Lorentz transformation of temperature and effective temperature

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We derive the Lorentz transformation of temperature by using holography. The Lorentz transformation of temperature has been under debate for more than a century, and three different transformations have been proposed. Here we show that the correct transformation is that proposed by Einstein and Planck. We obtain a relativistic thermal distribution function from holography as well. The distribution function is Lorentz invariant and is consistent with the covariant first law of thermodynamics proposed by Einstein and Planck. The same method is applicable to the effective temperature in nonequilibrium steady states. We derive the boost transformation of the effective temperature. This enables us to distinguish the boost effect from the many-body effect in the effective temperature of a test particle dragged in a heat bath.

1. INTRODUCTION

Lorentz transformation of temperature has been under debate for more than a century [1, 2]. So far, three different transformations have been proposed as follows. Let us consider a thermal equilibrium system at finite temperature. Suppose that the temperature is observed by two observers: the observer $O_0$ is in the rest frame of the equilibrium system, and the observer $O$ moves at constant velocity $-\vec{v}$ relative to $O_0$. In general, the observers $O_0$ and $O$ may measure different temperatures $T_0$ and $T$, respectively. They should be related through the Lorentz transformation. Up to now, three different transformations have been proposed: $T = \gamma^\ell T_0$ ($\ell = \pm 1, 0$), where $\gamma = 1/\sqrt{1 - v^2}$ is the Lorentz factor in the unit of $c = 1$. The transformation with $\ell = -1$ was proposed by Einstein and Planck [3, 4]; that with $\ell = 1$ was proposed by Ott [5]; that with $\ell = 0$ was given by Landsberg [6] and van Kampen [7]. All these proposals are based on thermodynamics. The disagreement of the transformation originates from the ambiguity of the relativistic definition of heat.

To put an end to the controversy, we need a method to derive the Lorentz transformation of temperature without relying on thermodynamics. For this purpose, we employ holography, which is a map between a quantum gauge theory and a classical gravity theory [8–10]. In the gravity side, the temperature is directly read from the geometry as the Hawking temperature. Therefore the Lorentz transformation of temperature is uniquely obtained by the Lorentz transformation
of geometry, without dealing with relativistic thermodynamics. We will show that the correct Lorentz transformation of temperature is that proposed by Einstein and Planck [3, 4]. Although the typical microscopic theories handled in holography may not be realized in nature, our results for the equilibrium temperature are general for any equilibrium systems, since temperature is independent of the details of the microscopic theory.

In general, the Lorentz transformation introduces the center-of-mass velocity $\vec{v}$ of the thermal equilibrium system. The Lorentz-covariant first law of thermodynamics that includes $\vec{v}$ has been proposed by Einstein and Planck [3, 4] as

$$dE = TdS - PdV + \vec{v} \cdot d\vec{k},$$

(1.1)

where $S$, $P$, $V$ and $\vec{k}$ are the entropy, the pressure, the volume and the momentum of the system, respectively. Here, $E$ is the total energy that is the internal energy plus the (macroscopic) kinetic energy of the system. We will derive the distribution function (2.8) that is consistent with (1.1).

An advantage of holography is its applicability to nonequilibrium physics [11]. For nonequilibrium steady states (NESSs), an effective temperature can be defined as the ratio between the fluctuation and the dissipation [12]. However, the generalization of thermodynamics for NESSs is not clear. In the framework of holography, the effective temperature of a NESS is obtained as the Hawking temperature of the effective geometry in the gravity dual [13–16]. Then, the Lorentz transformation of the effective temperature is unambiguously obtained from the transformation of the effective geometry. In this work, we use the natural unit $\hbar = c = k_B = 1$.

2. LORENTZ TRANSFORMATION OF TEMPERATURE IN EQUILIBRIUM SYSTEMS

Let us consider an equilibrium system of gauge particles at finite temperature. The gauge theory is a strongly-coupled large-$N_c$ gauge theory in $d$ dimensions. The coordinates of the $d$-dimensional spacetime are $(t, \vec{x})$, where $t$ is the time, and $\vec{x} = (x^1, \cdots, x^{d-1})$ denotes the space-like coordinates. To simplify the notation, we refer to $x^1$ as $x$.

The gravity dual of the system is realized as a $(d + 1)$-dimensional black-hole geometry [10] when the system is in the deconfinement phase. The metric is given by

$$ds^2 = g_{tt}(u)dt^2 + g_{xx}(u)dx^2 + g_{uu}(u)du^2 + \sum_{\mu \neq t,x,u} g_{\mu\mu}(u)(dx^\mu)^2.$$  

(2.1)

In the vicinity of the horizon $u = u_H$, $g_{tt}$ and $g_{uu}$ are expanded as $g_{tt} = -a(u - u_H) + O((u - u_H)^2)$
and \( g_{uu} = b/(u - u_H) + \mathcal{O}(1) \), respectively, and the other components are positive and finite. In our convention, \( g_{tt} < 0 \) and \( g_{uu} > 0 \) outside the horizon. The Hawking temperature \( T_0 \) is obtained as

\[
T_0 = \frac{1}{4\pi} \sqrt{\frac{a}{b}}. \tag{2.2}
\]

The Hawking temperature \( T_0 \) is identified with the temperature of the gauge theory side. Let the coordinate system \((t, \vec{x})\) be the frame of the observer \( O_0 \), who measures the temperature as \( T_0 \). In this frame, the probability distribution of the Hawking radiation is given by \( \exp \left[ -\frac{E_0}{T_0} \right] \), where \( E_0 \) is the energy of the \( i \)-th microscopic state. The distribution function is isotropic, and hence we call this frame the rest frame.

Now, we consider an observer \( O \) moving in the \( x \)-direction at constant velocity \( \beta \) relative to the rest frame. Let the observer \( O \) measure the temperature as \( T \). To obtain the temperature \( T \), we perform a boost transformation

\[
\Lambda'_{\mu} = \Lambda_{\nu} \delta^{\nu}_{\mu},
\]

where

\[
\Lambda^t = \Lambda_x = \gamma(\beta), \quad \Lambda^t = \Lambda_x = -\gamma(\beta), \quad \gamma(\beta) = 1/\sqrt{1 - \beta^2}. \tag{2.3}
\]

The line element in the boosted frame is given by

\[
ds^2 = g'_{tt} dt'^2 + 2g'_{tx} dt' dx' + g'_{xx} dx'^2 + g_{uu} du^2 + \sum_{\mu \neq t, x, u} g_{\mu\mu} (dx')^2. \tag{2.4}
\]

We can diagonalize the metric (2.4) as

\[
ds^2 = \left( g'_{tt} - \frac{g'_{tx}^2}{g'_{xx}} \right) dt'^2 + g'_{xx} \left( dx' + \frac{g'_{tx}}{g'_{xx}} dt' \right)^2 + g_{uu} du^2 + \sum_{\mu \neq t, x, u} g_{\mu\mu} (dx')^2. \tag{2.5}
\]

In the vicinity of the horizon, the coefficient of \( dt'^2 \) in (2.5) is expanded as

\[
g'_{tt} - \frac{g'_{tx}^2}{g'_{xx}} = -a'(u - u_H) + \mathcal{O}((u - u_H)^2), \tag{2.6}
\]

where \( a' = a/\gamma^2 \). Hence the Hawking temperature in the boosted frame is given by

\[
T = \frac{1}{4\pi} \sqrt{\frac{a'}{b}} = \frac{T_0}{\gamma(\beta)}. \tag{2.7}
\]

This shows that the moving observer \( O \) measures the lower temperature \( T \) than \( T_0 \). Note that (2.7) is uniquely obtained, and the result agrees with that proposed by Einstein and Planck [3, 4].
We can generalize the result (2.7) by replacing $\beta$ with $\vec{\beta}$ in a general direction, since the system is rotational invariant at the rest frame.

From the viewpoint of the observer $O$, the probability distribution of the Hawking radiation (see Appendix B) is given by

$$\exp \left( -\frac{E_i - \vec{v} \cdot \vec{k}_i}{T} \right),$$

where $E_i$ and $\vec{k}_i$ are the energy and the momentum of the $i$-th microscopic state, and $\vec{v} = -\vec{\beta}$ is the velocity of the system. The transformation (2.7) shows that $\gamma(v)T$ is Lorentz invariant, and we know $\gamma(v)(E_i - \vec{v} \cdot \vec{k}_i)$ is also Lorentz invariant. Hence (2.8) is Lorentz invariant. The probability distribution should be scalar, and hence we should employ (2.8) rather than $\exp \left[ -\frac{E_i}{T} \right]$ in the relativistic theory. This distribution is consistent with the relativistic distribution proposed in the previous works [17, 18].

We emphasize that the rest frame, where the distribution function is isotropic, exists for each thermal equilibrium system. In general, the distribution (2.8) is characterized by $\vec{v}$ and is anisotropic. However, we can always perform a boost transformation with $\vec{v}$ and go back to the rest frame. Hence, we refer to $T_0$ as the “rest temperature”. This is an analogue of the rest mass $m_0$. Now, the Lorentz invariance of $\gamma(v)T = T_0$ is naturally understood. The rest temperature $T_0$ characterizes each equilibrium system.

We can interpret the velocity $\vec{v}$ in analogy to the chemical potential in a finite density system. If we compare (2.8) with the grand canonical distribution at finite chemical potential $\mu$, $\exp \left[ -\frac{(E_i - \mu N_i)}{T} \right]$ with $N_i$ being the number of the particles of $i$-th microscopic state, $\vec{v}$ can be understood as a “chemical potential for momentum”. Then our spectrum (2.8) suggests that the term $\vec{v} \cdot d\vec{k}$ in (1.1) is understood in analogy with the term $\mu dN$, where $N$ is the number of the particles in the system.

For later use, we consider the new observer $\tilde{O}$ moving at velocity $\tilde{\beta}$ in the $x$-direction relative to the observer $O$. The additional boost transformation with $\tilde{\beta}$ on top of (2.3) yields

$$\tilde{T} = \frac{T}{(1 - v\tilde{\beta}) \gamma(\tilde{\beta})},$$

where $\tilde{T}$ is the temperature measured by the observer $\tilde{O}$. If we replace $(v, \tilde{\beta})$ by $(0, \beta)$, we return to (2.7) again. In addition, we can generalize (2.9) by replacing $v\tilde{\beta}$ with the inner product $\vec{v} \cdot \vec{\beta}$. 
3. BOOST TRANSFORMATION OF EFFECTIVE TEMPERATURE IN NONEQUILIBRIUM STEADY STATES

From now on, we consider boost transformation of effective temperature in nonequilibrium steady states (NESSs). A NESS is a nonequilibrium state where the macroscopic variables do not evolve in time. To realize a NESS, we need three ingredients: a heat bath, a system in study, and an external force. The external force drives the system out of equilibrium, but the heat produced in the system dissipates into the heat bath, so that the system is steady from the macroscopic point of view. Even in a NESS, it is possible to define a “temperature” as the ratio between the fluctuation and the dissipation; it is referred to as effective temperature [12].

Let us consider a NESS in a conductor system at finite charge density $\rho$, where both positively and negatively charged carriers are immersed in a heat bath. We apply the external electric field $E$ along the $x$-direction. Let the constant current $J$ flows in the same direction. We consider the boost transformation along the electric field in this work, so that the magnetic field is not induced for simplicity.

The conductor system can be described in holography [19]. We assume that the system is described from the viewpoint of the observer $O_0$. Then, the heat bath is mapped to the black hole geometry (2.1). The sector of the charge carriers is mapped to a D-brane which is defined in the framework of superstring theory. The effective action of the D-brane is described by using the induced metric $h_{ab}$ and the U(1) gauge field strength $F_{ab}$ (see Appendix C).

The effective temperature is studied in the conductor system by using holography [14–16, 20]. To obtain the effective temperature, we consider the linearized equations of motion for fluctuations that are governed by an effective geometry of the D-brane. The effective geometry has an effective horizon when the electric field is turned on, in our setup. Let us pay attention to the case that the effective metric is given by the open-string metric, $G_{ab} = h_{ab} - (2\pi\alpha')^2(Fh^{-1}F)_{ab}$. We consider the linearized equations governed by the open-string metric in the vicinity of the effective horizon, $u = u_*$. Then, the effective temperature is given by the Hawking temperature, which is evaluated at the effective horizon.

More specifically, let us pay attention to the $3 \times 3$ part of the open-string metric on the $(t, x, u)$ coordinates. In our setup, $G_{ab}$ has off-diagonal components: $G_{ut} = G_{tu} = -(2\pi\alpha')^2F_{tx}h^{xx}F_{xu}$ and $G_{xt} = G_{tx} = -(2\pi\alpha')^2F_{tu}h^{uu}F_{ux}$ appear by the existence of $F_{xt} = \mathcal{E}$, $F_{ux} = Jf_1$, and $F_{ut} = \rho f_2$, where $f_1$ and $f_2$ are finite functions. This metric $G_{ab}$ can be diagonalized in the vicinity of $u_*$ by
using the following coordinate transformation [16, 21]:

\[
\begin{pmatrix}
\frac{dt}{d\hat{t}} \\
\frac{dx}{d\hat{x}} \\
\frac{du}{d\hat{u}}
\end{pmatrix} =
\begin{pmatrix}
1 & \frac{Q^{i\varphi} du}{du} \\
\frac{G_{xx} dx + Q^{i\varphi} dt + Q^{\varphi \varphi} du}{du} & \frac{du}{du}
\end{pmatrix},
\tag{3.1}
\]

where \(Q^{a\varphi}_b\) are defined by

\[
Q^{i\varphi}_u = \frac{G_{tx}G_{xu} - G_{xx}G_{lu}}{G^2_{tx} - G_{xx}G^l_t},
\]
\[
Q^{\varphi \varphi}_u = \frac{G_{xt}}{G_{xx}} |_{u = u_*}, \quad Q^{\varphi \varphi}_t = \frac{G_{xu}}{G_{xx}}.
\tag{3.2}
\]

The metric \(G_{ab} = [(Q^{-1})^T G Q^{-1}]_{ab}\) on the new coordinates is given by

\[
G_{ii} = \frac{G_{tt}G_{xx} - G^2_{xt}}{G_{xx}}, \quad G_{\hat{t}\hat{t}} = G_{xx},
\]
\[
G_{\hat{u}\hat{u}} = \frac{-\text{det} G}{g^2_{xx} (G_{tt}G_{xx} - G^2_{xt})} + O(1),
\tag{3.3}
\]

where \(\text{det} G\) is the determinant of \(G_{ab}\). \(G_{tt}G_{xx} - G^2_{xt}\) is of the order of \(u - u_*\), and vanishes at \(u = u_*\). The off-diagonal components are \(G_{i\hat{t}} = O(u - u_*)\), \(G_{i\hat{u}} = O(u - u_*)\), and \(G_{\hat{t}\hat{u}} = O(1)\) at \(u = u_*\).

The Hawking temperature is read from \(G_{ii}\) and \(G_{\hat{u}\hat{u}}\) in the same way as (2.2):

\[
T_* = \frac{1}{4\pi} \sqrt{\frac{a_*}{b_*}},
\tag{3.4}
\]

where \(a_*\) and \(b_*\) are defined by \(G_{ii} = -a_* (u - u_*) + O((u - u_*)^2)\) and \(G_{\hat{u}\hat{u}} = b_* / (u - u_*) + O(1)\) in the vicinity of \(u_*\). The effective temperature \(T_*\) is measured by the moving observer \(O_0\).

Next, let us consider the effective temperature \(T'_*\) measured by the moving observer \(O\). We perform the boost transformation of the open-string metric \(G_{ab}\) as \(G'_{ab} = (\Lambda^{-1})^c_a (\Lambda^{-1})^d_b G_{cd}\), where \(\Lambda^a_b\) are given by (2.3). Then we diagonalize \(G'_{ab}\) in the same way as we did in (3.1), (3.2) and (3.3). One finds that the “time-time” component of the diagonalized metric \(G'_{ii}\) is expanded as \(G'_{ii} = -a' (u - u_*) + O((u - u_*)^2)\) in the vicinity of the effective horizon, where \(a' = a_* / (1 - v_m \beta)^2 \gamma^2(\beta)\) with \(v_m = -Q^{\varphi \varphi}_t = -G_{xt} / G_{xx} |_{u = u_*}\). Then the effective temperature \(T'_*\) in the boosted frame is obtained as

\[
T'_* = \frac{1}{4\pi} \sqrt{\frac{a'}{b_*}} = \frac{T_*}{(1 - v_m \beta) \gamma(\beta)},
\tag{3.5}
\]

Note that (3.5) is similar to (2.9).
Let us consider the meaning of the similarity between (3.5) and (2.9). The distribution function of the fluctuations in the NESS was computed \[21\] as

\[
\exp \left( \frac{E_i - v_m k_i}{T_*} \right),
\]

(3.6)

where \(E_i\) and \(k_i\) are the energy and the \(x\) component of the momentum of the fluctuation in consideration, respectively. If we perform a boost transformation with \(\beta = v_m\), the distribution function (3.6) goes to \(\exp \left[ -E_{*0i}/T_{*0} \right]\), where \(E_{*0i}\) is the energy of the \(i\)-th state in the new frame. We have defined \(T_{*0} = T'_*|_{\beta = v_m} = \gamma(v_m)T_*\), which is boost invariant in parallel with (2.7). We call this frame the “rest frame of the NESS”, since the distribution function is isotropic. Compared to (2.8), the distribution function (3.6) states that the system has a relative velocity \(v_m\) with respect to the rest frame of the NESS. This is consistent with the interpretation of \(v_m\) from the phenomenological viewpoint \[21\], where \(v_m\) is referred to as the mean velocity of NESS and is defined as the average of the velocities of the charge carriers. Regarding \(v_m\) as the relative velocity with respect to the rest frame of the NESS, we can understand (3.5) in parallel with that of the equilibrium temperature (2.9).

4. CONCLUSIONS AND DISCUSSIONS

We have uniquely derived the Lorentz transformation of temperature (2.7), (2.9), and the boost transformation of effective temperature (3.5) without any assumptions on thermodynamics. In the equilibrium system, the correct transformation of temperature (2.7) is that proposed by Einstein and Planck [3, 4]. We have found that \(\gamma(v)T\) is invariant under the transformation. We have obtained the Lorentz invariant distribution (2.8) that is consistent with the covariant first law of thermodynamics [3, 4] (1.1): \(dE = TdS - PdV + \vec{v} \cdot d\vec{k}\). The heat bath velocity \(\vec{v}\) can be interpreted as the momentum chemical potential in analogy with the grand canonical ensemble. We have defined the rest frame of the heat bath as the frame where the heat bath velocity is zero. In NESSs, the boost transformation of effective temperature (3.5) is essentially the same as that of temperature (2.9). The product \(\gamma(v_m)T_*\) is boost invariant. From the distribution function (3.6), we have interpreted \(v_m\) as the macroscopic mean velocity of the NESS, that agrees with the mean velocity obtained phenomenologically [21]. We have defined the rest frame of NESS where the mean velocity is zero. The rest frame of NESS is different from that of the heat bath in general.

We have two remarks. First, let us consider a four-vector \(\beta^\mu_T\) which is defined by

\[
\beta^\mu_T = \frac{u^\mu}{\gamma(v)T} = \frac{(1, \vec{v})}{T},
\]

(4.1)
where $u^\mu$ is the four-velocity of the heat bath given by $u^\mu = \gamma(v)(1, \vec{v})$. One finds the following relationship holds:

$$\beta^2_{T0} = -\eta_{\mu\nu}\beta^\mu_T\beta^\nu_T,$$

where $\beta_{T0} = 1/T_0$ and $\eta_{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$. This relation is the alternative expression of $T_0 = \gamma(v)T$. Using (4.1), we can rewrite (2.8) as $\exp(\eta_{\mu\nu}\beta^\mu_Tk^\nu_i)$, where $k^\nu_i = (E_i, \vec{k}_i)$. Equation (4.2) is analogous to $m_0^2 = -p^2$ where $m_0$ is the rest mass. In the same manner, we can define $\beta^\mu_{T*} = u^\mu_m/(\gamma(v_m)T_*)$ in the NESSs, where $u^\mu_m = \gamma(v_m)(1, v_m, 0, \cdots, 0)$. We obtain the relation $\beta^2_{T*0} = -\eta_{\mu\nu}\beta^\mu_{T*}\beta^\nu_{T*}$, where $\beta_{T*0} = 1/T_{*0}$. We can rewrite (3.6) as $\exp(\eta_{\mu\nu}\beta^\mu_{T*}k^\nu_i)$.

Secondly, we can apply the boost transformation of temperature to estimate the contribution of many-body dynamics to the effective temperature. In ref. [15], it was considered that a test particle moving at constant velocity $v_{\text{test}}$ in a thermal bath of temperature $T_0$, and its effective temperature $T_*(v_{\text{test}})$ was given by $T_* = (1 - v^2_{\text{test}})^{-p}(1 + Cv^2_{\text{test}})^{\frac{1}{2}}T_0$, where $C = \frac{1}{2}(q + 3 - p + n\frac{3-p}{1-p})$ and $p, q, n$ are integers that characterize the models. It was pointed out that $T_*$ can be lower than $T_0$. (See also ref. [13].) However, if we compare $T_*(v_{\text{test}})$ with $T_0/\gamma(v_{\text{test}})$, we find that $T_*(v_{\text{test}}) \geq T_0/\gamma(v_{\text{test}})$ for all the cases of the models considered in ref. [15], under the condition that the heat capacity of the heat bath is positive ($p < 5$) and the combinations of $p, q$ and $n$ are consistent with the superstring theory. The equality holds at $v_{\text{test}} = 0$. The inequality states that, although $T_*(v_{\text{test}})$ can be lower than the heat bath temperature, $T_*(v_{\text{test}})$ is still higher than the temperature that would be observed if the test particle were just adiabatically boosted to the same velocity. This suggests that the interactions between the test particle and the heat bath contributes to raise the effective temperature from $T_0/\gamma(v_{\text{test}})$. It is important to study the relationship between the effective temperature and the many-body dynamics of the test particle and the heat bath. When the system is relativistic, taking account of the boost transformation of temperature is quite important.

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Appendix A: Covariant first law of thermodynamics

We consider an equilibrium system at velocity \( \vec{v} \) and momentum \( \vec{k} \). The conservation law of energy is given by

\[
dE = dS + dW,
\]

where \( E \) is the total energy of the system, \( T \) and \( S \) are the temperature and the entropy of the system, respectively. \( dW \) is the work given from the external system, which is

\[
dW = -PdV + \vec{v} \cdot d\vec{k},
\]

where \( P \) and \( V \) are the pressure and the volume of the system, respectively. Note that the external force can be used to increase the kinetic energy of the system that is represented by the last term of (A.2). Then we obtain

\[
dE = TdS - PdV + \vec{v} \cdot d\vec{k}.
\]

Let us rewrite (A.3) in the following way

\[
\gamma(dE - \vec{v} \cdot d\vec{k}) = \gamma(TdS - PdV),
\]

where \( \gamma = 1/\sqrt{1 - \vec{v}^2} \). Note that the left-hand side is given by \( u_\mu dp^\mu \) where \( u_\mu \) is the four-velocity \((-\gamma, \gamma \vec{v})\) and \( dp^\mu = (dE, d\vec{k})\): the left-hand side is Lorentz invariant.

Our results in the main text show that \( \gamma T \) is a scalar. Since the entropy is determined by the number of states, which is also a scalar, \( \gamma TdS \) is Lorentz invariant. Then we conclude that \( PdV \) is Lorentz invariant as well. We know that \( \gamma dV \) is invariant under the Lorentz transformation. This means that \( P \) alone is a scalar. This is natural, because the pressure is the force over the area: both of them transform as a vector. This is how our results are consistent with the first law (A.3) proposed by Einstein and Planck [3, 4]. In other words, our results propose that the definition of the relativistic heat is given by (A.1).

Appendix B: Hawking temperature in the boosted frame

Here, we derive the Hawking temperature measured by the moving observer \( O \), who moves in the \( x \)-direction at velocity \( \beta \) relative to the rest frame of the system. From the viewpoint of the observer \( O \), the metric \( g'_{\mu\nu} \) is given by (2.4) in the main text, which has the off-diagonal component
In this coordinate, we consider a scalar field \( \phi(t, \vec{x}, u) \). Suppose that \( \phi(t, \vec{x}, u) \) is a free massless scalar field for simplicity. Then \( \phi(t, \vec{x}, u) \) obeys the equation of motion,

\[
\left( g^{tt} \partial_t^2 - 2 g^{tx} \partial_t \partial_x + g^{xx} \partial_x^2 \right) \phi + g^{uu} \partial_u^2 + \sum_{\mu \neq t, x, u} g^{\mu \mu} \partial_\mu^2 \phi = 0,
\]

where \( g^{tt} = (\Lambda^t)^2 g^{tt} + (\Lambda^x)^2 g^{xx} \), \( g^{tx} = \Lambda^t \Lambda^x g^{tt} + \Lambda^x \Lambda^x g^{xx} \), and \( g^{xx} = (\Lambda^x)^2 g^{tt} + (\Lambda^x)^2 g^{xx} \) with

\[
\Lambda^t = \Lambda^x = \gamma(\beta), \quad \Lambda^x = -\gamma(\beta) \beta, \quad \gamma(\beta) = 1 / \sqrt{1 - \beta^2}.
\]

To diagonalize the \((t, x)\) components of the metric, we define a new coordinate \((\tau, y)\) on which

\[
\begin{pmatrix}
\partial_\tau \\
\partial_y
\end{pmatrix} = \begin{pmatrix}
\partial_t - \frac{g'_{tx}}{g_{xx}} \partial_x \\
\partial_x 
\end{pmatrix}.
\]

Then the components of the diagonalized metric are given by

\[
G^{\tau \tau} = \frac{g'_{xx}}{g_{tt} g_{xx} - g_{tx}^2}, \quad G^{yy} = \frac{1}{g_{xx}^2},
\]

and the other components are the same as those of \( g^{\mu \nu} \).

Here, we follow the approach given in ref. [22], where the spectrum of the radiation is derived by considering a tunneling effect in the semi-classical approximation. First, we employ the tortoise coordinate \( dU \equiv du / (u - u_H) \), where \( u_H \) is the location of the horizon. Then, we consider the equation (B.1) in the vicinity of the horizon, which is approximated as

\[
\left[ -\frac{1}{a'(u - u_H)} \partial_\tau^2 + \frac{1}{b(u - u_H)} \partial_\tau \right] \phi = 0,
\]

where the constant \( a' \) and \( b \) are given by \( G_{\tau \tau} = -a'(u - u_H) + \mathcal{O}((u - u_H)^2) \) and \( g_{uu} = b / (u - u_H) + \mathcal{O}(1) \), respectively. Employing the WKB approximation, we obtain the distribution of the scalar field as

\[
\exp \left( -\frac{E - v k}{T} \right),
\]

where \( v = -\beta \), \( E \) and \( k \) are the energy and the momentum of the scalar field measured by the moving observer \( O \). The Hawking temperature \( T \) is given by \( 4\pi T = \sqrt{a'/b} \) from the viewpoint of the observer \( O \). The result (B.6) is generalized by replacing the velocity \( \beta \) in the \( x \)-direction with
in a general direction, since the rotational symmetry exists in the equilibrium system at the rest frame. Then, the distribution function is given by
\[ \exp \left( -\frac{E_i - \vec{v} \cdot \vec{k}_i}{T} \right), \]
where \( E_i \) and \( \vec{k}_i \) are the energy and the momentum of the \( i \)-th microscopic state from the viewpoint of the observer moving at the velocity \( \vec{\beta} \).

**Appendix C: Gravity dual of NESS**

Here, we describe a homogeneous NESS, where the macroscopic quantities are independent of the coordinates \((t, \vec{x})\). To realize the NESS, we need three ingredients: the heat bath, the system in study, and the external force. Let us consider a conductor system which has both positively and negatively charged carriers immersed in the heat bath in the presence of a constant electric field. Suppose that the dissipation balances the energy given by the constant external field at late times, where the NESS is realized. We can describe a homogeneous NESS in the conductor system, using the holography [19].

We consider a ten-dimensional black-hole spacetime, since the superstring theory is formulated in ten dimensions. The metric is given by (2.1) in the main text with \( d = 9 \). We assume that the spacetime is decomposed into the following subspaces: a \((\tilde{d} + 1)\)-dimensional spacetime \((t, x^1, \cdots, x^{\tilde{d}-1}, u)\), where \( \tilde{d} < 9 \); a \((9 - \tilde{d})\)-dimensional compact space \((x^{\tilde{d}+1}, \cdots, x^9)\) whose metric is finite. This black hole geometry is dual to the heat bath. Now we are in the frame of the observer \( O_0 \).

The sector of the charge carriers is described by a \( D(q+1+n) \)-brane [21, 23], where \( \tilde{d} \geq q \geq 1 \). We employ the probe approximation. Let \( \zeta^a \) denote the coordinate system on the worldvolume of the brane. We employ the static gauge, \( \zeta^a = (t, u, \vec{x}, \vec{\Omega}) \): the worldvolume extends in \( \vec{x} = (x^1, \cdots, x^q) \) and the \( u \)-direction, and wraps an \( n \)-dimensional subspace \( \vec{\Omega} \) of the \((9 - \tilde{d})\)-dimensional compact manifold, where \( 9 - \tilde{d} \geq n \). The Dirac-Born-Infeld action for the probe \( D(q+1+n) \)-brane is given by
\[ S_{D(q+1+n)} = -T_{D(q+1+n)} \int d^{\tilde{d}+2+n} \zeta \]
\[ \times e^{-\phi} \sqrt{-\det (h_{ab} + 2\pi\alpha' F_{ab})}, \]
where \( T_{D(q+1+n)} \) is the tension of the brane. The dilaton field \( \phi \) depends only on \( u \). The constant \( \alpha' \) is related to the ’t Hooft coupling \( \lambda \). For example, \( \lambda = 2N_c T_{Dp}(2\pi\alpha')^{-2} \) when the bulk geometry
is the Dp-brane background [24], where $N_c$ is the rank of the gauge group. The induced metric $h_{ab}$ and the U(1) gauge field strength $F_{ab}$ on the brane are defined by $h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$ and $F_{ab} = \partial_a A_b - \partial_b A_a$, where $X^\mu(\zeta)$ represents the configuration of the D$(q + 1 + n)$-brane, and $A_a(\zeta)$ is the U(1) gauge field. In our analysis, the Wess-Zumino term does not affect the result, and hence we can neglect it.

We assume that $F_{ab}$ and $X^\mu$ depend only on $u$. For the gauge field, we take the following ansatz: $A_t$ depends on only $u$; $A_x(t, u) = -\mathcal{E}t + a_x(u)$; the other components of the gauge field are switched off in our gauge choice. The constant $\mathcal{E}$ is the external electric field acting on the charge carriers.

Let us consider the dynamics of the gauge fields. The Lagrangian density $\mathcal{L}$ is given by $\mathcal{L} = T_{D(q+1+n)} \int d^n \zeta e^{-\phi} \sqrt{-\det(h_{ab} + 2\pi\alpha' F_{ab})}$. The Gubser-Klebanov-Polyakov-Witten prescription gives the following identifications [9, 10, 19]:

$$\frac{\delta \mathcal{L}}{\delta A_t'} = \rho, \quad \frac{\delta \mathcal{L}}{\delta A_x'} = J,$$

where the prime denotes $\partial/\partial u$, and $\rho$ and $J$ are the expectation values of the charge density and the current density, respectively. A reality condition for the gauge fields yields the relation between $J$ and $\mathcal{E}$,

$$J = \frac{2\pi\alpha'}{h_{xx}} \sqrt{\rho^2 + \tilde{V}^2} \mathcal{E} \bigg|_{u = u_*},$$

where $u_*$ is defined by $|h_{tt}(u_*)| h_{xx}(u_*) = (2\pi\alpha')^2 \mathcal{E}^2$ [19]. Here, $\tilde{V} = 2\pi\alpha' T_{D(q+1+n)} \int d^n \zeta e^{-\phi} h_{xx}^{q/2} \sqrt{h_\Omega}$, where $\sqrt{h_\Omega}$ is the volume element of the worldvolume along the compact directions, which may depend on the brane configuration. (See also ref. [21].) This means that $\sqrt{h_\Omega}$, hence $\tilde{V}$, contains dynamical degrees of freedom as well as $h_{uu}$ does, and they are determined by the Euler-Lagrange equations.

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