1 INTRODUCTION

The classical stochastic control problem under partial information, as, for instance, described in the book of A. Bensoussan [2], can be formulated as a control problem for Zakai equation, whose solution is the unnormalized conditional probability distribution of the state of the system, which is not directly accessible. Zakai equation is a stochastic Fokker-Planck equation. Therefore, the mathematical problem to be solved is very similar to that met in Mean Field Control theory. Since Mean Field Control theory is much posterior to the development of Stochastic Control with partial information, the tools, techniques and concepts obtained in the last decade, for Mean Field Games and Mean field type Control theory, have not been used for the control of Zakai equation. It is the objective of this work to connect the two theories. Not only, we get the power of new tools, but also we get new insights for the problem of stochastic control with partial information. For mean field theory, we get new interesting applications, but also new problems. The possibility of using direct methods is, of course, quite fruitful. Indeed, if Mean Field Control Theory...
is a very comprehensive and powerful framework, it leads to very complex equations, like the Master equation, which is a nonlinear infinite dimensional P.D.E., for which general theorems are hardly available, although an active research in this direction is performed, see P. Cardialaguet, F. Delarue, J.M. Lasry, P.L. Lions [3]. Direct methods are particularly useful to obtain regularity results. We will develop in detail the linear quadratic regulator problem, but because we cannot just consider the gaussian case, well know results, like the separation principle are not available. An interesting and important result is available in the literature, due to A. Makowsky, [4]. It describes the solution of Zakai equation for linear systems with general initial condition (non-gaussian). Curiously, this result had not been exploited for the control aspect, in the literature. We show that the separation principle can be extended for quadratic pay-off functionals, but the Kalman filter is much more complex than in the gaussian case. Finally we compare our work to the work of Bandini, Corso, Fuhrman and Pham [1] and we show that the example E. Bandini et al. provided does not cover ours. Our system remains nonlinear in their setting.

2 STOCHASTIC CONTROL WITH PARTIAL INFORMATION

2.1 THE PROBLEM

We describe the problem formally, without making precise the assumptions. The state of the system \( x(t) \in \mathbb{R}^n \) is solution of a diffusion

\[
dx = g(x,v)dt + \sigma(x)dw
\]

\( x(0) = \xi \)

so, we assume that there exists a probability space \( \Omega, \mathcal{A}, P \) on which are constructed a random variable \( \xi \) and a standard Wiener process in \( \mathbb{R}^n \), which is independent of \( \xi \). There is a control \( v(t) \) in the drift term, with values in \( \mathbb{R}^m \). Since, we cannot access to the state \( x(t) \), which is not observable, it cannot be defined by a feedback on the state, nor adapted to the state. Formally, we have an observation equation

\[
dz = h(x)dt + db(t)
\]

in which \( z(t) \), with values in \( \mathbb{R}^d \), represents the observation and \( b(t) \) is also a Wiener process, independent of the pair \( (\xi, w(.)) \). The function \( h(x) \) corresponds to the measurement of the state \( x \) and \( b(t) \) captures a measurement error. So the control \( v(t) \) should be adapted to the process \( z(t) \), not a feedback of course. It
is well known that this construction is ill-posed. Indeed, the control is adapted to the observation, which depends also on the state, which depends on the control. It is a chicken and egg effect, that is usually solved by the Girsanov theorem, at the price of constructing appropriately the Wiener process \( b(t) \). In practice, we construct on \((\Omega, \mathcal{A}, P)\) three objects, \( \xi, w(\cdot), z(\cdot) \). The processes \( w(\cdot), z(\cdot) \) are independent Wiener processes on \( \mathbb{R}^n, \mathbb{R}^d \) respectively and \( \xi \) is independent of these two processes. We set

\[
\mathcal{F}^t = \sigma(\xi, w(s), z(s), s \leq t) \quad \mathcal{Z}^t = \sigma(z(s), s \leq t)
\]

the filtrations on \((\Omega, \mathcal{A}, P)\) generated by \((\xi, w(\cdot), z(\cdot))\) and \( z(\cdot) \) respectively. The process \( z(\cdot) \) is the observation process, but it is defined externally. We can then choose the control \( v(\cdot) \) as a process with values in \( \mathbb{R}^m \), which is adapted to the filtration \( \mathcal{Z}^t \). So, it is perfectly well defined, as well as the process \( x(\cdot) \) solution of (2.1). In fact in (2.1) \( v(\cdot) \) is fixed, like \( \xi \) and \( w(\cdot) \), and we assume that we can solve the S.D.E. (2.1) in a strong sense. So \( x(\cdot) \) is well defined. Here comes Girsanov theorem. We define the scalar \( P, \mathcal{F}^t \) martingale \( \eta(t) \), solution of the equation

\[
d\eta(t) = \eta(t) h(x(t)).dz(t), \quad \eta(0) = 1 \quad (2.3)
\]

This martingale allows to define a new probability on \( \Omega, \mathcal{A} \), denoted \( P^{v(\cdot)} \) to emphasize the fact that it depends on the control \( v(\cdot) \). It is given by the Radon-Nikodym derivative

\[
\frac{dP^{v(\cdot)}}{dP}|_{\mathcal{F}^t} = \eta(t) \quad (2.4)
\]

Finally, we define the process

\[
b^{v(\cdot)}(t) = z(t) - \int_0^t h(x(s))ds \quad (2.5)
\]

which also depends on the control decision. We take a finite horizon \( T \), to fix ideas. Making the change of probability from \( P \) to \( P^{v(\cdot)} \) and considering the probability space \((\Omega, \mathcal{F}^T, P^{v(\cdot)})\), then \( b^{v(\cdot)} \) appears as a standard Wiener process, which is independent of \( w(\cdot) \) and \( \xi \). Therefore, (2.5) is a template of (2.2) as far as probability laws are concerned. We can then rigorously define the control problem (without the chicken and egg effect)

\[
J(v(\cdot)) = E^{v(\cdot)}[\int_0^T f(x(t), v(t))dt + f_T(x(T))]
\]
in which the functions $f(x,v)$ and $f_T(x)$ represent the running cost and the final cost contributing to the pay off functional to be minimized. The notation $E^{v(.)}$ refers to the expected value with respect to the probability law $P^{v(.)}$.

**Remark 1.** The previous presentation, which is currently the common one to formalize stochastic control problems with partial information, has a slight drawback, in comparison with the description of the problem with full information. With full information, there is no $Z^t$ and the underlying filtration $\mathcal{F}^t = \sigma(\xi, w(.))$ is accessible. A control $v(.)$ is a stochastic process adapted to $\mathcal{F}^t$. We call it open-loop, because it is externally defined (this should not be confused with the practice in engineering to call open-loop controls, those which are deterministic functions of time). But, since the state $x(t)$ is also accessible, we can also consider controls, defined by feedbacks built on the state. In spite of the difference in the definition, the class of feedback controls is contained in that of open-loop controls. Indeed, after constructing the trajectory corresponding to a feedback, we feed the feedback with that trajectory. We get an open-loop control, leading to the same cost. The interesting feature of cost functionals of the type (2.6) is that the optimal open-loop control is defined by a feedback. So restricting ourselves to the subclass of feedback controls does not hurt. This is very important, when we formulate the control problem in the framework of mean-field theory. In mean-field theory, we must define the control with a feedback. Surprisingly, open-loop controls and feedback controls will lead to different solutions. In the case of partial information, we have unfortunately no choice. There is no feedback, since the state is not accessible. With the formulation above, the observation filtration $Z^t$ is externally defined, and the control is open-loop, since it is externally defined as a process adapted to $Z^t$. It is important to have this discussion in mind, when we formulate the problem with mean-field theory.

### 2.2 CONTROL OF ZAKAI EQUATION

Note first that the functional (2.6) can be written as

$$ J(v(.)) = E[\int_0^T \eta(t)f(x(t), v(t))dt + \eta(T)f_T(x(T))] \tag{2.7} $$

This is obtained by using the Radon-Nikodym derivative (2.4) and the martingale property of $\eta(t)$. We next recall the classical nonlinear filtering theory result. Let $\Psi(x)$ be any bounded continuous function. We want to express the conditional expectation $E^{v(.)}[\Psi(x(t))|Z^t]$ of the random variable $\Psi(x(t))$ with respect to the $\sigma$-algebra $Z^t$, on the probability space $\Omega, \mathcal{A}, P^{v(.)}$. We have the basic result of nonlinear filtering theory

$$ E^{v(.)}[\Psi(x(t))|Z^t] = \frac{E[\eta(t)\Psi(x(t))|Z^t]}{E[\eta(t)|Z^t]} = \frac{\int_{\mathbb{R}^n} \Psi(x)q(x, t)dx}{\int_{\mathbb{R}^n} q(x, t)dx} \tag{2.8} $$
where \( q(x,t) \) is called the un-normalized conditional probability density of the random variable \( x(t) \) with respect to the \( \sigma \)-algebra \( \mathcal{Z}^t \). The conditional probability itself is given by \( \frac{q(x,t)}{\int_{\mathbb{R}^n} q(\xi,t) d\xi} \). The function \( q(x,t) \) is a random field adapted to the filtration \( \mathcal{Z}^t \). It is the solution of a stochastic P.D.E.

\[
dq + A^* q(x,t) dt + \text{div} \left( g(x,v(t))q(x,t) \right) dt - q(x,t) h(x).dz(t) = 0 \tag{2.9}
\]

\[ q(x,0) = q_0(x) \]

in which \( A^* \) is the second order differential operator

\[
A^* \varphi(x) = - \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) \varphi(x))
\]

which is the dual of

\[
A \varphi(x) = - \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}
\]

with \( a(x) = \frac{1}{2} \sigma \sigma^*(x) \). The initial condition \( q_0(x) \) is the probability density of \( \xi \). We suppose that \( \xi \) has a probability density. The random field \( q(x,t) \) depends on \( v(.) \) and is thus denoted \( q^{v(\cdot)}(x,t) \). From (2.8) and (2.7) we can write the pay-off \( J(v(\cdot)) \) as

\[
J(v(\cdot)) = E\left[ \int_0^T \int_{\mathbb{R}^n} q^{v(\cdot)}(x,t)f(x,v(t))dxdt + \int_{\mathbb{R}^n} q^{v(\cdot)}(x,T)f_T(x)dx \right] \tag{2.10}
\]

The minimization of \( J(v(\cdot)) \) is a stochastic control problem for a dynamic system whose evolution is governed by the stochastic P.D.E. \( (2.9) \).

Remark 2. We can elaborate more on the difference between feedback controls and open-loop controls, as addressed in Remark 1, by considering equation (2.9) describing the evolution of the state \( q(x,t) \). In this equation \( v(t) \) is a stochastic process adapted to the filtration \( \mathcal{Z}^t \), so it is fixed with respect to the space variable \( x \).

3 MEAN FIELD APPROACH

3.1 PRELIMINARIES

We define the value function
\[ \Phi(q_0, 0) = \inf_{v(\cdot)} J(v(\cdot)) \]  

(3.1)

and following the main concept of Dynamic Programming, we embed this value function into a family parametrized by initial conditions \( q, t \), where \( q \) denotes an unnormalized probability density on \( \mathbb{R}^n \). We also make precise the choice of the functional space in which the function \( q(x) \) lies. To fix ideas, we take \( q \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) and \( q(x) \geq 0 \). We shall assume that

\[ \int_{\mathbb{R}^n} |x|^2 q(x) dx < +\infty \]  

(3.2)

Considering functionals on \( L^2(\mathbb{R}^n) \), \( \Psi(q) \), we say that it is Gateaux differentiable, with Gateaux derivative \( \frac{\partial \Psi}{\partial q}(q)(x) \) if the function \( t \to \Psi(q + t\tilde{q}) \) is differentiable with the formula

\[ \frac{d}{dt} \Psi(q + t\tilde{q}) = \int_{\mathbb{R}^n} \frac{\partial \Psi}{\partial q}(q + t\tilde{q})(x) \tilde{q}(x) dx, \forall \tilde{q}(\cdot) \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \tilde{q}(x) \geq 0 \]  

(3.3)

We shall assume that with \( q \times x \to \frac{\partial \Psi}{\partial q}(q)(x) \) is continuous, satisfying

\[ |\frac{\partial \Psi}{\partial q}(q)(x)| \leq c(q)(1 + |x|^2) \]  

(3.4)

such that \( c(q) \) is continuous and bounded on bounded subsets of \( L^2(\mathbb{R}^n) \). We also need the concept of second order Gateaux derivative. The second order Gateaux derivative is a functional \( \frac{\partial^2 \Psi}{\partial q^2}(q)(\xi, \eta) \) such that the function \( t \to \Psi(q + t\tilde{q}) \) is twice differentiable in \( t \) and

\[ \frac{d^2}{dt^2} \Psi(q + t\tilde{q}) = \int_{\mathbb{R}^n} \frac{\partial^2 \Psi}{\partial q^2}(q + t\tilde{q})(\xi, \eta) \tilde{q}(\xi)\tilde{q}(\eta) d\xi d\eta \]  

(3.5)

Moreover, the function \( q, \xi, \eta \to \frac{\partial^2 \Psi}{\partial q^2}(q)(\xi, \eta) \) is continuous satisfying

\[ |\frac{\partial^2 \Psi}{\partial q^2}(q)(\xi, \eta)| \leq c(q)(1 + |\xi|^2 + |\eta|^2) \]  

(3.6)

with \( c(q) \) continuous bounded on bounded subsets of \( L^2(\mathbb{R}^n) \). From formula (3.5), it is clear that we can choose \( \frac{\partial^2 \Psi}{\partial q^2}(q)(\xi, \eta) \) to be symmetric in \( \xi, \eta \). Set

\[ f(t) = \Psi(q + t\tilde{q}) \]

Then, combining the above assumptions, we can assert that \( f(t) \) is \( C^2 \). Therefore, we have the identity
\[
f(1) = f(0) + f'(0) + \int_0^1 \int_0^1 t f''(st) ds dt
\]

which leads to the formula

\[
\Psi(q + \hat{q}) = \Psi(q) + \int_{R^n} \frac{\partial \Psi}{\partial q}(q(x)) \hat{q}(x) dx + \int_0^1 \int_{R^n} \frac{\partial^2 \Psi}{\partial q^2}(q + st\hat{q})(\xi, \eta) \hat{q}(\xi) \hat{q}(\eta) d\xi d\eta
\]  

(3.7)

### 3.2 Bellman Equation

We consider the control problem with initial conditions \( q, t \)

\[
dq + A^*q(x, s) ds + \text{div} \ (g(x, v(s))q(x, s)) ds - q(x, s) h(x). dz(s) = 0, \ s > t
\]

(3.8)

\[
q(x, t) = q(x)
\]

\[
J_{q,t}(v(.)) = E\left[ \int_t^T \int_{R^n} q^{(1)}(x, s) f(x, v(s)) dx ds + \int_{R^n} q^{(1)}(x, T) f_T(x) dx \right]
\]

(3.9)

and define the value function

\[
\Phi(q, t) = \inf_{v(.)} J_{q,t}(v(.))
\]

(3.10)

Assuming that the value function has derivatives

\[
\frac{\partial \Phi}{\partial t}(q, t), \frac{\partial \Psi}{\partial q}(q(x), \frac{\partial^2 \Psi}{\partial q^2}(q)(\xi, \eta)
\]

then, by standard arguments, we can check formally that \( \Phi(q, t) \) is solution of the Bellman equation

\[
\frac{\partial \Phi}{\partial t} - \int_{R^n} A \frac{\partial \Phi}{\partial q}(q, t)(x) q(x) dx +
\]

\[
+ \frac{1}{2} \int_{R^n} \int_{R^n} \frac{\partial^2 \Phi}{\partial q^2}(q, t)(\xi, \eta) q(\xi) q(\eta) h(\xi). h(\eta) d\xi d\eta +
\]

\[
+ \inf_v \int_{R^n} q(x) \left( f(x, v) + D_x \frac{\partial \Phi}{\partial q}(q, t)(x), g(x, v) \right) dx = 0
\]

\[
\Phi(q, T) = \int_{R^n} f_T(x) q(x) dx
\]
The optimal open-loop control is obtained by achieving the infimum in (3.11). We derive a functional \( \hat{v}(q,t) \), which is a feedback in \( q \) but not in \( x \). We can then feed the Zakai equation (3.8) with this feedback to get the optimal state equation

\[
\begin{align*}
 dq + A^* q(x,s) ds + \text{div} \left( g(x, \hat{v}(q,s)) q(x,s) \right) ds - q(x,s) h(x). dz(s) &= 0, \ s > t \\
 q(x,t) &= q(x)
\end{align*}
\]  

(3.12)

Once we solve this stochastic P.D.E. we obtain the optimal state \( \hat{q}(s) := \hat{q}(x,s) \). We then define the control \( \hat{v}(s) = \hat{v}(\hat{q}(s),s) \), which is indeed adapted to the filtration \( \mathcal{F}^x_s = \sigma(z(\tau) - z(t), \ t \leq \tau \leq s) \). This is the optimal open-loop control.

### 3.3 THE MASTER EQUATION

The functional \( \hat{v}(q,t) \) defined above depends on the function \( \frac{\partial \Phi}{\partial q}(q,t)(x) \) denoted \( U(x,q,t) \). So, it is convenient to denote by \( \hat{v}(q,U) \) the vector \( v \) which achieves the minimum of

\[
\inf_v \int_{\mathbb{R}^n} q(x) \left( f(x,v) + D_x U(x,q).g(x,v) \right) dx
\]  

(3.13)

in which we omit to write explicitly the argument \( t \). Bellman equation (3.11) can be written as

\[
\begin{align*}
 \frac{\partial \Phi}{\partial t} - \int_{\mathbb{R}^n} A_x U(x,q) q(x) dx + \\
 \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\partial^2 \Phi}{\partial q^2}(q,t)(\xi,\eta)q(\xi)q(\eta) h(\xi).h(\eta) d\xi d\eta + \\
 + \int_{\mathbb{R}^n} q(x) \left( f(x,\hat{v}(q,U)) + D_x U(x,q).g(x,\hat{v}(q,U)) \right) dx &= 0
\end{align*}
\]  

(3.14)

It is also convenient to set

\[
V(q,t)(x,y) = \frac{\partial^2 \Phi}{\partial q^2}(q,t)(x,y)
\]  

(3.15)

Therefore, Bellman equation reads

\[
\begin{align*}
 \frac{\partial \Phi}{\partial t} - \int_{\mathbb{R}^n} A_x U(x,q) q(x) dx + \\
 \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\partial^2 \Phi}{\partial q^2}(q,t)(\xi,\eta)q(\xi)q(\eta) h(\xi).h(\eta) d\xi d\eta + \\
 + \int_{\mathbb{R}^n} q(x) \left( f(x,\hat{v}(q,U)) + D_x U(x,q).g(x,\hat{v}(q,U)) \right) dx &= 0
\end{align*}
\]  

(3.16)
$$\Phi(q,T) = \int_{\mathbb{R}^n} f_T(x)q(x)dx$$

The Master equation is an equation for $U(x,q,t)$. It is obtained by differentiating (3.16) with respect to $q$. We obtain, formally

$$\frac{\partial U}{\partial t} - A_x U - \int_{\mathbb{R}^n} A_\xi V(q,t)(x,\xi)q(\xi)d\xi +$$

$$+ h(x).\int_{\mathbb{R}^n} V(q,t)(x,\xi)h(\xi)q(\xi)d\xi + \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\partial V}{\partial q}(q,t)(\xi,\eta)(x)h(\xi),h(\eta)q(\xi)q(\eta)d\xi d\eta +$$

$$+ f(x,\hat{v}(q,U)) + D_x U.g(x,\hat{v}(q,U)) + \int_{\mathbb{R}^n} D_\xi V(q,t)(\xi,x)g(\xi,\hat{v}(q,U))q(\xi)d\xi = 0$$

$$U(x,q,T) = f_T(x)$$

Note that

$$\frac{\partial V}{\partial q}(q,t)(\xi,\eta)(x) = \frac{\partial^3 \Phi}{\partial q^3}(q,t)(x,\xi,\eta)$$

which is symmetric in the arguments $(x,\xi,\eta)$

### 3.4 SYSTEM OF HJB-FP EQUATIONS

In Mean field theory approach, the Master equation is the key equation. However, it is an infinite-dimensional nonlinear P.D.E. Direct approaches are very limited. The most convenient approach is to use ideas similar to the classical method of characteristics. This amounts to solving a system of forward-backward finite dimensional stochastic P.D.E. Since it is forward-backward the initial conditions matter. We shall consider that the initial time is 0, for convenience. It should be any time $t \in [0,T]$. This system is called Hamilton-Jacobi-Bellman for the backward equation and Fokker-Planck for the forward one. The Fokker-Planck equation is the Zakai equation in which we insert the optimal feedback $\hat{v}(q,U)$. So we get

$$dq + A^*q(x,t)dt + \text{div} \ (g(x,\hat{v}(q,U))q(x,t)) \ dt - q(x,t)h(x).dz(t) = 0$$

(3.19)
The functional $U(x,q,t)$ used in (3.19) is the functional solution of the master equation (3.17). We call simply $q(t)$ the solution of (3.19). We then set

$$u(x,t) = U(x,q(t),t)$$ (3.20)

We use the notation $\hat{v}(q_t, u_t)$ to represent the functional $\hat{v}(q,U)$ in which the arguments $q,U$ are replaced by $q(.,t)$ and $U(.,q(.,t),t) = u(.,t)$. We note $q_t = q(.,t), u_t = u(.,t)$ to simplify. The functional $\hat{v}(q_t,u_t)$ achieves the infimum of

$$\hat{v}(q_t,u_t) = \operatorname{Arg min}_v \int_{\mathbb{R}^n} q(x,t) (f(x,v) + D_xu(x,t)g(x,v)) \, dx$$ (3.21)

The next step is to obtain the equation for $u(x,t)$. It is a long and tedious calculation, obtained in taking the Ito differential of the random field defined by (3.20). We give the result as follows

$$-\, du + (Au - Du.g(x,\hat{v}(q_t,u_t)))dt = f(x,\hat{v}(q_t,u_t)) - K(x,t).(dz(t) - h(x)dt)$$ (3.22)

$$u(x,T) = f_T(x)$$

where $K(x,t)$ is defined by the formula

$$K(x,t) = \int_{\mathbb{R}^n} V(q_t,t)(x,\xi)h(\xi)q(\xi,t) \, d\xi$$ (3.23)

In fact, we do not need to compute $K(x,t)$ by formula (3.23), which would require the knowledge of $V(q_t,t)(x,\xi)$, thus solving the master equation. From the theory of backward stochastic P.D.E, the random field $K(x,t)$ is required by the condition of adaptativity of $u(x,t)$. So the solution of (3.22) is not just $u(x,t)$ but the pair $(u(x,t),K(x,t))$, and we can expect uniqueness. The equation (3.22) is the HJB equation. It must be coupled with the FP equation (3.19) written as follows

$$dq + A^*q(x,t)dt + \operatorname{div} (g(x,\hat{v}(q_t,u_t))q(x,t)) \, dt - q(x,t)h(x) \, dz(t) = 0$$ (3.24)

$$q(x,0) = q(x)$$
recalling also (3.21). So the pair (3.22), (3.24) is the pair of HJB-FP equations. Since $q(x,0) = q(x)$ we can assert

$$u(x,0) = U(x,q,0) \quad (3.25)$$

Therefore we can compute $U(x,q,0)$ by solving the system of HJB-FP equations and using formula (3.25). Of course $u(x,t) \neq U(x,q,t)$. To compute $U(x,q,t)$ we have to write the system (3.22), (3.24) on the interval $(t,T)$ instead of $(0,T)$. In that sense, the system of HJB-FP equations (3.22), (3.24) is a method of characteristics to solve the master equation (3.17). Besides the optimal optimal feedback $\hat{v}(q,U(.,q,t),t)$ can be derived from the system of HJB-FP equations. Indeed,

$$\hat{v}(q,U(.,q,0),0) = \hat{v}(q,u(.,0))$$

and setting the initial condition of the system of HJB-FP equations at $t$ instead of 0 yields $\hat{v}(q,U(.,q,t),t)$. To compute the value function, we have to rely on Bellman equation (3.16). Let us compute $\frac{\partial \Phi}{\partial t}(q,0)$, by using (3.16). The only term which is not known is $\int_{R^n} \int_{R^n} V(q,0)(\xi,\eta)q(\xi)q(\eta)h(\xi).h(\eta)d\xi d\eta$. However from (3.26) we can write

$$\int_{R^n} \int_{R^n} V(q,0)(\xi,\eta)q(\xi)q(\eta)h(\xi).h(\eta)d\xi d\eta = \int_{R^n} h(\xi).K(\xi,0)q(\xi)d\xi \quad (3.26)$$

Collecting results we can write the formula

$$\frac{\partial \Phi}{\partial t}(q,0) = \int_{R^n} A_x u(x,0)q(x)dx - \frac{1}{2} \int_{R^n} h(x).K(x,0)q(x)dx \quad (3.27)$$

$$- \int_{R^n} q(x) (f(x,\hat{v}(q,u(.,0))) + D_xu(x,0).g(x,\hat{v}(q,u(.,0)))) dx$$

In a similar way we can define $\frac{\partial \Phi}{\partial t}(q,t)$ for any $t$ and any $q$. Since we know $\Phi(q,T)$ we obtain $\Phi(q,t)$ for any $t$. So solving the system of HJB-FP equations provides all the information on the value function and on the optimal feedback.

4 WEAK FORMULATION OF ZAKAI EQUATION

4.1 WEAK FORMULATION AND LINEAR DYNAMICS

In this section, we consider Zakai equation as follows
\[
dq + A^*q(x,t)dt + \text{div} \left( g(x, v(t))q(x,t) \right) dt - q(x,t) h(x).dz(t) = 0 \quad (4.1)
\]

\[
q(x,0) = q(x)
\]
in which \(v(t)\) is a fixed process adapted to the filtration \(\mathcal{Z}^t = \sigma(z(s), s \leq t)\). If \(\psi(x,t)\) is a deterministic function of \(x, t\) which is \(C^2\) in \(x\) and \(C^1\) in \(x\), we deduce immediately from (4.1), by simple integration by parts that

\[
\int_{\mathbb{R}^n} dq\psi(x,t) = \int_{\mathbb{R}^n} q(x,t)[-A\psi(x,t) + g(x, v(t)).D\psi(x,t)]dt + \int_{\mathbb{R}^n} q(x,t)\psi(x,t)h(x).dz(t)
\]

and thus

\[
\int_{\mathbb{R}^n} q(x,t)\psi(x,t)dx = \int_{\mathbb{R}^n} q(x)\psi(x,0)dx +
\]

\[
\int_0^t \int_{\mathbb{R}^n} q(x,s)(\frac{\partial\psi}{\partial s} - A\psi(x,s) + g(x, v(s)).D\psi(x,s))dxds + \int_0^t \int_{\mathbb{R}^n} q(x,s)\psi(x,s)h(x).dz(s)
\]

which is the weak formulation of Zakai equation. Note that the formulation (4.1) (strong form) and the weak form (4.2) are not equivalent. We may have a weak solution and not a strong solution.

### 4.2 LINEAR SYSTEM AND LINEAR OBSERVATION

We want to solve Zakai equation in the following case

\[
g(x, v) = Fx + Gv, \quad \sigma(x) = \sigma \quad (4.3)
\]

\[
h(x) = Hx
\]

In general, this case is associated to an initial probability \(q(x)\), which is gaussian. In our approach, we cannot take a special \(q(x)\). It must remain general, because it is an argument of the value function and of the solution of the master equation. When we solve the system of HJB-FP equations, we can take \(q(x)\) gaussian, but then we cannot use this method to obtain the solution of the master equation or of Bellman equation.

For a given control \(v(t)\) which is a process adapted to \(\mathcal{Z}^t\), Zakai equation reads

\[
dq - \text{tra}D_x^2q(x,t) dt + \text{div} \left( Fx + Gv(t))q(x,t) \right) dt - q(x,t) Hx.dz(t) = 0 \quad (4.4)
\]
\[ q(x, 0) = q(x) \]

where \( a = \frac{1}{2} \sigma \). Makowsky [4] has shown that this equation has an explicit solution, that we describe now, in a weak form. We first need some notation. We introduce the matrix \( \Sigma(t) \) solution of the Riccati equation

\[
\frac{d\Sigma}{dt} + \Sigma(t)H^*H\Sigma(t) - F\Sigma(t) - \Sigma(t)F^* = 2a \tag{4.5}
\]

\[ \Sigma(0) = 0 \]

We then define the matrix \( \Phi(t) \) solution of the differential equation

\[
\frac{d\Phi}{dt} = (F - \Sigma(t)H^*H)\Phi(t) \tag{4.6}
\]

\[ \Phi(0) = I \]

and

\[
S(t) = \int_0^t \Phi^*(s)H^*H\Phi(s)ds \tag{4.7}
\]

We then introduce stochastic processes \( \beta(t) \) and \( \rho(t) \) adapted to the filtration \( \mathcal{F}_t \), defined by the equations

\[
d\beta(t) = (F\beta(t) + Gv(t))dt + \Sigma(t)H^*(dz - H\beta(t)dt) \tag{4.8}
\]

\[ \beta(0) = 0 \]

\[
d\rho(t) = \Phi^*(t)H^*(dz(t) - H\beta(t)dt) \tag{4.9}
\]

\[ \rho(0) = 0 \]

The process \( \beta(t) \) is the Kalman filter for the linear system [4,3] with a deterministic initial condition, equal to 0. If we set
we obtain the Kalman filter for the same linear dynamic system, with intial condition \( x \). It satisfies the equation

\[
d_t m(x, t) = (Fm(x, t) + Gv(t))dt + \Sigma(t)H^* (dz - Hm(x, t)dt)
\]

\[
m(x, 0) = x
\]

Finally we introduce the martingale \( \theta(x, t) \) defined by

\[
d_t \theta(x, t) = \theta(x, t) Hm(x, t).dz(t)
\]

\[
\theta(x, 0) = 1
\]

whose solution is the exponential

\[
\theta(x, t) = \exp \left( \int_0^t Hm(x, s).dz(s) - \frac{1}{2} \int_0^t |Hm(x, s)|^2 ds \right)
\]

4.3 FORMULAS

We can state the following result, due to A. Makowsky \[4\], whose proof can be found in \[2\]

**Proposition 3.** For any test function \( \psi(x, t) \), we have

\[
\int_{R^n} q(x, t)\psi(x, t)dx = \int_{R^n} \theta(x, t) \left( \int_{R^n} \psi(m(x, t) + \Sigma(t)\frac{1}{2}\xi, t) \frac{\exp - |\xi|^2}{2}d\xi \right) q(x)dx
\]

**Proof.** Equality \[(4.14)\] is true for \( t = 0 \). Let us set

\[
\mathcal{L}\psi(x, t) = \frac{\partial\psi}{\partial s} - A\psi(x, t) + g(x, v(t)).D\psi(x, t)
\]

According to \[(4.12)\] it is thus sufficient to show that

\[
d_t \theta(x, t) \left( \int_{R^n} \psi(m(x, t) + \Sigma(t)\frac{1}{2}\xi, t) \frac{\exp - |\xi|^2}{2}d\xi \right) = \theta(x, t) \int_{R^n} \mathcal{L}\psi(m(x, t) + \Sigma(t)\frac{1}{2}\xi, t) \exp - \frac{|\xi|^2}{2}d\xi dt +
\]
\[ + \theta(x, t) \int_{\mathbb{R}^n} \psi(m(x, t) + \Sigma(t)^{1/2} \xi, t) H(m(x, t) + \Sigma(t)^{1/2} \xi) \exp \left( - \frac{|\xi|^2}{2} \right) d\xi \, dz(t) \]

This is done through a tedious calculation, whose details can be found in [2]. □

We shall derive from (4.14) a more analytic formula. We first set

\[ \nu(t) = \int_{\mathbb{R}^n} q(x, t) \, dx = \int_{\mathbb{R}^n} \theta(x, t) q(x) \, dx \] (4.15)

Hence from (4.12)

\[ d\nu(t) = \int_{\mathbb{R}^n} \theta(x, t) Hm(x, t) q(x) \, dx \, dz(t) \]

But from (4.14) we see that

\[ \int_{\mathbb{R}^n} q(x, t) x \, dx = \int_{\mathbb{R}^n} \theta(x, t) m(x, t) q(x) \, dx \] (4.16)

Therefore

\[ d\nu(t) = H \int_{\mathbb{R}^n} q(x, t) x \, dx \, dz(t) = \nu(t) H\hat{x}(t) \, dz(t) \] (4.17)

where we have set

\[ \hat{x}(t) = \frac{\int_{\mathbb{R}^n} q(x, t) x \, dx}{\int_{\mathbb{R}^n} q(x, t) \, dx} \] (4.18)

Referring to (2.8) we see that

\[ \hat{x}(t) = E^{v(t)}[x(t)|Z^t] \]

the conditional mean of the process \( x(t) \) defined by \( dx = (Fx(t) + Gv(t)) \, dt + \sigma \, dw \) (4.19)

\[ x(0) = \xi \]

with respect to the filtration \( Z^t \) on the probability space \( (\Omega, \mathcal{A}, P^v(\cdot)) \). It is thus the Kalman filter in this probabilistic set up. We shall derive the form of its evolution in the sequel. Now, from (4.17) we can assert that
\[
\nu(t) = \exp\left\{ \int_0^t H \hat{x}(s).dz(s) - \frac{1}{2} \int_0^t |H \hat{x}(s)|^2 ds \right\} \int_{R^n} q(x)dx
\]

(4.20)

recalling that, see (4.15), \( \nu(0) = \int_{R^n} q(x)dx \). Next, from (4.13) and (4.10), we have

\[
\theta(x, t) = \exp\left( \int_0^t H(\Phi(s)x + \beta(s)).dz(s) - \frac{1}{2} \int_0^t |H(\Phi(s)x + \beta(s)).|^2 ds \right)
\]

\[= \gamma(t) \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho(t)) \]

with

\[
\gamma(t) = \exp\left\{ \int_0^t H \beta(s).dz(s) - \frac{1}{2} \int_0^t |H \beta(s)|^2 ds \right\}
\]

and recalling the definition of \( S(t) \) and \( \rho(t) \), see (4.7) and (4.9). From (4.15) we obtain

\[
\nu(t) = \gamma(t) \int_{R^n} \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho(t)) q(x)dx
\]

Combining results, we can assert that

\[
\theta(x, t) = \nu(t) \frac{\exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho(t))}{\int_{R^n} \exp -\frac{1}{2}(\xi^* S(t)\xi - 2\xi^* \rho(t)) q(\xi)d\xi}
\]

(4.21)

Next, using (4.16) and (4.10) we have

\[
\int_{R^n} q(x,t)x dx = \int_{R^n} \theta(x,t)(\Phi(t)x + \beta(t))q(x)dx
\]

\[= \Phi(t) \int_{R^n} \theta(x,t)x q(x)dx + \beta(t) \int_{R^n} \theta(x,t) q(x)dx \]

therefore, from (4.18) we obtain also

\[
\hat{x}(t) = \Phi(t) \frac{\int_{R^n} \theta(x,t)x q(x)dx}{\int_{R^n} \theta(x,t) q(x)dx} + \beta(t)
\]

(4.22)

Let us introduce the deterministic function of arguments \( \rho \in R^n \) and \( t \)

\[
b(\rho, t) = \frac{\int_{R^n} x \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho) q(x)dx}{\int_{R^n} \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho) q(x)dx}
\]

(4.23)

then (4.22) can be written
\[ \hat{x}(t) = \Phi(t)b(\rho(t), t) + \beta(t) \]  

(4.24)

We can finally state the main formula for the unnormalized conditional probability \( q(x, t) \)

**Theorem 4.** The unnormalized conditional probability \( q(x, t) \) is given by

\[
\int_{\mathbb{R}^n} q(x, t)\psi(x, t)dx = \frac{\nu(t)}{\int_{\mathbb{R}^n} \exp -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t))q(x)dx} \int_{\mathbb{R}^n} \exp \left( -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t)) \right) x \psi(x, t)dx
\]

(4.25)

\[
\left[ \int_{\mathbb{R}^n} \psi(\hat{x}(t) + \Phi(t)(x - b(\rho(t), t))) + \Sigma(t)^{\frac{1}{2}}\xi, t) \frac{\exp -\frac{|\xi|^2}{2}}{(2\pi)^{n/2}} \right] q(x)dx
\]

4.4 **SUFFICIENT STATISTICS**

We see, from formula (4.25) that the unnormalized conditional probability \( q(x, t) \) is completely characterized by two processes \( \hat{x}(t) \) and \( \rho(t) \), which are stochastic processes adapted to \( \mathcal{Z}^t \) with values in \( \mathbb{R}^n \). So it is important to obtain their evolution. We need to introduce a new function \( B(\rho, t) \) similar to \( b(\rho, t) \) defined by the following formula

\[
B(\rho, t) = \int_{\mathbb{R}^n} xx^* \exp -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t)) q(x)dx
\]

and we define

\[
\Gamma(\rho, t) = \Sigma(t) + \Phi(t)(B(\rho, t) - b(\rho, t)b^*(\rho, t))\Phi(t)
\]

(4.27)

We are going to show that

**Proposition 5.** The pair \( \hat{x}(t), \rho(t) \) is solution of the following system of S.D.E.

\[
d\hat{x}(t) = (F\hat{x}(t) + Gv(t))dt + \Gamma(\rho(t), t)H^*(dz(t) - H\hat{x}(t)dt)
\]

(4.28)

\[
\hat{x}(0) = \frac{\int_{\mathbb{R}^n} xq(x)dx}{\int_{\mathbb{R}^n} q(x)dx}
\]

\[
d\rho(t) = \Phi^*(t)H^*(dz(t) - H(\hat{x}(t) - \Phi(t)b(\rho(t), t)))
\]

(4.29)
\[ \rho(0) = 0 \]

**Proof.** The pair \( \rho(t), \beta(t) \) satisfies (4.8), (4.9) and \( \dot{x}(t) \) satisfies (4.23) therefore

\[
d\dot{x}(t) = d\beta(t) + \frac{d\Phi(t)}{dt} b(\rho(t), t)dt + \Phi(t)db(\rho(t), t) \tag{4.30}
\]

Next we use

\[
D_\rho b(\rho, t) = B(\rho, t) - b(\rho, t)b^*(\rho, t) \tag{4.31}
\]

\[
\text{tr} \left( D^2_\rho b(\rho, t)L \right) = -B(\rho, t)(L + L^*) b(\rho, t) - \text{tr} \left( B(\rho, t)L \right) b(\rho, t) + 2b(\rho, t)b^*(\rho, t)Lb(\rho, t) + \tag{4.32}
\]

\[
+ \int_{\mathbb{R}^n} x x^* Lx \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho) q(x)dx
\]

for any matrix \( L \). Therefore

\[
\frac{1}{2} \text{tr} \left( D^2_\rho b(\rho, t)\Phi^*(t)H^*H\Phi(t) \right) = -B(\rho, t)\Phi^*(t)H^*H\Phi(t)b(\rho, t) - \frac{1}{2} \text{tr} \left( B(\rho, t)\Phi^*(t)H^*H\Phi(t) \right) b(\rho, t) +
\]

\[
+ \text{tr} \left( b(\rho, t)b^*(\rho, t)\Phi^*(t)H^*H\Phi(t) \right) b(\rho, t) + \frac{1}{2} \int_{\mathbb{R}^n} x x^* \Phi^*(t)H^*H\Phi(t) x \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho) q(x)dx
\]

Also

\[
\frac{\partial b(\rho, t)}{\partial t} = \frac{1}{2} \text{tr} \left( B(\rho, t)\Phi^*(t)H^*H\Phi(t) \right) b(\rho, t) - \frac{1}{2} \int_{\mathbb{R}^n} x x^* \Phi^*(t)H^*H\Phi(t) x \exp -\frac{1}{2}(x^* S(t)x - 2x^* \rho) q(x)dx
\]

Hence

\[
\frac{\partial b(\rho, t)}{\partial t} + \frac{1}{2} \text{tr} \left( D^2_\rho b(\rho, t)\Phi^*(t)H^*H\Phi(t) \right) = -B(\rho, t)\Phi^*(t)H^*H\Phi(t)b(\rho, t) +
\]

\[
+ \text{tr} \left( b(\rho, t)b^*(\rho, t)\Phi^*(t)H^*H\Phi(t) \right) b(\rho, t) \tag{4.34}
\]

We can then compute \( db(\rho(t), t) \), making use of (4.34), (4.31) and (4.29). We obtain
\[ db(\rho(t), t) = (B(\rho(t), t) - b(\rho(t), t)b^*(\rho(t), t))\Phi^*(t)H^*(dz(t) - H\dot{x}(t)dt) \] (4.35)

Using (4.6), (4.30) and (4.35) we obtain easily (4.28), recalling the definition of \( \Gamma(\rho, t) \), see (4.27). The relation (4.29) follows immediately from (4.9) and (4.24). The proof is complete. \( \square \)

We also have the following interpretation of \( \Gamma(\rho(t), t) \) as the conditional variance of the process \( x(t) \)

**Proposition 6.** We have the formula

\[ \Gamma(\rho(t), t) = \frac{\int_{\mathbb{R}^n} xx^* q(x, t)dx}{\int_{\mathbb{R}^n} q(x, t)dx} - \hat{x}(t)\hat{x}(t)^* \] (4.36)

**Proof.** We use (4.25) to write

\[ \frac{\int_{\mathbb{R}^n} xx^* q(x, t)dx}{\int_{\mathbb{R}^n} q(x, t)dx} = \frac{1}{\int_{\mathbb{R}^n} \exp -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t))q(x)dx} \int_{\mathbb{R}^n} \exp -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t)) \times \]

\[ \left[ \int_{\mathbb{R}^n} (\hat{x}(t) + \Phi(t)(x - b(\rho(t), t)) + \Sigma(t)\frac{1}{2}\xi)(\hat{x}(t) + \Phi(t)(x - b(\rho(t), t)) + \Sigma(t)\frac{1}{2}\xi)^* \frac{\exp -|\xi|^2}{(2\pi)^{2n}} d\xi \right] q(x)dx \]

\[ = \frac{1}{\int_{\mathbb{R}^n} \exp -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t))q(x)dx} \int_{\mathbb{R}^n} \exp -\frac{1}{2}(x^*S(t)x - 2x^*\rho(t)) \times \]

\[ \left[ ((\hat{x}(t) + \Phi(t)(x - b(\rho(t), t)))((\hat{x}(t) + \Phi(t)(x - b(\rho(t), t))))^* + \Sigma(t) = \hat{x}(t)\hat{x}(t)^* + \Phi(t)(B(\rho(t), t) - b(\rho(t), t))(B(\rho(t), t) - b(\rho(t), t))^*\Phi(t)^* + \Sigma(t) \right. \]

\[ = \hat{x}(t)\hat{x}(t)^* + \Gamma(\rho(t), t) \]

which is (4.36). \( \square \)

### 4.5 THE GAUSSIAN CASE

We first begin by giving the characteristic function of the unnormalized probability density (Fourier transform) denoted

\[ \hat{Q}(\lambda, t) = \int_{\mathbb{R}^n} q(x, t) \exp i\lambda^* x dx = \] (4.37)

\[ \nu(t) \exp \left[ -\frac{1}{2} \lambda^* \Sigma(t) \lambda + i\lambda^*(\hat{x}(t) - \Phi(t)b(\rho(t), t)) \right] \times \]
\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} (x^* S(t)x - 2x^* \rho(t)) \right) q(x) dx
\]
\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{2} (x^* S(t)x - 2x^* \rho(t)) \right) q(x) dx
\]

The gaussian case corresponds to an initial value of the system (4.19) which is gaussian

\[
q(x) = \frac{\exp \left( -\frac{1}{2} (x - \bar{x}_0)^* P_0^{-1} (x - \bar{x}_0) \right)}{(2\pi)^{\frac{n}{2}} |P_0|^{\frac{1}{2}}}
\]  

(4.38)

where we have assumed the initial variance \( P_0 \) to be invertible, to simplify calculations. Using (4.23) we obtain

\[
b(\rho, t) = (S(t) + P_0^{-1})^{-1} (\rho + P_0^{-1} \bar{x}_0)
\]  

(4.39)

\[
B(\rho, t) = b(\rho, t)b(\rho, t)^* + (S(t) + P_0^{-1})^{-1}
\]  

(4.40)

Therefore, from (4.27) we obtain

\[
\Gamma(\rho, t) = \Sigma(t) + \Phi(t)(S(t) + P_0^{-1})^{-1}\Phi(t)^*
\]

\[= P(t)\]

(4.41)

which is independent of \( \rho \). An easy calculation shows that \( P(t) \) is the solution of the Riccati equation

\[
\frac{dP}{dt} + PH^*HP - FP - PF^* = 2a
\]

(4.42)

\[
P(0) = P_0
\]

and \( \hat{x}(t) \) is then the classical Kalman filter

\[
d\hat{x}(t) = (F\hat{x}(t) + Gv(t))dt + P(t)H^*(dz(t) - H\hat{x}(t)dt
\]

(4.43)

\[
\hat{x}(0) = \bar{x}_0
\]

To obtain \( q(x, t) \), we use the characteristic function (4.37). An easy calculation yields

\[
\hat{Q}(\lambda, t) = v(t) \exp [i\lambda^* \hat{x}(t) - \frac{1}{2} \lambda^* P(t) \lambda]
\]

(4.44)
which is the characteristic function of a gaussian random variable with mean \( \hat{x}(t) \) and variance \( P(t) \). Recall that it is a conditional probability given \( Z^t \).

## 5 LINEAR QUADRATIC CONTROL PROBLEM

### 5.1 SETTING OF THE PROBLEM

We want to apply the theory developed in section \( \Xi \) to the linear dynamics and linear observation \( \Xi \Xi \), with a quadratic cost

\[
f(x, v) = x^* M x + v^* N v
\]

\[
f_T(x) = x^* M_T x
\]

in which \( M, M_T \) are \( n \times n \) symmetric positive semi-definite matrices and \( N \) is a \( m \times m \) symmetric positive definite matrix. We want to solve the control problem \( \Xi \Xi \), \( \Xi \Xi \) in this case. We write it as follows

\[
d q - \text{tra} D_x^2 q(x, t) \, dt + \text{div} (Fx + Gv(t))q(x, t)) \, dt - q(x, t) Hx. dz(t) = 0 \quad 5.2
\]

\[
q(x, 0) = q(x)
\]

\[
J(v(.) ) = E\left[ \int_0^T \int_{\mathbb{R}^n} q^{\delta}(x, t)(x^* M x + v(t)^* N v(t))dx dt + \int_{\mathbb{R}^n} q^{\delta}(x, T) x^* M_T x dx \right] \quad 5.3
\]

In the sequel we will drop the index \( v(.) \) in \( q(x, t) \).

### 5.2 APPLICATION OF MEAN FIELD THEORY

We begin by finding the function \( \hat{v}(q, U) \) defined by \( \Xi \Xi \). We have to solve the minimization problem

\[
\inf_v [v^* N v \int_{\mathbb{R}^n} q(x)dx + Gv. \int_{\mathbb{R}^n} D_x U(x, q) q(x)dx] \quad 5.4
\]

which yields

\[
\hat{v}(q, U) = \frac{1}{2} N^{-1} G^* \int_{\mathbb{R}^n} D_x U(x, q) q(x)dx \int_{\mathbb{R}^n} q(x)dx \quad 5.5
\]
and thus

$$\inf_v [v^* N v \int_{\mathbb{R}^n} q(x) dx + G v \int_{\mathbb{R}^n} D_x U(x, q) q(x) dx] = \quad (5.6)$$

$$-\frac{1}{4} \int_{\mathbb{R}^n} D_x U(x, q) q(x) dx \cdot GN^{-1} G^* \int_{\mathbb{R}^n} D_x U(x, q) q(x) dx \int_{\mathbb{R}^n} q(x) dx$$

We consider the value function

$$\Phi(q, t) = \inf_v \int_{\mathbb{R}^n} q(x) dx$$

and we write Bellman equation (3.16)

$$\frac{\partial \Phi}{\partial t} + \text{tr} \int_{\mathbb{R}^n} D^2 U(x, q, t) q(x) dx +$$

$$+ \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} V(q, t)(\xi, \eta) H_{\xi} H_{\eta} q(\xi) q(\eta) d\xi d\eta + \int_{\mathbb{R}^n} x^* M x q(x) dx + \int_{\mathbb{R}^n} F_x D_x U(x, q, t) q(x) dx$$

$$- \frac{1}{4} \int_{\mathbb{R}^n} D_x U(x, q, t) q(x) dx \cdot GN^{-1} G^* \int_{\mathbb{R}^n} D_x U(x, q, t) q(x) dx \int_{\mathbb{R}^n} q(x) dx = 0$$

$$\Phi(q, T) = \int_{\mathbb{R}^n} x^* M T x q(x) dx$$

in which we recall the notation

$$U(x, q, t) = \frac{\partial \Phi(q, t)}{\partial q}(x), \quad V(q, t)(x, y) = \frac{\partial^2 \Phi(q, t)}{\partial q^2}(x, y)$$

We can next write the Master equation (3.17), which is the equation for $U(x, q, t)$. We get

$$\frac{\partial U}{\partial t} + \text{tr} a D^2 U(x, q, t) + \text{tr} a \int_{\mathbb{R}^n} D^2 V(q, t)(x, \xi) q(\xi) d\xi$$

$$+ H x H \int_{\mathbb{R}^n} V(q, t)(x, \xi) q(\xi) d\xi + \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\partial V(q, t)}{\partial q}(\xi, \eta)(x) H_{\xi} H_{\eta} q(\xi) q(\eta) d\xi d\eta$$

$$+ x^* M x + F_x D_x U(x, q, t) + \int_{\mathbb{R}^n} F_x D_x V(q, t)(x, \xi) q(\xi) d\xi +$$
\[
\begin{align*}
+ \frac{1}{4} \int_{\mathbb{R}^n} D_x U(x,q,t) q(x) dx. G N^{-1} G^* \int_{\mathbb{R}^n} D_x U(x,q,t) q(x) dx - \frac{1}{2} D_x U(x,q,t). G N^{-1} G^* \int_{\mathbb{R}^n} D_x U(x,q,t) q(x) dx \\
- \frac{1}{2} \int_{\mathbb{R}^n} D_x V(q,t)(x,\xi) q(\xi) d\xi. G N^{-1} G^* \int_{\mathbb{R}^n} D_x U(x,q,t) q(x) dx \\
\int_{\mathbb{R}^n} q(\xi) d\xi = 0
\end{align*}
\]

\[U(x,q,T) = x^* M_T x\]

5.3 SYSTEM OF HJB-FP EQUATIONS

We now write the system of HJB-FP equations (3.22), (3.24). We look for a pair \(u(x,t),q(x,t)\) adapted random fields solution of the coupled system

\[
\begin{align*}
- d_t u &= \left( \text{tr} a D_x^2 u + x^* M x + F x. D_x u(x,t) + \frac{1}{4} \int_{\mathbb{R}^n} D_x u(\xi,t) q(\xi,t) d\xi. G N^{-1} G^* \int_{\mathbb{R}^n} D_x u(\xi,t) q(\xi,t) d\xi \right) \\
- \frac{1}{2} D_x u(x,t). G N^{-1} G^* \int_{\mathbb{R}^n} D_x u(\xi,t) q(\xi,t) d\xi \\
\int_{\mathbb{R}^n} q(\xi,t) d\xi = 0
\end{align*}
\]

\[d_t q = \left( \text{tr} a D_x^2 q - \text{div}[(F x - \frac{1}{2} G N^{-1} G^* \int_{\mathbb{R}^n} D_x u(\xi,t) q(\xi,t) d\xi) q(x,t)] \right) dt
\]

\[q(x,0) = q(x)\]

The random field \(K(x,t)\) can be expressed by

\[K(x,t) = H \int_{\mathbb{R}^n} \xi V(q_t)(x,\xi) q(\xi,t) d\xi\]

The key result is that we can solve this system of equations explicitly and obtain the optimal control. We introduce the matrix \(\pi(t)\) solution of the Riccati equation

\[
\frac{d\pi}{dt} + \pi F + F^* \pi - \pi G N^{-1} G^* \pi + M = 0
\]

\[\pi(T) = M_T\]
We next introduce the function $Z(x, \rho, t)$ solution of the deterministic linear P.D.E.

$$
\frac{\partial Z}{\partial t} + D_x Z (F - \Gamma(\rho, t) H^* H)x + D_\rho Z \Phi^*(t) H^* H (\Phi(t)b(\rho, t) + x) + \nabla + D_x^2 Z (2a + \Gamma(\rho, t) H^* H \Gamma(\rho, t)) + \frac{1}{2} \text{tr} D_\rho^2 Z \Phi^*(t) H^* H \Phi(t) - \text{tr} D_{x\rho}^2 Z \Phi^*(t) H^* H \Gamma(\rho, t) + x^* \pi(t) G^{-1} G^* \pi(t)x + 2 \text{tr} a \pi(t) = 0
\tag{5.14}
$$

$$
Z(x, \rho, T) = 0
\tag{5.15}
$$

We next introduce the pair of adapted processes $\hat{x}(t), \rho(t)$ solution of the system of SDE

$$
d\hat{x} = (F - G N^{-1} G^* \pi(t)) \hat{x}(t)dt + \Gamma(\rho(t), t) H^* (dz(t) - H \hat{x}(t)dt)
\tag{5.16}
$$

$$
\hat{x}(0) = \bar{x}_0
\tag{5.17}
$$

$$
d\rho = \Phi^*(t) H^* (dz(t) - H(\hat{x}(t) - b(\rho(t), t)))
\tag{5.18}
$$

They are built on a convenient probability space on which $z(t)$ is a standard Wiener process with values in $\mathbb{R}^d$. We associate to the pair $\hat{x}(t), \rho(t)$ the unnormalized conditional probability $q(x, t)$ defined by Zakai equation

$$
dq = \left( \text{tr} a D_z^2 q - \text{div}[(F x - G N^{-1} G^* \pi(t) \hat{x}(t))q(x, t)] \right) dt + q(x, t) H x . dz(t)
\tag{5.19}
$$

$$
q(x, 0) = q(x)
\tag{5.20}
$$

We next define the random field

$$
u(x, t) = x^* \pi(t)x + Z(x - \hat{x}(t), \rho(t), t)
\tag{5.21}
$$
We state the main result of the paper

**Theorem 7.** We have the property

\[
\int_{\mathbb{R}^n} D_x Z(x - \hat{x}(t), \rho(t), t) q(x,t) dx = 0, \text{ a.s. }, \forall t
\]  

(5.19)

and \( u(x,t), q(x,t) \) defined by (5.18), (5.17) are solution of (5.10), (5.11).

The optimal control is given by

\[
\hat{v}(t) = \hat{v}(q(t), u(t)) = -N^{-1}G^* \pi(t) \hat{x}(t)
\]  

(5.20)

**Proof.** We first prove (5.19). We differentiate (5.14) in \( x \), to obtain

\[
\frac{\partial}{\partial t} D_x Z + D_x^2 Z(F - \Gamma(\rho, t)H^* H) x + (F^* - H^* H \Gamma(\rho, t)) D_x Z +
\]  

(5.21)

\[
+ D_{xx}^2 \Phi^*(t) H^* H (\Phi(t) b(\rho, t) + x) + H^* H \Phi(t) D_{xx} Z +
\]  

\[
+ \frac{1}{2} \text{tr} D_x^2 D_x Z (2\alpha + \Gamma(\rho, t) H^* H \Gamma(t)) + \frac{1}{2} \text{tr} D_{xx}^2 D_x Z \Phi^*(t) H^* H \Phi(t) - \text{tr} D_{xx}^2 D_x Z \Phi^*(t) H^* H \Gamma(\rho, t) +
\]  

\[
+ 2\pi(t) GN^{-1} G^* \pi(t) x = 0
\]  

\[
Z(x, \rho, T) = 0
\]

We next consider \( q(x,t) \) defined by (5.17). A long calculation then shows that

\[
d_t \int_{\mathbb{R}^n} D_x Z(x - \hat{x}(t), \rho(t), t) q(x,t) dx + (F^* - H^* H \Gamma(\rho(t), t) + H^* H \Phi(t)) \int_{\mathbb{R}^n} D_x Z(x - \hat{x}(t), \rho(t), t) q(x,t) dx dt +
\]  

(5.22)

\[
+ \left( \int_{\mathbb{R}^n} [-D_x^2 Z(x - \hat{x}(t), \rho(t), t) \Gamma(\rho(t), t) + D_{xx}^2 Z(x - \hat{x}(t), \rho(t), t) \Phi^*(t) + D_x Z(x - \hat{x}(t), \rho(t), t) x^*] q(x,t) \right) H^* dz(t)
\]

\[
\int_{\mathbb{R}^n} D_x Z(x - \hat{x}(T), \rho(T), T) q(x,T) dx = 0
\]

From this relation it follows that
\[ \int_{\mathbb{R}^n} D_x Z(x - \hat{x}(t), \rho(t), t) q(x, t) dx \exp \int_0^t (F^* - H^* H \Gamma(\rho(s), s) + H^* H \Phi(s)) ds \]

is a \( Z^t \) martingale. Since it vanishes at \( T \), it is 0 at any \( t \), a.e. Hence (5.19) is obtained. Consider \( u(x, t) \) by formula (5.18), therefore, using (5.19) we get

\[ \frac{\int_{\mathbb{R}^n} D_\xi u(\xi, t) q(\xi, t) d\xi}{\int_{\mathbb{R}^n} q(\xi, t) d\xi} = 2\pi(t) \hat{x}(t) \] (5.23)

To check (5.10) we have to check

\[ -d_t u = [\text{tr} aD_x^2 u + x^* M x + F_x D_x u + \hat{x}(t)^* \pi(t) G N^{-1} G^* \pi(t) \hat{x}(t) - \\
-D_x u G N^{-1} G^* \pi(t) \hat{x}(t)] dt - K(x, t). (dz(t) - H x dt) \] (5.24)

\[ u(x, T) = x^* M_T x \]

Note that the final condition is trivially satisfied. We can check (5.24) by direct calculation. We obtain also the value of \( K(x, t) \)

\[ K(x, t) = H[-\Gamma(\rho(t), t) D_x Z(x - \hat{x}(t), \rho(t), t) + \Phi(t) D_\rho Z(x - \hat{x}(t), \rho(t), t)] \] (5.25)

So we have proved that \( u(x, t), q(x, t) \) is solution of the system of HJB-FP equations (5.10), (5.11). The result (5.20) is an immediate consequence of (5.23). The proof is complete.

5.4 COMPLEMENTS

The result (5.20) is important. It shows that the optimal control of the problem (5.2), (5.3) follows the celebrated “Separation Principle”. We recall that in the deterministic case, the optimal control, which is necessarily an open-loop control can be obtained by a linear feedback on the state. Open loop and feedback controls are equivalent. The separation principle claims that in the partially observed case, the optimal open loop control (adapted to the observation process) can be obtained by the same feedback as in the deterministic case, replacing the nonobservable state by its best estimate, the Kalman filter. The fact that the separation principle holds is well known when the initial state follows a gaussian distribution. We have proven that it holds in general. What drives the separation principle is the linearity of the dynamics and of the observation and the fact that the cost is quadratic. The gaussian assumption does not play any role. A significant simplification occurs in the gaussian case, regarding the computation of the Kalman filter. In
the gaussian case, the Kalman filter solves a single equation. In general, the Kalman filter is coupled to another statistics \( \rho(t) \) and the pair \( \hat{x}(t), \rho(t) \) must be obtained simultaneously.

We proceed by obtaining the value function \( \Phi(q, t) \). In fact, we obtain \( \Phi(q, 0) \). The same procedure must be repeated at any time \( t \). First, we have

\[
\frac{\partial \Phi(q, 0)}{\partial q}(x) = u(x, 0)
\]

\( (5.26) \)

\[
= x^* \pi(0) x + Z(x - \int_{R^n} \frac{\xi(q) d\xi}{\int_{R^n} q(\xi)} , 0, 0)
\]

We next obtain \( \frac{\partial \Phi}{\partial t}(q, 0) \) by formula \( (3.27) \). We obtain

\[
\frac{\partial \Phi}{\partial t}(q, 0) = -\text{tr} a \int_{R^n} D_x^2 u(x, 0) q(x) dx - \frac{1}{2} \int_{R^n} Hx. K(x, 0) q(x) dx -
\]

\( (5.27) \)

\[
- \int_{R^n} [x^* M x + \hat{\nu}(0)^* N \hat{\nu}(0) + D_x u(x, 0). (F x + G \hat{\nu}(0))] q(x) dx
\]

\[
= -2 \text{tr} a \pi(0) - \text{tr} \left( \int_{R^n} D_x^2 Z(x - \bar{x}_0, 0, 0) q(x) dx \right) - \frac{1}{2} \int_{R^n} Hx. H(-\Gamma(0, 0) D_x Z(x - \bar{x}_0, 0, 0) + D_x Z(x - \bar{x}_0, 0, 0)) q(x) dx -
\]

\[
- \int_{R^n} Fx. D_x Z(x - \bar{x}_0, 0, 0) q(x) dx + (\bar{x}_0^* \pi'(0) \bar{x}_0 - \text{tr} \Gamma(0, 0)(M + 2 \pi(0) F)) \int_{R^n} q(x) dx
\]

where

\[
\bar{x}_0 = \frac{\int_{R^n} x q(x) dx}{\int_{R^n} q(x) dx}
\]

We can finally obtain the value of \( \Phi(q, 0) \). Since we know the optimal control for the problem \( (5.2), (5.3) \) we have

\[
\Phi(q, 0) = J(\hat{\nu}(\cdot)) =
\]

\[
= E \left[ \int_0^T \int_{R^n} q(x, t)(x^* M x + \hat{x}(t)^* \pi(t) G N^{-1} G^* \pi(t) \hat{x}(t)) dt + \int_{R^n} q(x, T) x^* M_T x dx \right]
\]

where \( q(x, t) \) is the solution of \( (5.17) \). Therefore also

\[
\Phi(q, 0) = E \int_0^T \nu(t)[\hat{x}(t)^* (M + \pi(t) G N^{-1} G^* \pi(t)) \hat{x}(t) + \text{tr} \Gamma(\rho(t), t)] dt +
\]

\( (5.28) \)
\[ + E[\nu(T)(\dot{x}(T)^* M_T \dot{x}(T) + \text{tr} M \Gamma(T), T)] \]

The triple \( \dot{x}(t), \rho(t), \nu(t) \) is solution of the system of S.D.E. (5.15), (5.16) and

\[ d\nu(t) = \nu(t) H \dot{x}(t).dz(t) \quad (5.29) \]

\[ \nu(0) = \int_{\mathbb{R}^n} q(x)dx \]

From the probabilistic formula (5.28) it is easy to derive an analytic formula as follows

\[ \Phi(q, 0) = (\bar{x}_0^* \pi(0) \bar{x}_0 + \mu(0, 0)) \int_{\mathbb{R}^n} q(x)dx \quad (5.30) \]

where \( \mu(\rho, t) \) is the solution of the linear P.D.E.

\[ \frac{\partial \mu}{\partial t} + D_\rho \mu. \Phi^*(t) H^* H \Phi(t) b(\rho, t) + \]

\[ + \frac{1}{2} \text{tr} D^2_\rho \mu \Phi^*(t) H^* H \Phi(t) + \text{tr}(M + \pi(t) \Gamma(t) H^* H) \Gamma(t) = 0 \]

\[ \mu(\rho, T) = \text{tr} M_T \Gamma(T) \]

We can apply these results in the gaussian case. We first solve the P.D.E s (5.14) and (5.31). We recall that \( \Gamma(\rho, t) = P(t) \), then \( Z(x, \rho, t) \) and \( \mu(\rho, t) \) are independent of \( \rho \) and as easily seen

\[ Z(x, \rho, t) = x^* \Lambda(t)x + \beta(t) \quad (5.32) \]

with

\[ \frac{d\Lambda}{dt} + \Lambda(t)(F - P(t)H^*H) + (F^* - H^*HP(t))\Lambda(t) + \pi(t) G^{-1} G^* \pi(t) = 0 \quad (5.33) \]

\[ \Lambda(T) = 0 \]

\[ \beta(t) = \int_t^T \text{tr} \Lambda(s)(2a + P(s)H^*HP(s))ds \quad (5.34) \]
Similarly \( \mu(\rho, t) = \mu(t) \) given by

\[
\mu(t) = \int_t^T \text{tr}\pi(s)P(s)H^*HP(s)ds + \text{tr} MTP(T)
\]  

(5.35)

Then

\[
\frac{\partial \Phi(q, 0)}{\partial q}(x) = x^*\pi(0)x + (x - \bar{x}_0)^*\Lambda(0)(x - \bar{x}_0) + \beta(0)
\]  

(5.36)

\[
\frac{\partial \Phi}{\partial t}(q, 0) = -2\text{tr}(a + P(0)F)(\pi(0) + \Lambda(0))
\]  

(5.37)

\[
+ \bar{x}_0^*\pi'(0)\bar{x}_0 - \text{tr} P(0)M + \text{tr}P(0)H^*HP(0)\Lambda(0)
\]

\[
\Phi(q, 0) = \bar{x}_0^*\pi(0)\bar{x}_0 + \mu(0)
\]  

(5.38)

### 6 COMPARAISON

#### 6.1 OBJECTIVES

We compare in this section our work with the approach of E. Bandini, A. Cosso, M. Fuhrmann, H. Pham [1]. They consider a general problem of stochastic control with partial information, to which our problem can be reduced. Their set up leads to a conditional probability, (hence normalized) solution of a linear stochastic PDE, which they call DMZ equation (for Duncan-Mortensen-Zakai equation). They formulate a control problem for this infinite dimensional state equation, for which they write a Bellman equation. The solution is a functional on the Wasserstein space of probability measures, since indeed the state is a probability.

When we formulate our problem in their set up, our Zakai equation cannot be their DMZ equation, since we have not a probability, but an un-normalized probability. To make the comparison easy we keep our model, but we follow the set up of [1]. We explain the difference between the two equations, and also the difference between our Bellman equation and their Bellman equation. Although our problem can appear as a particular case of [1], it is at the price of complicating it, which turns out to be not suitable. The discussion will explain the reasons. E. Bandini et al. provide an example with linear equations, which does not cover ours. In the set up of E. Bandini et al., our system remains nonlinear, which is also a consequence of the complication of the approach. We remain formal in our presentation, since we want to discuss the concepts and compare the methods.
6.2 USE OF THE SET UP OF [1]

In the set up of [1], we consider the pair \( x(t), \eta(t) \) solution of the system

\[
\frac{dx}{dt} = g(x,v)dt + \sigma(x)dw \tag{6.1}
\]

\[
x(0) = x_0
\]

\[
d\eta(t) = \eta(t)h(x(t)).dz(t) \tag{6.2}
\]

\[
\eta(0) = \eta_0
\]

in which \( w(\cdot), z(\cdot) \) are independent Wiener processes and \( x_0, \eta_0 \) are random variables independent of \( w(\cdot), z(\cdot) \). We observe only the process \( z(\cdot) \). The DMZ equation introduced by [1] is the equation for the conditional probability of the pair \( x(t), \eta(t) \) given the \( \sigma- \) algebra \( \mathcal{Z}^t = \sigma(z(s), s \leq t) \). In (6.1) the control \( v(t) \) is simply adapted to \( \mathcal{Z}^t \).

If \( \varphi(\eta,x,t) \) is a deterministic function on \( \mathbb{R}^{n+1} \times \mathbb{R}^+ \), we are interested in the process \( \rho(\varphi)(t) = E[\varphi(\eta(t), x(t), t)| \mathcal{Z}^t] \). It is the solution of the DMZ equation. We use the notation

\[
A_x \varphi(\eta,x,t) = -\text{tr}(D^2_x \varphi(\eta,x,t)a(x)) \tag{6.3}
\]

with \( a(x) = \frac{1}{2}\sigma(x)\sigma^*(x) \). We note

\[
A_x^* \varphi(\eta,x,t) = -\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) \varphi(\eta,x,t)) \tag{6.4}
\]

We next define the operators

\[
\mathcal{L}^{v(\cdot)} \varphi(\eta,x,t) = D_x \varphi(\eta,x,t).g(x,v(t)) - A_x \varphi(\eta,x,t) + \frac{1}{2} \eta^2 h(x)^2 \frac{\partial^2 \varphi(\eta,x,t)}{\partial \eta^2} \tag{6.5}
\]

\[
\mathcal{M} \varphi(\eta,x,t) = \eta \frac{\partial \varphi(\eta,x,t)}{\partial \eta} h(x) \tag{6.6}
\]

Then the DMZ equation is
\[ dp(\varphi)(t) = \rho \frac{\partial \varphi}{\partial t} + L^{v(\cdot)} \varphi(t) dt + \rho(M \varphi)(t) dz(t) \] (6.7)

\[ \rho(\varphi)(0) = E \varphi(\eta_0, x_0, 0) \]

In the sequel we assume the existence of a density \( p(\eta, x, t) \) which is the joint conditional probability density of \( \eta(t), x(t) \) given \( \mathcal{Z}^t \). It is defined by

\[ \rho(\varphi)(t) = \int p(\eta, x, t) \varphi(\eta, x, t) d\eta dx \] (6.8)

The conditional probability density is the solution of the stochastic P.D.E.

\[ dp + [A^*_x p(\eta, x, t) + \text{div}(g(x, v(t)) p(\eta, x, t))] dt - \frac{1}{2} h(x)^2 \frac{\partial^2}{\partial \eta^2} (\eta^2 p(\eta, x, t))] dt = -h(x) \frac{\partial}{\partial \eta} (\eta p(\eta, x, t)) dz(t) \] (6.9)

\[ p(\eta, x, 0) = p_0(\eta, x) \]

It is easy to check that

\[ q(x, t) = \int \eta p(\eta, x, t) d\eta \] (6.10)

is the solution of Zakai equation (2.9), provided that the initial condition

\[ q_0(x) = \int \eta p_0(\eta, x) d\eta \] (6.11)

It is then clear, that although \( p(\eta, x, t) \) is indeed a probability density, \( q(x, t) \) is not. Conversely, if we start with \( q_0(x) \) and want to solve Zakai equation (2.9), we can use (6.10) by looking for \( p(\eta, x, t) \) solution of the DMZ equation (6.9). We need to take the initial condition

\[ p_0(\eta, x) = \delta(\eta) - \int q_0(\xi) d\xi \otimes \frac{q_0(x)}{\int q_0(\xi) d\xi} \] (6.12)

This is not a probability density, so we need to use the weak formulation, to proceed.

We get some kind of interesting quandary. Using the set up [1] we can use probability measures, the Wasserstein topology and the lifting method of P.L. Lions, but the price to pay is to increase the dimension
by 1, with a nonlinearity. If we stay with the traditional set up, we have to work with unnormalized probability densities. If we can work with densities, it is not a serious drawback, but otherwise we have to find an alternative to the Wasserstein space and the lifting procedure, and it is not clear how to proceed.

We can, of course, consider Kushner equation, instead of Zakai equation, whose solution is a probability. But Kushner equation is nonlinear, conversely to Zakai equation.

6.3 BELLMAN EQUATION

We can extend the comparison at the level of Bellman equation. We consider $p^{\varepsilon,(\cdot)}(\eta, x, s), s \geq t$ solution of

\begin{equation}
\begin{aligned}
dp + [A^*_x p(\eta, x, s) + \\
+ \text{div}(g(x, v(s)) p(\eta, x, s)) - \frac{1}{2} |h(x)|^2 \frac{\partial^2}{\partial \eta^2} (\eta^2 p(\eta, x, s))] ds
\end{aligned}
\end{equation}

\begin{equation}
p(\eta, x, t) = p(\eta, x)
\end{equation}

where $p(\eta, x)$ is a probability density, denoted in the sequel by $p$. We can then define the payoff $J_{p,t}(v(\cdot))$ by

\begin{equation}
J_{p,t}(v(\cdot)) = E[\int_t^T \int p^{\varepsilon,(\cdot)}(\eta, x, s) \eta f(x, v(s)) d\eta dx ds + \int p^{\varepsilon,(\cdot)}(\eta, x, T) \eta f_T(x) d\eta dx]
\end{equation}

and we define the value function

\begin{equation}
\Psi(p, t) = \inf_{v(\cdot)} J_{p,t}(v(\cdot))
\end{equation}

We can write Bellman equation corresponding to this problem. Indeed,

\begin{equation}
\begin{aligned}
\frac{\partial \Psi}{\partial t} + \int [-A_x \frac{\partial \Psi}{\partial p}(p, t)(\eta, x) + \\
+ \frac{1}{2} |h(x)|^2 \eta^2 \frac{\partial^2}{\partial \eta^2} (p, t)(\eta, x) ] p(\eta, x) d\eta dx + \\
+ \inf_v \int [\eta f(x, v) + D_x \frac{\partial \Psi}{\partial p}(p, t)(\eta, x) g(x, v)] p(\eta, x) d\eta dx = 0
\end{aligned}
\end{equation}

\begin{equation}
\Psi(p, T) = \int \eta f_T(x) p(\eta, x) d\eta dx
\end{equation}
If we compare with the Bellman equation (3.11) rewritten with the current notation (with argument an unnormalized probability) we obtain

\[
\frac{\partial \Phi}{\partial t} + \int [-A_x \frac{\partial \Phi}{\partial q}(q,t)(x)q(x)dx + \frac{1}{2} \int \int \frac{\partial^2 \Phi}{\partial q^2}(q,t)(x;\tilde{x})h(x).h(\tilde{x})q(\tilde{x})q(x)dxd\tilde{x} + 
\]

\[
\inf_v \int [f(x,v) + D_x \frac{\partial \Phi}{\partial q}(q,t)(x).g(x,v)]q(x)dx = 0
\]

\[
\Phi(q,T) = \int f_T(x)q(x)dx
\]

Equations (6.16) and (6.17) are linked by the formulas

\[
\Psi(p,t) = \Phi(\int \eta p(\eta,.)d\eta, t)
\]

(6.18)

\[
\Phi(q,t) = \Psi(\delta(., - \int q(\xi)d\xi) \otimes \frac{q(\xi)}{\int q(\xi)d\xi}, t)
\]

(6.19)

### 6.4 THE LINEAR CASE

If we go back to the linear case (4.3), we get

\[
dx = (Fx + Gv)dt + \sigma dw
\]

\[
d\eta = \eta.Hx.dz
\]

therefore, in the set up [1], we still have a nonlinear system. Therefore, we cannot use the linear case of [1]. This explains why our formulas are completely different. The fact that we have an explicit solution of the system of HJB-FP equations does not imply that we have an explicit solution of Bellman equation. This is consistent with the spirit of the method of characteristics.

### References

[1] E. Bandini, A. Corso, M. Fuhrman, H. Pham, Randomized filtering and Bellman equation in Wasserstein space for part incraseial observation control problem, arXiv, sept. 2016

[2] A. Bensoussan, *Stochastic Control of Partially Observable Systems*, Cambridge University Press, 1992
[3] P. Cardialaguet, F. Delarue, J.M. Lasry, P.L. Lions, The Master Equation and the Convergence Problem in Mean Field Games, HAL archives, September, 2015

[4] A. Makowsky, Filtering Formulae for Partially Observed Linear Systems with Non-Gaussian Initial Conditions, Stochastics, 16, 1-24