We review recent results on new physical models constructed as $P\bar{T}$-symmetrical deformations or extensions of different types of integrable models. We present non-Hermitian versions of quantum spin chains, multi-particle systems of Calogero–Moser–Sutherland type and nonlinear integrable field equations of Korteweg–de Vries type. The quantum spin chain discussed is related to the first example in the series of the non-unitary models of minimal conformal field theories. For the Calogero–Moser–Sutherland models, we provide three alternative deformations: a complex extension for models related to all types of Coxeter/Weyl groups; models describing the evolution of poles in constrained real-valued field equations of nonlinear integrable systems; and genuine deformations based on antilinearly invariant deformed root systems. Deformations of complex nonlinear integrable field equations of Korteweg–de Vries type are studied with regard to different kinds of $P\bar{T}$-symmetrical scenarios. A reduction to simple complex quantum mechanical models currently under discussion is presented.

1. Introduction

Until fairly recently [1], non-Hermitian systems have been mostly viewed as not self-consistent descriptions of dissipative systems. However, in contrast to the previous misconception, it is by now well understood that Hamiltonians admitting an antilinear symmetry may be used to define consistent classical, quantum mechanical and quantum field theoretical systems. Various techniques have been developed to achieve this. Central to this is construction of metric operators such that certain quantities in the models can be viewed as physical observables [2–11]. In particular, it was found that such types of models possess real energy spectra in the...
large sectors in their parameter space, despite being non-Hermitian. The explanation for this property can be traced back to an observation made by Wigner more than 50 years ago [12], who noticed that operators invariant under antilinear transformations possess either real eigenvalues or eigenvalues occurring in complex conjugate pairs depending on whether their eigenfunctions also respect this symmetry or not, respectively. A very explicit example of such a symmetry is a simultaneous parity transformation \( P \) and time reversal \( T \). This \( PT \)-symmetry is trivially verified, for instance, for Hamiltonian operators \( H \), but less obvious for the corresponding wave functions \( \psi \) due to the fact that often they are not known explicitly. When

\[
[H, PT] = 0 \quad \text{and} \quad PT \psi = \psi
\]

(1.1)

hold, one speaks of a \( PT \)-symmetric system, but when only the first relation holds, one speaks of spontaneously broken \( PT \)-symmetry and when none of the relations in (1.1) holds of broken \( PT \)-symmetry. Here, we will view the \( PT \)-operator in a wider sense and refer to it loosely as \( PT \), even when it is not strictly a reflection in time and space, but when it is an antilinear involution satisfying

\[
PT(\alpha \psi + \beta \phi) = \alpha^* PT \psi + \beta^* PT \phi \quad \text{for} \quad \alpha, \beta \in \mathbb{C} \quad PT^2 = I.
\]

(1.2)

Very often synonymously used, even though conceptually quite different, are the notions of quasi-Hermiticity [2,13,14] and pseudo-Hermiticity [4,15,16]. These concepts refer more directly to the properties of the metric operator, and their subtle difference is often overlooked, even though this is very important as they allow for different types of conclusions. In the quasi-Hermitian case, the metric operator is positive and Hermitian, but not necessarily invertible. It was shown [2,13,14] that in this case, the existence of a definite metric is guaranteed, and the eigenvalues of the Hamiltonian are real. The pseudo-Hermitian scenario, which is dealing with an invertible Hermitian, but not necessarily positive metric, is less appealing as the eigenvalues are only guaranteed to be real, but no definite conclusions can be reached with regard to the existence of a definite metric. Thus, in this latter case, the status and consistency of the corresponding quantum theory remain inconclusive.

Even though some fundamental questions remain partially unanswered, such as the puzzle concerning the uniqueness of the metric or the question of what constitutes a good set of ingredients to formulate a consistent physical theory, the understanding is general in a very mature state. So far, it could be used to revisit some old theories, which had either been discarded as being non-physical or had considerable gaps in their treatment, and put them on more solid ground. Another interesting possibility that had opened up through these studies is the formulation of entirely new models based on non-Hermitian Hamiltonians that, however, possess the desired \( PT \)-symmetry. In other words, one may use the \( PT \)-symmetry to deform or extend previously studied models and thus obtain large sets of entirely unexplored theories. In principle, this kind of programme can be carried out in any area of physics. Here, we will explore how these ideas can be used in the context of integrable models. We will not report on how well-established methods from integrable systems can be applied to study non-Hermitian quantum mechanical models [17], even though we will report some scenarios in which they naturally emerge as reduced integrable systems [18]. Instead, we present how these ideas have been used so far to formulate and study new models previously overlooked as they would have been regarded as non-physical owing to their non-Hermitian nature. We present results on standard types of integrable models, a quantum spin chain, multi-particle systems of Calogero type and nonlinear wave equations of Korteweg–de Vries (KdV) type.

The construction principle is fairly simple. By identifying some antisymmetric operators \( O \) in the system, we seek a deformation map \( \delta_\varepsilon \) of the form

\[
PT: O \leftrightarrow -O \quad \Rightarrow \quad \delta_\varepsilon : O \mapsto -i(iO)^\varepsilon,
\]

(1.3)

with \( \varepsilon \) being a deformation parameter such that the non-deformed model is recovered in the limit \( \varepsilon \rightarrow 1 \). Alternatively, one can also just add \( PT \)-symmetric terms to the original system and regard them as perturbations.
2. \(\mathcal{PT}\)-symmetrically deformed quantum spin chains

Quantum spin chains constitute a good starting point because, being just finite matrix models, they can be viewed in many ways as the easiest integrable models. We present here a model that has been considered first by von Gehlen [19], an Ising quantum spin chain in the presence of a magnetic field in the \(z\)-direction as well as a longitudinal imaginary field in the \(x\)-direction. The corresponding Hamiltonian for a chain of length \(N\) acting on a Hilbert space of the form \((\mathbb{C}^2)^\otimes N\) is given by

\[ H(\lambda, \kappa) = -\frac{1}{2} \sum_{j=1}^{N} (\sigma_j^x + \lambda \sigma_j^x \sigma_{j+1}^x + i \kappa \sigma_j^x), \quad \lambda, \kappa \in \mathbb{R}. \]  

(2.1)

We used the standard notation for the \(2^N \times 2^N\) matrices \(\sigma_i^{x,y,z} = \mathbb{I} \otimes \ldots \otimes \sigma^x \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I}\) with Pauli matrices

\[ \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  

(2.2)

describing spin \(\frac{1}{2}\) particles as the \(i\)th factor acting on site \(i\) of the chain. This model is of interest as it can be viewed [20] as a perturbation of the \(\mathcal{M}_{5,2}\) model in the \(\mathcal{M}_{p,q}\) series of minimal conformal field theories [21]. It is the simplest non-unitary model in this infinite class of models, which are all characterized by the condition \(p - q > 1\), and whose corresponding Hamiltonians are all expected to be non-Hermitian. The \(\mathcal{PT}\)-symmetry of the model was exploited in Castro-Alvaredo & Fring [22].

(a) Different versions of \(\mathcal{PT}\)-symmetry

Let us first identify the \(\mathcal{PT}\)-symmetry for the Hamiltonian (2.1). Non-Hermitian spin chains have first been studied in this regard in Korff & Weston [23], where the parity operator \(\mathcal{P}': \sigma_i^{x,y,z} \rightarrow \sigma_{N+1-i}^{x,y,z}\) was interpreted quite literally as a reflection about the centre of the chain. Viewing \(T\) as a standard complex conjugation, \(\mathcal{PT}\) is then easily identified as a symmetry of the XXZ-spin chain Hamiltonian \(H_{XXZ}\). However, it is seen immediately that this operator is not a symmetry of the Hamiltonian \(H(\lambda, \kappa)\) in (2.1). Defining instead the operator [22]

\[ \mathcal{P} := \prod_{i=1}^{N} \sigma_i^z, \quad \text{with} \quad \mathcal{P}^2 = \mathbb{1}^\otimes N, \]  

(2.3)

as an analogue to the parity operator, we may carry out a site-by-site reflection

\[ \mathcal{P} : (\sigma_i^x, \sigma_i^y, \sigma_i^z) \rightarrow (-\sigma_i^x, -\sigma_i^y, \sigma_i^z) \quad \text{and} \quad T : (\sigma_i^x, \sigma_i^y, \sigma_i^z) \rightarrow (\sigma_i^x, -\sigma_i^y, \sigma_i^z). \]  

(2.4)

It is then easy to verify that this operator is a symmetry of \(H(\lambda, \kappa)\), i.e. we have \([\mathcal{PT}, H(\lambda, \kappa)] = 0\). Clearly, this \(\mathcal{PT}\)-operator acts antilinearly satisfying (1.2), and is therefore a viable candidate for our purposes. In analogy to (2.3), it is then also suggestive to define

\[ \mathcal{P}_x := \prod_{i=1}^{N} \sigma_i^x \quad \text{and} \quad \mathcal{P}_y := \prod_{i=1}^{N} \sigma_i^y, \]  

(2.5)

which act as

\[ \mathcal{P}_{x/y} : (\sigma_i^x, \sigma_i^y, \sigma_i^z) \rightarrow (\pm \sigma_i^x, \mp \sigma_i^y, -\sigma_i^z). \]  

(2.6)

One can verify that \([\mathcal{P}_{x/y} T, H(\lambda, \kappa)] \neq 0\) and \([\mathcal{P}_{x/y} T, H_{XXZ}] = 0\). Similar properties can be observed for the non-Hermitian quantum spin chain [24]

\[ H_{DG} = \sum_{i=1}^{N} (\kappa_{zz} \sigma_i^z \sigma_{i+1}^z + \kappa_x \sigma_i^x + \kappa_y \sigma_i^y), \]  

(2.7)
with ℂ ∈ ℝ and ℂ ∈ ℂ. Clearly, when ℂ or ℂ ∈ ℝ, the Hamiltonian H_{DG} is not Hermitian, but once again, one can find suitable symmetry operators. We notice that [PT, H] = 0, whereas [PT, H] = 0 for ℂ, ℂ ∈ ℂ and [PT, H] = 0 for ℂ, ℂ ∈ ℂ.

Below, we will encounter further ambiguities in the definition of the antilinear symmetry, which will all manifest in the non-uniqueness of the metric operator and therefore in the definition of the physics described by these models. For the Hamiltonians H_{XXZ} and H_{DG}, the consequences of this fact are yet to be explored.

(b) The two-site model

It is instructive to commence with the simplest example for which all quantities of interest can be computed explicitly in a very transparent way. We specify at first the length of the chain to be N = 2 and without loss of generality fix the boundary conditions to be periodic σ^X_{N+1} = σ^X_1. The Hamiltonian (2.1) then acquires the simple form of a non-Hermitian (4 × 4) matrix

\[ H_2(λ, κ) = -\frac{1}{2}[σ_z ⊗ I + I ⊗ σ_z + 2λσ_x ⊗ σ_x + iκ(I ⊗ σ_x + σ_x ⊗ I)] \]  

(2.8)

and

\[ H_2(λ, κ) = -\begin{pmatrix} -1 & iκ/2 & iκ/2 & λ \\ iκ/2 & 0 & λ & iκ/2 \\ iκ/2 & λ & 0 & iκ/2 \\ λ & iκ/2 & iκ/2 & -1 \end{pmatrix}. \]  

(2.9)

The characteristic polynomial for (2.9) factorizes into a first- and a third-order polynomial such that the eigenvalues acquire a simple analytic form. Defining the domain

\[ U_{PT} = {λ, κ : κ^6 + 8λ^2κ^4 - 3κ^4 + 16λ^2κ^2 + 20λ^2κ^2 + 3κ^2 - λ^2 - 1 ≤ 0} \]  

(2.10)

in the parameter space, the PT-symmetry is unbroken in the sense described by (1.1) when (λ, κ) ∈ U_{PT}. The four real eigenvalues are evaluated in this case to

\[ ε_1 = λ, \quad ε_2 = 2q^{1/2} \cos \left( \frac{θ}{3} \right) - \frac{λ}{3} \quad \text{and} \quad ε_3/4 = 2q^{1/2} \cos \left( \frac{θ + \pi}{3} + \frac{π}{3} \right) - \frac{λ}{3}, \]  

(2.11)

where

\[ θ = \arccos \left( \frac{r}{q^{3/2}} \right), \quad q = \frac{1}{9} \left( 3 + 4λ^2 - 3κ^2 \right) \quad \text{and} \quad r = \frac{λ}{27} \left( 18κ^2 + 8λ^2 + 9 \right). \]  

(2.12)

We depict the eigenvalues in figure 1 for some fixed λ or κ and varying κ or λ, respectively.

We observe the typical behaviour for PT-symmetric systems, namely that two eigenvalues start to coincide at the exceptional point [25] when κ and λ are situated on the boundary of U_{PT}. Going beyond those values, the PT-symmetry is spontaneously broken, and the two merged eigenvalues develop into a complex conjugate pair. This is, of course, a phenomenon prohibited for standard Hermitian systems by the Wigner–von Neumann non-crossing rule [26].

For the Hamiltonian (2.8), one can compute explicitly the left |Φ_n⟩ and right eigenvectors |Ψ_n⟩, forming a bi-orthonormal basis

\[ ⟨Ψ_n | Φ_m⟩ = δ_{nm} \quad \text{and} \quad \sum_n |Φ_n⟩⟨Ψ_n| = I, \]  

(2.13)

and verify that indeed for the spontaneously broken regime, the second relation in (1.1) does not hold, see Castro-Alvaredo & Fring [22] for the concrete expressions. We then have all the ingredients to compute the metric operator ρ and define the inner product ⟨.|.⟩_ρ := ⟨.|ρ.|⟩ with
regard to which the Hamiltonian (2.8) is Hermitian,
\[
\langle \psi | H \phi \rangle_\rho = \langle H \psi | \phi \rangle_\rho.
\] (2.14)
Computing the signature \(s = (s_1, s_2, \ldots, s_n)\) from
\[
\mathcal{P} | \Phi_n \rangle = s_n | \Psi_n \rangle, \quad \text{with } s_n = \pm 1,
\] (2.15)
we may evaluate the so-called \(C\)-operator introduced in Bender et al. [3],
\[
C := \sum_n s_n | \Phi_n \rangle \langle \Psi_n |,
\] (2.16)
and hence the metric operator \(\rho\), which also relates the Hamiltonian to its conjugate
\[
\rho := \mathcal{P} C \quad \text{and} \quad H^\dagger \rho = \rho H.
\] (2.17)

The explicit expressions can be found in Castro-Alvaredo & Fring [22], from which one can verify explicitly that the metric operator is Hermitian, positive and invertible. Thus, the Hamiltonian (2.8) is quasi-Hermitian as well as pseudo-Hermitian. From the expression for \(\rho\), we can also obtain the so-called Dyson map \(\eta\) [27], by taking the positive square root \(\eta = \sqrt{\rho}\). This operator serves to construct an isospectral Hermitian counterpart \(h\) to \(H\) by its adjoint action. For (2.8), we find
\[
h_2(\lambda, \kappa) = \eta H_2(\lambda, \kappa) \eta^{-1} = \sum_{s=x,y,z} \nu_s \sigma_s \otimes \sigma_s + \mu_z (\sigma_z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_z).
\] (2.18)
The constants \(\nu_x, \nu_y, \nu_z\) and \(\mu_z\) can be found in Castro-Alvaredo & Fring [22].

(c) Perturbative computation for the \(N\)-site model

It is clear that when proceeding to longer spin chains, it becomes increasingly complex to compute the earlier-mentioned operators such that exact computation becomes less transparent and can only be carried out with great effort. However, we may also gain considerable insights by resorting to a perturbative analysis. For this purpose, we separate the Hamiltonian into its Hermitian and non-Hermitian part as \(H(\lambda, \kappa) = h_0(\lambda) + i \kappa h_1\), where \(h_0\) and \(h_1\) are both Hermitian, with \(\kappa\) being a real coupling constant as, for instance, introduced in (2.1). Assuming next that the inverse of the metric exists and that it can be parametrized as \(\rho = e^\theta\), the second equation in (2.17)
can be written as

\[ H^1 = e^{iH}e^{-q} = H + [q, H] + \frac{1}{2} [q, [q, H]] + \frac{1}{3!} [q, [q, [q, H]]] + \cdots. \] (2.19)

Presuming further that the metric can be perturbatively expanded as

\[ q = \sum_{k=1}^{\infty} \kappa^{2k-1} q_{2k-1}, \] (2.20)

we obtain the following equations order-by-order in \( \kappa \):

\begin{align*}
[h_0, q_1] &= 2i \hbar_1, \quad (2.21) \\
[h_0, q_3] &= \frac{1}{6} [q_1, [q_1, h_1]], \quad (2.22) \\
\text{and} \quad [h_0, q_5] &= \frac{1}{6} [q_1, [q_3, h_1]] + \frac{1}{6} [q_3, [q_1, h_1]] - \frac{i}{360} [q_1, [q_1, [q_1, h_1]]]. \quad (2.23)
\end{align*}

It is clear from (2.21)–(2.23) that at each order, one new unknown quantity enters the computation for which we can solve our equations, i.e. in (2.21), we solve for \( q_1 \) for known \( h_0 \) and \( \hbar_1 \), in (2.22) for \( q_3 \), in (2.23) for \( q_5 \), etc. This process can be continued up to any desired order of precision, see Figueira de Morisson Faria & Fring [8] for further general details on perturbation theory.

Proceeding in this manner, we compute the Dyson operator as described earlier and determine the Hermitian counterpart. For \( N = 3 \), we obtain

\[ h = \mu_{xx} S_{xx}^3 + \mu_{yy} S_{yy}^3 + \mu_{zz} S_{zz}^3 + \mu_{xx} \mu_{zz} S_{xx} S_{zz} + \mu_{xx} \mu_{yy} S_{xx} S_{yy} + \mu_{xx} \mu_{zz} S_{xx} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} S_{xx} S_{yy} S_{zz}, \] (2.24)

where for convenience we introduced a new notation

\[ S_{a_1 a_2 \ldots a_p}^N := \sum_{k=1}^{N} \sigma_k^{a_1} \sigma_{k+1}^{a_2} \ldots \sigma_{k+p-1}^{a_p}, \quad \text{for} \quad a_i = x, y, z, u; \quad i = 1, \ldots, p \leq N. \] (2.25)

The coefficients \( \mu_{xx}, \ldots, \mu_{zz} \) are real functions of the couplings \( \lambda \) and \( \kappa \). We denote here \( \sigma^u = \mathbb{I} \) to allow for the possibility of non-local, i.e. not nearest neighbour, interactions. In fact, they do occur when we increase the length of the chain by one site. For \( N = 4 \), we compute

\[ h = \mu_{xx} S_{xx}^4 + \mu_{yy} S_{yy}^4 + \mu_{zz} S_{zz}^4 + \mu_{xx} \mu_{yy} S_{xx} S_{yy} + \mu_{xx} \mu_{zz} S_{xx} S_{zz} + \mu_{yy} \mu_{zz} S_{yy} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} S_{xx} S_{yy} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} \mu_{xy} S_{xx} S_{yy} S_{xy} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} \mu_{xy} \mu_{yz} S_{xx} S_{yy} S_{xy} S_{yz} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} \mu_{xy} \mu_{yx} S_{xx} S_{yy} S_{xy} S_{yx} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} \mu_{xy} \mu_{yx} \mu_{yz} \mu_{zx} S_{xx} S_{yy} S_{xy} S_{yx} S_{yz} S_{zx} S_{zz} + \mu_{xx} \mu_{yy} \mu_{zz} \mu_{xy} \mu_{yx} \mu_{yz} \mu_{zx} \mu_{zy} S_{xx} S_{yy} S_{xy} S_{yx} S_{yz} S_{zx} S_{zy} S_{zz}. \] (2.26)

We observe that the first non-local interaction terms proportional to \( S_{xx}^4, S_{xy}^4 \) and \( S_{zz}^4 \) emerge in this model. Thus, we encounter a very typical feature of non-Hermitian \( \mathcal{PT} \)-symmetric Hamiltonian systems, whereas the non-Hermitian Hamiltonian is fairly simple, and its Hermitian isospectral counterpart is quite complicated involving non-nearest neighbour interactions. An additional feature not present for chains of smaller length is the fact that some of the \( \lambda \)-dependence of the coefficients \( \mu_{xx}, \ldots, \mu_{zz} \) is no longer polynomial and gives rise to singularities.

This model exhibits the basic feature, but clearly there is plenty of scope left for further analysis. More explicit analytic formulae should be computed for \( \eta, \rho \) and \( \hbar \) for longer chains, models with higher spin values should be considered and further members of the class belonging to the perturbed \( \mathcal{M}_{p,q} \) series of minimal conformal field theories should be studied. Interesting recent results on other non-Hermitian quantum spin chains may be found in other studies [28,29].
3. $\mathcal{PT}$-symmetrically deformed Calogero-type models

$\mathcal{PT}$-deformed versions of multi-particle systems of Calogero type have been obtained so far in three quite different ways, as simple extensions, as constrained field equations or as genuine deformations.

(a) Extended Calogero–Moser–Sutherland models

The most direct and simplest way to obtain $\mathcal{PT}$-symmetrically extended versions of Calogero–Moser–Sutherland (CMS) models is to add a $\mathcal{PT}$-symmetric term to the original model as proposed in Fring [30],

$$\mathcal{H}_{\mathcal{PTCMS}} = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha^2 V(\alpha \cdot q) + \frac{1}{2} \tilde{g}_\alpha f(\alpha \cdot q) \alpha \cdot p,$$

(3.1)

with coupling constants $g, \tilde{g} \in \mathbb{R}$, canonical variables $q, p \in \mathbb{R}^{\ell+1}$ for an $(\ell + 1)$-dimensional representation of the roots $\alpha$ of some arbitrary root system $\Delta$, which is left invariant under the Coxeter group. The potential may take on different forms $V(x) = f^2(x)$ defined by means of the function $f(x) = 1/x$, $f(x) = 1/\sinh x$ or $f(x) = 1/\sin x$. The model in (3.1) is a generalization of an extension of the $A_\ell$ and $B_\ell$ Calogero model, i.e. $f(x) = 1/x$, for a specific representation of the roots as suggested by Basu-Mallick and co-workers [31–33]. The $\mathcal{PT}$-symmetry of $\mathcal{H}_{\mathcal{PTCMS}}$ is easily verified. In Fring [30], it was shown that for $f(x) = 1/x$, one may rewrite the Hamiltonian in (3.1), such that it becomes the standard Hermitian Calogero Hamiltonian with shifted momenta $p \to p + i\mu$,

$$\mathcal{H}_{\mathcal{PTCMS}} = \frac{1}{2} (p + i\mu)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \tilde{g}_\alpha^2 V(\alpha \cdot q),$$

(3.2)

where $\mu := \frac{1}{2} \sum_{\alpha \in \Delta} \tilde{g}_\alpha f(\alpha \cdot q) \alpha$, and the coupling constants have been redefined to $\tilde{g}_\alpha^2 := g_\alpha^2 + \alpha^2 \mu^2$ for $\alpha \in \Delta_s$ and $\tilde{g}_\alpha^2 := \tilde{g}_\alpha^2 + \alpha^2 \mu^2$ for $\alpha \in \Delta_l$, where $\Delta_l$ and $\Delta_s$ refer to the root system of the long and short roots, respectively. This manipulation is based on the not obvious identity $\mu^2 = \alpha^2 \tilde{g}_\alpha^2 \sum_{\alpha \in \Delta_s} V(\alpha \cdot q) + \alpha^2 \tilde{g}_\alpha^2 \sum_{\alpha \in \Delta_l} V(\alpha \cdot q)$, which is valid only for rational potentials. Even then, it has not yet been proved in a case-independent manner, but verified for many examples on a case-by-case basis [30].

For the rational potential, it is straightforward to obtain the Dyson map $\eta = e^{-q\mu}$, which relates the standard Hermitian Calogero model to the non-Hermitian model (3.1) by an adjoint action $\mathcal{H}_{\mathcal{PT}} = \eta^{-1} \mathcal{H}_{\mathcal{CMS}} \eta$. The integrability of the rational version of $\mathcal{H}_{\mathcal{PTCMS}}$ follows then from the existence of the Lax pair $L_C$ and $M_C$ obeying the Lax equation $\dot{L}_{\mathcal{PT}} = [L_{\mathcal{PT}}, M_{\mathcal{PT}}]$, which may be obtained from the standard Calogero Lax pair [34] as $L_{\mathcal{PT}}(p) = \eta^{-1} L_C(p) \eta = L_C(p + i\mu)$ and $M_{\mathcal{PT}}(p) = \eta^{-1} M_C(p) \eta = M_C(p + i\mu)$. By expanding the shifted kinetic term in (3.2), we obtain

$$\mathcal{H}_{\mathcal{PTCMS}} = \frac{1}{2} p^2 + \frac{1}{2} \sum_{\alpha \in \Delta} \tilde{g}_\alpha^2 V(\alpha \cdot q) + i\mu \cdot p - \frac{1}{2} \mu^2.$$

(3.3)

By the reasoning provided, it follows that this model is integrable for all of the earlier-mentioned potentials, whereas the model without the $\mu^2$ term is integrable only for rational potentials.

(b) From constraint field equations to $\mathcal{PT}$-deformed Calogero models

Another more surprising way to obtain particle systems of complex Calogero type arises from considering real-valued field solutions for some nonlinear equations. Making an ansatz in a form
of a real-valued field,
\[ u(x, t) = \frac{\lambda}{2} \sum_{k=1}^{\ell} \left( \frac{i}{x - z_k(t)} - \frac{i}{x - z_k^2(t)} \right), \quad \lambda \in \mathbb{R}, \]  
(3.4)

it was shown more than 30 years ago [35,36] that this constitutes an \( \ell \)-soliton solution for the Benjamin–Ono equation
\[ u_t + uu_x + \lambda H u_{xx} = 0, \]  
(3.5)

with \( Hu(x) \) denoting the Hilbert transform \( Hu(x) = P/\pi \int_{-\infty}^{\infty} u(x)/(z - x) \, dz \), provided the poles \( z_k \) in (3.4) obey the complex \( A_\ell \) Calogero equation of motion,
\[ \ddot{z}_k = \frac{\lambda^2}{2} \sum_{j \neq k} (z_j - z_k)^{-3}, \quad z_k \in \mathbb{C}. \]  
(3.6)

Clearly, for different types of nonlinear equations, the constraining equation might be of a more complicated form. However, we may consistently impose additional constraints by making use of the following theorem of Airault et al. [37].

**Theorem 3.1.** Given a Hamiltonian \( H(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) \) with flow
\[ x_i = \frac{\partial H}{\partial \dot{x}_i} \quad \text{and} \quad \dot{x}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n, \]  
(3.7)

and conserved charges \( I_i \) in involution with \( H \), i.e. vanishing Poisson brackets \( \{I_i, H\} = 0 \). Then, the locus of \( \text{grad} \; I = 0 \) is invariant with regard to time evolution. Thus, it is permitted to restrict the flow to that locus provided it is not empty.

Making now the ansatz
\[ v(x, t) = \lambda \sum_{k=1}^{\ell} (x - z_k(t))^{-2}, \quad \lambda \in \mathbb{R}, \]  
(3.8)

one can show that this solves the Boussinesq equation
\[ v_{tt} = a(v^2)_{xx} + b v_{xxxx} + v_{xx}, \quad a, b \in \mathbb{R}, \]  
(3.9)

if and only if \( b = \frac{1}{12}, \lambda = -a/2 \) and the poles \( z_k \) obey the constraining equations
\[ \ddot{z}_k = 2 \sum_{j \neq k} (z_j - z_k)^{-3} \iff \ddot{z}_k = -\frac{\partial H_{\text{Cal}}}{\partial z_i} \]  
(3.10)

and
\[ \ddot{z}_k^2 = 1 - \sum_{j \neq k} (z_j - z_k)^{-2} \iff g \text{rad}(I_3 - I_1) = 0. \]  
(3.11)

Here, \( I_3 = \sum_{j=1}^{\ell} \frac{\dot{z}_j^3}{3} + \sum_{k \neq j} \dot{z}_j(z_j - z_k)^2 \) and \( I_1 = \sum_{j=1}^{\ell} \dot{z}_j \) are two conserved charges in the \( A_\ell \) Calogero model. Thus, in comparison with the previous example (3.4)–(3.5), we have to satisfy additional constraints (3.11) besides the equations of motion of the \( A_\ell \) Calogero model. However, according to theorem 3.1, this is still a consistent system of equations, provided equations (3.10) and (3.11) possess any non-trivial solution. Only very few solutions have been found so far. The simplest two-particle solution has already been reported in Airault et al. [37],
\[ z_1 = \kappa + \sqrt{(t + \tilde{\kappa})^2 + \frac{1}{4}} \quad \text{and} \quad z_2 = \kappa - \sqrt{(t + \tilde{\kappa})^2 + \frac{1}{4}}. \]  
(3.12)

In this case, the Boussinesq solution acquires the form
\[ v(x, t) = 2\lambda \frac{(x - \kappa)^2 + (t + \tilde{\kappa})^2 + 1/4}{[(x - \kappa)^2 - (t + \tilde{\kappa})^2 - 1/4]^2}, \]  
(3.13)

Note that \( v(x, t) \) is still a real solution. However, without any complication, we may change \( \kappa \) and \( \tilde{\kappa} \) to be purely imaginary, in which case, and only in this case, (3.13) becomes a solution for the
\( \mathcal{H}'_{\mathcal{P}\mathcal{T}} \text{-symmetric equation (3.9) in the sense that } \mathcal{P}\mathcal{T} : x \rightarrow -x, t \rightarrow -t \text{ and } v \rightarrow v. \) A three-particle solution was reported in Assis & Fring \[38\], which exhibits an interesting solitonic behaviour in the complex plane. In that case, no real solution could be found and, once again, one was forced to consider complex particle systems. For more particles, different types of algebras or other types of nonlinear equations, these investigations have not yet been carried out.

\section*{(c) Deformed Calogero–Moser–Sutherland models}

Let us now consider the CMS models with an additional confining potential,
\[
\mathcal{H}'_{\mathcal{P}\mathcal{T}\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta} (\alpha \cdot \bar{\alpha})^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot \bar{\alpha}), \quad m, g_\alpha \in \mathbb{R},
\]
and also deform the coordinates \( q \rightarrow \bar{q} \). Considering at first, the \( \mathcal{A}_2 \) case for a standard three-dimensional representation for the simple \( \mathcal{A}_2 \) roots \( \alpha_1 = \{1, -1, 0\}, \alpha_2 = \{0, 1, -1\} \), we deformed the coordinates as
\[
q_1 \rightarrow \bar{q}_1 = q_1 \cosh \varepsilon + i \sqrt{3}(q_2 - q_3) \sinh \varepsilon, \quad (3.15)
\]
\[
q_2 \rightarrow \bar{q}_2 = q_2 \cosh \varepsilon + i \sqrt{3}(q_3 - q_1) \sinh \varepsilon \quad (3.16)
\]
and
\[
q_3 \rightarrow \bar{q}_3 = q_3 \cosh \varepsilon + i \sqrt{3}(q_1 - q_2) \sinh \varepsilon, \quad (3.17)
\]
such that the relevant terms in the potential become
\[
\alpha_1 \cdot \bar{q} = q_{12} \cosh \varepsilon - \frac{i}{\sqrt{3}}(q_{13} + q_{23}) \sinh \varepsilon, \quad (3.18)
\]
\[
\alpha_2 \cdot \bar{q} = q_{23} \cosh \varepsilon - \frac{i}{\sqrt{3}}(q_{21} + q_{31}) \sinh \varepsilon \quad (3.19)
\]
and
\[
(\alpha_1 + \alpha_2) \cdot \bar{q} = q_{13} \cosh \varepsilon + \frac{i}{\sqrt{3}}(q_{12} + q_{32}) \sinh \varepsilon, \quad (3.20)
\]
with the abbreviation \( q_{ij} := q_i - q_j \). We observe for this example the following antilinear involutory symmetries:
\[
S_1 : q_1 \leftrightarrow q_2, \quad q_3 \leftrightarrow q_3, \quad i \rightarrow -i \quad (3.21)
\]
and
\[
S_2 : q_2 \leftrightarrow q_3, \quad q_1 \leftrightarrow q_1, \quad i \rightarrow -i. \quad (3.22)
\]

At this stage, this deformation appears to be somewhat ad hoc. In fact, it arose in recent studies \[39,40\] from the physical motivation to eliminate singularities in the potential when solving the separable Schrödinger equation for the Hamiltonian \( \mathcal{H}'_{\mathcal{P}\mathcal{T}\text{CMS}} \). It was noted that the new non-Hermitian model could be defined on less separated configuration space. Although, in general, one had to restrict the models to distinct Weyl chambers and analytically continue the wave functions across their boundaries with the inclusion of some chosen phase, this is no longer necessary in the deformed models. In addition, the new models possess a modified energy spectrum with real eigenvalues, which we attribute to the fact that the theory is invariant with respect to the antilinear transformations \( S_1 \) and \( S_2 \). Motivated by this success, one may attempt to find a more direct systematic mathematical procedure to deform the coordinates rather than the indirect implication resulting from the separability of the Schrödinger equation. In any case, the latter approach would be entirely unpractical for models related to higher rank Lie algebras.
We notice first that the Hamiltonian (3.14) also results from deforming the roots involved. For the $A_2$ case, we may take the simple roots
\[
\tilde{\alpha}_1 = \alpha_1 \cosh \varepsilon + i \frac{1}{\sqrt{3}} \sinh \varepsilon (\alpha_1 + 2\alpha_2)
\]
and
\[
\tilde{\alpha}_2 = \alpha_2 \cosh \varepsilon - i \frac{1}{\sqrt{3}} \sinh \varepsilon (2\alpha_1 + \alpha_2),
\]
and rewrite (3.14) equivalently as
\[
\mathcal{H}_{PTCMS} = \frac{\mu^2}{2} + \frac{m^2}{16} \sum_{\tilde{\alpha} \in \Delta_\varepsilon} (\tilde{\alpha} \cdot \cdot q)^2 + \frac{1}{2} \sum_{\tilde{\alpha} \in \Delta_\varepsilon} \bar{g}_\tilde{\alpha} V(\tilde{\alpha} \cdot q), \quad m, g_\tilde{\alpha} \in \mathbb{R}.
\]

Now, the symmetries (3.21)–(3.22) can be identified equivalently for the roots. We note
\[
\sigma^\varepsilon_1 : \tilde{\alpha}_1 \leftrightarrow -\tilde{\alpha}_1, \quad \tilde{\alpha}_2 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \leftrightarrow q_1 \leftrightarrow q_2, \quad q_3 \leftrightarrow q_3, \quad i \rightarrow -i
\]
and
\[
\sigma^\varepsilon_2 : \tilde{\alpha}_2 \leftrightarrow -\tilde{\alpha}_2, \quad \tilde{\alpha}_1 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \leftrightarrow q_2 \leftrightarrow q_3, \quad q_1 \leftrightarrow q_1, \quad i \rightarrow -i.
\]
This observation has been taken as the basis for the formulation of a systematic construction procedure leading to antilinearly invariant, and therefore potentially physical, models [41–43]. The dynamical variables, or possibly more general fields, appear in the dual space of some roots with respect to the standard inner product. Because these root spaces are naturally equipped with various symmetries due to the fact that by construction, they remain invariant under the action of the entire Weyl group $W$, it is by far easier and systematic to identify the antilinear symmetries directly in the root spaces rather than in the configuration space. Once they have been identified, they can be transformed to the latter.

The aim is therefore to construct complex extended antilinearly invariant root systems that we denote by $\tilde{\Delta}(\varepsilon)$. The proposed procedure consists of constructing two maps, which may be obtained in any order. In one step, we extend the representation space of the entire Weyl group with the crucial property for our purposes, namely to guarantee that it is left invariant under an antilinear involutory map
\[
\sigma : \tilde{\Delta}(\varepsilon) \rightarrow \tilde{\Delta}(\varepsilon) \quad \text{and} \quad \tilde{\alpha} \mapsto \omega \tilde{\alpha}.
\]
This means the map in (3.29) satisfies $\sigma : \tilde{\alpha} = \mu_1 \alpha_1 + \mu_2 \alpha_2 \mapsto \mu_1^\ast \omega \alpha_1 + \mu_2^\ast \omega \alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$ and also $\sigma \circ \sigma = 1$. In order to facilitate the construction we make some further additional assumptions.

(i) The operator $\omega$ decomposes as
\[
\omega = \tau \bar{\omega} = \bar{\omega} \tau,
\]
with $\bar{\omega} \in \mathcal{W}$, $\bar{\omega}^2 = \mathbb{I}$ and $\tau$ being a complex conjugation. This will guarantee that $\sigma$ is antilinear.

(ii) There are at least two different maps $\omega_i$ with $i = 1, \ldots, \kappa \geq 2$. This assumption simplifies the solution procedure.

(iii) There exists a similarity transformation of the form
\[
\omega_i := \theta_i \bar{\omega}_i \theta_i^{-1} = \tau \bar{\omega}_i, \quad \text{for } i = 1, \ldots, \kappa \geq 2.
\]
from the constraints (i)–(v), we may obtain \( \theta^\star \) for our construction. An immediate consequence of (iii) is that \( \sigma^\star \) denoting quantities in and acting on the dual space by *.

Naturally, we can also identify an antilinear involutory map \( \sigma^\star \) and (3.25), we would also like to construct a dual map \( \delta^* \) for \( \delta \) acting on the coordinate space with \( q = \{q_1, \ldots, q_n\} \) or possibly fields. We therefore define

\[
\delta^* : \mathbb{R}^n \rightarrow \tilde{\Lambda}^*(\varepsilon) = \mathbb{R}^n \oplus i\mathbb{R}^n \quad \text{and} \quad x \mapsto \tilde{x} = \theta^*_i x,
\]

(3.34) denoting quantities in and acting on the dual space by *. Thus, assuming \( \theta^\varepsilon \) has been constructed from the constraints (i)–(v), we may obtain \( \theta^\varepsilon^* \) by solving the \( \ell \) equations

\[
(\tilde{a}_i \cdot x) = ((\theta^\varepsilon \alpha)_i \cdot x) = (\alpha_i \cdot \theta^*_\varepsilon x) = (\alpha_i \cdot \tilde{x}), \quad \text{for } i = 1, \ldots, \ell,
\]

(3.35) involving the standard inner product. This means \( (\theta^\varepsilon^\varepsilon)^{-1} \alpha_i = (\theta^\varepsilon \alpha)_i \). Note that in general \( \theta^\varepsilon \neq \theta^\varepsilon^* \).

Naturally, we can also identify an antilinear involutory map

\[
\omega^\star : \tilde{\Lambda}^*(\varepsilon) \rightarrow \tilde{\Lambda}^*(\varepsilon) \quad \text{and} \quad \tilde{x} \mapsto \omega^\star \tilde{x},
\]

(3.36) corresponding to \( \omega \), but acting in the dual space. Concretely, we need to solve for this the \( \kappa \times \ell \) relations

\[
(\omega_i \tilde{a}_j) \cdot x = \omega_i^\star \omega_j^\star \tilde{x}, \quad \text{for } i = 1, \ldots, \kappa; \quad j = 1, \ldots, \ell,
\]

(3.37) for \( \omega^\star \) with given \( \omega_i \).

In Fring & Smith [41–43], many solutions to the set of constraints (i)–(v) were constructed. A particular systematic construction can be found when we take \( \kappa = 2 \) in requirement (ii) and identify \( \omega_1 = \sigma_- \) and \( \omega_2 = \sigma_+ \). The maps \( \sigma_\pm \) factorize the Coxeter element in a unique way,

\[
\sigma := \sigma_- \sigma_+, \quad \text{with} \quad \sigma_\pm := \prod_{i \in V_\pm} \sigma_i,
\]

(3.38) where the \( \sigma_i \) are simple Weyl reflections \( \sigma_i(x) := x - 2(x \cdot \alpha_i / \alpha_i^2) \alpha_i \) associated to each simple root for \( 1 \leq i \leq \ell \equiv \text{rank} V \). The two sets \( V_\pm \) are defined by means of a bi-coloration of the Dynkin diagrams consisting of associating values \( c_i = \pm 1 \) to its vertices in such a way that no two vertices with the same values are linked together. The consequence of this labelling is that \( [\sigma_i, \sigma_j] = 0 \) for \( i, j \in V_+ \) or \( i, j \in V_- \) such that the factorization in (3.38) becomes unique. Clearly, \( \sigma_\pm^2 = \mathbb{I} \) is required for our construction. An immediate consequence of (iii) is that \( \sigma \) and \( \theta^\varepsilon \) commute, such that the following ansatz captures all possible cases based on the assumption stated before (3.38):

\[
\theta^\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon) \sigma^k, \quad \text{with} \quad \lim_{\varepsilon \rightarrow 0} c_k(\varepsilon) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0, \end{cases} \quad c_k(\varepsilon) \in \mathbb{C}.
\]

(3.39) The upper limit in the sum results from the fact that \( \sigma^h = \mathbb{I} \), with \( h \) denoting the Coxeter number. Invoking also the remaining constraints allows us to determine the functions \( c_k(\varepsilon) \). For the \( A_3 \)
Weyl group invariant root system, this yields, for instance, the following three deformed simple roots:

\[
\begin{align*}
\tilde{\alpha}_1 &= \cosh \varepsilon \alpha_1 + (\cosh \varepsilon - 1)\alpha_3 - i \sqrt{2} \cosh \varepsilon \sinh \left(\frac{\varepsilon}{2}\right) (\alpha_1 + 2\alpha_2 + \alpha_3), \\
\tilde{\alpha}_2 &= (2 \cosh \varepsilon - 1)\alpha_2 + 2i \sqrt{2} \cosh \varepsilon \sinh \left(\frac{\varepsilon}{2}\right) (\alpha_1 + \alpha_2 + \alpha_3) , \\
\text{and} & \\
\tilde{\alpha}_3 &= \cosh \varepsilon \alpha_3 + (\cosh \varepsilon - 1)\alpha_1 - i \sqrt{2} \cosh \varepsilon \sinh \left(\frac{\varepsilon}{2}\right) (\alpha_1 + 2\alpha_2 + \alpha_3).
\end{align*}
\]

In some cases, we were even able to provide closed formulae for entire subseries. For instance, for $A_{4n-1}$, we found a closed expression for the deformation matrix in the form

\[
\theta_\varepsilon = r_0 \mathbb{1} + (1 - r_0)\sigma^{2n} + i \sqrt{r_0^2 - r_0}(\sigma^n - \sigma^{-n}) .
\]

A possible choice for the function $r_0$ is $r_0 = \cosh \varepsilon$. It was also shown in Fring & Smith [41] that it is impossible to construct solutions for (i)-(v) for certain Weyl groups based on the factorization (3.38), such as for instance $B_{2n+1}$. However, in Fring & Smith [42,43], it was demonstrated that one can slightly alter the procedure by choosing different factors instead and constructing solutions based on an ansatz similar to (3.39). For $B_{2n+1}$, we found closed expressions of the form

\[
\tilde{\alpha}_{2j-1} = \cosh \varepsilon \alpha_{2j-1} + i \sinh \varepsilon \left(\alpha_{2j-1} + 2 \sum_{k=2j}^\ell \alpha_k\right), \quad \text{for } j = 1, \ldots, n
\]

and

\[
\begin{align*}
\tilde{\alpha}_{2j} &= \cosh \varepsilon \alpha_{2j} - i \sinh \varepsilon \left(\sum_{k=2j}^{2j+2} \alpha_k + 2 \sum_{k=2j+3}^\ell 2\alpha_k\right), \quad \text{for } j = 1, \ldots, n-1, \\
\tilde{\alpha}_{\ell-1} &= \cosh \varepsilon (\alpha_{\ell-1} + \alpha_{\ell}) - \alpha_{\ell} - i \sinh \varepsilon (\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_{\ell}), \\
\text{and} & \\
\tilde{\alpha}_\ell &= \alpha_\ell.
\end{align*}
\]

In this case, the dual deformation matrix, which acts on the coordinates (3.35), takes on a very familiar form and turns out to be composed of pairwise complex rotations,

\[
\theta_\varepsilon^* = \frac{R}{N} = \begin{pmatrix} R & 0 \\
\varepsilon & R \\
0 & R \\
\varepsilon & R \\
1 & R \\
\end{pmatrix}, \quad \text{with } R = \begin{pmatrix} \cosh \varepsilon & \sinh \varepsilon \\
-\sinh \varepsilon & \cosh \varepsilon \\
\end{pmatrix}.
\]

Having constructed various deformed root spaces that, by construction, are equipped with an antilinear involutory symmetry, we may then consider various models formulated in terms of roots, such as $\mathcal{H}_{PT\CMS}$ defined in (3.25). We then encounter several interesting new features in these models. Because many of the key identities needed for the solution procedure are identical in terms of roots or deformed roots, we may adopt similar solution techniques as in the undeformed case, such as separating variables. As a crucial new feature, we find that the energy spectrum is modified and admits new real solutions when compared with the undeformed model. For instance, for the $G_2$ case, the undeformed energies $E_n = 2|\omega|(2n + \lambda + 1)$ with $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}^+$ become

\[
E_{nm}^\pm = 2|\omega|(2n + 6(k^s_{ij} + k^l_{ij} + m) + 1), \quad \text{for } n, m \in \mathbb{N}_0,
\]

with $k^s_{ij} = (1 + \sqrt{1 + 4\alpha^2_{ij}})/4$.

For the $B_{2n+1}$ case, we can support these observations with the explicit construction of Dyson maps as introduced in (2.18) and the metric operators (2.17). For the models based on the deformed roots (3.44)-(3.45), the Dyson map is simply $\eta = \eta_1 \eta_2 \ldots \eta_{(\ell-2)(\ell-1)}$, with $\eta_{ij} = e^{-\varepsilon(x_i y_j - x_j y_i)}$, such that the metric operator becomes $\rho = \eta^2$. 
A further novelty in the deformed models is that the wave functions are regularized by means of the deformation such that many singularities disappear. In particular, this means that these models can be defined usually on the entire space $\mathbb{C}^n$, whereas the undeformed models could only be defined in certain Weyl chambers. The continuation from a chamber to its neighbouring one was achieved by introducing a phase factor by hand, thus selecting particular statistics. The deformed models, on the other hand, have these phase factors already built in as a property of the model. For instance, in the $A_3$ case, we find that the four-particle wave function obeys

$$\psi(q_1, q_2, q_3, q_4) = e^{i\pi s} \psi(q_2, q_4, q_1, q_3).$$

These properties can be read off easily from the action of the generalized $\mathcal{PT}$-symmetry on the deformed roots, translated to the dual space that is to the coordinates and then to the parametrization of the wave function. We note that the phase factor emerges as an intrinsic property rather than as an imposition. We illustrate the relation (3.48) as follows:

$$w \quad x \quad y \quad z = e^{i\pi s} \quad w \quad x \quad y \quad z.$$

In the deformed models, we can even allow some of the particles to occupy the same place and scatter them with single particles

$$x \quad y \quad z = e^{i\pi s} \quad x \quad y \quad z.$$

We can also scatter pairs of particles resulting in the exchange of one of the particles

$$x \quad y = e^{i\pi s} \quad x \quad y,$$

and we may even scatter triplets with a single particle

$$x \quad y = e^{i\pi s} \quad x \quad y.$$

It is clear that these models have very interesting new properties, and there are still various open issues left worthwhile investigating. More explicit solutions for spectra and wave functions should be constructed; the important questions of whether the deformed models are still integrable should be settled; Dyson maps, the metric operators and Hermitian counterparts should be constructed such that more observables of the models can be studied. The constructed root systems could be used to formulate entirely new models of different kinds to Calogero systems.

4. $\mathcal{PT}$-symmetrically deformed nonlinear wave equations

The prototype integrable system of nonlinear wave type is the KdV equation [44],

$$u_t + \beta uu_x + \gamma u_{xxx} = 0, \quad \beta, \gamma \in \mathbb{C},$$

resulting from a Hamiltonian density

$$\mathcal{H}_{\text{KdV}}[u] = -\frac{\beta}{6} u^3 + \frac{\gamma}{2} u_x^2.$$

The system admits two different types of $\mathcal{PT}$-symmetries,

$$\mathcal{PT}_\pm: x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto \pm u, \quad \text{for } \beta, \gamma \in \mathbb{R},$$

on August 13, 2017
The asymptotic conditions are chosen to be
\[ \lim_{\zeta \to \pm \infty} u(\zeta) = \left(1 - i\right)/4, \] the speed of the wave as \( c = 1 \) and the constants of the model as \( \beta = 2 + 2i \) and \( \gamma = 3 \).

Figure 2. The two different Riemann sheets for rational solutions with broken \( \mathcal{PT} \)-symmetry of the \( \mathcal{H}_{1/2}^+ \) model with different values of purely complex initial conditions whose imaginary part is indicated. The asymptotic conditions are chosen to be
\[ \lim_{\zeta \to \pm \infty} u(\zeta) = \left(1 - i\right)/4, \] the speed of the wave as \( c = 1 \) and the constants of the model as \( \beta = 2 + 2i \) and \( \gamma = 3 \).

(Online version in colour.)

which have been exploited only recently [18,45–47]. According to the deformation prescription (1.3), we can now deform the Hamiltonian density (4.2) in two alternative ways,
\[ \delta^+ \varepsilon : u_x \mapsto u_{x \varepsilon} \quad \text{or} \quad \delta^- \varepsilon : u \mapsto u_{\varepsilon}, \] respectively, depending on whether we assume \( u(x,t) \) to be \( \mathcal{PT} \)-symmetric or \( \mathcal{PT} \)-antisymmetric. The deformed models, suitably normalized, are then defined by the densities
\[ \mathcal{H}^+_\varepsilon = -\frac{\beta}{6} u^3 - \frac{\gamma}{1 + \varepsilon} (iu_x)^{\varepsilon+1} \quad \text{and} \quad \mathcal{H}^-_\varepsilon = \frac{\beta}{(1 + \varepsilon)(2 + \varepsilon)} (iu)^{\varepsilon+2} + \frac{\gamma \varepsilon^2}{2 u_x^2}, \] with corresponding equations of motions
\[ u_t + \beta u u_x + \gamma (u_{x \varepsilon})_{xx} = 0 \quad \text{and} \quad u_t + i\beta u x + \gamma u_{xxx} = 0. \] The \( \mathcal{PT} \)-symmetry can be exploited to ensure the reality for expressions such as the energy on a certain interval \([-a,a]\),
\[ E = \int_{-a}^{a} \mathcal{H}[u(x)] \, dx = \oint_{\Gamma} \mathcal{H}[u(x)] \frac{du}{u_x}. \] One would expect this expression to be real for the unbroken symmetric regime. However, in Cavaglia et al. [18], some unexpected cases with real energies were also found, for which the \( \mathcal{PT} \)-symmetry was entirely broken (4.3), i.e. for the Hamiltonian and for its solutions. This possibly indicates the existence of different realizations.

A further characteristic feature of the deformed models is that, in general, one has to view them on various Riemann sheets. An example for a travelling wave solution parametrized by \( \zeta = x - ct \), with \( c \) denoting the wave speed, for broken \( \mathcal{PT} \)-symmetry is presented in figure 2.

The branch cut at \(-\infty - i/4 \) to \((1-i)/4 \) is passed from above in figure 2a to below in figure 2b. The trajectories for the \( \mathcal{PT} \)-symmetric and broken \( \mathcal{PT} \)-symmetric case look qualitatively very similar; the major difference being that the fixed point has moved away from the real axis, thus leading to a loss of symmetry.

Viewing the systems as two-dimensional models, the nature of the fixed points has been investigated systematically by exploiting the fact that their characteristic behaviour is completely classified in dependence on the different types of eigenvalues for the Jacobian. In Cavaglia
et al. [18], it was found that they may even undergo Hopf bifurcations in these systems, passing from a star node over a centre to a focus. This feature was derived for the $\mathcal{PT}$-symmetric as well as for the broken $\mathcal{PT}$-symmetric regime. In particular, this also means that we encounter closed trajectories, despite the fact that the $\mathcal{PT}$-symmetry is broken. We depict an example in figure 3 for different values of $c$, $\beta$ and $\gamma$.

An interesting relation between these type of deformations and some simple complex quantum mechanical models was pointed out in Cavaglia et al. [18]. As a special case of this general observation, we consider here the $\mathcal{H}_2^{-}$ model with Hamiltonian density

$$\mathcal{H}_2^{-} [u] = \frac{\beta}{12} u^4 + \frac{\gamma}{2} u_x^2, \quad (4.8)$$

and make contact with the model studied in Anderson et al. [48]. As explained in Cavaglia et al. [18], integrating (4.6) twice with respect to $\zeta$, we obtain

$$u_\zeta^2 = \frac{2}{\gamma} \left( \kappa_2 + \kappa_1 u + \frac{c}{2} u^2 + \beta \frac{1}{12} u^4 \right), \quad (4.9)$$
with integration constants $\kappa_1, \kappa_2 \in \mathbb{C}$. Identifying $u \to x$ and $\zeta \to t$ for the travelling wave equation, together with the constraints

$$\kappa_1 = -\gamma \tau, \quad \kappa_2 = \gamma E_x, \quad \beta = -3\gamma g \quad \text{and} \quad c = -\gamma \omega^2, \quad (4.10)$$

converts the derivative of equation (4.9) into Newton’s equation,

$$\ddot{x} + \tau + \omega^2 x + gx^3 = 0, \quad (4.11)$$

for the quartic-harmonic oscillator of form

$$H_{\text{quartic}} = \frac{1}{2} p^2 + \tau x + \frac{\omega^2}{2} x^2 + \frac{g}{4} x^4. \quad (4.12)$$

One may now directly translate some of the properties of the system (4.8) to the quantum mechanical model (4.12). The special choice $\kappa_1 = \tau = 0$ for the integration constants imply that one is considering asymptotically vanishing waves with $\lim_{\zeta \to \infty} u(\zeta) = 0$ and with Neumann boundary conditions $\lim_{\zeta \to \infty} u_x(\zeta) = \sqrt{2E_x}$, where $H_{\text{quartic}} = E_x$. Accordingly, the energy in the classical analogue of a complex classical particle corresponds to an integration constant in the nonlinear wave equation context multiplied by one of the coupling constants in the latter model. This is of course different from the energy as defined in (4.7), which also leads to different conclusions regarding the reality of these quantities resulting from the various $\mathcal{P}\mathcal{T}$-symmetric scenarios.

In a similar way, the complex seminal [1] cubic harmonic oscillator

$$H_{\text{cubic}} = \frac{1}{2} p^2 + \frac{1}{2} x^2 + igx^3, \quad (4.13)$$

treated also in Bender et al. [49], simply results from integrating the KdV equation twice with the identification $\kappa_1 = 0, \kappa_2 = \gamma E_x, \beta = -i6cg$ and $\gamma = -c$.

It appears to be unlikely that the models are still integrable as, in general, they do not pass the Painlevé test [50,51]. Similar studies have also been carried out for other types of nonlinear wave equations as, for instance, for deformed Ito systems in Cavaglia et al. [18]. It was even shown that one can $\mathcal{P}\mathcal{T}$-symmetrically deform the supersymmetric version of the KdV equation (4.1) while still preserving its supersymmetry [47].

Evidently, many features remain still unexplored, and it would be very interesting to extend these studies to a larger range of values for the deformation parameter, to other nonlinear field equations such as Burgers, Bussinesque, Kadomtsev Petviashvili, generalized shallow water equations, extended KdV equations with compacton solutions, etc.

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