Multipartite entanglement control via Quantum Zeno Effect

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Abstract

We develop a protocol based on $2M$ pairwise interacting qubits, which through Quantum Zeno Effect controls the entanglement distribution of the system. We also show that if the coupling constants are different the QZE may be used to achieve perfect entanglement swap.

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Introduction

Entangled states as Quantum Mechanics was constructed were in the heart of feverous debates about apparent paradoxes such as EPR (Einstein, Podolsky e Rosen)\cite{1}. Nowadays the status of entangled states changed radically from Gedanken experiments to actual tools for various technological improvements, such as in the area of quantum computing\cite{2}. From the point of view of realizing quantum computation one of the most difficult and important tasks is the entanglement control of a large series of systems.

Recently a system composed by four qubits interacting pairwise has been exhaustively studied. Many important features of entanglement dynamics were clarified by this analytic model, namely: sudden death of entanglement\cite{3,4}, relation between energy and entanglement\cite{5}, pairwise concurrence dynamics\cite{6}, entanglement invariant for this model\cite{7} and entanglement protection\cite{8}.

In the present contribution we extend that study to a system composed of $M$ pairs of qubits interacting pairwise, whose entanglement distribution control is performed via the QZE (Quantum Zeno Effect). We basically present two applications for the scheme:

a) We show that systems of four qubits whose autonomous dynamics have different coupling constants may be slowed down by QZE in such a way that the swap is completed as if the coupling constants were the same.

b) We show that generic entangled states in the $2M$ qubits system can be transferred from any partition to any other directly, independently of the distance between the partitions. This is achieve, again through QZE.

The system

Let us consider a multi-partite system composed by $2M$ qubits interacting pairwise, with the hamiltonian given by

$$H = \sum_{k=1}^{M} H_{akAK},$$

where

$$H_{akAK} = \hbar \omega_{ak} \sigma_z^{ak} + \hbar \omega_{Ak} \sigma_z^{Ak} + g_k \left( \sigma_z^{ak} \sigma_z^{Ak} + \sigma_z^{ak} \sigma_z^{Ak} \right),$$

$g_k$ is the coupling constant of the $k$-th pair. To avoid unnecessary complications let us consider $\omega_{ak} = \omega_{Ak} = \omega$. Notice that all the terms in equation (1) commute with each other. Therefore, the global system evolution can be separated in evolutions of each pair and written as:
\[ |\psi(T)\rangle = e^{-iT/\hbar} |\psi(0)\rangle = (e^{-iH_{a_1A_1}T/\hbar} \otimes e^{-iH_{a_2A_2}T/\hbar} \otimes \ldots \otimes e^{-iH_{a_M A_M}T/\hbar}) |\psi(0)\rangle. \] (3)

If the initial state is an entangled state, the unitary evolution will distribute the entanglement through the system. To give an example of this entanglement dynamics suppose that

\[ |\psi(0)\rangle = \frac{1}{\sqrt{M}} (|a_10\rangle + |a_20\rangle + \ldots + |a_M0\rangle) |A_10\rangle + \ldots + |a_10\rangle |A_20\rangle + \ldots + |a_M0\rangle |A_M0\rangle \]

and

\[ g_j = g \text{ (where } j = 1, 2, \ldots M) \]

Initially there is a maximally entangled state on the part that contains the qubits \( \{a_k\} \), and a factorized state on partition \( \{A_k\} \). The evolution of \( |\psi(0)\rangle \) governed by the Hamiltonian (1) at time \( T = \frac{\pi}{2g} \) results on an entanglement swap between parts \( a \) and \( A \). Notice that the entanglement distribution induced by the free evolution is dynamical and one has not much control over it.

In this contribution, we present a protocol that allows for a more incisive control of the entanglement distribution on the system. By performing Zeno-like measurements on one component of any pair, it is possible to inhibit the dynamics of this pair. Therefore, a controlled evolution can be build up by \( N \) steps of free interactions followed by appropriate projections. For example, to control the \( j \)-th pair of \( |\psi(0)\rangle \) we must perform \( N \) projective measurements on one of the qubits \( a_j \) or \( A_j \). This controlled evolution can be written as:

\[ (P_j e^{-iT/\hbar})^N |\psi(0)\rangle = \left[ e^{-iH_{a_1A_1}T/\hbar} \otimes \ldots \otimes (P_j e^{-iH_{a_jA_j}T/\hbar})^N \otimes \ldots \otimes e^{-iH_{a_M A_M}T/\hbar} \right] |\psi(0)\rangle, \] (4)

where \( P_j \) projects one of the qubits \( a_j \) or \( A_j \) in its initial state, therefore by QZE the dynamics of the chosen pair is inhibited when \( N \to \infty \). Notice that \( P_j \) acts only on the subsystem of the \( j \)-th pair, so the evolution of all the other pairs of qubits is free. This monitored evolution allow us to control the entanglement distribution on the system, selecting the pairs of qubits that will be free to evolve and the ones that will have their evolution frozen.

**Applications:**

a) **Double Jaynes-Cummings Model**

In this section we present an explicit calculation for the time evolution of two pairs of qubits when one of them is subjected to \( N \) projective measurements. The model for two qubits interacting pairwise is refereed to as Double Jaynes-Cummings model and the Hamiltonian that governs the free evolution of this system is (1) with \( M = 2 \).

Let us consider the initial state...
\[ |\psi(0)\rangle = (\alpha_0|1_{a_1}0_{a_2}\rangle + \beta_0|0_{a_1}1_{a_2}\rangle) |0_{A_1}, 0_{A_2}\rangle, \] (5)

The free evolution of (5) and its entanglement dynamics were studied in Ref. [5, 7]. An interesting aspect of this free evolution is the probability of entanglement swap, i.e., when \( g_1 = g_2 \) the entangled state, initially prepared in one partition is completely transferred to the other partition of the system when the evolution time is \( t = \frac{j\pi}{2g} \) (\( j \) is an odd number). The requirement for the coupling constants to be equal \( (g_1 = g_2) \) may bring some difficulties for empirical implementations of this entanglement swap. We show that if the dynamics is controlled by QZE, the entanglement swap can be obtained even with different coupling constants. To control the dynamics let us introduce the projector \( P_2 = I_{a_1} \otimes I_{a_2} \otimes I_{A_1} \otimes |0_{A_2}\rangle\langle 0_{A_2}| \), which acts on the subsystem \( A_2 \) projecting in its initial state.

The vector state submitted to the controlled evolution (evolution divided by \( N \) projective measurements on \( A_2 \)) is given by

\[ |\psi(t)\rangle^N = \alpha_0 e^{-iH_{a_1A_1}N\tau/\hbar} |1_{a_1}0_{A_1}\rangle \otimes |0_{a_20_{A_2}}\rangle + \beta_0 |0_{a_1}0_{A_1}\rangle \otimes \left( P_2 e^{-iH_{a_2A_2}\tau/\hbar} \right)^N |1_{a_20_{A_2}}\rangle, \] (6)

where \( N\tau = t \) and

\[
\left( P_2 e^{-iH_{a_2A_2}\tau/\hbar} \right)^N = \left( \cos^N(g_2t) |1_{a_2}\rangle \langle 1_{a_2}| + |0_{a_2}\rangle \langle 0_{a_2}| \right) \otimes |0_{A_2}\rangle \langle 0_{A_2}| \]
\[-i \tan(g_2t) \cos^N(g_2t) |1_{a_2}\rangle \langle 0_{a_2}| \otimes |0_{A_2}\rangle \langle 1_{A_2}|. \] (7)

The vector state after the controlled evolution can be written as:

\[ |\psi(t)\rangle^N = \frac{1}{\sqrt{\alpha_0^2 [1 - \cos^2N(g_2\tau)] + \cos^2N(g_2\tau)}} (|\mu(t)\rangle |0_{A_1}, 0_{A_2}\rangle - i|\nu(t)\rangle |1_{A_1}, 0_{A_2}\rangle), \] (8)

where

\[ |\mu(t)\rangle = \alpha_0 \cos(g_1t) |1_{a_1}, 0_{a_2}\rangle + \beta_0 \cos^N(g_2\tau) |0_{a_1}, 1_{a_2}\rangle, \] (9)

\[ |\nu(t)\rangle = \alpha_0 \sin(g_1t) |0_{a_1}, 0_{a_2}\rangle. \] (10)

Taking the limit \( N \to \infty \) in eq. (11):
\[
\lim_{N \to \infty} |\psi(t)|^N = [\alpha_0 \cos(g_1 t)|1_{a_1}, 0_{a_2}\rangle + \beta_0|0_{a_1}, 1_{a_2}\rangle]|0_{A_1}, 0_{A_2}\rangle - i\alpha_0 \sin(g_1 t)|0_{a_1}, 0_{a_2}\rangle|1_{A_1}, 0_{A_2}\rangle,
\]

(11)

To obtain the entanglement swap in this system, that has different coupling constants for the pairs, we must inhibit, through QZE, the excitation transfer of one pair for a certain period of time. If \(g_1 < g_2\) (\(g_2 < g_1\)) the excitation transfer in the pair \((a_2, A_2)\) (\((a_1, A_1)\)) is faster than the transfer in \((a_1, A_1)\) (\((a_2, A_2)\)). The transfer in the faster pair, \((a_2, A_2)\) (\((a_1, A_1)\)), must be inhibited for a period of time given by \(T = \frac{\pi}{2} \left( \frac{1}{g_2} - \frac{1}{g_1} \right)\) \(T = \frac{\pi}{2} \left( \frac{1}{g_1} - \frac{1}{g_2} \right)\). Therefore, the evolution that allows for the entanglement swap is composed by two parts. In the first part, which happens for the period of time \(T\), the evolution of one pair (the fastest pair) is inhibited by QZE, while the other pair evolves freely. In the second part of the evolution both pairs evolve freely. The total time of evolution must correspond to a \(\pi\) pulse for the slowest pair. The fastest pair freezing in the first part of the total evolution allows the complete excitation transfer in both pairs to take place at the same exact time, this coincidence is an essential factor for the entanglement swap.

Another interesting consequence of the partial control is that the concurrence \([9]\) for the qubits \(a_1\) and \(a_2\) after \(N\) projective measurements \([8]\), given by

\[
C_{a_1, a_2}^N(t) = \frac{2|\alpha_0 \beta_0 \cos(g t) \cos^N(g \tau)|}{|\alpha_0|^2 + |\beta_0|^2 \cos^{2N}(g \tau)};
\]

(12)

becomes, in the limit \(N \to \infty\), identical to the concurrence calculated in Ref.\([10]\), where the entanglement dynamics between an isolated atom and a Jaynes-Cummings atom is studied

\[
\lim_{N \to \infty} C_{a_1, a_2}^N(t) = 2|\alpha_0 \beta_0 \cos(g t)|.
\]

(13)

Therefore using QZE, one can extract the entanglement dynamics of the system studied in Ref.\([10]\) from a double Jaynes-Cummings system. We consider the initial state \((\alpha_0|1_{a_1}, 0_{a_2}\rangle + \beta_0|0_{a_1}, 1_{a_2}\rangle)|0_{A_1}, 0_{A_2}\rangle\) in the calculation, but the same results can be shown for \((\alpha_0|1_{a_1}, 1_{a_2}\rangle + \beta_0|0_{a_1}, 0_{a_2}\rangle)|0_{A_1}, 0_{A_2}\rangle\) as an initial state.

b) Transferring entangled states

In this section we show how to transfer the entanglement from one partition of the \(2M\) qubits system interacting pairwise, to any other partition of this system, using QZE and unitary evolution. For simplicity let us consider in the calculation an eight qubits system
(the generalization for 2M qubits is straightforward). The eight qubits are coupled pairwise and the time evolution is governed by the Hamiltonian in equation (1) (with $M = 4$). The system is prepared in the initial state

$$|\psi(0)\rangle = (|\phi^+\rangle + |\psi\rangle) \otimes |0_{A_1}0_{A_2}0_{A_3}0_{A_4}\rangle,$$

where

$$|\phi^+_{a_1,a_2,a_3,a_4}\rangle = c_1|1_{a_1}1_{a_2}1_{a_3}1_{a_4}\rangle + c_6|0_{a_1}0_{a_2}0_{a_3}0_{a_4}\rangle,$$

$$|\psi_{a_1,a_2,a_3,a_4}\rangle = c_2|1_{a_1}0_{a_2}0_{a_3}0_{a_4}\rangle + c_3|0_{a_1}1_{a_2}0_{a_3}0_{a_4}\rangle +$$

$$c_4|0_{a_1}0_{a_2}1_{a_3}0_{a_4}\rangle + c_5|0_{a_1}0_{a_2}0_{a_3}1_{a_4}\rangle,$$

(15)

(16)

Notice that a general entangled state is prepared in the partition $\{a_k\}$. Now, suppose we want to transfer it to the partition composed by the qubits $a_3 - a_4 - A_1 - A_2$. The procedure is simple, let the qubits $a_1, a_2, A_1$ and $A_2$ undergo a $\pi$ pulse with time evolution governed by the Hamiltonian in equation (11) (with $M = 4$), inhibiting the evolution of qubits $a_3 - a_4 - A_3 - A_4$ by QZE. This dynamics gives us the state

$$|\psi\left(\frac{\pi}{2g}\right)\rangle = -c_1|0_{a_1}0_{a_2}1_{a_3}1_{a_4}\rangle|1_{A_1}A_20_{A_3}0_{A_4}\rangle + c_2|0_{a_1}0_{a_2}0_{a_3}0_{a_4}\rangle|1_{A_1}0_{A_2}0_{A_3}0_{A_4}\rangle +$$

$$c_3|0_{a_1}0_{a_2}0_{a_3}0_{a_4}\rangle|0_{A_1}1_{A_2}0_{A_3}0_{A_4}\rangle + c_4|0_{a_1}0_{a_2}1_{a_3}0_{a_4}\rangle|0_{A_1}0_{A_2}0_{A_3}0_{A_4}\rangle +$$

$$c_5|0_{a_1}0_{a_2}0_{a_3}1_{a_4}\rangle|0_{A_1}0_{A_2}0_{A_3}0_{A_4}\rangle + c_6|0_{a_1}0_{a_2}0_{a_3}0_{a_4}\rangle|0_{A_1}0_{A_2}0_{A_3}0_{A_4}\rangle,$$

which can be written as:

$$|\psi\left(\frac{\pi}{2g}\right)\rangle = |0_{a_1}0_{a_2}\rangle\left(|\phi^-_{a_3,a_4,A_1,A_2}\rangle + |\psi_{a_3,a_4,A_1,A_2}\rangle\right)|0_{A_3}0_{A_4}\rangle,$$

where

$$|\phi^-_{a_3,a_4,A_1,A_2}\rangle = -c_1|1_{a_3}1_{a_4}1_{A_1}1_{A_2}\rangle + c_6|0_{a_3}0_{a_4}0_{A_1}0_{A_2}\rangle,$$

$$|\psi_{a_3,a_4,A_1,A_2}\rangle = c_2|0_{a_3}0_{a_4}1_{A_1}0_{A_2}\rangle + c_3|0_{a_3}0_{a_4}0_{A_1}1_{A_2}\rangle +$$

$$c_4|1_{a_3}0_{a_4}0_{A_1}0_{A_2}\rangle + c_5|0_{a_3}1_{a_4}0_{A_1}0_{A_2}\rangle,$$

(18)

(19)

(20)
It is clear that we transfer an entangled state from the subsystem $a_1 - a_2 - a_3 - a_4$ to the subsystem $a_3 - a_4 - A_1 - A_2$.

In summary we have constructed a protocol for an extended system and shown how the use of QZE may help control the entanglement distribution and entanglement swap. A further study in this regard is actually finding a system where this protocol may be implemented and to discuss the role of deleterious environment effects which will affect the number of effective qubits.

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