Algebraic algorithm for the computation of one-loop Feynman diagrams in lattice QCD with Wilson fermions

Giuseppe Burgio
Dipartimento di Fisica and INFN – Sezione di Parma
Università degli Studi di Parma
I-43100 Parma, ITALIA
Internet: BURGIO@PARMA.INFN.IT

Sergio Caracciolo
Dipartimento di Fisica and INFN – Sezione di Lecce
Università degli Studi di Lecce
I-73100 Lecce, ITALIA
Internet: CARACCIO@UX1SNS.SNS.IT

Andrea Pelissetto
Dipartimento di Fisica and INFN – Sezione di Pisa
Università degli Studi di Pisa
I-56100 Pisa, ITALIA
Internet: PELISSET@IBMTH1.DIFI.UNIPI.IT
PELISSET@IPIFIDPT.DIFI.UNIPI.IT

December 1, 2021

Abstract
We describe an algebraic algorithm which allows to express every one-loop lattice integral with gluon or Wilson-fermion propagators in terms of a small number of basic constants which can be computed with arbitrary high precision. Although the presentation is restricted to four dimensions the technique can be generalized to every space dimension. Various examples are given, including the one-loop self-energies of the quarks and gluons and the renormalization constants for some dimension-three and dimension-four lattice operators. We also give a method to express the lattice free propagator for Wilson fermions in coordinate space as a linear function of its values in eight points near the origin. This is an essential step in order to apply the recent methods of Lüscher and Weisz to higher-loop integrals with fermions.

PACS: 11.15.Ha 12.38.Gc
1 Introduction

Perturbation theory plays an important role in our present understanding of quantum field theory. In particular on the lattice Feynman diagram computations are performed to obtain such quantities as ratios of $\Lambda$-parameters, non-universal coefficients of $\beta$-functions or renormalization constants of lattice operators. Due to the loss of Lorentz invariance lattice calculations are usually particularly involved and thus in order to obtain reliable results one has to make use of computer symbolic programs. To implement this strategy an important step is the simplification of the lattice integrals appearing in the Feynman diagrams of the theory. In [1] we presented a general technique which allows to express every one-loop bosonic integral at zero external momentum in terms of two unknown basic quantities which could be computed numerically with high precision. This method allows the complete evaluation of every diagram with gluon propagators.

In this paper we want to generalize the technique to deal with integrals with both gluonic and fermionic propagators. For the fermions we use the Wilson action [2]. Notice that our method depends only on the structure of the propagator and thus it can be applied in calculations with the standard Wilson action as well as with the improved clover action [3]. We show that every integral at zero external momentum can be expressed in terms of a small number of basic quantities (nine for purely fermionic integrals, thirteen for integrals with bosonic and fermionic propagators). The advantage of this procedure is twofold: first of all every Feynman diagram can be computed in a completely symbolic way making it easier to perform checks and verify cancellations; moreover the basic constants can be easily computed with high precision and thus the numerical error on the final result can be reduced at will. Although the presentation is restricted to four dimensions the technique can be generalized to every space dimension.

A second important application of our method is connected with the use of coordinate-space methods for the evaluation of higher-loop Feynman diagrams. This technique, introduced by Lüscher and Weisz [4], is extremely powerful and allows a very precise determination of two- and higher-loop integrals. One of the basic ingredients of this method is the computation of the free propagator in coordinate space. We will give an algorithm which allows the analytic determination of the free fermionic propagator in $x$-space in terms of eight basic constants which can be reinterpreted as the values of the propagator near the origin.

The paper is organized as follows: in section 2 we review the computation of the continuum limit of lattice integrals, showing that the calculation can be split in two parts: the evaluation of a subtracted continuum integral and of a certain number of lattice integrals with zero external momentum which are then discussed in the following sections. In section 3 we present the method in the bosonic case, simplifying the strategy discussed in [1]. Section 4 represents the core of the paper and gives the detailed algorithm for fermionic and mixed bosonic-fermionic integrals. In both these sections we need to introduce an infrared cut-off to regularize the integrals for $k = 0$. We choose to introduce a mass $m$; other possibilities are discussed in section 5 where the connection with dimensionally regularized integrals is presented. Finally in
section 6 we present a few examples: we give the analytic expressions for the lattice gluon and fermion self-energy, for the renormalization constants of dimension-three bilinear fermion operators and a computation of the renormalization constants for the operators which show up in the energy-momentum tensor and which are relevant in the computation of the structure functions which appear in the deep-inelastic scattering [3]. Finally we discuss an algorithm for the computation of the free lattice propagators.

2 Continuum limit of lattice integrals

In this Section we want to discuss the computation of one-loop Feynman diagrams on the lattice. In general each graph has the form

\[ G(\{p_i\}) = \int \frac{d^4q}{(2\pi)^4} F(q; \{p_i\}) \quad (2.1) \]

where \( \{p_i\} \) is a set of external momenta. Of course one is not interested in the exact computation of (2.1) but only in its value in the continuum limit. If the integral is ultraviolet-convergent one can simply substitute \( F(q; \{p_i\}) \) with its continuum counterpart and obtain a continuum convergent integral. Let us now suppose that (2.1) is divergent and, for simplicity, that there is only one external momentum. If every propagator is massive so that \( F(q; \{p_i\}) \) is finite for any set of momenta going to zero, one can use the general technique of BPHZ [4] which have been generalized on the lattice by Reisz [7] writing

\[ G(p) = \int \frac{d^4q}{(2\pi)^4} [F(q; p) - (T^{n_F} F)(q; p)] + \int \frac{d^4q}{(2\pi)^4} (T^{n_F} F)(q; p) \equiv G^c(p) + G^L(p) \quad (2.2) \]

where \( n_F \) is the degree of the divergence of the integral and

\[ (T^{n_F} F)(q; p) = \sum_{k=0}^{n_F} \frac{1}{k!} p_{\mu_1} \cdots p_{\mu_k} \left[ \frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_k}} F(q; p) \right]_{p=0} \quad (2.3) \]

The first integral in (2.2) is ultraviolet-finite [4] and thus one can take the continuum limit. Thus all the effects of the lattice regularization remain only in the second term, which is simply a polynomial in the external momentum with coefficients given by lattice zero-momentum integrals.

If the integrand contains massless propagators one has to be more careful: indeed an expansion around \( p = 0 \) can give rise to infrared divergences. A simple way out consists in introducing an intermediate infrared regularization: one can use the dimensional regularization [3, 4] working in dimension \( d > 4 \), or introduce a mass in the propagators. In both cases \( G^c(p) \) and \( G^L(p) \) will be singular for \( d \to 4 \) or \( m \to 0 \) but, of course, the singularity will cancel when summing up the two terms.

In conclusion the computation of the continuum limit of (2.1) splits in the computation of two different quantities: a continuum ultraviolet-finite integral and a certain number of zero-momentum lattice integrals whose computation will be discussed in the next sections.
3 Bosonic integrals

In this section we discuss the evaluation of the most general one-loop lattice integral at zero external momentum with bosonic propagators. It is very easy to see that any such integral can be written as a linear combination of terms of the form

$$B(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{D_B(k, m)^p}$$ (3.1)

where $p$ and $n_i$ are positive integers, $\hat{k}_\mu = 2 \sin(k_\mu/2)$ and

$$D_B(k, m) = \hat{k}^2 + m^2.$$ (3.2)

is the inverse bosonic propagator. In the following when one of the arguments $n_i$ is zero it will be omitted as argument of $B$.

We will review here a general technique for expressing these integrals in terms of three constants \[1\].

We will first generalize (3.1) by considering the following more general integrals

$$B_\delta(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{\hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t}}{D_B(k, m)^{p+\delta}}$$ (3.3)

where $p$ is an arbitrary integer (not necessarily positive) and $\delta$ a real number which is introduced in order to avoid singular cases at intermediate stages of the computation and which will be set to zero at the end.

The first result we want to prove is that each integral $B_\delta(p; n_x, n_y, n_z, n_t)$ can be reduced through purely algebraic manipulations to a sum of integrals of the same type with $n_x = n_y = n_z = n_t = 0$.

Indeed the integrals $B_\delta$ satisfy the following recursion relations:

$$B_\delta(p; 1) = \frac{1}{4} [B_\delta(p - 1) - m^2 B_\delta(p)]$$

$$B_\delta(p; x, 1) = \frac{1}{3} [B_\delta(p - 1; x) - B_\delta(p; x + 1) - m^2 B_\delta(p; x)]$$

$$B_\delta(p; x, y, 1) = \frac{1}{2} [B_\delta(p - 1; x, y) - B_\delta(p; x + 1, y) - B_\delta(p; x, y + 1) - m^2 B_\delta(p; x, y)]$$

$$B_\delta(p; x, y, z, 1) = B_\delta(p - 1; x, y, z) - B_\delta(p; x + 1, y, z) - B_\delta(p; x, y + 1, z) - B_\delta(p; x, y, z + 1) - m^2 B_\delta(p; x, y, z)$$ (3.4)

which can be obtained using the trivial identity

$$D_B(k, m) = \sum_{i=1}^{4} \hat{k}_i^2 + m^2.$$ (3.5)

Furthermore, when $r > 1$, we can write

$$\frac{\langle \hat{k}_w \rangle^r}{D_B(k, m)^{p+\delta}} = 4 \frac{\langle \hat{k}_w \rangle^{r-1}}{D_B(k, m)^{p+\delta}} + 2 \frac{\langle \hat{k}_w \rangle^{r-2}}{p + \delta - 1} \sin k_w \frac{\partial}{\partial k_w} \frac{1}{D_B(k, m)^{p+\delta-1}}$$ (3.6)
Then, integrating by parts, we obtain the recursion relation:

\[
\mathcal{B}_\delta(p; \ldots, r) = \frac{r - 1}{p + \delta - 1} \mathcal{B}_\delta(p - 1; \ldots, r - 1) - \frac{4r - 6}{p + \delta - 1} \mathcal{B}_\delta(p - 1; \ldots, r - 2) + 4 \mathcal{B}_\delta(p; \ldots, r - 1)
\]  

(3.7)

Let us notice that for \( p \neq 1 \) this recursion relation is regular for \( \delta \to 0 \). For \( p = 1 \) instead the coefficients of \( \mathcal{B}_\delta(0; \ldots) \) diverge as \( 1/\delta \). This means that to compute \( \mathcal{B}_\delta(1; \ldots) \) for \( \delta = 0 \), we need to compute \( \mathcal{B}_\delta(0; \ldots) \) including terms of order \( \delta \). Since by the application of the previous recursion relations to \( \mathcal{B}_\delta(0; \ldots) \) we generate \( \mathcal{B}_\delta(-1; \ldots) \), then \( \mathcal{B}_\delta(-2; \ldots) \) and so on and their coefficients are finite for \( \delta \to 0 \), we see that in general we need to compute all integrals \( \mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) \) with \( p \leq 0 \) up to \( O(\delta^2) \).

The previous relations allow to reduce every integral \( \mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) \) to a sum of the form

\[
\mathcal{B}_\delta(p; n_x, n_y, n_z, n_t) = \sum_{r = p - n_x - n_y - n_z - n_t}^{p} a_r(m, \delta) \mathcal{B}_\delta(r)
\]

(3.8)

The \( m \)-dependence of \( a_r(m, \delta) \) is very simple: it is indeed a polynomial in \( m^2 \). Let us now discuss the \( \delta \)-dependence. If \( p \leq 0 \) only nonpositive values of \( r \) are allowed in (3.8) with coefficients which are regular for \( \delta \to 0 \). For \( p > 0 \) the situation is more complicated: for \( r \geq 1 \), \( \lim_{\delta \to 0} a_r(m, \delta) \) is finite while for \( r \leq 0 \) \( a_r(m, \delta) \) may behave as \( 1/\delta \) when \( \delta \) goes to zero. As we already observed, this means that we need to compute \( \mathcal{B}_\delta(r), r \leq 0 \), including terms of order \( \delta \).

Now let us show that all \( \mathcal{B}_\delta(p) \) can be expressed in terms of a finite number of them. Although this can be shown for generic values of the mass\(^1\) many simplifications occur if one restricts the attention to the massless case, i.e. if one considers the limit \( m^2 \to 0 \) keeping only the non-vanishing terms. Let us thus discuss the limit \( m^2 \to 0 \) of \( \mathcal{B}_\delta(p) \).

Simple power-counting shows that the integrals \( \mathcal{B}_\delta(p), p \leq 1 \), are finite. To compute the divergent part of \( \mathcal{B}_\delta(p) \), \( p \geq 2 \), let us start from the well-known representation

\[
\mathcal{B}_\delta(p) = \frac{1}{2^{p+\delta}\Gamma(p+\delta)} \int_0^{\infty} d\lambda \lambda^{p+\delta-1} e^{-m^2\lambda/2-4\lambda} I_0(\lambda)^4
\]

(3.9)

where \( I_0(\lambda) \) is a modified Bessel function \( [10] \). We introduce constants \( b_i \) that are defined by the asymptotic expansion for large \( x \) of \( I_0(x)^4 \) as

\[
I_0(x)^4 \approx e^{4x} \sum_{i=0}^{\infty} \frac{b_i}{x^{i+2}}
\]

(3.10)

The constants \( b_i \) are rational numbers multiplied by \( 1/\pi^2 \). Then the divergent part of \( \mathcal{B}_\delta(p) \) is given by

\[
\frac{1}{\Gamma(p+\delta)} \sum_{i=2}^{p} \frac{b_{i-2} \Gamma(p+\delta-i)}{2^i (m^2)^{p+\delta-i}}
\]

(3.11)

\(^1\)The analogous procedure in two dimensions is presented in Appendix A.3 of [9].
Since here \( p \) is positive we can set \( \delta = 0 \). However in the next section we will be interested also in the divergent part proportional to \( \delta \). Thus expanding in \( \delta \) and neglecting finite terms, up to \( O(\delta^2) \), we get\[\frac{1}{\Gamma(p)} \sum_{i=2}^{p-1} \frac{b_{i-2}(p-i)}{2^i(m^2)^{p-i}} - \frac{b_{p-2}}{2\Gamma(p)} \log m^2 \]

\[\delta \left[ \frac{1}{\Gamma(p)} \sum_{i=2}^{p-1} \frac{b_{i-2}(p-i)}{2^i(m^2)^{p-i}} \left( \log m^2 + \sum_{k=p-i}^{p-1} \frac{1}{k} \right) \right] - \frac{b_{p-2}}{2\Gamma(p)} \left( \frac{1}{2} \log m^2 + \sum_{k=1}^{p-1} \frac{1}{k} \right) \log m^2 + O(\delta^2) \]

(3.12)

Let us now go back to (3.8) rewriting it as

\[B_\delta(p; n_x, n_y, n_z, n_t) = \sum_r a_r(0,\delta)B_\delta(r) + \sum_r (a_r(m,\delta) - a_r(0,\delta))B_\delta(r) \]

(3.13)

and let us consider the limit \( m^2 \to 0 \). It is clear that in the second sum only \( B_\delta(r) \) with \( r \geq 3 \) can contribute, since only these integrals have power divergences for \( m^2 \to 0 \). As the values of \( r \) in the sums satisfy \( r \geq p \), we find that the second sum contributes only for \( p \geq 3 \). Thus we get for \( p \leq 2 \)

\[B_\delta(p; n_x, n_y, n_z, n_t) = \sum_r a_r(0,\delta)B_\delta(r) + O(m^2) \]

(3.14)

while, for \( p > 2 \),

\[B_\delta(p; n_x, n_y, n_z, n_t) = \sum_r a_r(0,\delta)B_\delta(r) + R(m,\delta) + O(m^2) \]

(3.15)

where \( R(m;\delta) \) is a polynomial in \( 1/m^2 \) whose coefficients, for \( \delta \to 0 \), are rational numbers multiplied by \( 1/\pi^2 \).

Let us finally find the last recursion relations. Let us start from the trivial identity

\[B_\delta(p; 1, 1, 1, 1) - 4B_\delta(p+1; 2, 1, 1, 1) - m^2B_\delta(p+1; 1, 1, 1, 1) = 0 \]

(3.16)

and let us apply the previous procedure to reduce each term to the form (3.14) and (3.13). We thus get a non trivial relation involving \( B_\delta \) of the form

\[\sum_{r=p-4}^{p} b_r(p;\delta)B_\delta(r) + S(p; m,\delta) = 0 \]

(3.17)

\(^2\)Notice that we are dealing here with two different limits, \( m^2 \to 0 \) and \( \delta \to 0 \). In general they do not commute and thus it is necessary to specify the correct order in which they are taken. Here we first consider \( \delta \to 0 \) at fixed \( m \) and then we let \( m \) go to zero.

\(^3\)Since we need to compute \( B_\delta(r), r \leq 0 \), including terms of order \( \delta \), in the computation of the various terms, one must keep the contributions of order \( \delta \) if \( p < 0 \), while for \( p \geq 0 \) it is enough to expand the identity to order \( O(\delta) \).
where $S(p; m, \delta) = 0$ for $p \leq 2$ while for $p > 2$ is a polynomial in $1/m^2$ which is finite for $\delta \to 0$. We will use this identity to express all $B_\delta(p)$ in terms of $B_\delta(r)$, $0 \leq r \leq 3$. Indeed we can solve (3.17) in terms of $B_\delta(p)$ and thus we get a relation which expresses it in terms of $B_\delta(p-1), \ldots, B_\delta(p-4)$. We will use this relation for $p \geq 4$. On the other hand we can solve (3.17) in terms of $B_\delta(p-4)$ and then shift $p \to p+4$. In this way we obtain a relation which expresses $B_\delta(p)$ in terms of $B_\delta(p+1), \ldots, B_\delta(p+4)$. We use this recursion for $p \leq -1$. Applying recursively these two relations we get finally ($p \neq 0, 1, 2, 3$)

$$B_\delta(p) = \sum_{r=0}^{3} c_r(p; \delta) B_\delta(r) + T(p; m, \delta)$$

(3.18)

where $T(p; m, \delta)$ is a polynomial in $1/m^2$. A direct analysis of (3.17) shows the following properties:

1. if $p \geq 4$, $c_0(p; \delta) = O(\delta^4)$;
2. if $p \leq -1$, $c_r(p; \delta) = O(\delta)$ for $1 \leq r \leq 3$;
3. if $p \leq -1$, $T(p; m, \delta)$ is of order $\delta$, while for $p \geq 4$ it is finite for $\delta \to 0$.

Substituting back in (3.14) or (3.15) we get

$$B_\delta(p; n_x, n_y, n_z, n_t) = A(\delta) B_\delta(0) + B(\delta) B_\delta(1) + C(\delta) B_\delta(2) + D(\delta) B_\delta(3) + E(m, \delta)$$

(3.19)

where $E(m, \delta)$ is a polynomial in $1/m^2$.

We can now go back to the original integral (3.4) (notice that we are only interested in the case $p > 0$). Because of the second property of the coefficients $c_r(p; \delta)$ and the property of $T(p; m, \delta)$, we immediately see that $B(\delta)$, $C(\delta)$, $D(\delta)$ and $E(m, \delta)$ are finite for $\delta \to 0$. Then, as the l.h.s. is obviously finite for $\delta \to 0$, also $A(\delta)$ is finite for $\delta \to 0$. Since $B(0) = 1$, we have finally

$$B(p; n_x, n_y, n_z, n_t) = A(0) + B(0) B(1) + C(0) B(2) + D(0) B(3) + E(m, 0)$$

(3.20)

We want now to make contact with our previous work where all results were expressed in terms of three constants, $Z_0$, $Z_1$ and $F_0$ defined by

$$Z_0 = B(1)|_{m=0}$$

(3.21)

$$Z_1 = \frac{1}{4} B(1; 1, 1)|_{m=0}$$

(3.22)

$$F_0 = \lim_{m \to 0} [16\pi^2 B(2) + \log m^2 + \gamma_E]$$

(3.23)

It is clear how to rewrite $B(1)$ and $B(2)$ in terms of $F_0$ and $Z_0$. For $B(3)$ a short calculation gives

$$B(3) = \frac{1}{32\pi^2 m^2} - \frac{1}{128\pi^2} \left( \log m^2 + \gamma_E - F_0 \right) - \frac{1}{1024} - \frac{13}{1536\pi^2} + \frac{Z_1}{256}$$

(3.24)
Table 1: Numerical values of the three constants $Z_0$, $Z_1$ and $F_0$.

|   |   |   |   |
|---|---|---|---|
| $Z_0$ | 0.15493390231060214084837208 |
| $Z_1$ | 0.1077813539874001343391550 |
| $F_0$ | 4.369225233874758 |

An additional simplification occurs if the original integral is finite. In this case the log $m^2$ terms must cancel. They appear only in $B(2)$ and $B(3)$ and always in the combination $(\log m^2 + \gamma_E - F_0)$. Thus the cancellation of log $m^2$ implies also the cancellation of $\gamma_E$ and $F_0$. All finite integrals are thus functions of $Z_0$ and $Z_1$ only. Numerical values of the constants are reported in Table 1.

It is interesting to notice that the same technique can be used for bosonic integrals in $d$ dimensions. The basic recursions can be trivially generalized as well as the identity (3.10). For generic $d$ we find that all integrals can finally be expressed in terms of $B_3(1), \ldots, B_3(d-1)$, i.e. in terms of $(d-1)$ constants, reducing to $(d-2)$ for infrared finite integrals.

### 4 Integrals with bosonic and Wilson-fermion propagators

We want now to discuss the computation of general integrals with fermionic and bosonic propagators at zero external momenta. Define

$$ F(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{k_x^{2n_x} k_y^{2n_y} k_z^{2n_z} k_t^{2n_t} }{D_F(k, m_f)^p D_B(k, m_b)^q} \tag{4.1} $$

where $p$, $q$ and $n_i$ are positive integers, $D_B(k, m_b)$ is defined in (3.2) and

$$ D_F(k, m_f) = \sum_i \sin^2 k_i + \frac{r_W^2}{4} (\hat{k}^2) + m_f^2 \tag{4.2} $$

is the denominator appearing in the propagator for Wilson fermions\footnote{To be precise, $D_F(k, m_f)$ is the denominator in the propagator for Wilson fermions only for $m_f = 0$. For $m_f \neq 0$ the correct denominator would be

$$ \hat{D}_F(k, m_f) = \sum_i \sin^2 k_i + \left( \frac{r_W}{2} \hat{k}^2 + m_f \right)^2. \tag{4.3} $$

However in our discussion $m_f$ will only play the role of an infrared regulator and thus it does not need to be the true fermion mass. The definition (4.2) is easier to handle than (4.3). For a discussion of integrals using (4.3) see Section 5.2.}. In the following when one of the arguments $n_i$ is zero it will be omitted as an argument of $F$. Following
the strategy we have used in the purely bosonic case we will first generalize the integrals introducing

$$\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{k_x^{2n_x} k_y^{2n_y} k_z^{2n_z} k_t^{2n_t}}{D_F(k, m_f)^{p+\delta} D_B(k, m_b)^{q}}$$  \hspace{1cm} (4.4)$$

Here we consider both \( p \) and \( q \) as arbitrary integers (not necessarily positive). The parameter \( \delta \) is used in the intermediate steps of the calculation and will be set to zero at the end.

To simplify the discussion we will only consider the case \( r_W = 1 \) but the technique can be applied to every value of \( r_W \). Moreover we will restrict our attention to the massless case, i.e. we will consider the integrals \( \mathcal{F}_\delta \) in the limit \( m_b = m_f^{\equiv m} \to 0 \).

In the following we will present a procedure that allows to compute iteratively a generic \( \mathcal{F}_\delta \) in terms of a finite number of them: precisely every \( \mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) \) with \( q \leq 0 \) can be expressed in terms of \( \mathcal{F}_\delta(1, 0) \), \( \mathcal{F}_\delta(1, -1) \), \( \mathcal{F}_\delta(1, -2) \), \( \mathcal{F}_\delta(2, 0) \), \( \mathcal{F}_\delta(2, -1) \), \( \mathcal{F}_\delta(2, -2) \), \( \mathcal{F}_\delta(3, -2) \), \( \mathcal{F}_\delta(3, -3) \) and \( \mathcal{F}_\delta(3, -4) \); the integral \( \mathcal{F}_\delta(2, 0) \) appears only in infrared-divergent integrals. If \( q > 0 \) the result contains three additional constants together with the bosonic quantities \( Z_0 \), \( Z_1 \) and \( F_0 - \gamma_E \).

Our procedure works in four steps:

1. First step: we express each integral \( \mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) \) in terms of \( \mathcal{F}_\delta(p, q) \) only.

2. Second step: we express every \( \mathcal{F}_\delta(p, q) \) in terms of \( \mathcal{F}_\delta(r, s) \) with \( 0 \leq r \leq 3 \), arbitrary \( s \) or \( r \leq -1 \) and \( s = 1, 2, 3 \) or \( r \geq 4 \) and \( s = 0, -1, -2 \). This is obtained by a systematic use of the identity

$$\mathcal{I}_1(p, q) \equiv \mathcal{F}_\delta(p, q; 1, 1, 1, 1) - 4\mathcal{F}_\delta(p, q+1; 2, 1, 1, 1) - m^2\mathcal{F}_\delta(p, q+1; 1, 1, 1, 1) = 0 \hspace{2cm} (4.5)$$

3. Third step: we express the remaining \( \mathcal{F}_\delta(p, q) \) in terms of \( \mathcal{F}_\delta(r, s) \) with \( r = 3 \), \( -4 \leq s \leq 0 \) or \( r = 2 \), \( -4 \leq s \leq 2 \) or \( r = 1 \), \( -4 \leq s \leq 4 \) or \( r = 0 \), \( -4 \leq s \leq 6 \) or \( r = -1 \) and \( s = 2 \). This is obtained by a systematic use of the identity

$$\mathcal{I}_2(p, q) \equiv \mathcal{F}_\delta(p, q; 1, 1, 1, 1) - \mathcal{F}_\delta(p+1, q-1; 1, 1, 1, 1) + \mathcal{F}_\delta(p+1, q; 3, 1, 1, 1) - \frac{1}{4} \left[ \mathcal{F}_\delta(p+1, q-2; 1, 1, 1, 1) - 2m^2\mathcal{F}_\delta(p+1, q-1; 1, 1, 1, 1) \right] + m^4\mathcal{F}_\delta(p+1, q; 1, 1, 1, 1) = 0 \hspace{1cm} (4.6)$$

4. Fourth step: the identities \( \mathcal{I}_1(p, q) \) and \( \mathcal{I}_2(p, q) \) are used to provide additional relations between the remaining integrals. We end up with the result we have quoted above.

We want to notice here two basic facts. First of all, as in the bosonic case, the structure of the recursion relations, will force us to compute all \( \mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) \) including terms of order \( \delta \) when \( p \leq 0 \). As a consequence we will require \( \mathcal{I}_1(p, q) \) and
$I_0(p, q)$ to be satisfied up to $O(\delta^2)$ specifically for $p \leq 0$ and $p \leq -1$ and up to $O(\delta)$ in the opposite case, i.e. for $p \geq 1$ and $p \geq 0$ respectively.

Let us finally mention some general properties of all the recursion relations we will consider: in all cases we will give results of the form

$$\mathcal{F}_\delta(p, q; \ldots) = \sum_{rs} \alpha_{pq;rs}(\delta)\mathcal{F}_\delta(r, s) + \mathcal{P}_{pq}(m, \delta) + O(m^2)$$

(4.7)

where $\alpha_{pq;rs}(\delta)$ and $\mathcal{P}_{pq}(m, \delta)$ will always have the following properties:

1. if $q \leq 0$, then $\alpha_{pq;rs}(\delta) = 0$ for $s > 0$. In other words we will express purely fermionic integrals in terms of integrals of the same type;

2. for $\delta \to 0$, we have $\alpha_{pq;rs}(\delta) \sim O(1/\delta)$ for $p > 0$ and $r \leq 0$, $\alpha_{pq;rs}(\delta) \sim O(1)$ for $p > 0$, $r > 0$ and $p \leq 0$, $r \leq 0$ and $\alpha_{pq;rs}(\delta) \sim O(\delta)$ for $p \leq 0$ and $r > 0$;

3. for $\delta \to 0$ we have $\mathcal{P}_{pq}(m, \delta) = \mathcal{P}_{pq}^{(1)}(m) + O(\delta)$ for $p > 0$ and $q \leq 0, \mathcal{P}_{pq}(m, \delta) = \delta \mathcal{P}_{pq}^{(1)}(m) + O(\delta^2)$ for $p \leq 0$ and $q \leq 0$; for $p > 0$ and $q > 0$ we have

$$\mathcal{P}_{pq}(m, \delta) = \frac{1}{\delta}(1 - \delta \log m^2)\mathcal{P}_{pq}^{(1)}(m) + \mathcal{P}_{pq}^{(2)}(m) + O(\delta)$$

(4.8)

while for $p \leq 0$ and $q > 0$ we have

$$\mathcal{P}_{pq}(m, \delta) = (1 - \delta \log m^2)\mathcal{P}_{pq}^{(1)}(m) + \delta \mathcal{P}_{pq}^{(2)}(m) + O(\delta^2)$$

(4.9)

in all cases $\mathcal{P}_{pq}^{(1)}(m)$ and $\mathcal{P}_{pq}^{(2)}(m)$ are polynomials in $1/m^2$ whose coefficients are rational numbers multiplied by $1/\pi^2$.

### 4.1 First step: the basic identities

We will now show that in a purely algebraic way all integrals can be reduced to a sum of $\mathcal{F}_\delta(p; q)$. Indeed it is easy to see that these integrals satisfy the following recursion relations (in giving these recursions we keep $m_b \neq m_f$ and $r_W$ generic)

$$\mathcal{F}_\delta(p, q; 1) = \frac{1}{4} \left[ \mathcal{F}_\delta(p, q - 1) - m_b^2 \mathcal{F}_\delta(p, q) \right]$$

$$\mathcal{F}_\delta(p, q; x, 1) = \frac{1}{3} \left[ \mathcal{F}_\delta(p, q - 1; x) - m_b^2 \mathcal{F}_\delta(p, q; x) - \mathcal{F}_\delta(p, q; x + 1) \right]$$

$$\mathcal{F}_\delta(p, q; x, y, 1) = \frac{1}{2} \left[ \mathcal{F}_\delta(p, q - 1; x, y) - m_b^2 \mathcal{F}_\delta(p, q; x) - \mathcal{F}_\delta(p, q; x + 1, y) \right]$$

$$\mathcal{F}_\delta(p, q; x, y, z, 1) = \mathcal{F}_\delta(p, q - 1; x, y, z) - m_b^2 \mathcal{F}_\delta(p, q; x, y, z) - \mathcal{F}_\delta(p, q; x + 1, y, z)$$

$$- \mathcal{F}_\delta(p, q; x + 1, z) - \mathcal{F}_\delta(p, q; x, y + 1, z) - \mathcal{F}_\delta(p, q; x, y, z + 1)$$

(4.10)

which can be obtained from the trivial identity

$$D_B(k, m_b) = \sum_i \tilde{k}_i^2 + m_b^2$$

(4.11)
A second recursion relation is obtained from the identity

\[
\sum_i \hat{k}_i^4 = 4(D_B(k, m_b) - D_F(k, m_f) - m_b^2 + m_f^2) + r_W^2(D_B(k, m_b) - m_b^2)^2
\]  

(4.12)

In this way we get

\[
\mathcal{F}_\delta(p, q; 2) = \mathcal{F}_\delta(p, q - 1) - \mathcal{F}_\delta(p - 1, q) + (m_f^2 - m_b^2)\mathcal{F}_\delta(p, q)
\]

\[
+ \frac{r_W^2}{4} \left[ \mathcal{F}_\delta(p, q - 2) - 2m_b^2\mathcal{F}_\delta(p, q - 1) + m_b^4\mathcal{F}_\delta(p, q) \right]
\]

\[
\mathcal{F}_\delta(p, q; 2) = \frac{4}{3} \left[ \mathcal{F}_\delta(p, q - 1; x) - \mathcal{F}_\delta(p - 1, q; x) + (m_f^2 - m_b^2)\mathcal{F}_\delta(p, q; x) - \frac{1}{4} \mathcal{F}_\delta(p, q; x + 2) \right]
\]

\[
+ \frac{r_W^2}{3} \left[ \mathcal{F}_\delta(p, q - 2; x) - 2m_b^2\mathcal{F}_\delta(p, q - 1; x) + m_b^4\mathcal{F}_\delta(p, q; x) \right]
\]

\[
\mathcal{F}_\delta(p, q; 2) = 2 \left[ \mathcal{F}_\delta(p, q - 1; x, y) - \mathcal{F}_\delta(p - 1, q; x, y) + (m_f^2 - m_b^2)\mathcal{F}_\delta(p, q; x, y) - \frac{1}{4} \mathcal{F}_\delta(p, q; x + 2, y) \right]
\]

\[
- \frac{1}{4} \mathcal{F}_\delta(p, q; x, y + 2) + \frac{r_W^2}{2} \left[ \mathcal{F}_\delta(p, q - 2; x, y) - 2m_b^2\mathcal{F}_\delta(p, q - 1; x, y) + m_b^4\mathcal{F}_\delta(p, q; x, y) \right]
\]

\[
\mathcal{F}_\delta(p, q; 2) = 4 \left[ \mathcal{F}_\delta(p, q - 1; x, y, z) - \mathcal{F}_\delta(p - 1, q; x, y, z) + (m_f^2 - m_b^2)\mathcal{F}_\delta(p, q; x, y, z) - \frac{1}{4} \mathcal{F}_\delta(p, q; x + 2, y, z) \right]
\]

\[
- \frac{1}{4} \mathcal{F}_\delta(p, q; x, y + 2, z) - \frac{1}{4} \mathcal{F}_\delta(p, q; x, y, z + 2) + \frac{r_W^2}{4} \left[ \mathcal{F}_\delta(p, q - 2; x, y, z) - 2m_b^2\mathcal{F}_\delta(p, q - 1; x, y, z) + m_b^4\mathcal{F}_\delta(p, q; x, y, z) \right]
\]  

(4.13)

Finally let us notice that we can write, for \( r \geq 3 \)

\[
\frac{(\hat{k}_w^2)^r}{D_F(k, m_f)^{p+\delta}} = \frac{4(\hat{k}_w^2)^{r-1} - 4(2 + r_W^2 \hat{k}_w^2)(\hat{k}_w^2)^{r-3}\sin^2 k_w}{D_F(k, m_f)^{p+\delta}} - \frac{4(\hat{k}_w^2)^{r-3}}{p + \delta - 1} \sin k_w \frac{\partial}{\partial k_w} D_F(k, m_f)^{p-1+\delta}
\]  

(4.14)

Integrating by parts we obtain the recursion relation (to be applied for \( r \geq 3 \))

\[
\mathcal{F}_\delta(p, q; \ldots, r)
\]

\[
= 6\mathcal{F}_\delta(p, q; \ldots, r - 1) - 8\mathcal{F}_\delta(p, q; \ldots, r - 2) - 4r_W^2 \mathcal{F}_\delta(p, q - 1; \ldots, r - 2) + 4r_W^2 m_b^2\mathcal{F}_\delta(p, q; \ldots, r - 2) + \ldots
\]

11
\[
\begin{align*}
&\frac{r_W^2}{4} \mathcal{F}_\delta(p, q - 1; \ldots, r - 1) - \frac{r_W^2}{4} m_b^2 \mathcal{F}_\delta(p; \ldots, r - 1) + \\
&\frac{4}{p + \delta - 1} \left[ -2q \mathcal{F}_\delta(p - 1, q + 1; \ldots, r - 2) + \frac{q}{2} \mathcal{F}_\delta(p - 1, q + 1; \ldots, r - 1) + \\
&(2r - 5) \mathcal{F}_\delta(p - 1, q; \ldots, r - 3) - \frac{1}{2}(r - 2) \mathcal{F}_\delta(p - 1, q; \ldots, r - 2) \right] \quad (4.15)
\end{align*}
\]

Notice that when this recursion is used for \( p = 1 \), terms of the form \( \mathcal{F}_\delta(0, \ldots)/\delta \) are generated: as we already noticed in the introduction to the section, this forces us to compute purely bosonic integrals. Expanding \( \Delta \) gives:

\[
\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \sum_{r=p-k+1} \sum_{s=q-k} a_{rs}(m, \delta) \mathcal{F}_\delta(r, s) \quad (4.16)
\]

where \( k = (n_x + n_y + n_z + n_t) \). It is easy to see, from the structure of the recursion relations, that, the coefficients \( a_{rs}(m, \delta) \) have the properties mentioned at the beginning of the section.

As in the bosonic case we can simplify this expression if we consider the limit \( m \to 0 \). Let us first compute the divergent part of the integrals \( \mathcal{F}_\delta(p, q) \) (of course we must have \( p + q \geq 2 \)). Let us define \( \Delta D_F(k, m) \equiv D_F(k, m) - D_B(k, m) \). Then let us rewrite

\[
\mathcal{F}_\delta(p, q) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{D_F(k, m)^{p+\delta} D_B(k, m)^q} \\
- \frac{1}{D_B(k, m)^{p+q+\delta}} \sum_{l=0}^{p+q-2} \binom{-p - \delta}{l} \left( \frac{\Delta D_F(k, m)}{D_B(k, m)} \right)^l \\
+ \sum_{l=0}^{p+q-2} \binom{-p - \delta}{l} \int \frac{d^4k}{(2\pi)^4} \frac{\Delta D_F(k, m)^l}{D_B(k, m)^{p+q+l+\delta}} \quad (4.17)
\]

It is easy to see that the first term is finite. Thus if we want to compute the divergences of \( \mathcal{F}_\delta(p, q) \) we can limit ourselves to consider the second term which contains purely bosonic integrals. Expanding \( \Delta D_F(k, m)^l \) we see that we need to compute the divergent part of purely bosonic integrals \( \mathcal{B}_\delta(r; n_x, n_y, n_z, n_t) \). If the original integral in (4.17) has \( p > 0 \) we need only the \( \delta \)-independent divergent part, while for \( p \leq 0 \) also the terms of order \( \delta \) are needed. The computation of the divergent part of \( \mathcal{B}_\delta(r; n_x, n_y, n_z, n_t) \) can be done in different ways. One possibility is using the recursion relations of the previous section and the expression for the divergent part of \( \mathcal{B}_\delta(r) \) reported in (3.12). One can also attack the problem directly. Indeed if we define constants \( b_i(n_x, n_y, n_z, n_t) \) by the asymptotic expansion

\[
\prod_{i=x,y,z,t} \left[ \frac{d^n}{d\alpha^n_i} (e^{-\alpha} I_0(\alpha)) \right] = (-\alpha)^{-n_x-n_y-n_z-n_t} \sum_{i=0}^\infty \frac{b_i(n_x, n_y, n_z, n_t)}{\alpha^{i+2}} \quad (4.18)
\]
and \( q = r - n_x - n_y - n_z - n_t \), the divergent part of \( \mathcal{B}_\delta(r; n_x, n_y, n_z, n_t) \) (of course for \( q \geq 2 \)) is given by

\[
\sum_{i=2}^{q-1} b_{i-2}(n_x, n_y, n_z, n_t) \frac{\Gamma(q-i)}{2^{q-2} \Gamma(p)} \frac{1}{m^{2q-2i}} \left( 1 - \delta \log m^2 - \delta \sum_{k=q-i}^{p-1} \frac{1}{k} \right)
- b_{q-2}(n_x, n_y, n_z, n_t) \frac{1}{2^{q-2} \Gamma(p)} \log m^2 \left( 1 - \frac{\delta}{2} \log m^2 - \delta \sum_{k=1}^{p-1} \frac{1}{k} \right)
\] (4.19)

From this expression we immediately see that the divergent part of \( \mathcal{F}_\delta(p, q) \) has the generic form

\[
\mathcal{D}^{(1)}(m)(1 - \delta \log m^2) + \delta \mathcal{D}^{(2)}(m) + \log \text{terms} + O(\delta^2) \] (4.20)

where \( \mathcal{D}^{(1)}(m) \) and \( \mathcal{D}^{(2)}(m) \) are polynomials in \( 1/m^2 \) whose coefficients are rational numbers multiplied by \( 1/\pi^2 \) and “log terms” indicates terms which diverge as a power of \( \log m^2 \).

Exactly as in the bosonic case, the knowledge of the divergent part of \( \mathcal{F}_\delta(p, q) \) can be used to simplify (4.16): indeed whenever an integral is multiplied by the infrared regulator we can substitute it with its divergent part. Thus we can rewrite (4.16) as

\[
\mathcal{F}_\delta(p, q; n_x, n_y, n_z, n_t) = \sum_{r=p-k+1}^{p} \sum_{s=q-k}^{q+k} a_{rs}(0, \delta) \mathcal{F}_\delta(r, s) + \mathcal{R}(m, \delta) + O(m^2) \] (4.21)

If \( p > 0 \) and \( q \leq 0 \), as only terms with \( s \leq 0 \) can appear in (4.16), the only \( \mathcal{F}_\delta(r, s) \) that can contribute to \( \mathcal{R}(m, \delta) \) have \( r \geq 3 \). Since \( a_{rs}(m, \delta) \) is finite for \( \delta \to 0 \), we see that \( \mathcal{R}(m, \delta) \) is a polynomial in \( 1/m^2 \), finite for \( \delta \to 0 \). For \( p > 0 \) and \( q > 0 \) also \( \mathcal{F}_\delta(r, s) \) with \( r \leq 0 \) can contribute to \( \mathcal{R}(m, \delta) \). As \( a_{rs}(m, \delta) \) can behave as \( 1/\delta \) we have (see (4.20))

\[
\mathcal{R}(m, \delta) = \frac{1}{\delta} \left( 1 - \delta \log m^2 \right) \mathcal{R}^{(1)}(m) + \mathcal{R}^{(2)}(m) + O(\delta) \] (4.22)

where \( \mathcal{R}^{(1)}(m) \) and \( \mathcal{R}^{(2)}(m) \) are polynomials in \( 1/m^2 \) whose coefficients are rational numbers multiplied by \( 1/\pi^2 \).

For \( p \leq 0 \) and \( q \leq 0 \) it is easy to see that \( \mathcal{R}(m, \delta) = 0 \) as only finite integrals appear in the r.h.s. of (4.16) while for \( p \leq 0 \) and \( q > 0 \) we have

\[
\mathcal{R}(m, \delta) = \left( 1 - \delta \log m^2 \right) \mathcal{R}^{(1)}(m) + \delta \mathcal{R}^{(2)}(m) + O(\delta^2) \] (4.23)

Thus in all cases the function \( \mathcal{R}(m, \delta) \) has the form stated in the introduction to the section.

### 4.2 Second step: the identity \( \mathcal{I}_1(p, q) \)

We will obtain here a new set of recursion relations using the identity \( \mathcal{I}_1(p, q) \). Applying the previous recursion relations we can write each term in (4.24) as in (4.21) obtaining a non trivial relation of the form

\[
\sum_{r,s} f_{rs}(p, q; \delta) \mathcal{F}_\delta(r, s) + \mathcal{R}_\delta(p, q; m, \delta) = 0 \] (4.24)
where \( p - 4 \leq r \leq p \). Following our discussion of the bosonic case we will use (4.24) to obtain new recursion relations. Let us first notice that in (4.24) there is only one term with \( r = p - 4 \). It has \( s = q + 4 \) and

\[
f_{p-4,q+4}(p, q; \delta) = -\frac{32(q + 1)(q + 2)(q + 3)}{(p + \delta - 1)(p + \delta - 2)(p + \delta - 3)}
\]

(4.25)

Thus, if \( q \neq -1, -2, -3 \) we can solve (4.24) in terms of \( \mathcal{F}_\delta(p - 4, q + 4) \). Shifting \( p \to p + 4 \) and \( q \to q - 4 \) we can express \( \mathcal{F}_\delta(p, q) \), \( q \neq 1, 2, 3 \) in terms of \( \mathcal{F}_\delta(r, s) \) with \( p + 1 \leq r \leq p + 4 \). We can then use this relation to express all integrals \( \mathcal{F}_\delta(p, q) \), \( p \leq -1 \), \( q \neq 1, 2, 3 \) in terms of \( \mathcal{F}_\delta(r, s) \) with either \( 0 \leq r \leq 3 \), \( s \) arbitrary or \( 1 \leq s \leq 3 \) and \( p \leq -1 \).

Two observations are in order:

1. A careful analysis of the recursion shows that in the result the coefficients of \( \mathcal{F}_\delta(r, s) \) with \( r = 1, 2, 3 \) are of order \( \delta \). This property is very important: indeed it guarantees that when substituting these expressions in (4.21) the coefficients of \( \mathcal{F}_\delta(p, q) \) with positive \( p \) are finite for \( \delta \to 0 \). This property would not be true if we were using the relation to eliminate also \( \mathcal{F}_\delta(r, s) \) with \( r \geq 0 \).

2. If we are considering \( \mathcal{F}_\delta(p, q) \) with \( q \leq 0 \) then the result is expressed only in terms of \( \mathcal{F}_\delta(r, s) \) with \( 0 \leq r \leq 3 \) and \( s \leq 0 \).

We could also try to use the same identity to obtain recursion relations which express \( \mathcal{F}_\delta(p, q) \) in terms of \( \mathcal{F}_\delta(r, s) \) with \( r < p \). To do this we should try to solve the identity for the \( \mathcal{F}_\delta(r, s) \) with the highest value of \( r \), namely \( r = p \). However in this case there are three terms with \( r = p \), namely \( \mathcal{F}_\delta(p, q - 2) \), \( \mathcal{F}_\delta(p, q - 1) \) and \( \mathcal{F}_\delta(p, q) \) and thus we cannot obtain a recursion which decreases the value of \( p \). We will thus proceed in a different way. We will solve the identity for \( \mathcal{F}_\delta(p, q - 2) \), shifting \( q \to q + 2 \). This is always possible as \( f_{p,q-2}(p, q; \delta) = -1 \). In this way we obtain a recursion relation which expresses \( \mathcal{F}_\delta(p, q) \) in terms of \( \mathcal{F}_\delta(r, s) \) with \( r < p \) and \( \mathcal{F}_\delta(p, s) \) with \( s > q \). We will use this recursion to eliminate recursively all the integrals with \( q \leq -3 \) and \( p \geq 4 \). The choice of stopping at \( q = -3 \) guarantees that only integrals \( \mathcal{F}_\delta(r, s) \) with \( s \leq 0 \) are generated.

A third recursion relation can finally be obtained by solving (4.24) in terms of \( \mathcal{F}_\delta(p, q) \). This is also always possible as \( f_{p,q}(p, q; \delta) = 256 \). In this way we can eliminate all integrals \( \mathcal{F}_\delta(p, q) \) with \( p \geq 4 \) and \( q > 0 \).

In conclusion, using the identity \( T_1(p, q) \) we can rewrite every \( \mathcal{F}_\delta(p, q) \) as

\[
\mathcal{F}_\delta(p, q) = \sum_{rs} c_{rs}(p, q; \delta) \mathcal{F}_\delta(r, s) + S(p, q; m, \delta)
\]

(4.26)

where in the sum we have either \( 0 \leq r \leq 3 \), \( s \) arbitrary, or \( r \leq -1 \) and \( s = 1, 2, 3 \) or \( r \geq 4 \) and \( s = -2, -1, 0 \). It is easy to see that the properties mentioned at the beginning of the section are satisfied by \( c_{rs}(p, q; \delta) \) and \( S(p, q; m, \delta) \).
4.3 Third step: the identity $\mathcal{I}_2(p, q)$

Let us now obtain a new set of recursion relations which allow to reduce the remaining $\mathcal{F}_\delta(p, q)$ in terms of a finite set of integrals. We will use here the second identity $\mathcal{I}_2(p, q)$. We will discuss separately four different regions:

1. $q \leq 0$, $0 \leq p \leq 3$;
2. $q > 0$, $0 \leq p \leq 3$;
3. $q = 1, 2, 3$, $p \leq -1$;
4. $q = -2, -1, 0$, $p \geq 4$.

4.3.1 The strip $q \leq 0$, $0 \leq p \leq 3$

Let us first consider the integrals $\mathcal{F}_\delta(p, q)$ with $q \leq 0$ and $0 \leq p \leq 3$.

We start from $\mathcal{I}_2(2, q + 4)$. We can use the previous relations to obtain an identity which involves only $\mathcal{F}_\delta(r, s)$ with $0 \leq r \leq 3$. If $q \leq -5$ we can solve it for $\mathcal{F}_\delta(3, q)$ expressing it in terms of $\mathcal{F}_\delta(r, s)$ with $r \leq 2$ or $r = 3$ and $s > q$. Notice that by stopping at $q = -5$ all integrals are expressed in terms of $\mathcal{F}_\delta(r, s)$ with $s \leq 0$.

We want now to obtain a relation for $\mathcal{F}_\delta(2, q)$. In this case we start from $\mathcal{I}_2(1, q + 4)$. Then we can use the relations of the first two steps to obtain a recursion relation which involves only $\mathcal{F}_\delta(r, s)$, $0 \leq r \leq 3$. Then we use the previous relation to eliminate $\mathcal{F}_\delta(3, q - 2)$, $\mathcal{F}_\delta(3, q - 1)$ and $\mathcal{F}_\delta(3, q)$. At the same time also $\mathcal{F}_\delta(2, q - 2)$ and $\mathcal{F}_\delta(2, q - 1)$ cancel. If $q \leq -5$, this relation can then be solved in terms of $\mathcal{F}_\delta(2, q)$, the result containing only $\mathcal{F}_\delta(r, s)$ with $s \leq 0$.

In a completely analogous way we can derive recursions for $\mathcal{F}_\delta(1, q)$ and $\mathcal{F}_\delta(0, q)$. In the first case we start from $\mathcal{I}_2(0, q + 6)$. We first apply the step-one and step-two relations, then use the previous relations to eliminate $\mathcal{F}_\delta(3, q - 2)$, $\mathcal{F}_\delta(3, q - 1)$, $\mathcal{F}_\delta(3, q)$ and $\mathcal{F}_\delta(2, q)$. If $q \leq -5$ we solve for $\mathcal{F}_\delta(1, q)$. Finally starting from $\mathcal{I}_2(-1, q + 6)$, eliminating $\mathcal{F}_\delta(3, q - 4)$, $\mathcal{F}_\delta(3, q - 3)$, $\mathcal{F}_\delta(3, q - 2)$, $\mathcal{F}_\delta(2, q - 2)$ and $\mathcal{F}_\delta(1, q - 2)$ we get a relation for $\mathcal{F}_\delta(0, q)$ valid for $q \leq -5$.

Using iteratively these four relations we are now able to express every $\mathcal{F}_\delta(p, q)$, $0 \leq p \leq 3$, $q \leq -5$, in terms of $\mathcal{F}_\delta(r, s)$ with $0 \leq r \leq 3$, $-4 \leq s \leq 0$.

4.3.2 The strip $q > 0$, $0 \leq p \leq 3$

Here we want to obtain relations analogous to the previous ones but which decrease the value of $q$. The procedure is identical to the previous one.

We first consider $\mathcal{I}_2(2, q + 1)$ and we use the step-one and step-two relations to obtain an identity involving only $\mathcal{F}_\delta(r, s)$, $0 \leq r \leq 3$, then we solve for $\mathcal{F}_\delta(3, q)$. We use this relation for $q > 0$.

Analogously from $\mathcal{I}_2(1, q + 1)$ and $\mathcal{I}_2(0, q + 1)$ we get relations for $\mathcal{F}_\delta(2, q)$ and $\mathcal{F}_\delta(1, q)$ respectively valid for $q > 2$ and $q > 4$. Finally we consider $\mathcal{I}_2(-1, q + 1)$: we apply the step-one and step-two substitutions, then we use the previous relations to eliminate $\mathcal{F}_\delta(3, q - 3)$, $\mathcal{F}_\delta(2, q - 2)$ and $\mathcal{F}_\delta(1, q - 1)$ and finally solve for $\mathcal{F}_\delta(0, q)$. This
relation is valid for \( q > 6 \). In this way we express all \( \mathcal{F}_\delta(p,q) \), \( 0 \leq p \leq 3 \), \( q > 0 \) in terms of \( \mathcal{F}_\delta(r,s) \), \( 0 \leq r \leq 3 \), \( s \leq 2(3-r) \).

4.3.3 The strip \( p \leq -1 \), \( q = 1,2,3 \)

Here we consider \( \mathcal{I}_2(p+3,1) \), apply the step-one and step-two relations to obtain an identity involving only \( \mathcal{F}_\delta(r,s) \) with \( s \leq 3 \), then we solve for \( \mathcal{F}_\delta(p,3) \). This relation is valid for \( p \leq -1 \). In the same way starting from \( \mathcal{I}_2(p+2,2) \) we get a relation for \( \mathcal{F}_\delta(p,2) \), \( p \leq -2 \). Finally starting from \( \mathcal{I}_2(p+1,3) \), after eliminating \( \mathcal{F}_\delta(p-1,3) \), we get an identity for \( \mathcal{F}_\delta(p,1) \), \( p \leq -1 \). Thus all integrals in this region but \( \mathcal{F}_\delta(-1,2) \) can be rewritten in terms of \( \mathcal{F}_\delta(r,s) \), \( r \geq 0 \).

4.3.4 The strip \( p \geq 4 \), \( q = -2, -1, 0 \)

Here we start from \( \mathcal{I}_2(p-1,0) \). We apply the step-one relations and then the step-two relations to eliminate \( \mathcal{F}_\delta(p,-4) \) and \( \mathcal{F}_\delta(p,-3) \), then we solve for \( \mathcal{F}_\delta(p,-2) \). This relation is valid for all \( p \geq 4 \). Analogously, starting from \( \mathcal{I}_2(p-1,-1) \), eliminating \( \mathcal{F}_\delta(p,r) \), \( -5 \leq r \leq -2 \), we get a relation for \( \mathcal{F}_\delta(p-1), p \geq 4 \). Finally starting from \( \mathcal{I}_2(p,-2) \), eliminating \( \mathcal{F}_\delta(p+1,r) \), \( -6 \leq r \leq -1 \) and \( \mathcal{F}_\delta(p,r) \), \( -4 \leq r \leq -1 \) we get a relation for \( \mathcal{F}_\delta(p,0), p \geq 4 \). In this way all \( \mathcal{F}_\delta(p,q) \) in this strip get rewritten in terms of \( \mathcal{F}_\delta(r,s), 0 \leq r \leq 3, s \leq 0 \).

4.4 The fourth step: the last relations

In the preceding step we showed that all integrals can be rewritten in terms of a finite number of them. However we have not used the identities \( \mathcal{I}_1(p,q) \) and \( \mathcal{I}_2(p,q) \) for all possible values of \( p \) and \( q \). For instance we never used \( \mathcal{I}_1(p,q) \) for \( p \leq 3 \) and \( q = -1, -2, -3 \) or \( p \geq 4 \) and \( q = 0 \); much larger is the number of cases where the second identity has not been used. We thus checked systematically if there were other values of \( p, q \) for which the two identities were not trivially satisfied, thus providing relations which could be used to further decrease the number of independent integrals.

Using \( \mathcal{I}_1(p,q) \) with \( (p,q) \) getting the values \( (3, -1), (2, -1), (1, -1), (0, -1), (3, -2), (2, -2), (1, -2), (1, -3) \) and \( \mathcal{I}_2(2,0), \mathcal{I}_2(3,-2) \) we obtain relations for \( \mathcal{F}_\delta(3,[0,-1]), \mathcal{F}_\delta(2,[-3,-4]), \mathcal{F}_\delta(1,[-3,-4]) \) and \( \mathcal{F}_\delta(0,[-1,-2,-3,-4]) \). These relations are reported in the appendix. Notice that each integral is expressed in terms of \( \mathcal{F}_\delta(r,s) \) with \( s \leq 0 \). We stress that the identities we used to derive the relations were chosen arbitrarily; other choices could have been equally used. However, once these relations have been computed we have found that \( \mathcal{I}_1(p,q) \), (resp. \( \mathcal{I}_2(p,q) \)), is identically satisfied for all values of \( q \) and for all \( p < 0 \) (resp. \( p \leq 0 \)).

In a completely analogous way we have found that the identities \( \mathcal{I}_1(4,0) \) and \( \mathcal{I}_2(p,q) \) with \( (p,q) \) getting the values \( (3,2), (3,3), (1,2), (1,3), (0,4), (0,5) \) are not yet satisfied. We have used them to get relations for \( \mathcal{F}_\delta(2,[1,2]), \mathcal{F}_\delta(1,[3,4]), \mathcal{F}_\delta(0,[4,5,6]) \). They are reported in the appendix.

Collecting everything together we get

\[
\mathcal{F}_\delta(p,q;n_x, n_y, n, n_t) = \sum_{r,s} d_{rs}(\delta) \mathcal{F}_\delta(r,s) + \mathcal{T}(m, \delta) \tag{4.27}
\]
where \( d_{rs}(\delta) \) and \( T(p, q; m, \delta) \) have the form explained at the beginning of the section.

Now, if \( p > 0 \) and \( q \leq 0 \) the second sum extends over ten values of \((r, s)\), i.e. \((0, 0)\), \((1, t)\), \((2, t)\) and \((3, t + 2)\) with \(0 \leq t \leq 2\). Let us consider the limit \( \delta \to 0 \). The polynomial \( T(p, q; m, \delta) \) is finite in this limit and the same is true for all \( d_{rs}(p, q; \delta) \) with \( r > 0 \). Thus the only coefficient which could have a \( 1/\delta \) divergence is \( d_{00}(p, q; \delta) \).

However the result is finite for \( \delta \to 0 \) and thus also this coefficient is finite in this limit. We can thus set simply \( \delta = 0 \) and use the fact that \( F(0, 0) = 1 + O(\delta) \) to get

\[
F(p, q; n_x, n_y, n_t) = \sum_{r > 0; s} d_{rs}(p, q; 0) F(r, s) + d_{00}(p, q; 0) + T(p, q; m, 0) \tag{4.28}
\]

Finally let us consider the case \( p > 0 \) and \( q > 0 \). In this case in the sum on the r.h.s of (4.27) we can also have \( F(−1, 2), F(0, 1), F(0, 2), F(0, 3), F(1, 1) \) and \( F(1, 2) \); \( T(p, q; m, \delta) \) has the form

\[
1/\delta T^{(1)}(p, q; m)(1 − \delta \log m^2) + T^{(2)}(p, q; m) + O(\delta) \tag{4.30}
\]

where \( T^{(1)}(p, q; m) \) and \( T^{(2)}(p, q; m) \) are polynomials in \( 1/m^2 \). Let us consider the limit \( \delta \to 0 \). The only coefficients that may behave as \( 1/\delta \) are those with \( r \leq 0 \). Writing in this case

\[
d_{rs}(\delta) = \frac{1}{\delta} d^{(1)}_{rs} + d^{(2)}_{rs} + O(\delta) \tag{4.31}
\]

the cancellation of the \( 1/\delta \) terms gives the equation

\[
d^{(1)}_{03} B(3) + d^{(1)}_{02} B(2) + (d^{(1)}_{01} + 2d^{(1)}_{−1,2}) B(1) + d^{(1)}_{00} + T^{(1)}(m) = 0 \tag{4.32}
\]
where we have used \( F_\delta(0, q) = B(q) + O(\delta) \) and
\[
F_\delta(-1, 2) = B(1) - B(2; 2) + 1/4 + O(\delta) = 2B(1) + O(\delta)
\] (4.33)

In all cases we have found that (4.32) is satisfied in a trivial way, i.e.
\[
d_{03}^{(1)} = d_{02}^{(1)} = d_{00}^{(1)} = 0
\]
\[
d_{01}^{(1)} + 2d_{-1,2}^{(1)} = 0
\]
\[
T^{(1)}(m) = 0
\] (4.34)

This is not completely surprising: indeed cancellation of the terms divergent for \( m \rightarrow 0 \) requires
\[
d_{03}^{(1)} = -8d_{02}^{(1)} \quad \text{and} \quad T^{(1)}(m) = -d_{03}^{(1)} 32 \pi^2 m^2 + t \pi^2
\] (4.35)

where \( t \) is a rational number. Then (4.32) becomes
\[
d_{01}^{(1)} F_\delta(0, 1) + d_{-1,2} F_\delta(-1, 2) = \frac{1}{2} d_{01}^{(1)} (2 F_\delta(0, 1) - F_\delta(-1, 2)) + (2 d_{-1,2} + d_{01}) B(1) + O(\delta)
\] (4.37)

showing that \( F_\delta(0, 1) \) and \( F_\delta(-1, 2) \) appear in the result only in the fixed combination \( 2 F_\delta(0, 1) - F_\delta(-1, 2) \). In conclusion, for \( q > 0 \) the result beside the nine integrals which appear for \( q \leq 0 \) contains also the bosonic constants \( Z_0, Z_1, F_0 - \gamma_E \) and the integrals \( F(1, 1), F(1, 2) \) and \( \lim_{\delta \rightarrow 0} (2 F_\delta(0, 1) - F_\delta(-1, 2)) / \delta \). As in the bosonic case, instead of these three quantities we have parameterized our results in terms of three infrared-finite integrals. We introduce:
\[
Y_1 = \frac{1}{8} F(1, 1; 1, 1, 1)
\]
\[
Y_2 = \frac{1}{16} F(1, 1; 1, 1, 1, 1)
\]
\[
Y_3 = \frac{1}{16} F(1, 2; 1, 1, 1)
\] (4.38)

Their numerical values are reported in Table 2. The relation with the original integrals is:
\[
\lim_{\delta \rightarrow 0} \frac{1}{\delta} (2 F_\delta(0, 1) - F_\delta(-1, 2)) = \frac{1}{4} - 12 Y_1 - 3 Z_0 + 2 F(1, 0)
\] (4.39)
\[ F(1, 1) = -\frac{1}{16\pi^2} (\log m^2 + \gamma_E - F_0) - \frac{1}{192} + \frac{1}{16\pi^2} + Y_0 - \frac{1}{4} Y_1 + \frac{1}{16} Y_2 \]
\[ + \frac{1}{768} F(1, -2) + \frac{1}{192} F(1, -1) + \frac{59}{192} F(1, 0) + \frac{1}{768} F(2, -2) - \frac{25}{48} F(2, -1) \] (4.40)
\[ F(1, 2) = \frac{1}{32\pi^2 m^2} - \frac{19}{12288} + \frac{307}{18432\pi^2} + \frac{3}{64} Y_0 - \frac{19}{256} Y_1 + \frac{19}{1024} Y_2 + \frac{1}{8} Y_3 \]
\[ + \frac{1}{768} Z_0 + \frac{19}{49152} F(1, -2) + \frac{53}{36864} F(1, -1) + \frac{187}{4096} F(1, 0) \]
\[ + \frac{5497}{147456} F(2, -2) - \frac{293}{9216} F(2, -1) - \frac{35}{6144} F(3, -4) - \frac{19}{512} F(3, -3) \]
\[ - \frac{173}{2304} F(3, -2) \] (4.41)

As in the bosonic case, let us notice that \( \log m^2 \) appears always in the fixed combination \( \log m^2 + \gamma_E - F_0 \). Thus in finite integrals \( F_0 - \gamma_E \) does not appear.

To conclude this section we want to add a few remarks on the numerical evaluation of the constants appearing in Table 2. A direct evaluation of the integrals does not provide accurate results: we have thus used a different procedure inspired by the work of [4]. To evaluate the constants \( F(p, q) \) with \( q \leq 0 \) we have considered the integrals \( J_q = F(1, -q) \) with \( 6 \leq q \leq 13 \). To compute them we have first calculated the sums

\[ J_{q,L} = \frac{1}{L^4} \sum_k \frac{D_B(k, 0)^q}{D_F(k, 0)} \] (4.42)

where \( k \) runs over the points \( k = (2\pi/L)(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}, n_4 + \frac{1}{2}), 0 \leq n_i < L \) for various values of \( L \) between 50 and 100. Then we tried to extrapolate \( J_{q,L} \) using the form

\[ J_{q,L} = J_q \left( 1 + \frac{a}{L^{2q+2}} \right) \] (4.43)

In all cases the correction turned out to be completely negligible. In practice \( J_q \) could be determined with a relative error \( \lesssim 10^{-25} \). Then each \( J_q \) was expressed in terms of the basic constants. We obtained in this way 8 equations which were solved giving the results of Table 2.

Analogously to compute \( Y_0, Y_1, Y_2 \) and \( Y_3 \), we computed numerically \( F(1, 1, 8), F(2, 1, 9), F(3, 1, 10) \) and \( F(5, 2, 11) \) and then solved the corresponding equations.

### 5 Integrals in other infrared regularizations

In the preceding two sections we have discussed the computation of bosonic and mixed bosonic-fermionic integrals using as infrared regulator a mass \( m \). Here we want to discuss other types of infrared regularization: first we will consider the dimensional regularization [8] and then we will consider mixed bosonic-fermionic integrals with the exact Wilson-fermion propagator (4.3).
5.1 Dimensional regularization

In this case we consider the integrals

\[
B^{DR}(p; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^d} \frac{k_{2n_x} k_{2n_y} k_{2n_z} k_{2n_t}}{D_B(k, 0)^p} \tag{5.1}
\]

\[
F^{DR}(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^d} \frac{k_{2n_x} k_{2n_y} k_{2n_z} k_{2n_t}}{D_F(k, 0)^p D_B(k, 0)^q} \tag{5.2}
\]

It is easy to see that the basic recursion relations (3.4), (3.7), (4.10), (4.13), (4.15) can be easily generalized to dimensionally-regularized integrals. The relation (3.16) and (1.3), (1.6) are instead intrinsically four-dimensional identities. The reason we have made the computation using a mass as a regulator. Now, we will show how to compute (5.1) and (5.2) from their mass-regularized counterparts.

Let us begin with the bosonic case, considering \( B(p; n_x, n_y, n_z, n_t) \). If \( q \equiv p - n_x - n_y - n_z - n_t < 2 \) the integral is finite and thus independent of the infrared regulator. For \( q \geq 2 \) let us rewrite

\[
B(p; n_x, n_y, n_z, n_t) = \frac{1}{2p\Gamma(p)} \int_0^\infty d\alpha \alpha^{p-1} e^{-\alpha m^2/2} \prod_{i = \{x,y,z,t\}} (-2)^{n_i} \frac{d^{n_i}}{d\alpha^{n_i}} (e^{-\alpha} I_0(\alpha)) \tag{5.3}
\]

Then define

\[
F(p; n_x, n_y, n_z, n_t) = \frac{1}{2p\Gamma(p)} \int_0^1 d\alpha \alpha^{p-1} \prod_{i = \{x,y,z,t\}} (-2)^{n_i} \frac{d^{n_i}}{d\alpha^{n_i}} (e^{-\alpha} I_0(\alpha))
\]

\[
+ \frac{1}{2p\Gamma(p)} \int_1^\infty d\alpha \alpha^{p-1} \left\{ \prod_{i = \{x,y,z,t\}} (-2)^{n_i} \frac{d^{n_i}}{d\alpha^{n_i}} (e^{-\alpha} I_0(\alpha)) \right\}
\]

\[
- \left( \frac{2}{\alpha} \right)^{n_x + n_y + n_z + n_t} \sum_{i=0}^{q-2} \frac{b_i(n_x, n_y, n_z, n_t)}{\alpha^{i+2}} \tag{5.4}
\]

where \( b_i(n_x, n_y, n_z, n_t) \) are defined by (4.18). Then for \( m^2 \to 0 \) we have

\[
B(p; n_x, n_y, n_z, n_t) = F(p; n_x, n_y, n_z, n_t)
\]

\[
+ \frac{1}{\Gamma(p)} \sum_{i=2}^{q-1} b_{i-2}(n_x, n_y, n_z, n_t) \left[ \frac{\Gamma(q-i)}{2^i} \frac{1}{m^{2q-2i}} - \frac{1}{2^i} \frac{1}{q-i} \right]
\]

\[
- \frac{1}{2q\Gamma(p)} b_{q-2}(n_x, n_y, n_z, n_t) \left( \log \frac{m^2}{2} + \gamma_E \right) \tag{5.5}
\]

Analogously we have

\[
B^{DR}(p; n_x, n_y, n_z, n_t) = \frac{1}{2p\Gamma(p)} \int_0^\infty d\alpha \alpha^{p-1} \left( e^{-\alpha} I_0(\alpha) \right)^{d-4} \prod_{i = \{x,y,z,t\}} (-2)^{n_i} \frac{d^{n_i}}{d\alpha^{n_i}} (e^{-\alpha} I_0(\alpha)) \tag{5.6}
\]
then, by comparison, we have

\[- \frac{2}{\epsilon} \frac{1}{2^{\Gamma(p)}} b_q(n_x, n_y, n_z, n_t) \left( 1 + \epsilon c_q(n_x, n_y, n_z, n_t) \right) + F(p; n_x, n_y, n_z, n_t) \quad (5.7)\]

where \( c_i(n_x, n_y, n_z, n_t) \) are defined by

\[
\prod_{i=x,y,z,t} \left[ (\alpha)^{n_i} \frac{d^{m_i}}{d\alpha^{n_i}} (e^{-\alpha I_0(\alpha)}) \right] \log(e^{-\alpha I_0(\alpha)} \sqrt{\alpha})
\]

\[- \sum_{i=0}^{\infty} \frac{1}{\alpha^{i+2}} b_i(n_x, n_y, n_z, n_t)c_i(n_x, n_y, n_z, n_t) \quad (5.8)\]

Then, by comparison, we have

\[ B^{DR}(p; n_x, n_y, n_z, n_t) = B(p; n_x, n_y, n_z, n_t) \]

\[- \frac{1}{\Gamma(p)} \sum_{i=2}^{q-1} b_{i-2}(n_x, n_y, n_z, n_t) \left( \frac{\Gamma(q-i)}{2^i m^{2q-2i}} - \frac{1}{2^q (q-i)} \right) \]

\[- \frac{1}{2\Gamma(p)} b_q(n_x, n_y, n_z, n_t) \left( -\log m^2 - \gamma_E + \frac{2}{\epsilon} + 2c_q(n_x, n_y, n_z, n_t) \right) \quad (5.9)\]

A simplification occurs if the integral is logarithmically infrared divergent, i.e. if \( q = 2 \). In this case, as \( c_0(n_x, n_y, n_z, n_t) = \frac{1}{2} \log 2\pi \), we have

\[ B^{DR}(p; n_x, n_y, n_z, n_t) = B(p; n_x, n_y, n_z, n_t) \]

\[- \frac{1}{4\Gamma(p)} b_0(n_x, n_y, n_z, n_t) \left[ -\log m^2 - \gamma_E + \frac{2}{\epsilon} + \log 4\pi \right] \quad (5.10)\]

Thus, for these integrals, we can use a very simple recipe to go from the mass regularization to the dimensional one: simply substitute in each integral \( \log m^2 + \gamma_E \) with \( 2/\epsilon + \log 4\pi \).

Let us now consider the fermionic case. Again we should consider only the case \( Q \equiv p + q - n_x - n_y - n_z - n_t \geq 2 \). Then let us rewrite

\[ F^{DR}(p, q; n_x, n_y, n_z, n_t) = \int \frac{d^d k}{(2\pi)^d} \left[ \frac{\hat{k}_x^{2q+2n} \hat{k}_y^{2n} \hat{k}_z^{2n} \hat{k}_t^{2m}}{D_F(k, 0)^p D_B(k, 0)^q} \right. \]

\[ - \frac{\hat{k}_x^{2q+2n} \hat{k}_y^{2n} \hat{k}_z^{2n} \hat{k}_t^{2m}}{D_B(k, 0)^{p+q}} \sum_{l=0}^{Q-2} \left( \frac{-p}{l} \right) \left( \frac{\Delta D_F(k, 0)}{D_B(k, 0)} \right)^l \]

\[ + \sum_{l=0}^{Q-2} \left( \frac{-p}{l} \right) \int \frac{d^d k}{(2\pi)^d} \left[ \frac{\hat{k}_x^{2q+2n} \hat{k}_y^{2n} \hat{k}_z^{2n} \hat{k}_t^{2m}}{D_B(k, 0)^{p+q+l}} \Delta D_F(k, 0)^l \right. \]

\[ \left. \frac{\Delta D_F(k, 0)^l}{D_B(k, 0)^{p+q+l}} \right] \quad (5.11)\]

where \( \Delta D_F(k, m) \equiv D_F(k, 0) - D_B(k, m) \). The first integral in the is clearly finite. We can thus set \( \epsilon = 0 \). Then it is easy to see that we can add everywhere a mass, i.e. substitute \( D_B(k, 0) \) with \( D_B(k, m) \) and analogously for \( D_F(k, 0) \), without changing
its value in the limit \( m \to 0 \). Thus we get finally

\[
\mathcal{F}^{DR}(p, q; n_x, n_y, n_z, n_t) = \mathcal{F}(p, q; n_x, n_y, n_z, n_t) + \sum_{l=0}^{Q-2} \left( -p \right)^l \left\{ \int \frac{d^4k}{(2\pi)^4} \hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t} \Delta D_F(k, 0)^l \right\} D_B(k, 0)^{p+q+l}
\]

\[
- \int \frac{d^4k}{(2\pi)^4} \hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t} \Delta D_F(k, m)^l \right\} D_B(k, m)^{p+q+l}
\]

(5.12)

In the last term purely bosonic integrals are involved and we have already discussed how to compute the difference between their value in the two regularizations. Notice that, as in the bosonic case, logarithmically divergent integrals can be dealt with easily: simply substitute \((2/\varepsilon + \log 4\pi)\) to \((\log m^2 + \gamma_E)\).

### 5.2 Massive Wilson-fermion-propagator integrals

Here we want to consider integrals of the form

\[
\hat{F}(p, q; n_x, n_y, n_z, n_t) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t} \hat{D}_F(k, m)^p \hat{D}_B(k, m)^q
\]

(5.13)

where \( \hat{D}_F(k, m) \) is defined in (4.3). When \( Q = p + q - n_x - n_y - n_z - n_t \geq 2 \) the integrals diverge for \( m \to 0 \). We will now relate them to \( F \). Indeed we can rewrite

\[
\hat{F}(p, q; n_x, n_y, n_z, n_t) = F(p, q; n_x, n_y, n_z, n_t) + \sum_{l=1}^{2Q-4} \left( -p \right)^l \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \hat{k}_x^{2n_x} \hat{k}_y^{2n_y} \hat{k}_z^{2n_z} \hat{k}_t^{2n_t} \Delta \hat{D}_F(k, m)^l \right\} D_B(k, m)^{p+q+l} + O(m)
\]

(5.14)

where \( \Delta \hat{D}_F(k, m) \equiv \hat{D}_F(k, m) - D_F(k, m) = m\hat{k}^2 \).

Let us notice that if the integral is logarithmically divergent \( (Q = 2) \), for \( m \to 0 \) we have \( \hat{F} = F \). For integrals with \( Q > 2 \) we see from the explicit expression (5.14) the the divergent part is now a polynomial in \( 1/m \) instead of \( 1/m^2 \). For this reason the expression for \( \hat{F} \) are in general more cumbersome than those involving \( F \) and this is why we have studied integrals with (4.2) instead of (4.3).

### 6 Applications

In this section, as an application of our method we will give a few examples. In the first subsection we will report analytic expressions for various renormalization constants whose value is reported in the literature only in numerical form or is expressed in terms of cumbersome integrals. Then we will briefly discuss the computation of the fermionic propagator.
6.1 Analytic expressions

6.1.1 Fermionic self-energy

We want to give here the expression for the fermionic self-energy. The first computation for the Wilson action in Feynman gauge was given in [11] and it was subsequently corrected in [12]. The fermionic self-energy at one loop has the generic form

\[
\Sigma^{\text{LAT}}(p^2, m^2) = g^2 N_f^2 - \frac{1}{2N} \left( \delta m + i \not{p} \Sigma_1(p^2, m^2) + m \Sigma_2(p^2, m^2) \right) \quad (6.1)
\]

For \( r_W = 1 \), in Feynman gauge, in terms of our basic integrals we have

\[
\delta m = -Z_0 - 2F(1, 0) \approx -0.3257141174 \quad (6.2)
\]

\[
\Sigma_1(p^2, m^2) = \frac{1}{16\pi^2} (2G(p^2 a^2, m^2 a^2) + \gamma_E - F_0) + \frac{1}{8} Z_0 + \frac{1}{192} - \frac{1}{32\pi^2} - Y_0 + \frac{1}{4} Y_1 - \frac{1}{16} Y_2 + 12 Y_3 - \frac{1}{768} F(1, 2) - \frac{1}{192} F(1, -1) + \frac{109}{192} F(1, 0)
\]

\[
\approx \frac{1}{8\pi^2} G(p^2 a^2, m^2 a^2) + 0.0877213749 \quad (6.3)
\]

\[
\Sigma_2(p^2, m^2) = \frac{1}{4\pi^2} (F(p^2 a^2, m^2 a^2) + \gamma_E - F_0) + \frac{1}{48} - \frac{1}{4\pi^2} - 4Y_0 + Y_1 - \frac{1}{4} Y_2 - \frac{1}{192} F(1, -2) - \frac{1}{48} F(1, -1) - \frac{83}{48} F(1, 0) - \frac{1}{192} F(2, -2) + \frac{49}{12} F(2, -1) \approx \frac{1}{4\pi^2} F(p^2 a^2, m^2 a^2) + 0.0120318529 \quad (6.4)
\]

where

\[
F(p^2 a^2, m^2 a^2) = \int_0^1 dx \log[(1 - x)(p^2 x + m^2 a^2)] \quad (6.5)
\]

\[
G(p^2 a^2, m^2 a^2) = \int_0^1 dx x \log[(1 - x)(p^2 x + m^2 a^2)] \quad (6.6)
\]

6.1.2 Gluonic self-energy

Let us now consider the gluonic self-energy which was firstly computed in [13, 8]. The contribution of the fermions, for \( r_W = 1 \), can be expressed in terms of our basic integrals as

\[
\Pi_{\mu
u}^f(p, m) = \frac{N_f}{2} \frac{g^2}{\sqrt{2\pi}} \left[ \frac{1}{12\pi^2} \left( 6H(p^2 a^2, m^2 a^2) + \gamma_E - \log 4\pi \right) + L \right] \quad (6.7)
\]

where \( N_f \) is the number of fermions which are in the fundamental representation of \( SU(N) \), \( m \) the fermion mass (for simplicity we assume all fermions to have the same

\[ \text{footnote}^5 \text{ Notice however that formula (3.15) in } [12] \text{ contains a misprint: the correct result is given in formula (10b) of } [13]. \]
\[ L = -\frac{1}{9} - \frac{1}{12\pi^2}(F_0 - \log 4\pi) - \frac{4}{3}Y_0 + \frac{1}{36}F(1, -2) \\
+ \frac{1}{18}F(1, -1) - \frac{7}{6}F(1, 0) + \frac{5}{24}F(2, -2) + \frac{2}{3}F(2, -1) \] (6.8)

and

\[
H(p^2 a^2, m^2 a^2) = \int_0^1 dx x(1-x) \log[x(1-x)p^2 a^2 + m^2 a^2] 
\] (6.9)

Numerically \( L \approx 0.0031048512 \) in agreement with [8].

For \( p \to 0 \) we have

\[ \Pi_f(p, m) = N_f \frac{g^2(p^2 \delta_{\mu\nu} - p_\mu p_\nu)}{12\pi^2} \left[ \frac{1}{6} \log m^2 - 0.013391999 + O(p^2) \right] \] (6.10)

while for \( m = 0 \) we have

\[ \Pi_f(p, 0) = N_f \frac{g^2(p^2 \delta_{\mu\nu} - p_\mu p_\nu)}{12\pi^2} \left[ \frac{1}{6} \log p^2 - 0.027464385 \right] \] (6.11)

### 6.1.3 Renormalization constants for bilinear fermion operators with the clover action

We want to give here the renormalization constants of bilinear fermion operators with the clover action [3, 14], using the explicit expressions in terms of lattice integrals of [13]. We define local operators

\[ O^{LATT,loc}(x) = \overline{\psi}(x) \Gamma \psi(x) \] (6.12)

and improved operators

\[ O^{LATT,imp}(x) = \overline{\psi}(x) \Gamma \psi(x) + \frac{r_W}{2} \sum_\mu \left( D_\mu \overline{\psi}(x) \gamma_\mu \Gamma \psi(x) - \overline{\psi}(x) \Gamma \gamma_\mu D_\mu \psi(x) \right) \] (6.13)

where \( \Gamma \) is a Dirac matrix and

\[ D_\mu \psi(x) = \frac{1}{2} \left[ U_\mu(x) \psi(x + \mu) - U_\mu^+(x - \mu) \psi(x - \mu) \right] \] (6.14)

For each operator we compute a finite renormalization constant \( Z \) such that

\[ \langle f | O^{CONT}(x) | i \rangle = Z \langle f | O^{LATT}(x) | i \rangle \] (6.15)

where \( f \) and \( i \) are arbitrary external states. In the continuum we adopt the \( \overline{MS} \)-scheme with scale \( \mu = 1/a \). We write at one loop

\[ Z = 1 + g^2 \frac{N^2 - 1}{8N} \Delta Z \] (6.16)

Expressions for \( \Delta Z \), in Feynman gauge, are reported in [15] in terms of quite complicated integrals. The expression for \( r_W = 1 \) in terms of our basic integrals is reported in Table 3 and Table 4. The final numerical values are in agreement with those of [15].
\begin{table}
\begin{center}
\begin{tabular}{|c|ccccc|}
\hline
 & $\Delta Z_{Id}$ & $\Delta Z_{\gamma_5}$ & $\Delta Z_{\gamma_\mu}$ & $\Delta Z_{\gamma_5\gamma_\mu}$ & $\Delta Z_{\sigma_{\mu\nu}}$
\hline
1 & $\frac{31}{48}$ & $\frac{7}{48}$ & $\frac{3}{32}$ & $\frac{11}{32}$ & $\frac{23}{144}$
\hline
$1/\pi^2$ & $-\frac{223}{128}$ & $-\frac{139}{128}$ & $-\frac{19}{64}$ & $-\frac{5}{8}$ & $\frac{73}{384}$
\hline
$(F_0 - \gamma_E)/4\pi^2$ & $-3$ & $-3$ & $0$ & $0$ & $1$
\hline
$Y_0$ & $-12$ & $-12$ & $-2$ & $-2$ & $\frac{4}{3}$
\hline
$Y_1$ & $-18$ & $-6$ & $-\frac{23}{2}$ & $-\frac{35}{2}$ & $-\frac{46}{3}$
\hline
$Y_2$ & $-\frac{1}{4}$ & $-\frac{1}{4}$ & $\frac{3}{8}$ & $\frac{3}{8}$ & $\frac{7}{12}$
\hline
$Y_3$ & $-48$ & $-48$ & $-48$ & $-48$ & $-48$
\hline
$Z_0$ & $6$ & $2$ & $3$ & $5$ & $4$
\hline
$Z_1$ & $-\frac{3}{4}$ & $-\frac{3}{4}$ & $-\frac{3}{4}$ & $-\frac{3}{4}$ & $-\frac{3}{4}$
\hline
$\mathcal{F}(1,-2)$ & $-\frac{5}{96}$ & $\frac{7}{96}$ & $\frac{15}{128}$ & $\frac{7}{128}$ & $\frac{1}{9}$
\hline
$\mathcal{F}(1,-1)$ & $-\frac{391}{192}$ & $-\frac{139}{192}$ & $-\frac{39}{32}$ & $-\frac{15}{8}$ & $\frac{923}{576}$
\hline
$\mathcal{F}(1,0)$ & $-\frac{53}{8}$ & $-\frac{99}{8}$ & $-\frac{1031}{96}$ & $-\frac{755}{96}$ & $-\frac{665}{72}$
\hline
$\mathcal{F}(2,-2)$ & $\frac{65}{192}$ & $-\frac{271}{192}$ & $\frac{235}{128}$ & $\frac{123}{128}$ & $\frac{971}{576}$
\hline
$\mathcal{F}(2,-1)$ & $-\frac{407}{8}$ & $\frac{117}{8}$ & $\frac{631}{24}$ & $-\frac{155}{24}$ & $\frac{1387}{72}$
\hline
$\mathcal{F}(3,-3)$ & $\frac{77}{32}$ & $\frac{105}{32}$ & $\frac{7}{16}$ & $\frac{49}{16}$ & $\frac{329}{96}$
\hline
$\mathcal{F}(3,-2)$ & $\frac{1057}{16}$ & $-\frac{203}{16}$ & $-\frac{259}{8}$ & $\frac{7}{8}$ & $-\frac{413}{16}$
\hline
\text{Total} & $-0.4891266$ & $-0.5669576$ & $-0.3882898$ & $-0.3493743$ & $-0.2819883$
\hline
Ref. \cite{15} & $-0.495$ & $-0.573$ & $-0.388$ & $-0.349$ & $-0.280$
\hline
\end{tabular}
\end{center}
\end{table}

Table 3: Results for $\Delta Z_\Gamma$ for the local operators (6.12) : we report here the coefficients of the various constant appearing in the result. “Total” is the numerical value of $\Delta Z_\Gamma$. 

|   | $\Delta Z_{Id}$ | $\Delta Z_{\gamma_5}$ | $\Delta Z_{\gamma_\mu}$ | $\Delta Z_{\gamma_5\gamma_\mu}$ | $\Delta Z_{\sigma_{\mu\nu}}$ |
|---|----------------|---------------------|---------------------|---------------------|---------------------|
| 1 | $-\frac{151}{48}$ | $\frac{7}{48}$ | $\frac{19}{24}$ | $-\frac{41}{48}$ | $\frac{11}{24}$ |
| $1/\pi^2$ | $-\frac{279}{128}$ | $-\frac{139}{128}$ | $-\frac{3}{16}$ | $-\frac{47}{64}$ | $\frac{101}{384}$ |
| $(F_0 - \gamma_E)/4\pi^2$ | $-3$ | $-3$ | 0 | 0 | 1 |
| $Y_0$ | $-12$ | $-12$ | $-2$ | $-2$ | $\frac{4}{3}$ |
| $Y_1$ | $-18$ | $-6$ | $\frac{1}{2}$ | $-\frac{11}{2}$ | $\frac{2}{3}$ |
| $Y_2$ | $-\frac{1}{4}$ | $-\frac{1}{4}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{7}{12}$ |
| $Y_3$ | $-48$ | $-48$ | $-48$ | $-48$ | $-48$ |
| $Z_0$ | 6 | 2 | $-1$ | 1 | $-\frac{4}{3}$ |
| $Z_1$ | $-\frac{3}{4}$ | $-\frac{3}{4}$ | $-\frac{3}{4}$ | $-\frac{3}{4}$ | $-\frac{3}{4}$ |
| $\mathcal{F}(1, -2)$ | $\frac{4}{3}$ | $\frac{7}{36}$ | $-\frac{11}{36}$ | $\frac{77}{192}$ | $-\frac{23}{192}$ |
| $\mathcal{F}(1, -1)$ | $-\frac{329}{192}$ | $-\frac{139}{192}$ | $\frac{77}{384}$ | $-\frac{113}{384}$ | $\frac{11}{384}$ |
| $\mathcal{F}(1, 0)$ | $\frac{157}{8}$ | $-\frac{99}{8}$ | $-\frac{1373}{96}$ | $\frac{163}{96}$ | $-\frac{173}{96}$ |
| $\mathcal{F}(2, -2)$ | $\frac{4321}{192}$ | $-\frac{271}{192}$ | $-\frac{2737}{384}$ | $\frac{1855}{384}$ | $-\frac{323}{384}$ |
| $\mathcal{F}(2, -1)$ | $-\frac{1823}{8}$ | $\frac{117}{8}$ | $\frac{1669}{24}$ | $-\frac{1217}{24}$ | $\frac{3511}{72}$ |
| $\mathcal{F}(3, -2)$ | $\frac{4713}{16}$ | $-\frac{203}{16}$ | $-\frac{179}{2}$ | $\frac{513}{8}$ | $-\frac{3067}{48}$ |
| Total | $-0.2634473$ | $-0.5669576$ | $-0.2493438$ | $-0.0975887$ | $-0.0591138$ |
| Ref. [13] | $-0.269$ | $-0.573$ | $-0.249$ | $-0.0973$ | $-0.0570$ |

Table 4: Results for $\Delta Z_\Gamma$ for the improved operators (6.13): we report here the coefficients of the various constants appearing in the result. "Total" is the numerical value of $\Delta Z_\Gamma$. 
6.1.4 Renormalization constants for fermionic energy-momentum tensor

We want now to compute the renormalization constants for the dimension-four operators which appear in the first moment of the deep-inelastic-scattering structure functions and in the definition of the energy-momentum tensor. We will consider two operators whose continuum formal limit is

\[ \mathcal{O}^{(1)}_{\mu\nu} = \frac{1}{4} \left[ \bar{\psi} \gamma_\mu D_\nu \psi - D_\mu \bar{\psi} \gamma_\nu \psi + (\mu \leftrightarrow \nu) \right] \]  

\[ \mathcal{O}^{(2)}_{\mu\nu} = F^a_{\mu\alpha} F^a_{\nu\alpha} \]  

(6.17)  

(6.18)

For their explicit definition in terms of lattice operators we refer to [16], formula (5.22) and [1], formulae (3.14)/(3.16). We want now to compute new operators so that

\[ \langle f| \hat{\mathcal{O}}^{(i), \text{LATT}}_{\mu\nu} |i \rangle = \langle f| \mathcal{O}^{(i), \text{CONT}}_{\mu\nu} |i \rangle \]  

(6.19)

for arbitrary states \( f \) and \( i \). In the continuum we adopt the \( \overline{\text{MS}} \)-scheme with scale \( \mu = 1/a \). To define \( \hat{\mathcal{O}}^{(i), \text{LATT}}_{\mu\nu} \) we must consider all possible mixings with operators of dimension less than or equal to four. Here we will restrict our attention to the gluonic sector. We will write at one loop

\[ \hat{\mathcal{O}}^{(i), \text{LATT}}_{\mu\nu} = \mathcal{O}^{(i)}_{\mu\nu} + g^2 \Delta^{(i)}_{\mu\nu} \]  

(6.20)

where

\[ \Delta^{(i)}_{\mu\nu} = \Delta^{(i)}_1 \left( F^a_{\mu\alpha} F^a_{\nu\alpha} - \frac{1}{4} F^2 \delta_{\mu\nu} \right)^{\text{LATT}} + \Delta^{(i)}_2 \left( \frac{1}{4} F^2 \delta_{\mu\nu} \right)^{\text{LATT}} \]  

\[ + \Delta^{(i)}_3 \delta_{\mu\nu} \left( F^a_{\mu\alpha} F^a_{\nu\alpha} - \frac{1}{4} F^2 \delta_{\mu\nu} \right)^{\text{LATT}} \]  

(6.21)

The superscript \( \text{LATT} \) indicates that we use here some lattice operator with the given continuum limit: the explicit discretization is however irrelevant at one loop. For \( r_W = 1 \) the constants \( \Delta^{(1)}_i \) are reported in table 5. For the gluonic operator (6.18) we write \( \Delta^{(2)}_i = \Delta^{(2g)}_i + N_f \Delta^{(2f)}_i \). The constants \( \Delta^{(2g)}_i \) have been computed in [1], see formula (6.11). As for the constants \( \Delta^{(2f)}_i \), we have \( \Delta^{(2f)}_3 = 0, \Delta^{(2f)}_2 = \Delta^{(2f)}_1 \). The explicit expression of \( \Delta^{(2f)}_1 \) for \( r_W = 1 \) is given in table 3.

We can also easily compute the renormalization constants for the operators which are the trace of \( \mathcal{O}^{(1)}_{\mu\nu} \) and \( \mathcal{O}^{(2)}_{\mu\nu} \), i.e. for

\[ \mathcal{O}^{(3)} = \frac{1}{2} \sum_\mu \left( \bar{\psi} \gamma_\mu D_\mu \psi - D_\mu \bar{\psi} \gamma_\mu \psi \right) \]  

(6.22)

\[ \mathcal{O}^{(4)} = F^a_{\alpha\beta} F^a_{\alpha\beta} \]  

(6.23)

---

The calculation in [1] considers minimal subtraction in the continuum. If one considers the \( \overline{\text{MS}} \)-scheme, one should use the formulae of [1], sect. 6, with \( Y = (F_0 - \gamma_E)/\pi^2 \).
|               | \( \Delta_1^{(1)} \) | \( \Delta_2^{(1)} \) | \( \Delta_3^{(1)} \) | \( \Delta_1^{(2f)} \) |
|---------------|----------------|----------------|----------------|----------------|
| 1             | \( \frac{2}{27} \) | \(- \frac{1}{108} \) | \(- \frac{11}{108} \) | \( \frac{1}{9} \) |
| \( 1/\pi^2 \) | \(- \frac{11}{144} \) | \(- \frac{1}{288} \) | \( \frac{19}{96} \) | 0 |
| \( (F_0 - \gamma_E)/4\pi^2 \) | \(- \frac{1}{3} \) | 0 | 0 | \( \frac{1}{3} \) |
| \( Y_0 \)     | \(- \frac{4}{3} \) | 0 | 0 | \( \frac{4}{3} \) |
| \( \mathcal{F}(1, -2) \) | \(- \frac{1}{54} \) | \( \frac{1}{432} \) | \( \frac{11}{432} \) | \(- \frac{1}{36} \) |
| \( \mathcal{F}(1, -1) \) | \(- \frac{2}{27} \) | \( \frac{25}{432} \) | \( \frac{47}{432} \) | \(- \frac{1}{18} \) |
| \( \mathcal{F}(1, 0) \) | \( \frac{13}{9} \) | \( \frac{7}{36} \) | \(- \frac{137}{36} \) | \( \frac{7}{6} \) |
| \( \mathcal{F}(2, -2) \) | \(- \frac{1}{54} \) | \(- \frac{97}{216} \) | \( \frac{43}{216} \) | \(- \frac{5}{24} \) |
| \( \mathcal{F}(2, -1) \) | \(- \frac{62}{9} \) | \( \frac{13}{6} \) | \( \frac{359}{18} \) | \(- \frac{2}{3} \) |
| \( \mathcal{F}(3, -4) \) | 0 | \( \frac{13}{288} \) | \(- \frac{13}{288} \) | 0 |
| \( \mathcal{F}(3, -3) \) | 0 | \( \frac{1}{8} \) | \(- \frac{17}{72} \) | 0 |
| \( \mathcal{F}(3, -2) \) | \( \frac{149}{18} \) | \(- \frac{113}{36} \) | \( - \frac{791}{36} \) | 0 |
| **Total**     | 0.00826199 | -0.01058036 | -0.00963232 | 0.01339200 |

Table 5: Results for \( \Delta_{(i)}^{(j)} \); we report here the coefficients of the various constant appearing in the result. "Total" is the numerical value. All constants must be multiplied by \( T_f \) defined by \( \text{Tr } T^aT^b = T_f \delta^{ab} \). For fermions in the fundamental representation of \( SU(N) \) we have \( T_f = \frac{1}{2} \).
Considering again only the gluonic sector we can write
\[
\hat{O}^{(3)} = \mathcal{O}^{(3)} + g^2 \Delta^{(3)} \mathcal{O}^{(4)} \quad (6.24)
\]
\[
\hat{O}^{(4)} = \left(1 + g^2 \Delta^{(4)}\right) \mathcal{O}^{(4)} \quad (6.25)
\]
We have
\[
\Delta^{(3)} = \Delta^{(1)} + \frac{1}{24\pi^2} T_f \quad (6.26)
\]
\[
\Delta^{(4)} = \Delta^{(2g)} + \Delta^{(2f)} - \frac{11N}{96\pi^2} \quad (6.27)
\]
where \(T_f\) is defined by \(\text{Tr} \ T^a T^b = T_f \delta^{ab}\) \((T_f = \frac{1}{2}\) for fermions in the fundamental representation of \(SU(N)\)). The last numbers in \(\Delta^{(3)}\) and \(\Delta^{(4)}\) are related to the anomaly of the energy-momentum tensor. Indeed define
\[
T_{\mu\nu} = \sum_f \left(\hat{O}_{\mu\nu}^{(1),f} - \frac{1}{4} \delta_{\mu\nu} \hat{O}^{(3),f}\right) + \hat{O}_{\mu\nu}^{(2)} - \frac{1}{4} \delta_{\mu\nu} \hat{O}^{(4)} \quad (6.28)
\]
where the first sum is over the \(N_f\) fermion species. Then
\[
T_{\mu\mu} = g^2 \left(\frac{11N}{96\pi^2} - \frac{1}{24\pi^2} N_f T_f\right) F^2 \quad (6.29)
\]
in agreement with [21, 22].

Finally we want to compare our results with those of Capitani and Rossi [17]. An easy calculation gives:
\[
B_{gg} = -16\pi^2 \Delta^{(2g)} - \frac{4}{3} \approx -17.778285 + \frac{2\pi^2}{N^2} \quad (6.30)
\]
\[
B_{gg}^f = -16\pi^2 \Delta^{(2f)} - \frac{20}{9} T_f \approx -2.16850094 (2T_f) \quad (6.31)
\]
\[
B_{gg} = 16\pi^2 \Delta^{(1)} - \frac{8}{9} T_f \approx 0.20789614 (2T_f) \quad (6.32)
\]

As already noticed in [17] our final result for \(B_{gg}\) agrees with theirs. We are also in perfect agreement for \(B_{gg}^f\), while we differ for \(B_{gg}\) which is reported in [17] to be \(B_{gg} = 0.019\) for \(T_f = \frac{1}{2}\).

---

7 As before there are additional mixings with dimension-three and dimension-four fermionic operators. For a numerical evaluation of these mixings see [17, 18].

8 The additional terms which appear in \(\Delta^{(3)}\) and \(\Delta^{(4)}\) are due to the fact that in dimensionless regularization \(\delta_{\mu\nu} N(O_{\mu\nu}) \neq N(\delta_{\mu\nu} O_{\mu\nu})\) [14, 20].

9 There are however some misprints in Tables 10 and 11 of [17]: in Table 10 the contribution of “Sails” is \(-85/(144\pi^2) - 7/(24\pi^2)(2/e + \log 4\pi - \gamma_E)\) and “Total I - J” is \((-1/(12\pi^2))\), while in Table 11 in “Sails” \(-7/(9\pi^2)\) should be replaced by \(-3/(16\pi^2)\) and in “Total J”, \(-11/(18\pi^2)\) should be \(-1/(48\pi^2)\). We are also in disagreement with their Table 2, where we get \(B_{gg} = -4/3, B_{gg}^f = -20T_f/9, B_{gg} = -8T_f/9\). However a recomputation confirms our results [23].

10 This discrepancy has been recently understood and the new result is in agreement with ours [23].
We want also to correct here the results which appeared in [24]. Indeed the constants $Z_5$, $Z_6$ and $Z_7$ appearing in Table 1 should be $Z_5 = 1 + g^2 1.02165$, $Z_6 = -g^2 0.65205$ and $Z_7 = g^2 0.25034$. The different result for $Z_5$ and $Z_7$ was due to a numerical mistake in the evaluation of the photon contribution (the exact result is in [1], sec. 5). There are also minor misprints: the sign on $F^2$ in (16), (17) and (18) of [24] should be “minus”. An analogous sign should be changed in section 7 of [16].

### 6.2 Calculation of the propagators

Recently a very efficient numerical method to evaluate higher-loop integrals has been presented in [4]. An essential ingredient in the method is the exact calculation of the free bosonic propagator. In [4] an algorithm was introduced which allowed to express its values in $x$-space in terms of the values at $x = (0,0,0,0)$ and $x = (1,1,0,0)$. Here we will discuss how to use our method to obtain an algorithm for the bosonic and the Wilson-fermion propagator.

In general we will consider

\[
G_B(p; x) = \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{D_B(k, m)^p} \quad (6.33)
\]

\[
G_F(p; q; x) = \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{D_F(k, m)^p D_B(k, m)^q} \quad (6.34)
\]

The computation of these two quantities is in principle straightforward: indeed, using the symmetry $k_\mu \rightarrow -k_\mu$ we can rewrite $e^{ikx}$ as $\prod_\mu \cos(k_\mu x_\mu)$ and then express $\cos(k_\mu x_\mu)$ as a polynomial in $k_\mu^2$. In this way $G_B(p; x)$ and $G_F(p; q; x)$ are expressed in terms of $B$ and $\hat{F}$ and thus, for $m \rightarrow 0$, using the results of the previous Sections, we can express $G_B(p; x)$ in terms of $Z_0$ and $Z_1$ (and $F_0$ if $p \geq 2$) and $G_F(p; q; x)$ in terms of the eight constants $F(1, [-2, -1, 0])$, $F(2, [-1, -2])$ and $F(3, [-4, -3, -2])$ if $q \leq 0$ and $p = 1$, to which we must add $F(2, 0)$ if $q \leq 0$, $p \geq 2$, and $Y_0$, $Y_1$, $Y_2$, $Y_3$, $Z_0$ and $Z_1$ if $q > 0$. Of course, in a completely equivalent way, in the bosonic case we can use instead of $Z_0$ and $Z_1$ the values of $G_B(1; x)$ at two different values of $x$. For instance, as in [4], we could choose

\[
G_B(1; (0,0,0,0)) = Z_0 \quad (6.35)
\]

\[
G_B(1; (1,1,0,0)) = Z_0 + Z_1 - \frac{1}{4} \quad (6.36)
\]

Analogously in the fermionic case we could express $G_F(p; q; x)$ for $q \leq 0$ in terms of the values of $G_F(1; 0; x)$ at eight different points: a possible choice, with all the points in a hypercube of side two, is given by $(0,0,0,0), (1,0,0,0), (2,0,0,0), (2,1,0,0), (2,2,0,0), (2,2,1,0), (2,2,2,0)$ and $(1,1,1,1)$.

11 Notice that also the reverse is true: the integrals $B$ and $\hat{F}$ can be rewritten in terms of $G_B$ and $G_F$ using the identity $(\hat{k}^2)^n = (-1)^m (e^{ik/2} - e^{-ik/2})^{2n}$. Loosely speaking $G_B$ and $B$ (and analogously $G_F$ and $\hat{F}$) are two different “basis” in the space of bosonic (respectively mixed bosonic-fermionic) lattice integrals.
From a practical point of view, one can simplify the algorithm by implementing the “integration-by-part” recursion relations (3.7) and (4.15) directly on the propagators. To apply our technique we will generalize (6.33) and (6.34) by considering “integration-by-part” recursion relations (3.7) and (4.15) directly on the propagators.

\[
G_B(p; x) = \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{D_B(k, m)^{p+\delta}} \quad (6.37)
\]

\[
G_F(p, q; x) = \int \frac{dk}{(2\pi)^4} \frac{e^{ikx}}{D_F(k, m)^{p+\delta}D_B(k, m)^q} \quad (6.38)
\]

For these two quantities it is very easy to obtain recursion relations. We obtain in the two cases:

\[
G_B(p; x + \mu) = G_B(p; x - \mu) - \frac{x_\mu}{p + \delta - 1} G_B(p - 1; x) \quad (6.39)
\]

\[
G_F(p, q; x + 2\mu) = G_F(p, q; x - 2\mu) - r_W^2 (G_F(p, q - 1; x + \mu) - G_F(p, q - 1; x - \mu)) + (m^2 r_W^2 - 2m) (G_F(p, q; x + \mu) - G_F(p, q; x - \mu))
- \frac{2}{p - 1 + \delta} [x_\mu G_F(p - 1, q; x) + q (G_F(p - 1, q + 1; x + \mu) - G_F(p - 1, q + 1; x - \mu))] \quad (6.40)
\]

where \(\mu\) is a lattice unit vector. Using these two relations it is possible to express any element \(G_B(p; x)\) in terms of \(G_B(p'; x')\) where \(x'\) is an element of the unit hypercube (i.e. \(x'_\mu = 0\) or 1). Analogously in the fermionic case we express every \(G_F(p, q; x)\) in terms of \(G_F(p', q'; x')\) with \(x'\) belonging to the hypercube of side two, i.e. \(x'_\mu = 0, 1\) and 2. These last quantities can then be easily expressed in terms of \(B_\delta\) or \(F_\delta\) and then the procedure we have been presented in the previous Sections can be applied.

In the bosonic case, the algorithm we have described is less efficient than the one introduced by [23]: indeed for \(p = 1\), starting from (6.39), Vohwinkel obtains a recursion relation which involves only terms with \(p = 1\) and which thus avoids the necessity of introducing \(\delta\) and nonpositive values of \(p\). In the fermionic case however we have not been able to implement the same trick.

Finally let us notice that once (6.34) has been computed one can easily obtain the fermion propagator

\[
\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} -i(\sum_\mu \gamma_\mu \sin k_\mu) + \frac{1}{2r_W k^2 + m} \overline{D_F(k, m)} \quad (6.41)
\]

as

\[
\Delta_F(x) = -\frac{1}{2} \sum_\mu \gamma_\mu (G_F(1, 0; x + \mu) - G_F(1, 0; x - \mu)) - \frac{r_W}{2} \sum_\mu (G_F(1, 0; x + \mu) + G_F(1, 0; x - \mu)) + (4r_W + m)G_F(1, 0; x) \quad (6.42)
\]

To conclude the section let us discuss the numerical evaluation of the fermionic propagator. As in the bosonic case [4], the expressions for \(G_F(p, q; x)\) in terms of our
basic constants are numerically unstable for $|x| \to \infty$: a numerical error in the basic constants gets amplified in $G_F(p, q; x)$ as $|x| \to \infty$. This problem has a standard way out: if the expressions are unstable going outward from the origin, they will be stable in the opposite direction: thus, if we want to compute the propagator for $|x| < d$, for some fixed $d$, we choose eight points with $|x| \approx d$ (say $y_1, \ldots, y_8$) and then we express the propagator for $|x| < d$ in terms of $G_F(1, 0; y_i)$, $i = 1, \ldots, 8$. The new expressions are numerically stable: the numerical error on $G_F(1, 0; y_i)$ gets reduced when we compute the propagator for $|x| \to 0$. As noticed in [4] the instability of the recursion can also be used to provide precise estimates for the basic constants.

We have thus used this method to obtain an independent numerical estimate of the first eight constants of table 2 considering the set of eight points $X(n) \equiv \{(n, [0 - 3], 0, 0), (n + 1, [0 - 3], 0, 0), \ldots\}$. If one estimates the propagator $G_F(1, 0; x)$ at $X(n)$ with an accuracy of 1%, one reproduces the results of Table 2 with an accuracy of approximately $10^{-6}$ (respectively $10^{-8}$) with $n = 7$ ($n = 9$ resp.).

It is also possible to apply the same procedure to $G_F(1, q; x)$ with $q < 0$. The main advantage is that, using larger negative values of $q$, one can obtain more precise estimates of $G_F(1, q; x)$ at the set of points $X(n)$. With $q = -2$ and $n = 7$, using the method presented at the end of section 4.3 (i.e. evaluating the integrals by computing discrete sums of the form (4.42) with $L = 50 - 100$), we obtain the values of $G_F(1, -2; x)$ at the points $X(n)$ with a relative precision of $10^{-6}$ and the final estimates of the basic constants with a precision of $10^{-9}$. If one needs better estimates one can obtain them by simply increasing both $n$ and $-q$. The precision can in this way be increased at will. Using $n = 26$ and $q = -3$ we checked the first eight constants of table 2.

### Acknowledgments

We want to thank Pietro Menotti for collaborating in the early stages of this work. We also thank Stefano Capitani and Gian Carlo Rossi for useful correspondence.

### A Final relations

In this appendix we report the relations which have been found in the fourth step of our reduction procedure for the fermionic integrals. The relations for $F_\delta(p, q), q \leq 0$ are

$$F_\delta(0, -4) = 7336 F_\delta(0, 0) + \delta \left[ \frac{910433}{144\pi^2} - \frac{1024087}{36} + \frac{325123}{144} F_\delta(1, -2) ight]$$
$$+ \frac{2538989}{72} F_\delta(1, -1) + \frac{501607}{6} F_\delta(1, 0) - \frac{14398667}{36} F_\delta(2, -2) + \frac{3075865}{9} F_\delta(2, -1)$$
$$+ \frac{1919221}{48} F_\delta(3, -4) + \frac{7343317}{12} F_\delta(3, -3) - \frac{20889553}{18} F_\delta(3, -2) \right]$$

$$F_\delta(0, -3) = 704 F_\delta(0, 0) + \delta \left[ \frac{9029}{9\pi^2} - \frac{12392}{27} - \frac{1313}{27} F_\delta(1, -2) \right]$$
For positive $q$ we have

$$
F_\delta(2, 1) = \frac{427}{18432 \pi^2} + \frac{1}{48} F_\delta(0, 2) - \frac{1}{384} F_\delta(0, 1) + F_\delta(1, 2) - \frac{5}{12} F_\delta(1, 1)
$$

$$
- \frac{19}{768} F_\delta(1, 0) - \frac{1}{9216} F_\delta(1, -1) + \frac{13}{48} F_\delta(2, 0) + \frac{35}{288} F_\delta(2, -1) + \frac{85}{2304} F_\delta(2, -2)
$$

$$
- \frac{35}{6144} F_\delta(3, -4) - \frac{19}{512} F_\delta(3, -3) - \frac{173}{2304} F_\delta(3, -2)
$$

$$
F_\delta(2, 2) = \frac{1}{96 \pi^2 m^4} - \frac{15}{1024 \pi^2 m^2} + \frac{2677}{11059200 \pi^2} - \frac{1}{30} F_\delta(0, 3) + \frac{41}{2880} F_\delta(0, 2)
$$
\[ -\frac{67}{46080} F_{\delta}(0, 1) + \frac{221}{480} F_{\delta}(1, 2) - \frac{2383}{11520} F_{\delta}(1, 1) + \frac{403}{18432} F_{\delta}(1, 0) \\
+ \frac{23}{276480} F_{\delta}(1, -1) + \frac{2681}{11520} F_{\delta}(2, 0) - \frac{4493}{276480} F_{\delta}(2, -1) - \frac{391}{13824} F_{\delta}(2, -2) \\
- \frac{276480}{276480} F_{\delta}(3, -2) + \frac{15360}{15360} F_{\delta}(3, -3) + \frac{36864}{36864} F_{\delta}(3, -4) \]  
(A.12)

\[ F_{\delta}(1, 3) = \frac{1}{96\pi^2 m^4} - \frac{1024\pi^2 m^2}{m^2} \frac{58982400\pi^2}{58982400\pi^2} - \frac{13}{320} F_{\delta}(0, 3) + \frac{1}{3840} F_{\delta}(0, 2) \\
+ \frac{47}{92160} F_{\delta}(0, 1) + \frac{149}{960} F_{\delta}(1, 2) + \frac{391}{7680} F_{\delta}(1, 1) + \frac{169}{294912} F_{\delta}(1, 0) \\
+ \frac{7}{245760} F_{\delta}(1, -1) - \frac{37}{1280} F_{\delta}(2, 0) + \frac{323}{245760} F_{\delta}(2, -1) - \frac{119}{12288} F_{\delta}(2, -2) \\
- \frac{491520}{491520} F_{\delta}(3, -2) + \frac{40960}{40960} F_{\delta}(3, -3) + \frac{32768}{32768} F_{\delta}(3, -4) \]  
(A.13)

\[ F_{\delta}(1, 4) = \frac{1}{192\pi^2 m^6} + \frac{1}{3840\pi^2 m^4} + \frac{43}{69120\pi^2 m^2} - \frac{230795603}{230795603} \]  

\[ \frac{1}{1935360} F_{\delta}(0, 3) - \frac{3870720}{3870720} F_{\delta}(0, 2) + \frac{1720320}{1720320} F_{\delta}(0, 1) - \frac{533}{645120} F_{\delta}(1, 2) \\
+ \frac{164753}{516960} F_{\delta}(1, 1) - \frac{990904320}{990904320} F_{\delta}(1, 0) - \frac{19}{55050240} F_{\delta}(1, -1) - \frac{25471}{860160} F_{\delta}(2, 0) \\
+ \frac{1486356480}{1486356480} F_{\delta}(2, -1) + \frac{323}{2752512} F_{\delta}(2, -2) - \frac{17909419}{2972712960} F_{\delta}(3, -2) \\
- \frac{1083}{9175040} F_{\delta}(3, -3) - \frac{1048576}{1048576} F_{\delta}(3, -4) \]  
(A.14)

\[ F_{\delta}(0, 4) = (1 - \delta \log m^2) \left[ \frac{1}{96\pi^2 m^4} - \frac{19}{4608\pi^2 m^2} + \frac{1}{9216\pi^2} \right] \\
+ \frac{31}{144} F_{\delta}(0, 3) - \frac{13}{1152} F_{\delta}(0, 2) + \frac{1}{9216} F_{\delta}(0, 1) \\
+ \delta \left[ -\frac{5}{576\pi^2 m^4} + \frac{61}{18432\pi^2 m^2} - \frac{8993600\pi^2}{8993600\pi^2} + \frac{347}{14400} F_{\delta}(0, 3) \\
- \frac{83}{2560} F_{\delta}(0, 2) + \frac{137}{184320} F_{\delta}(0, 1) - \frac{689}{2880} F_{\delta}(1, 2) + \frac{1139}{23040} F_{\delta}(1, 1) \\
- \frac{147456}{415} F_{\delta}(1, 0) + \frac{23}{4423680} F_{\delta}(1, -1) - \frac{329}{11520} F_{\delta}(2, 0) + \frac{13283}{1105920} F_{\delta}(2, -1) \\
- \frac{391}{221184} F_{\delta}(2, -2) - \frac{30479}{2211840} F_{\delta}(3, -2) + \frac{437}{245760} F_{\delta}(3, -3) + \frac{161}{589824} F_{\delta}(3, -4) \right] \]  
(A.15)

\[ F_{\delta}(0, 5) = (1 - \delta \log m^2) \left[ \frac{1}{192\pi^2 m^6} + \frac{1}{1536\pi^2 m^4} - \frac{415}{442368\pi^2 m^2} + \frac{11}{442368\pi^2} \right] \\
+ \frac{523}{13824} F_{\delta}(0, 3) - \frac{289}{110592} F_{\delta}(0, 2) + \frac{25}{884736} F_{\delta}(0, 1) \\
+ \delta \left[ -\frac{7}{2304\pi^2 m^6} - \frac{92160\pi^2 m^4}{92160\pi^2 m^4} + \frac{1769472\pi^2 m^2}{1769472\pi^2 m^2} + \frac{9988315545600\pi^2}{9988315545600\pi^2} \\
+ \frac{634603}{7741440} F_{\delta}(0, 3) - \frac{157898120}{371589120} F_{\delta}(0, 2) + \frac{61477}{371589120} F_{\delta}(0, 1) - \frac{172157}{2580480} F_{\delta}(1, 2) \\
+ \frac{61931520}{1321205760} F_{\delta}(1, 1) - \frac{7297}{356725552} F_{\delta}(1, 0) - \frac{356725552}{356725552} F_{\delta}(1, -1) \right] \]
\[ \mathcal{F}_\delta(0,6) = (1 - \delta \log m^2) \left[ \begin{array}{c} 1 \left( \frac{1}{320 \pi^2 m^8} + \frac{380 \pi^2 m^6}{17836277760} + \frac{1}{20480 \pi^2 m^4} - \frac{3538940 \pi^2 m^2}{891813888} \right) \\
\frac{1}{7053481} \mathcal{F}_\delta(3, -2) - \frac{1}{138643} \mathcal{F}_\delta(3, -3) - \frac{1}{339738624} \mathcal{F}_\delta(3, -4) \\
+ \delta \left[ -\frac{9}{903503350441} \mathcal{F}_\delta(0, 1) - \frac{1}{230400 \pi^2 m^6} - \frac{74410051}{1223059046400} \mathcal{F}_\delta(2, -1) + \frac{8561600 \pi^2 m^4}{3838603} \mathcal{F}_\delta(1, -1) + \frac{4954521600}{65256251} \mathcal{F}_\delta(2, -2) + \frac{9795427451}{428070666240} \mathcal{F}_\delta(3, -2) \\
- \frac{3838603}{36485} \mathcal{F}_\delta(2, -1) + \frac{1}{163074539520} \mathcal{F}_\delta(3, -3) - \frac{393738624}{339738624} \mathcal{F}_\delta(3, -4) \right] \right] \] 

(A.16)
[12] H. W. Hamber and C. M. Wu, Phys. Lett. **133B** (1983) 351.

[13] P. Weisz, Phys. Lett. **100B** (1981) 331.

[14] G. Heatlie, G. Martinelli, C. Pittori, G. C. Rossi and C.T. Sachrajda, Nucl. Phys. **B352** (1991) 266.

[15] E. Gabrielli, G. Martinelli, C. Pittori, G. Heatlie and C. T. Sachrajda, Nucl. Phys. **B362** (1991) 475.

[16] S. Caracciolo, G. Curci, P. Menotti and A. Pelissetto, Ann. Phys. (NY) **197** (1990) 119.

[17] S. Capitani and G. C. Rossi, Nucl. Phys. **B433** (1995) 351.

[18] M. Göckeler, R. Horsley, E.-M. Ilgenfritz, H. Perlt, A. Schiller, P. Rakow and G. Schierholz, [hep-lat/9603006](http://arxiv.org/abs/hep-lat/9603006).

[19] P. Breitenlohner and D. Maison, Comm. Math. Phys. **52** (1977) 11, 39, 55.

[20] G. Bonneau, Nucl. Phys. **B167** (1980) 261 and **B171** (1980) 477.

[21] S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. **D15** (1977) 1712.

[22] J. C. Collins, A. Duncan and S. D. Joglekar, Phys. Rev. **D16** (1977) 438.

[23] S. Capitani and G. C. Rossi, private communication.

[24] S. Caracciolo, G. Curci, P. Menotti and A. Pelissetto, Phys. Lett. **B228** (1989) 375.

[25] C. Vohwinkel, unpublished, cited in [4].