Enhancement of particle creation in nonlinear resonant cavities

Dmitrii A. Trunin

1 Moscow Institute of Physics and Technology, 141701, Institutskiy pereulok, 9, Dolgoprudny, Russia

(Dated: December 29, 2022)

The rate of particle creation in a resonantly oscillating cavity is known to be approximately constant at large evolution times. Employing the Schwinger-Keldysh diagrammatic technique, we show that nonlinear interactions generate nonzero quantum averages and significantly enhance this rate. To illustrate this phenomenon, we consider a massless scalar field with a quartic interaction in a one-dimensional cavity with perfectly reflecting walls oscillating at twice the fundamental frequency.

I. INTRODUCTION

In nonstationary quantum systems, the notions of particle and vacuum state cannot be fixed once and forever, so the initial vacuum fluctuations can convert into real particles and produce measurable stress-energy fluxes "from nothing". This observation underlies many celebrated phenomena, including cosmological particle production [1–4], Schwinger [4, 5], Hawking [6–8], and Unruh [9–11] effects. Furthermore, the creation of real particles from vacuum fluctuations was recently observed in experiments with superconducting quantum circuits that model a resonant cavity with nonuniformly moving walls [12, 13] or time-dependent refractive index [14]. This phenomenon is known as the dynamical Casimir effect (DCE) and provides us with one of the most convenient testbeds for nonstationary quantum field theory [15–17].

All these phenomena, including the DCE, are usually studied in the tree-level approximation, where interactions between the quantum fields are believed to be negligible [2–4]. However, this approximation is generally valid only for relatively small evolution times. On the contrary, at large evolution times, interactions build up, and loop corrections to the energy level density and correlated pair density cannot be ignored [18, 19]. In this case, the correct expressions for the created particle number and stress-energy tensor are recovered only after the resummation of the leading secularly growing loop corrections. Moreover, the resummed expressions might significantly differ from the tree-level results even if interactions are very weak. The examples of such a resummation in various nonequilibrium systems can be found in [20–26].

Recently, the DCE was also shown to suffer from the secularly growing loop corrections. In Refs. [27, 28], the secular growth was established at the two-loop level. In Ref. [29], this observation was extended to an arbitrary loop order, and the leading secularly growing corrections were resummed using a simplified quantum-mechanical Hamiltonian. Nevertheless, this approach was restricted to weak deviations from the stationarity, i.e., to such motions that produce only a small number of particles. At the same time, all experimental implementations of the DCE are based on resonant cavities, where the number and total energy of created particles rapidly grow with time [30–42]. Furthermore, some of these implementations are essentially nonlinear [14, 43, 44]. Hence, it is important to extend the results of [27–29] to such nonlinear resonant cavities and check whether the interactions drastically affect the particle creation in the most feasible theoretical model of the DCE.

In this paper, we show that loop corrections significantly enhance the particle production in nonlinear resonant cavities. We consider a simple yet widespread case of resonant motion, in which the frequency of oscillations equals twice the frequency of the fundamental mode. It is remarkable that for this motion, correlation functions and quantum averages are approximately "two-loop exact", i.e., determined by the "setting sun" diagram with two classical vertices. This indicates that the late-time evolution of correlation functions and number of created particles in the resonant DCE is governed by the classical statistical approximation even if the initial quantum state is close to vacuum.

II. FREE FIELDS

We begin by considering a free scalar field in a one-dimensional resonant cavity with perfectly reflecting walls (we assume $c = \hbar = 1$):

\[
(\partial_t^2 - \partial_x^2) \phi(t, x) = 0, \quad \phi[t, L(t)] = \phi[t, R(t)] = 0, \quad (1)
\]

where functions $L(t)$ and $R(t)$ determine the positions of the left and right mirror, respectively. To fix the definitions of particle and vacuum in the asymptotic past, we assume that both mirrors are static, $L(t) = 0$ and

---

1 The characteristic time that distinguishes these regimes is usually inversely proportional to a positive power of the coupling constant. In a closed quantum system on a stationary background, this time roughly coincides with the thermalization time [18].

2 We consider the one-dimensional case for two reasons. First, calculations in this case are simpler due to the absence of transverse momentum. Second, the experimental implementations of the DCE [12–14] essentially work with one-dimensional scalars.
\( R(t) = \Lambda \), for \( t < 0 \). For such initial conditions, the quantized field is decomposed as follows [45–48]:

\[
\hat{\phi}(t, x) = \sum_{n=1}^{\infty} \left[ \hat{a}_n^\text{in} f_n^\text{in}(t, x) + \text{H.c.} \right].
\]

(2)

Here, operators \( \hat{a}_n^\text{in} \) and \( (\hat{a}_n^\text{in})^\dagger \) satisfy the bosonic commutation relations, and the in-mode functions \( f_n^\text{in}(t, x) \) are written in terms of auxiliary functions \( G(z) \) and \( F(z) \):

\[
f_n^\text{in}(t, x) = \frac{i}{\sqrt{4\pi n}} e^{-i\pi n G(t+x) - e^{-i\pi n F(t-x)}}. \]

(3)

These functions solve the generalized Moore’s equations:

\[
G[t + L(t)] - F[t - L(t)] = 0,
\]

\[
G[t + R(t)] - F[t - R(t)] = 2,
\]

and satisfy the initial conditions \( G(z \leq \Lambda) = F(z \leq 0) = z/\Lambda \) to ensure that the in-modes have a positive definite energy in the asymptotic past:

\[
f_n^\text{in}(t, x) = \frac{1}{\sqrt{4\pi n}} e^{-i\pi n t} \sin \frac{\pi n x}{\Lambda} \quad \text{for} \quad t < 0. \]

(5)

We can also define the in-vacuum as the state that is annihilated by all annihilation operators:

\[
\hat{a}_n^\text{in}|0\rangle = 0 \quad \text{for all} \quad n. \]

(6)

If the mirrors return to their initial positions and stay at rest after some interval of time \( T \), we can also introduce another field decomposition:

\[
\hat{\phi}(t, x) = \sum_{n=1}^{\infty} \left[ \hat{a}_n^\text{out} f_n^\text{out}(t, x) + \text{H.c.} \right],
\]

(7)

where operators \( \hat{a}_n^\text{out} \) and \( (\hat{a}_n^\text{out})^\dagger \) again satisfy the bosonic commutation relations, and out-mode functions \( f_n^\text{out}(t, x) \) have a positive definite energy in the asymptotic future:

\[
f_n^\text{out}(t, x) = \frac{1}{\sqrt{4\pi n}} e^{-i\pi n t} \sin \frac{\pi n x}{\Lambda} \quad \text{for} \quad t > T.
\]

(8)

In general, in- and out-modes do not coincide, correspond to different vacua, and are related via the canonical Bogoliubov transformation [2–4]:

\[
f_n^\text{out} = \sum_k \left[ \alpha_{kn}^* f_n^\text{in} - \beta_{kn} (f_k^\text{in})^* \right],
\]

\[
\hat{a}_n^\text{out} = \sum_k \left[ \alpha_{kn} \hat{a}_k^\text{in} + \beta_{kn} (\hat{a}_k^\text{in})^\dagger \right].
\]

(9)

The Bogoliubov coefficients:

\[
\alpha_{nk} = \frac{\langle f_n^\text{in} | f_k^\text{out} \rangle}{\langle f_n^\text{in} | f_n^\text{in} \rangle} \quad \text{and} \quad \beta_{nk} = -\frac{\langle f_k^\text{in} | f_n^\text{out} \rangle}{\langle f_k^\text{in} | f_k^\text{in} \rangle},
\]

(10)

are determined by the motion of mirrors in the interval \( 0 < t < T \) and calculated using the Klein-Gordon inner product:

\[
(u, v) = -i \int^{R(t)}_{L(t)} [u\partial_t v^* - v^*\partial_t u] \, dx.
\]

(11)

Note that we can also consider the modes that have a positive definite energy at a fixed moment \( 0 < t < T \) and similarly calculate the corresponding Bogoliubov coefficients.

Now, it is straightforward to see that the expectation values of the operators \( (\hat{a}_n^\text{out})^\dagger \hat{a}_q^\text{out} \) and \( \hat{a}_n^\text{out} \hat{a}_q^\text{out} \), which have physical meaning at \( t > T \), are not zero, thus indicating a difference between the in- and out-vacuum states:

\[
n_{pq}^\text{out} = \langle 0 | (\hat{a}_p^\text{out})^\dagger \hat{a}_q^\text{out} | 0 \rangle = \sum_{n,k} \beta_{np}^* \beta_{nq} \sum_{n,k} \left[ \alpha_{np}^* \alpha_{kq} + \beta_{np}^* \beta_{nq} \beta_{kp} \right] n_{nk}^\text{in} + \sum_{n,k} \beta_{np} \alpha_{kq} n_{nk}^\text{in} + \sum_{n,k} \alpha_{np} \beta_{kq} n_{nk}^\text{in},
\]

(12)

\[
\kappa_{pq}^\text{out} = \langle 0 | \hat{a}_p^\text{out} \hat{a}_q^\text{out} | 0 \rangle = \sum_{n,k} \alpha_{np} \beta_{nq} \sum_{n,k} \left[ \beta_{np}^* \alpha_{kq} + \beta_{nq}^* \beta_{kp} \right] n_{nk}^\text{in} + \sum_{n,k} \alpha_{np} \alpha_{kq} n_{nk}^\text{in} + \sum_{n,k} \beta_{np}^* \beta_{kq} n_{nk}^\text{in}.
\]

(13)

We keep the initial values of \( n_{nk}^\text{in} = \langle 0 | (\hat{a}_n^\text{in})^\dagger \hat{a}_n^\text{in} | 0 \rangle \) and \( n_{nk}^\text{in} = \langle 0 | \hat{a}_n^\text{in} \hat{a}_k^\text{in} | 0 \rangle \) for generality, although they are zero for a vacuum initial state. These quantum averages are referred to as the energy level density and correlated pair density, and the diagonal part \( n_p = n_{pp} \) (no sum) has the meaning of the number of particles populating the \( p \)-th mode. Both quantities are experimentally measurable when the cavity is static and frequently used to track changes in the vacuum state (e.g., see [14]). Moreover, they determine such observables as correlation functions and stress-energy fluxes.

### III. RESONANT PUMPING

The most experimentally feasible and theoretically illustrative type of resonant motion is the one that occurs at twice the frequency of the fundamental mode:

\[
L(t) = 0, \quad R(t) = \Lambda \left[ 1 + \epsilon \sin \left( \frac{2\pi t}{\Lambda} \right) \right],
\]

(14)

where \( \epsilon \ll 1 \) and \( 0 < t < T = \Lambda \tau_f, \tau_f \in \mathbb{N} \), cf. [12–14, 33]. Due to that reason, we focus on this particular type of
motion in the present paper and defer its generalizations to future work.

For resonant motion (14), the auxiliary functions coincide, \( G(t) = F(t) \), and are given by the following approximate expression valid for times \( 1/\epsilon \ll t/\Lambda \ll 1/\epsilon^2 \) \cite{30–32}:

\[
G(t) \approx \frac{t}{\Lambda} - \frac{1}{\pi} \arctan \left[ \frac{1 - \zeta(t)}{1 + \zeta(t)} \right] + \frac{1}{\pi} \ln \left( 1 + \frac{1}{\zeta(t)^2} \right) + O(\epsilon),
\]

where we introduced a short notation for \( \zeta(t) = e^{-2\pi \epsilon t/\Lambda} \).

Note that in the considered time interval, the function \( G(t) \) rapidly approaches a staircase profile with an exponentially small width of the stair riser. For practical purposes, this profile can be approximated by a piecewise linear function:

\[
G(t) \approx \begin{cases} 
\tau + 2\delta \xi + \delta, & \text{as } -\frac{1}{2} \leq \xi < -\delta, \\
\tau + \frac{1}{2} + \frac{1}{2\delta} \xi^2 + \delta \xi, & \text{as } -\delta \leq \xi < \delta, \\
\tau + 1 + 2\delta \xi - \delta, & \text{as } \delta \leq \xi < \frac{1}{2}.
\end{cases}
\]

Here, we parametrize the argument as \( \tau + 1/2 + \xi, \tau \in \mathbb{N}, \xi \in [-1/2,1/2) \), and approximate the half-width of the \( n \)-th stair riser as \( \delta = \frac{1}{2} e^{-2\pi \epsilon t/\Lambda} \) (so, essentially, \( \tau, \xi, \) and \( \delta \) are functions of \( t \)).

We emphasize that the parameter \( \delta \) has an important physical meaning: it determines the threshold frequency of modes that are affected by the mirror motion. Indeed, let us show that the Bogoliubov coefficients rapidly decay for mode numbers larger than \( 1/\delta(t) \) in the interval \( 1/\epsilon \ll t/\Lambda \ll 1/\epsilon^2 \). Substituting the in-modes (3) into the definitions (10), integrating them by parts, and employing the Moore’s equations (4), we get the following integral representation for the coefficients \( \alpha_{nk} \) and \( \beta_{nk} \):

\[
\frac{\beta_{nk}}{\alpha_{nk}} = \frac{1}{2} \sqrt{\frac{k}{n}} \int_{t/\Lambda - 1}^{t/\Lambda + 1} e^{-i\pi n G(\Lambda z) + i\pi k z} dz.
\]

Note that \( |\alpha_{nk}| \approx |\alpha_{k|n}| \) and \( |\beta_{nk}| \approx |\beta_{k|n}| \). These approximate identities are proved by integrating (17) by parts and keeping in mind that the inverse function is equal to

\[
G^{-1}(z)/\Lambda \approx G(\Lambda z + \Lambda/2) - 1/2
\]

for \( \epsilon \ll 1 \) and \( z \gg 1/\epsilon \).

First, we numerically estimate integral (17) for arbitrary values of \( n \) and \( k \), see Fig. 1. These numerical results imply that the Bogoliubov coefficients exponentially decay as

\[
\frac{\beta_{nk}}{\alpha_{nk}} \sim e^{-(n+k)\delta} \quad \text{for } n, k \gg 1/\delta.
\]

So, in any expressions involving the Bogoliubov coefficients, summations over frequencies are effectively cut off at \( n = 1/\delta \) and \( k = 1/\delta \).

Second, we analytically calculate the integral (17) for moderate frequencies employing the approximation (16):

\[
\frac{\beta_{nk}}{\alpha_{nk}} \approx \frac{1}{\pi} \frac{1 - (-1)^n}{(n + 2k\delta)(k + 2n\delta)},
\]

for \( n, k \ll 1/\delta \).

Finally, keeping in mind relations (19)–(20) and assuming \( n_{pq}^{in} = 0, n_{pq}^{in} = 0 \), we estimate the energy level density and correlated pair density for a free theory (1) in the initial vacuum state:

\[
n_{pq}^{out} \approx \kappa_{pq}^{out} \approx \frac{2}{\pi} \frac{1 - (-1)^{p+q-2}}{(p+q-2)/2} \frac{1}{\sqrt{pq} \Lambda},
\]

for \( p, q \ll 1/\delta \) and \( n_{pq}^{out} \approx \kappa_{pq}^{out} \approx 0 \) otherwise. In particular, this approximate identity reproduces the rate of particle creation established in [30–33]:

\[
\frac{d}{dt} n_{pq}^{out} \approx \frac{2}{\pi} \frac{1 - (-1)^p}{p} \frac{\epsilon t}{\Lambda}
\]

for \( p \ll 1/\delta \), which confirms the validity of approximations (19), (20).

\section{IV. LOOP CORRECTIONS}

Now, let us turn on interactions, i.e., consider a non-linear generalization of the free model (1):

\[
(\partial_t^2 - \partial_x^2) \phi(t, x) = -\frac{\partial V(\phi)}{\partial \phi},
\]

where \( V(\phi) = \sum_{\alpha=3}^{\infty} \lambda_\alpha \phi^\alpha \). To be specific, we first discuss the quartic interaction:

\[
(\partial_t^2 - \partial_x^2) \phi(t, x) = -\lambda \phi^2(t, x),
\]

and then generalize the results to arbitrary potentials.
In nonstationary situations, interactions lead to two separate effects [18]. On one hand, they affect the particle production during the period of nonstationarity (0 < t < T in the model (14)), i.e., generate non-trivial loop corrections to the initial quantum averages \( n_{pq}^{in} \) and \( \kappa_{pq}^{in} \). On the other hand, interactions force the system to thermalize in the asymptotically static regions (at least, in dimensions higher than two). In this paper, we are interested in the particle production during the mirror’s motion, so we consider the initial vacuum state and assume that the coupling constant \( \Lambda \) is adiabatically turned on after \( t = t_0 \) < 0 and abruptly turned off after \( t = T \). In other words, we estimate the corrections to \( n_{pq}^{out} \) and \( \kappa_{pq}^{out} \) in the interacting theory by the moment \( t = T \). The further evolution of these quantum averages, which, we believe, describes the thermalization of the system, will be studied elsewhere.

In general, loop corrections to \( n_{pq}^{in} \) and \( \kappa_{pq}^{in} \) are conveniently calculated in the Schwinger-Keldysh diagram technique [49-54]. In the particular model (24), this technique contains two interaction vertices:

\[
-i\lambda \int_{t_0}^{T} dt \int_{L(t)}^{R(t)} dx \phi_{cl}^{\dagger} \phi_{cl}, \quad -i\lambda \int_{t_0}^{T} dt \int_{L(t)}^{R(t)} dx \phi_{q}^{\dagger} \phi_{q},
\]

and three propagators:

\[
G_{12}^{K(\text{Keldysh})} = -i\langle \phi_{cl}(t_1, x_1) \phi_{cl}(t_2, x_2) \rangle,
\]

\[
G_{12}^{R(\text{retarded})} = -i\langle \phi_{cl}(t_1, x_1) \phi_{q}(t_2, x_2) \rangle,
\]

\[
G_{12}^{A(\text{advanced})} = -i\langle \phi_{q}(t_1, x_1) \phi_{cl}(t_2, x_2) \rangle,
\]

where angle brackets denote the averaging w.r.t. some initial state, and \( \phi_{cl} \) and \( \phi_{q} \) correspond to the classical and quantum components of the field following the notations of the Schwinger-Keldysh technique. At the tree level, retarded and advanced propagators characterize the particle spectrum and do not depend on the initial state:

\[
iG_{12}^{R,\text{free}} = iG_{21}^{A,\text{free}} = \theta(t_1 - t - 2) \sum_n \left( n_{1,1,n}^{in} f_{2,n}^{in} - \text{H.c.} \right),
\]

where we introduce the short notation \( f_{a,n}^{in} = f_{n}^{in}(t_a, x_a) \). On the contrary, the tree-level Keldysh propagator is determined by initial quantum averages of interest to us:

\[
iG_{12}^{K,\text{free}} = \sum_{p,q} \left[ \left( \frac{\delta_{pq}}{2} + n_{pq}^{in} \right) f_{1,p}^{in} f_{2,q}^{in} + \kappa_{pq}^{in} f_{1,p}^{in} f_{2,q}^{in} + \text{H.c.} \right].
\]

Furthermore, propagators approximately preserve the form (27), (28) if both their points are taken to the future infinity while the difference is kept finite [18, 19, 51]:

\[
\frac{t_1 + t_2}{2} \gg \frac{1}{\lambda \Lambda} \quad \text{and} \quad \frac{t_1 + t_2}{2} \gg |t_1 - t_2|.
\]

This property allows us to extract the corrected quantum averages in the interacting theory from the exact Keldysh propagator in the limit in question.

To estimate the loop resummed Keldysh propagator in the interacting model (24) with resonantly moving mirrors (14) in the limit (29), we make four crucial observations.

First, we map the trajectories (14) to stationary ones:

\[
t + x = G^{-1}(\tau + \xi), \quad t - x = G^{-1}(\tau - \xi),
\]

in all internal vertices of diagrams that describe loop corrections to the Keldysh propagator, e.g.:

\[
V = -i\lambda \int_{t_0}^{T} dt \int_{L(t)}^{R(t)} dx \ f_{m}^{in}(t, x) f_{n}^{in}(t, x) f_{p}^{in}(t, x) f_{q}^{in}(t, x)
\]

\[
= -i\lambda \int_{t_0}^{T} dt \int_{0}^{1} d\xi \ d\tau \ G^{-1}(\tau - \xi) G^{-1}(\tau + \xi) \times
\]

\[
e^{-i\pi(m+n+p+q)\tau} \sin(\pi m \xi) \sin(\pi n \xi) \sin(\pi p \xi) \sin(\pi q \xi). \]

Second, we expect that the leading contribution to the loop corrections come from large evolution times: \( t/\Lambda = \tau \gg 1/\epsilon \). Keeping in mind identities (16) and (18), we approximate the \( G^{-1} \) with a piecewise-linear function:

\[
V \approx -i\lambda \Lambda^2 \int_{1/\epsilon}^{\tau_f} dt \int_{0}^{1} d\xi \ d\tau \ G^{-1}(\tau - \xi) G^{-1}(\tau + \xi) \times
\]

\[
e^{-i\pi(m+n+p+q)\tau} \sin(\pi m \xi) \sin(\pi n \xi) \sin(\pi p \xi) \sin(\pi q \xi). \]

Here, \( \delta_\xi(\tau) = 1/2 \delta_\xi \) if \( |\tau - s - 1/2| < \delta_\xi \) for some \( s \in \mathbb{N} \), \( \delta_\xi(\tau) = 2\delta_\xi \) otherwise, and \( \delta_\xi = \frac{1}{4} e^{-2\pi \epsilon s} \).

Third, the exponential decay of the Bogoliubov coefficients (19) and the relation (9) between the in-modes and the positive energy modes at a moment \( t \) imply that sums over the virtual momenta are effectively cut off\(^3\) at mode numbers \( n \sim 1/\delta_\xi \). Roughly speaking, we can exclude high-energy modes from consideration because they are unaffected by the mirror motion and are not involved in the particle creation\(^4\). Keeping in mind this cutoff, we replace the functions \( \delta_\xi(\tau) \) with exact delta-functions and reduce integrals (32) to sums:

\[
V \approx -i\lambda \Lambda^2 \sum_{s=1/\epsilon}^{\tau_f} g_m^s g_n^s g_p^s g_q^s
\]

where we introduce the notation for the “remnant” of the initial mode \( f_{n}^{in}(t, x) \):

\[
f_{n}^{in}(t, x) \to g_n^s = -i \frac{(-1)^s}{\sqrt{\pi n}} \left( \frac{1}{2} \right).
\]

\(^3\) More precisely, sums over larger momenta give negligible corrections to (35) if we consider times \( 1/\Lambda^2 \ll t/\Lambda \ll 1/(\Lambda^2)^2 \).

\(^4\) However, we cannot use \( 1/\delta_\xi \) as an ultraviolet regulator for other purposes, e.g., renormalization of mass and coupling constant.
The “two-loop exactness” of quantum averages (35) and (36) resembles the “one-loop exactness” of scalar electrodynamics on a strong electric field background [24, 25]. In that case, \( n_{pq} \) and \( \kappa_{pq} \) are also determined by the first nontrivial loop correction and uniformly grow at large evolution times due to the symmetry of mode functions.

We also emphasize that the argumentation of this section is equally applicable to general analytic potentials\(^6\), \( V(\phi) = \sum_{a=3}^{\infty} \lambda_a \phi^a \). Repeating it for each power of \( \phi \), we obtain an analog of Eq. (36):

\[
\frac{\Delta n_{pq}^{\text{out}}}{n_{pq}^{\text{out}}} \approx \frac{\Delta \kappa_{pq}^{\text{out}}}{\kappa_{pq}^{\text{out}}} \approx \sum_{a=2}^{\infty} \frac{8}{2a+1} \left( \frac{\Lambda T}{\lambda} \right)^2 \left( \frac{\epsilon T}{\Lambda} \right)^{2a} ,
\]

where we took into account that the odd powers are suppressed in the limit \( 1/\lambda \Lambda^2 \ll T/\lambda \ll 1/(\lambda \Lambda^2)^2 \) and \( 1/\epsilon \ll T/\lambda \ll 1/\epsilon^2 \). In particular, for the experimentally and theoretically interesting potential

\[
V(\phi) = \lambda \left[ \cos(\phi) + \phi^2/2 - 1 \right],
\]

we get \( \lambda_{2a} = \lambda(-1)^a/(2a)! \) and

\[
\Delta n_{pq}^{\text{out}} \approx \Delta \kappa_{pq}^{\text{out}} \sim \left( \Lambda \lambda T \right)^2 \left[ \sinh \left( \frac{\epsilon T}{\Lambda} \right) - \frac{\epsilon T}{\Lambda} \right].
\]

Thus, for interactions (38), number of particles created in each mode grows exponentially instead of linearly.

V. DISCUSSION

We have shown that the energy level density and the correlated pair density, which are generated during resonant oscillations of the cavity walls in an interacting theory, are significantly enhanced compared to a free theory. We emphasize that the enhancement factor (36) or (37) is determined by the coupling constants \( \lambda_a \) and relative amplitude of wall oscillations \( \epsilon \), but does not depend on the mode numbers \( p \) and \( q \) for the low-laying modes. Hence, the number of particles created in each mode, the total number of particles, and their total energy are enhanced by the same factor (36).

In particular, such an enhancement implies that the number of particles created in even modes is small in both linear and nonlinear models, as would be expected for resonant pumping [30–33]. At the same time, note that the number of particles created in separate modes grows slower than exponential (unless the interaction term is specially chosen), which distinguishes the resonant DCE from other parametric resonances.

Note that the leading loop corrections to the correlation functions and quantum averages (36)–(37) are given

\[\text{Figure 2. Loop corrections to the Keldysh propagator that do not contain internal retarded/advanced propagators. Solid lines denote the tree-level Keldysh propagators, half-dashed lines denote the retarded/advanced propagators.}\]

\[\text{Fourth, now, it is straightforward to see that any diagrams containing virtual retarded/advanced propagators are approximately zero. On one hand, if retarded propagator (27) connects two internal vertices, both} f_{in}^{\text{in}} \text{ and } f_{2,n}^{\text{in}} \text{ are eventually replaced with their “remnants” (34). On the other hand, functions } g_n^{\text{a}} \text{ are purely imaginary, and their product is purely real, so the r.h.s. of (27) is approximately zero. Therefore, as long as we are interested only in the leading contribution to the exact Keldysh propagator in the interacting theory, we can consider only such loop diagrams where internal vertices are connected by the Keldysh propagators alone (see Fig. 2). Furthermore, the “tadpole” diagrams (Fig. 2a–c) can be absorbed into the renormalized mass and do not contribute to the evolution of the initial state and particle creation [27–29]. Hence, we need to estimate only the “setting sun” diagram (Fig. 2d). Keeping in mind approximations (32)–(34) and employing decomposition (28), we obtain the leading correction to } n_{pq}^{\text{in}} \text{ and } \kappa_{pq}^{\text{in}}:\]

\[
\Delta n_{pq}^{\text{in}} \approx \frac{3}{5\pi} \left[ \frac{1}{2\sqrt{pq}} \right] \left( \Lambda \lambda T \right)^2 \left( \frac{\epsilon T}{\Lambda} \right)^3 ,
\]

for \( p, q < 1/\delta_{pq} \sim e^{2\pi \epsilon T/\Lambda} \).

Finally, we substitute \( \Delta n_{pq}^{\text{in}} \) and \( \Delta \kappa_{pq}^{\text{in}} \) into Eqs. (12)–(13) and take into account Eq. (21) to determine the relative correction to the quantum averages \( n_{pq}^{\text{out}} \) and \( \kappa_{pq}^{\text{out}} \), which are physically meaningful in the asymptotic future:

\[
\frac{\Delta n_{pq}^{\text{out}}}{n_{pq}^{\text{out}}} \approx \frac{\Delta \kappa_{pq}^{\text{out}}}{\kappa_{pq}^{\text{out}}} \approx \frac{12}{5} \left( \Lambda \lambda T \right)^2 \left( \frac{\epsilon T}{\Lambda} \right)^4 .
\]

So, loop contributions to the energy level density and correlated pair density significantly exceed the tree-level expressions in the time interval \( 1/\lambda \Lambda^2 \ll T/\lambda \ll 1/(\lambda \Lambda^2)^2 \) and \( 1/\epsilon \ll T/\Lambda \ll 1/\epsilon^2 \).

\[\text{\footnote{This reasoning is based on the behavior of } G(z), \text{ so it does not apply directly to higher-order resonances or nonresonant motions.}}\]
by the “setting sun” diagrams with two classical vertices\(^7\). Such a behavior indicates that the non-perturbative dynamics may be well described by the classical statistical approximation, so higher-order corrections can be calculated within the semiclassical approach \([19, 55–58]\). However, we emphasize that classical statistical approximation is valid only when all modes are highly populated. In the DCE with the initial vacuum state, this condition is indeed fulfilled at large evolution times, but violated at the very beginning of evolution. This observation distinguishes the DCE from the standard applications of the semiclassical approach.

Our results can be extended in several possible directions. First, the enhancement (36)–(37) encourages a careful measurement of the large-time asymptotics of \(\eta^\text{out}\) and \(\eta^\text{out}_X\) in experimental implementations of the DCE \([12–15]\) (see also Refs. \([59–62]\) for other nonlinear models of the DCE). We emphasize that this enhancement is noticeable only at evolution times \(T/A \gg 1/\Lambda^2\). Although such evolution times are by two orders larger than the typical measurement time in existing experiments, we expect them to be achieved in the future modifications \([29]\).

Second, the nonlinear DCE is very similar to an interacting scalar field in a rapidly expanding universe \([63–68]\) or its condensed matter analogs \([69–75]\). On the one hand, loop corrections to the Keldysh propagator grow equally rapidly with time for the DCE and for the light scalar field in an expanding Poincaré patch of de Sitter space. On the other hand, condensed matter analogs of particle creation in an expanding universe are closely related to parametric instabilities and can be described using a semiclassical approach. So, it is interesting to study the calculations of Sec. IV in light of this relation.

Finally, it is promising to extend the results of this paper to resonant oscillations different from (14) and study the evolution of quantum averages (35) in the asymptotic future \(t > T\).

**ACKNOWLEDGMENTS**

We are grateful to Emil Akhmedov and Damir Sadakov for valuable discussions and proofreading of the paper. We also thank an anonymous referee for helpful comments. This work was supported by the Russian Ministry of education and science and by the grant from the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”.

![ArXiv Preprint](https://arxiv.org/pdf/1503.02907v1)

---

\(^7\) That is, vertices with only one quantum component, e.g., the first vertex in (25).
[20] D. Krotov and A. M. Polyakov, Infrared sensitivity of unstable vacua, Nucl. Phys. B 849, 410 (2011) [arXiv:1012.2107].
[21] A. M. Polyakov, Infrared instability of the de Sitter space, arXiv:1209.4135.
[22] E. T. Akhmedov, Lecture notes on interacting quantum fields in de Sitter space, Int. J. Mod. Phys. D 23, 1430001 (2014) [arXiv:1309.2557].
[23] E. T. Akhmedov and F. Bascone, Quantum heating as an alternative of reheating, Phys. Rev. D 97, 045013 (2018) [arXiv:1710.06118].
[24] E. T. Akhmedov, N. Astrakhantsev, and F. K. Popov, Secularly growing loop corrections in strong electric fields, J. High Energy Phys. 09 (2014) 071 [arXiv:1405.5285].
[25] E. T. Akhmedov and F. K. Popov, A few more comments on secularly growing loop corrections in strong electric fields, J. High Energy Phys. 09 (2015) 085 [arXiv:1412.1554].
[26] D. A. Trunin, Particle creation in nonstationary large N quantum mechanics, Phys. Rev. D 104, 045001 (2021) [arXiv:2105.01647].
[27] E. T. Akhmedov and S. O. Alexeev, Dynamical Casimir effect and loop corrections, Phys. Rev. D 96, 065001 (2017) [arXiv:1707.02242].
[28] L. A. Akopyan and D. A. Trunin, Dynamical Casimir effect in nonlinear vibrating cavities, Phys. Rev. D 103, 065005 (2021) [arXiv:2012.02129].
[29] D. A. Trunin, Nonlinear dynamical Casimir effect at weak nonstationarity, Eur. Phys. J. C 82, no.5, 440 (2022) [arXiv:2108.07747].
[30] V. V. Dodonov, A. B. Klimov, and D. E. Nikonov, Quantum phenomena in resonators with moving walls, J. Math. Phys. 34, 2742 (1993).
[31] D. A. R. Dalvit and F. D. Mazzitelli, Renormalization group approach to the dynamical Casimir effect, Phys. Rev. A 57, 2113 (1998) [quant-ph/9710048].
[32] D. A. R. Dalvit and F. D. Mazzitelli, Creation of photons in an oscillating cavity with two moving mirrors, Phys. Rev. A 59, 3049 (1999) [quant-ph/9810092].
[33] V. V. Dodonov and A. B. Klimov, Generation and detection of photons in a cavity with a resonantly oscillating boundary, Phys. Rev. A 53, 2664 (1996).
[34] C. K. Law, Effective Hamiltonian for the radiation in a cavity with a moving mirror and a time-varying dielectric medium, Phys. Rev. A 49, 433 (1994).
[35] O. Méplan and C. Gignoux, Exponential growth of the energy of a wave in a 1D vibrating cavity: Application to the quantum vacuum, Phys. Rev. Lett. 76, 408 (1996).
[36] A. Lambrecht, M. T. Jaekel, and S. Reynaud, Motion induced radiation from a vibrating cavity, Phys. Rev. Lett. 77, 615 (1996) [quant-ph/9606029].
[37] C. K. Cole and W. C. Schieve, Radiation modes of a cavity with a moving boundary, Phys. Rev. A 52, 4405 (1995).
[38] L. Li and B. Z. Li, Geometrical method for the generalized Moore equations of a one-dimensional cavity with two moving mirrors, Chin. Phys. Lett. 19, 1061 (2002).
[39] V. V. Dodonov, Resonance photon generation in a vibrating cavity, J. Phys. A 31, 9835 (1998) [quant-ph/9810077].
[40] R. Schützhold, G. Plunien, and G. Soff, Trembling cavities in the canonical approach, Phys. Rev. A 57, 2311 (1998) [quant-ph/9709008].
[41] Ying Wu, K. W. Chan, M.-C. Chu, and P. T. Leung, Radiation modes of a cavity with a resonantly oscillating boundary, Phys. Rev. A 59, 1662 (1999).
[42] M. Crocc, D. A. R. Dalvit, and F. D. Mazzitelli, Resonant photon creation in a three-dimensional oscillating cavity, Phys. Rev. A 64, 013808 (2001) [quant-ph/0012040].
[43] T. Weiβl, B. Küng, E. Dumur, A. K. Feofanov, I. Matei, C. Naud, O. Buisson, F. W. J. Hekking, W. Guichard, Kerr coefficients of plasma resonances in Josephson junction chains, Phys. Rev. B 92, 104508 (2015) [arXiv:1505.05845].
[44] Yu. Krupko et al., Kerr nonlinearity in a superconducting Josephson metamaterial, Phys. Rev. B 98, 094516 (2018) [arXiv:1807.01499].
[45] G. T. Moore, Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity, J. Math. Phys. (N.Y.) 11, 2679 (1970).
[46] B. S. DeWitt, Quantum field theory in curved space-time, Phys. Rep. 19, 295 (1975).
[47] P. C. W. Davies and S. A. Fulling, Radiation from a moving mirror in two-dimensional space-time: conformal anomaly, Proc. R. Soc. A 348, 393 (1976).
[48] P. C. W. Davies and S. A. Fulling, Radiation from moving mirrors and from black holes, Proc. R. Soc. A 356, 237 (1977).
[49] J. S. Schwinger, Brownian motion of a quantum oscillator, J. Math. Phys. 2, 407 (1961).
[50] L. V. Keldysh, Diagram technique for nonequilibrium processes, Zh. Eksp. Teor. Fiz. 47, 1515 (1964) [Sov. Phys. JETP 20, 1018 (1965)].
[51] A. Kamenev, Field Theory of Non-Equilibrium Systems (Cambridge University Press, Cambridge, England, 2011); A. Kamenev, Many-body theory of non-equilibrium systems, cond-mat/0412296.
[52] J. Rammer, Quantum field theory of non-equilibrium states (Cambridge University Press, Cambridge, England, 2007).
[53] L. D. Landau and E. M. Lifshitz, Physical Kinetics (Pergamon Press, Oxford, 1981), Vol. 10.
[54] P. I. Arseev, On the nonequilibrium diagram technique: derivation, some features, and applications, Phys. Usp. 58, 1159 (2015).
[55] J. Berges and T. Gasenzer, “Quantum versus classical statistical dynamics of an ultracold Bose gas,” Phys. Rev. A 76, 033604 (2007) [cond-mat/0703163].
[56] G. Aarts and J. Berges, “Classical aspects of quantum fields far from equilibrium,” Phys. Rev. Lett. 88, 041603 (2002) [hep-ph/0107129].
[57] G. Aarts and J. Smit, “Classical approximation for time dependent quantum field theory: Diagrammatic analysis for hot scalar fields,” Nucl. Phys. B 511, 451 (1998) [hep-ph/9707342].
[58] A. A. Radovskaya and A. G. Semenov, “Semiclassical approximation meets Keldysh–Schwinger diagrammatic technique: scalar φ⁴,” Eur. Phys. J. C 81, no.8, 704 (2021) [arXiv:2003.06395].
[59] A. V. Dodonov and V. V. Dodonov, Dynamical Casimir effect via modulated Kerr or higher-order nonlinearities, Phys. Rev. A 105, 037309 (2022) [arXiv:2201.07625].
[60] R. Román-Ancheyta, C. González-Gutiérrez, and J. Récamier, Influence of the Kerr nonlinearity in a single nonstationary cavity mode, J. Opt. Soc. Am. B 34, 1170 (2017).
[61] I. M. d. Sousa and A. V. Dodonov, Microscopic toy model for the cavity dynamical Casimir effect, J. Phys. A 48, 245302 (2015) [arXiv:1504.02413].
[62] Y. N. Srivastava, A. Widom, S. Sivasubramanian, and M. Pradeep Ganesh, Dynamical Casimir effect instabilities, Phys. Rev. A 74, 032101 (2006).
[63] E. T. Akhmedov, U. Moschella, and F. K. Popov, Characters of different secular effects in various patches of de Sitter space, Phys. Rev. D 99, 086009 (2019) [arXiv:1901.07293].
[64] E. T. Akhmedov, U. Moschella, K. E. Pavlenko, and F. K. Popov, Infrared dynamics of massive scalars from the complementary series in de Sitter space, Phys. Rev. D 96, 025002 (2017) [arXiv:1701.07226].
[65] E. T. Akhmedov and P. A. Anempodistov, Loop corrections to cosmological particle creation, Phys. Rev. D 105, 105019 (2022) [arXiv:2204.01388].
[66] E. T. Akhmedov, K. V. Bazarov, and D. V. Diakonov, Quantum fields in the future Rindler wedge, Phys. Rev. D 104, 085008 (2021) [arXiv:2106.01791].
[67] F. Gautier and J. Serreau, Infrared dynamics in de Sitter space from Schwinger-Dyson equations, Phys. Lett. B 727, 541 (2013) [arXiv:1305.5705].
[68] F. Gautier and J. Serreau, Scalar field correlator in de Sitter space at next-to-leading order in a 1/N expansion, Phys. Rev. D 92, 105035 (2015) [arXiv:1509.05546].
[69] F. Jain, S. Weinfurtner, M. Visser, and C. W. Gardiner, “Analogue model of a FRW universe in Bose-Einstein condensates: Application of the classical field method,” Phys. Rev. A 76, 033616 (2007) [arXiv:0705.2077].
[70] I. Carusotto, R. Balbinot, A. Fabbri, and A. Recati, “Density correlations and dynamical Casimir emission of Bogoliubov phonons in modulated atomic Bose-Einstein condensates,” Eur. Phys. J. D 56, 391 (2010) [arXiv:0907.2314].
[71] T. V. Zache, V. Kasper, and J. Berges, “Inflationary preheating dynamics with two-species condensates,” Phys. Rev. A 95, no.6, 063629 (2017) [arXiv:1704.02271].
[72] A. Chatrchyan, K. T. Geier, M. K. Oberthaler, J. Berges, and P. Hauke, “Analog cosmological reheating in an ultracold Bose gas,” Phys. Rev. A 104, no.2, 023302 (2021) [arXiv:2008.0290].
[73] S. Butera and I. Carusotto, “Particle creation in the spin modes of a dynamically oscillating two-component Bose-Einstein condensate,” Phys. Rev. D 104, no.8, 083503 (2021) [arXiv:2105.09349].
[74] V. S. Barroso, A. Geelmuyden, Z. Fifer, S. Erne, A. Avgoustidis, R. J. A. Hill, and S. Weinfurtner, “Primary thermalisation mechanism of Early Universe observed from Faraday-wave scattering on liquid-liquid interfaces,” arXiv:2207.02199.
[75] S. Eckel, A. Kumar, T. Jacobson, I. B. Spielman, and G. K. Campbell, “A rapidly expanding Bose-Einstein condensate: an expanding universe in the lab,” Phys. Rev. X 8, no.2, 021021 (2018) [arXiv:1710.06800].