On the Arnold’s conjecture on hyperbolic homogeneous polynomials

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Abstract

The Hessian Topology is a subject with interesting relations with some classical problems of analysis and geometry [2], [13], [3]. In this article we prove a conjecture on this subject stated by V.I. Arnold in [1] and [4], concerning the number of connected components of hyperbolic homogeneous polynomials of degree n. The proof is constructive and provides models. Our approach uses index properties at isolated singularities of hyperbolic quadratic differential forms and combinatorial properties of recurrent functions.

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1 Introduction

A well known classification of the points on a smooth surface in $\mathbb{R}^3$ is given in terms of the contact of maximal order of the tangent lines with the surface at each point. A point $p$ of a surface is elliptic if all lines tangent to the surface at $p$ have a contact of order 2 with the surface at that point. It is hyperbolic if there are exactly two straight lines having a contact of order at least 3 with the surface at that point. These lines are known as asymptotic lines. A point $p$ is parabolic if it has exactly one asymptotic line. It is possible that all the tangent lines at a point be asymptotic lines. In this case the point is named a degenerate parabolic point. The concept of generic surface can be stated in terms of this type of contact in such a way that a generic surface has the following structure: The sets of elliptic and hyperbolic points form a union of disjoint domains on the surface whose boundary is a smooth curve.

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constituted by parabolic points and referred to as the *parabolic curve* of the surface, \([10]\).

Let us now consider \(H^n[x, y] \subset \mathbb{R}[x, y]\) the set of real homogeneous polynomials of degree \(n \geq 1\) in two variables. The graph of any \(f \in H^n[x, y]\) contains the origin of \(\mathbb{R}^3\). The polynomial \(f\) is called *hyperbolic* (*elliptic*) if its graph is a surface with only hyperbolic (*elliptic*) points off the origin. The subset of \(H^n[x, y]\) constituted by hyperbolic polynomials is a topological subspace of \(\mathbb{R}[x, y]\) denoted by \(Hyp(n)\). The connectedness of this space has been studied as part of the subject known as the Hessian Topology introduced in \([2]\), \([13]\), \([3]\) and named by V. I. Arnold in \([1]\) and \([4]\) (problems 2000-1, 2000-2, 2001-1, 2002-1). In fact, in reference \([1]\) it is shown that this property of the space depends on the degree of the polynomials that constitute it. That is, \(Hyp(3)\) and \(Hyp(4)\) are connected subspaces meanwhile \(Hyp(6)\) is a disconnected one. According to this, V.I. Arnold stated the following conjecture \([1]\), p.1067 and \([4]\), p.139:

"The number of connected components of the space of hyperbolic homogeneous polynomials of degree \(n\) increases as \(n\) increases (at least as a linear function of \(n\))."

Moreover, the connectedness of the space of hyperbolic functions defined as follows was also studied in \([1]\). Let \(n\) be a real number and \((r, \varphi)\) polar coordinates in the real plane. Let us define \(Hyp^\infty(n)\) the space of smooth functions \(F : S^1 \to \mathbb{R}\) such that the function \(f(r, \varphi) = r^n F(\varphi)\) referred to as a *homogeneous function of degree \(n\)* is hyperbolic, namely, its graph is constituted by only hyperbolic points off the origin. This space has infinitely number of connected components and is closely related with the space of hyperbolic homogeneous polynomials.

In the present article we prove this conjecture, Theorem 13. In fact, we present a constructive proof. That is, we provide a good amount of examples of hyperbolic polynomials lying on different connected components in order to guarantee the required increasing growth of the number of these components in terms of the degree. Following \([1]\), we consider the field of asymptotic lines on the graph of the polynomial. This field of lines has a unique singularity whose index is a convenient invariant of the connected component. Now, let us describe the ideas we provide in order to prove the conjecture using this approach. First, we determine inequality (1) involving a pair of polynomials \(P\) and \(Q\), and prove that this semi-algebraic condition implies the topological property of index preservation of the singularities of the fields of asymptotic lines of \(P\) and \(PQ\), if they are hyperbolic and \(Q\) is
elliptic, Corollary 7. Moreover, this theorem holds not only for polynomials but for general hyperbolic functions. Second, we point out that among the families of polynomials analyzed in [1] there is no one providing an idea of the growth of the number of connected components of \( H_{yp}(n) \) in terms of \( n \), so we define a new family, using a family \( \mathcal{P} \) analyzed in this reference. Family \( \mathcal{P} \) constituted by hyperbolic homogeneous polynomials of degree \( m \in \mathbb{N} \) has one and only one hyperbolic polynomial of degree \( m \), for each \( m > 2 \). Its field of asymptotic lines has a unique singularity at the origin with index \( \frac{2m-2}{2} \). On the other hand, the above new family of polynomials satisfies the following property: for a fixed degree \( n \) it contains polynomials of this degree isotopic to those of degree \( m \) lying in family \( \mathcal{P} \) such that \( m \leq n \) and \( m \equiv n \) (mod 2). Since the index is an invariant of the connected component they belong to different components. In order to satisfy this property we define its elements as product polynomials \( PQ \), where \( P \in \mathcal{P}, Q \) is elliptic, and satisfy together inequality (1). Let us observe that proving this inequality is equivalent to solving certain combinatorial equations involving technical formulae and recurrent functions that are not in the literature, Theorem 11. We end the article by providing a qualitative description of the foliation of the asymptotic lines of the graphs of these polynomials, Corollary 15.

2 Preliminaries

If the surface is the graph of a real valued smooth function \( f \) on the plane, the image of the parabolic curve in the \( xy \)-plane under the projection \( \pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2, (x, y, z) \mapsto (x, y) \) will be referred to as the Hessian curve of \( f \).

The directions determined by the projections of the asymptotic lines on the \( xy \)-plane under this projection are the solutions of the following quadratic differential equation:

\[
f_{xx}(x, y)dx^2 + 2f_{xy}(x, y)dxdy + f_{yy}(x, y)dy^2 = 0,
\]

where the quadratic differential form on the left will be denoted by \( II_f(x, y) \) and referred to as the second fundamental form of \( f \). Its discriminant defined as

\[
\Delta_{II_f} = f_{xy}^2 - f_{xx}f_{yy},
\]

allows us to characterize the type of point in the graph of \( f \). That is, \((p, f(p))\) is hyperbolic (elliptic) if \( \Delta_{II_f}(p) \) is positive (negative). The point is parabolic if \( \Delta_{II_f}(p) = 0 \) and \( II_f(p) \) does not vanish. We say that \((p, f(p))\)
is a degenerate parabolic point if \( II_f(p) = 0 \). In this case we say that this quadratic form has a singularity at \( p \).

Let \( M \) be an orientable smooth surface and \( X \) be a differentiable field of lines tangent to \( M \) with an isolated singularity at the point \( p \). Take a simple closed curve \( \Gamma : [0,1] \to M \), such that \( p \) is the only singularity of \( X \) in the closure of the region determined by \( \Gamma \) containing \( p \). Thus, consider the restriction of this field of lines to the image of \( \Gamma \). Moreover, take along the image of \( \Gamma \) any differentiable vector field \( Y \) without singularities, for instance a standard coordinate vector field. The total change of the angle between the oriented field of lines \( X \) and \( Y \) after going around once \( \Gamma \) in the positive sense (with respect to the orientation of the surface) is the index of the field of lines \( X \) at \( p \). This number is independent of the choice of \( \Gamma \) and \( Y \), moreover, it has the form \( \text{ind}_p(X) = \frac{n}{2}, \ n \in \mathbb{Z} \) [9]. If a tangent field of lines on \( M \) is integrable the set of its integral curves will be referred to as its foliation. Let \( X \) and \( Y \) be two integrable fields of lines on \( M \). We say that \( X \) and \( Y \) are topologically equivalent if there exists an homeomorphism \( H : M \to M \) which transforms the integral curves of the foliation of \( X \) into the integral curves of the foliations of \( Y \) [12].

A quadratic differential

\[
\omega(x, y) = A(x, y)dx^2 + 2B(x, y)dxdy + C(x, y)dy^2,
\]

on the punctured \( xy \)-plane, \( \mathbb{R}^2 \setminus \{(0,0)\} \) denoted by \( \mathbb{R}^2^* \), is smooth if the coefficient functions \( A, B, C : \mathbb{R}^2^* \to \mathbb{R} \) are smooth. If its discriminant \( \Delta_\omega = B^2 - AC \) at a point \( p \) is positive we will say that the quadratic form is hyperbolic at \( p \). The quadratic form will be called hyperbolic if it is so at every point of its domain. In the sequel we will consider smooth hyperbolic quadratic differential forms whose coefficient functions extend continuously at the origin with values \( A(0,0) = B(0,0) = C(0,0) = 0 \). This defines a continuous extension of \( \omega \) to the plane with the origin as a unique singularity. The local classification of the solution curves defined by this type of quadratic differential forms satisfying some generic conditions at the singular point have been studied by several authors, see for instance [5], [6] and [14].

The second fundamental form of a hyperbolic homogeneous polynomial, and generally that of a hyperbolic homogeneous function \( f \) of degree \( n \) are examples of this kind of smooth hyperbolic quadratic differential forms. Thus, \( II_f \) defines two asymptotic lines at each point of \( \mathbb{R}^2^* \). Moreover, it defines two continuous asymptotic fields of lines without singularities on \( \mathbb{R}^2^* \) that extend to the origin. These fields of lines are topologically equivalent.
Therefore, their indexes at the origin coincide. Consequently, this index will be called the index of the field of asymptotic lines at the origin, and it will be denoted by $i_0(II_f)$.

A hyperbolic isotopy between two smooth hyperbolic quadratic differential forms $\omega$ and $\delta$ on $\mathbb{R}^2$ that extend themselves to the origin with a singularity is a smooth map

$$
\Psi : \mathbb{R}^2 \times [0, 1] \to \mathcal{Q}, \quad (x, y, t) \mapsto \Psi_t(x, y),
$$

where $\mathcal{Q}$ is the space of real quadratic forms on the plane and the following conditions hold: $\Psi_0(x, y) = \omega(x, y)$, $\Psi_1(x, y) = \delta(x, y)$ and $\Psi_t(x, y)$ is a smooth hyperbolic quadratic differential form on $\mathbb{R}^2$ which extends at the origin with a singularity. In this case we will say that $\omega$ and $\delta$ are hyperbolic isotopic.

If the second fundamental forms of two hyperbolic homogeneous polynomials of degree $n$ are hyperbolic isotopic, in fact, they are topologically equivalent. Therefore, the indexes of their fields of asymptotic lines at the origin coincide.

There is a natural application $II$, defined on the space of hyperbolic homogeneous polynomials of degree $n$, whose image lies on the space of smooth hyperbolic quadratic differential forms, associating to each polynomial its second fundamental form. Given two hyperbolic homogeneous polynomials of degree $n$, $f$ and $g$ lying in the same connected component $C$, we have a smooth curve $\gamma : [0, 1] \to C$ such that $\gamma(0) = f$ and $\gamma(1) = g$. Then, the application $\Psi_t = II \circ \gamma(t)$ defines a hyperbolic isotopy between $II_f$ and $II_g$. Therefore, let us state the following:

**Proposition 1** If two hyperbolic homogeneous polynomials of degree $n$ lie on the same connected component, the indexes of their fields of asymptotic lines at the origin coincide.

### 3 The index of the field of asymptotic lines at the origin of a hyperbolic homogeneous polynomial

In the following analysis we consider a polynomial $f \in \mathbb{R}[x, y]$ as a Hamiltonian function with Hamiltonian vector field $\nabla f = (f_y, -f_x)$, on $\mathbb{R}^2$. The field of Hessian matrices, $Hessf = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ determines at each point $p \in \mathbb{R}^2$ a bilinear form. That is,

$$
Hessf_p : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}
$$
\[ Hess_f_p(X, Y) = X(Hess(p))Y^t, \]

where \( X, Y \in \mathbb{R}^2 \) and the index \( t \) means the transpose of the vector \( Y \).

Thus, for any homogeneous polynomial \( P \in H^n[x, y] \) with non-null Hessian matrix we define the following application:

\[
\nabla P HessP : H^m[x, y] \to H^{2n+m-4}_0[x, y],
Q \mapsto \nabla P HessP \nabla Q^t,
\]

where \( H^{2n+m-4}_0[x, y] = H^{2n+m-4} \cup \{0\} \).

A straightforward computation implies that

\[
\nabla P HessP \nabla Q^t = P_{xx}P_{yy}Q_{yy} + P_{yy}P_{xx}Q_{xx} - P_{xy}(P_{x}Q_{y} + P_{y}Q_{x}).
\]

The following inequality plays an important role in the proof of Theorem 2.

\[
\nabla P HessP \nabla Q^t(p) \leq 0, \quad p \in \mathbb{R}^2.
\] (1)

In fact, if it holds at each point \( p \) of the plane it guarantees that the isotopy applied to the second fundamental forms of the polynomials considered in the proof of this theorem preserves the index of the asymptotic lines at the origin.

Let us present an interpretation of this condition. Endow \( \mathbb{R}^2 \) with the standard orientation and interior product \( <\cdot, \cdot> \). Let us define the map:

\[
GradP : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (P_x(x, y), P_y(x, y)).
\]

Since the derivative of this map at \( p_0 \) is \( D_{p_0}GradP = HessP_{p_0} \), if we assume that \( \nabla Q^t(p_0) \) is not in the kernel of \( HessP_{p_0} \), we have that the image by this map of the level curve of the polynomial \( Q \) at \( p_0 \) has tangent vector \( HessP_{p_0} \nabla Q^t(p_0) \). Using the parallel translation of \( \mathbb{R}^2 \), we can suppose that this curve intersects the level curve of the polynomial \( P \) at \( p_0 \). Thus, inequality (1) holds at each point of the plane, if and only if the oriented angle of intersection of these curves at each point where the polynomial function \( P \) is not singular lies on the interval \( \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \).

**Theorem 2** Let \( P, Q \) be homogeneous polynomials such that \( P \) is hyperbolic, \( Q \) is elliptic and the product \( f = PQ \) is hyperbolic. Suppose also that \( Q \) is positive on \( \mathbb{R}^2 \) and inequality (1) holds at each point of the plane. Then, \( hf \) and \( hp \) are hyperbolic isotopic.

**Proof.** Let us present the proof divided in lemmas. A straightforward computation shows the following
Lemma 3 The discriminant of the quadratic differential form \( Q_{II\text{ }p+2dPdQ} \) has the following expression:

\[
\Delta_{Q_{II\text{ }p+2dPdQ}} = -Q^2 \det(\text{Hess}P) + 4\Delta_{dPdQ} - 2Q(\nabla PHessP\nabla Q').
\] (2)

Lemma 4 Suppose that \( P \) is a hyperbolic homogeneous polynomial, \( Q \) is a positive polynomial on \( \mathbb{R}^2 \) and assume that inequality (1) holds. Then,

a) The quadratic differential form \( Q_{II\text{ }p+2dPdQ} \) is hyperbolic on \( \mathbb{R}^2 \).

b) The quadratic differential forms \( Q_{II\text{ }p+2dPdQ} \) and \( Q_{II\text{ }p} \) are hyperbolic isotopic.

Proof. a) Since \( P \) is a hyperbolic polynomial and 

\[
\Delta_{dPdQ} = \frac{1}{4}(P_y Q_x - P_x Q_y)^2,
\]
then inequality (1) implies that the right side of equation (2) is positive on \( \mathbb{R}^2 \), that is, \( Q_{II\text{ }p+2dPdQ} \) is hyperbolic on \( \mathbb{R}^2 \).

b) Considering the isotopy \( \Psi_t(x, y) = Q_{II\text{ }p+2tPdQ}(x, y) \), \( t \in [0, 1] \), we can see from equation (2) that the discriminant of the quadratic differential form \( Q_{II\text{ }p+2tPdQ}(x, y) \) is

\[
\Delta_{\Psi_t} = -Q^2 \det(\text{Hess}P) + t^2\Delta_{dPdQ} - 2tQ(\nabla PHessP\nabla Q'),
\]
which is positive on \( \mathbb{R}^2 \). This implies that \( \Psi_t(x, y) \) is a hyperbolic isotopy. \( \square \)

In order to prove the following lemma we point out the next easy

Remark 5 Let \( a, b, c \) be real numbers such that \( a + b + c > 0 \), \( a > 0 \) and \( c \leq 0 \). Then \( a + tb + t^2c > 0 \) for \( t \in (0, 1) \).

Lemma 6 Let \( \omega, \delta \) be two smooth quadratic differential forms on \( \mathbb{R}^2 \) vanishing at the origin. Suppose that \( \omega, \omega + \delta \) are hyperbolic and \( \delta \) is non-hyperbolic at each point of \( \mathbb{R}^2 \). Then, the quadratic forms \( \omega \) and \( \omega + \delta \) are hyperbolic isotopic.

Proof. Let \( \omega(x, y) = \omega_1 dx^2 + 2\omega_2 dxdy + \omega_3 dy^2 \) and \( \delta(x, y) = \delta_1 dx^2 + 2\delta_2 dxdy + \delta_3 dy^2 \) be two quadratic differential forms. Consider the isotopy

\[
\Psi_t(x, y) = \omega(x, y) + t\delta(x, y).
\]

The discriminant of \( \Psi_t(x, y) \) is

\[
\Delta_{\Psi_t} = \omega_2^2 - \omega_1 \omega_3 + t(2\omega_2 \delta_2 - \omega_1 \delta_3 \omega_3 \delta_1) + t^2(\delta_2^2 - \delta_1 \delta_3).
\]
Because $\omega$ and $\omega + \delta$ are hyperbolic on $\mathbb{R}^2$, we have that

$$\omega_2^2 - \omega_1 \omega_3 > 0 \quad \text{and} \quad \omega_2^2 - \omega_1 \omega_3 + (2 \omega_2 \delta_2 - \omega_1 \delta_3 \omega_3 \delta_1) + (\delta_2^2 - \delta_1 \delta_3) > 0,$$

on $\mathbb{R}^2$. Moreover, since $\delta_2^2 - \delta_1 \delta_3 \leq 0$ the fact that $\Delta \Psi_t(x,y)$ is negative at each point on $\mathbb{R}^2$ follows from Remark 5. □

End of the proof of Theorem 2. Let us take $\omega = QII_p + 2dPdQ$ and $\delta = PII_Q$. Observe that Lemma 4(a) ensures that $\omega$ is hyperbolic on $\mathbb{R}^2$. Moreover, since $f = PQ$ is hyperbolic we conclude, by Lemma 6, that there exists a hyperbolic isotopy between the quadratic forms $II_f = PII_Q + QII_P + 2dPdQ$ and $QII_p + 2dPdQ$. Then, Lemma 4(b) implies that $II_f$ and $II_P$ are hyperbolic isotopic. □

Corollary 7 Assume that $f$ and $P$ satisfy the hypothesis of Theorem 2. Then the fields of asymptotic lines of these polynomials extended to the origin with a singularity are topologically equivalent on $\mathbb{R}^2$ and their indexes at the origin coincide.

Remark 8 We observe that Corollary 7 is true in a more general setting. Namely, let us consider instead of homogeneous polynomials, a pair of real valued differentiable functions $P$ and $Q$ vanishing at the origin. Suppose that they satisfy the hyperbolicity and ellipticity hypothesis of the statement in $\mathbb{R}^2$, respectively. Assume that $Q$ is positive in $\mathbb{R}^2$ and the origin is a degenerate parabolic point of $f = PQ$ and $P$. Suppose that inequality (1), stated in this case on the class of differentiable functions holds at each point of the plane. Then, the fields of asymptotic lines of these functions are topologically equivalent on $\mathbb{R}^2$ and their indexes at the origin coincide.

4 Proof of the conjecture

For clearness sake we present the strategy proof of the conjecture: In order to determine the desired number of connected components of the hyperbolic homogenous polynomials in terms of the degree, the goal is to find for each $n \in \mathbb{N}$ a big enough number of polynomials in $Hyp(n)$ whose fields of asymptotic lines at the origin have different indexes. For this, we first consider the well known family $P$ of hyperbolic homogeneous polynomials of degree $m$, whose elements $P^m$ described below, define fields of asymptotic lines at the origin of indexes $\frac{2-m}{2}, m > 2$, respectively, [11] p.1037. Observe that each element of the family determines only one connected component
of $Hyp(m)$. On the other hand, for each $n \geq 4$ we consider an elliptic homogeneous polynomial $Q^{2k}$ of degree $2k$ in such a way that the family of polynomials $\{P^m Q^{2k}\}$, where $1 \leq k \leq \frac{n}{2} - 1$ and $n = m + 2k$ is contained in $Hyp(n)$, Proposition 10. The corresponding family of fields of asymptotic lines have indexes $\frac{2-m}{2}$ at the origin, respectively. That is, the contribution of $Q^{2k}$ to the index at the origin of the field of asymptotic lines of $P^m Q^{2k}$ is null. To prove that this index property holds for this type polynomials we apply Corollary 7. Namely, we need to prove that any pair of polynomials $P \in \{P^m\}$ and $Q \in \{Q^{2k}\}$ satisfies condition (1), Theorem 11.

Let us describe the two families of homogeneous polynomials that we will use in the proof.

In [1], V. I. Arnold proves that the polynomials of degree $m$

$$P(x, y) = r^{m-k} \operatorname{Re}(x + iy)^k,$$

where $r = \sqrt{x^2 + y^2}$, $k^2 > m$, $k \leq m$ and $m - k$ is even, are hyperbolic homogeneous polynomials and $i_0(II_P) = \frac{2-k}{2}$. Taking, in particular $k = m \geq 2$ these polynomials get the form

$$P^m(x, y) = \sum_{j=0}^{\frac{m}{2}} (-1)^j \binom{m}{2j} x^{m-2j} y^{2j}, \quad \text{if } m \text{ is even},$$

and

$$P^m(x, y) = \sum_{j=0}^{\frac{m-1}{2}} (-1)^j \binom{m}{2j} x^{m-2j} y^{2j}, \quad \text{if } m \text{ is odd},$$

with (in both cases) $i_0(II_{P^m}) = \frac{2-m}{2}$.

Now, let us consider the family of polynomials

$$Q^{2k}(x, y) = (x^2 + y^2)^k,$$

where $k$ is a positive integer.

A direct computation of the discriminant of the form $II_{Q^{2k}}$ implies

**Proposition 9** For $k \geq 1$ the polynomial $Q^{2k}$ is elliptic.

**Proposition 10** Let $k, m \in \mathbb{N}$ such that $m > \max\{2, k\}$. Then, the homogeneous polynomial $f(x, y) = P^m(x, y)Q^{2k}(x, y)$ is hyperbolic.
Proof. Consider a homogeneous polynomial \( f(x, y) \) of degree \( n \), such that in polar coordinates \((r, \varphi)\), \( f(r, \varphi) = r^n F(\varphi) \), where \( F(\varphi) \) is a trigonometric function. The following hyperbolicity condition stated by V. I. Arnold in [1] (Theorem 1 p.1031) guarantees that \( f \) is hyperbolic if and only if the function \( F \) satisfies

\[
n^2 F^2 + n F F'' - (n-1)(F')^2 < 0. \quad (3)
\]

In our case, \( f(r, \varphi) = r^{m+2k} \cos(m\varphi) \). Thus, the left-hand side of this inequality has the expression

\[
n^2 F^2 + n F F'' - (n-1)(F')^2 = \cos^2(m\varphi)[4k(m+k)] - m^2(m+2k-1). \quad (4)
\]

Note that the right-side of (4) is negative since

\[
4k(m+k) < m^2(m+2k-1). \quad \square
\]

**Theorem 11** Let \( k, m \in \mathbb{Z} \) such that \( k \geq 0 \) and \( m \geq 2 \). Then, the polynomials \( P^m, Q^{2k} \) satisfy inequality (1).

The most important part of the proof of this theorem is a consequence of some combinatorial relations which are not in the literature. Now, we prove them. We present only the case when \( m \) is even because the odd case is analogous. Let us begin by proving the formulae below.

**Lemma 12** Let \( m \geq 2 \) be a natural number. For each integer number \( 0 \leq j \leq \frac{m-2}{2} \) consider the combinatorial functions

\[
A(j) = (-1)^j \left[ \left( \begin{array}{c} m-1 \\ 2j \end{array} \right) + \sum_{k=0}^{j-1} \left[ \left( \begin{array}{c} m-1 \\ 2k \end{array} \right) \left( \begin{array}{c} m-1 \\ 2j-2k \end{array} \right) - \left( \begin{array}{c} m-1 \\ 2k+1 \end{array} \right) \left( \begin{array}{c} m-1 \\ 2j-2k-1 \end{array} \right) \right] \right],
\]

\[
B = (-1)^{\frac{m}{2}} \left[ \left( 1 - m + \sum_{k=0}^{\frac{m-2}{2}} \left( \begin{array}{c} m-1 \\ 2k+1 \end{array} \right) \left[ \left( \begin{array}{c} m-1 \\ 2k+2 \end{array} \right) - \left( \begin{array}{c} m-1 \\ 2k \end{array} \right) \right] \right) \right],
\]

\[
C(j) = (-1)^{\frac{m}{2}+j-1} \left[ - \left( \begin{array}{c} m-1 \\ 2j-1 \end{array} \right) + \sum_{k=0}^{\frac{m-1}{2}} \left[ \left( \begin{array}{c} m-1 \\ 2k+1 \end{array} \right) \left( \begin{array}{c} m-1 \\ 2k \end{array} \right) \left( \begin{array}{c} m-1 \\ 2j \end{array} \right) - \left( \begin{array}{c} m-1 \\ 2k+2j-1 \end{array} \right) \right] \right].
\]
Then, they can be reduced to the following expressions.

\[ A(j) = \binom{m-1}{j}, \quad (5) \]

\[ B = \binom{m-1}{\frac{m}{2}}, \quad (6) \]

\[ C(j) = \binom{m-1}{j + \frac{m}{2} - 1}. \quad (7) \]

**Proof.**

Let us begin by proving (5). Using several times the formula

\[ (m-k) \binom{m}{k} = m \binom{m-1}{k}, \quad (8) \]

(derived from the absorption identity [7]), the expression (5) becomes

\[ (-1)^j \left[ \binom{m-1}{2j} + \sum_{k=0}^{j-1} \frac{4k-2j+1}{m} \left( \binom{m}{2k+1} \binom{m}{2j-2k} \right) \right] = \binom{m-1}{j}. \quad (9) \]

By the formula for the alternating sum of consecutive binomial coefficients,

\[ (-1)^r \binom{m-1}{r} = \sum_{k=0}^{r} (-1)^k \binom{m}{k}, \quad (10) \]

the expression (9) results

\[ (-1)^j \sum_{k=0}^{j-1} \frac{4k-2j+1}{m} \left( \binom{m}{2k+1} \binom{m}{2j-2k} \right) = \sum_{k=1}^{j} (-1)^{k+1} \binom{m}{j+k}. \]

For the following reduction it is useful to write down the sum of both sides of the last expression in two parts. For the left side sum, chose the first part as the sum from the lowest value of \( k \) up to \( \frac{j}{2} \) (odd) if \( j \) is even (odd), and the second part containing the remaining terms. For the right side sum, take the first part as the sum of even terms and the second part as the sum of odd terms. Then, by associating corresponding terms of both sides we obtain the equation

\[ \sum_{k=1}^{j} (-1)^k \binom{m}{j+k} \left[ 1 + \frac{1 - 2k}{m} \binom{m}{j-k+1} \right] = 0. \quad (11) \]
We present now a proof of (11) based on some recurrence relations. It was given by C. Merino López [11].

By the alternating sum (10) we have

\[ m \sum_{k=0}^{j} (-1)^k \binom{m}{j+k} = \begin{cases} \binom{m-2j}{2j} - \binom{m}{j} \binom{m}{j} & \text{if } j \text{ is even} \\ -\binom{m-2j}{2j} - \binom{m}{j} \binom{m}{j} & \text{if } j \text{ is odd} \end{cases} \]

By formula (8) the last equality becomes

\[ m \sum_{k=0}^{j} (-1)^k \binom{m}{j+k} = \begin{cases} \binom{m-1}{2j+1} \binom{m}{2j+1} - \binom{m}{j+1} \binom{m}{j+1} & \text{if } j \text{ is even} \\ -\binom{m-1}{2j} - \binom{m}{j} \binom{m}{j} & \text{if } j \text{ is odd} \end{cases} \]

Replacing the last equality in (11) we obtain

\[ \sum_{k=1}^{j+1} (-1)^{k+1} (2k-1) \binom{m}{j+k} \binom{m}{j-k+1} = (j+1) \binom{m}{j+1}. \]

Denote by \( T(m,j) \) the left side of the last expression. Using the Stifel identity \( \binom{m}{j} = \binom{m-1}{j} + \binom{m-1}{j-1} \), we verify that \( T(m,j) \) satisfies the recurrence relation

\[ T(m,j) = T(m-1,j) + T(m-1,j-1) + \binom{m-1}{j}^2 + 2 \sum_{k=1}^{j} (-1)^k \binom{m-1}{j-k} \binom{m-1}{j+k}. \] (12)

Now, using again the Stifel formula for the function

\[ F(m-1,j) = \binom{m-1}{j}^2 + 2 \sum_{k=1}^{j} (-1)^k \binom{m-1}{j-k} \binom{m-1}{j+k}, \]

we have that it verifies the recurrence relation

\[ F(m,j) = F(m-1,j) + F(m-1,j-1). \] (13)

Considering the Stifel identity, we remark that the expression \( \binom{m}{j} \) satisfies also the recurrence relation (13). Because its initial values are the same
that those of (13) we conclude that $F(m, j) = \binom{m}{j}$. So, the recurrence relation (12) becomes

$$T(m, j) = T(m-1, j) + T(m-1, j-1) = \binom{m-1}{j}.$$  

But this relation is also satisfied by $(j+1)\left(\binom{m}{j+1}\right)$ and moreover, their initial values are the same. Then $T(m, j) = (j+1)\left(\binom{m}{j+1}\right)$. This proves (11).

Now, let us prove equation (7). By formula (8) the expression $C(j)$ results

$$C(j) = (-1)^{\frac{m}{2}+j-1} \left[\sum_{k=0}^{m-j} \frac{m-4k-2j-1}{m} \binom{m}{2k+1} \binom{m}{2k+2j}\right].$$

So, the expression (7) becomes

$$(-1)^{\frac{m}{2}+j-1} \left[\sum_{k=0}^{m-j} \frac{m-4k-2j-1}{m} \binom{m}{2k+1} \binom{m}{2k+2j}\right] = \binom{m-1}{j + \frac{m}{2} - 1}.$$  

Because $m$ is even we replace $m = 2r$ in both sides of the last expression. Moreover, we consider the change $r - j = n$ to obtain

$$(-1)^{2r-n-1} \left[\sum_{k=0}^{n} \frac{2n-4k-1}{2r} \binom{2r}{2k+1} \binom{2r}{2k+2-2n}\right] = \binom{2r-1}{2r-n-1}.$$  

Using in the last expression the symmetry identity $\binom{a}{b} = \binom{a}{a-b}$ [7], and replacing to $2r$ by $m$ it results

$$(-1)^n \left[\sum_{k=0}^{n} \frac{4k-2n+1}{m} \binom{m}{2k+1} \binom{m}{2n-2k}\right] = \binom{m-1}{n}.$$  

Note that the corresponding term $k = n$ on the left is $\binom{m-1}{2n}$. Since expression (15) becomes (9), equation (7) is proved.

We conclude by proving equation (6). Equation (8) implies that

$$\binom{m-1}{2k+2} - \binom{m-1}{2k} = \frac{m-4k-3}{m} \binom{m}{2k+1} \binom{m}{2k+2}.$$
Then $B(j)$ results

$$(-1)^{m+j} \left( \sum_{k=0}^{m-1} \frac{(m-4k-3)}{m} \left( \begin{array}{c} m \\ 2k+1 \end{array} \right) \left( \begin{array}{c} m \\ 2k+2 \end{array} \right) \right).$$

So, we must prove

$$(-1)^{m+j} \left( \sum_{k=0}^{m-1} \frac{(m-4k-3)}{m} \left( \begin{array}{c} m \\ 2k+1 \end{array} \right) \left( \begin{array}{c} m \\ 2k+2 \end{array} \right) \right) = \left( \begin{array}{c} m-1 \\ \frac{m}{2} \end{array} \right).$$

But, when we replace $j = 1$ in (14), we retrieve (16).

**Proof of Theorem 11.** In order to prove that inequality (11) holds for the polynomials $P^m$ and $Q^{2k}$ we consider the polynomial expression of $\nabla P^m \text{Hess} P(\nabla Q^{2k})^t$ and prove the following

$$P_x(Q_y P_{xy} - Q_x P_{yy}) + P_y(Q_x P_{xy} - Q_y P_{xx}) = 2k m^2 (m-1)(x^2 + y^2)^{k+m-2}.$$

Since $m$ is even, a straightforward computation shows that

$$Q_y P_{xy} - Q_x P_{yy} = 2k(x^2 + y^2)^{k-1} \left[ \sum_{j=0}^{m-1} (-1)^j \frac{(m-1)m!}{(2j)!(m-2j-1)!} x^{m-2j-1} y^{2j} \right].$$

Now, we multiply both sides of the last expression by $P_x$. The product $P_x(Q_y P_{xy} - Q_x P_{yy})$ results

$$2k(x^2 + y^2)^{k-1} m^2 (m-1) \left[ \sum_{j=0}^{m-1} (-1)^j \left( \begin{array}{c} m-1 \\ 2j \end{array} \right) x^{m-2j-1} y^{2j} \right]^2.$$

Developing the squared factor of the last expression we have

$$P_x(Q_y P_{xy} - Q_x P_{yy}) = 2k (x^2 + y^2)^{k-1} m^2 (m-1)$$

$$\left[ x^{2m-2} + \sum_{j=1}^{m-1} \left( \sum_{k=0}^{j} (-1)^j \left( \begin{array}{c} m-1 \\ 2k \end{array} \right) \left( \begin{array}{c} m-1 \\ 2j-2k \end{array} \right) \right) x^{2m-2j-2} y^{2j} + \right.$$  

$$\sum_{j=1}^{m-1} \left( \sum_{k=0}^{m-j-1} (-1)^{m-j-1} \left( \begin{array}{c} m-1 \\ 2k+2j \end{array} \right) \left( \begin{array}{c} m-1 \\ m-2k-2 \end{array} \right) \right) x^{m-2j} y^{m-2j-2} \right].$$

(17)
Now, we shall compute the expression $P_y(Q_x P_{xy} - Q_y P_{xx})$. After doing some elemental simplifications we have

$$Q_x P_{xy} - Q_y P_{xx} = 2k(x^2 + y^2)^{k-1} \left[ \sum_{r=0}^{m-1} (-1)^{r+1} m(m-1) \left( \frac{m-1}{2r+1} \right) x^{m-2r-2} y^{2r+1} \right].$$

Now, consider the product of the last expression by $P_y$. So, the product $P_y(Q_x P_{xy} - Q_y P_{xx})$ results in

$$2k(x^2 + y^2)^{k-1}(m-1)m^2y^2 \left[ \sum_{j=0}^{m-1} (-1)^{j+1} \left( \frac{m-1}{2j+1} \right) x^{m-2j-2} y^{j} \right]^2.$$

Developing the squared term we obtain

$$P_y(Q_x P_{xy} - Q_y P_{xx}) = 2k(x^2 + y^2)^{k-1}m^2(m-1) \left[ y^{m-2} + \sum_{j=1}^{m} \left( \sum_{k=0}^{j-1} (-1)^{j+1} \left( \frac{m-1}{2k+1} \right) \left( \frac{m-1}{2j-2k-1} \right) \right) x^{2m-2j-2} y^{2j} + \right.$$

$$\left. \sum_{j=1}^{m-1} \left( \sum_{k=0}^{m-j} (-1)^{m-j} \left( \frac{m-1}{2k+2j-1} \right) \left( \frac{m-1}{m-2k-1} \right) \right) x^{m-2j} y^{m+2j-2} \right] \quad (18)$$

Adding the expressions (17) and (18) we obtain

$$P_x(Q_y P_{xy} - Q_x P_{yy}) + P_y(Q_x P_{xy} - Q_y P_{xx}) =$$

$$= 2k(x^2 + y^2)^{k-1}m^2(m-1) \left[ x^{2m-2} + \left( \sum_{j=1}^{m-1} A(j)x^{2m-2j-2} y^{2j} \right) + Bx^{m-2} y^{m}$$

$$+ \left( \sum_{j=2}^{m-1} C(j)x^{m-2j} y^{m+2j-2} \right) + y^{2m-2} \right]. \quad (19)$$
Replacing (5), (6) and (7) in (19) we conclude that

\[ P_x (Q_y P_{xy} - Q_x P_{yy}) + P_y (Q_x P_{xy} - Q_y P_{xx}) = \]

\[ = 2k (x^2 + y^2)^{k-1} m^2 (m-1) \left( \sum_{j=0}^{m-1} \binom{m-1}{j} x^{2m-2j-2} y^{2j} + \right) \]

\[ + \left( \frac{m-1}{m} \right) x^{m-2} y^m + \sum_{r=m+1}^{m-2} \binom{m-1}{r} x^{2m-2r-2} y^{2r} + y^{2m-2} \right]. \]

Putting together all the terms lying inside of the square brackets we have

\[ P_x (Q_y P_{xy} - Q_x P_{yy}) + P_y (Q_x P_{xy} - Q_y P_{xx}) = \]

\[ = 2k (x^2 + y^2)^{k-1} m^2 (m-1) \left( \sum_{j=0}^{m-1} \binom{m-1}{j} x^{2m-2j-2} y^{2j} \right) \].

Observe that the expression exposed inside of the square brackets is the binomial \((x^2 + y^2)^{m-1}\). So, finally

\[ P_x(Q_y P_{xy} - Q_x P_{yy}) + P_y(Q_x P_{xy} - Q_y P_{xx}) = 2k m^2 (m-1) (x^2 + y^2)^{k+m-2}. \]

\[ \square \]

**Theorem 13** The Arnold’s conjecture is true. In fact, the number of connected components of \( Hyp(n) \) is at least \( \left\lfloor \frac{n^2}{2} \right\rfloor \).

**Proof.** Proposition 10 asserts that the homogeneous polynomial \( f^{m+2k} = P^m Q^{2k} \), where \( k \geq 1 \), \( m > max\{2, k\} \) is hyperbolic, meanwhile Proposition 9 ensures that the polynomial \( Q^{2k} \) is elliptic. Moreover, Theorem 11 implies that they satisfy inequality (1). So, by Corollary 7 we conclude that \( i_0(II_{f^{m+2k}}) = \frac{2-m}{2} \). Let \( n \geq 3 \) be a natural number. Now, we shall determine the number of pairs \((k, m) \in \mathbb{N} \times \mathbb{N}\) such that \( k \geq 1 \), \( m > max\{2, k\} \) and \( 2k + m = n \). (Table 1)

- If \( n \) is even, the set of pairs is \( \{(k, n-2k) : k \geq 1, m > max\{2, k\}\} = \{(k, n-2k) : k = 1, \ldots, \frac{n}{2} - 2\}. \) Moreover, since \( i_0(II_{f^{m+2k}}) = \frac{2-m}{2} = k + 1 - \frac{n}{2} \), then each one of these polynomials belongs to different connected component of \( Hyp(n) \). Adding the connected component determined by the polynomial \( P^m \) of degree \( n \), we conclude that the number of connected components of \( Hyp(n) \) is at least \( \frac{n}{2} - 1 \).
• If \( n \) is odd, the set of pairs is \( \{(k, n - 2k) : k \geq 1, m > \max\{2, k\}\} = \{(k, n - 2k) : k = 1, \cdots, \frac{n}{2} - \frac{3}{2}\} \). Moreover, since \( i_0 \left( \Pi_{f^{m+2k}} \right) = k + 1 - \frac{n}{2} \), each one of these polynomials belongs to different connected component of \( Hyp(n) \). Adding the connected component determined by the polynomial \( P^n \) of degree \( n \), we conclude that the number of connected components of \( Hyp(n) \) is at least \( \frac{n-1}{2} \). \( \square \)

| \( n = \text{deg of } f^{m+2k} \) | \( k \) | \( m \) | \( i_0(\Pi_{f^{m+2k}}) = \frac{2-m}{2} \) | \( \text{Low bound for the number of components} \) |
|----------------|-----|-----|----------------|----------------|
| 3              | 0   | 3   | -1/2           | 1              |
| 4              | 0   | 4   | -1             | 1              |
| 5              | 0   | 5   | -3/2           | 2              |
|                | 1   | 3   | -1/2           | 2              |
| 6              | 0   | 6   | -2             | 2              |
|                | 1   | 4   | -1             | 2              |
| 7              | 0   | 7   | -5/2           | 3              |
|                | 1   | 5   | -3/2           | 3              |
|                | 2   | 3   | -1/2           | 3              |
| 8              | 0   | 8   | -3             | 3              |
|                | 1   | 6   | -2             | 3              |
|                | 2   | 4   | -1             | 3              |

Table 1: Hyperbolic homogeneous polynomials up to degree 8.

**Remark 14** The number of connected components for degrees \( n = 3, 4 \) and \( 5 \), was determined with a different approach in [1].

Let us provide a qualitative description of the foliation of the field of asymptotic lines of the polynomials \( f^{2k+m} \), see [S], p. 161.

**Corollary 15** The foliation of the field of asymptotic lines of the polynomials \( f^{2k+m} \), \( k \geq 0, m \geq 3 \) on \( \mathbb{R}^2 \) has only one singularity at the origin where \( m \) separatrices pass through dividing the plane in \( m \) hyperbolic sectors.

**Proof.** Since the second fundamental forms of \( f^{2k+m} \) and \( P^m \) are hyperbolic isotopic their fields of asymptotic lines are topologically equivalent. That is, it is enough to describe the foliation corresponding to \( P^m \). Considering the natural identification of \( \mathbb{R}^2 \) with the complex plane, it is easy to define a hyperbolic isotopy between the second fundamental form of \( P^m \), when \( m \)
is even, and the hyperbolic quadratic differential form $\text{Im}(z^{m-2})dz^2$, where $z = x + iy$, $dz = dx + idy$ and $\text{Im}(z^{m-2})dz^2$ means the imaginary part of the quadratic differential form, described by Hopf in [9]. If $m$ is odd, the proof follows by noting that the second fundamental form of the polynomial $P^m$ composed with the reflection $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(u, v) = (v, u)$ is equal to the quadratic form $m(m - 1) \text{Im}(z^{m-2})dz^2$.

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