On an integrable reduction of the Dirac equation

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Abstract

A symmetry reduction of the Dirac equation is shown to yield the system of ordinary differential equations whose integrability by quadratures is closely connected to the stationary mKdV hierarchy.

Consider the Dirac equation of an electron

\[ i \sum_{\mu=0}^{3} \gamma_\mu \psi_{x_\mu} - \left( e \sum_{\mu=0}^{3} \gamma_\mu A_\mu + m \right) \psi = 0, \]  

(1)

moving in the electric field

\[ A_0 = A_0(x_3), \quad A_1 = A_2 = A_3 = 0. \]

In the above formulae \( \gamma_\mu \) are 4 \( \times \) 4 Dirac matrices, \( \psi = \psi(x_0, x_1, x_2, x_3) \) is a four-component complex-valued function and \( e, m \) are constants.

The form of the vector-potential \( A_\mu \) imply the following Ansatz for the spinor field \( \psi(x) \):

\[ \psi(x) = \varphi(x_3). \]

Inserting this expression into the Dirac equation \( [1] \) yields system of ordinary differential equations (ODEs) for the four-component function \( \varphi(x_3) \)

\[ \varphi' - (ie \gamma_3 \gamma_0 A_0 - im \gamma_3) \varphi = 0. \]  

(2)

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Denoting
\[ x = 2x_3, \quad V(x) = eA_0(x_3), \]
\[ J_1 = \frac{1}{2}\gamma_0, \quad J_2 = \frac{i}{2}\gamma_3, \quad J_3 = \frac{i}{2}\gamma_3\gamma_0, \]
we rewrite (2) in the following form:
\[ \mathcal{L}\varphi \equiv (D_x - V(x)J_3 - mJ_2)\varphi = 0, \quad D_x = \frac{d}{dx}. \quad (3) \]

Note that the $4 \times 4$ matrices $J_1, J_2, J_3$ fulfill the commutation relations of the Lie algebra $so(3)$
\[ [J_a, J_b] = J_c, \quad (a, b, c) = \text{cycle } (1, 2, 3). \quad (4) \]

To integrate system of ODEs (3) we will make use of its symmetry properties. The general routine for calculating symmetry group admitted by a differential equation is the infinitesimal Lie method. This method makes it possible to reduce the problem of constructing the maximal symmetry group to integrating some linear system of partial differential equations (called determining equations). The general solution of the latter gives rise to the maximal transformation group admitted by the equation under study (for more detail, see, e.g. [1, 2]). However direct application of the Lie method to system (3) is inefficient, since integration of the corresponding determining equations is, in fact, equivalent to integration of the initial system of ODEs (3).

That is why to be able to integrate the determining equations one has inevitably to impose some \textit{a priori} restrictions on the choice of symmetry operators. Our idea is to look for a symmetry operator $Q$ admitted by system (3) in the form of an $n$th order polynomial in $m$ with matrix coefficients. First, we consider in some detail the case when $n = 3$ and then give the results on calculating Lie symmetries obtained for the case of an arbitrary $n \in \mathbb{N}$.

Thus, we adopt for a symmetry operator the following Ansatz:
\[ Q = \sum_{k=1}^{3} (a_k(x) + b_k(x) m + c_k(x) m^2 + d_k(x) m^3)J_k, \quad (5) \]
where $a_k, b_k, c_k, d_k$ are some smooth complex-valued functions.
Inserting the expression for $Q$ into the invariance criterion $[L, Q] = 0$ and splitting with respect to the powers of $m$ and then with respect to linearly independent matrices $J_1, J_2, J_3$ we get the system of determining equations for the functions $a_k, b_k, c_k, d_k$

\[
\begin{align*}
  d_1 &= 0, \quad d_3 = 0, \\
  d_2' - Vd_1 &= 0, \quad d_1' + Vd_2 - c_3 = 0, \quad d_3' - c_1 = 0, \\
  c_2' - Vc_1 &= 0, \quad c_1' + Vc_2 - b_3 = 0, \quad d_3' - b_1 = 0, \\
  b_2' - Vb_1 &= 0, \quad b_1' + Vb_2 - a_3 = 0, \quad c_3' - a_1 = 0, \\
  a_2' - Va_1 &= 0, \quad a_1' + Va_2 = 0, \quad a_3' = 0.
\end{align*}
\]

Integrating the above system of ODEs yields

\[
\begin{align*}
  d_1 &= 0, \quad d_2 = C_1, \quad d_3 = 0, \\
  c_1 &= 0, \quad c_2 = C_2, \quad c_3 = C_1 V, \\
  b_1 &= C_1 V', \quad b_2 = \frac{1}{2} C_1 V^2 + C_3, \quad b_3 = C_2 V, \\
  a_1 &= C_2 V', \quad a_2 = \frac{1}{2} C_2 V^2 + C_4, \quad a_3 = C_1 (V'' + \frac{1}{2} V^3) + C_3 V,
\end{align*}
\]

where $C_1, C_2, C_3, C_4$ are arbitrary constants and furthermore the potential $V(x)$ has to satisfy the following nonlinear ODEs

\[
C_1 (V''' + \frac{3}{2} V^2 V') + C_3 V' = 0, \quad C_2 (V'' + \frac{1}{2} V^3) + C_4 V = 0.
\]

Thus we have established that if the function $V(x)$ is a solution of the stationary mKdV equation

\[
C_1 (V''' + \frac{3}{2} V^2 V') + C_3 V' = 0, \quad (6)
\]

then the initial system of ODEs (3) admits the Lie symmetry

\[
Q = C_1 J_2 m^3 + C_1 V J_3 m^2 + C_1 V' J_1 m + \left( \frac{1}{2} C_1 V^2 + C_3 \right) J_2 m + \left( C_1 (V'' + \frac{1}{2} V^3) + C_3 V \right) J_3.
\]

This symmetry solves the problem of integrability of system of ODEs (3) by quadratures due to the assertion given below.
Lemma 1 Let the system of ODEs

\[ \mathcal{L}\psi \equiv \left( \frac{d}{dx} + f_a(x)Q_a \right) \psi = 0, \quad (7) \]

where \( Q_1, Q_2, Q_3 \) are constant matrices forming a basis of the Lie algebra \( \text{so}(3) \), admit a Lie symmetry

\[ X = \sum_{a=1}^{3} g_a(x)Q_a. \]

Then it is integrable by quadratures.

**Proof.** Making a change of dependent variables

\[ \psi \rightarrow \tilde{\psi} = \mathcal{V}(x)\psi, \quad \mathcal{V}(x) = \exp\left\{ \sum_{a=1}^{3} h_a(x)Q_a \right\} \]

we can always transform the operator \( X \) to become

\[ \tilde{X} = \mathcal{V}^{-1}X\mathcal{V} = g(x)Q_1, \quad g(x) \neq 0 \]

and what is more this transformation preserves the structure of system (7). The invariance criterion \([\tilde{\mathcal{L}}, \tilde{X}] = 0\), where

\[ \tilde{\mathcal{L}} = \mathcal{V}^{-1}\mathcal{L}\mathcal{V} = \sum_{a=1}^{3} \tilde{f}_a(x)Q_a, \]

implies that

\[ g(\tilde{f}_2Q_3 - \tilde{f}_3Q_2) - g'Q_1 = 0. \]

As the matrices \( Q_1, Q_2, Q_3 \) are linearly independent, hence it follows that

\[ \tilde{f}_2 = 0, \quad \tilde{f}_3 = 0, \quad g = \text{const}. \]

Consequently, the transformed system of ODEs necessarily takes the form

\[ \left( \frac{d}{dx} + \tilde{f}_1Q_1 \right) \psi = 0 \]

and is evidently integrable by quadratures. The lemma is proved.
Hence, we get a remarkable fact: if $V(x)$ is a solution of the stationary mKdV equation (8), then system of ODEs (4) is integrable by quadratures.

Now we turn to the case when a Lie symmetry is looked for as a polynomial in $m$ of an arbitrary order $n$

$$Q = \sum_{k=0}^{n} \sum_{a=1}^{3} f_a^k(x)J_a m^{n-k}.$$ 

The invariance criterion $[\mathcal{L}, Q] = 0$ yields the following system of determining equations for the coefficients of the operator $Q$:

$$f_0^0 = 0, \quad f_3^0 = 0, \quad (f_3^k)' - f_1^{k+1} = 0, \quad (f_1^k)' - V f_2^k = 0, \quad (f_2^k)' + V f_1^k - f_3^{k+1} = 0, \quad k = 0, \ldots, n - 1,$$

$$(f_0^n)' = 0, \quad (f_1^n)' - V f_2^n = 0, \quad (f_1^k)' + V f_2^k = 0.$$

We have obtained two classes of solutions of the above system of ODEs which are given below

1. $n = 2N + 1, \quad N \in \mathbb{N}$,

$$f_1^0 = 0, \quad f_2^0 = 1, \quad f_3^0 = 0, \quad f_1^{2k+1} = f_2^{2k+1} = 0, \quad f_3^{k+1} = R_k, \quad k = 1, \ldots, N,$$

$$f_1^{2k+2} = D_x R_k, \quad f_1^{2k+1} = (V - D_x^{-1} V') R_k,$$

$$f_3^{2k+2} = 0, \quad k = 0, \ldots, N - 1$$

and the equation

$$D_x R_N = 0 \quad \text{(8)}$$

holds.

2. $n = 2N + 2, \quad N \in \mathbb{N}$

$$f_1^0 = 0, \quad f_2^0 = 1, \quad f_3^0 = 0, \quad f_1^{2k+1} = f_2^{2k+1} = 0, \quad f_3^{k+1} = R_k, \quad k = 1, \ldots, N,$$

$$f_1^{2k+2} = D_x R_k, \quad f_1^{2k+1} = (V - D_x^{-1} V') R_k,$$

$$f_3^{2k+2} = 0, \quad k = 0, \ldots, N$$

and the equation

$$R_{N+1} = 0$$
holds.

In the above formulae we make use of the following notations

\[ R_k = \sum_{j=0}^{k} C_j (D_x^2 + V^2 - V D_x^{-1} V')^j V, \quad k = 0, \ldots, N + 1, \]

where \( C_0, \ldots, C_{N+1} \) are arbitrary real constants and \( D_x^{-1} \) is the inverse of \( D_x \).

A reader familiar with the soliton theory will immediately recognize the operator \( \mathcal{X} = D_x^2 + V^2 - V D_x^{-1} V_x \) as the recursion operator for the mKdV equation \[3, 4\]

\[ V_t + V_{xxx} + \frac{3}{2} V^2 V_x = 0. \]

Acting repeatedly with the operator \( \mathcal{X} \) on the trivial conserved density \( I_0 = V \) we get the whole set of conserved densities of the mKdV equation. Next, the operator

\[ \mathcal{Y} = D_x \mathcal{X} D_x^{-1} = D_x^2 + V^2 + V_x D_x^{-1} V \]

is the second recursion operator for the mKdV equation. Its repeated action on the trivial Lie symmetry \( S_0 = V_x \) yields the whole hierarchy of the higher symmetries of the mKdV equation. Hence it follows, in particular, that the condition (8) is rewritten in the form

\[ \sum_{k=0}^{N} C_k S_k = 0, \quad S_k = \mathcal{Y}^k V'. \] (9)

The above equation is nothing else than the higher stationary mKdV equation. Provided \( N = 1 \) it reduces to the standard stationary mKdV equation (3).

Hence, due to Lemma 1 it follows the validity of the following assertion.

**Theorem 1** Let the function \( V(x) \) satisfy the higher stationary mKdV equation (8) with some fixed \( N \) and \( C_0, \ldots, C_N \). Then, the system of ODEs (3) is integrable by quadratures.

It is a common knowledge that the stationary mKdV hierarchy is reduced to the stationary KdV hierarchy with the help of the Miura transformation (see, e.g. \[4, 5\]). Furthermore, the latter are integrated in terms of \( \theta \)-functions.
Consequently, the system of ODEs (2) is also integrable by quadratures thus giving rise to exact solutions of the initial Dirac equation (1).

Thus symmetry analysis of a very simple reduction of the Dirac equation (there are quite a few of much more sophisticated reductions, see [7]) reveals such important elements of the inverse scattering technique for the stationary mKdV equation as the recursion operators, infinite number of conserved densities, the hierarchy of higher symmetries of the mKdV equation which makes the integration of system (2) very rich in results. For those involved into the inverse scattering business this is not surprising at all, since the above used procedure is just an inversion of the famous approach to analysis of solitonic equations via the Lax pair [8]. More exactly, we reduce the problem of integrating linear equation to solving the nonlinear one. The crucial point is that this nonlinear differential equation can be integrated which enables us to construct the general solution of the initial linear equation.

Now it seems reasonable to carry out systematic analysis of all reductions of the Dirac equation by inequivalent subgroups of its symmetry group. We believe that in this way other hierarchies of higher solitonic equations will be obtained. This situation is in some analogy to symmetry reductions of the self-dual Yang-Mills equations yielding almost all known integrable solitonic equations (see, e.g. [9]).

One more important point is that the Lie transformation groups generated by symmetries $Q$ obtained above are not subgroups of the maximal symmetry group $C(1, 3) \otimes U(1)$ admitted by initial system (1). These symmetries correspond to conditional symmetry of the Dirac equation (1). Let us note that conditional symmetry of the linear and nonlinear Dirac equations was studied in [7, 10, 11].

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