A Bombieri–Vinogradov-type result for exponential sums over Piatetski-Shapiro primes

Stoyan Ivanov Dimitrov

Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Bulgaria
(e-mail: sdimitrov@tu-sofia.bg)

Received April 24, 2022; revised September 11, 2022

Abstract. In this paper, we establish a theorem of Bombieri–Vinogradov type for exponential sums over Piatetski-Shapiro primes $p = [n^{1/\gamma}]$ with $865/886 < \gamma < 1$.

MSC: 11L07, 11L20

Keywords: Bombieri–Vinogradov-type result, exponential sum, Piatetski-Shapiro primes

1 Introduction and statement of the result

Let $\mathbb{P}$ denote the set of all prime numbers. In 1953, Piatetski-Shapiro [10] have shown that for any fixed $11/12 < \gamma < 1$, the set

$$\mathbb{P}_\gamma = \{ p \in \mathbb{P} \mid p = [n^{1/\gamma}] \text{ for some } n \in \mathbb{N} \}$$

is infinite. The prime numbers of the form $p = [n^{1/\gamma}]$ are called the Piatetski-Shapiro primes of type $\gamma$. Denote

$$P_\gamma(X) = \sum_{\substack{p \leq X \\atop p = [n^{1/\gamma}]}} 1.$$  

Piatetski-Shapiro states that $P_\gamma(X) \sim X^{\gamma}/\log X$ for $11/12 < \gamma < 1$. The best results up to now belong to Rivat and Sargos [11] with $P_\gamma(X) \sim X^{\gamma}/\log X$ for $2426/2817 < \gamma < 1$ and to Rivat and Wu [12] with $P_\gamma(X) \gg X^{\gamma}/\log X$ for $205/243 < \gamma < 1$. On the other hand, the celebrated Bombieri–Vinogradov theorem is an extremely important result in analytic number theory and has various applications. It concerns the distribution of primes in arithmetic progressions, averaged over a range of moduli and states that for $A > 0$,

$$\sum_{d \leq \sqrt{X}/(\log X)^{A+\delta}} \max_{y \leq X} \max_{(a, d) = 1} \left| \sum_{p \leq y \atop p \equiv a \pmod{d}} \log p - \frac{y}{\varphi(d)} \right| \ll \frac{X}{\log^A X},$$

where $\varphi$ is Euler’s function.
Recently Li, Zhang, and Xue [7] obtained a mean value theorem of Bombieri–Vinogradov type for Piatetski-Shapiro primes. More precisely, they showed that if $85/86 < \gamma < 1$ and $\alpha \neq 0$ is a fixed integer, then for any $A > 0$ and any sufficiently small $\varepsilon > 0$,

$$\sum_{d \leq X^\gamma} \left| \sum_{p \leq X \atop (d,a)=1} 1 - \frac{1}{\varphi(d)} P_\gamma(X) \right| \ll \frac{X^{\gamma}}{\log^A X},$$

where

$$\theta = \frac{129}{4} \gamma - \frac{255}{8} - \varepsilon.$$

Weaker results were previously obtained by Peneva [9], Wang and Cai [13], and Lu [8].

Recently, the author [3] established a new Bombieri–Vinogradov-type result for exponential sums over primes. Let $1 < c < 3$, $c \neq 2$, $0 < \mu < 1$, and $A > 0$. Then for $|t| < X^{1/4-\varepsilon}$, we have the inequality

$$\sum_{d \leq \sqrt{X/(\log X)^{6A+341/3}}} \max_{y \leq X} \max_{(a,d)=1} \left| \sum_{\mu y \leq p \leq y \atop p \equiv a \pmod{d}} e(tp^c) \log p - \frac{1}{\varphi(d)} \int_\mu^y e(tx^c) \, dx \right| \ll \frac{X}{\log^A X}.$$

Motivated by these investigations, in this paper, we establish a new Bombieri–Vinogradov-type result for exponential sums over Piatetski-Shapiro primes. More precisely, we establish the following theorem.

**Theorem 1.** Let $865/886 < \gamma < 1$, $a \in \mathbb{Z} \setminus \{0\}$, $1 < c < 3$, $c \neq 2$, and $A > 0$. Then for $|t| < X^{1/4-\varepsilon}$ and any sufficiently small $\varepsilon > 0$, we have the inequality

$$\sum_{d \leq \sqrt{X/(\log X)^{6A+341/3}}} \sum_{p \leq X \atop p \equiv a \pmod{d}} p^{1-\gamma} e(tp^c) \log p - \frac{\gamma}{\varphi(d)} \int_2^X e(ty^c) \, dy \ll \frac{X}{\log^A X},$$

where

$$\theta = \theta(\gamma) = \frac{443}{55} \gamma - \frac{173}{22} - \varepsilon.$$

**Remark.** From Theorem 1 it follows that $\theta(\gamma) \to 21/110$ as $\gamma \to 1$.

## 2 Notations

Let $X$ be a sufficiently large positive number. By $p$ with or without subscript we will always denote prime numbers. The notation $m \sim M$ means that $m$ runs through the interval $(M, 2M]$. As usual, $\varphi$ is Euler’s function, $\tau(n)$ denotes the number of positive divisors of $n$, and $\Lambda$ is von Mangoldt’s function. By $(d, a)$ we denote the greatest common divisor of $d$ and $a$. Instead of $m \equiv n \pmod{k}$, then for simplicity, we write $m \equiv n(k)$. By $|t|$, $\{t\}$, and $||t||$ we denote the integer part of $t$, the fractional part of $t$, and the distance from $t$ to the nearest integer, respectively. Moreover, $e(t) = \exp(2\pi it)$ and $\psi(t) = \{t\} - 1/2$. The letter $\varepsilon$ denotes an arbitrary small positive number, not the same in all appearances. Throughout this paper, we suppose
that $865/886 < \gamma < 1$. Denote

$$\theta = \frac{443}{55} \gamma - \frac{173}{22} - \varepsilon, \quad (2.1)$$

$$D = X^\theta. \quad (2.2)$$

3 Preliminary lemmas

**Lemma 1.** Let $1 < c < 3$, $c \neq 2$, $0 < \mu < 1$, and $A > 0$. Then for $|t| < X^{1/4-c}$, we have the inequality

$$\sum_{d \leq \sqrt{X}/(\log X)^{(6A+34)/3}} \max_{y \leq X} \max_{(a,d)=1} \left| \sum_{\mu y < p \leq y} e(tp^c) \log p - \frac{1}{\varphi(d)} \int_1^y e(tx^c) \, dx \right| \leq \frac{X}{\log^A X}. \quad (2.2)$$

**Proof.** See [3, Lemma 18]. □

**Lemma 2.** Let $|f^{(m)}(u)| \asymp YX^{1-m}$ for $1 \leq X < u \leq X_1 \leq 2X$ and $m \geq 1$. Then

$$\left| \sum_{X < n \leq X_1} e(f(n)) \right| \leq Y^{\varepsilon}X^A + Y^{-1},$$

where $(\varepsilon, \lambda)$ is any exponent pair.

**Proof.** See [4, Chap. 3]. □

**Lemma 3.** For every $\varepsilon > 0$, $(13/84 + \varepsilon, 55/84 + \varepsilon)$ is an exponent pair.

**Proof.** See [2, Thm. 6].

**Lemma 4.** Let $G$ be an arithmetic function. Then

$$\left| \sum_{n \leq N} A(n)G(n) \right| \leq N^\varepsilon \max \left| \sum_{m \sim N} \sum_{l \sim M} a(m)b(l)G(ml) \right|, \quad (3.1)$$

where the maximum is taken over all bilinear forms with coefficients satisfying

$$M \leq N \quad (3.2)$$

and one of the three cases

$$|a(m)| \leq 1, \quad b(l) = 1, \quad (3.3)$$

$$|a(m)| \leq 1, \quad b(l) = \log l, \quad (3.4)$$

$$|a(m)| \leq 1, \quad |b(l)| \leq 1. \quad (3.5)$$

**Proof.** See [5]. □

The sums in cases (3.3) and (3.4) are called sums of type I and are denoted by $S_I$. The sums in case (3.5) are called sums of type II and are denoted by $S_{II}$. 

Lith. Math. J., 62(4):435–446, 2022.
Lemma 5. Let \( a, b, c \) be real numbers such that

\[
0 < a < 1, \quad b < \frac{2}{3}, \quad 0 < b < c < 1, \quad 1 - c < c - b, \quad 1 - a < \frac{c}{2}.
\]

(3.6)

Then (3.1) still holds when (3.2) is replaced by the conditions

\[
\begin{align*}
M & \leq N^a \quad \text{for type I sums,} \\
N^b & \leq M \leq N^c \quad \text{for type II sums.}
\end{align*}
\]

Proof. See [1, Prop. 1]. □

Lemma 6. Let \( 1/2 < \gamma < 1, \ H \geq 1, \ N \geq 1, \) and \( \Delta > 0. \) Denote by \( \mathcal{N}(\Delta) \) the number of solutions of the inequality

\[
|h_1n_1^\gamma - h_2n_2^\gamma| \leq \Delta, \quad h_1, h_2 \sim H, \ n_1, n_2 \sim N.
\]

Then

\[
\mathcal{N}(\Delta) \ll \Delta H N^{2-\gamma} + H N \log(H N).
\]

Proof. See [6]. □

4 Outline of the proof

We recall that

\[
|t| \leq X^{1/4-c}.
\]

(4.1)

To prove Theorem 1, we need to establish the following estimates of exponential sums:

\[
\sum_{d \leq D} \left| \sum_{p \leq X, p \equiv a \pmod{d}} p^{1-\gamma} \left((p + 1)^\gamma - p^\gamma\right) e\left(tp^\gamma\right) \log p - \frac{\gamma}{\varphi(d)} \int_2^X e\left(yt^\gamma\right) \frac{dy}{y} \right| \ll \frac{X}{\log^A X},
\]

(4.2)

\[
\sum_{d \leq D} \left| \sum_{p \leq X} p^{1-\gamma} \left(\psi\left(-(p + 1)^\gamma\right) - \psi\left(-p^\gamma\right)\right) e\left(tp^\gamma\right) \log p \right| \ll \frac{X}{\log^A X}.
\]

(4.3)

Estimate (4.2) immediately follows from Lemma 1. It remains to prove (4.3). Using the simplest splitting up argument, we obtain that to prove (4.3), it suffices to prove that

\[
\sum_{d \leq D} \left| \sum_{n \sim N \pmod{a}} A(n) e\left(tn^\gamma\right) \left(\psi\left(-(n + 1)^\gamma\right) - \psi\left(-n^\gamma\right)\right) \right| \ll \frac{X^{\gamma}}{\log^A X}
\]

(4.4)

for any \( N \leq X. \) Consider the case where

\[
N \leq X^{1-\varepsilon}.
\]
We have
\[
\sum_{d \leq D \atop (d,a)=1} \left| \sum_{n \sim N \atop n \equiv a \pmod{d}} A(n)e(tn^\gamma) \left( \psi(-n+1) - \psi(n) \right) \right| \\
\ll \sum_{d \leq D \atop (d,a)=1} \left| \sum_{n \sim N \atop n \equiv a \pmod{d}} A(n)e(tn^\gamma) \left( (n+1)^\gamma - n^\gamma \right) \right| \\
+ \sum_{d \leq D \atop (d,a)=1} \left| \sum_{n \sim N \atop n \equiv a \pmod{d}} A(n)e(tn^\gamma) \left( \lfloor n^\gamma \rfloor - \lfloor (n+1)^\gamma \rfloor \right) \right| \\
\ll (\log X) \sum_{n \sim N} n^{\gamma-1}(\tau(n-a) + (\log X) \sum_{n \sim N} \tau(n-a) \\
\ll N^{\gamma+\varepsilon/2} \ll \frac{X^\gamma}{\log^A X}.
\]

Henceforth we assume that
\[X^{1-\varepsilon} \leq N \leq X. \tag{4.5}\]

Obviously, if
\[\theta \leq \frac{1-\varepsilon}{2}, \tag{4.6}\]
then (2.2), (4.5), and (4) gives
\[N^\theta \leq D \leq N^{\theta+\varepsilon/2}. \tag{4.7}\]

We recall the well-known expansions
\[
\psi(t) = -\sum_{1 \leq |k| \leq H} \frac{e(kt)}{2\pi i k} + O\left( \min\left(1, \frac{1}{H||t||}\right) \right), \tag{4.8}
\]
\[
\min\left(1, \frac{1}{H||t||}\right) = \sum_{k=-\infty}^{\infty} b_k e(kt),
\]
where
\[|b_k| \ll \min\left( \frac{\log 2H}{H}, \frac{1}{|k|}, \frac{H}{|k|^2} \right).\]

Using (4.8) for the left-hand side of (4.4), we deduce
\[
\sum_{d \leq D \atop (d,a)=1} \left| \sum_{n \sim N \atop n \equiv a \pmod{d}} A(n)e(tn^\gamma) \left( \psi(-(n+1)^\gamma) - \psi(-n^\gamma) \right) \right| \ll \Gamma_1 + \Gamma_2 + \Gamma_3. \tag{4.9}
\]
where

\[
\Gamma_1 = \sum_{d \leq D} \sum_{1 \leq h \leq H \atop (d,a) = 1} \frac{1}{h} \left\{ \sum_{n \sim N \atop n \equiv a \pmod{d}} A(n) e\left(\frac{tn^\gamma}{h} \right) \left( e\left(-hn^\gamma\right) - e\left(-h(n+1)^\gamma\right) \right) \right\},
\]

(4.10)

\[
\Gamma_2 = \sum_{d \leq D} \sum_{n \sim N \atop (d,a) = 1} A(n) \sum_{k = -\infty}^{\infty} b_k e\left(kn^\gamma\right),
\]

(4.11)

\[
\Gamma_3 = \sum_{d \leq D} \sum_{n \sim N \atop (d,a) = 1} A(n) \sum_{k = -\infty}^{\infty} b_k e\left(k(n+1)^\gamma\right).
\]

(4.12)

5 Upper bounds of \( \Gamma_i, i = 1, 2, 3 \)

We now present estimates of \( \Gamma_2, \Gamma_3, \) and \( \Gamma_1 \), which will yield Theorem 1.

First, arguing as in [7], for sums (4.11) and (4.12), we have that

\[
\Gamma_2, \Gamma_3 \ll \frac{X^\gamma}{\log^A X},
\]

(5.1)

provided that

\[
H = N^{1-\gamma + \varepsilon}, \quad \gamma > \frac{1}{2} + \theta.
\]

(5.2)

Now we estimate \( \Gamma_1 \). Proceeding as in [7], for sum (4.10), we obtain

\[
\Gamma_1 \ll X^{\gamma - 1} \sum_{n \sim N} A(n) G(n),
\]

(5.3)

where

\[
G(n) = \sum_{1 \leq h \leq H} F_h(n) e\left(\frac{tn^\gamma}{h} - hn^\gamma\right),
\]

(5.4)

\[
F_h(n) = \sum_{d \leq D \atop (d,a) = 1, d \mid n-a} c(d, h), \quad |c(d, h)| = 1.
\]

Therefore it remains to show that

\[
\left| \sum_{n \sim N} A(n) G(n) \right| \ll \frac{N}{\log^A N}.
\]

(5.5)

To apply Lemma 5, we need to find the upper bounds of the sums \( S_I \) and and \( S_{II} \) of types I and II, respectively.

**Lemma 7.** Let \( 1 \leq d \leq X < X_1 \leq 2X \). Then

\[
\sum_{X < n \leq X_1 \atop n \equiv a \pmod{d}} e(h_1 n^\gamma + h_2 n^\gamma) \ll \min \left( X d^{-1}, (|h_1| d X^{-1} + |h_2| d X^{-1})^{-1} + d^{\varepsilon - \lambda} |h_1| \|X^{\varepsilon - \varepsilon + \lambda} + d^{\varepsilon - \lambda} |h_2| \|X^{\varepsilon - \varepsilon + \lambda} \right),
\]

where \((\varepsilon, \lambda)\) is any exponent pair.
Proof. Let \( b \) be an integer such that \( 1 \leq b \leq d \) and \( b \equiv a(d) \). Then

\[
\sum_{X < n \leq X_1, n \equiv a(d)} e(h_1 n^c + h_2 n^\gamma) = \sum_{(X-b)/d < m \leq (X_1-b)/d} e(h_1(b+md)^c + h_2(b+md)^\gamma). \tag{5.6}
\]

Denote

\[
f(y) = h_1(b+yd)^c + h_2(b+yd)^\gamma.
\]

We have

\[
|f^{(m)}(y)| \asymp (|h_1|dX^{c-1} + |h_2|dX^{\gamma-1}) \left( \frac{X}{d} \right)^{1-m} \tag{5.7}
\]

for \( X < y \leq X_1 \) and \( m \geq 1 \). Using (5.6), (5.7), the trivial estimation, and Lemma 2 with any exponent pair \((\alpha, \lambda)\), we establish the statement of the lemma.

\[\square\]

Lemma 8. Assume that

\[M \ll N^{(31/3)\gamma-19/2-\theta-\varepsilon}. \tag{5.8}\]

Then \( S_I \ll N^{1-\varepsilon} \).

Proof. Splitting the range of \( l \) into dyadic subintervals of the form \((L, 2L]\) so that \( ML \sim N \), changing the order of summation, using (4.1), (4.5), (4.7), (5.2), (5.8), and Lemma 7 with the exponent pair \(A^4B(0, 1) = (1/62, 57/62)\), we deduce

\[
S_I \ll (\log X)^2 \sum_{1 \leq h \leq H} \sum_{d \leq D} c(d, h) \sum_{m \sim M} a(m) \sum_{l \sim L, ml \sim N, ml \equiv a(d)} e\left( t(ml)^c - h(ml)^\gamma \right)
\]

\[
\ll (\log X)^2 \sum_{1 \leq h \leq H} \sum_{d \leq D} \sum_{m \sim M} \sum_{l \sim L, ml \sim N, ml \equiv a(d)} \left| e(t(ml)^c - h(ml)^\gamma) \right|
\]

\[
\ll (\log X)^2 \sum_{1 \leq h \leq H} \sum_{d \leq D} \sum_{m \sim M} \left( (|t|dMN^{c-1} + hdMN^{\gamma-1})^{-1} 
\right.
\]

\[
\left. + d^{-28/31} t^{1/62} M^{-28/31} N^{(c+56)/62} + d^{-28/31} h^{1/62} M^{-28/31} N^{(c+56)/62} \right)
\]

\[
\ll (\log X)^2 \sum_{1 \leq h \leq H} \sum_{d \leq D} \sum_{m \sim M} (h^{-1}d^{-1}M^{-1}N^{1-\gamma})
\]

\[
+ d^{-28/31} N^{(1/4-\varepsilon)/62} M^{-28/31} N^{(c+56)/62} + d^{-28/31} h^{1/62} M^{-28/31} N^{(c+56)/62} \right)
\]

\[
\ll (\log X)^4 \left( N^{1-\gamma} + H^{225/248} D^{3/31} + H^{63/62} M^{3/31} N^{(c+56)/62} D^{3/31} \right)
\]

\[
\ll (\log X)^4 \left( N^{1-\gamma} + H^{63/62} M^{3/31} N^{(c+56)/62} D^{3/31} \right) \ll N^{1-\varepsilon}. \tag{5.9}
\]

Bearing in mind (5.8) and (5.9), we establish the statement of the lemma.
Lemma 9. Assume that

\[ 0 < \theta \leq \frac{21}{110} - \varepsilon, \quad (5.10) \]
\[ \frac{865}{886} + \frac{55}{443} \theta + \varepsilon < \gamma < 1, \quad (5.11) \]

Then \( S_{II} \ll N^{1-\varepsilon}. \)

Proof. Splitting the range of \( l \) and \( h \) into dyadic subintervals of the form \( (L, 2L] \) and \( (K, 2K] \) so that \( ML \sim N \) and

\[ \frac{1}{2} \leq K \leq H, \quad (5.13) \]

we obtain

\[ S_{II} \ll (\log X)^2 \sum_{m \sim M} \left| \sum_{l \sim L, h \sim K} \sum_{m \sim N} b(l)F_h(ml)e(t(ml)^{\varepsilon} - h(ml)^{\gamma}) \right|. \]

It is easy to see that \( 0 < hl^{\gamma} \leq 4KL^{\gamma} \). Let \( T \) be a parameter that will be determined later. We decompose the pairs \( (h, l) \) into the sets \( \mathcal{G}_y \) \( (1 \leq y \leq T) \) defined by

\[ \mathcal{G}_y = \left\{ (h,l) \mid h \sim K, l \sim L, \frac{4KL^{\gamma}(y - 1)}{T} < hl^{\gamma} \leq \frac{4KL^{\gamma}y}{T} \right\}. \]

Hence

\[ S_{II} \ll (\log X)^2 \sum_{1 \leq y \leq T} \sum_{m \sim M} \left| \sum_{(h,l) \in \mathcal{G}_y} \sum_{m \sim N} b(l)F_h(ml)e(t(ml)^{\varepsilon} - h(ml)^{\gamma}) \right|. \]

Using Cauchy's inequality, we get

\[ |S_{II}|^2 \ll (\log X)^4 TM \sum_{1 \leq y \leq T} \sum_{m \sim M} \left| \sum_{(h,l) \in \mathcal{G}_y} b(l)F_h(ml)e(t(ml)^{\varepsilon} - h(ml)^{\gamma}) \right|^2 \]
\[ \ll (\log X)^4 TM \]
\[ \times \sum_{1 \leq y \leq T} \sum_{(h_1,l_1) \in \mathcal{G}_y} \sum_{(h_2,l_2) \in \mathcal{G}_y} \left| \sum_{m \sim M} F_{h_1}(ml_1)F_{h_2}(ml_2)e\left((l_1^{\varepsilon} - l_2^{\varepsilon})tm^{\varepsilon} - (h_1l_1^{\gamma} - h_2l_2^{\gamma})m^{\gamma}\right) \right| \]
\[ \ll (\log X)^4 TM \sum_{h_1 \sim K} \sum_{h_2 \sim K} \sum_{l_1 \sim L} \sum_{l_2 \sim L} |\Theta(t)|, \quad \alpha = h_1l_1^{\gamma} - h_2l_2^{\gamma}, \quad (5.14) \]

where

\[ \Theta(t) = \sum_{m \sim M} F_{h_1}(ml_1)F_{h_2}(ml_2)e\left((l_1^{\varepsilon} - l_2^{\varepsilon})tm^{\varepsilon} - \alpha m^{\gamma}\right). \quad (5.15) \]
If the system of congruences
\[ l_1 m \equiv a (d_1), \quad l_2 m \equiv a (d_2) \] (5.16)
has no solution, then \( \Theta(t) = 0 \). Assume that system (5.16) has a solution. Then there exists an integer \( f = f(l_1, l_2, a, d_1, d_2) \) such that \( (f, |d_1, d_2|) = 1 \) and (5.16) is equivalent to \( m \equiv f (|d_1, d_2|) \). From (5.4), (5.15), and the last consideration it follows that
\[
\Theta(t) = \sum_{d_1 \leq D} c(d_1, h_1) \sum_{d_2 \leq D} c(d_2, h_2) \sum_{m \sim M} \sum_{m \equiv f (|d_1, d_2|)} e\left((l_1^c - l_2^c)tm^c - \alpha m^\gamma\right)
\ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} \sum_{m \sim M} \sum_{m \equiv f (|d_1, d_2|)} e\left((l_1^c - l_2^c)tm^c - \alpha m^\gamma\right). \tag{5.17}
\]

According to Lemma 3,
\[
A\left(\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon\right) = \left(\frac{13}{194} + \varepsilon, \frac{152}{194} + \varepsilon\right)
\]
is an exponent pair. Now (5.17) and Lemma 7 with exponent pair \((13/194 + \varepsilon, 152/194 + \varepsilon)\) imply
\[
\Theta(t) \ll \sum_{d_1 \leq D} \sum_{d_2 \leq D} \min\left(\frac{M}{|d_1, d_2|}, \Omega\right), \tag{5.18}
\]
where
\[
\Omega = \left(M^{-1}[d_1, d_2]|(l_1^c - l_2^c)t| + M^{-1}\alpha\right)^{-1}
+ \left[d_1, d_2\right]^{-139/194}\left|(l_1^c - l_2^c)t\right|^{13/194}M^{13\varepsilon/194+13\varepsilon/194}
+ \left[d_1, d_2\right]^{-139/194}\left|\alpha\right|^{13/194}M^{13\varepsilon/194+13\varepsilon/194}. \tag{5.19}
\]

Denote
\[
E_1 = \sum_{h_1 \sim K} \sum_{h_2 \sim K} \sum_{l_1 \sim L} \sum_{l_2 \sim L} \sum_{d_1 \leq D} \sum_{d_2 \leq D} M \left|\frac{d_1, d_2}{d_1, d_2}\right|, \tag{5.20}
\]
\[
E_2 = \sum_{h_1 \sim K} \sum_{h_2 \sim K} \sum_{l_1 \sim L} \sum_{l_2 \sim L} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left(M^{-1}[d_1, d_2]|(l_1^c - l_2^c)t| + M^{-1}\alpha\right)^{-1}, \tag{5.21}
\]
\[
E_3 = \sum_{h_1 \sim K} \sum_{h_2 \sim K} \sum_{l_1 \sim L} \sum_{l_2 \sim L} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left[d_1, d_2\right]^{-139/194}\left|(l_1^c - l_2^c)t\right|^{13/194}M^{13\varepsilon/194+139/194}, \tag{5.22}
\]
\[
E_4 = \sum_{h_1 \sim K} \sum_{h_2 \sim K} \sum_{l_1 \sim L} \sum_{l_2 \sim L} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \left[d_1, d_2\right]^{-139/194}\left|\alpha\right|^{13/194}M^{13\varepsilon/194+139/194}. \tag{5.23}
\]
If $|\alpha| \leq M^{-\gamma}$, then

$$\frac{M}{[d_1, d_2]} \leq \frac{M^{1-\gamma}}{[d_1, d_2]|\alpha|},$$

and the contribution of $M[d_1, d_2]^{-1}$ to $|S_{II}|^2$ is

$$E'_1 \ll (\log X)^7 TM^2 N(M^{-\gamma}),$$

where we used the elementary estimate

$$\sum_{d_1 \leq D} \sum_{d_2 \leq D} \frac{1}{|d_1, d_2|} \ll (\log D)^3.$$  

(5.25)

If $|\alpha| \geq M^{-\gamma}$, then

$$\frac{M}{[d_1, d_2]} \geq \frac{M^{1-\gamma}}{[d_1, d_2]|\alpha|},$$

and the contribution of $(M^{-1}[d_1, d_2]|(l_1^2 - l_2^2)t| + M^{-1}[d_1, d_2]|\alpha|)^{-1}$ to $|S_{II}|^2$ is

$$E'_2 \ll (\log X)^{8} TM^{2-\gamma} \max_{M^{-\gamma} \leq \Delta \leq 4KL^7T^{-1}} \frac{N(\Delta)}{\Delta},$$

(5.26)

where we used estimate (5.25) again.

Further, the total contribution of the term $[d_1, d_2]^{-139/194}|(l_1^2 - l_2^2)t|^{13/194} M^{13\gamma/194+139/194}$ to $|S_{II}|^2$ is

$$E'_3 \ll (\log X)^4 TM L^{13\gamma/194} M^{13\gamma/194+139/194} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \frac{1}{[d_1, d_2]^{139/194}} N(4KL^7T^{-1})$$

$$\ll (\log X)^4 TL^{13\gamma/194} M^{13\gamma/194+333/194} N(1/4-c)(13/194) D^{55/194} N(4KL^7T^{-1}),$$

(5.27)

where we used (4.1), (4.5), and the upper bound

$$\sum_{d_1 \leq D} \sum_{d_2 \leq D} \frac{1}{[d_1, d_2]^{139/194}} \ll \sum_{1 \leq r \leq D} \sum_{k_1 \leq D/r} \sum_{k_2 \leq D/r} \frac{1}{r^{139/194} k_1^{139/194} k_2^{139/194}} \ll \sum_{1 \leq r \leq D} \frac{1}{r^{139/194}} \left( \frac{D}{r} \right)^{110/194} \ll D^{55/194}.$$  

(5.28)

It remains to note that the total contribution of the term $[d_1, d_2]^{-139/194}|\alpha|^{13/194} M^{13\gamma/194+139/194}$ to $|S_{II}|^2$ is

$$E'_4 \ll (\log X)^4 TM M^{13\gamma/194+139/194} |\alpha|^{13/194} \sum_{d_1 \leq D} \sum_{d_2 \leq D} \frac{1}{[d_1, d_2]^{139/194}} N(4KL^7T^{-1})$$

$$\ll (\log X)^4 TM^{13\gamma/194+333/194} (KL^7T^{-1})^{13/194} D^{55/194} N(4KL^7T^{-1}),$$

(5.29)

where we used the upper bound (5.28) again.
Bearing in mind (5.14), (5.18)–(5.24), (5.26), (5.27), and (5.29), by Lemma 6 we deduce

\[
|S_{III}|^2 \ll \left( \log X \right)^7 T M^2 N(M^{-\gamma}) + \left( \log X \right)^6 T M^{2-\gamma} \max_{M^{-\gamma} \leq A \leq 4KL^{-T^{-1}}} \frac{N(\Delta)}{A}
\]

\[
+ \left( \log X \right)^4 TL^{13\gamma/194} M^{13\gamma/194+333/194} N^{1/14-\varepsilon}(13/194) D^{55/194} N(4KL^\gamma T^{-1})
\]

\[
+ \left( \log X \right)^4 TM^{13\gamma/194+333/194} \left( KL^{-2\gamma T^{-1}} \right)^{13/194} D^{55/194} N(4KL^\gamma T^{-1})
\]

\[
\ll \left( \log X \right)^6 T M^{2-\gamma} \max_{M^{-\gamma} \leq A \leq 4KL^{-T^{-1}}} \left( KL^{2-\gamma} + A^{-1} KL \log X \right)
\]

\[
+ \left( \log X \right)^4 \left( TN^{13/776} M^{333/194} D^{55/194} + T^{181/194} M^{333/194} D^{55/194} K^{13/194} N^{13\gamma/194} \right)
\]

\[
\times \left( K^2 L^2 T^{-1} + KL \log X \right)
\]

\[
\ll \left( \log X \right)^9 \left( TK M^{2-\gamma} + TK N M + K^2 N^{1565/776} M^{-55/194} D^{55/194} + TK N^{789/776} M^{139/194} D^{55/194} + T^{-13/194} K^{401/194} N^{13\gamma/194+2} M^{-55/194} D^{55/194} + T^{181/194} K^{207/194} N^{13\gamma/194+1} M^{139/194} D^{55/194} \right). \quad (5.30)
\]

We take \( T \) such that

\[
TK M = T^{-13/194} K^{401/194} N^{13\gamma/194+2} M^{-55/194} D^{55/194}.
\]

Therefore

\[
T = \left[ N^{(13\gamma+194)/207} M^{-83/69} K D^{55/207} \right] + 1. \quad (5.31)
\]

Now (4.7), (5.2), (5.10)–(5.13), (5.30), and (5.31) yield

\[
|S_{III}| \ll \left( K^2 N^{608-194\gamma}/207 M^{-83/69} D^{55/207} + K^2 N^{(13\gamma+401)/207} M^{-14/69} D^{55/207} + K^{2N^{1565/776} M^{-55/194} D^{55/194}} + K^{2N^{10088\gamma+313867}/160632} M^{-6511/13386} D^{22055/40158} + K^{2N^{26\gamma+388}/207} M^{-28/69} D^{110/207} + K^{N^{2-\gamma}} + K N M + K N^{789/776} M^{139/194} D^{55/194} + K^{2N^{207/194} M^{13\gamma/194+2} M^{139/194} D^{55/194}} \right)^{1/2}
\]

\[
\ll \left( N^{(1022-608\gamma)/207} M^{-83/69} D^{55/207} + N^{(815-401\gamma)/207} M^{-14/69} D^{55/207} + N^{(3117-1552\gamma)/776} M^{-55/194} D^{55/194} + N^{(635131-311176\gamma)/160632} M^{-6511/13386} D^{22055/40158} + N^{(802-388\gamma)/207} M^{-28/69} D^{110/207} + N^{4-\gamma} + N^{2-\gamma} M + N^{(1565-776\gamma)/776} M^{139/194} D^{55/194} + N^{(401-194\gamma)/194} M^{139/194} D^{55/194} \right)^{1/2}
\]

\[
\ll N^{1-\varepsilon}.
\]

The lemma is proved. \( \Box \)

We are now in a good position to establish the validity of (5.5).

In Lemma 5, we take

\[
a = \frac{31}{3} \gamma - \frac{19}{2} - \theta - \varepsilon, \quad b = \frac{313}{44} - \frac{388}{55} \gamma + \theta + \varepsilon, \quad c = \gamma - \varepsilon.
\]

The direct verification ensures inequalities (3.6), (5.2), and (5.11). From (2.1) and Lemmas 5, 8, and 9 it follows that (5.5) holds.
Bearing in mind (4.5), (5.3), and (5.5), we deduce that

\[ \Gamma_1 \ll \frac{X^\gamma}{\log^A X}. \]  

(5.32)

Proof of Theorem 1. The proof now follows from (4.9), (5.1), and (5.32). □

References

1. A. Balog and J.P. Friedlander, A hybrid of theorems of Vinogradov and Piatetski-Shapiro, *Pac. J. Math.*, 156:45–62, 1992.
2. J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Am. Math. Soc.*, 30:205–224, 2017.
3. S.I. Dimitrov, A ternary Diophantine inequality by primes with one of the form \( p = x^2 + y^2 + 1 \), *Ramanujan J.*, 59(2):571–607, 2022.
4. S.W. Graham and G. Kolesnik, *Van der Corput’s Method of Exponential Sums*, Cambridge Univ. Press, New York, 1991.
5. D.R. Heath-Brown, Prime numbers in short intervals and a generalized Vaughan identity, *Can. J. Math.*, 34:1365–1377, 1982.
6. D.R. Heath-Brown, The Piatetski-Shapiro prime number theorem, *J. Number Theory*, 16:242–266, 1983.
7. J. Li, M. Zhang, and F. Xue, An additive problem over Piatetski-Shapiro primes and almost-primes, *Ramanujan J.*, 57:1307–1333, 2022.
8. Y. Lu, An additive problem on Piatetski-Shapiro primes, *Acta Math. Sin., Engl. Ser.*, 34:255–264, 2018.
9. T. Peneva, An additive problem with Piatetski-Shapiro primes and almost-primes, *Monatsh. Math.*, 140:119–133, 2003.
10. I.I. Piatetski-Shapiro, On the distribution of prime numbers in sequences of the form \([f(n)]\), *Mat. Sb.*, 33:559–566, 1953.
11. J. Rivat and P. Sargos, Nombres premiers de la forme \([n^c]\), *Can. J. Math.*, 53:414–433, 2001.
12. J. Rivat and J. Wu, Prime numbers of the form \([n^c]\), *Glasg. Math. J.*, 43(2):237–254, 2001.
13. X. Wang and Y. Cai, An additive problem involving Piatetski-Shapiro primes, *Int. J. Number Theory*, 7:1359–1378, 2011.