New formulas constraining static anisotropy in late time cosmology

L. G. Gomes

Abstract

In this paper we show how the cosmological anisotropy would be constrained if we assume the hypothesis of a spatially flat spacetime filled with a perfect fluid. It satisfies simple nonlinear algebraic relations, which involve just the observational parameters and are independent of further considerations on the fluid. To obtain them, we rely on a new approach to the perfect fluid Bianchi-I spacetimes for which the metric is described explicitly in terms of the components of the energy-momentum tensor. The final result is a valuable framework to probe the observational validity of the isotropy in the Copernican Principle.

1 Introduction

The question of the large scale isotropy in our Universe has long been in debate. On observational grounds, there are many factors favoring it, as the highly isotropic Cosmic Microwave Background (CMB) observations [8] and the substantial evidence for no direction dependence in the Hubble constant [11]. However, as the data are analyzed from different perspectives, some doubts emerge that cosmological anisotropies could in fact exist [2] (see also the comments and references in [7]). Even on theoretical grounds it can be the subject of discordance, for if we assume that the initial conditions for the evolution of our universe were exactly isotropic, then we will have to evoke some version of the anthropic principle in order to explain it [4].

In this paper, amid those different perspectives for the large scale behavior of the cosmic evolution, we present a novel tool to probe the cosmological anisotropy: a set of formulas constraining the behavior of the observational anisotropic parameters in any spatially flat spacetime filled with a perfect
fluid, which are presented in their most general forms (36) and (38), together with their linearized versions (41) and (42).

In order to obtain the aforementioned formulas, we develop the $\rho$-coordinate system approach to our spacetime, which is a coordinate representation where the energy density plays the role of time. To be more specific, we start with the most general spatially flat spacetime, a non-diagonal Bianchi-I model, and assume it is filled with a perfect fluid [5]. This means that the energy-momentum tensor can be represented in a frame adapted to the symmetry as

$$T^\nu_0 = -\rho(t) \delta^\nu_0 \quad \text{and} \quad T^i_k = p(t) \delta^i_k.$$  \hfill (1)

We also take the Einstein’s equations to hold, and therefore the metric $-dt^2 + \gamma_{ik}(t) dx^i dx^k$ can be diagonalized. As the energy density is not constant through time, $\dot{\rho} \neq 0$, it can be taken as a new time-like coordinate. Hence, the metric can be written in the form

$$g = -\left(\frac{1}{\dot{\rho}}\right)^2 d\rho^2 + a_1(\rho)^2 dx^2 + a_2(\rho)^2 dy^2 + a_3(\rho)^2 dz^2.$$  \hfill (2)

The central result in this paper is to give explicit formulas for the components of $g$, a kind of ”integral version” of the Einstein’s equations. We do it the section 2, together with some of its first applications. In the section 3 we use them to determine the deceleration and jerk along the principal directions, both of them as formulas of the physical quantities only. In the section 4 we show that their direction dependence involves simple algebraic constraints among the observational parameters. Therefore, they give us the necessary relations any anisotropic observation should satisfy, as far as it can be considered cosmological in nature and the spacetime close to a spatially flat one filled with a perfect fluid. In the last section we make some final remarks.

2 Bianchi-I revisited: the $\rho$-coordinates.

Let us consider our spacetime spatially flat and filled with a perfect fluid, just like in the equations (11) and (12). We define the Hubble parameter in

\footnote{We can always set the initial conditions $\gamma_{ik}(t_0)$ and $\dot{\gamma}_{ik}(t_0)$ to have a simultaneously diagonal form, implying that $T_{ij}(t_0) = p(t_0)\gamma_{ik}(t_0)$ is also diagonal. Hence, the perfect fluid Einstein’s equations have $\gamma_{ik} = 0$ for $i \neq k$ as the unique solution. This is not possible if the energy-momentum tensor has non-vanishing off-diagonal components.}
each principal direction just as usual: \( H_i = \dot{a}_i/a_i \), \( i = 1, 2, 3 \). In the same way, to the isotropic scale factor \( a = (a_1 a_2 a_3)^{\frac{1}{3}} \) corresponds the isotropic Hubble function
\[
H = \frac{\dot{a}}{a} = \frac{1}{3} (H_1 + H_2 + H_3)
\]
They play the role of the conventional parameters in the isotropic context.

As we settle the initial conditions \( a_k(t_0) = 1 \) for \( k = 1, 2, 3 \), we follow the scheme presented in [1] to obtain
\[
a_k(t) = a(t) e^{b(t) \sin \alpha_k}, \quad b(t) := \int_{t_0}^{t} \lambda(s) H(s) \, ds,
\]
where each phase \( \alpha_k \) is a constant determined by
\[
\alpha_1 = \alpha, \quad \alpha_2 = \alpha + \frac{2\pi}{3}, \quad \alpha_3 = \alpha - \frac{2\pi}{3}.
\]
Indeed, the fact that \( \alpha \) is constant is a consequence of the Einstein’s equations for the perfect fluid case. We also have the relations
\[
\sum_{k=1}^{3} \sin \alpha_k = 0, \quad \sum_{k=1}^{3} \sin^2 \alpha_k = \frac{3}{2} \quad \text{and} \quad \sum_{k=1}^{3} \sin^3 \alpha_k = -\frac{3}{4} \sin(3\alpha).
\]
The function \( \lambda(t) \) is the deviation parameter, that determines the way the Hubble factor changes with the direction:
\[
H_k = (1 + \lambda \sin \alpha_k) H \quad k = 1, 2, 3.
\]
The deviation and the phases can be calculated as
\[
\lambda = \sqrt{\frac{2}{3} \sum_{k=1}^{3} \left( \frac{H_k - H}{H} \right)^2} \quad \text{and} \quad \sin \alpha_k = \frac{H_k - H}{\lambda H}.
\]
The remaining equations are the anisotropic "Friedmann" one
\[
3 \left( 4 - \lambda^2 \right) H^2 = 4 \rho
\]
and the conservation
\[
\dot{\rho} = -3H (\rho + p).
\]
We assume a non-constant energy density \( \rho \), which allows us to set \( \rho \) as a valid time coordinate. Hence, a very interesting representation shows up
where the Einstein’s field equations can be put in an ”integrated” form. Stating it more precisely, it is straightforward to check that, given any function $p = p(\rho)$ defined in an interval where $p(\rho) \neq -\rho$, the metric

$$g = -\frac{(4 - \lambda^2) \, d\rho^2}{12 \rho (\rho + p)^2} + a^2 \sum_{k=1}^{3} e^{2b \, \sin \alpha_k} \, dx_i^2,$$

has the Einstein tensor

$$(G^\mu_\nu) = \text{diag}\{ -\rho, p(\rho), p(\rho), p(\rho) \},$$

provided that the isotropic scale factor is

$$\ln a(\rho) = -\frac{1}{3} \int_{\rho_0}^{\rho} \frac{d\rho'}{\rho' + p(\rho')},$$

and its counterpart in each principal direction is determined by

$$b(\rho) = -\frac{1}{3} \int_{\rho_0}^{\rho} \frac{\lambda(\rho') \, d\rho'}{\rho' + p(\rho')}.$$

where

$$\lambda(\rho) = \frac{2 \lambda_0}{\sqrt{\lambda_0^2 + (4 - \lambda_0^2) \, e^{-G(\rho)}}}$$

and

$$G(\rho) = \int_{\rho_0}^{\rho} \frac{\rho' - p(\rho') \, d\rho'}{\rho' + p(\rho')} \rho'.$$

As we return to the proper time coordinate $t$, we get

$$t = t_0 \pm \int_{\rho_0}^{\rho(t)} \sqrt{\frac{4 - \lambda(\rho)^2}{12 \rho} \, \frac{d\rho}{\rho + p(\rho)}},$$

with $\pm$ the signal of $-H$. It is clear that the isotropic model is recovered if, and only if, the initial deviation vanishes ($\lambda_0 = 0$).

In order to avoid misunderstandings about these formulas, observe that the whole scheme presented so far stands (locally) for any Bianchi-I space-time whenever $\dot{\rho} \neq 0$. But in the barotropic case, where a fixed equation of state $p = p(\rho)$ is fixed a priori, it provides us the general solution of the Einstein’s equations. In this case, the function $\rho(t)$ is implicitly obtained through the formula (16). According to the conservation (10), the sole excluded case turns out to be the vacuum with a cosmological constant, $p = -\rho$.
(H = 0 is also vacuum). We leave the mathematical aspects of such solutions to be treated elsewhere, since it goes beyond the scope of this paper. Notwithstanding, for the general ”non-barotropic” fluid, the whole scheme is not necessarily a true ”exact solution”. Indeed, they represent a kind of an ”integral” version of the ”differential” Einstein’s equations, which in turn, depending on the situation, might not help in finding any exact solution at all. Their true strength is to provide a natural framework for a physical analysis in such spacetimes, despite how an exact solution would look like. Furthermore, this seems to be the first time that a cosmological model has the Einstein’s equations compared in simplicity to those of the standard isotropic ones, which is a significant achievement for opening up new possibilities in cosmology.

As another application, suppose that the matter content satisfies the restricted dominant energy condition |p| ≤ K ρ, 0 ≤ K < 1. Then $e^{G(\rho)} \leq \left(\frac{\rho}{\rho_0}\right)^\alpha$, for $0 < \rho \leq \rho_0$, and $\left(\frac{\rho}{\rho_0}\right)^\alpha \leq e^{G(\rho)}$, for $\rho_0 \leq \rho$, with $\alpha := \frac{1-K}{1+K}$. Therefore, by the formula (15), the isotropic limit $\lambda \to 0$ is approached if, and only if, $\rho \to 0^+$ and it happens as fast as

$$0 < \rho \ll \rho_0 \Rightarrow |\lambda| \lesssim \left(\frac{\rho}{\rho_0}\right)^\frac{\alpha}{2}. \quad (17)$$

This justifies why the Bianchi-I spacetimes are relevant in late time cosmology, at least as a phenomenological model, when we are close to the isotropic limit in a low energy scale.

3 The observational parameters

Concerning the observable parameters, there are many possibilities to exploit with those new set of formulas. For example, from the anisotropic Friedmann equation (9) with $\rho > 0$ we readily conclude that $|\lambda| < 2$, that is, in the expanding scenario $H > 0$ the Hubble expansion in each direction is bounded as

$$(1 - 2|\sin \alpha_k|) \ H < H_k < (1 + 2|\sin \alpha_k|) \ H \quad (18)$$

while the inequalities invert during the contraction.

The second and third order parameters, the deceleration and the jerk, are constrained on direction. Since the wave initially set to propagate parallel to a principal axis in a diagonal Bianchi-I spacetime will keep its path straight [9], along them we have the same interpretation and methods of the
corresponding isotropic model. Hence, along those directions we have the deceleration
\[ q_k = - \frac{1}{H_k^2 a_k} \frac{d^2 a_k}{dt^2} = - \left( 1 + \frac{\dot{H}_k}{H_k^2} \right) \] (19)
and the jerk
\[ j_k = \frac{1}{H_k^3 a_k} \frac{d^3 a_k}{dt^3} = (1 + 2 q_k) q_k - \frac{\ddot{q}_k}{H_k} \] (20)
just as in the standard model, where their values today, at the instant \( t_0 \), compose the Taylor polynomial up to third order of the luminous distance as a function of the redshift [14]. Note that now we have possibly four different behaviours: one for each principal direction, that involves the scale factors \( a_1(t), a_2(t) \) and \( a_3(t) \), and one for their isotropic counterpart, which involves \( a(t) \).

In the calculations we use \( \frac{1}{H^2} \frac{d}{dt} = -3 \rho (1 + w) \frac{d}{d\rho} \), a consequence of the conservation equation (10), where \( w = p/\rho \). Hence, from the equations (9) and (15) we get
\[ \frac{\dot{\lambda}}{\lambda H} = -\frac{3}{8} (1 - w) (4 - \lambda^2) \quad \left( w = \frac{p}{\rho} \right) \] (21)
and
\[ \frac{\dot{H}}{H^2} = -\frac{3}{8} \left( 4(1 + w) + (1 - w)\lambda^2 \right). \] (22)

3.1 The deceleration parameter

In order to calculate the deceleration parameter, we use the relations (7), (19), (21) and (22) to get
\[ q_k = \frac{q + \lambda \sin \alpha_k - \lambda^2 \sin^2 \alpha_k}{(1 + \lambda \sin \alpha_k)^2}. \] (23)
where \( q \) corresponds to the isotropic deceleration, calculated with the scale factor \( a(t) \). Processing the sum \( \sum_k q_k (1 + \lambda \sin \alpha_k)^2 \) in the equation (23) and using the relations (6), we arrive in the expression that relates the mean and the isotropic accelerations:
\[ q = \langle q \rangle + \frac{2}{3} \sum_{k=1}^{3} q_k \sin \alpha_k + \frac{\lambda^2}{2} \left( 1 + \frac{2}{3} \sum_{k=1}^{3} q_k \sin^2 \alpha_k \right), \] (24)
3 The observational parameters

where

\[ \langle q \rangle = \frac{1}{3}(q_1 + q_2 + q_3). \]  \hfill (25)

This tells us how to determine \( q \) from the observations. From another point of view, as we proceed to calculate \( q \) from the isotropic analogous of formula (19), we obtain

\[ q = \frac{1}{2}(1 + 3w) + \frac{3}{8}(1 - w) \lambda^2. \]  \hfill (26)

Note that it gives the right isotropic limit \((1 + 3w)/2\) for \( \lambda \to 0 \) \hfill (12). It determines \( w \) as

\[ w = \frac{8q - 3\lambda^2 - 4}{3(4 - \lambda^2)}. \]  \hfill (27)

3.2 The jerk parameter

To find the jerks \( j_1, j_2, j_3 \) along the principal directions we use the equations (20), (21), (22), (23) and (27), such that if we denote \( j \) for its isotropic counterpart obtained from \( a(t) \), we arrive to the expression

\[ j_k = j + 3 \lambda \sin \alpha_k - 6 \lambda^2 \sin^2 \alpha_k + \lambda^3 \sin^3 \alpha_k \]  \hfill (28)

As we open \( \sum_k j_k(1 + \lambda \sin \alpha_k)^3 \) using the relations (6), we determine the isotropic jerk \( j \) from the observed ones as

\[ j = \langle j \rangle + c_1 \lambda + (c_2 + 3) \lambda^2 + \left( \frac{c_3}{3} + \frac{\sin(3\alpha)}{4} \right) \lambda^3, \]  \hfill (29)

where the mean value of the jerk is \( \langle j \rangle = \frac{1}{3}(j_1 + j_2 + j_3) \) and

\[ c_\ell = \sum_{k=1}^{3} j_k \sin^\ell \alpha_k. \]  \hfill (30)

If we take the time derivative of the equation (27), we obtain

\[ \dot{w} = \frac{8H}{3(4 - \lambda^2)} \left( q(1 + 2q) - j + 2 \lambda^2 \frac{(2 - q)^2}{4 - \lambda^2} \right). \]  \hfill (31)

In terms of \( dp/d\rho = w - \dot{w}/(3H(1 + w)) \), we can put it as

\[ \frac{dp}{d\rho} = \frac{p}{\rho} - \frac{4}{3} \left( \frac{4(2q^2 + q - j) + (8 - 9q + j) \lambda^2}{(4 - \lambda^2)(4 + 4q - 3\lambda^2)} \right). \]  \hfill (32)
In the time $t_0$, as we apply the formulas (32) to the isotropic limit $\lambda = 0$, we get the result found in the reference [12]:

$$\kappa_0 = \left( \frac{dp}{d\rho} \right)_{t=t_0} = -\frac{1 - j_0}{3(1 + q_0)}.$$  \hspace{1cm} (33)

4 On the cosmological anisotropy and its observational constraints

As we look for anisotropies in the sky, it is most important that we keep track of its origins, for we want to put apart the cosmological effects from the local ones due to our surrounding inhomogeneities or proper motion. Therefore it is worthwhile pursuing constraints that would involve only the observable quantities in order to characterize the cosmological anisotropy presented in our model.

First of all, let us divide the observational quantities as the first order parameters, $H_k, H, \lambda, \alpha_k$, the second order ones, $q_k, q$, and so on. The constraints will follow the same characterization: the second order ones will involve only observational parameters up to second order, etc. It is clear that they are not independent, for the $H_k$’s and $q_k$’s represent the fundamental data from which the others are built. For instance, the three formulas in (23) are not independent, but just two of them, for they are used in (24) to determine the isotropic acceleration $q$. Therefore, we should expect only two independent second order constraints. The same is true for the $j_k$’s and $j$, leading to only two third order constraints. As a straightforward generalization, one could follow this receipt and build higher order terms.

Before analyzing the constraints imposed by the perfect fluid Bianchi-I model, we shall be concerned with the existence of anisotropies:

$$\lambda \neq 0.$$  \hspace{1cm} (34)

If this in fact happens, then we could verify it in almost any orthogonal frame through which we examine the sky, for this would not be restricted to the principal directions only. In this way, it turns to be easier to check and the first step to make in order to look for the anisotropy. At the moment of writing, the condition of isotropy from the observations of type Ia supernovae at galaxies with redshift $z \leq 1$ is still under scrutiny [2] [3] [7] [11].

As we assume the existence of cosmological anisotropies, the quest of the determination of the three principal axis would demand a complete analysis...
considering the entire angular distribution of the observational parameters, which requires a quite more involved mathematical argumentation \[6 \text{[10]}.\] Despite the model used, one could expect them to be mutually orthogonal, since they must be related to the principal directions of the space metric, but not necessarily constant in time.\[2] The general Bianchi-I non-diagonal geometry can deal with that. Indeed, even when the fluid is presented in a diagonal form but is not perfect, the phase \(\alpha\) is time dependent, so that the principal directions change with time. This shear effect is very interesting in many aspects. In particular, while in the perfect fluid case (\(\dot{\alpha} = 0\)) we have \(\lambda H\) decreasing as \(a^{-6}\), and therefore diminishing our hope to find the relics of an early anisotropy, if the shear is present, \(\lambda H\) can have a very different behavior, and the anisotropy can in fact decrease slower than \(a^{-3}\)\[1]. Hence, if a kind of static anisotropy is observed in the sky, it should rather be an asymptotic stage of an earlier epoch when the principal directions where dynamic. Therefore, the (possibly asymptotic) perfect fluid Bianchi-I spacetimes can describe the late time anisotropy, if any at all, as deep in the sky as we can consider the principal directions as static, that is, following the formula (8), as deep as we observe the relation

\[
\frac{d}{dt} \left( \frac{(H_k - H)^2}{\sum_{\ell=1}^{3} (H_\ell - H)^2} \right) = 0 \quad k = 1, 2, 3. \tag{35}
\]

We shall deal with a more general setting in future works, covering the full angular spectrum and considering the direction dynamics.

The two independent second order constraints are obtained as we consider the difference of the expressions \(H^2 q_k (1 + \lambda \sin \alpha_k)^2\) in (23) and take \(\lambda \sin \alpha_k = (H_k - H)/H\). They are equivalent to

\[
Q_1 = Q_2 = Q_3 = H^2 q , \tag{36}
\]

with (no sum in the index \(k\))

\[
Q_k = H_k^2 q_k + (H - H_k) (2H - H_k) . \tag{37}
\]

For the third order ones, we exploit the expression (28) for \(j_k (1 + \lambda \sin \alpha_k)^3\) to obtain the independent constraints

\[
J_1 = J_2 = J_3 = H^3 j , \tag{38}
\]

\[\text{It could also depend on the position, in which case we would have to consider inhomogeneous models.}\]
where (no sum in the index \( k \))

\[
J_k = H_k^3 j_k + (H - H_k) \left( 10 H^2 - 8 H H_k + H_k^2 \right).
\]  

(39)

Note that both of them describe a pattern that any cosmological static anisotropy should satisfy, as long as we assume an universe sufficiently close to space flatness and the fluid almost perfect. Furthermore, they do not depend on the equation of state of this fluid. Therefore, as \( \lambda \neq 0 \) characterizes the existence of the cosmological anisotropies, the existence of three mutually orthogonal directions such that the constraints (36) and (38) are satisfied would be a strong indicative that the universe is indeed of the Bianchi-I type with \( \dot{a} = 0 \). On the other hand, any claim that \( \lambda \neq 0 \) should cope with those constraints if, in the background, it is assuming a spatially flat model.

For small \( \delta H_k = H_k - H \), \( \delta q_k = q_k - \langle q \rangle \) and \( \delta j_k = j_k - \langle j \rangle \), as we denote \( \delta H = \sqrt{\delta H_1^2 + \delta H_2^2 + \delta H_3^2} \) and analogously \( \delta q \) and \( \delta j \), by inspection of the formulas (24) and (29) with the aid of the first relation in (6), we have

\[
q = \langle q \rangle + O^2(\delta H, \delta q) \quad \text{and} \quad j = \langle j \rangle + O^2(\delta H, \delta j).
\]  

(40)

Hence, in the linear regime we have \( q = \langle q \rangle \), \( j = \langle j \rangle \), the linearized version of the second order constraints (36),

\[
\delta q_k = (1 - 2\langle q \rangle) \frac{\delta H_k}{H},
\]  

(41)

and the third order ones obtained as we linearize the equation (38),

\[
\delta j_k = 3 \left( 1 - \langle j \rangle \right) \frac{\delta H_k}{H}.
\]  

(42)

It is worth noting that the both the constraints (36) and (38), as well as their linearized versions (41) and (42), are valid for any instant of time \( t \), and not only at \( t_0 \).

5 Final remarks

The light observed from the distant galaxies is very important to determine the rate of expansion and the degree of isotropy of our universe in late time cosmology. Since extracting this kind of information from the observational data is far from a trivial question, in this paper we have added a new tool
to help accomplishing this aim. We have derived the nonlinear algebraic formulas (36) and (38), and their linearized versions (41) and (42), showing how the anisotropy data would be constrained if we assume the hypothesis of a spatially flat spacetime filled with a perfect fluid. They are simple relations involving just the observational parameters and independent of further considerations on the fluid. For the time being, we have, for the sake of simplicity, concentrated only on the deviations from the isotropy in their principal directions. The full angular spectrum will be covered in a latter work.

References

[1] E. Bittencourt, L.G. Gomes, R. Klippert, Bianchi-I cosmology from causal thermodynamics, Class. Quantum Grav. 34, 045010 (2017).

[2] R.G. Cai and Z. L. Tuo, Direction dependence of the deceleration parameter, J. Cosmol. Astropart. Phys. 2012-02 (2012).

[3] Jacques Colin, Roya Mohayaee, Mohamed Rameez, S. Sarkar, Evidence for anisotropy of cosmic acceleration, Astron. & Astrophysics, 631, L13 (2019).

[4] C.B. Collins, S W. Hawking, Why is the universe isotropic?, The Astrophysical Journal 180 317 (1973).

[5] G. F. R. Ellis, R. Maartens and M.A.H. MacCallum, Relativistic Cosmology, Cambridge University Press (2012).

[6] P. Fleury, C. Pitrou, J.P. Uzan, Light propagation in a homogeneous and isotropic universe, Phys. Rev. D 91, 043511 (2015).

[7] H. Lin, S. Wang, Z. Chang, X. Li, Testing the isotropy of the Universe by using the JLA compilation of type-Ia supernovae, Mon. Not. R. Astron. Soc. 456, Issue 2, 18811885 (2016).

[8] D. Saadeh, S. M. Feeney, A. Pontzen, H. V. Peiris, J. D. McEwen, How Isotropic is the Universe?, Phys. Rev. Lett. 117, 131302, (2016).

[9] A. Sagnotti, B. Zwiebach, Electromagnetic waves in a Bianchi type-I universe, Phys. Rev. D 24, 305 (1981).
5 Final remarks

[10] P.T. Saunders, *Observations in homogeneous model universes*, Mon. Not. R. Astr. Soc. **141**, 427 (1968).

[11] J. Soltis, A. Farahi, D. Huterer, C. M. Liberato II, *Percent-Level Test of Isotropic Expansion Using Type Ia Supernovae*, Phys. Rev. Lett. **122**, 091301 (2019).

[12] M. Visser, *Jerk, snap and the cosmological equation of state*, Class. Quantum Grav. **21**, 2603 (2004).

[13] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, *Exact solutions to Einstein’s field equations*, 2nd edition, Cambridge University Press (2003).

[14] S. Weinberg, *Cosmology*, Oxford University Press (2008).