The physical meaning of the de Sitter invariants

Ion I. Cotăescu *
West University of Timişoara,
V. Parvan Ave. 4, RO-300223 Timişoara
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Abstract

We study the Lie algebras of the covariant representations transforming the matter fields under the de Sitter isometries. We point out that the Casimir operators of these representations can be written in closed forms and we deduce how their eigenvalues depend on the field’s rest energy and spin. For the scalar, vector and Dirac fields, which have well-defined field equations, we express these eigenvalues in terms of mass and spin obtaining thus the principal invariants of the theory of free fields on the de Sitter spacetime. We show that in the flat limit we recover the corresponding invariants of the Wigner irreducible representations of the Poincaré group.

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*E-mail: cota@physics.uvt.ro
1 Introduction

In general relativity, apart from the general conservation laws associated to the relativistic and gauge covariance \[1, 2\], the isometries play an important role giving rise to the principal conserved physical quantities \[3\]. The best example is the de Sitter spacetime \[4, 5\] which has \(SO(1,4)\) isometries and, moreover, allows one to analytically solve the equations of the usual matter fields variously coupled to the de Sitter gravity. These opportunities encouraged many authors to develop either a de Sitter quantum mechanics \[6\] or a quantum field theory \[7\] in an axiomatic manner, exploiting the high symmetry of this manifold but avoiding the standard steps of the canonical theory.

We preferred another strategy, considering the quantum theory of fields on curved spacetimes based on the canonical quantization. This is a Lagrangian tetrad-gauge covariant field theory in non-holonomic orthogonal (local) frames where the fields with spin half can be correctly defined \[8\].

Under such circumstances the theory of the spacetime symmetries must be completed with the tools able to control simultaneously the transformations of both the holonomic and non-holonomic frames. For this reason we introduced the external symmetry transformations which combine isometries and gauge transformations such that the tetrad fields remain invariant under isometries \[9\]. We obtained thus the external symmetry group which is the universal covering group of the isometry one. Moreover, we pointed out that the matter fields transform under isometries according to the covariant representations (CR) of this group induced by the linear representations of the gauge group corresponding to the metric of the (pseudo) Euclidean model of the curved manifold. We have shown that the basis-generators of these CRs are given by the Carter and McLenaghan formula \[10\], derived initially for the Dirac field and generalized then for any CR \[9, 11\]. These operators are important since they commute with those of the free field equations playing thus the role of conserved quantities.

Our approach is helpful on the four-dimensional de Sitter spacetime where all the usual free field equations can be analytically solved while the isometries give rise to the rich \(so(1,4)\) algebra. Hereby we selected various sets of commuting operators determining the quantum modes of the scalar \[12\], vector \[13, 14\] and Dirac \[15, 16\] fields.

We must stress that many of our results differ from those of Refs \[6\] where the tetrad-gauge covariance is neglected focusing only on the usual
linear representations of the isometry group. In this way one obtains a quantum mechanics on the five-dimensional Minkowski manifold in which the de Sitter spacetime is embedded. The advantage therein is that the fields obey five-dimensional homogeneous equations and transform according to unitary and irreducible representations (UIR) which have well-known properties [17]. Unfortunately, these representations do not play the main role in our gauge-covariant field theory since this is a four-dimensional theory in the non-holonomic frames of the de Sitter spacetime where the CRs of the isometry group are induced by the finite-dimensional representations of the gauge group. For this reason we believe that the study of the Casimir operators of our CRs could complete the analyze of the de Sitter invariants offered by the five-dimensional theory [6].

The present paper is devoted to this problem. We calculate first the $SO(1, 4)$ generators of the CRs in the de Sitter local chart with FRW line element and proper time where we choose the diagonal gauge. These generators are differential operators with spin parts given by the generators of the finite-dimensional representations of the $SL(2, \mathbb{C})$ gauge group that induce the CRs. Furthermore, we write down the Casimir operators of the CRs and analyze their properties pointing out that their eigenvalues can be expressed in terms of rest energy and spin when the particles are at rest. A remarkable identity relating these Casimir operators is derived for the CRs with unique spin in the sense of the $SL(2, \mathbb{C})$ gauge symmetry. In the flat limit this identity gives just the second invariant of the Wigner UIRs of the Poincaré group [18, 19]. These results combined with the definitions of the masses given by the field equations enable us to write down the de Sitter invariants of the basic fields in terms of mass and spin, as in special relativity, but with supplemental terms which vanishes in the flat limit.

We start in the second section with a short review of our theory of external symmetry following to briefly present in the next one the Poincaré invariants determined by the mass and spin according to the Wigner theory of the UIRs of this group [18]. The principal new results are presented in section 4 where we calculate the $SO(1, 4)$ generators of the CRs deriving the corresponding Casimir operators in closed forms and finding their eigenvalues for the particles at rest. Moreover, we establish the mentioned identity among these operators and the spin ones that holds in many cases when the spin is unique. In the next section we deduce the de Sitter invariants of the scalar, vector and Dirac fields minimally coupled to the de Sitter gravity. Our conclusions are presented in the last section. The Appendix is devoted to
the classical conserved quantities which can be compared with the quantum ones.

2 External symmetry

The theory of the matter fields with spin on the pseudo-Riemannian spacetime \((M, g)\) can be formulated considering simultaneously both the holonomic and non-holonomic frames. The holonomic frames are local charts of coordinates \(x^\mu\), labeled by natural indices, \(\mu, \nu, \ldots = 0, 1, 2, 3\). In a given tetrad-gauge, the tetrad fields \(e_\mu\) and \(\hat{e}^\mu\), which define the non-holonomic orthogonal frames and the corresponding coframes, are labeled by local indices, \(\hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3\). These fields have the usual duality, \(\hat{e}_\alpha^\mu e^\alpha_\nu = \delta^\mu_\nu\), \(\hat{e}_\alpha^\mu e^\beta_\nu = \delta^\beta_\alpha\), and orthonormalization, \(e_\mu \cdot e_\nu = \eta_{\mu\nu}\), \(\hat{e}_\mu \cdot \hat{e}_\nu = \eta^{\mu\nu}\), properties.

The metric tensor \(g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}_{\mu}^{\hat{\alpha}} \hat{e}_{\nu}^{\hat{\beta}}\) raises or lowers the natural indices while for the local indices we have to use the flat metric \(\eta = \text{diag}(1, -1, -1, -1)\) of the Minkowski spacetime \((M_0, \eta)\) which is the pseudo-Euclidean model of \((M, g)\).

The metric \(\eta\) remains invariant under the transformations of the group \(O(1, 3)\) which includes as a subgroup the Lorentz group, \(L^{\uparrow}_+\), whose universal covering group is \(SL(2, \mathbb{C})\). In the usual covariant parametrization, with the real parameters, \(\omega^{\hat{\alpha}\hat{\beta}} = -\omega^{\hat{\beta}\hat{\alpha}}\), the transformations \(A(\omega) = \exp(-\frac{i}{2} \omega^{\hat{\alpha}\hat{\beta}} S_{\hat{\alpha}\hat{\beta}}) \in SL(2, \mathbb{C})\) depend on the covariant basis-generators of the \(sl(2, \mathbb{C})\) Lie algebra, \(S_{\hat{\alpha}\hat{\beta}}\), which are the principal spin operators generating all the spin terms of other operators. This parametrization offers, in addition, the advantage of a simple expansion of the matrix elements in local bases, \(\Lambda_{\hat{\alpha}\hat{\beta}}^{\mu\nu}[A(\omega)] = \delta_\mu^{\hat{\alpha}} + \omega^{\hat{\alpha}_\mu}_{\hat{\beta}} + \cdots\), of the transformations \(\Lambda[A(\omega)] \in L^{\uparrow}_+\) associated to \(A(\omega)\) through the canonical homomorphism [19].

Assuming now that \((M, g)\) is orientable and time-orientable we can restrict ourselves to consider \(G(\eta) = L^{\uparrow}_+\) as the gauge group of the Minkowski metric \(\eta\) [3]. This is the structure group of the principal fiber bundle whose basis is \(M\) while the group \(\text{Spin}(\eta) = SL(2, \mathbb{C})\) represents the structure group of the spin fiber bundle [3, 8] which is well-defined when some special conditions are fulfilled as in the case of the de Sitter manifold that is globally hyperbolic [20]. The matter fields, \(\psi(\rho) : M \to \mathcal{V}(\rho)\), are locally defined over \(M\) with values in the vector spaces \(\mathcal{V}(\rho)\) carrying finite-dimensional representations \(\rho\) of the group \(SL(2, \mathbb{C})\) which, in general, are reducible as direct sums of irreducible ones, \((j_+, j_-)\) [19]. These representations are non-unitary
and, consequently, in order to assure the global unitary properties of the field theory one must use invariant hermitian forms called often relativistic scalar products\footnote{We proposed a general solution of this problem in arXiv:math-ph/9904029 but this paper was never published because of its 'horrific' notations.} which play a similar role in special or general relativity.

The choice of the representation $\rho$ determines the form of the covariant derivatives of the field $\psi(\rho)$ in local frames,

$$D^{(\rho)}_{\dot{A}} = e^\mu_{\dot{A}} D^{(\rho)}_\mu = \dot{\partial}_{\dot{A}} + \frac{i}{2} \rho(\hat{S}_\gamma^\beta) \hat{\gamma}^\beta_{\dot{A}\dot{B}},$$

(1)

that depend, in addition, on the local derivatives \( \dot{\partial}_{\dot{A}} = e^\mu_{\dot{A}} \partial_\mu \) and the connection coefficients in local frames, \( \hat{\gamma}^\beta_{\dot{A}\dot{B}} = e^\sigma_{\dot{A}} e^\rho_{\dot{B}} (e^\gamma_{\dot{A}} \Gamma^\gamma_{\dot{A}\dot{B}} - e^\sigma_{\dot{A}} e^\rho_{\dot{B}}) \), which assure the covariance of the whole theory under tetrad-gauge transformations produced by the automorphisms $A \in SL(2, \mathbb{C})$ of the spin fiber bundle. This is the general framework of the theories involving fields with half integer spin which can not be formulated in natural frames.

A special difficulty of the field theory in local frames arises from the fact that the tetrad fields transform under isometries in a non-covariant manner because of their natural indices. For this reason we proposed the theory of external symmetry in which each isometry transformation is coupled to a gauge one able to correct the position of the local frames such that the whole transformation should preserve not only the metric but the tetrad-gauge too \[9\]. Thus, for any isometry transformation $x \to x' = \phi_\xi(x) = x + \xi^a k_a(x) + ...$, depending on the parameters $\xi^a$ \((a, b, ..., = 1, 2, ...N)\) of the isometry group $I(M)$, one must perform the gauge transformation $A_\xi$ defined as

$$\Lambda_{\dot{A}}[A_\xi(x)] = \hat{e}^{\dot{A}}_{\mu} [\phi_\xi(x)] \frac{\partial \phi_\xi^{\mu}(x)}{\partial x^\nu} e_{\nu}^\beta(x),$$

(2)

with the supplementary condition $A_{\xi=0}(x) = 1 \in SL(2, \mathbb{C})$. Then the transformation laws of our fields are

\[
(A_\xi, \phi_\xi) : \quad e(x) \rightarrow e'(x') = e[\phi_\xi(x)], \\
\hat{e}(x) \rightarrow \hat{e}'(x') = \hat{e}[\phi_\xi(x)], \\
\psi(\rho)(x) \rightarrow \psi'(\rho)(x') = \rho[A_\xi(x)] \psi(\rho)(x). (3)
\]

We have shown that the pairs \((A_\xi, \phi_\xi)\) constitute a well-defined Lie group we called the external symmetry group of $(M, g)$, denoted by $S(M)$, pointing
out that this is just the universal covering group of \(I(M)\) \[9\]. For small values of \(\xi^a\), the \(SL(2,C)\) parameters of \(A_\xi(x) \equiv A[\omega_\xi(x)]\) can be expanded as

\[\omega^{\hat{\alpha}\hat{\beta}}(x) = \xi^a \Omega^{\hat{\alpha}\hat{\beta}}_a(x) + \cdots,\]

in terms of the functions

\[\Omega^{\hat{\alpha}\hat{\beta}}_a \equiv \frac{\partial \omega^{\hat{\alpha}\hat{\beta}}}{\partial \xi^a} |_{\xi=0} = \left( e_{\mu}^{\hat{\alpha}} k^\mu_{a,\nu} + e_{\nu}^{\hat{\alpha}} k^\mu_{a,\mu} \right) e^{\nu}_{\chi} \eta^{\hat{\lambda}\hat{\beta}}(4)\]

which depend on the Killing vectors \(k_a = \partial_{\xi^a} \phi_\xi |_{\xi=0}\) associated to the parameters \(\xi^a\). We note that the functions (4) are skew-symmetric, \(\Omega^{\hat{\alpha}\hat{\beta}}_a = -\Omega^{\hat{\beta}\hat{\alpha}}_a\), only when \(k_a\) are Killing vectors \[9\].

The last of equations (3) defines the operator-valued representations \(T^{(\rho)}: (A_\xi, \phi_\xi) \to T^{(\rho)}_\xi\) of the group \(S(M)\) which are called the covariant representations (CR) induced by the finite-dimensional representations \(\rho\) of the group \(SL(2,C)\). The covariant transformations,

\[T^{(\rho)}_\xi(\rho_\xi(x)] = \rho[A_\xi(x)]\psi^{(\rho)}(x),\]

leave the field equation invariant since their basis-generators \[9\],

\[X^{(\rho)}_a = i \partial_{\xi^a} T^{(\rho)}_\xi |_{\xi=0} = -i k^\mu_{a,\nu} \partial_{\mu} + \frac{1}{2} \Omega^{\hat{\alpha}\hat{\beta}}_a \rho(S_{\hat{\alpha}\hat{\beta}}),\]

commute with the operator of the field equation and satisfy the commutation rules \([X^{(\rho)}_a, X^{(\rho)}_b] = ic_{abc} X^{(\rho)}_c\) determined by the structure constants, \(c_{abc}\), of the algebras \(s(M) \sim i(M)\). In other words, the operators \(6\) are the basis-generators of a CR of the \(s(M)\) algebra induced by the representation \(\rho\) of the \(sl(2,C)\) algebra.

We note that these generators are proportional to the Kosmann’s Lie derivatives \[21\] associated to the Killing vectors \(k_a\). They can be put in covariant form either in non-holonomic frames \[9\] or even in holonomic ones \[11\], generalizing thus the formula given by Carter and McLenaghan for the Dirac field \[10\].

A specific feature of the CRs is that their generators have, in general, point-dependent spin terms which do not commute with the orbital parts. However, there are tetrad-gauges in which at least the generators of a subgroup \(G \subset I(M)\) may have point-independent spin terms commuting with the orbital parts. Then we say that the restriction to \(G\) of the CR \(T^{(\rho)}\) is manifest covariant \[9\]. Obviously, if \(G = I(M)\) then the whole representation \(T^{(\rho)}\) is manifest covariant.
3 The Poincaré invariants

In this paper we study the physical meaning of the de Sitter invariants of the fields which transforms according to the CRs $T^{(\rho)}$. In the flat limit these invariants must coincide with the well-known Poincaré invariants which are completely determined by the mass and spin of the matter field. For this reason it deserves to briefly review the properties of these invariants focusing only on the massive case.

On the Minkowski flat spacetime $(M_0, \eta)$ the fields $\psi^{(\rho)}$ transform under isometries according to manifest covariant representations in inertial (local) frames defined by $\delta^{\mu}_{\nu} = \hat{\delta}^{\mu}_{\nu} = \hat{\delta}^{\mu}_{\nu}$. The isometries are just the transformations $x \to x' = \Lambda[A(\omega)]x - a$ of the Poincaré group $I(M_0) = \mathcal{P}_+^\uparrow = T(4) \otimes L_+^\uparrow$ whose universal covering group is $S(M_0) = \tilde{\mathcal{P}}_+^\uparrow = T(4) \otimes SL(2, \mathbb{C})$. Both these groups are semidirect products where the translations form the normal Abelian subgroup $T^{(4)}$.

The manifest covariant representations, $T^{(\rho)} : (A(\omega), a) \to T^{(\rho)}_{\omega, a}$, of the group $S(M_0)$ have the transformation rules

$$ (T^{(\rho)}_{\omega, a}(\psi^{(\rho)}))(x) = \rho[A(\omega)]\psi^{(\rho)}\{A[A(\omega)]^{-1}(x + a)\}, \quad (7) $$

and the well-known basis-generators of the $s(M_0)$ algebra,

$$ \hat{P}_\mu \equiv \hat{X}^{(\mu)}_{(\mu)} = i\partial_\mu, \quad (8) $$

$$ \hat{j}^{(\rho)}_{\mu\nu} \equiv \hat{X}^{(\rho)}_{(\mu\nu)} = (\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\alpha}x^\alpha\partial_\mu) + S^{(\rho)}_{\mu\nu}, \quad (9) $$

which have point-independent spin parts denoted from now by $S^{(\rho)}_{\mu\nu}$ instead of $\rho(S_{\mu\nu})$. Here it is convenient to separate the energy operator, $\hat{H} = \hat{P}_0$, and to write the $sl(2, \mathbb{C})$ generators as

$$ \hat{j}^{(\rho)}_i = \frac{1}{2}\varepsilon_{ijk}\hat{j}^{(\rho)}_{jk} = -i\varepsilon_{ijk}x^j\partial_k + S^{(\rho)}_i, \quad S^{(\rho)}_i = \frac{1}{2}\varepsilon_{ijk}S^{(\rho)}_{jk}, \quad (10) $$

$$ \hat{K}^{(\rho)}_i = \hat{j}^{(\rho)}_{0i} = i(x^i\partial_t + t\partial_i) + S^{(\rho)}_{0i}, \quad i, j, k = 1, 2, 3, \quad (11) $$

denoting $\hat{S}^2 = S_iS_i$ and $\hat{S}^2_0 = S_{0i}S_{0i}$. Thus we lay out the standard basis of the $s(M_0)$ algebra, $\{\hat{H}, \hat{P}_i, \hat{j}^{(\rho)}_i, \hat{K}^{(\rho)}_i\}$.

The invariants of the manifest covariant fields are the eigenvalues of the Casimir operators of the representations $T^{(\rho)}$ that read

$$ \hat{C}_1 = \hat{P}_\mu\hat{P}^\mu, \quad \hat{C}_2^{(\rho)} = -\eta_{\mu\nu}\hat{W}^{(\rho)}_\mu\hat{W}^{(\rho)}_\nu, \quad (12) $$
where the Pauli-Lubanski operator \( [19] \),
\[
\hat{W}^{(\rho)}_{\mu} = -\frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \hat{P}_\nu \hat{j}^{(\rho)}_{\alpha\beta},
\]
(13)
has the components
\[
\hat{W}^{(\rho)}_{0} = \hat{J}^{(\rho)}_i \hat{P}_i = S^{(\rho)}_i \hat{P}_i, \quad \hat{W}^{(\rho)}_{i} = \hat{H} \hat{j}^{(\rho)}_i + \varepsilon_{ijk} \hat{K}^{(\rho)}_j \hat{P}_k,
\]
(14)
resulting from equations (8) and (9) where we take \( \varepsilon^{0123} = -\varepsilon_{0123} = -1 \).

The first invariant (12a) deals with the mass condition,
\[
\hat{P}_2^2 \psi^{(\rho)} = m^2 \psi^{(\rho)},
\]
fixing the orbit in the momentum spaces on which the Fourier transform, \( \hat{\psi}^{(\rho)} \), of the covariant field \( \psi^{(\rho)} \) is defined. In general, the field \( \hat{\psi}^{(\rho)} \) transforms under isometries according to a reducible CR, \( \hat{T}^{(\rho)} \), which is equivalent with a direct sum of Wigner UIRs, \( (\pm m, s) \), induced by the subgroup \( T(4) \oplus SU(2) \subset SL(2, \mathbb{C}) \) [18, 19]. Consequently, each subspace \( \mathcal{V}_s \subset \mathcal{V}^{(\rho)} \) of spin \( s \) carries its own UIRs \( (\pm m, s) \) for which the second Casimir operator yields the eigenvalue \[19\]
\[
\hat{C}_2^{(s)} \sim m^2 s (s + 1).
\]
(15)
These invariants can be easily calculated in the rest frames where \( \hat{P}_i \sim 0 \) so that \( \hat{W}^{(\rho)}_0 \sim 0 \) and \( \hat{W}^{(\rho)}_i \sim \hat{H} \hat{S}^{(\rho)}_i \) while the mass condition gives the rest energy \( \hat{H} \sim E_0 = \pm m \). As a matter of fact, the Wigner UIRs of the massive particles (and antiparticles) are determined by \( E_0 \) and \( s \).

The advantage of this approach is that the UIRs \( (\pm m, s) \) are unitary with respect to a scalar product in momentum representation which corresponds to the relativistic scalar products of the covariant fields \( \psi^{(\rho)} \). Moreover, they point out the mass and spin as being the principal invariants of the theory of free fields and help one to keep under control the spin content of the covariant representations. When \( \hat{T}^{(s)} \) is equivalent only with the UIRs \( (\pm m, s) \) one says that the covariant field has the unique spin \( s \).

4 The de Sitter invariants

The Wigner theory in momentum representation works in any (pseudo) Euclidean manifold whose isometry group is the semidirect product between an orthogonal subgroup and a normal Abelian one [18]. Since the isometry group of the Sitter manifold does not have this property, we must study the CRs of this group in the configuration space, following step by step the method presented in the second section.
4.1 Killing vectors

Let us consider \((M, g)\) be the de Sitter spacetime defined as the hyperboloid of radius \(1/\omega^2\) in the five-dimensional flat spacetime \((M^5, \eta^5)\) of coordinates \(z^A\) (labeled by the indices \(A, B, \ldots = 0, 1, 2, 3, 4\)) and metric \(\eta^5 = \text{diag}(1, -1, -1, -1, -1)\) \(^4\). The local charts of coordinates \(\{x\}\) can be easily introduced on \((M, g)\) giving the set of functions \(z^A(x)\) which solve the hyperboloid equation,

\[
\eta^5_{AB} z^A(x) z^B(x) = -\frac{1}{\omega^2}.
\]  

(16)

In this manner \((M, g)\) is defined as a homogeneous space of the pseudo-orthogonal group \(SO(1, 4)\) which is in the same time the gauge group of the metric \(\eta^5\) and the isometry group, \(I(M)\), of the de Sitter spacetime. The group of the external symmetry, \(S(M) = \text{Spin}(\eta^5) = Sp(2, 2)\), has the Lie algebra \(s(M) = sp(2, 2) \sim so(1, 4)\) for which we use the covariant real parameters \(\xi^{AB} = -\xi^{BA}\). Then, the orbital basis-generators of the natural representation of the \(s(M)\) algebra (carried by the space of the scalar functions over \(M^5\)) have the standard form

\[
L^5_{AB} = i \left[ \eta^5_{AC} z^C \partial_B - \eta^5_{BC} z^C \partial_A \right] = -i K_{(AB)}^C \partial_C
\]

(17)

which allows us to derive the corresponding Killing vectors of \((M, g)\), \(k_{(AB)}\), using the identities \(k_{(AB)} d x^\mu = K_{(AB)C} d z^C\).

We assume that \((M, g)\) is equipped with the local chart \(\{t, \vec{x}\}\) of Cartesian coordinates defined by the functions

\[
z^0(x) = \chi(x) e^{\omega t}, \quad \chi(x) = \frac{1}{2\omega} (1 + \omega^2 \vec{x}^2 - e^{-2\omega t}),
\]

(18)

\[
z^4(x) = -\chi(x) e^{\omega t} + \frac{1}{\omega} e^{\omega t}, \quad z^i(x) = e^{\omega t} x^i,
\]

(19)

giving rise to the FRW line element

\[
d s^2 = \eta^5_{AB} d z^A d z^B = g_{\mu\nu} d x^\mu d x^\nu = d t^2 - e^{2\omega t} (d \vec{x} \cdot d \vec{x}).
\]

(20)

\(^2\)We denote by \(\omega\) the Hubble de Sitter constant since \(H\) is reserved for the Hamiltonian operator.
In this chart the Killing vectors of the de Sitter symmetry have the components:

\[ k^0_0 = k^0_4 = x^i, \quad k^i_0 = k^0_i - \frac{1}{\omega} \delta^i_j = \omega x^i x^j - \delta^i_4 \chi, \quad (21) \]
\[ k^0_{(ij)} = 0, \quad k^i_{(ij)} = \delta^i_j x^i - \delta^j_i x^j; \quad k^0_{(04)} = -\frac{1}{\omega}, \quad k^i_{(04)} = x^i. \quad (22) \]

The form of the basis-generators of our CRs depends on these components and on the choice of the tetrad-gauge. The simplest gauge is the diagonal one in which the non-vanishing components of the tetrad fields are

\[ \hat{e}^0_0 = e^0_0 = 1, \quad \hat{e}^i_j = e^{\omega t} \delta^i_j, \quad e^i_j = e^{-\omega t} \delta^i_j. \quad (23) \]

### 4.2 Generators of covariant representations

According to the general theory, the generators of the CRs \( T^{(\rho)} \) of the group \( S(M) = Sp(2, 2) \), induced by the representations \( \rho \) of the \( SL(2, \mathbb{C}) \) group, constitute CRs of the \( sp(2, 2) \) algebra induced by the representations \( \rho \) of the \( sl(2, \mathbb{C}) \) algebra. Therefore, their commutation relations are determined by the structure constant of the group \( Sp(2, 2) \) and the principal invariants are the Casimir operators of the CRs which can be derived as those of the algebras \( sp(2, 2) \sim so(1, 4) \).

In the covariant parametrization of the \( sp(2, 2) \) algebra adopted here, the generators \( X^{(\rho)}_{(AB)} \) corresponding to the Killing vectors \( k_{(AB)} \) result from equation (13) and the functions (14) with the new labels \( a \rightarrow (AB) \). Using then the Killing vectors (21) and (22) and the tetrad-gauge (23) of the chart \( \{ t, \vec{x} \} \), after a little calculation, we find first the \( sl(2, \mathbb{C}) \) generators. These are the total angular momentum,

\[ J^{(\rho)}_i \equiv \frac{1}{2} \varepsilon_{ijk} X^{(\rho)}_{(jk)} = -i \varepsilon_{ijk} x^j \partial_k + S^{(\rho)}_i, \quad (24) \]

and the generators of the Lorentz boosts

\[ K^{(\rho)}_i \equiv X^{(\rho)}_{(0i)} = i x^i \partial_t + i \chi(x) \partial_i - i \omega x^i x^j \partial_j + e^{-\omega t} S^{(\rho)}_0 + \omega S^{(\rho)}_{ij} x^j, \quad (25) \]

where \( \chi \) is defined by equation (18b). In addition, there are three generators,

\[ R^{(\rho)}_i \equiv X^{(\rho)}_{(i4)} = -K^{(\rho)}_i + \frac{1}{\omega} i \partial_t, \quad (26) \]
which play the role of a Runge-Lenz vector, in the sense that \( \{J_i, R_i\} \) generate a \( so(4) \) subalgebra. The energy (or Hamiltonian) operator \[15\],

\[
H \equiv \omega X^{(\rho)}_{(04)} = i\partial_t - i\omega x^i\partial_i ,
\]

is given by the Killing vector \( k_{(04)} \) which is time-like only for \( \omega|\vec{r}|e^{\omega t} \leq 1 \). Fortunately, this condition is accomplished everywhere inside the light-cone of an observer at rest in \( \vec{x} = 0 \). Therefore, despite of some doubts appeared in literature \[22\], the operator \( H \) is correctly defined.

The generators introduced above form the basis \( \{H, J_i^{(\rho)}, K_i^{(\rho)}, R_i^{(\rho)}\} \) of the covariant representation of the \( sp(2,2) \) algebra with the following commutation rules:

\[
\begin{align*}
[J_i^{(\rho)}, J_j^{(\rho)}] &= i\varepsilon_{ijk} J_k^{(\rho)} , \\
[J_i^{(\rho)}, K_j^{(\rho)}] &= i\varepsilon_{ijk} K_k^{(\rho)} , \\
[K_i^{(\rho)}, J_j^{(\rho)}] &= -i\varepsilon_{ijk} J_k^{(\rho)} , \\
[R_i^{(\rho)}, J_j^{(\rho)}] &= i\varepsilon_{ijk} R_k^{(\rho)} , \\
[R_i^{(\rho)}, K_j^{(\rho)}] &= i\varepsilon_{ijk} R_k^{(\rho)} .
\end{align*}
\]

and

\[
\begin{align*}
[H, J_i^{(\rho)}] &= 0 , \\
[H, K_i^{(\rho)}] &= i\omega R_i^{(\rho)} , \\
[H, R_i^{(\rho)}] &= i\omega K_i^{(\rho)} .
\end{align*}
\]

In some applications it is useful to replace the operators \( \vec{R}^{(\rho)} \) and \( \vec{K}^{(\rho)} \) by the momentum operator \( \vec{P} \) and its dual, \( \vec{Q}^{(\rho)} \), whose components are defined as

\[
P_i = \omega (R_i^{(\rho)} + K_i^{(\rho)}) = i\partial_i , \quad Q_i^{(\rho)} = \omega (R_i^{(\rho)} - K_i^{(\rho)}) .
\]

We obtain thus the basis \( \{H, P_i, Q_i^{(\rho)}, J_i^{(\rho)}\} \) with the new commutators

\[
\begin{align*}
[H, P_i] &= i\omega P_i , \\
[H, Q_i^{(\rho)}] &= -i\omega Q_i^{(\rho)} , \\
[J_i^{(\rho)}, P_j] &= i\varepsilon_{ijk} P_k , \\
[J_i^{(\rho)}, Q_j^{(\rho)}] &= i\varepsilon_{ijk} Q_k^{(\rho)} , \\
[Q_i^{(\rho)}, P_j] &= 2i\omega \delta_{ij} H + 2i\omega^2 \varepsilon_{ijk} J_k^{(\rho)} , \\
[Q_i^{(\rho)}, Q_j^{(\rho)}] &= [P_i, P_j] = 0 .
\end{align*}
\]

Another basis is of the Poincaré type being formed by \( \{H, P_i, J_i^{(\rho)}, K_i^{(\rho)}\} \). This has the commutation rules given by equations (28a), (29a), (30a), (33a), (34a) and \( [P_i, K_j^{(\rho)}] = i\delta_{ij} H - i\omega \varepsilon_{ijk} J_k^{(\rho)} , \) while the commutator (31a) has to be rewritten as \( [H, K_i^{(\rho)}] = iP_i - i\omega K_i^{(\rho)} . \)
The last two bases bring together the conserved energy (27) and momentum (32a) which are the only genuine orbital operators, independent on $\rho$. What is specific for the de Sitter symmetry is that these operators can not be put simultaneously in diagonal form since, according to equation (33a), they do not commute to each other. Therefore, there are no mass-shells [15].

4.3 Casimir operators

The first invariant of the CR $T^{(\rho)}$ is the quadratic Casimir operator

$$C_1^{(\rho)} = -\omega^2 \frac{1}{2} X^{(\rho)}_{(AB)} X^{(\rho)}_{(AB)}$$

which can be calculated according to equations (24)-(27) and (32). After a few manipulation we obtain its definitive expression

$$C_1^{(\rho)} = E_{KG} + 2i\omega e^{-\omega t} S_{0i}^{(\rho)} \partial_i - \omega^2 (\vec{S}^{(\rho)})^2,$$

depending on the Klein-Gordon operator of the scalar field,

$$E_{KG} = -\partial_t^2 - 3\omega \partial_t + e^{-2\omega t} \Delta, \quad \Delta = \vec{\partial}^2.$$

The second Casimir operator,

$$C_2^{(\rho)} = -\eta_{AB} W^{(\rho)} A W^{(\rho)} B,$$

is written with the help of the five-dimensional vector-operator $W^{(\rho)}$ whose components read [6]

$$W^{(\rho)} A = \frac{1}{8} \omega \varepsilon^{ABCDE} X^{(\rho)}_{(BC)} X^{(\rho)}_{(DE)},$$

where $\varepsilon^{01234} = 1$ and the factor $\omega$ assures the correct flat limit. After a little calculation we obtain the concrete form of these components,

$$W_0^{(\rho)} = \omega \vec{J}^{(\rho)} \cdot \vec{R}^{(\rho)},$$

$$W_i^{(\rho)} = H J_i^{(\rho)} + \omega \varepsilon_{ijk} K_j^{(\rho)} R_k^{(\rho)},$$

$$W_4^{(\rho)} = -\omega \vec{J}^{(\rho)} \cdot \vec{K}^{(\rho)},$$

12
which indicate that $W^{(\rho)}$ plays an important role in theories with spin, similar to that of the Pauli-Lubanski operator (13) of the Poincaré symmetry. For example, the helicity operator is now $W_0^{(\rho)} - W_4^{(\rho)} = S_i^{(\rho)} P_i$.

Replacing then the components (44)-(46) in equation (42) we are faced with a complicated calculation but which can be performed using algebraic codes under Maple. Thus we obtain the closed form of the second Casimir operator,

$$C_2^{(\rho)} = -(\vec{S}^{(\rho)})^2 (\partial_t^2 + 3 \omega \partial_t + 2 \omega^2) + 2 e^{-\omega t} (i S_0^{(\rho)} - \varepsilon_{ijk} S_i^{(\rho)} S_j^{(\rho)} \partial_k \partial_t$$

$$- e^{-2\omega t} \left( (\vec{S}_0^{(\rho)})^2 \Delta - (S_i^{(\rho)} S_j^{(\rho)} + S_0^{(\rho)} S_0^{(\rho)}) \partial_i \partial_j \right)$$

$$+ 2 i \omega e^{-\omega t} (S_i^{(\rho)} S_j^{(\rho)} S_j^{(\rho)} + S_0^{(\rho)} S_0^{(\rho)}) \partial_k$$, (47)

which represents a step forward to a complete theory of the de Sitter invariants of the covariant fields.

Hence, we derived the general expressions of both the Casimir operators of the CRs $T^{(\rho)}$ of the group $Sp(2,2)$ induced by the finite-dimensional representations $\rho$ of the $SL(2,\mathbb{C})$ group. In general, these operators are complicated and can not be related to each other in an arbitrary frame and for any representation $\rho$. On the other hand, here we do not have a theory of reducibility in momentum representation similar to the Wigner one. Therefore, we must restrict ourselves to study these invariants in the configuration space or at most by using momentum expansions in which only $P_i$ are diagonal while $H$ does not do that.

It is interesting to look for the invariants of the particles at rest in the chart $\{t, \vec{x}\}$. These have the vanishing momentum ($P_i \sim 0$) so that $H$ acts as $i\partial_t$ and, therefore, it can be put in diagonal form its eigenvalues being just the rest energies, $E_0$. Then, for each subspace $V_s \subset V^{(\rho)}$ of given spin, $s$, we obtain the eigenvalues of the first Casimir operator,

$$C_1^{(\rho)} \sim E_0^2 + 3i\omega E_0 - \omega^2 s(s + 1),$$

using equations (40) and (41) while those of the second Casimir operator,

$$C_2^{(\rho)} \sim s(s + 1)(E_0^2 + 3i\omega E_0 - 2\omega^2),$$

result from equation (47). These eigenvalues are real numbers so that the rest energies, $E_0 = \Re E_0 - \frac{3i\omega}{2}$, must be complex numbers whose imaginary parts are due to the decay produced by the de Sitter expansion. The above
results indicate that the CRs are reducible to direct sums of UIRs of the principal series \([17], (p, q), \) with \(p = s\) and \(q(1 - q) = (\Re E_0)^2 + \frac{1}{4} \Re\) \([6]\). What is new here is that we meet only one type of UIRs, denoted by \(\Re E_0, s\), which are completely determined by the rest energy and the spin defined as in special relativity. In the flat limit these UIRs tend to the corresponding Wigner ones.

Another important point is to study the covariant fields with unique spin in the sense of the \(SL(2, \mathbb{C})\) symmetry. In this case, we must select the representations \(\rho\) with unique spin, \((s, 0)\) or \((0, s)\) and, obviously, \((s, 0) \oplus (0, s)\), for which we have to replace \(S^{(\rho)}_{\omega i} = \pm iS^{(\rho)}_{i}\) in equation \([17]\) finding the remarkable identity

\[
C_2^{(s)} = C_1^{(s)}(\vec{S}(s))^2 - 2\omega^2(\vec{S}(s))^2 + \omega^2[(\vec{S}(s))^2]^2.
\]

(50)

Analyzing particular fields, as for example the vector field presented in the next section, we see that this identity holds even for other representations \(\rho\) for which the uniqueness of the spin is guaranteed by supplemental constraints. Moreover, whether the particles are at rest then equations \([48]\) and \([49]\) lead to the identity \([50]\) for each subspace \(\mathcal{V}_s \subset \mathcal{V}_{(\rho)}\) separately. This suggests that the CRs with unique spin \(s\) are equivalent with the UIRs \(\Re E_0, s\).

However, despite of the above results, it remains to find the concrete form of the transformations among the CRs and the UIRs we found here. Another problem is the relation between the rest energy and mass since in the de Sitter case we do not have a general rule analogous to the Poincaré mass condition (providing \(E_0 = \pm m\)). This impediment leads to ambiguities as we shall see in the next section where we show that the masses defined by the usual field equations give rise to different rest energies for the scalar, vector and Dirac fields minimally coupled to the de Sitter gravity.

Finally, we specify that the physical interpretation adopted here is plausible since in the flat limit we recover the usual physical meaning of the Poincaré generators. We observe that the generators \([24]\) are independent on \(\omega\) having the same form as in the Minkowski case, \(J^{(\rho)}_k = \hat{J}_k^{(\rho)}\). The other generators have the limits

\[
\lim_{\omega \to 0} H = \hat{H} = i\partial_t, \quad \lim_{\omega \to 0} (\omega R^{(\rho)}_i) = \hat{P}_i = i\partial_t, \quad \lim_{\omega \to 0} K^{(\rho)}_i = \hat{K}_i^{(\rho)},
\]

(51)

which means that the basis \(\{H, P_i, \hat{J}_i^{(\rho)}, \hat{K}_i^{(\rho)}\}\) of the algebra \(s(M) = sp(2, 2)\) tends to the basis \(\{\hat{H}, \hat{P}_i, \hat{J}_i^{(\rho)}, \hat{K}_i^{(\rho)}\}\) of the \(s(M_0)\) algebra when \(\omega \to 0\). More-
over, the Pauli-Lubanski operator (13) is the flat limit of the five-dimensional vector-operator (43) since
\[
\lim_{\omega \to 0} W_0^{(\rho)} = \hat{W}_0^{(\rho)}, \quad \lim_{\omega \to 0} W_i^{(\rho)} = \hat{W}_i^{(\rho)}, \quad \lim_{\omega \to 0} W_4^{(\rho)} = 0. \tag{52}
\]
Under such circumstances the limits of our invariants read
\[
\lim_{\omega \to 0} C_1^{(\rho)} = \hat{C}_1 = \hat{\rho}^2, \quad \lim_{\omega \to 0} C_2^{(\rho)} = \hat{C}_2^{(\rho)}, \tag{53}
\]
indicating that their physical meaning may be related to the mass and spin of the matter fields in a similar manner as in special relativity.

5 Covariant physical fields

In order to investigate how the de Sitter invariants may depend on mass and spin we must focus only on the fields satisfying equations that can be seen as defining the mass in each particular case separately. These are the usual fields with unique spin, i.e., the scalar, (vector) Proca and Dirac fields, whose Casimir operators obey the identity (50).

5.1 The massive scalar field

The simplest example is the massive scalar field $\phi$ minimally coupled to the de Sitter gravity. This satisfies the Klein-Gordon equation
\[
\frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} g^{\mu \alpha} \partial_\alpha \phi) + m^2 \phi = 0, \quad g = |\det(g_{\mu \nu})|, \tag{54}
\]
which can be put in the form $\mathcal{E}_{KG} \phi = m^2 \phi$ using the operator (41). Since $\rho = (0, 0)$ there are only genuine orbital generators commuting with $\mathcal{E}_{KG}$ and the de Sitter invariants $\hat{C}_1 = \mathcal{E}_{KG} \sim m^2$ and $\hat{C}_2 = 0$.

The rest energy of the scalar field deduced from equation (48) reads
\[
E_0 = -\frac{3i\omega}{2} \pm \omega k_s, \quad k_s = \sqrt{\frac{m^2}{\omega^2} - \frac{9}{4}}. \tag{55}
\]
We remind the reader that $ik_s$ is the index of the time-dependent Hankel functions of the plane wave solutions of the Klein-Gordon equation [12]. This
result can be obtained directly by solving the mentioned equation for the rest particle with \( P_i \sim 0 \).

The conclusion is that the de Sitter invariants of the scalar field minimally coupled to gravity define the UIRs \([\pm \omega k_s, 0]\) and coincide to the Poincaré invariants of the Wigner UIRs \((\pm m, 0)\) of the flat case. For other types of coupling this property does not hold because of the supplemental terms introduced by these couplings [5].

### 5.2 The massive Dirac field

The massive Dirac field \( \psi \) on the de Sitter spacetime \((M, g)\) is defined as a spinor field which transforms under gauge transformations according to the spinor representation \( \rho_s = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) of the \( SL(2, \mathbb{C}) \) group. The Dirac matrices (labeled by local indices) satisfy the identities \( \{ \gamma^\alpha, \gamma^\beta \} = 2 \eta^{\alpha \beta} \mathbf{1}_{4 \times 4} \) and give rise to the generators \( \rho_s(S_{\mu \nu}) = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \). The covariant Dirac equation, \( \mathcal{E}_D \psi = m \psi \), of the spinor field minimally coupled to gravity, is governed by the Dirac operator which has the form

\[
\mathcal{E}_D = i \gamma^0 \partial_t + ie^{-\omega t} \gamma^i \partial_i + \frac{3i\omega}{2} \gamma^0 ,
\]

in the chart \( \{ t, \vec{x} \} \) and the gauge (23). The generators defined by equations (24)-(27) commute with \( \mathcal{E}_D \) offering one a large collection of operators among them various sets of commuting operators were used for deriving different Dirac quantum modes in four dimensions [15, 16]. Notice that generalizations to higher dimensions were also proposed [23].

The first de Sitter invariant is derived using equations (40) and (56) which lead to the identity

\[
\mathcal{C}_1^{(\rho_s)} = \mathcal{E}_D^2 + \frac{3}{2} \omega^2 \mathbf{1}_{4 \times 4} \sim m^2 + \frac{3}{2} \omega^2 .
\]

This result and equation (48) yield the rest energy of the Dirac field,

\[
E_0 = -\frac{3i\omega}{2} \pm m ,
\]

which has a natural simple form where the decay (first) term is added to the usual rest energy of special relativity. A similar result can be obtained by solving the Dirac equation with vanishing momentum.
The second invariant results from equations (50) and (57) if we take into account that \((\vec{S}^{(\rho_s)})^2 = \frac{3}{16} \mathbf{1}_{4 \times 4}\). Thus we find

\[
C_2^{(\rho_s)} = \frac{3}{4} E_D^2 + \frac{3}{16} \omega^2 \mathbf{1}_{4 \times 4} \sim \frac{3}{4} \left( m^2 + \frac{1}{4} \omega^2 \right) = \omega^2 (s + 1) \nu_+ \nu_- ,
\]

where \(s = \frac{1}{2}\) is the spin and \(\nu_\pm = \frac{1}{2} \pm \frac{im}{\omega}\) are the indices of the Hankel functions giving the time modulation of the Dirac spinors of the momentum basis [15].

These invariants define the UIRs \([\pm m, \frac{1}{2}]\). In the flat limit we recover the well-known results

\[
\lim_{\omega \to 0} C_1^{(\rho_s)} \sim m^2 , \quad \lim_{\omega \to 0} C_2^{(\rho_s)} \sim \frac{3}{4} m^2 ,
\]

(60)
corresponding to the Wigner UIRs \((\pm m, \frac{1}{2})\).

5.3 The Proca field

The theory of the Proca (massive vector) field \(A\), minimally coupled to the gravity of the de Sitter spacetime, is based on the Proca equation in natural frames,

\[
\frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} g^{\nu \alpha} g^{\mu \beta} F_{\alpha \beta}) + m^2 A^\mu = 0 ,
\]

(61)
where \(F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the field strength. The Lorentz condition,

\[
\partial_\mu (\sqrt{g} A^\mu) = 0 ,
\]

(62)
which is mandatory for \(m \neq 0\), guarantees the uniqueness of the spin \(s = 1\). Recently we presented the complete quantum theory of the Proca field in the de Sitter moving chart with conformal time [13]. However, here we must consider the vector field in non-holonomic frames where we can exploit the results of the previous section.

For outlining the theory in non-holonomic frames we start with the column matrix \(A = [A^0, A^1, A^2, A^3]^T\) formed by the components \(A^\alpha = \tilde{e}_\alpha^\mu A^\mu\) in the local frames defined by the gauge (23) in the chart \(\{t, \vec{x}\}\). Then, after a little calculation, we can put the Proca equation (61) in the form \(\mathcal{E} P A = m^2 A\)
defining the Proca operator

\[
\mathcal{E}_P = \begin{bmatrix}
e^{-2\omega t} \Delta & e^{-\omega t} \partial_t (\partial_t + \omega) & e^{-\omega t} \partial_2 (\partial_t + \omega) & e^{-\omega t} \partial_3 (\partial_t + \omega) \\
-e^{-\omega t} \partial_1 (\partial_t + \omega) & \mathcal{E} - e^{-2\omega t} \partial_1^2 & -e^{-2\omega t} \partial_1 \partial_2 & -e^{-2\omega t} \partial_1 \partial_3 \\
-e^{-\omega t} \partial_2 (\partial_t + \omega) & -e^{-2\omega t} \partial_1 \partial_2 & \mathcal{E} - e^{-2\omega t} \partial_2^2 & -e^{-2\omega t} \partial_2 \partial_3 \\
-e^{-\omega t} \partial_3 (\partial_t + \omega) & -e^{-2\omega t} \partial_1 \partial_3 & -e^{-2\omega t} \partial_2 \partial_3 & \mathcal{E} - e^{-2\omega t} \partial_3^2 
\end{bmatrix} \tag{63}
\]

where \( \mathcal{E} = \mathcal{E}_{KG} - 2\omega^2 \). In addition, we introduce the line matrix-operator

\[
\mathcal{L} = \begin{bmatrix}
\partial_t + 3\omega & e^{-\omega t} \partial_1 & e^{-\omega t} \partial_2 & e^{-\omega t} \partial_3
\end{bmatrix}
\tag{64}
\]

which helps us to write the Lorentz condition (62) simply as \( \mathcal{L} A = 0 \).

The vector field \( A \) is defined on the carrier space \( \mathcal{V}_{(\rho_v)} = \mathcal{V}_0 \oplus \mathcal{V}_1 \) of the usual vector representation \( \rho_v \) of \( \frac{1}{2}, \frac{1}{2} \) which is irreducible. Therefore, the generators (24)-(27) have to be calculated by using the well-known matrices of the representation \( \rho_v \) [19]. The spin terms of the rotation generators act in the three-dimensional subspace \( \mathcal{V}_1 \) (with \( s = 1 \)) while the elements of \( \mathcal{V}_0 \) behave as scalars. The Lorentz condition which selects the spin \( s = 1 \) without eliminating the scalars must guarantee that the Casimir operators have the same action on both the subspaces of \( \mathcal{V}_{(\rho_v)} \). Indeed, a straightforward calculation leads to the following identities

\[
\begin{align*}
\mathcal{C}_1^{(\rho_v)} &= \mathcal{E}_P + \mathcal{D} \mathcal{L} , \\
\mathcal{C}_2^{(\rho_v)} &= 2(\mathcal{C}_1^{(\rho_v)} - \mathcal{D} \mathcal{L}) = 2 \mathcal{E}_P ,
\end{align*}
\tag{65, 66}
\]

where we use the column matrix-operator

\[
\mathcal{D} = \begin{bmatrix}
-\partial_t & e^{-\omega t} \partial_1 & e^{-\omega t} \partial_2 & e^{-\omega t} \partial_3
\end{bmatrix}^T
\tag{67}
\]

Hereby we observe that the Lorentz condition (\( \mathcal{L} A = 0 \)) allows us to drop out the term \( \mathcal{D} \mathcal{L} \) remaining with the eigenvalues

\[
\mathcal{C}_1^{(\rho_v)} \sim m^2 \quad \mathcal{C}_2^{(\rho_v)} \sim 2m^2 ,
\tag{68}
\]

which coincide to those of the Wigner UIRs \((\pm m, 1)\) of the flat case.

The scalar component \( A^0 \) vanishes when the particle is at rest such that the rest energy has to be calculated only for \( s = 1 \). Using equation (48) or solving directly the Proca equation we obtain the rest energy

\[
E_0 = -\frac{3i\omega}{2} \pm \omega k_v , \quad k_v = \sqrt{\frac{m^2 - 1}{4}} . \tag{69}
\]

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This means that the Proca field transforms according to the UIRs $[\pm \omega k_v, s]$. As in the scalar case, $i k_v$ is the index of the Hankel functions of the plane wave solutions of the Proca equation [13].

6 Concluding remarks

We succeeded here to derive the generators of the CRs of the $Sp(2, 2)$ group induced by the finite-dimensional representations of the $SL(2, \mathbb{C})$ group. Moreover, we expressed the Casimir operators of these representations in closed forms focusing on the properties of their eigenvalues that represent the principal invariants produced by the de Sitter symmetry.

Our main goal was to find the physical meaning of these invariants analyzing how their concrete values depend on mass, spin and the de Sitter Hubble constant. Our new results are independent on the local chart and the gauge we use representing the universal values of the de Sitter invariants summarized in the next table.

| field      | spin | $C_1^{(\rho)}$ | $C_2^{(\rho)}$ |
|------------|------|---------------|---------------|
| Klein-Gordon | 0    | $m^2$         | 0             |
| Dirac      | $\frac{1}{2}$ | $m^2 + \frac{3}{2} \omega^2$ | $\frac{3}{4} m^2 + \frac{3}{16} \omega^2$ |
| Proca      | 1    | $m^2$         | $2 m^2$       |

The conclusion is that the invariants of the scalar and vector fields minimally coupled to the de Sitter gravity obey the flat rule ($m^2$ and $m^2 s(s+1)$ respectively). The Dirac field behaves in a different manner since its invariants have supplemental terms depending on $\omega$ which vanishes in the flat limit.

The results presented here open the way to a complete theory of the CRs on the de Sitter manifolds. The principal problem which remains to be solved by further investigations is to find the transformations which assure the equivalence of our CRs with the UIRs of given mass and spin. We hope that this could be achieved by combining our results with those of the five-dimensional theory [6].

Another open problem is the relation between the mass and the rest energy which does not comply with a common rule as we deduce from equations (55), (58) and (69). In fact, only the real part of the rest energy of the Dirac field respects the usual rule ($\pm m$) while the scalar and vector fields have different rest energies in the minimal coupling. This is somewhat strange.
since the rest energy of the classical de Sitter geodesic motion is just that of special relativity, $E_0 = m$ (as it is shown in Appendix). However, the group theoretical methods are not able to enlighten this point since this relies on more general conjectures such as the definition of the field equations and the selection of the appropriate couplings among the covariant fields and gravity.

Finally, we note that the three examples we analyzed above are not enough for drawing general conclusions concerning the de Sitter invariants of the tensor fields of any rank or to speak about a spinor anomaly. We can say only that the mass and spin of the covariant fields defined on the de Sitter manifold have the same meaning and play a similar role as in special relativity giving rise to the principal invariants of the theory of free fields.

**Appendix: Classical conserved quantities**

It is a simple exercise to integrate the geodesic equations and to find the conserved quantities on a geodesic trajectory of the de Sitter background. These are proportional with $k_{(AB)}\mu u^\mu$ (where $u^\mu = \frac{dx^\mu}{ds}$) and can be derived by using the Killing vectors $\text{(21)}$ and $\text{(22)}$. We assume that in the chart $\{t, \vec{x}\}$ the particle of mass $m$ has the conserved momentum $\vec{p}$ of components $p^i = \omega m(k_{(0)i}\mu - k_{(4)i}\mu)u^\mu$ so that we can write

$$u^0 = \frac{dt}{ds} = \sqrt{1 + \frac{\vec{p}^2}{m^2}e^{-2\omega t}}; \quad u^i = \frac{dx^i}{ds} = \frac{p^i}{m}e^{-2\omega t},$$

(70)

using the notation $p = |\vec{p}|$. Hereby we deduce the trajectory,

$$x^i(t) = x^i_0 + \frac{p^i}{\omega p^2} \left( \sqrt{m^2 + p^2e^{-2\omega t_0}} - \sqrt{m^2 + p^2e^{-2\omega t}} \right),$$

(71)

of a particle passing through the point $\vec{x}_0$ at time $t_0$.

Furthermore, we calculate the other conserved quantities, i.e. the energy,

$$E = \omega \vec{x}_0 \cdot \vec{p} + \sqrt{m^2 + p^2e^{-2\omega t_0}},$$

(72)

the angular momentum $\vec{\ell} = \vec{x}_0 \wedge \vec{p}$ and the vectors

$$\vec{k} = -\vec{r} - \frac{1}{\omega} \vec{p} = \vec{x}_0 E - \vec{p} \chi(t_0, \vec{x}_0),$$

(73)

corresponding to the operators $\text{(25)}$ and $\text{(26)}$ for $P_t \rightarrow -p^t$ and $H \rightarrow E$ and replacing $t \rightarrow t_0$ and $\vec{x} \rightarrow \vec{x}_0$ including in $\chi(x)$ given by equation $\text{(18)}$). All
these conserved quantities obey the identity $E^2 - \omega^2(\vec{l}^2 + \vec{r}^2 - \vec{k}^2) = m^2$ which represents the classical version of the first invariant (38). The second invariant vanishes since $W = 0$ as was expected in this spinless case.

When the particle is at rest, staying in $\vec{x}(t) = \vec{x}_0$ with $\vec{p} = 0$, then the non-vanishing conserved quantities are the rest energy $E_0 = m$ and $k_0 = -r_0 = m\vec{x}_0$. Thus we see that the classical rest energy on the de Sitter spacetime is the same as in special relativity.

References

[1] M. Ferraris and M. Francaviglia, in Mechanics, Analysis and Geometry: 200 Years after Lagrange Editor: M. Francaviglia (Elsevier Sci. Pub. B. V., 1991).

[2] L. Fatibene, M. Ferraris and M. Francaviglia, J. Math. Phys. 34, 1644 (1994); L. Fatibene, M. Ferraris M. Francaviglia and M. Godina, Gen. Relat. and Grav. 30, 1371 (1998).

[3] R. M. Wald, General Relativity (Univ. of Chicago Press: Chicago and London 1984).

[4] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Wiley, New York, 1972).

[5] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge 1982).

[6] J.-P. Gazeau and M.V. Takook, J. Math. Phys. 41, 4920 (2000); P. Bartesaghi, J.-P. Gazeau, U. Moschella and M. V. Takook, Class. Quantum. Grav. 18, 4373 (2001); T. Garidi, J.-P. Gazeau and M. Takook, J.Math.Phys. 44, 3838 (2003); J.-P. Gazeau and M. Lachize-Rey, arXiv:0802.3441.

[7] B. Allen and T. Jacobson, Commun. Math. Phys. 103, 669 (1986); N. C. Tsamis and R. P. Woodard, J. Math. Phys. 48, 042306 (2007), gr-qc/0608069; O. Bertolami and D. F. Mota, Phys. Lett. B 444, 96 (1999), gr-qc/9811087; T. Prokopec, O. Törnkvist and R. P. Woodard, Phys Rev. Lett. 89, 101301 (2002); T. Prokopec, N. C. Tsamis and R. P. Woodard, Phys. Rev. D 79, 043423 (2008).
[8] H. B. Lawson Jr. and M.-L. Michaelson, *Spin Geometry* (Princeton Univ. Press. Princeton, 1989).

[9] I. I. Cotăescu, *J. Phys. A: Math. Gen.* **33**, 9177 (2000).

[10] B. Carter and R. G. McLenaghan, *Phys. Rev. D* **19**, 1093 (1979).

[11] I. I. Cotăescu, *Europhys. Lett.* **86**, 20003 (2009).

[12] I. I. Cotăescu, C. Crucean and A. Pop, *Int. J. Mod. Phys. A* **23**, 2463 (2008).

[13] I. I. Cotăescu, *Gen. Relat. and Grav.* **42**, 861 (2010).

[14] I. I. Cotăescu and C. Crucean, to appear in *Progr. Theor. Phys.* **124** (2010).

[15] I. I. Cotăescu, *Phys. Rev. D* **64**, 084008 (2002).

[16] I. I. Cotăescu, R. Racoceanu and C. Crucean, *Mod. Phys. Lett. A* **21**, 1313 (2006); I. I. Cotăescu and C. Crucean, *Int. J. Mod. Phys. A* **23**, 3707 (2008).

[17] J. Dixmier, *Bull. Soc. Math. France* **89**, 9 (1961); B. Takahashi, *Bull. Soc. Math. France* **91**, 289 (1963).

[18] G. Mackey, *Ann. Math.* **44**, 101 (1942)

[19] W.-K. Tung, *Group Theory in Physics* (World Sci., Philadelphia, 1984).

[20] R. Geroch, *J. Math. Phys.* **9**, 1739 (1968).

[21] Y. Kossmann, *Comptes Rendus Acad. Sc. Paris, serie A* **264**, 344 (1967); *id.* **262**, 289 (1966); *id.* **262**, 394 (1966); Y. Kossmann, *Ann. di Matematica Pura et Appl.* **91** (1972).

[22] E. Witten, [hep-th/0106109](http://arxiv.org/abs/hep-th/0106109).

[23] J. F. Koksma and T. Protopop, *Class. Quantum Grav.* **26**, 125003 (2009).