AN EXTENSION OF HERON’S FORMULA

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Abstract. This paper introduces an extension of Heron’s formula to approximate area of cyclic n-gons where the error never exceeds $\frac{\pi}{6} - 1$.

1. Introduction

A cyclic n-gon is a polygon with n vertices all located on the perimeter of a given circle. Triangles and cyclic quadrilaterals are examples of cyclic n-gons. We are interested in finding a formula for the area of a cyclic n-gon in terms of its sides. For $n = 3$ and $n = 4$ the formula is known, namely Heron’s and Brahmagupta’s formulas. Heron’s formula calculates the area of a given triangle in terms of its sides:

$$S = \sqrt{P(P-a)(P-b)(P-c)}$$

where $a, b, c$ are sides, $P = \frac{1}{2}(a + b + c)$ is the semi-perimeter, and $S$ is the area of the triangle. Brahmagupta’s formula calculates the area of a cyclic quadrilateral in terms of its four sides:

$$S = \sqrt{(P-a)(P-b)(P-c)(P-d)}$$

where $a, b, c, d$ are sides, $P = \frac{1}{2}(a + b + c + d)$ and $S$ is the area of the cyclic quadrilateral.

Naturally we can ask if there is any similar formula to calculate area of a cyclic n-gon in terms of its sides. In this paper, we will introduce a natural extension of Heron and Brahmagupta’s formulas. It turns out that this generalized formula is not exactly equal to the area of the cyclic n-gon, but it can approximate the area with a small error which never exceeds $\frac{\pi}{6} - 1$.

The area of a general convex quadrilateral could be obtained by formulas which have terms similar to Heron’s terms [1]. It is also known that the area of a cyclic n-gon raised to power two and then multiplied with a factor sixteen, is a monic polynomial whose other coefficients are polynomials in the sides of n-gon [2], [3], [4]. Therefore, the formula suggested in this paper of area an n-gon in terms of its sides provides with an estimation for Heron polynomials [5].

2. Area of n-gons in terms of its sides

Imagine a cyclic n-gon with sides $x_1, x_2, ..., x_n$, area $S_n$ and semi-perimeter

$$P_n = \frac{1}{2}(x_1 + x_2 + ... + x_n).$$

A natural extension of Heron or Brahmagupta’s formula is

$$S_n = \sqrt{P(4-n)(P-x_1)(P-x_2)...(P-x_n)}.$$

Here $n \geq 3$. Obviously, when $n = 3$, we get Heron’s formula and when $n = 4$, we get Brahmagupta’s.

Another good aspect of this suggested formula is that if the length of one of the sides of n-gon, say

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$x_1 = 0$, then we obtain an $(n - 1)$-gon. The new formula consistently calculates the area of resulting $(n - 1)$-gon in two ways:

$$S_n = \sqrt{P^{(4-n)}(P - 0)(P - x_2)...(P - x_n)} = \sqrt{P^{(4-(n-1))}(P - x_2)...(P - x_n)} = S_{n-1}.$$  

To understand if this formula is correct to calculate the area of $n$-gon, we first compute the area in terms of central angles:

Connect all vertices to the center of circle, which inscribes the $n$-gon. Denote the central angles $\alpha_1, \alpha_2, ..., \alpha_n$. So, $\sum_{i=1}^{n} \alpha_i = 2\pi$ (Figure 1.)

\[
A_n = \frac{1}{2} \sum_{i=1}^{n} h_i x_i \\
= \frac{1}{4} \sum_{i=1}^{n} \cot \frac{\alpha_i}{2} \frac{x_i^2}{2} \\
= \frac{1}{4} \sum_{i=1}^{n} 4R^2 \sin^2 \frac{\alpha_i}{2} \cot \frac{\alpha_i}{2} \\
= R^2 \sum_{i=1}^{n} \sin \frac{\alpha_i}{2} \cos \frac{\alpha_i}{2} \\
= \frac{R^2}{2} \sum_{i=1}^{n} \sin \alpha_i.
\]
Here, we use the fact that \( h_i = \frac{1}{2} x_i \cot \frac{\alpha_i}{2} \) and \( x_i = 2R \sin \frac{\alpha_i}{2} \) where \( R \) is the radius of the circle. Next, we calculate generalized Heron’s formula for the area of a cyclic n-gon in terms of central angles:

\[
S_n = \sqrt{P(4-n)(P-x_1)(P-x_2)...(P-x_n)}
\]

\[
= P^2 \sqrt{(1 - \frac{x_1}{P})(1 - \frac{x_2}{P})...(1 - \frac{x_n}{P})}
\]

\[
= P^2 \prod_{i=1}^{n} [(1 - \frac{x_i}{P})]^\frac{1}{2}
\]

\[
= \frac{1}{4} (\sum_{i=1}^{n} x_i)^2 \prod_{i=1}^{n} [(1 - \frac{2x_i}{\sum_{i=1}^{n} x_i})]^\frac{1}{2}
\]

\[
= \frac{1}{4} A.R^2(\sum_{i=1}^{n} \sin \frac{\alpha_i}{2})^2 \prod_{i=1}^{n} [(1 - \frac{2\sin \frac{\alpha_i}{2}}{\sum_{i=1}^{n} \sin \frac{\alpha_i}{2}})]^\frac{1}{2}
\]

\[
= R^2(\sum_{i=1}^{n} \sin \frac{\alpha_i}{2})^2 \prod_{i=1}^{n} [(1 - \frac{2\sin \frac{\alpha_i}{2}}{\sum_{i=1}^{n} \sin \frac{\alpha_i}{2}})]^\frac{1}{2}
\]

Now we define a new function \( D_n \) by dividing the exact area formula \( A_n \) from the suggested area formula \( S_n \):

\[
D_n = \frac{S_n}{A_n}
\]

\[
= \frac{R^2(\sum_{i=1}^{n} \sin \frac{\alpha_i}{2})^2 \prod_{i=1}^{n} [(1 - \frac{2\sin \frac{\alpha_i}{2}}{\sum_{i=1}^{n} \sin \frac{\alpha_i}{2}})]^\frac{1}{2}}{\left( \frac{2\alpha_i}{\sum_{i=1}^{n} \sin \alpha_i} \right)^2 \prod_{i=1}^{n} [(1 - \frac{2\sin \frac{\alpha_i}{2}}{\sum_{i=1}^{n} \sin \frac{\alpha_i}{2}})]^\frac{1}{2}}
\]

When all \( \alpha_i \), except four, are zero the n-gon is in fact a cyclic quadrilateral and thus \( D_n = 1 \) by Brahmagupta’s formula. Let’s try to find critical values of \( D_n \) as a function of \( n \)-variables \( \alpha_1, \alpha_2, ..., \alpha_n \) with respect to the condition \( \sum_{i=1}^{n} \alpha_i = 2\pi \). By the Lagrange Multiplier method, this function will be optimized when

\[
\frac{\partial D_n}{\partial \alpha_1} = \frac{\partial D_n}{\partial \alpha_2} = ... = \frac{\partial D_n}{\partial \alpha_n}.
\]

Since \( D_n \) is a symmetric function with respect to all \( \alpha_i \), the solution of system of differential equations will happen at \( \alpha_1 = \alpha_2 = ... = \alpha_n \).

Denote \( D_n \) with \( D_n^{eg} \) when all \( \alpha_i \) are equal. This will give us the value of \( D_n \) for a regular n-gon:

\[
D_n^{eg} = \frac{S_n^{eg}}{A_n^{eg}}
\]

\[
= \frac{n^2}{4} (1 - \frac{2}{n})^\frac{1}{2}
\]

\[
= \frac{n^2}{4} \cot \frac{\pi}{n}
\]

\[
= n(\tan \frac{\pi}{n})(1 - \frac{2}{n})^\frac{1}{2}
\]

where here \( x \) is the length of each side of the regular n-gon. Consider the sequence \( \{x_n\}_{n=3}^{\infty} \) where \( x_n = n(\tan \frac{\pi}{n})(1 - \frac{2}{n})^\frac{1}{2} \). The first two terms \( x_3 \) and \( x_4 \) are equal 1 by Heron’s and Brahmagupta’s formulas but \( x_5 = 1.013 \). Therefore, equal values of \( \alpha_i \) must maximize \( D_5 \). This proves that the sequence is increasing for \( n > 4 \). The reason for this behavior is that an \( (n-1) \)-gon could be considered
as a limit case of an \(n\)-gon, and therefore the maximum value of \(D_n\) must be bigger than of maximum value of \(D_{n-1}\). So, equal values of \(\alpha_i\) must maximize \(D_n\). Also,

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \left(n \tan \frac{\pi}{n}\right) \lim_{n \to \infty} \left(1 - \frac{2}{n}\right)^{\frac{\pi}{2}} = \frac{\pi}{e} = \frac{\pi}{e}.
\]

This shows that our suggested extension of Heron’s formula approximates area of cyclic n-gons where the error never exceeds \(\frac{\pi}{2} - 1 \approx 0.1557\). In Figure 2, we can see the curve of the function

\[
f(x) = x \tan \frac{\pi}{x} \left(1 - \frac{2}{x}\right)^{\frac{\pi}{2}}
\]

for \(x \geq 2\).

![Figure 2](image)

3. Comments

The area of a general convex quadrilateral could be obtained by formulas which have terms similar to Heron’s terms \[1\]. It is also known that the area of a cyclic n-gon raised to power two and then multiplied with a factor sixteen, is a monic polynomial whose other coefficients are polynomials in the sides of \(n\)-gon \[2, 3, 4\]. Therefore, the formula suggested in this paper of area an \(n\)-gon in terms of its sides provides with an estimation for Heron polynomials \[5\].

References

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