Upper bounds for the dominant dimension of Nakayama and related algebras

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Abstract

Optimal upper bounds are provided for the dominant dimensions of Nakayama algebras and more generally algebras $A$ with an idempotent $e$ such that there is a minimal faithful injective-projective module $eA$ and such that $eAe$ is a Nakayama algebra. This answers a question of Abrar and proves a conjecture of Yamagata in this case.

Keywords: Dominant dimension, Nakayama algebras, representation theory of finite dimensional algebras, Gorenstein dimension.

Introduction

The dominant dimension $\text{domdim}(A)$ of a finite dimensional algebra $A$ is defined as $\text{domdim}(A) := \sup\{n | I_i \text{ is projective for } i = 0, 1, \ldots, n\} + 1$, if $I_0$ is projective, and $\text{domdim}(A) := 0$, if $I_0$ is not projective, where $(I_i)_{i=0,1,...}$ are the terms of the minimal injective resolution of the right regular module $A$: $0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$.

Note that the dominant dimension is invariant under Morita equivalence and also under field extensions (see [Mue] Lemma 5). Thus we can assume from now on that all algebras are basic and split over the field unless stated otherwise. For this reason we assume throughout this article that algebras are given by quiver and relations if not stated otherwise. One of the most famous conjectures in the representation theory of finite dimensional algebras is the Nakayama conjecture. This conjecture states that the dominant dimension of a nonselfinjective finite dimensional algebra is always finite (see [Nak]). A stronger conjecture was given in [Yam], where Yamagata conjectured that the dominant dimension is bounded by a function depending on the number of simple modules of a nonselfinjective algebra. Since the finiteness of the dominant dimension of a nonselfinjective algebra follows as a corollary of the finiteness of the finitistic dimension, the Nakayama conjecture is true for many classes of algebras. Examples include algebras with representation dimension smaller than or equal to 3 (see [IgTo]). In contrast to that, explicit optimal bounds or values for the dominant dimension are rarely known.
Problem
For a given class of connected nonselfinjective algebras, find optimal bounds for the dominant dimensions.

In [Abr] Theorem 1.2.3., Abrar shows that the dominant dimension of connected quiver algebras with an acyclic quiver is bound by the number of projective-injective indecomposable modules and that this bound is optimal for this class of algebras. One conjecture about the optimal bound of the dominant dimension for nonselfinjective Nakayama algebras was given by Abrar in [Abr] as Conjecture 4.3.21:

Conjecture (Abrar)
Let $A$ be a nonselfinjective Nakayama algebra with $n \geq 3$ simple modules. Then

$$\text{domdim}(A) \leq 2n - 3.$$ 

In [Abr], Abrar calculated the dominant dimension for many Nakayama algebras and there the biggest value attained by a nonselfinjective Nakayama algebra with $n$ simple modules was $2n - 3$, which lead him to his conjecture.

We have four main results. The structure of the results is as follows: result 2 corrects and proves the conjecture of Abrar. Result 2 is a consequence of the much more general result 1, which also proves Yamagata’s conjecture for the class of algebras given in result 1. Our methods also give a bound on the finitistic dominant dimension, defined below, which is result 3. Result 4 gives an explicit formula for the dominant dimension for Nakayama algebras that are Morita algebras in the sense of Kerner and Yamagata (see [KerYam]). Result 4 is used to show that the bound is optimal.

Result 1
(see [1.3.10]) Let $A$ be a finite dimensional nonselfinjective algebra with dominant dimension at least 1 and minimal faithful injective-projective module $eA$. Let $s$ be the number of nonisomorphic indecomposable injective-projective modules in mod-$A$ and assume that $eAe$ is a Nakayama algebra. Then the dominant dimension of $A$ is bounded by $2s$.

We will prove that for every Nakayama algebra $A$ and every idempotent $e \in A$, $eAe$ is again a Nakayama algebra. So we can apply result 1 in the case that $A$ itself is a Nakayama algebra, and we get the following answer to the conjecture of Abrar:

Result 2
(see [1.4.1] and [2.1.5]) Let $A$ be a nonselfinjective Nakayama algebra with $n \geq 2$ simple modules. Then $\text{domdim}(A) \leq 2n - 2$ and the bound is optimal.

We also introduce the finitistic dominant dimension $\text{fdomdim}(A)$ of an algebra $A$, which is defined as the supremum of all dominant dimensions of all modules having finite dominant dimension. We prove the following for Nakayama algebras, which also gives an alternative proof of Abrar’s conjecture:
Result 3
(see 1.5.3) Let $A$ be a nonselfinjective Nakayama algebra with $n \geq 2$ simple modules. Then $\text{fdomdim}(A) \leq 2n - 2$.

In the last section an explicit formula is given for the dominant dimensions of Nakayama algebras that are also Morita algebras as defined in [KerYam]. In the case of a Nakayama algebras that are also gendo-symmetric algebras (defined in [FanKoe]) the dominant and Gorenstein dimensions have a surprising graph theoretical interpretation. This is used to construct a gendo-symmetric Nakayama algebra with $n \geq 2$ simple modules and with dominant dimension equal to $2n - 2$. We give here the result for gendo-symmetric Nakayama algebras, and refer to section 2 of this paper for the general case and details. In the following $\equiv_n$ denotes equality mod $n$.

Result 4
(see 2.1.4 and 2.2.5) Let $A$ be a symmetric Nakayama algebra with Loewy length $w \equiv_n 1$ and $n$ simple modules. Let $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^r e_{x_i} A/e_{x_i} J^{w-1}$ with the $x_i$ pairwise different for all $i \in \{1, \ldots, r\}$. The $x_i$ in the quiver of $A$ are called special points. Then $B := \text{End}_A(M)$ is a Nakayama algebra and the following holds:

$$\text{domdim}(B) = 2 \inf \{s \geq 1 \mid \exists i, j : x_i + s \equiv_n x_j \}$$

So the dominant dimension is just twice the (directed) graph theoretical minimal distance of two special points which appear in $M$. Furthermore $B$ has Gorenstein dimension

$$2 \sup \{u_i \mid u_i = \inf \{b \geq 1 \mid \exists j : x_i + b \equiv_n x_j \} \},$$

which is twice the maximal distance between two special points.

In forthcoming work we will also give formulae to calculate the finitistic dominant dimension of Nakayama algebras that are Morita algebras. There the finitistic dominant dimension of nonselfinjective gendo-symmetric Nakayama algebras will be shown to be equal to the Gorenstein dimension of that algebra.

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0 Preliminaries

In this article all algebras are finite-dimensional $K$-algebras, for an arbitrary field $K$, and all modules are finitely generated right modules, unless stated otherwise. We will also assume that our algebras will be connected. $J$ will always denote the Jacobson radical of an algebra. When talking about Nakayama algebras, we assume that they are given by quivers and relations (meaning that they are basic and split algebras). As explained in the introduction, this is not really a restriction since the dominant dimension is invariant under Morita equivalence and field extensions. A Nakayama algebra with an acyclic quiver is called LNakayama algebra (L for line) and with a cyclic quiver CNakayama algebra (C for circle).

For connected CNakayama algebras with $n$ simple modules the simple modules are numbered from 0 to $n-1$ clockwise (corresponding to $e_iA$, the projective indecomposable modules at the point $i$). $\mathbb{Z}/n$ denotes the cyclic group of order $n$ and $l_r(x)$ the length of the projective indecomposable right module at the point $x$ (so $l_r$ is a function from $\mathbb{Z}/n$ to the natural numbers). $l_l(x)$ gives the length of the projective indecomposable left module at $x$. For basic facts about Nakayama algebras see for example the chapter about serial rings in [AnFul] § 32. Recall that the lengths of the projective indecomposable modules determine the Nakayama algebra uniquely. We often denote $l_r(i)$ by $c_i$ and $l_l(i)$ by $d_i$.

In the case of a non-selfinjective CNakayama algebra, one can order the $c_i$ such that $c_{n-1} = c_0 + 1$ and $c_i - 1 \leq c_{i+1}$ for $0 \leq i \leq n - 2$ and then $(c_0, c_1, ..., c_{n-1})$ is called the Kupisch series of the Nakayama algebra. A Nakayama algebra $A$ is selfinjective iff the $c_i$ are all equal and the quiver of $A$ is a circle. Every indecomposable module of a Nakayama algebra is uniserial, which means that the chain of submodules of an indecomposable module coincides with its radical series. Thus one can write every indecomposable module of a Nakayama algebra as a quotient of an indecomposable projective module by a radical power of the projective module. Two Nakayama algebras $A$ (with Kupisch series $(c_0, c_1, ..., c_{n-1})$) and $B$ (with Kupisch series $(C_0, C_1, ..., C_{m-1})$) are said to be in the same difference class, if $n = m$ and $c_i \equiv_n C_i$ for all $i = 0, 1, ..., n - 1$. Given a Nakayama algebra with $n$ simple modules, the largest number of the $c_i$ minus the smallest number is less than $n$. Therefore there are only finitely many difference classes of Nakayama algebras with a fixed number of simple modules. $D := \text{Hom}_K(-, K)$ denotes the $K$-duality of an algebra $A$ over the field $K$. We denote by $S_i = e_iA/e_iJ$, $P_i = e_iA$ and $I_i = D(Ae_i)$ the simple, indecomposable projective and indecomposable injective module, respectively, at the point $i$.

The dominant dimension $\text{domdim}(M)$ of a module $M$ with a minimal injective resolution $(I_i)$ is defined as:

$\text{domdim}(M) := \sup \{n | I_i \text{ is projective for } i = 0, 1, ..., n \} + 1$, if $I_0$ is projective, and

$\text{domdim}(M) := 0$, if $I_0$ is not projective.

The codominant dimension of a module $M$ is defined as the dominant dimension of the dual module $D(M)$. The dominant dimension of a finite dimensional algebra is defined as the dominant dimension of the regular module. So $\text{domdim}(A) \geq 1$ means that the
injective hull of the regular module $A$ is projective. Algebras with dominant dimension larger than or equal to 1 are called QF-3 algebras. All Nakayama algebras are QF-3 algebras (see [Abr], Proposition 4.2.2 and Proposition 4.3.3). For more information on dominant dimensions and QF-3 algebras, we refer to [Ta]. By an acyclic algebra we denote quiver algebras whose quiver is acyclic. The quiver of a CNakayama algebra:

$$Q = \begin{array}{c}
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\end{array}$$

The quiver of an LNakayama algebra:

$$Q = \begin{array}{c}
\circ \\
\downarrow \\
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\circ \\
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\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\downarrow \\
\circ \\
\end{array}$$

1 Nakayama algebras

In this article we prove, besides other things, that the dominant dimension of a nonselfinj ective Nakayama algebra $A$ is bounded by $2s$, where $s$ is the number of nonisomorphic projective-injective indecomposable modules of $A$. Later we will provide examples which show that the number $2s$ is attained by some Nakayama algebras with $s$ nonisomorphic projective-injective indecomposable modules. Then we can correct and prove a sharpened version of a conjecture of Abrar, who conjectured that the dominant dimension of a Nakayama algebra with $n$ simple modules is bounded by $2n - 3$ (see [Abr]). We will in fact show that the correct bound is $2n - 2$ and this value is attained in an example.

1.1 Resolutions for Nakayama algebras

In this subsection results about Nakayama algebras will be collected. $A$ will always be a Nakayama algebra with $n$ simple modules and indices of primitive idempotents are integers modulo $n$.

1.1.1 Minimal projective resolutions and minimal injective resolutions over Nakayama algebras

Let $M := e_i A / e_i J^k$ be an indecomposable module of $A$. The projective cover of $M$ is obviously $e_i A$ and $\Omega(M) = e_i J^k$. Then $\Omega(e_i J^k) = e_{i+k} J^{l_r(i)-k}$, since

$$\text{top}(e_i J^k) = e_i J^k / e_i J^{k+1} \cong S_{i+k},$$

$$\dim(e_i J^k) = l_r(i) - k$$

and

$$\dim(e_{i+k} J^{l_r(i)-k}) = l_r(i + k) - (l_r(i) - k),$$

5
which determines $\Omega(e_i J^k)$ uniquely. To see this, recall that the submodules of $e_{i+k} A$ form a chain and $\dim(e_i J^k) + \dim \Omega(e_i J^k) = \dim(e_{i+k} A) = t_i(i + k)$. We set $j := i + k$ and define a function $f : \mathbb{Z}/n \to \mathbb{Z}/n, f(x) := x + t_i(x)$, which seems to be due to [Gus]. Then the minimal projective resolution of $M$ has the following form by repeating the above process ($f^e$ denotes the function $f$ taken $e$ times for a natural number $e \geq 0$):

$$
\cdots \to e_{f^2(j)^A} \to e_{f(j)^A} \to e_{f_{1}(j)^A} \to e_{j^A} \to e_i^A \to M \to 0
$$

If we denote $e_X J^y$ for short by $(x, y) \in \mathbb{Z}/n \times \mathbb{N}$, then $\Omega(e_X J^y) = (x + y, c_x - y)$. Like this we can calculate the syzygies successively with a simple formula depending only on the Kupisch series of the Nakayama algebra. Dually, we get a minimal injective resolution (with $k = c_i$, if $M$ is projective): We have soc$(M) = S_{i+k-1}$ (the simple module corresponding to the point $i + k - 1$). Therefore, the injective hull of $M$ is $D(Ae_{i+k-1})$ and we get $\Omega^{-1}(M) = D(J^k e_{i+k-1})$ and $\Omega^{-1}(D(J^k e_{i+k-1})) = D(J^{i(i+k-1) - k} e_{i-1})$, again by comparing dimensions and using that submodules form a chain. Defining $g : \mathbb{Z}/n \to \mathbb{Z}/n$ as $g(x) := x - t_i(x)$, the minimal injective resolution of $M$ looks like this by repeating the above process:

$$
0 \to M \to D(Ae_{j-1}) \to D(Ae_{j-2}) \to \cdots \to D(Ae_{j-1}) \to D(Ae_{j-i-1}) \to D(Ae_{j-i-1}) \to D(Ae_{j-i-1}) \to \cdots
$$

If we denote $D(J^y e_x)$ for short by $[x, y] \in \mathbb{Z}/n \times \mathbb{N}$ then we get that $\Omega^{-1}(D(J^y e_x)) = [x - y, d_x - y]$. Like this we can calculate the cosyzygies successively.

Now we specialize to selfinjective Nakayama algebras with Loewy length $k$. We give the minimal projective resolution of a general indecomposable nonprojective module $M$ and a formula for Ext$^i(M, M)$ for arbitrary $i \geq 1$. Without loss of generality, we can assume that $M = e_0 J^s$, for $1 \leq s \leq k - 1$. The minimal projective resolution of $M$ then looks like this:

$$
\cdots \to e_{(i+1)k+s}^A \xrightarrow{L_{(i+1)k+s}} e_{(i+1)k+s}^A \xrightarrow{L_{ik+k+s}} e_{ik+k+s}^A \xrightarrow{L_{ik+k}} e_{ik+k}^A \to e_{ik}^A \to e_{i}^A \to \cdots
$$

Here, we denote by $L_{x,y}$ the left multiplication by $w_{x,y}$, where $w_{x,y}$ is the unique path starting at $x$ and having length $y$. If we apply the functor Hom$(-, M)$ to this minimal projective resolution (with $M$ deleted), then we get the complex:

$$
0 \to e_0 J^s e_x \xrightarrow{R_{x,k-s}} e_0 J^s e_k \xrightarrow{R_{k,s}} e_0 J^s e_{k+s} \to \cdots
$$

Here, $R_{x,y}$ is right multiplication by $w_{x,y}$. We see that $R_{i+k+s, k-s} = 0$ for all $i \geq 1$, since
we map paths of length at least \( s \) to paths of length at least \( k \) (and \( J^k = 0 \)). Therefore, we have for all \( i \geq 1 \):
\[
\text{Ext}^{2i-1}(M, M) = \ker(R_{ik,s}) \neq 0, \text{ iff there is a path of length larger than or equal to } k - s \text{ in } e_0 J^s e_{ik} \text{ and } \text{Ext}^{2i}(M, M) = e_0 J^s e_{ik+s}/\text{Im}(R_{ik,s}).
\]

1.1.2 Length of the indecomposable left modules

The length of the indecomposable projective left module \( Ae_x \) at a vertex \( x \) (and, therefore, the length of the indecomposable injective right module at \( x \)) satisfies:
\[
\ell_l(x) = \inf \{ k | k \geq \ell_r(x - k) \}.
\]
The values of \( \ell_l(x) \) are a permutation of the values of \( \ell_r(x) \) and are determined uniquely by the lengths of the projective indecomposable right modules.

**Proof.** See [Ful] Theorem 2.2. \( \square \)

1.1.3 Structure of indecomposable injective modules

Let \( M := e_i A/e_i J^m \) be an indecomposable module of the Nakayama algebra \( A \) with \( m = \dim(M) \leq c_i \). Then \( M \) is injective iff \( c_i - 1 \leq m \).

**Proof.** See [AnFul] Theorem 32.6. \( \square \)

Calculating dominant dimensions of Nakayama algebras

The following theorem shows that the dominant dimension of a given Nakayama algebra depends only on its difference class:

**1.1.4 Theorem**

Let \( A \) be a Nakayama algebra with \( n \) simple modules and \( M = e_i A/e_i J^k \) be an \( A \)-module. The dominant dimension of \( M \) depends only on the difference class of \( A \) and on the \( i \) mod \( n \) and \( k \) mod \( n \). Especially, the dominant dimension of \( A \) depends only on the difference class of \( A \).

**Proof.** We may assume that \( M \) is not injective. First we see that in a given difference class of Nakayama algebras, \( e_i A \) is injective iff \( c_i - 1 \leq c_i \), so the position of the injective-projective modules doesn’t depend on the choice of \( A \) inside a given difference class.

In order to determine the dominant dimension of \( M \), we calculate a minimal injective resolution \( (I_i) \) and the cosyzygies of \( M \) by the above formulae in 1.1.1. Note that \( \Omega^{-1}(M) = D(J^k e_{i+k-1}) \) and that calculating syzygies of modules of the form \( [x,y] = D(J^y e_x) \) is done by \( \Omega^{-1}[x,y] = [x - y, d_x - y] \). If \( \Omega^{-1}(M) = D(J^y e_q) \), then \( I_{j-1} \cong D(Ae_q) \). We see that all those calculations only depend on \( i, k \) mod \( n \) and the difference class (which determines the \( d_i \) mod \( n \)) of the algebra. Now there are two cases to consider:

Case 1: \( \Omega^j(M) \neq 0 \) for every \( j \geq 1 \). Then by the above the indices of the \( I_i \) do not depend on the difference class and \( i \) mod \( n \) and \( k \) mod \( n \). Thus the calculation of the dominant dimension of \( M \) is also independent of the difference class and \( i \) mod \( n \) and \( k \) mod \( n \).

7
Case 2: Assume now that $\Omega^{-j}(M) = 0$ in one algebra of a given difference class, but $\Omega^{-j}(M) \neq 0$ in another algebra in the given difference class for a module $M$ of the form $e_i A/e_j J^k$, for some $j \geq 1$. When $\Omega^{-j}(M) = 0$ happens for some $j \geq 1$ for the first time, there must have been an $I_l$ with $l \leq j - 1$, which is not projective. Otherwise we would have a minimal injective resolution $(I_i)$, with the properties that all terms are also projective and that its ending has the following form:

$$\cdots \rightarrow I_{j-2} \xrightarrow{f} I_{j-1} \rightarrow 0.$$  

Therefore, the surjective map $f$ between projective modules would be split, contradicting the minimality of the resolution. So calculating the dominant dimension of $M$ involves only those terms $I_l$ for $1 \leq l \leq j - 1$ in the minimal injective resolution of $M$ until $\Omega^{-j}(M) = 0$ happens for the first time. But those terms in the injective resolution depend only on the difference class of $A$ and $i \mod n$ and $k \mod n$ and so does the dominant dimension of $M$.

\[\blacksquare\]

1.1.5 Example

We calculate the dominant dimension of a Nakayama algebra $A$ in the difference class of Nakayama algebras with Kupisch series $(c_0, c_1, c_2) = (3k + 2, 3k + 2, 3k + 3)$, for $k \geq 0$. First we calculate the dimension of the injective indecomposable modules with the help of 1.1.2:

$$l_i(0) = \inf\{s \geq 3k + 2 \mid s \geq l_r(-s)\} = 3k + 2$$ and likewise $l_i(1) = 3k + 3$ and $l_i(2) = 3k + 2$.

Thus $(d_0, d_1, d_2) = (3k + 2, 3k + 3, 3k + 2)$. With $\soc(e_1 A) = e_1 J^{3k+1} \cong S_2$, it follows that $e_1 A$ embeds into $D(Ae_2)$. But, since $e_1 A$ and $D(Ae_2)$ have the same dimension, they are isomorphic.

With $\soc(e_2 A) = e_2 J^{3k+2} \cong S_1$, it follows that $e_2 A$ embeds into $D(Ae_1)$ and as above both are isomorphic, because they have the same dimension. Thus the projective-injective indecomposable modules are $e_1 A \cong D(Ae_2)$ and $e_2 A \cong D(Ae_1)$. Now it's enough to look at an injective resolution of $e_0 A$. Since $\soc(e_0 A) = e_0 J^{3k+1} \cong S_1$, $e_0 A$ embeds into $D(Ae_1)$ with cokernel equal to $D(J^{3k+2} e_1) = [1, 3k + 2]$. Then $\Omega^{-1}([1, 3k + 2]) = [1 - (3k + 2), d_1 - (3k + 2)] = [2, 1]$ and $\Omega^{-1}([2, 1]) = [2 - 1, d_2 - 1] = [1, 3k + 1]$ and $\Omega^{-1}([1, 3k + 1]) = (1 - (3k + 1), d_3 - (3k + 1)) = [0, 2]$. The minimal injective resolution of $e_0 A$ starts as follows:

$$0 \rightarrow e_0 A \rightarrow D(Ae_1) \rightarrow D(Ae_2) \rightarrow D(Ae_1) \rightarrow D(Ae_0) \rightarrow \cdots.$$  

Since $D(Ae_0)$ is not projective, the dominant dimension of $e_0 A$ is equal to 3, as is the dominant dimension of $A$, since $e_0 A$ is the only indecomposable projective and not injective module. Note that if $A$ has Kupisch series $(2, 2, 3)$, then $D(J^2 e_0) = [0, 2]$=0, while for $k \geq 1$, that module is nonzero. Also note that the Gorenstein dimension is not independent of the difference class of the Nakayama algebra: If $A$ has Kupisch series $(2, 2, 3)$, then, by the above, the Gorenstein dimension is equal to the dominant dimension and finite. But, if $A$ has Kupisch series $(3k + 2, 3k + 2, 3k + 3)$ for a $k \geq 1$, then continuing as above, one gets: $\Omega^{-1}([0, 2]) = [1, 3k]$, $\Omega^{-1}([1, 3k]) = [1, 3]$, $\Omega^{-1}([1, 3]) = [1, 3k + 2] = \Omega^{-1}(e_0 A)$, and the resolution gets periodic and is, therefore, infinite.
1.2 Gorenstein-projective modules

In this section, $A$ denotes a finite dimensional algebra. See [Che] Section 2, for an elementary introduction to Gorenstein homological algebra. We take our definitions and lemmas from this source.

1.2.1 Definition
A complex $P^\bullet: \ldots \to P^{n-1} d^{n-1} \to P^n \overset{d^n}{\to} P^{n+1} \to \ldots$ of projective $A$-modules is called totally acyclic, if it is exact and the complex $\text{Hom}(P^\bullet, A)$ is also exact. An $A$-module $M$ is called Gorenstein-projective, if there is a totally acyclic complex of projective modules such that $M = \ker(d^0)$. We denote by $A$-gproj the full subcategory of $\text{mod-}A$ of Gorenstein-projective modules and we denote by $\perp_A$ the full subcategory of $\text{mod-}A$ of all modules $N$ with $\text{Ext}^i(N, A) = 0$, for all $i \geq 1$. $D(A)^\perp$ denotes the subcategory of $\text{mod-}A$ of all modules $N$ with $\text{Ext}^i(D(A), N) = 0$ for all $i \geq 1$.

1.2.2 Lemma
(see [Che] Corollary 2.1.9. and 2.2.17.)
Let $A$ be a finite dimensional algebra and $M$ an $A$-module.
1. $A$-gproj $\subseteq \perp_A$.
2. An $A$-module $N$ is in $A$-gproj, in case there is an $n$, such that $\text{Ext}^i(N, A) = 0$, for all $i = 1, \ldots, n$, and $\Omega^n(N) = N$.
3. If $\text{Ext}^i(N, A) = 0$ for all $i = 1, \ldots, d$ and $\Omega^d(N)$ is Gorenstein-projective, then also $N$ is Gorenstein-projective.

1.2.3 Lemma
If $A$ is a Nakayama algebra, then $A$-gproj $= \perp_A$.

Proof. We know that $A$-gproj $\subseteq \perp_A$. Now let $M \in \perp_A$ with $M$ indecomposable. Since all syzygies over a Nakayama algebra of an indecomposable module are also indecomposable and since there is only a finite number of indecomposable modules, there exist numbers $k, n$ with $\Omega^n(\Omega^k(M)) = \Omega^k(M)$. Since we also have $\Omega^k(M) \in \perp_A$ by the formula $\text{Ext}^i(\Omega^k(M), A) = \text{Ext}^{i+k}(M, A) = 0$, we know that $\Omega^k(M)$ is Gorenstein-projective by 2. of the above lemma. Now by 3. of the above lemma also $M$ is Gorenstein-projective.

1.2.4 Lemma
Let $M$ be an indecomposable Gorenstein-projective $A$-module which is not projective. Then there is an exact sequence $0 \to M \to P \to N \to 0$ such that $P$ is projective.

Proof. By the definition of Gorenstein-projective, $M$ can be embedded in a projective module $P$.

1.2.5 Corollary
An indecomposable injective and Gorenstein projective module is projective.
1.3 CoGen-dimension and dominant dimension

1.3.1 Definition
For a finite dimensional algebra $A$ and a module $M$ we define $\phi_M$ as
\[
\phi_M := \inf \{ r \geq 1 | Ext_A^r(M, M) \neq 0 \}
\]
with the convention $\inf(\emptyset) = \infty$. We call a module $M$ which is a generator and a cogenerator for short a CoGen. We also define $\Delta_A := \sup \{ \phi_M | M \text{ is a nonprojective CoGen} \}$.

We remark that for a nonselfinjective algebra $A$
\[
\Delta_A = \inf \{ r \geq 1 | Ext_A^r(D(A), A) \neq 0 \}, \text{ and for a selfinjective algebra } A
\]
\[
\Delta_A = \sup \{ \phi_M | M \text{ is a non-projective indecomposable } A\text{-module} \}.
\]

1.3.2 Theorem of Mueller
(see [Mue]) If $M$ is a CoGen of $A$, then the dominant dimension of $B := End_A(M)$ is equal to $\phi_M + 1$.

1.3.3 Nakayama conjecture
The Nakayama conjecture states that every nonselfinjective finite dimensional algebra has finite dominant dimension.

As a corollary of Mueller’s theorem, the Nakayama conjecture is equivalent to the finiteness of $\Delta_A$, for every finite dimensional algebra $A$.

1.3.4 Yamagata conjecture
Yamagata (in [Yam]) states the even stronger conjecture that the dominant dimensions of nonselfinjective algebras with a fixed number of simple modules are bounded by a function of the number of simple modules of $A$.

In this section, we will prove Yamagata’s conjecture in case $eAe$ is a Nakayama algebra or a quiver algebra with an acyclic quiver, when $eA$ is the minimal faithful injective-projective $A$-module.

1.3.5 Lemma
Let $A$ be a nonselfinjective connected algebra of finite injective dimension $g = injdim(A)$. Then $\Delta_A \leq g$.

Proof. We have
\[
g = injdim(A) = projdim(D(A)) = \sup \{ r \geq 1 | Ext_A^r(D(A), A) \neq 0 \} \geq \inf \{ r \geq 1 | Ext_A^r(D(A), A) \neq 0 \}
\]
\[
= \Delta_A,
\]
where we used $projdim(M) = \sup \{ r \geq 1 | Ext_A^r(M, A) \neq 0 \}$, in case $M$ has finite projective dimension.

The following generalizes and gives an easier proof of Theorem 1.2.3 of [Abr], which states 3. of the following Corollary.
1.3.6 Corollary
1. Let $A$ be an connected acyclic algebra with $d \geq 2$ simple modules. Then $\text{gldim}(A) \leq d - 1$ and, therefore, $\Delta_A \leq d - 1$.
2. Let $A$ be a QF-3 algebra with $s$ projective-injective indecomposable modules such that $e Ae$ is acyclic, where $eA$ is the minimal faithful injective-projective module. Then $\text{domdim}(A) \leq s$.
3. Let $A$ be an acyclic algebra with $s$ indecomposable injective-projective modules, then $\text{domdim}(A) \leq s - 1$.
4. For an LNakayama algebra $A$ with $n$ simple modules, the following holds: $\Delta_A \leq n - 1$.

Proof. 1. For an elementary proof of $\text{gldim}(A) \leq d - 1$, see e.g. [Farn]. Then $\Delta_A \leq d - 1$ follows from the previous lemma, since the equality $\text{gldim}(A) = \text{injdim}(A)$ holds, in case $A$ has finite global dimension.
2. By Mueller’s theorem and the previous part.
3. This holds, since with $A$ also $eAe$ is acyclic for every idempotent $e \in A$.
4. This is clear by 3., since LNakayama algebras are acyclic.

We give the following example of a nonacyclic algebra $A$ such that $eAe$ is acyclic to show that the above is really a generalisation of Theorem 1.2.3 of [Abr].

1.3.7 Example
Take any acyclic endowild algebra $C$ (this means that for every finite dimensional algebra $R$, there is a finite dimensional $C$-module $N$ with $R \cong \text{End}_C(N)$). Examples of such algebras $C$ are wild hereditary algebras over an algebraically closed field, see [SimSko3] page 329) and a $C$-module $M$ such that $\text{End}_C(M)$ is not acyclic. We claim that then the algebra $\Lambda = \text{End}_C(B(C \oplus D(C) \oplus M))$ is not acyclic, where $B(X)$ denotes the basic version of a module $X$. Denoting here by $e$ the projection from $B(C \oplus D(C) \oplus M)$ onto $B(M)$, the algebra $e\Lambda e \cong \text{End}_C(B(M))$ is not acyclic and thus $\Lambda$ is not acyclic.

For the main lemma, recall the following result:

1.3.8 Lemma
Let $A$ be a finite dimensional algebra, $N$ be an indecomposable $A$-module and $S$ a simple $A$-module. Let $(P_i)$ be a minimal projective resolution of $N$ and $(I_i)$ a minimal injective resolution of $N$.
1. For $l \geq 0$, $\text{Ext}^l(N, S) \neq 0$ iff $S$ is a quotient of $P_l$.
2. For $l \geq 0$, $\text{Ext}^l(S, N) \neq 0$ iff $S$ is a submodule of $I_l$.

Proof. See [Ben] Corollary 2.5.4.

1.3.9 Main lemma
Let $A$ be a finite dimensional nonselfinjective Nakayama algebra with $n \geq 2$ simple modules. Let $N$ be an $A$-module and $S$ a simple $A$-module.
1. Assume that $\text{Ext}^l(N, S) \neq 0$ for some $l \geq 1$. Then $\inf\{s \geq 1|\text{Ext}^s(N, S) \neq 0\} \leq 2n - 2$. 

11
2. Assume that $\Ext^l(S, N) \neq 0$ for some $l \geq 1$. Then $
abla\inf\{s \geq 1|\Ext^s(S, N) \neq 0\} \leq 2n - 2$.

Proof. We only prove 1. since 2. follows dually. We can assume that $\Ext^1(N, S) = 0$, since the result is obvious in case $\Ext^1(N, S) \neq 0$. So in the following we look at the problem of determining the smallest possible finite value $s \geq 2$ with respect to the following properties:

$\Ext^s(N, S) \neq 0$, but $\Ext^l(N, S) = 0$, for $i = 1, ..., s - 1$, for an indecomposable module $N$. $\Ext^s(N, S) \neq 0$ simply means that in the minimal projective resolution $(P_i)$ of $N$ there is a direct summand of $P_s$ isomorphic to the projective cover of $S = S_r := e_rA/e_rJ$ by the previous lemma. By [1.1.1] the minimal projective resolution has the form:

$$
\cdots \rightarrow e_{f_{r+1}(i)}A \rightarrow e_{f_{r}(i)}A \rightarrow \cdots \rightarrow e_{f_{1}(i)}A \rightarrow e_{i}A \rightarrow e_{i}A \rightarrow N \rightarrow 0
$$

Claim 1. $f$ is not surjective.

Proof: If $f$ were surjective, it would be bijective and, because of $\soc(e_iA) = S_{f(i) - 1}$, for every $i$, $A$ would be selfinjective (see [SkoYam] Chapter IV. Theorem 6.1.), contradicting our assumption that $A$ is not selfinjective. So Claim 1 is proved.

Now, $\Ext^s(N, S) \neq 0$, for some $u \geq 1$, tells us $e_rA \cong P_u$.

Claim 2. The smallest index $i$ with $e_rA \cong P_i$ must be smaller than or equal to $2n - 2$.

Proof: Since $f$ is a mapping from a finite set to a finite set, there is a minimal number $w$ with $\Im(f^w) = \Im(f^{w+1})$. Define $X := \Im(f^w)$. Note that the cardinality of $X$ is smaller than or equal to $n - w$, since $f : \mathbb{Z}/n \rightarrow \mathbb{Z}/n$ is not surjective and therefore the number of elements in $\Im(f^i)$ decreases by at least 1 as long as $i < w$ in the sequence $\Im(f) \supset \Im(f^2) \supset \cdots \supset \Im(f^w)$.

$f$ is a bijection from $X$ to $X$, since $f|_X : X \rightarrow X$ is surjective (and $X$ a finite set). When we have reached

$$
\cdots \rightarrow e_{f^{w+1}(i)}A \rightarrow e_{f^{w+1}(i)}A \rightarrow e_{f^{w}(i)}A \rightarrow e_{f^{w}(i)}A \rightarrow \cdots,
$$

$f$ acts as a cyclic permutation on the $\{f^l(i)\}$ and $\{f^l(j)\}$ for $l \geq w$. Note that after the term $e_{f^{w+n-w-1}(i)}A = e_{f^{n-1}(i)}A = P_{2n-2}$ (recall that $X$ has cardinality at most $n - w$) some index $q \in X$ must exist such that $P_{2n-1} = e_qA$ and this indecomposable projective module $e_qA$ is also isomorphic to $P_u$ for an $u \leq 2n - 2$. Therefore, the index $r$ must occur in the first $2n - 2$ terms, since later there are only indices which already occurred before.

1.3.10 Theorem

Let $A$ be an algebra with dominant dimension larger than 1 and the property that $eA$ is a minimal faithful projective injective module and $eAe$ is a Nakayama-algebra with $s$ simple modules. Then:
domdim(A) ≤ ∆_{eA} + 1 ≤ 2s.

Especially, Yamagata’s conjecture 1.3.4 is true in this special case.

Proof. We first prove the following lemma:

1.3.11 Lemma
For a Nakayama algebra with $n$ simple modules, the following holds: $\Delta_A ≤ 2n - 1$.

We split the proof of this lemma in two cases: one case of a nonselfinjective Nakayama algebra and in the other case the Nakayama algebra is selfinjective.

Case 1
For a nonselfinjective CNakayama algebra $A$ with $n$ simple modules, $\Delta_A ≤ 2n - 1$.

Proof. There is an injective indecomposable module $I$ of the form $I = eA/soc(eA)$: The structure of indecomposable injective modules 1.1.3 tells us that the module $I = eA/soc(eA)$ is injective, when $i$ is chosen such that $c_{i-1} ≤ c_i - 1$ and $e := e_i$. This module is not Gorenstein-projective by Corollary 1.2.3 since it is injective, but not projective. By Lemma 1.2.3, $I$ is not in $\perp A$. Therefore, there is a smallest index $k ≥ 1$ with $Ext^k(I, A) ≠ 0$. Thus also $Ext^k(D(A), A) ≠ 0$, which implies the theorem, if we show that $k ≤ 2n - 1$. If $Ext^1(I, A) ≠ 0$, there is nothing to prove. So we assume that $Ext^1(I, A) = 0$. Denote $soc(eA)$ by $S = S_r$ (which means that the projective cover of $S$ is $e_rA$). In general we have $Ext^i(S, A) = Ext^i(\Omega(I), A) = Ext^{i+1}(I, A)$. Therefore, we will look in the following for the smallest index $s$ with $Ext^s(S, A) ≠ 0$. But by the main lemma inf $\{s ≥ 1 | Ext^s(S, A) ≠ 0\} ≤ 2n - 2$. Because of $Ext^s(S, A) = Ext^s(\Omega(I), A) = Ext^{s+1}(I, A)$ and $2n - 2 + 1 = 2n - 1$ we proved case 1.

For the next case, recall the results of 1.1.1 about calculating minimal projective resolutions and $Ext^i(M, M)$ for an indecomposable module $M$ in a selfinfective Nakayama algebra.

Case 2
A selfinjective Nakayama algebra $A$ with $n$ simple modules, satisfies:

$\Delta_A ≤ 2n - 1$.

Proof. To prove this, we have to show that $\phi_M ≤ 2n - 1$ for all nonprojective indecomposable modules $M$.

We can assume that $A$ has Loewy length $k$ and $M = e_0J^s$, with $1 ≤ s ≤ k - 1$. We consider two cases:

First case; $k$ is a zero divisor in $\mathbb{Z}/n$. Then there is a $q$ with $kq \equiv_n n \equiv_n 0$ and $1 ≤ q ≤ n - 1$. We know that $\Omega^{2i}(M) = e_{ik}J^s$ and, therefore, $\Omega^{2i}(M) = e_{ik}J^s = e_0J^s = M$. 13
Consequently, $\text{Ext}^2(M, M) = \text{Hom}(\Omega^2(M), M) = \text{Hom}(M, M) \neq 0$.

Second case: $k$ is not a zero divisor in $\mathbb{Z}/n$ and, therefore, a unit. We have $\text{Ext}^{2i-1}(M, M) = \ker(R_{ik,s}) \neq 0$, iff there is a path of length larger than or equal to $k - s$ in $e_0J^*e_{ik}$. But, for $i = 1, \ldots, n$, the integers $ik$ are all different from one another mod $n$. This is why there surely is a path of length larger than or equal to $k - s$ in $e_0J^*e_{ik}$ for some $i \leq n$.

Now we return to the proof of 1.3.10. Combining Case 1 and Case 2, we have proved the lemma 1.3.11. To get a proof of theorem 1.3.10, we use Mueller’s theorem 1.3.2 and the fact that the number of non-isomorphic indecomposable projective-injective modules equals the number of simple modules of $eAe$ to get that

$$\text{domdim}(A) \leq \Delta_{eAe} + 1 \leq 2s.$$  

This finishes the proof of 1.3.10.

1.4 Dominant dimension of Nakayama algebras

We will prove the bound of result 2 in this section:

1.4.1 Theorem

Let $B$ be a nonselfinjective Nakayama algebra with $s$ projective-injective indecomposable modules. Then the dominant dimension of $B$ is bounded above by $2s$.

Note that $s$ in the previous theorem is always bounded by $n - 1$, when $n$ denotes the number of simple $B$-modules. We will also show in the next section that there is a Nakayama algebra such that the maximal value $2(n - 1)$ (if $B$ has $n$ simple modules) is attained, see Corollary 2.1.4. Therefore the maximal possible value of the dominant dimension of a Nakayama algebra with $n$ simple modules is $2(n - 1)$. This corrects and proves a conjecture of Abrar, who conjectured that the maximal value is $2n - 3$ (see [Abr] Conjecture 4.3.21).

1.4.2 Lemma

If $B$ is a Nakayama algebra, then $A := eBe$ is a Nakayama algebra for every idempotent $e$ of $B$.

Proof. If $J$ is the radical of $B$, the radical of $eBe$ is $eJe$ (see [Lam] Theorem 21.10). If $e = e_1 + \ldots + e_n$, with primitive orthogonal idempotents $e_i$ of $B$, then those $e_i$ are a complete system of primitive orthogonal idempotents in $eBe$. We have $e_i(rad(eBe))/(rad^2(eBe)))e_j = e_iJ\alpha \beta/(e_iJ\alpha \beta)$, where $\alpha$ is a path (of length larger than or equal to 1) from $i$ to a point in $e$ and $\beta$ is a path (of length larger than or equal to 1) from a point in $e$ to $j$. We see that there is at most one arrow starting at $i$ and at most one arrow ending at $j$.
in the quiver of $eBe$, since $e_i(\text{rad}(eBe)/(\text{rad}^2(eBe)))e_j = e_iJe_j/(e_iJeJe_i) \neq 0$, iff there is no $e_k$, which is a summand of $e$, between $e_i$ and $e_j$. But the property that there is at most one arrow starting at $e_i$ and at most one arrow ending at $e_j$ in the quiver of $eBe$ for all points $e_i, e_j$ characterises Nakayama algebras.

Therefore, Theorem 1.4.1 follows from 1.3.10 and the theorem of Mueller, since $A := eBe$ is a Nakayama algebra with $s$ simple modules and $\Delta_A \leq 2s - 1$. So we have by Mueller’s theorem: \( \text{domdim}(B) = \Delta_A + 1 \leq 2s \).

1.5 Finitistic dominant dimension of Nakayama algebras

Using the main lemma, we show in this section that we can give a bound of the finitistic dominant dimension for Nakayama algebras.

1.5.1 Definition
The finitistic dominant dimension of a finite dimensional algebra $A$ is

\[
\text{fdomdim}(A) := \sup\{\text{domdim}(M) | \text{domdim}(M) < \infty \}
\]

1.5.2 Example
If $A$ has global dimension $g$, then $\text{fdomdim}(A) \leq g$, since for every noninjective module $M$ $\text{domdim}(M) \leq \text{injdim}(M) \leq g$ holds.

The following theorem gives again the bound $2n - 2$ for the dominant dimension of Nakayama algebras.

1.5.3 Theorem
Let $A$ be a nonselfinjective Nakayama algebra with $n \geq 2$ simple modules. Then $\text{fdomdim}(A) \leq 2n - 2$.

Proof. Clearly we can assume that $A$ is a CNakayama algebra, since an LNakayama algebra has global dimension at most $n - 1$. So assume now that $A$ is a CNakayama algebra and $M$ an indecomposable $A$-module with finite dominant dimension. Note that $\text{domdim}(M) = \inf\{i | \text{Ext}^i(S, M) \neq 0 \text{ for a simple module } S \text{ with nonprojective injective hull}\}$. We can assume that $M$ has dominant dimension larger than or equal to 1. Let $S$ be a simple module with nonprojective injective envelope such that $\text{Ext}^i(S, M) \neq 0$ for an $i \geq 1$. Then by the main lemma $\inf\{s \geq 1 | \text{Ext}^s(S, N) \neq 0\} \leq 2n - 2$ and thus $\text{domdim}(M) \leq 2n - 2$. \qed

1.5.4 Example
Take the CNakayama algebra $A$ with Kupisch series $(3s + 1, 3s + 2, 3s + 2), s \geq 1$. We first calculate the Gorenstein dimension and the dominant dimension of $A$ and then the finitistic dominant dimension of $A$. First note that $e_1A \cong D(Ae_2)$ is injective. Also $e_2A \cong D(Ae_0)$ is injective. The only noninjective indecomposable projective module is then $e_0A$ and the only nonprojective injective indecomposable module is $D(Ae_1)$. We have the following injective resolution:

\[
0 \rightarrow e_0A \rightarrow D(Ae_0) \rightarrow D(Ae_2) \rightarrow D(Ae_1) \rightarrow 0.
\]
Thus the dominant dimension and the Gorenstein dimension of $A$ are both 2. Now take an indecomposable module $M = e_a A/e_a J^k$ and calculate the minimal injective presentation of $M$ using the method from 1.1.1: $0 \rightarrow M \rightarrow D(Ae_{a+k-1}) \rightarrow D(Ae_{a-1})$. Thus $M$ has dominant dimension larger than or equal to 2 iff $a + k - 1 \in \{0,2\}$ mod 3 and $a - 1 \in \{0,2\}$ mod 3 iff $(a = 0 \text{ mod } 3 \text{ and } k \in \{0,1\} \text{ mod } 3)$ or $(a = 1 \text{ mod } 3 \text{ and } k \in \{0,2\} \text{ mod } 3)$. The following table gives the relevant values of the dominant dimensions:

| $a$ | 0 | 1 |
|-----|---|---|
| $k \equiv 0 \text{ mod } 3$ | 4 | 2 |
| $k \equiv 1 \text{ mod } 3$ | 2 | - |
| $k \equiv 2 \text{ mod } 3$ | - | 3 |

Thus the finitistic dominant dimension equals 4, while the finitistic dimension equals the Gorenstein dimension which is 2.

2 Nakayama algebras which are Morita algebras and their dominant and Gorenstein dimension

In this section we calculate the dominant dimension of all Nakayama algebras that are Morita algebras and give the promised example of a non-selfinjective Nakayama algebra having $n$ simple modules and dominant dimension $2n - 2$. We also show how to calculate the Gorenstein dimension of such algebras and give a surprising interpretation of the dominant and Gorenstein dimension for gendo-symmetric Nakayama algebras.

2.1 Calculating the dominant dimensions of Nakayama algebras that are Morita algebras

2.1.1 Definition
A finite dimensional algebra $B$ is called a Morita algebra, if it is isomorphic to the endomorphism ring of a module $M$, which is a generator of a selfinjective algebra $A$ (see [KerYam]). If $A$ is even symmetric, then $B$ is called a gendo-symmetric algebra (see [FanKoe]).

The following is a special case of a result of Yamagata in [Yam2].

2.1.2 Theorem (Yamagata)
Let $A$ be a nonsemisimple selfinjective Nakayama algebra with Loewy length $w$ and $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} A/e_{x_i} J^{k_i}$. Then $B := \text{End}_A(M)$ is a basic non-selfinjective Nakayama algebra, iff all the $x_i$ are pairwise different and $k_i = w - 1$ for all $i \in \{1, ..., r\}$ and $r \geq 1$.

Keep this notation for $B$ and call points of the form $x_i$ special points.
2.1.3 Proposition

Let $r \geq 1$. Let $A$ be a selfinjective Nakayama algebra with Loewy length $w$ and $n$ simple modules. Let $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} A/e_{x_i}J^{w-1}$ with the $x_i$ different for all $i \in \{1, ..., r\}$. Then:

\[
\text{domdim}(B) = \phi_M + 1 = \inf\{k \geq 1 \mid \exists x_i, x_j : \text{Ext}^k(e_{x_i}A/e_{x_i}J^{w-1}, e_{x_j}A/e_{x_j}J^{w-1}) \neq 0\} + 1
\]

\[
= \inf\{k \geq 1 \mid \exists x_i, x_j : x_j + w - 1 \equiv_n x_i + \left[\frac{k+1}{2}\right]w - g_k\} + 1.
\]

Here, we set $g_k = 1$, if $k$ is even, and $g_k = 0$, if $k$ is odd. $[l]$ is equal to $l$, if $l$ is an integer, and otherwise equal to the smallest integer larger than $l$ (for example $[1.5] = 2$).

**Proof.** Note that the first equality is by Mueller’s theorem and we just have to show the last equality. Lemma 1.3.7 says that for a module $M$ and a simple module $S$ 

\[
\text{Ext}^i(M, S) \neq 0 \iff S \text{ is a direct summand of the top of the module } P_i,
\]

where $P_i$ is the $i$-th term in a minimal projective resolution of $M$. Note that

\[
\text{Ext}^k(e_{x_i}A/e_{x_i}J^{w-1}, e_{x_j}A/e_{x_j}J^{w-1}) = \text{Ext}^k(e_{x_i}J^{w-1}, e_{x_j}J^{w-1}),
\]

which is what we want to calculate.

Observe that $e_{x_i}J^{w-1} \cong S_{x_i+w-1}$ is a simple module. Using \[1.1.1\] the minimal projective resolution of $e_{x_i}J^{w-1}$ then looks like this:

\[
\cdots \longrightarrow e_{x_i+(i+1)w-1}A \longrightarrow e_{x_i+(i+1)w}A \longrightarrow e_{x_i+iw-1}A \longrightarrow e_{x_i+iw}A \longrightarrow 0
\]

Thus the $k$-th term in the minimal projective resolution of $e_{x_i}J^{w-1}$ is equal to

\[
P_k = e_{x_i+[\frac{k+1}{2}]w-g_k}A.
\]

Then $\text{Ext}^k(e_{x_i}A/e_{x_i}J^{w-1}, e_{x_j}A/e_{x_j}J^{w-1}) \neq 0$, iff $x_j + w - 1 \equiv_n x_i + [\frac{k+1}{2}]w - g_k$, for a $k \geq 1$.

2.1.4 Corollary

Let $A$ be a symmetric Nakayama algebra with Loewy length $w \equiv_n 1$ and $n$ simple modules. Let $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} A/e_{x_i}J^{w-1}$ with the $x_i$ different for all $i \in \{1, ..., r\}$. Then for $B = \text{End}_A(M)$:

\[
\text{domdim}(B) = 2 \inf\{s \geq 1 \mid \exists i, j : x_i + s \equiv_n x_j\}
\]

So the dominant dimension is just twice the (directed) graph theoretical minimal distance of two special points which appear in $M$. 

17
Proof. The formula takes for \( w \equiv_n 1 \) an especially nice form:

\[
\text{domdim}(B) = \inf\{k \geq 1 \mid \exists i, j : x_i + \left\lfloor \frac{k + 1}{2} \right\rfloor - g_k \equiv_n x_j\} + 1.
\]

For \( k = 2s + 1 \) and \( k = 2s + 2 \) the value of \( \left\lfloor \frac{k + 1}{2} \right\rfloor - g_k \) is the same. This means that the infimum is attained at an odd number of the form \( k = 2s - 1 \) and the formula simplifies to

\[
\text{domdim}(B) = \inf\{2s - 1 \geq 1 \mid \exists i, j : x_i + s \equiv_n x_j\} + 1 = 2 \inf\{s \geq 1 \mid \exists i, j : x_i + s \equiv_n x_j\}.
\]

We can now state the corrected conjecture of Abrar as the next proposition by showing that the bound is optimal:

2.1.5 Proposition
A non-selfinjective Nakayama algebra with \( n \geq 2 \) simple modules has its dominant dimension bounded above by \( 2n - 2 \) and this bound is optimal for every \( n \geq 2 \), that is, there exists a non-selfinjective Nakayama algebra with \( n \) simple modules and dominant dimension \( 2n - 2 \). This Nakayama algebra has \( n - 1 \) injective-projective indecomposable modules.

Proof. The bound was given in Theorem 1.4.1. Using the previous corollary, we take the algebra \( C = \text{End}_A(A \oplus e_0A/e_0Jw^{-1}) \), where \( A \) is a symmetric Nakayama algebra with \( n - 1 \) simple modules and Loewy length \( w \). Then \( C \) is a Nakayama algebra with \( n \) simple modules and dominant dimension \( 2n - 2 \).

We give another application, showing how to construct algebras of arbitrary dominant dimension larger or equal to two:

2.1.6 Corollary
Let \( w \equiv_n 2 \) and \( B \) as above. Then \( B \) has dominant dimension

\[
\text{domdim}(B) = \inf\{k \geq 1 \mid \exists x_i, x_j : x_j \equiv_n x_i + k\} + 1.
\]

Proof. This follows, since in case \( w \equiv_n 2 \):

\[
\left\lfloor \frac{k + 1}{2} \right\rfloor w - g_k = k + 1 \quad \text{and therefore:}
\]

\[
\text{domdim}(B) = \inf\{k \geq 1 \mid \exists x_i, x_j : x_j + w - 1 \equiv_n x_i + \left\lfloor \frac{k + 1}{2} \right\rfloor w - g_k\} + 1 =
\]

\[
\inf\{k \geq 1 \mid \exists x_i, x_j : x_j + 1 \equiv_n x_i + k + 1\} + 1 =
\]

\[
\inf\{k \geq 1 \mid \exists x_i, x_j : x_j \equiv_n x_i + k\} + 1.
\]

So in this case the dominant dimension is simply equal to one plus the minimal distance of two special points. Like this, one can construct a family of Nakayama algebras with dominant dimension an arbitrary number larger than or equal to two.
2.2 Gorenstein dimensions of Nakayama algebras which are Morita algebras

We first recall definitions and standard facts about approximations. Note that by maps we always mean $A$-homomorphisms, when we speak about modules.

2.2.1 Definition

Let $M$ and $N$ be $A$-modules. Recall that a map $g : M \rightarrow N$ is called right minimal in case $gh = g$ implies that $h$ is an isomorphism for any map $h : M \rightarrow M$. A map $f : M_0 \rightarrow X$, with $M_0 \in \text{add}(M)$, is called a right add$(M)$-approximation of $X$ iff the induced map $\text{Hom}(N, M_0) \rightarrow \text{Hom}(N, X)$ is surjective for every $N \in \text{add}(M)$. Note that in case $M$ is a generator, such an $f$ must be surjective. When $f$ is a right minimal homomorphism, we call it a minimal right add$(M)$-approximation. Note that minimal right add$(M)$-approximations always exist for finite dimensional algebras. The kernel of such a minimal right add$(M)$-approximation $f$ is denoted by $\Omega_M(X)$. Inductively we define $\Omega_0^M(X) := X$ and $\Omega_n^M(X) := \Omega_M(\Omega_{n-1}^M(X))$. The add$(M)$-resolution dimension of a module $X$ is defined as:

$$M\text{-resdim}(X) := \inf\{n \geq 0 | \Omega_n^M(X) \in \text{add}(M)\}.$$

We use the following Proposition 3.11. from [CheKoe] in order to calculate the Gorenstein dimensions:

2.2.2 Proposition

Let $A$ be a finite dimensional algebra and $M$ a CoGen of mod-$A$ and define $B := \text{End}_A(M)$. Let $B$ have dominant dimension $z + 2$, with $z \geq 0$. Then, for the right injective dimension of $B$ the following holds:

$$\text{injdim}(B) = z + 2 + M\text{-resdim}(\tau_{z+1}(M) \oplus D(A)).$$

Here we use the common notation $\tau_{z+1} = \tau \Omega^z$, introduced by Iyama (see [Iya]). We note that the Gorenstein symmetry conjecture (which says that the injective dimensions of $A$ and $A^{op}$ are the same) is known to hold for algebras with finite finitistic dimension (see [ARS] page 410, conjecture 13), and thus for Nakayama algebras which are our main examples. Therefore, we will only look at the right injective dimension at such examples.

We now fix our notation as in the previous section: $A$ is a selfinjective Nakayama algebra with $n$ simple modules, Loewy length $w$ and $M = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^r e_x A/e_x J^{w-1}$. Using the same notation as in the above theorem, we note that $B$ is derived equivalent to $C = \text{End}_A(A \oplus N)$ (see [HuX] Corollary 1.3. (2)), with the semisimple module $N = \Omega^1(M) = \bigoplus_{i=1}^r e_x J^{w-1}$. We also set $W := A \oplus N$ and we fix that notation for the rest of this section.

2.2.3 Lemma

The above mentioned derived equivalence between $B$ and $C$ preserves dominant dimension and Gorenstein dimension.
Proof. In [HuXi] Corollary 1.2., it is proved that such a kind of derived equivalence preserves dominant dimension and finitistic dimension. If the Gorenstein dimension is finite, it is equal to the finitistic dimension. Since a derived equivalence also preserves the finiteness of Gorenstein dimension, the result follows.

We see that we need to know how to calculate minimal right add($W$)-approximations of an arbitrary module in a selfinjective Nakayama algebra. For this we have the following lemma:

2.2.4 Lemma
Let $e_a J^y$ be an arbitrary non-projective indecomposable module in the selfinjective Nakayama algebra $A$ and assume that this module is not contained in add($N$).
1. If $a \neq x_i$ for all $i = 1, \ldots, r$, then the projective cover $e_{a+y} A \to e_a J^y \to 0$ is a minimal right add($W$)-approximation of $e_a J^y$.
2. If there is an $x_i$ with $a = x_i$, then we have the following short exact sequence:

$$0 \to e_{x_i+y} J^{w-(y+1)} \to e_{x_i+y} A \oplus e_{x_i} J^{w-1} \to e_{x_i} J^y \to 0.$$ 

Here, the map $e_{x_i+y} A \oplus e_{x_i} J^{w-1} \to e_{x_i} J^y$ is the sum of the projective cover of $e_{x_i} J^y$ and the socle inclusion of $e_{x_i} J^{w-1}$ in $e_{x_i} J^y$. Then the surjective map in the above short exact sequence is a minimal right add($W$)-approximation.

Proof. 1. The projective cover is clearly minimal. The kernel of the projective cover is $e_{a+y} J^{w-y}$ and we have to show $\text{Ext}^1(Z, e_{a+y} J^{w-y}) = 0$ for every $Z \in \text{add}(W)$. Since $W$ is a direct sum of simple and projective modules, this simply means that $I_1$ (the first term in a minimal injective resolution of $e_{a+y} J^{w-y}$) has a socle, which does not lie in add($W$). But this is true because of $I_1 = e_a A$ and our assumption in i).
2. Again, the minimality is obvious. At first we show that the short exact sequence exists. What is left to show is that the kernel is really $e_{x_i+y} J^{w-(y+1)}$. With

$$e_{x_i} J^y \cong e_{x_i+y} A / e_{x_i+y} J^{w-y}$$

and

$$e_{x_i} J^{w-1} \cong e_{x_i+y} J^{w-y-1} / e_{x_i+y} J^{w-y}$$

we see that the map of interest has up to isomorphism the following form:

$$f : e_{x_i+y} A \oplus e_{x_i+y} J^{w-y-1} / e_{x_i+y} J^{w-y} \to e_{x_i+y} A / e_{x_i+y} J^{w-y}.$$ 

We have $f(w_1, w_2) = w_1 + w_2$, when $\overline{w}$ denotes the residue class of an element $w$. A basis of the kernel is thus given by the elements

$$\{(\phi_{x_i+y,w-1},0) \mid w - 1 \geq l \geq w - y\} \cup \{(\phi_{x_i+y,w-y-1},-\phi_{x_i+y,w-y-1})\},$$

when we denote by $\phi_{c,d}$ the unique path starting at $c$ and having length $d$.

A basis of the socle of the kernel is given by $(\phi_{x_i+y,w-1},0)$ and thus the kernel is isomorphic to $e_{x_i+y} J^{w-(y+1)}$ (by comparing dimension and socle). We now have to show
that the induced map $\text{Hom}(G, e_x J^{w-1} \oplus e_x+y A) \to \text{Hom}(G, e_x J^y)$ is surjective for every $G \in \text{add}(W)$. Note that we can assume that $G$ has no simple summands $S$ which are not isomorphic to $e_x J^{w-1}$, since we would have $\text{Hom}(S, e_x J^y) = 0$ then. With this assumption we get

$$
\text{Ext}^1(G, e_x+y J^{w-(y+1)}) = 0, \text{ iff } \text{Ext}^1(e_x J^{w-1}, e_x+y J^{w-(y+1)}) = 0,
$$

and this is true, since the minimal injective presentation of $e_x+y J^{w-(y+1)}$ is the following:

$$0 \to e_x+y J^{w-(y+1)} \to e_x+y A \to e_{x-1} A.$$ 

Then $\text{Ext}^1(G, e_x+y J^{w-(y+1)}) = 0$ and thus, the induced map

$$\text{Hom}(G, e_x J^{w-1} \oplus e_x+y A) \to \text{Hom}(G, e_x J^y)$$

is surjective.\phantom{ }\phantom{\Box}\phantom{2.2.5 Theorem}

Now we will use this result to calculate the Gorenstein dimensions of gendo-symmetric Nakayama algebras. We note that for a simple module $S$, $\tau_{z+1}(S)$ is always a simple module, if the dominant dimension of $B$ is even. It is a radical of a projective indecomposable module, if the dominant dimension of $B$ is odd. So, in order to calculate the Gorenstein dimension, it is enough to calculate the minimal right $\text{add}(W)$-resolutions for modules of the form $(a, w-1)$ and $(a, 1)$ for a point $a$. A diagram of the form

$$
\begin{array}{ccc}
A' & \xrightarrow{1} & B' \\
\downarrow & & \downarrow \\
B' & \xrightarrow{2} & C'
\end{array}
$$

means that the kernel of a $W$-approximation of the indecomposable nonprojective module $A' = e_a J^k$ is $B'$, in case $e_a J^{w-1}$ is not a summand of $W$ (always corresponding to the arrow with a 1), and the kernel is $C'$ otherwise (always corresponding to an arrow with a 2).

So for a general module $(a, k) = e_a J^k$, not in $\text{add}(W)$, the diagram looks as follows in the first step:

$$
\begin{array}{ccc}
(a, k) & \xrightarrow{1} & (a+k, w-k) \\
\downarrow & & \downarrow \\
\phantom{(a+k, w-k)} & & \phantom{(a+k, w-k)}
\end{array}
$$

We also set $B' = \text{stop}$, if $B'$ is a summand of $W$. Dots like $\cdots$ indicate that it is clear how the resolution continues from this point on.

2.2.5 Theorem

Let $w \equiv_n 1$ (which is equivalent to $A$ being a symmetric Nakayama algebra). Then $B$ has Gorenstein dimension

$$2 \sup\{u_i \mid u_i = \inf\{b \geq 1 \mid \exists j : x_i + b \equiv_n x_j\}\},$$

which is two times the maximal distance between two special points.
Proof. By 2.1.4, $B$ has dominant dimension equal to $2 \inf \{ s \geq 1 \mid \exists i, j : x_i + s \equiv_n x_j \}$, which is equal to two times the smallest distance of two special points. Denote by $d$ the smallest distance between two special points. Using 2.2.2 and $\tau \cong \Omega^2$ (since $A$ is symmetric), the Gorenstein dimension is equal to $2d + W \text{-resdim}(\Omega^{2d}(W))$, with $W = \bigoplus_{i=0}^{n-1} e_i A \oplus \bigoplus_{i=1}^{r} e_{x_i} J^{w-1}$. Note that $\Omega^{2d}(W) = \bigoplus_{i=1}^{r} e_{x_i, d} J^{w-1}$ and so we have to calculate $W \text{-resdim}(\Omega^{2d}(W))$. Since the resolution dimension of a direct sum of modules equals the supremum of the resolution dimensions of the indecomposable summands, it is enough to look at a resolution of a single simple module of the form $(x_j + d, w - 1)$:

![Resolution Diagram]

Considering this diagram, we see now that the resolution finishes exactly when the kernel is of the form $(x_j + d + i, w - 1)$ with the smallest $i \geq 0$ such that $e_{x_j + d + i} J^{w-1}$ is a summand of $W$. This takes $2i$ steps. Now the result is clear.

It follows that the dominant dimension (Gorenstein dimension) of a nonsymmetric gendo-symmetric Nakayama algebra $A$ can be calculated purely graph theoretically: It is two times the minimal (two times the maximal) distance of special points in the quiver of the symmetric Nakayama algebra $e Ae$, when $e$ is a primitive idempotent, such that $eA$ is a minimal faithful projective-injective module of $A$.

Combining these results, we get the following geometric characterisation when the dominant dimension equals the Gorenstein dimension for a nonselfinjective gendo-symmetric Nakayama algebra:

2.2.6 Corollary

In the situation of the above theorem, $\text{injdim}(B) = \text{domdim}(B)$ iff all the special points in $M$ have the same distance from one another.

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24