IMPROVED APPROXIMATIONS OF RESOLVENTS IN HOMOGENIZATION OF FOURTH ORDER OPERATORS

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We study homogenization of fourth order elliptic operators $A_\varepsilon$ in divergence form with $\varepsilon$-periodic coefficients in $\mathbb{R}^d$ and obtain an $\varepsilon^2$-order approximation of the resolvents $(A_\varepsilon + 1)^{-1}$ in the energy operator ($L^2 \to H^2$)-norm as $\varepsilon \to 0$. Bibliography: 16 titles.

1 Introduction

In the space $\mathbb{R}^d$, $d \geq 2$, we consider the fourth order differential operator

$$A_\varepsilon = \sum_{i,j,s,t} D_{ij}(a_{ijst}(x/\varepsilon)D_{st}), \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad \varepsilon \in (0,1), \quad (1.1)$$

where the sum is taken over $i, j, s, t$ from 1 to $d$, $a(x) = \{a_{ijst}(x)\}$ is a 1-periodic measurable fourth order tensor satisfying the symmetry, positive definiteness, and boundedness conditions. In particular, these conditions provide the ellipticity of the operator $A_\varepsilon$. Such operators appear in the theory of elastic thin plates made of periodic composite materials. The case $d = 2$ corresponds to thin plates in the three-dimensional space.

In applications to composites, the operator $A_\varepsilon$ has nonsmooth coefficients. For example, for a two-phase elastic composite with homogenous isotropic phases and contrast moduli of elasticity the coefficients $a_{ijst}(x)$ are constant on each phase, but have a jump on the phase interface. Moreover, the geometry of phases can be arbitrary, so that their boundary are not necessarily smooth. Therefore, it is important to study operators of the form (1.1) with measurable coefficients.

If we deal with an elastic thin plate made of a composite material, then the presence of a small parameter $\varepsilon$ in periodic coefficients of the operator $A_\varepsilon$ reflects the fine cell structure of the composite: due to small periodicity cells, the composite medium becomes strongly inhomogeneous. However, the opposite phenomenon is observed for sufficiently small $\varepsilon$: a medium can be strongly inhomogeneous at microlevel, but well described at macrolevel by rather effective constant characteristics (tensors) corresponding to a homogeneous medium, called an effective
or homogenized medium. In the language of operators, this phenomenon means that \( A_\varepsilon \) are close to the homogeneous operator \( \hat{A} \) corresponding to the homogenized medium. Homogenization theory provides formulas for homogenized characteristics (tensors) and establishes the closeness (in a certain sense) of the original and homogenized models for small \( \varepsilon \).

We are interested in operator homogenization estimates providing approximations of the resolvent \((A_\varepsilon + 1)^{-1}\) of the original operator in different operator norms with different accuracy with respect to the small parameter \( \varepsilon \). In particular, the uniform resolvent convergence of \( A_\varepsilon \) to \( \hat{A} \) and the estimate for the convergence rate

\[
\|(A_\varepsilon + 1)^{-1} - (\hat{A} + 1)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\varepsilon
\]

were proved in [1, 2]. Here, the constant \( C \) depends only on the dimension \( d \) and the ellipticity constant for the tensor \( a(x) \), the homogenized operator \( \hat{A} \) is of the same ellipticity class as the original operator, but has the simpler form

\[
\hat{A} = \sum_{i,j,s,t} D_{ij}(\hat{a}_{ijst}D_{st}),
\]

where the tensor \( \hat{a} = \{\hat{a}_{ijst}\} \) is constant. In fact, the following stronger result was proved in [1, 2]: the resolvent of the operator (1.1) was approximated by the sum \((\hat{A} + 1)^{-1} + \varepsilon^2\mathcal{K}_\varepsilon\) of the resolvent of the homogenized operator \( \hat{A} \) and the correcting operator \( \mathcal{K}_\varepsilon \) with the estimate

\[
\|(A_\varepsilon + 1)^{-1} - (\hat{A} + 1)^{-1} - \varepsilon^2\mathcal{K}_\varepsilon\|^2_{L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} \leq C\varepsilon,
\]

where the constant \( C \) depends only on the dimension \( d \) and the ellipticity constant of the tensor \( a(x) \), whereas the operator \( \mathcal{K}_\varepsilon \) is defined in terms of solutions to an auxiliary problem on the periodicity cell (the unit cube). This problem is introduced to determine the homogenized tensor \( \hat{a} \). We note that

\[
\|\varepsilon^2\mathcal{K}_\varepsilon\|_{L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} \leq C,
\]

\[
\|\mathcal{K}_\varepsilon\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C,
\]

where the constants on the right-hand side are of the same type as in (1.4). By the second inequality in (1.5), we can derive (1.2) from (1.4) by weakening the norm and then transferring the term \( \varepsilon^2\mathcal{K}_\varepsilon \) to the remainder.

Due to a more delicate application of \( H^2 \)-estimates (1.4) in [3], it becomes possible to improve the \( L^2 \)-estimate (1.2) with respect to powers of \( \varepsilon \). In fact, we have

\[
\|(A_\varepsilon + 1)^{-1} - (\hat{A} + 1)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C\varepsilon^2,
\]

where the constant \( C \) is of the same form as in (1.2). Without the symmetry condition on the tensor, it is shown [4] that

\[
(A_\varepsilon + 1)^{-1} = (\hat{A} + 1)^{-1} + \varepsilon\mathcal{K}_1 + O(\varepsilon^2),
\]

in the operator \( L^2 \)-norm, where \( \mathcal{K}_1 \) does not contain \( \varepsilon \)-periodic factors and is independent of \( \varepsilon \).

Our goal is to obtain a more accurate (in comparison with (1.4)) \( \varepsilon^2 \)-order approximation of the resolvent of \( A_\varepsilon \) in the operator \( (L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)) \)-norm. We construct the approximation

\[
(\hat{A}_\varepsilon + 1)^{-1} + \varepsilon^2\mathcal{K}_2(\varepsilon) + \varepsilon^3\mathcal{K}_3(\varepsilon),
\]
2 Statement of the Problem and the Main Result

2.1. We consider the fourth order equation with parameter \( \varepsilon \in (0, 1) \)
\[
 u^\varepsilon \in H^2(\mathbb{R}^d), \quad A_\varepsilon u^\varepsilon + u^\varepsilon = f, \quad f \in L^2(\mathbb{R}^d),
\]
\[
 A_\varepsilon = D^*a_\varepsilon(x)D, \quad a_\varepsilon(x) = a(x/\varepsilon),
\]
where \( a(y) = \{a_{ijst}(y)\} \) is a measurable real-valued periodic fourth order tensor with periodicity cell \( Y = [-1/2, 1/2]^d \) that satisfies the symmetry and ellipticity conditions
\[
a_{ijst} = a_{stij}, \quad a_{ijst} = a_{jist},
\]
\[
\lambda \xi \cdot \xi \leq a(\cdot)\xi \cdot \xi \leq \lambda^{-1} \xi \cdot \xi
\] (2.2)
for any symmetric matrix \( \xi = \{\xi_{ij}\} \). Here, \( \lambda > 0 \) is the ellipticity constant, \( \xi \cdot \eta \) denotes the product of matrices \( \xi = \{\xi_{ij}\} \), and \( \eta = \{\eta_{ij}\} \), i.e., \( \xi \cdot \eta = \xi_{ij}\eta_{ij} \). Throughout the paper, we adopt the summation convention over repeated indices from 1 to \( d \), unless otherwise stated.

By the symmetry property (2.2), the action of the tensor \( a \) on a symmetric matrix \( \xi \) is the symmetric matrix \( \eta = a\xi \). For an arbitrary matrix \( \xi \) we have \( a\xi = a\xi^s \), where \( \xi^s = 1/2(\xi + \xi^T) \) is the symmetric part of \( \xi \); moreover, \( a(\xi + \xi^T) = 2a\xi \) which will be often used below to simplify formulas.

To describe the differential operators in (2.1), we use the notation \( D\varphi = \nabla^2\varphi = \{D_{ij}\}_{i,j=1}^d \),
\[
 D_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j},
\]
so that the operator \( D \) sends scalar functions to matrix-valued functions. Then the adjoint \( D^* = \nabla^*\nabla^* = \text{div div} \) formally acts on a matrix \( \eta = \{\eta_{ij}\} \) by the rule
\[
 D^*\eta = D_{ij}\eta_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\eta_{ij},
\]
reducing a matrix to a scalar. The equality
\[
 D^*\eta = f,
\] (2.3)
where \( \eta \in L^2(\mathbb{R}^d)^{d \times d} \) and \( f \in L^2(\mathbb{R}^d) \), is understood in the sense of the theory of distributions in \( \mathbb{R}^d \), i.e., in the sense of the integral identity
\[
 \int_{\mathbb{R}^d} \eta \cdot D\varphi dx = \int_{\mathbb{R}^d} f\varphi dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).
\] (2.4)
Thus, the operator $A_\varepsilon$ can be written in the form

$$A_\varepsilon = D_{ij}(a_{ij\varepsilon}(x)D_{st}), \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},$$

i.e., the same expression as in (1.1), if we take into account the summation convention over repeated indices.

**Example 2.1.** Let the tensor $a$ act on a matrix $\xi$ by the formula

$$a\xi = (\alpha \text{Tr } \xi)E,$$

where $\text{Tr } \xi = \xi_{ii}$ is the trace of the matrix $\xi$, $E$ is the identity matrix, and $\alpha$ is a scalar function. It is obvious that $a\xi \cdot \xi = \alpha(\text{Tr } \xi)^2$ and the trace $\text{Tr } \xi$ of the matrix $\xi = D\varphi$ is expressed in terms of the Laplacian since

$$\text{Tr } D\varphi = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = \Delta \varphi.$$

Therefore,

$$aD\varphi \cdot D\varphi = \alpha \Delta \varphi \Delta \varphi.$$

For the tensor $a$ we have an equation of the form (2.1) with the operator

$$A_\varepsilon = \Delta \alpha(\varepsilon/\varphi)\Delta.$$

By (2.3) and (2.4), the solution to Equation (2.1) is understood in the sense of the integral identity

$$\int_{R^d} (a_\varepsilon(x)Du^\varepsilon \cdot D\varphi + u^\varepsilon \varphi)dx = \int_{R^d} f\varphi dx \quad \forall \varphi \in C_0^\infty(R^d),$$

where test functions can be taken in the space $H^2(R^d)$. By the Riesz representation theorem for linear functionals in a Hilbert space, a solution to Equation (2.1) exists and is unique; moreover,

$$\|u^\varepsilon\|_{H^2(R^d)} \leq c(\lambda)\|f\|_{L^2(R^d)}.$$

As known, one can introduce the $H^2(R^d)$-norm

$$\|\varphi\|_{H^2(R^d)}^2 = \int_{R^d} (|\varphi|^2 + |D\varphi|^2)dx,$$

equivalent to the standard norm in $H^2(R^d)$.

**2.2.** The following equation similar to Equation (2.1), but with constant coefficients is said to be the *limiting* or *homogenized* equation for (2.1) as $\varepsilon \to 0$:

$$u \in H^2(R^d), \quad \hat{A}u + u = f, \quad f \in L^2(R^d), \quad \hat{A} = D^*\hat{\alpha}D,$$

where the constant fourth order tensor $\hat{\alpha}$ is defined in (3.5) below and satisfies the conditions (2.2). Using the notation introduced in Subsection 2.1, we can write the operator $\hat{A}$ in the form

$$\hat{A} = D_{ij}\hat{\alpha}_{ijst}D_{st} = \hat{\alpha}_{ijst}D_{ij}D_{st}, \quad D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},$$
which agrees with (1.3). Since the coefficients in (2.5) are constant and satisfy the ellipticity condition (2.2), one can use the Fourier transform to show that the solution $u$ to Equation (2.5) belongs to the space $H^4(\mathbb{R}^d)$ and satisfies the estimate

$$\|u\|_{H^4(\mathbb{R}^d)} \leq C\|f\|_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(d, \lambda).$$

By the results of [5], the solutions to the problems (2.1) and (2.5) are connected by

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^2(\mathbb{R}^d)} = 0 \quad \forall f \in L^2(\mathbb{R}^d).$$

So, it is reasonable to regard Equation (2.5) as the limiting equation for (2.1). Since $u^\varepsilon = (A_\varepsilon + 1)^{-1}f$ and $u = (\hat{A} + 1)^{-1}f$, whereas the resolvents $(A_\varepsilon + 1)^{-1}$ and $(\hat{A} + 1)^{-1}$ can be regarded as operators acting in $L^2(\mathbb{R}^d)$, we can call (2.7) the strong resolvent $L^2(\mathbb{R}^d)$-convergence of the operators $A_\varepsilon$ to the limiting operator $\hat{A}$. The known results asserting the operator convergence (with estimates for the convergence rate) in (1.2) or (1.6) confirm that $\hat{A}$ is the limiting operator for $A_\varepsilon$. In approximations of the resolvent $(A_\varepsilon + 1)^{-1}$ indicated in (1.4) and (1.7), the zeroth approximation is constructed from the resolvent of the operator $\hat{A}$, and the same is true for correcting terms. Owing to these facts, it is reasonable to call $\hat{A}$ the limiting operator.

Our goal is to obtain an $\varepsilon^2$-order approximation of the resolvent $(A_\varepsilon + 1)^{-1}$ in the operator $(L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d))$-norm as in (1.4). To construct the zeroth approximation, one should take, instead of $(\hat{A} + 1)^{-1}$, the resolvent $(\hat{A}_\varepsilon + 1)^{-1}$ of the more complicated operator $\hat{A}_\varepsilon$ defined by

$$\hat{A}_\varepsilon = D_{ij}(\hat{a}_{ijrst}D_{st} + \varepsilon \hat{b}^{rst}_{ij}D_{rst}), \quad D_{rst} = \frac{\partial^3}{\partial x_r \partial x_s \partial x_t}$$

or, in terms of $D$ and $D^*$,

$$\hat{A}_\varepsilon u = D^*(\hat{a}Du + \varepsilon \hat{b}^{rst}D_{rst}u), \quad D_{rst} = \frac{\partial^3}{\partial x_r \partial x_s \partial x_t},$$

where $\hat{b}^{rst} = \{\hat{b}^{rst}_{ij}\}_{i,j}$ are introduced in (3.11) for all $r, s, t$ from 1 to $d$.

2.3. We introduce a new version of the homogenized equation with the operator (2.8)

$$(\hat{A}_\varepsilon + 1)\hat{u}^\varepsilon = f, \quad f \in L^2(\mathbb{R}^d).$$

Since the coefficients are constant, the solution can be constructed by using the Fourier transform. Applying the Fourier transform to Equation (2.10), we get

$$(1 + \Lambda(\xi) - i\varepsilon \Lambda_0(\xi))F[\hat{u}^\varepsilon] = F[f], \quad i = \sqrt{-1},$$

where $f(x) \to F[f](\xi)$ denotes the Fourier transform,

$$\Lambda(\xi) = \bar{a}_{pqst}\xi_p\xi_q\xi_s\xi_t, \quad \Lambda_0(\xi) = \hat{b}^{rst}_{pq}\xi_p\xi_q\xi_r\xi_s\xi_t.$$

Since the coefficients $\bar{a}_{pqst}$ and $\hat{b}^{rst}_{pq}$ are real, from (2.11)–(2.12) we successively derive

$$(1 + \Lambda(\xi))F[\hat{u}^\varepsilon] \leq |F[f]|,$$

$$\int_{\mathbb{R}^d} (1 + \Lambda(\xi))^2|F[\hat{u}^\varepsilon]|^2 d\xi \leq \int_{\mathbb{R}^d} |F[f]|^2 d\xi.$$
By the ellipticity of the tensor \( \hat{a}_{pqst} \) and the Plancherel identity, from the last relation we get the \( \varepsilon \)-uniform estimate
\[
\| \hat{u}^\varepsilon \|_{H^4(\mathbb{R}^d)} \leq C \| f \|_{L^2(\mathbb{R}^d)},
\]
(2.13)
where the constant \( C \) depends only on the ellipticity constant \( \lambda \) in (2.2) and the dimension \( d \). Thus, we have obtained an analog of the elliptic estimate (2.6).

2.4. An approximation to the solution to Equation (2.1) is taken in the form
\[
v^\varepsilon(x) = u^\varepsilon(x) + \varepsilon^2 U^\varepsilon_2(x) + \varepsilon^3 U^\varepsilon_3(x),
\]
where
\[
U^\varepsilon_2(x) = N^{ij}(x/\varepsilon)D_iD_j u^\varepsilon(x), \quad U^\varepsilon_3(x) = N^{ijk}(x/\varepsilon)D_iD_jD_k u^\varepsilon(x), \quad D_i = \frac{\partial}{\partial x_i},
\]
(2.15)
\[
u^\varepsilon(x) = \Theta^\varepsilon \hat{u}^\varepsilon(x), \quad \Theta^\varepsilon = S^\varepsilon S^\varepsilon, \quad S^\varepsilon \text{ is the Steklov smoothing operator,}
\]
(2.16)
\[N^{ij}(y), N^{ijk}(y), \hat{u}^\varepsilon(x) \text{ are the solutions to the problems (3.1), (3.9) (2.10) respectively for all } i, j, k \text{ from 1 to } d.\]
We recall that for \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^d) \) the Steklov (averaging) smoothing operator is defined by
\[
(S^\varepsilon \varphi)(x) = \int_Y \varphi(x - \varepsilon \omega) d\omega, \quad Y = [-1/2, 1/2]^d.
\]
(2.17)

We formulate the main result of the paper. The proof is given in Section 5.

**Theorem 2.1.** The difference between the solution to the problem (2.1) and the function constructed in (2.14)–(2.16) satisfies the estimate
\[
\| u^\varepsilon - v^\varepsilon \|_{H^2(\mathbb{R}^d)} \leq C \varepsilon^2 \| f \|_{L^2(\mathbb{R}^d)},
\]
(2.18)
where the constant \( C \) depends only on the dimension \( d \) and the ellipticity constant \( \lambda \) in (2.2).

Together with \( v^\varepsilon \), we consider the following approximation of simpler structure:
\[
\hat{v}^\varepsilon(x) = \hat{u}^\varepsilon(x) + \varepsilon^2 N^{ij}(x/\varepsilon)D_iD_j S^\varepsilon \hat{u}^\varepsilon(x) + \varepsilon^3 N^{ijk}(x/\varepsilon)D_iD_jD_k S^\varepsilon S^\varepsilon \hat{u}^\varepsilon(x),
\]
(2.19)
where the zeroth approximation does not involve new smoothing operators, the first and second correctors contain the smoothing operators \( S^\varepsilon \) defined by (2.17) or its iteration \( S^\varepsilon S^\varepsilon \) respectively, whereas \( \hat{u}^\varepsilon, N^{ij}, N^{ijk} \) are as above. Using Theorem 2.1 and based only on the smoothing properties (cf. Section 4), we derive the estimate
\[
\| u^\varepsilon - \hat{v}^\varepsilon \|_{H^2(\mathbb{R}^d)} \leq C \varepsilon^2 \| f \|_{L^2(\mathbb{R}^d)},
\]
(2.20)
where the constant \( C \) is of the same type as in (2.18).

It is possible to write the estimate (2.18) or (2.19) in the operator terms, i.e., for the difference between the resolvent \( (A_\varepsilon + 1)^{-1} \) and its approximations. For example, from (2.19) and (2.20) we obtain the approximation (1.8) in the energy operator norm
\[
\|(\hat{A}_\varepsilon + 1)^{-1} + \varepsilon^2 \mathcal{K}_2(\varepsilon) + \varepsilon^3 \mathcal{K}_3(\varepsilon) - (A_\varepsilon + 1)^{-1}\|_{L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} \leq C \varepsilon^2,
\]
(2.21)
where the structure of \( \mathcal{K}_2(\varepsilon) \) and \( \mathcal{K}_3(\varepsilon) \) is restored from the correctors in (2.19):
\[
\mathcal{K}_2(\varepsilon) f(x) = N^{ij}(x/\varepsilon) S^\varepsilon D_{ij} \hat{u}^\varepsilon(x), \quad \hat{u}^\varepsilon(x) = (\hat{A}_\varepsilon + 1)^{-1} f(x), \quad D_{ij} = D_iD_j,
\]
(2.22)
\[
\mathcal{K}_3(\varepsilon) f(x) = N^{ijk}(x/\varepsilon) S^\varepsilon S^\varepsilon D_{ijk} \hat{u}^\varepsilon(x), \quad \hat{u}^\varepsilon(x) = (\hat{A}_\varepsilon + 1)^{-1} f(x), \quad D_{ijk} = D_iD_jD_k.
\]
Remark 2.1. Homogenized operators of a higher order than the order of the original operator are considered, for example, in [6], where the improved approximations of the operator exponential in the operator $L^2$-norm are studied for the nonstationary diffusion problem in a periodic medium and the order of the homogenized operator is determined by the approximation order $m$ and increases with growth of $m$.

3 Auxiliary Problems

3.1. We introduce the problem on the periodicity cell $Y$:

$$N^{ij} \in \tilde{H}^2_{\text{per}}(Y), \quad D_y^*[a(y)(D_y N^{ij}(y) + e^{ij})] = 0, \quad i, j = 1, \ldots, d. \quad (3.1)$$

Here, the tensor $a(y)$ is the same as in Subsection 2.1 and the matrix $e^{ij} = \{e_{sh}^{ij}\}_{s,h}$ has entries $e_{sh}^{ij} = \delta^i_s \delta^j_h$, where $\delta^i_s$ denotes the Kronecker symbol, $\tilde{H}^2_{\text{per}}(Y) = \{\varphi \in H^2_{\text{per}}(Y) : \langle \varphi \rangle = 0\}$ are the Sobolev spaces of 1-periodic functions with zero mean

$$\langle \varphi \rangle = \int_Y \varphi dy.$$

The relation

$$D_y^* b(y) = 0, \quad b \in L^2_{\text{per}}(Y)^{d\times d}, \quad (3.2)$$

means the integral identity

$$\langle b \cdot D\varphi \rangle = 0 \quad \forall \varphi \in C^\infty_{\text{per}}(Y). \quad (3.3)$$

where test functions can be taken in $\tilde{H}^2_{\text{per}}(Y)$. Matrices satisfying this relations are called solenoidal.

As known in homogenization theory, the equality (3.2) can be also interpreted in the sense of the theory of distribution in $\mathbb{R}^d$, i.e., in addition to (3.3) the following integral identity holds:

$$\int_{\mathbb{R}^d} b \cdot D\varphi dx = 0 \quad \forall \varphi \in C^\infty_0(\mathbb{R}^d).$$

By the above conventions, the solution to the problem (3.1) satisfies the integral identity

$$\langle a D N^{ij} \cdot D\varphi \rangle = -\langle ae^{ij} \cdot D\varphi \rangle \quad \forall \varphi \in \tilde{H}^2_{\text{per}}(Y).$$

The existence of a unique solution to Equation (3.1) can be easily proved by the Riesz theorem. For this purpose it is convenient to introduce the $\tilde{H}^2_{\text{per}}(Y)$-norm

$$\|\varphi\|^2_{\tilde{H}^2_{\text{per}}(Y)} = \langle |D\varphi|^2 \rangle$$

which is equivalent to the standard norm in view of the Poincaré inequality

$$\langle |v|^2 \rangle \leq C_P \langle |\nabla v|^2 \rangle \quad \forall v \in H^1(Y), \langle v \rangle = 0$$

for any $\varphi \in \tilde{H}^2_{\text{per}}(Y)$ and $\nabla \varphi$. 494
For the solution to the problem (3.1) from the integral identity it follows that

\[ \|N^{ij}\|_{H^2_{\perp}(Y)}^2 \leq C, \quad i, j = 1, \ldots, d, \] (3.4)

where the constant \( C \) depends only on \( d \) and \( \lambda \).

The fourth order tensor \( \tilde{a} \) in (2.5) is given by

\[ \tilde{a}e^{ij} = \langle a(\cdot)(D_yN_{ij}(\cdot) + e^{ij}) \rangle, \quad i, j = 1, \ldots, d. \] (3.5)

The tensor \( \tilde{a} \) inherits the symmetry and positive definiteness properties (2.2) of the original tensor \( a \), which can be proved in the same way as a similar fact in homogenization of second order elliptic equations in divergence form (cf., for example, [5, 7, 8]).

### 3.2. The proof of the following lemma can be found in [1, 2, 9].

**Lemma 3.1.** Let \( g = \{g_{st}\}_{s,t} \in L^2_{\perp}(Y)^{d \times d} \) be a symmetric matrix such that \( (g) = 0, D^*g = 0 \). Then there is a family of matrices \( G^{st} = \{G^{st}_{km}\}_{k,m} \in H^2_{\perp}(Y)^{d \times d} \) with \( s \) and \( t \) from 1 to \( d \) such that for all \( s, t, k, m \) from 1 to \( d \) the following assertions hold:

(i) \( G^{st}_{km} = G^{st}_{mk} \) (symmetry property),

(ii) \( G^{st}_{km} = -G^{km}_{st} \) (skew-symmetry property),

(iii) \( \|G^{st}\|_{H^2(Y)^{d \times d}} \leq c(d)\|g\|_{L^2(Y)^{d \times d}} \),

(iv) \( g_{st} = D^*G^{st} \).

Lemma 3.1 can be applied to the matrix-valued function

\[ g^{ij} := a(D_yN^{ij} + e^{ij}) - \tilde{a}e^{ij} \] (3.6)

caused by the problem (3.1). Indeed, by (3.1) and (3.4),

\[ D^*g^{ij} = 0, \quad (g^{ij}) = 0. \] (3.7)

Therefore, each entry of the matrix \( g^{ij} = \{g^{ij}_{st}\}_{s,t} \) is represented in terms of \( G^{ij, st} = \{G^{ij, st}_{km}\}_{k,m} \in H^2_{\perp}(Y)^{d \times d} \); moreover,

\[ g^{ij}_{st} = D^*G^{ij, st}, \quad G^{ij, st}_{km} = -G^{km, st}_{ij}, \quad \|G^{ij, st}\|_{H^2(Y)^{d \times d}} \leq c(d, \lambda) \] (3.8)

for all from 1 to \( d \).

### 3.3. Based on the solutions \( N^{ij} \) to the problem (3.1) and the matrix potentials \( G^{ij, st} \) in (3.8), we introduce the following family of problems on the cell:

\[ N^{ijk} \in H^2_{\perp}(Y), \quad D^*_y(a(D_yN^{ijk}) + 2a(\nabla N^{ij} \times e^k) + 2\partial_m G^{ij, km}) = 0, \quad i, j, k = 1, \ldots, d. \] (3.9)

Here, \( e^1, \ldots, e^d \) are the vectors of canonical basis for the space \( \mathbb{R}^d \), \( \partial_m = \partial/\partial y_m \) and \( \alpha \times \beta = \{\alpha_p \beta_q\}_{p,q} \) is the matrix composed of the coordinates of \( d \)-dimensional vectors \( \alpha = \{\alpha_p\}_p \) and \( \beta = \{\beta_q\}_q \), where \( \alpha = \nabla N^{ij} \) and \( \beta = e^k \).

For any triple \( i, j, k \) from 1 to \( d \) the problem (3.9) has a unique solution and it is connected with the matrix

\[ g^{ijk} = a(D_yN^{ijk}) + 2a(\nabla N^{ij} \times e^k) + 2\partial_m G^{ij, km} - b^{ijk}, \] (3.10)
where

$$b^{ijk} = \langle a(D_y N^{ijk}) + 2a(\nabla N^{ij} \times e^k) \rangle. \quad (3.11)$$

We can apply Lemma 3.1 to the matrix-valued function $g^{ijk} = \{g^{ijk}_{st}\}_{s,t}$ since

$$D^* g^{ijk} = 0, \quad \langle g^{ijk} \rangle = 0. \quad (3.12)$$

Hence we have a representation for entries of this matrix in terms of the matrix potential $G^{ijk, st} = \{G^{ijk}_{pq}_{st}\}_{p,q} \in \tilde{H}^2_{\mathrm{per}}(Y)^{d \times d}$. Hence

$$g^{ijk}_{st} = D^* G^{ijk, st}, \quad G^{ijk}_{pq}_{st} = -G^{ijk}_{pq}_{st}, \quad \|G^{ijk, st}\|_{H^2(Y)^{d \times d}} \leq c(d, \lambda) \quad (3.13)$$

for all indices from 1 to $d$.

### 3.4. The following assertion is a consequence of Lemma 3.1.

**Lemma 3.2.** Let a 1-periodic matrix $g(y) = \{g_{st}(y)\}_{s,t}$ satisfy the assumptions of Lemma 3.1, and let $G^{st}(y) = \{G^{st}_{km}(y)\}_{k,m} \in \tilde{H}^2_{\mathrm{per}}(Y)^{d \times d}$, where $s, t = 1, \ldots, d$, is a family of matrix potentials with properties (i)--(iv) listed in Lemma 3.1. Then

$$g_{st}(x/\varepsilon) \Phi(x) = \varepsilon^2 D^*(G^{st}(x/\varepsilon) \Phi(x)) - \varepsilon^2 G^{st}(x/\varepsilon) \cdot D \Phi(x) - 2\varepsilon(\nabla g^{st}(x/\varepsilon)) \cdot \nabla \Phi(x) \quad (3.14)$$

for any $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, where $s, t$ run from 1 to $d$ and the matrix

$$M(x) = \{D^*(G^{st}(x/\varepsilon) \Phi(x))\}_{s,t} \quad (3.15)$$

is solenoidal, i.e., $D^* M = 0$ (in the sense of the theory of distributions in $\mathbb{R}^d$).

**Proof.** By assumption, $g_{st}(y) = D^* G^{st}(y)$ and $g_{st}(x/\varepsilon) \Phi(x) = D^* (\varepsilon^2 G^{st}(x/\varepsilon)) \Phi(x)$. To obtain (3.14), it suffices to recall the rule of applying the operator $D^*$ to the product of a symmetric matrix $B = \{B_{ij}\}_{i,j}$ and a scalar $\Phi$:

$$D^*(\Phi B) = \Phi D^* B + B \cdot D \Phi + 2(\nabla B) \cdot \nabla \Phi,$$

where $\nabla B = \{\partial B_{ij}/\partial x_j\}_i$ is a vector and the dot means the inner product of matrices or vectors.

The matrix $M(x)$ is solenoidal because of the integral identity

$$\int_{\mathbb{R}^d} \varphi D^* M \, dx = 0 \quad \forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$$

obtained from the chain of integral identities

$$\int_{\mathbb{R}^d} \varphi D^* M \, dx = \int_{\mathbb{R}^d} D \varphi \cdot M \, dx \overset{(3.15)}{=} \int_{\mathbb{R}^d} D_{st} \varphi(x) D^*(G^{st}(x/\varepsilon) \Phi(x)) \, dx$$

$$= \int_{\mathbb{R}^d} D_{st} \varphi(x) D_{km} (G^{st}_{km}(x/\varepsilon) \Phi(x)) \, dx = \int_{\mathbb{R}^d} D_{km} D_{st} \varphi(x) G^{st}_{km}(x/\varepsilon) \Phi(x) \, dx,$$

and the pointwise identity $D_{km} D_{st} \varphi(x) G^{st}_{km}(x/\varepsilon) = 0$ since the family $G^{st}(y) = \{G^{st}_{km}(y)\}_{k,m}$ is skew-symmetric, i.e., $G^{st}_{km} = -G^{km}_{st}$. \hfill $\square$
4 Smoothing Operator

4.1. Steklov smoothing operator. We consider the Steklov smoothing operator

\[ S^\varepsilon \varphi(x) = \int_Y \varphi(x - \varepsilon \omega) d\omega, \quad Y = [-1/2, 1/2]^d, \]  

possessing the following simple properties:

\[ \|S^\varepsilon \varphi\|_{L^2(\mathbb{R}^d)} \leq \|\varphi\|_{L^2(\mathbb{R}^d)}, \]  

\[ \|S^\varepsilon \varphi - \varphi\|_{L^2(\mathbb{R}^d)} \leq (\sqrt{d}/2)\varepsilon \|\nabla \varphi\|_{L^2(\mathbb{R}^d)}. \]  

It is obvious that \( S^\varepsilon (\nabla \varphi) = \nabla (S^\varepsilon \varphi) \). The following smoothing property (proved in [9]–[11]) plays an important role in our method.

**Lemma 4.1.** If \( \varphi \in L^2(\mathbb{R}^d), b \in L^2_{\text{per}}(Y) \) and \( b_\varepsilon(x) = b(\varepsilon^{-1}x) \), then \( b_\varepsilon S^\varepsilon \varphi \in L^2(\mathbb{R}^d) \) and

\[ \|b_\varepsilon S^\varepsilon \varphi\|_{L^2(\mathbb{R}^d)} \leq (b^2)^{1/2} \|\varphi\|_{L^2(\mathbb{R}^d)}. \]  

The estimate (4.3) can be specified in the case of higher regularity:

\[ \|S^\varepsilon \varphi - \varphi\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^2 \|
abla^2 \varphi\|_{L^2(\mathbb{R}^d)} \quad \forall \varphi \in H^2(\mathbb{R}^d), \quad C = \text{const}(d). \]  

(4.5)

It can be proved in the same way as (4.3) by using the Taylor expansion of \( \varphi(x - \varepsilon \omega) \) at a point \( x \) with a suitable remainder. By duality, (4.5) implies

\[ \|S^\varepsilon \varphi - \varphi\|_{H^{-2}(\mathbb{R}^d)} \leq C\varepsilon^2 \|\varphi\|_{L^2(\mathbb{R}^d)} \quad \forall \varphi \in L^2(\mathbb{R}^d), \quad C = \text{const}(d). \]  

(4.6)

We refer the reader to [12]–[14] for other properties of the Steklov smoothing operator which can be used to obtain homogenization estimates for operators.

4.2. Smoothing with arbitrary kernel. We consider the smoothing operator

\[ \Theta^\varepsilon \varphi(x) = \int_{\mathbb{R}^d} \varphi(x - \varepsilon \omega) \theta(\omega) d\omega, \]  

(4.7)

where the smoothing kernel \( \theta \in L^\infty(\mathbb{R}^d) \) is compactly supported, \( \theta \geq 0 \), and

\[ \int_{\mathbb{R}^d} \theta(x) dx = 1. \]

The estimates (4.2)–(4.4) for the usual Steklov smoothing operator remain valid for the general smoothing operator (4.7) provided that the right-hand side contains constants depending not only on the dimension \( d \), but also on the kernel \( \theta \). If \( \theta \) is even, then \( \Theta^\varepsilon \) possesses properties similar to (4.5) and (4.6).

For the kernel \( \theta \) of higher regularity we have the following assertion proved in [15, 14].

**Lemma 4.2.** Assume that \( \theta \) is a piecewise \( C^k \)-smooth function, \( k \) is a natural number, \( b \in L^2_{\text{per}}(Y) \), \( b_\varepsilon(x) = b(x/\varepsilon) \), and \( \varphi \in L^2(\mathbb{R}^d) \). Then

\[ \|\Theta^\varepsilon \nabla^k \varphi\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{-k} \|\varphi\|_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(\theta), \]  

(4.8)

\[ \|b_\varepsilon \Theta^\varepsilon \nabla^k \varphi\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon^{-k} (b^2)^{1/2} \|\varphi\|_{L^2(\mathbb{R}^d)}, \quad C = \text{const}(\theta). \]  

(4.9)
It is obvious that the Steklov smoothing operator $S^\varepsilon$ is defined by (4.7). In this case, the smoothing kernel is the characteristic function $\theta_0(x)$ of the cube $Y = [-1/2, 1/2]^d$.

The double Steklov smoothing operator $(S^\varepsilon)^2 = S^\varepsilon S^\varepsilon$ is a smoothing operator equal to the convolution $\theta_1 = \theta_0 \ast \theta_0$. The convolution is computed in [14], where it is also shown that $\theta_1$ is a piecewise $C^1$-smooth function. Consequently, $\Theta^\varepsilon = (S^\varepsilon)^2$ possess the properties (4.8) and (4.9) for $k = 1$. Furthermore, since $\theta_1$ is even, the estimates (4.5) and (4.6) hold for the double smoothing.

5 \textit{H}^2-\textit{Approximation}

\textbf{Proof of Theorem 2.1.} 1. Denote by $b_\varepsilon$ or $(b)_\varepsilon$ the $\varepsilon$-periodic function of the variable $x$ obtained from a 1-periodic function $b(y)$ by substitution $y = x/\varepsilon$, i.e.,

\[ b_\varepsilon(x) = b(x/\varepsilon). \]  \hspace{1cm} (5.1)

For example, $a_\varepsilon(x) = a(x/\varepsilon)$, $N^\varepsilon_{ij}(x) = N_{ij}(x/\varepsilon)$, $(aD N^\varepsilon_{ij}) = (a(y) D_y N^\varepsilon_{ij}(y))|_{y = x/\varepsilon}$ and so on. In what follows, we systematically use the differentiation formula for two-scale functions

\[ D\Phi(x, x/\varepsilon) = [D_x \Phi(x, y) + \varepsilon^{-2} D_y \Phi(x, y) + \varepsilon^{-1}(\nabla_x \times \nabla_y) \Phi(x, y) + \varepsilon^{-1}(\nabla_y \times \nabla_x) \Phi(x, y)]|_{y = x/\varepsilon}, \]  \hspace{1cm} (5.2)

where the differential operator $(\nabla_x \times \nabla_y)$ sends a scalar function $\Phi(x, y)$ to the matrix of mixed second order derivatives $\left\{ \frac{\partial^2 \Phi}{\partial x_i \partial y_j} \right\}_{i,j}$. By (5.2), we can formally calculate

\[ D(u(x) + \varepsilon^2 N^\varepsilon_{ij}(x/\varepsilon) D_i D_j u(x)) \]
\[ = Du(x) + D_y N^\varepsilon_{ij}(y) D_i D_j u(x) + \varepsilon(\nabla_y N^\varepsilon_{ij}(y)) \times (\nabla D_i D_j u(x)) \]
\[ + \varepsilon(\nabla D_i D_j u(x)) \times (\nabla_y N^\varepsilon_{ij}(y)) + \varepsilon^2 N^\varepsilon_{ij}(y) D D_i D_j u(x), \]  \hspace{1cm} (5.3)

where $y = x/\varepsilon$ and, in the case of a symmetric tensor $a(y)$,

\[ a(x/\varepsilon) D(u(x) + \varepsilon^2 N^\varepsilon_{ij}(x/\varepsilon) D_i D_j u(x)) \]
\[ = a(y)(D_y N^\varepsilon_{ij}(y) + e^{ij}) D_i D_j u(x) + 2\varepsilon a(y)(\nabla_y N^\varepsilon_{ij}(y)) \times (\nabla D_i D_j u(x)) \]
\[ + \varepsilon^2 a(y) N^\varepsilon_{ij}(y) D D_i D_j u(x), \]  \hspace{1cm} (5.4)

where $y = x/\varepsilon$ and the matrix $e^{ij}$ was introduced above.

2. Let us estimate the discrepancy of the approximation (2.14) in Equation (2.1), starting with the following representation:

\[ A_{\varepsilon} v^\varepsilon + v^\varepsilon - f = (A_{\varepsilon} + 1) v^\varepsilon - (\hat{A}_{\varepsilon} + 1) w^\varepsilon + (f^\varepsilon - f) = (A_{\varepsilon} v^\varepsilon - \hat{A}_{\varepsilon} u^\varepsilon) + (v^\varepsilon - u^\varepsilon) + (f^\varepsilon - f), \]

where we take into account the equality

\[ (\hat{A}_{\varepsilon} + 1) u^\varepsilon = f^\varepsilon, \quad f^\varepsilon = \Theta^\varepsilon f, \]  \hspace{1cm} (5.5)
obtained by applying the smoothing operator $\Theta^\varepsilon$ to Equation (2.10):

$$\Theta^\varepsilon(\hat{A}_\varepsilon \hat{u}^\varepsilon + \hat{u}^\varepsilon) = \Theta^\varepsilon f \Rightarrow (\hat{A}_\varepsilon + 1)\Theta^\varepsilon \hat{u}^\varepsilon = \Theta^\varepsilon f,$$

which yields (5.5) in the notation $\Theta^\varepsilon \hat{u}^\varepsilon = u^\varepsilon$ introduced in (2.16). Then

$$A_\varepsilon v^\varepsilon + v^\varepsilon - f \overset{(2.14),(2.1),(2.9)}{=} D^*(a_\varepsilon D(u^\varepsilon + \varepsilon^2 U_2^\varepsilon + \varepsilon^3 U_3^\varepsilon) - \hat{a} Du^\varepsilon - \varepsilon b^{\varepsilon, st} D_{r, st}u^\varepsilon)$$

$$+ (\varepsilon^2 U_2^\varepsilon + \varepsilon^3 U_3^\varepsilon) + (f^\varepsilon - f). \quad (5.6)$$

To analyze (5.6), we set

$$R_\varepsilon := a_\varepsilon D(u^\varepsilon + \varepsilon^2 U_2^\varepsilon + \varepsilon^3 U_3^\varepsilon) - \hat{a} Du^\varepsilon - \varepsilon b^{\varepsilon, st} D_{r, st}u^\varepsilon \quad (5.7)$$

and introduce the notation

$$z_{ij} = D_i D_j u^\varepsilon, \quad z_{ijk} = D_i D_j D_k u^\varepsilon, \ldots, z_{ijkpq} = D_i D_j D_k D_p D_q u^\varepsilon. \quad (5.8)$$

Taking into account (5.1), (5.8) and making calculations similar to (5.3) and (5.4), we get

$$R_\varepsilon \overset{(5.7),(2.15)}{=} a_\varepsilon[Du^\varepsilon + z_{ij}(DN)^{ij}\varepsilon + 2\varepsilon(\nabla N)^{ij}\varepsilon \times \nabla z_{ij} + \varepsilon^2 N_{ij}^\varepsilon Dz_{ij}$$

$$+ \varepsilon z_{ijk}(DN)^{ijk}\varepsilon + 2\varepsilon^2(\nabla N)^{ijk}\varepsilon \times \nabla z_{ijk} + \varepsilon^3 N_{ijk}^\varepsilon Dz_{ijk} - \hat{a} Du^\varepsilon - \varepsilon b^{\varepsilon, ij} z_{ij}. \quad (5.9)$$

Collecting terms with the same power $\varepsilon^n$, $n \geq 0$, we find

$$R_\varepsilon = (a(DN)^{ij} + e^{ij}) - \hat{a} e^{ij}\varepsilon z_{ij} + \varepsilon[(a(DN)^{ijk} + 2a(\nabla N)^{ij} \times e^k) - b^{ijk}\varepsilon z_{ijk}]$$

$$+ \varepsilon^2[(aN)^{ij}\varepsilon Dz_{ij} + 2a(\nabla N)^{ijk}\varepsilon \times e^m z_{ijkm}] + \varepsilon^3[(aN)^{ijk}\varepsilon Dz_{ijk}]$$

$$:= R_\varepsilon^0 + R_\varepsilon^1 + R_\varepsilon^2 + R_\varepsilon^3,$$

where the vectors $e^m$ with $m$ from 1 to $d$ form the canonical basis for the space $\mathbb{R}^d$. The term $R_\varepsilon^0$ contains the oscillating factor $(a(DN)^{ij} + e^{ij}) - \hat{a} e^{ij}\varepsilon z_{ij}$, and can be written as $R_\varepsilon^0 = g^{ij}\varepsilon z_{ij}$, where the 1-periodic matrix $g^{ij} = \{g^{ij}_{st}\}_{i, j}$ can be represented in terms of the matrix potential (cf. (3.8)) $g^{ij}_{st} = D^* G^{ij, st}$ for all $i, j, s, t$ from 1 to $d$. By Lemma 3.2, we can transform

$$(g^{ij}_{st})\varepsilon z_{ij} = (D^* G^{ij, st})\varepsilon z_{ij} = D^*(e^2 G^{ij, st})\varepsilon z_{ij} = -2\varepsilon D^* G^{ij, st} \cdot \nabla z_{ij}$$

$$= D^*(e^2 G^{ij, st})\varepsilon z_{ij} = -2\varepsilon D^* G^{ij, st} \cdot e^k\varepsilon z_{ijk} + e^2 G^{ij, st} \cdot Dz_{ij},$$

where each matrix $\{D^*(e^2 G^{ij, st})\}_{s, t}$ (without summation over $i$ and $j$) satisfies (3.15) and, consequently, is solenoidal. Therefore,

$$D^* R^0_\varepsilon = D^* (g^{ij}_\varepsilon z_{ij}) = D_{st}((g^{ij}_{st})\varepsilon z_{ij}) = -D_{st}(2\varepsilon (D^* G^{ij, st} \cdot e^k)\varepsilon z_{ijk} + e^2 G^{ij, st} \cdot Dz_{ij}),$$

where

$$(D^* G^{ij, st} \cdot e^k)\varepsilon z_{ijk} = (\partial_m G^{ij, st}_{km})\varepsilon z_{ijk} = -(\partial_m G^{ij, km}_{st})\varepsilon z_{ijk},$$

$$G^{ij, st} \cdot Dz_{ij} = (G^{ij, st}_{km})\varepsilon z_{ijkm} = -(G^{ij, km}_{st})\varepsilon z_{ijkm}.$$
since the potential \( \{g_{km}^{ij, st}\}_{k,m} \) is skew-symmetric. Consequently,
\[
D^* R_\varepsilon = D_{st}(2\varepsilon(\partial_m G_{st}^{ij,km})\varepsilon z_{ijk} + \varepsilon^2 (G_{st}^{ij,km})\varepsilon z_{ijkm})
\[
= D^*(2\varepsilon(\partial_m G_{st}^{ij,km})\varepsilon z_{ijk} + \varepsilon^2 (G_{st}^{ij,km})\varepsilon z_{ijkm}).
\]
(5.10)

From (5.9) and (5.10) we obtain the expansion in \( \varepsilon^n \) without terms with the zeroth power of \( \varepsilon \):
\[
D^* R_\varepsilon = \varepsilon D^*[(a(DN^{ij,k} + 2a_\varepsilon(\nabla N^{ij} \times \varepsilon^k) + 2\partial_m G_{st}^{ij,km} - b^{ij,k})\varepsilon z_{ijk}] \\
+ \varepsilon^2 D^*[(a(N^{ij})\varepsilon Dz_{ij} + 2a_\varepsilon(\nabla N^{ijk} \times \varepsilon^m)\varepsilon z_{ijkm} + (G^{ij,km})\varepsilon z_{ijkm}] \\
+ \varepsilon^3 D^*[(a(N^{ijk})\varepsilon Dz_{ijk}] .
\]
(5.11)

The term with power of \( \varepsilon \) in (5.11) contains the oscillating factor
\[
(a(DN^{ij,k} + 2a_\varepsilon(\nabla N^{ij} \times \varepsilon^k) + 2\partial_m G_{st}^{ij,km} - b^{ij,k})\varepsilon)^{(5.10)} = g_{\varepsilon}^{ijk}
\]
and can be written as
\[
\varepsilon D^*[g_{\varepsilon}^{ijk}z_{ijk}] = \varepsilon D_{st}[(g_{st}^{ij})\varepsilon z_{ijk}];
\]
(5.12)
morover, the 1-periodic matrix \( g_{\varepsilon}^{ij} = \{g_{st}^{ij}\}_{s,t} \) possesses the properties (3.12) and, according to
Lemma 3.1, is represented as \( g_{st}^{ij} = D^*g^{ij, st}_{\varepsilon} \) (cf. (3.13)1) for all \( i, j, k, s, t \) from 1 to \( d \).

By Lemma 3.2, we can write
\[
(g_{st}^{ij,k})\varepsilon z_{ijk} = (D^*g^{ij, st}_{\varepsilon})\varepsilon z_{ijk} = D^*(\varepsilon^2 G_{\varepsilon}^{ij, st}z_{ijk}) - \varepsilon^2 G_{\varepsilon}^{ij, st}\cdot Dz_{ijk} - 2\varepsilon(\text{div}G_{\varepsilon}^{ij, st})\varepsilon \cdot \nabla z_{ijk}
\]
\[
= D^*(\varepsilon^2 G_{\varepsilon}^{ij, st}z_{ijk}) - 2\varepsilon(\text{div}G_{\varepsilon}^{ij, st} \cdot \epsilon^p)\varepsilon z_{ijkp} - \varepsilon^2 G_{\varepsilon}^{ij, st}\cdot Dz_{ijk},
\]
where each matrix \( \{D^*(\varepsilon^2 G_{\varepsilon}^{ij, st}z_{ijk})\}_{s,t} \) (without summation over \( i, j, \) and \( k \)) satisfies (3.15)
and, consequently, is solenoidal. Hence (5.12) can be written as
\[
\varepsilon D^*[g_{\varepsilon}^{ijk}z_{ijk}] = 2\varepsilon^2 D_{st}(\partial_q G_{st}^{ij,kpq})\varepsilon z_{ijkp} + \varepsilon^3 D_{st}(G_{st}^{ij, kpq})\varepsilon z_{ijkpq}
\]
\[
= D^*[2\varepsilon^2 (\partial_q G_{\varepsilon}^{ij,kpq})\varepsilon z_{ijkp} + \varepsilon^3 G_{\varepsilon}^{ij,kpq}z_{ijkpq}],
\]
(5.13)
where we simplify the formulas since the matrix potentials are skew-symmetric, as in the proof of (5.10).

Thus, the expansion (5.11) after the transformation (5.13) is written as
\[
D^* R_\varepsilon = \varepsilon D^*[2(\partial_q G^{ij,kpq})\varepsilon z_{ijkp} + (a N^{ij})\varepsilon Dz_{ij} + 2a_\varepsilon(\nabla N^{ijk} \times \varepsilon^m)\varepsilon z_{ijkm}
\]
\[
+ (G^{ij,km})\varepsilon z_{ijkm}] + \varepsilon^3 D^*[G_{\varepsilon}^{ij,kpq}z_{ijkpq} + (a N^{ijk})\varepsilon Dz_{ijk}] 
\]
(5.14)

3. From (5.6), (5.7), and the identity \( f = (A_\varepsilon + 1)u^\varepsilon \) we obtain an equation satisfied by the difference \( v^\varepsilon - u^\varepsilon \)
\[
A_\varepsilon(v^\varepsilon - u^\varepsilon) + (v^\varepsilon - u^\varepsilon) = D^* R_\varepsilon + r_\varepsilon + (f^\varepsilon - f) =: F^\varepsilon,
\]
(5.15)
where, in the notation (5.1) and (5.8),
\[
r_\varepsilon = \varepsilon^2 U_2^\varepsilon + \varepsilon^3 U_3^\varepsilon \quad \text{(5.15)} \\
= \varepsilon^2 N_{ij}^\varepsilon z_{ij} + \varepsilon^3 N_{ijk}^\varepsilon z_{ijk},
\]
(5.16)
It is easy to show that
\[ \| F^\varepsilon \|_{H^{-2}(\mathbb{R}^d)} \leq C\varepsilon^2 \| f \|_{L^2(\mathbb{R}^d)}. \] (5.17)
Here, we use Lemma 4.1 and 4.2 to analyze \( D^* R \varepsilon \) and \( r \varepsilon \) in the sum in (5.15) and an estimate of the form (4.6) for the double smoothing to analyze the term \( (f, \varepsilon - f) \).

By the inequality (5.17), from the energy estimate
\[ \| z^\varepsilon \|_{H^2(\mathbb{R}^d)} \leq c \| F^\varepsilon \|_{H^{-2}(\mathbb{R}^d)}, \quad c = \text{const}(d, \lambda), \]
for the solutions \( z^\varepsilon = v^\varepsilon - u^\varepsilon \) to the equation \((A_\varepsilon + 1) z^\varepsilon = F^\varepsilon; z^\varepsilon \in H^2(\mathbb{R}^d)\), we derive (2.18). Theorem 2.1 is proved. \( \square \)

We comment the use of Lemmas 4.1 and 4.2 in the proof of the estimate (5.17). We note that \( D^* R \varepsilon \) and \( r \varepsilon \) are the sums of six and two terms respectively (cf. (5.14) and (5.16)). We apply Lemma 4.1 to all terms in (5.14) except for the last two ones and Lemma 4.2 to the two removed terms in (5.14), containing \( \varepsilon^3 \). Here, the role of the oscillating factor \( b(y) \) is played by the functions
\[ N^{ij}, N^{ijk}, \nabla N^{ijk}, G^{ij,km}, G^{ijk,pq}, \nabla G^{ijk,pq} \] (5.18)
where \( i, j, k, p, q \) run from 1 to \( d \). By the estimate (3.4) for \( N^{ij} \), all functions in (5.18) belong to \( L^2_{\text{per}}(Y) \) (cf. Section 3); moreover, for any \( b \) in (5.18) we have \( \| b \| \leq C \), where the constant \( C \) depends only on \( d \) and \( \lambda \).

Recalling (5.8), we see that the functions \( z_{ij}, z_{ijk}, z_{ijkp} \) in (5.14) and (5.16), with \( i, j, k, p \) from 1 to \( d \), have the form \( S^\varepsilon \varphi \), where \( \varphi \in L^2(\mathbb{R}^d) \); moreover,
\[ \| \varphi \|_{L^2(\mathbb{R}^d)} \leq \| \hat{u}^\varepsilon \|_{H^4(\mathbb{R}^d)} \leq C \]
in view of (2.13). Here, the constant depends only on \( d \) and \( \lambda \). Therefore, we can apply Lemma 4.1 to the terms containing \( z_{ij}, z_{ijk}, z_{ijkp} \). For example, since \( z_{ijkm} = S^\varepsilon (S^\varepsilon D_i D_j D_k D_m \hat{u}^\varepsilon) \) in view of (5.8) and (2.16), we have
\[ \|(G^{ij,km})_\varepsilon z_{ijkm}\|_{L^2(\mathbb{R}^d)} \leq C \| f \|_{L^2(\mathbb{R}^d)}, \]
where, at the last step, we take into account the property (4.2) of the smoothing operator \( S^\varepsilon \) and the \( L^2 \)-estimate for \( G^{ij,km} \).

We apply Lemma 4.2 to the terms in (5.14) containing \( z_{ijkpq} \) or \( D z_{ij} \). Then one factor \( \varepsilon \) in these terms vanishes as a compensation of the \( \varepsilon \)-uniform estimate. Thus, in spite of the presence of \( \varepsilon^3 \), these terms have the smallness order \( O(\varepsilon^2) \). For example, \( z_{ijkpq} = \Theta^\varepsilon(D_q \varphi) \), where \( \varphi = D_i D_j D_k D_p \hat{u}^\varepsilon \). Therefore,
\[ \varepsilon \|(G^{ijk,pq})_\varepsilon z_{ijkpq}\|_{L^2(\mathbb{R}^d)} \leq C \| f \|_{L^2(\mathbb{R}^d)} \]
since the kernel of the smoothing operator \( \Theta^\varepsilon = S^\varepsilon S^\varepsilon \) is piecewise \( C^1 \)-smooth. Finally, we take into account the \( L^2 \)-estimate for \( G^{ijk,pq} \).

**Remark 5.1.** To simplify the exposition, we constructed the improved approximation in the energy operator norm for the resolvent of an elliptic operator of order more than two by considering a scalar selfadjoint fourth order operator with real coefficients. However, the proposed method can be used in more general situations with matrix nonselfadjoint operators of an even order \( 2m \geq 4 \) with complex coefficients provided that a coercivity condition like the Garding inequality holds. Such operators were studied in [2, 4, 16].
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