A symmetry classification for a class of
(2 + 1)-nonlinear wave equation

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Abstract

In this paper, a symmetry classification of a (2 + 1)-nonlinear wave
equation \( u_{tt} - f(u)(u_{xx} + u_{yy}) = 0 \) where \( f(u) \) is a smooth function on
\( u \), using Lie group method, is given. The basic infinitesimal method
for calculating symmetry groups is presented, and used to determine
the general symmetry group of this (2 + 1)-nonlinear wave equation.

1 Introduction

It is well known that the symmetry group method plays an important role
in the analysis of differential equations. The history of group classification
methods goes back to Sophus Lie. The first paper on this subject is [1],
where Lie proves that a linear two-dimensional second-order PDE may admit
at most a three-parameter invariance group (apart from the trivial infinite
parameter symmetry group, which is due to linearity). He computed the
maximal invariance group of the one-dimensional heat conductivity equation
and utilized this symmetry to construct its explicit solutions. Saying it the
modern way, he performed symmetry reduction of the heat equation. Nowa-
days symmetry reduction is one of the most powerful tools for solving nonlin-
ear partial differential equations (PDEs). Recently, there have been several

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generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [2] developed the method of partially invariant solutions. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study.

For many nonlinear systems, there are only explicit exact solutions available. These solutions play an important role in both mathematical analysis and physical applications of the systems. There are a number of papers to study \( (1 + 1) \)-nonlinear wave equations from the point of view of Lie symmetries method. First, for solving some of the physical problems, the quasi-linear hyperbolic equation with the form

\[ u_{tt} = [f(u)u_x]_x, \quad (1) \]

in [3] and later its the generalized cases

\[ u_{tt} = [f(x, u)u_x]_x, \quad u_{tt} = [f(u)u_x + g(x, u)]_x, \quad (2) \]

in [4] and [5], respectively, are investigated. Also the most important classes of the \( (1 + 1) \)-nonlinear wave equations with the forms

\[ v_{tt} = f(x, v_x)v_{xx} + g(x, v_x), \quad u_{tt} = f(x, u)u_{xx} + g(x, u), \quad (3) \]

can be found in two attempts [6] and [7] respectively. An alternative form of Eq. (1) was also investigated by Oron and Rosenau [8] and Suhubi and Bakkaloglu [9]. The equations

\[ u_{tt} = F(u)u_{xx}, \quad u_{tt} + K(u)u_t = F(u)u_{xx}, \quad u_{tt} + K(u)u_t = F(u)u_{xx} + H(u)u_x, \quad (4) \]

are classified in [10, 11, 12], respectively. Lahno et al. [13] presented the most extensive list of symmetries of the equations

\[ u_{tt} = u_{xx} + F(t, x, u, u_x), \quad (5) \]

by using the infinitesimal Lie method, the technique of equivalence transformations, and the theory of classification of abstract low-dimensional Lie algebras. There are also some papers [14, 15, 16] devoted to the group classification of the equations of the following form:

\[ u_{tt} = F(u_{xx}), \quad u_{tt} = F(u_x)u_{xx} + H(u_x), \quad u_{tt} + u_{xx} = g(u, u_x), \quad (6) \]
Studies have also been made for \((2+1)\)-nonlinear wave equation with constant coefficients \([17, 18, 19]\). In the special case the \((2 + 1)\)-dimensional nonlinear wave equation

\[
    u_{tt} = u^n(u_{xx} + u_{yy}),
\]

is investigated in \([20]\). The goal of this paper is to investigate the Lie symmetries for some class of \((2 + 1)\)-nonlinear wave equation

\[
    u_{tt} - f(u)(u_{xx} + u_{yy}) = 0,
\]

where \(f(u)\) is a arbitrary smooth function of the variable \(u\). Clearly, in Eq. \((8)\) case of \(f_u = 0\) namely \(f(u) = constant\) is not interest because this case reduces the wave equation to a linear one. Similarly technics were applicable for some classes of the nonlinear heat equations in \([21, 22]\).

2 Symmetry Methods

Let a partial differential equation contains one dependent variable and \(p\) independent variables. The one-parameter Lie group of transformations

\[
    \begin{align*}
        \bar{x}_i &= x_i + \epsilon \xi_i(x, u) + O(\epsilon^2); \\
        \bar{u} &= u + \epsilon \varphi(x, u) + O(\epsilon^2),
    \end{align*}
\]

where \(i = 1, \ldots, p\), and \(\xi_i = \frac{\partial \varphi}{\partial u}\) \(|_{\epsilon=0}\), acting on \((x, u)\)-space has as its infinitesimal generator

\[
    \mathbf{v} = \xi_i \frac{\partial}{\partial x_i} + \varphi \frac{\partial}{\partial u}, \quad i = 1, \ldots, p.
\]

Therefore, the characteristic of the vector field \(\mathbf{v}\) given by \((10)\) is the function

\[
    Q(x, u^{(1)}) = \varphi(x, u) - \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial u}{\partial x_i}.
\]

The symmetry generator associated with \((10)\) given by

\[
    \mathbf{v} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u}.
\]
The second prolongation of $v$ is the vector field
\[ v^{(2)} = v + \varphi_x \frac{\partial}{\partial u_x} + \varphi_y \frac{\partial}{\partial u_y} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xy} \frac{\partial}{\partial u_{xy}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{yy} \frac{\partial}{\partial u_{yy}} + \varphi^{yt} \frac{\partial}{\partial u_{yt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}}. \] (13)

with coefficient
\[ \varphi^i = D_i Q + \xi u_{xi} + \eta u_{yi} + \tau u_{ti}, \] (14)
\[ \varphi^{ij} = D_i (D_j Q) + \xi u_{x_{ij}} + \eta u_{y_{ij}} + \tau u_{t_{ij}}, \] (15)

where $Q = \varphi - \xi u_x - \eta u_y - \tau u_t$ is the characteristic of the vector field $v$ given by (12) and $D_i$ represents total derivative and subscripts of $u$ are derivative with respect to the respective coordinates. $i$ and $j$ in above could be $x$, $y$ or $t$ coordinates. By the theorem 6.5. in [23], $v^{(2)}[u_{tt} - f(u)(u_{xx} + u_{yy})] = 0$ whenever
\[ u_{tt} - f(u)(u_{xx} + u_{yy}) = 0. \] (16)

Since
\[ v^{(2)}[u_{tt} - f(u)(u_{xx} + u_{yy})] = \varphi^{tt} - \varphi f_u (u_{xx} + u_{yy}) - f(u)(\varphi^{xx} + \varphi^{yy}), \]

therefore
\[ \varphi^{tt} - \varphi f_u (u_{xx} + u_{yy}) - f(u)(\varphi^{xx} + \varphi^{yy}) = 0. \] (17)

Using the formula (15) we obtain coefficient functions $\varphi^{xx}, \varphi^{yy}, \varphi^{tt}$ as
\[ \varphi^{xx} = D_x^2 Q + \xi u_{xxx} + \eta u_{yxx} + \tau u_{txx}, \] (18)
\[ \varphi^{yy} = D_y^2 Q + \xi u_{yyy} + \eta u_{yx} + \tau u_{tyy}, \] (19)
\[ \varphi^{tt} = D_t^2 Q + \xi u_{qtt} + \eta u_{ytt} + \tau u_{ttt}, \] (20)

where the operators $D_x$, $D_y$ and $D_t$ denote the total derivatives with respect to $x$, $y$ and $t$:
\[ D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} + u_{xt} \frac{\partial}{\partial u_t} + \ldots \]
\[ D_y = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{yx} \frac{\partial}{\partial u_x} + u_{yt} \frac{\partial}{\partial u_t} + \ldots \] (21)
\[ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + u_{ty} \frac{\partial}{\partial u_y} + \ldots \]
and by substituting them into invariance condition (17), we are left with a polynomial equation involving the various derivatives of \( u(x, y, t) \) whose coefficients are certain derivatives of \( \xi, \eta, \tau, \varphi \). Since \( \xi, \eta, \tau, \varphi \) only depend on \( x, y, t, u \) we can equate the individual coefficients to zero, leading to the complete set of determining equations:

\[
\begin{align*}
\xi & = \xi(x, t) \quad (22) \\
\eta & = \eta(y, t) \quad (23) \\
\tau & = \tau(x, y, t) \quad (24) \\
\varphi & = \alpha(x, y, t)u + \beta(x, y, t) \quad (25) \\
\tau_t & = \varphi_u = \alpha(x, y, t), \quad (26) \\
\xi_{tt} & = f(u)(\xi_{xx} - 2\varphi_{xu}) \quad (27) \\
\eta_{tt} & = f(u)(\eta_{yy} - 2\varphi_{yu}) \quad (28) \\
\tau_{tt} & = f(u)(\tau_{xx} + \tau_{yy}) + 2\varphi_{tu} \quad (29) \\
f_u\varphi & = 2f(u)(\xi_t - \tau_t) \quad (30) \\
f_u\varphi & = 2f(u)(\eta_t - \tau_t) \quad (31) \\
f(u)\varphi & = \xi_t \quad (32) \\
f(u)\tau_x & = \eta_t \quad (33) \\
f(u)\tau_y & = \varphi_u = f(u)(\varphi_{xx} + \varphi_{yy}) \quad (34)
\end{align*}
\]

### 3 classification of symmetries of the model

In this section we start to classify the symmetries of the nonlinear wave equation (8). To fined a complete solution of the above system we consider Eq. (30) and with assumption \( f_u \neq 0 \) we rewrite:

\[
\varphi = 2\frac{f}{f_u}(\xi_t - \tau_t) \quad (35)
\]

Note the case of \( f(u) = constant \) explained in introduction. Two general cases are possible:

\[
\begin{align*}
i) \quad & \frac{f}{f_u} = c, \quad (36) \\
ii) \quad & \frac{f}{f_u} = g(u), \quad (37)
\end{align*}
\]
where \( c \) is a constant.

### 3.1 Case (i)

In this case with integrating from Eq. (36) with respect to \( u \) to obtain

\[
f(u) = Ke^\frac{u}{c},
\]

where \( K \) is an integration constant. Then the Eq. (35) reduce to

\[
\varphi = 2c(\xi_x - \tau_t).
\]

With substituting (39) into (26)-(33) we have

\[
\begin{align*}
\xi(x) &= c_1 x + c_2; \\
\eta(y) &= c_1 y + c_3; \\
\tau(t) &= c_4 t + c_5; \\
\varphi &= 2c(c_1 - c_4).
\end{align*}
\]

where \( c_i, i = 1, \ldots, 5, \) are arbitrary constants. The Lie symmetry generator for Eq. (8) in this case (i) is

\[
v = (c_1 x + c_2) \frac{\partial}{\partial x} + (c_1 y + c_3) \frac{\partial}{\partial y} + (c_4 t + c_5) \frac{\partial}{\partial t} + 2c(c_1 - c_4) \frac{\partial}{\partial u}.
\]

Therefore the symmetry algebra of the \((2 + 1)\)-nonlinear wave equation (8) is spanned by the vector fields

\[
\begin{align*}
v_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2c \frac{\partial}{\partial u}; \\
v_2 &= \frac{\partial}{\partial x}; \\
v_3 &= \frac{\partial}{\partial y}; \\
v_4 &= t \frac{\partial}{\partial t} - 2c \frac{\partial}{\partial u}; \\
v_5 &= \frac{\partial}{\partial t}.
\end{align*}
\]

The commutation relations satisfied by generators (42) in the case (i) are shown in table 1. The invariants associated with the infinitesimal generator \( v_1 \) are obtained by integrating the characteristic equation:

\[
\begin{align*}
\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{2c}.
\end{align*}
\]

and have the forms

\[
r = \frac{y}{x}, \quad s = t, \quad \text{and} \quad \omega(r, s) = u(x, y, t) - 2c \ln x,
\]

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Table 1: Commutation relations satisfied by infinitesimal generators in Cases (i) and (ii)

| [v_i, v_j] | v_1 | v_2 | v_3 | v_4 | v_5 |
|------------|-----|-----|-----|-----|-----|
| v_1        | 0   | -v_2| -v_3| 0   | 0   |
| v_2        | v_1 | 0   | 0   | 0   | 0   |
| v_3        | v_3 | 0   | 0   | 0   | 0   |
| v_4        | 0   | 0   | 0   | 0   | -v_5|
| v_5        | 0   | 0   | 0   | v_5 | 0   |

With substituting (44) into (16) to determine the form of the function $\omega$ to obtain

$$\omega_{ss} = Ke^\frac{u}{2}((1 + r^2)\omega_{rr} + 2r\omega_r - 2c),$$ (45)

By solving this partial differential equation we obtain the reduced equation

$$\omega(r, s) = \zeta_1(r) + \zeta_2(s),$$ (46)

where $\zeta_1$ and $\zeta_2$ satisfy in following second-order differential equations

$$\ddot{\zeta}_1(r^2 + 1) + c_1e^{-\zeta_1^2} + 2(r\dot{\zeta}_1 - c) = 0; \quad \ddot{\zeta}_2 + Kc_1e^{\frac{u}{2}} = 0,$$ (47)

with $c_1, c, K$, arbitrary constants. The characteristic equation associated with $v_4$ is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{t} = \frac{du}{-2c},$$ (48)

which generate the invariants $x, y, t^{-2c}e^{-u}$. Then the similarity solution is chosen to have the form

$$u(x, y, t) = 2c\ln\frac{h(x, y)}{t}.$$ (49)

By substituting (49) into (16) to determine the form of the function $h$ to obtain

$$\frac{1}{K} - h(x, y)(h_{xx} + h_{yy}) + h_x^2 + h_y^2 = 0,$$ (50)
which has the solution
\[ h(x, y) = mx + py + q; \quad m^2 + p^2 = K^{-1}, \] (51)
where \( m, p, q \) are arbitrary constants. For the remaining infinitesimal generators \( v_2, v_3, v_5 \), the invariants associated are the arbitrary functions \( \lambda(y, t, u) \), \( \mu(x, t, u) \), and \( \nu(x, y, u) \) respectively.

3.2 Case (ii)

In this case we classify solution of the wave equation (8), with assumption \( g_u \neq 0 \). With substituting (34) into (27)-(28), since \( \xi, \eta \) and \( \tau \) are not dependent to \( u \), therefore from
\[ u(x, y, t, u) = 2g(u)(\xi_x - \tau_t), \] (52)
and also from (8) and (25), we conclude
\[ g(u) = e_1u + e_2, \] (53)
where \( e_1 \neq 0 \) and \( e_2 \) are arbitrary constants. Now we substitute (53) into (37) and rewrite
\[ \frac{f_u}{f} = \frac{1}{e_1u + e_2}. \] (54)
Therefore by integrating from (54) with respect to \( u \) we have
\[ f(u) = L(e_1u + e_2)^{-1}, \] (55)
where \( L \) is an integration constant. Now by considering Eq. (22)-(34), it is not hard to find that the components \( \xi, \eta, \tau \) and \( \varphi \) of infinitesimal generators become
\[ \begin{align*}
\xi(x) &= c_1x + c_2; \\
\eta(y) &= c_1y + c_3; \\
\tau(t) &= c_4t + c_5; \\
\varphi &= 2e_1(c_1 - c_4)u + 2e_2(c_1 - c_4),
\end{align*} \] (56)
where \( c_i, i = 1, \ldots, 5 \), are arbitrary constants. From above the five infinitesimal generators can be constructed:
\[ \begin{align*}
v_1 &= x\partial_x + y\partial_y + (2e_1u + 2e_2)\partial_u; \\
v_2 &= \partial_x; \\
v_3 &= \partial_y; \\
v_4 &= t\partial_t - 2(e_1u + e_2)\partial_u; \\
v_5 &= \partial_t.
\end{align*} \] (57)
It is easy to check that the infinitesimal generators (57) from a closed Lie algebra whose it's corresponding commutation relations are coincided with obtained results in table 1. For generator $v_1$, the associated equations are
\[
\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{0} = \frac{du}{2(e_1u + e_2)},
\]
which generate the invariants $p = \frac{y}{x}$, $q = t$, and $\vartheta(p, q) = (u + \frac{e_2}{e_1})x^{-2e_1}$. Consequently, the similarity solution is chosen to have the form
\[
\vartheta(x, y, t) = \vartheta(t, \frac{y}{x})x^{2e_1} + \frac{e_2}{e_1}.
\]
We substitute (60) into (16) to obtain following partial differential equation
\[
\vartheta_{pp} = L\vartheta_{qq} + 2q\vartheta_q(1 - 2e_1) + 2e_1(2e_1 - 1)\vartheta,
\]
as an example, for particular case $e_1 = 1$, the solution of (60) is
\[
\vartheta(p, q) = \varsigma_1(p) \cdot \varsigma_2(q),
\]
where $\varsigma_1(p)$ and $\varsigma_2(q)$ satisfy in second order equations
\[
\ddot{\varsigma}_1 - c\varsigma_1 = 0; \quad (q^2 + 1)\ddot{\varsigma}_2 - 2q\dot{\varsigma}_2 + 2\varsigma_2 - cL^{-1} = 0,
\]
where $c$ is an arbitrary constant. Also characteristic equation corresponding generator $v_4$ is
\[
\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{t} = \frac{du}{-2(e_1u + e_2)},
\]
and so
\[
u(x, y, t) = l(x, y)t^{-2e_1} - \frac{e_2}{e_1}.
\]
Substitute (64) into (16), $l(x, y)$ satisfies in the following equation:
\[
L(l_{xx} + l_{yy})l^{e_1^{-1} - 1}) - 2e_1(2e_1 + 1) = 0.
\]
For the remaining infinitesimal generators $v_2$, $v_3$, $v_5$, the invariants associated are the arbitrary functions $r(y, t, u)$, $m(x, t, u)$, and $n(x, y, u)$ respectively.
4 Conclusion and new ideas

In this paper we have obtained some particular Lie point symmetries group of the $(2 + 1)$-nonlinear wave equation $u_{tt} - f(u)(u_{xx} + u_{yy}) = 0$ where $f(u)$ is a smooth function on $u$, by using here the classical Lie symmetric method. In section 2, the complete set of determining equations was obtained by substituting the equations (18), (19) and (20) in invariance condition (17) and then in section 3, we classify the symmetries of this nonlinear wave equation by assumption two cases in (36) and (37) to consider $\frac{f}{f_u}$ is a constant or is a smooth function with respect to $u$ and $f_u \neq 0$. The commutation relations satisfied by infinitesimal generators in two cases are given in table 1, and their invariants associated with the infinitesimal generators are obtained. This method is suitable for preliminary group classification of some class of nonlinear wave equations [6, 7].

There are some classes of $(2 + 1)$-nonlinear wave equations that will be investigated by both classical or nonclassical symmetries method similarly whose we do for classical case. For examples

$$u_{tt} - f(x, u)(u_{xx} + u_{yy}) = 0,$$  \hspace{1cm} (66)
$$u_{tt} - f(x, u_x)(u_{xx} + u_{yy}) = 0,$$  \hspace{1cm} (67)

or generalized case

$$u_{tt} - f(x, y, u, u_x)(u_{xx} + u_{yy}) = 0,$$  \hspace{1cm} (68)

are interested.

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