A MOYAL QUANTIZATION OF THE CONTINUOUS TODA FIELD

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ABSTRACT

Since the lightcone self dual spherical membrane, moving in flat target backgrounds, has a direct correspondence with the SU(∞) Nahm equations and the continuous Toda theory, we construct the Moyal deformations of the self dual membrane in terms of the Moyal deformations of the continuous Toda theory. This is performed by using the Weyl-Wigner-Moyal quantization technique of the 3D continuous Toda field equation, and its associated 2D continuous Toda molecule, based on Moyal deformations of rotational Killing symmetry reductions of Plebanski first heavenly equation associated with 4D Self Dual Gravity.

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I. Introduction

The quantization program of the 3D continuous Toda theory (2D Toda molecule) is a challenging enterprise that we believe would enable us to understand many of the features of the quantum dynamics and spectra of the quantum self dual membrane [1]. This is based on the observation that the light-cone-gauge (spherical) supermembrane moving in a D dimensional flat spacetime background has a correspondence with a D − 1 SU(∞) Super Yang-Mills theory, dimensionally reduced to one temporal dimension; i.e. with a SU(∞) supersymmetric gauge quantum mechanical model (matrix model) [6].

It was shown in [1] that exact (particular) solutions of the D = 11 light-cone (spherical) supermembrane, related to the D = 10 SU(∞) SYM theory, could be constructed based on a particular class of reductions of the SYM equations from higher dimensions to four dimensions [25]. In particular, solutions of the D = 10 YM equations given by the D = 10 YM potentials, A_μ, can be obtained in terms of the D = 4 YM potentials, A_i, that obey the D = 4 Self Dual YM equations. Dimensional reductions of the latter SU(∞) SDYM equations to one temporal dimension are equivalent to the SU(∞) Nahm equations in the temporal gauge A_0 = 0.

Finally, the embedding of the continuous SU(∞) Toda equation into the SU(∞) Nahm equations was performed by the author [1] based on the connection between the D = 5 self dual membrane and the SU(2) Toda molecule/chain equations [2]. A continuous Toda theory in connection to self dual gravity was also found by Chapline and Yamagishi [7] in the description of a three-dimensional version of anyon superconductivity. Based on the theory of gravitational instantons a 3D model describing the condensation of quasiparticles (chirons) with properties related to fractional statistics was found.

The classical Toda theory can be obtained also from a rotational Killing symmetry reduction [3] of the 4D Self Dual Gravitational (SDG) equations expressed in terms of Plebanski first heavenly form that furnish (complexified) self dual metrics of the form:
\[ ds^2 = (\partial_{x^i} \partial_{\tilde{x}^j} \Omega) dx^i d\tilde{x}^j \text{ for } x^i = y, z; \tilde{x}^j = \tilde{y}, \tilde{z} \text{ and } \Omega \text{ is Plebanski first heavenly form. The latter equations can, in turn, be obtained from a dimensional reduction of the 4D } SU(\infty) \text{ Self Dual Yang Mills equations (SDYM), an effective } 6D \text{ theory} [4,5] \text{ and references therein. The Lie algebra } su(\infty) \text{ was shown to be isomorphic ( in a basis dependent limit) to the Lie algebra of area preserving diffeomorphisms of a 2D surface, } sdiff(\Sigma) \text{ by Hoppe [6]. It is for this reason that a WWM quantization of the reductions of Plebanski first heavenly equation will be used in this letter.}

The Toda theory also appears in the construction of noncritical } W_\infty \text{ strings . It was shown [1] that the expected critical dimension of the (super) membrane } D = 11, 27, \text{ was closely related to the number of spacetime dimensions of an anomaly-free noncritical ( super) } W_\infty \text{ string . A BRST analysis revealed that a very special sector of the membrane spectrum should have a relationship to the first unitary minimal model of a ( super) } W_N \text{ algebra adjoined to a critical ( super) } W_N \text{ string spectrum in the } N \rightarrow \infty \text{ limit. The study of the bosonic and supersymmetric case furnished } D = 27, 11 \text{ respectively. Noncritical } W_N \text{ strings are constructed by coupling } W_N \text{ conformal matter to } W_N \text{ gravity. By integrating out the matter sector the effective induced } W_N \text{ gravity action, in the conformal gauge, is obtained and it takes precisely the same form of a Toda action for the scalar fields [26]. The same action can be obtained from a constrained WZNW model by a quantum Drinfeld-Sokolov reduction process of a } SL(N, R) \text{ Kac-Moody algebra at level } k. \text{ Each of this Toda actions posseses a } W_N \text{ symmetry.}

It is important to emphasize that the effective ( super) Toda theory associated with noncritical ( super) } W_\infty \text{ strings in } D = 27, 11 \text{ dimensions is not the same Toda theory which appears in the embedding process of the continuous Toda theory into the } D = 4 \text{ } SU(\infty) \text{ Nahm equations [1]. However, one may fix, assign or choose the coupling constant of the latter Toda theory in terms of the former Toda theory if one wishes [1] with the proviso that the } 4D \text{ theory is anomaly free as well ( this needs to be verified). In this case the coupling constant turns out to be imaginary. Imaginary couplings are typical of Affine Toda theories [27] that correspond to massive but integrable deformations of conformal two-dim theories. Soliton solutions have been found and evidence of the conjectured relation between solitons in one Toda theory and fundamental particles of the dual Toda theory, at the quantum level, has been verified as well. For real couplings, another duality has also been found. The } S \text{ matrix of the Toda theory for one Kac-Moody algebra at coupling } \beta \text{ is equal to the } S \text{ matrix of the dual Toda theory for the dual Kac-Moody algebra with coupling } (1/\beta). \text{ For these reasons, the Toda models may be an appropriate laboratory to test the conjecture dualities of string, M,F,..theory.}

The masses of the solitons of these Affine Toda models can be understood in terms of a Higgs-like mechanism associated to the spontaneous breakdown of the conformal symmetry of a more general theory, the so-called Conformal Affine Toda models (CAT) [28] . Massless solitons travelling with speed less than } c \text{ have been found [29]. These CAT models can also be obtained by Hamiltonian reduction of the two-loop WZNW models [30]. Kac-Moody extensions of the area-preserving diffs algebra of the membrane have been constructed [31] and a Chern-Simmons gauge theory with gauge fields valued in an affine Kac-Moody algebra, after a Hamiltonian reduction, yields also the CAT models [32]. Once again, the connection between membrane and Toda models can be established via
these two-loop WZNW models.

Having outline some of the most salient features between Self Dual Gravity, membranes, $W_\infty$ strings, $SU(\infty)$ Nahm equations and the Toda models we embark into the quantization program of the continuous $SU(\infty)$ Toda theory. We remarked in [1] that a Killing symmetry reduction of the 4D Quantized Self Dual Gravity, via the $W_\infty$ co-adjoint orbit method performed by [7,8], gives a quantized Toda theory. In this letter we shall present a more direct quantization method and quantize the Toda theory using the Weyl-Wigner-Moyal prescription (WWM). A WWM description of the $SU(\infty)$ Nahm equations was carried out by [9] and a correspondence between BPS magnetic monopoles and hyper Kahler metrics was provided. There is a one-to-one correspondence between solutions of the Bogomolny equations with appropriate boundary conditions and solutions of the $SU(2)$ Nahm equations. As emphasized in [9], BPS monopoles are solutions of the Bogomolny equations whose role has been very relevant in the study of $D$ 3-branes realizations of $N = 2\ D = 4$ super YM theories in IIB superstrings [10]; $D$ instantons constructions [11]; in the study of moduli spaces of BPS monopoles and origins of “mirror” symmetry in 3D [12]; in constructions of self dual metrics associated with hyper Kahler spaces [13,14], among others.

Using our results of [15] based on [9,16] we show in II that a WWM [17] quantization approach yields a straightforward quantization scheme for the 3D continuous Toda theory (2D Toda molecule). Supersymmetric extensions can be carried out following [4] where we wrote down the supersymmetric analog of Plebanski equations for SD Supergravity. Simple solutions are proposed. There are fundamental differences between our results and those which have appeared in the literature [9]. Among these are (i) One is not taking the limit of $\hbar \to 0$ while having $N = \infty$ in the classical $SU(N)$ Nahm equations. (ii) We are working with the $SU(\infty)$ Moyal-Nahm equations and not with the $SU(2)$ Moyal-Nahm equations. Hence, we have $\hbar \neq 0; N = \infty$ simultaneously. (iii) The connection with the self dual membrane and $W_\infty$ algebras was proposed in [1] by the author. The results of [9] become very useful in the implementation of the WWM quantization program and in the embedding of the $SU(2)$ Moyal-Nahm solutions [9] into the $SU(\infty)$ Moyal-Nahm equations studied in the present work.

In the next section a Moyal quantization of the continuous Toda theory is performed and in III we decribe how the $SU(\infty)$ Moyal-Nahm equations are related to a WWM formalism of the Brockett equation associated with the continuum $\mathbb{Z}$-graded Lie algebras [18]. For $q$ deformations of the self dual membrane we refer to [33]. We expect that a $q$-Moyal deformation program of the self dual membrane might yield important information about how to quantize the full membrane theory beyond the self dual exactly integrable sector and that the particle/soliton spectrum of the Conformal Affine Toda models will shed some light into the particle content of the more general theory.

**II The Moyal Quantization of the Continuous Toda Theory**

In this section we shall present the Moyal quantization of the continuous Toda theory. The Moyal deformations of the rotational Killing symmetry reduction of Plebanski self dual gravity equations in 4D were given by the author in [15] based on the results of [16]. Starting with:
\[ \Omega(y, \tilde{y}, z, \tilde{z}; \kappa) \equiv \sum_{n=0}^{\infty} (\kappa/\tilde{y})^n \Omega_n(r, z, \tilde{z}). \]  

(1)

where each \( \Omega_n \) is only a function of the complexified variables \( r \equiv y\tilde{y} \) and \( z, \tilde{z} \). Our notation is the same from [15]. A real slice may be taken by setting \( \tilde{y} = y, \tilde{z} = z, \ldots \) The Moyal deformations of Plebanski’s equation read:

\[ \{ \Omega_z, \Omega_y \}_\text{Moyal} = 1. \]  

(2)

where the Moyal bracket is taken w.r.t the \( \tilde{z}, \tilde{y} \) variables. In general, the Moyal bracket may defined as a power expansion in the deformation parameter, \( \kappa \):

\[ \{ f, g \}_{\tilde{y}, \tilde{z}} \equiv [\kappa^{-1} \sin \kappa(\partial_{\tilde{y}} \partial_{\tilde{z}} - \partial_{\tilde{y}} \partial_{\tilde{z}})]fg. \]  

(3)

with the subscripts under \( \tilde{y}, \tilde{z} \) denote derivatives acting only on \( f \) or on \( g \) accordingly.

We begin by writing down the derivatives w.r.t the \( y, \tilde{y} \) variables when these are acting on \( \Omega \)

\[ \partial_y = \frac{1}{y} r \partial_r, \quad \partial_{\tilde{y}} = \frac{1}{\tilde{y}} r \partial_r. \]  

(4)

\[ \partial_y \partial_{\tilde{y}} = r \partial_r^2 + \partial_r, \quad \partial_{\tilde{y}}^2 = \left( \frac{1}{\tilde{y}} \right)^2 (r^2 \partial_r^2 + r \partial_r). \]

\[ \partial_{\tilde{y}}^3 = \left( \frac{1}{\tilde{y}} \right)^3 (r^3 \partial_r^3 + r^2 \partial_r^2 - r \partial_r). \]

Hence, the Moyal bracket (2) yields the infinite number of equations after matching, order by order in \( n \), powers of \( (\kappa/\tilde{y}) \):

\[ \{ \Omega_{0z}, \Omega_{0y} \}_\text{Poisson} = 1 \Rightarrow (r \Omega_{0r})_r \Omega_{0zz} - r \Omega_{0rz} \Omega_{0r\tilde{z}} = 1. \]  

(5)

\[ \Omega_{0zz}[-\Omega_{1r} + (r \Omega_{1r})_r] - r \Omega_{1rz} \Omega_{0rz} + \Omega_{1z \tilde{z}}(r \Omega_{0r})_r + \Omega_{0rz}(\Omega_{1z} - r \Omega_{1rz}) = 0. \]

the subscripts represent partial derivatives of the functions \( \Omega_n(r = y\tilde{y}, z, \tilde{z}) \) for \( n = 0, 1, 2, \ldots \) w.r.t the variables \( r, z, \tilde{z} \) in accordance with the Killing symmetry reduction conditions. The first equation, after a nontrivial change of variables, can be recast as the \( sl(\infty) \) continual Toda equation as demonstrated [2,3]. The remaining equations are the Moyal deformations. The symmetry algebra of these equations is the Moyal deformation of the classical \( w_\infty \) algebra which turns out to be precisely the centerless \( W_\infty \) algebra as shown by [19]. Central extensions can be added using the cocycle formula in terms of logarithms of derivative operators [20] giving the \( W_\infty \) algebra first built by [21].
From now on in order not to be confused with the notation of [5] we shall denote for 
\( \tilde{\Omega}(y', \tilde{y}', z', \tilde{z}; \kappa) \) to be the solutions to eq-(2). The authors [5] used \( \Omega(z + \tilde{y}, \tilde{z} - y, q, p; h) \) as solutions to the Moyal deformations of Plebanski equation. The dictionary from the results of [15], given by eqs-(1-5), to the ones used by the authors of [5] is obtained from the relation:

\[
\{ \tilde{\Omega}_{z'}, \tilde{\Omega}_{y'} \} \tilde{z}', \tilde{y}' = \{ \Omega_w, \Omega_{\tilde{w}} \} q, p = 1. \quad \kappa = \bar{h} \quad w = z + \tilde{y} \quad \tilde{w} = \tilde{z} - y. \tag{6a}
\]

For example, the four conditions : \( \tilde{\Omega}_{z'} = \Omega_w; \tilde{\Omega}_{y'} = \Omega_{\tilde{w}} \) and \( \tilde{z}' = q; \tilde{y}' = p \) are one of many which satisfy the previous dictionary relation (6a). One could perform a deformed-canonical transformation from \( \tilde{z}', \tilde{y}' \) to the new variables \( q, p \) iff the Moyal bracket \( \{ q, p \} = 1. \) Clearly, the simplest canonical transformation is the one chosen above. The latter four conditions yield the transformation rules from \( \tilde{\Omega} \) to \( \Omega. \) The change of coordinates:

\[
\tilde{z}' = q. \quad \tilde{y}' = p. \quad z' = z'(w, \tilde{w}, q, p|\Omega). \quad y' = y'(w, \tilde{w}, q, p|\Omega). \tag{6b}
\]

leads to:

\[
z' = w + f(p, q). \quad y' = \tilde{w} + g(p, q).
\]

once one sets:

\[
\tilde{\Omega}[z'(w, \tilde{w}, ...); y'(w, \tilde{w}, ...); \tilde{z}' = q; \tilde{y}' = p] = \Omega(w, \tilde{w}, q, p). \tag{6c}
\]

for \( \tilde{\Omega}, \Omega \) obeying eqs-(2,6a). The implicitly defined change of coordinates by the four conditions stated above is clearly dependent on the family of solutions to eqs-(2,6a). It is highly nontrivial. The reason this is required is because the choice of variables must be consistent with those of [9] to implement the WWM formalism. For example, choosing \( \Omega = \Omega_o = z'\tilde{z}' + y'\tilde{y}' \) as a solution to the eqs-(2,5) yields for (6b):

\[
z' = w + \lambda q. \quad y' = \tilde{w} - \lambda p. \tag{6d}
\]

The reality conditions on \( w, \tilde{w} \) may be chosen to be : \( \tilde{w} = \bar{w} \) which implies \( \tilde{z} = \bar{z}; \tilde{y} = -\bar{y}. \) It differs from the reality condition chosen for the original variables. It is important to remark as well that the variables \( p, q \) are also complexified and the area-preserving algebra is also : the algebra is \( su^*(\infty) \) [4].

Now we can make contact with the results of [5,9]. In general, the expressions that relate the 6D scalar field \( \Theta(z, \tilde{z}, y, \tilde{y}, q, p; h) \) to the 4D \( SU(\infty) \) YM potentials become, as a result of the dimensional reduction of the effective 6D theory to the 4D SDG one, the following [4,5] :

\[
\partial_z \Theta = \partial_y \Theta = \partial_w \Theta. \quad \partial_{\tilde{y}} \Theta = -\partial_{\tilde{z}} \Theta = -\partial_{\tilde{w}} \Theta. \tag{7a}
\]

with \( \kappa \equiv h \) and \( w = z + \tilde{y}; \bar{w} = \tilde{z} - y. \) Eqs-(7a) are basically equivalent to the integrated dimensional reduction condition:
\[ \Theta(z, \tilde{z}, y, \tilde{y}, q, p; h) = \Omega(z + \tilde{y}, \tilde{z} - y, q, p; h) \equiv \Omega(w, \tilde{w}, q, p; h) \equiv \sum_{n=0}^{\infty} (h)^n \Omega_n(r = w\tilde{w}; q, p). \]

which furnishes the Moyal-deformed YM potentials:

\[ A_{\tilde{z}}(\tilde{y}, w, \tilde{w}, q, p; h) = \partial_{\tilde{w}} \Omega(w, \tilde{w}, q, p; h) + \frac{1}{2} \tilde{y}. \quad A_{\tilde{y}}(\tilde{z}, w, \tilde{w}, q, p; h) = \partial_w \Omega(w, \tilde{w}, q, p; h) - \frac{1}{2} \tilde{z}. \]

(8)

One defines the linear combination of the YM potentials:

\[ A_{\tilde{z}} - A_y = A_{\tilde{w}}, \quad A_{\tilde{y}} + A_z = A_w \] (9)

The new fields are denoted by \( A_w, A_{\tilde{w}} \). After the following gauge conditions are chosen \( A_z = 0, A_y = 0 \), [5], it follows that \( A_{\tilde{z}} = A_{\tilde{w}} \) and \( A_{\tilde{y}} = A_w \).

For every solution of the infinite number of eqs-(5) by successive iterations, one has the corresponding solution for the YM potentials given by eqs-(8) that are associated with the Moyal deformations of the Killing symmetry reductions of Plebanski first heavenly equation. Therefore, YM potentials obtained from (5) and (8) encode the Killing symmetry reduction. In eq-(14) we shall see that the operator equations of motion corresponding to the Moyal quantization process of the Toda theory involves solely the operator \( \hat{\Omega} \). However, matters are not that simple because to solve the infinite number of equations (5) iteratively is far from trivial. The important fact is that in principle one has a systematic way of solving (2).

The authors [9] constructed solutions to the Moyal deformations of the \( SU(2)/SL(2) \) Nahm’s equations employing the Weyl-Wigner-Moyal (WWM) map which required the use of known representations of \( SU(2)/SL(2) \) Lie algebras [22] in terms of operators acting in the Hilbert space, \( L^2(R^1) \). Also known in [9] were the solutions to the classical \( SU(2)/SL(2) \) Nahm equations in terms of elliptic functions. The “classical” \( \hbar \to 0 \) limit of the WWM quantization of the \( SU(2) \) Nahm equations was equivalent to the \( N \to \infty \) limit of the classical \( SU(N) \) Nahm equations and, in this fashion, hyper Kahler metrics of the type discussed by [13,14] were obtained.

Another important conclusion that can be inferred from [5,9] is that one can embed the WWM-quantized \( SU(2) \) solutions of the Moyal-deformed \( SU(2) \) Nahm equations found in [9] into the \( SU(\infty) \) Moyal-deformed Nahm equations and have, in this way, exact quantum solutions to the Moyal deformations of the 2D continuous Toda molecule which was essential in the construction of the quantum self dual membrane [1]. Since a dimensional reduction of the \( W_\infty \oplus \hat{W}_\infty \) algebra is the symmetry algebra of the 2D effective theory, algebra that was coined \( U_\infty \) in [1], one can generate other quantum solutions by \( U_\infty \) co-adjoint orbit actions of the special solution found by [9]. One has then recovered the Killing symmetry reductions of the Quantum 4D Self Dual Gravity via the \( W_\infty \) co-adjoint orbit method [7,8].

The case displayed here is the converse. We do not have (as far as we know) \( SU(\infty) \) representations in \( L^2(R^1) \). However, we can in principle solve (5) iteratively. The goal is now to retrieve the operator corresponding to \( \Omega(w, \tilde{w}, q, p; h) \).
The WWM formalism [17] establishes the one-to-one map that takes self-adjoint operator-valued quantities, $\hat{\Omega}(w, \tilde{w})$, living on the 2D space parametrized by coordinates, $w, \tilde{w}$, and acting in the Hilbert space of $L^2(R^1)$, to the space of smooth functions on the phase space manifold $\mathcal{M}(q, p)$ associated with the real line, $R^1$. The map is defined:

$$\Omega(w, \tilde{w}, q, p; \sqrt{\hbar}) \equiv \int_{-\infty}^{\infty} d\xi < q - \frac{\xi}{2} |\hat{\Omega}(w, \tilde{w})| q + \frac{\xi}{2} > \exp\left[i\xi p\sqrt{\hbar}\right].$$

(10a)

Since the l.h.s of (10a) is completely determined in terms of solutions to eq-(2) after the iteration process in (5) and the use of the relation (6), the r.h.s is also known: the inverse transform yields the expectation values of the operator:

$$< q - \frac{\xi}{2} |\hat{\Omega}(w, \tilde{w})| q + \frac{\xi}{2} > = \int_{-\infty}^{\infty} dp \Omega(w, \tilde{w}, q, p; \hbar) \exp[-\frac{i\xi p}{\hbar}].$$

(10b)

i.e. all the matrix elements of the operator $\hat{\Omega}(w, \tilde{w})$ are determined from (10b), therefore the operator $\hat{\Omega}$ can be retrieved completely. The latter operator obeys the operator analog of the zero curvature condition, eq-(14), below. The authors in [23] have discussed ways to retrieve distribution functions, in the quantum statistical treatment of photons, as expectation values of a density operator in a diagonal basis of coherent states. Eq-(10b) suffices to obtain the full operator without the need to recur to the coherent (overcomplete) basis of states.

It is well known by now that the SDYM equations can be obtained as a zero curvature condition [24]. In particular, eq-(2). The operator valued extension of the zero-curvature condition reads:

$$\partial_{\tilde{z}}\hat{A}_{\tilde{y}} - \partial_{\tilde{y}}\hat{A}_{\tilde{z}} + \frac{1}{i\hbar}[\hat{A}_{\tilde{y}}, \hat{A}_{\tilde{z}}] = 0.$$  

(11)

which is the WWM transform of the original Moyal deformations of the zero curvature condition:

$$\partial_{\tilde{z}}A_{\tilde{y}}(\tilde{z}, q, p, w, \tilde{w}; \hbar) - \partial_{\tilde{y}}A_{\tilde{z}}(\tilde{y}, q, p, w, \tilde{w}; \hbar) + \{A_{\tilde{y}}, A_{\tilde{z}}\}_{q,p} = 0.$$  

(12)

This is possible due to the fact that the WWM formalism, the map $\mathcal{W}^{-1}$ preserves the Lie algebra commutation relation:

$$\mathcal{W}^{-1}\left(\frac{1}{i\hbar}[\hat{O}^i, \hat{O}^j]\right) \equiv \{O^i, O^j\}_{\text{Moyal}}.$$  

(13)

The latter equations (11,12) can be recast entirely in terms of $\Omega(w, \tilde{w}, q, p, \hbar)$ and the operator $\hat{\Omega}(w, \tilde{w})$ after one recurs to the relations $A_{\tilde{z}} = A_{\tilde{w}}; A_{\tilde{y}} = A_w$ (9) and the dimensional reduction conditions (7): $\partial_{\tilde{z}} = \partial_{\tilde{w}}; \partial_{\tilde{y}} = \partial_w$. Hence, one arrives at the main result of this section:

$$\frac{1}{i\hbar}[\hat{\Omega}_w, \hat{\Omega}_{\tilde{w}}] = \hat{1} \leftrightarrow \{\Omega_w, \Omega_{\tilde{w}}\}_{\text{Moyal}} = 1.$$  

(14)
i.e. the operator \( \hat{\Omega} \) obeys the operator equations of motion encoding the quantum dynamics. The carets denote operators. The operator form of eq-(14) was possible due to the fact that the first two terms in the zero curvature condition (12) are:

\[
\frac{\partial^2 \rho}{\partial \tilde{z} \partial z} = \frac{\partial^2 e^\rho}{\partial t'^2} \cdot \rho = \rho(z, \tilde{z}, t). \tag{16}
\]

At this stage we should point out that one should not confuse the variables \( z, \tilde{z}, t \) of eq-(16) with the previous \( z, \tilde{z} \) coordinates and the ones to be discussed below. The operator valued expression in (14) encodes the Moyal quantization of the continuous Toda field. The original continuous Toda equation is \([2,3,18]\):

\[
\partial_z A_{\tilde{y}} - \partial_{\tilde{y}} A_z = -1. \tag{15}
\]

as one can verify by inspection from the dimensional reduction conditions in (7) and after using (8).

The operator valued expression in (14) encodes the Moyal quantization of the continuous Toda field. The original continuous Toda equation is \([2,3,18]\):

\[
\frac{\partial^2 \rho}{\partial \tilde{z} \partial z} = \frac{\partial^2 e^\rho}{\partial t'^2} \cdot \rho = \rho(z, \tilde{z}, t). \tag{16}
\]

The mapping of the effective 3D fields \( \Omega_n(r \equiv w \tilde{w}, q, p) \) appearing in the power expansion (7b) into the \( u_n(r' \equiv t\tilde{t}; q', p') \), furnishes the Moyal deformed continuous Toda equation.

The map of the zeroth-order terms, \( \Omega_o(r, q, p) \rightarrow u_0(r', q', p') \) is the analog of the map that \([2,3]\) found to show how a rotational Killing symmetry reduction of the (undeformed) Plebanski equation leads to the ordinary continuous Toda equation (the first equation in the series appearing in (5)). Roughly speaking, to zeroth-order, having a function \( \Omega(r, z, \tilde{z}) \), one introduces a new set of variables \( t \equiv r \partial_r \Omega(r, z, \tilde{z}); s \equiv \partial_z \tilde{\Omega} \) and \( \tilde{w} = z; w = \tilde{z} \). After one eliminates \( s \) and defines \( r \equiv e^u \), one gets the field \( u = u(t, w, \tilde{w}) \) which satisfies the continuous Toda equation, as a result of the elimination of \( s \), iff the original \( \Omega(r, z, \tilde{z}) \) obeyed the Killing symmetry reduction of Plebanski’s equation to start with. The transformation from \( \Omega \) to \( u \) is a Legendre-like one.

Order by order in powers of \( (h)^n \) one can define:

\[
t = t_o + \hbar t_1 + \hbar^2 t_2 \ldots + \hbar^n t_n, \quad t_n \equiv \frac{r \partial \Omega_n(r, z, \tilde{z})}{\partial r}. \quad n = 0, 1, 2, \ldots
\]

\[
s = s_o + \hbar s_1 + \hbar^2 s_2 \ldots + \hbar^n s_n, \quad s_n \equiv \frac{\partial \Omega_n(r, z, \tilde{z})}{\partial \tilde{z}}. \quad n = 0, 1, 2, \ldots \tag{18}
\]

this can be achieved after one has solved iteratively eqs-(5) to order \( n \) for every \( \Omega_n(r, z, \tilde{z}) \); with \( n = 0, 1, 2, \ldots \). After eliminating \( s_0, s_1, s_2 \ldots s_n \), to order \( n \), one has for analog of the original relation: \( r = e^u \) the following:

\[
r = r(t = t_o + \hbar t_1 \ldots + \hbar^n t_n; z = \tilde{w}; \tilde{z} = w) \equiv e^{u_o + \hbar u_1 \ldots \hbar^n u_n}. \tag{19}
\]

eq-(19) should be viewed as:
\[ e^u = 1 + (u_o + \hbar u_1, ..., + \hbar^n u_n) + \frac{1}{2!} (u_o + \hbar u_1, ..., + \hbar^n u_n)^2 + \ldots \]

\[ r = r(t_0) + \frac{\partial r}{\partial t}(t_0)(\hbar t_1, ..., + \hbar^n t_n) + \frac{1}{2!} \frac{\partial^2 r}{\partial t^2}(t_0)(\hbar t_1, ..., + \hbar^n t_n)^2 + \ldots \]

\[ u_0 = u_0(t_0; z, \bar{z}). u_1 = u_1(t_0 + \hbar t_1; z, \bar{z}); \ldots; u_n = u_n(t_0 + \hbar t_1 + \hbar^2 t_2, ..., + \hbar^n t_n; z, \bar{z}). \quad (20) \]

This procedure will allow us, 
order by order in powers of \((\hbar)^n\), after eliminating \(s_0, s_1, s_2, \ldots\) to find the corresponding equations involving the functions \(u_n(t, w, \bar{w})\) iff the set of fields \(\Omega_n\) obeyed eqs-(5) to begin with. It would be desirable if one could have a master Legendre-like transform from the function \(\Omega(r, z; \bar{z}; \hbar)\) to the \(u(t, w, \bar{w}; \hbar)\) that would generate all the equations in one stroke. i.e. to have a compact way of writing the analog of eqs-(2,5) for the field \(u = \sum_n \hbar^n u_n\). In III we will define such transform for the 2D Toda molecule case.

A further dimensional reduction of the 3D continuous Moyal-Toda equation corresponds to the deformed 2D continuous Toda molecule equation which can be embedded into the Moyal deformations of the \(SU(\infty)\) Nahm’s equations. The ansatz which furnished the embedding of the 2D continuous Toda molecule into the ordinary \(SU(\infty)\) Nahm’s equations in connection to the quantization of the self dual membrane was studied in [1]. In the next section we will show how the master Legendre-like transform between eqs-(2,5) and the continuous Toda theory can be achieved by using the Brockett equation [18]. The supersymmetric extensions follow from the results of [4] where we wrote down the Plebanski analog of 4D Self Dual Supergravity.

To conclude this section: A WWM formalism is very appropriate to Moyal quantize the continuous Toda theory which we believe is the underlying theory behind (a sector of) the self dual membrane. Due to the variable entanglement of the original Toda equation, given by the first equation in the series of eqs-(5), one has to use the dictionary relation (6) that allows to use the WWM formalism of [9] in a straightforward fashion.

III. The \(SU(\infty)\) Moyal-Nahm equations

We will study in this section the Moyal-Nahm equations in connection to the 2D Toda molecule. To begin with, the Moyal bracket of two YM potentials \(A_y, \bar{A}_{\bar{y}}\), for example, can be expanded in powers of \(\hbar\) as [16]:

\[ \sum_{s=0}^{\infty} \frac{(-1)^2 \hbar^{2s}}{(2s + 1)!} \sum_{l=0}^{2s+1} (-1)^l (C_l^{2s+1}) [\partial_q^{2s+1-l} \partial_p^l A_y][\partial_p^{2s+1-l} \partial_q^l A_{\bar{y}}]. \quad (21) \]

where \(C_l^{2s+1}\) are the binomial coefficients.

The crucial difference between the solutions of the \(SU(2)\) Moyal-Nahm eqs [9] and the \(SU(\infty)\) Moyal-Nahm case is that one must have an extra explicit dependence on another variable, \(t\), for the YM potentials. For example, expanding in powers of \(\hbar\), the YM potentials involved in the \(SU(\infty)\) Moyal-Nahm equations are:

\[ A^i(\tau, t, q, p; \hbar) \equiv \sum_{n=0}^{\infty} \hbar^n A^i_n(\tau, t, q, p). \quad (22) \]
In this fashion, in the continuum limit, $N \to \infty$, we expect to have the continuous Moyal-Toda molecule and be able to embed it into the $SU(\infty)$ Moyal-Nahm equations. In this fashion, the relationship to the Moyal deformations of (a sector of) the self dual membrane will be established. Similar results hold in the supersymmetric case. It was for this reason that the highest weight representations of $W_{\infty}$ algebras should provide important information about the self dual membrane spectra [1]. Dimensional reduction of $W_{\infty} \oplus \bar{W}_{\infty}$ act as the spectrum generating algebra.

The continuous Toda molecule equation as well as the usual Toda system may be written in the double commutator form of the Brockett equation [18]:

$$\frac{\partial L(\tau, t)}{\partial \tau} = [L, [L, H]].$$

$L$ has the form

$$L \equiv A_+ + A_- = X_0(-iu) + X_{+1}(e^{\rho/2}) + X_{-1}(e^{\rho/2}),$$

with the connections $A_{\pm}$ taking values in the subspaces $G_0 \oplus G_{\pm 1}$ of some $Z$-graded continuum Lie algebra $G = \oplus_m G_m$ of a novel class. $H = X_0(\kappa)$ is a continuous limit of the Cartan element of the principal $sl(2)$ subalgebra of $G$. The functions $\kappa(\tau, t), u(\tau, t), \rho(\tau, t)$ satisfied certain equations given in [18].

To implement the WWM prescription, one may consider the case when $G$ is a group of unitary operators acting in the Hilbert space of square integrable functions on the line, $L^2(R^1)$. Then, $G$ is now the associated (continuum) Lie algebra of self-adjoint operators acting in the Hilbert space, $L^2(R^1)$. The operator-valued (acting in the Hilbert space) quantities depend on the two coordinates, $\tau, t$ and obey the operator version of the Brockett equation:

$$\frac{\partial \hat{L}(\tau, t)}{\partial \tau} = \frac{1}{i\hbar} [\hat{L}, \frac{1}{i\hbar} [\hat{L}, \hat{H}]], \leftrightarrow \frac{\partial \mathcal{L}}{\partial \tau} = \{\mathcal{L}, \{\mathcal{L}, \mathcal{H}\}\}. \quad (25)$$

where $\mathcal{L}(\tau, t, q, p; \hbar), \mathcal{H}(\tau, t, q, p; \hbar)$ are the corresponding elements in the phase space after performing the WWM map. The main problem with this approach is that we do not have representations of the continuum $Z$-graded Lie algebras in the Hilbert space, $L^2(R^1)$ and, consequently, we cannot evaluate the matrix elements $\langle q - \frac{\xi}{2} | \hat{L}(\tau, \tau)|q + \frac{\xi}{2} \rangle; \langle \hat{H} \rangle$. For this reason we have to recur to the iterations in (5) and insert the solutions into (10a,10b). The quantities $\mathcal{L}, \mathcal{H}$ are defined as:

$$\mathcal{L}(\tau, t, q, p; \hbar) \equiv \int_{-\infty}^{\infty} d\xi \langle q' - \frac{\xi}{2} | \hat{L}(\tau, t)|q' + \frac{\xi}{2} \rangle > \exp\left[i\frac{\xi p'}{\hbar}\right]. \quad (26)$$

$$\mathcal{H}(\tau, t, q, p; \hbar) \equiv \int_{-\infty}^{\infty} d\xi \langle q' - \frac{\xi}{2} | \hat{H}(\tau, t)|q' + \frac{\xi}{2} \rangle > \exp\left[i\frac{\xi p'}{\hbar}\right]. \quad (27)$$

the latter matrix elements, if known, suffice to determine the quantities $\mathcal{L}$ and $\mathcal{H}$ associated with the $2D$ continuous Toda molecule equation. Despite not knowing the explicit operator form of $\hat{L}(\tau, t)$ and $\hat{H}(\tau, t)$ acting in the Hilbert space, $L^2(R^1)$, one may still write down the
2D Toda equation in the double-commutator Brockett form \[18\] and embed the continuous Moyal-Toda molecule into the \(SU(\infty)\) Moyal-Nahm equations as follows:

First of all one must have:

\[
A_1(t, \tau, p, q; \hbar) = \sum_{n=0}^{\infty} \hbar^n A_1^n(t, \tau, p, q), \quad A_2(t, \tau, p, q; \hbar) = \sum_{n=0}^{\infty} \hbar^n A_2^n(t, \tau, p, q).
\]

\[
A_3(t, \tau, p, q; \hbar) = \sum_{n=0}^{\infty} \hbar^n A_3^n(t, \tau, p, q).
\] (28)

Secondly, we rewrite the \(SU(\infty)\) Moyal-Nahm equations as:

\[
\frac{\partial^2 A_3}{\partial \tau^2} = \{A_1, \{A_3, A_1\}\}_{Moyal} + \{\{A_2, A_3\}, A_2\}_{Moyal}.
\] (29)

and finally, given \(\mathcal{L}(\tau, t, q, p; \hbar)\) and \(\mathcal{H}(\tau, t, q, p; \hbar)\) one may impose the correspondence with eqs-(23-25):

\[
\{A_1, \{A_3, A_1\}\}_{Moyal} + \{\{A_2, A_3\}, A_2\}_{Moyal} \leftrightarrow \{\mathcal{L}, \{\mathcal{L}, \mathcal{H}\}\}_{Moyal}.
\] (30)

\[
\frac{\partial^2 A_3(q, p, \tau; \hbar)}{\partial \tau^2} \leftrightarrow \frac{\partial}{\partial \tau} \mathcal{L}(\tau, t, q, p; \hbar).
\] (73)

Eq-(29) is obtained by a straight dimensional reduction of the original 4D \(SU(\infty)\) SDYM equations, which is an effective 6D theory, to the final equations depending on four coordinates \((q, p, \tau, t)\). The temporal gauge condition \(A_0 = 0\) is required. Whereas the r.h.s of (30) is obtained through a sequence of reductions from the effective 6D theory \(\rightarrow 4D\ SDG \rightarrow 3D\) continuous Toda and, finally, to the continuous 2D Toda molecule as shown in II. The gauge conditions \(A_y = A_z = 0\) are required (see [5]) and a WWM formalism is performed in order to recover quantities depending on the two extra phase space variables, \(p, q\).

Eqs-(30) represent the embedding of the 2D continuous Moyal-Toda into the \(SU(\infty)\) Moyal-Nahm equations. It is desirable to write down explicitly the 2D continuous Moyal-Toda molecule equation. For this reason we shall discuss briefly the results of [16] where the Moyal-Nahm equations admit a reduction to the Toda chain.

The \(SU(2)\) Moyal-Nahm equations are of the form:

\[
\frac{\partial X}{\partial \tau} = \{Y, Z\}, \quad \frac{\partial Y}{\partial \tau} = \{Z, X\}, \quad \frac{\partial Z}{\partial \tau} = \{X, Y\}.
\] (31)

where the bracket is taken w.r.t the \(t, \tau\) variables. A slight change of notation is chosen in order to be consistent with the previous notation. The ansatz:

\[
X = h(t, \tau) \cos p, \quad Y = h(t, \tau) \sin p, \quad Z = f(t, \tau).
\] (32)
allowed [16] to resum the infinite series in the Moyal bracket. After the field redefinition $e^{\rho/2} = h(t, \tau)$, one obtains the Toda chain equation upon elimination of the function $f(t, \tau)$:

$$\frac{\partial^2 \rho}{\partial \tau^2} = -\left[\frac{\Delta - \Delta^{-1}}{h}\right]^2 e^\rho \Rightarrow \frac{\partial^2}{\partial t^2} e^\rho. \quad \rho = \rho(t, \tau). \quad (33)$$

The operators in the r.h.s of (33) are defined [16] as the shift operators: $\Delta f = f(t + \bar{h})$. $\Delta^{-1} f = f(t - \bar{h})$. In the classical $\bar{h} = 0$ limit one recovers the continuum Toda chain. The operator term in the r.h.s, when $\bar{h} = 0$, is exactly the continuum limit of the Cartan $SU(N)$ matrix:

$$K(t, t') = \frac{\partial^2}{\partial t \partial t'} \delta(t - t') \quad [18].$$

(34)

From section II one knows how to obtain (in principle, by iterations) solutions to the Moyal deformations of the effective 2D Toda molecule equations, starting from solutions to the Moyal deformations of the rotational Killing symmetry reductions of Plebanski first heavenly equation. This is attained after one has performed the Legendre-like transform from the $\Omega_n(r, \tau = z + \bar{z})$ fields to the $u_n(t, \tau = w + \bar{w})$ for $n = 0, 1, 2, ...$. Based on eqs (34), can one write an ansatz which encompasses the Legendre-like transform of the infinite number of eqs-(2,5) into one compact single equation involving the Moyal-deformed field: $\rho(t, \tau; \bar{h})$?

Assuming that the Legendre-like transform equations are recast in terms of the single $\rho(\tau, t; \bar{h})$ field whose expansion in powers of $\bar{h}$:

$$\rho(\tau, t; \bar{h}) \equiv \sum_{n=0}^{\infty} \bar{h}^n \rho_n(\tau, t). \quad (35)$$

A plausible guess, based on eqs(32-34), is to write as a tentative Moyal deformation of the continuous Toda molecule:

$$D^2(\tau; \bar{h}) \rho(t, \tau; \bar{h}) = -\sum_{n=0}^{\infty} \bar{h}^n \tilde{K}_n e^{\rho(t, \tau; \bar{h})}. \quad (36)$$

with $\tilde{K}_n$ is a collection of higher order differential operators w.r.t the $t$ variable. When $\rho$ is deformed in powers of $\bar{h}$ like in (35) this should also change the values of the $X, Y, Z$ quantities in the ansatz (32) and, in turn, it will change the value of (33). The derivative $D(\tau; \bar{h})$ is some deformation of the ordinary derivative w.r.t the $\tau$ variable. For example, one could use the Jackson derivative for the $q$ parameter $q = e^{\bar{h}}$.

$$D_z f(z) = \frac{f(z) - f(qz)}{(1 - q)z}. \quad q = e^{\bar{h}}. \quad (37)$$

The value of $\tilde{K}_0$ is given by the r.h.s of eq-(34). A plausible choice for the operators $\tilde{K}_n$ could be of the form:
\[ \mathcal{D}^2(\tau; h) \rho(t, \tau; h) = - \sum_{n=0}^{\infty} a_n h^n [\frac{\Delta - \Delta^{-1}}{h}]^{n+2} e^{\rho(t, \tau; h)}. \]  

(38)

However, this does not mean that (36,38) are the correct equation. The correct Legendre-like transform is the one provided by eqs-(30) in terms of the Brockett double commutator form.

The real test to verify whether or not (36,38) is indeed the correct continuous Moyal-Toda chain equation is to study the Legendre-like transform of the infinite number of eqs-(2,5) and see whether or not it agrees with eqs-(30). One could use eqs-(36,38) as the defining master Legendre-transform. The question still remains if such transform is compatible and consistent with eqs-(2,5) and with eqs-(18-20). And, furthermore, whether or not it agrees with eqs-(30) as well. Clearly one has to integrate eqs-(30) w.r.t the \( q, p \) variables in order to have a proper match of the \( \tau, t \) variables appearing in eqs-(36,38).

This difficult question is currently under investigation.

Concluding, the candidate master Legendre-like transform that takes the original Moyal-Plebanski equation \( \{ \Omega_w, \tilde{\Omega}_w \}_M = 1 \) for all fields, \( \Omega_n; \ n = 0, 1, 2, ... \), after the Killing symmetry and dimensional reductions, into the Moyal-Toda equations is the one given by eqs-(30). The number of variables matches exactly. The only difficulty is to write down representations of continuum \( \mathbb{Z} \)-graded Lie algebras in \( L^2(R^l) \). Eq-(30) encode the master Legendre-like transform between those reductions of the Moyal-Plebanski equations that are linked to the embeddings of the 2D Moyal-Toda molecule equation into the \( SU(\infty) \) Moyal-Nahm equations. The study of the geometry associated with these Moyal deformations has been given by [16,34].

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