DEGENERATING CURVES AND SURFACES: FIRST RESULTS

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ABSTRACT. Let $S \rightarrow \mathbb{A}^1$ be a smooth family of surfaces whose general fibre is a smooth surface of $\mathbb{P}^3$ and whose special fibre has two smooth components, intersecting transversally along a smooth curve $R$. We consider the Universal Severi-Enriques variety $V$ on $S \rightarrow \mathbb{A}^1$. The general fibre of $V$ is the variety of curves on $S_t$ in the linear system $|O_{S_t}(n)|$ with $k$ cusps and $\delta$ nodes as singularities. Our problem is to find all irreducible components of the special fibre of $V$. In this paper, we consider only the cases $(k, \delta) = (0, 1)$ and $(k, \delta) = (1, 0)$. In particular, we determine all singular curves on the special fibre of $S$ which, counted with the right multiplicity, are a limit of 1-cuspidal curves on the general fibre of $S$.

1. Introduction

In this section we introduce our problem and we fix notation. Let $F$ be the pencil of surfaces of the complex projective space $\mathbb{P}^3(\mathbb{C}) := \mathbb{P}^3$, generated by a general surface $S_d$ of degree $d \geq 2$ and a reducible surface $S_0 = S_{d-1} \cup \pi$, where $S_{d-1}$ is a general surface of degree $d-1$ and $\pi$ is a general plane, intersecting $S_{d-1}$ along a smooth curve $R$. The singular locus of the total space $A$ of $F$ consists of the $d(d-1)$ points $p_1, \ldots, p_{d(d-1)}$ of the special fibre which are intersection of $S_d$ and $R = S_{d-1} \cap \pi$. Moreover, at every such a point, $A$ has a rational double singularity. Let $S \rightarrow \mathbb{A}^1$ be the smooth family of surfaces obtained by smoothing $A$ and by contracting the exceptional components on $\pi$. The general fibre $S_t$ of $S$ is isomorphic to the general fibre of $A$, while the special fibre of $S$ is $S_0 = A \cup B$ where $A = S_{d-1}$ and $B$ is the blowing-up of $\pi$ at the $d(d-1)$ singular points of $A$. In particular, we have that $R_B^2 = (d - 1)^2 - d(d - 1) = -(d - 1) = -R_A^2$. From now on, we shall indicate by $E_1, \ldots, E_{d(d-1)}$ the exceptional curves of $B$.

Now, by denoting by $H$ the pull-back to $S$ of the hyperplane divisor of $\mathbb{P}^3$, by $H_t$ the restriction of $H$ to the fibre $S_t$ of $S$ and by $p_a(d, n)$ the arithmetic genus of a divisor in $|O_{S_t}(nH_t)|$, we consider the locally closed set, in the Zariski topology, $W_{nH,k,\delta} \subset |O_S(nH)| \times (\mathbb{A}^1 \setminus \{0\})$ defined by

$$W_{nH,k,\delta} = \{(D, t) \mid S_t \text{ is smooth, } D \cap S_t := D_t \in |O_{S_t}(nH_t)\mid \text{ is irreducible}\}$$

given $g = p_a(d, n) - k - \delta$ having $\delta$ nodes and $k$ cusps as singularities.

From now on, we will often denote by $|O_{S_t}(n)|$ the linear series $|O_{S_t}(nH_t)|$. Now, if $H_{S|\mathbb{A}^1}$ is the relative Hilbert Scheme of the family $S \rightarrow \mathbb{A}^1$ and if $H_n$ is the irreducible component of $H_{S|\mathbb{A}^1}$ whose fibre over $t$, with respect to the natural morphism $H_{S|\mathbb{A}^1} \rightarrow \mathbb{A}^1$, is $|O_{S_t}(nH_t)|$, one defines a natural rational map

$$f : W_{nH,k,\delta} \rightarrow H_n \subset H_{S|\mathbb{A}^1},$$

sending the point $([D], t)$ to the point $[D_t] \in H_n$, parametrizing the curve $D_t = D \cap S_t$. We denote by

$$V_{nH,k,\delta} \subset H_n$$

the closure in the Zariski topology of the image of $W_{nH,k,\delta}$ in $H_n$, with respect to $f$. Since the general fibre $V_t$ of $V_{nH,k,\delta}$ is the Severi-Enriques variety $V_t = \cdots$
\( V_{nH,k,δ}^S \) of irreducible curves on \( S_t \) in the linear system \(|nH|\), with \( δ \) nodes and \( k \) cusps as singularities, we call \( V_{nH,k,δ}^S \) the Universal Severi-Enriques Variety on \( S \) of irreducible curves in the linear system \(|nH|\), with \( δ \) nodes and \( k \) cusps. The problem we are interested is the following.

**Problem 1.1** (Main Problem). Let \( V_0 \) be the special fibre of \( V_{nH,k,δ}^S \). Then every its irreducible component \( V_0^i \) will have dimension equal to \( \dim V_{nH,k,δ}^S - 1 = \dim(V_1) \). We want to determinate all irreducible components \( V_0^i \) of \( V_0 = \sum_i m_i V_0^i \), with their respective multiplicity \( m_i \).

This problem has been already studied by Ran in [4], [5] and [6] in the case \( k = 0 \) and \( S \) equal to a family whose general fibre \( S_t \) is a plane and whose special fibre is \( S^0 = A \cup B \), where \( A \simeq \mathbb{P}^2 \) and \( B \simeq \mathbb{F}_1 \). Our paper is strongly inspired to Ran’s works but our approach is very different. The techniques we use to find all possible irreducible components of the special fibre of the Universal Enriques-Severi variety have been introduced by Ciliberto e Miranda in [1] and the aim of this paper is also to show how the notion of limit linear system can be useful for studying the problem (1.1). Unfortunately, if these kinds of methods are very helpful in order to determine all irreducible components of \( V_0 \subset V_{nH,k,δ}^S \), they are not enough to find their respective multiplicity. Here we consider only the cases \((k, δ) = (0, 1)\) and \((k, δ) = (1, 0)\). In both these cases, by only using Ciliberto and Miranda techniques, we are able to describe all irreducible components of \( V_0 \) and to compute their respective “geometric multiplicity”, which we are going to define.

**Definition 1.2.** Let \( V \) be an irreducible component of the special fibre \( V_0 \) of the Universal Severi-Enriques variety \( V_{nH,k,δ}^S \). The geometric multiplicity of \( V \)in \( V_0 \) is the minimal integer \( m \) such that there exist local analytic \( m \)-multisections passing through the general element \([D]\) of \( V \) and intersecting the general fibre \( V_t \) of \( V_{nH,k,δ}^S \) at \( m \) general points.

**Remark 1.3.** Notice that the geometric multiplicity of an irreducible component \( V_0^i \) of \( V_0 \) coincides with the usual multiplicity if the Universal Severi-Enriques Variety is smooth at the general element of \( V_0^i \). When \((k, δ) = (1, 0)\) or \((k, δ) = (0, 1)\), the geometric multiplicity of every irreducible \( V \) of \( V_0 \) is in fact the multiplicity of \( V \) in \( V_0 \) and, by counting every irreducible component with multiplicity equal to geometric multiplicity, we can obtain a recursion formula for the degree of \( V_{nH,1,0}^S \) and \( V_{nH,0,1}^S \) for every \( d \geq 2 \). Actually, the well known formulas of the number of one-nodal curves in a pencil and one-cuspidal curves in a net can be used to prove that in these cases geometric multiplicity and multiplicity coincide. To compute explicitly the tangent space of the Universal Severi-Enriques Variety at the general point of every its irreducible component is not trivial, even in the cases \((k, δ) = (0, 1)\) and \((k, δ) = (1, 0)\) and it requires very different methods from those we use in this paper. For this reason, we prefer to approach this problem in a future and more general paper. We just want to say that geometric multiplicity and multiplicity may be different even for families of curves with a very few singularities. For example, the special fibre \( V_0 \) of \( V_{nH,0,2} \), has an component (parametrizing curves having two ”simple tacnodes” and nodes at the intersection points with the singular locus \( R \) of \( S_0 \)) having, by our computations, geometric multiplicity 2 but multiplicity 4, according to Ran’s results.

Section 2 is devoted to the case \((k, δ) = (0, 1)\) whereas section 3 is devoted to the case \((k, δ) = (1, 0)\). We want to stress that, even if our proof is different, the results of section 2 are already known and can be deduced, with some further observations, by theorems 2.1 and 2.2 of [3]. On the contrary, as far as we know, the analysis we do of the special fibre \( V_0 \) of \( V_{nH,1,0}^S \) is completely new. Techniques we use in this paper can be applied to any smooth family of surfaces whose special fibre is...
irreducible and also to families of curves with singularities different from nodes or cusps. We conclude this section by introducing some terminology.

1.1. **Terminology.** Let \( \mathcal{X} \to \mathbb{A}^1 \) be any family of surfaces obtained from \( S \to \mathbb{A}^1 \) by a finite number of base changes and blow-ups and let \( f : \mathcal{X} \to \mathbb{P}^3 \) be the natural morphism from \( \mathcal{X} \) to \( \mathbb{P}^3 \). By abusing notation, we will denote always by \( H \) the pull-back to \( \mathcal{X} \) of the hyperplane divisor class of \( \mathbb{P}^3 \). Similarly, if \( \mathcal{Y} \to \mathbb{A}^1 \) is a family of surfaces obtained from \( \mathcal{X} \to \mathbb{A}^1 \) by a finite number of base changes and blow-ups and \( L \subset \mathcal{X}_0 \) is a curve lying in an irreducible component of the special fibre of \( \mathcal{X} \), then we will usually denote by the same symbol \( L \) the proper transform of \( L \subset \mathcal{X} \) in \( \mathcal{Y} \). Moreover, if \( S \subset \mathbb{P}^3 \) is a surface of degree \( n \) and \( C_1 = f^* S \cap \mathcal{X} \) is the curve cut out on the fibre \( \mathcal{X}_t \) of \( \mathcal{X} \) by the pull-back of \( S \) via \( f \), from now on we will say that \( C_1 \) is cut out on \( \mathcal{X} \) by \( S \). Furthermore, if \( W \subset |O_A(n)| \) and \( V \subset |O_B(n)| \) are two projective subvarieties and \( W \subset H^0(A, O_A(n)) \) and \( V \subset H^0(B, O_B(n)) \) are the affine cones associated to \( W \) and \( V \) respectively, then, by abusing notation, we shall denote by \( V \times |O_B(n)| W \) the projective variety \( \mathbb{P}(W \times H^0(R, O_{R(n)})) \) \( V \subset |O_{A\cup B}(n)| \). Finally, we will indicate by \( F_n \) the Hirzebruch surface \( F_n = \mathbb{P}(O_{p^3} \oplus O_{p^1}(n)) \).

2. **The case of one-nodal curves.**

In this section we consider the case \( k = 0 \) and \( \delta = 1 \), determining all irreducible components of \( \mathcal{V}_0 \) with the respective geometric multiplicity (see definition [12]). Notice that, in this case, the Severi variety \( \mathcal{V}_1 = \mathcal{V}^{S}_{nH,0.1} \) on the general fibre is irreducible, it has the expected dimension, it is smooth at every point corresponding to a one-nodal curve and, finally, it coincides with the locus of \( [nH] \) parametrizing singular curves. Moreover, before we describe the special fibre \( \mathcal{V}_0 = \mathcal{V}^{S}_{nH,0.1} \), notice that the restriction of the linear system \( [nH] \) to the special fibre \( S_0 = A \cup B \) of \( S \) is

\[
|\mathcal{O}_{A\cup B}(n)| \cong \mathbb{P}(H^0(A, O_A(n)) \times H^0(R, O_{R(n)})) H^0(B, O_B(n))),
\]

and, by Bertini Theorem, the general element of \( |\mathcal{O}_{A\cup B}(n)| \) is a curve smooth outside \( R \) and with nodes at its intersection points with \( R \). By using notation introduced in the previous section, we also denote by \( W_A(d, n) \) and \( W_B(d, n) \) the divisors of \( |\mathcal{O}_{A\cup B}(n)| \) defined as

\[
W_A(d, n) := V^A_{nH,0.1} \times |O_{R(n)}| |O_B(n)| \quad \text{and} \quad W_B(d, n) := |O_A(n)| \times |O_{R(n)}| V^B_{nH,0.1}.
\]

Notice that the general element of \( W_A(d, n) \) corresponds to a curve cut out on \( A \cup B \subset \mathbb{P}^3 \) by a surface of degree \( n \) tangent to \( A \) at a general point. Moreover, if \( (d, n) \neq (2, 1), (d, n) \neq (3, 1), d \geq 2 \) and \( n \geq 1 \) then the general element of \( W_A(d, n) \) corresponds to a curve \( D = D_A \cup D_B \) where \( D_A \) is an irreducible one-nodal curve intersecting transversally \( R \) and \( D_B \) is a smooth curve on \( B \), not intersecting \( E_i \), for every \( i \), and intersecting transversally \( R \) at the points \( A \cap R \). Similarly, for \( d, n \geq 2 \) the general element of \( W_B(d, n) \) corresponds to a curve \( D = D_A \cup D_B \) such that \( D_A \subset A \) is smooth, intersecting transversally \( R \), and \( D_B \) is a one-nodal curve (which is irreducible for \( (d, n) \neq (d, 3) \)), not intersecting \( E_i \) for every \( i \), and such that \( D_B \cap R = D_A \cap R \). In particular, \( W_A(d, n) \) and \( W_B(d, n) \) are both irreducible divisors of \( |\mathcal{O}_{A\cup B}(n)| \). Finally, we denote by \( W_{E_i}(d, n) \), \( i = 1, \ldots, (d-1) \) and \( T(d, n) \) the irreducible divisors of \( |\mathcal{O}_{A\cup B}(n)| \) defined as

\[
W_{E_i}(d, n) = \{ D \in |\mathcal{O}_{A\cup B}(n)| \text{ such that } E_i \subset D \},
\]

where \( E_1, \ldots, E_{d(d-1)} \) are the exceptional divisors of \( B \), and

\[
T(d, n) = \{ D = D_A \cup D_B \in |\mathcal{O}_{A\cup B}(n)| |\text{ such that } D_A \text{ and } D_B \text{ are tangent to } R \text{ at a general point} \}.
\]

By using the terminology introduced at the end of the previous section, notice that \( W_{E_i}(d, n) \) parametrizes curves on \( A \cup B \) cut out by surfaces of degree \( n \) in \( \mathbb{P}^3 \) passing through \( p_i \), for every \( i = 1, \ldots, d(d-1) \), whereas \( T(d, n) \) parametrizes curves on
A \cup B$ cut out by surfaces of degree $n$ of $\mathbb{P}^3$ tangent to $R$ at its general points. In particular, if $[D = D_A \cup D_B] \in T(d, n)$ is a general point, then $D_A$ and $D_B$ are smooth, tangent to $R$ at only one point and they will be transverse to $R$ outside this point, except for the case $(d, n) \neq (2, 1)$, when $D_A = D_B = R$.

**Lemma 2.1.** $W_A(d, n)$ is an irreducible component of $V_0$ of geometric multiplicity 1, for every $(d, n) \neq (2, 1)$. Similarly, $W_B(d, n)$ is an irreducible component of $V_0$ of geometric multiplicity 1, if $(d, n) \geq (2, 2)$. Moreover, if $V$ is an irreducible component of $V_0$ of geometric multiplicity 1 and whose general element corresponds to a curve $D = D_A \cup D_B$ which does not contain any $E_i \subset B$, then $V$ coincides with $W_A(d, n)$ or $W_B(d, n)$.

**Proof.** Assume that $(d, n) \neq (2, 1)$, let $\gamma$ be a section of $S \to \mathbb{A}^1$ passing through a general point $p$ of $A$ and let $S \sim nH$ be a general divisor singular along $\gamma$. Notice that such a divisor exists and, by generality, its general fibre $S \cap S_t$ is a 1-nodal curve. Let $X = Bl_\gamma S$ with exceptional divisor $\Gamma$. Then, if $S'$ is the proper transform of $S$ on $X$, we have that $S' \sim nH - 2\Gamma$. Now, denoting by $X_t$ the fibre over $t$ of $X$, consider the exact sequence

$$0 \to \mathcal{O}_X(nH - 2\Gamma - X_t) \to \mathcal{O}_X(nH - 2\Gamma) \to \mathcal{O}_{X_t}(nH - 2\Gamma) \to 0.$$ 

Since the fibres of the family $X \to \mathbb{A}^1$ are linearly equivalent, we have that the dimension of the image of the map $H^0(X, \mathcal{O}_X(nH - 2\Gamma)) \xrightarrow{r_t} H^0(X_t, \mathcal{O}_{X_t}(nH - 2\Gamma))$ doesn’t depend on $t$. Moreover, since for $t$ general the map $r_t$ is surjective, we have that $\dim(\text{Im}(r_t)) = \dim(\text{Im}(r_0)) = h^0(X_t, \mathcal{O}_{X_t}(nH - 2\Gamma)) = h^0(X_t, \mathcal{O}_{X_t}(nH)) - 3 = h^0(X_t, \mathcal{O}_{X_t}(nH - 2\Gamma))$. So, every curve of $A \cup B$ with a node at the point $\gamma \cap A$ is a limit of a one-nodal curve in the linear system $|nH|$ on $S_t$. By the generality of the point $\gamma \cap A$ on $A$ we have that $W_A(d, n)$ is an irreducible component of $V_0$ of geometric multiplicity 1. In order to prove the lemma for $W_B(d, n)$ repeat the same argument as used for $W_A(d, n)$.

Now, let $V$ be an irreducible component of $V_0$, whose general element $[D]$ corresponds to a curve $D = D_A \cup D_B$ which does not contain any $E_i \subset B$ and such that there exists an analytic neighborhood $U$ of 0 and an analytic section $\Delta$ of the universal Severi variety $V_{nH, [U]}^S$, passing through $[D]$. Then, the singular locus of the family of curves $C \to \Delta$, naturally parametrized by $\Delta$, is a section of $S_{[U]} \to \Delta$ and it must intersect the special fibre at a smooth point $q$, i.e. at a point $q \notin R = A \cap B$. If $q \in A$ then $D_A$ is singular, $[D] \in W_A(d, n)$ and $V = W_A(d, n)$. Otherwise $V = W_B(d, n)$.

**Remark 2.2.** By the previous lemma, the other possible irreducible components of $V_0$ “are produced” by degenerations of the general element $[D_t]$ of $V_t$ such that as $D_t$ goes to $A \cup B$, the node of $D_t$ specializes to a point of the singular locus $R$ of $A \cup B$.

**Lemma 2.3.** If $(d, n) \neq (2, 1)$, then $T(d, n)$ is an irreducible component of $V_0$ of geometric multiplicity 2.

**Proof.** Let $p \in R = A \cup B = S_0$ be a point such that $p \notin E_i$, for every $i$. We consider a double covering of our family $S \to \mathbb{A}^1$ totally ramified at its special fibre $S' \to \mathbb{A}^1$ isomorphic to the special fibre $S_0$ of $S$ and we still denote it by $A \cup B$. But, if $xy = t$ is the local equation of $S$ at $p$, the local equation of $S'$ at $p$ will be $xy = t^2$. In particular, $S'$ is singular along the singular
locus $R = A \cap B$ of the special fibre. If we blow-up the ambient space along $R$, the proper transform $\mathcal{X}$ of $S'$ is smooth, the general fibre of $\mathcal{X}$ is isomorphic to the general fibre of $S'$ while the special fibre of $\mathcal{X}$ is $\mathcal{X}_0 = A \cup E \cup B$, where $E \cong \mathbb{P}_{4-1}$ is the intersection of $X$ with the exceptional divisor. The inverse image of $p$ to $X$ is a fibre of $E$ which we denote by $F$. Now, if $U$ is an analytic neighborhood of $0 \in \mathbb{A}^1$ small enough, let $\gamma$ be a section of the family $\mathcal{X}|_U$ and let $C \subset \mathcal{X}|_U$ be a general divisor with nodal singularities along $\gamma$ and linearly equivalent to $nH$, where we still denote by $H$ the pull-back to $X$ of the hyperplane divisor. Notice that such a divisor $C$ exists for every $n \geq 1$ and $d \geq 2$, because, for every $n \geq 1$ and $d \geq 2$, on the general fibre $\mathcal{X}_t$ of $\mathcal{X}$, there exist curves, linearly equivalent to $nH|_{\mathcal{X}_t} = nH_t$ with only a node at a general point $q_t$ and no further singularities. We want to understand the kind of singularities of the special fibre $C_0$ of $C$. To this aim, let $\pi : \mathcal{Y} \to \mathcal{X}|_U$ be the blowing-up of $\mathcal{X}|_U$ along $\gamma$, with exceptional divisor $\Theta$. The special fibre of $\mathcal{Y}$ is now given by $A \cup \mathcal{E}' \cup B$, where $\mathcal{E}'$ is the blowing-up of $E$ at $\gamma \cap F = \gamma \cap E$, with new exceptional divisor $E_0 = \Gamma \cap \mathcal{E}'$. In particular, $F_{\mathcal{E}'} = -1$ and, if $C'$ is the proper transform of $C$ on $\mathcal{Y}$, then $C' \sim nH - 2\Gamma$.

Now, if $(d, n) \neq (2, 1)$, denoting by $F_{\mathcal{E}'}$, the pull-back to $\mathcal{E}'$ of the linear equivalence class of the fibre of $E$, we have that the divisor $(nH - 2\Gamma)|_{\mathcal{E}'} \sim F_{\mathcal{E}'} - 2E_0$ is not effective. Since $C'$ is effective we deduce that $\mathcal{E}' \subset C'$ and, in particular, $R_1 \subset C'|_A$ and $R_2 \subset C'|_B$. So the point $[C] \in |O_{A\cup B}(n)|$, corresponding to the curve $C_A \cup C_B$, where $C_A = C|_A$ and $C_B = C|_B$, belongs to the variety $T(2, 1)$ and cannot be general in any irreducible component of $\mathcal{V}_0$.

If $(d, n) \neq (2, 1)$, then the divisor $(nH - 2\Gamma)|_{\mathcal{E}'} \sim (n(d - 1) - 2)F_{\mathcal{E}'} + 2F$. In particular, $F$ is contained in the divisor $(nH - 2\Gamma)|_{\mathcal{E}'}$, with multiplicity at least 2 and $C' \cdot F = -2$. We want to compute the multiplicity $\alpha$ of $C'$ along $F$. Let then $\pi' : Z \to \mathcal{Y}$ be the blowing-up of $\mathcal{Y}$ along $F$. The special fibre of $Z$ is $A' \cup \mathcal{E}' \cup B' \cup \Theta$, where $\Theta$ is the new exceptional divisor and $A'$ and $B'$ are the blowups of $A$ and $B$ at the point $F \cap A$ and $F \cap B$, with exceptional divisor $\Theta \cap A'$ and $\Theta \cap B'$ respectively. By the triple point formula

$$\Theta \cdot \mathcal{E}' \cdot (A' + \mathcal{E}' + B' + \Theta) = 0$$

we deduce that $(F^2)_\Theta = -(F_{\mathcal{E}'})^2 - 2 = -1$ and, in particular, $\Theta \cong \mathbb{F}_1$ and $F = \mathcal{E}' \cap \Theta$ is the exceptional divisor of $\Theta$. Moreover, denoting by $C''$ the pullback to $Z$ of $C'$ and by $F_\Theta$ the linear equivalence class of the fibre of $\Theta$, we have that

$$C''|_\Theta \sim (2\alpha - 2)F_\Theta + \alpha F.$$ 

Since $C''|_\Theta$ must be an effective divisor, we have that $2\alpha - 2 \geq 0$, i.e. $\alpha \geq 1$.

Now, for every $\alpha \geq 1$, we have that the image into $\mathcal{X}|_U$ of a divisor in $Z$, linearly equivalent to $nH - 2\Gamma - \alpha \Theta$, is a divisor linearly equivalent to $nH$ and with double singularities along $\gamma$. Since we have taken a general divisor $C$ in $\mathcal{X}|_U$, with these properties, we may assume that $\alpha$ is the minimum integer in order that $C''|_\Theta$ is effective, i.e. $\alpha = \text{mult}_F(C') = 1$ and $C''|_\Theta = F = C'|_{\mathcal{E}'}$. This implies that $C''|_{Z|A'}$ must intersect the exceptional divisor $\Theta \cap A'$ with multiplicity one at the points $F \cap A$. Thus, recontracting $\Theta$ and going back to $C' \subset \mathcal{Y}$, we have that $C'|_A$ must pass through the point $F \cap R_1$ and it must be smooth and tangent to $R_1$ at this point. At the same way, $C'|_B$ passes through $F \cap R_2$ with multiplicity 1 and it is tangent to $R_2$ at this point.

Now, let $D$ be the image of $C$ into $S|_{\nu_2(U)}$. If $t$ is a general point of $\mathbb{A}^1$ and $\{t_1, t_2\} = \nu_2^{-1}(t)$, then the fibre of $D$ over $t$ is $D_t = C_{t_1} \cup C_{t_2}$ and, in particular, it is the union of two one-nodal curves in the linear system $|nH_t|$. Whereas, the special fibre $D_0$ of $D$ is the curve, counted with multiplicity 2, image of $C_0$ under the contraction of $\mathcal{E}$. Thus $D_0 = 2C_A \cup 2C_B$, where $C_A = C \cap A$ and $C_B = C \cap B$.

From what we proved before, the point $x = [C_A \cup C_B] \in T(d, n) \cap \mathcal{V}_0$ and the curve
in $\mathcal{H}_n$ corresponding to $\mathcal{D}$ is a local bisection of $\mathcal{V}_{nH,0,1}$ passing through $[D]$ and intersecting the general fibre at 2 general points.

Now we want to prove that the point $x$ is general in $T(d,n)$. To this aim let $Z^1$ be the blowing-up of $Z$ along $F = \Theta \cap \mathcal{E}'$. Now the special fibre of $Z^1$ is $Z_0^1 = A'' \cup \mathcal{E}' \cup B'' \cup \Theta \cup 2\Theta_1$, where $\Theta_1 \simeq \mathbb{F}_0$ is the new exceptional divisor and $A''$ and $B''$ are the proper transforms of $A'$ and $B'$ respectively. Moreover, denoting by $\mathcal{C}''$ the proper transform of $\mathcal{C}''$ on $Z_1$, we have that

$$\mathcal{C}'' \sim nH - 2\Gamma'' - \Theta - 2\Theta_1,$$

where $\Gamma''$ is the proper transform of $\Gamma$. Now, by denoting by $D$ the divisor $D = \Theta + 2\Theta_1 + 2\Gamma''$, we consider the exact sequence

$$(1) \quad 0 \rightarrow \mathcal{O}_{Z^1}(nH - D) \rightarrow \mathcal{O}_{Z^1}(nH - D) \rightarrow \mathcal{O}_{Z_1}(nH - D) \rightarrow 0$$

where $Z_1^1$ is any fibre of $Z^1$. By arguing as in the previous lemma, we see that the image of the map $\rho_0 : H^0(\mathcal{O}_{Z^1}(nH - D)) \rightarrow H^0(\mathcal{O}_{Z_1}(nH - D))$ is equal to $h^0(\mathcal{O}_{Z_0}(nH)) - 3 = h^0(A \cup B, \mathcal{O}_{A \cup B}(n)) - 3$. Hence, if we denote by $T(n)_p \subset |\mathcal{O}_p(nH)|$ the locus of surfaces of $\mathbb{P}^3$, passing through $p$ and tangent to $R = S^d - 1 \cap \pi$ at $p$, then all divisors in the linear series $|\mathcal{O}_{Z_0}(nH - D)|$ are cut out on $Z_0^1$ by surfaces parametrized by a divisor in $T(n)_p$. We denote this divisor by $D_q$ where $q = \gamma \cap F$. How to characterize $D_q$? In order to answer this question we contract $\Theta_1$, $\Theta$ and finally $\Gamma$, and we come back to the family of surfaces $\mathcal{X}^*_U \rightarrow U$. Now, the fibre $F$ of $\mathcal{E}$ can be identified with a double cover of the fibre over $p$ of the projectivized normal bundle to $R$ in $\mathbb{P}^3$. In other words, every point $r$ of $F$ corresponds to a plane $H_r$ in $\mathbb{P}^3$ containing the line $T_p R$ and there are exactly two points of $F$ corresponding to the same plane. From the hypothesis that the singular locus $\gamma$ of $\mathcal{C}$ intersects the fibre $F$ at $q$, we deduce that the image curve of $C_0$ in $\mathbb{P}^3$ is cut out on $S^d - 1 \cup \pi$ from a surface of degree $n$ passing through $p$ and tangent at $p$ to the hyperplane $H_q$ corresponding to $q \in F$. The locus of surfaces with these properties is the divisor $D_q \subset T(n)$. By the generality of $q$ in $F$, we conclude that $T(d,n)$ is an irreducible component of $\mathcal{V}_0$. To see that there are not local sections of $\mathcal{V}_{nH,0,1}$ passing through the general point $[D]$ of $T(d,n)$ use the previous lemma. \quad \Box

Now we want to see which is the limit curve on $S^d - 1 \cup \pi$ if we consider a degeneration of a general one nodal curve on the general fibre of the pencil $\mathcal{F}$, specializing to $S^d - 1 \cup \pi$ in such a way that the node comes to a base point of the pencil. Computations we are going to do are partially contained in theorem 2.2 of [3].

**Remark 2.4.** Let $p_1 \in \mathbb{P}^3$ be the base point of our pencil $\mathcal{F}$ corresponding to the exceptional divisor $E_1 \subset B$. Let $\gamma \subset \mathbb{P}^3$ be a general curve passing through $p_1$ and, in particular, transversally intersecting $S^d - 1$ at $l(d - 1)$ different points $p_1, q_2, \ldots, q_{l(d-1)}$, where $l$ is the degree of $\gamma$. If $\mathcal{A} \subset \mathbb{A}^3 \times \mathbb{A}^1$ is the total space of $\mathcal{F}$ and $g : \mathcal{A} \rightarrow \mathbb{A}^3$ is the natural map, then, locally working at $(p_1, 0)$, we may suppose that $(p_1, 0) = (0, 0, 0, 0)$, that $\mathcal{A}$ has equation

$$xy - tz = 0$$

and, finally, that $g^* \gamma$ has equations

$$\begin{cases} xy - tz = 0 \\ x - z = 0 \\ y - z = 0. \end{cases}$$

Notice that $(p_1, 0)$ is a rational double point of $\mathcal{A}$ and that the pull-back curve $g^* \gamma \subset \mathcal{A}$ has two irreducible components passing through $(p_1, 0)$,

$$(2) \quad \gamma_1 : \begin{cases} z = 0 \\ x = 0 \end{cases} \quad \text{and} \quad \gamma_2 : \begin{cases} z = t \\ x = t \\ y = t, \end{cases}$$
transversally intersecting at \((0, 0, 0, 0)\) and where \(\gamma_1 = g^*(p_1)\) is nothing else but the section of \(\mathcal{A}\) corresponding to the inverse image of the point \(p_1 \in \mathbb{P}^3\) (which is a base point of \(\mathcal{F}\)). If we denote by \(\pi : \mathcal{A} \to \mathcal{A}\) the restriction to the proper transform of \(\mathcal{A}\) of the blowing-up of \(\mathcal{A} \times \mathbb{A}^1\) at \((p_1, 0)\), then the special fibre of \(\mathcal{A}\) is given by 

\[
\mathcal{A}_0 = \mathcal{A} \cup \Theta_1 \cup \tilde{B},
\]

where \(\tilde{B}\) is the blowing up of \(\mathcal{A} = S^{d-1}\) at \(p_1\), \(\tilde{B}\) is the blowing up of \(\pi\) at \(p_1\) and \(\Theta_1\) is the quadric cut out on \(\mathcal{A}\) by the exceptional divisor. By identifying \(\Theta_1\) with the “projectivized” tangent cone of \(\mathcal{A}\) at \((0, 0, 0, 0)\) we see that the curves \(\pi^* \gamma_1\) and \(\pi^* \gamma_2\) intersect \(\mathcal{A}_0\) at two different points \(q_1\) and \(q_2\), corresponding to the tangent directions of \(\gamma_1\) and \(\gamma_2\) at \((0, 0, 0, 0)\) respectively. Moreover the tangent direction to \(\gamma_2\) at \((0, 0, 0, 0)\) is determined by the tangent direction to \(\gamma_1\) at \(p_1\) in \(\mathbb{P}^3\).

In particular, if we set \(E_1 = \Theta_1 \cap \tilde{B}\) and \(E'_1 = \Theta_1 \cap \tilde{A}\), then \(q_1\) and \(q_2\) don’t lie on \(E_1 \cup E'_1\) because \(\gamma_1\) and \(\gamma_2\) are not tangent to \(\pi\) or \(S^{d-1}\) at \((0, 0, 0, 0)\). Finally, notice that, the tangent space to \(\mathcal{A} \subset \mathbb{P}^3 \times \mathbb{A}^1\) at every point \((0, 0, 0, t)\), with \(t \neq 0\), is given by the hyperplane \(H_z\) of equation \(z = 0\), and the curve \(\gamma_1\) is contained in \(H_z\).

Hence, the two lines \(H_1\) and \(H'_1\) of \(\Theta_1\) with \(H_1 \in |E_1|\) and \(H'_1 \in |E'_1|\), intersecting at \(q_1\), are nothing else but the projectivization of the intersection of \(H_z\) with the tangent cone to \(\mathcal{A}\) at \((0, 0, 0, 0)\).

**Lemma 2.5.** \(W_{E_i}(d,n)\) is an irreducible component of \(\mathcal{V}_0\) of geometric multiplicity 1, for every \(i = 1, \ldots, d(d-1)\).

**Proof.** We will prove the lemma for \(i = 1\). We use the notation introduced in Remark 24. We blow-up the ambient space at their singular points and we contract all the exceptional components of the proper transform of \(\mathcal{A}\), different from \(\Theta_1\), on \(\tilde{B}\). We denote by \(\tilde{S} \to \mathbb{A}^1\) the family of surfaces obtained in this way. \(\tilde{S}\) is nothing else but the blowing-up of \(S\) along \(E_1\). We denote by \(A' \cup \Theta_1 \cup B\) its special fibre. Now, let \(S \subset \tilde{S}\) be a general divisor, linearly equivalent to \(nH - \alpha \Theta_1\), with nodal singularities along \(\gamma_2\). We ask which is the minimum \(\alpha\) such that \(S|_{\Theta_1}\) is effective. Now, \(S|_{\Theta_1} \sim -\alpha \Theta_1|_{\Theta_1} \sim \alpha E_1 + \alpha E'_1\). Thus the minimum \(\alpha\) such that \(S|_{\Theta_1}\) is effective is \(\alpha = 1\). Then take \(S \sim nH - \Theta_1\). From the fact that \(S\) is nodal along \(\gamma_2\), we have that \(S|_{\Theta_1}\) has at least a node at \(\gamma_2 \cap \Theta_1\). But \(S|_{\Theta_1} \sim E_1 + E'_1\) and hence \(S|_{\Theta_1} = H_2 \cup H'_2\), where \(H_2 \in |E_1|\) and \(H'_2 \in |E'_1|\) are the two lines passing through \(q_2\). In particular, we find that the curve \(\tilde{S} \cap (A' \cup \Theta_1 \cup B)\) is cut out from a surface of \(\mathbb{P}^3\) passing through \(p_1\) and tangent to \(p_1\) at a plane \(H_{q_2}\), determined by the point \(q_2\). Moreover, if we blow-up \(\tilde{S}\) along \(\gamma_2\) and we denote by \(\mathcal{E}_2\) the exceptional divisor, by \(Z\) the resulting family of surfaces and by \(\Theta'_1\) the proper transform of \(\Theta_1\), then, by arguing as in the last part of the proof of lemma 24, you can prove that the image of the restriction map

\[
r_0 : H^0(\mathcal{O}_Z(nH - 2\Gamma_2 - \Theta'_1)) \to H^0(\mathcal{O}_{\tilde{S}, \Theta_1 \cup B}(nH - 2\Gamma_2 - \Theta'_1))
\]

has dimension \(\dim(\mathcal{O}_{\tilde{S}, nH}) = 3\), and, in particular, \(r_0\) is surjective. So, the general curve on \(\tilde{S}_0 = A \cup B\) cut out by a surface of \(\mathbb{P}^3\) passing through \(p_1\) and tangent to \(p_1\) at a plane \(H_{q_2}\), is a limit of a general one-nodal curve on \(\tilde{S}_0\) in the linear system \(|nH|\). Finally, by using the generality of the point \(q_2\) in \(E_1\) and by contracting \(\Theta_1\) on \(\tilde{B}\) we conclude our proof. \(\square\)

**Theorem 2.6.** Let \(\mathcal{V}_0\) be the special fibre of \(\mathcal{V}_{nH,0,1}\). Then, the irreducible components of \(\mathcal{V}_0\) are

- \(W_A(d,n)\), \(W_B(d,n)\) and \(W_{E_i}(d,n)\), for \(i = 1, \ldots, d(d-1)\), with geometric multiplicity 1 and \(T(d,n)\) with geometric multiplicity 2, if \(d,n \geq 2\);

- \(W_{E_i}(d,n)\) and \(W_{E_k}(d,n)\) with geometric multiplicity 1, if \((d,n) = (2,1)\);

- \(W_A(d,n)\) and \(W_{E_k}(d,n)\), for \(i = 1, \ldots, d(d-1)\), with geometric multiplicity 1 and \(T(d,n)\) with geometric multiplicity 2, if \((d,n) = (d,1)\) and \(d \geq 3\).
Proof. From what we proved before, we have only to show that \( T(d, n), W_A(d, n), W_B(d, n) \) and \( W_{E_i}(d, n) \), for \( i = 1, \ldots, d(d-1) \), are the only irreducible components of the special fibre \( V_0 \) of \( V_{nH,0,1} \).

Case 1. Let \( V \subset V_0 \) be an irreducible component whose general element \([D]\) corresponds to a curve containing \( E_i \). Thus \( V \subset W_{E_i}(d, n) \). But \( \text{dim}(V) = \text{dim}(W_{E_i}(d, n)), \) so \( V = W_{E_i}(d, n) \).

Case 2. Let \( V \) be an irreducible component of \( V_0 \) whose general element \([D]\) corresponds to a curve \( D = D_A \cup D_B \) which is singular at a smooth point of \( A \cup B \) and which does not contain \( E_i \), for every \( i \). If \( D_A \) is singular, thus \( V = W_A(d, n) \) and \( D_B \) is smooth. If \( D_B \) is singular, then \( V = W_B(d, n) \) and \( D_A \) is smooth.

Case 3. Assume that \( V \) is an irreducible component of \( V_0 \) different from \( W_A(d, n), W_B(d, n) \) and \( W_{E_i}(d, n) \), for every \( i \). Thus, from what we observed above, if \( D = D_A \cup D_B \) is the curve corresponding to the general element \([D]\) of \( V \), then \( D_A \) and \( D_B \) are both smooth curves. If \( U \subset A^1 \) is a sufficiently small analytic neighborhood and \( C \subset V_{nH,0,1} | V \) is a general \( m \)-multisection passing through the point \([D]\), then the family of curves \( D \to C \), naturally parametrized by \( C \), has special fibre \( D_0 = mD \) and general fibre \( D_i = D_0 \cup \cdots \cup D_i \), equal to the union of \( m \) one-nodal irreducible curves in the linear system \([O_S(n)]\). If we make a base change of order \( m \) and we smooth the total space of the obtained family, we get a family \( X \to A^1 \) which is a cover of order \( m \) of \( S \to A^1 \), totally ramified along its special fibre. In particular, the special fibre of \( X \) is \( X_0 = A \cup E_1 \cup \cdots \cup E_{m-1} \cup B \), where every \( E_i \) is a \( P^1 \)-bundle on an irreducible curve \( R_i \) isomorphic to \( R \), for \( i = 1, \ldots, m \), with \( A \) intersecting \( E_1 \) along \( R_1, \ldots, E_i \), intersecting \( E_{i-1} \) along \( R_i \neq R_1, \ldots, E_{m-1} \) intersecting \( B \) along a section \( R_m \cong R \), with \( R_m \neq R_{m-1} \). The image into \( P^3 \) of a section \( \gamma \) of \( X \), intersecting the special fibre at a point \( q \in E_i \), is an analytic curve intersecting \( A \) with multiplicity \( m - i \) at the point \( p \in R \), image of \( q \), and \( B \) with multiplicity \( i \) at the same point. Now, we have \( m \) irreducible divisors \( D^1 \subset X \), \( i = 1, \ldots, m \), in the linear system \([nH]\), mapped to \( D \subset S \), via the morphism \( X \to S \). For our purposes, it is enough to consider only one of these divisors, say \( D^1 \). All fibres of \( D^1 \) are now reduced. The general fibre of \( D^1 \) is an irreducible one-nodal curve, corresponding to a general element of \( \mathcal{V}_1 \), while \( D^1 \) cuts on \( X_0 \) a connected Cartier divisor which restricts to \( D_A \) on \( A \), to \( D_B \) on \( B \) and a union of fibres, counted with the right multiplicity, on every \( E_i \). The singular locus of \( D^1 \) is a section of \( X \), which we denote by \( \gamma_1 \).

Now, by the hypothesis that \( D_B \) does not contain \( E_i \), for every \( i \), we have that \( \gamma_1 \) does not intersect \( X_0 \) at a point on \( E_i \) or on the fibre \( F_{i,j} \) of \( E_i \), whose image into \( S_0 \) is the point \( E_i \cap R \), for every \( i \) and \( j \). To see this, let \( S \sim nH \) be a divisor whose singular locus \( \gamma \) intersects \( E_i \) at a smooth point of \( X_0 \) and let \( \mathcal{Y} \) be the blowing-up of \( X \) along \( \gamma \), with exceptional divisor \( \Gamma \). If \( S' \) is the proper transform of \( S \) on \( \mathcal{Y} \), then \( S' \cap E_i = -2T E_i = -2 \), so \( E_i \subset S' \) and hence \( E_i \subset D_B \subset S \subset X \). Similarly, if \( \gamma \) intersects \( X_0 \) at a smooth point of \( F_{i,j} \), then, by arguing as before, we see that \( F_{i,j} \subset S \). Moreover, by blowing-up along \( F_{i,j} \), if \( j < m - 1 \), we find that \( F_{i,j+1} \subset S \) and so on, until we get that \( E_i \subset S \).

Finally, by the hypothesis that \( D_A \) and \( D_B \) are smooth and from what we proved above, the curve \( \gamma_1 \) must intersect \( X_0 \) at a smooth point, say \( q_1 \), lying on a fibre
$F_i$ of $E_i$, whose image point $p$ in $\mathbb{P}^3$ is not a base point of the pencil $\mathcal{F}$. Let $\mathcal{Y}$ be the blowing-up of $X$ along $\gamma_1$ and let us denote by $\Gamma_1$ the new exceptional divisor. The proper transform, which we still denote by $F_i$, of $F_i$ on $Y_0$ is now a $(-1)$-curve on the proper transform $E_i'$ of $E_i$. Now, if $D'$ is the proper transform of $D$, by the hypothesis that $D'$ is singular along $\gamma_1$, we have that $D' - \Gamma_1$ intersects $F_i$ with multiplicity at least 2 in the divisor $D'|_{E_i} \sim 2F_i + cF_{E_i}$, where $c = n(d-1) - 2$ and $F_{E_i}$ is the linearly equivalence class of the fibre of $E_i$. By using that $D'$ is a Cartier divisor, we find that $D_A = D'|_A$ and $D_B = D'|_B$, intersect $R$ with multiplicity at least 2 at the point $p$ of $R$ corresponding to the fibre $F_i$. Since $D_A$ and $D_B$ are both smooth curves, it follows that the curve $D_A \cup D_B \subset A \cup B$ is parametrized by a point $[D_A \cup D_B] \in T(d, n)$. This proves the statement.

\textbf{Corollary 2.7.} Let $V_{nH,k,\delta}^S$ be the Universal Severi-Enriques variety introduced in the first section. Let $V_0$ be the special fibre of $V_{nH,k,\delta}$ and let $[D] \in V_0$ be any point, corresponding to a curve $D = D_A \cup D_B \subset A \cup B$. Assume that $D_A$ and $D_B$ intersect transversally $R = A \cap B$ at a point $p$, in such a way that $D$ has a node at $p$. Let $U \subset \mathbb{A}^1$ be an analytic neighborhood small enough of $0 \in \mathbb{A}^1$ and let $C$ be a general local $m$-multisection of $V_{nH,k,\delta}$ passing through $[D]$. Denote by $\gamma : D \to C$ the family of curves naturally parametrized by $C$ and by $D_i$ the general fibre of $D_i$, with irreducible components $D_i^1, \ldots, D_i^m$. Then, the point $p$ is not a limit of any singular point of $D_i^1$, for every $i = 1, \ldots, m$. 

\textbf{Proof.} If we make a base change of order $m$ and we repeat the same argument as in Case 3 of Lemma 2.7, we see that, if $p$ is limit of a singular point of $D_i$, then $p$ is contained with multiplicity at least 2 in the divisor $R \cap D' = R \cap D_A$. Finally, notice that this is true for every point $p \in R$, also if $p = E_i \cap R$, for some $i = 1, \ldots, d(d-1)$. 

\section{The case of one-cuspidal curves}

In this section we want to determine all irreducible components of the special fibre $V_0$ of $V_{nH,k=1,\delta=0}$ with the respective geometric multiplicities (see definition 1.2). We will assume $d, n \geq 2$, in such a way that on the general surface $S_i \subset \mathbb{P}^3$ of the pencil $\mathcal{F}$ there exist irreducible curves in the linear system $|\mathcal{O}_{S_i}(n)|$ with a cusp at the general point of $S_i$ and no further singularities. In particular, under the hypothesis $d, n \geq 2$, we have that $V_{nH,1,0}^{S_i}$ is a non-empty, irreducible subvariety of codimension 2 of $|\mathcal{O}_{S_i}(n)|$.

\textbf{Lemma 3.1.} Assume that $d \geq 2$ and $n \geq 3$. Then

$$|\mathcal{O}_A(n)| \times |\mathcal{O}_B(n)| \mathcal{V}_{nH,1,0}^B$$

is an irreducible component of $V_0$ of multiplicity 1. If $[D = D_A \cup D_B]$ is its general element, then $D_A$ and $D_B$ are irreducible, they intersect transversally $R$, $D_A$ is smooth and $D_B$ has only one cusp as singularity. In particular, $D$ does not contain any exceptional divisor $E_i \subset B$, with $i = 1, \ldots, d(d-1)$.

\textbf{Proof.} If $d \geq 2$ and $n = 2$, then the variety $|\mathcal{O}_A(n)| \times |\mathcal{O}_B(n)| \mathcal{V}_{nH,1,0}^B$ has dimension smaller that $\dim([|\mathcal{O}_{A \cup B}(2)|] - 2$ and, in particular it cannot be an irreducible component of $V_0$. On the contrary, when $d \geq 2$ and $n \geq 3$ we have that $\dim([|\mathcal{O}_{A \cup B}(n)|] - 2 = \dim(V_{nH,1,0}^{S_i})$. We leave to the reader to verify that the general element $[D = D_A \cup D_B]$ of $|\mathcal{O}_A(n)| \times |\mathcal{O}_B(n)| \mathcal{V}_{nH,1,0}^B$ corresponds to a curve as in the statement. To prove that $[D]$ is a limit of the general element of $V_0$, consider in the linear series $|\mathcal{O}_S(n)|$ the family of divisors singular along a section $\gamma$ of $S$ passing through a general point $p$ of $B$ and see that it cuts out on $A \cup B$ a family of curves of codimension 4 in $|\mathcal{O}_{A \cup B}(n)|$, so it cuts out on $A \cup B$ all curves with a
cusp at \( p \in B \) in the linear system \(|nH|\). By the generality of \( p \), the statement is proved. \( \square \)

**Lemma 3.2.** Assume that \( d, n \geq 2 \) and \((d, n) \neq (2, 2)\). Then

\[
V_{nH,1,0}^A \times |O_{nH}| \quad |O_B(n)|
\]

is an irreducible component of \( V_0 \) of geometric multiplicity 1. If \(|D = D_A \cup D_B|\) is its general element, the curve \( D_A \cup D_B \) does not contain any exceptional divisor \( E_i \subset B \), the curve \( D_A \) and \( D_B \) are irreducible, they intersect transversally \( R \), \( D_B \) is smooth and \( D_A \) has only one cusp as singularity. Finally, \( V_{nH,1,0}^A \times |O_{nH}| \quad |O_B(n)| \) and \(|O_A(n)| \times |O_{nH}| \quad V_{nH,1,0}^B \) are the only irreducible components of \( V_0 \) of geometric multiplicity 1 and whose general element does not contain any \( E_i \).

**Proof.** The proof is left to the reader. \( \square \)

**Remark 3.3.** By the previous Lemmas and by Corollary 2.4 the other possible irreducible components of the special fibre \( V_0 \) of \( Y_{nH,1,0} \) “are produced” by degenerations of the general element \([D_t]\) of \( V_t \) such that as \( D_t \) goes to \( A \cup B \), the cusp of \( D_t \) specializes to the a point of the singular locus \( R \) of \( A \cup B \).

Let \( F(d, n) \subset |O_A \cup B(n)| \) be defined as the Zariski closure of the quasi projective variety

\[
\{[D] \mid D_A \text{ and } D_B \text{ intersect } R \text{ at a general point } p \text{ with multiplicity 3 and they are smooth at } p\}.
\]

Notice that \( F(d, n) \) parametrizes curves cut out on \( A \cup B \) by surfaces of degree \( n \) intersecting \( R \) with multiplicity 3 at a general point \( p \) of \( R \). If \( d = n = 2 \), then \( F(2, 2) \) is cut out on \( A \cup B \) by quadrics containing \( R \) and you can verify that \( \dim(F(2, 2)) < \dim(|O_{A \cup B}(n)|) - 2 \). If \( d, n \geq 2 \) and \((d, n) \neq (2, 2)\), then \( \dim(F(d, n)) = \dim(|O_{A \cup B}(n)|) - 2 \) and moreover the general element \( F(d, n) \) corresponds to a curve \( D = D_A \cup D_B \) such that \( D_A \) and \( D_B \) are smooth and they intersect transversally \( R \) at further \( d(d - 1) - 3 \) general points, outside the multiplicity 3 intersection point.

**Lemma 3.4.** Let \((d, n) \geq (2, 2)\), then \( F(d, n) \) is a non-empty irreducible component of the special fibre \( V_0 \) of \( Y_{nH,1,0} \) if and only if \((d, n) \neq (2, 2)\). The geometric multiplicity of \( F(d, n) \) is 2.

**Proof.** Step 1. Let \( \mathcal{X} \) be the normalization of the double cover of \( S \to \mathbb{A}^1 \) ramified at the special fibre

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathbb{P}^3 \\
S' \downarrow h & & \\
\mathbb{A}^1 & \xrightarrow{\nu_2} & \mathbb{A}^1
\end{array}
\]

as in the proof of lemma 2.3. If \( U \) is an analytic neighborhood of \( 0 \in \mathbb{A}^1 \) small enough, let \( \gamma \) be a section of the family \( \mathcal{X}'|_U \), intersecting the special fibre \( X_0 = A \cup E \cup B \) at a point \( q \) lying on a general fibre \( F \) of \( E \). Now, let \( Y \) be the blowing-up of \( \mathcal{X} \) at \( q \), with new exceptional divisor \( T \simeq \mathbb{P}^2 \). The special fibre of \( Y \) now is \( A \cup E' \cup B \cup T \), where \( E' \) is the blowing-up of \( E \) at \( q \), \( F_2 = -1 \), \( T \) intersects \( E' \) along a curve which is a line on \( T \) and a \(-1\)-curve on \( E' \) and it has not intersections with \( A \) and \( B \). The proper transform \( \tilde{\gamma} \) of \( \gamma \) now will intersect \( T \) at a general point \( q_0 \). Let us consider a general effective divisor \( C \subset Y|_U \) linearly equivalent to \( nH - 3T \) and with cuspidal singularities along \( \tilde{\gamma} \). Notice that such a divisor \( C \) exists and its
general fibre is an irreducible curve with only a cusp as singularity at $q_t$. Now, if $(d, n) = (2, 2)$, then $(nH - 3T)|_{\nu}$ is not effective. This implies that $E'$ is contained in the base locus of $|nH - 3T|$ and, if $C_A \cup C_B \subset A \cup B$ is the special fibre of the image of $C$ to $S_r$, then $R = C_A \cap C_B$. Finally, the point $[C_A \cap C_B]$ belongs to $F(2, 2)$ but it is not general in any irreducible component of $V_0$. Assume that $d, n \geq 2$ and $(d, n) \neq (2, 2)$. Then $(nH - 3T)|_{\nu}$ is an effective divisor containing $F$ with multiplicity at least three. Moreover, $C|_T$ will be a cubic with at least a cusp at $q_{0}$.

Step 2. Let $Y'$ be the blowing-up of $Y$ along $\gamma$ and denote by $\Gamma$ the new exceptional divisor. The special fibre of $Y'$ is $A \cup E' \cup B \cup T'$, where $T'$ is the blowing-up of $T$ at $q_0$, with exceptional divisor $E_0 = T' \cap \Gamma$. Moreover, notice that $\Gamma$ intersects every fibre of $Y'$ along a $-1$-curve $E_1$. By the hypothesis that $C \subset Y$ has cuspidal singularities along $\gamma$, we have that the proper transform $C'$ of $C$ is linearly equivalent to $nH - 3T' - 2\Gamma$ and the general fibre $C'_j$ of $C'$ intersects $E_i$ at only one point $q'_i$ and it smooth and tangent to $E_i$ at this smooth point. The points $q'_i$ determine a section $\psi$ of $Y'$, contained in $\Gamma$, intersecting the special fibre of $Y'$ at a general point $q'_0 \in T' \cap \Gamma$. If we blow-up $Y'$ along $\psi$, we denote by $Y''$ the obtained family of surfaces and by $\Psi$ the new exceptional divisor, then the proper transform $C''$ of $C$ will be linearly equivalent to $nH - 3T'' - 2\Gamma - 3\Psi$. Now observe that $C''F = -3T''F = -3$, and hence $F \subset C$. We set $\alpha = \text{mult}_F(C) = \text{mult}_F(C'')$.

Step 3. Let $Y^1$ be the blowing-up of $Y''$ along $F$. We denote by $\Theta_1$ the new exceptional divisor. Now, the special fibre of $Y^1$ is given by $A_1 + B_1 + \Theta_1 + E' + T_1$, where $A_1$, $B_1$ and $T_1$ are the blow-ups of $A$, $B$ and $T''$ at $F \cap A$, $F \cap B$, and $F \cap T''$ respectively. Then, by the triple point formula

$$\Theta_1(A_1 + B_1 + T_1 + E' + \Theta_1) = 0,$$

we find that $F^2_{\Theta_1} = -2$ and hence $\Theta_1 \simeq \mathbb{F}_2$. Moreover, if $C_1$ is the proper transform of $C''$, then, denoting by $F_{\Theta_1}$ the linear equivalence class of the fibre of $\Theta_1$, we find that

$$C_1|_{\Theta_1} \sim -3F_{\Theta_1} - \alpha\Theta_1|_{\Theta_1} \sim -3F_{\Theta_1} + \alpha(3F_{\Theta_1} + F) \sim (3\alpha - 3)F_{\Theta_1} + \alpha F.$$

Now $C_1|_{\Theta_1}$ must be effective, so $\alpha \geq 1$. Moreover, since $C \subset Y$ is general among divisors linearly equivalent to $nH - 3T$ and with cuspidal singularities along $\gamma$, we may assume that $\alpha$ is the minimum integer in order that $C_1|_{\Theta_1}$ is effective, i.e. $\alpha = \text{mult}_F(C) = 1$ and $C_1|_{\Theta_1} = F$. Again, notice that $F \subset C_1$, because $FC_1 = -3T_1 - \Theta_1F = -3 - (\Theta_1)^2E' = -3 + 1 = -2$. Moreover, $C_1$ must be smooth along $F$.

Step 4. Let $Y^2$ be the blowing-up along $F$ of $Y^1$. We denote by $\Theta_2$ the new exceptional divisor. Now the special fibre $Y^2_{\Theta_2}$ of $Y^2$ is

$$Y^2_{\Theta_2} = A_2 \cup E' \cup B_2 \cup T_2 \cup \Theta_1 \cup 2\Theta_2,$$

where $A_2$, $T_2$ and $B_2$ are the blowing-up of $A_1$, $T_1$ and $B_1$ at $F \cap A_1$, $F \cap T_1$ and $F \cap B_1$ respectively. Now, by the triple point formula, $F^2_{\Theta_2} = -1$ and $\Theta_2 \simeq \mathbb{F}_1$. Moreover, if $C_2$ is the proper transform of $C_1$, then $C_2|_{\Theta_2} \sim -2F_{\Theta_2} + \frac{1}{2}(3F_{\Theta_2} + F + F_{\Theta_2} + F)$, where $F_{\Theta_2}$ is the linear equivalence class of the fibre of $\Theta_2$. So, $C_2|_{\Theta_2} = F$.

Step 5. Finally, let $Y^3$ be the blowing-up of $Y^2$ along $F$ and let $\Theta_3 \simeq \mathbb{F}_0$ be the new exceptional divisor. The special fibre of $Y^3$ now is

$$Y^3_{\Theta_3} = A_3 + E' + T_3 + B_3 + \Theta_1 + 2\Theta_2 + 3\Theta_3,$$

where $A_3$, $T_3$ and $B_3$ are the proper transforms of $A_2$, $T_2$ and $B_2$. Moreover, if you denote by $C_3$ the proper transform of $C_2$ then $C_3 \sim nH - 3T_3 - 2\Gamma - 3\Psi - \Theta_1 - 2\Theta_2 - 3\Theta_3$ and the linear system $|C_3|_{\Theta_3} = |F|$ is the ruling determined by $F$ and it does not contain $F$ in its base locus.

Now, let $D$ be the image of $C \subset Y^3_\nu$ into $S^*_\nu(U)$ and let $D_0 = 2C_A \cup 2C_B$, where $C_A = C \cap A$ and $C_B = C \cap B$, be the special fibre of $D$. What we proved above shows that the point $x = [C_A \cup C_B] \in F(d, n) \cap V_0$. In particular, the family $D$
corresponds, into the relative Hilbert Scheme $\mathcal{H}_n$, to an analytic local bisection of $\mathcal{V}^S_{n,n+1,0}$ intersecting the special fibre $\mathcal{V}_0$ at the point $x$ and the general fibre $\mathcal{V}_i$ at two general points.

**Step 6.** Now we want to prove that the point $x$ is general in $F(d,n)$. To this aim, recall that $F(d,n) \subset |O_{A\cup B}(n)|$ parametrizes curves cut out on $A \cup B$ by surfaces $S_n \subset \mathbb{P}^3$ intersecting $R$ with multiplicity three at a general point. These surfaces are parametrized by a codimension two subvariety $\mathcal{F}(n) \subset |O_{p}(n)|$. Now, going back to the family of surfaces $\mathcal{X}$ of Step 1, let $p$ be the point of $R$ corresponding to the fibre $F$ of $E$. As we already observed in Lemma 2.3 $F$ can be identified with a double cover of the projectivization of the fibre over $p$ of the normal bundle to $R$ in $\mathbb{P}^3$. Equivalently, $F$ is a double cover of the parameter space of planes of $\mathbb{P}^3$ containing the tangent line $T_pR$ to $R$ at $p$. Let $H_q$ be the plane corresponding to $q = \gamma \cap F$. Since the singular locus of the image of $C \subset Y|_U$ into $\mathcal{X}|_U$ intersects $F$ at $q$, then the special fibre $C_0 = C \cap Y_0$ of $C$ is cut out on $Y_0$ by a surface $S_n$ intersecting $R$ at $p$ with multiplicity three and tangent to $H_p$ at $p$. In other words, if we denote by $F(d,n)_{p,H_p}$ the codimension 4 subvariety of $F(d,n)$, parametrizing curves cut out by surfaces of $\mathbb{P}^3$ of degree $n$ intersecting with multiplicity three $R$ at the point $p$ and tangent to the plane $H_q$, then $x \in F(d,n)_{p,H_p}$.

**Step 7.** Now, let us denote by $D_{\gamma,\Psi}$ the linear equivalence class of the divisor $3T_2 + 2T + 3\Psi + \Theta_1 + 2\Theta_2 + 3\Theta_3 \subset \mathbb{P}^3$ and by $\mathcal{V}_3$ the fibre of $\mathcal{Y}^3$ over $t \in U \subset \mathbb{A}$. Then, by arguing as in the proof of lemma 2.3 we find that the dimension of the image $\text{Im}(r_0) := \mathcal{W}_{\gamma,\Psi}$ of the map

$$r_0 : H^0(\mathcal{Y}^3, O_{\mathcal{Y}^3}(nH - D_{\gamma,\Psi})) \to H^0(\mathcal{V}^3_0, O_{\mathcal{V}^3_0}(nH - D_{\gamma,\Psi}))$$

is

$$h^0(\mathcal{Y}^3_0, O_{\mathcal{Y}^3_0}(nH - D_{\gamma,\Psi})) = h^0(\mathcal{V}^3_0, O_{\mathcal{V}^3_0}(nH - 2T - 3\Psi)) = h^0(\mathcal{Y}^3_0, O_{\mathcal{Y}^3_0}(nH)) - 5.$$

We want to prove that the family $\mathcal{V}_{\gamma,\Psi} \subset |O_{A\cup B}(nH)|$ of image divisors of divisors in $\mathcal{W}_{\gamma,\Psi}$, with respect to the natural morphism $\mathcal{Y}^3 \to S$, has dimension $h^0(\mathcal{Y}_1, O_{\mathcal{Y}_1}(nH)) - 5 = h^0(S_1, O_{S_1}(nH)) - 5$. To this aim, notice that, from what we have proved until now, at the Step 2, the restricted linear system $|C|_T$ is the linear system $\mathcal{F}_{\gamma,\Psi}$ of cubics having a flex at the point $F \cap T$ and a cusp, with cuspidal tangent line $R_0$ determined by the section $\psi$ of $\Gamma$, at the point $\gamma \cap T$. So, $\text{dim}(\mathcal{F}_{\gamma,\Psi}) \geq 1$. Actually, it is easy to show that $\mathcal{F}_{\gamma,\Psi}$ is a pencil whose all fibres are irreducible and moreover, by using Proposition 2.1 of [2], one can prove that, if $C_1$ and $C_2$ are two cubics of $\mathcal{F}_{\gamma,\Psi}$, then $C_1$ and $C_2$ intersect with multiplicity exactly three at the point $F \cap T$. In particular, by using notation of Step 5, the proper transforms $\tilde{C}_1$ and $\tilde{C}_2$ of $C_1$ and $C_2$ to $T_3$ intersect the exceptional divisor $\Theta_3 \cap T_3$ at two different points $r_1$ and $r_2$. If $S_1$ and $S_2$ are two divisors in the family $\mathcal{V}_{\gamma,\Psi} \subset |nH - (3T_3 - 2T + 3\Psi + \Theta_1 + 2\Theta_2 + 3\Theta_3)|$ such that $S_i|_{T_3} = \tilde{C}_i$, then the intersection points $S_1|_{A_3} \cap \Theta_3$ and $S_2|_{A_3} \cap \Theta_3$ are different and they are determined by $r_1$ and $r_2$. More precisely, $S_i|_{A_3} \cap \Theta_3 = F_{r_i} \cap A_3$, where $F_{r_i}$ is the line of the ruling $|F|$ of $\Theta_3$, passing through $r_i$, $i = 1, 2$. We deduce, in particular, that there are not two divisors in $\mathcal{W}_{\gamma,\Psi}$ restricting to the same divisor on $A$ and on $B$ and to two different cubics on $T_3$. So, $\mathcal{V}_{\gamma,\Psi} \subset F(d,n) \cap \mathcal{V}_0$ has dimension $\text{dim}(\mathcal{V}_{\gamma,\Psi}) = \text{dim}(\mathcal{W}_{\gamma,\Psi}) = \text{dim}(|O_{S_0}(nH)|) - 5$.

Moreover, let us consider, at the Step 2, a general section $\psi_1$ of $\Gamma \subset \mathcal{Y}'$, intersecting $T'$ at a point on $\Gamma \cap T'$ different from $\psi \cap T' \in \Gamma \cap T'$. We want to prove that $\mathcal{V}_{\gamma,\Psi}$ and $\mathcal{V}_{\gamma,\Psi_1}$ are different subvarieties of $F(d,n) \cap \mathcal{V}_0 \subset |O_{A\cup B}(nH)|$. To this aim, let $\mathcal{Y}''$ be the blowing-up of $\mathcal{Y}'$ along $\psi$ and $\psi_1$, let $\Psi$ and $\Psi_1$ the new exceptional divisors, let us repeat all blow-ups of Steps 3, 4 and 5 and let us use the same notation. Now, if $D$ and $D_1$ are two irreducible cubics belonging respectively to the pencils $\mathcal{F}_{\gamma,\Psi}$ and $\mathcal{F}_{\gamma,\Psi_1}$ on $T$, then $D$ and $D_1$ intersect with multiplicity 4 at
\(\tilde{\gamma} \cap T\) and with multiplicity \(m\), with \(3 \leq m \leq 5\) at the point \(F \cap T\). Moreover, for any cubic \(C\) in the pencil \(F_{\Gamma, \Psi}\), there exists only one cubic \(C_1\) in the linear system \(F_{\Gamma, \Psi}\), intersecting \(C\) with multiplicity at least 4 in \(F \cap T\). The proper transforms \(\tilde{C}\) and \(\tilde{C}_1\) to \(T_3\) of \(C\) and \(C_1\) will intersect at a point \(r\) of the exceptional divisor \(\Theta_3 \cap T_3\) with multiplicity at most 2. Let \(S\) and \(S_1\) be any two divisors in \(\mathcal{Y}_3\), belonging respectively to the linear series \([nH - (3T_3 + 2\Gamma + 3\Psi + \Theta_1 + 2\Theta_2 + 3\Theta_3)]\) and \([nH - (3T_3 + 2\Gamma + 3\Psi + \Theta_1 + 2\Theta_2 + 3\Theta_3)]\), and such that \(S|_{T_3} = \tilde{C}\) and \(S_1|_{T_3} = \tilde{C}_1\). We want to prove that the curves \(S \cap (A_3 \cup B_3)\) and \(S_1 \cap (A_3 \cup B_3)\) cannot be equal.

Assume that \(\text{mult}_{r}(\tilde{C} \cap \tilde{C}_1) = 1\). Let \(\mathcal{Y}_4\) be the blowing-up of \(\mathcal{Y}_3\) along the fibre \(F_r\) of \(\Theta_3\) passing through the point \(r\). (Notice that \(F_r = S \cap \Theta_3 = \Theta_3 \cap S_1\)). Let \(T_4, A_4\), and \(B_4\) be the proper transforms of \(T_3, A_3\), and \(B_3\) and let \(\Theta_4\) be the new exceptional divisor. Now, \(\Theta_4\) is isomorphic to \(F_1\) and it is contained in the special fibre \(\mathcal{Y}_4^S\) of \(\mathcal{Y}_4^S\) with multiplicity 3 and \(F_r^2 = -1\). Moreover, if \(S'\) and \(S'_1\) are the proper transforms of \(S\) and \(S_1\) in \(\mathcal{Y}_4\) then

\[
S'|_{\Theta_4} \sim (nH - 3T_4 - 2\Gamma - 3\Psi - \Theta_1 - 2\Theta_2 - 3\Theta_3 - 4\Theta_4)|_{\Theta_4}
\]

\[
\sim -3T_4|_{\Theta_4} - 3\Theta_3|_{\Theta_4} + \frac{4}{3}(3\Theta_3 + T_4 + A_4 + B_4)|_{\Theta_4}
\]

\[
\sim H_{\Theta_4}
\]

\[
\sim S'_1|_{\Theta_4},
\]

where \(H_{\Theta_4}\) is the linear equivalence class of a line on \(\Theta_4\). The two lines \(\mathcal{R} = S' \cap \Theta_4\) and \(\mathcal{R}_1 = S'_1 \cap \Theta_4\) intersect \(T_4\) at two different points by the hypothesis that \(\text{mult}_{r}(\tilde{C} \cap \tilde{C}_1) = 1\) on \(T_3\). If \(R\) and \(R_1\) intersect \(A_4\) at the same point, and hence \(S \cap A_3\) and \(S_1 \cap A_3\) are tangent at \(F_r \cap A_3\), then \(R\) and \(R_1\) must intersect \(B_4\) at two different points. In particular, \(S \cap B_3\) and \(S' \cap B_3\) intersect transversally at \(F_r \cap B_3\) and so they are different curves. Assume now that \(\text{mult}_{r}(\tilde{C} \cap \tilde{C}_1) = 2\) on \(T_3\). Then, when we blow-up along \(F_r\), by using the same notation, the lines \(\mathcal{R} = S' \cap \Theta_4\) and \(\mathcal{R}_1 = S'_1 \cap \Theta_4\) intersect \(T_4\) at the same point. If \(R \neq R_1\) we find that \(S \cap (A_3 \cup B_3)\) and \(S_1 \cap (A_3 \cup B_3)\) intersect transversally at \(F_r \cap A_3\) and \(F_r \cap B_3\) and, in particular, they are different curves. If \(R = R_1\) let \(\mathcal{Y}_4\) be the blowing-up of \(\mathcal{Y}_3\) along \(R\). The new exceptional divisor \(\Theta_5\) is still an \(F_1\) contained in the special fibre of \(\mathcal{Y}_5\) with multiplicity 6. Moreover, the proper transforms \(S''\) and \(S''_1\) of \(S'\) and \(S'_1\) will intersect \(\Theta_5\) along two lines \(R\) and \(R_1\) respectively. Now, \(\tilde{R}\) and \(\tilde{R}_1\) must be different by the hypothesis that \(\text{mult}_{r}(\tilde{C} \cap \tilde{C}_1) = 2\) on \(T_3\). This implies, in particular, that the curves \(S'' \cap A_5 \cup B_5\) and \(S''_1 \cap A_5 \cup B_5\) are different. This proves that \(V_{\Gamma, \Psi}\) and \(V_{\Gamma, \Psi, 1}\) are two different subvarieties \(F(d, n) \cap \mathcal{Y}_0\) of dimension \(\dim([O_{S_1}(nH)]) - 5\).

It follows that \(F(d, n) \cap \mathcal{Y}_0\) contains the codimension 4 subvariety \(F(d, n)_{p, H_0}\) of \(F(d, n)\). By using now the generality of \(q = \gamma \cap F\) in \(F \subset \mathcal{Y}\) and the generality of the fibre \(F\) on \(\mathcal{E}\), we see that \(F(d, n)\) is an irreducible component of \(\mathcal{Y}_0\). Finally, the fact that there are not local analytic sections of \(\mathcal{Y}_{nH, 1.0}\) passing through the general element of \(F(d, n)\) follows from lemma 3.1.

\[\square\]

**Lemma 3.5.** Assume \(d, n \geq 2\). Let \(V\) be an irreducible component of geometric multiplicity 2 of the special fibre \(\mathcal{Y}_0\) of \(\mathcal{Y}_{nH, 1.0}\) whose general element \([D]\) corresponds to a curve \(D = D_A \cup D_B\), such that \(D_B\) does not contain \(E_i\), for every \(i \leq d(d - 1)\). Then \(V = F(d, n)\). In particular, for \(d = n = 2\), there are not irreducible components of \(\mathcal{Y}_0\) of geometric multiplicity 2.
Proof. Under the hypothesis $d, n \geq 2$, let $V$ be an irreducible component of $\mathcal{V}_0$ as in the statement and let $[D = DA \cup DB]$ its general element. Let

$$
\begin{array}{ccc}
\mathcal{X} & \overset{h}{\longrightarrow} & \mathcal{S} \\
\mathbb{A}^1 & \overset{\nu_2}{\longrightarrow} & \mathbb{A}^1
\end{array}
$$

be the smooth double cover of $\mathcal{S}$, totally ramified at the special fibre, which we already described in Lemmas 2.3 and 3.4. Let $S \sim nH$ be a general divisor, such that $S \cap X_i$ is a general 1-cuspidal curve on $X_i$ and such that $S \cap A = DA$ and $S \cap B = DB$. The singular locus $\gamma$ of $S$ is a section of $\mathcal{X}$. By the hypothesis that there are not local analytic sections of $V_{nH,1,0}$ passing through the general element of $V$, we have that $V \neq V_{nH,1,0}^A \times |O_A(n)| \cup |O_B(n)|$ and $V \neq |O_A(n)| \times |O_A(n)| V_{nH,1,0}^B$, and so $\gamma \cap X_0$ is not a smooth point of $A \cup B$ lying on $A$ or $B \cup \bigcup_{i=1}^{d-1} E_i$.

By the hypothesis that $DB$ does not contain $E_i$, for every $i$, by arguing as in Case 3 of Lemma 2.6 we have that $\gamma$ does not intersect $X_0$ at a point on $E_i$ or on the fibre $F_i$ of $E$ passing through the point $E_i \cap E$.

Hence, $\gamma$ intersects $X_0$ at a smooth point $q \in F \subset E$, where $F$ is any fibre of $E$ different from $F_i$, for every $i$. As in the Step 1 of the previous Lemma, let $\gamma'$ be the blowing-up of $\mathcal{X}$ at $q$ with exceptional divisor $T$ and let $\hat{\gamma}$ be the proper transform of $\gamma$. Now, the proper transform $S'$ of $S$ is linearly equivalent to $nH - mT$, where $m$ is the multiplicity of $S$ at $q$. Since $S'|T$ is a plane curve of degree $m$ which must have at least a cusp at $\hat{\gamma} \cap T$, we have that $m \geq 2$.

If $d = n = 2$ and $m \geq 2$ then $\gamma' \subset nH - mT$. In particular, $R_1 \subset S|A = DA$ and $R_2 \subset S|B = DB$ and the point $[D]$ is not general in any irreducible component of $\mathcal{V}_0$.

Suppose that $m \geq 4$ and $(d, n) \neq (2, 2)$. Then $S'|_{\gamma'}$ contains $F$ with multiplicity $m \geq 4$ and so $DA \cup DB$ is cut out on $A \cup B$ from a surface of $\mathbb{P}^3$ intersecting $R = S'^d \cap \pi$ with multiplicity at least four at the point $p$ image of $F \subset E$. In particular, $[DA \cup DB]$ is general in a subvariety $W \subset |O_{A \cup B}(n)|$ of codimension at least $4 - 1 = 3$ and it can not be general in any irreducible component of $\mathcal{V}_0$.

If $m = 3$ and $(d, n) \neq (2, 2)$, then $V = F(d, n)$ by the previous lemma.

Assume that $m = 2$ and $d, n \geq 2$. Then, $S'|T = 2R$, where $R$ is a line passing through the point $\hat{\gamma} \cap T$. Now, by $FS' = -2$ we have that $S'|_{\gamma'}$ contains $F$ with multiplicity $r \geq 2$. Since $S'$ is a Cartier divisor, $R$ is the line generated by $F \cap T$ and $\hat{\gamma} \cap T$ and $F$ is contained with multiplicity exactly 2 in the divisor $S'|_{E'}$. Now, if $Y'$ is the blowing-up of $Y'$ along $F$, the proper transform $S''$ of $S'$ restricts on the new exceptional divisor $\Theta_1 \simeq \mathbb{F}_2$ to an effective divisor linearly equivalent to $-2F_{\Theta_1} + \alpha(3F_{\Theta_1} + F)$, where we may assume that $\alpha$ is the minimal integer in order that $S''|_{\Theta_1}$ is effective and it intersects with multiplicity two $T' \cap \Theta_1$ at the point $R' \cap \Theta_1$, where $R'$ and $T'$ are the proper transforms of $R$ and $T$. So $\alpha = 2$ and $DA$ and $DB$ have a double point at the point $p \in R$ corresponding to the fibre $F$. Also in this case $DA \cup DB$ is general in a subvariety $W \subset |O_{A \cup B}(n)|$ of codimension at least $4 - 1 = 3$, and so $[D] = [DA \cup DB]$ can not be general in any irreducible component of $\mathcal{V}_0$. \hfill $\Box$

In order to describe the other irreducible components of $\mathcal{V}_0 \subset \mathcal{V}_{nH,1,0}$ we need to introduce some notation. Let $S_A(d, n) = V_{nH,0,1}^{A, R}$ be the Zariski closure of the locally closed set

$$
\{ [DA] | DA \text{ has a node at a general point } p \text{ of } R \} \subset |O_A(nH)|
$$
and let $S_B(d, n) = V_{nH, 0.1}^{B, R} \subset |O_B(n)|$ be defined at the same way. Moreover, let $T_A(d, n)$ be the Zariski closure of the locally closed set

$$\{(D_A) \mid D_A \text{ is tangent to } R \text{ at a general point } \subset |O_A(nH)|\}$$

and let $T_B(d, n) \subset |O_B(n)|$ be defined at the same way.

**Lemma 3.6.** For every $d, n \geq 2$, we have that $S_A(d, n) \times |O_R(n)| T_B(d, n)$ and $T_A(d, n) \times |O_R(n)| S_B(d, n)$ are two irreducible components of the special fibre $V_0$ of $V_{nH, 1.0}$ of geometric multiplicity 3.

Before proving the lemma notice that $S_A(d, n) \times |O_R(n)| T_B(d, n)$ parametrizes curves cut out on $A \cup B$ by surfaces of degree $n$ in $\mathbb{P}^3$ tangent to $S^{d-1}$ and transverse to $\pi$ at the general point $p$ of $R = S^{d-1} \cap \pi$. In particular, its general element $[D]$ corresponds to a curve $D_A \cup D_B$, such that $D_A$ is a one nodal curve, with the node at a general point $p$ of $R$, intersecting transversally $R$ outside $p$ at points different from $E_i \cap R$, for every $i$ and $D_B$ is a smooth curve tangent to $R$ at $p$ and such that $D_B \cap R = D_A \cap R$. Similarly for $T_A(d, n) \times |O_R(n)| S_B(d, n)$.

**Proof.** We prove the lemma for $S_A(d, n) \times |O_R(n)| T_B(d, n)$. The other case is the same if you substitute $A$ with $B$. Let $p \in R = A \cap B \subset S_0$ be a point different from $E_i \cap R$, for every $i$. By using the notation of Theorem 2.6 we denote by $\mathcal{Y}$ the desingularization of the triple cover of $S$, totally ramified along its special fibre $\mathcal{Y}_0 = A \cup E_1 \cup E_2 \cup B$ and by $F_i$ the fibre of $E_i$, with $i = 1, 2$, over the point $p = (0, 0, 0, 0)$ of $S$. Let now $U \subset \mathbb{A}^1$ be an analytic neighborhood of 0 and let $\gamma \subset \mathcal{Y}|U$ be a local section passing through a general point $q_0^1$ of $F_1$.

*Step 1.* Let $\mathcal{Y}$ be the blowing-up of $\mathcal{Y}$ at the point $q_0^1 = \gamma \cap \mathcal{Y}_0$, with new exceptional divisor $T$ and with special fibre $\mathcal{Y}_0$. Let us denote by $\gamma$ the proper transform of $\gamma$ to $\mathcal{Y}$ and let $\mathcal{C} \subset \mathcal{Y}|U$

be a general divisor such that $\mathcal{C} \sim nH - 2T$ and the general fibre of $\mathcal{C}$ is reducible with a cusp as singularity at the intersection point $p_1^1 = \gamma \cap \mathcal{Y}_1$. Notice that, since we are assuming $d, n \geq 2$, such a divisor exists. Which kind of singularities may appear in the special fibre of $\mathcal{C}$? First of all we observe that $\mathcal{C}|_T = 2L$, where $L$ is the line generated by the points $\gamma \cap T$ and $F_1 \cap T$ and so $\mathcal{C}|_{\mathcal{Y}_0}$ contains $F_1$ with multiplicity exactly 2. Now, if we recontract $T$, the image $C$ of $\mathcal{C}$ in $\mathcal{Y}$ is a family of curves with cuspidal singularities along $\gamma$ and such that $C|_{\mathcal{Y}_0}$ contains $F_1$ with multiplicity exactly 2. Let $\mathcal{Y}_1$ be the blowing-up of $\mathcal{Y}$ along $\gamma$. If $\Gamma$ is the new exceptional divisor, then $\Gamma$ is a $\mathbb{P}^1$-bundle over $\gamma$ intersecting every fibre $\mathcal{Y}_t^1$ of $\mathcal{Y}_1$ along a curve $E_t$ which is the exceptional divisor on $\mathcal{Y}_t$ and a fibre on $\Gamma$. The special fibre of $\mathcal{Y}_1$ is $\mathcal{Y}_0^1 = A \cup E_1^\prime \cup E_2 \cup B$, where $E_1^\prime$ is the blowing-up of $E_1$ at $\gamma \cap E_1$, with exceptional divisor $E_0$. By the hypothesis that $C \subset \mathcal{Y}$ has cuspidal singularities along $\gamma$, we have that the proper transform $C'$ of $C$ in $\mathcal{Y}_1$ is linearly equivalent to $nH - 2\Gamma$ and the general fibre $C'_t$ of $C'$ is tangent at $E_t$ at a smooth point $q_t^1$. Since $C'|_{\mathcal{Y}_0}$ contains $F_1$ with multiplicity exactly 2, the limit point $q_0^2$ of $q_t^1$ will be the intersection point of $E_0$ with $F_1$. We denote be $\phi$ the section described by points $q_t^1$. Notice that $\Gamma \cap C' = 2\phi$ and moreover $C'_1 F_1 = -2$.

*Step 2.* Let us set $\alpha_1 = \text{mult}_F C'$ and let $\mathcal{Y}^2$ be the blowing-up of $\mathcal{Y}_1$ along $F_1$. The special fibre of $\mathcal{Y}^2$ is $\mathcal{Y}_0^2 = A' \cup E_1^\prime \cup E_2 \cup \Theta_1 \cup \Theta_1$, where $\Theta_1$ is the new exceptional divisor and $A'$ and $E_1^\prime$ are the blowing-up of $A$ and $E_2$ at $A \cap F_1$ and $E_2 \cap F_1$ respectively. By the triple point formula, the new exceptional divisor $\Theta_1$ is an $\mathbb{F}_1$, with exceptional divisor $F_1 = \Theta_1 \cap E_1^\prime$. Now, since $\phi$ intersects $E_1^\prime$ transversally in $\mathcal{Y}_1$ at $q_0^1 \in F_1$, the proper transform $\phi'$ of $\phi$ in $\mathcal{Y}^2$, contained in the proper transform $\Gamma'$ of $\Gamma$, must intersect $\Theta_1$ at a point $q_1$ lying on the fibre $F_{\Theta_0, q_0^2}$ of $\Theta_1$, corresponding to the point $q_0^2$. If $C''$ is the proper transform of $C'$, then $C''|_{\Theta_1}$ must
be an effective divisor passing through \( q_1 \) and tangent to \( F_{\Theta_1, q_1} \) at \( q_1 \). Finally, by generality, \( \alpha_1 \) is the minimum integer such that \( C''|_{\Theta_1, q_1} \) satisfies these properties.

Now, if \( F_{\Theta_1} \) is the linearly equivalence class of the fibre of \( \Theta_1 \), then

\[
C''|_{\Theta_1} \sim -2F_{\Theta_1} + \alpha_1(A' + \varepsilon_1 + \varepsilon_2)|_{\Theta_1} \\
\sim (2\alpha_1 - 2)F_{\Theta_1} + \alpha_1 F_1.
\]

From what we observed before, we may assume \( \alpha_1 = 2 \) and hence \( C''|_{\Theta_1} \) is a conic tangent to \( F_{\Theta_1, q_1} \) at \( q_1 \) and verifying one more property. Indeed, since \( C'' \sim nH - 2T_2 \) we have that \( C''F_2 = -2\Theta_1 F_2 = -2 \), and in particular \( F_2 \subset C'' \). More precisely, \( C''|_{\Theta_1} \) contains \( F_2 \) with multiplicity 2, because, as we observed at the previous Step, \( C''|_{\Theta_1} \subset Y^1 \) contains \( F_1 \) with multiplicity 2 and \( C''|_{\Theta_1} \) is a Cartier divisor.

Step 3. In order to understand the type of singularity of \( C''|_B \) at the point \( F_2 \cap B \), let \( Y^3 \) be the blowing-up of \( Y^2 \) along \( F_2 \). The special fibre of \( Y^3 \) is now

\[
Y^3_0 = A' + \varepsilon' + \Theta'_1 + \varepsilon'_2 + B' + \Theta_2,
\]

where \( \Theta_2 \) is the new exceptional divisor and \( B' \) and \( \Theta'_1 \) are the blowing-up of \( B \) and \( \Theta_1 \) at \( B \cap F_2 \) and \( \Theta_1 \cap F_2 \) respectively. Again, by the triple point formula, \( \Theta_2 \) is isomorphic to \( F_1 \) with exceptional divisor \( F_2 = \Theta_2 \cap \varepsilon'_2 \). Now, if we set \( \alpha_2 = \text{mult}_{F_2}C'' \), then \( \alpha_2 \) is the minimum integer in order that \( C''|_{\Theta_2} \) is effective. By arguing as before, we find that

\[
C''|_{\Theta_2} \sim (2\alpha_2 - 2)F_{\Theta_2} + \alpha_2 F_2,
\]

where \( F_{\Theta_2} \) is the linearly equivalence class of the fibre of \( \Theta_2 \). Thus \( \alpha_2 = 1 \) and \( C''|_{\Theta_2} \) is equal to \( F_2 \) and, again using that \( C''|_{\Theta_2} \) is a Cartier divisor, \( C''|_{B} \) must contain the point \( F_2 \cap B \) with multiplicity 1. In particular, we have that, at the Step 2, the divisor \( C''|_{\Theta_1} \) is a smooth conic tangent to the fibre \( \Theta_1 \cap \varepsilon_2 \) at the point \( F_2 \cap \Theta_1 \) and to the fibre \( F_{\Theta_1, q_1} \) at the point \( q_1 \). So, the divisor \( C \subset Y|_U \)

cuts on \( A \) a curve \( C_A = C|_A \) with a node at \( F_1 \cap A \) and on \( B \) a curve \( C_B = C|_B \) which is smooth and simply tangent to \( R_2 \) at the point \( F_2 \cap B \).

Moreover, if \( [C] \in |O_{AL,B}(n)| \) is the point corresponding to the curve \( C = C_A \cup C_B \), then \( [C] \in V_0 \cap S_A(d, n) \times |O_{nH}| T_B(d, n) \) and there exists a local analytic trisection of \( V_{nH, 1, 1, 0} \) passing through \( [C] \) and intersecting the general fibre of \( V_{nH, 1, 1, 0} \) at three general points.

Step 4. We want to prove that \( C \) is a general point of \( S_A(d, n) \times |O_{nH}| T_B(d, n) \).

Let \( Y^4 \) be the blowing-up of \( Y^3 \) along \( F_2 \) and \( \varphi' \), with new exceptional divisors \( \Theta_3 \) and \( \Phi \) respectively.

The special fibre of \( Y^4 \) is

\[
Y^4_0 = A' + \varepsilon' + \varepsilon' + \Theta''_1 + \Theta_2 + 2\Theta_3 + B'',
\]

where \( \Theta''_1 \) and \( B'' \) are the proper transforms of \( \Theta'_1 \) and \( B' \) respectively. By the triple point formula, \( \Theta_3 \simeq F_0 \) and \( F_2 \) is a line on \( \Theta_3 \). Moreover, if \( \tilde{C} \) is the proper transform of \( C'' \) in \( Y^4 \), then \( \tilde{C} \sim nH - 2T_2 - 3\Phi - 2\Theta''_1 - \Theta_2 - 2\Theta_3 \) and \( \tilde{C}|_{\Theta_3} \sim F_2 \). Now, denoting by \( \Theta_1^i \) the general fibre of \( Y^4 \) and by \( D_{g_2} \) the linear equivalence class of the divisor \( 2T_2 - 3\Phi + 2\Theta''_1 + \Theta_2 + 2\Theta_3 \), by arguing as in Lemma 2.1, we see that the image \( W_{q_1} \) of the restriction map

\[
r_0 : |O_{Y^4}(nH - D_{g_2})| \rightarrow |O_{Y^3}(nH - D_{g_2})|
\]

has dimension \( \dim(|O_{Y^4}(nH)|) - 5 \). Moreover, you can easily verify that also the restriction map

\[
F : W_{q_1} \rightarrow |O_{\Theta''_1}(nH - D_{g_2})|
\]

is surjective and, hence, the pencil \( |O_{\Theta''_1}(nH - D_{g_2})| \) cuts on the curve \( \Theta''_1 \cap A' \) a \( g_2 \), which we denote by \( L_{q_1} \), whose ramification points are \( F_1 \cap A' \) and \( R_{q_1} \), where \( R_{q_1} \) is the intersection point of the fibre \( \Theta''_1 \cap A' \) and the proper transform \( L_{q_1} \) on \( \Theta''_1 \) of the line on \( \Theta_1 \) generated by the points \( q_1 \) and \( F_2 \cap \Theta_1 \).
Step 5. Now, notice that, under the natural map \( \psi_0^1 : Y_0^1 \to S_0 \) contracting all exceptional components of \( \psi_0^1 \), the variety \( W_{q_1} \) is mapped injectively to a codimension 5 subvariety \( \mathcal{V}_{q_1} \) of \( |O_{S(n)}| = |O_{A,n}(n)| \). This follows from the fact that there are not two divisors \( S_1 \) and \( S_2 \) in the linear system \( |O_{Y_0}(nH - D_{q_1})| \) cutting out the same divisor on \( A' \) and \( B'' \) and two different conics to \( \Theta' \), because the conic cut out by \( S_1 \) on \( \Theta' \) determine the intersection points \( S_i \cap A' \cap \Theta'' \). Finally, notice that \( V_{q_1} \subset V_0 \cap S_A(d,n) \times |O_{S(n)}| T_B(d,n) \).

Now, if \( p_1 \) is another general point of the fibre \( F_{\Theta_1,q_1} \), corresponding to the intersection of \( \Theta_1 \) with another general section \( \psi_1 \) of \( \Gamma' \), and, if blow-up along \( p_1 \) and we consider the varieties \( W_{p_1} \) and \( V_{p_1} \), then \( W_{p_1} \neq W_{q_1} \) and \( V_{p_1} \neq V_{q_1} \). Indeed, by the previous Step, the linear series \( L_{q_1} \) and \( L_{p_1} \) are different because they have different ramification points. So, for every point \( q_1 \in F_{\Theta_1,q_1} \), the variety \( V_{q_1} \) is contained in a codimension 4 subvariety \( \mathcal{V}_{q_1} \) of \( |O_{A,n}(n)| \), contained in \( V_0 \cap S_A(d,n) \times |O_{S(n)}| T_B(d,n) \) and parameterizing all curves on \( A \cup B \) which are image of curves cut out on \( X_0 = A \cup E_1 \cup E_2 \cup B \) by divisors in the lineal system \( nH \) with cuspidal singularity along \( \gamma \).

Moreover, if \( \beta \) is another section of \( Y \), intersecting \( Y_0 \) at a smooth point \( x_0^1 \in F_1 \subset E_1 \), with \( x_0^1 \neq \Theta_1 \), and we construct the related variety \( Y_\beta \subset |O_{A,n}(nH)| \) then \( V_{\beta} \neq V_\gamma \). To see this, let us come back to Step 1. Let \( S_\beta \) and \( S_\gamma \) be two divisor in the lineal system \( |O_{Y}(n)| \) with cuspidal singularities along \( \beta \) and \( \gamma \), respectively. Let \( Y' \) be the blow-up of \( Y \) along \( \beta \) and \( \gamma \) and after along \( F_1 \). We denote by \( S'_\beta \) and \( S'_\gamma \) the proper transforms of \( S_\beta \) and \( S_\gamma \) on \( Y' \) and again by \( \Theta_1 \) the exceptional divisor of the blowing-up along \( F_1 \). We know that \( S'_{\beta}|_{E_1} = C_{\beta} \) and \( S'_{\gamma}|_{E_1} = C_{\gamma} \) are two irreducible conics tangent to \( E_2 \cap \Theta_1 \). Moreover, \( C_{\beta} \) and \( C_{\gamma} \) cannot coincide because they are tangent to two different fibres of \( \Theta_1 \), but they can intersect \( A' \) at the same points, where \( A' \) is again the proper transform of \( A \) on \( Y' \). Assume that \( C_\beta \cap A' = C_\gamma \cap A' = \{r_1,r_2\} \). Then \( C_\beta \) and \( C_\gamma \) intersect with multiplicity exactly 2 at the point \( F_2 \cap \Theta_1 \), by the Bezout theorem. Now, since when we blow-up twice \( F_2 \), the last exceptional divisor is isomorphic to \( F_0 \) and the pull-back of \( S' \) and \( S' \) will restrict to a line in \( F_2 \) on \( \Theta_1 \), it follows that the curves \( S'_{\beta}|_B = S_{\beta}|_B \) and \( S'_{\gamma}|_B = S_{\gamma}|_B \) intersect with multiplicity exactly two at the point \( F_2 \cap B \). In particular, \( S_{\beta}|_B \neq S_{\beta}|_B \).

We have proved that the locus \( W_{F_1} \) parametrizing curves on \( Y_0 \), cut out by divisors in the linear system \( |O_Y(n)| \) with cuspidal singularities along a section of \( Y \), intersecting \( Y_0 \) at a smooth point of \( F_1 \), has dimension \( \dim(W_{F_1}) = \dim(|O_{Y_0}(n)|) - 3 \) and it is mapped one to one to the variety \( V_p \subset V_0 \cap S_A(d,n) \times |O_{n(n)}| T_B(d,n) \) parametrizing divisors in \( |O_{A,n}(n)| \) cut out by surfaces tangent to \( A \) and transverse to \( B \) at \( p \). By the generality of \( p \) on \( R \), we find that \( S_A(d,n) \times |O_{n(n)}| T_B(d,n) \) is an irreducible component of \( V_0 \). The fact that there are not local analytic sections or bisections of \( V_{nH,1,0} \) passing through the general element of \( S_A(d,n) \times |O_{n(n)}| T_B(d,n) \) follows by the previous lemmas of this section.

**Lemma 3.7.** Let \( V \) be an irreducible component of \( V_0 \) of geometric multiplicity \( m \geq 3 \), whose general element \( [D] \) corresponds to a curve \( D \subset A \cup B \) not containing \( E_i \) for every \( i \). Then \( V \) is equal to \( S_A(d,n) \times |O_{n(n)}| T_B(d,n) \) or \( S_A(d,n) \times |O_{n(n)}| S_B(d,n) \).

**Proof.** Let \( V \) be an irreducible component of \( V_0 \) as in the statement. By the generality of \( [D] \) in \( V \), if \( \sum_i m_i p_i \), with \( p_i \neq p_j \) if \( i \neq j \), is the divisor cut out by \( D \) on \( R \), then we have that \( m_i \leq 3 \) for every \( i \). Indeed, if \( W \subset |O_{A,B}(n)| \) is an irreducible component of the locally closed set \( \{ |C| = C_A \cup C_B \subset |O_{A,B}(n)| \text{ such that } C \cap R = \sum_i m_i p_i, \text{ with } m_i \geq 4 \text{ for some } i \} \), then \( \dim(W) \leq \dim(|O_{A,B}(n)|) - 4 + 1 = \dim(|O_{A,B}(n)|) - 3 \) and so \( W \) cannot be an irreducible component of \( V_0 \). Moreover, if \( D \cap R = \sum_i m_i p_i \) and for some \( i \) we
have that \( m_i = 3 \) then \( D_A \) and \( D_B \) must be both smooth at \( p_i \), because otherwise \( \dim(V) \leq \dim(|O_{A \cup B}(n)|) - 4 + 1 \). So, in this case we have that \( [D] \in F(d, n) \) and hence \( V = F(d, n) \) and \( m = 2 \).

Hence, we may assume that, if \( D \cap R = \sum_i m_i p_i \), then \( m_i \leq 2 \) for every \( i \). Now, let

\[
\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\psi} & \mathbb{P}^3 \\
\downarrow^f & & \downarrow^g \\
S & \xrightarrow{h} & S'
\end{array}
\]

be the covering of order \( m \) of \( S \) totally ramified at the special fibre, which we already introduced in the Lemma \ref{lem:covering}. By using the same notation as in Lemma \ref{lem:covering}, let \( X_0 = A \cup \cdots \cup E_i \cup \cdots \cup B \) be the special fibre of \( X \) and let \( D \subset X \) be a divisor, linearly equivalent to \( nH \), such that \( D \cap A = D_A, D \cap B = D_B \) and \( D \cap X_i \) is a general 1-cuspidal curve on the fibre in \( |O_{X_i}(n)| \). By the hypothesis that \( D_B \) does not contain any exceptional divisor \( E_i \), by using the argument of Lemma \ref{lem:classes2} we have that the singular locus \( \gamma \) of \( D \) intersects \( X_0 \) at a point \( q \), lying on a fibre \( F_i \), whose image point \( p \) in \( S_0 \) is not \( E_i \cap R \), for every \( l \leq d(d - 1) \). Now let \( X' \) be the blowing-up of \( X \) along \( \gamma \) with exceptional divisor \( \Gamma \) and special fibre \( X'_0 = A \cup \cdots \cup E'_i \cup \cdots \cup B \), where \( E'_i \) is the blowing-up of \( E_i \) at \( q \). We denote by \( D' \) the proper transform of \( D \) in \( X' \). By the hypothesis that, if \( D \cap R = \sum_i m_i p_i \), then \( m_i \leq 2 \) for every \( i \), the divisor \( D'_{|E'_i} \) contains \( F_i \) with multiplicity exactly 2.

This implies that \( D' \cap \Gamma = 2\psi \), where \( \psi \) is a section of \( \Gamma \) intersecting \( E_i \) at the point \( q_1 = F_i \cap \Gamma \). If \( \alpha_i = \text{mult}_{F_i} D_1 \) and if \( \Sigma_2 \) is the blowing-up of \( X' \) along \( F_i \), with new exceptional divisor \( \Theta_i \sim F_1 \), then

\[
D^2_{|\Theta_i} \sim -2F_1 + \alpha_i(2F_1 + F_i),
\]

where \( D^2 \) is the proper transform of \( D' \) in \( X'' \), \( F_{|\Theta_i} \) is the linear equivalence class of the fibre of \( \Theta_i \) and \( F_1 \) is the \((-1)\)-curve on \( \Theta_i \). Now, if \( \Gamma' \) and \( \psi' \) are the proper transforms of \( \Gamma \) and \( \psi \) to \( X'' \) and \( F_{|\Theta_i} = \Gamma' \cap \Theta_i \), then \( D^2_{|\Theta_i} \) must be an effective divisor intersecting \( F_{|\Theta_i} \) at the point \( \psi' \cap \Theta_i \). So \( \alpha_i = 2 \) and \( D^2_{|\Theta_i} \) is a conic.

**Case 1.** If \( i = 1 \), i.e. \( E_i = E_1 \subset X_0 \) is the \( \mathbb{P}^1 \)-bundle intersecting \( A \), then the special fibre of \( X'' \) is \( X''_0 = A' \cup E'_1 \cup \Theta_1 \cup \cdots \cup B \), where \( A' \) is the blowing-up of \( A \) at the point \( F_1 \cap A \) with exceptional divisor \( A' \cap \Theta_1 \). From what we have proved above, we have that \( D^2 \) cuts on \( A' \cap \Theta_1 \) a divisor of degree 2 and so \( D_A \subset A \) has a double point at \( p = F_1 \cap A \). Now, if \( D_B \subset B \) is smooth at the point \( p \), then the point \( [D_A \cup D_B] \) belongs to \( S_A(d, n) \times |O_{n}(n)| T_B(d, n) \), so \( V = S_A(d, n) \times |O_{n}(n)| T_B(d, n) \) and, by the previous lemma, the geometric multiplicity \( m \) of \( V \) is 3. If \( D_B \) is singular at \( p \) then the point \([D]\) cannot be general in any irreducible component \( V \) of \( V_0 \).

**Case 2.** If \( i = m - 1 \) then, by substituting \( A \) with \( B \) in the previous case, we find that \( V = T_A(d, n) \times |O_{n}(n)| S_B(d, n) \) and \( m = 3 \).

**Case 3.** Assume that \( m \geq 4 \) and \( i \geq 2 \). Also in this case, we will prove that at least one of the curves \( D_A \) and \( D_B \) is singular at \( p \). We denote by \( F_{i-1} \) the fibre of \( E_{i-1} \) passing through \( F_i \cap E_{i-1} \) and so on, in such a way that \( F_i \cup \cdots \cup F_{m-1} \) is a connected chain of fibres, with \( F_0 \subset E_{i-1} \), contained in \( D_{|X_0} \) with multiplicity 2 and whose image in \( S_0 \) is the point \( p \in R \). Now, the conic \( D^2_{|\Theta_i} \) must intersect with multiplicity 2 the fibre \( F_{|\Theta_i} \) at the point \( \psi' \cap \Theta_i \), the fibre \( E_{i+1} \cap \Theta_i \) at the point \( F_{i+1} \cap \Theta_i \) and the fibre \( E_{i-1} \cap \Theta_i \) at the point \( F_{i-1} \cap \Theta_i \). So the points \( \psi' \cap \Theta_i \), \( F_{i+1} \cap \Theta_i \) and \( F_{i-1} \cap \Theta_i \) belong to a line \( L_i \) and \( D^2_{|\Theta_i} = 2L_i \). Now, let \( X'' \) be the blowing-up of \( X'' \) along \( F_{i-1}, F_{i+1} \) and \( \psi' \), with exceptional divisors \( \Theta_{i-1}, \Theta_{i+1} \).
and $\Psi$. We denote by $\Theta'_i$ the proper transform of $\Theta_i$ in $X^3$. Now, $L^2_{i|\Theta'_i} = -2$.

Moreover, by repeating always the same argument, we see that $\Theta_{i-1} \simeq F_i \simeq \Theta_{i+1}$ and, denoting by $D^3$ the proper transform of $D^2$ in $X^3$, we have that $D^3 \cap \Theta_{i+1}$ and $D^3 \cap \Theta_{i-1}$ are two conics intersecting respectively the fibres $\Theta'_i \cap \Theta_{i+1}$ and $\Theta'_i \cap \Theta_{i-1}$ at the points $L_i \cap \Theta_{i+1}$ and $L_i \cap \Theta_{i-1}$ with multiplicity $2$. In particular, denoting by $\Gamma''$ the proper transform of $\Gamma'$ in $X^3$ and by $E$ the $(-1)$-curve $E = \Psi \cap \Theta'_i$, we have that

$$D^3 \simeq nH - 2\Gamma'' - 3\Psi - 2\Theta'_i - 2\Theta_{i-1} - 2\Theta_{i+1},$$

$$D^3|_{\Theta'_i} = 2L_i + E$$

and

$$D^3 L_i = -(3 + 2(\Theta_{i-1} + \Theta_{i+1}))L_i - 4 = (2L_i + E)L_i = -3.$$

So, if $X^4$ is the blowing-up of $X^3$ along $L_i$ and if we denote by $\Theta_{L_i}$ the new exceptional divisor, then $\Theta_{L_i} \simeq F_0$ and $L^2_{i|\Theta_{L_i}} = 0$. Moreover, denoting by $D^4$ the proper transform of $D^3$ and by $|F_{\Theta_{L_i}}|$ the ruling of $\Theta_{L_i}$, different from $|L_i|$, we find that

$$D^4|_{\Theta_{L_i}} \simeq -3F_{\Theta_{L_i}} + \alpha_{L_i}(2F_{\Theta_{L_i}} + L_i),$$

where $\alpha_{L_i} = \text{mult}_{L_i} D^3$. Since $D^3|_{\Theta_{L_i}}$ must be an effective divisor, we find that $\alpha_{L_i} = 2$ and $D^4|_{\Theta_{L_i}} \simeq F_{\Theta_{L_i}} + 2L_i$. Now we want to prove that $D^4|_{\Theta_{L_i}}$ contains the fibre $F_E \in |F_{\Theta_{L_i}}|$ passing through the point $E \cap \Theta_{L_i}$. To see this, let $\tilde{X}^4$ be the blowing-up of $X^4$ along $E$. If $\Theta_E$ is the new exceptional divisor, then $\Theta_E \simeq F_0$ and, denoting by $D^4$ the pull-back of $D^4$ to $X^4$, then

$$D^4 \simeq nH - 2\Gamma'' - 3\Psi - 2\Theta'_i - 2\Theta_{i-1} - 2\Theta_{i+1} - 4\Theta_{L_i} - 5\Theta_{E},$$

where $\Theta'_{L_i}$ is the proper transform of $\Theta_{L_i}$ in $\tilde{X}^4$. In particular,

$$F_E D^4 = -4(\Theta'_i|_{L_i} - F_E F_E = +4\Theta_E F_E - 5\Theta_{E} F_E = -1.$$

By using that $(F_E)^2_{\Theta_{L_i}} = -1$, we have that $F_E \subset \tilde{X}^4$ and so, recontracting $\Theta_E$, $F_E$ is contained in $D^4|_{\Theta_{L_i}}$. Now, the fact that $D^3$ is singular along $L_i$ implies that the two conics $D^3|_{\Theta_{i-1}} = C_{i-1}$ and $D^3|_{\Theta_{i+1}} = C_{i+1}$ are singular respectively at the points $L_i \cap \Theta_{i-1}$ and $L_i \cap \Theta_{i+1}$. If $i = 2$ it follows that $D_A$ is singular at $p$ and, similarly, if $i = m - 2$ then $D_B$ is singular at $p$. Assume now that $2 < i < m - 2$. Then $C_{i-1} = 2L_{i-1}$ and $C_{i+1} = 2L_{i+1}$, where $L_{i-1} \subset \Theta_{i-1} \subset X^3$ is the line joining $F_{i-2} \cap \Theta_{i-1}$ and $L_i \cap \Theta_{i-1}$ and, similarly, $L_{i+1} \subset \Theta_{i+1}$ is the line joining $F_{i+2} \cap \Theta_{i+1}$ and $L_i \cap \Theta_{i+1}$. We will prove now that $D^3$ is singular along $L_{i-1}$. By using the same argument, you can verify that $D^3$ is singular also along $L_{i+1}$. First we observe that, by the equality $D^3|_{\Theta_{i=1}} = C_{i-1} = 2L_{i-1}$, it follows that, in $X^4$, the points $L_i \cap \Theta_{L_i}$ and $L_{i+1} \cap \Theta_{L_i}$ stay on the same line $F_{L_i} \in |L_i|$ of $\Theta_{L_i} \simeq F_0$ and $D^4|_{\Theta_{L_i}} = 2F_{L_i} + F_E$. Now let $X^5$ be the blowing-up of $X^4$ along $F_{L_i}$ with new exceptional divisor $\Theta_{i-2} \simeq F_1$ and let $D^5$ be the proper transform of $D^4$ in $X^5$. If we denote by $\Theta'_{i-2}$ the proper transform of $\Theta_{i-2} \subset X^3$ in $X^5$, we have that $(F_{i-2})^2_{\Theta'_{i-2}} = -1$ and $(L_{i-1})^2_{\Theta'_{i-2}} = -1$. Moreover, by using that $F_{i-2} D^4 = -2$, we find that the restricted linear system $D^5|_{\Theta'_{i-2}}$ is a conic, intersecting with multiplicity $2$ the fibre $\Theta'_{i-1} \cap \Theta_{i-2}$ at $L_{i-1} \cap \Theta_{i-2}$. In particular,

$$D^5 \simeq nH - 2\Gamma'' - 3\Psi - 2\Theta'_i - 2\Theta_{i-1} - 2\Theta_{i+1} - 4\Theta_{L_i} - 2\Theta_{L_{i-2}},$$

and $L_{i-1} D^5 = (-2\Theta_{i-2} - 2\Theta_{L_i} + 2\Theta_{L_{i-2}} + 2\Theta_{L_{i+2}})L_{i-1} = -2$. It is enough to blow-up along $L_{i-1}$ to see that $D^5$ is singular along $L_{i-1}$ and so $D^5|_{\Theta'_{i-2}}$ is a conic singular at $L_{i-1} \cap \Theta_{i-2}$. If $i = 2$ this implies that $D_A$ is singular at $p$. If $i = m - 2$, then $D^5|_{\Theta_{i-2}} = 2L_{i-2}$, where $L_{i-2}$ is the line joining $L_{i-1} \cap \Theta_{i-2}$ and $F_{i-3} \cap \Theta_{i-2}$. Now we will prove that $D^5$ is singular along $L_{i-2}$. To this aim, let $X^6$ be the blowing-up of $X^5$ along $L_{i-1}$ and $F_{i-3}$ with exceptional divisors $\Theta_{L_{i-1}} \simeq F_1$ and $\Theta_{i-3} \simeq F_1$. 
Now, denoting by $\mathcal{D}^6$ and $\Theta'_{i-2}$ the proper transform of $\mathcal{D}^5$ and $\Theta_{i-2}$ on $\mathcal{X}^6$, we have that,

- $\mathcal{D}^6|_{\Theta_L}$ is the double line $F_{L_{i-1}}$ joining $F_{L_i} \cap \Theta_{L_{i-1}}$ and $L_{i-2} \cap \Theta_{L_{i-1}}$;

- $\mathcal{D}^6|_{\Theta_{i-3}}$ is a conic intersecting with multiplicity 2 the fibre $\Theta'_{i-2} \cap \Theta_{i-3}$ at the point $L_{i-2} \cap \Theta_{i-3}$;

- $(L_{i-2})^2_{\Theta_{i-2}} = -1$.

In particular we find that

$$\mathcal{D}^6 \sim nH - 2\Gamma'' - 3\Psi - 2\Theta'_{i-1} - 2\Theta_{i-2} - 4\Theta_{L_{i-1}} - 2\Theta_{i-3}$$

and $L_{i-2} \mathcal{D}^6 = L_{i-2} (-2\Theta_{i-2} - 4\Theta_{L_{i-1}} - 2\Theta_{i-3}) = -2 - 4 + 4 = -2$. Now it is enough to blowing-up along $L_{i-2}$ to see that $\mathcal{D}^6$ is singular along $L_{i-2}$. If $i - 3 = 1$ this implies that $D_A$ is singular at $p$. If $i - 3 > 1$, then $\mathcal{D}^6|_{\Theta_{i-3}} = 2L_{i-3}$ where $L_{i-3}$ is the line joining $F_{i-4} \cap \Theta_{i-3}$ and $F_{i-2} \cap \Theta_{i-3}$. Moreover, by the same argument as used to prove that $\mathcal{D}_5$ is singular along $L_{i-2}$, you can verify that $\mathcal{D}^6$ is singular along $L_{i-3}$. By repeating this argument at the end you prove that $\mathcal{D}$ is singular along $F_1$ and so $D_A$ has a double point at $p$. At the same way, you can prove that also $D_B$ has a double at $p$. Hence, if $2 < i < m$ both curves $D_A$ and $D_B$ are singular at $p$ and $[D]$ cannot be general in any irreducible component of $\mathcal{V}_0$. 

**Corollary 3.8.** Let $\mathcal{V}^S_{nH,k,\delta}$ be the Universal Severi-Enriques Variety introduced in the first section. Let $\mathcal{V}_0$ be the special fibre of $\mathcal{V}^S_{nH,k,\delta}$ and let $[D] \in \mathcal{V}_0$ be any point, corresponding to a curve $D = D_A \cup D_B \subset A \cup B$. Assume that $D_A$ and $D_B$ are smooth and simply tangent to $R = A \cap B$ at a point $p$, with $p \neq E_i \cap R$, for every $i$. Assume that $[D]$ is a point of geometric multiplicity $m$ of $\mathcal{V}_0$. Let $U \subset A^1$ be an analytic neighborhood small enough of $0 \in A^1$ and let $\Delta$ be a general local $m$-multisection of $\mathcal{V}^S_{nH,k,\delta}$ passing through $[D]$. Denote by $\mathcal{D} \to \Delta$ the family of curves naturally parametrized by $\Delta$ and by $\mathcal{D}_i$ the general fibre of $\mathcal{D}$, with irreducible components $\mathcal{D}_i^1, \ldots, \mathcal{D}_i^m$. Then, the point $p$ is not a limit of any cusp of $\mathcal{D}_i^1$, for every $i = 1, \ldots, m$.

**Proof.** This follows by the proof of Lemma 2.3 if $m = 2$ and Cases 1, 2 and 3 of the previous Lemma if $m \geq 3$. □

Now, we denote by $T_{E_i}(d, n) \subset |O_B(n)|$ the closure, in the Zariski topology, of the locally closed set

$$\{ [D_B] | D_B = E_i \cup D'_B, \text{ where } D'_B \sim nH - E_i \text{ is smooth and passes through } E_i \cap R \}$$

where $E_1, \ldots, E_{d(d-1)}$ are the exceptional divisors of $B$.

**Lemma 3.9.** The variety

$$T_A(d, n) \times_{|O_B(n)|} T_{E_i}(d, n)$$

is an irreducible component of the special fibre $\mathcal{V}^0$ of $\mathcal{V}_{nH,1,\delta}$ of geometric multiplicity 3 and it is the only irreducible component of $\mathcal{V}_0$ whose general element corresponds to a curve containing $E_i$, for every $i = 1, \ldots, d(d-1)$.

**Proof.** Since the proof is the same for every $i$, we assume $i = 1$.

Let $V$ be an irreducible component of $\mathcal{V}_0$, whose general element $[D = D_A \cup D_B]$ corresponds to a curve $D_A \cup D_B$ containing $E_1$. First of all, we want to prove that

$$V \text{ has geometric multiplicity at least equal to 3.}$$

**Case 1.** Assume that the geometric multiplicity of $V$ is 1. Then, if $\Delta$ is a general local analytic curve passing through $[D]$ and $\mathcal{D} \to \Delta$ is the family of curves naturally parametrized by $\Delta$, we have that the special fibre of $\mathcal{D}$ is $\mathcal{D}_0 = D_A \cup D_B$ and the
singular locus $\gamma$ of $D$ intersects $E_1$ at a smooth point $q_1 \in E_1$ of $\mathcal{S}_0$. Now, let $\mathcal{X}$ be the blowing-up of $S$ along $E_1$ with exceptional divisor $\Theta_1 \simeq \mathbb{P}_0$. The pull-back $\gamma'$ of $\gamma$ to $\mathcal{Y}$ now intersects $\Theta_1$ at a general point $q'_1$. Moreover, if $\mathcal{D}'$ is the proper transform of $D$ in $\mathcal{Y}$, then $\mathcal{D}' \sim nH - \alpha_1 \Theta_1$, where $\alpha_1 = \text{mult}_E D$ and $\mathcal{D}'$ has cuspidal singularities along $\gamma'$. Since $\mathcal{D}'|_{\Theta_1} \sim (nH - \alpha_1 \Theta_1)|_{\Theta_1} \sim \alpha_1 F_1 + \alpha_1 F_2$, where $|F_1|$ and $|F_2|$ are the two rulings of $\Theta_1$, we have that the minimal $\alpha_1$ such that $\mathcal{D}'|_{\Theta_1}$ is a cusp at $q'_1$ is $\alpha_1 = 2$. This implies that $D_A$ has a double point at $E_1 \cap A$ and $D_B = 2E_1 + D_B'$, where $D_B' \sim nH - 2E_1$. So $D$ is cut out on $A \cup B$ by a surface $S_n \subset \mathbb{P}^3$ singular at the point $p_1$ corresponding to the exceptional divisor $E_1$. Thus $|D|$ cannot be general in any irreducible component of $\mathcal{V}_0$.

Case 2 Assume that $V$ has multiplicity two. Let $\mathcal{X}$ be the normalization of the double covering of $\mathcal{S}$ totally ramified at its special fibre $X_0 = A \cup E \cup B$. The proper transform of $D$ on $X_0$, which we still denote by $D$ is the connected Cartier divisor which restricts to $D_A$ on $A$, to $D_B$ on $B$ and to a union of fibres on $E$. Now, in $\mathcal{X}$, we can find divisors $\mathcal{D} \sim nH$, such that $\mathcal{D}|_{X_0} = D_0 = D$, the general fibre $D_1$ is a general one-cuspidal curve on $X_1$, and the singular locus of $\mathcal{D}$ is a section $\gamma$ of $\mathcal{X}$ intersecting $X_0$ at a smooth point $q_1$ lying on $E_1$ or on the fibre $F_1$ of $E$ intersecting $E_1$.

Case 2.1 Assume that $q_1 \in E_1$. Let $\mathcal{Y}$ be the blowing-up of $\mathcal{X}$ along $E_1$, with new exceptional divisor $\Theta_1 \simeq \mathbb{P}_0$ and special fibre $A \cup E' \cup \Theta_1 \cup B$. Now, the pull-back of $F_1$ to $\mathcal{Y}$, which we still denote by $F_1$, is a $(-1)$-curve intersecting transversally $\Theta_1$, whereas the pull-back of $\gamma$ is a curve $\gamma'$ intersecting $\Theta_1$ at a general point. Moreover, $F_1 \mathcal{D}' = \alpha \Theta_1 F_1 = -\alpha$, where $\alpha = \text{mult}_E D$ and $\mathcal{D}'$ is the proper transform of $D$ on $\mathcal{Y}$. So $F_1 \subset \mathcal{D}'$ and, denoting by $|H_1|$ and $|H_2|$ are the two rulings of $\Theta_1$, we have that $\mathcal{D}'|_{\Theta_1} \sim \alpha (H_1 + H_2)$ is an effective divisor with a cusp at $\gamma' \cap \Theta_1$ and intersecting with multiplicity two $\Theta_1 \cap E'$ at the point $F_1 \cap \Theta_1$. The minimal $\alpha$ such that these two conditions are verified is $\alpha = 2$ and so $\mathcal{D}'|_B$ intersects $E_1$ at two points and, by contracting $\Theta_1$ on $E_1$, we find that $D_B = 2E_1 + D_B'$, where $D_B' \sim nH - 2E_1$. Moreover, by blowing-up twice $\mathcal{Y}$ along $F_1$, we see that $\mathcal{D}'|_A = D_A$ is smooth and tangent to $E' \cap A$ at $F_1 \cap A$. So $D_A \cup D_B$ is cut out by a surface in $\mathbb{P}^3$ tangent to $B$ at the base point $p_1$ corresponding to $E_1$. It follows that $|D|$ is general in a family of codimension at least 3 in $|O_{A \cup B}(n)|$ and it cannot be general in any irreducible component of $\mathcal{V}_0$.

Case 2.2 Assume that $q_1 \in F_1$. Then, let $\mathcal{Y}$ be the blowing-up of $\mathcal{X}$ along $\gamma$ with exceptional divisor $\Gamma$ and special fibre $A \cup E' \cup B$. Now, $\mathcal{D}' \cap F_1 = -2F_1 = -2$. In particular, if $\mathcal{D}'$ is the proper transform of $D$, then $F_1$ is contained in $\mathcal{D}'|_E'$ with multiplicity $m \geq 2$. If $m \geq 3$ then $D_A$ intersects $E'$ with multiplicity $m \geq 3$ at $F_1 \cap A$ and so $|D|$ cannot be general in any irreducible component of $\mathcal{V}_0$. Hence $m = 2$ and, if we blow-up $\mathcal{Y}$ along $F_1$ and we denote by $\Theta_1$ the new exceptional divisor and by $\mathcal{D}''$ the proper transform of $\mathcal{D}'$, we find that $\mathcal{D}''|_{\Theta_1}$ is a conic, tangent to the fibre $\Gamma' \cap \Theta_1$, where $\Gamma'$ is the pull-back of $\Gamma$. Moreover, since $\mathcal{D}''|_{E_1} = -2$, we have that $E_1 \subset \mathcal{D}''$ and the conic $\mathcal{D}''|_{\Theta_1}$ passes through $E_1 \cap \Theta_1$. In particular, re-contracting $\Theta_1$, we find that $D_A$ has a double point at $F_1 \cap A$ whereas $D_B = E_1 \cup D_B'$, where $D_B' \sim nH - E_1$. So $D_A \cup D_B$ is cut out on $A \cup B$ by a surface $S_n \subset \mathbb{P}^3$ tangent to $A$ at the point $p_1$ corresponding to the exceptional divisor $E_1$, and it cannot be general in any irreducible component of $\mathcal{V}_0$.

Claim (3) has been proved. Now we observe that, if $V$ is an irreducible component of $\mathcal{V}_0$, whose general element $[D]$ corresponds to a curve $D = D_A \cup D_B$ containing $E_1$, then the singularity of $D$ at the point $E_1 \cap R$ must impose at most two conditions to the linear system $|O_{A \cup B}(n)|$. So we have one of the following two cases:

1. $D$ has a node at $E_1 \cap R$, in particular $D_A$ meets transversally $R$ at $E_1 \cap R$ and $D_B = E_1 \cup D_B'$, where $D_B'$ does not contain $E_1 \cap R$;
(2) $D_A$ is tangent to $R$ at $E_1 \cap R$ and $D_B = D_B' \cup E_1$, where $D_B' \sim nH - E_1$ passes through $E_1 \cap R$. In particular, $V = T_A(d,n) \times_{\mathcal{O}_{\mathbb{P}^n}(n)} T_{E_1}(d,n)$.

We want to prove that the case (4) cannot occur. To this aim let $m$ be the geometric multiplicity of $V$ and let $Y$ be the finite covering of degree $m$ of $S$ which we already introduced in Lemma 2.6. Let $\mathcal{Y}_0 = A \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{m-1} \cup B$ be its special fibre. We denote by $F_1 \cup \cdots \cup F_m$ the connected chain of fibres such that $F_i \subset \mathcal{E}_i$ and $F_{m-1} \cap B = E_1 \cap R$. Let $\mathcal{D} \sim nH$ be a general divisor in $Y$ cutting out $D_A$ on $A$ and $D_B$ on $B$ with cuspidal singularity along a section $\gamma$ of $Y$. We already know that $\gamma$ must intersect $\mathcal{Y}_0$ at a smooth point $q$ lying on $E_1$ or on $F_i$, for some $i$. If $q \in E_1$ then, arguing as in Case 2.1, we see that $D_B$ contains $E_1$ with multiplicity 2. If $q \in F_i$, then $D|_{\mathcal{Y}_0}$ contains every $F_i$ with multiplicity $r \geq 2$ and so $D_A \cap R = D_B \cap R$ contains $E_1 \cap R$ with multiplicity $r \geq 2$. This proves that case (4) cannot occur and $V = T_A(d,n) \times_{\mathcal{O}_{\mathbb{P}^n}(n)} T_{E_1}(d,n)$.

Now, we will show that, actually,

(4) $T_A(d,n) \times_{\mathcal{O}_{\mathbb{P}^n}(n)} T_{E_1}(d,n)$ is an irreducible component of $V_0$ of multiplicity 3.

Assume $m = 3$ and let $Y$, $D$ and $F_i \subset \mathcal{E}_i$ be as before. We denote by $\gamma$ a section of $Y$ intersecting $\mathcal{Y}_0 = A \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup B$ at a smooth point $q \in F_2$.

Step 1. Let $S \subset Y$ be a general divisor in $|\mathcal{O}_Y(nH)|$ with cuspidal singularities along $\gamma$ and such that $S$ contains the fibres $F_1$ and $F_2$ with multiplicity exactly 2. Let $\mathcal{Y}_1$ be the blowing up of $Y$ along $\gamma$ with new exceptional divisor $\Gamma$. Denote by $S^1$ the proper transform of $S$ in $\mathcal{Y}_1$ and by $\mathcal{Y}_0^1 = A \cup \mathcal{E}_1 \cup S^1 \cup B$ the special fibre of $\mathcal{Y}_1$, where $S^1$ is the blowing up of $\mathcal{E}_2$ at $q$. Then $\mathcal{Y}_1 \sim nH - 2\Gamma$, $S^1 \mathcal{F}_2 = -2$ and, by the hypothesis that $S$ contains $F_2$ with multiplicity exactly 2, $S^1$ will be tangent to $\Gamma$ along a smooth section $\psi$ of $\mathcal{Y}_1$, intersecting $\mathcal{Y}_0^1$ at the point $F_2 \cap \Gamma$, which we still denote by $q$.

Step 2. Let $\mathcal{Y}_2$ be the blowing-up of $\mathcal{Y}_1$ along $F_2$, with new exceptional divisor $\Theta_2$. Denote by $\mathcal{Y}_2^1 = A' \cup \mathcal{E}'_1 \cup \mathcal{E}'_2 \cup \Theta_2 \cup B$ the special fibre of $\mathcal{Y}_2^1$, where $\mathcal{E}'_2$ is the blowing up of $\mathcal{E}'_2$ at $q$. By using that $F_2^2 \cap \mathcal{E}'_2 = -1$, we have that $\Theta_2 \simeq \mathbb{F}_1$ and $F_2^2 \Theta_2 = -1$. Moreover, denoting by $S^2$ the proper transform of $S^1$ in $\mathcal{Y}_2^1$, we have that

$$S^2|_{\Theta_2} \sim -2F_{\Theta_2} + \alpha(2F_{\Theta_2} + F_2),$$

where $F_{\Theta_2}$ is the linearly equivalence class of the fibre of $\Theta_2$. Now, denoting by $\psi'$ the proper transform in $\mathcal{Y}_2^1$ of $\psi$ and by $q'$ the intersection point of $\psi'$ with $\Theta_2$, we have that $S^2|_{\Theta_2}$ is an effective divisor

(1) tangent to the fibre $F_q \subset |F_{\Theta_2}|$, passing through $q$, at the point $q'$;

(2) intersecting the fibre $\mathcal{E}'_1 \cap \Theta_2$ with multiplicity 2 at the point $F_1 \cap \Theta_2$.

So, $\alpha = 2$, $S^2|_{\Theta_2}$ is a conic verifying (1) and (2) and $S^2 \sim nH - 2\Gamma' - 2\Theta_2$, where $\Gamma'$ is the proper transform of $\Gamma$ in $\mathcal{Y}_2^1$. Moreover, the pull-back of $E_1$ to $\mathcal{Y}_2^1$, which we still denote by $E_1$, is a $(-2)$-curve transversally intersecting $\Theta_2 \cap B'$ at a point $q_1$. So, $E_1 S^2 = -2\Theta_2 E_1 = -2$, $E_1 \subset S^2$ and the conic $S^2|_{\Theta_2}$ passes through $q_1$. Now, if $\mathcal{Y}_3$ is the blowing-up of $\mathcal{Y}_2^1$ along $E_1$, with new exceptional divisor $\Theta_{E_1}$, then $(E_1)_2^2 \Theta_{E_1} = 1$ and $\Theta_{E_1} \simeq \mathbb{F}_1$. Moreover, the proper transform $\Theta_3$ of $\Theta_2$ in $\mathcal{Y}_3$ is the blowing-up of $\Theta_2$ at $q_1$ with new exceptional divisor $\Theta_{E_1} \cap \Theta_2$. Finally, denoting by $S^3$ the proper transform of $S^2$ in $\mathcal{Y}_3$, we have that

$$S^3|_{\Theta_{E_1}} \sim -2F_{\Theta_{E_1}} + E_1 + F_{\Theta_{E_1}} \sim E_1 - F_{\Theta_{E_1}},$$

where $F_{\Theta_{E_1}}$ is the linearly equivalence class of a fibre of $\Theta_{E_1}$ and, as we already observed, $E_1$ is a line. Hence,

$$S^3|_{\Theta_{E_1}}$$

is the exceptional divisor of $\Theta_{E_1}$.

This implies that
(3) $S^2|_{\Theta_2}$ is a conic verifying properties $\mathbf{(1)}$ and $\mathbf{(2)}$ and tangent at $q_1$ to a fixed line $r_1 \subset \Theta_2$ which does not depend on the sections $\gamma$ or $\psi$ and on the divisor $S$.

Now, the family of conics on $\Theta_2$ tangent to $E'_1 \cap \Theta_2$ at $F_1 \cap \Theta_2$ and to $r_1$ at $q_1$ is a pencil $G$ cutting out on the fibre $F'_q$ a $g^2_2$ with ramification points $x_q^1$ and $x_q^2$. It follows that

$$q' = x_q^i$$

where $C_q^i$ is the only conic of the pencil $G$ tangent to $F'_q$ at $x_q^i$. Finally, if $Y^4$ is the blowing-up of $Y^3$ once along $\psi$ and twice along $F_1$, with new exceptional divisors $F, \Theta_1 \simeq F$ and $T_1 \simeq F_0$, then it is easy to see that

$$S^4 \in |\mathcal{O}_{Y^4}(nH - 2\Gamma' - 3\Psi - 2\Theta'_2 - \Theta_{E_1} - \Theta_1 - 2T_1)|,$$

where $S^4$ and $\Theta'_2$ are the proper transforms of $S^3$ and $\Theta'_2$ to $Y^4$. In particular, $[D_A \cup D_B] \in T_A(d, n) \times |\mathcal{O}_{E_1}(n)| T_{E_1}(d, n)$ and, more precisely, $S$ cuts on $A$ a curve $D_A$ smooth and tangent to $R$ at $E_1 \cap R$ and on $B$ a curve $D_B = D_B' \cup E_1$, with $D_B' \sim nH - E_1$ passing through $E_1 \cap R$ and having fixed tangent direction at $E_1 \cap R$.

Now, by using the notation above, what we proved implies that, if $\gamma$ is a section of $Y$ intersecting $F_2$ at a general point $q$ and $\psi$ is a general section of $\Gamma \subset Y^3$ passing through $q = F_2 \cap \Gamma$, by denoting by $Z^1$ the blowing up of $Y^3$ along $\psi$, we have that, for every divisor $S$ in the linear series $|\mathcal{O}_{Z^1}(nH - 2\Gamma' - 3\Psi)|$, the restriction of $S$ to the special fibre of $Z^1$ contains the fibres $F_1$ and $F_2$ with multiplicity at least 2. Moreover, always by using the notation above, if $\psi \subset \Gamma \subset Y^3$ is a section such that the proper transform $\psi'$ of $\psi$ in $Y^2$ intersects $\Theta_2$ at $x_q^1$ or $x_q^2$, if $Y^2_2$ is the special fibre of $Y^4$ and $D$ is the linear equivalence class of the divisor $2\Gamma' + 3\Psi + 2\Theta'_2 + \Theta_{E_1} + \Theta_1 + 2T_1 \subset Y^4$, then the image $W_\gamma$ of the restriction map

$$r_0 : H^0(Y^4, \mathcal{O}_{Y^4}(nH - D)) \rightarrow H^0(Y^4_0, \mathcal{O}_{Y^4_0}(nH - D))$$

is a linear system of dimension $\text{dim}(|\mathcal{O}_{A \cup B}(nH)|) - 5$. Now, since by $[\mathbf{1}]$, all divisors in $|\mathcal{O}_{Y^4}(nH - D)|$ restrict to the same divisor on $\Theta'_2$ and $T_1$, the image $U_\gamma \subset T_A(d, n) \times |\mathcal{O}_{E_1}(n)| T_{E_1}(d, n) \cap \mathcal{V}_0 \subset |\mathcal{O}_{A \cup B}(n)|$ of $W_\gamma$, through the natural morphism $|\mathcal{O}_{Y^4_0}(nH - D)| \rightarrow |\mathcal{O}_{A \cup B}(n)|$, has still dimension $\text{dim}(|\mathcal{O}_{A \cup B}(nH)|) - 5$. Moreover, we stress that, denoting by $\tilde{C}_q^i$ and $A''$ the proper transforms of $C_q^i$ and $A$ to $Y^4$, by using $[\mathbf{2}]$ and by using that $S^4|_{T^1}$ is the fibre $F_{q,i}$ in the ruling $F_1$, we have that the general divisor in the linear system $W_\gamma$ cuts $T_1 \cap A''$ at the point $y_{q,i} = F_{q,i} \cap A''$ and $\Theta_q' \cap B'$ at the point $x_{q,i} = \tilde{C}_q^i \cap B'$, different from $E_1 \cap \Theta'_2$. Now, in order to prove that $T_A(d, n) \times |\mathcal{O}_{E_1}(n)| T_{E_1}(d, n)$ is an irreducible component of $\mathcal{V}_0$, it is enough to prove that $U_\gamma$ is not contained in any irreducible component $V$ of $\mathcal{V}_0$ which we have found previously in this Section. We will prove this only for $V = V_{nH,1,0}$ and $d \geq 2$. In the other cases, you can use a similar argument. To prove that $U_\gamma$ is not contained in $V$ it is enough to prove that the general element of $W_\gamma$ does not corresponds to a curve with a cusp at a point of $A''$. To see this, observe that, by using the notation of the introduction, by the generality of $\pi \subset \mathbb{P}^3$, the point $p_1 \subset S^{d-1}$ corresponding to $E_1 \subset B$ is a general point of $S^{d-1}$ and the tangent line to $R$ at $p_1$ is a general tangent line to $S^{d-1}$ at $p_1$. This implies that the family of curves $V$ with a cusp on $A''$ in the linear system $|\mathcal{O}_{Y^4_0}(nH - D)|$ is irreducible of dimension $\text{dim}(|\mathcal{O}_{A \cup B}(n)|) - 2 - 3$, but it is not a linear system, so it cannot coincide with $W_\gamma$. \[\square\]

Previous Lemmas of this sections imply the following theorem.

**Theorem 3.10.** Let $V_0$ be the special fibre of $V_{nH,1,0}$. Assume that $n \geq 3$ and $d \geq 2$. Then, the irreducible components of $V_0$ are
$V_{nH,1,0}^A \times |O_{R(n)}| |O_B(n)|$ with geometric multiplicity 1;

$|O_A(n)| \times |O_{R(n)}| V_{nH,1,0}^B$ with geometric multiplicity 1;

$F(d,n)$ with geometric multiplicity 2;

$S_A(d,n) \times |O_{R(n)}| T_B(d,n), T_A(d,n) \times |O_{R(n)}| S_B(d,n), T_A(d,n) \times |O_{R(n)}| T_{E_i}(d,n)$, for $i = 1, \ldots, d(d-1)$, with geometric multiplicity 3.

If $n = 2$ and $d \geq 3$ the description of $V_0$ is as in the previous case, except for the fact that, in this case, $|O_A(n)| \times |O_{R(n)}| V_{nH,1,0}^B$ does not appear. Finally, if $d = n = 2$ then the irreducible components of $V_0$ are $S_A(d,n) \times |O_{R(n)}| T_B(d,n), T_A(d,n) \times |O_{R(n)}| S_B(d,n), T_A(d,n) \times |O_{R(n)}| T_{E_1}(d,n), T_A(d,n) \times |O_{R(n)}| T_{E_2}(d,n)$, all with geometric multiplicity 3.

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