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Boston University
Dynamical glass in weakly nonintegrable Klein-Gordon chains

Carlo Danieli,1 Thudiyangal Mithun,1 Yagmur Kati,1,2 David K. Campbell,3 and Sergej Flach1,4
1Center for Theoretical Physics of Complex Systems, Institute for Basic Science, Daejeon 34126, Republic of Korea
2Basic Science Program, Korea University of Science and Technology (UST), Daejeon 34143, Republic of Korea
3Department of Physics, Boston University, Boston, Massachusetts 02215, USA
4New Zealand Institute for Advanced Study, Massey University, Auckland 02215, New Zealand

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Integrable many-body systems are characterized by a complete set of preserved actions. Close to an integrable limit, a nonintegrable perturbation creates a coupling network in action space which can be short or long ranged. We analyze the dynamics of observables which become the conserved actions in the integrable limit. We compute distributions of their finite time averages and obtain the ergodization time scale \( T_e \) on which these distributions converge to \( \delta \) distributions. We relate \( T_e \) to the statistics of fluctuation times of the observables, which acquire fat-tailed distributions with standard deviations \( \sigma^+ \) dominating the means \( \mu^+ \) and establish that \( T_e \sim (\sigma^+)^2/\mu^+ \). The Lyapunov time \( T_L \) (the inverse of the largest Lyapunov exponent) is then compared to the above time scales. We use a simple Klein-Gordon chain to emulate long- and short-range coupling networks by tuning its energy density. For long-range coupling networks \( T_L \approx \sigma^+ \), which indicates that the Lyapunov time sets the ergodization time, with chaos quickly diffusing through the coupling network. For short-range coupling networks we observe a dynamical glass, where \( T_L \) grows dramatically by many orders of magnitude and greatly exceeds the Lyapunov time, which satisfies \( T_L \lesssim \mu^+ \). This effect arises from the formation of highly fragmented inhomogeneous distributions of chaotic groups of actions, separated by growing volumes of nonchaotic regions. These structures persist up to the ergodization time \( T_e \).

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I. INTRODUCTION

Ergodicity and mixing are central concepts in statistical mechanics. Both properties characterize the temporal evolution of a dynamical system: ergodicity demands that a solution visits almost all states of the available phase space; mixing requires that, as the system evolves, any choice of two open sets of available states will eventually overlap [1]. With mixing being a necessary condition for ergodicity, both constitute fundamental aspects of the phenomenon of thermalization. In particular, both properties imply that the available phase space does not fragment into inaccessible open sets, and that the infinite time average of any observable matches its phase space average. The latter feature typically acts as a definition of ergodicity. The search for the violation of ergodicity, mixing, and thermalization gave rise to some of the most important discoveries in mathematics and statistical physics. Several of these results were found for weakly nonintegrable systems: models of \( N \) degrees of freedom whose Hamiltonians \( H = H_0 + \varepsilon H_1 \) consist of an integrable part \( H_0 \), a nonintegrable perturbation \( H_1 \), and the perturbation strength \( \varepsilon \). In 1954 Kolmogorov proved the existence for small enough perturbation strength \( 0 < \varepsilon < \varepsilon_0 \) of sets with nonzero measure of infinite time stable solutions which are confined to \( N \)-dimensional manifolds of the phase space (later labeled KAM tori) [2]. His work was extended by Arnold [3] and Moser [4] to larger classes of Hamiltonian systems, leading to the celebrated KAM theorem. Almost concomitantly with the discovery made by Kolmogorov, a numerical test on a small chain of harmonic oscillators in the presence of weak anharmonic coupling failed to show the expected equipartition of energy along the chain (original report in Ref. [5], reviews in Refs. [6–8]). This computer experiment, performed by Fermi, Pasta, Ulam and Tsingou (FPUT), and the attempts to explain its striking outcomes, led to the discovery of solitons [9,10] and to remarkable advances in the theory of Hamiltonian chaos [11–13]. In particular, it was found that a large number of weakly nonintegrable lattices possess families of exact time periodic solutions whose actions turn into the ones of the integrable system \( H_0 \) for \( \varepsilon = 0 \). Depending on the limit being considered, these solutions are called either (1) discrete breathers [14–17] and show typically exponential localization of energy in real space or (2) \( q \)-breathers which show exponential localization of energy in normal mode space [18,19]. Although forming a set of zero measure, these coherent solutions can impact the dynamical properties of a many-degrees-of-freedom system, since a generic trajectory can spend long times in their neighborhoods in phase space [20–26]. These events (also labeled excursions out of equilibrium) have been experimentally studied in the context of semiconductor lasers [27], superfluids [28], microwave cavities [29], optical fibers [30], and arrays of waveguides [31,32], among others.

The impact of these excursions on the ergodic properties has been studied in the past for spin glasses [33] and time-continuous random walks [34–37]. Recently an efficient numerical method to quantify the impact of the above excursions out of equilibrium in weakly nonintegrable Hamiltonian systems was proposed in Refs. [38,39]. That scheme chooses the actions at the integrable limit as the relevant observables and subsequently tracks their temporal fluctuations. The resulting distributions of fluctuation times and their finite time average distributions permit one to extract a novel ergodization time
scale $T_E$ [40]. The dependence of $T_E$ on the strength of nonintegrable perturbation $\bar{\epsilon}$, in particular its divergence for vanishing $\bar{\epsilon}$, will signal the approaching of the integrable limit.

In this work we show that a nonintegrable perturbation $H_I$ can span different classes of interaction networks among the actions of the integrable limit $H_0$. These classes differ by network type being long range (respectively, short range). This distinction allows us to show that the Lyapunov time $T_L$—the inverse largest Lyapunov exponent—controls the ergodization dynamics and time scales for long-range networks, whereas it does not in the short-range case. We use a simple model—the Klein-Gordon (KG) chain for our computational studies. This model—as well as many other systems—exhibits all the above qualitatively different integrable limits. The paper is organized as follows. In the next section, we introduce weakly nonintegrable Hamiltonian systems and define long and short networks of actions. We then present our numerical studies of the different integrable limits of the Klein-Gordon chain. In the concluding section, we recap and discuss our results. A series of Appendixes provides technical details.

II. WEAKLY NONINTEGRABLE HAMILTONIAN SYSTEMS AND NETWORKS OF ACTIONS

Consider a Hamiltonian $H$ with $N$ degrees of freedom,

$$H = H(p, q), \quad (1)$$

where $q = (q_1, \ldots, q_N)$ are the position coordinates and $p = (p_1, \ldots, p_N)$ are the conjugate momenta. These coordinates belong to the $2N$-dimensional phase space $X = \mathbb{R}^N \times \mathbb{R}^N$. The equations of motion are

$$\dot{p}_n = -\frac{\partial H}{\partial q_n}, \quad \dot{q}_n = \frac{\partial H}{\partial p_n}. \quad (2)$$

An integral of motion $I$ (e.g., the Hamiltonian energy $H$) is a quantity that is conserved along the solutions of Eqs. (2). The existence of $\ell$ integrals of motion implies that a trajectory is confined to a codimension $\ell$ submanifold, called the available phase space. A Hamiltonian $H$ is called integrable if there exists a canonical transformation $(p, q) = \phi(J, \theta)$ that maps the conjugate coordinates $(p, q)$ into action-angle coordinates $(J, \theta)$ such that

$$H(\phi(J, \theta)) \equiv H_0(J). \quad (3)$$

so that the Hamiltonian $H_0$ depends only on the actions $\{J_n\}_{n=1}^N$. The existence of such a canonical transformation $\phi$ is ensured by the Liouville-Arnold theorem [41]. The equations of motion (2) of an integrable system expressed in action-angle coordinates read

$$J_n = -\frac{\partial H_0}{\partial \theta_n} = 0, \quad \theta_n = \frac{\partial H_0}{\partial J_n} = \omega_n(J). \quad (4)$$

Solutions of Eqs. (4) yield constant actions $J_n(t)$ and time-periodic angles $\theta_n(t)$ that wind on $N$-dimensional tori $T^N$. Let us consider a general Hamiltonian $H$ and define the energy density $\epsilon = H/N$. If in regimes of small or large $\epsilon$ some of its terms become negligible with respect to an otherwise integrable reminder $H_0$, we say that $H$ possesses an integrable limit. This can be realized by considering a Hamiltonian of the form

$$H = H_0 + \bar{\epsilon} H_I, \quad (6)$$

where $H_0$ is integrable, and $H_I$ is a nonintegrable perturbation whose strength is controlled by a small parameter $0 < \bar{\epsilon} \ll 1$. Then Eqs. (2) read

$$\dot{\theta}_n = \omega_n(J) + \bar{\epsilon} V_n(J, \theta), \quad \dot{J}_n = -\bar{\epsilon} W_n(J, \theta). \quad (7)$$

where $V_n = \partial H_1/\partial J_n$ and $W_n = \partial H_1/\partial \theta_n$. We call the system in Eq. (6) weakly nonintegrable. For $\bar{\epsilon} \neq 0$, the term $W_n$ links the time dynamics of an action $J_n$ to all or a subset of actions and angles of the system. In a typical case, each action $J_n$ is connected to a number $R_n$ of groups of actions, $\{G_m\}_{m=1}^N$, each one formed by $L_{m,n}$ actions $G_m = \{J_{n,m}(t)\}_{m=1}^{L_{m,n}}$. It then follows that a nonintegrable perturbation defines a network between the actions $\{J_n\}_{n=1}^N$, where $R_n$ and $L_{m,n}$ depend on $H_1$. We henceforth distinguish networks according to how the number of groups of actions linked by the perturbation $H_I$ depends on the number of degrees of freedom, $N$. Let us define the coupling range $R = \max[R_n/n \ll N]$. We can then distinguish the following two cases:

- **Long-range network (LRN).** The coupling range $R$ increases with the number $N$ of degrees of freedom of the system, $R = g(N)$, for a certain monotonic function $g$.
- **Short-range network (SRN).** The coupling range $R$ is finite and independent from the number $N$ of degrees of freedom of the system.

III. THE MODEL

We consider a class of classical translation-invariant interacting many-body systems described by the Hamiltonian

$$H = \sum_{n=1}^{N} \left[ \frac{p_n^2}{2} + V(q_n) + \bar{\epsilon} W(q_{n+1} - q_n) \right], \quad (8)$$

where $V$ is a local potential and $W$ is an interaction potential, with $V(0) = W(0) = V'(0) = W'(0) = 0, V''(0), W''(0) > 0$. We focus on the KG chain, for which

$$V(q) = \frac{q^2}{2}, \quad W(q) = \frac{q^4}{4}. \quad (9)$$

The equations of motion (2) read

$$\dot{q}_n = -q_n - q_n^3 + \epsilon(q_{n+1} + q_{n-1} - 2q_n). \quad (10)$$

Let us discuss several different integrable limits using $\epsilon$ and the energy density $\epsilon$ as control parameters. For $\epsilon = \text{const}$, $\epsilon \to 0$ and equally $\bar{\epsilon} \to \infty$, $\epsilon = \text{const}$ the system reaches an integrable set of decoupled oscillators, with the interaction potential $W$ acting as the perturbation $H_1$ in Eq. (6), and the network is SRN. In contrast, $\epsilon \to 0$, $\epsilon = \text{const}$ is a LRN integrable limit, since the quartic term in the local potential $V$ becomes negligible with respect to the remaining quadratic ones. In this limit, the KG chain reduces to a chain of harmonic oscillators. The term $q_n^2/4$ (transformed to Fourier
space) couples all the normal modes to each other and yields a long-range network. We note that the KG Hamiltonian possesses only one conserved quantity $H$. Equation (10) will be integrated in time using symplectic schemes (see Appendix A).

IV. METHODS

We follow the dynamics of time-dependent observables $J$ which become conserved at the chosen integrable limit. Ergodicity away from the integrable limit would imply that their infinite time average equals their statistical average $\langle J \rangle$:

$$\lim_{T \to \infty} J_T = \lim_{\tau \to \infty} = \frac{1}{T} \int_0^T J(t) dt = \langle J \rangle. \quad (11)$$

We numerically compute the finite time averages $J_T$ for $M$ different trajectories and obtain their distribution $\rho(J; T)$. This distribution is characterized by a first moment $\mu_J$ and a nonzero variance $\sigma_\mu^2(T)$. From Eq. (11) it follows that $\rho(J; T \to \infty) = \delta(J - \langle J \rangle)$. We study this convergence by computing the dimensionless squared coefficient of variation (also called the fluctuation index):

$$q(T) = \frac{\sigma_\mu^2(T)}{\mu_J^2}. \quad (12)$$

Following Ref. [40], we extract an ergodization time scale $T_E$ as

$$q(T) \sim \begin{cases} q(0) & \text{for } T \ll T_E, \\ \frac{1}{\mu_J^2} & \text{for } T \gg T_E. \end{cases} \quad (13)$$

We further study the fluctuation statistics of $J(t)$ by computing the piercing times $t^\pm$ at which $J(t) = \langle J \rangle$. The excursion times

$$\tau^\pm(i) = t_i^\pm + t_i^\mp \quad (14)$$

are distinguished between $J(t) > \langle J \rangle$ (excursion) and $J(t) < \langle J \rangle$ (return). We then compute numerically the distributions $P_\pm^\pm$ of the excursion times $\tau^\pm$, as well as their averages $\mu_{\tau^\pm}$ and their standard deviations $\sigma_{\tau^\pm}$ (see Appendix B for details). In particular, we focus on the $\tau^+$ events (see Appendix C). These time scales dictate the ergodization time scales $T_E$, according to Ref. [40],

$$T_E \sim \tau_q^+ = \frac{\sigma_{\tau_q}^2}{\mu_{\tau_q}^2}, \quad (15)$$

when neglecting correlations between different events (see Appendix E for details). Finally, we relate the ergodization time $T_E$ to the Lyapunov time,

$$T_\Lambda = 1/\Lambda, \quad (16)$$

defined as the inverse of the largest Lyapunov exponent $\Lambda$ and numerically obtained via the tangent method (see Appendix F for details). The dynamics of a nonintegrable system will be essentially identical to that of an integrable (but usually unknown) approximation precisely up to the time scale $T_\Lambda$.

V. LONG-RANGE NETWORK

Let us consider the small energy limit $h \to 0$, $\varepsilon = \text{const}$ of the KG chain. The integrable Hamiltonian $H_0$ consists of a chain of harmonic oscillators

$$H_0 = \sum_{n=1}^{N} \left[ \frac{p_n^2}{2} + \frac{q_n^2}{2} + \frac{\varepsilon}{2} (q_{n+1} - q_n)^2 \right]. \quad (17)$$

The nonintegrable perturbation is then given by

$$\tilde{H}_1 = \sum_{n=1}^{N} \frac{q_n^4}{4}. \quad (18)$$

We choose fixed boundary conditions $p_0 = p_{N+1} = q_0 = q_{N+1} = 0$, in analogy with the small energy limit of the Fermi-Pasta-Ulam (FPU) chain [5], in order to remove degeneracies of eigenmode frequencies.

We use the canonical transformation to normal mode momenta and coordinates $(p_k, q_k)$

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \sqrt{\frac{2}{N+1}} \sum_{n=1}^{N} \begin{pmatrix} p_n \\ q_n \end{pmatrix} \sin \left( \frac{\pi nk}{N+1} \right), \quad (19)$$

for $k = 1, \ldots, N$. This transformation diagonalizes the Hamiltonian $H_0 = \sum_{k=1}^{N} E_k$ in Eq. (17), where the normal mode energies $E_k$ are

$$E_k = \frac{p_k^2 + \Omega_k^2 q_k^2}{2}, \quad \omega_k = 2 \sin \left( \frac{\pi k}{2(N+1)} \right), \quad (20)$$

for $\Omega_k = \sqrt{1 + \varepsilon \omega_k^2}$. The equations of motion (10) in the normal mode coordinates [Eq. (19)] then read

$$\dot{q}_k + \Omega_k^2 q_k = -\frac{1}{2(N+1)} \sum_{i,j} A_{k,i,j} q_i q_j Q_i Q_j, \quad (21)$$

where

$$A_{k,i,j} = \frac{\delta_{k+1,i} \delta_{k+1,j} - \delta_{k+1,i} \delta_{k+1,j}}{\sqrt{J_i J_j J_k}}, \quad (22)$$

represents the coupling between the Fourier coordinates $Q_k$. Using the canonical transformation

$$Q_k = \sqrt{2J_k} \sin \theta_k, \quad P_k = \Omega_k \sqrt{2J_k} \cos \theta_k, \quad (23)$$

it follows that

$$J_k = -\frac{1}{\Omega_k} \sum_{i,j} A_{k,i,j} q_i q_j Q_i Q_j, \quad (24)$$

where the coefficients $A_{k,i,j}$ depend on the angles $\{\theta_j\}_k$.

$$A_{k,i,j} = \frac{A_{k,i,j} q_i q_j}{2(N+1)} \cos \theta_i \sin \theta_j \sin \theta_i \sin \theta_j, \quad (25)$$

For each action $J_k$, the sum on the right-hand side of Eq. (24) involves $R_k = N^2$ groups of variables $\{J_l\}$ due to the constrain enforced by Eq. (22). Each of the $m \leq R_k$ groups is formed by $L_k \times m$ actions. Hence the coupling range $R_k$—the integer which counts the number of connections $R = \max\{R_k | k \leq N\}$—is $R = N^2$, and this limit leads to a long-range network of actions $J_k$, similar to the FPU case discussed in Ref. [42]. We use the normal mode energies $E_k = \Omega_k^2 J_k$ as the time-dependent observables, which are statistically distinguishable.
In both cases, the asymptotic decay edge affects the outcome up to a common scaling prefactor (see δ function profile, and varying the cutoff condition does not show substantial convergence to its limiting value ±$\epsilon$=0.1. For $h$→0, $\sigma_{r,k}$ is not well defined and $\mu_{r,k}$ is not accurately defined; the fat tails of the distributions of excursion times are controlled by the Lyapunov time. To illustrate this, we show $T_k$, $T_L$ and $A_{\tau_k}$ in units of $T_L$, and as a function of $T_L$ in Figs. 1(c) and 1(d).

VI. SHORT-RANGE NETWORK

We use periodic boundary conditions $p_1=p_{N+1}$, $q_1=q_{N+1}$. In the limit of weak coupling $\epsilon \ll 1$, $h=\epsilon=\text{const}$ (respectively, $h \gg 1$, $\epsilon=\text{const}$) the system of equations (8) and (9) is close to an integrable limit with an integrable Hamiltonian $H$.
The nonintegrable perturbation is then given by

\[ H = \frac{h}{\Lambda} \sum_{\tau=1}^{N} \left[ \sum_{n=1}^{N} \left( \frac{p_n^2}{2} + q_n^2 + q_n^4 \right) \right]. \]  

(26)

The nonintegrable perturbation is then given by

\[ \tilde{\epsilon} H_1 = \frac{\epsilon}{2} \sum_{n=1}^{N} (q_n - q_{n-1})^2. \]  

(27)

\( H_1 \) couples only nearest-neighboring oscillators, leading to a SRN of actions. As in Ref. [44], we choose

\[ L_a = \frac{p_a^2}{2} + V(q_a) + \frac{\epsilon}{4} \left( (q_{a+1} - q_a)^2 + (q_a - q_{a-1})^2 \right) \]  

(28)

as the time-dependent observables, which become conserved in both integrable limits. Due to translation invariance, the observables \( I_a \) are statistically equivalent, fluctuating around the energy density \( h \). Since their distributions of finite time averages and of fluctuation times are identical, we extract measurements from all sites and use them for the computation of the distributions (see Appendix I). That allows us to reduce the number of trajectories studied.

In Fig. 4 we show \( q(T) \). Again \( q(T) \sim q(0) \) for \( T \ll T_E \), and \( q(T) \sim T_E/T \) for \( T \gg T_E \) in accord with Eq. (13). The ergodization time \( T_E \) is extracted by rescaling and fitting the curves (see Appendix G for details).

We compute the excursion times \( \tau_\gamma^+ \) of the observables \( I_a \), and their distributions \( P^+ \). In Fig. 5 we show \( P^+ \) for different \( h \) with \( \epsilon = 0.05 \) with \( N = 2^8 \). We notice that the distributions acquire fat tails, with an intermediate power-law trend \( \tau^{-2} \) which extends as \( h \) grows. This is substantiated in the inset, where the local derivative \( \gamma(\tau) \equiv d \log_{10} P^+ / d \log_{10} \tau \) is shown. We also show in Fig. 5 the distributions \( P^+ \) computed
FIG. 5. (a) $P_\tau(\tau)$ obtained for (left to right) $h=0.42$ (red), $h=1.09$ (magenta), $h=2.5$ (green), and $h=5.31$ (violet) for $N=2^3$. For $h=2.5$, $P_\tau(\tau)$ is shown for $N=2^7$ (orange) and $N=2^{10}$ (blue) - see text for details. The dashed line guides the eye and indicates the algebraic decay. Inset (b): $\gamma(\tau)\equiv d(\log_{10} P_\tau)/d(\log_{10} \tau)$. Here $\epsilon = 0.05$ and $T = 10^{10}$.

for a given $h = 2.5$ and $\epsilon = 0.05$ and different system sizes $N = 2^7$, $2^9$, and $2^{10}$ (orange, green, and blue curves) to confirm the absence of finite size corrections. We finally compute the average $\mu_\tau^+$ and the standard deviation $\sigma_\tau^+$.

In Fig. 6 we compare all computed time scales for the large energy density regime $h \gg 1$, $\epsilon = 0.05$ [Fig. 6(a)] and the weak-coupling regime $\epsilon \ll 1, h = 5$ [Fig. 6(b)]. We found that $T_E$ grows over several orders of magnitude. The standard deviation $\sigma_\tau^+$ outgrows the average $\mu_\tau^+$ as the integrable limit is approached. We plot $A\tau_\tau^+$ for $A = 132$ (see Ref. [43] for details) and observe very good agreement with $T_E$.

Finally we compare $T_E, \mu_\tau^+, \sigma_\tau^+$, and $A\tau_\tau^+$ with the Lyapunov time $T_\Lambda$ (shown in black stars) as a function of $h$ in Fig. 6(a) and $\epsilon$ in Fig. 6(b). In contrast to the long-range-network results where $T_\Lambda \approx \sigma_\tau^+$, in both short-range-network cases $T_\Lambda \lesssim \mu_\tau^+ \ll \sigma_\tau^+$. Consequently, $T_E \approx 10^9$ [at $h = 10$ for given $\epsilon = 0.05$ in Fig. 6(a)], and at $\epsilon = 0.1$ for $h = 5$ in Fig. 6(b)] and $T_\Lambda \approx 10$, leaving a gap of eight orders of magnitude in time to be understood. In Fig. 6(c), we confirm the above statements by showing $T_E, \mu_\tau^+, \sigma_\tau^+$, and $A\tau_\tau^+$ in units of $T_\Lambda$, as a function of $T_\Lambda$. The observed widening temporal gap between $T_\Lambda$ and $T_E$ is similar to the short-range-network studies of a classical chain of Josephson junctions in Ref. [40] and signals the emergence of a dynamical glass [40].

VII. CONCLUSION

Our studies of the microcanonical dynamics of Klein-Gordon chains with up to 1024 degrees of freedom show that the distributions of time average $\rho(\mathcal{T}_n; T)$ of integrable limit actions tend towards $\delta$ functions in the large-$T$ limit. Consequently the extracted ergodization times $T_E$ increase upon approaching the integrable limits but retain finite values at finite distance from the limits. We also computed the statistics of fluctuation times of $J_\rho(t)$. We found that both their average $\mu_\tau^+$ and standard deviation $\sigma_\tau^+$ diverge in the same integrable limits, as well as their dimensionless ratio $\sigma_\tau^+ / \mu_\tau^+$ which indicates the emergence of fat tails. Assuming the statistical independence of fluctuation events, it follows that $\sigma_\tau^+ / \mu_\tau^+ \sim T_E$ which was confirmed in all studied cases. Similar observations were obtained for classical chains of Josephson junctions [40], raising the interesting question of how general our findings are.

We studied two different types of integrable limits, characterized by long-range (respectively, short-range) networks between the actions $J_n$ spanned by the nonintegrable perturbation. While the above findings appear to be generic for both cases, the comparison of the Lyapunov time $T_\Lambda$ with the ergodization time scales shows remarkable differences between the two types of networks. For long-range networks, we find that $T_\Lambda \approx \sigma_\tau^+$ and consequently $T_E \sim T_\Lambda^2$. On the contrary, for short-range networks $T_\Lambda \approx \mu_\tau^+$ (see, e.g., Fig. 7). The latter observation is again in line with similar results obtained for
classical chains of Josephson junctions [40], indicating the emergence of a dynamical glass.

Both types of networks are characterized by a finite coordination number $L$ introduced in Sec. II. In the case of the KG model, this coordination number amounts to $L = 2$ for the SRN case and $L = 4$ for the LRN case. The Lyapunov time corresponds to the time scale on which the dynamics of resonant interacting multiplets of actions becomes chaotic, in accordance with Chirikov overlap criterion studies (see, e.g., Refs. [45–49]). Close to the corresponding integrable limit, the probability and corresponding density of resonant multiplets will diminish. In the case of a LRN, chaotic dynamics of any group of $L$ resonant actions will still couple into the whole network. In contrast, in the case of SRN, the chaotic dynamics of a group of $L$ resonant actions will couple only to its nearest neighbors, leaving the dynamics of the majority of all actions almost regular and unchanged. We conjecture that the rapidly widening gap between the ergodization time $T_E$ and the Lyapunov time $T_A$ for SRN is due to slow processes of diffusion, or meandering, or percolation, of resonance through the network, as observed also for classical chains of Josephson junctions [40].

Many open questions remain. Among the most pressing ones are pushing the limits of our calculation to $N \to \infty$ of the SRN case where the finite size effects on $q(T)$ disappear. On the other hand, the considered large lattice for this case, $N = 1024$, hints that $T_E$ remains the same even in the limit $N \to \infty$. Other challenges concern collecting evidence that the above observed scenarios of many-body Hamiltonian dynamics approaching integrable limits is generic, as well as the impact of quantization on the slow ergodization dynamics.

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APPENDIX B: DISTRIBUTION $P_k$ OF THE EXCURSION TIMES

In the numerical calculations of the PDF $P_k$ of excursions out of equilibrium $\tau$, we consider the interval $\mathcal{I}_b = [10^2, 10^9]$ to which the excursion times belong, and we separate this into $M$ bins of logarithmic width $[52]$. Hence, the bins $B_i$ are defined as

$$\mathcal{I}_b = \bigcup_{i=1}^{M} B_i = \bigcup_{i=1}^{M} [10^{2+i(k-1)}, 10^{2+i(k+\varepsilon)}] \quad (B1)$$

with $k = (b - 2)/\log(10)$. From our plots, we exclude all the bins that count less than 100 events, as these should be considered as statistically not relevant. If one reduces this number, then the range of the tail of the computed distribution increases; however, it is at the expense of less statistical significance, and thus with strongly fluctuating tails. The smoothness of the tails is by itself enough evidence for the statistical significance of our data, and the chosen cutoff at 100 events ensures a numerical error of less than 10%. Let us remark that this cutoff is applied only to the numerical reconstruction of the distributions, but not to the calculation of the first moment $\mu_1^\pm$ and the standard deviation $\sigma_1^\pm$, where instead all detected events are included.

APPENDIX C: AVERAGE AND STANDARD DEVIATION OF $\tau_n^\pm$

In Fig. 8 we show the averages and the standard deviations of both $\tau_n^+$ and $\tau_n^-$. The four plots [Figs. 8(a)–8(d)] correspond to the four cases discussed numerically in the main text, namely, $\mu_1^+$ and $\sigma_1^+$ for the low-energy regime with $k = 10$ [Fig. 8(a)] and for $k = 32$ [Fig. 8(b)], corresponding to Figs. 3(a) and 3(b), respectively; and $\mu_1^\pm$ and $\sigma_1^\pm$ for the large-energy limit [Fig. 8(c)] and anticontinuum limit [Fig. 8(d)] corresponding to Figs. 6(a) and 6(b), respectively. In all these cases, we observe that the average $\mu_1^\pm$ of $\tau^+$ (red diamonds) shows a divergence trend similar to $\mu_1^+$ of $\tau^-$ (green squares). Similarly, the standard deviation $\sigma_1^\pm$ of $\tau^+$ (blue triangles) shows a divergence trend like that of $\sigma_1^+$ of $\tau^-$ (orange circles). We then focus on the “$+$” events only, $\tau^+$, which we recall are events during which $J_n > (J_{n})_x$.

APPENDIX D: ENSEMBLE OF INITIAL CONDITIONS

Let us assume that there exists only one conserved quantity of the system (the total energy $H$). We define the initial condition at $t = 0$ as zeroing the position coordinates $q_i = 0$ and distributing the half squares of kinetic energy $p_i^2/2$, according to the following distribution $P_i$ defined for a positive real number $C > 0$:

$$P_i(z) = C e^{-Cz}, \quad z \in [0, \infty]. \quad (D1)$$

From this, for a uniform distribution of random numbers $w(r)$ distributed in the range $[0,1]$, one can get $z = -\log(w(r))/C$. This leads to a set of initial momenta coordinates $p_i^{(1)}$ at $t = 0$, where the sign is a discrete random variable, $a_n = \pm 1$ with distribution $P_2(a_n = \pm 1) = 0.5$. Here

$$p_1^{(1)} = a_n \sqrt{2z}. \quad (D2)$$

The total energy $E_T$ of the system is

$$E_T = \frac{1}{2} \sum_{n=1}^{N} (p_n^{(1)})^2. \quad (D3)$$

We then set a chosen energy density $h$ by the following rescaling:

$$p_n^{(2)} = \sqrt{\frac{hN}{E_T}} p_n^{(1)}. \quad (D4)$$

The resulting momentum coordinates $p_n^{(2)}$ with the position coordinates $q_i = 0$ fixed at zero are evolved in time for $T_{IC} = 10^9$ using the SABoC integrator with time step $\tau = 10^{-2}$, which keeps the relative energy error at $\Delta E \sim 10^{-9}$. We choose $T_{IC} = 10^9$ since it exceeds by one order of magnitude the largest Lyapunov time $T_L$ we observed ($T_L \approx 10^4$, observed in Fig. 3 for $h = 0.01$ and $\varepsilon = 1$). No qualitative differences in the resulting measured time scales were noticed if larger prerun times $T_{IC}$ were chosen. The result of this time evolution is then taken as an initial condition for our simulation. Then, $M$ draws of the distributions $P_1$ and $P_2$ yield an ensemble of $M$ initial conditions.

APPENDIX E: ASYMPTOTIC BEHAVIOR OF $q(T)$

In Eq. (15) we indicated that the ergodization time $T_E$ of an observable $J_n$ is proportional to the ratio between the variance $(\sigma_T^+)^2$ and the average $\mu_T^+$ of its excursions out of equilibrium:

$$T_E \sim \frac{\sigma_T^+}{\mu_T^+}. \quad (E1)$$

We here derive this relation, which is based on the approximation of the time evolution of $J_n$ with telegraphic random signal
Away from the integrable limit, an action $J_n$ becomes a time-dependent variable $J_n(t) = \langle J_n \rangle_t + \phi_n(t)$, where $\phi_n$ is a continuous function fluctuating around zero. At the piercing times $t_i^n$ of the action $J_n$, it follows that $\phi_n(t_i^n) = 0$. Then, the time average of $J_n$, here indicated as $B_n(T)$, is

$$B_n(T) = \frac{1}{T} \int_0^T J_n(t) \, dt = \langle J_n \rangle_t + \frac{1}{T} \int_0^T \phi_n(t) \, dt \approx \langle J_n \rangle_t + D_n(T).$$

(E2)

The interval $[0, T]$ consists of $M_n^+$ events $\tau_n^+$ and $M_n^-$ of $\tau_n^-$, plus uncompleted initial and final events of duration $\bar{I}_n$ and $\bar{E}_n$, respectively [56]. Let us observe that both $\bar{I}_n$ and $\bar{E}_n$ are not distributed according to the distribution $P_n$ of excursion times $\tau_n^\pm$. Nevertheless, for large values of $M_n^+$ and $M_n^-$ we neglect their contribution. Hence, for all $k$ it holds that

$$T = \sum_{i=1}^{M_n^+} \tau_n^+(i) + \sum_{i=1}^{M_n^-} \tau_n^-(i).$$

(E3)

The numbers $M_n^\pm$ of events $\tau_n^\pm$ are distributed according to the distributions $P_{\bar{I}_n}^\pm$ and $P_{\bar{E}_n}^\pm$ and have an average $\mu_{\bar{I}_n}^\pm$ and $\mu_{\bar{E}_n}^\pm$. In Fig. 9(a), we show that the average $\mu_{\bar{E}_n}^\pm$ of the $\tau_n^- \sim T / \mu_{\bar{I}_n}^+ \to \infty$, where $\mu_{\bar{E}_n}^\pm$ is the average of $\tau_n^- \tau_n^-$ computed for the short-range case of the KG chain using the observables $J_n$ in Eq. (28). From Eq. (E3), we can write the integral over the interval $[0, T]$ in Eq. (E2) as a sum of integrals over the excursion times

$$\int_0^T \phi_n(t) \, dt = \sum_{i=1}^{M_n^+} \int_{t_i^n}^{t_{i+1}^+} \varphi_n^+(t) \, dt - \sum_{i=1}^{M_n^-} \int_{t_i^-}^{t_{i+1}^-} \varphi_n^-(t) \, dt$$

$$= \alpha_n \sum_{i=1}^{M_n^+} \tau_n^+(i) - \beta_n \sum_{i=1}^{M_n^-} \tau_n^-(i),$$

(E4)

where $\alpha_n$ and $\beta_n$ are defined as

$$\alpha_n = \frac{\sum_{i=1}^{M_n^+} \int_{t_i^n}^{t_{i+1}^+} \varphi_n^+(t) \, dt}{\sum_{i=1}^{M_n^+} \tau_n^+(i)}, \quad \beta_n = \frac{\sum_{i=1}^{M_n^-} \int_{t_i^-}^{t_{i+1}^-} \varphi_n^-(t) \, dt}{\sum_{i=1}^{M_n^-} \tau_n^-(i)}.$$

(E5)

These coefficients $\alpha_n$ and $\beta_n$ are distributed by the distributions $\rho_{\alpha_n}$ and $\rho_{\beta_n}$, respectively. Let us here define their averages $\alpha$ and $\beta$. We then approximate Eq. (E4) by the telegraphic noise signal

$$\int_0^T \phi_n(t) \, dt \approx \alpha \sum_{i=1}^{M_n^+} \tau_n^+(i) - \beta \sum_{i=1}^{M_n^-} \tau_n^-(i)$$

$$\equiv \alpha S_n^+ - \beta S_n^-.$$  

(E6)

Let us now consider the limit of $q(T)$ for $T \to \infty$. Due to the continuity of $\phi_n$ and the finiteness of all moments of the excursion times $\tau_n^\pm$, the term $D_n(T)$ in Eq. (E2) converges to zero as $T \to \infty$. Then, it follows that

$$\lim_{T \to \infty} q(T) \equiv \frac{1}{\langle J_n \rangle_t} \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi_n(t) \, dt \approx \frac{1}{\langle J_n \rangle_t} \lim_{T \to \infty} \sigma_n^2(T).$$

(E7)

Recalling the properties of the variance for a constant $A$,

$$\sigma_{\Delta \phi_n}^2 = A^2 \sigma_n^2, \quad \sigma_{\Delta \phi_n + \Delta \phi_n}^2 = \sigma_n^2,$$

(E8)

from Eq. (E2) it follows that

$$\sigma_{\Delta \phi_n}^2 = \frac{\sigma_n^2}{T^2}.$$

(E9)

We can restrict to the $\tau_n^+$ events only in Eq. (E6) by adding and subtracting $\beta S_n^+$. It follows that

$$D_n(T) = \frac{1}{T} \int_0^T \phi_n(t) \, dt = \frac{1}{T} [\alpha S_n^+ - \beta (S_n^+ + S_n^-)]$$

$$\equiv \frac{1}{T} [(\alpha + \beta) S_n^+ - \beta (S_n^+ + S_n^-)]$$

$$= \frac{1}{T} [(\alpha + \beta) S_n^+ - \beta T] = \frac{\alpha + \beta}{T} S_n^+ - \beta.$$  

(E10)

By Eq. (E8), the variance of $B_n$ is

$$\sigma_{\Delta B_n}^2 = \frac{(\alpha + \beta)^2}{T^2} \sigma_n^2.$$  

(E11)

The excursion times $\tau_n^+$ are identically distributed variables. Assuming these to be independent events, the variance $\sigma_n^2$ of the sum of the head events $S_n^+$ in Eq. (E10) is the product of the variance $(\sigma_n^2)^2$ of the head events multiplied by the number of events, $M_n^+$:

$$S_n^+ \equiv \sum_{i=1}^{M_n^+} \tau_n^+(i) \Rightarrow \sigma_{\Delta S_n^+}^2 = M_n^+ (\sigma_n^2)^2.$$  

(E12)

For $T \gg \mu_n^+$, we expect that $\mu_n^+ \sim T / \mu_n^+$. In Fig. 9(a) we report the averages $\mu_n^+$ of $\tau_n^+$ and $\mu_n^-$ of the number $M_n^+$ and the duration $\tau_n^+$ of the events detected by all the observables $J_n$ in Eq. (28) for $h = 5$ and $\varepsilon = 0.05$. This plot confirms the above expectation.

In Eq. (E12) we approximate the variable $M_n^+$ by its average $\mu_n^+$, which leads to

$$\sigma_{\Delta B_n}^2 \sim (\alpha + \beta)^2 \frac{(\sigma_n^2)^2}{\mu_n^+} \frac{1}{T}.$$  

(E13)
Ultimately, this results in the formula
\[ q(T) \sim \frac{(\alpha + \beta)^2 (\sigma^+_n)^2}{(J_n^\alpha)^2} \frac{1}{\mu^+_n T}. \] (E14)

In Fig. 9(b) we show the energy density \( h \) dependence of the ratio \( (\alpha + \beta)^2 / (J_n^\alpha)^2 \) introduced in Eq. (E14) computed for the short-range case of the KG chain using the observables \( J_n \) in Eq. (28) for \( \varepsilon = 0.05 \). Over three orders of magnitude, the ratio \( (\alpha + \beta)^2 / (J_n^\alpha)^2 \) fluctuates between 2 and 2.5. Hence this ratio in Eq. (E14) does not contribute to the asymptotic trend of the fluctuation parameter \( q(T) \) as the system approaches an integrable limit. It therefore follows that the time scale \( (\sigma^+_n)^2 / \mu^+_n \) is proportional to the ergodization time \( T_E \) defined in Eq. (13), as stated in Eq. (15).

**APPENDIX F: LARGEST LYAPUNOV EXPONENT \( \Lambda \)**

We compute the largest Lyapunov exponent \( \Lambda \) by considering a small amplitude deviation vector \( w(t) = (\delta q(t), \delta p(t)) \) of a trajectory. We then numerically solve the variational equations
\[ \dot{w}(t) = [J_{2n} \cdot D_H^2(x(t))] \cdot w(t) \] (F1)
associated with a Hamiltonian \( H \) using the tangent method [57–59], where \( D_H^2 \) is the Hessian matrix and \( J_{2n} \) the symplectic matrix. Solving Eq. (F1) using the SABA2C integration scheme presented in Appendix A yields extended resolvent operators \( e^{\Delta t L_{av}} \), \( e^{\Delta t L_{av}} \), and \( e^{\Delta t L_{cv}} \) in Eqs. (A3), (A4), and (A7), where \( (\delta q_n, \delta p_n) \) at the time \( t \) are simultaneously integrated to \( (\delta q_n', \delta p_n') \) at the time \( t + \Delta t \). The additional equations for \( e^{\Delta t L_{av}} \) are
\[ e^{\Delta t L_{av}} : \begin{cases} \delta q_n' = \delta q_n + \delta p_n \Delta t \\ \delta p_n' = \delta p_n \end{cases} \] (F2)
while for \( e^{\Delta t L_{av}} \) they are
\[ e^{\Delta t L_{av}} : \begin{cases} \delta q_n' = \delta q_n + \delta p_n \Delta t \\ \delta p_n' = \delta p_n + \left\{ \varepsilon (\delta q_{n+1} + \delta q_{n-1}) - 1 + 3 \delta q_n^2 + 2 \varepsilon [\delta q_n'] \right\} \Delta t. \] (F3)

For the correction term, we get
\[ e^{\Delta t L_{cv}} : \begin{cases} \delta q_n' = \delta q_n \\ \delta p_n' = \delta p_n + [\gamma_n \delta q_n + \gamma_{n+1} \delta q_{n+1} + \gamma_{n+2} \delta q_{n+2} + \gamma_{n-1} \delta q_{n-1} + \gamma_{n-2} \delta q_{n-2}] \Delta t, \end{cases} \] (F4)
where
\[ \gamma_n = -2 \left[ \left[ 1 + 3 \delta q_n^2 + 2 \varepsilon \right] + 6 \varepsilon \delta q_n (1 + \delta q_n^2) \right] + \varepsilon (2 \delta q_n - \delta q_{n+1} - \delta q_{n-1}) + 2 \delta q_n^2, \] (F5)
\[ \gamma_{n+1} = 2 \delta q_n + 4 \varepsilon + 3 \delta q_n^2 + 2 \varepsilon \delta q_n + \delta q_n^2, \]
\[ \gamma_{n-1} = 2 \delta q_n + 4 \varepsilon + 3 \delta q_n^2 + 2 \varepsilon \delta q_n + \delta q_n^2, \]
\[ \gamma_{n+2} = -2 \delta q_n, \]
\[ \gamma_{n-2} = -2 \delta q_n. \]

As mentioned above, in both resolvents \( e^{\Delta t L_{av}} \) and \( e^{\Delta t L_{cv}} \) in Eqs. (F3) and (F4) the boundary conditions have to be applied: fixed boundary conditions for the LRN cases, and periodic boundary conditions for the SRN cases. The largest Lyapunov exponent \( \Lambda \) is computed by considering the limit
\[ \Lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|w(t)\|}{\|w(0)\|}, \] (F6)
where \( \| \cdot \| \) is a norm of the vector \( w \). An extended presentation of this method applied to the KG chain can be found in Ref. [60].

**APPENDIX G: MEASUREMENT OF THE COEFFICIENT \( T_E \)**

In both Figs. 3 and 6 we have shown the behavior of the ergodization time \( T_E \) as the system approaches the integrable limit. In the cases shown in Fig. 6, \( T_E \) was determined by first extracting the prefactor of the power-law regression of
the black curve of both plots [corresponding to $h = 0.01$ in Fig. 4(a) and to $\varepsilon = 0.8$ in Fig. 4(b)]. We fix these as the two basic cases. Then, for all the higher $h$ or lower $\varepsilon$ cases, respectively, we rescaled the integration time $T \rightarrow T/x$ of the curve $q$ by a factor $x$, and selected the proper $x$ for which the rescaled curves align with their corresponding basic cases. The ergodization time $T_E$ of each curve is finally obtained by multiplying its selected $x$ with the ergodization time $T_E$ of the corresponding basic case. In Fig. 10 we present the time evolution of the parameter $q$ shown in Fig. 6 rescaled by the ergodization time $T_E$, to show the alignment between the curves. In the cases shown in Fig. 3, the intermediate plateau exhibited by the time evolution of the index $q$ prevented us from obtaining a proper $1/T$ fitting and the rescaling of the integration time $T$ using the techniques described above. Then, the ergodization time $T_E$ was determined by the introduction of a cutoff at $q = 10^{-1}$ (violet dashed horizontal line in Fig. 1). We then test the reliability of this cutoff procedure on two SRN cases. In Fig. 10 we show the ergodization time $T_E$ extracted by rescaling (orange squares) and by cutoff at $q = 10^{-1}$ (blue squares). The two sets of measurements show good agreement. We do not find significant changes in the scaling on $T_E$ if different cutoff values are used (e.g., the value 0.075).

**APPENDIX H: DISTRIBUTION $\rho$ OF THE FINITE TIME AVERAGE**

We here show the time dependence of the distribution $\rho$, corresponding to two cases of the weak coupling $\varepsilon \ll 1$.

**APPENDIX I: DISTRIBUTION $P_n^+$ IN THE SRN CASE**

In Fig. 5 we plot the distribution function $P^+$ of the events $\tau^+$ obtained by combining the events detected by all the observables $J_n$ in Eq. (28) in one unique distribution. In Fig. 12 we show the distribution $P_n^+$ computed for the observables $I_1, I_{16}, I_{32}$ and the distribution $P^+$ for all $n$ combined for three different energies $h = 1.01$ (green), $h = 2.88$ (blue), and $h = 12.9$ (black). In the three cases, the distributions $P_n^+$ obtained with a single observable $I_n$ overlap with each other and with the distribution $P^+$ obtained by combining the events detected by all $I_n$. The curves cannot be distinguished. This suggests that, due to translation invariance, the excursions out of equilibrium of each action $I_n$ in Eq. (28) at a given distance from an integrable limit are governed by the same distribution.

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