Harder–Narasimhan filtration for rank 2 tensors and stable coverings

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Abstract. We construct a Harder–Narasimhan filtration for rank 2 tensors, where there does not exist any such notion \textit{a priori}, as coming from a GIT notion of maximal unstability. The filtration associated to the 1-parameter subgroup of Kempf giving the maximal way to destabilize, in the GIT sense, a point in the parameter space of the construction of the moduli space of rank 2 tensors over a smooth projective complex variety, does not depend on a certain integer used in the construction of the moduli space, for large values of the integer. Hence, this filtration is unique and we define the Harder–Narasimhan filtration for rank 2 tensors as this unique filtration coming from GIT. Symmetric rank 2 tensors over smooth projective complex curves define curve coverings lying on a ruled surface, hence we can translate the stability condition to define stable coverings and characterize the Harder–Narasimhan filtration in terms of intersection theory.

Keywords. Harder–Narasimhan filtration; geometric invariant theory; tensors; curve coverings; moduli space; Kempf.

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1. Introduction

This article is part of the research developed in the author’s Ph.D. thesis (c.f. [15]) and it is a continuation of [3]. In a moduli problem, usually, we impose a notion of stability for the objects in order to obtain a moduli space with good properties. When constructing the moduli space using geometric invariant theory (GIT), a notion of GIT stability for the orbits appears and, to obtain the moduli space it is shown, at some point, that both notions of stability do coincide.

Harder and Narasimhan construct a canonical filtration (c.f. [9]) for unstable sheaves, named after them, which maximally contradicts the definition of stability we impose in the construction of the moduli space. On the other hand, there has been some results in the literature, trying to find the best way of destabilizing an orbit in the GIT sense (c.f. [6, 10, 11]). The GIT stability is checked by 1-parameter subgroups, by the classical Hilbert–Mumford criterion, and it turns out that there exists, up to some rescaling, a unique 1-parameter subgroup giving a notion of maximal GIT unstability. That special
unique 1-parameter subgroup produces a filtration in a natural way, which we call Kempf filtration, based on results of [10] (c.f. § 4). The immediate question is whether the Harder–Narasimhan filtration and the Kempf filtration do coincide.

In [3], the authors develop an idea to answer positively the previous question and establish a correspondence between both filtrations, based on rewriting a function (which appears in [10] and is maximized by the special 1-parameter subgroup) in a more geometrical way, to show that the Kempf filtration satisfies certain convexity properties (c.f. § 5), similar to the properties which characterize the Harder–Narasimhan filtration. In this article, the author translates that idea to the case of rank 2 tensors over a smooth complex projective variety of arbitrary dimension.

We call a rank 2 tensor the pair given by \((E, \varphi : \overline{E} \otimes \cdots \otimes \overline{E} \rightarrow M)\), where \(E\) is a coherent torsion free sheaf of rank 2 over a smooth projective complex variety \(X\), and \(M\) is a line bundle over \(X\). Apart from being a geometrical object by itself, in the case when \(X\) is a smooth projective complex curve and \(\varphi\) is symmetric they define degree \(s\) coverings of \(X\) lying on the ruled surface \(\mathbb{P}(E)\), by considering the vanishing locus of the morphism (c.f. § 8). Hence, a notion of stability for rank 2 tensors defines a notion of stable or unstable covering of an algebraic curve embedded into a ruled surface.

Besides, the importance of this rank 2 tensors case is that, for other moduli problems previously considered in [3, 14, 15] (e.g. sheaves, holomorphic pairs, Higgs sheaves, quiver representations), there is already a notion of Harder–Narasimhan filtration for these objects, constructed analogously to the case of sheaves (c.f. Theorem 1.3.4 of [8]). However, for rank 2 tensors there is no notion \textit{a priori} of Harder–Narasimhan filtration because we do not know, in principle, how to define a quotient tensor (c.f. Remark 7.4), which is needed to construct the filtration recursively, after finding a maximally destabilizing subobject.

Rank 2 tensors are a particular case of tensors or decorated sheaves (c.f. [2] and [4]). After this work was completed, it appeared in a paper by Pustetto (c.f. [12]) where it was proved that the existence of a maximal destabilizing object for tensors which are \(\epsilon\)-semistable and for those which are \(k\)-semistable of rank \(\leq 3\). Both \(\epsilon\) and \(k\)-stability notions are much easier to check (they only need to be checked on subsheaves instead of filtrations) hence the calculation of the quantity (2.2) is more straightforward. Indeed, it is precisely the rank 2 case where the stability notion considered in this article coincides with \(\epsilon\) and \(k\)-stability (c.f. Lemma 46 of [12]), hence stability is checked by subobjects (c.f. Definition 2.2) and it is expected, as usual, the existence of a maximal destabilizing subobject.

The main technical difficulty here is to prove that the Kempf filtration does not depend on the choice of a certain integer made during the construction of the moduli space, for large values of the integer (c.f. Theorem 4.2, proof in § 6). Finally, we define the Harder–Narasimhan filtration as the unique filtration (after proving the independence of this integer) giving maximal unstability from the GIT point of view (c.f. § 7).

When the tensor is defined over a smooth projective complex curve, we can characterize the Harder–Narasimhan filtration in terms of intersection theory for ruled surfaces (c.f. § 8). As we pointed out before, an unstable tensor will define an unstable covering of the curve, and the Harder–Narasimhan filtration can be reinterpreted as a section of a ruled surface whose intersection numbers maximize certain quantity. The Harder–Narasimhan filtration can be a useful tool to study the moduli space of such coverings.
In principle, the ideas in this article can be used in different moduli problems to show that the filtration giving maximal unstability from the GIT point of view and the Harder–Narasimhan filtration do coincide in cases where the latter is previously known, or to define a notion of Harder–Narasimhan filtration, otherwise. For example, in [14], a similar correspondence is proven for representations of a finite quiver in the category of finite dimensional vector spaces over an algebraically closed field of arbitrary characteristic. For tensors in general, a notion of Harder–Narasimhan filtration is unknown, and rank 2 tensors is a particular example of tensors, for which this method can be implemented.

2. Stability for rank 2 tensors

Let \( X \) be a smooth complex projective variety of dimension \( n \). Let \( E \) be a coherent torsion free sheaf over \( X \) of rank 2. Let \( M \) be a line bundle over \( X \). We call a rank 2 tensor the pair consisting of
\[
(E, \varphi : E \otimes \cdots \otimes E \to M),
\]
where the morphism \( \varphi \) is not identically zero. These objects are particular cases of the ones in Definition 1.1 of [2] for arbitrary \( s, c = 1, b = 0, R = \text{Spec} \mathbb{C} \) and \( D = M \).

A weighted filtration \((E_\bullet, n_\bullet)\) of a sheaf \( E \) is a filtration
\[
0 \subset E_1 \subset E_2 \subset \cdots \subset E_t \subset E_{t+1} = E, \tag{2.1}
\]
and rational numbers \( n_1, n_2, \ldots, n_t > 0 \). We denote \( r_i = \text{rk} (E_i) \). Let \( \gamma \) be a vector of \( \mathbb{C}^r \) defined as \( \gamma = \sum_{i=1}^t n_i \gamma^{(\text{rk } E_i)} \), where
\[
\gamma^{(k)} := (k - r, \ldots, k - r, k, \ldots, k) \quad (1 \leq k < r).
\]
Hence, the vector is of the form
\[
\gamma = (\gamma_1, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_2, \ldots, \gamma_{t+1}, \ldots, \gamma_{t+1}),
\]
where \( n_i = \frac{\gamma_{i+1} - \gamma_i}{r} \).

Now let \( I = \{1, \ldots, t + 1\}^{\times s} \) be the set of all multi-indexes \( I = (i_1, \ldots, i_s) \) and define
\[
\mu(\varphi, E_\bullet, n_\bullet) = \min_{I \in \mathcal{I}} \{ \gamma_{i_1} + \cdots + \gamma_{i_s} : \varphi|E_{i_1} \otimes \cdots \otimes E_{i_s} \neq 0 \}. \tag{2.2}
\]
We assume that \( \varphi \) is not identically zero, then (2.2) is well defined. Let \( I_0 \) be the multi-index giving minimum in (2.2). We will denote by \( \epsilon_i(\varphi, E_\bullet, n_\bullet) \) (or just \( \epsilon_i(E_\bullet) \) if the rest of the data is clear from the context) the number of elements \( k \) of multi-index \( I \) such that \( r_k \leq r_i \). Let us call \( \epsilon^i(E_\bullet) = \epsilon_{i+1}(E_\bullet) - \epsilon_i(E_\bullet) \). Using a calculation made in [2, 15], we can rewrite (2.2) as
\[
\mu(\varphi, E_\bullet, n_\bullet) = \sum_{i=1}^t n_i (s \gamma_{i} - \epsilon_i(E_\bullet) r) . \tag{2.3}
\]
Let $\delta$ be a polynomial of degree at most $\dim X - 1 = n - 1$ and positive leading coefficient. If $P_1$ and $P_2$ are two polynomials, we write $P_1 < P_2$ if $P_1(m) < P_2(m)$ for $m \gg 0$, and analogously for ‘$\leq$’ and ‘$\preceq$’.

**DEFINITION 2.1** (Definition 1.3 of [2])

We say that $(E, \varphi, u)$ is $\delta$-semistable if for all weighted filtrations $(E_\bullet, n_\bullet)$ of $E$,

$$\sum_{i=1}^i n_i((r P_{E_i} - r_i P_E)) + \delta (s r_i - \epsilon_i(E_\bullet)) \preceq 0.$$  

(2.4)

We say that $(E, \varphi, u)$ is $\delta$-stable if we have a strict inequality in (2.4) for every weighted filtration. If $(E, \varphi, u)$ is not $\delta$-semistable we say that it is $\delta$-unstable.

It suffices to check the condition in Definition 2.1 over filtrations with $\text{rk} E_i < \text{rk} E_{i+1}$. Hence, as the rank of $E$ is 2, the only filtrations we have to check are one-step filtrations, i.e. subsheaves of rank 1, and we can rewrite the stability condition as follows:

**DEFINITION 2.2**

A rank 2 tensor $(E, \varphi)$ is $\delta$-semistable if for every rank 1 subsheaf $L \subset E$,

$$(2P_L - P_E) + \delta(s - 2\epsilon(L)) \preceq 0,$$

(2.5)

where $\epsilon(L)$ is the number of times that 1 appears in the multi-index $(i_1, \ldots, i_s)$ giving the minimum in (2.2) (notice that $L$ plays the role of $E_1$ in (2.1)) and $P_E, P_L$ are the Hilbert polynomials of $E$ and $L$ respectively. If the inequality is strict for every $L$, we say that $(E, \varphi)$ is $\delta$-stable. If $(E, \varphi)$ is not $\delta$-semistable, we say that it is $\delta$-unstable.

**3. Moduli space of rank 2 tensors**

We recall the main points of the construction of the moduli space for tensors with fixed determinant $\det(E) \cong \Delta$ of degree $d$ and $\text{rk} E = 2$. The construction for tensors in general appears in [2], following Simpson’s method, and it is also included in section 1.2 of [15], following Gieseker’s method. Recall that our case can be obtained by setting $c = 1, b = 0$, arbitrary $s$, $R = \text{Spec} \mathbb{C}$ and $D = M$, line bundle over $X \times R \simeq X$.

Let $V$ be a vector space of dimension $p := h^0(E(m))$, where $m$ is a suitable large integer (in particular, $E(m)$ is generated by global sections and $h^i(E(m)) = 0$ for $i > 0$). Given an isomorphism $V \cong H^0(E(m))$ we obtain a point

$$(\tilde{Q}, \tilde{\Phi}) \in \mathbb{P}(\text{Hom}(\wedge^r V, A)) \times \mathbb{P}(\text{Hom}(V^\otimes s, B)).$$

If we change the isomorphism $\det(E) \cong \Delta$, we obtain a different point in the line defined by $Q$. Similarly, if we change the isomorphism $V \cong H^0(E(m))$ by a homothecy, we obtain a different point in the line defined by $Q$. In both cases, the point $\tilde{Q}$ in the projective space is the same. The same applies for $\tilde{\Phi}$. If we fix once and for all a basis of $V$, then giving an isomorphism between $V$ and $H^0(E(m))$ is equivalent to giving a basis of $H^0(E(m))$. A change of basis is given by an element of $\text{GL}(V)$, but, since a homothecy
does not change the point \((\tilde{Q}, \tilde{\Phi})\), when we want to get rid of this choice it is enough to divide by the action of \(\text{SL}(V)\).

A weighted filtration \((V_\bullet, n_\bullet)\) of \(V\) is a filtration
\[
0 \subset V_1 \subset V_2 \subset \cdots \subset V_t \subset V_{t+1} = V,
\tag{3.1}
\]
and rational numbers \(n_1, n_2, \ldots, n_t > 0\). Similarly to weighted filtrations of \(E\) (c.f. (2.1)), this is equivalent to giving a 1-parameter subgroup \(\Gamma : \mathbb{C}^* \to \text{SL}(V)\) represented by the vector
\[
\Gamma = (\Gamma_1, \ldots, \Gamma_1, \Gamma_2, \ldots, \Gamma_2, \ldots, \Gamma_{t+1}, \ldots, \Gamma_{t+1}),
\]
up to conjugacy by an element of the parabolic subgroup defined by the filtration, where
\[
n_i = \frac{\Gamma_{i+1} - \Gamma_i}{\dim V_i} \text{ and define } V^i := V_i / V_{i-1} \text{ (c.f. [3, 15])}.
\]

By the Hilbert–Mumford criterion (c.f. Theorem 2.1 of [11]), a point \((\tilde{Q}, \tilde{\Phi})\) is GIT semistable with respect to the natural linearization on \(O(a_1, a_2)\) if and only if, for all weighted filtrations, it is
\[
\mu(\tilde{Q}, V_\bullet, n_\bullet) + \frac{a_2}{a_1} \mu(\tilde{\Phi}, V_\bullet, n_\bullet) \leq 0.
\]

The second summand of the expression is given by
\[
\mu(\Phi, V_\bullet, n_\bullet) = \min_{I \in \mathcal{I}} \{ \Gamma_{\dim V_{i_1}} + \cdots + \Gamma_{\dim V_{i_s}} : \Phi|_{V_{i_1} \otimes \cdots \otimes V_{i_s} \neq 0} \}. \tag{3.2}
\]

If \(I = (i_1, \ldots, i_s)\) is the multi-index giving a minimum in (3.2), we will analogously denote by \(\epsilon_i(\Phi, V_\bullet, n_\bullet)\) (or just \(\epsilon_i(\Phi)\) if the rest of the data is clear from the context) the number of elements \(k\) of the multi-index \(I\) such that \(\dim V_k \leq \dim V_i\). Let \(\epsilon^i(\Phi) = \epsilon_i(\Phi) - \epsilon_{i-1}(\Phi)\).

Given a weighted filtration of \(V \simeq H^0(E(m))\) as in (3.1), we denote by \(E_{V_i}\) the subsheaf of \(E\) generated by \(V_i\), and let \(r_i = \text{rk } E_{V_i}\) be its rank. Similarly, we denote by \(E_{V_i}\) the sheaf generated by \(V^i = V_i / V_{i-1}\) and let \(r_i = \text{rk } E_{V_i} = r_i - r_{i-1}\) be its rank.

Using the calculation of the numerical function to apply Mumford criterion for GIT stability (c.f. Proposition 1.2.29 of [15]), we can state the following:

**PROPOSITION 3.1**

A point \((\tilde{Q}, \tilde{\Phi})\) is GIT \(a_2/a_1\)-semistable if for all weighted filtrations \((V_\bullet, n_\bullet)\),
\[
\sum_{i=1}^{t} n_i (r \dim V_i - r_i \dim V) + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\Phi) \dim V) \leq 0.
\]

The following result can be found in Theorem 4.5.3 of [4]. See also Theorem 3.6 of [2] and Theorem 1.2.31 of [15] for the construction of a moduli space of tensors following Simpson [13] and Gieseker [1] respectively, whose notations we follow in this article.
Theorem 3.2. Let \((E, \varphi)\) be a tensor. There exists an \(m_0\) such that, for \(m \geq m_0\) the associated point \((\bar{Q}, \Phi)\) is GIT \(a_2/a_1\)-semistable if and only if the tensor is \(\delta\)-semistable, where

\[
\frac{a_2}{a_1} = \frac{r \delta(m)}{P_E(m) - s \delta(m)}.
\]

4. Kempf theorem

Let \(X\) be a smooth complex projective variety of dimension \(n\) and let \(\delta\) be a polynomial of degree at most \(\text{dim } X - 1 = n - 1\) and positive leading coefficient. Let \((E, \varphi)\) be a \(\delta\)-unstable rank 2 tensor. Let \(m_0\) be an integer as in Theorem 3.2 (i.e. such that the \(\delta\)-stability and the GIT stability coincide) and also such that \(E\) is \(m_0\) regular (choosing a larger integer, if necessary). Choose an integer \(m \geq m_0\) and let \(V\) be a vector space of dimension \(P_E(m) = h^0(E(m))\).

By the Hilbert–Mumford criterion, stability of an orbit in the parameter space where a group acts can be checked through 1-parameter subgroups, which turns out to be the checking of the positivity of some quantity (c.f. Proposition 3.1). The natural question which arises is whether there exists a best way to destabilize a point in the sense of GIT, i.e. whether there exists a best 1-parameter subgroup which maximizes that quantity. There are results in the literature (c.f. [6, 10, 11]) studying the possibility of finding the best 1-parameter subgroup moving most rapidly toward the origin, i.e. giving a notion of GIT maximal unstability. We will make use of [10] for our purposes.

Given a 1-parameter subgroup, or equivalently a weighted filtration, i.e. a filtration of vector subspaces \(0 \subset V_1 \subset \cdots \subset V_{t+1} = V\) and rational numbers \(n_1, \cdots, n_t > 0\), we define the following function

\[
\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^t n_i (r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\Phi) \dim V))}{\sqrt{\sum_{i=1}^{t+1} \dim V_i \Gamma_i^2}},
\]

which we call a *Kempf function* for this problem, i.e. a function whose numerator coincides with the numerical function in Proposition 3.1 and the denominator is a norm, a bilinear symmetric invariant form \(||\Gamma||\) in the space of 1-parameter subgroups (c.f. definition of length in [10]). As the group \(SL(V)\) is simple, every bilinear symmetric invariant form is a multiple of the Killing norm, hence it is unique up to scalar. Note that the norm is chosen in order to avoid the rescaling of the weights when asking for the maximum of the function. In other words, the choice of a norm calibrates the speed of the 1-parameter subgroups.

The result of Kempf states that, given a GIT unstable point, i.e. a point for which there exists any 1-parameter subgroup making the quantity in Proposition 3.1 positive (the numerator of the Kempf function), there exists a unique parabolic subgroup containing a unique 1-parameter subgroup in each maximal torus, giving maximum for the Kempf function. In terms of filtrations, there exists a unique weighted filtration giving maximum for the Kempf function. Therefore, we rewrite Theorem 2.2 of [10] for this case:

Theorem 4.1. There exists a unique weighted filtration

\[
0 \subset V_1 \subset \cdots \subset V_{t+1} = V
\]
and rational numbers \( n_1, \ldots, n_t > 0 \), up to multiplication by a scalar, called the Kempf filtration of \( V \), such that the Kempf function \( \mu(V_\bullet, n_\bullet) \) achieves the maximum among all filtrations and positive weights \( n_i > 0 \).

Let
\[
0 \subset V_1 \subset \cdots \subset V_{t+1} = V
\]
be the Kempf filtration of \( V \) (c.f. Theorem 4.1), and let
\[
0 \subseteq (E_1^m, \varphi|_{E_1^m}) \subseteq (E_2^m, \varphi|_{E_2^m}) \subseteq \cdots \subseteq (E_t^m, \varphi|_{E_t^m}) \subseteq (E_{t+1}^m, \varphi|_{E_{t+1}^m}) = (E, \varphi)
\]
be the \( m \)-Kempf filtration of the rank 2 tensor \((E, \varphi)\), where \( E_i^m \subset E \) is the subsheaf generated by \( V_i \) under the evaluation map. Note that the subsheaves do depend on the integer \( m \) we have chosen during the process of constructing the moduli space.

For a given \( m \), the \( m \)-Kempf filtration represents the maximal way of destabilizing a \( \delta \)-unstable tensor from the GIT point of view. In this case, there is no notion, \( a \) \( p \) \( r \) \( i \) \( o \) \( r \) \( i \), of a Harder–Narasimhan filtration. Hence, the filtration we obtain from GIT, once we prove that it does not depend on \( m \), will define by uniqueness a notion of Harder–Narasimhan filtration.

In the following we will prove this theorem, in an analogous way as it was done in [3] for sheaves.

**Theorem 4.2.** There exists an integer \( m' \gg 0 \) such that the \( m \)-Kempf filtration of the \( \text{rk} \ 2 \) tensor \((E, \varphi)\) does not depend on \( m \), for \( m \geq m' \).

### 5. Results on convexity

Now we recall the results from section 2 of [3] about convexity. We study a function on a convex set, and how to maximize it. It will turn out to be that this function will be in correspondence with the Kempf function and we will use these results to figure out properties about the Kempf filtration.

Endow \( \mathbb{R}^{t+1} \) with an inner product \((\cdot, \cdot)\) defined by a diagonal matrix
\[
\begin{pmatrix}
  b^1 & 0 & & \\
  & \ddots & & \\
  0 & & & b^{t+1}
\end{pmatrix},
\]
where \( b^i \) are positive integers. Let
\[
\mathcal{C} = \{ x \in \mathbb{R}^{t+1} : x_1 < x_2 < \cdots < x_{t+1} \},
\]
and let \( v = (v_1, \ldots, v_{t+1}) \in \mathbb{R}^{t+1} - \{0\} \) verifying \( \sum_{i=1}^{t+1} v_i b^i = 0 \). Define the function
\[
\mu_v : \mathcal{C} - \{0\} \to \mathbb{R} \cap \{0\} \quad \Gamma \mapsto \mu_v(\Gamma) = \frac{(\Gamma, v)}{||\Gamma||}
\]
and note that \( \mu_v(\Gamma) = ||v|| \cdot \cos(\Gamma, v) \). Then, the function \( \mu_v(\Gamma) \) does not depend on the norm of \( \Gamma \) and takes the same value on every point of the ray spanned by each \( \Gamma \).
Assuming that there exists $\Gamma \in \mathcal{C}$ verifying $\mu_v(\Gamma) > 0$, we would like to find a vector $\Gamma \in \mathcal{C}$ maximizing $\mu_v$. We set $w^i = -b^i v_i$, $w_1 = w^1 + \ldots + w^i$ and $b_i = b^1 + \ldots + b^i$ and draw a graph joining the points with coordinates $(b_i, w_i)$, where the slope of each segment is given by $-v_i$ (thin line in figure 1). Now draw the convex envelope of this graph (thick line in figure 1), whose coordinates are denoted by $(b_i, \tilde{w}_i)$, and let us define $\Gamma_i = -\tilde{w}_i + \tilde{w}_{i-1}$. Hence, $-\Gamma_i$ are the slopes of the convex envelope graph of $v$. We call $\Gamma_v$ to the vector defined by the coordinates $\Gamma_i$.

**Theorem 5.1.** The vector $\Gamma_v$ defined in this way (c.f. Figure 1) gives a maximum for the function $\mu_v$ on its domain.

### 6. The $m$-Kempf filtration stabilizes with $m$

In this section, we will prove Theorem 4.2 through a series of partial results. Given a $\delta$-unstable rank 2 tensor $(E, \varphi)$ we have the $m$-Kempf filtration of $(E, \varphi)$ (c.f. (4.2)). To this filtration we associate a graph, in order to apply the previous results on convexity.

**DEFINITION 6.1**

Let $m \geq m_0$. Given $0 \subset V_1 \subset \ldots \subset V_{t+1} = V$, a filtration of vector spaces of $V$, let

$$v_{m,i} = m^{n+1} \cdot \frac{1}{\dim V^i \dim V} \left[ r^i \dim V - r \dim V^i \right] + a_2 \left( \epsilon^i(\Phi) \dim V - s \dim V^i \right),$$

$$b_m^i = \frac{1}{m^n} \dim V^i > 0,$$

$$w_m^i = -b_m^i \cdot v_{m,i} = m \cdot \frac{1}{\dim V} \left[ r \dim V^i - r^i \dim V \right] + a_2 \left( s \dim V^i - \epsilon^i(\Phi) \dim V \right).$$
Also let

\[ b_{m,i} = b_1^m + \cdots + b_i^m = \frac{1}{m^n} \dim V_i , \]
\[ w_{m,i} = w_1^m + \cdots + w_i^m = m \cdot \frac{1}{\dim V} \left[ r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\Phi) \dim V) \right] . \]

We call the graph defined by points \((b_{m,i}, w_{m,i})\) the graph associated to the filtration \(V_{\bullet} \subset V\).

Now we prove a crucial lemma which will let us relate the Kempf function with the function in Theorem 5.1, in order to prove Theorem 4.2. The lemma strongly uses the assumption on the rank 2 of the tensor, the reason why the result cannot be analogously extended in more generality. A discussion about the issues when applying the method for rank 3 can be read in section 2.5 of [15].

**Lemma 6.2.** The symbols \(\epsilon_i(\Phi) = \epsilon_i(\Phi, V_{\bullet}, n_{\bullet})\) do not depend on the weights \(n_{\bullet}\).

Therefore, the graph associated to the filtration only depends on the data \(V_{\bullet} \subset V\), not the weights \(n_{\bullet}\).

**Proof.** Note that \(\text{rk } E_1 \geq 1\) because it is generated by, at least, a non zero global section.

Suppose that \(\text{rk } E_1^m = \text{rk } E_2^m = \cdots = \text{rk } E_k^m = 1\) and \(\text{rk } E_{k+1}^m = \cdots = \text{rk } E_t^m = \text{rk } E = 2\). Then, for example, \(E_1^m\) coincide with \(E_2^m\) on an open set and, generically, the behavior with respect to \(\varphi\) is the same, i.e.

\[ \Phi|_{V_1 \otimes \cdots \otimes V_1} = 0 \Leftrightarrow \varphi|_{E_1^m \otimes \cdots \otimes E_1^m} = 0 \Leftrightarrow \varphi|_{E_2^m \otimes E_1^m \otimes \cdots \otimes E_1^m} = 0 . \]

Therefore, the values \(\epsilon_i(\Phi, V_{\bullet}, n_{\bullet})\) only depend on the filters \(E_i^m\) but not on the specific values of the \(\Gamma_i\). In fact, they will only depend on \(\Gamma_1\) and \(\Gamma_{k+1}\), because they are the minimal ones among the filters of the same rank (c.f. (2.2) and (3.2)). In this case we will just write \(\epsilon_i(\Phi, V_{\bullet})\) or \(\epsilon_i(\Phi)\), when the filtration is clear from the context. \(\square\)

Next, we shall identify the Kempf function in Theorem 4.1

\[ \mu(V_{\bullet}, n_{\bullet}) = \frac{\sum_{i=1}^{t+1} n_i (r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\Phi) \dim V))}{\sqrt{\sum_{i=1}^{t+1} \dim V_i \Gamma_i^2}} = \frac{\sum_{i=1}^{t+1} \Gamma_i \dim V_i - r \dim V_i + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\Phi) \dim V - s \dim V_i))}{\sqrt{\sum_{i=1}^{t+1} \dim V_i \Gamma_i^2}} , \]

where \(n_i = \frac{\Gamma_{i+1} - \Gamma_i}{\dim V_i}\), with the function in Theorem 5.1. Precisely, we use Lemma 6.2 to assure that the data of the filters \(V_{\bullet} \subset V\), and the data of the weights \(n_{\bullet}\) are independent, so we can maximize the Kempf function with respect to each of them, independently, as in Theorem 5.1.

**Proposition 6.3**

For every integer \(m\), the following equality holds:

\[ \mu(V_{\bullet}, n_{\bullet}) = m^{(-\frac{3}{2}-1)} \cdot \mu_v^m(\Gamma) \]

between the Kempf function on Theorem 4.1 and the function in Theorem 5.1.
Proof. By Lemma 6.2, we can fix a vector \( v_m \) and look for the maximum of the function \( \mu_{v_m} \) among the corresponding convex cone. \( \square \)

In the following, we will omit the subindex \( m \) for the numbers \( v_{m,i}, b_{m,i}, w_{m,i} \) in the definition of the graph associated to the filtration of vector spaces, where it is clear from the context.

Now, we recall (c.f. [3, 15]) two lemmas encoding the convexity properties of the graph associated to the Kempf filtration. They will be used in the following, to show properties shared by the possible filters \( E_i^m \) appearing in the different \( m \)-Kempf filtrations.

**Lemma 6.4** (Lemma 3.4 of [3] or Lemma 2.1.15 of [15]). Let \( 0 \subset V_1 \subset \cdots \subset V_{t+1} = V \) be the Kempf filtration of \( V \) (c.f. Theorem 4.1). Let \( v = (v_1, \ldots, v_{t+1}) \) be the vector of the graph associated to this filtration by Definition 6.1. Then 
\[
v_1 < v_2 < \cdots < v_t < v_{t+1},
\]
i.e., the graph is convex.

**Lemma 6.5** (Lemma 3.5 of [3] or Lemma 2.1.16 of [15]). Let \( 0 \subset V_1 \subset \cdots \subset V_{t+1} = V \) be the Kempf filtration of \( V \) (c.f. Theorem 4.1). Let \( W \) be a vector space with \( V_i \subset W \subset V_{i+1} \) and consider the new filtration \( V'_i \subset V \), \( 0 \subset V'_1 \subset \cdots \subset V'_{t+1} = V \). Then, \( v'_{i+1} \geq v_{i+1} \). We say that the Kempf filtration is the convex envelope of every refinement.

**Lemma 6.6** (Corollary 1.7 of [13] or Lemma 2.2 of [7]). Let \( r > 0 \) be an integer. Then there exists a constant \( B \) with the following property: for every torsion free sheaf \( E \) with \( 0 < \text{rk} (E) \leq r \), we have
\[
h^0(E) \leq \frac{1}{g^{n-1}n!} \left( (\text{rk}(E) - 1)(\mu_{\text{max}}(E) + B)_+^n + (\mu_{\text{min}}(E) + B)_+^n) \right),
\]
where \( g = \deg O_X(1), [x]_+ = \max\{0, x\} \), and \( \mu_{\text{max}}(E) \) (respectively \( \mu_{\text{min}}(E) \)) is the maximum (resp. minimum) slope of the Mumford-semistable factors of the Harder–Narasimhan filtration of \( E \).

We denote \( P_{O_X}(m) = \frac{\alpha_n}{n!} m^n + \frac{\alpha_{n-1}}{(n-1)!} m^{n-1} + \cdots + \frac{\alpha_1}{1!} m + \frac{\alpha_0}{0!} \) the Hilbert polynomial of \( O_X \), then \( \alpha_n = g \). Let
\[
P(m) = \frac{rg}{n!} m^n + \frac{d + r\alpha_{n-1}}{(n-1)!} m^{n-1} + \cdots.
\]
be the Hilbert polynomial of the sheaf \( E \), where \( d \) is the degree and \( r \) is the rank. Let us call \( A = d + r\alpha_{n-1} \), so
\[
P(m) = \frac{rg}{n!} m^n + \frac{A}{(n-1)!} m^{n-1} + \cdots.
\]
Let us define
\[ C = \max \left\{ r|\mu_{\text{max}}(E)| + \frac{d}{r} + r|B| + |A| + s\delta_{n-1}(n-1)! + 1, 1 \right\}, \quad (6.2) \]
a positive constant, where \( \delta_{n-1} \) is the leading coefficient of the polynomial \( \delta(m) \), of degree \( \leq n - 1 \) (if \( \deg \delta < n - 1 \), set \( \delta_{n-1} = 0 \)).

**PROPOSITION 6.7**

Given a sufficiently large \( m \), each filter in the \( m \)-Kempf filtration of the \( r \)k 2 tensor \( (E, \varphi) \) has slope \( \mu(E^m_i) \geq \frac{d}{r} - C \).

**Proof.** The proof follows analogously to Proposition 4.8 of [3]. Choose an \( m_1 \) such that
\[ [\mu_{\text{max}}(E) + gm + B]_+ = \mu_{\text{max}}(E) + gm + B \]
and
\[ \left[ \frac{d}{r} - C + gm + B \right]_+ = \frac{d}{r} - C + gm + B. \]
Let \( m_2 \) be such that \( P_E(m) - s\delta(m) > 0 \) for \( m \geq m_2 \). Now consider \( m \geq \max\{m_0, m_1, m_2\} \) and let
\[ 0 \subseteq (E^m_1, \varphi|_{E^m_1}) \subseteq (E^m_2, \varphi|_{E^m_2}) \subseteq \cdots \subseteq (E^m_t, \varphi|_{E^m_t}) \subseteq (E^m_{t+1}, \varphi|_{E^m_{t+1}}) = (E, \varphi) \]
be the \( m \)-Kempf filtration.

Suppose that we have a filter \( E^m_i \subseteq E \), of rank \( r_i \) and degree \( d_i \), such that \( \mu(E^m_i) < \frac{d}{r} - C \). The subsheaf \( E^m_i(m) \subseteq E(m) \) satisfies the estimate in Lemma 6.6,
\[
h^0(E^m_i(m)) \leq \frac{1}{g^{n-1}n!}((r_i - 1)((\mu_{\text{max}}(E^m_i) + gm + B)_+)^n + ((\mu_{\text{min}}(E^m_i) + gm + B)_+)^n) .
\]

Given that \( \mu_{\text{max}}(E^m_i) \leq \mu_{\text{max}}(E) \) and \( \mu_{\text{min}}(E^m_i) \leq \mu(E^m_i) \leq \frac{d}{r} - C \), and using the choice of \( m \),
\[
h^0(E^m_i(m)) \leq \frac{1}{g^{n-1}n!} \left( (r_i - 1)(\mu_{\text{max}}(E) + gm + B)^n + \left( \frac{d}{r} - C + gm + B \right)^n \right) = G(m),
\]
where
\[
G(m) = \frac{1}{g^{n-1}n!} \left[ r_i g^m n^m + n g^{n-1} (r_i - 1) \mu_{\text{max}}(E) + \frac{d}{r} - C + r_i B \right] m^{n-1} + \cdots .
\]
By Definition 6.1, to the $m$-Kempf filtration we associate the graph given by
\[
w_j = w^1 + \cdots + w^j = m \cdot \frac{1}{\dim V} \left[ r \dim V_j - r_j \dim V + \frac{a_2}{a_1} (s \dim V_j - \epsilon_j(\Phi) \dim V) \right].
\]

We will get a contradiction by showing that $w_i < 0$. Indeed, if $w_i < 0$ there is a $j < i$ such that $-v_j < 0$. Hence, as the graph is convex by Lemma 6.4 the rest of the slopes of the graph are negative, $-v_k < 0$, $k \geq i$. Then $w_i > w_{i+1} > \cdots w_{t+1}$ and $w_{t+1} < 0$. But it is
\[
w_{t+1} = m \cdot \frac{1}{\dim V} \left[ r \dim V_{t+1} - r_{t+1} \dim V + \frac{a_2}{a_1} (s \dim V_{t+1} - \epsilon_{t+1}(\Phi) \dim V) \right] = 0,
\]
because $r_{t+1} = r$, $V_{t+1} = V$ and $\epsilon_{t+1}(\Phi) = s$, then the contradiction.

Since $E_i^m(m)$ is generated by $V_i$ under the evaluation map, it is $\dim V_i \leq H^0(E_i^m(m))$, hence
\[
w_i = \frac{m}{\dim V} \left[ r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\Phi) \dim V) \right]
\leq \frac{m}{P_E(m)} \left[ r h^0(E_i^m(m)) - r_i P_E(m) + \frac{r \delta(m)}{P_E(m) - s \delta(m)} (s h^0(E_i^m(m)) - \epsilon_i(\Phi) P_E(m)) \right]
\leq m \frac{[P_E(m) - s \delta(m)(r G(m) - r_i P_E(m)) + (r \delta(m))(s G(m) - \epsilon_i(\Phi) P_E(m))]}{P_E(m)(P_E(m) - s \delta(m))}.
\]

Then, $w_i < 0$ is equivalent to
\[
\Psi(m) = (P_E(m) - s \delta(m))(r G(m) - r_i P_E(m)) + (r \delta(m))(s G(m) - \epsilon_i(\Phi) P_E(m)) < 0,
\]
and $\Psi(m) = \xi_{2n} m^{2n} + \xi_{2n-1} m^{2n-1} + \cdots + \xi_1 m + \xi_0$ is a $(2n)$th-order polynomial, whose higher order coefficient is
\[
\xi_{2n} = (P_E(m) - s \delta(m))_n (r G(m) - r_i P_E(m))_n + (r \delta(m))_n (s G(m) - \epsilon_i(\Phi) P_E(m))_n
= (P_E(m) - s \delta(m))_n \left( r_i \frac{g}{n!} - r_i \frac{g}{n!} \right) + 0 = 0.
\]

The $(2n - 1)$th-order coefficient is
\[
\xi_{2n-1} = (P_E(m) - s \delta(m))_n (r G(m) - r_i P_E(m))_{n-1} + (r \delta(m))_{n-1} (s G(m) - \epsilon_i(\Phi) P_E(m))_{n-1}
= \frac{rg}{n!} \left( r G_{n-1} - r_i \frac{A}{(n - 1)!} \right) + r \delta_{n-1} \left( s \frac{rg}{n!} - \epsilon_i(\Phi) \frac{rg}{n!} \right),
\]
where $G_{n-1}$ is the $(n-1)$th-coefficient of the polynomial $G(m)$,

$$G_{n-1} = \frac{1}{n!} r^n (r_i - 1) \mu_{\text{max}}(E) + \frac{d}{r} - C + r_i B$$

$$= \frac{1}{(n-1)!} (r_i - 1) \mu_{\text{max}}(E) + \frac{d}{r} - C + r_i B$$

$$\leq \frac{1}{(n-1)!} (r_i - 1) |\mu_{\text{max}}(E)| + \frac{d}{r} - C + r_i |B|$$

$$\leq \frac{1}{(n-1)!} (r_i - 1) |\mu_{\text{max}}(E)| + \frac{d}{r} - C + r_i |B| \leq \frac{-|A|}{(n-1)!} - s \delta_{n-1},$$

where the last inequality comes from the definition of $C$ in (6.2). Then

$$\xi_{2n-1} < \frac{rg}{n!} \left( r \left( \frac{-|A|}{(n-1)!} - s \delta_{n-1} \right) - r_i \frac{A}{(n-1)!} \right)$$

$$+ r \delta_{n-1} \left( \frac{r_i g s}{n!} - \epsilon_i(\Phi) \frac{rg}{n!} \right)$$

$$= \frac{rg}{n!} \left[ \left( \frac{-|A| - r_i A}{(n-1)!} \right) - r s \delta_{n-1} + \delta_{n-1} (r_i s - \epsilon_i(\Phi) r) \right]$$

$$= \frac{rg}{n!} \left[ \left( \frac{-|A| - r_i A}{(n-1)!} \right) + \delta_{n-1} (-r s + r_i s - \epsilon_i(\Phi) r) \right]$$

$$< \frac{rg}{n!} \delta_{n-1} (-r s + r_i s - \epsilon_i(\Phi) r).$$

because $-r|A| - r_i A < 0$. Last expression is always negative because, if $r_i < r$,

$$-r s + r_i s - \epsilon_i(\Phi) r = -(r - r_i) s - \epsilon_i(\Phi) r \leq \epsilon_i(\Phi) r \leq 0,$$

with equality if and only if $r_i = r$, and if $r_i = r$, then $\epsilon_i(\Phi) = s$, and

$$-r s + r_i s - \epsilon_i(\Phi) r = -r s < 0.$$

Hence, it is $\xi_{2n-1} < 0$.

Therefore $\Psi(m) = \xi_{2n-1} m^{2n-1} + \cdots + \xi_1 m + \xi_0$ with $\xi_{2n-1} < 0$, so there exists an integer $m_3$ such that for $m \geq \{m_0, m_1, m_2, m_3\}$ we have $\Psi(m) < 0$ and $w_i < 0$, then the contradiction.

Once we have seen that all possible filters in the different $m$-Kempf filtrations have their numerical invariants bounded, and all of them are subsheaves of the same sheaf, we can prove the following:

**PROPOSITION 6.8**

There exists an integer $m_4$ such that for $m \geq m_4$ the sheaves $E^m_i$ and $E^{m,i} = E^m_i/E^{m-1}_i$ are $m_4$-regular. In particular, their higher cohomology groups, after twisting with $O_X(m_4)$, vanish and they are generated by global sections.

**Proof.** See Proposition 3.9 of [3].

**PROPOSITION 6.9**

Let $m \geq m_4$. For each filter $E^m_i$ in the $m$-Kempf filtration of the rk 2 tensor $(E, \varphi)$, we have $\dim V_i = h^0(E^m_i(m))$, therefore $V_i \cong H^0(E^m_i(m)).$
Proof. Let \( V_* \subseteq V \) be the Kempf filtration of \( V \) (c.f. Theorem 4.1) and let \((E^m_\bullet, \varphi|_{E^m_\bullet}) \subseteq (E, \varphi)\) be the \( m \)-Kempf filtration of \((E, \varphi)\). We can construct two filtrations:

\[
0 \subset \cdots \subset V_i \subset V_{i+1} \subset V_{i+2} \subset \cdots \subset V
\]

\[
\cap \quad \cap \quad \cap \quad \cap
\]

\[
H^0(E^m_i(m)) \subset H^0(E^m_{i+1}(m)) \subset H^0(E^m_{i+2}(m))
\]

(6.3)

and

\[
0 \subset \cdots \subset V_i \subset H^0(E^m_i(m)) \subset V_{i+1} \subset \cdots \subset V
\]

\[
\cap \quad \cap \quad \cap \quad \cap
\]

\[
V'_i \quad V'_{i+1} \quad V'_{i+2}
\]

(6.4)

to be in situation of Lemma 6.5, where \( W = H^0(E^m_i(m)) \), filtration \( V_* \) is (6.3) and filtration \( V'_* \) is (6.4).

Now, the graph associated to the filtration \( V_* \) is given, by Definition 6.1, by the points

\[
(b_i, w_i) = \left( \frac{\dim V_i}{m^n}, \frac{m}{\dim V} \left( r \dim V_i - r_i \dim V + \frac{a_2}{a_1} (s \dim V_i - \epsilon_i(\tilde{\Phi}, V_* \dim V) \right) \right),
\]

the slopes \(-v_i\) of the graph given by

\[
-v_i = \frac{w^i}{b^i} = \frac{w_i - w_{i-1}}{b_i - b_{i-1}} = \frac{m^{n+1}}{\dim V} \left( r - \frac{r_i}{\dim V_i} \frac{\dim V}{\dim V_i} \right)
\]

\[
+ \frac{a_2}{a_1} \left( s - \epsilon^i(\tilde{\Phi}, V_* \dim V) \right)
\]

\[
\leq \frac{m^{n+1}}{\dim V} \left( r + s \frac{a_2}{a_1} \right) = R,
\]

and equality holds if and only if \( r^i = 0 \) (note that \( r^i = 0 \) implies \( \epsilon^i(\tilde{\Phi}, V_* = 0) \).

The new point which appears in the graph of the filtration \( V'_* \) is

\[
Q = \left( \frac{h^0(E^m_i(m))}{m^n}, \frac{m}{\dim V} \left( r h^0(E^m_i(m)) - r_i \dim V \right)
\]

\[
+ \frac{a_2}{a_1} (s h^0(E^m_i(m)) - \epsilon_i(\tilde{\Phi}, V_* \dim V)) \right).
\]

Note that

\[
\epsilon_j(\tilde{\Phi}, V'_*) = \epsilon_j(\tilde{\Phi}, V_*), \quad j \leq i
\]

\[
\epsilon_j(\tilde{\Phi}, V'_*) = \epsilon_{j-1}(\tilde{\Phi}, V_*), \quad j > i.
\]

(6.5)

This is the reason why we write \( \epsilon_i(\tilde{\Phi}, V'_*) \) instead of \( \epsilon_i(\tilde{\Phi}, V'_*) \).

The slope of the segment between \((b_i, w_i)\) and \( Q \) is, similarly,

\[
-v'_i = \frac{m^{n+1}}{\dim V} \left( r + s \frac{a_2}{a_1} \right) = R.
\]
By Lemma 6.4, the graph is convex, so $v_1 < v_2 < \cdots < v_{t+1}$. Besides, $r^1 = r_1 > 0$, then $-R < v_1$. This is because $E$ is torsion free, hence $E^m_1 \subset E$ also has no torsion, and a rank 0 torsion free sheaf is the zero sheaf. On the other hand, the graph associated to $V_\bullet \subset V$ is a refinement of the one associated to the Kempf filtration $V_\bullet \subset V$, then we apply Lemma 6.5 and get $v'_i \geq v_i$. Hence,

$$-R < v_1 < v_2 < \cdots < v_i \leq v'_i = -R,$$

which is a contradiction.

Therefore, $\dim V_i = h^0(E^m_i(m))$, for every filter in the $m$-Kempf filtration. \(\square\)

**COROLLARY 6.10**

Let $m \geq m_4$. For every filter $E^m_i$ in the $m$-Kempf filtration of the rk 2 tensor $(E, \varphi)$, it is $r^i > 0$. Therefore, the $m$-Kempf filtration consists of a rank 1 subsheaf, $0 \subset (L^m, \varphi|_{L^m}) \subset (E, \varphi)$.

**Proof.** By Proposition 6.9, $r^i = 0$ is equivalent to $-v_i = R$. Then, $r^1 = r_1 > 0$ and $-R < v_1 < v_2 < \cdots < v_{t+1}$ imply the statement. \(\square\)

For any $m \geq m_4$, by Corollary 6.10 there is only one filter $(L^m, \varphi|_{L^m})$ in the $m$-Kempf filtration and, by Proposition 6.8, $L^m$ is $m_4$-regular. Hence, $L^m(m_4)$ is generated by the subspace $H^0(L^m(m_4)) \subset H^0(E(m_4))$ by the evaluation map. Note that the dimension of the vector space $H^0(E(m_4))$ does not depend on $m$.

We call $m$-type of the $m$-Kempf filtration to the Hilbert polynomial $P_{L^m}$. Once we fix $V \simeq H^0(E(m_4))$ whose dimension does not depend on $m$, all possible filtrations of $V$ are parametrized by a finite-type scheme, hence the set of possible $m$-types

$${\mathcal P} = \{P_{L^m}\}$$

is finite, for all integers $m \geq m_4$.

Rewrite the graph associated to the $m$-Kempf filtration (c.f. Definition 6.1)

$$v_{m,i} = \frac{m^{n+1}}{\dim V^i \cdot \dim V} \left[ r^i \dim V - r \dim V^i + \frac{a_2}{a_1} (\epsilon^i(\Phi) \dim V - s \dim V^i) \right],$$

$$b_{m}^i = \frac{1}{m^n} \cdot \dim V^i,$$

as

$$v_{m,i} = \frac{m^{n+1}}{P_m^i(m) P(m)} \left[ r^i P(m) - r P_m^i(m) + \frac{r \delta(m)}{P(m) - s \delta(m)} (\epsilon^i(\Phi) P(m) - s P_m^i(m)) \right],$$

$$b_{m}^i = \frac{1}{m^n} \cdot P_m^i(m),$$

by Propositions 6.8 and 6.9.
Note that, by Corollary 6.10, the graph has only two slopes given by

\[
v_{m,1} = \frac{m^{n+1}}{P_L(m)P(m)} \left[ P(m) - 2P_L(m) \right]
+ \frac{2\delta(m)}{P(m) - \epsilon_L m P(m)} \left( (\epsilon_L - s P_L(m)) \right),
\]

\[
v_{m,2} = \frac{m^{n+1}}{P_{E/L}(m)P(m)} \left[ P(m) - 2P_{E/L}(m) \right]
+ \frac{2\delta(m)}{P(m) - \epsilon_L m P(m)} \left( (s - \epsilon_L)P(m) - s P_{E/L}(m) \right),
\]

where \(\epsilon(L^m)\) is the number of times that the subsheaf \(L^m\) appears on the minimal multi-index (c.f. (3.2)).

The set

\[ A = \{ \Theta_m : m \geq m_4 \}, \]

where

\[ \Theta_m(l) = (\mu_{v_m(l)}(\Gamma_{v_m(l)}))^2 = ||v_m(l)||^2, \]

is finite because the set of \(m\)-types, \(P\), is. We say that \(f_1 < f_2\) for two rational functions, if the inequality \(f_1(l) < f_2(l)\) holds for \(l \gg 0\). Let \(K\) be the maximal function in the finite set \(A\), with respect to the defined ordering. The function \(K\) verifies that there exists an integer \(m_5\) such that, for all \(m \geq m_5\), it is \(\Theta_m = K\) (c.f. Lemma 5.2 of [3]).

**Proposition 6.11**

Let \(l_1\) and \(l_2\) be integers with \(l_1 \geq l_2 \geq m_5\). Then, the \(l_1\)-Kempf filtration of \(E\) is equal to the \(l_2\)-Kempf filtration of \(E\).

**Proof.** By construction, the filtration

\[ 0 \subset H^0(L^{l_1}(l_1)) \subset H^0(E(l_1)) \]

is the \(l_1\)-Kempf filtration of \(V \simeq H^0(E(l_1))\). Now consider the filtration \(V' \subset V \simeq H^0(E(l_1))\) defined as follows:

\[ 0 \subset H^0(L^{l_2}(l_1)) \subset H^0(E(l_1)). \]

We have to prove that (6.7) is, in fact, the \(l_1\)-Kempf filtration of \(V \simeq H^0(E(l_1))\).

Given that \(l_1, l_2 \geq m_5\), we have \(\Theta_{l_1} = \Theta_{l_2} = K\). Hence, \(\Theta_{l_1}(l_1) = \Theta_{l_2}(l_1)\) and, by uniqueness of the Kempf filtration (c.f. Theorem 4.1), filtrations (6.6) and (6.7) do coincide. Since, in particular, \(l_1, l_2 \geq m_4\), \(L^{l_1}\) and \(L^{l_2}\) are \(l_1\)-regular by Proposition 6.8. Hence, \(L^{l_1}(l_1)\) and \(L^{l_2}(l_1)\) are generated by their global sections \(H^0(L^{l_1}(l_1))\) and \(H^0(L^{l_2}(l_1))\), respectively. By the previous argument, \(H^0(L^{l_1}(l_1)) = H^0(L^{l_2}(l_1))\), therefore \(L^{l_1}(l_1) = L^{l_2}(l_1)\) and, tensoring with \(O_X(-l_1)\), this implies that the subsheaves \(L^{l_1} \subset E\) and \(L^{l_2} \subset E\) coincide.

Therefore, Theorem 4.2 follows from Proposition 6.11. Hence, eventually, the Kempf filtration of the rk 2 tensor \((E, \varphi)\) does not depend on the integer \(m\).
DEFINITION 6.12

If \( m \geq m_5 \), the \( m \)-Kempf filtration of the rk 2 tensor \((E, \varphi)\)

\[
0 \subset (L, \varphi|_L) \subset (E, \varphi)
\]
is called the Kempf filtration or the Kempf subsheaf of \((E, \varphi)\).

7. Harder–Narasimhan filtration for rk 2 tensors

Kempf theorem (c.f. Theorem 4.1) says that, given an integer \( m \) and \( V \cong H^0(E(m)) \), there exists a unique weighted filtration of vector spaces \( V_\bullet \subseteq V \) which gives maximum for the Kempf function

\[
\mu(V_\bullet, n_\bullet) = \frac{\sum_{i=1}^{t+1} \frac{r_i}{\dim V}(r_i \dim V - r \dim V + \frac{a_2}{\partial_1}(\epsilon^i(\tilde{\Phi}) \dim V - s \dim V^i))}{\sqrt{\sum_{i=1}^{t+1} \dim V^i \Gamma^2_i}}.
\]

This filtration induces a unique rank 1 subsheaf \( L \subset E \) called the Kempf subsheaf of the rk 2 tensor \((E, \varphi)\). By Proposition 6.11, the subsheaf \( L \) does not depend on \( m \), for \( m \geq m_5 \).

The Kempf function is a function on \( m \) (c.f. Proposition 6.3). Consider the function

\[
K(m) = m^{\frac{a_2}{\partial_1} + 1} \cdot \mu(V_\bullet, m_\bullet) = \mu_{v_m}(\Gamma).
\]

Set \( \gamma_i = \frac{\partial_1}{P} \Gamma_i \), then \( \frac{\gamma_{i+1} - \gamma_i}{\partial_1} = n_i \) and \( \sum r_i \gamma_i = \gamma_1 + \gamma_2 = 0 \), which gives \( \gamma_1 = -n_1 \), \( \gamma_2 = n_1 \). Making the substitutions for \( m \) sufficiently large,

\[
\begin{align*}
\dim V_1 &= \dim V^1 = h^0(L(m)) = P_L(m), \\
\dim V^2 &= \dim V - \dim V_1 = h^0(E/L(m)) = P_{E/L}(m).
\end{align*}
\]

we get

\[
K(m) = m^{\frac{a_2}{\partial_1} + 1} \cdot \sum_{i=1}^{2} \frac{\gamma_i}{r}[\frac{r^i P - r P^i}{P - s\delta} + \frac{r\delta}{P - s\delta}(\epsilon^i P - s P^i)]
\]

\[
\times \frac{\sqrt{\sum_{i=1}^{2} P^i P^2_{\gamma_i} \gamma_i^2}}{\sqrt{P(P - s\delta)}}[2P_L - P_E + \delta(s - 2\epsilon(L))],
\]

where we set \( P = P_E(m), P^1 = P_L(m), P^2 = P_{E/L}(m), \epsilon^1 = \epsilon(L), \epsilon^2 = s - \epsilon(L) \).

Note that \( \epsilon^i = \epsilon^i(\tilde{\Phi}) = \epsilon^i(\varphi) \) and recall

\[
\frac{a_2}{\partial_1} = \frac{r\delta}{P - s\delta}.
\]

Substituting, we get

\[
K(m) = m^{\frac{a_2}{\partial_1} + 1} \cdot \frac{1}{P - s\delta}
\times \frac{-n_1[2(\delta\epsilon^1 - P^1) + (P - \delta s)] + n_1[2(\delta\epsilon^2 - P^2) + (P - \delta s)]}{\sqrt{P^1 n_1^2 + P^2 n_1^2}}
\]

\[
= m^{\frac{a_2}{\partial_1} + 1} \cdot \frac{r}{\sqrt{P(P - s\delta)[2P_L - P_E + \delta(s - 2\epsilon(L))]}},
\]
Note that the unique weight \( n_1 \) does not appear in the function later from the substitutions, as it was expected from a one-step filtration. Also note that the denominator of the function \( K \) is positive (c.f. choice of \( m_2 \) in proof of Proposition 6.7). Hence, we can state the following theorem.

**Theorem 7.1.** Given a \( \delta \)-unstable \( \text{rk} \, 2 \) tensor \((E, \varphi : E \otimes \cdots \otimes E \to M)\), there exists a unique line subsheaf \( L \subset E \) which gives maximum for the polynomial function

\[
K(m) = 2PL(m) - PE(m) + \delta(m)(s - 2\epsilon(L)) .
\]

If \( X \) is a one dimensional complex projective variety, i.e. a smooth projective complex curve, we can simplify the function \( K \). Recall that, by Riemann–Roch, the Hilbert polynomial of a sheaf \( E \) of rank \( r \) and degree \( d \) over a curve of genus \( g \) is

\[
P_E(m) = rm + d + r(1 - g) ,
\]
and the polynomial \( \delta(m) \) becomes a positive constant that we will denote by \( \tau \). In this case, a coherent torsion free sheaf of rank 2 is a vector bundle of rank 2 over \( X \), and the Kempf subsheaf will be a line subbundle.

**Theorem 7.2.** Given a \( \tau \)-unstable \( \text{rk} \, 2 \) tensor \((E, \varphi : E \otimes \cdots \otimes E \to M)\) over a smooth projective complex curve, there exists a unique line subbundle \( L \subset E \) which maximizes the quantity

\[
2 \deg L - \deg E + \tau(s - 2\epsilon(L)) .
\]

Note that, if the tensor is unstable, such quantity will be positive, and the graph corresponding to the filtration will be a cusp which is a convex graph.

If we define the corrected Hilbert polynomials of \((E, \varphi)\) and \((L, \varphi|_L)\) as

\[
\tilde{P}_E = P_E - \delta s , \quad \tilde{P}_L = P_L - \delta \epsilon(L) ,
\]
we can rewrite the notion of stability for \( \text{rk} \, 2 \) tensors (c.f. Definition 2.5): a \( \text{rk} \, 2 \) tensor \((E, \varphi)\) is \( \delta \)-unstable if there exists a line subsheaf \( L \subset E \) such that

\[
\frac{\tilde{P}_L}{\text{rk} \, L} > \frac{\tilde{P}_E}{\text{rk} \, E} \iff \tilde{P}_L > \frac{\tilde{P}_E}{2} .
\]

Theorem 7.1 establishes that there exists a unique subsheaf, the Kempf subsheaf, maximizing certain polynomial function. This is equivalent to contradict, in a maximal way, the definition of stability (c.f. Definition 2.5). Therefore, we can define a notion of a Harder–Narasimhan filtration for \( \delta \)-unstable \( \text{rk} \, 2 \) tensors as this unique line subsheaf which maximally contradicts GIT stability.

**DEFINITION 7.3**

If \((E, \varphi)\) is a \( \delta \)-unstable \( \text{rk} \, 2 \) tensor, there exists a unique \( \text{rk} \, 1 \) subsheaf maximizing

\[
2 \cdot \tilde{P}_L - \tilde{P}_E > 0 .
\]
We call

$$0 \subset (L, \varphi|_L) \subset (E, \varphi)$$

the **Harder–Narasimhan filtration** of \( (E, \varphi) \), and we call \( L \) the **Harder–Narasimhan subsheaf** of \( (E, \varphi) \).

**Remark 7.4.** We do not know, in principle, how to define a quotient tensor \( (E/L, \bar{\varphi}|_{E/L}) \), because we do not know, \textit{a priori}, how to define \( \bar{\varphi}|_{E/L} \). This is why we cannot talk about quotient tensors.

Given the exact sequence of sheaves, \( 0 \to L \to E \to E/L \to 0 \), we define the corrected Hilbert polynomial of the quotient as \( \tilde{P}_{E/L} = \tilde{P}_E - \tilde{P}_L \), and we have, trivially, the additivity of the corrected polynomials on exact sequences of sheaves. This way we can consider that Definition 7.3 contains the analogous to the conditions of the classical Harder–Narasimhan filtration for sheaves, in the case of \( \text{rk} \ 2 \) tensors. Indeed,

$$2 \cdot \tilde{P}_L - \tilde{P}_E > 0 \iff \tilde{P}_L > \tilde{P}_{E/L} ,$$

and the semistability of \( (L, \varphi|_L) \) and \( (E/L, \bar{\varphi}|_{E/L}) \) (whichever definition of \( \bar{\varphi}|_{E/L} \) we impose), would follow trivially from the fact that they are rank 1 tensors.

Therefore, Definition 7.3 gives a notion of a Harder–Narasimhan filtration with the properties we would expect it to have.

**8. Stable coverings of a projective curve**

In this section we use the previous notions for \( \text{rk} \ 2 \) tensors over curves where the morphism is symmetric, and the Definition 7.3 of the Harder–Narasimhan subsheaf, to define stable coverings of a projective curve and, for the unstable ones, a maximally destabilizing object, in terms of intersection theory.

In the following, we shall consider tensors \( (E, \varphi) \) where \( E \) is a \( \text{rk} \ 2 \) vector bundle over a smooth complex projective curve \( X \), and

$$\varphi : \underbrace{E \otimes \cdots \otimes E}_{s \ \text{times}} \to M$$

is a symmetric non degenerate morphism. We call it a **symmetric non degenerate rank 2 tensor**. The non degeneracy condition means that \( \varphi \) induces an injective morphism

$$E \hookrightarrow (E \otimes \cdots \otimes E)^{\vee} \otimes M .$$

Let \( \tau \) be a positive real number. Let \( \mathcal{P}(E) \) be the projective space bundle of the vector bundle \( E \), which is a ruled algebraic surface (c.f. Section V.2 of [5]).

The morphism \( \varphi \) is, fiberwise, a symmetric multilinear map

$$\varphi : \underbrace{V \otimes \cdots \otimes V}_{s \ \text{times}} \to \mathbb{C} ,$$

$$\Psi : \underbrace{M \otimes \cdots \otimes M}_{(s-1) \ \text{times}} \to \mathcal{P}(E) .$$
where $V \simeq \mathbb{C}^2$. Then, $\varphi_x$ factors through $\text{Sym}^s(V)$, isomorphic to the $(s+1)$-dimensional vector space of homogeneous polynomials of degree $s$ in two variables. Hence, fiberwise, $\varphi$ can be represented by a polynomial

$$\varphi_x \equiv \sum_{i=0}^{s} a_i(x)X_0^iX_1^{s-i} \quad (8.1)$$

which vanishes on $s$ points in $\mathbb{P}(V) \simeq \mathbb{P}^1_{\mathbb{C}}$. Therefore, as $\varphi$ varies on $X$, it defines a degree $s$ covering $\mathbb{P}(E) \supset X' \rightarrow X$.

Suppose that $(E, \varphi)$ is a $\tau$-unstable rk 2 tensor. Then, by Theorem 7.2, there exists a line subbundle $L \subset E$, the Harder–Narasimhan subbundle, giving maximum for the quantity

$$2 \deg(L) - \deg(E) + \tau(s - 2\epsilon(L)). \quad (8.2)$$

The subbundle $L$ can be seen as a section of $\mathbb{P}(E)$, each fiber $L_x$ corresponding to a point $P = \{L_x\} \in \mathbb{P}^1_{\mathbb{C}}$. Recall from Definition 2.2 that $\epsilon(L) = k$ if $\varphi|_{L \otimes (k+1) \otimes E \otimes (s-k-1)} = 0$ and $\varphi|_{L \otimes k \otimes E \otimes (s-k)} \neq 0$. Note that here we use the symmetry of the morphism $\varphi$. Therefore, $\epsilon(L) = k$ means that, generically, $P = \{L_x\}$ is a zero of multiplicity $s - k$ and, by definition of the covering $X' \rightarrow X$, $s - \epsilon(L)$ is, exactly, the number of branches of $X'$ which generically do coincide with the section defined by $L$, counted with multiplicity.

We can find in [1] the classical example of classifying a configuration of points in $\mathbb{P}^1_{\mathbb{C}}$ up to the action of $\text{PGL}(2)$. There, a homogeneous polynomial of degree $N$, $P = \sum_i a_i X_0^iX_1^{N-i}$, is unstable if it contains a linear factor of degree greater than $N/2$. Now, observe that the restriction of a rk 2 tensor to a point $x \in X$ in (8.1), passing to the projectivization $\mathbb{P}(E)$ hence fibers are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, is precisely one of the homogeneous polynomials in [1]. Fiberwise, the morphism $\varphi$ defines a set of $s$ points in $\mathbb{P}^1_{\mathbb{C}}$. See that, from the point of view of [1], letting $s = N$, the set of points is unstable if there exists a point with multiplicity greater that $\frac{s}{2}$.

Then, as $s - \epsilon(L)$ is the multiplicity of the point defined by the line $L_x$ (the fiber of the Harder–Narasimhan subbundle over $x$), in the set of $s$ points defined by the morphism $\varphi$, following the previous argument, this point $\{L_x\}$ will destabilize the set if

$$s - \epsilon(L) > \frac{s}{2} \Leftrightarrow s - 2\epsilon(L) > 0,$$

which is the second summand in (8.2). Hence, the positivity of $s - 2\epsilon(L)$ is equivalent to the line bundle $L$ defining a point in the fiber $\mathbb{P}^1_{\mathbb{C}}$, which coincides with one of the zeroes of $\varphi$ in the fiber, and such that the zero has multiplicity greater that $\frac{s}{2}$.

To conclude, we can say that the expression (8.2) consists of two summands weighted by the parameter $\tau$. First one, $2 \deg(L) - \deg(E)$, is measuring the stability of the vector bundle $E$. Second one, $s - 2\epsilon(L)$, is measuring the stability of the morphism or, with the previous observations, the generic stability of the set of points defined in $\mathbb{P}^1_{\mathbb{C}}$, fiberwise, as in [1], when varying along the covering. Therefore, an object destabilizing a rk 2 tensor is an object which contradicts these two stabilities, weighted by $\tau$, and the Harder–Narasimhan subbundle is the unique one which maximally does, for a $\tau$-unstable tensor.
The sets of points in each fiber defined by $\varphi$ give a covering of degree $s$,

$$\mathbb{P}(E) \supset X' \to X.$$  

In the following, we rewrite the stability of the sets of points, fiberwise, as stability for the covering, using intersection theory for ruled surfaces.

**PROPOSITION 8.1** (Proposition V.2.8 of [5])

*Given a ruled surface $\mathbb{P}(E)$, there exists $E' \simeq E \otimes N$, with $N$ line bundle, such that $H^0(E') \neq 0$ but for all line bundles $N'$ with negative degree we have $H^0(E' \otimes N') = 0$. Therefore, $\mathbb{P}(E) = \mathbb{P}(E')$ and the integer $e = -\deg E'$ is an invariant of the ruled surface. Furthermore, in this case, there exists a section $\sigma_0 : X \to \mathbb{P}(E')$ with image $C_0$, such that $L(C_0) \simeq \mathcal{O}_X(1)$.***

**DEFINITION 8.2**

Let $(E, \varphi : [E \otimes \cdots \otimes E] \to M)$ be a symmetric non degenerate rank 2 tensor over $X$. We call $(E', \varphi')$ an *associated normalized tensor* where $E = E \otimes N$, $N$ a line bundle as in Proposition 8.1, and $\varphi'$ is the induced morphism given by

$$\varphi' : (E')^\otimes s = E^\otimes s \otimes N^\otimes s \to M \otimes N^\otimes s,$$

and extending by the identity on $N^\otimes s$.

**PROPOSITION 8.3**

*The quantity in (8.2) is an invariant for all associated normalized tensors. Hence, $(E, \varphi)$ is $\tau$-unstable if and only if an associated normalized tensor $(E', \varphi')$ is $\tau$-unstable.***

**Proof.** Let $N$ be a line bundle over $X$, as in Proposition 8.1. If we change $E$ by $E' = E \otimes N$, then we have the line subbundle $L \otimes N \subset E'$ (by exactness of the tensor product with locally free sheaves), and

$$\deg(E') = \deg(E \otimes N) = \deg(E) + 2\deg(N),$$

$$\deg(L \otimes N) = \deg(L) + \deg(N),$$

so the quantity $2\deg(L) - \deg(E)$ is invariant by tensoring $E$ with a line bundle.

Also note that, after defining

$$\varphi' : (E')^\otimes s = E^\otimes s \otimes N^\otimes s \to M \otimes N^\otimes s,$$

it is $\epsilon'(L \otimes N) = \epsilon(L)$. Hence, the quantity

$$2\deg(L) - \deg(E) + \tau(s - 2\epsilon(L))$$

remains the same for associated normalized tensors. $\square$
Let $\mathbb{P}(E')$ be a ruled surface with $E'$ normalized as in Proposition 8.1. Let $\sigma : X \to \mathbb{P}(E)$ be a section, and let $D = \text{im} \sigma$ be a divisor on $\mathbb{P}(E)$. It can be proved that $\deg(\sigma) = -e - C_0 \cdot D$, with these conventions (c.f. Proposition V.2.9 of [5]). Note that the section $C_0$ depends on the line bundle $N$ in Proposition 8.1, but the number $\deg(\sigma) = -e - C_0 \cdot D$ does not. Let us define, by analogy, $\epsilon(\sigma) = \epsilon(D)$ as the number of branches of $X'$ which generically do coincide with $D$, the divisor defined by $\sigma$, counted with multiplicity.

**DEFINITION 8.4**

Let $f : X' \to X$ be a covering defined by a normalized symmetric non degenerate rank 2 tensor $(E, \varphi)$, $X' \subset \mathbb{P}(E)$. Let $C_0$ be the image of a section $\sigma_0 : X \to \mathbb{P}(E)$ such that $\mathcal{L}(C_0) \simeq \mathcal{O}_X(1)$. Let $\tau$ be a positive number. We say that $f$ is $\tau$-unstable if there exists a section $\sigma : X \to \mathbb{P}(E)$ with image $D$, i.e. there exists a line subbundle $L \subset E$, such that the following holds:

$$-2C_0 \cdot D - e + \tau(s - 2\epsilon(D)) > 0.$$  

(8.3)

**PROPOSITION 8.5**

*Let $\tau$ be a positive number. A symmetric non degenerate rank 2 tensor $(E, \varphi)$ is $\tau$-unstable if and only if the associated covering $f : X' \to X$ is $\tau$-unstable.*

*Proof.* By Proposition 8.3, we can suppose that $(E, \varphi)$ is normalized and $\tau$-unstable. Then, we just have to notice that expression (8.3) corresponds to (8.2) by the previous discussion, and does also not change by passing to a normalized associated tensor.  

Finally, as we announced, we do characterize the Harder–Narasimhan filtration, in this case, in terms of intersection theory. This last theorem follows from the previous results.

**Theorem 8.6.** *If $f : X' \to X$ is a degree $s$ covering coming from a symmetric non degenerate rank 2 tensor $(E, \varphi)$ which is $\tau$-unstable, then there exists a unique section $\sigma : X \to \mathbb{P}(E)$ with image $D$, giving maximum for

$$-2C_0 \cdot D - e + \tau(s - 2\epsilon(D)).$$

We call $\sigma$ the Harder–Narasimhan section of the covering.*

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