BIDE - SIDE EXPONENTIAL AND
MOMENT INEQUALITIES FOR TAILS OF
DISTRIBUTIONS OF POLYNOMIAL MARTINGALES.

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In this paper non-asymptotic exponential estimates are derived for the tail distribution of polynomial martingale differences in terms unconditional tails distributions of summands. Applications are considered in the theory of polynomials on independent random variables, to the theory of \( U \) - statistics, multiply martingale series and in the theory of weak compactness measures on the Banach spaces.  

1. Introduction. Notations. Statement of problem. Let \((\Omega, F, P)\) be a probability space, \(\xi(i, 1), \xi(i, 2), \ldots, \xi(i, d)\) be a family of centered (\(E\xi(i, m) = 0\)) martingale - differences on the basis of the same flow \(\sigma\) - fields (filtration) \(F(i) : F(0) = \{\emptyset, \Omega\}, F(i) \subset F(i + 1) \subset F, \forall m = 1, 2, \ldots, d \Rightarrow \xi(0, m) = 0; |\xi(i, m)| < \infty\), and for every \(i \geq 0, m = 1, 2, \ldots, d\), \(\forall k = 0, 1, \ldots, i - 1 \Rightarrow E\xi(i, m)/F(k) = 0; E\xi(i, m)/F(i) = \xi(i, m) \mod P\),

\[ I = I(d) = \{(i_1, i_2, \ldots, i_d)\}, I(d, n) - the set of indexes I of the form \(I(d, n) = \{(i_1, i_2, \ldots, i_d)\} : 1 \leq i_1 < i_2 \ldots < i_{d-1} < i_d \leq n, J(d) = J(d, n) - the subset of I(d, n) - the set of indexes of the form \(J(d, n) = J(d) = \{(i_1, i_2, \ldots, i_{d-1}, n)\}\) such that \(1 \leq i_1 < i_2 \ldots < i_{d-1} \leq n - 1, b(I) = b(i_1, i_2, \ldots, i_d)\) is a \(d\) - dimensional numerical non-random sequence,

\[
\xi(I) = \prod_{m=1}^{d} \xi(i_m, m), \xi(J) = \prod_{m=1}^{d-1} \xi(i_m, m), \sigma^2(i, m) = D\xi(i, m),
\]

\[ Q_d = Q(d, n, \{\xi(\cdot, \cdot)\}) = Q(d, n) = \sum_{I \in I(d, n)} b(I) \xi(I) - (1.0) \]

be a homogeneous polynomial (random polynomial) power \(d\) on the variables \(\{\xi(i, m)\}\) "without diagonal members", (on the other hand, multiply stochastic integral on the discrete martingale measure, martingale transform), \(n\) is an integer number: \(n = 1, 2, \ldots, \infty\); in the case \(n = \infty\) \(Q(d, \infty)\) should be understood as a limit \(Q(d, \infty) = \lim_{n \to \infty} Q(d, n)\) (with probability 1).

Note than

\[ DQ(d, n) = \sum_{I \subset I(d, n)} b^2(I) \prod_{m=1}^{d} \sigma^2(i_m, m); \]

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hence, if
\[
\sum_{I \subset I(d, \infty)} b^2(I) \prod_{m=1}^{d} \sigma^2(i_m, m) < \infty,
\]
then by virtue of theorem of D.Doob \(Q(d, \infty)\) exists. In particular, if
\[
\sup_{i,m} D\xi(i, m) = \sup_{i,m} \sigma^2(i, m) < \infty, \quad \sum_{I \subset I(d, \infty)} b^2(I) < \infty,
\]
then condition (1.1) is satisfied.

We shall denote
\[
B(d, n) = B = \{b(I) : \sum_{I \in I(d, n)} b^2(I) = 1\}, \quad n \leq \infty
\]
and assume that \(b \in B = B(d, n)\).

If \(b(\cdot) \in B, \sigma^2(i, m) = 1\), then \(DQ_d = 1\).

Another notations. For any random variable \(\tau\) we define
\[
T(\tau, x) = \max(\mathbf{P}(\tau > x), \mathbf{P}(\tau < -x)), \quad x > 0,
\]
the tail of distribution \(\tau\),
\[
T_{Q(d,n)}(x) = T_Q(x) = T(Q_d, x).
\]

Our goal is to make non-asymptotic uniform over \(b \in B\) estimate of the tails of random variables \(T(Q_d, x)\) and of the moments \(\mathbf{E}|Q_d|^p\) in the terms of unconditional tails \(T(\xi(i, m), x)\) and moments \(\mathbf{E}|\xi(i, m)|^p\) of summands \(\{\xi(i, m)\}\).

To interest our readers, we shall formulate now two simple results. Denote as \(G(q), q > 0\) the set of all random variables \(\{\eta\}\), which are defined on the our probability space \(\{\Omega, F, \mathbf{P}\}\), such that
\[
\exists K = \text{const} \in (0, \infty], \forall x > 0 \Rightarrow T(\eta, x) \leq \exp(-x/K)^q).
\]
It follows from theory of \(G -\) spaces ([11], p. 31 - 37) that in according to the norm
\[
||\eta||_q := \sup_{m \geq 1} \left[ |\eta|_m \right] m^{-1/q}, \quad |\eta|_m \overset{def}{=} [\mathbf{E}|\eta|^m]^{1/m}
\]
the set \(G(q)\) is the (full) Banach space which is isomorphic to the Orlicz space on the probability space \((\Omega, F, \mathbf{P})\) with \(N -\) function \(N(u) = \exp(|u|^q) - 1\) [11, p. 35 - 37].

In the case \(q = \infty\) the space \(G(\infty)\) consists of all bounded \((mod \mathbf{P})\) variables, and the norm \(G(\infty)\) is equivalent to the classical \(L_\infty\) norm
\[
|\eta|_\infty = \text{vraisup}_{\omega \in \Omega} |\eta(\omega)|.
\]
Generalization: denote $G(q, r)$, $q > 0, r \in (-\infty, +\infty)$ the set of all random variables $\{\eta\}$ with finite norm:

$$|||\eta|||_{q,r} = \sup_{p \geq 2} |\eta|_p \ p^{-1/q} \ \log^{-r} p;$$

it is known that $|||\eta|||_{q,r} < \infty$ if and only if

$$\exists K = \text{const} > 0, \ T(\eta, x) \leq \exp \left( -\frac{(x/K)^q (\log(F + x/K))^{-qr}}{C_1(q,r)} \right),$$

and for some $C_1, C_2 = C_1(q, r), C_2(q, r) \in (0, \infty), C_1 \leq C_2$

$$C_1(q, r) K \leq |||\eta|||_{q,r} \leq C_2(q, r) K,$$

where

$$F = F(q, r) = 1, \ r \leq 0; \ F(q, r) = \exp(q), \ r > 0.$$

*Throughout this paper the letter $C_j(\cdot)$ will denote various constants which may differ from one formula to the next even within a single string of estimates and which do not depend upon $n$. We make no attempt to obtain the best values for these constants.*

Another example (concise). Let us introduce the other space of random variables $\Psi(C, \beta)$, $\beta = \text{const} > 0$. By definition, this space consist of all random variables $\{\eta\}$ with finite norm

$$|||\eta|||_{C,\beta} \overset{\text{def}}{=} \sup_{p \geq 1} |\eta|_p \exp \left( -Cp^\beta \right); \ \eta \in \bigcup_{C \geq 0} \Psi(C, \beta) \Leftrightarrow$$

$$\exists C_1 > 0, \ T(\eta, x) \leq \exp \left( -C_1(\log(1 + x))^{1+1/\beta} \right).$$

For instance assume that there exist constants $K(m) \in (0, \infty)$, $q(m) > 0$ so that

$$\forall x \geq 0$$

$$\sup_i T(\xi(i, m), x) \leq \exp \left( -\frac{(x/K(m))^q(m)}{C_1(q, r)} \right), \quad (1.4)$$

or briefly $\max_{m=1,2,\ldots,d} \sup_{i \leq n} |||\xi(i, m)|||_{q(m)}/K(m) < \infty$. We define $K = \prod_{m=1}^d K(m), 1/\infty = 0, M = M(d, q) =$

$$M(d, q) = M(d; q(1), q(2), \ldots, q(d)) = \left( d/2 + \sum_{m=1}^d (1/q(m)) \right)^{-1},$$

$$S(\vec{q}, x) = S(q(1), q(2), \ldots, q(d); x) = \sup_{b \in B} \sup_{\{\xi(i,m)\}} T_Q(x),$$

where interior sup is calculated over all families of centered martingale - differences $\{\xi(i, m)\}$ satisfying condition (1.4).

**Theorem 1.** There exist two constants $C_1 = C_1(\vec{q}), C_2 = C_2(\vec{q}), 0 < C_2 < C_1 < \infty$ so that for all $x > x_0 = \text{const} > 0$.
\[
\exp \left( -\left[ x/(C_1 K) \right]^M \right) \leq S(q, x) \leq \exp \left( -\left[ x/(C_2 K) \right]^M \right). \tag{1.5}
\]

On the other hand, the right inequality in our result (1.5) may be rewritten in the terms of \( G(q) \) — spaces as

\[
\sup_{\{\xi(i,m) : ||\xi(i,m)||_{q(m)} < \infty\}} \|Q_d\|_{M(d,q)}/ \prod_{m=1}^{d} \sup_{i \leq n} ||\xi(i,m)||_{q(m)} \leq C_3(q), \quad \tag{1.6}
\]

and the left part of Theorem 1 denotes that inequality (1.6) can not be improved.

We can improve the result (1.5) in so-called ”independent case”, i.e. if all the variables \( \{\xi(i,m)\} \) are independent. Let us consider a more generally centered polynomial (with ”diagonal members”) degree \( d \) of a view \( R_d = R_d(n) = R_d(n, \{\xi(i,s)\}) = \)

\[
\sum_{1 \leq i_1 < i_2 < \ldots < i_d \leq n} b(i_1, i_1, \ldots, i_1; i_2, i_2, \ldots, i_2; \ldots, i_d, i_d, \ldots, i_d) \times \]

\[
\prod_{l=1}^{d} \left( \xi^{k(l)}(i_l, l) - m(k(l), i_l, l) \right),
\]

where \( m(k,i,l) = E\xi^k(i,l) \), \( k(l) \in \{0,1,\ldots,d\} \), so that \( \sum_i k(l) = d \),

\[k(l) = \text{card}\{i_l \ \text{in} \ b(i_1, i_1, \ldots, i_1; \ldots, i_1, \ldots, i_l, \ldots, i_d, i_d, \ldots, i_d)\},\]

and if some \( k(l) = 0 \), then by definition,

\[
\xi^{k(l)}(i_l, l) - m(k(l), i_l, l) = 1.
\]

We again suppose that \( b \in B; \) (the sequence \( b(I) \) may be not symmetric.) The multiply series for \( R_d \) in the case \( n = \infty \) converge with probability 1, for instance, if \( b \in B \) and the correspondence even moments are bounded:

\[
\sup_{\{i,l\}} m(2d, i, l) < \infty. \tag{1.7}
\]

For example if \( d = 2 \) then \( R_d \) has a view

\[
R_2 = \sum_{1 \leq i < j \leq n} b(i, j)\xi(i)\xi(j) + \sum_{i \leq n} b(i, i)(\xi^2(i) - E\xi^2(i)) +
\]

\[
\sum_{1 \leq i < j \leq n} c(i, j)\eta(i)\eta(j) + \sum_{i \leq n} c(i, i)(\eta^2(i) - E\eta^2(i)) +
\]

\[
\sum_{1 \leq i, j \leq n} a(i, j)\xi(i)\eta(j).
\]

If \( d = 3 \), then
\[ R_3 = \sum_{1 \leq i < j < k \leq n} \sum_{1 \leq i < k \leq n} b(i, j, k)\xi(i)\eta(j)\tau(k) + \]
\[ \sum_{1 \leq i < k \leq n} b(i, i, k)(\xi^2(i) - \mathbb{E}\xi^2(i))\tau(k) + \sum_{i} b(i, i, i)(\xi^3(i) - \mathbb{E}\xi^3(i)) + \ldots. \]

Here \( \{\xi(i), \eta(j), \tau(k)\} \) are independent sequences of independent random variables. Denote

\[ U(q, x) = U(q(1), q(2), \ldots, q(d); x) = \sup_{b \in B} \sup_{\{\xi(i, m) \in G(q(m))\}} T(R_d, x), \]

where \( \sup_{\{\xi(i, m) \in G(q(m))\}} \) is calculated over all totally independent random variables \( \{\xi(i, m)\} \) such that \( \exists q(m) > 0, K(m) > 0 \Rightarrow \exists m = 1, 2, \ldots, d \)

\[ \sup_{i} T(\xi(i, m), x) \leq \exp \left( -\left(\frac{x}{K(m)}\right)^{q(m)} \right), \quad x > 0, \]

or, equally

\[ \sup_i \|\xi(i, m)\|_{q(m)} \leq C K(m), \quad C = \text{const} < \infty. \]

Put \( K = \prod_{m=1}^{d} K(m) \), \( N(q) = 2q/(q + 2) \) by \( q \in (0, 1] \), \( N(q) = \min(q, 2) \) if \( q > 1 \), and define a family of function \( N(\bar{q}) = N_d(\bar{q}) = N_d(q(1), q(2), \ldots, q(d)) \) by the following recursion: \( N_1(q) = N(q) \), (initial condition),

\[ N_{d+1}^{(k)}(q(1), \ldots, q(d), q(d + 1)) = \left[ \frac{d - 1}{2} + \sum_{m=1,2,\ldots,d:m \neq k} \frac{1}{q(m)} + \frac{1}{N(q(k))} \right]^{-1}, \]

\[ N_{d+1}(q) = N_{d+1}(q(1), q(2), \ldots, q(d), q(d + 1)) = \max_{k=1,2,\ldots,d+1} N_{d+1}^{(k)}(q(1), q(2), \ldots, q(d + 1)). \]

**Theorem 2.** There exists a function \( C_3 = C_3(\bar{q}) \in (0, \infty) \) such that \( \forall x \geq 2 \)

\[ U(q(1), q(2), \ldots, q(d), x) \leq \exp \left( -C_3(\bar{q}) \left(\frac{x}{K}\right)^{N_d(q)} \right). \quad (1.8) \]

In the terms of \( G(q) \) spaces the proposition (1.8) may be rewritten as

\[ \|R_q\|_{N_d(q)} \leq C_4(d, \bar{q}) \prod_{m=1}^{d} \sup_{i \leq n} \|\xi(i, m)\|_{q(m)}. \]

For example assume that \( q(m) = q = \text{const} > 0 \) and denote

\[ \gamma(d, q) = 2q/[d(q + 2)], \quad q \in (0, 1], \]

\[ \gamma(d, q) = 2q/[2d + q(d - 1)], \quad q \in (1, 2], \]
\[ \gamma(d, q) = 2q/(dq + 2(d - 1)), \quad q \in (2, \infty]. \]

In this case \( N_d(q) = \gamma(d, q) \), and we receive the following result: if \( \xi(i, m) \) are totally independent, centered and

\[ T(\xi(i, m), x) \leq \exp(-x^q), \quad x \geq 0, \]

then by all \( x \geq x_0 \) and \( \text{dim}(q, q, \ldots, q) = d \Rightarrow \)

\[ U(q, q, \ldots, q, x) \leq \exp\left(-C_3 \cdot x^{\gamma(d, q)}\right). \quad (1.9) \]

**Theorem 3.** (Low bounds for \( U(q, q, \ldots, q, x) \)). There exists a function \( C_4 = C_4(d, q) \in (0, \infty) \) such that for all \( x \geq 1 \)

\[ U(q, q, \ldots, q, x) \geq \exp\left(-C_4 \cdot x^{\min(q, 2)/d}\right). \quad (1.10) \]

Obviously upper estimations (1.9) and low (1.10) ”almost” coincides: at \( q \to 0^+ \) or by \( q = \infty \) for all \( d \), and by \( d = 1, q \in (1, \infty] \). If for example \( \{\xi(i, s)\} \) are independent Rademacher series: \( P(\xi(i, s) = 1) = P(\xi(i, s) = -1) = 1/2 \) and \( \sum b^2(I) < \infty \), we deduce the well-known result ([28, p. 78], [31]) as a particular case:

\[ \exists \varepsilon > 0; \quad \mathbb{E}\exp\left(\varepsilon|Q_d|^{2/d}\right) < \infty. \]

Further we shall formulate and prove more generally results.

There are many publications about the limit theorem, moment and exponential inequalities for tail distributions of martingales and polynomials from independent variables. Semiinvariant inequalities for random polynomials are received in the book [15, p. 100 -103] in the case Gaussian limit distribution. The case \( d = 2 \) is considered in paper [30] and it is proved that in all rearrangement invariant space \( X \), for example \( X = G(q) \),

\[ \| \sum \sum_{1 \leq i \neq j \leq n} b(i, j)\xi(i,1)\xi(j,1)\|_X \approx \sqrt{\sum \sum b^2(i, j)}. \]

Consequently, the norm \( \sqrt{\sum \sum b^2(i, j)} \) is optimal. We can explain this from identity \( |\sum b(i)\xi(i)|^2 = D \sum b^2(i)\xi(i) = \sum b^2(i) \), where \( \{\xi(i)\} \) are sequence of centered martingale - differences with condition \( D\xi(i) = 1 \).

Non-uniform estimations \( T_Q \) are obtained in the works [1], [2], [3], [6], [7] etc. They are received in the terms of conditional expectations \( \mathbb{E}_g(\xi(i))/F(i - 1) \) of a view, for example:

\[ P(\exists n, \sum_{i=1}^n \xi(i) > x, \sum_{i=1}^n \mathbb{E}\xi^2(i)/F(i - 1) \leq y) \leq \exp(-x^2/(2(y + Cx))), \quad x, y > 0, \]
i.e. without sequences of coefficients $b(I)$ as in classical Bernstein - Bennett estimations. But these estimations are not convenient in the practice.

So-called "decoupling method" for calculation of order of magnitude expectation

$$\mathbb{E}f\left(\sum_{i,j=1}^{n} g_{i,j}(\xi(i),\eta(j))\right)$$

is described in the articles [2], [4] and in other publications, but only for a function $f$, belonging $\Delta_2$ class. See another publications in references.

The limit theorems for martingales are well-known [20, p. 58]. The limit theorems are received in [24] for symmetric polynomials, for example of a view:

$$Q_d = \sum_{1\leq i_1 < i_2 < \ldots < i_d \leq n} \prod_{s=1}^{d} \xi(i_s),$$

where $\{\xi(i)\}$ are i.i.d. random variables and it is proved that under some conditions at $n \to \infty$ in the sense of distribution convergence

$$Q_d(n) / \sqrt{DQ_d(n)} \overset{d}{\to} I(h),$$

where $I(h)$ is a multiply stochastic integral

$$I(h) = \int \int \ldots \int_{\mathbb{R}^d} h(\lambda_1, \lambda_2, \ldots, \lambda_d) \prod_{s=1}^{d} Z(d\lambda_s),$$

$Z(\cdot)$ - white Gaussian measure: $E Z(A) = 0$, $E Z(A) Z(B) = mes(A \cap B)$, $h \in L_2(\mathbb{R}^d), h \neq 0$.

Our estimations are formulated in the very simple terms only unconditional individual (marginal) tails of summand distributions, are very convenient for using, generalized in the multidimensional case $d > 1$, and are non-improved essentially (see, for example, theorem 1 (1.5) and inequalities (1.9), (1.10)).

2. Main results: exponential estimations. We shall introduce some notation used in following sections. Let $T(x)$ and $G(x), x > 0$ be two tail-functions, i.e. $T(0) = G(0) = 1$, monotonically decreasing right continuous and so that $T(\infty) = G(\infty) = 0$. We denote

$$T \vee G(x) = \min(4 \inf_{y>0} (T(y) + G(x/y)), 1).$$

The function $T \vee G(x)$ has the following sense: if $T(\xi, x) \leq T(x)$, $T(\eta, x) \leq G(x)$ then

$$T(\xi \cdot \eta, x) \leq T \vee G(x).$$

(2.1)

For example if $\forall \ x \geq 0 \ T(\xi, x) \leq Y_1 \exp\left(-\frac{(x/A)^{q(1)}}{A}\right)$, $T(\eta, x) \leq Y_2 \exp\left(-\frac{(x/B)^{q(2)}}{B}\right)$ for some $q(1), q(2), A, B = const > 0, Y_1, Y_2 = const \geq 1$, then $\forall x \geq 0 \Rightarrow

$$T(\xi \cdot \eta, x) \leq 8 \max(Y_1, Y_2) \exp\left(-\frac{(x/(AB))^{q(1)q(2)/(q(1)+q(2))}}{(AB)}\right).$$
More generally, if
\[ T(\xi, x) \leq \exp \left( -x^{q(1)}(\log(F(q(1), r(1)) + x))^{r(1)} \right), \]
\[ T(\eta, x) \leq \exp \left( -x^{q(2)}(\log(F(q(2), r(2)) + x))^{r(2)} \right), \]
then \( T(\xi \cdot \eta, x) \leq \min(1, 4 \exp(-C p(3) \log(F(q(3), r(3)) + x))^{r(3)}), \)
where
\[ q(3) = q(1)q(2)/(q(1) + q(2)), \quad r(3) = [q(1)r(2) + q(2)r(1)]/(q(1) + q(2)). \]

On the other hand, according to the language of \( G(q, r) \) spaces: for some \( C = \text{C(q(1), q(2), r(1), r(2))} \in (0, \infty) \Rightarrow \)
\[ \|\xi \cdot \eta\|_{q(3), r(3)} \leq C\|\xi\|_{q(1), r(1)} \cdot \|\eta\|_{q(2), r(2)/q(2)}. \]

If \( \xi, \eta \) are independent then the estimation (2.2) is exact.

Further, let us denote for the tail function \( T(\cdot) \) the following operator (non-linear)
\[ W[T](x) = \min \left( 1, \inf_{v>0} \left[ \exp(-x^2/(8v^2)) - \int_v^\infty x^2 \, dT(x) \right] \right), \]
if there exists the second moment \( |\int_0^\infty x^2 \, dT(x)| < \infty \).

**Lemma 1.** Let \( d = 1, \ \xi(i) = \xi(i, 1) \) be a sequence of martingale differences with filtration \( \{F(i)\} \) and let \( T(\xi(i), x) \leq T(x) \), where \( T(x) \) is some tail function. Then for all \( x \geq 2 \)
\[ \sup_{b \in B} T \left( \sum_i b(i)\xi(i), \ x \right) \leq W[T](x). \]  

**Proof.** Without loss of generality we can assume that \( i = 1, 2, \ldots, n; \) where \( n < \infty \). We shall use the so-called "truncation" method [10]. The following inequality for finite martingale differences is known, i.e. when \( \text{vrai} \max_{\omega} |\xi(i)| = c(i) < \infty \) (see [8], [25])
\[ T(\sum_i \xi(i), x) \leq \exp \left( -x^2/(2 \sum c^2(i)) \right). \]  

Note that in [25] the factor 2 is omitted in the denominator of the exponent index in the corresponding formulation.

Put \( X = \sum_i b(i)\xi(i), \ Y(1, i) = b(i)\xi(i)\chi(|\xi(i)| \leq v), \)
\[ Y(i) = Y(1, i) - \text{E}Y(i, i)/F(i - 1), \]
\[ Z(1, i) = b(i)\xi(i)\chi(|\xi(i)| > v), \ v = v(x) > 0, \]
\[ Z(i) = Z(1, i) = \text{E}Z(1, i)/F(i - 1), \]
\[\chi(A) = 1, \ \omega \in A, \ \chi(A) = 0, \ \omega \notin A, \ A \in F.\]

We can write \(X = X_1 + X_2\), where \(X_1 = \sum Y(i), X_2 = \sum Z(i)\), and we obtain according to the inequality (2.4) and since \(|Z(i)| \leq |b(i)|v\):

\[T(X_1, x) \leq \exp \left( -x^2/(2 \sum b^2(i)v^2) \right) = \exp \left( -x^2/2v^2 \right).\]

Further, if \(b(i) \neq 0\) then \(DZ^2(i)b^{-2}(i) = EZ^2(i)b^{-2}(i) \leq -\int_v^\infty y^2 dT(\xi(i), y) = v^2 T(\xi(i), v) + 2 \int_v^\infty yT(\xi(i), y) dy \leq v^2 T(v) + 2 \int_v^\infty yT(y) dy = -\int_v^\infty y^2 dT(y).\)

Since \(|Z(\cdot)|\) are not correlated, we see that

\[EX^2 \leq -\sum_i b^2(i) \int_v^\infty y^2 dT(y) = -\int_v^\infty y^2 dT(y).\]

By virtue of Chebyshev inequality

\[T(X_2, x) \leq \left| \int_v^\infty y^2 dT(y) \right| / x^2.\]

Now our statement it follows from the simple inequality that

\[P(X > x) \leq P(X_1 > x/2) + P(X_2 > x/2), \ x \geq 2,\]

after the minimization on \(v\). This completes the proof of lemma 1.

For example, assume \(T(\xi(i), x) \leq Y \exp(-(x/K)^q), \ K, x, q \in (0, \infty), \ Y = const \geq 1\). Denote (in time, in this section)

\[\delta = \delta(q) = (\min(q/2, 1))^{1/q}; \ q \in (0, 2] \Rightarrow \]

\[\beta(q) = \max \left( (1/q)\Gamma(2/q), \ (e/q)\left[2/(eq)\right]^{2/q} \right), \]

\[\beta(q) := \sup_{\nu \geq 0} \exp(v^q) \int_v^\infty x \exp(-x^q) \ dx \leq \Gamma(2/q)/(qe)\]

in the case \(q > 2\); here \(\Gamma(\cdot)\) is the Gamma - function. After some calculation we find for all values \(x \geq 0\):

\[\sup_{b \in B} T(Q_1, x) \leq W[T](x) \leq (1 + 2\beta(q) Y) \exp \left( -x/(K\delta)^{2q/(2+q)} \right).\]

Analogously assume that \(\sup_i ||\xi(i)||_{q,r} = K < \infty, q > 0, r \in (-\infty, \infty)\) or equally \(x \geq 2 \Rightarrow T(\xi(i), x) \leq \exp (-(x/K)^q \log(F(q, r) + x/K)^r)\).
Let us introduce the following vector - function $L(q, r) = \{L_1(q, r), L_2(q, r)\}$ of two variables $(q, r)$:

$$L_1(q, r) = \frac{2q}{q + 2}, \quad L_2(q, r) = \frac{2r}{q + 2}.$$  

We see that for $x > 0$: $\sup_{b \in B} T(\sum_i b(i) \xi(i), x) \leq \exp \left( -\frac{x}{(C_1 K)} L_1(\log(F(L_1, L_2)) + \frac{x}{(C_1 K)}) \right), \quad (2.5)$

where $L(i) = L(i; q, r), \ i = 1, 2$; or

$$\sup_{b \in B} ||\sum_i b(i) \xi(i)||_{L_1(q, r), L_2(q, r)} \leq C_2(q, r) \sup_i ||\xi(i)||_{q, r}.$$  

The proposition (2.5) is new even for independent variables $\{\xi(i)\}$ in the case $q \in (0, 1]$.

**Theorem 4. (Martingale case).** Let us denote $T_m(x) = \sup_i T(\xi(i, m), x)$ and assume that $\lim_{x \to \infty} T_m(x) = 0$, and define the sequence $T^{(s)}$ of tail functions in the following way:

$$T^{(1)}(x) = W[T_d](x),$$  

(initial condition), and for $s = 2, 3, \ldots, d - 1$ by recurrent equation

$$T^{(s+1)}(x) = W \left[ T_{s+1} \lor T^{(s)} \right](x).$$  

**Statement:**

$$\sup_{b \in B} T(Q_d, x) \leq T^{(d)}(x). \quad (2.6)$$

**Proof.** We shall prove the statement (2.6) by means of induction over $d$. The basis of induction $(d = 1)$ in (2.6) is proved in lemma 1. Further, the sequence $\{Q_d(n), F(n)\}$ is again martingale with correspondence martingale - differences

$$\zeta(n) = Q_d(n) - Q_d(n - 1) = \xi(n, d) \sum_{I \in J(d, n)} b(I) \xi(I) =$$

$$= \xi(n, d) \sum_{1 \leq i_1 < i_2 < \ldots < i_d-1 \leq n-1} b(i_1, i_2, \ldots, i_d-1, n) \prod_{m=1}^{d-1} \xi(i_m, m) =$$

$$= \left[ \xi(n, d) \times \sqrt{\sum_{I \in J(d, n)} b^2(I)} \right] \times \left[ \sum_{I \in J(d, n)} \frac{\sum_{J \in J(d, n)} \xi(J) b(I)}{\sqrt{\sum_{I \in J(d, n)} b^2(I)}} \right] \equiv$$

$$\equiv [\eta(n, d)] \times [\tau(n, d)].$$

We get in according of induction statement:

$$T(\tau(n, d), x) \leq T^{(d-1)}(x).$$
We deduce by virtue of the formula (2.1) for the product \( \eta(d, n) \times \tau(d, n) \)

\[
T(\eta(d, n) \cdot \tau(d, n), x) \leq (T_d \lor T^{(d-1)}) (x).
\]

Again using the statement of lemma 1 for a martingale differences \( \eta(d, n) \) \( \tau(d, n) \) we obtain the statement of theorem 4.

If for example

\[
T(\xi(i, m), x) \leq \exp \left( -(x/K(m))^{q(m)} \right),
\]

After some calculations we shall receive the statement of theorem 1.

Let us consider now "independent" case. Define

\[
\varphi_m(\lambda) = \sup \max_{i} \pm \log E \exp(\pm \lambda \xi(i, m)).
\]

This definition is non-trivial:

\[
\exists \lambda_0 \in (0, \infty], \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \varphi_m(\lambda) < \infty \tag{2.7}
\]

only if the sequence \( \{\xi(i, m)\} \) satisfies the uniform Kramer condition.

If the condition (2.7) is valid, we introduce the functions \( \chi_m(\lambda) \) by formula

\[
\chi_m(\lambda) = \sup \left\{ \sum_{j=1}^{\infty} \varphi_m(\lambda y(j)), \{y(j)\} : \sum_{j=1}^{\infty} y^2(j) \leq 1 \right\}.
\]

Note that if the function \( z \to \varphi'_m(z)/z \) is finite and monotonic on the right-hand half-line then

\[
\chi_m(\lambda) = \sup_n n \varphi_m(\lambda/\sqrt{n}). \tag{2.8}
\]

Namely:

\[
\chi_m(\lambda) = \sup_n \max_{\{y(j)\}} \sum_{j=1}^{n} \varphi_m(\lambda y(j)),
\]

where the internal maximum is taken over all finite sequences \( \{y(j)\} \) of length \( n \) so that \( \sum_{j=1}^{n} y^2(j) = 1 \); applying the Lagrange factor method for calculation we find that the maximum is attained, in particular, on non-negative sequences, some component of which equals zero, while the positive components are equal.

For instance if \( \forall |\lambda| \geq 1 \) \( \varphi_m(\lambda) = |\lambda|^q \) for some \( q = const > 1 \), then \( \chi_m(\lambda) \sim C|\lambda|^{\max(2,q)} \), \( |\lambda| \to \infty \).

**Lemma 2.** Let the variables \( \{\xi(i, m)\} \) be centered, independent, and assume that the condition (2.7) is satisfied, then

\[
\sup_{b \in B} T \left( \sum_i b(i) \xi(i, m), x \right) \leq \exp(-\chi^*_m(x)), \tag{2.9}
\]

where

\[
\chi^*_m(x) = \sup_{\lambda} (\lambda x - \chi(\lambda)) -
\]
is the so-called Young-Fenchel transform.

**Proof.** Taking into account the independence and assuming $\lambda > 0$ we obtain:

$$E \exp(\lambda \sum_{i=1}^{n} b(i)(i,m)) = \prod_{i=1}^{n} E \exp(\lambda b(i)(i,m)) \leq \exp \left( \sum_{i} \varphi_{m}(\lambda b(i)) \right) \leq \exp \chi_{m}(\lambda)$$

by virtue of definition of the function $\chi_{m}(\cdot)$. The proposition of lemma 2 it follows now from Chebyshev inequality. See in detail [11, p.24.]

**Corollary 1.** We obtain syntesing propositions of lemma 2 and lemma 1, that in our assumptions

$$\sup_{b \in B} T(\sum_{i} b(i) \xi(i,m), x) \leq \min\{W[T_{m}](x), \exp (-\chi_{m}^{*}(x))\} \overset{\text{def}}{=} W[T_{m}](x).$$

For instance suppose that $T(\xi(i,1), x) \leq \exp (-x^{q})$, $q > 0$, and recall that the function $N(q)$ at $q \in (0,1]$ is equal to $2q/(2 + q)$ and at $q \in (1, \infty]$ $\Rightarrow N(q) = \min(q,2)$. Then we get at $x \geq 1$

$$\sup_{b \in B} T(\sum_{i} b(i) \xi(i,1), x) \leq \exp \left( -C_{7}(q)x^{N(q)} \right).$$

It is proved in [11, p. 50] that the exponent $\min(q,2)$ in the case $q > 1$ is non-improvable.

More generally assume that $\exists q > 0, r \in (-\infty, \infty) \Rightarrow \forall x > 0$

$$T(\xi(i,1), x) \leq \exp (-C r^{q}(\log(F(q,r) + x))^{r}).$$

Introduce the following vector-function $N(q,r) = (N(1; q,r), N(2; q,r))$:

a) at $q \in (0,1)$ or $q = 1, r < 0$ $\Rightarrow$

$$\Rightarrow N(1; q,r) = 2q/(q + 2), \quad N(2; q,r) = 2r/(q + 2);$$

b) at $q = 1, r \geq 0$ or $q \in (1,2)$, or $q = 2$, $r < 0$ $\Rightarrow$

$$\Rightarrow N(1; q,r) = q, \quad N(2; q,r) = r;$$

c) at $q = 2, r \geq 0$ or $q > 2$ $\Rightarrow$

$$\Rightarrow N(1; q,r) = 2, \quad N(2; q,r) = 0.$$

**Proposition:** $\sup_{b \in B} T(\sum_{i} b(i) \xi(i,1), x) \leq$

$$\exp \left( -Cs(q,r)x^{N(1; q,r)} (\log(F(1; q,r), N(2; q,r)) + x) \right)^{N(2; q,r)}.$$

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**Theorem 5.** Assume in addition that the r.v. \( \{\xi(i,m)\} \) are independent. Let us define the sequence of tail functions by the following initial condition and recursion:

\[
T^{(1)}(x) = W[T_d](x),
\]

\[
T^{(m+1)}(x) = W \left[ T_{m+1} \lor T^{(m)} \right](x), \quad m = 1, 2, \ldots, d - 1.
\]

**Proposition:**

\[
\sup_{b \in B} T(Q_d, x) \leq T^{(d)}(x). \tag{2.10}
\]

**Proof.** For the sum

\[
Q(d, n) = \sum_{I \in I(d, n)} b(I) \prod_{m=1}^{d} \xi(i_m, m),
\]

i.e. without diagonal members, in the case \( d = 1 \) our result is the content of lemma 2. The recursion is proved likewise the proof of theorem 4.

**Proof** of theorem 2. We must prove that the non-diagonal members in expansion for \( R_d \) have "little" tails:

\[
\sup_{b \in B} T(R_d - Q_d, x) \leq \exp \left( -C(q)x^{U+\varepsilon} \right), \quad x \geq 1, \tag{2.11}
\]

for some \( \varepsilon = \text{const} \geq 0 \) as long as \( T(\xi + \eta, x) \leq T(\xi, x/2) + T(\eta, x/2) \). The unique non-trivial case is if the family \( \{\xi(ii,m)\}, m \leq d - 2 \) does not depend on the \( \{\xi(k, d-1)\} \) and \( \xi(i, d-1) = \xi(i, d) \). Let us denote

\[
A = \sum_{I \in J(d, n)} b(I)\xi(i_1, 1)\xi(i_2, 2) \ldots, \xi(i_{d-2}, d-2)[\xi^2(i_d, d) - m(2, i_d, d)].
\]

We receive:

\[
T(\xi^2(i, d) - m(2, i, d), x) \leq \exp \left( -C(d)x^{q/2} \right), \quad x \geq 1.
\]

Thus

\[
T(A, x) \leq \exp \left( -C x^{N(q(1), q(2), \ldots q(d-2), q/2)} \right), \quad q = q(d).
\]

As long as

\[
1/N(q(1), q(2), \ldots, q(d-2), q/2) = (d-2)/2 + \sum_{j=1}^{d-2} [1/q(j)] + 1/L(q/2),
\]

\[
1/N(q(1), q(2), \ldots, q(d-2), q, q) = (d-1)/2 + \sum_{j=1}^{d-2} [1/q(j)] + 1/q + 1/L(q),
\]

It is enough to prove the following inequality: \( \forall q > 0 \Rightarrow \)

\[
1/2 + 1/q + 1/L(q) - 1/L(q/2) \geq 0. \tag{2.12}
\]
After the consideration of all cases \( q \in (0, 1], q \in (1, 2], q \in (2, 4] \) and \( q > 4 \) we can see that the inequality (2.12) is valid.

*Low bounds for tails in the martingale cases (theorem 1).* Assume \( d = 2; \) the general case provides analogously.

Let us introduce two martingale - differences:

\[
\xi(i) = \tau \nu(i), \quad \eta(j) = \rho \theta(j),
\]

where all variables \( \tau, \nu(i), \rho, \theta(j) \) are independent and \( \nu(i), \theta(j) \) are Rademacher series:

\[
P(\theta(j) = 1) = P(\theta(j) = -1) = P(\nu(i) = 1) = P(\nu(i) = -1) = \frac{1}{2}
\]

and \( \tau \geq 0, \rho \geq 0, \)

\[
T(\tau, x) = \exp(-x^{q(1)}), \quad T(\rho, x) = \exp(-x^{q(2)}),
\]

and let us introduce at \( n \geq 2 \) the probability

\[
P_n(x) = P\left(\sum_{1 \leq i < j \leq n} \tau \rho \nu(i) \theta(j)/\sqrt{n(n-1)} > x\right), \quad x > 1.
\]

We see that for \( y, z \geq 1, \) \( x/(yz) > 1 :\)

\[
P_n(x) \geq P\left(\sum \nu(i) \theta(j)/\sqrt{n(n-1)} > x/(yz)\right) \exp\left(-y^{q(1)}\right) \exp\left(-z^{q(2)}\right).
\]

As long as (see [24])

\[
\lim_{n \to \infty} P\left(\sum_{1 \leq i < j \leq n} \nu(i) \theta(j)/\sqrt{n(n-1)} > x/(yz)\right) = P(I(g) > x/(yz)),
\]

where \( I(g) \) is a two - dimensional stochastic integral (1.11) with some non - trivial function \( g(\cdot) \in L_2(R^2) \). Since

\[
P(I(g) > x/(yz)) \geq \exp\left(-C(g)x/(yz)\right),
\]

we obtain:

\[
\lim_{n \to \infty} P_n(x) \geq \exp\left(-C(g)x/(yz) - y^{q(1)} - z^{q(2)}\right).
\]

Since

\[
\sup_n P_n(x) \geq \lim_{n \to \infty} P_n(x),
\]

after the maximization the right - side of inequality (2.14) over \( y, z \), we deduce the left inequality in the theorem 1.
In order to prove the low bounds for tails distributions in the independent case (theorem 3), we must consider two examples. Taking \( Q(d,n) = \prod_{i=1}^{d} \xi(i) \), where \( \xi(i) \) are independent, symmetrical and such that
\[
T(\xi(i), x) = \exp(-x^q), \ x \geq 0,
\]
we deduce after some calculations:
\[
T(Q(d,n), x) = \mathbb{P}(\prod_{i=1}^{d} \xi(i) > x) \geq \exp\left(-C(d,q)x^{q/d}\right).
\]
The sequence \( Q(n,d) \) converge in distribution at \( n \to \infty \) by appropriate choice of sequences of coefficients \( b(I) = b(I; n) \) to the \( d \)−multiple stochastic integral over the Gaussian orthogonal stochastic measure \( Z(\cdot) \) (1.11):
\[
Q(d,n)/\sqrt{D} Q(d,n) \to I(h) = \int_{\mathbb{R}^d} h(y)Z(dy),
\]
with tail behavior \( T(I(h), x) \geq \exp\left(-Cx^{2/d}\right) \). Consequently,
\[
\sup_{b \in B} \sup_{\{\xi(i,m)\}_{|\xi(i,m)|\leq |q(m)|}} T(Q_d, x) \geq \exp\left(-Cx^{\min(2,q)/d}\right).
\]

3. Moment estimations. We shall derive here the moment estimations for \( Q_d \) in martingale and independent cases in the terms of unconditional moments, more exactly, \( L_p \) norms:
\[
|\eta|_p = \mathbb{E}^{1/p}|\eta|^p
\]
of summands:
\[
\mu_m(p) \overset{d\!ef}{=} \sup_i \mathbb{E}^{1/p}|\xi(i,m)|^p = \sup_i |\xi(i,m)|_p, \ p \geq 2.
\]
We define the sequence \( \gamma(d), d = 1,2,\ldots \) by the following initial condition: \( \gamma(1) = \sqrt{2} \) and recurrent equation:
\[
\gamma(d+1) = \sqrt{2}(1 + 1/d)^d\gamma(d).
\]
In particular, \( \gamma(2) = 4, \gamma(3) = 9\sqrt{2} \). Since \( (1 + 1/d)^d < e \), we conclude: \( \forall d \geq 3 \Rightarrow \gamma(d) \leq 9 \exp((d-3)+(\log 2)(d-2)/2) \).

**Theorem 6.** If \( \forall m = 1,2,\ldots,d \ \mu_m(d \cdot p) < \infty \), then
\[
\sup_{b \in B} |Q_d|^p \leq \gamma(d) p^d \prod_{n=1}^{d} \mu_m(d \cdot p). \tag{3.1}
\]

**Proof.** We start our consideration from the case \( d = 1 \). The beginning is like one in [10], but further instead of convexity we intend to employ Hölder inequality.

Note that it can be assumed that \( p > 2 \) (the case \( p = 2 \) is trivial) and \( \forall i \leq n \Rightarrow b(i) \neq 0 \). Further, the sequence \( b(i) \xi(i), \xi(i) \overset{d\!ef}{=} \xi(i,1) \) is also the sequence
of martingale differences relative to the initial filtration. Using the Burkholder inequality ([20], p. 78 - 81; [31]) we obtain
\[ |\sum b(i) \xi(i)|^p_p \leq (p\sqrt{2})^p \mathbb{E} \left[ \sum b^2(i) \xi^2(i) \right]^{p/2}. \] (3.2)

Let \( a(i) \) be some positive non-random sequence, the choice of which we shall clarify below. By virtue of Hölder inequality, in which we substitute
\[ \beta = p/2, \quad \alpha = \beta/(\beta - 1) = p/(p - 2), \]
we see get
\[ \sum b^2(i)\xi^2(i) = \sum a(i)b^2(i) \cdot [\xi^2(i)/a(i)] \leq \left[ \sum (b^2(i)a(i))^\alpha \right]^{1/\alpha} \cdot \left[ \sum (\xi^2(i)/a(i))^\beta \right]^{1/\beta}. \]

Let us choose \( a(i) \) by the formula
\[ a(i) = |b(i)|^{-4/p}, \]
then we receive
\[ \left[ \sum (b^2(i)a(i))^\alpha \right]^{p/(2\alpha)} = \left( \sqrt{\sum b^2(i)} \right)^{p-2}; \]
\[ \mathbb{E} \left[ \sum (\xi^2(i)/a(i))^\beta \right]^{p/2\beta} \leq \mathbb{E} \sum |\xi(i)|^p b^2(i) \leq \mu_1(p) \sum b^2(i). \]

Substituting the last inequality into (3.2) we obtain the proposition of theorem 6.

**Corollary 2.** Let us denote
\[ K_M(1, p) = \sup_{b \in B} |\sum b(i)\xi(i)|_p/\mu(p), \]
where the upper bound is calculated over all sequences of centered martingale differences \( \{\xi(i)\} \) with finite absolute moments of order \( p \). It follows from [3.1] that \( K_M(1, p) \leq p\sqrt{2} \). But it is proved in [13], [22] etc. that if \( \{\xi(i)\} \) are independent symmetrical identically distributed, then the fraction in the right-hand part can has an estimate from below of the form \( 0.87 p/ \log p \). Thus
\[ 0.87 p/ \log p \leq K_M(1, p) \leq \sqrt{2} p. \]

Therefore our estimation can’t be essentially improved.

**Remark 1.** It the other terms of conditional expectations \( \mathbb{E}\xi^2(i, m)/F(i - 1), \mathbb{E} \max_{i \leq n} |\xi(i, m)|^p \) the upper bound for the constant which is like to our \( K_M(1, p) \) is obtained in [5] then growth by \( p \to \infty \) like \( p/ \log p \). Our result can not be obtained from [5].

The final estimate for \( K_M(1, p) \), i.e. the exact grows for \( K_M(1, p) \) by \( p \to \infty \) is now unknown.
We shall prove the general case $d > 1$ by induction over $d$. We get for the corresponding martingale differences $\zeta(n) = \tau(n,d)\eta(n,d)$ of a martingale $(Q(n), F(n))$ on the basis H"older inequality and induction statement:

$$|\zeta(n)|_p \leq |\eta(n,d)|_{pd} |\tau(n,d)|_{pd/(d-1)} = \mu_d(pd) \cdot |Q(d-1)|_{pd/(d-1)} \leq \mu_d(pd) \cdot (pd/(d-1))^{d-1} \gamma(d-1) \sqrt{\sum_{I \in J(d)} b^2(I) \prod_{m=1}^d \mu_m(p(d-1)d/(d-1))} = \gamma(d-1)[pd/(d-1)]^{d-1} \prod_{m=1}^d \mu_m(pd) \sqrt{\sum_{I \in J(d)} b^2(I)}.$$

We receive from the one-dimensional case:

$$|Q(d,n)|_p \leq p\sqrt{2} \gamma(d-1) [pd/(d-1)]^{d-1} \prod_{m=1}^d \mu_m(pd) \times \sqrt{\sum_{I \in I(d,n)} b^2(I)} = \gamma(d) p^d \prod_{m=1}^d \mu_m(pd).$$

(Recall that $b \in B \iff \sum_{I \in I(d,n)} b^2(I) = 1$.)

**Corollary 3.** Let us denote

$$K_M(d,p) = \sup_n \sup_{\{\xi(i,m)\} : |\xi(i,m)|_{d,p} < \infty} |Q_d|_p / \prod_{m=1}^d \mu_m(d \cdot p),$$

where sup is calculated over all families of sequences of centered martingale differences $\{\xi(i,m)\}$ with condition $|\xi(i,m)|_{d,p} < \infty$. It follows from theorem 6 that $K_M(d,p) \leq \gamma(d) p^d$. We shall prove now the low bounds for $K_M(d,p)$: at $d \geq 2$

$$K_M(d,p) \geq C^d p^{d/2},$$

here $C$ is an absolute constant. Let us choose the family of independent Rademacher series:

$$P(\xi(i,m) = \pm 1) = 1/2,$$

then $\mu_m(d \cdot p) = 1$. Introduce at $n \geq d$ the random variables

$$\overline{Q}_d(n) = \sum_{I \subset I(d,n)} \prod_{m=1}^d \xi(i_m,m)/[n(n-1) \ldots (n-d+1)].$$

We conclude relaying on the main result of the article [24]: at $n \to \infty \Rightarrow \overline{Q}_d \overset{d}{\to} I(h)$ in the sense of distribution convergence, where $I(h)$ is the multiply stochastic integral (1.11) with tail behavior

$$T(I(h), x) \geq \exp \left(-C_1 x^{2/d}\right), \ x \geq C_2.$$
It is easy to verify that the moment convergence in [24] is true. Hence

\[ K_M(d, p) \geq \lim_{n \to \infty} |Q_d|_p \geq C_3^d p^{d/2}. \]

Now we shall consider again ”independent case”, under condition: for some \( p \geq 2 \)

\[ \mu_m(p) \overset{\text{def}}{=} \sup_{i \leq n} |\xi(i, m)|_p < \infty, \ m = 1, 2, \ldots, d. \]

**Theorem 7.** If all the variables \( \{\xi(i, m)\} \) are independent, centered and so that for some \( p \geq 2 \) \( \Rightarrow \mu_m(p) < \infty \ \forall m = 1, 2, \ldots, d \), then

\[ \sup_{b \in B} |Q_d|_p \leq 2^{d/2} p^d \left( \prod_{m=1}^d \mu_m(p) \right) / \log p. \quad (3.3) \]

**Proof** is the same as in the theorem 6. In the case \( d = 1 \) the proposition (3.3) is provided in [13]; we can rewrite this result on the form

\[ |\sum b(i)\xi(i, 1)|_p \leq \sqrt{\frac{2}{\mu_1(p)}} \frac{\log p}{d} \]

Since the variables \( \{\xi(i, m)\} \) are independent we see that for \( \zeta(n) = \eta(n, d) \tau(n, d) \) the following inequality holds:

\[ |\zeta(n)|_p = |\eta(n, d)|_p |\tau(n, d)|_p \leq \mu_d(p)|Q(d-1, n)|_p. \]

It follows from induction statement that:

\[ |Q(d, n)|_p \leq 2^{d/2} p \mu_d(p) p^{d-1} \prod_{m=1}^{d-1} \sum_{I \in d(n)} b^2(I) / \log p = \]

\[ 2^{d/2} p^d \prod_{m=1}^d \mu_m(p) \sqrt{\sum_{I \in d(n)} b^2(I) / \log p}. \]

Note that our result (3.3) improves the same estimations in article [35].

**Corollary 4.** Let us denote in the independent case

\[ K_I(p, d) = \sup_n \sup_{b \in B} \sup_{\{\xi(i, s)\} : |\xi(i, m)|_p < \infty} |Q(d, n)|_p / \prod_{m=1}^d \mu_m(p). \]

Proposition: for some absolute constant \( C_4 \in (0, \infty) \)

\[ C_4^d p^d \log^{-d} p \leq K_I(p, d) \leq 2^{d/2} p^d / \log p. \]

**Proof** the low bounds: the moment estimations are derived in [21], [29] for symmetrical polynomials on independent identical symmetrically distributed variables:

\[ |Q(d, n)|_p / \left( \sqrt{\mathbb{D}Q(d, n)} \mu^d(p) \right) \geq C_4^d p^d / (\log^d p). \]
Our hypothesis: \( p \to \infty \Rightarrow K_t(p, d) \simeq C(d) \frac{p^d}{(\log p)^d} \).

**Corollary 5.** We want make the comparison between exponential and moment estimations in the independent case. Let us introduce, more exactly, for any Orlicz space of random variables \( W \) with norm \( ||\eta||_W \) on the our probability space the uniform tail of distribution

\[
P_W(x) = \sup_{||\xi(i,1)||_W \leq 1} \sup_{b \in B} T(Q_1, x).
\]

There are two methods of estimation of \( P_W(x) \): ”exponential estimations” and ”moment estimations”. Namely, if a norm \( || \cdot ||_W \) in a \( W \) space is equivalent to the norm of a view

\[
|||\eta|||_W = \sup_{p \geq 2} |\eta|_p/\psi(p),
\]

where \( \psi(\cdot) \) is a monotonically increasing continuous positive function, \( \psi(\infty) = \infty \), for example norms in the spaces \( L_p, G(q), G(q, r), \Psi(\beta) \), then by means of theorem 7 we can estimate all moments of \( Q_1 \) and further by means of Chebyshev inequality the tail of distribution \( Q(d, n) \). For the spaces \( L_p, p \geq 2 \) the moment method gives better result; in the case \( W = \Psi(\beta) \) both methods give the same result:

\[
P_W(x) \leq \exp \left( -C(\beta)(\log(1 + x))^{1+1/\beta} \right);
\]

for the spaces \( G(q, r) \) the the exponential method is better.

Let us consider the following example. Assume that for some \( r = \text{const}, q = \text{const} > 0 \)

\[
\sup_{i, m} T(\xi(i, m), x) \leq \exp \left( -x^q(\log(F(q, r) + x)^r) \right), \ x > 0.
\]

Denote \( V(q, d) = 2q/(d(q + 2)) \) and define the sequence \( r(d) \) by the recurrent equation:

\[
r(d + 1) = [r(d)q + V(d, q)r]/[V(d, q)q + 2V(d, q) + 2q]
\]

with initial condition \( r(1) = 2r/(q + 2) \). Then in the martingale case

\[
\sup_{b \in B} T(Q_d, x) \leq \exp \left( -C(d, q)x^{V(d, q)}(\log(F(V(d, q), r(d)) + x)^{r(d)} \right).
\]

If \( \sup_i, \sup_m T(\xi(i, m) \leq \exp \left( -\log(1 + x) \right)^{1+1/\beta}), \ \beta > 0, \) then

\[
\sup_{b \in B} T(Q_d, x) \leq \exp \left( -C_1(d, \beta)(\log(1 + x))^{1+1/\beta} \right).
\]

**Remark 2.** We can derive similarly using Doob’s inequalities the moment and exponential estimates for the distribution of the variables

\[
\max_{k=1,2,...,n} Q(d, k), \ \max_{k=1,2,...,n} |Q(d, k)|,
\]
Remark 3. It is easy to receive the generalization of our inequalities in the so-called martingale fields $\xi(i,m) = \xi(i,m)$; see definitions and some preliminary results in [10].

4. Application to the theory of $U$ - statistics. In this section we shall apply our estimations (2.6), (3.3) etc. in the theory of $U$ - statistics. For more detail about using of martingale technique based on the Hoeffding martingale representation for $U$ - statistic see in [37], [16]. Our results improve also somewhat the estimations in [16], [21] etc.

Let $\{\xi(i)\}$, $i = 1, 2, \ldots, n$ be independent identically distributed random variables with values in the fixed measurable space $\{X, S\}$, $\Phi(x_1, x_2, x_3, \ldots, x_d)$, $d < n$ be a symmetrical measurable non-trivial numerical function (kernel) of $d$ variables: $\Phi : X^d \to R^1$,

$$U(n) = U(n; \Phi, d) = \sum_{i \in I(d,n)} \Phi(\xi(i(1)), \xi(i(2)), \ldots, \xi(i(d)))\left(\begin{array}{c}n \\ d \end{array}\right)$$

be a so-called $U$ - statistic. Denote $Dim \Phi = d$,

$$\Phi = \Phi(\xi(1), \xi(2), \ldots, \xi(d)), \ r = rank U \in [1, 2, \ldots, d - 1];$$

$$T(\{\Phi, d\}, x) \overset{def}{=} \sup_{n > d} T((U(n) / \sqrt{DU(n)}), x).$$

Assume that $E\Phi = 0$, $D\Phi \in (0, \infty)$ and all the moments $\Phi$ which is written below there exist; otherwise the results are trivial. Recall here for readers convenience the so-called martingale representation for the centered $U$ - statistic (see [16], p. 26; [37]):

$$U(n) = \sum_{k=r}^{d} \left(\begin{array}{c}k \\ d \end{array}\right) U(n; k), \ U(n; k) = \sum_{i \in I(n,k)} g_k(\check{\xi}(I))\left(\begin{array}{c}n \\ k \end{array}\right),$$

where $\check{\xi}(I) = \{\xi(i(1)), \xi(i(2)), \ldots, \xi(i(d))\}$, $\mu(A) = P(\xi(i) \in A)$, $\int_X f(y)\delta_x(dy) = f(x)$, $g_k(x_1, x_2, \ldots, x_d) = \Phi_k = g_k[\Phi](x_1, x_2, \ldots, x_d) = \int_{X^d} \Phi(y_1, y_2, \ldots, y_d) \prod_{l=1}^{k} (\delta_{x_l}(dy_l) - \mu(dy_l)) \prod_{l=k+1}^{d} \mu(dy_l)$.

It is well known ([16], [37]) that the sequence $S(k) = S^{(n)}(k) = C(k, n) U(n, k)$, $k \leq n$ relatively some filtration $F(k) = F^{(n)}(k)$ is a martingale:

$$ES(l)/F(k) = S(k), \ k \in [1, l],$$

and that by $n \to \infty \Rightarrow DU(n) \asymp n^{-r}$.

Theorem 8.

$$|U(n)/\sqrt{DU(n)}|_p \leq C^d \ p^d \ |\Phi|_p / \log p. \quad (4.1)$$
Proof. The case \(d = 1\) was consider in the section 3; now we shall use the method of induction over \(d\); for simplicity we shall investigate only the case \(d = 2\).

Assume for beginning that the (non-trivial) kernel \(\Phi\) is degenerate: 

\[
\text{D}\Phi(\xi(1), \xi(2))/\xi(1) = 0 \pmod{p}, \quad (\Leftrightarrow r \geq 2).
\]

The sequence 

\[
\binom{n}{2} U(n) = \sum_{1 \leq i < j \leq n} \Phi(\xi(i), \xi(j))
\]

is a martingale relative to some \(\sigma\)-flow (filtration) with correspondence martingale-differences

\[
\zeta(n) = \sum_{i=1}^{n-1} \Phi(\xi(i), \xi(n)),
\]

Fixing the value of \(\xi(n)\), and using the induction statement, and denoting \(\mu(A) = P(\xi(i) \in A)\), at \(p, n \geq 2\), we obtain:

\[
|\zeta(n)|^p_p = E|\zeta(n)|^p = \int_X \mu(dx) E\left| \sum_{i=1}^{n-1} \Phi(\xi(i), x) \right|^p \leq \int_X \mu(dx) (n-1)^{p/2}(p\sqrt{2}/\log p)^p E|\Phi(\xi(1), x)|^p = (n-1)^{p/2}(p\sqrt{2}/\log p)^p |\Phi|^p_p.
\]

Hence 

\[
|\zeta(n)|_p \leq \sqrt{2n-2}\left(\frac{p}{\log p}\right)|\Phi|^p_p.
\]

We can prove our proposition in the case degenerate kernel substituting into the inequality (1.6) the least estimation.

Now we shall consider the general (and non-trivial) case \(r = 1\). We denote 

\[
E\Phi(\xi(1), \xi(2))/(\xi(2) = x) = g(x),
\]

\[
\Phi^0(x, y) = \Phi(x, y) - g(x) - g(y) = g_1[\Phi](x, y).
\]

Then \(Eg(\xi(i)) = 0\) and \(\Phi^0\) is a degenerate kernel. It follows from Jensen inequality for conditional expectation that 

\[
|g(\xi(i))|_p \leq |\Phi|^p_p, \quad \text{and, hence} \quad |\Phi^0|^p_p \leq C(d)|\Phi|^p_p.
\]

Let us write the Hoeffding decomposition for \(U\) - statistic: 

\[
\sqrt{n} U(n) = \frac{2}{\sqrt{n} (n-1)} \sum_{1 \leq i < j \leq n} \Phi^0(\xi(i), \xi(j)) + [4\sqrt{n}/(n-1)] \left( \sum_{i=1}^{n-1} g(\xi(i)) \right) - \sqrt{n}/(n-1).
\]

Using triangle inequality and our proposition for degenerate statistics, we obtain:

\[
|\sqrt{n} U(n)|_p \leq C_1 n^{-1/2} p^2 |\Phi|^p_p/\log p + C_2 p |g(\xi(i))|_p/\log p \leq 21
\]
\[ C_3 p^2 \frac{|\Phi|_p}{\log p}; \]

here \( C_{1,2,3} = C_{1,2,3}(d). \)

We shall derive now the exponential bounds for tail of distribution \( T(\Phi, d, x) \).

**Theorem 9.** Assume that for some \( K \in (0, \infty), q > 0, r \in \mathbb{R}^1 \Rightarrow \)

\[ T(\Phi, x) \leq \exp \left( -\left( x/K \right)^q \left( \log(1 + x/K) \right)^{-qr} \right), \quad x \geq 0. \quad (4.2) \]

Then \( T(\{\Phi, d\}, x) \leq \)

\[ \exp \left( -C(d, q, r)\frac{(x/K)^{q/(q+1)}(\log(1 + x/K))^{-(r-1)/q}}{q+1} \right). \quad (4.3) \]

**Proof.** We can assume \( K = 1 \). It follows from condition (4.2) that:

\[ |\Phi|_p \leq C_1 p^{1/q} \log^r p, \quad p \geq 2. \]

We conclude using theorem 8:

\[ |U(n)/\sqrt{\text{D}(n)}|_p \leq C_2 p^{d+1/q} \log^{r-1} p. \]

We deduce (4.3) returning to the tail of probability.

Now we shall receive the refined but more cumbersome exponential bounds for tail distribution \( T(\{\Phi, d\}, x) \). We shall deduce the recurrent equation (on the dimension \( d = \text{Dim} \Phi \)) for one in the spirit of the section 2. Let us denote

\[ t(d, k, r) = 1/\left( (d-r+1) \binom{d}{k} \right) \]

and for almost all (mod \( \mu \)) values \( z \in X : g_{k,(z)}[\Phi_k] = \)

\[ \Phi_{k,(z)} = \Phi_{k,(z)}(x_1, x_2, \ldots, x_{d-1}) \overset{\text{def}}{=} \Phi_k(x_1, x_2, \ldots, x_{d-1}, z). \]

Note that

\[ T(\{\Phi, d\}, x) \leq \sum_{k=r}^{d} T(\{g_k[\Phi], d\}, t(d, k, r) \cdot x). \quad (4.4) \]

Consequently, it is sufficient to estimate this distribution only for degenerate kernels \( \Phi_k(\cdot) \), i.e. if \( r = \text{rank} U \geq 2 \), as long as the arbitrary kernel \( \Phi(\cdot) \) may be represented as a linear combination of degenerate kernels \( \Phi_k ([16], \text{p. 26}). \)

**Theorem 10.** The tails \( T(\{\Phi_k, d\}, x) = T(\{g_k[\Phi], d\}, x) \) may be estimated as

\[ T(\{g_k[\Phi], d\}, x) \leq L(\{g_k[\Phi], d\}, x), \quad (4.5) \]

where the functions \( L(\{g_k[\Phi], d\}, x) \) satisfy the following system of a recurrent equations:

\[ L(\{g_k[\Phi], d\}, x) = W \left[ \int_X \mu(dz) L(\{g_k,(z)[\Phi], d-1\}, x) \right], \quad d \geq 2, \quad (4.6) \]
with initial condition
\[ L(\{\Phi, 1\}, x) = \mathcal{W}[T(\Phi)](x). \] (4.7)

**Proof** (briefly; in the case \( d = 2 \)). Define the sequence of functions \( L(\{g_k[\Phi], d\}, x) \) by means of the equations (4.6) and (4.7). Then the estimation (4.5) by \( d = 1 \) it follows from the definition of operator that \( \mathcal{W}(\cdot) \), as long as the variables \( \{\Phi(\xi(i))\} \) are independent.

Further, since the kernel \( g_k[\Phi] \) is degenerate, we deduce that the sequence \( \Delta(n) = \sum_{I \in J(d,n)} \Phi_k(\vec{\xi}(I), z) \) for almost all values \( z, z \in X \) with respect to some filtration is a martingale with correspondence martingale - differences \( \zeta(n) \).

Using the proof of theorem 4 and omitting some calculations we obtain the inequality (4.5).

5. Applications to the stochastic integration. A). Let \( (Z(t), F(t)), t \geq 0 \) be a left - continuos in the \( L_2(\Omega, \mathcal{F}) \) sense centered square integrable martingale relatively the flow of \( \sigma - \) fields (filtration) \( F(t) : F(0 + 0) = F(0) = \{\emptyset, \Omega\}, F(t - 0) = F(t), t > 0, Z(0) = 0 \). Denote for \( 0 \leq a < c \leq \infty \) \( \nu([a,c)) = D(Z(c) - Z(a)) \); then \( \nu(\cdot) \) may be continued to the measure (may be unbounded) on the Borel subsets on half - line \([0, \infty)\).

Let \( b(t), t \geq 0 \) be a non - random measurable function belonging to the space \( L_2(R_1^+, \nu) : ||b(\cdot)||^2(L_2(\nu)) \overset{def}{=} \int_{[0,\infty)} b^2(t) \nu(dt) < \infty \).

We define for some \( q > 0 \)
\[ ||Z||(Lip(q), \nu) = \sup_{\nu([a,c)) \in (0,\infty)} ||Z(c) - Z(a)||_q / \{\sqrt{\nu(c) - \nu(a)}\}, \]
and consider a stochastic integral
\[ [b; Z] \overset{def}{=} \int_{[0,\infty)} b(t)dZ(t). \]

**Theorem 11.** If for some \( q > 0 \) \( ||Z||(Lip(q), \nu) < \infty \), then for some \( C(q) \in (0, \infty) \)
\[ ||b; Z||_{M(1,q)} \leq C(q)||b||(L_2(\nu)) \cdot ||Z||(Lip(q), \nu). \] (5.1)

**Proof.** It is enough to prove our proposition (5.1) for a simple functions \( b(\cdot) \), i.e of a view \( b(t) = \)
\[ = \sum_{k=0}^{K} b(t(k))\chi(t \in [t(k), t(k + 1))], \quad 0 \leq t(0) < t(1) < \ldots < t(K + 1) < \infty. \]
Here \( \{t(k)\} \) is any non-random sequence such that \( \nu(t(k+1)) - \nu(t(k)) > 0 \). For those functions we can write: \( [b; Z] = \)
\[
\sum_{k=0}^{K} b(t(k)) \{ Z(t(k+1)) - Z(t(k)) \} = \sum_{k=0}^{K} b(t(k)) \sqrt{\nu([t(k), t(k+1)])}
\]
\[
\{ Z(t(k+1)) - Z(t(k)) \}/\{ \sqrt{\nu([t(k), t(k+1)])} \}.
\] (5.2)

We deduce by virtue of theorem 1, using the proposition of theorem 1 for the sequence \( b(i) := b(t(i)) \sqrt{\nu(t(i+1)) - \nu(t(i))} \) and choosing a martingale differences \( \{\xi(i)\} \) as
\[
\xi(i) = \{ Z(t(i+1)) - Z(t(i)) \}/\{ \sqrt{\nu([t(i), t(i+1)])} \}:
\]
\[
||[b; Z]||_{M(1,q)} \leq C(q) \sqrt{\sum_{k=0}^{K} b^2(t(k)) \nu([t(k), t(k+1)]) \cdot \sup_{i} ||\xi(i)||_{q} = C(q) ||b||_{L_2(\nu)} ||Z||(Lip(q), \nu).
\]

On the other hand, we can write denoting \( (b; Z)_q = C(q) ||b||_{L_2(\nu)} \times ||Z||(Lip(q), \nu) \) for enough greatest values \( x > x_0 \) the inequality:
\[
\exp \left( -[x/(C_1(b; Z)_q)]^{M(1,q)} \right) \leq \sup_{b(\cdot); ||b||=1} \sup_{Z:||Z||(Lip(q), \nu) < \infty} \sup_{T([b; Z], x) \leq T([b; Z], x)} \exp \left( -[x/(C_2(b; Z)_q)]^{M(1,q)} \right).
\]

Low bounds follow from the left inequality of Theorem 1 (1.5).

It is easy to see that we can receive analogous result for multiply \( d \)-dimensional stochastic integral of a kind \( [b; Z_1, Z_2, \ldots, Z_d] = \)
\[
\int \int \ldots \int_{0 \leq t(1) < t(2) < \ldots < t(d) < \infty} b(t(1), t(2), \ldots, t(d)) \prod_{m=1}^{d} Z_m(dt(m)),
\]
where \( b(t(1), t(2), \ldots, t(d)) \) are non-random measurable square integrable functions, \( Z_m(t) \) are the square integrable centered left-continuous martingales with the correspondences measures \( \nu_m([a, c]) = D(Z_m(c) - Z_m(a)) \) and values \( ||Z_m||(Lip(q(m), \nu_m)) : \exists C(q) \in (0, \infty) \Rightarrow ||[b; Z_1, Z_2, \ldots, Z_d] ||_{M(d, \bar{q})} \leq C(\bar{q}) ||b||_{L_2(R^d, \prod_{m=1}^{d} \nu_m)} \prod_{m=1}^{d} ||Z(m)||_{Lip(q(m), \nu_m)} \).

This result may be improved in the case when all martingales \( Z_m \) are independent and have independent increments as in the theorems 2,3. For Gaussian martingales \( Z_m(t) \) in ([15], p.119) there is more exactly result. But we consider the other problem.
B). Let again \((Z(t), F(t)), t \geq 0\) be a square integrable centered continuous (with probability one) martingale with correspondence quadratic variation \(<Z, Z>_t; <Z, Z>_t = <Z, Z>_0\). Let us consider the \(d\) - dimensional multiply stochastic integral 
\[
[1; Z, Z, \ldots, Z, (d)] = \int \int \ldots \int_{0 \leq t(1) < t(2) \ldots < t(d) \leq 1} \prod_{m=1}^{d} dZ(t(m)).
\]

It is proved in the article [39] the following generalization of the classical Burkholder - Davis - Gundy inequality (in our terms and notation):
\[
||[1; Z, Z, \ldots, Z]||_p \leq A(d, p) < Z, Z >^{1/2} ||_p,
\]
\[
A(d, p) \overset{df}{=} (1 + 1/p)^d (dp)^{d/2} / d! \leq C(d/p)^{d/2},
\]
where \(p \geq 2\) and \(C\) is absolute constant.

**Theorem 12.** Assume that for some \(q > 0\)
\[
|| < Z, Z > ||_q < \infty. \tag{5.4}
\]
Then

1. \(||[1; Z, (d)]||_{M(d,q)} \leq C(d, q) || < Z, Z >^{1/2} ||_q^d. \tag{5.5}\)
2. If (5.4) holds for some \(q > 2\), then taking now \(\beta = \beta(q) = (q + 2)/(q - 2) > 0\) we assert that the integral series \(1 + \sum_{d=1}^{\infty} [1; Z, (d)]\) for Dolean exponent \(E_D(Z) = \exp(Z(1 - 0.5 < Z, Z >)\) convergent in the \(\Psi(C, \beta(q))\), \(\exists C \in (0, \infty)\) norm:
\[
||E_D(Z)||\Psi(\beta(q)) \leq 1 + \sum_{d=1}^{\infty} ||[1; Z, (d)]||\Psi(\beta(q)) < \infty. \tag{5.6}\)

**Remark 4.** The case \(q = 2\) is considered in [39].

**Proof.** We shall assume without loss of generality \(|| < Z, Z >^{1/2} ||_q \leq 1/C_2\), where \(C_2\) is a "great" constant, so that
\[
\forall p \geq 2 \Rightarrow | < Z, Z >^{1/2} ||_p \leq p^{1/q}.
\]
We receive using (5.4) and Stirling formula
\[
||[1; Z, (d)]||_p \leq C^d p^{d/2} d^{-d/2} p^{d/q} \leq C^d p^{1/M(d,q)} d^{d/q - d/2}. \tag{5.7}
\]
Hence
\[
||[1; Z, (d)]||_{M(d,q)} \leq C^d d^{-d/2} < \infty,
\]
and hence we deduce (5.6).

Further, assume \(q > 2\). We can write from (5.4) \(\forall p \geq 2:\)
\[
|E_D(Z)|_p \leq 1 + \sum_{d=1}^{\infty} ||[1; Z, (d)]||_p \leq C + \sum_{d=1}^{\infty} p^{d(0.5 + 1/q)} d^{-d(0.5 - 1/q)} \leq
\]

25
\[ \exp \left( C p^{(q+2)/(q-2)} \right) = \exp \left( C p^{\beta(q)} \right). \]

Now the estimation (5.6) it follows from the definition of \( \Psi(\beta, C) \) - spaces.

For instance assume that (5.4) holds at \( q = +\infty \), i.e.
\[ \text{vrainax}_{\omega} | < Z, Z > | < \infty. \]
Then \( \beta = 1 \), hence
\[ T(E_D(Z), x) \leq \exp \left( -C \log^2 (1 + x) \right), \ x > 0. \quad (5.8) \]

In the case when \( Z(t) \) is the Wiener martingale, estimation (5.8) is exact as long as \( < Z, Z > = 1 \) and \( E_D(Z) = \exp(Z(1) - 0.5) \).

6. Applications to the martingale sums. We shall consider in this section the tail behavior of the centered martingale sums (1.0)
\[ Q_d = \sum_{I \in I(d,n)} b(I)\xi(I), \]
where \( n \leq \infty, \sup_{I \leq n} ||\xi(i, m)||_{q(m)} < \infty \), or, equally, \( \exists K(m), q(m) \in (0, \infty] \) such that
\[ \sup_{i \leq n} T(\xi(i, m), x) \leq \exp \left( -\left( x/K(m) \right)^{q(m)} \right), \quad x > 0, \quad (6.1) \]
if the speed of convergence \( b(I) \to 0 \) by \( I \to Z^d_+ \) is rapid.

We obtain in the sections 2,3 the estimation of tail \( T(Q_d, x) \) under condition \( \sum b^2(I) < \infty \). But if
\[ \sum_{I \in I(d,n)} |b(I)| < \infty, \quad (6.2) \]
it follows from triangular inequality for the \( G \) - norms that
\[ ||Q_d||_G \leq C \sum_{I \in I(d,n)} |b(I)| \prod_{m=1}^d \sup_{i \leq n} ||\xi(i, m)||_{q(m)} < \infty, \]
\[ G = \left( \sum_{m=1}^d 1/q(m) \right)^{-1} \in (0, \infty), \quad K = \prod_{m=1}^d K(m), \]
as long as \( \sup_I ||\xi(I)||_G \leq C(\bar{q}) \prod_{m=1}^d K(m) \). We receive in this case, i.e. if both conditions (6.1), (6.2) are satisfied:
\[ T(Q_d, x) \leq C_1 \exp \left( -C_2 (x/K)^G \right), \]
and this estimation is not improvable, e.g. for the polynomial \( Q_d = \prod_{m=1}^d \xi(1, m) \).

Thus, we can assume:
\[ \sum_{I \in I(d,n)} |b(I)| = \infty, \quad \sum_{I \in I(d,n)} b^2(I) < \infty. \quad (6.3) \]
(Note that the condition (6.3) is not trivial only if \( n = \infty \)). We introduce a two measures on the subsets \( I(d,n) \):

\[
\mu_1(A) = \sum_{I \in A} |b(I)|, \quad \mu_2(A) = \sum_{I \in A} b^2(I),
\]

and introduce for \( \lambda = \text{const} > 0 \) the functions \( A(\lambda) = \{ I : |b(I)| \leq \lambda \}, \quad B(\lambda) = \{ I : |b(I)| > \lambda \}, \quad a_1(\lambda) = \mu_1(A(\lambda)), \quad a_2(\lambda) = \sqrt{\mu_2(B(\lambda))} \).

**Theorem 13.** For some \( C_{1,2} = C_{1,2}(b(\cdot)) \in (0, \infty) \quad T(Q_d, x) \leq \]

\[
C_1 \inf_{\lambda > 0} \left( \exp \left( -C_2 \left( x/(a_1(\lambda)K) \right) \right) \right)^G + \exp \left( -C_2 \left( x/(a_2(\lambda)K) \right)^{M(d,q)} \right).\]

**Theorem 14.** If in addition all the variables \( \{ \xi(i,m) \} \) are independent then \( T(R_d, x) \leq \]

\[
C_1 \inf_{\lambda > 0} \left( \exp \left( -C_2 \left( x/(a_1(\lambda)K) \right) \right) \right)^G + \exp \left( -C_2 \left( x/(a_2(\lambda)K) \right)^{N_2(q)} \right).\]

**Proof.** We shall assume without loss of generality \( ||\xi(i,m)||_{q(m)} = 1 \). Let \( A \) be some subset of \( I(d,n) \) and \( \overline{A} = I(d,n) \setminus A \). We write: \( Q_d = Q_d(1) + Q_d(2) \), where

\[
Q_d(1) = \sum_{I \in A} b(I) \xi(I), \quad Q_d(2) = \sum_{I \in \overline{A}} b(I) \xi(I).
\]

We get for \( Q_d(1) \) — sum from triangle inequality:

\[
||Q_d(1)||_G \leq C_3(q) \sum_{I \in A} |b(I)| ||\xi(I)||_G \leq C_4(q) \mu_1(A),
\]

hence

\[
T(Q_d(1), x) \leq C_5(q) \exp \left( -C_6(x/\mu_1(A))^G \right).
\]

We obtain for the second sum \( Q_d(2) \) by virtue of theorem 1:

\[
||Q_d(2)||_M \leq C_7 \sqrt{\sum_{I \in \overline{A}} b^2(I)} = C_7 \sqrt{\mu_2(\overline{A})}.
\]

The proposition of the theorem 13 follows from the elementary inequality

\[
T(Q_d, x) \leq T(Q_d(1), x/2) + T(Q_d(2), x/2)
\]

after the minimization over set \( A \); it is easily to see that the optimal choosing \( A \) has a view \( A = A(\lambda) \) for some \( \lambda \geq 0 \). The proof of theorem 14 is like to one.

**Example.** Assume that \( q(1) = q(2) = \ldots = q(d) = q = \text{const} \in (0, \infty) \), so that

\[
T(\xi(i,m), x) \leq \exp (-x^q), \quad x \geq 0;
\]
and assume also $|b(I)| \leq C|I|^{-\alpha}$, $\alpha > d/2$; here

$$|I| = |(i(1), i(2), \ldots, i(d))| = \sqrt{\sum_{m=1}^{d} i^2(m)}.$$  

We deduce from theorem 13 (in the martingale case): if $\alpha \in (d/2, d)$, then

$$T(\sum b(I)\xi(I), x) \leq C_1(d, q, \alpha) \exp \left(-C_2(d, q, \alpha)x^{d/(q(d-\alpha)+d)} \right).$$  

If $\alpha = d$, then we conclude

$$T(Q_d, x) \leq C_1 \exp \left(-C_2 x^{G} \right), \quad x \geq 3.$$  

We conclude in the case $\alpha > d$

$$T(Q_d, x) \leq C_1 \exp \left(-C_2x^{G} \right), \quad x \geq 0.$$  

If $d = 1$ and $q = +\infty$, i.e. if $\exists K \in (0, \infty), \forall x > K \Rightarrow T(\xi(i, m), x) = 0$, yields a well-known result (see, e.g. [28], p. 33 - 37.)

Now we shall consider the moment estimations for $Q_d$ in our case. Assume that for some $p \geq 2$

$$\sup_{i} |\xi(i, m)|_p \leq 1,$$

and that all the variables $\{\xi(i, m)\}$ are independent.

**Theorem 15.**

$$|Q_d|_p \leq C(b(\cdot)) \inf_{\lambda > 0} \left( a_1(\lambda) + a_2(\lambda)p^d/\log p \right).$$

**Proof** is the same as in theorem 13. We obtain for the sum $Q_d(1)$ using triangular inequality:

$$|Q_d(1)|_p \leq C \sum_{I \in A(\lambda)} |b(I)||\xi(I)|_p \leq C\mu(A(\lambda)) = Ca_1(\lambda).$$

We deduce from theorem 7:

$$|Q_d(2)|_p \leq C \sqrt{\sum_{I \in A(\lambda)} b^2(I) p^d/\log p} = Ca_2(\lambda) p^d/\log p.$$  

**Example.** Assume again in addition $|b(I)| \leq C|I|^{-\alpha}$, where $\alpha > d/2$. Then if $\alpha \in (d/2, d)$, then

$$|Q_d|_p \leq C(d, \alpha)p^{2(d-\alpha)}(\log p)^{2(1-\alpha/d)}.$$  

If $\alpha = d$ then

$$|Q_d|_p \leq C(d) \log p;$$
and in the case \( \alpha > d \) \( \Rightarrow \)
\[
\sup_{p \geq 2} |Q_d|_p \leq C(d, \alpha) < \infty.
\]

**Theorem 16.** Assume now that all the sequences \( \{\xi(i, m), F(i)\}, m = 1, 2, \ldots, d \)
are centered martingale - differences ("martingale case") such that
\[
\max_m \sup_i |\xi(i, m)|_{d, p} \leq 1
\]
and \( b \in B \). Then
\[
|Q_d|_p \leq C(b(\cdot), d) \inf_{\lambda > 0} \left( a_1(\lambda) + a_2(\lambda)p^d \right).
\]

For example assume again in addition (in the "martingale case") \( b(I) \sim C|I|^{-\alpha}, \exists \alpha > d/2 \). It follows from theorem 16 (if \( \mu_m(d \cdot p) \leq 1, m = 1, 2, \ldots, d \)
for some \( p \geq 2 \)) that if \( \alpha \in (d/2, d) \),
\[
|Q_d|_p \leq C(d, \alpha) p^{2(d-\alpha)}.
\]

In cases \( \alpha = d \) and \( \alpha > d \) we obtain (in our condition \( \mu(d \cdot p) < \infty \)) the same results
as in independent case (see Examples in the theorem 15).

Now we shall investigate the case
\[
\sum_{I \subset I(d, \infty)} b^2(I) \prod_{m=1}^d \sigma^2(i_m, m) = \infty.
\]

In particular, the series for \( Q_d \) may divergent. Let us consider now the naturally
normed multiply sum
\[
\theta_n = \sum_{I \subset I(d, n)} \frac{\xi(I)}{\left[ \sum_{I \subset I(d, n)} \prod_{m=1}^d \sigma^2(i_m, m) \right]},
\]
so that \( \mathbf{E}\theta_n = 0, \mathbf{D}\theta_n = 1 \).

**Lemma 3.** 1. If
\[
\sup_{i, m} T(\xi(i, m)/\sigma(i, m), x) \leq \exp(-x^q), q, x \geq 0,
\]
then at \( x > 2 \)
\[
\sup_n T(\theta_n, x) \leq \exp \left( -Cx^{M(d, q)} \right).
\]

2. If in addition the variables \( \xi(i, m) \) are independent, then
\[
\sup_n T(\theta_n, x) \leq \exp \left( -Cx^{N(d, q)} \right).
\]

3. If
\[
\sup_{i, m} |\xi(i, m)|_{p, d}/\sigma(i, m) \leq 1,
\]
then
\[ \sup_n |\theta_n|_p \leq Cp^d. \]

4. If in additional to the 3 \( \{\xi(i, m)\} \) are independent and
\[ \sup_{i, m} |\xi(i, m)|_p / \sigma(i, m) \leq 1, \]
then
\[ \sup_n |\theta_n|_p \leq Cp^d / \log p. \]

**Proof** is very simple. Substituting \( \xi(i, m) = \sigma(i, m) \nu(i, m) \) and choosing
\[ b(I) = \prod_{m=1}^d \sigma(i_m, m)/\left( \sqrt{\sum_{I \subset I(d, n)} \prod_{m=1}^d \sigma^2(i_m, m)} \right), \]
we can write
\[ \theta_n = \sum_{I \subset I(d, n)} b(I)\nu(I), \ b \in B(d, n). \]

We receive using our estimations (1.5), (1.8), (3.1) and (3.3) the proposition of lemma 3.

Our result may be considered as some addition to the limit theorem for martingales (see, for example, [20], p.58.)

7. **Applications to the weak compactness.** Assume that the multidimensional sequence of coefficients \( \{b(I)\} \) dependent on some parameter \( t; t \in V, V \) is an arbitrary set: \( b(I) = b(I, t) \). Suppose
\[ \sup_{t \in V} \sum_{I \in I(d, \infty)} b^2(I, t) < \infty, \] (7.1)
and introduce the following distance between two arbitrary points \( t_1, t_2 \in V : \)
\[ r_1(t_1, t_2) = \sqrt{\sum_{I \in I(d, \infty)} (b(I, t_1) - b(I, t_2))^2}. \]

We shall consider a random field
\[ \tau(t) = \sum_{I \in I(d, \infty)} b(I, t)\xi(I), \] (7.2)
(series with random coefficients), where \( \{\xi(i, m)\} \) is again a sequence of martingale differences so that for some \( q(m) > 0 \)
\[ \sup_i \sup_{t \in V} ||\xi(i, m, t)||_{q(m)} < \infty. \]
Theorem 17. Assume that the metric space \((V, r_1)\) is full and
\[
\int_0^1 H^{1/M(d,q)}(V, r_1, \varepsilon) \, d\varepsilon < \infty,
\]
where \(H(V, r_1, \varepsilon)\) is so-called metric entropy of space \(V\) on the distance \(r_1:\)
\[
H(V, r_1, \varepsilon) = \log N(V, r_1, \varepsilon), \quad N(V, r_1, \varepsilon) =
= \inf \{\text{card}\{t_i : \cup_i \{t : r_1(t, t_i) \leq \varepsilon\} = V\}\}.
\]
Then the series (7.2) convergent uniformly on the \(t, t \in V\) with probability 1,
\[
P(\tau(\cdot) \in C(V, r_1)) = 1, \quad (C(V, r_1) \text{ denote the space of all } r_1 - \text{continuos functions}
\]
\(f : V \to R,\) and for some \(C_{10} > 0\)
\[
T(\sup_{t \in V} |\tau(t)|, x) \leq \exp \left(-C_{10} x^{M(d,q)}\right), \quad x \geq 2.
\]
Proof. It follows from theorem 1 that:
\[
\sup_{t \in V} ||\tau(t)||_{M(d,q)} < \infty.
\]
Further, since
\[
\tau(t_1) - \tau(t_2) = \sum_{I \in I(d, \infty)} [b(I, t_1) - b(I, t_2)]\xi(I),
\]
we deduce analogously:
\[
||\tau(t_1) - \tau(t_2)||_{M(d,q)} \leq C_{11} \sqrt{\sum_{I \in I(d, \infty)} [b(I, t_1) - b(I, t_2)]^2} = C_{11} r_1(t_1, t_2).
\]
Our proposition it follows from the propositions (11), p. 195 - 196, (28), p. 303 - 306.)
Assume now that the coefficients \(b(I)\) are constants, i.e. does not dependent
on \(\omega \in \Omega, \ t \in V,\) but the martingale - differences \(\xi(i, m)\) are separable functions
depending still on some parameter (parameters) \(t, t \in V:\)
\[
\xi(i, m) = \xi(i, m, t), \quad t \in V,
\]
i.e. \(\xi(i, m, \cdot)\) are separable random fields. Suppose that for some \(q(m) \in (0, \infty]\)
\[
\sup_{t \in V} \sum_{m=1}^d \sup_i ||\xi(i, m, t)||_{q(m)} < \infty.
\]
Let us introduce the following distance between \(t_1, t_2 \in V:\)
\[
r_2(t_1, t_2) = \sum_{m=1}^d \sup_i ||\xi(i, m, t_1) - \xi(i, m, t_2)||_{q(m)} < \infty.
\]
Theorem 18. Suppose that the metric space $(V, r_2)$ is full and that
\[
\int_0^1 H^{1/M(d,q)}(V, r_2, \varepsilon) \, d\varepsilon < \infty.
\]
Then the series
\[
\zeta(t) = \sum_{I \in I(d, \infty)} b(I) \xi(I, t)
\]
convergent uniformly on the parameter $t$, $t \in V$, the random field $\zeta(t)$ belong to the space $C(V, r_2)$ with probability 1, and
\[
T(\sup_{t \in V} |\zeta(t)|, x) \leq \exp \left( -C_1 x^{M(d,q)} \right), \quad x \geq 1.
\]

Proof of theorem 18 is full analogous to the theorem 17; we need only to consider the difference
\[
\zeta(t_1) - \zeta(t_2) = \sum_{I \in I(d, \infty)} b(I) \left[ \xi(I, t_1) - \xi(I, t_2) \right].
\]
and use the theorem 1 and identity
\[
\prod_{m=1}^d y(m) - \prod_{m=1}^d x(m) = \sum_{m=1}^d (y(m) - x(m)) \cdot \prod_{k \in A(m)} y(k) \cdot \prod_{l \in B(m)} x(l),
\]
where $A(m), B(m)$ are the sets if indexes so that $A(m) \cap B(m) = \emptyset$, $\text{card } A(m) + \text{card } B(m) = d - 1$.

Assume that the coefficients $b(I)$ are functions on some parameter $\alpha$; $\alpha \in \{\alpha\}$, $b = b_\alpha(I)$, and, for instance,
\[
\sup_{\alpha} \sum_I b_\alpha^2(I) < \infty,
\]
\[
\int_0^1 H^{1/M(d,q)}(V, r_2, \varepsilon) \, d\varepsilon < \infty,
\]
then the family of distributions on the space $C(V, r_2)$
\[
\mu_\alpha(A) = P(\zeta_\alpha(\cdot) \in A),
\]
$A$ – is some Borel subset $C(V, r_2)$, are weakly compact; here
\[
\zeta_\alpha(t) = \sum_{I \in I(d, \infty)} b_\alpha(I) \xi(I, t).
\]

Remark 5. Analogous results may be obtained in the terms of Majorizings Measures (see, for example, [28], p. 314 - 318.)

Remark 6. Probably it is very interesting to generalize our results on the so-called non-commutative case, in the spirit of article [40].

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