Vibration and stress analysis in the presence of structural uncertainty

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Abstract. At medium to high frequencies the dynamic response of a built-up engineering system, such as an automobile, can be sensitive to small random manufacturing imperfections. Ideally the statistics of the system response in the presence of these uncertainties should be computed at the design stage, but in practice this is an extremely difficult task. In this paper a brief review of the methods available for the analysis of systems with uncertainty is presented, and attention is then focused on two particular “non-parametric” methods: statistical energy analysis (SEA), and the hybrid method. The main governing equations are presented, and a number of example applications are considered, ranging from academic benchmark studies to industrial design studies.

1. Introduction

There has been much recent interest in predicting the response of structures that have uncertain or random properties, and this is particularly true in the field of vibro-acoustics. In predicting stress, vibration, and interior noise a broad frequency range is of interest and the response of an engineering structure such as a car or an aeroplane can be highly sensitive to small manufacturing variations, particularly at higher frequencies. Ideally this sensitivity should be identified and quantified at the design stage, so that the performance of the complete set of produced articles can be predicted, rather than just the performance of a nominally perfect design. Perhaps the most direct way to consider the effect of system uncertainty on the response is to adopt a parametric model. This approach consists of (i) identifying the uncertainties in the key physical properties (or “parameters”) of the system, and (ii) propagating this uncertainty through the governing equations of motion to predict the uncertainty in the response. However both parts of this procedure can present severe practical difficulties. With regard to issue (i), the parametric model would ideally consist of the detailed statistical distribution of the system properties (material properties, dimensions, rigidity of joints, etc), but it can be extremely difficult to acquire this data, or even to identify which of the system properties are statistically significant. This problem can be overcome to some extent by adopting a less precise parametric model of the system uncertainties, consisting of perhaps bounds [1], or fuzzy-number descriptions [2], but the difficulty of propagating this uncertainty through the equations of motion, issue (ii), still remains. Typical industrial structures are described by finite element models that may have thousands or millions of degrees of freedom, and so the propagation of parametric uncertainty using Monte Carlo simulations, or more analytical means, can be highly computationally intensive. It is clear that parametric modelling is a very challenging research area, and progress is being made by the introduction of innovate Monte Carlo methods [3] or alternative propagation methods [2].
Despite the foregoing discussion, the fact that engineering systems can be large and complex is not necessarily to the disadvantage of the analyst. In fact, very complex systems can sometimes be easier to analyse than less complex systems. Of course, care is needed in the use of the term “complex” in this context; here, by “more complex” the intention is to say “more random”. To consider the response of a single plate for example, the introduction of a small degree of randomness will cause the natural frequencies, the mode shapes, and the plate response to be random, and it can be challenging to work out precisely the statistics of these quantities. The results obtained will be dependent on the statistics of the random input parameters, and thus a very detailed analytical model will be required.

If, on the other hand, the plate is subjected to a high degree of randomness, then it is known that the natural frequencies (excluding the lower frequencies) approach a standard distribution described by the Gaussian Orthogonal Ensemble (GOE) [4-6], regardless of the statistical details of the input parameters. In this case the prediction of the natural frequency statistics is unnecessary (since they are known) and the statistics of the response can be calculated in a relatively straightforward way [7]. This opens up the possibility of non-parametric models of uncertainty for complex systems, i.e. models which avoid the need to describe the detailed statistical distributions of the underlying physical parameters. One of the earliest non-parametric modelling approaches developed in engineering is known as statistical energy analysis (SEA) [8], in which the governing equations are based on the flow of vibrational energy between regions of the system known as “subsystems”. By adopting a single (energy) degree of freedom for each subsystem, relatively efficient models of the system can be derived, and the mean energy response can be computed without the need for parametric modelling, large finite element models, or Monte Carlo simulations. The statistical assumptions employed in SEA are valid only at high frequencies, and significant current research effort is being directed at the development alternative, more general, non-parametric approaches. Much of this work is based on novel applications of random matrix theory (for example, [9]), or, as described below, the combination of SEA with other approaches.

Over the past decade, much research into the dynamic analysis of uncertain engineering systems has been motivated by the “mid-frequency” problem, where neither the finite element method (FE) nor statistical energy analysis (SEA) provide a satisfactory solution. At mid-frequencies the system is too complex for FE to apply (too random, with too many degrees of freedom), and not complex enough for SEA apply (not random enough for the averaging required by SEA to apply, so that very high response variances arise which cannot be predicted by standard SEA). The obvious approach is therefore to try and combine FE and SEA into a single model, so that advantage can be taken of the strengths of each method. This type of hybrid model has been proposed by many researchers, and the pioneering work of Soize [10] on fuzzy structure theory highlighted many general underlying principles. An early attempt to produce a hybrid model consisted of partitioning the response of a system component into a “long wavelength” contribution and a “short wavelength” contribution [11]. The long wavelength part was then included in an FE model, and the short wavelength part in an SEA model. A computational difficulty with this approach for general systems is that the coupling between the FE and SEA parts of the model require the evaluation of some rather difficult integrals. An alternative method was then developed in which the response of the component is partitioned not into different wavelength components, but into a “direct field” and a “reverberant field” [12]. One advantage of this approach is that the coupling between the SEA and FE parts of the model can be affected simply and efficiently through a “diffuse field reciprocity relation” [13,14]. Also, the method can be combined with aspects of random matrix theory to predict not only the mean, but also the variance of the system response [15].

In this paper attention is focused on two analysis methods: SEA and the hybrid FE-SEA approach. The basic concepts of SEA are described in Section 2, while the hybrid FE-SEA method is described in Section 3. An outline derivation of the key equations is given in each case, although the cited references provide a much more rigorous discussion of the various approximations involved. A number of example applications is given for each method.
2. Statistical Energy Analysis (SEA)

2.1. The ensemble mean response
In SEA a complex system is represented as an assembly of subsystems, and the aim is to find the ensemble mean of the vibrational energy of each subsystem. The governing equations, which arise from a consideration of power balance, have the form (see for example [8])

\[ \omega\eta_j E_j + \sum_k \omega n_j P_{\text{in},k} (E_j / n_j - E_k / n_k) = P_{\text{in},j} \quad \text{or} \quad \hat{C}\hat{E} = P_{\text{in}} \] (1a,b)

where, for subsystem \( j \), \( E_j \) is the ensemble averaged vibrational energy (which may also be averaged over a frequency band), \( \eta_j \) is the loss factor, \( n_j \) is the modal density, and \( P_{\text{in},j} \) is the power input from external sources. The term \( \eta_{jk} \) is the coupling loss factor between subsystems \( j \) and \( k \), and \( \omega \) is the vibration frequency. In Eq. (1b), \( \hat{E}_j = E_j / n_j \), and the entries of the matrix \( C \) and the vector \( P_{\text{in}} \) can readily be deduced from Eq. (1a). The assumptions that underpin Eq. (1) have been discussed, for example, by Mace [16], Lyon and DeJong [17], and Fahy [18].

2.2. The ensemble variance
In reference [19] an expression has been derived for the variance of the energy of the \( j \)th subsystem within the context of SEA. The result has the form

\[ \text{Var}(E_j) = \sum_k (C^{-1}_{jk})^2 \text{Var}(P_{\text{in},k}) + \sum_k \sum_{s \neq k} [(C^{-1}_{jk} - C^{-1}_{js}) \hat{E}_j]^2 \text{Var}(C_{\text{in},ks}) \] (2)

where \( C^{-1}_{jk} \) represents the \( jk \)th entry of the inverse of the matrix \( C \) that appears in Eq. (1b), and the two “Var” terms are given by

\[ \text{Var}(P_{\text{in},k}) = P_{\text{in},k}^2 r^2 (\alpha_k, m_k, B_k') \quad \text{and} \quad \text{Var}(C_{\text{in},ks}) = C_{\text{in},ks}^2 r^2 (\alpha_s, m_s, B_s') \] (3,4)

\[ m_k' = \omega n_k \eta_k', \quad \eta_k' = 1 / (\omega n_k C^{-1}_{jk}), \quad B_k' = \frac{\Delta}{\omega n_k} \] (5-7)

where \( \Delta \) is the frequency bandwidth over which the energy is averaged (\( \Delta=0 \) for a purely harmonic analysis). The function \( r \) that appears in Eqs (3) and (4) is given by [7,20]

\[ r^2 (\alpha, m, B) = \frac{\alpha - 1}{\alpha m} \left( \frac{1}{B^2} \right) \left[ 2B \tan^{-1} (B) - \ln \left( 1 + B^2 \right) \right] + \frac{1}{\alpha m} \left( \frac{1}{B^2} \right) \ln \left( 1 + B^2 \right) . \] (8)

This is an approximate result arising from simplifying certain integrals that appear in the underlying derivations. A more exact result is available for the case \( B=0 \):

\[ r^2 (\alpha, m, 0) = \frac{1}{\alpha m} \left\{ \alpha - 1 + \frac{1}{2\alpha m} \left[ 1 - e^{-2\pi m} \right] + E_i(\pi m) \left[ \cosh (\pi m) - \frac{1}{\pi m} \sinh (\pi m) \right] \right\} . \] (9)

Equation (2) can readily be applied as a post-processing step at the end of a standard SEA analysis. The only “non-SEA” parameters that appear in Eqs (2)-(9) are the coefficients \( \alpha_k \) and \( \alpha_s \), and these
are readily found for various types of power input ($\alpha_k$) and various types of coupling between subsystems ($\alpha_{ks}$). The derivation of Eqs. (2)-(9) is fully described in reference [19] and will not be repeated here, other than to state that the main assumptions behind the equations are: (i) each subsystem exhibits statistical overlap, and the statistics of the mode shapes and natural frequencies are described by the Gaussian Orthogonal Ensemble (GOE), and (ii) the subsystems are weakly coupled, as normally required in SEA.

2.3. Example applications

An example of the application of equations (1-9) is shown in Figure 1. The results concern two plates that are coupled along a common edge. Plate 1 is subject to a point load, and Monte Carlo simulations have been performed on an ensemble of systems generated by adding small masses in random locations. The irregular curves in the figure represent the Monte Carlo results, while the smooth curves are the predictions arising from the foregoing theory. Results are shown for the mean and relative variance of the bending energy in each plate, and good agreement with theory is demonstrated. It can be noted that while the driven plate has the largest mean energy, the relative variance is greater for the second plate. This is because the response of the second plate is affected by the randomness in both plates, while the first plate response is little affected by the second plate. This type of behaviour is further illustrated in Figure 2 for a three plate system: the relative variance increases with increasing distance from the excited plate.

![Figure 1: Mean and relative variance for the bending energies in a two plate system [19].](image1)

![Figure 2: Mean and relative variance for the bending energies in a three plate system [19].](image2)
3. The hybrid method

3.1. Introduction and key concepts
As explained in the previous section, SEA is based on modelling a structure as a collection of subsystems. It is assumed that each subsystem has random properties across the ensemble, and that the subsystem natural frequencies are sufficiently random for the tenets of SEA to apply. In many practical structures this condition is not met: in an aircraft structure for example, the skin panels may well have highly random modes, but the frames do not, being modally sparse and relatively deterministic. Similarly, parts of an automotive structure, such as the door pillars and side rails, are stiff and modally sparse, while the door panels and roof have many modes. In such cases it would be advantageous to combine the finite element method (or some other deterministic modelling method) with SEA, and this is achieved through the hybrid method described in the following sections.

By way of background, it can be noted that the literature on mid and high frequency vibration is extensive, and there have been many efforts to combine deterministic and statistical analysis methods into a single model. Beylaev and Palmov [21] were among the first researchers to include locally averaged statistical behaviour into a global deterministic model of a system. This type of approach was developed much further by the pioneering work of Soize [10] on fuzzy structure theory, and Langley and Bremner [11] developed a hybrid FE-SEA method based on these ideas combined with a wavelength partitioning scheme. A hybrid FE-SEA method ideally combines the low frequency performance of the FE method with the high frequency performance of SEA to produce a robust method that can be applied across the whole frequency range. However, the coupling of FE and SEA in a single model is difficult because the methods differ in two ways: (i) FE is based on dynamic equilibrium while SEA is based on the conservation of energy flow, and (ii) FE is a deterministic method while SEA is inherently statistical. More recently Shorter and Langley [12] have developed a new method of effecting this coupling, which is based on wave concepts rather then the modal type of approach employed in reference [11]. At the heart of the method is a reciprocity result [13] regarding the forces exerted at the boundaries of an SEA subsystem; this centres on the concept of a “direct field” dynamic stiffness matrix, which is explained in what follows.

As discussed above, in the mid-frequency range some components of a complex structure (for example thin panels) display short wavelength vibrations and are sensitive to the effects of random uncertainties, while others (for example beams) show little variation in their dynamic properties and are essentially deterministic. In the hybrid method proposed by Shorter and Langley [12], the deterministic components are modelled by using the finite element method, while the random components are modelled as SEA subsystems. A key feature of the method is the concept of a “direct field” or “power absorbing” dynamic stiffness matrix associated with each SEA subsystem. Consider for example a thin plate that is excited at the boundaries. The excitation generates waves that propagate through the plate and are reflected repeatedly at the boundaries; the total dynamic stiffness matrix of the plate, phrased in terms of the edge degrees of freedom, has contributions from all of these reflections. Suppose now that the response is viewed in two parts:

1) The contribution from the initial generated waves, prior to any boundary reflections. This can be called the “direct field”.

2) The contribution from waves produced on the first and all subsequent reflections. This can be called the “reverberant field”.

The direct field dynamic stiffness matrix can be defined as that resulting from the presence of the direct field waves – this matrix corresponds to “power absorbing” behaviour, in the sense that the direct field waves all propagate energy away from the boundaries. Such a matrix can be found analytically for each of the subsystems by a variety of methods. For example, consider a thin plate with four straight boundaries. If this plate is part of a larger FE model, then the degrees of freedom of
the boundary can be described in terms of the nodal degrees of freedom of the FE mesh. To employ the hybrid method we need to find the direct field dynamic stiffness matrix associated with the edge degrees of freedom of the plate. This can be done by considering each straight boundary in turn, and taking it to form a segment of the edge of a semi-infinite plate. Motion of the boundary will generate waves into the semi-infinite plate, and for a given boundary motion the generated waves can be found by Fourier transform techniques. Calculation of the boundary forces associated with the generated waves then allows the dynamic stiffness matrix to be constructed: i.e., strictly the dynamic stiffness matrix of a segment of the edge of a semi-infinite plate, when the motion of the segment is described by FE nodal degrees of freedom. This is our required “direct field” dynamic stiffness matrix for the plate edge, and repeating the process for each of the plate edges will give the total direct field dynamic stiffness matrix for the plate subsystem. It is important to note that the direct field dynamic stiffness matrix can also be viewed as the ensemble average of the full dynamic stiffness matrix when averaged over random boundary reflections. Given the concept of the direct field dynamic stiffness matrix, the hybrid FE-SEA equations can now be described.

3.2. The hybrid equations
The starting point for the hybrid method is to identify those parts of the system response that will be described by SEA subsystems. The remaining part of the system (which can be considered to be the “deterministic” part) is then modelled by using the FE method. For example, it might be decided that the bending motions of the panels of a car have a short wavelength of deformation and will be described using SEA subsystems. The bending degrees of freedom of these panels will then be omitted from the FE model of the system, at all points other than the panel boundaries. The relevant “direct field” dynamic stiffness matrix is then added to the FE model at the panel boundaries, and this augmented FE model is then used in the subsequent analysis. If the degrees of freedom of the deterministic part are labelled \( q \), then the governing equations of motion (for harmonic vibration of frequency \( \omega \) say) will have the form

\[
D_{\text{tot}} q = f + \sum_k D^{(k)}_{\text{rev}} f^{(k)} + \sum_k D^{(k)}_{\text{dir}}.
\]

The summation is over the number of SEA subsystems in the model, and \( D^{(k)}_{\text{dir}} \) represents the direct field dynamic stiffness matrix associated with subsystem \( k \). Furthermore, \( D_{\text{tot}} \) is the dynamic stiffness matrix given by the finite element model of the deterministic part of the system, \( f \) is the set of external forces applied to this part of the system, and \( f^{(k)}_{\text{rev}} \) represents the force arising from the reverberant field in subsystem \( k \), which is not accounted for in \( D^{(k)}_{\text{dir}} \). The matrix \( D_{\text{tot}} \) is the dynamic stiffness matrix of the FE model (excluding the SEA subsystem degrees of freedom), when augmented by the direct field dynamic stiffness matrix of each SEA subsystem. It should be noted that equations (10) and (11) are exact – all that has been done is to split the forces arising from the SEA subsystems into a direct field part, which is accounted for by \( D^{(k)}_{\text{dir}} \), and a reverberant part which is carried to the right hand side of equation (10). The following result (Shorter and Langley [13]) is central to the development of the hybrid method

\[
S^{(k),\text{rev}}_{gg} \equiv E\left[f^{(k)}_{\text{rev}} f^{(k)\text{rev}}\right] = \left(\frac{4E_k}{\omega \pi n_k}\right) \text{Im}\{D^{(k)}_{\text{dir}}\}.
\]

Here \( E_k \) and \( n_k \) are respectively the (ensemble average) vibrational energy and the modal density of the \( k \)th subsystem. Equation (12) implies that the cross-spectral matrix of the force exerted by the
reverberant field is proportional to the resistive part of the direct field dynamic stiffness matrix, which
is a form of diffuse field reciprocity statement. Now from equation (10), the response \( \mathbf{q} \) can be
expanded in the form

\[
\mathbf{q} = \mathbf{q}_d + \sum_k \mathbf{q}^{(k)}_d, \quad \mathbf{q}_d = \mathbf{D}^{-1}_{\text{tot}} \mathbf{F}, \quad \mathbf{q}^{(k)} = \mathbf{D}^{-1}_{\text{tot}} \mathbf{F}^{(k)}.
\] (13)

Now the time averaged power input to the direct field of subsystem \( j \) can be written as

\[
P_{\text{in},j} = (\omega/2) \text{Im}\left\{ \mathbf{q}^\top \mathbf{D}_{\text{dir}}^{(j)} \mathbf{q} \right\} = (\omega/2) \sum_{rs} \text{Im}\left\{ \mathbf{D}_{\text{dir},rs}^{(j)} \mathbf{S}_{rs} \mathbf{D}_{\text{tot}}^{(j)^\top} \right\},
\] (14)

where it has been noted that the dynamic stiffness matrix is symmetric. If the various contributions \( \mathbf{q}^{(k)} \) that appear in equation (13) are taken to be uncorrelated and of zero mean, then equations (12)-(14) yield

\[
P_{\text{in},j} = P_{\text{ext}}^{\text{in}} + \sum_k \omega \eta_{jk} n_j (E_j / n_j).
\] (15)

where

\[
P_{\text{ext}}^{\text{in}} = (\omega/2) \sum_{rs} \text{Im}\left\{ \mathbf{D}_{\text{dir},rs}^{(j)} \left( \mathbf{D}^{-1}_{\text{tot}} \mathbf{S}_{rs} \mathbf{D}_{\text{tot}}^{(j)^\top} \right) \right\},
\] (16)

\[
\omega \eta_{jk} n_j = (2 / \pi) \sum_{rs} \text{Im}\left\{ \mathbf{D}_{\text{dir},rs}^{(j)} \left( \mathbf{D}^{-1}_{\text{tot}} \text{Im}\left\{ \mathbf{D}_{\text{dir}}^{(k)} \mathbf{D}_{\text{tot}}^{(k)^\top} \right\} \right) \right\}.
\] (17)

Given that the dynamic stiffness matrices are symmetric, it is readily shown from equation (17) that
reciprocity holds, in the sense that \( \eta_{jk} n_j = \eta_{jk} n_k \). As will be shown in what follows, the terms \( \eta_{jk} \)
are equivalent to the coupling loss factors that appear in SEA.

The power output from the reverberant field in subsystem \( j \) can be written as the sum of: (i) the
power dissipated through damping, (ii) the power transferred to the other subsystems, and (iii) the
power dissipated in the deterministic system due to the response \( \mathbf{q}^{(j)} \). Thus

\[
P_{\text{out},j} = \omega \eta_j E_j + \sum_k \omega \eta_{jk} n_k (E_j / n_j) + \omega \eta_{d,j} E_j.
\] (18)

where

\[
\omega \eta_{d,j} = (\omega/2E_j) \text{Im}\left\{ \mathbf{q}^{(j)^\top} \mathbf{D}_j \mathbf{q}^{(j)} \right\} = \left( \frac{2}{\pi n_j} \right) \sum_{rs} \text{Im}\left\{ \mathbf{D}_{d,rs} \left( \mathbf{D}^{(j)^\top}_{\text{tot}} \text{Im}\left\{ \mathbf{D}_{\text{dir}}^{(k)} \mathbf{D}_{\text{tot}}^{(k)^\top} \right\} \right) \right\}.
\] (19)

Equations (21) and (25) then lead to the following energy balance equation for subsystem \( j \)

\[
\omega (\eta_j + \eta_{d,j}) E_j + \sum_k \omega \eta_{jk} n_j (E_j / n_j - E_k / n_k) = P_{\text{in},j}^{\text{ext}}.
\] (20)

Furthermore, the cross-spectral matrix of the response \( \mathbf{q} \) can be derived from equations (12) and (13),
which yields
Equations (20) and (21) form the two main equations of the hybrid method. It is clear that these equations couple FE and SEA methodologies: equation (20) has precisely the form of SEA, but the coupling loss factors $\eta_{jk}$ and loss factors $\eta_{d,j}$ are calculated by using the FE model (augmented by the direct field dynamic stiffness matrices) via equations (17) and (19); furthermore, equation (21) has the form of a standard deterministic FE analysis, but additional forces arise from the reverberant energies in the subsystems. If no SEA subsystems are included then the method becomes purely FE; on the other hand, if only the junctions between the SEA subsystems are modelled by FE, then the method becomes purely SEA, with a novel method of computing the coupling loss factors.

3.3. Academic benchmark example

The first example of the application of the hybrid method is taken from reference [22]. Two thin plates are coupled via a complex beam, as shown in Figure 3. Two models of the system are considered: (i) a detailed finite element model, (ii) a hybrid model in which the beam and the in-plane motion of the plates is modeled by FE, while the bending motion of the plates is modeled using two SEA subsystems (one for each plate). The detailed finite element model has been randomized by adding small masses to the plates in random locations, and an ensemble of 200 structures has been created in this way. The detailed finite element model has 12000 degrees of freedom, while the hybrid model has 3500 finite element degrees of freedom, while the hybrid model has 3500 finite element degrees of freedom and two SEA subsystems.

![Figure 3: Two plates coupled by a beam. Left hand diagram – detailed FE model; right hand diagram – FE model used as part of the hybrid model.](image)

The lower panel is subjected to rain-on-the roof excitation. It has been found that the hybrid method requires around 1/100 of the computer time needed to perform 20 Monte Carlo simulations using the detailed model. The predicted response of the system is shown in Figure 4, where the ensemble average yielded by the detailed by the detailed FE model is compared with the results yielded by the hybrid method. It can be seen that the hybrid method yields very good response predictions; the driven plate is fairly insensitive to the dynamics of the coupling beam, while the non-driven plate shows peaks corresponding to the beam resonances. Other benchmark example applications of the method are given in references [12] and [22], and variance predictions using an extended version of the method are presented in reference [15].
3.4. Industrial example

The hybrid method has been implemented in the software package VA One [23], and this has been used on many industrial problems. Example applications can be found in references [24-26], covering automotive, aerospace, and rail vehicles. The application of the method to an automotive structure is shown in Figure 5 [27]. Stiff elements of the car, such as the rails, engine supports, and shock towers, are modelled using FE, whereas flexible components such as the roof, windows, doors, and the interior acoustic cavity, are represented by SEA subsystems. The use of SEA subsystems removes a huge number of degrees of freedom from the finite element model, since the flexible components are very modally dense, and require a very fine mesh if they are to be modelled using the FE method.

Typical results for the interior noise level are shown in Figure 6. The results for trimmed and untrimmed versions of the vehicle are included in the graphs. The numerical results are in general within 3 to 5 dB of the experimental results, correctly accounting for the effect of noise control treatments applied in the trimmed version. The results for the trimmed version are actually in better agreement with experiment than for the bared version, and this can be traced to better characterization of the interior cavity damping; in the absence of trim, the cavity damping is very low, and difficult to estimate. Full details of the hybrid model for this case can be found in reference [27].
4. Conclusions
This paper has consider various method for the dynamic analysis of structures in the presence of manufacturing uncertainties. Parametric and non-parametric approaches have been briefly reviewed in the introduction, and then a detailed presentation of two non-parametric methods has been given: statistical energy analysis (SEA) and the hybrid method. Each of these methods can be used to estimate the mean and variance of stress, vibration, and noise levels within a complex system, with SEA being suited to high-frequencies and the hybrid method to mid-frequencies. At low frequencies the statistics of the response of a system can be very sensitive to the statistics of the uncertain input parameters, and parametric methods are more suitable than the two methods described here. Work is continuing on combining the hybrid method with parametric models of uncertainty within the FE component of the method.

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