CKM matrix and flavor symmetries

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Abstract
Following the way proposed recently by Hernandez and Smirnov, we seek possible residual symmetries in the quark sector with a focus on the von Dyck groups. We begin with two extreme cases in which both $\theta_{13}$ and $\theta_{23}$ or only $\theta_{13}$ are set to zero. Then, cases where all the Cabibbo-Kobayashi-Maskawa parameters are allowed to take nonzero values are explored. The $Z_7$ symmetry is favorable to realize only the Cabibbo angle. On the other hand, larger groups are necessary in order to be consistent with all the mixing parameters. Possibilities of embedding the obtained residual symmetries into the $\Delta(6N^2)$ series are also briefly discussed.
1 INTRODUCTION

Theoretical understanding of the observed quark and lepton mixings is one of the longstanding issues in the standard model (SM). Indeed, various attempts to understand them have been done by imposing symmetries, using some dynamics or assuming extra dimensions.

Neutrino oscillation experiments show large mixing angles in the lepton sector except \( \theta_{13} \). In the approximation with \( \theta_{13} \approx 0 \), the tribimaximal mixing matrix is a quite interesting ansatz for the lepton sector \([2]\). The tribimaximal mixing matrix has certain symmetries. Then, a number of studies have been carried out to derive it by using non-Abelian discrete flavor symmetries (see for review Refs. \([3, 4, 5]\)). In those studies, first a non-Abelian flavor symmetry \( G_{f}^{(\ell)} \) for the lepton sector is assumed. Then, such a symmetry is broken to \( G_{\ell} (G_{\nu}) \) in the mass terms of the charged lepton (neutrino) sector. It was also found that inherent symmetries \( G_{\nu} = Z_{2} \times Z_{2} \) and \( G_{\ell} = Z_{3} \) in a certain basis are important to derive the tribimaximal mixing matrix \([6]\).

Recent neutrino experiments show that \( \theta_{13} \neq 0 \) \([7, 8]\). However, the above approach to use flavor symmetries is still interesting to derive experimental values of lepton mixing angles (see e.g. Ref. \([9]\)), although we need some modifications. For example, \( G_{\ell} \) was often extended from \( Z_{3} \) to \( Z_{m} \).

Hernandez and Smirnov developed a model-independent way of screening out possible flavor symmetries in the lepton sector to derive experimental values \([10]\). (See also Ref. \([11]\).) They figured out necessary conditions for ensuring that the inherent symmetries \( G_{\nu} = Z_{2} \times Z_{2} \) and \( G_{\ell} = Z_{3} \) can be embedded into a single discrete group. In particular, the lepton mixing angles are written interestingly in terms of a small number of integers by requiring that the product \( g \) between the \( Z_{m} \) element in \( G_{\ell} \) and the \( Z_{2} \) in \( G_{\nu} \) should satisfy \( g^{p} = 1 \), that is, \( g \) is a \( Z_{p} \) element.

The mixing angles in the quark sector, the Cabibbo-Kobayashi-Maskawa (CKM) matrix, are another issue to study. These patterns are quite different from those in the lepton sector. These mixing angles except the Cabibbo angle are rather small. There is no ansatz to figure out underlying symmetries behind the CKM matrix like the tribimaximal mixing ansatz. Here we study systematically the symmetries of the CKM matrix following Hernandez-Smirnov’s analysis on the lepton sector. We assume that the flavor symmetry \( G_{f}^{(q)} \) in the quark sector is broken to \( G_{u} (G_{d}) \) in the mass terms of the up-quark (down-quark) sector, where \( G_{u} = Z_{n} \) and \( G_{d} = Z_{m} \). Then, following Hernandez and Smirnov, we assume that the product \( g \) between the \( Z_{m} \) element in \( G_{d} \) and the \( Z_{n} \) in \( G_{u} \) should satisfy \( g^{p} = 1 \), that is, \( g \) is a \( Z_{p} \) element. That leads to constraints on three angles and one phase in the CKM matrix depending on \( m, n \) and \( p \). Then, we would figure out what Abelian symmetries are important to realize the CKM matrix as \( Z_{2} \times Z_{2} \) and \( Z_{3} \) symmetries are important to realize the tribimaximal mixing matrix. Such an analysis is useful to investigate candidates for the quark flavor symmetry \( G_{f}^{(q)} \), which includes \( Z_{n}, Z_{m} \) and \( Z_{p} \).

This paper is organized as follows. In Sec. \([2]\) we apply Hernandez-Smirnov’s analysis to the quark sector and derive conditions on the CKM parameters. For some special cases, we consider the conditions and derive possible residual symmetries in Sec. \([3]\).
of embedding the residual symmetries into the $\Delta(6N^2)$ groups are discussed in Sec. 4. We summarize our discussions in Sec. 5.

2 APPLICATION TO THE QUARK SECTOR

In this section, following the way proposed by Hernandez and Smirnov, we apply their method to the quark sector. Of the SM Lagrangian, only the quark mass terms are relevant to our discussion, which are written by

$$-\mathcal{L} = \bar{U}_R \hat{M}_U U_L + \bar{D}_R M_D D_L + H.c.,$$

where $U_{L,R} \equiv (u, c, t)^T_{L,R}$ and $D_{L,R} \equiv (d, s, b)^T_{L,R}$ are the up- and down-type quarks, respectively, and $\hat{M}_U \equiv \text{diag}\{m_u, m_c, m_t\}$ is the diagonalized up quark mass matrix. The down quark mass matrix, in this basis, can be taken to be

$$M_D = V_{\text{CKM}} \hat{M}_D V_{\text{CKM}}^\dagger,$$

where $V_{\text{CKM}}$ is the CKM matrix, and $\hat{M}_D \equiv \text{diag}\{m_d, m_s, m_b\}$. The CKM matrix is determined by the four parameters, $(\theta_{12}, \theta_{13}, \theta_{23}, \delta)$, and we obey the standard parametrization:

$$V_{\text{CKM}} = \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
    -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
    s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix},$$

where $s_{ij}$ and $c_{ij}$ represent $\sin \theta_{ij}$ and $\cos \theta_{ij}$, respectively. In what follows, we will use the following central values given in Ref. [1]:

$$(\sin \theta_{12}, \sin \theta_{23}, \sin \theta_{13}, \cos \delta) = (0.225, 0.0412, 0.00341, 0.355).$$

In general, each quark mass term inherently and maximally respects three $U(1)$ symmetries corresponding to three generations. For instance, in the basis of Eq. (2.1), the down quark mass term is invariant under transformations

$$D_R \to S_D D_R, \quad D_L \to S_D D_L,$$

where

$$S_0 \equiv \text{diag}\{e^{i\phi_d}, e^{i\phi_s}, e^{i\phi_b}\}, \quad S_D = V_{\text{CKM}} S_0 V_{\text{CKM}}^\dagger.$$

We regard $S_D$ as an element of a residual symmetry $G_d$ arising from spontaneous breaking of a full flavor symmetry $G_f^{(q)}$ at a high energy scale. In this work, for simplicity, we assume that $G_f^{(q)}$ and $G_d$ are discrete subgroups of $SU(3)$ and that $G_d$ is $Z_m$. Then $S_D$ satisfies

$$S_D^m = 1.$$

The phases in $S_D$ are described as

$$\phi_d \equiv 2\pi \frac{k_d}{m}, \quad \phi_s \equiv 2\pi \frac{k_s}{m}, \quad \phi_b \equiv 2\pi \frac{k_b}{m},$$

where $k_d, k_s, k_b$ are integers.
where \((k_d, k_s, k_b)\) represent \(Z_m\) charges and are restricted to be

\[
k_d + k_s + k_b \equiv 0 \mod m,
\]

because of the assumption \(G_d \subset SU(3)\). Similarly, in the up quark mass term, we postulate \(G_u = Z_n\) and define \(Z_n\) transformations as

\[
U_L \rightarrow TU_L, \quad U_R \rightarrow TU_R,
\]

where

\[
T \equiv \text{diag}\{e^{i\phi_u}, e^{i\phi_c}, e^{i\phi_t}\},
\]

\[
T^n = \mathbb{1},
\]

\[
\phi_u \equiv 2\pi \frac{k_u}{n}, \quad \phi_c \equiv 2\pi \frac{k_c}{n}, \quad \phi_t \equiv 2\pi \frac{k_t}{n},
\]

and, from the assumption \(G_u \subset SU(3)\),

\[
k_u + k_c + k_t \equiv 0 \mod n.
\]

In order for both \(G_u\) and \(G_d\) to be residual symmetries of \(G_f(q)\), products of \(S_D\) and \(T\) must also belong to \(G_f(q)\) and have a finite order. Hence, we here introduce a new element

\[
W \equiv S_D T,
\]

and require that \(W\) is an element of \(Z_p\), leading to

\[
W^p = (S_D T)^p = \mathbb{1}.
\]

Thus, in total, \(G_f(q)\) should contain three elements, \(S_D, T\) and \(W\), satisfying

\[
S_D^m = T^n = W^p = \mathbb{1},
\]

and groups composed of such elements are called von Dyck group. The groups have the finite number of elements if \(1/n + 1/m + 1/p > 1\), whereas it is infinite in the case of \(1/n + 1/m + 1/p \leq 1\).

The requirement of \(W^p = \mathbb{1}\) gives two constraints on the CKM matrix elements. In order to demonstrate that, let us consider the trace of \(W\). On one hand, \(\text{tr}[W]\) can directly be written down with \(W = V_{\text{CKM}} S_0 V^\dagger_{\text{CKM}} T\). On the other hand, \(\text{tr}[W]\) is equal to a sum of the eigenvalues of \(W\). From \(W^p = \mathbb{1}\), the sum, \(a\), is given by

\[
a = e^{2\pi i q_1/p} + e^{2\pi i q_2/p} + e^{2\pi i q_3/p}, \quad q_1 + q_2 + q_3 \equiv 0 \mod p,
\]

\[\text{Note that since Eqs. (2.9) and (2.14) lead to } \det S_0 = \det S_D = \det T = 1, \text{ these } Z_m \text{ and } Z_n \text{ symmetries are anomaly free (see Ref. } [12] \text{ and references therein).} \]
where \((q_1, q_2, q_3)\) represent \(Z_p\) charges. As a result, one obtains

\[
tr[W] = e^{i\phi_u} \left[ |V_{ud}|^2 e^{i\phi_d} + |V_{us}|^2 e^{i\phi_s} + |V_{ub}|^2 e^{i\phi_b} \right] \\
+ e^{i\phi_c} \left[ |V_{cd}|^2 e^{i\phi_d} + |V_{cs}|^2 e^{i\phi_s} + |V_{cb}|^2 e^{i\phi_b} \right] \\
+ e^{i\phi_t} \left[ |V_{td}|^2 e^{i\phi_d} + |V_{ts}|^2 e^{i\phi_s} + |V_{tb}|^2 e^{i\phi_b} \right] = a, \tag{2.19}
\]

where \(V_{ij}\) stands for the \(ij\) element of the CKM matrix. Since \(a\) is a complex parameter, Eq. (2.19) generally yields two constraints on the CKM matrix elements with a fixed integer set \((n, k_u, k_c, m, k_d, k_s, p, q_1, q_2)\). Since the CKM matrix is characterized by the four parameters \(\theta_{12}, \theta_{13}, \theta_{23}, \delta\), the above two constraints cannot determine the whole CKM matrix. However, by setting some of the CKM parameters by hand, one can predict the remaining CKM parameters. In the following section, under several cases, we systematically search the integer sets \((n, k_u, k_c, m, k_d, k_s, p, q_1, q_2)\), which predict the experimental data.

### 3 QUARK MIXING

#### 3.1 \(\theta_{13} = \theta_{23} = 0\) case

By fixing at least two of the four CKM parameters by hand, one can predict the remaining CKM parameters with Eq. (2.19) for given \((n, k_u, k_c, m, k_d, k_s, p, q_1, q_2)\). Since \(\theta_{13}\) and \(\theta_{23}\) are much smaller than \(\theta_{12}\), they are often taken to be vanishing at zeroth order in some flavor models. We here adopt the same stance and concentrate on predicting the Cabibbo angle \(\theta_{12}\). Note that we do not aim at precisely reproducing the Cabibbo angle and others. Small perturbations are expected to occur, because a breaking scale of the full flavor symmetry is supposed to be very high.

In the case of \(\theta_{13} = \theta_{23} = 0\), the Dirac phase \(\delta\) also disappears from the CKM matrix, then Eq. (2.19) is reduced to be

\[
e^{i(\phi_u + \phi_d)} + e^{i(\phi_s + \phi_b)} + e^{i(\phi_t + \phi_c)} - s_{12}^2(e^{i\phi_d} - e^{i\phi_s})(e^{i\phi_d} - e^{i\phi_s}) = a. \tag{3.1}
\]

This yields two conditions on \(\theta_{12}\):

\[
\sin^2 \theta_{12} = \frac{-\text{Re}[a] + \cos(\phi_b + \phi_t) + \cos(\phi_d + \phi_u) + \cos(\phi_s + \phi_c)}{\cos(\phi_d + \phi_u) - \cos(\phi_d + \phi_c) - \cos(\phi_s + \phi_u) + \cos(\phi_s + \phi_c)}, \tag{3.2}
\]

from the real part while

\[
\sin^2 \theta_{12} = \frac{-\text{Im}[a] + \sin(\phi_b + \phi_t) + \sin(\phi_d + \phi_u) + \sin(\phi_s + \phi_c)}{\sin(\phi_d + \phi_u) - \sin(\phi_d + \phi_c) - \sin(\phi_s + \phi_u) + \sin(\phi_s + \phi_c)}, \tag{3.3}
\]

from the imaginary part. In general, these conditions are independent of each other and must simultaneously be satisfied. However, in some cases, restrictions on \(\theta_{12}\) become reduced. For example, when the denominator of Eq. (3.2) [or Eq. (3.3)] is zero, the real (or the imaginary) part of Eq. (3.1) becomes free from \(\theta_{12}\), which implies that Eq. (3.2)
[or Eq. (3.3)] gives no constraint on $\theta_{12}$. Furthermore, in certain cases, both conditions, Eqs. (3.2) and (3.3), provide no constraints on $\theta_{12}$, e.g. the denominators are vanishing in both conditions; we will omit such cases in what follows.

We perform a numerical search for a possible integer set $(n, k_u, k_c, m, k_d, k_s, p, q_1, q_2)$ up to $(n, m, p) = (6, 6, 12)$

Figure 1 shows $\sin \theta_{12}$ near the experimental value as a function of $p$. We also list the seven nearest solutions in Table 1 and find that all the solutions are within the region of $0.21 < \sin \theta_{12} < 0.24$. In the table, only the case of $(n, m, p) = (2, 2, 7)$ can be a finite von Dyck group, which is $D_7$ and discussed in Ref. [13], while all the other cases compose infinite von Dyck groups. Some of the latter cases may be embedded into finite subgroups of infinite von Dyck groups [14] [10].

Let us study more on the case with $n = m = 2$. We consider a special case with $n = m = 2$, $k_u = k_s = q_1 = 0$, and $k_c = k_d = 1$. Then, the mixing angle in Eq. (3.2) can be expressed by a simple form of $\sin^2 \theta_{12} = \cos^2 (\pi q_2 / p)$. The mixing angle in this form can be derived by the finite group $\Delta(6N^2)$ as discussed in Sec. 4. Experimentally, the center value $\sin \theta_{12}$ is 0.225, so we need $q_2/p = 0.428$. Also, the range $\sin \theta_{12} = [0.224, 0.226]$ corresponds to $q_2/p = [0.427, 0.428]$. To express this value approximately by integer sets, the combination of $p = 7$ and $q_2 = 3$ is the simplest, i.e. $3/7 = 0.4286$, as it appeared in our numerical analysis. For $p < 89$, there is no integer to fit better than $p = 7$ except $(p, q_2) = (7r, 3r)$. Taking greater number for $p$, we can better fit to the experimental value, for instance $(p, q_2) = (89, 38), (96, 41), (103, 44)$, i.e. $38/89 = 0.4270, 41/96 = 0.4271, 44/103 = 0.4272$. Thus, $p = 7$ is much simpler to fit the experimental value in this scheme and it seems that the $Z_{7r}$ symmetry is favorable.

![Figure 1: Distribution of $\sin \theta_{12}$ as a function of $p$ up to $(n, m, p) = (6, 6, 12)$ while setting $\theta_{23} = \theta_{13} = 0$. The bold line is the experimental value, $\sin \theta_{12} = 0.225$.](image)

\footnote{When $n = 2$ or $m = 2$, one needs to introduce an additional $Z_2$ symmetry to keep the mass matrix diagonal in their diagonal basis. The additional $Z_2$ can be chosen not to affect $\theta_{12}$.}
Table 1: While setting $\theta_{23} = \theta_{13} = 0$ and up to $(n, m, p) = (6, 6, 12)$, we pick out seven solutions close to the experimental value; all of them are within the range of $0.21 < \sin \theta_{12} < 0.24$. Integer sets which give the same $\sin \theta_{12}$ displayed here are omitted.

| $\sin \theta_{12}$ | $n$ | $m$ | $p$ | $k_u$ | $k_c$ | $k_d$ | $k_s$ | $q_1$ | $q_2$ | $(\text{Re}[a], \text{Im}[a])$ |
|-------------------|----|----|----|------|------|------|------|------|------|-----------------------------|
| 0.223             | 2  | 2  | 7  | 0    | 1    | 1    | 0    | 0    | 3    | $(-0.802, 0)$               |
| 0.211             | 2  | 5  | 6  | 0    | 1    | 1    | 4    | 1    | 2    | $(1.00, 1.73)$              |
| 0.216             | 3  | 5  | 7  | 1    | 2    | 1    | 4    | 0    | 3    | $(-0.802, 0)$               |
| 0.216             | 3  | 5  | 12 | 0    | 1    | 3    | 2    | 7    | 8    | $(-1.37, -2.37)$            |
| 0.237             | 3  | 6  | 11 | 0    | 1    | 1    | 3    | 0    | 2    | $(1.83, 0)$                 |
| 0.221             | 4  | 4  | 10 | 0    | 1    | 1    | 0    | 2    | 3    | $(1.00, 1.90)$              |
| 0.237             | 5  | 5  | 8  | 0    | 2    | 3    | 0    | 0    | 3    | $(-0.414, 0)$               |

3.2 $\theta_{13} = 0$ case

Next, we lift the restriction of $\theta_{23} = 0$. Then, the expression of $\text{tr}[W]$ is given as follows:

$$
e^{i(\phi_u + \phi_d)} + e^{i(\phi_c + \phi_s)} + e^{i(\phi_t + \phi_b)} - s_{12}^2(e^{i\phi_u} - e^{i\phi_c})(e^{i\phi_d} - e^{i\phi_s})$$

$$- s_{23}^2(e^{i\phi_c} - e^{i\phi_t})(e^{i\phi_s} - e^{i\phi_b}) - s_{12}^2 s_{23}^2(e^{i\phi_c} - e^{i\phi_t})(e^{i\phi_d} - e^{i\phi_s}) = a. \quad (3.4)$$

As in Sec. 3.1, we focus only on solutions in which $\theta_{12}$ and $\theta_{23}$ are uniquely determined. In addition, we here omit solutions predicting $\theta_{23} = 0$ since they are included in the results of Sec. 3.1.

Similar to Sec. 3.1, we search for possible integer sets up to $(n, m, p) = (6, 6, 12)$. We plot $\sin \theta_{23}$ as a function of $\sin \theta_{12}$ in Fig. 2 and pick out seven solutions close to the experimental values in Table 2. All the solutions in Table 2 satisfy $0.15 < \sin \theta_{12} < 0.30$ and $0.05 < \sin \theta_{23} < 0.10$. From Fig. 2, it can be seen that $\sin \theta_{12}$ can be near the experimental value, whereas most of the obtained $\sin \theta_{23}$ are much larger than its experimental value. Moreover, solutions which are marginally consistent with the experimental value of $\sin \theta_{23}$ are obtained for relatively large $(n, m, p)$. These tendencies may suggest that larger $(n, m, p)$ are necessary for reproducing a realistic $\sin \theta_{23}$ as well as $\sin \theta_{12}$.

3.3 Setting $\theta_{13}$ and $\delta$ to the experimental values

In the previous subsections, we set the small CKM angles to zero. We here consider a more realistic case by setting $\theta_{13}$ and $\delta$ to their experimental values: $(\sin \theta_{13}, \cos \delta) = (0.00341, 0.355)$. The results up to $(n, m, p) = (6, 6, 12)$ are displayed in Fig. 3. Comparing with Fig. 2, one can find that some of the solutions appear in both figures. In

3 However, the opposite is not always true. For example, the solution of $(n, m, p) = (2, 2, 7)$ in Table 1 does not derive $\theta_{23} = 0$ in Eq. (3.3). This is because the terms associated with $\theta_{23}$ are dropped from Eq. (3.4), and thus $\theta_{23}$ is not determined.
Figure 2: Distribution of $\sin \theta_{23}$ as a function of $\sin \theta_{12}$ up to $(n, m, p) = (6, 6, 12)$ while setting $\theta_{13} = 0$. Their experimental values are depicted as the large red point at $(\sin \theta_{12}, \sin \theta_{23}) = (0.225, 0.0412)$.

| $(\sin \theta_{12}, \sin \theta_{23})$ | $n$ | $m$ | $p$ | $k_u$ | $k_c$ | $k_d$ | $k_s$ | $q_1$ | $q_2$ | $(\text{Re}[a], \text{Im}[a])$ |
|-----------------------------|-----|-----|-----|-------|-------|-------|-------|-------|-------|-----------------------------|
| $(0.206, 0.0666)$          | 4   | 5   | 11  | 0     | 1     | 4     | 2     | 6     | 7     | $(-1.20, -1.95)$            |
| $(0.206, 0.0652)$          | 5   | 4   | 11  | 1     | 3     | 0     | 3     | 2     | 4     | $(-1.20, 1.95)$             |
| $(0.203, 0.0651)$          | 5   | 4   | 11  | 4     | 2     | 0     | 1     | 6     | 7     | $(-1.20, -1.95)$            |
| $(0.291, 0.0588)$          | 5   | 6   | 11  | 1     | 3     | 0     | 5     | 2     | 4     | $(-1.20, 1.95)$             |
| $(0.198, 0.0950)$          | 5   | 6   | 11  | 2     | 0     | 2     | 1     | 1     | 2     | $(1.11, 0.460)$             |
| $(0.291, 0.0615)$          | 6   | 5   | 11  | 0     | 1     | 4     | 2     | 6     | 7     | $(-1.20, -1.95)$            |
| $(0.198, 0.0953)$          | 6   | 5   | 11  | 2     | 1     | 2     | 0     | 1     | 2     | $(1.11, 0.460)$             |

Table 2: While setting $\theta_{13} = 0$ and up to $(n, m, p) = (6, 6, 12)$, we pick out seven solutions close to the experimental values. The solutions are all within $0.15 < \sin \theta_{12} < 0.30$ and $0.05 < \sin \theta_{23} < 0.10$. Solutions predicting $\sin \theta_{23} = 0$ are excluded.
addition to such solutions, there also exist solutions which do not appear in Fig. 2. Seven solutions close to the experimental values are picked up in Table 3. It seems that the tendency for large groups to generate realistic values still holds, while the combination $(p, q_2) = (7, 3)$ is included as one of the simplest ones again. Nevertheless, as variety has come to the solution, one can find favorable solutions more easily. Note that similar results are obtained by fixing two other CKM parameters.

Figure 3: Legend is the same as Fig. 2, but for $(\sin \theta_{13}, \cos \delta) = (0.00341, 0.355)$.

| $(\sin \theta_{12}, \sin \theta_{23})$ | $n$ | $m$ | $p$ | $k_u$ | $k_c$ | $k_d$ | $k_s$ | $q_1$ | $q_2$ | $(\text{Re}[a], \text{Im}[a])$ |
|---------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----------------|
| $(0.223, 0.00341)$             | 4   | 4   | 7   | 1   | 3   | 1   | 3   | 0   | 3   | ($-0.802, 0$)   |
| $(0.221, 0.000522)$            | 4   | 4   | 10  | 1   | 2   | 0   | 3   | 2   | 3   | ($-1.00, 1.90$) |
| $(0.273, 0.0408)$              | 2   | 5   | 12  | 0   | 1   | 2   | 3   | 1   | 5   | ($-1.00, 1.00$) |
| $(0.273, 0.0563)$              | 5   | 4   | 10  | 2   | 4   | 1   | 3   | 5   | 7   | ($-1.00, -1.90$) |
| $(0.271, 0.0898)$              | 5   | 6   | 10  | 1   | 2   | 2   | 1   | 5   | 6   | ($-1.00, -1.18$) |
| $(0.291, 0.0173)$              | 2   | 6   | 11  | 0   | 1   | 1   | 2   | 0   | 2   | ($1.83, 0$)     |
| $(0.288, 0.0159)$              | 5   | 6   | 11  | 1   | 4   | 5   | 1   | 0   | 1   | ($2.68, 0$)     |

Table 3: Legend is the same as Table 2, but for $(\sin \theta_{13}, \cos \delta) = (0.00341, 0.355)$.
3.4 $m = 2$ case

Lastly, we consider the case of $m = 2$ as the situation is slightly different from the other cases. In this case, Eq. (2.19) turns out to be

$$|V_{ui}|^2 = \frac{\text{Re}[a] \cos \frac{\phi_u + \phi_t}{2} + \cos \frac{\phi_u + \phi_t - 2\phi_u}{2} + \text{Im}[a] \sin \frac{\phi_u + \phi_t}{2}}{4 \sin \frac{\phi_u - \phi_t}{2} \sin \frac{\phi_u - \phi_t}{2}},$$

$$|V_{ci}|^2 = \frac{\text{Re}[a] \cos \frac{\phi_u + \phi_t}{2} + \cos \frac{\phi_u + \phi_t - 2\phi_u}{2} + \text{Im}[a] \sin \frac{\phi_u + \phi_t}{2}}{4 \sin \frac{\phi_u - \phi_t}{2} \sin \frac{\phi_u - \phi_t}{2}},$$

$$|V_{ti}|^2 = \frac{\text{Re}[a] \cos \frac{\phi_u + \phi_t}{2} + \cos \frac{\phi_u + \phi_t - 2\phi_u}{2} + \text{Im}[a] \sin \frac{\phi_u + \phi_t}{2}}{4 \sin \frac{\phi_u - \phi_t}{2} \sin \frac{\phi_u - \phi_t}{2}},$$

as derived in Ref. [10] for the lepton sector. As a result, one can directly constrain the $i$th column of the CKM matrix without inputting the CKM parameters by hand. Note that the unitary condition, $|V_{ui}|^2 + |V_{ci}|^2 + |V_{ti}|^2 = 1$, has been used, and the subscript $i$ reflects the position of a positive sign in $S_0$, e.g. $S_0 = \text{diag} \{1, -1, -1\}$ for $i = d$. In the following, we take $i = s$, for the experimental values of the corresponding column are slightly large.

Results up to $(n, p) = (6, 12)$ are similar to those in Sec. 3.3. Thus, we here vary $(n, p)$ up to $(20, 20)$ and show that large groups can derive a much more realistic CKM mixing. For each $n$, in Fig. 4 we pick out a solution which best reproduces the observed CKM mixing. Furthermore, we select the seven nearest solutions from Fig. 4 and list them in Table 4. As can be seen, one can easily find solutions near the experimental values with large $n$ and $p$.

| $(|V_{us}|, |V_{cs}|, |V_{ts}|)$ | $n$ | $p$ | $k_u$ | $k_c$ | $q_1$ | $q_2$ | $(\text{Re}[a], \text{Im}[a])$ |
|-----------------|-----|-----|------|------|-----|-----|----------------|
| (0.269, 0.962, 0.0444) | 7   | 19  | 0    | 2    | 4   | 5   | (-0.823, 1.80) |
| (0.218, 0.975, 0.0510) | 12  | 19  | 3    | 8    | 11  | 12  | (-1.31, -2.18) |
| (0.248, 0.968, 0.0322) | 13  | 19  | 5    | 7    | 10  | 11  | (-1.08, -1.25) |
| (0.190, 0.981, 0.0468) | 15  | 13  | 1    | 10   | 7   | 9   | (-1.21, -2.17) |
| (0.275, 0.960, 0.0444) | 17  | 18  | 5    | 11   | 10  | 11  | (-1.21, -1.85) |
| (0.215, 0.975, 0.0507) | 18  | 20  | 6    | 11   | 11  | 12  | (-1.17, -1.71) |
| (0.219, 0.975, 0.0345) | 19  | 15  | 6    | 1    | 1   | 2   | (1.89, 0.199)   |

Table 4: From Fig. 4, we pick out seven solutions close to the experimental values.

4 EMBEDDING INTO $\Delta(6N^2)$

We have considered the von Dyck group. Constraining to some special cases, it is possible to embed the obtained residual symmetries into a finite group, for example, $\Delta(6N^2)$. We
Figure 4: $|V_{us}|$, $|V_{cs}|$, and $|V_{ts}|$ are plotted as a function of $n$ up to $(n, p) = (20, 20)$ while assuming $m = 2$. The black diamonds, blue squares, and red circles represent $|V_{us}|$, $|V_{cs}|$, and $|V_{ts}|$, respectively. The horizontal lines describe the experimental values: $|V_{us}|$, $|V_{cs}|$, $|V_{ts}| = (0.225, 0.973, 0.0404)$. For each $n$, we pick out a solution which derives the nearest mixing to the experimental values.

stress that this group is also useful to provide the mixing angles of leptons under the experimentally allowed region.

In this section, we focus on the case with $\theta_{13} = \theta_{23} = 0$ as studied in Sec. 3.1. For the purpose of embedding residual symmetries into a smaller $\Delta(6N^2)$ series, we redraw the plots in Fig. 4 as a function of the least common multiples (LCMs) of $(n, m, p)$ to Fig. 5.

For example, in Figure 5, the solution with LCM$(n, m, p) = 12$ corresponds to

$$(\sin \theta_{12}, n, m, p) = (0.259, 2, 2, 12)$$

which is also found in Table I of Ref. [10]. This solution can be derived by $\Delta(6 \times 12^2)$ and it realizes the value by $\sin \theta_{12} = \frac{1}{2} |e^{2\pi i/12} + e^{\pi i}|$. The detail of this derivation is explained later. Another example is LCM$(n, m, p) = 14$ with $\sin \theta_{12} = 0.223$, and we can derive it by $\Delta(6 \times 7^2)$ with the value of $\sin \theta_{12} = \frac{1}{2} |e^{2\pi i/7} + e^{8\pi i/7}|$.

At first, let us simply review the group theory of $\Delta(6N^2)$. [15, 4]. Elements of this group are given by the products of $a, a', b$, and $c$ satisfying $a^N = a'^N = b^3 = c^2 = e$. In addition, they obey $b a^x a^y b^2 = a^{-x+y} a^{x-y}$, $b^2 a^x a^y b = a^{-y} a^{x+y}$, and $c b^x c = b^{2x}$. All the group elements are expressed by $g = b^x c^y a^z d^y$.

To derive the Cabibbo angle, we use the $Z_2$ elements of $\Delta(6N^2)$, namely, elements satisfying $g^2 = e$. We can write down $Z_2$ elements for fixed $z$ and $w$. For $(z, w) = (0, 0)$, $(1, 0)$, and $(2, 0)$, $g^2$ is given as $a^{2x} d^{2y}$, $b^2 a^{x-y} d^x$, and $b d^{y-x}$, respectively. Then $Z_2$ elements exist in $(z, w) = (0, 0)$, which are $a^{N/2}$, $a'^{N/2}$, and $a^{N/2} a'^{N/2}$ when $N$ is even.
Figure 5: Distribution of $\sin \theta_{12}$ as a function of the least common multiples of $(n, m, p)$ up to $(6, 6, 12)$ in the case of $\theta_{13} = \theta_{23} = 0$.

Similarly, for $(z, w) = (0, 1), (1, 1), \text{and} (2, 1)$, $g^2$ is given as $g^2 = a^{x-y}a^{y}, a^{y}a^{2y}$, and $a^{2x}a^{x}$, respectively. Then $Z_2$ elements are written by $ca^{x}a^{y}, bca^{x}$, and $b^2ca^{y}$.

Here we identify $Z_2$ symmetries in down and up quark mass terms with $bca^{x}$ and $bca^{y}$, respectively. In the diagonalizing basis of the up quark mass terms, matrix representations are expressed as

$$S_D = \begin{pmatrix}
\cos \frac{2\pi}{N}(x - y) & i \sin \frac{2\pi}{N}(x - y) & 0 \\
-i \sin \frac{2\pi}{N}(x - y) & -\cos \frac{2\pi}{N}(x - y) & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (4.2)$$

and $T = \text{diag}\{1, -1, -1\}$. Setting $S_0 = \text{diag}\{-1, 1, -1\}$, the CKM matrix is obtained as

$$V_{\text{CKM}} = \begin{pmatrix}
(1 - \rho^{x-y})/2 & (1 + \rho^{x-y})/2 & 0 \\
(1 + \rho^{x-y})/2 & (1 - \rho^{x-y})/2 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (4.3)$$

where $\rho = e^{2\pi i/N}$. Since the overall phase is irrelevant, we obtain $\sin \theta_{12} = \cos \frac{\pi(x - y)}{N}$. This formula is the same as the one derived in Sec. 3.1, i.e., $\sin \theta_{12} = \cos(\pi q_2/p)$. When $N = 7$ and $x - y = 3$, it yields $|V_{us}| = 0.223$, which is very close to the experimental value. As commented in Sec. 3.1, there is no integer $N$ for $N < 89$, which fits better than $N = 7$ except $(N, x - y) = (7r, 3r)$. Thus, the $\Delta(6N^2)$ with $N = 7r$ seems favorable. Figure 6 shows $\sin \theta_{12}$ for several values of $N$ and $x - y$.

Next, we consider mixing angles of the lepton sector by using other group elements. We find that $Z_3$ elements are useful to explain large mixing angles of leptons. Explicitly, $Z_3$ elements of the group are $a^{iN/3}a^{jN/3}, ba^{x}a^{y}$, and $b^2a^{x}a^{y}$ where $i, j = 0, 1, 2$, and $N/3$.
Figure 6: $\sin \theta_{12}$ versus $N$ are plotted. $N$ is the number of $\Delta(6N^2)$ that is taken from 3 to 20. The continuous line describes the experimental value, $\sin \theta_{12} = 0.225$.

is assumed to be the integer. This time, we take the charged lepton $Z_3$ symmetry as $T' \equiv ba^{-x}a'^{-x-y}$. This representation can be written and diagonalized by

$$T' = \begin{pmatrix} 0 & \rho^{-y} & 0 \\ 0 & 0 & \rho^{x+y} \\ \rho^{-x} & 0 & 0 \end{pmatrix} = U_{T'}, \quad U_{T'} = \frac{1}{\sqrt{3}} \begin{pmatrix} \rho^x & \omega \rho^{x+y} & \omega^2 \rho^{x+y} \\ \rho^{x+y} & \omega \rho^{x+y} & \omega \rho^{x+y} \\ 1 & 1 & 1 \end{pmatrix},$$

(4.4)

where $\omega = e^{2\pi i/3}$. For the neutrino $Z_2$ symmetry, we choose $S' \equiv b^2 ca'^{x'}$ so that

$$S' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\rho^{x'} \\ 0 & -\rho^{x'} & 0 \end{pmatrix} = U_{S'}, \quad U_{S'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \rho^{x'}/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & \rho^{x'}/\sqrt{2} \end{pmatrix}.$$

(4.5)

Then the mixing matrix of the lepton sector becomes

$$U_{PMNS} = U_{T'}^\dagger U_{S'} = \begin{pmatrix} \frac{1 - \rho^{-x-y-x'}}{\sqrt{6}} & \frac{\rho^x}{\sqrt{6}} & \frac{\rho^{-x-y} + \rho^x}{\sqrt{6}} \\ \frac{1 - \omega \rho^{-x-y-x'}}{\sqrt{3}} & \frac{\omega^2 \rho^x}{\sqrt{6}} & \frac{\omega \rho^{-x-y} + \rho^x}{\sqrt{6}} \\ \frac{1 - \omega^2 \rho^{-x-y-x'}}{\sqrt{3}} & \frac{\omega \rho^{-x-y} + \rho^x}{\sqrt{3}} & \frac{\omega^2 \rho^{-x-y} + \rho^x}{\sqrt{3}} \end{pmatrix}.$$  

(4.6)

The same mixing matrix has been analyzed by [17, 18] and it predicts the sum rules $\theta_{23} \approx 45^\circ \mp \theta_{13}/\sqrt{2}$. Also, it realizes the trimaximal mixing $\sin^2 \theta_{12} \approx 1/3$ for small $\theta_{13}$. In particular, since $\sin \theta_{13}$ can be written as $\sqrt{2} \cos(\pi(x+y+x')/N)/\sqrt{3}$, small $\theta_{13}$ is realized by $(x+y+x')/N \approx 1/2$. When $N = 7$ and $x+y+x' = 3$, we have $\sin^2 \theta_{13} = 0.128$ which is close to the experimental value. As a result, the $\Delta(6N^2)$ for $N = 7r$ is interesting to realize the mixing angles for both the quark and lepton sectors.
5 SUMMARY

We have applied the model-independent formalism, which was recently developed by Hernandez and Smirnov, to the quark sector and sought possible residual symmetries with a focus on the von Dyck groups. In the case of $\theta_{13} = \theta_{23} = 0$, the Cabibbo angle $\theta_{12}$ can be close to its experimental values for small $n$, $m$, and $p$. In particular, the combination between the $Z_2$ and $Z_7$ symmetries seems favorable to realize the realistic value of $\theta_{12}$. Furthermore, these residual symmetries can originate from finite groups such as $D_N$ and $\Delta(N^2)$ with $N = 7r$.

Indeed, we have also discussed possibilities of embedding the obtained residual symmetries into the $\Delta(N^2)$ series. It is found that $\Delta(N^2)$ for $N = 7r$ would be favorable to realize the Cabibbo angle and also interesting from the viewpoint of the mixing angles for the lepton sector.

In contrast, relatively large $n$, $m$, and $p$ are needed in order to reproduce all the quark mixing parameters at the same time. The von Dyck groups with such large integers correspond to not finite groups, but infinite ones. However, they may be embedded into finite subgroups of infinite von Dyck groups (see e.g. Ref. [14]). The work is in progress, and we would like to postpone this issue to our next publication. At any rate, we have shown which combinations of residual symmetries lead to favorable results. That can become a starting point to investigate the full flavor symmetry hiding behind the quark and lepton mass matrices.

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