On completion of the cone of CP linear maps with respect to the energy-constrained diamond norm

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Abstract

The completion of the cone of CP linear maps between Banach spaces of trace class operators w.r.t. the metric induced by the energy-constrained diamond norm are described. This completion consists of linear maps defined on the linear span of states with finite energy and characterized by the Stinespring-like representation with operators (unbounded, in general) belonging to the special class – the completion of the set of all bounded operators w.r.t. the operator $E$-norms [arXiv:1806.05668].

It is shown that the sets of quantum channels and quantum operations are complete w.r.t. the metric induced by the energy-constrained diamond norm.

Some properties of maps belonging to the completed cone are described. In particular, the corresponding generalization of the Kretschmann-Schlingemann-Werner theorem is obtained.

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1 Introduction

The norm of complete boundedness (typically called the diamond norm) on the set of completely positive (CP) linear maps between Banach spaces of trace class operators is widely used in the quantum theory [1, 21, 22]. The corresponding distance between quantum channels can be treated as a measure of distinguishability between these channels by quantum measurements [22, Ch.9]. Nevertheless, the topology (convergence) generated by the diamond norm distance is too strong for description of physical perturbations of infinite-dimensional quantum channels [17, 24]. This is explained, briefly speaking, by the fact that the definition of the diamond norm does not separate input states with finite energy (which can be produced in a physical experiment) and unrealizable states with infinite energy. To take this "energy discrimination" of quantum states into account the energy-constrained diamond norms (ECD norms) are introduced independently in [17, 24], where it is shown that these norms induce an appropriate metric and topology on the set of quantum channels and operations. In particular, ECD norms induce the strong convergence topology on the set of quantum channels and operations provided that the input system Hamiltonian is of discrete type [17].

The ECD norms turned out to be a useful tool for quantitative continuity analysis of basic capacities of energy-constrained infinite-dimensional channels [17, 24]. These norms are also used in study of quantum dynamical semigroups [2, 24].

The cone \( \mathcal{F}(A, B) \) of CP linear maps between Banach spaces \( \mathcal{F}(\mathcal{H}_A) \) and \( \mathcal{F}(\mathcal{H}_B) \) of trace class operators is complete w.r.t. the diamond norm metric, but it is not complete w.r.t. the ECD norm metric induced by an unbounded Hamiltonian \( G \) of the system \( A \). In this paper we describe the completion \( \hat{\mathcal{F}}_G(A, B) \) of the cone \( \mathcal{F}(A, B) \) w.r.t. the ECD norm metric induced by any positive operator \( G \) on \( \mathcal{H}_A \). It turned out that this completion \( \hat{\mathcal{F}}_G(A, B) \) is closely related to the completion \( \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_{BE}) \) of the set \( \mathcal{B}(\mathcal{H}_A, \mathcal{H}_{BE}) \) of all bounded linear operators from the input Hilbert space \( \mathcal{H}_A \) to the output-environment Hilbert space \( \mathcal{H}_{BE} \) w.r.t. the operator E-norm induced by same operator \( G \) [16]. Namely, the set \( \hat{\mathcal{F}}_G(A, B) \) consists of linear maps defined on the linear span of states with finite energy and characterized by the Stinespring-like representation with operators in \( \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_{BE}) \).

The results from [16] concerning operator E-norms allow to show that a diamond norm bounded subset of \( \mathcal{F}(A, B) \) is complete w.r.t the ECD norm if and only if it is closed w.r.t this norm. This implies completeness of the sets of quantum channels and quantum operations w.r.t. the ECD norm.

We prove that maps in the completed cone \( \hat{\mathcal{F}}_G(A, B) \) are characterized by the Kraus representation with operators in \( \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_B) \) and generalize the Kretschmann-Schlingemann-Werner theorem (obtained in [9]) to maps from the cone \( \hat{\mathcal{F}}_G(A, B) \).

We give a nonconstructive description of the completion \( \hat{\mathcal{Y}}_G(A, B) \) of the real linear space \( \mathcal{Y}(A, B) \) of all Hermitian-preserving completely bounded linear maps from \( \mathcal{F}(\mathcal{H}_A) \) to \( \mathcal{F}(\mathcal{H}_B) \) w.r.t. the ECD norm. We prove that the positive cone in \( \hat{\mathcal{Y}}_G(A, B) \) coincides with the cone \( \hat{\mathcal{F}}_G(A, B) \) if the operator \( G \) has a discrete spectrum of finite multiplicity.
2 Preliminaries

2.1 Basic notations

Let \( \mathcal{H} \) be a separable infinite-dimensional Hilbert space, \( \mathfrak{B}(\mathcal{H}) \) – the algebra of all bounded operators on \( \mathcal{H} \) with the operator norm \( \| \cdot \| \) and \( \mathfrak{T}(\mathcal{H}) \) – the Banach space of all trace-class operators on \( \mathcal{H} \) with the trace norm \( \| \cdot \|_1 \) (the Schatten class of order 1) \([5, 13]\). Let \( \mathfrak{T}_+(\mathcal{H}) \) be the cone of positive operators in \( \mathfrak{T}(\mathcal{H}) \) and \( \mathfrak{S}(\mathcal{H}) \) its subset consisting of operators with unit trace (called quantum states). Extreme points of \( \mathfrak{S}(\mathcal{H}) \) are 1-rank projectors called pure states.

Trace-class operators will be usually denoted by the Greek letters \( \rho, \sigma, \omega, \ldots \) in contrast to other linear operators (bounded and unbounded) denoted by the Latin letters. The Greek letters \( \phi, \varphi, \psi, \ldots \) will be used for vectors in a Hilbert space.

Denote by \( I_{\mathcal{H}} \) the unit operator on a Hilbert space \( \mathcal{H} \) and by \( \text{Id}_{\mathcal{H}} \) the identity transformation of the Banach space \( \mathfrak{T}(\mathcal{H}) \).

The Bures distance between operators \( \rho \) and \( \sigma \) in \( \mathfrak{T}_+(\mathcal{H}) \) is defined as

\[
\beta(\rho, \sigma) = \sqrt{\|\rho\|_1 + \|\sigma\|_1 - 2F(\rho, \sigma)},
\]

where

\[
F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2
\]

is the fidelity of \( \rho \) and \( \sigma \). The following relations between the Bures distance and the trace-norm distance hold (cf. \([7, 22]\))

\[
\frac{\|\rho - \sigma\|_1}{\sqrt{\|\rho\|_1 + \|\sigma\|_1}} \leq \beta(\rho, \sigma) \leq \sqrt{\|\rho - \sigma\|_1}.
\]

If quantum systems \( A \) and \( B \) are described by Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) then the bipartite system \( AB \) is described by the tensor product of these spaces, i.e. \( \mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B \). A state in \( \mathfrak{S}(\mathcal{H}_{AB}) \) is denoted by \( \rho_{AB} \), its marginal states \( \text{Tr}_{\mathcal{H}_B}\rho_{AB} \) and \( \text{Tr}_{\mathcal{H}_A}\rho_{AB} \) are denoted respectively by \( \rho_A \) and by \( \rho_B \) (here \( \text{Tr}_{X}\rho_{AB} \equiv \text{Tr}_{\mathcal{H}_X}\rho_{AB} \)).

We will pay a special attention to the class of unbounded densely defined positive operators on \( \mathcal{H} \) having discrete spectrum of finite multiplicity. In Dirac’s notations any such operator \( G \) can be represented as follows

\[
G = \sum_{k=0}^{+\infty} E_k |\tau_k\rangle \langle \tau_k|
\]

on the domain \( \mathcal{D}(G) = \{ \varphi \in \mathcal{H} | \sum_{k=0}^{+\infty} E_k^2 |\tau_k\rangle \langle \varphi||^2 < +\infty \} \), where \( \{\tau_k\}_{k=0}^{+\infty} \) is the orthonormal basis of eigenvectors of \( G \) corresponding to the nondecreasing sequence \( \{E_k\}_{k=0}^{+\infty} \) of eigenvalues tending to \( +\infty \). We will use the following (cf. \([24]\))

**Definition 1.** An operator \( G \) having representation (4) is called discrete.
2.2 Operator E-norms

Denote by \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) the Banach space of all bounded linear operators from a Hilbert space \( \mathcal{H} \) to a Hilbert space \( \mathcal{K} \). We will use special norms on \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) described below.

Let \( G \) be a positive operator on \( \mathcal{H} \) with a dense domain \( D(G) \) such that

\[
\inf \{ \langle \phi | G | \phi \rangle \mid \phi \in D(G), \| \phi \| = 1 \} = 0.
\]

(5)

The operator E-norms on \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) are defined as

\[
\| A \|_{E}^{G} = \sup_{\rho \in \mathcal{S}(\mathcal{H}): \text{Tr} G \rho \leq E} \sqrt{\text{Tr} A \rho A^*}, \quad E > 0,
\]

(6)

where the supremum is over all states \( \rho \) in \( \mathcal{S}(\mathcal{H}) \) such that \( \text{Tr} G \rho \leq E \). These norms on \( \mathcal{B}(\mathcal{H}) \) are studied in detail in [16], all the results presented therein are generalized to the case of \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) with obvious modifications (see Remark 1 in [16]).

One of the basic properties of the family of E-norms is the concavity of the non-decreasing nonnegative function \( E \mapsto \| A \|_{E}^{G} \) for any \( A \) in \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) which shows that

\[
\| A \|_{E_1}^{G} \leq \| A \|_{E_2}^{G} \leq \sqrt{E_2/E_1} \| A \|_{E_1}^{G}, \quad \text{for any } E_2 > E_1 > 0.
\]

(7)

Hence for given operator \( G \) all the norms \( \| \cdot \|_{E}^{G}, \quad E > 0 \), are equivalent on \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \).

Different operators \( G \) induce E-norms generating different topologies (types of convergence) on \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \). In particular, E-norms induced by any discrete unbounded operator \( G \) (Def.1) generates the strong operator topology on bounded subsets of \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) [16, Proposition 2].

Condition (5) implies that the supremum in definition (6) can be taken over all positive operators in the unit ball of \( \mathcal{S}(\mathcal{H}) \) (not only states) such that \( \text{Tr} G \rho \leq E \) [16, Proposition 3]. It follows that

\[
\| A \|_{E}^{G} = \sup \left\{ \sqrt{\sum_i \| A \phi_i \|^2} \mid \{ \phi_i \} \subset \mathcal{V}_G : \sum_i \| \phi_i \|^2 \leq 1, \sum_i \| \sqrt{G} \phi_i \|^2 \leq E \right\},
\]

(8)

where \( \{ \phi_i \} \) is a finite or countable collection of vectors in the set

\[
\mathcal{V}_G = \mathcal{D}(\sqrt{G}) = \{ \phi \in \mathcal{H} \mid \langle \phi | G | \phi \rangle < +\infty \}.
\]

(9)

By using operator E-norms one can obtain a generalised version of the Kretschmann-Schlingemann-Werner theorem [9]. This version allows to quantify continuity of the Stinespring representation of CP linear maps w.r.t. different topologies on the set of CP linear maps and the corresponding Stinespring operators [16, Theorem 3].

The set \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) is not complete w.r.t. the norm \( \| \cdot \|_{E}^{G} \) (provided that \( G \) is an unbounded operator). To describe the completion of \( \mathcal{B}(\mathcal{H}, \mathcal{K}) \) w.r.t. this norm denote

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\(^1\)The original Kretschmann-Schlingemann-Werner theorem quantifies continuity of the Stinespring representation of CP linear maps w.r.t. the diamond norm topology on the set of CP linear maps and the operator norm topology on the set of Stinespring operators.
by $L_G(H, K)$ the set of all linear (bounded or unbounded) operators $A$ from $H$ to $K$ whose domain $D(A)$ contains the set $V_G$ defined in (9).

We will say that operators $A$ and $B$ in $L_G(H, K)$ are $G$-equivalent if they coincide on the set $V_G$. In what follows we will identify $G$-equivalent operators.

To avoid the notion of adjoint operator\(^2\) we will define the operator $A\rho B^*$ for any $A, B \in L_G(H)$ and any finite rank operator $\rho$ such that $TrG\rho < \infty$ as follows

$$A\rho B^* = \sum_i |\alpha_i\rangle \langle \beta_i|, \text{ where } |\alpha_i\rangle = A|\varphi_i\rangle, |\beta_i\rangle = B|\varphi_i\rangle,$$

provided that $\rho = \sum_i |\varphi_i\rangle \langle \varphi_i|$ (by using Schrodinger’s mixture theorem (see [1] Ch.8)) it is easy to show that the r.h.s. of (10) does not depend on this decomposition of $\rho$.

Thus, for any $A \in L_G(H, K)$ we may define $\|A\|_E^G$ (as a nonnegative number or $+\infty$) by the same formula (9), where the supremum is taken over all finite rank states satisfying the inequality $TrG\rho \leq E$. Equivalent definition of $\|A\|_E^G$ is given by formula (8) provided that condition (3) is valid. It is shown in [16] that the linear subspace

$$\mathcal{B}_G(H, K) = \{A \in L_G(H, K) | \|A\|_E^G < +\infty\}$$

of $L_G(H, K)$ equipped with the norm $\|\cdot\|_E^G$ is a Banach space (not depending on $E$).

The set $\mathcal{B}(H, K)$ is naturally embedded into $\mathcal{B}_G(H, K)$ as a nonclosed linear subspace, its closure $\mathcal{B}_G(H, K)$ (i.e. the completion of $\mathcal{B}(H, K)$ w.r.t. the norm $\|\cdot\|_E^G$) coincides with the proper linear subspace of $\mathcal{B}_G(H, K)$ determined by the condition

$$\|A\|_E^G = o(\sqrt{E}) \text{ as } E \to +\infty.$$

We will use the following characterization of elements of $\mathcal{B}_G^0(H, K)$ presented in [16] Theorem 3]. For a separable Hilbert space $H_R$ introduce the sets:

$$V_G \otimes H_R = \{\varphi \otimes \psi \in V_G \otimes \psi \in H_R, \langle \varphi | G \otimes \psi \rangle \leq E\}, \text{ i.e. } V_G \otimes H_R$$

is the linear span of all vectors $\varphi \otimes \psi$, where $\varphi \in V_G$, $\psi \in H_R$, and\(^3\)

$$V_{G \otimes I_R,E} = \{\eta \in H \otimes H_R | \langle \eta | G \otimes I_R | \eta \rangle \leq E\}, \text{ E > 0.}\) (14)

**Lemma 1.** An operator $A \in \mathcal{B}_G(H, K)$ belongs to the set $\mathcal{B}_G^0(H, K)$ if and only if for any Hilbert space $H_R$ the operator $A \otimes I_R$ naturally defined on the set $V_G \otimes H_R$ has a unique linear extension to the set $V_{G \otimes I_R,E} = \bigcup_{E>0} V_{G \otimes I_R,E}$ which is uniformly continuous on $V_{G \otimes I_R,E}$ for any $E > 0$. This extension denoted also by $A \otimes I_R$ belongs to the space $\mathcal{B}_G^0(\otimes R \otimes H_R, K \otimes H_R)$ and $\|A \otimes I_R\|_{E \otimes I_R} = \|A\|_E^G$.

For any $A \in \mathcal{B}_G^0(H, K)$ the boundedness and continuity properties of the operator $A \otimes I_R$ on $V_{G \otimes I_R,E}$ are described by the inequalities

$$\|A \otimes I_R | \eta \| \leq \|A\|_E^G \text{ and } \|A \otimes I_R | \theta - A \otimes I_R | \psi \| \leq f_A(E, \varepsilon)$$

\(^2\)We make no assumptions about closability of unbounded operators, so adjoint operators may not exist as densely defined operators [13].

\(^3\)Here and in what follows we write $I_X$ instead of $I_{H_X}$ (where $X = A, B, E, R$) to simplify notations.
valid for any \( \eta \) in the unit ball of \( \mathcal{V}_{G \otimes I_R, E} \) and any \( \theta, \vartheta \) in \( \mathcal{V}_{G \otimes I_R, E} \) such that \( \| \theta - \vartheta \| \leq \varepsilon \), where \( f_A(E, \varepsilon) = \varepsilon \| A \|_E^G / \varepsilon^2 \) is a function vanishing as \( \varepsilon \rightarrow 0^+ \) by condition (12).

Lemma 2. Let \( A \) and \( B \) be any operators in \( \mathcal{B}_{G}(\mathcal{H}, \mathcal{K}) \) and

\[
\mathcal{E}_{G,E} = \{ \rho \in \mathcal{T}_+(\mathcal{H}) \mid \text{Tr} \rho \leq 1, \text{Tr} G \rho \leq E \}, \quad E > 0.
\]

A) For any \( \rho \) in \( \mathcal{E}_{G,E} \) the operator \( A \rho B^* \) is correctly defined by the formula

\[
A \rho B^* = \sum_i |A \varphi_i \rangle \langle B \varphi_i|,
\]

where \( \rho = \sum_i | \varphi_i \rangle \langle \varphi_i | \) is any decomposition of \( \rho \) into 1-rank operators, and

\[
\| \text{Tr} A \rho B^* \| \leq \| A \|_E^G \| B \|_E^G.
\]

for any \( \rho \) in \( \mathcal{E}_{G,E} \) such that \( \| \rho - \sigma \|_1 \leq \varepsilon \), where \( f_X \) is the function defined in Lemma [7]

Proof. A) If \( \rho = \sum_i | \varphi_i \rangle \langle \varphi_i | \) and \( \{ \psi_i \} \) is a set of orthogonal unit vectors in a separable Hilbert space \( \mathcal{H}_R \) then \( | \eta \rangle = \sum_i | \varphi_i \rangle \otimes | \psi_i \rangle \) is a vector in \( \mathcal{V}_{G \otimes I_R} \) such that \( \rho = \text{Tr}_R | \eta \rangle \langle \eta | \). By Lemma 1 the operators \( A \otimes I_K \) and \( B \otimes I_K \) naturally defined on the set \( \mathcal{V}_{G \otimes \mathcal{H}_R} \) have linear extensions to the set \( \mathcal{V}_{G \otimes I_R} \) which is uniformly continuous on the set \( \mathcal{V}_{G \otimes I_R, E} \) for any \( E > 0 \). By using this extension and the well known relation between different purifications of a given state [7, 22], it is easy to show that the r.h.s. of formula (16) does not depend on the representation \( \rho = \sum_i | \varphi_i \rangle \langle \varphi_i | \).

By using the Cauchy-Schwarz inequality we obtain

\[
| \text{Tr} A \rho B^* | = \left| \sum_i \langle B \varphi_i | A \varphi_i \rangle \right| \leq \sum_i \| A \varphi_i \| \| B \varphi_i \| \leq \sqrt{\sum_i \| A \varphi_i \|^2 \sqrt{\sum_i \| B \varphi_i \|^2}}.
\]

It follows from (18) that the r.h.s. of this inequality does not exceed \( \| A \|_E^G \| B \|_E^G \).

B) Let \( \rho \) and \( \sigma \) be operators in \( \mathcal{E}_{G,E} \) such that \( \| \rho - \sigma \|_1 \leq \varepsilon \). If \( \mathcal{H}_R \cong \mathcal{H} \) then one can find vectors \( \theta \) and \( \vartheta \) in \( \mathcal{V}_{G \otimes I_R, E} \) such that \( \rho = \text{Tr}_R | \theta \rangle \langle \theta | \), \( \sigma = \text{Tr}_R | \vartheta \rangle \langle \vartheta | \) and \( \| \theta - \vartheta \| \leq \sqrt{\varepsilon} \). By noting that the trace norm does not increase under partial trace and by using the inequality \( \| | \alpha \rangle \langle \beta | - | \varphi \rangle \langle \psi | \|_1 \leq \| \alpha \| \| \beta - \psi | + || \psi \| \| \alpha - \varphi \| \) and Lemma 1 we obtain

\[
\| A \rho B^* - A \sigma B^* \|_1 \leq \sqrt{\varepsilon} \| A \|_E^G / \varepsilon \| B \otimes I_R \| + \sqrt{\varepsilon} \| B \|_E^G / \varepsilon \| A \otimes I_R \|. \]

This implies (18), since \( \| A \otimes I_R \| \leq \| A \|_E^G \) and \( \| B \otimes I_R \| \leq \| B \|_E^G \) by Lemma 1. □
3 The main results

For a completely positive (CP) linear map \( \Phi : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B) \) the Stinespring theorem (cf.\[19\]) implies existence of a separable Hilbert space \( \mathcal{H}_E \) and an operator \( V_\Phi : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E \) such that

\[
\Phi(\rho) = \text{Tr}_E V_\Phi \rho V_\Phi^*, \quad \rho \in \mathcal{T}(\mathcal{H}_A),
\]

where \( \text{Tr}_E \) denotes the partial trace over \( \mathcal{H}_E \). If \( \Phi \) is trace-preserving (correspondingly, trace-non-increasing) then \( V_\Phi \) is an isometry (correspondingly, contraction) [7, Ch.6].

The norm of complete boundedness of a linear map between the algebras \( \mathfrak{B}(\mathcal{H}_B) \) and \( \mathfrak{B}(\mathcal{H}_A) \) (cf. [11]) induces (by duality) the diamond norm

\[
\| \Phi \|_{\diamond} = \sup_{\omega \in \mathcal{T}(\mathcal{H}_{AR}), \| \omega \|_1 \leq 1} \| \Phi \otimes \text{Id}_R(\omega) \|_1
\]

on the set of all linear maps between Banach spaces \( \mathcal{T}(\mathcal{H}_A) \) and \( \mathcal{T}(\mathcal{H}_B) \), where \( \mathcal{H}_R \) is a separable Hilbert space and \( \mathcal{H}_{AR} = \mathcal{H}_A \otimes \mathcal{H}_R \) [1]. If \( \Phi \) is a Hermitian preserving map then the supremum in (20) can be taken over the set \( \mathfrak{G}(\mathcal{H}_{AR}) \) [21, Ch.3].

The diamond norm is widely used in the quantum theory, but the convergence induced by this norm is too strong for description of physical perturbations of infinite-dimensional quantum channels: there exist quantum channels with close physical parameters such that the diamond norm distance between them equals to 2 [24]. The reason of this inconsistency is pointed briefly in the Introduction. By taking it into account the energy-constrained diamond norms (called ECD norms in what follows)

\[
\| \Phi \|_{G, E} = \sup_{\omega \in \mathfrak{G}(\mathcal{H}_{AR}) : \text{Tr} G \omega_A \leq E} \| \Phi \otimes \text{Id}_R(\omega) \|_1, \quad E > 0,
\]

on the set \( \mathfrak{G}(A, B) \) of Hermitian-preserving linear maps from \( \mathcal{T}(\mathcal{H}_A) \) to \( \mathcal{T}(\mathcal{H}_B) \) are introduced independently in [17] and [24] (here \( G \) is a positive operator on the space \( \mathcal{H}_A \) satisfying condition [3] treated as a Hamiltonian of a quantum system \( A \) [1]).

In [24] it is shown that for any given \( \Phi \in \mathfrak{G}(A, B) \) the nondecreasing nonnegative function \( E \mapsto \| \Phi \|_{G, E} \) is concave on \( \mathbb{R}_+ \). This implies that

\[
\| \Phi \|_{0, E_1}^G \leq \| \Phi \|_{0, E_2}^G \leq (E_2/E_1) \| \Phi \|_{0, E_1}^G \quad \text{for any } E_2 > E_1 > 0.
\]

Hence for given operator \( G \) all the norms \( \| \cdot \|_{G, E}, \quad E > 0 \), are equivalent on \( \mathfrak{G}(A, B) \).

In [17, 24] it is shown that the ECD norms induce adequate metric on the set of infinite-dimensional quantum channels which is consistent with the energy separations of quantum states. In particular, this metric has an operational interpretation in terms of discriminating quantum channels with test states of bounded energy [24]. If \( G \) is a discrete unbounded operator (see Def[11]) then any of the norms (21) generates the

\[\text{Slightly different energy-constrained diamond norm is used in [12].}\]
strong convergence on the set of quantum channels \[ \text{This holds, for example, if } G \text{ is the Hamiltonian of a multi-mode quantum oscillator [17 Ch.12].} \]

Denote by \( \mathfrak{F}(A,B) \) the cone of all CP linear maps from \( \mathfrak{T}(H_A) \) into \( \mathfrak{T}(H_B) \). The cone \( \mathfrak{F}(A,B) \) is complete w.r.t. the metric induced by the diamond norm but not complete w.r.t. the metric induced by the ECD norm (provided that \( G \) is an unbounded operator). Our aim is to describe the completion of the cone \( \mathfrak{F}(A,B) \) and of its important subsets w.r.t. the metric induced by the ECD norm.

Let \( H_E \) be a separable Hilbert space and \( V \) an arbitrary operator in the Banach space \( \mathfrak{B}_E^0(H_A,H_{BE}) \) (defined in Section 2.2). Lemma 2 implies that \( \rho \mapsto V \rho V^* \) is an affine map from the set \( \mathfrak{G}_G(H_A) \) of all states \( \rho \) in \( \mathfrak{S}(H_A) \) with finite energy \( \text{Tr} G \rho \) into the cone \( \mathfrak{T}(H_{BE}) \) correctly defined by the formula

\[
V \rho V^* = \sum_i |V \varphi_i \rangle \langle V \varphi_i|,
\]

where \( \rho = \sum_i |\varphi_i \rangle \langle \varphi_i| \) is any decomposition of \( \rho \) into 1-rank operators. So, we may define the affine map

\[
\Phi(\rho) = \text{Tr}_E V \rho V^* \quad (24)
\]

from the set \( \mathfrak{G}_G(H_A) \) into the cone \( \mathfrak{T}(H_B) \) which can be extended to a unique linear map from the linear span \( \mathfrak{T}_G(H_A) \) of \( \mathfrak{G}_G(H_A) \) into the space \( \mathfrak{T}(H_B) \).

We will say that two maps having form \((24)\) are \( G \)-equivalent if they coincide on the set \( \mathfrak{G}_G(H_A) \). In what follows we will identify \( G \)-equivalent maps.

Let \( \mathfrak{F}_G(A,B) \) be the set of all maps having form \((24)\) for some separable Hilbert space \( H_E \) and operator \( V \in \mathfrak{B}_E^0(H_A,H_{BE}) \).[5] By the Stinespring representation \[ (19) \] the cone \( \mathfrak{F}(A,B) \) is naturally embedded into the set \( \mathfrak{F}_G(A,B) \) (the map \( \Phi \) defined in \((24)\) belongs to the cone \( \mathfrak{F}(A,B) \) if and only if the operator \( V \) belongs to the subset \( \mathfrak{B}(H_A,H_{BE}) \) of \( \mathfrak{B}_G^0(H_A,H_{BE}) \)).

**Definition 2.** The operator \( V \) in any representation \((24)\) of a map \( \Phi \in \mathfrak{F}_G(A,B) \) will be called representing operator for this map.

Let \( \Phi \) be any map in \( \mathfrak{F}_G(A,B) \) with representing operator \( V \in \mathfrak{B}^0_G(H_A,H_{BE}) \). By Lemma 1 for any separable Hilbert space \( H_R \) the operator \( V \otimes I_R \) belongs to the space \( \mathfrak{B}^0_{G \otimes I_R}(H_{AR},H_{BER}) \) and \( \| V \otimes I_R \|^E_{G \otimes I_R} = \| V \|^E_E \). By Lemma 2 the map

\[
\Theta(\omega) = \text{Tr}_E [V \otimes I_R] \omega [V \otimes I_R]^* \quad (25)
\]

is well defined on the set \( \mathfrak{C}_{G \otimes I_R}(H_{AR}) \) and uniformly continuous on the set \( \mathfrak{C}_{G \otimes I_R,E} \) for any \( E > 0 \). It follows that this map does not depend on the representing operator \( V \). Indeed, let \( V' \in \mathfrak{B}^0_G(H_A,H_{BE}) \) be another representing operator for \( \Phi \) and \( \Theta' \) the
corresponding map \( \Phi \). It suffices to show that \( \Theta \) and \( \Theta' \) coincide at any pure state \( |\varphi\rangle \langle \varphi| \) in \( S_{G \otimes I_R}(H_{AR}) \). Any such state can be represented as

\[
|\varphi\rangle \langle \varphi| = \sum_{i,j} |\alpha_i\rangle \langle \alpha_j| \otimes |\beta_i\rangle \langle \beta_j|,
\]

for some collections \( \{\alpha_i\} \) and \( \{\beta_i\} \) of orthogonal vectors in \( H_A \) and \( H_R \) such that

\[
\sum_i \langle \alpha_i|G|\alpha_i\rangle \leq E, \quad \sum_i ||\alpha_i||^2 = 1 \quad \text{and} \quad ||\beta_j|| = 1 \quad \text{for all} \quad j
\]

for some finite \( E \). The fist inequality in (26) implies that all the vectors \( \alpha_i \) belong to the set \( V_G \) defined in (25). Hence, all the operators \( |\alpha_i\rangle \langle \alpha_j| \) belong to the set \( \mathfrak{F}_G(H_A) \). For any given \( n \) let

\[
|\varphi_n\rangle \langle \varphi_n| = \sum_{i,j=1}^n |\alpha_i\rangle \langle \alpha_j| \otimes |\beta_i\rangle \langle \beta_j|.
\]

Then

\[
\Theta(|\varphi_n\rangle \langle \varphi_n|) = \Theta'(|\varphi_n\rangle \langle \varphi_n|) = \sum_{i,j=1}^n \Phi(|\alpha_i\rangle \langle \alpha_j|) \otimes |\beta_i\rangle \langle \beta_j|.
\]

Since the state \( |\varphi\rangle \langle \varphi| \) and all the operators \( |\varphi_n\rangle \langle \varphi_n| \) belong to the set \( \mathfrak{C}_G \otimes I_{R,E} \); the continuity of the maps \( \Theta \) and \( \Theta' \) on this set implies that \( \Theta(|\varphi\rangle \langle \varphi|) = \Theta'(|\varphi\rangle \langle \varphi|) \).

Thus, for any map \( \Phi \) in \( \mathfrak{F}_G(A,B) \) and any separable Hilbert space \( H_R \) there is a unique map \( \Phi \otimes \text{Id}_R \) in \( \mathfrak{F}_G(AR,BR) \) defined in (28). This property can be treated as \textit{complete positivity} of \( \Phi \). It implies that for any map \( \Phi \) in the real linear span of \( \mathfrak{F}_G(A,B) \) we may define the ECD-norm \( \|\Phi\|^G_{E,E} \) by formula (21). So, for any two maps \( \Phi \) and \( \Psi \) in \( \mathfrak{F}_G(A,B) \) we may define the distance

\[
D_E^G(\Phi, \Psi) \doteq \|\Phi - \Psi\|^G_{E,E}, \quad E > 0.
\]

It is clear that \( D_E^G \) is a metric on the set \( \mathfrak{F}_G(A,B) \) for any \( E > 0 \) coinciding with the ECD-norm metric on the cone \( \mathfrak{F}^G(A,B) \).

Note that \( \mathfrak{F}_G(A,B) \) is a cone as well. Indeed, if \( \Phi \in \mathfrak{F}_G(A,B) \) then the map \( \lambda \Phi \) obviously belongs to the set \( \mathfrak{F}_G(A,B) \) for any positive \( \lambda \). If \( \Phi_k(\rho) = \text{Tr}_{E_k} V_k \rho V_k^* \), where \( V_k \in \mathfrak{B}_G(H_A, H_{BE_k}) \), \( k = 1, 2 \), then

\[
(\Phi_1 + \Phi_2)(\rho) = \text{Tr}_{E} \tilde{V} \rho \tilde{V}^*, \quad \rho \in \mathfrak{F}_G(H_A),
\]

where \( H_{\tilde{E}} = H_{E_1} \oplus H_{E_2} \) and \( \tilde{V} \) is the operator from \( V_G \) into \( H_{BE_1} \oplus H_{BE_2} \) defined by setting \( \tilde{V}|\varphi\rangle = V_1|\varphi\rangle \oplus V_2|\varphi\rangle \) for any \( \varphi \in V_G \). It is easy to see that \( \tilde{V} \) belongs to the set \( \mathfrak{B}_G(H_A, H_{BE}) \). Thus, the map \( \Phi_1 + \Phi_2 \) belongs to the set \( \mathfrak{F}_G(A,B) \).

In what follows we will assume that the cone \( \mathfrak{F}_G(A,B) \) is equipped with the metric \( D_E \) defined in (27).
Theorem 1. The cone $\mathfrak{F}_G(A, B)$ coincides with the completion of the cone $\mathfrak{F}(A, B)$ w.r.t. the metric $D^G_E$ induced by the ECD norm. Any map $\Phi$ in $\mathfrak{F}_G(A, B)$ with representing operator $V \in \mathfrak{B}_G^0(H_A, H_{BE})$ is approximated in the metric $D^G_E$ by the sequence of maps

$$\Phi_n(\rho) = \text{Tr}_E V P_n \rho P_n V^*$$

from the cone $\mathfrak{F}(A, B)$ determined by a sequence $\{E_n\} \subset \mathbb{R}_+$ tending to $+\infty$, where $P_n$ is the spectral projector of $G$ corresponding to the interval $[0, E_n]$. Quantitatively,

$$D^G_E(\Phi_n, \Phi) \leq 2\sqrt{E/E_n} \|V\|_{E/E_n}^G \|V\|_E^G \quad \text{for all } n \text{ such that } E_n \geq E. \quad (28)$$

A diamond norm bounded subset of $\mathfrak{F}(A, B)$ is complete w.r.t. the metric $D^G_E$ if and only if it is closed w.r.t. this metric.

The last assertion of Theorem 1 implies the following

Corollary 1. The set of quantum channels and the set of quantum operations are complete w.r.t. the metric induced by the ECD norm.

Proof of Theorem 1. Show first that $\mathfrak{F}_G(A, B)$ is a complete metric space. Assume that $\{\Phi_n\}$ is a Cauchy sequence in $\mathfrak{F}_G(A, B)$. Take a subsequence $\{\Phi_{n_k}\}$ such that $D^G_E(\Phi_{n_k}, \Phi_{n_{k+1}}) \leq 4^{-k}$. Lemma 3 below and the right inequality in (31) imply existence of a separable Hilbert space $H_E$ and a sequence $\{V_k\}$ of operators in $\mathfrak{B}_G^0(H_A, H_{BE})$ such that $\Phi_{n_k}(\rho) = \text{Tr}_E V_k \rho V_k^*$ for all $\rho \in \mathfrak{F}_G(A, B)$ and

$$\|V_k - V_{k+1}\|_E^G \leq \sqrt{D^G_E(\Phi_{n_k}, \Phi_{n_{k+1}})} \leq 2^{-k} \quad \forall k.$$ 

Thus, $\{V_k\}$ is a Cauchy sequence in the Banach space $\mathfrak{B}_G^0(H_A, H_{BE})$ and hence it has a limit $V_0 \in \mathfrak{B}_G^0(H_A, H_{BE})$. Lemma 4 below implies that the subsequence $\{\Phi_{n_k}\}$ converges to the map $\Phi_0(\rho) = \text{Tr}_E V_0 \rho V_0^*$ w.r.t. the metric $D^G_E$. Hence the whole sequence $\{\Phi_n\}$ converges to the map $\Phi_0$ w.r.t. the metric $D^G_E$ as well.

To show the density of $\mathfrak{F}(A, B)$ in $\mathfrak{F}_G(A, B)$ it suffices to prove (28), since Lemma 4 in [16] implies that all the operators $VP_n$ are bounded and hence all the maps $\Phi_n$ belong to the cone $\mathfrak{F}(A, B)$.

For given $n$ let $\rho$ be a finite rank state such that $\text{Tr} G \rho \leq E$ and $x_n = 1 - \text{Tr} P_n \rho > 0$. Let $\tilde{P}_n = I_A - P_n$ and $\rho_n = x_n^{-1} \tilde{P}_n \rho \tilde{P}_n$. We have

$$\text{Tr} V \tilde{P}_n \rho \tilde{P}_n V^* = x_n \text{Tr} V \rho_n V^* \leq x_n \left[\|V\|_{E/x_n}^G\right]^2 \leq (E/E_n) \left[\|V\|_{E/E_n}^G\right]^2.$$ 

The first inequality follows from the definition of the $E$-norm and the inequality $\text{Tr} G \rho_n \leq E/x_n$, the second one follows from concavity of the function $E \mapsto \left[\|V\|_{E}^G\right]^2$, Lemma 3 below and the inequality $x_n \leq E/E_n$ (which holds, since $\text{Tr} G \rho \leq E$). The above estimate implies that $\|V - V P_n\|_E^G \leq \sqrt{E/E_n} \|V\|_{E/E_n}^G$. So, inequality (28) follows from Lemma 4 below (since it is easy to see that $\|V P_n\|_E^G \leq \|V\|_E^G$ for all $n$). \(\square\)

7Since the operator $V$ satisfies condition (12), the r.h.s. of (28) tends to zero as $n \to +\infty$. 

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Lemma 3. \[23\] If \( f \) is a concave nonnegative function on \([0, +\infty)\) then for any positive \( x < y \) and any \( z \geq 0 \) the inequality \( x f(z/x) \leq y f(z/y) \) holds.

Lemma 4. Let \( \Phi \) and \( \Psi \) be any maps in \( \Sigma_G(A, B) \) with representing operators \( V_\Phi \) and \( V_\Psi \) in \( \mathfrak{B}_G^0(H_A, H_{BE}) \). Then

\[
D_E^G(\Phi, \Psi) \leq \|V_\Phi - V_\Psi\|_E^G \left( \|V_\Phi\|_E^G + \|V_\Psi\|_E^G \right).
\]

Proof. Let \( \omega \) be any state in \( \mathcal{S}(H_{AR}) \) such that \( \text{Tr} G \omega_A \leq E \). We have

\[
\|\Phi \otimes \text{Id}_R(\omega) - \Psi \otimes \text{Id}_R(\omega)\|_1 \leq \|V_\Phi \otimes I_R \cdot \omega \cdot V_\Phi^* \otimes I_R - V_\Psi \otimes I_R \cdot \omega \cdot V_\Psi^* \otimes I_R\|_1
\]

\[
\leq \|(V_\Phi - V_\Psi) \otimes I_R \cdot \omega \cdot V_\Phi^* \otimes I_R\|_1 + \|V_\Psi \otimes I_R \cdot \omega \cdot (V_\Phi^* - V_\Psi^*) \otimes I_R\|_1
\]

\[
\leq \|(V_\Phi - V_\Psi) \otimes I_R\|_E^{\otimes I_R} \|V_\Phi \otimes I_R\|_E^{\otimes I_R} + \|(V_\Phi - V_\Psi) \otimes I_R\|_E^{\otimes I_R} \|V_\Psi \otimes I_R\|_E^{\otimes I_R}
\]

\[
\leq \|V_\Phi - V_\Psi\|_E^G \|V_\Phi\|_E^G + \|V_\Phi - V_\Psi\|_E^G \|V_\Psi\|_E^G.
\]

The first and the second inequalities follow from the properties of the trace norm (the non-increasing under partial trace and the triangle inequality), the third inequality follows from Lemma 2, the last one – from Lemma 1.

The Bures distance between any maps \( \Phi \) and \( \Psi \) in \( \mathfrak{F}(A, B) \) is defined as

\[
\beta(\Phi, \Psi) = \sup_{\omega \in \mathfrak{S}(H_{AR})} \beta(\Phi \otimes \text{Id}_R(\omega), \Psi \otimes \text{Id}_R(\omega)), \tag{29}
\]

where \( \beta \) in the r.h.s. is the Bures distance between positive trace class operators defined in \([11]\) and \( H_R \) is an infinite-dimensional separable Hilbert space \([3, 9]\).

Following \([14]\) consider the energy-constrained Bures distance

\[
\beta_E(\Phi, \Psi) = \sup_{\omega \in \mathfrak{S}(H_{AR}), \text{Tr} H_{AR} \omega_A \leq E} \beta(\Phi \otimes \text{Id}_R(\omega), \Psi \otimes \text{Id}_R(\omega)), \quad E > 0, \tag{30}
\]

between any maps \( \Phi \) and \( \Psi \) in \( \mathfrak{F}(A, B) \). The arguments before \([23]\) show that this distance is well defined by the same formula for any maps \( \Phi \) and \( \Psi \) from the cone \( \mathfrak{F}_G(A, B) \).

Remark 1. For \( \Phi \) and \( \Psi \) in \( \mathfrak{F}_G(A, B) \) the infimum in \((30)\) can be taken only over pure states \( \omega \). This follows from the freedom of choice of \( R \), since the Bures distance between positive trace class operators defined in \([11]\) does not increase under partial trace: \( \beta(\rho, \sigma) \geq \beta(\rho_A, \sigma_A) \) for any \( \rho \) and \( \sigma \) in \( \mathfrak{S}_+(H_{AR}) \) \([7, 21, 22]\). Inequality \([3]\) and definition \([23]\) imply that

\[
\frac{D_E^G(\Phi, \Psi)}{\sqrt{\|\Phi\|_{\infty, E}^G + \|\Psi\|_{\infty, E}^G}} \leq \beta_E(\Phi, \Psi) \leq \sqrt{D_E^G(\Phi, \Psi)} \tag{31}
\]

for any maps \( \Phi \) and \( \Psi \) in \( \mathfrak{F}_G(A, B) \).
Lemma 5. Let $\{\Phi_n\}$ be a sequence of maps in $\mathfrak{F}_G(A,B)$. There exist a separable Hilbert space $\mathcal{H}_E$ and a sequence $\{V_n\}$ of operators in $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE})$ such that

$$\Phi_n(\rho) = \text{Tr}_E V_n \rho V_n^*, \rho \in \mathfrak{F}_G(\mathcal{H}_A), \quad \text{and} \quad \|V_{n+1} - V_n\|_E^2 = \beta_E(\Phi_{n+1}, \Phi_n) \quad \forall n.$$ 

If all the maps $\Phi_n$ lie in $\mathfrak{F}(A,B)$ then all the operators $V_n$ can be taken in $\mathfrak{B}(\mathcal{H}_A, \mathcal{H}_{BE})$.

Proof. We may assume that all the maps $\Phi_n$ have representing operators $V_n$ in $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE})$, where $\mathcal{H}_E$ is an infinite-dimensional separable Hilbert space. Let $\mathcal{H}_E = \bigoplus_{k=1}^{+\infty} \mathcal{H}_E^k$, where $\mathcal{H}_E^k$ is a copy of $\mathcal{H}_E$ for each $k$. Let $\tilde{V}_1$ be an operator from $V_G \subset \mathcal{H}_A$ into $\mathcal{H}_B \otimes \mathcal{H}_E$ defined by setting

$$\tilde{V}_1(\varphi) = V_1(\varphi) \oplus |0\rangle \otimes |0\rangle \ldots$$

for any $\varphi \in V_G$, where the first and the second summands here lie, respectively, in $\mathcal{H}_B \otimes \mathcal{H}_E$ and $\mathcal{H}_B \otimes \mathcal{H}_E^2$ and $C_2$ is a contraction in $\mathfrak{B}(\mathcal{H}_E)$.

Assume now that $\Phi = \Phi_2$, $\Psi = \Phi_3$ and $V_0 = (I_B \otimes C_2) V_2 \oplus (I_B \otimes \sqrt{I_E - C_2^* C_2}) V_2$ is an operator from $V_G$ into $\mathcal{H}_B \otimes (\mathcal{H}_E \oplus \mathcal{H}_E)$ representing the map $\Phi$. Then Lemma 6 below implies existence of an operator $\tilde{V}_3$ from $V_G \subset \mathcal{H}_A$ into $\mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\|\tilde{V}_2 - \tilde{V}_1\|_E^2 = \beta_E(\Phi_3, \Phi_2).$$

This operator is defined by setting

$$\tilde{V}_2(\varphi) = (I_B \otimes C_2) V_2(\varphi) \oplus (I_B \otimes \sqrt{I_E - C_2^* C_2}) V_2(\varphi) \oplus |0\rangle \otimes |0\rangle \ldots$$

for any $\varphi \in V_G$, where the first and the second summands here lie, respectively, in $\mathcal{H}_B \otimes (\mathcal{H}_E^2 \oplus \mathcal{H}_E^3)$ and $\mathcal{H}_B \otimes (\mathcal{H}_E^3 \oplus \mathcal{H}_E^2)$, $U_3$ is some unitary transformation of $\mathcal{H}_E$ onto $\mathcal{H}_E^2 \oplus \mathcal{H}_E \oplus \mathcal{H}_E^3$ and $C_3$ is a contraction in $\mathfrak{B}(\mathcal{H}_E^2)$.

By using this way based on Lemma 5 sequentially we obtain operators $\tilde{V}_n$, $n \geq 4$, from $V_G \subset \mathcal{H}_A$ into $\mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\|\tilde{V}_{n+1} - \tilde{V}_n\|_E^2 = \beta_E(\Phi_{n+1}, \Phi_n) \quad \forall n.$$ 

The operator $\tilde{V}_n$ is defined by setting

$$\tilde{V}_n(\varphi) = (I_B \otimes C_n U_{2n-2}) V_n(\varphi) \oplus (I_B \otimes \sqrt{I_{E_{2n-2}} - C_n^* C_n U_{2n-2}}) V_n(\varphi) \oplus |0\rangle \otimes |0\rangle \ldots$$

for any $\varphi \in V_G$, where the first and the second summands here lie, respectively, in $\mathcal{H}_B \otimes (\mathcal{H}_E^2 \oplus \ldots \oplus \mathcal{H}_E^{2n-2})$ and $\mathcal{H}_B \otimes (\mathcal{H}_E^{2n-2} \oplus \ldots \oplus \mathcal{H}_E^2)$, $U_{2n-2}$ is some unitary transformation of $\mathcal{H}_E$ onto the direct sum $\mathcal{H}_{E_{2n-2}}$ of $2^{n-2}$ copies of $\mathcal{H}_E$ and $C_n$ is a contraction in $\mathfrak{B}(\mathcal{H}_{E_{2n-2}})$.

By the construction $\Phi_n(\rho) = \text{Tr}_E \tilde{V}_n \rho (\tilde{V}_n)^*$ for all $\rho \in \mathfrak{F}_G(\mathcal{H}_A)$ and any $n$. $\square$

The following lemma describes a generalization of the basic construction from the proof of Theorem 1 in $\mathfrak{F}$. 

Lemma 6. Let $\Phi$ and $\Psi$ be any maps in $\mathfrak{F}_G(A,B)$ with representing operators $V_0$ and $V_0$ in $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE})$, where $\mathcal{H}_E$ is a separable Hilbert space. Let $\mathcal{H}_E = \mathcal{H}_E^1 \oplus \mathcal{H}_E^2$, where $\mathcal{H}_E^1$ and $\mathcal{H}_E^2$ are copies of $\mathcal{H}_E$ and $C$ is a contraction in $\mathfrak{B}(\mathcal{H}_{E_1}, \mathcal{H}_{E_1}) \cong \mathfrak{B}(\mathcal{H}_E)$. 


The operators $\tilde{V}_\psi$ and $\tilde{V}_\psi$ from $V_G \subset H_A$ into $H_B \otimes H_{E_1} \oplus H_B \otimes H_{E_2}$ defined by setting

\[
\tilde{V}_\psi |\varphi\rangle = V_\psi |\varphi\rangle \oplus |0\rangle, \quad \tilde{V}_\psi^C |\varphi\rangle = (I_B \otimes C)V_\psi |\varphi\rangle \oplus \left( I_B \otimes \sqrt{I_E - C^*C} \right) V_\psi |\varphi\rangle \quad (32)
\]

for any $\varphi \in V_G$ (where we assume that the operators $V_\phi$ and $V_\psi$ act from $V_G$ to $H_B \otimes H_{E_1}$ and $H_B \otimes H_{E_2}$ correspondingly) are representing the maps $\Phi$ and $\Psi$. By inequality (31) to prove the lemma it suffices to show that $\inf_{C \in B^0_G(H_A, H_{BE})} \|\tilde{V}_\psi^C - \tilde{V}_\psi\|^2_E = \beta_E(\Phi, \Psi)$, where $B^0_1(H_E)$ is the unit ball of $B(H_E)$, and that this infimum is attainable.

Denote by $\mathcal{S}_{G,E}$ the set of states $\rho$ in $\mathcal{S}(H_A)$ such that $\text{Tr}_G \rho \leq E$. It is easy to see that

\[
\|\tilde{V}_\psi^C - \tilde{V}_\psi\|^2_E = \sup_{\rho \in \mathcal{S}_{G,E}} \sqrt{\text{Tr}\Phi(\rho) + \text{Tr}\Psi(\rho) - 2\Re \text{Tr}(I_B \otimes C)V_\phi \rho V_\psi^*},
\]

where $V_\phi \rho V_\psi^*$ is the trace class operator well defined for any state $\rho$ in $\mathcal{S}_{G,E}$ in accordance with Lemma 2A.

By Lemma 2B the function $\rho \mapsto V_\phi \rho V_\psi^*$ is affine on the set $\mathcal{S}_{G,E}$ for any $E > 0$ and takes values in $\Sigma_+(H_{BE})$. This and the $\sigma$-weak compactness of the unit ball $B_1(H_E)$ (cf.3) make it possible to apply Ky Fan’s minimax theorem (cf.18) to change the order of the optimization as follows

\[
\inf_{C \in B^0_1(H_E)} \|\tilde{V}_\psi^C - \tilde{V}_\psi\|^2_E = \inf_{C \in B^0_1(H_E)} \sup_{\rho \in \mathcal{S}_{G,E}} \sqrt{\text{Tr}\Phi(\rho) + \text{Tr}\Psi(\rho) - 2\Re \text{Tr}(I_B \otimes C)V_\phi \rho V_\psi^*} = \sup_{\rho \in \mathcal{S}_{G,E}} \inf_{C \in B^0_1(H_E)} \sqrt{\text{Tr}\Phi(\rho) + \text{Tr}\Psi(\rho) - 2\Re \text{Tr}(I_B \otimes C)V_\phi \rho V_\psi^*} = \sup_{\rho \in \mathcal{S}_{G,E}} \sqrt{|\text{Tr}(I_B \otimes C)V_\phi \rho V_\psi^*| - 2 \sup_{C \in B^0_1(H_E)} |\text{Tr}(I_B \otimes C)V_\phi \rho V_\psi^*|}.
\]

Note that all the above infima are attainable, since the expression under the first square root is a $\sigma$-weak continuous function of $C$ on the $\sigma$-weak compact set $B_1(H_E)$.

By Lemma 1 for any Hilbert space $H_R$ the operators $V_\phi \otimes I_R$ and $V_\psi \otimes I_R$ are well defined and uniformly continuous on the set $\{ \eta \in H_A \otimes H_R | \langle \eta | G \otimes I_R | \eta \rangle \leq E \}$. Hence for any state $\rho$ in $\mathcal{S}_{G,E}$ we have

\[
\sup_{C \in B^0_1(H_E)} |\text{Tr}(I_B \otimes C)V_\phi \rho V_\psi^*| = \sup_{C \in B^0_1(H_E)} |\langle V_\psi \otimes I_R | \varphi \rangle | I_B \otimes C | V_\phi \otimes I_R \varphi | \rangle |, \quad (33)
\]

where $\varphi$ is a purification of $\rho$, i.e. a vector in $H_A \otimes H_R$ such that $\text{Tr}_R |\varphi\rangle \langle \varphi | = \rho$.

Since the vectors $V_\phi \otimes I_R |\varphi\rangle$ and $V_\psi \otimes I_R |\varphi\rangle$ in $H_{BER}$ are purifications of the operators $\Phi \otimes \text{Id}_R(\omega)$ and $\Psi \otimes \text{Id}_R(\omega)$ in $\mathcal{T}(H_{BR})$, where $\omega = |\varphi\rangle \langle \varphi |$, and the set $B_1(H_E)$ contains all isometries in $B(H_E)$, Uhlmann’s theorem [20,22] implies that the quantity in the r.h.s. of (33) is not less than the square root of the fidelity of these operators.
defined in (2). On the other hand the r.h.s. of (33) is not greater than the square root of the fidelity, since \( \hat{V}_\psi \otimes I_R |\psi\) and \( \hat{V}_\psi \otimes I_R |\varphi\) are purifications of the operators \( \Phi \otimes \text{Id}_R(\omega) \) and \( \Psi \otimes \text{Id}_R(\omega) \) as well and
\[
\langle V_\psi \otimes I_R \varphi |I_B \otimes C|V_\psi \otimes I_R \varphi \rangle = \langle V_\psi \otimes I_R \varphi |V_\psi \otimes I_R \varphi \rangle.
\]
Thus, since \( \text{Tr} \Phi \otimes \text{Id}_R(\omega) = \text{Tr} \Phi(\rho) \) and \( \text{Tr} \Psi \otimes \text{Id}_R(\omega) = \text{Tr} \Psi(\rho) \), we have
\[
\inf_{C \in \mathcal{B}_1(H_E)} \| \hat{V}_\psi^C - \hat{V}_\psi \|_E^G = \sup_{\omega \in \mathcal{G}_{G,E}} \sqrt{\text{Tr} \hat{\Phi}(\omega) + \text{Tr} \hat{\Psi}(\omega) - 2\sqrt{F(\hat{\Phi}(\omega), \hat{\Psi}(\omega))}}
\]
\[
= \sup_{\omega \in \mathcal{G}_{G,E}} \beta(\hat{\Phi}(\omega), \hat{\Psi}(\omega)) = \beta_E(\Phi, \Psi),
\]
where \( \hat{\Theta} = \Theta \otimes \text{Id}_R, \Theta = \Phi, \Psi \), the second equality follows from the definition of the Bures distance and the third one – from Remark 1. \( \Box \)

4 Properties of maps in \( \mathcal{F}_G(A, B) \)

4.1 General properties and equivalent definition of \( \mathcal{F}_G(A, B) \)

By definition a liner map \( \Phi : \mathfrak{L}_G(H_A) \mapsto \mathfrak{L}(H_B) \) belongs to the cone \( \mathcal{F}_G(A, B) \) if there exist a separable Hilbert space \( H \) and an operator \( V \) in \( \mathcal{B}_G(H_A, H_{BE}) \) such that
\[
\Phi(\rho) = \text{Tr}_E V \rho V^*, \quad \rho \in \mathfrak{L}_G(H_A),
\]
where the operator \( V \rho V^* \) is defined in (13). By Lemma 2 the map \( \Phi \) is uniformly continuous on the set \( \mathfrak{E}_{G,E} = \{ \rho \in \mathfrak{L}_+(H_A) | \text{Tr} G \rho \leq E, \text{Tr} \rho \leq 1 \} \) for any \( E \geq 0 \).

A generalization of this property is presented in the following proposition, which also gives a characterization of the cone \( \mathcal{F}_G(A, B) \) in the case of discrete operator \( G \).

**Proposition 1.** An arbitrary map \( \Phi \) in \( \mathcal{F}_G(A, B) \) has the following properties:

a) For any separable Hilbert space \( H \) the map \( \Phi \otimes \text{Id}_R \) naturally defined on the set \( \mathfrak{L}_G(H_A) \otimes \mathfrak{L}(H_R) \) has a linear extension to the set \( \mathfrak{L}_{G \otimes I_R}(H_{AR}) \) such that:

- \( \Phi \otimes \text{Id}_R(\omega) \geq 0 \) for any positive operator \( \omega \) in \( \mathfrak{L}_{G \otimes I_R}(H_{AR}) \);
- the map \( \Phi \otimes \text{Id}_R \) is uniformly continuous on the set \( \mathfrak{E}_{G \otimes I_R,E} \) for any \( E > 0 \).

b) \( \| \Phi \|^G_{G,E} = o(E) \) as \( E \to +\infty \).

If \( G \) is a discrete operator (Def.1) then any linear map \( \Phi : \mathfrak{L}_G(H_A) \mapsto \mathfrak{L}(H_B) \) with property a) belongs to the cone \( \mathcal{F}_G(A, B) \), i.e. it has the representation (34).

**Remark 2.** The last assertion of Proposition 1 remains valid with the weakened version of property a), in which the uniform continuity of the map \( \Phi \otimes \text{Id}_R \) is replaced by continuity of this map on the set \( \mathfrak{E}_{G \otimes I_R,E} \).

\( ^8 \)\( \mathfrak{L}_G(H_A) \) is the linear span of all states \( \rho \in \mathfrak{S}(H_A) \) with finite energy \( \text{Tr} G \rho \).

\( ^9 \)\( \mathfrak{L}_G(H_A) \otimes \mathfrak{L}(H_R) \) is the linear span of all operators \( \rho \otimes \sigma \), where \( \rho \in \mathfrak{L}_G(H_A) \) and \( \sigma \in \mathfrak{L}(H_R) \).

\( ^{10} \)We denote this extension by the same symbol \( \Phi \otimes \text{Id}_R \).
Proof. In Section 3 (after Def.2) it is shown that for any separable Hilbert space $\mathcal{H}_R$ the map
\[
\mathcal{I}_{G \otimes I_R}(\mathcal{H}_{AR}) \ni \omega \mapsto \operatorname{Tr}_E[V \otimes I_R] \omega [V \otimes I_R]^*
\]
is well defined and doesn’t depend on V (for given $\Phi$). Since $V \otimes I_R \in \mathfrak{B}^0_{G \otimes I_R}(\mathcal{H}_{AR}, \mathcal{H}_{BER})$, this map naturally denoted by $\Phi \otimes \operatorname{Id}_R$ belongs to the cone $\mathfrak{K}_G(AR, BR)$. By Lemma 2 the map $\Phi \otimes \operatorname{Id}_R$ is uniformly continuous on the set $\mathcal{C}_{G \otimes I_R,E}$ for any $E > 0$.

Since $\|V\|_G^2 = o(\sqrt{E})$ as $E \to +\infty$ for any $V$ in $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE})$, we have
\[
\|\Phi\|_{G,E}^2 = \left[\|V\|_G^2\right]^2 = o(E) \quad \text{as} \quad E \to +\infty.
\]

The last assertion of the proposition and Remark 2 follow from Theorem 3B in Section 5 (proved independently). $\square$

If $V_1$ and $V_2$ are arbitrary operators in $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE})$, where $\mathcal{H}_E$ is any separable Hilbert space, then Lemma 2 shows that the formula
\[
\Psi(\rho) = \operatorname{Tr}_E V_1 \rho V_2^*
\]
correctly defines a linear map form $\mathfrak{I}_G(\mathcal{H}_A)$ into $\mathfrak{I}(\mathcal{H}_B)$ which is uniformly continuous on the set $\mathcal{C}_{G,E}$ for any $E > 0$.

**Proposition 2.** A map $\Psi$ belongs to the linear span of $\mathfrak{K}_G(A, B)$ if and only if it has representation (35) for some Hilbert space $\mathcal{H}_E$ and operators $V_1, V_2$ in $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE})$.

Proof. If $\Psi$ belongs to the linear span of $\mathfrak{K}_G(A, B)$ then it can be represented as follows
\[
\Psi = (\Phi_1 - \Phi_2) + i(\Phi_3 - \Phi_4), \quad \Phi_k \in \mathfrak{K}_G(A, B), \quad k = 1, 2, 3, 4.
\]

Let $V_k \in \mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE_k})$ be a representing operator for the map $\Phi_k$. Then the operators $W_1$ and $W_2$ from $\mathcal{V}_G$ to $\mathcal{H}_{BE} = \mathcal{H}_B \otimes (\mathcal{H}_{E_1} \oplus \mathcal{H}_{E_2} \oplus \mathcal{H}_{E_3} \oplus \mathcal{H}_{E_4})$ defined by settings
\[
W_1|\varphi\rangle = V_1|\varphi\rangle \oplus -V_2|\varphi\rangle \oplus iV_3|\varphi\rangle \oplus -iV_4|\varphi\rangle, \quad W_2|\varphi\rangle = V_1|\varphi\rangle \oplus V_2|\varphi\rangle \oplus V_3|\varphi\rangle \oplus V_4|\varphi\rangle
\]
for any $\varphi \in \mathcal{V}_G$, belong to the space $\mathfrak{B}^0_G(\mathcal{H}_A, \mathcal{H}_{BE_k})$. It is easy to see that $\Psi(\rho) = \operatorname{Tr}_{E_k} W_1 \rho W_2^*$ for any $\rho \in \mathfrak{I}_G(\mathcal{H}_A)$.

If $\Psi$ has representation (35) then $\Psi = \frac{1}{4}(\Phi_1 - \Phi_2 + i\Phi_3 - i\Phi_4)$, where
\[
\Phi_1(\rho) = \operatorname{Tr}_E(V_1 + V_2) \rho (V_1 + V_2)^*, \quad \Phi_2(\rho) = \operatorname{Tr}_E(V_1 - V_2) \rho (V_1 - V_2)^*,
\]
\[
\Phi_3(\rho) = \operatorname{Tr}_E(V_1 + iV_2) \rho (V_1 + iV_2)^*, \quad \Phi_4(\rho) = \operatorname{Tr}_E(V_1 - iV_2) \rho (V_1 - iV_2)^*
\]
are maps in $\mathfrak{K}_G(A, B)$. $\square$
4.2 Kraus representation

A CP linear map $\Phi : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ is characterized by the Kraus representation

$$\Phi(\rho) = \sum_k V_k \rho V_k^*, \quad \rho \in \mathcal{T}(\mathcal{H}_A),$$

(36)

where $\{V_k\}$ is a finite or countable collection of operators in $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ such that $\|\sum_k V_k^* V_k\| = \|\Phi\|_\diamond$ [7, 21, 22]. A similar characterization of maps in the cone $\mathcal{F}_G(A, B)$ is presented in the following

**Proposition 3.** A map $\Phi$ belongs to the cone $\mathcal{F}_G(A, B)$ if and only if it can be represented as

$$\Phi(\rho) = \sum_k V_k \rho V_k^*, \quad \rho \in \mathcal{S}_G(\mathcal{H}_A),$$

(37)

where $\{V_k\}$ is a finite or countable collection of operators in $\mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_B)$ such that

$$\|\{V_k\}\|_E^G = \sup_{\rho \in \mathcal{S}(\mathcal{H}_A) : \Tr_G \rho \leq E} \sum_k \Tr V_k \rho V_k^* = o(E) \quad \text{as} \quad E \to +\infty.$$  

(38)

The quantity $\|\{V_k\}\|_E^G$ coincides with the ECD norm $\|\Phi\|_{\diamond, E}$ defined in (27) [11]

**Proof.** If $\Phi \in \mathcal{F}_G(A, B)$ then there exist a Hilbert space $\mathcal{H}_E$ and an operator $V$ in $\mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_BE)$ such that $\Phi(\rho) = \Tr_E V \rho V^*$ for all $\rho$ in $\mathcal{S}_G(\mathcal{H}_A)$.

Let $\{\tau_k\}$ be an orthonormal basis in $\mathcal{H}_E$. By Lemma [7] below for each $k$ there is an operator $V_k$ in $\mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_B)$ such that

$$\langle \psi \otimes \tau_k | V | \varphi \rangle = \langle \psi | V_k | \varphi \rangle \quad \text{for all} \quad \varphi \in \mathcal{V}_G \subset \mathcal{H}_A \text{ and } \psi \in \mathcal{H}_B.$$  

It is easy to see that this relation implies that

$$\Tr V \rho V^* = \sum_k \Tr V_k \rho V_k^* \quad \text{for any } \rho \in \mathcal{T}_+(\mathcal{H}_A) \text{ with finite } \Tr_G \rho$$

(39)

and that $V | \varphi \rangle = \sum_k V_k | \varphi \rangle \otimes | \tau_k \rangle$ for any $\varphi \in \mathcal{V}_G$ (the convergence of the last series follows from the equality $\sum_k \|V_k \varphi\|^2 = \|V \varphi\|^2$ which is a partial case of (39)).

Thus, for any $\varphi \in \mathcal{V}_G$ we have

$$V | \varphi \rangle \langle \varphi | V^* = \sum_{k,j} | V_k \varphi \rangle \langle V_j \varphi | \otimes | \tau_k \rangle \langle \tau_j |.$$  

This implies that (37) holds for any finite rank operator in $\mathcal{T}_+(\mathcal{H}_A)$ with finite $\Tr_G \rho$.

Let $\rho_0$ be an infinite rank state in $\mathcal{S}(\mathcal{H}_A)$ with finite energy $E_0 = \Tr_G \rho_0$ and $\{\rho_n\}$ a sequence of finite rank operators in $\mathcal{T}_+(\mathcal{H}_A)$ such that $\Tr_G \rho_n \leq E_0$ and $\Tr \rho_n \leq 1$.

---

\(^{11}\) $\mathcal{S}_G(\mathcal{H}_A)$ is the subset of $\mathcal{S}(\mathcal{H}_A)$ consisting of states $\rho$ with finite energy $\Tr_G \rho$.

\(^{12}\) The set $\mathcal{V}_G$ is defined in [9].
This allows to show that \( \rho \) relation in (40) show the validity of this representation with \( \varepsilon > 0 \) nondecreasing, it follows from Dini’s lemma that for any sequence can be constructed by using the spectral decomposition of \( \rho_0 \). Lemma 23 implies that

\[
\lim_{n \to +\infty} V \rho_n V^* = V \rho_0 V^* \quad \text{and} \quad \lim_{n \to +\infty} V_k \rho_n V_k^* = V_k \rho_0 V_k^* \quad \forall k, \quad (40)
\]

where the limits are w.r.t. the norm \( \| \cdot \|_1 \). The first relation in (40) and (39) show that

\[
\lim_{n \to +\infty} \sum_k \text{Tr} V_k \rho_n V_k^* = \sum_k \text{Tr} V_k \rho_0 V_k^*.
\]

The second relation in (40) implies that the function \( f_m(\rho) = \sum_{k=1}^m \text{Tr} V_k \rho V_k^* \) is continuous on the compact set \( \{ \rho_n \}_{n \geq 0} \) for each finite \( m \). Since, the sequence \( \{ f_m \} \) is nondecreasing, it follows from Dini’s lemma that for any \( \varepsilon > 0 \) there is \( m \) such that

\[
\left\| \sum_{k>m} V_k \rho_n V_k^* \right\|_1 \leq \sum_{k>m} \text{Tr} V_k \rho_n V_k^* \leq \varepsilon \quad \text{for all } n \geq 0.
\]

This allows to show that

\[
\lim_{n \to +\infty} \sum_k V_k \rho_n V_k^* = \sum_k V_k \rho_0 V_k^*.
\]

Since representation (37) holds with \( \rho = \rho_n \) for all \( n > 0 \), this relation and the first relation in (10) show the validity of this representation with \( \rho = \rho_0 \).

Since \( V \in \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_{BE}) \), relation (38) follows from equality (39).

Let \( \Phi \) be a map with representation (37) and \( \mathcal{H}_E \) a Hilbert space whose dimension coincides with the cardinality of the set \( \{ V_k \} \). Since condition (38) implies that \( \sum_k \| V_k \varphi \|^2 < +\infty \) for any \( \varphi \in \mathcal{V}_G \), we may define for given basis \( \{ \tau_k \} \) in \( \mathcal{H}_E \) an operator \( V \) from \( \mathcal{V}_G \) into \( \mathcal{H}_{BE} \) by setting \( V \varphi = \sum_k V_k \varphi \otimes | \tau_k \rangle \) for any \( \varphi \in \mathcal{V}_G \).

This definition of \( V \) implies that equality (39) holds for any finite rank state \( \rho \) with finite \( \text{Tr} G \rho \). This equality and condition (38) show that \( V \) is an operator in \( \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_{BE}) \).

By using the arguments from the first part of the proof one can show the coincidence of \( \text{Tr} E V \rho V^* \) and \( \sum_k V_k \rho V_k^* \) for any state \( \rho \) in \( \mathcal{S}_G(\mathcal{H}_A) \).

The last assertion of the proposition follows from equality (39), since it implies that \( \| \{ V_k \} \|_E^G = \| V \|_E^G \) for any \( E > 0 \).

**Lemma 7.** If \( V \) is an operator in \( \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_{BE}) \) then for any unit vector \( \tau \) in \( \mathcal{H}_E \) there is a unique operator \( V_\tau \) in \( \mathcal{B}_G^0(\mathcal{H}_A, \mathcal{H}_B) \) such that \( \| V_\tau \|_E^G \leq \| V \|_E^G \) and

\[
\langle \psi \otimes \tau | V \varphi \rangle = \langle \psi | V_\tau \varphi \rangle \quad \text{for all } \varphi \in \mathcal{V}_G \text{ and } \psi \in \mathcal{H}_B. \quad (41)
\]

**Proof.** Let \( \mathcal{H}_A^0 = P_n(\mathcal{H}_A) \), where \( P_n \) is the spectral projector of \( G \) corresponding the interval \([0, n]\). By Lemma 4 in [16] the operator \( V \) is bounded on \( \mathcal{H}_A^n \). Hence, the relation \( \langle \psi \otimes \tau | V | \varphi \rangle = \langle \psi | V_\tau^n | \varphi \rangle \), \( \varphi \in \mathcal{H}_A^0, \psi \in \mathcal{H}_B \) defines an operator \( V_\tau^n \) in \( \mathcal{B}(\mathcal{H}_A^n, \mathcal{H}_B) \). It is clear that \( V_\tau^n+1 | \mathcal{H}_A^n = V_\tau^n \) for all \( n \). So, we may define a linear
obtained in [9] states that

The Kretschmann-Schlingemann-Werner theorem (the KSW-theorem in what follows)

Consider the sequence \( \{V_\tau P_n\} \). By noting that (11) holds with \( V \) and \( V_\tau \) replaced, respectively, by \( VP_n \) and \( V_\tau P_n \) for all \( n \), it is easy to show that

\[
\|V_\tau P_n - V_\tau P_m\|_E^G \leq \|VP_n - VP_m\|_E^G
\]

for all \( m \) and \( n \). Since \( V \in \mathcal{B}_E^0(\mathcal{H}_A,\mathcal{H}_{BE}) \), the sequence \( \{VP_n\} \) converges to the operator \( V \) w.r.t. the norm \( \| \cdot \|_E \) [16, Remark 5]. So, it is a Cauchy sequence w.r.t. the norm \( \| \cdot \|_E^G \) and the above inequality implies that \( \{V_\tau P_n\} \) is a Cauchy sequence in \( \mathcal{B}(\mathcal{H}_A,\mathcal{H}_B) \) w.r.t. this norm. Hence the last sequence has a limit in \( \mathcal{B}_E^0(\mathcal{H}_A,\mathcal{H}_B) \) which will be also denoted by \( V_\tau \) (it is clear that this limit is an extension of the operator \( V_\tau \) defined above to the set \( V_G \)). To prove relation (11) it suffice to note that it holds for any \( \varphi \in \mathcal{H}_A^*, \psi \in \mathcal{H}_B \), since \( \lim_{n \to \infty} VP_n|\varphi\rangle = V|\varphi\rangle \) and \( \lim_{n \to \infty} V_\tau P_n|\varphi\rangle = V_\tau|\varphi\rangle \) for any \( \varphi \in V_G \) by Lemma 11. \( \square \)

4.3 Generalized version of the Kretschmann-Schlingemann-Werner theorem

The Kretschmann-Schlingemann-Werner theorem (the KSW-theorem in what follows) obtained in [9] states that

\[
\frac{\|\Phi - \Psi\|_o}{\sqrt{\|\Phi\|_o} + \sqrt{\|\Psi\|_o}} \leq \inf_{V_\Phi, V_\Psi} \|V_\Phi - V_\Psi\| = \beta(\Phi, \Psi) \leq \sqrt{\|\Phi - \Psi\|_o},
\]

for any maps \( \Phi \) and \( \Psi \) in the cone \( \mathcal{F}(A,B) \), where the infimum is over all common Stinespring representations

\[
\Phi(\rho) = \text{Tr}_E V_\Phi\rho V_\Phi^* \quad \text{and} \quad \Psi(\rho) = \text{Tr}_E V_\Psi\rho V_\Psi^*.
\]

(42)

and \( \beta(\Phi, \Psi) \) is the Bures distance between the maps \( \Phi \) and \( \Psi \) defined in (29).

Lemmas 4 and 6 imply the following generalized version of the KSW theorem.

**Theorem 2.** Let \( G \) be a positive operator in \( \mathcal{H}_A \) satisfying condition (3), \( \| \cdot \|_E^G \), \( \| \cdot \|_{G,E}^o \) and \( \beta_E \) the operator E-norm, the ECD norm and the energy constrained Bures distance defined, respectively, in (4), (27) and (30). Let \( \mathcal{H}_E \) be an infinite-dimensional separable Hilbert space.

For any maps \( \Phi \) and \( \Psi \) in \( \mathcal{F}_G(A,B) \) the following relations holds,

\[
\frac{\|\Phi - \Psi\|_{G,E}^o}{\sqrt{\|\Phi\|_{G,E}^o} + \sqrt{\|\Psi\|_{G,E}^o}} \leq \inf_{V_\Phi, V_\Psi} \|V_\Phi - V_\Psi\|_{E,G}^o \leq \beta_E(\Phi, \Psi) \leq \sqrt{\|\Phi - \Psi\|_{G,E}^o},
\]

where the infimum is over all operators \( V_\Phi \) and \( V_\Psi \) in \( \mathcal{B}_E^0(\mathcal{H}_A,\mathcal{H}_{BE}) \) representing the maps \( \Phi \) and \( \Psi \), i.e. such that (12) holds for any state \( \rho \) in \( \mathcal{S}(\mathcal{H}_A) \) with finite \( \text{Tr} G \rho \).

**Note:** In contrast to the original KSW theorem mentioned above and to the E-version of this theorem presented in [16 Section 3], Theorem 2 do not assert that \( \inf_{V_\Phi, V_\Psi} \|V_\Phi - V_\Psi\|_{E,G}^0 = \beta_E(\Phi, \Psi) \) for any maps \( \Phi \) and \( \Psi \) in \( \mathcal{F}_G(A,B) \).
5 On completion of the set of Hermitian-preserving completely bounded linear maps w.r.t. the ECD norm

Let \( \mathcal{Y}(A, B) \) be the real linear space of all Hermitian-preserving completely bounded linear maps from \( \mathfrak{H}(H_A) \) into \( \mathfrak{H}(H_B) \). The space \( \mathcal{Y}(A, B) \) endowed with the diamond norm (20) is a real Banach space, but \( \mathcal{Y}(A, B) \) is not complete w.r.t. the ECD norm (21) if \( G \) is an unbounded operator. In this section we describe the real Banach space \( \mathcal{Y}_G(A, B) \) containing the completion of \( \mathcal{Y}(A, B) \) w.r.t. the ECD norm, which coincides with this completion if \( G \) is a discrete unbounded operator (Def.1).

Let \( \mathcal{Y}_G(A, B) \) be the set of all linear maps \( \Phi \) from the subset \( \mathfrak{H}_G(H_A) \subset \mathfrak{H}(H_A) \) into \( \mathfrak{H}(H_B) \) with the following properties:

1) \( \Phi(\rho^*) = [\Phi(\rho)]^* \) for all \( \rho \) in \( \mathfrak{H}_G(H_A) \), i.e. \( \Phi \) is Hermitian preserving;

2) For any separable Hilbert space \( H_R \) the map \( \Phi \otimes \text{Id}_R \) naturally defined on the set \( \mathfrak{H}_G(H_A) \otimes \mathfrak{H}(H_R) \) has a linear extension to the set \( \mathfrak{H}_G(H_{AR}) \), which is continuous on the set \( \mathfrak{C}_{G\otimes I_R,E} = \{ \omega \in \mathfrak{F}(H_{AR}) \mid \text{Tr} G \omega_A \leq E, \text{Tr} \omega \leq 1 \} \) for any \( E > 0 \).

By using the arguments after (25) it is easy to show that the extension of \( \Phi \otimes \text{Id}_R \) mentioned in property 2 is unique.

Let \( \mathcal{Y}_G^+(A, B) \) be the subset of \( \mathcal{Y}_G(A, B) \) consisting of maps \( \Phi \) such that \( \Phi \otimes \text{Id}_R(\omega) \) is positive for any positive \( \omega \in \mathfrak{H}_G(H_{AR}) \) and any system \( R \). It is clear that \( \mathcal{Y}_G^+(A, B) \) is a cone in \( \mathcal{Y}_G(A, B) \).

It is easy to see that \( \mathcal{Y}_G(A, B) \) is a real linear space and that \( \| \cdot \|_{\omega,E}^G \) is a norm on \( \mathcal{Y}_G(A, B) \) for any \( E > 0 \). By repeating the arguments in [24] one can show that for any given \( \Phi \in \mathcal{Y}_G(A, B) \) the nondecreasing nonnegative function \( E \mapsto \| \Phi \|_{\omega,E}^G \) is concave on \( \mathbb{R}_+ \). It implies relations (22) which show the equivalence of all the norms \( \| \cdot \|_{\omega,E}^G \), \( E > 0 \), on the set \( \mathcal{Y}_G(A, B) \) for given operator \( G \). In what follows we will assume that the space \( \mathcal{Y}_G(A, B) \) is endowed with the norm \( \| \cdot \|_{\omega,E}^G \) for some \( E > 0 \).

**Theorem 3.** A) \( \mathcal{Y}_G(A, B) \) is a real Banach space containing the set \( \mathcal{Y}(A, B) \).

B) **If** \( G \) **is a discrete operator (Def.1) then** \( \mathcal{Y}_G(A, B) \) **is the completion of** \( \mathcal{Y}(A, B) \) w.r.t. **the norm** \( \| \cdot \|_{\omega,E}^G \) **and** \( \mathcal{Y}_G^+(A, B) = \mathfrak{H}_G(A, B) \)**

**Proof.** A) Note first that condition (5) implies that

\[
\| \Phi \otimes \text{Id}_R(\omega) \|_1 \leq \| \Phi \|_{\omega,E}^G
\]

(43)

for any \( \Phi \) in \( \mathcal{Y}_G(A, B) \) and any operator \( \omega \) in \( \mathfrak{F}(H_{AR}) \) such that \( \text{Tr} G \omega_A \leq E \) and \( \text{Tr} \omega \leq 1 \), where \( H_R \) is a separable Hilbert space. Indeed, let \( \omega \) be such an operator and

\[\mathfrak{H}(H_A)\] is the linear span of all states \( \rho \) in \( \mathfrak{H}(H_A) \) with finite energy \( \text{Tr} G \rho \).

\[\mathfrak{H}_G(A, B)\] is the completion of the cone \( \mathfrak{H}(A, B) \) of CP maps in \( \mathcal{Y}(A, B) \) w.r.t. the norm \( \| \cdot \|_{\omega,E}^G \).
$r = \text{Tr}\omega$. Then $\hat{\omega} = r^{-1}\omega$ is a state such that $\text{Tr}G\hat{\omega}_A \leq E/r$. So, by using concavity of the function $E \mapsto \|\Phi\|_{G,E}^G$ on $\mathbb{R}_+$ mentioned above and Lemma 3, we obtain

$$\|\Phi \otimes \text{Id}_R(\omega)\|_1 = r\|\Phi \otimes \text{Id}_R(\hat{\omega})\|_1 \leq r\|\Phi\|_{G,E/r}^G \leq \|\Phi\|_{G,E}^G.$$  

Let $\{\Phi_n\}$ be a Cauchy sequence in $\mathfrak{Y}_G(A,B)$ and $R$ an infinite-dimensional quantum system. Then for any operator $\omega$ in $\mathfrak{T}_{G \otimes I_R}(\mathcal{H}_{AR})$ the sequences $\{\Phi_n \otimes \text{Id}_R(\omega)\}$ and $\{\Phi_n(\omega_A)\}$ are Cauchy sequences in $\mathfrak{T}(\mathcal{H}_{BR})$ and $\mathfrak{T}(\mathcal{H}_B)$ correspondingly. Hence they have limits which will be denoted, respectively, by $\Theta(\omega)$ and $\Phi(\omega_A)$. By this way we define the Hermitian-preserving linear maps $\Theta : \mathfrak{T}_{G \otimes I_R}(\mathcal{H}_{AR}) \to \mathfrak{T}(\mathcal{H}_{BR})$ and $\Phi : \mathfrak{T}_G(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$.

Note that

$$\lim_{n \to +\infty} \sup_{\omega \in \mathcal{C}_{G \otimes I_R,E}} \|\Phi_n \otimes \text{Id}_R(\omega) - \Theta(\omega)\|_1 = 0,$$

(44)

where $\mathcal{C}_{G \otimes I_R,E} = \{\omega \in \mathfrak{T}_+(\mathcal{H}_{AR}) | \text{Tr}G\omega_A \leq E, \text{Tr}\omega \leq 1\}$. Indeed, if this relation does not hold then (by passing to a subsequence) we may assume that there is $\varepsilon > 0$ and a sequence $\{\omega_n\} \subset \mathcal{C}_{G \otimes I_R,E}$ such that $\|\Phi_n \otimes \text{Id}_R(\omega_n) - \Theta(\omega_n)\|_1 \geq \varepsilon$. By choosing $n$ such that $\|\Phi_n - \Phi_m\|_{G,E} \leq \varepsilon/2$ for all $m > n$, we have

$$\|\Phi_n \otimes \text{Id}_R(\omega_n) - \Theta(\omega_n)\|_1 \leq \|\Phi_n \otimes \text{Id}_R(\omega_n) - \Phi_m \otimes \text{Id}_R(\omega_n)\|_1 + \|\Phi_m \otimes \text{Id}_R(\omega_n) - \Theta(\omega_n)\|_1.$$  

It follows from (43) that the first term in the r.h.s. of this inequality does not exceed $\|\Phi_n - \Phi_m\|_{G,E} \leq \varepsilon/2$, while the second one can be made less than $\varepsilon/2$ by choosing sufficiently large $m$. This contradicts the above assumption.

By definition of $\mathfrak{Y}_G(A,B)$ all the functions $\omega \mapsto \Phi_n \otimes \text{Id}_R(\omega)$ are continuous on the set $\mathcal{C}_{G \otimes I_R,E}$ for each $E > 0$. So, relation (44) implies that the function $\omega \mapsto \Theta(\omega)$ is also continuous on the set $\mathcal{C}_{G \otimes I_R,E}$ for each $E > 0$.

By the definitions of $\Theta$ and $\Phi$ the map $\Theta$ coincides with the map $\Phi \otimes \text{Id}_R$ on the set $\mathfrak{T}_G(\mathcal{H}_A) \otimes \mathfrak{T}(\mathcal{H}_B)$. Thus, $\Theta$ is the extension of $\Phi \otimes \text{Id}_R$ mentioned in property 2 of the above definition of $\mathfrak{Y}_G(A,B)$.

Relation (44) implies that $\|\Phi_n - \Phi\|_{G,E}^G$ tends to zero as $n \to +\infty$.

B) We have to show that any map $\Phi$ in $\mathfrak{Y}_G(A,B)$ can be approximated by a sequence of maps in $\mathfrak{Y}(A,B)$ w.r.t. the norm $\|\cdot\|_{G,E}^G$ provided that the operator $G$ has representation (4).

For given natural $n$ denote by $\mathcal{H}_A^n$ the linear span of the vectors $\tau_0, \ldots, \tau_{n-1}$, i.e. $\mathcal{H}_A^n$ is the subspace of $\mathcal{H}_A$ corresponding to the minimal $n$ eigenvalues of $G$ (taking the multiplicity into account). Denote by $P_n$ the projector onto $\mathcal{H}_A^n$. Consider the quantum channel $\Pi_n(\rho) = \rho P_n + [\text{Tr}\hat{P}_n]\langle\tau_0|\langle\tau_0|$, where $\hat{P}_n = I_A - P_n$.

Let $\Phi$ be a map in $\mathfrak{Y}_G(A,B)$. Note first that the map $\Phi_n = \Phi \circ \Pi_n$ belongs to the set $\mathfrak{Y}(A,B)$ for any natural $n$. Indeed, for arbitrary state $\omega$ in $\mathfrak{T}(\mathcal{H}_{AR})$ we have

$$\|\Phi_n \otimes \text{Id}_R(\omega)\|_1 = \|\Phi \otimes \text{Id}_R(\Pi_n \otimes \text{Id}_R(\omega))\|_1 \leq \|\Phi\|_{G,E_n}^G,$$

where the inequality follows from definition of the norm $\|\cdot\|_{G,E}^G$, since $\text{Tr}G\Pi_n(\rho) \leq E_n$ for any $\rho \in \mathfrak{T}(\mathcal{H}_A)$. So, it follows from the definition of the diamond norm that

$$\|\Phi_n\|_0 \leq \|\Phi\|_{G,E_n}^G < +\infty \quad \forall n.$$
Let $\tilde{G}$ be a positive operator on $\mathcal{H}_R$ isomorphic to the operator $G$. By Lemma 8 below we have

$$\|\Phi - \Phi_n\|_{G,E}^G = \sup_{\omega \in \mathcal{G}(G)} \|\Phi \otimes \text{Id}_R(\omega - \Pi_n \otimes \text{Id}_R(\omega))\|_1, \quad (45)$$

where $\mathcal{G}_{G,G,E} = \{\omega \in \mathcal{G}(\mathcal{H}_{AR}) | \text{Tr}G\omega_A \leq E, \text{Tr}\tilde{G}\omega_R \leq E\}$.

Let $\tilde{P}_n = I_A - P_n$. By using the inequality

$$\|\tilde{P}_n \otimes I_R \cdot \omega \| \leq \sqrt{\text{Tr} \tilde{P}_n \otimes I_R} \omega$$

easily proved for any $\omega \in \mathcal{G}(\mathcal{H}_{AR})$ via the operator Cauchy-Schwarz inequality (see the proof of Lemma 11.1 in [7]) and by noting that $\text{Tr}G\rho \leq E$ implies $\text{Tr}\tilde{P}_n\rho \leq E/E_n$ for any $\rho \in \mathcal{G}(\mathcal{H}_A)$ we obtain

$$\|\omega - \Pi_n \otimes \text{Id}_R(\omega)\|_1 \leq 2\|P_n \otimes I_R \cdot \omega \cdot \tilde{P}_n \otimes I_R\|_1 + \|\tilde{P}_n \otimes I_R \cdot \omega \cdot \tilde{P}_n \otimes I_R\|_1 \quad (46)$$

for any state $\omega$ in $\mathcal{G}(\mathcal{H}_{AR})$ such that $\text{Tr}G\omega_A \leq E$.

By the Lemma in [6] the set of states $\rho$ satisfying the inequality $\text{Tr}G\rho \leq E$ is compact for any $E > 0$. So, Corollary 6 in [8] implies that the set $\mathcal{G}_{G,G,R,E}$ in (45) is a compact subset of $\mathcal{G}_{G\otimes I_R,E}$ for any $E > 0$. It follows that the continuous function $\omega \mapsto \Phi \otimes \text{Id}_R(\omega)$ is uniformly continuous on $\mathcal{G}_{G,G,R,E}$. Thus, estimate (46) implies that the r.h.s. of (45) tends to zero as $n \to +\infty$.

To prove that $\mathcal{G}_+^+(A,B) = \mathcal{G}_G(A,B)$ we have to show that any map $\Phi$ in $\mathcal{G}_+^+(A,B)$ belongs to $\mathcal{G}_G(A,B)$. We will use the sequence $\{\Phi_n\}$ constructed before. In this case it consists of CP maps. Since the sequence $\{\Phi_n\}$ converges to the map $\Phi$ w.r.t. the ECD norm, it is a Cauchy sequence w.r.t. this norm. So, it contains a subsequence $\{\Phi_{n_k}\}$ such that $\|\Phi_{n_{k+1}} - \Phi_{n_k}\|_{G,E} \leq 4^{-k}$. By Lemma 5 in Section 3 there exist a separable Hilbert space $\mathcal{H}_E$ and a sequence $\{V_k\}$ of operators in $\mathcal{B}(\mathcal{H}_A,\mathcal{H}_BE)$ such that

$$\Phi_{n_k}(\rho) = \text{Tr}_E V_k \rho V_k^*, \rho \in \mathcal{G}(\mathcal{H}_A), \quad \text{and} \quad \|V_{k+1} - V_k\|_E^G = \beta_E(\Phi_{n_k}, \Phi_{n_k}) \leq 2^{-k} \forall k.$$ 

It follows that $\{V_k\}$ is a Cauchy sequence in $\mathcal{B}(\mathcal{H}_A,\mathcal{H}_BE)$ w.r.t. to the norm $\| \cdot \|_E^G$. Hence it has a limit $V$ in $\mathcal{B}_G^0(\mathcal{H}_A,\mathcal{H}_BE)$. By using Lemma 4 in Section 3 it is easy to show that $\Phi(\rho) = \text{Tr}_E V \rho V^*$ for any $\rho \in \mathcal{G}(\mathcal{H}_A)$. It means that $\Phi \in \mathcal{G}_G(A,B)$.

Lemma 8. Let $\tilde{G}$ be a positive operator on $\mathcal{H}_R$ isomorphic to the operator $G$. Then

$$\|\Phi\|_{G,E}^G = \sup_{\omega \in \mathcal{G}(G)} \|\Phi \otimes \text{Id}_R(\omega)\|_1$$

for any $\Phi$ in $\mathcal{G}_G(A,B)$, where $\mathcal{G}_{G,G,E} = \{\omega \in \mathcal{G}(\mathcal{H}_{AR}) | \text{Tr}G\omega_A \leq E, \text{Tr}\tilde{G}\omega_R \leq E\}$.

Proof. Since the system $R$ in definition (21) is assumed arbitrary, the supremum in (21) can be taken over all pure states $\omega$ in $\mathcal{G}(\mathcal{H}_{AR})$ satisfying the condition $\text{Tr}G\omega_A \leq E$. 

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Since for any such pure state $\omega$ the partial states $\omega_A$ and $\omega_R$ are isomorphic, by applying local unitary transformation of the system $R$ this state can be transformed into a state $\omega'$ belonging to the set $\mathcal{E}_{G,G,E}$. It suffices to note that $\|\Phi \otimes \text{Id}_R(\omega)\|_1 = \|\Phi \otimes \text{Id}_R(\omega')\|_1$. □

It is well known that any map $\Phi$ in $\mathcal{Y}(A,B)$ can be represented as $\Psi_1 - \Psi_2$, where $\Psi_1$ and $\Psi_2$ are maps in $\mathcal{F}(A,B)$ \cite{11, 21}. Theorem 3B gives a reason for the following

**Conjecture.** Any map $\Phi$ in $\mathcal{Y}_G(A,B)$, where $G$ is a positive discrete unbounded operator, can be represented as $\Phi = \Psi_1 - \Psi_2$, where $\Psi_1$ and $\Psi_2$ are maps in $\mathcal{F}_G(A,B)$ \cite{15}.

Theorem 3B implies the following characterisation of the class of maps in $\mathcal{Y}_G(A,B)$ which have such representation.

**Proposition 4.** Let $G$ be a positive discrete unbounded operator on $\mathcal{H}_A$ (Def.\cite{17}). A map $\Phi$ in $\mathcal{Y}_G(A,B)$ can be represented as $\Phi = \Psi_1 - \Psi_2$, where $\Psi_1$ and $\Psi_2$ are maps in $\mathcal{F}_G(A,B)$, if and only if there exist linear maps $\Lambda_1$ and $\Lambda_2$ from $\mathcal{F}_G(\mathcal{H}_A)$ into $\mathcal{F}(\mathcal{H}_B)$ such that the map

$$\Theta(\rho) = \left[\begin{array}{c}
\Lambda_1(\rho) & \Phi(\rho) \\
\Phi(\rho) & \Lambda_2(\rho)
\end{array}\right]$$

belongs to the cone $\mathcal{Y}_{G}^+(A,B \oplus B)$, where $\mathcal{H}_{B \oplus B} = \mathcal{H}_B \oplus \mathcal{H}_B$.

**Proof.** By Theorem 3B $\mathcal{Y}_G(A,B \oplus B) = \mathcal{F}_G(A,B \oplus B)$. So, if $\Theta \in \mathcal{Y}_G(A,B \oplus B)$ then there exist a Hilbert space $\mathcal{H}_E$ and an operator $V$ in $\mathcal{B}_G(\mathcal{H}_A, \mathcal{H}_E \otimes (\mathcal{H}_B^1 \oplus \mathcal{H}_B^2))$ such that

$$\Theta(\rho) = \text{Tr}_E V \rho V^*, \ \rho \in \mathcal{F}_G(\mathcal{H}_A).$$

Denote by $P_1$ and $P_2$ the projectors onto the first and the second summands in $\mathcal{H}_B^1 \oplus \mathcal{H}_B^2$ correspondingly. Then the operators $V_1 = (P_1 \otimes I_E)V$ and $V_2 = (P_2 \otimes I_E)V$ belong, respectively, to the spaces $\mathcal{B}_G(\mathcal{H}_A, \mathcal{H}_B^1 \otimes \mathcal{H}_E)$ and $\mathcal{B}_G(\mathcal{H}_A, \mathcal{H}_B^2 \otimes \mathcal{H}_E)$. By using these operators one can represent the map $\Theta$ as follows

$$\Theta(\rho) = \left[\begin{array}{cc}
\text{Tr}_E V_1 \rho V_1^* & \text{Tr}_E V_1 \rho V_2^* \\
\text{Tr}_E V_2 \rho V_1^* & \text{Tr}_E V_2 \rho V_2^*
\end{array}\right].$$

Hence $\Phi(\rho) = \text{Tr}_E V_1 \rho V_2^*$. Since the map $\Phi$ is Hermitian-preserving, it follows that $\Phi = (\Psi_+ - \Psi_-)/4$, where

$$\Psi_+(\rho) = \text{Tr}_E (V_1 + V_2) \rho (V_1 + V_2)^* \quad \text{and} \quad \Psi_- = \text{Tr}_E (V_1 - V_2) \rho (V_1 - V_2)^*$$

are maps from the cone $\mathcal{F}_G(A,B)$.

If $\Phi = \Psi_1 - \Psi_2$ then it is easy to see that the map

$$\Theta(\rho) = \left[\begin{array}{cc}
(\Psi_1 + \Psi_2)(\rho) & \Phi(\rho) \\
\Phi(\rho) & (\Psi_1 + \Psi_2)(\rho)
\end{array}\right]$$

belongs to the cone $\mathcal{Y}_G^+(A,B \oplus B)$. □

\footnote{I would be grateful for any comments concerning this conjecture.}
The above conjecture is indirectly supported by the following proposition, since the property stated in this proposition holds for any map $\Phi$ belonging to the linear span of $\mathfrak{H}(A,B)$ by Proposition 1 in Section 4.

**Proposition 5.** If a map $\Phi$ belongs to the completion of the space $\mathfrak{H}(A,B)$ w.r.t. the ECD norm then $\|\Phi\|_{G,E} = o(E)$ as $E \to +\infty$.

**Proof.** There is a sequence $\{\Phi_n\}$ of maps in $\mathfrak{H}(A,B)$ such that $\|\Phi_n - \Phi\|_{G,E_0}$ tends to zero as $n \to +\infty$ any given $E_0 > 0$. By using the triangle inequality and relations (22) we obtain

$$\left| \frac{\|\Phi_n\|_{G,E}}{E} - \frac{\|\Phi\|_{G,E}}{E} \right| \leq \frac{\|\Phi_n - \Phi\|_{G,E}}{E} \leq \frac{\|\Phi_n - \Phi\|_{0,\phi}}{E_0}$$

for arbitrary $E > E_0$. Since $\|\Phi_n\|_{G,E} = o(E)$ as $E \to +\infty$ for all $n$ (in fact, all the functions $E \mapsto \|\Phi_n\|_{0,E}$ are bounded), the above inequality implies that $\|\Phi\|_{G,E} = o(E)$ as $E \to +\infty$. □

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