Unpredictability, Information, and Chaos

Carlton M. Caves\textsuperscript{(a)} and Rüdiger Schack\textsuperscript{(b)}

\textsuperscript{(a)}Center for Advanced Studies, Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131-1156, USA
Telephone: (505)277-8674
FAX: (505)277-1520
E-mail: caves@tangelo.phys.unm.edu

\textsuperscript{(b)}Department of Mathematics, Royal Holloway, University of London, Egham, Surrey TW10 0EX, United Kingdom
E-mail: r.schack@rhbnc.ac.uk

\textbf{ABSTRACT}

A source of unpredictability is equivalent to a source of information: unpredictability means not knowing which of a set of alternatives is the actual one; determining the actual alternative yields information. The degree of unpredictability is neatly quantified by the information measure introduced by Shannon. This perspective is applied to three kinds of unpredictability in physics: the absolute unpredictability of quantum mechanics, the unpredictability of the coarse-grained future due to classical chaos, and the unpredictability of open systems. The incompatibility of the first two of these is the root of the difficulty in defining quantum chaos, whereas the unpredictability of open systems, it is suggested, can provide a unified characterization of chaos in classical and quantum dynamics.

Please direct all correspondence to C. M. Caves at the above address.
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I. UNPREDICTABILITY AND INFORMATION

Unpredictability. To discuss it, we need a framework. What is it? How is it described, and—if we intend a scientific discussion—how is it quantified? The goal in this introductory section is to suggest a framework for discussing and describing unpredictability.

Suppose one is interested in a particular set of alternatives. Unpredictability for these alternatives means simply that one can’t say reliably which alternative is the actual one. Several examples provide reference points for the discussion.

- A horse race where a gambler bets on the winner. The relevant alternatives for the gambler are the horses in the race.
- A game of frisbee golf played in a gusty wind. The alternatives of interest to a participant are the possible paths of the frisbee—more precisely, histories of the frisbee’s center-of-mass position and orientation as the frisbee is buffeted by the wind.
- A New Mexico farmer growing chiles. The chiles are infested by a pest that can be eliminated by applying a biological agent. The catch is that the agent is rendered ineffective if it rains within a day of application, not an unlikely event during the thunderstorm season of July and August. The relevant alternatives for the farmer are whether or not it rains on a given day.
- A photon incident on a polarizing beam splitter. An experimenter, by adjusting the orientation of the beam splitter, selects two orthogonal linear polarizations, which are sent along different paths by the beam splitter. The alternatives of interest to the experimenter are these two orthogonal linear polarizations.

In these examples, there is an element of unpredictability, an uncertainty as to which alternative turns out to be the actual one. Though prediction is often thought of as having to do with future behavior—our examples have this flavor—in this article we do not include any temporal relationship in the notions of predictability and unpredictability. The key element in unpredictability is uncertainty about the actual alternative among a collection of alternatives. There can be uncertainty about past as well as future events. For example, the horse race might have been run last night; not knowing the outcome, two friends could bet on the winner today, facing the same uncertainty about the outcome as they would have encountered before the race.

How is unpredictability described mathematically? By assigning probabilities to the alternatives. At the horse race the bettor assigns probabilities based on what he knows about the horses. These probabilities are betting odds: if the bettor assigns a probability $1/n$ for a horse to win, it means that he is willing to place a bet on that horse at odds of $n$ to 1 or better. The frisbee golfer, applying his experience to the present condition of the wind, assigns probabilities intuitively to various possible histories of the frisbee’s motion. The New Mexico farmer can assign a probability for rain based on his own experience and observations or, perhaps better, obtain a probability for rain from the National Weather Service or from private weather-forecasting concerns. The experimenter with the polarizing beam splitter assigns probabilities to the two linear polarizations based on what he knows.
about the photon. Suppose, for example, that the beam splitter is oriented at 45° to the vertical; the two output paths then correspond to linear polarizations at 45° and 135° to the vertical. If the experimenter knows that the photon is vertically polarized, then the rules of quantum mechanics dictate that he assign equal probabilities to the two output polarizations.

These probabilities are Bayesian probabilities.1–4 Bayesian probabilities apply to a single realization of the alternatives, as is evident in the examples. They are subjective in that their values depend on what one knows. Sometimes, as in the case of quantum mechanics, there are explicit rules for converting one’s knowledge into a probability assignment; in other cases, the values of the probabilities represent little more than hunches.

Reliable prediction requires probabilities that are close to 0 or 1. If probabilities for a single realization are not close to 0 or 1, it is still possible to make reliable predictions, if one has available a large ensemble of independent, identical realizations, all described by the same probabilities. In such a large ensemble, the frequencies of occurrence of the various alternatives can be predicted reliably to be close to the probabilities. Indeed, this is the reason that ensembles are used—to convert unpredictability for a single realization into reliable predictions for frequencies within a large ensemble of independent, identically distributed realizations.

Because the scientific enterprise requires precise predictions, scientific experiments often use a large ensemble of independent, identically distributed realizations. Sometimes, as in the case of the photon, such an ensemble is cheap, making it easy to perform precise experimental tests of probability assignments. In other cases, as in clinical trials of new drugs, it is difficult to assemble an appropriate ensemble, the large cost being justified only by the need for a precise test. Because of the importance of ensembles in scientific experiments, scientists are vulnerable to the notion that probabilities have meaning only as frequencies in a large ensemble.5 Yet probabilities for a single realization are used routinely in rational decision making. The example of the horse race shows this: the bettor determines his strategy for a single race from the probabilities that he assigns to that race.

The most misunderstood aspect of Bayesian probability theory is the relation between the probabilities for a single realization and frequencies within an ensemble of realizations. Because this relation is so important, yet so often misunderstood, it deserves a brief digression here. An ensemble has a bigger space of alternatives than does a single realization. The probabilities on this larger space do not follow from the probabilities assigned to individual realizations within the ensemble. Assigning probabilities to the ensemble alternatives involves considerations beyond those needed to assign probabilities to individual realizations.4,6 In particular, if there are correlations among the members of the ensemble, the probabilities assigned to the ensemble should reflect these correlations. When there are correlations, the probabilities for an individual realization generally do not predict frequencies within the ensemble. A simple example is that of a biased coin, the
bias known exactly except for its sign, which is unknown. The single-realization probabilities for heads and tails, which are clearly equal, do not predict the many-toss frequencies. The lesson here is that *Bayesian probabilities for individual realizations do not necessarily predict frequencies within an ensemble of realizations.*

There are special cases, however, for which the probabilities for individual realizations *do* predict reliably the frequencies within an ensemble. One of these special cases is the case of a large ensemble of independent, identically distributed realizations. This successful prediction of frequencies, important though it is, should not lead to the erroneous conclusion that probabilities are equivalent to frequencies within a large ensemble. Relations between probabilities and frequencies cannot be posited; they must be derived from probability theory itself.

The New Mexico farmer illustrates more generally how probabilities for a single realization enter into rational decision making, providing a simple example of *decision theory.*

For simplicity, assume that to be effective the biological agent must be applied on a particular day during the development of the pest. The Weather Service provides the farmer probabilities $p_R$, for rain that day, and $p_{R/} = 1 - p_R$, for no rain. If the farmer decides that it is not going to rain, he buys the biological agent and applies it to his chiles; if he decides that it is going to rain, he doesn’t. Yet how is the farmer to decide? Though the probabilities describe completely his uncertainty, he can’t make a decision on the basis of the probabilities alone. In addition to the probabilities, he must consider the costs of his actions. Given the costs, he can use the probabilities to make a rational decision.

Suppose that the pest, if left unchecked, reduces the value of the crop by $a$ dollars, and suppose that to buy and apply the biological agent costs $b$ dollars. If the farmer doesn’t buy the agent, then, rain or shine, his cost is the $a$ dollars for a damaged crop. If the farmer buys and applies the agent, his cost is $a + b$ dollars if it rains and $b$ dollars if it doesn’t; the average cost of buying and applying the agent is thus $(a+b)p_R + bp_{R/} = ap_R + b$. The farmer should take the action that has lower average cost. Hence, he should buy and apply the agent if $ap_R + b < a$ or, equivalently, if $p_R > b/a$. The costs determine how confident the farmer should be that it will not rain before it is prudent for him to buy and apply the biological agent.

Reliable prediction corresponds to probabilities near 0 or 1; other probability distributions describe varying degrees of unpredictability. How can the degree of unpredictability be quantified? A natural measure comes from information theory. The information missing toward specification of the actual alternative is given by the *Shannon information:*

$$H = - \sum_j p_j \log_2 p_j ,$$

where $p_j$ is the probability for alternative $j$ (the use of base-2 logarithms means that information is measured in bits). An alternative that has unit probability can be predicted definitely to occur; in this case there is no missing information, i.e., $H = 0$. At the other extreme, if the alternatives are equally likely, then the situation is maximally unpredictable;
the missing information is maximal, with value $H = \log_2 J$ bits, where $J$ is the number of alternatives. This is the case, for example, when the vertically polarized photon is incident on the beam splitter oriented at $45^\circ$; the equal probabilities for the two output paths correspond to 1 bit of missing information. For probability distributions between the two extremes, the Shannon information takes on intermediate values.

One can acquire the missing information by observing which alternative is the actual one. Wait for the finish of the horse race, and see which horse wins. Observe the frisbee to see which path it takes. Wait to see it if rains. Wait for the photon to pass through the polarizing beam splitter, and determine which direction it goes. This sort of observation is, of course, not prediction. It is worth stressing that the reason it is not prediction is not the element of “waiting” in our examples; the reason is that the missing information is acquired by observing the alternatives themselves.

In contrast to observing the alternatives, it is often the case that uncertainty about the actual alternative can be reduced or even eliminated entirely by gathering information about factors that influence the alternatives or determine the actual alternative. For example, the frisbee golfer, by combining detailed observations of the wind velocity immediately upstream (initial conditions) with a model of the local terrain (boundary conditions), could integrate the coupled equations for the wind and the frisbee’s motion, thus allowing him to predict with certainty the frisbee’s path. The New Mexico farmer might persuade the National Weather Service to gather sufficiently fine-grained meteorological data (initial and boundary conditions) so that it could integrate the hydrodynamic equations for the atmosphere, thereby predicting thunderstorms a day in advance. In the spirit of these examples, it is tempting to posit in general an independent record of the missing information, outside the realization of interest, an “archive” that records the actual alternative. The archive stores the missing information in an encoded form, which must be decoded through, for example, integration of an appropriate partial differential equation. Nonetheless, if one has access to the archive and can decode the information stored there, predictability can be restored.

It is useful to compare this description of unpredictability with that of a noiseless communication channel.9–11 A transmitter prepares one of several alternative messages and sends it down a channel to a receiver, which reads the message. How much information is communicated from transmitter to receiver? The transmitter prepares the alternative messages with various probabilities. The receiver is unable to predict which message it will receive and thus acquires, on average, an amount of information given by the Shannon information constructed from the message probabilities. The transmitter retains a record of which message it sent, perhaps in an encoded form; by consulting the transmitter directly, one can eliminate the uncertainty about which message is sent down the channel.

The archive where a record of the actual alternative is stored is like a transmitter: it is a source both of unpredictability and of information. Indeed, the lesson is that a source of unpredictability is the same as a source of information. By gaining access to the archive,
one can acquire the missing information about which alternative is the actual one, thereby restoring predictability.

In the examples of frisbee golf and the New Mexico farmer, the missing information is available in the initial conditions and boundary conditions that determine a unique solution to a set of differential equations. In the horse-race example, the bettor can improve his prediction by gathering data about the previous performances of the horses and about specific conditions on the race day. We don’t know if the remaining unpredictability can be eliminated by gathering yet more information, because our understanding of the factors that enter into determining the winner of a horse race is incomplete. To be more precise, we don’t know whether there is a mathematical model that specifies what information needs to be collected and how that information is to be decoded so as to predict the winner with something approaching certainty. Because we have no complete mathematical model of a horse race, we place its unpredictability, for the present, outside the scope of scientific inquiry.

The case of the photon is the most interesting. Within quantum mechanics there is no archive that can be consulted to determine the photon’s path through the beam splitter, no identifiable transmitter of the bit of information that specifies the photon’s linear polarization. The bit pops into existence out of nowhere. Yet, unlike the horse race, where there is no complete mathematical model, quantum mechanics is thought to be a complete theory, which provides the framework for all fundamental physical law, a framework in which the probabilities are intrinsic. Quantum-mechanical probabilities cannot be eliminated by gathering more information about the photon’s state, for to say that the photon is vertically polarized is already a maximal quantum description of its state.

We are in a position now to characterize how fundamental a source of unpredictability is. It seems sensible to say that the more difficult it is to consult the archive and acquire the missing information necessary for predictability, the more fundamental is the unpredictability. In this regard quantum unpredictability is in a class by itself: there is no archive that stores an independent record of the information that is acquired in a quantum measurement. Quantum unpredictability is a consequence of information without a source; it cannot be eliminated by consulting an archive because there is no archive. Unpredictability without an information source is so fundamental that we reserve for it the appellation absolute unpredictability.

In the case of the New Mexico thunderstorm, the missing information is available as initial data, but it is is very difficult to obtain because of the phenomenon of classical chaos, which means that the coarse-grained past does not predict the coarse-grained future. To predict a phenomenon on the coarse-grained scale of a thunderstorm requires initial data on a much finer scale—indeed, a scale that is exponentially finer in the time over which one desires a reliable prediction.

Although the data required to predict the coarse-grained dynamics of a chaotic system are very difficult to obtain, that they are obtainable in principle cautions that care should
be exercised in characterizing the difficulty of the task. In the spirit of decision theory, what ought to be done is to compare costs: the cost of consulting the archive, obtaining the missing information, and thereby eliminating the unpredictability should be compared with the cost of not having the missing information and thereby having to deal with the resulting unpredictability. If the cost of obtaining the missing information exceeds the benefit of having it, we can point to a fundamental reason for “allowing” the unpredictability. Indeed, a direct comparison of costs is the only way we know of to quantify how fundamental a source of unpredictability is. We refer to unpredictability for which the cost of obtaining the missing information far exceeds the benefit of having it as strong unpredictability. Having introduced the notions of absolute unpredictability and strong unpredictability, we can continue to use—and encourage others to use—the phrase “fundamental unpredictability” in any other fashion desired.

We develop these ideas in the next section, with a discussion of three sources of unpredictability in physics: the absolute unpredictability of quantum mechanics, the unpredictability of the coarse-grained future of a classically chaotic Hamiltonian system, and unpredictability that arises when a physical system is coupled to a perturbing environment. Our main interest is how these three kinds of unpredictability are related to chaos in classical and quantum dynamics. The first two sources of unpredictability have already been discussed; their incompatibility lies at the heart of the difficulty in formulating a description of chaos in quantum dynamics. A further difficulty is that the unpredictability of the coarse-grained future of a chaotic system does not lend itself to a meaningful comparison of the costs and benefits of obtaining the missing information. The third kind of unpredictability, due to environmental perturbation, can be used to put chaos in classical and quantum dynamics on the same footing, as we show in the last section of the article. In particular, we describe a new characterization of classical chaos in terms of sensitivity to environmental perturbation, a characterization in which costs and benefits can be compared directly, with classical chaos emerging as a strong source of unpredictability. We indicate briefly how this same way of characterizing chaos can be applied to quantum dynamics.

II. SOURCES OF UNPREDICTABILITY IN PHYSICS

In this section we focus on scientific unpredictability, specifically, unpredictability in physics. One doesn’t have to look far to find such unpredictability; we need only look for any place where physicists employ a probabilistic description.

The obvious place to look is quantum mechanics, which in our present understanding provides the framework in which fundamental physical laws are formulated. In a quantum-mechanical description, even if one has maximal information about a physical system, i.e., knows its quantum state exactly, one nonetheless cannot predict the results of most measurements. We can extend the photon example to provide a vivid illustration of this. Suppose that the photon, initially polarized along the vertical axis, is incident on a series of polarizing beam splitters whose orientations alternate between 45° and vertical. At
each beam splitter there is a bit of missing information about which direction the photon goes. It appears that the photon is an inexhaustible source of information, yet within the conventional formulation of quantum mechanics, there is no source for this information, no archive that can be consulted to predict which path the photon will take through the series of beam splitters. This is what we have designated an absolute source of unpredictability.

One might expect the entire discussion in this Workshop to be focused on the absolute unpredictability of quantum mechanics. That it isn’t requires explanation, and we can suggest two reasons. The first is that an alternative theory in which there is an identifiable source for the missing information has serious drawbacks, a fact made evident by 30 years of work on Bell inequalities.\textsuperscript{12} The desire for a source for the missing information—a quantum archive—is strong. From the early days of quantum mechanics, many physicists have found it unreasonable to have intrinsic unpredictability—unpredictability without a source of information—and they have posited the existence of “hidden variables” that constitute a quantum archive.\textsuperscript{12,13} The hidden variables, though they are presently and perhaps permanently inaccessible, would provide enough information, could they be consulted, to eliminate quantum unpredictability.

The catch is the following: the Bell inequalities show that if a hidden-variable theory is to agree with the statistical predictions of quantum mechanics—and, as experiments show, if it is to agree with observation—the hidden variables must be nonlocal. The archive cannot have independent subarchives for different subsystems (for example, a subarchive within the apparatus that observes each subsystem), but rather must be one enormous record that commands and correlates the behavior of everything in the Universe.\textsuperscript{[1]} The necessity for hidden-variable theories to be nonlocal makes them considerably less attractive—depending on one’s taste, even less attractive than the absolute unpredictability of quantum mechanics. Yet if it turns out that the fundamental constituents of matter exist in more dimensions than the four of our familiar spacetime, as in string theory, then locality within four-dimensional spacetime might lose much of the force we presently attach to it.

The second reason, mentioned in Hartle’s introductory lecture at this Workshop, is this: the present Universe is enormously complex, its particularities describable only by a great deal of information; if the fundamental physical laws and the initial conditions are simple, where does all that information come from? In a hidden-variable theory, the complexity of the present Universe is a revelation of the details of the hidden variables; because the hidden variables can be thought of as part of the initial conditions, the initial conditions necessarily become complex. It is somehow more appealing to imagine that the laws and initial conditions are simple and that there is no archive in which is written an independent record of the complexity of the present Universe; quantum mechanics obliges

\textsuperscript{[1]} This conclusion about the nonlocality of a hidden-variable archive is true even if the archive is forever inaccessible. Indeed, the notion of a nonlocal archive is perhaps easier to swallow if it is inaccessible.
by making the complexity almost wholly a consequence of the unpredictability of quantum rolls of the dice.

What about unpredictability in classical physics? Nonlinear classical systems—here restricted to Hamiltonian systems—can display a kind of unpredictability that comes from classical chaos.\textsuperscript{14} Classical chaos is usually characterized in terms of the unpredictability of phase-space trajectories. Consider the points along a phase-space trajectory at a discrete sequence of uniformly spaced times. The points are never given to infinite precision; any finite precision corresponds to a gridding of phase space into coarse-grained cells. The sequence of finite-precision points, coarse grained both on phase space and in time, is what is meant by a coarse-grained trajectory. For a classically chaotic system, coarse-grained initial data do not predict a unique coarse-grained trajectory; more precisely, to predict a unique coarse-grained trajectory requires initial data that become exponentially finer in the time over which the prediction is desired.

It is instructive to review the mathematical formulation of classical chaos in which the initial data appear explicitly as a source of unpredictability and information. Consider a classical system whose motion is restricted to a compact phase-space region, represented as a square in Figure 1. Grid this phase space into coarse-grained cells of uniform volume $\mathcal{V}$. The coarse-grained initial data (at time $t = 0$) are that the initial phase-space point lies somewhere within a particular coarse-grained cell. This corresponds to a phase-space density that is uniform on the initial cell. Under a chaotic Hamiltonian evolution, the phase-space density spreads across phase space, creating a pattern of uniform density (see Figure 1), which occupies the same volume as the initial cell and which develops structure on finer and finer scales as the evolution proceeds.

At each of the discrete times $t$, the evolved pattern can be partitioned into all its separate intersections with the initial grid. Each piece of the partition corresponds to at least one coarse-grained trajectory that issues from the initial cell and terminates in that piece at time $t$. It turns out that we introduce no error into the present discussion by pretending that each piece of the partition corresponds to a unique coarse-grained trajectory. Label the various pieces by an index $j$, and let $\mathcal{V}_j$ be the phase-space volume of the $j$th piece. The probability for the corresponding coarse-grained trajectory is $q_j = \mathcal{V}_j / \mathcal{V}$, the fraction of the original phase-space volume occupied by the $j$th piece. The information needed to specify a particular coarse-grained trajectory out to time $t$ is given by the Shannon information constructed from the probabilities $q_j$. For a chaotic evolution, in the limit of large $t$, this information grows linearly in time:

$$- \sum_j q_j \log_2 q_j \sim K t .$$  \hspace{1cm} (2)

The linear rate of information increase, $K$, called the Kolmogorov-Sinai (KS) or metric entropy,\textsuperscript{15} quantifies the degree of classical chaos.\textsuperscript{[2]}

\textsuperscript{[2]} \hspace{1cm} The definition of KS entropy given here is not quite right, because $K$ can depend on
The information (2) is missing information about which is the actual coarse-grained trajectory. The missing information can be obtained from the actual initial condition. The correspondence can be made explicit in the following way. The \( j \)th piece of the partition of the evolved pattern corresponds to a region of volume \( V_j \) within the initial cell; this region, which has probability \( q_j \), is the region of initial conditions that lead to the coarse-grained trajectory that terminates in the \( j \)th piece of the partition. Thus at each time \( t \) the initial cell is partitioned into initial-condition regions, each of which gives rise to a particular coarse-grained trajectory out to time \( t \).

Imagine gridding the initial cell into very fine cells of uniform volume \( \Delta v \), cells so fine that they are much finer than the initial-condition regions for all times of interest. The information needed to specify a particular fine-grained cell within the initial coarse-grained cell—this is the entropy of the initial phase-space density—is \( \log_2 (V/\Delta v) \). This information, when written as

\[
\log_2 \frac{V}{\Delta v} = - \sum_j q_j \log_2 q_j + \sum_j q_j \log_2 \frac{V_j}{\Delta v},
\]

illustrates how the initial data act as an archive for the coarse-grained trajectory. The first term on the right is the information needed to specify the initial-condition region for a particular coarse-grained trajectory. The second term is the further information needed to specify a fine-grained cell within an initial-condition region. The total information needed to specify a fine-grained cell is the sum of these two terms. As a chaotic evolution proceeds, more and more of the information needed to specify a fine-grained initial cell is required to predict a particular coarse-grained trajectory.

A crude, but instructive picture of what is happening is that the number of pieces in the partition of the evolved pattern grows as \( 2^{Kt} \), each piece having roughly the same phase-space volume and, hence, the same probability \( q_j = 2^{-Kt} \). As the evolution proceeds, the corresponding partition of the initial cell becomes exponentially finer, consisting of roughly \( 2^{Kt} \) initial-condition regions, and the information needed to specify a particular region grows linearly in time. A coarse-grained trajectory can be regarded as a progressive unveiling of finer and finer details of the actual initial data within the initial coarse-grained cell.

How fundamental is the chaotic unpredictability of coarse-grained trajectories? It’s not absolute unpredictability, as in quantum mechanics, because it is easy to identify the source of information that must be consulted to eliminate the unpredictability. The unpredictability of a coarse-grained trajectory is due wholly to a lack of knowledge of the initial conditions. The source of information is thus the initial conditions, which when the choice of initial cell. The quantity that grows asymptotically as \( Kt \) is really the average of the information on the left side of Eq. (2) over all initial cells. We ignore this distinction here, thereby assuming implicitly that the chaotic system has roughly constant Lyapunov exponents over the accessible region of phase space.
decoded through the equations of motion, yield a unique prediction for the coarse-grained trajectory. Of course, if the required initial data are so fine that they are at the quantum level on phase space, then the unpredictability of the coarse-grained trajectory becomes sensitive to the absolute unpredictability of quantum mechanics. Classical chaos then serves as an amplifier of quantum unpredictability to a classical level.\(^{16}\)

Suppose we wish to assess how fundamental is the unpredictability of a chaotic coarse-grained trajectory. Since the initial data are a source for the missing information needed to predict a coarse-grained trajectory, we ought to compare the cost of obtaining the necessary initial data with the cost of the unpredictability that comes from not having the required data. Here a problem arises: it is difficult to formulate this comparison in a way that is intrinsic to the system under consideration, because the costs generally depend on factors external to the system. Take the New Mexico farmer as an example. The cost of acquiring the required initial data depends on the level of technology used in gathering the meteorological data. Worse, the cost of unpredictability is highly dependent on who assesses the cost. It may be important to the farmer to know whether it will rain, but the one of us who lives in Albuquerque generally doesn’t care much whether there is a thunderstorm on a particular summer day. When he does care, a ten-minute warning, easily obtained by looking out the window, is generally sufficient.

Classical physics has none of the absolute unpredictability of quantum mechanics. Does quantum mechanics have any of the sensitivity to initial conditions that is displayed by classically chaotic systems? There is no sensitivity to initial conditions in the evolution of the quantum state vector: the unitarity of quantum evolution implies that it preserves inner products, so the “distance” between state vectors remains unchanged during quantum evolution. Suppose one looks for sensitivity to initial conditions in the “coarse-grained trajectory” of some observable like position or momentum. Such a coarse-grained quantum trajectory is constructed by periodically making coarse-grained measurements of the observable. The problem is that the measurements generally introduce the absolute unpredictability of quantum mechanics, making the coarse-grained trajectory unpredictable for reasons that are essentially independent of the quantum dynamics. One ends up studying not any sort of sensitivity to initial conditions in quantum dynamics, but rather the absolute unpredictability of quantum mechanics.

The incompatibility of the absolute unpredictability of quantum mechanics with the classical unpredictability due to sensitivity to initial conditions is the chief difficulty in formulating a description of quantum chaos. What is needed is a description of chaos that, avoiding the absolute unpredictability of quantum mechanics and the classical sensitivity to initial conditions, is formulated in terms of a form of unpredictability that is common to classical and quantum physics. Notice, in particular, that instead of trying first to formulate a description of quantum chaos, the primary task is to reformulate the description of classical chaos, albeit in a way that is equivalent to the standard characterization in terms of the unpredictability of coarse-grained trajectories. We have suggested and investigated such a new way to characterize chaos, which we introduce here by considering yet a third
source of unpredictability, the unpredictability of an open system, i.e., a system that is coupled to a perturbing environment.\textsuperscript{17–22}

In investigating this third source of unpredictability, an essential tool is the entropy of a physical system. We introduce the notion of entropy, in both classical and quantum physics, as the missing information about the system’s fine-grained state.\textsuperscript{23,24} For a classical system, suppose that phase space is gridded into very fine-grained cells of uniform volume $\Delta v$, labeled by an index $j$. If one doesn’t know which cell the system occupies, one assigns probabilities $p_j$ to the various cells; equivalently, in the limit of infinitesimal cells, one can use a phase-space density $\rho(X_j) = p_j/\Delta v$. The \textit{classical entropy} (measured in bits),

$$ H = -\sum_j p_j \log_2 p_j = -\int dX \rho(X) \log_2 (\rho(X) \Delta v) , $$

(4)

is the missing information about which fine-grained cell the system occupies. For example, throughout this article we use as initial data a phase-space density that is uniform on a coarse-grained cell of volume $V$; the corresponding entropy is $\log_2 (V/\Delta v)$. In quantum mechanics the fine-grained alternatives are normalized state vectors in Hilbert space. From a set of probabilities for various state vectors, one can construct a density operator

$$ \hat{\rho} = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j| , $$

(5)

where the state vectors $|\psi_j\rangle$ are the eigenvectors of $\hat{\rho}$, with eigenvalues $\lambda_j$. The normalization of the density operator, $\text{tr}(\hat{\rho}) = 1$, implies that the eigenvalues make up a normalized probability distribution. The \textit{von Neumann entropy} of $\hat{\rho}$ (measured in bits),

$$ H = -\text{tr}(\hat{\rho} \log_2 \hat{\rho}) = -\sum_j \lambda_j \log_2 \lambda_j , $$

(6)

can be thought of as the missing information about which eigenvector the system is in.

Entropy quantifies the degree of unpredictability about the system’s fine-grained state. What makes it such an important quantity is that there is a readily identifiable cost, intrinsic to the system, for the inability to predict the system’s fine-grained state. Suppose that the system exists in the presence of a heat reservoir at temperature $T$, so that all exchanges of energy with the system that are not in the form of useful work must ultimately be exchanged with the reservoir as heat. Then each bit of entropy reduces the useful work that can be extracted from the system by $k_B T \ln 2$, where $k_B$ is the Boltzmann constant. (The factor $k_B \ln 2$ is a change of units; it translates entropy from bits to conventional thermodynamic units.) The cost of missing information is a reduction in the useful work that can be extracted from the system.

Entropy remains unchanged under Hamiltonian dynamical evolution, both classically and quantum mechanically. Classically this follows from the preservation of phase-space volume under Hamiltonian evolution; quantum mechanically it follows from the unitarity
of Hamiltonian evolution, which preserves the eigenvalues of the density operator. Suppose, however, that the system is coupled to a perturbing environment. The interaction disturbs the system’s evolution; averaging over the disturbance generally causes the system’s entropy to increase. This is a standard mechanism for entropy increase, the increase quantifying the decreasing ability to predict the fine-grained state of the system. In this case it is obvious that the source of unpredictability—the source of information—is the perturbing environment. By observing the environment, one can determine aspects of the perturbation, thus reducing the entropy increase of the system and rendering the fine-grained state of the system more predictable.

The rub is that the information acquired by observing the environment has a thermodynamic cost, too, a cost paid when the information is erased. For erasure into a heat reservoir at temperature $T$, this Landauer erasure cost\textsuperscript{25,26} is $k_B T \ln 2$ per bit, exactly the same as the cost of missing information. The Landauer erasure cost exorcises Maxwell demons.\textsuperscript{27–30} A demon observes a system directly, thereby decreasing the system’s entropy—according to the demon, the system’s fine-grained state becomes more predictable—and increasing the amount of work that the demon can extract from the system. The demon can’t win, however, because the entropy reduction, averaged over the possible results of the observation, is equal to the amount of information acquired from the observation; hence the erasure cost of the acquired information cancels the increase in available work. Turned on its head, this line of argument shows that if the Second Law of Thermodynamics is to be maintained against the demon’s attack, acquired information must have a thermodynamic cost of $k_B T \ln 2$ per bit, as was first noted by Szilard\textsuperscript{31}; Landauer\textsuperscript{25} realized that the cost is paid when the information is erased.

A demon observes the system directly. Here we contemplate something different: making inferences about the system by observing the environment that interacts with it. This difference is crucial for two reasons. First, when observing the environment, there is no necessary balance between the entropy reduction and the amount of acquired information; this permits a nontrivial comparison between the cost of acquiring the information from the environment and the cost of not having it. Second, by observing only the environment, we are considering a kind of unpredictability that can be formulated in the same way in both classical and quantum physics; this allows a meaningful comparison of classical and quantum dynamics. Both these reasons deserve further discussion.

To discuss the first reason, it is useful to introduce the notation that we use in comparing costs. Averaging over the perturbing environment causes the system’s entropy to increase by an amount $\Delta H_0$. By observing the environment, one can make the system’s entropy increase smaller. We let $\Delta I$ be the amount of information acquired from the observation, and we let $\Delta H \leq \Delta H_0$ be the corresponding entropy increase of the system, averaged over the possible results of the observation. The reduction in the system’s entropy as a consequence of the observation is $\Delta H_0 - \Delta H$. The acquired information, which has a thermodynamic cost of $\Delta I k_B T \ln 2$, buys an increase in available work of $(\Delta H_0 - \Delta H) k_B T \ln 2$. Because entropy and acquired information weigh the same in the
balance of thermodynamic cost, we can compare directly the cost of acquiring the information with the benefit of having it just by comparing the amount of acquired information, $\Delta I$, with the entropy reduction it purchases, $\Delta H_0 - \Delta H$.

If one doesn’t observe the environment, one acquires no information, i.e., $\Delta I = 0$, and the entropy reduction is zero, i.e., $\Delta H = \Delta H_0$. A very coarse observation of the environment gathers very little information and yields very little entropy reduction. The entropy reduction can be made progressively larger by making progressively more detailed observations, which gather more and more information about the environmental perturbation. The entropy reduction cannot exceed the information acquired, i.e.,

$$\Delta I \geq \Delta H_0 - \Delta H$$  \hspace{1cm} (7)

—this is another expression of the Second Law—but generally the entropy reduction is smaller, and can be much smaller, than the information acquired from the observation.

How can such an imbalance occur? Classically, the reduction in entropy that comes from observing the environment can be pictured roughly in the following way. Averaging over the perturbing environment yields a phase-space density—this we call the average density—that occupies a phase-space volume bigger than the initial volume by a factor of $2^{\Delta H_0}$. The observation determines a phase-space density, within the average density, that occupies a volume smaller than the average density by a factor of $2^{\Delta H_0 - \Delta H}$. If the results of the observation corresponded to a set of (equally likely) nonoverlapping densities that fit within the average density, then, there being about $2^{\Delta H_0 - \Delta H}$ of these nonoverlapping densities, the information acquired from the observation would be roughly $\log_2(2^{\Delta H_0 - \Delta H}) = \Delta H_0 - \Delta H$. Generally, however, the results of the observation correspond to overlapping densities within the average density, of which there can be many more than the number of nonoverlapping densities. Consequently, the information $\Delta I$ can be much larger than the entropy reduction. The discussion in the next section indicates how a proliferation of overlapping perturbed densities arises as a consequence of chaotic classical dynamics, the result being an exponential imbalance between information and entropy reduction.

The same explanation for a potential imbalance between acquired information and entropy reduction works quantum mechanically, with the average phase-space density replaced by an average density operator and with the nonoverlapping and overlapping densities replaced by orthogonal and nonorthogonal density operators. The potential for an imbalance in quantum mechanics arises because an observation of the environment generally determines one of a set of nonorthogonal density operators, of which many more can contribute to the average density operator than can orthogonal density operators.

For open systems strong unpredictability can now be seen to mean that the cost of acquiring information from the environment and thus making the fine-grained state more predictable is much greater than the benefit of having that predictability, i.e.,

$$\Delta I \gg \Delta H_0 - \Delta H$$  \hspace{1cm} (8)
We say that a system is *hypersensitive to perturbation* if, for the optimal way of observing the environment, it displays this strong unpredictability for all values of $\Delta H$. We have used hypersensitivity to perturbation to characterize classical and quantum chaos. A rigorous analysis\textsuperscript{21} in symbolic dynamics\textsuperscript{15} shows that classically chaotic systems display an *exponential hypersensitivity to perturbation*, for which

$$\Delta I \sim 2^{Kt}(\Delta H_0 - \Delta H).$$

The acquired information becomes exponentially larger than the entropy reduction, with the exponential rate of increase given by the KS entropy of the chaotic dynamics. In the next section we present a heuristic version of the symbolic dynamics analysis.

What allows one to compare costs directly, in a way that is intrinsic to the system, is the connection of missing and acquired information to thermodynamics and statistical physics. Indeed, this connection provides a statistical-physics motivation for our approach. Why does one average over the environment and allow the entropy of a system to increase? Usually one gives an excuse: the environment is said to be so complicated that averaging over it is the only practical way to proceed. We reject such apologetics in favor of a direct comparison of costs. When a system is hypersensitive to perturbation, so that the acquired information far exceeds the entropy reduction, it is thermodynamically highly unfavorable to try to reduce the system entropy by gathering information from the environment. The thermodynamically advantageous course is to average over the perturbing environment, thus allowing the system entropy to increase. Thus *the strong unpredictability of classically chaotic open systems provides a justification for the entropy increase of the Second Law of Thermodynamics.*

Return now to the second reason for considering observations of the environment. Classically we are dealing with the ability to predict a phase-space density when a system is disturbed by a perturbing environment; quantum mechanically we are dealing with the ability to predict a state vector when a system is so disturbed. Thus we deal with a kind of unpredictability that is common to classical and quantum physics.

The key to placing classical and quantum unpredictability on the same footing is to put aside phase-space trajectories, dealing instead with phase-space densities classically and with state vectors quantum mechanically. When the system is unperturbed, the evolution of classical phase-space densities is governed by the Liouville equation, and the evolution of quantum state vectors by the Schrödinger equation. Neither the Liouville equation nor the Schrödinger equation displays sensitivity to initial conditions: the overlap of phase-space densities is preserved by the canonical transformations of Liouville evolution, and the overlap of state vectors is preserved by the unitary transformations of Schrödinger evolution. Moreover, there is none of the absolute unpredictability of quantum mechanics, because we are considering deterministic Schrödinger evolution rather than the unpredictable results of measurements on the system. To this predictable unperturbed evolution, we add unpredictability by including a perturbing environment, and we ask how the perturbation degrades the ability to predict phase-space densities classically and to predict state vectors...
quantum mechanically. The source of unpredictability in both cases is the disturbance introduced by interaction with the environment.

Before going on to our heuristic argument for classical hypersensitivity, it is important to mention that we model the perturbing environment by adding a stochastic term to the system Hamiltonian. Such a stochastic perturbation can be realized as a particular kind of coupling to an environment: one couples the system to conserved quantities of an environment; different values of the conserved quantities specify the various realizations of the stochastic perturbation. Such a stochastic Hamiltonian is by no means the most general kind of coupling to an environment. What is missed by the stochastic model are classical correlation with the environment and quantum entanglement with the environment.

III. UNPREDICTABILITY AND CHAOS

We turn now to a discussion of hypersensitivity to perturbation in classically chaotic systems. Our objective is not to give a rigorous analysis, but rather to capture the flavor of why classically chaotic systems display exponential hypersensitivity to perturbation. For systems that have a symbolic dynamics, a rigorous analysis has been given, and the reader intent on rigor or just interested in a more thorough formulation of the problem is referred to that analysis.

Consider a classical Hamiltonian system, which is globally chaotic on a compact region of phase space, the degree of chaos characterized by the KS entropy $K$. Recall the picture of the system dynamics that was introduced in Section II (see Figure 1). The initial data, that the system occupies a coarse-grained cell of volume $V$, correspond to an initial phase-space density that has the uniform value $V^{-1}$ over the initial cell. Under the chaotic dynamics the initial coarse-grained cell is stretched and folded to form an increasingly intricate pattern on phase space. The evolving pattern has the same volume $V$ as the initial coarse-grained cell, and the phase-space density is uniform over the evolving pattern, with value $V^{-1}$.

The dynamics of the pattern can be described crudely as an exponential expansion in half the phase-space dimensions and an exponential contraction in the other half of the phase-space dimensions. The exponential rate of expansion or contraction in a particular phase-space dimension is given by a typical Lyapunov exponent $\lambda = K/D$, where $2D$ is the dimension of phase space. The expansion in the expanding dimensions means that the phase-space pattern spreads over roughly $(2^\lambda)^D = 2^{Kt}$ coarse-grained cells at time $t$. The width of the pattern in a contracting dimension is roughly $2^{-\lambda t}V^{1/2D}$.

We now imagine perturbing this evolution stochastically. The perturbation is modeled as a diffusion on phase space, characterized by a diffusion constant $D$. Such a perturbation is described by adding a stochastic term to the system Hamiltonian: during each small time interval the system evolves according to its own Hamiltonian plus an additional Hamiltonian selected randomly from a continuous set of possible perturbing Hamiltonians. For a particular temporal sequence of perturbing Hamiltonians—we call such a realization of the perturbation a *perturbation history*—the phase-space pattern is disturbed in a particular
way (see Figure 1). The resulting perturbed pattern is different from the unperturbed pattern, but it occupies the same volume $V$ as the unperturbed pattern, and the perturbed phase-space density has the uniform value $\nu^{-1}$ on the perturbed pattern.

Averaging over the possible perturbed patterns—i.e., averaging over all the perturbation histories—yields an average phase-space density, as shown in Figure 1. The diffusion “smears out” the unperturbed pattern into an average density that occupies a larger volume than the unperturbed pattern.

Formally the evolution of the phase-space density is described by a Liouville equation that has a stochastic contribution to the Hamiltonian. Each perturbation history corresponds to a particular realization of the stochastic term in this Liouville equation and yields a particular perturbed pattern. The equation that governs the evolution of the average density is obtained by averaging the stochastic Liouville equation over all perturbation histories. The resulting evolution equation, a Fokker-Planck equation on phase space, has a systematic term that describes the unperturbed Hamiltonian evolution and a diffusion term that describes the perturbation. The perturbation is characterized by its strength and by how it is correlated across phase space. Both of these aspects of the perturbation are important for our discussion.

An important time emerges from the interplay between the unperturbed dynamics and the diffusion. During a typical Lyapunov time $\lambda^{-1}$, the diffusion smears out the average density by an amount $\sqrt{D/\lambda}$ in each phase-space dimension. In the expanding dimensions this smearing is overwhelmed by the expansion, but in the contracting dimensions it becomes important once the width of the unperturbed phase-space pattern becomes comparable to the amount of diffusion in a Lyapunov time. After this time the diffusion balances the exponential contraction of the dynamics, with the result that the average density ceases to contract in the contracting dimensions. We say that the perturbation becomes effective at a time given roughly by

$$t_{\text{eff}} \sim \frac{1}{2\lambda} \log_2 \left( \frac{\lambda \nu^{1/D}}{D} \right).$$

No matter how weak the perturbation, the exponential contraction of the chaotic dynamics eventually renders the perturbation effective, typically within several Lyapunov times.

After the perturbation becomes effective, the average phase-space density continues to expand exponentially in the expanding dimensions, but this expansion is no longer balanced by contraction in the contracting dimensions. Thus the average density occupies an exponentially increasing phase-space volume $V_0 \sim 2^{K(t-t_{\text{eff}})}\nu$—i.e., a factor of $2^{K(t-t_{\text{eff}})}$ larger than the phase-space volume occupied by the unperturbed pattern—and the entropy of the average density increases as

$$\Delta H_0 \sim K(t - t_{\text{eff}}) \quad \text{for } t \gtrsim t_{\text{eff}}.$$  

Once the perturbation becomes effective, the entropy increase $\Delta H_0$ of the average density is determined by the KS entropy of the system dynamics, not by some property of the perturbation. We assume for the remainder of the discussion that $t \gtrsim t_{\text{eff}}$. 

The unpredictability quantified by the entropy increase (11) has an obvious source in the stochastic perturbation. The perturbation histories constitute an archive that can be consulted to reduce the entropy increase. Acquiring information about the perturbation means finding out something about which perturbation history is the actual one. Our task is to estimate how much information $\Delta I$ about the perturbation is required to reduce the entropy increase to $\Delta H \leq \Delta H_0$. Determining the actual perturbation history specifies a particular perturbed pattern, which, since it occupies the same phase-space volume as the unperturbed pattern, has the same entropy, so that the entropy increase $\Delta H$ is kept to zero.

For a diffusive perturbation, however, determining the actual perturbation history means singling out one history from an infinite number of histories and thus requires an infinite amount of information. We are more interested here in acquiring partial information about the perturbation, which means determining that the actual perturbation history lies in some class of perturbation histories. Suppose the corresponding class of perturbed patterns, when averaged together, produces a partial density that occupies a phase-space volume $2^{\Delta H} \mathcal{V} = 2^{-(\Delta H_0 - \Delta H)} \mathcal{V}_0$, so that the corresponding entropy increase is $\Delta H$. We must estimate how much information is required to determine that the actual perturbation history lies in such a class.

To make such an estimate, we need to say something about how the diffusive perturbation is correlated across phase space. For simplicity, let us assume that the perturbation is well correlated across a coarse-grained cell, but is essentially uncorrelated across scales larger than a coarse-grained cell. At any particular time, the main effect of the perturbation is the diffusion during the last Lyapunov time, because the effects of the perturbation more than a few Lyapunov times in the past are suppressed by the exponential contraction. To keep the entropy increase to $\Delta H$, one must acquire enough information about the perturbation histories so that the corresponding partial density occupies a phase-space volume that is a factor of $2^{\Delta H_0 - \Delta H}$ smaller than the volume occupied by the average density.

Now we come to the key point. Within each coarse-grained cell occupied by the unperturbed pattern at time $t$, there are $2^{\Delta H_0 - \Delta H}$ possible “slots” for the partial density. Since the perturbation is essentially independent from one coarse-grained cell to the next, one must acquire enough information, in each coarse-grained cell, to determine in which of these slots the perturbed pattern actually lies. This means acquiring $\Delta H_0 - \Delta H$ bits of information about the stochastic perturbation for each coarse-grained cell occupied by the unperturbed pattern at time $t$. Since the unperturbed pattern spreads over about $2^{Kt}$ coarse-grained cells at time $t$, the total amount of information that must be acquired to keep the entropy increase to $\Delta H$ is roughly

$$\Delta I \sim 2^{Kt}(\Delta H_0 - \Delta H) \quad \text{for } t \gtrsim t_{\text{eff}}. \quad (12)$$

Equation (12) is the main result of our heuristic argument. Its content is the following: once the perturbation becomes effective, a classically chaotic system displays an exponential hypersensitivity to perturbation. A simple example of the argument leading
to Eq. (12) is worked out in the caption of Figure 1. Although Eq. (12) is derived here from a crude, heuristic argument, it is confirmed by a rigorous analysis of systems that have a symbolic dynamics.\(^{21}\)

There are limits to the validity of Eq. (12). First, Eq. (12) is no longer valid for times large enough that the partition of the unperturbed pattern begins to have more than one piece in each coarse-grained cell, because then the perturbations of two such pieces are correlated. Second, Eq. (12) goes bad when the allowed entropy increase \(\Delta H\) is sufficiently small—somewhere between 0 and \(D\) bits—because one is then required to keep track of the perturbed pattern on scales finer than the width of the unperturbed pattern. The information \(\Delta I\) then counts perturbation histories as distinct if the corresponding perturbed patterns differ only on scales finer than the finest scale set by the system dynamics. As already indicated, for a diffusive perturbation \(\Delta I\) becomes infinite as \(\Delta H\) goes to zero because a diffusive perturbation has an infinite number of realizations. The result is that when \(\Delta H\) is sufficiently small, the information \(\Delta I\) reveals properties of the stochastic perturbation—essentially the number of perturbation histories that differ on very fine scales.

The flip side of the coin is that for \(\Delta H > D\), the information-entropy relation (12) is, like the entropy increase of Eq. (11), a property of the system dynamics, not a property of the perturbation. This is evident from the fact that both the entropy increase of Eq. (11) and the information-entropy relation of Eq. (12) are determined by the KS entropy and are independent of the strength of the perturbation, provided the perturbation is strong enough to become effective before the unperturbed pattern spreads over all the coarse-grained cells.

The further entropy increase \(\Delta H_0 - \Delta H\) beyond the allowed increase \(\Delta H\) is a logarithmic measure of the number of nonoverlapping partial densities of volume \(2^{-(\Delta H_0 - \Delta H)} V_0\) that fit within the average density of volume \(V_0\). In contrast, the information \(\Delta I\) is a logarithmic measure of the much greater number of overlapping partial densities produced by the perturbation. The proliferation of overlapping partial densities is a consequence of the chaotic dynamics, which spreads the unperturbed pattern over an exponentially increasing number of coarse-grained cells, in each of which the perturbation acts essentially independently.

The notion of hypersensitivity to perturbation can be applied directly to quantum dynamics. The initial data specify a state vector that is localized on phase space. This state vector evolves under the influence of an unperturbed dynamics and a stochastic perturbation. Each perturbation history leads to a particular state vector at time \(t\). Averaging over the perturbation yields an average density operator that corresponds to an entropy increase \(\Delta H_0\). Computer simulations indicate that quantum systems whose classical limit is chaotic display hypersensitivity to perturbation,\(^{22}\) in that the information \(\Delta I\) about the perturbation required to reduce the entropy increase to \(\Delta H\) far exceeds the entropy reduction \(\Delta H_0 - \Delta H\).
The mechanism for classical hypersensitivity is that the chaotic dynamics spreads the phase-space pattern over an exponentially increasing number of phase-space cells, in each of which the perturbation acts independently. The result is a proliferation of overlapping perturbed phase-space patterns. Our simulations suggest a similar mechanism for quantum hypersensitivity. The chaotic quantum dynamics spreads the state vector over an exponentially increasing number of quantum phase-space cells, each of which is a state vector localized on phase space. Since the evolution is unitary Schrödinger evolution, the spreading creates a coherent superposition of these localized state vectors. The stochastic perturbation changes amplitudes and phases within this superposition, thereby creating a proliferation of nonorthogonal state vectors, which are distributed randomly over the space spanned by the localized state vectors in the superposition. This proliferation of nonorthogonal state vectors is responsible for quantum hypersensitivity.

Our computer simulations indicate that, in contrast to the classical situation, quantum hypersensitivity can occur for a perturbation that is correlated across all of phase space. The essential difference seems to be that quantum mechanically the perturbation can act on the phases in the quantum superposition, a mechanism not available to a classical perturbation. Should this speculation be correct, quantum hypersensitivity would emerge as a distinctly quantum-mechanical phenomenon, similar to classical hypersensitivity, yet subtly different because of quantum superposition. Indeed, one could say that in the case of classical hypersensitivity, the perturbation generates classical information that is stored in an ensemble of overlapping phase-space patterns, whereas in the case of quantum hypersensitivity, the perturbation generates quantum information that is stored in an ensemble of nonorthogonal state vectors.

Though our simulations are not sufficient to verify this mechanism for quantum hypersensitivity, they do suggest models that might be simple enough to elucidate the nature of quantum hypersensitivity analytically. Such models, together with further computer simulations, are the focus of our current work, the goal of which is to develop a deeper understanding of the unpredictability of open quantum systems.

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**FIGURE CAPTION**

Figure 1. Cartoon of classically chaotic Hamiltonian dynamics. Phase space is represented as a two-dimensional square gridded into coarse-grained cells. The initial data are that the system begins in the shaded coarse-grained cell shown in (a); these initial data correspond to a uniform phase-space density over the shaded cell. Under chaotic Hamiltonian evolution, the phase-space density spreads across phase space, creating a pattern of uniform density, shown as the central dark line in (b); the evolved pattern occupies the same phase-space volume as the initial cell and develops structure on finer and finer scales as the evolution proceeds. The chaotic dynamics is characterized by the Kolmogorov-Sinai (KS) or metric entropy, denoted by $K$. A crude picture is that the evolved pattern spreads over $2^{Kt}$ coarse-grained cells at time $t$. To analyze hypersensitivity to perturbation, we assume that the evolution is perturbed stochastically by a diffusive perturbation that is essentially independent from one coarse-grained cell to the next. A typical perturbed pattern is shown in (b) as the dashed line that is twined about the unperturbed pattern. The average density, shown as the shaded region in (b), is obtained by averaging over all the perturbed patterns. In this example the average density occupies a phase-space volume that is about four times as large as the volume occupied by the unperturbed pattern, corresponding to an entropy increase of $\Delta H_0 \sim \log_2 4 = 2$ bits. To reduce the entropy increase to $\Delta H = 1$ bit, one must answer the following question: in each coarse-grained cell, on which side of the unperturbed pattern does the perturbed pattern lie? Answering this question requires giving $\Delta H_0 - \Delta H \sim 1$ bit of information for each coarse-grained cell occupied by the unperturbed pattern and thus requires a total amount of information given by

$$\Delta I \sim 2^{Kt}(\Delta H_0 - \Delta H) \sim 2^{Kt} \text{ bits}.$$
