THE CALABI INVARIANT AND THE LEAST NUMBER OF PERIODIC SOLUTIONS OF LOCALLY HAMILTONIAN EQUATIONS

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Abstract. In this paper we prove a lower bound for the least number of one-periodic solutions of nondegenerate locally Hamiltonian equations on compact symplectic manifolds in terms of the Betti numbers of the Novikov homology associated to the Calabi invariant of the locally Hamiltonian equations. Our result improves lower bounds obtained by Lê-Ono and Ono for the least number of nondegenerate locally Hamiltonian symplectic fixed points. Our result also generalizes the homological Arnold conjecture that has been proved by Fukaya-Ono and Liu-Tian.

Contents

1. Introduction 2
2. Floer-Novikov chain complexes on compact weakly monotone symplectic manifolds 5
3. The Betti numbers of Floer-Novikov homology 10
3.1. Novikov ring $\Lambda_R^{\theta,\omega}$ revisited 10
3.2. Admissible family of nondegenerate Hamiltonian functions 14
3.3. Invariance of the Betti numbers of Floer-Novikov chain complexes 23
3.4. Computing the Betti numbers of $HFN_* (\pi^*(H), \mathbb{F})$ 31
4. Proof of Theorem 1.3 35
5. Concluding remarks 35
Acknowledgement 36
References 36

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1. Introduction

Periodic solutions, after stationary points, are simplest objects in the qualitative theory of dynamical systems. In this paper we study one-periodic solutions of locally Hamiltonian systems on compact symplectic manifolds $(M^{2n},\omega)$. Given a symplectic form $\omega$ on $M^{2n}$, there is an isomorphism $L_\omega: TM^{2n} \rightarrow T^*M^{2n}$ that satisfies the following equation:

$$\langle L_\omega(V), W \rangle := -\omega(V, W)$$

for all $V, W \in TM^{2n}$.

Recall that a vector field $V$ on $M^{2n}$ is said to be locally Hamiltonian, if $L_\omega(V)$ is a closed 1-form on $M^{2n}$. A dynamical system on $M^{2n}$

$$\frac{d}{dt}x(t) = V_t(x(t))$$

is called locally Hamiltonian, if $V_t$ is a locally Hamiltonian vector field on $M^{2n}$ for all $t$. We refer the reader to [Tarasov2008, Chapter 16] for classical examples of locally Hamiltonian systems and to [Farber2004], [FJ2003] for topological consideration of flows generated by time independent locally Hamiltonian vector fields.

Let $\varphi_t : M^{2n} \rightarrow M^{2n}$ be the flow generated by the locally Hamiltonian vector fields $V_t$ in (1.1). Clearly, the set of the fixed points of the time-one map $\varphi_1$ is in 1-1 correspondence with the set of one-periodic solutions of (1.1), i.e. those solutions $x(t)$ with $x(0) = x(1)$. If we are interested only in one-periodic solutions of (1.1) we can assume w.l.o.g. that $V_t$ is one-periodic in $t$, i.e. $V_t = V_{t+1}$ for all $t$ [LO1995].

An important invariant of the time-one map $\varphi_1$ is its Calabi invariant, defined as follows [Banyaga1978]:

$$\text{Cal}(\varphi_1) := \left[ \int_0^1 L_\omega(V_t) \, dt \right] \in H^1(M^{2n}, \mathbb{R}).$$

In [LO1995, Deformation Lemma 2.1] Lê-Ono showed that there exist a one-periodic Hamiltonian function $H \in C^\infty(S^1 \times M^{2n})$ and a closed 1-form $\theta \in \Omega^1(M^{2n})$ such that the time-one map $\varphi_1$ associated with (1.1) is the solution at time $t = 1$ of the following equation

$$\frac{d}{dt} \varphi_t(x) = L_{\omega}^{-1}(\theta + dH_t)(\varphi_t(x)), \quad \varphi_0 = \text{Id}$$

(1.2)

where $H_t(x) := H(t, x)$, and $[\theta] = \text{Cal}(\varphi_1)$. Henceforth the set of one-periodic solutions of (1.1) coincides with the set of one-periodic solutions of the following equation

$$\frac{d}{dt}x(t) = L_{\omega}^{-1}(\theta + dH_t)(x(t)).$$

(1.3)
Thus, in the present paper we consider only one-periodic solutions of locally Hamiltonian equations of the form (1.3). We shall also call [θ] the Calabi invariant of the equation (1.3).

A one-periodic solution of (1.3) is called nondegenerate, if the fixed point x(0) of the associated time-one map ϕ1 is nondegenerate, or equivalently, \( \det(Id - d\varphi_1(x(0))) \neq 0 \). A locally Hamiltonian equation (1.3) is called nondegenerate, if all one-periodic solutions of (1.3) are nondegenerate. Since nondegenerate fixed points of a diffeomorphism are isolated, a nondegenerate locally Hamiltonian equation on a compact symplectic manifold \( M^{2n} \) has only a finite number of one-periodic solutions.

In [LO1995] Lê-Ono introduced Floer-Novikov chain complexes associated with nondegenerate locally Hamiltonian equations of the form (1.3) on compact weakly monotone symplectic manifolds \( (M^{2n}, \omega) \), see also section 2 below. As a result, Lê-Ono obtained the following.

**Proposition 1.1.** ([LO1995, Main Theorem]) Let \((M, \omega)\) be a closed symplectic manifold of dimension \(2n\) which satisfies the following condition

\[ c_1|\pi_2(M) = \lambda \omega|\pi_2(M), \; \lambda \neq 0, \]

and if \( \lambda < 0 \), the minimal Chern number \( N \) satisfies \( N > n - 3 \). Suppose \( \varphi_1 \) is the time-one map of the flow associated to (1.3). If all the fixed points of \( \varphi_1 \) are nondegenerate, then the number of fixed points of \( \varphi_1 \) is at least the sum of the Betti numbers of the Novikov homology over \( \mathbb{Z}_2 \) associated to the Calabi invariant of \( \varphi_1 \).

The restriction of Proposition 1.1 to the class of positively or negatively monotone symplectic manifolds is caused by the difficulty in computing the Floer-Novikov cohomology which depends on the Calabi invariant \([\theta] \in H^1(M^{2n}, \mathbb{R})\) of \( \varphi_1 \). In [Ono2005] Ono refined the energy estimate in [LO1995] in order to show that Floer-Novikov chain complexes can be defined over Novikov rings that are smaller than the one defined in [LO1995]. Using in addition the construction of the Kuranishi structure proposed by Fukaya-Ono in [FO1999], he proved another variant of Proposition 1.1 as follows.

**Proposition 1.2.** ([Ono2005, Theorem 1.1]) Let \((M^{2n}, \omega)\) be a compact symplectic manifold. Suppose \( \varphi_1 \) is the time-one map of the flow associated to (1.3). If all fixed points of \( \varphi \) are nondegenerate, the number of fixed points \( \text{Fix}(\varphi) \) of \( \varphi \) is not less than \( \sum_{\nu} \min -\text{nov}^\nu(M) \).

Let us recall the definition of \( \min -\text{nov}^\nu(M) \) introduced by Ono in [Ono2005]. For \( a \in H^1(M, \mathbb{R}) \) we denote by \( HN^*(M; a) \) the Novikov cohomology over \( \mathbb{Q} \) associated with \( a \). The function \( a \mapsto \text{rank} HN^*(M; a) \)
attains the absolute minimum at generic $a$. Denote by $\min_{-\text{nov}}^p(M)$ the minimum of rank $HN^p(M; a)$, which we call the $p$-th minimal Novikov number [Ono2005]. The number $\min_{-\text{nov}}^p(M)$ appears because it is also difficult to control the family of Floer-Novikov chain complexes as $\theta$ varies.

Denote by $\mathcal{P}(\omega, \theta, H)$ the set of all contractible one-periodic solutions of (1.3). The goal of this paper is to prove the following.

**Theorem 1.3 (Main Theorem).** Let $(M^{2n}, \omega)$ be a compact symplectic manifold. Assume that all the contractible one-periodic solutions of (1.3) are nondegenerate. Then the cardinal of the set $\mathcal{P}(\omega, \theta, H)$ is at least the sum of the Betti numbers of the Novikov homology over $\mathbb{Q}$ associated to the Calabi invariant of $\omega$. If moreover $(M^{2n}, \omega)$ is weakly monotone, then for any field $\mathbb{F}$ the cardinal of the set $\mathcal{P}(\omega, \theta, H)$ is at least the sum of the Betti numbers of the Novikov homology over $\mathbb{F}$ associated to the Calabi invariant of $\omega$.

We obtain from Theorem 1.3 immediately the following generalization of Propositions 1.1, 1.2.

**Corollary 1.4.** Let $(M^{2n}, \omega)$ be a compact symplectic manifold. Suppose $\varphi_1$ is the time-one map of the flow associated to (1.3). If all fixed points of $\varphi$ are nondegenerate, the number of fixed points $\text{Fix}(\varphi)$ of $\varphi$ is at least the sum of the Betti numbers of the Novikov homology over $\mathbb{Q}$ associated to the Calabi invariant of $\varphi$. If $(M^{2n}, \omega)$ is weakly monotone, then we can replace the sum of the Betti numbers of the Novikov homology over $\mathbb{Q}$ by the sum of the Betti numbers of the Novikov homology over $\mathbb{F}$ for any field $\mathbb{F}$.

**Remark 1.5.** Corollary 1.4 also generalizes different versions of the homological version of the Arnold conjecture for Hamiltonian symplectic fixed points that have been proved by Ono [Ono1995] for compact weakly monotone symplectic manifolds, and by Fukaya-Ono [FO1999] and Liu-Tian [LT1998] for general compact symplectic manifolds.

To prove Theorem 1.3 in the case $(M^{2n}, \omega)$ is weakly monotone, we first show that the underlying Novikov ring $\Lambda_{\theta, \omega}$ of Novikov-Floer chain complexes with coefficient in an integral domain $R$ is an integral domain (Proposition 3.3). Thus the Betti numbers of the Floer-Novikov homology groups are well-defined. Then we introduce notions of an admissible family of nondegenerate (multi-valued) Hamiltonian functions and its good neighborhood, which are generalization and formalization of the notion of special deformations of a nondegenerate symplectic isotopy that has been introduced in [LO1995] and refined in [Ono2005].
Using a good neighborhood of an admissible family of nondegenerate (multi-valued) Hamiltonian functions we compare the Betti numbers of two “close” Floer-Novikov chain complexes, using and extending results and ideas in [LO1995, Ono2005]. Then we compute the Betti numbers of the homology of a refined Floer chain complex on the minimal covering \( \tilde{M}^{2n} \) of \( M^{2n} \) associated with \([\theta]\), using standard arguments in Floer theory.

Our paper is organized as follows. In section 2 we recall the construction of Floer-Novikov chain complexes on compact weakly monotone symplectic manifolds, following [LO1995]. In section 3 we compute the Betti numbers of the Floer-Novikov homology with coefficient in an integral domain \( R \) in the case of compact weakly monotone symplectic manifold \( (M^{2n}, \omega) \). In section 4 we prove our main theorem. Finally in section 5 we discuss some open problems.

2. Floer-Novikov chain complexes on compact weakly monotone symplectic manifolds

In this section we summarize the construction of Floer-Novikov chain complexes on compact weakly monotone symplectic manifolds \( (M^{2n}, \omega) \), following [LO1995].

We always assume in this paper that the equation (1.3) is nondegenerate. We identify the set \( \mathcal{P}(\omega, \theta, H) \) of contractible one-periodic solutions of (1.3) with the zero-set of the following closed 1-form

\[
d\mathcal{A}_{\theta,H}(x, \xi) = \int \omega(\dot{x}, \xi) + (\theta + dH_t)(x(t))(\xi).
\]

We will restrict ourselves to the component \( \mathcal{L}M^{2n} \) of contractible loops on \( M^{2n} \). We construct an associated covering space \( \tilde{\mathcal{L}}\tilde{M}^{2n} \) of \( \mathcal{L}M^{2n} \) such that the pull back of \( d\mathcal{A}_{\theta,H} \) on this cover is an exact 1-form. Consider the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{\mathcal{L}}\tilde{M}^{2n} & \xrightarrow{j} & \mathcal{L}M^{2n} \\
\downarrow \tilde{\Pi} & & \downarrow \Pi \\
\tilde{\mathcal{L}}\tilde{M}^{2n} & \xrightarrow{j} & \mathcal{L}M^{2n} \\
\end{array}
\quad \xrightarrow{\tilde{e}} \quad \begin{array}{ccc}
\tilde{M}^{2n} \\
\downarrow \tilde{\Pi} \\
M^{2n}
\end{array}
\]

Here \( \tilde{M}^{2n} \) denotes the covering space of \( M^{2n} \) associated to the period homomorphism of \( \theta, I_{\theta} : \pi_1(M^{2n}) \to \mathbb{R} \). In other words the covering transformation group of \( \tilde{M} \) is isomorphic to the quotient group

\[
\Gamma_1 = H_1(M^{2n}, \mathbb{Z})/\ker I_{\theta}.
\]
In the above diagram $\tilde{e}$ denotes the evaluation map $x \mapsto x(0)$ and $\tilde{\mathcal{LM}}^{2n}$ is the covering of $\mathcal{LM}^{2n}$ whose deck transformation group is $\Gamma_2 = \frac{\pi_2(M^{2n})}{\text{ker}\, I_c \cap \text{ker}\, I_\omega}$, where the homomorphisms $I_c, I_\omega: \pi_2(M^{2n}) \to \mathbb{R}$ are defined by evaluating $c_1$ and $[\omega]$ respectively. Thus, an element of $\tilde{\mathcal{LM}}^{2n}$ is represented by an equivalence class of pairs $(\tilde{x}, \tilde{w})$, where $\tilde{x}$ is a loop in $\tilde{M}^{2n}$, $\tilde{w}$ is a disk in $\tilde{M}^{2n}$ bounding $\tilde{x}$. The pair $(\tilde{x}, \tilde{w})$ is equivalent to $(\tilde{y}, \tilde{v})$ if and only if $\tilde{x} = \tilde{y}$ and the values of $I_c$ and $I_\omega$ are zero on $w\#(-v)$, where $w = \pi(\tilde{w}), v = \pi(\tilde{v})$. The covering transformation group $\Gamma$ acts as follows

\[(\gamma_1 \oplus \gamma_2)[\tilde{x}, \tilde{w}] = [\gamma_1 \cdot \tilde{x}, A_2 \# \gamma_1 \cdot \tilde{w}],\]  

(2.2)

where $A_2$ is any representative of $\gamma_2$ in $\pi_2(M^{2n})$. Since the torsion part of $\pi_2(M^{2n})$ lies in the intersection $\text{ker}\, I_c \cap \text{ker}\, I_\omega$, we obtain the following

**Lemma 2.1.** (cf. [LO1995, Lemma 2.2]) The covering transformation group $\Gamma$ of $\tilde{\mathcal{LM}}^{2n} \to \mathcal{LM}^{2n}$ is the direct sum of the finitely generated torsion free abelian groups $\Gamma_1$ and $\Gamma_2$.

Observe that there exists a unique up to a constant Hamiltonian $\tilde{H} \in C^\infty(S^1 \times \tilde{M}^{2n})$ such that

\[d\tilde{H}_t = \pi^*(\theta + dH_t)\]  

(2.3)

for all $t \in S^1$. For the sake of simplicity, we also denote by $\omega$ the symplectic form $\pi^*(\omega)$ on $M^{2n}$. Clearly, the time-dependent Hamiltonian flow on $\tilde{M}^{2n}$ generated by $\tilde{H}$ is the pull-back of the original symplectic flow on $M^{2n}$. In particular, the set of contractible one-periodic solutions

\[\mathcal{P}(\tilde{H}) := \mathcal{P}(\omega, 0, \tilde{H})\]

coincides with the set $\pi^{-1}(\mathcal{P}(\omega, \theta, H))$. Furthermore, $\tilde{\mathcal{P}}(\tilde{H}) := \tilde{\mathcal{P}}(\mathcal{P}(\tilde{H}))$ is the critical set of the following action functional

\[A_{\tilde{H}}([\tilde{x}, \tilde{w}]) = -\int_D \tilde{w}^*\omega + \int_0^1 \tilde{H}(t, \tilde{x}(t)) \, dt.\]

Denote by $\mathcal{J}(M^{2n}, \omega)$ the set of all smooth compatible almost complex structures on $(M^{2n}, \omega)$. Let $\mathcal{J}_{\text{reg}}(M^{2n}, \omega) \subset \mathcal{J}(M^{2n}, \omega)$ be the subset of regular compatible almost complex structures, see [HS1994] and Remark [2.5] for a short explanation. For $J \in \mathcal{J}_{\text{reg}}(M^{2n}, \omega)$ we also denote by $J$ the lifted almost complex structure on $\tilde{M}^{2n}$. Let us denote by
\( g \) the associated Riemannian metric on \( \tilde{M} \). Using \( \omega(X, Y) = g(JX, Y) \) we obtain

\[
X_{\tilde{H}_t} := L^{-1}_\omega(d\tilde{H}_t) = J\nabla \tilde{H}_t
\]

where \( \nabla \) denotes the gradient w.r.t the Riemannian metric \( g \). We now consider the space \( \mathcal{M}([\tilde{x}^-, \tilde{w}^-], [\tilde{x}^+, \tilde{w}^+]; \tilde{H}, J) \) of connecting orbits \( \tilde{u} : \mathbb{R} \times S^1 \to \tilde{M}^{2n} \) satisfying the equation of \( L_2 \)-gradient flow on \( \mathcal{L}\tilde{M}^{2n} \):

\[
\partial_{s,t}\tilde{H}(\tilde{u}) = \frac{\partial \tilde{u}}{\partial s} + J(u)(\frac{\partial \tilde{u}}{\partial t} - X_{\tilde{H}_t}(\tilde{u})) = 0, \quad (2.4)
\]

with the following boundary conditions

\[
\lim_{s \to \pm \infty} \tilde{u}(s, t) = \tilde{x}^\pm(t) \in \mathcal{P}(\tilde{H}) \quad (2.5)
\]

\[
[\tilde{x}^-, \tilde{w}^- \# \tilde{u}] = [\tilde{x}^+, \tilde{w}^+]. \quad (2.6)
\]

The following energy identity for \( u \in \mathcal{M}([\tilde{x}^-, \tilde{u}^-], [\tilde{x}^+, \tilde{u}^+]; \tilde{H}, J) \) is crucial in the theory of Floer(-Novikov) chain complexes:

\[
E(\tilde{u}) = \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial \tilde{u}}{\partial s} \right|^2 dt ds = \mathcal{A}_{\tilde{H}}([\tilde{x}^-, \tilde{w}^-]) - \mathcal{A}_{\tilde{H}}([\tilde{x}^+, \tilde{w}^+]). \quad (2.7)
\]

The dimension of the space of connecting orbits is computed as follows

\[
\dim \mathcal{M}([\tilde{x}^-, \tilde{w}^-], [\tilde{x}^+, \tilde{w}^+]; \tilde{H}, J) = \mu([\tilde{x}^-, \tilde{w}^-]) - \mu([\tilde{x}^+, \tilde{w}^+]),
\]

where \( \mu([\tilde{x}, \tilde{w}]) \) is the Conley-Zehnder index of \( [\tilde{x}, \tilde{w}] \). The Conley-Zehnder index satisfies the following identity:

\[
\mu([\tilde{x}, A\# \tilde{w}]) - \mu([\tilde{x}, \tilde{w}]) = 2c_1(A) \text{ for } A \in \pi_2(M^{2n}).
\]

Let \( N \) be the minimal Chern number of \( (M^{2n}, \omega) \) and \( \tilde{x} \in \tilde{\mathcal{P}}(\tilde{H}) \). We will write \( \mu(\tilde{x}) = k \in \mathbb{Z}_{2N} := \mathbb{Z}/2N \) if there is a bounding disk \( \tilde{w} \) such that \( \mu([\tilde{x}, \tilde{w}]) = k \mod 2N \).

For \( k \in \mathbb{Z}_{2N} \) we set

\[ \tilde{\mathcal{P}}_k(\tilde{H}) := \{ [\tilde{x}, \tilde{u}] \in \tilde{\mathcal{P}}(\tilde{H}) | \mu([\tilde{x}, \tilde{u}]) = k \}. \]

Let \( R \) be an integral domain. We define the Floer-Novikov chain groups \( CFN_*(\tilde{H}, R) \) as follows.

\[
CFN_k(\tilde{H}, R) := \{ \sum \xi_{[\tilde{x}, \tilde{w}]} : [\tilde{x}, \tilde{w}] , [\tilde{x}, \tilde{w}] \in \tilde{\mathcal{P}}_k(\tilde{H}) , \xi_{[\tilde{x}, \tilde{w}]} \in R \}
\]

for all \( c \in R \) there is only finite number of \( [\tilde{x}, \tilde{w}] \) such that \( \xi_{[\tilde{x}, \tilde{w}]} \neq 0 \) \& \( \mathcal{A}_{\tilde{H}}([\tilde{x}, \tilde{w}]) > c \}

Set

\[
\Gamma^0 := \frac{\ker I_c}{\ker I_c \cap \ker I_\omega} \subset \Gamma_2,
\]

\[
\Gamma^0 := \Gamma_1 \oplus \Gamma^0_2 \subset \Gamma.
\]
Then $\Gamma^0$ is a finitely generated torsion free abelian group.

Following [LO1995], we denote by $\Lambda^R_{\theta,\omega}$ the upward completion of the group ring $R[\Gamma^0]$ w.r.t. the weight homomorphism $\Psi_{\theta,\omega} := I_\theta \oplus -I_\omega$, which we also call the Novikov ring. More precisely,

$$\Lambda^R_{\theta,\omega} := \{ \sum \lambda_g \cdot g, g \in \Gamma^0, \lambda_g \in R \mid \text{for all } c \in \mathbb{R} \text{ there is only finite number of } g \text{ such that } \lambda_g \neq 0 \& \Psi_{\theta,\omega}(g) < c \} \quad (2.8)$$

The Novikov ring $\Lambda^R_{\theta,\omega}$ is a commutative ring with unit. It acts on $\text{CFN}_*(\tilde{H}, R)$ in the following way. For $\lambda = \sum \lambda_g \cdot g \in \Lambda^R_{\theta,\omega}$ and for $\xi = \sum \xi_{[\tilde{x}, \tilde{w}]} \cdot [\tilde{x}, \tilde{w}] \in \text{CFN}_*(\tilde{H}, R)$ we let

$$(\lambda \ast \xi) := \sum (\lambda \ast \xi)_{[\tilde{x}, \tilde{w}]} [\tilde{x}, \tilde{w}]$$

where

$$(\lambda \ast \xi)_{[\tilde{x}, \tilde{w}]} := \sum_{g \in \Gamma^0} \lambda_g \xi_{-g[\tilde{x}, \tilde{w}]}.$$ 

**Lemma 2.2.** ([LO1995, Lemma 4.2]) For any $k \in \mathbb{Z}_{2N}$, the chain group $\text{CFN}_k(\tilde{H}, R)$ is a finitely generated free module over the commutative ring $\Lambda^R_{\theta,\omega}$. The rank of this module is the cardinal of the set $\mathcal{P}_k(\omega, \theta, H)$.

**Remark 2.3.** Lemma 2.2 reflects the fact that the ground ring $\Lambda^R_{\theta,\omega}$ for the chain group $\text{CFN}_*(\tilde{H}, R)$ depends only on the cohomology class $[\theta]$ such that $\pi^*(\theta) = d\tilde{H}$. We now express this fact in a slightly different way. Using the compactness of $M$ and the finiteness of $\mathcal{P}(\omega, \theta, H)$, we characterize the chain group $\text{CFN}_*(\tilde{H}, R)$ as follows. Let $\tilde{H}' = \tilde{H} + \pi^*(df)$ where $f \in C^\infty(S^1 \times M^{2n})$. Then it is not hard to see

$$\text{CFN}_k(\tilde{H}, R) := \{ \sum \xi_{[\tilde{x}, \tilde{w}]} \cdot [\tilde{x}, \tilde{w}], [\tilde{x}, \tilde{w}] \in \tilde{\mathcal{P}}_k(\tilde{H}), \xi_{[\tilde{x}, \tilde{w}]} \in R \mid \text{for all } c \in \mathbb{R} \text{ there is only finite number of } [\tilde{x}, \tilde{w}] \text{ such that } \xi_{[\tilde{x}, \tilde{w}]} \neq 0 \& \mathcal{A}_{\tilde{H}}([\tilde{x}, \tilde{w}]) > c \}.$$ 

We regard $R$ as a right $\mathbb{Z}$-module, and we denote by $1_R$ the unit of $R$. For a generator $[\tilde{x}, \tilde{w}]$ in $\text{CFN}_k(\tilde{H}, R)$, we define the boundary operator $\partial_{(J, \tilde{H})}$ as follows.

$$\partial_{(J, \tilde{H})}([\tilde{x}, \tilde{w}]) := \sum_{\mu([\tilde{y}, \tilde{v}]) = k-1} 1_R \cdot \nu([\tilde{x}, \tilde{w}], [\tilde{y}, \tilde{v}])[\tilde{y}, \tilde{v}],$$

where $\nu([\tilde{x}, \tilde{w}], [\tilde{y}, \tilde{v}])$ denotes the algebraic number of the solutions in the space $\mathcal{M}([\tilde{x}, \tilde{w}], [\tilde{y}, \tilde{v}]; \tilde{H}, J)/\mathbb{R}$, where $\mathbb{R}$ acts by translation in variable $s$. It is known that

$$\partial_{(J, \tilde{H})}([\tilde{x}, \tilde{w}]) \in \text{CFN}_{k-1}(\tilde{H}, R). \quad (2.9)$$
Observe that $\partial_{(J,\tilde{H})}$ is invariant under the action of $\Gamma^0$. Taking into account (2.9), this allows us to extend $\partial_{(J,\tilde{H})}$ as a $\Lambda^R_{\theta,\omega}$-linear map from $\text{CFN}_k(\tilde{H}, R)$ to $\text{CFN}_{k-1}(\tilde{H}, R)$. Using the standard gluing argument and the weak compactness argument, see e.g. [AD2014, McDS2004, Schwarz1995], we deduce that $\partial^2_{(J,\tilde{H})} = 0$. The chain complex $(\text{CFN}_*(\tilde{H}, R), \partial_{(J,\tilde{H})})$ is called the Floer-Novikov chain complex associated with $(\tilde{H}, J)$. For $k \in \mathbb{Z}_{2N}$, the homology group

$$HFN_k(\tilde{H}, J, R) = \frac{\ker \partial_{J,\tilde{H}} \cap \text{CFN}_k(\tilde{H}, R)}{\text{im} \partial_{J,\tilde{H}}(\text{CFN}_{k+1}(\tilde{H}, R))}$$

is called the $k$th Floer-Novikov homology group of the pair $(\tilde{H}, J)$ with coefficients in $\Lambda^R_{\theta,\omega}$. The following theorem shows that the Floer homology groups are invariant under exact deformations.

**Proposition 2.4.** (cf. [LO1995, Theorem 4.3]) For generic pairs $(\tilde{H}^\alpha, J^\alpha)$, $(\tilde{H}^\beta, J^\beta)$ such that $\tilde{H}^\alpha - \tilde{H}^\beta = \pi^*(H^{\alpha,\beta})$ for some $H^{\alpha,\beta} \in C^\infty(S^1 \times M^{2n})$ there exists a natural chain homotopy equivalence

$$\Phi^{\beta,\alpha} : (\text{CFN}_*(\tilde{H}^\alpha, R), \partial_{(J^\alpha,\tilde{H}^\alpha)}) \to (\text{CFN}_*(\tilde{H}^\beta, R), \partial_{(J^\beta,\tilde{H}^\beta)}).$$

**Remark 2.5.** In our simplified exposition of the theory of Floer-Novikov homology we did not specify the regularity condition posed on a compatible almost complex structure $J \in J_{\text{reg}}(M^{2n}, \omega)$ and we also omit a $J$-regularity condition on a nondegenerate Hamiltonian $\tilde{H}$. These conditions have been introduced in [HS1994] for compact weakly monotone symplectic manifolds $(M^{2n}, \omega)$ and extended in [LO1995] for regular coverings of $(M^{2n}, \omega)$. Roughly speaking, a compatible almost complex structure $J$ is called regular, if the moduli space of $J$-holomorphic spheres realizing a homology class $A \in H_2(M^{2n}, \mathbb{Z})$ is a manifold for any $A$. Given a regular compatible almost complex structure $J$, a nondegenerate Hamiltonian $\tilde{H} \in C^\infty(S^1 \times \tilde{M})$ is called $J$-regular, if the following three conditions hold.

1. The set of points in $M^{2n}$ that lie on contractible orbits in $\mathcal{P}(\tilde{H})$ does not intersect with the set $M_1(J)$ consisting of points lying on $J$-holomorphic spheres of Chern index at most 1.
2. The space of connecting orbits defined by (2.4), (2.5), (2.6) is a finite dimensional manifold.
3. The set of points in $M^{2n}$ that lie on the connecting orbits of relative Conly-Zehnder index at most 2 does not intersect with the set $M_0(J)$ consisting of points lying on $J$-holomorphic spheres of Chern index at most 0.
By Proposition 2.4, the Floer-Novikov homology group $HF_{\ast}(\tilde{H}, J, R)$ depends only on $\tilde{H}, R$. So we shall abbreviate it as $HF_{\ast}(\tilde{H}, R)$. We also abbreviate $(CF_{\ast}(\tilde{H}, R), \partial_{(J, \tilde{H})})$ as $CF_{\ast}(\tilde{H}, J, R)$ if it does not cause a confusion.

3. The Betti numbers of Floer-Novikov homology

In this section we restrict ourselves to the case of weakly monotone symplectic manifolds $(M^{2n}, \omega)$ with minimal Chern number $N$. We fix a covering $\tilde{M}$ associated with a class $[\theta] \in H^1(M^{2n}, \mathbb{R})$. First we show that the Novikov ring $\Lambda_{\theta, \omega}^{R}$ is an integral domain for any integral domain $R$ (Proposition 3.3). Hence the Betti numbers of the Floer-Novikov homology are well-defined, see (3.7). Then we prove that the Betti numbers of the Floer-Novikov homology $HF_{\ast}(\tilde{H}, R)$ do not depend on the choice of $\tilde{H}$ (Theorem 3.16). For this purpose we first show that the chain complex $(CF_{\ast}(\tilde{H}), \partial_{(J, \tilde{H})})$ is an extension by scalars of a chain complex with the same generators but defined on a proper sub-ring of the Novikov ring (Theorem 3.12). Hence the Betti numbers of the Floer-Novikov homology $HF_{\ast}(\tilde{H}, R)$ are equal to the Betti numbers of the “smaller” Floer-Novikov homology groups (Proposition 3.15). Then we introduce the notions of an admissible family of nondegenerate (multi-valued) $J$-regular Hamiltonian functions and its good neighborhoods and study their properties (Definitions 3.9, 3.11, Theorem 3.12, Proposition 3.15). Using the obtained results, we prove that the Betti numbers of the Floer-Novikov homology $HF_{\ast}(\tilde{H}, R)$ locally do not depend on the “weight” of their Calabi invariant (Proposition 3.17). Finally we compute the Betti number of the Floer-Novikov homology group $HF_{\ast}(\tilde{H}, R)$, where $\tilde{H}$ is a lift of a nondegenerate Hamiltonian on $M^{2n}$, using the Piunikhin-Salamon-Schwarz construction (Theorem 3.27, Corollary 3.30).

3.1. Novikov ring $\Lambda_{\theta, \omega}^{R}$ revisited. Given a ring $R$, a group $\Gamma$ and a homomorphism $\phi : \Gamma \to \mathbb{R}$ we denote by $R((\Gamma, \phi))$ the upward completion of the group ring $R[\Gamma]$ w.r.t. the weight homomorphism $\phi$. More precisely, as in (2.8), we define

$$R((\Gamma, \phi)) := \{ \sum \lambda_g \cdot g, g \in \Gamma, \lambda_g \in R \mid \text{for all } c \in \mathbb{R} \text{ there is only finite number of } g \text{ such that } \lambda_g \neq 0 & \phi(g) < c \} \ (3.1)$$

If $\Gamma$ is a subgroup of $\mathbb{R}$, $e : \Gamma \to \mathbb{R}$ is the natural embedding, then we abbreviate $R((\Gamma, e))$ as $R((\Gamma))$. 
In this paper we consider only commutative rings \( R \) with unit and without zero divisor, i.e. \( R \) are integral domains.

**Lemma 3.1.** Assume that \( \Gamma \) is a torsion free finitely generated abelian group and \( \phi : \Gamma \to \mathbb{R} \) is a homomorphism. Then there is a subgroup \( \Gamma_\phi \subset \Gamma \) such that
\[
\Gamma = \ker \phi \oplus \Gamma_\phi.
\]

**Proof.** Let \( t_1, \ldots, t_n \in \mathbb{R} \) be linearly independent generators of the subgroup \( \phi(\Gamma) \subset \mathbb{R} \). Pick elements \( \gamma_i \in \Gamma \) such that \( \phi(\gamma_i) = t_i \).

Let \( \Gamma_\phi \) be the subgroup in \( \Gamma \) that is generated by \( \{ \gamma_i | i \in [1, n] \} \). Clearly \( \Gamma_\phi \cap \ker \phi = 0 \). We shall show that \( \Gamma = \ker \phi \oplus \Gamma_\phi \).

Then we have
\[
\gamma - \sum_i a_i \gamma_i \in \ker \phi.
\]

This completes the proof of Lemma 3.1. \( \square \)

**Proposition 3.2.** Assume that \( \Gamma \) is a torsion free finitely generated abelian group and \( \phi : \Gamma \to \mathbb{R} \) is a homomorphism. Then we have a ring isomorphism \( R((\Gamma, \phi)) = R[\ker \phi]((\phi(\Gamma))) \).

**Proof.** Using Lemma 3.1 we write \( \Gamma = \ker \phi \oplus \Gamma_\phi \). By definition any element \( \lambda \in R((\Gamma, \phi)) \) can be written as follows
\[
\lambda = \sum_{ij} \lambda_{(\alpha_i, \beta_j)} \cdot (\alpha_i + \beta_j), \quad \lambda_{(\alpha_i, \beta_j)} \in R, \ \alpha_i \in \ker \phi, \ \beta_j \in \Gamma_\phi \text{ such that } \phi(\beta_j) = C < \infty.
\]

It follows that given \( \lambda \in R((\Gamma, \phi)) \), for each \( \beta_j \in \Gamma_\phi \) there is only a finite number of \( \lambda_{(\alpha_i, \beta_j)} \) such that \( \lambda_{(\alpha_i, \beta_j)} \cdot (\alpha_i + \beta_j) \) is a term in \( \lambda \). Hence
\[
\sum_i \lambda_{(\alpha_i, \beta_j)} \cdot \alpha_i \in R[\ker \phi].
\]

Now we define a map
\[
R((\Gamma, \phi)) \xrightarrow{\phi^*} R[\ker \phi]((\phi(\Gamma))), \quad \lambda \mapsto \sum_j (\sum_i \lambda_{(\alpha_i, \beta_j)} \cdot \alpha_i) \cdot \phi(\beta_j). \quad (3.2)
\]

It is straightforward to verify that \( \phi^* \) is a ring homomorphism, since \( \Gamma \) is abelian.

Since the restriction of \( \phi \) to the subgroup \( \Gamma_\phi \) is a monomorphism, from (3.2) we conclude that \( \phi^* \) is a ring monomorphism.
Now let \( \delta \in R[\ker \phi]((\phi(\Gamma))) \). We write
\[
\delta = \sum_i \delta_i \cdot \phi(\beta_i) | \beta_i \in \Gamma_\phi, \ delta_i \in R[\ker \phi] \text{ such that }
\]
for any \( C \in \mathbb{R} \# \{ \delta_i | \delta_i \neq 0 \text{ and } \phi(\beta_i) < C \} < \infty. \) (3.3)

We write \( \delta_i = \sum_{j=1}^{N(i)} \delta_{ij} \cdot \alpha_{ij} \), where \( \delta_{ij} \in R, \alpha_{ij} \in \ker \phi. \) Then
\[
\delta = \sum_{ij} \delta_{ij} \cdot \alpha_{ij} \cdot \phi(\beta_i).
\]

Since for each \( i \) the number of \( \alpha_{ij} \) is finite, we obtain from (3.3)
\[
\# \{ \delta_{ij} | \delta_{ij} \neq 0 \text{ and } \phi(\alpha_{ij} \cdot \beta_i) < C \} < \infty. \) (3.4)

Using (3.4), we define a map
\[
R[\ker \phi]((\phi(\Gamma))) \xrightarrow{\Phi} R((\Gamma, \phi)), \delta \mapsto \sum_i \sum_{ij} \delta_{ij} (\alpha_{ij} + \beta_i).
\]

Set \( \delta_{(\alpha_{ij}, \beta_i)} := \delta_{ij}. \) Since \( \Gamma \) is abelian, \( \phi_* \) is a ring homomorphism. Observing that \( \phi^* \circ \phi_* = Id, \) we conclude that \( \phi^* \) is an epimorphism. Hence \( \phi^* \) is an isomorphism. This proves Proposition 3.2. \( \square \)

**Proposition 3.3.** The ring \( \Lambda_{\theta,\omega}^R \) is an integral domain.

**Proof.** By Proposition 3.2
\[
\Lambda_{\theta,\omega}^R = R[\ker \Psi_{\theta,\omega}]((\Psi_{\theta,\omega}(\Gamma_0))).
\]

Since \( \Gamma_0 \) is a finitely generated torsion free abelian group, \( \ker \Psi_{\theta,\omega} \) is a finitely generated torsion free abelian group. Hence the group ring \( R[\ker \Psi_{\theta,\omega}] \) is an integral domain, see [Passman1986] Chapter 3 for a survey on Kaplansky’s zero-divisor conjecture, or see Corollary 3.5 below. To complete the proof of Proposition 3.3 we need the following lemma, which has been formulated in [HS1994].

**Lemma 3.4.** Assume that \( R \) is an integral domain and \( \Gamma \) is a finitely generated torsion free abelian group. Then \( R((\phi(\Gamma))) \) is an integral domain for any homomorphism \( \phi : \Gamma \to \mathbb{R}. \)

**Proof.** Since Hofer and Salamon omit a proof of Lemma 3.4, which, in fact, can be constructed from their arguments in [HS1994] §4, for the sake of reader’s convenience we present here a proof. Note that \( \phi(\Gamma) \) is a finitely generated torsion-free abelian group. Let \( m \) be the rank of \( \phi(\Gamma). \) We identify elements of \( \phi(\Gamma) \) with \( t^k, \) where \( t \) is a formal variable and \( k = (k_1, \cdots, k_m) \in \mathbb{Z}^m. \) We say that \( k > k' \) if \( \phi(t^k) > \phi(t^{k'}). \)
This defines a total ordering on $\phi(\Gamma)$, which is compatible with the multiplication on $\phi(\Gamma)$:

$$k > k' \implies k + k'' > k' + k'' \text{ for all } k'', k'' \in \mathbb{Z}^m.$$

Then

$$R((\phi(\Gamma))) = \left\{ \sum_{i=1}^{\infty} a_i t^{k_i} \mid a_i \in R, a_1 \neq 0, k_i < k_{i+1} \right\}.$$

Assume the opposite, i.e. there exist elements $A, B \in R((\phi(\Gamma)))$ such that $A \cdot B = 0$ but $A \neq 0$ and $B \neq 0$. Using (3.39) we write

$$A = \sum_{i=0}^{\infty} a_i t^{k_i}, \quad B = \sum_{i=0}^{\infty} b_i t^{l_i},$$

where

$$k_0 < k_1 < \cdots, \text{ and } l_0 < l_1 < \cdots$$

and $a_0 \neq 0$ and $b_0 \neq 0$. Since $A \cdot B = 0$ implies $a_0 \cdot b_0 = 0$, taking into account the fact that $R$ is an integral domain, we conclude that $a_0 = 0$ or $b_0 = 0$. This contradicts our assumption that $a_0 \neq 0$ and $b_0 \neq 0$. Hence the proof of Lemma 3.4 is completed.

Corollary 3.5. Assume that $R$ is an integral domain and $G$ is a finitely generated torsion free abelian group. Then $R[G]$ is an integral domain.

Proof. Since $G$ is a finitely generated torsion free abelian group, there is a monomorphism $\phi : G \to \mathbb{R}$. Lemma 3.4 implies that $R((G, \phi))$ is an integral domain. Since $\phi$ is a monomorphism, $R[G]$ is a subring of $R((G, \phi))$. It follows that $R[G]$ is also an integral domain.

Clearly Proposition 3.2 follows from Lemma 3.4 and the fact that $R[\ker \Psi_{\theta,\omega}]$ is an integral domain.

Recall that the rank of a module $L$ over an integral domain $A$ is defined to be the dimension of the vector space $F(A) \otimes_A L$ over the field of fractions $F(A)$ of $A$. By Proposition 3.3, $\Lambda_R^{\theta,\omega}$ is an integral domain. By Lemma 2.2 the chain group $CFN_k(\tilde{H}, R)$, and hence Floer-Novikov homology group $HF_N_k(\tilde{H}, R)$ are left modules over $\Lambda_R^{\theta,\omega}$. Thus we define the Betti numbers $b_i(HF_N(\tilde{H}, R))$ as follows

$$b_i(HF_N(\tilde{H}, J, R)) = \dim_{F(\Lambda_R^{\theta,\omega})}(F(\Lambda_R^{\theta,\omega}) \otimes_{\Lambda_R^{\theta,\omega}} HF_N(\tilde{H}, J, R)).$$

Lemma 3.6. Assume that $F = F(R)$ is the field of fractions of an integral domain $R$. Then

(i) $F(\Lambda_R^{\theta,\omega}) = F(\Lambda_F^{\theta,\omega})$. 


(ii) \( b_i(HFN_*(\tilde{H}, J, R)) \leq \text{rank} \left( CFN_i(\tilde{H}, R) \right) \).

(iii) \( b_i(HFN_*(\tilde{H}, J, R)) = b_i(HFN_*(\tilde{H}, J, F)) \).

Proof. (i) Since \( R \subset \mathbb{F} \) we have \( F(\Lambda^R_{\theta, \omega}) \subset F(\Lambda^F_{\theta, \omega}) \). To show that \( F(\Lambda^R_{\theta, \omega}) \supset F(\Lambda^F_{\theta, \omega}) \) it suffices to observe that \( F(\Lambda^R_{\theta, \omega}) \supset \mathbb{F} \). This proves the first assertion of Lemma 3.6.

(ii) The second assertion of Lemma 3.6 follows from the universal coefficient theorem. Since the field of fractions \( F(\Lambda^R_{\theta, \omega}) \) is a flat right \( \Lambda^R_{\theta, \omega} \)-module, we obtain

\[
b_i(HFN_*(\tilde{H}, J, R)) = \dim_{F(\Lambda^R_{\theta, \omega})} H_i(F(\Lambda^F_{\theta, \omega}) \otimes_{\Lambda^R_{\theta, \omega}} CFN_*(\tilde{H}, J, R)) \leq \text{rank} \left( CFN_i(\tilde{H}, R) \right).\]

(iii) The last assertion of Lemma 3.6 follows from the first assertion, the universal coefficient theorem (3.28), taking into account the identity

\[
F(\Lambda^F_{\theta, \omega}) \otimes_{\Lambda^R_{\theta, \omega}} CFN_*(\tilde{H}, J, R) = F(\Lambda^F_{\theta, \omega}) \otimes_{\Lambda^R_{\theta, \omega}} CFN_*(\tilde{H}, J, F).\]

This completes the proof of Lemma 3.6.

3.2. Admissible family of nondegenerate Hamiltonian functions. Our goal in this subsection is to prove Theorem 3.12, which is an improvement of Lemma 2.2 and will be needed for our computation of the Betti numbers of \( HFN_*(\tilde{H}, R) \) later. Let \( \theta \) be a representative of \([\theta]\) and let us pick a function \( h^\theta \) on \( \tilde{M} \) such that

\[
dh^\theta = \pi^*(\theta). \tag{3.8}\]

For \( \lambda \in \mathbb{R} \) we set

\[
C^\infty_{(\lambda)}(S^1 \times \tilde{M}^{2n}) := \{ f \in C^\infty(S^1 \times \tilde{M}^{2n}) | f = \lambda \cdot h^\theta + \pi^*(\bar{f}) \}
\]

for some \( \bar{f} \in C^\infty(S^1 \times M^{2n}) \).

The number \( \lambda \) will be called the weight of \( \tilde{H} \in C^\infty_{(\lambda)}(S^1 \times \tilde{M}^{2n}) \).

Let

\[
C^\infty_{(s)}(S^1 \times \tilde{M}^{2n}) := \bigcup_{\lambda \in \mathbb{R}} C^\infty_{(\lambda)}(S^1 \times \tilde{M}^{2n}).
\]

Note that if \( J \) is regular, \( \tilde{H} \in C^\infty_{(s)}(S^1 \times \tilde{M}) \) is nondegenerate and \( J \)-regular, then the Floer-Novikov chain complex \( (CFN_*(\tilde{H}, R), \partial_{(J, \tilde{H})}) \) is well defined.

Remark 3.7. (1) If \( \tilde{H} \in C^\infty_{(s)}(S^1 \times \tilde{M}) \) then the time-dependent Hamiltonian vector field \( X_{\tilde{H}} \) is invariant under the covering transformation group \( \Gamma_1 \) of \( \tilde{M}^{2n} \).
(2) By Proposition 2.4, if \( H \) and \( H' \) belong to the same space \( C^\infty_\gamma(S^1 \times \tilde{M}^{2n}) \), the chain complexes \( (CFN_\gamma(\tilde{H}, R), \partial(\tilde{J}, \tilde{R})) \) and \( (CFN_\gamma(\tilde{H'}, R), \partial(\tilde{J}, \tilde{R}')) \) are chain homotopy equivalent. Hence the Floer-Novikov homology \( H \) w.l.o.g. we assume that \( \lambda \in \mathbb{R} \) such that \( \tilde{H} \in C^\infty_\gamma(S^1 \times \tilde{M}) \).

For \( \tilde{H} \in C^\infty_\gamma(S^1 \times \tilde{M}^{2n}) \) we set

\[
CFN^0_\gamma(\tilde{H}, R) := \{ \sum g_i \cdot a_i \mid g_i \in R, a_i \in Crit_k(A_{\tilde{H}}) \}.
\]

Then \( CFN^0_\gamma(\tilde{H}, R) \) is a \( R/[\Gamma^0] \)-module.

For \( 0 \leq \tau_1 \leq \tau_2 \in \mathbb{R} \) and for \( \tilde{H} \in C^\infty_\gamma(S^1 \times \tilde{M}^{2n}) \) we set

\[
\Lambda^R_{\theta(\tau_1, \tau_2), \omega} := \Lambda^R_{\theta, \tau_1, \omega} \cap \Lambda^R_{\theta, \tau_2, \omega},
\]

\[
CFN^\tau_{\gamma(\tau_1, \tau_2)}(\tilde{H}, R) := \Lambda^R_{\theta(\lambda-\tau_1, \lambda+\tau_2), \omega} \otimes_{R/[\Gamma^0]} CFN^0_\gamma(\tilde{H}, R).
\]

The chain complex \( CFN^\tau_{\gamma(\tau_1, \tau_2)}(\tilde{H}, R) \) will play important role in our proof of the Main Theorem, so we shall describe them more carefully in Lemma 3.8 below.

For a function \( \tilde{H} \in C^\infty_\gamma(S^1 \times \tilde{M}) \) and for \( \tau \in \mathbb{R} \) we set

\[
\tilde{H}(\tau) := \tilde{H} + \tau \cdot h^\theta.
\]

**Lemma 3.8.** We have

\[
\Lambda^R_{\theta(\tau_1, \tau_2), \omega} = \bigcap_{\tau \in [\tau_1, \tau_2]} \Lambda^R_{\theta, \tau, \omega},
\]

\[
CFN^\tau_{\gamma(\tau_1, \tau_2)}(\tilde{H}, R) := \{ \sum \xi_{[\bar{x}, \bar{w}]} \cdot [\bar{x}, \bar{w}], [\bar{x}, \bar{w}] \in \tilde{P}_k(\tilde{H}), \xi_{[\bar{x}, \bar{w}]} \in R \}
\]

for all \( c \in \mathbb{R} \) for any \( \tau \in (\tau_1, \tau_2) \)

\[
\# \{ [\bar{x}, \bar{w}] \mid \xi_{[\bar{x}, \bar{w}]} \neq 0 \& \mathcal{A}_{\tilde{H}(\tau)}([\bar{x}, \bar{w}]) > c \}
\]

is finite

\[
(3.11)
\]

**Proof.** W.l.o.g. we assume that \( \tau_1 < \tau_2 \). Since \( \Lambda^R_{\theta(\tau_1, \tau_2), \omega} \supset \bigcap_{\lambda \in [\tau_1, \tau_2]} \Lambda^R_{\theta, \lambda, \omega} \), to prove (3.10) it suffices to show that

\[
\Lambda^R_{\theta(\tau_1, \tau_2), \omega} \subset \bigcap_{\lambda \in [\tau_1, \tau_2]} \Lambda^R_{\theta, \lambda, \omega}.
\]

Let \( \lambda \in \Lambda^R_{\theta(\tau_1, \tau_2), \omega} \). Then we write (cf. (2.8))

\[
\lambda = \sum \lambda_g \cdot g, g \in \Gamma^\circ, \lambda_g \in R \text{ s.t. for all } c \in \mathbb{R}
\]

\[
\# \{ g \mid \lambda_g \neq 0 \& \Psi_{\theta, \tau, \omega}(g) < c \} < \infty \text{ and}
\]

\[
\# \{ g \mid \lambda_g \neq 0 \& \Psi_{\theta, \tau, \omega}(g) < c \} < \infty.
\]

\[
(3.13)
\]
Now assume that (3.12) does not hold. This implies that (3.10) does not hold for some $c \in \mathbb{R}$ and some $\tau \in (\tau_1, \tau_2)$, i.e. there exists $\lambda = \sum \lambda_g \cdot g \in \Lambda^R_{\theta(\tau_1, \tau_2), \omega}$ such that

$$\#\{g \mid \Psi_{\theta, \tau, \omega}(g) < c\} = \infty.$$  \hfill (3.14)

Set

$$a = \frac{\tau_2 - \tau}{\tau_2 - \tau_1}, \quad b = \frac{\tau - \tau_1}{\tau_2 - \tau_1}.$$  

Then $0 < a, b < 1$ and

$$\Psi_{\theta, \tau, \omega} = a\Psi_{\theta, \tau_1, \omega} + b\Psi_{\theta, \tau_2, \omega}.$$  

Since $\Psi_{\theta, \tau, \omega}(g) < c$ then either

$$\Psi_{\theta, \tau_1, \omega}(g_i) < \frac{c}{2a} \quad \hfill (3.15)$$

or

$$\Psi_{\theta, \tau_2, \omega}(g_i) < \frac{c}{2b} \quad \hfill (3.16)$$

Denote by $\Gamma^0(\lambda, \tau_1)$ (resp. $\Gamma^0(\lambda, \tau_2)$) the set of all $g \in \Gamma^0$ such that $\lambda_g \neq 0$ and $g$ satisfies (3.15) (resp. (3.16)). The above argument implies

$$\{g \mid \lambda_g \neq 0 \& \Psi_{\theta, \tau, \omega}(g) < c\} \subset \Gamma^0(\lambda, \tau_1) \cup \Gamma^0(\lambda, \tau_2).$$

Since $\Gamma^0(\lambda, \tau_1)$ and $\Gamma^0(\lambda, \tau_2)$ are finite sets by (3.13), it follows that (3.14) cannot happen. This proves the first assertion of Lemma 3.8.

The second assertion of Lemma 3.8 follows from the first one. This completes the proof of Lemma 3.8. \hfill \Box

In the remainder of this subsection we shall prove a family version of [Ono2005, Proposition 4.5], which improves Lemma 2.2. We assume that $J$ is a regular compatible almost complex structure on $(M^{2n}, \omega)$.

First we describe a special set of admissible perturbations of a non-degenerate $J$-regular Hamiltonian function $\tilde{H}_0 = \lambda \cdot h^\theta + \pi^*(H_0)$, where $H_0 \in C^\infty(S^1 \times M^{2n})$. Let $\{x_1, \cdots, x_k\}$ be the set of one-periodic orbits of the locally Hamiltonian equation associated to $\lambda \cdot \theta$ and $H_0$. Then $\{\tilde{x}_i = \pi^{-1}(x_i)\}$ are one-periodic orbits of the Hamiltonian flow generated by $\tilde{H}_0$. Let $d$ denote the distance on $\tilde{M}$ induced from the Riemannian metric $\pi^*(g_J)$. We define the distance between 1-periodic orbits $\tilde{x}$ and $\tilde{y}$ as follows

$$\rho(\tilde{x}, \tilde{y}) := \int_0^1 d(\tilde{x}(t), \tilde{y}(t)) dt.$$  

It is easy to see that $\rho(\tilde{x}, \tilde{y}) = 0$ iff $\tilde{x} = \tilde{y}$ and

$$\rho(\tilde{x}, \tilde{y}) \leq \rho(\tilde{x}, \tilde{z}) + \rho(\tilde{z}, \tilde{y}).$$
Since $\tilde{H}_0$ is nondegenerate, there exists a positive number $\varepsilon(\tilde{H}_0) > 0$ such that
\[
\frac{1}{4} \max_{t \in S^1} d(\tilde{x}(t), \tilde{y}(t)) \geq \frac{1}{4} \rho(\tilde{x}, \tilde{y}) > \varepsilon(\tilde{H}_0)
\] (3.17)
for distinct orbits $\tilde{x}, \tilde{y} \in \mathcal{P}(\tilde{H}_0)$. Let $U_i, i \in [1, k]$, be the $\varepsilon$-tubular neighborhood of the graph $G_{x_i}$ of $x_i$ in $S^1 \times M^{2n}$. Then $U_i$ are mutually disjoint. Set $\tilde{U}_i := \pi^{-1}(U_i)$.

**Definition 3.9.** A family $F := \{ \tilde{H}_\chi \mid \chi \in [0, 1] \}$ of nondegenerate Hamiltonian functions in $C^\infty(S^1 \times M^{2n})$ will be called admissible, if

1. The map $\chi \mapsto \tilde{H}_\chi$ is continuous in the $C^1$-topology induced on $F$,
2. $\mathcal{P}(\tilde{H}_\chi) = \mathcal{P}(\tilde{H}_0)$ for all $\chi \in [0, 1]$.

The parameter space $[0, 1]$ of an admissible family $F$ can be replaced by any compact interval $[\delta, \delta'] \subset \mathbb{R}$, e.g. by reparametrization of $[\delta, \delta']$. To make the exposition simple, we consider in this subsection only admissible families with parameter $\chi \in [0, 1]$.

Given a function $\tilde{H}_\chi$ in an admissible family $F$ of nondegenerate $J$-regular Hamiltonian functions we set
\[
\mathcal{U}_c(\tilde{H}_\chi) := \{ \tilde{H}_\chi + \pi^*(h_\chi) \mid h_\chi \in C^\infty(S^1 \times M^{2n}), ||h_\chi||_\varepsilon < c, (h_\chi)|_{U_i} = 0 \forall i \}
\]
where
\[
||h||_\varepsilon := \sum_{k=0}^{\infty} \varepsilon_k ||h||_{C^k(S^1 \times M^{2n})}.
\]
Here $\varepsilon_k > 0$ is a sufficiently rapidly decreasing sequence [Floer1988], see also [AD2014, §8.3] for a detailed discussion. (In particular, we borrow the condition $(h_\chi)|_{U_i} = 0$ from [AD2014, p.233].)

We also fix a vector space $\mathbb{R}^{N_0}$ and an isometric embedding $(M^{2n}, g_J)$ into the Euclidean space $\mathbb{R}^{N_0}$. This shall simplify notations of different norms on different bundles over submanifolds in $M^{2n}$.

Further, we set $\mathcal{P}(F) := \mathcal{P}(\tilde{H}_0)$ and
\[
\varepsilon(F) := \varepsilon(\tilde{H}_0)
\]
where $\varepsilon(\tilde{H}_0)$ is the constant in (3.17).

The following Lemma is a family version of [LO1995, Lemma 5.2]. It contains key estimates (3.18), (3.19), which we shall exploit later in Subsection 3.3.
Lemma 3.10. Assume that $\mathcal{F} := \{\tilde{H}_x| x \in [0,1]\}$ is an admissible family of nondegenerate Hamiltonian functions in $C^\infty_0(S^1 \times \tilde{M}^{2n})$. There exist a positive number $c := c(\mathcal{F}) > 0$ and a positive number $\delta_1 = \delta_1(\mathcal{F}) > 0$ such that for any $x \in [0,1]$ the following statement hold.

(i) Let $\sigma(t)$ be a smooth contractible loop on $\tilde{M}^{2n}$ with

$$\max_{t} d(\sigma(t), \tilde{x}(t)) > \varepsilon(\mathcal{F})$$

for any $\tilde{x} \in \mathcal{P}(\mathcal{F})$. Then for any $\tilde{H}'_x \in \mathcal{U}_c(\tilde{H}_x)$ we have

$$||\tilde{\sigma} - X_{\tilde{H}'_x}(G_\sigma)||_{L^2(S^1, \mathbb{R}^{N_0})} > \delta_1(\mathcal{F}).$$

(ii) For any $x \in [0,1]$ and any $\tilde{H}'_x \in \mathcal{U}_c(\tilde{H}_x)$ we have

$$\mathcal{P}(\tilde{H}'_x) = \mathcal{P}(\mathcal{F}).$$

Proof. (i) Assume the opposite, i.e. there exist the following sequences

(1) $\{c_j \in \mathbb{R}^+| \lim_{j \to \infty} c_j = 0\}$,

(2) $\{\chi(j) \in [0,1]| \lim_{j \to \infty} \chi(j) = \chi(\infty) \in [0,1]\}$,

(3) $\{\tilde{H}_{\chi(j)} \in \mathcal{U}_c(\tilde{H}_{\chi(j)})\}$,

(4) $\{\sigma_j \in \mathcal{L}M^{2n}| \max_{t} d(\sigma_j(t), x(t)) > \varepsilon(\mathcal{F}) \text{ for all } x \in \mathcal{P}(\mathcal{F}) \text{ and }$

$$\lim_{j \to \infty} ||\tilde{\sigma}_j - X_{\tilde{H}_{\chi(j)}}(G_{\sigma_j})||_{L^2(S^1, \mathbb{R}^{N_0})} = 0\}.$$

By Definition 3.9(1),

$$\lim_{j \to \infty} ||X_{\tilde{H}'_{\chi(j)}} - X_{\tilde{H}_{\chi(\infty)}}||_{C^0(S^1 \times \tilde{M}^{2n})} = 0.$$ 

Hence

$$\lim_{j \to \infty} ||\tilde{\sigma}_j - X_{\tilde{H}_{\chi(\infty)}}(G_{\sigma_j})||_{L^2(S^1, \mathbb{R}^{N_0})} = 0.$$ 

By Lemma 5.1 in [LO1995], the last relation implies that a subsequence of $\{\sigma_j\}$ converges to some contractible orbit $x \in \mathcal{P}(\tilde{H}_{\chi(\infty)}) = \mathcal{P}(\mathcal{F})$. This is a contradiction, since $\max_{t} d(\sigma_j(t), x(t)) > \varepsilon(\mathcal{F})$. The proof of Lemma 3.10(i) is completed.

(ii) Assume that there is a contractible orbit $\sigma(t)$ of a Hamiltonian function $\tilde{H}'_x \in \mathcal{U}_c(\tilde{H}_x)$ such that $\sigma \notin \mathcal{P}(\mathcal{F})$. If the graph of $\sigma(t)$ belongs to some neighborhood $U_i \subset S^1 \times \tilde{M}^{2n}$ then $\sigma = \tilde{x}_i$, since $(\tilde{H}'_x)|_{U_i} = (\tilde{H}_x)|_{U_i}$. If not then

$$\max_{t} d(\sigma(t), \tilde{x}_i(t)) > \varepsilon(\mathcal{F})$$
for any \( \tilde{x} \in \mathcal{P}(\mathcal{F}) \). By the assertion proved above \( \sigma(t) \) cannot be an orbit of the flow generated by \( \tilde{H}_\chi \). We arrive at a contradiction. This completes the proof of Lemma \ref{3.10}. □

**Definition 3.11.** For an admissible family \( \mathcal{F} := \{ \tilde{H}_\chi | \chi \in [0, 1] \} \) we set

\[
U(\mathcal{F}) := \cup_{\chi \in [0, 1]} U_c(\tilde{H}_\chi).
\]

where \( c(\mathcal{F}) \) is the constant in Lemma \ref{3.10}. We call \( U(\mathcal{F}) \) a good neighborhood of \( \mathcal{F} \).

**Theorem 3.12.** Assume that \( J \) is a regular compatible almost complex structure and \( \mathcal{F} := \{ \tilde{H}_\chi \in C^\infty(M), \chi \in [0, 1] \} \) is an admissible family of nondegenerate Hamiltonian functions. Assume that \( \tilde{H}_0 \) is \( J \)-regular.

(i) For each \( \chi \in [0, 1] \) the set \( U^r_c(\tilde{H}_\chi) \) of \( J \)-regular Hamiltonian functions is dense in \( U_c(\tilde{H}_\chi) \) provided with the topology generated by the Banach norm \( ||.||_\varepsilon \).

(ii) There exists a positive number \( \tau := \tau(\mathcal{F}) > 0 \) with the following property. Let \( \tilde{H}_\mu \in U(\mathcal{F}) \) is \( J \)-regular. Then for any \( [\tilde{x}, \tilde{w}] \in \text{Crit}(A_{\tilde{H}_\mu}) \) we have

\[
\partial(\tilde{J}, \tilde{H}_\mu)([\tilde{x}, \tilde{w}]) \in \text{CFN}^{(-\tau, \tau)}(\tilde{H}_\mu, R).
\]

Consequently \( \text{CFN}^{(-\tau, \tau)}(\tilde{H}_\mu, R), \partial(\tilde{J}, \tilde{H}_\mu) \) is a chain complex.

**Proof.** (i) The \( J \)-regularity of \( \tilde{H}_0 \) ensures that any nondegenerate \( \tilde{H}_\mu \in U_c(\mathcal{F}) \) satisfies the requirement (1) in Remark \ref{2.5} for the \( J \)-regularity, since \( \mathcal{P}(\tilde{H}_\mu) = \mathcal{P}(\tilde{H}_0) \) by \ref{3.10}. Clearly, the nondegeneracy condition of \( \tilde{H}_\mu \in U_c(\mathcal{F}) \) defines a open and dense subset of \( U_c(\mathcal{F}) \). To prove that the requirements (2), (3) of the \( J \)-regularity also define a dense subset in \( U_c(\tilde{H}_\chi) \) we use the standard transversality argument in the proof of Theorems 3.2, 3.3 in \[HS1994\], see also the proof of Theorems 3.2, 3.3 in \[LO1995\]. So we omit the proof.

(ii) Our proof of the second assertion uses many ideas in the proof of \[Ono2005\] Proposition 4.5.

First we prove the following two Lemmas containing uniform estimates for the proof of Theorem \ref{3.12}

We set

\[
e(\mathcal{F}) := \min(4\varepsilon^2(\mathcal{F}), \frac{\delta^2(\mathcal{F})}{2}).
\]

**Lemma 3.13.** (cf. \[Ono2005\] Lemma 3.2) Suppose that \( \tilde{H}_\mu \in U(\mathcal{F}) \) is \( J \)-regular, \( -\infty < R_1 < R_2 < \infty, \tilde{x}_1, \tilde{x}_2 \in \mathcal{P}(\tilde{H}_\mu) = \mathcal{P}(\mathcal{F}) \) are
distinct one-periodic orbits and \( u \in \mathcal{M}([\bar{x}_1, \bar{w}], [\bar{x}_2, \bar{v}], \bar{H}_\mu, J) \) satisfies \( \max_t d(u(R_i, t), \bar{x}_i(t)) \leq \varepsilon(F) \). Then
\[
\int_{R_1}^{R_2} \int_0^1 |\frac{\partial u}{\partial s}|^2 \, ds \, dt > e(F).
\]

**Lemma 3.14.** (cf. [Ono2005, Lemma 3.4], cf. [LO1995, Lemma 3.5])
Suppose that \( \bar{H}_\mu \in U(F) \) is \( J \)-regular and \( \bar{x}, \bar{y} \in \mathcal{P}(\bar{H}_\mu) = \mathcal{P}(F) \) are distinct one-periodic orbits. For any \( u, v \in \mathcal{M}([\bar{x}, \bar{w}], [\bar{y}, \bar{v}], \bar{H}_\mu, J) \) we have
\[
E(u) = \int_{-\infty}^{\infty} \int_0^1 |\frac{\partial u}{\partial s}|^2 \, ds \, dt \geq \frac{\delta_1(F)}{2} \rho(\bar{x}, \bar{y}).
\]

**Proof of Lemma 3.14.** Our proof is a refinement of the proof of [Ono2005, Lemma 3.2]. W.l.o.g. we may assume that
1. \( \max_t d(u(R_i, t), \bar{x}_i(t)) = \varepsilon(F) \),
2. For any \( r \in (R_1, R_2) \) and for \( i = 1, 2 \) we have
\[
\max_{t \in (R_1, R_2)} d(u(r, t), \bar{x}_i(t)) > \varepsilon(F).
\]

**Case 1:** \( R_2 - R_1 \leq 1 \). Using the Cauchy-Schwarz inequality we obtain
\[
\int_{R_1}^{R_2} \int_0^1 |\frac{\partial u}{\partial s}|^2 \, ds \, dt \geq \int_{R_1}^{R_2} (\int_0^1 |\frac{\partial u}{\partial s}| \, ds)^2 \, dt \geq \int_{R_1}^{R_2} (d(u(R_1, t), u(R_2, t)))^2 \, dt.
\]
Applying the Cauchy-Schwarz inequality again, we obtain from the above inequality
\[
\int_{R_1}^{R_2} \int_0^1 |\frac{\partial u}{\partial s}|^2 \, ds \, dt \geq \rho(u(R_1, -), u(R_2, -))^2
\]
\[
\geq (\rho(\bar{x}_1, \bar{x}_2) - \rho(\bar{x}_1, u(R_1, -)) - \rho(\bar{x}_2, u(R_2, -))^2
\]
\[
\geq (\rho(\bar{x}_1, \bar{x}_2) - 2\varepsilon(F))^2 - 4\varepsilon^2(F),
\]
since \( \rho(\bar{x}_i, u(R_i, -)) \leq \max_t d(u(R_i, t), \bar{x}_i(t)) = \varepsilon(F) \) and by (3.17)
\[
\rho(\bar{x}_1, \bar{x}_2) - 2\varepsilon(F) \geq 2\varepsilon(F).
\]
Since \( 4\varepsilon^2(F) \geq e(F) \), Lemma 3.13 holds in Case 1.

**Case 2:** \( R_2 - R_1 > 1 \). Assume that Lemma 3.13 does not hold. Since \( R_2 - R_1 > 1 \) there exists \( r \in (R_1, R_2) \) such that
\[
\int_{0}^{1} |\frac{\partial u(s, t)}{\partial s}|^2 (r, t) \, dt < e(F) \leq \frac{\delta_1(F)}{2}.
\]
Since \( u \) is a connecting orbit associated with the Hamiltonian \( \tilde{H}_\mu \) we obtain from (3.20)

\[
|| \frac{\partial u}{\partial t}(r,.) - X_{\tilde{H}_\mu}(G_u(r,.)) ||_{L^2(S^1, R^N)} < \delta_1(\mathcal{F}).
\]

By Lemma 3.10 taking into account the assumption (2) at the beginning of the proof of Lemma 3.13, (3.21) cannot happen. Hence Lemma 3.13 also holds in Case 2. This completes the proof of Lemma 3.13.

**Proof of Lemma 3.14.** We repeat the proof of [Ono2005, Lemma 3.4], which is a refinement of the proof of Lemma [LO1995, Lemma 3.5], and we make precise the meaning of the constant \( \delta \) in the statement of [Ono2005, Lemma 3.4].

Let \( u \in \mathcal{M}([\tilde{x}, \tilde{w}],[\tilde{y}, \tilde{v}], \tilde{H}_\mu, J) \). Since \( E(u) < \infty \), by Lemma 3.13 there are finitely many real numbers \(-\infty < R_1 < \cdots < R_k + < R_{k+} < +\infty \) and one-periodic solutions \( \tilde{x}_0 = \tilde{x}, \tilde{x}_1, \cdots, \tilde{x}_k = \tilde{y} \) such that

1. \( \max_i d(\tilde{x}_{i-1}(t), u(R_i^{-}, t)) = \max_i d(\tilde{x}_{i}(t), u(R_i^{+}, t)) = \varepsilon(\mathcal{F}) \),
2. \( \max_i (u(s,t), \tilde{z}(t)) > \varepsilon(\mathcal{F}) \) for \( s \in (R_i^{-}, R_i^{+}) \) and \( \tilde{z} \in \mathcal{P}(\mathcal{F}) \).

First we estimate

\[
E_{R_i^{+} - R_i^{-}}^{R_i^{+}}(u) := \int_{R_i^{-}}^{R_i^{+}} \int_{0}^{1} |\frac{\partial u}{\partial s}|^2 ds dt
\]

\[
= \int_{R_i^{-}}^{R_i^{+}} \sqrt{\int_{0}^{1} |\frac{\partial u}{\partial s}(s, t) - X_{\tilde{H}_\mu}(t, u(s, t))|^2 dt}^2 ds.
\]

Applying the Cauchy-Schwarz inequality, we obtain

\[
E_{R_i^{+} - R_i^{-}}^{R_i^{+}}(u) \geq \frac{1}{R_i^{+} - R_i^{-}} \left( \int_{R_i^{-}}^{R_i^{+}} \sqrt{\int_{0}^{1} |\frac{\partial u}{\partial s}(s, t) - X_{\tilde{H}_\mu}(t, u(s, t))|^2 dt} ds \right)^2.
\]

Combining the property (2) with Lemma 3.10 taking into account that \( u \) is a connecting orbit associated with the Hamiltonian \( \tilde{H}_\mu \), we obtain from (3.22)

\[
E_{R_i^{+} - R_i^{-}}^{R_i^{+}}(u) \geq \delta_1(\mathcal{F}) \left( \int_{R_i^{-}}^{R_i^{+}} \sqrt{\int_{0}^{1} |\frac{\partial u}{\partial s}|^2 dt} ds \right).
\]

Applying the Cauchy-Schwarz inequality again, we obtain

\[
E_{R_i^{+} - R_i^{-}}^{R_i^{+}}(u) \geq \delta_1(\mathcal{F}) \int_{R_i^{-}}^{R_i^{+}} \int_{0}^{1} |\frac{\partial u}{\partial s}| dt ds \geq \delta_1(\mathcal{F})(\rho(\tilde{x}_{i-1}, \tilde{x}_i) - 2\varepsilon(\mathcal{F})).
\]
Using (3.17), we obtain
\[ E_{R^+_{\tilde{r}^{-}}} (u) \geq \delta_1 (\mathcal{F}) (\rho(\tilde{x}_{i-1}, \tilde{x}_i) - 2\varepsilon (\mathcal{F})) \geq \frac{\delta_1 (\mathcal{F})}{2} \rho(\tilde{x}_{i-1}, \tilde{x}_i). \]
Hence
\[ E(u) \geq \sum_{i=1}^{k} E_{R^+_{\tilde{r}^+}} (u) \geq \delta_1 (\mathcal{F})^2 \rho(\tilde{x}, \tilde{y}). \]
This completes the proof of Lemma 3.14. \( \square \)

Continuation of the proof of Theorem 3.12 (ii) Now assume that \( \tilde{H}_\mu \in U(\mathcal{F}) \) is \( J \)-regular. We set
\[ \tau = \tau (\mathcal{F}) := \frac{\delta_1 (\mathcal{F})}{4 \| \theta \|_{C^0}}. \] (3.23)
Recall that \( h^\theta \) is defined in (3.8) and \( \tilde{H}_\mu^{(\lambda)}([\tilde{y}, \tilde{v}]) \) is defined in (3.9). To prove Theorem 3.12 it suffices to show that for any \( C \in \mathbb{R} \) and any \([\tilde{x}, \tilde{w}]\) with \( \mu([\tilde{x}, \tilde{w}]) = k \) we have
\[ \# \{ u \in \mathcal{M}([\tilde{x}, \tilde{w}], [\tilde{y}, \tilde{v}]) | \mu([\tilde{y}, \tilde{v}]) = k - 1, A_{\tilde{H}_\mu^{(\lambda)}}([\tilde{y}, \tilde{v}]) > C \} < \infty \] (3.24)
for \( \lambda = \tau \) and for \( \lambda = -\tau \). We write
\[ A_{\tilde{H}_\mu^{(\pm\tau)}} ([\tilde{y}, \tilde{v}]) = A_{\tilde{H}_\mu^{(\pm\tau)}} ([\tilde{x}, \tilde{w}]) + A_{\tilde{H}_\mu} ([\tilde{y}, \tilde{v}]) \]
\[ + \int_{0}^{1} \pm \tau \cdot h^\theta (\tilde{y}(t)) dt - A_{\tilde{H}_\mu} ([\tilde{x}, \tilde{w}]) - \int_{0}^{1} \pm \tau \cdot h^\theta (\tilde{x}(t)) dt. \] (3.25)
Taking into account (3.23) and Lemma 3.14 we obtain
\[ | \int_{0}^{1} \pm \tau \cdot h^\theta (\tilde{x}(t)) dt - \int_{0}^{1} \pm \tau \cdot h^\theta (\tilde{y}(t)) dt | \]
\[ \leq | \tau \cdot \theta |_{C^0} \cdot \rho(\tilde{x}, \tilde{y}) < \frac{E(u)}{2}. \] (3.26)
We obtain from (3.25) and (3.26), taking into account the energy identity (2.7)
\[ A_{\tilde{H}_\mu^{(\pm\tau)}} ([\tilde{y}, \tilde{v}]) > C \]
\[ \implies A_{\tilde{H}_\mu} ([\tilde{y}, \tilde{v}]) > C - A_{\tilde{H}_\mu^{(\pm\tau)}} ([\tilde{x}, \tilde{w}]) + A_{\tilde{H}_\mu} ([\tilde{x}, \tilde{w}]) - E(u) \]
\[ > C - A_{\tilde{H}_\mu^{(\pm\tau)}} ([\tilde{x}, \tilde{w}]) + A_{\tilde{H}_\mu} ([\tilde{x}, \tilde{w}]) - A_{\tilde{H}_\mu} ([\tilde{y}, \tilde{v}]) \]
\[ \implies \frac{A_{\tilde{H}_\mu} ([\tilde{y}, \tilde{v}])}{2} \geq C - A_{\tilde{H}_\mu^{(\pm\tau)}} ([\tilde{x}, \tilde{w}]) + \frac{A_{\tilde{H}_\mu} ([\tilde{x}, \tilde{w}])}{2}. \] (3.27)
Since the RHS of (3.27), which depends only on $C$ and on $[\bar{x}, \bar{w}]$, is bounded form below, there is only finite numbers of $[\bar{y}, \bar{v}]$ that satisfies (3.27), because $[\bar{y}, \bar{v}]$ enters in $\partial_{J, \bar{H}_p}$. This yields (3.24) and completes the proof of Theorem 3.12. □

From now on we abbreviate the notation $(CFN^{(-\tau, \tau)}_s(\bar{H}, R), \partial_{(J, \bar{H})})$ as $CFN^{(-\tau, \tau)}_s(\bar{H}, J, R)$.

**Proposition 3.15.** Let $N$ be the minimal Chern number of a compact symplectic manifold $(M^{2n}, \omega)$ and $U(\mathcal{F})$ a good neighborhood of an admissible family $\mathcal{F}$ of nondegenerate Hamiltonian functions in $C^\infty_\alpha(S^1 \times \bar{M})$. Assume that $\bar{H} \in U(\mathcal{F})$ is a $J$-regular for some regular compatible almost complex structure $J$ on $(M^{2n}, \omega)$. For any $i \in \mathbb{Z}_{2N}$ we have

$$b_i(HFN_s(\bar{H}, J, R)) = b_i(HFN^{(-\tau, \tau)}_s(\bar{H}, J, R)),$$

where $\tau = \tau(\mathcal{F})$ is defined in (3.23).

**Proof.** Since the field of fractions $F(\Lambda^R_{\theta(\tau_1, \tau_2)})$ is a flat right $\Lambda^R_{\theta(\tau_1, \tau_2)}$-module, we obtain from (3.7), using the universal coefficient theorem

$$b_i(HFN_s(\bar{H}, J, R)) = \dim_{F(\Lambda^R_{\theta(\tau_1, \tau_2)})} H_i(F(\Lambda^R_{\theta, \omega}) \otimes \Lambda^R_{\theta, \omega} CFN_s(\bar{H}, J, R)).$$

Using Theorem 3.12 we obtain from (3.28)

$$b_i(HFN_s(\bar{H}, J, R)) = \dim_{F(\Lambda^R_{\theta(\tau_1, \tau_2)})} H_i(F(\Lambda^R_{\theta, \omega}) \otimes \Lambda^R_{\theta(\tau_1, \tau_2), \omega} CFN^{(-\tau, \tau)}_s(\bar{H}, J, R))$$

$$= \dim_{F(\Lambda^R_{\theta(\tau_1, \tau_2), \omega})} H_i(F(\Lambda^R_{\theta(\tau_1, \tau_2), \omega}) \otimes \Lambda^R_{\theta(\tau_1, \tau_2), \omega} CFN^{(-\tau, \tau)}_s(\bar{H}, J, R))$$

$$= \dim_{F(\Lambda^R_{\theta(\tau_1, \tau_2), \omega})} H_i(F(\Lambda^R_{\theta(\tau_1, \tau_2), \omega}) \otimes \Lambda^R_{\theta(\tau_1, \tau_2), \omega} CFN^{(-\tau, \tau)}_s(\bar{H}, J, R)).$$

This completes the proof of Proposition 3.15. □

### 3.3. Invariance of the Betti numbers of Floer-Novikov chain complexes

In this subsection we assume that $\mathbb{F}$ is a field. The goal of this subsection is to prove the following.

**Theorem 3.16.** The Betti numbers $b_i(HFN_s(\bar{H}, \mathbb{F}))$ do not depend on the choice of $\bar{H} \in C^\infty_\alpha(S^1 \times \bar{M}^{2n})$.

**Proof.** By Remark 3.7 the Betti numbers $b_i(HFN_s(\bar{H}, \mathbb{F}))$ depend only on $\lambda$, where $\bar{H} \in C^\infty_\lambda(S^1 \times \bar{M}^{2n})$. Thus, to prove Theorem 3.16, it suffices to prove the following.
Proposition 3.17. Assume that $J$ is regular. Let $\tilde{H} \in \mathcal{C}^{\infty}_{(\lambda)}(S^1 \times \tilde{M})$ be nondegenerate and $J$-regular. Then there exists a number $\delta = \delta(\tilde{H}) > 0$ such that for any $\chi \in (-\delta, \delta)$ there is a nondegenerate $J$-regular Hamiltonian function $\tilde{H}_{\mu(\chi)} \in \mathcal{C}^{\infty}_{(\lambda+\chi)}(S^1 \times \tilde{M}^{2n})$ with

$$b_i(HFN_*(\tilde{H}, \mathbb{F})) = b_i(HFN_*(\tilde{H}_\chi, \mathbb{F}))$$

for all $i \in \mathbb{Z}_{\geq 2}$.

Proof. Let $\tilde{H} \in \mathcal{C}^{\infty}_{(\lambda)}(S^1 \times \tilde{M})$. We shall construct an admissible family $\mathcal{F} \ni \tilde{H}$ and find an open interval $(-\delta, \delta)$ such that for any $\chi \in (-\delta, \delta)$ there is a function $\tilde{H}_{\mu(\chi)} \in \mathcal{C}^{\infty}_{(\lambda+\chi)}(S^1 \times \tilde{M}^{2n}) \cap U(\mathcal{F})$ which satisfies the required property in Proposition 3.17. This will be done in 6 steps.

Since $\tilde{H} \in \mathcal{C}^{\infty}_{(\lambda)}(S^1 \times \tilde{M})$, there is a function $H \in \mathcal{C}^{\infty}(S^1 \times \tilde{M}^{2n})$ such that

$$\tilde{H} - \lambda \cdot h^0 = \pi^*(H).$$

Step 1. In this step we construct a “linear part” of the desired admissible family $\mathcal{F} \ni \tilde{H}$ of Hamiltonians for the proof of Proposition 3.17. The main point of this step is Lemma 3.18.

Denote by $U_i := U_i(G_{x_i})$ the open $\varepsilon$-tubular neighborhood of the graph $G_{x_i} \subset S^1 \times M^{2n}$ of the periodic solution $x_i \in \mathcal{P}(\tilde{H})$, where $\varepsilon = \varepsilon(\tilde{H})$ satisfies the inequality in (3.17)

$$\frac{1}{4} \max_{t \in S^1} d(x_i(t), x_j(t)) \geq \frac{1}{4} \rho(x_i, x_j) > \varepsilon(\tilde{H})$$

if $i \neq j$.

Let $p : S^1 \times M^{2n} \to M^{2n}$ and $q : S^1 \times M^{2n} \to S^1$ denote the projections onto the second and the first component respectively. We set

$$T^{0,1}(S^1 \times M^{2n}) := p^*(T^* M^{2n}), \quad T^{1,0}(S^1 \times M^{2n}) := q^*(T^* S^1).$$

Then we have $T^*(S^1 \times M^{2n}) = T^{0,1}(S^1 \times M^{2n}) \oplus T^{1,0}(S^1 \times M^{2n})$. This yields a direct decomposition

$$\Omega^1(S^1 \times M^{2n}) = \Omega^{0,1}(S^1 \times M^{2n}) \oplus \Omega^{1,0}(S^1 \times M^{2n}),$$

where

$$\Omega^{0,1}(S^1 \times M^{2n}) := \{ \zeta \in \Omega^1(S^1 \times M^{2n}) | \zeta(t, x) \in T^{0,1}(S^1 \times M^{2n}) \},$$

$$\Omega^{1,0}(S^1 \times M^{2n}) := \{ \zeta \in \Omega^1(S^1 \times M^{2n}) | \zeta(t, x) \in T^{1,0}(S^1 \times M^{2n}) \}. $$
For a function $H \in C^\infty(S^1 \times M^{2n})$ denote by $d_x H$ the projection of $dH$ on the component $\Omega^{0,1}(S^1 \times M^{2n})$. Denote by $p$ the natural projection $S^1 \times M^{2n} \to M^{2n}$.

**Lemma 3.18.** There exists a 1-form $\eta \in \Omega^{0,1}(S^1 \times M^{2n})$ such that the following conditions hold:

1) $\eta - p^*(\theta) = d_x H$, for some $H \in C^\infty(S^1 \times M^{2n})$,
2) $\eta_{|U_i} = 0$ for all $i \in [1,k]$.

**Proof.** Lemma 3.18 has been used in [LO1995] without a (detailed) proof. For the reader’s convenience we present a detailed proof here.

Since $d_x p^*(\theta) = d_x \theta = 0$, and $x_i(t)$ is a contractible curve in $M^{2n}$, there exists a function $H_i \in C^\infty(U_i)$ such that

$$p^*(\theta)_{|U_i} = d_x H_i.$$  \hspace{1cm} (3.29)

Since $U_i$ are mutually disjoint, there exists a function $H \in C^\infty(S^1 \times M^{2n})$ such that

$$H_{|U_i} = H_i.$$  \hspace{1cm} (3.30)

Now we set $\eta = p^*(\theta) - d_x H$. Then $\eta$ satisfies the first condition in Lemma 3.18. By (3.29), (3.30) we have

$$\eta_{|U_i} = d_x H_i - d_x H_i = 0.$$

Thus $\eta$ also satisfies the second condition of Lemma 3.18. This completes the proof of Lemma 3.18. \hfill \square

**Step 2.** In this step, using $\eta$ in Lemma 3.18, we construct “the action” of the desired admissible family $\mathcal{F}$. The main point of this step is Lemma 3.20.

First we choose a positive number $\delta_1 = \delta_1(\tilde{H})$ from following Lemma, which is a special case of Lemma 3.10.

**Lemma 3.19.** There exists a positive number $\delta_1 = \delta_1(\tilde{H}) > 0$ such that

$$\|\dot{\sigma} - X_{\tilde{H}}(G_{\sigma})\|_{L^2(S^1, \mathbb{R}^{2n})} > \delta_1$$

for any loop $\sigma(t)$ in $\tilde{M}^{2n}$ satisfying $\max_{t} d(\sigma(t), x(t)) > \varepsilon(\tilde{H})$ for any $x \in \mathcal{P}(\tilde{H})$.

Let $\eta_t := \eta(t, \cdot)$. Denote by $\phi_t^\eta$ the symplectic flow on $M^{2n}$ that is generated by the time-depending symplectic vector field $L^{-1}_\omega(\eta_t)$ with $\phi_0^\eta = Id$.

Now we choose a small positive number $0 < \delta_2 = \delta_2(\tilde{H}, \eta) < \delta_1/3$ such that

$$\|\pi_*(X_{\tilde{H}_t}) - d\phi_t^\eta(\pi_*(X_{\tilde{H}_t}))\|_{C^0(M^{2n})} < \delta_1/3$$  \hspace{1cm} (3.31)
and
\[ \|X_{c,\eta}\|_{C^0(M^{2n})} < \frac{\delta_1}{3}, \tag{3.32} \]
for any \( c \in [-\delta_2, \delta_2] \) and any \( t \in [0,1] \). The number \( \delta_2 \) exists, since \([0,1] \times M^{2n}\) is compact and \( d\phi_t^0\eta = I_d \) for all \( t \in [0,1] \).

Denote by \( \varphi_t \) the symplectic flow on \( M^{2n} \) generated by the time-dependent vector field \( \pi_*(X_{\tilde{H}}) \).

**Lemma 3.20.** The symplectic flows \( \varphi_t \) and \( \phi_t^{c,\eta} \circ \varphi_t \) have the same contractible one-periodic orbits for all \( c \in [-\delta_2, \delta_2] \).

**Proof.** Lemma 3.20 and our proof stem from analogous arguments in Ono 2005 §3.2. Assume the opposite, i.e. there is a number \( c \in [-\delta_2, \delta_2] \) and a contractible one-periodic orbit \( \sigma(t) \) of the flow \( \phi_t^{c,\eta} \circ \varphi_t \) which is not a one-periodic orbit of \( \varphi_t \). We abbreviate \( L_\omega^{-1}(\eta) \) as \( X_\eta \).

We compute
\[ \frac{d}{dt}(\phi_t^{c,\eta} \circ \varphi_t(x)) = d\phi_t^{c,\eta}(\pi_*(X_{\tilde{H}}))(\varphi_t(x)) + X_{c,\eta}(\phi_t^{c,\eta} \circ \varphi_t(x)) \tag{3.33} \]
Now let \( x = \sigma(0) \). If \( (t, \sigma(t)) \subset U_i \) then by (3.33)
\[ \frac{d}{dt}(\phi_t^{c,\eta} \circ \varphi_t(\sigma(0))) = d\phi_t^{c,\eta}(\pi_*(X_{\tilde{H}}))(\varphi_t(x)). \tag{3.34} \]
Since \( \eta_{U_i} = 0 \), it is not hard to conclude from (3.34) that
\[ \sigma(t) := \phi_t^{c,\eta}(\varphi_t(\sigma(0)) = \varphi_t(\sigma(0)). \]
Hence \( \sigma(t) \) is also an one-periodic orbit of \( \varphi_t \), which is a contradiction. This implies that \( (t, \sigma(t)) \) does not lie in \( U_i \) for any \( i \). Hence
\[ \max_i d(\sigma(t), x_i(t)) > \varepsilon(\tilde{H}) \]
for any \( i \). Combining with Lemma 3.19 we obtain
\[ \|\pi_*(X_{\tilde{H}})(\sigma(t)) - \hat{\sigma}(t)\|_{L^2(S^1,\mathbb{R}^{N_0})} > \delta_1(\tilde{H}). \tag{3.35} \]
Using (3.33) we obtain the following inequalities, taking into account the inequalities (3.35), (3.32), (3.31),
\[ \|\frac{d}{dt}(\phi_t^{c,\eta} \circ \varphi_t)(\sigma(0)) - \hat{\sigma}(t)\|_{L^2(S^1,\mathbb{R}^{N_0})} \]
\[ = \|d\phi_t^{c,\eta}(\pi_*(X_{\tilde{H}}))(\varphi_t(\sigma(0))) + X_{c,\eta}(\sigma(t)) - \hat{\sigma}(t)\|_{L^2(S^1,\mathbb{R}^{N_0})} \]
\[ \geq \|\pi_*(X_{\tilde{H}})(\sigma(t)) - \hat{\sigma}(t)\|_{L^2(S^1,\mathbb{R}^{N_0})} - \|X_{c,\eta}\|_{C^0(M^{2n})} \]
\[ - \|\pi_*(X_{\tilde{H}})(\sigma(t)) - d\phi_t^{c,\eta}(\pi_*(X_{\tilde{H}}))(\sigma(t))\|_{L^2(S^1,\mathbb{R}^{N_0})} \]
\[ \geq \delta_1 - \frac{\delta_1}{3} - \frac{\delta_1}{3} > 0. \]
We arrive at a contradiction. This completes the proof of Lemma 3.20. □

Step 3 In this step we construct the desired admissible family $\mathcal{F} \ni \tilde{H}$. The main point of this step is Lemma 3.21.

Assume that $\chi \in [-\delta_2, \delta_2]$. Then for all $t \in [0, 1]$

$$[L^{-1}_\omega (d\phi_t^{\chi, \eta}(\pi_*(X_{\tilde{H}_t}(\phi_t(x))) - \pi_*(X_{\tilde{H}_t}(\phi_t^{\chi, \eta} \circ \varphi_t(x))) = 0 \in H^1(M^{2n}, \mathbb{R}).$$

Hence there exists a unique function $h_0^\chi \in C^\infty(S^1 \times M^{2n})$ such that for a given point $x_0 \in M^{2n}$

$$h_0^\chi(x_0) = 0$$

and $\tilde{H} + \chi \cdot h^\theta + \pi^*(h_0^\chi)$ generates the Hamiltonian isotopy whose projection on $M^{2n}$ is the isotopy $\phi_t^{\chi, \eta} \circ \varphi_t$. We set

$$\tilde{h}_\chi := \chi \cdot h^\theta + \pi^*(h_0^\chi), \quad (3.36)$$

$$\tilde{H}_\chi := \tilde{H} + \tilde{h}_\chi. \quad (3.37)$$

From Lemma 3.20 we obtain immediately the following, observing that the nondegeneracy of $\tilde{H}$ is an open property.

**Lemma 3.21.** There is a positive number $\delta_3(\tilde{H}) \leq \delta_2(\tilde{H}, \eta)$ such that the following statement holds. The family $\mathcal{F}(\tilde{H}) := \{\tilde{H}_\chi \in C^\infty(S^1 \times M^{2n})| \chi \in [-\delta_3(\tilde{H}), \delta_3(\tilde{H})]\}$ is an admissible family of nondegenerate Hamiltonian functions.

Step 4. In this step we shall choose first candidates for $\delta(\tilde{H})$ and a $J$-regular nondegenerate Hamiltonian $\tilde{H}_{\mu(\chi)}$ for the proof of Proposition 3.17.

Set

$$\delta_4 := \min\{\delta_3(\tilde{H}) + \tau(\mathcal{F})\}.$$

Then

$$\delta_4 \leq \min\{\tau(\mathcal{F}) - \chi, \tau(\mathcal{F}) + \chi\} \quad \text{for all } \chi \in [-\delta_4, \delta_4]. \quad (3.38)$$

**Lemma 3.22.** For any $\chi \in [-\delta_4, \delta_4]$ and any $J$-regular Hamiltonian function $\tilde{H}_{\mu(\chi)} \in U^c_{\text{reg}}(\tilde{H}_\chi)$ the following assertion holds. Let $[\tilde{x}, \tilde{w}] \in \text{Crit}(\mathcal{A}_{\tilde{H}_{\mu(\chi)}})$. Then

$$\partial_{J, \tilde{H}_{\mu(\chi)}}(\tilde{x}, \tilde{w}) \in \Lambda^F_{(\lambda - \delta_4, \lambda + \delta_4), \omega} \otimes_{F[\Gamma^0]} CFN^0_{\lambda}(\tilde{H}_{\mu(\chi)}), F).$$

Furthermore

$$b_1(CFN^0_{\lambda}(\tilde{H}_{\mu(\chi)}), J, F) = b_1(\Lambda^F_{(\lambda - \delta_4, \lambda + \delta_4), \omega} \otimes_{F[\Gamma^0]} CFN^0_{\lambda}(\tilde{H}_{\mu(\chi)}), J, F).$$
Proof. Let $\tilde{H}_{\mu(\chi)} \in U_{c}^{reg}(\tilde{H}_{\chi})$. We set $\tau := \tau(F)$. By Theorem 3.12 we have

$$\partial_{J,\tilde{H}_{\mu(\chi)}}(x, \tilde{w}) \in CFN_{s}^{-\tau,\tau}(\tilde{H}_{\mu(\chi)}, F).$$  \hspace{1cm} (3.39)

Since $\tilde{H}_{\mu(\chi)} \in C_{(\lambda+\chi)}^{\infty}(S^{1} \times \tilde{M}^{2n})$, by definition we have

$$CFN_{s}^{-\tau,\tau}(\tilde{H}_{\mu(\chi)}, F) = \Lambda_{\theta(\lambda+\chi-\tau, \lambda+\chi+\tau), \omega} \otimes_{\mathbb{F}[\sigma]} CFN_{s}^{0}(\tilde{H}_{\mu(\chi)}, F).$$  \hspace{1cm} (3.40)

From (3.38) we obtain

$$\lambda + \chi - \tau \leq \lambda - \delta_{4} \leq \lambda + \delta_{4} \leq \lambda + \chi + \tau.$$ \hspace{1cm} (3.41)

Using Lemma 3.8 we deduce from (3.39), (3.40) and (3.40) the first assertion of Lemma 3.22 immediately. The second assertion follows from the first assertion and Proposition 3.15. This completes the proof of Lemma 3.22. \hspace{1cm} \square

Step 5 In this step we shrink the chosen interval $[-\delta_{4}, \delta_{4}]$ to a smaller sub-interval $[-\delta, \delta]$ and shrink the good neighborhood $U(F')$ to a “better” neighborhood $U_{c}(F')$, where $F'$ is the subfamily of $F$ with parameter $\chi \in [-\delta, \delta]$. This is necessary for the proof of different energy estimates, which we use in establishing a chain map between the chain complexes arising from the map in (3.49).

First, $\delta = \delta(\tilde{H})$ must satisfy the following two conditions.

$$||\delta \cdot \theta||_{C^{0}(S^{1} \times \tilde{M}^{2n})} < \frac{\delta_{1}(\tilde{H})}{12},$$  \hspace{1cm} (3.42)

$$||d(h^{0}_{\chi} - h^{0}_{\chi'})||_{C^{0}(S^{1} \times \tilde{M}^{2n})} < \frac{\delta_{1}(\tilde{H})}{12}, \text{ for any } \chi, \chi' \in [-\delta, \delta].$$ \hspace{1cm} (3.43)

Further, we define

$$U_{c}(\tilde{H}_{\chi}) := U_{c}(\tilde{H}_{\chi}) \cap \{ \tilde{H}_{\chi} + \pi^{*}(h_{\chi}) ||d\tilde{h}_{\chi}||_{C^{0}(S^{1} \times \tilde{M}^{2n})} < \frac{\delta_{1}(\tilde{H})}{12} \}. $$ \hspace{1cm} (3.44)

Lemma 3.23. Let $\chi \in [-\delta, \delta]$, $\tilde{H}_{\mu(0)} \in U_{c}(\tilde{H}_{0})$ and $\tilde{H}_{\mu(\chi)} \in U_{c}(\tilde{H}_{\chi})$. Then there exists $\tilde{h}_{\mu(\chi)} \in C_{(s)}^{\infty}(S^{1} \times \tilde{M}^{2n})$ such that

$$\tilde{H}_{\mu(\chi)} = \tilde{H}_{\mu(0)} + \tilde{h}_{\mu(\chi)},$$

$$||d\tilde{h}_{\mu(\chi)}||_{C^{0}(S^{1} \times \tilde{M}^{2n})} < \frac{\delta_{1}(\tilde{H})}{3}.$$  \hspace{1cm} (3.45)

In Lemma 3.23, under the norm $||d\tilde{h}_{\mu(\chi)}||_{C^{0}(S^{1} \times \tilde{M}^{2n})}$ we mean the norm $||\theta'||_{C^{0}(S^{1} \times \tilde{M}^{2n})}$ where $d\tilde{h}_{\mu(\chi)} = \pi^{*}(\theta')$. The 1-form $\theta'$ exists uniquely, since $\tilde{h}_{\mu(\chi)} \in C_{(s)}^{\infty}(S^{1} \times \tilde{M}^{2n})$. 

Proof. By (3.44), (3.36), (3.37), we have
\[
\tilde{H}_{\mathbf{x}(\mathbf{0})} - \tilde{H}_{\mathbf{x}(\mathbf{0})} = (H_{\mathbf{x}} + \pi^*(h_{\mathbf{x}})) - (H_0 + \pi^*(h_0))
\]
\[
= h^0_{\mathbf{x}} - h^0_0 + \pi^*(h_{\mathbf{x}}) - \pi^*(h_0)
\]
= \chi \cdot h^0 + (\pi^*(h_{\mathbf{x}}) - \pi^*(h_0)) + \pi^*(h_{\mathbf{x}}) - \pi^*(h_0).
\]
Now using (3.42, (3.44) and (3.43) we obtain
\[
|d\mu_{\mathbf{x}(\mathbf{0})}|_{C^0(S^1 \times M^{2n})} < 4 \cdot (\delta_1(\tilde{H})) = \frac{\delta_1(\tilde{H})}{3},
\]
what is required to prove. □

To simplify notations we shall re-denote in the remainder of this subsection \(c'\) as \(c\) and we let
\[
CFN^*_\text{red}(\tilde{H}_{\mu(\mathbf{x})}, \mathcal{F}) := \Lambda_{\mathbf{x}(\lambda - \delta, \lambda + \delta), \omega} \otimes_{\mathbb{F}[\Gamma^0]} CFN^*_\mu(\tilde{H}_{\mu(\mathbf{x})}, \mathcal{F})
\]
for any \(\mathbf{x} \in [-\delta, \delta]\).

Now we are ready to define a linear mapping between \(CFN^*_\text{red}(\tilde{H}_{\mu(\mathbf{x})}, \mathcal{F})\) and \(CFN^*_\text{red}(\tilde{H}_{\mu(\mathbf{0})}, \mathcal{F})\) for \(\mathbf{x} \in [-\delta, \delta]\). Let \(\phi(s)\) be a monotone increasing smooth function on \([-R, R]\) which vanishes near \(-R\) and equals 1 near \(R\). Taking into account Lemma 3.23 we set with \((J_s, \tilde{H}_{s,t}) = (J, (\tilde{H}_{\mu(\mathbf{0})})_t)\) for \(s < -R\),
\[
(J_s, \tilde{H}_{s,t}) = (J, (\tilde{H}_{\mu(\mathbf{x})})_t) \text{ for } s > R,
\]
\[
\tilde{H}_{s,t} = (\tilde{H}_{\mu(\mathbf{0})})_t + \phi(s) \cdot (\tilde{h}_{\mu(\mathbf{x})})_t.
\]

We consider the space \(\mathcal{M}([\tilde{x}, \tilde{w}^-], [\tilde{y}, \tilde{w}^+], \tilde{H}_{s,t}, J_s)\) of the solution \(\tilde{u} : \mathbb{R} \times S^1 \to \tilde{M}^{2n}\) of the following Floer chain map equation (cf: (2.4), 2.5, 2.6)
\[
\frac{\partial u}{\partial s} + J_s(u)(\frac{\partial u}{\partial t}) - X_{\tilde{H}_{s,t}} = 0,
\]
with the following boundary conditions
\[
\lim_{s \to -\infty} = \tilde{x} \in \mathcal{P}(\mathcal{F}),
\]
\[
\lim_{s \to \infty} = \tilde{y} \in \mathcal{P}(\mathcal{F}),
\]
\[
[\tilde{x}, \tilde{w}^- \# \tilde{u}] = [\tilde{y}, \tilde{w}^+]
\]

Lemma 3.24. \(\mathcal{M}([\tilde{x}, \tilde{w}^-], [\tilde{y}, \tilde{w}^+], \tilde{H}_{s,t}, J_s)\) is a finite set if \(\mu([\tilde{x}, \tilde{w}^-]) = \mu([\tilde{y}, \tilde{w}^+])\).
Proof. We define the energy of a solution $\tilde{u} \in \mathcal{M}([\tilde{x}, \tilde{w}^-], [\tilde{y}, \tilde{w}^+], \tilde{H}_{s,t}, J_s)$ as follows

$$E(u) = \int_{-\infty}^{\infty} \int_{0}^{1} |\frac{\partial u}{\partial s}|^2 dt ds.$$ 

The following estimate has been obtained in [LO1995].

**Lemma 3.25.** ([LO1995, Lemma 5.4]) Assume that $|\tilde{h}_{\mu(\chi)}|_{C^1(S^1 \times M^{2n})} < \delta_1/3$. For a solution $\tilde{u}$ of (3.45) satisfying (3.46, 3.47) and (3.48) we have

$$E(\tilde{u}) \leq 3(A_{\tilde{H}_{\mu(\chi)}}([\tilde{x}, \tilde{w}^+]) - A_{\tilde{H}_{\mu(\chi)}}([\tilde{y}, \tilde{w}^-])).$$

Recall that our choice of $\tilde{h}_{\mu(\chi)}$ satisfies the estimate in Lemma 3.23 and hence the condition of Lemma 3.25 is fulfilled. Hence the energy of a solution $\tilde{u} \in \mathcal{M}([\tilde{x}, \tilde{w}^-], [\tilde{y}, \tilde{w}^+], \tilde{H}_{s,t}, J_s)$ is uniformly bounded. The weak compactness theorem yields Lemma 3.24 immediately.

Now we define a map $\psi: \mathcal{P}(\tilde{H}) \rightarrow CFN^{red}_{*}(\tilde{H}_{\mu(\chi)}; \mathbb{F})$ as follows

$$\psi(\tilde{x}) := \sum_{\mu(\tilde{y}) = \mu(\tilde{x})} m(\tilde{x}, \tilde{y}) \cdot \tilde{y},$$

where $m(\tilde{x}, \tilde{y})$ denotes the algebraic cardinality of $\mathcal{M}(\tilde{x}, \tilde{y}, \tilde{H}_{s,t}, J_s)$.

Lemma 3.24 and the existence of a coherent orientation on the moduli space of the solutions of the Floer chain map equation imply that the number $m(\tilde{x}, \tilde{y})$ is well defined, but there are possibly infinitely many terms in the RHS of (3.49). So we need the following

**Lemma 3.26.** Given $\tilde{x} \in \mathcal{P}(\tilde{H})$ and a number $C \in \mathbb{R}$ there exists only finitely many $\tilde{y}$ in the RHS of (3.49) such that $A_{\tilde{H}_{\tau}}(\tilde{y}) > C$ for any $\tau \in [-\delta, \delta]$.

**Proof.** This Lemma is an analogue of the relation (3.24). As the proof of (3.24) is based on the uniform energy estimate in Lemma 3.14 our proof is based on the following estimate in [Ono2005, Lemma 3.5] for a solution $\tilde{u} \in \mathcal{M}([\tilde{x}, \tilde{w}^-], [\tilde{y}, \tilde{w}^+], \tilde{H}_{s,t}, J_s)$, which states that

$$A_{\tilde{H}_{\tau}}([\tilde{x}, \tilde{w}^-]) - A_{\tilde{H}_{\tau}}([\tilde{y}, \tilde{w}^+]) > \frac{\delta_1(F)}{6} \rho(\tilde{x}, \tilde{y}).$$

(3.50)

Repeating the argument in the proof of (3.24), using (3.50) instead of Lemma 3.14 we obtain immediately Lemma 3.26. This completes the proof of Lemma 3.26.

□
Step 6. In this step we complete the proof of Proposition 3.17. By Lemmas 3.24 3.26 the map \( \psi \) in (3.49) extends uniquely as a chain map, which we also denote by \( \psi : \text{CFN}_{\ast}^\text{red}(\tilde{H}_{\mu(0)}, \mathbb{F}) \rightarrow \text{CFN}_{\ast}^\text{red}(\tilde{H}_{\mu(\chi)}, \mathbb{F}) \).

We need to show that \( \psi \) is a chain homotopy equivalence. The proof of the chain homotopy equivalence is proceeded using standard arguments as in the proof of [LO1995, Theorem 5.3], see also [Ono2005, Theorem 4.6]. Combining with Lemma 3.22, this completes the proof of Proposition 3.17.

As we have remarked, Proposition 3.17 yields Theorem 3.12.

3.4. Computing the Betti numbers of \( HFN_{\ast}(\pi^{\ast}(H), \mathbb{F}) \). Let \( \mathbb{F} \) be a field. In this subsection we compare the sum of the Betti numbers of the Floer-Novikov homology \( HFN_{\ast}(\pi^{\ast}(H), \mathbb{F}) \) with the sum of the Betti numbers of the Novikov homology \( HN_{\ast}(M; [\theta], \mathbb{F}) \). It is known that the latter ones can be computed via the refined Morse complex \( CM_{\ast}(\pi^{\ast}(f), \partial^{\text{Morse}}, \mathbb{F}) \) with coefficients in \( \mathbb{F} \) of a lifted Morse function \( f \in C^\infty(M^{2n}) \), see e.g. [Farber2004] [Pajitnov2006]. Recall that \( \Lambda^F_{0,\omega} \) is the completion of \( \mathbb{F}[\Gamma^0] \) w.r.t. the weight homomorphism \( 0 \oplus -I_\nu \), see (2.3). By Proposition 3.2, \( \Lambda^F_{0,\omega} = \mathbb{F}[\Gamma_1][((-I_\nu(\Gamma^0_\omega))] \). Thus \( \Lambda^F_{0,\omega} \) is the underlying ring of the Floer-Novikov chain complex \( \text{CFN}_{\ast}(\pi^{\ast}(H), \mathbb{F}) \), which is also called a refined Floer chain complex by Ono-Pajitnov [OP2014], see also [LO2015] for further discussion.

Theorem 3.27. Assume that \( J \) is a regular compatible almost complex structure and \( H \in C^\infty(S^1 \times M^{2n}) \) is nondegenerate and \( J \)-regular. There is a chain isomorphism between the Floer-Novikov chain complex \( \text{CFN}_{\ast}(\pi^{\ast}(H), J, \mathbb{F}) \) and the \( \mathbb{Z}_{2N} \)-graded extended Novikov chain complex \( (\Lambda^F_{0,\omega} \otimes_{\mathbb{F}[\Gamma_1]} \tau_{2N}(CM_{\ast+n}(f)), I_\nu \otimes \partial^{\text{Morse}}), \) where \( \tau_{2N} \) denotes the natural projection of \( \mathbb{Z} \)-grading to \( \mathbb{Z}_{2N} \)-grading. Consequently \( \sum b_i(HFN_{\ast}(\pi^{\ast}(H), \mathbb{F}) = \sum b_i(HN_{\ast}(M; [\theta], \mathbb{F})). \)

Theorem 3.27 is a partial case of Theorem 2.8 in [OP2014] which considers refined Floer complexes over \( \mathbb{Z} \), see also Theorem 2.9 in [OP2014] concerning general compact symplectic manifolds. Ono and Pajitnov omit the proof of [OP2014, Theorems 2.8, 2.9], which can be done in the same way as in the Floer homology theory. For the reader convenience we shall outline the proof of Theorem 3.27 using the Piunikhin-Salamon-Schwarz scheme for comparing Floer chain complexes with Morse chain complexes [PSS1996]. We use the analytical framework developed by Oh-Zhu in [OZ2011] and by Lu in [Lu2004], see also [OZ2012] for another analytical approach.
Outline of the proof of Theorem 3.27. Let $H \in C^\infty(S^1 \times M^{2n})$ be a nondegenerate Hamiltonian and $J$-regular for some regular compatible almost complex structure $J$. We shall define a chain map

$$\Phi : (\Lambda^F_{0,\omega} \otimes \mathbb{F}[\Gamma_1] \tau_{2N}(CM_s(\pi^*(f))), Id \otimes \partial^{\text{Morse}}) \rightarrow CFN_s(\pi^*(H), J, \mathbb{F})$$

and a chain map

$$\Psi : CFN_s(\pi^*(H), J) \rightarrow (\Lambda^F_{0,\omega} \otimes \mathbb{F}[\Gamma_1] \tau_{2N}(CM_s(\pi^*(f))), Id \otimes \partial^{\text{Morse}})$$

by defining the value of $\Phi$ and $\Psi$ on the generators of each involved module, and then extending $\Phi$ and $\Psi$ linearly over the ring $\Lambda^F_{0,\omega}$.

Recall that the generators of $\Lambda^F_{0,\omega} \otimes \mathbb{F}[\Gamma_1] \tau_{2N}(CM_s(\pi^*(f)))$ are critical points of $\tilde{f}$, which we shall denote by $\tilde{p}$. Note that $\tilde{p} := \pi^{-1}(p)$, where $p$ are critical points of $f$. The generators of $CFN_s(\pi^*(H), J)$ are the critical points $[\tilde{x}, \tilde{w}] \in \tilde{P}(\pi^*(H))$ of the action functional $A_{\pi^*(H)}$. Similarly we have $\tilde{P}(\pi^*(H)) = \pi^{-1}(\tilde{P}(H))$.

We define $\Phi$ by the following formula

$$\Phi(\tilde{p}) := \sum_{[\tilde{x}, \tilde{w}] \in \tilde{P}(\pi^{-1}(H))} m(\tilde{p}, [\tilde{x}, \tilde{w}]; A)[\tilde{x}, \tilde{w}]e^{-A}, \quad (3.51)$$

where $A \in \Gamma_0$. We define $\Psi$ as follows

$$\Psi([\tilde{x}, \tilde{w}]) := \sum_{\tilde{p} \in \text{Crit}(f)} n([\tilde{x}, \tilde{w}], [\tilde{p}, A])\tilde{p} \cdot e^A, \quad (3.52)$$

where $A \in \Gamma_0$. The coefficient $m(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$ (resp. $n([\tilde{x}, \tilde{w}], [\tilde{p}, A])$) will be defined as the algebraic cardinal of the moduli space $\mathcal{M}(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$ (resp. $\mathcal{M}([\tilde{x}, \tilde{w}], [\tilde{p}, A])$) of spiked disks that consist of (perturbed) holomorphic disks equivalent to $[A\# - \tilde{w}]$ and bounding $\tilde{x}$ and of a spike which is a gradient flow line from $\tilde{p}$ to a holomorphic disk in consideration.

In the first step we shall describe the moduli spaces of involved spiked disks. Then we shall show that $\Phi$ and $\Psi$ are well-defined and $\Gamma^0$-invariant and therefore extensible over $\Lambda^F_{0,\omega}$. Finally we shall show that $\Psi$ and $\Phi$ are chain homotopy equivalences.

**Description of the moduli space $\mathcal{M}(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$.** Let $\hat{\Sigma}_+$ (resp. $\hat{\Sigma}_-$) denote the the punctured sphere with one marked point $o_+$ (resp. $o_-$) and one positive puncture $e_+$ (resp. $e_-$). We fix an identification

$$\hat{\Sigma}_{\pm} \setminus \{o_{\pm}\} \cong \mathbb{R} \times S^1 \quad (3.53)$$

and denote by $(\tau, t)$ the corresponding coordinates so that $\{\pm \infty\} \times S^1$ and $\{\pm \infty\} \times S^1$ correspond to $o_+$ and $e_+$ respectively.
Set
\[ K_\pm(s, t, x) := \beta_\pm(s) \cdot \pi^*(H)(t, x), \]
where \( \beta_\pm: \mathbb{R} \to [0, 1] \) is a smooth cut-off function given by
\[
\beta_+(s) = 0 \text{ for } s \leq 0, \\
\beta_+(s) = 1 \text{ for } s \geq 1, \\
\beta_-(s) := \beta_+(-s)
\]
such that
\[
0 < s'_+(\tau) < 2 \text{ for } s \in (0, 1).
\]

For a regular compatible almost complex structure \( J \) on \( M^{2n} \) we define a perturbed Cauchy-Riemann operator \( \bar{\partial}(K_\pm, J) \) acting on the space of smooth mappings \( u_\pm: \hat{\Sigma}_\pm \to \tilde{M} \) as follows.
\[
\bar{\partial}(K_\pm, J)u_\pm := \partial_s u + J(u)(\partial_t u - \beta_\pm(s)X_{\pi^*(H)}).
\]
Assume that \( \bar{\partial}(K_\pm, J)u_\pm = 0 \) and the energy
\[
E(u_\pm) := \int_{\infty}^{\infty} |\partial_s u_\pm|^2_{g_J} \, ds dt < +\infty.
\]
Then \( u_\pm \) extends to \( \hat{\Sigma}_\pm \), since nearby the marked point \( o_\pm \) the perturbation term \( K_\pm \) vanishes.

We set
\[
M_\pm(\tilde{p}, [\tilde{x}, \tilde{w}]; A) = \{ (\chi_\pm, u_\pm)|u_\pm: \hat{\Sigma}_\pm \to \tilde{M}, [u_\pm\#\tilde{w}] = A, \\
\quad u_\pm(\pm\infty, t) = \tilde{x}(t), \bar{\partial}(K_\pm, J)u_\pm = 0, \\
\quad \dot{\chi}_\pm = \nabla(\tilde{f}(\chi_\pm)), \chi_\pm(\mp\infty) = \tilde{p}, \chi_\pm(0) = u_\pm(o_\pm) \}.
\]
In the above expression we require that
\[
m(\tilde{x}, \tilde{w}) = n - (\mu(\tilde{f})(p)
\]
so that
\[
\dim M_\pm(\tilde{p}, [\tilde{x}, \tilde{w}]; A) = 0.
\]
Then we set
\[
m(\tilde{p}, [\tilde{x}, \tilde{w}]; A) := \#M_+(\tilde{p}, [\tilde{x}, \tilde{w}]; A_+), \quad (3.54)
\]
Similarly we set
\[
n([\tilde{x}, \tilde{w}], [\tilde{p}, A]) := \#M_-(\tilde{p}, [\tilde{x}, \tilde{w}]; A), \quad (3.55)
\]

**Theorem 3.28.** The maps \( \Phi \) and \( \Psi \) defined in (3.54) and (3.55) are well-defined and they are \( \Gamma_1 \)-equivariant. Furthermore, the compositions \( \Psi \circ \Phi \) and \( \Phi \circ \Psi \) are chain homotopy equivalent to the identity.
Outline of the proof. The first assertion of Theorem 3.28 follows from the compactness and the coherent orientability of the moduli spaces $\mathcal{M}_\pm(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$. The compactness is proved by using an upper estimation for the energy of the perturbed holomorphic disks $u_\pm$ involved in the moduli spaces in consideration. Recall that

$$E(u_\pm) := E(u_\pm|_{\Sigma_\pm}\{o_\pm\}) = \int_{-\infty}^{\infty} \int_{0}^{1} |\partial_\tau u|^2_{g_J} dtd\tau. \quad (3.56)$$

Lemma 3.29. Assume that $u_+ \in \mathcal{M}_+(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$. Then

$$E(u) \leq \omega(A) - \mathcal{A}_H([x, w]) + 2 \max_{(t,x) \in S^1 \times M^{2n}} H(t, x).$$

Proof. We write

$$E(u_+) = \int_{-\infty}^{\infty} \int_{0}^{1} |\partial_\tau u|^2_{g_J} dtd\tau = \int_{-\infty}^{\infty} \int_{0}^{1} \langle \partial_\tau u, \partial_\tau u + \beta(\tau) \nabla H(t, u) \rangle dtd\tau$$

$$= \omega(A) - \omega(\tilde{w}) - \int_{-\infty}^{\infty} \int_{0}^{1} \beta(\tau) \frac{d}{ds} H(t, u) dtd\tau$$

$$= \omega(A) - \mathcal{A}_H([\tilde{x}, \tilde{w}]) + \int_{0}^{1} \beta'(\tau) \int_{0}^{1} H(t, u) dtd\tau,$$

since $\beta'(s) = 0$ for $s \in \mathbb{R} \setminus [0,1]$. Taking into account $0 \leq \beta_+(s)' \leq 2$, we obtain Lemma 3.29 immediately. \hfill \Box

Lemma 3.29 provides the weak compactness of the moduli space $\mathcal{M}_+(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$. In the same way we obtain the weak compactness of the moduli space $\mathcal{M}_-(\tilde{p}, [\tilde{x}, \tilde{w}]; A)$. The regularity of $J$ and the $J$-regularity of $H$ yield the compactness of the moduli spaces in consideration. The coherent orientation of the moduli space is defined as in the \cite{FH1993}.

The second assertion of Theorem 3.28 follows from the fact, that on the covering space $\tilde{M}$ all objects under consideration are $\Gamma_1$-invariant. Finally the last assertion of Theorem 3.28 is proved in the same way as in the Floer homology case, so we omit the proof.

We obtain immediately from Theorems 3.16, 3.27 the following.

Corollary 3.30. For any nondegenerate $J$-regular Hamiltonian function $\tilde{H} \in C^\infty(S^1 \times \tilde{M})$ we have

$$\sum_i b_i(\text{HF} \nu_*(\tilde{H}, \mathbb{F})) = \sum_i b_i(\text{HN}_*(M, [\theta]), \mathbb{F}).$$
4. Proof of Theorem 1.3

The second assertion of Theorem 1.3 follows immediately from Corollary 3.30 and Lemmas 3.6(ii), 2.2.

To prove the first assertion of Theorem 1.3 we consider the Floer-Novikov chain complexes and their homology with coefficients in $\mathbb{Q}$ as in [Ono2005, FO1999]. We also refer the reader to [FOOO2015] for the latest account of the Fukaya-Oh-Ohta-Ono theory of Kuranishi structures and their applications. Let us rapidly recall the construction of the Floer-Novikov chain complexes on general compact symplectic manifolds. First we compactify the quotient spaces $M([\tilde{x},w^{-}],[\tilde{y},w^{+}])/\mathbb{R}$. The compactified space $\overline{M}([\tilde{x},w^{-}],[\tilde{y},w^{+}])$ is obtained from the quotient space $M([\tilde{x},w^{-}],[\tilde{y},w^{+}])/\mathbb{R}$ by adding stable connecting orbits as in [FO1999, Definitions 19.9, 19.10, p.1018-1019]. Here we do not require the regularity of a compatible almost complex structure $J$ and the $J$-regularity of a Hamiltonian function $\tilde{H} \in C^\infty(S^1 \times \tilde{M}^{2n})$. There exists a natural Kuranishi structure on $\overline{M}([\tilde{x},w^{-}],[\tilde{y},w^{+}])$. Using abstract multi-valued perturbation technique, we can define the “algebraic cardinality” $\langle [\tilde{x},w^{-}],[\tilde{y},w^{+}] \rangle \in \mathbb{Q}$ when $\mu([\tilde{x},w^{-}]) = \mu([\tilde{y},w^{+}]) = 1$ and set

$$\partial_{J,\tilde{H}}[\tilde{x},w^{-}] := \sum \langle [\tilde{x},w^{-}],[\tilde{y},w^{+}] \rangle [\tilde{y},w^{+}],$$

where $[\tilde{y},w^{+}]$ runs over the set of critical points of $A_{\tilde{H}}$ such that $\mu([\tilde{x},w^{-}]) = 1$. It is known that $\partial^2_{J,\tilde{H}} = 0$. The resulting homology is called the Floer-Novikov homology of the Floer-Novikov chain complex $CFN_n(\tilde{H},J,\mathbb{Q})$. We also know that the Floer-Novikov homology is invariant under Hamiltonian isotopy [Ono2005, Theorem 3.1].

To prove the invariance of the Floer-Novikov homology $HFN_n(\tilde{H},\mathbb{Q})$ where $\tilde{H} \in C^\infty(S^1 \times \tilde{M}^{2n})$ we use a simplified argument in the previous subsection. Namely it suffices to use an admissible family $\mathcal{F}'$ defined in Step 5 (but not the “better” neighborhood $U_c(\mathcal{F})$ which contains $J$-regular Hamiltonian functions). With $\mathcal{F}'$ we have all necessary energy estimates without taking care on the $J$-regularity of perturbed Hamiltonians $\tilde{H}'$. This completes the proof of Theorem 1.3.

5. Concluding remarks

1. One of main technical difficulties in the computation of the Floer-Novikov homology is the variation of the isomorphism type of the underlying Novikov ring $\mathbb{Z}_{\tilde{\theta}}$ under the variation of $[\theta]$ inside its conformal class $\mathbb{R} \cdot [\theta]$. For example, when $R$ is a field, by Proposition 3.3, the
Novikov ring $R_{\theta,\omega}$ is a field, if and only if $\ker \Psi_{\theta,\omega} = 0$. The important idea that the Floer-Novikov chain complex can be defined over a proper sub-ring of $R_{\theta,\omega}$, which is constant for a small variation of $[\theta]$ in its conformal class, has been appeared first in [Ono2005]. If there is a completion of $R[\Gamma^0]$ which is an integral domain and contains both different Novikov rings $R_{\theta,\omega}^F, R_{c,\theta,\omega}^F$ as its sub-rings, the proof of the Main Theorem can be simplified.

2. In [Seidel1997, Seidel2002] Seidel defined his version of Floer homology of a symplectomorphism $\phi$ as the Floer homology of the symplectic fibration obtained from the torus mapping of $\phi$. It is not clear how Seidel’s version of Floer-Novikov homology is related to our construction, especially how to recognize the Calabi invariant of $\phi$, if $\phi$ is symplectic isotopic to the identity.

3. Based on [LO2001] we conjecture that we could remove the restriction of the field $\mathbb{Q}$ in our Main Theorem.

4. In [LO2015] we develop other aspects of the theory of Floer-Novikov chain complexes to obtain new lower bounds for the number of symplectic fixed points.

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