Mixed $f$-divergence and inequalities for log-concave functions

Umut Caglar and Elisabeth M. Werner

**Abstract**

Mixed $f$-divergences, a concept from information theory and statistics, measure the difference between multiple pairs of distributions. We introduce them for log-concave functions and establish some of their properties. Among them are affine invariant vector entropy inequalities, like new Alexandrov–Fenchel-type inequalities and an affine isoperimetric inequality for the vector form of the Kullback Leibler divergence for log-concave functions.

Special cases of $f$-divergences are mixed $L_\lambda$-affine surface areas for log-concave functions. For those, we establish various affine isoperimetric inequalities as well as a vector Blaschke Santaló-type inequality.

1. **Introduction**

Affine invariant notions have had a transformative effect in convex geometry, for example, [14, 25, 28, 46, 60]. One reason for this is that there are powerful inequalities associated to those notions. See, for instance, [14, 16, 23, 27, 28, 30, 55, 56]. Within the last few years, amazing connections have been discovered between some of these affine invariant notions and concepts from information theory, for example, [13, 15, 18, 29, 31, 32, 43], leading to a totally new point of view and introducing a whole new set of tools in the area of convex geometry. In particular, it was observed in [53] that one of the most important affine invariant notions, the $L_p$-affine surface area for convex bodies [27, 50], is Rényi entropy from information theory and statistics. Rényi entropies are special cases of $f$-divergences. Consequently, those were then introduced for convex bodies and their corresponding entropy inequalities have been established in [54]. An $f$-divergence (see below for the precise definition) is a function that measures the difference between (probability) densities. Aside from Rényi entropies, for example, the relative entropy or Kullback–Leibler divergence [19] and the Bhattacharyya distance [5] are examples of $f$-divergences.

Much effort has been devoted lately to extend concepts and inequalities that hold for convex bodies to the corresponding ones for classes of functions. A natural analog for a convex body is a log-concave function. For those, functional analogs of important inequalities have been proved, such as the Blaschke Santaló inequality [2, 4, 11, 20] and the affine isoperimetric inequality [3]. In [7], $f$-divergences were introduced for log-concave functions. This new concept yielded entropy inequalities which are stronger than the already existing ones, the reverse log-Sobolev and the reverse Poincare inequalities of [3].

Now we develop these ideas even further and introduce the mixed $f$-divergence for log-concave functions. For convex bodies, these were introduced and developed in [57]. Mixed $f$-divergence, which is important in applications, such as statistical hypothesis testing and classification, see, for example, [35, 40, 61], measures the difference between multiple pairs of (probability) distributions. Examples include, for example, the Matusita’s affinity [33, 34], the Toussaint’s affinity [51], the information radius [48] and the average divergence [47]. Mixed
$f$-divergence is an extension of the classical $f$-divergence and can be viewed as a vector form of classical $f$-divergence. For a vector $\vec{\varphi} = (\varphi_i)_{1 \leq i \leq n}$ consisting of log-concave functions $\varphi_i : \mathbb{R}^n \to [0, \infty)$ and a vector $\vec{f} = (f_i)_{1 \leq i \leq n}$ consisting of concave or convex functions $f_i : (0, \infty) \to \mathbb{R}_+$, we define the mixed $f$-divergence for $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$ by

$$D_f(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) = \prod_{i=1}^{n} \left[ \varphi_i f_i \left( \frac{\mathbf{e}^{(\nabla \varphi_i, x)_i}/\varphi_i}{\varphi_i^2} \det[\text{Hess}(- \log \varphi_i)] \right) \right]^{1/n} \, dx. \quad (1)$$

If all $\varphi_i$ are the same and all $f_i$ are the same, then we recover the $f$-divergences of [7]. Like those, the new expressions are $\text{SL}(n)$ invariant. Here, $\nabla \varphi$ denotes the gradient and $\text{Hess}(\varphi) = (\partial^2 \varphi / \partial x_i \partial x_j)_{1 \leq i, j \leq n}$ is the Hessian of $\varphi$.

One of the difficulties, to introduce this notion, was to find the right expression for the densities. A passage from functions to convex bodies and back, lets us achieve this goal and it can be seen that the expressions (1) appear naturally. This is demonstrated in [7].

The study of mixed $f$-divergences leads us to obtain new linear, respectively, affine invariant entropy inequalities, among them new Alexandrov–Fenchel-type inequalities for log-concave functions. Alexandrov–Fenchel inequality is a fundamental result in geometry. It is arguably one of the strongest inequality in this area as many important inequalities such as the Brunn–Minkowski inequality and Minkowski’s first inequality follow from Alexandrov–Fenchel inequality (see, for example, [45]). Different generalization of Alexandrov–Fenchel inequalities for log-concave functions can be found in, for example, [38]. Various vector entropy inequalities are consequences of this new Alexandrov–Fenchel inequality, for instance the following upper bound for the vector form of the $f$-divergence in terms of the classical $f$-divergences

$$[D_f(P_{\vec{\varphi}}, Q_{\vec{\varphi}})]^n \leq \prod_{k=1}^{n} D_{K_L}(P_{\varphi_k}, Q_{\varphi_k}),$$

and an affine isoperimetric inequality for the vector form of the relative entropy for normalized log-concave functions,

$$D_{K_L}(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) \leq \log(2\pi)^n.$$ 

We refer to Theorem 5 and Corollary 9 for the detailed statements and the corresponding equality characterizations. While for the classical Alexandrov–Fenchel inequality for convex bodies the equality characterizations are not known in general, such equality characterizations can be established for these new Alexandrov–Fenchel inequalities for log-concave functions. To do so, we use, among other things, the matrix version of the Brunn–Minkowski inequality and recently established unique solutions of certain Monge Ampère differential equations [58].

Mixed $L_\lambda$-affine surface areas for a vector $\vec{\varphi}$ of log-concave functions, denoted by $\text{as}_\lambda(\vec{\varphi})$, are special cases of mixed $f$-divergences. This new definition corresponds, on the level of convex bodies, to the mixed $L_\lambda$-affine surface areas (see, for example, [26, 56, 59]), a generalization of $L_\mu$-affine surface areas. We refer to, for example, [16, 24, 25, 27, 36, 49, 50, 52–55] for more information on $L_\lambda$-affine surface area for convex bodies. The $L_\mu$-affine surface areas for functions were introduced in [6].

We establish several affine isoperimetric inequalities for these quantities. Among them is a vector Blaschke Santaló-type inequality for log-concave functions with barycenter at 0,

$$\text{as}_\lambda(\vec{\varphi}) \text{as}_\lambda(\vec{\varphi}^\circ) \leq (2\pi)^n.$$

Here, $\varphi^\circ$ is the dual function of $\varphi$, defined in (16) and $\lambda \in [0, 1]$.

Please note that all the definitions and results hold, with obvious modifications, for $s$-concave functions as well. We refer the reader to [7] for that.

Throughout the paper, we will assume that the convex or concave functions $f : (0, \infty) \to \mathbb{R}$ and the log-concave functions $\varphi : \mathbb{R}^n \to [0, \infty)$ have enough smoothness and integrability
properties so that the expressions considered in the statements make sense, that is, we will always assume that \( \varphi \) and \( \varphi^\circ \in C^2 \cap L^1(\mathbb{R}^n, dx) \), where \( C^2 \) denotes the twice continuously differentiable functions, and that

\[
P \prod_{i=1}^n \left[ \varphi_i f_i \left( \frac{e^{(\nabla \varphi_i \cdot x) / \varphi_i}}{\varphi_i^2} \det(\text{Hess}(-\ln \varphi_i)) \right) \right] \in L^1(\mathbb{R}^n, dx).
\]  

(2)

2. Mixed f-divergence

2.1. Background on mixed f-divergence

In information theory, probability theory and statistics, an \( f \)-divergence is a function that measures the difference between two (probability) distributions. This notion was introduced by Csiszár [9], and independently Morimoto [41] and Ali and Silvery [1].

Let \( (X, \mu) \) be a finite measure space and let \( P = p \mu \) and \( Q = q \mu \) be (probability) measures on \( X \) that are absolutely continuous with respect to the measure \( \mu \). Let \( f : (0, \infty) \to \mathbb{R} \) be a convex or a concave function. The \( \ast \)-adjoint function \( f^\ast : (0, \infty) \to \mathbb{R} \) of \( f \) is defined by

\[
f^\ast(t) = tf(1/t), \quad t \in (0, \infty).
\]  

(3)

It is obvious that \((f^\ast)^\ast = f\) and that \( f^\ast \) is again convex if \( f \) is convex, respectively, concave if \( f \) is concave. Then the \( f \)-divergence \( D_f(P, Q) \) of the measures \( P \) and \( Q \) is defined by

\[
D_f(P, Q) = \int_X f \left( \frac{p}{q} \right) q \, d\mu.
\]  

(4)

It is a generalization of well-known divergences, such as, the variational distance, the Kullback–Leibler divergence or relative entropy, the Rényi divergence and many more. More on \( f \)-divergence can be found in, for example, [12, 21, 22, 42, 44, 54, 57].

For applications, such as statistical hypothesis test and classification, it is important to have extension of \( f \)-divergence from two (probability) measures to multiple (probability) measures, see, for example, [35, 40, 61].

For \( 1 \leq i \leq n \), let \( P_i = p_i \mu \) and \( Q_i = q_i \mu \) be probability measures on \( X \) that are absolutely continuous with respect to the measure \( \mu \). We also assume that the density functions \( p_i \) and \( q_i \) are nonzero almost everywhere with respect to \( \mu \). Denote by

\[
\bar{P} = (P_1, P_2, \ldots, P_n), \quad \bar{Q} = (Q_1, Q_2, \ldots, Q_n).
\]

We use \( \bar{p} \) and \( \bar{q} \) to denote the density vectors for \( \bar{P} \) and \( \bar{Q} \), respectively,

\[
\frac{d\bar{P}}{d\mu} = \bar{p} = (p_1, p_2, \ldots, p_n), \quad \frac{d\bar{Q}}{d\mu} = \bar{q} = (q_1, q_2, \ldots, q_n).
\]

For \( 1 \leq i \leq n \), let \( f_i : (0, \infty) \to \mathbb{R}_+ \) be either convex or concave functions. Denote by \( \bar{f} \) the vector \( \bar{f} = (f_1, f_2, \ldots, f_n) \) and the \( \ast \)-adjoint vector of \( \bar{f} \) by \( \bar{f}^\ast = (f_1^\ast, f_2^\ast, \ldots, f_n^\ast) \). The mixed \( f \)-divergence for \((\bar{f}, \bar{P}, \bar{Q})\) is defined in [57] as

\[
D_f(\bar{P}, \bar{Q}) = \int_X \prod_{i=1}^n \left[ f_i \left( \frac{p_i}{q_i} \right) q_i \right]^{1/n} \, d\mu.
\]  

(5)

If \( f_i = f \), \( P_i = P \) and \( Q_i = Q \), for all \( 1 \leq i \leq n \), then the mixed \( f \)-divergence becomes the classical \( f \)-divergence, defined in (4).

Similarly, the mixed \( f \)-divergence for \((\bar{f}, \bar{Q}, \bar{P})\) is

\[
D_f(\bar{Q}, \bar{P}) = \int_X \prod_{i=1}^n \left[ f_i \left( \frac{q_i}{p_i} \right) p_i \right]^{1/n} \, d\mu.
\]  

(6)
It is obvious that $D_f(P, Q) = D_f(Q, P)$. Therefore, it is enough to consider $D_f(P, Q)$, which we will do throughout the paper.

We now present some examples. For more examples and properties, see [57].

**Examples.** (1) For $1 \leq i \leq n$, let $f_i(t) = |t - 1|$. Then the mixed $f$-divergence becomes the mixed total variation of $P$ and $Q$, defined by Werner and Ye in [57],

$$D_f(P, Q) = \int_X \prod_{i=1}^n |p_i - q_i|^{1/n} d\mu. \quad (7)$$

(2) For $1 \leq i \leq n$, let $f_i(t) = \log t$. Then the mixed $f$-divergence is mixed Kullback–Leibler divergence or the mixed relative entropy of $P$ and $Q$ (see [57]),

$$D_{KL}(P, Q) = D_{f_i}(P, Q) = \int_X \prod_{i=1}^n \left[ q_i \log \frac{p_i}{q_i} \right]^{1/n} d\mu, \quad (8)$$

where for $a \in \mathbb{R}^n$, $a_+ = (\max\{a_1, 0\}, \max\{a_2, 0\}, \ldots, \max\{a_n, 0\})$. Recall that Kullback–Leibler divergence or relative entropy from $P$ to $Q$ is defined as (see, for example, [8])

$$D_{KL}(P\|Q) = \int_X q \log \frac{p}{q} d\mu. \quad (9)$$

2.2. Mixed $f$-divergence for log-concave functions

A function $\varphi : \mathbb{R}^n \to [0, \infty)$ is log concave, if it is of the form $\varphi(x) = e^{-\psi(x)}$, where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a convex function. For $1 \leq i \leq n$, we put

$$q_{\varphi_i} = \varphi_i \quad \text{and} \quad p_{\varphi_i} = \varphi_i^{-1} e^{(\nabla \varphi_i x)/\varphi_i} \det[Hess(- \log \varphi_i)]. \quad (10)$$

We use the expressions (10) to define the mixed $f$-divergences for log-concave functions. These quantities are the proper ones to use in order to define divergences for log-concave functions. This was shown in [7].

**Definition 1.** Let $f_i : (0, \infty) \to \mathbb{R}_+$ be convex and/or concave functions and let $\varphi_i : \mathbb{R}^n \to [0, \infty)$ be log-concave functions. Then the mixed $f$-divergence for $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$ is

$$D_f(P_{\vec{\varphi}}, Q_{\vec{\varphi}}) = D_f((P_{\varphi_1}, \ldots, P_{\varphi_n}), (Q_{\varphi_1}, \ldots, Q_{\varphi_n})) = \left[ \prod_{i=1}^n f_i \left( \frac{p_{\varphi_i}}{q_{\varphi_i}} \right) q_{\varphi_i} \right]^{1/n} dx$$

$$= \left[ \prod_{i=1}^n \varphi_i f_i \left( \frac{e^{(\nabla \varphi_i x)/\varphi_i}}{\varphi_i^2} \det[Hess(- \log \varphi_i)] \right) \right]^{1/n} dx. \quad (11)$$

**Remarks and Examples.** (i) If we let $f_i = f$ and $\varphi_i = \varphi$, $1 \leq i \leq n$, then we obtain the usual $f$-divergence for log-concave functions, $D_f(P_\varphi, Q_\varphi)$, defined in [7],

$$D_f(P_\varphi, Q_\varphi) = \int \varphi f \left( \frac{e^{(\nabla \varphi x)/\varphi}}{\varphi^2} \det[Hess(- \log \varphi)] \right) dx. \quad (12)$$

Thus, Definition 1 extends the definition (12) of $f$-divergence for a log-concave function of [7] and consequently the inequalities and identities given below generalize the ones given in [7]. This is our motivation for Definition 1.
(ii) Similarly to (11),
\[
D_f(Q_φ, P_φ) = D_f((Q_{φ_1}, \ldots, Q_{φ_n}), (P_{φ_1}, \ldots, P_{φ_n})) = \left[ \prod_{i=1}^{n} \left[ f_i \left( \frac{q_{φ_i}}{p_{φ_i}} \right) \right] \right]^{1/n} dx
\]
\[
= \left[ \prod_{i=1}^{n} \left[ \varphi_i^{-1} e^{(\nabla φ_i, x)/\varphi_i} \det[-\text{Hess}(\log φ_i)] f_i \left( \frac{φ_i^2 e^{-(\nabla φ_i, x)/φ_i}}{\det[\text{Hess}(-\log φ_i)]} \right) \right] \right]^{1/n} dx.
\]

(iii) If we write a log-concave function as \( φ = e^\psi \), \( ψ \) convex, then (11) becomes
\[
D_f(P_φ, Q_φ) = \left[ \prod_{i=1}^{n} \left[ e^{-ψ_i} f_i (e^{2ψ_i} - (\nabla ψ_i, x) \det[\text{Hess}(ψ_i)]) \right] \right]^{1/n} dx.
\] (13)

(iv) For \( 1 \leq i \leq n \), let \( A_i \) be a \((n \times n)\) positive-definite matrix, \( c_i > 0 \) be a constant and let \( φ_i(x) = c_i e^{-(1/2)(A_i x, x)} \). Then
\[
D_f(P_φ, Q_φ) = \frac{(2nπ)^{n/2}}{(\det(\sum_{i=1}^{n} A_i))^{1/2}} \left[ \prod_{i=1}^{n} \left[ c_i f_i \left( \frac{\det(A_i)}{c_i^2} \right) \right] \right]^{1/n}.
\] (14)
In particular, if \( A_i = A \) for all \( i \), where \( A \) is a \((n \times n)\) positive-definite matrix, then (14) becomes
\[
D_f(P_φ, Q_φ) = \frac{(2π)^{n/2}}{\sqrt{\det(A)}} \left[ \prod_{i=1}^{n} \left[ c_i f_i \left( \frac{\det(A)}{c_i^2} \right) \right] \right]^{1/n}.
\] (15)

**Proposition 2.** For \( 1 \leq i \leq n \), let \( f_i : (0, \infty) \to \mathbb{R}_+ \) be convex and/or concave functions and let \( φ_i : \mathbb{R}^n \to [0, \infty) \) be log-concave functions. Then \( D_f(P_φ, Q_φ) \) is invariant under self-adjoint \( SL(n) \) maps.

**Proof.** Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) be a self-adjoint, \( SL(n) \) invariant linear map.
\[
D_f(P_{φ_{10A}}, \ldots, P_{φ_{n0A}}), (Q_{φ_{10A}}, \ldots, Q_{φ_{n0A}}))
\]
\[
= \left[ \prod_{i=1}^{n} \left[ φ_i(Ax) f_i \left( φ_i(Ax)/\varphi_i(Ax) \right)^2 \det[\text{Hess}(-\log φ_i(Ax))] \right] \right]^{1/n} dx
\]
\[
= \frac{1}{\det(A)} \left[ \prod_{i=1}^{n} \left[ φ_i f_i \left( \frac{\det(A)}{φ_i^2} \right) \right] \right]^{1/n} dx
\]
\[
= D_f((P_{φ_1}, \ldots, P_{φ_n}), (Q_{φ_1}, \ldots, Q_{φ_n})). \] \( \square \)

Recall that for a function \( φ : \mathbb{R}^n \to [0, \infty) \), the dual function \( φ^* \) (see [2]) is defined by
\[
φ^*(y) = \inf_{x \in \mathbb{R}^n} \left[ \frac{e^{-φ(x,y)}}{φ(x)} \right].
\]

If \( φ \) is a log-concave function, that is, \( φ(x) = e^{-ψ(x)} \) with \( ψ : \mathbb{R}^n \to \mathbb{R} \) convex, then this duality notion is connected with the Legendre transform \( ψ^* (y) = \sup_{x \in \mathbb{R}^n} \{ (x, y) - ψ(x) \} \),
\[
φ^*(y) = e^{-ψ^*(y)}.
\] (16)

For special forms of the log-concave functions \( φ_i \), we have the following duality formula. This is the functional counterpart to the one proved in [59] for convex bodies and for special \( f \).
Theorem 3. For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be convex and/or concave functions and let $\varphi_i = \lambda_i \varphi$, for some log-concave function $\varphi : \mathbb{R}^n \to [0, \infty)$ and $\lambda_i > 0$. Then

$$D_f(P_{\varphi}, Q_{\varphi}) = D_{f_i}(P_{\varphi_i}, Q_{\varphi_i}).$$

Proof. We write $\varphi = e^{-\psi}$, $\psi$ convex, and let $\psi^*(y)$ be the Legendre transform of $\psi$. Please note that when $\psi$ is a $C^2$ strictly convex function, then

$$\psi(x) + \psi^*(y) = \langle x, y \rangle$$

if and only if $y = \nabla \psi(x)$ if and only if $x = \nabla \psi^*(y)$.

It follows that

$$\forall y \in \mathbb{R}^n, \quad \psi(\nabla \psi^*(y)) = \langle y, \nabla \psi^*(y) \rangle - \psi^*(y)$$

and

$$\nabla \psi \circ \nabla \psi^* = \nabla \psi^* \circ \nabla \psi = \text{Id},$$

so that for any $x, y \in \mathbb{R}^n$,

$$\text{Hess} \psi(\nabla \psi^*(y)) \text{Hess} \psi^*(y) = \text{Id} = \text{Hess} \psi^*(\nabla \psi(x)) \text{Hess} \psi(x).$$

Using equations (18)–(20), the change of variable $x = \nabla \psi^*(y)$ gives

$$D_{f_i}(P_{\varphi_i}, Q_{\varphi_i})$$

$$= \int \prod_{i=1}^{n} \left[ \frac{\varphi_i f_i^* \left( \frac{e^{(\nabla \varphi_i, x) / \varphi_i}}{\varphi_i^2} \text{det}[\text{Hess}(-\log \varphi_i)] \right)^{1/n}}{\lambda_i \varphi} \right]^{1/n} \ dx$$

$$= \int \prod_{i=1}^{n} \left[ \frac{e^{(\nabla \varphi_i, x) / \varphi_i} \det[\text{Hess}(-\log \varphi_i)] f_i \left( \frac{\lambda_i^2 e^{-(\nabla \varphi_i, x) / \varphi_i}}{\text{det}[\text{Hess}(-\log \varphi)]} \right)^{1/n} \ dx}{\lambda_i \varphi} \right]^{1/n}$$

$$= \frac{1}{\lambda_1 \cdots \lambda_n} \prod_{i=1}^{n} \frac{\det[\text{Hess}(\nabla \psi^*(y))] e^{(\psi(\nabla \psi^*(y)) - \langle y, \nabla \psi^*(y) \rangle)}}{\text{det}(\text{Hess}(\psi^*(y)))}^{1/n}$$

$$\times \prod_{i=1}^{n} \left[ f_i \left( \frac{\lambda_i^2 e^{-2\psi(\nabla \psi^*(y)) - \langle y, \nabla \psi^*(y) \rangle}}{\text{det}(\text{Hess}(\nabla \psi^*(y)))} \right) \right]^{1/n} \text{det}(\text{Hess}(\psi^*(y))) \ dy$$

$$= \frac{1}{\lambda_1 \cdots \lambda_n} \prod_{i=1}^{n} \left[ \frac{e^{-\psi(\nabla \psi^*(y))} f_i (\lambda_i^2 \text{det}(\text{Hess}(\psi^*(y))) e^{-\langle y, \nabla \psi^*(y) \rangle} + 2\psi^*(y))^{1/n}}{\text{det}(\text{Hess}(\psi^*(y)))} \right]^{1/n} \ dy$$

$$= \frac{1}{\lambda_1 \cdots \lambda_n} \prod_{i=1}^{n} \left[ \frac{\varphi_i^0 f_i \left( \frac{\lambda_i^2 \text{det}(\text{Hess}(-\log \varphi)) e^{(\nabla \varphi_i, x) / \varphi_i^0}}{(\varphi_i^0)^2} \right)}{\lambda_i \varphi} \right]^{1/n} \ dx$$

$$= D_f(P_{\varphi^0}, Q_{\varphi^0}).$$

The last part follows from the fact that $(\lambda \varphi)^0 = \varphi^0 / \lambda$, for $\lambda \in \mathbb{R}$, $\lambda \neq 0$. \hfill \Box

Remark 4. If $f_i = f$ and $\lambda_i = 1$, that is, $\varphi_i = \varphi$ for all $i = 1, \ldots, n$, then $D_f(P_{\varphi^0}, Q_{\varphi^0}) = D_{f_i}(P_{\varphi^0}, Q_{\varphi^0})$. This was proved in [7].

The classical Alexandrov–Fenchel inequality for mixed volumes of convex bodies is one of the most important results in convex geometry. We refer the reader to, for example, [45] for
the details and prove now an Alexandrov–Fenchel-type inequality for mixed $f$-divergences for log-concave functions. The proof is similar to one given in [57]. We include it for completeness. We use the following notation.

For $1 \leq m \leq n-1$ and $k > n - m$, we put

$$\vec{f}_{m,k} = (f_1, f_2, \ldots, f_{n-m}, f_k, \ldots, f_k),$$

$$P_{\varphi_{m,k}} = (P_{\varphi_1}, \ldots, P_{\varphi_{n-m}}, P_{\varphi_k}, \ldots, P_{\varphi_k}), \quad Q_{\varphi_{m,k}} = (Q_{\varphi_1}, \ldots, Q_{\varphi_{n-m}}, Q_{\varphi_k}, \ldots, Q_{\varphi_k}).$$

Following [17], we say that two functions $f$ and $g$ are effectively proportional if there are constants $a$ and $b$, not both zero, such that $af = bg$. Functions $f_1, \ldots, f_m$ are effectively proportional if every pair $(f_i, f_j), 1 \leq i, j \leq m$ is effectively proportional. A null function is effectively proportional to any function.

Moreover, for $1 \leq m \leq n-1$, we let

$$h_0(x) = \prod_{i=1}^{n-m} \left[ \varphi_i f_i \left( e^{\langle \nabla \varphi_i, x \rangle / \varphi_i} \varphi_i^2 \det[\text{Hess}(\log \varphi_i)] \right)^{1/n} \right]$$

(21)

and for $j = 0, \ldots, m-1$,

$$h_{j+1}(x) = \left[ \varphi_{n-j} f_{n-j} \left( e^{\langle \nabla \varphi_{n-j}, x \rangle / \varphi_{n-j}} \varphi_{n-j}^2 \det[\text{Hess}(\log \varphi_{n-j})] \right)^{1/n} \right].$$

(22)

Then an Alexandrov–Fenchel-type inequality holds for log-concave functions, namely, we have the following theorem.

**Theorem 5.** For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be either all convex or all concave functions and let $\varphi_i : \mathbb{R}^n \to [0, \infty)$ be log-concave functions. Then, for $1 \leq m \leq n-1$,

$$[D_f(P_{\varphi}, Q_{\varphi})]_m^m \leq \prod_{k=n-m+1}^n D_{f_{m,k}}(P_{\varphi_{m,k}}, Q_{\varphi_{m,k}}).$$

Equality holds if and only if one of the functions $h_j^{1/m}, 1 \leq j \leq m$, is null or all are effectively proportional.

If $m = n$, then

$$[D_f(P_{\varphi}, Q_{\varphi})]^n \leq \prod_{k=1}^n D_{f_k}(P_{\varphi_k}, Q_{\varphi_k}).$$

Equality holds if and only if one of the functions $h_j, 1 \leq j \leq n$, is null or all are effectively proportional.

**Remark 6.** In particular, equality holds in Theorem 5, if (i) all $\varphi_i$ coincide and $f_i = \lambda_i f$ for some positive convex function $f$ and $\lambda_i > 0, i = n-m+1, \ldots, n$, or (ii) $f_i = \lambda_i f$, for some positive convex function $f$, for some $\lambda_i > 0, \varphi_i = a_i \varphi$, for some positive, log-concave function $\varphi$, for some $a_i, i = n-m+1, \ldots, n$ and $f$ is homogeneous of degree $\alpha \in [0, 1)$. 


Proof. We first treat the case $m = n$. By Hölder’s inequality, for example, \cite{17},

$$D_f(P_\varphi, Q_\varphi) = \int \prod_{i=1}^{n} \left[ \varphi_i f_i \left( \frac{e^{(\nabla \varphi_i, x)} / \varphi_i}{\varphi_i^2 \det[\text{Hess}(-\log \varphi_i)]} \right)^{1/n} \right] \, dx$$

$$\leq \prod_{i=1}^{n} \left[ \left[ \varphi_i f_i \left( \frac{e^{(\nabla \varphi_i, x)} / \varphi_i}{\varphi_i^2 \det[\text{Hess}(-\log \varphi_i)]} \right)^{1/n} \right] \right] \, dx$$

$$= \prod_{i=1}^{n} (D_f(P_\varphi, Q_\varphi))^{1/n}.$$

Let now $m \leq n - 1$. Again, by Hölder’s inequality,

$$[D_f(P_\varphi, Q_\varphi)]^m = \left( \int \prod_{j=0}^{m-1} (h_0(x) h_{j+1}^m(x))^{1/m} \, dx \right)^m$$

$$\leq \prod_{j=0}^{m-1} \left( h_0(x) h_{j+1}^m(x) \right) \, dx = \prod_{k=n-m+1}^{n} D_{f_{m,k}}(P_{\varphi_{m,k}}, Q_{\varphi_{m,k}}).$$

In both cases, characterization of equality follows from the equality characterization in Hölder’s inequality, for example, \cite{17}.

The following entropy inequality is a consequence of Theorem 5.

**Theorem 7.** For $1 \leq i \leq n$, let $f_i : (0, \infty) \to \mathbb{R}_+$ be concave functions and let $\varphi_i : \mathbb{R}^n \to [0, \infty)$ be log-concave functions. Then

$$[D_f(P_\varphi, Q_\varphi)]^n \leq \prod_{i=1}^{n} f_i \left( \frac{\int \varphi_i^2 \, dx}{\int \varphi_i \, dx} \right) \left( \int \varphi_i \, dx \right).$$

Equality holds if and only if $\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}$ where $c_i$ is a positive constant and $A$ is a $(n \times n)$ positive-definite matrix.

**Proof.** The inequality follows immediately from Theorem 5 for $m = n$ and \cite[Theorem 1]{7}, which says that for a concave function $f : (0, \infty) \to \mathbb{R}$ and a log-concave function $\varphi : \mathbb{R}^n \to [0, \infty)$, we have

$$D_f(P_\varphi, Q_\varphi) \leq f \left( \frac{\int \varphi^2 \, dx}{\int \varphi \, dx} \right) \left( \int \varphi \, dx \right).$$

It was proved in \cite{58}, that equality holds in (24) if and only if $\varphi(x) = c e^{-(1/2)(Ax,x)}$ where $c > 0$ is a constant and $A$ is a $(n \times n)$ positive-definite matrix.

We now treat the equality characterization. Using (15), it is easy to check that equality holds if $\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}$, $1 \leq i \leq n$. On the other hand, if equality holds in (23), then in particular,

$$\prod_{i=1}^{n} D_{f_i}(P_\varphi, Q_\varphi) = \prod_{i=1}^{n} f_i \left( \frac{\int \varphi_i^2 \, dx}{\int \varphi_i \, dx} \right) \left( \int \varphi_i \, dx \right).$$

Thus, equality holds in particular for all $i$ in the entropy inequality (24), which, by the equality characterization of \cite{58}, means that for all $i$, $\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}$, where $c_i$ is a positive constant and $A_i$ is a $(n \times n)$ positive-definite matrix. Thus, also using (14), the equality holds if and only if $\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}$. \hfill \Box
condition leads to the following identity:

\[
\prod_{i=1}^{n} (\det(A_i))^{1/n} = \frac{\det(\sum_{i=1}^{n} A_i)}{n^n}.
\] (25)

The Brunn Minkowski inequality for matrices \[8, 10, 39\] says that for positive-definite matrices \(A_i, 1 \leq i \leq n\), one has

\[
\left( \det \left( \sum_{i=1}^{n} A_i \right) \right)^{1/n} \geq \sum_{i=1}^{n} \left( \det(A_i) \right)^{1/n},
\] (26)

with equality if and only if all \(A_i = \lambda_i A\) for some positive-definite matrix \(A\) and scalars \(\lambda_i \geq 0\).

It follows from the geometric arithmetic mean inequality that

\[
\left( \sum_{i=1}^{n} \left( \det(A_i) \right)^{1/n} \right)^n \geq n^n \prod_{i=1}^{n} \left( \det(A_i) \right)^{1/n},
\]

with equality if and only if \(\det(A_i) = \det(A_j)\), for all \(i, j\). With (26), we get altogether,

\[
\prod_{i=1}^{n} \left( \det(A_i) \right)^{1/n} \leq \frac{1}{n^n} \left( \sum_{i=1}^{n} \left( \det(A_i) \right)^{1/n} \right)^n \leq \frac{1}{n^n} \left( \det \left( \sum_{i=1}^{n} A_i \right) \right).
\] (27)

By assumption, equality (25) holds. Therefore, we have equality in both, the geometric arithmetic mean inequality and the Brunn Minkowski inequality which means that for all \(i, A_i = \lambda_i A\), for some \(\lambda > 0\), for some positive-definite matrix \(A\). Hence we have that \(\varphi_i(x) = c_i e^{-(Ax,x)/2}\).

If we let \(f_i(t) = \log t, 1 \leq i \leq n\), in Theorem 7, then we obtain the following corollaries. We use again the notation \(a_+ = (\max\{a_1, 0\}, \max\{a_2, 0\}, \ldots, \max\{a_n, 0\})\), for \(a \in \mathbb{R}^n\).

**Corollary 8.** For \(1 \leq i \leq n\), let \(\varphi_i : \mathbb{R}^n \to [0, \infty)\) be log-concave functions. Then

\[
[D_{KL}(P_{\varphi}, Q_{\varphi})]^n \leq \prod_{i=1}^{n} \left[ \log \left( \frac{\int_{\mathbb{R}^n} \varphi_i dx}{\int_{\mathbb{R}^n} \varphi_i^0 dx} \right) \right] + \left( \int_{\mathbb{R}^n} \varphi_i dx \right).
\] (28)

Equality holds if and only if \(\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}\), where \(c_i\) is a positive constant and \(A\) is a \((n \times n)\) positive-definite matrix.

**Corollary 9.** For \(1 \leq i \leq n\), let \(\varphi_i : \mathbb{R}^n \to [0, \infty)\) be log-concave functions such that \(\int x \varphi_i dx = 0\) for all \(i\). Then

\[
[D_{KL}(P_{\varphi}, Q_{\varphi})]^n \leq \prod_{i=1}^{n} \left[ \log \left( \frac{\int_{\mathbb{R}^n} (2\pi)^n}{\int_{\mathbb{R}^n} \varphi_i dx^2} \right) \right] + \left( \int_{\mathbb{R}^n} \varphi_i dx \right).
\] (29)

Equality holds if and only if \(\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}\), where \(c_i\) is a positive constant and \(A\) is a \((n \times n)\) positive-definite matrix.

**Proof.** The functional Blaschke Santaló inequality \[2, 4, 11, 20\] says that for a log-concave function \(\varphi\) with barycenter at 0, that is, \(\int x \varphi dx = 0\), one has

\[
\left( \int \varphi dx \right) \left( \int \varphi^0 dx \right) \leq (2\pi)^n,
\]
with equality if and only if there exists a positive-definite matrix $A$ and $c > 0$ such that $\varphi(x) = c e^{-\langle Ax, x \rangle / 2}$. We apply the functional Blaschke Santaló inequality on the right-hand side of (28) to each $\varphi_i$ and get inequality (29).

Using (15), it is easy to see that equality holds in (29) if $\varphi_i(x) = c_i e^{-\langle (1/2)A_i x, x \rangle}$, where $c_i$ is a positive constant and $A_i$ is a $(n \times n)$ positive-definite matrix. On the other hand, if equality holds in (29), then equality holds in particular for all $i$ in the functional Blaschke Santaló inequality which means that for all $i$, $\varphi_i(x) = c_i e^{-\langle (1/2)A_i x, x \rangle}$, where $c_i$ is a positive constant and $A_i$ is a $(n \times n)$ positive-definite matrix. Thus, as above in the proof of Theorem 7, the equality condition again leads to the identity

$$\prod_{i=1}^{n} (\det(A_i))^{1/n} = \frac{\det(\sum_{i=1}^{n} A_i)}{n^n}$$

and we conclude as above. \hfill \Box

3. The $i$th mixed $f$-divergence for log-concave functions

Throughout this section, let $f_1, f_2 : (0, \infty) \to \mathbb{R}^+$ be either convex or concave functions. As above, let $(X, \mu)$ be a finite measure space and, for $l = 1, 2$, let $P_l = p_l \mu$ and $Q_l = q_l \mu$ be measures on $X$ that are absolutely continuous with respect to the measure $\mu$. Denote $\tilde{f} = (f_1, f_2)$, $\tilde{P} = (P_1, P_2)$ and $\tilde{Q} = (Q_1, Q_2)$.

The $i$th mixed $f$-divergence was introduced in [57]. We refer the reader to [57] for properties and examples and only give the definition.

**Definition 10.** Let $i \in \mathbb{R}$. The $i$th mixed $f$-divergence for $(\tilde{f}, \tilde{P}, \tilde{Q})$ is defined in [57] as

$$D_{\tilde{f}}(\tilde{P}, \tilde{Q}; i) = \int_X \left[ f_1 \left( \frac{p_1}{q_1} \right) q_1 \right]^{i/n} \left[ f_2 \left( \frac{p_2}{q_2} \right) q_2 \right]^{(n-i)/n} d\mu. \quad (30)$$

As before, for $l = 1, 2$, we let

$$q_{\varphi_l} = \varphi_l \quad \text{and} \quad p_{\varphi_l} = \varphi_l^{-1} e^{\langle \nabla \varphi_l, x \rangle / \varphi_l} \det[\text{Hess}(-\log \varphi_l)] \quad (31)$$

and use Definition 10 with $q_l = q_{\varphi_l}$ and $p_l = p_{\varphi_l}$, $l = 1, 2$, and get the $i$th mixed $f$-divergences for log-concave functions.

**Definition 11.** Let $f_1, f_2 : (0, \infty) \to \mathbb{R}^+$ be either convex or concave functions and let $\varphi_1, \varphi_2 : \mathbb{R}^n \to [0, \infty)$ be log-concave functions. Let $i \in \mathbb{R}$. Then the $i$th mixed $f$-divergence of $\tilde{\varphi} = (\varphi_1, \varphi_2)$ is

$$D_{\tilde{f}}(P_{\varphi_1}, P_{\varphi_2}, (Q_{\varphi_1}, Q_{\varphi_2}); i) = \int \left[ f_1 \left( \frac{p_1}{q_1} \right) q_1 \right]^{i/n} \left[ f_2 \left( \frac{p_2}{q_2} \right) q_2 \right]^{(n-i)/n} dx$$

$$= \int \left[ \varphi_1 f_1 \left( \frac{e^{\langle \nabla \varphi_1, x \rangle / \varphi_1}}{\varphi_1^2} \det[\text{Hess}(-\log \varphi_1)] \right) \right]^{i/n}$$

$$\times \left[ \varphi_2 f_2 \left( \frac{e^{\langle \nabla \varphi_2, x \rangle / \varphi_2}}{\varphi_2^2} \det[\text{Hess}(-\log \varphi_2)] \right) \right]^{(n-i)/n} dx.$$

If we let $q_l = q_{\varphi_l}$ and $p_l = p_{\varphi_l}$, $l = 1, 2$, then the following proposition is an immediate consequence of [57, Proposition V.I]. We also denote

$$P_{\tilde{\varphi}} = (P_{\varphi_1}, P_{\varphi_2}), \quad Q_{\tilde{\varphi}} = (Q_{\varphi_1}, Q_{\varphi_2}).$$
Proposition 12. Let \( f_1, f_2 : (0, \infty) \rightarrow \mathbb{R}_+ \) be either convex or concave functions and let \( \varphi_1, \varphi_2 : \mathbb{R}^n \rightarrow [0, \infty) \) be log-concave functions. If \( j \leq i \leq k \) or \( k \leq i \leq j \), then
\[
D_f(P_\varphi, Q_\varphi; i) \leq [D_f(P_\varphi, Q_\varphi; j)]^{(k-i)/(k-j)} \times [D_f(P_\varphi, Q_\varphi; k)]^{(i-j)/(k-j)}.
\]
Equality holds trivially if \( i = k \) or \( i = j \). Otherwise, equality holds if and only if one of the functions \( f_l(p_{\varphi_1}/q_{\varphi_1}), l = 1, 2 \) is null or are effectively proportional.

The next corollary follows immediately from Proposition 12 and (24).

Corollary 13. Let \( \varphi_1, \varphi_2 : \mathbb{R}^n \rightarrow [0, \infty) \) be log-concave functions and let \( f_1, f_2 : (0, \infty) \rightarrow \mathbb{R}_+ \). If \( f_1, f_2 \) are concave and \( 0 \leq i \leq n \), then
\[
[D_f(P_\varphi, Q_\varphi; i)]^n \leq f_1 \left( \frac{\int \varphi_1^i \, dx}{\int \varphi_1 \, dx} \right) \left( \frac{\int \varphi_2 \, dx}{\int \varphi_2 \, dx} \right) \left( \frac{\int \varphi_2 \, dx}{\int \varphi_2 \, dx} \right)^{-n-i}.
\]
If (i) \( f_1 \) is convex, \( f_2 \) is concave and \( i \geq n \), or (ii) \( f_1 \) is concave, \( f_2 \) is convex and \( i \leq 0 \), then the inequality is reversed.

Equality holds trivially if \( i = 0 \) or \( i = n \). Otherwise, equality holds if and only if \( \varphi_1 = c_l e^{-(1/2)(Ax,x)}, l = 1, 2 \), where \( c_l \) is a positive constant and \( A \) is a \((n \times n)\) positive-definite matrix.

Proof. We give the proof in the first case. The others are done similarly. Let \( k = 0 \) and \( j = n \) (or \( j = 0 \) and \( k = n \)) in Proposition 12. By (24),
\[
[D_f(P_\varphi, Q_\varphi; i)]^n \leq [D_{f_1}(P_{\varphi_1}, Q_{\varphi_1})]^i \times [D_{f_2}(P_{\varphi_2}, Q_{\varphi_2})]^{n-i} \leq f_1 \left( \frac{\int \varphi_1^i \, dx}{\int \varphi_1 \, dx} \right) \left( \frac{\int \varphi_2 \, dx}{\int \varphi_2 \, dx} \right)^i f_2 \left( \frac{\int \varphi_2 \, dx}{\int \varphi_2 \, dx} \right) \left( \frac{\int \varphi_2 \, dx}{\int \varphi_2 \, dx} \right)^{n-i}.
\]

It is easy to see that equality holds if \( \varphi_l = c_l e^{-(1/2)(Ax,x)}, l = 1, 2 \), where \( c_l \) is a positive constant and \( A \) is a \((n \times n)\) positive-definite matrix. On the other hand, if equality holds in the inequality, then in particular, equality holds in (24), which means that \( \varphi_l = c_l e^{-(1/2)(A_l x,x)}, l = 1, 2 \), where \( c_l \) are positive constants and \( A_l \) are \((n \times n)\) positive-definite matrices. Thus, equality in the inequality leads to the following identity:
\[
\text{det} \left( \frac{i}{n} A_1 + \left(1 - \frac{i}{n}\right) A_2 \right) = (\text{det} A_1)^i/\text{det} A_2)^{1-i/n}.
\]
We conclude again, by the Brunn Minkowski inequality for matrices [8, 10, 39], that \( A_1 = A_2 \).

Remark 14. In particular, if we let \( f_1(t) = f_2(t) = \log(t) \) in Corollary 13, then we obtain similar results for the \( i \)th mixed Kullback–Leibler divergence, as in Corollary 8.

4. Applications to special functions: mixed \( L_\lambda \)-affine surface area

Now we consider special functions \( f \) and obtain special cases of mixed \( f \)-divergences for log-concave functions.
For \( i = 1, \ldots, n \), we let \( f_i(t) = t^\lambda, \ -\infty < \lambda < \infty \), and we obtain the mixed \( L_\lambda \)-affine surface area, denoted by \( a_\lambda(\vec{\varphi}) \), for log-concave functions \( \varphi_i \),

\[
a_\lambda(\vec{\varphi}) = \left[ \prod_{i=1}^{n} \left( \varphi_i \left( e^{(\nabla \varphi_i,x)/\varphi_i} \frac{\det(\text{Hess}(-\log \varphi_i))}{\varphi_i^2} \right) \right) \right]^{1/n} dx,
\]

or, writing \( \varphi_i(x) = e^{-\psi_i(x)} \), \( \psi_i \) convex,

\[
a_\lambda(\vec{\varphi}) = \left[ \prod_{i=1}^{n} \left( e^{(2\lambda-1)\psi_i(x) - \lambda(x, \nabla \psi_i(x))} \right) \right]^{1/n} dx.
\]

In particular, \( a_\lambda(\vec{\varphi}) = \int (\varphi_1 \cdots \varphi_n)^{1/n} dx \). Please note that for any \( \vec{\varphi} \), we have \( a_\lambda(\vec{\varphi}) \geq 0 \). Moreover, by Proposition 2, the \( a_\lambda(\vec{\varphi}) \) are invariant under self-adjoint \( \text{SL}(n) \) maps.

**Remarks.** (i) If we let \( \varphi_i = \varphi \) for \( i = 1, \ldots, n \), then we recover the \( L_\lambda \)-affine surface area, \( a_\lambda(\varphi) \), defined in [6] (see also [7]),

\[
a_\lambda(\varphi) = \int \varphi \left( e^{(\nabla \varphi,x)/\varphi} - \det(\text{Hess}(-\log \varphi)) \right) \lambda^{1/n} dx.
\]

(ii) For \( 1 \leq i \leq n \), let \( A_i \) be a \((n \times n)\) positive-definite matrix, \( c_i > 0 \) be a constant and let \( \varphi_i(x) = c_i e^{-(1/2)(A_i,x,x)} \). Then,

\[
a_\lambda(\vec{\varphi}) = \frac{(2n\pi)^{n/2}}{(\det(\sum_{i=1}^{n} A_i))^{1/2}} \prod_{i=1}^{n} [c_i^{1-2\lambda}(\det(A_i))^{\lambda}]^{1/n}.
\]

We also give a definition for \( a_{\infty}(\vec{\varphi}) \) and \( a_{-\infty}(\vec{\varphi}) \), similarly as it was done for the \( L_\lambda \)-affine surface area [7] (see also [37]).

\[
a_{\infty}(\vec{\varphi}) = \max_{x} \prod_{i=1}^{n} \left[ e^{(\nabla \varphi_i,x)/\varphi_i} \frac{\det(\text{Hess}(-\log \varphi_i))}{\varphi_i^2} \right]^{1/n} \quad \text{and} \quad a_{-\infty}(\vec{\varphi}) = \frac{1}{a_{\infty}(\vec{\varphi})}.
\]

The following two propositions are direct consequences of Theorems 3 and 5.

**Proposition 15.** Let \( \varphi_i : \mathbb{R}^n \to [0, \infty) \) be log-concave functions such that \( \varphi_i = a_i \varphi \) for some log-concave function \( \varphi : \mathbb{R}^n \to [0, \infty) \) and \( a_i > 0 \), \( i = 1, \ldots, n \). Then

\[
a_\lambda(\varphi) = a_{1-\lambda}(\varphi^{\circ}).
\]

Proposition 15 is generalization of the duality \( a_\lambda(\varphi) = a_{1-\lambda}(\varphi^{\circ}) \), proved in [6].

In the next proposition, we use, for \( k > n - m \), the notation

\[
a_\lambda(\vec{\varphi}_{m,k}) = \left[ \prod_{i=1}^{n-m} \left[ \varphi_i \left( e^{(\nabla \varphi_i,x)/\varphi_i} \frac{\det(\text{Hess}(-\log \varphi_i))}{\varphi_i^2} \right) \right]^{\lambda} \right]^{(n-m)/n} dx.
\]
Proposition 16. For $1 \leq i \leq n$, let $\varphi_i : \mathbb{R}^n \to [0, \infty)$ be log-concave functions and let $-\infty < \lambda < \infty$. Then, if $1 \leq m \leq n - 1$,

$$[\text{as}_\lambda(\vec{\varphi})]^m \leq \prod_{k=n-m+1}^{n} \text{as}_\lambda(\vec{\varphi}_{m,k}).$$

In particular, if $m = n$,

$$[\text{as}_\lambda(\vec{\varphi})]^n \leq \prod_{k=1}^{n} \text{as}_\lambda(\varphi_k).$$

The equality characterization is the same as in Theorem 5.

Next, we prove affine isoperimetric inequalities for the mixed $L_\lambda$-affine surface area.

Proposition 17. For $1 \leq i \leq n$, let $\varphi_i : \mathbb{R}^n \to [0, \infty)$ be log-concave functions such that $\varphi_i$ has barycenter at 0. If $\lambda \in [0, 1]$, then

$$\left( \frac{\text{as}_\lambda(\vec{\varphi})}{\text{as}_\lambda(g, \ldots, g)} \right)^n \leq \prod_{i=1}^{n} \left( \frac{\int \varphi_i}{\int g} \right)^{1-2\lambda},$$

(38)

where $g(x) = e^{-\|x\|^2/2}$. Equality holds if and only if $\varphi_i = c_i e^{-(1/2)(Ax,x)}$ where $c_i > 0$, $1 \leq i \leq n$, and $A$ is a $(n \times n)$ positive-definite matrix.

Proof. By Proposition 16,

$$\left( \frac{\text{as}_\lambda(\vec{\varphi})}{\text{as}_\lambda(g, \ldots, g)} \right)^n \leq \prod_{i=1}^{n} \frac{\text{as}_\lambda(\varphi_i)}{\text{as}_\lambda(g)} \leq \prod_{i=1}^{n} \left( \frac{\int \varphi_i}{\int g} \right)^{1-2\lambda}.$$  

The last part follows from a corollary in [6], which says that for a log-concave function $\varphi : \mathbb{R}^n \to [0, \infty)$ with barycenter at 0,

$$\frac{\text{as}_\lambda(\varphi)}{\text{as}_\lambda(g)} \leq \left( \frac{\int \varphi}{\int g} \right)^{1-2\lambda}.$$  

(39)

It was proved in [6] that equality holds if and only if $\varphi(x) = c e^{-(1/2)(Ax,x)}$ where $c > 0$ is a constant and $A$ is a $(n \times n)$ positive-definite matrix.

Using (35), it is easy to see that equality holds in (38) if $\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}$, where $c_i$ is a positive constant and $A$ is a $(n \times n)$ positive-definite matrix. On the other hand, if equality holds in (38), then equality holds in particular, for all $i$, in the inequality (39) which means that for all $i$, $\varphi_i(x) = c_i e^{-(1/2)(Ax,x)}$, where $c_i$ is a positive constant and $A_i$ is a $(n \times n)$ positive-definite matrix. Thus, as before, the equality condition again leads to the identity

$$\prod_{i=1}^{n} \left( \text{det}(A_i) \right)^{1/n} = \frac{\text{det}(\sum_{i=1}^{n} A_i)}{n^n}$$

and we conclude as before. \hfill \Box

We also have a Blaschke Santaló-type inequality.

Proposition 18. For $1 \leq i \leq n$, let $\varphi_i : \mathbb{R}^n \to [0, \infty)$ be log-concave functions such that $\varphi_i$ has barycenter at 0. If $\lambda \in [0, 1]$, then

$$\text{as}_\lambda(\vec{\varphi}) \text{as}_\lambda(\vec{\varphi^*}) \leq (2\pi)^n.$$  

(40)
Equality holds if and only if \( \varphi_i = c_i e^{-(1/2)(Ax,x)} \) where \( c_i > 0, 1 \leq i \leq n \), and \( A \) is a \((n \times n)\) positive-definite matrix.

**Proof.** By Proposition 16,

\[
[as_\lambda(\varphi)as_\lambda(\varphi^\circ)]^n \leq \prod_{i=1}^n as_\lambda(\varphi_i)as_\lambda(\varphi_i^\circ).
\]

The following Blaschke Santaló-type inequality was proved in [6]:

\[
as_\lambda(\varphi)as_\lambda(\varphi^\circ) \leq (2\pi)^n,
\]

(41)

where \( \varphi \) is a log-concave function with barycenter at 0. It was proved in [6] that equality holds if and only if \( \varphi(x) = ce^{-(1/2)(Ax,x)} \) where \( c > 0 \) is a constant and \( A \) is a \((n \times n)\) positive-definite matrix. Thus, the statement of the theorem follows. By the duality formula (35) and (37), it is easy to see that equality holds in (40) if \( \varphi_i(x) = c_i e^{-(1/2)(Ax,x)} \). On the other hand, if equality holds in (40), then equality holds in particular, for all \( i \), in the inequality (41) which means that for all \( i \), \( \varphi_i(x) = c_i e^{-(1/2)(Ax,x)} \), where \( c_i \) is a positive constant and \( A_i \) is a \((n \times n)\) positive-definite matrix. Note that for \( \varphi_i(x) = c_i e^{-(1/2)(Ax,x)} \), the dual function is \( \varphi_i^\circ(x) = c_i^{-1} e^{-(1/2)(A_i^{-1}x,x)} \). Thus, also using (35), the equality condition leads to the following identity:

\[
(det(A_1 + \cdots + A_n)det(A_1^{-1} + \cdots + A_n^{-1}))^{1/2} = n^n.
\]

(42)

Therefore, by (27), we must have for all \( i, A_i = AA_i \) for some \( \lambda > 0 \) and for some positive-definite matrix \( A \). Hence we have that \( \varphi_i(x) = c_i e^{-(Ax,x)/2} \).

The next proposition gives a monotonicity behavior of the mixed \( L_\lambda \)-affine surface area. The proofs follow by Hölder’s inequality (see also [7]).

**Proposition 19.** Let \( \alpha \neq \beta, \lambda \neq \beta \) be real numbers. Let \( \varphi_1, \ldots, \varphi_n : \mathbb{R}^n \rightarrow [0, \infty) \) be log-concave functions.

1. If \( 1 \leq (\alpha - \beta)/(\lambda - \beta) < \infty \), then \( as_\lambda(\varphi) \leq (as_\alpha(\varphi))^{(\lambda-\beta)/(\alpha-\beta)}(as_\beta(\varphi))^{(\alpha-\lambda)/(\alpha-\beta)} \).
2. If \( 1 \leq \alpha/\lambda < \infty \), then \( as_\lambda(\varphi) \leq (as_\alpha(\varphi))^{\lambda/\alpha}(\int(\varphi_1 \cdots \varphi_2)^{1/n})^{(\alpha-\lambda)/\alpha} \).
3. If \( \beta \geq \lambda \), then \( as_\lambda(\varphi) \leq (as_\infty(\varphi))^{\lambda/\beta} as_\beta(\varphi) \).

If \( (\alpha - \beta)/(\lambda - \beta) = 1 \) in (i), respectively, \( \alpha/\lambda = 1 \) in (ii), then \( \alpha = \lambda \) and equality holds trivially in (i), respectively, (ii). Equality also holds if for \( 1 \leq i \leq n \), \( \varphi_i(x) = c_i e^{-(1/2)(A_i,x,x)} \), where \( c_i \) is a positive constant and \( A_i \) is a \((n \times n)\) positive-definite matrix.

It follows from Proposition 19(ii) that for \( 0 < \lambda \leq \alpha \),

\[
0 \leq \left( \frac{as_\lambda(\varphi)}{\int(\varphi_1 \cdots \varphi_n)^{1/n} dx} \right)^{1/\lambda} \leq \left( \frac{as_\alpha(\varphi)}{\int(\varphi_1 \cdots \varphi_n)^{1/n} dx} \right)^{1/\alpha},
\]

which means that for \( \lambda > 0 \) the function \( (as_\lambda(\varphi)/\int(\varphi_1 \cdots \varphi_n)^{1/n} dx)^{1/\lambda} \) is bounded below by 0 and is increasing for \( \lambda > 0 \). Therefore, the limit

\[
\Omega_{\varphi} = \lim_{\lambda \downarrow 0} \left( \frac{as_\lambda(\varphi)}{\int(\varphi_1 \cdots \varphi_n)^{1/n} dx} \right)^{1/\lambda}
\]

exists and the quantity \( \Omega_{\varphi} \) is invariant under self-adjoint \( SL(n) \) maps. This quantity was first introduced by Paouris and Werner in [43] for convex bodies, then by Caglar and Werner [7] for log-concave functions using \( L_\lambda \)-affine surface area. It also follows from Proposition 19(ii)
that for \( \lambda < 0 \), the function \( \lambda \rightarrow (as_{\lambda}(\vec{\varphi})/\int (\varphi_1 \cdots \varphi_n)^{1/n} dx)^{1/\lambda} \) is increasing. Therefore, \( \lim_{\lambda \to 0}(as_{\lambda}(\vec{\varphi})/\int (\varphi_1 \cdots \varphi_n)^{1/n} dx)^{1/\lambda} \) exists and, in fact, is equal to \( \Omega_{\vec{\varphi}} \).

The quantity \( \Omega_{\vec{\varphi}} \) is related to the relative entropy as follows.

**Proposition 20.** Let \( \varphi_i : \mathbb{R}^n \to [0, \infty) \) be log-concave functions, \( i = 1, \ldots, n \). Then

\[
\Omega_{\vec{\varphi}} = \exp \left[ \frac{D_{KL}(P_{\Pi_{i=1}^n \varphi_i^{1/n}}, Q_{\Pi_{i=1}^n \varphi_i^{1/n}})}{\int \prod_{i=1}^n \varphi_i^{1/n} dx} + \log \left( \frac{\prod_{i=1}^n (\det\text{Hess}(-\log \varphi_i))^{1/n}}{(1/n) \sum_{i=1}^n \text{Hess}(-\log \varphi_i)} \right) \right],
\]

where

\[
d\mu = \frac{\prod_{i=1}^n \varphi_i^{1/n} dx}{\int \prod_{i=1}^n \varphi_i^{1/n} dx}.
\]

**Proof.** By definition and de l’Hôpital,

\[
\Omega_{\vec{\varphi}} = \lim_{\lambda \to 0} \left( \frac{as_{\lambda}(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{1/n} dx} \right)^{1/\lambda} = \lim_{\lambda \to 0} \exp \left( \frac{1}{\lambda} \log \left( \frac{as_{\lambda}(\vec{\varphi})}{\int (\varphi_1 \cdots \varphi_n)^{1/n} dx} \right) \right).
\]

Now we treat the exponent further. As \( q_{\Pi_{i=1}^n \varphi_i^{1/n}} = \prod_{i=1}^n \varphi_i^{1/n} = \prod_{i=1}^n \varphi_i^{1/n} \),

\[
p_{\Pi_{i=1}^n \varphi_i^{1/n}} = \prod_{i=1}^n \left( e^{(1/n)(\langle \nabla \varphi_i, x \rangle / \varphi_i)} \right) \varphi_i^{1/n} \text{det} \left[ \text{Hess} \left( -\log \prod_{i=1}^n \varphi_i^{1/n} \right) \right],
\]

we get that

\[
\int \prod_{i=1}^n \varphi_i^{1/n} \log \left[ \prod_{i=1}^n \left( e^{(1/n)(\langle \nabla \varphi_i, x \rangle / \varphi_i)} \right) \varphi_i^{1/n} \text{det} \left[ \text{Hess} \left( -\log \varphi_i \right) \right] \right] dx
\]

\[
= \int q_{\Pi_{i=1}^n \varphi_i^{1/n}} \log \left[ \prod_{i=1}^n \left( \frac{p_{\Pi_{i=1}^n \varphi_i^{1/n}}}{q_{\Pi_{i=1}^n \varphi_i^{1/n}}} \right) \frac{\prod_{i=1}^n (\text{det}\text{Hess}(-\log \varphi_i))^{1/n}}{\prod_{i=1}^n \text{det}\text{Hess}(-\log \varphi_i)} \right] dx
\]

\[
= D_{KL}(P_{\Pi_{i=1}^n \varphi_i^{1/n}} || Q_{\Pi_{i=1}^n \varphi_i^{1/n}}) + \int \prod_{i=1}^n \varphi_i^{1/n} \log \left[ \prod_{i=1}^n (\text{det}\text{Hess}(-\log \varphi_i))^{1/n} \right] \text{det} \left[ \frac{1}{n} \sum_{i=1}^n \text{Hess}(-\log \varphi_i) \right].
\]

**Corollary 21.** Let \( \varphi_i : \mathbb{R}^n \to [0, \infty) \) be log-concave functions, \( i = 1, \ldots, n \). Then

\[
\log(\Omega_{\vec{\varphi}}) \leq \frac{D_{KL}(P_{\Pi_{i=1}^n \varphi_i^{1/n}} || Q_{\Pi_{i=1}^n \varphi_i^{1/n}})}{\int \prod_{i=1}^n \varphi_i^{1/n} dx}.
\]

If \( n = 1 \), then equality holds trivially. Otherwise, equality holds if and only if one of \( \text{Hess}(-\log \varphi_i) \), \( 1 \leq i \leq n \), is null or all are effectively proportional.

**Proof.** For \( i = 1, \ldots, n \), we put \( H_i = \text{Hess}(-\log \varphi_i) \). Then, by Proposition 20,

\[
\Omega_{\vec{\varphi}} = \exp \left[ \frac{D_{KL}(P_{\Pi_{i=1}^n \varphi_i^{1/n}} || Q_{\Pi_{i=1}^n \varphi_i^{1/n}})}{\int \prod_{i=1}^n \varphi_i^{1/n} dx} \right] \exp \left[ \int \log \left( \frac{\prod_{i=1}^n (\det H_i)^{1/n}}{\text{det}[(1/n) \sum_{i=1}^n H_i]} \right) d\mu \right].
\]
It is easy to see that equality holds if \( n = 1 \). Otherwise, by (27),
\[
\prod_{i=1}^{n} (\det(H_i))^{1/n} \leq \frac{1}{n^n} \left( \det \left( \sum_{i=1}^{n} H_i \right) \right) = \det \left( \frac{1}{n} \sum_{i=1}^{n} H_i \right),
\]
with equality if and only if for all \( i \), \( H_i = \lambda H \), for some \( \lambda > 0 \) and \( H = \text{Hess}(\log \varphi) \), for some log concave \( \varphi \). Therefore,
\[
\Omega_{\varphi} \leq \exp \left[ \frac{D_{KL} \left( P_{\prod_{i=1}^{n} \varphi_i^{1/n}} || Q_{\prod_{i=1}^{n} \varphi_i^{1/n}} \right)}{\int \prod_{i=1}^{n} \varphi_i^{1/n} \, dx} \right],
\]
with equality if and only if for all \( i \), \( \text{Hess}(\log \varphi_i) = \text{Hess}(\log \varphi) \), for some log concave \( \varphi \), that is, \( \text{Hess}(\log \varphi_i) \) are all effectively proportional.

**Corollary 22.** Let \( \varphi_i : \mathbb{R}^n \to [0, \infty) \) be log-concave functions, \( i = 1, \ldots, n \).

(i) Then \( \Omega_{\varphi} \leq \frac{(\text{as}_\lambda(\varphi))}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} \), for all \( \lambda > 0 \) and \( \Omega_{\varphi} \geq \frac{(\text{as}_\lambda(\varphi))}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} \), for all \( \lambda < 0 \).

(ii) Let \( \varphi_i = a_i \varphi \) for some log-concave function \( \varphi : \mathbb{R}^n \to [0, \infty) \) and \( a_i > 0 \). Then
\[
\Omega_{\varphi} \leq 1.
\]

(iii) Let \( \varphi_i = a_i \varphi \) for some log-concave function \( \varphi : \mathbb{R}^n \to [0, \infty) \) and \( a_i > 0 \). Then
\[
\Omega_{\varphi} = \lim_{\alpha \to 1} \left( \frac{\text{as}_\alpha(\varphi)}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} \right)^{1/(1-\alpha)}.
\]

Equality holds in (i) and (ii) if \( \varphi_i = c_i e^{-(1/2)(Ax, x)} \) where \( c_i > 0 \), \( 1 \leq i \leq n \), and \( A \) is a \( (n \times n) \) positive-definite matrix.

**Proof.** Condition (i) is deduced immediately from the monotonicity behavior of the function \( \lambda \to \frac{(\text{as}_\lambda(\varphi))}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} \) and the definition of \( \Omega_{\varphi} \).

By (i) and Proposition 15,
\[
\Omega_{\varphi} \leq \frac{\text{as}_1(\varphi)}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} = \frac{\text{as}_0(\varphi)}{\text{as}_1(\varphi)}, \quad \Omega_{\varphi} \leq \frac{\text{as}_0(\varphi)}{\text{as}_1(\varphi)}.
\]

(iii) We use the duality formula (37). By definition
\[
\Omega_{\varphi} = \lim_{\alpha \to 1} \left( \frac{\text{as}_\alpha(\varphi)}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} \right)^{1/(1-\alpha)} = \lim_{\lambda \to 0} \left( \frac{\text{as}_{1-\lambda}(\varphi)}{\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx} \right)^{1/\lambda}.
\]

Therefore, \( \Omega_{\varphi} = \lim_{\alpha \to 1} (\text{as}_\alpha(\varphi))/\mathcal{J}(\varphi_1 \cdots \varphi_n)^{1/n} \, dx)^{1/(1-\alpha)} \).

We define the \( i \)th mixed \( L_\lambda \)-affine surface area \( \text{as}_{\lambda,i}(\varphi) \) of \( \varphi = (\varphi_1, \varphi_2) \) by
\[
\text{as}_{\lambda,i}(\varphi) = \left[ \varphi_1 \left( \frac{e^{(\nabla \varphi_1, x)/\varphi_1}}{\varphi_1^2} \det[\text{Hess}(\log \varphi_1)] \right)^{1/n} \right] \left[ \varphi_2 \left( \frac{e^{(\nabla \varphi_2, x)/\varphi_2}}{\varphi_2^2} \det[\text{Hess}(\log \varphi_2)] \right)^{1/(n-1)/n} \right] \, dx.
\]
Clearly, for all \( \lambda \), as\(_{\lambda,0}(\vec{\varphi}) = as_{\lambda}(\varphi_2) \) and as\(_{\lambda,n}(\vec{\varphi}) = as_{\lambda}(\varphi_1) \). Moreover, as\(_{0,n}(\vec{\varphi}) = \int \varphi_1 \, dx \) and as\(_{1,n}(\vec{\varphi}) = \int \varphi_1^2 \, dx \) (see [7]). We also give a definition for as\(_{\infty,i}(\vec{\varphi}) \) and as\(_{\infty,i}(\vec{\varphi}) \).

\[
\text{as}_{\infty,i}(\vec{\varphi}) = \max_x \left[ \frac{e^{(\nabla \varphi_1 \cdot x) / \varphi_1^2}}{\varphi_1^2} \det[\text{Hess}(-\log \varphi_1)] \right]^{i/n} \prod_{l=1}^{m-1} \left[ \frac{e^{(\nabla \varphi_2 \cdot x) / \varphi_2^2}}{\varphi_2^2} \det[\text{Hess}(-\log \varphi_2)] \right]^{(n-i)/n},
\]

\[\text{as}_{\infty,i}(\vec{\varphi}) = \frac{1}{\text{as}_{\infty,i}(\vec{\varphi})}.\]

It is easy to see that these expressions are invariant under symmetric linear transformations with determinant 1.

**Remarks.**

(i) It follows from Proposition 15 that as\(_{1-\lambda,i}(\vec{\varphi}) = as_{\lambda,i}(\varphi^5) \) where \( \vec{\varphi} = (\varphi_1, \varphi_2) \) such that \( \varphi_1 = a\varphi_2, \ a > 0 \).

(ii) Let \( \varphi_1(x) = c_1 e^{-(1/2)(A_1 x, x)} \), where \( c_1 \) is a positive constant and \( A_1 \) is a \((n \times n)\) positive-definite matrix for \( l = 1, 2 \). Then,

\[
as_{\lambda,i}(\vec{\varphi}) = (c_1^2 c_2^{n-i})^{(1-2\lambda)/n} ((\det(A_1))^i (\det(A_2))^{n-i})^{\lambda/n} \frac{(2\pi)^n/2}{(\det(iA_1 + (n-i)A_2))^{1/2}}.
\]

The next proposition is identical to Proposition 19 and the proof follows by Hölder’s inequality.

**Proposition 23.** Let \( i \in \mathbb{R} \) and \( \alpha \neq \beta, \ \lambda \neq \beta \) be real numbers. Let \( \varphi_1, \varphi_2 : \mathbb{R}^n \to [0, \infty) \) be log-concave functions.

(i) If \( 1 \leq (\alpha - \beta) / (\lambda - \beta) < \infty \), then \( \text{as}_{\lambda,i}(\vec{\varphi}) \leq (\text{as}_{\alpha,i}(\vec{\varphi}))^{(\lambda - \beta) / (\alpha - \beta)} (\text{as}_{\beta,i}(\vec{\varphi}))^{(\alpha - \lambda) / (\alpha - \beta)} \).

(ii) If \( 1 \leq \alpha / \lambda < \infty \), then \( \text{as}_{\lambda,i}(\vec{\varphi}) \leq (\text{as}_{\alpha,i}(\vec{\varphi}))^{\lambda / \alpha} (\int \varphi_1^i \varphi_2^{n-i})^{(\alpha - \lambda) / \alpha} \).

(iii) If \( \beta < \lambda \), then \( \text{as}_{\lambda,i}(\vec{\varphi}) \leq (\text{as}_{\alpha,i}(\vec{\varphi}))^{\lambda - \beta} \text{as}_{\beta,i}(\vec{\varphi}) \).

If \( (\alpha - \beta) / (\lambda - \beta) = 1 \) in (i), respectively, \( \alpha / \lambda = 1 \) in (ii), then \( \alpha = \lambda \) and equality holds trivially in (i), respectively, (ii). Equality also holds if \( \varphi_1(x) = c_1 e^{-(1/2)(A_1 x, x)} \), where \( c_1 \) is a positive constant and \( A_1 \) is a \((n \times n)\) positive-definite matrix for \( l = 1, 2 \).

The following proposition is a direct consequence of Proposition 12.

**Proposition 24.** Let \( \varphi_1, \varphi_2 : \mathbb{R}^n \to [0, \infty) \) be log-concave functions. If \( j \leq i \leq k \) or \( k \leq i \leq j \), then

\[
as_{\lambda,i}(\vec{\varphi}) \leq \left[ \text{as}_{\lambda,j}(\vec{\varphi}) \right]^{(k-i)/(k-j)} \times \left[ \text{as}_{\lambda,k}(\vec{\varphi}) \right]^{(i-j)/(k-j)}.
\]

Equality holds trivially if \( i = k \) or \( i = j \). Otherwise, equality holds if and only if one of the functions \( \varphi_1((e^{(\nabla \varphi_1 \cdot x) / \varphi_1^2} / \varphi_1^2) \det[\text{Hess}(-\log \varphi_1)])^\lambda, l = 1, 2 \), is null or they are effectively proportional.

In Proposition 24, if we let \( j = 0 \) and \( k = n \), then for all \( \lambda \) and \( 0 \leq i \leq n \)

\[
\left[ \text{as}_{\lambda,i}(\vec{\varphi}) \right]^n \leq \left[ \text{as}_{\lambda,n}(\varphi_2) \right]^{n-i} \left[ \text{as}_{\lambda}(\varphi_1) \right]^i.
\]

If we let \( i = 0 \) and \( j = n \), then for all \( \lambda \) and \( k \leq 0 \)

\[
\left[ \text{as}_{\lambda,k}(\vec{\varphi}) \right]^n \geq \left[ \text{as}_{\lambda}(\varphi_2) \right]^{n-k} \left[ \text{as}_{\lambda}(\varphi_1) \right]^k.
\]
From inequality (45) and an inequality of [6], already quoted here as inequality (41), one gets for functions with barycenter at 0,

\[
[as_{\lambda,i}(\varphi_1, \varphi_2)]^n [as_{\lambda,i}(\varphi_1^\circ, \varphi_2^\circ)]^n \leq [as_{\lambda}(\varphi_2)][as_{\lambda}(\varphi_1^\circ)]^{n-i}[as_{\lambda}(\varphi_1)]^i \leq (2\pi)^n
\]

holds true for all \( \lambda \in [0,1] \) and \( 0 \leq i \leq n \). Hence, we have proved the following proposition which also follows directly from Proposition 18.

**Proposition 25.** Let \( \varphi_1, \varphi_2 \) be log-concave functions with barycenter at 0. If \( \lambda \in [0,1] \) and \( 0 \leq i \leq n \), then

\[
[as_{\lambda,i}(\varphi_1)as_{\lambda,i}(\varphi_2^\circ)] \leq (2\pi)^n.
\]

Equality holds if and only if \( \varphi_l = c_l e^{-(1/2)(Ax,x)} \) where \( c_l > 0 \), \( l = 1,2 \), and \( A \) is a \((n \times n)\) positive-definite matrix.

**Proof.** The inequality follows from above. Using (44) and the duality formula \( as_{1-\lambda,i}(\varphi) = as_{\lambda,i}(\varphi^\circ) \), it is easy to see that equality holds in (47) if \( \varphi_l = c_l e^{-(1/2)(Ax,x)} \) where \( c_l > 0 \) and \( A \) is a \((n \times n)\) positive-definite matrix. On the other hand, if equality holds in (47), then equality holds in particular, for \( l = 1,2 \), in the inequality (41) which means that, \( \varphi_l(x) = c_l e^{-(1/2)(Ax,x)} \), where \( c_l \) is a positive constant and \( A_l \) is a \((n \times n)\) positive-definite matrix. Note that for \( \varphi_l(x) = c_l e^{-(1/2)(Ax,x)} \), the dual function is \( \varphi_l^\circ(x) = c_l^{-1} e^{-(1/2)(A_l^{-1}x,x)} \). Thus, also using (44), the equality condition leads to the following identity:

\[
(det(iA_1 + (n-i)A_2)det(iA_1^{-1} + (n-i)A_2^{-1}))^{1/2} = n^n.
\]

Therefore, by (27), we must have \( A_1 = A_2 \). Hence we have that \( \varphi_l(x) = c_l e^{-(Ax,x)/2} \).

**References**

1. M. S. Ali and D. Silvey, ‘A general class of coefficients of divergence of one distribution from another’, *J. R. Stat. Soc. Ser. B* 28 (1966) 131–142.
2. S. Artstein-Avidan, B. Klartag and V. Milman, ‘The Santaló point of a function, and a functional form of Santaló inequality’, *Mathematika* 51 (2004) 33–48.
3. S. Artstein-Avidan, B. Klartag, C. Schütt and E. Werner, ‘Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality’, *J. Funct. Anal.* 262 (2012) 4181–4204.
4. K. Ball, ‘Isometric problems in linear and sections of convex sets’, PhD Dissertation, University of Cambridge, 1986.
5. A. Bhattacharyya, ‘On some analogues to the amount of information and their uses in statistical estimation’, *Sankhya* 8 (1946) 1–14.
6. U. Caglar, M. Fradelizi, O. Guédon, J. Lehec, C. Schütt and E. Werner, ‘Functional version of \( L_p \)-affine surface area and entropy inequalities’, Preprint, 2014, arXiv:1402.3250.
7. U. Caglar and E. M. Werner, ‘Divergence for \( s \)-concave and log concave functions’, *Adv. Math.* 257 (2014) 219–247.
8. T. Cover and J. Thomas, *Elements of information theory*, 2nd edn, Wiley-Interscience (Wiley, Hoboken, NJ, 2006).
9. I. Csiszár, ‘Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffischen Ketten’, *Publ. Math. Inst. Hungar. Acad. Sci. Ser. A* 8 (1963) 84–108.
10. K. Fan, ‘On a theorem of Weyl concerning eigenvalues of linear transformations’, *Proc. Natl Acad. Sci. USA* 36 (1950) 31–35.
11. M. Fradelizi and M. Meyer, ‘Some functional forms of Blaschke–Santaló inequality’, *Math. Z.* 256 (2007) 379–395.
12. D. García-García and R. C. Williamson, ‘Divergences and risks for multiclass experiments’, *Proceedings of Annual Conference on Learning Theory*, Edinburgh, 23 (2012) 28.1–28.20.
13. R. J. Gardner, ‘The Brunn–Minkowski Inequality’, *Bull. Amer. Math. Soc.* 39 (2002) 355–405.
14. R. J. Gardner and G. Zhang, ‘Affine inequalities and radial mean bodies’, *Amer. J. Math.* 120 (1998) 505–528.
28. E. Lutwak, D. Yang and G. Zhang, ‘Information theoretic inequalities for contoured probability distributions’, IEEE Trans. Inform. Theory 48 (2002) 2377–2383.
29. C. Haberl and F. Schuster, ‘General Lp affine isoperimetric inequalities’, J. Differential Geom. 83 (2009) 1–26.
30. G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, 2nd edn (Cambridge University Press, Cambridge, 1952).
31. J. Jenkinson and E. Werner, ‘Relative entropies for convex bodies’, Trans. Amer. Math. Soc., 366 (2014) 2889–2906.
32. S. Kullback and R. Leibler, ‘On information and sufficiency’, Ann. Math. Statist. 22 (1951) 79–86.
33. J. Lehec, ‘A simple proof of the functional Santaló inequality’, Duke Math. J. 119 (2003) 159–188.
34. M. Ludwig and M. Reitzner, ‘A characterization of affine surface area’, Adv. Math. 147 (1999) 138–172.
35. M. Ludwig and M. Reitzner, ‘A classification of SL(n) invariant valuations’, Ann. of Math. 172 (2010) 1223–1271.
36. E. Lutwak, ‘Mixed affine surface area’, J. Math. Anal. Appl. 125 (1987) 351–360.
37. E. Lutwak, ‘The Brunn–Minkowski–Firey theory II: affine and geominimal surface areas’, Adv. Math. 118 (1996) 244–294.
38. E. Lutwak, D. Yang and G. Zhang, ‘Sharp affine Lp Sobolev inequalities’, J. Differential Geom. 62 (2002) 17–38.
39. E. Lutwak, D. Yang and G. Zhang, ‘The Cramer–Rao inequality for star bodies’, Duke Math. J. 112 (2002) 59–81.
40. E. Lutwak, D. Yang and G. Zhang, ‘Volume inequalities for subspaces of Lp’, J. Differential Geom. 68 (2004) 159–184.
41. E. Lutwak, D. Yang and G. Zhang, ‘Moment-entropy inequalities’, Ann. Probab. 32 (2004) 757–774.
42. E. Lutwak, D. Yang and G. Zhang, ‘Cramer–Rao and moment-entropy inequalities for Rényi entropy and generalized Fisher information’, IEEE Trans. Inform. Theory 51 (2005) 473–478.
43. K. Matušita, ‘On the notion of affinity of several distributions and some of its applications’, Ann. Inst. Statist. Math. 19 (1967) 181–192.
44. K. Matušita, ‘Some properties of affinity and applications’, Ann. Inst. Statist. Math. 23 (1971) 137–155.
45. M. L. Menéndez, J. A. Pardo, L. Pardo and K. Zografos, ‘A preliminary test in classification and probabilities of misclassification’, Statistics 39 (2005) 183–205.
46. M. Meyer and E. Werner, ‘The Santaló-regions of a convex body’, Trans. Amer. Math. Soc. 350 (1998) 4569–4591.
47. M. Meyer and E. Werner, ‘On the p-affine surface area’, Adv. Math. 152 (2000) 288–313.
48. V. Milman and L. Rotem, ‘Mixed integrals and related inequalities’, J. Funct. Anal. 264 (2013) 570–604.
49. H. Minkowski, ‘Diskontinuitätsbereich für arithmetische Äquivalenz’, J. für Math. 129 (1950) 220–274.
50. D. Morales, L. Pardo and K. Zografos, ‘Informational distances and related statistics in mixed continuous and categorical variables’, J. Statist. Plann. Inference 75 (1998) 47–63.
51. T. Morimoto, ‘Markov processes and the H-theorem’, J. Phys. Soc. Japan 18 (1963) 328–331.
52. F. Österreicher and I. Vajda, ‘A new class of metric divergences on probability spaces and its applicability in statistics’, Ann. Inst. Statist. Math. 55 (2003) 639–653.
53. G. Paouris and E. Werner, ‘Relative entropy of cone measures and Lp centroid bodies’, Proc. London Math. Soc. (3) 104 (2012) 253–286.
54. D. Morales, L. Pardo and K. Zografos, ‘Informational distances and related statistics in mixed continuous and categorical variables’, J. Statist. Plann. Inference 75 (1998) 47–63.
55. T. Morimoto, ‘Markov processes and the H-theorem’, J. Phys. Soc. Japan 18 (1963) 328–331.
56. F. Schuster, ‘Crofton measures and Minkowski valuations’, Duke Math. J. 154 (2010) 1–30.
57. A. Scarrò, ‘Informational divergence and the dissimilarity of probability distributions’, Calcolo 18 (1981) 293–302.
58. R. Ribeiro, ‘Information radius’, Probab. Theory Related Fields 14 (1969) 149–160.
59. C. Schütt and E. Werner, ‘The convex floating body’, Math. Scand. 66 (1990) 275–290.
60. C. Schütt and E. Werner, ‘Surface bodies and p-affine surface area’, Adv. Math. 187 (2004) 98–145.
61. G. T. Toussaint, ‘Some properties of Matusita’s measure of affinity of several distributions’, Ann. Inst. Statist. Math. 26 (1974) 389–394.
62. E. Werner, ‘On Lp-affine surface areas’, Indiana Univ. Math. J. 56 (2007) 2305–2324.
63. E. Werner, ‘Rényi divergence and Lp-affine surface area for convex bodies’, Adv. Math. 230 (2012) 1040–1059.
64. E. Werner, ‘f-Divergence for convex bodies’, Proceedings of the ‘Asymptotic Geometric Analysis’ Workshop (Fields Institute, Toronto, 2012).
65. E. Werner and D. Ye, ‘New Lp-affine isoperimetric inequalities’, Adv. Math. 218 (2008) 762–780.
66. E. Werner and D. Ye, ‘Inequalities for mixed p-affine surface area’, Math. Ann. 347 (2010) 703–737.
57. E. Werner and D. Ye, ‘On mixed f-divergence for multiple pairs of measures’, Preprint, 2013, arXiv:1304.6792.
58. E. Werner and T. Yolcu, ‘Equality characterization and stability for entropy inequalities’, Preprint, 2013, arXiv:1312.4148.
59. D. Ye, ‘Inequalities for general mixed affine surface areas’, J. London Math. Soc. 85 (2012) 101–120.
60. G. Zhang, ‘Intersection bodies and Busemann–Petty inequalities in R^n’, Ann. of Math. 140 (1994) 331–346.
61. K. Zografos, ‘f-dissimilarity of several distributions in testing statistical hypotheses’, Ann. Inst. Statist. Math. 50 (1998) 295–310.

Umut Caglar  
Department of Mathematics  
Case Western Reserve University  
Cleveland, OH 44106  
USA  
umut.caglar@case.edu

Elisabeth Werner  
Department of Mathematics  
Case Western Reserve University  
Cleveland, OH 44106  
USA

and

Université de Lille 1  
UFR de Mathématique  
59655 Villeneuve d’Ascq  
France  
elisabeth.werner@case.edu