Synchronization and semistability analysis of the Kuramoto model of coupled nonlinear oscillators

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Abstract—An interesting problem in synchronization is the study of coupled oscillators, wherein oscillators with different natural frequencies synchronize to a common frequency and equilibrium phase difference. In this paper, we investigate the stability and convergence in a network of coupled oscillators described by the Kuramoto model. We consider networks with finite number of oscillators, arbitrary interconnection topology, non-uniform coupling gains and non-identical natural frequencies. We show that such a network synchronizes provided the underlying graph is connected and certain conditions on the coupling gains are satisfied. In the analysis, we consider as states the phase and angular frequency differences between the oscillators, and the resulting dynamics possesses a continuum of equilibria. The synchronization problem involves establishing the Lyapunov stability of the fixed points and showing convergence of trajectories to these points. The synchronization result is established in the framework of semistability theory.

Index Terms—Kuramoto oscillator, synchronization, global convergence, direction cone, nontangency, semistability.

I. INTRODUCTION

Synchronization as a control problem in networked multi-agent systems has always assumed great importance. In this context, the study of synchronization in natural and engineered systems is of significant value, as it would serve to further the understanding of the phenomenon. The Kuramoto model, in which oscillators in the network spontaneously synchronize for coupling gains above a certain value, is of particular interest in this regard. The Kuramoto model has been used in the past to study a variety of systems, biological systems such as networks of pacemaker cells in the heart, circadian pacemaker cells in the brain, laser arrays and superconducting Josephson junctions 1.

The study of synchronization in networked dynamical systems entails an analysis of stability of synchronized states and convergence of trajectories to these states. It is also important to obtain necessary and sufficient conditions for synchronization and characterize the basins of attraction corresponding to the synchronized states.

The interest in synchronization problem can be gauged by the vast body of related literature in the area of networked dynamical systems. In the paper 2, the authors discuss synchronization in pulse-coupled biological oscillators. The article 3 reviews work on the Kuramoto model. Although published years ago, it serves to be a very useful starting point. Subsequently, in 4, the authors prove local exponential convergence for the Kuramoto model of coupled oscillators, individual oscillators having identical natural frequencies with uncertainties, from a control-theoretic viewpoint, while in 5, the authors derive a lower bound on the coupling gain for the onset of synchronization and a lower bound on the same gain which is sufficient for synchronization, with exponential convergence. The authors in 6 present a review of the work on synchronization in the Kuramoto model. In 7, the necessary and sufficient conditions on the critical coupling to achieve synchronization in the Kuramoto model are derived and in 8 the authors analyze the non-uniform Kuramoto model and its equivalence to the classical swing equation in power networks. In 9, the authors analyze the linear stability of the phase-locked state in the Kuramoto model. Robustness of the phase-locking in the Kuramoto model subjected to time-varying natural frequencies is analyzed in 10. In a recent development the authors in 11 have shown the exponential synchronization convergence of Kuramoto oscillator in the presence of a pacemaker. The pacemaker, also called as pinner, supplies reference phase to the rest of the network and the objective is to synchronize the phases of the rest of the network to the reference phase. In 12, the authors illustrate the mechanism of desynchronization with decreasing coupling strength in a 3-oscillator network described by the Kuramoto model, and present numerical illustrations for the \( N = 5 \) case. In 13, the authors use proportional mean-field feedback control to achieve desynchronization in the Kuramoto model for oscillators with small natural frequencies. Analysis of limit cycles in interconnected oscillators, using a method based on energy exchange, considering the oscillator networks as open systems is presented in 14. As for relevant work in synchronization of multi-agent systems, the authors in 15 study synchronization in networks of identical linear systems. In a recent paper 16, the authors derive protocols for robust synchronization in uncertain linear multi-agent systems.

Lyapunov stability of an equilibrium point guarantees that for initial conditions close to the equilibrium point, the trajectories remain arbitrarily close to it, while asymptotic stability guarantees the convergence of trajectories to the equilibrium point, for initial conditions in its neighbourhood. Naturally, asymptotic stability implies Lyapunov stability. The two key ideas are those of stability (in the sense of Lyapunov) and convergence of solutions. The concept of semistability finds relevance in systems that possess a continuum of (or non-isolated) equilibria. In any neighbourhood of a non-isolated equilibrium point there exists another equilibrium point, which implies
that such an equilibrium point cannot be asymptotically stable. However, for initial conditions in the neighbourhood of such an equilibrium point, it is still of interest to study the stability and convergence of solutions (to possibly different equilibrium points). The concept of semistability of an equilibrium point essentially encompasses these two notions, of stability (in the sense of Lyapunov) of the equilibrium point, and convergence of solutions.

We analyze the stability in the Kuramoto model from a control theoretic viewpoint. For the problem of synchronization in the Kuramoto model, we consider the dynamics of phase and angular frequency differences between oscillators in the network, by a linear transformation from the space of phase angles and angular frequencies of oscillators in the network. The natural frequencies of the individual oscillators in such a setting influence the initial conditions of the system, which possesses a continuum of equilibria. The set of all initial conditions in the Kuramoto model corresponds to an equivalent subset of initial conditions of the system. The problem of synchronization of oscillators in the network now reduces to that of convergence of trajectories to limit points in the equilibrium set, and the stability of the limit points. Semistability, as a notion of stability is appropriate in this context. We consider the general case of the Kuramoto model with non-identical natural frequencies, non-uniform (but symmetric) coupling with the underlying graph corresponding to the network being connected. We provide conditions on the coupling strengths and establish that for all initial phases in a compact set contained in a half-circle, the oscillators synchronize. We further show that the set \([-\pi/2, \pi/2]\) is an attracting set in \((-\pi, \pi]\) for the phase differences, and through semistability analysis establish that the trajectories converge to stable fixed points, which form the contribution of this paper.

The outline of the paper is as follows. In subsection III-B we introduce the necessary background material on semistability of continuous systems. In section III we introduce the Kuramoto model, define various notions of synchronization, and follow it up with the formulation of the problem for the case of two coupled oscillators for illustrative purposes. In section IV we derive semistability and global synchronization results for the two-oscillator case. Section V contains the analysis for the general case of networks with finite \(N\) coupled oscillators, with non-uniform coupling, arbitrary natural frequencies and arbitrary interconnection topology. In section VI we present simulation results and conclude with a summary of results and contributions in Section VII.

### A. Notations and preliminaries

Let \(\|\cdot\|\) denote the 2-norm on \(\mathbb{R}^n\). The unit circle on \(\mathbb{R}^2\) is given by \(S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}\) and we denote by \(\Gamma^n \triangleq S^1 \times S^1 \times \cdots \times S^1\) the \(n\)-torus. Let \(\overline{K}\) denote the closure of \(K\), where \(K \subseteq \mathbb{R}^n\). Two subsets \(A\) and \(B\) of \(\mathbb{R}^n\) are said to be separated if both \(\overline{A} \cap B\) and \(A \cap \overline{B}\) are empty. A set \(K \subseteq \mathbb{R}^n\) is said to be connected if \(K\) is not a union of two non-empty separated sets. The connected component of \(K\) is the maximal connected subset of \(K\). A set \(K\) is called convex if \(\alpha x + (1 - \alpha)y \in K\) for all \(x, y \in K\) and \(\alpha \in [0, 1]\). The convex hull of \(K\), denoted by \(\text{conv} K\), is the intersection of all convex sets containing \(K\) and \(K\) denote the union of convex hulls of the connected components of \(K\). The cone generated by \(K\) is denoted by \(\text{co} K\). Let \(\text{Col}(A)\) denote the column space of a matrix \(A\), where \(\text{Null}(A)\) denotes its nullspace. For \(X = (x_1, x_2, \ldots, x_n) \in \Gamma^n\), \(\sin(X) \triangleq (\sin(x_1), \sin(x_2), \ldots, \sin(x_n))\). We denote by \(1_N\) an \(N\)-dimensional vector with all components equal to one.

### B. Background on semistability of continuous systems

Consider the dynamical system described by

\[
\dot{x}(t) = f(x(t)),
\]

where \(f : D \to \mathbb{R}^n\) is continuous on an open and connected set \(D \subseteq \mathbb{R}^n\). By Peano’s existence theorem [16], there exists a \(t_1 > 0\) such that on \([0, t_1]\), the differential equation \((\text{1})\) possesses a continuous solution \(x : [0, t_1] \to D\). Further we assume that the solution is \(C^1\) and unique. Let \(\psi(t, x_0)\) denote the solution of \((\text{1})\) that exists for all \(t \in [0, \infty)\) and satisfies the initial condition \(x(0) = x_0\). These assumptions imply that the map \(\psi : [0, \infty) \times D \to D\) is continuous, satisfies \(\psi(0, x_0) = x_0\) and possesses the semi-group property \(\psi(t_1, \psi(t_2, x)) = \psi(t_1 + t_2, x)\) for all \(t_1, t_2 \geq 0\) and \(x \in D\).

Given \(t \in \mathbb{R}\), we denote the map \(\psi(t, \cdot) : D \to D\) by \(\psi_t\). A point \(x_0 \in D\) satisfying \(f(x_0) = 0\) or \(\psi(t, x_0) = x_0\) for all \(t \geq 0\) is an equilibrium point of \((\text{1})\). The collection of all equilibrium points of \((\text{1})\) is the set of equilibria, denoted by \(E\).

The system is said to possess a continuum of equilibria if the set \(E\) has no isolated points. We list only the key definitions from [17] related to semistability analysis and for details, we refer the reader to the same reference.

**Definition 1.1:** The system \((\text{1})\) is convergent if, for every \(x \in \mathbb{R}^n\), \(\lim_{t \to \infty} \psi(t, x)\), exists, is a singleton.

**Definition 1.2:** An equilibrium point \(x \in \mathbb{R}^n\) is semistable if there exists an open neighbourhood \(U \subseteq \mathbb{R}^n\) of \(x\) such that, for every \(z \in U\), \(\lim_{t \to \infty} \psi(t, z)\), exists and is Lyapunov stable.

**Definition 1.3:** Given \(x \in G\), \(G \subseteq \mathbb{R}^n\), the direction cone \(F_x\) of \(f\) at \(x\) relative to \(G\) is the intersection of all sets of the form \(\text{coco}(f(U) \setminus \{0\})\), where \(U \subseteq G\) is a relatively open neighbourhood of \(x\).

If \(K\) is a smooth submanifold of \(\mathbb{R}^n\), then \(T_xK\) is the usual tangent space [18] to \(K\) at \(x\). The vector field \(f\) is nontangent to the set \(K\) at the point \(x \in K\) if \(T_xK \cap F_x \subseteq \{0\}\).

The following results from [17] establishes semistability result based on the sufficient condition of nontangency.

**Corollary 1.1:** Let \(G \subseteq \mathbb{R}^n\) be positively invariant. Suppose \(V : G \to \mathbb{R}\) is a continuous function such that \(V\) is defined on \(G\). Let \(x \in V^{-1}(0)\) be a local maximizer of \(V\) relative to \(G\) and a local minimizer of \(V\) relative to the set \(K \triangleq G \setminus V^{-1}(0)\). Then

(i) If \(f\) is nontangent to \(V^{-1}(V(x))\) at \(x\) relative to \(G\), then \(x\) is a Lyapunov stable equilibrium relative to \(G\).

(ii) If \(f\) is nontangent to \(V^{-1}(0)\) at \(x\) relative to \(G\), then \(x\) is a Lyapunov stable equilibrium relative to \(G\).
(iii) If there exists a relatively open neighbourhood \( \mathcal{U} \subseteq \mathcal{G} \) of \( x \) such that every equilibrium in \( \mathcal{U} \) is a local minimizer of \( V \) relative to \( \mathcal{K} \) and \( f \) is nontangent to \( V^{-1}(0) \) at every point in \( \mathcal{U} \cap V^{-1}(0) \) relative to \( \mathcal{G} \), then \( x \) is a semistable equilibrium relative to \( \mathcal{G} \).

(iv) If \( x \) is an isolated point of the set \( V^{-1}(0) \), then \( x \) is an asymptotically stable equilibrium relative to \( \mathcal{G} \).

II. KURAMOTO MODEL

The Kuramoto model \([4]\) for a group of \( N \) oscillators with symmetric coupling between them is governed by

\[
\dot{\theta}_i = \omega_i + \sum_{j=1, j \neq i}^{N} \frac{K_{ij}}{N} \sin(\theta_j - \theta_i)
\]

where \( \theta_i \in S^1 \) is the phase of the \( i \)-th oscillator, \( \omega_i \) is its natural frequency and \( K_{ij} > 0 \) is the coupling gain. Both \( \omega_i \) and \( K_{ij} \) are assumed to be constants.

We need the notion of phase-locking/exact synchronization \([9]\) associated with \([2]\). A solution \( \theta^* \) of (2) is said to synchronize if and only if

\[
\dot{\theta}_i^* - \dot{\theta}_j^* \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \forall \ i, j = 1, \ldots, N
\]

and, further it is said to exactly synchronize if

\[
\theta_i^* - \theta_j^* \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \forall \ i, j = 1, \ldots, N.
\]

Further, the network \([2]\) is said to globally synchronize if

\[
\dot{\theta}_i^* - \dot{\theta}_j^* \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \forall \ i, j = 1, \ldots, N
\]

for all initial conditions \( \theta_i(0), i = 1, \ldots, N \). The dynamics of the Kuramoto network with two oscillators is first considered.

By letting \( \Delta \omega = \omega_1 - \omega_2, K_{12} = K \) and \( \Delta \theta = \theta_1 - \theta_2 \), the dynamics of the phase difference between the two oscillators is

\[
\dot{\Delta \theta} = \Delta \omega - K \sin(\Delta \theta) = -K \Delta \theta \cos(\Delta \theta).
\]

Defining \( x_1 = \Delta \theta \) and \( x_2 = \dot{x}_1 \), the state-space representation of the system is given by

\[
\dot{x} = f(x) = \begin{pmatrix}
x_2 \\
-Kx_2 \cos x_1
\end{pmatrix}
\]

and the associated set of equilibria of (4) is \( \mathcal{E} = \{(x_1, x_2) : x_2 = 0\} \). A phase-locking solution corresponds to an equilibrium solution \( x^* \in \mathcal{E} \) of the dynamics (4). In the following section we present semistability analysis for the two oscillator case.

III. SEMISTABILITY ANALYSIS OF THE TWO OSCILLATOR KURAMOTO MODEL

In \([17]\), the authors derived nontangency-based Lyapunov function results to show convergence and semistability for continuous systems. These results do not require the sign definiteness of the Lyapunov function. Instead, they need the derivative of the Lyapunov function to be nonpositive and the equilibrium to be a local minimizer of the Lyapunov function on the set of points where the Lyapunov function derivative is negative. The weaker assumptions on Lyapunov function in showing semistability are supplemented by considering nontangency of the vector field to invariant or negatively invariant subsets of the zero-level subset of the Lyapunov function derivative. We apply the nontangency based semistability results to the Kuramoto model.

Define \( \mathcal{E}_1 = (-\pi/2, \pi/2) \subseteq \mathcal{E} \); the connected component of the equilibria that is stable. We characterize a set \( \mathcal{G} \) that contains \( \mathcal{E}_1 \) and is positively invariant. Let \( x_2 = h(x_1), x_1 \in (-\pi/2, \pi/2) \), where \( h \) is a smooth function, define the upper and lower boundary of the set \( \mathcal{G} \). The upper and lower boundaries correspond to the trajectories of the system \([4]\) with appropriate initial conditions. On differentiating \( x_2 = h(x_1) \) along the trajectories of (4), we obtain

\[
\dot{x}_2 = \frac{\partial h}{\partial x_1} x_2
\]

which on integrating,

\[
h(x_1) = -K \sin x_1 + C
\]

where, \( C = \Delta \omega \) is the constant of integration. Using the limiting boundary conditions \( (x_1, x_2) = (\pi/2, 0) \) for the upper boundary and \( (x_1, x_2) = (-\pi/2, 0) \) for the lower boundary, \( \mathcal{G} \) is defined as

\[
\mathcal{G} = \{(x_1, x_2) : x_1 \in (-\pi/2, \pi/2), x_2 \in [-K(1 + \sin x_1), K(1 - \sin x_1)]\}.
\]

The vector field plot of the system along with the set \( \mathcal{G} \) is shown in Figure 1 where trajectories are trapped in \( \mathcal{G} \). Through the following Lemma we show that \( \mathcal{G} \) is positively invariant.

**Lemma 3.1:** The set \( \mathcal{G} \) is positively invariant along the trajectories of (4).

**Proof:** The boundary of \( \mathcal{G} \) can be expressed as \( \partial \mathcal{G} = B_{x_1} \cup B_{x_2} \).

\[
B_{x_1} = \begin{cases}
B_{x_1}^+ = \{ (x_1, x_2) : x_1 = \frac{\pi}{2} \}, \\
B_{x_1}^- = \{ (x_1, x_2) : x_1 = -\frac{\pi}{2} \}
\end{cases}
\]

\[
B_{x_2} = \{ x : x_1 \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], x_2 = C - K \sin(x_1), C = \pm K \}
\]

The normal to the boundary \( B_{x_1}^+ \) is \( e_1 \). The condition

\[
e_1^T \begin{bmatrix}
x_2 \\
-Kx_2 \cos(x_1)
\end{bmatrix} \leq 0
\]

ensures that the trajectories at \( B_{x_1}^+ \) point into \( \mathcal{G} \). We have

\[
e_1^T \begin{bmatrix}
x_2 \\
-Kx_2 \cos(x_1)
\end{bmatrix} = x_2.
\]

From the definition of \( \mathcal{G} \), \( -K - K \sin(x_1) \leq x_2 \leq K - K \sin(x_1) \). Therefore, at the boundary of \( B_{x_1}^+ \), \( x_1 = \frac{\pi}{2} \), \(-2K \leq x_2 \leq 0 \). The same can be shown for the boundary \( B_{x_1}^- \).

The boundary \( B_{x_2} \) is a level set of the form \( \eta(x_1, x_2) = x_2 + K \sin(x_1) = \pm K \). The normal to the boundary \( B_{x_2} \) is \( \nabla_x \eta = [K \cos(x_1)] \). The dot product of the normal with
the vector field $f$ is $\nabla_x \eta \left[ -\frac{x_2}{K \cos(x_1)} \right] = 0$. The vector field $f$ is tangential to the boundary $B_{x_2}$. Hence the claim. \hfill \blacksquare

Stability is thus ensured when the phase difference lies in an open half-circle and difference in angular frequencies satisfy $|\Delta \omega| \leq K$.

Remark 3.1: Note that exact synchronization is ensured only when the oscillators have identical natural frequencies, that is, $\Delta \omega = 0$.

We next show that every equilibrium in $E_1$ is semistable.

A. Direction cone, tangent cone and nontangency

A primary step in establishing the semistability result following the approach given in [17] is to verify the sufficient condition of nontangency. This further requires the computation of the direction cone and the tangent cone. In hitherto published results [17], [19] the direction cone is obtained by computing the limiting direction set or by expressing the vector field as a span of a set of linearly independent vectors. If the aforementioned approaches fail in characterizing the direction cone, then the direction cone is computed by invoking the definition. The application of the definition [13] requires an explicit computation of the image set $f(\mathcal{U}) \setminus \{0\}$, which is tedious and can be circumvented by an outer bounding set. Further, by avoiding the intersection over all open sets $\mathcal{U}$ in $\text{coco} \{f(\mathcal{U}) \setminus \{0\}\}$, leads to a superset of the direction cone. The resulting set, we call it as an outer estimate of the direction cone, denoted by $\hat{\mathcal{F}}_x$. The construction of this estimate is made clear by working out the direction cone through the Kuramoto model.

Let $x = [a \ 0]^T \in E$, where $a \in \mathbb{R}$ and $\mathcal{U} \subset \mathcal{G}$ be a open and bounded neighbourhood of $x$. We first consider the case $a \in (0, \pi/2)$. Define $\mathcal{U} = \{z \in \mathbb{R}^2 : \|z - x\|_\infty < \epsilon\}$, where $\epsilon > 0$ is such that $a + \epsilon < \pi/2$. Then $\mathcal{U}$ consists of the following connected components:

$A_1 = \{(x_1, x_2) : a - \epsilon < x_1 < a + \epsilon, 0 < x_2 < \epsilon\}$
$A_2 = \{(x_1, x_2) : a - \epsilon < x_1 < a + \epsilon, -\epsilon < x_2 < 0\}$
$A_3 = \{(x_1, x_2) : a - \epsilon < x_1 < a + \epsilon, x_2 = 0\}$.

Now, $f(\mathcal{U}) \setminus \{0\} = f(A_1) \cup f(A_2) \subset A^+ \cup A^-$ where,

$A^- = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < -\epsilon, -K \cos(a - \epsilon) < \frac{y_2}{y_1} < -K \cos(a + \epsilon), -K \epsilon \cos(a - \epsilon) < y_2 < 0\}$,

$A^+ = \{(y_1, y_2) \in \mathbb{R}^2 : -\epsilon < y_1 < 0, -K \cos(a - \epsilon) < \frac{y_2}{y_1} < -K \cos(a + \epsilon), 0 < y_2 < K \epsilon \cos(a - \epsilon)\}$.

By performing similar analysis for the case $a \in (-\pi/2, 0)$, for every $x \in E_1$, and for every $a \in (-\pi/2, \pi/2)$ the outer estimate of the direction cone (see Figure 2) is

$\hat{\mathcal{F}}_x = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 = py_1, p \in [-K \cos(a + \epsilon), -K \cos(a - \epsilon)]\}$.

For every $x \in E$, the tangent cone is $T_x E = \{(c, 0) : c \in \mathbb{R}\}$. It now follows that for every $x \in E_1$, the intersection of tangent cone with the outer estimate of the direction cone is $\{0\}$. The same fact is captured in Figure 2. Hence, the nontangency condition holds for every $x \in E_1$. The following result establishes the semistability of $E_1$.

Proposition 3.1: Every equilibrium in $E_1$ of $\mathcal{G}$ is semistable relative to $\mathcal{G}$.

Proof: Consider the continuously differentiable function $V_1 : \mathcal{G} \rightarrow \mathbb{R}$ defined by $V_1(x) = \frac{x^2}{2}$. The derivative of $V_1$ along the trajectories of $\mathcal{G}$ is $V_1'(x) = -x_2 \cos x_1 \leq 0$ for every $x \in \mathcal{G}$. The set of points where the derivative of $V_1$ is zero is given by $V_1^{-1}(0) = \{(x_1, x_2) \in \mathcal{G} : x_2 = 0\}$. The largest negatively invariant subset of $V_1^{-1}(0)$ is $E_1$. Let $\mathcal{K} \triangleq \mathcal{G} \setminus V_1^{-1}(0)$. Clearly, every equilibrium $z \in E_1$ is a local maximizer of $V_1$ relative to $\mathcal{G}$ and a local minimizer of $V_1$ relative to $\mathcal{K}$.

Consider an equilibrium $x \in E_1$. There exists a relatively open neighbourhood $\mathcal{V} \subset \mathcal{G}$ of $x$ such that $\mathcal{V} \cap E = \mathcal{V} \cap E_1$. It now follows that every $z \in \mathcal{V} \cap E$ is a local maximizer of $V_1$ relative to $\mathcal{G}$ and a local minimizer of $V_1$ relative to $\mathcal{K}$.
Moreover, \( f \) is nontangent to \( V \cap E \) at every point \( z \in V \cap E \) relative to \( G \). Now, by applying (iii) of Corollary [11], every \( x \in E_1 \) is semistable relative to \( G \).

We end this section with the proof for the global synchronization in the two-oscillator network.

**B. Global synchronization**

We first note that when \(|\Delta \omega| \leq K\), all initial conditions of (2) belong to the non-empty set \( R \supseteq Q \cup G \), where \( Q \) is defined by

\[
Q = \{ x : x_1 \in (-\pi, \pi] \setminus [-\pi/2, \pi/2], x_2 = \Delta \omega - K \sin(x_1), |\Delta \omega| \leq K \}.
\]

The global synchronization result is established through the following proposition.

**Proposition 3.2:** The oscillators in (2) synchronize for every initial condition \( \theta_1(0), \theta_2(0) \), when \(|\Delta \omega| \leq K\).

**Proof:** We first recall from Proposition 3.1 that every equilibrium in \( E_1 \) is semistable relative to \( G \) and that the vector field \( f \) at every point in \( B_{x_1} \setminus E \) points into \( G \). It suffices to show that there exists \( T \geq 0 \) such that \( T_\tau(R) \cap (G) \neq \emptyset \), where \( T_\tau \) denotes the solution to (4) corresponding to the initial condition \( x(0) = x_0 \). The set \( T_\tau \cap G \) is empty if and only if there exist stable fixed points and/or closed orbits in \( Q \). Suppose ad absurdum the set \( T_\tau \cap G \) is empty.

It can be easily verified that the fixed points in \( Q \), given by \( E \cap Q = \{ (x_1, x_2) : x_1 \in (-\pi, \pi]/[-\pi/2, \pi/2], x_2 = 0 \} \) are unstable.

Moreover,

\[
\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -K \cos(x_1) > 0
\]

in \( Q \). Hence, by Bendixson’s theorem [20], there are no closed orbits lying entirely in the simply connected set \( Q \), leading to a contradiction.

We next consider an \( N \)-oscillator network (for any finite \( N \)) with arbitrary interconnection topology, non-uniform coupling strengths and arbitrary, but constant natural frequencies. We first identify a positively invariant set that contains the equilibrium set of interest (stable equilibria), and proceed to derive the non-tangency based semistability result.

**IV. N-Oscillator Network**

A system of \( N \) coupled Kuramoto oscillators with is considered here

\[
\dot{\theta}_i = \omega_i + \sum_{j=1,j \neq i}^{N} \frac{K_{ij}}{N} \sin(\theta_j - \theta_i)
\]

where \( \theta_i \in S^1 \) is the phase of the \( i \)-th oscillator, \( \omega_i \) is its natural frequency and \( K_{ij} \in IR \) is the coupling gain. We assume symmetric coupling \( (K_{ij} = K_{ji}) \) between a pair of oscillators and \( \omega_i \) is a constant. Define \( \Upsilon \supseteq \{ (K_{i1}, K_{i2}, \ldots, K_{i(N-1),N}) : K_{ij} \geq 0, i \neq j \} \). Let \( K = \{ K_{12}, K_{13}, \ldots, K_{(N-1),N} \} \). The set \( \Upsilon \) represents all possible interconnection topologies.

If the oscillators are considered as nodes with every node connected to every other node, then the nodes form a graph with \( N \) vertices and \( e = \binom{N}{2} \) edges. The incidence matrix \( B \in IR^{N \times e} \) of an oriented graph \( G \) is constructed such that \( B_{ij} = -1 \) if the edge is incoming to the vertex \( i \), \( B_{ij} = 1 \) if the edge is outgoing to the vertex \( i \), and \( 0 \) otherwise. We consider the incidence matrix corresponding to an all-to-all connectivity and generate every possible interconnection topology by an appropriate choice of \( K \). In the rest of the paper, \( B \) has the fixed form

\[
B = \begin{bmatrix}
1 & 1 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & -1
\end{bmatrix}.
\]

The objective of our analysis is to obtain conditions on the interconnection topology such that the oscillator network [6] synchronizes.

Rewriting (4) in vector form

\[
\dot{X} = F(X, V) = \left( \begin{array}{c}
\dot{X} \\
\dot{V}
\end{array} \right) = F(X, V) = \left( \begin{array}{c}
V \\
G(X)V
\end{array} \right)
\]

where \( G(X) = -B^\top BK \) diag(cos(X)).

**Remark 4.1:** Since rank\((B^\top) = N - 1\), the states \( x_i, i = N, \ldots, e \) are linear combinations of \( x_i, i = 1, \ldots, N - 1 \).

**Remark 4.2:** Since \( X = B^\top \theta \) and Null\((B^\top) = \text{span}\{1_N\}\), we note that \( V = 0 \) in (8) corresponds to the synchronized condition in (7).

The following result will be useful in the ensuing section.

**Lemma 4.1:** For all \( X \in \left( -\pi/2, \pi/2 \right)^c \cap \text{Col}(B^\top), V \in \text{Col}(B^\top), V^\top G(X)V \leq 0 \).

**Proof:**

\[
V^\top G(X)V = -V^\top BK \text{ diag(cos}(X))V = -\dot{\theta}^\top BB^\top BK \text{ diag(cos}(X))B^\top \dot{\theta} = -\dot{\theta}^\top BB^\top BP(X)B^\top \dot{\theta}
\]

Note that \( P(X) = K \text{ diag(cos}(X)) \) is a positive semi-definite diagonal matrix. Since \( BB^\top B = NB \), we have \( -\dot{\theta}^\top BB^\top BP(X)B^\top \dot{\theta} = -N\dot{\theta}^\top BP(X)B^\top \dot{\theta} \leq 0 \). The set of equilibria of (8) is \( E = \{ (X, V) : V = 0 \} \). Linearizing (8) about an equilibrium point \((X^*, 0)\),

\[
A
\]

\[
= \begin{bmatrix}
0 & I_{e \times e} \\
0 & G(X)^\top
\end{bmatrix}
\]

where \( \dot{X} = X - X^* \) and \( \dot{V} = V \). Further, for all \( X \in \left( -\pi/2, \pi/2 \right)^c \), all the non-zero eigenvalues of \( A \) are negative.
Thus, the equilibrium set of interest for semistability analysis is \( \mathcal{E}_x = \left( -\pi/2, \pi/2 \right]^e \subset \mathcal{E} \). Note that \( X \in \text{Col}(B^T) \subset R^e, V \in \text{Col}(B^T) \subset R^e \) and \( \text{rank}(B^T) = N - 1 \).

To proceed with the analysis, we define a set \( \mathcal{H} \) that is positively invariant through the following Lemma. Let \( e_i \in \mathbb{R}^e \) be the \( i \)th basis vector from the canonical basis.

**Lemma 4.2:** The set
\[
\mathcal{H} = \{ (X, V) : X \in \left( -\pi/2, \pi/2 \right]^e \cap \text{Col}(B^T), V \in J \}
\]
is positively invariant along the trajectories of (3), where
\[
\mathcal{J} = \{ V \in \mathbb{R}^e : V = (B^T \omega - B^T BK \sin(X)) ; X \in \left( -\pi/2, \pi/2 \right]^e \cap \text{Col}(B^T) ;
|e_i^T B^T \omega| \leq \frac{2}{N} \bar{K}_i + \frac{1}{N} \sum_{j \neq i}^e |(B^T B)_{ij}| \bar{K}_j \sin |x_j|,
\]
i = 1, \ldots, e \}.

**Proof:** Note that there are \( 2e \) bounding surfaces for \( X \in \left( -\pi/2, \pi/2 \right]^e \). The \( i \)th bounding surface is characterized by \( x_i = \frac{\pi}{2} i = 1, \ldots, e \). The boundary of \( \mathcal{H} \) is given by \( \partial \mathcal{H} = B_X \cup B_V \), where \( B_X = \cup_{i=1}^e B_{X_i} \) with
\[
B_{X_i} = \left\{ \left( X, V \right) : x_i = \frac{\pi}{2} \right\}, i = 1, \ldots, e \text{ and } B_{V_i} = \left\{ \left( X, V \right) : x_i = -\frac{\pi}{2} \right\}, i = e + 1, \ldots, 2e.
\]
and \( B_V \) to be defined later in the proof. The outward normal to the bounding surface characterized by \( x_i = \frac{\pi}{2} \) is \( e_i \). For \( \mathcal{H} \) to be positively invariant,
\[
[e_i^T \ 0] \begin{bmatrix} V \\ G(X)V \end{bmatrix} \leq 0
\]on the bounding surface. Inequality (10) yields \( e_i^T V \leq 0 \). For \( V = B^T \omega - B^T BK \sin(X) \), we get
\[
e_i^T B^T \omega \leq e_i^T B^T BK \sin(X)
\]
where, \( x_i = \frac{\pi}{2} \), and imposing the condition \( x_j \in [-\pi/2, \pi/2] \), we observe that
\[
x_j \in \begin{cases}
0, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 1 \\
-\frac{\pi}{2}, 0 & \text{if } (B^T B)_{ij} = -1 \\
-\frac{\pi}{2}, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 0.
\end{cases}
\]
Similarly, the outward normal to the bounding surface characterized by \( x_i = -\frac{\pi}{2} \) is \( -e_i \). Inequality (10) is
\[
-e_i^T (B^T \omega - B^T BK \sin(X)) \leq 0
\]
for \( x_i = -\frac{\pi}{2} \), and
\[
x_j \in \begin{cases}
0, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 1 \\
-\frac{\pi}{2}, 0 & \text{if } (B^T B)_{ij} = -1 \\
-\frac{\pi}{2}, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 0.
\end{cases}
\]
Condition (12) can be rewritten as
\[
e_i^T B^T \omega \geq -e_i^T B^T BK \sin(X)
\]
where, \( x_i = \frac{\pi}{2} \), and
\[
x_j \in \begin{cases}
0, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 1 \\
-\frac{\pi}{2}, 0 & \text{if } (B^T B)_{ij} = -1 \\
-\frac{\pi}{2}, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 0.
\end{cases}
\]
From (11) and (13), it follows that
\[
|e_i^T B^T \omega| \leq e_i^T B^T BK \sin(X)
\]where, \( x_i = \frac{\pi}{2} \), and
\[
x_j \in \begin{cases}
0, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 1 \\
-\frac{\pi}{2}, 0 & \text{if } (B^T B)_{ij} = -1 \\
-\frac{\pi}{2}, \frac{\pi}{2} & \text{if } (B^T B)_{ij} = 0.
\end{cases}
\]
Equation (14) can be expanded as follows
\[
|e_i^T B^T \omega| \leq e_i^T B^T BK \sin(X)
\]
where, \( x_j \in [-\frac{\pi}{2}, \frac{\pi}{2}] \). From (15), a conservative bound (for which the set \( \mathcal{H} \) is positively invariant) can be obtained as
\[
|e_i^T B^T \omega| \leq \frac{2}{N} \bar{K}_i \leq \min(e_i^T B^T BK \sin(X)).
\]
This yields the following sufficient condition on the coupling strength
\[
\bar{K}_i \geq \frac{N}{2} |e_i^T B^T \omega|
\]
where \( \bar{K}_i \) is the coupling strength corresponding to the \( i \)th edge and \( |e_i^T B^T \omega| \) is the magnitude of the difference in natural frequencies between the oscillators at the vertices corresponding to the \( i \)th edge.

The boundary \( B_V \) is a level set of the form \( \eta(X, V) = \frac{1}{2} (V + B^T BK \sin(X))^T (V + B^T BK \sin(X)) = C_i \), where
\[
C_i = \omega_i^2 B^T B \omega_i, \quad \text{and } \omega_i \text{ satisfies } |e_i^T B^T \omega_i| = \frac{2K_i}{N}, \quad i = 1, \ldots, e.
\]
The normal to the level set of \( \eta(X, V) \) is
\[
\nabla_{(X, V)} \eta = (V + B^T BK \sin(X))^T \left[ B^T BP(X) \right] \left[ V \right] G(X)V
\]
where, \( X \) and \( V \) are the state variables at the edge of the boundary \( B_V \). The dot product of the vector field \( F \) in (8) and the normal \( \nabla_{(X, V)} \eta \), at any point on the boundary \( B_V \) is given by
\[
(V + B^T BK \sin(X))^T \left[ B^T BP(X) \right] \left[ V \right] G(X)V
= (V + B^T BK \sin(X))^T (-G(X)V + G(X)V) = 0.
\]
Therefore, for every point on the boundary \( B_V \), the vector field \( F \) is tangential to it. Therefore, the set \( \mathcal{H} \) is positively invariant.

The bound on the critical values of the coupling strengths below which the trajectories of (8) will not be bounded by \( \mathcal{H} \)
for any initial condition, are given by

\[ |e_i^T B^T \omega| \leq \max(e_i^T B^T BK \sin(X)) \]

\[ = \max\left(\frac{2}{N} \tilde{K}_i + \frac{1}{N} \sum_{j \neq i} (B^T B)_{ij} \tilde{K}_j \sin(x_j)\right) \]

\[ < \frac{2}{N} \tilde{K}_i + \frac{1}{N} \sum_{j \neq i} (|B^T B|)_{ij} \tilde{K}_j, \quad (17) \]

For networks with uniform coupling (\( \tilde{K}_i = K_0 \)), (17) reduces to

\[ \max(|e_i^T B^T \omega|) \leq \frac{2K_0}{N} + \frac{K_0}{N} \sum_{j \neq i} |(B^T B)_{ij}| \]

\[ = \frac{K_0}{N} \left(2 + \left(\frac{N}{2} - 1 - \left(\frac{N}{2}\right) \right)\right) \]

\[ = \frac{2K_0(N-1)}{N}, \quad (18) \]

which can be re-expressed as

\[ K_0 = \frac{N}{2(N-1)} \max(|e_i^T B^T \omega|) = \frac{N}{2(N-1)} \|B^T \omega\|_{\infty}. \quad (19) \]

The bound on the critical coupling gain \( K_0 \), in (19) is same as that derived by Jadbabaei et. al in [3] for the onset of synchronization in uniform networks with all-to-all coupling. The bounds for critical coupling derived in this paper are a generalization of this result for networks with non-uniform coupling and arbitrary interconnection topology. In summary, we have derived the necessary and sufficient conditions on the coupling strengths for the positive invariance set \( \mathcal{H} \) (the set of phase angles for which \( (\theta_i - \theta_j) \in [-\pi/2, \pi/2] \)). The following subsection utilizes this result to prove the semistability of (8), therefore synchronization in the Kuramoto model (6).

\[ \text{A. Nontangency and semistability} \]

For every \( X \in \mathcal{E}_s \), the tangent cone is \( T_X \mathcal{E}_s = \{ (\gamma d, 0) : d \in \mathcal{E}_s, \gamma \in \mathbb{R} \} \). For an equilibrium point \( (X, 0) \in \mathcal{E}_s \), let \( D \) be an open and bounded neighbourhood of \( X \) defined as

\[ D = \{(x, v) \in \mathbb{R}^e \times \mathbb{R}^e : ||(X, 0) - (x, v)|| < \epsilon(X), \epsilon(X) > 0 \} \subset \mathcal{H}. \]

Then, an outer estimate of the direction cone is given by

\[ \tilde{\mathcal{F}}_X = \{\lambda v, G(x)v : (x, v) \in D, \lambda > 0\}. \]

We next show the nontangency between the vector field \( F \) and the set of equilibria \( \mathcal{E}_s \) holds through the following Lemma.

**Lemma 4.3:** For every \( X \in \mathcal{E}_s \), \( T_X \mathcal{E}_s \cap \tilde{\mathcal{F}}_X = \{0\} \) if the network graph corresponding to (6) is connected.

**Proof:** From the second line of (8),

\[ \tilde{V} = G(X)V = -B^T BK \text{diag}(\cos(X))B^T \dot{\theta}. \]

We claim that if the network graph corresponding to (6) is connected, \( \tilde{V} = 0 \) if and only if \( V = 0 \). If \( \tilde{V} = 0 \implies V = G(X)V = 0 \). Conversely,

\[ \tilde{V} = -B^T B P^T \dot{\theta} = 0 \]

\[ \implies \theta \in \text{Null}(B^T B P^T) \]

\[ \implies \theta \in \text{Null}(B^T B P^T) = \text{Null}(B P B^T). \]

Since \( P(X) \) is a positive semi-definite diagonal matrix, we let \( P = P^1 \frac{P^2}{2}, \) where \( P^1 = \sqrt{T} \) and \( P^2 = \frac{P^1}{2}. \) Hence, \( \text{Null}(B P B^T) = \text{Null}((B P^1)(B P^2)^T) = \text{Null}(P^1 B^T). \)

Therefore,

\[ \dot{\theta} \in \text{Null}(B^T B P B^T) \implies \dot{\theta} \in \text{Null}(P^1 B^T) \]

\[ \implies \dot{\theta} \in \text{Null}(B^T) \) or \( B^T \dot{\theta} \in \text{Null}(P^2) = \text{Null}(P) \). Thus,

\[ \text{Null}(P) = \text{Span}\{e_i \in \mathbb{R}^e : \tilde{K}_i = 0\} \]

\[ \text{Col}(B^T) = \text{Span}\{B_i \in \mathbb{R}^e : 1 \leq i \leq N - 1\} \]

where \( B_i \) is the \( i \)-th row vector of \( B \). We note that for connected graphs, \( \tilde{V} = 0 \) if and only if \( V = 0 \) (\( \text{Col}(B^T) \cap \text{Null}(P) = \emptyset \) and only if the graph is connected).

The intersection of \( T_X \mathcal{E}_s \) with \( \tilde{\mathcal{F}}_X \) yields

\[ \lambda v = \gamma d \]

\[ \lambda G(x)v = 0 \]

Since \( G(x)v = 0 \) if and only if \( v = 0 \) it follows that \( T_X \mathcal{E}_s \cap \tilde{\mathcal{F}}_X = \{0\} \).

The following result establishes the semistability of (8).

**Proposition 4.1:** Every equilibrium in \( \mathcal{E}_s \) of (8) is semistable relative to \( \mathcal{H} \).

**Proof:** Consider the continuously differentiable function \( V_2 : \mathcal{H} \to \mathbb{R} \) defined by \( V_2(x) = \frac{1}{2} \|x\|^2 \). The derivative of \( V_2 \) along the trajectories of (8) is \( V_2 = \dot{\theta}^T \dot{G}(x)V \leq 0 \) for every \( (X, V) \in \mathcal{H} \), which follows from Lemma 4.1. The set of points where the derivative of \( V_2 \) is zero is given by \( V_2^{-1}(0) = \{(X, V) : X \in \mathcal{H} : V = 0\} \). This claim follows from Lemma 4.1 and the arguments in Lemma 4.3. The largest negatively invariant subset of \( V_2^{-1}(0) \) is \( \mathcal{E}_s \). Let \( \mathcal{L} \triangleq \mathcal{H}\backslash V_2^{-1}(0) \). Clearly, every equilibrium \( Z \in \mathcal{E}_s \) is a local maximizer of \( V_2 \) relative to \( \mathcal{L} \) and a local minimizer of \( V_2 \) relative to \( \mathcal{L} \).

Consider an equilibrium \( X \in \mathcal{E}_s \). There exists a relatively open neighbourhood \( U \subset \mathcal{H} \) of \( X \) such that \( U \cap \mathcal{E} = \{X\} \cap \mathcal{E} \). It now follows that every \( Z \in U \cap \mathcal{E} \) is a local maximizer of \( V_2 \) relative to \( \mathcal{L} \) and a local minimizer of \( V_2 \) relative to \( \mathcal{L} \). Moreover, from Lemma 4.3, the vector field \( F \) in (8) is nontangent to \( U \cap \mathcal{E} \) at every point \( Z \in \mathcal{E} \) relative to \( \mathcal{L} \). Now, by applying (iii) of Corollary 1.1, every \( X \in \mathcal{E}_s \) is semistable relative to \( \mathcal{H} \).

We next show that \( \mathcal{H} \) is an attracting set in \( \mathcal{M} \), where

\[ \mathcal{M} \triangleq \{(X, V) : X \in (-\pi + \delta, -\pi - \delta)_e \cap \text{Col}(B^T), \delta > 0, V \in \mathcal{J}\}. \]

**Proposition 4.2:** There exists a non-empty, connected, compact and positively invariant set \( \mathcal{N} \subset \mathcal{M} \) containing \( \mathcal{H} \) such that \( \mathcal{H} \) is an attracting set of (8), if the coupling gains satisfy the condition

\[ \frac{1}{N} \sum_{i=1}^e |e_i^T B^T \omega| + \frac{1}{N} \sum_{i=1}^e |\tilde{K}_i| = (N-2)\Delta_m \]

(20)

where the non-zero \( \tilde{K}_i \)s satisfy \( \tilde{K}_i \geq (N-2) \Delta_m \), and \( \Delta_m = (\tilde{K}_{\max} - \tilde{K}_{\min}) \).
Pre-multiplying (21) by defined over $V$ where function of the form
\[ \sum_{i=1}^{e} \{(\text{sign} (x_i)) v_i \text{ if } x_i \neq 0 \} \]
The set-valued Lie derivative \[^{21}\], denoted by $\hat{L}_F V_3(X)$ of $V_3$ is obtained as follows.
\[ \hat{L}_F V_3(X) = \left\{ \begin{array}{ll}
\sum_{i=1}^{e} \{(\text{sign} (x_i)) v_i \text{ if } x_i \neq 0 \forall i \in \{1, \ldots, e\} \\
\emptyset \text{ if } v_i \neq 0 \text{ when } x_i = 0 \\
\sum_{i=1}^{e} \{(\text{sign} (x_i)) v_i \text{ if } v_i = 0 \text{ when } x_i = 0.}
\end{array} \right. \]
where $V = (v_1, \ldots, v_e)^T$. $\mathcal{H}$ is an attracting set in $\mathcal{M}$ if $\hat{L}_F V_3(X) < 0 \forall X \in \mathcal{M} \setminus \mathcal{H}$ and $V \neq 0$, which is true if the following condition holds.
\[ \sum_{i=1}^{e} (\text{sign} (x_i)) e_i^T B^T \omega - \frac{1}{N} \left[ \sum_{j=1}^{e} (B^T B)_{ij} \text{sign} (x_j) K_j \right] \times \text{sign} (x_i) \sin |x_i| < 0 \]
where $\sin(|X|) \overset{\triangle}{=} (|\sin x_1|, \ldots, |\sin x_e|)^T$.
\[ \frac{1}{N} \sum_{i=1}^{e} \sum_{j=1}^{e} (B^T B)_{ij} \text{sign} (x_j) K_j \sin x_1 \]
\[ > \frac{1}{N} \sum_{i=1}^{e} (2K_i - (N - 2)\Delta_m) \sin |x_i| \]
\[ > \frac{1}{N} \sum_{i=1}^{e} |e_i^T B^T \omega| \]
\[ > \sum_{i=1}^{e} (\text{sign} (x_i)) e_i^T B^T \omega \]
which yields the sufficient condition \[^{20}\]. Finally, the set $\mathcal{N}$ is characterized as
\[ \mathcal{N} \overset{\triangle}{=} \{(X, V) : X \in (\pi + \delta, \pi - \delta]^e \cap \text{Col}(B^T), \sum_{i=1}^{e} |x_i| - (N - 1)\pi \leq 0, V \in \mathcal{J}\} \]

We end this section with the derivation of the synchronized frequency through the following Lemma.

Lemma 4.4: For all initial conditions in $\mathcal{N}$, the angular frequencies of $\mathcal{I}$ synchronize to the mean of the natural frequencies of the oscillators.

Proof: Through Propositions \[^{4.1}\] and \[^{4.2}\] it was established that for all initial conditions in $\mathcal{N}$, the oscillators synchronize, which corresponds to $V = 0$ in (3). This further implies that
\[ V = B^T \omega - B^T BK \sin(X^*) = 0 \] (21)
where $(X^*, 0)$ is the fixed point of (3). From (7), we obtain
\[ \Omega^* 1_{\mathcal{N}} \overset{\triangle}{=} \lim_{k \to \infty} \hat{\theta}(t) = \omega - BK \sin(X^*)\]

Pre-multiplying (21) by $\frac{1}{N} B$, we obtain $\frac{1}{N} B B^T K \sin(X^*) = BK \sin(X^*) = \frac{1}{N} B B^T \omega$. Equation (22) reduces to $\Omega^* 1_{\mathcal{N}} = (I - \frac{1}{N} B B^T) \omega = (\frac{1}{N} \sum_{i=1}^{N} \omega_i) 1_{\mathcal{N}}$. To summarize our results in this section, we have established that there always exists a set of coupling strengths for all natural frequencies of the oscillators, such that for all initial conditions $X(0) \in \mathcal{N}$ the oscillators synchronize, provided the network graph is connected.

V. Simulation Results

We first consider an open-chain network with $N = 3$ and non-uniform coupling $K = \text{diag}\{3, 2, 0\}$ and natural frequencies $\omega = (1, 2, 3)$. With $X \overset{\triangle}{=} (x_1, x_2, x_3)$, the reduced-order system (note that $x_3 = x_2 - x_1$) can rewritten as
\[ \dot{x}_1 = -1 - 6 \sin(x_1) - 2 \sin(x_2) \]
\[ \dot{x}_2 = -2 - 3 \sin(x_1) - 4 \sin(x_2) \]
The stream plot for this case is shown in Figure 3. In the interval $X \in (\pi, \pi]^3$, there exists only one fixed point at $(x_1, x_2) = (0, -\pi/6)$ that corresponds to the synchronized state. In this case, we observe that $\mathcal{N} = \mathcal{M}$. We next consider a network (see Figure 4) with $N = 5$ and non-uniform coupling and natural frequencies $\omega = (1, 2, 3, 4, 5)$. The plot
of angular frequencies versus time is shown in Figure 5 for the initial condition \( \theta(0) = (-2\pi/3, 2\pi/3, \pi/3, -\pi/6, 0) \), where the frequencies synchronize to the mean \( \Omega^* = 3 \text{ rad/s} \).

![Graph showing time-response of angular frequencies](image)

**Fig. 5.** Time-response of angular frequencies

### VI. Conclusions

The objective of this work was to obtain convergence results for the Kuramoto model, which was presented in the framework of semistability theory. For illustrative purpose, these results were obtained for the two-oscillator case. In arriving at the semistability result, the nontangency between the vector field and the tangent space of the set of equilibria was established by using a novel method for obtaining an estimate of the outer bound of the direction cone. In the \( N \)-oscillator case we consider networks with connected graphs, arbitrary interconnection topology, non-uniform coupling strengths and non-identical natural frequencies. We establish that such a network synchronizes under certain conditions on the coupling gains which have been explicitly derived.

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