ON THE NUMBER OF TILTING MODULES OVER A CLASS OF
AUSLANDER ALGEBRAS

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Abstract. Let Λ be a radical square zero algebra of a Dynkin quiver and let Γ be the Auslander
algebra of Λ. Then the number of tilting right Γ-modules is $2^{m-1}$ if Λ is of $A_m$ type for $m \geq 1$.
Otherwise, the number of tilting right Γ-modules is $2^{m-3} \times 14$ if Λ is either of $D_m$ type for $m \geq 4$
or of $E_m$ type for $m = 6, 7, 8$.

1. Introduction

Tilting theory has been essential in the representation theory of finite dimensional algebras
since 1970s (see [BGP, BB, HR]). Tilting modules play an important role in tilting theory. So,
it is interesting but difficult to classify the tilting modules over a given algebra. There are many
algebraists working on this topics. Brüstle, Hille, Ringel and Röhrle [BHRR] classified the tilting
modules over the Auslander algebra of $K[x]/(x^n)$. Iyama and Zhang [IZ1] studied tilting modules
over Auslander-Gorenstein algebras. Geuenich [G] studied tilting modules of finite projective
dimension for the Auslander algebra of $K[x]/(x^n)$. Zhang [Z2] showed the number of tilting
modules over the Auslander algebra of radical square zero Nakayama algebras. Xie, Gao and
Huang [XGH] studied the number of tilting modules over the Auslander algebras of radical cube
zero Nakayama algebras. For more recent development on tilting modules, we refer to [AT, K, PS].

In 2014, Adachi Iyama and Reiten [AIR] introduced the notion of $τ$-tilting modules as general-
izations of tilting modules in terms of mutations. This help us be able to get the tilting modules
in terms of support $τ$-tilting modules. Therefore, it is important to classify support $τ$-tilting mod-
ules for a given algebra. Adachi [A1] classified support $τ$-tilting modules over Nakayama algebras;
Adachi [A2] and Zhang [Z1] studied $τ$-rigid modules over algebras with radical square zero; Mizuno
[M] classified $τ$-tilting modules over preprojective algebras of Dynkin type; Iyama and Zhang [IZ2]
classified $τ$-tilting modules over the Auslander algebra of $K[x]/(x^n)$. For more recent development
on $τ$-tilting modules, we can refer to [AH, DL, KK, W, Z1, Z2].

In this paper, we study tilting modules over the Auslander algebras of radical square zero
algebras of Dynkin quiver in terms of $τ$-tilting theory. By using a bijection over Auslander-
Gorenstein algebras built by Iyama and the second author [IZ1], we can get the number of tilting
modules over the Auslander algebras of radical square zero algebras of Dynkin quivers, which
extends the results in [Z2]. More precisely, we prove the following main result.

Theorem 1.1. Let Λ be a radical square zero algebra of a Dynkin quiver and let Γ be the Auslander
algebra of Λ. Then the number of tilting right Γ-modules is $2^{m-1}$ if Λ is of $A_m$ type for $m \geq 1$.
Otherwise, the number of tilting right Γ-modules is $2^{m-3} \times 14$ if Λ is either of $D_m$ type for $m \geq 4$
or of $E_m$ type for $m = 6, 7, 8$.

We show the organization of this paper as follows: In Section 2, we recall some basic preliminaries
on tilting modules, $τ$-tilting modules and Auslander algebras; In Section 3, we prove the main
results and give examples to show the main results.

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Throughout this paper, all the algebras are finite dimensional basic algebras over an algebraically closed field $K$ and all modules are finitely generated right modules. For a tilting module, we mean the classical tilting module. We use $\tau$ to denote the Auslander-Reiten translation functor. For an algebra $\Lambda$, we use $\text{mod}\Lambda$ to denote the category of finitely generated right $\Lambda$-modules.

2. Preliminaries

In this section, we recall definitions and basic facts on tilting modules, $\tau$-tilting modules, and Auslander algebras.

For a module $M$, we use $\text{pd}_\Lambda M$ and $|M|$ to denote the projective dimension of $M$ and the number of indecomposable direct summand of $M$, respectively. Now we recall the definition of a tilting module [HR].

Definition 2.1. Let $\Lambda$ be an algebra and $T \in \text{mod}\Lambda$, $T$ is called a tilting module if the following are satisfied:

1. $\text{pd}_\Lambda T \leq 1$.
2. $\text{Ext}_\Lambda^i(T,T) = 0$, for $i \geq 1$.
3. $|T| = |\Lambda|$.

Now we recall the definition of $\tau$-tilting modules introduced in [AIR].

Definition 2.2. Let $\Lambda$ be an algebra and $M \in \text{mod}\Lambda$.

1. We call $M$ $\tau$-rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$.
2. $M$ is called a $\tau$-tilting module if $M$ is $\tau$-rigid and $|M| = |\Lambda|$.
3. We call $M$ in $\text{mod}\Lambda$ support $\tau$-tilting if there exists an idempotent $e$ of $\Lambda$ such that $M$ is a $\tau$-tilting $(\Lambda/(e))$-module.

The following lemma [AIR, Proposition 2.4] on $\tau$-rigid modules is important.

Lemma 2.3. Let $X$ be in $\text{mod}\Lambda$ with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0$. Then $X$ is $\tau$-rigid if and only if the map $\text{Hom}_\Lambda(P_0, X) \xrightarrow{d_1^*} \text{Hom}_\Lambda(P_1, X)$ is surjective, where $d_1^* = \text{Hom}_\Lambda(d_1, X)$.

We also need the following definition of support $\tau$-tilting modules.

Definition 2.4. Let $(M, P)$ be a pair with $M \in \text{mod}\Lambda$ and $P$ projective.

1. We call $(M, P)$ a $\tau$-rigid pair if $M$ is $\tau$-rigid and $\text{Hom}_\Lambda(P, M) = 0$.
2. We call $(M, P)$ a support $\tau$-tilting (respectively, almost complete support $\tau$-tilting) pair if $(M, P)$ is $\tau$-rigid and $|M| + |P| = |\Lambda|$ (respectively, $|M| + |P| = |\Lambda| - 1$).

For an algebra $\Lambda$, we use $\text{sr}$-$\text{tilt}\Lambda$ to denote the set of isomorphism class of support $\tau$-tilting modules over $\Lambda$. Denote by $Q(\text{sr}$-$\text{tilt}\Lambda)$ the support $\tau$-tilting quiver of $\Lambda$. The following lemma in [AIR, Corollary 2.38] is useful in this paper.

Lemma 2.5. If $Q(\text{sr}$-$\text{tilt}\Lambda)$ has a finite connected component $C$, then $Q(\text{sr}$-$\text{tilt}\Lambda) = C$.

In the following we recall the definition of Auslander algebras in [ARS].

Definition 2.6. An algebra $\Lambda$ is called an Auslander algebra if $\text{gl.dim}\Lambda \leq 2$ and $E_i(\Lambda)$ is projective for $i = 0, 1$, where $E_i(\Lambda)$ is the $(i + 1)$-th term in a minimal injective resolution of $\Lambda$.

It is shown in [ARS] that there is one to one bijection between Auslander algebras and algebras of finite representation type in [ARS]. Let $\Lambda$ be an algebra of finite representation type, and $M$ an additive generator of $\text{mod}\Lambda$. Then we call $\Gamma = \text{End}_\Lambda M$ the Auslander algebra of $\Lambda$.

For an algebra $\Lambda$, we use $\text{tilt}\Lambda$ to denote the set of isomorphism classes of tilting modules in $\text{mod}\Lambda$. The following theorem on tilting modules over Auslander algebras in [Z2] is essential in this paper. For more details on this bijection map we refer to [IZ1, J].
Theorem 2.7. Let \( \Lambda \) be an Auslander algebra and \( e \) be an idempotent such that \( e\Lambda \) is the additive generator of projective-injective modules. Then there is a bijective between the set \( \text{tilt}\Lambda \) of tilting modules over \( \Lambda \) and the set \( \tau\text{-tilt}\Lambda/(e) \) of support \( \tau \)-tilting modules over \( \Lambda/(e) \).

Now we recall the definition of Dynkin algebras as follows.

Definition 2.8. We call an algebra \( \Lambda \) of Dynkin type if the quiver of \( \Lambda \) is one of the following quivers:

- \( A_m \) (\( m \geq 1 \)):
  \[
  1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{m-2}} m-1 \xrightarrow{a_{m-1}} m
  \]

- \( D_m \) (\( m \geq 4 \)):
  \[
  1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \xrightarrow{a_4} \cdots \xrightarrow{a_{m-2}} m-1 \xrightarrow{a_{m-1}} m
  \]

- \( E_6 \):
  \[
  3 \xrightarrow{a_3} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 4 \xrightarrow{a_4} 5 \xrightarrow{a_5} 6
  \]

- \( E_7 \):
  \[
  3 \xrightarrow{a_3} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 4 \xrightarrow{a_4} 5 \xrightarrow{a_5} 6 \xrightarrow{a_6} 7
  \]

- \( E_8 \):
  \[
  3 \xrightarrow{a_3} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 4 \xrightarrow{a_4} 5 \xrightarrow{a_5} 6 \xrightarrow{a_6} 7 \xrightarrow{a_7} 8
  \]

In the following we recall properties of the block decomposition of an algebra from [Al] P92-93, Theorem 1, Proposition 2.

Proposition 2.9. Let \( A \) be an algebra and \( M \in \text{mod}A \). Then
(1) \( A \) has a unique decomposition into a direct sum of indecomposable subalgebras, that is \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_r \).
(2) \( M \) has a unique decomposition as \( M = M_1 \oplus M_2 \oplus \cdots \oplus M_r \) with \( M_i \in \text{mod}A_i \) and \( M_iA_j = 0 \).
(3) \( \text{Hom}_A(M_i, M_j) = 0 \) for any \( i \neq j \).

3. Main results

In this section, we show the number of tilting modules over Auslander algebras of radical square zero Dynkin algebras. By a straight calculation, one gets the Auslander algebras of radical square zero Dynkin algebras as follows.

Proposition 3.1. (1) Let \( \Gamma \) be the Auslander algebra of a radical square zero algebra of type \( A_m \). Then \( \Gamma \) is given by the quiver \( Q_1 \):
  \[
  1 \xleftarrow{a_1} 2 \xleftarrow{a_2} \cdots \xleftarrow{a_{2m-3}} 2m-2 \xleftarrow{a_{2m-2}} 2m-1
  \]
  with the relations:
  \[ a_{2k-1}a_{2k} = 0 (1 \leq k \leq m-1). \]
(2) Let $\Gamma$ be the Auslander algebra of a radical square zero algebra of type $D_m$. Then $\Gamma$ is given by the quiver $Q_2$:

\[
\begin{array}{c}
1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{2m-2}} \xrightarrow{a_{2m-6}} \xrightarrow{a_{2m-5}} 2m - 5 \xrightarrow{a_{2m-4}} \xrightarrow{a_{2m-3}} 2m - 4 \xrightarrow{a_{2m-2}} \xrightarrow{a_{2m-1}} 2m - 1 \xrightarrow{a_{2m}} 2m - 3 \xrightarrow{a_{2m-6}} 2m - 6 \xrightarrow{a_{2m-5}} 2m - 5 \xrightarrow{a_{2m-4}} 2m - 4 \xrightarrow{a_{2m-3}} 2m - 2 \xrightarrow{a_{2m-2}} 2m - 3 \xrightarrow{a_{2m-1}} 2m - 1
\end{array}
\]

with the relations: $a_1a_2 = 0, a_3a_4 = 0, \ldots, a_{2m-7}a_{2m-6} = 0, a_{2m-5}a_{2m-3} = a_{2m-4}a_{2m-2}, a_{2m-3}a_{2m-1} = 0, a_{2m-2}a_{2m} = 0$.

(3) Let $\Gamma_m$ be the Auslander algebra of a radical square zero algebra of type $E_m$ with $m = 6, 7, 8$. Then $\Gamma_m$ is given by the quiver $Q_6$:

\[
\begin{array}{c}
1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \xrightarrow{a_4} 5 \xrightarrow{a_5} 6 \xrightarrow{a_6} 7 \xrightarrow{a_7} 8 \xrightarrow{a_8} 9 \xrightarrow{a_9} 10 \xrightarrow{a_{10}} 11 \xrightarrow{a_{11}} 12 \xrightarrow{a_{12}} 13 \xrightarrow{a_{13}} 14
\end{array}
\]

with the relations: $a_1a_2 = 0, a_3a_4 = 0, a_5a_7 = a_8a_9, a_7a_9 = 0, a_8a_{10} = 0, a_{11}a_{12} = 0$. $Q_7$:

\[
\begin{array}{c}
1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} 4 \xrightarrow{a_4} 5 \xrightarrow{a_5} 6 \xrightarrow{a_6} 7 \xrightarrow{a_7} 8 \xrightarrow{a_8} 9 \xrightarrow{a_9} 10 \xrightarrow{a_{10}} 11 \xrightarrow{a_{11}} 12 \xrightarrow{a_{12}} 13 \xrightarrow{a_{13}} 14
\end{array}
\]

with the relations: $a_1a_2 = 0, a_3a_4 = 0, a_5a_6 = 0, a_7a_9 = a_8a_{10}, a_9a_{11} = 0, a_{10}a_{12} = 0, a_{13}a_{14} = 0$. $Q_8$:

\[
\begin{array}{c}
1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_7} 8 \xrightarrow{a_8} 9 \xrightarrow{a_9} 10 \xrightarrow{a_{10}} 11 \xrightarrow{a_{11}} 12 \xrightarrow{a_{12}} 13 \xrightarrow{a_{13}} 14 \xrightarrow{a_{14}} 15 \xrightarrow{a_{15}} 16
\end{array}
\]

with the relations: $a_1a_2 = 0, a_3a_4 = 0, a_5a_6 = 0, a_7a_8 = 0, a_9a_{11} = a_{10}a_{12}, a_{11}a_{13} = 0, a_{12}a_{14} = 0, a_{15}a_{16} = 0$.

To prove the main results we need the following proposition in [Z2].

**Proposition 3.2.** Let $\Lambda$ be a semi-simple algebra with $n$ simple modules. Then the number of support $\tau$-tilting $\Lambda$-modules is $2^n$.

The following proposition on the support $\tau$-tilting modules over direct sums of algebras is essential in this paper.
Proposition 3.3. Let $\Lambda$ be an algebra which can be decomposed as a direct sum of two subalgebras, that is, $\Lambda = \Lambda_1 \oplus \Lambda_2$.

1. For any $M \in \text{mod}\Lambda$, $M$ can be decomposed as $M_1 \oplus M_2$ with $M_i \in \text{mod}\Lambda_i$ for $i = 1, 2$.
2. For any $M \in \text{mod}\Lambda$ with the decomposition $M = M_1 \oplus M_2$, $M$ is a support $\tau$-tilting module if both $M_1 \in \text{mod}\Lambda_1$ and $M_2 \in \text{mod}\Lambda_2$ are support $\tau$-tilting modules.
3. For any $M \in \text{mod}\Lambda$ with the decomposition $M = M_1 \oplus M_2$, $M$ is a $\tau$-tilting module if both $M_1 \in \text{mod}\Lambda_1$ and $M_2 \in \text{mod}\Lambda_2$ are $\tau$-tilting modules.
4. If $|\text{sr-tilt}\Lambda_1| = m$ and $|\text{sr-tilt}\Lambda_2| = n$, then $|\text{sr-tilt}\Lambda| = mn$.

Proof. (1) This is a straight result of Proposition 2.6.
(2) We divide the proof into three steps.
(a) We show $M_1 \oplus M_2$ is $\tau$-rigid.
Let $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M_1 \rightarrow 0$ be a minimal projective presentation of $M_1 \in \text{mod}\Lambda_1$ and let $Q_1 \xrightarrow{d'_1} Q_0 \xrightarrow{d'_0} M_2 \rightarrow 0$ be a minimal projective presentation of $M_2 \in \text{mod}\Lambda_2$. By Proposition 2.6 one gets a minimal projective presentation of $M = M_1 \oplus M_2$ as follows:

\[ P_1 \oplus Q_1 \xrightarrow{d_1 \oplus d'_1} P_0 \oplus Q_0 \xrightarrow{d_0 \oplus d'_0} M_1 \oplus M_2 \rightarrow 0 \quad (1) \]

Since $M_1$ is support $\tau$-tilting module, then the map $\text{Hom}_{\Lambda_1}(P_0, M_1) \xrightarrow{(d_1, M_1)} \text{Hom}_{\Lambda_1}(P_1, M_1)$ is surjective by Lemma 2.3. Similarly, one gets the map $\text{Hom}_{\Lambda_2}(Q_0, M_2) \xrightarrow{(d'_1, M_2)} \text{Hom}_{\Lambda_2}(Q_1, M_2)$ is surjective since $M_2$ is support $\tau$-tilting module. Applying $\text{Hom}_{\Lambda}(-, M_1 \oplus M_2)$ to (1), by Proposition 2.6 we have the following exact sequence

\[ 0 \rightarrow \text{Hom}_{\Lambda}(M_1 \oplus M_2, M_1 \oplus M_2) \xrightarrow{F_0} \text{Hom}_{\Lambda}(P_0 \oplus Q_0, M_1 \oplus M_2) \xrightarrow{F_1} \text{Hom}_{\Lambda}(P_1 \oplus Q_1, M_1 \oplus M_2), \]

where $F_i = (d_i, M_1) \oplus (d'_i, M_2)$ for $i = 0, 1$. Notice that both the map $(d_1, M_1)$ and the map $(d'_1, M_2)$ are surjective, then $(d_1, M_1) \oplus (d'_1, M_2)$ is surjective. By Lemma 2.3 then $M_1 \oplus M_2$ is $\tau$-rigid.

(b) Denote by $(M_1, P)$ and $(M_2, Q)$ the support $\tau$-tilting pair in $\text{mod}\Lambda_1$ and $\text{mod}\Lambda_2$. We show that $(M_1 \oplus M_2, P \oplus Q)$ is a $\tau$-rigid pair in $\text{mod}\Lambda$.

Since $(M_1, P)$ is a support $\tau$-tilting pair in $\text{mod}\Lambda_1$, then $\text{Hom}_{\Lambda_1}(P, M_1) = 0$ holds. Similarly, one gets $\text{Hom}_{\Lambda_2}(Q, M_2) = 0$. Then by Proposition 2.7 $\text{Hom}_{\Lambda}(P \oplus Q, M_1 \oplus M_2) \cong \text{Hom}_{\Lambda_1}(P, M_1) \oplus \text{Hom}_{\Lambda_2}(Q, M_2) \not\cong 0$, so $(M_1 \oplus M_2, P \oplus Q)$ is $\tau$-rigid pair in $\text{mod}\Lambda$ by (a).

(c) We show $(M_1 \oplus M_2, P \oplus Q)$ is a support $\tau$-tilting pair.

Since $(M_1, P)$ is a support $\tau$-tilting pair, one gets $|M_1| + |P| = |\Lambda_1|$. Similarly, the fact $(M_2, Q)$ is a support $\tau$-tilting pair in $\text{mod}\Lambda_2$ implies $|M_2| + |Q| = |\Lambda_2|$. So one gets $|M_1| + |M_2| + |P| + |Q| = |\Lambda_1| + |\Lambda_2| = |\Lambda|$. Then by (b) $(M_1 \oplus M_2, P \oplus Q)$ is a support $\tau$-tilting pair in $\text{mod}\Lambda$. And hence $M_1 \oplus M_2$ is a support $\tau$-tilting module.

(3) This is a straight result of (2).

(4) Let $|\text{sr-tilt}\Lambda_1| = m$, $|\text{sr-tilt}\Lambda_2| = n$, by (2) one gets a finite connected component $\mathcal{C}$ of the quiver of $Q(\text{sr-tilt}\Lambda)$. Then by Lemma 2.3 we get that $|\text{sr-tilt}\Lambda| = mn$. \qed

The following lemma is also useful.

Lemma 3.4. Let $\Lambda$ be a algebra with $Q$:

\[
\begin{array}{ccc}
4 & \downarrow & 3 \\
\downarrow \leftarrow & & \leftarrow \\
3 & \leftarrow & 5
\end{array}
\]

Then the number of support $\tau$-tilting $\Lambda$-modules is 14.
Theorem 3.5. Let $\Gamma$ be the Auslander algebra of a radical square zero of type $A_m$ with $m \geq 1$. Then the number of tilting $\Gamma$-modules is $2^{m-1}$.

Proof. By Proposition 3.1 we can get the quiver of the algebra $\Gamma$. We also get the indecomposable projective-injective modules are as follows: $P(2) = I(1), P(4) = I(2), P(6) = I(4), \cdots, P(2m-2) = I(2m-6), P(2m-1) = I(2m-3), P(2m) = I(2m-4)$. Take the idempotent $e = e_2 + e_4 + e_6 + \cdots + e_{2m-6} + e_{2m-2} + e_{2m-1} + e_{2m}$. Then $\Gamma/(e)$ is a direct sum of a semi-simple algebra with $m-3$ vertices and an algebra in Lemma 3.3. Then by Theorem 2.7, Proposition 3.2, Proposition 3.3 and Lemma 3.4 one can get the number of the tilting modules $\Gamma$ is $2^{m-3} \times 14$.

Now we show the number of tilting modules over the Auslander algebras of radical square zero algebras of type $E_m$ for $m = 6, 7, 8$.

Theorem 3.6. Let $\Lambda$ be a radical square zero algebra of type $D_m$ with $m \geq 4$ and let $\Gamma$ be the Auslander algebra of $\Lambda$. Then the number of tilting $\Gamma$-module is $2^{m-3} \times 14$.

Proof. By Proposition 3.1 we can get the quiver of the algebra $\Gamma$. We also get the indecomposable projective-injective modules are as follows: $P(2) = I(1), P(4) = I(2), P(6) = I(4), \cdots, P(2m-2) = I(2m-6), P(2m-1) = I(2m-3), P(2m) = I(2m-4)$. Take the idempotent $e = e_2 + e_4 + e_6 + \cdots + e_{2m-6} + e_{2m-2} + e_{2m-1} + e_{2m}$. Then $\Gamma/(e)$ is a direct sum of a semi-simple algebra with $m-3$ vertices and an algebra in Lemma 3.3. Then by Theorem 2.7, Proposition 3.2, Proposition 3.3 and Lemma 3.4 we get the number of the tilting modules $\mod \Gamma$ is $2^{m-3} \times 14$.
I(4), P(10) = I(6), P(11) = I(9), P(13) = I(8), P(14) = I(13). Take the idempotent \( e = e_2 + e_4 + e_6 + e_{10} + e_{11} + e_{13} + e_{14} \). And hence the quotient algebra \( \Gamma/\langle e \rangle \) is a direct sum of a semi-simple algebra with 4 vertices and an algebra in Lemma 3.3. Then by Theorem 2.7, Proposition 3.2, Proposition 3.3 and Lemma 3.4, we get the number of tilting modules in \( \text{mod} \Gamma \) is \( 2^4 \times 14 = 224 \).

If \( \Lambda \) is of type \( E_6 \), then by Proposition 3.1, we get the quiver of \( \Gamma \). Moreover, one gets the indecomposable projective-injective modules as follows: \( P(2) = I(1), P(4) = I(2), P(6) = I(4), P(8) = I(6), P(12) = I(8), P(13) = I(11), P(15) = I(10), P(16) = I(15) \). Take the idempotent \( e = e_2 + e_4 + e_6 + e_8 + e_{12} + e_{13} + e_{15} + e_{16} \). So the quotient algebra \( \Gamma/\langle e \rangle \) is a direct sum of a semi-simple algebra with 5 vertices and an algebra in Lemma 3.3. Then by Theorem 2.7, Proposition 3.2, Proposition 3.3 and Lemma 3.4, we can get the number of the tilting modules in the algebra \( \Gamma \) is \( 2^5 \times 14 = 448 \).

At the end of this paper, we give some examples to show our main results.

**Example 3.8.** Let \( \Lambda \) be a radical square zero algebra of type \( A_3 \). Then Auslander algebra \( \Gamma \) of \( \Lambda \) is given by the quiver \( Q : 1 \overset{u_1}{\longleftarrow} 2 \overset{4}{\longrightarrow} 3 \overset{u_3}{\longleftarrow} 4 \overset{u_5}{\longrightarrow} 7 \overset{u_6}{\leftarrow} 5 \overset{u_7}{\longrightarrow} 6 \overset{u_8}{\leftarrow} 8 \) with the relations: \( u_1u_2 = 0, u_3u_5 = u_4u_6, u_5u_7 = 0, u_6u_8 = 0 \). Then the tilting \( \Gamma \)-modules are follows:

\[
T_1 = I, T_2 = P(5) \oplus P(4) \oplus S(4) \oplus P(2) \oplus P(1) \\
T_3 = P(5) \oplus P(4) \oplus P(3) \oplus P(2) \oplus S(2) \\
T_4 = P(5) \oplus P(4) \oplus S(4) \oplus P(2) \oplus S(2) \\
2^{m-1} = 2^{3-1} = 4
\]

**Example 3.9.** Let \( \Lambda \) be a radical square zero algebra of type \( D_4 \). Then Auslander algebra \( \Gamma \) is given by the quiver \( Q \):

\[
1 \overset{u_1}{\longleftarrow} 2 \overset{u_2}{\longleftarrow} 3 \overset{u_3}{\leftarrow} 4 \overset{u_5}{\longrightarrow} 7 \overset{u_6}{\leftarrow} 5 \overset{u_7}{\longrightarrow} 6 \overset{u_8}{\leftarrow} 8
\]

with the relations: \( u_1u_2 = 0, u_3u_5 = u_4u_6, u_5u_7 = 0, u_6u_8 = 0 \).

The number of support \( \tau \) tilting \( \Gamma/\langle e \rangle \)-modules are as follows:

\[
T_1 = 1 \oplus 5, T_2 = 0 \oplus 5 \\
T_3 = 1 \oplus 3, T_4 = 0 \oplus 3 \\
T_5 = 1 \oplus 4, T_6 = 0 \oplus 4 \\
T_7 = 1 \oplus 0, T_8 = 0 \oplus 0 \\
T_9 = 1 \oplus 5 \oplus 4, T_{10} = 0 \oplus 5 \oplus 4 \\
T_{11} = 1 \oplus 5 \oplus [^5_3], T_{12} = 0 \oplus 5 \oplus [^5_3] \\
T_{13} = 1 \oplus 4 \oplus [^4_3], T_{14} = 0 \oplus 4 \oplus [^4_3] \\
T_{15} = 1 \oplus 3 \oplus [^3_3], T_{16} = 0 \oplus 3 \oplus [^3_3] \\
T_{17} = 1 \oplus 3 \oplus [^3_3], T_{18} = 0 \oplus 3 \oplus [^3_3] \\
T_{19} = 1 \oplus [^4_3 \oplus [^4_3 \oplus [^5_3 \oplus 5 \oplus 4, T_{20} = 0 \oplus [^4_3 \oplus [^4_3 \oplus [^5_3 \oplus 5 \oplus 4 \\
T_{21} = 1 \oplus 3 \oplus [^4_3 \oplus [^5_3 \oplus 4, T_{22} = 0 \oplus 3 \oplus [^4_3 \oplus [^5_3 \\
T_{23} = 1 \oplus [^4_3 \oplus [^5_3 \oplus 5 \oplus [^3_3, T_{24} = 0 \oplus [^4_3 \oplus [^5_3 \oplus 5 \oplus [^3_3 \\
T_{25} = 1 \oplus [^4_3 \oplus [^5_3 \oplus 4, T_{26} = 0 \oplus [^4_3 \oplus [^5_3 \oplus 4 \\
T_{27} = 1 \oplus [^4_3 \oplus [^5_3 \oplus 5, T_{28} = 0 \oplus [^4_3 \oplus [^5_3 \oplus 5 \\
2^{m-3} \times 14 = 2^{1-3} \times 14 = 28
\]

By Theorem 2.7, then the number of tilting \( \Gamma \)-modules is 28 (\( = 2^{m-3} \times 14 \)).
Example 3.10. Let $\Lambda$ be a radical square zero algebra of type $E_6$. Then Auslander algebra $\Gamma$ is given by the quiver $Q$:

$\xymatrix{1 \ar[r]^{u_1} & 2 \ar[r]^{u_2} & 3 \ar[r]^{u_3} & 4 \ar[r]^{u_4} & 5 \ar[ld]_{u_5} \ar[rd]^{u_7} \ar[rd]^{u_9} \ar[ld]_{u_6} \ar[r] & S \ar[ld]_{u_6} \ar[rd]^{u_9} \ar[ld]_{u_5} & 6 \ar[r]_{u_7} & 9 \ar[ld]_{u_6} \ar[rd]^{u_9} \ar[ld]_{u_5} & 8 \ar[r]_{u_7} & 10 \ar[r]_{u_{11}} & 11 \ar[r]_{u_{12}} & 12}$

with the relations: $u_1u_2 = 0, u_3u_4 = 0, u_5u_7 = u_6u_8, u_7u_9 = 0, u_8u_{10} = 0, u_{11}u_{12} = 0$.

The number of support $\tau$ tilting $\Gamma/(e)$-modules are as follows:

- $T_1 = 1 \oplus 3 \oplus 7 \oplus 0, T_2 = 0 \oplus 3 \oplus 7 \oplus 0$
- $T_3 = 1 \oplus 3 \oplus 5 \oplus 10, T_4 = 0 \oplus 3 \oplus 5 \oplus 10$
- $T_5 = 1 \oplus 3 \oplus 6 \oplus 10, T_6 = 0 \oplus 3 \oplus 6 \oplus 10$
- $T_7 = 1 \oplus 3 \oplus 0 \oplus 10, T_8 = 0 \oplus 3 \oplus 0 \oplus 10$
- $T_9 = 1 \oplus 3 \oplus 7 \oplus 10, T_{10} = 0 \oplus 3 \oplus 7 \oplus 10$
- $T_{11} = 1 \oplus 3 \oplus 5 \oplus 0, T_{12} = 0 \oplus 3 \oplus 5 \oplus 0$
- $T_{13} = 1 \oplus 3 \oplus 6 \oplus 0, T_{14} = 0 \oplus 3 \oplus 6 \oplus 0$
- $T_{15} = 1 \oplus 3 \oplus 0 \oplus 0, T_{16} = 0 \oplus 3 \oplus 0 \oplus 0$
- $T_{17} = 1 \oplus 0 \oplus 7 \oplus 0, T_{18} = 0 \oplus 0 \oplus 7 \oplus 0$
- $T_{19} = 1 \oplus 0 \oplus 5 \oplus 0, T_{20} = 0 \oplus 0 \oplus 5 \oplus 0$
- $T_{21} = 1 \oplus 0 \oplus 6 \oplus 0, T_{22} = 0 \oplus 0 \oplus 6 \oplus 0$
- $T_{23} = 1 \oplus 0 \oplus 0 \oplus 0, T_{24} = 0 \oplus 0 \oplus 0 \oplus 0$
- $T_{25} = 1 \oplus 0 \oplus 7 \oplus 10, T_{26} = 0 \oplus 0 \oplus 7 \oplus 10$
- $T_{27} = 1 \oplus 0 \oplus 5 \oplus 10, T_{28} = 0 \oplus 0 \oplus 5 \oplus 10$
- $T_{29} = 1 \oplus 0 \oplus 6 \oplus 10, T_{30} = 0 \oplus 0 \oplus 6 \oplus 10$
- $T_{31} = 1 \oplus 0 \oplus 0 \oplus 10, T_{32} = 0 \oplus 0 \oplus 0 \oplus 10$
- $T_{33} = 1 \oplus 3 \oplus 7 \oplus 6 \oplus 0, T_{34} = 0 \oplus 3 \oplus 7 \oplus 6 \oplus 0$
- $T_{35} = 1 \oplus 0 \oplus 7 \oplus 6 \oplus 0, T_{36} = 0 \oplus 0 \oplus 7 \oplus 6 \oplus 0$
- $T_{37} = 1 \oplus 3 \oplus 7 \oplus 6 \oplus 10, T_{38} = 0 \oplus 3 \oplus 7 \oplus 6 \oplus 10$
- $T_{39} = 1 \oplus 0 \oplus 7 \oplus 6 \oplus 10, T_{40} = 0 \oplus 0 \oplus 7 \oplus 6 \oplus 10$
- $T_{41} = 1 \oplus 3 \oplus 5 \oplus 7 \oplus [5 \oplus 7] \oplus 0, T_{42} = 0 \oplus 3 \oplus 5 \oplus [5 \oplus 7] \oplus 0$
- $T_{43} = 1 \oplus 3 \oplus 5 \oplus [6 \oplus 5] \oplus 0, T_{44} = 0 \oplus 3 \oplus 5 \oplus [6 \oplus 5] \oplus 0$
- $T_{45} = 1 \oplus 3 \oplus 7 \oplus [7 \oplus 5] \oplus 0, T_{46} = 0 \oplus 3 \oplus 7 \oplus [7 \oplus 5] \oplus 0$
- $T_{47} = 1 \oplus 3 \oplus 6 \oplus [6 \oplus 5] \oplus 0, T_{48} = 0 \oplus 3 \oplus 6 \oplus [6 \oplus 5] \oplus 0$
- $T_{49} = 1 \oplus 0 \oplus 5 \oplus [5 \oplus 7] \oplus 0, T_{50} = 0 \oplus 0 \oplus 5 \oplus [5 \oplus 7] \oplus 0$
- $T_{51} = 1 \oplus 0 \oplus 5 \oplus [6 \oplus 5] \oplus 0, T_{52} = 0 \oplus 0 \oplus 5 \oplus [6 \oplus 5] \oplus 0$
- $T_{53} = 1 \oplus 0 \oplus 7 \oplus [7 \oplus 5] \oplus 0, T_{54} = 0 \oplus 0 \oplus 7 \oplus [7 \oplus 5] \oplus 0$
- $T_{55} = 1 \oplus 0 \oplus 6 \oplus [6 \oplus 5] \oplus 0, T_{56} = 0 \oplus 0 \oplus 6 \oplus [6 \oplus 5] \oplus 0$
- $T_{57} = 1 \oplus 0 \oplus 5 \oplus [5 \oplus 7] \oplus 10, T_{58} = 0 \oplus 0 \oplus 5 \oplus [5 \oplus 7] \oplus 10$
- $T_{59} = 1 \oplus 3 \oplus 5 \oplus [6 \oplus 5] \oplus 10, T_{60} = 0 \oplus 3 \oplus 5 \oplus [6 \oplus 5] \oplus 10$
- $T_{61} = 1 \oplus 3 \oplus 7 \oplus [7 \oplus 5] \oplus 10, T_{62} = 0 \oplus 3 \oplus 7 \oplus [7 \oplus 5] \oplus 10$
- $T_{63} = 1 \oplus 3 \oplus 6 \oplus [6 \oplus 5] \oplus 10, T_{64} = 0 \oplus 3 \oplus 6 \oplus [6 \oplus 5] \oplus 10$
- $T_{65} = 1 \oplus 0 \oplus 5 \oplus [5 \oplus 7] \oplus 10, T_{66} = 0 \oplus 0 \oplus 5 \oplus [5 \oplus 7] \oplus 10$
- $T_{67} = 1 \oplus 0 \oplus 5 \oplus [6 \oplus 5] \oplus 10, T_{68} = 0 \oplus 0 \oplus 5 \oplus [6 \oplus 5] \oplus 10$
By Theorem 2.7, the number of tilting \( \Gamma \)-modules is 112 (= 2^{6-3} \times 14).

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