Quantum Phase Transition in A Multi-connected Superconducting Jaynes-Cummings Lattice

Kangjun Seo and Lin Tian

School of Natural Sciences, University of California Merced, Merced, California 95343, USA

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The connectivity and tunability of superconducting qubits and resonators provide us with an appealing platform to study the many-body physics of microwave excitations. Here we present a multi-connected Jaynes-Cummings lattice model which is symmetric with respect to the qubit-resonator couplings. Our calculation shows that this model exhibits a Mott insulator-superfluid-Mott insulator phase transition, featured by a reentry to the Mott insulator phase, at commensurate filling. The phase diagrams in the grand canonical ensemble are also derived, which confirm the incompressibility of the Mott insulator phase. Different from a general-purposed quantum computer, it only requires two operations to demonstrate this phase transition: the preparation and the detection of the commensurate many-body ground state. We discuss the realization of these operations in the superconducting circuit.

I. INTRODUCTION

The past few years have witnessed stimulating progress in the study of superconducting quantum devices [1–3]. Quantum logic operations with fidelity exceeding 99.9% and quantum error correction codes were recently realized [4–6]. By experimenting with various designs of the superconducting qubits and resonators, decoherence times on the scale of several tens of microseconds have been achieved in both 3-dimensional and planar circuits [7–9]. In several designs, such as the Xmon qubit, one qubit can be simultaneously connected to multiple resonators and control wires, which significantly improves the scalability and tunability of the superconducting systems [9–13]. In the aspect of detection, quantum-limited amplifiers were developed to conduct phase-sensitive measurement of the amplitude of the microwave field and test quantum coherence effects at the single-photon level [14, 15].

The technological advancements in superconducting devices provide us with an appealing platform to explore many-body correlations. Analog and digital quantum simulators [16, 17] of the superconducting systems have been proposed for numerous many-body effects, including phase transitions in the quantum spin systems [18–25], topological effects [26–29], electron-phonon physics [30, 31], and even high-energy physics [32–34]. The implementation of these simulators can help us understand many-body phenomena that are hard to solve with traditional condensed matter techniques. Given the connectivity and tunability of the superconducting devices, we can also construct many-body Hamiltonians that do not exist in the real world, but carry novel many-body correlations. One such model is the so-called coupled cavity array (CCA) model, which is composed of an array of cavities each connected to neighboring cavities. Each cavity couples to a non-linear medium, such as a qubit or a number of impurity atoms. In the pioneer works of Refs. [35–40], it was shown that the CCA exhibits the Mott insulator (MI)-to-superfluid (SF) phase transition for the cavity-polaritons, due to its resemblance to the Bose-Hubbard (BH) model [41, 42]. The CCA has been thoroughly compared to the BH model in Refs. [36, 40]. Experimental efforts towards realizing the CCA with superconducting devices have also been conducted [43, 44].

In this work, stimulated by recent experimental progress, we present a multi-connected Jaynes-Cummings (JC) lattice model that demonstrates novel quantum phase transition for the cavity-polaritons. This model is constructed with arrays of qubit-resonator systems, where each qubit is connected to multiple resonators by exploiting the unique connectivity of planar superconducting qubits. In contrast to the CCA [35–40], there is no direct coupling between the resonators. Instead, the qubit-resonator couplings in this multi-connected model serve both as onsite Hubbard interaction and as photon hopping. By varying one of the control parameters, this system first makes a transition from the MI phase to the SF phase at commensurate filling, similar to the CCA and BH models. More interestingly, as the parameter continuously varying, this system makes another transition from the SF phase to reenter the MI phase. This is confirmed by our calculation of the single-particle density matrix and the energy gap using the exact diagonalization method [45, 46]. We also obtain the phase diagrams of this model in the grand canonical ensemble at zero temperature, which indicate the incompressibility of the MI phase and the closing of the energy gap in the SF phase [47]. The reentry to the MI phase is due to the symmetry between the qubit-resonator couplings in this model. One advantage of this system, compared with a generic-purpose quantum computer [16], is that it only requires two operations to demonstrate the phase transition: the preparation of the many-body ground state at commensurate filling and the detection of such state, both of which can be realized with current technology. This multi-connected JC lattice can be extended to two-dimensional patterns and more complicated configurations to study many-body correlations in bosonic systems. The nonequilibrium dynamics of the cavity-polaritons can also be investigated in this setup.

The paper is organized as follows. In Sec. II, we present the multi-connected JC lattice model and its construction with superconducting qubits and resonators. The effective Hubbard interaction and photon hopping are analyzed in the limiting case of drastically-different coupling constants. We calculate the single-particle density matrix and the energy gap of this multi-connected model at commensurate filling using the exact diagonalization method in Sec. III. Then, in Sec. IV, this
II. MULTI-CONNECTED JC LATTICE

A. Model Hamiltonian

A multi-connected superconducting JC lattice is depicted in Fig. 1 (a). The building block of this one-dimensional lattice is made of a superconducting qubit denoted by \( Q \), and a superconducting resonator denoted by \( R \). The qubit \( Q \), couples to neighboring resonators \( R \) and \( R_{i-1} \) with coupling strengths \( g_r \) and \( g_{r_1} \), respectively. This configuration can be extended to a two-dimensional checkerboard pattern of alternative qubits and resonators. The total Hamiltonian for this model can be written as \( H = \sum_i (H^i_{\text{int}} + H^i_{\text{JC}}) \), where

\[
H^i_{\text{int}} = \omega_\epsilon a^\dagger_i a_i + \frac{\omega_\epsilon}{2} \sigma^z_i \tag{1}
\]

is the noninteracting Hamiltonian of one repeating unit and

\[
H^i_{\text{JC}} = g_r (a^\dagger_i \sigma^+_i + \sigma^+_i a_i) + g_{r_1} (a^\dagger_{i-1} \sigma^+_i + \sigma^+_i a_{i-1}) \tag{2}
\]

describes the JC couplings between a qubit and its neighboring resonators [48]. Here \( \omega_\epsilon \) is the angular frequency of the resonator modes, \( \omega_\epsilon \) is the energy level splitting of the qubits, \( a_i (a'_i) \) is the annihilation (creation) operator of the resonator mode \( R \), and \( \sigma^+_i, \sigma^-_i \) are the Pauli operators of the qubit \( Q \). We set \( h = 1 \) for convenience of discussion.

The repeating units in our model are connected via qubit-resonator couplings. This is in sharp contrast to the CCA, where neighboring resonators couple directly to each other via a hopping Hamiltonian \( -\sum (a^\dagger_i a_{i+1} + a^\dagger_{i+1} a_i) \) [35-40]. As we will show, the qubit-resonator couplings in our model play both the role of onsite interaction and the role of photon hopping. A key feature of this model is that the system is invariant with respect to the exchange of the couplings \( g_r \) and \( g_{r_1} \). Hence, the unit cell can be defined in two ways, either with \( Q_i \) and \( R_i \) or with \( Q_i \) and \( R_{i-1} \) in one cell, as shown in Fig. 1 (b).

This multi-connected JC model can be realized with superconducting qubits and resonators developed in recent state-of-the-art experiments. One promising system is the so-called Xmon qubit, which excels in connectivity, controllability, and decoherence time [9, 10]. This qubit can be connected to multiple resonators and control wires with tunable couplings. It also demonstrates a decoherence time exceeding 40\( \mu \)s. In our discussions, we choose the control parameters to be in range of \( g_{r_1}/2\pi \in [0, 300] \) MHz, the resonator detuning \( \Delta/2\pi \in [-1, 1] \) GHz with \( \Delta = \omega_\epsilon - \omega_\zeta \), and \( \omega_\zeta/2\pi = 10 \) GHz.

B. Limiting case: \( g_1 \ll g_r \) (or \( g_r \ll g_r_1 \))

We start with the simple case of \( g_1 = 0 \), i.e., each repeating unit as defined by the top part of Fig. 1 (b) is isolated from each other with a vanishing coupling between \( Q_i \) and \( R_{i-1} \). Note that the opposite limit of \( g_r \ll g_r_1 \) can be studied similarly due to the symmetry between \( g_r \) and \( g_{r_1} \). The total Hamiltonian in this limit has the form of \( H = \sum_i H^i_{\text{JC}} \) with

\[
H^i_{\text{JC}} = \omega_\epsilon a^\dagger_i a_i + \frac{\omega_\epsilon}{2} \sigma^z_i + g_r (a^\dagger_i \sigma^+_i + \sigma^+_i a_i). \tag{3}
\]

The Hilbert space of each unit cell is spanned by the basis states \( \{ |n_i, \sigma_i \rangle \} \) with \( n_i \) being the microwave photon number of the resonator mode and \( \sigma_i = \uparrow, \downarrow \) being the qubit state at site \( i \). The lowest eigenstate of \( H^i_{\text{JC}} \) is \( |0_i, \uparrow \rangle \) with the energy \( -\omega_\epsilon/2 \). All other eigenstates, denoted by \( |n_i, \pm \rangle \) with \( n_i > 0 \), are polariton doublets in the subspace of \( |n_i - 1, \uparrow \rangle, |n_i, \downarrow \rangle \), and contain both photon and qubit excitations. The eigenenergies of the states \( |n_i, \pm \rangle \) are \( E_{n_i, \pm} = (n_i - 1/2) \omega_\epsilon \pm \Omega_{\text{eff}}(\Delta)/2 \) with \( \Omega_{\text{eff}}(\Delta) = \sqrt{\Delta^2 + 4 g_r^2 n_i} \), depending on the detuning \( \Delta \) [48].

The qubit-resonator coupling \( g_r \) generates nonlinearity in the polariton states. In Appendix A, we present an analysis of the nonlinearity involving only the lower-polariton states. The nonlinearity can be viewed as an effective Hubbard interaction for the polariton modes. Our results, different from that in Refs. [36, 40], are in good agreement with the energy gap shown in Fig. A1. For the low-lying states \( |1_i, \uparrow \rangle \) and \( |2_i, \downarrow \rangle \), the interaction strength \( U = (2 - \sqrt{2}) g_r \) at \( \Delta = 0 \); and \( U = (\Delta + |\Delta|)/2 \) for \( |\Delta| \gg g_r \), demonstrating drastically-different behavior for large positive and negative detunings.

Next, we introduce a small but finite coupling strength \( g_{r_1} \) that satisfies the condition \( g_{r_1} \ll g_r \). This coupling can be viewed as a perturbation that induces hopping of a polariton excitation between adjacent unit cells with the conservation of the total excitation number, e.g., the nonzero matrix element

\[
\langle 0_{i-1}, \downarrow, | \langle 2_{i-1}, \downarrow | a^\dagger_{i-1} a_{i-1} |1_{i-1}, \uparrow \rangle |1_i, \downarrow \rangle = -1/2 \sqrt{2} \tag{4}
\]
is associated with the hopping of an excitation at site \( i - 1 \) to site \( i \) with a hopping strength \( i \propto g_{r_1} \).
The total Hamiltonian of the multi-connected JC lattice thus contains two competing elements for a MI-to-SF phase transition [47]: onsite interaction and hopping between neighboring sites, both originated from the qubit-resonator couplings. With $g_l \ll g_r$ (or vice versa), the system is dominated by the onsite interaction and is expected to be in a MI phase. With the increase of $g_l$, the kinetic energy of the polariton mode eventually overcomes that of the Hubbard interaction, and the system enters a SF phase. Given the symmetry of $g_l$ and $g_r$, the two couplings play similar roles when their strengths become comparable, each contributing to the onsite interaction as well as the hopping term. In the following sections, we will study the quantum phase transition of this model in detail.

III. PHASE TRANSITION AT COMMENSURATE FILLING

Define the operator $\hat{N} = \sum_i (a_i^\dagger a_i + \sigma_i^+ \sigma_i^-)$ as the total excitation number of the lattice, containing both photon and qubit excitations. Because $[H, \hat{N}] = 0$, the total excitation number is a good quantum number. For a bosonic system, the MI phase occurs at commensurate filling, i.e., the excitation number $N$ is a multiple of the lattice size $M$. Here we study the many-body phases of the multi-connected JC lattice with a fixed excitation number $N$ and $N/M$ being an integer. We apply the exact diagonalization method to find the precise ground state of this model [45, 46]. The natural choice of the basis vectors is all possible configurations of the state $\vert \psi \rangle = \vert n_1, \sigma_1 \rangle \vert n_2, \sigma_2 \rangle \cdots \vert n_M, \sigma_M \rangle$ that satisfies $\sum_i (n_i + \delta_i) = N$, where $\delta_i$ refers to the qubit excitation at site $i$ with $\delta_i = 0$ (1) for $\sigma_i = \downarrow (\uparrow)$. The Hamiltonian in the $N$-excitation subspace can be written as a sparse matrix on these basis vectors. Using a Lanczos-type algorithm, the low-lying eigenstates, in particular, the ground state, can be obtained.

A. Single-particle density matrix

For a system of fixed particle number, $(G \vert a_i \rangle \langle G) \equiv 0$ and cannot be utilized as an order parameter, where $\vert G \rangle$ is the many-body ground state. Instead, we calculate the normalized single-particle density matrix [49]

$$\rho_1(i,j) = \langle G \vert a_i^\dagger a_j \vert G \rangle / \langle G \vert a_i^\dagger a_i \vert G \rangle,$$

(5)

to characterize the phase transition of the multi-connected JC lattice. This matrix is generically Hermitian. Because of the lattice translational and reflectional invariances of the ground state, $\rho_1(i,j) = \rho_1(i+k, j+k)$ for an arbitrary integer $k$; and $\rho_1(i,i) = \rho_1(j,j)$. The matrix $\rho_1(i,j)$ is hence real, symmetric and cyclic. Below, we replace $\rho_1(i,j)$ by the notation $\rho_1(|i-j\rangle)$. Nonzero off-diagonal matrix element $\rho_1(|i-j\rangle \rightarrow |i-j\rangle)$, the so-called off-diagonal long range order (ODLRO) [50], implies a condensation of the bosons and is a strong indicator of superfluidity in the system. In an insulating phase, on the other hand, $\rho_1(|i-j\rangle \rightarrow |i-j\rangle) \rightarrow 0$ for large $|i-j|$.

We calculate $\rho_1(|i-j\rangle$ using the exact diagonalization method for a lattice of $M = 8$ and total excitation number $N = 8$ under the periodic boundary condition. For such a lattice, the maximal lattice distance $x_{\max} = 4$. Our results show that even for a small-size system, this method can reveal the essential feature of the MI-to-SF phase transition. In Fig. 2 (a), $\rho_1(x)$ is plotted versus the lattice distance $x$ for three sets of couplings $(g_l, g_r)$. For $(g_l, g_r) = (5, 295) \text{ MHz}$, i.e., with $g_l \ll g_r$, $\rho_1(x)$ decreases to zero as the lattice distance increases to $x = x_{\max}$. This indicates the absence of ODLRO and the system being in an insulating phase. As analyzed in Sec. II B, in this limit, the coupling $g_r$ provides strong Hubbard interaction; while $g_l$ only induces small hopping. By slightly increasing $g_l$ to $25 \text{ MHz}$ and decreasing $g_r$ to $275 \text{ MHz}$, $\rho_1(x)$ increases but still vanishes at $x = x_{\max}$. In contrast, for $(g_l, g_r) = (150, 150) \text{ MHz}$, $\rho_1(x)$ remains finite at the maximal lattice distance $x_{\max}$. Both couplings $g_{l,r}$ now generate hopping and onsite repulsion that are comparable in strength. The system hence embodies ODLRO.

The dependence of $\rho_1(x_{\max})$ on the coupling $g_r$ is shown in Fig. 2 (b) for three values of $g_l$. For each $g_l$, $\rho_1(x_{\max})$ decreases to zero when $g_l \ll g_r$ and $g_r \gg g_l$; and it reaches a large maximum when $g_l \sim g_r$. Hence, by continuously changing the coupling $g_r$ at a given $g_l$, the ground state evolves from a MI phase to a SF phase, and then reenters the MI phase. This is a unique feature of this multi-connected model, rooted in the symmetry with respect to the two couplings. In Fig. 2 (c), $\rho_1(x_{\max})$ is plotted as functions of $g_l$. For three detunings, which verifies the symmetry of the couplings. It also indicates that the detuning plays an important role in the phase transition. With a negative detuning, the system becomes more “photon”-like with a reduced effective interaction, as discussed in Appendix A. The SF phase then becomes more favorable and exists in a broader parameter regime. With a positive detuning, on the other hand, the system becomes more “spin”-like with a larger effective interaction, and the SF regime is narrowed.
The energy gap is another important quantity to study the critical behavior of quantum phase transition. It is also directly associated with the inverse of the compressibility of the many-body phases. Let $E_{N+} = E(N + 1) - E(N)$ ($E_{N-} = E(N) - E(N - 1)$) be the energy difference of adding (removing) one excitation to a system of $N$ excitations, where $E(N)$ is the ground state energy for a system with $N$ polaritons. The energy gap is defined as $E_{gp} = E_{N+} - E_{N-}$ [46]. In the MI phase at commensurate filling, $E_{gp}$ is finite due to the Hubbard interaction; while in the SF phase, $E_{gp}$ vanishes.

We calculate the energy gap $E_{gp}$ of the multi-connected model at the filling factor $N/M = 1$. In Fig. 3 (a), $E_{gp}$ is plotted as a function of $1/M$. Due to the finite-size effect, the energy gap remains open for a finite lattice in all regimes of the couplings. For $g_r \ll g_l$ or $g_r \gg g_l$, $E_{gp}$ is nearly independent of the size of the system; whereas for $g_r$ comparable to $g_l$, $E_{gp}$ strongly depends on $M$. We thus extrapolate the energy gap to the thermodynamic limit with $M \to \infty$ using a fourth-degree polynomial of $M$. The extrapolated gap $E_{gp}^{0}$, plotted in Fig. 3 (b) versus the coupling $g_r$ at fixed $g_l$'s, clearly bears the feature of a MI-to-SF phase transition. In the regime of $g_r \ll g_l$, where a MI phase is predicted, the gap $E_{gp}^{0}$ is open. With the increase of $g_r$, $E_{gp}^{0}$ decreases and eventually closes when $g_r$ becomes comparable to $g_l$, with this system entering a SF phase. As $g_r$ further increases towards $g_r \gg g_l$, $E_{gp}^{0}$ opens again after a finite interval of zero gap, and the system reenters the MI phase. The energy gap in the limit of $g_r \ll g_l$ and $g_r \gg g_l$ can be well explained by the simple analysis of an effective Hubbard interaction, presented in detail in Appendix A and Fig. A1.

The reentrance to the MI phase is an interesting feature of this multi-connected JC lattice and is due to the symmetry of $g_l$ and $g_r$. At zero detuning, this many-body phase transition is solely determined by the ratio $g_r/g_l$. For $g_r/g_l < \beta_c$ or $g_r/g_l > \beta_c^{-1}$, with $\beta_c$ being the critical point, the system is in the MI phase; and in the intermediate regime, the system is in the SF phase. From Fig. 3 (b), we estimate that $\beta_c \approx 2/3$. Note that our numerical method cannot yield an accurate value of this critical point. It can be shown that the phase transition at $\Delta \neq 0$ also embodies this feature.

### IV. PHASE TRANSITION IN GRAND CANONICAL ENSEMBLE

The quantum phase transition in the CCA is often studied in the grand canonical ensemble (GCE) [35–38], where the excitation density (filling factor) is directly associated with the many-body phase and its compressibility. Here we extend the exact diagonalization method used in Sec. III to study the multi-connected JC lattice in the GCE [45]. Consider the free energy $\hat{F} = \hat{H} - \mu \hat{N}$ at a given chemical potential $\mu$ and define $\langle G \rangle$ as the ground state of the free energy $\hat{F}$. In the GCE, the total excitation number $N$ is a function of the chemical potential, and can be obtained from the ground-state wave function by $N(\mu) = \langle \hat{N} \rangle(\mu)$. The basis vectors in this calculation are: $|\phi\rangle = |n_1, \sigma_1\rangle|n_2, \sigma_2\rangle \cdots |n_M, \sigma_M\rangle$ with $\sum (n_i + \delta_i) \leq N_{\max}$ for a lattice of $M$ sites.

The maximal total excitation number $N_{\max}$ is chosen to include all possible basis vectors at the given chemical potential; and $N(\mu) \leq N_{\max}$. Note that the chemical potential, as discussed in previous works, is not a directly controllable parameter of this system [40].

#### A. Excitation density

We calculate the many-body ground state of a lattice with $M = 6$. The chemical potential is in a range that yields an excitation density $n \in [0, 2]$ with $n = N/M$. In Fig. 4 (a), the density $n$ is plotted as a function of the chemical potential at $\Delta = 0$. For the couplings $(g_l, g_r) = (5, 295), (25, 275)$ MHz, the density first increases with $\mu$ by small discrete steps of $\delta n = 1/M$ to reach a broad plateau of $n = 1$ at a critical chemical potential $\mu_c(n = 1)$, as indicated by the solid circle. At $\mu > \mu_c(n = 1)$, indicated by the solid square, the density starts increasing again to reach a plateau of $n = 2$. The discreteness of the small steps is due to the finite size of this system, where the ground state always has fixed (integer) number of total excitations. The excitation number increases with the chemical potential one at a time, which gives the discrete density increment of $\delta n$. For $(g_l, g_r) = (150, 150)$ MHz, in contrast, no such plateau exists, and $n$ increases continuously with $\mu$ in small steps. These plateaus at the commensurate fillings imply the incompressibility of the state, which is an important feature of the MI phase [47]. The critical chemical potentials $\mu_c(n)$ hence correspond to the boundaries between commensurate and incommensurate densities, and as a consequence, between the MI and the SF phases. The single-particle density matrix $\rho_1(x_{\max})$ is plotted in Fig. 4 (b). When the chemical potential is within the plateaus, $\rho_1(x_{\max})$ is reduced to a small value (for this finite size system), indicating the vanishing of the ODLRO; whereas $\rho_1(x_{\max})$ shows a large value outside the plateaus, indicating the presence of superfluidity.

#### B. Phase diagrams

The critical chemical potentials $\mu_c(n)$ discussed above can be used to define the phase boundaries of the multi-connected
JC lattice [36]. To derive the phase boundaries in the thermodynamic limit, we calculate $\mu_\pm(n)$ for finite lattices with $M = 3, 4, 5, 6$, respectively, and then extrapolate the results to $M \to \infty$ to derive $\mu_\pm^0(\lambda)$. In Fig. 4 (c), $\mu_\pm^0(n)$ are plotted versus the logarithmic ratio $\lambda = \log(g_r/g_i)$ with $g_r + g_i = 300$ MHz at $\Delta = 0$ and versus $\Delta$ with $g_r = 150$ MHz. Here $\mu_\pm^0(n)$ are solid (dashed) at the Mott lobes; dotted in the SF phase. Yellow (orange) lobes: $n = 1$ ($n = 2$).

to multiple resonators and control wires [9, 10]. The detuning can be adjusted by applying dc field to tune the energy level splitting of the qubits. Tunable coupling in the qubit-resonator systems has been tested in several experimental works [10–13]. By varying one of the couplings (Fig. 3 (b)), the MI-SF-MI phase transition could be demonstrated.

Compared with a general-purpose quantum computer [16], this analog quantum simulator only requires two operations to be realized: 1. the preparation of the many-body ground state at selected control parameters and filling factor; 2. the detection of this ground state. Below we study the implementation of these operations and discuss the effects of quantum errors.

### A. State preparation

The MI-SF-MI phase transition studied in Sec. III occurs in the ground state of this multi-connected model at integer filling. We present a scheme to prepare the $N$-excitation ground state with $N/M = 1$. This approach can be extended to prepare states with higher integer fillings. Our procedure contains two steps: 1. flipping of the state of the superconducting qubits; 2. adiabatically transferring the system to the proper ground state using a Landau-Zener process.

![FIG. 4.](image)

(a) The density $n$ and (b) $\rho_1(x_{max})$ versus $\mu - \omega_c$ for a lattice of $M = 6$ at $\Delta = 0$. Blue dot-dashed curve: $(g_r, g_i) = (5, 295)$ MHz; green solid: $(25, 275)$ MHz; and red dashed: $(150, 150)$ MHz. The circles (squares) mark $\mu_+(n)$ ($\mu_-(n)$). (c) and (d) $\mu_\pm^0(n)$ versus $\lambda = \log(g_r/g_i)$ with $g_r + g_i = 300$ MHz and $\Delta = 0$ and versus $\Delta$ with $g_r = 150$ MHz. Here $\mu_\pm^0(n)$ are solid (dashed) at the Mott lobes; dotted in the SF phase. Yellow (orange) lobes: $n = 1$ ($n = 2$).

![FIG. 5.](image)

(a) $E_x$ versus $\Delta$ for a lattice of $M = 8$ and $N/M = 1$ and (b) $E_x^0$ for $M \to \infty$. Blue circle: $(g_r, g_i) = (5, 295)$ MHz; green triangle: $(25, 275)$ MHz; and red square: $(150, 150)$ MHz.

We first discuss the excitation energy $E_x$ between the first excited state and the ground state of a lattice with $N = M$ excitations. The dependence of $E_x$ on the detuning is plotted in Fig. 5 (a) for a lattice of $M = 8$. Here $E_x$ continuously increases with $\Delta$ and exhibits a linear dependence on $\Delta$ at large positive detuning. Due to the finite size effect, for $|\Delta|$ comparable to the couplings, the excitation energy remains sizable regardless of the many-body phase. When extrapolated to the thermodynamic limit with $M \to \infty$, however, $E_x$ is reduced to very small value in the regime of the SF phase and remains sizable for the MI phase, as shown in Fig. 5 (b).

For state preparation, we first adjust the qubit energy to obtain a large positive detuning with $\Delta \gg g_r, g_i$. Here the qubits are nearly decoupled from the resonators. The initial state of this system can be written as $|01, \downarrow\rangle|02, \uparrow\rangle\cdots|0M, \downarrow\rangle$ with $N = 0$ excitation. By applying an ac driving field to generate a Rabi oscillation, the qubits are flipped to the state $|\uparrow\rangle$, and the system state becomes $|01, \uparrow\rangle|02, \uparrow\rangle\cdots|0M, \downarrow\rangle$. This state contains $N = M$ excitations and is the ground state of the multi-connected JC lattice in the limit of large positive

### V. REALIZATION

In Sec. II A, we briefly discussed the realization of the multi-connected JC lattice with superconducting qubits and resonators. Our model works in practical parameter regimes within reach of current technology. Recent experiments have shown that superconducting qubits can couple simultaneously

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detuning. Next, we adiabatically reduce the detuning to a target value, which is in a regime of interest to the study of the quantum phase transition. With the Landau-Zener theorem [52], the final state is the many-body ground state at the target detuning. The time interval for the adiabatic process is determined by the excitation energy $E_x$, which remains a sizable value in all parameter regimes, e.g., $E_x = 66$ MHz for $g_1 = g_2 = 150$ MHz and $\Delta = 0$, for a finite lattice of $M = 8$. The state preparation can hence be implemented within tens of nanoseconds, much shorter than the decoherence time of the qubits and the resonators, and would not be seriously affected by the environmental noise.

Because of the small anharmonicity in certain superconducting qubits, such as the transmon and the Xmon, the higher states in the qubit circuits can affect the state preparation scheme [53]. Let the third quantum state in a qubit be $|\tilde{e}\rangle$ and the energy level splitting between the states $|\uparrow\rangle$ and $|\tilde{e}\rangle$ be $\omega'$. In a typical transmon (Xmon), the anharmonicity is $\sim 5\%$ of $\omega_c$, yielding $(\omega_c - \omega')/2\pi \sim 500$ MHz. During the Rabi flip, the ac field generates nonzero coupling between $|\uparrow\rangle$ and $|\tilde{e}\rangle$ which is of the same order of magnitude as the Rabi frequency $\Omega$ for the spin-flip operation. To avoid leakage to the state $|\tilde{e}\rangle$, it requires that $\Omega \ll (\omega_c - \omega')$, which puts a constraint on the spin-flip time. By choosing $\Omega/2\pi = 50$ MHz, the spin flip can be realized in a practical time scale of 3 ns.

B. Detection

The phase transition can be characterized by measuring the quadrature correlation of the resonator modes at sites $i$ and $i + x_{\text{max}}$. Consider a quadrature component $X_i = a_i + a_i^\dagger$ for the resonator mode $a_i$. The correlation of the quadratures $\langle X_i \cdot X_j \rangle$ can be detected by measuring the amplitude of the microwave field of both resonators and making a statistical average on the measured quadrature products. Such measurement has been utilized to study photon coherence and correlation in recent experiments [14]. To achieve a faithful measurement of the many-body state, it requires that a single run during the measurement takes place in a time interval much shorter than the decoherence time of the qubits and the resonators. For a finite system with fixed number of excitations, $\langle a_i^\dagger a_j^\dagger \rangle \equiv 0$ and $\rho_1(i,j)$ is symmetric to $i$ and $j$. We then have $\langle X_i \cdot X_{i + x_{\text{max}}} \rangle = 2\rho_1(x_{\text{max}})$. As discussed in Sec. III A, $\rho_1(x_{\text{max}})$ carries the signature of the many-body phases and can be used to study the quantum phase transition.

VI. CONCLUSION

To conclude, stimulated by recent experimental progress in superconducting quantum devices, we studied the many-body phases of a multi-connected JC lattice model. We showed that a MI-SF-MI phase transition can be observed for cavity polaritons at commensurate filling. Different from the CCA model studied in previous works, our model embodies a symmetry with respect to the qubit-resonator couplings, which is at the root of the reentry to the MI phase. Our results for the single-particle density matrix and the energy gap confirm our analysis of an effective Hubbard interaction. Phase diagrams in the grand canonical ensemble are obtained, where the incompressibility of the MI phase is verified. We also studied the realization of this model with superconducting devices, presenting robust schemes for state preparation and detection. This model can be extended to two-dimensional qubit-resonator arrays and other more complicated configurations to study the many-body physics of microwave excitations. It also provides an interesting perspective to study the nonequilibrium dynamics of the cavity polaritons in this setup.

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APPENDIX A HUBBARD INTERACTION IN JC MODEL

With $g_1 = 0$, the multi-connected JC lattice is an array of isolated qubit-resonator systems each described by the JC model. The qubit-resonator coupling generates nonlinearity in the JC model. We connect this nonlinearity to an effective Hubbard interaction for the polaritons with a simple analysis.

The eigenstates $|n_i, \pm\rangle$ are the lower- and upper- polariton states with excitation number $n_i = \langle a_i^\dagger a_i \rangle$, and the state $|0\rangle$ contains no excitation. Note that the excitation number $n_i$ is a good quantum number in this model. We assume that the excitations fill the lower-polariton states only. Denote the energy to add $n_i$ excitations to this system as $\Delta \varepsilon_n = \varepsilon_{n,\downarrow} - \varepsilon_{0,\downarrow}$. We derive

$$\Delta \varepsilon_n = n_i \omega_c - \Delta/2 - \Omega_n(\Delta)/2$$

with $\Omega_n(\Delta) = \sqrt{\Delta^2 + 4g_c^2 n_i}$, using the expression for the eigenenergy in Sec. IIB.

![Figure A1](image)

**FIG. A1.** $E_{\Delta0}^n$ (solid) and effective Hubbard $U$ (dashed) versus $\Delta$. (a) $(g_1, g_2) = (5, 295)$ MHz; (b) $(25, 275)$ MHz; and (c) $(150, 150)$ MHz.

Assume that the lower-polariton states can be described by an effective Hamiltonian $H_{\text{eff}} = \omega_p p_i^\dagger p_i + (U/2)p_i^\dagger p_i p_j$, where $p_i$ is the annihilation operator of the polariton mode and $U$ is the strength of an onsite Hubbard interaction. Under this Hamiltonian, the energy of $n_i$ excitations is $\Delta \varepsilon_n = n_i \omega_p + Un_i(n_i - 1)/2$. For $n_i = 1$, $\Delta \varepsilon_n = \omega_p$. The effective interaction for $n_i$ and $n_i + 1$ excitations can then be derived.
as \( U = (\Delta \varepsilon_{n+1} - \Delta \varepsilon_n - \Delta \varepsilon_1)/n_i \). Combining this result with Eq. (A1), we find the effective Hubbard interaction for the JC model as

\[
U = \left[ \Delta - \Omega n_{n+1}(\Delta) + \Omega n_n(\Delta) + \Omega n_{1}(\Delta) \right]/2n_i, \tag{A2}
\]

depending on the coupling strength \( g_r \), the detuning \( \Delta \) and the excitation number \( n_i \). For the low-lying states \( |1, -\rangle \) and \( |2, -\rangle \), which correspond to the lower-polariton states with one and two excitations, we have

\[
U = \frac{\Delta}{2} + \sqrt{\Delta^2 + 4g_r^2} - \frac{1}{2} \sqrt{\Delta^2 + 8g_r^2}. \tag{A3}
\]

This result is different from that in previous works using similar analysis \([36, 40]\).

At \( \Delta = 0 \), \( U = (2 - \sqrt{2})g_r \), determined by the coupling \( g_r \). In the limiting case of \( |\Delta| \gg g_r \), \( U = (\Delta + |\Delta|)/2 \), i.e.,

\[
U = \begin{cases} 
0, & \Delta < 0; \\
\Delta, & \Delta > 0. 
\end{cases} \tag{A4}
\]

For large negative detuning, the effective interaction vanishes. This is because the lower-polariton states in this regime are approximately photon-number states with equal energy level spacing. For large positive detuning, the interaction increases with the detuning. This offers us a convincing explanation of the behavior of the energy gap at the filling factor \( N/M = 1 \). In Fig. A1, we plot the effective interaction \( U \) in comparison with the extrapolated energy gap \( E_{gp}^0 \) studied in Sec. III B. In the regime of \( g_l \ll g_r \) and \( g_r \ll g_i \), the effective \( U \) agrees very well with \( E_{gp}^0 \). This confirms the validity of our analysis for the effective interaction.

We want to emphasize that this simple analysis only gives us a rough picture of the effective onsite interaction in the JC model, which decreases with the excitation number \( n_i \). The JC model bears many properties that are distinctly different from that of the onsite Hubbard model.

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