Bisector optimal ball-end tool orientation

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Abstract. For five-axis CNC machining with ball-end tool, smooth variation of tool orientation is desirable for stable rotary axes movement with constant inclination angle between surface normal and tool axis so as to obtain constant cutting speed. According to the Darboux frame, the orientation for a ball-end tool can be decomposed into two components, one of which along the surface normal is fixed and the other on the tangent plane is undetermined for possible optimization. Based on geometrical analysis for the latter component, a novel bisector optimization approach is proposed to optimize variation of the tool orientation. Simulations are conducted and the bisector optimization approach has exhibited smoother variation of tool orientation along the whole path.

1. Introduction
In five-axis computer numerical control (CNC) machining, the tool orientation control has been always an important problem followed with many researches since unfitted choice of tool orientation can lead excessive motion of rotary axes, gouging and poor quality of machined surface. Recently, smooth tool orientation have been concerned by many studies for stable rotary axes movement. Given several surface points and corresponding key tool axes, radial basis function (RBF) interpolation algorithm and RBF-spline approximation algorithm [1] are developed to construct smooth tool orientation field function to improve the surface quality and dynamics for 5-axis finish machining. Besides considering the collision avoidance, the smoother tool orientation is also concerned with graph analysis in [2] through application of Dijkstra’s algorithm. To smooth rotary axes movements for ball-end milling, the progressive iterative approximation method [3] is proposed based on a gradient-based optimization. Through dividing the surface region by using Gaussian curvature and mean, Li [4] planned the tool orientation orderly and reduced the change rate of tool orientation. Zhang [5] analyses the influence of tool lead and tilt angles, and steps over on the maximum cutting fore and form error, then the smooth tool orientation can be obtained to realize a high efficiency and accuracy machining.

Based on the rotation-minimizing frame (RMF), a rigorous solution for smooth movement of ball-end tool axis is proposed by Farouki [6] on a smooth surface without considering other problems such as maximization of material removal rate, collision avoidance and gouging, etc. Following the paper, an alternative method [7] based on the tractrix are presented to minimize the change rate of tool orientation, then the Dual RMF method [8] are proposed to minimize the variation of tool axis along the path curve for the flat-end tool and the angular acceleration of rotary axes are reduced. In this paper, based on a general geometry analysis to the tool orientation, another alternative rigorous
solution named bisector optimization for smoothing the rotary of the ball-end tool axis is presented. Section 2 introduces the problem with relative geometry. Then an overview of the RMF method is given in section 3. In section 4 the bisector optimization is presented. In section 5, experiments are implemented to exhibit the efficiency of our technique compared with the RMF method. In the end, the conclusion is given.

2. The problem with relative geometry

Supposing a curve path $C(\xi) = S(u(\xi), v(\xi))$ on a smooth work-piece surface $S(u, v)$ in figure 1, the surface is regular and so as the curve on the respective domains, $\xi$ is the path parameter, $\sigma(\xi)$ is the rate change of arc length, $\sigma(\xi) = |C'(\xi)|$, a cutting path point $P$ in the local coordinate system $L$ as showed in figure 1, $n$ is the unit surface normal, $t$ is the unit surface tangent or the feed direction, $k = t \times n$, $(t, k, n)$ is defined as the well-known frame-Darboux frame, and the relationship can be expressed as follow.

$$\begin{bmatrix} t' \\ k' \\ n' \end{bmatrix} = \sigma \begin{bmatrix} 0 & -\kappa_g & \kappa_n \\ \kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} t \\ k \\ n \end{bmatrix}$$  \hspace{1cm} (1)

Figure 1. Tool axis orientation along a tool path on a surface.

where $\kappa_g$, $\kappa_n$ and $\tau_g$ are the geodesic curvature, normal curvature, and geodesic torsion respective, and $\kappa_g$, $\kappa_n$ and $\tau_g$ are satisfied with the follow equations.

$$\kappa_n = \frac{n \cdot t'}{\sigma}, \quad \kappa_g = -\frac{k \cdot t'}{\sigma}, \quad \tau_g = -\frac{k \cdot n'}{\sigma}$$  \hspace{1cm} (2)

Obviously, a unit vector is orthogonal to its derivative. In the Darboux frame, the tool orientation $T$ which maintains a constant inclination angle $\alpha$ can be decomposed into two parts, one is named $m$ on the tangent surface and the other is parallel with the surface normal and it can be expressed by $n \cos \alpha$, therefore the tool orientation can be represented as follow.

$$T = n \cos \alpha + m \sin \alpha$$  \hspace{1cm} (3)

where

$$m = t \cos \beta + k \sin \beta$$  \hspace{1cm} (4)

$\beta$ is the tilt angle between $m$ and $t$ which is positive in the direction of rotation of $m$ onto $t$. For the ball-end tool machine, the inclination angle is supposed to be constant so as to maintain a stable cutting speed along the cutting path to render the stable cutting process and improve the quality of
machined surface. Under this limitation, the variation of tool orientations between two successive cutting points is expected to be minimized to enhance the stability of rotary axes movement.

3. The rotation-minimizing frame method

The RMF method [6] is proposed to minimize the variation of tool orientation through eliminating unnecessary rotary motion with respect to the surface normal for the tool based on the rotation-minimizing frame. The RMF method is consistent with the concept of the parallel transport. At the beginning, the RMF definition is overviewed. Given an orthogonal frame \( F(f_1, f_2, f_3) \) moving along a space curve \( C(\xi) \) on the surface \( S(u,v) \), the frame is a rotation-minimizing frame if the following condition can be satisfied:

\[
    f_2' \times f_1 = 0 , \quad f_3' \times f_1 = 0
\]

where \( f_2 \) and \( f_3 \) have no instantaneous rotation about \( f_1 \) when the frame is moving along the path curve, namely the derivatives of \( f_2 \) and \( f_3 \) are parallel to \( f_1 \). Herein, \( f_3 \) is named the rotary vector, \( f_2 \) and \( f_3 \) are named basic vectors which are the least variation vectors on the surface constituted by \( f_2 \) and \( f_3 \).

In the Darboux frame, a RMF \((n, m, t)\) is constructed to provide a basic vector as the component of tool orientation on the tangent plane with the RMF method, therefore \( m \) has no rotary motion with the surface normal when the tool axis is moving along the path curve. According to equation (2) and equation (4), the derivative of \( m \) with respect to the curve parameter \( \xi \) can be written as follow:

\[
    m' = -(\beta' - \sigma \kappa_s) \sin \beta t + (\beta' - \sigma \kappa_s) \cos \beta k + \sigma (\kappa_n \cos \beta + \tau_s \sin \beta) n
\]

Since \( m \) is a basic vector on the tangent surface, it must be satisfied with \( m' \parallel n \) which is equal to \( m' \times t = 0 \) and \( m' \times k = 0 \), then the follow conclusion can be obtained.

\[
    \beta' - \sigma \kappa_s = 0
\]

The integration of equation (7) is \( \beta(\xi) = \beta_0 + \int_0^\xi \sigma \kappa_s d\xi \) where \( \beta_0 \) is an integration constant and the derivative of \( m \) can be simplified as follow.

\[
    m' = \sigma (\kappa_n \cos \beta + \tau_s \sin \beta) n
\]

It can be observed that the derivative of \( m \) is parallel with the surface normal and \( m \) is also the parallel transport vector which has the least variation on the tangent plane, then the variation of tool orientation can be smaller than the Struz method [8] in which the tool vector \( T' \) in figure 1) is lying on the surface constituted by \( n \) and \( t \) and maintaining fixed inclination angle with surface normal. The derivative of tool orientation can be calculated with equation (2), equation (3) and equation (8).

\[
    T' = n' \cos \alpha + m' \sin \alpha
\]

\[
    = \sigma (\kappa_n \cos \beta + \tau_s \sin \beta) \sin \alpha n - \sigma \kappa_s \cos \alpha t - \sigma \tau_s \cos \kappa \cos k
\]

The magnitude of \( T' \) are computed as follow.

\[
    |T'| = \sigma \sqrt{\kappa_n^2 + \tau_s^2} \cos^2 \alpha + (\kappa_n \cos \beta + \tau_s \sin \beta)^2 \sin^2 \alpha
\]

In the RMF method, the strategy of \( m' \parallel n \) is adopted as the choice for the parallel transport vector to determine how the component of the tool vector on the tangent plane should be varied. However, this cannot be the only strategy for deciding \( m' \).

4. Bisector optimization based on general geometry

In this section, the general definition for the derivative of \( m \) is deduced and based on it, the magnitude of \( m' \) is computed, then through the application of the basic inequality in mathematics, a bisector optimization measure is proposed to minimize the variation of tool orientation, the procedure is presented as the follow.
Figure 2. The derivative of $m$ (the component of tool vector on the tangent plane) in the Darboux frame.

As showed in figure 2, since $m \cdot m' = 0$, the derivative of $m$ can be expressed with no limitation as follow:

$$m' = \lambda n + \mu h$$

(11)

where $h = k \cos \beta - tsin \beta$, $\lambda$ and $\mu$ are coefficients to be solved, $n$ and $h$ are orthogonal. Combined equation (11) with equation (6), $\lambda$ and $\mu$ can be calculated.

$$\lambda = \sigma (\kappa \cos \beta + \tau_s \sin \beta)$$

(12)

$$\mu = \beta' - \sigma \kappa$$

(13)

Therefore, the derivative of $m$ can be rewritten.

$$m' = \sigma (\kappa \cos \beta + \tau_s \sin \beta) n + (\beta' - \sigma \kappa) h$$

(14)

The derivative of tool orientation can be represented as follow.

$$T' = \sigma (\kappa \cos \beta + \tau_s \sin \beta) \sin \alpha n +$$

$$\left[ \sigma (\kappa \sin \beta - \tau_s \cos \beta) \cos \alpha + (\beta' - \sigma \kappa) \sin \alpha \right] h -$$

$$\sigma (\kappa \cos \beta + \tau_s \sin \beta) \cos \alpha m$$

(15)

The magnitude of $T'$ is:

$$|T'| = \sqrt{[\sigma (\kappa \cos \beta + \tau_s \sin \beta)]^2 + [\sigma (\kappa \sin \beta - \tau_s \cos \beta) \cos \alpha + (\beta' - \sigma \kappa) \sin \alpha]^2}$$

(16)

According to the equation (14), the limitation of the RMF method is $\beta' - \sigma \kappa = 0$, then the derivative of $m$ has only one component along the surface normal, namely the coefficient of $h$ is equal to zero, hence the RMF method can be thought as a special optimal measure. Moreover, the magnitude of $m'$ can be calculated.

$$|m'| = \sqrt{\sigma^2 (\kappa \cos \beta + \tau_s \sin \beta)^2 + (\beta' - \sigma \kappa)^2}$$

(17)

Linking the equation (17) with the basic inequality in mathematics that the arithmetic mean of two positive real numbers is greater than or equal to their geometric mean, the magnitude of $m'$ has the inequality of $\sqrt{\lambda^2 + \mu^2} \geq \sqrt{2 |\lambda \mu|}$ and the minimal value for $|m'|$ can be obtained when $\lambda = \mu$, namely,

$$\beta' = \sigma \kappa + \sigma (\kappa \cos \beta + \tau_s \sin \beta)$$

(18)
The integration of equation (18) is \( \beta = \beta_0 + \int_{0}^{\xi} \sigma \kappa \pm \sigma (\kappa \cos \beta + \tau \sin \beta) d\xi \), \( \beta_0 \) is an integration constant. In the figure 2, it can be observed obviously that the derivative of \( m \) is parallel with the angular bisector of the right angle between \( n \) and \( h \), which means that the angle \( \gamma \) between \( m \) and \( h \) is 45°.

The bisector optimization (BO) measure should likely balance the rotary variation along the surface normal and the tangent surface for the component on the tangent plane based on the general geometry analysis and the application with basic inequality in mathematics while the RMF method is consistent with the parallel transport vector through eliminating the unnecessary rotary motion with respect to the surface normal for the component on the tangent plane. In the experiments section, the two methods (BO method and RMF method) are implemented to have a comparison of their performance on minimizing the change rate of tool orientation.

5. Experiments

The tilt angle can be calculated by the numerical method [6] for both of two methods (RMF method and BO method), and then the change rate of tool orientation can be calculated for the comparison of two methods according to equation (10) and equation (16) respectively. In the examples, the path curve marked with green line in figures is defined by \( C(\xi) = [u(\xi), v(\xi)] \) with the grid size \( \Delta \xi = 0.01 \). The data related to the BO method is marked with red colour while the RMF one is marked with blue colour.

5.1 The Torus

The surface is defined as \( S(u,v) = [(2 + \cos v) \cos u, (2 + \cos v) \sin u, \sin v] \) where \( 0 \leq u, v \leq 2\pi \) and the tool path is \( C(\xi) = ([\frac{\pi \xi}{2}], [\frac{\pi \xi}{2}], \xi) \) with \( \xi \in [0,1] \). The initial tool orientation is fixed by \( \alpha_0 = 45^\circ \) and \( \beta_0 = 0^\circ \). In figure 3, at the start point the BO method and the RMF method have the same \( |T'| \), and \( |T'| \) of the BO method is lower than the RMF method’s in the area \( \xi \in (0,0.6) \) while \( |T'| \) of the RMF method is smaller in the area \( \xi \in (0.6,1) \). On the whole, the BO method performances better than the RMF method on reducing the change rate of tool orientation and it can be verified from the figure 4. At the beginning, the tool vector of two methods are overlapped and maintain a constant inclination angle with the surface normal, and then the tool axis varies along with the path curve, the tool vector of the BO method experiences less variation till the end of path compared with the RMF method.
5.2 The Dome

The surface is defined as \( S(u, v) = [u, v, 1 - u^2 - v^2] \) where \(-1 \leq u, v \leq 1\) and the tool path is \( C(\xi) = [\xi, \xi^2] \) with \( \xi \in [0, 1] \). The initial angles \((\alpha_0, \beta_0)\) are the same as the torus example. In figure 5, except the small field \( \xi \in [0, 0.15] \) where the RMF method has smaller change rate of tool orientation, the BO method is always providing smaller \( |T'| \) in the large domain left. This leads to obvious less variation of tool axis along the curve path when it moves along the path curve which can be observed from the figure 6.
5.3 The bi-cubic spline surface
A bi-cubic spline surface is employed to validate the effectivity of the BO method compared with the RMF method. The bi-cubic spline surface is a typical spline surface which is widely applied for free-form approximation and modelling. The surface is defined as follow.
\[
S = \begin{pmatrix}
    x = [u^3 \ u^2 \ u \ 1](M \ Mx M^T) [v^3 \ v^2 \ v \ 1]^T \\
    y = [u^3 \ u^2 \ u \ 1](M \ My M^T) [v^3 \ v^2 \ v \ 1]^T \\
    z = [u^3 \ u^2 \ u \ 1](M \ Mz M^T) [v^3 \ v^2 \ v \ 1]^T
\end{pmatrix}
\]

where
\[
\begin{bmatrix}
100 & 200 & 300 & 400 \\
110 & 210 & 300 & 380 \\
130 & 250 & 330 & 410 \\
150 & 280 & 360 & 480
\end{bmatrix}
\begin{bmatrix}
100 & 200 & 300 & 400 \\
300 & 280 & 280 & 320 \\
250 & 200 & 180 & 200 \\
350 & 250 & 250 & 280
\end{bmatrix}
\begin{bmatrix}
35 & 130 & 230 & 150 \\
20 & 40 & 60 & 80 \\
35 & 36 & 46 & 66 \\
30 & 50 & 70 & 55
\end{bmatrix}
\]

\[
M = \frac{1}{6} \begin{bmatrix}
-1 & 3 & -3 & 3 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{bmatrix}, \quad 0 \leq u, v \leq 1.
\]

The path curve is given by \( C(\xi) = [\xi, \xi] \) with \( \xi \in [0,1] \). Here, the initial condition is \( \alpha_0 = 20^\circ \) and \( \beta_0 = 10^\circ \). In figure 7, the \( |T'| \) curves of two methods are similarly, and in the end curves, they are nearly overlapping. On the whole, the BO method still provides smaller \( |T'| \) in most domain. The figure 8 exhibits the variation of tool axis for the two methods, obviously, the tool orientation of the BO method has smoother tool orientations than the RMF method along the path curve.

Figure 7. The change rate of tool orientation on the bi-cubic spline surface: RMF method and BO method.
Figure 8. The tool vectors on the Dome: RMF method and BO method.

6. Conclusion
A novel approach is proposed to smooth the orientation of ball-end tool in five-axis CNC machining. The component of tool vector on the tangent plane in the Darboux frame is the focus and the objective of optimization. A geometrical analysis is conducted on the component and a general mathematic formulation is obtained. The bisector optimization (BO) comes out with the application of the basic inequality in mathematics. Experiments have shown the variation of tool axis is smoother with the bisector optimization compared with the RMF method. The BO algorithm tends to balance the variations of the component on the tangent plane while the RMF method eliminates the rotary motion about the surface normal. This work open a new venue for tool orientation in five-axis CNC machining and has extensive potentials in robotics, computer animation and computer vision.

Acknowledgements
This work is supported by National Natural Science Foundation of China (No.51975231).

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