Frequency dependent conductivity of vortex cores
in type II superconductors

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The recent optical transmission experiment of Karrai et al. has probed the localized quasiparticle excitations of vortices in the superconductor YBa$_2$Cu$_3$O$_7$. In this paper we develop a microscopic description of single vortex dynamics, based on the Bogoliubov-deGennes equations and self-consistency through the gap equation, to determine the response of vortices to a transverse time dependent electric field. It is applicable to the low temperature, clean, extreme type II limit. In the limit of large planar mass anisotropy it simplifies. Thus it may be especially relevant to materials such as NbSe$_2$ and high temperature superconductors. Of particular interest is the response of the vortex at frequencies near the minigap, $\Delta^2/E_F$, where $\Delta$ is the bulk energy gap and $E_F$ is the fermi energy. A dissipative equation of motion for vortex cores valid at non-zero frequencies is derived. We give a clear microscopic meaning to the vortex drag parameter. The expected dipole transition between quasi-particle states localized at the core is hidden because of the self-consistent nature of the vortex potential. Instead the vortex itself moves and has a resonance at the frequency of the transition. We calculate the conductivity of vortices as a function of frequency. A analogy is made to the Mattis-Bardeen result for the electrodynamic response of bulk superconductors that, unless translation invariance is broken, single particle
properties are ‘invisible’ to a long-wavelength probe. However we show that upon adding a translation invariance breaking term the dipole transition re-appears. This approach may eventually form the basis of a microscopic theory of vortex pinning at non-zero frequency.

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I. INTRODUCTION

The condensed state of type II superconductors can possess a topological defect, a vortex, at which there is a zero in the order parameter. The existence of these defects is made energetically favourable upon the application of a sufficiently large magnetic field or current. Associated with these defects are quasiparticle states with wavefunctions localized near the vortex cores and energies within the bulk energy gap. These states are superpositions of electrons and holes and one may think of them as trapped in the vortex by continual Andreev reflections due to the spatially varying order parameter. Their quantitative description in a simple limit was formulated some time ago. Caroli, deGennes and Matricon [1] and later Bardeen et al. [2] calculated the energies and wavefunctions of these discrete levels using the Bogoliubov-deGennes (BdG) equations.

The electrodynamical response of type II superconductors in the mixed state is determined by the dynamics of vortices. It is usual to approach this by saying that the phase slip from vortex motion is associated with voltage drops and hence gives rise to dissipation in the superconducting state. Another, less conventional, way to look at things is to note that the low energy excitations of the system are associated with the states inside the bulk energy gap which are localized near vortex cores. For most practical applications, however, vortex pinning is the most important effect. But in the case of very clean systems (possibly probed at high frequencies) one is interested in dissipative flux flow. The microscopic calculations mentioned above helped form a basis for theories of vortex motion based on the idea of a ‘normal’ core such as those of Bardeen and Stephen [3], Nozieres and Vinen [4] and others [5].

Even in the presence of weak pinning, flux flow may be relevant if pinning is overcome at high enough temperatures (but below \( T_c \)) or high enough frequencies. Gittleman and Rosenblum [6,7] studied dissipation in the mixed state of conventional superconductors as a function of frequency and noticed that there was a crossover to essentially free vortex behaviour over one or two orders of magnitude centered about some frequency. What is
significant for this work is that the crossover frequency region appears to be of the same
order of magnitude as the expected spacing between discrete levels in vortex cores. That
spacing is of order $\Delta^2/E_F$ or $\hbar^2/m\xi^2$ (where $\Delta$ is the bulk gap, $E_F$ is the Fermi energy and $\xi$ is
the coherence length) and follows from applying the uncertainty principle to a quasiparticle
confined to a vortex core of size $\xi$. Unless impurities radically alter the vortex size this
should be the characteristic energy scale. The comparison is made in table I where some
unpublished data of Schleger and Hardy [8] are included. The fact that the de-pinning and
level spacing frequencies are roughly comparable means that any microscopic understanding
of the de-pinning transition as a function of frequency should take into account the discrete
spectrum of states inside vortex cores.

Caroli and Matricon [9] discussed possible ways of observing the discrete structure within
vortex cores through ultrasonic and nuclear magnetic relaxation. None of these methods has
so far succeeded. A few years ago Hess et al. [11] probed the gross structure of vortices in
NbSe$_2$ by scanning tunneling microscopy (STM). The results were qualitatively explained
by theories [13] based upon the Eilenberger equation approach of Kramer and Pesch [12]. The
STM experiments however had a resolution of about 0.1meV whereas the energy separation
of the discrete levels is expected to be about 10mK. STM experiments probing the vortex
cores of high temperature superconductors (where the energy scale is expected to be of
order 1meV or more) would be extremely desirable but have not yet been accomplished. In
passing we note that in charge density wave systems similar microscopic vortex structure
and possible experimental implications were proposed by Maki and Huang [10].

Recent experiments carried out by Karrai et al. [14] have measured the optical trans-
mission of the mixed state of YBa$_2$Cu$_3$O$_7$ at low temperature. They observed a knee in the
normalized transmission coefficient in the frequency range $50–100\text{cm}^{-1}$ and attributed it to
a quasiparticle resonance. Some features are of interest. First the resonance frequency is at
least 3 or 4 times higher than expected from a simple microscopic calculation. Second, the
fitted relaxation rate of the resonance is very large: of the same order of magnitude as the
resonant frequency itself, $\tau^{-1} \sim 50\text{cm}^{-1} \sim 10^{12}\text{s}^{-1}$. The is roughly the same order of magni-
tude as the relaxation rate of YBa$_2$Cu$_3$O$_7$ as measured by microwave surface impedance [15]. The experiment is performed at a temperature $T = 2$K, over an order of magnitude lower than the resonant frequency which allows us to deal with only the lowest energy excitations. One would like to know whether the observed effect is due to the intrinsic response of free vortices, or whether it is more important to consider defects, with which the vortex cores would be highly correlated.

In a previous paper [16] the response of quasiparticles in cores of vortices (in the clean, low temperature limit) to a long wavelength electromagnetic field was considered. It was concluded that the motion of the vortex itself had to be taken into account because it was a self-consistent potential. In this paper this idea is re-derived in a different fashion and expanded in order to be directly relevant to infrared measurements in high temperature superconductors. The paper is organized as follows: In section II we introduce the formalism associated with the Bogoliubov quasiparticles and the BdG equation. In sections III and IV we describe displaced and moving vortices respectively using this formalism. In section V we derive the equation of motion, calculate the response of vortices to a long wavelength electromagnetic wave and discuss its consequences. We discuss some effects of pinning in section VI. It is followed by a discussion and summary in sections VII and VIII.

II. MICROSCOPIC FORMALISM

The full Eliashberg-Gorkov formalism of BCS theory is, of course, intractable for the problem of a moving vortex. Instead we consider the simpler BdG equation for s-wave superconductors. This equation is valid in the case of a short range instantaneous pairing interaction. It may also be derived in the quasiclassical limit $k_F\xi >> 1$ where $k_F$ is the fermi momentum and $\xi$ is the coherence length. That limit is the leading order result in the formulation of Eilenberger [17]. We shall supplement the BdG equations with a local gap equation $\Delta (r) = V \langle c_{\uparrow} (r) c_{\downarrow} (r) \rangle$ which determines how the vortex itself moves (self-consistently). Of course high temperature superconductors stretch the assumptions made.
here. The coherence length is only about 5–10 lattice constants parallel to the Cu-O planes so that $k_F \xi \gg 1$ is barely satisfied. The discreteness of the lattice may also come into play. Real superconductors are of course non-local and high temperature superconductors may even have d-type symmetry of the order parameter (for which the microscopic structure of a vortex has not yet been calculated). Also, in general, the pairing interaction may be retarded but since we are concerned with very low frequencies relative to the gap, some relevance to real materials may remain.

The eigenfunctions for Bogoliubov quasiparticles are two component objects which can be thought of as the electron and hole amplitudes of the quasiparticle,

$$\psi(r) = \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}.$$  \hspace{1cm} (2.1)

$\psi$ satisfies a Schrödinger equation which is just the BdG equation,

$$i\hbar \frac{\partial}{\partial t} \psi(r) = \sigma^z \left[ \frac{1}{2m} \left( p - \sigma^z \frac{e}{c} A \right)^2 - E_F \right] \psi(r) + \begin{pmatrix} 0 & \Delta(r) \\ \Delta^*(r) & 0 \end{pmatrix} \psi(r),$$  \hspace{1cm} (2.2)

where $\sigma^z$ is a Pauli matrix, $E_F$ is the chemical potential, and $\Delta = |\Delta(r - r_0)| \exp (-i \theta(r - r_0))$. $\theta(r - r_0)$ is the angle about the center of the vortex $r_0$ measured from the $\hat{x}$ axis. By convention the $\hat{z}$ direction shall be in the direction parallel to the vortex.

Going to second quantized formalism the conventional definition \[2\] of the quasiparticle creation operators is,

$$\left( \begin{array}{c} \Gamma_{\mu\uparrow}^\dagger \\ \Gamma_{\mu\downarrow}^\dagger \end{array} \right) \equiv \int d^3r \left( \begin{array}{cc} c_{\uparrow}^\dagger(r) & c_{\downarrow}(r) \\ c_{\downarrow}^\dagger(r) & -c_{\uparrow}(r) \end{array} \right) \psi_\mu(r).$$  \hspace{1cm} (2.3)

where $\mu$ is an index for the low energy solutions of this equation and $c_{\uparrow}$, $c_{\downarrow}$ denote spin up and spin down electron operators respectively. It is conventional to define the creation operators only for positive energy states (they are a complete set of states). However it will be convenient for us to use the complete set of states which includes both positive and negative energy states but eliminates the spin degeneracy. That is, we shall use, for $\epsilon_\mu > 0$,
and for $\epsilon_\mu < 0$,

$$\gamma_\mu \equiv \Gamma^{\dagger}_{-\mu_\downarrow},$$

(2.5)

where $-\mu$ refers to the corresponding time reversed positive energy state. This is possible because if $(u_\mu, v_\mu)$ is a solution for energy $\epsilon_\mu$ then $(v_\mu^* - u_\mu^*)$ is a solution for energy $-\epsilon_\mu$.

Finally, as we shall see below, we have a ladder of states from negative to positive energy which are filled for $\epsilon_\mu < 0$. The inverse transformation from electrons to quasiparticles is then

$$c^{\dagger}_{\uparrow}(r) = \sum_\mu \gamma^\dagger_{\mu_\uparrow} u^\dagger_\mu(r),$$

$$c^{\dagger}_{\downarrow}(r) = \sum_\mu \gamma^\dagger_{\mu_\downarrow} v_\mu(r).$$

(2.6)

We shall be interested in the extreme type II limit with $H << H_{c2}$. This can certainly be satisfied in the high temperature superconductors. Vortices are well separated and there is no quasiparticle tunneling between vortices (although see reference [20] for a treatment of that situation). The magnetic field applied to create the vortices may be ignored in the BdG equation. Because the magnetic field is spread out over an area $\lambda^2$ it’s importance compared to the phase of the order parameter is reduced by $\xi^2/\lambda^2$ where $\xi$ is the coherence length and $\lambda$ is the penetration depth.

The low energy eigenfunctions for fixed $k_z, \mu << k_{F\perp} \xi$, and the radial coordinate $r << \xi$ are

$$\psi_{\mu}(r) = \left(\frac{k_F}{2\pi \xi L_z}\right)^2 e^{i k_z z} \left( e^{i(\mu - \frac{1}{2}) \phi} J_{\mu - \frac{1}{2}}(k_{F\perp} r) \right) \left( e^{i(\mu + \frac{1}{2}) \phi} J_{\mu + \frac{1}{2}}(k_{F\perp} r) \right)$$

(2.7)

with angular momentum index $\mu = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots$ and $k_{F\perp}$ referring to the Fermi momentum projected onto the $k_x, k_y$ plane. At distances of order $\xi$ from the vortex center the wavefunctions begin to decay exponentially [1,9]. The energies as calculated by Kramer and Pesch [12], who analytically accounted for some self-consistency effects due to the gap equation $\Delta(r) = V \langle c_{\uparrow}(r) c_{\downarrow}(r) \rangle$, are
\[ \epsilon_\mu = \frac{2\mu \Delta_0^2}{k_F v_F \cos^2 \Theta} \ln\left( \frac{\pi}{2} \frac{\xi_0 \cos \Theta}{\xi_1} \right), \quad \cos \Theta \equiv k_{F\perp}/k_F. \]  

(2.8)

The zero of energy is the Fermi energy.

The logarithmic factor is a simple rescaling of energies. It was derived by Kramer and Pesch and arises from self-consistency of the vortex solution. At low temperature, energy is favoured over entropy and so the vortex pinches together. This increases level spacing, pushing down the energies of occupied states and reducing the occupation of excited states. This factor will not be important for this paper but its effects were explored by Bardeen and Sherman \[18\] and Larkin and Ovchinnikov \[19\].

In this paper, in order to simplify matters, we will suppose that the Fermi surface is open and nearly cylindrical. That is, the system is nearly two dimensional and the dispersion in the (by convention \( \hat{z} \)) direction perpendicular to the planes is very small, \( i.e. \) the vortex is aligned perpendicular to the planes. This is a rather good approximation for materials such as NbSe\(_2\) and the copper-oxide high temperature superconductors. Moreover there is another factor which helps. Near two-dimensionality is equivalent to \( k_{F\perp} = k_F \cos \Theta \) varying little with \( k_z \). As one can see from Eq. (2.8), the dispersion in the \( k_z \) direction enters through the \( \cos \Theta \) in the denominator. Each angular momentum level broadens into a band. Nevertheless, the one-dimensional density of states associated with the variation of this cosine diverges at \( \Theta \sim 0 \) and this strengthens the approximation of neglecting the \( k_z \) dispersion.

III. DISPLACED VORTEX

The quasiparticle states are a complete set of states so that once the density matrix in that basis is specified the state of the system is defined. Conversely it is possible to describe a displaced vortex in this basis because the order parameter is completely specified in terms of the quasiparticles through the gap equation. By using Eqn. (2.8) and substituting into the gap equation we have,
\[
\Delta(r) = -V \sum_{\mu,\nu} \langle \gamma^\dagger_\nu \gamma_\mu \rangle v^*_\nu(r) u_\mu(r).
\] \hspace{1cm} (3.1)

As an aside recall that in the well-known Ginzburg-Landau solution for the order-parameter profile of a vortex core in an s-wave superconductor we have \(\Delta(r) \sim r\), as \(r \to 0\). From the series representation of the Bessel functions, \(J_n(x) \sim x^{|n|}\) for small \(x\) and integer \(n\). From Eqs. (2.7) and (3.1) we can see that this property is satisfied by the vortex on a microscopic level, but it is only the \(\mu = \pm 1/2\) states that contribute to the linear in \(r\) component of the order parameter. At temperatures higher than \(\Delta^2/k_B E_F\) the occupation of \(\mu = \pm 1/2\) states is about the same and the linear in \(r\) component is washed out because \(J_{-n}(x) = (-1)^n J_n(x)\).

Suppose the vortex is displaced by a small amount \(\delta r_0\) in a direction which makes angle \(\phi_0\) with the \(x\)-axis. Let the occupation be diagonal before displacement, that is, \(\langle \gamma^\dagger_\nu \gamma_\mu \rangle = \delta_{\mu\nu} f(\epsilon_\nu)\), where \(f(\epsilon)\) is the fermi distribution function. The changes \(\delta u_\mu(r) = -\delta r_0 \cdot \nabla u_\mu(r)\) (and similarly for \(v_\mu(r)\)) may be approximated by assuming that \(u\) and \(v\) behave roughly like Bessel functions. This is explained in detail in the appendix. In the undisplaced coordinates, to linear order in \(\delta r_0\), and for small \(r\),

\[
\delta \Delta(r) \approx -V \frac{\delta r_0 k_F^+}{2} \sum_\nu (f(\epsilon_\nu) - f(\epsilon_{\nu+1})) \times \left[ e^{i\phi_0} v^*_\nu+1(r) u_\nu(r) + e^{-i\phi_0} v^*_\nu(r) u_{\nu+1}(r) \right].
\] \hspace{1cm} (3.2)

It is important to note that small displacements can be described in terms of changes in the quasiparticle occupation or density matrix. The above displacement can be represented by the change

\[
\delta \langle \gamma^\dagger_{\nu+1} \gamma_\nu \rangle = \frac{\delta r_0 k_F^+}{2} [f(\epsilon_\nu) - f(\epsilon_{\nu+1})] e^{i\phi_0}, \hspace{1cm} \forall \nu
\] \hspace{1cm} (3.3)

(and the Hermitian conjugate) in the quasiparticle density matrix using the undisplaced basis.

**IV. MOVING VORTEX**

In this section we consider a vortex moving with a small velocity and show, using a Galilean transformation, how it can be approximately described by quasiparticle excitations
in the stationary frame. Suppose that the vortex moves with velocity $v$. Then using the invariance of the BdG equation the eigenfunctions transform like

$$ u(r) \rightarrow u(r - vt)e^{imv \cdot r} $$

$$ v(r) \rightarrow v(r - vt)e^{-imv \cdot r} $$

(4.1)

The gap transforms like

$$ \Delta(r) \rightarrow \Delta(r - vt)e^{2imv \cdot r} $$

(4.2)

while the chemical potential transforms as

$$ E_F \rightarrow E_F + \frac{1}{2}mv^2 - \frac{e}{c}v \cdot A $$

(4.3)

assuming the gauge field $A$ is constant. From the usual gauge invariance there is an arbitrariness as to whether to put the extra phase in $A$ or $\Delta$. For what follows we shall take $A = 0$.

We wish to evaluate a general component $\langle \gamma^\dagger_{\nu+1} \gamma_\nu \rangle$ of the density matrix in the stationary frame. But the quasiparticle operators (for which the corresponding density matrix is taken to be diagonal) in the moving frame look like

$$ \tilde{\gamma}_\nu^\dagger = \int d^3r c_\nu^\dagger(r)u_\nu(r - vt)e^{imv \cdot r} + c_\nu(r)v_\nu(r - vt)e^{-imv \cdot r} $$

(4.4)

for $\nu > 0$ and

$$ \tilde{\gamma}_\nu^\dagger = \int d^3r - c_\nu^\dagger(r)v_\nu^\star(r - vt)e^{imv \cdot r} + c_\nu(r)u_\nu^\star(r - vt)e^{-imv \cdot r} $$

(4.5)

for $\nu < 0$. Let us consider a vortex which is moving but not displaced. That is, we take $t = 0$ in the above expressions. Also we retain only terms up to first order in the velocity. Using the definitions of Eqs. (4.4) and (4.5) and expanding the exponential we can rewrite the anti-commutator as

$$ \{ \gamma_\mu, \gamma^\dagger_\nu \} \approx \delta_{\mu\nu} + \int d^3r \psi^\dagger_\mu(r)\sigma^z(imv \cdot r)\psi_\nu(r) $$

(4.6)
Using the results of appendix, we may evaluate the matrix element in Eq. (4.6) and find
\[ \left\{ \gamma_\mu, \gamma_\nu^\dagger \right\} \approx \delta_{\mu\nu} + \frac{\hbar v k_{F\perp}}{2i(\epsilon_\mu - \epsilon_\nu)} \left( e^{i\theta_0} \delta_{\mu,\nu-1} - e^{-i\theta_0} \delta_{\mu,\nu+1} \right). \]  
\hspace{1cm} (4.7)

where \( \theta_0 \) is the angle that \( \mathbf{v} \) makes with \( \hat{x} \).

Now we are able to calculate the density matrix for a moving vortex, but in the basis of eigenstates of a stationary vortex. To linear order in the velocity we have
\[ \langle \gamma_\nu^\dagger \gamma_\nu \rangle \approx \mp [f(\epsilon_n) - f(\epsilon_{n\pm1})] \frac{\hbar v k_{F\perp} e^{\mp i\theta_0}}{2i(\epsilon_{\nu\pm1} - \epsilon_\nu)}. \] 
\hspace{1cm} (4.8)

Note the similarity in form of this expression with that of a displaced vortex, Eq. (3.3). Comparing the two we find that the corresponding displacement is \( \delta r = \hbar v / (\Delta^2 / E_F) \) at an angle \( \pi/2 \) relative to the velocity direction. The naive gap equation is not satisfied for a moving vortex! The quasiparticles are displaced relative to the center of the vortex and have a non-equilibrium distribution as seen from the lattice frame.

At this point we should note the effect of dispersion in the \( k_z \) direction on the energy denominator in Eq. (4.8). From Eq. (2.8) we see that it varies as \( \cos^{-2}\Theta \) as we change \( k_z \). Therefore the displacement discussed above is not the same for states of different \( k_z \). In fact it will be clear that the different \( k_z \) components of the vortex will not move together. We shall ignore this for now and assume a rigid motion of the vortex but this is one place where a relaxation of the two-dimensionality assumption will complicate matters. Vortex bending is another matter and will be discussed in section VI in conjunction with pinning.

V. RESPONSE TO EXTERNAL FIELD

A. VORTEX EQUATION OF MOTION

We shall be interested in the linear response of a vortex to a uniform time-varying supercurrent or electromagnetic wave. In a previous paper \hspace{1cm} (10) an equation of motion was derived for a vortex in the presence of this perturbation. The derivation was a bit cumbersome because it relied on tracking the evolution of the quasiparticle density-matrix as a function
of time for small times and then extrapolating to all times. In this paper we shall derive
the same result by an entirely different means which takes advantage of the results derived
in the previous two sections.

Consider a long wavelength electromagnetic wave incident in the \( \hat{z} \) direction, \( \mathbf{A} \), with
polarization making angle \( \theta_0 \) with the \( \hat{x} \) axis and perpendicular to the vortex line itself. The
perturbation to the Hamiltonian is

\[
- \frac{e}{mc} \mathbf{A} \cdot \mathbf{p} = - \frac{i \hbar e}{mc} A \left\{ \sin \left[ \theta - \left( \theta_0 + \frac{\pi}{2} \right) \right] \frac{\partial}{\partial r} + \cos \left[ \theta - \left( \theta_0 + \frac{\pi}{2} \right) \right] \frac{\partial}{r \partial \theta} \right\}.
\]

(5.1)

The vortex will move with some velocity \( \mathbf{v}_L \) in the presence of a background superfluid
velocity \( \mathbf{v}_S \equiv -e\mathbf{A}/mc \) resulting from the applied field. These velocities will be assumed
uniform in the \( \hat{z} \) direction (along the length of the vortex). Such an assumption would be
valid if the distance that the wave penetrates the superconductor (the smaller of the London
penetration depth and the skin depth) were long and the distance between pins were also
long compared to the coherence length.

First let us present a simple example to illustrate how we shall calculate the motion of
the vortex in this paper. The basic idea is to calculate time derivatives of the quasiparticle
density-matrix and use the results of sections III and IV to identify the corresponding
motions of the vortex.

From the appendix we find that the matrix element of Eq. (5.1) is approximately

\[
\int d^3 r \psi_{\mu}^\dagger (-\frac{e}{mc} \mathbf{A} \cdot \mathbf{p}) \psi_\mu = \frac{\hbar k_F A}{2mc} e^{\mp i(\theta_0 + \frac{\pi}{2})} \equiv W_\mp.
\]

(5.2)

If \( \mathbf{A} \) were time dependent and this were the only perturbation then we would have

\[
\frac{d}{dt} \langle \gamma_{\mu}^\dagger \gamma_{\mu-1} \rangle (t) = [f(\epsilon_\mu) - f(\epsilon_{\mu-1})] \times \left\{ -i \frac{W_\mp}{\hbar} \right\}
\]

(5.3)

and its Hermitian conjugate.

Compare this result with Eq. (3.3) and assume that there is a time dependence in \( \delta \mathbf{r}_0 \).
Making the identification \( d(\delta \mathbf{r}_0)/dt \equiv \mathbf{v}_L \) we see that \( \mathbf{v}_L = \mathbf{v}_S \). This is the well known
result, that, in the absence of dissipation or other forces the vortex moves with the same
velocity as the background superfluid. This is the expected result from Galilean invariance or from the fact that in wave mechanics, a uniform gauge field, $\mathbf{A}$, boosts the group velocity of all waves by a velocity $v_S = -(e/mc)\mathbf{A}$.

Let us now proceed with the derivation of the full equation of motion. Suppose that the incident electromagnetic field $\mathbf{A}$ is time dependent but still uniform. It will be convenient to perform a gauge/Galilean transformation which is a boost by velocity $v_S$ and eliminates $\mathbf{A}$. Under this transformation,

$$
\begin{align*}
    u(r) &\rightarrow u(r) e^{i(e/\hbar c)\mathbf{A} \cdot \mathbf{r}}, \\
    v(r) &\rightarrow v(r) e^{-i(e/\hbar c)\mathbf{A} \cdot \mathbf{r}}, \\
    \Delta(r) &\rightarrow \Delta(r) e^{2i(e/\hbar c)\mathbf{A} \cdot \mathbf{r}}.
\end{align*}
$$

The time dependence of $\mathbf{A}$ rewritten as $\mathbf{E} = \dot{\mathbf{A}}/c$ results in a perturbation $\sigma^z e\mathbf{E} \cdot \mathbf{r}$. Now we are left with $\mathbf{A} = 0$ and a vortex which has velocity $\mathbf{v}_L - \mathbf{v}_S$. The quasiparticle density matrix has off-diagonal elements

$$
\langle \gamma^{\dagger}_{\nu} \gamma_{\nu \pm 1} \rangle = \mp \left[ f(\epsilon_{\nu}) - f(\epsilon_{\nu \pm 1}) \right] \frac{|\mathbf{v}_L - \mathbf{v}_S| k_F \mp i\phi_0}{2i(\epsilon_{\nu \pm 1} - \epsilon_{\nu})}.
$$

The angle $\phi_0$ is the angle of $\mathbf{v}_L - \mathbf{v}_S$ with respect to the $\hat{x}$ axis. Here we remark that the Fermion occupation $f(\epsilon_{\nu})$ does not change to linear order in velocity. Because of the presence of a gap the energy goes quadratically with the velocity.

Now the time evolution of the off-diagonal elements of the density matrix coming from the difference $\epsilon_{\nu} \neq \epsilon_{\nu \pm 1}$ corresponds to the motion of the vortex at velocity $\mathbf{v}_L - \mathbf{v}_S$ as derived in section IV. However a moving vortex also has a relative displacement between the order parameter and the quasiparticles relative to what the gap equation for a stationary vortex would give. This displacement is

$$
\delta r'_{0} = \frac{\hbar |\mathbf{v}_L - \mathbf{v}_S|}{\epsilon_{\nu \pm 1} - \epsilon_{\nu}}
$$

in a perpendicular direction $\phi'_0 = \phi_0 - \pi/2$. Thus the displaced vortex itself will affect the quasiparticle density matrix through a perturbation $\delta \Delta = -\delta \mathbf{r}'_{0} \cdot \nabla \Delta$. Its matrix element is
\[ W_{\mu\nu} = \int d^3r \, \psi_{\mu}^\dagger(r) \begin{pmatrix} 0 & \delta\Delta \\ \delta\Delta^* & 0 \end{pmatrix} \psi_{\nu}(r) \] (5.7)

In the appendix we show that it has the value

\[ W_{\mu\nu} = -\frac{\delta r_0}{2i}k_{F\perp} \delta_{\mu,\nu+1}(\epsilon_{\nu} - \epsilon_{\mu})e^{\pm i(\phi_0^0 + \frac{\pi}{2})}. \] (5.8)

The contribution of the relative displacement \( \delta r_0 \) to the evolution of the density matrix is

\[ \frac{d}{dt} \langle \gamma_{\mu}^\dagger \gamma_{\mu-1} \rangle = [f(\epsilon_{\mu}) - f(\epsilon_{\mu-1})] \times \left\{ -iW_{\mu-1,\mu} \right\} = \frac{1}{2i} |v_L - v_S| k_{F\perp} e^{i\phi_0}. \] (5.9)

Comparing with Eq. (4.8) which relates velocity to off-diagonal elements of the density matrix we may interpret the results in terms of an acceleration

\[ a = |v_L - v_S| (\epsilon_{\nu+1} - \epsilon_{\nu})/\hbar \] (5.10)

at an angle \( \phi_0 - \pi/2 \), or, in other words,

\[ a = \frac{\Delta^2}{\hbar E_F} (v_L - v_S) \times \hat{z}. \] (5.11)

As for the electric field we require the matrix element (using the results from the appendix)

\[ \int \psi_{\mu}^\dagger e \mathbf{E} \cdot \mathbf{r} \sigma^z \psi_{\nu} \approx \frac{eE k_{F\perp}}{2m(\epsilon_{\nu+1} - \epsilon_{\nu})} \left( e^{i\phi_0} \delta_{\mu,\nu-1} - e^{-i\phi_0} \delta_{\mu,\nu+1} \right), \] (5.12)

where \( \phi_0 \) is now the angle of \( \mathbf{E} \). From the electric field then we have a contribution

\[ \frac{d}{dt} \langle \gamma_{\mu}^\dagger \gamma_{\mu} - 1 \rangle = [f(\epsilon_{\mu}) - f(\epsilon_{\mu-1})] \times \frac{eE k_{F\perp}}{2im\hbar(\epsilon_{\nu} - \epsilon_{\nu-1})} e^{i\phi_0}. \] (5.13)

The corresponding acceleration, by comparing with Eq. (1.8) is

\[ a = -\frac{eE}{m} = \mathbf{v}_S \] (5.14)

The last influence on the vortex motion we shall consider in detail is dissipation. Suppose that we are in the lattice rest frame and looking at vortex moving with velocity \( v_L \). We
see that there is apparently a non-equilibrium off-diagonal component whose magnitude is proportional to $v_L$ and whose phase gives the direction of $v_L$. Let us assume that, due to scattering from things that break translation invariance such as impurities or phonons, this off-diagonal component has a lifetime $\tau$ over which it decays and re-appears as a thermalized (diagonal), isotropic contribution to the density matrix (i.e. not carrying current). In other words we assume that the core can maintain thermal equilibrium with the lattice unlike, say, the case of $^3$He in which the cores of vortices can heat up. The parameter $\tau$ which will appear in the vortex equation of motion in the drag term and in the conductivity has a clear microscopic meaning. It is the lifetime of low energy quasiparticle states.

This theory of the dissipation then gives an additional component to the time derivative of the vortex velocity,

$$\dot{v}_L = -\frac{1}{\tau}v_L.$$  \hspace{1cm} (5.15)

Putting all the contributions together we have the final equation of motion,

$$\dot{v}_L = \dot{v}_S + \frac{\Delta_0^2}{\hbar E_F}(v_L - v_S) \times \hat{z} - \frac{1}{\tau}v_L.$$  \hspace{1cm} (5.16)

It is useful to compare the present expression with that of Nozieres-Vinen [4]. Their equation is derived by balancing forces. There is the ‘magnus’ force $(\hbar n/2)(v_S - v_L) \times \hat{z}$ where $n$ is the (superfluid) number density of electrons. Comparing this to the corresponding coefficient in the present equation of motion, taking $n = k_{F\perp}^2/2\pi$ and the in-plane coherence length $\xi_{\perp} = \hbar v_{F\perp}/\pi\Delta$, one may extract a ‘mass’ of the vortex $M \sim m_{\perp} (k_{F\perp}\xi_{\perp})^2/4$ per unit length in the $\hat{z}$ direction. This expression is perhaps a microscopic justification for the concept of a ‘normal core’ of size $\xi$, even at very low temperatures when there is a gap in the density of states of single particles. This is should be contrasted with another definition of the mass of a vortex as recently discussed by Duan and Legget [21]. This definition is based upon the change in energy of a moving vortex at order $v^2$ which has been ignored in this paper.

Let us pause and summarize the situation. We began by asking whether the application of a long-wavelength electromagnetic field could cause dipole transitions between the
discrete quasiparticle states in the cores of vortices. Bearing in mind the approximation of rigid vortex motion and other assumptions we find that an applied electric field, instead of causing dipole transitions, causes the density matrix to evolve off-diagonal matrix elements corresponding to vortex motion itself (after applying the gap equation). The vortex does not stand still and allow a dipole transition to take place, as the core of an atom. The difference is that the vortex is a self-consistent potential. Thus the experimental observation of such quasiparticle resonances will not be a simple dipole resonance (except in the presence of pins which will be discussed in section VI). The STM tunneling experiment differs from the present case in that the tunneling process introduces an extra particle which is not correlated with the quasiparticles in the vortex whilst the effect of an applied electric field is to change the off-diagonal components of the density matrix.

A simple understanding of the equation of motion comes from looking at the homogeneous solution. It can be written

$$
\begin{pmatrix}
v_{Lx} \\
v_{Ly}
\end{pmatrix}
= e^{-t/\tau}
\left[
\begin{array}{c}
a_+ e^{i\Omega_0 t} \\
a_- e^{-i\Omega_0 t}
\end{array}
\right]
\begin{pmatrix}
1 \\ i
\end{pmatrix}
+ a_+ e^{i\Omega_0 t} \\
a_- e^{-i\Omega_0 t}
\begin{pmatrix}
1 \\
-i
\end{pmatrix}
\right)

(5.17)
$$

which corresponds to moving in circles, with a definite sense of rotation, at a frequency \(\Omega_0 \equiv \Delta^2/E_F\). This is the sort of state one hopes to excite with the external probe. Therefore experimentally it would be important to probe this system with polarized waves. In the next subsection we discuss what happens when we do so.

**B. DISSIPATION IN A SINGLE VORTEX**

It is very illuminating to calculate the dissipation in a single vortex as a function of frequency and polarization for constant magnitude \(v_S\). In the next subsection we shall calculate the conductivity for a finite density of vortices. First it is necessary to extract the paramagnetic current carried by the vortex core. The paramagnetic current is

$$
J(r) = \sum_\sigma c_\sigma(r)^\dagger c_\sigma(r)
$$

(5.18)
Substituting for the electron operators using the inverse transformation Eq. (2.6), and using the results of the appendix to evaluate matrix elements of \( \nabla \) we find that

\[
\int \hat{n} \cdot \mathbf{J} = \frac{i k_{F\perp}}{2m} \sum_{\nu} \left( \gamma_{\nu+1}^\dagger \gamma_{\nu} e^{-i \phi_0} - \gamma_{\nu}^\dagger \gamma_{\nu+1} e^{i \phi_0} \right),
\]

where \( \hat{n} \) is some arbitrary unit vector pointing in direction \( \phi_0 \).

Now we may calculate the (spatially averaged or zero momentum) paramagnetic current associated with a vortex motion by comparing Eqs. (5.19) and (4.8). It is

\[
\langle \int \frac{\mathbf{v}_L}{v_L} \cdot \mathbf{J(r)} \rangle = \frac{v_L \hbar^2 k_{F\perp}^2}{2m(\Delta^2/E_F)}
\]

per unit length in the \( \hat{z} \) direction (for a non-cylindrical Fermi surface we would average over \( k_z \)).

Let us calculate now the dissipation as a function of frequency for a single vortex. For simplicity we shall keep the amplitude \( v_S \) constant but we shall allow it to have an arbitrary polarization. The steady state solution of Eq. (5.16) for \( v_S(t) = v_S(0) \exp(i \omega t) \) is

\[
v_{Lx} - v_{Sy} = \left[ \frac{\Omega_0 \tau v_S x - (1 + i \omega \tau) v_S y}{(1 + i \omega \tau)^2 + (\Omega_0 \tau)^2} \right],
\]

and another equation with \( x,y \) interchanged and \( \Omega_0 \rightarrow -\Omega_0 \).

Eq. (5.16) at zero frequency was introduced by deGennes and Matricon [22]. This had the drawback of not allowing for small conductivities seen in flux flow experiments [23] and was discarded by Nozieres and Vinen in favour of an equation of motion where the dissipation acts on \( v_S \) rather than \( v_L \). Nevertheless the solution Eq. (5.21) agrees with the Nozieres-Vinen equation [4] at low frequency and low dissipation. It is probably not valid except in the clean limit where the levels are well defined. Moreover, because of the the frequency dependent term there is a ‘resonance’ at \( \omega \approx \Omega_0 \) with a width \( \tau^{-1} \). If one simply added a \( \dot{v}_S \) term to the Nozieres-Vinen equation one would obtain an unphysical divergence at that frequency. In addition, even though Eq. (5.19) was not derived in the large dissipation limit, that limit makes sense for the present equation of motion. The vortex simply stops moving. The Nozieres-Vinen equation and its relatives have divergent behaviour because in
these models the current through the vortex core is forced to be equal to the background transport velocity $v_S$.

There are two contributions to the dissipation which come from a current (either the background superfluid or that due to vortex motion) that is in phase with an electric field (time derivative of the supercurrent or transverse motion of the vortex). The first is transverse vortex motion in phase with the supercurrent. For clarity let $v_{Sy} = v_{Sx} \exp i \theta = v_S / \sqrt{2}$. The parameter $\theta$ simply controls the polarization. There is an induced voltage per vortex $\hat{z} \times \vec{v}_L (\hbar / 2e)$. The supercurrent density $v_S ne$ gives dissipation

$$-N_v (h n / 2) \text{Re}(v_{Lx}^* v_{Sy} - v_{Ly}^* v_{Sx}),$$

where $N_v$ is the vortex density. Using Eq. (5.21)

$$\text{Re}(v_{Lx}^* v_{Sy} - v_{Ly}^* v_{Sx}) = -v_S^2 \Omega_0 \tau \frac{1-(\omega \tau)^2+(\Omega_0 \tau)^2}{[1-(\omega \tau)^2+(\Omega_0 \tau)^2]^2+4(\omega \tau)^2} + \sin \theta v_S^2 \omega \tau \frac{1+(\omega \tau)^2-(\Omega_0 \tau)^2}{[1-(\omega \tau)^2+(\Omega_0 \tau)^2]^2+4(\omega \tau)^2}.$$

(5.22)

This first term contributes at low frequencies (relative to $\Delta^2 / E_F$) and corresponds to the usual phase slip dissipation mechanism for flux flow.

The second source is the current due to vortex motion which is in phase and parallel with the applied electric field. A straightforward calculation gives the average current density due to vortex motion to be $2v_L (E_F / \Delta)^2 N_v e$ per unit length in the $\hat{z}$ direction. With the electric field $\mathcal{E} = (m / e) \dot{\vec{v}}_S$ the dissipation is $2m (E_F / \Delta)^2 N_v \text{Re}(v_{Ly}^* i \omega v_{Sy} + v_{Lx}^* i \omega v_{Sx})$. Using Eq. (5.21), we obtain

$$\text{Re}(v_{Ly}^* i \omega v_{Sy} + v_{Lx}^* i \omega v_{Sx}) = v_S^2 \omega^2 \tau \frac{1+(\omega \tau)^2-(\Omega_0 \tau)^2}{[1-(\omega \tau)^2+(\Omega_0 \tau)^2]^2+4(\omega \tau)^2} - \sin \theta v_S^2 \Omega_0 \omega \tau \frac{1-(\omega \tau)^2+(\Omega_0 \tau)^2}{[1-(\omega \tau)^2+(\Omega_0 \tau)^2]^2+4(\omega \tau)^2}.$$

(5.23)

This second part contributes mostly to high frequencies. It may be understood better by letting $\Omega_0 \to 0$. In that case we recover a Drude-like expression $1/[1 + (\omega \tau)^2]$ multiplied by the volume of the vortex cores. This analogous to the well known zero frequency result for flux flow which is that as $H \to H_{c2}$, the resistance in the vortex cores multiplied by the core volume matches the resistance for the normal state above $H_{c2}$.
For clarity and in order to add together these two terms we shall make the oversimplified but well-defined choice that superfluid density be equal to the density of all electrons, \( n = \frac{k_{F,\perp}^2}{2\pi} \) (again, per unit length in the \( \hat{z} \) direction) and \( E_F = \frac{\hbar^2 k_{F,\perp}^2}{2m} \) independent of \( k_z \).

With this definite choice we may then plot the dissipation. In Fig. (1) we have plotted the dissipation for right and left circularly polarized \( \mathbf{v}_S \). For one of the polarizations we see dissipation but no special behaviour near the characteristic frequency of about \( \Omega_0 \) (depending on the magnitude of \( \tau \)). In the other polarization the vortex does respond at the characteristic frequency but essentially as an anti-resonance. There is a minimum in dissipation.

\section*{C. Conductivity}

In this subsection we calculate the experimentally accessible conductivity at momentum \( q = 0 \). The usual straightforward way of calculating the conductivity would be to take the gauge invariant current-current correlation operator, re-express it in terms of Bogoliubov operators using Eq. (2.6) (allowing for time dependence of the fermion operators) and evaluate its expectation value. As usual the (time) Fourier transform of this object will have poles at the excitation frequencies of the system. The lowest quasi-particle-quasi-hole excitation would contribute a pole ostensibly around \( \Omega_0 \). That would lead to a contribution proportional to \( \delta(\omega - \Omega_0) \) in the real part of the conductivity.

However that is not the whole story because of the ‘residual’ or ‘final-state’ interaction between these quasiparticles. It can modify the energy from what one expects given the effective single-quasiparticle energy spectrum. Very generally, when symmetry breaking and long-range order occurs in a Fermionic system resulting in an energy gap for single-particle excitations these residual interactions can mix particle-hole excitations with the collective (Goldstone) mode by creating a bound state in the particle-hole channel. Another way to look at this is to say that while a mean field or Hartree order parameter is constructed so that \textit{single} particle excitations have positive energy and do not scatter, when a particle \textit{and}
a hole are present the change in the order parameter due to the hole affects the particle and vice versa.

Since the position of a vortex is arbitrary (the real magnetic field that creates it is arbitrarily uniform), the overall energy of a vortex should be unchanged if it is translated by some amount. If this amount is small compared to the coherence length (and this is certainly the case in the experimental situation), then it can be described by one quasiparticle-quasihole pair as given in section [1]. Thus one expects from translational invariance that the residual interaction will reduce the quasiparticle-quasihole excitation energy to nearly zero.

Unfortunately, an explicit demonstration of final-state effects is difficult because in order to get precisely zero energy one must possess precise self-consistent order parameter and single-particle states. This calculation would require detailed numerical work even for a simple zero-range pairing interaction and would be hopeless for any realistic model. Nevertheless, let us show, analytically, in a simple model, that the correction to the excitation energy is of order \(-\Delta^2/(E_F\ln[2\omega_c/\Delta])\). We begin with a simple zero-range pairing interaction

\[-V \int dr c_\uparrow(r)c_\uparrow(r)c_\downarrow(r)c_\downarrow(r).\]

In this case the residual interaction is

\[-V \int dr \left(c_\uparrow(r)c_\uparrow(r) - \Delta^*(r)\right)\left(c_\downarrow(r)c_\downarrow(r) - \Delta(r)\right).\]

This expression, converted to Bogoliubov operators, is diagonal for the simplest \(\mu = -1/2 \rightarrow +1/2\) excitation and has the expectation value

\[-V \int dr \left(|u_{1/2}|^2 + |v_{1/2}|^2\right)^2.\]

Now using that fact that u and v have spatial extent \(\xi \sim \hbar v_F/\Delta\) and the simple BCS expression \(1 = VN_0\arcsinh(\omega_c/\Delta)\) we arrive at the above estimate. The main conclusion is that, in the vortex core, it is possible for the residual interaction to give negative corrections of the same order of magnitude as the ‘bare’ excitation energy. In charged superconductors the
collective mode is ‘plasmonized’ by the coulomb repulsion. That is, there is a large positive correction to the energy. However this probably does not happen to the quasiparticle-quasihole excitation in a vortex because the corresponding translation of the vortex should not create a charge density fluctuation.

In this paper we follow an alternative route and explicitly allow the order parameter to be dynamic, self-consistent and to affect the time evolution of the quasiparticles. It is, in a sense, a conserving approximation because it respects the translational invariance of the problem.

To begin the calculation of conductivity we enumerate the sources of electric field and current density and consider only the uniform $q = 0$ component. The electric field due to the time derivative of the background supercurrent is $\mathbf{E} = (i\omega m/e)\mathbf{v}_S$. There is also the electric field due to transverse vortex motion, $N_v(h/2e)\hat{z} \times \mathbf{v}_L$, where $N_v$ is the areal vortex density. The trivial piece of the current density is simply the supercurrent $n_se\mathbf{v}_S$. Then there is the $q = 0$ component of the current density due to vortex motion, from Eq. (5.20),

$$J_{q=0} = N_v \frac{e^2 h^2 k_{F\perp}^2 (\mathbf{v}_L - \mathbf{v}_S)}{2m(\Delta^2/E_F^2)}.$$  

(5.27)

We have added the term $-\mathbf{v}_S$ so that the total current is $n_se\mathbf{v}_S$ when $\mathbf{v}_L = \mathbf{v}_S$. The next step is to write both the total electric field and current density in terms of $\mathbf{v}_S$ using the vortex equation of motion Eq. (5.21). Then the conductivity tensor, $\sigma_{ab}$, may be found by comparing those two expressions and using the definition $J_a = \sigma_{ab}E_b$. The final result for the conductivity tensor is rather complicated. To simplify it we again take $n_s = k_{F\perp}^2/2\pi$, $E_F = h^2 k_{F\perp}^2/2m$ and define $\Phi = N_v h/2m\Omega_0$ ($\Phi$ is roughly $H/H_{c2}$). The longitudinal and Hall conductivities are

$$\sigma_{xx} = \mathcal{F}\left[\left(i\frac{\omega}{\Omega_0} + \frac{\Omega_0 \tau}{D} \Phi\right) \left(1 - \Phi \frac{1 + i\omega \tau}{D}\right) - \Phi^2 \frac{\Omega_0 \tau}{D} \left(1 - \frac{1 + i\omega \tau}{D}\right)\right]$$  

(5.28)

and

$$\sigma_{xy} = \mathcal{F}\left[\Phi \left(1 - \frac{1 + i\omega \tau}{D}\right) \left(1 - \Phi \frac{1 + i\omega \tau}{D}\right) + \Phi \frac{\Omega_0 \tau}{D} \left(\frac{i\omega}{\Omega_0} + \frac{\Omega_0 \tau}{D} \Phi\right)\right]$$  

(5.29)
where $D \equiv (1 + i\omega\tau)^2 + \Omega_0^2\tau^2$ and

\[
\mathcal{F} = \frac{ne^2}{m\Omega_0} \left[ \left( \frac{i\omega}{\Omega_0} + \frac{\Omega_0\tau}{D} \Phi \right)^2 + \Phi^2 \left( 1 - \frac{1 + i\omega\tau}{D} \right)^2 \right]^{-1}.
\]

(5.30)

We are interested in the term in the conductivity proportional to $\Phi$ or the number of vortices when $\Phi$ is small. However some care has to be taken with that limit because in the low frequency limit the resistivity is proportional to $\Phi$. Now, if we throw away terms of order $\Phi^2$ we have

\[
\sigma_{xx} \sim \frac{ne^2}{m\Omega_0} \frac{1 - \Phi\frac{1 + i\omega\tau}{D}}{i\omega\Omega_0 + \frac{\Omega_0\tau}{D}\Phi}.
\]

(5.31)

Note that in the $\omega \to 0$ limit this has the correct value $\Phi^{-1} ne^2 \tau / m$. In order to bring out the structure, we take the $\tau \to \infty$ limit and obtain

\[
\sigma_{xx} \sim \frac{ne^2}{im\omega} \left[ 1 - \frac{i\omega^2\Phi}{\omega\tau(\Omega_0^2 - \omega^2)} \right] \left[ 1 - \frac{i\Omega_0^2\Phi}{\omega\tau(\Omega_0^2 - \omega^2)} \right]^{-1}.
\]

(5.32)

At ‘resonance’, $\omega \sim \Omega_0$, the longitudinal conductivity becomes, in this limit, completely imaginary. This agrees with the results of the previous subsection.

Similarly we may look analytically at $\text{Im}\sigma_{xy}$ in the small $\Phi$ limit. $\text{Im}\sigma_{xy}$ gives the polarization dependent dissipation. In this limit it has the value

\[
\text{Im}\sigma_{xy} \sim \frac{2ne^2\Phi\Omega_0\tau}{m\omega} \left[ (1 - \omega^2\tau^2 + \Omega_0^2\tau^2)^2 + (4\omega\tau)^2 \right]^{-1}.
\]

(5.33)

This expression has a pole-like feature at $\omega \approx \Omega_0$ but it is weak because its value at the maximum goes as $1/\tau$ and therefore doesn’t sharpen as $\tau$ increases.

In Fig. (2) we plot the real and imaginary parts of the longitudinal and Hall conductivities as a function of frequency without the above approximations. They have been multiplied by $\omega$ or $\omega^2$ in order to bring out their behaviour near $\omega \sim \Omega_0$. In general the conductivity at the characteristic frequency does not show dramatic behaviour. What is important is that $\text{Re}\sigma_{xx}$ does not show the expected peak from a naive dipole resonance and $\text{Im}\sigma_{xy}$ does not show a strong peak as expected from a polarization sensitive dipole resonance. There is structure at low frequencies which depends on the value of $\Phi$ but is not important to
the present discussion. It is related to the problem with taking the low frequency limit. At some point the resistivity becomes proportional to $\Phi$ instead of the conductivity. It is also the point at which the electric field due to transverse vortex motion is comparable to that of the time derivative of the gauge field.

VI. PINNING

As mentioned in the introduction, in most practical cases pinning comes into play. In this section we introduce the effects of pinning phenomenologically and show that a dipole-like resonance is recovered. First let us return to the homogeneous solution of the equation of motion Eq. (5.17). It should be possible to cause these modes to absorb energy, however the equation of motion we derived apparently does not allow it to happen resonantly. It would seem that one requires a perturbation to 'bump' the system into one of these excited states. That will be the purpose of the pinning center.

The main feature of pinning sites or bending of vortices is that there is a non-translation-invariant restoring force present. Thus we propose to study the effect of modifying, in a very general way, the equation of motion, Eq. (5.16) into

$$\dot{v}_L = \dot{v}_S + \Omega_0(v_L - v_S) \times \hat{z} - \frac{1}{\tau} v_L - \alpha^2 r_0.$$  

(6.1)

Here $r_0$ is the position of the vortex and $\alpha$ is the characteristic frequency of a harmonic well we have introduced at the origin.

Solving this equation (and keeping only terms to order $1/\tau$) we find the steady state solution to be

$$\begin{pmatrix} v_{Lx} \\ v_{Ly} \end{pmatrix} = i\omega \begin{pmatrix} (\alpha^2 - \omega^2)^2 + \frac{2i\omega}{\tau} (\alpha^2 - \omega^2) - \omega^2 \Omega_0^2 \end{pmatrix}^{-1} \begin{pmatrix} v_Sx \\ v_Sy \end{pmatrix}$$  

(6.2)

$$\times \begin{pmatrix} i\omega (\alpha^2 + \Omega_0^2 - \omega^2 + i\omega/\tau) & -\Omega_0 (\alpha^2 + i\omega/\tau) \\ \Omega_0 (\alpha^2 + i\omega/\tau) & i\omega (\alpha^2 + \Omega_0^2 - \omega^2 + i\omega/\tau) \end{pmatrix} \begin{pmatrix} v_Sx \\ v_Sy \end{pmatrix}.$$  

(6.3)
Let us now also take $\alpha \ll \Omega_0$. In that case the quantity in square brackets gives a resonance at $\omega \approx \Omega_0 + \alpha^2/\Omega_0$. The term most important at resonance comes from the imaginary part of the square bracket. Let us evaluate Eq. (6.3) close to resonance by setting $\omega = \Omega_0$ in other parts of the expression to obtain

$$\begin{pmatrix} v_{Lx} \\ v_{Ly} \end{pmatrix} = \frac{\alpha^2 \tau/(2\Omega_0)}{1 + (\omega - \Omega_0)^2\tau^2} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_{Sx} \\ v_{Sy} \end{pmatrix}. \tag{6.4}$$

The complete expression for the conductivity is messy and perhaps not worth writing down for this simple model. However in the spirit of subsection V B we may look at the dissipation in a single vortex. Setting $v_{Sy} = v_{Sx} \exp i\theta = v_s/\sqrt{2}$ we may repeat the calculations leading to Eq. (5.22) and Eq. (5.23) and obtain

$$\text{Re}(v_{Lx}^* v_{Sy} - v_{Ly}^* v_{Sx}) = f (1 + \sin \theta) v_S^2 \tag{6.5}$$

and

$$\text{Re}(v_{Ly}^* i\omega v_{Sy} + v_{Lx}^* i\omega v_{Sx}) = -\omega f (1 + \sin \theta) v_S^2 \tag{6.6}$$

where

$$f = \frac{\alpha^2 \tau/(2\Omega_0)}{1 + (\omega - \Omega_0)^2\tau^2} \tag{6.7}$$

gives the resonance structure which corresponds to the usual dipole resonance. Note that $f$ is proportional to $\alpha^2$. The dipole absorption is proportional to the strength of the translation-invariance-breaking perturbation. This perturbation increases somewhat the resonant frequency but need not dominate. The resonance is also fully polarization dependent and turns off when $\sin \theta = -1$, or, at the correct choice of handedness of the circular polarization.

**VII. DISCUSSION**

One of the main conclusions of this work, that in a pure system the single-quasiparticle properties of a vortex are invisible to a long wavelenth probe, is very reminiscent of the result
of Mattis and Bardeen [25] regarding the long wavelength electromagnetic response near the bulk gap frequency. For vortices, if $\tau \to \infty$ we have $v_L = v_S$ and there is no dissipation. Mattis and Bardeen found that, in the bulk, the conductivity in the long wavelength limit $\sigma(q, \omega \sim 2\Delta) \to 0$ as $q \to 0$ unless there are impurities present to allow the violation of momentum conservation. There is a basic picture which is shared by these two situations. A quasi-particle-quasi-hole excitation can look like a translation of the condensate as a whole and therefore the system responds differently from what one expects.

In this paper we have calculated various quantities without the need to choose an appropriate ‘size’ for the core. In previous works [3,4] this was required in order to fix the coefficient of the ‘Lorentz’ or ‘magnus’ term. The core size was chosen so that at $H = H_{c2}$ the core volume would be the total volume of the system. Here there is no particular reason to extrapolate to the superconducting phase boundary.

The low temperature limit studied here may present some simplifications. A good definition of the ‘core’ of the vortex would be those states closest to zero energy which move at velocity $v_L$. Some intermediate states would have to be adjusted some other way to match the bulk velocity $v_S$ and conserve current. They will be states further away from the chemical potential and at low temperature perhaps not important for calculating dissipation. This point is discussed further in the appendix. It is not clear at present exactly how crucial it is. The equation of motion of Bardeen-Stephen is derived assuming that the full transport current $v_S$ flows through the vortex core and this core is necessarily defined by some surface or boundary region. From the low temperature microscopic point of view the full transport current does not flow through the core. That is because it is unfavourable to pay the discrete amount of energy required to make charge fluctuations in the core. Without these charge fluctuations the paramagnetic current in the core is determined by the velocity of the core because there aren’t any other states available in the vortex core. At higher temperatures, it is favourable to allow charge fluctuations in the core and thus it is possible to increase the current flowing there.

Future work should concentrate on detailing the microscopic theory of pinning and vortex
bending since that would be most important for the experiments. It would be very nice if the de-pinning frequency correlation presented in table I could be explained. It would also be useful to work out the quasiparticle structure of a vortex in a d-wave superconductor. That involves a non-local BdG equation. One could also check how the discreteness of the lattice affects the microscopic structure. In general, calculations of the microscopic structure will be sensitive to the nature of the pairing. Thus knowledge of the microscopic structure of vortices may give us some insight into the mechanism of high temperature superconductivity.

VIII. CONCLUSION

In this paper we have studied the microscopic dynamics of vortices as a function of frequency. We have derived an equation of motion and calculated the electrodynamic response at \( q = 0 \). For free vortices we find that the single-particle character is essentially invisible unless translation invariance is broken. Thus the clear, albeit broad, response in optical transmission of Karrai et al. is perhaps due to pinned vortices. Another interesting possibility is that interaction with the discrete lattice is responsible. The possibly d-wave nature of the paired state in high temperature superconductors should not affect this conclusion. We believe that the large width observed experimentally may come from intrinsic scattering from whatever breaks translation invariance.

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MATRIX ELEMENTS

Throughout this paper we use matrix elements of the quasiparticle wavefunctions. They are difficult to calculate because the detailed behaviour of the wavefunctions depends non-
trivially on the radial dependence of $|\Delta(r)|$ which in turn is determined self-consistently. In this appendix we derive approximate matrix elements of $\mathbf{r}$ and $\nabla$ between the low energy quasiparticle states. The approximation we make is that the main contribution to these matrix elements comes from close to the center of the vortex where the functions $u(\mathbf{r})$ and $v(\mathbf{r})$ can be approximated by Bessel functions and the order parameter $\Delta(\mathbf{r})$ is small. It is assumed that matrix elements of the low energy states are not very sensitive to the exact behaviour of the wavefunctions near the core boundary because the functions $u$ and $v$ are suppressed exponentially when the magnitude $|\Delta(\mathbf{r})|$ becomes substantial. Boundary conditions have entered implicitly through the energies of the quasiparticle levels.

We shall make use of the Bessel function identities

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_{\nu}(z),$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2\frac{d}{dz}J_{\nu}(z).$$

(1)

and suppose that $u(\mathbf{r})$ and $v(\mathbf{r})$ satisfy similar identities at least where they contribute most to the matrix element. To be specific, in the calculation of matrix elements of $\hat{\mathbf{n}} \cdot \nabla = \cos(\theta - \theta_0)\frac{\partial}{\partial r} - \sin(\theta - \theta_0)\frac{\partial}{r \partial \theta}$

we use

$$\frac{\partial}{\partial r}u_{\mu} = \frac{k_{F\perp}}{2} (e^{i\theta}u_{\mu-1} - e^{-i\theta}u_{\mu+1})$$

$$\frac{1}{r}\frac{1}{\partial \theta}u_{\mu} = -\frac{k_{F\perp}}{2i} (e^{i\theta}u_{\mu-1} + e^{-i\theta}u_{\mu+1})$$

(2)

and exactly the same equations with $u$ replaced by $v$. $\theta$ is the polar angle. The matrix elements of

$$\hat{\mathbf{n}} \cdot \nabla = \cos(\theta - \theta_0)\frac{\partial}{\partial r} - \sin(\theta - \theta_0)\frac{\partial}{r \partial \theta}$$

(3)

where $\hat{n}$ is a unit vector pointing at angle $\theta_0$ to the x-axis, are thus

$$\int d^3\mathbf{r} \psi_{\mu \pm 1}^\dagger \hat{\mathbf{n}} \cdot \nabla \psi_{\mu} = \frac{k_{F\perp}}{2} e^{\mp i(\theta_0 + (\pi/2))}$$

(4)

and zero for states differing by other angular momenta. The lack of dependence on $\mu$ is a result of the approximations we have made.
Another approximation we have made is to only consider matrix elements between low energy states. We consider in this paper mostly operators that contain angular momentum \( l = \pm 1 \) and therefore amongst the low energy states it is clear which matrix elements are non-zero. The approximation made in Eq. (2) does not affect that. There are, however, high energy scattering states (of energy \( \Delta \) or higher) which have non-zero matrix elements. These have been ignored. It is not clear whether this is a safe approximation or not.

Now let us consider the matrix elements of \( \sigma^z \hat{n} \cdot \mathbf{r} \). To do this we consider the commutator \([H, \sigma^z \hat{n} \cdot \mathbf{r}]\) where \( H \) is the BdG Hamiltonian with \( A = 0 \). We have

\[
\int \psi_\mu^\dagger \sigma^z \hat{n} \cdot \mathbf{r} \psi_\nu = \frac{1}{\epsilon_\mu - \epsilon_\nu} \int \psi_\mu^\dagger [H, \sigma^z \hat{n} \cdot \mathbf{r}] \psi_\nu \\
= \frac{1}{\epsilon_\mu - \epsilon_\nu} \int \psi_\mu^\dagger \left[ -\frac{i}{m} \hat{n} \cdot \mathbf{p} + 2 \hat{n} \begin{pmatrix} 0 & -\Delta \\ \Delta^* & 0 \end{pmatrix} \right] \psi_\nu
\]

The second term in the square brackets is a smaller contribution because whenever \( u \) and \( v \) are substantial, \(|\Delta|\) is small so in our approximation we shall ignore it. We are then left with the matrix element of the momentum operator which we have just evaluated above. The matrix element is

\[
-\frac{\hbar k F_{\perp}}{2m(\epsilon_\mu - \epsilon_\nu)} \left( e^{i\theta_0} \delta_{\mu,\nu-1} - e^{-i\theta_0} \delta_{\mu,\nu+1} \right). \tag{6}
\]

In section \([\text{V}]\) we encounter the matrix element

\[
W_{\mu\nu} = \int d^3 \mathbf{r} \psi_\mu^\dagger(\mathbf{r}) \begin{pmatrix} 0 & \delta \Delta \\ \delta \Delta^* & 0 \end{pmatrix} \psi_\nu(\mathbf{r}) \tag{7}
\]

where \( \delta \Delta = -\delta \mathbf{r}_0' \cdot \nabla \Delta \). It may be found by utilizing translation invariance if \( \mu \neq \nu \). Let \( \delta \psi_\nu(\mathbf{r}) = -\delta \mathbf{r}_0' \cdot \nabla \psi_\nu(\mathbf{r}) \) be the change in the eigenfunctions upon displacing them by an amount \( \delta \mathbf{r}_0' \). Then substituting \( \delta \psi_\nu \) and \( \delta \Delta \) into the Schrödinger equation, Eq. (2.2), subtracting the same equation with \( \delta \mathbf{r}_0' = 0 \), keeping terms to first order in \( \delta \mathbf{r}_0' \), multiplying on the left by \( \psi_\mu(\mathbf{r}) \) and integrating by parts results in

\[
W_{\mu\nu} = (\epsilon_\nu - \epsilon_\mu) \int d^3 \mathbf{r} \psi_\mu^\dagger(\mathbf{r}) \delta \psi_\nu(\mathbf{r}) \tag{8}
\]
This can be further simplified making the assumption again that $u(r)$ and $v(r)$ resemble Bessel functions in the relevant spatial region yielding

$$
\delta r'_0 \cdot \nabla \psi_\mu(r'_0) = \frac{\delta r k_{F\perp}}{2i} \left[ e^{i(\phi'_0 + \frac{\pi}{2})} \psi_{\mu-1}(r) + e^{-i(\phi'_0 + \frac{\pi}{2})} \psi_{\mu+1}(r) \right]
$$

(9)

where $\delta r$ makes an angle $\phi$ with the $\hat{x}$ axis. In terms of the displacement $\delta r'_0$ the matrix element is,

$$
W_{\mu\nu} = -\frac{\delta r'_0 k_{F\perp}}{2i} \delta_{\mu,\nu \mp 1} (\epsilon_\nu - \epsilon_\mu) e^{\pm i(\phi'_0 + \frac{\pi}{2})}.
$$

(10)
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FIGURES

FIG. 1. (a) Dissipation (normalized to the high frequency limit) as a function of frequency (normalized to $\Omega_0$) when $\Omega_0\tau = 1$. Solid line: circular polarization of $v_S$; dashed line: opposite sense of circular polarization. (b) The same quantities but for $\Omega_0\tau = 3$.

FIG. 2. (a) Real part of the longitudinal conductivity multiplied by frequency $\omega$ (for clarity), $\text{Re} \omega \sigma^{xx}$, as a function of frequency (normalized to $\Omega_0$). Solid line: $\Phi = 0.1$; dashed line: $\Phi = 0.2$. (b) Same plots for the imaginary part multiplied by $\omega^2$ for clarity, $\text{Im} \omega^2 \sigma^{xx}$. (c) Real part of the Hall conductivity, $\text{Re} \omega^2 \sigma^{xy}$. (d) Imaginary part, $\text{Im} \omega^2 \sigma^{xy}$.
TABLE I. Observed crossover frequencies for vortex de-pinning and quasiparticle energy level separations in vortex cores ($\Delta^2/E_F$, where $\Delta$ is the bulk gap and $E_F$ is the Fermi energy) for various samples of low temperature superconductors.

| Sample   | $T_c$ (K) | Observed frequency (MHz) | Energy level separation (MHz) |
|----------|-----------|--------------------------|-------------------------------|
| PbIn     | 1.7       | 4                        | 2                             |
| PbIn     | 1.7       | 5                        | 2                             |
| NbTa     | 4.2       | 15                       | 20                            |
| PbIn     | 1.7       | 7                        | 2                             |
| PbIn     | 1.7       | 8                        | 2                             |
| NbTa     | 4.2       | 26                       | 20                            |
| NbSe$_2$ | 7.2       | 100                      | 1000                          |