Numerical solution of integral equations using Bernoulli wavelets

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Abstract
In this paper, Bernoulli wavelets operational matrix method is developed for the numerical solution of integral equations. Properties of Bernoulli wavelets and its function approximation is discussed. Firstly, the Bernoulli wavelets operational matrix of integration is generated. This operational matrix is employed for solving integral equations. Next, this technique converts the integral equation into system of algebraic equations and then solving these equations to obtain the Bernoulli wavelet coefficients. The accuracy of the proposed method is justified through the Illustrative examples and the obtained solutions are compared with those of exact solutions. Error analysis is presented to show the efficiency of the proposed method.

Keywords
Integral equations, Bernoulli wavelet, Bernoulli numbers, Bernoulli polynomials.

AMS Subject Classification
11B68, 65T60, 31B10.

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1. Introduction
Wavelet is newly emerging area in the field of mathematics. Wavelets are extensively used for signal processing in communications and physics, and it is one of the best mathematical tools.[8] For describing knowledge models integral equations are important tools in applied mathematics. Since in many cases, the exact solution of integral equations does not exist, the numerical approximation of these equations become necessary. There are various methods for approximating these equations and different basis functions[1, 4, 10], are used.

Bernoulli wavelet is a basis function and is an effective mathematical method for obtaining numerical solution of various types of problems with little additional work. The functions in the Bernoulli wavelet family are polynomials, and therefore their integrals and derivatives of any order exist. So, these functions can be applied for solving a wide variety of differential equations and integral equations [5]. In this paper we consider the Fredholm integral equations:

\[ y(x) = f(x) + \int_a^b k_1(x,t)y(t)dt, \]

Volterra integral equations:

\[ y(x) = f(x) + \int_a^x k_1(x,t)y(t)dt, \]

Volterra-Fredholm integral equations,

\[ y(x) = f(x) + \int_a^x k_1(x,t)y(t)dt + \int_a^b k_2(x,t)y(t)dt. \]

These integral equations are solved using Bernoulli wavelets operational matrix of integration. This paper is organized as follows. Section 2, gives the properties of Bernoulli wavelets. In section 3, we derive the Bernoulli wavelets operational matrix of integration. Section 4, gives the Bernoulli wavelets method of solution. In section 5, some examples are presented to show the efficiency of the presented method. Finally, in section 6, conclusion is drawn.
2. Bernoulli wavelets

Bernoulli wavelets \( \psi_{n,m}(x) = \psi(k,n,m,x) \) have four arguments \([3]: n = 1, 2, \ldots, 2^{k-1}, k \) is assumed to be any positive integer, \( m \) is the order for Bernoulli polynomials and \( t \) is the normalized time. They are defined on the interval \([0, 1]\) as follows:

\[
\psi_{nm}(x) = \begin{cases} 2^{k-1}\tilde{B}_m(2^{k-1}x - n + 1), & \frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}
\]

with

\[
\tilde{B}_m(x) = \begin{cases} 1, & m = 0 \\ \frac{1}{\sqrt{\left((-1)^{m-1}(m!)^2\right)/(2m!)}} & m > 0 \end{cases}
\]

where \( m = 0, 1, 2, \ldots, M - 1 \) and \( n = 1, 2, \ldots, 2^{k-1} \). The coefficient \( \sqrt{\left((-1)^{m-1}(m!)^2\right)/(2m!)\alpha_{2m}} \) is for normality, the dilation parameter is \( a = 2^{-(k-1)} \) and the translation parameter is \( b = (n-1)2^{-(k-1)} \). Here, \( B_m(x) \) are the Bernoulli polynomials of order \( m \) and \( \alpha_m \) are Bernoulli numbers.

### Function approximation

Suppose \( f(x) \in L^2(0,1) \) is expanded in terms of the Bernoulli wavelets as,

\[
f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x). \tag{2.2}\]

Truncating the above infinite series, we get,

\[
f(t) \approx \sum_{n=1}^{M} \sum_{m=0}^{2^{k-1}-1} c_{nm} \psi_{nm}(x) = C^T \psi(x) = f_\tilde{m}(x), \tag{2.3}\]

where, \( C \) and \( \psi(x) \) are \( m \times \tilde{m} = 2^{k-1}M \) matrices given by

\[
C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2k-1,0}, \ldots, c_{2k-1,M-1}]^T, \tag{2.4}\]

and

\[
\psi(x) = [\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1M-1}(x), \psi_{20}(x), \ldots, \psi_{2M-1}(x), \ldots, \psi_{2k-1,0}(x), \ldots, \psi_{2k-1,M-1}(x)]^T. \tag{2.5}\]

For instance, the first six Bernoulli wavelet bases for \( k = 2, M = 3 \) are:

\[
\begin{align*}
\psi_{10}(x) &= \sqrt{2}, \\
\psi_{11}(x) &= \sqrt{6}(4x - 1), \\
\psi_{12}(x) &= \sqrt{10}(24x^2 - 12x + 1), \\
\psi_{20}(x) &= \sqrt{2}, \\
\psi_{21}(x) &= \sqrt{6}(4x - 3), \\
\psi_{22}(x) &= \sqrt{10}(24x^2 - 36x + 13).
\end{align*}
\]

We use the collocation point \( x_i = \frac{(i-0.5)}{2^{k-1}M}, i = 1, 2, \ldots, 2^{k-1}M \).

For instance for \( k = 2, M = 3 \), from (2.5), we have

\[
\psi(x) = \begin{bmatrix} \psi_{10}(x_i) \\ \psi_{11}(x_i) \\ \psi_{12}(x_i) \\ \psi_{20}(x_i) \\ \psi_{21}(x_i) \\ \psi_{22}(x_i) \end{bmatrix} = \begin{bmatrix} \psi_{10}(x_i) \\ \psi_{11}(x_i) \\ \psi_{12}(x_i) \\ \psi_{20}(x_i) \\ \psi_{21}(x_i) \\ \psi_{22}(x_i) \end{bmatrix} = \begin{bmatrix} \psi_1(x_i) \\ \psi_2(x_i) \\ \psi_3(x_i) \\ \psi_4(x_i) \\ \psi_5(x_i) \\ \psi_6(x_i) \end{bmatrix}. \tag{2.6}\]

3. Bernoulli wavelets operational matrix of integration

Now we derive the operational matrix of Bernoulli wavelets as follows:

\[
\int_0^x \psi(t)dt = P\psi(x), \tag{3.1}\]

where \( P \) is a \( \tilde{m} \times \tilde{m} \) matrix and is called the operational matrix of integration of Bernoulli wavelets. In particular, for \( M = 3 \) and \( k = 2 \), we have

\[
\psi(x) = [\psi_{10}(x), \psi_{11}(x), \psi_{12}(x), \psi_{20}(x), \psi_{21}(x), \psi_{22}(x)]^T, \tag{3.2}\]

\[
\int_0^x \psi_{10}(t)dt = \begin{cases} \sqrt{2}x, & 0 \leq x < 1/2, \\
\sqrt{2}, & 1/2 \leq x < 1, \end{cases} \tag{3.3}\]

\[
\int_0^x \psi_{11}(t)dt = \begin{cases} 2\sqrt{6}x^2 - 6x, & 0 \leq x < 1/2, \\
0, & 1/2 \leq x < 1, \end{cases} \tag{3.4}\]

\[
\int_0^x \psi_{12}(t)dt = \begin{cases} 8\sqrt{10}x^3 - 6\sqrt{10}x^2 + \sqrt{10}x, & 0 \leq x < 1/2, \\
0, & 1/2 \leq x < 1, \end{cases} \tag{3.5}\]

\[
\int_0^x \psi_{20}(t)dt = \begin{cases} 0, & 0 \leq x < 1/2, \\
\sqrt{2} - \sqrt{2}, & 1/2 \leq x < 1, \end{cases} \tag{3.6}\]

\[
\int_0^x \psi_{21}(t)dt = \begin{cases} 0, & 0 \leq x < 1/2, \\
\sqrt{2} - \sqrt{2}, & 1/2 \leq x < 1, \end{cases} \tag{3.7}\]

\[
\int_0^x \psi_{22}(t)dt = \begin{cases} 0, & 0 \leq x < 1/2, \\
\sqrt{2} - \sqrt{2}, & 1/2 \leq x < 1, \end{cases} \tag{3.8}\]
\[ \int_{0}^{x} \psi_{21}(t) dt = \begin{cases} 0, & 0 \leq x < 1/2, \\ 2\sqrt{6}x^2 - 3\sqrt{6}x + \sqrt{6}, & 1/2 \leq x < 1, \end{cases} \]

Using equations (3.3) to (3.8) and omitting \( \psi_{13}(x) \) and \( \psi_{23}(x) \), we get

\[ \int_{0}^{x} \psi(x) \psi^T(x) F \psi(x), \] (3.9)

where, \( \psi(x) \) is the Taylor wavelet coefficient matrix and \( F \) is an \( m \times m \) matrix given by

\[ F = \psi(x) \psi^{-1}(x), \] (3.10)

where \( \tilde{F} = \text{diag}(\psi^{-1}(x) F) \). Also, for a \( m \times \hat{m} \) matrix \( X \), we have

\[ \psi^T(X) \psi(x) = \hat{X}^T \psi(x), \] (3.11)

where, \( \hat{X}^T = V \psi^{-1}(x) \) and \( V = \text{diag}(\psi^T(x) X \psi(x)) \) is an \( \hat{m} \)-vector.

**Remark 3.1.** For a \( \hat{m} \)-vector \( F \), we have

\[ \psi(x) \psi^T(x) F = \tilde{F} \psi(x), \] (3.12)

where \( \psi(x) \) is the Taylor wavelet coefficient matrix and \( \tilde{F} \) is an \( \hat{m} \times \hat{m} \) matrix given by

\[ \tilde{F} = \psi(x) F \psi^{-1}(x), \] (3.13)

where \( \tilde{F} = \text{diag}(\psi^{-1}(x) \tilde{F}) \). Also, for a \( \hat{m} \times \hat{m} \) matrix \( X \), we have

\[ \psi^T(X) \psi(x) = \hat{X}^T \psi(x), \] (3.14)

where, \( \hat{X}^T = V \psi^{-1}(x) \) and \( V = \text{diag}(\psi^T(x) X \psi(x)) \) is an \( \hat{m} \)-vector.

**4. Bernoulli wavelet method of Solution**

**Fredholm integral equations**

Consider the Fredholm integral equation,

\[ y(x) = f(x) + \int_{a}^{b} k_1(x,t)y(t) dt. \] (4.1)

For the sake of simplicity, without loss of generality, we assume that \((a,b) = (0,1)\). Approximating \( f(x) \), \( y(x) \) and \( k_1(x,t) \), with respect to Bernoulli wavelets as follows:

\[ y(x) \simeq C^T \psi(x) = C \psi^T(x), \] (4.2)

where \( C \) is given in equation (2.4) and is the unknown vector to be determined.

\[ f(x) \simeq F^T \psi(x) = F \psi^T(x), \] (4.3)

and

\[ k_1(x,t) \simeq \psi^T(x) K_1 \psi(t) = \psi^T(t) K_1^T \psi(x), \] (4.4)

where \( C \) and \( F \) are Bernoulli wavelet coefficient vectors and \( K_1 \) is the Bernoulli wavelet matrix. Substituting (4.2), (4.3) and (4.4) in (4.1), we get

\[ C^T \psi(x) = F^T \psi(x) + C^T \left( \int_{0}^{x} \psi(x) \psi^T(x) \right) K_1 \psi(x). \]

Using the relation \( \int_{0}^{1} \psi(x) \psi^T(x) \) \( dx = 1 \), we get

\[ C^T \psi(x) = F^T \psi(x) + C^T K_1 \psi(x), \]

where \( \tilde{C} \) is a \( \hat{m} \times \hat{m} \) matrix. Using the operational matrix of Bernoulli wavelets, we get

\[ C^T \psi(x) = F^T \psi(x) + C^T K_1 \psi(x), \]

hence

\[ C^T - C^T K_1 = F^T. \] (4.5)

Solving this linear system of equations, we get the unknown vector \( C \). Substituting this unknown vector in equation (4.2), we get the solution the Fredholm integral equation given in equation (4.1).

**Volterra integral equations**

Consider the Volterra-Fredholm integral equation,

\[ y(x) = f(x) + \int_{a}^{x} k_1(x,t)y(t) dt. \] (4.6)

For the sake of simplicity, without loss of generality, we assume that \( \alpha = 0 \). Approximating \( f(x) \), \( y(x) \) and \( k_1(x,t) \), with respect to Bernoulli wavelets as follows:

\[ y(x) \simeq C^T \psi(x) = C \psi^T(x), \] (4.7)

where \( C \) is given in equation (2.4) and is the unknown vector to be determined.

\[ f(x) \simeq F^T \psi(x) = F \psi^T(x), \] (4.8)

and

\[ k_1(x,t) \simeq \psi^T(x) K_1 \psi(t) = \psi^T(t) K_1^T \psi(x), \] (4.9)

where \( C \) and \( F \) are Bernoulli wavelet coefficient vectors and \( K_1 \) is the Bernoulli wavelet matrix. Substituting (4.7), (4.8) and (4.9) in (4.6), we get

\[ C^T \psi(x) = F^T \psi(x) + \psi^T(x) \left( \int_{0}^{x} \psi(x) \psi^T(x) C \right) ds. \]
Using the operational matrix of integration of Bernoulli wavelets, we get
\[ C^T \psi(x) = F^T \psi(x) + \psi^T(t) K_1^T \left( \int_0^t C \psi(x) ds \right), \]
where \( C \) is a \( \hat{m} \times \hat{m} \) matrix. Using the operational matrix of Bernoulli wavelets, we get
\[ C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \hat{C} \hat{P} \psi(t). \]
Let \( X_1 = K_1^T \hat{C} \hat{P} \). Again using remark 3.1, we get
\[ C^T \psi(x) - \hat{X}_1^T \psi(x) = F^T \psi(x), \]
where \( \hat{X}_1 \) is a \( \hat{m} \times \hat{m} \) matrix and are linear functions of \( C \) and this equation is applicable for all \( t \in [0, 1] \), hence
\[ C^T - \hat{X}_1^T = F^T. \] (4.10)

Solving this linear system of equations we get the unknown vector \( C \). Substituting this unknown vector in equation (4.7), we get the solution the Volterra integral equation given in equation (4.6).

**Volterra-Fredholm integral equations**

We use the operational matrix of integration of Bernoulli wavelets for the numerical solution of Volterra-Fredholm integral equations. Here we consider the Volterra-Fredholm integral equation,
\[ y(x) = f(x) + \int_a^x k_1(x,t)y(t) dt + \int_a^b k_2(x,t)y(t) dt. \] (4.11)

For the sake of simplicity, without loss of generality, we assume that \( (a,b) = (0,1) \). Approximating \( f(x) \), \( y(x) \) and \( k_i(x,t) \), \( i = 1,2 \) with respect to Bernoulli wavelets as follows:
\[ y(x) \approx C^T \psi(x) = C \psi^T(x), \] (4.12)
where \( C \) is given in equation (2.4) and is the unknown vector to be determined.
\[ f(x) \approx F^T \psi(x) = F \psi^T(x), \] (4.13)
\[ k_1(x,t) \approx \psi^T(x) K_1 \psi(t) = \psi^T(t) K_1^T \psi(x), \] (4.14)
and
\[ k_2(x,t) \approx \psi^T(x) K_2 \psi(t) = \psi^T(t) K_2^T \psi(x), \] (4.15)
where \( C \) and \( F \) are Bernoulli wavelet coefficient vectors and \( K_1 \) and \( K_2 \) are Bernoulli wavelet matrices. Substituting (4.12), (4.13), (4.14) and (4.15) in (4.11), we get
\[ C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \left( \int_0^x C \psi(x) ds \right) \]
\[ + C^T \left( \int_0^1 \psi(x) \psi^T(x) ds \right) K_2 \psi(x). \]
Using the relation \( \int_0^1 \psi(x) \psi^T(x) dx = 1 \) and the above remark, we get
\[ C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \left( \int_0^1 C \psi(x) ds \right) + C^T K_2 \psi(x), \]
where \( C \) is a \( \hat{m} \times \hat{m} \) matrix. Using the operational matrix of Bernoulli wavelets, we get
\[ C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \hat{C} \hat{P} \psi(t) + C^T K_2 \psi(x). \]
Let \( X_1 = K_1^T \hat{C} \hat{P} \). Again using remark 3.1, we get
\[ C^T \psi(x) - \hat{X}_1^T \psi(x) = F^T \psi(x), \]
where \( \hat{X}_1 \) is a \( \hat{m} \times \hat{m} \) matrix and are linear functions of \( C \) and this equation is applicable for all \( t \in [0, 1] \), hence
\[ C^T - \hat{X}_1^T - C^T K_1 = F^T. \] (4.16)

Solving this linear system of equations, we get the unknown vector \( C \). Substituting this unknown vector in equation (4.12), we get the solution the Volterra-Fredholm integral equation given in equation (4.11).

**5. Numerical Results**

**Example 5.1.** First, consider the Fredholm-integral equation[5]
\[ y(x) = f(x) + \int_0^1 k_1(x,t)y(t) dt, \] (5.1)
where
\[ f(x) = \sin(2\pi x), \]
and
\[ k_1(x,t) = \cos(x). \]

Exact solution is \( y(x) = \sin(2\pi x) \).

Using the method described in section 4 and the collocation points, we get the Bernoulli wavelets solution for \( k = 2 \) and \( M = 3 \) as,
\[
\begin{array}{cccccccc}
  x & 0.5/6 & 0.5/6 & 0.5/6 & 0.5/6 & 0.5/6 & 0.5/6 & 0.5/6 \\
  y(x) & 0.5 & 1 & -0.5 & -1 & -0.5 \\
\end{array}
\]
Table 1 show the absolute error of example 1, for the different values of \( k \) and \( M \) and Comparison of Exact and Bernoulli wavelets solution of the example 1, for \( k = 2 \) and \( M = 4 \), is drawn in figure 1.

**Example 5.2.** Next, consider the Volterra-integral equation[7].
\[ y(x) = f(x) + \int_0^x k_1(x,t)y(t) dt, \] (5.2)
We use the method described in section 4. Table 2 shows wavelets approximate values of example 2 for $k$ and $M$ shows that the accuracy improves with increasing the values of $k$ and $M$.

Example 5.3. Lastly, consider the Volterra-Fredholm integral equation:

$$y(x) = f(x) + \int_0^1 k_1(x,t)y(t)dt + \int_0^1 k_2(x,t)y(t)dt,$$

proceeding as in example 1 and using the method described in section 4 we obtain the following results. Table 3 shows the absolute error of example 3 for different values of $k$ and $M$.

where

$$f(x) = e^{-x} - e^x (x - 1),$$

and

$$k_1(x,t) = e^{x+t},$$

and

$$k_2(x,t) = -e^{x+t}.$$

Exact solution is $y(x) = e^{-x}$.

Table 2. Absolute errors of the example 5.2.

| $x$   | $k = 2$ and $M = 2$ | $k = 2$ and $M = 3$ | $k = 2$ and $M = 4$ |
|-------|-------------------|-------------------|-------------------|
| 0.125 | 3.3929e-03        | 9.5507e-04        | 2.4204e-04        |
| 0.25  | 2.4228e-03        | 1.9812e-04        | 4.6315e-04        |
| 0.375 | 1.4527e-03        | 5.4593e-04        | 4.8674e-04        |
| 0.5   | 1.8545e-03        | 1.4124e-03        | 2.8156e-04        |
| 0.625 | 2.2564e-03        | 1.6710e-03        | 1.0259e-04        |
| 0.75  | 2.2142e-03        | 1.9544e-04        | 1.7878e-04        |
| 0.875 | 2.3535e-03        | 1.6092e-03        | 2.4581e-04        |

Table 3. Absolute errors of the example 5.3.

| $x$   | $k = 2$ and $M = 2$ | $k = 2$ and $M = 3$ | $k = 2$ and $M = 4$ |
|-------|-------------------|-------------------|-------------------|
| 0.125 | 3.3850e-04        | 6.1821e-06        | 7.0865e-06        |
| 0.25  | 5.0551e-04        | 5.9817e-06        | 1.3708e-05        |
| 0.375 | 6.7252e-04        | 5.7207e-06        | 2.0863e-05        |
| 0.5   | 8.4929e-04        | 8.4584e-06        | 3.4099e-05        |
| 0.625 | 1.0261e-03        | 1.6206e-05        | 4.6213e-05        |
| 0.75  | 3.8549e-03        | 3.0974e-05        | 7.7161e-05        |
| 0.875 | 6.4195e-03        | 9.1110e-05        | 1.9745e-04        |

6. Conclusion

The Bernoulli wavelets operational method is developed for the numerical solution of integral equations. The present method reduces an integral equation into a set of algebraic equations. For instance in the illustrative example 5.1, our results are higher accuracy with exact ones, subsequently other examples are also same in the nature. The numerical result shows that the accuracy improves with increasing the $k$ and $M$ for better accuracy. Error analysis justifies the efficiency, validity and applicability of the present technique.
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**Figure 3.** Exact and Bernoulli wavelets solution of the example 5.3 for and $k = 2$ and $M = 4$

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