Algebraic stacks whose number of points over finite fields is a polynomial

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1 Introduction

The aim of this article is to investigate the cohomology (ℓ-adic as well as Betti) of schemes, and more generally of certain algebraic stacks, $\mathcal{X}$, say, that are proper and smooth over the spectrum of $\mathbb{Z}[1/n]$ for some $n \geq 1$ and have the property that there exists a polynomial $P$ with coefficients in $\mathbb{Q}$ such that for every finite field $\mathbb{F}_q$ of characteristic not dividing $n$ we have $\#\mathcal{X}(\mathbb{F}_q) = P(q)$. For the precise definitions and conditions the reader is invited to read the rest of the article, at least up to the statement of Theorem 2.1. Under those conditions, we prove that for all prime numbers $l$ the étale cohomology $H(\mathcal{X}_{\mathbb{Q},\text{ét}}, \mathbb{Q}_l)$, considered as a representation of the absolute Galois group of $\mathbb{Q}$, is as expected: zero in odd degrees, and, after semi-simplification, a direct sum of $\mathbb{Q}_l(-i)$ in degree $2i$, with the number of terms equal to the coefficient $P_i$ of $P$. For $n = 1$ we prove that $H(\mathcal{X}_{\mathbb{Q},\text{ét}}, \mathbb{Q}_l)$ is semi-simple. Our main tools here are Behrend’s Lefschetz trace formula in [Beh93] and ℓ-adic Hodge theory combined with the fact that $\mathbb{Z}$ has no nontrivial unramified extensions. Finally, using comparison theorems from ℓ-adic Hodge theory, we obtain a corollary which says that, under the extra assumption that the coarse moduli space of $\mathcal{X}$ is a quotient by a finite group, the Betti cohomology $H(\mathcal{X}(\mathbb{C}), \mathbb{Q})$ with its Hodge structure is as expected: zero in odd degree, and $\mathbb{Q}(-i)^{P_i}$ in degree $2i$.

The results in this article are motivated by a question by Carel Faber on potential applications to some moduli stacks $\mathcal{M}_{g,n}$ of stable $n$-pointed curves of genus $g$. These stacks are proper and smooth over $\mathbb{Z}$, and they do also satisfy the extra hypotheses of the corollary by results of Pikaart and Boggi [PB00]. We are told that $\#\mathcal{M}_{g,n}(\mathbb{F}_q)$ is a polynomial in $q$ for all pairs of the form $(0, n)$ with $n \geq 3$, $(1, n)$ with $1 \leq n \leq 10$, $(2, n)$ with $0 \leq n \leq 5$ (probably even up to $n = 9$), and $(3, n)$ with $0 \leq n \leq 3$ (and probably more). For genus 2 and 3 these results are due to Jonas Bergström.

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As this article is motivated by its application to certain \( \mathcal{M}_{g,n} \), we have not made an effort to make our results as general as possible. In particular, we have not tried to generalise comparison theorems from \( l \)-adic Hodge theory from schemes to stacks.

We hope that this article will be of help to those computing the rational Hodge structure on the cohomology of certain \( \mathcal{M}_{g,n} \). Counting points, using a suitable stratification, could be easier than having to compute the cohomology, using the same stratification. We apologise for our lack of expertise in the fields of algebraic stacks and \( l \)-adic Hodge theory. Readers with more competence in these areas will probably find the contents of this article rather straightforward and the proofs too elaborate. But there seems to be a lack of ‘well-known facts’ in the literature and we have tried hard to give precise references and proofs understandable also to the non-expert.

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Terminology, conventions

Concerning stacks, our terminology is that of [LMB00]. In particular, algebraic stacks are by definition quasi-separated.

Let \( k \) be a finite field. If \( \mathcal{X} \) is a Deligne-Mumford stack of finite type over \( \mathbb{Z} \), define its number of points over \( k \) to be

\[
\# \mathcal{X}(k) = \sum_{\xi} \frac{1}{|\text{Aut}(\xi)|},
\]

where the sum is over representatives of isomorphism classes of objects in \( \mathcal{X}(k) \). Here \( \text{Aut}(\xi) \) denotes the finite group of automorphisms of \( \xi \).

If \( G \) is a topological group, by a \( G \)-representation (over \( \mathbb{Q}_l \)) we shall mean a continuous representation of \( G \) on a finite dimensional \( \mathbb{Q}_l \)-vector space equipped with the \( l \)-adic topology. We use the same notation for a representation and its underlying vector space. For any \( G \) and \( n \geq 0 \), the symbol \( \mathbb{Q}_l^n \) denotes the trivial \( n \)-dimensional \( G \)-representation.

Updates

After version 1 of this text appeared on arxiv, some related results that use the relations between point counting over finite fields, étale cohomology and Hodge numbers have been brought to our attention. In [Ito03], Ito uses these tools to reprove that the stringy \( E \)-function does not depend on the choice of a resolution of singularities. In [KL02] Kisin and Lehrer use point counting over finite fields to determine the character of a finite group.
acting on the cohomology of a variety. In [KL05] they study varieties, not necessarily proper or smooth, whose number of points over finite fields is given by a polynomial, as in our case, and give two sufficient conditions for the cohomology to be of Tate type. A result like our Lemma 4.2, and its application to global Galois representations, is in the article [KW03] by Kisin and Wortmann. In [BT05] Bergström and Tommasi have adapted our proofs to the equivariant case for the action by a finite group; their article shows that \( M_4 \) and all strata of its boundary satisfy the polynomiality condition. Bergström has also written two other articles ([Ber06a] and [Ber06b]) involving the polynomiality condition for various moduli spaces of curves.

The first author has proved, in [Bog08, Cor. 8.12], the de Rham comparison theorem from \( p \)-adic Hodge theory for Deligne-Mumford stacks that are smooth and proper over complete discrete valuation fields with perfect residue field of characteristic \( p \). This enables us to remove the extra condition that the coarse moduli space is a certain quotient space in Corollary 5.3 of the last section of this article (see the footnote there for more detail).

We thank Moret-Bailly for bringing to our attention the fact that our proof gives a more general result than the one in the statement of the original theorem. Part of the conclusions of the main theorem are valid under the assumption that the stack is smooth and proper over a dense open part of \( \text{Spec} \mathbb{Z} \); originally, we had the stronger assumption that it had to be smooth and proper over the whole of \( \text{Spec} \mathbb{Z} \).

2 Results

**Theorem 2.1** Let \( \mathcal{X} \) be a Deligne-Mumford stack over \( \mathbb{Z} \). Let \( d \geq 0 \) and assume that \( \mathcal{X} \) is proper, smooth and of pure relative dimension \( d \) over some non-empty open subscheme \( U \) of \( \text{Spec} \mathbb{Z} \). Let \( S \) be a set of primes of Dirichlet density 1. Assume —

\[
(*) \quad \text{there exists a polynomial } P(t) = \sum_{i \geq 0} P_i t^i, \text{ with } P_i \in \mathbb{Q}, \text{ such that } \\
\# \mathcal{X}(\mathbb{F}_{p^n}) = P(p^n) + o(p^{nd/2}) \quad (n \to \infty)
\]

for all \( p \in S \).

Then the degree of \( P(t) \) is \( d \), and there exists a unique such polynomial satisfying \( P_i = P_{d-i} \) for all \( 0 \leq i \leq d \). Suppose \( P(t) \) is of this form. Then it has non-negative integer coefficients and satisfies \( \# \mathcal{X}(\mathbb{F}_{p^n}) = P(p^n) \) for all primes \( p \) in \( U \) and all \( n \geq 1 \). Furthermore, for all primes \( l \) and all \( i \geq 0 \) there is an isomorphism of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)-representations

\[
H^i(\mathcal{X}_{\overline{\mathbb{Q}}, \text{ét}}, \mathbb{Q}_l)^{\text{ss}} \simeq \begin{cases} 0, & \text{if } i \text{ is odd;} \\
\mathbb{Q}_l(-i/2)^{P_{i/2}}, & \text{if } i \text{ is even.}
\end{cases}
\]

Here \( H^i(\mathcal{X}_{\overline{\mathbb{Q}}, \text{ét}}, \mathbb{Q}_l)^{\text{ss}} \) denotes the semi-simplification of \( H^i(\mathcal{X}_{\overline{\mathbb{Q}}, \text{ét}}, \mathbb{Q}_l) \). If \( U = \text{Spec} \mathbb{Z} \), then \( H^i(\mathcal{X}_{\overline{\mathbb{Q}}, \text{ét}}, \mathbb{Q}_l)^{\text{ss}} \simeq H^i(\mathcal{X}_{\overline{\mathbb{Q}}, \text{ét}}, \mathbb{Q}_l) \).
We remark that part of the theorem can also be stated in terms of the coarse moduli space associated to \( \mathcal{X} \). Indeed, the number of points over a finite field of a Deligne-Mumford stack equals the number of points of its coarse moduli space; and furthermore, the cohomologies of both spaces with coefficients in a \( \mathbb{Q} \)-algebra are the same.

### 3 Some Results on Stacks

Let us make two technical remarks.

In [LMB00, §18] the theory of constructible sheaves of \( \mathbb{Z}/l^n\mathbb{Z} \)-modules over the smooth-étale site of an algebraic \( S \)-stack is developed, where \( S \) is a scheme. There is a straightforward extension of this theory to constructible \( l \)-adic sheaves, e.g., by working with projective systems of \( \mathbb{Z}/l^n\mathbb{Z} \)-modules modulo torsion in the usual way. We will use this without further comments. Associated to a Deligne-Mumford stack \( \mathcal{X} \) are its étale topos (denoted \( \mathcal{X}_{\text{ét}} \)) and its smooth-étale topos. They however give the same cohomology theory of constructible sheaves (see [LMB00, §12], especially Prop. 12.10.1). This justifies the fact that we will only work with the étale topos of a Deligne-Mumford stack, but freely cite results stated in terms of the other topos.

The scheme counterpart of the following proposition is classical. By lack of a precise reference, we have included a proof for the case of stacks.

**Proposition 3.1** Let \( \mathcal{X} \) be a Deligne-Mumford stack which is smooth and proper over \( \mathbb{Z}_p \). For every prime \( l \neq p \) and every \( i \geq 0 \), the canonical map of Gal(\( \overline{\mathbb{Q}}_p/\mathbb{Q}_p \))-representations

\[
H^i(\mathcal{X}_{\overline{\mathbb{Q}}_p, \text{ét}}, \mathbb{Q}_l) \to H^i(\mathcal{X}_{\mathbb{Q}_p, \text{ét}}, \mathbb{Q}_l)
\]

is an isomorphism. In particular, \( H^i(\mathcal{X}_{\overline{\mathbb{Q}}_p, \text{ét}}, \mathbb{Q}_l) \) is unramified.

**Proof.** Denote by \( \mathbb{Q}_p^{nr} \) the maximal unramified extension of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}}_p \) and let \( \mathbb{Z}_p^{nr} \) be its ring of integers. Set \( S = \text{Spec}(\mathbb{Z}_p^{nr}) \) and denote by \( s \) resp. \( \eta \) its closed resp. generic point. Let \( \overline{\eta} \to \eta \) correspond to \( \mathbb{Q}_p^{nr} \to \overline{\mathbb{Q}}_p \). Consider the natural morphisms

\[
\mathcal{X}_{\overline{\eta}} \to \mathcal{X}_S \leftarrow \mathcal{X}_s.
\]

These maps induce continuous morphisms between the associated étale sites.

Let \( (U, u) \) be an étale neighbourhood of \( \mathcal{X}_S \) and let \( j_U \) be the pull-back of \( j \) along \( u \). By [LMB00 18.2.1(i)], for every \( q \) we have \( (R^qj_*\mathbb{Q}_l)_U = R^q(j_U)_*\mathbb{Q}_l \). As \( U \) is smooth, it follows that \( j_*\mathbb{Q}_l = \mathbb{Q}_l \) and \( R^qj_*\mathbb{Q}_l = 0 \) if \( q \neq 0 \). Hence the Leray spectral sequence gives an isomorphism \( H^i(\mathcal{X}_S, \mathbb{Q}_l) \to H^i(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_l) \). But on the other hand, \( H^i(\mathcal{X}_S, \mathbb{Q}_l) \) is naturally isomorphic to \( H^i(\mathcal{X}_s, \mathbb{Q}_l) \); this follows from the proper base change theorem (LMB00, 18.5.1) for \( \mathcal{X}_S \) over \( S \) and the fact that \( S \) is strictly local. \( \square \)
The next topic is Poincaré duality for the $l$-adic cohomology of certain stacks (see Prop. 3.3 below). We will obtain this by considering the cohomologies of their associated coarse moduli spaces.

Let $\mathcal{X}$ be a separated Deligne-Mumford stack of finite type over an algebraically closed field of characteristic zero. We will denote by $\overline{\mathcal{X}}$ its coarse moduli space and by $q: \mathcal{X} \to \overline{\mathcal{X}}$ the corresponding mapping. Note that we can cover $\overline{\mathcal{X}}$ by étale charts $U$ such that the pull-back of $U$ in $\mathcal{X}$ is the quotient stack of an algebraic space by a finite group ($[LMB00, Rem. 6.2.1]$).

**Lemma 3.2** For every $i$ the pull-back map

$$q^*: H^i(\overline{\mathcal{X}}, \mathbb{Q}_l) \to H^i(\mathcal{X}, \mathbb{Q}_l)$$

is an isomorphism.

**Proof.** The lemma follows from the Leray spectral sequence once we have shown that the canonical map $\mathbb{Q}_l \to Rq_*\mathbb{Q}_l$ is an isomorphism. This question is étale local on $\overline{\mathcal{X}}$ and therefore we may assume that $\mathcal{X} = [V/G]$ for some algebraic space $V$ equipped with an action by a finite group $G$. Denote by $p: V \to \mathcal{X}$ the canonical morphism. Note that $\mathbb{Q}_l \simeq (p_*\mathbb{Q}_l)^G$. As $p$ and $qp$ are finite and $\mathbb{Q}[G]$ is a semi-simple $\mathbb{Q}$-algebra, we obtain

$$Rq_*\mathbb{Q}_l \simeq Rq_*(p_*\mathbb{Q}_l)^G \simeq ((qp)_*\mathbb{Q}_l)^G \simeq \mathbb{Q}_l.$$

Now suppose that $\mathcal{X}$ is defined over $\mathbb{C}$ and smooth. Consider the complex analytic space $\overline{\mathcal{X}}^\text{an}$ associated to $\overline{\mathcal{X}}$. It can naturally be equipped with the structure of a $V$-manifold, i.e., locally $\overline{\mathcal{X}}^\text{an}$ is the quotient of a connected manifold by a finite group; c.f., $[Ste77]$.

**Proposition 3.3** Suppose $\mathcal{X}$ is a Deligne-Mumford stack which is smooth and proper over $\overline{\mathbb{Q}}$ of pure dimension $d$ for some $d \geq 0$.

i) Suppose $\mathcal{X}$ is integral. For an integer $i$, consider the cup product mapping

$$H^i(\mathcal{X}, \mathbb{Q}_l) \otimes H^{2d-i}(\mathcal{X}, \mathbb{Q}_l) \to H^{2d}(\mathcal{X}, \mathbb{Q}_l).$$

Then the right-hand side is one-dimensional and the pairing thus obtained is perfect.

ii) Suppose $\mathcal{X}$ is smooth and proper $\overline{\mathbb{Q}}$-scheme of pure dimension $d$ and let $f: X \to \mathcal{X}$ be a $\overline{\mathbb{Q}}$-morphism which is surjective and generically finite. Then for all $i$, the induced map

$$f^*: H^i(\mathcal{X}, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$$

is injective.
Proof. By Lemma 3.2 and the comparison theorem between Betti and étale cohomology, it suffices to show that in case (i)
\[ H^i(\overline{X}^{an}, \mathbb{Q}) \otimes H^{2d-i}(\overline{X}^{an}, \mathbb{Q}) \rightarrow H^{2d}(\overline{X}^{an}, \mathbb{Q}) \]
is a perfect pairing; and in case (ii) that
\[ H^i(\overline{X}^{et}, \mathbb{Q}_l) \rightarrow H^i(X^{et}, \mathbb{Q}_l) \]
is injective. Now the singular cohomology of a $V$-manifold satisfies Poincaré duality ([Ste77]), from which these statements follow. □

4 Proof of the Main Theorem

We will now prove Thm. 2.1. In this section, all cohomology is with respect to the étale sites. We begin with an analytic lemma used in the course of the proof.

Lemma 4.1 Let us be given the following integers: $d \geq 0$, $d \leq r \leq 2d$, and for $0 \leq i \leq r$ also $d_i \geq 0$. Furthermore, let $p > 1$ be a real number, let $P_i \in \mathbb{Q}$ for all $i \geq 0$, with $P_i = 0$ for $i$ large, and let $\alpha_{i,j} \in \mathbb{C}$ for $0 \leq i \leq r$ and $1 \leq j \leq d_i$. Assume $|\alpha_{i,j}| = p^{i/2}$ and
\[ \sum_{i=0}^{r} (-1)^i \sum_{1 \leq j \leq d_i} \alpha_{i,j}^n = \sum_{i \geq 0} P_i p^{ni} + o(p^{nd/2}) \quad (n \rightarrow \infty). \tag{2} \]
Then $d_i = 0$ for $i \geq d$ odd, $P_i = 0$ for $i > r/2$, while for $d/2 \leq i \leq r/2$ we have $P_i = d_{2i}$; for these $i$ also $\alpha_{2i,j} = p^i$.

Proof. The lemma follows by induction on $r$. Indeed, assume that either $r = d$ or that the lemma holds for $r - 1$. As $|\sum_{1 \leq j \leq d_i} \alpha_{i,j}^n| \leq d_i p^{in/2}$, we have
\[ |\sum_{i=0}^{r} (-1)^i \sum_{1 \leq j \leq d_i} \alpha_{i,j}^n| = |\sum_{1 \leq j \leq d_r} \alpha_{r,j}^n| + o(p^{nr/2}) \quad (n \rightarrow \infty) \]
and also, using (2), that $P_i = 0$ for $i > r/2$.

Note that if $z$ is an element of a finite product $(S^1)^s$ of complex unit circles, then the closure of $\{z^n \mid n \geq 1\}$ contains the unit element. Hence for every $\epsilon > 0$ there exists an infinite subset $N \subseteq \mathbb{N}$ such that for all $n \in N$ and for all $i$ and $j$ we have $|(\alpha_{i,j} p^{-i/2})^n - 1| < \epsilon$ and in particular
\[ |\sum_{1 \leq j \leq d_r} (\alpha_{r,j} p^{-r/2})^n - d_r| < \epsilon', \tag{3} \]
with $\epsilon' = d_r \epsilon$.

Now first suppose $r$ is odd. Then
\[ |\sum_{1 \leq i \leq d_r} \alpha_{i,j}^n| = o(p^{nr/2}) \quad (n \rightarrow \infty), \]
which by (3) implies \( d_r = 0 \). Therefore, if \( r = d \) we are done, while if \( r > d \) we can apply the induction hypotheses.

So from now on suppose \( r \) is even. From (2) follows

\[
\left| \sum_{1 \leq j \leq d_r} \alpha_{r,j}^n - P_r/2p^{nr/2} \right| = o(p^{nr/2}) \quad (n \to \infty),
\]

or equivalently:

\[
\lim_{n \to \infty} \left| \sum_{1 \leq j \leq d_r} (\alpha_{r,j}p^{-r/2})^n - P_r/2 \right| = 0.
\]

Together with (3) this implies \( d_r = P_r/2 \). In turn this easily leads to \( \alpha_{r,j} = p^{r/2} \).

Now subtract \( P_r/2p^{r/2} = \sum_{1 \leq j \leq d_r} \alpha_{r,j}^n \) from (2) and if \( r > d \) apply the induction hypotheses. \( \square \)

Let \( \mathcal{X}, d, U \) and \( S \) be as in Thm. 2.1 and let \( P(t) = \sum_{i \geq 0} P_i t^i \) be a polynomial satisfying (\#). Without loss of generality we assume that \( P_i = P_{d-i} \) for all \( 0 \leq i \leq d \) and that \( S \) is contained in \( U \). We also fix a prime \( l \).

By [LMB00, Thm. 16.6] and resolution of singularities, there exists a smooth and proper \( \mathbb{Q} \)-scheme \( X \) of pure dimension \( d \) and a surjective and generically finite map \( f: X \to \mathcal{X}_\mathbb{Q} \). By removing a finite number of primes from \( S \) if necessary, we may assume that \( X_{\mathbb{Q}_p} \) extends to a smooth scheme over \( \mathbb{Z}_p \) for all \( p \in S \). As a consequence, for all \( p \neq l \) in \( S \) the representation \( H^i(X_{\mathbb{F}_p}, \mathbb{Q}_l) \) is unramified and we can consider the action of Frobenius. By [Del74], the eigenvalues of Frobenius have complex absolute value \( p^{i/2} \). Using Prop. 3.3, we obtain the same conclusions for the subrepresentation \( H^i(X_{\mathbb{F}_p}, \mathbb{Q}_l) \).

Fix a prime \( p \neq l \) in \( S \). By Behrend’s Lefschetz trace formula (see [Beh93] or [LMB00, Thm. 19.3.4]),

\[
\sum_{i \geq 0} (-1)^i \text{Tr}(\text{Frob}^n, H^i(X_{\mathbb{F}_p}, \mathbb{Q}_l)) = \#\mathcal{X}(\mathbb{F}_p^n)
\]

for all \( n \geq 1 \). Let \( \alpha_{i,1}, \ldots, \alpha_{i,d_i} \) be the complex roots of the characteristic polynomial of Frobenius. Applying (\#) and Prop. 3.3 formula (4) becomes

\[
\sum_{i=0}^{2d} (-1)^i \sum_{1 \leq j \leq d_i} \alpha_{i,j}^n = P(p^n) + o(p^{nd/2}) \quad (n \to \infty).
\]

From Lemma 4.1 we now obtain that \( P(t) \) has degree \( d \) and for all \( d \leq i \leq 2d \) we have that for \( i \) even \( P_{i/2} = d_i \) and \( \alpha_{i,j} = p^{i/2} \), while \( d_i = 0 \) for \( i \) odd. Using Poincaré duality (Prop. 3.3) we obtain the same conclusions for all \( i \). (Note that \( P(t) \) is defined in such a way that \( P_i = P_{d-i} \) for \( 0 \leq i \leq d/2 \).)

Hence \( H^i(X_{\mathbb{F}_p}, \mathbb{Q}_l) \) vanishes for any odd \( i \), while for all even \( i \) it has dimension \( P_{i/2} \); in particular, the coefficients of \( P(t) \) are non-negative integers. Furthermore,

\[
\text{Tr}(\text{Frob}, H^{2i}(X_{\mathbb{F}_p}, \mathbb{Q}_l)) = \text{Tr}(\text{Frob}, Ql(-i)^{P_i}).
\]
This holds for all primes $p$ in the set $S \setminus \{l\}$. But a semi-simple, almost everywhere unramified, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation over $\mathbb{Q}_l$ is determined by the trace of Frobenius on a set of primes of Dirichlet density 1 (a proof of this is outlined in [DDT97 Prop. 2.6]). We conclude that the semi-simplification of $H^2(\mathcal{X}_\overline{\mathbb{Q}}, \mathbb{Q}_l)$ is isomorphic to $\mathbb{Q}_l(-i)^{P_i}$.

By Prop. 3.1 $H^2(\mathcal{X}_\overline{\mathbb{Q}}, \mathbb{Q}_l)$ is unramified for every prime $p \neq l$ in $U$. As (5) then holds for all these primes, Behrend’s Lefschetz trace formula gives $\#\mathcal{X}(\mathbb{F}_p^\infty) = P(p^n)$. Changing $l$, we see that this formula is valid for every prime $p$ in $U$.

Now assume $U = \text{Spec} \mathbb{Z}$. All that remains to be proved is that $H^2(\mathcal{X}_\overline{\mathbb{Q}}, \mathbb{Q}_l)$ is semi-simple, or equivalently, that its $i$th Tate twist $H = H^2(\mathcal{X}_\overline{\mathbb{Q}}, \mathbb{Q}_l)(i)$ is semi-simple. Note that $H$ is unramified outside $l$ and that the semi-simplification of $H$ is isomorphic to the trivial representation $\mathbb{Q}_l^{P_i}$.

By [LMB00 16.6] and [Del96], there exists a finite extension $K$ of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$, a proper, semi-stable scheme $X$ of pure relative dimension $d$ over the ring of integers of $K$ and a surjective and generically finite $K$-morphism $f : X_K \to \mathcal{X}_K$. By [Tsu02 Thm. 1.1], $H^2(\mathcal{X}_\overline{\mathbb{Q}}, \mathbb{Q}_l)$ is a semi-stable representation of $\text{Gal}(\overline{\mathbb{Q}}_l/K)$. So it follows from Prop. 3.3 that $H^2(\mathcal{X}_\overline{\mathbb{Q}}, \mathbb{Q}_l)$ and hence also $H$ are potentially semi-stable.

**Lemma 4.2** Let $n \geq 1$ be an integer. Consider a short exact sequence of $\text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$-representations

$$0 \to \mathbb{Q}_l^n \to V \to \mathbb{Q}_l \to 0. \quad (6)$$

If $V$ is potentially semi-stable, then $V$ is unramified.

**Proof.** By assumption, there is a finite extension $K$ of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$, such that the restriction of $V$ to $G_K := \text{Gal}(\overline{\mathbb{Q}}_l/K)$ is semi-stable. Fix such a $K$ and denote by $K_0$ its maximal unramified subfield relative to $\mathbb{Q}_l$. Denote by $\sigma$ the automorphism of $K_0$ obtained by lifting the automorphism $x \mapsto x^l$ of the residue field of $K_0$.

We will briefly recall some theory about semi-stable representations and filtered $(\varphi, N)$-modules; for more details, see [CF00].

Denote by $\text{Rep}_{\text{st}}(G_K)$ the category of semi-stable representations of $G_K$ over $\mathbb{Q}_l$ and denote by $\text{Rep}_{\text{imr}}(G_K)$ its full subcategory of unramified representations. Let $\text{MF}_{K}^f(\varphi, N)$ be the category of (weakly) admissible filtered $(\varphi, N)$-modules over $K$. An object of $\text{MF}_{K}^f(\varphi, N)$ is a finite dimensional $K_0$-vector space $E$ equipped with a $\sigma$-semi-linear bijective $\varphi : E \to E$, a nilpotent endomorphism $N$ of $E$ and an exhaustive and separating descending filtration $\text{Fil}^i E_K$ on $E_K := K \otimes_{K_0} E$. We must have $N \varphi = l \varphi N$ and furthermore there is a certain admissibility condition to be satisfied (c.f., [CF00 §3]).

All above categories are Tannakian (so in particular they are all abelian $\mathbb{Q}_l$-linear $\otimes$-categories). Fontaine has constructed a functor $D_{\text{st}, K}$ from $\text{Rep}_{\text{st}}(G_K)$ to $\text{MF}_{K}^f(\varphi, N)$ and the main result of [CF00] is that this is an equivalence of Tannakian categories.

To the trivial one-dimensional representation $\mathbb{Q}_l$ corresponds $D_{\text{st}, K}(\mathbb{Q}_l)$, which is just $K_0$ equipped with the trivial maps $\varphi = \sigma$, $N = 0$ and filtration determined by $\text{Fil}^0 K = K$ and $\text{Fil}^1 K = 0$. By abuse of notation we will denote $D_{\text{st}, K}(\mathbb{Q}_l)$ also by $K_0$. 

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We obtain natural maps of $\mathbb{Q}_l$-vector spaces

$$\text{Ext}^1_{\text{Rep}_{\text{unr}}(G_K)}(\mathbb{Q}_l, \mathbb{Q}_l^n) \rightarrow \text{Ext}^1_{\text{Rep}(G_K)}(\mathbb{Q}_l, \mathbb{Q}_l^n) \rightarrow \text{Ext}^1_{\text{MF}_{\mathbb{K}}(\varphi,N)}(K_0, K_0^n),$$

where the second map is the isomorphism induced by $D_{\text{st},K}$. We need to show that $i$ is an isomorphism. Since $\text{Ext}^1$ is additive in the second variable, it suffices to treat the case $n = 1$. We will show that the dimension of $\text{Ext}^1_{\text{Rep}_{\text{unr}}(G_K)}(\mathbb{Q}_l, \mathbb{Q}_l)$ is at least as big as the dimension of the $\mathbb{Q}_l$-vector space $\text{Ext}^1_{\text{MF}_{\mathbb{K}}(\varphi,N)}(K_0, K_0)$.

First we consider the extensions in $\text{MF}_{\mathbb{K}}(\varphi,N)$. Let

$$0 \rightarrow K_0 \rightarrow E \rightarrow K_0 \rightarrow 0$$

be a short exact sequence in $\text{MF}_{\mathbb{K}}(\varphi,N)$. Choosing a splitting of the short exact sequence of vector spaces underlying (7), we write $E$ be a short exact sequence in $\text{MF}_{\mathbb{K}}(\varphi,N)$, and one checks that it is in fact $\mathbb{Q}_l$-linear. Take $x \in K_0$ and let $L$ be the automorphism of $K_0 \oplus K_0$ given by $(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix})$. If we equip the source with $\varphi$ and the target with $\varphi'$, then $L$ induces an equivalence of the associated extensions if $\varphi' = L^{-1}\varphi L$. It follows that $j(\alpha) = j(\alpha + \sigma(x) - x)$, so the kernel of $j$ is a sub-$\mathbb{Q}_l$-vector space of $K_0^n$ of codimension 1; this implies that $\dim_{\mathbb{Q}_l}\text{Ext}^1_{\text{MF}_{\mathbb{K}}(\varphi,N)}(K_0, K_0) \leq 1$.

But on the other hand, $\dim_{\mathbb{Q}_l}\text{Ext}^1_{\text{Rep}_{\text{unr}}(G_K)}(\mathbb{Q}_l, \mathbb{Q}_l) = 1$. To see this, suppose $V$ is unramified and sits in an extension of $\mathbb{Q}_l$ by $\mathbb{Q}_l$. Taking a suitable basis for $V$, the action of the Galois group is given by $(\begin{smallmatrix} 1 & \eta \\ 0 & 1 \end{smallmatrix})$, where $\eta$ is an unramified character $\text{Gal}(\overline{\mathbb{Q}}_l/K) \rightarrow \mathbb{Q}_l$. In other words, $\eta$ is a morphism of groups $\hat{\mathbb{Z}} \rightarrow \mathbb{Q}_l$ which, being continuous, must factor through $\mathbb{Z}_l$. But $\text{Hom}(\mathbb{Z}_l, \mathbb{Q}_l)$ is one-dimensional.

Thus we obtain that $V$ in (8) is unramified when restricted to $\text{Gal}(\overline{\mathbb{Q}}_l/K)$. Then if $g$ is an element of the inertia subgroup of $\text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$, the $[K: \mathbb{Q}_l]$-th power of $g$ must act trivially. But on the other hand $g$ must act unipotently, as $V$ sits in the short exact sequence (8). Hence $g$ itself must act trivially and $V$ is an unramified representation of $\text{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$. □

Now let

$$0 = H_0 \subset H_1 \subset \cdots \subset H_{F_1} = H$$
be a Jordan-Hölder filtration of $H$. Hence $H_j/H_{j-1} \simeq \mathbb{Q}_l$ and each $H_j$ is unramified outside $l$ and potentially semi-stable at $l$. Clearly, $H_1 \simeq \mathbb{Q}_l$; assume that $H_j \simeq \mathbb{Q}_l^j$ for some $j$ ($1 \leq j \leq P_l - 1$). Then $H_{j+1}$ is everywhere unramified by the above lemma. However, as Minkowski’s theorem says that $\mathbb{Q}$ has no non-trivial unramified extensions, we conclude that $H_{j+1}$ is isomorphic to $\mathbb{Q}_l^{j+1}$. So by induction, $H \simeq \mathbb{Q}_l^P$. This finishes the proof of Thm. 2.1.

5 The Hodge structure

Let $V$ be an integral, regular, quasi-projective $\mathbb{Q}$-scheme. Consider a finite group $G$ acting on $V$ from the right. Denote by $f: V \to Q$ the canonical projection to the quotient scheme $Q = V/G$. Note that there is a natural $G$-action on the module $f_\ast \Omega_V^{\ast}/Q$.

We are interested in the cohomology of the quotient space $Q$ and in particular in the Hodge structure of the singular cohomology of its associated analytic space. For this we will use [Ste77]. In order to be able to apply some results of [Ste77] we will need the following result:

Proposition 5.1 Let $\Sigma \subset Q$ be a closed subset of codimension $\geq 2$ containing all singular points of $Q$. Let $j: U \hookrightarrow Q$ be the open subscheme complementary to $\Sigma$. Then, for all $p$, there is a canonical isomorphism

$$j_\ast \Omega^p_U/Q \rightarrow (f_\ast \Omega^p_V/Q)^G$$

of $\mathcal{O}_Q$-modules.

Proof. For any $\mathbb{Q}$-scheme $Z$, we abbreviate $\Omega_Z := \Omega_{Z/Q}^p$. Put $W = f^{-1}(U)$ and let $g: W \to U$ be the restriction of $f$. Note that $G$ acts on $W$ with quotient $U$. The canonical map $g^\ast \Omega_U \to \Omega_W$ induces a morphism

$$\alpha: \Omega_U \rightarrow (g^\ast \Omega_U)^G \rightarrow (g_\ast \Omega_W)^G.$$ 

This map is an isomorphism in the stalk at the generic point. To define $[S]$, we apply $j_\ast$ to $\alpha$ to obtain a map from $j_\ast \Omega_U$ to the sheaf $j_\ast (f_\ast \Omega_V)^G$. This last sheaf is isomorphic to $(f_\ast \Omega_V)^G$, as a consequence of the fact that if $Z$ is a regular $\mathbb{Q}$-variety and $z: Z' \hookrightarrow Z$ is a dense open subset with complement of codimension $\geq 2$, then $z_\ast \Omega_{Z'} \simeq \Omega_Z$. Another consequence of this fact is that the map $[S]$ thus defined is independent of the choice of $\Sigma$. In particular, we have reduced the problem to showing that for any point $\eta \in U$ whose closure has codimension 1, the map $\alpha$ is an isomorphism in the stalk at $\eta$.

The question being local for the étale topology on $U$ and $V$, it suffices to consider, for $d \geq 1$ and $n \geq 0$, the action of $\mu_d$ on $\mathbb{A}^n_Q$, with the quotient map $\mathbb{A}^n_Q \to \mathbb{A}^n_Q$ mapping $(a_1, \ldots, a_n)$ to $(a_1^d, a_2^d, \ldots, a_n^d)$. The results follows from an easy calculation.  

Let $j: U \hookrightarrow Q$ be the smooth locus of $Q$. As a consequence of Prop. 5.1 there is a canonical isomorphism between the De Rham complexes

$$j_\ast \Omega^{\ast}_{U/Q} \rightarrow (f_\ast \Omega^{\ast}_{V/Q})^G$$
Therefore, for each $i$ we obtain an isomorphism

$$H^i(Q, j_* \Omega^\bullet_{U/Q}) \sim H^i(Q, (f_* \Omega^\bullet_{V/Q})^G) \sim H^i_{DR}(V/Q)^G. \quad (9)$$

This last isomorphism follows from the fact that $f$ is finite and that taking cohomology with $\mathbb{Q}$-coefficients commutes with taking invariants under a finite group action. The Hodge filtration on the De Rham complex induces filtrations on the vector spaces in (9) and the isomorphisms in (9) respect those filtrations.

Fix a prime $l$. Denote by $D_{DR}$ the Fontaine functor (see e.g., [Tsu02]) from the category of $\text{Gal}(\mathbb{Q}_l/\mathbb{Q}_l)$-representations over $\mathbb{Q}_l$ to the category of finite dimensional filtered $\mathbb{Q}_l$-vector spaces.

**Proposition 5.2** In the above situation, suppose furthermore that $Q$ is proper over $\mathbb{Q}$. For every $i$, there is an isomorphism of filtered vector spaces

$$D_{DR}(H^i(Q_{\overline{Q}, \text{ét}}, \mathbb{Q}_l)) \sim H^i(Q, j_* \Omega^\bullet_U/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l.$$

**Proof.** As $V$ is proper and smooth, the comparison theorem (see e.g., [Tsu02 Thm. A1]) states that $D_{DR}(H^i(V_{\overline{Q}, \text{ét}}, \mathbb{Q}_l))$ and $H^i_{DR}(V/\mathbb{Q}_l)$ are isomorphic. This isomorphism preserves the $G$-invariants. Using (9) we obtain the isomorphism of the proposition. \[\square\]

With these prerequisites, we are finally ready to prove the corollary to the main theorem:

**Corollary 5.3** In the situation of Theorem 2.1 (with $U$ a non-empty open subscheme of $\text{Spec} \mathbb{Z}$), suppose that the coarse moduli space $\mathcal{X}$ of $\mathcal{X}_Q$ is the quotient of a smooth projective $\mathbb{Q}$-scheme by a finite group $[1]$.

Then for each $i$, there is an isomorphism of $\mathbb{Q}$-Hodge structures

$$H^i(\mathcal{X}(\mathbb{C}), \mathbb{Q}) \simeq \begin{cases} 0, & \text{if } i \text{ is odd}, \\ \mathbb{Q}(-i/2)^{P_{i/2}}, & \text{if } i \text{ is even}, \end{cases}$$

where the left hand side is equipped with the canonical Hodge structure of [Del74ii].

**Proof.** Consider a short exact sequence

$$0 \to W' \to W \to W'' \to 0$$

in the category of representations of $\text{Gal}(\mathbb{Q}_l/\mathbb{Q}_l)$ over $\mathbb{Q}_l$. If $W$ is a de Rham representation, then the corresponding sequence

$$0 \to D_{DR}(W') \to D_{DR}(W) \to D_{DR}(W'') \to 0$$

\[1\]This last condition can be omitted, by invoking [Bog08 Cor. 8.12] directly after the statement, in the proof, that $D_{DR}(W) \simeq D_{DR}(W^{ss})$ if $W$ is de Rham.
is a short exact sequence on the underlying vector spaces and all maps are strict with respect to the filtrations (see [Fon94, 3.4, 3.7 and 3.8]). It follows that $D_{DR}(W) \simeq D_{DR}(W^{ss})$ if $W$ is de Rham.

Without loss of generality we assume that $\overline{X}$ is integral. Note that $H^i(X_{\mathbb{Q}_l}, \mathbb{Q}_l)$ is isomorphic to $H^i(\overline{X}_{\mathbb{Q}_l}, \mathbb{Q}_l)$ by Lemma 3.2 and note that these representations are de Rham. Therefore, combining Thm. 2.1 and Prop. 5.2 it suffices to exhibit an isomorphism of filtered vector spaces

$$H^i(\overline{X}(\mathbb{C}), \mathbb{C}) \simto H^i(\overline{X}, j_*\Omega^\bullet_{U/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C},$$

where $j: U \to \overline{X}$ is the smooth locus of $\overline{X}$. This is done in [Ste77, Thm. 1.12]. □

References

[Beh93] K. Behrend: The Lefschetz trace formula for algebraic stacks. Invent. Math., 112, 127–149 (1993).

[Ber06a] J. Bergström: Equivariant counts of points of the moduli spaces of pointed hyperelliptic curves. Arxiv, math.AG/0611813.

[Ber06b] J. Bergström: Cohomology of moduli spaces of curves of genus three via point counts. Arxiv, math.AG/0611815

[BT05] J. Bergström and O. Tommasi: The rational cohomology of $\overline{M}_4$. Arxiv, math.AG/0506502.

[Bog08] T. van den Bogaart: The de Rham comparison theorem for Deligne-Mumford stacks. arXiv:0809.1242

[CF00] P. Colmez and J.-M. Fontaine: Construction des représentations $p$-adiques semi-stables. Invent. Math., 140, 1–43 (2000).

[DDT97] H. Darmon, F. Diamond and R. Taylor: Fermat’s last theorem. In: Elliptic Curves, Modular Forms & Fermat’s Last Theorem (Hong Kong 1993). Internat. Press, Cambridge, 2–140 (1997).

[Del74i] P. Deligne: La conjecture de Weil, I. Publ. Math. IHES, 43, 273–307 (1974).

[Del74ii] P. Deligne: Théorie de Hodge, III. Publ. Math. IHES, 44, 5–77 (1974).

[Fon94] J.-M. Fontaine: Représentations $p$-adiques semi-stables. Périodes $p$-adiques. Astérisque 223, SMF, 113–184 (1994).

[DeJ96] A.J. de Jong: Smoothness, semi-stability and alterations. Publ. Math. IHES, 83, 51–93 (1996).
[Ito03] T. Ito: *Stringy Hodge numbers and $p$-adic Hodge theory*. Comp. Math., **140**, pp. 1499–1517 (2004).

[KL02] M. Kisin and G.I. Lehrer: *Equivariant Poincaré polynomials and counting points over finite fields*. J. Algebra **247**, pp. 435–451 (2002).

[KL05] M. Kisin and G.I. Lehrer: *Eigenvalues of Frobenius and Hodge numbers*. In preparation.

[KW03] M. Kisin and S. Wortmann: *A note on Artin motives*. Math. Res. Lett. **10**, pp. 375–389 (2003).

[LMB00] G. Laumon and L. Moret-Bailly: *Champs algébriques*. Springer-Verlag, Berlin (2000).

[PB00] M. Boggi and M. Pikaart: *Galois covers of moduli of curves*. Comp. Math., **120**, pp. 171–191 (2000).

[Ste77] J.H.M. Steenbrink: *Mixed Hodge structure on the vanishing cohomology*. In: *Real and Complex Singularities (Oslo 1976)*. Sijthoff and Noordhoff, Alphen aan den Rijn, 525–563 (1977).

[Tsu02] T. Tsuji: *Semi-stable conjecture of Fontaine-Jannsen : a survey*. In: *Cohomologies $p$-adiques et applications arithmétiques (II)*. Asterisque, **279**, 323–370 (2002).