RIGIDITY OF AREA-MINIMIZING 2-SPERES IN $n$-MANIFOLDS WITH POSITIVE SCALAR CURVATURE

JINTIAN ZHU

Abstract. We prove that the least area of non-contractible immersed spheres is no more than $4\pi$ in any oriented compact manifold with dimension $n + 2 \leq 7$ which satisfies $R \geq 2$ and admits a map to $S^2 \times T^n$ with nonzero degree. We also prove a rigidity result for the equality case. This can be viewed as a generalization of the result in [2] to higher dimensions.

Keywords: Rigidity, Area-minimizing 2-spheres, Scalar Curvature

1. INTRODUCTION

Throughout this paper, $(M^n, \bar{g})$ will be an oriented closed Riemannian manifold with a nontrivial second homotopy group $\pi_2(M)$. We say that an immersed sphere $\Sigma$ in $M$ is non-contractible, if it is not homotopic to a single point. Since $\pi_2(M) \neq \{0\}$, there exists at least one non-contractible immersed sphere. Denote

(1.1) $\mathcal{F}_M := \{ \Sigma \subset M : \Sigma \text{ is a non-contractible immersed sphere} \}$

and

(1.2) $\mathcal{A}_{S^2}(M, \bar{g}) := \inf_{\Sigma \in \mathcal{F}_M} A_{\bar{g}}(\Sigma),$

where $A_{\bar{g}}(\Sigma)$ represents the area of $\Sigma$ with the induced metric from $\bar{g}$. We also use $R_{\bar{g}}$ to represent the scalar curvature of the metric $\bar{g}$ in the following context.

In the paper [2], H. Bray, S. Brendle and A. Neves proved the following result:

Theorem 1.1. If $(M^3, \bar{g})$ satisfies $R_{\bar{g}} \geq 2$, then we have $\mathcal{A}_{S^2}(M, \bar{g}) \leq 4\pi$. Furthermore, the equality implies that the universal covering $(\hat{M}, \hat{g})$ is $S^2 \times \mathbb{R}$.

It is natural to ask whether such result still holds for the dimension $n \geq 4$. Unfortunately, the result cannot be true in general. An easy example is as follows. Let $M = S^2(r_1) \times S^m(r_2)$ with $m \geq 3$ such that the scalar curvature of $M$ is 2. With appropriate value for $r_2$ to be taken, the scalar curvature of $S^m(r_2)$ can be arbitrarily close to 2, from which it follows that the area of $S^2(r_1)$ can be arbitrarily large. This yields that no upper bound depending only on the scalar curvature can be obtained for the least area of non-contractible spheres in this case. Hopefully, if we impose further conditions on $(M^n, \bar{g})$, such result can still be true. Namely, we have the following theorem:
Theorem 1.2. For $n + 2 \leq 7$, let $(M^{n+2}, g)$ be an oriented closed Riemannian manifold with $R_g \geq 2$, which admits a non-zero degree map $f : M \to S^2 \times T^n$. Then $A_{S^2}(M, g) \leq 4\pi$. Furthermore, the equality implies that the universal covering $(\hat{M}, \hat{g})$ is $S^2 \times \mathbb{R}^n$.

Remark 1.3. The restriction $n + 2 \leq 7$ comes from the regularity issue of area-minimizing hypersurface.

Our inspiration for Theorem 1.2 comes from M. Gromov’s work in [4]. He proved that

Theorem 1.4 ([4], P. 653, and P. 679). If $(V^n, g)$ is an over-torical band with $R_g \geq n(n-1)$ and $n \leq 8$, then

$$\text{width}(V, g) \leq \frac{2\pi}{n}.$$  

When the equality holds, the universal covering of the extreme band is $\mathbb{R}^{n-1} \times O(n-1)$-invariant.

In the above theorem, an over-torical band means a Riemannian manifold $(V, g)$ with boundary $\partial V = \partial V_+ \sqcup \partial V_-$, which admits a nonzero map $f : (V, \partial V_+) \to (T^{n-1} \times [-1,1], T^{n-1} \times \{\pm 1\})$. The width of $(V, g)$ is defined to be the distance between $\partial V_+$ and $\partial V_-$. If one just takes $V$ to be the torical band $T^{n-1} \times [-1,1]$, then the theorem actually tells us that the lower positive bound of the scalar curvature gives an upper bound for the least distance transverse to $T^{n-1}$ slices. From this point of view, our theorem deals with the area case in a similar spirit of M. Gromov’s result.

We also recall that R. Schoen and S.T. Yau proved the following theorem:

Theorem 1.5 ([7], Corollary 2). Let $M$ be a closed Riemannian manifold which admits a nonzero degree to $T^n$ with $n \leq 7$. Then the only possible metric with nonnegative scalar curvature is the flat one.

Based on this, our theorem seems to have a similar spirit of this result as well. If one writes $T^n$ as $T^2 \times T^{n-2}$ and considers the above theorem as one for the torus case, then our theorem can be thought as a quantitative one for the sphere case.

We have to say that Theorem 1.2 is one of the few rigidity results in high dimensions. Actually, the relationship of the scalar curvature and minimal 2-surfaces is far from clear when the dimension of the ambient manifold is greater than 4, although there have been various results in 3-dimensional case. Apart from Theorem 1.1, we mention that there are similar rigidity results for area-minimizing projective planes and min-max spheres in three dimensional manifolds. We refer readers to [1] and [5] for further information. It is a very interesting topic to prove some counterparts of these results in higher dimensions.

In high dimension case, there are some new challenges in the proof of Theorem 1.2. To show the upper bound for the least area of non-contractible immersed spheres, it is no longer valid to analyzing the second variation of area for an area-minimizing sphere in the class of non-contractible spheres. The first trouble is that an area-minimizing sphere may fail to be immersed due to possible branched points, which however can be ruled out in dimension 3. While even if the area-minimizing sphere is immersed, we cannot obtain enough control on its area from the second variation formula due to higher codimensions.
To overcome these challenges, instead of taking an area-minimizing sphere, we construct directly a non-contractible embedded one with area no more than $4\pi$. To be precise, we apply the Torical Symmetrization argument from [4] to find a slicing (see Definition 2.1)

$$(\Sigma_2, g_2) \mapsto (\Sigma_3, g_3) \mapsto \cdots \mapsto (\Sigma_{n+1}, g_{n+1}) \mapsto (\Sigma_{n+2}, g_{n+2}) = (M, \tilde{g}),$$

where $\Sigma_2$ appears to be a non-contractible embedded sphere and $\Sigma_2 \times T^n$ admits a warped product metric

$$\tilde{g}_2 = g_2 + \sum_{i=2}^{n+1} u_i^2 dt_i^2$$

with $R_{\tilde{g}_2} \geq 2$. In this case, we are able to show $A_{\tilde{g}_2}(\Sigma_2) \leq 4\pi$ from a straightforward calculation, which then gives $A_{S^2}(M, \tilde{g}) \leq 4\pi$. To present a more clear picture, let me briefly illustrate how to obtain a desired slicing from the Torical Symmetrization argument. The essential step is as follows. Suppose that we are given a Riemannian manifold $(\bar{M}, \bar{g})$, which admits a nonzero degree map to $S^2 \times T^n$, then from geometric measure theory we can find an area-minimizing hypersurface $(\bar{M}, \bar{\tilde{g}})$ in the pull-back homology class of $S^2 \times T^{n-1}$. Taking the first eigenfunction $\bar{u}$ of the Jacobi operator $\bar{J}$, we can obtain a new manifold $(\bar{M}', \bar{\tilde{g}}')$ with $\bar{M}' = \bar{M} \times S^1$ and $\bar{g}' = \bar{u}^2 dt^2 + \bar{\tilde{g}}$, which satisfies $R_{\tilde{g}'} \geq R_{\tilde{g}}$. The advantage of this construction is a lift in symmetry. That is, if $(\bar{M}, \bar{\tilde{g}})$ is $T^m$-symmetric, then $(\bar{M}', \bar{\tilde{g}}')$ is $T^{m+1}$-symmetric. Using the step stated above for several times and following the procedure as shown below, we can obtain the desired slicing.

![Diagram](image)

**Figure 1.** We start with $(\Sigma_{n+2}, \tilde{g}_{n+2}) = (M, \tilde{g})$ to find an area-minimizing surface $(\tilde{\Sigma}_{n+1}, \tilde{g}_{n+1})$. From $(\tilde{\Sigma}_{n+1}, \tilde{g}_{n+1})$ we construct the warped product $(\tilde{\Sigma}_{n+1}, \tilde{g}_{n+1})$ with $\tilde{\Sigma}_{n+1} = \Sigma_{n+1} \times S^1$, and then we use this as a new starting manifold to find another area-minimizing surface $(\tilde{\Sigma}_n, \tilde{g}_n)$. By warped product, we obtain $(\tilde{\Sigma}_n, \tilde{g}_n)$. ... After repeating again and again, we obtain $\tilde{\Sigma}_k$ for all $k \geq 2$. From higher and higher symmetry, we can write $\tilde{\Sigma}_k = \Sigma_k \times T^{n+2-k}$, and these $\Sigma_k$ provide us the desired slicing.

The proof for the rigidity in the equality case turns out to be very subtle. From $A_{S^2}(M, \tilde{g}) = 4\pi$, it is easy to show that $(\Sigma_2, g_2)$ is isometric to the standard sphere, therefore it is enough to show that there exist local isometries $\Phi_k : \Sigma_k \times \mathbb{R} \to \Sigma_{k+1}$ for $k$ from 2 to $n+1$. To obtain this result, we choose to prove a stronger one that there exists a local isometry $\tilde{\Phi}_k : \Sigma_k \times \mathbb{R} \to \Sigma_{k+1}$. For simplicity, we will only explain our idea in the case $k = 2$, since the rest cases follow from the same argument. Through a calculation, it is direct to show $R_{\tilde{g}_2} = R_{\tilde{g}_2} = 2$ from $A_{S^2}(M, \tilde{g}) = 4\pi$. From the construction, this implies that $\tilde{\Sigma}_2$ is totally geodesic and has vanished normal Ricci curvature in $\tilde{\Sigma}_3$. To show the existence of $\tilde{\Phi}_2$, we adopt the idea from [3] Theorem 1.6] to construct an area-minimizing foliation around $\tilde{\Sigma}_2$. From the inverse function theorem, it is not
difficult to show the existence of a foliation \( \{ \Sigma_{2,t} \}_{-\epsilon \leq t \leq \epsilon} \) with constant mean curvature. The difficulty here is to show that the mean curvatures \( \tilde{H}_{2,t} \) have the correct sign such that each slice in the foliation is also area-minimizing. At this stage, we follow the idea from [4, page 679] to rule out the worried situation. Assuming that one of the mean curvatures \( \tilde{H}_{2,t} \) has a wrong sign, such as \( \tilde{H}_{2,t_0} > 0 \) for \( t_0 > 0 \), by minimizing an appropriate brane functional in the form of

\[ B = A(\tilde{\Sigma}) - \delta V(\tilde{\Omega}), \]

we can find a soap bubble \( (\hat{\Sigma}_k, \hat{g}_k) \) in \( \bar{\Sigma}_{k+1} \) such that \( \hat{\Sigma}_k \times S^1 \) has a warped product metric

\[ \hat{g}_{k+1} = \hat{g}_k + u_k^2 dt_{k+1}^2 \]

with \( R_{\hat{g}_{k+1}} \geq 2 + \delta^2 \). With the Torical Symmetrization argument applied to \( \hat{\Sigma}_k \times S^1 \), there will be a non-contractible embedded sphere in \( (M, \bar{g}) \) with area strictly less than \( 4\pi \), which leads to a contradiction. Consequently, each surface in the constructed foliation must be area-minimizing. Therefore they are totally geodesic and have vanished normal Ricci curvature as well. From this, it is quick to show that \( \bar{\Sigma}_3 \) splits as \( \bar{\Sigma}_2 \times (-\epsilon, \epsilon) \) around \( \bar{\Sigma}_2 \). Through a continuous argument, we can obtain the local isometry \( \tilde{\Phi}_2 \).

In the following, the article will be organized as follows. In section 2, we use the torical symmetrization argument to construct a non-contractible embedded sphere with area no more than \( 4\pi \). In section 3, we prove the rigidity for the case of equality.

**Acknowledgments.** This research is partially supported by the NSFC grants No. 11671015 and 11731001. The author also would like to thank Professor Yuguang Shi for many encouragements and discussions.

## 2. Torical Symmetrization

In this section, we show how to use Torical Symmetrization argument to construct an embedded non-contractible sphere with area no more than \( 4\pi \). For the convenience to state our results, we introduce the following definition first.

**Definition 2.1.** A sequence of isometric immersions

\[ (\Sigma_0, g_0) \varphi_1 \rightarrow (\Sigma_1, g_1) \varphi_2 \rightarrow \cdots \varphi_m \rightarrow (\Sigma_m, g_m) \]

is called a slicing if each immersion \( \varphi_i \) is codimension one for \( i = 1, 2, \ldots, m \).

In the following, we denote the projection map from \( S^2 \times T^i \) to \( S^2 \times T^j \) by \( \pi^i_j \) for any \( i \geq j \). Using the Torical Symmetrization argument, we can obtain the following proposition:

**Proposition 2.2.** For \( n + 2 \leq 7 \), let \( (M^{n+2}, \bar{g}) \) be an oriented closed Riemannian manifold with \( R_{\bar{g}} \geq 2 \), which admits a non-zero degree map \( f : M \rightarrow S^2 \times T^n \). Then there exists a slicing

\[ (\Sigma_2 \approx S^2, g_2) \varphi_1 \rightarrow (\Sigma_3, g_3) \varphi_2 \rightarrow \cdots \varphi_m \rightarrow (\Sigma_m, g_m) \]

such that the map \( f_k = \pi^n_k \circ f|_{\Sigma_k} : \Sigma_k \rightarrow S^2 \times T^{k-2} \) has non-zero degree. Furthermore, there exist positive functions \( u_k : \Sigma_k \rightarrow \mathbb{R}, k = 2, 3, \ldots, n + 1 \), such that the manifolds \( (\Sigma_k, \bar{g}_k) \) with
\[ \bar{\Sigma}_k = \Sigma_k \times T^{n+2-k} \text{ and} \]

\[ \tilde{g}_k = g_k + \sum_{i=k}^{n+1} (u_i|_{\Sigma_k})^2 d\ell_{i+1}^2 \]

satisfies \( R_{\tilde{g}_k} \geq 2 \), and that \((\bar{\Sigma}_k, \tilde{g}_k)\) is area-minimizing in \((\Sigma_{k+1}, \tilde{g}_{k+1})\), where \( \bar{\Sigma}_k = \Sigma_k \times T^{n+1-k} \) and

\[ \tilde{g}_k = g_k + \sum_{i=k+1}^{n+1} (u_i|_{\Sigma_k})^2 d\ell_{i+1}^2. \]

**Proof.** We are going to construct the desired slicing with an induction argument. For convenience, let us denote \((\Sigma_{n+2}, \bar{g}_{n+2}, u_{n+2}, f_{n+2}) = (M, \bar{g}, 1, f)\). Without loss of generality, we assume that \( f \) is a smooth function. Now we state how to obtain \((\Sigma_k, g_k, u_k, f_k)\) from \((\Sigma_{k+1}, g_{k+1}, u_{k+1}, f_{k+1})\) for \( k = n + 1, n, \ldots, 2 \). Define the map

\[ F_{k+1} = (f_{k+1}, id) : \Sigma_{k+1} \times T^{n+1-k} \rightarrow S^2 \times T^n = S^2 \times T^{k-1} \times T^{n+1-k}. \]

According to Sard’s theorem, there exists a \( \theta_{k-1} \) such that the preimage

\[ S = F_{k+1}^{-1}(S^2 \times T^{k-2} \times \{\theta_{k-1}\} \times T^{n+1-k}) = f_{k+1}^{-1}(S^2 \times T^{k-2} \times \{\theta_{k-1}\}) \times T^{n+1-k} \]

is a smooth 2-sided embedded hypersurface. It is easy to see that \( \deg \pi_{n-1}^n \circ f_{k+1}|_S = \deg f_{k+1} \neq 0 \), from which we know that \( [S] \) represents a nontrivial homology class. By geometric measure theory, there exists a smooth embedded, 2-sided, area-minimizing hypersurface \((\bar{\Sigma}_k, \tilde{g}_k)\) in the homology class \([S]\) in Riemannian manifold \((\bar{\Sigma}_{k+1}, \tilde{g}_{k+1})\), where \( \tilde{g}_k \) is the induced metric and \( \bar{\Sigma}_k = \Sigma_{k+1} \times T^{n+1-k}, \quad \tilde{g}_k = g_k + \sum_{i=k+1}^{n+1} (u_i|_{\Sigma_{k+1}})^2 d\ell_{i+1}^2. \)

We claim that \( \bar{\Sigma}_k \) is \( T^{n+1-k} \)-invariant. Notice that \((\bar{\Sigma}_{k+1}, \tilde{g}_{k+1})\) is \( T^{n+1-k} \)-invariant, taking isometric \( S^1 \)-action \( \rho_{i,\theta} \), \( i = k+2, k+3, \ldots, n+2 \), such that

\[ \rho_{i,\theta} ((x, t_{k+2}, \ldots, t_i, \ldots, t_{n+2})) = (x, t_{k+2}, \ldots, t_i + \theta, \ldots, t_{n+2}), \quad x \in \Sigma_{k+1}, \]

then \( \rho_{i,\theta} \) induces a smooth variation vector field \( X = \phi \nu \) on \( \bar{\Sigma}_k \), where \( \nu \) is a fixed unit normal vector field. If \( \phi \) is not identical to zero, from the fact that \( \bar{\Sigma}_k \) is area-minimizing and \( \rho_{i,\theta} \) is isometry, we see that \( \phi \) is the first eigenfunction of the Jacobi operator \( \tilde{J}_k \), which implies that \( \phi \) has a fixed sign. This yields that the algebraic intersection number of the circle \( S^1_i \) and \( \bar{\Sigma}_k \) is nonzero. However, notice that \( S^1_i \) lies on the hypersurface \( S \), the algebraic intersection of \( S^1_i \) and \( S \) is zero, which leads to a contradiction. Therefore, we have \( \phi \equiv 0 \) and that \( \bar{\Sigma}_k \) is \( \rho_{i,\theta} \)-invariant. Ranging \( i \) from \( k+2 \) to \( n+2 \), we know that \( \bar{\Sigma}_k \) is \( T^{n+1-k} \)-invariant, and it follows that \( \Sigma_k = \Sigma_{k+1} \times T^{n+1-k} \). Denote \( f_k = \pi_{k-2}^n \circ f|_{\Sigma_k} = \pi_{k-2}^{k-1} \circ f_{k+1}|_{\Sigma_k} \). Notice that \( \bar{\Sigma}_k \) is homologic to \( S \), we have

\[ \deg f_k = \deg \pi_{n-1}^n \circ f_{k+1}|_{\Sigma_k} = \deg \pi_{n-1}^n \circ f_{k+1}|_S = \deg f_{k+1} = \deg f_{k+1} \neq 0. \]

\(^1\)In this paper, area-minimizing means to have the least area in the homotopy class, although the hypersurface may have some much stronger area-minimizing property.
It is possible that $\Sigma_k$ have several connected components, but we can choose one of these components, still denoted by $\Sigma_k$, satisfying $\deg f_k \neq 0$. In this case, it follows from [3, Lemma 33.4] that $\tilde{\Sigma}_k = \Sigma_k \times T^{n+1-k}$ is still area-minimizing. Let $g_k$ be the induced metric of $\Sigma_k$ from $(\Sigma_{k+1}, g_{k+1})$, then

$$\tilde{g}_k = g_k + \sum_{i=k+1}^{n+1} (u_i|_{\Sigma_k})^2 dt_{i+1}^2,$$

and the last statement in the proposition follows easily.

In the following, we construct the desired function $u_k$. Since $\tilde{\Sigma}_k$ is area-minimizing, the Jacobi operator

$$(2.2) \quad \tilde{J}_k = -\Delta_{\tilde{g}_k} - (\text{Ric}_{g_{k+1}}(\tilde{\nu}_k, \tilde{\nu}_k) + \|\tilde{A}_k\|^2) = -\Delta_{\tilde{g}_k} - \frac{1}{2} \left( R_{g_{k+1}}|_{\tilde{\Sigma}_k} - R_{\tilde{g}_k} + \|\tilde{A}_k\|^2 \right)$$

is nonnegative, where $\tilde{\nu}_k$ and $\tilde{A}_k$ are the unit normal vector field and the corresponding second fundamental form. Denote $\tilde{u}_k$ to be the first eigenfunction of $\tilde{J}_k$ with respect to the first eigenvalue $\tilde{\lambda}_{1,k} \geq 0$. Since the first eigenspace is dimension one, $\tilde{u}_k$ is also $T^{n+1-k}$-invariant, and hence $\tilde{u}_k$ can be viewed as a function $u_k$ over $\Sigma_k$. Given the smooth metric $\tilde{g}_k = g_k + \sum_{i=k+1}^{n+1} (u_i|_{\Sigma_k})^2 dt_{i+1}^2$, on $\tilde{\Sigma}_k$, direct calculation shows

$$(2.3) \quad R_{\tilde{g}_k} = R_{\tilde{g}_k} - 2\Delta_{\tilde{g}_k} \frac{\tilde{u}_k}{\tilde{u}_k} = R_{g_{k+1}}|_{\Sigma_k} + \|\tilde{A}_k\|^2 + 2\tilde{\lambda}_{1,k} \geq 2.$$

Through an induction argument from $k = n + 1$ to $k = 2$, we can obtain the desired slicing

$$(\Sigma_2, g_2) \varphi (\Sigma_3, g_3) \varphi \cdots \varphi (\Sigma_{n+1}, g_{n+1}) \varphi (\Sigma_{n+2}, g_{n+2}) = (M, \tilde{g}).$$

The only thing needs to be verified is that $\Sigma_2$ is a 2-sphere. Otherwise, $\Sigma_2$ is a surface with genus $g \geq 1$, and then there exists a map $\tilde{f} : \Sigma_2 \times T^n \to T^{n+2}$ with nonzero degree. Combined with [7, Corollary 2], this contradicts to the fact $R_{g_2} \geq 2$. \hfill $\square$

The following lemma yields an upper bound for the area of $\Sigma_2$.

**Lemma 2.3.** Let $(S^2 \times T^n, g)$ be a Riemannian manifold with $R_g \geq 2$ and

$$(2.4) \quad \tilde{g} = g + \sum_{i=2}^{n+1} u_i^2 dt_{i+1}^2,$$

where $g$ is a smooth metric on $S^2$ and $u_i : S^2 \to \mathbb{R}$ are positive smooth functions for $i = 2, 3, \ldots, n + 1$. Then we have

$$A_{\tilde{g}}(S^2) \leq 4\pi,$$

where equality holds if and only if $(S^2, g)$ is isometric to the standard sphere and $u_i$ are positive constants for $i = 2, 3, \ldots, n + 1$. 


Proof. Through direct calculation (refer to [6] Lemma 2.5), we have

\[ R_g = R_g - 2 \sum_{i=2}^{n+1} u_i^{-1} \Delta_g u_i - 2 \sum_{2 \leq i < j \leq n+1} \langle \nabla_g \log u_i, \nabla_g \log u_j \rangle. \]

Integrating over \( S^2 \), we obtain

\[ 2A_g(S^2) \leq \int_{S^2} R_g \, d\mu_g = 8\pi - \sum_{i=2}^{n+1} \int_{S^2} |\nabla_g \log u_i|^2 \, d\mu_g - \int_{S^2} \sum_{i=2}^{n+1} \nabla_g \log u_i \, d\mu_g, \]

where we have used the fact

\[ u_i^{-1} \Delta_g u_i = \Delta_g \log u_i + |\nabla_g \log u_i|^2, \quad i = 2, 3, \ldots, n+1. \]

Then it is clear that \( A_g(S^2) \leq 4\pi \). When the equality holds, we see that \( u_i \) are positive constants for \( i = 2, 3, \ldots, n+1 \), and then \( R_g = R_g \geq 2 \). From the Gauss-Bonnet formula, we obtain \( R_g \equiv 2 \), which implies that \( (S^2, g) \) is isometric to the standard sphere. \( \square \)

**Corollary 2.4.** Let \((M, \tilde{g})\) be as in Theorem 1.2. Then we have \( A_{S^2}(M, \tilde{g}) \leq 4\pi \).

Proof. From Proposition 2.2, we can find an embedded sphere \( \Sigma_2 \) such that \( \pi_0^* \circ f|\Sigma_2 : \Sigma_2 \to S^2 \) has nonzero degree. This implies that \( \Sigma_2 \) is non-contractible. By Lemma 2.3, we know \( A_g(\Sigma_2) \leq 4\pi \). Therefore, \( A_{S^2}(M, \tilde{g}) \leq A_g(\Sigma_2) \leq 4\pi \). \( \square \)

3. The Case of Equality

In this section, \((M^{n+2}, \tilde{g})\) is always an oriented closed Riemannian manifold with dimension \( n + 2 \leq 7 \) and scalar curvature \( R_g \geq 2 \), which admits a non-zero degree map \( f : M \to S^2 \times T^n \).

Also, we use

\[ (\Sigma_2, g_2) \vdash (\Sigma_3, g_3) \vdash \cdots \vdash (\Sigma_{n+1}, g_{n+1}) \vdash (\Sigma_{n+2}, g_{n+2}) = (M, \tilde{g}) \]

to denote an arbitrary slicing constructed as in Proposition 2.2, and we adapt the same notation there.

The first lemma below is the starting point for our rigidity result.

**Lemma 3.1.** If \( A_{S^2}(M, \tilde{g}) = 4\pi \), then \((\Sigma_2, g_2)\) is isometric to the standard sphere, \( R_{g_2} = 2 \) and \( u_i|\Sigma_2 \) are positive constants for \( i = 2, 3, \ldots, n+1 \).

Proof. It follows from Proposition 2.2 that \((\Sigma_2, g_2)\) satisfies \( R_{g_2} \geq 2 \), where

\[ \tilde{\Sigma}_2 = \Sigma_2 \times T^n \quad \text{and} \quad \tilde{g}_2 = g_2 + \sum_{i=2}^{n+1} (u_i|\Sigma_2)^2 \, dt_{i+1}^2. \]

If \( A_{S^2}(M, \tilde{g}) = 4\pi \), from Lemma 2.3 we have \( A_{g_2}(\Sigma_2) = 4\pi \). As a result, \((\Sigma_2, g_2)\) is isometric to the standard sphere, and \( u_i|\Sigma_2 \) are positive constants. Therefore, we have \( R_{g_2} = 2 \). \( \square \)

In the following, we prove the infinitesimal rigidity for \( \tilde{\Sigma}_k \).

**Lemma 3.2.** Fix some \( 2 \leq k \leq n+1 \), if \( R_{g_k} \equiv 2 \) and \( u_i|\Sigma_k \) are positive constants for \( i = k, k+1, \ldots, n+1 \), then \( \tilde{\Sigma}_k \) is totally geodesic in \( \tilde{\Sigma}_{k+1} \) and satisfies \( \text{Ric}_{g_{k+1}}(\tilde{\nu}_k, \tilde{\nu}_k) = 0 \).
Proof. Recall from (2.3) that
\[ R_{g_k} \geq R_{g_{k+1}}|_{\Sigma_k} + \|\tilde{A}_k\|^2 + 2\tilde{\lambda}_{1,k}, \]
combined with the facts \( R_{g_k} = 2 \), \( R_{g_{k+1}} \geq 2 \) and \( \tilde{\lambda}_{1,k} \geq 0 \), we obtain \( R_{g_{k+1}}|_{\Sigma_k} \equiv 2 \), \( \tilde{A}_k \equiv 0 \) and \( \tilde{\lambda}_{1,k} = 0 \). Since \( u_k \) is a positive constant on \( \Sigma_k \), we also know \( R_{g_k} \equiv 2 \), which implies \( \text{Ric}_{g_{k+1}}(\tilde{\nu}_k, \tilde{\nu}_k) = 0 \). \( \square \)

As in [2] Proposition 5], we use the infinitesimal rigidity of \( \tilde{\Sigma}_k \) to construct a foliation of constant mean curvature hypersurfaces around \( \tilde{\Sigma}_k \) in \( \tilde{\Sigma}_{k+1} \).

**Lemma 3.3.** If \( \tilde{\Sigma}_k \) is an area-minimizing hypersurface in \( \tilde{\Sigma}_{k+1} \) with vanished second fundamental form and normal Ricci curvature, then we can construct a local foliation \( \{\tilde{\Sigma}_{k,t}\}_{-\epsilon \leq t \leq \epsilon} \) in \( \tilde{\Sigma}_{k+1} \) such that \( \tilde{\Sigma}_{k,t} \) are of constant mean curvatures and \( \tilde{\Sigma}_{k,0} = \tilde{\Sigma}_k \). Furthermore, \( \tilde{\Sigma}_{k,t} \) is \( T^{n+1-k} \)-invariant.

**Proof.** Consider the map
\[ (3.1) \quad \psi : C^{2,\alpha}(\tilde{\Sigma}_k) \to \tilde{C}^{\alpha}(\tilde{\Sigma}_k) \times \mathbb{R}, \quad f \mapsto (\tilde{H}_f - \int_{\tilde{\Sigma}_k} \tilde{H}_f \, d\mu, \int_{\tilde{\Sigma}_k} f \, d\mu), \]
where \( \tilde{H}_f \) is the mean curvature of the graph over \( \tilde{\Sigma}_k \) with graph function \( f \) and the space \( \tilde{C}^{\alpha}(\tilde{\Sigma}_k) = \{ \phi \in C^{\alpha}(\tilde{\Sigma}_k) : \int_{\tilde{\Sigma}_k} \phi \, d\mu = 0 \} \).

Since \( \text{Ric}_{g_{k+1}}(\tilde{\nu}_k, \tilde{\nu}_k) = 0 \) and \( \tilde{A}_k = 0 \), it is clear that the linearized operator of \( \psi \) at \( f = 0 \)
\[ (3.2) \quad D\psi|_0 : C^{2,\alpha}(\tilde{\Sigma}_k) \to \tilde{C}^{\alpha}(\tilde{\Sigma}_k) \times \mathbb{R}, \quad g \mapsto (-\Delta g, \int_{\tilde{\Sigma}_k} g \, d\mu) \]
is invertible. From the inverse function theorem, we can find a family of function \( f_t : \tilde{\Sigma}_k \to \mathbb{R} \) with \( t \in (-\epsilon, \epsilon) \) with the following properties:
- The functions \( f_t \) satisfies \( f_0 \equiv 0 \),
- The graphs \( \tilde{\Sigma}_{k,t} \) over \( \tilde{\Sigma}_k \) with graph function \( f_t \) has constant mean curvature.

From (3.3), with the value of \( \epsilon \) decreased a little bit, the speed \( \partial_t f_t \) will be positive everywhere, from which it follows that the graphs \( \tilde{\Sigma}_{k,t} \) form a foliation around \( \tilde{\Sigma}_k \) for \( -\epsilon \leq t \leq \epsilon \). The \( T^{n+1-k} \)-invariance of \( \tilde{\Sigma}_{k,t} \) is due to the uniqueness from the inverse function theorem. \( \square \)

We are ready to prove the existence of local isometries \( \tilde{\Phi}_k \).

**Proposition 3.4.** If \( A_{g_2}(M, \bar{g}) = 4\pi \), then for any slicing constructed as in Proposition 2.2 and any \( 2 \leq k \leq n+1 \), we have \( R_{g_k} = 2 \) and that \( u_i|_{\Sigma_k} \) are positive constants for \( i \) from \( k \) to \( n+1 \). In addition, there exists a local isometry \( \tilde{\Phi}_k : \tilde{\Sigma}_k \times \mathbb{R} \to \tilde{\Sigma}_{k+1} \).

**Proof.** From Lemma 3.1, for any slicing constructed as in Proposition 2.2 we have \( R_{g_2} = 2 \) and that \( u_i|_{\Sigma_k} \) are positive constants for \( i \) from \( 2 \) to \( n+1 \). For the rest cases, we are going to apply
the induction argument. Assuming that we already obtain \( R_{\tilde{g}_k} \equiv 2 \) and that \( u_i|_{\Sigma_k} \) are positive constants for \( i \) from \( k \) to \( n + 1 \), using Lemma 3.2 we know that \( \tilde{\Sigma} \) is totally geodesic and has vanished normal Ricci curvature in \( \tilde{\Sigma}_{k+1} \). From Lemma 3.3 there exists a constant mean curvature foliation \( \{ \tilde{\Sigma}_{k,t} \}_{-\epsilon \leq t \leq \epsilon} \) in \( \tilde{\Sigma}_{k+1} \) with \( \tilde{\Sigma}_{k,0} = \tilde{\Sigma}_k \). Denote \( \tilde{H}_{k,t} \) to be the mean curvature of \( \tilde{\Sigma}_{k,t} \), we claim that \( \tilde{H}_{k,t} \equiv 0 \) for any \( t \in (-\epsilon, \epsilon) \). It suffices to prove this for nonnegative \( t \), since the other side follows from the same argument. Suppose that the consequence is not true, then there exists a \( t_0 \) such that \( \tilde{H}_{k,t_0} > 0 \), since otherwise \( \tilde{H}_{k,t} \leq 0 \) and the area-minimizing property of \( \tilde{\Sigma}_t \) will imply \( \tilde{H}_{k,t} \equiv 0 \). Denote \( \tilde{\Omega}_{k,t_0} \) to be the region enclosed by \( \tilde{\Sigma}_{k,t_0} \) and \( \tilde{\Sigma}_k \), and define the brane functional

\[
B(\tilde{\Omega}) = A(\partial \tilde{\Omega}\setminus \tilde{\Sigma}_k) - \delta V(\tilde{\Omega}), \quad 0 < \delta < \tilde{H}_{k,t_0},
\]

where \( \tilde{\Omega} \) is any Borel subset of \( \tilde{\Omega}_{k,t_0} \) with finite perimeter and \( \tilde{\Sigma}_k \subset \partial \tilde{\Omega} \). Due to \( 0 < \delta < \tilde{H}_{k,t_0} \), the hypersurfaces \( \tilde{\Sigma} \) and \( \tilde{\Sigma}_{k,t_0} \) serve as barriers, therefore we can find a Borel set \( \tilde{\Omega}_{k,t_0} \) minimizing the brane functional \( B \), for which \( \tilde{\Sigma}_k = \partial \tilde{\Omega}_{k,t_0}\setminus \tilde{\Sigma}_k \) is a smooth 2-sided hypersurface disjoint from \( \tilde{\Sigma}_k \) and \( \tilde{\Sigma}_{k,t_0} \). A similar argument as in Proposition 2.2 gives \( \tilde{\Sigma}_k \) is \( T^{n+1-k} \)-invariant, so we can write \( \tilde{\Sigma}_k \) as \( \tilde{\Sigma}'_k \times T^{n+1-k} \), where \( \tilde{\Sigma}'_k \) is a hypersurface in \( (\Sigma_{k+1}, g_{k+1}) \). Denote the induced metric of \( \tilde{\Sigma}'_k \) by \( \hat{g}'_k \), then the induced metric of \( \tilde{\Sigma}_k \) is

\[
\hat{g}_k = \hat{g}'_k + \sum_{i=k+1}^{n+1} (u_i|_{\tilde{\Sigma}'_k})^2 dT^2_{i+1}.
\]

Since \( \tilde{\Sigma}_k \) is \( B \)-minimizing, it has constant mean curvature \( \delta \) and it is \( B \)-stable. So the Jacobi operator

\[
\hat{J}_k = -\Delta \hat{g}_k - (\text{Ric}_{\hat{g}_{k+1}}(\hat{\nu}_k, \hat{\nu}_k) + \|\hat{A}_k\|^2) = -\Delta \hat{g}_k - \frac{1}{2}(R_{\hat{g}_{k+1}} - R_{\hat{g}_k} + \delta^2 + \|\hat{A}_k\|^2)
\]

is nonnegative, where \( \hat{\nu}_k \) is the unit normal vector field of \( \tilde{\Sigma}_k \) and \( \hat{A}_k \) is the corresponding second fundamental form. Taking the first eigenfunction \( \hat{u}_k \) of \( \hat{J}_k \) with respect to the first eigenvalue \( \hat{\lambda}_{1,k} \geq 0 \) and defining the following metric

\[
\hat{g}_{k+1} = \hat{g}_k + \hat{u}_k^2 dT^2_{k+1},
\]

on \( \tilde{\Sigma}_k \times S^1 \), we obtain

\[
R_{\hat{g}_{k+1}} \geq R_{\hat{g}_{k+1}} + \delta^2 + \|\hat{A}_k\|^2 + 2\hat{\lambda}_{1,k} \geq 2 + \delta^2.
\]

Since \( \tilde{\Sigma}_k \) is homologous to \( \Sigma_k \), the map \( \tilde{F}_k = \pi^n_{n-1} \circ F_{k+1}|_{\tilde{\Sigma}_k} \) has non-zero degree. If \( \tilde{\Sigma}_k \) is not connected, we may choose a suitable component as \( \tilde{\Sigma}_k \) such that \( \tilde{F}_k \) has non-zero degree. As a result, the map \( \tilde{f}_k = \pi^n_{k-2} \circ f|_{\tilde{\Sigma}'_k} \) has non-zero degree. Repeating what we have done as in Proposition 2.2, we can find a slicing

\[
(\tilde{\Sigma}_2', \hat{g}'_2) \varsubsetneq (\tilde{\Sigma}_3', \hat{g}'_3) \varsubsetneq \cdots \varsubsetneq (\tilde{\Sigma}_k', \hat{g}'_k) \varsubsetneq (\Sigma_{k+1}, g_{k+1}) \varsubsetneq \cdots \varsubsetneq (\Sigma_{n+2}, g_{n+2}) = (M, g)
\]
and functions $\hat{u}_i : \hat{\Sigma}'_i \to \mathbb{R}$, $i = 2, 3, \ldots, k$, such that the scalar curvature $R_{\hat{g}_2} \geq 2 + \delta^2$ for

$$
\hat{g}_2 = \hat{g}'_2 + \sum_{i=2}^{k} (\hat{u}_i|_{\hat{\Sigma}'_2})^2 dt^2_i + \sum_{i=k+1}^{n+1} (u_i|_{\hat{\Sigma}'_2})^2 dt^2_i + 1.
$$

Also, we have $\deg \pi_0^0 \circ f|_{\hat{\Sigma}_2} \neq 0$. Therefore, $\hat{\Sigma}_2$ is a non-contractible embedded sphere in $M$. It follows from Lemma 2.3 that $A(\hat{\Sigma}_2) \leq 4\pi(1 + \delta^2/2)^{-1} < 4\pi$, which leads to a contradiction.

Since $\hat{H}_{k,t}$ vanishes for any $t \in (-\epsilon, \epsilon)$, there holds $A(\hat{\Sigma}_{k,t}) \equiv A(\hat{\Sigma}_k)$, which yields that $\hat{\Sigma}_{k,t}$ is also area-minimizing. Replacing $\hat{\Sigma}_{k,t}$, we can also obtain a slicing as in Proposition 2.2. Combined with the induction hypothesis and Lemma 3.2, $\hat{\Sigma}_{k,t}$ is totally geodesic and has vanished normal Ricci curvature.

In the following, we show that there exists a local isometry $\tilde{\Phi}_k : \tilde{\Sigma}_k \to \tilde{\Sigma}_{k+1}$. Denote $\tilde{V}_k$ to be the normal variation vector field of the foliation $\tilde{\Sigma}_k, \tilde{\Sigma}_k, \tilde{\Sigma}_{k,t}$ and $\tilde{\Phi}_k : \tilde{\Sigma}_k \times (-\epsilon, \epsilon) \to \tilde{\Sigma}_{k+1}$ to be the flow generated by $\tilde{V}_k$. Notice that $\tilde{\Sigma}_{k,t}$ are $T^{n+1-k}$-invariant, $\tilde{\Sigma}_k, \tilde{\Sigma}_{k,t}$ can be written as $\Sigma_{k,t} \times T^{n+1-k}$ and $\tilde{V}_k$ can be viewed as the normal variation vector field $V_k$ of $\tilde{\Sigma}_k, \tilde{\Sigma}_{k,t}$ on $\Sigma_{k+1}$. Let $\tilde{\Phi}_k : \Sigma_k \times (-\epsilon, \epsilon) \to \Sigma_{k+1}$ be the flow generated by $V_k$, then $\tilde{\Phi}_k = (\Phi_k, id)$. It is clear that $\tilde{\Phi}_k$ is an embedding around $\Sigma_k$ and the pull-back of the metric $\tilde{g}_{k+1}$ is

$$
\tilde{\Phi}_k^* (\tilde{g}_{k+1}) = \phi^2 dt^2 + \tilde{\Phi}_k^* (\hat{g}_{k,t}) = \phi^2 dt^2 + \Phi_k^* (g_{k,t}) + \sum_{i=k+1}^{n+1} (u_i|_{\Sigma_{k+1}})^2 dt^2_i + 1,
$$

where $\phi > 0$ is the lapse function, $\hat{g}_{k,t}$ and $g_{k,t}$ are the induced metrics of $\Sigma_{k,t}$ and $\Sigma_{k,t}$, respectively. Since $\tilde{\Sigma}_k, \tilde{\Sigma}_k, \tilde{\Sigma}_{k,t}$ is totally geodesic, we have $\partial_t \tilde{\Phi}_k^* (\hat{g}_{k,t}) = 2\phi \hat{h}_{k,t} = 0$, which yields $\tilde{\Phi}_k^* (\hat{g}_{k,t}) \equiv \hat{g}_k$ and that $u_i|_{\Sigma_{k,t}} = u_i|_{\Sigma_k}$ are positive constants for $i$ from $k+1$ to $n+1$. Also, from stability we have $\tilde{F} \phi = -\Delta \phi = 0$, which gives $\phi(\cdot, t) \equiv \text{const}$. Let

$$
\rho(t) = \int_0^t \phi(\cdot, s) ds,
$$

then $\tilde{\Phi}_k^* (\hat{g}_{k+1}) = \rho^2 + \hat{g}_k$. Therefore, $\tilde{\Phi}_k : \tilde{\Sigma}_k \times (-\epsilon, \epsilon) \to \tilde{\Sigma}_{k+1}$ is a local isometry. Through a continuous argument as in [2, Proposition 11], we conclude that there exists a local isometry $\tilde{\Phi}_k : \tilde{\Sigma}_k \times \mathbb{R} \to \tilde{\Sigma}_{k+1}$. As a result, we obtain $R_{\hat{g}_{k+1}} = R_{\hat{g}_k} = 2$ and that $u_i|_{\Sigma_{k+1}}$ are positive constants for $i$ from $k+1$ to $n+1$. Now, the proposition follows easily from the induction argument.

From the proof above, we have the following corollary:

**Corollary 3.5.** If $A_{\mathbb{S}^2}(M, \hat{g}) = 4\pi$, then for any slicing constructed as in Proposition 2.2 there exist local isometries $\Phi_k : \Sigma_k \times \mathbb{R} \to \Sigma_{k+1}$ for $k = 2, 3, \ldots, n+1$.

We now prove the main theorem.

**Proof of Theorem 1.2.** From Corollary 2.4, we have $A_{\mathbb{S}^2}(M, \hat{g}) \leq 4\pi$. If the equality holds, from Proposition 2.2, Lemma 3.1 and Corollary 3.5, we can find a slicing

$$(\Sigma_2, g_2) \rhd (\Sigma_3, g_3) \rhd \cdots \rhd (\Sigma_{n+1}, g_{n+1}) \rhd (\Sigma_{n+2}, g_{n+2}) = (M, \hat{g})$$

such that \((\Sigma_2, g_2)\) is isometric to the standard sphere and that there exist local isometries \(\Phi_k : \Sigma_k \to \Sigma_{k+1}\) for \(k = 2, 3, \ldots, n + 1\). Denote \(\Phi\) to be the composition of the following maps
\[
S^2 \times \mathbb{R}^n = \Sigma_2 \times \mathbb{R}^n \xrightarrow{(\Phi_2, id)} \Sigma_3 \times \mathbb{R}^{n-1} \xrightarrow{(\Phi_3, id)} \cdots \xrightarrow{\Phi_{n+1}} \Sigma_{n+2} = M,
\]
then \(\Phi\) is a local isometry from \(S^2 \times \mathbb{R}^n\) to \(M\). Since \(S^2 \times \mathbb{R}^n\) is complete, \(\Phi\) is a covering map. It is clear that \(S^2 \times \mathbb{R}^n\) is the universal covering of \((M, \bar{g})\). □

References

[1] Hubert Bray, Simon Brendle, Michael Eichmair, and André Neves, Area-minimizing projective planes in 3-manifolds, Comm. Pure Appl. Math. 63 (2010), no. 9, 1237–1247. MR 2675487
[2] Hubert Bray, Simon Brendle, and André Neves, Rigidity of area-minimizing two-spheres in three-manifolds, Communications in Analysis and Geometry 18 (2010), no. 4, 821–830. MR 2765731
[3] Alessandro Carlotto, Otis Chodosh, and Michael Eichmair, Effective versions of the positive mass theorem, Invent. Math. 206 (2016), no. 3, 975–1016. MR 3573977
[4] Misha Gromov, Metric inequalities with scalar curvature, Geom. Funct. Anal. 28 (2018), no. 3, 645–726. MR 3816521
[5] Fernando C. Marques and André Neves, Rigidity of min-max minimal spheres in three-manifolds, Duke Math. J. 161 (2012), no. 14, 2725–2752. MR 2993139
[6] Richard Schoen and Shing-Tung Yau, Positive scalar curvature and minimal hypersurface singularities, preprint, arxiv:1704.05490.
[7] ———, On the structure of manifolds with positive scalar curvature, Manuscripta Math. 28 (1979), no. 1-3, 159–183. MR 535700
[8] Leon Simon, Lectures on geometric measure theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 756417

Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China

E-mail address: zhujt@pku.edu.cn