SYMMETRY AND MONOTONICITY OF SINGULAR SOLUTIONS TO p-LAPLACIAN SYSTEMS INVOLVING A FIRST ORDER TERM

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ABSTRACT. We consider positive singular solutions (i.e. with a non-removable singularity) of a system of PDEs driven by p-Laplacian operators and with the additional presence of a nonlinear first order term. By a careful use of a rather new version of the moving plane method, we prove the symmetry of the solutions. The result is already new in the scalar case.

1. Introduction

The study of symmetry properties of solutions of PDEs is a long-standing topic which dates back to the foundational works of Alexandrov [1], Serrin [24], Gidas, Ni and Nirenberg [17] and Berestycki and Nirenberg [2]. The common point among such papers is the use of the celebrated moving plane method in the case of semilinear elliptic equations. This powerful technique relies heavily on the validity of (weak and strong) maximum or comparison, principles. The relevance of the ideas which are at the core of the aforementioned papers is clearly witnessed by the huge number of contributions which are now available in the literature, whose aim has been to extend the results above mentioned to several different cases in the local framework: quasilinear equations (see, e.g., [7, 8, 9, 10]) and cooperative elliptic systems (see, e.g., [4, 6, 11, 12, 13, 27]).

A slightly different line of research concerns the study of symmetry properties of singular solutions. In this case, we refer, e.g., to [5, 25] for the case of point-singularity. More recently, Sciunzi [22] developed a new method which allows to study symmetry properties of solutions which are singular on sets of small capacity.

The aim of this paper is to focus on systems of PDEs driven by the p-Laplacian with the additional presence of a nonlinear gradient term. To be more precise, we will consider solutions \(u_i \in C^{1,\alpha}(\Omega \setminus \Gamma)\) \((i = 1, \ldots, m)\) to the following system of PDEs

\[
\begin{cases}
-\Delta_p u_i + a_i(u_i)|\nabla u_i|^q = f_i(u_1, \ldots, u_m) & \text{in } \Omega \setminus \Gamma \\
u_i > 0 & \text{in } \Omega \setminus \Gamma \\
u_i = 0 & \text{on } \partial \Omega.
\end{cases}
\]

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Here, $\Omega \subseteq \mathbb{R}^N$ is a smooth and bounded domain and for suitable smooth functions
\[-\Delta_p v := \text{div} \left( |\nabla v|^{p-2} \nabla v \right)\]
denotes the $p$-Laplacian. Generally, solutions to equations involving the $p$-Laplace operator are of class $C^{1,\alpha}$. This assumption is natural according to classical regularity results [14, 19, 26]. We point out that, in this paper, we deal with singular solutions that are $C^{1,\alpha}$ far from the critical set $\Gamma$. We consider solutions with a non-removable singularity: we mean that solutions possibly do not admit a smooth extension in $\Omega$, i.e. it is not possible to find $\tilde{u}_i \in W^{1,p}_0(\Omega)$ such that $u_i \equiv \tilde{u}_i$ in $\Omega \setminus \Gamma$. Indeed, without any a priori assumption, the gradient of the solutions possibly blows up near the critical set and hence each equations of (S) may exhibit both a degenerate and a singular nature at the same time.

Before stating our main result, we first introduce the main structural assumptions we require on problem (S); these assumptions will be tacitly understood in the sequel.

**Assumptions.** Throughout what follows, we assume that
\[(h_\Omega): \Omega \text{ is a bounded domain in } \mathbb{R}^N \text{ (with } N \geq 2), \text{ which is convex in the } x_1\text{-direction and symmetric with respect to the hyperplane}\]
\[\Pi_0 := \{x \in \mathbb{R}^N : x_1 = 0\};\]
\[(h_\Gamma): \Gamma \subseteq \Omega \cap \Pi_0 \text{ is a compact set satisfying } \text{Cap}_{p_k}(\Gamma) = 0, \text{ where }\]
\[p := \max_{1 \leq k \leq m} p_k;\]
\[(h_{p,q}): \text{ for every } i = 1, \ldots, m, \text{ we have}\]
\[
\frac{2N + 2}{N + 2} < p_i \leq N \quad \text{and} \quad \max\{p_i - 1, 1\} \leq q_i < p_i
\]
\[(1.1)\]
or
\[2 \leq p_i \leq N \quad \text{and} \quad q_i = p_i.\]
\[(h_a): \text{ the functions } a_i(\cdot) \text{ are locally Lipschitz-continuous on } \mathcal{I} := [0, +\infty), \text{ that is, for every } M > 0 \text{ there exists a constant } L = L_M > 0 \text{ such that}\]
\[|a_i(t) - a_i(s)| \leq L|t - s| \quad \text{for every } t, s \in [0, M];\]
\[(h_f): \text{ the functions } f_i(\cdot) \text{ are of class } C^1 \text{ on } \mathcal{I}^m; \text{ moreover, we assume that}\]
\[(a) \ f_i(t_1, \ldots, t_m) > 0 \text{ for every } t_1, \ldots, t_m > 0;\]
\[(b) \ \partial_{t_k} f_i(\cdot) \geq 0 \text{ for every } 1 \leq i, k \leq m \text{ with } i \neq k.\]
We remark that, without loss of generality, we are considering the $x_1$-direction being the operator invariant with respect to translations and rotations.

Then, we properly define what we mean by a weak solution to problem (S).

**Definition 1.1.** We say that a vector-valued function
\[u = (u_1, \ldots, u_m) \in C^{1,\alpha}(\Omega \setminus \Gamma; \mathbb{R}^m)\]
is a weak solution to (S) if it satisfies the properties listed below:
for every $i = 1, \ldots, m$ and every $\varphi \in C^1_c(\Omega \setminus \Gamma)$, one has
\begin{equation}
\int_\Omega |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \varphi \rangle \, dx + \int_\Omega a_i(u_i)|\nabla u_i|^{q_i} \varphi \, dx = \int_\Omega f_i(u_1, \ldots, u_m) \varphi \, dx.
\end{equation}

(2) $u_i > 0$ pointwise in $\Omega \setminus \Gamma$ (for every $i = 1, \ldots, m$);
(3) $u_i = 0$ on $\partial\Omega$ (for every $i = 1, \ldots, m$).

Our main results is the following

**Theorem 1.2.** Let assumptions $(h_\Omega)$-to-$(h_f)$ be in force, and let $u \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m)$ be a weak solution to problem $(S)$. Then, the following facts hold:

(i) $u$ is symmetric with respect to the hyperplane $\Pi_0$, namely
\[ u(x_1, x_2, \ldots, x_N) = u(-x_1, x_2, \ldots, x_N) \text{ in } \Omega. \]

(ii) $u$ is non-decreasing in the $x_1$-direction in the set $\Omega_0 = \{x_1 < 0\}$ and moreover
\begin{equation}
\partial_{x_1} u_i > 0 \text{ in } \Omega_0,
\end{equation}
for every $i \in \{1, \ldots, m\}$.

We remark that, although the technique that we will develop to prove Theorem 1.2 works for any $p_i > N$, the result is stated in the range (1.1) since there are no sets (different from the empty-set) of zero $p_i$-capacity when $p_i > N$.

We now want to spend a few comments about Theorem 1.2. Firstly, we notice that Theorem 1.2 extends the results in [15] to the case of singular (non regular) solutions. The main novelty of our result concerns the singular case $p_i < 2$ (for some $i = 1, \ldots, m$). In order to explain this, let us consider the case of a scalar equation with $a(u) \equiv 0$: our problem boils then down to the case considered in [16] (which is therefore our benchmark case), that is,
\begin{equation}
\begin{cases}
-\Delta_p u = f(u) & \text{in } \Omega \setminus \Gamma \\
u > 0 & \text{in } \Omega \setminus \Gamma \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\end{equation}

In [16] the Authors, already in the scalar case, require some a priori assumptions on the nonlinearity $f$ involved in the problem, while here we are able to remove the growth a priori assumption made in [16] on the nonlinearities $f_i$. This aspect is strictly related with a technical issue that has to be faced when using the *integral version* of the moving plane method. In particular, one needs to apply a weighted Sobolev inequality due to Trudinger [28], whose validity depends on proper summability conditions of the weight, which happens to be of the form $|\nabla u|^{p-2}$. In [16] this condition is proved to be satisfied after a nice study of the behavior of the gradient of the solution near the singular set $\Gamma$, which is based on a subtle growth estimate proved in [21], thanks to a priori assumption on the nonlinearity $f$ involved in the problem. Since we are lacking an analogous estimate for our solutions (even in the simpler case $a_i \equiv 0$ for every $i = 1, \ldots, m$), we are not even entitled to profit of the weighted Sobolev inequality of Trudinger. Nevertheless, we can avoid such a priori assumptions by exploiting some a priori regularity results that we prove
in this paper, see Lemma 2.3 and Lemma 2.4 below. We stress that this is a considerable simplification of the proof in [16] and it is crucially based on the new Lemma 2.4.

Now, by scrutinizing the proof of Theorem 1.2, one can easily recognize that a key ingredient for our argument is the fact that, under assumption \((h_{p,q})\), we have
\[
p_i^* = \frac{Np_i}{N - p_i} \geq 2 \quad \text{(for all } i = 1, \ldots, m)\]
However, since the above inequality is equivalent to require that \(p_i \geq 2N/(N + 2)\), it is natural to wonder why we require \(p_i\) to satisfy the worse lower bound
\[
p_i > \frac{2N + 2}{N + 2} =: \beta_N.
\]
The main reason behind this choice is that, when \(p \leq \beta_N\), we do not have at our disposal a Strong Comparison Principle for the operator \(-\Delta_p u + a(u)|\nabla u|^q\) which allows us to do not take care of the critical set \(\mathcal{Z} = \{\nabla u = 0\}\) that is, in general, a crucial point when one deals with the \(p\)-Laplace operator.

In view of this fact, in order to establish Theorem 1.2 when
\[
\frac{2N}{N + 2} \leq p_i \leq \frac{2N + 2}{N + 2}
\]
we would need to recover a technical lemma analogous to [16, Lemma 3.2], which also requires suitable estimates for the second-order derivatives of the solution of \((S)\). Even if these estimates are available in our setting (see [15, Theorem 2.1]), we prefer to avoid this technicality here: indeed, we believe that the main novelties of our work are both to consider systems involving general first order terms and the possibility of considering the case \(p_i < 2\) without the need of prescribing precise (a priori) growth estimates for the solution of \((S)\). To the best of our knowledge, our result is new also in the case of a single scalar equation.

The paper is organized as follows: we prove some technical results in Section 2 that we will exploit in Section 3 to prove Theorem 1.2.

2. Notation and preliminary results

In this first section we collect all the relevant notation which shall be used throughout the paper; moreover, we review some results already known in the literature which shall be fundamental for the proof of our Theorem 1.2. We will adopt the symbol \(|A|\) to denote the \(N\)-dimensional Lebesgue measure of a measurable set \(A \subseteq \mathbb{R}^N\).

2.1. Capacity. Due to the important role played by assumption \((h_{\Gamma})\) in our arguments, we briefly recall a few notions and results concerning the Sobolev capacity.

Let \(K \subseteq \mathbb{R}^N\) be a compact set, and let \(\mathcal{O} \subseteq \mathbb{R}^N\) be an open set such that \(K \subseteq \mathcal{O}\). Given any \(1 < r \leq N\), the \(r\)-capacity of the condenser \(\mathcal{C} := (K, \mathcal{O})\) is defined as
\[
\text{Cap}_r(K, \mathcal{O}) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla \phi|^r \, dx : \phi \in C_0^\infty(\mathcal{O}) \text{ and } \phi \geq 1 \text{ on } K \right\}.
\]
We then say that $K$ has *vanishing $r$-capacity*, and we write $\text{Cap}_r(K) = 0$, if
$$\text{Cap}_r(K, O) = 0 \quad \text{for every open set } O \supset K.$$ 
As is reasonable to expect, compact sets with vanishing $r$-capacity have to be *very small*; the next theorem shows that this is actually true in the sense of Hausdorff measure.

**Theorem 2.1.** Let $K \subseteq \mathbb{R}^N$ be a compact set. Then, the following assertions hold.

1. If $\text{Cap}_r(K) = 0$, then $\mathcal{H}^s(K) = 0$ for every $s > N - r$.
2. If $r < N$ and $\mathcal{H}^{N-r}(K) < \infty$, then $\text{Cap}_r(K) = 0$.

For a proof of Theorem 2.1, we refer the Reader to [18, Sec. 2.24].

**Corollary 2.2.** Let $1 < r_1 < r_2 \leq N$ and let $K \subseteq \mathbb{R}^N$ be compact. Then,
$$\text{Cap}_{r_2}(K) = 0 \implies \text{Cap}_{r_1}(K) = 0.$$

*Proof.* Since, by assumption, $\text{Cap}_{r_2}(K) = 0$, by Theorem 2.1-(1) we have $\mathcal{H}^s(K) = 0$ for every $s > N - r_2$; in particular, since $r_1 < r_2$, we derive that
$$r_1 < N \quad \text{and} \quad \mathcal{H}^{N-r_1}(K) = 0.$$ 
Using this fact and Theorem 2.1-(2), we then conclude that $\text{Cap}_{r_1}(K) = 0$. \hfill $\square$

On account of Corollary 2.2, if $\Gamma \subseteq \Omega \cap \Pi_0$ is as in Theorem 1.2, we have
$$\text{Cap}_{p_i}(\Gamma) = 0 \quad \text{for every } i = 1, \ldots, m.$$

Let now $K \subseteq \mathbb{R}^N$ be a fixed compact set *with vanishing $r$-capacity*, and let $O \subseteq \mathbb{R}^N$ be an open set containing $K$. We aim to show that it is possible to construct a family of functions in $\mathbb{R}^N$, say $\{\psi_\varepsilon\}_\varepsilon$, which satisfies the following properties:

1. $\psi_\varepsilon \in \text{Lip}(\mathbb{R}^N)$ and $0 \leq \psi_\varepsilon \leq 1$ pointwise in $\mathbb{R}^N$;
2. there exists an open neighborhood $\mathcal{V}_\varepsilon \subseteq O$ of $K$ such that
$$\psi_\varepsilon \equiv 0 \text{ on } \mathcal{V}_\varepsilon;$$
3. $\psi_\varepsilon(x) \to 1$ as $\varepsilon \to 0^+$ for a.e. $x \in \mathbb{R}^N$;
4. there exists a constant $C_0 > 0$, only depending on $r$, such that
$$\int_{\mathbb{R}^N} |\nabla \psi_\varepsilon|^r \, dx \leq C_0 \varepsilon.$$

In fact, let $\varepsilon_0 = \varepsilon_0(K, O) > 0$ be such that
$$B_\varepsilon := \{x \in \mathbb{R}^N : \text{dist}(x, K) < \varepsilon\} \subseteq O \quad \text{for every } 0 < \varepsilon < \varepsilon_0.$$ 
Since $\text{Cap}_r(K) = 0$ and $B_\varepsilon \subseteq \mathbb{R}^N$ is an open neighborhood of $K$, for every $\varepsilon \in (0, \varepsilon_0)$ there exists a smooth function $\phi_\varepsilon \in C^\infty_c(B_\varepsilon)$ such that
$$\phi_\varepsilon \geq 1 \text{ on } K \quad \text{and} \quad \int_{\mathbb{R}^N} |\nabla \phi_\varepsilon|^r \, dx < \varepsilon.$$ 
We then consider the Lipschitz functions
(i) \( T(s) := \max\{0; \min\{s; 1\}\} \) (for \( s \in \mathbb{R} \)),
(ii) \( g(t) := \max\{0; 1 - 2t\} \) (for \( t \geq 0 \))

and we define, for every \( \varepsilon \in (0, \varepsilon_0) \),

\[
(2.3) \quad \psi_{\varepsilon} : \mathbb{R}^N \to \mathbb{R}, \quad \psi_{\varepsilon}(x) := g(T(\varphi_{\varepsilon}(x))).
\]

Clearly, \( \psi_{\varepsilon} \in \text{Lip}(\mathbb{R}^N) \) and \( 0 \leq \psi_{\varepsilon} \leq 1 \) in \( \mathbb{R}^N \); moreover, by (2.2) we have

\[
\int_{\mathbb{R}^N} |\nabla \psi_{\varepsilon}| \, dx \leq 2^r \int_{\mathbb{R}^N} |\nabla \phi_{\varepsilon}| \, dx \leq 2^r \varepsilon.
\]

Hence, the family \( \{\psi_{\varepsilon}\}_\varepsilon \) satisfies properties (1) and (4) (with \( C_0 = 2^r \)). As for the validity of property (2) we observe that, by the explicit definitions of \( T \) and \( g \), we have

\[
\psi_{\varepsilon} \equiv 0 \quad \text{on} \quad \mathcal{V}_{\varepsilon} := \{\phi_{\varepsilon} > 1/2\};
\]

as a consequence, since \( \mathcal{V}_{\varepsilon} \) is an open neighborhood of \( K \) and since \( \mathcal{V}_{\varepsilon} \subseteq \mathcal{B}_{\varepsilon} \subseteq \mathcal{O} \) (remind that \( \phi_{\varepsilon} \geq 1 \) on \( K \) and \( \text{supp}(\phi_{\varepsilon}) \subseteq \mathcal{B}_{\varepsilon} \)), we immediately conclude that also property (2) is satisfied. Finally, the validity of property (3) follows from the fact that

\[
(2.4) \quad \psi_{\varepsilon} \equiv g(0) = 1 \quad \text{on} \quad \mathbb{R}^N \setminus \mathcal{B}_{\varepsilon}.
\]

Indeed, since the family \( \{\mathcal{B}_{\varepsilon}\}_\varepsilon \) shrinks to \( K \) as \( \varepsilon \to 0^+ \), by (2.4) we get

\[
\psi_{\varepsilon}(x) \to 1 \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{for every} \quad x \in \mathbb{R}^N \setminus K;
\]

from this, since \( |K| = \mathcal{H}^N(K) = 0 \) (remind that \( \text{Cap}_p(K) = 0 \) and see Theorem 2.1), we immediately conclude that the family \( \{\psi_{\varepsilon}\}_\varepsilon \) satisfies also property (3), as claimed.

Throughout what follows, we will repeatedly use the family \( \{\psi_{\varepsilon}\}_\varepsilon \) with different choices of \( K \) and \( \mathcal{O} \); hence, to simplify the notation, we shall refer to this family as a cut-off family for the compact set \( K \), related with the open set \( \mathcal{O} \).

### 2.2. Notations for the moving plane method.

Let \( u \in C^{1,\alpha}(\overline{\Omega} \setminus \Gamma; \mathbb{R}^m) \) be a weak solution to problem \( (S) \). For every fixed \( \lambda \in \mathbb{R} \), we denote by \( R_{\lambda} \) the reflection trough the hyperplane \( \Pi_{\lambda} := \{x \in \mathbb{R}^N : x_1 = \lambda\} \), that is,

\[
(2.5) \quad R_{\lambda}(x) = x_{\lambda} := (2\lambda - x_1, x_2, \ldots, x_N);
\]

accordingly, we introduce the vector-valued function

\[
(2.6) \quad u_{\lambda}(x) = (u_{1,\lambda}(x), \ldots, u_{m,\lambda}(x)) := u(x_{\lambda}), \quad \text{for} \quad x \in R_{\lambda}(\overline{\Omega} \setminus \Gamma).
\]

We point out that, since \( u \) solves \( (S) \), it is easy to see that

1. \( u_{\lambda} \in C^{1,\alpha}(R_{\lambda}(\overline{\Omega} \setminus \Gamma); \mathbb{R}^m) \);
2. for every \( i = 1, \ldots, m \) and every \( \varphi \in C^1_c(R_{\lambda}(\Omega \setminus \Gamma)) \), one has

\[
(2.7) \quad \int_{R_{\lambda}(\Omega)} |\nabla u_{i,\lambda}|^{p_i-2} \langle \nabla u_{i,\lambda}, \nabla \varphi \rangle \, dx + \int_{R_{\lambda}(\Omega)} a_i(u_{i,\lambda}) |\nabla u_{i,\lambda}|^{p_i} \varphi \, dx = \int_{R_{\lambda}(\Omega)} f_i(u_{\lambda}) \varphi \, dx;
\]

3. \( u_{i,\lambda} > 0 \) pointwise \( R_{\lambda}(\Omega \setminus \Gamma) \) (for every \( i = 1, \ldots, m \));
4. \( u_{i,\lambda} \equiv 0 \) on \( R_{\lambda}(\partial \Omega) \) (for every \( i = 1, \ldots, m \)).
To proceed further, we let
\[ (2.8) \quad \varrho = \varrho_\Omega := \inf_{x \in \Omega} x_1 \]
and we observe that, since \( \Omega \) is bounded and symmetric with respect to \( \Pi_0 \), we certainly have \( -\infty < \varrho < 0 \). As a consequence, for every \( \lambda \in (\varrho, 0) \) we can set
\[ (2.9) \quad \Omega_\lambda := \{ x \in \Omega : x_1 < \lambda \}. \]
We explicitly point out that, since \( \Omega \) is convex in the \( x_1 \)-direction, we have
\[ (2.10) \quad \Omega_\lambda \subseteq R_\lambda(\Omega) \cap \Omega. \]
Finally, for every \( \lambda \in (\varrho, 0) \) we define the function
\[ w_\lambda(x) = (w_{1,\lambda}(x), \ldots, w_{m,\lambda}(x)) := (u - u_\lambda)(x), \quad \text{for } x \in (\Omega \setminus \Gamma) \cap R_\lambda(\Omega \setminus \Gamma). \]
On account of \((2.10)\), \( w_\lambda \) is surely well-posed on \( \Omega_\lambda \setminus R_\lambda(\Gamma) \).

2.3. Preliminary results. After these preliminaries, we devote the remaining part of this section to collect some auxiliary results which shall be useful for the proof of Theorem 1.2. In what follows, we tacitly inherit all the notation introduced so far.

To begin with, we recall some identities between vectors in \( \mathbb{R}^N \) which are useful in dealing with quasilinear operators: for every \( s > 1 \) there exist constants \( C_1, \ldots, C_4 > 0 \), only depending on \( s \), such that, for every \( \eta, \eta' \in \mathbb{R}^N \), one has
\[ \begin{align*}
\langle |\eta|^{s-2}\eta - |\eta'|^{s-2}\eta', \eta - \eta' \rangle & \geq C_1(|\eta| + |\eta'|)^{s-2}|\eta - \eta'|^2, \\
|\eta|^{s-2}\eta - |\eta'|^{s-2}\eta' & \leq C_2(|\eta| + |\eta'|)^{s-2}|\eta - \eta'|, \\
\langle |\eta|^{s-2}\eta - |\eta'|^{s-2}\eta', \eta - \eta' \rangle & \geq C_3|\eta - \eta'|^s \quad \text{(if } s \geq 2), \\
|\eta|^{s-2}\eta - |\eta'|^{s-2}\eta' & \leq C_4|\eta - \eta'|^{s-1} \quad \text{(if } 1 < s < 2). \\
\end{align*} \]
We refer, e.g., to [7] for a proof of \((2.11)\).

We then establish the following fundamental lemma.

**Lemma 2.3.** Let assumptions \((h_\Omega)\)-to-\((h_f)\) be in force. Let \( i \in \{1, \ldots, m\} \) be fixed, and let \( \lambda \in (\varrho, 0) \) be such that
\[ R_\lambda(\Gamma) \cap \overline{\Omega_\lambda} \neq \emptyset. \]
Then, there exists a constant \( c = c_i > 0 \) such that
\[ (2.12) \quad \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq c_i. \]

**Proof.** We first notice that, since \( \lambda < 0 \), by assumption \((h_\Gamma)\) we have \( \overline{\Omega_\lambda} \cap \Gamma = \emptyset \). Thus, the function \( u = (u_1, \ldots, u_m) \) is of class \( C^{1, \alpha} \) on \( \overline{\Omega_\lambda} \), and we can set
\[ M = M_u := \max_{1 \leq j \leq m} \left( \|u_j\|_{L^\infty(\overline{\Omega_\lambda})} + \|\nabla u_j\|_{L^\infty(\overline{\Omega_\lambda})} \right) < +\infty. \]
Moreover, since \( u_i \) and \( u_{i,\lambda} \) are non-negative, we have

\[
0 \leq w_{i,\lambda}^+ = (u_i - u_{i,\lambda})^+ \leq u_i \leq M \quad \text{pointwise in } \overline{\Omega} \setminus R_{\lambda}(\Gamma).
\]

Now, to prove (2.12) we distinguish two cases.

(i) \( \max\{1, p_i - 1\} \leq q_i < p_i \);  
(ii) \( q_i = p_i \).

**Case (i).** First of all we observe that, since \( R_{\lambda} \) is a bijective linear map and since \( \Gamma \) has vanishing \( p_i \)-capacity (see (2.1)), the compact set \( \Gamma_{\lambda} := R_{\lambda}(\Gamma) \) satisfies

\[
\text{Cap}_{p_i}(\Gamma_{\lambda}) = 0.
\]

As a consequence, if \( O_{\lambda} \subseteq \mathbb{R}^N \) is a fixed open neighborhood of \( \Gamma_{\lambda} \), we can choose a cut-off family \( \{\psi_{\varepsilon}\}_{\varepsilon<\varepsilon_0} \) for \( \Gamma_{\lambda} \), related with the open set \( O_{\lambda} \). This means, precisely, that

1. \( \psi_{\varepsilon} \in \text{Lip}(\mathbb{R}^N) \) and \( 0 \leq \psi_{\varepsilon} \leq 1 \) pointwise in \( \mathbb{R}^N \);  
2. there exists an open neighborhood \( \mathcal{V}_{\varepsilon} \subseteq O_{\lambda} \) of \( \Gamma_{\lambda} \) such that

\[
\psi_{\varepsilon} \equiv 0 \text{ on } \mathcal{V}_{\varepsilon};
\]

3. \( \psi_{\varepsilon}(x) \to 1 \) as \( \varepsilon \to 0^+ \) for a.e. \( x \in \mathbb{R}^N \);  
4. there exists a constant \( C_0 > 0 \), independent of \( \varepsilon \), such that

\[
\int_{\mathbb{R}^N} |\nabla \psi_{\varepsilon}|^{p_i} \, dx \leq C_0 \varepsilon.
\]

We now define, for every \( \varepsilon \in (0, \varepsilon_0) \), the map

\[
\varphi_{i,\varepsilon}(x) := \begin{cases} 
  w_{i,\lambda}^+(x) \psi_{\varepsilon}^{p_i}(x) = (u_i - u_{i,\lambda})^+(x) \psi_{\varepsilon}^{p_i}(x), & \text{if } x \in \Omega_{\lambda}, \\
  0, & \text{otherwise}.
\end{cases}
\]

We then claim that the following assertion hold:

(i) \( \varphi_{i,\varepsilon} \in \text{Lip}(\mathbb{R}^N) \);  
(ii) \( \text{supp}(\varphi_{i,\varepsilon}) \subseteq \Omega_{\lambda} \) and \( \varphi_{i,\varepsilon} \equiv 0 \) near \( \Gamma_{\lambda} \).

In fact, since \( u_i \in C^{1,\alpha}(\overline{\Omega}_{\lambda}) \) and \( u_{i,\lambda} \in C^{1,\alpha}(\overline{\Omega}_{\lambda} \setminus \Gamma_{\lambda}) \), we readily see that

\[
w_{i,\lambda}^+ \in \text{Lip}(\overline{\Omega}_{\lambda} \setminus V) \text{ for every open set } V \supseteq \Gamma_{\lambda};
\]

from this, since \( \psi_{\varepsilon} \in \text{Lip}(\mathbb{R}^N) \) and \( \psi_{\varepsilon} \equiv 0 \) on \( \mathcal{V}_{\varepsilon} \supseteq \Gamma_{\lambda} \), we get \( \varphi_{i,\varepsilon} \in \text{Lip}(\overline{\Omega}_{\lambda}) \). On the other hand, since \( \varphi_{i,\varepsilon} \equiv 0 \) on \( \partial \Omega_{\lambda} \), we easily conclude that \( \varphi_{i,\varepsilon} \in \text{Lip}(\mathbb{R}^N) \), as claimed. As for assertion (ii), it follows from the definition of \( \varphi_{i,\varepsilon} \), jointly with the fact that

\[
\psi_{\varepsilon} \equiv 0 \text{ on } \mathcal{V}_{\varepsilon} \supseteq \Gamma_{\lambda}.
\]
On account of properties (i)-(ii) of $\varphi_{i,\varepsilon}$, a standard density argument allows us to use $\varphi_{i,\varepsilon}$ as a test function both in (1.2) and (2.7), obtaining

\[
\begin{align*}
\int_{\Omega_{\lambda}} \langle |\nabla u_i|^{p_i} - |\nabla u_{i,\lambda}|^{p_i} \nabla u_i, \nabla \varphi_{i,\varepsilon} \rangle \, dx \\
+ \int_{\Omega_{\lambda}} (a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i}) \varphi_{i,\varepsilon} \, dx \\
= \int_{\Omega_{\lambda}} (f_i(u) - f_i(u_{\lambda})) \varphi_{i,\varepsilon} \, dx.
\end{align*}
\]  

(2.14)

By unraveling the very definition of $\varphi_{i,\varepsilon}$, we then obtain

\[
\begin{align*}
\int_{\Omega_{\lambda}} \langle |\nabla u_i|^{p_i} - |\nabla u_{i,\lambda}|^{p_i} \nabla u_i, \nabla w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \rangle \, dx \\
+ p_i \int_{\Omega_{\lambda}} \langle |\nabla u_i|^{p_i} - |\nabla u_{i,\lambda}|^{p_i} \nabla \psi_{\varepsilon} \rangle w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i-1} \, dx \\
+ \int_{\Omega_{\lambda}} (a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i}) w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx \\
= \int_{\Omega_{\lambda}} (f_i(u) - f_i(u_{\lambda})) w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx.
\end{align*}
\]  

(2.15)

We now observe that the integral in the right-hand side of (2.15) is actually performed on the set $A_{i,\lambda} := \{ x \in \Omega_{\lambda} : u_i \geq u_{i,\lambda} \} \setminus \Gamma_{\lambda}$; moreover, we have

\[
0 \leq u_{i,\lambda}(x) \leq u_i(x) \leq M \quad \text{for every } x \in A_{i,\lambda}.
\]  

(2.16)

The right hand side of (2.15) can be arranged as follows

\[
\begin{align*}
\int_{\Omega_{\lambda}} (f_i(u) - f_i(u_{\lambda})) w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx &= \int_{\Omega_{\lambda}} [f_i(u_1, \ldots, u_m) - f_i(u_{1,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx \\
&= \int_{\Omega_{\lambda}} [f_i(u_1, u_2, \ldots, u_m) - f_i(u_{1,\lambda}, u_2, \ldots, u_m)] w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx \\
&+ \int_{\Omega_{\lambda}} [f_i(u_{1,\lambda}, u_2, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx \\
&+ \int_{\Omega_{\lambda}} [f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{i,\lambda}, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{i,\lambda}, \ldots, u_m)] w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx \\
&+ \int_{\Omega_{\lambda}} [f_i(u_{1,\lambda}, u_{2,\lambda}, u_{3,\lambda}, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, u_{3,\lambda}, \ldots, u_{m,\lambda})] w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i} \, dx.
\end{align*}
\]  

(2.17)
By (2.17) we have

\begin{equation}
\int_{\Omega,\lambda} (f_i(u) - f_i(u_{\lambda})) w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx = \int_{\Omega,\lambda} \left[ f_i(u_1, u_2, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) \right] w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx
\end{equation}

\begin{align*}
&= \int_{\Omega,\lambda} \frac{f_i(u_1, u_2, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_1 - u_{1,\lambda}} (u_1 - u_{1,\lambda}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
&\quad + \int_{\Omega,\lambda} \frac{f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_2 - u_{2,\lambda}} (u_2 - u_{2,\lambda}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
&\quad + \int_{\Omega,\lambda} \frac{f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_{i} - u_{i,\lambda}} (u_{i} - u_{i,\lambda}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
&\quad + \int_{\Omega,\lambda} \frac{f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_{m} - u_{m,\lambda}} (u_{m} - u_{m,\lambda}) w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx
\end{align*}

\begin{align*}
&\leq \int_{\Omega,\lambda} \frac{f_i(u_1, u_2, \ldots, u_m) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_1 - u_{1,\lambda}} (u_1 - u_{1,\lambda})^+ w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
&\quad + \int_{\Omega,\lambda} \frac{f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_{2} - u_{2,\lambda}} (u_{2} - u_{2,\lambda})^+ w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
&\quad + \int_{\Omega,\lambda} \frac{f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_{i} - u_{i,\lambda}} (u_{i} - u_{i,\lambda})^+ w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
&\quad + \int_{\Omega,\lambda} \frac{f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda}) - f_i(u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{m,\lambda})}{u_{m} - u_{m,\lambda}} (u_{m} - u_{m,\lambda})^+ w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx
\end{align*}

\begin{align*}
&\leq \sum_{j=1}^{m} L_j \int_{\Omega,\lambda} w_{j,\lambda}^+ w_{j,\lambda}^+ \psi_\varepsilon^{p_i} \, dx,
\end{align*}

where in the last inequality we thought

\begin{equation}
\begin{aligned}
u_j - u_{j,\lambda} &:= w_{j,\lambda}^+ - w_{j,\lambda}^-
\end{aligned}
\end{equation}

and we used the cooperativity assumption of each $f_j$ for $j \neq i$, together with the fact that $f_j$ is of class $C^1$ (see assumption (h$_f$)). Here, $L_j > 0$ is the Lipschitz constant of $f_j$ on $[0, M] \times \cdots \times [0, M]$. Hence, resuming the computations above, we have

\begin{equation}
\begin{aligned}
\int_{\Omega,\lambda} (f_i(u) - f_i(u_{\lambda})) w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx &\leq \sum_{j=1}^{m} L_j \int_{\Omega,\lambda} w_{j,\lambda}^+ w_{j,\lambda}^+ \psi_\varepsilon^{p_i} \, dx.
\end{aligned}
\end{equation}
On the other hand, by using the estimates in (2.11), we get
\[
\int_{\Omega} (|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla w_{i,\lambda}^+ \psi_\varepsilon^{p_i})\,dx
\]
\[
\geq C_1 \int_{\Omega} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\,dx.
\] (2.20)
Gathering together (2.19) and (2.20), from (2.15) we then obtain
\[
C_1 \int_{\Omega} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i}\,dx
\leq \int_{\Omega} (|\nabla u_i|^{p_i-2}\nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2}\nabla u_{i,\lambda}, \nabla \psi_\varepsilon w_{i,\lambda}^\pm \psi_\varepsilon^{p_i-1})\,dx
\leq p_i \int_{\Omega} \left( |a_i(u_i)|^{p_i-2}\nabla u_i - |a_i(u_{i,\lambda})|^{p_i-2}\nabla u_{i,\lambda}, |\nabla \psi_\varepsilon| w_{i,\lambda}^\pm \psi_\varepsilon^{p_i-1} \right)\,dx
+ \int_{\Omega} \sum_{j=1}^m L_j \int_{\Omega} w_{i,\lambda}^+ w_{i,\lambda} \psi_\varepsilon^{p_i}\,dx
\leq p_i \int_{\Omega} \left( |a_i(u_i)|^{p_i-2}\nabla u_i - |a_i(u_{i,\lambda})|^{p_i-2}\nabla u_{i,\lambda}, |\nabla \psi_\varepsilon| w_{i,\lambda}^\pm \psi_\varepsilon^{p_i-1} \right)\,dx
+ \int_{\Omega} \sum_{j=1}^m L_j \int_{\Omega} w_{i,\lambda}^+ w_{i,\lambda} \psi_\varepsilon^{p_i}\,dx.
\] (2.21)
To proceed further, we now turn to provide ad-hoc estimates for the integrals in the right-hand side of (2.21). To this end, we first introduce the notation
\[
F_i := \sum_{j=1}^m L_j \int_{\Omega} w_{j,\lambda}^+ w_{i,\lambda} \psi_\varepsilon^{p_i}\,dx.
\]
Moreover, we split the set \(\Omega_\lambda\) as \(\Omega_\lambda = \Omega_\lambda^{(1)} \cup \Omega_\lambda^{(2)}\), where
\[
\Omega_\lambda^{(1)} = \{ x \in \Omega_\lambda \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| < 2|\nabla u_i| \}
\quad \text{and} \quad
\Omega_\lambda^{(2)} = \{ x \in \Omega_\lambda \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| \geq 2|\nabla u_i| \}.
\]
Then it follows that
- By the definition of \(\Omega_\lambda^{(1)}\), one has
\[
|\nabla u_i| + |\nabla u_{i,\lambda}| < 3|\nabla u_i|,
\] (2.22)
Accordingly, we define
\begin{align}
(2.23) \quad & \frac{1}{2} |\nabla u_{i,\lambda}| \leq |\nabla u_{i,\lambda}| - |\nabla u_i| \leq |\nabla w_{i,\lambda}| \leq |\nabla u_{i,\lambda}| + |\nabla u_i| \leq \frac{3}{2} |\nabla u_{i,\lambda}|.
\end{align}

We then turn to estimate all the above integrals. In what follows, we denote by the same $C$ any positive constant which is independent of $\varepsilon$ (but possibly depending on $i$).

- Estimate of $P_{i,1}$. If $1 < p_i < 2$, from (2.11), (2.22) and by Hölder’s inequality we obtain
\begin{align}
(2.24) \quad & P_{i,1} := \int_{\Omega_{i,\lambda}^{(1)}} \left| \nabla u_i \right|^{p_i - 2} \left| \nabla u_i \right| - \left| \nabla u_{i,\lambda} \right|^{p_i - 2} \left| \nabla u_{i,\lambda} \right| \left| \nabla \psi_\varepsilon \right| w_{i,\lambda}^+ \psi_\varepsilon^{p_i - 1} \, dx \\
& P_{i,2} := \int_{\Omega_{i,\lambda}^{(2)}} \left| \nabla u_i \right|^{p_i - 2} \left| \nabla u_i \right| - \left| \nabla u_{i,\lambda} \right|^{p_i - 2} \left| \nabla u_{i,\lambda} \right| \left| \nabla \psi_\varepsilon \right| w_{i,\lambda}^+ \psi_\varepsilon^{p_i - 1} \, dx \\
& I_{i,1} := \int_{\Omega_{i,\lambda}^{(1)}} \left| a_i(u_i) - a_i(u_{i,\lambda}) \right| \left| \nabla u_{i,\lambda} \right|^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
& I_{i,2} := \int_{\Omega_{i,\lambda}^{(2)}} \left| a_i(u_i) - a_i(u_{i,\lambda}) \right| \left| \nabla u_{i,\lambda} \right|^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
& J_{i,1} := \int_{\Omega_{i,\lambda}^{(1)}} \left| \nabla u_i \right|^{q_i} \left| \nabla u_{i,\lambda} \right|^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx \\
& J_{i,2} := \int_{\Omega_{i,\lambda}^{(2)}} \left| \nabla u_i \right|^{q_i} \left| \nabla u_{i,\lambda} \right|^{q_i} w_{i,\lambda}^+ \psi_\varepsilon^{p_i} \, dx.
\end{align}
If \( p_i \geq 2 \), from (2.11) and the weighted Young’s inequality we have
\[
P_{i,1} \leq C_2 \int_{\Omega^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+| |\nabla \psi_\varepsilon|^{p_i-1} w_{i,\lambda}^+ \, dx
\]
\[
\leq C_\delta \int_{\Omega^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^{p_i} \, dx
\]
\[
+ \frac{C}{\delta} \int_{\Omega^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^2 \, dx.
\]
Using (2.22) and Hölder’s inequality, we obtain
\[
P_{i,1} \leq C_\delta \int_{\Omega^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^{p_i} \, dx
\]
\[
+ \frac{C}{\delta} \left( \int_{\Omega^{(1)}} |\nabla u_i|^{p_i} |\nabla \psi_\varepsilon|^{p_i} \, dx \right)^{\frac{p_i-2}{p_i}} \left( \int_{\Omega^{(1)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx \right)^{\frac{2}{p_i}}.
\]
(2.26)

- Estimate of \( P_{i,2} \). If \( 1 < p_i < 2 \), using the weighted Young’s inequality and (2.23) we get
\[
P_{i,2} \leq C_4 \int_{\Omega^{(2)}} |\nabla w_{i,\lambda}^+|^{p_i-1} |\nabla \psi_\varepsilon|^{p_i-1} w_{i,\lambda}^+ \, dx
\]
\[
\leq C_\delta \int_{\Omega^{(2)}} |\nabla w_{i,\lambda}^+|^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx
\]
\[
\leq C_\delta \int_{\Omega^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} w_{i,\lambda}^+ \, dx + \frac{C}{\delta} \int_{\Omega^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx
\]
\[
\leq C_\delta \int_{\Omega^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx
\]
\[
\leq C_\delta \int_{\Omega^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx.
\]
(2.27)
If \( p_i \geq 2 \), by the weighted Young’s inequality and (2.11) we deduce that

\[
P_{i,2} \leq C_i \int_{\Omega_{\lambda}^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+| |\nabla \psi_\varepsilon|^{p_i-1} w_{i,\lambda}^+ \, dx
\]

\[
\leq C \delta \int_{\Omega_{\lambda}^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{\frac{p_i(p_i-2)}{p_i-1}} |\nabla w_{i,\lambda}^+|^{\frac{p_i}{p_i-1}} \psi_\varepsilon^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega_{\lambda}^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx
\]

\[
= C \delta \int_{\Omega_{\lambda}^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{\frac{p_i(p_i-2)}{p_i-1}} |\nabla w_{i,\lambda}^+|^2 |\nabla w_{i,\lambda}^+|^{\frac{p_i}{p_i-1}} \psi_\varepsilon^{p_i} \, dx
\]

\[
+ \frac{C}{\delta} \int_{\Omega_{\lambda}^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx.
\]

Using (2.23) and noticing that

\[
\frac{p_i}{(p_i - 1)} - 2 \leq 0,
\]

we obtain the following estimate

\[
P_{i,2} \leq C \delta \int_{\Omega_{\lambda}^{(2)}} |\nabla u_{i,\lambda}|^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega_{\lambda}^{(2)}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx
\]

(2.28)

In the second line of (2.28) we exploited the fact that, since \( p_i \geq 2 \) then

\[
|\nabla u_{i,\lambda}|^{p_i-2} \leq (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}.
\]

Collecting (2.25), (2.26), (2.27) and (2.28) we deduce that for \( p_i \geq 2 \) it holds

\[
P_{i,1} + P_{i,2} \leq C \delta \int_{\Omega_{\lambda}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx
\]

\[
+ \frac{C}{\delta} \left( \int_{\Omega_{\lambda}} |\nabla u_i|^{p_i} \psi_\varepsilon^{p_i} \, dx \right)^{\frac{p_i-2}{p_i}} \left( \int_{\Omega_{\lambda}} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+)^{p_i} \, dx \right)^{\frac{2}{p_i}}
\]

\[
+ \frac{C}{\delta} \int_{\Omega_{\lambda}} |\nabla \psi_\varepsilon|^{p_i} \, dx \quad \text{(for every } \delta > 0),
\]

where \( C > 0 \) is a suitable constant depending on \( p_i, \lambda, \Omega \) and \( M \).

In the same way, if \( 1 < p_i < 2 \), we deduce that

\[
P_{i,1} + P_{i,2} \leq C \delta \int_{\Omega_{\lambda}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx
\]

\[
+ C \left( \int_{\Omega_{\lambda}} |\nabla u_i|^{p_i} \, dx \right)^{\frac{p_i-1}{p_i}} \left( \int_{\Omega_{\lambda}} |\nabla \psi_\varepsilon|^{p_i} \, dx \right)^{\frac{1}{p_i}}
\]

\[
+ \frac{C}{\delta} \int_{\Omega_{\lambda}} |\nabla \psi_\varepsilon|^{p_i} \, dx \quad \text{(for every } \delta > 0),
\]
where $C > 0$ is a suitable constant depending on $p_i, \lambda, \Omega$ and $M$.

In both cases, taking into account (2.13) and the properties of $\psi_\varepsilon$, we derive the estimate

$$P_{i,1} + P_{i,2} \leq C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla u_{i,\lambda}|^2\psi_{\varepsilon}^{p_i} \, dx + C\delta \varepsilon^{1/p_i},$$

holding true for every choice of $\delta > 0$.

- Estimate of $I_{i,1}$. Since the integral $I_{i,1}$ is actually performed on the set

$$A_{i,\lambda} = \{u_i \geq u_{i,\lambda}\} \setminus \Gamma_\lambda,$$

from (2.16), assumption $(h_a)$ and the definition of $\Omega^{(1)}_\lambda$ we immediately obtain

$$I_{i,1} \leq L \int_{\Omega^{(1)}_\lambda} |\nabla u_{i,\lambda}|^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx \leq 2^{q_i} L \int_{\Omega^{(1)}_\lambda} |\nabla u_i|^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx$$

(since $u_i \in C^{1,\alpha}(\overline{\Omega_\lambda})$ and $0 \leq \psi_\varepsilon \leq 1$)

$$\leq C \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2 \, dx. \tag{2.30}$$

- Estimate of $I_{i,2}$. First of all, since also the integral $I_{i,2}$ is actually performed on $A_{i,\lambda}$, we can use again estimate (2.16) and assumption $(h_a)$, obtaining

$$I_{i,2} \leq L \int_{\Omega^{(2)}_\lambda} |\nabla u_{i,\lambda}|^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx \leq L \int_{\Omega^{(2)}_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i} |w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx =: (\ast).$$

From this, since $q_i < p_i$, using (2.23) and by the Young’s inequality we obtain

$$\ast \leq L \int_{\Omega^{(2)}_\lambda} [(|\nabla u_i| + |\nabla u_{i,\lambda}|)^{q_i} \psi_\varepsilon^{q_i}] \cdot [w_{i,\lambda}^+|^{2\psi_\varepsilon^{p_i-q_i}} \psi_\varepsilon] \, dx$$

$$\leq C\delta \int_{\Omega^{(2)}_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_\varepsilon^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega^{(2)}_\lambda} |w_{i,\lambda}^+|^{2\psi_\varepsilon^{p_i-q_i}} \psi_\varepsilon^{p_i} \, dx$$

(since $0 \leq \psi_\varepsilon \leq 1$ and $p_i/(p_i - q_i) > 1$)

$$\leq C\delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2}|\nabla w_{i,\lambda}^+|^2 \psi_\varepsilon^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega_\lambda} |w_{i,\lambda}^+|^2 \psi_\varepsilon \, dx \quad \text{(for every } \delta > 0).$$
- **Estimate of** $J_{i,1}$. We first observe that, since $0 \leq u_i \leq M$ pointwise in $\overline{\Omega}_\lambda$, by exploiting assumption $(h_a)$ and the Mean Value theorem, we obtain the following estimate:

\[
J_{i,1} \leq C \int_{\Omega^{(1)}} \left| \nabla u_i \right|^q \leq \left| \nabla u_{i,\lambda} \right|^{q_i} w_{i,\lambda}^+ \psi_{\varepsilon}^p dx
\]

\[
\leq C \int_{\Omega^{(1)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{q_i-1} \left| \nabla w_{i,\lambda}^+ \right| w_{i,\lambda}^+ \psi_{\varepsilon}^p dx
\]

\[
\leq C \int_{\Omega^{(1)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{2q_i-p_i} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{\frac{p_i-2}{2}} \left| \nabla w_{i,\lambda}^+ \right| w_{i,\lambda}^+ \psi_{\varepsilon}^p dx
\]

\[
\leq C \delta \int_{\Omega^{(1)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{2q_i-p_i} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{p_i-2} \left| \nabla w_{i,\lambda}^+ \right|^2 \psi_{\varepsilon}^p dx + \frac{C}{\delta} \int_{\Omega^{(1)}} \left| w_{i,\lambda}^+ \right|^2 dx.
\]

Finally, using (2.22), since $q_i \geq \max\{p_i - 1, 1\}$ and therefore $q_i \geq p_i/2$, we conclude that

\[
(2.32) \quad J_{i,1} \leq C \delta \int_{\Omega^{(1)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{p_i-2} \left| \nabla w_{i,\lambda}^+ \right|^2 \psi_{\varepsilon}^p dx + \frac{C}{\delta} \int_{\Omega^{(1)}} \left| w_{i,\lambda}^+ \right|^2 dx,
\]

and this estimate holds for every choice of $\delta > 0$.

- **Estimate of** $J_{i,2}$. Using once again the fact that $0 \leq u_i \leq M$ in $\overline{\Omega}_\lambda$, and taking into account assumption $(h_a)$, we get

\[
J_{i,2} \leq C \int_{\Omega^{(2)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{q_i} w_{i,\lambda}^+ \psi_{\varepsilon}^p dx =: (\bullet).
\]

By Young’s inequality and estimate (2.23) we obtain

\[
(\bullet) = C \int_{\Omega^{(2)}} \left[ \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{q_i} \psi_{\varepsilon}^q \right] \left[ w_{i,\lambda}^+ \psi_{\varepsilon}^{p_i-q_i} \right] \psi_{\varepsilon}^p dx
\]

\[
\leq C \delta \int_{\Omega^{(2)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{p_i-2} \left| \nabla w_{i,\lambda}^+ \right|^2 \psi_{\varepsilon}^p dx + \frac{C}{\delta} \int_{\Omega^{(2)}} \left| w_{i,\lambda}^+ \right|^2 dx
\]

\[
\leq C \delta \int_{\Omega^{(2)}} \left( \left| \nabla u_i \right| + \left| \nabla u_{i,\lambda} \right| \right)^{p_i-2} \left| \nabla w_{i,\lambda}^+ \right|^2 \psi_{\varepsilon}^p dx + \frac{C}{\delta} \int_{\Omega^{(2)}} \left| w_{i,\lambda}^+ \right|^2 dx,
\]

where we used the fact $p_i - (p_i - q_i) \geq 2$ (because $q_i \geq p_i/2$) and $0 \leq u_i \leq M$ in $\overline{\Omega}_\lambda$; this estimate holds for every $\delta > 0$.

- **Estimate of** $F_i$. Since $0 \leq \psi_{\varepsilon} \leq 1$, by Young’s inequality we immediately get

\[
(2.34) \quad F_i \leq \sum_{j=1}^{m} L_j \left( \int_{\Omega_{\lambda}} \left| w_{i,\lambda}^+ \right|^2 dx + \int_{\Omega_{\lambda}} \left| w_{j,\lambda}^+ \right|^2 dx \right) \leq C \sum_{j=1}^{m} \int_{\Omega_{\lambda}} \left| w_{j,\lambda}^+ \right|^2 dx.
\]
Thanks to all the above estimates, we can finally complete the proof of (2.12): in fact, by gathering together (2.29)-to-(2.34), from (2.21) we infer that

\begin{equation}
C_1 \int_{\Omega} \left( |\nabla u_i| + |\nabla u_{i,\lambda}| \right)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi \, dx \leq C \delta \int_{\Omega} \left( |\nabla u_i| + |\nabla u_{i,\lambda}| \right)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \psi \, dx + C \delta^{1/p_i} + \frac{C}{\delta} \int_{\Omega} |w_{i,\lambda}^+|^2 \, dx + C \sum_{j=1}^{m} \int_{\Omega} |w_{j,\lambda}^+|^2 \, dx,
\end{equation}

and this estimates holds for every \( \delta > 0 \). As a consequence, if we choose \( \delta \) sufficiently small and if we let \( \varepsilon \to 0^+ \) with the aid of Fatou’s lemma, we obtain

\begin{equation}
\int_{\Omega} \left( |\nabla u_i| + |\nabla u_{i,\lambda}| \right)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq C \sum_{j=1}^{m} \int_{\Omega} |w_{j,\lambda}^+|^2 \, dx,
\end{equation}

where \( C > 0 \) is a suitable constant depending on \( p_i, q_i, a_i, f_i, \lambda, \Omega, M \). This, together with the estimate in (2.13), immediately implies the desired (2.12).

Case (ii). Let us define the function

\[ A_i(t) = \int_{0}^{t} a_i(s) \, ds. \]

Using the family of cut-off function \( \{ \psi_{\varepsilon} \}_{\varepsilon \in \varepsilon_0} \) defined in the previous case, we now define in a very similar way, for every \( \varepsilon \in (0, \varepsilon_0) \), the maps

\[ \varphi_{i,\varepsilon}^{(1)}(x) := \begin{cases} e^{-A_i(u_i(x))} w_{i,\lambda}^+(x) \psi_{\varepsilon}(x), & \text{if } x \in \Omega_{\lambda}, \\ 0, & \text{otherwise}, \end{cases} \]

and

\[ \varphi_{i,\varepsilon}^{(2)}(x) := \begin{cases} e^{-A_i(u_{i,\lambda}(x))} w_{i,\lambda}^+(x) \psi_{\varepsilon}(x), & \text{if } x \in \Omega_{\lambda}, \\ 0, & \text{otherwise}. \end{cases} \]

It is possible to prove that, see e.g. [15],

(i) \( \varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)} \in \text{Lip}(\mathbb{R}^N) \);

(ii) \( \text{supp}(\varphi_{i,\varepsilon}^{(1)}) \subseteq \Omega_{\lambda}, \text{supp}(\varphi_{i,\varepsilon}^{(2)}) \subseteq \Omega_{\lambda} \) and \( \varphi_{i,\varepsilon}^{(1)} \equiv \varphi_{i,\varepsilon}^{(2)} \equiv 0 \) near \( \Gamma_{\lambda} \).

Hence, taking into account properties (i)-(ii) of \( \varphi_{i,\varepsilon}^{(1)} \) and \( \varphi_{i,\varepsilon}^{(2)} \), a standard density argument allows us to use \( \varphi_{i,\varepsilon}^{(1)} \) and \( \varphi_{i,\varepsilon}^{(2)} \) as a test functions respectively in (1.2) and (2.7). We then
substitute the latter to the former getting

\[
\int_{Q_t} \left( |\nabla u_i|^{p_i-2} \nabla u_i, \nabla \phi_{i,\varepsilon}^{(1)} \right) dx - \int_{Q_t} \left( |\nabla u_{i,\lambda}|^{p_i-2} \nabla u_{i,\lambda}, \nabla \phi_{i,\varepsilon}^{(2)} \right) dx \\
+ \int_{Q_t} a_i(u_i)|\nabla u_i|^{p_i-2} \nabla u_i \cdot \nabla \phi_{i,\varepsilon}^{(1)} dx - \int_{Q_t} a_i(u_{i,\lambda}) |\nabla u_{i,\lambda}|^{p_i} \phi_{i,\varepsilon}^{(2)} dx \\
= \int_{Q_t} f_i(u) \phi_{i,\varepsilon}^{(1)} dx - \int_{Q_t} f_i(u) \phi_{i,\varepsilon}^{(2)} dx.
\]

Now, we use the explicit expression of both \( \phi_{i,\varepsilon}^{(1)} \) and \( \phi_{i,\varepsilon}^{(2)} \) to get

\[
- \int_{Q_t} a_i(u_i) e^{-A_i(u_i)} |\nabla u_i|^{p_i} \psi_i^{\varepsilon} \psi_i dx + \int_{Q_t} e^{-A_i(u_i)} |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon} dx \\
+ p_i \int_{Q_t} e^{-A_i(u_i)} |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon-1} dx \\
+ \int_{Q_t} a_i(u_i,\lambda) e^{-A_i(u_i,\lambda)} |\nabla u_{i,\lambda}|^{p_i} \psi_i^{\varepsilon} \psi_i dx - \int_{Q_t} e^{-A_i(u_i,\lambda)} |\nabla u_{i,\lambda}|^{p_i-2} \langle \nabla u_{i,\lambda}, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon} dx \\
- p_i \int_{Q_t} e^{-A_i(u_i,\lambda)} |\nabla u_{i,\lambda}|^{p_i-2} \langle \nabla u_{i,\lambda}, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon-1} dx \\
+ \int_{Q_t} f_i(u) e^{-A_i(u_i)} \psi_i^{\varepsilon} \psi_i dx - \int_{Q_t} f_i(u) e^{-A_i(u_i,\lambda)} \psi_i^{\varepsilon-1} dx.
\]

After a simplification, we add on both sides the term

\[
\int_{Q_t} e^{-A_i(u_i,\lambda)} |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon} dx,
\]

and, in the left hand side, we add and subtract the term

\[
p_i \int_{Q_t} e^{-A_i(u_i,\lambda)} |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon-1} dx.
\]

Rearranging the terms, we find

\[
I_{\lambda} = \int_{Q_t} \left( e^{-A_i(u_i,\lambda)} - e^{-A_i(u_i)} \right) |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon} dx \\
+ p_i \int_{Q_t} \left( e^{-A_i(u_i,\lambda)} - e^{-A_i(u_i)} \right) |\nabla u_i|^{p_i-2} \langle \nabla u_i, \nabla \psi_i^{\varepsilon} \rangle \psi_i^{\varepsilon-1} dx \\
+ p_i \int_{Q_t} e^{-A_i(u_i,\lambda)} |\nabla u_i,\lambda|^{p_i-2} \nabla u_i, \nabla \psi_i \rangle \psi_i^{\varepsilon-1} dx \\
+ \int_{Q_t} \left( e^{-A_i(u_i)} f_i(u) - e^{-A_i(u_i,\lambda)} f_i(u) \right) \psi_i^{\varepsilon} dx.
\]
Arguing similarly to Case (i), see (2.20), the left hand side can be estimated from below with

\begin{equation}
C \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|_2^2 \psi_\varepsilon^{p_i} \, dx.
\end{equation}

Indeed, in the set \( \Omega_\lambda \cap \text{supp}(w_{i,\lambda}^+) \), one has \( e^{-A_i(u_{i,\lambda})} \geq C > 0 \).

We now focus on the right hand side which we firstly bound from above passing to the absolute values, getting

\[
\int_{\Omega_\lambda} |e^{-A_i(u_{i,\lambda})} - e^{-A_i(u_i)}| |\nabla u_i|^{p_i-1} |\nabla w_{i,\lambda}^+|_2^2 |\psi_\varepsilon|^{p_i} \, dx
\]
\[
+ p_i \int_{\Omega_\lambda} |e^{-A_i(u_{i,\lambda})} - e^{-A_i(u_i)}| |\nabla u_i|^{p_i-1} |\nabla \psi_\varepsilon|^{p_i} \, dx
\]
\[
+ p_i \int_{\Omega_\lambda} e^{-A_i(u_{i,\lambda})} \left( |\nabla u_{i,\lambda}|^{p_i-2} |\nabla u_{i,\lambda}| - |\nabla u_i|^{p_i-2} |\nabla u_i| \right) |\nabla \psi_\varepsilon|^{p_i} \, dx
\]
\[
+ \int_{\Omega_\lambda} \left( e^{-A_i(u_i)} f_i(u) - e^{-A_i(u_{i,\lambda})} f_i(u_{i,\lambda}) \right) w_{i,\lambda}^+ \psi_\varepsilon \, dx =: J_{i,1} + J_{i,2} + J_{i,3} + J_{i,4}.
\]

For the reader’s convenience, we recall that, in this case, we are assuming \( 2 \leq p_i \leq N \). We have the following

- **Estimate of \( J_{i,1} \).** Using the lipschitzianity of \( t \mapsto e^{-A_i(t)} \), we get that there exists a positive constant \( C > 0 \) such that

\[
J_{i,1} \leq C \int_{\Omega_\lambda} |\nabla u_i|^{p_i-1} |\nabla w_{i,\lambda}^+|_2^2 |\psi_\varepsilon|^{p_i} \, dx.
\]

Now, by exploiting the weighted Young’s inequality, we obtain that for every \( \delta > 0 \) it holds the following

\begin{equation}
J_{i,1} \leq C \delta \int_{\Omega_\lambda} |\nabla u_i|^{2p_i-2} |\nabla w_{i,\lambda}^+|_2^2 |\psi_\varepsilon|^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega_\lambda} (w_{i,\lambda}^+) \, dx
\end{equation}

\[
\leq C \delta \int_{\Omega_\lambda} (|\nabla u_i|^{p_i} + |\nabla u_{i,\lambda}|^{p_i})^{p_i-2} |\nabla w_{i,\lambda}^+|_2^2 |\psi_\varepsilon|^{p_i} \, dx + \frac{C}{\delta} \int_{\Omega_\lambda} (w_{i,\lambda}^+) \, dx
\]

where in the last step \( C > 0 \) depends also on \( \|\nabla u\|_{L^\infty(\Omega_\lambda)} \).

- **Estimate of \( J_{i,2} \).** Using once again the lipschitzianity of \( t \mapsto e^{-A_i(t)} \) together with the boundedness of \( \nabla u \) in \( \Omega_\lambda \), and then exploiting the Hölder’s inequality with exponents \( (p_i, p_i/(p_i - 1)) \), we get

\begin{equation}
J_{i,2} \leq C \left( \int_{\Omega_\lambda} |\nabla \psi_\varepsilon|^{p_i} (w_{i,\lambda}^+) \, dx \right)^{1/p_i} \left( \int_{\Omega_\lambda} \psi_\varepsilon^{p_i} \, dx \right)^{p_i/(p_i - 1)}.
\end{equation}

- **Estimate of \( J_{i,3} \).** We notice first that \( e^{-A_i(u_{i,\lambda})} \leq 1 \). Arguing similarly as we did in the computations that led to (2.25) and subsequently to (2.29), we deduce:

\[
J_{i,3} \leq C \delta \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|_2^2 |\psi_\varepsilon|^{p_i} \, dx + C_{\delta} \varepsilon^{1/p_i} \quad \text{(for every} \quad \delta > 0).\]
- Estimate of $\mathcal{J}_{i,4}$. We first consider the function

$$g_i(t) := e^{-A(t_i)} f_i(t), \quad t \in \mathcal{I}^m,$$

which is still a $C^1$ function satisfying the cooperativity condition $\partial_{u_k} g_i \geq 0$ on $\mathcal{I}^m$ for every $k \neq i$. Hence, we can repeat the computations made in (2.17) up to (2.19) to get

$$(2.41) \quad \mathcal{J}_{i,4} \leq \sum_{j=1}^{m} L_g \int_{\Omega_\lambda} w_{i,\lambda}^+ w_{i,\lambda}^+ \psi_i^p dx \leq C \sum_{j=1}^{m} \int_{\Omega_\lambda} |w_{i,j,\lambda}^+|^2 dx.$$ 

Putting everything together, we can conclude as in the previous Case (i). Hence, we obtain the desired (2.12) for every $i = 1, \ldots, m$. \hfill \square

We conclude this section by proving the following useful lemma.

**Lemma 2.4.** Let assumptions (h$_\Omega$)-to-(h$_f$) be in force. Let $i \in \{1, \ldots, m\}$ be fixed, $1 < p_i < 2$, and let $\lambda \in (\varrho, 0)$. Then, there exists a constant $c = c_i > 0$ such that

$$(2.42) \quad \int_{\Omega_\lambda} |\nabla w_{i,\lambda}^+|^{p_i} \psi_i^{p_i} \, dx \leq c \left( \int_{\Omega_\lambda} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^{2} dx \right)^{\frac{p_i}{2}}.$$

**Proof.** Let us define the set $\Omega_\lambda^+ := \Omega_\lambda \cap \text{supp}(w_{i,\lambda}^+)$ and let $\psi_\varepsilon$ be the cut-off function defined in (2.3). Using Hölder’s inequality with conjugate exponents $2/(2 - p_i), 2/p_i$, we obtain

$$(2.43) \quad \int_{\Omega_\lambda^+} |\nabla w_{i,\lambda}^+|^{p_i} \psi_i^{p_i} \, dx \leq \int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_i^{p_i} \, dx \leq \left( \int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_i^{p_i} \, dx \right)^{\frac{p_i}{2}} \left( \int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{2} \psi_i^{p_i} \, dx \right)^{\frac{2-p_i}{2}}.$$

Now we are going to give an estimate to the first term in the right hand side of (2.43), i.e. the term

$$\mathcal{I} := \int_{\Omega_\lambda^+} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_i^{p_i} \, dx.$$ 

In the same spirit of the previous lemma, we split the set $\Omega_\lambda$ as $\Omega_\lambda = \Omega_\lambda^{(1)} \cup \Omega_\lambda^{(2)}$, where

$$\Omega_\lambda^{(1)} = \{ x \in \Omega_\lambda^+ \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| \leq 2|\nabla u_i| \} \quad \text{and} \quad \Omega_\lambda^{(2)} = \{ x \in \Omega_\lambda^+ \setminus \Gamma_\lambda : |\nabla u_{i,\lambda}(x)| \geq 2|\nabla u_i| \}.$$ 

Hence

$$\mathcal{I} = \int_{\Omega_\lambda^{(1)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_i^{p_i} \, dx + \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i} \psi_i^{p_i} \, dx =: \mathcal{I}_1 + \mathcal{I}_2.$$
- Estimate of $I_1$. Since $\psi_\varepsilon^{p_i} \leq 1$, using (2.22) we immediately get

\begin{equation}
I_1 \leq C \int_{\Omega_\lambda^{(1)}} |\nabla u_i|^{p_i} \, dx, \tag{2.44}
\end{equation}

- Estimate of $I_2$. Using (2.23), we obtain

\begin{align}
I_2 &= \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla u_{i,\lambda}|^{2} \psi_\varepsilon^{p_i} \, dx \\
&\leq C \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^{+}|^{2} \psi_\varepsilon^{p_i} \, dx \tag{2.45}
\end{align}

Collecting (2.44) and (2.45) in order to estimate $I$, we have

\[ I \leq C \left( \int_{\Omega_\lambda^{(1)}} |\nabla u_i|^{p_i} \, dx + \int_{\Omega_\lambda^{(2)}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^{+}|^{2} \psi_\varepsilon^{p_i} \, dx \right). \]

Now, since each $\nabla u_i \in L^\infty(\Omega_\lambda)$ for every $\lambda < 0$ and the second term is bounded by Lemma 2.3, we deduce that

\[ I \leq C, \]

where $C$ is a positive constant that does not depends on $\varepsilon$. Since $\psi_\varepsilon^{p_i} \leq 1$, the inequality (2.43) becomes

\begin{equation}
\int_{\Omega_\lambda^{+}} |\nabla w_{i,\lambda}^{+}|^{p_i} \psi_\varepsilon^{p_i} \, dx \leq C \left( \int_{\Omega_\lambda^{+}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^{+}|^{2} \, dx \right)^{\frac{p_i}{2}} \tag{2.46}
\end{equation}

Finally by Fatou’s lemma from (2.46), we get (2.42). \hfill \Box

3. Proof of Theorem 1.2

Thanks to all the results established so far, we are ready to give the

Proof of Theorem 1.2. Our approach relies on a suitable adaptation of the integral version of the moving plane method. We consider the set

\[ \Lambda := \{ \eta \in (\varrho,0) : u_i \leq u_{i,\lambda} \text{ on } \Omega_\lambda \setminus R_\lambda(\Gamma) \text{ for all } \lambda \in (\varrho,\eta] \text{ and for all } i = 1, \ldots, m \}, \]

and we claim that the following facts hold:

(a) $\Lambda \neq \emptyset$;

(b) setting $\lambda_0 := \sup(\Lambda)$, one has $\lambda_0 = 0$.

Proof of (a). First of all we observe that, since $\Gamma$ is compact and contained in $\Omega \cap \Pi_0$, it is possible to find a small $\tau_0 > 0$ such that $\varrho + \tau_0 < 0$ and

\[ R_\lambda(\Gamma) \cap \Omega_\lambda = \emptyset \quad \text{for every } \lambda \in I_0 := (\varrho, \varrho + \tau_0). \]
In particular, for every \( \lambda \in I_0 \) we have \( u, u_\lambda \in C^{1,\alpha}(\overline{\Omega}_\lambda) \). On the other hand, since both \( u_i \) and \( u_{i,\lambda} \) are non-negative (for every \( i = 1, \ldots, m \)), it is immediate to recognize that

\[ u_i \leq u_{i,\lambda} \text{ on } \partial\Omega_\lambda \quad \text{for every } i = 1, \ldots, m. \]

Now, it is possible to start the moving plane procedure using [15, Proposition 2.5]. For the reader’s convenience we state such a result adapted to our context. The proof is exactly the same and therefore we skip it.

**Proposition 3.1.** Let assumptions \((h_a)\) and \((h_f)\) be in force, and suppose that \( p_i > 1 \) and \( q_i \geq \max\{p_i - 1, 1\} \) for every \( i = 1, \ldots, m \).

In addition, suppose that \( u, u_\lambda \in C^{1,\alpha}(\overline{\Omega}_{\varphi+\tau_0}) \). Then, there exists a number \( \delta > 0 \), depending on \( m, p_i, q_i, a_i, f_i, \|u_i\|_{L^\infty(\Omega_{\varphi+\tau_0})}, \|\nabla u_i\|_{L^\infty(\Omega_{\varphi+\tau_0})} \) and \( \|\nabla \tilde{u}_i\|_{L^\infty(\Omega_{\varphi+\tau_0})} \), with the following property: if \( \Omega_\lambda \subseteq \Omega_{\varphi+\tau_0} \) is such that \( |\Omega_\lambda| \leq \delta \), and if

\[ u_i \leq u_{i,\lambda} \text{ on } \partial\Omega_\lambda \quad \text{for every } i = 1, \ldots, m, \]

then \( u_i \leq u_{i,\lambda} \) in \( \Omega_\lambda \) for every \( i = 1, \ldots, m \).

Therefore, by possibly shrinking \( \tau_0 \) in such a way that \( |\Omega_\lambda| \leq \delta \) for every \( \lambda \in I_0 \), we derive that \( u_i \leq u_{i,\lambda} \) in \( \Omega_\lambda \) for every \( i = 1, \ldots, m \). Hence, \( \eta := \varphi + \tau_0 \in \Lambda \), and thus

\[ \Lambda \neq \emptyset. \]

**Proof of (b).** On account of (a), \( \lambda_0 \) is well-defined and \( \lambda_0 \leq 0 \). Arguing by contradiction, we then suppose that \( \lambda_0 < 0 \), and we prove that there exists some \( \tau_0 > 0 \) such that

\[ u_i \leq u_{i,\lambda_0} \text{ on } \Omega_\lambda \setminus R_\lambda(\Gamma) \quad \text{for all } i = 1, \ldots, m \text{ and } \lambda \in (\lambda_0, \lambda_0 + \tau_0]. \tag{3.1} \]

Since (3.1) is clearly in contrast with the very definition of \( \lambda_0 \), we can conclude that

\[ \lambda_0 = 0. \]

In order to establish the needed (3.1), we proceed as follows: first of all, since both \( u \) and \( u_\lambda \) are continuous on \( \Omega_\lambda \setminus R_\lambda(\Gamma) \), we observe that

\[ u_i \leq u_{i,\lambda_0} \text{ on } \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) \quad \text{for every } i = 1, \ldots, m. \tag{3.2} \]

We then claim that, as a consequence of (3.2), one actually has

\[ u_i < u_{i,\lambda_0} \text{ on } \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma) \quad \text{for every } i = 1, \ldots, m. \tag{3.3} \]

To prove (3.3), we arbitrarily fix \( i \in \{1, \ldots, m\} \).

We first point out that, since \( \Omega_{\lambda_0} = \Omega \cap \{x_1 < \lambda_0\} \) is connected and the compact set \( \Gamma_{\lambda_0} := R_{\lambda_0}(\Gamma) \) has vanishing \( p_i \)-capacity (as the same is true of \( \Gamma \), see assumption \((h_f)\)), it is not difficult to check that \( \Omega_{\lambda_0} \setminus \Gamma_{\lambda_0} \) is connected (see, e.g., [3, Lemma 2.4]); thus, owing to assumptions \((h_f)\), we deduce that it holds the following distributional inequality

\[ -\Delta_{p_i} u_i + a_i(u_i) |\nabla u_i|^{q_i} + \Lambda_i u_i \leq -\Delta_{p_i} u_{i,\lambda_0} + a_i(u_{i,\lambda_0}) |\nabla u_{i,\lambda_0}|^{q_i} + \Lambda_i u_{i,\lambda_0} \quad \text{in } \Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}, \]
where $\Lambda_i$ is a positive constant. As a consequence, since $u_i \leq u_{i,\lambda_0}$ in $\Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}$, we can apply the Strong Comparison Principle (see, e.g., [20, Theorem 1.2]), ensuring that

\begin{equation}
(3.4) \quad \text{either } u_i \equiv u_{i,\lambda_0} \text{ or } u_i < u_{i,\lambda_0} \text{ in } \Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}.
\end{equation}

On the other hand, since $u$ solves $(S)$, we have $u_i - u_{i,\lambda_0} = -u_{i,\lambda_0} < 0$ on $\partial \Omega_{\lambda_0} \cap \partial \Omega$. This immediately gives (3.3), since the alternative $u_i \equiv u_{i,\lambda_0}$ in $\Omega_{\lambda_0} \setminus \Gamma_{\lambda_0}$ obviously cannot be achieved because of the Dirichlet boundary condition and the fact that $u_i > 0$ in $\Omega \setminus \Gamma$.

Now we have fully established (3.3), we can continue with the proof of (3.1). To begin with, we arbitrarily fix a compact set $K \subseteq \Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$ and we observe that, since $R_{\lambda_0}(\Gamma)$ is compact, we can find $\tau_0 = \tau_0(K) > 0$ so small that $K \subseteq \Omega_{\lambda} \setminus R_{\lambda}(\Gamma)$ for every $\lambda_0 \leq \lambda \leq \lambda_0 + \tau_0$.

Moreover, since $u$ is continuous on $K$, a simple uniform-continuity argument based on (3.3) shows that, by possibly shrinking $\tau_0$, we also have (for all $i = 1, \ldots, m$)

\begin{equation}
(3.5) \quad u_i < u_{i,\lambda} \text{ on } K \quad \text{for every } \lambda_0 \leq \lambda \leq \lambda_0 + \tau_0.
\end{equation}

We now turn to prove that, for every $\lambda_0 \leq \lambda \leq \lambda_0 + \tau_0$ and every $i = 1, \ldots, m$, one can find a constant $C_i > 0$, depending of $p_i, q_i, a_i, f_i, \lambda, \Omega, M$ (with $M = M_u := \max_{1 \leq j \leq m} (\|u_j\|_{L^\infty(\Omega_{\rho+\tau_0})} + \|\nabla u_j\|_{L^\infty(\Omega_{\rho+\tau_0})}) < +\infty$), such that

\begin{equation}
(3.6) \quad \int_{\Omega_{\lambda} \setminus K} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq C \sum_{j=1}^m \int_{\Omega_{\lambda} \setminus K} |w_{j,\lambda}^+|^2 \, dx.
\end{equation}

Taking (3.6) for granted for a moment, let us show how this integral estimate can be used in order to prove (3.1). First of all, by taking the sum in (3.6) for $i = 1, \ldots, m$, we get

\begin{equation}
(3.7) \quad \sum_{i=1}^m \int_{\Omega_{\lambda} \setminus K} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla w_{i,\lambda}^+|^2 \, dx \leq C' \sum_{i=1}^m \int_{\Omega_{\lambda} \setminus K} |w_{i,\lambda}^+|^2 \, dx.
\end{equation}

Now, we have to distinguish the singular case from the degenerate one. To this end, let us suppose (up to a rearrangement of the sum in both sides of the inequality (3.7)) that

\[ \frac{(2N+2)}{(N+2)} < p_i \leq 2 \quad \text{for every } i = 1, \ldots, m', \]

for some $1 \leq m' \leq m$. In this case, we have that $p_i^* > 2$. Applying the Hölder inequality with conjugate exponents $((p_i^* - 2)/p_i^*, p_i^*/2)$ and the Sobolev inequality to the first $m'$
terms of the right hand side of (3.7), we deduce that

\begin{equation}
\sum_{i=1}^{m'} \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}^+| \right)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \, dx + \sum_{i=m'+1}^{m} \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}^+| \right)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \, dx
\end{equation}

\begin{equation}
\leq C \sum_{i=1}^{m'} \int_{\Omega \setminus \mathcal{K}} |w_{i,\lambda}^+|^2 \, dx + C \sum_{i=m'+1}^{m} \int_{\Omega \setminus \mathcal{K}} |w_{i,\lambda}^+|^2 \, dx
\end{equation}

\begin{equation}
\leq C \sum_{i=1}^{m'} \left| \Omega \setminus \mathcal{K} \right| \frac{p_i^* - 2}{p_i} \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}^+| \right)^{\frac{p_i - 2}{p_i}} |\nabla w_{i,\lambda}^+|^2 \, dx
\end{equation}

\begin{equation}
+ C \sum_{i=m'+1}^{m} C_p(\Omega \setminus \mathcal{K}) \int_{\Omega \setminus \mathcal{K}} |\nabla u_i|^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \, dx
\end{equation}

\begin{equation}
\leq C \sum_{i=1}^{m'} \left| \Omega \setminus \mathcal{K} \right| \frac{p_i^* - 2}{p_i} \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}^+| \right)^{\frac{p_i - 2}{p_i}} |\nabla w_{i,\lambda}^+|^2 \, dx
\end{equation}

\begin{equation}
+ C \sum_{i=m'+1}^{m} C_p(\Omega \setminus \mathcal{K}) \int_{\Omega \setminus \mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}^+|)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \, dx,
\end{equation}

where $C_p(\Omega \setminus \mathcal{K})$ is the Poincaré constant that tends to zero, when the Lebesgue measure $|\Omega \setminus \mathcal{K}|$ tends to zero. From (3.9), up to redefine constants we have

\begin{equation}
\sum_{i=1}^{m} \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}^+| \right)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \, dx
\end{equation}

\begin{equation}
\leq C(\Omega \setminus \mathcal{K}) \sum_{i=1}^{m} \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}^+| \right)^{p_i - 2} |\nabla w_{i,\lambda}^+|^2 \, dx,
\end{equation}
where $C(\Omega_\lambda \setminus \mathcal{K})$ tends to zero, when the Lebesgue measure $|\Omega_\lambda \setminus \mathcal{K}|$ tends to zero. Now, we choose the compact set $\mathcal{K}$ sufficiently large such that

$$C(\Omega_\lambda \setminus \mathcal{K}) < 1.$$ 

From this fact we immediately deduce that $u_i \leq u_{i,\lambda}$ for each $\lambda_0 < \lambda \leq \lambda_0 + \tau$ and this gives a contradiction with the definition of $\lambda_0$. Therefore $\lambda_0 = 0$ and the thesis (i) is proved. Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. To prove the thesis (ii), we observe that the monotonicity of the solution is in fact implicit in the moving plane method and in particular we get that $\partial_{x_1} u_i \geq 0$ in $\Omega_0$. To get (1.3) it is sufficient to apply the Strong Maximum Principle [15, Theorem 2.3].

Hence, we are left to prove (3.6). To this end, for every fixed $\lambda \in (\lambda_0, \lambda_0 + \tau_0]$ we choose an open neighborhood $\mathcal{O}_\lambda \subseteq \Omega_\lambda \setminus \mathcal{K}$ of $\Gamma_\lambda = R_\lambda(\Gamma)$, and a cut-off family $\{\psi_{\varepsilon}\}_{\varepsilon < \varepsilon_0}$ for $\Gamma_\lambda$ related with $\mathcal{O}_\lambda$. This means, precisely, that

1. $\psi_{\varepsilon} \in \text{Lip}(\mathbb{R}^N)$ and $0 \leq \psi_{\varepsilon} \leq 1$ pointwise in $\mathbb{R}^N$;
2. there exists an open neighborhood $\mathcal{V}_\lambda^\varepsilon \subseteq \mathcal{O}_\lambda$ of $\Gamma_\lambda$ such that $\psi_{\varepsilon} \equiv 0$ on $\mathcal{V}_\lambda^\varepsilon$;
3. $\psi_{\varepsilon}(x) \rightarrow 1$ as $\varepsilon \rightarrow 0^+$ for a.e. $x \in \mathbb{R}^N$;
4. there exists a constant $C_0 > 0$, independent of $\varepsilon$, such that

$$\int_{\mathbb{R}^N} |\nabla \psi_{\varepsilon}|^p_i \, dx \leq C_0 \varepsilon.$$ 

We also fix $i \in \{1, \ldots, m\}$, and we distinguish two cases.

(i) $q_i < p_i$;
(ii) $q_i = p_i$.

Case (i). In this case, for every $\varepsilon \in (0, \varepsilon_0)$ we consider the map

$$\varphi_{i,\varepsilon}(x) := \begin{cases} w^+_{i,\lambda}(x) \psi_{\varepsilon}^{p_i}(x) = (u_i - u_{i,\lambda})^+(x) \psi_{\varepsilon}^{p_i}(x), & \text{if } x \in \Omega_\lambda, \\ 0, & \text{otherwise.} \end{cases}$$ 

As already recognized in Lemma 2.3, $\varphi_{i,\varepsilon}$ satisfies the following properties:

(i) $\varphi_{i,\varepsilon} \in \text{Lip}(\mathbb{R}^N)$;
(ii) $\text{supp}(\varphi_{i,\varepsilon}) \subseteq \Omega_\lambda$ and $\varphi_{i,\varepsilon} \equiv 0$ near $\Gamma_\lambda$.

Moreover, since we know that $u_i < u_{i,\lambda}$ on $\mathcal{K}$, we also have

$$w^+_{i,\lambda} = (u_i - u_{i,\lambda})^+ \equiv 0 \text{ on } \mathcal{K}. \quad (3.11)$$
As a consequence, a standard density argument allows us to use \( \varphi_{i,\varepsilon} \) as a test function both in (1.2) and (2.7), from (3.11) we obtain
\[
\int_{\Omega\setminus\mathcal{K}} \langle |\nabla u_i|^{p_i-2} \nabla u_i - |\nabla u_{i,\lambda}|^{p_i-2} \nabla u_{i,\lambda}, \nabla \varphi_{i,\varepsilon} \rangle \, dx
\]
\[+ \int_{\Omega\setminus\mathcal{K}} (a_i(u_i)|\nabla u_i|^{q_i} - a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i}) \varphi_{i,\varepsilon} \, dx \]
\[= \int_{\Omega\setminus\mathcal{K}} (f_i(u) - f_i(u_\lambda)) \varphi_{i,\varepsilon} \, dx. \]
Starting from this identity, and proceeding exactly as in the proof of Lemma 2.3 - Case (i), we obtain the following estimate, which is the analog of (2.36):
\[
(3.12) \quad \int_{\Omega\setminus\mathcal{K}} (|\nabla u_i| + |\nabla u_{i,\lambda}|)^{p_i-2} |\nabla u_{i+1,\lambda}|^2 \, dx \leq C \sum_{j=1}^{m} \int_{\Omega\setminus\mathcal{K}} |u_{j+1,\lambda}|^2 \, dx
\]
(here, \( C > 0 \) is a suitable constant depending on \( p_i, q_i, a_i, f_i, \lambda, \Omega, M \) (with \( M = M_\lambda := \max_{1 \leq j \leq m} (\|u_j\|_{L^\infty(\Omega_{\rho+\eta})} + \|\nabla u_j\|_{L^\infty(\Omega_{\rho+\eta})}) < +\infty) \)). But, in this case, (3.12) coincides with (3.6).

**Case (ii).** In this case, for every \( \varepsilon \in (0, \varepsilon_0) \) we consider the maps
\[
\varphi_{i,\varepsilon}^{(1)}(x) := \begin{cases} e^{-A_i(u_i(x))} w_{i,\lambda}^+(x) \psi_{\varepsilon}(x), & \text{if } x \in \Omega_\lambda, \\
0, & \text{otherwise}, \end{cases}
\]
\[
\varphi_{i,\varepsilon}^{(2)}(x) := \begin{cases} e^{-A_i(u_{i,\lambda}(x))} w_{i,\lambda}^+(x) \psi_{\varepsilon}(x), & \text{if } x \in \Omega_\lambda, \\
0, & \text{otherwise}, \end{cases}
\]
where \( A_i(t) := \int_0^t a_i(s) \, ds \). Also in the present case, we have already recognized in the proof of Lemma 2.3 that \( \varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)} \) satisfy the following properties:

(i) \( \varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)} \in \text{Lip}(\mathbb{R}^N) \);

(ii) \( \text{supp}(\varphi_{i,\varepsilon}^{(1)}) \subseteq \Omega_\lambda, \text{supp}(\varphi_{i,\varepsilon}^{(2)}) \subseteq \Omega_\lambda \) and \( \varphi_{i,\varepsilon}^{(1)} \equiv \varphi_{i,\varepsilon}^{(2)} \equiv 0 \) near \( \Gamma_\lambda \).

As a consequence, a standard density argument shows that it is possible to use \( \varphi_{i,\varepsilon}^{(1)}, \varphi_{i,\varepsilon}^{(2)} \) as test functions in (1.2) and (2.7), respectively. By subtracting the resulting identities, and by taking into account (3.11), we then obtain
\[
\int_{\Omega\setminus\mathcal{K}} \langle |\nabla u_i|^{p_i-2} \nabla u_i, \nabla \varphi_{i,\varepsilon}^{(1)} \rangle \, dx - \int_{\Omega\setminus\mathcal{K}} \langle |\nabla u_{i,\lambda}|^{p_i-2} \nabla u_{i,\lambda}, \nabla \varphi_{i,\varepsilon}^{(2)} \rangle \, dx
\]
\[+ \int_{\Omega\setminus\mathcal{K}} a_i(u_i)|\nabla u_i|^{q_i} \varphi_{i,\varepsilon}^{(1)} \, dx - \int_{\Omega\setminus\mathcal{K}} a_i(u_{i,\lambda})|\nabla u_{i,\lambda}|^{q_i} \varphi_{i,\varepsilon}^{(2)} \, dx \]
\[= \int_{\Omega\setminus\mathcal{K}} f_i(u) \varphi_{i,\varepsilon}^{(1)} \, dx - \int_{\Omega\setminus\mathcal{K}} f_i(u_\lambda) \varphi_{i,\varepsilon}^{(2)} \, dx. \]
Starting from this identity, and proceeding exactly as in the proof of Lemma 2.3 - Case (ii), we obtain the following estimate, which is again the analogue of (2.36):

\[
(3.13) \quad \int_{\Omega \setminus \mathcal{K}} \left( |\nabla u_i| + |\nabla u_{i,\lambda}| \right)^{p_i-2} |\nabla w_{i,\lambda}|^2 \, dx \leq C \sum_{j=1}^{m} \int_{\Omega \setminus \mathcal{K}} |w_{j,\lambda}|^2 \, dx
\]

(here, \( C > 0 \) is a suitable constant depending on \( p_i, q_i, a_i, f_i, \lambda, \Omega, M \) (with \( M = M_u := \max_{1 \leq j \leq m} \left( \|u_j\|_{L^\infty(\mathcal{P}_{\rho+\tau_0})} + \|\nabla u_j\|_{L^\infty(\mathcal{P}_{\rho+\tau_0})} \right) < +\infty \)). But it (3.13) coincides with (3.6) in this case. \( \square \)

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