UNITAL $A_\infty$-CATEGORIES

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Abstract. We prove that three definitions of unitality for $A_\infty$-categories suggested by the first author, by Kontsevich and Soibelman, and by Fukaya are equivalent.

1. Introduction

Over the past decade, $A_\infty$-categories have experienced a resurgence of interest due to applications in symplectic geometry, deformation theory, non-commutative geometry, homological algebra, and physics.

The notion of $A_\infty$-category is a generalization of Stasheff’s notion of $A_\infty$-algebra [12]. On the other hand, $A_\infty$-categories generalize differential graded categories. In contrast to differential graded categories, composition in $A_\infty$-categories is associative only up to homotopy that satisfies certain equation up to another homotopy, and so on. The notion of $A_\infty$-category appeared in the work of Fukaya on Floer homology [1] and was related to mirror symmetry by Kontsevich [5]. Basic concepts of the theory of $A_\infty$-categories have been developed by Fukaya [2], Keller [4], Lefèvre-Hasegawa [6], Lyubashenko [8], Soibelman [11].

The definition of $A_\infty$-category does not assume the existence of identity morphisms. The use of $A_\infty$-categories without identities requires caution: for example, there is no sensible notion of isomorphic objects, the notion of equivalence does not make sense, etc. In order to develop a comprehensive theory of $A_\infty$-categories, a notion of unital $A_\infty$-category, i.e., $A_\infty$-category with identity morphisms (also called units), is necessary. The obvious notion of strictly unital $A_\infty$-category, despite its technical advantages, is not quite satisfactory: it is not homotopy invariant, meaning that it does not translate along homotopy equivalences. Different definitions of (weakly) unital $A_\infty$-category have been suggested by the first author [8, Definition 7.3], by Kontsevich and Soibelman [6, Definition 4.2.3], and by Fukaya [2, Definition 5.11]. We prove that these definitions are equivalent. The main ingredient of the proofs is the Yoneda Lemma for unital (in the sense of Lyubashenko) $A_\infty$-categories proven in [8, Appendix A], see also [10, Appendix A].

2. Preliminaries

We follow the notation and conventions of [8], sometimes without explicit mentioning. Some of the conventions are recalled here.

Throughout, $k$ is a commutative ground ring. A graded $k$-module always means a $\mathbb{Z}$-graded $k$-module.

Key words and phrases. $A_\infty$-category, unital $A_\infty$-category, weak unit.
A graded quiver $\mathcal{A}$ consists of a set $\text{Ob}\mathcal{A}$ of objects and a graded $k$-module $\mathcal{A}(X,Y)$, for each $X,Y \in \text{Ob}\mathcal{A}$. A morphism of graded quivers $f : \mathcal{A} \rightarrow \mathcal{B}$ of degree $n$ consists of a function $\text{Ob}\ f : \text{Ob}\mathcal{A} \rightarrow \text{Ob}\mathcal{B}$, $X \mapsto Xf$, and a $k$-linear map $f = f_{X,Y} : \mathcal{A}(X,Y) \rightarrow \mathcal{B}(Xf,Yf)$ of degree $n$, for each $X,Y \in \text{Ob}\mathcal{A}$.

For a set $S$, there is a category $\mathcal{Q}/S$ defined as follows. Its objects are graded quivers whose set of objects is $S$. A morphism $f : \mathcal{A} \rightarrow \mathcal{B}$ in $\mathcal{Q}/S$ is a morphism of graded quivers of degree 0 such that $\text{Ob}\ f = \text{id}_S$. The category $\mathcal{Q}/S$ is monoidal. The tensor product of graded quivers $\mathcal{A}$ and $\mathcal{B}$ is a graded quiver $\mathcal{A} \otimes \mathcal{B}$ such that
\[
(\mathcal{A} \otimes \mathcal{B})(X,Z) = \bigoplus_{Y \in S} \mathcal{A}(X,Y) \otimes \mathcal{B}(Y,Z), \quad X,Z \in S.
\]

The unit object is the discrete quiver $kS$ with $\text{Ob}\ kS = S$ and
\[
(kS)(X,Y) = \begin{cases} 
k & \text{if } X = Y, \\0 & \text{if } X \neq Y, \end{cases} \quad X,Y \in S.
\]

Note that a map of sets $f : S \rightarrow R$ gives rise to a morphism of graded quivers $k f : kS \rightarrow kR$ with $\text{Ob}\ k f = f$ and $(kf)_{X,Y} = \text{id}_k$ is $X = Y$ and $(kf)_{X,Y} = 0$ if $X \neq Y$, $X,Y \in S$.

An augmented graded cocategory is a graded quiver $\mathcal{C}$ equipped with the structure of an augmented counital coassociative coalgebra in the monoidal category $\mathcal{Q}/\text{Ob}\mathcal{C}$. Thus, $\mathcal{C}$ comes with a comultiplication $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$, a counit $\epsilon : \mathcal{C} \rightarrow k \text{Ob}\mathcal{C}$, and an augmentation $\eta : k \text{Ob}\mathcal{C} \rightarrow \mathcal{C}$, which are morphisms in $\mathcal{Q}/\text{Ob}\mathcal{C}$ satisfying the usual axioms. A morphism of augmented graded cocategories $f : \mathcal{C} \rightarrow \mathcal{D}$ is a morphism of graded quivers of degree 0 that preserves the comultiplication, counit, and augmentation.

The main example of an augmented graded cocategory is the following. Let $\mathcal{A}$ be a graded quiver. Denote by $TA$ the direct sum of graded quivers $T^n\mathcal{A}$, where $T^n\mathcal{A} = \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ is the $n$-fold tensor product of $\mathcal{A}$ in $\mathcal{Q}/\text{Ob}\mathcal{A}$; in particular, $T^0\mathcal{A} = k \text{Ob}\mathcal{A}$, $T^1\mathcal{A} = \mathcal{A}$, $T^2\mathcal{A} = \mathcal{A} \otimes \mathcal{A}$, etc. The graded quiver $TA$ is an augmented graded cocategory in which the comultiplication is the so-called ‘cut’ comultiplication $\Delta_0 : TA \rightarrow TA \otimes TA$ given by
\[
f_1 \otimes \cdots \otimes f_n \mapsto \sum_{k=0}^n f_1 \otimes \cdots \otimes f_k \otimes f_{k+1} \otimes \cdots \otimes f_n,
\]
the counit is given by the projection $\text{pr}_0 : TA \rightarrow T^0\mathcal{A} = k \text{Ob}\mathcal{A}$, and the augmentation is given by the inclusion $\text{in}_0 : k \text{Ob}\mathcal{A} = T^0\mathcal{A} \hookrightarrow TA$.

The graded quiver $TA$ admits also the structure of a graded category, i.e., the structure of a unital associative algebra in the monoidal category $\mathcal{Q}/\text{Ob}\mathcal{A}$. The multiplication $\mu : TA \otimes TA \rightarrow TA$ removes brackets in tensors of the form $(f_1 \otimes \cdots \otimes f_m) \otimes (g_1 \otimes \cdots \otimes g_n)$. The unit $\eta : k \text{Ob}\mathcal{A} \rightarrow TA$ is given by the inclusion $\text{in}_0 : k \text{Ob}\mathcal{A} = T^0\mathcal{A} \hookrightarrow TA$.

For a graded quiver $\mathcal{A}$, denote by $s\mathcal{A}$ its suspension, the graded quiver given by $\text{Ob}s\mathcal{A} = \text{Ob}\mathcal{A}$ and $(s\mathcal{A}(X,Y))^n = \mathcal{A}(X,Y)^{n+1}$, for each $n \in \mathbb{Z}$ and $X,Y \in \text{Ob}\mathcal{A}$. An $A_{\infty}$-category is a graded quiver $\mathcal{A}$ equipped with a differential $b : Ts\mathcal{A} \rightarrow Ts\mathcal{A}$ of degree 1 such that $(Ts\mathcal{A}, \Delta_0, \text{pr}_0, \text{in}_0, b)$ is an augmented differential graded cocategory. In other terms, the equations
\[
b^2 = 0, \quad b\Delta_0 = \Delta_0(b \otimes 1 + 1 \otimes b), \quad b\text{pr}_0 = 0, \quad \text{in}_0 b = 0
\]
hold true. Denote by
\[ b_{mn} \overset{\text{def}}{=} [T^m sA \overset{\text{in}_m}{\longrightarrow} T sA \overset{b}{\longrightarrow} T sA \overset{\text{pr}_n}{\longrightarrow} T^n sA] \]
matrix coefficients of \( b \), for \( m, n \geq 0 \). Matrix coefficients \( b_{m1} \) are called components of \( b \) and abbreviated by \( b_n \). The above equations imply that \( b_0 = 0 \) and that \( b \) is unambiguously determined by its components via the formula
\[ b_{mn} = \sum_{p+k+q=m} 1^{\otimes p} \otimes b_k \otimes 1^{\otimes q} : T^m sA \to T^n sA, \quad m, n \geq 0. \]
The equation \( b^2 = 0 \) is equivalent to the system of equations
\[ \sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q})b_{p+1+q} = 0 : T^m sA \to sA, \quad m \geq 1. \]

For \( A_\infty \)-categories \( A \) and \( B \), an \( A_\infty \)-functor \( f : A \to B \) is a morphism of augmented differential graded cocategories \( f : T sA \to T sB \). In other terms, \( f \) is a morphism of augmented graded cocategories and preserves the differential, meaning that \( fb = bf \).

Denote by
\[ f_{mn} \overset{\text{def}}{=} [T^m sA \overset{\text{in}_m}{\longrightarrow} T sA \overset{f}{\longrightarrow} T sB \overset{\text{pr}_n}{\longrightarrow} T^n sB] \]
matrix coefficients of \( f \), for \( m, n \geq 0 \). Matrix coefficients \( f_{m1} \) are called components of \( f \) and abbreviated by \( f_m \). The condition that \( f \) is a morphism of augmented graded cocategories implies that \( f_0 = 0 \) and that \( f \) is unambiguously determined by its components via the formula
\[ f_{mn} = \sum_{i_1 + \cdots + i_n = m} f_{i_1} \otimes \cdots \otimes f_{i_n} : T^m sA \to T^n sB, \quad m, n \geq 0. \]
The equation \( fb = bf \) is equivalent to the system of equations
\[ \sum_{i_1 + \cdots + i_n = m} (f_{i_1} \otimes \cdots \otimes f_{i_n})b_n = \sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q})f_{p+1+q} : T^m sA \to sB, \]
for \( m \geq 1 \). An \( A_\infty \)-functor \( f \) is called strict if \( f_n = 0 \) for \( n > 1 \).

3. Definitions

3.1. Definition (cf. [2, 4]). An \( A_\infty \)-category \( A \) is strictly unital if, for each \( X \in \text{Ob}\, A \), there is a \( k \)-linear map \( X i_0^A : k \to (sA)^{-1}(X, X) \), called a strict unit, such that the following conditions are satisfied: \( X i_0^A b_1 = 0 \), the chain maps \( (1 \otimes X i_0^A)b_2, -(X i_0^A \otimes 1)b_2 : sA(X, Y) \to sA(X, Y) \) are equal to the identity map, for each \( X, Y \in \text{Ob}\, A \), and \((\cdots \otimes X i_0^A \otimes \cdots)b_n = 0 \) if \( n \geq 3 \).

For example, differential graded categories are strictly unital.

3.2. Definition (Lyubashenko [8, Definition 7.3]). An \( A_\infty \)-category \( A \) is unital if, for each \( X \in \text{Ob}\, A \), there is a \( k \)-linear map \( X i_0^A : k \to (sA)^{-1}(X, X) \), called a unit, such that the following conditions hold: \( X i_0^A b_1 = 0 \) and the chain maps \( (1 \otimes X i_0^A)b_2, -(X i_0^A \otimes 1)b_2 : sA(X, Y) \to sA(X, Y) \) are homotopic to the identity map, for each \( X, Y \in \text{Ob}\, A \). An arbitrary homotopy between \( (1 \otimes X i_0^A)b_2 \) and the identity map is called a right unit homotopy. Similarly, an arbitrary homotopy between \(-(X i_0^A \otimes 1)b_2 \) and the identity map
is called a left unit homotopy. An \( A_{\infty} \)-functor \( f : \mathcal{A} \to \mathcal{B} \) between unital \( A_{\infty} \)-categories is unital if the cycles \( x f_0^A f_1 \) and \( x f_0^B \) are cohomologous, i.e., differ by a boundary, for each \( X \in \text{Ob}\mathcal{A} \).

Clearly, a strictly unital \( A_{\infty} \)-category is unital.

With an arbitrary \( A_{\infty} \)-category \( \mathcal{A} \) a strictly unital \( A_{\infty} \)-category \( \mathcal{A}^u \) with the same set of objects is associated. For each \( X, Y \in \text{Ob}\mathcal{A} \), the graded \( k \)-module \( s\mathcal{A}^u(X, Y) \) is given by

\[
s\mathcal{A}^u(X, Y) = \begin{cases} 
  s\mathcal{A}(X, Y) & \text{if } X \neq Y, \\
  s\mathcal{A}(X, X) \oplus k x_i^u & \text{if } X = Y,
\end{cases}
\]

where \( x_i^u \) is a new generator of degree \(-1\). The element \( x_i^u \) is a strict unit by definition, and the natural embedding \( e : \mathcal{A} \hookrightarrow \mathcal{A}^u \) is a strict \( A_{\infty} \)-functor.

3.3. Definition (Kontsevich–Soibelman \([\mathcal{K}], \text{Definition 4.2.3}\)). A weak unit of an \( A_{\infty} \)-category \( \mathcal{A} \) is an \( A_{\infty} \)-functor \( U : \mathcal{A}^u \to \mathcal{A} \) such that

\[
[A \hookrightarrow \mathcal{A}^u \overset{U}{\to} \mathcal{A}] = \text{id}_\mathcal{A}.
\]

3.4. Proposition. Suppose that an \( A_{\infty} \)-category \( \mathcal{A} \) admits a weak unit. Then the \( A_{\infty} \)-category \( \mathcal{A} \) is unital.

Proof. Let \( U : \mathcal{A}^u \to \mathcal{A} \) be a weak unit of \( \mathcal{A} \). For each \( X \in \text{Ob}\mathcal{A} \), denote by \( x_i^u \) the element \( x_i^u U_1 \in s\mathcal{A}(X, X) \) of degree \(-1\). It follows from the equation \( U_1 b_1 = b_1 U_1 \) that \( x_i^u b_1 = 0 \). Let us prove that \( x_i^u \) are unit elements of \( \mathcal{A} \).

For each \( X, Y \in \text{Ob}\mathcal{A} \), there is a \( k \)-linear map

\[
h = (1 \otimes Y_0) U_2 : s\mathcal{A}(X, Y) \to s\mathcal{A}(X, Y)
\]

of degree \(-1\). The equation

\[
(1 \otimes b_1 + b_1 \otimes 1) U_2 + b_2 U_1 = U_2 b_1 + (U_1 \otimes U_1) b_2
\]

implies that

\[
-b_1 h + 1 = h b_1 + (1 \otimes X_0 b_1) : s\mathcal{A}(X, Y) \to s\mathcal{A}(X, Y),
\]

thus \( h \) is a right unit homotopy for \( \mathcal{A} \). For each \( X, Y \in \text{Ob}\mathcal{A} \), there is a \( k \)-linear map

\[
h' = -(X_0 \otimes 1) U_2 : s\mathcal{A}(X, Y) \to s\mathcal{A}(X, Y)
\]

of degree \(-1\). Equation \((3.1)\) implies that

\[
b_1 h' - 1 = -h' b_1 + (X_0 \otimes 1) u_2 : s\mathcal{A}(X, Y) \to s\mathcal{A}(X, Y),
\]

thus \( h' \) is a left unit homotopy for \( \mathcal{A} \). Therefore, \( \mathcal{A} \) is unital. \( \square \)

3.5. Definition (Fukaya \([\mathcal{K}], \text{Definition 5.11}\)). An \( A_{\infty} \)-category \( \mathcal{C} \) is called homotopy unital if the graded quiver

\[
\mathcal{C}^+ = \mathcal{C} \oplus k \mathcal{C} \oplus s k \mathcal{C}
\]

(with \( \text{Ob}\mathcal{C}^+ = \text{Ob}\mathcal{C} \)) admits an \( A_{\infty} \)-structure \( b^+ \) of the following kind. Denote the generators of the second and the third direct summands of the graded quiver \( s\mathcal{C}^+ = s\mathcal{C} \oplus s k \mathcal{C} \oplus s^2 k \mathcal{C} \) by \( x_i^{0u} = 1s \) and \( j_X = 1s^2 \) of degree respectively \(-1\) and \(-2\), for each \( X \in \text{Ob}\mathcal{C} \). The conditions on \( b^+ \) are:

1. for each \( X \in \text{Ob}\mathcal{C} \), the element \( X i_0^c \) is contained in \( s\mathcal{C}(X, X) \);
(2) $\mathcal{C}^+$ is a strictly unital $A_\infty$-category with strict units $x_i^{\mathcal{C}^+}$, $X \in \text{Ob } \mathcal{C}$;
(3) the embedding $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is a strict $A_\infty$-functor;
(4) $(s\mathcal{C} \oplus s^2k\mathcal{C})^n b^+ \subset s\mathcal{C}$, for each $n > 1$.

In particular, $\mathcal{C}^+$ contains the strictly unital $A_\infty$-category $\mathcal{C}^{su} = \mathcal{C} \oplus k\mathcal{C}$. A version of this definition suitable for filtered $A_\infty$-algebras (and filtered $A_\infty$-categories) is given by Fukaya, Oh, Ohta and Ono in their book [3, Definition 8.2].

Let $\mathcal{D}$ be a strictly unital $A_\infty$-category with strict units $i_0^D$. Then it has a canonical homotopy unital structure $(\mathcal{D}^+, b^+)$. Namely, $j^D_X b^+_1 = x_i^{\mathcal{C}^{su}} - x_i^D$, and $b^+$ vanishes for each $n > 1$ on each summand of $(s\mathcal{D} \oplus s^2k\mathcal{D})^n$ except on $s\mathcal{D}^{\infty}$, where it coincides with $b^+_0$. Verification of the equation $(b^+)^2 = 0$ is a straightforward computation.

3.6. Proposition. An arbitrary homotopy unital $A_\infty$-category is unital.

Proof. Let $\mathcal{C} \subset \mathcal{C}^+$ be a homotopy unital category. We claim that the distinguished cycles $x_i^C \in \mathcal{C}(X, X)[1]^{-1}$, $X \in \text{Ob } \mathcal{C}$, turn $\mathcal{C}$ into a unital $A_\infty$-category. Indeed, the identity 
$$(1 \otimes b^+_1 + b^+_1 \otimes 1)h^+_2 + b^+_1b^+_1 = 0$$
applied to $s\mathcal{C} \otimes j^C$ or to $j^C \otimes s\mathcal{C}$ implies
$$(1 \otimes b^+_1)h^+_2 = 1 + (1 \otimes j^C)b^+_2b^+_1 + b^+_1(1 \otimes j^C)b^+_2 : s\mathcal{C} \rightarrow s\mathcal{C},$$
$$(i^C \otimes 1)h^+_2 = -1 + (j^C \otimes 1)b^+_2b^+_1 + b^+_1(j^C \otimes 1)b^+_2 : s\mathcal{C} \rightarrow s\mathcal{C}.$$ Thus, $(1 \otimes j^C)b^+_2 : s\mathcal{C} \rightarrow s\mathcal{C}$ and $(j^C \otimes 1)b^+_2 : s\mathcal{C} \rightarrow s\mathcal{C}$ are unit homotopies. Therefore, the $A_\infty$-category $\mathcal{C}$ is unital. \qed

The converse of Proposition 3.6 holds true as well.

3.7. Theorem. An arbitrary unital $A_\infty$-category $\mathcal{C}$ with unit elements $i^C_0$ admits a homotopy unital structure $(\mathcal{C}^+, b^+)$ with $j^C_0b^+_1 = i^{\mathcal{C}^{su}}_0 - i^C_0$.

Proof. By [10, Corollary A.12], there exists a differential graded category $\mathcal{D}$ and an $A_\infty$-equivalence $\phi : \mathcal{C} \rightarrow \mathcal{D}$. By [10, Remark A.13], we may choose $\mathcal{D}$ and $\phi$ such that $\text{Ob } \mathcal{D} = \text{Ob } \mathcal{C}$ and $\text{Ob } \phi = \text{id}_{\text{Ob } \mathcal{C}}$. Being strictly unital $\mathcal{D}$ admits a canonical homotopy unital structure $(\mathcal{D}^+, b^+)$. In the sequel, we may assume that $\mathcal{D}$ is a strictly unital $A_\infty$-category equivalent to $\mathcal{C}$ via $\phi$ with the mentioned properties. Let us construct simultaneously an $A_\infty$-structure $b^+$ on $\mathcal{C}^+$ and an $A_\infty$-functor $\phi^+ : \mathcal{C}^+ \rightarrow \mathcal{D}^+$ that will turn out to be an equivalence.

Let us extend the homotopy isomorphism $\phi_1 : s\mathcal{C} \rightarrow s\mathcal{D}$ to a chain quiver map $\phi^+_1 : s\mathcal{C}^+ \rightarrow s\mathcal{D}^+$. The $A_\infty$-equivalence $\phi : \mathcal{C} \rightarrow \mathcal{D}$ is a unital $A_\infty$-functor, i.e., for each $X \in \text{Ob } \mathcal{C}$, there exists $v_X \in \mathcal{D}(X, X)[1]^{-2}$ such that $x_i^{\mathcal{C}^+} - x_i^D \phi^+_1 = v_X b^+_1$. In order that $\phi^+$ be strictly unital, we define $x_i^{\mathcal{C}^{su}} \phi^+_1 = x_i^{\mathcal{D}^{su}}$. We should have 
$$j^C_X \phi^+_1 + b^+_1 = j^C_X b^+_1 \phi^+_1 = x_i^{\mathcal{C}^{su}} \phi^+_1 - x_i^D \phi^+_1 = x_i^{\mathcal{D}^{su}} - x_i^D + x_i^D - x_i^D \phi^+_1 = (j^C_X + v_X)b^+_1,$$ so we define $j^C_X \phi^+_1 = J^D_X + v_X$.

We claim that there is a homotopy unital structure $(\mathcal{C}^+, b^+)$ of $\mathcal{C}$ satisfying the four conditions of Definition 3.6 and an $A_\infty$-functor $\phi^+ : \mathcal{C}^+ \rightarrow \mathcal{D}^+$ satisfying four parallel conditions:

(1) the first component of $\phi^+$ is the quiver morphism $\phi^+_1$ constructed above;
(2) the $A_\infty$-functor $\phi^+$ is strictly unital;

(3) the restriction of $\phi^+$ to $\mathcal{C}$ gives $\phi$;

(4) $(s\mathcal{C} \oplus s^2\mathbb{R})^{\otimes n}\phi^+_n \subset s\mathcal{D}$, for each $n > 1$.

Notice that in the presence of conditions (2) and (3) the first condition reduces to $j_X^\mathcal{C}(\phi^+_1) = j_X^\mathcal{P} + v_X$, for each $X \in \text{Ob} \mathcal{C}$.

Components of the $(1,1)$-coderivation $b^+ : Ts\mathcal{C}^+ \to Ts\mathcal{C}^+$ of degree 1 and of the augmented graded cocategory morphism $\phi^+ : Ts\mathcal{C}^+ \to Ts\mathcal{D}^+$ are constructed by induction.

We already know components $b^+_1$ and $\phi^+_1$. Given an integer $n \geq 2$, assume that we have already found components $b^+_m$, $\phi^+_m$ of the sought $b^+$ and $\phi^+$ for $m < n$ such that the equations

$$(b^+)^2_m = 0 : T^m s\mathcal{C}^+(X, Y) \to s\mathcal{C}^+(X, Y), \quad (3.2)$$

$$(\phi^+ b^+)_m = (b^+ \phi^+_m)_m : T^m s\mathcal{C}^+(X, Y) \to s\mathcal{D}^+(X f, Y f) \quad (3.3)$$

are satisfied for all $m < n$. Define $b^+_n$, $\phi^+_n$ on direct summands of $T^n s\mathcal{C}^+$ which contain a factor $i_0^{sa}$ by the requirement of strict unitality of $\mathcal{C}^+$ and $\phi^+$. Then equations (3.2), (3.3) hold true for $m = n$ on such summands. Define $b^+_n$, $\phi^+_n$ on the direct summand $T^n s\mathcal{C} \subset T^n s\mathcal{C}^+$ as $b^+_n$ and $\phi^+_n$. Then equations (3.2), (3.3) hold true for $m = n$ on the summand $T^n s\mathcal{C}$. It remains to construct those components of $b^+$ and $\phi^+$ which have $j^\mathcal{C}$ as one of their arguments.

Extend $b_1 : s\mathcal{C} \to s\mathcal{C}$ to $b'_1 : s\mathcal{C}^+ \to s\mathcal{C}^+$ by $i_0^{sa} b'_1 = 0$ and $j^\mathcal{C} b'_1 = 0$. Define $b^+_1 = b^+_1 - b'_1 : s\mathcal{C}^+ \to s\mathcal{C}^+$. Thus, $b^+_1 s^{sa} = 0$, $j^\mathcal{C} s^{sa} = i_0^{sa} - i_0^{sa}$ and $b^+_1 = b'_1 + b^+_1$. Introduce for $0 \leq k \leq n$ the graded subquiver $N_k \subset T^n (s\mathcal{C} \oplus s^2\mathbb{R})$ by

$$N_k = \bigoplus_{l_0 + l_1 + \cdots + l_k = n} T^{p_0} s\mathcal{C} \otimes j^\mathcal{C} \otimes T^{p_1} s\mathcal{C} \otimes \cdots \otimes j^\mathcal{C} \otimes T^{p_k} s\mathcal{C}$$

stable under the differential $d^N_k = \sum_{p + 1 + q = n} 1^{\otimes p} \otimes b'_l \otimes 1^{\otimes q}$, and the graded subquiver $P_l \subset T^n s\mathcal{C}^+$ by

$$P_l = \bigoplus_{l_0 + l_1 + \cdots + l_l = n} T^{p_0} s^{su} \otimes j^\mathcal{C} \otimes T^{p_1} s^{su} \otimes \cdots \otimes j^\mathcal{C} \otimes T^{p_l} s^{su}.$$

There is also the subquiver

$$Q_k = \bigoplus_{l=0}^{k} P_l \subset T^n s\mathcal{C}^+$$

and its complement

$$Q\perp_k = \bigoplus_{l=k+1}^{n} P_l \subset T^n s\mathcal{C}^+.$$

Notice that the subquiver $Q_k$ is stable under the differential $d^Q_k = \sum_{p + 1 + q = n} 1^{\otimes p} \otimes b^+_l \otimes 1^{\otimes q}$, and $Q\perp_k$ is stable under the differential $d^Q\perp_k = \sum_{p + 1 + q = n} 1^{\otimes p} \otimes b'_l \otimes 1^{\otimes q}$. Furthermore, the image of $1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c} : N_k \to T^n s\mathcal{C}^+$ is contained in $Q_{k-1}$ for all $a, c \geq 0$ such that $a + 1 + c = n$.

Firstly, the components $b^+_n$, $\phi^+_n$ are defined on the graded subquivers $N_0 = T^n s\mathcal{C}$ and $Q_0 = T^n s^{su}$. Assume for an integer $0 < k \leq n$ that restrictions of $b^+_n$, $\phi^+_n$ to $N_l$ are already
found for all \( l < k \). In other terms, we are given \( b^+_n : Q_{k-1} \to s\mathcal{C}^+ \), \( \phi^+_n : Q_{k-1} \to s\mathcal{D} \) such that equations (3.2), (3.3) hold on \( Q_{k-1} \). Let us construct the restrictions \( b^+_n : N_k \to s\mathcal{C} \), \( \phi^+_n : N_k \to s\mathcal{D} \), performing the induction step.

Introduce a (1,1)-coderivation \( \tilde{b} : Ts\mathcal{C}^+ \to Ts\mathcal{C}^+ \) of degree 1 by its components
\[
(0, b^+_1, \ldots, b^+_{n-1}, pr_{Q_{k-1}}b^+_n|_{Q_{k-1}}, 0, \ldots).
\]
Introduce also a morphism of augmented graded cocategories \( \phi : Ts\mathcal{C}^+ \to Ts\mathcal{D}^+ \) with \( \text{Ob} \tilde{\phi} = \text{Ob} \phi \) by its components
\[
(\phi^+_1, \ldots, \phi^+_{n-1}, pr_{Q_{k-1}} \cdot \phi^+_n|_{Q_{k-1}}, 0, \ldots).
\]
Here \( pr_{Q_{k-1}} : T^n s\mathcal{C}^+ \to Q_{k-1} \) is the natural projection, vanishing on \( Q_{k-1}^\perp \). Then \( \lambda \overset{\text{def}}{=} \tilde{b}^2 : Ts\mathcal{C}^+ \to Ts\mathcal{C}^+ \) is a (1,1)-coderivation of degree 2 and \( \nu \overset{\text{def}}{=} -\tilde{\phi}b^+ + \tilde{b}\tilde{\phi} : Ts\mathcal{C}^+ \to Ts\mathcal{D}^+ \) is a \((\tilde{\phi}, \tilde{\phi})\)-coderivation of degree 1. Equations (3.2), (3.3) imply that \( \lambda_m = 0, \nu_m = 0 \) for \( m < n \). Moreover, \( \lambda_n, \nu_n \) vanish on \( Q_{k-1} \). On the complement the \( n \)-th components equal
\[
\lambda_n = \sum_{1 < r < n} (1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c})b^+_{a+1+c} + \sum_{a+1+c = n} (1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c})b^+_n : Q_{k-1}^\perp \to s\mathcal{C}^+,
\]
\[
\nu_n = - \sum_{1 < r < n} (\phi^+_1 \otimes \cdots \otimes \phi^+_r) b^+_r + \sum_{a+1+c = n} (1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c}) \phi^+_n + \\
\sum_{a+1+c = n} (1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c}) \tilde{\phi}^+_n : Q_{k-1}^\perp \to s\mathcal{D}.
\]
The restriction \( \lambda_n|_{N_k} \) takes values in \( s\mathcal{C} \). Indeed, for the first sum in the expression for \( \lambda_n \) this follows by the induction assumption since \( r > 1 \) and \( a + 1 + c > 1 \). For the second sum this follows by the induction assumption and strict unitality if \( n > 2 \). In the case of \( n = 2, k = 1 \) this is also straightforward. The only case which requires computation is \( n = 2, k = 2 \):
\[
(j^c \otimes j^c)(1 \otimes b^-_1 + b^-_1 \otimes 1) \tilde{b} = j^c - (j^c \otimes j^c) b^-_2 = j^c - (j^c \otimes j^c) b^+_2,
\]
which belongs to \( s\mathcal{C} \) by the induction assumption.

Equations (3.2), (3.3) for \( m = n \) take the form
\[
-b^+_n b^+_1 - \sum_{a+1+c = n} (1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c}) b^+_n = \lambda_n : N_k \to s\mathcal{C}, \quad (3.4)
\]
\[
\phi^+_n b^+_1 - \sum_{a+1+c = n} (1^{\otimes a} \otimes b^+_1 \otimes 1^{\otimes c}) \phi^+_n - b^+_n \phi^+_1 = \nu_n : N_k \to s\mathcal{D}. \quad (3.5)
\]
For arbitrary objects \( X, Y \) of \( \mathcal{C} \), equip the graded \( k \)-module \( N_k(X, Y) \) with the differential
\[
d^{N_k} = \sum_{p+q = n} 1^{\otimes p} \otimes b^+_1 \otimes 1^{\otimes q} \]
and denote by \( u \) the chain map
\[
C_k(N_k(X, Y), s\mathcal{C}(X, Y)) \to C_k(N_k(X, Y), s\mathcal{D}(X, Y)), \quad \lambda \mapsto \lambda \phi^+_1.
\]
Since \( \phi^+_1 \) is homotopy invertible, the map \( u \) is homotopy invertible as well. Therefore, the complex \( \text{Cone}(u) \) is contractible, e.g. by [8, Lemma B.1], in particular, acyclic. Equations
(3.4) and (3.5) have the form $-b_n^+d = \lambda_n, \phi_n^+d + b_n^+u = \nu_n$, that is, the element $(\lambda_n, \nu_n)$ of 
\[ C^2_0(N_k(X, Y), sC(X, Y)) \oplus C^1_0(N_k(X, Y), sD(X\phi, Y\phi)) = \text{Cone}^1(u) \]
has to be the boundary of the sought element $(b_n^+, \phi_n^+)$ of 
\[ C^1_0(N_k(X, Y), sC(X, Y)) \oplus C^0_0(N_k(X, Y), sD(X\phi, Y\phi)) = \text{Cone}^0(u). \]
These equations are solvable because $(\lambda_n, \nu_n)$ is a cycle in Cone$^1(u)$. Indeed, the equations to verify $-\lambda_n d = 0, \nu_n d + \lambda_n u = 0$ take the form 
$$-\lambda_n b_1 + \sum_{p+1+q=n} (1^{\otimes p} \otimes b_1^{\otimes q})\lambda_n = 0 : N_k \rightarrow sC,$$
$$\nu_n b_1 + \sum_{p+1+q=n} (1^{\otimes p} \otimes b_1^{\otimes q})\nu_n - \lambda_n \phi_1 = 0 : N_k \rightarrow sD.$$ 
Composing the identity $-\lambda\bar{b} + b\lambda = 0 : T^n sC^+ \rightarrow T^r sC^+$ with the projection $pr_1 : T^r sC^+ \rightarrow sC^+$ yields the first equation. The second equation follows by composing the identity $\nu b^+ + b\nu - \lambda \phi = 0 : T^r sC^+ \rightarrow T^r sD^+$ with $pr_1 : T^r sD^+ \rightarrow sD^+$. 
Thus, the required restrictions of $b_n^+, \phi_n^+$ to $N_k$ (and to $Q_k$) exist and satisfy the required equations. We proceed by induction increasing $k$ from 0 to $n$ and determining $b_n^+, \phi_n^+$ on the whole $Q_n = T^n sC^+$. Then we replace $n$ with $n+1$ and start again from $T^{n+1} sC$. Thus the induction on $n$ goes through. 

3.8. Remark. Let $(\mathcal{C}^+, b^+)$ be a homotopy unital structure of an $A_\infty$-category $\mathcal{C}$. Then the embedding $A_\infty$-functor $\iota : \mathcal{C} \rightarrow \mathcal{C}^+$ is an equivalence. Indeed, it is bijective on objects. By [3], Theorem 8.8 it suffices to prove that $\iota_1 : s\mathcal{C} \rightarrow s\mathcal{C}^+$ is homotopy invertible. And indeed, the chain quiver map $\pi_1 : s\mathcal{C}^+ \rightarrow s\mathcal{C}, \pi_1|_{s\mathcal{C}} = \text{id}, x_i^{0^+} \pi_1 = x_i^{0}, j_X^+ \pi_1 = 0$, is homotopy inverse to $\iota_1$. Namely, the homotopy $h : s\mathcal{C}^+ \rightarrow s\mathcal{C}^+, h|_{s\mathcal{C}} = 0, x_i^{0^+} h = j_X^+$, $j_X^+ h = 0$, satisfies the equation $\text{id}_{s\mathcal{C}^+} - \pi_1 \cdot \iota_1 = \bar{b}_1^+ + b_1^+ h$.

The equation between $A_\infty$-functors 
\[ [\mathcal{C} \xrightarrow{\phi^+} \mathcal{D}^+] = [\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\iota^0} \mathcal{D}^+] \]
obtained in the proof of Theorem 3.7 implies that $\phi^+$ is an $A_\infty$-equivalence as well. In particular, $\phi_1^+$ is homotopy invertible.

The converse of Proposition 3.4 holds true as well, however its proof requires more preliminaries. It is deferred until Section 3.

4. DOUBLE CODERIVATIONS

4.1. Definition. For $A_\infty$-functors $f, g : A \rightarrow B$, a double $(f, g)$-coderivation of degree $d$ is a system of $k$-linear maps 
\[ r : (T^sA \otimes T^sA)(X, Y) \rightarrow T^sB(Xf, Yg), \quad X, Y \in \text{Ob} A, \]
of degree $d$ such that the equation 
\[ r\Delta_0 = (\Delta_0 \otimes 1)(f \otimes r) + (1 \otimes \Delta_0)(r \otimes g) \quad (4.1) \]
holds true.
Equation \((1.1)\) implies that \(r\) is determined by a system of \(k\)-linear maps \(r \, \text{pr}_1 : T^m \mathcal{A} \otimes T^n \mathcal{A} \to s \mathcal{B}\) with components of degree \(d\)

\[
r_{n,m} : s \mathcal{A}(X_0, X_1) \otimes \cdots \otimes s \mathcal{A}(X_{n+m-1}, X_{n+m}) \to s \mathcal{B}(X_0 f, X_{n+m} g),
\]

for \(n, m \geq 0\), via the formula

\[
r_{n,m;k} = (r|_{T^n \mathcal{A} \otimes T^m \mathcal{A}}) \, \text{pr}_k : T^n s \mathcal{A} \otimes T^m s \mathcal{A} \to T^k s \mathcal{B},
\]

\[
r_{n,m;k} = \sum_{i_1 + \cdots + i_p + i = n, j_1 + \cdots + j_q + j = m} f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \cdots \otimes g_{j_q}.
\]

\[(4.2)\]

This follows from the equation

\[
r \Delta_0^{(l)} = \sum_{p+1+q = l} (\Delta_0^{p+1} \otimes \Delta_0^{q+1})(f \otimes r \otimes g^{\otimes q}) : T^m \mathcal{A} \otimes T^n \mathcal{A} \to (T s \mathcal{B})^{\otimes l},
\]

\[(4.3)\]

which holds true for each \(l \geq 0\). Here \(\Delta_0^{(0)} = \varepsilon, \Delta_0^{(1)} = \text{id}, \Delta_0^{(2)} = \Delta_0\) and \(\Delta_0^{(l)}\) means the cut comultiplication iterated \(l - 1\) times.

Double \((f, g)\)-coderivations form a chain complex, which we are going to denote by \((\mathcal{D}(\mathcal{A}, \mathcal{B}))(f, g), B_1\). For each \(d \in \mathbb{Z}\), the component \(\mathcal{D}(\mathcal{A}, \mathcal{B}))(f, g)^d\) consists of double \((f, g)\)-coderivations of degree \(d\). The differential \(B_1\) of degree 1 is given by

\[
r B_1 \overset{\text{def}}{=} rb - (-)^d (1 \otimes b + b \otimes 1) r,
\]

for each \(r \in \mathcal{D}(\mathcal{A}, \mathcal{B}))(f, g)^d\). The component \([r B_1]_{n,m}\) of \(r B_1\) is given by

\[
\sum_{i_1 + \cdots + i_p + i = n, j_1 + \cdots + j_q + j = m} (f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \cdots \otimes g_{j_q}) b_{p+1+q}
\]

\[- (-)^r \sum_{a+k+c = n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m}) r_{a+1+c,m}
\]

\[- (-)^r \sum_{u+i+v = m} (1^{\otimes n+u} \otimes b_i \otimes 1^{\otimes v}) r_{n,u+1+v},
\]

\[(4.4)\]

for each \(n, m \geq 0\). An \(A_\infty\)-functor \(h : \mathcal{B} \to \mathcal{C}\) gives rise to a chain map

\[
\mathcal{D}(\mathcal{A}, \mathcal{B}))(f, g) \to \mathcal{D}(\mathcal{A}, \mathcal{C}))(fh, gh), \quad r \mapsto rh.
\]

The component \([r h]_{n,m}\) of \(rh\) is given by

\[
\sum_{i_1 + \cdots + i_p + i = n, j_1 + \cdots + j_q + j = m} (f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \cdots \otimes g_{j_q}) h_{p+1+q},
\]

\[(4.5)\]

for each \(n, m \geq 0\). Similarly, an \(A_\infty\)-functor \(k : \mathcal{D} \to \mathcal{A}\) gives rise to a chain map

\[
\mathcal{D}(\mathcal{A}, \mathcal{B}))(f, g) \to \mathcal{D}(\mathcal{D}, \mathcal{B}))(kf, kg), \quad r \mapsto (k \otimes k)r.
\]
The component $[(k \otimes k)r]_{n,m}$ of $(k \otimes k)r$ is given by
\[
\sum_{i_1 + \ldots + i_p = n, j_1 + \ldots + j_q = m} (k_{i_1} \otimes \cdots \otimes k_{i_p} \otimes k_{j_1} \otimes \cdots \otimes k_{j_q})r_{p,q},
\]
for each $n, m \geq 0$. Proofs of these facts are elementary and are left to the reader.

Let $\mathcal{C}$ be an $A_\infty$-category. For each $n \geq 0$, introduce a morphism
\[
\nu_n = \sum_{i=0}^{n} (-)^{n-i}(1^\otimes i \otimes \varepsilon \otimes 1^{\otimes n-i}) : (Ts\mathcal{C})^{\otimes n+1} \to (Ts\mathcal{C})^{\otimes n},
\]
in $\mathcal{Q}/\text{Ob}\mathcal{C}$. In particular, $\nu_0 = \varepsilon : Ts\mathcal{C} \to \mathbb{k}\text{Ob}\mathcal{C}$. Denote $\nu = \nu_1 = (1 \otimes \varepsilon) - (\varepsilon \otimes 1) : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}$ for the sake of brevity.

4.2. Lemma. The map $\nu : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}$ is a double $(1, 1)$-coderivation of degree 0 and $\nu B_1 = 0$.

Proof. We have:
\[
(\Delta_0 \otimes 1)(1 \otimes \nu) + (1 \otimes \Delta_0)(\nu \otimes 1) = (\Delta_0 \otimes 1)(1 \otimes 1 \otimes \varepsilon) - (\Delta_0 \otimes 1)(1 \otimes \varepsilon \otimes 1) + (1 \otimes \Delta_0)(1 \otimes \varepsilon \otimes 1) - (1 \otimes \Delta_0)(\varepsilon \otimes 1 \otimes 1)
\]
due to the identities
\[
(\Delta_0 \otimes 1)(1 \otimes \varepsilon \otimes 1) = 1 \otimes 1 = (1 \otimes \Delta_0)(1 \otimes \varepsilon \otimes 1) : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}.
\]

This computation shows that $\nu : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}$ is a double $(1, 1)$-coderivation. Its only non-vanishing components are $X,Y\nu_{1,0} = 1 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ and $X,Y\nu_{0,1} = 1 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$, $X,Y \in \text{Ob}\mathcal{C}$.

Since $\nu B_1$ is a double $(1, 1)$-coderivation of degree 1, the equation $\nu B_1 = 0$ is equivalent to its particular case $\nu B_1 \text{pr}_1 = 0$, i.e., for each $n, m \geq 0$
\[
\sum_{0 \leq i \leq n, \ 0 \leq j \leq m} (1^{\otimes n-i} \otimes \nu_{i,j} \otimes 1^{\otimes m-j})b_{n-i+1+m-j} - \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m})\nu_{a+1+c,m}
\]
\[- \sum_{a+c+l+v=m} (1^{\otimes a+u} \otimes b_t \otimes 1^{\otimes v})\nu_{n,u+1+v} = 0 : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \to s\mathcal{C}.
\]

It reduces to the identity
\[
\chi(n > 0)b_{n+m} - \chi(m > 0)b_{n+m} - \chi(m = 0)b_n + \chi(n = 0)b_m = 0,
\]
where $\chi(P) = 1$ if a condition $P$ holds and $\chi(P) = 0$ if $P$ does not hold. \hfill \Box

Let $\mathcal{C}$ be a strictly unital $A_\infty$-category. The strict unit $i_0^c \mathcal{C}$ is viewed as a morphism of graded quivers $i_0^c : \mathbb{k}\text{Ob}\mathcal{C} \to s\mathcal{C}$ of degree $-1$, identity on objects. For each $n \geq 0$, introduce a morphism of graded quivers
\[
\xi_n = [(Ts\mathcal{C})^{\otimes n+1} \xrightarrow{1 \otimes i_0^c \mathcal{C} \otimes \cdots \otimes 1} Ts\mathcal{C} \otimes s\mathcal{C} \otimes Ts\mathcal{C} \otimes \cdots \otimes s\mathcal{C} \otimes Ts\mathcal{C} \xrightarrow{\mu(2n+1)} Ts\mathcal{C}],
\]
of degree \(-n\), identity on objects. Here \(\mu^{(2n+1)}\) denotes composition of \(2n + 1\) composable arrows in the graded category \(Ts\mathcal{C}\). In particular, \(\xi_0 = 1 : Ts\mathcal{C} \to Ts\mathcal{C}\). Denote \(\xi = \xi_1 = (1 \otimes i_0^c \otimes 1)\mu^{(3)} : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}\) for the sake of brevity.

4.3. Lemma. The map \(\xi : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}\) is a double \((1, 1)\)-coderivation of degree \(-1\) and \(\xi B_1 = \nu\).

Proof. The following identity follows directly from the definitions of \(\mu\) and \(\Delta_0\):

\[
\mu \Delta_0 = (\Delta_0 \otimes 1)(1 \otimes \mu) + (1 \otimes \Delta_0)(\mu \otimes 1) - 1 : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}.
\]

It implies

\[
\mu^{(3)} \Delta_0 = (\Delta_0 \otimes 1 \otimes 1)(1 \otimes \mu^{(3)}) + (1 \otimes 1 \otimes \Delta_0)(\mu^{(3)} \otimes 1) \\
+ (1 \otimes \Delta_0 \otimes 1)(\mu \otimes \mu) - (1 \otimes 1)(\mu \otimes 1) : Ts\mathcal{C} \otimes Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}\quad (4.7)
\]

Since \(i_0^c \Delta_0 = i_0^c \otimes \eta + \eta \otimes i_0^c : \kappa \text{Ob }\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}\), it follows that

\[
(1 \otimes i_0^c \Delta_0 \otimes 1)(\mu \otimes \mu) - (1 \otimes (i_0^c \otimes 1))\mu - ((1 \otimes i_0^c)\mu \otimes 1) = 0 : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}.
\]

Equation (4.7) yields

\[
(1 \otimes i_0^c \otimes 1)\mu^{(3)} \Delta_0 = (\Delta_0 \otimes 1)(1 \otimes (1 \otimes i_0^c \otimes 1)\mu^{(3)}) + (1 \otimes \Delta_0)((1 \otimes i_0^c \otimes 1)\mu^{(3)} \otimes 1),
\]

i.e., \(\xi = (1 \otimes i_0^c \otimes 1)\mu^{(3)} : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}\) is a double \((1, 1)\)-coderivation. Its the only non-vanishing components are \(\chi_{0,0} = \chi i_0^c \in s\mathcal{C}(X, X), X \in \text{Ob }\mathcal{C}\).

Since both \(\xi B_1\) and \(\nu\) are double \((1, 1)\)-coderivations of degree 0, the equation \(\xi B_1 = \nu\) is equivalent to its particular case \(\xi B_1 \text{pr}_1 = \nu \text{pr}_1\), i.e., for each \(n, m \geq 0\)

\[
\sum_{0 \leq p \leq n} \sum_{0 \leq q \leq m} (1^{\otimes n-p} \otimes \xi_{p,q} \otimes 1^{\otimes m-q})b_{n-p+1,m-q} + \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m})\nu_{a+1,c,m}
\]

\[
+ \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v})\nu_{n,u+1,v} = \nu_{n,m} : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \to s\mathcal{C}.
\]

It reduces to the the equation

\[
(1^{\otimes n} \otimes i_0^c \otimes 1^{\otimes m})b_{n+1,m} = \nu_{n,m} : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \to s\mathcal{C},
\]

which holds true, since \(i_0^c\) is a strict unit. \(\square\)

Note that the maps \(\nu_n, \xi_n\) obey the following relations:

\[
\xi_n = (\xi_{n-1} \otimes 1)\xi, \quad \nu_n = (1^{\otimes n} \otimes \varepsilon) - (\nu_{n-1} \otimes 1), \quad n \geq 1. \quad (4.8)
\]

In particular, \(\xi_n \varepsilon = 0 : (Ts\mathcal{C})^{\otimes n+1} \to \kappa \text{Ob }\mathcal{C}\), for each \(n \geq 1\), as \(\xi \varepsilon = 0\) by equation (4.3).
4.4. Lemma. The following equations hold true:

\[ \xi_n \Delta_0 = \sum_{i=0}^{n} (1^\otimes i \otimes \Delta_0 \otimes 1^\otimes n-i)(\xi_i \otimes \xi_{n-i}), \quad n \geq 0, \]  

(4.9)

\[ \xi_n b - (-)^n \sum_{i=0}^{n} (1^\otimes i \otimes b \otimes 1^\otimes n-i)\xi_n = \nu_n \xi_{n-1}, \quad n \geq 1. \]  

(4.10)

Proof. Let us prove (4.9). The proof is by induction on \( n \). The case \( n = 0 \) is trivial. Let \( n \geq 1 \). By (4.8) and Lemma 4.3,

\[ \xi_n \Delta_0 = (\xi_{n-1} \otimes 1)\xi_0 = (\xi_{n-1} \Delta_0 \otimes 1)(1 \otimes \xi) + (\xi_{n-1} \otimes \Delta_0)(\xi \otimes 1). \]

By induction hypothesis,

\[ \xi_{n-1} \Delta_0 = \sum_{i=0}^{n-1} (1^\otimes i \otimes \Delta_0 \otimes 1^\otimes n-1-i)(\xi_i \otimes \xi_{n-1-i}), \]

therefore

\[ \xi_n \Delta_0 = \sum_{i=0}^{n-1} (1^\otimes i \otimes \Delta_0 \otimes 1^\otimes n-i)(\xi_i \otimes \xi_{n-1-i} \otimes 1)(1 \otimes \xi) + \sum_{i=0}^{n-1} (1^\otimes i \otimes \Delta_0 \otimes 1^\otimes n-i)(\xi_i \otimes \xi_{n-i}), \]

since \( (\xi_{n-1-i} \otimes 1)\xi = \xi_{n-i} \) if \( 0 \leq i \leq n-1 \).

Let us prove (4.10). The proof is by induction on \( n \). The case \( n = 1 \) follows from Lemma 4.3. Let \( n \geq 2 \). By (4.8) and Lemma 4.3,

\[ \xi_n b - (-)^n \sum_{i=0}^{n} (1^\otimes i \otimes b \otimes 1^\otimes n-i)\xi_n \]

\[ = (\xi_{n-1} \otimes 1)\xi b - (-)^n \sum_{i=0}^{n-1} ((1^\otimes i \otimes b \otimes 1^\otimes n-1-i)\xi_{n-1} \otimes 1)\xi - (-)^n (1^\otimes n \otimes b)(\xi_{n-1} \otimes 1)\xi \]

\[ = - (\xi_{n-1} b \otimes 1)\xi - (\xi_{n-1} \otimes b)\xi + (\xi_{n-1} \otimes 1)\nu \]

\[ + (-)^{n-1} \sum_{i=0}^{n-1} ((1^\otimes i \otimes b \otimes 1^\otimes n-1-i)\xi_{n-1} \otimes 1)\xi + (\xi_{n-1} \otimes b)\xi \]

\[ = (\xi_{n-1} \otimes 1)\nu - \left( [\xi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^\otimes i \otimes b \otimes 1^\otimes n-1-i)\xi_{n-1}] \otimes 1 \right) \xi. \]

By induction hypothesis

\[ \xi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^\otimes i \otimes b \otimes 1^\otimes n-1-i)\xi_{n-1} = \nu_{n-1} \xi_{n-2}, \]
therefore
\[ \xi_n b - (-)^n \sum_{i=0}^n (1 \otimes i \otimes b \otimes 1 \otimes \cdots \otimes 1) \xi_n = (\xi_{n-1} \otimes 1) \nu - (\nu_{n-1} \xi_{n-2} \otimes 1) \xi. \]

Since by (4.8),
\[ (\xi_{n-1} \otimes 1) \nu - (\nu_{n-1} \xi_{n-2} \otimes 1) \xi = (\xi_{n-1} \otimes \varepsilon) - (\xi_{n-1} \xi) - (\nu_{n-1} \otimes 1) \xi_{n-1} \]
\[ = (1 \otimes \varepsilon) \xi_{n-1} - (\nu_{n-1} \otimes 1) \xi_{n-1} = \nu \xi, \]
equation (4.10) is proven. \( \square \)

5. AN AUGMENTED DIFFERENTIAL GRADED COCATEGORY

Let now \( \mathcal{C} = \mathcal{A}^{\text{su}} \), where \( \mathcal{A} \) is an \( A_{\infty} \)-category. There is an isomorphism of graded \( k \)-quivers, identity on objects:
\[ \zeta : \bigoplus_{n \geq 0} (T s \mathcal{A})^{\otimes n+1}[n] \rightarrow T s \mathcal{A}^{\text{su}}. \]
The morphism \( \zeta \) is the sum of morphisms
\[ \zeta_n = [(T s \mathcal{A})^{\otimes n+1}[n] \xrightarrow{s^{-n}} (T s \mathcal{A})^{\otimes n+1} \xrightarrow{e^{\otimes n+1}} (T s \mathcal{A}^{\text{su}})^{\otimes n+1} \xrightarrow{\xi_n} T s \mathcal{A}^{\text{su}}], \] (5.1)
where \( e : \mathcal{A} \hookrightarrow \mathcal{A}^{\text{su}} \) is the natural embedding. The graded quiver
\[ \mathcal{E} \overset{\text{def}}{=} \bigoplus_{n \geq 0} (T s \mathcal{A})^{\otimes n+1}[n] \]
adopts a unique structure of an augmented differential graded cocategory such that \( \zeta \) becomes an isomorphism of augmented differential graded cocategories. The comultiplication \( \Delta : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{E} \) is found from the equation
\[ \left[ \mathcal{E} \xrightarrow{\zeta} T s \mathcal{A}^{\text{su}} \xrightarrow{\Delta_{\mathcal{A}}} T s \mathcal{A}^{\text{su}} \otimes T s \mathcal{A}^{\text{su}} \right] = \left[ \mathcal{E} \xrightarrow{\Delta} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\zeta \otimes \zeta} T s \mathcal{A}^{\text{su}} \otimes T s \mathcal{A}^{\text{su}} \right]. \]
Restricting the left hand side of the equation to the summand \( (T s \mathcal{A})^{\otimes n+1}[n] \) of \( \mathcal{E} \), we obtain
\[ \zeta_n \Delta_0 = s^{-n} e^{\otimes n+1} \xi_n \Delta_0 \]
\[ = s^{-n} \sum_{i=0}^n (e^{\otimes i} \otimes e \Delta_0 \otimes e^{\otimes n-i}) (\xi \otimes \xi_{n-i}) : (T s \mathcal{A})^{\otimes n+1}[n] \rightarrow T s \mathcal{A}^{\text{su}} \otimes T s \mathcal{A}^{\text{su}}, \]
by equation (4.8). Since \( e \) is a morphism of augmented graded cocategories, it follows that
\[ \zeta_n \Delta_0 = s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (e^{\otimes i+1} \xi_i \otimes e^{\otimes n-i+1} \xi_n - i) \]
\[ = s^{-n} \sum_{i=0}^n (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i}) (s^i \otimes s^{n-i}) (\xi_i \otimes \xi_n - i) : (T s \mathcal{A})^{\otimes n+1}[n] \rightarrow T s \mathcal{A}^{\text{su}} \otimes T s \mathcal{A}^{\text{su}}. \]
This implies the following formula for $\tilde{\Delta}$:

$$\tilde{\Delta} |_{(TsA)^{\otimes n+1}[n]} = s^{-n} \sum_{i=0}^{n} (1^\otimes i \otimes \Delta_0 \otimes 1^\otimes n-i)(s^i \otimes s^{n-i}) : (TsA)^{\otimes n+1}[n] \to \bigoplus_{i=0}^{n} (TsA)^{\otimes i+1}[i] \otimes (TsA)^{\otimes n-i+1}[n-i].$$

(5.2)

The counit of $\mathcal{E}$ is $\tilde{\varepsilon} = [\mathcal{E} \xrightarrow{pr_n} TsA \xrightarrow{\varepsilon} \mathbb{k} \text{Ob} A = \mathbb{k} \text{Ob} \mathcal{E}]$. The augmentation of $\mathcal{E}$ is $\tilde{\eta} = [\mathbb{k} \text{Ob} \mathcal{E} = \mathbb{k} \text{Ob} A \xrightarrow{\eta} \xrightarrow{T \text{Ob} A} \xrightarrow{\text{in}} \mathcal{E}]$. The differential $\tilde{b} : \mathcal{E} \to \mathcal{E}$ is found from the following equation:

$$[\mathcal{E} \xrightarrow{\tilde{\varepsilon}} TsA^{su} \xrightarrow{\tilde{b}} TsA^{su}] = [\mathcal{E} \xrightarrow{\tilde{b}} \xrightarrow{\tilde{\varepsilon}} TsA^{su}].$$

Let $\tilde{b}_{n,m} : (TsA)^{\otimes n+1}[n] \to (TsA)^{\otimes m+1}[m]$, $n, m \geq 0$, denote the matrix coefficients of $\tilde{b}$. Restricting the left hand side of the above equation to the summand $(TsA)^{\otimes n+1}[n]$ of $\mathcal{E}$, we obtain

$$\zeta_n b = s^{-n} e^{\otimes n+1} \zeta_n b = s^{-n} e^{\otimes n+1} \nu_n \xi_{n-1} + (-)^n s^{-n} \sum_{i=0}^{n} (e^i \otimes eb \otimes e^{\otimes n-i}) \xi_n : (TsA)^{\otimes n+1}[n] \to TsA^{su},$$

by equation (5.1). Since $e$ preserves the counit, it follows that

$$e^{\otimes n+1} \nu_n = \nu_n e^{\otimes n} : (TsA)^{\otimes n+1} \to (TsA^{su})^{\otimes n}.$$  

Furthermore, $e$ commutes with the differential $b$, therefore

$$\zeta_n b = s^{-n} \nu_n s^{-1} (s^{-(n-1)} e^{\otimes n} \xi_{n-1}) + (-)^n s^{-n} \sum_{i=0}^{n} (1^\otimes i \otimes b \otimes 1^\otimes n-i)s^n (s^{-n} e^{\otimes n+1} \xi_n)$$

$$= s^{-n} \nu_n s^{-1} \xi_{n-1} + (-)^n s^{-n} \sum_{i=0}^{n} (1^\otimes i \otimes b \otimes 1^\otimes n-i)s^n \xi_n : (TsA)^{\otimes n+1}[n] \to TsA^{su}.$$  

We conclude that

$$\tilde{b}_{n,n} = (-)^n s^{-n} \sum_{i=0}^{n} (1^\otimes i \otimes b \otimes 1^\otimes n-i)s^n : (TsA)^{\otimes n+1}[n] \to (TsA)^{\otimes n+1}[n],$$

(5.3)

for $n \geq 0$, and

$$\tilde{b}_{n,n-1} = s^{-n} \nu_n s^{-1} : (TsA)^{\otimes n+1}[n] \to (TsA)^{\otimes n}[n-1],$$

(5.4)

for $n \geq 1$, are the only non-vanishing matrix coefficients of $\tilde{b}$.

Let $g : \mathcal{E} \to TsB$ be a morphism of augmented differential graded cocategories, and let $g_n : (TsA)^{\otimes n+1}[n] \to TsB$ be its components. By formula (5.2), the equation $g\Delta_0 =$
\( \tilde{\Delta}(g \otimes g) \) is equivalent to the system of equations
\[
g_n \Delta_0 = s^{-n} \sum_{i=0}^{n} (1 \otimes \Delta_0 \otimes 1^{\otimes n-i}) (s^n g_i \otimes s^{n-i} g_{n-i}) : (T_sA)^{\otimes n+1}[n] \to T_sB \otimes T_sB,
\]
for \( n \geq 0 \).

The equation \( g\varepsilon = \varepsilon(k \text{Ob } g) \) is equivalent to the equations \( g_0 \varepsilon = \varepsilon(k \text{Ob } g_0), \ g_n \varepsilon = 0, \ n \geq 1 \). The equation \( \tilde{\eta}g = (k \text{Ob } g)\eta \) is equivalent to the equation \( \eta g_0 = (k \text{Ob } g_0)\eta \). By formulas (5.3) and (5.4), the equation \( gb = \tilde{\eta}g \) is equivalent to \( g_0b = b g_0 : T_sA \to T_sB \) and
\[
g_n b = (-)^n s^{-n} \sum_{i=0}^{n} (1 \otimes b \otimes 1^{\otimes n-i}) s^n g_i + s^{-n} \nu_n s^{n-1} g_{n-1} : (T_sA)^{\otimes n+1}[n] \to T_sB,
\]
for \( n \geq 1 \).

Introduce \( k \)-linear maps \( \phi_n = s^n g_n : (T_sA)^{\otimes n+1}(X, Y) \to T_sB(Xg, Yg) \) of degree \( -n \), \( X, Y \in \text{Ob } A, n \geq 0 \). The above equations take the following form:
\[
\phi_n \Delta_0 = \sum_{i=0}^{n} (1 \otimes \Delta_0 \otimes 1^{\otimes n-i}) (\phi_i \otimes \phi_{n-i}) : (T_sA)^{\otimes n+1} \to T_sB \otimes T_sB,
\]
for \( n \geq 1 \);
\[
\phi_n b = (-)^n \sum_{i=0}^{n} (1 \otimes b \otimes 1^{\otimes n-i}) \phi_i + \nu_n \phi_{n-1} : (T_sA)^{\otimes n+1} \to T_sB,
\]
for \( n \geq 1 \);
\[
\phi_0 \Delta_0 = \Delta_0(\phi_0 \otimes \phi_0), \quad \phi_0 \varepsilon = \varepsilon, \quad \phi_0 b = b \phi_0,
\]
\[
\phi_n \varepsilon = 0, \quad n \geq 1.
\]

Summing up, we conclude that morphisms of augmented differential graded cocategories \( \mathcal{E} \to T_s \mathcal{B} \) are in bijection with collections consisting of a morphism of augmented differential graded cocategories \( \phi_0 : T_sA \to T_sB \) and of \( k \)-linear maps \( \phi_n : (T_sA)^{\otimes n+1}(X, Y) \to T_sB(Xg, Yg) \) of degree \( -n \), \( X, Y \in \text{Ob } A, n \geq 1 \), such that equations (5.3), (5.4), and (5.5) hold true.

In particular, \( A_\infty \)-functors \( f : A_{su} \to B \), which are augmented differential graded cocategory morphisms \( T_s A_{su} \to T_s B \), are in bijection with morphisms \( g = \zeta f : \mathcal{E} \to T_s \mathcal{B} \) of augmented differential graded cocategories. With the above notation, we may say that to give an \( A_\infty \)-functor \( f : A_{su} \to B \) is the same as to give an \( A_\infty \)-functor \( \phi_0 : A \to B \) and a system of \( k \)-linear maps \( \phi_n : (T_sA)^{\otimes n+1}(X, Y) \to T_sB(Xg, Yg) \) of degree \( -n \), \( X, Y \in \text{Ob } A, n \geq 1 \), such that equations (5.3), (5.4) and (5.5) hold true.

5.1. Proposition. The following conditions are equivalent.

(a) There exists an \( A_\infty \)-functor \( U : A_{su} \to A \) such that
\[
[A \xrightarrow{\varepsilon} A_{su} \xrightarrow{U} A] = \text{id}_A.
\]
(b) There exists a double \((1, 1)\)-coderivation \(\phi : TsA \otimes TsA \to TsA\) of degree \(-1\) such that \(\phi B_1 = \nu\).

\textit{Proof.} (a)⇒(b) Let \(U : A^{su} \to A\) be an \(A_\infty\)-functor such that \(eU = \text{id}_A\), in particular \(\text{Ob}U = \text{id} : \text{Ob}A^{su} = \text{Ob}A \to \text{Ob}A\). It gives rise to the family of \(k\)-linear maps \(\phi_n = s^n\zeta_n U : (TsA)^{\otimes n+1}(X, Y) \to Ts\mathcal{B}(X, Y)\) of degree \(-n\), \(X, Y \in \text{Ob}A\), \(n \geq 0\), that satisfy equations (5.5), (5.6) and (5.8). In particular, \(\phi_0 = eU = \text{id}_A\). Equations (5.5) and (5.8) for \(n = 1\) read as follows:

\[
\phi_1\Delta_0 = (\Delta_0 \otimes 1)(\phi_0 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes \phi_0) = (\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes 1),
\]

\[
\phi_1 b = (1 \otimes b + b \otimes 1)\phi_1 + \nu_1 \phi_0 = (1 \otimes b + b \otimes 1)\phi_1 + \nu.
\]

In other words, \(\phi_1\) is a double \((1, 1)\)-coderivation of degree \(-1\) and \(\phi_1 B_1 = \nu\).

(b)⇒(a) Let \(\phi : TsA \otimes TsA \to TsA\) be a double \((1, 1)\)-coderivation of degree \(-1\) such that \(\phi B_1 = \nu\). Define \(k\)-linear maps

\[
\phi_n : (TsA)^{\otimes n+1}(X, Y) \to TsA(X, Y), \quad X, Y \in \text{Ob}A,
\]

of degree \(-n\), \(n \geq 0\), recursively via \(\phi_0 = \text{id}_A\) and \(\phi_n = (\phi_{n-1} \otimes 1)\phi_n\), \(n \geq 1\). Let us show that \(\phi_n\) satisfy equations (5.5), (5.6) and (5.8). Equation (5.8) is obvious: \(\phi_n \varepsilon = (\phi_{n-1} \otimes 1)\phi\varepsilon = 0\) as \(\phi\varepsilon = 0\) by (4.3). Let us prove equation (5.5) by induction. It holds for \(n = 1\) by assumption, since \(\phi_1 = \phi\) is a double \((1, 1)\)-coderivation. Let \(n \geq 2\). We have:

\[
\phi_n\Delta_0 = (\phi_{n-1} \otimes 1)\phi_1\Delta_0
\]

\[
= (\phi_{n-1} \otimes 1)((\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes 1))
\]

\[
= (\phi_{n-1}\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_{n-1} \otimes 1)\phi_1 \otimes 1).
\]

By induction hypothesis,

\[
\phi_{n-1}\Delta_0 = \sum_{i=0}^{n-1}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-1-i})(\phi_i \otimes \phi_{n-1-i}),
\]

so that

\[
\phi_n\Delta_0 = \sum_{i=0}^{n-1}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-1-i})(\phi_i \otimes \phi_{n-1-i} \otimes 1)(1 \otimes \phi_1)
\]

\[
+ (1^{\otimes n} \otimes \Delta_0)((\phi_{n-1} \otimes 1)\phi_1 \otimes 1)
\]

\[
= \sum_{i=0}^{n}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\phi_i \otimes \phi_{n-i}),
\]

since \((\phi_{n-1-i} \otimes 1)\phi_1 = \phi_{n-i}, \ 0 \leq i \leq n - 1\).
Let us prove equation (5.6) by induction. For \( n = 1 \) it is equivalent to the equation \( \phi B_1 = \nu \), which holds by assumption. Let \( n \geq 2 \). We have:

\[
\phi_n b - (-)^n \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n
\]

\[
= (\phi_{n-1} \otimes 1) \phi b - (-)^n \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} \otimes 1) \phi - (-)^n (1^{\otimes n} \otimes b)(\phi_{n-1} \otimes 1) \phi
\]

\[
= - (\phi_{n-1} b \otimes 1) \phi - (\phi_{n-1} \otimes b) \phi + (\phi_{n-1} \otimes 1) \nu
\]

\[
+ (-)^{n-1} \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} \otimes 1) \phi + (\phi_{n-1} \otimes b) \phi
\]

\[
= (\phi_{n-1} \otimes 1) \nu - \left( [\phi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1}] \otimes 1 \right) \phi.
\]

By induction hypothesis,

\[
\phi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i}) \phi_{n-1} = \nu_{n-1} \phi_{n-2},
\]

therefore

\[
\phi_n b - (-)^n \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n = (\phi_{n-1} \otimes 1) \nu - (\nu_{n-1} \phi_{n-2} \otimes 1) \phi.
\]

Since by (4.8)

\[
(\phi_{n-1} \otimes 1) \nu - (\nu_{n-1} \phi_{n-2} \otimes 1) \phi = (\phi_{n-1} \otimes \varepsilon) - (\phi_{n-1} \varepsilon \otimes 1) - (\nu_{n-1} \otimes 1) \phi_{n-1}
\]

\[
= (1^{\otimes n} \otimes \varepsilon) \phi_{n-1} - (\nu_{n-1} \otimes 1) \phi_{n-1} = \nu \phi_{n-1},
\]

and equation (5.6) is proven.

The system of maps \( \phi_n, n \geq 0 \), corresponds to an \( A_\infty \)-functor \( U : A^{su} \to A \) such that \( \phi_n = s^n \zeta_n U, n \geq 0 \). In particular, \( eU = \phi_0 = id_A \). \( \square \)

5.2. Proposition. Let \( A \) be a unital \( A_\infty \)-category. There exists a double \((1, 1)\)-coderivation \( h : TsA \otimes TsA \to TsA \) of degree \(-1\) such that \( hB_1 = \nu \).

Proof. Let \( A \) be a unital \( A_\infty \)-category. By [10, Corollary A.12], there exist a differential graded category \( \mathcal{D} \) and an \( A_\infty \)-equivalence \( f : A \to \mathcal{D} \). The functor \( f \) is unital by [8, Corollary 8.9]. This means that, for every object \( X \) of \( A \), there exists a \( k \)-linear map \( x v_0 : k \to (s\mathcal{D})^{-2}(Xf, Xf) \) such that \( x f_0^A f_1 = x f_0^D + x v_0 b_1 \). Here \( x f_0^D \) denotes the strict unit of the differential graded category \( \mathcal{D} \).

By Lemma 1.3, \( \xi = (1 \otimes i_0^D \otimes 1) \mu : Ts\mathcal{D} \otimes Ts\mathcal{D} \to Ts\mathcal{D} \) is a \((1, 1)\)-coderivation of degree \(-1\). Let \( \iota \) denote the double \((f, f)\)-coderivation \((f \otimes f)\xi\) of degree \(-1\). By Lemma 1.3,

\[
\iota B_1 = (f \otimes f)(\xi B_1) = (f \otimes f) \nu = \nu f.
\]
By Lemma 4.2, the equation $\nu B_1 = 0$ holds true. We conclude that the double coderivations $\nu \in \mathcal{D}(\mathcal{A}, \mathcal{A})(\text{id}_\mathcal{A}, \text{id}_\mathcal{A})^0$ and $\iota \in \mathcal{D}(\mathcal{A}, \mathcal{D})(f, f)^{-1}$ satisfy the following equations:

$$\nu B_1 = 0, \quad (5.9)$$
$$\iota B_1 - \nu f = 0. \quad (5.10)$$

We are going to prove that there exist double coderivations $h \in \mathcal{D}(\mathcal{A}, \mathcal{A})(\text{id}_\mathcal{A}, \text{id}_\mathcal{A})^{-1}$ and $k \in \mathcal{D}(\mathcal{A}, \mathcal{D})(f, f)^{-2}$ such that the following equations hold true:

$$h B_1 = \nu,$$
$$hf = \iota + kB_1.$$

Let us put $\chi h_{0,0} = \chi i^n_0$, $\chi k_{0,0} = \chi v_0$, and construct the other components of $h$ and $k$ by induction. Given an integer $t \geq 0$, assume that we have already found components $h_{p,q}$, $k_{p,q}$ of the sought $h$, $k$, for all pairs $(p, q)$ with $p + q < t$, such that the equations

$$(h B_1 - \nu)_{p,q} = 0 : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{p+q-1}, X_{p+q}) \to s\mathcal{A}(X_0, X_{p+q}), \quad (5.11)$$
$$(k B_1 + \iota - hf)_{p,q} = 0 : s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{p+q-1}, X_{p+q}) \to s\mathcal{D}(X_0 f, X_{p+q} f) \quad (5.12)$$

are satisfied for all pairs $(p, q)$ with $p + q < t$. Introduce double coderivations $\tilde{h} \in \mathcal{D}(\mathcal{A}, \mathcal{A})(\text{id}_\mathcal{A}, \text{id}_\mathcal{A})$ and $\tilde{k} \in \mathcal{D}(\mathcal{A}, \mathcal{D})(f, f)$ of degree $-1$ resp. $-2$ by their components: $\tilde{h}_{p,q} = h_{p,q}$, $\tilde{k}_{p,q} = k_{p,q}$ for $p + q < t$, all the other components vanish. Define a double $(1, 1)$-coderivation $\lambda = \tilde{h} B_1 - \nu$ of degree 0 and a double $(f, f)$-coderivation $\kappa = \tilde{k} B_1 + \iota - \tilde{h} f$ of degree $-1$. Then $\lambda_{p,q} = 0$, $\kappa_{p,q} = 0$ for all $p + q < t$. Let non-negative integers $n$, $m$ satisfy $n + m = t$. The identity $\lambda B_1 = 0$ implies that

$$\lambda_{n,m} b_1 - \sum_{l=1}^{n+m} (1^\otimes l - 1 \otimes b_1 \otimes 1^\otimes n+m-l) \lambda_{n,m} = 0.$$

The $(n, m)$-component of the identity $\kappa B_1 + \lambda f = 0$ gives

$$\kappa_{n,m} b_1 + \sum_{l=1}^{n+m} (1^\otimes l - 1 \otimes b_1 \otimes 1^\otimes n+m-l) \kappa_{n,m} + \lambda_{n,m} f_1 = 0.$$

The chain map $f_1 : \mathcal{A}(X_0, X_{n+m}) \to s\mathcal{D}(X_0 f, X_{n+m} f)$ is homotopy invertible as $f$ is an $\mathcal{A}_\infty$-equivalence. Hence, the chain map $\Phi$ given by

$$\mathcal{C}^\bullet_k(N, s\mathcal{A}(X_0, X_{n+m})) \to \mathcal{C}^\bullet_k(N, s\mathcal{D}(X_0 f, X_{n+m} f)), \quad \lambda \mapsto \lambda f_1,$$

is homotopy invertible for each complex of $k$-modules $N$, in particular, for $N = s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{n+m-1}, X_{n+m})$. Therefore, the complex $\text{Cone}(\Phi)$ is contractible, e.g. by [8, Lemma B.1]. Consider the element $(\lambda_{n,m}, \kappa_{n,m})$ of

$$\mathcal{C}^0_k(N, s\mathcal{A}(X_0, X_{n+m})) \oplus \mathcal{C}^{-1}_k(N, s\mathcal{D}(X_0 f, X_{n+m} f)).$$

The above direct sum coincides with $\text{Cone}^{-1}(\Phi)$. The equations $-\lambda_{n,m} d = 0$, $\kappa_{n,m} d + \lambda_{n,m} \Phi = 0$ imply that $(\lambda_{n,m}, \kappa_{n,m})$ is a cycle in the complex $\text{Cone}(\Phi)$. Due to acyclicity of
Cone(Φ), (λₙ,m, κₙ,m) is a boundary of some element (hₙ,m, −kₙ,m) of Cone⁻²(Φ), i.e., of
\[ C^{-1}_k(N, sA(X_0, X_{n+m})) \oplus C^{-2}_k(N, D(X_0 f, X_{n+m} f)). \]
Thus, −kₙ,m d + hₙ,m f₁ = κₙ,m, −hₙ,m d = λₙ,m. These equations can be written as follows:
\[ -hₙ,m b₁ - \sum_{u+1+v=n+m} (1^{⊗u} b₁ 1^{⊗v}) hₙ,m = (\tilde{h}B₁ - ν)ₙ,m; \]
\[ -kₙ,m b₁ + \sum_{u+1+v=n+m} (1^{⊗u} b₁ 1^{⊗v}) kₙ,m + hₙ,m f₁ = (\tilde{k}B₁ + t - \tilde{h}f)ₙ,m. \]
Thus, if we introduce double coderivations \( \overline{h} \) and \( \overline{k} \) by their components:
\[ \overline{h}_{p,q} = h_{p,q}; \overline{k}_{p,q} = k_{p,q} \]
for \( p + q \leq t \) (using just found maps if \( p + q = t \)) and 0 otherwise, then these coderivations satisfy equations (5.11) and (5.12) for each \( p, q \) such that \( p + q \leq t \).
Induction on \( t \) proves the proposition. \( \square \)

5.3. **Theorem.** Every unital \( A_∞ \)-category admits a weak unit.

**Proof.** The proof follows from Propositions 5.1 and 5.2. \( \square \)

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