Uncertain SEIAR model for COVID-19 cases in China

Lifen Jia · Wei Chen

Accepted: 7 September 2020 / Published online: 24 September 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
The Susceptible-Exposed-Infectious-Asymptomatic-Removed (SEIAR) epidemic model is one of most frequently used epidemic models. As an application of uncertain differential equations to epidemiology, an uncertain SEIAR model is derived which considers the human uncertainty factors during the spread of an epidemic. The parameters in the uncertain epidemic model are estimated with the numbers of COVID-19 cases in China, and a prediction to the possible numbers of active cases is made based on the estimates.

Keywords Uncertainty theory · Uncertain differential equation · Uncertain SEIAR model · COVID-19 · Parameter estimation

1 Introduction
Epidemics have always been a threat to human health. To describe and predict the spread of an epidemic, various models have been built such as the SIS model, SIR model, SEIR model and SEIAR model. Since the establishment of stochastic differential equation theory in 1950s, stochastic epidemic models have been investigated for the reasons of indeterminate factors during the spread process of an epidemic. For example, Gray et al. (2011) presented a stochastic SIS model, and gave some conditions for extinction and persistence of the disease. Ji et al. (2012) discussed a stochastic SIR model, and proved the stability conditions of disease-free equilibrium. Artalejo et al. (2015) presented a stochastic SEIR model, and studied the evolution of the epidemic before its extinction. As we know, stochastic differential equations are applicable to dynamic systems with random factors rather than human uncertainty, so it is questionable whether they can properly describe epidemic systems which are heavily affected by human behaviors.
As a branch of mathematics for modelling human uncertainty, the uncertainty theory was founded by Liu (2007) and perfected by Liu (2009). Within the framework of uncertainty theory, the concept of uncertain differential equation was proposed by Liu (2008) to describe dynamic systems with human uncertainty. Chen and Liu (2010) gave a sufficient condition for an uncertain differential equation to have a unique solution, and Yao et al. (2013) proved some stability theorems about uncertain differential equations. The structure of the solution of an uncertain differential equation was found by Yao and Chen (2013), based on which various numerical methods have been designed to solve uncertain differential equations, such as Yang and Ralescu (2015), Gao (2016), and Zhang et al. (2017). In order to estimate the parameters in uncertain differential equations based on observed data, Yao and Liu (2020) presented the method of moments, which was extended to the generalized method of moments by Liu (2020). In addition, the least squares estimation and the maximum likelihood estimation for uncertain differential equations were presented by Sheng et al. (2020) and Liu and Liu (2020), respectively.

As an application of uncertain differential equations to finance, uncertain finance theory was extended during the past years. For example, Liu (2009) assumed the price of a stock follows a lognormal uncertainty distribution, and derived the European option pricing formulas. Chen and Gao (2013) described the short-term interest rate with uncertain differential equations, and investigated the pricing problems of zero-coupon bonds. Uncertain differential equations have also been applied to optimal control (Zhu 2010), game theory (Yang and Gao 2015), population growth model (Sheng et al. 2017) and pharmacokinetics model (Liu and Yang 2020), etc.

The investigation of uncertain epidemic models was initialized by Li et al. (2017), where an uncertain SIS model was constructed and the disease-free equilibrium was discussed. After that, the uncertain SIS model was generalized to the cases with standard incidence and demography (Fang et al. 2018) and with nonlinear incidence and demography (Li and Teng 2019). However, applications of these SIS models are limited because the groups of exposed individuals and recovered individuals with potential immunity are not taken into consideration in these models. In this paper, we aim to build a more general epidemic model by means of uncertain differential equations that is called uncertain SEIAR model. Groups of susceptible individuals, exposed individuals, symptomatically infected individuals, asymptotically infected individuals, and removed individuals are all considered in such a model, and its application to the COVID-19 cases in China is also provided. The rest of this paper is organized as follows. Section 2 introduces some basic concepts related to uncertain differential equations. Section 3 derives the uncertain SEIAR model, and gives the uncertain SEIR model and the uncertain SIR model as degenerated forms of uncertain SEIAR model. Section 4 estimates the parameters in the uncertain epidemic model by using the numbers of COVID-19 cases in China, and Sect. 5 predicts the possible numbers of COVID-19 active cases in China by means of the uncertain epidemic model with those estimated parameters. Finally, some conclusion are made in Sect. 6.
2 Preliminary

The uncertain measure $\mathcal{M}$ is a set function from a measurable space $(\Gamma, \mathcal{L})$ to the interval $[0, 1]$ which satisfies the normality, duality, subadditivity and product axioms. An uncertain variable $\xi$ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers.

**Definition 1** (Liu 2007) The uncertainty distribution $\Phi$ of an uncertain variable $\xi$ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number $x$.

An uncertain variable $\xi$ is called normal if it has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \mathbb{R}$$

denoted by $\mathcal{N}(e, \sigma)$. A normal uncertainty distribution is called standard if $e = 0$ and $\sigma = 1$.

**Definition 2** (Liu 2007) Let $\xi$ be an uncertain variable, and let $k$ be a positive integer. Then the $k$-th moment of $\xi$ is

$$E[\xi^k] = \int_{-\infty}^{+\infty} \mathcal{M}\left\{\xi^k \geq x\right\} dx - \int_{-\infty}^{0} \mathcal{M}\left\{\xi^k \leq x\right\} dx$$

provided that at least one of the two integrals is finite.

The first and second moments of a standard normal uncertain variable $\mathcal{N}(0, 1)$ are 0 and 1, respectively.

**Definition 3** (Liu 2009) An uncertain process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
(ii) $C_t$ has stationary and independent increments;
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable $\mathcal{N}(0, t)$.

**Definition 4** (Liu 2009) Let $X_t$ be an uncertain process, and let $C_t$ be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$ 

Then Liu integral of $X_t$ with respect to $C_t$ is defined as

$$\int_a^b X_t dC_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$
provided that the limit exists almost surely and is finite.

**Definition 5** (Liu 2008) Suppose that \( C_t \) is a Liu process, and \( f \) and \( g \) are two measurable functions. Then

\[
\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t
\]

(1)

is called an uncertain differential equation.

The integral form of the uncertain differential equation (1) is

\[
X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s.
\]

### 3 Uncertain SEIAR model

In an SEIAR model, the population is divided into 5 groups, namely, susceptible individuals, exposed individuals, symptomatically infected individuals, asymptotically infected individuals, and removed individuals (recovered or dead). Susceptible individuals will become exposed individuals after contact with symptomatically infected individuals or asymptotically infected individuals; Exposed individuals will possibly become symptomatically infected individuals or asymptotically infected individuals; Symptomatically infected individuals and asymptotically infected individuals will be removed in some proportions; Removed individuals will not become susceptible individuals again. For simplicity, the numbers of individuals in these 5 groups at the time \( t \) are denoted by \( S_t, E_t, I_t, A_t \) and \( R_t \) in the uncertain SEIAR model, respectively.

#### 3.1 Equation of susceptible individuals

Let \( \beta_{1t} \) denote the contact rate between a susceptible individual and a symptomatically infected individual, and \( \beta_{2t} \) denote the contact rate between a susceptible individual and an asymptotically infected individual. Considering the human uncertainty during the contact process, we assume

\[
\beta_{1t} = \beta_I + \sigma_1 \cdot \text{“Noise”}
\]

where \( \beta_I \) is a nonnegative number, and “Noise” is a normal uncertain variable \( \mathcal{N}(0, 1) \). Representing the “Noise” by

\[
\frac{C_{1,t+\Delta t} - C_{1t}}{\Delta t}
\]

where \( C_{1t} \) is a Liu process, we further have

\[
\beta_{1t} = \beta_I + \sigma_1 \frac{C_{1,t+\Delta t} - C_{1t}}{\Delta t}.
\]
Similarly, the contact rate $\beta_{2t}$ can be represented as

$$
\beta_{2t} = \beta_A + \sigma_2 \frac{C_{2,t+\Delta t} - C_{2t}}{\Delta t}
$$

where $C_{2t}$ is a Liu process, and $\beta_A$ and $\sigma_2$ are two nonnegative numbers. Then the increment of the number of susceptible individuals during an infinitesimal time interval $[t, t + \Delta t]$ is

$$
S_{t+\Delta t} - S_t = -(\beta_{1t} S_t I_t \Delta t + \beta_{2t} S_t A_t \Delta t) - \sigma_1 S_t I_{t+\Delta t} - \sigma_2 S_t A_{t+\Delta t} - C_{2t}.
$$

For a time interval $[0, t]$ with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$, the increment of the number of susceptible individuals during such an interval is

$$
S_t - S_0 = \sum_{i=0}^{n-1} (S_{t_{i+1}} - S_{t_i})
= -\sum_{i=0}^{n-1} (\beta_{1t} S_{t_i} I_{t_i} + \beta_A S_{t_i} A_{t_i}) (t_{i+1} - t_i) - \sum_{i=0}^{n-1} \sigma_1 S_{t_i} I_{t_{i+1}} - C_{1t_i}
- \sigma_2 S_{t_i} A_{t_{i+1}} - C_{2t_i}
\rightarrow - \int_0^t (\beta_{1t} S_{t} I_{t} + \beta_A S_{t} A_{t}) d\tau - \sigma_1 \int_0^t S_{t} I_{t} dC_{1\tau} - \sigma_2 \int_0^t S_{t} A_{t} dC_{2\tau}
$$

as

$$
\max_{0 \leq i \leq n-1} |t_{i+1} - t_i| \rightarrow 0.
$$

The above integral equation can be rewritten as an uncertain differential equation

$$
dS_t = - (\beta_{1t} S_t I_t + \beta_A S_t A_t) dt - \sigma_1 S_t I_{t} dC_{1t} - \sigma_2 S_t A_{t} dC_{2t}.
$$

(2)

### 3.2 Equation of exposed individuals

Let $\nu_{1t}$ denote the rate that exposed individuals become symptomatically infected individuals, and $\nu_{2t}$ denote the rate that exposed individuals become asymptptomatically infected individuals. Similar to the assumptions about the contact rates in Sect. 3.1, we assume that

$$
\nu_{1t} = \nu_I + \sigma_3 \frac{C_{3,t+\Delta t} - C_{3t}}{\Delta t}
$$

$$
\nu_{2t} = \nu_A + \sigma_4 \frac{C_{4,t+\Delta t} - C_{4t}}{\Delta t}
$$
and

\[ v_{2t} = v_A + \sigma_4 \frac{C_{4,t+\Delta t} - C_{4t}}{\Delta t} \]

where \( C_{3t} \) and \( C_{4t} \) are two Liu processes, and \( v_I, v_A, \sigma_3 \) and \( \sigma_4 \) are some nonnegative numbers. Then the increment of the number of exposed individuals during an infinitesimal time interval \([t, t + \Delta t]\) is

\[
E_{t+\Delta t} - E_t = (S_t - S_{t+\Delta t}) - (v_{1t} + v_{2t}) E_t \Delta t
\]

\[= (S_t - S_{t+\Delta t}) - (v_I + v_A) E_t \Delta t - \sigma_3 E_t (C_{3,t+\Delta t} - C_{3t})
\]

\[- \sigma_4 E_t (C_{4,t+\Delta t} - C_{4t}).\]

For a time interval \([0, t]\) with a partition \(0 = t_0 < t_1 < \cdots < t_n = t\), the increment of the number of exposed individuals during such an interval is

\[
E_t - E_0 = \sum_{i=0}^{n-1} (E_{t_{i+1}} - E_{t_i})
\]

\[= S_0 - S_t - \sum_{i=0}^{n-1} (v_I + v_A) E_{t_i} (t_{i+1} - t_i) - \sum_{i=0}^{n-1} \sigma_3 E_{t_i} (C_{3_{t_{i+1}}} - C_{3t_i})
\]

\[- \sum_{i=0}^{n-1} \sigma_4 E_{t_i} (C_{4_{t_{i+1}}} - C_{4t_i})
\]

\[\rightarrow \int_0^t (\beta_I S_t I_t + \beta_A S_t A_t) d\tau + \sigma_1 \int_0^t S_t I_t dC_{1\tau} + \sigma_2 \int_0^t S_t A_t dC_{2\tau}
\]

\[- (v_I + v_A) \int_0^t E_t d\tau - \sigma_3 \int_0^t E_t dC_{3\tau} - \sigma_4 \int_0^t E_t dC_{4\tau}
\]

as

\[
\max_{0 \leq i \leq n-1} |t_{i+1} - t_i| \rightarrow 0.
\]

The above integral equation can be rewritten as an uncertain differential equation

\[
dE_t = (\beta_I S_t I_t + \beta_A S_t A_t - v_I E_t - v_A E_t) dt + \sigma_1 S_t I_t dC_{1t} + \sigma_2 S_t A_t dC_{2t}
\]

\[- \sigma_3 E_t dC_{3t} - \sigma_4 E_t dC_{4t}.
\]

(3)

### 3.3 Equation of symptomatically infected individuals

Let \( v_{1t} \) denote the rate that exposed individuals become symptomatically infected individuals, and \( \mu_{1t} \) denote the removed (recovered or dead) rate of symptomatically
infected individuals. As the assumption in Sect. 3.2, we have

\[ v_{1t} = v_t + \sigma_3 \frac{C_{3, t+\Delta t} - C_{3t}}{\Delta t}. \]

Similarly, we assume that

\[ \mu_{1t} = \mu_t + \sigma_5 \frac{C_{5, t+\Delta t} - C_{5t}}{\Delta t} \]

where \( C_{5t} \) is a Liu process, and \( \mu_t \) and \( \sigma_5 \) are two nonnegative numbers. Then the increment of the number of symptomatically infected individuals during an infinitesimal time interval \([t, t + \Delta t]\) is

\[ I_{t+\Delta t} - I_t = v_{1t} E_t \Delta t - \mu_{1t} I_t \Delta t + (v_I E_t - \mu_I I_t) \Delta t + \sigma_3 E_t (C_{3, t+\Delta t} - C_{3t}) - \sigma_5 I_t (C_{5, t+\Delta t} - C_{5t}). \]

For a time interval \([0, t]\) with a partition \(0 = t_0 < t_1 < t_2 < \cdots < t_n = t\), the increment of the number of symptomatically infected individuals during such an interval is

\[ I_t - I_0 = \sum_{i=0}^{n-1} (I_{t_{i+1}} - I_{t_i}) \]

\[ = \sum_{i=0}^{n-1} (v_I E_{t_i} - \mu_I I_{t_i}) (t_{i+1} - t_i) + \sum_{i=0}^{n-1} \sigma_3 E_{t_i} (C_{3t_{i+1}} - C_{3t_i}) \]

\[ - \sum_{i=0}^{n-1} \sigma_5 I_{t_i} (C_{5t_{i+1}} - C_{5t_i}) \]

\[ \rightarrow \int_0^t (v_I E_{\tau} - \mu_I I_{\tau}) d\tau + \sigma_3 \int_0^t E_{\tau} dC_{3\tau} - \sigma_5 \int_0^t I_{\tau} dC_{5\tau} \]

as

\[ \max_{0 \leq i \leq n-1} |t_{i+1} - t_i| \rightarrow 0. \]

The above integral equation can be rewritten as an uncertain differential equation

\[ dI_t = (v_I E_t - \mu_I I_t) dt + \sigma_3 E_t dC_{3t} - \sigma_5 I_t dC_{5t}. \]

(4)

3.4 Equation of asymptptomatically infected individuals

Let \( v_{2t} \) denote the rate that exposed individuals become asymptptomatically infected individuals, and \( \mu_{2t} \) denote the removed (recovered or dead) rate of asymptptomatically
infected individuals. As the assumption in Sect. 3.2, we have

\[ v_{2t} = v_A + \sigma_4 \frac{C_{4,t+\Delta t} - C_{4t}}{\Delta t}. \]

Similarly, we assume that

\[ \mu_{2t} = \mu_A + \sigma_6 \frac{C_{6,t+\Delta t} - C_{6t}}{\Delta t} \]

where \( C_{6t} \) is a Liu process, and \( \mu_A \) and \( \sigma_6 \) are two nonnegative numbers. Then the increment of the number of asymptomatically infected individuals during an infinitesimal time interval \([t, t + \Delta t]\) is

\[
A_{t+\Delta t} - A_t = v_{2t} E_t \Delta t - \mu_{2t} A_t \Delta t \\
= (v_A E_t - \mu_A A_t) \Delta t + \sigma_4 E_t (C_{4,t+\Delta t} - C_{4t}) - \sigma_6 A_t (C_{6,t+\Delta t} - C_{6t}).
\]

For a time interval \([0, t]\) with a partition \(0 = t_0 < t_1 < t_2 < \cdots < t_n = t\), the increment of the number of asymptomatically infected individuals during such an interval is

\[
A_t - A_0 = \sum_{i=0}^{n-1} (A_{t_{i+1}} - A_{t_i}) \\
= \sum_{i=0}^{n-1} (v_A E_{t_i} - \mu_A A_{t_i})(t_{i+1} - t_i) + \sum_{i=0}^{n-1} \sigma_4 E_{t_i} (C_{4t_{i+1}} - C_{4t_i}) \\
- \sum_{i=0}^{n-1} \sigma_6 A_{t_i} (C_{6t_{i+1}} - C_{6t_i}) \\
\to \int_0^t (v_A E_\tau - \mu_A A_\tau) d\tau + \sigma_4 \int_0^t E_\tau dC_{4\tau} - \sigma_6 \int_0^t A_\tau dC_{6\tau}
\]

as

\[
\max_{0 \leq i \leq n-1} |t_{i+1} - t_i| \to 0.
\]

The above integral equation can be rewritten as an uncertain differential equation

\[
dA_t = (v_A E_t - \mu_A A_t)dt + \sigma_4 E_t dC_{4t} - \sigma_6 A_t dC_{6t}. \tag{5}
\]

### 3.5 Equation of removed individuals

Let \( \mu_1 t \) denote the removed rate of symptomatically infected individuals, and \( \mu_2 t \) denote the removed rate of asymptomatically infected individuals. As the assumptions
in Sects. 3.3 and 3.4, we have

$$\mu_{1t} = \mu_I + \sigma_5 \frac{C_{5,t+\Delta t} - C_{5t}}{\Delta t}$$

and

$$\mu_{2t} = \mu_A + \sigma_6 \frac{C_{6,t+\Delta t} - C_{6t}}{\Delta t}.$$  

The increment of the number of removed individuals during an infinitesimal time interval \([t, t + \Delta t]\) is

$$R_{t+\Delta t} - R_t = \mu_{1t} I_t \Delta t + \mu_{2t} A_t \Delta t$$

$$= (\mu_I I_t + \mu_A A_t) \Delta t + \sigma_5 I_t (C_{5,t+\Delta t} - C_{5t}) + \sigma_6 A_t (C_{6,t+\Delta t} - C_{6t}).$$

For a time interval \([0, t]\) with a partition \(0 = t_0 < t_1 < t_2 < \cdots < t_n = t\), the increment of the number of removed individuals during such an interval is

$$R_t - R_0 = \sum_{i=0}^{n-1} (R_{t_{i+1}} - R_{t_i})$$

$$= \sum_{i=0}^{n-1} (\mu_I I_{t_i} + \mu_A A_{t_i}) (t_{i+1} - t_i) + \sum_{i=0}^{n-1} \sigma_5 I_{t_i} (C_{5t_{i+1}} - C_{5t_i})$$

$$+ \sum_{i=0}^{n-1} \sigma_6 A_{t_i} (C_{6t_{i+1}} - C_{6t_i})$$

$$\to \int_0^t (\mu_I I_{\tau} + \mu_A A_{\tau}) \, d\tau + \sigma_5 \int_0^t I_{\tau} \, dC_{5\tau} + \sigma_6 \int_0^t A_{\tau} \, dC_{6\tau}$$

as

$$\max_{0 \leq i \leq n-1} |t_{i+1} - t_i| \to 0.$$

The above integral equation can be rewritten as an uncertain differential equation

$$dR_t = (\mu_I I_t + \mu_A A_t) \, dt + \sigma_5 I_t \, dC_{5t} + \sigma_6 A_t \, dC_{6t}.$$(6)

### 3.6 Uncertain SEIAR model

Based on the equations in Sects. 3.1–3.5, we propose the uncertain SEIAR model
to describe an epidemic system with uncertain information, where $S_t$, $E_t$, $I_t$, $A_t$ and $R_t$ denote the numbers of susceptible individuals, exposed individuals, symptomatically infected individuals, asymptotically infected individuals, and removed individuals, respectively, $C_{i_t}$, $i = 1, 2, \ldots, 6$ are Liu processes, $\beta_l$, $\beta_A$, $\nu_l$, $\nu_A$, $\mu_l$, $\mu_A$ and $\sigma_i$, $i = 1, 2, \ldots, 6$ are some nonnegative numbers.

The uncertain SEIAR model (7) degenerates to an uncertain SEIR model if the group of asymptotically infected individuals is neglected. Setting $A_t = 0$ and $\nu_A = \sigma_A = 0$ in the model (7), we get the uncertain SEIR model

$$
\begin{aligned}
&dS_t = -(\beta_l S_t I_t + \beta_A S_t A_t) dt - \sigma_1 S_t I_t dC_{1t} - \sigma_2 S_t A_t dC_{2t} \\
&dE_t = (\beta_l S_t I_t + \beta_A S_t A_t - \nu_l E_t - \nu_A E_t) dt + \sigma_1 S_t I_t dC_{1t} + \sigma_2 S_t A_t dC_{2t} - \sigma_3 E_t dC_{3t} - \sigma_4 E_t dC_{4t} \\
&dI_t = (\nu_l E_t - \mu_l I_t) dt + \sigma_3 E_t dC_{3t} - \sigma_5 I_t dC_{5t} \\
&dA_t = (\nu_A E_t - \mu_A A_t) dt + \sigma_4 E_t dC_{4t} - \sigma_6 A_t dC_{6t} \\
&dR_t = (\mu_l I_t + \mu_A A_t) dt + \sigma_5 I_t dC_{5t} + \sigma_6 A_t dC_{6t}
\end{aligned}
$$

Furthermore, the uncertain SEIR model (8) degenerates to an uncertain SIR model if the group of exposed individuals is neglected. In this case, all the exposed individuals are regarded as infected individuals. Replacing $\nu_l E_t dt + \sigma_3 E_t dC_{3t}$ with $\beta_l S_t I_t dt + \sigma_1 S_t I_t dC_{1t}$ in the model (8), we get the uncertain SIR model

$$
\begin{aligned}
&dS_t = -\beta_l S_t I_t dt - \sigma_1 S_t I_t dC_{1t} \\
&dI_t = (\beta_l S_t I_t - \mu_l I_t) dt + \sigma_1 S_t I_t dC_{1t} - \sigma_5 I_t dC_{5t} \\
&dR_t = \mu_l I_t dt + \sigma_5 I_t dC_{5t}
\end{aligned}
$$

4 Parameter estimation with COVID-19 cases

Based on the COVID-19 cases in Mainland China, we estimate the parameters in the uncertain epidemic models in this section. The focus of this research is on the number of active cases, so we perform the parameter estimation with respect to the uncertain SIR model for simplicity. Beside, we accept the following stipulation in order to further simplify the parameter estimation process.

Stipulation: The number of susceptible individuals is a constant that is 1.4 billion.

According to National Bureau of Statistics of the People’s Republic of China, the population of Mainland China is about 1.40005 billion, while the number of confirmed cases in mainland China is 82,052 as of April 11, 2020. Hence, we stipulate that the number of susceptible individuals is 1.4 billion for simplicity.
Following the above stipulation, the uncertain SIR model (9) is simplified to an IR model

\[
\begin{aligned}
\frac{dI_t}{dt} &= (\beta I_t S_0 - \mu I_t)dt + \sigma_1 S_0 I_t dC_{1t} - \sigma_5 I_t dC_{5t} \\
\frac{dR_t}{dt} &= \mu I_t dt + \sigma_5 I_t dC_{5t}
\end{aligned}
\] (10)

where \( S_0 = 1.4 \) billion is the number of susceptible individuals, \( I_t \) and \( R_t \) denote the numbers of symptomatically infected individuals and removed individuals, respectively, \( C_{1t} \) and \( C_{5t} \) are two Liu processes which are assumed to be independent for the purpose of parameter estimation, and \( \beta, \mu, \sigma_1 \) and \( \sigma_5 \) are nonnegative parameters to be estimated.

The following parameter estimation process mainly follows the method of moments for uncertain differential equations by Yao and Liu (2020) and for multi-factor uncertain differential equations by Liu and Yang (2020). Consider the equation of removed individuals

\[
\frac{dR_t}{dt} = \mu I_t dt + \sigma_5 I_t dC_{5t}
\] (11)

in the uncertain IR model (10). A Euler difference scheme of the uncertain differential equation (11) is

\[
R_{t_{i+1}} - R_{t_i} = \mu I_{t_i} (t_{i+1} - t_i) + \sigma_5 I_{t_i} (C_{5t_{i+1}} - C_{5t_i})
\]

which can be rewritten as

\[
\frac{R_{t_{i+1}} - R_{t_i} - \mu I_{t_i} (t_{i+1} - t_i)}{\sigma_5 I_{t_i} (t_{i+1} - t_i)} = \frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}
\]

According to the definition of Liu process, the right expression

\[
\frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}
\]

is a standard normal uncertain variable \( \mathcal{N}(0, 1) \). Hence, when \( R_{t_i} \) and \( I_{t_i} \) are assigned the numbers of closed cases and active cases on the \( i \)-th day, respectively, the left expression

\[
\frac{R_{t_{i+1}} - R_{t_i} - \mu I_{t_i} (t_{i+1} - t_i)}{\sigma_5 I_{t_i} (t_{i+1} - t_i)}
\]

can be regarded as samples of a standard normal uncertain variable \( \mathcal{N}(0, 1) \). Note that the first and second moments of a standard normal uncertain variable \( \mathcal{N}(0, 1) \) are 0 and 1, respectively. The estimates \( \mu^*_I \) and \( \sigma^*_5 \) of \( \mu \) and \( \sigma_5 \) solve the system of
equations

\[
\begin{aligned}
&\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{R_{t_{i+1}} - R_{t_i} - \mu_I I_i (t_{i+1} - t_i)}{\sigma_5 I_i (t_{i+1} - t_i)} = 0 \\
&\frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{R_{t_{i+1}} - R_{t_i} - \mu_I I_i (t_{i+1} - t_i)}{\sigma_5 I_i (t_{i+1} - t_i)} \right)^2 = 1.
\end{aligned}
\]

That is,

\[
\mu_I^* = \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{R_{t_{i+1}} - R_{t_i}}{I_i (t_{i+1} - t_i)}, \tag{12}
\]

\[
\sigma_5^* = \left( \frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{R_{t_{i+1}} - R_{t_i}}{I_i (t_{i+1} - t_i)} - \mu_I^* \right)^2 \right)^{1/2}. \tag{13}
\]

Now consider the equation of symptomatically infected individuals

\[
dI_t = (\beta I S_0 I_t - \mu_I I_t) dt + \sigma_1 S_0 I_t dC_{1t} - \sigma_5 I_t dC_{5t}
\]

in the uncertain IR model (10). Substituting \( \mu_I \) and \( \sigma_5 \) with \( \mu_I^* \) and \( \sigma_5^* \) that have been determined in Eqs. (12) and (13), we get

\[
dI_t = (\beta I S_0 I_t - \mu_I^* I_t) dt + \sigma_1 S_0 I_t dC_{1t} - \sigma_5^* I_t dC_{5t}. \tag{14}
\]

A Euler difference scheme of the uncertain differential equation (14) is

\[
I_{t_{i+1}} - I_{t_i} = (\beta I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i) + \sigma_1 S_0 I_{t_i} (C_{1t_{i+1}} - C_{1t_i}) - \sigma_5^* I_{t_i} (C_{5t_{i+1}} - C_{5t_i})
\]

which can be rewritten as

\[
\frac{I_{t_{i+1}} - I_{t_i} - (\beta I S_0 I_{t_i} - \mu_I^* I_{t_i})(t_{i+1} - t_i)}{(\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i})(t_{i+1} - t_i)} = \frac{\sigma_1 S_0 I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} \frac{C_{1t_{i+1}} - C_{1t_i}}{t_{i+1} - t_i} - \frac{\sigma_5^* I_{t_i}}{\sigma_1 S_0 I_{t_i} + \sigma_5^* I_{t_i}} \frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}.
\]

It follows from the independence and definition of Liu processes \( C_{1t} \) and \( C_{5t} \) that

\[
\frac{C_{1t_{i+1}} - C_{1t_i}}{t_{i+1} - t_i}
\]

and

\[
\frac{C_{5t_{i+1}} - C_{5t_i}}{t_{i+1} - t_i}
\]
Table 1  Numbers of active cases of COVID-19 in mainland China from February 12 to April 11, 2020.  
Source COVID-19 Cases Reports by the National Health Commission of the People’s Republic of China  
National Health Commission of the People’s Republic of China (2020)

| Date  | Cases       |
|-------|-------------|
| 52,526| 55,748      |
| 51,606| 54,965      |
| 27,433| 32,652      |
| 12,094| 13,526      |
| 4735  | 1138        |

are independent standard normal uncertain variables with a common uncertainty distribution \( N(0, 1) \). Then according to the operational law of uncertain variables, we have

\[
\frac{\sigma_1 S_0 I_t}{\sigma_1 S_0 I_t + \sigma_s^2 I_t} C_{I_{t+1}} - C_{I_t} \sim N \left( 0, \frac{\sigma_1 S_0 I_t}{\sigma_1 S_0 I_t + \sigma_s^2 I_t} \right) = N(0, 1).
\]

Hence, when \( I_t \) are assigned the numbers of active cases on the \( i \)-th day, the left expression

\[
\frac{I_{t+1} - I_t - (\beta I S_0 I_t - \mu I^* I_t)(t_{i+1} - t_i)}{(\sigma_1 S_0 I_t + \sigma_s^* I_t)(t_{i+1} - t_i)}
\]

can be regarded as samples of a standard normal uncertain variable \( N(0, 1) \). Then by means of the method of moments, the estimates \( \beta_I^* \) and \( \sigma_1^* \) of \( \beta_I \) and \( \sigma_1 \) solve the system of equations

\[
\begin{cases}
\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{I_{t_{i+1}} - I_t - (\beta_I S_0 I_t - \mu_I^* I_t)(t_{i+1} - t_i)}{(\sigma_1 S_0 I_t + \sigma_s^* I_t)(t_{i+1} - t_i)} = 0 \\
\frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{I_{t_{i+1}} - I_t - (\beta_I S_0 I_t - \mu_I^* I_t)(t_{i+1} - t_i)}{(\sigma_1 S_0 I_t + \sigma_s^* I_t)(t_{i+1} - t_i)} \right)^2 = 1.
\end{cases}
\]

That is,

\[
\beta_I^* = \frac{1}{S_0} \left( \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{I_{t_{i+1}} - I_t}{I_t(t_{i+1} - t_i)} + \mu_I^* \right), \quad (15)
\]

\[
\sigma_1^* = \frac{1}{S_0} \left( \left( \frac{1}{n-1} \sum_{i=1}^{n-1} \left( \frac{I_{t_{i+1}} - I_t}{I_t(t_{i+1} - t_i)} - (\beta_I^* S_0 - \mu_I^*) \right)^2 \right)^{1/2} - \sigma_s^* \right). \quad (16)
\]

Now, we estimate the parameters \( \beta_I, \sigma_1, \mu_I, \) and \( \sigma_s \) in the uncertain IR model (10) based on the numbers of active cases (Table 1) and closed cases (Table 2) in...
Table 2  Numbers of closed cases of COVID-19 in mainland China from February 12 to April 11, 2020.  

| Date       | Cases |
|------------|-------|
| 25th June  | 7278  |
| 26th June  | 8103  |
| 27th June  | 9619  |
| 28th June  | 11,084|
| 29th June  | 12,614|
| 30th June  | 14,420|
| 1st July   | 16,380|
| 2nd July   | 18,273|
| 3rd July   | 20,500|
| 4th July   | 23,004|
| 5th July   | 25,330|
| 6th July   | 27,326|
| 7th July   | 29,986|
| 8th July   | 32,460|
| 9th July   | 35,239|
| 10th July  | 38,905|
| 11th July  | 41,837|
| 12th July  | 44,495|
| 13th July  | 47,374|
| 14th July  | 50,147|
| 15th July  | 52,837|
| 16th July  | 55,057|
| 17th July  | 56,768|
| 18th July  | 58,474|
| 19th July  | 60,162|
| 20th July  | 61,719|
| 21st July  | 63,033|
| 22nd July  | 64,633|
| 23rd July  | 65,962|
| 24th July  | 67,287|
| 25th July  | 68,730|
| 26th July  | 70,110|
| 27th July  | 70,962|
| 28th July  | 71,905|
| 29th July  | 72,838|
| 30th July  | 73,665|
| 31st July  | 74,398|
| 1st Aug    | 74,995|
| 2nd Aug    | 75,505|
| 3rd Aug    | 75,973|
| 4th Aug    | 76,436|
| 5th Aug    | 76,931|
| 6th Aug    | 77,338|
| 7th Aug    | 77,880|
| 8th Aug    | 78,266|
| 9th Aug    | 78,748|
| 10th Aug   | 79,074|
| 11th Aug   | 79,357|
| 12th Aug   | 79,550|
| 13th Aug   | 79,726|
| 14th Aug   | 80,077|
| 15th Aug   | 80,293|
| 16th Aug   | 80,409|
| 17th Aug   | 80,498|
| 18th Aug   | 80,612|
| 19th Aug   | 80,705|
| 20th Aug   | 80,791|
| 21st Aug   | 80,864|
| 22nd Aug   | 80,914|

Mainland China from February 12 to April 11, 2020, which are released in the COVID-19 cases reports by National Health Commission of the People’s Republic of China National Health Commission of the People’s Republic of China (2020). Details about the computation procedure are given as below.

Step 1  Set $S_0 = 1.4 \times 10^9$ and $n = 60$.

Step 2  Set February 12 as the first day, February 13 as the second day, and so forth. That is, $t_i = i$, and $I_{t_i}$ and $R_{t_i}$ are the $i$-th numbers in Tables 1 and 2, respectively, for $i = 1, 2, \cdots, 60$. For example, $I_{t_3} = 56873$ and $R_{t_5} = 12614$.

Step 3  Compute $\mu^*_I$ based on Eq. (12).

Step 4  Compute $\sigma^*_1$ based on Eq. (13).

Step 5  Compute $\beta^*_I$ based on Eq. (15).

Step 6  Compute $\sigma^*_5$ based on Eq. (16).

Following the above procedure, we get the estimates of the parameters $\beta_I, \sigma_1, \mu_I$ and $\sigma_5$ in the uncertain IR model (10) that are

$$\beta^*_I = 1.1745 \times 10^{-11}, \quad \sigma^*_1 = 7.7866 \times 10^{-12}, \quad \mu^*_I = 0.0785, \quad \sigma^*_5 = 0.0296.$$ 

Hence, the uncertain IR model for COVID-19 in Mainland China from February 12 to April 11, 2020 is

$$\begin{align} 
\frac{dI_t}{dt} &= -0.0620 \cdot I_t dt + 0.0109 \cdot I_t dC_{1t} - 0.0296 \cdot I_t dC_{5t}, \\
\frac{dR_t}{dt} &= 0.0785 \cdot I_t dt + 0.0296 \cdot I_t dC_{5t}. 
\end{align}$$ 

(17)

5 Prediction of the confirmed cases

Based on the uncertain IR model (10) with the estimated parameters, we predict the possible numbers of active cases ($I_t$) in this section.

Consider the equation of symptomatically infected individuals

$$dI_t = (\beta^*_I S_0 I_t - \mu^*_I I_t)dt + \sigma^*_1 S_0 I_t dC_{1t} - \sigma^*_5 I_t dC_{5t}$$

$\copyright$ Springer
which has a solution

\[ I_t = I_0 \exp \left( (\beta^*_I S_0 - \mu^*_I)t + \sigma^*_1 S_0 C_{1t} - \sigma^*_5 C_{5t} \right). \]

Please note that its uncertainty distribution is

\[ \Phi_t(x) = \mathbb{M}\{I_t \leq x\} = \mathbb{M}\{\sigma^*_1 S_0 C_{1t} - \sigma^*_5 C_{5t} \leq \ln x - (\beta^*_I S_0 - \mu^*_I)t - \ln I_0\} . \]

Since \( C_{1t} \) and \( C_{5t} \) are independent Liu processes with a common uncertainty distribution \( \mathcal{N}(0, t) \), we have

\[ \Phi_t(x) = \left( 1 + \exp \left( \frac{\pi((\beta^*_I S_0 - \mu^*_I)t + \ln I_0 - \ln x)}{\sqrt{3}(\sigma^*_1 S_0 + \sigma^*_5 t)} \right) \right)^{-1} . \] (18)

A prediction interval of possible numbers that \( I_t \) may take at the time \( s \) with a confidence \( \alpha \) is \([i_L, i_U]\) where \((i_L, i_U)\) is the optimal solution of the optimization problem

\[
\begin{aligned}
\min_{i_L, i_U} & \quad i_U - i_L \\
\text{subject to:} & \quad \Phi_s(i_U) - \Phi_s(i_L) \geq \alpha \\
& \quad i_L \text{ and } i_U \text{ are positive integers.}
\end{aligned}
\] (19)

Below are the details about the procedure to compute the prediction interval of active cases numbers based on the optimization model (19). For simplicity, we take the computation of prediction interval with confidence \( \alpha = 0.95 \) on April 15 as an example.

**Step 1** Set April 11, 2020 as the initial day, and set \( I_0 = 1138 \) in Eq. (18) which is the number of active cases on April 11, 2020. Then for April 15, i.e., the fourth day from the initial day, we have \( t = 4 \).

**Step 2** Set

\[ \beta^*_I = 1.1745 \times 10^{-11}, \quad \sigma^*_1 = 7.7866 \times 10^{-12}, \quad \mu^*_I = 0.0785, \quad \sigma^*_5 = 0.0296 \]

in Eq. (18) which have already been obtained in Sect. 3 based on the numbers of active cases and closed cases in Mainland China from February 12 to April 11, 2020. Then we get

\[ \Phi_4(x) = \left( 1 + \exp \left( \frac{\pi \cdot 6.7889}{0.1619} - \ln x \right) \right)^{-1} . \]
Step 3 Set $\alpha = 0.95$, and solve the optimization problem

$$\begin{align*}
\min_{x_1 > 0, x_2 > 0} & \quad x_2 - x_1 \\
\text{subject to:} & \quad \Phi_4(x_2) - \Phi_4(x_1) \geq \alpha
\end{align*}$$

by using the function “fmincon” in Matlab toolbox. The optimal solution is $(x_1, x_2) = (617.6, 1199.6)$.

Step 4 Noting that $i_L$ and $i_U$ are positive integers in the optimization problem (19), we have $i_L = 617$ and $i_U = 1200$ as $\Phi_4(1200) - \Phi_4(618) < 0.95$ and $\Phi_4(1199) - \Phi_4(617) < 0.95$.

Hence, with 95% confidence, the number of active cases on April 15 will be no greater than 1200 and no less than 617.

6 Conclusion

By means of uncertain differential equations, an uncertain SEIAR model was derived to describe the spread of an epidemic. Specifically, with the COVID-19 cases data released by the National Health Commission of the People’s Republic of China, the parameters in an uncertain epidemic model were estimated by following the method of moments. Furthermore, a method to predict the numbers of active cases was presented. Further research may consider the influence of vaccination to susceptible individuals, and design efficient vaccination strategies based on the uncertain SEIAR model.

Acknowledgements This research was supported by the Project of High-level Teachers in Beijing Municipal Universities in the Period of 13th Five-year Plan (No. CIT&TCD20190338), the Humanity and Social Science Foundation of Ministry of Education of China (No. 19YJAZH005), the Young Academic Innovation Team of Capital University of Economics and Business (No. QNTD202002), and the special fund of basic scientific research business fees of Beijing Municipal University of Capital University of Economics and Business (No. XRZ2020016).

References

Artalejo, J. R., Economou, A., & Lopez-Herrero, M. J. (2015). The stochastic SEIR model before extinction: Computational approaches. *Applied Mathematics and Computation*, 265, 1026–1043.

Chen, X., & Liu, B. (2010). Existence and uniqueness theorem for uncertain differential equations. *Fuzzy Optimization and Decision Making*, 9(1), 69–81.

Chen, X., & Gao, J. (2013). Uncertain term structure model of interest rate. *Soft Computing*, 17(4), 597–604.

Fang, J., Li, Z., Yang, F., & Zhou, M. (2018). Solution and $\alpha$-path of uncertain SIS epidemic model with standard incidence and demography. *Journal of Intelligent & Fuzzy Systems*, 35(1), 927–935.

Gao, R. (2016). Milne method for solving uncertain differential equations. *Applied Mathematics and Computation*, 274, 774–785.

Gray, A., Greenhalgh, D., Hu, L., Mao, X., & Pan, J. (2011). A stochastic differential equation SIS epidemic model. *SIAM Journal on Applied Mathematics*, 71(3), 876–902.

Ji, C., Jiang, D., & Shi, N. (2012). The behavior of an SIR epidemic model with stochastic perturbation. *Stochastic Analysis and Applications*, 30(5), 755–773.

Li, Z., Sheng, Y., Teng, Z., & Miao, H. (2017). An uncertain differential equation for SIS epidemic model. *Journal of Intelligent & Fuzzy Systems*, 33(4), 2317–2327.
Li, Z., & Teng, Z. (2019). Analysis of uncertain SIS epidemic model with nonlinear incidence and demography. *Fuzzy Optimization and Decision Making, 18*(4), 475–491.

Liu, B. (2007). *Uncertainty Theory* (2nd ed.). Berlin: Springer.

Liu, B. (2008). Fuzzy process, hybrid process and uncertain process. *Journal of Uncertain Systems, 2*(1), 3–16.

Liu, B. (2009). Some research problems in uncertainty theory. *Journal of Uncertain Systems, 3*(1), 3–10.

Liu, Y., & Liu, B. (2020). Estimating unknown parameters in uncertain differential equation by maximum likelihood estimation, Technical Report.

Liu, Z. (2020). Generalized moment estimation for uncertain differential equations, Technical Report.

Liu, Z., & Yang, Y. (2020). Pharmacokinetic model based on multifactor uncertain differential equation, Technical Report.

National Health Commission of the People’s Republic of China. (2020). Coronavirus disease (COVID-19) cases reports. http://www.nhc.gov.cn/xcs/yqtb/list_gzbd.shtml, Accessed April 12, 2020.

Sheng, Y., Gao, R., & Zhang, Z. (2017). Uncertain population model with age-structure. *Journal of Intelligent & Fuzzy Systems, 33*(2), 853–858.

Sheng, Y., Yao, K., & Chen, X. (2020). Least squares estimation in uncertain differential equations. *IEEE Transactions on Fuzzy Systems*. https://doi.org/10.1109/TFUZZ.2019.2939984.

Yang, X., & Gao, J. (2015). Linear-quadratic uncertain differential game with application to resource extraction problem. *IEEE Transactions on Fuzzy Systems, 24*(4), 819–826.

Yang, X., & Ralescu, D. A. (2015). Adams method for solving uncertain differential equations. *Applied Mathematics and Computation, 270*, 993–1003.

Yao, K., & Chen, X. (2013). A numerical method for solving uncertain differential equations. *Journal of Intelligent & Fuzzy Systems, 25*(3), 825–832.

Yao, K., Gao, J., & Gao, Y. (2013). Some stability theorems of uncertain differential equation. *Fuzzy Optimization and Decision Making, 12*(1), 3–13.

Yao, K., & Liu, B. (2020). Parameter estimation in uncertain differential equations. *Fuzzy Optimization and Decision Making, 19*(1), 1–12.

Zhang, Y., Gao, J., & Huang, Z. (2017). Hamming method for solving uncertain differential equations. *Applied Mathematics and Computation, 313*, 331–341.

Zhu, Y. (2010). Uncertain optimal control with application to a portfolio selection model. *Cybernetics and Systems: An International Journal, 41*(7), 535–547.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.