COUNTABLE APPROXIMATION OF TOPOLOGICAL G-MANIFOLDS, II: LINEAR LIE GROUPS G

QAYUM KHAN

Abstract. Let $G$ be a matrix group. Topological $G$-manifolds with Palais-proper action have the $G$-homotopy type of countable $G$-CW complexes (3.2). This generalizes E Elfving’s dissertation theorem for locally linear $G$-manifolds (1996). Also we improve the Bredon–Floyd theorem from compact groups $G$.  

1. Equivariant cohomology manifolds

We observe a generalization of the Bredon–Floyd theorem [Bor60, VII:2.2] to noncompact groups, by adapting the circle of ideas within Floyd’s initial argument.

Theorem 1.1. Let $G$ be a Lie group. Let $M$ be a cohomology manifold over $\mathbb{Z}$ with Cartan-proper $G$-action. Any compact set in $M$ has only finitely many orbit types.

The definition of a $\mathbb{Z}$-cohomology manifold is given in [Bor60, I:3.3]. In particular, $M$ is locally compact and Hausdorff, without assuming separable or metrizable. Proper in the sense of Cartan is in [tD87, I:3.17] and of Palais is in [Pal61, 1.2.2].

Proof. Assume not. Then there exists an infinite sequence $\{x_i\}_{i=0}^{\infty}$ in some compact subset $K$ of $M$ such that no two of the isotropy groups $G_{x_i}$ are conjugate in $G$. Since the action is Cartan, $C := \{g \in G \mid gK \cap K \neq \emptyset\}$ is compact [tD87, I:3.21]. In particular, since each $G_{x_i}$ is a closed subset of $C$, each $G_{x_i}$ is compact. Recall that the set $\text{Cpt}(G)$ of nonempty compact subsets of a metric space $(G, d)$ admits the Hausdorff–Pompeiu metric $d_{HP}$, which is compact if the ambient metric space is compact [Mun00, 45.7]. Then the infinite sequence $\{G_{x_i}\}_{i=0}^{\infty}$ in the compact metric space $(\text{Cpt}(C), d_{HP})$ has a convergent subsequence, which we may reindex to be the original. By continuity of multiplication and inversion in $G$, the compact subset $H := \text{lim}_{i \to \infty} G_{x_i}$ is a subgroup of $G$. Thus $H$ is a Lie group [Lee13, 20.10].

Let $U$ be a compact neighborhood of the neutral element in $G$. On the one hand, since $H$ is a compact Lie group acting on the $\mathbb{Z}$-cohomology manifold $M$, by the Bredon–Floyd theorem [Bor60, VII:2.2], the compact set $UK \subset M$ supports only finitely many $H$-orbit types. On the other hand, by the Montgomery–Zippin neighboring-subgroups theorem [Pal61, 4.2], there is a neighborhood $N$ of $H$ in $G$ so any subgroup of $G$ contained in $N$ is $U$-conjugate to a subgroup of $H$. Since $H$ is a limit, there exists $i_0$ such that $G_{x_i} \subset N$ for all $i \geq i_0$. Re-index so that $i_0 = 0$. Then there exists $u_i \in U$ such that $G_{u_ix_i} = u_iG_{x_i}u_i^{-1} \subset H$ for each $i$. Note $\{u_i x_i\}_{i=0}^{\infty}$ is an infinite sequence in $UK$ such that no two $G_{u_ix_i}$ are $G$-conjugate hence not $H$-conjugate, contradicting that $UK$ has only finitely many $H$-orbit types.  

Date: August 24, 2018.
2. Equivariant absolute neighborhood retracts

Recall that $X$ is a $G$-ANR for the class $\mathcal{C}$ ($\mathcal{C}$-absolute $G$-neighborhood retract) if $X$ belongs to $\mathcal{C}$ and, for any closed $G$-embedding of $X$ into a member of $\mathcal{C}$, there is a $G$-neighborhood of $X$ with $G$-retraction to $X$. More generally, $X$ is a $G$-ANE for the class $\mathcal{C}$ ($\mathcal{C}$-absolute $G$-neighborhood extensor) if, for any member $B$ of $\mathcal{C}$ and closed subset $A$ of $B$ and any $G$-map $A \rightarrow X$, there exists a $G$-extension $U \rightarrow X$ from some $G$-neighborhood $U$ of $A$ in $B$. Notice a $G$-ANE need not belong to $\mathcal{C}$.

Not long ago, S Antonyan [Ant05, 5.7] made equivariant O Hanner’s open-union theorem (see [Hu65, III:8.3]), providing a local-to-global principle for $G$-extensors.

**Theorem 2.1** (Antonyan). Let $G$ be a locally compact Hausdorff group. Let $\mathcal{C}$ be a subclass of the class $G$-$\mathcal{P}$ of paracompact Palais $G$-spaces with paracompact orbit space. Any union of open $G$-subsets that are $G$-ANEs for $\mathcal{C}$ is also a $G$-ANE for $\mathcal{C}$.

Equivariant CW structures were found over very general groups, using the nerves of locally finite coverings of neighborhoods in certain $G$-Banach spaces [AE09, 1.1].

**Theorem 2.2** (Antonyan–Elving). Let $G$ be a locally compact Hausdorff group. Suppose that $X$ is a $G$-ANR for the class $G$-$\mathcal{M}$ of $G$-metrizable Palais $G$-spaces. Then $X$ has the equivariant homotopy type of a $G$-CW complex with Palais action.

**Remark 2.3.** Observe that the class $G$-$\mathcal{M}$ is a subclass of $G$-$\mathcal{P}$, as follows. Let $X$ be a member of $G$-$\mathcal{M}$. Since $X$ is $G$-metrizable, the orbit space $X/G$ has an induced metric given by an infimum. Then, since both $X$ and $X/G$ are metrizable, by the Rudin–Stone theorem [Mun00, 41.4], both $X$ and $X/G$ are paracompact.

As classes, observe $\mathcal{C} \cap G$-ANE($\mathcal{C}$) $\subseteq$ G-ANE($\mathcal{C}$); a converse is [AAMP14, 6.3].

**Theorem 2.4** (Antonyan–Antonyan–Martín-Peinador). Let $G$ be a locally compact Hausdorff group. Then $G$-ANE($G$-$\mathcal{M}$) $=$ $G$-$\mathcal{M}$ $\cap$ $G$-ANE($G$-$\mathcal{M}$).

The following technical notion over compact groups was introduced in [Jaw81]. We restate from [AAMRVB17, 2.2] the generalization over noncompact groups.

**Definition 2.5** (Jaworowski). Let $G$ be a Lie group. A Palais $G$-space $X$ has finite structure if it has only finitely many orbit types and, for each orbit type $(H)$, the quotient map $X_{(H)} \rightarrow X_{(H)}/G$ is a $G/H$-bundle with only finitely many local trivializations. Here $(H)$ is the conjugacy class of $H$ in $G$. $X_{(H)} := \{x \in X \mid (G_x) = (H)\}$ is the $(H)$-stratum, $G_x := \{g \in G \mid gx = x\}$ is an isotropy group.

In the following recent theorem [AAMRVB17, 6.1], Jaworowski–Lashof’s criterion for $G$-ANRs [Jaw81] is generalized from compact Lie groups $G$ to linear ones.

**Theorem 2.6** (Antonyan–Antonyan–Mata-Romero–Vargas-Betancourt). Let $G$ be a linear Lie group. Let $X$ be a $G$-metrizable Palais $G$-space with finite structure. Then $X$ is a $G$-ANR for the class of $G$-metrizable Palais $G$-spaces, if and only if $X^H$ is an ANR for the class of metrizable spaces for each closed subgroup $H$ of $G$.

Here $X^H := \{x \in X \mid \forall g \in H : gx = x\}$ denotes the $H$-fixed subspace of $X$.

**Remark 2.7.** Any compact Lie group is linear: it has a homomorphic embedding into $GL_n(\mathbb{R})$ for some $n$. This is a special case of the following consequence of the Peter–Weyl theorem: any compact topological group $G$ embeds into a product of unitary groups; if $G$ has no small subgroups this product is finite; see [Kha18, 4.1].
3. Equivariant topological manifolds

**Theorem 3.1.** Let $G$ be a linear Lie group. Let $M$ be a cohomology manifold over $\mathbb{Z}$ that is both separable and metrizable. Suppose $M$ has Palais $G$-action and the fixed set $M^H$ is an ANR for the class of metrizable spaces for each closed subgroup $H$ of $G$. Then $M$ is $G$-homotopy equivalent to a countable proper $G$-CW complex.

**Proof.** Let $M$ be a $\mathbb{Z}$-cohomology manifold. Since $M$ is separable and locally compact, there exists an increasing infinite sequence $\{M_i\}_{i=0}^{\infty}$ of open sets in $M$ whose union is $M$ and whose closures $\overline{M}_i$ in $M$ are compact. By Theorem 1.1, the compact set $\overline{M}_i$, hence $M_i$, has only finitely many conjugacy classes of isotropy group. The $G$-saturation $GM_i = \bigcup_{g \in G} gM_i$ is also open [tD87, I:3.1(i)] and has only finitely many $G$-orbit types. Since $(GM_i)^H = GM_i \cap M^H$ is open in the ANR $M^H$, we have that $(GM_i)^H$ is also an ANR by Hanner's global-to-local principle [Hu65, III:7.9].

Since $G$ is a Lie group and $GM_i$ is a Palais $G$-space, by Palais' slice theorem [Pal61, 2.3.1, 2.1.2], $GM_i$ has a covering $\mathcal{T}_i$ by $G$-tubes of varying orbit types. Furthermore, since $(GM_i)/G = \overline{M}_i/G$ is compact, $\mathcal{T}_i$ can be assumed finite. The stratum $(GM_i)_i$ of $GM_i \subset GM_i$ has a single orbit type, so restriction of $\mathcal{T}_i$ to it gives a finite covering by local trivializations of a $G/H$-fiber bundle with structure group $G$. So the Palais $G$-space $GM_i$ has finite structure. By Palais' metrization theorem [Pal61, 4.3.4], the separable metrizable $M$, hence $GM_i$, is $G$-metrizable. Since $G$ is linear, $GM_i$ is a $G$-ANR for $G\setminus M$ (2.6), hence is a $G$-ANE for $G\setminus M$ (2.4).

Thus, by Remark 2.3 and Theorem 2.1, $M = \bigcup_{i \in I} GM_i$ is a $G$-ANE for $G\setminus M$. Then, since $M$ is also member of $G\setminus M$, $M$ is a $G$-ANE for $G\setminus M$. Therefore, by Theorem 2.2, we conclude $M$ has the $G$-homotopy type of a proper $G$-CW complex.

We now make some remarks on how to guarantee only countably many $G$-cells. The proof of Theorem 2.2 starts in [AARM09, 5.2], with a closed $G$-embedding of $X$ into a $G$-normed linear space $L$ with Palais action on some $G$-neighborhood. Specifically, those authors take $L = E \times N$ [AARM09, 3.10], which is valid for any $G$-metrizable Palais $G$-space $X$. Since our $X = M$ is locally compact, alternatively use the simpler and more classical $G$-Banach space $L = C_0(X)$, where

$$C_0(X) := \{ f \in C(X) \mid \forall \varepsilon > 0, \exists \text{ compact } K \subset X, \forall x \in X - K : |f(x)| < \varepsilon \}$$

$$\|f\| := \sup\{|f(x)| : x \in X\},$$

which is well-defined.

Indeed, E Elfvings in [Elf01, Propositions 2.3] showed the existence of a Kurotowski-like $G$-embedding of $X$ into $C_0(X) - \{0\}$ on which the continuous $G$-action is Palais.

Since $X$ is separable, there exists a countable dense subset $\Delta \subset X$. Since $X$ is locally compact, the Alexandroff one-point compactification $X^*$ exists. Since $X$ is second-countable, so is $X^*$, hence $X^*$ admits a metric $d$ by the Urysohn metrization theorem [Mun00, 34.1]. Consider the countable collection $\Delta_d \subset C(X^*)$ defined by

$$\Delta_d := \{1\} \cup \{ d(-, p) \in C(X^*) : p \in \Delta \}.$$

Since $\Delta_d$ contains a nonzero constant function and separates points because $\Delta$ is dense in $X^*$, by the Stone–Weierstrass theorem [Sto48, Corollary 3, p174], the countable subring $\mathbb{Q} \langle \Delta_d \rangle$ is dense in $C(X^*)$. Hence $C_0(X) \subset C(X^*)$ is separable.

Then the $G$-neighborhood $U$ of $X$ in $C_0(X) - \{0\}$, on which the $G$-retraction $U \rightarrow X$ is defined, is Lindelöf, as it is separable and metrizable. So in the proof of [AE09, Proposition 5.2], the rich $G$-normal cover $\mathcal{U}$ with index set $G \times M$ can be assumed to have $M$ a countable set. The geometric $G$-nerve $K(\mathcal{U})$ is indexed [AE09, p166] by certain finite subsets of $M$. Thus the semisimplicial $G$-space $K(\mathcal{U})$
has only countably many $G$-cells, according to the proof of [AE09, Theorem 5.3], which relies on S Illman [Ill00] and this in turn involves only countably many $G$-cells for a smooth $G$-manifold. Finally, since [AE09, Proposition 5.2] states that $K(1)$ dominates $X$, by a $G$-version of Mather’s trick (see second paragraph of [Kha18, Proof 2.5]), the $G$-CW complex for $X = M$ has only countably many $G$-cells. □

Finally, we generalize [Kha18, 2.5] from $G$ being compact. Note that the manifold must be noncompact if $G$ is noncompact in order for the action to be Cartan-proper.

**Corollary 3.2.** Let $G$ be a linear Lie group. Any topological $G$-manifold with Palais action has the equivariant homotopy type of a countable proper $G$-CW complex.

Here, by topological $G$-manifold [Kha18, 2.2], we mean the $H$-fixed subspace is a topological $(C^0)$ manifold for each closed subgroup $H$ of a topological group $G$. Herein, a topological manifold shall be separable, metrizable, and locally euclidean.

**Proof.** Let $M$ be a topological $G$-manifold with Palais action. By Hanner’s local-to-global principle [Hu65, III:8.3], each manifold $M^H$ is an ANR for the class of metrizable spaces. Also $M$ is separable, metrizable, and a $Z$-cohomology manifold. Therefore we are done by Theorem 3.1. □

Thus more tractible are its Davis–Lück $G$-spectral homology groups [DL98, 3.7, 4.3], since we conclude countability of the $G$-CW complex that left-approximates.

**Corollary 3.3.** Let $G$ be a linear Lie group. Let $f : M \longrightarrow N$ be a $G$-map between topological $G$-manifolds with Palais actions. Then $f$ is a $G$-homotopy equivalence if and only if $f^H : M^H \longrightarrow N^H$ is a homotopy equivalence for each closed $H$ of $G$.

**Proof.** This is immediate from Corollary 3.2 and the corresponding theorem for $G$-CW complexes [tD87, II:2.7], which is proven using $G$-obstruction theory. □

In particular, we generalize the main result of Elfving’s thesis [Elf96, 4.20]. The definition of locally linear, along with some discussion, is found in [Kha18, 3.6, 3.7]. Note any smoothable action is locally linear, but not vice versa; see [Bre72, VI:9.6].

**Corollary 3.4 (Elfving).** Let $G$ be a linear Lie group. Let $M$ be a locally linear $G$-manifold with Palais action. If $M$ has only finitely many orbit types, then $M$ has the equivariant homotopy type of a $G$-CW complex.

**Proof.** This special case now follows immediately from Corollary 3.2. □

4. **Examples that are not locally linear**

We continue the three families of uncountable examples of [Kha18, 3.1, 3.2, 3.3]. The purpose here is to show there do exist topological $G$-manifolds that are not locally linear when $G$ is a noncompact linear Lie group with torsion. (All principal bundles are trivial if $G$ is connected torsionfree, such as $G = R$ for complete flows.)

Their common trick is that the diagonal action will become Palais [Pal61, 1.3.3], even though it is not on the first factor, using a homogenous space $G/H$ with $H$ compact for the second factor. These $G/H$ are exactly those with transitive Palais $G$-action. The transitivity on the second factor guarantees the same quotient space as the first’s. Any $C^1$ Palais action by a Lie group is $C^\omega$ [Ill95, Ill03]; ours are $C^0$.

Indeed there is no contradiction to Palais’ slice theorem [Pal61, 2.3.1, 2.1.2]. There does exist a $G_e$-slice for each point $x$ of the Palais $G$-manifolds, but not all the slices are euclidean, and this is why in particular these slices are not $G_e$-linear.
Example 4.1 (Bing). Consider the double $D := E \cup_A E$ of the non-simply connected side $E$ in $S^3$ of the Alexander horned sphere $A \approx S^2$, whose embedding is not locally flat. This double has obvious involution $r_B$ that interchanges the two pieces and leaves the horned sphere fixed pointwise. Bing showed $D$ is homeomorphic to $S^3$ [Bin52]. Thus $r_B$ minus a fixed point (so on $\mathbb{R}^3$) negatively answers a question of Montgomery [Eil49, 39b], asking if the action is conjugate to an isometric one.

Consider the Lie group $G = \text{Isom}(\mathbb{R}) = \mathbb{R} \times_{-1} O_1$, a closed subgroup of $GL_2(\mathbb{R})$

$$\left\langle \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\rangle.$$  

Define a non-Cartan action of $G$ on $S^3$ by epimorphism to $O_1 \cong \langle r_B \rangle \leq \text{Homeo}(S^3)$. As noted above, the diagonal action of $G$ on the product of $S^3$ and the homogenous space $\mathbb{R} = G/O_1$ is Palais. Then Corollary 3.2 applies to the topological $G$-manifold $S^3 \times \mathbb{R}$. In the orbit space $(S^3 \times \mathbb{R})/G = S^3/r_B = E$, the stratum $A$ is not locally cofibrant, so the $C^0$ action of $G$ on the 4-manifold $S^3 \times \mathbb{R}$ cannot be locally linear.

For each $n \geq 3$, Lininger [Lin70, 9, 10] applies [Bin64] to produce uncountably many inequivalent involutions on $S^n$ with fixed set an $(n+1)$-sphere and quotient not a manifold-with-boundary, so none is equivalent to a locally linear action. They arise from uncountably many Cantor’s space $2^k$; in the form of multiparameter Antoine necklaces, the $n = 4$ case is due to Sher [She68].

Example 4.2 (Montgomery–Zippin). Adaptation of Bing’s 1952 idea produces an involution $r_{MZ}$ of $S^3$ whose fixed set is an embedded circle $K$ that is not locally flat [MZ54, §2]. In Example 4.1, replacing $r_B$ and $A$ with $r_{MZ}$ and $K$ works verbatim. Note $r_{MZ}$ preserves orientation and was first to negatively answer the $C^0$ version of a question of Paul A Smith [Eil49, 36], asking if the fixed circle is unknotted.

Alford gave uncountably many involutions fixing a wild circle [Alf66].

Higher codimension-two examples are provided by Lininger. He uses rotation of the Alexander horned sphere $A$ in 4-space to obtain a semifree $U_1$-action on $S^4$ with fixed set a 2-sphere [Lin70, 7]. More generally, using Bing’s later techniques [Bin64], he obtains uncountably many semifree $U_1$-actions on $S^n$ whose fixed set is an $(n-2)$-sphere and quotient not a manifold-with-boundary [Lin70, 8, 10].

Example 4.3 (Lininger). For each $k \geq 3$, there are uncountably many inequivalent free $U_1$-actions on $S^{2k-1}$ whose quotients are not $C^0$ manifolds [Lin69, Remark 2]. At the root of Lininger’s work are Andrews–Curtis decomposition spaces [AC62]: non-euclidean quotients $Q$ by a wild arc, any of whose product with $\mathbb{R}$ is euclidean.

Consider the Lie group $G = \text{Isom}^+(\mathbb{C}) = \mathbb{C} \times U_1$, a closed subgroup of $GL_2(\mathbb{C})$

$$\left\langle \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \mid c \in \mathbb{C}, u \in U_1 \right\rangle.$$  

Define a non-Cartan action of $G$ on $S^{2k-1}$ by epimorphism to $U_1$ then use Lininger. The diagonal action of $G$ on the product of $S^{2k-1}$ and homogeneous space $\mathbb{C} = G/U_1$ is Palais, as well as free. The orbit space $(S^{2k-1} \times \mathbb{C})/G = S^{2k-1}/U_1$ is not a topological manifold, though the projection from $S^{2k-1} \times \mathbb{C}$ is a principal $G$-bundle. In particular, none in this uncountable family of free $G$-actions can be locally linear.

The same holds for $G = U_1 \times G'$ with $G'$ a linear Lie group and $M = S^{2k-1} \times G'$.

We end with a family of examples whose linear Lie group $G$ is arbitrarily large.
Example 4.4 (Lininger). For each $n > k + 1 \geq 3$, there are uncountably many inequivalent semifree $SO_k$-actions on $S^{n}$ whose fixed set is a wild $(n-k-1)$-sphere [Lin70, 11]. Again, the construction arises from the quotient by any wild arc [AC62].

The Lie group $G = \text{Isom}^+(\mathbb{R}^k) = \mathbb{R}^k \rtimes SO_k$ is a closed subgroup of $GL_{2k}(\mathbb{R})$:

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} r \\ 0 \\ 1 \end{pmatrix} \right\} \cap \mathbb{R}^k, \ r \in SO_k.$$

Define a non-Cartan action of $G$ on $S^n$ by epimorphism to $SO_k$ then use Lininger. The diagonal action of $G$ on the product of $S^n$ and homogeneous space $\mathbb{R}^k = G/SO_k$ is Palais. The orbit space $(S^n \times \mathbb{R}^k)/G = S^n/SO_k$ is not a manifold-with-boundary. In particular, none in this uncountable family of semifree $G$-actions is locally linear.

Acknowledgements. I thank Christopher Connell for various basic discussions.

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Department of Mathematics  Saint Louis University  St Louis MO 63103 USA

E-mail address: khanq@slu.edu