Three-body equations of motion in successive post-Newtonian approximations

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Abstract
There are periodic solutions to the equal-mass three-body (and \(N\)-body) problem in Newtonian gravity. The figure-eight solution is one of them. In this paper, we discuss its solution in the first and second post-Newtonian approximations to general relativity. To do so we derive the canonical equations of motion in the ADM gauge from the three-body Hamiltonian. We then integrate those equations numerically, showing that quantities such as the energy, linear and angular momenta are conserved down to numerical error. We also study the scaling of the initial parameters with the physical size of the triple system. In this way we can assess when general relativistic results are important and we determine that this occurs for distances of the order of 100\(M\), with \(M\) the total mass of the system. For distances much closer than those, presumably the system would completely collapse due to gravitational radiation. This sets up a natural cut-off to Newtonian \(N\)-body simulations. The method can also be used to dynamically provide initial parameters for subsequent full nonlinear numerical simulations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The closest star to the solar system, Alpha Centauri, is a triple system, so is Polaris and HD 188753. Triple stars and black holes are common in globular clusters [1, 2], and galactic discs. Triple black hole mergers can be formed in galaxy merger [3] and a triple quasar, representing a triple supermassive black hole system, has been recently discovered [4].

The 2005 breakthroughs in numerical relativity [5–7], not only provided a solution to the long standing two-body problem in general relativity, but it also proved applicable to the
black hole–neutron star binaries [8] and recently to the three (and N)-black-holes systems [9, 10].

In general, the solution of three-body problem in Newtonian gravity can be chaotic. There are however, periodic orbits in the problem of three equal masses on a plane. In this idealized set up, one of the most surprising solution is a figure-eight orbit. The three bodies chase each other forever around a fixed eight-shaped curve. This was found first by Moore [11] and discussed with the proof of the existence in [12]. Heggie [13] also estimates the probability for such systems to occur in a galaxy and finds that these are very unlikely to occur.

Because of effects such as the perihelion shift, it was unclear if the figure-eight orbits would exist in a low post-Newtonian expansion, even if it consists of only conservative terms. Imai et al [14] succeeded in obtaining the figure-eight solution in a first post-Newtonian order approximation by finding the general relativistic corrections to the Newtonian initial conditions. In [15] they also estimated the periodic gravitational waves from this system.

[14] used the Euler–Lagrange equations of motion in an approximation to first post-Newtonian order. In our paper we instead assume the Hamiltonian formulation to derive the equations of motion. We start from the Hamiltonian given in [16] (with typos corrected in our appendix). We derive the equations of motion in this formalism, which are different from those used in [14] and have the virtue of explicitly satisfying the Hamiltonian (and momenta) constraints.

The paper is organized as follows. In section 2, we summarize the equations of motion to be solved numerically in order to obtain the figure-eight orbits. The starting point is the three-body Hamiltonian in the first post-Newtonian approximation (1PN). In section 3, we discuss the initial conditions for the figure-eight solutions. We study the scaling relation between the orbital radius and the linear momenta. From this analysis, we can estimate when general relativistic effects are important. In section 4, we extend our calculation to the second post-Newtonian order and in section 5, we summarize the results of this paper and discuss some remaining problems. The 2PN three-body Hamiltonian is explicitly given in the appendix. Throughout this paper, we use units in which $c = G = 1$.

2. Equations of motion

As we mentioned in the introduction, the Newtonian configuration that leads to orbital braid figures can also be obtained within the Lagrangian approach, in the first post-Newtonian approximation, by finding the appropriate corrections to the initial data [14].

Here we will consider the Hamiltonian formulation since it generates equations of motion that conserve the energy, linear and angular momenta. This is crucial to provide initial data parameters for subsequent full numerical evolutions and to help monitor the accuracy of our numerical integrator. This is crucial to reach high accuracy in the numerical integrations, which is needed to keep good track of the orbital motion, that in the three-body problem might be chaotic.

The Hamiltonian ($H = H_N + H_{1 PN} + H_{2 PN}$) for the three-body problem in the second post-Newtonian approximation is given next. Note that since gravitational radiation only enters at 2.5PN order and higher, the current analysis applies to a conservative system.

The Newtonian Hamiltonian is given by

$$H_N = \frac{1}{2} \sum_a m_a \frac{p_a^2}{m_a} - \frac{1}{2} \sum_{a,b \neq a} \frac{m_a m_b}{r_{ab}},$$

(1)
and to the first post-Newtonian order by

\[ H_{1\text{PN}} = -\frac{1}{8} \sum_a m_a \left( \frac{p_a^2}{m_a^2} \right)^2 - \frac{1}{4} \sum_{a,b \neq a} m_a m_b \left( \frac{6 p_a^2}{m_a^2} - 7 \frac{p_a \cdot p_b}{m_a m_b} - \frac{\vec{n}_{ab} \cdot \vec{p}_a}{m_a m_b} \left( \vec{n}_{ab} \cdot \vec{p}_b \right) \right) \]

(2)

where \( a, b \) and \( c \) run over 1, 2 and 3. We have used the notations; \( x_{ab} = x_a - x_b, r_{ab} = |x_{ab}|, \vec{n}_{ab} = x_{ab}/r_{ab}, p_a^2 = \vec{p}_a \cdot \vec{p}_a \) and the dot (\( \cdot \)) means the scalar product. The Hamiltonian for the second post-Newtonian order is given in appendix A.

We then obtain the canonical equations

\[ (p_i)_t = \frac{\partial H}{\partial (q_a)_i}, \quad (q_a)_t = -\frac{\partial H}{\partial (p_a)_i}, \]

(3)

where \( i \) denotes \( x, y \) or \( z \).

Explicitly, the equations of motion for the first post-Newtonian order are given for the particle 1 by

\[ \frac{\partial}{\partial t} x_1 = \frac{p_1}{m_1} - \frac{1}{2} \left( p_1 \cdot 1 \right) p_1 - \frac{1}{2} m_1 m_2 \frac{6 p_1}{m_1^2} - 7 \frac{p_1 \cdot p_2}{m_1 m_2} - \frac{\vec{n}_{12} \cdot \vec{p}_1}{m_1 m_2} \left( \vec{n}_{12} \cdot \vec{p}_2 \right) \]

(4)

\[ \frac{\partial}{\partial t} p_1 = -\frac{x_{12}}{r_{12}} \left( \frac{m_1 m_2}{r_{12}^3} - \frac{m_2 m_3}{r_{12}^3 r_{23}} + \frac{m_1 m_2}{r_{12}^3 r_{31}} \right) + \frac{1}{2} m_1 m_2 \left( 3 \left( p_1 \cdot p_1 \right) - \frac{3 (p_1 \cdot p_2)}{m_1^2} \right) \]

\[ + \frac{x_{12}}{r_{12}} \left( \frac{m_3 m_1}{r_{31}^2} - \frac{m_2 m_3}{r_{31}^2 r_{23}} + \frac{m_3 m_1}{r_{31}^2 r_{12}} \right) + \frac{1}{2} m_3 m_1 \left( 3 \left( p_1 \cdot p_3 \right) - \frac{3 (p_1 \cdot p_1)}{m_3^2} \right) \]

\[ - \frac{x_{12}}{r_{12}} \left( \frac{m_3 m_1}{r_{31}^2} - \frac{m_2 m_3}{r_{31}^2 r_{23}} + \frac{m_3 m_1}{r_{31}^2 r_{12}} \right) + \frac{1}{2} m_3 m_1 \left( 3 \left( p_1 \cdot p_3 \right) - \frac{3 (p_1 \cdot p_1)}{m_3^2} \right) \]

(5)

where to obtain the equations of motion for the particle 2 (and 3), we change the subscripts as \( \{1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1\} \) (and \( \{1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2\} \)), respectively.

We solved the above equations numerically for three-body problems using a \texttt{dsolve/numeric} command with an absolute error tolerance \( = 10^{-7} \) and a relative error tolerance \( = 10^{-6} \) in \texttt{Maple} 8 with typical runs of a few seconds on a Laptop. Since we use the canonical momentum in the calculation, the Hamiltonian \( H \), the total linear momentum \( \vec{P} = \sum p_a \) and angular momentum \( \vec{L} = \sum x_a \times p_a \) are conserved quantities. These represent useful checks of the accuracy of the numerical runs.

3. The first post-Newtonian corrections

In the Newtonian case, a figure-eight motion can be obtained from the following initial conditions \[14], i.e., the positions \( \vec{l} \) and linear momenta \( \vec{p} \):
Here, we set $m_1 = m_2 = m_3 = m = 1$. For the above initial condition, the total linear momentum and angular momentum are zero (see figure 1).

At the 1PN order, we also impose the total linear momentum $P = 0$ and the total angular momentum $L = 0$. By these conditions, we find that each linear momentum is given by the relations

$$p_3 = -2p_1 = -2p_2.$$  

Therefore, when we give the positions of the three particles, it is then only necessary to search numerically for $p_3$. In order to obtain $p_3$, we make some iterative computations until the figure-eight is reproduced for a few orbits.

In figures 2 and 3 we show the relative error of the Hamiltonian conservation:

$$\Delta H(t) = \frac{H(t) - H(0)}{H(0)}.$$  

Figure 2 is estimated by using the orbit calculated in [14] and we observe that they lead to violations of the order of $3 \times 10^{-3}$. The explanation of the violations is discussed in appendix B. Figure 3 is derived by using the canonical equations derived in our paper and they display errors of the order of $10^{-6}$, growing linearly in time due to the propagation of numerical errors triggered by initial roundoff.

Next we will discuss the scaling behavior of $p_3$ when we change the initial separation as $l \to \lambda l$, and hence the size of the orbit. Note that $p_3 \to \lambda^{-1/2} p_3$ in the Newtonian limit as can be easily derived from the Hamiltonian in (1) or the equations of motion (4).

In table 1, we summarize our numerical findings for the 1PN initial conditions for $\lambda$ from 1 to 100. We note that $p_3$ with $\lambda = 1$ is different from the value which are derived from the initial velocity of [14]. The value $\theta$ in the table is the inclination angle of the principal axes. The principal axes of the 1PN figure-eight motion are not along the $x$ and $y$ axes [14].

In figure 4, the figure-eight rescaled orbits with $\lambda = 1, 10$ and 100 are shown. Here, in order to display the general relativistic effects, we have used the coordinates: $(x_\alpha(t)/\lambda, y_\alpha(t)/\lambda)$. We have chosen here the $x$-axis as the principal axis. We observe that the superposition of the $\lambda = 10$ and $\lambda = 100$ suggests that at those scales the general relativistic
effects are very small while for $\lambda < 1$ they are dominant, but remainder gauge effects may also mask this effect because the orbits are not gauge invariant. A cleaner analysis can be made directly looking at the initial linear momenta scaling.
Figure 4. Figure-eight motions. We show \( \lambda = 1 \) (solid line), \( \lambda = 10 \) (dashed line) and \( \lambda = 100 \) (dotted line). On the right panel a zoom-in of the differences.

Table 1. The initial conditions and inclination angle.

| \( \lambda \) | \( (p_3)_x \) | \( (p_3)_y \) | \( \theta \) (radian) |
|-------|---------|---------|-----------|
| 1.00  | -0.09811067089 | -0.09490870640 | 0.01535863098 |
| 2.00  | -0.06754964265 | -0.06392246619 | 0.007238984240 |
| 5.00  | -0.04209168100 | -0.03934705365 | 0.002786451510 |
| 10.00 | -0.02961805051 | -0.02758150399 | 0.001351084509 |
| 20.00 | -0.02089989478 | -0.01941808121 | 0.00006871250545 |
| 50.00 | -0.01319661317 | -0.01225031026 | 0.0002447024114 |
| 100.00| -0.009328862000| -0.008654573162| 0.0001269692928 |

By using the results of the runs in table 1, we propose a fitting formula for \( |p_3| \) inspired again in the 1PN Hamiltonian or the equations of motion

\[
|p_3|^2(\lambda) = \frac{0.01617387234}{\lambda} + \frac{0.002042558971}{\lambda^2} + \frac{0.0004169461512}{\lambda^3}.
\]  

(8)

In figure 5, we show the fitting function.

Independently in the Newtonian calculations, the \( \lambda \)-\( |p_3| \) relation can be obtained from the initial condition in (6) as

\[
|p_3|^2_N(\lambda) = \frac{0.1271642973}{\lambda^{1/2}}.
\]  

(9)

Note that relative difference \( |p_3|^2 \) between the Newtonian and the first post-Newtonian calculations:

\[
\frac{|p_3|^2_N(\lambda) - |p_3|^2_N(\lambda)}{|p_3|^2_N(\lambda)},
\]

is 7% for \( \lambda = 1 \), 0.6% for \( \lambda = 10 \) and 0.07% for \( \lambda = 100 \).

4. Second post-Newtonian corrections

It is interesting to verify if this kind of orbit also exists in the second post-Newtonian approximation to general relativity, since they incorporate further effects of the curvature, but yet not gravitational radiation. The calculations are done by using the same method as for the first post-Newtonian order. In table 2, we summarize the initial conditions for each \( \lambda \), from 1 to 100. We show the numerical errors as measured through the Hamiltonian non-conservation in figure 6.
We find that we can approximate $|p_3|$ by the fitting formula

$$|p_3|^2_{\text{fit}}(\lambda) = \frac{0.016 177 170 62}{\lambda} + \frac{0.002 004 172 619}{\lambda^2} + \frac{0.000 315 195 9703}{\lambda^3} + \frac{0.000 152 477 0451}{\lambda^4}. \quad (10)$$

There is a significant difference between the coefficient of $1/\lambda^3$ in (8) and (10). This is due to second post-Newtonian corrections entering in this coefficient, as we can verify from the form of the Hamiltonian. In figure 7, we show the fitting function.

Finally, we summarize the results by showing the difference between the Newtonian, first and second post-Newtonian results in figure 8. The second post-Newtonian effect is small but clearly not negligible for $\lambda = 1$.

5. Discussion

In this paper we have used the figure-eight orbits as a theoretical lab to test the properties of the low post-Newtonian expansions of general relativity. In this case we have found that those closed orbits exist for three (and presumably $N$) bodies. We have provided an
improved first-post-Newtonian order formalism for deriving the equations of motion that satisfy the Hamiltonian (and the linear and angular momenta) constraint to round-off error. The subsequent numerical evolution is well behaved for more than $t \sim 10000 m$. We have also extended this analysis to the 2PN corrections, still giving a conservative system of equations. In the process of finding the figure-eight solutions by trial of different initial momenta we also showed (numerically) the stability of the orbit against small perturbations.

This method is particularly useful to determine, dynamically (as an alternative to determine them through families of initial data [17]), initial orbital parameters for subsequent full
numerical evolution [9], when the holes are close enough that general relativistic effects can no longer be ignored. Note that our method fully takes into account the three-body post-Newtonian interactions unlike other simulations that approximate the problem in successive two-body problems [18].

It is interesting to note here that the scaling fits (10) give a practical way to determine when relativistic or Newtonian approaches are appropriate. In a convergent expansion the successive orders should be much smaller. Setting that the next correction be no bigger than, lets say, 10%, one can establish a value of \( \lambda \), giving us a physical scale, as follows. For \( \lambda = 1 \) we have that the ratio of the first coefficient, 0.016 177 170 62 (Newtonian) to the second coefficient 0.002 004 172 619 (first post-Newtonian) is nearly 0.12/\( \lambda \), and the second coefficient to the third one 0.000 315 195 9703 (dominated by second post-Newtonian) is approximately 0.16/\( \lambda \). This indicates that post-Newtonian corrections are more important than those of binary systems in circular orbit. At a given orbital radius, the post-Newtonian effects on the three-body problem are about twice as much as the binary case. Besides a crucial difference between the three-body and two-body problems is that the former can lead to chaotic orbital behavior, i.e. a small deviation in the initial data can lead to large deviations in the subsequent orbital dynamics making small post-Newtonian corrections more relevant. This aspect holds whenever gravitational radiation is not important. In this regime one also observes a large (conservative) exchange of energy among the triplet components.

For \( \lambda = 1 \) the distance between the initial bodies is 200m, which indicates that for nearly 67\( M \) with \( M \approx 3m \) the total mass of the system has strong post-Newtonian effects. For \( \lambda \gg 1 \) Newtonian gravity should describe the system accurately, while for \( \lambda \ll 1 \) general relativistic effects should be very important, eventually leading to the total collapse of the system. It is interesting to remark here that most of the \( N \)-body codes use some sort of regularization of the Newtonian gravity for very close encounters [19], instead the natural way to regularize these close encounters [9] is given by the general theory of relativity, and as we show here, the post-Newtonian corrections are already non-negligible at separations of the order of 100\( M \). In any case, for most of the astrophysical encounters this is way too short a distance, but it can obviously be reached in systems involving black holes and compact neutron stars.

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Appendix A. The second post-Newtonian three-body Hamiltonian

In this appendix, we give explicitly the Hamiltonian for the three-body problem at second post-Newtonian order in the ADM gauge since there are some typos in the summation of [16]. The equations of motion used in our paper can be derived straightforwardly from this Hamiltonian, but are too cumbersome to write down here.

\[
H_{2\text{PN}} = \frac{1}{16} \sum_a m_a \left( \frac{p_{a}^2}{m_a^3} \right)^3 + \frac{1}{16} \sum_{a,b} m_a m_b \frac{m_a}{r_{ab}} \left[ 10 \left( \frac{p_a^2}{m_a^2} \right)^2 - 11 \frac{p_a^2 p_b^2}{m_a^2 m_b^2} - 2 (\frac{p_a \cdot p_b}{m_a m_b})^2 \right]
+ \frac{5}{8} \sum_{a,b} m_a m_b \frac{m_a^2}{r_{ab}} \left[ 2 (\frac{p_a \cdot p_{ac}}{m_a m_c})^2 - 14 (\frac{p_{ac} \cdot p_{b}}{m_a m_c})^2 \right]
+ \frac{1}{4} \sum_{a,b} m_a m_b \frac{m_a^3}{r_{ab}^2} \left[ \frac{p_a^2}{m_a^2} + \frac{p_b^2}{m_b^2} - 2 \frac{p_a \cdot p_b}{m_a m_b} \right]
+ \frac{1}{2} \sum_{a,b} m_a m_b \frac{m_a^3}{r_{ab}^3} \left[ \frac{m_a}{r_{ab} + r_{bc} + r_{cb}} \right] \left( n_{ab}^i + n_{ac}^i \right) \left( n_{ac}^j + n_{cb}^j \right) \left[ \frac{g_{pai} p_{cj}}{m_a m_c} - 16 g_{paj} p_{ci} \right] \\
- \frac{3}{2} \frac{p_a \cdot p_b - (\frac{p_{ab} \cdot p_a}{m_a} (\frac{p_{ab} \cdot p_b}{m_b})}{m_a m_b r_{ab}^2} - \frac{1}{4} \frac{m_a^3 m_b}{r_{ab}^3} + \frac{1}{2} \sum_{a,b} m_a m_b \frac{m_a^3 m_b}{r_{ab}^3}
- \frac{3}{8} \sum_{a,b} m_a m_b m_c \frac{m_a^3 m_b}{r_{ab}^3} + \frac{3}{8} \sum_{a,b} m_a m_b m_c \frac{m_a^3 m_b}{r_{ab}^3}
- \frac{1}{64} \sum_{a,b} m_a m_b m_c \frac{18 r_{ab}^2 r_{ac}^2}{r_{bc}^2} - 60 \frac{r_{ab}^2 r_{ac}^2}{r_{bc}^2} - 24 \frac{r_{ab}^2 r_{ac}^2 (r_{ab} + r_{bc}) + 60 r_{ab} r_{ac}^2}{r_{ab}^3 r_{ac}^3}
+ 56 r_{ab}^2 r_{bc}^2 - 72 r_{ab} r_{bc}^3 + 35 r_{bc}^4 - 6 r_{ab}^4} - \frac{1}{4} \sum_{a,b} m_a^2 m_b^2 r_{ab}^3.
\]

(A.1)
Interesting enough after the completion of our paper a new paper revisiting the problem provides a closed simple form of the $N$-body Hamiltonian in the post-Minkowskian approximation [20].

Appendix B. Approximated equation of motion

In this appendix, we consider the violations of the Hamiltonian conservation shown in figure 2. This does not arise from numerical error, but is related to some approximation as sketched below.

When we consider the Lagrangian formulation, Euler–Lagrange equations are used

$$\frac{d}{dt} \left( \frac{\partial L}{\partial (\dot{q}_a)_i} \right) = \frac{\partial L}{\partial (q_a)_i},$$

where $L$ is the Lagrangian and $(\dot{q}_a)_i = d(q_a)_i/dt$. We start from the Lagrangian equivalent to the Hamiltonian given in (A.1), i.e.,

$$L = \sum_a p_a \cdot \dot{q}_a - H.$$  \hspace{1cm} (B.2)

For example, we obtain the following equation for the $x$ direction of the particle 1.

$$P(x_a, x_b) \ddot{x}_1 + Q(x_a, x_b) \dot{y}_1 + R(x_a, x_b) \ddot{z}_1 + S(x_a, x_b) \ddot{x}_2 + \cdots = F(x_a, x_b),$$

where we have sorted the above equation by the accelerations $(\ddot{q}_a)_i$. Note that the leading order of the function $P$ is proportional to $m_1$ in the Newtonian order, and the leading order of $Q, R$ and $S$ arises from the 1PN terms of the Lagrangian. This is so because they are derived from the second and the fourth (and higher) order terms of the velocities, respectively. The function $F$ does not include any acceleration. If we numerically integrate the above equations without modification, we obtain the same conservation of the Hamiltonian as the Hamiltonian formulation. But, one usually takes

$$m_1 \ddot{x}_1 = F_{1x}(x_a, x_b),$$

as the equation of motion. In order to obtain the above type of equation from the Euler–Lagrange formulation, we need a truncation at the given post-Newtonian approximation. In practice, to remove the accelerations $\ddot{y}_1, \ddot{z}_1, \ddot{x}_2$ etc., we use another component of the Euler–Lagrange equations which has the leading term $m_1 \ddot{y}_1, m_1 \ddot{z}_1, m_2 \ddot{x}_2$ etc., respectively. By using these substitutions, the equation includes higher post-Newtonian terms which should be truncated at the given post-Newtonian approximation. Therefore, the integration of (B.4) does not conserve the Hamiltonian.

A way to make the Lagrangian and Hamiltonian formulations agree is to consider a generalized Lagrangian depending not only on velocities and positions but also on accelerations, and then replace accelerations in the final equations of motion (see section 9.2 of [21] for a review).

References

[1] Gultekin K, Miller M C and Hamilton D P 2003 AIP Conf. Proc. 686 135 (Preprint astro-ph/0306204)
[2] Miller M C and Hamilton D P 2002 Preprint astro-ph/0202298
[3] Valtonen M J 1996 Mon. Not. R. Astron. Soc. 278 186
[4] Djorgovski S G, Courbin F, Meylan G, Sluse D, Thompson D J, Mahabal A and Glikman E 2007 Preprint astro-ph/0701155
[5] Pretorius F 2005 Phys. Rev. Lett. 95 121101 (Preprint gr-qc/0507014)
[6] Campanelli M, Lousto C O, Marronetti P and Zlochower Y 2006 Phys. Rev. Lett. 96 111101 (Preprint gr-qc/0511048)

[7] Baker J G, Centrella J, Choi D I, Koppitz M and van Meter J 2006 Phys. Rev. Lett. 96 111102 (Preprint gr-qc/0511103)

[8] Faber J A, Baumgarte T W, Etienne Z B, Shapiro S L and Taniguchi K 2007 Phys. Rev. D 76 104021 (Preprint arXiv:0708.2436)

[9] Campanelli M, Lousto C O and Zlochower Y 2008 Phys. Rev. D 77 101501 (Preprint arXiv:0710.0879)

[10] Lousto C O and Zlochower Y 2008 Phys. Rev. D 77 024034 (Preprint arXiv:0711.1165)

[11] Moore C 1993 Phys. Rev. Lett. 70 3675

[12] Chenciner A and Montgomery R 2000 Ann. Math. 152 881

[13] Heggie D C, Hut P and McMillan S L W 2000 Mon. Not. R. Astron. Soc. 318 L61 (Preprint astro-ph/9604016)

[14] Imai T, Chiba T and Asada H 2007 Phys. Rev. Lett. 98 201102 (Preprint gr-qc/0702076)

[15] Chiba T, Imai T and Asada H 2007 Mon. Not. R. Astron. Soc. 377 269 (Preprint astro-ph/0609773)

[16] Chäfer G 1987 Phys. Lett. A 123 336

[17] Campanelli M, Dettwyler M, Hannam M and Lousto C O 2006 Phys. Rev. D 74 087503 (Preprint astro-ph/0509814)

[18] Aarseth S J 2007 Mon. Not. R. Astron. Soc. 378 285 (Preprint astro-ph/0701612)

[19] Aarseth S J 2003 Gravitational N-Body Simulations (Cambridge: Cambridge University Press)

[20] Ledvinka T, Schäfer G and Bicak J 2008 Phys. Rev. Lett. 100 251101 (Preprint arXiv:0807.0214)

[21] Blanchet L 2006 Living Rev. Rel. 9 4 (Preprint gr-qc/0202016)