \mu_{n-1} \]

\[ + \mu_{n-1} \mathbb{E}[Y^*(t_{n-2})]e^{\int_{t_{n-2}}^{t_{n-1}} r(t) - \beta(t) \, dt} + \frac{\mu_{n-1} \lambda_{n-1}^*}{u_{n-1} + P_n(t_{n-1})} \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right]. \]

Note that the coefficient of \( \lambda_{n-1}^* \) is

\[ \lambda_{n-1}^* = \frac{P_n(t_{n-1}) + \mu_{n-1} e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt}}{\mu_{n-1} + P_n(t_{n-1})} > 0, \]

which indicates that there exists a unique solution for \( \lambda_{n-1}^* : \)

\[ \lambda_{n-1}^* = \frac{P_n(t_{n-1}) + \mu_{n-1} e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt}}{P_n(t_{n-1}) + \mu_{n-1} e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt}} + \mu_{n-1} \left( \frac{P_n(t_{n-1}) \rho_n - g_n(t_{n-1})}{P_n(t_{n-1}) + \mu_{n-1} e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt}} \right) \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right] \]

\[ + \mu_{n-1} \mathbb{E}[Y^*(t_{n-2})]e^{\int_{t_{n-2}}^{t_{n-1}} r(t) - \beta(t) \, dt} \left( \frac{\mu_{n-1} \lambda_{n-1}^*}{u_{n-1} + P_n(t_{n-1})} \right) \left[ 1 - e^{-\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt} \right] e^{\int_{t_{n-2}}^{t_{n-1}} \beta(t) \, dt}. \quad (3.23) \]
Combining equations (3.22) and (3.23), we have,

\[
\mu_{N-1} = \frac{[1 + P_N(t_{N-1})p_N - g_N(t_{N-1})] \left( e^{\int_{t_{N-2}}^{t_{N-1}} \beta(t)dt} - 1 \right)}{\mathbb{E}[Y^*(t_{N-1})] - \mathbb{E}[Y^*(t_{N-2})] e^{\int_{t_{N-2}}^{t_{N-1}} \beta(t)dt}}
\]

(3.24)

Again, based on constrained condition (2.3) of \( \mathbb{E}[Y^*(t_{N-1})] = L_{N-1} \), \( \mathbb{E}[Y^*(t_{N-2})] = L_{N-2} \) and condition (3.16), we can solve \( \lambda^*_N \) and \( \mu_{N-1} > 0 \).

Similar to the case \( i = N - 1 \), we can solve \( \lambda^*_i, \mu_i \), \( i = 1, 2, \cdots, N - 1 \) step by step from \( N - 1 \) to 1,

\[
\lambda_i^* = \frac{P_{i+1}(t_i) + \mu_i}{P_{i+1}(t_i) + \mu_i e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}} + \mu_i \left( \frac{P_{i+1}(t_i)p_{i+1} - g_{i+1}(t_i)}{P_{i+1}(t_i) + \mu_i e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}} \right) \left[ 1 - e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt} \right] + \frac{P_{i+1}(t_i) + \mu_i e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}}{P_{i+1}(t_i) + \mu_i e^{-\int_{t_{i-1}}^{t_i} \beta(t)dt}} \mathbb{E}[Y^*(t_{i-1})] e^{\int_{t_{i-1}}^{t_i} \beta(t)dt},
\]

(3.25)

and

\[
\mu_i = \frac{[1 + P_{i+1}(t_i)p_{i+1} - g_{i+1}(t_i)] \left( e^{\int_{t_{i-1}}^{t_i} \beta(t)dt} - 1 \right)}{\mathbb{E}[Y^*(t_i)] - \mathbb{E}[Y^*(t_{i-1})] e^{\int_{t_{i-1}}^{t_i} \beta(t)dt}}
\]

(3.26)

Therefore, the optimal strategy \( \pi^*(\cdot) \) of cost functional (3.3) is an optimal strategy of cost functional (3.2). This completes the proof. \( \square \)

**Remark 3.2.** The conditions (3.16) and (3.18) guarantee that cost functional (3.2) has an optimal strategy with the parameters \( \lambda^* \) and \( \mu \). However, we haven’t given the condition for \( L_i, \ i = 0, 1, \cdots, N \) to guarantee \( \mu_i > 0, \ i = 1, 2, \cdots, N \) which satisfies conditions (3.16). In the following section, we give the condition for
$L_1, L_2$ to guarantee $\mu_1, \mu_2 > 0$ for the case $N = 2$ and solve $\lambda^* = (\lambda_1^*, \lambda_2^*)$, $\mu = (\mu_1, \mu_2)$ by $\E[Y^\pi(t_1)] = L_1$, $\E[Y^\pi(t_2)] = L_2$.

4 Explicit solution and simulation

In this section, we consider a simple example with $N = 2$ which is used to verify the results in Theorem 3.2 and investigate an explicit solution for the parameters $\lambda^* = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, $(\E[Y^\pi(t_1)], \E[Y^\pi(t_2)])$, and variance $(\text{Var}[Y^\pi(t_1)], \text{Var}[Y^\pi(t_2)])$. Furthermore, we compare our multi-time state mean-variance model with classical mean-variance model.

4.1 Explicit solution

We suppose there are two assets, one bond and one stock, which are traded in the market. Let $d = 1$, $n = 1$, $N = 2$, the bond satisfies,

$$
\begin{align*}
\frac{d R(t)}{R(t)} &= r(t) dt, \quad t > 0, \\
R(0) &= a_0 > 0,
\end{align*}
$$

and the stock asset is described by,

$$
\begin{align*}
\frac{d S(t)}{S(t)} &= b(t) S(t) dt + \sigma(t) S(t) dW(t), \quad t > 0, \\
S(0) &= s_0 > 0.
\end{align*}
$$

Our target is to minimize the following multi-time state mean-variance problem:

$$
J(\pi(\cdot)) = \sum_{i=1}^{2} \left( \frac{\mu_i}{2} \text{Var}[Y^\pi(t_i)] - \E[Y^\pi(t_i)] \right),
$$

(4.1)
and a tractable auxiliary problem is given as follows:

\[
\hat{J}(\pi(\cdot)) = \sum_{i=1}^{2} \mathbb{E}\left[\frac{\mu_i}{2} Y^x(t_i)^2 - \lambda_i Y^x(t_i)\right].
\] (4.2)

Based on the results in Theorem 3.2 and formulas (3.20) and (3.25), we set

\[
\lambda_2^* = e^{\int_{t_1}^{t_2} \beta(t)dt} + \mu_2 \mathbb{E}[Y^x(t_1)]e^{\int_{t_1}^{t_2} r(t)dt},
\]

\[
\lambda_1^* = \frac{P_2(t_1) + \mu_1}{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t_2} \beta(t)dt}} + \mu_1 \left(\frac{P_2(t_1) \rho_2 - g_2(t_1)}{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t_2} \beta(t)dt}}[1 - e^{-\int_{t_1}^{t_2} \beta(t)dt}]\right)
\]

\[
+ \frac{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t_2} \beta(t)dt}}{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t_2} \beta(t)dt}} ye^{\int_{t_1}^{t_2} [r(t) - \beta(t)]dt}.
\]

The optimal strategy of model (4.2) is given as follows:

\[
\pi^*(t) = \frac{b(t) - r(t)}{\sigma(t)^2} \left[\frac{\lambda_i^*}{\mu_i} - \frac{g_i(t)}{P_i(t)}\right] e^{-\int_{t}^{t_i} r(t)dt} - Y^x(t),
\] for \(t_{i-1} < t \leq t_i, i = 1, 2,

and

\[
\mathbb{E}[Y^x(t_2)] = \frac{e^{\int_{t_1}^{t_2} \beta(t)dt} - 1}{\mu_2} + \mathbb{E}[Y^x(t_1)]e^{\int_{t_1}^{t_2} r(t)dt};
\]

\[
\mathbb{E}[Y^x(t_1)] = ye^{\int_{t_1}^{t_2} [r(t) - \beta(t)]dt} + \left(\frac{\lambda_i^*}{\mu_i} - \frac{P_2(t_1) (\rho_1 - \rho_2) + g_2(t_1)}{P_2(t_1) + \mu_1}\right)(1 - e^{-\int_{t_1}^{t_2} \beta(t)dt});
\]

\[
\beta(t) = \left(\frac{b(t) - r(t)}{\sigma(t)}\right)^2, \quad t \leq t_2,
\] (4.3)

and \((P_2(t_1), g_2(t_1))\) satisfies the following Riccati equations,

\[
\begin{aligned}
& \frac{dP_2(t)}{dt} = [\beta(t) - 2r(t)]P_2(t) dt, \\
& P_2(t_2) = \mu_2 + P_3(t_2), \quad t_1 \leq t < t_2,
\end{aligned}
\] (4.4)
and related equations,

\[
\begin{cases}
    dg_2(t) = [ (\beta(t) - r(t))g_2(t) - \rho_2 r(t) P_2(t) ] dr, \\
    g_2(t_2) = g_3(t_2) + P_3(t_2)(\rho_2 - \rho_3), \quad t_1 \leq t < t_2,
\end{cases}
\] (4.5)

where \( P_3(t_2) = 0, \ g_3(t_2) = 0, \ \rho_3 = 0. \) By a simple calculation, we can obtain that

\[
P_2(t_1) = \mu_2 e^{\int_{t_1}^{t} [2r(\tau) - \beta(\tau)] d\tau},
\]

\[
\frac{\lambda_2^*}{\mu_2} - \frac{g_2(t_2)}{P_2(t_2)} = \frac{\lambda_2^*}{\mu_2};
\]

\[
\frac{\lambda_1^*}{\mu_1} - \frac{g_1(t_1)}{P_1(t_1)} = \frac{\lambda_1^* + \lambda_2^* e^{\int_{t_1}^{t} [2r(\tau) - \beta(\tau)] d\tau}}{\mu_1 + \mu_2 e^{\int_{t_1}^{t} [2r(\tau) - \beta(\tau)] d\tau}}.
\] (4.6)

Combining formulas (4.3) and (4.6), it follows that

\[
\mathbb{E}[Y^*(t_1)] = y e^{\int_{t_1}^{t} [r(\tau) - \beta(\tau)] d\tau} + \frac{\lambda_1^* + \lambda_2^* e^{\int_{t_1}^{t} [2r(\tau) - \beta(\tau)] d\tau}}{\mu_1 + \mu_2 e^{\int_{t_1}^{t} [2r(\tau) - \beta(\tau)] d\tau}} \left[ 1 - e^{-\int_{t_1}^{t} \beta(\tau) d\tau} \right];
\]

\[
\mathbb{E}[Y^*(t_2)] = \frac{e^{\int_{t_1}^{t} \beta(\tau) d\tau} - 1}{\mu_2} + \mathbb{E}[Y^*(t_1)] e^{\int_{t_1}^{t} r(\tau) d\tau};
\]

\[
\lambda_2^* = e^{\int_{t_1}^{t} \beta(\tau) d\tau} + \mu_2 \mathbb{E}[Y^*(t_1)] e^{\int_{t_1}^{t} r(\tau) d\tau};
\]

\[
\lambda_1^* = \frac{P_2(t_1) + \mu_1}{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t} \beta(\tau) d\tau}} + \mu_1 \left( \frac{P_2(t_1) \rho_2 - g_2(t_1)}{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t} \beta(\tau) d\tau}} \left[ 1 - e^{-\int_{t_1}^{t} \beta(\tau) d\tau} \right] + \frac{P_2(t_1) + \mu_1}{P_2(t_1) + \mu_1 e^{\int_{t_1}^{t} \beta(\tau) d\tau}} y e^{\int_{t_1}^{t} [r(\tau) - \beta(\tau)] d\tau} \right). \] (4.7)

In the following, we set \( T = 2, \ y = 1, \ t_1 = 1, \) and \( t_2 = 2. \) Let \( r(t) = r, \ b(t) = \)
$b$, $\sigma(t) = \sigma$, $\beta(t) = \beta$, where $0 \leq t \leq T$. From formulas (4.7), we have

\[
\mathbb{E}[Y^*(1)] = e^{r-\beta} + \frac{\lambda_1^* + \lambda_2^* e^{r-\beta}}{\mu_1 + \mu_2 e^{2r-\beta}} (1 - e^{-\beta});
\]

\[
\mathbb{E}[Y^*(t_2)] = \frac{e^{\beta} - 1}{\mu_2} + \mathbb{E}[Y^*(t_1)] e^r;
\]

\[
\lambda_2^* = e^{\beta} + \mu_2 \mathbb{E}[Y^*(1)] e^r;
\]

\[
\lambda_1^* = \mu_1 \left( \frac{\lambda_2^* (e^r - e^{r-\beta})}{\mu_2 e^{2r} + \mu_1} + \frac{\mu_2 e^{2r} + \mu_1}{\mu_2 e^{2r} + \mu_1} \right) + \mu_2 e^{2r} + \mu_1 e^\beta.
\]

**Remark 4.1.** Let $\mathbb{E}[Y^*(1)] = L_1$, $\mathbb{E}[Y^*(2)] = L_2$, and

\[
L_2 > L_1 e^r > e^{2r};
\]

\[
(L_2 - L_1 e^r) e^\beta > (L_1 - e^r) e^r.
\]

Note that, the condition $L_2 > L_1 e^r > e^{2r}$ guarantees that the constraints on the mean values $\mathbb{E}[Y^*(1)] = L_1$, $\mathbb{E}[Y^*(2)] = L_2$ are bigger than the return which is invested into the risk-free asset bond, while the condition $(L_2 - L_1 e^r) e^\beta > (L_1 - e^r) e^r$ guarantees the parameter $\mu_1 > 0$ in technique.

Applying formulas (4.8), by a simple calculation, one obtains

\[
\mu_2 = \frac{e^\beta - 1}{L_2 - L_1 e^r};
\]

\[
\lambda_2^* = \frac{L_2 e^\beta - L_1 e^r}{L_2 - L_1 e^r};
\]

\[
\mu_1 = \frac{(e^\beta - 1)(e^\beta + \lambda_2^* e^r) - (L_1 e^\beta - e^r) e^{2r} \mu_2}{(L_1 - e^r) e^\beta};
\]

\[
\lambda_1^* = \mu_1 \left( \frac{\lambda_2^* (e^r - e^{r-\beta})}{\mu_2 e^{2r} + \mu_1} + \frac{\mu_2 e^{3r-\beta} + \mu_1 e^r}{\mu_2 e^{2r} + \mu_1} \right) + \frac{\mu_2 e^{2r} + \mu_1 e^\beta}{\mu_2 e^{2r} + \mu_1}.
\]
Based on Theorem 3.2, applying the formula (4.6), we can obtain the related optimal strategy for the multi-time state mean-variance model (4.1) with the constraints on means $\mathbb{E}[Y^*(1)] = L_1$, $\mathbb{E}[Y^*(2)] = L_2$,

$$\pi^*(t) = \begin{cases} 
\frac{b - r}{\sigma^2} \left[ \lambda_1^* + \lambda_2^* e^{r(t-1)} e^{r(t-1)} - Y^*(t) \right], & 0 \leq t \leq 1; \\
\frac{b - r}{\sigma^2} \left[ \frac{\lambda_2^*}{\mu_2} e^{r(t-2)} - Y^*(t) \right], & 1 < t \leq 2.
\end{cases}$$

(4.11)

Thus, $\mathbb{E}[Y^*(\cdot)]$ satisfies

$$\mathbb{E}[Y^*(t)] = \begin{cases} 
\frac{e^{(r-\beta)t} + \lambda_1^* + \lambda_2^* e^{r(t-1)} e^{r(t-1)} e^{r(t-1)} - e^{(r-\beta)t-1}, & 0 \leq t \leq 1; \\
\mathbb{E}[Y^*(1)] e^{(r-\beta)(t-1)} + \frac{\lambda_2^*}{\mu_2} [e^{r(t-2)} - e^{r(t-2)-\beta(t-1)}], & 1 < t \leq 2,
\end{cases}$$

and from Lemma 3.1, the variance of $Y^*(\cdot)$ at $t_1 = 1$, $t_2 = 2$ are given as follows:

$$\text{Var}(Y^*(1)) = \frac{\left(\mathbb{E}[Y^*(1)] - e^r\right)^2}{e^\beta - 1};$$

(4.12)

$$\text{Var}(Y^*(2)) = \text{Var}(Y^*(1)) e^{2r-\beta} + \frac{\left(\mathbb{E}[Y^*(2)] - \mathbb{E}[Y^*(1)] e^r\right)^2}{e^\beta - 1}.$$

Now, we show the results of case $N = 1$ which is the classical mean-variance model:

$$\mathbb{E}[Y^*(2)] = L_2;$$

$$\mu = \frac{e^{2\beta} - 1}{L_2 - e^{2r}};$$

$$\lambda^* = \frac{L_2 e^{2\beta} - e^{2r}}{L_2 - e^{2r}},$$

where the related optimal strategy is

$$\pi^*(t) = \frac{b - r}{\sigma^2} \left[ \frac{\lambda^*}{\mu} e^{r(t-2)} - Y^*(t) \right], 0 \leq t \leq 2.$$  

(4.13)
The mean $\mathbb{E}[Y^\#(\cdot)]$ and variance $\text{Var}(Y^\#(\cdot))$ satisfy

$$
\begin{align*}
\mathbb{E}[Y^\#(t)] &= e^{(r - \beta)t} + \frac{\lambda e^{\beta(t-2)}}{\mu}[1 - e^{-\beta t}]; \\
\text{Var}(Y^\#(t)) &= \frac{(\mathbb{E}[Y^\#(t)] - e^r)^2}{e^{2\beta t} - 1}, \quad 0 \leq t \leq 2, \\
\end{align*}
$$

(4.14)

where $\mathbb{E}[Y^\#(2)] = L_2$. Based on formulas (4.12) and (4.14), we have the following comparison results for $(\text{Var}(Y^*(1)), \text{Var}(Y^*(2)))$ and $(\text{Var}(Y^\#(1)), \text{Var}(Y^\#(2)))$:

**Corollary 4.1.** Suppose $L_1$ and $L_2$ satisfy condition (4.9), one obtains

$$
\text{Var}(Y^*(1)) < \text{Var}(Y^\#(1));
$$

$$
\text{Var}(Y^*(2)) > \text{Var}(Y^\#(2)).
$$

Furthermore, if

$$
\frac{L_2 + e^{2(r - \beta)} + 1}{e^{r - \beta} + e^r + e^{-r}} \leq L_1 < \frac{L_2 + e^{2r - \beta}}{e^{r - \beta} + e^r},
$$

we have

$$
\text{Var}(Y^*(1)) + \text{Var}(Y^*(2)) < \text{Var}(Y^\#(1)) + \text{Var}(Y^\#(2)).
$$

**Proof:** By equality (4.14), we have

$$
\mathbb{E}[Y^\#(1)] = \frac{L_2 e^{\beta - r} + e^r}{e^\beta + 1}.
$$

Applying the condition $(L_2 - L_1 e^\beta) e^\beta > (L_1 - e^r) e^r$ in (4.9), one obtains

$$
e^r < \mathbb{E}[Y^*(1)] = L_1 < \frac{L_2 e^{\beta - r} + e^r}{e^\beta + 1} = \mathbb{E}[Y^\#(1)],
$$

it follows that,

$$
\text{Var}(Y^*(1)) < \text{Var}(Y^\#(1)).
$$
By formula (4.12), we have

\[
\text{Var}(Y^\ast (2)) = \frac{(L_1 - e^r)^2}{e^{\beta - 1}} + \frac{(L_2 - L_1 e^r)^2}{e^{\beta - 1}} \\
= \frac{[e^{2r-\beta} + e^{2r}]L_1^2 - 2e^r[L_2 + e^{2r-\beta}]L_1 + L_2^2 + e^{4r-\beta}}{e^{\beta - 1}} \\
= \frac{e^{2r-\beta} + e^{2r}}{e^{\beta - 1}} [L_1 - \frac{L_2 e^{\beta - r} + e^r}{e^{\beta} + 1}]^2 + \frac{(L_2 - e^{2r})^2}{e^{2\beta - 1}}.
\]

From equality (4.14), one obtains

\[
\text{Var}(Y^\#(2)) = \frac{(L_2 - e^{2r})^2}{e^{2\beta - 1}}.
\]

It follows that,

\[
\text{Var}(Y^\ast (2)) > \text{Var}(Y^\#(2)).
\]

Furthermore, we have

\[
\text{Var}(Y^\ast (1)) + \text{Var}(Y^\ast (2)) \\
= \frac{(L_1 - e^r)^2}{e^{\beta - 1}} [e^{2r-\beta} + 1] + \frac{(L_2 - L_1 e^r)^2}{e^{\beta - 1}} \\
= \frac{[e^{2r-\beta} + e^{2r} + 1]L_1^2 - 2e^r[L_2 + e^{2r-\beta} + 1]L_1 + L_2^2 + e^{4r-\beta} + e^{2r}}{e^{\beta - 1}}.
\]

It follows that \(\text{Var}(Y^\ast (1)) + \text{Var}(Y^\ast (2))\) admits the minimum values at

\[
L_1 = \frac{L_2 + e^{2r-\beta} + 1}{e^{r-\beta} + e^r + e^{-r}},
\]

Again, applying condition (4.9), we have

\[
\frac{L_2 + e^{2r-\beta} + 1}{e^{r-\beta} + e^r + e^{-r}} < \frac{L_2 + e^{2r-\beta}}{e^{r-\beta} + e^r} = \mathbb{E}[Y^\#(1)].
\]

Notice that, if

\[
L_1 = \frac{L_2 + e^{2r-\beta}}{e^{r-\beta} + e^r},
\]
one obtains,

\[ \text{Var}(Y^*(1)) + \text{Var}(Y^*(2)) = \text{Var}(Y^#(1)) + \text{Var}(Y^#(2)). \]

Thus if

\[ \frac{L_2 + e^{2r-\beta} + 1}{e^{r-\beta} + e^{r} + e^{-r}} \leq L_1 < \frac{L_2 + e^{2r-\beta}}{e^{r-\beta} + e^{r}}, \]

we have

\[ \text{Var}(Y^*(1)) + \text{Var}(Y^*(2)) < \text{Var}(Y^#(1)) + \text{Var}(Y^#(2)). \]

This completes the proof. \( \square \)

4.2 Simulation analysis

Let \( r = 0.04, \ b = 0.12, \ \sigma = 0.2, \ \beta = 0.16, \) we show the simulation results of the case \( N = 2, \) and case \( N = 1, \) where case \( N = 1 \) is same with the classical continuous time mean-variance model.
In Figure 1, we take $L_2 = e^{5r}$, $L_1 = e^{2.1r}$ which satisfies conditions (4.9). The expectations of $Y^*(\cdot)$ and $Y^\#(\cdot)$ are given as follows, respectively,

$$
\mathbb{E}[Y^*(t)] = \begin{cases} 
  e^{(r-\beta)t} + \frac{\lambda_1^* + \lambda_2^* e^{r-\beta}}{\mu_1 + \mu_2 e^{2r-\beta}} [e^{r(t-1)} - e^{(r-\beta)(t-1)}], & 0 \leq t \leq 1; \\
  \mathbb{E}[Y^*(1)] e^{(r-\beta)(t-1)} + \frac{\lambda_2^*}{\mu_2} [e^{r(t-2)} - e^{r(t-2)-\beta(t-1)}], & 1 < t \leq 2,
\end{cases}
$$

and

$$
\mathbb{E}[Y^\#(t)] = e^{(r-\beta)t} + \frac{\lambda^* e^{r(t-2)}}{\mu} [1 - e^{-\beta t}], \quad 0 \leq t \leq 2.
$$

From conditions (4.9), we obtain $\mathbb{E}[Y^*(1)] = L_1 < \mathbb{E}[Y^\#(1)]$ and thus,

$$
\mathbb{E}[Y^*(t)] < \mathbb{E}[Y^\#(t)], \quad 0 < t < 2.
$$
These results show that if we want to minimize the variances of the wealth at times $t_1 = 1$, $t_2 = 2$ together, the means of the investment portfolio may be smaller than that of classical mean-variance model in continuous time.

Figure 2: Comparing the values $Y^*(\cdot)$ and $Y^#(\cdot)$

In Figure 2, we plot the values $Y^*(\cdot)$ and $Y^#(\cdot)$ in pathwise. The left one shows that the pathwise of the function $Y^*(\cdot)$ along with $\mathbb{E}[Y^*(t)]$, while the right one shows that of $Y^#(\cdot)$. We can see that the variance of $Y^*(1)$ is bigger than that of $Y^*(1)$, and the variance of $Y^*(2)$ is almost the same as that of $Y^*(2)$. These phenomena verify the results of Corollary 4.1. In addition, in Figure 1, we can see that $\mathbb{E}[Y^*(t)] < \mathbb{E}[Y^#(t)]$, $0 < t \leq 1$, while Figure 2 shows that the variance of $Y^*(\cdot)$ is smaller than that of $Y^#(\cdot)$ before time 1.
In Figure 3, we plot the function of $\mathbb{E}[\text{MD}_{Y^*}]$ along with $\theta \in [1.145, 2.665]$, where

$$\text{MD}_{Y^*}^h = \text{esssup} \{ z \mid z = Y^*(t) - Y^*(s), \ 0 \leq t \leq s \leq h \},$$

$0 < h \leq 2$, and

$$L_1 = e^{\theta r}, \quad \mathbb{E}[Y^*(1)] = e^{2.665 r}.$$ 

We can see that $\mathbb{E}[\text{MD}_{Y^*}^t]$ is decreasing with $\theta \in [1.145, 2.505]$, increasing with $\theta \in [2.505, 2.665]$ and thus decreasing with $L_1 \in [e^{1.145 r}, e^{2.505 r}]$, increasing with $L_1 \in [e^{2.505 r}, e^{2.665 r}]$, while $\mathbb{E}[\text{MD}_{Y^*}^t]$ is increasing with $L_1 \in [e^{1.145 r}, e^{2.665 r}]$, where $t_1 = 1$, $t_2 = 2$. 

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5 Conclusion

For given $0 = t_0 < t_1 < \cdots < t_N = T$, to reduce the variance of the mean-variance model at the multi-time state $(Y^\pi(t_1), \cdots, Y^\pi(t_N))$, we propose a multi-time state mean-variance model with a constraint on the multi-time state mean value. In the proposed model, we solve the multi-time state mean-variance model by introducing a new sequence of Riccati equations.

Our main results are as follows:

- We can use the multi-time state mean-variance model to manage the risk of the investment portfolio along the multi-time $0 = t_0 < t_1 < \cdots < t_N = T$.

- A new sequence of Riccati equations which are connected by a jump boundary condition are introduced, based on which we find an optimal strategy for the multi-time state mean-variance model.

- Furthermore, the relationship of the means and variances of this multi-time state mean-variance model is established and is similar to the classical mean-variance model.

- An example is employed to show that minimizing the variances for multi-time state can affect the average value of Maximum-Drawdown of the investment portfolio.

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