In Fe-based superconductors, the nematic order and fluctuations attract great attention as one of the essential properties of the electronic states. A schematic phase diagram of BaFe$_2$As$_2$ as a function of carrier doping $y$ is shown in Fig. 1: For $y > 0$ (e-doping), the non-magnetic orthorhombic ($C_2$) phase transition occurs at $T_S$, and the antiferro (AF) spin order is realized at $T_N (\lesssim T_S)$ in the $C_2$ phase. In Ba(Fe$_{1-y}$Co$_y$)$_2$As$_2$ ($y = x$), both the structural and magnetic quantum critical points (QCPs) are very close, and strong magnetic fluctuations are observed near the QCPs by NMR [1]. In addition, strong nematic susceptibility that couples to the shear modulus $C_{66}$ and the Raman quadrupole susceptibility $\chi_{Raman}^{x^2-y^2}$ due to the Aslamazov-Larkin vertex correction, the nematic-type orbital fluctuations are induced, and they enhance both $1/C_{66}$ and $\chi_{Raman}^{x^2-y^2}$ strongly. However, $\chi_{Raman}^{x^2-y^2}$ remains finite even at the structure transition temperature $T_S$, because of the absence of the band Jahn-Teller effect and the Pauli (=intra-band) contribution, as proved in terms of the linear response theory. The present study clarifies that origin of the nematicity in Fe-based superconductors is the nematic-orbital order/fluctuations.

FIG. 1: (color online) (a)Schematic phase diagram of Fe-based superconductors. (b)Fermi surfaces for $y = 0$. The weight of $dx_z$ orbital is stressed by green circles. (c) Relation $\lambda_{\text{photon}} \gg \lambda_{\text{ac}}$. (d) Particle-hole excitation continuum.

In this paper, we analyze both $C_{66}$ and $\chi_{Raman}^{x^2-y^2}$, both of which are key experiments to uncover the nematic order parameter. It is found that both $C_{66}$ and $\chi_{Raman}^{x^2-y^2}$ are enhanced by the orbital fluctuations due to Aslamazov-Larkin type VC (AL-VC). However, $\chi_{Raman}^{x^2-y^2}$ is less singular since the band Jahn-Teller (band-JT) effect and the Pauli (=intra-band) quadrupole susceptibilities does not contribute to $\chi_{Raman}^{x^2-y^2}$. Since both $C_{66}$ and $\chi_{Raman}^{x^2-y^2}$ are explained satisfactorily, the orbital nematic scenario is essential for many Fe-based superconductors.
As for the pairing mechanism, at present, both the spin fluctuation mediated \(s_\pm\) wave state [22–24] and orbital fluctuation mediated \(s_{++}\) wave state [25, 26] have been discussed. When both fluctuations coexist, nodal \(s\)-wave state can be realized [27]. The \(s_{++}\)-wave state is consistent with the robustness of \(T_c\) against impurities [28, 29] and broad hump structure in the inelastic neutron scattering [30, 31]. The self-consistent vertex correction (SC-VC) method [13, 26] predicts the developments of ferro- and AF-orbital fluctuations, and the freezing of the latter fluctuations would explain the nematic order at \(T^* \sim 200\text{K} (\gg T_S)\) [32, 33].

First, we discuss the susceptibility at \(k \approx 0\) with respect to the quadrupole order parameter \(\hat{O}_{x^2-y^2} \equiv n_{xz} - n_{yz}\) in the Hubbard model. For \(U = U' + 2J\), it is approximately given as [13, 16]

\[
\chi_{x^2-y^2}(k) = 2\Phi(k)/(1 - (U - 5J)\Phi(k)),
\]

where \(k = (k, \omega)\), and \(\Phi(k) \equiv \chi^{(0)}(k) + X(k)\) is the intra-orbital (within \(d_{xz}\) orbital) irreducible susceptibility: \(\chi^{(0)}(k)\) is the non-interacting susceptibility and \(X(k)\) is the VC for the charge channel. The orbital nematic order \(n_{xz} \neq n_{yz}\) occurs when the charge Stoner factor \(\alpha_c = (U - 5J)\Phi(0)\) reaches unity, which is realized near the magnetic QCP since the AL-VC is proportional to the square of the magnetic correlation length [13, 16, 17].

Next, we discuss the “total” quadrupole susceptibility in real systems, by including the realistic quadrupole interaction due to the acoustic phonon for the orthorhombic distortion. According to Ref. [34], it is given as \(-g_{\text{ac}}(k)\hat{O}_{x^2-y^2} \equiv \hat{O}_{x^2-y^2}(-k)\), where \(\hat{O}_{x^2-y^2}\) is the quadrupole operator, and \(g_{\text{ac}}(k) = g (v_{ac}(k)/\omega)^2/(v_{ac}(k)/\omega)^2 - 1\) is the phonon propagator multiplied by the coupling constants. \(v_{ac}\) is the phonon velocity. Since the Migdal’s theorem tells that the effect of \(g\) on the irreducible susceptibility is negligible, the total susceptibility is

\[
\chi_{x^2-y^2}^{\text{tot}}(k) = \chi_{x^2-y^2}(k)/(1 - g_{\text{ac}}(k)\chi_{x^2-y^2}(k)).
\]

Now, we discuss the acoustic and optical responses based on the total susceptibility (2), by taking notice that any susceptibilities in metals are discontinuous at \(\omega = |k| = 0\). Since the elastic constant is measured under the static \((\omega = 0)\) strain with long wavelength \((|k| \rightarrow 0)\), \(C_{66}\) is given as

\[
C_{66}^{-1} \sim 1 + \lim_{k \rightarrow 0} g_{\text{ac}}(k, 0)\chi_{x^2-y^2}^{\text{tot}}(k, 0) = \frac{1}{1 - g \chi_{k-\text{lim}}},
\]

where \(\chi_{k-\text{lim}} \equiv \lim_{k \rightarrow 0} \chi_{x^2-y^2}(k, 0)\) is called the \(k\)-limit, and the relation \(g_{\text{ac}}(k) = g\) for \(\omega = 0\) is taken into account. The structure transition occurs when \(C_{66}^{-1}\) diverges. When the AL-VC is negligible, \(\chi_{k-\text{lim}}\) is as small as \(\chi_{k-\text{lim}}^{(0)}\). Even in this case, \(C_{66}^{-1}\) can diverge when \(g\) is very large, which is known as the band-JT effect. However, the band-JT mechanism cannot explain the strong enhancement of \(\chi_{R\text{aman}}^{x^2-y^2}\), as we will clarify later. In fact, the fitting of experimental data in the present paper indicates that the softening of \(C_{66}\) is mainly given by the AL-VC. The relation \(1/g \sim \chi_{k-\text{lim}} \gg \chi_{k-\text{lim}}^{(0)}\) is satisfied in Fe-based superconductors.

Next, we derive the optical response in the DC limit, measured by using the low-energy photon with \(k = (k, \omega = c|k|)\) and \(\omega \rightarrow 0\). Considering that the photon velocity \(c\) is much faster than the Fermi velocity \(v_F\) and \(v_{ac}\), it is given as

\[
\chi_{x^2-y^2}^{\text{Raman}} \sim \lim_{\omega \rightarrow 0} \chi_{x^2-y^2}^{\text{tot}}(0, \omega) = \chi_{\omega-\text{lim}}.
\]

where \(\chi_{\omega-\text{lim}} = \lim_{\omega \rightarrow 0} \chi_{x^2-y^2}(0, \omega)\) is called the \(\omega\)-limit [35, 36]. Since \(g_{\text{ac}}(k)\) is zero for \(|\omega/k| = c\), the band-JT effect does not contribute to the Raman susceptibility. The physical explanation is that the acoustic phonons cannot be excited by photons because of the mismatch of the wavelengths \(\lambda_{\text{photon}} \gg \lambda_{\text{ac}}\) for the same \(\omega\) as shown in Fig. 1 (c). Also, since \(\omega \gg v_F\), low-energy photon cannot induce the intraband particle-hole excitation as understood from the location of the particle-hole continuum shown in Fig. 1 (d). This fact leads to the relationship “\(\chi_{\omega-\text{lim}}\) is smaller than \(\chi_{k-\text{lim}}\)” as we discuss mathematically later. For the charge quadrupole susceptibility, this relationship holds even if the quasiparticle lifetime is finite due to impurity scattering; see the Supplemental Material [37]. Therefore, \(\chi_{x^2-y^2}^{\text{Raman}}\) remains finite at \(T \sim T_S\) although \(C_{66}^{-1}\) diverges at \(T_S\), consistently with experiments [20, 21].

**FIG. 2:** (color online) (a) \(X_{\text{lim}}/T\) and (b) \(X_{\omega-\text{lim}}/T\) as functions of \(T\). Their \(T\)-dependences originates from \(|\Lambda_Q^{k-(\omega))^{-1}}|^2\) since \(\xi^2 \propto 1/(1 - \alpha_s)\) is fixed.

Hereafter, we perform the numerical calculation of the quadrupole susceptibility in the five-orbital model. The unit of energy is eV unless otherwise noted. First, we discuss the \(k\)-limit and \(\omega\)-limit of the bare bubble made of two \(d_{xz}\)-orbital Green functions. They are connected by the following relation:

\[
\chi_{k-\text{lim}}^{(0)} = \chi_{\omega-\text{lim}}^{(0)} + \sum_{\alpha} \left(-\frac{\partial F_{\alpha}}{\partial \lambda_{\alpha}}\right) \left(\chi_{k-\text{lim}}^{(0)}\right)^2,
\]
where \( z^2_R = |(xz, k|\alpha, k)|^2 \leq 1 \) is the weight of the \( dx_z \)-orbital on band \( \alpha \), and \( f_k^\alpha = (\exp((c_k^\alpha - \mu)/T) + 1)^{-1} \).

In Eq. (5), \( \chi_{\omega \text{-lim}}^{(0)} \) is given by

\[
\chi_{\omega \text{-lim}}^{(0)} = \sum_{\alpha \neq \beta} \left( f_k^\alpha - f_k^\beta \right) \left( c_k^\alpha - c_k^\beta \right)
\]

is the spin Stoner factor. Thus, we obtain the relationship \( X_{k \text{-lim}} / T \sim T^{-0.5} \xi^2 \), in which the factor \( T^{-0.5} \) originates from the strong \( T \)-dependence of \( |Q|_{\text{lim}}^{k}^2 \). We also show the temperature dependence of \( X_{\omega \text{-lim}} / T \) in Fig. 2 (b): The relation \( X_{\omega \text{-lim}} / T \sim (b - T) \xi^2 \) is realized due to the \( T \)-dependence of \( |Q|_{\text{lim}}^{k}^2 \) [40]. Therefore, the relationship \( X_{k \text{-lim}} > X_{\omega \text{-lim}} \) is confirmed by the present calculation.

Here, we perform the fitting of experimental data. To reduce the number of fitting parameters, we put \( x_{s} - y_{s} \approx 2 \Phi \) by assuming \( (U - 5J) \sim 0 \), which would be justified since the relation \( J / U \sim 0.15 \) is predicted by the first principle study [41]. Also, we put \( \Phi \approx X \) by assuming that \( X > \chi(0) \). Then, Eqs. (3) and (4) are simplified as

\[
C_{66}^{-1} \propto 1/(1 - 2gX_{k \text{-lim}}), \tag{9}
\]

\[
\chi_{x_{s} - y_{s}}^{Raman} \propto X_{\omega \text{-lim}}, \tag{10}
\]

where \( X_{k \text{-lim}} \equiv a_0 T_0 \xi^2 \) and \( X_{\omega \text{-lim}} \equiv b_0 (b - T) \xi^2 \). According to Fig. 2, \( a \sim 0.5 \) and \( b \sim 0.1 \) for \( T > 0.01 \).

First, we fit the data of \( C_{66}^{\text{exp}} \), which is normalized by the shear modulus due to phonon anharmonicity (=33% Co-Ba122 data) given in Ref. [5]. We putting \( a = 0.5 \), and the remaining fitting parameters are \( h = 2ga_0 \) and \( \theta \). Figure 3 (a) shows the fitting result for
Ba(Fe\textsubscript{1−x}Co\textsubscript{x})\textsubscript{2}As\textsubscript{2}: The “dotted line C\textsubscript{66}′′ is the fitting result of C\textsubscript{66}\textsuperscript{exp} under the constraint C\textsubscript{66} = 0 at T = T\textsubscript{S}. We fix \( h = 2.16 \) for all \( x \), and change \( \theta \) from 116K to −30K. The “broken line C\textsubscript{66}′′” is the fitting for \( x = 0 \sim 0.09 \) without the constraint, by using \( h = 2.67 \). Thus, both fitting methods can fit the T- and x-dependences of C\textsubscript{66}\textsuperscript{exp} very well by choosing only \( \theta(x) \) with a fixed \( h \). Figure 3 (b) shows the obtained \( \theta(x) \) by C\textsubscript{66}′-fitting \( (x = 0 \sim 0.043) \) and by C\textsubscript{66}′′-fitting \( (x = 0.06, 0.09) \), as explained above. The obtained \( \theta(x) \) is very close to \( \theta_{\text{Raman}} \) given by the Curie-Weiss fitting of 1/\( T_1 \)\( T \) \cite{1}, which manifests the importance of the AL-VC. Also, \( \theta_{\text{Raman}} \) is given by the Raman spectroscopy \cite{20}.

In this paper, we showed that Raman susceptibility at \( \omega = 0 \) is enlarged by the AL-VC. The present theory predicts that the \( \omega \)-dependence of the AC Raman susceptibility follows \( \chi_{\text{Raman}}(x, \omega) \sim X(0, \omega) \sim (1 - i\omega / \Gamma)^{-1} \), and \( \Gamma \) is approximately \( \sim \omega_{\text{F}} \). However, \( \Gamma \) could be modified by the \( \omega \)-dependence of \( |\Delta_{\text{F}}(k)|^2 \).

In summary, we presented a unified explanation for the softening of C\textsubscript{66} and enhancement of C\textsubscript{Raman} based on the five-orbital model. Both 1/C\textsubscript{66} and \( \chi_{\text{Raman}} \) are enhanced by the nematic-type orbital fluctuations induced by the AL-VC. However, \( \chi_{\text{Raman}} \) remains finite even at the structure transition temperature \( T\textsubscript{S} \), because of the absence of the band-JT effect and the Pauli (=intra-band) contribution. The present study clarified that the origin of the nematicity, which is a central issue in Fe-based superconductors, is the nematic-orbital order/fluctuations.

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[40] The analytic expression of $\Lambda^\omega_{\alpha \beta \gamma}$ is given as

$$\Lambda^\omega_{\alpha \beta \gamma} = \sum_{\alpha, \beta, \gamma} \sum_{k} \frac{1}{\epsilon_k - \epsilon_k'} \left( \frac{f_{k}^{\beta}}{\epsilon_k - \epsilon_k'} - \frac{f_{k}^{\alpha}}{\epsilon_k - \epsilon_k} \right) + \frac{f_{k}^{\gamma}}{\epsilon_k - \epsilon_k'} \left( \frac{\epsilon_k - \epsilon_k'}{\epsilon_k'} \right) \zeta_{\alpha \beta \gamma} = \left( \frac{\epsilon_k - \epsilon_k'}{\epsilon_k'} \right)^2 \zeta_{\alpha \beta \gamma}.$$
[SUPPLEMENTAL MATERIAL]: RELATIONSHIP \( \chi_{k\text{-limit}} > \chi_{\omega\text{-limit}} \) IN THE PRESENCE OF IMPURITIES

In the main text, we have studied the \( k \)-limit and \( \omega \)-limit of the quadrupole susceptibility \( \chi_{x^2-y^2}(q,\omega) \), and found that the relationship \( \chi_{k\text{-limit}} > \chi_{\omega\text{-limit}} \) is satisfied. The basis of this relationship is that the intra-band Pauli term is absent in both \( \chi^{(0)}(k) \) and \( \chi^{(0)}(q) \). However, the relationship \( \chi_{k\text{-limit}} > \chi_{\omega\text{-limit}} \) is not trivial when the scattering processes exist. Here, we calculate both \( \chi^{(0)}(k) \) and \( \chi^{(0)}(q) \) in the presence of the local nonmagnetic impurities based on the \( T \)-matrix approximation in the five-orbital model. For the charge quadrupole susceptibility, the relationship \( \chi_{k\text{-limit}} > \chi_{\omega\text{-limit}} \) is confirmed even in the presence of impurities.

We assume that the impurity potential \( I \) is diagonal in the orbital basis. (We write \( d_{x^2}, d_{xz}, d_{yz}, d_{xy}, d_{x^2-y^2} \) orbitals as 1, 2, \( \cdots \), 5, respectively.) Then, the \( T \)-matrix in the orbital basis is given as

\[
\hat{T}(\epsilon_n) = I(1 - \sum q \hat{G}(q,\epsilon_n))^{-1}
\]

where \( \epsilon_n = (2n + 1)\pi T \) and the Green function is \( \hat{G}(q,\epsilon_n) = (i\epsilon_n + \mu - \hat{H}_0^q - \Sigma^{\text{imp}}(\epsilon_n)) \), and

\[
\Sigma^{\text{imp}}(\epsilon_n) = \pi n_{\text{imp}} \hat{T}(\epsilon_n)
\]

is the impurity self-energy when the impurity concentration is \( n_{\text{imp}}(\ll 1) \). The Bethe-Salpeter equation for the one-particle operator \( \hat{O} \) is

\[
\hat{L}^{\text{imp}}(k;\epsilon_n) = \hat{O} + n_{\text{imp}} \sum q \hat{T}(\omega_l + \epsilon_n)\hat{G}(q + k) \\
\times \hat{L}^{\text{imp}}(k;\epsilon_n)\hat{G}(q)\hat{T}(\epsilon_n)
\]

where \( q = (q,\epsilon_n) \) and \( k = (k,\omega_l) \). We will show the significant role of the VC given by the second term; \( \hat{L}^{\text{imp}} - \hat{O} \).

First, we study the impurity effect on the bare-bubble \( \chi^{(0)}(k) \) for the \( O_{x^2-y^2} \) quadrupole. The impurity effect is divided into the (i) self-energy correction (12) and (ii) vertex correction (13). Only (i) is taken into account, the bare-bubble within the \( d_{xz} \)-orbital is given as

\[
\chi^{(0),\Sigma}(k) = -T \sum q G_{2,2}(k + q) G_{2,2}(q)
\]

where \( G \) includes the self-energy, and the suffix 2 in \( G \) represents the \( d_{xz} \)-orbital. If both (i) and (ii) is taken into account, it is given as

\[
\chi^{(0),\text{true}}(k) = -T \sum_{q,m,m'} \hat{L}^{\text{imp}}_{m,m'}(k;\epsilon_n) G_{m',2}(k + q) G_{2,m}(q)
\]

for \( \hat{O} = \hat{O}_{x^2-y^2} \) in Eq. (13), where \( l, m = 1 \sim 5 \) represents the \( d \)-orbital. \( \chi^{(0),\text{true}} \) gives the correct susceptibility for \( n_{\text{imp}} > 0 \), whereas \( \chi^{(0),\Sigma} \) is incorrect.

\[
\Sigma^{\text{imp}}(\epsilon_n) = \pi n_{\text{imp}} \hat{T}(\epsilon_n)
\]

FIG. 5: (color online) (a) \( \chi_{\omega\text{-limit}}^{(0)} \) and \( \chi_{\text{true}}^{(0)} \), (b) \( \chi_{\omega\text{-limit}}^{(0)} \) and \( \chi_{\text{true}}^{(0)} \), (c) \( \chi_{\omega\text{-limit}}^{(0)} \) and \( \chi_{\text{true}}^{(0)} \), and (d) \( \chi_{\omega\text{-limit}}^{(0)} \) and \( \chi_{\text{true}}^{(0)} \) as functions of \( n_{\text{imp}} \). The correct results are given in (b) and (d). Here, \( \omega \)-limit values are obtained by extrapolating the data at \( \omega_l \) with \( l = 1 \sim 10 \) to the real axis numerically.

Here, we discuss the susceptibilities in the \( k \)-limit and \( \omega \)-limit. Using Eq. (14) or (15), the former is simply given as \( \chi_{k\text{-limit}} = \chi(k,\omega_l) \) at \( l = 0 \) and \( k = 0 \). Here, we derive the latter numerically by extrapolating the data at \( \omega_l \) with \( l = 1 \sim 10 \) to the real axis. This procedure is successful at sufficiently low temperatures. Figure 5 (a) and (b) represent the numerically obtained \( \chi^{(0),\Sigma} \) and \( \chi^{(0),\text{true}} \) for \( I = 1 \), respectively. We fix \( T = 3 \) meV and \( n = 6.0 \). In (a), \( \chi_{\omega\text{-limit}} \) quickly increases with \( n_{\text{imp}} \), and it is almost equal to \( \chi^{(0),\text{true}} \) for \( n_{\text{imp}} > 0.01 \). In (b), in contrast, \( \chi_{\omega\text{-limit}} \) does not reach the \( k \)-limit value even for \( n_{\text{imp}} \sim 0.1 \). In both (a) and (b), impurity effect on the \( k \)-limit value is very small. Since \( \chi^{(0),\text{true}} \) gives the true susceptibility, we conclude that the relationship \( \chi^{(0)}_{k\text{-limit}} > \chi^{(0)}_{\omega\text{-limit}} \) is satisfied even for \( n_{\text{imp}} > 0 \).

In Fig. 5 (a), \( \chi^{(0),\Sigma} \) approaches to the \( k \)-limit value.
for $n_{\text{imp}} > 0$, since the intra-band Pauli term also contributes to the $\omega$-limit ($k = 0$ and $\omega \to 0$) due to the broadening of the quasiparticle spectrum caused by $\text{Im} \Sigma$. However, the impurity three-point vertex $\hat{L}^{\text{imp}}(k; \epsilon_n)$ takes large value for $\omega_l \cdot (\epsilon_n + \omega_l) < 0$, and it suppresses the Pauli term. These effects exactly cancel for conserved quantities: For this reason, the charge and spin susceptibilities become zero in the $\omega$-limit even for $n_{\text{imp}} > 0$. Although $O_{x^2-y^2}$ is not conserved, the VC in $\hat{L}^{\text{imp}}(k; \epsilon_n)$ is nonzero in the present model, and therefore the relationship $\chi^{(0), \text{true}}_k > \chi^{(0), \text{true}}_\omega$ is satisfied.

Next, to discuss the AL-VC, we calculate $\Lambda^{k(\omega)-\lim}_Q$ at $Q = (0, \pi)$ introduced in the main text (Eq. (8)) in the presence of impurities ($I = +1$), by which the AL-VC is given as $X_{k(\omega)-\lim} \sim T|\Lambda^{k(\omega)-\lim}_Q|^2 \sum_k \chi^s(k)^2$. Figure 5 (a) shows the numerically obtained $\Lambda^{Q(\omega)-\lim}_{Q, \Sigma}$, in which only $\Sigma^{\text{imp}}$ is included. We see that $\Lambda^{Q(\omega)-\lim}_{Q, \Sigma}$ increases with $n_{\text{imp}}$, and coincides with the $k$-limit value just for $n_{\text{imp}} \gtrsim 0.01$. We also calculate $\Lambda^{k(\omega)-\lim}_{Q, \text{true}}$, in which both $\Sigma^{\text{imp}}$ and $L^{\text{imp}}$ are taken into account properly. In this case, $\Lambda^{Q(\omega)-\lim}_{Q, \text{true}}$ does not reach the $k$-limit value even for $n_{\text{imp}} \gtrsim 0.1$ thanks to the VC in $L^{\text{imp}}$. Since $\Lambda^{k(\omega)-\lim}_{Q, \text{true}}$ gives the true vertex function, the relation of the AL-VC $X^{k(\omega)-\lim}_k > X^{\omega-\lim}_\text{true}$ is confirmed even for $n_{\text{imp}} > 0$.

In summary, we confirmed that the relationship $\chi^{k-\lim}_k > \chi^{\omega-\lim}_\omega$ is satisfied in the presence of impurities, by taking both $\Sigma^{\text{imp}}$ and $L^{\text{imp}}$ into account correctly. In other words, although the relation $\chi^{k-\lim}_k \approx \chi^{\omega-\lim}_\omega$ is obtained by including $\Sigma^{\text{imp}}$ only, it is an artifact due to the neglect of the VC in $L^{\text{imp}}$. (Since $L^{\text{imp}} = \hat{O}$ for the charge current $\hat{O} = v_k$, such discontinuity will be absent for the conductivity.) In real compounds, the Raman vertex $\hat{R}_{x^2-y^2}$ is very complex and momentum dependent. In this paper, we take the momentum average of $\hat{R}_{l,l}^{x^2-y^2}$ ($l = 2, 3$), and consider the constant Raman vertex $\hat{O}_{x^2-y^2}$ to simplify the discussion. In the present multiorbital model, $L^{\text{imp}}$ does not vanish even if the $k$-dependence of $\hat{R}_{x^2-y^2}$ is taken into account, so the relationship $\chi^{k-\lim}_k > \chi^{\omega-\lim}_\omega$ should be satisfied for $n_{\text{imp}} > 0$. 