Pricing Illiquid Options with Nth Liquid Proxies Using Mixed Dynamic-Static Hedging

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Various applications of this approach are possible, e.g. credit risk modeling, dynamic-static framework of exotic or illiquid commodity, equity, or FX options.
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A mixed dynamic-static hedging strategy may be reasonable or attractive in some cases:

- Static hedging using liquid vanilla options is an attractive alternative to dynamic hedging in the underlying for some problems, e.g. for barrier options
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- Assume some market where the following instruments can be traded:
  - a risk-free zero-coupon bond $B_0$
  - a risky non-defaultable index $S$ (e.g. CDX.NA or S&P 500)
  - a set of liquid bonds $B_{Y_i}$ issued by firms $Y_i$, $i = 1, N$, with market prices $p_{Y_i}$

- A financial model for pricing and hedging an illiquid credit $B_Z$ amounts to computing its price $p_Z$ in terms of all $p_{Y_i}$ and $S$ at $t = 0$, along with optimal hedges.
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Further technical details for setting our framework

Payoff example - Merton’s model

Assume that the firm is run by equity holders. At time $T$, equity holders pay the face value of the debt if the asset value $V_T \geq D$. The remaining amount goes to equity holders. If $V_T < D$, bond holders take over the firm and receive a "recovery" of $V_T$, while equity holders get nothing. Mathematically, this means the following payoffs

$$B_T = \min(V_T, D) = D - \max(D - V_T, 0)$$
$$S_T = \max(V_T - D, 0)$$

Correlations: Are the credit and stock of a firm strongly correlated?

Some evidence of negative co-dependence at most, weak correlation!
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Merton model and market incompleteness

- The Merton model allows one to hedge debt with equity (the two are perfectly correlated). In practice, a perfect hedging of credit with equity is not possible, as they are not 100% correlated.
- We need a version of the Merton model for an incomplete market where a credit risk cannot be perfectly hedged.
- One approach is to de-correlate equity and credit for a given firm. Another approach, more aligned with our objectives, is to consider a Nth-firm Merton-like setting for firms $Y_i$, $i = 1, N$ and $Z$, and compute the price and hedge ratios for the debt of $Z$ in terms of the liquid debts of $Y_i$ (plus the value of index $S$).
- The latter choice keeps the perfect correlation of the Merton model between the equity and credit for each firm, but it does not use equity to hedge credit. Instead, we hedge the illiquid $Z$-bond as described in the previous item.
Derivatives pricing in incomplete markets

- For incomplete markets, risk in derivatives positions cannot be completely eliminated. As a result, both the pricing and hedging are not unique, and depend on investor’s risk preferences and a chosen hedging/pricing scheme.
- A number of approaches are available:
  - Pricing by computation of a “market-implied” pricing measure $Q$: parametric or non-parametric methods (e.g. MaxEnt).
  - Pricing and hedging by risk minimization e.g. a quadratic risk minimization (Schweizer and Follmer 1994, Bouchaud and Potters 2000).
  - Utility-based indifference pricing (Hodges and Neuberger 1989, Davis et al 1993).

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Utility-based pricing

- An often choice of investor’s utility function is to construct it as a function of total terminal wealth, e.g. $U(W_T)$. The ultimate goal then would be to determine the "value" $V = \mathbb{E}[U(W_T)]$ for a given choice of investment strategy in risky and riskless assets.

- Some popular choices for the utility function: $U(x) = -e^{-\gamma x}$ (exponential utility), or $U(x) = x^{1-R}/(1-R)$ (power utility). We pick the former as it is more tractable.
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Optimal static hedge for 2 assets: Y and Z

- Merton (1969):

\[ V^0(t, x, y, z) = \exp \left( -\gamma xe^{r\tau} - \frac{1}{2} \eta_s^2 \tau \right), \quad \tau = T - t \]

where \( \eta_s = \frac{\mu_s - r}{\sigma_s} \). Let the value function with the composite option be

\[ V^{B\alpha}(t, x, y, z) = \exp \left( -\gamma xe^{r\tau} - \frac{1}{2} \eta_s^2 \tau \right) \Phi(y, z, \tau) \]

where \( \Phi(y, z, \tau) \) is a function to be calculated. Then the option price is

\[ g^{\alpha^*}_Z = \max_\alpha \left( -\frac{1}{\gamma} \log \Phi + \alpha p_y \right) \]

The maximum is unique as the RHS is a concave function of \( \alpha \).

- A similar setting is presented by Ilhan and Sircar (IS) (2004) for indifference hedging of barrier options with a stochastic volatility model.
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- Merton (1969):

\[
V^0(t, x, y, z) = \exp \left( -\gamma xe^{r \tau} - \frac{1}{2} \eta_s^2 \tau \right), \quad \tau = T - t
\]

where \( \eta_s = \frac{\mu_s - r}{\sigma_s} \). Let the value function with the composite option be

\[
V^{B_\alpha}(t, x, y, z) = \exp \left( -\gamma xe^{r \tau} - \frac{1}{2} \eta_s^2 \tau \right) \Phi(y, z, \tau)
\]

where \( \Phi(y, z, \tau) \) is a function to be calculated. Then the option price is

\[
g^{\alpha^*}_Z = \max_\alpha \left( -\frac{1}{\gamma} \log \Phi + \alpha p_y \right)
\]

The maximum is unique as the RHS is a concave function of \( \alpha \).

- A similar setting is presented by Ilhan and Sircar (IS) (2004) for indifference hedging of barrier options with a stochastic volatility model.
The HJB equation

Let all assets’ prices \( F_t : \{ S_t, Z_t, Y_1...Y_N \} \) follow a GBM with time-dependent drifts

\[
dF_{i,t} = F_{i,t} [\mu_i(t)dt + \sigma_i dW_{i,t}], \quad i = x, z, y_1...y_N
\]

and also assume that the correlation matrix is constant. Let’s use notation \( y_0 \equiv Z \). Let \( \theta \) be the investment strategy in the index. Then the HJB equation reads

\[
V_t + \sup_{\theta} L^\theta V = 0
\]

where \( L^\theta \) is the Markov generator. The formal solution for \( \theta \) is:

\[
\theta^* = -\frac{\eta_s V_x + \sum_{i=0}^N \rho_{xy_i} \sigma_i V_{xy_i}}{V_{xx}}, \quad \eta_s = \frac{\mu_s - r}{\sigma_s}, \quad y_i = \log(Y_{i,t}/K_{y_i}), \quad x = \log S_t,
\]

where \( K_{y_i} \) are the corresponding strikes. Plug this into the HJB and use the ansatz \( V(t, x, y, z) = -e^{-\gamma x} e^{\int_0^\tau r(k)dk} G(y_0, ... y_N, \tau = T - t) \).

Function \( G \) satisfies a non-linear PDE (with \( \hat{\mu}_i = \mu_i - \frac{1}{2} \sigma_i^2 - \eta_s \rho_{xy_i} \sigma_i \))

\[
G_\tau = \sum_{i=0}^N \hat{\mu}_i G_{y_i} + \frac{1}{2} \sum_{i=0}^N \sum_{j=0}^N \rho_{ij} \sigma_i \sigma_j G_{y_i,y_j} - \frac{1}{2} \left( \sum_{i=0}^N \rho_{xy_i} \sigma_i G_{y_i} \right)^2 \frac{1}{G} - \frac{1}{2} \eta_s G
\]
The HJB equation and "adiabatic" variables

- We want to find a map \( y = (y_0...y_N) \rightarrow u = (u_0...u_N) \) to simultaneously diagonalize the Hessian matrix and the quadratic term.

- In matrix notation, let \( A \) be the Hessian matrix, i.e. \( A = \|\rho_{ij}\sigma_i\sigma_j\| \). Let \( a \) be a vector \( a = (\rho_{xyi}\sigma_i) \). Let \( R \) be a transformation matrix, i.e. \( u = R^T y \). Then we need to find such an \( R \) which obeys

\[
R^T A R = \lambda, \quad a R = B,
\]

where \( \lambda \) is some diagonal matrix, \( B = (1, 0...0) \).
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Algorithm: Take a diagonal matrix \( D = \begin{bmatrix} d_0 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_N & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \)

Solve an eigenvalues problem \( R^{-1}DAR = \Lambda \). Then we can show that \( R^T A R = \lambda \)

Solve a system of non-linear algebraic equations \( a R = B \) wrt to \( d_i, \ i = 1, N \).
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  Solve a system of non-linear algebraic equations \( a R = B \) wrt to \( d_i, i = 1, N \).
HJB in adiabatic variables - can be easily done numerically

```matlab
function g = coef()
clc;
Rho = [1.0, 0.2, 0.3, 0.35; ... 0.2, 1.0, 0.8, 0.4; ... 0.3, 0.8, 1.0, 0.7; ... 0.35,0.4, 0.7, 1.0];
Sigma = [0.3, 0.25, 0.35, 0.23];
Rhox = [0.23, 0.34, 0.45, 0.6];
x0 = [0.3,0.35, 0.3];
N = length(Sigma);
A = Rho.*(Sigma'*Sigma);
aa = Sigma.*Rhox;
function f = func(xx)
    [V,lambda] = eig(diag([xx,-1])*A);
g = aa*V;
f = g(2:N);
end
options = ...
    optimset('TolX',1.e-12,'TolFun',1.e-16,'MaxIter',40000,'MaxFunEvals',40000);
[x,fval,exitflag,output] = fsolve(@func,x0,options);
```

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Thus, we finally use a change of independent variables

\[ u = R^T y + \tau M, \quad M = \left( \frac{1}{\tau} \int_0^\tau \hat{\mu}_0(k) dk, \ldots, \frac{1}{\tau} \int_0^\tau \hat{\mu}_N(k) dk \right), \]

and dependent variable

\[ G(y_0 \ldots y_N, \tau) = e^{-\frac{1}{2} \int_0^\tau \eta_s^2(k) dk} \Phi(u_0 \ldots u_N, \tau) \]

to obtain

\[ \Phi_{\tau} = \frac{1}{2} \sum_{i=0}^N a_i \Phi_{y_i, y_i} - \frac{1}{2} b_0 \frac{\Phi^2}{\Phi} \]

This can also be written as \( \Phi_{\tau} = \sum_{i=0}^N L_i \Phi \)

\[ L_0 \Phi = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi - \frac{1}{2} b_0 \frac{\Phi^2}{\Phi}, \quad L_i = \frac{1}{2} a_i \frac{\partial^2}{\partial y_i^2}, \quad i = 1, N \]
HJB equation - cont.

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\[ u = R^T y + \tau M, \quad M = \left( \frac{1}{\tau} \int_0^\tau \hat{\mu}_0(k) dk, \ldots, \frac{1}{\tau} \int_0^\tau \hat{\mu}_N(k) dk \right), \]

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This can also be written as \( \Phi_\tau = \sum_{i=0}^{N} \mathcal{L}_i \Phi \)

\[ \mathcal{L}_0 \Phi = \frac{1}{2} a_0 \frac{\partial^2 \Phi}{\partial y_0^2} - \frac{1}{2} b_0 \frac{\Phi^2}{\Phi}, \quad \mathcal{L}_i = \frac{1}{2} a_i \frac{\partial^2 \Phi}{\partial y_i^2}, \quad i = 1, N \]
Splitting

- In case of two assets \( y_0, y_1 \) (or \( Z \) and \( Y \)) the HJB equation can be solved by using an asymptotic expansion proposed in (Halperin, Itkin, ”Pricing options on illiquid assets with liquid proxies using utility indifference and dynamic-static hedging”, 2012, Submitted to Quantitative Finance).

- Henderson & Liang, 2011 studied a problem of computing counterparty risk of derivatives in incomplete markets in a set of one traded and multiple non-traded assets. They also used splitting (different). However, their method is of the first order in time.
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- In general a $N$-th dimensional variant of Strang’s splitting can be used which is $O(\Delta \tau^2)$. For instance, for linear operators, formally solve the equation.

$$
\Phi_\tau = \sum_i \mathcal{L}_i \Phi \rightarrow \Phi(\tau + \Delta \tau) = e^{\Delta \tau \sum_i \mathcal{L}_i} \Phi(\tau)
$$

and apply a generalized BCH formula

$$
e^{\Delta t \sum_i \mathcal{L}_i} = e^{\frac{\Delta t}{2} \mathcal{L}_0} e^{\frac{\Delta t}{2} \mathcal{L}_1} ... e^{\frac{\Delta t}{2} \mathcal{L}_{N-1}} e^{\Delta t \mathcal{L}_N} e^{\frac{\Delta t}{2} \mathcal{L}_{N-1}} ... e^{\frac{\Delta t}{2} \mathcal{L}_0} + O(\Delta t^2)
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e^{\Delta t \sum_i L_i} = e^{\frac{\Delta t}{2} L_0} e^{\frac{\Delta t}{2} L_1} \ldots e^{\frac{\Delta t}{2} L_{N-1}} e^{\Delta t L_N} e^{\frac{\Delta t}{2} L_{N-1}} \ldots e^{\frac{\Delta t}{2} L_0} + O(\Delta t^2)
\]
Splitting - cont.

Similar for nonlinear operators, (see Thalhammer, Koch 2010). We can represent the previous equation as

$$\Phi_{\tau} = \sum_i \mathcal{L}_i \Phi = \mathcal{L}_0 \Phi + \mathcal{L}_{1N} \Phi,$$

and use Strang splitting. Explicitly this means that at each time step we have to solve a system of three equations

$$\Phi^{*}_{\theta} = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi^{*} - \frac{1}{2} b_0 \frac{\Phi^{*,2}_{y_0}}{\Phi^{*}}, \quad \theta \in [0, \Delta \tau / 2], \quad (1)$$

$$\Phi^{**}_{\theta} = \mathcal{L}_{1N} \Phi^{**}, \quad \theta \in [0, \Delta \tau]$$

$$\Phi^{***}_{\theta} = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi^{***} - \frac{1}{2} b_0 \frac{\Phi^{***,2}_{y_0}}{\Phi^{***}}, \quad \theta \in [0, \Delta \tau / 2]$$

In what follows, we choose a specific payoff of the form $\Pi_{Y_i} = \min(Y_i, K_{y_i})$ that corresponds to a portfolio of bonds of firms $Y_i$ with strikes $K_{y_i}$ within the Merton credit-equity model. Then the terminal condition for $\Phi(y_0, ..., y_N, \tau)$ reads

$$\Phi(y_0, ..., y_N, 0) = \exp \left[ -\gamma \left( K_{y_0} e^{y_0} - \sum_{i=1}^{N} \alpha_i K_{y_i} e^{y_i} \right) \right]$$

where $y_i^- = \min(y_i, 0)$. 
Solution of HJB equation

Splitting - cont.

Similar for nonlinear operators, (see Thalhammer, Koch 2010). We can represent the previous equation as

\[ \Phi_{\tau} = \sum_i \mathcal{L}_i \Phi = \mathcal{L}_0 \Phi + \mathcal{L}_{1N} \Phi, \quad \mathcal{L}_{1N} = \sum_{i=1}^{N} \mathcal{L}_i \]

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\[ \Phi_{\theta}^* = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi_{\theta}^* - \frac{1}{2} b_0 \frac{\Phi_{\theta}^*,2}{\Phi_{\theta}^*}, \quad \theta \in [0, \Delta \tau/2], \quad (1) \]

\[ \Phi_{\theta}^{**} = \mathcal{L}_{1N} \Phi_{\theta}^{**}, \quad \theta \in [0, \Delta \tau] \]

\[ \Phi_{\theta}^{***} = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi_{\theta}^{***} - \frac{1}{2} b_0 \frac{\Phi_{\theta}^{***,2}}{\Phi_{\theta}^{***}}, \quad \theta \in [0, \Delta \tau/2] \]

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Splitting - cont.

- Recall we have to solve the following equations

\[ \Phi^*_\tau = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi^* - \frac{1}{2} b_0 \frac{\Phi^*.,^2}{\Phi^*}, \quad \Phi^*(0) = \Phi(\tau - \Delta \tau), \quad T = \tau - \Delta \tau / 2, \]

\[ \Phi^{**}\tau = \mathcal{L}_0^N \Phi^{**}, \quad \Phi^{**}(0) = \Phi^*(\tau - \Delta \tau / 2), \quad T = \tau, \]

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- The second equation in (2) is a $N$-dimensional heat equation. It can be solved using Fast Gauss Transform.
Splitting - cont.

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\[
\Phi^{**} = L_{0N} \Phi^{**}, \quad \Phi^{**}(0) = \Phi^*(\tau - \Delta\tau/2), \quad T = \tau,
\]

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- The first and third equations: change of variables (Cole-Hopf transformation) $\tilde{\tau} = a_0 \tau$, $\Phi = \Phi \frac{1}{1-(b_0/a_0)}$ also reduces them to the heat equation which can be solved using FGT.
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- Thus, using a grid with \( M \) points (could be non-uniform) in all dimensions we need to solve task 2 for \( M^N \) target points using a \( N \) dimensional FGT. Total complexity: \( \approx O(M^N) \), \( M \gg N \) operations. Same as FD, but better local space error.
Splitting - cont.

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  \[ \Phi_\tau^* = \frac{1}{2} a_0 \frac{\partial^2}{\partial y_0^2} \Phi^* - \frac{1}{2} b_0 \frac{\Phi^*_y}{\Phi^*}, \quad \Phi^*(0) = \Phi(\tau - \Delta\tau), \quad T = \tau - \Delta\tau/2, \]

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Computation of optimal hedges and option price

As the solution of the HJB equation can be found numerically pretty fast, hedge calculation and option pricing is almost straightforward:

- Find the price of "composite option" $g^\alpha_Z$ from the solution of HJB equation and indifference price equation
- Compute the optimal static static hedge $\alpha^* = (\alpha_1^*...\alpha_N^*)$ from equation
  $$g^\alpha_{Z} = \arg \max_{\alpha_1...\alpha_N} \left( -\frac{1}{\gamma} \log \Phi + \sum_{i=1}^{N} \alpha_i \rho_y i \right)$$
- Compute the optimal dynamic hedge in the index $S$ from the HJB equation, which is
  $$\theta^* = -\frac{\eta_s V_x + \sum_{i=0}^{N} \rho_{xy} \sigma_i V_{xy_i}}{V_{xx}}$$
  $$V(x, y, \tau) = -e^{-\gamma x e^{\int_{0}^{\tau} r(k)dk}} \left( e^{-\frac{1}{2} \int_{0}^{\tau} \eta_s^2(k)dk} \Phi(u, \tau), \quad u = R^T y \right).$$
- The option price is now $g^\alpha_{Z}^*$
Numerical example
We consider $N = 2$ and provide two sets of tests with parameters

| Test | $\mu_x$ | $\sigma_x$ | $r$  | $\rho_{yz}$ | $K_z$ | $\mu_z$ | $\sigma_z$ | $\rho_{xz}$ | $z_0$ | $K_y$ | $\mu_y$ | $\sigma_y$ | $\rho_{xy}$ | $y_0$ |
|------|---------|-----------|-----|-------------|------|---------|-----------|-------------|------|-------|---------|-----------|-------------|------|
| 1    | 0.04    | 0.25      | 0.02| 0.8         | 110  | 0.05    | 0.2       | 0.4         | 100  | 90    | 0.03    | 0.3       | 0.3         | 100  |
| 2    | 0.04    | 0.25      | 0.02| 0.8         | 110  | 0.05    | 0.3       | 0.3         | 50   | 90    | 0.03    | 0.3       | 0.2         | 100  |

**Table:** Initial parameters used in test calculations.

Below is a 3D plot of the value function $V(u, v)$ is presented for the initial parameters marked in Table 1 as 'Test 1'. We also use $\gamma = 0.03$, $\alpha = 1$. It is seen that $V(u, v)$ quickly goes to constant outside of a narrow region around $u = 0$ and $v = 0$. 
Analytic vs numerical method.

Next is the value function $V$ as computed for $v_0 = 1.22$ and $T = 10$ yrs. Here the first plot presents comparison of the numerical solution with zero and '0+1' approximations. The second plot compares the zero and first order approximation.

Results obtained with a second set of parameters (Test 2 in the Table 1) are shown below.
Analytic vs numerical method.

Here we present price $g_Z^\alpha$ computed in tests 1,2 as a function of $u$, where $p_Y$ is Black-Scholes put price with parameters of the corresponding tests.

Next the optimal hedge $\alpha_*$ is computed based on indifference pricing equation which was solved using Brent’s method.
Analytic vs numerical method.

Finally, price $g_Z^{\alpha}$ is presented as a function of $\alpha$ for various $\gamma$. It is seen that this function is convex which was first showed in Ilhan & Sircar, 2006 in a different setting.
Conclusion

- We proposed a framework for pricing derivatives written on illiquid asset using a mixed dynamic-static hedging in a proxy index and $N$ proxy options - the Hedged Incomplete-market Merton (HIM) model.
- Applicable for different asset classes and different payoffs.
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