TOPOLOGIES ON CENTRAL EXTENSIONS OF VON NEUMANN ALGEBRAS

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Abstract

Given a von Neumann algebra $M$ we consider the central extension $E(M)$ of $M$. We introduce the topology $t_c(M)$ on $E(M)$ generated by a center-valued norm and prove that it coincides with the topology of convergence locally in measure on $E(M)$ if and only if $M$ does not have direct summands of type II. We also show that $t_c(M)$ restricted on the set $E(M)_h$ of self-adjoint elements of $E(M)$ coincides with the order topology on $E(M)_h$ if and only if $M$ is a $\sigma$-finite type I$_{fin}$ von Neumann algebra.
1 Introduction

In the series of papers [1]-[5] we have considered derivations on the algebra $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra $M$, and on various subalgebras of $LS(M)$. A complete description of derivations has been obtained in the case of von Neumann algebras of type I and III. A comprehensive survey of recent results concerning derivations on various algebras of unbounded operators affiliated with von Neumann algebras is presented in [4]. A general form of automorphisms on the algebra $LS(M)$ in the case of von Neumann algebras of type I has been obtained in [5]. In proof of the main results of the above papers the crucial role is played by the co-called central extensions of von Neumann algebras and also by various topologies considered in [3].

Let $M$ be an arbitrary von Neumann algebra with the center $Z(M)$ and let $LS(M)$ denote the algebra of all locally measurable operators with respect $M$. We consider the set $E(M)$ of all elements $x$ from $LS(M)$ for which there exists a sequence of mutually orthogonal central projections $\{z_i\}_{i \in I}$ in $M$ with $\bigvee_{i \in I} z_i = 1$, such that $z_ix \in M$ for all $i \in I$. It is known [3] that $E(M)$ is a *-subalgebra in $LS(M)$ with the center $S(Z(M))$, where $S(Z(M))$ is the algebra of all measurable operators with respect to $Z(M)$, moreover, $LS(M) = E(M)$ if and only if $M$ does not have direct summands of type II.

A similar notion (i.e. the algebra $E(\mathcal{A})$) for arbitrary *-subalgebras $\mathcal{A} \subset LS(M)$ was independently introduced by M.A. Muratov and V.I. Chilin [7]. The algebra $E(M)$ is called the central extension of $M$. It is known ([3], [7]) that an element $x \in LS(M)$ belongs to $E(M)$ if and only if there exists $f \in S(Z(M))$ such that $|x| \leq f$. Therefore for each $x \in E(M)$ one can define the following vector-valued norm $||x|| = \inf\{f \in S(Z(M)) : |x| \leq f\}$. This center-valued norm naturally generates a topology on $E(M)$ which denoted by $t_c(M)$.

In this paper we study the relationship between the topology $t_c(M)$ on $E(M)$ generated by the above center-valued norm, the topology $t(M)$ – of convergence locally in measure, and the order topology $t_o(M)$ on $E(M)_h$. We prove that $t_c(M)$ coincides with the topology $t(M)$ on $E(M)$ if and only if $M$ does not have direct summands of type II. We show that $t_c(M)$ coincides with the order topology on $E(M)_h$ if and only if $M$ is a $\sigma$-finite type I$_{fin}$ algebra.

2 Central extensions of von Neumann algebras

Let $H$ be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. Consider a von Neumann algebra $M$ in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in $M$.

A linear subspace $D$ in $H$ is said to be affiliated with $M$ (denoted as $D_{\eta}M$), if
$u(D) \subset D$ for every unitary $u$ from the commutant
\[ M' = \{ y \in B(H) : xy = yx, \forall x \in M \} \]
of the von Neumann algebra $M$.

A linear operator $x$ on $H$ with the domain $D(x)$ is said to be affiliated with $M$ (denoted as $x\eta M$) if $D(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in D(x)$.

A linear subspace $D$ in $H$ is said to be strongly dense in $H$ with respect to the von Neumann algebra $M$, if
1) $D\eta M$;
2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow 1$, $p_n(H) \subset D$ and $p_n^+ = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$, where $1$ is the identity in $M$.

A closed linear operator $x$ acting in the Hilbert space $H$ is said to be measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and $D(x)$ is strongly dense in $H$. Denote by $S(M)$ the set of all measurable operators with respect to $M$ (see [10]).

A closed linear operator $x$ in $H$ is said to be locally measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in $M$ such that $z_n \uparrow 1$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$ (see [11]).

It is well-known [6], [11] that the set $LS(M)$ of all locally measurable operators with respect to $M$ is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains $S(M)$ as a solid *-subalgebra.

Let $(\Omega, \Sigma, \mu)$ be a measure space and from now on suppose that the measure $\mu$ has the direct sum property, i.e. there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set $B$ with zero measure such that $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on $(\Omega, \Sigma, \mu)$ equipped with the topology of convergence in measure.

Consider the algebra $S(Z(M))$ of operators which are measurable with respect to the center $Z(M)$ of the von Neumann algebra $M$. Since $Z(M)$ is an abelian von Neumann algebra it is *-isomorphic to $L^\infty(\Omega, \Sigma, \mu)$ for an appropriate measure space $(\Omega, \Sigma, \mu)$. Therefore the algebra $S(Z(M))$ coincides with $Z(LS(M))$ and can be identified with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on $(\Omega, \Sigma, \mu)$.

The basis of neighborhoods of zero in the topology of convergence locally in measure on $L^0(\Omega, \Sigma, \mu)$ consists of the sets
\[ W(A, \varepsilon, \delta) = \{ f \in L^0(\Omega, \Sigma, \mu) : \exists B \in \Sigma, B \subseteq A, \mu(A \setminus B) \leq \delta, \]
\[ f \cdot \chi_B \in L^\infty(\Omega, \Sigma, \mu), |||f \cdot \chi_B|||_{L^\infty(\Omega, \Sigma, \mu)} \leq \varepsilon \} , \]

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where \( \varepsilon, \delta > 0 \), \( A \in \Sigma \), \( \mu(A) < +\infty \), and \( \chi_B \) is the characteristic function of the set \( B \in \Sigma \).

Recall the definition of the dimension functions on the lattice \( P(M) \) of projection from \( M \) (see [6], [10]).

By \( L_+ \) we denote the set of all measurable functions \( f : (\Omega, \Sigma, \mu) \to [0, \infty] \) (modulo functions equal to zero \( \mu \)-almost everywhere).

Let \( M \) be an arbitrary von Neumann algebra with the center \( Z(M) \equiv L^\infty(\Omega, \Sigma, \mu) \).

Then there exists a map \( D : P(M) \to L_+ \) with the following properties:

(i) \( d(e) \) is a finite function if only if the projection \( e \) is finite;
(ii) \( d(e + q) = d(e) + d(q) \) for \( p, q \in P(M), eq = 0 \);
(iii) \( d(uu^*) = d(u^*u) \) for every partial isometry \( u \in M \);
(iv) \( d(ze) = zd(e) \) for all \( z \in P(Z(M)), e \in P(M) \);
(v) if \( \{e_\alpha\}_{\alpha \in J}, e \in P(M) \) and \( e_\alpha \uparrow e \), then

\[
d(e) = \sup_{\alpha \in J} d(e_\alpha).
\]

This map \( d : P(M) \to L_+ \), is a called the dimension functions on \( P(M) \).

Recall that for an element \( x \in LS(M) \) the projection defined as

\[
c(x) = \inf \{z \in P(Z(M)) : zx = x \}
\]

is called the central cover of \( x \).

Remark 2.1. Let \( M \) be a type I von Neumann algebra. If \( p, q \in P(M) \) abelian projections are faithful (i.e. with \( c(p) = c(q) = 1 \),) then the property (iii) implies that \( 0 < d(p)(\omega) = d(q)(\omega) < \infty \) for \( \mu \)-almost every \( \omega \in \Omega \). Therefore replacing \( d \) by \( d(p)^{-1}d \) we can assume that \( d(p) = c(p) \) for every faithful abelian projection \( p \in P(M) \). Thus for all \( e \in P(M) \) we have that \( d(e) \geq c(e) \).

The basis of neighborhoods of zero in the topology \( t(M) \) of convergence locally in measure on \( LS(M) \) consists (in the above notations) of the following sets

\[
V(A, \varepsilon, \delta) = \{x \in LS(M) : \exists p \in P(M), \exists z \in P(Z(M)), xp \in M, \]
\[
||xp||_M \leq \varepsilon, \ z^\perp \in W(A, \varepsilon, \delta), \ d(zp^\perp) \leq \varepsilon z \},
\]

where \( \varepsilon, \delta > 0 \), \( A \in \Sigma \), \( \mu(A) < +\infty \).

The topology \( t(M) \) is metrizable if and only if the center \( Z(M) \) is \( \sigma \)-finite (see [3]).

Given an arbitrary family \( \{z_i\}_{i \in I} \) of mutually orthogonal central projections in \( M \) with \( \bigvee_{i \in I} z_i = 1 \) and a family of elements \( \{x_i\}_{i \in I} \) in \( LS(M) \) there exists a unique element \( x \in LS(M) \) such that \( z_i x = z_i x_i \) for all \( i \in I \). This element is denoted by \( x = \sum_{i \in I} z_i x_i \).
We denote by \( E(M) \) the set of all elements \( x \) from \( LS(M) \) for which there exists a sequence of mutually orthogonal central projections \( \{ z_i \}_{i \in I} \) in \( M \) such that \( z_i x \in M \) for all \( i \in I \), i.e.

\[
E(M) = \{ x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i \in I} z_i = 1, z_i x \in M, i \in I \},
\]

where \( Z(M) \) is the center of \( M \).

It is known [3] that \( E(M) \) is \(-\)subalgebras in \( LS(M) \) with the center \( S(Z(M)) \), where \( S(Z(M)) \) is the algebra of all measurable operators with respect to \( Z(M) \), moreover, \( LS(M) = E(M) \) if and only if \( M \) does not have direct summands of type II.

A similar notion (i.e. the algebra \( E(A) \)) for arbitrary \(-\)subalgebras \( A \subset LS(M) \) was independently introduced recently by M.A. Muratov and V.I. Chilin [7]. The algebra \( E(M) \) is called the central extension of \( M \).

It is known (3, 7) that an element \( x \in LS(M) \) belongs to \( E(M) \) if and only if there exists \( f \in S(Z(M)) \) such that \( |x| \leq f \). Therefore for each \( x \in E(M) \) one can define the following vector-valued norm

\[
||x|| = \inf\{ f \in S(Z(M)) : |x| \leq f \}
\]

and this norm satisfies the following conditions:

1) \( ||x|| \geq 0; ||x|| = 0 \iff x = 0; \)
2) \( ||fx|| = |f||x||; \)
3) \( ||x + y|| \leq ||x|| + ||y||; \)
4) \( ||xy|| \leq ||x|| ||y||; \)
5) \( ||xx^*|| = ||x||^2 \)

for all \( x, y \in E(M), f \in S(Z(M)) \).

3 Topologies on the central extensions of von Neumann algebras

Let \( M \) be an arbitrary von Neumann algebra with the center \( Z(M) \equiv L^\infty(\Omega, \Sigma, \mu) \). On the space \( E(M) \) we consider the following sets:

\[
O(A, \varepsilon, \delta) = \{ x \in E(M) : ||x|| \in W(A, \varepsilon, \delta) \},
\]

(3.1)

where \( \varepsilon, \delta > 0, A \in \Sigma, \mu(A) < +\infty. \)

The following proposition gives elementary properties of the sets \( O(A, \varepsilon, \delta) \), which immediately follow from the corresponding properties of the sets \( V(A, \varepsilon, \delta) \) (see [6] Proposition 3.5.1)).

**Proposition 3.1.** Let \( \varepsilon, \varepsilon_j > 0 \) and \( \delta, \delta_j > 0, j = 1, 2, A \in \Sigma, \mu(A) < \infty. \) Then
i) $\lambda O(A, \varepsilon, \delta) = O(A, |\lambda|\varepsilon, \delta)$ \quad $\lambda \in \mathbb{C}, \lambda \neq 0$;

ii) $O(A, \varepsilon, \delta_1) \subseteq O(A, \varepsilon_2, \delta_2)$ \quad $\varepsilon_1 \leq \varepsilon_2, \delta_1 \leq \delta_2$;

iii) $O(A, \varepsilon_1, \delta_1) + O(A, \varepsilon_2, \delta_2) \subseteq O(A, \varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$;

iv) $O(A, \varepsilon, \delta_1)O(A, \varepsilon_2, \delta_2) \subseteq O(A, \varepsilon\varepsilon_2, \delta_1 + \delta_2)$;

v) $O^*(A, \varepsilon, \delta) = O(A, \varepsilon, \delta)$, where $O^*(A, \varepsilon, \delta) = \{x^* : x \in O(A, \varepsilon, \delta)\}$;

vi) $\bigcap\{O(A, \varepsilon, \delta) : \varepsilon > 0, \delta > 0, A \in \Sigma, \mu(A) < \infty\} = \{0\}$.

From Proposition 3.1 it follows that the system of sets

$$\{x + O(A, \varepsilon, \delta)\}, \quad (3.2)$$

where $x \in E(M), \varepsilon > 0, \delta > 0, A \in \Sigma, \mu(A) < \infty$, defines on $E(M)$, a Hausdorff vector topology $t_c(M)$, for which the sets $(3.2)$ form the base of neighborhoods of the element $x \in E(M)$. Moreover in this topology the involution is continuous and the multiplication is jointly continuous, i.e. $(E(M), t_c(M))$ is a topological *-algebra.

From [4, Proposition 5.3] it follows that $(E(M), t_c(M))$ is complete.

Thus we obtain the following result.

**Proposition 3.2.** i) $(E(M), t_c(M))$ is a complete topological *-algebra;

ii) $M$ is a $t_c(M)$-dense in $E(M)$.

**Proof.** i) is proved above.

ii). Let $x \in E(M)$ and $A \in \Sigma, \mu(A) < \infty, \varepsilon, \delta > 0$. For $n \in \mathbb{N}$ put

$$B_n = \{\omega \in A : ||x||(\omega) \leq n\}.$$

Since $\mu(A \setminus B_n) \to 0$ as $n \to \infty$ there is $k \in \mathbb{N}$ such that $\mu(A \setminus B_k) \leq \varepsilon$. Put $x_k = \chi_{B_k}x$. Then

$$||x_k|| \leq k1$$

and

$$||x - x_k||\chi_{B_k} = ||x\chi_{B_k} - x_k\chi_{B_k}|| = ||x\chi_{B_k} - x\chi_{B_k}|| = 0.$$

Thus $x_k + O(A, \varepsilon, \delta)$. This means that $\overline{M}_{t_c(M)} = E(M)$. The proof is complete.

**Remark 3.3.** Note that if $M$ is a commutative von Neumann algebra then $||x|| = |x|$ for each $x \in E(M)$, and therefore $O(A, \varepsilon, \delta) = W(A, \varepsilon, \delta)$ for all $\varepsilon, \delta > 0$, $A \in \Sigma, \mu(A) < + \infty$. Hence the topology $t_c(M)$ on $E(M)$ coincides with the topology of convergence locally in measure $t(M)$.

If $M$ is a factor, then $E(M) = M$ and $t_c(M) = t_{\| \cdot \|_M}$, where $t_{\| \cdot \|_M}$ uniform topology on $M$.

**Proposition 3.4.** i) A net $\{p_n\} \subset P(M)$ converges to zero with respect to the topology $t_c(M)$ if and only if $c(p_n) \xrightarrow{t(Z(M))} 0$, where $t(Z(M))$ is the topology of convergence locally in measure on $Z(M)$.
ii) A net \( \{x_\alpha\} \subset E(M) \) converges to zero with respect to the topology \( t_e(M) \) if and only if \( e_\lambda^+(|x_\alpha|) t_{\lambda(M)} \to 0 \) for any \( \lambda > 0 \), where \( \{e_\lambda(|x_\alpha|)\} \) is a spectral projections family for the operator \( x_\alpha \).

Proof. i) The proof immediately follows from the definition of the topology \( t_e(M) \) and the equality \( ||p|| = c(p), p \in P(M) \).

ii) Let \( x_\alpha \xrightarrow{t_e(M)} 0 \) and \( \lambda > 0 \). Take any \( A \in \Sigma, \mu(A) < \infty, 0 < \varepsilon < \lambda/2, \delta > 0 \). Since \( ||x_\alpha|| \xrightarrow{\mu(M)} 0 \), then there exists \( \alpha_0 \) such that \( ||x_\alpha|| \in W(A, \varepsilon, \delta) \) for each \( \alpha \geq \alpha_0 \). Therefore there exists \( B_\alpha \in \Sigma, B_\alpha \subseteq A \) such that \( \mu(A \setminus B_\alpha) \leq \delta, ||||x_\alpha|||\chi_{B_\alpha}||M \leq \varepsilon \).
Thus \( ||x_\alpha||\chi_{B_\alpha}||M \leq \varepsilon \), i.e. \( |x_\alpha|\chi_{B_\alpha} \leq \varepsilon \chi_{B_\alpha} \). Since \( \varepsilon < \lambda/2 \) then from the last inequality we have that \( c(e_\lambda^+(|x_\alpha|))\chi_{B_\alpha} = 0 \). The inequality \( \mu(A \setminus B_\alpha) \leq \delta \) implies that \( c(e_\lambda^+(|x_\alpha|)) \in W(A, \varepsilon, \delta) \), i.e. \( c(e_\lambda^+(|x_\alpha|)) \xrightarrow{\mu(M)} 0 \). Thus \( e_\lambda^+(|x_\alpha|) \xrightarrow{t_e(M)} 0 \).

Now let \( e_\lambda^+(|x_\alpha|) \xrightarrow{t_e(M)} 0 \) and \( 0 < \varepsilon < 1, \delta > 0 \). Then \( c(e_\lambda^+(|x_\alpha|)) \xrightarrow{\mu(M)} 0 \). Therefore there exists \( \alpha_0 \) such that \( c(e_\lambda^+(|x_\alpha|)) \in W(A, \varepsilon, \delta) \) for all \( \alpha \geq \alpha_0 \). Hence there exists \( B_\alpha \in \Sigma, B_\alpha \subseteq A \) such that \( \mu(A \setminus B_\alpha) \leq \delta, ||c(e_\lambda^+(|x_\alpha|))\chi_{B_\alpha}||M \leq \varepsilon < 1 \). Thus \( c(e_\lambda^+(|x_\alpha|))\chi_{B_\alpha} = 0 \), i.e. \( |x_\alpha|\chi_{B_\alpha} \leq \varepsilon \chi_{B_\alpha} \). Therefore

\[
|||x_\alpha|||\chi_{B_\alpha}||M \leq \varepsilon
\]

and

\[
\mu(A \setminus B_\alpha) \leq \delta.
\]

Thus \( ||x_\alpha|| \in W(A, \varepsilon, \delta) \), i.e. \( ||x_\alpha|| \xrightarrow{t_e(M)} 0 \). Therefore \( x_\alpha \xrightarrow{t_e(M)} 0 \). The proof is complete.

Let \( t(M) \) denote the topology on \( E(M) \) induced by the topology \( t(M) \) from \( LS(M) \).

**Proposition 3.5.** The topology \( t_e(M) \) is stronger than the topology \( t(M) \) of convergence locally in measure.

*Proof.** It is sufficient to show that

\[
O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta).
\]

(3.3)

Let \( x \in O(A, \varepsilon, \delta) \), i.e. \( ||x|| \in W(A, \varepsilon, \delta) \). Then there exists \( B \in \Sigma \) such that

\[
B \subseteq A, \mu(A \setminus B) \leq \delta,
\]

and

\[
||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu), \quad |||x|||\chi_B||M \leq \varepsilon.
\]

Put \( z = p = \chi_B \). Then \( ||xp|| = ||x\chi_B|| = ||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu) \), i.e. \( xp \in M \) and moreover \( ||xp||M \leq \varepsilon \). Since \( \mu(A \setminus B) \leq \delta \) and \( z^\perp \chi_B = \chi_B^2 \chi_B = 0 \), one has \( z^\perp \in W(A, \varepsilon, \delta) \). Therefore

\[
||xp||M \leq \varepsilon, \quad z^\perp \in W(A, \varepsilon, \delta), \quad zp^\perp = \chi_B \chi_B^2 = 0
\]
and hence \( x \in V(A, \varepsilon, \delta) \), i.e. \( O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta) \). The proof is complete. \( \square \)

**Proposition 3.6.** If \( M \) is a type I or III von Neumann algebra and \( 0 < \varepsilon < 1 \), then

\[
O(A, \varepsilon, \delta) = V(A, \varepsilon, \delta).
\]

**Proof.** From above (3.3) we have that \( O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta) \). Therefore it is sufficient to show that \( V(A, \varepsilon, \delta) \subset O(A, \varepsilon, \delta) \).

Let \( x \in V(A, \varepsilon, \delta) \). Then there exist \( p \in P(M) \) and \( z \in P(Z(M)) \) such that

\[
|\chi_A^p| = 1 \quad \text{and} \quad d(zp^\perp) \leq \varepsilon z.
\]

If \( M \) is of type I then Remark [2.1] implies that \( d(zp^\perp) \geq c(zp^\perp) \). Now from \( d(zp^\perp) \leq \varepsilon z \) it follows that \( c(zp^\perp) \leq \varepsilon z \). From \( 0 < \varepsilon < 1 \) we obtain that \( zp^\perp = 0 \).

If \( M \) is of type III then the finiteness of the projection \( zp^\perp \) implies that \( zp^\perp = 0 \).

Thus \( z = zp \). Put \( z = \chi_E \) for an appropriate \( E \in \Sigma \). Since \( z^\perp \in W(A, \varepsilon, \delta) \) one has that \( \chi_{\Omega \setminus E} \in W(A, \varepsilon, \delta) \). Thus there exists \( B \in \Sigma \) such that \( B \subset A \), \( \mu(A \setminus B) \leq \delta \), \( |\chi_{\Omega \setminus E} \chi_B| \leq \varepsilon < 1 \). Hence \( \chi_B \leq \chi_E \). So we obtain

\[
||x|| \chi_B \leq ||x|| \chi_E = ||x|| z = ||xz|| = ||xz p|| = ||xp|| \leq \varepsilon.
\]

This means that \( x \in O(A, \varepsilon, \delta) \). The proof is complete. \( \square \)

Proposition 3.6 implies that following

**Theorem 3.7.** If \( M \) is a type I or III von Neumann algebra then the topologies \( t(M) \) and \( t_c(M) \) coincide.

**Proposition 3.8.** If \( M \) is of type II then \( t(M) < t_c(M) \).

**Proof.** Since \( M \) is a type II then there exists a decreasing sequence of projections \( \{p_n\} \) in \( M \) such that \( c(p_n) = 1 \) and \( d(p_n) = \frac{1}{2^k} \) for all \( n \in \mathbb{N} \). Then \( \{p_n\} \) converges to zero with respect to the topology locally in measure. Indeed take any neighborhood of zero \( V(A, \varepsilon, \delta) \) in the topology \( t(M) \). Put \( z = 1, p = p_k^\perp \), where the number \( k \) is such that \( \frac{1}{2^k} < \varepsilon \). For \( n \geq k \) we have that

\[
p_n p = p_n p_k^\perp = (p_n p_k) p_k^\perp = 0,
\]

\[
z^\perp \in W(A, \varepsilon, \delta)
\]

and

\[
d(zp^\perp) = d(p_k) = \frac{1}{2^k} \leq \varepsilon z.
\]

This means that \( p_n \in V(A, \varepsilon, \delta) \) for all \( n \geq k \), i.e. \( \{p_n\} \) converges to zero with respect to the topology locally in measure.

On the other hand the equality \( c(p_n) = 1 \) implies that \( ||p_n|| = 1 \). Thus a sequence \( \{p_n\} \) does not converges to zero in the topology \( t_c(M) \). Hence \( t(M) < t_c(M) \). The proof is complete. \( \square \)
Theorem 3.7 and Proposition 3.8 imply the following result which describes the class of von Neumann algebras $M$ for which the topologies $t(M)$ and $t_o(M)$ coincide.

**Theorem 3.9.** The following conditions on a given von Neumann algebra $M$ are equivalent:

(i) $t(M) = t_o(M)$;

(ii) $M$ does not have direct summands of type II.

By $E(M)_h$ we denote the set of all self-adjoint elements in $E(M)$. A net $\{x_\alpha\}_{\alpha \in I} \subset E(M)_h$ is called $(o)$-convergent to $x \in E(M)_h$ (denoted $x_\alpha \overset{(o)}{\to} x$), if there exist nets $\{a_\alpha\}_{\alpha \in I}$ and $\{b_\alpha\}_{\alpha \in I}$ in $E(M)_h$, such that $a_\alpha \leq x_\alpha \leq b_\alpha$ for each $\alpha \in I$ and $a_\alpha \uparrow x$, $b_\alpha \downarrow x$. The strongest topology on $E(M)_h$ for which $(o)$-convergence of nets implies their convergence in the topology is called the order topology, or the $(o)$-topology, and is denoted by $t_o(M)$.

Let $t_{ch}(M)$ (respectively $t_h(M)$) denote the topology on $E(M)_h$ induced by the topology $t_c(M)$ (respectively $t(M)$) from $E(M)$.

We now describe class of von Neumann algebras $M$ for which the topologies $t_c(M)$ and $t_o(M)$ coincide.

**Theorem 3.10.** (i) $t_{ch}(M) \leq t_o(M)$ if and only if $M$ is of type $I_{fin}$;

(ii) $t_{ch}(M) = t_o(M)$ if and only if $M$ is a $\sigma$-finite type $I_{fin}$ algebra.

**Proof.** (i) Let $t_{ch}(M) \leq t_o(M)$. If the algebra $M$ does not has type $I_{fin}$ then there exists a nonzero projection $z \in P(Z(M))$ and a sequence of mutually orthogonal projections $\{p_n\}_{n=1}^\infty$ in $M$ with $c(p_n) = z, n \in \mathbb{N}$. Then $p_n \overset{(o)}{\to} 0$, and therefore $p_n \overset{t_{ch}(M)}{\to} 0$. Hence $||p_n|| \overset{t_o(M)}{\to} 0$. Since $||p_n|| = c(p_n) = z$ it follows that $z = 0$, this is a contradiction with $z \neq 0$. Hence $M$ is a type $I_{fin}$ algebra.

Conversely let $M$ be a type $I_{fin}$ algebra. Then by [3, Proposition 1.1] we have that $LS(M) = E(M)$. Thus theorem 3.9 implies that $t_{ch}(M) = t_h(M)$. Since $t_h(M) \leq t_o(M)$ (see [8, Theorem 1 (i)]) then $t_{ch}(M) \leq t_o(M)$.

(ii) If $t_{ch}(M) = t_o(M)$ then $M$ is a type $I_{fin}$ algebra (see (i)). Again using the theorem 3.9 we have that $t_{ch}(M) = t_h(M)$. Thus $t_h(M) = t_o(M)$. Now by [8, Theorem 1 (ii)] follows that $M$ is a $\sigma$-finite algebra.

Conversely let $M$ be a $\sigma$-finite type $I_{fin}$ algebra. Then by theorem 3.9 we have that $t_{ch}(M) = t_h(M)$ and by [8, Theorem 1 (ii)] we obtain that $t_h(M) = t_o(M)$. Hence

$$t_{ch}(M) = t_h(M) = t_o(M).$$

The proof is complete. □

Theorem 3.10 yields the following corollary.

**Corollary 3.11.** The following assertions are true:
(i) If $M$ is a $\sigma$-finite von Neumann algebra but is not type $I_{\text{fin}}$, then $t_o(M) < t_h(M)$;
(ii) If $M$ is not a $\sigma$-finite von Neumann algebra but is type $I_{\text{fin}}$, then $t_h(M) < t_o(M)$.

**Proposition 3.12.** The topology $t_c(M)$ is locally convex if and only if $M$ is $^*$-isomorphic to the $C^*$-product $\bigoplus_{j \in J} M_j$, where $M_j$ are factors.

**Proof.** Let $t_c(M)$ be a locally convex topology on $E(M)$. Since $t_c(M)$ induces the topology $t(Z(M))$ on $Z(E(M)) = S(Z(M))$, we have that $(S(Z(M)), t(Z(M)))$ is a locally convex space. It follows from [9, 12, Ch. V, §3] that $Z(M)$ is an atomic von Neumann algebra. Hence, the algebra $M$ is $^*$-isomorphic to the $C^*$-product $\bigoplus_{j \in J} M_j$, where $M_j$ are factors for all $j \in J$.

Conversely, let $M = \bigoplus_{j \in J} M_j$, where $M_j$ are factors. Then

$$E(M_j) = M_j, t_c(M) = t_{\| \cdot \|_{M_j}}, E(M) = \prod_{j \in J} M_j$$

and, hence the topology $t_c(M)$ is a Tychonoff product of the normed topologies $t_{\| \cdot \|_{M_j}}$, that is, $t_c(M)$ is a locally convex topology. The proof is complete. \( \Box \)

Similarly, we obtain the following

**Proposition 3.13.** The topology $t_c(M)$ can be normed if and only if $M = \bigoplus_{j=1}^n M_j$, where $M_j$ are factors, $j = 1, n, n \in \mathbb{N}$.

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