Statistical Guarantees of Generative Adversarial Networks
for Distribution Estimation

Minshuo Chen, Wenjing Liao, Hongyuan Zha, Tuo Zhao*

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Abstract

Generative Adversarial Networks (GANs) have achieved great success in unsupervised learning. Despite the remarkable empirical performance, there are limited theoretical understandings on the statistical properties of GANs. This paper provides statistical guarantees of GANs for the estimation of data distributions which have densities in a Hölder space. Our main result shows that, if the generator and discriminator network architectures are properly chosen (universally for all distributions with Hölder densities), GANs are consistent estimators of the data distributions under strong discrepancy metrics, such as the Wasserstein distance. To our best knowledge, this is the first statistical theory of GANs for Hölder densities. In comparison with existing works, our theory requires minimum assumptions on data distributions. Our generator and discriminator networks utilize general weight matrices and the non-invertible ReLU activation function, while many existing works only apply to invertible weight matrices and invertible activation functions. In our analysis, we decompose the error into a statistical error and an approximation error by a new oracle inequality, which may be of independent interest.

1 Introduction

The generative adversarial networks (GANs) proposed in Goodfellow et al. (2014) utilize two neural networks competing with each other to generate new samples with the same distribution as the training data. They have been successful in many applications including producing photorealistic images, improving astronomical images, and modding video games (Reed et al., 2016; Ledig et al., 2017; Schawinski et al., 2017; Brock et al., 2018; Volz et al., 2018; Radford et al., 2015; Salimans et al., 2016).

From the viewpoint of statistics, GANs have stood out as an important unsupervised method for learning target data distributions. Different from explicit distribution estimations, e.g., density estimation, GANs implicitly learn the data distribution and act as samplers to generate new fake samples mimicking the data distribution (see Figure 1).

*Minshuo Chen, Tuo Zhao are affiliated with School of Industrial and Systems Engineering at Georgia Tech; Wenjing Liao is affiliated with School of Mathematics at Georgia Tech; Hongyuan Zha is affiliated with School of Computational Science and Engineering at Georgia Tech. Emails: {mchen393, wliao60, tourzhao@gatech.edu; zha@cc.gatech.edu}.
To estimate a data distribution $\mu$, GANs solve the following minimax optimization problem

$$(g^*, f^*) \in \arg \min_{g \in G} \max_{f \in F} \mathbb{E}_{z \sim \rho}[f(g(z))] - \mathbb{E}_{x \sim \mu}[f(x)],$$  

where $G$ denotes a class of generators, $F$ denotes a symmetric class (if $f \in F$, then $-f \in F$) of discriminators, and $z$ follows some easy-to-sample distribution $\rho$, e.g., uniform or Gaussian distributions. The estimator of $\mu$ is given by the pushforward distribution of $\rho$ under $g^*$, denoted by $(g^*)_\# \rho$.

The inner maximization problem of (1) is an Integral Probability Metric (IPM, Müller (1997)), which quantifies the discrepancy between two distributions $\mu$ and $\nu$ with respect to the symmetric function class $F$:

$$d_F(\mu, \nu) = \sup_{f \in F} \mathbb{E}_{x \sim \mu}[f(x)] - \mathbb{E}_{y \sim \nu}[f(y)].$$

Accordingly, GANs are essentially minimizing an IPM between the generated distribution and the data distribution. IPM unifies many standard discrepancy metrics. For example, when $F$ is taken to be all 1-Lipschitz functions, $d_F(\cdot, \cdot)$ is the Wasserstein distance; when $F$ is taken to be all indicator functions, $d_F(\cdot, \cdot)$ is the total variation distance; when $F$ is taken as the discriminator network, $d_F(\cdot, \cdot)$ is the so-called “neural net distance” (Arora et al., 2017).

GANs parameterize the generator and discriminator classes $G$ and $F$ by deep neural networks (ReLU activation is considered in this paper) denoted by $G = G_{\text{NN}}$ and $F = F_{\text{NN}}$, which consist of functions given by a feedforward ReLU network of the following form

$$f(x) = W_L \cdot \text{ReLU}(W_{L-1} \cdots \text{ReLU}(W_1x + b_1) \cdots + b_{L-1}) + b_L,$$

where the $W_i$’s and $b_i$’s are weight matrices and intercepts respectively, and ReLU denotes the rectified linear unit ($\text{ReLU}(a) = \max\{0, a\}$). ReLU networks are widely used in computer vision, speech recognition, natural language processing, etc. (Nair and Hinton, 2010; Glorot et al., 2011; Maas et al., 2013). These networks can ease the notorious vanishing gradient issue during training, which commonly arises with sigmoid or hyperbolic tangent activations (Glorot et al., 2011; Goodfellow et al., 2016).

When $n$ samples of $\mu$ are given as $\{x_i\}_{i=1}^n$, one can replace $\mu$ in (1) by its empirical counterpart $\hat{\mu}_n(x) = \frac{1}{n} \sum_{i=1}^n 1\{x = x_i\}$, and (1) becomes

$$(g^*_\theta, f^*_\omega) \in \arg \min_{g_\theta \in G_{\text{NN}}} \max_{f_\omega \in F_{\text{NN}}} \mathbb{E}_{z \sim \rho}[f_\omega(g_\theta(z))] - \frac{1}{n} \sum_{i=1}^n f_\omega(x_i),$$
where $\theta$ and $\omega$ are parameters in the generator and discriminator networks, respectively. The empirical estimator of $\mu$ given by GANs is the pushforward distribution of $\rho$ under $g^*_\theta$, denoted by $(g^*_\theta)_\#\rho$.

In contrast to the prevalence of GANs in applications, only very limited works study the theoretical properties of GANs (Arora et al., 2017; Bai et al., 2018; Liang, 2018; Singh et al., 2018; Thanh-Tung et al., 2019). Here we focus on the following fundamental questions from a theoretical point of view:

1) What types of distributions can be learned by GANs?

2) If the distribution can be learned, what is the statistical rate of convergence?

This paper shows that, if the generator and discriminator network architectures are properly chosen, GANs can effectively learn distributions with Hölder densities supported on proper domains. Specifically, we consider a data distribution $\mu$ supported on a compact domain $\mathcal{X} \subset \mathbb{R}^d$ with $d$ being the data dimension. We assume $\mu$ has a density lower bounded away from 0 on $\mathcal{X}$, and the density belongs to the Hölder class $\mathcal{H}^\alpha$.

In order to learn $\mu$, we choose proper generator and discriminator network architectures — we specify the width and depth of the network, total number of neurons, and total number of weight parameters (details are provided in Section 2). Roughly speaking, the generator is chosen to be flexible enough to approximate the data distribution, and the discriminator is powerful enough to distinguish the generated distribution from the data distribution.

Let $(g^*_\theta, f^*_\omega)$ be the optimal solution to (3), and then $(g^*_\theta)_\#\rho$ is the generated data distribution as an estimation of $\mu$. Our main results can be summarized as, for any $\beta \geq 1$,

$$\mathbb{E}\left[d_{\mathcal{H}^\beta}( (g^*_\theta)_\#\rho, \mu) \right] = \tilde{O}(n^{-\frac{\beta}{2\beta+2d}} \log n),$$

where the expectation is taken over the randomness of samples, and $\tilde{O}$ hides constants and polynomials in $\beta$ and $d$.

To our best knowledge, this is the first statistical theory of GANs for Hölder densities. It shows that the Hölder IPM between the generated distribution and the data distribution converges at a rate depending on the Hölder index $\beta$ and dimension $d$. When $\beta = 1$, our theory implies that GANs can estimate any distribution with a Hölder density under the Wasserstein distance. It is different from the generalization bound in Arora et al. (2017) under the weaker neural net distance.

In our analysis, we decompose the distribution estimation error into a statistical error and an approximation error by a new oracle inequality. A key step is to properly choose the generator network architecture to control the approximation error. Specifically, the generator architecture allows an accurate approximation to a data transformation $T$ such that $T_\#\rho = \mu$. The existence of such a transformation $T$ is guaranteed by the optimal transport theory (Villani, 2008), and holds universally for all the data distributions with Hölder densities.

In comparison with existing works (Bai et al., 2018; Liang, 2018; Singh et al., 2018), our theory holds with minimum assumptions on the data distributions and does not require invertible generator networks (all the weight matrices have to be full-rank, and the activation function needs to be the invertible leaky ReLU activation). See Section 4 for a detailed comparison.
Notations: Given a real number $\alpha$, we denote $\lfloor \alpha \rfloor$ as the largest integer smaller than $\alpha$ (in particular, if $\alpha$ is an integer, $\lfloor \alpha \rfloor = \alpha - 1$). Given a vector $v \in \mathbb{R}^d$, we denote its $\ell_2$ norm by $\|v\|_2$, the $\ell_\infty$ norm as $\|v\|_\infty = \max_i |v_i|$, and the number of nonzero entries by $\|v\|_0$. Given a matrix $A \in \mathbb{R}^{d_1 \times d_2}$, we denote its number of nonzero entries by $\|A\|_0$. Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we denote its $\ell_\infty$ norm as $\|f\|_\infty = \sup x |f(x)|$. For a multivariate transformation $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and a given distribution $\rho$ in $\mathbb{R}^d$, we denote the pushforward distribution as $T_\# \rho$, i.e., for any measurable set $\Omega$, $T_\# \rho(\Omega) = \rho(T^{-1}(\Omega))$.

2 Statistical Theory

We consider a data distribution $(\mathcal{X}, \mu)$ supported on a subset $\mathcal{X} \subset \mathbb{R}^d$. We assume the distribution $\mu$ has a density function $p_\mu$. Suppose we can easily generate samples from some easy-to-sample distribution $(Z, \rho)$, such as the uniform distribution.

Before we proceed, we make the following assumptions.

**Assumption 1.** The domains $\mathcal{X}$ and $Z$ are compact, i.e., there exists a constant $B > 0$ such that for any $x \in \mathcal{X}$ or $x \in Z$, $\|x\|_\infty \leq B$.

**Assumption 2.** The density function $p_\mu$ belongs to the H"older class $\mathcal{H}^\alpha(\mathcal{X})$ with H"older index $\alpha > 0$ in the interior of $\mathcal{X}$, such that

1. for any positive integer $s < \lfloor \alpha \rfloor$ and $x$ in the interior of $\mathcal{X}$, $|\partial^s p_\mu(x)| \leq 1$;
2. for any $x, y$ in the interior of $\mathcal{X}$,

$$|\partial^{\lfloor \alpha \rfloor} p_\mu(x) - \partial^{\lfloor \alpha \rfloor} p_\mu(y)| \leq \|x - y\|_2^{\alpha - \lfloor \alpha \rfloor},$$

where $\partial^s$ denotes the $s$-th order derivative of $\mu$. Meanwhile, $p_\mu$ is lower bounded on $\mathcal{X}$, i.e., $p_\mu(x) \geq \tau$ whenever $x \in \mathcal{X}$ for some constant $\tau > 0$.

**Assumption 3.** The easy-to-sample distribution $\rho$ has a $C^\infty$ (infinitely smooth) density $p_\rho$.

H"older densities have been widely studied in density estimation (Wasserman, 2006; Tsybakov, 2008). $p_\mu$ being lower bounded is a technical assumption common in literature (Moser, 1965; Caffarelli, 1996). Assumption 3 is always satisfied, since $\rho$ is often taken as the uniform distribution.

We consider the following two sampling scenarios:

**Scenario 1.** The support $\mathcal{X}$ is convex.

**Scenario 2.** The support $Z = \mathcal{X}$ is open and its boundary satisfies some smoothness condition.

The condition in either scenario guarantees the existence of a H"older transformation $T$ such that $T_\# \rho = \mu$ (see Section 3). In **Scenario 1**, one can simply take $\rho$ as the uniform distribution on $[0, 1]^d$ such that $Z = [0, 1]^d$. In **Scenario 2**, $\mathcal{X}$ needs to be known as a priori information, since we need to take samples on $Z = \mathcal{X}$.
Given Assumptions 1–3, for both Scenarios 1 and 2, we represent the generator network architecture as

$$
\mathcal{G}_{\text{NN}}(R, \kappa, L, p, K) = \{g = [g_1, \ldots, g_d]^\top : \mathbb{R}^d \mapsto \mathbb{R}^d \mid g \text{ in form (2)} \text{ with } L \text{ layers and width bounded by } p, \\
\|g_i\|_\infty \leq R, \|W_i\|_{0, \infty, \infty} \leq \kappa, \|b_i\|_\infty \leq \kappa, \sum_{i=1}^L \|W_i\|_0 + \|b_i\|_0 \leq K, \text{ for } i = 1, \ldots, L, \}
$$

and the discriminator network architecture as

$$
\mathcal{F}_{\text{NN}}(\bar{R}, \bar{\kappa}, \bar{L}, \bar{p}, \bar{K}) = \{f : \mathbb{R}^d \mapsto \mathbb{R} \mid f \text{ in form (2)} \text{ with } \bar{L} \text{ layers and width bounded by } \bar{p}, \\
\|f\|_{\infty} \leq \bar{R}, \|W_i\|_{0, \infty, \infty} \leq \bar{\kappa}, \|b_i\|_\infty \leq \bar{\kappa}, \sum_{i=1}^{\bar{L}} \|W_i\|_0 + \|b_i\|_0 \leq \bar{K}, \text{ for } i = 1, \ldots, \bar{L}, \}
$$

where $$\|\cdot\|_0$$ denotes the number of nonzero entries in a vector or a matrix, and $$\|A\|_{0, \infty, \infty} = \max_{i,j} |A_{ij}|$$ for a matrix $$A$$.

We show that under either Scenarios 1 or 2, the generator can universally approximate the data distributions.

**Theorem 1.** (a) For any data distribution $$(X, \mu)$$ and easy-to-sample distribution $$(Z, \rho)$$ satisfying Assumptions 1–3, under either Scenario 1 or 2, there exists a transformation $$T \in \mathcal{H}_{d+1}$$ such that $$T_{\#} \rho = \mu$$.

(b) Let $$X$$ and $$Z$$ be fixed under either Scenario 1 or 2. Given any $$\epsilon \in (0, 1)$$, there exists a generator network with parameters

$$
R = B, \ \kappa = \max\{1, B\}, \ \bar{L} = O\left(\log \frac{1}{\epsilon}\right), \ p = O\left(d e^{-\frac{d}{p\epsilon}}\right), \ \text{and} \ K = O\left(d e^{-\frac{d}{p\epsilon}} \log \frac{1}{\epsilon}\right), \ (4)
$$

such that for any data distribution $$(X, \mu)$$ and easy-to-sample distribution $$(Z, \rho)$$ satisfying Assumptions 1–3, if the weight parameters of this network are properly chosen, then it yields a transformation $$g_\theta$$ satisfying

$$
\max_{z \in Z} \|g_\theta(z) - T(z)\|_\infty \leq \epsilon.
$$

We next state our statistical estimation error in terms of the Hölder IPM between $$(g_\theta^*)_{\#} \rho$$ and $$\mu$$, where $$g_\theta^*$$ is the optimal solution of GANs in (3).

**Theorem 2.** Suppose Assumptions 1–3 hold. For any $$\beta \geq 1$$, consider either Scenario 1 or 2, and choose $$\epsilon = n^{-\frac{\beta}{2\beta + d}}$$ in Theorem 1 (b) for the generator architecture and

$$
\bar{R} = Bd, \ \bar{\kappa} = \max\{1, Bd\}, \ \bar{L} = O\left(\frac{\beta}{2\beta + d} \log n\right), \ \bar{p} = O\left(n^{\frac{d}{2\beta + d}}\right), \ \text{and} \ \bar{K} = O\left(\frac{\beta}{2\beta + d} n^{\frac{d}{2\beta + d}} \log n\right)
$$

for the discriminator architecture. Then we have

$$
\mathbb{E}\left[d_{\mathcal{H}_{\#}}((g_\theta^*)_{\#} \rho, \mu)\right] = \tilde{O}\left(n^{-\frac{\beta}{2\beta + d}} \log^2 n\right).
$$

Theorem 2 demonstrates that GANs can effectively learn data distributions, with a convergence rate depending on the smoothness of the function class in IPM and the dimension $$d$$. Here are some remarks:
1. Both networks have uniformly bounded outputs. Such a requirement can be achieved by adding an additional clipping layer to the end the network, in order to truncate the output in the range \([-R, R]\). We utilize

\[
g(a) = \max\{-R, \min\{a, R\}\} = \text{ReLU}(a - R) - \text{ReLU}(a + R) - R.
\]

2. This is the first statistical guarantee for GANs estimating data distributions with Hölder densities. Existing works require restrictive assumptions on the data distributions (e.g., the density can be implemented by an invertible neural network).

In the case that only \(m\) samples from the easy-to-sample distribution \(\rho\) can be obtained, GANs solve the following alternative minimax problem

\[
\min_{g_0 \in \mathcal{G}_{\text{NN}}} \max_{f_0 \in \mathcal{F}_{\text{NN}}} \frac{1}{m} \sum_{i=1}^{m} f_0(g_0(z_i)) - \frac{1}{n} \sum_{j=1}^{n} f_0(x_j),
\]

(5)

We slightly abuse the notation to denote \((g^*_0, f^*_0)\) as he optimal solution to (5). We show in the following corollary that GANs retain similar statistical guarantees for distribution estimation with finite generated samples.

**Corollary 1.** Suppose Assumptions 1 – 3 hold. We choose

\[
L = O\left(\frac{\alpha + 1}{2(\alpha + 1) + d} \log m\right), \quad p = O\left(\frac{d(\alpha + 1)}{2(\alpha + 1) + d} m^{\frac{d}{2(\alpha + 1)} - 1} \log m\right), \quad K = O\left(\frac{d(\alpha + 1)}{2(\alpha + 1) + d} m^{\frac{d}{2(\alpha + 1)} - 1} \log m\right),
\]

\[
R = B, \quad \kappa = \max\{1, B\},
\]

for the generator network, and the same architecture as in Theorem 2 for the discriminator network. Then we have

\[
\mathbb{E}\left[d_{H}^{\beta}\left((g_0^*)^\# \rho, \mu\right)\right] \leq \tilde{O}\left(n^{-\frac{\beta}{2d\alpha + d}} + m^{-\frac{\alpha + 1}{2(\alpha + 1)d}}\right).
\]

Here \(\tilde{O}\) hides logarithmic factors in \(n, m\). As it is often cheap to obtain a large amount of samples from \(\rho\), the statistical convergence rate is dominated by \(n^{-\frac{\beta}{2d\alpha + d}}\) whenever \(m \geq n^{\frac{d(\alpha + 1)}{2(\alpha + 1)d}} \cdot \frac{\log n}{\log \log n}\).

### 3 Proof of Distribution Estimation Theory

#### 3.1 Proof of Theorem 1

We begin with distribution transformations and function approximation theories using ReLU networks.

**Transformations between Distributions.** Let \(\mathcal{X}, \mathcal{Z}\) be subsets of \(\mathbb{R}^d\). Given two probability spaces \((\mathcal{X}, \mu)\) and \((\mathcal{Z}, \rho)\), we aim to find a transformation \(T : \mathcal{Z} \mapsto \mathcal{X}\), such that \(T(z) \sim \mu\) for \(z \sim \rho\). In general, \(T\) may not exist nor be unique. Here we assume \(\mu\) and \(\rho\) have Hölder densities \(p_\mu\) and \(p_\rho\), respectively. The Monge map and Moser’s coupling ensure the existence of a Hölder transformation \(T\).
• **Monge Map.** Monge map finds an optimal transformation between two distributions that minimizes certain cost function. It has wide applications in economics (Santambrogio, 2010; Galichon, 2017) and machine learning (Ganin and Lempitsky, 2014; Courty et al., 2016). We assume that the support $\mathcal{X}$ is convex and the densities $p_\mu$ and $p_\rho$ belong to the Hölder space with index $\alpha$, i.e., $p_\mu, p_\rho \in \mathcal{H}^\alpha$ and are bounded below by some positive constant.

The Monge map $T$ is the solution to the following optimization problem

$$T \in \arg\min_T \mathbb{E}_{z \sim \rho} [c(z, T(z))],$$

subject to $T \# \rho = \mu$.

where $c$ is a cost function. (6) is known as the Monge problem. When $\mathcal{X}$ is convex and the cost function is quadratic, the solution to (6) satisfies the Monge-Ampère equation (Monge, 1784). Caffarelli (1992b,a, 1996) and Urbas (1988, 1997) proved the regularity of $T$ independently, using different sophisticated tools. Their main result is summarized in the following lemma.

**Lemma 1 (Caffarelli (1992b)).** In Scenario 1, suppose Assumptions 1 – 3 hold. Then there exists a transformation $T : \mathcal{Z} \mapsto \mathcal{X}$ such that $T \# \rho = \mu$. Moreover, this transformation $T$ belongs to the Hölder class $\mathcal{H}^{\alpha+1}$.

• **Moser’s Coupling.** Moser’s coupling extends to nonconvex supports, which was first proposed in Moser (1965) to transform densities supported on the same compact and smooth manifold without boundary. Later, Greene and Shiohama (1979) established results for noncompact manifolds. Moser himself also extended the results to the case where the supports are open sets with boundary (Dacorogna and Moser, 1990). We summarize the main result.

**Lemma 2 (Theorem 1 in Dacorogna and Moser (1990)).** In Scenario 2, suppose Assumptions 1 – 3 hold. Assume $\partial \mathcal{X}$ (the boundary of $\mathcal{X}$) is $\mathcal{H}^{\alpha+3}$. Then there exists a transformation $T : \mathcal{Z} \mapsto \mathcal{X}$ such that $T \# \rho = \mu$. Moreover, this transformation $T$ belongs to the Hölder class $\mathcal{H}^{\alpha+1}$.

Such a transformation $T$ can be explicitly constructed. Specifically, let $u(x)$ solve the Poisson equation $\Delta u(x) = p_\mu(x) - p_\rho(x)$, where $\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian. We construct a vector field $\xi(t, x) = \frac{\nabla u(x)}{(1 - p_\mu(x) + p_\rho(x))}$ and define $T(x) = \int_0^1 \xi(t, x) d\tau$. Note that $\xi(t, x)$ is well defined, in that $p_\rho$ and $p_\mu$ are bounded below. Using the conservation of mass formula, one checks that $T \# \rho = \mu$ (Chapter 1, Villani (2008)), and $T \in \mathcal{H}^{\alpha+1}$.

**Function Approximation by ReLU Networks.** The representation abilities of neural networks are studied from the perspective of universal approximation theories (Cybenko, 1989; Hornik, 1991; Chui and Li, 1992; Barron, 1993; Mhaskar, 1996). Recently, Yarotsky (2017) established a universal approximation theory for ReLU networks, where the network attains the optimal size and is capable of approximating any Hölder/Sobolev functions. The main result is summarized in the following lemma.

**Lemma 3.** Given any $\delta \in (0, 1)$, there exists a ReLU network architecture such that, for any $f \in \mathcal{H}^\beta([0, 1]^d)$ for $\beta \geq 1$, if the weight parameters are properly chosen, the network yields a function $\hat{f}$ for the approximation of $f$ with $\|\hat{f} - f\|_\infty \leq \delta$. Such a network has
1) no more than \( c (\log \frac{1}{\delta} + 1) \) layers, and
2) at most \( c'\delta^{-\frac{1}{\beta}} (\log \frac{1}{\delta} + 1) \) neurons and weight parameters, where the constants \( c \) and \( c' \) depend on \( d, \beta, \) and \( f \).

Lemma 3 is a direct result of Theorem 1 in Yarotsky (2017), which is originally proved for Sobolev functions. Proof for Hölder functions can be found in Chen et al. (2019a). The high level idea consists of two steps: 1) Approximate the target function \( f \) using a weighted sum of local Taylor polynomials; 2) Implement each Taylor polynomial using a ReLU network. The second step can be realized, since polynomials can be implemented only using the multiplication and addition operations. It is shown that ReLU network can efficiently approximate the multiplication operation (Proposition 3 in Yarotsky (2017)). We also remark that all the weight parameters in the network constructed in Lemma 3 are bounded by 1.

Theorem 1 is obtained by combining Lemmas 2 – 3. In Scenario 1, we can take \( Z = [0,1]^d \) for simplicity, and then apply Lemma 3. More generally, if \( Z = [-B, B]^d \), we define a scaling function \( \phi(z) = (z + B1)/(2B) \in [0,1]^d \) for any \( z \in Z \), where 1 denotes a vector of 1’s. For any data transformation \( T \), we rewrite it as \( T \circ \phi^{-1}(\phi(\cdot)) \) so that it suffices to approximate \( T \circ \phi^{-1} \) supported on \([0,1]^d \). When \( Z \) is a subset of \([0,1]^d \) with a positive measure, especially in Scenario 2, we can apply the same proof technique of Lemma 3 to extend the approximation theory in Lemma 3 to \( \mathcal{H}_\beta(Z) \) (Hölder functions defined on \( Z \)).

Lemma 2 yields a data transformation \( T \in \mathcal{H}^{d+1} \) with \( T \mu = \mu \). We invoke Lemma 3 to construct the generator network architecture. Denote \( T = [T_1, \ldots, T_d]^T \) where \( T_i : \mathcal{X} \rightarrow \mathbb{R}, i = 1, \ldots, d \). We then approximate each coordinate mapping \( T_i \): For a given error \( \delta \in (0,1) \), \( T_i \) can be approximated by a ReLU network with \( O(\log \frac{1}{\delta}) \) layers and \( O(\delta^{-\frac{1}{2\beta}} \log \frac{1}{\delta}) \) neurons and weight parameters. Finally \( T \) can be approximated by \( d \) such ReLU networks.

### 3.2 Proof of Theorem 2

We first show a new oracle inequality, which decomposes the distribution estimation error as the generator approximation error \( \mathcal{E}_1 \), discriminator approximation error \( \mathcal{E}_2 \), and statistical error \( \mathcal{E}_3 \).

**Lemma 4.** Let \( \mathcal{H}_\beta(\mathcal{X}) \) be the Hölder function class defined on \( \mathcal{X} \) with Hölder parameter \( \beta \geq 1 \). Define \( \mathcal{H}_\beta^{\infty} = \{ f \in \mathcal{H}_\beta : |f(x) - f(y)| \leq \|x - y\|_\infty \} \). Then

\[
d_{\mathcal{H}_\beta}(g_0, \mu, \mu) \leq \inf_{g_0 \in \mathcal{G}_{NN}} d_{\mathcal{H}_\beta}(g_0, \mu, \mu) + 4 \sup_{f \in \mathcal{H}_\beta} \inf_{f_\omega \in \mathcal{F}_{NN}} \|f - f_\omega\|_\infty + d_{\mathcal{H}_\beta}(\mu, \mu_\infty) + d_{\mathcal{F}_\infty}(\mu_\infty, \mu_\infty).
\]

**Proof Sketch of Lemma 4.** The proof utilizes the triangle inequality. The first step introduces the empirical data distribution as an intermediate term:

\[
d_{\mathcal{H}_\beta}(g_0, \mu, \mu) \leq d_{\mathcal{H}_\beta}(g_0, \mu_\infty, \mu) + d_{\mathcal{H}_\beta}(\mu_\infty, \mu)
\]
We replace the first term on the right-hand side by the training loss of GANs:
\[
d_{H^\beta}((g^*_\theta)\sharp\rho,\mu_n) = d_{H^\beta}((g^*_\theta)\sharp\rho,\mu_n) - d_{\mathcal{F}_{NN}}((g^*_\theta)\sharp\rho,\mu_n) + d_{\mathcal{F}_{NN}}((\hat{g}_\theta)\sharp\rho,\mu_n)
\leq 2 \inf_{f_\omega \in \mathcal{F}_{NN}} \sup_{f \in H^\beta} \|f - f_\omega\|_\infty + d_{\mathcal{F}_{NN}}((\hat{g}_\theta)\sharp\rho,\mu_n).
\]

Note that \(\inf_{f_\omega \in \mathcal{F}_{NN}} \sup_{f \in H^\beta} \|f - f_\omega\|_\infty\) reflects the approximation error of the discriminator.

To finish the proof, we apply the triangle inequality on \(d_{\mathcal{F}_{NN}}((\hat{g}_\theta)\sharp\rho,\mu_n)\):
\[
d_{\mathcal{F}_{NN}}((\hat{g}_\theta)\sharp\rho,\mu_n) = \inf_{g_\theta \in \mathcal{G}_{NN}} d_{\mathcal{F}_{NN}}((g_\theta)\sharp\rho,\mu_n)
\leq \inf_{g_\theta \in \mathcal{G}_{NN}} d_{\mathcal{F}_{NN}}((g_\theta)\sharp\rho,\mu) + d_{\mathcal{F}_{NN}}((\hat{g}_\theta)\sharp\rho,\mu).
\]

The last step is to break the coupling between the discriminator and generator class by invoking the auxiliary function class \(\mathcal{H}_{\infty}^\beta\).
\[
d_{\mathcal{F}_{NN}}((g_\theta)\sharp\rho,\mu) = d_{\mathcal{F}_{NN}}((g_\theta)\sharp\rho,\mu) - d_{\mathcal{H}_{\infty}^\beta}((g_\theta)\sharp\rho,\mu) + d_{\mathcal{H}_{\infty}^\beta}((\hat{g}_\theta)\sharp\rho,\mu)
\leq 2 \inf_{f_\omega \in \mathcal{F}_{NN}} \sup_{f \in \mathcal{H}_{\infty}^\beta} \|f - f_\omega\|_\infty + d_{\mathcal{H}_{\infty}^\beta}((\hat{g}_\theta)\sharp\rho,\mu)
\leq 2 \inf_{f_\omega \in \mathcal{F}_{NN}} \sup_{f \in \mathcal{H}_{\infty}^\beta} \|f - f_\omega\|_\infty + d_{\mathcal{H}_{\infty}^\beta}((\hat{g}_\theta)\sharp\rho,\mu),
\]

where the last inequality follows from \(\mathcal{H}_{\infty}^\beta \subset \mathcal{H}_{\infty}^\beta\). The oracle inequality is obtained by combining all the previous ingredients. See details in Appendix A.

We next bound each error term separately. \(\mathcal{E}_1\) and \(\mathcal{E}_2\) can be controlled by proper choices of the generator and discriminant architectures. \(\mathcal{E}_3\) can be controlled using empirical process theories (Van Der Vaart and Wellner, 1996; Győrfi et al., 2006).

**Bounding Generator Approximation Error \(\mathcal{E}_1\).** We answer this question: Given \(\epsilon_1 \in (0,1)\), how can we properly choose \(\mathcal{G}_{NN}\) to guarantee \(\mathcal{E}_1 \leq \epsilon_1\)? Later, we will pick \(\epsilon_1\) based on the sample size \(n\), and Hölder indexes \(\beta\) and \(\alpha\).

**Lemma 5.** Let \(\mathcal{X}\) and \(\mathcal{Z}\) be fixed under either Scenario 1 or 2. Given any \(\epsilon_1 \in (0,1)\), there exists a ReLU network architecture \(\mathcal{G}_{NN}(R,\kappa,L,p,K)\) with parameters given by (4) with \(\epsilon = \epsilon_1\) such that, for any data distribution \((\mathcal{X},\mu)\) and easy-to-sample distribution \((\mathcal{Z},\rho)\) satisfying Assumptions 1 – 3, if the weight parameters of this network are properly chosen, then it yields a transformation \(g_\theta\) satisfying \(d_{\mathcal{H}_{\infty}^\beta}((g_\theta)\sharp\rho,\mu) \leq \epsilon_1\).

**Proof Sketch of Lemma 5.** For any given \(\delta \in (0,1)\), Theorem 1 implies that the chosen network architecture can yield a data transformation \(g_\theta\) satisfying \(\max_{z \in \mathcal{Z}} \|g_\theta(z) - T(z)\|_\infty \leq \delta\). Here \(T\) is the data transformation given by the Monge map or the Moser’s coupling so that it satisfies \(T\sharp\rho = \mu\).

The remaining step is to choose \(\delta\) so that \(d_{\mathcal{H}_{\infty}^\beta}((g_\theta)\sharp\rho,\mu) \leq \epsilon_1\). Using the definition of \(\mathcal{H}_{\infty}^\beta\), we

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derive
\[
    d_{\mathcal{H}^\delta}((g_0)_\#\rho, \mu) = d_{\mathcal{H}^\delta}((\tilde{g}_0)_\#\rho, \tilde{T}_\#\rho)
\]
\[
= \sup_{f \in \mathcal{H}^\delta} \mathbb{E}_{z \sim \rho}[f(g_0(z))] - \mathbb{E}_{z \sim \rho}[f(T(z))]
\]
\[
\leq \mathbb{E}_{z \sim \rho}[\|g_0(z) - T(z)\|_\infty]
\]
\[
\leq \delta.
\]

The proof is complete by choosing \( \delta = \epsilon_1 \). The details are provided in Appendix B.

* Bounding Discriminator Approximation Error \( \mathcal{E}_2 \). Analogous to the generator, we pre-
define an error \( \epsilon_2 \in (0, 1) \), and determine the discriminator architecture.

The discriminator is expected to approximate any function \( f \in \mathcal{H}^{\delta}(\mathcal{X}) \). It suffices to consider \( \mathcal{H}^{\delta}(\mathcal{X}) \) with a bounded diameter. The reason is that IPM is invariant under linear translations, i.e.,
\[
d_\mathcal{F} = d_{\mathcal{F} + c} \quad \text{for any constant } c,
\]
where \( \mathcal{F} + c = \{f + c : f \in \mathcal{F}\} \). Therefore, we may assume there exists \( x_0 \in \mathcal{X} \) such that \( f(x_0) = 0 \) for all \( f \in \mathcal{H}^{\delta}(\mathcal{X}) \). By the Hölder continuity and the compactness of the support \( \mathcal{X} \), we have for any \( f \in \mathcal{H}^{\delta} \), \( \|f\|_\infty \leq \max_x \|\nabla f(x)\|_2 \sqrt{d B} \leq B d \).

**Lemma 6.** Given any \( \epsilon_2 \in (0, 1) \), there exists a ReLU network architecture \( \mathcal{F}_{\text{NN}}(\bar{R}, \bar{\kappa}, \bar{L}, \bar{\rho}, \bar{K}) \) with
\[
\bar{L} = O\left(\log \frac{1}{\epsilon_2^2}\right), \quad \bar{\rho} = O\left(\epsilon_2^{-\frac{3}{2}}\right), \quad \bar{K} = O\left(\epsilon_2^{-\frac{3}{2}} \log \frac{1}{\epsilon_2}\right), \quad \bar{R} = dB, \quad \text{and } \bar{\kappa} = \max(1, dB),
\]
such that, for \( f \in \mathcal{H}^{\delta}(\mathcal{X}) \), if the weight parameters are properly chosen, this network architecture yields a function \( f_\omega \) satisfying \( \|f_\omega - f\|_\infty \leq \epsilon_2 \).

*Proof Sketch of Lemma 6.* Lemma 3 immediately yields a network architecture for uniformly approximating functions in \( \mathcal{H}^{\delta}(\mathcal{X}) \). Let the approximation error be \( \epsilon_2 \). Then the network architecture consists of \( O\left(\log \frac{1}{\epsilon_2}\right) \) layers and \( O\left(\epsilon_2^{-\frac{3}{2}} \log \frac{1}{\epsilon_2}\right) \) total number of neurons and weight parameters.

To this end, we can establish that for any \( f \in \mathcal{H}^{\delta}(\mathcal{X}) \), identity \( \inf_{f_\omega \in \mathcal{F}_{\text{NN}}} \|f - f_\omega\|_\infty \leq \epsilon_2 \) holds.

* Bounding Statistical Error \( \mathcal{E}_3 \). The statistical error term is essentially the concentration of empirical data distribution \( \hat{\mu}_n \) to its population counterpart. Given a symmetric function class \( \mathcal{F} \), we show \( \mathbb{E}[d_\mathcal{F}(\hat{\mu}_n, \mu)] \) scales with the complexity of the function class \( \mathcal{F} \).

**Lemma 7.** For a symmetric function class \( \mathcal{F} \) with \( \sup_{f \in \mathcal{F}} \|f\|_\infty \leq L \) for some constant \( L \), we have
\[
\mathbb{E}[d_\mathcal{F}(\hat{\mu}_n, \mu)] \leq 2 \inf_{0 < \delta < L} \left(2\delta + \frac{12}{\sqrt{n}} \int_{\delta}^{L} \sqrt{\log \mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_\infty)} d\epsilon\right),
\]
where \( \mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_\infty) \) denotes the \( \epsilon \)-covering number of \( \mathcal{F} \) with respect to the \( \ell_\infty \) norm.

*Proof Sketch of Lemma 7.* The proof utilizes the symmetrization technique and Dudley’s entropy integral, with details provided in Appendix C. In short, the first step relates \( \mathbb{E}[d_\mathcal{F}(\hat{\mu}_n, \mu)] \) to the
Rademacher complexity of $\mathcal{F}$:

$$
\mathbb{E}[d_\mathcal{F}(\tilde{\mu}, \mu)] = \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \mathbb{E}_y[f(y)] \right] \\
\leq \mathbb{E}_x \mathbb{E}_y \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} (f(x_i) - f(y_i)) \right] \\
\overset{(i)}{=} \mathbb{E}_x \mathbb{E}_y \mathbb{E}_\xi \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f(x_i) - f(y_i)) \right] \\
= 2 \mathbb{E}_x \mathbb{E}_\xi \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f(x_i) \right],
$$

where $y_i$’s are independent copies of $x_i$’s. Equality (i) holds due to symmetrization. The proof then proceeds with Dudley’s chaining argument (Dudley, 1967).

Now we need to find the covering number of Hölder function class and that of the discriminator networks. Classical results show that the $\delta$-covering number of $\mathcal{H}_\beta$ is bounded by $\log N(\delta, \mathcal{H}_\beta, \| \cdot \|_\infty) \leq c \left( \frac{1}{\delta} \right)^{d_{\beta}^2}$ with $c$ being a constant depending on the diameter of $\mathcal{H}_\beta$ (Nickl and Pötscher, 2007).

On the other hand, the following lemma quantifies the covering number of $\mathcal{F}_{NN}$.

**Lemma 8.** The $\delta$-covering number of $\mathcal{F}_{NN}(\bar{R}, \bar{\kappa}, \bar{L}, \bar{p}, \bar{K})$ satisfies the upper bound

$$
N(\delta, \mathcal{F}_{NN}(\bar{R}, \bar{\kappa}, \bar{L}, \bar{p}, \bar{K}), \| \cdot \|_\infty) \leq \left( \frac{\bar{L}(\bar{p}B + 2)(\bar{\kappa}\bar{p})^{L-1}}{\delta} \right)^{\bar{K}}.
$$

**Proof Sketch of Lemma 8.** The detailed proof is in Appendix D. Since each weight parameter in the network is bounded by a constant $\bar{\kappa}$, we construct a covering by partition the range of each weight parameter into a uniform grid. Consider $f, f' \in \mathcal{F}_{NN}(\bar{R}, \bar{\kappa}, \bar{L}, \bar{p}, \bar{K})$ with each weight parameter differing at most $h$. By an induction on the number of layers in the network, we show that the $\ell_\infty$ norm of the difference $f - f'$ scales as

$$
\|f - f'\|_\infty \leq h\bar{L}(\bar{p}B + 2)(\bar{\kappa}\bar{p})^{L-1}.
$$

As a result, to achieve a $\delta$-covering, it suffices to choose $h$ such that $h\bar{L}(\bar{p}B + 2)(\bar{\kappa}\bar{p})^{L-1} = \delta$. Therefore, the covering number is bounded by

$$
N(\delta, \mathcal{F}_{NN}(\bar{R}, \bar{\kappa}, \bar{L}, \bar{p}, \bar{K}), \| \cdot \|_\infty) \leq \left( \frac{\bar{K}}{h} \right)^{\bar{K}} \leq \left( \frac{\bar{L}(\bar{p}B + 2)(\bar{\kappa}\bar{p})^{L}}{\delta} \right)^{\bar{K}}.
$$

The proof is complete. \qed
Combining Lemma 7 and the covering numbers, the statistical error can be bounded by

$$
\mathbb{E}\left[d_{H^\theta}(\widehat{\mu}, \mu) + d_{\mathcal{F}_{NN}}(\mu, \widehat{\mu})\right] \leq 4 \inf_{\delta_1 \in (0, Bd)} \left( \delta_1 + \frac{6}{\sqrt{n}} \int_{\delta_1}^{Bd} \sqrt{\log \mathcal{N}(\epsilon, H^\theta, \|\cdot\|_\infty)} d\epsilon \right) \\
+ 4 \inf_{\delta_2 \in (0, Bd)} \left( \delta_2 + \frac{6}{\sqrt{n}} \int_{\delta_2}^{Bd} \sqrt{\log \mathcal{N}(\epsilon, \mathcal{F}_{NN}, \|\cdot\|_\infty)} d\epsilon \right) \\
\leq (i) \inf_{\delta_1 \in (0, Bd)} \left( \delta_1 + \frac{6}{\sqrt{n}} \int_{\delta_1}^{Bd} e^{\left(1 - \epsilon^2\right)\left(\frac{2}{n} + 1\right)} d\epsilon \right) \\
+ 4 \inf_{\delta_2 \in (0, Bd)} \left( \delta_2 + \frac{6}{\sqrt{n}} \int_{\delta_2}^{Bd} K \log \frac{L(\bar{p} B + 2)(\bar{p} B)^L}{\epsilon} d\epsilon \right).
$$

We find that the first infimum in step (i) is attained at $\delta_1 = n^{-\frac{d}{2}}$. It suffices to take $\delta_2 = \frac{1}{n}$ in the second infimum. By omitting constants and polynomials in $\beta$ and $d$, we derive

$$
\mathbb{E}\left[d_{H^\theta}(\widehat{\mu}, \mu) + d_{\mathcal{F}_{NN}}(\mu, \widehat{\mu})\right] \leq \tilde{O}\left( \frac{1}{n} + n^{-\frac{d}{2}} + \frac{1}{\sqrt{n}} \sqrt{KL \log (nL\bar{p})} \right).
$$

• **Balancing the Approximation Error and Statistical Error.** Combining the previous three ingredients, we can establish, by the oracle inequality (Lemma 4),

$$
\mathbb{E}\left[d_{H^\theta}(g^*_\theta \| \rho, \mu)\right] \leq \tilde{O}\left( \epsilon_1 + \epsilon_2 + \frac{1}{n} + n^{-\frac{d}{2}} + \sqrt{\frac{KL \log (nL\bar{p})}{n}} \right) \\
\leq \tilde{O}\left( \epsilon_1 + \epsilon_2 + \frac{1}{n} + n^{-\frac{d}{2}} + \sqrt{\frac{\epsilon_2^{-d} \log \frac{1}{\epsilon_2} \log \frac{1}{n \epsilon_2^{-d}}} {n}} \right).
$$

We choose $\epsilon_1 = n^{-\frac{d}{2}}$, and $\epsilon_2$ satisfying $\epsilon_2 = n^{-\frac{1}{2}} \epsilon_2^{-\frac{d}{2}}$, i.e., $\epsilon_2 = n^{-\frac{d}{2}}$. This gives rise to

$$
\mathbb{E}\left[d_{H^\theta}(g^*_\theta \| \rho, \mu)\right] \leq \tilde{O}\left( n^{-\frac{d}{2}} \log^2 n \right).
$$

4 **Comparison with Related Works**

The statistical properties of GANs have been studied in several recent works (Arora et al., 2017; Bai et al., 2018; Liang, 2018; Jiang et al., 2018; Thanh-Tung et al., 2019). Among these works, Arora et al. (2017) studied the generalization error of GANs. Lemma 1 of Arora et al. (2017) shows that GANs can not generalize under the Wasserstein distance and the Jensen-Shannon divergence unless the sample size is $\tilde{O}(e^{-\text{Polynomial}(d)})$, where $c$ is the generalization gap. Alternatively, they defined a surrogate metric “neural net distance” $d_{\mathcal{F}_{NN}}(\cdot, \cdot)$, where $\mathcal{F}_{NN}$ is the class of discriminator networks. They showed that GANs generalize under the neural net distance, with sample complexity of $\tilde{O}(e^{-2})$. These results have two limitations: 1). The sample complexity depends on some unknown parameters of the discriminator network class (e.g., the Lipschitz constant of discriminators with respect to parameters); 2). A small neural net distance does not necessarily implies
that two distributions are close (see Corollary 3.2 in Arora et al. (2017)). In contrast, our results are explicit in the network architectures, and show the statistical convergence of GANs under the Wasserstein distance ($\beta = 1$).

Some followup works attempted to address the first limitation in Arora et al. (2017). Specifically, Thanh-Tung et al. (2019) explicitly quantified the Lipschitz constant and the covering number of the discriminator network. They improved the generalization bound in Arora et al. (2017) based on the framework of Bartlett et al. (2017). Whereas the bound has an exponential dependence on the depth of the discriminator. Jiang et al. (2018) further showed a tighter generalization bound under spectral normalization of the discriminator. The bound has a polynomial dependence on the size of the discriminator. These generalization theories are derived under the assumption that the generator can well approximate the data distribution with respect to the neural net distance. Nevertheless, how to choose such a generator remains unknown.

Other works (Bai et al., 2018; Liang, 2018) studied the estimation error of GANs under the Wasserstein distance for a special class of distributions implemented by a generator, while the discriminator is designed to guarantee zero bias (or approximation error). Bai et al. (2018) showed that for certain generator classes, there exist corresponding discriminator classes with a strong distinguishing power against the generator. Particular examples include two-layer ReLU network discriminators (half spaces) for distinguishing Gaussian distributions/mixture of Gaussians, and $(L + 2)$-layer discriminators for $(L + 1)$-layer invertible generators. In these examples, if the data distribution can be exactly implemented by some generator, then the neural net distance can provably approximate the Wasserstein distance. Consequently, GANs can generalize under the Wasserstein distance. This result is specific to certain data distributions, and the generator network needs to satisfy restrictive assumptions, e.g., all the weight matrices and the activation function must be invertible.

Another work in this direction is Liang (2018), where the estimation error of GANs was studied under the Sobolev IPMs. Liang (2018) considered both nonparametric and parametric settings. In the nonparametric setting, no generator and discriminator network architectures are chosen, so that the bias of the distribution estimation remains unknown. As a result, the bound cannot provide an explicit sample complexity for distribution estimation. The parametric setting in (Liang, 2018) is similar to the one in Bai et al. (2018), where all weight matrices in the discriminator are full rank, and the activation function is the invertible leaky ReLU function. This ensures that the generator network is invertible, and the log density of the generated distribution can be calculated. The discriminator is then chosen as an $(L + 2)$-layer feedforward network using the dual leaky ReLU activation. The main result in Corollary 1 shows that the squared Wasserstein distance between the GAN estimator and the data distribution converges at a rate of $O(\sqrt{pL/n})$, where $p$ is the width of the generator (discriminator) network. This result requires strong assumptions on the data distribution and the generator, i.e., the generator needs to be invertible and the data distribution needs to be exactly implementable by the generator.

Apart from the aforementioned results, Liang (2017); Singh et al. (2018) studied nonparametric density estimation under Sobolev IPMs. Later Uppal et al. (2019) generalized the result to Besov IPMs. The main results are similar to Liang (2018) in the nonparametric setting. The
bias of the distribution estimation was assumed to be small, the generator and discriminator network architectures are provided to guarantee this. Our main result is also in the nonparametric setting, but the generator and discriminator network architectures are explicitly chosen to learn distributions with Hölder densities.

5 Discussions

Curse of Dimensionality and Low-Dimensional Geometric Structures in Data. To estimate distributions with $\mathcal{H}^\alpha$ densities $\mu$, the minimax optimal rate (Liang, 2018) under the $\mathcal{H}^\beta$ IPM loss reads

$$\inf_{\tilde{\mu}_n} \sup_{\mu \in \mathcal{H}^\alpha} \mathbb{E}[d_{\mathcal{H}^\beta}(\tilde{\mu}_n, \mu)] \gtrsim n^{-\frac{\alpha \beta}{2\alpha + \beta} + \frac{1}{2}},$$

where $\tilde{\mu}_n$ is any estimator of $\mu$ based on $n$ data points. The minimax rate suggests that the curse of data dimensionality is unavoidable regardless of the approach.

The empirical performance of GANs, however, can mysteriously circumvent such a curse of data dimensionality. This largely owes to the fact that practical data sets often exhibit low-dimensional geometric structures. Many images, for instance, consist of projections of a three-dimensional object followed by some transformations, such as rotation, translation, and skeleton. This generating mechanism induces a small number of intrinsic parameters (Hinton and Salakhutdinov, 2006; Osher et al., 2017; Chen et al., 2019b). Several existing works show that neural networks are adaptive to low-dimensional data structures in function approximation (Shaham et al., 2018; Chui and Mhaskar, 2016; Chen et al., 2019a) and regression (Chen et al., 2019b). It is worthwhile to investigate the performance of GANs for learning distributions supported on low-dimensional sets.

Convolutional Filters. Convolutional filters (Krizhevsky et al., 2012) are widely used in GANs for image generating and processing. Empirical results show that convolutional filters can learn hidden representations that align with various patterns in images (Zeiler and Fergus, 2014; Zhou et al., 2018), e.g., background, objects, and colors. An interesting question is whether convolutional filters can capture the aforementioned low-dimensional structures in data.

Smoothness of Data Distributions and Regularized Distribution Estimation. Theorem 2 indicates a convergence rate independent of the smoothness of the data distribution. The reason behind is that the empirical data distribution $\tilde{\mu}_n$ cannot inherit the same smoothness as the underlying data distribution. This limitation exists in all previous works (Liang, 2017; Singh et al., 2018; Uppal et al., 2019). It is interesting to investigate whether GANs can achieve a faster convergence rate (e.g., attain the minimax optimal rate).

From a theoretical perspective, Liang (2018) suggested to first obtain a smooth kernel estimator from $\tilde{\mu}_n$, and then replace $\tilde{\mu}_n$ by this kernel estimator to train GANs. In practice, kernel smoothing is hardly used in GANs. Instead, regularization (e.g., entropy regularization) and normalization (e.g., spectral normalization and batch-normalization) are widely applied as implicit approaches to encourage the smoothness of the learned distribution. Several empirical studies of GANs suggest that divergence-based and mutual information-based regularization can stabilize
the training and improve the performance (Che et al., 2016; Cao et al., 2018) of GANs. We leave it as future investigation to analyze the statistical properties of regularized GANs.

**Computational Concerns.** Our statistical guarantees hold for the global optimizer of (3), whereas solving (3) is often difficult. In practice, it is observed that larger neural networks are easier to train and yield better statistical performance (Zhang et al., 2016; Jacot et al., 2018; Du et al., 2018a; Allen-Zhu et al., 2018; Du et al., 2018b; Li and Liang, 2018; Arora et al., 2019; Allen-Zhu et al., 2019; Du and Hu, 2019). This is referred to as overparameterization. Establishing a connection between computation and statistical properties of GANs is an important direction.

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Supplementary Materials for Statistical Guarantees of GANs for Distribution Estimation

A Proof of Lemma 4

**Proof.** We use the triangle inequality to expand the left-hand side:

\[
\begin{align*}
\phi H^\theta ((g_\theta^*) \sharp \rho, \mu) & \leq \phi H^\theta ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) + \phi H^\theta (\widehat{\mu}_n, \mu) \\
& = \phi F_{\NN} ((g_\theta^*) \sharp \rho, \mu) + \phi H^\theta ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) - \phi F_{\NN} ((g_\theta^*) \sharp \rho, \mu) + \phi H^\theta (\widehat{\mu}_n, \mu) \\
& \leq \phi F_{\NN} ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) + 2 \sup_{f \in H^\theta} \inf_{f_\omega \in F_{\NN}} \|f - f_\omega\|_\infty + \phi H^\theta (\widehat{\mu}_n, \mu),
\end{align*}
\]

where step (i) follows from the triangle inequality, and step (ii) is obtained by rewriting \(\phi H^\theta ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) - \phi F_{\NN} ((g_\theta^*) \sharp \rho, \widehat{\mu}_n)\) as

\[
\begin{align*}
\phi H^\theta ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) - \phi F_{\NN} ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) &= \sup_{f \in H^\theta} \left[ \mathbb{E}_{x \sim \nu_n} f(x) - \mathbb{E}_{x \sim \mu} f(x) \right] - \sup_{f_\omega \in F_{\NN}} \left[ \mathbb{E}_{x \sim \nu_n} f(x) - \mathbb{E}_{x \sim \mu} f(x) \right] \\
& = \sup_{f \in H^\theta} \inf_{f_\omega \in F_{\NN}} \left[ \mathbb{E}_{x \sim \nu_n} f(x) - \mathbb{E}_{x \sim \mu} f(x) \right] \\
& = \sup_{f \in H^\theta} \inf_{f_\omega \in F_{\NN}} \mathbb{E}_{x \sim \nu_n} [f(x) - f_\omega(x)] - \mathbb{E}_{x \sim \mu} [f(x) - f_\omega(x)] \\
& \leq 2 \sup_{f \in H^\theta} \inf_{f_\omega \in F_{\NN}} \|f - f_\omega\|_\infty.
\end{align*}
\]

Now we bound \(\phi F_{\NN} ((g_\theta^*) \sharp \rho, \widehat{\mu}_n)\) using a similar triangle inequality trick:

\[
\phi F_{\NN} ((g_\theta^*) \sharp \rho, \widehat{\mu}_n) = \inf_{g_\theta \in \Theta_{\NN}} \phi F_{\NN} ((g_\theta) \sharp \rho, \widehat{\mu}_n) \\
\leq \inf_{g_\theta \in \Theta_{\NN}} \phi F_{\NN} ((g_\theta) \sharp \rho, \mu) + \phi F_{\NN} (\mu, \widehat{\mu}_n) \\
= \inf_{g_\theta \in \Theta_{\NN}} \phi F_{\NN} ((g_\theta) \sharp \rho, \mu) - \phi H^\mu ((g_\theta) \sharp \rho, \mu) + \phi H^\mu ((g_\theta) \sharp \rho, \mu) + \phi F_{\NN} (\mu, \widehat{\mu}_n) \\
\leq 2 \sup_{f \in H^\theta} \inf_{f_\omega \in F_{\NN}} \|f - f_\omega\|_\infty + \inf_{g_\theta \in \Theta_{\NN}} \phi H^\mu ((g_\theta) \sharp \rho, \mu) + \phi F_{\NN} (\mu, \widehat{\mu}_n),
\]

where the last inequality holds by the identity \(\mu_{\infty} \subset H^\beta\). Substituting the above ingredients into (7), we have

\[
\phi H^\theta ((g_\theta^*) \sharp \rho, \mu) \leq \inf_{g_\theta \in \Theta_{\NN}} \phi H^\mu ((g_\theta) \sharp \rho, \mu) + 4 \sup_{f \in H^\theta} \inf_{f_\omega \in F_{\NN}} \|f - f_\omega\|_\infty + \phi H^\mu (\widehat{\mu}_n, \mu) + \phi F_{\NN} (\mu, \widehat{\mu}_n).
\]

\[\square\]
B Proof of Lemma 5

Proof. Consider Scenario 1 first. Without loss of generality, we assume $Z = [0,1]^d$. Otherwise, we can rescale the domain to be a subset of $[0,1]^d$. By Monge map (Lemma 1), there exists a mapping $T = [T_1,\ldots,T_d] : Z \mapsto X$ such that $T_#\nu = \mu$. Such a mapping is Hölder continuous, i.e., each coordinate mapping $T_i$ for $i = 1,\ldots,d$ belongs to $H^{\alpha+1}$. We approximate each function $T_i$ using the network architecture identified in Lemma 3. Specifically, given approximation error $\delta \in (0,1)$. There exists a network architecture with no more than $c(\log\frac{1}{\delta}+1)$ layers and $c'\delta^{-\frac{d}{\alpha+1}}(\log\frac{1}{\delta}+1)$ neurons and weight parameters, such that with properly chosen weight parameters, yields an approximation $\tilde{T}_i$ of $T_i$ satisfying $\|\tilde{T}_i - T_i\|_{\infty} \leq \delta$. Applying this argument $d$ times, we form an approximation $g_\theta = [\tilde{T}_1,\ldots,\tilde{T}_d]$ of $T$. We show $(g_\theta)_\#\rho$ satisfies the following IPM bound

$$d_{H^{\beta}_{\infty}}((g_\theta)_\#\rho,\mu) = d_{H^{\beta}}((g_\theta)_\#\rho,T_\#\rho)$$

$$= \sup_{f \in H^{\beta}} E_{x \sim (g_\theta)_\#\rho}[f(x)] - E_{y \sim T_\#\rho}[f(y)]$$

$$= \sup_{f \in H^{\beta}} E_{z \sim \rho}[f(g_\theta(z))] - E_{z \sim \rho}[f(T(z))]$$

$$\leq E_{z \sim \rho}[\|g_\theta(z) - T(z)\|_{\infty}]$$

$$\leq E_{z \sim \rho}[\|\tilde{T}_1(z) - T_1(z),\ldots,\tilde{T}_d(z) - T_d(z)\|_{\infty}]$$

$$\leq \delta.$$

Therefore, choosing $\delta = \epsilon_1$ gives rise to $d_{H^{\beta}_{\infty}}((g_\theta)_\#\rho,\mu) \leq \epsilon_1$.

For Scenario 2. The proof is nearly identical, except that we replace the Monge map by Moser’s coupling. As shown in Lemma 2, the Moser’s coupling between $\rho$ and $\mu$ is also Hölder continuous with modulus of continuity $\alpha + 1$. \qed

C Proof of Lemma 7

Proof. The proof utilizes the symmetrization technique and Dudley’s entropy integral, which can be found in empirical process theory (Dudley, 1967; Van Der Vaart and Wellner, 1996). We prove here for completeness. Let $y_1,\ldots,y_n$ be i.i.d. samples from $\mu$, independent of $x_i$’s. By symmetriza-
tion, we derive

$$\mathbb{E}[d_{\mathcal{F}}(\bar{\mu}_n, \mu)] = \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \mathbb{E}_{y \sim \mu} [f(y)] \right]$$

$$= \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \mathbb{E}_{y_i \sim \mu} \left( \frac{1}{n} \sum_{i=1}^{n} f(y_i) \right) \right]$$

$$\leq \mathbb{E}_{x} \mathbb{E}_{y} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} (f(x_i) - f(y_i)) \right]$$

$$= \mathbb{E}_{x} \mathbb{E}_{y} \mathbb{E}_{\xi} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f(x_i) - f(y_i)) \right]$$

$$= 2 \mathbb{E}_{x} \mathbb{E}_{\xi} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f(x_i) \right],$$

where $\xi_i$'s are i.i.d. Rademacher random variables, i.e., $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$. The next step is to discretize the function space $\mathcal{F}$. Let $\{\delta_i\}_{i=1}^{k}$ be a decreasing series of real numbers with $\delta_{i+1} < \delta_i$. We construct a collection of coverings on $\mathcal{F}$ under the function $\ell_\infty$ norm with accuracy $\delta_i$. Denote the $\delta_i$-covering number as $N(\delta_i, \mathcal{F}, \|\cdot\|_\infty)$. For a given $f$, denote the closest element (in the $\ell_\infty$ sense) to $f$ in the $\delta_i$ covering as $f^{(i)}$ for $i = 1, \ldots, k$. We expand $\mathbb{E}_{x,\xi} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f(x_i) \right]$ as a telescoping sum as

$$\mathbb{E}_{x,\xi} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f(x_i) \right] \leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f(x_i) - f^{(k)}(x_i)) \right]$$

$$+ \sum_{j=1}^{k-1} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f^{(j+1)}(x_i) - f^{(j)}(x_i)) \right]$$

$$+ \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f^{(1)}(x_i) \right].$$

We choose $\delta_1 = \text{diam}(\mathcal{F})$, i.e., the diameter of the class $\mathcal{F}$. Then $f^{(1)}$ can be arbitrarily picked from $\mathcal{F}$. Therefore, the last term $\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f^{(1)}(x_i) \right] = 0$ since $\xi_i$'s are symmetric. The first term $\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f(x_i) - f^{(k)}(x_i)) \right]$ can be bounded by Cauchy-Schwarz inequality:

$$\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f(x_i) - f^{(k)}(x_i)) \right] \leq \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \left( \sum_{i=1}^{n} \xi_i^2 \right) \left( \sum_{i=1}^{n} (f(x_i) - f^{(k)}(x_i))^2 \right) \right]$$

$$\leq \delta_k.$$  

We now bound each term in the telescoping sum $\sum_{j=1}^{k-1} \mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f^{(j+1)}(x_i) - f^{(j)}(x_i)) \right]$. Observe

$$\|f^{(j+1)} - f^{(j)}\|_\infty = \|f^{(j+1)} - f - f^{(j)}\|_\infty \leq \|f^{(j+1)} - f\|_\infty + \|f - f^{(j)}\|_\infty \leq \delta_{j+1} + \delta_j.$$
By Massart’s lemma (Mohri et al., 2018), we have
\[
\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i (f^{(j+1)}(x_i) - f^{(j)}(x_i)) \right] \leq \frac{(\delta_{j+1} + \delta_j) \sqrt{2 \log(N(\delta_{j+1}, \mathcal{F}, ||\cdot||_\infty), N(\delta_{j+1}, \mathcal{F}, ||\cdot||_\infty))}}{\sqrt{n}}
\]
\[
\leq \frac{2(\delta_{j+1} + \delta_j) \sqrt{\log N(\delta_{j+1}, \mathcal{F}, ||\cdot||_\infty)}}{\sqrt{n}}.
\]

Collecting all the terms, we establish
\[
\mathbb{E}_{x, \xi} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f(x_i) \right] \leq \delta_k + 2 \sum_{j=1}^{k-1} \frac{(\delta_{j+1} + \delta_j) \sqrt{\log N(\delta_{j+1}, \mathcal{F}, ||\cdot||_\infty)}}{\sqrt{n}}.
\]

It suffices to set \( \delta_{j+1} = \frac{1}{2} \delta_j \). Invoking the identity \( \delta_{j+1} + \delta_j = 6(\delta_{j+1} - \delta_{j+2}) \), we derive
\[
\mathbb{E}_{x, \xi} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \xi_i f(x_i) \right] \leq \delta_k + 12 \sum_{j=1}^{k-1} \frac{(\delta_{j+1} - \delta_{j+2}) \sqrt{\log N(\delta_{j+1}, \mathcal{F}, ||\cdot||_\infty)}}{\sqrt{n}}
\]
\[
\leq \delta_k + \frac{12b}{\sqrt{n}} \int_{\delta_{j+1}}^{\delta_j} \sqrt{\log N(\epsilon, \mathcal{F}, ||\cdot||_\infty)} d\epsilon
\]
\[
\leq \inf_\delta 2\delta + \frac{12b}{\sqrt{n}} \int_{\delta}^{\delta_1} \sqrt{\log N(\epsilon, \mathcal{F}, ||\cdot||_\infty)} d\epsilon.
\]

By the assumption, we pick \( \delta_1 = L \) and set the \( \delta_1 \)-covering with only one element \( f = 0 \). This yields the desired result
\[
\mathbb{E} [d_\mathcal{F}(\hat{\mu}_0, \mu)] \leq 2 \inf_{0 < \delta < L} \left( 2\delta + \frac{12b}{\sqrt{n}} \int_{\delta}^{L \delta} \sqrt{\log N(\epsilon, \mathcal{F}, ||\cdot||_\infty)} d\epsilon \right).
\]

\[\square\]

D Proof of Lemma 8

Proof. To construct a covering for \( \mathcal{F}_{\text{NN}}(\mathcal{R}, \mathcal{K}, \mathcal{L}, \mathcal{P}, \mathcal{K}) \), we discretize each weight parameter by a uniform grid with grid size \( h \). To simplify the presentation, we omit the bar notation in this proof. Recall we write \( f_\omega \in \mathcal{F}_{\text{NN}}(R, \kappa, L, p, K) \) as \( f_\omega = W_L \cdot \text{ReLU}(W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1}) + b_L \). Let \( f_\omega, f_\omega' \in \mathcal{F} \) with all the weight parameters at most \( h \) from each other. Denoting the weight matrices in \( f_\omega \) and \( f_\omega' \) as \( W_L, \ldots, W_1, b_L, \ldots, b_1 \) and \( W'_L, \ldots, W'_1, b'_L, \ldots, b'_1 \), respectively, we bound the \( \ell_\infty \) difference \( \|f_\omega - f_\omega'\|_\infty \) as
\[
\|f_\omega - f_\omega'\|_\infty = \|W_L \cdot \text{ReLU}(W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1}) + b_L
\]
\[
- (W'_L \cdot \text{ReLU}(W'_{L-1} \cdots \text{ReLU}(W'_1 x + b'_1) \cdots + b'_{L-1}))\|_\infty
\]
\[
\leq \|b_L - b'_L\|_\infty + \|W_L - W'_L\|_1 \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1}\|_\infty
\]
\[
+ \|W_L\|_1 \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1} - (W'_{L-1} \cdots \text{ReLU}(W'_1 x + b'_1) \cdots + b'_{L-1})\|_\infty
\]
\[
\leq h + hp \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1}\|_\infty
\]
\[
+ \kappa p \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1} - (W'_{L-1} \cdots \text{ReLU}(W'_1 x + b'_1) \cdots + b'_{L-1})\|_\infty.
\]
We further expand the first term on the right-hand side as (7) in the proof of Lemma 4 yields

Proof.

We show an alternative oracle inequality for finite generated samples as follows. Inequality

\[ E \text{ Proof of Corollary 1} \]

We derive the following bound on \( \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1}\|_\infty \):

\[
\|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1}\|_\infty \leq \|W_{L-1}(\cdots \text{ReLU}(W_1 x + b_1) \cdots)\|_\infty + \|b_{L-1}\|_\infty
\]

\[
\leq \|W_{L-1}\|_1 \|W_{L-2}(\cdots \text{ReLU}(W_1 x + b_1) \cdots) + b_{L-2}\|_\infty + \kappa
\]

\[
\leq \kappa p \|W_{L-2}(\cdots \text{ReLU}(W_1 x + b_1) \cdots) + b_{L-2}\|_\infty + \kappa
\]

\[
\leq (\kappa p)^{-1} B + \kappa \sum_{i=0}^{l-3} (\kappa p)^i
\]

\[
\leq (\kappa p)^{-1} B + \kappa (\kappa p)^{l-2},
\]

where (i) is obtained by induction and \( \|x\|_\infty \leq B \). The last inequality holds, since \( \kappa p > 1 \). Substituting back into the bound for \( \|f_\omega - f'_\omega\|_\infty \), we have

\[
\|f_\omega - f'_\omega\|_\infty \leq \kappa p \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1} - (W_1' \cdots \text{ReLU}(W_1' x + b_1') \cdots + b_{L-1}')\|_\infty
\]

\[
+ h + h \left[(\kappa p)^{-1} B + \kappa (\kappa p)^{l-2}\right]
\]

\[
\leq \kappa p \|W_{L-1} \cdots \text{ReLU}(W_1 x + b_1) \cdots + b_{L-1} - (W_1' \cdots \text{ReLU}(W_1' x + b_1') \cdots + b_{L-1}')\|_\infty
\]

\[
+ h(pB + 2)(\kappa p)^{l-1}
\]

\[
\leq (\kappa p)^{l-1} \|W_1 x + b_1 - W_1' x - b_1'\|_\infty + h(L - 1)(pB + 2)(\kappa p)^{l-1}
\]

\[
\leq hL(pB + 2)(\kappa p)^{l-1},
\]

where (i) is obtained by induction. We choose \( h \) satisfying \( hL(pB + 2)(\kappa p)^{l-1} = \delta \). Then discretizing each parameter uniformly into \( \kappa/h \) grids yields a \( \delta \)-covering on \( \mathcal{F}_{\text{NN}} \). Therefore, the covering number is upper bounded by

\[
\mathcal{N}(\delta, \mathcal{F}_{\text{NN}}(R, \kappa, L, p, K), \|\cdot\|_\infty) \leq \left(\frac{\kappa}{h}\right)^\# \text{ of nonzero parameters}.
\]

\[
(8)
\]

E Proof of Corollary 1

Proof. We show an alternative oracle inequality for finite generated samples as follows. Inequality

(7) in the proof of Lemma 4 yields

\[
d_{\mathcal{H}^\theta}(\mathcal{F}_{\text{NN}}(\mathcal{G}_\theta^\ast \rho, \mu_\infty)) + 2 \sup_{f \in \mathcal{H}^\theta} \inf_{f_\omega \in \mathcal{F}_{\text{NN}}} \|f - f_\omega\|_\infty + d_{\mathcal{H}^\theta}(\mu_\infty, \mu).
\]

We further expand the first term on the right-hand side as

\[
d_{\mathcal{F}_{\text{NN}}}(\mathcal{G}_\theta^\ast \rho, \mu_\infty) \leq d_{\mathcal{F}_{\text{NN}}}(\mathcal{G}_\theta^\ast \rho, \mu_\infty) + d_{\mathcal{F}_{\text{NN}}}(\mu_\infty, \mu_\infty).
\]
By the optimality of \( g_\theta^* \), for any \( g_\theta \in \mathcal{G}_{NN} \), we have

\[
d_{\mathcal{F}_{NN}}((g_\theta^*)_\sharp \widehat{p}_m, \widehat{m}_n) \leq d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, \widehat{m}_n)
\]
\[
\leq d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, (g_\theta)_\sharp \rho) + d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \rho, \mu) + d_{\mathcal{F}_{NN}}(\mu, \widehat{m}_n)
\]
\[
\leq d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \rho, \mu) + \sup_{g_\theta \in \mathcal{G}_{NN}}{d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, (g_\theta)_\sharp \rho) + d_{\mathcal{F}_{NN}}(\mu, \widehat{m}_n)}
\]
\[
\leq d_{\mathcal{H}_\rho}(\mu, \rho) + 2 \sup_{f \in \mathcal{H}^\rho} \inf_{f \in \mathcal{F}_{NN}}{\|f - f_\omega\|_\infty + 2d_{\mathcal{F}_{NN}}(\mu, \widehat{m}_n)}
\]
\[
\quad + \sup_{g_\theta \in \mathcal{G}_{NN}}{d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, (g_\theta)_\sharp \rho)},
\]

where the last inequality follows the same argument in the proof of Lemma 7. Combining all the inequalities together, we have

\[
d_{\mathcal{H}_\rho}(\mu, \rho) \leq \inf_{g_\theta \in \mathcal{G}_{NN}}{d_{\mathcal{H}_\rho}(\mu, \rho)} + 4 \sup_{f \in \mathcal{H}^\rho} \inf_{f \in \mathcal{F}_{NN}}{\|f - f_\omega\|_\infty + 2d_{\mathcal{F}_{NN}}(\mu, \widehat{m}_n)} + d_{\mathcal{H}_\rho}(\mu, \widehat{m}_n)
\]
\[
\quad + \sup_{g_\theta \in \mathcal{G}_{NN}}{d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, (g_\theta)_\sharp \rho) + d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \rho, (g_\theta)_\sharp \widehat{p}_m)}.
\]

Given Theorem 2, we only need to bound the extra statistical error terms \( \sup_{g_\theta \in \mathcal{G}_{NN}}{d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, (g_\theta)_\sharp \rho)} \) and \( d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \rho, (g_\theta)_\sharp \widehat{p}_m). \) In fact, Lemma 7 and Lemma 8 together imply

\[
\sup_{g_\theta \in \mathcal{G}_{NN}}{d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \widehat{p}_m, (g_\theta)_\sharp \rho)} \leq \widetilde{O}\left(\frac{1}{m} + \frac{1}{\sqrt{m}} \sqrt{\bar{K} \bar{L} \log(mL)} + KL \log(mL)\right)
\]
\[
d_{\mathcal{F}_{NN}}((g_\theta)_\sharp \rho, (g_\theta)_\sharp \widehat{p}_m) \leq \widetilde{O}\left(\frac{1}{m} + \frac{1}{\sqrt{m}} \sqrt{KL \log(mL)}\right),
\]

where the first inequality is obtained by taking \( \mathcal{F} = \mathcal{F}_{NN} \circ \mathcal{G}_{NN} \) in Lemma 7, and its covering number is upper bounded by the product of the covering numbers of \( \mathcal{F}_{NN} \) and \( \mathcal{G}_{NN} \). Putting together, the estimation error \( d_{\mathcal{H}_\rho}(g_\theta^* \rho, \mu) \) can be bounded analogously to Theorem 2 as

\[
\mathbb{E}\left[d_{\mathcal{H}_\rho}(g_\theta^* \rho, \mu)\right] \leq \widetilde{O}\left(\epsilon_1 + \epsilon_2 + \frac{1}{n} + \frac{1}{m} + n^{-\frac{\rho}{2}} + \sqrt{\frac{\epsilon_2}{n}} + \sqrt{\frac{\epsilon_1 + \frac{1}{n} + \frac{1}{m}}{\epsilon_2}} + \frac{\epsilon_2}{m}\right).
\]

It suffices to choose \( \epsilon_2 = n^{-\frac{\rho}{2m}} \) and \( \epsilon_1 = m^{-\frac{a+1}{2m+1}} \), which yields

\[
\mathbb{E}\left[d_{\mathcal{H}_\rho}(g_\theta^* \rho, \mu)\right] \leq \widetilde{O}\left(n^{-\frac{\rho}{2m}} + m^{-\frac{a+1}{2m+1}} + \sqrt{\frac{n^{-\frac{d}{m}}}{m}}\right).
\]

\[\square\]