Some Results on Nearly Cosymplectic Manifolds

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Abstract

The object of this paper is to study Ricci solitons under some curvature conditions in nearly cosymplectic manifolds.

1. Introduction

Cosymplectic manifold is an odd dimensional counterpart of a Kähler manifold which is defined by Lipperman and Blair 1967 [9]. In parallel with Olzak's work [1], [2], Endo investigated the geometry of nearly cosymplectic manifolds [3].

Ricci soliton is a special solution to the Ricci flow introduced by Hamilton [10] in the year 1982. In [12], Sharma initiated the study of Ricci solitons in contact Riemannian geometry. Later, Tripathi [13], Nagaraja et al. [11] and others extensively studied Ricci solitons in contact metric manifolds. Ricci soliton in Riemannian manifold \((M, g)\) is a natural generalization of an Einstein metric and is defined as a triple \((g, V, \lambda)\) with \(g\) a Riemannian metric, \(V\) a vector field and \(\lambda\) a real scalar such that

\[
(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0
\]

where \(S\) is the Ricci tensor of \(M\) and \(L_V\) denoted the Lie derivative operator along the vector field \(V\). The Ricci soliton is said to be shrinking, steady and expanding accordingly as \(\lambda\) is negative, zero and positive respectively.

In [16], [19], authors studied the properties of generalized recurrent manifolds where as the properties of generalized \(\varphi\)-recurrent manifolds have studied in [8], [16], [17] and [18].

In this paper we study some curvature conditions such that \(\varphi\)-recurrent, pseudo-projective \(\varphi\)-recurrent, concircular \(\varphi\)-recurrent and Ricci recurrent which characterize Ricci solitons in nearly cosymplectic manifolds.

2. Preliminaries

2.1. Nearly Cosymplectic Manifolds

Let \((M, \varphi, \xi, \eta, g)\) be an \((2n + 1)\)–dimensional almost contact Riemannian manifold, where \(\varphi\) is a type of \((1, 1)\)–tensor field, \(\xi\) is the structure vector field, \(\eta\) is a \(1\)–form and \(g\) is the Riemannian metric. It is well known that the \((\varphi, \xi, \eta, g)\)–structure satisfies the conditions [7] for any vector fields \(X\) and \(Y\) on \(M\),

\[
\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X)
\]

\[
\eta(\varphi X) = 0, \quad \varphi\xi = 0,
\]

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\[ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \]

A nearly cosymplectic manifold is an almost contact metric manifold \((M, \varphi, \xi, \eta, g)\) such that
\[ (\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 0, \]
for all vector fields \(X, Y\). Clearly, this condition is equivalent to \((\nabla_X \varphi)X = 0\). It is known that in a nearly cosymplectic manifold the Reeb vector field \(\xi\) is Killing and satisfies \(\nabla_\xi \xi = 0\) and \(\eta\) is a contact form \(\nabla_\xi \eta = 0\). The tensor field \(h\) of type \((1, 1)\) defined by
\[ \nabla_X \xi = hX, \]
is skew symmetric and anticommutes with \(\varphi\). It satisfies
\[ h\xi = 0, \quad \eta \circ \varphi = 0, \]
and the following formulas hold [3], [4]
\[ g((\nabla_W R)(X, Y)Z, hZ) = \eta(Y)g(h^2 X, \varphi Z) - \eta(X)g(h^2 Y, \varphi Z), \]
\[ tr(h^2) = \text{constant}, \]
\[ R(Y, Z)\xi = \eta(Y)h^2 Z - \eta(Z)h^2 Y, \]
\[ S(Z, \xi) = -tr(h^2)\eta(Z), \]
where \(R, S, Q\) and \(\eta\) are the Riemannian curvature tensor type of \((1, 3)\), the Ricci tensor of type \((0, 2)\), the Ricci operator defined by \(g(QX, Y) = S(X, Y)\).

Let \((g, V, \lambda)\) be a Ricci soliton in a nearly cosymplectic manifold \(M\). Taking \(V = \xi\) then from (2.4) and (1.1), we have
\[ S(X, Y) = -\lambda g(X, Y). \]
The above equation yields
\[ QX = -\lambda X, \]
\[ S(X, \xi) = \lambda \eta(X), \]
\[ r = -\lambda n. \]

Also by definition of covariant derivative, we have
\[ (\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \]

3. \(\varphi\)-Recurrent Nearly Cosymplectic Manifolds

**Definition 3.1.** A nearly cosymplectic manifold is said to be \(\varphi\)-recurrent manifold [14] if there exist a non-zero 1–form \(A\) such that
\[ \varphi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z \]
for arbitrary vector fields \(X, Y, Z, W\).

Let us consider a \(\varphi\)-recurrent nearly cosymplectic manifold. By virtue of (2.1) and (3.1), we have
\[ -(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z. \]

**Theorem 3.2.** Let given Ricci soliton on nearly cosymplectic manifolds. Then there is not exist \(\varphi\)-recurrent nearly cosymplectic manifold.
Proof. Contracting (3.2) with \( U \), we obtain
\[
-g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) = A(W)g(R(X, Y)Z, U).
\]
(3.3)

Let \( e_i \) (\( i = 1, 2, \ldots, 2n + 1 \)), be an orthonormal basis of the tangent space at any point of the manifold. Taking \( X = U = e_i \) in (3.3) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get
\[
-(\nabla_W S)(Y, Z) = A(W)S(Y, Z).
\]
(3.4)

Replacing \( Z \) by \( \xi \) in (3.4) and using (2.7), we have
\[
-(\nabla_W S)(Y, \xi) = -tr(h^2)A(W)\eta(Y).
\]
(3.5)

Using (2.7) and (2.4) in (2.12), we obtain
\[
(\nabla_W S)(Y, \xi) = -[S(Y, hW) + tr(h^2)g(Y, hW)].
\]
(3.6)

In view of (3.5) and (3.6), we have
\[
S(Y, hW) = -tr(h^2)[g(Y, hW) + A(W)\eta(Y)].
\]
(3.7)

Taking \( Y = \xi \) in (3.7), we get
\[
S(\xi, hW) = -tr(h^2)[g(Y, hW) + A(W)\eta(\xi)].
\]
(3.8)

Using (2.1), (2.5) and (2.8) in (3.8), we find
\[
-\lambda g(hW, \xi) = tr(h^2)A(W),
\]
\[
tr(h^2)A(W) = 0,
\]
\[
A(W) = 0.
\]

This is a contradiction. \( \square \)

4. Generalized \( \varphi \)-Recurrent Nearly Cosymplectic Manifolds

**Definition 4.1.** A nearly cosymplectic manifold is said to be generalized \( \varphi \)-recurrent manifold if its curvature tensor \( R \) satisfies the relation
\[
\varphi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)\{g(Y, Z)X - g(X, Z)Y\},
\]
(4.1)

where \( A \) and \( B \) are 1-forms and non-zero and these are defined by
\[
A(W) = g(W, \rho_1), \quad B(W) = g(W, \rho_2),
\]
and \( \rho_1, \rho_2 \) are unit vector fields associated with 1-forms \( A, B \) respectively.

**Theorem 4.2.** In a generalized \( \varphi \)-recurrent strictly nearly cosymplectic manifold \( (M, g) \), the associated vector fields \( \rho_1 \) and \( \rho_2 \) of the 1-forms \( A \) and \( B \) respectively are co-directional.

**Proof.** In consequence of (2.1), equation (4.1) becomes
\[
-(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z + B(W)\{g(Y, Z)X - g(X, Z)Y\},
\]
(4.2)

from which it follows by taking inner product with \( U \) that
\[
-g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) = A(W)g(R(X, Y)Z, U) + B(W)\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}.
\]
(4.2)

Let \( \{e_i\}, i = 1, 2, \ldots, 2n + 1 \) be an orthonormal basis of the tangent space at any point of the manifold. Then putting \( X = U = e_i \) in (4.2) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get
\[
-(\nabla_W S)(Y, Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = A(W)S(Y, Z) + 2nB(W)g(Y, Z).
\]
(4.3)

Again replacing \( Z \) by \( \xi \) in (4.3) and using (2.1) and (2.7), we get
\[
-(\nabla_W S)(Y, \xi) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) = R(W)\{\lambda \xi g\} + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) - trh^2A(W) + 2nB(W)\}\eta(Y).
\]
(4.4)

The second term of left hand side in (4.4) with (2.1) takes the form
\[
\sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) = \eta((\nabla_W R)(\xi, Y)\xi)\eta(\xi) = g((\nabla_W R)(\xi, Y)\xi, \xi).
\]
(4.5)
Using (2.4), (2.5) and (2.6) in (4.5), we obtain
\[ g((\nabla_W R)(\xi, Y)\xi, \xi) = 0. \] (4.6)

In view of (4.6), (4.4) becomes
\[ (\nabla_W S)(Y, \xi) = \{tr(h^2)A(W) - 2nB(W)\} \eta(Y). \] (4.7)

The equation (2.12) with (2.4) and (2.7) takes the form
\[ (\nabla_W S)(Y, \xi) = -tr(h^2)g(Y, hW) - S(Y, hW). \] (4.8)

From equations (4.7) and (4.8), we find
\[ -tr(h^2)g(Y, hW) - S(Y, hW) = (tr(h^2)A(W) - 2nB(W))\eta(Y). \] (4.9)

Replacing \( Y \) by \( \xi \) then using (2.5) in (4.9) we have
\[ A(W) = \frac{2n}{tr(h^2)}B(W). \]

This means that the vector fields \( \rho_1 \) and \( \rho_2 \) of the \( 1 \)-forms are co-directional. \( \square \)

5. Ricci-Recurrence Nearly Cosymplectic Manifold

**Theorem 5.1.** Let given Ricci soliton on nearly cosymplectic manifolds. Then there is not exist Ricci recurrent nearly cosymplectic manifold.

**Proof.** A nearly cosymplectic manifold is said to be Ricci-recurrent manifold if there exist a non-zero \( 1 \)-form \( A \) such that
\[ (\nabla_W S)(Y, Z) = A(W)S(Y, Z). \] (5.1)

Replacing \( Z \) by \( \xi \) in (5.1) and using (2.7), we have
\[ (\nabla_W S)(Y, \xi, \xi) = -tr(h^2)A(W)\eta(Y). \] (5.2)

Using (2.4) and (2.7) in (2.12), we obtain
\[ (\nabla_W S)(Y, \xi) = -[S(Y, hW) + tr(h^2)g(y, hW)]. \] (5.3)

In view of (5.2) and (5.3), we have
\[ S(Y, hW) = tr(h^2)g(Y, hW) + tr(h^2)A(W)\eta(Y). \] (5.4)

Taking \( Y = \xi \) in (5.4), we get
\[ A(W) = 0. \]

It contradicts that \( A \neq 0 \). Thus, the proof is completed. \( \square \)

6. Pseudo-projective \( \varphi \)-recurrent Nearly Cosymplectic Manifold

In a nearly cosymplectic manifold \( M \), the pseudo-projective curvature tensor \( \tilde{P} \) is given by [20]
\[ \tilde{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{2n+1} \left( \frac{a}{2n} + b \right) \eta(Y, Z)X - g(X, Z)Y \] (6.1)

where \( a \) and \( b \) are constants such that \( a, b \neq 0 \).

**Theorem 6.1.** Ricci soliton in a pseudo-projective \( \varphi \)-recurrent nearly cosymplectic manifold \((M, g)\) with \( 1 \)-form non-zero \( A \) depends on the sign of \( tr(h^2) \).

**Proof.** A nearly cosymplectic manifold is said to be pseudo-projective \( \varphi \)-recurrent manifold if there exists a non-zero \( 1 \)-form \( A \) such that
\[ \varphi^2((\nabla_W \tilde{P})(X, Y)Z) = A(W)\tilde{P}(X, Y)Z, \] (6.2)

for arbitrary vector fields \( X, Y, Z, W \). Let us consider a pseudo-projective \( \varphi \)-recurrent nearly cosymplectic manifold. By virtue of (2.1) and (6.2), we have
\[ -(\nabla_W \tilde{P})(X, Y)Z + \eta((\nabla_W \tilde{P})(X, Y)Z)\xi = A(W)\tilde{P}(X, Y)Z. \] (6.3)

Contracting (6.3) with \( U \), we obtain
\[ -g((\nabla_W \tilde{P})(X, Y)Z, U) + \eta((\nabla_W \tilde{P})(X, Y)Z)\eta(U) = A(W)g(\tilde{P}(X, Y)Z, U). \] (6.4)
Let $e_i (i = 1, 2, ..., 2n + 1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (6.4) and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$\langle \nabla_W S \rangle (Y, Z) = A(W)\{S(Y, Z) - \frac{r}{2n+1}g(Y, Z)\}. \tag{6.5}$$

Replacing $Z$ by $\xi$ in (6.5) and using (2.1) and (2.7), we have

$$\langle \nabla_W S \rangle (Y, \xi) = -A(W)\{tr(h^2) - \frac{r}{2n+1}\} \eta(Y). \tag{6.6}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$\langle \nabla_W S \rangle (Y, \xi) = -[S(Y, hX) + tr(h^2)g(Y, hX)]. \tag{6.7}$$

In view of (6.6) and (6.7), we have

$$S(Y, hX) = A(W)\{tr(h^2) + \frac{r}{2n+1}\} \eta(Y) - tr(h^2)g(Y, hX).$$

Taking $Y = \xi$ and using (2.5), (2.8), (2.11) we get

$$A(W)\{tr(h^2) - \frac{\lambda n}{2n+1}\} = 0.$$

for non-zero $A(W)$ we find

$$\lambda = \frac{tr(h^2)(2n+1)}{n}.$$

Hence, the proof is completed. \hfill \Box

### 7. Concircular $\varphi$–Recurrent Nearly Cosymplectic Manifold

The Concircular curvature tensor of $(M, g)$ is given by [21]

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)} [g(Y, Z)X - g(X, Z)Y]. \tag{7.1}$$

**Definition 7.1.** A nearly cosymplectic manifold is said to be concircular $\varphi$–recurrent manifold if there exist a non-zero 1–form $A$ such that

$\varphi^2(\langle \nabla_W \tilde{C} \rangle(X, Y)Z) = A(W)\tilde{C}(X, Y)Z. \tag{7.2}$

for arbitrary vector fields $X, Y, Z, W$.

**Theorem 7.2.** Ricci soliton in a concircular $\varphi$–recurrent nearly cosymplectic manifold $M$ with 1–form non-zero $A$ depends on the sign of $tr(h^2)$.

*Proof.* Let us consider a concircular $\varphi$–recurrent nearly cosymplectic manifold. By virtue of (2.1) and (7.2), we have

$$-\langle \nabla_W \tilde{C} \rangle(X, Y)Z + \eta(\langle \nabla_W \tilde{C} \rangle(X, Y)Z) \xi = A(W)\tilde{C}(X, Y)Z. \tag{7.3}$$

Contracting (7.3) with $U$, we obtain

$$-g(\langle \nabla_W \tilde{C} \rangle(X, Y)Z, U) + \eta(\langle \nabla_W \tilde{C} \rangle(X, Y)Z)\eta(U) = A(W)g(\tilde{C}(X, Y)Z, U). \tag{7.4}$$

Let $e_i (i = 1, 2, ..., 2n + 1)$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (7.4) and taking summation over $i, 1 \leq i \leq 2n + 1$, we get

$$\langle \nabla_W S \rangle (Y, Z) = -A(W)\{S(Y, Z) - \frac{r}{2n+1}g(Y, Z)\}. \tag{7.5}$$

Replacing $Z$ by $\xi$ in (7.5) and using (2.1) and (2.7), for a constant $r$, we have

$$\langle \nabla_W S \rangle (Y, \xi) = A(W)\eta(Y)\{tr(h^2) + \frac{r}{2n+1}\}. \tag{7.6}$$

Using (2.7) and (2.4) in (2.12), we obtain

$$\langle \nabla_W S \rangle (Y, \xi) = -[S(Y, hW) + tr(h^2)g(Y, hW)]. \tag{7.7}$$

In view of (7.6) and (7.7), we have

$$S(Y, hW) = -\{tr(h^2) + \frac{r}{2n+1}\}A(W)\eta(Y) - tr(h^2)g(Y, hW). \tag{7.8}$$

Taking $Y = \xi$, and using (2.5) and (2.8) a characteristic vector field in (7.8), we get

$$A(W)\{tr(h^2) + \frac{r}{2n+1}\} = 0. \tag{7.9}$$

Using (2.11) in (7.9), for non-vanishing $A$, we have

$$\lambda = \frac{tr(h^2)(2n+1)}{n}.$$ 

So, we have desired result. \hfill \Box
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