On a generalization of the Fay-Sato identity for KP Baker functions and its application to constrained hierarchies

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Abstract

Some new formulas for the KP hierarchy are derived from the differential Fay identity. They proved to be useful for the $k$-constrained hierarchies providing a series of determinant identities for them. A differential equation is introduced which is called “universal” since it plays an important role for all the $k$-constrained hierarchies. In the cases $k = 1, 2$ and $3$ explicit formulas are presented, in all the others recurrence relations are given which enable one to obtain the identities.
0. Introduction. This paper is a result of our discussions about the meaning and the significance of a proposition in the article [1] (Proposition 2). We found out that a broader problem can be posed and solved using a more general language. Virtually, we suggest a new type of problems for the KP and, especially, \( k \)-constrained KP hierarchies. The explicit identities are obtained for the smallest values of \( k \), \( k = 1, 2 \) and \( 3 \); for higher values we have recurrence formulas, i.e. a recipe to get new identities. Calculations become cumbersome. We hope that there exists a more general approach allowing one to circumvent these complicated calculations at all.

The so-called differential Fay identity for the KP hierarchy (proven by Sato for the most general \( \tau \) functions) is well-known (see, e.g., [3]). It can be written in the following form using the Baker function \( \hat{w} \):

\[
G F(t_1, t_2, ...) = F(t_1 + 1/z_2, t_2 + 1/(2z_2^2), t_3 + 1/(3z_2^3), ...)
\]

where \( z_2 \) is another parameter and let \( G_1 = G - 1 \). Then

\[
(z_1 - z_2)G_1 \hat{w} = -G_1 \hat{w}' + G_1 w_1 \cdot G_1 \hat{w} + G_1 w_1 \cdot \hat{w} - \hat{w}'.
\]

(0.1)

Here and further \( \hat{w} \) is always \( \hat{w}(t, z_1) \) and \( w_1 \) is the first non-trivial coefficient in the expansion \( \hat{w} = \sum_0^\infty w_i z_1^{-i} \) (recall that \( w_0 = 1 \)). Let

\[
H = -\partial + G_1 w_1 + \hat{w} \cdot \text{res}_{z_1}
\]

be an operator that acts on series \( \sum a_{ij} z_1^{-i} z_2^{-j} \). Eq.(0.1) can be represented as

\[
(z_1 - z_2)n^{(1)} = H(n^{(1)}) - \hat{w}', \quad \text{where } n^{(1)} = G_1 \hat{w}.
\]

(0.3)

In fact, this identity is equivalent to the hierarchy itself since, expanding it in powers of \( z_1^{-1} \) and \( z_2^{-1} \), all the \( \partial \)\-terms can be obtained.

Let us think on the last term of (0.3) as on \( -\partial_1 \hat{w} \). We are going to generalize this identity by showing that one can construct quantities \( n^{(k)} = \sum_1^\infty n_j^{(k)} z_2^{-j} \) where \( n_j^{(k)} = P_j^{(k)} \hat{w} \) and \( P_j^{(k)} \) are differential operators in \( \partial_t \) such that

\[
(z_1 - z_2)n^{(k)} = H(n^{(k)}) - \partial_k \hat{w}, \quad \partial_k = \partial/\partial t_k.
\]

(0.4)

Explicit formulas for \( n^{(k)} \) are obtained only for \( k = 1, 2, 3 \). In the case of larger \( k \) we give the recurrence formulas for \( n_j^{(k)} \).

These identities have an interesting application in the theory of constrained KP hierarchies. The \( k \)-constraint for the KP hierarchy is \( L^k = L^k_+ + q \partial^{-1} r \). An important role for all the \( k \)-constraints plays an equation which we call “universal”. This is

\[
f' + 1 - z_2 f + Q f - G_1 w_1 \cdot f = 0
\]

(0.5)

where \( Q \) is any function. The universal equation has the only solution in the form of a series in \( z_2^{-1} \), \( Q = \sum_0^\infty f_j z_2^{-j} \). The coefficients \( f_j \) are functions, one can find that \( f_0 = z_2^{-1} + Q z_2^{-2} + ... \). It will be shown that for the \( k \)-constrained hierarchy the quantities \( n^{(k)}/\partial_k \hat{w} \) with the above \( n^{(k)} \) satisfy the universal equation with \( Q = q'/q \). Therefore \( n^{(k)}/\partial_k \hat{w} = f_Q \) and they do not
depend on $z_1$. The quantity $n^{(k)}$ happens to be a product of a series in $z_1^{-1}$ and a series in $z_2^{-1}$. Hence, the relations $P_j^{(k)} w_i = f_j \cdot \partial_k w_i$, $j \geq k$ hold where $P_j^{(k)}$ are the above differential operators. This implies that the matrix $P_j^{(k)} w_i$ is of rank 1 and all of its second order minors vanish. This supplies one with infinitely many identities. The first of them, with $j_1 = 1$ and $j_2 = 2$ and arbitrary $i_1$ and $i_2$, form a necessary and sufficient condition that a Baker function of the KP hierarchy satisfies the $k$th constraint.

1. The KP hierarchy. Let us recall some terminology. Let

$$L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + ...$$

be a pseudodifferential operator, then

$$\partial_n L = [L^n, L], \text{ where } \partial_n = \partial / \partial t_n$$

are the equations of the KP hierarchy,

$$L = \hat{w}(t, \partial) \partial^{-1}(t, \partial), \quad \hat{w}(t, \partial) = \sum_{k=0}^{\infty} w_k(t) \partial^{-k}, \quad w_0 = 1,$$

and

$$\psi(t, z) = \hat{w}(t, z) \exp \xi = \hat{w}(t, \partial) \exp \xi \text{ where } \xi(t, z) = \sum_{1}^{\infty} t_k z^k$$

is the (formal) Baker function.

It is easy to see that the equations

$$\partial_n \hat{w}(t, \partial) = -L^n \hat{w}(t, \partial)$$

imply the above hierarchy equations for $L$. Then $\psi$ satisfies the equations

$$L \psi = z \psi, \quad \partial_n \psi = L^n \psi.$$

The Schur polynomials are defined by

$$\exp \xi = \sum_{0}^{\infty} p_n(t) z^n, \quad t = (t_1, t_2, ...).$$

Let $\tilde{\partial} = (\partial_1, \partial_2/2, \partial_3/3, ...)$. Then $p_n(\tilde{\partial})$ are differential operators denoted by $p_n^\tilde{\partial}$.

2. The $\tau$-function and the differential Fay identity. The $\tau$-function is defined by the equality

$$\hat{w}(t, z) = \frac{\tau(t - [z^{-1}])}{\tau(t)}$$

where $\tau(t) = \tau(t_1, t_2, ...)$ and $\tau(t - [z^{-1}]) = \tau(t_1 - 1/z, t_2 - 1/(2z^2), t_3 - 1/(3z^3), ...)$. The existence of such a function is not obvious and must be proven. An important step in the proof is establishment of a relation

$$\frac{\hat{w}(t - [z_1^{-1}], z_1)}{\hat{w}(t, z_1)} = \frac{\hat{w}(t - [z_2^{-1}], z_2)}{\hat{w}(t, z_2)}$$

(2.2)
which, in its turn, easily follows from (2.1). We need further a corollary of this equality that can be obtained expanding it in powers of $z_1^{-1}$:

$$\frac{\hat{w}'(t, z_2)}{\hat{w}(t, z_2)} = -w_1(t - [z_2^{-1}]) + w_1(t). \quad (2.3)$$

The relation (2.2) states that $\hat{w}(t - [z_2^{-1}], z_1)/\hat{w}(t, z_1)$ is symmetric with respect to $z_1$ and $z_2$ and $(z_2 - z_1)\hat{w}(t - [z_2^{-1}], z_1)/\hat{w}(t, z_1)$ is skew symmetric. We can do a more precise statement.

**Lemma.** A relation

$$(z_2 - z_1)\frac{\hat{w}(t - [z_2^{-1}], z_1)}{\hat{w}(t, z_1)} = -\frac{\hat{w}'(t, z_1)}{\hat{w}(t, z_1)} + \frac{\hat{w}'(t, z_2)}{\hat{w}(t, z_2)} + z_2 - z_1$$

holds.

**Proof.** As it is easy to check, this is nothing but the differential Fay identity (see, e.g., [3])

$$\partial \tau(t - [z_2^{-1}])\tau(t - [z_2^{-1}]) - \tau(t - [z_2^{-1}])\partial \tau(t - [z_2^{-1}])$$

$$= (z_2 - z_1)\{\tau(t - [z_2^{-1}])\tau(t - [z_2^{-1}]) - \tau(t - [z_2^{-1}])\tau(t - [z_2^{-1}])\tau(t)\}$$

expressed in terms of $\hat{w}(t, z)$ instead of $\tau(t)$. This identity follows from Sato’s bilinear identity. □

Shifting the argument $t \mapsto t + [z_2^{-1}]$ and taking into account (2.3) one gets

$$(z_1 - z_2)\{\hat{w}(t + [z_2^{-1}], z_1) - \hat{w}(t, z_1)\}$$

$$= -\hat{w}'(t + [z_2^{-1}], z_1) + \{w_1(t + [z_2^{-1}]) - w_1(t)\}\hat{w}(t + [z_2^{-1}], z_1).$$

For simplicity, let us introduce notations: $\hat{w}$ is always $\hat{w}(t, z_1)$, $\mathbf{G}$ will denote the shift $\mathbf{G}\hat{w} = \hat{w}(t + [z_2^{-1}], z_1)$. Thus, the statement of the lemma takes the form

$$(z_1 - z_2)(\mathbf{G}\hat{w} - \hat{w}) = -\mathbf{G}\hat{w}' + (\mathbf{G}w_1 - w_1)\mathbf{G}\hat{w}.$$ 

This equation contains all the equations of the hierarchy. They can be obtained by an expansion of the equation in powers of $z_2^{-1}$ taking into account that $\mathbf{G} = \exp(\sum_{k=1}^{\infty} \partial_k z_2^{-k}/k) = \sum_{n=0}^{\infty} p_n^+ \cdot z_2^{-n}$:

$$z_1\hat{w}' = p_2^+ \hat{w} - \hat{w}'' + w_1' \hat{w}, \quad (2.4)$$

$$z_1p_2^+ \hat{w} = p_3^+ \hat{w} - p_2^+ \hat{w}'+ w_1' \hat{w}' + p_2^+ w_1 \cdot \hat{w}, \quad (2.5)$$

$$z_1p_3^+ \hat{w} = p_4^+ \hat{w} - p_3^+ \hat{w}'+ p_3^+ w_1 \cdot \hat{w} + p_2^+ w_1 \cdot \hat{w}' + w_1' \cdot p_3^+ \hat{w} \quad (2.6)$$

and, generally,

$$z_1p_j^+ \hat{w} = p_{j+1}^+ \hat{w} - p_j^+ \hat{w}'+ \sum_{l=0}^{j-1} p_{j-l}^+ w_1 \cdot p_l^+ \hat{w}. \quad (2.7)$$
3. New identities.

Let

\[ G_n = G - \sum_{i=0}^{n-1} p_i^+ z_2^{-i}. \]

In other words, \( G_n \) is the operator \( G \) minus a part of its asymptotic in \( z_2^{-1} \). Let

\[ H = -\partial + G_1 w_1 + \hat{w} \cdot \text{res}_{z_1}. \]

The statement of the lemma of Sect. 2 can be presented as

\[(z_1 - z_2)G_1 \hat{w} = -G_1 \hat{w}' - \hat{w}' + G_1 w_1 \cdot G_1 \hat{w} + G_1 w_1 \cdot \hat{w}.\]

Taking into account that \( \text{res}_{z_1} G_1 \hat{w} = G_1 w_1 \), we find the following form of the Fay identity,

\[(z_1 - z_2)G_1 \hat{w} = H(G_1 \hat{w}) - \hat{w}'. \tag{3.1} \]

**Proposition 1.** The equation

\[(z_1 - z_2)n^{(k)} = H(n^{(k)}) - \partial k \hat{w} \tag{3.2} \]

has a unique solution

\[ n^{(k)} = \sum_{i=0}^{\infty} n_j^{(k)} z_2^{-j}, \quad n_j^{(k)} = P_j^{(k)} \hat{w} \]

where \( P_j^{(k)} \) are differential operators in \( \partial \) having a form \( \sum f^{\alpha_1, \ldots, \alpha_s} p_{\alpha_1}^+ \cdots p_{\alpha_s}^+ \) being \( \alpha_1, \ldots, \alpha_s > 0, s > 0 \). Coefficients \( f \) do not depend on \( z_1 \).

In the case \( k = 1, 2, \) and 3 the explicit expression for \( n^{(k)} \) are

\[ n^{(1)} = G_1 \hat{w}, \tag{3.3a} \]

\[ n^{(2)} = 2z_2 G_2 \hat{w} - (\partial - G_1 w_1)n_1, \tag{3.3b} \]

\[ n^{(3)} = 3z_2^2 G_3 \hat{w} - (\partial - G_1 w_1)m_2 - z_2 (\partial - G_1 w_1) G_2 \hat{w}. \tag{3.3c} \]

**Proof.** The equation (3.2) is equivalent to a set of recurrence relations

\[ n_j^{(k)} = z_1 n_{j-1}^{(k)} + (n_{j-1}^{(k)})' - \sum_{m=1}^{j-2} p_m^+ w_1 \cdot n_{j-1-m}^{(k)} - n_{j-1,j-1}^{(k)} \hat{w}, \quad n_1^{(k)} = \partial k \hat{w} \tag{3.4} \]

where \( n_{-1,j-1}^{(k)} = \text{res}_{z_1} n_{j-1}^{(k)} \) and the term with a sum is absent when \( j - 2 < 1 \). One can use induction. For \( j = 1 \) the statement is true: one can prove a formula

\[ \partial_k = k \sum_{\alpha_1 + \ldots + \alpha_s = k} (-1)^{s-1} \cdot p_{\alpha_1}^+ \cdots p_{\alpha_s}^+ \]

where the sum runs over all decompositions of \( k \) into sums of positive (all \( \alpha_l > 0 \)) integers.

\[ \sum_{l=1}^{\infty} \partial_{\alpha_1} z_2^{-k}/k = \ln \exp \sum_{l=1}^{\infty} \partial_{\alpha_1} z_2^{-k}/k = \ln G = \ln(1 + G_1) = \sum_{l=1}^{\infty} \frac{(-1)^{s-1} G_1^s}{s}. \]

Expanding in powers of \( z_2^{-1} \), one arrives at the required formula.
Let the statement be true for $n_{j-1}^{(k)}$. One can eliminate $z_1n_{j-1}^{(k)}$ from Eq.(3.4) with the help of the Fay identity (2.7). We find that $n_{j-1}^{(k)}$ has a needed form, however without guarantee that there are no terms where all the $\alpha_i$'s vanish. If this were the case, $n_{j-1}^{(k)}$ would contain terms of zero degree in $z_1^{-1}$. On the other hand, expanding (3.4) in powers of $z_1^{-1}$ and taking the zero degree term, one has: $n_{j-1}^{(k)}|_{z_1} = n_{-1,j-1}^{(k)} - n_{-1,j-1}^{(k)} = 0$. This proves the proposition for all the $j$'s and $k$'s. The uniqueness easily follows from the recurrence formula (3.4).

The additional statement (3.3) for $k = 1$ is just the Fay identity (3.1). The others ($k = 2$ and 3) can be obtained by straightforward but cumbersome calculations, see Appendix. □

4. The universal equation. Let $Q$ be a function. We call the equation

$$f' + 1 - z_2 f + Qf - G_1 w_1 f = 0$$  (4.1)

universal since it plays a fundamental role for all the constrained hierarchies, as we will see below. This equation has exactly one solution in the form of a series $f_0 = \sum_{j=1}^{\infty} f_j z_2^{-j}$. The coefficients $f_j$ are functions. It is easy to see that $f_1 = 1$, $f_2 = Q$, $f_3 = Q' + Q^2 - w_1'$ etc.

5. $k$-constraint. The $k$-constraint is $L_+^k = q\partial^{-1} r$ where $q$ and $r$ satisfy differential equations

$$\partial_n q = L_+^n q, \quad \partial_n r = -L_+^{n+1} r$$  (5.1)

and $L_+^{n+1}$ is a formal adjoint operator. It is well-known that this constraint is compatible with the hierarchy equations. The equation for $\hat{w}(t, \partial)$: $\partial_n \hat{w}(t, \partial) = -L_+^n \hat{w}(t, \partial)$ transforms for $n = k$ to

$$\partial_k \hat{w}(t, \partial) = -q \partial^{-1} r \hat{w}(t, \partial).$$  (5.2)

Dividing the last equation by $q$, and multiplying on the left by $\partial$, one gets

$$\partial \circ \partial_k (\hat{w}(t, \partial)) - \frac{q'}{q} \partial_k (\hat{w}(t, \partial)) = -qr \hat{w}(t, \partial).$$

The parentheses in the l.h.s. mean that the operator $\partial_k$ acts on $\hat{w}(t, \partial)$, $q' = \partial(q)$. Rewriting this as

$$\partial_k (\hat{w}(t, \partial)) \partial + \partial_k (\hat{w}'(t, \partial)) - \frac{q'}{q} \partial_k \hat{w}(t, \partial) = -qr \hat{w}(t, \partial),$$

one obtains for $\hat{w}(t, z_1)$ the following:

$$z_1 \partial_k (\hat{w}(t, z_1)) + \partial_k (\hat{w}'(t, z_1)) - \frac{q'}{q} \partial_k \hat{w}(t, z_1) = -qr \hat{w}(t, z_1).$$

Finally, let $Q = q'/q$ and $R = qr$, then

$$\partial_k (\hat{w}'(t, z_1)) + (z_1 - Q) \partial_k (\hat{w}(t, z_1)) + R \hat{w}(t, z_1) = 0.$$  (5.3)

The first term of expansion in powers of $z_1^{-1}$ yields $R = -\partial_k w_1$.

6. Conditions that $\hat{w}$ belongs to a constrained hierarchy. Suppose one wants to find necessary and sufficient conditions that a Baker function $\psi(t, z_1) = \hat{w}(t, z_1) \exp \xi(t, z_1)$
of the KP hierarchy belongs to the $k$-constrained subhierarchy. (With some abuse of terminology, we also call $\hat{w}$ a Baker function). This problem can be solved easily. If $\hat{w}$ belongs to the constrained hierarchy then Eq. (5.3) holds and
\[
\frac{\partial_k \hat{w}' + z_1 \partial_k \hat{w} - \partial_k w_1 \cdot \hat{w}}{\partial_k \hat{w}} = Q, \tag{6.1}
\]
and therefore the left-hand side of this equation is independent of $z_1$. Conversely, let the expression (6.1) where $\hat{w}$ is a Baker function of the KP hierarchy not depend on $z_1$. Then letting $R = -\partial_k w_1$ and solving the equations $Q = q'/q$ and $R = qr$ with respect to $q$ and $r$, one easily obtains (5.2). It is known (see [2]) that this implies that $q$ and $r$ satisfy Eq. (5.1), i.e., $\hat{w}$ belongs to the $k$-constrained hierarchy. Thus, the following criterion holds:

**Proposition 2.** A Baker function of the KP hierarchy belongs to the $k$-constrained hierarchy if and only if the expression (6.1) is independent of $z_1$.

The term with $z_1$ in (6.1) can be eliminated with the aid of (2.4-7). E.g., for $k = 1$ using (2.4) we have
\[
\partial_1 \hat{w}' + z_1 \partial_1 \hat{w} - w_1' \hat{w} = \hat{w}'' + z_1 \hat{w}' - w_1' \hat{w}' = p_2^+ \hat{w}
\]
and the condition reads: $p_2^+ \hat{w}/\hat{w}'$ must be independent of $z_1$. The same also can be expressed as
\[
\left| \begin{array}{cc}
w_{i_1}' & w_{i_2}' \\
p_2^+ w_{i_1} & p_2^+ w_{i_2}
\end{array} \right| = 0 \tag{6.2}
\]
where $i_1$ and $i_2$ are arbitrary.

In the case $k = 2$ we have
\[
\partial_3 \hat{w} - z_1 \partial_2 \hat{w} = p_3^+ \hat{w} - w_1' \hat{w}' - \partial_2 w_1 \cdot \hat{w}
\]
Then the expression (6.1) takes the form, using (2.5),
\[
Q = \frac{\partial_2 \hat{w}' + \partial_3 \hat{w} - p_3^+ \hat{w} - w_1' \hat{w}'}{\partial_2 \hat{w}} = 2p_3^+ \hat{w} - 2p_2^+ \hat{w}' + w_1' \hat{w}'
\]
We also have used here the known explicit expressions for Schur polynomials:
\[
p_2^+ = \frac{1}{2}(\partial^2 + \partial), \quad p_3^+ = \frac{1}{3}\partial_3 + p_2^+ \partial - \frac{1}{3}\partial^3.
\]
Independence of $z_1$ of the expression $Q$ is equivalent to
\[
\left| \begin{array}{cc}
\partial_2 w_{i_1} & \partial_3 w_{i_2} \\
(\partial_3 + \partial_2 \partial - p_3^+ + w_1' \partial)w_{i_1} & (\partial_3 + \partial_2 \partial - p_3^+ + w_1' \partial)w_{i_2}
\end{array} \right| = 0
\]
or
\[
\left| \begin{array}{cc}
(2p_2^+ - \partial^2)w_{i_1} & (2p_2^+ - \partial^2)w_{i_2} \\
(2p_3^+ \hat{w} - p_2^+ \hat{w}' + w_1' \hat{w}')w_{i_1} & (2p_3^+ \hat{w} - p_2^+ \hat{w}' + w_1' \hat{w}')w_{i_2}
\end{array} \right| = 0. \tag{6.3}
\]
For $k = 3$, as it can be shown, the condition is
where

\[ P_{1}^{(3)} = 3p_{3}^+ - 3p_{2}^+ \partial + \partial^3, \quad P_{2}^{(3)} = 3p_{1}^+ - 3p_{3}^+ \partial + p_{2}^+ \partial^2 + 3w_{1}p_{2}^+ - 2w_{1}' \partial^2 - w_{1}'' \partial. \]

All this is simple and straightforward and would not deserve a special discussion if not the following generalization. As it will be shown in the next sections, infinite sequences of determinant identities can be written in each of cases \( k = 1, k = 2 \) and \( k = 3 \) such that (6.2), (6.3) and (6.4) are but first terms of them. A similar statement is also true for any \( k \).

7. Main results about constrained hierarchies. Here we make a statement and derive some corollaries of it. In the next section we give a proof.

**Proposition 3.** If \( n^{(k)} \) are those defined in Sect.3 then for a \( k \)-constrained hierarchy we have

\[ \frac{n^{(k)}}{\partial_{k}w} = f_{Q}, \quad k = 1, 2, 3, \ldots \tag{7.1} \]

Here \( f_{Q} \) is the solution of the universal equation, Sect.4, and \( Q = q'/q \). Hence the left-hand side of the equation (7.1) does not depend on \( z_{1} \).

What we have done actually, is that we constructed some expressions having the asymptotic \( z_{2}^{-1} + Qz_{2}^{-2} + \ldots \) with a property: if the first nontrivial term \( Q \) does not depend on \( z_{1} \) then neither does the whole series.

The numerators of these expressions can be expanded in \( z_{2}^{-1} \):

\[ n^{(k)} = \sum_{j=1}^{\infty} P_{j}^{(k)} \hat{w} \cdot z_{2}^{-j} \]

where \( P_{j}^{(k)} \) are differential operators. Namely,

\[ P_{j}^{(1)} = p_{j}^+, \quad P_{j}^{(2)} = \sum_{l=1}^{j-1} p_{l}^+ w_{1} \cdot p_{j-l}^+ - p_{j}^+ \partial + 2p_{j+1}^+ \]

and

\[ P_{j}^{(3)} = \sum_{\alpha, \beta, \gamma > 0, \alpha + \beta + \gamma = j} p_{\alpha}^+ w_{1} \cdot p_{\beta}^+ w_{1} \cdot p_{\gamma}^+ - \sum_{\alpha=1}^{j-1} (2p_{\alpha}^+ w_{1} \cdot p_{j-\alpha}^+ \partial + p_{\alpha}^+ w_{1}' \cdot p_{j-\alpha}^+) + 3 \sum_{l=1}^{j-1} p_{l}^+ w_{1} \cdot p_{j+1-l}^+ \partial + 3p_{j+2}^+. \]

**Corollary 4.** The equalities

\[ P_{j}^{(k)} w_{i} = f_{j} \cdot \partial_{k}w_{i}, \quad j \geq 1 \]
Corollary 5. There are identities
\[ \begin{vmatrix} p_{j_1}^{(k)} w_{i_1} & p_{j_2}^{(k)} w_{i_2} \\ p_{j_1}^{(k)} w_{i_1} & p_{j_2}^{(k)} w_{i_2} \end{vmatrix} = 0 \]
where \( i_1, i_2, j_1 \) and \( j_2 \) are arbitrary \( \geq 1 \).

The equalities (6.2-4) are the first of these identities, with the least possible \( j_1 \) and \( j_2 \).

8. Proof of the proposition 3. First we prove that the left-hand side of Eq.(7.1) satisfies a slightly different equation
\[ f' = -1 + z_2 f + G_1 w_1 \cdot f - Q f - \frac{\partial_k w_1}{\partial_k \hat{w}} (f - f_0) \tag{8.1} \]
where \( f_0 \) is the limiting value of \( f \) when \( z_1 \to \infty \): \( f_0 = n^{(k)}_{-1}/\partial_k w_1 \), where \( n^{(k)}_{-1} = \text{res}_{z_1} n^{(k)} \). If we manage to prove this, then, passing to the limit \( z_1 \to \infty \) in the equation (8.1), we get for \( f_0 \):
\[ f'_0 = -1 + z_2 f_0 + G_1 w_1 f_0 - Q f_0. \]
Subtracting this equation from Eq.(8.1), we obtain for \( g = f - f_0 \):
\[ g' = z_2 g + G_1 w_1 \cdot g - Q g - \frac{\partial_k w_1}{\partial_k \hat{w}} g. \]
This is a homogeneous equation, \( g \) is a double series in \( z_1^{-1} \) and \( z_2^{-1} \). Let \( g = \sum_j g_j z_2^{-j} \) where \( g_j \) is the first non-zero coefficient assuming that it exists. Then expanding the equation for \( g \) in powers of \( z_2^{-1} \) one gets for the coefficient of \( z_2^{-j-r+1} \): \( g_{j+r} = 0 \) in contradiction to the assumption. This means that \( g \equiv 0 \). Then \( f = f_0 \), \( f \) does not depend on \( z_1 \) and satisfies the universal equation, as required. Thus, it suffices to prove that the left-hand side of Eq.(7.1), which will be denoted as \( f \), satisfies Eq.(8.1).

We have \( f = n^{(k)}/\partial_k \hat{w} \). What equation for \( n^{(k)} \) does follow from Eq.(7.1)? Firstly,
\[ f' = \frac{(n^{(k)})'}{\partial_k \hat{w}} - f \cdot \frac{\partial_k \hat{w}'}{\partial_k \hat{w}}. \]
Eliminating \( \partial_k \hat{w}' \) with the help of the constraint equation (5.3) one obtains
\[ f' = \frac{(n^{(k)})'}{\partial_k \hat{w}} + (z_1 - Q)f - \frac{\partial_k w_1}{\partial_k \hat{w}} f. \]
Substituting this expression for \( f' \) in (8.1) one gets the equation for numerators
\[ (z_1 - z_2) n^{(k)} = -(n^{(k)})' + G_1 w_1 \cdot n_k + n^{(k)}_{-1} \hat{w} - \partial_k \hat{w}. \tag{8.2} \]
Here \( n^{(k)}_{-1} = f_0 \cdot \partial_k w_1 = \text{res}_{z_1} n^{(k)} \). This equation is nothing less than the equation of the proposition 1, and \( n^{(k)} \)'s are its solutions. \( \square \)
Acknowledgement. We are thankful to the Mathematisches Forschungsinstitut Oberwolfach whose RiP program helped us to accomplish this work.

Appendix. Proof of the Proposition 1 for \( k = 2 \) and 3.

Recall the definition of the operator \( H \):

\[
H(v) = - (\partial - G_1 w_1) v + v_{-1} \hat{w} = -v' + G_1 w_1 \cdot v + v_{-1} \hat{w},
\]

here

\[
v_{-1} = \text{res}_{z_1}(v).
\]

Thus, for instance,

\[
\begin{align*}
n_{-1}^{(1)} &= G_1 w_1, \\
n_{-1}^{(2)} &= 2z_2 G_2 w_1 - (\partial - G_1 w_1)n_{-1}^{(1)}, \\
n_{-1}^{(3)} &= 3z_2^2 G_3 w_1 - (\partial - G_1 w_1)n_{-1}^{(2)} - z_2(\partial - G_1 w_1)G_2 w_1,
\end{align*}
\]

We prepare a few properties of \( H \). For any function \( v \)

\[
H(G_1 w_1 \cdot v) = G_1 w_1 \cdot H(v) - G_1 w_1' \cdot v
\]

and

\[
-(\partial - G_1 w_1)H(v) = H(-(\partial - G_1 w_1)v) - v_{-1} \hat{w}'.
\]

The following lemmata will be frequently needed.

**Lemma 1.** Let

\[
(z_1 - z_2)v = H(v) - u
\]

then

\[
(z_1 - z_2)(-(\partial - G_1 w_1)v) = H(-(\partial - G_1 w_1)v) - v_{-1} \hat{w}' + (\partial - G_1 w_1)u.
\]

This is obvious.

**Lemma 2.** The following holds:

\[
\begin{align*}
(z_1 - z_2)G_1 \hat{w} &= H(G_1 \hat{w}) - p_1^+ \hat{w}, \\ 
(z_1 - z_2)z_2 G_2 \hat{w} &= H(z_2 G_2 \hat{w}) + (-p_2^+ \hat{w} + G_1 w_1 \cdot p_1^+ \hat{w}), \\ 
(z_1 - z_2)z_2^2 G_3 \hat{w} &= H(z_2^2 G_3 \hat{w}) + z_2(G_1 w_1 \cdot p_1^+ \hat{w}) \\
&+ (-p_3^+ \hat{w} + G_1 w_1 \cdot p_2^+ \hat{w} - w'_1 p_1^+ \hat{w}),
\end{align*}
\]

**Proof.** One must replace \( G_2 \hat{w} \) by \( G_1 \hat{w} - z_2^{-1} \hat{w}' \) and \( G_3 \hat{w} \) by \( G_1 \hat{w} - z_2^{-1} \hat{w}' - z_2^{-2} p_2^+ \hat{w} \), then apply equations (3.1) and (2.4-6). Finally, one must go back to \( G_2 \hat{w} \) and \( G_3 \hat{w} \). The calculations are not difficult. \( \square \)
Now we are proving the proposition. In the case \( k = 1 \) there is nothing to prove, the required equation coincides with the Fay identity (3.1).

In the case \( k = 2 \) Lemma 2 yields

\[
(z_1 - z_2)2z_2G_2\hat{w} = 2H(z_2G_2\hat{w}) + 2(-p^+_2\hat{w} + 2G_1w_1 \cdot p^+_1\hat{w}) .
\]

Lemma 1, starting from (3.1), implies

\[
(z_1 - z_2)(-\partial - G_1w_1)n^{(1)} = H(\partial - G_1w_1)n^{(1)}
\]

\[
- n^{(1)}_1\hat{w}' + (\partial - G_1w_1)\hat{w}'
\]

\[
= H(\partial - G_1w_1)n^{(1)} - 2G_1w_1\hat{w}' + \hat{w}''.
\]

A sum of two last equations is exactly what was to prove.

Now, let \( k = 3 \). Applying Lemma 2 again, we have

\[
(z_1 - z_2)3z_2^2G_3\hat{w} = H(3z_2^2G_3\hat{w}) + 3z_2(G_1w_1 \cdot p^+_1\hat{w}) + 3(-p^+_3\hat{w} + G_1w_1 \cdot p^+_2\hat{w} - w'p^+_1\hat{w}),
\]

while Lemma 1 and the equation we have just proven for \( k = 2 \) yield:

\[
(z_1 - z_2)(-\partial - G_1w_1)n^{(2)} = H(\partial - G_1w_1)n^{(2)}
\]

\[
- n^{(2)}_1\hat{w}' + (\partial - G_1w_1)\partial_2\hat{w}.
\]

Finally we use the equation for \( G_2\hat{w} \) and apply Lemma 1 to it:

\[
(z_1 - z_2)z_2(-\partial - G_1w_1)G_2\hat{w} = H(z_2(-\partial - G_1w_1)G_2\hat{w})
\]

\[
- z_2G_2w_1 \cdot \hat{w}'
\]

\[
+ (\partial - G_1w_1)(p^+_2\hat{w} - G_1w_1 \cdot \hat{w}').
\]

On addition of the last three equations we obtain:

\[
(z_1 - z_2)n^{(3)} = H(n^{(3)})
\]

\[
- 3w'_1\hat{w}' + 3z_2(G_1w_1 \cdot \hat{w}' - G_2w_1 \cdot \hat{w}')
\]

\[
- 3p^+_3\hat{w} + \partial\partial_2\hat{w} + \partial p^+_2\hat{w}
\]

\[
+ 2G_1w_1 \cdot p^+_2\hat{w} - G_1w_1 \cdot \partial_2\hat{w} - G_1w_1 \cdot \hat{w}'',
\]

which gives the claim by using obvious identities for the Schur polynomials \( p_2 \) and \( p_3 \) and the equality \( G_1w_1 - G_2w_1 = z_2^{-1}p^+_1w_1 \).

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