Infinite-channel deep stable convolutional neural networks

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Abstract

The interplay between infinite-width neural networks (NNs) and classes of Gaussian processes (GPs) is well known since the seminal work of Neal (1996). While numerous theoretical refinements have been proposed in the recent years, the interplay between NNs and GPs relies on two critical distributional assumptions on the NN’s parameters: A1) finite variance; A2) independent and identical distribution (iid). In this paper, we consider the problem of removing A1 in the general context of deep feed-forward convolutional NNs. In particular, we assume iid parameters distributed according to a stable distribution and we show that the infinite-channel limit of a deep feed-forward convolutional NNs, under suitable scaling, is a stochastic process with multivariate stable finite-dimensional distributions. Such a limiting distribution is then characterized through an explicit backward recursion for its parameters over the layers. Our contribution extends results of Favaro et al. (2020) to convolutional architectures, and it paves the way to expand exciting recent lines of research that rely on classes of GP limits.

1. Introduction

Fully-connected neural networks (NNs) are defined by an interleaved application of affine transforms and non-linear functions evaluated element-wise. By associating a distribution to the parameters of a NN, it is natural to consider the NN as a probabilistic model. Modern NNs typically operate in the over-parametrized regime, with millions of parameters representing a standard setting. Consequently, the analysis of the properties of these highly-parametrized probabilistic NNs, as well as their inference, are highly non-trivial. As a way forward, the seminal work by Neal (1996) established the equivalence between a certain class of shallow probabilistic NNs and corresponding classes of limiting Gaussian processes (GPs) when the NN’s width, hence the dimensionality of its parameters, becomes infinite. It is thus possible to apply standard inference algorithms to the limiting GPs, which are seen as suitable approximations of corresponding overparameterized NNs. Similarly, analyzing the properties of the limiting GPs’ kernels can shed light on the behaviour of overparameterized NNs. Recently, the interplay between NNs and GPs has been extended to the context of deep fully-connected NNs (Lee et al., 2018; Matthews et al., 2018; Bracale et al., 2021) and of deep convolutional NNs (Novak et al., 2018; Garriga-Alonso et al., 2018).

Two common assumptions on the NN’s parameters’ distribution are at the basis of the interplay between NNs and GPs: A1) finite variance; A2) independent and identical distribution (iid). In this paper, we extend the main results of Novak et al. (2018) and Garriga-Alonso et al. (2018) to the more general assumption of iid parameters distributed according to a stable distribution, effectively removing A1. More precisely, we study the infinite-channel limit of CNNs under the following general setting: i) the CNN is deep, namely is composed of multiple layers; ii) biases and scaled weights are iid according to a centered symmetric stable distribution; iii) the number of convolutional channels in each network’s layers goes to infinity jointly on the layers; iv) the convergence in distribution is established jointly for multiple inputs, namely the convergence concerns the class of finite-dimensional distributions of the CNN viewed as a stochastic process. The use of stable distributions, which includes the Gaussian distribution as special case, is natural in this setting. Indeed, stable distributions are the most general distribution class toward which infinite sums of iid random variables can converge in distribution. See the monograph by Samoradnitsky & Taqqu (1994) for a detailed account on stable distributions.

Through this paper we define a deep CNN (DCNN) of the form

\[
\begin{align*}
    f^{(1)} &= W^{(1)} \ast x + b^{(1)} \\
    f^{(l)} &= W^{(L)} \ast \phi(f^{(l-1)}) + b^{(l)}, \quad l = 2, \ldots, L,
\end{align*}
\]

(1)

where \( l = 1, \ldots, L \) indexes the \( L \) layers, \( \{W^{(l)}\}_{l=1}^{L} \) are the weights, \( \{b^{(l)}\}_{l=1}^{L} \) are the biases and \( \phi \) is an activation function.
function. We show that the infinite-channel limit of the DCNN, under suitable scaling on the weights, is a stochastic process whose finite-dimensional distributions are multivariate stable distributions (Samorodnitsky & Taqqu, 1994). This limiting result extends the recent study of Favaro et al. (2020) to the more general context of convolutional architectures. The limiting process is referred to as the convolutional stable process (CSP). Our result contributes to the general theory of DCNNs, and it paves the way to extend some recent research directions that rely on classes of infinite-channel GP limits.

The paper is structured as follows. In Section 2 we introduce the notation and all the definitions that are preliminary to our main result. In Section 3 we define a DCNN jointly on K distinct inputs, and we specify the distributional assumptions on the model parameters. In Section 4 we compute the limiting distributions jointly over K inputs, whereas in Section 5 we establish the distribution, again over K inputs, arising from the projection of all spatial features to a single output vector (i.e. a readout layer). In Section 6 we briefly discuss our main result, and we present some concluding remarks.

2. Preliminaries

We denote by [n] the set \{1, \ldots, n\}, for any n ∈ N. For a fixed size S ∈ N, we call S-tensor of dimension \(D_S := D_1 \times \cdots \times D_S\), where \(D_j \in \mathbb{N}\) for each \(j \in [S]\), an element \(A \in \mathbb{R}^{D_S}\). For ease of notation, hereafter we write \([D_S] := [D_1] \times \cdots \times [D_S]\) and \([D_S] := \prod_{j=1}^{S} D_j\). For any \(d = (d_1, \ldots, d_S) \in [D_S]\), we denote by \(A_{d} \in \mathbb{R}\) the component of \(A\) at position \(d\) and the norm \(\|A\|_\alpha = \sum_{d \in [D_S]} |A_{d}|^\alpha\). In particular, we write \(A_{d(\cdot, \cdot, \cdot)}\) meaning the \((S-1)\)-tensor of dimension \(1 \times D_1 \times \cdots \times D_S\) consisting on the \(d_1\)-th position of the first dimension of \(A\) and all the other positions of \(A\). An integral with respect to \(d(A_{d(\cdot, \cdot, \cdot)})\) means an integral in the flattened tensor \(A\), that is with respect to \(d(\text{vec}(A))\), resulting an integral with \([D_S]\) variables of integration. \(1_{(D_S)}\) indicates the tensor of all 1s in \(\mathbb{R}^{D_S}\) and \(1_{([D_S])}(d)\) the tensor with 1 in the \(d\)-th entry and zero otherwise.

In order to give a compact definition of DCNNs compactly, we need to introduce three operations between tensors. The first operation is the known Frobenius (inner) product \(\otimes\) between two S-tensors \(A, B \in \mathbb{R}^{F}\). This is defined as follows

\[
A \otimes B = B \otimes A = \sum_{e \in [E]} A_e B_e \in \mathbb{R}.
\]

The second operation is denoted by \(\square\), and it is the square product under \(\diamond\), where \(\diamond\) is an operation that operates within tensors of the same size into \(\mathbb{R}\) (e.g. the Frobenius product). In particular, the square product under \(\diamond\) is the map

\[
\square : \mathbb{R}^{D_S} \times \mathbb{R}^{E_{s'}} \times \mathbb{R}^{E_{s'}} \rightarrow \mathbb{R}^{D_S},
\]

i.e.

\[
(A, B) \mapsto A \diamond B,
\]

where each position \(d \in [D_S]\) of \(A \diamond B\) is defined by \(A_{d(\cdot, \cdot)} B_{d(\cdot, \cdot)} \in \mathbb{R}\). We observe that when \(S = S' = 1\) the square product under the Frobenius operation, i.e. \(\square\), coincides precisely with the matrix-vector product and the Frobenius product degenerates into the Euclidean inner product, i.e. \(\otimes = \langle \cdot \rangle_{E_1}\). For ease of notation, hereafter we write \(\square\) to be \(\diamond\), and we refer to this operation as the square product.

Finally, the third operation is denoted by \(\triangle\), and we refer to it as the bias product, since it will be applied in biases. This is the map

\[
\triangle : \mathbb{R}^{D_S} \times \mathbb{R}^{E_{s'}} \rightarrow \mathbb{R}^{D_S} \times \mathbb{R}^{E_{s'}},
\]

i.e.

\[
(A, B) \mapsto A \triangle B,
\]

where \(A \triangle B = (\mathcal{A}B_{b})_{b \in [E_{s'}]}\). Operations \(\otimes, \sqrt[\alpha]{\times}, \square\) and \(\triangle\) are applied in Section 3 to give a compact definition of DCNN.

Given an arbitrary operation \(\bullet\) between tensors, when we specify dimensions as a superscript of \(\bullet\) it means that the operation \(\bullet\) is applied through all dimensions except for the dimensions specified in the superscript. For instance, \((P,K)\)

\[\square\]

is the square product applied through all dimensions except for the dimensions \(P\) and \(K\). Moreover, when we specify dimensions as a subscript of \(\bullet\) it means that the operation \(\bullet\) is applied only to the dimensions specified in the subscript. For instance, the operator \(\square\) is the square product applied only to the dimensions \(P\) and \(K\). Now, let fix \(0 < \alpha \leq 2\) and \(\sigma > 0\) and define the following stable distributions.

Definition 1. A \(\mathbb{R}\)-valued random variable \(A\) is distributed according to a stable distribution with index \(\alpha\), skewness \(\tau \in [-1,1]\), scale \(\sigma\) and shift \(\mu \in \mathbb{R}\), and we write \(A \sim \text{St}(\alpha, \tau, \sigma, \mu)\), if the characteristic function of \(A\) is of the form

\[
\varphi_A(t) = \mathbb{E}[e^{iT_A}] = e^{\psi(t)}, \quad t \in \mathbb{R}
\]

where

\[
\psi(t) = \begin{cases} 
-\sigma^\alpha |t|^\alpha [1 + i \sigma \tan(\frac{\pi \alpha}{2}) \text{sign}(t)] + i \mu & \alpha \neq 1 \\
-\sigma |t| [1 + i \frac{2}{\pi} \text{sign}(t) \log(|t|)] + i \mu & \alpha = 1
\end{cases}
\]

Property 1.2.16 of Samorodnitsky & Taqqu (1994) shows that if \(A \sim \text{St}(\alpha, \tau, \sigma, \mu)\) with \(0 < \alpha < 2\) then \(\mathbb{E}[|A|^r] < \infty\) for \(0 < r < \alpha\), and \(\mathbb{E}[|A|^\alpha] = \infty\) for \(r \geq \alpha\). Moreover by property 1.2.2 of Samorodnitsky & Taqqu (1994), \(\text{St}(\alpha, \tau, \sigma, \mu) \sim \sigma \text{St}(\alpha, \tau, 1, \mu)\).
Definition 2. A $\mathbb{R}$-valued random variable $A$ is distributed according to a symmetric $\alpha$-stable distribution with scale parameter $\sigma$, and we write $A \sim \text{St}(\alpha, \sigma)$, if it is stable with $\beta = \mu = 0$, i.e. the characteristic function of $A$ is of the form

$$\varphi_A(t) = E[e^{itA}] = e^{-\sigma |t|^{\alpha}}, \quad t \in \mathbb{R}$$

Definition 3. A $\mathbb{R}^{D_1}$-valued random vector $A$ is distributed according to the symmetric $D$-dimensional $\alpha$-stable distribution with scale (finite) spectral measure $\Gamma$ on the domain $S^{D-1}$ = $\{ z \in \mathbb{R}^D : \|z\| = 1 \}$, and we write $A \sim \text{St}_D(\alpha, \Gamma)$, if the characteristic function of $A$ is of the form

$$\varphi_A(t) = E[e^{i(t \cdot A)}] = e^{-\int_{S^{D-1}} |(t \cdot z)|^{\alpha} \Gamma(\text{d}z)}, \quad t \in \mathbb{R}^D$$

Definition 4. A $\mathbb{R}^{D_2}$-valued random $S$-tensor $A$ is distributed according to a symmetric $D_s$-dimensional $\alpha$-stable distribution with scale (finite) spectral measure $\Gamma$ on the domain $S^{D_s-1}$ = $\{ z \in \mathbb{R}^{D_s} : \|z\| = 1 \}$, and we write $A \sim \text{St}_{D_s}(\alpha, \Gamma)$, if the characteristic function of $A$ is of the form

$$\varphi_A(t) = E[e^{i(t \otimes A)}] = e^{-\int_{S^{D_s-1}} |(t \otimes z)|^{\alpha} \Gamma(\text{d}z)}, \quad t \in \mathbb{R}^{D_s}$$

where $\otimes$ is considered flattened.

3. Stable convolutional networks

3.1. Shallow CNN

Convolutions can be defined over an arbitrary number of spacial dimensions. Illustrative examples of convolutions are e.g., convolutions taking as input a time-series (1D), an image (2D) or a video (3D). Inputs to a convolution have an additional channel dimension: a time-series can be multivariate, the colour of an image at a given pixel is described by a 3-elements RGB vector. Hence the input to a convolution is a tensor $x \in \mathbb{R}^{C \times P_1 \times \cdots \times P_s}$ where $S$ is the number of spacial dimensions for each $s = 1, \ldots, S$, $P_s$ is the size of the $s$-th spacial dimension, and $C$ is the number of channels. For instance, for an RGB image $C = 3$, $S = 2$ and $P_1, P_2$ respectively represent the image width and height.

The defining property of a convolution is that the same collection of weights (filters) is applied to multiple patches extracted from the input tensor $x$. The filter size must thus agree with the extracted patches sizes. There is great flexibility in defining the specific details of a given convolutional transform, including its striding, padding, and dilation characteristics. See Dumoulin & Visin (2016), and references therein, for a detailed account. In this paper, we consider the following general setting. We define the filters $W \in \mathbb{R}^{C' \times C \times G_1 \times \cdots \times G_s}$ where, for each $s = 1, \ldots, S, G_s \leq P_s$, and the equality makes the network a $S$-dimensional fully connected NN) is the filter size across the $s$-th space dimension and $C'$ is the number of output channels. We also define a bias term $b \in \mathbb{R}^{C'}$. A convolution transform over $x$ results in an output tensor $y \in \mathbb{R}^{C' \times P'_1 \times \cdots \times P'_S}$ where $P'_s$ is the size of the $s$-th spacial output dimension. $P'_s$ depends on both $P_s$ and on the convolution characteristics. In particular, we write

$$y = W \ast x + b,$$

where the product $\ast$ is defined as follows. A patch extracted from $x$ at output position $p$ is $x_{*p} \in \mathbb{R}^{C \times G_1 \times \cdots \times G_S}$, $x_{*p} = x_{1:p_1; \cdots; S:p_S}$ where $*: p \mapsto sp$ is a function that depends on the moving window chosen in the structure and returns the positions of $x$ associated with the corresponding output position $p$. When extraction happens outside of $x$, i.e. when an input position $i$ is such that $i < 1$ or $i > P$, the padded values (often a constant) are taken as input. The convolution transform at output position $p$ is thus given by $y_p \in \mathbb{R}^{C'}$, $y_p = Wx_{*p} + b$. Finally $y \in \mathbb{R}^{C'}$ is obtained by stacking $y_p$ over the $P'_1 \times \cdots \times P'_S$ output positions, i.e. the equation (2) is defined as

$$y = Wx_{*p} + b_{(p)}.$$

As an illustrative example, we consider the simplest case of a 1D convolution with 1 input and output channels, a filter size of 3, unitary striding, and 0-padding of 1 position to the beginning and end of the input $x \in \mathbb{R}^{1 \times P}$. Then $W \in \mathbb{R}^{1 \times 1 \times 3}$, $b \in \mathbb{R}$ and $y_{1:p} = Wx_{1:p-1:p+1} + b$ for each $p = 1, \ldots, P'$, that is keeping in mind padding: $y_{1:1} = W[0, x_{1:2}] + b$, $y_{1:2} = Wx_{1:3} + b$, $\ldots$, $y_{1:P'} = W[ x_{1:P'-1:P'} , 0 ] + b$.

3.2. DCNN

By means of the definitions given in Section 2, we extend (3) to the more general setting of DCNNs. In particular, a DCNN is defined by means of multiple layers of convolutional transforms followed by the application of an activation function $\phi$, which is applied element-wise. Optionally skip connections are present in the case of residual networks and local/global max/average pooling frequently features in competitive convolutional architectures. For ease of exposition, hereafter we consider the case where all layers have the same number of channels $C^{(1)} = \cdots = C^{(L)} = C$. Now, let define

$$\begin{cases}
\mathbf{P}^{(l)} = \mathbf{P}^{(l)}_{S^{(l)}} = P_1^{(l)} \times \cdots \times P_S^{(l)} & l \in [L] \cup \{0\} \\
\mathbf{G}^{(l)} = \mathbf{G}^{(l)}_{S^{(l)-1}} = G_1^{(l)} \times \cdots \times G_{S^{(l)-1}} & l \in [L].
\end{cases}$$

Then, a plain DCNN of $L$ layers with single input and all positions $p^{(l)} \in [\mathbf{P}^{(l)}]$ is described by the following recur-
where we could have obtained the filters and the bias terms do not depend on each channel depending only on the first \( l \) channel for each layer and each channel depends only on the first \( (l-1) \) channels of the previous layer. For each \( l \in [L] \), \( f_{\ast(p(l))} \in \mathbb{R}^{C(l-1) \times G(l)} \) is a patch extracted from \( f^{(l-1)} \) at output position \( p(l) \in \mathbb{P}(l) \) and is defined by \( f_{\ast(p(l))} = f^{(l-1)}_{(l-1),(C(l-1))} \ast_{g(l)} \), where the patch operator \( \ast_{g(l)} : \mathbb{P}(l) \rightarrow (Z_1 \times \cdots \times Z_{G(l)})^{G(l)} \), for each \( p(l) \) returns \( g(l) \) that is a \( S(l-1) \)-dimension tensor of dimension \( G(l) \) containing some spatial indexes of \( f^{(l-1)} \) depending on the moving window chosen in the structure (padding, stride etc). More precisely, for each \( g(l) \in G(l) \), \( g(l) = (i_1, \ldots, i_{G(l)}) \in Z_1 \times \cdots \times Z_{G(l)} \) we could have \( i_{S(l-1)} < 1 \) or \( i_{S(l-1)} > P_{S(l-1)} \) for some \( S(l-1) \in [S(l-1)] \) because extraction could happens outside of \( f^{(l-1)} \). Hence, the following possibilities are allowed:

1) there exists \( s(l-1) \in [S(l-1)] \) such that \( i_{s(l-1)} < 1 \) or \( i_{s(l-1)} > P_{s(l-1)} \); in particular, in that case \( \left(f_{\ast(p(l))}^{(l-1)}\right)_{(i_{s(l-1)},g(l))} = f^{(l-1)}_{(i_{s(l-1)},g(l))} = 0 \) for each \( c(l-1) \in [C(l-1)] \);

2) \( i_{s(l-1)} \in [P_{s(l-1)}] \) for all \( s(l-1) \in [S(l-1)] \), and there exists \( p(l) \in \mathbb{P}(l) \) such that it holds true that \( g(l) = p(l) \); in particular, in that case \( \left(f_{\ast(p(l))}^{(l-1)}\right)_{(i_{s(l-1)},g(l))} = f^{(l-1)}_{(i_{s(l-1)},p(l))} = 0 \) for each \( c(l-1) \in [C(l-1)] \).

### 3.3. Explicit definition jointly for \( K \) inputs

Consider \( K \) inputs \( x^{(k)} \in \mathbb{R}^{C^{(k)} \times P^{(k)}} \) for \( k \in [K] \) and define \( x^{(1:K)} = (x^{(1)}, \ldots, x^{(K)})^T \in \mathbb{R}^{C^{(1)} \times P^{(1)} \times K} \). For each \( x^{(k)} \) the convolutional structure remains constant, that is the filters and the bias terms do not depend on \( k \). Since they do not even depend on the positions, we can rewrite (4) with \( K \) inputs as

\[
\begin{align*}
  f^{(0)}_{\ast(1:K)} &= x^{(1:K)}, \\
  f^{(1)}_{\ast(1:K)} &= f^{(l)}(x^{(1:K)}), \\
  f^{(l)}_{\ast(p(l))} &= W^{(l)} \ast_{g(l)} \left( f^{(l-1)}_{\ast(p(l))} \right) + b^{(l)} \square \mathbb{I}(p(l) \times K),
\end{align*}
\]

with \( f^{(l)}_{\ast(p(l))} \in \mathbb{R}^{\times P^{(l)} \times K} \) for each \( l \in [L] \), and where we have defined \( x^{(l)}_{\ast(K)}, g(l) \in \mathbb{R}^{C^{(l)} \times G^{(l)} \times P^{(l)} \times K} \) and \( f^{(l-1)}_{\ast(1:K)} \in \mathbb{R}^{C^{(l-1)} \times G^{(l-1)} \times P^{(l-1)} \times K} \) for \( l = 2, \ldots L \) as follows:

\[
\begin{align*}
  x^{(l)}_{\ast(1:K)} &= f^{(l)}_{\ast(p(l))}, \\
  f^{(l-1)}_{\ast(p(l))} &= f^{(l-1)}_{\ast(p(l))} \square \mathbb{I}(p(l) \times K).
\end{align*}
\]

### 3.4. Assumptions

Let the random weights \( W^{(l)}_{\ast(c(l)),(c(l-1)),g(l)} \) be independent of the biases \( b^{(l)}_{\ast(c(l)),g(l)} \) for each \( l \in [L], c(l) \geq 1 \) and \( g(l) \in G(l) \) and such that

\[
W^{(l)}_{\ast(c(l)),(c(l-1)),g(l)} \overset{\text{id}}{\sim} \text{St}(\alpha, \sigma_w), \quad b^{(l)}_{\ast(c(l)),g(l)} \overset{\text{id}}{\sim} \text{St}(\alpha, \sigma_b)
\]

Let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be a non-linearity with a finite number of discontinuities and such that it satisfies the envelope condition

\[
|\phi(s)| \leq a + b|s|^\beta
\]

for every \( s \in \mathbb{R} \) and some \( a, b > 0 \) and \( \beta < 1 \). For each \( l \in [L] \), we are interested in the limiting distribution of \( f^{(l)}_{\ast(1:K)} = f^{(l)}(x^{(1:K)}), g(l) \) as \( C \rightarrow \infty \), especially for \( l = L \). Let \( f^{(L)}_{\ast(1:K)} \in \mathbb{R}^{\times P^{(L)} \times K} \) be this limiting distribution. In particular, the distribution \( f^{(L)}_{\ast(1:K)} \) is the joint limiting distribution over all (infinite) channels, positions and inputs. As first step, fixed a channel \( c(l) \geq 1 \), we establish the limiting distribution of \( f^{(L)}_{\ast(c(l)),g(l)} \in \mathbb{R}^{P^{(L)} \times K} \). Note that fully connected NNs are recovered under the assumption \( P(l) = 1 \) for each \( l \in [L] \cup \{0\} \) and the patch extraction corresponds to the whole input for each convolutional transform.

### 4. Infinite-channel limits

For a fixed channel \( c(l) \geq 1 \), we compute the limiting distribution of \( f^{(L)}_{\ast(c(l)),g(l)} \) where, from (5), we can rewrite it as

\[
\begin{align*}
  f^{(0)}_{\ast(1:K)} &= x^{(1:K)}, \\
  f^{(1)}_{\ast(1:K)} &= f^{(l)}(x^{(1:K)}), \\
  f^{(l)}_{\ast(p(l))} &= W^{(l)} \ast_{g(l)} \left( f^{(l-1)}_{\ast(p(l))} \right) + b^{(l)} \square \mathbb{I}(p(l) \times K),
\end{align*}
\]
\[
\begin{aligned}
\begin{cases}
 f^{(1)(1;K)}_{(c^{(1)};z)} = W^1_{(c^{(1)};z)}(P^{(1)} K) x^1_{(1;K)} + b^{(1)} c_{(1;K)} + I_{P^{(1)} K}, \\
 f^{(l)(1;K)}_{(c^{(l)};z)} = \frac{1}{C^{(l)}} W^l_{(c^{(l)};z)}(P^{(l)} K) \phi(f^{(l-1)(1;K)}),
\end{cases}
\end{aligned}
\]
where the last equation holds for \( l = 2, \ldots, L \) and \( f^{(l)(1;K)}_{(c^{(l)};z)} \in \mathbb{R}^{P^{(l)} K} \) for each \( l \in [L] \). In order to establish the convergence of the finite-dimensional distributions, we rely on Theorem 5.3 of Kallenberg (2002) which takes into account the point-wise convergence of the corresponding characteristic functions. Hereafter, we denote by \( d \) the convergence in distribution (or weak convergence) of a sequence of random variables and by \( n \) the convergence in distribution (or weak convergence) of a sequence of probability measures. Moreover, let \( \xrightarrow{d} \) and \( \xrightarrow{a.s.} \) denote the convergence in probability and the almost sure convergence, respectively.

Our proof is an alternative to the Strong Law of Large Numbers for stable random variables. The key point of the proof lies in recognizing the exchangeability of the sequence \( f^{(1)(1;K)}_{(c^{(1)};z)} \) which allows us to apply the de Finetti theorem.

4.1. First layer

We start by establishing the distribution of \( f^{(1)(1;K)}_{(c^{(1)};z)} \), that is the distribution of a generic node in the first layer jointly on the positions and the inputs. In particular, for each \( l \in [L] \), let consider a function \( \Psi^{(l)} : \mathbb{R}^{P^{(l)} K} \rightarrow \mathbb{R} \) defined as follows

\[
\Psi^{(l)}(z) := \begin{cases}
\frac{1}{2} \delta\left(\frac{z}{\|y\|_2}\right) + \frac{1}{2} \delta\left(-\frac{z}{\|y\|_2}\right), & 0 \neq z \in \mathbb{R}^{P^{(l)} K} \\
0, & 0 = z \in \mathbb{R}^{P^{(l)} K}
\end{cases}
\]
where the function \( \delta \) denotes the standard Dirac delta function.

**Theorem 1.** \( f^{(1)(1;K)}_{(c^{(1)};z)} \sim Sf_{P^{(1)} K}(\alpha, \Gamma^{(1)}) \), where

\[
\Gamma^{(1)} = \|\sigma_b I_{P^{(1)} K}\|^\alpha \Phi^{(1)}(I_{P^{(1)} K}) + \sum_{(c^{(0)}, g^{(1)}) \in C^{(0)} G^{(1)}} \|\sigma_{c^{(0)}}(x_{(1;K)}^{(1)})_{(c^{(0)}, g^{(1)})}\|^\alpha \times \Phi^{(1)}\left(x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)})\right)
\]
where

\[
(x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)}) = \left[\left(x_{(1;K)}^{(k)}(c^{(0)}, g^{(1)})\right]_{(g^{(1)}, k) \in [P^{(1)} K]}\right.
\]

Here, we present a sketch of the proof of Theorem 1, and we defer to the Supplementary Material A for the complete proof.

**Proof.** We compute the characteristic function of \( f^{(1)(1;K)}_{(c^{(1)};z)} \).

For each \( l \in [L] \), each entry in \( f^{(l)(1;K)}_{(c^{(l)};z)} \) contains the same set of random variables \( \Phi^{(l)}_{(c^{(l)};z)} := \{W^{(l)}_{(c^{(l-1)}, g^{(l)})}, c^{(l-1)} \in [C^{(l-1)}], g^{(l)} \in [G^{(l)}]\} \cup \{(0, 0)\} \) and all the elements in \( \Phi^{(l)}_{(c^{(l)};z)} \) are iid by assumption. It means that \( \mathbb{E}[\exp\sum_{(c^{(0)}, g^{(1)})} f^{(1)(1;K)}_{(c^{(1)};z)}] = \prod_{(c^{(0)}, g^{(1)})} \mathbb{E}[\exp\ldots] \) and thus we obtain

\[
\begin{aligned}
\varphi(\Gamma^{(1)})(\{f^{(l)(1;K)}_{(c^{(l)};z)}\}) & = \mathbb{E}\left[\exp\left\{it^{(1)} \otimes f^{(1)(1;K)}_{(c^{(1)};z)}\right\}\right] \\
& = e^{-\sigma^{(0)}(t^{(1)} \otimes I_{P^{(1)} K})^\alpha} \times \prod_{(c^{(0)}, g^{(1)}) \in [C^{(0)} G^{(1)}]} e^{-\sigma^{(1)}(t^{(1)} \otimes (x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)})\right)^\alpha}.
\end{aligned}
\]

By rewriting

\[
\begin{aligned}
\|t^{(1)} \otimes I_{P^{(1)} K}\|^\alpha & = \|I_{P^{(1)} K}\|^\alpha \|t^{(1)} \otimes I_{P^{(1)} K}\|^\alpha \\
\|x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)})\|^\alpha & = \|x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)})\|^\alpha \|x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)})\|^\alpha
\end{aligned}
\]
we conclude by observing that \( \|\sigma_{c^{(0)}} I_{P^{(1)} K}\|^\alpha \) and \( \|x_{(1;K)}^{(1)}(c^{(0)}, g^{(1)})\|^\alpha \) belong to \( \mathbb{R}^{P^{(1)} K} \). This completes the proof.

4.2. Further layers conditioned on the previous one

Fix \( l = 2, \ldots, L \). We establish the distribution of the conditional random variable \( f^{(l)(1;K)}_{(c^{(l)};z)} | f^{(l-1)(1;K)}_{(c^{(l-1)};z)} \), that is the distribution of a generic node in a layer \( l \) conditioned on the \( C \) channels of the previous layer jointly on the positions and the inputs.

**Theorem 2.** For each \( l = 2, \ldots, L, \)

\[
f^{(l)(1;K)}_{(c^{(l)};z)} | f^{(l-1)(1;K)}_{(c^{(l-1)};z)} \sim Sf_{P^{(l)} K}(\alpha, \Gamma^{(l)}),
\]
where

\[
\Gamma^{(l)}_C = \|\sigma_b I_{P^{(l)} K}\|^\alpha \Phi^{(l)}(I_{P^{(l)} K}) + \frac{1}{C^{(l-1)}} \sum_{(c^{(l-1)}, g^{(l)}) \in [C \times G^{(l)}]} \|\sigma_{c^{(l-1)}} \times \phi(f^{(l-1)(1;K)}_{(c^{(l-1)};z)})\|^\alpha \times \Phi^{(l)}(\phi(f^{(l-1)(1;K)}_{(c^{(l-1)};z)})\right)
\]

Here, we present a sketch of the proof of Theorem 2, and we defer to the Supplementary Material B for the complete proof.
Thus we compute the limiting distribution for all the others $q^{(l-1)}$ with $C_{\Gamma}^\alpha$.

Here we omit the sketch of the proof of Theorem 2, as it is a step-by-step parallel of the proof of Theorem 1. The reader is referred to the Supplementary Material B for the full proof.

4.3. Unconditioned layers and limit as $C$ grows to infinity

For each $l \in [L]$, we determine the limiting distribution of $f^{(l)(1:K)}_{(e^{(l)}.; :)}$ as $C \to \infty$. First, observe that for $l = 1$, being $f^{(1)(1:K)}_{(e^{(1)}.; :)}$ referred only to the $C_0$ channels of the input layer, we have that $f^{(1)(1:K)}_{(e^{(1)}.; :)} \to S_{P^{(1)} \times \mathcal{K}}(\alpha, \Gamma_\infty^{(1)})$ constantly as $C \to \infty$, where we have defined $\Gamma_\infty^{(1)} = \Gamma^{(1)}$. Thus we compute the limiting distribution for all the others layers.

**Theorem 3.** For each $l = 2, \ldots, L$, $f^{(l)(1:K)}_{(e^{(l)}.; :)} \to f^{(l)(1:K)}_{\infty(e^{(l)}.; :)} \sim S_{P^{(l)} \times \mathcal{K}}(\alpha, \Gamma_\infty^{(l)})$ as $C \to \infty$, where

$$
\Gamma_\infty^{(l)} = \|\sigma_b 1_{(P^{(l)} \times \mathcal{K})}\|^\alpha \Psi^{(l)} \left( 1_{(P^{(l)} \times \mathcal{K})} \right) + \\
+ \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|\sigma_b \phi(f_{g^{(l)}})\|^\alpha \times
\Psi^{(l)} \left( \phi(f_{g^{(l)}}) \right) g^{(l-1)}(d f_{g^{(l)} \in \mathcal{G}^{(l)}})
$$

where $f_{g^{(l)}} \in \mathbb{R}^{P^{(l)} \times \mathcal{K}}$ for each $g^{(l)} \in \mathcal{G}^{(l)}$ and $g^{(l-1)} = S_{P^{(l-1)} \times \mathcal{K}}(\alpha, \Gamma_\infty^{(l-1)})$.

Here, we present a sketch of the proof of Theorem 3, and we defer to the Supplementary Material C for the complete proof.

**Proof.** Fix $l = 2, \ldots, L$ and, for each $C$, let $h^{(l)}_{\infty}$ denote the de Finetti random probability measure of the exchangeable sequence $(f^{(l)(1:K)}_{(e^{(l)}.; :)})_{e^{(l)}., \geq 1}$, i.e. $(f^{(l)(1:K)}_{(e^{(l)}.; :)})_{e^{(l)}., h^{(l)}_{\infty} \sim h^{(l-1)}_{\infty}}$. Consider the induction hypothesis that as $C \to \infty$

$$
h^{(l-1)}_{\infty} \to q^{(l-1)}
$$

where $q^{(l-1)} = S_{P^{(l-1)} \times \mathcal{K}}(\alpha, \Gamma_\infty^{(l-1)})$ and the finite measure $\Gamma_\infty^{(l-1)}$ will be specified. In order to compute the characteristic function of $f^{(l)(1:K)}_{(e^{(l)}.; :)}$ we follow the steps A)-D):

A) First we condition with respect to the first $C$ channels of the previous layer;

B) Step A) is useful because $f^{(l)(1:K)}_{(e^{(l)}.; :)}| f^{(l-1)(1:K)}_{(e^{(l-1)}.; :)}$ is well known by Theorem 2;

C) Then we condition with respect to $h^{(l-1)}_{\infty}$;

D) Step C) is useful because $(f^{(l-1)(1:K)}_{(e^{(l-1)}.; :)}| h^{(l-1)}_{\infty}) \to h^{(l-1)}_{\infty}$.

Now we show explicitly these steps. For $l > 1$ and any $t^{(l)} := f^{(l)(k)}_{(e^{(l)}.; :)} \in \mathbb{R}^{P^{(l)} \times \mathcal{K}}$, we can write that

$$
\varphi f^{(l)(1:K)}_{(e^{(l)}.; :)} (t^{(l)})
$$

$$
A) \quad \mathbb{E} \left[ \mathbb{E} \left[ \exp \left\{ i t^{(l)} \times f^{(l)(1:K)}_{(e^{(l)}.; :)} \right\} \left| f^{(l-1)(1:K)}_{(e^{(l-1)}.; :)} \right. \right] \right]
$$

$$
B) \quad \mathbb{E} \left[ \exp \left\{ - \int_{\mathbb{R}^{P^{(l)} \times \mathcal{K}}} \left| t^{(l)} \times f^{(l)(1:K)}_{(e^{(l)}.; :)} \right|^\alpha d s^{(l)} \right\} \right]
$$

$$
= \mathbb{E} \left[ \left( \exp \left\{ - \sigma^\alpha C^\alpha \times \sum_{e^{(l-1)}, h^{(l-1)}_{\infty}} \left| t^{(l)} \times \phi(f^{(l)(1:K)}_{(e^{(l-1))., :)}) \right|^\alpha \right\} \right) h^{(l-1)}_{\infty} \right]
$$

$$
C) \quad \mathbb{E} \left[ \left( \exp \left\{ - \sigma^\alpha C^\alpha \times \sum_{e^{(l-1)}, h^{(l-1)}_{\infty}} \left| t^{(l)} \times \phi(f^{(l)(1:K)}_{(e^{(l-1))., :)}) \right|^\alpha \right\} \right) h^{(l-1)}_{\infty} \right] C
$$

$$
D) \quad \mathbb{E} \left[ \left( \exp \left\{ - \sigma^\alpha C^\alpha \times \sum_{e^{(l-1)}, h^{(l-1)}_{\infty}} \left| t^{(l)} \times \phi(f^{(l)(1:K)}_{(e^{(l-1))., :)}) \right|^\alpha \right\} \right) h^{(l-1)}_{\infty} \right] C
$$

where $f_{g^{(l)}} \in \mathbb{R}^{P^{(l)} \times \mathcal{K}}$ for each $g^{(l)} \in \mathcal{G}^{(l)}$ and we have defined

$$
y_C(f_{g^{(l)}} \in \mathcal{G}^{(l)}) := \frac{\sigma^\alpha C}{C} \sum_{g^{(l)} \in \mathcal{G}^{(l)}} | t^{(l)} \times \phi(f_{g^{(l)}}) |^\alpha.
$$

Hereafter we show the limiting distribution of $f^{(l)(1:K)}_{(e^{(l)}.; :)}$ as $C \to \infty$. The following technical lemmas (Supplementary Material C) hold:

L1) $\sup_{C} \int \sum_{g^{(l)}} \| \phi(f_{g^{(l)}}) \|^\alpha h^{(l-1)}_{\infty}(d f_{g^{(l)}} \in \mathcal{G}^{(l)}) < \infty$;

L1.1) there exists $\epsilon > 0$ such that, $\sup_{C} \mathbb{E} \left[ \left| \left| \phi(f^{(l)(1:K)}_{(e^{(l-1)), g^{(l)}}}) \right|^{\alpha+\epsilon} \right| h^{(l-2)}_{\infty} \right] < \infty$;

L2) $\int \sum_{g^{(l)}} | t^{(l)} \times \phi(f_{g^{(l)}}) |^\alpha h^{(l-1)}_{\infty}(d f_{g^{(l)}} \in \mathcal{G}^{(l)}) \to \int \sum_{g^{(l)}} | t^{(l)} \times \phi(f_{g^{(l)}}) |^\alpha q^{(l-1)}(d f_{g^{(l)}) \in \mathcal{G}^{(l)})$ as $C \to \infty$. 






L3) \[ \sum_{g(1)} \phi(f(g(1)))^\alpha = \lim_{C \to \infty} \int_{g(2) \in [G(2)]^\alpha} \left( 1 + \sum_{g(1) \in [G(1)]^\alpha} \phi(f(g(1)))^\alpha \right) \]

Lemma L1 and L1.1 allow us to prove L2 and L3, while the latters will be used later. By Lagrange theorem, for any \( C \) there exists \( \theta_C \in (0,1) \) such that (9) can be rewritten as follows

\[ \varphi(\psi_{(c(1),c(2)),l}) (t^{(l)}) = e^{-\sigma^2 \psi_{t^{(l)}}(n)} \frac{1}{1 + \sum_{g(1) \in [G(1)]^\alpha} \phi(f(g(1)))^\alpha} \]

The last integral tends to 0 in probability as \( C \to \infty \) by L3. Finally, by using the definition of the exponential function \( e^x = \lim_{n \to \infty} (1 + x/n)^n \), and by means of L2, we get, as \( C \to \infty \)

\[ \varphi(\psi_{(c(1),c(2)),l}) (t^{(l)}) \rightarrow \exp \left\{ -\sigma^2 \| (p^{(l)} \times K) \|^{\alpha} \right\} \]

Finally, we obtain the weak limit as \( \xi \rightarrow \infty \) of \( C \).

Theorem 4. For each \( l \in [L] \), \( f^{(l)}_{(c(1),c(2)),l} = f^{(l)}_{(c(1),c(1)),l} \phi_{c(1),c(2)} \) is \( \mathbb{S}_{\mathbb{P}_{K}}^{c(1) \times K}(\alpha, \Gamma^{(l)}_{\infty}) \), where the spectral measure \( \Gamma^{(l)}_{\infty} \) is

\[ \Gamma^{(l)}_{\infty} = \| (p^{(l)} \times K) \|^{\alpha} \phi(t^{(l)}) \left( \mathbb{I}_{(p^{(l)} \times K)} \right) + \int_{g(1) \in [G(1)]^\alpha} \sigma^2 \phi(f(g(1)))^\alpha \times \Psi(t^{(l)} \phi(f(g(1)))) \left( (d f(g(1)) \in [G(1)]^\alpha) \right). \]

This completes the proof.
where

\[ \Delta_{\infty}^{(l)} = \|z\|_\alpha \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|\sigma_\omega \phi(f_{g^{(l)}})\|_\alpha \times \Psi^{(l)}(\phi(f_{g^{(l)}}))q^{(l-1)}(d f_{g^{(l)}} \in \mathcal{G}^{(l)}) \].

This completes the proof. \( \Box \)

5. Readout layer on positions

Focus the attention on the last layer \( L \). We found that \( f^{(L)}(x^{(1):K}) \rightarrow f^{(L)}_{\infty}(x^{(1):K}) \) that is a convergence of a sequence of \( \mathbb{R}^{\infty \times p^{(L)} \times K} \) valued random variables. To gather information on the positions \( p^{(L)} \) we consider a linear combination with respect to \( p^{(L)} \), i.e. we project the \( \infty \times p^{(L)} \times K \) dimensional vectors \( f^{(L)}(x^{(1):K}) = f^{(L)}(x^{(1):K}), C \) into a \( \infty \times K \) dimensional vectors, and we take the limit as \( C \rightarrow \infty \). More precisely, for \( l \in [L] \), fix \( u \in \mathbb{R}^{p^{(l)}} \) such that \( u \otimes I_{p^{(l)}}(1) = 1 \) and define the transformation \( T_{u^{(l)}} : \mathbb{R}^{\infty \times p^{(l)} \times K} \rightarrow \mathbb{R}^{\infty \times K}, (a, b, c) \mapsto (a, u \otimes b, c) \). In other words \( T_{u^{(l)}}(l) \equiv u \otimes \cdot \). We want to establish the convergence of \( T_{u^{(l)}}(l) f^{(1):(1):K} \) as \( C \rightarrow \infty \).

For \( l = 1 \) we get

\[
T_{u^{(1)}}(l) f^{(1):(1):K}) = T_{u^{(1)}}(W^{(1)}(p^{(1):K}) \otimes x^{(1):K} + b^{(1)} \triangle I_{p^{(1)} \times K})
\]

and, for \( l > 1 \),

\[
T_{u^{(l)}}(l) f^{(1):(1):K}) = T_{u^{(l)}}(\frac{1}{C^{1/\alpha}} W^{(l)}(p^{(l):K}) \otimes \phi(f_{\star}^{(l-1):(1):K})) + b^{(1)} \triangle I_{p^{(1)} \times K})
\]

and, for \( l > 1 \),

\[
T_{u^{(l)}}(l) f^{(1):(1):K}) = T_{u^{(l)}}(\frac{1}{C^{1/\alpha}} W^{(l)}(p^{(l):K}) \otimes \phi(f_{\star}^{(l-1):(1):K})) + b^{(1)} \triangle I_{p^{(1)} \times K})
\]

Then, by following the same steps of Theorem 4 we obtain that

\[
T_{u^{(l)}}(l) f^{(1):(1):K}) \rightarrow \bigotimes_{e^{(l)} = 1}^{\infty} \text{St}_K(\alpha, \Gamma_{\infty}^{(l)}(u)),
\]

where

\[
\Gamma_{\infty}^{(l)}(u) = \|\sigma_b \ I_{(K)} \|^* \Psi^{(l)}(I_{(K)}) + \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|\sigma_\omega u \otimes \phi(f_{g^{(l)}})\|_\alpha \times \delta^{(l)}(u \otimes \phi(f_{g^{(l)}}))q^{(l-1)}(d f_{g^{(l)}} \in \mathcal{G}^{(l)})
\]

and with the function \( \delta^{(l)} : \mathbb{R}^K \rightarrow \mathbb{R} \) being defined as follows

\[
\delta^{(l)}(z) := \begin{cases} \frac{1}{2} \delta\left(\frac{z}{\pi \|\|}\right) + \frac{1}{2} \delta\left(-\frac{z}{\pi \|\|}\right) & 0 \neq z \in \mathbb{R}^K \\ 0 & z = 0 \in \mathbb{R}^K \end{cases}
\]

and \( f_{g^{(l)}} \in \mathbb{R}^{p^{(l)} \times K} \) and \( q^{(l)} = \text{St}_{p^{(l)} \times K}(\alpha, \Gamma_{\infty}^{(l)}) \) for \( l \in [L] \), being \( \Gamma_{\infty}^{(l)} \) defined in theorem 3.

6. Discussion

In this paper, we have showed that an infinite-channel deep convolutional NN, under suitable scaling of NN’s parameters, defines a stochastic process whose finite-dimensional distributions are multivariate stable distributions. The limiting distribution can be evaluated by an explicit backward recursion for its parameters over the layers of the NN. Moreover, we have established the finite-dimensional distributions arising from the NN readout layer. Our contribution may be viewed as a generalization of Novak et al. (2018) and Garriga-Alonso et al. (2018) to the context of iid parameters distributed as a stable distributions, or as a generalization of Favaro et al. (2020) to the context of convolutional architectures.

Our study paves the way to several interesting directions for future work, both on the theoretical and methodological side. First, our main result are at the basis to establish a neural tangent kernel limit (Arora et al., 2019) for stable distributed NN’s parameters in convolutional architectures. Second, our study is limited to the finite-dimensional distributions of the NN layers. Formally, this is not sufficient in order to guarantee the convergence of the NN as a random function of the input space, i.e. to establish a functional limit theorem for the NN. In this respect, further effort is needed to extend the functional limit theorem of Bracale et al. (2021) in the setting of stable distributions for NN’s parameters and convolutional architectures for any \( 0 < \alpha < 2 \). In particular, a functional perspective of our result would provide estimates on the smoothness proprieties of the limiting stochastic processes. Finally, as in the work of Favaro et al. (2020), it remains open the problem of devising efficient inference algorithms that allows to apply the stochastic processes introduced in this paper to current computer vision problems.
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Supplementary Materials

During the proofs we will use without any mention the following inequality:

**Lemma 1.** For any real values \( \alpha, z_1, \ldots, z_n \geq 0 \) there exists a constant \( C = C(\alpha, n) \) such that

\[
(z_1 + \cdots + z_n)^\alpha \leq C(z_1^\alpha + \cdots + z_n^\alpha)
\]

**Proof.** Let \( Z = \max\{z_1, \ldots, z_n\} \). Thus we get

\[
(z_1 + \cdots + z_n)^\alpha \leq (nZ)^\alpha \leq n^\alpha (z_1^\alpha + \cdots + z_n^\alpha)
\]

In particular \( C = C(\alpha, n) = n^\alpha \).

We give an important intuition of the reason why the proofs that will follow work. Intuitively we will provide an alternative proof of the strong law of large numbers for stable random variables using the de Finetti theorem regarding the exchangeability of sequences of random variables. To this end we will have to require that the expected value of the stochastic process is finite. Using the above Lemma 1, from assumption (7) we get

\[
|\phi(s)|^\alpha \leq (a + b|s|^\beta)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha |s|^{\beta \alpha})
\]

When \( s \) is \( \alpha \) stable distributed with any skewness, scale and shift parameters, then

\[
\mathbb{E}[|\phi(s)|^\alpha] \leq 2^\alpha a^\alpha + 2^\beta b^\alpha \mathbb{E}[|s|^{\beta \alpha}] < \infty
\]

which is finite since \( \beta < 1 \) thus \( \beta \alpha < \alpha \). Then, the assumption \( \beta < 1 \) is essential to guarantee the existence of the expected value of the stochastic process. However, this assumption also allows us to apply Jensen’s inequality in the following sense: for any positive random variable \( s \),

\[
\mathbb{E}[s^\beta] \leq (\mathbb{E}[s])^\beta
\]

We will use these inequalities repeatedly during the proofs.

**Supplementary Material A**

We prove that \( f_{(c^{(1)}, \cdot)}^{(1)1} \sim \text{St}_{\mathbf{P}^{(1)} \times K}(\alpha, \Gamma^{(1)}) \), where

\[
\Gamma^{(1)} = \|\sigma_\beta \mathbb{I}_{\mathbf{P}^{(1)} \times K} \|^\alpha \Psi^{(1)}(\mathbb{I}_{\mathbf{P}^{(1)} \times K}) + \sum_{(c^{(0)}, g^{(1)}) \in [C^{(0)} \times G^{(1)}]} \|\sigma_\omega(x^{(1)K}_{\cdot})(x^{(0)}, g^{(1)})\|^\alpha \Psi^{(1)} \left( x^{(1)K}_{\cdot}(x^{(0)}, g^{(1)}) \right)
\]
Proof. For $l = 1$, from definition (8) and assumption (6) we have that, for any $t^{(1)} := \{t^{(1)(k)}_{p^{(1)}} \mid \{(p^{(1)}, k) \in p^{(1)} \times K\} \in \mathbb{R}^{p^{(1)} \times K}$,

\[
\varphi_{f^{(1)(1;K)}_{c^{(1)}}}(t^{(1)}) = \mathbb{E} \left[ \exp \left\{ i t^{(1)} \otimes f^{(1)(1;K)}_{c^{(1)}} \right\} \right] = \mathbb{E} \left[ \exp \left\{ i t^{(1)} \otimes \left( W^{(1)}_{c^{(1)}, c^{(1)}} (p^{(1)}, K) \otimes x^{(1;K)} + \delta^{(1)}_{c^{(1)}} I_{p^{(1)} \times K} \right) \right\} \right] = \mathbb{E} \left[ \exp \left\{ i W^{(1)}_{c^{(1)}, c^{(1)}} (p^{(1)}, K) \otimes \left( t^{(1)} (c^{(0)}, G^{(1)}) \otimes x^{(1;K)} \right) + \delta^{(1)}_{c^{(1)}} t^{(1)} \otimes I_{p^{(1)} \times K} \right\} \right] = \mathbb{E} \left[ \exp \left\{ \delta^{(1)}_{c^{(1)}} t^{(1)} \otimes I_{p^{(1)} \times K} \right\} \right] \times \prod_{(e^{(0)}, g^{(1)}) \in [C^{(0)} \times G^{(1)}]} \mathbb{E} \left[ \exp \left\{ i W^{(1)}_{e^{(1)}, e^{(0)}, g^{(1)}} (t^{(1)} \otimes (x^{(1;K)}))_{(e^{(0)}, g^{(1)})} \right\} \right] = e^{- \sigma^{(1)}_{c^{(1)}} t^{(1)} \otimes I_{p^{(1)} \times K}} \prod_{(e^{(0)}, g^{(1)}) \in [C^{(0)} \times G^{(1)}]} e^{- \sigma^{(1)}_{e^{(0)}, g^{(1)}} t^{(1)} \otimes (x^{(1;K)}_{e^{(0)}, g^{(1)}})} = \exp \left\{ - \sigma^{(1)}_{c^{(1)}} t^{(1)} \otimes I_{p^{(1)} \times K} \right\}^{\alpha} - \sigma^{(1)}_{c^{(1)}} \sum_{(e^{(0)}, g^{(1)}) \in [C^{(0)} \times G^{(1)}]} \left| t^{(1)} \otimes (x^{(1;K)}_{e^{(0)}, g^{(1)}}) \right|^{\alpha} + \sigma^{(1)}_{c^{(1)}} \sum_{(e^{(0)}, g^{(1)}) \in [C^{(0)} \times G^{(1)}]} \left| (x^{(1;K)}_{e^{(0)}, g^{(1)}}) \right|^{\alpha} \right\} = \exp \left\{ - \int_{\mathbb{R}^{p^{(1)} \times K}} \left| t^{(1)} \otimes s^{(1)} \right|^{\alpha} \Gamma^{(1)}(1) (ds^{(1)}) \right\}
\]

where $(x^{(1;K)}_{e^{(0)}, g^{(1)}}) = \left[ (x^{(k)}_{e^{(1)}, e^{(0)}, g^{(1)}}) \right]_{\{(e^{(1)}, k) \in p^{(1)} \times K\}}$ and

\[
\Gamma^{(1)} = \left\| \sigma_{c^{(1)}} I_{p^{(1)} \times K} \right\|^{\alpha} \left( I_{p^{(1)} \times K} \right) + \sigma^{(1)}_{c^{(1)}} \sum_{(e^{(0)}, g^{(1)}) \in [C^{(0)} \times G^{(1)}]} \left| (x^{(1;K)}_{e^{(0)}, g^{(1)}}) \right|^{\alpha} \Psi^{(1)}_{c^{(1)}} \left( (x^{(1;K)}_{e^{(0)}, g^{(1)}}) \right)
\]

\[\square\]

Supplementary Material B

We prove that for each $l = 2, \ldots, L$, $f^{(l)(1;K)}_{c^{(l)}, c^{(l-1)}} \sim \text{St}_{p^{(l)} \times K}(\alpha, \Gamma^{(l)})$, where

\[
\Gamma^{(l)}_C = \left\| \sigma_{c^{(l)}} I_{p^{(l)} \times K} \right\|^{\alpha} \Psi^{(l)}_{c^{(l)}} \left( I_{p^{(l)} \times K} \right) + \frac{1}{C} \sum_{(c^{(l-1)}, g^{(l)}) \in [C \times G^{(1)}]} \left\| \sigma_{c^{(l)}} \phi^{(l)(1;K)}_{c^{(1)}, c^{(l-1)}} \right\|^{\alpha} \Psi^{(l)}_{c^{(l)}} \left( \phi^{(l)(1;K)}_{c^{(1)}, c^{(l-1)}} \right)
\]

Proof. For $l \geq 2$, from definition (8) and assumption (6) we have that, for any $t^{(l)} := \{t^{(l)(k)}_{p^{(l)}} \mid \{(p^{(l)}, k) \in p^{(l)} \times K\} \in \mathbb{R}^{p^{(l)} \times K}$, it holds...
$\varphi(f^{(l-1)(1)}) (C)$

Thus we compute the limit for all the others layers. We prove that for each $l$,

$$
\varphi \left( f^{(l-1)(1)} \right) (C) \to E \left[ \exp \left\{ \frac{1}{C^l} W^{l} (C) \otimes \phi(f^{(l-1)(1)}) + b^{(l)} \right\} \right]
$$

$$
= E \left[ \exp \left\{ \frac{1}{C^l} W^{l} (C) \otimes \phi(f^{(l-1)(1)}) + b^{(l)} \right\} \right]
$$

$$
= E \left[ \exp \left\{ \frac{1}{C^l} W^{l} (C) \otimes \phi(f^{(l-1)(1)}) + b^{(l)} \right\} \right]
$$

$$
= E \left[ \exp \left\{ \frac{1}{C^l} W^{l} (C) \otimes \phi(f^{(l-1)(1)}) + b^{(l)} \right\} \right]
$$

$$
= \prod_{(l-1),g^{(l)}} E \left[ \exp \left\{ \frac{1}{C^l} W^{l} (C) \otimes \phi(f^{(l-1)(1)}) + b^{(l)} \right\} \right]
$$

$$
= e^{-s^a_t (t_0) \otimes \mathbb{1}_{P^{(0) \times K}}} \prod_{(l-1),g^{(l)}} e^{-s^a_t (t_0) \otimes \mathbb{1}_{P^{(0) \times K}}} \left| t^{(l)} \otimes \phi(f^{(l-1)(1)}) (C, g^{(l)}) \right|^a
$$

$$
= \prod_{(l-1),g^{(l)}} \exp \left\{ - s^a_t (t_0) \otimes \mathbb{1}_{P^{(0) \times K}} \right\} - \frac{s^a_t (t_0) \otimes \mathbb{1}_{P^{(0) \times K}}}{C}
$$

$$
= \prod_{(l-1),g^{(l)}} \exp \left\{ - s^a_t (t_0) \otimes \mathbb{1}_{P^{(0) \times K}} \right\} - \frac{s^a_t (t_0) \otimes \mathbb{1}_{P^{(0) \times K}}}{C}
$$

$$
= \prod_{(l-1),g^{(l)}} \exp \left\{ \int_{S^{(l) \times K}} [t^{(l)} \otimes s^{(l)}] \sigma^C(l) (d s^{(l)}) \right\}
$$

$$
\Gamma^C = \|s_0 \mathbb{1}_{P^{(0) \times K}}\|^{\alpha \Psi(l)} (\mathbb{1}_{P^{(0) \times K}}) +
$$

$$
+ \frac{1}{C} \sum_{(l-1),g^{(l)}} \|s_0 \phi(f^{(l-1)(1)}) (C, g^{(l)})\|^{\alpha \Psi(l)} (\phi(f^{(l-1)(1)}) (C, g^{(l)}))
$$

$$
\text{with} \ (f^{(l-1)(1)}) = \left[ (f^{(l-1)(k)}) \right]_{(l),k \in \{P^{(0) \times K}\}}.
$$

**Supplementary Material C**

Here we compute the limit distribution of $f^{(l)} (C)$ as $C \to \infty$. Note that for $l = 1$, being $f^{(1)(1)}$ referred only to the $C^{(0)}$ channels of the input layer, then $f^{(1)(1)} (C) \to \text{St}_{P^{(0) \times K}} (\alpha, \Gamma^{(1)})$ constantly as $C \to \infty$, where we have defined $\Gamma^{(1)} = \Gamma^{(1)}$. Thus we compute the limit for all the others layers. We prove that for each $l = 2, \ldots, L$, $f^{(l)(1)} (C) \to \text{St}_{P^{(0) \times K}} (\alpha, \Gamma^{(l)})$ as $C \to \infty$, where

$$
\Gamma^{(l)} = \|s_0 \mathbb{1}_{P^{(0) \times K}}\|^{\alpha \Psi(l)} (\mathbb{1}_{P^{(0) \times K}}) +
$$

$$
+ \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|s_0 \phi(f^{(l)(k)})\|^{\alpha \Psi(l)} (\phi(f^{(l)(k)})) q^{(l-1)} (d f^{(l)(g)} \in \mathcal{G}^{(l)})
$$

$$
\text{with} \ f^{(l)(k)} \in \mathbb{R}^{P^{(0) \times K}} \text{ for each } g^{(l)} \in \mathcal{G}^{(l)} \text{ and } q^{(l-1)} = \text{St}_{P^{(l-1) \times K}} (\alpha, \Gamma^{(l-1)}).
For the proof we will need the following proposition which is a direct consequence of Exercise 2.3.4 of (Samoradnitsky & Taqqu, 1994).

**Proposition 1.** If \( A \sim St_D(\alpha, \Gamma) \) then for each \( u \in \mathbb{R}^D \) the 1-dimensional r.v. \( \langle u, A \rangle \sim St(\alpha, \tau(u), \sigma(u), \mu(u)) \) where

\[
\sigma(u) = \left( \int_{S^{D-1}} |\langle u, s \rangle|^\alpha \Gamma(ds) \right)^{1/\alpha} \\
\tau(u) = \sigma(u)^{-1} \int_{S^{D-1}} |\langle u, s \rangle| \text{sign}(\langle u, s \rangle) \Gamma(ds) \\
\mu(u) = \begin{cases} 
0 & \alpha \neq 1 \\
-\frac{2}{\alpha} \int_{S^{D-1}} \langle u, s \rangle \log(\|\langle u, s \rangle\|) \Gamma(ds) & \alpha = 1
\end{cases}
\]

Proof. Fix \( l = 2, \ldots, L \) and, for each \( C \), let \( h^{(l)}_C \) denote the de Finetti random probability measure of the exchangeable sequence \( (f^{(l)}_*(1:K))_{(c^{(l)},\epsilon)} \), i.e. \( (f^{(l)}_*(1:K))_{(c^{(l)},\epsilon)} h^{(l)}_C \overset{iid}{\sim} h^{(l)}_C \). Consider the induction hypothesis that as \( C \to \infty \)

\[
h^{(l-1)}_C \overset{w}{\to} q^{(l-1)}
\]

where \( q^{(l-1)} = St_{p^{(l-1)} \times K}(\alpha, \Gamma^{(l-1)}) \) and the finite measure \( \Gamma^{(l-1)} \) will be specified. For \( l > 1 \) and any \( t^{(l)} := [t^{(l)}_p, t^{(l)}_g] \in \mathbb{R}^{p^{(l)} \times K} \),

\[
\begin{align*}
\hat{q}^{(f^{(1)}(1:K))_{(c^{(l)},\epsilon)}}(t^{(l)}) &= E \left[ \exp \left\{ i t^{(l)} \otimes f^{(1)}(1:K) \right\} \right] \\
&= E \left[ \exp \left\{ i t^{(l)} \otimes f^{(1)}(1:K) \right\} f^{(l-1)}(1:K) \right] \\
&= E \left[ \exp \left\{ -\int_{|p^{(l)} \times K|} |t^{(l)} \otimes s^{(l)}| \alpha \Gamma^{(l)}_C (ds^{(l)}) \right\} \right] \\
&= \exp \left\{ -\sigma^\alpha C \sum_{(c^{(l)},\epsilon),g^{(l)}} |t^{(l)} \otimes \phi(f^{(l-1)}(1:K))_{(c^{(l)},\epsilon),g^{(l)})| \alpha \right\} \\
&\times \left[ \exp \left\{ -\sigma^\alpha C \sum_{(c^{(l)},\epsilon),g^{(l)}} |t^{(l)} \otimes \phi(f^{(l-1)}(1:K))_{(c^{(l)},\epsilon),g^{(l)})| \alpha \right\} \right] \left( h^{(l-1)}_C \right) \\
&\times \left[ \left( \int_{|g^{(l)}|} \exp \left\{ -\sigma^\alpha C \sum_{g^{(l)}} |t^{(l)} \otimes \phi(f^{(l)}_g)| \alpha \right\} h^{(l-1)}_C (dg^{(l)}) \right) C \right] \\
\end{align*}
\]

where \( f^{(l)}_g \in \mathbb{R}^{p^{(l)} \times K} \) for each \( g \in G^{(l)} \). Hereafter we show the limiting behaviour. In order to do this we need the following lemmas:

L1) For each \( l = 2, \ldots, L \), \( \sup_C \int \sum_{g \in G^{(l)}} \| \phi(f^{(l)}_g) \| \alpha h^{(l-1)}_C (dg) < \infty \)

L1.1) There exists \( \epsilon > 0 \) such that, \( \sup_C E[|\sum_{g \in G^{(l)}} \| \phi(f^{(l-1)}(1:K))_{(c^{(l)},\epsilon),g^{(l)})\|^{\alpha+\epsilon}|h^{(l-2)}_C| < \infty \) for each \( l = 2, \ldots, L \)

L2) \( \int \sum_{g \in G^{(l)}} |t^{(l)} \otimes \phi(f^{(l)}_g)| \alpha h^{(l-1)}_C (dg) \overset{p}{\to} \int \sum_{g \in G^{(l)}} |t^{(l)} \otimes \phi(f^{(l)}_g)| \alpha q^{(l-1)}(dg) \) as \( C \to \infty \)

L3) \( \int \sum_{g \in G^{(l)}} \| \phi(f^{(l)}_g) \| \alpha [1 - \exp\left\{ -\frac{\alpha}{C} \sum_{g \in G^{(l)}} |t^{(l)} \otimes \phi(f^{(l)}_g)| \alpha \right\}] h^{(l-1)}_C (dg) \overset{p}{\to} 0 \) as \( C \to \infty \)
.1. Proof of L1

For \( l = 2 \), for each \( c^{(1)} \geq 1 \), from assumptions (6) and (7) and from Lemma 1 we get

\[
\mathbb{E} \left[ \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \| \phi(f^{(1)}_{c^{(1)}, g^{(2)}}) \|^{\alpha} \right] \\
= \mathbb{E} \left[ \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathcal{P}^{(2)}} \sum_{k \in [K]} \| \phi(f^{(1)}_{c^{(1)}, g^{(2)}}) \|^{\alpha} \right] \\
= \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathcal{P}^{(2)}} \sum_{k \in [K]} \mathbb{E} \left[ \| \phi(f^{(1)}_{c^{(1)}, g^{(2)}}) \|^{\alpha} \right] \\
\leq \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathcal{P}^{(2)}} \sum_{k \in [K]} \mathbb{E} \left[ (a + b)(f^{(1)}_{c^{(1)}, g^{(2)}})^{\beta} \right]^{\alpha} \\
\leq 2^\alpha \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathcal{P}^{(2)}} \sum_{k \in [K]} \mathbb{E} \left[ (f^{(1)}_{c^{(1)}, g^{(2)}})^{\alpha \beta} \right] \\
< \infty
\]

where we used that \( (f^{(1)}_{c^{(1)}, g^{(2)}})_{c^{(1)}, g^{(2)}} \), by Proposition 1 is distributed according to a stable distribution with index \( \alpha \) (and some skewness, scale and shift parameters) and then, being \( \alpha \beta < \alpha \), \( \mathbb{E} \left[ (f^{(1)}_{c^{(1)}, g^{(2)}})^{\alpha \beta} \right] < +\infty \). Now assuming that L1 is true for \( l - 2 \) we prove that it is true for \( l - 1 \). First, from assumptions (6) and (7) and from Lemma 1, we compute the following

\[
\mathbb{E} \left[ \sum_{g^{(1)} \in \mathcal{G}^{(1)}} \| \phi(f^{(l-1)}_{c^{(l-1)}, g^{(1)}}) \|^{\alpha} \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right] \right] \\
= \mathbb{E} \left[ \sum_{g^{(1)} \in \mathcal{G}^{(1)}} \sum_{p^{(1)} \in \mathcal{P}^{(1)}} \sum_{k \in [K]} \| \phi(f^{(l-1)}_{c^{(l-1)}, g^{(1)}}) \|^{\alpha} \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right] \right] \\
= \sum_{g^{(1)} \in \mathcal{G}^{(1)}} \sum_{p^{(1)} \in \mathcal{P}^{(1)}} \sum_{k \in [K]} \mathbb{E} \left[ \| \phi(f^{(l-1)}_{c^{(l-1)}, g^{(1)}}) \|^{\alpha} \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right] \right] \\
\leq \sum_{g^{(1)} \in \mathcal{G}^{(1)}} \sum_{p^{(1)} \in \mathcal{P}^{(1)}} \sum_{k \in [K]} \mathbb{E} \left[ (a + b)(f^{(l-1)}_{c^{(l-1)}, g^{(1)}})^{\beta} \right]^{\alpha} \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right] \\
\leq 2^\alpha \| \mathcal{G}^{(1)} \| \| \mathcal{P}^{(1)} \| K \alpha^\alpha + 2^\alpha b^\alpha \sum_{g^{(1)} \in \mathcal{G}^{(1)}} \sum_{p^{(1)} \in \mathcal{P}^{(1)}} \sum_{k \in [K]} \mathbb{E} \left[ (f^{(l-1)}_{c^{(l-1)}, g^{(1)}})^{\alpha \beta} \right] \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right]
\]

Recall that for each \( p^{(l-1)} \in \mathcal{P}^{(l-1)} \), \( (f^{(l-1)}_{c^{(l-1)}, g^{(1)}})_{c^{(l-1)}, g^{(1)}} \) could be equal to 0 or there exists an unique position \( p^{(l-1)} \in \mathcal{P}^{(l-1)} \) such that \( (f^{(l-1)}_{c^{(l-1)}, g^{(1)}})_{c^{(l-1)}, g^{(1)}} = f^{(l-1)}_{c^{(l-1), p^{(l-1)}}} \), thus we get

\[
\mathbb{E} \left[ \sum_{g^{(1)} \in \mathcal{G}^{(1)}} \| \phi(f^{(l-1)}_{c^{(l-1)}, g^{(1)}}) \|^{\alpha} \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right] \right] \\
\leq 2^\alpha \| \mathcal{G}^{(1)} \| \| \mathcal{P}^{(1)} \| K \alpha^\alpha + 2^\alpha b^\alpha \sum_{p^{(l-1)} \in \mathcal{P}^{(l-1)}} \sum_{k \in [K]} \mathbb{E} \left[ f^{(l-1)}_{c^{(l-1), p^{(l-1)}}} \right]^{\alpha \beta} \left[ f^{(l-2)}_{c^{(l)}, c^{(1)}} \right]
\]

Moreover, from theorem 2 we know that \( f^{(l-1)}_{c^{(l-1)}, c^{(1)}} f^{(l-2)}_{c^{(l)}, c^{(1)}} \sim \text{St}_{\mathcal{P}^{(l-1)}} (\alpha, \Gamma^{(l-1)}) \) and from proposition 1, denoted \( U(p^{(l-1)}, k) = 1_{(p^{(l-1)} \times K)}((p^{(l-1)}, k)) \), for each \( (p^{(l-1)}, k) \in \mathcal{P}^{(l-1)} \times K \) we have \( f^{(l-1)}_{c^{(l-1), p^{(l-1)}}} f^{(l-2)}_{c^{(l)}, c^{(1)}} = \).
\( U(p^{(l-1)}, k) \otimes f^{(l-1)(1:K)}(c^{(l-1)}, :) \sim \mathbf{St}(\alpha, \tau(U(p^{(l-1)}, k)), \sigma(U(p^{(l-1)}, k)), \mu(U(p^{(l-1)}, k))) \sim \sigma(U(p^{(l-1)}, k)) \mathbf{St}(\alpha, \tau(U(p^{(l-1)}, k)), 1, \mu(U(p^{(l-1)}, k))). \) Since \( \beta \alpha < \alpha \) we get

\[
\mathbb{E} \left[ \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \| \phi(f^{(l-1)(1:K)}_c(c^{(l-1)}, g^{(l)})) \| \left| \mathcal{F}^{(l-2)(1:K)}_1 \right| \right] \\
\leq 2^\alpha |\mathcal{G}^{(l)}| |\mathbf{P}^{(l)}| K \alpha^\alpha + \\
+ 2^\alpha b^\alpha \sum_{p^{(l-1)} \in \mathbf{P}^{(l-1)}} \sum_{k \in [K]} \sigma(U(p^{(l-1)}, k))^{\beta \alpha} \mathbb{E} \left[ \mathbf{St}(\alpha, \tau(U(p^{(l-1)}, k)), 1, \mu(U(p^{(l-1)}, k)))^{\beta \alpha} \right] \\
\leq 2^\alpha |\mathcal{G}^{(l)}| |\mathbf{P}^{(l)}| K \alpha^\alpha + \\
+ 2^\alpha b^\alpha \mathcal{M} \sum_{p^{(l-1)} \in \mathbf{P}^{(l-1)}} \sum_{k \in [K]} \left( \int_{\mathbb{S}^{(l-1)} \times K_{l-1}} \left| U(p^{(l-1)}, k) \otimes s \right|^\alpha \Gamma^{(l-1)}(d s) \right) \beta \\
\leq 2^\alpha |\mathcal{G}^{(l)}| |\mathbf{P}^{(l)}| K \alpha^\alpha + \\
+ 2^\alpha b^\alpha \mathcal{M} \sum_{p^{(l-1)} \in \mathbf{P}^{(l-1)}} \sum_{k \in [K]} \left( \int_{\mathbb{S}^{(l-1)} \times K_{l-1}} \left| U(p^{(l-1)}, k) \otimes s \right|^\alpha \Gamma^{(l-1)}(d s) \right)^\beta \tag{11}
\]

where \( \mathcal{M} = \max_{(p^{(l-1)}, k) \in \mathbf{P}^{(l-1)} \times k} \mathbb{E} \left[ \mathbf{St}(\alpha, \tau(U(p^{(l-1)}, k)), 1, \mu(U(p^{(l-1)}, k)))^{\beta \alpha} \right] < +\infty. \) Then,
where we used the Jensen’s inequality. Moreover,

\[
\mathbb{E} \left[ \int_{\mathbb{R}^{p(l-1) \times k(l-1)}} U(p(l-1), k) \otimes \mathbb{I}^{p(l-1)}(d s) \right] h_{C(l-2)}^{(l-2)}
\]

\[= \mathbb{E} \left[ \sigma_b^\alpha |U(p(l-1), k) \otimes \mathbb{I}(p(l-1) \times K(l-1))|^\alpha + \right.
\]

\[+ \frac{\sigma_b^\alpha}{C} \sum_{(c(l-2), g(l-1)) \in [C \times G(l-1)]} |U(p(l-1), k) \otimes \phi(f_{p(l-1)}^{(l-2)(1):K(l-1)})|^\alpha \left|h_{C(l-2)}^{(l-2)}\right|
\]

\[= \mathbb{E} \left[ \sigma_b^\alpha + \frac{\sigma_b^\alpha}{C} \sum_{(c(l-2), g(l-1)) \in [C \times G(l-1)]} |\phi(f_{p(l-1)}^{(l-2)(1):K(l-1)})|^\alpha \left|h_{C(l-2)}^{(l-2)}\right|
\]

\[= \sigma_b^\alpha + \frac{\sigma_b^\alpha}{C} \sum_{(c(l-2), g(l-1)) \in [C \times G(l-1)]} \mathbb{E} \left[ |\phi(f_{p(l-1)}^{(l-2)(1):K(l-1)})|^\alpha \left|h_{C(l-2)}^{(l-2)}\right|ight]
\]

\[= \sigma_b^\alpha + \frac{\sigma_b^\alpha}{C} \sum_{(c(l-2), g(l-1)) \in [C \times G(l-1)]} \int |\phi(f_{g(l-1)})|^\alpha h_{C(l-2)}^{(l-2)}(d f_{g(l-1)} \in [G(l-1)])
\]

\[= \sigma_b^\alpha + \sigma_b^\alpha \int \sum_{g(l-1) \in [G(l-1)]} |\phi(f_{g(l-1)})|^\alpha h_{C(l-2)}^{(l-2)}(d f_{g(l-1)} \in [G(l-1)])
\]

\[
(12)
\]

Note that we have used the inequality $|x| \leq \|x\|$ and that $(f_{p(l-1)}^{(l-2)(1):K(l-1)} h_{C(l-2)}^{(l-2)}) \overset{\text{id}}{\sim} h_{C(l-2)}^{(l-2)}$ (with respect to $c(l-1) \geq 1$). Putting together (11) and (12) we have shown that

\[
\sup_C \mathbb{E} \left[ \sum_{g(l) \in [G(l)]} |\phi(f_{p(l-1)}^{(l-1)(1):K(l-1)})|^\alpha h_{C(l-2)}^{(l-2)} \right]
\]

\[
\leq 2^\alpha |G(l)| |P(l)| K_\alpha^\alpha +
\]

\[+ 2^\alpha b^\alpha M \sum_{p(l-1) \in [P(l-1)]} \sum_{k \in [K]} \left( \sigma_b^\alpha + \sigma_b^\alpha \sup_C \int \sum_{g(l-1) \in [G(l-1)]} |\phi(f_{g(l-1)})|^\alpha h_{C(l-2)}^{(l-2)}(d f_{g(l-1)} \in [G(l-1)]) \right)^{\beta}
\]

which is finite by induction hypothesis. Now we conclude:

\[
\sup_C \int \sum_{g(l) \in [G(l)]} |\phi(f_{g(l)})|^\alpha h_{C(l-1)}^{(l-1)}(d f_{g(l)} \in [G(l)])
\]

\[= \sup_C \mathbb{E} \left[ \sum_{g(l) \in [G(l)]} |\phi(f_{p(l-1)}^{(l-1)(1):K(l-1)})|^\alpha h_{C(l-1)}^{(l-1)} \right]
\]

\[= \sup_C \mathbb{E} \left[ \sup_C \mathbb{E} \left[ \sum_{g(l) \in [G(l)]} |\phi(f_{p(l-1)}^{(l-1)(1):K(l-1)})|^\alpha h_{C(l-1)}^{(l-2)} h_{C(l-1)}^{(l-1)} \right] \right]
\]

\[< +\infty
\]

that is finite by previous step.
2. Proof of L1.1

The proof of L1.1) follows by induction, and along lines similar to the proof of L1). In particular, let $\epsilon > 0$ be such that $\beta(\alpha + \epsilon) < \alpha$. It exists since $\beta < 1$. For $l = 2$, for each $c^{(1)} \geq 1$

$$\mathbb{E}\left[ \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \|\phi(f_{\cdot}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}\|^{\alpha + \epsilon} \right]$$

$$= \mathbb{E}\left[ \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathfrak{P}^{(2)}} \sum_{k \in [K]} |\phi(f_{\cdot p^{(2)}}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}|^{\alpha + \epsilon} \right]$$

$$= \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathfrak{P}^{(2)}} \sum_{k \in [K]} \mathbb{E} \left[ |\phi(f_{\cdot p^{(2)}}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}|^{\alpha + \epsilon} \right]$$

$$\leq \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathfrak{P}^{(2)}} \sum_{k \in [K]} \mathbb{E} \left[ (a + b|f_{\cdot p^{(2)}}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}|^{\beta} \right]$$

$$\leq 2^{(\alpha + \epsilon)} |\mathcal{G}^{(2)}| |\mathfrak{P}^{(2)}| \mathcal{K}^{a(\alpha + \epsilon)} + (2b)^{(\alpha + \epsilon)} \sum_{g^{(2)} \in \mathcal{G}^{(2)}} \sum_{p^{(2)} \in \mathfrak{P}^{(2)}} \sum_{k \in [K]} \mathbb{E} \left[ |f_{\cdot p^{(2)}}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}|^{\beta} \right]$$

$$< \infty$$

where we used that $(f_{\cdot p^{(2)}}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}$, by Proposition 1 is distributed according to a stable distribution with index $\alpha$ (and some skewness, scale and shift parameters) and then, being $(\alpha + \epsilon) \beta < \alpha$, $\mathbb{E} \left[ |f_{\cdot p^{(2)}}^{(1)}(1;K))_{(c^{(1)},g^{(2)})}|^{\beta} \right] < +\infty$. Moreover the bound is uniform with respect to $C$ since the law is invariant with respect to $C$. Now assuming that L1.1) is true for $l - 2$ we prove that it is true for $l - 1$. As in the previous lemma, we can write the following inequality

$$\mathbb{E}\left[ \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|\phi(f_{\cdot}^{(l-1)}(1;K))_{(c^{(l-1)},g^{(l)})}\|^{\alpha + \epsilon} |\mathcal{H}_{C}^{(l-2)}\right]$$

$$= \mathbb{E}\left[ \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|\phi(f_{\cdot}^{(l-1)}(1;K))_{(c^{(l-1)},g^{(l)})}\|^{\alpha + \epsilon} |\mathcal{H}_{C}^{(l-2)}\right]$$

$$\leq 2^{(\alpha + \epsilon)} |\mathcal{G}^{(l)}||\mathfrak{P}^{(l)}| \mathcal{K}^{a(\alpha + \epsilon)} +$$

$$+ 2^{(\alpha + \epsilon)} \mathcal{M} \sum_{p^{(l-1)} \in \mathfrak{P}^{(l-1)}} \sum_{k \in [K]} \mathbb{E} \left[ \int \int_{[p^{(l-1)} \times K^{(l-1)}]} |U(p^{(l-1)}, k \otimes s)^{\alpha + \epsilon} \Gamma^{(l-1)}(d s) |^{\beta} |\mathcal{H}_{C}^{(l-2)}\right]$$

$$\leq 2^{(\alpha + \epsilon)} |\mathcal{G}^{(l)}||\mathfrak{P}^{(l)}| \mathcal{K}^{a(\alpha + \epsilon)} +$$

$$+ 2^{(\alpha + \epsilon)} \mathcal{M} \sum_{p^{(l-1)} \in \mathfrak{P}^{(l-1)}} \sum_{k \in [K]} \left( \mathbb{E} \left[ \int \int_{[p^{(l-1)} \times K^{(l-1)}]} |U(p^{(l-1)}, k \otimes s)^{\alpha + \epsilon} \Gamma^{(l-1)}(d s) |^{\beta} |\mathcal{H}_{C}^{(l-2)}\right] \right)^{\beta}$$

Moreover, following the same steps as in the previous lemma (just replacing $\alpha + \epsilon$ instead of $\alpha$), we get

$$\mathbb{E} \left[ \int \int_{[p^{(l-1)} \times K^{(l-1)}]} |U(p^{(l-1)}, k \otimes s)^{\alpha + \epsilon} \Gamma^{(l-1)}(d s) |^{\beta} |\mathcal{H}_{C}^{(l-2)}\right]$$

$$= \sigma_b^{\alpha + \epsilon} + \sigma_w^{\alpha + \epsilon} \int \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} |\phi(f_{g^{(l-1)}})|^{\alpha + \epsilon} |\mathcal{H}_{C}^{(l-2)}(d f_{g^{(l-1)}})$$
and

\[ \int \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} \| \phi(f_{g^{(l-1)}}) \|^{\alpha + \epsilon} h_{C}^{(l-2)}(\text{d} f_{g^{(l-1)}} \in \mathcal{G}^{(l-1)}) \]

\[ = \mathbb{E} \left[ \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} \| \phi(f^{(l-2)}(1;K))_{(c^{(l-2)},g^{(l-1)})} \|^{\alpha + \epsilon} h_{C}^{(l-2)} \right] \]

\[ = \mathbb{E} \mathbb{E} \left[ \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} \| \phi(f^{(l-2)}(1;K))_{(c^{(l-2)},g^{(l-1)})} \|^{\alpha + \epsilon} h_{C}^{(l-3)} \right] h_{C}^{(l-2)} \]

\[ \leq \mathbb{E} \mathbb{E} \left[ \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} \| \phi(f^{(l-2)}(1;K))_{(c^{(l-2)},g^{(l-1)})} \|^{\alpha + \epsilon} h_{C}^{(l-3)} \right] h_{C}^{(l-2)} \]

Thus taking the sup\_C in (13), by previous inequalities, it is less or equal than

\[ 2^{(\alpha + \epsilon)} G^{(l)} \| \mathbb{P}^{(l)} \| K a^{(\alpha + \epsilon)} + 2^{(\alpha + \epsilon)} b^{(\alpha + \epsilon)} M \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} \sum_{k \in [K]} \left( \sigma_{b}^{\alpha + \epsilon} + \right. \]

\[ + \sigma_{a}^{\alpha + \epsilon} \mathbb{E} \left[ \sup_{C} \mathbb{E} \left[ \sum_{g^{(l-1)} \in \mathcal{G}^{(l-1)}} \| \phi(f^{(l-2)}(1;K))_{(c^{(l-2)},g^{(l-1)})} \|^{\alpha + \epsilon} h_{C}^{(l-3)} \right] h_{C}^{(l-2)} \right]^{\beta} \]

which is bounded by hypothesis induction.

.3. Proof of L2

By induction hypothesis, \( h_{C}^{(l-1)} \) converges to \( h^{(l-1)} \) in distribution with respect to the weak topology. Since the limit law is degenerate on \( h^{(l-1)} \) (in the sense that it provides a.s. the distribution \( q^{(l-1)} \)), then for every sub-sequence \( (C') \) there exists a sub-sequence \( (C'') \) such that \( h_{C''}^{(l-1)} \) converges a.s. By the induction hypothesis, \( h^{(l-1)} \) is absolutely continuous with respect to the Lebesgue measure. Since \( \phi \) is almost everywhere continuous, and by L1.1) uniformly integrable with respect to \( (h_{C}^{(l-1)}) \) then we can write the following

\[ \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} |t^{(l)} \otimes \phi(f_{g^{(l)}})|^{\alpha} h_{C''}^{(l-1)}(\text{d} f_{g^{(l)}} \in \mathcal{G}^{(l)}) \overset{a.s.}{\rightarrow} \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} |t^{(l)} \otimes \phi(f_{g^{(l)}})|^{\alpha} q^{(l-1)}(\text{d} f_{g^{(l)}} \in \mathcal{G}^{(l)}) \]

Thus

\[ \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} |t^{(l)} \otimes \phi(f_{g^{(l)}})|^{\alpha} h_{C}^{(l-1)}(\text{d} f_{g^{(l)}} \in \mathcal{G}^{(l)}) \overset{p}{\rightarrow} \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} |t^{(l)} \otimes \phi(f_{g^{(l)}})|^{\alpha} q^{(l-1)}(\text{d} f_{g^{(l)}} \in \mathcal{G}^{(l)}) \]

as \( C \rightarrow +\infty \).

.4. Proof of L3

Let \( \epsilon > 0 \) as in L1.1), \( r = \frac{\alpha + \epsilon}{\alpha} \) and \( q \) such that \( \frac{1}{r} + \frac{1}{q} = 1 \). Thus, by Hölder inequality and by the fact that, being \( q > 1 \), for every \( y \geq 0 \) it holds that \( (1 - e^{-y})^{q} \leq (1 - e^{-y}) \leq y \), we get
\[ \int \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \| \phi(f_{g^{(i)}}) \|^\alpha [1 - \exp\{-\frac{\sigma^\alpha_C}{C} \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha] h_C^{(l-1)}(d f_{g^{(i)}}) \]

\leq \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \left[ \int \| \phi(f_{g^{(i)}}) \|_{\alpha r} h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha r}} \left[ \int [1 - \exp\{-\frac{\sigma^\alpha_C}{C} \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha] h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha}}

\leq \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \left[ \int \| \phi(f_{g^{(i)}}) \|_{\alpha r} h_C^{(l-1)}(d f_{g^{(i)}}) \right]^{\frac{1}{\alpha r}} \left[ \frac{\sigma^\alpha_C}{C} \int \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha}}

\leq \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \left[ \int \| \phi(f_{g^{(i)}}) \|_{\alpha r} h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha r}} \left[ \| t^{(l)} \|^{\alpha r} \frac{\sigma^\alpha_C}{C} \int \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \| \phi(f_{g^{(i)}}) \| h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha}}

= \left( \| t^{(l)} \|^{\alpha r} \frac{\sigma^\alpha_C}{C} \right)^{\frac{1}{\alpha r}} \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \left[ \int \| \phi(f_{g^{(i)}}) \|_{\alpha r} h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha r}} \left[ \int \sum_{g^{(i)} \in \mathbf{G}^{(i)}} \| \phi(f_{g^{(i)}}) \| h_C^{(l-1)}(d f_{g^{(i)})} \right]^{\frac{1}{\alpha}} \xrightarrow{p} 0

\text{as } C \to \infty \text{ by L1) and L1.1).}

.5. Conclusion

By Lagrange theorem for \( y > 0 \) there exists \( \theta \in (0, 1) \) such that \( e^{-y} = 1 - y + y(1 - e^{-y}) \). In our case, for \( y = y_C(f_{g^{(i)} \in \mathbf{G}^{(i)}}) = \frac{\sigma^\alpha_C}{C} \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha \), for any \( C \) there exists a \( \theta_C \in (0, 1) \) such that, from (10), the follow equality holds

\[ \varphi_{(f^{(1,K)})}^{(l)}}(t^{(l)}) = e^{-\sigma^\alpha_C |t^{(l)} \otimes I_{(p^{(i)} \times K)}|^\alpha} \mathbb{E} \left[ \left( \int e^{-y_C(f_{g^{(i)} \in \mathbf{G}^{(i)}})} h_C^{(l-1)}(d f_{g^{(i)} \in \mathbf{G}^{(i)}}) \right)^C \right]

= e^{-\sigma^\alpha_C |t^{(l)} \otimes I_{(p^{(i)} \times K)}|^\alpha} \mathbb{E} \left[ \left( 1 - \int y_C(f_{g^{(i)} \in \mathbf{G}^{(i)}}) h_C^{(l-1)}(d f_{g^{(i)} \in \mathbf{G}^{(i)}}) \right) + \int y_C(f_{g^{(i)} \in \mathbf{G}^{(i)}}) \left[ 1 - e^{-\theta_C y_C(f_{g^{(i)} \in \mathbf{G}^{(i)}})} h_C^{(l-1)}(d f_{g^{(i)} \in \mathbf{G}^{(i)}}) \right]^C \right]

= \exp \left\{ - \sigma^\alpha_C |t^{(l)} \otimes I_{(p^{(i)} \times K)}|^\alpha \right\} \times \left\{ 1 - \frac{\sigma^\alpha_C}{C} \int \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha h_C^{(l-1)}(d f_{g^{(i)} \in \mathbf{G}^{(i)}}) + \frac{\sigma^\alpha_C}{C} \int \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha \left[ 1 - \exp \left\{ - \theta_C \frac{\sigma^\alpha_C}{C} \sum_{g^{(i)} \in \mathbf{G}^{(i)}} |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha \right\} \right] \times h_C^{(l-1)}(d f_{g^{(i)} \in \mathbf{G}^{(i)}}) \right\}^C \]

The last integral tends to 0 as \( C \to \infty \) since by Cauchy inequality \( |t^{(l)} \otimes \phi(f_{g^{(i)}})|^\alpha \leq \| t^{(l)} \| \| \phi(f_{g^{(i)}) \|^\alpha \) we have
We prove that for each \( l \in [L] \), \( f^{(l)}(x) = f^{(l)}(x(1) \mid K) \xrightarrow{d} \mathbb{R}^{\infty} \) and define \( \Gamma^{(l)} = S_{l^{(1)} \times K} \). The symbol \( \otimes \) denotes the product measure. The proof follows by the Cramér-Wold theorem for finite-dimensional projection of \( f^{(l)}(x(1) \mid K) \) for which it is sufficient to prove the large \( C \) asymptotic behavior of any linear combination of the \( f^{(l)}(x(\cdot) \mid K) \)s for \( c^{(l)} \in \mathcal{L} \subset \mathbb{N} \). See, e.g. (Billingsley, 1999) for details.

**Proof.** Following the notation of (Matthews et al., 2018), consider a finite linear combination of the function values without the bias, i.e. fix \( z = (z^{(l)})_{l \in \mathcal{L}} \) and define

\[
\mathcal{T}^{(l)}(x^{(1) \mid K}, C^{(l-1)}) = \sum_{e^{(l)} \in \mathcal{L}} z^{(e^{(l)})} [f^{(l)}(x^{(1) \mid K}) - f^{(l)}(x^{(1) \mid K})].
\]
The case \( l = 1 \) is easy since it does not depend on \( C \), indeed we get

\[
\mathcal{T}^{(1)}(\mathcal{L}, z, x^{(1):K}, C^{(0)}) = \sum_{c^{(1)} \in \mathcal{L}} z_{c^{(1)}} W_{(c^{(1):}, \ldots)}^{(1)} (p^{(i)}_1, K) \otimes x^{(1):K}_c
\]

and, following the same steps as in Theorem 1, for any \( t^{(1)} := [f^{(1)(k)}_{p^{(i)}}]_{(p^{(i)}, k) \in [p^{(i)} \times K]} \in \mathbb{R}^{p^{(i)} \times K} \), called \( \| z \| = \sum_{c^{(1)} \in \mathcal{L}} |z_{c^{(1)}_c}|^\alpha \), we get

\[
\varphi \left( T^{(1)}(\mathcal{L}, z, x^{(1):K}, C^{(0)}) \right) (t^{(1)}) = \mathbb{E} \left[ \exp \left\{ i \sum_{c^{(1)} \in \mathcal{L}} z_{c^{(1)}} W_{(c^{(1):}, \ldots)}^{(1)} (p^{(i)}_1, K) \otimes \left( t^{(1)}(0) \right) (c^{(0)} \times G^{(1)}) \right\} \right]
\]

\[
\varphi \left( T^{(1)}(\mathcal{L}, z, x^{(1):K}, C^{(0)}) \right) (t^{(1)}) = \mathbb{E} \left[ \exp \left\{ - \sum_{c^{(1)} \in \mathcal{L}} \sigma^{\alpha}(0) \left| \left( t^{(1)}(0) \right) (G^{(1)}) \right|^\alpha \right\} \right]
\]

\[
\varphi \left( T^{(1)}(\mathcal{L}, z, x^{(1):K}, C^{(0)}) \right) (t^{(1)}) = \mathbb{E} \left[ \exp \left\{ - \sum_{c^{(1)} \in \mathcal{L}} \sigma^{\alpha}(0) \left| \left( t^{(1)}(0) \right) (G^{(1)}) \right|^\alpha \right\} \right]
\]

where \( \Delta_{C}^{(1)} \) coincides with \( \Gamma^{(1)} \) just replacing \( \sigma_b \leftarrow 0 \) and \( \sigma_x \leftarrow \sigma_x \| z \|. \) Thus since the characteristic function does not depend on \( C \) we get (as \( C \to \infty \))

\[
\mathcal{T}^{(1)}(\mathcal{L}, z, x^{(1):K}, C^{(0)}) \overset{d}{\to} \mathbb{S}_{p^{(1)} \times K} (\alpha, \Delta_{\infty}^{(1)})
\]

where

\[
\Delta_{\infty}^{(1)} = \Delta_{\infty}^{(1)} = \| z \|^{\alpha} \sum_{c^{(0)}, g^{(1)} \in [C^{(0)} \times G^{(1)}]} \| \sigma^{\alpha}_{x^{(1):K}} \|_{(c^{(0)}, g^{(1)})} \|^{\alpha} \Psi^{(1)}_{x^{(1):K}}(c^{(0)}, g^{(1)})
\]

For \( l = 2, \ldots, L \),

\[
\mathcal{T}^{(l)}(\mathcal{L}, z, x^{(1):K}, C) = \sum_{c^{(1)} \in \mathcal{L}} \frac{z_{c^{(1)}}^{(1)}}{C^{(1):\alpha}} W_{(c^{(1):}, \ldots)}^{(l)} (p^{(i)}_1, K) \otimes \phi(f^{(l-1)(1):K})
\]

and, following the same steps as in Theorem 2, for any \( t^{(l)} := [f^{(l)(k)}_{p^{(i)}}]_{(p^{(i)}, k) \in [p^{(i)} \times K]} \in \mathbb{R}^{p^{(i)} \times K} \), called \( \| z \| = \sum_{c^{(1)} \in \mathcal{L}} |z_{c^{(1)}_c}|^\alpha \), we get

\[
\varphi \left( T^{(l)}(\mathcal{L}, z, x^{(1):K}, C) \right) (t^{(l)}) = \mathbb{E} \left[ \exp \left\{ - \sum_{c^{(1)} \in \mathcal{L}} \sigma^{\alpha}(0) \left| \left( t^{(l)}(0) \right) (G^{(1)}) \right|^\alpha \right\} \right]
\]
where \( \Delta_C^{(l)} \) coincides with \( \Gamma_C^{(l)} \) just replacing \( \sigma_b \leftarrow 0 \) and \( \sigma_\omega \leftarrow \sigma_\omega \|z\| \). Now, proceeding as in Theorem 3, we get the weak limit as \( C \rightarrow +\infty \), i.e.

\[
\mathcal{T}^{(l)}(L, z, x^{(1:L)}, C) \xrightarrow{d} \text{St}_{\mathcal{P}(l) \times \mathcal{K}}(\alpha, \Delta^{(l)}_\infty)
\]

where

\[
\Delta^{(l)}_\infty = \|z\|^\alpha \int \sum_{g^{(l)} \in \mathcal{G}^{(l)}} \|\sigma_\omega \phi(f_{g^{(l)}})\|^{\alpha} \Phi^{(l)}(\phi(f_{g^{(l)}})) q^{(l-1)}(d f_{g^{(l)}} \in \mathcal{G}^{(l)})
\]

This completes the proof.