ON THE STABLE RANK OF ALGEBRAS OF OPERATOR FIELDS OVER AN N-CUBE

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Abstract. Let \( A \) be a unital maximal full algebra of operator fields with base space \([0, 1]^k\) and fibre algebras \( \{A_t\}_{t \in [0, 1]^k} \). We show that the stable rank of \( A \) is bounded above by the quantity \( \sup_{t \in [0, 1]^k} sr(C([0, 1]^k) \otimes A_t) \). Here the symbol “sr” means stable rank. Using the above estimate, we compute the stable ranks of the \( C^* \)-algebras of the (possibly higher rank) discrete Heisenberg groups.

1. Introduction

Rieffel [12] introduced the notion of stable rank for \( C^* \)-algebras as the noncommutative version of complex dimension of ordinary topological spaces. It turns out that the stable rank of a unital \( C^* \)-algebra is the same as its Bass stable rank (see [7]). The purpose of this paper is to study stable rank for continuous field \( C^* \)-algebras. Our main result is

Theorem 1.1. Let \( A \) be a unital maximal full algebra of operator fields with base space \([0, 1]^k\) and fibre algebras \( \{A_t\}_{t \in [0, 1]^k} \). Then the stable rank of \( A \) satisfies the following inequality:

\[
   sr(A) \leq \sup_{t \in [0, 1]^k} sr(C([0, 1]^k) \otimes A_t).
\]

One of the key ingredients of the proof of the above result is Nistor’s notion of absolute connected stable rank (see [10]). Recall that for a unital \( C^* \)-algebra \( A \), the absolute connected stable rank of \( A \) is numerically the same as the stable rank of the tensor product \( C[0, 1] \otimes A \).

As an application of our theorem, we will compute the stable ranks of the universal \( C^* \)-algebras of the (possibly higher rank) discrete Heisenberg groups. Recall that for a positive integer \( n \), the discrete Heisenberg group of rank \( 2n + 1 \) is the group of all \( 2n + 1 \) by \( 2n + 1 \) upper triangular matrices with integer entries, with ones on the diagonal, and zero entries on all but the first row, last column and the diagonal. The rank \( 2n + 1 \) discrete Heisenberg group is naturally a lattice subgroup of the \( 2n + 1 \)-dimensional Heisenberg Lie group.

The stable ranks of the universal \( C^* \)-algebras of various type I Lie groups have been extensively studied (see [13], [15], [16]). In particular, we point out that for a simply connected nilpotent Lie group \( G \), the stable rank of the universal \( C^* \)-algebra \( C^*(G) \) of \( G \) has been computed by Sudo and Takai (see [15]). Roughly speaking, they showed that the stable rank of \( C^*(G) \) is controlled by the ordinary topological dimension of the space of one-dimensional representations of \( G \).

Recently, the stable ranks of a class of non-type I solvable Lie groups (which include the Mautner group) have also been computed (see [14]).

Our computations for the discrete Heisenberg groups constitute a class of interesting nontrivial examples of the stable rank of the universal \( C^* \)-algebra of a non-type I discrete group.

In a later paper, we will apply the techniques developed in this paper to compute the stable ranks of arbitrary finitely generated, torsion-free two-step nilpotent groups.

A general reference for stable rank of \( C^* \)-algebras is [12]. General references for algebras of operator fields are [5], [8] and [17].

In what follows, for a \( C^* \)-algebra \( A \), the notation “sr(\( A \))” will always mean the stable rank of \( A \). If, in addition, \( A \) is unital and \( N \) is a positive integer, then \( L_{QN}(A) \) is the set of \( N \)-tuples \((a_1, a_2, ..., a_N)\) in \( A^N \) such that \( \sum_{i=1}^{N} (a_i)^* a_i \) is an invertible element in \( A \).

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2. Main results

**Lemma 2.1.** Let $\mathcal{A}$ be a unital maximal full algebra of operator fields with base space $[0, 1]$ and fibre algebras $\{\mathcal{A}_t\}_{t \in \mathcal{F}}$. Then the stable rank of $\mathcal{A}$ satisfies the inequality

$$sr(\mathcal{A}) \leq \sup_{t \in [0, 1]} sr(C[0, 1] \otimes \mathcal{A}_t).$$

*Proof.* In this proof, we will always let “$\mathcal{F}$” denote the continuity structure for the continuous field decomposition of $\mathcal{A}$ in the hypothesis.

Suppose that $M = \sup_{t \in [0, 1]} sr(C[0, 1] \otimes \mathcal{A}_t)$ is a finite number. Let an $M$-tuple $(a_1, a_2, \ldots, a_M) \in \mathcal{A}_M$ and a positive real number $\epsilon > 0$ be given. Since $[0, 1]$ is compact, let $I_1, I_2, \ldots, I_N$ be a finite set of open intervals which cover $[0, 1]$ and for each $i = 1, 2, \ldots, N$, let $(f_{i,1}, f_{i,2}, \ldots, f_{i,M})$ be an $M$-tuple in $\mathcal{A}_M^M$ such that

1. $\sum_{j=1}^M f_{i,j}(t)^* f_{i,j}(t)$ is an invertible element of $\mathcal{A}_i$ for $t \in I_i$. Here the operator field $t \mapsto f_{i,j}(t)$ for $t \in [0, 1]$, is the representation of $f_{i,j}$ as a continuous operator field in $\mathcal{A}$. And
2. $\|f_{i,j} - a_j\| < \epsilon$, for all $i, j$.

We can choose such intervals $I_i$ and such elements $f_{i,j}$, since our hypothesis implies that the stable rank of each fibre $\mathcal{A}_i$ is less than or equal to $M$, and by the existence and continuity of operator fields in a full algebra of operator fields (see the definition of full algebra of operator fields in [5], [8] or [17]). Also, one needs to use the fact that any element of a unital $C^*$-algebra that is sufficiently close to the unit is invertible. Finally, to make $f_{i,j}$ uniformly within $\epsilon$ of $a_j$ (for all $i, j$), one needs the maximality of the algebra of operator fields $\mathcal{A}$ (see [8] Proposition 1 and [17] Theorem 1.1).

For simplicity, we will assume that there are only two intervals $(I_1$ and $I_2)$ and only two $M$-tuples $(f_1, f_2, \ldots, f_M)$ and $(f_2, f_2, \ldots, f_M)$). We may additionally assume that neither interval is contained in the closure of the other, and that their intersection is a continuum. Our goal is to “connect” the two $M$-tuples over the intersection $\overline{I_1 \cap I_2}$ to get an element of $LGM(\mathcal{A})$ which approximates $(a_1, a_2, \ldots, a_M)$ within $\epsilon$. The proof for more than two intervals is an iteration of this procedure (after appropriate contraction, removal or addition of intervals...).

Our procedure for “connecting” the two $M$-tuples over $\overline{I_1 \cap I_2}$ will involve constructing a sequence of strictly increasing points $\{t_n\}_{n=1}^\infty$ in $I_1 \cap I_2$, and constructing sequences of operator fields $\{a_j^n\}_{n=1}^\infty$ $(j = 1, 2, \ldots, M)$ satisfying:

1. $a_j^n$ is a continuous operator field in $\mathcal{A}(n)$, for all $j, n$. Here $\mathcal{A}(n)$ is the unital maximal full algebra of operator fields gotten by restricting $\mathcal{A}$ to the interval $[t_n, t_{n+1}]$. (In particular, the continuity structure for $\mathcal{A}(n)$ is gotten by taking the restriction to $[t_n, t_{n+1}]$ of all the fields in $\mathcal{F}$).
2. $a_j^n$ is within $\epsilon$ of the restriction of $a_j$ to $[t_n, t_{n+1}]$, for all $j, n$.
3. $a_j^n(t_1) = f_{1,j}(t_1)$ and $a_j^{n+1}(t_{n+1}) = a_j^n(t_{n+1})$, for all $j, n$.
4. $a_j^{n+1}$ is within $\epsilon/2^n$ of the restriction of $f_{2,j}$ to $[t_{n+1}, t_{n+2}]$, for all $j, n$. And
5. $(a_{1,j}^n, a_{2,j}^n, \ldots, a_{M,j}^n)$ is in $LGM(\mathcal{A}(n))$ or equivalently, $\sum_{j=1}^M a_j^n(t)^* a_j^n(t)$ is an invertible element of $\mathcal{A}_i$ for all $t \in [t_n, t_{n+1}]$, for all $n$ (the proof of equivalence is a small spectral theory argument which uses the fact that $[t_n, t_{n+1}]$ is compact).

Henceforth, we will let "(*)" denote conditions (1) - (5).

By [10] Lemma 2.4 and our hypothesis for $M$, we have that $LGM(\mathcal{A}_t) \cap \{(b_1, b_2, \ldots, b_M) \in \mathcal{A}_M^M : \|a_i(t) - b_i\| < \epsilon \forall i\}$ is a nonempty connected open set for all $t \in [0, 1]$. Hence, fixing $t_1 \in I_1 \cap I_2$, there are continuous paths $\gamma_j : [0, 1] \to LGM(\mathcal{A}_{t_1})$, $j = 1, 2, \ldots, M$, such that a) $\gamma_j(0) = f_{1,j}(t_1)$ and $\gamma_j(1) = f_{2,j}(t_1)$ for all $j$ and b) $\gamma_j(s)$ is within $\epsilon$ of $a_j(t_1)$ for all $s \in [0, 1]$ and for all $j$.

Note that since $[0, 1]$ is compact, $(\gamma_1, \gamma_2, \ldots, \gamma_M)$ is in $LGM([0, 1] \otimes \mathcal{A}_{t_1})$. Now consider the unital maximal full algebra of operator fields $\tilde{\mathcal{A}}$, which has base space $[0, 1]$ and fibre algebras $\{C[0, 1] \otimes \mathcal{A}_t\}_{t \in [0, 1]}$. The continuity structure $\tilde{\mathcal{F}}$ for $\tilde{\mathcal{A}}$ consists of all operator fields of the form $t \mapsto \sum_{i=1}^N f_i \otimes c_i(t)$, $t \in [0, 1]$. Here the $f_i$s are in $C[0, 1]$, the $c_i$s are continuous operator fields in $\tilde{\mathcal{A}}$ (with respect to the continuity structure $\tilde{\mathcal{F}}$), and $N$ is a nonnegative integer. Now since $\mathcal{A}$ is a full algebra of operator fields, the second variable of $\gamma_j(., t)$ ranges over the base space of $\tilde{\mathcal{A}}$, and for $t \in [0, 1]$, $\gamma_j(., t)$ is an element of the fibre algebra $C[0, 1] \otimes \mathcal{A}_t$ of $\mathcal{A}$.

One can show that the operator field $t \mapsto \gamma_j(1, t)$ is a continuous operator field in $\mathcal{A}$ (with continuity structure $\tilde{\mathcal{F}}$) for $j = 1, \ldots, M$. Also, one may view the field $t \mapsto a_j(t)$ as an operator field in $\tilde{\mathcal{A}}$ in the natural
way for $j = 1, \ldots, M$. From these and the fact that (as elements of $C[0,1] \otimes \mathcal{A}_1$) $\gamma_j(., t_1)$ is within $\epsilon$ of $a_j(t_1)$, we find an open neighbourhood $V$ of $t_1$ with $V \subset I_1 \cap I_2$ such that for all $t \in V$, a) as elements of $C[0,1] \otimes \mathcal{A}_1$, $\gamma_j(., t)$ is within $\epsilon$ of $a_j(t)$, $j = 1, 2, \ldots, M$, and b) $\gamma_j(1, t)$ is within $\epsilon/2$ of $f_{2,j}(t)$. Furthermore, by continuity and since elements in a unital $C^*$-algebra which are sufficiently close to the unit are invertible, we may assume that $V$ is sufficiently small so that for all $t \in V$, $\sum_{j=1}^M \gamma_j(., t) \gamma_j(., t)$ is an invertible element of $C[0,1] \otimes \mathcal{A}_1$.

One can see that for any continuous function $g : [0,1] \to [0,1]$, the operator field $t \mapsto \gamma_j(g(t), t)$ is a continuous field in $\mathcal{A}$, for $j = 1, 2, \ldots, M$. Hence, we can pick $t_2 \in V$ such that $t_1 < t_2$; and we can let $\alpha_j(t) = df \gamma_j((t - t_1)/(t_2 - t_1))$ for all $t \in [t_1, t_2]$, $j = 1, 2, \ldots, M$. This will give us the first members of the sequences in (*).

Now we can repeat almost exactly the same argument as before, replacing $t_1$ by the point $t_2$ and replacing $f_{1,j}(t_1)$ by $\alpha_j(t_2)$. Hence, by the induction hypothesis, for each $j = 1, 2, \ldots, M$, the fibre algebra $\alpha_j^2$, $j = 1, 2, \ldots, M$, which will be the next members of the sequences in (*). We need only note two minor modifications that are needed in the argument: a) firstly, one has to use the fact that for all $t$, the set

$$Lg_M(\mathcal{A}_1) \cap \{(b_1, b_2, \ldots, b_M) \in \mathcal{A}_1^M : \|a_j(t) - b_j\| < \epsilon \text{ and } \|f_{2,j}(t) - b_j\| < \epsilon/2, \text{ for all } j\}$$

is a connected open set (see [10] Lemma 2.4), and b) when choosing the corresponding neighbourhood about $t_2$, one must make it sufficiently small so that the corresponding quantities which result will also be sufficiently small in order to fulfill condition (4) in (*).

Repeating this process ad infinitum (making the appropriate modifications at each step), we get a sequence of points $\{t_n\}_{n=1}^\infty$ and sequences of continuous operator fields $\{\alpha_n^j\}_{n=1}^\infty$, $j = 1, 2, \ldots, M$, which fulfill the conditions in (*). Now let $t = lim_{n \to \infty} t_n$. For $j = 1, 2, \ldots, M$, let $a_j$ be the continuous operator field in $\mathcal{A}$ defined by

1. $a_j(t) = f_{1,j}(t)$ for $t \in [0, t_1]$,
2. $a_j(t) = \alpha_n^j(t)$ for $t \in [t_n, t_{n+1}]$, and
3. $a_j(t) = f_{2,j}(t)$ for all $t \in [t, 1]$. 

Then $(\alpha_1, \alpha_2, \ldots, \alpha_M) \in Lg_M(\mathcal{A})$, and for all $j = 1, 2, \ldots, M$, $\alpha_j$ approximates $a_j$ within $\epsilon$.

\[ \square \]

Proof of Theorem 1.1. We proceed by induction. The base case $k = 1$ has already been dealt with in Lemma 2.1. We now do the induction step, supposing that $k \geq 2$. By [8] Theorem 4, let $\pi : Prim(\mathcal{A}) \to [0,1]^k$ be the continuous open surjection corresponding to the continuous field decomposition of $\mathcal{A}$ in the hypothesis (Here $Prim(\mathcal{A})$ is the primitive ideal space of $\mathcal{A}$).

Since $k \geq 2$, the map $p : [0,1]^k \to [0,1]^{k-1}$, given by projecting onto the first $k-1$ coordinates, is a continuous open surjection. Hence, the composition $p \circ \pi : Prim(\mathcal{A}) \to [0,1]^{k-1}$ is a continuous open surjection. Hence by [8] Theorem 4, we can realize $\mathcal{A}$ as a unital maximal full algebra of operator fields with base space $[0,1]^{k-1}$ and fibre algebras, say, $\{\mathcal{B}_r\}_{r \in [0,1]^{k-1}}$. Hence by the induction hypothesis, $sr(\mathcal{A}) \leq sup_{r \in [0,1]^{k-1}}sr(C([0,1]^{k-1}) \otimes \mathcal{B}_r)$.

But for each $s$, the fibre algebra $\mathcal{B}_s$ can be realized as a maximal full algebra of operator fields with base space $p^{-1}(s) = \{s\} \times [0,1]$ and fibre algebras $\{\mathcal{A}_r\}_{r \in \{s\} \times [0,1]}$ (By [8] Theorem 4, $\mathcal{B}_s$ is isomorphic to $\mathcal{A}/I$ where $I = \cap (p \circ \pi)^{-1}(s)$). Using this fact, one can construct the natural continuous open surjection of $Prim(\mathcal{B}_s)$ onto $\{s\} \times [0,1]$, $s \in [0,1]^{k-1}$. Let us suppose that this continuous field decomposition of $\mathcal{B}_s$ is given by a continuity structure $\mathcal{G}$. Then $C([0,1]^{k-1}) \otimes \mathcal{B}_s$ can be realized as a unital maximal full algebra of operator fields with base space $[0,1]$ and fibre algebras $\{C([0,1]^{k-1}) \otimes \mathcal{A}_r\}_{r \in \{s\} \times [0,1]}$. Here the continuity structure consists of operator fields of the form $r \mapsto \sum_{i=1}^N f_i \otimes b_i(r)$, for $r \in \{s\} \times [0,1]$. Here the $f_i$s are in $C([0,1]^{k-1})$, the $b_i$s are continuous fields in $\mathcal{B}_s$ (with respect to the continuity structure $\mathcal{G}$) and $N$ is a nonnegative integer.

Hence by the induction hypothesis, for each $s$, we have that $sr(C([0,1]^{k-1}) \otimes \mathcal{B}_s) \leq sup_{r \in \{s\} \times [0,1]}sr(C([0,1]^{k-1}) \otimes \mathcal{A}_r)$. It follows, then, that $sr(\mathcal{A}) \leq sup_{s \in [0,1]^k}sr(C([0,1]^k) \otimes \mathcal{A}_s)$.

We note that the statements of Lemma 2.1 and Theorem 1.1 would still be true if the unit interval $[0,1]$ was replaced by the circle $S^1$ (and if the $k$-cube $[0,1]^k$ was replaced by the $k$-torus $T^k$). The proofs would be exactly the same. In other words, we have that
Corollary 2.2. Let $A$ be a unital maximal full algebra of operator fields with base space the $k$-torus $\mathbb{T}^k$ and fibre algebras $\{A_t\}_{t \in \mathbb{T}^k}$. Then the stable rank of $A$ satisfies

$$sr(A) \leq \sup_{t \in \mathbb{T}^k} sr(C([0,1]^k) \otimes A_t).$$

Theorem 2.3. Let $H^Z_{2n+1}$ be the discrete Heisenberg group of rank $2n+1$. Let $C^*(H^Z_{2n+1})$ be the universal $C^*$-algebra of $H^Z_{2n+1}$. Then $sr(C^*(H^Z_{2n+1})) = n + 1$.

Proof. By [12] Proposition 1.7 and Theorem 4.3, and the fact that $C^*(T^{2n})$ is a quotient of $C^*(H^Z_{2n+1})$, the stable rank of $C^*(H^Z_{2n+1})$ is greater than or equal to $n + 1$. By [1] and [9] Theorem 3.4, $C^*(H^Z_{2n+1})$ can be realized as a unital maximal full algebra of operator fields with base space the 1-torus $T$ and fibre algebras, say, $\{A_t\}_{t \in \mathbb{T}}$. Hence, by the Corollary 2.2, the stable rank of $C^*(H^Z_{2n+1})$ satisfies $sr(C^*(H^Z_{2n+1})) \leq \sup_{t \in \mathbb{T}} sr(C([0,1] \otimes A_t)$.

Now by [1] and [9] Theorem 3.4, each fibre algebra $A_t$ can be realized as a unital maximal full algebra of operator fields with base space a torus with dimension less than or equal to $2n$ (the zero-dimensional torus being a point) and fibre algebras, say, $\{B^t_s\}_{s \in \mathbb{T}^l_t}$, where $\mathbb{T}^l_t$ is the base in the continuous field decomposition of $A_t$ ($l \leq 2n$). Moreover, for each $t \in \mathbb{T}$ and for each $s \in \mathbb{T}^l_t$, $B^t_s$ is isomorphic to either a full matrix algebra $M_n(C)$ or $M_n(C) \otimes (\mathbb{O}^n \otimes A_t)$, where $\mathbb{O}^n \otimes A_t$ is the $n$ times tensor product of a fixed irrational rotation algebra $A_t$ with irrational rotation $\theta$. Now by [4] each irrational rotation algebra can be decomposed as an inductive limit of building blocks of the form $M_n(C(T)) \oplus M_n(C(T))$ (the integers $m$ and $n$ get arbitrarily large as we move along building blocks in the inductive limit). Hence, since $C([0,1] \otimes A_t$ can be realized as a maximal full algebra of operator fields with base space $\mathbb{T}^l_t$ and fibre algebras $\{C[0,1] \otimes B^t_s\}_{s \in \mathbb{T}^l_t}$, it follows, by Corollary 2.2, [12] Proposition 1.7 and Theorems 5.1 and 6.1, that $sr(C([0,1] \otimes A_t) \leq n + 1$, for all $t$. Hence, $sr(C^*(H^Z_{2n+1})) = n + 1$.

\[ \square \]

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