Generation of Werner states and preservation of entanglement in a noisy environment

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We study the influence of noisy environment on the evolution of two-atomic system in the presence of collective damping. Generation of Werner states as asymptotic stationary states of evolution is described. We also show that for some initial states the amount of entanglement is preserved during the evolution.

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I. INTRODUCTION

Entanglement of quantum states is the most non-classical feature of quantum systems and one of the key resources in quantum information theory [1]. In real quantum systems, inevitable interactions with surrounding environment may lead to decoherence resulting in degradation of entanglement. Since entanglement once it has been lost, cannot be restored by local operations, it is important to understand the process of disentanglement to control the effects of noise and to preserve as much entanglement as possible. The main motivation of the investigations presented in the paper is to study the possibility of the preservation of entanglement in the model of two two-level atoms interacting with environment with maximal noise (the case of thermal noise will be discussed elsewhere). In the Markovian approximation, the influence of that kind of environment on a single atom is described by dynamical semi-group with Lindblad generator in which the transition rates from ground state to excited state of the atom and vice versa, are equal to decay rate $\Gamma_0$. This noisy dynamics is related to the following limiting procedure: temperature of thermal reservoir tends to infinity, whereas the coupling strength goes to zero $\Gamma \to 0$ [2, 3]. On the other hand, the collective properties of two - atomic systems can alter the decay process compared with the single atom. It was already shown by Dicke [4] in the case of spontaneous emission and environment in the vacuum state that there are states with enhanced emission rates (superradiant states) and such that the emission rate is reduced (subradiant states). In the latter case, two-atom system can decohere slower compared with individual atoms and some amount of initial entanglement can be preserved or even created by the indirect interaction between atoms $\text{[5, 6, 7, 8]}$.

In the model discussed in the paper, the collective dynamics of two atoms is described by damping rate $\Gamma$ which depends on interatomic distance. When the atoms are separated by a large distance, one can assume that they are located inside two independent environments and $\Gamma = 0$. In that case, the noisy dynamics brings all initial states into unique asymptotic state which is maximally mixed. Moreover, all entangled states disentangle in finite time $\text{[9]}$. When atoms are confined in a region smaller than the radiation wave length, the collective damping rate is close to the decay rate, so we can use the approximation $\Gamma = \Gamma_0$. In that case, similarly as in the Dicke model, the antisymmetric state (singlet state) is decoupled from the environment and therefore is stable. This is the main physical reason why entanglement can be preserved in spite of the influence of noisy environment.

In this paper we are mainly interested in robust entanglement, so we study long time (asymptotic) behaviour of dynamical semi-group. We also not discuss here the case of arbitrary separation of atoms, since in that case all states disentangle asymptotically. Small distance separation modeled by the condition $\Gamma = \Gamma_0$, leads to the interesting semi-group which is not ergodic: asymptotic stationary states depend on initial conditions and can be parametrized by fidelity of initial state with respect to singlet state. We show that some of these asymptotic states are entangled. They belong to the important class of Werner states $\text{[10]}$. In particular, we prove that if fidelity is greater then 1/2, the asymptotic Werner state is entangled. So collective damping can produce correlations between atoms which partially overcome the effect of decoherence, but in contrast to the zero temperature case $\text{[5, 6, 7, 8]}$, this process cannot create entanglement. Our noisy dynamics has also another remarkable property: there are initial entangled states for which entanglement is preserved during the evolution, although the process of decoherence takes place, resulting in decreasing of purity. As we prove, asymptotic entanglement depends only on the overlap of the initial state with singlet state but not on its entanglement. So initial states with the same entanglement can behave differently with respect to the noise. This opens the possibility of protecting some entanglement of the initial state by performing local operations (which do not change entanglement) to maximize its overlap with singlet state.

When the interatomic separation is small, coupling by the dipol – dipol interaction plays a significant role. It causes the entanglement between two atoms to oscillate in time, so for some period initial entanglement may even increase, but the noise decreases the amplitude of these oscillations, and asymptotically its contribution to the preservation of entanglement vanishes.

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Since we mainly study here the role of noise in the time evolution of entanglement, we are not discussing in detail the interesting problem of the influence of dipol–dipol coupling on this evolution. Similarly, the experimental side of the problems studied here is beyond the scope of the present paper, where we investigate the theoretical model of the compound system interacting with maximally noised environment.

II. NOISY DYNAMICS OF TWO-ATOMIC SYSTEM

Time evolution of a density matrix of two two-level atoms A and B interacting with a noisy environment of the type discussed in the introduction, can be described by the following master equation

$$\frac{d\rho}{dt} = -i[H,\rho] + L_N\rho$$  \hspace{1cm} (II.1)

Here

$$H = \omega_0 \sum_{j=A,B} \sigma^j_3 + \sum_{j,k=A,B, j \neq k} \Omega_{jk}\sigma^j_+\sigma^k_-$$  \hspace{1cm} (II.2)

and

$$L_N\rho = \frac{1}{2} \sum_{j,k=A,B} \Gamma_{jk} (2\sigma^j_-\rho\sigma^k_+ - \sigma^j_+\rho\sigma^k_- - \rho\sigma^j_-\sigma^k_+ + \rho\sigma^j_+\sigma^k_-)$$  \hspace{1cm} (II.3)

where

$$\sigma^j_\pm = \sigma^j_+ \otimes I, \quad \sigma^j_\pm = I \otimes \sigma^j_\pm, \quad \sigma^A_3 = \sigma_3 \otimes I, \quad \sigma^B_3 = I \otimes \sigma_3$$

and we identify ground state $|0\rangle$ and excited state $|1\rangle$ of the atom A or B with vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $\mathbb{C}^2$. In the hamiltonian term (II.2), $\omega_0$ is the frequency of the transition $|0\rangle \rightarrow |1\rangle$ and $\Omega_{AB} = \Omega_{BA} = \Omega$ describes interatomic coupling by the dipol-dipol interaction. On the other hand, noisy dynamics is given by the generator (II.3) with

$$\Gamma_{AA} = \Gamma_{BB} = \Gamma_0$$  \hspace{1cm} (II.4)

and

$$\Gamma_{AB} = \Gamma_{BA} = \Gamma$$  \hspace{1cm} (II.5)

In the equality (II.4), $\Gamma_0$ is a decay rate of individual atom. The parameter $\Gamma$ in the equality (II.5) describes the collective damping rate of two atoms interacting with a noisy environment. In our model, $\Gamma$ satisfies

$$\Gamma = g(R) \Gamma_0$$

where $g(R)$ is the function of the distance $R$ between atoms such that $g(R) \rightarrow 1$ when $R \rightarrow 0$. Notice that (II.3) can be rewritten as

$$L_N\rho = \Gamma_0 \left[ \sigma^A_+ \rho \sigma^A_- + \sigma^B_+ \rho \sigma^B_- + \sigma^A_- \rho \sigma^A_+ + \sigma^B_- \rho \sigma^B_+ - 2\rho \right]$$

$$+ \frac{1}{2} \Gamma \left[ 2\sigma^A_+ \rho \sigma^B_- + 2\sigma^A_- \rho \sigma^B_+ - \sigma^A_- \rho \sigma^A_+ - \sigma^B_- \rho \sigma^B_+ - \sigma^A_+ \rho \sigma^A_- - \sigma^B_+ \rho \sigma^B_- \right]$$

$$+ \frac{1}{2} \Gamma \left[ 2\sigma^B_+ \rho \sigma^A_- + 2\sigma^B_- \rho \sigma^A_+ - \sigma^B_- \rho \sigma^B_+ - \sigma^A_- \rho \sigma^A_+ - \sigma^B_+ \rho \sigma^B_- \right]$$  \hspace{1cm} (II.6)

The master equation (II.1) describing the time evolution of a two-atomic system in a noisy environment can be used to obtain the equations for matrix elements of any density matrix. To simplify calculations one can work in the basis of so called collective states in the Hilbert space $\mathbb{C}^4$ (II.1). If

$$f_1 = |1\rangle \otimes |1\rangle, \quad f_2 = |1\rangle \otimes |0\rangle, \quad f_3 = |0\rangle \otimes |1\rangle, \quad f_4 = |0\rangle \otimes |0\rangle$$

then this basis containing excited state, ground state and symmetric and antisymmetric combination of the product states, is defined as follows

$$|e\rangle = f_1, \quad |g\rangle = f_4, \quad |s\rangle = \frac{1}{\sqrt{2}} (f_2 + f_3), \quad |a\rangle = \frac{1}{\sqrt{2}} (f_2 - f_3)$$  \hspace{1cm} (II.8)
In the basis of collective states, two-atom system can be treated as single four-level system with ground state \(|g\rangle\), excited state \(|e\rangle\) and two intermediate states \(|s\rangle\) and \(|a\rangle\). From \([II.1]\) it follows that the matrix elements with respect to the basis \(|e\rangle, |s\rangle, |a\rangle, |g\rangle\) of the state \(\rho\) satisfy

\[
\begin{align*}
\frac{d\rho_{aa}}{dt} &= -(\Gamma_0 - \Gamma)(2\rho_{aa} - \rho_{gg} - \rho_{ee}) \\
\frac{d\rho_{sa}}{dt} &= -\left(\Gamma_0 + \Gamma\right)(2\rho_{sa} - \rho_{gg} - \rho_{ee}) \\
\frac{d\rho_{ee}}{dt} &= -2\Gamma_0\rho_{ee} + \left(\Gamma_0 + \Gamma\right)\rho_{ss} + \left(\Gamma_0 - \Gamma\right)\rho_{aa} \\
\frac{d\rho_{gg}}{dt} &= -2\Gamma_0\rho_{gg} + \left(\Gamma_0 + \Gamma\right)\rho_{ss} + \left(\Gamma_0 - \Gamma\right)\rho_{aa} \\
\frac{d\rho_{ee}}{dt} &= -(2\Gamma_0 + 4i\omega_0)\rho_{eg} \\
\frac{d\rho_{aa}}{dt} &= -(2\Gamma_0 - 2i\Omega)\rho_{as} \\
\frac{d\rho_{ae}}{dt} &= -(\Gamma_0 - \Gamma)\rho_{eg} + \left[\left(\Gamma + i\Omega\right) - 2(\Gamma_0 - i\omega_0)\right]\rho_{ae} \\
\frac{d\rho_{ag}}{dt} &= -(\Gamma_0 - \Gamma)\rho_{ga} + \left[\left(\Gamma + i\Omega\right) - 2(\Gamma_0 + i\omega_0)\right]\rho_{ag} \\
\frac{d\rho_{se}}{dt} &= (\Gamma_0 + \Gamma)\rho_{ge} - \left[\left(\Gamma + i\Omega\right) + 2(\Gamma_0 - 2i\omega_0)\right]\rho_{se} \\
\frac{d\rho_{sg}}{dt} &= (\Gamma_0 + \Gamma)\rho_{eg} - \left[\left(\Gamma + i\Omega\right) + 2(\Gamma_0 + 2i\omega_0)\right]\rho_{sg}
\end{align*}
\]

and for the remaining matrix elements one can use hermiticity of \(\rho\). Notice that in \([II.9]\), equations \([II.9a] - [II.9d]\), \([II.9e] - [II.9j]\) and \([II.9g] - [II.9i]\) are decoupled and can be solved independently. Observe also that four-level system prepared in symmetric state \(|s\rangle\) decays with enhanced rate \(\Gamma_0 + \Gamma\), whereas antisymmetric initial state \(|a\rangle\) leads to reduced rate \(\Gamma_0 - \Gamma\). When two atoms are confined in a region smaller than the resonant wave length, we can put \(\Gamma = \Gamma_0\) (see e.g. \([I.1]\)) so antisymmetric state \(|a\rangle\) is completely decoupled from the environment and is decoherence-free state of the two-atomic system.

One can also check that equations \([II.9]\) describe two types of time evolution of the system, depending on the relations between \(\Gamma\) and \(\Gamma_0\). When \(\Gamma < \Gamma_0\), there is a unique asymptotic state of the system, which is maximally mixed state \(|\frac{I}{4}\rangle\). In that case the relaxation process brings all initial states of two atoms into the state with maximal entropy. In the present note, we analyse the case when \(\Gamma = \Gamma_0\) and show that asymptotic stationary states depend on initial conditions and can be parametrized by matrix elements \(\rho_{aa}\) of the initial states.

\section*{III. SMALL DISTANCE SEPARATION AND GENERATION OF WERNER STATES}

When \(\Gamma = \Gamma_0\), equations \([II.9]\) simplify considerably and one can check that solutions of \([II.9e] - [II.9j]\) asymptotically vanish. Nontrivial contribution to the asymptotic stationary states comes from the matrix elements \(\rho_{aa}, \rho_{ss}, \rho_{ee}\) and \(\rho_{gg}\) satisfying equations \([II.9a] - [II.9d]\), which in that case can be written as follows

\[
\begin{align*}
\frac{d\rho_{aa}}{dt} &= 0 \\
\frac{d\rho_{sa}}{dt} &= 2\Gamma_0(\rho_{ee} + \rho_{gg} - 2\rho_{ss}) \\
\frac{d\rho_{ee}}{dt} &= 2\Gamma_0(\rho_{ss} - \rho_{ee}) \\
\frac{d\rho_{gg}}{dt} &= 2\Gamma_0(\rho_{ss} - \rho_{ee})
\end{align*}
\]
The system of equations [III.1] – [III.4] can be solved and we obtain

\[
\begin{align*}
\rho_{aa}(t) &= \rho_{aa}(0) \\
\rho_{ss}(t) &= \frac{1}{3} (1 - \rho_{aa}(0)) + \frac{1}{3} e^{-6r_{\text{of}}'} (\rho_{aa}(0) + 3\rho_{ss}(0) - 1) \\
\rho_{ee}(t) &= \frac{1}{3} (1 - \rho_{aa}(0)) + \frac{1}{6} e^{-6r_{\text{of}}'} (1 - \rho_{aa}(0) - 3\rho_{ss}(0)) + \frac{1}{2} e^{-2r_{\text{of}}'} (\rho_{aa}(0) + \rho_{ss}(0) + 2\rho_{ee}(0) - 1) \\
\rho_{gg}(t) &= \frac{1}{3} (1 - \rho_{aa}(0)) + \frac{1}{6} e^{-6r_{\text{of}}'} (1 - \rho_{aa}(0) - 3\rho_{ss}(0)) + \frac{1}{2} e^{-2r_{\text{of}}'} (\rho_{aa}(0) + \rho_{ss}(0) + 2\rho_{gg}(0) - 1)
\end{align*}
\]

Thus

\[
\lim_{t \to \infty} \rho_{ss}(t) = \frac{1}{3} (1 - \rho_{aa}(0))
\]

and similarly for \( \rho_{ee}(t) \) and \( \rho_{gg}(t) \). So the stationary asymptotic states \( \rho_\infty \) are parametrized by \( \rho_{ss}(0) = \langle a | \rho | a \rangle \), where \( \langle a | \rho | a \rangle = F \) is the fidelity of the initial state \( \rho \) with respect to the state \( |a\rangle \) (or the overlap of \( \rho \) with singlet state \( |a\rangle \)). In the canonical basis [II.7], \( \rho_\infty \) has the form

\[
\rho_\infty = \begin{pmatrix}
\frac{1-F}{3} & 0 & 0 & 0 \\
0 & \frac{1+2F}{6} & \frac{1-4F}{6} & 0 \\
0 & \frac{1-4F}{6} & \frac{1+2F}{6} & 0 \\
0 & 0 & 0 & \frac{1-F}{3}
\end{pmatrix}
\]

Notice that for some values of parameter \( F \), the state \( \rho_\infty \) is entangled. If we compute its concurrence given by the well known formula [12] [13]

\[
C(\rho) = \max(0, 2\lambda_{\text{max}}(\hat{\rho}) - \text{tr} \hat{\rho})
\]

where \( \lambda_{\text{max}}(\hat{\rho}) \) is the maximal eigenvalue of \( \hat{\rho} \) and

\[
\hat{\rho} = \sqrt{\rho} \hat{\rho} \sqrt{\rho}, \quad \hat{\rho} = (\sigma_2 \otimes \sigma_2) \bar{\rho} (\sigma_2 \otimes \sigma_2)
\]

with \( \bar{\rho} \) denoting complex conjugation of the matrix \( \rho \), then we obtain

\[
C(\rho_\infty) = \begin{cases}
0, & F \leq 1/4 \\
2F - 1, & F > 1/4
\end{cases}
\]

So we see that for any initial state \( \rho \) with fidelity \( F \), there exist asymptotic state \( \rho_\infty \) such that:

1. if \( F \in [0, 1/4] \), \( \rho_\infty \) is separable and can be written as

\[
\rho_\infty = \frac{1}{4} \begin{pmatrix}
1 + \frac{p}{3} & 0 & 0 & 0 \\
0 & 1 - \frac{p}{3} & -\frac{2p}{3} & 0 \\
0 & -\frac{2p}{3} & 1 - \frac{p}{3} & 0 \\
0 & 0 & 0 & 1 + \frac{p}{3}
\end{pmatrix}, \quad p = 1 - 4F
\]

2. if \( F = \frac{1}{4} \), \( \rho_\infty = \frac{3}{4} \mathbb{I} 

3. if \( F \in (1/4, 1] \), \( \rho_\infty \) is equal to the Werner state

\[
W_a = (1-p) \frac{I}{4} + p |a\rangle \langle a|
\]

with \( p = \frac{4F - 1}{3} \). It is separable for \( F \in (1/4, 1/2] \) and entangled for \( F > 1/2 \) with concurrence \( C(\rho_\infty) = 2F - 1 \).
This result shows that all initially entangled states with fidelity greater than 1/2 preserve some amount of their entanglement during the interaction with maximally noisy environment and evolve into Werner states with the same fidelity. The class of Werner states has interesting properties: they interpolate between maximally entangled and maximally mixed states, for that class it was shown that some mixed entangled states can satisfy Bell inequalities [10], they have maximal possible entanglement with respect to the non-local unitary transformations and local and non-local general operations [14]. Werner states can be also applied in entanglement teleportation via mixed states [15].

Let us stress that the asymptotic behavior of the initial state depends only on its overlap with the singlet state \(|a\rangle\) and not on its entanglement. There are many states with the same entanglement for which fidelity varies from 0 to maximal value. This is for example the case of maximally entangled pure states. So initial states with the same entanglement can behave very differently with respect to the noise. The states which are more “similar” to the singlet state are more stable. Thus to protect the initial dynamics will be discussed on explicit examples in the next section.

**IV. SOME EXAMPLES**

**A. Pure separable initial states**

Let \(\rho = |\Psi \otimes \Phi\rangle\langle \Psi \otimes \Phi|\) for \(\Psi, \Phi \in \mathbb{C}^2\). Since for this state

\[
F = \frac{1}{2} (1 - |\langle \Psi, \Phi \rangle|^2) \tag{IV.1}
\]

0 \(\leq F \leq \frac{1}{2}\) and all asymptotic states are separable. Depending on the value of \(|\langle \Psi, \Phi \rangle|^2\) we have the following possibilities:

1. if \(\frac{1}{2} < |\langle \Psi, \Phi \rangle|^2 \leq 1\), then the asymptotic state \(\rho_\infty\) is equal to the state (III.10) with \(p = 2|\langle \Psi, \Phi \rangle|^2 - 1\).

2. if \(|\langle \Psi, \Phi \rangle|^2 = \frac{1}{2}\), then \(\rho_\infty = \mathbb{I}/4\).

3. if \(0 \leq |\langle \Psi, \Phi \rangle|^2 < \frac{1}{2}\), then \(\rho_\infty\) is equal to separable Werner state \(W_a\) given by (III.11) with \(p = \frac{1 - 2|\langle \Psi, \Phi \rangle|^2}{3}\). Notice that in contrast to the zero temperature case, where purely incoherent dissipative process can lead to the creation of entanglement [5, 6, 7, 8] in the present model initial separable states remain separable.

**B. Pure maximally entangled initial states**

Consider now the class of maximally entangled states [16]

\[
P(a, \theta_1, \theta_2) = \frac{1}{2} \left( \begin{array}{cccc}
    a^2 & a\sqrt{1-a^2}e^{-i\theta_1} & a\sqrt{1-a^2}e^{i\theta_2} & -a^2e^{-i(\theta_1+\theta_2)} \\
    a\sqrt{1-a^2}e^{i\theta_1} & 1-a^2 & 1-a^2e^{i(\theta_1-\theta_2)} & -a\sqrt{1-a^2}e^{i\theta_2} \\
    a\sqrt{1-a^2}e^{i\theta_2} & 1-a^2e^{-i(\theta_1-\theta_2)} & 1-a^2 & -a\sqrt{1-a^2}e^{-i\theta_1} \\
    -a^2e^{i(\theta_1+\theta_2)} & -a\sqrt{1-a^2}e^{i\theta_2} & -a\sqrt{1-a^2}e^{i\theta_1} & a^2 \\
\end{array} \right) \tag{IV.2}
\]

where \(a \in [0,1], \theta_1, \theta_2 \in [0,2\pi]\). All states from the class (IV.2) have concurrence equal to 1. On the other hand

\[
F = \frac{1}{2} (1 - a^2)(1 - \cos(\theta_1 - \theta_2)) \tag{IV.3}
\]

One can check that fidelity \(F\) can take all values from 0 to 1 depending on parameters \(a\) and \(\theta = \theta_1 - \theta_2\). In particular \(F > \frac{1}{2}\) inside the set \(E\) on the \((a, \theta)\) - plane, given by

\[
E = \{(a, \theta) : 0 \leq a \leq \frac{1}{\sqrt{2}}, \arccos \frac{a^2}{a^2-1} < \theta < 2\pi - \arccos \frac{a^2}{a^2-1}\} \tag{IV.4}
\]

Outside this set, \(F < \frac{1}{2}\). On the curve

\[
\theta = \arccos \frac{2a^2 - 1}{2(a^2 - 1)}, \quad a \in [0, \sqrt{3}/2] \tag{IV.5}
\]
the fidelity is equal to $\frac{1}{4}$ (see FIG.1). Thus all initial states $P(a, \theta_1, \theta_2)$ with $(a, \theta_1 - \theta_2) \in E$ evolve into entangled Werner states, whereas states with $(a, \theta_1 - \theta_2)$ outside $E$ become separable. When these parameters lie on the curve (IV.5), the dynamics brings corresponding initial maximally entangled states into maximally mixed state $\frac{I_4}{4}$. For all initial states $P(a, \theta_1, \theta_2)$ with $(a, \theta_1 - \theta_2) \in E$, asymptotic concurrence is smaller then 1, except antisymmetric state $|a\rangle$ which is stable.

C. Some mixed initial states

1. Mixed separable states

If $\rho$ is mixed separable state, then it can be written as

$$\rho = \sum_k s_k P_k, \quad s_k \geq 0, \quad \sum_k s_k = 1 \quad (IV.6)$$

where $P_k$ are pure separable states. Since for any pure separable state, fidelity is not greater then 1/2, by (IV.6) the same is true for all mixed separable states. So they evolve into separable asymptotic states.

2. Werner states

Let

$$|\pm\rangle = \frac{1}{\sqrt{2}}(f_1 \pm f_4)$$

Besides $W_a$ define also the states

$$W_s = (1 - p)\frac{I_4}{4} + p |s\rangle\langle s|, \quad W_{\pm} = (1 - p)\frac{I_4}{4} + p |\pm\rangle\langle \pm| \quad (IV.7)$$

One can check that for all states (IV.7)

$$F = \frac{1 - p}{4}$$
so $0 \leq F < \frac{1}{2}$ and they evolve to asymptotic separable state (III.10). On the other hand, the states (IV.7) are locally equivalent to $W_a$. If we define

$$U_s = \sigma_3 \otimes I_2, \quad U_+ = I_2 \otimes i\sigma_2, \quad U_- = I_2 \otimes \sigma_1$$

then

$$W_s = U_s W_a U_s^*, \quad W_+ = U_+ W_a U_+^*, \quad W_- = U_- W_a U_-^*$$

3. Bell-diagonal states

Let $\rho_B$ be the convex combination of pure states $|+\rangle, |-\rangle, |s\rangle$ and $|a\rangle$

$$\rho_B = p_1 |+\rangle \langle +| + p_2 |-\rangle \langle -| + p_3 |s\rangle \langle s| + p_4 |a\rangle \langle a|$$  \hspace{1cm} (IV.8)

It is known that if all $p_i \in [0, 1/2]$, $\rho_B$ is separable, while for $p_1 > 1/2$, $\rho_B$ is entangled with concurrence equal to $2p_1 - 1$ (similarly for $p_2, p_3, p_4$) \cite{17}. On the other hand, for states (IV.3)

$$F = p_4$$

so all states (separable or entangled) with $p_4 < 1/2$ become separable asymptotically. If $p_4 > 1/2$ then noisy dynamics produces asymptotic Werner state $W_a$ with concurrence equal to $2F - 1 = 2p_4 - 1$. In this case asymptotic entanglement is exactly equal to initial entanglement, so the amount of entanglement is preserved. In the next section, we discuss this interesting phenomenon for some class of initial states.

V. PRESERVATION OF ENTANGLEMENT FOR SOME NON-MAXIMALLY ENTANGLLED INITIAL STATES

Let us discuss now how entanglement of the asymptotic state can depend on initial entanglement. This problem has a simple solution in the case when initial entanglement is a function of fidelity $F$. Since $F$ is constant during the evolution, the final entanglement is exactly equal to its initial value. So the process of collective damping can preserve entanglement of some initial states. Simple examples of such states are described below. Let

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (V.1)

Notice that in (V.1) all matrix elements are real and $F = \frac{1}{2}(1 - 2\rho_{23})$. Moreover

$$C(\rho) = 2|\rho_{23}| = |1 - 2F|$$  \hspace{1cm} (V.2)

From our previous results it follows that:

a. if $\rho_{23} \geq 0$ i.e. $F \leq \frac{1}{2}$, then $C(\rho) = 2\rho_{23} \geq 0$, but $C(\rho_\infty) = 0$.

b. if $\rho_{23} < 0$ i.e. $F > \frac{1}{2}$, then $C(\rho) = -2\rho_{23} = 2F - 1$ and $C(\rho_\infty) = 2F - 1$, so

$$C(\rho_\infty) = C(\rho)$$  \hspace{1cm} (V.3)

Notice that the unitary operator $\sigma_3 \otimes I_2$ transforms the states (V.1) with $\rho_{23} > 0$ to the states with $\rho_{23} < 0$. So performing local operations on the initial state, we can protect its entanglement.

Consider also explicit examples of states for which the relation (V.3) holds. Let us take the class of pure states

$$\Psi = \cos \phi |0\rangle \otimes |1\rangle + \sin \phi |1\rangle \otimes |0\rangle, \quad \phi \in [0, \pi]$$  \hspace{1cm} (V.4)

One can check that for the class (V.4)

$$F = \frac{1}{2}(1 - \sin 2\phi)$$
One can check that for states (V.5), concurrence is given by

\[ C(\rho_{\infty}) = \max(0, -\sin 2\phi) \]

We see that all pure entangled states (V.4) with \( \phi \in [\pi/2, \pi] \), evolve into asymptotic Werner states, which have the same entanglement as initial states.

It is instructive to discuss evolution of initial states (V.4) also for finite times. Here the dipol – dipol interaction between atoms plays a significant role. For small separation of atoms the ratio \( \Omega / T_0 \) can be large [11, 12] and this coupling can induce transient increase (or decrease) of entanglement. But noise reduces the amplitude of those oscillations of concurrence and they vanish asymptotically. To see some details, we solve the equations (III.1) – (III.4) and (II.9f) for such initial states and obtain

\[
\rho(t) = \begin{pmatrix}
\rho_{11}(t) & 0 & 0 & 0 \\
0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\
0 & \rho_{23}(t) & \rho_{33}(t) & 0 \\
0 & 0 & 0 & \rho_{44}(t)
\end{pmatrix}
\]

(V.5)

where

\[
\rho_{11}(t) = \frac{1}{6}(1 + 2\rho_{23}) - \frac{1}{6}e^{-6\Gamma_0 t}(1 + 2\rho_{23})
\]

(V.6a)

\[
\rho_{22}(t) = \frac{1}{2}(1 - \rho_{23}) + \frac{1}{2}e^{-6\Gamma_0 t}(1 + 2\rho_{23}) + \frac{1}{2}e^{-2\Gamma_0 t}\cos 2\Omega t(\rho_{22} - \rho_{33})
\]

(V.6b)

\[
\rho_{23}(t) = \frac{1}{2}(1 - \rho_{23}) + \frac{1}{2}e^{-6\Gamma_0 t}(1 + 2\rho_{23}) - \frac{1}{2}e^{-2\Gamma_0 t}\cos 2\Omega t(\rho_{22} - \rho_{33})
\]

(V.6c)

\[
\rho_{44}(t) = \frac{1}{6}(1 + 2\rho_{23}) - \frac{1}{6}e^{-6\Gamma_0 t}(1 + 2\rho_{23})
\]

(V.6d)

\[
\rho_{23}(t) = \frac{1}{6}(4\rho_{23} - 1) + \frac{1}{6}e^{-6\Gamma_0 t}(1 + 2\rho_{23}) - \frac{i}{2}e^{-2\Gamma_0 t}\sin 2\Omega t(\rho_{22} - \rho_{33})
\]

(V.6e)

One can check that for states (V.5), concurrence is given by

\[
C(\rho(t)) = \max\left(0, 2\left(|\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\right)\right) = \max\left(0, 2\left(|\rho_{23}(t)| - \rho_{11}(t)\right)\right)
\]

(V.7)

so

\[
C(\rho(t)) = \max\left(0, 2\sqrt{A(t)^2 + B(t)^2\sin^2 2\Omega t} - 2\rho_{11}(t)\right)
\]

(V.8)

with

\[
A(t) = \frac{1}{6}\left(4\rho_{23} - 1 + e^{-6\Gamma_0 t}(1 + 2\rho_{23})\right), \quad B(t) = \frac{1}{2}e^{-2\Gamma_0 t}(\rho_{22} - \rho_{33})
\]

(V.9)

This shows that the concurrence as a function of time oscillates with period depending on the strength of dipol – dipol interaction but when \( \rho_{23} < 0 \), it tends to its initial value (FIG. 2). When the dipol – dipol interaction is absent or when the initial state is such that \( \rho_{22} = \rho_{33} = \frac{1}{2} \) and \( \rho_{23} < 0 \), then from (V.8) and (V.9) it follows that

\[
|\rho_{23}(t)| - \rho_{11}(t) = -\rho_{23}(t) - \rho_{11}(t)
\]

(V.10)

Since time-dependent terms in (V.10) cancel, it is equal to \(-2\rho_{23}\), and

\[
C(\rho(t)) = C(\rho)
\]

for all \( t \). When \( \rho_{23} > 0 \), then

\[
|\rho_{23}(t)| - \rho_{11}(t) = \rho_{23}(t) - \rho_{11}(t)
\]

(V.11)

and (V.10) depends on \( t \) in such a way that \( C(\rho(t)) \) monotonically goes to zero. The effect of preservation of entanglement for states (V.5) should be contrasted with the monotonically decreasing of purity of the state \( \rho \) defined by \( \text{tr} \rho^2 \), during the time evolution (II.1). As shown in [19], it is equivalent to the condition

\[
L_N(\mathbb{I}_4) = 0
\]

which can be simply checked to be true in our model.
VI. CONCLUSIONS

During the evolution of open quantum systems interacting with environment the process of decoherence usually results in degradation of entanglement. To preserve as much entanglement as possible one has to control the effects of noise. In this context, we have studied the model of compound system of two atoms influenced by so called maximal noise, which can be treated as the limiting case of thermal noise, when temperature goes to infinity. As we have shown, even in that case there are entangled states which are decoherence – free. Explicit examples are given by singlet state $|a\rangle$ and Werner state $W_a$. On the other hand, there are evolving states with a very interesting property: its asymptotic entanglement is exactly equal to the initial one, or even is stable during the time evolution. We have also shown that performing some local operations on initial states can help with protecting entanglement.

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