The Dequantization Transform
and Generalized Newton Polytopes

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Vienna, Preprint ESI 1587 (2005)

Supported by the Austrian Federal Ministry of Education, Science and Culture
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THE DEQUANTIZATION TRANSFORM AND GENERALIZED
NEWTON POLYTOPES

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ABSTRACT. For functions defined on $\mathbb{C}^n$ or $\mathbb{R}^n$ we construct a dequantization transform $f \mapsto \mathcal{f}$; this transform is closely related to the Maslov dequantization. If $f$ is a polynomial, then the subdifferential $\partial \mathcal{f}$ of $f$ at the origin coincides with the Newton polytope of $f$. For the semiring of polynomials with nonnegative coefficients, the dequantization transform is a homomorphism of this semiring to the idempotent semiring of convex polytopes with respect to the well-known Minkowski operations. Using the dequantization transform we generalize these results to a wide class of functions and convex sets.

1. Introduction. The Maslov dequantization and the dequantization transform

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers. The well-known max-plus algebra $\mathbb{R}_{\text{max}} = \mathbb{R} \cup \{-\infty\}$ is defined by the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$.

The max-plus algebra can be treated as a result of the Maslov dequantization of the semifield $\mathbb{R}_+$ of all nonnegative numbers, see, e.g., [1, 2]. The change of variables

$$x \mapsto u = h \log x,$$

where $h > 0$, defines a map $\Phi_h: \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$. Let the addition and multiplication operations be mapped from $\mathbb{R}_+$ to $\mathbb{R} \cup \{-\infty\}$ by $\Phi_h$, i.e. let

$$u \oplus_h v = h \log(\exp(u/h) + \exp(v/h)), \quad u \odot_v u = u + v,$$

$$0 = -\infty = \Phi_h(0), \quad 1 = 0 = \Phi_h(1).$$

It can easily be checked that $u \oplus_h v \to \max\{u, v\}$ as $h \to 0$. Thus we get the semifield $\mathbb{R}_{\text{max}}$ (i.e. the max-plus algebra) with zero $0 = -\infty$ and unit $1 = 0$ as a result of this deformation of the algebraic structure in $\mathbb{R}_+$.

The semifield $\mathbb{R}_{\text{max}}$ is a typical example of an idempotent semiring; this is a semiring with idempotent addition, i.e., $x \oplus x = x$ for arbitrary element $x$ of this semiring, see, e.g., [3].
The analogy with quantization is obvious; the parameter $\hbar$ plays the role of the Planck constant [2]. In fact the Maslov dequantization is the usual Schrödinger dequantization but for pure imaginary values of the Planck constant (see, e.g., [4]). The map $x \mapsto |x|$ and the Maslov dequantization for $\mathbb{R}_+$ give us a natural passage from the field $\mathbb{C}$ (or $\mathbb{R}$) to the max-plus algebra $\mathbb{R}_{\text{max}}$.

Let $X$ be a topological space. For functions $f(x)$ defined on $X$ we shall say that a certain property is valid almost everywhere (a.e.) if it is valid for all elements $x$ of an open dense subset of $X$.

Suppose $X$ is $\mathbb{C}^n$ or $\mathbb{R}^n$; denote by $\mathbb{R}^n_+$ the set $x = \{ (x_1, \ldots, x_n) \in X \mid x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \}$. For $x = (x_1, \ldots, x_n) \in X$ we set $\exp(x) = (\exp(x_1), \ldots, \exp(x_n))$; so if $x \in \mathbb{R}^n$, then $\exp(x) \in \mathbb{R}^n_+$.

Denote by $\mathcal{F}(\mathbb{C}^n)$ the set of all functions defined and continuous on an open dense subset $U \subset \mathbb{C}^n$ such that $U \supset \mathbb{R}^n_+$. In all the examples below we consider even more regular functions, which are holomorphic in $U$. It is clear that $\mathcal{F}(\mathbb{C}^n)$ is a ring (and an algebra over $\mathbb{C}$) with respect to the usual addition and multiplications of functions.

For $f \in \mathcal{F}(\mathbb{C}^n)$ let us define the function $\hat{f}_h$ by the following formula:
\begin{equation}
\hat{f}_h(x) = h \log |f(\exp(x/h))|,
\end{equation}
where $h$ is a (small) real parameter and $x \in \mathbb{R}^n$. Set
\begin{equation}
\hat{f}(x) = \lim_{h \to 0} \hat{f}_h(x),
\end{equation}
if the right-hand part of (3) exists almost everywhere. We shall say that the function $\hat{f}(x)$ is a dequantization of the function $f(x)$ and the map $f(x) \mapsto \hat{f}(x)$ is a dequantization transform. By construction, $\hat{f}_h(x)$ and $\hat{f}(x)$ can be treated as functions taking their values in $\mathbb{R}_{\text{max}}$. Note that in fact $\hat{f}_h(x)$ and $\hat{f}(x)$ depend only on the restriction of $f$ to $\mathbb{R}^n_+$; so in fact the dequantization transform is constructed for functions defined on $\mathbb{R}^n_+$ only. It is clear that the dequantization transform is generated by the Maslov dequantization and the map $x \mapsto |x|$. Of course, similar definitions can be given for functions defined on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

We shall see that if $f(x)$ is a polynomial, then there exists the dequantization $\hat{f}$ of this polynomial and the subdifferential $\partial \hat{f}$ of the function $\hat{f}$ coincides with the Newton polytope of the polynomial $f$.

It is well known that all the convex compact subsets in $\mathbb{R}^n$ form an idempotent semiring $\mathcal{S}$ with respect to the Minkowski operations: for $A, B \in \mathcal{S}$ the sum $A \oplus B$ is the convex hull of the union $A \cup B$; the product $A \odot B$ is defined in the following way: $A \odot B = \{ x \mid x = a + b, \text{ where } a \in A, b \in B \}$. In fact $\mathcal{S}$ is an idempotent linear space over $\mathbb{R}_{\text{max}}$ (see, e.g., [4]). Of course, the Newton polytopes in $V$ form a subsemiring $\mathcal{N}$ in $\mathcal{S}$. If $f, g$ are polynomials, then $\partial(fg) = \partial f \odot \partial g$; moreover, if $f$ and $g$ are “in general position”, then $\partial(f + g) = \partial f + \partial g$. For the semiring of all polynomials with nonnegative coefficients the dequantization transform is a homomorphism of this “traditional” semiring to the idempotent semiring $\mathcal{N}$.
functions and convex sets.

Hence we can suppose that \( \hat{f}(x) \) exists (and is defined on an open dense subset of \( V \)). By \( \mathcal{D}(\mathbb{C}^n) \) denote the set of all dequantizable functions and by \( \hat{\mathcal{D}}(V) \) denote the set \( \{ \hat{f} \mid f \in \mathcal{D}(\mathbb{C}^n) \} \). Recall that functions from \( \mathcal{D}(\mathbb{C}^n) \) (and \( \hat{\mathcal{D}}(V) \)) are defined almost everywhere and \( f = g \) means that \( f(x) = g(x) \) a.e., i.e., for \( x \) ranging over an open dense subset of \( \mathbb{C}^n \) (resp., of \( V \)). Denote by \( \mathcal{D}_+(\mathbb{C}^n) \) the set of all functions \( f \in \mathcal{D}(\mathbb{C}^n) \) such that \( f(x_1, \ldots, x_n) \geq 0 \) if \( x_i \geq 0 \) for \( i = 1, \ldots, n \); so \( f \in \mathcal{D}_+(\mathbb{C}^n) \) if the restriction of \( f \) to \( V_+ = \mathbb{R}_+^n \) is a nonnegative function. By \( \hat{\mathcal{D}}_+(V) \) denote the image of \( \mathcal{D}_+(\mathbb{C}^n) \) under the dequantization transform. We shall say that functions \( f, g \in \mathcal{D}(\mathbb{C}^n) \) are in general position whenever \( \hat{f}(x) \neq \hat{g}(x) \) for \( x \) running an open dense subset of \( V \).

**Theorem 1.** For functions \( f, g \in \mathcal{D}(\mathbb{C}^n) \) and any nonzero constant \( c \), the following equations are valid:

1) \( \hat{f} g = \hat{f} \hat{g} \);
2) \( |f| = \hat{f} \hat{c} = f; \hat{c} = 0 \);
3) \( (f + g)(x) = \max \{ \hat{f}(x), \hat{g}(x) \} \) a.e. if \( f \) and \( g \) are nonnegative on \( V_+ \) (i.e., \( f, g \in \mathcal{D}_+(\mathbb{C}^n) \)) or \( f \) and \( g \) are in general position.

Left-hand sides of these equations are well-defined automatically.

**Proof.** Statements 1) and 2) can be easily deduced from our basic definitions and formulas (2) and (3).

Let us prove statement 3). Set \( \hat{x}_h = \exp(x/h) \in V_+ \). Suppose \( f, g \in \mathcal{D}_+(\mathbb{C}^n) \); then \( |f| = \hat{f}, |g| = \hat{g}, |f + g| = f + g \) on \( V_+ \) and we have the following inequalities:

\[
\max \{ f(\hat{x}_h), g(\hat{x}_h) \} \leq (f + g)(\hat{x}_h) \leq 2 \max \{ f(\hat{x}_h), g(\hat{x}_h) \}.
\]

Hence

\[ h \log(\max \{ f(\hat{x}_h), g(\hat{x}_h) \}) \leq h \log((f + g)(\hat{x}_h)) \leq h \log 2 + h \log(\max \{ f(\hat{x}_h), g(\hat{x}_h) \}). \]

But \( h \log 2 \to 0 \) as \( h \to 0 \), and the logarithmic function is monotonic. Thus we have

\[ \max \{ \hat{f}(x), \hat{g}(x) \} \leq (\hat{f} + \hat{g})(x) \leq \max \{ \hat{f}(x), \hat{g}(x) \}, \]

which completes this part of our proof.

A similar idea can be used if functions \( f \) and \( g \) are in general position. Without loss of generality we can suppose that \( \hat{f}(x) < \hat{g}(x) \) almost everywhere in \( V \). Take an \( x \in V \) where this inequality holds. In this case there exists a positive number \( c \) such that \( h \log |f(\hat{x}_h)| + c < h \log |g(\hat{x}_h)| \) if the parameter \( h \) is small enough. Hence \( |f(\hat{x}_h)| \exp(c/h) < |g(\hat{x}_h)| \).
Note that $\exp(c/h) \to \infty$ as $h \to \infty$; hence $\exp(c/h) > 2$ if $h$ is small enough. Therefore we have $|f(\tilde{x}_h)| < (1/2)|g(\tilde{x}_h)|$ and $(1/2)|g(\tilde{x}_h)| < |f(\tilde{x}_h) + g(\tilde{x}_h)|$. On the other hand, we obviously have the inequality $|(f + g)(\tilde{x}_h)| < 2|g(\tilde{x}_h)|$. So we get

$$
\frac{1}{2}|g(\tilde{x}_h)| < |(f + g)(\tilde{x}_h)| < 2|g(\tilde{x}_h)|;
$$

from this and from formulas (2) and (3) it follows that

$$
\hat{g}(x) \leq (\widehat{f + g})(x) \leq \hat{g}(x) = \max\{\hat{f}(x), \hat{g}(x)\}.
$$

This concludes the proof. \(\square\)

**Corollary 1.** The set $\mathcal{D}_+(\mathbb{C}^n)$ has a natural structure of a semiring with respect to the usual addition and multiplication of functions taking their values in $\mathbb{C}$. The set $\hat{\mathcal{D}}_+(V)$ has a natural structure of an idempotent semiring with respect to the operations $(f \oplus g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$; elements of $\hat{\mathcal{D}}_+(V)$ can be naturally treated as functions taking their values in $\mathbb{R}_{\max}$. The dequantization transform generates a homomorphism from $\mathcal{D}_+(\mathbb{C}^n)$ to $\hat{\mathcal{D}}_+(V)$.

### 3. Generalized Polynomials and Simple Functions

For any nonzero number $a \in \mathbb{C}$ and any vector $d = (d_1, \ldots, d_n) \in V = \mathbb{R}^n$ we set $m_{a,d}(x) = a \prod_{i=1}^n x_i^{d_i}$; functions of this kind we shall call generalized monomials. Generalized monomials are defined a.e. on $\mathbb{C}^n$ and on $V_+$, but not on $V$ unless the numbers $d_i$ take integer or suitable rational values. We shall say that a function $f$ is a generalized polynomial whenever it is a finite sum of linearly independent generalized monomials. For instance, Laurent polynomials are examples of generalized polynomials.

As usual, for $x, y \in V$ we set $(x, y) = x_1 y_1 + \cdots + x_n y_n$. The following proposition is a result of a trivial calculation.

**Proposition 1.** For any nonzero number $a \in V = \mathbb{C}$ and any vector $d \in V = \mathbb{R}^n$ we have $(\overline{m_{a,d}})_h(x) = (d, x) + h \log |a|$.  

**Corollary 2.** If $f$ is a generalized monomial, then $\hat{f}$ is a linear function.

Recall that a real function $p$ defined on $V = \mathbb{R}^n$ is sublinear if $p = \sup \rho_n p_n$, where $\{p_n\}$ is a collection of linear functions. Sublinear functions defined everywhere on $V = \mathbb{R}^n$ are convex; thus these functions are continuous, see [6], Theorem 5.5 and Corollary 10.1.1. We discuss sublinear functions of this kind only. Suppose $p$ is a continuous function defined on $V$, then $p$ is sublinear whenever

1) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$;

2) $p(cx) = cp(x)$ for all $x \in V$, $c \in \mathbb{R}_+$.  

So if $p_1$, $p_2$ are sublinear functions, then $p_1 + p_2$ is a sublinear function.

We shall say that a function $f \in \mathcal{F}(\mathbb{C}^n)$ is simple, if its dequantization $\hat{f}$ exists and a.e. coincides with a sublinear function; by misuse of language, we shall denote this (uniquely defined everywhere on $V$) sublinear function by the same symbol $\hat{f}$.  

Recall that simple functions $f$ and $g$ are in general position if $\hat{f}(x) \neq \hat{g}(x)$ for all $x$ belonging to an open dense subset of $V$. In particular, generalized monomials are in general position whenever they are linearly independent.

Denote by $\text{Sim}(\mathbb{C}^n)$ the set of all simple functions defined on $V$ and denote by $\text{Sim}_+^+(\mathbb{C}^n)$ the set $\text{Sim}(\mathbb{C}^n) \cap \mathcal{D}_+(\mathbb{C}^n)$. By $\text{Sbl}(V)$ denote the set of all (continuous) sublinear functions defined on $V = \mathbb{R}^n$ and by $\text{Sbl}_+(V)$ denote the image $\widehat{\text{Sim}_+^+(\mathbb{C}^n)}$ of $\text{Sim}_+^+(\mathbb{C}^n)$ under the dequantization transform.

The following statements can be easily deduced from Theorem 1 and definitions.

**Corollary 3.** The set $\text{Sim}_+^+(\mathbb{C}^n)$ is a subsemiring of $\mathcal{D}_+(\mathbb{C}^n)$ and $\text{Sbl}_+(V)$ is an idempotent semiring with respect to the operations $(f \odot g)(x) = \max\{f(x), g(x)\}$, $(f \odot g)(x) = f(x) + g(x)$.

**Corollary 4.** Polynomials and generalized polynomials are simple functions.

We shall say that functions $f, g \in \mathcal{D}(V)$ are asymptotically equivalent whenever $\hat{f} = \hat{g}$; any simple function $f$ is an asymptotic monomial whenever $\hat{f}$ is a linear function. A simple function $f$ will be called an asymptotic polynomial whenever $\hat{f}$ is a sum of a finite collection of nonequivalent asymptotic monomials.

**Corollary 5.** Every asymptotic polynomial is a simple function.

**Example 1.** Generalized polynomials, logarithmic functions of (generalized) polynomials, and products of polynomials and logarithmic functions are asymptotic polynomials. This follows from our definitions and formula (2).

4. **Subdifferentials of sublinear functions**

We shall use some elementary results from convex analysis. These results can be found, e.g., in [5], ch. 1, §1.

For any function $p \in \text{Sbl}(V)$ we set

$$(4) \quad \partial p = \{v \in V \mid (v, x) \leq p(x) \ \forall x \in V\}.$$  

It is well known from convex analysis that for any sublinear function $p$ the set $\partial p$ is exactly the subdifferential of $p$ at the origin. The following propositions are also known in convex analysis.

**Proposition 2.** Suppose $p_1, p_2 \in \text{Sbl}(V)$, then

1) $\partial (p_1 + p_2) = \partial p_1 \odot \partial p_2 = \{v \in V \mid v = v_1 + v_2, \ \text{where} \ v_1 \in \partial p_1, v_2 \in \partial p_2\}$;

2) $\partial (\max\{p_1(x), p_2(x)\}) = \partial p_1 \odot \partial p_2$.

Recall that $\partial p_1 \odot \partial p_2$ is a convex hull of the set $\partial p_1 \cup \partial p_2$.

**Proposition 3.** Suppose $p \in \text{Sbl}(V)$. Then $\partial p$ is a nonempty convex compact subset of $V$.
Corollary 6. The map $p \mapsto \partial p$ is a homomorphism of the idempotent semiring $Sbl(V)$ (see Corollary 3) to the idempotent semiring $S$ of all convex compact subsets of $V$ (see Section 1 above).

5. NEWTON SETS FOR SIMPLE FUNCTIONS

For any simple function $f \in Sim(C^n)$ let us denote by $N(f)$ the set $\partial(\hat{f})$. We shall call $N(f)$ the Newton set of the function $f$.

Proposition 4. For any simple function $f$, its Newton set $N(f)$ is a nonempty convex compact subset of $V$.

This proposition follows from Proposition 3 and definitions.

Theorem 2. Suppose that $f$ and $g$ are simple functions. Then

1) $N(fg) = N(f) \odot N(g) = \{v \in V \mid v = v_1 + v_2 \text{ with } v_1 \in N(f), v_2 \in N(g)\};$

2) $N(f + g) = N(f) \oplus N(g)$, if $f_1$ and $f_2$ are in general position or $f_1, f_2 \in Sim_+(C^n)$ (recall that $N(f) \oplus N(g)$ is the convex hull of $N(f) \cup N(g)$).

This theorem follows from Theorem 1, Proposition 2 and definitions.

Corollary 7. The map $f \mapsto N(f)$ generates a homomorphism from $Sim_+(C^n)$ to $S$.

This statement follows from Theorem 1, Corollary 1, Corollary 6, and Theorem 2.

Proposition 5. Let $f = m_{a,d}(x) = \prod_{i=1}^{n} a_i x_i^{d_i}$ be a monomial; here $d = (d_1, \ldots, d_n) \in V = \mathbb{R}^n$ and $a$ is a nonzero complex number. Then $N(f) = \{d\}$.

This follows from Proposition 1, Corollary 2 and definitions.

Corollary 8. Let $f = \sum_{d \in D} m_{a,d}$ be a polynomial. Then $N(f)$ is the polytope $\oplus_{d \in D}\{d\}$, i.e. the convex hull of the finite set $D$.

This statement follows from Theorem 2 and Proposition 5. Thus in this case $N(f)$ is the well-known classical Newton polytope of the polynomial $f$.

Now the following corollary is obvious.

Corollary 9. Let $f$ be a generalized or asymptotic polynomial. Then its Newton set $N(f)$ is a convex polytope.

Example 2. Consider the one dimensional case, i.e., $V = \mathbb{R}$ and suppose $f_1 = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ and $f_2 = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$, where $a_n \neq 0$, $b_m \neq 0$, $a_0 \neq 0$, $b_0 \neq 0$. Then $N(f_1)$ is the segment $[0,n]$ and $N(f_2)$ is the segment $[0,m]$. So the map $f \mapsto N(f)$ corresponds to the map $f \mapsto \deg(f)$, where $\deg(f)$ is a degree of the polynomial $f$. In this case Theorem 2 means that $\deg(fg) = \deg f + \deg g$ and $\deg(f + g) = \max\{\deg f, \deg g\} = \max\{n, m\}$ if $a_i \geq 0$, $b_i \geq 0$ or $f$ and $g$ are in general position.

Remark. The above results can be extended to the infinite-dimensional case. This generalization will be the subject of another paper.
THE DEQUANTIZATION TRANSFORM

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