Models of Contractions

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Abstract. Massless, spinning particles on AdS can be thought of as composite objects consisting of Rac's and Di's. A better geometric understanding of this idea may be useful for furthering physical applications. We describe a model associated with contractions of massless representations of $SO(2,3)$ into the Poincaré group which may be useful for this purpose.

1. Deformations and Contractions

Let $G = (\mathcal{V}, [\cdot, \cdot])$ be an $n$-dimensional Lie algebra with underlying vector space $V$ over $\mathbb{R}$ or $\mathbb{C}$ and Lie bracket $[\cdot, \cdot]$. Relative to a fixed basis $\{e_1, e_2, \ldots, e_n\}$ of $V$, we have $[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$, where $c_{ij}^k$ are the structure constants of $G$. Let $\mathcal{M}_n$ be the space of all such Lie structures on $V$ i.e. the space of all equivalence classes of bilinear mappings

$$\mu : V \times V \to V \quad \left( [x, y] = \mu(x, y) \right)$$

such that $\mu(x, y) = -\mu(y, x)$ and

$$\sum_{\text{cyc}(x,y,z)} \mu(x, \mu(y, z)) = 0 \quad \text{(Jacobi identity)}$$

where two such maps are equivalent if they give isomorphic Lie algebras.

Let $\Lambda \in \mathbb{R}^+$, then a deformation of a given Lie algebra $G$ over $\mathbb{R}$ is a continuous mapping $\psi : [0, \Lambda] \to \mathcal{M}_n$ with $G = (V, \psi(0))$, i.e. $\psi(0)$ is the Lie structure of $G$, where continuous means continuous in the topology $\mathcal{M}_n$ inherits from the inclusion of the space of structure constants in $\mathbb{R}^{n^3}$ (i.e. in the Segal topology) [1]. $\psi$ is a trivial deformation if $\psi(t)$ for all $t \in [0, \Lambda]$ is equivalent to $\psi(0)$. Semisimple Lie algebras are rigid, i.e. they have no nontrivial deformations [2].

The process of deformation of a Lie algebra may be viewed, at least in special circumstances, as the inverse of contraction of a Lie algebra, which idea goes back to Segal [2] and Inönü and Wigner [3] and formalized by Saletan [4]. We now turn to this notion of contraction of a Lie algebra. Let $(\phi_{\lambda})_{\lambda \in \mathbb{R}^+}$ be a continuous family of surjective mappings $\phi_{\lambda} : V \to W$ where $W$ is another vector space of the same dimension as $V$. If the $\phi_{\lambda}$ are injective, we may define a new bracket on $V$ by

$$[x, y]_{\lambda} = \phi_{\lambda}^{-1}([\phi_{\lambda}(x), \phi_{\lambda}(y)]) \quad \forall \epsilon \in (0, 1], \quad \forall \ x, y \in V.$$
Set $G_\lambda = (V, [\cdot, \cdot]_\lambda)$ and assume that $G_1 = G$ so that $\phi_1$ defines an isomorphism of $G$ onto its image. If

$$\lim_{\lambda \to 0} [x, y]_\lambda := [x, y]_0$$

exists for all $x, y \in V$ then it defines a (possibly) new Lie algebra and we call this new Lie algebra, $G_0 = (V, [\cdot, \cdot]_0)$, a contraction of the Lie algebra $G$ or (simply) the \textit{contracted of} $G$.

In\-öni-\-Wigner (IW) contractions are a particular case of the above and are defined as follows. Let $H$ be a subalgebra of $G = (V, [\cdot, \cdot])$. Let $\mathcal{P} \subset G$ be the subspace complementary to $H$. Let Proj($H$) and Proj($\mathcal{P}$) be the projections onto $H$ and $\mathcal{P}$, respectively. Define $\phi_\lambda : \mathbb{R}^+ \to GL(V)$ by

$$\phi_\lambda = \text{Proj}(H) + \lambda \text{Proj}(\mathcal{P}) \ (\lambda \in \mathbb{R}^+).$$

Then $G_0$ as defined above in eqns. (1) and (2) exists and contains $\mathcal{P}$ as an abelian ideal with $H = G_0/\mathcal{P}$ so that

$$G_0 = H \ltimes \mathcal{P}$$

where $\ltimes$ denotes semidirect product of Lie algebras. Generalizations of IW contractions are given in [4], [5], [6], [7]. For attempts at classifications we refer to the recent work of Nesterenko and Popovych [8] and a criticism of their results in [9].

For contractions of Lie groups we recall the papers by Mickelsson and Niederle [10] and Dooley and Rice [11]. We follow [11]. Let $G$ be a noncompact semisimple Lie group $G$ with finite center. The Lie algebra of $G$ is $\mathfrak{g}$. Let $H$ be a closed subgroup with subalgebra $\mathfrak{h}$. The coset space $G/H$ is reductive if $H$ admits an $Ad_G(H)$ invariant complement $V$ in $\mathfrak{g}$. In this case, we can form the semidirect product $V \ltimes_s H$ with respect to the adjoint action of $H$ on $V$. In order to construct the global counterpart of $\phi_\lambda$, we define a family of mappings $\Phi_\lambda : V \times H \to G$ by $\Phi_\lambda(vh) = \exp_G(\lambda v) h$ for $\lambda \in \mathbb{R}^+$ where $\exp_G$ means exponential mapping from $G$ to $G$. Then the differential of the map $\Phi_\lambda$ at the identity is $\phi_\lambda$ of the IW contraction. Furthermore the Lie algebra of $V \ltimes_s H$ is precisely $G_0$ of the IW contraction [11]. For the case of $H$ equal to $K$, the maximal compact subgroup of $G$, $V \ltimes_s H$ is called the \textit{Cartan motion group} associated to $G$.

Now we turn to contractions of representations. We start with an infinitesimally unitarizable representation $(d\pi(G), H)$ of $G = (W, [\cdot, \cdot])$ on a fixed Hilbert space $H$ and a family of closed invertible linear transformations $(\Pi_\lambda)_{\lambda \in \mathbb{R}^+}$ of $H$ with $\Pi_1$ being the identity operator on $H$. We define a representation of $G_\lambda$ on $H$:

$$G_\lambda = (V, [\cdot, \cdot]_\lambda) : X \to d\pi_\lambda(X) = \Pi_\lambda^{-1} d\pi(\phi_\lambda(X)) \Pi_\lambda.$$  

For the representation condition we verify

$$[d\pi_\lambda(X), d\pi_\lambda(Y)] = \Pi_\lambda^{-1} [d\pi(\phi_\lambda(X)), d\pi(\phi_\lambda(Y))] \Pi_\lambda =$$

$$\Pi_\lambda^{-1} d\pi([\phi_\lambda(X), \phi_\lambda(Y)]) \Pi_\lambda = \Pi_\lambda^{-1} d\pi(\phi_\lambda([X, Y]_\lambda)) \Pi_\lambda = d\pi_\lambda([X, Y]_\lambda)$$

where in the last line we have made use of eqn. (1). The \textit{contracted} of the representation, $(d\pi(G), H)$ is:

$$d\pi_0(X) = \lim_{\lambda \to 0} d\pi_\lambda(X) \forall \ X \in G_0.$$  

At the group level, the analog of eqn. (7) is $\pi_0 = \lim_{\lambda \to 0} \pi \circ \Phi_\lambda$ which, at least in certain cases, can be given a precise meaning (cf. [11], [12], [13]).
2. The Conformal Group in n dimensions

Pseudo-Euclidean space in $n+2$ dimensions is $\mathbb{R}^{n+2}$ equipped with metric defined by quadratic form $Q(x) = x_0^2 + x_1^2 - x_2^2 - \ldots - x_n^2$ where $x = (x_0, x_1, x_2, \ldots , x_n) \in \mathbb{R}^{n+2}$.

We denote $n$ dimensional Minkowski space by $M_0$. Let $G = SO_0(2, n)$ denote the connected component of the group of linear transformations of $\mathbb{R}^{n+2}$ preserving the symmetric bilinear form which is associated to $Q(x)$ by polarization. We shall call $G$ the $n$-dimensional conformal group, and we denote the universal cover of $G$ by $G^\sim = SO_0(2, n)^\sim$. Let $\mathfrak{g}$ be the Lie algebra of $G$. $\mathfrak{g}$ is identified with the set of all matrices $(a_{ij})$ ($-1 \leq i, j \leq n$) such that $a_{ii} = 0$ $(0 \leq i \leq n)$, $a_{ij} = -a_{ji}$ $(1 \leq i \leq j \leq n)$, $a_{0,j} = a_{j,0}$ $(1 \leq j \leq n)$, $a_{-1,j} = a_{j,-1}$ $(1 \leq j \leq n)$ and $a_{-1,0} = -a_{0,-1}$. We define subalgebras $\mathfrak{k}$, $\mathfrak{k}'$, $\mathfrak{g}'$, $\mathfrak{a}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ as follows. Let $E_{ij}$ be the matrix such that the $(i,j)$ component is equal to 1 and the other components are all equal to 0. Let $L_{ij} = E_{ij} - E_{ji}$ $(1 \leq i \leq j \leq n)$, $L_{0i} = E_{i0} + E_{0i}$ $(1 \leq i \leq n)$, $L_{-1,i} = -E_{-1,i} - E_{i,-1}$ $(1 \leq i \leq n)$ and $L_{10} = L_{-1,0} = E_{-1,0}$. Let $\mathfrak{k}$ be the subalgebra spanned by: $L_{ij}$ $(1 \leq i, j \leq n - 1)$ and $L_{10}$; $\mathfrak{g}'$ be the subalgebra spanned by: $L_{ij}$ $(1 \leq i, j \leq n - 1)$, $L_{0i}$ $(1 \leq i \leq n - 1)$, $L_{-1,i}$ $(1 \leq i \leq n - 1)$ and $L_{1,0}$; $\mathfrak{g}$ be the subalgebra spanned by: $L_{ij}$ $(1 \leq i, j \leq n - 1)$; $\mathfrak{k}'$ be the subalgebra spanned by: $L_{ij}$ $(1 \leq i, j \leq n - 1)$, $L_{0i}$ $(1 \leq i \leq n - 1)$, $L_{-1,i}$ $(1 \leq i \leq n - 1)$; $A$ be the subalgebra spanned by $L_{-1,n}$; $N_+$ ($N_-$) be the subalgebra spanned by $P_i = \frac{1}{2}(L_{0,i} + L_{-1,i})$ $(0 \leq i \leq n - 1)$; $P_i = \frac{1}{2}(L_{0,i} - L_{-1,i})$ $(0 \leq i \leq n - 1)$. Denote the analytic subgroups of $G$ corresponding to $\mathfrak{k}$, $\mathfrak{k}'$, $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ by $K$, $H$, $\hat{H}$, $A$, $N_+$ and $N_-$, respectively. $H = SO_0(2, n - 1)$ ($\hat{H} = SO_0(1, n)$) is the anti-de Sitter (deSitter) subgroup, and we have an Iwasawa like decomposition of $G$ i.e. the map $H' \times A \times N_+ \longrightarrow G$ ($(H' \times A \times N_- \longrightarrow G)$ is an injective diffeomorphism onto an open, dense subset of $G$, where $H'$ is $SO_0(2, n - 1)$ ($H'$ is $SO(1, n)$) [14].

Consider the $n+1$ dimensional isotropic cone in $\mathbb{R}^{n+2}$ defined by
\[
C = \{ x \in \mathbb{R}^{n+2} | Q(x) = 0 \}.
\]

Let $\mathbb{R}^{n+2*}$ and $C^*$ be the sets of nonzero elements in $\mathbb{R}^{n+2}$ and $C$, respectively. Let $P \subset G$ be the stabilizer subgroup of $e = e_{-1} + e_0$ where $e_i = (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0)$ $\in \mathbb{R}^{n+2}$ (i.e. $e_i$ is the vector with 1 in the $i$th slot and zeros elsewhere). $P \cong SO_0(1, n - 1) \times_s N_+$, i.e. $P$ is the $n$ dimensional Poincaré group with Lie algebra $\mathfrak{p}$. The orbit of $e$ under $G$ is $C^*$. Hence $C^* \cong G/P$.

3. Results on Positive Energy Representations of the Conformal Groups

Let $\pi$ be a representation of $SO_0(2, n)^\sim$ and $d\pi$ be the associated representation of the Lie algebra, $so(2, n)$. $d\pi$ lifts to a representation of the enveloping algebra. Let $X$ stand for $d\pi(X)$ with $X \in \mathcal{E}(\mathfrak{g})$, the enveloping algebra of the complexification $\mathfrak{g}$ of $\mathfrak{g}$. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Delta$ be the root system of $(\mathfrak{g}, \mathcal{H})$. We define $\mathcal{G}^\Delta = \{ X \in \mathcal{G}[h, X] = \lambda(h)X \ \forall \ h \in \mathcal{H} \ \text{and} \ \lambda \in \Delta \}$. Set $\mathcal{N} = \Sigma_{\lambda>0} \mathcal{G}^\lambda$ and $\tilde{\mathcal{N}} = \Sigma_{\lambda<0} \mathcal{G}^\lambda$ and let $V$ be a $\mathcal{E}(\mathfrak{g})$ module with action $d\pi$ of $\mathcal{E}(\mathfrak{g})$ on $V$. Let: $a) \ V = \bigoplus_{\lambda} V_{\lambda}$ with $V_{\lambda} = \{ v \in V | H_{\lambda} v = \lambda(H_{\lambda}) v \};$ $b) \ \exists \ v_0 \in V \ \exists \ (i) X^+(X^-) v_0 = 0 \ \text{with} \ X^+ \in N \ (X^- \in \tilde{N});$ $\ (ii) H_{\lambda} v_0 = \lambda_0(H_{\lambda}) v_0;$ $\ (iii) \mathcal{E}(\mathfrak{g}) v_0 = V.$ If $d\pi$ comes from a unitary representation of $G^\sim$, then it is an infinitesimally unitarizable lowest (highest) weight representation with lowest (highest) weight $\lambda_0$.

An irreducible, infinitesimally unitarizable representation $d\pi$ of $G$ on a Hilbert space $\mathcal{H}$ is a representation with positive energy if
\[
C_{\pi} = \{ X \in \mathfrak{g} | id\pi(X) \ is \ a \ \text{nonnegative self adjoint operator} \}
\]
is a non-zero and proper cone in $\mathfrak{g}$. For a proof of the following well-known result see [15].
Lemma. Let \(d\sigma\) be an irreducible, infinitesmally unitarizable representation of \(\mathcal{G}\) on \(\mathcal{H}\). Then \(d\sigma\) is positive energy if and only if \(d\sigma\) is a highest (lowest) weight representation.

The Poincaré energy is \(i\mathbf{P}_0 = \frac{1}{2}(\mathbf{L}_{-10} + \mathbf{L}_{0n})\). A detailed proof of the following result can be found in [15].

Theorem. A representation \(d\sigma\) of \(so(2,n)\) is positive energy if and only if \(-d\sigma(i\mathbf{P}_0)\) is a positive, self-adjoint operator.

Classifications of positive energy representations of \(SO(2,n)\) follow from general results on hermitian symmetric spaces and their unitary highest weight modules established first by Jakobsen [16] and then later by Enright, Howe and Wallach [17]. For explicit and, in some cases, detailed results for \(SO(2,n)\) see refs. [18] and [19]. The massless representations of \(SO(2,n)\) have been completely described in ref. [20]. For explicit and detailed results on the highest weight representations in the \(SO(2,3)\) case see refs. [21] and [22] and for an exhaustive and explicit treatment of the \(SO(2,4)\) case we refer to the paper of Mack [23].

We state here the classification of the infinitesmally unitarizable, irreducible lowest weight representations of \(\mathcal{G} = so(2,3)\) given in ref. [22]. Their results are: let \(D(E_0, s_0)\) be a given such lowest weight representation with lowest weight \(\Lambda \ni \Lambda(H_1) = s_0\) and \(\Lambda(H_2) = E_0\). Then we have: i) \(D(E_0, s_0) = D(\frac{3}{2}, 0)\) (Rac); ii) \(D(E_0, s_0) = D(1, \frac{7}{2})\) (Di); iii) \(D(E_0 > \frac{3}{2}, s_0 = 0)\); iv) \(D(E_0 > 1, s_0 = \frac{1}{2})\); v) \(D(E_0 \geq s_0 + 1, s_0 \geq 1)\). The massless representations are: \(D(E_0 = s_0 + 1, s_0 \geq 1)\).

4. Contractions of massless \(SO(2,n-1)\) Representations into Representations of the Poincaré Group and a Deformation of the Poincaré Lie Algebra

Recall that a basis of \(\mathcal{H}\) is \(L_{ij}\) \((0 \leq i, j \leq n - 1)\), \(L_{-1i}\) \((0 \leq i \leq n - 1)\). We define the contraction mapping \(\phi_\lambda\) by \(\phi_\lambda(L_{ij}) = L_{ij\lambda}\); \(\phi_\lambda(L_{-1i}) = \lambda(2 - \lambda) L_{-1i}\). It is easy to see that \([L_{-1i}, L_{-1j}]_\lambda = \lambda^2(2 - \lambda)^2[L_{-1i}, L_{-1j}] \rightarrow 0\), \([L_{ij}, L_{-1k}]_\lambda = [L_{ij}, L_{-1k}]\), so that \([L_{ij}, L_{kl}]_\lambda = [L_{ij}, L_{kl}]\). Hence the contracted of \(\mathcal{H}\) is \(\mathcal{P}\).

Now we consider massless representations of \(\mathcal{H}\) and describe their contractions into representations of \(\mathcal{P}\). In ref. [20] the following result is established.

**Proposition.** If \(n\) is even, massless representations of \(SO(2,n)\) restrict to massless representations of \(SO(2,n-1)\), and their restriction is irreducible.

Let \((\pi(H), \mathcal{H})\) be the restriction of a massless representation of \(SO(2,n)\) to \(H\). Then

\[
d\pi(L_{-1i}) = \frac{1}{2} \left[ d\pi(P_i) + d\pi(\bar{P}_i) \right]
\]

where we also denote by \((\pi(G^\sim), \mathcal{H})\) the extension of the massless representation of \(H\) to \(SO(2,n)\). Let

\[
(\Pi \lambda \phi)(x) = \lambda^{(n-2)/2}\phi(\lambda x) \quad \text{for} \quad \phi(x) \in \mathcal{H}
\]

Define a representation of \(G'_\lambda\) on \(\mathcal{H}\) by

\[
d\pi_\lambda(X) = \Pi^{-1}\lambda \, d\pi(\phi_\lambda(X)) \Pi, \quad X \in G'_\lambda.
\]

Then (cf. ref. [20])

\[
d\pi_\lambda(L_{-1i}) = \left(1 - \frac{\lambda}{2}\right) d\pi(P_i) + \lambda^2 d\pi(\bar{P}_i) \leftarrow_{\lambda \rightarrow 0} d\pi(P_i).
\]
following element of $\mathcal{E}(\mathcal{G}')$: \[
Q_2 = \sum_{i=1}^{n-1} L^i_0 - \frac{1}{\tau} \sum_{i,j=1}^{n-1} L^i_j.
\] Let $\mathcal{K}(\mathcal{P})$ be the skew field of $\mathcal{P}$ and $\mathcal{K}(\mathcal{G}'_\lambda)$ be the skew field of $\mathcal{G}'_\lambda$ (see below for definition of $\mathcal{G}'_\lambda$). Define a commutative algebraic extension of $\mathcal{K}(\mathcal{P})$ by $\mathcal{K}(\mathcal{P})^{\text{ext}} = \left\{ a + bY \mid a, b \in \mathcal{K}(\mathcal{P}), Y^2 = \mathcal{P}^2 \right\}$ where $Y$ commutes with all elements of $\mathcal{K}(\mathcal{P})$ and $\mathcal{P}^2 = \sum_{k=0}^{n-1} \mathcal{P}_k \mathcal{P}_k$. Now define a mapping $\tau_\lambda$ from $\mathcal{G}'_\lambda$ to $\mathcal{K}(\mathcal{P})$ by
\[
\tau_\lambda(L_{\mu\nu}) = L_{\mu\nu}, \quad \tau_\lambda(L_{-1\mu}) = \frac{i\lambda}{2\tau} \left[ Q_2, P_\mu \right] + P_\mu.
\] The $\lambda^{-1}\tau_\lambda(L_{-1\mu})$ and $\tau_\lambda(L_{\mu\nu})$ satisfy the commutation relations of the generators of $\mathcal{G}'$. Thus, if we let $\tilde{Y} = Y$, then $\tau = \tau_{\lambda = 1}$ and define a commutative algebraic extension of $\mathcal{K}(\mathcal{G}'_\lambda)$ by
\[
\mathcal{K}(\mathcal{G}'_\lambda)^{\text{ext}} = \left\{ a + b\tilde{Y} + c\tilde{Y}^2 + d\tilde{Y}^3 \mid a, b, c, d \in \mathcal{K}(\mathcal{G}_\lambda) \right\}
\] where $\tilde{Y}$ commutes with all elements of $\mathcal{K}(\mathcal{G}'_\lambda)$. $\tau_\lambda$ can be extended to a homomorphism of $\mathcal{K}(\mathcal{G}'_\lambda)^{\text{ext}}$ into $\mathcal{K}(\mathcal{P})^{\text{ext}}$ in an obvious way, which, is surjective in the case $n = 4$ \cite{24}. Denote this extension also by $\tau = \tau_{\lambda = 1}$. Elements of $\mathcal{K}(\mathcal{G}'_\lambda)^{\text{ext}}$ have a tilde to keep them distinct from elements of $\mathcal{K}(\mathcal{P})^{\text{ext}}$, and we introduce $*$ structures on $\mathcal{K}(\mathcal{P})^{\text{ext}}$ and $\mathcal{K}(\mathcal{G}'_\lambda)^{\text{ext}}$.

Now let $\mathcal{V}$ be the underlying vector space of the Lie algebra of $\mathcal{P} = (V, [\cdot, \cdot]_0)$ and define a map $\tilde{\phi}_\lambda : \mathcal{P} \to \mathcal{K}(\mathcal{P})^{\text{ext}}$ by
\[
\tilde{\phi}_\lambda(L_{\mu\nu}) = \tau_\lambda(L_{\mu\nu}), \quad \tilde{\phi}_\lambda(P_\mu) = \tau_\lambda(L_{-1\mu}).
\] Due to $\ker(\tau)|_{\mathcal{E}(\mathcal{G}')} = 0$ (cf. ref. \cite{24}), $\tilde{\phi}_\lambda$ must be a vector space isomorphism onto its image and thus we can define a new Lie bracket on $\mathcal{P}$ by:
\[
[a, b]_\lambda = \tilde{\phi}_\lambda^{-1}([\tilde{\phi}_\lambda(a), \tilde{\phi}_\lambda(b)]) \forall a, b \text{ in } \mathcal{P}.
\] Thus, if we let $\psi : I \subset \mathcal{B} / \mathcal{M}_n$ be defined as $\psi(\lambda) = [\cdot, \cdot]_\lambda$ with $\lambda \in I$, then $\psi$ is a (nontrivial) deformation of $\mathcal{P}$, with $\psi(1)$ being the Lie structure of $\mathcal{G}'$.

The just described deformation of the Poincaré Lie algebra and related results, briefly described above, enables us to associate to certain representations of $\mathcal{P}$ representations of $so(2, 3)$ and visa versa (cf. \cite{24}, \cite{25}). For the massless representations $D(E_0 = s_0 + 1, s_0 \geq 1)$ of $SO_0(2, 3)$ we recall Proposition 4.1 of \cite{24}: $D(E_0 = s_0 + 1, s_0 \geq 1)$ give representations of $\mathcal{P}$ none of which are skew symmetric. In the proof of this proposition, which is given in \cite{24}, it is shown that the mass and spin of the associated representation of $\mathcal{P}$ are related by
\[
m^2 = \lambda^2(s_0 + \frac{1}{2})^2.
\] 5. The Realization of the Rac Representation on AdS

We define $n$-dimensional AdS (anti deSitter space) as
\[
AdS_n = \left\{ \xi^i_1 \in \mathcal{B}^{1+n} \mid \xi_1^2 + \xi_1^0 - \xi_1^2 - \ldots - \xi_{n-1}^2 = R_0^2 \right\}
\] Note the following typographical errors in Ref. \cite{24}. Everywhere in the paragraph below Proposition 4.1 there should be $s_0$ and not $s$ and, in the line just below the statement of Proposition 4.1, it should read $C_2 = 2(s_0^2 - 1) - I$. 

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with $R_0$ being the radius of the space. We also define $n$-dimensional de Sitter space:

$$V_n = \left\{ \xi_i \in \mathbb{R}^{n+1} \mid \xi_0^2 - \xi_1^2 - \xi_2^2 - \ldots - \xi_n^2 = - R_0^2 \right\}. \quad (19)$$

Natural coordinates on $n$-dimensional Minkowski space $M_n$ are denoted by $x_i$ i.e. $x_i$ denotes the $i$-th component of $(x_0, x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^n$. The $n$-dimensional Einstein universe into which all of these manifolds may be conformally embedded is defined as

$$\tilde{M}_n \simeq S^1 \times S^{n-1} = \left\{ u_i \in \mathbb{R}^{n+2} \mid u_0^2 + u_1^2 = u_1^2 + u_2^2 + \ldots + u_n^2 = R_0^2 \right\}. \quad (20)$$

The following embedding relations between these various manifolds can be obtained from ref. [26].

$$\xi_a = R_0 \frac{u_a}{u_{n-1}}, \quad \xi_a' = R_0 \frac{u_a}{u_n}, \quad x_i = R_0 \frac{2u_i}{(u_{n-1} + u_n)}. \quad (21)$$

It follows that $\xi_a = \frac{\xi_a'}{\xi_{n-1}}$. Using eqns. (21) it is possible to define, at least locally, various parallelizations [27] (or pictures) for representations of the isometry groups associated with these various manifolds. However, for some representations, their parallelized forms may be singular or ill-defined, as, for example, in the case of the Di and the Rac representations on Minkowski space [24].

Some useful differential operators on $\text{AdS}_n$ are $S = \sum_{i=1}^{n-1} \frac{\partial^2}{\partial \xi_i^2}$, $Q = \sum_{i=1}^{n-1} \epsilon_i \xi_i I$, $\Delta = \sum_{i=1}^{n-1} \epsilon_i \frac{\partial^2}{\partial \xi_i^2}$, where $\epsilon_i$ is +1 for $i = -1$ or 0 and -1 for $i = 1, \ldots, n-1$ and $I = \text{identity operator on } C^\infty(\text{AdS}_n)$. We have:

$$[\Delta, Q] = 4(S + \frac{n+1}{2} I), \quad [S, \Delta] = -2\Delta, \quad [S, Q] = 2Q, \quad (22)$$

and Euler’s theorem on homogeneous functions is

$$S \phi = \sigma \phi \quad \text{for } \phi \in C^\infty(\text{AdS}_n) \quad (23)$$

for $\phi$ homogeneous of degree $\sigma$. Let $y = (\xi_1^2 + \xi_2^2)^{\frac{1}{2}}$, $r = (\xi_1^2 + \xi_2^2 + \ldots + \xi_{n-1}^2)^{\frac{1}{2}}$ so that $y^2 = r^2 + R_0^2$ and set $-\sigma = -2 + \frac{n+1}{2}$.

The action of $L_{ab}$ in the left regular representation of $SO(2, n-1)$ on $\text{AdS}_n$ gives a representation of the Lie algebra by the differential operators:

$$L_{ab} = \xi_a \frac{\partial}{\partial \xi_b} - \xi_b \frac{\partial}{\partial \xi_a}. \quad (24)$$

Consider the space of all $\phi \in C^\infty(\text{AdS}_n)$ for which

$$\lim_{r \to \infty} r^{-\sigma} \phi(\xi') < \infty \quad (25)$$

and

$$\Delta \phi = 0, (S - \sigma) \phi = 0. \quad (26)$$

The subspace of such $\phi$ for which $\lim_{r \to \infty} r^{-\sigma} \phi(\xi') = 0$ is an invariant subspace for the action of the $L_{ab}$ and a representation of $SO_0(2, n-1)$ is realized on this quotient which for $n = 4$ is the scalar singleton representation of $SO_0(2, 3)$ i.e. the Rac representation [28]. Thus Rac fields are completely determined by their values at infinity in $\text{AdS}_n$. Equivalently, using the relations in eqn. (21), the Rac fields are given (at time $\xi_0 = 0$) by their values on the hypersphere $\xi_n = 0$ in de Sitter space or by their values on a hypersphere of radius $2R_0$ in Minkowski space.
6. A Model for Massless Representations of $SO_0(2,3)$

Flato and Fronsdal have established results which lead to the interpretation of massless particles as composites of singletons (Di’s and Rac’s) and they have initiated the development of a quantum field theory of interacting singletons on AdS with an aim towards a rigorous description of this interpretation [29]. In particular they have established the following result [30]:

$$D(\frac{1}{2}, 0) \otimes D(\frac{1}{2}, 0) = \sum_{s_0=0,1,\ldots}^{+\infty} D(s_0 + 1, s_0)$$

(27)

We present here a physical model (cf. Fig. 1) associated with their results. It consists of two particles executing circular motion about their geometric center. We want to use the results of [24] to show how this model serves as an aid in understanding the decomposition given by eqn. (27).

The momenta and energies of the two constituents in Fig. 1 are ($c = 1$):

$$p_1 = p_2 = \frac{m_0 v}{(1 - v^2)^{1/2}}, \quad E_1 = E_2 = \frac{m_0}{(1 - v^2)^{1/2}}.$$  

(28)

The total angular momentum of the system is

$$s = \frac{2m_0 vr}{(1 - v^2)^{1/2}},$$

(29)

The energy of the center of mass of the system is $E = E_1 + E_2$, which, according to Feynman et al. [32], must be identified with the observable mass $m$ of the composite system. We thus have

$$2m_0^2 = m_0^2 + m_0^2 = (E_1^2 - p_1^2) + (E_2^2 - p_2^2) = (E_1 + E_2)^2 - 2E_1 E_2 - p_1^2 - p_2^2$$

$$= E^2 - 2 \frac{m_0^2}{(1 - v^2)} - 2 \frac{m_0 v^2}{(1 - v^2)} = E^2 - 2 \frac{m_0^2(1 - v^2)}{(1 - v^2)} - \frac{4m_0 v^2}{(1 - v^2)}.$$  

(30)

This implies that

$$m^2 = 4m_0^2 + \frac{1}{r^2} s^2$$

(31)

with $E = m$.

The analysis leading to eqn. (31) can also be carried out quantum mechanically. We work in the tensor product of the Hilbert space for a single particle, which we take to be the Rac, i.e. we consider $D(\frac{1}{2}, 0) \otimes D(\frac{1}{2}, 0)$ and we introduce, in analogy to eqn. (28), $E = p_0 \otimes I + I \otimes p_0$ (where now $E$ is an operator) and $P = p \otimes I + I \otimes p$, where the $p_0$ and $p$ are the operators corresponding to the energy and magnitude of three momentum vector for one of the constituent particles.\(^2\) For the total angular momentum about the center, we have $S = \hat{s} \otimes I + I \otimes \hat{s}$, where $\hat{s}$ is the operator corresponding to the spin about the center of one of the particles. Using these definitions we obtain, after some calculations with tensor products, the following operator equation similar to eqn. (31):

$$E^2 = 4m_0^2 I \otimes I + \frac{\hat{s}^2}{r^2}.$$  

(32)

\(^2\)The $p_0$ and $p$ cannot in any simple way be related to the operators of the energy and the magnitude of three momentum vector in the representation of the Poincaré Lie algebra associated with the Rac representation. The reason for this is that the action, in the Rac representation, of the Poincaré translation generators, $P_\mu$, defined by Lemma 3.2 of ref. [24] are undefined as linear operators on the Rac representation space. In analogy to quantum field theory, we can view the action of the $P_\mu$ in the representation $D(\frac{1}{2}, 0)$ as the “bare momentum” operators and the $p_0$ and $p$ as the operators for the effective momentum of a constituent, with $m_0$ (which is equal to the eigenvalue of $\sqrt{p_0^2 - p^2}$ in $D(\frac{1}{2}, 0)$) being the effective Poincaré mass of one of the “quarks.”
In an arbitrary Lorentz frame, $E^2 \rightarrow -P^2 = -P_\mu P^\mu$, where the $P_\mu$ are defined by $P_\mu = d\pi(\tau^{-1}_\lambda(P_\mu))$ with $\pi$ being the representation of $SO_0(2,3)$ on $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$. Similarly $S^2 \rightarrow \frac{W}{r^2}$ with $W$ being the magnitude squared of the Pauli-Lubanski 4-vector in the representation, so that we obtain out of eqn.(32) the following:

$$-P^2 = \left(4m_0^2 \cdot I \otimes I + \frac{W}{r^2} P^2 \right). \tag{33}$$

Now it is an easy exercise with tensor products to show that $d\pi(iL_{-10})$, where $iL_{-10}$ is the anti-deSitter energy operator, is a positive, self-adjoint operator on $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$.\(^3\) Hence, since $\pi(SO_0(2,3))$ is unitary, it is fully reducible [33], and the theorem in section three then implies that $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$ must necessarily decompose into $SO_0(2,3)$ irreducible, positive energy representations.

To determine $m_0$ we argue as follows. On physical grounds $2m_0$ should be the Poincaré mass of the minimal energy state of the system [34]. The lowest weight in the tensor product representation $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$ is $(E = \frac{\sqrt{5}}{2} + \frac{3}{2}, s = 0)$ which is the lowest weight for the irreducible representation $D(1,0)$. $D(1,0)$ must therefore be one of the irreducible factors in the decomposition of $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$. Since the Poincaré mass associated with the representation $D(1,0)$ is $m = \frac{\sqrt{5}}{2}$ (cf. eqn. 17), we equate this with $2m_0$ to get $m_0 = \frac{\sqrt{5}}{4}$. Substituting this into eqn. (33) and setting $r\lambda = 1$ gives

$$-P^2 = \lambda^2 \left(\frac{1}{4} \cdot I \otimes I + \frac{W}{P^2} \right). \tag{34}$$

We know that the other irreducible factors in the decomposition of $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$ are associated with representations of the Poincaré Lie algebra, $\mathcal{P}$, provided we can show that zero lies in the resolvent set of the operator $d\pi(D)$ where $D$ is defined in Theorem 3.2 of [24]. The verification of this fact is similar to the calculations on page 77 of [35] and it is left to reader. Additionally, we have established the validity of the operator equation, eqn. (34), in the

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\(^3\) Let $\psi = \sum \alpha_i \phi_i \otimes \phi_j$ be an arbitrary vector in $D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)$ where the $\phi_i$ are an orthonormal basis of eigenvectors of $iL_{-10}$ in $D(\frac{1}{2},0)$. Then it is a routine computation to show that $(\psi, d\pi(iL_{-10} \otimes I \otimes iL_{-10}) \psi) > 0$.\(^3\)
representation space \(D(\frac{1}{2},0) \otimes D(\frac{1}{2},0)\), and we know that the spectral values of \(\frac{W}{2}\pi\) are \(s_0(s_0 + 1)\) with \(s_0 = 0, 1, 2, \ldots\), since this is so in the rest frame of the center of mass. Thus, we obtain out of eqn. (34)

\[
m^2 = \lambda^2(s_0 + \frac{1}{2})^2
\]

where \(m^2\) is the expectation value of \(-P^2\). But this is exactly the mass-spin relation for representations of the Poincaré Lie algebra associated with the massless representations \(D(s_0 + 1, s_0)\) of \(SO_0(2,3)\) stated in section 4, i.e. eqn. (17) with \(\lambda = \frac{1}{2\pi G}\). We do not know if these nonzero mass representations of the Poincaré Lie algebra are integrable to nonunitary representations of the Poincaré group. In general, there does not seem to be much explicit information about nonunitary representations of the Poincaré group [36].

7. References

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