On the Structure and Complexity of Rational Sets of Regular Languages

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Abstract. In a recent thread of papers, we have introduced FQL, a precise specification language for test coverage, and developed the test case generation engine Fshell for ANSI C. In essence, an FQL test specification amounts to a set of regular languages, each of which has to be matched by at least one test execution. To describe such sets of regular languages, the FQL semantics uses an automata-theoretic concept known as rational sets of regular languages (RSRLs). RSRLs are automata whose alphabet consists of regular expressions. Thus, the language accepted by the automaton is a set of regular expressions.

In this paper, we study RSRLs from a theoretic point of view. More specifically, we analyze RSRL closure properties under common set theoretic operations, and the complexity of membership checking, i.e., whether a regular language is an element of a RSRL. For all questions we investigate both the general case and the case of finite sets of regular languages. Although a few properties are left as open problems, the paper provides a systematic semantic foundation for the test specification language FQL.

1 Introduction

Despite the success of model checking and theorem proving, software testing has a dominant role in industrial practice. In fact, state-of-the-art development guidelines such as the avionic standard DO-178B [1] are heavily dependent on test coverage criteria. It is therefore quite surprising that the formal specification of coverage criteria has been a blind spot in the formal methods and software engineering communities for a long time.

In a recent thread of papers [2–7], we have addressed this situation and introduced the Fshell Query Language (FQL) to specify and tailor coverage criteria, together with Fshell, a tool to generate matching test suites for ANSI C programs. At the semantic core of FQL, test goals are described as regular expressions whose alphabet are the edges of the program control flow graph (CFG). For example, to cover a particular CFG edge $c$, one can use the regular expression $\Sigma^* c \Sigma^*$. Importantly, however, a coverage criterion usually contains not just a single test goal, but a (possibly large) number of test goals – e.g. all basic blocks of a program. FQL therefore employs regular languages which can
express sets of regular expressions. To this end, the alphabet contains not only the CFG edges but also \textit{postponed regular expressions} over these edges, written within quotes.

For example, "\(\Sigma^*\) (\(a+b+c+d\) )" describes the language \{"\(\Sigma^*\) a "\(\Sigma^*\), "\(\Sigma^*\) b "\(\Sigma^*\), "\(\Sigma^*\) c "\(\Sigma^*\), "\(\Sigma^*\) d "\(\Sigma^*\)\}. Each of these words is a regular expression that will then serve as a test goal. Following [8], we call such languages \textit{rational sets of regular languages (RSRL)}.

The goal of this paper is to initiate a systematic study of RSRLs from a theoretical point of view, considering closure properties and complexity of common set-theoretic operations. Thus, this paper is a first step towards a systematic foundation of FQL. RSRLs have a similar role for test specifications as relational algebra has for databases. In particular, a good understanding of set-theoretic operations is necessary for systematic algorithmic optimization and manipulation of test specifications. First results on query optimization for FQL have been obtained in [7].

A rational set of regular languages is given by a regular language \(L\) over alphabet \(\Delta\), and a \textit{regular language substitution} \(\phi : \Delta \to 2^{\Sigma^*}\), mapping each symbol \(\delta \in \Delta\) to a regular language \(\phi(\delta)\) over alphabet \(\Sigma\). We extend \(\phi\) to words \(w \in \Delta^+\) with \(\phi(\delta \cdot w) = \phi(\delta) \cdot \phi(w)\), and set \(\phi(L) = \bigcup_{w \in L} \phi(w)\) for \(L \subseteq \Delta^+\).

\textbf{Definition 1 (Rational Sets of Regular Languages, RSRLs [8])}. A set \(R\) of regular languages over \(\Sigma\) is called \textit{rational}, written \(R = (K, \phi)\), if there exists a finite alphabet \(\Delta\), a regular language \(K \subseteq \Delta^+\), and a regular language substitution \(\phi : \Delta^+ \to 2^{\Sigma^*}\), such that \(R = \{\phi(w) \mid w \in K\}\). The RSRL \(R\) is \textit{Kleene star free}, if \(K\) is given as Kleene star free regular expression.

Depending on context, we refer to \(R\) as a set of languages or as a pair \((K, \phi)\), but we always write \(L \in R\) iff \(\exists w \in K : L = \phi(w)\). Consider the above specification "\(\Sigma^*\) (\(a+b+c+d\) )" over base alphabet \(\Sigma = \{a, b, c, d\}\). To represent this specification as RSRL \(R = (K, \phi)\), we set \(\Delta = \{\delta_{\Sigma}\} \cup \Sigma\), containing a fresh symbol \(\delta_{\Sigma}\), for the quoted expression "\(\Sigma^*\)". We set \(K = L(\delta_{\Sigma}, (a+b+c+d) \delta_{\Sigma})\) with \(\phi(\delta_{\Sigma}) = \Sigma^*\) and \(\phi(\sigma) = \sigma\) for \(\sigma \in \Sigma\). Thus \(K\) contains the words \(\delta_{\Sigma}, a \delta_{\Sigma}, \ldots\) with \(\phi(\delta_{\Sigma}, a \delta_{\Sigma}, \ldots) = L(\Sigma^* \ a \Sigma^*) \in R\), as desired.

Note that the RSRL above is finite with exactly four elements. This is of course not atypical: in concrete testing applications, FQL generates finite sets of test goals, since it relies on \textit{Kleene star free} RSRLs only. For future applications, however, it is well possible to consider infinite sets of test goals e.g. for unbounded integer and real valued variables or for path coverage criteria which are either matched partially, or by abstract executions. In this paper, we are therefore considering the general, finite, and Kleene star free case.

\textbf{Example 2}. Consider the alphabets \(\Delta = \{\delta_1, \delta_2\}\) and \(\Sigma = \{a, b\}\). Then, (1) with \(\phi(\delta_1) = L(a^*), \phi(\delta_2) = \{ab\}\), and \(K = L(\delta_1 \delta_2 \delta_1)\), we obtain the rational set of regular languages \(\{L(a^*ab)^i a^* \mid i \in \mathbb{N}\}\); (2) with \(\phi(\delta_1) = \{a^i \mid i \geq 0\}\), \(\phi(\delta_2) = \{a\}\), and \(K = \{\delta_1 \delta_2^i \mid i \geq 0\}\), we obtain \(\phi(w_1) \supset \phi(w_2)\) for all \(w_1, w_2 \in K\) with \(|w_1| < |w_2|\); (3) with \(\phi(\delta_1) = \{a, e\}\), \(\phi(\delta_2) = \{aa\}\), and \(K = \{\delta_1 \delta_2^i \mid i \geq 0\}\), we have \(|\phi(w)| = 2\) and \(\phi(w) \cap \phi(w') = \emptyset\) for all \(w \neq w' \in K\).
In the finite case we make an additional distinction for the subcase where the regular expressions in $\Delta$, i.e., the set of postponed regular expressions, are fixed. This has practical relevance, because in the context of FQL, the results of the operations on RSRL will be better readable by engineers if $\Delta$ is unchanged.

Contributions and Organization. In Section 3, we investigate closure properties of general and finite RSRLs, considering the operators product, Kleene star, complement, union, intersection, set difference, and symmetric difference. We also consider the case of finite RSRLs with a fixed language substitution $\varphi$, as this case is of particular interest for testing applications. Next, in Section 4, we investigate the complexity of the decision problems equivalence, inclusion, and membership for Kleene star free RSRLs. We also give an algorithm for checking the membership in general and analyze its complexity. We close in Section 5 with a discussion on how our results reflect back to design decisions for FQL.

2 Related Work

Afonin et al. [8] introduced RSRL and studied the decidability of whether a regular language is contained in an RSRL and the decidability of whether an RSRL is finite. Although Afonin et al. shortly discuss possible upper bounds for the membership decision problem, their analysis is incomplete due to gaps in their algorithmic presentation. Pin introduced the term extended automata for RSRLs as an example for a formalism that can be expressed by equations [9], but did not investigate any of their properties. In our own related work on FQL [2, 10, 3–7], we deal with practical issues arising in testcase generation. Note that FQL uses a language layer on top of RSRLs which extracts the alphabet from the program using a convenient syntax. In conclusion, we are unaware of related work that considers the properties we study here.

Let us finally discuss other work whose terminology is similar to RSRLs without direct technical relation. Barceló et al. define rational relations, which are relations between words over a common alphabet, whereas we consider sets of regular languages [11]. Barceló et al. also investigate parameterized regular languages [12], where words are obtained by replacing variables in expressions with alphabet symbols. Metaregular languages deal with languages recognized by automata with a time-variant structure [13, 14]. Lattice Automata [15] only consider lattices that have a unique complement element, whereas RSRLs are not closed under complement (no RSRL has a RSRL as complement).

3 Closure Properties

Operators. We investigate the closure properties of RSRLs, considering standard set theoretic operators, such as union, intersection, and complement, and variants thereof, fitting RSRLs. In particular, we apply those operators also to pairs in the Cartesian product of RSRLs, and point-wise to each element in a RSRL and another given regular language.
Definition 3 (Operations on RSRL). Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be RSRLs and let $R$ be a regular language. Then, we define the following operations on RSRLs:

| Operation       | Definition                                                                 |
|-----------------|-----------------------------------------------------------------------------|
| Product         | $\mathcal{R}_1 \cdot \mathcal{R}_2 = \{ L_1 \cdot L_2 \mid L_1 \in \mathcal{R}_1, L_2 \in \mathcal{R}_2 \}$ |
| Kleene Star     | $\mathcal{R}_1^* = \bigcup_{i \in \mathbb{N}} \mathcal{R}_1^i$            |
| Point-wise      | $\mathcal{R}_1^\ddot{\cdot} = \{ L^\ddot{\cdot} \mid L \in \mathcal{R}_1 \}$ |
| Complement      | $\overline{\mathcal{R}_1} = \{ L \subseteq \Sigma^* \mid L \notin \mathcal{R}_1 \}$ |
| Point-wise      | $\overline{\mathcal{R}_1^\ddot{\cdot}} = \{ L \mid L \in \mathcal{R}_1 \}$ |
| Binary Operators| $\mathcal{R}_1 \cap \mathcal{R}_2, \mathcal{R}_1 \cup \mathcal{R}_2, \mathcal{R}_1 - \mathcal{R}_2$ (standard def.) |
| Point-wise      | $\mathcal{R}_1 \cup / \cap / - R = \{ L \cup / \cap / - R \mid L \in \mathcal{R}_1 \}$ |
| Cartesian       | $\mathcal{R}_1 \times \mathcal{R}_2 = \{ L_1 \cup / \cap / - L_2 \mid L_1 \in \mathcal{R}_1, L_2 \in \mathcal{R}_2 \}$ |
| Symmetric Difference | $\mathcal{R}_1 \Delta \mathcal{R}_2 = \{ L \mid L \in (\mathcal{R}_1 \cup \mathcal{R}_2) - (\mathcal{R}_1 \cap \mathcal{R}_2) \}$ |

Language Restrictions. We analyze three different classes of RSRLs for being closed under these operators: (1) General RSRLs, (2) finite RSRLs, and (3) finite RSRLs with a fixed language substitution $\varphi$. For closure properties, we do not distinguish between Kleene star free and finite RSRLs, since every finite RSRL is expressible as Kleene star free RSRL (however, given a RSRL with Kleene star, it is non-trivial to decide whether the given RSRL it finite or not [8]). Therefore, all closure properties for finite RSRLs apply to Kleene star free RSRLs as well. Hence, cases (2-3) correspond to FQL. Case (3) is relevant for usability in practice, allowing to apply the corresponding operators without constructing a new language substitution. This does not only significantly reduce the search space but also provides more intuitive results to users.

Theorem 4 (Closure Properties of RSRL). The following Table summarizes the closure properties for RSRLs.

| Operation       | General | Fixed Subst. |
|-----------------|---------|--------------|
| Product         | Prop. 7 | +            |
| Kleene Star     | Prop. 7 | +            |
| Point-wise      | Prop. 10| -            |
| Complement      | Prop. 11| -            |
| Point-wise      | Prop. 12| +            |
| Union           | Prop. 14| +            |
| Point-wise      | Prop. 16| -            |
| Cartesian       | Cor. 30 | -            |
| Intersection    | Prop. 17| ?            |
| Point-wise      | Prop. 18| -            |
| Cartesian       | Cor. 30 | -            |
| Difference      | Prop. 19| ?            |
| Point-wise      | Prop. 20| -            |
| Cartesian       | Cor. 30 | -            |
| Symmetric       | Prop. 21| ?            |
Unifying Alphabets. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be RSRLs over a common alphabet $\Sigma$ with $\mathcal{R}_i = (K_i, \varphi_i)$, $K_i \subseteq \Delta_i$, and $\varphi_i : \Delta_i \to 2^{\Sigma^*}$. Then we create a unified alphabet $\Delta = \{(i, \delta) \mid \delta \in \Delta_i \text{ with } i = 1, 2\}$ and a unified language substitution $\varphi : \Delta \to 2^{\Sigma^*}$ with $\varphi((i, \delta)) = \varphi_i(\delta)$. We obtain $\mathcal{R}_i = (K'_i, \varphi)$ where $K'_i$ is derived from $K_i$ by substituting each symbol $\delta \in \Delta_i$ with $(i, \delta) \in \Delta$. Hence without loss of generality, we fix the alphabets $\Delta$ and $\Sigma$ with language substitution $\varphi$, allowing our RSRLs only to differ in the generating languages $K_i$. When we discuss binary operators, we freely refer to RSRLs $\mathcal{R}_i = (K_i, \varphi)$ for $i = 1, 2$, in case of unary operators to $\mathcal{R} = (K, \varphi)$, and in case of point-wise operators to the regular language $R \subseteq \Sigma^*$.

General Observations. We exploit in our proofs some general observations on the cardinality of RSRLs. Moreover, we prove all closure properties of Cartesian binary operators by reducing the point-wise operators to the Cartesian one. For space reasons, we show this generic argument only in Appendix A.

Fact 5 (Finite Sets of Regular Languages are Rational) Every finite set of regular languages is rational.

Proof. For a finite set of regular languages $\mathcal{R}$, we set $\varphi(\delta_L) = L$ for all $L \in \mathcal{R}$, taking fresh symbols $\delta_L$. With $\Delta_\mathcal{R} = \{\delta_L \mid L \in \mathcal{R}\}$ we obtain $\mathcal{R} = (\Delta_\mathcal{R}, \varphi)$. $\square$

Fact 6 (Cardinality of RSRL) A RSRL contains at most countably many languages. In particular, $2^{2^{\Sigma^*}}$ is not a RSRL.

Proof. A RSRL $\mathcal{R} = (K, \varphi)$ is countable, as $K$ contains countably many words, and $|K| \geq |\mathcal{R}|$ holds. Since $2^{2^{\Sigma^*}}$ is uncountable, it is not a RSRL. $\square$

3.1 Product and Kleene Star

Proposition 7 (Closure of Product and Kleene Star). (1) $\mathcal{R}_1 \cdot \mathcal{R}_2$ is a RSRL, defined over the same substitution $\varphi$. If $\mathcal{R}_i$ are finite, then $\mathcal{R}_1 \cdot \mathcal{R}_2$ is also finite. (2) $\mathcal{R}^*$ is a RSRL. It is in general infinite even if $\mathcal{R}$ is finite.

Proof. (1) We construct $\mathcal{R}' = (K', \varphi)$ with $K' = K_1 \cdot K_2$ and obtain $\mathcal{R}_1 \cdot \mathcal{R}_2 = \mathcal{R}'$. (2) We construct $\mathcal{R}' = (K', \varphi')$ with $K' = K^* \setminus \{\varepsilon\}$ setting $\varphi'(\delta_\varepsilon) = \{\varepsilon\}$ and $\varphi'(\delta) = \varphi(\delta)$ otherwise, and obtain $\mathcal{R}^* = \mathcal{R}'$. Consider the finite RSRL $\mathcal{R} = \{\{a\}\}$, then, $\mathcal{R}^*$ is the infinite RSRL $\{\{a^i\} \mid i \geq 0\}$. $\square$

In the following we consider the set $S(L)$ of shortest words of a language $L$, disregarding $\varepsilon$, defined with $S(L) = \{w \in L \setminus \{\varepsilon\} \mid \exists w' \in L \setminus \{\varepsilon\} \text{ with } |w'| < |w|\}$. We also refer to the shortest words $S(\mathcal{R})$ of a RSRL $\mathcal{R}$ with $S(\mathcal{R}) = \bigcup_{L \in \mathcal{R}} S(L)$.

Lemma 8. Let $\varepsilon \in \varphi(\delta)$ hold for all $\delta \in \Delta$. Then, for each $w \in \Delta^+$ and shortest word $v \in S(\varphi(w))$, there exists a $\delta \in \Delta$ such that $v \in S(\varphi(\delta))$. 

5
Proof. We start with a little claim: Because of $\varepsilon \in \varphi(\delta)$ for all $\delta \in \Delta$, we have $\varphi(\delta_i) \subseteq \varphi(w)$ for $w = \delta_1 \ldots \delta_k$ and all $1 \leq i \leq k$.

Assume $v \in S(\varphi(w))$ with $v \notin \varphi(\delta)$ for all $\delta \in \Delta$. Then $v = v_1 \ldots v_k$ with $v_i \in \varphi(\delta_i)$, and since $v \notin \varepsilon$, $v_p \neq \varepsilon$ for some $1 \leq p \leq k$. We fix such a $p$. From the claim above, we get $v_p \in \varphi(\delta_p) \subseteq \varphi(w)$, leading to a contradiction: If $v \neq v_p$, then $v$ is not a shortest word in $\varphi(w) \setminus \{\varepsilon\}$, as $v_p$ is shorter. If $v = v_p$, we contradict our assumption with $v = v_p \in \varphi(\delta_p)$.

Thus, we have shown that there exists a $\delta$ with $v \in \varphi(\delta)$. It remains to show $v \in S(\varphi(\delta))$. Assuming that $v' \in \varphi(\delta) \setminus \{\varepsilon\}$ is shorter than $v$, we quickly arrive at a contradiction: $v' \in \varphi(\delta) \subseteq \varphi(w)$ from the claim above, implies that $v$ would not be a shortest word in $\varphi(w) \setminus \{\varepsilon\}$ in the first place, i.e., $v \notin S(\varphi(w))$. \qed

**Corollary 9.** Let $\varepsilon \in \varphi(\delta)$ hold for all $\delta \in \Delta$. Then the set of shortest words $S(R)$ is finite.

**Proof.** Lemma 8 states for each word $v \in S(R)$, we have $v \in S(\varphi(\delta))$ for some $\delta \in \Delta$. But there are only finitely many symbols $\delta \in \Delta$, each generating only finitely many shortest words in $\varphi(\delta) \setminus \{\varepsilon\}$. Hence $S(R)$ must be finite. \qed

**Proposition 10 (Closure of Point-wise Kleene Star).** (1) In general, $\hat{R}^*$ is not a RSRL. (2) If $R$ is finite, $\hat{R}^*$ is a finite RSRL. (3) In the latter case, expressing $\hat{R}^*$ requires a new language substitution $\phi$.

**Proof.** (1) Consider the RSRL $R = \{\{a^i\} \mid i \geq 1\}$ with $\hat{R}^* = \{L_i \mid i \geq 1\}$ with $L_i = \{a^{i+j} \mid j \geq 0\}$. Every language $L_i \in \hat{R}^*$ contains the empty word $\varepsilon = a^0$, and hence, $\varepsilon \in \varphi(\delta)$ for all $\delta \in \Delta$ (disregarding symbols $\delta$ not occurring in $K$).

Thus, Corollary 9 applies, requiring that the set of shortest words $S(\hat{R}^*)$ is finite. This leads to a contradiction, since $S(\hat{R}^*) = \{a^i \mid i \geq 1\}$ is infinite. (2) Since $R$ is finite, also $\hat{R}^*$ has to be finite and statement follows from Fact 5. (3) Consider the RSRL $R = \{\{a\}\}$, produced from $(K, \varphi)$ with $K = \{\delta_a\}$ and $\varphi(\delta_a) = a$.

Then, $\hat{R}^* = \{\{a^i \mid i \geq 0\}\}$, and since $\{a\} \neq \{a^i \mid i \geq 0\}$ we have to introduce a new symbol. \qed

**3.2 Complement**

**Proposition 11 (Non-closure under Complement).** Let $R$ be a rational set of regular languages. Then $\overline{R}$ is not a rational set of regular languages.

**Proof.** Fact 6 states that $R$ is countable while $2^\Sigma^*$ is uncountable. Hence, $2^\Sigma^* \setminus R$ is uncountable and is therefore inexpressible as RSRL. \qed

**Proposition 12 (Closure of Point-wise Complement).** (1) $\overline{R}$ is in general not a RSRL. (2) If $R$ is finite, $\overline{R}$ is a finite RSRL as well, (3) requiring, in general, a modified language substitution.

**Proof.** (1) Consider the RSRL $R = (K, \varphi)$ with $K = L(\delta^*)$ and $\varphi(\delta) = \{a, b\} = \Sigma$. Then we have $R = \{\Sigma^i \mid i \geq 1\}$. For $i \neq j$, we have $\Sigma^i \cap \Sigma^j = \emptyset$, and
consequently, $\Sigma^i \not\subseteq \Sigma^j$ and $\Sigma^j \not\subseteq \Sigma^i$. Furthermore, observe $\varepsilon \in \Sigma^i$ for each $i \geq 1$. Assume $R$ is a RSRL. Then, there are $K'$ and $\varphi'$ such that $R = (K', \varphi')$. Since $R'$ is infinite and $K'$ is regular, there exists a word $w \in K'$ with $w = uvz$ and $\varphi(v) \neq \{\varepsilon\}$ and $uv^iz \in K'$ for all $i \geq 1$. Because of $\varepsilon \in \Sigma^i = \varphi(uvwz)$ for some $p$, we obtain $\varepsilon \in \varphi(v)$ as well. But then, for all $i \geq 1$, $\varphi(uvwz) \subseteq \varphi(uw^iz)$, i.e., $\varphi(uvwz) = \Sigma^i \subseteq \Sigma^j = \varphi(uw^iz)$. This contradicts the observation that $\Sigma^i \not\subseteq \Sigma^j$.

(2) By Fact 5. (3) Let $R = (\{\delta_n\}, \varphi)$ with $\varphi(\delta_n) = \{a\}$. Then, $R' = \{\Sigma^* \setminus \{a\}\}$. But, $\{a\} \neq \Sigma^* \setminus \{a\}$. Therefore, we need a new symbol to represent $\Sigma^* \setminus \{a\}$. □

In contrast to complementation, some RSRLs have a point-wise complement which is a RSRL as well; first, this is true for all finite RSRLs, as shown above, but there are also some infinite RSRLs which have point-wise complement.

**Example 13.** The RSRL $R = (L(\delta\delta^n), \varphi)$ with $\varphi(\delta) = \{a, b, \varepsilon\}$ has the point-wise complement $R' = (L(\delta_1\delta_2\delta_2), \varphi')$ with $\varphi'(\delta_1) = \{a, b\}$ and $\varphi'(\delta_2) = L((a+b)^*)$.

### 3.3 Union

**Proposition 14 (Closure of Union).** The set $R_1 \cup R_2$ is a rational set of regular languages, expressible as $(K_1 \cup K_2, \varphi')$ without changing the substitution $\varphi$.

*Proof.* Regular languages are closed under union, hence the claim follows. □

The following set of regular languages is not rational. We will use it in the proof of Proposition 16 to show that, in general, RSRLs are not closed under point-wise union.

**Example 15.** Consider the set $M = \{\{b\} \cup \{a^i \mid 1 \leq i \leq n+1\} \mid n \in \mathbb{N}\} \subseteq 2^{\{a,b\}^*}$. $M$ contains infinitely many languages, therefore, any RSRL $R = (K, \varphi)$, with $M = R$, requires a regular language $K$ containing infinitely many words. By $L_n$ we denote the set $\{\{b\} \cup \{a^i \mid 1 \leq i \leq n+1\} \mid n \in \mathbb{N}\}$. Then, $L_0 \subseteq L_1 \subseteq \ldots L_{i-1} \subseteq L_i \subseteq L_{i+1} \subseteq \ldots$. There must be a word $w = uvz \in K$ such that $uv^iz \in K$, for all $i \geq 1$ (cf. pumping lemma for regular languages [16]). Furthermore, there must be such a word $w = uvz$ such that $\varphi(u) \neq \emptyset$, $\varphi(v) \neq \emptyset$, $\varphi(v) \neq \{\varepsilon\}$, and $\varphi(z) \neq \emptyset$. This is due to the fact that we have to generate arbitrary long words $a_i$. We can assume that $b \not\in \varphi(v)$ because otherwise $b^i \in \varphi(v^i)$, for all $i \geq 1$. Therefore, $a^k \in \varphi(v)$ for some $k \geq 1$. Since $b \in \varphi(uvwz)$ has to be true, we can assume w.l.o.g. that $b \in \varphi(u)$. But, then $ba^k \ldots \in \varphi(uvwz)$. This is a contradiction to the fact that, for all $n \geq 1$, $ba^k \ldots \notin L_n$.

**Proposition 16 (Closure of Point-wise Union).** (1) The set $R_1 \cup R$ is, in general, not a RSRL. (2) The set $R \cup R$ is a RSRL for finite $R$. (3) In the latter case, the resulting RSRL requires in general a different language substitution.

*Proof.* (1) Let $R = (L(\delta_1\delta_2), \varphi)$ with $\varphi(\delta_1) = \{a\}$ and $\varphi(\delta_2) = L(a + \varepsilon)$ and let $R = \{b\}$. Then, $R \cup R = \{\{b\} \cup \{a^i \mid 1 \leq i \leq n+1\} \mid n \in \mathbb{N}\}$ which is not a RSRL, as shown in Example 15. (2) By Fact 5. (3) Let $R = (\{\delta\}, \varphi)$ with $\Delta = \{\delta\}$, $\Sigma = \{a, b\}$, $\varphi(\delta) = \{a\}$ and let $R = \{b\}$. Then, $R \cup R = \{\{a, b\}\}$, which is inexpressible with $\varphi$. □
3.4 Intersection

Proposition 17 (Closure of Intersection). Let \( R_1 \) and \( R_2 \) be two finite RSRLs using the same language substitution \( \varphi \). Then, \( R_1 \cap R_2 \) is a finite RSRL which can be expressed using the language substitution \( \varphi \).

Proof. We can enumerate each word \( w_1 \in K_1 \) and check whether there is a word \( w_2 \in K_2 \) such that \( \varphi(w_1) = \varphi(w_2) \). If so, we keep \( w_1 \) in a new set \( K_3 = \{ w_1 \in K_1 \mid \exists w_2 \in K_2. \varphi(w_1) = \varphi(w_2) \} \) and \( (K_3, \varphi) = R_1 \cap R_2 \).

In general, RSRLs are not closed under point-wise intersection but they are closed under point-wise intersection when restricting to finite RSRLs.

Proposition 18 (Closure of Point-wise Intersection). (1) RSRL are not closed under point-wise intersection. (2) For finite \( R \) \( R \cap R \) is a finite RSRL, (3) in general requiring a different language substitution.

Proof. (1) Let \( R = (K, \varphi) \) with \( K = L(\delta \delta^*) \) and \( \varphi(\delta) = L(a + b) \), and set \( R = L(a^* + b) \). Then \( R \cap R = \{ b \} \cup \{ a^i \mid 1 \leq i \leq n + 1 \} \mid n \in \mathbb{N} \}. \) In Example 15, we showed that \( R \cap R \) is not a RSRL. (2) By Fact 5. (3) Let \( R = (K, \varphi) \) with \( K = \{ \delta \} \) and \( \varphi(\delta) = L(a + b) \), and set \( R = L(a^* + b) \). Then, \( R \cap R = \{ L(a + b) \} \) which in inexpressible via \( \varphi \).

3.5 Set Difference

Proposition 19 (Closure of Difference). For finite \( R_1 \) and \( R_2 \), \( R_1 - R_2 \) is a finite RSRL, expressible as \( (K_3, \varphi) \), for some \( K_3 \subseteq K_1 \).

Proof. Set \( K_3 = \{ w \in K_1 \mid \varphi(w) \in R_2 \} \) and the claim follows.

Proposition 20 (Closure of Point-wise Difference). (1) In general, \( R - R \) is not a RSRL. (2) \( R - R \) is a finite RSRL for finite \( R \), (3) requiring in general a different language substitution.

Proof. (1) Let \( R = (L(\delta_1 \delta_2^*), \varphi) \) with \( \varphi(\delta_1) = L(a + b) \) and \( \varphi(\delta_2) = L(a + b + \varepsilon) \). Let \( R = L(\delta_1 \delta_2^* \in \varphi(a + b)^* \in \varphi(a + b)^* ba(a + b)^*) \). Then, \( R - R = \{ b \} \cup \{ a^i \mid 1 \leq i \leq n + 1 \} \mid n \in \mathbb{N} \} \) which is not a RSRL (see Example 15). (2) By Fact 5. (3) Let \( R = (\{ \delta_a \}, \varphi) \) with \( \varphi(\delta_a) = \{ a \} \) and let \( R = \{ a \} \). Then, \( R - R = \{ \emptyset \} \), requiring a new symbol.

Proposition 21 (Closure of Symmetric Difference). Let \( R_1 \) and \( R_2 \) be finite RSRLs using the same language substitution \( \varphi \). Then, \( R_1 \Delta R_2 \) is a finite RSRL and can be expressed using the language substitution \( \varphi \).

Proof. The proof follows immediately from the closure properties of union, intersection, and difference.
Algorithm 1: membership\((R, K, \varphi)\)

\[
\begin{align*}
\text{input} & : \text{regular languages } R \subseteq \Sigma^*, K \subseteq \Delta^*, \\
& \quad \text{regular language substitution } \varphi \text{ with } \varphi(\delta) \subseteq \Sigma^* \text{ for all } \delta \in \Delta, \text{ and} \\
& \quad \text{all as regular expressions} \\
\text{returns} & : \text{true iff } \exists w \in K : \varphi(w) = R \text{ (i.e., iff } R \in (K, \varphi)) \\
1 & \text{foreach } M' \in \text{enumerate}(R, K, \varphi) \text{ do} \\
2 & \quad \text{if basiccheck}(R, M', \varphi) \text{ then return true;} \\
3 & \quad \text{return false};
\end{align*}
\]

4 Decision Problems

Given a a regular language \(R \subseteq \Sigma^*\) and a RSRL \(R = (K, \varphi)\) over the alphabets \(\Delta\) and \(\Sigma\), the membership problem is to decide whether \(R \in R\) holds. Given another \(R' = (K', \varphi')\), also over the alphabets \(\Delta'\) and \(\Sigma\), the inclusion problem asks whether \(R \subseteq R'\) holds, and the equivalence problem, whether \(R = R'\) holds.

Theorem 22 (Equivalence, Inclusion, and Membership for Kleene star free RSRLs). Membership, inclusion, and equivalence are PSPACE-complete for Kleene star free RSRLs.

This holds true, since in case of Kleene star free RSRLs, we can enumerate the regular expressions defining all member languages in PSPACE. Given the PSPACE-completeness of regular language equivalence, we compare a given regular expression with all member languages, solving the membership problem in PSPACE. Doing so for all languages of another RSRL solves the inclusion problem, and checking mutual inclusion yields an algorithm for equivalence. This approach does not immediately generalize to finite RSRLs, since finite RSRLs \(R = \{\varphi(w) \mid w \in K\}\) may be generated from an infinite \(K\) with Kleene stars.

In the general case, the situation is quite different: Previous work shows that the membership problem is decidable [8]. Taking this work as starting point, we give a first 2ExpSpace upper bound on the complexity of the problem. A corresponding lower bound is missing, however we expect the problem to be at least ExpSpace-hard. Due to space reasons, we only give an overview on the algorithm in the paper and must defer its details to the appendix. Finally, the decidability of inclusion and equivalence are still open problems.

4.1 Membership for general RSRLs

By definition, the membership problem is equivalent to asking whether there exists a \(w \in K\) with \(\varphi(w) = R\). For checking the existence of such a \(w\), we have to check possibly infinitely many words in \(K\) efficiently. To render this search feasible, we (A) rule out irrelevant parts of \(K\), and (B) treat subsets of \(K\) at once. This leads to the procedure \(\text{membership}(K, R, \varphi)\) shown in Algorithm 1, which first enumerates with \(M' \in \text{enumerate}(K, R, \varphi)\) a sufficient set of sublanguages...
We restrict the search space further by checking the minimal word length, i.e., we set $M = \phi^{-1}(\{w \mid \text{minlen}(w) = \text{minlen}(R)\})$. Furthermore, all subsets $M \subseteq M_\phi(R)$ are called rewritings of $R$, and if $\phi(M) = R$ holds, $M$ is called exact rewriting.

**Proposition 23 (Regularity of maximal rewritings [17])**. Let $\phi : \Delta \rightarrow 2^{\Sigma^*}$ be a regular language substitution. Then the maximal $\phi$-rewriting of a regular language $R \subseteq \Sigma^*$ is a regular language over $\Delta$.

As all words $w$ with $\phi(w) = R$ must be element of $M_\phi(R)$, we restrict our search to $M = M_\phi(R) \cap K$.

**(A.1) Maximal Rewriting.** To rule out all $w$ with $\phi(w) \not\subseteq R$, we rely on the notion of a maximal $\phi$-rewriting $M_\phi(R)$ of $R$, taken from [17]. $M_\phi(R)$ consists of the words $w$ with $\phi(w) \subseteq R$, i.e., we set $M_\phi(R) = \{w \in \Delta^+ \mid \phi(w) \subseteq R\}$. Furthermore, all subsets $M \subseteq M_\phi(R)$ are called rewritings of $R$, and if $\phi(M) = R$ holds, $M$ is called exact rewriting.

**(A.2) Minimal Word Length.** We restrict the search space further by checking the minimal word length, i.e., we compare the length of the respectively shortest word in $R$ and $\phi(w)$. If $R$ and $\phi(w)$ have different minimal word lengths, $R \neq \phi(w)$ holds, and hence, we rule out $w$. We define the minimal word length minlen($L$) of a language $L$ with minlen($L$) = min{$|w| \mid w \in L$}, leading to the definition of language strata.

**Definition 24 (Language Stratum).** Let $L$ be a language over $\Delta$, and $\phi : \Delta \rightarrow 2^{\Sigma^*}$ be a regular language substitution, then the $B$-stratum of $L$, denoted as $L[B, \phi]$, is the set of words in $L$ which generate via $\phi$ languages of minimal word length $B$, i.e., $L[B, \phi] = \{w \in L \mid \text{minlen}(\phi(w)) = B\}$.

Starting with $M = M_\phi(R) \cap K$, we restrict our search further to $M[\text{minlen}(R), \phi]$.

**(B) 1-Word Summaries.** It remains to subdivide $M[\text{minlen}(R), \phi]$ into finitely many subsets $M'$, which are then checked efficiently without enumerating their words $w \in M'$. Here, we only discuss the property of these subsets $M'$ which enables such an efficient check, and later we will describe an enumeration of those subsets $M'$. When we check a subset $M'$, we do not search for a single word $w \in M'$ with $\phi(w) = R$ but for a finite set $F \subseteq M'$ with $\phi(F) = R$. The soundness of this approach will be guaranteed by the existence of 1-word summaries: A language $M' \subseteq \Delta^*$ has 1-word summaries, if for all finite subsets
Proposition 25 (Membership Condition for Summarizable Languages).
Let $M' \subseteq \Delta^*$ be a regular language with 1-word summaries and $\varphi(M') \subseteq R$. Then there exists a $w \in M'$ with $\varphi(w) = R$ iff there exists a finite subset $F \subseteq M'$ with $\varphi(F) = \varphi(M') = R$.

Putting it together. First, combining A.2 and B, we obtain Lemma 26, to subdivide the search space $M[B, \varphi]$ into a set $\text{rep}(M, B, \varphi)$ of languages $M'$ with 1-word summaries. Second, in Theorem 27, building upon Lemma 26 and A.1, we fix $B = \text{minlen}(R)$ and iterate through these languages $M'$. We check each of them at once with our membership condition from Proposition 25. In terms of Algorithm 1, Lemma 26 provides the foundation for $\text{enumerate}(K, R, \varphi)$ and Proposition 25 underlies $\text{basiccheck}(R, M', \varphi)$.

Lemma 26 (Summarizable Language Representation, adapting [8]).
Let $M \subseteq \Delta^*$ be a regular language. Then, for each bound $B \geq 0$, there exists a family $\text{rep}(M, B, \varphi)$ of union-free regular languages $M' \in \text{rep}(M, B, \varphi)$ with 1-word summaries, such that $M[B, \varphi] \subseteq \bigcup_{M' \in \text{rep}(M, B, \varphi)} M' \subseteq M$ holds.

Theorem 27 (Membership Condition, following [8]). Let $R = (K, \varphi)$ be a RSRL and $\varphi : \Delta \to 2^{\Sigma^*}$ be a regular language substitution. Then, for a regular language $R \subseteq \Sigma^*$, we have $R \in R$, iff there exists an $M' \in \text{rep}(M, \varphi) \cap K, \text{minlen}(R), \varphi$ with a finite subset $F \subseteq M'$ with $\varphi(F) = \varphi(M') = R$.

We obtain the space complexity of membership, depending on the size of the expressions, representing the involved languages. More specifically, we use the expression sizes $||R||$ and $||K||$ and the summed size $||\varphi|| = \sum_{\delta \in \Delta} ||\varphi(\delta)||$ of the expressions in the co-domain of $\varphi$.

Theorem 28 (Membership($R, K, \varphi$) runs in 2ExpSpace). More precisely, it runs in $\text{DSpace}(||K||^r 2^{|R|+||\varphi||^s})$ for some constants $r$ and $s$.

5 Conclusion
Motivated by applications in testcase specifications with FQL, we have studied general and finite RSRLs. While we showed that general RSRLs are not closed under most common operators, finite RSRLs are closed under all operators except Kleene stars and complementation (Theorem 4). This shows that our restriction to Kleene star free and hence finite RSRLs in FQL results in a natural framework with good closure properties. Likewise, the proven PSPACE-completeness results for Kleene star free RSRLs provide a starting point to develop practical reasoning procedures for Kleene star free RSRLs and FQL. Experience with LTL model checking shows that PSPACE-completeness often leads to algorithms which are feasible in practice. In contrast, for general and
possibly infinite RSRLs, we have described a \( 2\text{ExpSpace} \) membership checking algorithm – leaving the question for matching lower bounds open. Nevertheless, reasoning on general RSRLs seems to be rather infeasible.

Last but not least, RSRLs give rise to new and interesting research questions, for instance the decidability of inclusion and equivalence for general RSRLs, and the closure properties left open in this paper. In our future work, we want to generalize RSRLs to other base formalisms. For example, we want \( \varphi \) to substitute symbols by context-free expressions, thus enabling FQL test patterns to recognize e.g. matching of parentheses or emptiness of a stack.

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We deal with Cartesian binary operators generically, by reducing the point-wise operators to the Cartesian one.

**Lemma 29 (Reducing Point-Wise to Cartesian Operators).** Let \( \circ \) be an arbitrary binary operator over sets, let \( \triangleright \in \{ \cup, \cap, - \} \), and let \( \otimes \in \{ \cup, \cap, \times \} \).

1. If \( R_1 \triangleright R \) is not closed under rational sets of regular languages, then the corresponding \( R_1 \otimes R_2 \) is not closed.
2. If \( R_1 \triangleright R \) is not closed under finite rational sets of regular languages with constant language substitution, even in presence of a symbol \( \delta_R \) with \( \varphi(\delta_R) = R \), then the corresponding \( R_1 \otimes R_2 \) is also not closed.

**Proof.** (1) If \( R_1 \triangleright R \) is not closed, we fix a violating pair \( R_1 \) and \( R \). Then we obtain \( R_1 \otimes R_2 = R_1 \triangleright R \) for \( R_2 = (\{\delta_R\}, \varphi) \) and \( \varphi(\delta_R) = R \). Since \( R_1 \triangleright R \) is not a RSRL, \( R_1 \otimes R_2 \) is not as well, and the claim follows. (2) If \( R_1 \triangleright R \) is inexpressible as a RSRL without introducing new symbols in \( \varphi \), even in presence of \( \delta_R \), then \( R_1 \otimes R_2 \) is also inexpressible without changing \( \varphi \). \( \square \)

Given Lemma 29, it is not surprising that point-wise and Cartesian operators behave for all discussed underlying binary operators identically, as shown in Theorem 4.

**Corollary 30 (Closure of Cartesian Binary Operators).** Let \( \otimes \in \{ \cup, \cap, \times \} \).

1. The set \( R_1 \otimes R_2 \) is, in general, not a rational set of regular languages.
2. The set \( R_1 \otimes R_2 \) is a rational set of regular languages if \( R_1 \) and \( R_2 \) are finite, (3) requiring in general a new language substitution.

**Proof.** (1) By Lemma 29 we reduce the point-wise case to the Cartesian case, covered by Propositions 16, 18, and 20 for union, intersection, and set difference, respectively. The claim follows. (2) Since all considered operators are closed for regular languages, the claim follows from Fact 5. (3) Again, with Lemma 29 we reduce the point-wise case to the Cartesian case. The lemma is applicable, as the examples in the proofs of Propositions 16, 18, and 20 are not jeopardized by a symbol \( \delta_R \) with \( \varphi(\delta_R) = R \). Hence the claim follows. \( \square \)
B Implementation and Proofs for Section 4

B.1 Implementing basiccheck($R, M', \varphi$)

Since Lemma 26 produces only languages $M' = N_1S_1N_2 \ldots N_mS_mN_{m+1}$ with 1-word summaries, we restrict our implementation to such languages and exploit these restrictions subsequently. So, given such a language $M'$ over $\Delta$, and a regular language substitution $\varphi : \Delta \rightarrow 2^{\Sigma^*}$, we need to check whether there exists a finite $F \subseteq M'$ with $\varphi(F) = \varphi(M') = R$. We implement this check with the procedure basiccheck($R, M', \varphi$), splitting the condition of Proposition 25 into two parts, namely (1) whether there exists a finite $F \subseteq M'$ with $\varphi(F) = \varphi(M')$, and (2) whether $\varphi(M') = R$ holds. While the latter condition amounts to regular language equivalence, the former requires distance automata as additional machinery.

**Definition 31 (Distance Automaton [8]).** A distance automaton over an alphabet $\Delta$ is a tuple $A = (\Delta, Q, \rho, q_0, F, d)$ where $(\Delta, Q, \rho, q_0, F)$ is an NFA and $d : \rho \rightarrow \{0, 1\}$ is a distance function, which can be extended to a function on words as follows. The distance function $d(\pi)$ of a path $\pi$ is the sum of the distances of all edges in $\pi$. The distance $\mu(w)$ of a word $w \in L(A)$ is the minimum of $d(\pi)$ for all paths $\pi$ accepting $w$.

A distance automaton $A$ is called limited if there exists a constant $U$ such that $\mu(w) < U$ for all words $w \in L(A)$.

In our check for (1), we build a distance automaton which is limited iff a finite $F$ with $\varphi(F) = \varphi(M')$ exists. Then, we rely on the PSPACE-decidability [18] of the limitedness of distance automata to check whether $F$ exists or not.

**Distance-automaton Construction.** Here, we exploit the assumption that $M'$ is a union-free language over $\Delta$: Given the regular expression defining $M'$, we construct the distance automaton $A_{M'}$ following the form of this regular expression:

- $\delta \in \Delta$: We construct the finite automaton $A_{\delta}$ with $L(A_{\delta}) = \varphi(\delta)$. We extend $A_\delta$ to a distance automaton by labeling each transition in $A_\delta$ with 0.
- $e \cdot f$: Given the distance automata $A_e = (Q_e, \Sigma, \rho_e, q_{0,e}, F_e, d_e)$ and $A_f = (Q_f, \Sigma, \rho_f, q_{0,f}, F_f, d_f)$, we set $A_{e \cdot f} = (Q_e \cup Q_f, \Sigma, \rho_e \cup \rho_f \cup \rho, q_{0,e}, F_e \cup F_f, d_{e \cdot f})$ where $\rho = \{(q, \varepsilon, q_{0,f}) | q \in F_e\}$ and $d_{e \cdot f} = d_e \cup d_f \cup \{(t, 0) | t \in \rho\}$, i.e., we connect each final state of $A_e$ to the initial state of $A_f$ and assign the distance 0 to these connecting transitions.
- $e^*$: We construct the distance automaton $A_e = (Q_e, \Sigma, \rho_e, q_{0,e}, F_e, d_e)$. Then, $A_{e^*} = (Q_e, \Sigma, \rho_e \cup \rho, q_{0,e}, F_e \cup \{q_{0,e}\}, d_{e^*})$, where $\rho = \{(q, \varepsilon, q_{0,e}) | q \in F_e\}$ and $d_{e^*} = d_e \cup \{(q, \varepsilon, p) | (q, \varepsilon, p) \in \rho\}$, i.e., we connect each final state of $A_e$ to the initial states of $A_e$ and assign the corresponding transitions the distance 1.

If the resulting distance automaton $A_{M'}$ is limited, then there exists a finite subset $F \subseteq M'$ such that $\varphi(F) = \varphi(M')$. This implies that (1) holds.
Algorithm 2: `basiccheck`\((R, M', \varphi)\)

**input**: regular languages \(R \subseteq \Sigma^*\), \(M' \subseteq \Delta^*\), and regular language substitution \(\varphi\) with \(\varphi(\delta) \subseteq \Sigma^*\) for all \(\delta \in \Delta\), all given as regular expressions

**requires**: \(M'\) is of form \(N_1 S_1 N_2 \ldots N_m S_m N_{m+1}\) with \(N_h, S_h \in \Delta^*\)

**returns**: \(\text{true}\) if \(\exists\) finite \(F \subseteq M'\) : \(\varphi(F) = \varphi(M') = R\)

1. build \(A_{M'}\);
2. if \(A_{M'}\) limited then
3. if \(\varphi(M') = R\) then return \(\text{true}\);
4. return \(\text{false}\);

Algorithm 3: `enumerate`\((R, K, \varphi)\)

**input**: regular languages \(R \subseteq \Sigma^*\), \(K \subseteq \Delta^*\), regular language substitution \(\varphi\) with \(\varphi(\delta) \subseteq \Sigma^*\) for all \(\delta \in \Delta\), and all given as regular expressions

**yields**: \(L \in \text{rep}(M, \text{minlen}(R), \varphi)\) for \(M = M_\varphi(R) \cap K\)

1. \(M := M_\varphi(R) \cap K\);
2. for \(L \in \text{unionfreedecomp}(M)\) do unfold\((L, \varphi, \text{minlen}(R))\);

So, given \(M', R\), and all languages in the domain of \(\varphi\) as regular expressions, `basiccheck`\((R, M', \varphi)\) in Algorithm 2 first builds \(A_{M'}\) (Line 1) and checks its limitedness (Line 2), amounting to condition (1). For condition (2), `basiccheck` verifies that \(\varphi(M')\) and \(R\) are equivalent (Line 3) and returns \(\text{true}\) if both checks succeed.

**Lemma 32** (`basiccheck`\((R, M', \varphi)\) runs in PSpace). `basiccheck`\((R, M', \varphi)\) runs in PSPACE, which is optimal, as it solves a PSPACE-complete problem.

### B.2 Implementing `enumerate`(\(K, R, \varphi\))

Our enumeration algorithm must produce the languages \(\text{rep}(M, B, \varphi)\), guaranteeing that all \(M' \in \text{rep}(M, B, \varphi)\) have 1-word summaries, and that \(M[B, \varphi] \subseteq \bigcup_{M' \in \text{rep}(M, B, \varphi)} M' \subseteq M\) holds (as specified by Lemma 26). To this end, we rely on a sufficient condition for the existence of 1-word summaries. First we show this condition with Proposition 33, before turning to the enumeration algorithm itself.

**Proposition 33** (Sufficient Condition for 1-Word Summaries). Let \(L\) be a union-free language over \(\Delta\), given as \(L = N_1 S_1^* N_2 \ldots N_m S_m^* N_{m+1}\), with words \(N_h \in \Delta^*\) and union-free languages \(S_h \subseteq \Delta^*\). If \(\varepsilon \in \varphi(w)\) for all \(w \in S_h\) and all \(S_h\), then \(L\) has 1-word summaries.

We are ready to design our enumeration algorithm, shown in Algorithm 3, and its recursive subprocedure in Algorithm 4. Both algorithms do not return a
Algorithm 4: unfold($L, \varphi, B$)

\begin{algorithmic}[1]
\State \textbf{input} : regular language $L = N_1 S_1^* N_2 \ldots N_m S_m^* N_{m+1} \subseteq \Delta^*$, 
\hspace{1cm} regular language substitution $\varphi$ with $\varphi(\delta) \subseteq \Sigma^*$ for all $\delta \in \Delta$, and 
\hspace{1cm} a bound $B$
\State \textbf{yields} : $L' \in \text{rep}(L, B, \varphi)$
\State \textbf{if} $\forall S_h \forall w \in S_h$ : $\varepsilon \in \varphi(w)$ \textbf{then} yield $L$;
\State \textbf{else}
\State \hspace{1cm} fix $S_h$ arbitrarily with $\exists w \in S_h$ : $\varepsilon \notin \varphi(w)$;
\State \hspace{1cm} $E := S_h \cap \Delta^*_e$; \hspace{1cm} // $\Delta_e = \{ \delta \in \Delta \mid \varepsilon \in \varphi(\delta) \}$
\State \hspace{1cm} $L_0 := N_1 S_1^* N_2 \ldots N_h E^* N_{h+1} \ldots N_m S_m^* N_{m+1}$;
\State \hspace{1cm} unfold($L_0, \varphi, B$);
\State \hspace{1cm} // $L_p := N_1 S_1^* N_2 \ldots N_h E^* \hat{E}_p S_h^* N_{h+1} \ldots N_m S_m^* N_{m+1}$ (see text)
\State \hspace{1cm} for $p \in \text{critical}(S_h)$ with $\minlen(\varphi(L_p)) \leq B$ \textbf{do} unfold($L_p, \varphi, B$);
\end{algorithmic}

result but yield their result as an enumeration: Upon invocation, both algorithms run through a sequence of \textbf{yield} statements, each time appending the argument of \textbf{yield} to the enumerated sequence. Thus, the algorithm never stores the entire sequence but only the stack of the invoked procedures.

Initializing the recursive enumeration, Algorithm 3 obtains the maximum recursive rewriting $M := M_\varphi(R) \cap K$ of $R$ (Line 1) and iterates over the languages $L$ in the union-free decomposition of $M$ (Line 2) to call for each $L$ the recursive procedure unfold, shown in Algorithm 4. In turn, Algorithm 4 takes a union free language $L = N_1 S_1^* N_2 \ldots N_m S_m^* N_{m+1}$ and a bound $B$ to unfold the Kleene-star expressions of $L$ until the precondition of Proposition 33 is satisfied or $\minlen(\varphi(L)) > B$.

More specifically, unfold exploits a rewriting, based on the following terms: Given a union free language $S_h$, let $E = S_h \cap \Delta^*_e$ with $\Delta_e = \{ \delta \in \Delta \mid \varepsilon \in \varphi(\delta) \}$ denote all words $w$ in $S_h$ with $\varepsilon \in \varphi(w)$ and let $\hat{E} = S_h \setminus E$. Since $\hat{E}$ is in general not union free, we need to split $\hat{E}$ further. To this end, we define $\text{ufs}(S_h, p)$ recursively for an integer sequence $p = \langle p_H, p_T \rangle$ with head element $p_H$ and tail sequence $p_T$. Intuitively, a sequence $p$ identifies a subexpression in $S_h$ by recursively selecting a nested Kleene star expression; $\text{ufs}(S_h, p)$ unfolds $S_h$ such that this selected expression is instantiated at least once. Formally, for $S_h = \alpha_1 \beta_1^* \alpha_2 \ldots \alpha_n \beta_n^* \alpha_{n+1}$ we set $\text{ufs}(S_h, \varepsilon) = S_h$ and $\text{ufs}(S_h, p) = \alpha_1 \ldots \alpha_{p_H} \beta_{p_H}^* \text{ufs}(\beta_{p_H}^* \alpha_{p_H+1} \ldots \alpha_{n+1})$. Consider $S_h = A^* (B^* C^*)^* D^*$ (with all $\alpha_i = \varepsilon$ for brevity), then we obtain $\text{ufs}(S_h, \langle 2, 1 \rangle) = A^* (B^* C^*)^* \text{ufs}(B^* C^* \langle 1 \rangle) (B^* C^*)^* D^*$ $\quad \quad = A^* (B^* C^*)^* (B^* \text{ufs}(B, \varepsilon) B^* C^*) (B^* C^*)^* D^*$ $\quad \quad = A^* (B^* C^*)^* (B^* \quad \quad \quad \quad \quad \quad \quad \quad (B^* \quad \quad \quad \quad \quad \quad \quad (B^* C^*)^*)^* D^*$ instantiating $B$ at position $\langle 2, 1 \rangle$ at least once. Let $\text{critical}(S_h)$ be integer sequences which identify a subexpression of $S_h$ which directly contain a symbol $\delta$ with $\varepsilon \notin \varphi(\delta)$ (and not only via another Kleene-star expression). Then, we write $\hat{E} = \bigcup_{p \in \text{critical}(S_h)} \hat{E}_p$, with $\hat{E}_p = \text{ufs}(S_h, p)$. This discussion leads to the following rewriting:
Proposition 34 (Rewriting for 1-Word Summaries). For every union free language \( S_h \), we have \( S_h = E^* \cup \bigcup_{p \in \text{critical}(S_h)} E^* F_p S_h \). All languages in the rewriting, i.e., \( E^* \) and \( E^* F_p S_h \), are union free, \( E^* \) has 1-word summaries, and minlen\( (S_h) \) < minlen\( (E^* F_p S_h) \) holds for all \( p \in \text{critical}(S_h) \).

If \( L \) already satisfies the precondition imposed by Proposition 33, Algorithm 4 yields \( L \) and terminates (Line 1). Otherwise, it fixes an arbitrary \( S_h \) violating this precondition and rewrites \( L \) recursively with Proposition 34 (Lines 3-7).

(1) **Termination:** In each recursive call, unfold either eliminates in \( L_0 \) an occurrence of a subexpression \( S_h \) violating the precondition of Proposition 33 (Line 6), or increases the minimum length in \( L_p \), eventually running into the upper bound \( B \) (Line 7).

(2) **Correctness:** Setting \( B = \infty \), unfold yields a possibly infinite sequence of union free languages which have 1-word summaries such that their union equals the original language \( L \). As the generation of these languages is based on the equality of Proposition 34 each rewriting step is sound and complete, leading to an infinite recursion tree whose leaves yield the languages in the sequence. The upper bound on minimum length only cuts off languages \( L_p \) producing words of minimum length beyond \( B \), i.e., \( L_p \cap L[B, \varphi] = \emptyset \), and in consequence, it is safe to drop \( L_p \), since we only need to construct \( \text{rep}(L, B, \varphi) \) with \( \text{rep}(L, B, \varphi) \supseteq L[B, \varphi] \).

### B.3 Proofs

**Proof (of Theorem 22).** PSPACE-Membership. We exploit for the PSPACE-membership of all three considered problems the same observations: (1) Given Kleene star free languages \( K \), we can enumerate in PSPACE all words \( w \in K \), and (2) we can check whether \( L(R) = L(\varphi(w)) \) holds, in PSPACE [19].

Thus, to check membership of \( R \) in \((K, \varphi)\), we enumerate all \( w \in K \) and check whether \( L(R) = L(\varphi(w)) \) holds for some \( w \) — if so, \( R \in \mathcal{R} \) is true. For checking the inclusion \( R' \subseteq R \), we enumerate all \( w' \in K' \) and search in a nested loop for a \( w \in K \) with \( L(\varphi(w)) = L(\varphi(w')) \). If such a \( w \) exists for all \( w' \), we have established \((K', \varphi') \subseteq (K, \varphi)\). We obtain PSPACE-membership for equivalence \((K', \varphi') = (K, \varphi)\) by checking both, \((K', \varphi') \subseteq (K, \varphi)\) and \((K, \varphi) \subseteq (K', \varphi')\).

**Hardness.** For hardness we reduce the PSPACE-complete problem whether a given regular expression \( X \subseteq \Sigma^* \) is equivalent to \( \Sigma^* \) [19] to all three considered problems: Given an arbitrary regular expressions \( X \), we set \( K = \{a\}, \varphi(a) = X \), \( K' = \{b\}, \varphi'(b) = \Sigma^* \), and \( R = \Sigma^* \). This gives us \( X = \Sigma^* \) iff \((K, \varphi) = (K', \varphi')\) (equivalence) iff \((K, \varphi) \subseteq (K', \varphi')\) (inclusion) iff \( R \in (K, \varphi) \) (membership).

**Proof (of Proposition 25).** (\( \Rightarrow \)) With \( w \in M' \) and \( \varphi(w) = R \), taking \( F = \{w\} \subseteq M' \), we obtain \( R = \varphi(w) = \varphi(F) \subseteq \varphi(M') \subseteq \varphi(M) \subseteq R \), as required.

(\( \Leftarrow \)) \( M' \) has 1-word summaries, hence there exists a \( w \in M' \) with \( \varphi(F) \subseteq \varphi(w) \), leading to \( R = \varphi(F) \subseteq \varphi(w) \subseteq \varphi(M') \subseteq \varphi(M) \subseteq R \), as required.

**Proof (of Lemma 26).** We prove the Lemma with Algorithm 4. unfold\( (L, \varphi, B) \) yields \( \text{rep}(L, B, \varphi) \) for union free languages \( L \), hence we obtain \( \text{rep}(M, B, \varphi) = \bigcup_{L \in \text{unionfreedecomplex}(M)} \text{unfold}(L, \varphi, B) \).
Proof (of Theorem 27). Most of the work for the proof of Theorem 27 is already achieved by the representation $\text{rep}(M, \text{minlen}(R), \varphi)$ of Lemma 26: The languages $M' \in \text{rep}(M, \text{minlen}(R), \varphi)$ are constructed to have 1-word summaries, which make the check whether there exists $w \in M'$ with $\varphi(w) = R$ relatively easy – this is the case iff there exists a finite subset $F \subseteq M'$ with $\varphi(F) = \varphi(M') = R$. We show both directions of the theorem statement individually.

$(\Rightarrow)$ Assume $R \in \mathcal{R}$: By Definition 1, there exists $w \in K$ with $R = \varphi(w)$, by Definition 23, we get $w \in M_\varphi(R)$, and hence $w \in M_\varphi(R) \cap K = M$. From $R = \varphi(w)$ and $\text{minlen}(R) = \text{minlen}(\varphi(w))$, we get $w \in M[\text{minlen}(R), \varphi]$. Since the maximal rewriting $M_\varphi(R)$ of a regular language $R$ is regular as well [17], and since regular languages are closed under intersection, we obtain the regularity of $M$, and hence, Lemma 26 applies. Thus, there exists an $M' \in \text{rep}(M, \text{minlen}(R), \varphi)$ with $w \in M'$, and via Proposition 25, we obtain for $F = \{w\} \subseteq M'$, $R = \varphi(F) = \varphi(M')$, as required.

$(\Leftarrow)$ Assume that there exists an $M' \in \text{rep}(M, \text{minlen}(R), \varphi)$ with a finite subset $F \subseteq M'$ with $\varphi(F) = \varphi(M') = R$. Then, via Proposition 25, we take the summary word $w \in M'$ for $F$, yielding $R \in \mathcal{R}$, as required. \hfill \Box

Proof (of Lemma 32). Membership. The construction of the automaton $A_{M'}$ (Line 1) runs in polynomial time and hence produces a polynomially sized distance automaton. Thus, the check for limitedness of $A_{M'}$ (Line 2) retains its PSPACE complexity [19]. Given $M'$, $R$, and all $\varphi(\delta)$ for $\delta \in \Delta$ as regular expressions, we can build a polynomially sized regular expression for $\varphi(M')$ by substituting $\varphi(\delta)$ for each occurrence of $\delta$ in $M'$. Then we check the equivalence of the regular expressions for $\varphi(M')$ and $R$ (Line 3), again keeping the original PSPACE complexity of regular expression equivalence [19]. This yields an overall PSPACE procedure.

Hardness. We reduce the PSPACE-complete problem of deciding whether a regular expression $X$ over $\Sigma$ is equivalent to $\Sigma^*$ [19] to a single basiccheck invocation – proving that basiccheck solves a PSPACE complete problem. Given an arbitrary regular expressions $X$, we set $M' = \{a\}$, $\varphi(a) = X$ and $R = \Sigma^*$. Then basiccheck$(R, M', \varphi)$ returns true iff $X$ is equivalent to $\Sigma^*$. \hfill \Box

Proof (of Proposition 33). We construct the desired word: Choose an arbitrary finite subset $F = \{f_1, \ldots, f_p\} \subseteq L$. Then each word $f_i \in F$ is of the form

$$f_i = N_1s_{1,i}N_2 \cdots N_m s_{m,i} N_{m+1}$$

with $s_{h,i} \in S_h^*$. We set $s_{h,F} = s_{h,1} \cdot s_{h,2} \cdots s_{h,p}$, and observe, because of $\varepsilon \in \varphi(w)$ for all $w \in S_h$ and $S_h$, $\varphi(s_{h,i}) = \varepsilon \varphi(s_{h,i}) \in \varphi(s_{h,1}) \cdots \varphi(s_{h,i-1}) \varphi(s_{h,i}) \varphi(s_{h,i+1}) \cdots \varphi(s_{h,p}) = \varphi(s_{h,F})$.

Thus we choose the summary word $w = N_1s_{1,F}N_2 \cdots N_{m}s_{m,F} N_{m+1}$ and obtain $\varphi(f_i) = \varphi(N_1s_{1,i}N_2 \cdots N_{m}s_{m,i}N_{m+1}) \subseteq \varphi(w)$, and hence $\varphi(F) \subseteq \varphi(w)$. \hfill \Box

Proof (of Proposition 34). We have $S_h^* = E^*(\tilde{E}E)^* E^* \cup E^* \tilde{E} E^*(\tilde{E}E)^* = E^* \cup E^* \tilde{E} S_h^*$ and find the desired result by substituting $\tilde{E} = \bigcup_{p \in \text{critical}(S_h)} E_p$, as discussed before Proposition 34.
(1) **Union freeness:** We construct the regular expression for $E^*$ by dropping all Kleene-starred subexpressions in $S^*_h$ which contain a symbol $\delta$ with $\varepsilon \not\in \varphi(\delta)$ (possibly producing the empty language), preserving union freeness. The construction of $\bar{E}_p$ only unrolls Kleene star expressions, also preserving the union freeness from $S_h$. (2) **1-word summaries for $E^*$:** For all $w \in E$, we have $\varphi(w) = \varepsilon$, since all symbols $\delta$ in $E$ have $\varepsilon \in \varphi(\delta)$. (3) **Increasing minimal length in $E^*\bar{E}_pS^*_h$:** Since $S^*_h$ is a subexpression of $E^*\bar{E}_pS^*_h$ the minimal length can only increase, and since $\bar{E}_p$ instantiates an expression with a symbol $\delta$ and $\varepsilon \not\in \varphi(\delta)$, it actually increases. □

**Proof of Theorem 28.** The proof is based on the complexity of the maximum rewriting from [17] and the complexity of unfold, shown first via Propositions 35 and 36, before proving the overall complexity of **enumerate** in Lemma 37. This Lemma, together with Lemma 32, leads to the desired theorem.

Recall the definition of $ufs$ before Proposition 34 for $L = \alpha_1 \beta_1^* \alpha_2 \ldots \alpha_n \beta_n^* \alpha_{n+1}$ and $p = (p_M | p_R)$ with $ufs(L, p) = \alpha_1 \ldots \alpha_{p_H} \beta_{p_H}^* \alpha_{p_H+1} \ldots \alpha_{n+1}$. We denote with $||L||$ the length of the regular expression representing $L$.

**Proposition 35 (An Upper Bound for $||ufs(L, p)||$).** Let $K$ be the maximum length of a Kleene star subexpression in $L$. Then $||ufs(L, p)|| = O(K||L||)$ holds.

**Proof.** $ufs$ duplicates $\beta_{p_H}$ of $L$ and continues recursively on a third copy of $\beta_{p_H}$. Since $ufs$ does not introduce new Kleene star subexpressions but only duplicates some, all Kleene star expressions occurring during the entire recursion are at most of length $K$. Hence, each recursive step of $ufs$ adds at most $2K$ to the entire expression, and because the Kleene star nesting depth of at most $L$, we obtain $||ufs(L, p)|| = O(K||L||)$. □

**Proposition 36 (unfold($L, \varphi, B$) runs in DSpace($B^2||L||^4 + ||\varphi||$)).**

**Proof.** In this proof, we denote with $L_{init}$ the language given in the first call to unfold, while $L$ denotes the language given to current call of unfold. We show the claim in three steps: (1) $||L|| = O(d||L_{init}||^2)$ holds at any point during the recursion, given $d$ is the number of recursive calls going through Line 7. First, recursive calls through Line 6 cannot increase the size of the expression, i.e., $||L_0|| \leq ||L||$, since we obtain $L_0$ by removing from $S^*_h$ all subexpressions directly containing a symbol $\delta$ with $\varepsilon \in \varphi(\delta)$ (and not only via another Kleene star expression). Thus, only recursive calls going through Line 7 possibly increase the size of the expression. Now, in such a call, we unroll a subexpression $S_h$ with $S^*_h = E^*\bar{E}_pS^*_h$ and $\bar{E} = ufs(S_h, p)$ for some integer sequence $p$. From Proposition 35, we have $||ufs(S_h, p)|| = O(K||S_h||)$. Since $ufs$ and unfold only duplicate already existing Kleene star subexpressions, we have both $||S_h|| \leq ||L_{init}||$ and $K \leq ||L_{init}||$, and hence $||ufs(S_h, p)|| = O(||L_{init}||^2)$. Together with $||E|| \leq ||S_h||$ and $||S_h|| \leq ||L_{init}||$, this leads to $||E^*\bar{E}_pS^*_h|| = O(||L_{init}||^2)$. $d$ recursive calls through Line 7 substitute $d$ subexpressions $S_h$ with $E^*\bar{E}_pS^*_h$ to unfold $L_{init}$ into $L$, each time adding $O(||L_{init}||^2)$ to the size of the expression representing $L$. Hence $||L|| = O(d||L_{init}||^2)$.
(2) $||L|| = \mathcal{O}(B||L_{init}||^2)$ holds for all recursive calls to unfold while computing unfold($L_{init}, \varphi, B$). unfold makes at most $B$ recursive steps through Line 7, since minlen($\varphi(L_p) >$ minlen($\varphi(L)$) holds (this is true, since $E_p$ in $L_p$ instantiates some $\delta$ with $\varepsilon \notin \varphi(\delta)$). Then the claim follows setting $d = B$.

(3) The total recursion depth of unfold is at most $\mathcal{O}(B||L_{init}||^2)$. In the previous claim, we saw that there are at most $B$ recursive calls through Line 7. It remains to give an upper bound for the calls through Line 6: In each such call, at least one Kleene star subexpression in $L$ is removed in substituting $E$ for $S_h$. At any point there are at most $||L|| = \mathcal{O}(B||L_{init}||^2)$ expressions in $L$, hence we get a maximum recursion depth of $\mathcal{O}(B||L_{init}||^2)$.

(4) The space required to compute unfold($L_{init}, \varphi, B$) is bounded by the depth of the recursion times the stack frame size, which is dominated by $||L||$, plus $||\varphi||$. This gives $\mathcal{O}((B||L_{init}||^2)^2 + ||\varphi||) = \mathcal{O}(B^2||L_{init}||^4 + ||\varphi||)$ as desired. □

Lemma 37 (enumerate($R, K, \varphi$) runs in DSpace($||K||^42^{2(||R||+||\varphi||)^k}$)).

Proof. The construction of $M = M_{\varphi}(R) \cap K$ yields an expression in the size $||K||2^{2(||R||+||\varphi||)^k}$ for some constant $l$ [17]. The union free decomposition yields possibly exponentially many union free languages, however, each of them has linear size, using the rewriting rules, $(A + B)C = AC + BC, A(B + C) = AB + AC, (A + B)(C + D) = AC + AD + BC + BD$, and $(A + B)^* = (A^* B^*)^*$. In practical implementations, however, one might prefer to generate less but larger individual expressions, employing e.g. [20]. With Proposition 36, we obtain the overall space complexity of enumerate with DSPACE($B^2||L||^4 + ||\varphi||$) for $B = \text{minlen}(R) \leq ||R||$ and $||L|| = ||K||2^{2(||R||+||\varphi||)^k}$. This leads to the desired result with DSPACE($||K||^42^{2(||R||+||\varphi||)^k}$) for some other constant $k$. □

Proof (of Theorem 28). The enumeration runs DSPACE($||K||^42^{2(||R||+||\varphi||)^k}$), producing expressions for basiccheck at most of the same size (Lemma 37). Since basiccheck is in PSPACE (Lemma 32), we obtain the overall complexity DSPACE($||K||^r2^{2(||R||+||\varphi||)^s}$) $\subseteq$ 2ExpSpace for some constants $r$ and $s$. □