ON THE FULL HOLONOMY GROUP OF SPECIAL LORENTZIAN MANIFOLDS

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Abstract. We study the full holonomy group of Lorentzian manifolds with a parallel null line bundle. We prove several results that are based on the classification of the restricted holonomy groups of such manifolds and provide a construction method for manifolds with disconnected holonomy which starts from a Riemannian manifold and a properly discontinuous group of isometries. Most of our examples are quotients of pp-waves with disconnected holonomy and without parallel vector field. Furthermore, we classify the full holonomy groups of solvable Lorentzian symmetric spaces and of Lorentzian manifolds with a parallel null spinor. Finally, we construct examples of globally hyperbolic manifolds with complete spacelike Cauchy hypersurfaces, disconnected full holonomy and a parallel spinor.

Keywords: Lorentzian manifolds, holonomy groups, isometry groups, parallel spinor fields, globally hyperbolic manifolds, pp-waves

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1. Introduction

The aim of this paper is to study the holonomy group of Lorentzian manifolds with a parallel bundle of null lines. The holonomy group of a semi-Riemannian manifold \((M,g)\) at a point \(p \in M\) is given as the group of parallel transports along loops\(^1\) based at \(p\),

\[
\text{Hol}_p(M, g) := \{ P_\gamma : T_p M \to T_p M \mid \gamma : [0, 1] \to M \text{ a curve with } \gamma(0) = \gamma(1) = p \}.
\]

Here \(P_\gamma\) denotes the parallel transport along \(\gamma\) with respect to the Levi-Civita connection \(\nabla^g\) of \(g\). The holonomy group is a subgroup of the orthogonal group \(O(T_p M, g_p)\), where \(T_p M\) is the tangent space of \(M\) at \(p\) and \(g_p\) the scalar product induced by the semi-Riemannian metric \(g\) on \(T_p M\).

Holonomy groups are not necessarily closed nor connected. The connected component \(\text{Hol}^0_p(M, g)\) is given by restricting the definition (1) to curves that can be contracted to the point \(p\). Indeed,

\[^1\]All curves we consider are piecewise smooth.
by contracting the loop \( \gamma \) we obtain a path in the holonomy group from \( P_\gamma \) to the identity. Hence, holonomy groups of simply connected manifolds are connected. The restricted holonomy group \( \text{Hol}_p^0(M, g) \) is a normal subgroup in the full holonomy group \( \text{Hol}_p(M, g) \). Moreover, the fundamental group of \( M \) surjects homomorphically onto their quotient,

\[
\pi_1(M, p) \twoheadrightarrow \text{Hol}_p(M, g)/\text{Hol}_p^0(M, g)
\]

(for a proof see for example [20, Chap. II, Sec. 4]). The holonomy group is a very powerful tool, for example, for determining parallel sections in geometric vector bundles. Knowing the holonomy group of a given semi-Riemannian manifold allows to find the solution to the partial differential equation for a parallel section by solving an algebraic problem, namely to determine the fixed vectors of the corresponding representation of the holonomy group. The parallel section is then obtained by parallel transporting the algebraic object at a point to the whole manifold and thus defining a global section. For example, for finding a parallel vector field one has to find a fixed vector under the holonomy group acting on the tangent space. For finding a parallel spinor field, one has to find a spinor that is fixed under the image of the holonomy group in the corresponding spinor group, or for simply connected manifolds, one that is fixed under the spin representation of the holonomy algebra. Another important example is the parallel complex structure of a Kähler manifold. Here the the holonomy group is reduced to the unitary group. These facts also show the importance of manifolds with special holonomy, e.g. Calabi-Yau manifolds, in string theory, where in some situations the underlying spacetime is required to have a covariantly constant, i.e. parallel, spinor field. For these reasons, a classification of possible holonomy groups of semi-Riemannian manifolds is a desirable result, but out of reach in full generality.

Classification results for holonomy groups are usually obtained only for the restricted holonomy group (see for example [7, 26, 23]). Such results are based on the Ambrose-Singer holonomy theorem [1] which states that the Lie algebra of the holonomy group is generated by curvature at every point in \( M \). More precisely, the Lie algebra of the holonomy group at \( p \) is generated as a vector space by the following linear maps of \( T_p M \)

\[
P_\gamma^{-1} \circ R_{\gamma(1)}(X, Y) \circ P_\gamma,
\]

where \( \gamma : [0, 1] \rightarrow M \) is a curve starting at \( p \), \( R_{\gamma(1)} \) the curvature tensor at \( \gamma(1) \), and \( X, Y \in T_{\gamma(1)} M \).

Since the Levi-Civita connection is torsion free, the curvature and hence all the maps in (3), satisfy the Bianchi identities. Hence, via the Ambrose-Singer holonomy theorem, the Bianchi identities impose strong algebraic conditions on the Lie algebra of the holonomy group which lead to classification results, but only for the restricted holonomy groups, and mostly under the assumption that it acts irreducibly, or at least indecomposably (see next paragraph for the definition). Similar classification results for full holonomy groups are out of reach. For example, due to the complete reducibility of the holonomy representation for Riemannian manifolds, the restricted holonomy group of a Riemannian manifold is always closed and hence compact. But the example given in [32] shows that \( \text{Hol}_p(M, g) \) can be non-compact even for compact Riemannian manifolds.

For Lorentzian manifolds the classification of restricted holonomy groups is obtained as follows: Using the splitting theorems by de Rham [14] and Wu [34] one can decompose every simply connected, complete Lorentzian manifold into a product of Riemannian manifolds and a Lorentzian manifold, all simply connected and complete, and with indecomposably acting holonomy group. By indecomposable we mean that the metric degenerates on every subspace of \( T_p M \) that is invariant under the holonomy group. Of course, for Riemannian manifolds this implies that the holonomy group acts irreducibly and one can apply Berger’s holonomy classification [7] to the Riemannian factors. The remaining Lorentzian factor is either flat, irreducible or indecomposable. On the one hand, the irreducible case is dealt with by the Berger’s list [7], on which \( \text{SO}^0(1, n - 1) \) is the only possible irreducible restricted holonomy group of Lorentzian manifolds. This also follows from the more fundamental result by Di Scala and Olmos in [15] that \( \text{SO}^0(1, n - 1) \) has no proper irreducible
subgroups. On the other hand, the classification in the indecomposable, non-irreducible case was achieved recently by Berard-Bergery and Ikemakhen [6], the third author [23], and Galaev [17]. We will explain the classification in the following paragraph and in Section 2.

We say that a Lorentzian manifold \((M, g)\) of dimension \((n + 2)\) has special holonomy, or simply is special if its restricted holonomy group acts indecomposably but is not equal to \(SO^0(1, n + 1)\), the connected component of the special orthogonal group in Lorentzian signature. As \(SO^0(1, n + 1)\) has no proper irreducible subgroups, this means that the representation of the restricted holonomy group of a special Lorentzian manifold cannot be irreducible but that the metric is degenerate on all invariant subspaces. Hence, there is a \(\text{Hol}^0_{p}(M, g)\)-invariant degenerate subspace \(W \subset T_p M\) which defines an invariant null line \(L := W \cap W^⊥\) and the restricted holonomy group is contained in the stabiliser in \(SO^0(T_p M, g_p)\) of this line \(L\). Identifying \(T_p M\) with \(\mathbb{R}^{1,n+1}\) by fixing a basis \((e_l, e_1, \ldots, e_n, e^*)\) in \(T_p M\) such that \(\ell \in L\) and the metric at \(p\) is of the form

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1_n & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

this stabiliser in \(O(1, n + 1)\) can be written as the parabolic subgroup

\[
P := \text{Stab}_{O(1,n+1)}(L) = (\mathbb{R}^* \times O(n)) \ltimes \mathbb{R}^n
\]

whose connected component is given by the stabiliser of \(L\) in \(SO^0(1, n + 1)\), i.e.,

\[
P^0 = \text{Stab}_{SO^0(1,n+1)}(L) = (\mathbb{R}^+ \times SO(n)) \ltimes \mathbb{R}^n.
\]

This defines three projections of \(P\) onto \(\mathbb{R}^*, O(n)\) and \(\mathbb{R}^n\), and of \(P^0\) onto \(\mathbb{R}^+, SO(n)\) and \(\mathbb{R}^n\) which we denote by

\[
pr_{\mathbb{R}} : P \to \mathbb{R}^*, \quad pr_{O(n)} : P \to O(n), \quad pr_{\mathbb{R}^n} : P \to \mathbb{R}^n.
\]

For Lorentzian manifolds with restricted holonomy group \(H^0\) acting indecomposably but not irreducibly in [23] it was shown that \(pr_{O(n)}(H^0) \subset SO(n)\) has to be the holonomy group of a Riemannian manifold. Using results in [6, see our Section 2] this gave a full classification of restricted holonomy groups of Lorentzian manifolds acting indecomposably and not irreducibly. Galaev [17] then extended previous existence results and verified that indeed all groups on the list can be realised as holonomy groups of Lorentzian manifolds. Together with the splitting theorems by de Rham [14] and Wu [34] and the fact mentioned above that \(SO^0(1, n + 1)\) has no proper irreducible subgroups, this yields the classification of restricted holonomy groups of Lorentzian manifolds.

Our first result about the full holonomy group is that it has the same \(\mathbb{R}^n\)-part as the restricted holonomy (see Proposition 1 for a more precise statement):

**Theorem 1.** Let \((M, g)\) be a Lorentzian manifold of dimension \((n + 2) > 2\) such that its restricted holonomy group \(\text{Hol}^0_{p}(M, g)\) acts indecomposably but not irreducibly. Then

1) the full holonomy group \(\text{Hol}_p(M, g)\) acts indecomposably but not irreducibly, and

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2This is in accordance with the terminology in the Riemannian setting where special holonomy usually refers to manifolds with restricted holonomy different from \(SO(n)\) but still acting irreducibly. Of course, in Riemannian signature irreducibility is the same as indecomposability, but in other signatures indecomposability is the property that is geometrically more important: If the restricted holonomy group acts decomposably, i.e. with a non-degenerate invariant subspace, then, without further assumptions on \(M\), the manifold is locally a semi-Riemannian product [34, Proposition 3].
2) there is a subset \( \Gamma \subset \mathbb{R}^* \times O(n) \) such that
\[
\text{Hol}(M, g) = \Gamma \cdot \text{Hol}^0(M, g).
\]

After recalling the basics on special Lorentzian geometry and proving this result, a large part of the paper is devoted to the construction of Lorentzian manifolds with disconnected holonomy. Our construction uses a method to obtain special Lorentzian manifolds of dimension \( (n+2) \) from Riemannian manifolds of dimension \( n \). Using this method for every group \( G \) that is a Riemannian holonomy group, connected or disconnected, we obtain special Lorentzian manifolds with holonomy
\[
G \ltimes \mathbb{R}^n, \ (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n, \ (\mathbb{R}^* \times G) \ltimes \mathbb{R}^n, \ (\mathbb{Z}_2 \times G) \ltimes \mathbb{R}^n
\]
(see Proposition 4 for details). Further examples are obtained as quotients of Lorentzian manifolds by a properly discontinuous group of isometries \( \Gamma \). To this end, in Proposition 3 of Section 4 we prove a generalisation of the fundamental formula (2) for general coverings \( \pi : \widetilde{M} \to M := \widetilde{M}/\Gamma \), which provides a surjective group homomorphism
\[
\Gamma \twoheadrightarrow \text{Hol}_p(M)/\text{Hol}_p(\widetilde{M}) \quad \sigma \mapsto [P_\gamma],
\]
where \( \gamma \) is a loop at \( p \) that, when lifted to a curve \( \tilde{\gamma} \) starting at \( \tilde{p} \), ends at \( \sigma^{-1}(\tilde{p}) \), and yields a formula of the parallel transport in \( M \) in terms of that in \( \widetilde{M} \). Applying this to our context in Theorem 3 and Corollary 1 leads to a variety of examples of special Lorentzian manifolds with disconnected holonomy in Section 6 including an example for which the quotient \( \text{Hol}/\text{Hol}^0 \) is infinitely generated. These various examples illustrate the possible differences between the full and the restricted holonomy group and feature a coupling between the \( O(n) \)-part and the \( \mathbb{R}^* \)-part of the holonomy group that is not present for the restricted holonomy group. Most of our examples are quotients of pp-waves.

One of the class of examples we consider are solvable Lorentzian symmetric spaces, so-called Cahen-Wallach spaces \([11, 10]\). In Proposition 6 we show that the full holonomy of a Cahen Wallach space is either connected, in which case it is equal to \( \mathbb{R}^n \), or given as \( \mathbb{Z}_2 \ltimes \mathbb{R}^n \), where the \( \mathbb{Z}_2 \) factor is generated by a reflection in \( O(n) \).

In the last part of the paper we consider the full holonomy of Lorentzian manifolds that admit a parallel spinor field. Such a spinor field induces a parallel vector field and hence the holonomy stabilises a vector, which, for indecomposable manifolds, has to be null, i.e. lightlike. First we show in Proposition 8 and Corollary 2 that, for a time- and space-orientable Lorentzian manifold with holonomy \( G \ltimes \mathbb{R}^n \), the existence of a spin structure with parallel spinors depends solely on \( G \). This result enables us to apply to the Lorentzian situation the classification of irreducible subgroups of \( O(n) \) stabilising a spinor and having \( SU(\frac{7}{2}), \ Sp(\frac{7}{2}), \ G_2 \) or \( \text{Spin}(7) \) as connected component. This classification was given by McInnes \([24]\) and Wang \([31]\) and it yields our

**Theorem 2.** Let \( (M, g) \) be a Lorentzian spin manifold of dimension \( (n+2) \geq 2 \) with full holonomy group \( H = \text{Hol}_p(M, g) \) with a parallel spinor. Assume that

(i) the connected component \( H^0 \) of the holonomy group \( H \) acts indecomposably, and

(ii) \( G^0 := \text{pr}_{O(n)}(H^0) \) acts irreducibly on \( \mathbb{R}^n \).

Then \( H \) is \( G \ltimes \mathbb{R}^n \), where \( G \subset O(n) \) is one of the groups listed in Theorem 4 and the dimension of parallel spinors on \( (M, g) \) is equal to the dimension \( N \) of spinors fixed under \( G \) as given in Theorem 4.

Finally, we study the existence problem for metrics with these holonomy groups and parallel spinors. We use a method developed in \([5]\) to construct globally hyperbolic Lorentzian manifolds with complete spacelike Cauchy hypersurfaces, parallel spinors and holonomy \( G \ltimes \mathbb{R}^n \) from Riemannian manifolds. We apply this method to examples given by Moroianu and Semmelmann in \([27]\) and obtain globally hyperbolic metrics with parallel spinors for the groups in Theorem 2.
2. Algebraic preliminaries

Let \((M, g)\) be a Lorentzian manifold of signature \((1, n + 1)\). The holonomy group as defined in (1) is an immersed Lie subgroup of \(O(T_p M, g_p)\) (for a proof see [20, Thm. II.4.2]). We denote its Lie algebra by \(\mathfrak{hol}_p(M, g)\). For connected manifolds, holonomy groups at different points are conjugated in \(O(1, n + 1)\) to each other. We assume from now on that all manifolds are connected. Hence we may omit the point \(p\) and consider holonomy groups only up to conjugation.

We will first derive some purely algebraic results which will imply Theorem 1 of the introduction. Let \(\mathbb{R}^{1,n+1}\) be the \((n + 2)\)-dimensional Minkowski space, in which we fix a basis \((\ell, e_1, \ldots, e_n, \ell^*)\) such that the Minkowski inner product is of the form (4). Let \(L\) be the null line spanned by \(\ell\). Furthermore let \(H \subset O(1, n + 1)\) be a subgroup and \(H^0\) a normal subgroup of \(H\). Obviously, \(H\) is contained in the normaliser in \(O(1, n + 1)\) of \(H^0\),

\[
H \subset \text{Nor}_{O(1, n+1)}(H^0).
\]

In this situation we prove:

**Lemma 1.** Let \(H^0 \subset O(1, n + 1)\) be a subgroup that acts indecomposably and stabilises the null line \(L\). Then the normaliser of \(H^0\) stabilises \(L\) as well, i.e.

\[
\text{Nor}_{O(1, n+1)}(H^0) \subset \text{Stab}_{O(1, n+1)}(L).
\]

In particular, if \(H \subset O(1, n + 1)\) is an immersed Lie group and \(H^0\) is the connected component of \(H\), then \(H\) stabilises \(L\) and acts indecomposably if \(H^0\) does.

**Proof.** Let \(g \in \text{Nor}_{O(1, n+1)}(H^0)\) and \(L = \mathbb{R} \cdot \ell\). Then for each \(h \in H^0\) we have that also \(\hat{h} := ghg^{-1} \in H^0\). Since \(H^0\) stabilises \(L\) there are \(\lambda\) such that \(\hat{h}(\ell) = \lambda \ell\). Multiplying this with \(g^{-1}\) gives \(hg^{-1}(\ell) = \lambda g^{-1}(\ell)\). Hence, \(g^{-1}(\ell)\) spans a null line that is fixed under all of \(H^0\). But since \(H^0\) was assumed to be indecomposable, \(L\) is the only line that is fixed by \(H^0\). Hence, \(g^{-1}(\ell) \in L\). □

**Remark 1.** We should remark that we can have immersed subgroups that fix \(L\) and acting indecomposably, but the connected component \(H^0\) acts decomposably. An example of this is given in Section 6.

From now on let \(H \subset O(1, n + 1)\) be an immersed subgroup with connected component

\[
H^0 \subset P^0 = \text{Stab}_{SO^0(1, n+1)}(L) = (\mathbb{R}^+ \times SO(n)) \ltimes \mathbb{R}^n
\]

in the stabiliser of a null line \(L\). Lemma 1 ensures that

\[
H \subset P = \text{Stab}_{O(1, n+1)}(L) = (\mathbb{R}^* \times O(n)) \ltimes \mathbb{R}^n
\]

and we can define

\[
G := \text{pr}_{O(n)}(H).
\]

If \(G^0\) denotes the connected component of \(G\) we have

\[
G^0 = \text{pr}_{O(n)}(H^0) = \text{pr}_{SO(n)}(H^0).
\]

We denote by \(\mathfrak{h} \subset \mathfrak{so}(1, n + 1)\) the Lie algebra of \(H^0\) and recall the classification of subalgebras of \(\mathfrak{so}(1, n + 1)\) that act indecomposably but not irreducibly given in [6]. If \(\mathfrak{h}\) is such a subalgebra, then it is contained in the Lie algebra of the stabiliser \(\mathfrak{p}\) of the null line \(L\), i.e. \(\mathfrak{h} \subset \mathfrak{p} := (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n\). We will write elements in \(\mathfrak{p}\) as triple \((a, X, v) \in \mathbb{R} \times \mathfrak{so}(n) \times \mathbb{R}^n\). Denote by \(\mathfrak{g}\) the projection of \(\mathfrak{h}\) onto \(\mathfrak{so}(n)\). Since \(\mathfrak{g}\) is reductive, it decomposes into its centre \(\mathfrak{z}\) and its derived Lie algebra \(\mathfrak{g}'\), i.e. \(\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'\). Then it was proven in [6] that, if \(\mathfrak{h}\) acts indecomposably, it is of one of the following types, the first two being uncoupled and the last two coupled:

**Type 1:** \(\mathfrak{h} = (\mathbb{R} \oplus \mathfrak{g}) \ltimes \mathbb{R}^n\),

**Type 2:** \(\mathfrak{h} = \mathfrak{g} \ltimes \mathbb{R}^n\),
Type 3: There exists an epimorphism \( \varphi : \mathfrak{z} \to \mathbb{R} \), such that \( \mathfrak{h} = (f \oplus \mathfrak{g}') \times \mathbb{R}^n \), where \( f := \text{graph } \varphi = \{(\varphi(Z), Z) | Z \in \mathfrak{z} \} \subset \mathbb{R} \oplus \mathfrak{z} \). Or, written in matrix form:

\[
\mathfrak{h} = \left\{ \begin{pmatrix} \varphi(Z) & v^t \\ 0 & Z + X \end{pmatrix} \right| Z \in \mathfrak{z}, X \in \mathfrak{g}', v \in \mathbb{R}^n \}
\]

Type 4: There exists a decomposition \( \mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k} \), \( 0 < k < n \), and an epimorphism \( \psi : \mathfrak{z} \to \mathbb{R}^k \), such that \( \mathfrak{h} = (f \oplus \mathfrak{g}') \times \mathbb{R}^{n-k} \) where \( f := \{(Z, \psi(Z)) | Z \in \mathfrak{z} \} = \text{graph } \psi \in \mathfrak{z} \oplus \mathbb{R}^k \).

Or, written in matrix form:

\[
\mathfrak{h} = \left\{ \begin{pmatrix} 0 & \psi(Z) & v^t \\ 0 & 0 & 0 \\ 0 & 0 & Z + X \end{pmatrix} \right| Z \in \mathfrak{z}, X \in \mathfrak{g}', v \in \mathbb{R}^{n-k} \}
\]

Note that, since \( \mathfrak{h} \) acts indecomposably, its projection onto \( \mathbb{R}^n \) is always all of \( \mathbb{R}^n \),

\[
\text{pr}_{\mathbb{R}^n}(\mathfrak{h}) = \mathbb{R}^n,
\]

for all four types. However, in the second coupled type, \( \mathfrak{h} \) does not contain \( \mathbb{R}^n \), only \( \mathbb{R}^{n-k} \). The connected Lie groups \( H^0 \) corresponding to \( \mathfrak{h} \) of types 1 and 2 are given as

\[
(R^+ \times G^0) \times \mathbb{R}^n \text{ or } G^0 \times \mathbb{R}^n.
\]

Denote by \( G^0 \) the connected subgroup in \( P \) that corresponds to the derived Lie algebra \( \mathfrak{g}' \) of \( \mathfrak{g} \). For the coupled type 3 we have that

\[
H^0 = (F^0 \times G^0) \times \mathbb{R}^n,
\]

where \( F^0 \) is the connected Lie group corresponding to the Lie algebra \( f = \{(\varphi(Z), Z) | Z \in \mathfrak{z} \} \subset \mathbb{R} \oplus \mathfrak{z} \).

For the last coupled type the connected component of \( H \) is given by

\[
H^0 = (F^0 \times G^0) \times \mathbb{R}^{n-k},
\]

where \( F^0 \) is the connected Lie group corresponding to the Lie algebra \( f = \{(Z, \psi(Z)) | Z \in \mathfrak{z} \} \subset \mathfrak{z} \oplus \mathbb{R}^k \). In all cases we have that \( \text{pr}_{\mathbb{R}^n}(H^0) = \mathbb{R}^n \) and \( G^0 := \text{pr}_{\text{SO}(n)}(H^0) \) is given by the the connected Lie subgroup in \( \text{SO}(n) \) corresponding to \( \mathfrak{g} \).

**Proposition 1.** Let \( H^0 \subset \text{SO}(n,1,n+1) \) be the connected component of an immersed Lie group \( H \subset \text{O}(1,n+1) \), and assume that \( H^0 \) acts indecomposably and not irreducibly. If \( G := \text{pr}_{\text{O}(n)}(H) \subset \text{O}(n) \) is the projection of \( H \) onto \( \text{O}(n) \) and \( G^0 := \text{pr}_{\text{SO}(n)}(H^0) \subset \text{SO}(n) \) its connected component, then, for the four different types, we have:

Type 1: \( H = (R^* \times G) \times \mathbb{R}^n \), or \( H = (R^+ \times G) \times \mathbb{R}^n \),

Type 2: \( H = \hat{G} \times \mathbb{R}^n \), where \( \hat{G} \subset R^* \times G \) with connected component \( G^0 \),

Type 3: There is a subset \( \Gamma \subset \mathbb{Z}_2 \times \mathbb{R}^* \times G \subset \mathbb{R}^* \times \text{O}(n) \) such that \( H = \Gamma \cdot H_0 \).

Type 4: There is a subset \( \Gamma \subset \mathbb{R}^* \times G \) such that \( H = \Gamma \cdot H_0 \).

**Proof.** In order to prove the statement, we show in all four cases,

\[
\text{for every } P \in H \text{ there is an element } Q \in H^0 \text{ such that } P \cdot Q \in \mathbb{R}^* \times \text{O}(n).
\]

For the first three types for which we have \( \mathbb{R}^n \subset H^0 \subset H \) the statement is obvious: Here we have

\[
P = \begin{pmatrix} a & v^t \\ 0 & A \end{pmatrix} \in H
\]

and find

\[
Q := \exp \begin{pmatrix} 0 & -a^{-1}v^t \\ 0 & a^{-1}v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -a^{-1}v^t \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^n \subset H^0
\]
This implies the form of $H$ in the types 1 and 2. For type 3 we can prove more. For an arbitrary 

$$P = \begin{pmatrix} \pm e^a & v^t & * \\ 0 & A & * \\ 0 & 0 & \pm e^{-a} \end{pmatrix} \in H$$

we choose $Z \in \mathfrak{z}$ such that $a + \varphi(Z) = 0$ and consider 

$$Q_1 := \exp \begin{pmatrix} \varphi(Z) & 0 & 0 \\ 0 & Z & 0 \\ 0 & 0 & -\varphi(Z) \end{pmatrix} = \begin{pmatrix} e^{\varphi(Z)} & 0 & * \\ 0 & \exp(Z) & * \\ 0 & 0 & e^{-\varphi(Z)} \end{pmatrix} \in \mathcal{F}_0 \subset H^0$$

and

$$Q_2 := \begin{pmatrix} 1 & \mp u^t e^Z & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^n \subset H^0.$$  

Then

$$P \cdot Q_1 \cdot Q_2 = \begin{pmatrix} \pm e^{a+\varphi(Z)} & v^t e^Z & * \\ 0 & A e^Z & * \\ 0 & 0 & \pm e^{-(a+\varphi(Z))} \end{pmatrix} \begin{pmatrix} 1 & \mp u^t e^Z & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & * \\ 0 & A e^Z & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \in \mathbb{Z}_2 \times \text{O}(n).$$

In the case, when $\mathbb{R}^n \nsubseteq H^0$, the Lie algebra $\mathfrak{h}$ of $H^0$ is of the second coupled type and we write

$$P = \begin{pmatrix} a & u^t & v^t & * \\ 0 & A & B & * \\ 0 & C & D & * \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \in H,$$  

with $u \in \mathbb{R}^k$ and $v \in \mathbb{R}^{n-k}$. Since the linear map $\psi : \text{pr}_{\mathfrak{s}(n)}(\mathfrak{h}) \to \mathbb{R}^k$ is surjective, we find an $X \in \mathfrak{s}(n-k)$ such that $\psi(X) = -a^{-1}u$. Then for

$$Q_1 := \exp \begin{pmatrix} 0 & 0 & -a^{-1}v^t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1}v \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -a^{-1}v & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H^0$$  

$$Q_2 := \exp \begin{pmatrix} 1 & \psi(X)^t & 0 & 0 \\ 0 & 0 & -\psi(X) & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \psi(X)^t & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & \exp(X) & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H^0$$

we obtain

$$P \cdot Q_1 \cdot Q_2 = \begin{pmatrix} a & u^t & 0 & * \\ 0 & A & B & * \\ 0 & C & D & * \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \cdot Q_2 = \begin{pmatrix} a & a\psi(X)^t + u^t & 0 & * \\ 0 & A & B \exp(X) & * \\ 0 & C & D \exp(X) & * \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \in \mathbb{R}^* \times \text{O}(n),$$

as $\psi(X) = -a^{-1}u$. This verifies (5) also for type 4 and proves the proposition.

Note that Lemma 1 and Proposition 1 imply Theorem 1 from the introduction when applied to the full and the restricted holonomy group of a Lorentzian manifold.
3. Null line bundle and screen bundle

In this section let $\text{Hol}_p(M, g)$ and $\text{Hol}^0_p(M, g)$ be the full and the restricted holonomy group of a Lorentzian manifold $(M, g)$ of dimension $n+2 > 2$, and let $\nabla$ denote the Levi-Civita connection of $g$. We assume that the restricted holonomy group acts indecomposably and not irreducibly. Then, from Theorem 1 we know that the same holds true for the full holonomy group. Hence, by the fundamental principle of holonomy by which holonomy invariant subspaces correspond to distributions on the manifold that are invariant under parallel transport [8, 10.19], the manifold admits a global distribution $\mathcal{L}$ of null lines that is invariant under parallel transport. Of course, also the distribution $\mathcal{L}^\perp$ whose fibres are orthogonal to the fibres of $\mathcal{L}$ is invariant under parallel transport. Hence, the tangent bundle is filtrated by parallel distributions

$$\mathcal{L} \subset \mathcal{L}^\perp \subset TM.$$ 

The Levi-Civita connection induces a linear connection $\nabla^\mathcal{L}$ on the bundle $\mathcal{L}$ by $\nabla^\mathcal{L} := \text{pr}_\mathcal{L} \circ \nabla$, where $\text{pr}_\mathcal{L}$ is the projection onto $\mathcal{L}$. Moreover, the metric $g$ and the Levi-Civita connection $\nabla$ induce a bundle metric $g^\mathcal{S}$ as well as a covariant derivative $\nabla^\mathcal{S}$ on the so-called screen bundle

$$S := \mathcal{L}^\perp/\mathcal{L} \to M$$

by $g^\mathcal{S}([X], [Y]) := g(X, Y)$ and $\nabla^\mathcal{S}_X [Y] := [\nabla_X Y]$, where $[.] : \mathcal{L}^\perp \to S = \mathcal{L}^\perp/\mathcal{L}$ denotes the canonical projection. The following Proposition shows the relation between the holonomy groups of $(\mathcal{L}, \nabla^\mathcal{L})$ and $(S, \nabla^\mathcal{S})$ and the projections of $\text{Hol}_p(M, g)$ onto $\mathbb{R}^*$ and $O(n)$, respectively.

**Proposition 2.** Let $(M, g)$ be a Lorentzian manifold with indecomposably, non-irreducibly acting restricted holonomy group, let $\mathcal{L}$ be the corresponding distribution of null lines and $S = \mathcal{L}^\perp/\mathcal{L}$ the screen bundle on $M$. Then:

1) $\text{Hol}_p(\mathcal{L}, \nabla^\mathcal{L}) = \text{pr}_\mathcal{L}^* (\text{Hol}_p(M, g))$.

2) The line bundle $\mathcal{L}$ is orientable, i.e., $\mathcal{L}$ admits a global nowhere vanishing section if, and only if, $\text{pr}_\mathcal{L} (\text{Hol}_p(M, g)) \subset \mathbb{R}^+$. This is equivalent to time-orientability of $(M, g)$.

3) The connection $\nabla^\mathcal{L}$ is flat if, and only if, $\text{pr}_\mathcal{L}^* (\text{Hol}^0_p(M, g)) = \{1\}$, and the line bundle $\mathcal{L}$ has a global parallel section iff $\text{pr}_\mathcal{L}^* (\text{Hol}_p(M, g)) = \{1\}$.

4) $\text{Hol}_p(S, \nabla^\mathcal{S}) = \text{pr}_{O(n)} (\text{Hol}_p(M, g))$ and $\text{Hol}^0_p(S, \nabla^\mathcal{S}) = \text{pr}_{O(n)} (\text{Hol}^0_p(M, g))$.

**Proof.** The first statement is obvious since the parallelity of $\mathcal{L}$ implies $P_\gamma^\mathcal{L} = P_\gamma^0|_{\mathcal{L}}$ for any curve $\gamma$, where $P_\gamma^\mathcal{L}$ is the parallel transport in $(\mathcal{L}, \nabla^\mathcal{L})$ and $P_\gamma^0$ that of $(M, g)$.

If $\mathcal{L}$ admits a global section $V \in \Gamma(\mathcal{L})$, then $P_{\gamma V}^\mathcal{L}(V(\gamma(0))) = \alpha(t)V(\gamma(t))$ for all curves $\gamma$, where $\alpha$ is a positive function with $\alpha(0) = 1$. Hence $\text{pr}_\mathcal{L}^* (\text{Hol}_p(M, g)) \subset \mathbb{R}^+$. Conversely, let $\text{pr}_\mathcal{L}^* (\text{Hol}_p(M, g)) \subset \mathbb{R}^+$. Then using the holonomy principle, the half-line $\mathbb{R}^+ \ell \subset \mathcal{L}_p$ provides a well defined field of directions $\mathcal{L}^+ \subset \mathcal{L}$. Thus, we have a covering $M = \bigcup_k U_k$ and local sections $V_k \in \Gamma(U_k, \mathcal{L})$ such that $V_k(x) \in \mathcal{L}^+_x$ for any $x \in U_k$. Using a partition of unity we derive a global nowhere vanishing section of $\mathcal{L}$. The orientability of $\mathcal{L}$ is equivalent to time-orientability of $(M, g)$.

To see this, we choose a splitting $s : S \to \mathcal{L}^\perp$ of the sequence

$$0 \to \mathcal{L} \to \mathcal{L}^\perp \to S \to 0.$$

Then $E := s(S) \subset \mathcal{L}^\perp$ and $E^\perp \subset TM$ is a subbundle of signature $(1, 1)$ with $\mathcal{L} \subset E^\perp$. Hence, the light-cone of $E^\perp_p$ at any $p \in M$ is a union of two lines, one of which is given by $\mathcal{L}_p$. Thus, we derive a second lightlike vector field $Z \in \Gamma(M, E^\perp)$ with $g(V, Z) = 1$. Then $\frac{1}{\sqrt{2}}(V - Z)$ is a nowhere vanishing timelike unit vector field and $(M, g)$ is time-orientable. On the other hand, any timelike unit vector field $T$ on $(M, g)$ defines a global field of null direction $\mathcal{L}^+ \subset \mathcal{L}$ by requiring $g(T, \mathcal{L}^+) > 0$, hence, $\mathcal{L}$ is orientable.

The third statement follows from the first one and standard facts of holonomy theory: $\nabla^\mathcal{L}$ is flat iff $\text{Hol}^0_p(\mathcal{L}, \nabla^\mathcal{L}) = \{1\}$, the line bundle $(\mathcal{L}, \nabla^\mathcal{L})$ admits a global parallel section if, and only if, $\text{Hol}_p(\mathcal{L}, \nabla^\mathcal{L}) = \{1\}$. 


Finally, we prove the fourth statement. Let $\gamma : [0,1] \to M$ be a loop around the point $p = \gamma(0) = \gamma(1) \in M$. We fix a complement $E \subset \mathcal{L}_p^\perp$ of $\mathcal{L}_p$ that is orthogonal to $\mathcal{L}_p$. Then the canonical projection $[\cdot] : \mathcal{L}_p^\perp \to \mathcal{S}_p$ becomes a linear isomorphism when restricted to $E$. For a fixed non-vanishing vector $\ell \in \mathcal{L}_p$ we denote by $V(t)$ the parallel displacement of $\ell$ along $\gamma$ with respect to the Levi-Civita connection $\nabla^g$ of $g$. We have to prove that the parallel transport $P^g_\gamma$ and the parallel transport $P^S_\gamma$ with respect to $\nabla^S$ commute with the canonical projection $[\cdot]$, 

\[ [P^g_\gamma(e)] = P^S_\gamma([e]), \]

for every $e \in E$. Indeed, if we write $P^S_{\gamma|p,e}([e]) = [U(t)]$ with $U(t) \in \mathcal{L}_{\gamma(t)}^\perp$ and $U(0) = e$, we have

\[ 0 = \nabla^S_{\gamma(t)}[U(t)] = \left[\nabla^g_{\gamma(t)} U(t)\right] \]

which implies that

\[ \nabla^g_{\gamma(t)} U(t) = f(t)V(t) \]

with a function $f : [0,1] \to \mathbb{R}$. Since $V(t)$ is parallel, the vector field

\[ U(t) - \int_0^t f(s)ds \cdot V(t) \]

is parallel along $\gamma$ with respect to $\nabla^g$ and equals $e$ in $t = 0$. Hence,

\[ [P^g_\gamma(e)] = \left[ U(1) - \int_0^1 f(s)ds \cdot V(1) \right] = [U(1)] = P^S_\gamma([e]). \]

This proves statement (6) and the proposition. \hfill \qed

Since $G^0_p := \text{pr}_{SO(n)}(\text{Hol}^0_p(M,g)) \subset SO(\mathcal{S}_p)$, as a subgroup of $SO(n)$, is compact, it acts completely reducible on

\[ \mathcal{S}_p = V_0 \oplus V_1 \oplus \ldots \oplus V_k \]

with $V_0$ trivial and $V_i$ irreducible for $i = 1, \ldots, k$, and moreover, using the Bianchi-identity, it can be shown (see [6] or [23]) that

\[ G^0_p = G_1 \times \ldots \times G_k \]

is a direct product of subgroups $G_i$ acting irreducibly on $V_i$ and trivial on $V_j$ for $j \neq i$. Furthermore, in [23] we have shown that $G^0_p$ acts as a Riemannian holonomy representation, i.e. $G^0_p$ is trivial or a product of the groups from the Berger list, i.e. of \n
\[ \text{SO}(n), \text{U}(m), \text{SU}(m), \text{Sp}(k), \text{Sp}(k) \cdot \text{Sp}(1), \text{Spin}(7), \text{and G}_2, \]

and of isotropy groups of Riemannian symmetric spaces. In section 7 we will use this in order to prove Theorem 2.

4. Holonomy groups and coverings

In the following we will consider manifolds that are given as a quotient by a group of diffeomorphisms. We recall the following facts (see for example [29, Chapter 7]): Let $\Gamma$ be a group of diffeomorphisms of a smooth manifold $M$. We say that $\Gamma$ is properly discontinuous, if

(PD1) each $p \in M$ has a neighborhood $U_p$ such that $\gamma(U) \cap U = \emptyset$ for all $\gamma \in \Gamma \setminus \{1\}$ and

(PD2) two points $p$ and $q$, which are not in the same orbit under $\Gamma$, have neighborhoods $U_p$ and $U_q$ such that $\gamma(U_p) \cap U_q = \emptyset$ for all $\gamma \in \Gamma$.

Clearly a properly discontinuous group acts freely on $M$. If $\Gamma$ is a properly discontinuous group of diffeomorphisms of $M$, the quotient space $M/\Gamma$ is a smooth manifold and the projection $\pi : M \to M/\Gamma$ is a smooth covering map [29, Chapter 7, Proposition 7.7]). If $\Gamma$ is a properly discontinuous group of isometries of a semi-Riemannian manifold $M$, then there is a unique metric on $M/\Gamma$, such that $\pi : M \to M/\Gamma$ is a semi-Riemannian covering ([29, Chapter 7, Corollary 7.12]).
Remark 2. Let $\Gamma_\lambda^\Omega$, $\lambda \in \mathbb{R} \setminus \{0\}$, be the group of diffeomorphisms of an appropriate domain $\Omega \subset \mathbb{R}^2$ generated by
$$\varphi_\lambda(v,u) := (e^{\lambda}v, e^{-\lambda}u), \quad (v,u) \in \Omega.$$  
For $\Omega^0 := \mathbb{R}^2 \setminus \{(0,0)\}$, the group $\Gamma_\lambda^{\Omega^0}$ fails to satisfy (PD2) for the points $p = (1,0), q = (0,1)$. For the halfspace $\Omega := \{(v,u) \in \mathbb{R}^2 \mid u > 0\}$, the group $\Gamma_\lambda^\Omega$ is properly discontinuous whereas the groups $\Gamma_\lambda^\Omega|_{\lambda_1,\lambda_2} \subset \text{Diff}(\Omega)$, generated by $\varphi_{\lambda_1}$ and $\varphi_{\lambda_2}$ for $\lambda_1, \lambda_2$ linearly independent over $\mathbb{Q}$, fail to satisfy (PD1) since there is a sequence of integers $(k_n, l_n) \in \mathbb{Z} \times \mathbb{Z}$ such that $0 \neq k_n\lambda_1 + l_n\lambda_2 \to \infty$.  

In the following section we will have to check that certain group actions on manifolds define quotients that are manifolds again. The following simple observation is useful. Its proof is straightforward.

**Definition 1.** Let $M_1$ and $M_2$ be two smooth connected manifolds and $\Gamma \subset \text{Diff}(M_1) \times \text{Diff}(M_2)$ a group of diffeomorphisms of $M_1 \times M_2$. We denote by $\Gamma_i := \text{proj}_i(\Gamma) \subset \text{Diff}(M_i)$ the projections and for $\sigma \in \Gamma_2$ by $\Gamma_\sigma \subset \Gamma_1$ the $\sigma$-section $\Gamma_\sigma := \{\gamma \in \Gamma_1 \mid (\gamma,\sigma) \in \Gamma\}$. We say that $\Gamma$ is of *quotient type*, if

(i) $\Gamma_2$ is properly discontinuous,
(ii) $\Gamma_{\text{id}_{M_2}} \subset \Gamma_1$ satisfies (PD1) and
(iii) $\Gamma_\sigma \subset \Gamma_1$ satisfies (PD2) for all $\sigma \in \Gamma_2$, 
(or if the same is true exchanging the role of $M_1$ and $M_2$).

Clearly, (iii) is satisfied if the section sets $\Gamma_\sigma \subset \Gamma_1$ are finite for all $\sigma \in \Gamma_2$.

**Lemma 2.** Let $\Gamma \subset \text{Diff}(M_1) \times \text{Diff}(M_2)$ be a group of quotient type. Then $\Gamma$ is properly discontinuous.

The fundamental relation (2) between the fundamental group and the quotient of the full holonomy by the restricted holonomy group is a special case of the following fact when applied to the universal covering. In the following when referring to semi-Riemannian manifold, for brevity of notation we do not mention explicitly the metrics. Furthermore, $\bar{P}_\gamma$ denotes the parallel transport along a curve $\bar{\gamma}$ in $\bar{M}$ and $P_\gamma$ along a curve $\gamma$ in $M$, both with respect to the Levi-Civita connections of the semi-Riemannian metrics on $\bar{M}$ and $M$, respectively.

**Proposition 3.** Let $\bar{M}$ be a connected semi-Riemannian manifold and $\Gamma$ a properly discontinuous group of isometries of $\bar{M}$ inducing the semi-Riemannian covering $\pi : \bar{M} \to M := \bar{M}/\Gamma$. Then, for any points $p \in M$ and $\bar{p}$ in the fibre $\pi^{-1}(p)$ we have:

(i) The holonomy group $\text{Hol}_p(\bar{M})$ injects homomorphically into $\text{Hol}_p(M)$ via $\iota : \bar{P}_\gamma \mapsto P_{\pi\circ\gamma}$, for $\bar{\gamma}$ a loop at $\bar{p}$, and the image is a normal subgroup.

(ii) The following map is a surjective group homomorphism, $\Phi : \Gamma \to \text{Hol}_p(M)/\text{Hol}_p(\bar{M})$
$$\gamma \mapsto [P_\gamma],$$
where $\gamma$ is a loop at $p$ that, when lifted to a curve $\bar{\gamma}$ starting at $\bar{p}$, ends at $\sigma^{-1}(\bar{p})$.

(iii) Let $\gamma$ be a loop in $M$ at $p \in M$. Then, using the identification of $T_pM$ with $T_{\bar{p}}\bar{M}$ by $d\pi_{\bar{p}}$, the parallel transport along $\gamma$ is given by
$$P_\gamma = d\sigma^{-1}_{(\bar{p})} \circ \bar{P}_\gamma,$$
where $\bar{\gamma}$ is the lift of $\gamma$ starting at $\bar{p}$ and $\bar{\gamma}(1) = \sigma^{-1}(\bar{p})$ with $\sigma \in \Gamma$. In particular,
$$\phi(\sigma) := d\sigma^{-1}_{(p)} \circ P_\gamma = (d\sigma^{-1}_{p})^{-1} \circ P_\gamma$$
is a representative of $\Phi(\sigma) \in \text{Hol}_p(M)/\text{Hol}_p(\bar{M})$. 


Proof. (i) Clearly, \( \iota \) is a group homomorphisms: If \( \tilde{\gamma} \) and \( \tilde{\delta} \) are loops at \( \tilde{p} \in \tilde{M} \), we have:

\[
\iota(\tilde{P}_\gamma \cdot \tilde{P}_\delta) = \iota(\tilde{P}_{\gamma \ast \delta}) = P_{\pi_0(\gamma \ast \delta)} = P_{\pi_0(\gamma) \ast P_{\pi_0(\delta)}} = \iota(\tilde{P}_\gamma) \cdot \iota(\tilde{P}_\delta).
\]

That \( \iota \) is injective follows from the fact that \( \pi \) is a local isometry, which implies that

\[
d\pi^{-1}_p \circ P_{\pi_0 \gamma} \circ d\pi_p = \tilde{P}_\gamma
\]

(see for example [29, p. 91]). To show that the image of \( \iota \) and \( \delta \) at \( \tilde{G} \) follows using the identification \( T_{\tilde{p}} \Gamma \) in \( \tilde{\Omega} \) in \( \tilde{\Omega} \). Hence, we have

\[
P_{\gamma} \cdot P_{\delta} = P_{\delta} \cdot P_{\delta^{-1} \gamma \ast \delta} = P_{\delta} \cdot P_{\pi_0(\delta^{-1} \gamma \ast \delta)}.
\]

Since \( \tilde{\delta}^{-1} \ast \tilde{\gamma} \ast \tilde{\delta} \) is a loop at \( \tilde{p} \), the image of \( \iota \) is normal.

(ii) First we have to verify that the map \( \Phi \) is well defined. For two curves \( \tilde{\gamma} \) and \( \tilde{\delta} \) starting at \( \tilde{p} \in \tilde{M} \) and ending at \( \sigma^{-1}(\tilde{p}) \) for a \( \sigma \in \Gamma \) we can write

\[
\tilde{P}_\delta = \tilde{P}_\gamma \cdot \tilde{P}_{\gamma^{-1} \ast \delta}
\]

which shows that

\[
P_{\pi_0 \delta} = P_{\pi_0 \gamma} \cdot P_{\pi_0(\gamma^{-1} \ast \delta)}.
\]

Noting that \( \tilde{\gamma} \) and \( \tilde{\delta} \) is a loop at \( \tilde{p} \) shows that the image of \( \sigma \) under \( \Phi \) does not depend on the chosen curve \( \gamma \).

Next we show that \( \Phi \) is a group homomorphism. Let \( \sigma_1 \) and \( \sigma_2 \) two elements of \( \Gamma \), and \( \gamma_1, \gamma_2 \) two loops at \( p \in M \) that lift to curves starting at \( \tilde{p} \) and ending at \( \sigma_1^{-1}(\tilde{p}) \) and \( \sigma_2^{-1}(\tilde{p}) \), respectively. Now, consider the curve \( \tilde{\gamma} := (\sigma_2^{-1} \circ \gamma_1) \ast \gamma_2 \) in \( \tilde{M} \). \( \gamma \) starts in \( \tilde{p} \) and ends in \( \sigma_2^{-1}(\sigma_1^{-1}(\tilde{p})) = (\sigma_1 \cdot \sigma_2)^{-1}(\tilde{p}) \) and projects to \( \gamma_1 \ast \gamma_2 \). Hence, we have

\[
\Phi(\sigma_1) \cdot \Phi(\sigma_2) = [P_{\gamma_1} \ast \gamma_2] = [P_{\pi_0 \gamma}] = \Phi(\sigma_1 \cdot \sigma_2).
\]

Finally, \( \Phi \) is surjective since \( \tilde{M} \) is connected.

(iii) Since \( \pi \) is a local isometry, we have \( P_\gamma = d\pi_{\gamma(1)} \circ \tilde{P}_\gamma \circ (d\pi_{\gamma(0)})^{-1} \). Then the statement follows using the identification \( T_p M \simeq T_{\tilde{p}} \tilde{M} \) by \( d\pi_p \) and \( d\pi_{\sigma^{-1}(\tilde{p})} = d\pi_{\tilde{p}} \circ d\sigma_{\sigma^{-1}(\tilde{p})} \).

\[\Box\]

5. Construction of Lorentzian manifolds with disconnected holonomy

The following proposition shows that special classes of non-connected subgroups of the stabilizer \( \text{Stab}_{O(1,n+1)}(L) \) can be realized as holonomy group of Lorentzian manifolds.

**Proposition 4.** Let \( G \subset O(n) \) be the holonomy group of an \( n \)-dimensional Riemannian manifold. Then the subgroups

\[
G \ltimes \mathbb{R}^n, \ (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n, \ (\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n, \ (\mathbb{Z}_2 \times G) \ltimes \mathbb{R}^n
\]

in \( \text{Stab}_{O(1,n+1)}(L) \) can be realized as holonomy groups of Lorentzian manifolds.

**Proof.** The first part of the proof follows [21] or [4], where it was given for the restricted holonomy group. We fix a Riemannian manifold \( (N, h) \) whose holonomy group is given by the not necessarily connected group \( G \). Furthermore we denote by \( (v, u) \) coordinates on \( \mathbb{R}^2 \) and fix an open domain \( \Omega \) in \( \mathbb{R}^2 \), for example \( \Omega = \mathbb{R}^2 \). Also we chose a smooth function \( f \in C^\infty(\Omega \times N) \) with the property that

\[
0 \neq \det \text{Hess}_p^h(f(v_0, u_0, \cdot)) \quad \text{for some} \ p \in N \text{ and} \ (v_0, u_0) \in \Omega.
\]

\[\footnote{Note, that by \( \gamma \ast \delta \) we denote the joint path that first runs through \( \delta \) and then through \( \gamma \).} \]
Then we define a Lorentzian manifold \((M,g)\) by
\[
(M = \Omega \times N, g^{f,h} = 2dvdu + 2fdu^2 + h).
\]

Computing the Levi-Civita connection \(\nabla^g\) of \(g\) we obtain as the only non-vanishing terms
\[
\nabla^g_X Y = \nabla^h_X Y,
\]
\[
\nabla^g_{\partial_u} X = \nabla^g_{\partial_u} \partial_u = df(X)\partial_v,
\]
\[
\nabla^g_{\partial_v} \partial_u = \partial_u(f)\partial_v - \text{grad}^h(f),
\]
\[
\nabla^g_{\partial_v} \partial_v = \nabla^g_{\partial_v} \partial_v = \partial_v(f)\partial_v
\]
for \(X,Y \in \Gamma(TN)\), \(\nabla^h\) and \(\text{grad}^h\) with respect to the Riemannian metric \(h\) on \(N\). This allows us to compute the parallel transport from a point \(q = (v_0, u_0, p)\) of a vector \(X_0 \in T_pN\) along a curve \(\delta = (v, u, \gamma) : [0, 1] \rightarrow M\) for \(\gamma\) a curve in \(N\) and \(\delta(0) = q\). Indeed, if \(X : [0, 1] \rightarrow TN\) is the vector field along \(\gamma\) that is the parallel transport of \(X_0 \in T_pN\) with respect to \(\nabla^h\) and the function \(\varphi : [0, 1] \rightarrow \mathbb{R}\) satisfies the ODE
\[
\varphi + \varphi \cdot \dot{u} \cdot \partial_v f \circ \delta + \dot{u} \cdot df(X) \circ \delta = 0,
\]
\[
\varphi(0) = 0,
\]
then the vector field,
\[
(12) \quad \varphi \cdot (\partial_v \circ \delta) + X
\]
is the parallel transport of \(X_0\) along \(\delta\). This shows that for the curve \(\delta\) we have that
\[
\text{pr}_{T_pN} \circ \mathcal{P}_\delta^{-1}|_{T_pN} = \mathcal{P}_\gamma^h.
\]

Hence, the \(O(n)\) projection of \(\text{Hol}_q(M, g)\) is given as \(G = \text{Hol}_p(N, h)\).

Furthermore, when computing the curvature \(R^g\) of \(g\) we get
\[
R^g(\partial_u, X)Y = \text{Hess}^h(f)(X, Y)\partial_v.
\]

Taking this at points \((v_0, u_0, p)\) where \(\det(\text{Hess}^h_p(f(v_0, u_0, \cdot)) \neq 0\) this shows that the holonomy algebra, and hence both, the restricted and the full holonomy group contain \(\mathbb{R}^n\).

Next we look at the \(\mathbb{R}\)-component of the holonomy group. Clearly, when \(\partial_v f = 0\), the vector field \(\partial_v\) is parallel and what we have shown so far implies that \(\text{Hol}(M, g) = G \times \mathbb{R}^n\). Otherwise, again by computing the curvature term,
\[
R^g(X, \partial_u)\partial_v = X(\partial_v(f))\partial_v
\]
and by choosing \(f\) such that \(\partial_v f\) is not constant on \(N\), we conclude that the holonomy algebra contains \(\mathbb{R}\) and hence the restricted holonomy contains \(\mathbb{R}^+\). Therefore, as \((M, g)\) is clearly time-orientable, the full holonomy group is equal \((\mathbb{R}^+ \times G) \ltimes \mathbb{R}^n\).

Using this result, we construct non-time-orientable Lorentzian manifolds with holonomy group \((\mathbb{R}^* \times G) \ltimes \mathbb{R}^n\) as well as with holonomy group \((\mathbb{Z}_2 \times G) \ltimes \mathbb{R}^n\). We set \(\Omega := \mathbb{R}^2 \setminus \{(0, 0)\}\), and choose a function \(f \in C^\infty(\Omega \times N)\) which satisfies \(f(-v, -u, p) = f(v, u, p)\) and condition (10). Then, for the Lorentzian manifold as defined in (11), the map
\[
(v, u, p) \mapsto \sigma(v, u, p) := (-v, -u, p)
\]
is an isometry of \((M, g^{f,h})\) and generates the group \(\mathbb{Z}_2\) in the isometry group of \((M, g^{f,h})\). This group acts properly discontinuous on \(M\) and hence \(M/\mathbb{Z}_2\) becomes a smooth Lorentzian manifold, which is not time-orientable. By Proposition 3, its holonomy is given as \((\mathbb{R}^* \times G) \ltimes \mathbb{R}^n\) if \(\partial_v f\) is not constant on \(N\), and as \((\mathbb{Z}_2 \times G) \ltimes \mathbb{R}^n\) if \(f\) is chosen such that \(\partial_v f = 0\). Indeed, if \(\delta\) is a loop in \(M/\mathbb{Z}_2\), starting and ending in \(\pi(0, 1, p)\), with a non-closed lift \(\delta\), then \(\delta\) is given by \(\delta(t) = (v(t), u(t), \gamma(t))\), starting at \((0, 1, p)\) and ending at \((0, -1, p) = \sigma^{-1}(0, 1, p)\), where \(\gamma\) is a loop in \(N\) based at \(p\). Then, by the above formulae, the parallel transport to \((0, -1, p)\) of the vector \(\partial_v\) and a vector \(X_0\) tangent to \(N\) at \((0, 1, p)\) along \(\delta\) is given as \(a\partial_v\) for a positive number \(a\), which is 1 when \(\partial_v f = 0\), and by
$P^h_\sigma(X_0) + b\partial v$, respectively. Hence, as $d\sigma_{(0,-1,p)}(\partial_v) = -\partial_v$ and $d\sigma_{(0,-1,p)}|_{TN} = \text{Id}$, formula (9) in Proposition 3 gives the result.

\begin{remark}
Lorentzian metrics of the form (11) are sometimes called pp-waves, for “plane-fronted waves”, or $Np$-waves, “$N$-fronted with parallel rays” (for example in [12] and [16]). Special classes of pp-waves are
\begin{itemize}
  \item[a)] \textit{pp-waves}, “plane fronted with parallel rays”, for which $h$ is flat,
  \item[b)] \textit{plane waves}, for which $h$ is flat and $f$ is a quadratic polynomial in the coordinates on $N$ with $u$-dependent coefficients, and
  \item[c)] \textit{Cahen-Wallach spaces}, [11, 10], which are Lorentzian symmetric spaces. Here $h$ is flat and $f$ is a quadratic polynomial in the coordinates on $N$ with constant coefficients.
\end{itemize}

Note that the construction for manifolds with holonomy $(\mathbb{Z}_2 \times G) \ltimes \mathbb{R}^n$ cannot be generalised directly to discrete subgroups of Lorentz boosts of $\mathbb{R}^{1,1}$. As we observed in remark 2, a group of Lorentz boosts $\Gamma$ generated by $\text{diag}(e^\lambda, e^{-\lambda}) \subset \text{SO}(1,1)$ will not act properly on $\mathbb{R}^{1,1} \setminus \{0\}$ and the quotient will not be Hausdorff. On the other hand, taking $\Omega$ smaller by removing a closed ball around the origin in $\mathbb{R}^{1,1}$, $\Omega$ would no longer be invariant under $\Gamma$. We will avoid these difficulties in the following construction by which we will obtain Lorentzian manifolds with disconnected holonomy group contained in the stabiliser of a null line from the following data:
\begin{enumerate}
  \item[1)] A simply connected Riemannian manifold $(N, h)$ together with a function $f$ on $M$ with the property (10).
  \item[2)] a properly discontinuous subgroup $\Gamma$ of isometries of the Lorentzian manifold $(\widetilde{M} := \Omega \times N, g^{f,h} := 2dvdu + 2fdu^2 + h)$.
\end{enumerate}

\begin{theorem}
Let $(N, h)$ be a Riemannian manifold, $\Omega$ an open domain in $\mathbb{R}^2$ with global coordinates $(v, u)$, $f$ a smooth function on $\Omega \times N$ with $\partial_v f = 0$ and satisfying property (10). Then every isometry $\sigma$ of the Lorentzian manifold

\begin{equation}
(\widetilde{M} = \Omega \times N, g^{f,h} = 2dvdu + 2fdu^2 + h)
\end{equation}

can be written as

\begin{equation}
(13) \quad \widetilde{\sigma}(v, u, p) = \left( a_\sigma v + \tau_\sigma(v, u, p), \frac{u}{a_\sigma} + b_\sigma, \nu_\sigma(v, u, p) \right),
\end{equation}

with certain $a_\sigma \in \mathbb{R}^*$, $b_\sigma \in \mathbb{R}$, $\tau_\sigma \in C^\infty(\widetilde{M})$ and $\nu_\sigma \in C^\infty(\widetilde{M}, N)$, such that

\begin{equation}
d\tau_\sigma(\partial_v) = d\nu_\sigma(\partial_v) = 0,
\end{equation}

and for each $(v, u) \in \Omega$, $\nu_\sigma(v, u, \cdot)$ is an isometry of the Riemannian manifold $(N, h)$.

Furthermore, let $\Gamma$ be a properly discontinuous group of isometries of $(\widetilde{M}, g^{f,h})$ and let $\pi : \widetilde{M} \rightarrow M := \widetilde{M}/\Gamma$ be the corresponding covering of the Lorentzian manifold $M$. Then, at a point $\pi(\tilde{q}) \in M$ for $\tilde{q} = (v_0, u_0, p) \in \widetilde{M}$, the map $\Phi : \Gamma \rightarrow \text{Hol}_q(M) / \text{Hol}_{\tilde{q}}(\widetilde{M})$ of Proposition 3, with the image written in the decomposition

\begin{equation}
\mathbb{R} \cdot \partial_v q_0 N \oplus \mathbb{R} \cdot (\partial_u - f \partial_v)(q_0)
\end{equation}

of $T_{\tilde{q}} \widetilde{M} \simeq T_{\pi(\tilde{q})} M$, is given by the representative

\begin{equation}
\Phi(\sigma) = \left( a_\sigma \quad 0 \quad 0 \right) \left( d\nu_{\sigma^{-1}}(q_0)^{-1} \circ P^h_{\sigma} \quad 0 \right) \left( 0 \quad a_\sigma \right) \in \Phi(\sigma),
\end{equation}

where
\begin{itemize}
  \item[i)] $\nu^0_{\sigma^{-1}} := \nu_{\sigma^{-1}}(v_0, u_0, \cdot)$ the isometry of $N$ defined by $\sigma^{-1}$ at $(v_0, u_0) \in \Omega$,
  \item[ii)] $P^h_{\sigma}$ denotes the parallel transport with respect to $h$ along some curve $\gamma : [0,1] \rightarrow N$ with $\gamma(0) = p$ and $\gamma(1) = \nu_{\sigma^{-1}}(v_0, u_0, p)$.
\end{itemize}

\end{remark}
In particular, the full holonomy group of the Lorentzian manifold \( M \) is given as
\[
\text{Hol}_{\pi(q)}(M) = \left\{ \hat{\phi}(\sigma) \middle| \sigma \in \Gamma \right\} \cdot \text{Hol}_{\pi}(N, h) \times \mathbb{R}^n.
\]

**Proof.** The construction of \( g^{f,h} \) and the conditions on \( f \) imply by Proposition 4 that the holonomy group of \((\tilde{M}, g^{f,h})\) is given by the group \(\text{Hol}_{\pi}(N, h) \times \mathbb{R}^n\). Furthermore, the vector field \( \partial_v \) is parallel and so is its push forward \( \sigma_* \partial_v \) by an isometry \( \sigma \). Indeed, for an isometry \( \sigma \) we have
\[
0 = \sigma_*(\nabla_X \partial_v) = \nabla_{(\sigma_*X)}(\sigma_*\partial_v)
\]
for all vector fields \( X \) on \( \tilde{M} \). By the conditions on \( f \), \( \tilde{M} \) is indecomposable, which implies that \( \sigma_* \partial_v \) is a constant multiple of \( \partial_v \), as otherwise \( \partial_v \) and \( \sigma_* \partial_v \) would span a non-degenerate parallel distribution. Hence, we write the isometry \( \sigma \) in components according to \( \tilde{M} = \Omega \times N \) and using global coordinates \((v, u)\) on \( \Omega \) as \( \sigma = (v \circ \sigma, u \circ \sigma, \nu_{\sigma}) \). Then, when \( \partial_v(v \circ \sigma) \) denotes the directional derivative \( \partial_v(v \circ \sigma)(p) = d(v \circ \sigma)_{p}(\partial_v(p)) \), we have
\[
\sigma_* \partial_v = \partial_v(v \circ \sigma) \circ \sigma^{-1} \cdot \partial_v,
\]
and that \( \partial_v(v \circ \sigma) \) is constant, i.e., there is a constant \( a_{\sigma} \in \mathbb{R}^n \) and a function \( \tau_{\sigma} \) of \( u \) and \( p \) in \( N \) such that \( v \circ \sigma(v, u, p) = a_{\sigma} v + \tau_{\sigma}(u, p) \). Also, because of
\[
d\sigma(\partial_v) = a_{\sigma} \partial_v + d(u \circ \sigma)(\partial_v)\partial_u + d\nu_{\sigma}(\partial_v)
\]
we get that \( u \circ \sigma \) and \( \nu_{\sigma} \) do not depend on \( v \). We also note that \( \sigma_*X \) is still orthogonal to \( \partial_v \) for \( X \in \Gamma(TN) \). This implies that \( d\sigma(X) \in \mathbb{R} \cdot \partial_v \oplus TN \) and thus that \( d(u \circ \sigma)(X) = 0 \) which shows that \( u \circ \sigma \) is a function only of the coordinate \( u \). We also note that
\[
1 = g(\partial_v, \partial_u) = \sigma^* g(\partial_v, \partial_u) = a_{\sigma} \frac{d}{du}(u \circ \sigma),
\]
which shows that \( u \circ \sigma = \frac{1}{a_{\sigma}} u + b_{\sigma} \) with a constant \( b_{\sigma} \in \mathbb{R} \).

Finally
\[
h(X, Y) = g(X, Y) = \sigma^* g(X, Y) = \nu_{\sigma}^* h(X, Y)
\]
for all \( X, Y \in TN \) shows that \( \nu_{\sigma}(v, u, \cdot) \) is an isometry of \((N, h)\). This proves formula (13).

In order to prove the result about the holonomy group, consider a curve
\[
\tilde{\delta}(t) = (v(t) := v \circ \tilde{\delta}(t), u(t) := u \circ \tilde{\delta}(t), \gamma(t))
\]
in \( \tilde{M} \) with \( \tilde{\delta}(0) = \tilde{q} = (v_0, u_0, p) \) and \( \tilde{\delta}(1) = \sigma^{-1}(\tilde{q}) \), \( \gamma \) a curve in \( N \) with \( \gamma(0) = p \) and \( \gamma(1) = \nu_{\sigma^{-1}}(\tilde{q}) \). Then the formulae in the proof of Proposition 4 imply that the parallel transport along \( \tilde{\delta}(t) = (v(t), u(t), \gamma(t)) \) is given in the decomposition
\[
\mathbb{R} \cdot \partial_v \oplus TN \oplus \mathbb{R} \cdot (\partial_u - f \partial_v)
\]
as
\[
\tilde{P}_{\tilde{\delta}} = \begin{pmatrix} 1 & * & * \\ 0 & P^h_{\gamma} & * \\ 0 & 0 & 1 \end{pmatrix}.
\]

Secondly, we have seen that the differential of \( \sigma \) at \( \sigma^{-1}(\tilde{q}) \) is given as
\[
d\sigma_{\sigma^{-1}(\tilde{q})} = \begin{pmatrix} a_{\sigma} & * \\ 0 & d^N \nu_{\sigma_{\sigma^{-1}(\tilde{q})}} & * \\ 0 & 0 & a_{\sigma}^{-1} \end{pmatrix} = \begin{pmatrix} a_{\sigma} & * \\ 0 & (dv_{\sigma^{-1}(\tilde{q})})^{-1} & * \\ 0 & 0 & a_{\sigma}^{-1} \end{pmatrix}.
\]

Having this, we can apply formula (8) in Proposition 3 directly. For a loop \( \delta : [0, 1] \to \tilde{M}/\Gamma \) at \( \pi(\tilde{q}) \), the lift is a curve \( \tilde{\delta} \) in \( \tilde{M} \) such that \( \tilde{\delta}(0) = \tilde{q} \) and \( \tilde{\delta}(1) = \sigma^{-1}(\tilde{q}) \) for a \( \sigma \in \Gamma \). Hence, by formula (8) in Proposition 3 the parallel transport along \( \delta \) is given by
\[
P_{\delta} = \begin{pmatrix} a_{\sigma} & * \\ 0 & (dv_{\sigma^{-1}(\tilde{q})})^{-1} \circ P^h_{\gamma} & * \\ 0 & 0 & a_{\sigma}^{-1} \end{pmatrix}.
\]
The same argument as in the proof of Proposition 1 shows, that we can use the matrix

\[ \hat{\phi}(\sigma) = \begin{pmatrix} a_\sigma & 0 & 0 \\ 0 & (db^0_{\sigma-1}|_p)^{-1} \circ P^h_\gamma & 0 \\ 0 & 0 & a_\sigma^{-1} \end{pmatrix} \]

as representative of the class \( \Phi(\sigma) \in Hol_q(M)/Hol_q(\tilde{M}) \). This proves the second statement. \( \square \)

The examples in the next section will be based on this Theorem. For most of them we will apply the following

**Corollary 1.** Let \((N,h)\) be an \(n\)-dimensional Riemannian manifold, \(\Gamma\) a properly discontinuous group of isometries of \((N,h)\), and \(f\) a \(\Gamma\)-invariant function on \(N\) satisfying property (10). Fix a not necessarily finite set of generators \((\gamma_1, \gamma_2, \ldots)\) of \(\Gamma\). Corresponding to these generators of \(\Gamma\) fix a sequence of integers \(m = (m_1, m_2, \ldots)\) and of real numbers \(\lambda := (\lambda_1, \lambda_2, \ldots)\).

1. Consider the Lorentzian metric

\[ g^{f,h} = 2dvdu + 2fdu^2 + h \]

on \(\tilde{M} := \mathbb{R}^2 \times N\). Suppose that the group \(\Gamma_m\) generated in the isometry group of \((\tilde{M}, g^{f,h})\) by the isometries

\[ \sigma_m(v,u,p) := ((-1)^{m_1}v, (-1)^{m_2}u, \gamma_i(p)) \]

for \(i = 1, 2, \ldots\) is of quotient type or restrict to \(\tilde{M}_0 := (\mathbb{R}^2 \setminus \{(0,0)\}) \times N\) otherwise. Then \(\Gamma_m\) is properly discontinuous and the holonomy group of the Lorentzian manifold \((M := \tilde{M}_0/\Gamma_m, g^{f,h})\) at \(\pi(v,u,p)\) is given as

\[ L\Gamma_m \cdot (\text{Hol}_p(N) \times \mathbb{R}^n) , \]

where \(L\Gamma_m\) is the group that is generated in \(O(1,n+1)\) by the linear maps

\[ \begin{pmatrix} (-1)^{m_1} & 0 & 0 \\ 0 & \phi(\gamma_i) & 0 \\ 0 & 0 & (-1)^{m_2} \end{pmatrix} \in \mathbb{Z}_2 \times \phi(\Gamma) , \]

where \(\phi(\gamma_i)\) is the representative for the class \(\Phi(\gamma_i) \in \text{Hol}_p(N/\Gamma)/\text{Hol}_p(N)\) as described in of Proposition 3. In particular, for appropriate choices of \(m\), \((M, g^{f,h})\) is not time-orientable, admits a parallel null line bundle but no parallel vector field.

2. Set \(\Omega := \{(v,u) \in \mathbb{R}^2 \mid u > 0\}\) and consider the Lorentzian metric

\[ g^{f/u^2,h} = 2dvdu + \frac{2}{u^2}fdu^2 + h \]

on \(\tilde{M} := \Omega \times N\). Suppose that the group \(\Gamma_\lambda\) generated in the isometry group of \((M, g^{f/u^2,h})\) by the isometries

\[ \sigma_\lambda(v,u,p) := (e^{\lambda_i}v, e^{\lambda_i}u, \gamma_i(p)) \]

for \(i = 1, 2, \ldots\) is of quotient type. Then \(\Gamma_\lambda\) is properly discontinuous and the holonomy group of the Lorentzian manifold \((M := \tilde{M}/\Gamma_\lambda, g^{f/u^2,h})\) at \(\pi(v,u,p)\) is given as

\[ L\Gamma_\lambda \cdot (\text{Hol}_p(N) \times \mathbb{R}^n) , \]

where \(L\Gamma_\lambda\) is the group that is generated in \(O(1,n+1)\) by the linear maps

\[ \begin{pmatrix} e^{\lambda_i} & 0 & 0 \\ 0 & \phi(\gamma_i) & 0 \\ 0 & 0 & e^{\lambda_i} \end{pmatrix} \in \mathbb{R}^+ \times \phi(\Gamma) . \]

In particular, \((M, g^{f,h})\) is time-orientable, admits a parallel null line bundle, but, for appropriate choices of \(\lambda\), admits no parallel vector field.
Proof. Since $\Gamma^\infty_m$ and $\Gamma^\infty$ are of quotient type, by Lemma 2 they are properly discontinuous as well. By the constructions of the Lorentzian metrics, they also act isometrically. Then the formula for the holonomy group of the Lorentzian quotient follows from formula (14) in Theorem 3. \hfill $\square$

Remark 4. Obviously, the constructions presented in this section always give non-compact examples. But in simple situations they can be modified in order to obtain compact Lorentzian manifolds by replacing $\Omega$ by a torus $S^1 \times S^1$ with a suitable metric. Start with a compact Riemannian manifold $(N, h)$ with holonomy $G$ and a function $f$ on $S^1 \times S^1 \times N$, even in the first two entries and satisfying condition (10). Then the Lorentzian metric on $M = S^1 \times S^1 \times N$ given by

$$g^{f,h} = 2d\phi d\theta + f d\theta^2 + h,$$

where $\phi$ and $\theta$ are standard angle coordinates on $S^1 \times S^1$, has holonomy $G \times \mathbb{R}^n$ or $(\mathbb{R}^+ \times G) \times \mathbb{R}^n$. Then we can consider the same involution on $M$ as in Proposition 4, in coordinates

$$(\phi, \theta, p) \mapsto (\phi + \pi, \theta + \pi, p),$$

to obtain compact Lorentzian manifolds with holonomy $(\mathbb{Z}_2 \times G) \times \mathbb{R}^n$ or $(\mathbb{R}^+ \times G) \times \mathbb{R}^n$.

This can be generalised to the situation when we have a cocompact properly discontinuous group $\Gamma^N$ of isometries of $(N, h)$ being generated by $\gamma_1, \gamma_2, \ldots$. We choose the function $f$ to be $\Gamma^N$-invariant and independent of $\theta$ and $\phi$, fix natural numbers $m = (m_1, m_2, \ldots)$ and consider the group $\Gamma^\infty_m$ of isometries of $(M, g^{f,h})$ which is generated by

$$(\phi, \theta, p) \mapsto (\phi + m_1 \pi, \theta + m_2 \pi, \gamma_i(p)),$$

for $i = 1, 2, \ldots$. Then $\Gamma^\infty_m$ is of quotient type and the holonomy of $M/\Gamma^\infty_m$ is contained in the group $(\mathbb{Z}_2 \times \text{Hol}(N/\Gamma^N)) \times \mathbb{R}^n$ but might have a coupling between the $\mathbb{Z}_2$ and the $\text{Hol}(N/\Gamma^N)$ part.

6. Examples with disconnected holonomy

In the following we will consider quotients of certain Lorentzian manifolds by a discrete group of isometries. These examples will illustrate some of the characteristic features of the holonomy groups that can be obtained by the construction given in Theorem 3.

6.1. Flat manifolds. We will start off with the flat case and give an example of a flat Lorentzian manifold with indecomposable, non irreducible full holonomy (see Remark 1). Let $\mathbb{R}^{1,n+1}$ be the Minkowski space and $E(1, n + 1) = O(1, n + 1) \times \mathbb{R}^{1,n+1}$ its isometry group. Any isometry $\gamma$ has the form $\gamma(x) = A_\gamma x + v_\gamma$, where $A_\gamma \in O(1, n + 1)$ and $v_\gamma \in \mathbb{R}^{n+2}$. For a discrete subgroup $\Gamma \subset E(1, n + 1)$ we denote by

$$L_\Gamma := \{A_\gamma \mid \gamma \in \Gamma \} \subset O(1, n + 1)$$

its linear part. Then, by Proposition 3, the full holonomy group of a flat space-time $\mathbb{R}^{1,n+1}/\Gamma$ is given by

$$\text{Hol}(\mathbb{R}^{1,n+1}/\Gamma) = L_\Gamma.$$ 

In many cases, the holonomy group of $\mathbb{R}^{1,n+1}/\Gamma$ stays trivial or acts decomposable. For example, if $\mathbb{R}^{1,n+1}/\Gamma$ is a complete homogeneous flat Lorentzian space, then $\Gamma$ is a group of translations, hence its holonomy group is trivial (see [33]). The same is true for non-complete flat homogeneous Lorentzian spaces (see [13] for the result). Looking at the affine classification of compact 3-dimensional flat space-times which are proper in the sense that their holonomy is contained in $SO^0(1,2)$ in [33, Sec. 3.6], one finds all with trivial, with decomposable as well as with indecomposable holonomy group. Typically, an indecomposable holonomy group appears, if $\Gamma$ contains an element $\gamma$, such that $A_\gamma$ has a lightlike eigenvector. We show this with a 4-dimensional example.
For a fixed $\theta \in \mathbb{R}$ we consider the matrix in $A_{\theta} \in \text{SO}(1,3)$:

$$A_{\theta} := \begin{pmatrix} 1 & -\cos \theta & \sin \theta & -\frac{1}{2} \\ 0 & \cos \theta & -\sin \theta & 1 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we used the basis $(\ell,e_1,e_2,e^*)$ as in the first section. Then, by $k \mapsto (A_{\theta})^k$ we obtain an immersion of the finite group $\mathbb{Z}_m$ into $O(1,3)$ if $\theta$ is a rational multiple of $\pi$, and an immersion of the group $\mathbb{Z}$ into $O(1,3)$ if $\theta$ is an irrational multiple of $\pi$. If we denote the image of this immersion by $H$, the connected component of $H$ is given by the identity and thus acts decomposable in a trivial way, but the group $H$ does not admit a non-degenerate invariant subspace, and hence it acts indecomposable. The group $H$ can be easily realized as the holonomy group of a flat space-time. For that we consider coordinates $(v,u,x,y)$ on $\mathbb{R}^4$ and the flat Minkowski metric

$$g_0 = 2dvdv + dxdx + dydy.$$

For $\theta \in \mathbb{R}$, let $\Gamma \subset E(1,3)$ be the discrete group of isometries generated by $\phi_{\theta}(v,u,x,y) = A_{\theta}(v,u,x,y) + (0,0,1,0)$.

$\Gamma$ acts freely and properly. Hence, the quotient $\mathbb{R}^{1,3}/\Gamma$ is a flat Lorentzian manifold with trivial restricted holonomy and full holonomy $H$.

6.2. pp-waves. The next examples will be quotients of Lorentzian manifolds which are usually referred to as pp-waves (see Remark 3). We will define them as follows: Let $\tilde{M}$ be an open set in $\mathbb{R}^{n+2}$ on which we fix global coordinates as $(v,u,x^1,\ldots,x^n)$. Let $f$ be a smooth function on $\tilde{M}$ that does not depend on the $v$ coordinate, i.e. $\partial_v f = 0$. A pp-wave metric on $\tilde{M}$ is a Lorentzian metric $g^f$ which, in these coordinates, is given as

$$(16) \quad g^f = 2dvdv + 2fdudu + \sum_{i=1}^{n} (dx^i)^2.$$

The vector fields

$$(17) \quad \partial_v, \partial_i, \partial_u - f\partial_v$$

form a global frame in which the metric $g^f$ is given as

$$g^f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here we used the obvious notation $\partial_v := \frac{\partial}{\partial v}$, $\partial_u := \frac{\partial}{\partial u}$ and $\partial_i := \frac{\partial}{\partial x^i}$, $i = 1,\ldots,n$. Here and in the following, all matrices are written with respect to the basis of the tangent spaces given by these vector fields at the corresponding points.

**Proposition 5.** The holonomy group of an $(n+2)$-dimensional pp-wave is abelian and if the matrix $(\partial_i \partial_j f)$ is non-degenerate at a point, the holonomy group is given by

$$\mathbb{R}^n \simeq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ 0 & 1 & -y \end{pmatrix} \right| y \in \mathbb{R}^n \right\}.$$

Conversely, any Lorentzian manifold with this holonomy group is locally a pp-wave.

**Proof.** pp-waves are a special case of the manifolds considered in Proposition 4. Hence the first statement follows from the proof of Proposition 4. For the converse statement see [21]. ⊓⊔
The formula (12) in the proof of Proposition 4 shows in addition that for a curve $\delta(t) = (v(t), u(t), \gamma(t))$ in a pp-wave, the parallel transport along $\delta$ is given by matrices of the form

$$
P_\delta = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.
$$

6.3. Lorentzian symmetric spaces. As a first class of manifolds that are covered by a pp-wave, we consider Lorentzian symmetric spaces. Let $(M, g)$ be an indecomposable Lorentzian symmetric space of dimension $n + 2 \geq 3$. Then its transvection group $G(M)$ is either solvable or semisimple [11]. In the latter case, $(M, g)$ is a space of constant sectional curvature $\kappa \neq 0$, hence its holonomy group acts irreducibly. The case of solvable transvection group was described by Cahen and Wallach in [11, 10], see also [28]. The simply-connected models are given by the following special pp-waves: Let $\Delta = (\lambda_1, \ldots, \lambda_n)$ be an $n$-tuple of real numbers $\lambda_j \in \mathbb{R}\setminus\{0\}$. Then the pp-waves $M_\Delta := (\mathbb{R}^{n+2}, g_\Delta)$, where

$$
g_\Delta := 2dvdu + \sum_{j=1}^{n} \lambda_j(x^j)^2 du^2 + \sum_{j=1}^{n} (dx^j)^2,
$$

are symmetric. If $\Delta = (\lambda_{\pi(1)}, \ldots, \lambda_{\pi(n)})$ is a permutation of $\Delta$ and $c > 0$, then $M_\Delta$ is isometric to $M_{\Delta c}$. By Proposition 5 the holonomy group of $M_\Delta$ is abelian and isomorphic to $\mathbb{R}^n$.

Any indecomposable solvable Lorentzian symmetric space $(M, g)$ of dimension $n + 2 \geq 3$ is isometric to a quotient $M_\Delta / \Gamma$, where $\Delta \in (\mathbb{R}\setminus\{0\})^n$ and $\Gamma$ is a discrete subgroup of the centralizer $Z_{I(M_\Delta)}(G(M_\Delta))$ of the transvection group $G(M_\Delta)$ in the isometry group $I(M_\Delta)$ of $M_\Delta$. For the centralizer $Z_\Delta := Z_{I(M_\Delta)}(G(M_\Delta))$ the following is known (see [9]):

1) If there is a positive $\lambda_i$ or if there are two numbers $\lambda_i, \lambda_j$ such that $\frac{\lambda_i}{\lambda_j} \notin \{q^2 \mid q \in \mathbb{Q}\}$, then $Z_\Delta = \{t_\alpha \mid \alpha \in \mathbb{R}\}$, where $t_\alpha$ is the translation $t_\alpha(v, x, u) = (v + \alpha, u, x)$.

2) If $\lambda_i = -k_i^2$ and $\frac{k_i}{k_j} \in \mathbb{Q}$ for all $i, j \in \{1, \ldots, n\}$, then $Z_\Delta$ is generated by $\{t_\alpha \mid \alpha \in \mathbb{R}\}$ and the isometry

$$
\varphi_{\beta}(v, u, x) = (v, u + \beta \pi, (-1)^{\beta k_1} x^1, \ldots, (-1)^{\beta k_n} x^n),
$$

where $\beta := \min \{r \in \mathbb{R}^+ \mid r k_i \in \mathbb{Z} \text{ for all } i \in \{1, \ldots, n\} \}$.

Clearly, any discrete subgroup in $Z_\Delta$ is generated by a translation $t_\alpha$ in the $v$-component and a power of $\varphi_{\beta}$. Hence, denote by $\Gamma_{m, \alpha}$ the discrete group of isometries generated by a translation $t_\alpha$ and by the isometry $\varphi_{\beta}^m$ with $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}_0$ ($m = 0$ in the first case), $\Gamma_{m, \alpha}$ is obviously properly discontinuous. Consequently, any indecomposable solvable Lorentzian symmetric space $M$ is isometric to $M_\Delta / \Gamma_{m, \alpha}$ for an appropriate $m$ and $\alpha$.

**Proposition 6.** Let $M_\Delta / \Gamma_{m, \alpha}$ be an indecomposable solvable Lorentzian symmetric space. Then the full holonomy group is given by

$$
\text{Hol}(M_\Delta / \Gamma_{m, \alpha}) \simeq \begin{cases} 
\mathbb{R}^n & \text{if } m \text{ even,} \\
\mathbb{Z}_2 \times \mathbb{R}^n = \{\text{Id}, S_m\} \times \mathbb{R}^n & \text{if } m \text{ odd,}
\end{cases}
$$

where in case of $m$ odd $S_m$ is the reflection $S_m = \text{diag}((-1)^{m k_1}, \ldots, (-1)^{m k_n}) \in \text{O}(n)$. In particular, any indecomposable solvable Lorentzian symmetric space is time-orientable and admits a global parallel lightlike vector field.

**Proof.** In order to determine the holonomy group of $M$ we apply formula (14) of Theorem 3. As the parallel transport in $M_\Delta$ is of the form (18), we only have to compute the differentials of the isometries in $\Gamma_{m, \alpha}$. But these differentials are clearly given by the identity or by

$$
d\varphi_{\beta}^m = \begin{pmatrix} 1 & * & * \\ 0 & S_m & * \\ 0 & 0 & 1 \end{pmatrix}.
$$
This proves the Proposition. □

6.4. Quotients of pp-waves. Now, by applying Theorem 3 and Corollary 1 we construct two types of quotients of pp-waves for which the full holonomy group has also elements in the dilatation part $\mathbb{R}^*$ of $\text{Stab}_{O(1,n+1)}(L)$. We fix linear independent vectors $(a_1, \ldots, a_p)$ in $\mathbb{R}^n$ and consider the discrete group of translations of $\mathbb{R}^n$,

$$\Gamma_{(a_1, \ldots, a_p)} := \{ t_a \mid a = \sum_{i=1}^p m_i a_i, \; m_i \in \mathbb{Z} \},$$

which is generated by the translations by the $a_i$. Furthermore, we fix a $\Gamma_{(a_1, \ldots, a_p)}$-invariant function $f \in C^\infty(\mathbb{R}^n)$ satisfying the property (10).

In the first example we consider the metric

$$\tilde{g}^f = 2dvdu + 2f(x)du^2 + \sum_{i=1}^n(dx^i)^2$$

on $\tilde{M} = \mathbb{R}^2 \times \mathbb{R}^n = \mathbb{R}^{n+2}$. Then, for $t_a \in \Gamma$ with $a = \sum_{i=1}^p m_i a_i$ we define a isometry $\varphi_a$ of $(\tilde{M}, g^f)$ by

$$\varphi_a(v, u, x) := ((-1)^i, m_i v, (-1)^j, m_j u, x + a).$$

and apply Theorem 3 to the properly discontinuous group

$$\Gamma := \{ \varphi_a \mid t_a \in \Gamma_{(a_1, \ldots, a_p)} \} \simeq \mathbb{Z}^p.$$

Then, by Corollary 1, the full holonomy group of the Lorentzian manifold $M = \mathbb{R}^{n+2}/\Gamma$ with the Lorentzian metric induced by $\tilde{g}^f$ is given by

$$\left\{ \begin{array}{ccc} \pm 1 & y^i & * \\ 0 & I_n & * \\ 0 & 0 & \pm 1 \end{array} \right\} \mid y \in \mathbb{R}^n \right\} \simeq \mathbb{Z}_2 \ltimes \mathbb{R}^n,$$

whereas the restricted holonomy is given by $\mathbb{R}^n$. In particular, $M$ is not time-oriented and admits a parallel lightlike line, but no parallel lightlike vector field.

In order to construct an example which is time-orientable we involve Lorentz boosts on $\mathbb{R}^{1,1}$ in this construction and consider now the pp-wave metric defined by $\frac{1}{u^2}f(x)$ on the half-space $\tilde{M} := \Omega \times \mathbb{R}^n$ with $\Omega = \{ (v, u) \in \mathbb{R}^2 \mid u > 0 \} \subset \mathbb{R}^{n+2}$, i.e.,

$$\tilde{g}^{f/u^2} = 2dvdu + \frac{2}{u^2}f(x)du^2 + (dx^1)^2 + \ldots + (dx^n)^2.$$

Let $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$ and define for $t_a \in \Gamma$, $a = \sum_i m_i a_i$, the isometries

$$\psi_a(v, u, x) := (e^{\sum_i m_i \lambda_i v}, e^{-\sum_i m_i \lambda_i} u, x + a).$$

on $(\tilde{M}, \tilde{g}^{f/u^2})$. The group $\Gamma := \{ \psi_a \mid t_a \in \Gamma_{(a_1, \ldots, a_p)} \} \simeq \mathbb{Z}^p$ is of quotient type, hence properly discontinuous, and $\tilde{g}^{f/u^2}$ induces a Lorentzian metric on the quotient $M := \tilde{M}/\Gamma$. Since $\lambda_1, \ldots, \lambda_p$ are independent over $\mathbb{Q}$, the epimorphism $F : \pi_1(M) \twoheadrightarrow \text{Hol}_p(\mathcal{L}, \nabla^\mathcal{L})$ is injective. Hence, the full holonomy group of the Lorentzian manifold $M = \tilde{M}/\Gamma$ is given by

$$\left\{ \begin{array}{ccc} e^{\sum_j k_j \lambda_j} & y^i & * \\ 0 & 1 & * \\ 0 & 0 & e^{-\sum_j k_j \lambda_j} \end{array} \right\} \mid y \in \mathbb{R}^n, k_j \in \mathbb{Z} \right\} \simeq \mathbb{Z}^p \ltimes \mathbb{R}^n,$$

whereas the restricted holonomy group is given by $\mathbb{R}^n$. In particular, the manifold $M$ is time-orientable admitting a parallel null line, but no parallel lightlike vector field.
6.5. Coupled holonomy. The aim of this section is to modify the examples of the previous section in a way so that the holonomy group features a coupling between the linear part in $\text{O}(n)$ and the scaling component in $\mathbb{R}^*$ in the sense that some group elements act simultaneously on $\mathbb{R}$ and on $\mathbb{R}^n$. The first examples are direct consequences of Corollary 1.

First we produce 4-dimensional examples. The first class of examples is based on 2-dimensional flat Riemannian space-forms which are either diffeomorphic to a Möbius strip or to a Kleinian bottle (for both cases, see [33, Chap. 2.2.5]). In case of the Möbius strip, this manifold is given by $\mathbb{R}^2/\Gamma^1$, where $\Gamma^1$ is infinite cyclic generated by $\gamma = (B, t_b) \in \text{O}(2) \times \mathbb{R}^2$, with a reflection $B$ fixing the non-zero vector $b \in \mathbb{R}^2$. In case of the Kleinian bottle this manifold is given by $\mathbb{R}^2/\Gamma^2$, where $\Gamma^2$ is generated by $\gamma_1 = (B, t_b)$ and $\gamma_2 = (I, t_a)$ in $\text{O}(2) \times \mathbb{R}^2$, and $B$ is again a reflection with $B(b) = b \neq 0$ and $B(a) = -a \neq 0$. For the construction of pp-waves, we fix two $\Gamma^i$-invariant functions $f_i$ on $\mathbb{R}^2$ with a non-degenerate Hessian.

Consider the pp-wave metrics
\[ \tilde{g}^{f_i} := 2dvdu + 2f_i(x^1, x^2)du^2 + (dx^1)^2 + (dx^2)^2 \]
on $\tilde{M} := \mathbb{R}^4$ and
\[ \tilde{g}^{f_i/u^2} := 2dvdu + \frac{2}{u^2}f_i(x^1, x^2)du^2 + (dx^1)^2 + (dx^2)^2 \]
on $\tilde{M} := \{(v, u, x^1, x^2) \mid u > 0\}$, both with holonomy $\mathbb{R}^2$. We fix numbers $m, m_1, m_2 \in \mathbb{N}$ and $\lambda, \lambda_1, \lambda_2$ in $\mathbb{R}$ and define the groups of isometries $\Gamma^1_m, \Gamma^2_{m_1, m_2}$ and $\Gamma^1_{\lambda_1, \lambda_2}$ as in Corollary 1, with respect to these numbers and the fixed generators $\gamma$ of $\Gamma^1$ and $\gamma_1$ and $\gamma_2$ of $\Gamma^2$. These groups are of quotient type. Then, by Corollary 1 we obtain the following holonomy groups for the quotients
\[ \text{Hol}(\tilde{M}/\Gamma^1_m, g^{f_i}) = \left\{ \begin{pmatrix} (-1)^{km} & w^t & * \\ 0 & B^k & * \\ 0 & 0 & (-1)^{km} \end{pmatrix} \mid w \in \mathbb{R}^2, k \in \mathbb{Z} \right\}, \]
\[ \text{Hol}(\tilde{M}/\Gamma^2_{m_1, m_2}, g^{f_i}) = \left\{ \begin{pmatrix} (-1)^{k_1m_1+k_2m_2} & w^t & * \\ 0 & B^{k_2} & * \\ 0 & 0 & (-1)^{k_1m_1+k_2m_2} \end{pmatrix} \mid w \in \mathbb{R}^2, k_1, k_2 \in \mathbb{Z} \right\}, \]
\[ \text{Hol}(\tilde{M}/\Gamma^1_{\lambda}, g^{f_i/u^2}) = \left\{ \begin{pmatrix} e^{k\lambda} & w^t & * \\ 0 & B^k & * \\ 0 & 0 & e^{-k\lambda} \end{pmatrix} \mid w \in \mathbb{R}^2, k \in \mathbb{Z} \right\}, \]
\[ \text{Hol}(\tilde{M}/\Gamma^2_{\lambda_1, \lambda_2}, g^{f_i/u^2}) = \left\{ \begin{pmatrix} e^{k_1\lambda_1+k_2\lambda_2} & w^t & * \\ 0 & B^{k_2} & * \\ 0 & 0 & e^{-k_1\lambda_1-k_2\lambda_2} \end{pmatrix} \mid w \in \mathbb{R}^2, k_1, k_2 \in \mathbb{Z} \right\}. \]

Thus, by different choices of the numbers $m, m_1, m_2$, and $\lambda, \lambda_1, \lambda_2$ we can realize the groups $\mathbb{Z}_2 \times \mathbb{R}^2$, $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \times \mathbb{R}^2$, $\mathbb{Z} \times \mathbb{R}^2$, and $(\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{R}^2$, where the integer parts are immersed into $(\mathbb{R}^* \times \text{O}(2))$ coupling both factors, as holonomy groups of 4-dimensional not time-orientable Lorentzian manifolds.

Now we start the construction with a group of isometries of $\mathbb{R}^2$ that is not properly discontinuous. Fix an angle $\theta$ and consider the group $\Gamma_\theta$ generated by the rotation $D_\theta$ by $\theta$. This group is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_p$ depending on whether $\theta$ is a rational multiple of $\pi$ or not. We fix a rotational invariant function $f$ on $\mathbb{R}^2$ with property (10) and consider again the pp-wave metric
\[ g^{f/u^2} = 2dvdu + \frac{2}{u^2}f + (dx^1)^2 + (dx^2)^2 \]
on $\tilde{M} = \Omega \times \mathbb{R}^2$ with $\Omega = \{(v, u) \in \mathbb{R}^2 \mid u > 0\}$. Then, for $\lambda, c \in \mathbb{R}$, the group generated by
\[ \varphi(v, u, x) := (e^{\lambda}v + c, e^{-\lambda}u, D_\theta(x)) \]
acts on $\tilde{M}$ by isometries of $g^{l/u^2}$ and is properly discontinuous. Indeed, since we restrict the action to \( \{ u > 0 \} \) the group of diffeomorphisms on $\Omega$ generated by

\[(v, u) \mapsto (e^{\lambda}v + c, e^{-\lambda}u)\]

is of quotient type and therefore properly discontinuous. Using this, we obtain that $\Gamma$ is of quotient type as well and Lemma 2 applies again. Hence, $\tilde{M}/\Gamma$ is a Lorentzian manifold with holonomy $\mathbb{Z} \times \mathbb{R}^n$ if $\lambda \neq 0$, i.e.,

\[
Hol(\tilde{M}/\Gamma, g^{l/u^2}) = \left\{ \begin{pmatrix} e^{k\lambda} & w' & \ast \\ 0 & D_\theta^2 & \ast \\ 0 & 0 & e^{-k\lambda} \end{pmatrix} \middle| w \in \mathbb{R}^2, k \in \mathbb{Z} \right\}.
\]

In a similar way we can construct non-time-orientable Lorentzian manifolds by taking $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\varphi(v, u, x) := (-v, -u, D_\theta)$. Here the holonomy $\mathbb{Z} \times \mathbb{R}^n$ or $\mathbb{Z}_p \times \mathbb{R}^n$ acts as $\mathbb{Z}_2$ in the $\mathbb{R}^*$ part.

**Remark 5.** Note that, if in the previous example we assume $\theta \neq \pi$, the orthogonal part $\text{pr}_{O(2)}(\text{Hol}(\tilde{M}/\Gamma, g^{l/u^2})) \simeq \Gamma_\theta$ cannot be realised as holonomy group of a complete Riemannian manifold. If there was a complete 2-dimensional Riemannian manifold $M^2$ with holonomy $\Gamma_\theta$, then $M^2$ would be a flat Euclidean space-form. But the holonomy groups of flat 2-dimensional space-forms are known to be trivial (for the plane, the cylinder and the torus) or generated by a reflection (for the Möbius strip and the Kleinian bottle) (see [33], Chap. 2.2.5). We do not know if this is also true if we drop the assumption of completeness.

This is in contrast to the situation when considering only the connected component of the holonomy group. Here it was proven in [23] that $\text{pr}_{SO(n)}(\text{Hol}^0)$ is always the connected holonomy of a Riemannian manifold, and, by results in Riemannian holonomy theory (an overview can be found in [19]), can be realised as holonomy of a *complete* Riemannian manifold.

In the same way, 5-dimensional examples can be constructed starting with flat 3-dimensional Riemannian space-forms. These are classified (see [33, Chap. 3.3.5]) and one can easily single out those which are generated by discrete groups $\Gamma$ with non-trivial linear part $L_\Gamma$. Then, similar constructions as above give various examples where the $\mathbb{R}^n$ and the $O(3)$ part of the holonomy is coupled. We show this only for the 3-manifolds of type $S^3 \mathbb{Z}$ in the list of 3-dimensional flat space-forms. Let $\theta \in (0, 2\pi)$ and let us consider the discrete group $\Gamma \subset \text{E}(3)$ generated by $(A, a_\theta) \in O(3) \times \mathbb{R}^3$, where $a \in \mathbb{R}^3$ is an eigenvector of $A$ and $A|_{a_\theta} = D_\theta$ is the rotation by the angle $\theta$. We fix a function $f \in C^\infty(\mathbb{R}^3)$ which is $\Gamma$-invariant and has a non-degenerate Hessian in a certain point and consider the pp-wave metric $g^l$ and $g^{l/u^2}$ on the appropriate $\tilde{M}$, where we choose the coordinates in such a way, that $a$ is a positive multiple of $\partial_1$. Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ be fixed numbers and denote by $\Gamma_m$ and $\Gamma_\lambda$ the properly discontinuous group of isometries on $\tilde{M}$ as in Corollary 1. Then the holonomy group of the quotients is given by

\[
\text{Hol}(\tilde{M}/\Gamma_m, g^l) = \left\{ \begin{pmatrix} (-1)^km & s & w' & \ast \\ 0 & 1 & 0 & \ast \\ 0 & 0 & D_\theta^k & \ast \\ 0 & 0 & 0 & (-1)^km \end{pmatrix} \middle| k \in \mathbb{Z}, s \in \mathbb{R}, w \in \mathbb{R}^2 \right\},
\]

\[
\text{Hol}(\tilde{M}/\Gamma_\lambda, g^{l/u^2}) = \left\{ \begin{pmatrix} e^{k\lambda} & s & w' & \ast \\ 0 & 1 & 0 & \ast \\ 0 & 0 & D_\theta^k & \ast \\ 0 & 0 & 0 & e^{-k\lambda} \end{pmatrix} \middle| k \in \mathbb{Z}, s \in \mathbb{R}, w \in \mathbb{R}^2 \right\}.
\]

Again, by appropriate choice of $\lambda$ and $\theta$ we can realise the groups $\mathbb{Z} \times \mathbb{R}^3$ and $\mathbb{Z}_{2\theta} \times \mathbb{R}^3$ as holonomy group of a 5-dimensional Lorentzian manifold, where $\mathbb{Z}$ and $\mathbb{Z}_{2\theta}$ are immersed into $\mathbb{R}^* \times O(3)$ coupling both factors.
6.6. An infinitely generated holonomy group. Again by applying Corollary 1 we will give an example of a 4-dimensional Lorentzian manifold with infinitely generated holonomy group arising as a quotient of a pp-wave. Consider $\mathbb{R}^2$ with the flat metric $h_0 = (dx^1)^2 + (dx^2)^2$ and restrict $h_0$ to the manifold $N := \mathbb{R}^2 \setminus \mathbb{Z}^2$. The fundamental group $\Gamma := \pi_1 N$ of $N$ is a free group infinitely generated by the loops going around the holes, but the holonomy group of $(N, h_0)$ is trivial. Fix a function $f$ on $N$ satisfying (10). The universal cover of $N$ is $\mathbb{R}^2$ and $N$ is the quotient of $\mathbb{R}^2$ by $\pi_1 N$. It is equipped with the $\Gamma$-invariant pull-back $h$ of the flat metric $h_0$ on $N$. Also the function $f$ pulls back to a $\Gamma$-invariant function on $\mathbb{R}^2$. We apply Corollary 1 to $(\mathbb{R}^2, h)$ and $\Gamma$.

Set $\Omega := \{(v, u) \in \mathbb{R}^2 \mid u > 0\}$ and define a Lorentzian metric on $M := \Omega \times \mathbb{R}^2$ by the usual procedure

$$
\tilde{g}_{f/u^2, h} = 2dvd\bar{u} + \frac{2}{u^2}fdu + h.
$$

Now we fix generators $(\gamma_1, \gamma_2, \ldots)$ of $\Gamma$ and real numbers $\lambda_\infty := (\lambda_1, \lambda_2, \ldots)$ which are linearly independent over $\mathbb{Q}$ and define the following isometries of $(M, g_{f/u^2, h})$:

$$
\varphi_i(v, u, x) = (e^{\lambda_i}v, e^{-\lambda_i}u, \gamma_i(x)).
$$

Let $\Gamma_\lambda$ be the group of isometries of $(\tilde{M}, \tilde{g}_{f/u^2, h})$ that is generated by the $\varphi_i$ for $i = 1, 2, \ldots$. Since the fundamental group $\Gamma = \pi_1 N$ acts as group of deck transformations of the universal covering $\pi : \mathbb{R}^2 \rightarrow N$, it is properly discontinuous. Moreover, since $\Gamma$ is a free group, the sections $(\Gamma_{\lambda})_\sigma \subset (\Gamma_\lambda)_1$ consist only of one element for all $\sigma \in \Gamma$. Hence, $\Gamma_{\lambda}$ is of quotient type and by Lemma 2 properly discontinuous, and the quotient $M = \tilde{M}/\Gamma_{\lambda}$ is a smooth Lorentzian manifold with the induced Lorentzian metric $g_{f/u^2, h}$. Then Corollary 1 shows that the holonomy group of $(M, g_{f/u^2, h})$ is generated by the following matrices

$$
\begin{pmatrix}
 e^{\lambda_i} & w^t & * \\
 0 & 1_2 & * \\
 0 & 0 & e^{-\lambda_i}
\end{pmatrix} \in O(1, 3),
$$

with $w \in \mathbb{R}^2$ and $\lambda_i$ one of the fixed real numbers. Since these were chosen linearly independent over $\mathbb{Q}$, the quotient $\text{Hol}_p(M)/\text{Hol}(\tilde{M}) = \text{Hol}_p(M)/\text{Hol}^0(M)$ and also the holonomy group of the induced connection on the null line bundle $\mathcal{L}$ is infinitely generated.

6.7. Examples with curved screen bundle. Using constructions in Riemannian geometry one obtains Lorentzian manifolds with parallel null line, curved screen bundle and disconnected holonomy with a coupling between the $\mathbb{R}^n$ and the $O(n)$ part.

The first construction is based on Riemannian manifolds that go back to Hitchin [18] and McInnes [24, 25] and which are described in detail in [27]. Let $N$ be the complete intersection of $m + 1$ quadrics in $\mathbb{C}P^{2p+1}$, defined as the common zero set of quadratic polynomials with strictly positive real coefficients, and endowed with the Riemannian metric $h$ from $\mathbb{C}P^{2p+1}$. Then $(N, h)$ is a simply connected Kähler manifold. On $N$ we have an involution defined by complex conjugation,

$$
\sigma([z_0, \ldots, z_{2p+1}]) := ([\overline{z}_0, \ldots, \overline{z}_{2p+1}]),
$$

which is an anti-holomorphic isometry of $(N, h)$. Then, if $\Gamma$ denotes the group of isometries generated by $\sigma$, in [27, Corollary 9] it is proven that $(N/\Gamma, h)$ is a compact $2p$-dimensional Riemannian manifold with holonomy $\mathbb{Z}_2 \ltimes SU(p)$. In addition, $(N/\Gamma, h)$ is Ricci flat and admits a parallel spinor field. Note that $\Phi(\sigma)$ is represented by the conjugation on the tangent space with respect to the complex structure given by the Kähler structure. We denote this involution by $\sigma_z$.

If we fix a $\Gamma$-invariant function $f$ on $N$ and, for $n \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$, perform both constructions as in Corollary 1, we obtain $(\mathbb{Z}_2 \ltimes SU(p)) \ltimes \mathbb{R}^{2p}$ and $(\mathbb{Z} \ltimes SU(p)) \ltimes \mathbb{R}^{2p}$ as holonomy groups, with
\( \mathbb{Z}_2 \) and \( \mathbb{Z} \) acting simultaneously on \( \mathbb{R} \) and SU\((p)\), i.e.,

\[
\text{Hol}(\mathcal{M}/T_m, g^{f,h}) = \left\{ \begin{pmatrix} (-1)^{km} & 0 & 0 \\ 0 & \sigma^k & 0 \\ 0 & 0 & (-1)^{km} \end{pmatrix} \middle| \ k \in \mathbb{Z}, \right\} \cdot (\text{SU}(p) \ltimes \mathbb{R}^{2p}),
\]

\[
\text{Hol}(\tilde{\mathcal{M}}/T_{\lambda}, g^{f/u^2,h}) = \left\{ \begin{pmatrix} e^{k\lambda} & 0 & 0 \\ 0 & \sigma^k & 0 \\ 0 & 0 & e^{-k\lambda} \end{pmatrix} \middle| \ k \in \mathbb{Z}, \right\} \cdot (\text{SU}(p) \ltimes \mathbb{R}^{2p}).
\]

Note that both manifolds no longer admit parallel spinors because they do not admit a parallel vector field. We will consider the full holonomy groups of Lorentzian manifolds with parallel spinors in the next section.

In [32] Wilking constructed a remarkable example of a compact Riemannian manifold with non-compact holonomy group. The manifold is given as a quotient of a 5-dimensional solvable Lie group \( S := \mathbb{R}^4 \ltimes \mathbb{R} \) with a left-invariant Riemannian metric by a cocompact subgroup \( \Gamma := \Lambda \ltimes \mathbb{Z} \) where \( \Lambda \) is a certain lattice of \( \mathbb{R}^4 \). Then \( S/\Gamma \) has restricted holonomy SO\((3) \subset \text{SO}(5)\), acting trivially on a 2-dimensional subspace of \( T_p S \), and SO\((3) \ltimes \mathbb{Z} \) as full holonomy group. Again, by the constructions in Corollary 1 we can produce 7-Lorentzian manifolds with parallel null line and full holonomy \((\mathbb{Z} \ltimes \text{SO}(3)) \ltimes \mathbb{R}^5 \) where \( \mathbb{Z} \) acts on \( \mathbb{R} \) and SO\((3)\) simultaneously, either as \( \mathbb{Z}_2 \) or as \( \mathbb{Z} \).

In the same way as described in Remark 4 we can even construct compact Lorentzian manifolds of the form \((S^1 \times S^1)/\mathbb{Z}_2 \times S/\Gamma \) with indecomposable holonomy \((\mathbb{Z}_2 \times \mathbb{Z} \ltimes \text{SO}(3)) \ltimes \mathbb{R}^5 \) or of the form \((S^1 \times S^1 \times S)/\Gamma_{m,n} \), where \( \Gamma_{m,n} \) is generated as in formula (15), and with holonomy contained in \((\mathbb{Z}_2 \times \mathbb{Z} \ltimes \text{SO}(3)) \ltimes \mathbb{R}^5 \), but possible with non-trivial \( \mathbb{Z}_2 \subset \mathbb{R}^5 \)-part which is now coupled to \( \mathbb{Z} \subset \text{O}(5) \).

### 7. Full Holonomy Groups of Lorentzian Manifolds with Parallel Null Spinor

A Lorentzian spin manifold \((M, g)\) of dimension \( n+2 > 2 \) is a time- and space-oriented Lorentzian manifold with a spin structure \( \bar{O}(M, g) \to O(M, g) \) that is a reduction with respect to the double cover \( \Lambda : \text{Spin}^0(1, n+1) \to \text{SO}^0(1, n+1) \), where \( O(M, g) \) is the bundle of time- and space-oriented frames over \( M \) orthonormal for \( g \). This allows us to write the tangent bundle as

\[
TM = O(M, g) \times_{\text{SO}^0(1, n+1)} \mathbb{R}^{1,n+1} = \bar{O}(M, g) \times_{\text{Spin}^0(1, n+1)} \mathbb{R}^{1,n+1},
\]

and defines the spinor bundle

\[
\Sigma = \bar{O}(M, g) \times_{\text{Spin}^0(1, n+1)} \Delta^{1,n+1},
\]

where \( \Delta^{1,n+1} \) is the spinor module. The spinor bundle is equipped with a metric of neutral signature, a Clifford multiplication

\[
\cdot : TM \times \Sigma \to \Sigma,
\]

and a covariant derivative \( \nabla^\Sigma \). All these structures are compatible with each other (for details see for example [3]).

Every section in the spinor bundle \( \varphi \in \Gamma(\Sigma) \) defines a causal vector field \( V_\varphi \in \Gamma(TM) \) via

\[
g(V_\varphi, Y) = -(Y \cdot \varphi, \varphi).
\]

Both \( \varphi \) and \( V_\varphi \) have the same zero set, and the spinor field is of zero length, a null spinor, if \( V_\varphi \) is of zero length, i.e. a null vector field. If the spinor bundle admits a section \( \varphi \) with \( \nabla^\Sigma \varphi = 0 \), for short, a parallel spinor, then \( V_\varphi \) is a parallel vector field as well.

Since a Lorentzian spin manifold is assumed to be time- and space-oriented, its full holonomy \( H := \text{Hol}_p(M, g) \) is contained in the connected component \( \text{SO}^0(1, n+1) \) of \( \text{SO}(1, n+1) \), but is not necessarily connected itself. Using Proposition 1 and results in [23] and [30] we can summarise what we know so far about the holonomy group in this situation:
Proposition 7. Let \((M, g)\) be a Lorentzian spin manifold with special holonomy. If \((M, g)\) admits a parallel spinor, then \((M, g)\) admits a parallel null vector field and its full holonomy group \(H\) is given as
\[
H = G \ltimes \mathbb{R}^n,
\]
with \(G \subset \text{SO}(n)\), and the connected component \(G^0\) of \(G\) is trivial or given as a direct product of some of the following groups
\[
\text{SU}(m), \text{Sp}(k), G_2, \text{Spin}(7).
\]

Proof. The parallel spinor on \((M, g)\) induces a parallel vector field, which, as \(H = \text{Hol}(M, g)\) acts indecomposably, has to be null. Hence, the full holonomy group fixes a null vector, \(H\) is a Riemannian holonomy group which is isomorphic to a subgroup in \(\text{Spin}(n)\) admitting a fixed spinor. Based on Berger’s list [7], these groups were determined by Wang [30]. Using Proposition 1 we obtain that \(H = G \ltimes \mathbb{R}^n\) with \(G \subset \text{SO}(n)\). \(\square\)

We will now generalize this result to the full holonomy group.

Proposition 8. Let \((M, g)\) be a Lorentzian manifold of dimension \((n + 2) > 2\) which is time- and space-orientable with indecomposable restricted holonomy group \(H^0 \subset (\mathbb{R}^+ \times \text{SO}(n)) \ltimes \mathbb{R}^n\). Then we have the following two implications:

1) If \((M, g)\) admits a spin structure with a parallel spinor field, then the full holonomy group \(H\) is given as \(H = G \ltimes \mathbb{R}^n\) with \(G \subset \text{SO}(n)\) and there exists a homomorphism \(\Phi : G \rightarrow \text{Spin}(n)\)
   \(\Phi = \text{Id}\) for \(\lambda : \text{Spin}(n) \rightarrow \text{SO}(n)\) the twofold cover, and
   there is a spinor \(w\) in the spinor module \(\Delta_n\) such that \(\Phi(\Delta_n)w = w\).

2) If the full holonomy group is \(H = G \ltimes \mathbb{R}^n\) with \(G \subset \text{SO}(n)\) and there is a homomorphism \(\Phi : G \rightarrow \text{Spin}(n)\) with \(\Phi = \text{Id}\) and \(\lambda : \text{Spin}(n) \rightarrow \text{SO}(n)\) the twofold cover, then \((M, g)\) has a spin structure with a parallel spinor.

Proof. Let \((M, g)\) be time- and space-oriented and \(H = \text{Hol}_H(M, g) \subset \text{SO}^0(1, n + 1)\) be its full holonomy group. The proof relies on the following three observations:

1) If \((M, g)\) admits a spin structure with a parallel spinor field, then there is a homomorphism \(\Psi : H \rightarrow \text{Spin}^0(1, n + 1)\) with \(\lambda \circ \Psi = \text{Id}_H\) and a spinor \(v \in \Delta_{1,n+1}\) such that \(\Psi(H)v = v\).

2) If there is a homomorphism \(\Psi : H \rightarrow \text{Spin}^0(1, n + 1)\) with \(\lambda \circ \Psi = \text{Id}_H\) and a spinor \(v \in \Delta_{1,n+1}\) with \(\Psi(H)v = v\), then \((M, g)\) has a spin structure with a parallel spinor on \((M, g)\).

3) If \(H = G \ltimes \mathbb{R}^n\), then there is a homomorphism \(\Psi : H \rightarrow \text{Spin}^0(1, n + 1)\) with \(\lambda \circ \phi = \text{Id}_H\) and a spinor \(v \in \Delta_{1,n+1}\) such that \(\Psi(H)v = v\) if and only if there is a homomorphims \(\Phi : G \rightarrow \text{Spin}(n)\) with \((1a)\) and \((1b)\).

The first observation was made by Wang in [31]. Indeed, if \((M, g)\) has a parallel spinor field, the holonomy group \(H\) of the spin connection fixes a spinor \(v \in \Delta_{1,n+1}\) and maps onto \(H\), hence it does not contain \(-1 \in \text{Spin}^0(1, n + 1)\). Therefore, \(\lambda|_H^{-1} : H \rightarrow \text{Spin}^0(1, n + 1)\) is an isomorphism and we can define \(\Psi := (\lambda|_H^{-1})^{-1} : H \rightarrow \text{Spin}^0(1, n + 1)\).

The second observation was made by Semmelmann and Moroianu in [27, Lemma 5] for Riemannian signature but their proof works in any signature. Indeed, if \(H\) is the holonomy bundle through a frame in \(O(M, g)\) and \(\Psi : H \rightarrow \text{Spin}^0(1, n + 1)\) is a homomorphism with \(\lambda \circ \Psi = \text{Id}\), we can define the spin structure by \(\text{O}(M, g) := H \times_H \text{Spin}^0(1, n + 1)\) which projects canonically onto \(O(M, g) = H \times \text{SO}(1, n + 1)\).
We have to proof the third observation. First assume that \( \Psi : H = G \times \mathbb{R}^n \rightarrow \text{Spin}^0(1, n + 1) \) is given. Since \( \Lambda \circ \Psi = \text{Id}_G \), the restriction of \( \Psi \) to \( G \) maps into \( \Lambda^{-1}(\text{SO}(n)) = \text{Spin}(n) \subset \text{Spin}^0(1, n + 1) \). Hence we can define

\[
\Phi := \Psi|_G : G \rightarrow \text{Spin}(n).
\]

Since \( \Lambda = \Lambda|_{\text{Spin}(n)} \), we also get that \( \Phi \circ \Lambda = \text{Id}_G \). Now, \( \mathbb{R}^n \subset H \) is a connected closed Abelian subgroup. When realising \( \text{Spin}^0(1, n + 1) \) in the Clifford algebra \( \mathcal{C}(1, n + 1) \), the image of \( \mathbb{R}^n \) under \( \Psi \) is given by

\[
\Psi(\mathbb{R}^n) = \{1 + \ell \cdot x \mid x \in \mathbb{R}^n\} \subset \text{Spin}^0(1, n + 1) \subset \mathcal{C}(1, n + 1),
\]

where \((\ell, e_1, \ldots, e_n, \ell^*)\) is a basis as in (4) and \( \mathbb{R}^n = \text{span}(e_1, \ldots, e_n) \). On the other hand, if \( \Phi : G \rightarrow \text{SO}(n) \) is given, we define

\[
\Psi : H = G \times \mathbb{R}^n \rightarrow \text{Spin}^0(1, n + 1)
\]

\[
g \cdot x \mapsto \Phi(g) \cdot (1 + \ell \cdot x).
\]

One can check that \( \Psi \circ \Lambda = \text{Id}_H \).

It remains to verify that there is fixed spinor \( v \in \Delta_{1,n+1} \) if, and only if, there is a fixed spinor \( w \in \Delta_n \). To this end we consider the Clifford algebra \( \mathcal{C}(1, 1) \) of the 2-dimensional space \( \mathbb{R} \ell \oplus \mathbb{R} \ell^* \) with the induced signature \((1, 1)\)-scalar product, fix a basis \((u_1, u_2)\) in \( \Delta_{1,1} \) satisfying \( \ell \cdot u_1 = \sqrt{2} u_2 \), \( \ell \cdot u_2 = 0 \), \( \ell^* \cdot u_1 = 0 \), \( \ell^* \cdot u_2 = -\sqrt{2} u_1 \) and assign to a spinor \( v \in \Delta_{1,n+1} \) two spinors \( v_1, v_2 \in \Delta_n \) by identifying

\[
\Delta_{1,n+1} \cong \Delta_n \otimes \Delta_{1,1}
\]

\[
v \mapsto v_1 \otimes u_1 + v_2 \otimes u_2.
\]

Then a computation shows that \((g \cdot a)(v) = v\) for all \( g \in \Psi(G) \) and \( a \in \Psi(\mathbb{R}^n) \) if, and only if,

\[
gv_2 - \sqrt{2}(g \cdot x)v_1 = v_2,
\]

\[
gv_1 = v_1
\]

for all \( g \in \Psi(G) \) and all \( x \in \mathbb{R}^n \). The first equation for \( g = 1 \) implies that \( x \cdot v_1 = 0 \) for all \( x \in \mathbb{R}^n \) and hence, \( v_1 = 0 \). Thus, \( v \in \Delta_{1,n+1} \) is fixed under \( \Psi(H) \) if, and only if, \( v = v_2 \otimes u_2 \) and \( v_2 \) is fixed under \( \Psi(G) = \Phi(G) \). This shows observation (7) and completes the proof.

\[
\text{Corollary 2. Let } H = G \ltimes \mathbb{R}^n \subset \text{SO}^0(1, n + 1) \text{ with } G \subset \text{SO}(n). \text{ Then the following vector spaces have the same dimension:}
\]

\[
i) \text{ spinors in } \Delta_n \text{ fixed under } G,
\]

\[
ii) \text{ spinors in } \Delta_{1,n+1} \text{ fixed under } H,
\]

\[
iii) \text{ parallel spinors fields on a Lorentzian manifold with holonomy group } H = G \ltimes \mathbb{R}^n.
\]

Finally, what is needed to complete a proof of Theorem 2 is a classification of subgroups of \( \text{SO}(n) \) that fix a spinor in \( \Delta_n \) and have connected component \( \text{SU}(m), \text{Sp}(k), G_2 \) or \( \text{Spin}(7) \). This result can be obtained from \([24]\) and \([31]\).

\[
\text{Theorem 4 (McInnes [24] and Wang [31]). Let } G \subset \text{SO}(n) \text{ be Lie group with connected component } G^0 \text{ equal to } \text{SU}(m), \text{Sp}(k), G_2 \text{ or } \text{Spin}(7) \text{ with a non-vanishing fixed spinor in } \Delta_n. \text{ Then } G \text{ is equal to one of the groups in the following table, in which } N \text{ is the dimension of spinors fixed under } G,
\]
Here

(1) $Q_{4d}$ is the double cover of the dihedral group $D_{2d}$ of order $2d$.

(2) $Sp(k) \cdot B_{4d}$ for $d = 6, 12, 30$, and $B_{4d}$ is the double cover in $Sp(1)$ of the polyhedral groups $P_{2d}$ in $SO(3)$, i.e. the tetrahedral group $P_{12}$, the octahedral group $P_{24}$, and the icosahedral group $P_{60}$.

(3) $\Gamma$ is an infinite subgroup of $U(1) \rtimes \mathbb{Z}_2$.

Steps in the proof. Since $G$ is contained in the normaliser of $G^0$ in $O(n)$, first we need a list of normalisers of the possible $G^0$'s. They can be found in [8, 10.114] with a correction made in [24] for the $SU(m)$-case. The cases in which $G^0$ is equal to $G_2$ or $Spin(7)$ are trivial, as both groups are equal to their own normaliser in $O(n)$. We are left with $G^0$ being $SU(m)$ or $Sp(k)$. Their normaliser in $O(n)$ is given as $U(m) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by complex conjugations, and as $Sp(k) \cdot Sp(1)$.

First assume that $G/G^0$ is finite. In [24], McInnes classified the possible holonomy groups of compact Ricci flat Riemannian manifolds. Since their fundamental group is finite, as the first, purely algebraic step in McInnes’ proof, possible subgroups $G$ in $SU(m) \rtimes \mathbb{Z}_2$ and $Sp(k) \cdot Sp(1)$ with finite quotients $G/SU(m)$ and $G/Sp(k)$ are listed. For $SU(m)$ they are of the form

\[
Z_{mr} \cdot SU(m) \quad \text{or} \quad (Z_{mr} \cdot SU(m)) \rtimes \mathbb{Z}_2,
\]

with a positive integer $r$ and with $Z_{mr} \in U(1)$. For $Sp(k)$ the list is longer:

(i) $\mathbb{Z}_r \cdot Sp(k)$, with $r$ odd,
(ii) $\mathbb{Z}_{2r} \cdot Sp(k)$, with $r$ even,
(iii) $Q_{4d} \cdot Sp(k)$, where $Q_{4d}$ is the double cover of the dihedral group $D_{2d}$ of order $2d$,
(iv) $B_{4d} \cdot Sp(k)$ for $d = 6, 12, 30$, and $B_{4d}$ is the double cover in $Sp(1)$ of the polyhedral groups $P_{2d}$ in $SO(3)$, i.e. the tetrahedral group $P_{12}$, the octahedral group $P_{24}$, and the icosahedral group $P_{60}$.

Using geometric arguments McInnes shortened this list to obtain all possible holonomy groups of compact Ricci flat Riemannian manifolds, but since we cannot apply geometric arguments for our purpose, we cannot use this shorter list. Instead we use results by Wang in [31], where the full holonomy groups of Riemannian manifolds — compact and non-compact — with parallel spinors are classified. Although in the compact case Wang can start from the shorter list obtained by

| $G^0$ | $n$ | $G$ | $N$ | conditions |
|-------|-----|-----|-----|------------|
| $SU(m)$ | $2m$ | $SU(m)$ | 2 | |
| $SU(m) \rtimes \mathbb{Z}_2$ | 1 | $m$ divisible by 4 |
| $Sp(k)$ | $4k$ | $Sp(k)$ | $k + 1$ | |
| $Sp(k) \rtimes \mathbb{Z}_d$ | $(k + 1)/d$ | $d > 1, d$ odd and divides $k + 1$ |
| $Sp(k) \cdot \mathbb{Z}_{2d}$ | $2 \lfloor k/2 \rfloor + 1$ | $k$ even, $1 < d \leq 2d$ |
| $Sp(k) \cdot Q_{4d}$ | $\lfloor k/2 \rfloor$ if $k$ odd | $k$ even, $1 < d \leq 2d$, $\lfloor k/2 \rfloor + 1$ if $k$ even |
| $Sp(k) \cdot B_{4d}$ | see ref. [31] | $k$ even and conditions in [31] |
| $Sp(k) \cdot \Gamma$ | 1 | $k$ even |
| $Spin_7$ | 8 | $Spin_7$ | 1 | |
| $G_2$ | 7 | $G_2$ | 1 | |
McInnes, for the non-compact case in [31, Proof of Theorem 4.1] only algebraic arguments can be used and the full list above has to be checked for the existence of fixed spinors. In the SU(m) case Wang shows that \( r = 0 \), that is, only SU(m) itself and SU(m) \( \ltimes \mathbb{Z}_2 \) remains. In the Sp(m) case Wang obtains the list in the table in the Theorem.

To conclude this section we consider the question whether there exist Lorentzian manifolds with the holonomy groups in Theorem 2 and special causality properties. In Proposition 4 we proved that starting with a Riemannian manifold \((N, g_N)\) with full holonomy group \( G \) and a function \( f \in C^\infty(\mathbb{R} \times N) \) such that \( \det(\text{Hess}^N(f))|_p \neq 0 \) at some point \( p \in \mathbb{R} \times N \), we obtain a Lorentzian manifold \( M := \mathbb{R}^2 \times N \) with the metric

\[
g^{f,h} = 2dvdu + 2fdu^2 + g_N
\]

with full holonomy \( G \ltimes \mathbb{R}^n \). Proposition 8 and Corollary 2 show, that in case of a spin manifold \((N, g_N)\), the Lorentzian manifold \((M, g^{f,h})\) is spin as well and the dimension of the spaces of parallel spinor fields on \((M, g^{f,h})\) and \((N, g_N)\) are the same. Moreover, for Lorentzian manifolds of type \((M, g^{f,h})\) various causality properties are known (see for example [12] and [16]). Let us quote here the following two results.

1) If \((N, g_N)\) is a complete Riemannian manifold, the function \( f \) does not depend on \( u \) and is at most quadratic at spacial infinity, i.e., there exist \( x_0 \in N \) and real constants \( r, c > 0 \) such that

\[
f(x) \leq c \cdot d_N(x_0, x)^2 \quad \text{for all } x \in N \text{ with } d_N(x_0, x) \geq r,
\]

then \((M, g^{f,h})\) is geodesically complete. Here \( d_N \) is the distance function of \((N, g_N)\).

2) If \((N, g_N)\) is a complete Riemannian manifold and the function \(-f\) is spacial subquadratic, i.e., there exist \( x_0 \in N \) and continuous functions \( p, c_1, c_2 \in C(\mathbb{R}, [0, \infty)) \) with \( p(u) < 2 \) such that

\[
-f(u, x) \leq c_1(u) d_N(x_0, x)^p(u) + c_2(u) \quad \text{for all } (u, x) \in \mathbb{R} \times N,
\]

then \((M, g^{f,h})\) is globally hyperbolic.

Of course, both conditions for \( f \) can be realized in addition to \( \det(\text{Hess}^N(f(u_0, x_0))) \neq 0 \). Hence, each of the groups in Theorem 2 can be realized as holonomy group of a Lorentzian manifold, and in addition, if the group \( G \) in Theorem 4 is the holonomy group of a complete Riemannian manifold, then \( H \) can be realized by a geodesically complete as well as by a globally hyperbolic Lorentzian manifold.

If one is interested in globally hyperbolic manifolds with complete or even compact space-like Cauchy surfaces, another construction based on Lorentzian cylinders is useful, which from the spin geometric point of view first was studied by Bär, Gauduchon and Moroianu in [2] and further developed in the context of special holonomy by the first author and Müller in [5]. Formulated for our situation the result is:

**Proposition 9 ([5]).** Let \((N, g_N)\) be an \( n \)-dimensional irreducible Riemannian spin manifold of dimension \( n \) with parallel spinors, \((F, g_F)\) the warped product \((F = \mathbb{R} \times N, g_F = ds^2 + e^{-4s}g_N)\) over \((N, g_N)\), \( C : TF \rightarrow TF \) a Codazzi tensor on \((F, g_F)\) with only positive eigenvalues and \( a \in \mathbb{R} \) a positive constant. Then the Lorentzian manifold \((M, g^C)\) given by

\[
M := (-a, \infty) \times \mathbb{R} \times N, \quad g^C := -dt^2 + (C + 2(t + a)\text{Id}_{TF})^*g_F
\]

has full holonomy

\[
\text{Hol}_{(0,0,p)}(M, g^C) = C^{-1} \circ \text{Hol}_p(N, g_N) \circ C \ltimes \mathbb{R}^n.
\]

Moreover, if \((N, g_N)\) is complete, then the Lorentzian manifold \((M, g^C)\) is globally hyperbolic and the space-like slices \( \{t\} \times F \), \( g^C_t = (C + 2(t + a)\text{Id}_{TF})^*g_F \) are complete Cauchy surfaces.

**Proof.** The proof in [5], Theorem 3, states the result for the reduced holonomy groups. Using Proposition 1 in addition, we obtain the result for the full holonomy group. \( \square \)
Explicit examples for Codazzi tensors $C$ on the warped product $((\mathbb{R} \times N, ds^2 + e^{-4s}g_N)$ are given in [5]. Take for example a bounded, strictly increasing function $f \in C^\infty(\mathbb{R})$ with $f(0) = 0$ and $f(s) < \lambda$ for all $s \in \mathbb{R}$. Then $C^f : \mathbb{R}\partial_s \oplus TN \to \mathbb{R}\partial_s \oplus TN$ given by

$$C^f := \begin{pmatrix} e^{2s}f'(s) & 0 \\ 0 & 2e^{2s}(\lambda - f(s))Id_{TN} \end{pmatrix}$$

is a Codazzi tensor on $(F, g_F)$ and the metric (21) is given by

$$g^{C^f} = -dt^2 + (e^{2s}f'(s) + 2a + 2t)^2ds^2 + 4(e^{-2s}t + e^{-2s}a + \lambda - f(s))^2g_N.$$

These two constructions reduce the problem of finding, for each $G$ in the table in Theorem 1, a Lorentzian manifold with holonomy $G \ltimes \mathbb{R}^n$ to the Riemannian case. First, one has to ensure the existence of Riemannian manifolds with holonomy group $G$. Then, for geodesically complete or globally hyperbolic Lorentzian metrics, one needs complete Riemannian manifolds with holonomy group $G$. For connected holonomy groups we can built on the deep existence results for complete and even compact Riemannian manifolds with special holonomy obtained by several authors (for an overview see [19]). Based on the examples with connected holonomy groups, Moroianu and Semmelmann in [27] constructed Riemannian manifolds with parallel spinor for each of the non-connected groups $G$ in the table in Theorem 2. For $SU(m) \ltimes \mathbb{Z}_2$ they construct a compact manifold, and for the remaining groups the metrics are obtained by removing points from compact spaces or by cone constructions, thus these metrics are not complete. This yields the following conclusion.

**Corollary 3.** For each of the groups $G$ in Theorem 2 there exist Lorentzian manifolds with holonomy $G \ltimes \mathbb{R}^n$ and parallel spinors. Moreover, for the connected groups $G$ and for $SU(m) \ltimes \mathbb{Z}_2$, there exist geodesically complete as well as globally hyperbolic Lorentzian manifolds with complete spacelike Cauchy surfaces and holonomy $G \ltimes \mathbb{R}^n$.

It would be interesting to know, if the groups $Sp(m) \times \mathbb{Z}_d$, and $Sp(m) \cdot \Gamma$ in Theorem 4 can be realized as holonomy group of a complete Riemannian manifold.

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