A first step phenomenology for the statistics of non-equilibrium fluctuations

Guillaume Attuel

1Laboratoire de Physique des Plasmas (CNRS UMR 7648), Ecole Polytechnique, 91128 Palaiseau cedex, France

(Dated: February 4, 2010)

Abstract

The paper assesses stationary probability distributions in out-of-equilibrium systems. In the phenomenology proposed, no free energy can be well defined. Fluctuations of Landau free energy couplings arise when the intrinsic chemical potential leads to intrinsic disorder. The relaxation is shown to take the form of a geometrical random process. Systems of this kind show criticality features as well as that of first order transitions, which encapsulate in the form of a generalized static fluctuation dissipation relation. This will help determine three classes of distributions, which are, by defining a Hurst exponent for the relaxation rate: the regular Maxwell-Boltzmann for all $H < \frac{1}{2}$; the usual scale free universal distributions with power law tails for $H \in [\frac{1}{2}, 1]$; and a new class. The latter lies in the intermediate case, when $H = \frac{1}{2}$. The distribution functions are scale free close to the origin, and cross-over to a Maxwell-Boltzmann asymptotic behavior, with both the scaling exponent and an effective temperature that depend on the magnitude of the intrinsic disorder.

PACS numbers: 05.40.-a, 05.65.+b, 05.70.Ln, 05.90.+m, 47.27.-i, 89.75.-k

*email: guillaume.attuel@gmail.com
I. INTRODUCTION

Non-Gaussianity has often been reported, and has appeared universal in a broad series of experimental situations, ranging from turbulent flows to magnetic fluctuations, as, for instance, in the 2D XY model at low temperatures, and it has been related to criticality [1].

If one thinks of the Landau theory of fluctuations [2], Maxwell-Boltzmann distributions describe monophasic fluctuations out of the vicinity of critical points, and for scales well above the correlation length. If metastable states are visited, it is in a way comparable to Arrhenius activation. On the contrary, in principle even at relatively low temperatures, out-of-equilibrium systems such as depicted in [1] show a kind of disorder that displays swift and sharps changes with high magnitude. This means clearly that the relaxation process is not in a linear regime.

Thus we need to seek out a different activation process.

A. Random walks

We start by following the remark that Maxwell-Boltzmann distributions can be taken from the limit central theorem by replacing time with the equilibrium time $t_{eq}$ (e.g. in a Ornstein-Uhlenbeck process) in the time dependent probability distribution $p(\delta v; t)$,

$$p(\delta v; t) \propto \frac{1}{\sqrt{D_v t}} \exp \left( -\frac{\delta v^2}{2D_v t} \right).$$

(1)

It amounts to making a prescription for the collection of such particles

$$\langle \delta v^2 \rangle = t_{eq} D_v$$

(2)

which is the static fluctuation dissipation relation (FDT). This results in

$$p_{eq}(\delta v) \propto \frac{1}{\sqrt{D_v t_{eq}}} \exp \left( -\frac{\delta v^2}{2D_v t_{eq}} \right).$$

(3)

From this, one already sees that it is not sufficient that $t_{eq} \to \infty$, but that regular diffusion ceases, in order to have non-Maxwellians. This means that the break down of the FDT relation eq.(2) is necessary.
Another prototypical case, that one expects to happen close to criticality, is a scaling random walk, or Levy walk. Its time dependent distribution is of the following form

\[ p(\delta v; t) \propto t^{-H} G \left( \frac{\delta v}{t^H} \right) \]  

(4)

with \( G \) being a universal function. In just the same way, one takes the asymptotic limit \( t \to \infty \), and assumes that scaling extends to the collection. Then, normalizing with respect to time, one gets straightforwardly

\[ p_{\text{stat}}(\delta v) \propto \delta v^{-(1+\frac{1}{H})}. \]  

(5)

So, there are at least two archetypes of probability distributions, depending on the nature of diffusivity in the system. One notices that the functional forms eq.(1) and eq.(4) are identical, which is due to scaling between spatial journey versus time. However, there is a radical change in the physics, since, going from the one case \( H > \frac{1}{2} \) to the other \( H < \frac{1}{2} \), scale invariance ceases, and especially the correlation length becomes finite for a Brownian walk.

The liquid gas critical point might be a good illustration. It is isolated in the phase diagram, and normal diffusion does not cease. One nevertheless observes opalescence, owing to the large fluctuations of the interface of the two phases. So, it lies in neither of the two already identified situations.

The question that naturally comes to mind after this can thus be formulated as follows: is the functional limit with respect to \( H \), of the probability distributions, in other words, of the universal function \( G \), continuous piecewise? The answer is no.

In the following, I attempt to show what these kind of large fluctuations have to do with phase transitions of the first order, in interplay with the vicinity of a critical point, which can be of large extent.

[1] Usually, one uses the fractional indexes \( \alpha \) and \( \beta \), which equal respectively 1 and 2 in the Brownian case. 

\[ H = \frac{\alpha}{\beta}. \]
B. Slow dynamics

On the one hand, break down of the FDT is a feature of slow dynamics. This term qualifies systems for which the correlation length depends on (usually grows with) time, generally a waiting time $t_w$ after some action, for instance, after the application of a magnetic field to a disordered magnet at low temperature \[3\]. It may be written in the form

$$\frac{d}{d\delta v^2} (t_{eq} D_v) = X(\delta v^2)$$  \hspace{1cm} (6)

where $X$ is a function, and where it is understood that the mean square depends slowly on time, $\delta v^2(t_w)$ \[4, 5\].

Now, if the large fluctuations are indeed associated with first order transitions, they imply latent heat exchanges, which means entropy exchanges. Thus, bearing in mind linear phenomenology, one may rightly suspect a global modification of dissipation, like a modification of FDT.

On the other hand, large fluctuations of interest are stationary, as in \[6–9\]. Slow dynamics may become stationary if the external field is being randomly triggered, for example. If this is so, a random waiting time would then be associated with each random step, and the break down of the FDT would be averaged over the realization of such a process. It is likely though, from eq.(6), that a regular FDT would result from this operation, be it only from the central limit theorem. This, then, is where the influence of a critical point is essential, in providing a new specification to this random triggering: scale invariance (i.e. long correlations). The external field amplitude scales with the correlation length, and by means of their functional dependence for slow dynamics, it also scales with the waiting time. The only way that such a process can occur is if the external field is in fact intrinsic.

We may now specify the form of the function $X$ thanks to this hypothesis.

II. AN EXAMPLE: BURGERS MODEL

A. Extrapolation of linear phenomenology

One dimensional systems are very unlikely in phase equilibrium, due to Landau’s argument \[2\]. Let us then consider a turbulent, compressible flow, in one dimension. There are
large fluctuations of velocity. Fluctuations of a certain magnitude have a certain spatial
and temporal extension, therefore one can readily name phases. The situation mimics an
inhomogeneous mixture of phases. With no other interaction, the chemical potential of each
single phase is basically identified in terms of the relative kinetic energy per particle

\[ \mu = \frac{1}{2} \delta v^2. \]  

(7)

The thermodynamic current tends to equalize the potential of the different phases. Linear
phenomenology relates it proportionally to the thermodynamic force

\[ \mathbf{F} = -\mathcal{L} \nabla \frac{\mu}{T} \]

with \( T = \frac{1}{2} \langle \delta v^2 \rangle \), and \( \mathcal{L} \) the phenomenological coefficient. The equation of phase motion is
the inviscid Burgers model[2]

\[ \frac{\partial}{\partial t} \delta v = -\lambda (\nabla \delta v) \delta v \]

(8)

with \( \lambda \equiv \frac{\mathcal{L}}{T} \).

As it is well known, a given perturbation may relax locally, but not globally towards some
equilibrium. If a noise source term is added here, the system is incapable of equilibrating
with it, instead develops long range correlations [10, 11] and no Gaussian distributions [12].
So we can see that the most simple case falls out of the scope of the hypothesis underlying
linear phenomenology and what it should be describing. The reason often alleged is that
actually no thermodynamic potential can be defined for eq.(8), so that the Burgers model
has become a prototype for non-equilibrium dynamics. Let us now contest this fact.

B. Dissipative function

First, it is quite contradictory to treat a thermodynamic phase like a Langevin particle,
by adding a noise source to the equilibration evolution of its chemical potential. In doing
so, one mixes a microscopic modelization with a macroscopic description. A more coherent
version should provide a macroscopic and self consistent mechanism that maintains the

---

[2] With only one phase, \( d\mu = 0 \), the equilibrium of the fluid is given by Bernoulli’s relation, which reads for
a uniform density \( \nabla T = \frac{k}{2} \nabla \delta v^2 \)
different $\delta v^2$ phases out-of-equilibrium. It is noticeable also that the confusion might be a clue for seeking scale invariance linking micro to macro scales, in such a mechanism.

Now, if one naively tries to find an effective free energy associated to eq.(8), it cancels out in the following way. If we look for a functional $F$, such that $\frac{\delta}{\delta(\delta v)} F = \lambda (\nabla \delta v) \delta v$, we try

$$F = \int dx \frac{1}{6} \nabla \delta v^3$$

but then the variation is in fact zero, after integration by parts,

$$\delta F = \frac{\lambda}{2} \int dx (\delta v) (\nabla \delta v^2 - \nabla \delta v^2) = 0.$$

The cubic term cannot contribute to the variation of $F$ because of its symmetries $\delta v \to -\delta v$, besides having no absolute minimum.

$x \to -x$

We may nonetheless consider with good reason that the non local gradient and the local field are independent variables.

Thus, instead of a cubic term we get an external field $h$ for the intrinsic thermodynamic force in the free energy

$$h \equiv F = \frac{\lambda}{2} \nabla \delta v^2.$$  

There will be some link between the two conjugate fields, but only statistically. In addition, symmetries have to be preserved statistically, so

$$\langle h \rangle = 0.$$

This in turn might yield $\langle \delta F \rangle = 0$, and with no work exchange, at first sight, would seem to violate the second principle, but at this stage we should simply recall that the system is out-of-equilibrium.

This treatment is a mean field description of the phases, because the intrinsic field $h$ represents their large scale interactions that should lead to reorganization. It could be more general than the above expression eq.(10). Finally, the choice that was made is not a priori unique. Instead of $h$, a complex "mass" $m^2 \equiv r_0$ could have been assigned

$$r_0 = \lambda \nabla \delta v$$
although it is less obvious then that its value adjusts self consistently, contrarily to what follows.

III. GENERICITY OF FIRST ORDER FLUCTUATIONS: FLUCTUATING FREE ENERGY

A. Degrees of freedom

A first order transition close to a critical point resembles formally a subcritical pitchfork bifurcation, if one thinks of a dynamical system. Lorenz’s famous system [13] is an example of the chaotic oscillations that occur in its vicinity. It could be that a particular phase, when forced from outside, be locally modeled by such a dynamical system, showing temporal chaos. But, it is not known what to expect when there is spatial coupling between different phases, because the number of degrees of freedom then scales with the system size.

Schematically, just as it is not possible to identify a potential energy establishing a deterministic path for one single particle, because of the infinite number of microscopic degrees of freedom, but only to reduce to a statistical picture through a thermodynamic potential, the above infinite number of degrees of freedom, for an infinite system size, make unlikely the existence of a thermodynamic potential, which should have established a deterministic path for the collection of the particles. We point out here more precisely that the noise which is always used to model randomness and energy supply, in domain growth for instance, with Langevin models such as the Burgers model, should not be thermal noise, but pretty much a "non equilibrium noise", operating directly in the free energy rather than acting externally on the local order parameters or phases.

In summary, an order parameter is indeed regarded as the state of a macro particle, under the influence of a random thermodynamic force. This means that no free energy minimum truly exists, and that the macroscopic state diffuses out-of-equilibrium. The original Langevin model would have as a straightforward pending, something like a random set of thermostat baths in interaction with each other.

The difference here is the necessarily self-generated nature of the noise. In a sense, one such macro particle must interact with itself, meaning that a macro particle designates a range from a single local phase to an encompassing of some phases. That is to say, a macro
particle understood as a set is in interaction with all of its subsets, and supersets. This is where scale invariance intuitively should come from.

B. Renormalization orbits

In Landau’s theory, the external field $h$ is considered to be a parameter, shifting the system away from absolute equilibrium, to an imposed equilibrium. If this parameter is intrinsic, as it is supposed to be here, it should enable the system to *spontaneously* leave absolute equilibrium. In this respect, it is valuable that the intrinsic field $h$ comprises space in a hidden way, in conformity with the meta thermodynamic like approach we are looking for.

Thus, within each set the value of $h$ should be different, as if in a mixture of sets. This leads the values of the Landau free energy couplings to be different from set to set too. In order to make this practical, we use the ideas of scale invariance and renormalization [14–16]. The way it works technically is simple: the free energy couplings undergo a subcritical pitchfork bifurcation, because of the finiteness of $h$, and therefore the order parameter does so too.

Consider initial conditions close to a critical state, and the effective Landau Free energy to be

$$F = F_{eq} + \frac{1}{2} r_0 \delta v^2 + B \delta v^4 - h \delta v.$$  

The renormalization flow after rescaling by a factor $b$, for the parameters $r_0$, $B$, and $h$, possesses one and a half degrees of freedom, as a dynamical system of equations. It is therefore likely to exhibit chaotic orbits as was noticed in [16]. If so, because of stretching and refolding the orbit visits the neighborhood of the critical point as well as far off regions, besides it is likely that the orbit cycles chaotically from one side of the stable (fast) mode variety to the other, therefore making the system go through first order transitions [17]. In other words, it shows hysteresis. It goes in a chaotic manner throughout each cycle from a critical state to a mean-field state.

It should be noted that the argument is slightly different from the typical linear shell to shell decimation procedure [16, 18], because it amounts to attributing to each single equilibrated phase a particular orbit cycle, with a typical cutoff correlation length, or extension length, associated with it. The cutoff (in correct units) corresponds to the inflexion point.
from where the orbit returns to the vicinity of the stable mode variety. We then obtain for
one single system sets of \( \{ r_i^0 \}, \{ B^i \}, \{ h^i \} \), and in particular \( \{ t^i_{eq} \} \), where the superscript \( i \) stands for one of each cycle, indifferently over space and time.

IV. A GENERALIZATION OF THE STATIC FLUCTUATION DISSIPATION RELATION

The return point is far from the critical point, so mean field theory should be a good
approximation there. At the same time, whenever the orbit goes by it, scale invariance
should hold. So, a natural idea is that both phenomenologies hold together.

A. Using mean field and scale invariance

First, let us go back to the mean field theory. When \( r_0 = 0 \), the (non linear) susceptibility
would be for \( h \neq 0 \)

\[
\chi^{-1} = 12B\delta v^2.
\] (11)

If we write instead of eq.(8), loosing space dependence,

\[
\partial_t \delta v = -h
\]

we would get with the help of eq.(11)

\[
\partial_t \delta v = -12\lambda B \delta v^3
\]

but the solutions are not valid for all times. This is a reason to give to such a relation as
eq.(11) a conditional average meaning, with \( h \) being a random variable instead.

Further, if we simply write the result of a diffusion process as

\[
\langle \delta v^2 \rangle \sim D_v t
\] (12)

where \( D_v = \int_0^\infty d\tau \langle h h_\tau \rangle \), because we would like a stationary process, it is essential that anomalous dimension arises in order to cancel out the secular divergence. In particular \( D_v t \)
should be taken as a whole, and depend on the value of \( \delta v^2 \) in the form of eq.(10).
Now that we have a mechanism which is self generating noise, we prefer to model the relaxation process with a stochastic 'mass' \( m^2 \equiv \tilde{r}_0 \), that we write (\( \mathcal{L} \) is taken constant with no loss of generality)

\[ \lambda \tilde{r}_0 \equiv \tilde{\lambda}. \]  

(13)

So relaxation in the system is modeled by a geometric process

\[ \partial_t \delta v = -\tilde{\lambda} \delta v. \]  

(14)

For instance, if \( \tilde{\lambda} \) had no relation with \( \delta v \), a finite variance, and was short range correlated, eq. (14) would be a multiplicative cascade, leading to a non stationary log-normal distribution \[19\]. Define a Hurst exponent coming from the conditional correlation function of \( \tilde{\lambda} \) as

\[ \int^t d\tau \int^t d\tau' \langle \tilde{\lambda} (\tau) \tilde{\lambda} (\tau') / \delta v \rangle \sim t^{2H}. \]

By integration of eq. (14) we obtain

\[ \langle \ln^2 \delta v(t) \rangle \sim t^{2H} \]

but a two time relation, reminding us of the slow dynamics, can also be found at a time \( \tau \), by integration over a much smaller time interval \( t \)

\[ \langle \delta v^2(\tau) \rangle (t) \sim \overline{\delta v^2(\tau, t)} t^{2H}. \]  

(15)

This can be regarded as intermediate thermalized states reached at times \( \tau \). We obviously want a real constant \( \langle \delta v^2 \rangle \), defining the amplitude of the stationary fluctuations. As already mentioned, the r.h.s. of eq. (15) is therefore considered inseparably, meaning that (dropping over line notation)

\[ \delta v^2(t) \sim t^{-2H}. \]

Whereas \( r_0 \to 0 \), this asymptotic in time prevents the averaged susceptibility to become infinite \( \langle T\chi \rangle \to \infty \). We then can work it out by replacing outrightly \( t^{-1} \) with \( \tilde{\lambda} \), very much as we have done so, in an equilibrium state, for the correlation time in eq. (2). This is almost
by the definition of the set of all possibilities for this random variable

\[ \{ \tilde{\lambda}' \} \equiv \{ \frac{1}{l_{eq}} \}. \]

Therefore, the fluctuating relaxation rate scales as

\[ \left \langle \lambda / \delta v \right \rangle \sim \delta v^{\frac{1}{H}}. \quad (16) \]

In particular \( \langle D_v t \rangle \) is now a constant, and the function \( X(A) \propto A^{-\frac{1}{2H}} \).

Actually, for the phenomenology of orbit cycles to hold, one needs a non zero overall dissipation (or instability) \( \lambda_0 \equiv \lambda r_0 \neq 0 \), although \( \lambda_0 \to 0 \). Thus, eq.\( \text{(16)} \) is rather

\[ \left \langle \tilde{\lambda} / \delta v \right \rangle \approx \lambda_0 \pm C_\pm \delta v^{\frac{1}{H}} \quad (17) \]

where \( C_\pm \) is related to \( B \). For instance, for a conserved quantity it describes locally the effects of the energy transfer from scale to scale within a physical range, where the effective diffusion rate is \( D_v \propto \tilde{\lambda}^{1-\frac{2}{z}} \) [20].

Finally, it is worth noticing that eq.\( \text{(17)} \) renders a kind of conditionally averaged Landau’s free energy, of the original form as soon as \( H = \frac{1}{2} \), which is indeed the case here. Otherwise because of eq.\( \text{(13)} \), the coherence of the phenomenology starting with Landau’s free energy might be questioned.

**B. A guesstimate on indexes**

Because near the critical point, it is the linear behavior that dominates, we can assume that

\[ \chi = h^{\frac{1}{2}} g \left( \frac{\epsilon}{h^{\frac{1}{2H}}} \right) \quad (18) \]

where \( r_0 \propto \epsilon \), and \( g(x) \to x^{-\gamma} \) as \( x \to \infty \). The interesting limit here is \( x \to 0 \), where to first order

\[ \lim_{x \to 0} g(x) = cte. \]
Now, by identification of the scale factors in eq. (18) and eq. (16), one finds the Hurst exponent

\[ H = \frac{\beta}{\gamma}. \]  

(19)

If the phenomenological coefficient \( L \) is not anomalous (what we have supposed here) then we also have

\[ \gamma = z\nu. \]

One has to be careful though that the critical indexes may not have their regular values. Indeed, in the linear hyperbolic region one can write \( r_0\Lambda^l \sim 1 \), where \( \Lambda \) stands for the eigenvalue and \( l \) for the iteration integer, as soon as the correlation length scale is reached. For one particular cycle of the chaotic orbit, the iteration integer starts from zero, and one can rewrite this as

\[ r_0^i\Lambda^{li} = a_i. \]

Letting down the superscripts for clarity, noting the correlation length \( \zeta = b^i\zeta_i \), where \( \zeta_i \) is a fixed number, if it happens that

\[ a \sim r_0^\alpha \]

then the critical index \( \nu \) will be modified by a factor \((1 - \alpha)\) as

\[ \nu = \frac{\ln (b) (1 - \alpha)}{\ln (\Lambda)}. \]  

(20)

Incidentally, it has been suggested in [17] that the broken symmetries due to \( h \) as well as the related disorder could change the indexes’ values, for instance from a 'bosonic' to a 'fermionic' mean field, for which \( \nu = 1 \) and \( z = 1 \), as was observed in [21].

Leaving this indetermination aside for the moment, interestingly, when \( H = \frac{1}{2} \) the system seems to be at the border of normal fluctuations, and from eq. (19), it seems to be related to the mean field values of the indexes \( \beta = \frac{1}{2}, \nu = \frac{1}{2} \) and \( z = 2 \) (or \( \nu = 1 \) and \( z = 1 \)). So, \( H = \frac{1}{2} \) could be related to marginality.
V. STATIONARY DISTRIBUTIONS

The previous generalization of the FDT allows the derivation of explicit forms for the probability distributions.

A. Functional form

Consider first when $H \in \left[\frac{1}{2}, 1\right]$, we have for the Fokker Planck criterion

$$\lim_{t \to 0} \frac{t^{2H}}{t} = 0.$$ 

Thus the Fokker Planck equation is reduced to $\frac{\partial}{\partial t} p = -\frac{\partial}{\partial \delta v} \left( \langle \tilde{\lambda} / \delta v \rangle \delta v p \right)$. This leads to the following stationary distribution, $p(\delta v) \propto \exp \left( - \int d\delta v I \right)$, where the integrand is $I = \langle \tilde{\lambda} / \delta v \rangle + \delta v \frac{\partial}{\partial \delta v} \langle \tilde{\lambda} / \delta v \rangle$. By using eq.(16) one finds a power law tail for the distribution in accordance with eq.(21)

$$p(\delta v) \propto \delta v^{-\left(1 + \frac{z\beta}{2}\right)}.$$ (21)

The exponent

$$1 + \frac{z\beta}{2}$$

is called the Gutenberg Richter exponent, and belongs here to the interval $[2, 3]$.

If $H < \frac{1}{2}$, one gets the usual Maxwellian distribution eq.(3).

Similarly, the intermediate case $H = \frac{1}{2}$ results in

$$p_{stat}(\delta v, \langle \tilde{\lambda}^2 \rangle) \propto \delta v^{-3 + \frac{z\beta}{2}} \exp \left( \mp C_{\pm} \frac{\lambda_0}{\langle \tilde{\lambda}^2 \rangle} \delta v^2 \right).$$ (22)

Close to the origin, the behavior is dominated by a power law. Interestingly, the slopes rely not only on the combination of the indexes $1 + \frac{1}{H}$, but also on the noise intensity $\langle \tilde{\lambda}^2 \rangle$. Asymptotically, one recovers a Maxwellian tail, with an effective temperature

$$T_{eff} \propto \langle \tilde{\lambda}^2 \rangle.$$ (23)

The cross-over goes to infinity with the effective temperature. Here, we have put back an
overall dissipative ‘mass’ $\lambda_0$, and the generalized relation, accounting for signs, looks like eq.\((17)\).

VI. CONCLUSION

I have presented a phenomenology for the determination of the statistics of out-of-equilibrium fluctuations, based upon the existence of a free energy, but where its couplings cannot be defined properly. These are indeed likely to cycle chaotically. Systems so governed go through second and first order transitions at the same time. A distribution (which resembles a chi square only in a particular case) that reminds us of both a scale free distribution at small amplitudes and a Maxwellian at larger amplitudes, with an effective temperature expressing the strength of intrinsic disorder, is found.

The interplay between first and second order transitions was first found in the study of the hysteretic behavior of some fluids, where it is due to instabilities of the Rayleigh type. I only mention here a common fluid modelization for avalanches in sand piles and for the resonance between waves and particles in a plasma \[17\], as this will be considered elsewhere.

I now would like to evoke further work that I think is worth while investigating, given this phenomenology. For instance, J. Sethna et al. \[22\] have also found a cross over behavior for the PDFs of avalanche amplitude in the random field Ising model, which bears a priori extrinsic disorder. S. T. Bramwell et al. \[23\] have performed precise calculations for the spin wave approximation of the XY model which establish evidence of a relation between the different moments for the Fokker Planck perturbative development. This could be related to some kind of Noether (or Ward) identity, in marginal systems, due to scale invariance symmetry, that we have set out here as a modified static FDT. I also question whether the phenomenology of chaotic cycles may help in the understanding of intermittency, which lies in the UV range, in developed turbulence.

[1] S. T. Bramwell, P. C. W. Holdsworth, and J. F. Pinton, letters to nature 396, 554 (1998).
[2] L. Landau and E. M. Lifchitz, Physique statistique (édition MIR Ellipses, 1994), 4th ed.
[3] L. Lundgren, P. Svedlinch, P. Nordblad, and O. Beckman, Phys. Rev. Lett. 51, 911 (1983).
[4] L. F. Cugliandolo and J. Kurchan, Phys. Rev. Lett. 71, 173 (1993).
[5] L. F. Cugliandolo, J. Kurchan, and L. Peliti, Phys. Rev. E 55, \(3898\) (1997).
[6] U. Frisch, *Turbulence* (Cambridge university press, 1995).
[7] B. Portelli, P. Holdworth, and J.-F. Pinton, Phys. Rev. Lett. 90, 104501 (2003).
[8] B. P. van Milligen et al., Phys. of Plasmas 12, 052507 (2005).
[9] P. Holdsworth, Priv. comm.;web (2006).
[10] D. Forster, D. Nelson, and M. Stephen, Phys. Rev A 16, 425 (1977).
[11] T. Hwa and M. Kardar, Phys. Rev. A 45, 7002 (1992).
[12] H. Chen, S. Chen, and R. H. Kraichnan, Phys. Rev. Lett. 63, 2657 (1989).
[13] E. N. Lorenz, Journal of the atmospheric sciences 20, 130 (1963).
[14] B. Widom, J. Chem. Phys. 43, 3892 (1965).
[15] L. P. Kadanoff et al., Rev. Mod. Phys. 39, 395 (1967).
[16] K. G. Wilson, Phys. Rev. B 4, 3174 3184 (1971).
[17] G. Attuel, Ph.D. thesis, LPP Polytechnique (2007).
[18] L. P. Kadanoff, *Statistical Physics: Statics, Dynamics and Renormalization* (World Scientific, 2000).
[19] A. N. Kolmogorov, Doklady p. 82 (1962).
[20] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).
[21] A. Ivanov, S. V. Vladimorov, and P. A. Robinson, Phys. Rev. E 71, 056406 (2005).
[22] O. Perković, K. Dahmen, and J. P. Sethna, Phys. Rev. Lett. 75, 4528 (1995).
[23] S. T. Bramwell et al., Phys. Rev. E 63, 041106 (2001).