BOUNDARY CROSS THEOREM IN DIMENSION 1

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Abstract. Let $X$, $Y$ be two complex manifolds of dimension 1 which are countable at infinity, let $D \subset X$, $G \subset Y$ be two open sets, let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$), and let $W$ be the 2-fold cross $((D \cup A) \times B) \cup (A \times (B \cup G))$. Suppose in addition that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and that $A$ and $B$ are of positive length. We determine the “envelope of holomorphy” $\hat{W}$ of $W$ in the sense that any function locally bounded on $W$, measurable on $A \times B$, and separately holomorphic on $(A \times G) \cup (D \times B)$ “extends” to a function holomorphic on the interior of $\hat{W}$.

1. Introduction

In this paper we consider a boundary version of the cross theorem in the spirit of the pioneer work of Malgrange–Zerner [16]. Epstein’s survey article [3] gives a historical discussion and motivation for this kind of theorems.

The first results in this direction are obtained by Komatsu [8] and Drużkowski [2], but only for some special cases. Recently, Gonchar [5, 6] has proved a more general result for the one-dimensional case. In recent works [10, 11], the authors are able to generalize Gonchar’s result to the higher dimensional case.

However, in all cases considered so far in the literature the hypotheses on the function to extend and its domain of definition are, in some sense, rather restrictive. Therefore, the main goal of this work is to establish some boundary cross theorems in more general (one-dimensional) cases with more optimal hypotheses. Perhaps, this will be a first step towards understanding the higher dimensional case in its full generality.

Our approach here is based on the previous work [10], the Gonchar–Carleman operator developed in [5, 6], a new result of Zeriahi [15] and a thorough geometric study of harmonic measures.

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2. Preliminaries

In order to recall the classical versions of the boundary cross theorem and to discuss in more detail our motivation, we need to introduce some notation and terminology. In fact, we keep the main notation from the previous work [10]. Here $E$ denotes the open unit disc in $\mathbb{C}$ and mes the linear measure (i.e. the one-dimensional Hausdorff measure). Throughout the paper, for a topological space $M, C(M)$ denotes the space of all continuous functions $f : M \to \mathbb{C}$ equipped with the sup-norm $|f|_M := \sup_M |f|$. Moreover, a function $f : M \to \mathbb{C}$ is said to be locally bounded on $M$ if, for any point $z \in M$, there are an open neighborhood $U$ of $z$ and a positive number $K = K_z$ such that $|f|_U < K$. Finally, for a complex manifold $\Omega$, $\mathcal{SH}(\Omega)$ (resp. $\mathcal{O}(\Omega)$) denotes the set of all subharmonic (resp. holomorphic) functions on $\Omega$.

In this work all complex manifolds are supposed to be countable at infinity.

2.1. Open set with partly Jordan-curve-like boundary. Let $X$ be a complex manifold of dimension 1. A Jordan curve in $X$ is the image $C := \{\gamma(t) : t \in [a, b]\}$ of a continuous one-to-one map $\gamma : [a, b] \to X$, where $a, b \in \mathbb{R}$, $a < b$. The set $\{\gamma(t) : t \in (a, b)\}$ is said to be the interior of the Jordan curve. A Jordan domain is the image $\{\Gamma(t), t \in E\}$ of a one-to-one continuous map $\Gamma : \partial D \to X$. A closed Jordan curve is the boundary of a Jordan domain.

Consider an open set $D \subset X$. Then $D$ is said to be Jordan-curve-like at a point $\zeta \in \partial D$ if there is a Jordan domain $U \subset X$ such that $\zeta \in U$ and $U \cap \partial D$ is the interior of a Jordan curve. Then $\zeta$ is said to be of type 1 if there is a neighborhood $V$ of $\zeta$ such that $V \cap D$ is a Jordan domain. Otherwise, $\zeta$ is said to be of type 2. We see easily that if $\zeta$ is of type 2, then there are an open neighborhood $V$ of $\zeta$ and two Jordan domains $V_1, V_2$ such that $V \cap D = V_1 \cup V_2$. Moreover, $D$ is said to be Jordan-curve-like on a subset $A$ of $\partial D$ if $D$ is Jordan-curve-like at all points of $A$.

Now let $D \subset X$ be an open set which is Jordan-curve-like on a set $A \subset \partial D$. In the remaining part of this subsection we will introduce various notions. We like to point out that these notions are intrinsic, i.e., they do not depend on any choice (of open neighborhoods, Jordan domains, conformal mappings ...) we made in their definitions.

$A$ is said to be Jordan-measurable if for every $\zeta \in A$ the following condition is fulfilled:

**Case 1: $\zeta$ is of type 1.** There are an open neighborhood $U = U_\zeta$ of $\zeta$ such that $U \cap D$ is a Jordan domain and a conformal mapping $\Phi = \Phi_\zeta$ from $U \cap D$ onto the unit disc $E$ which extends homeomorphically from $U \cap D$ onto $E$ such that $\Phi(U \cap D \cap A)$ is Lebesgue measurable on $\partial E$.

**Case 2: $\zeta$ is of type 2.** There are an open neighborhood $U = U_\zeta$ of $\zeta$ such that $U \cap D = U_1 \cup U_2$ with Jordan domains $U_1 = U_{1,\zeta}$, $U_2 = U_{2,\zeta}$, and conformal mappings $\Phi_j = \Phi_{j,\zeta}$ ($j = 1, 2$) from $U_j$ onto $E$ which extend homeomorphically from $U_j \cap A$ onto $E$ such that $\Phi_j(U_j \cap A)$ is Lebesgue measurable (on $\partial E$).

A Jordan-measurable set $A \subset \partial D$ is said to be of zero length if for all $\zeta \in A$, if one takes $U_\zeta$, $\Phi_\zeta$ when $\zeta$ is of type 1 (resp. $U_\zeta$, $\Phi_{j,\zeta}$ when $\zeta$ is of type 2) as
in the previous definition and notation, then \( \text{mes} \left( \Phi \zeta (U \cap D \cap A) \right) = 0 \) (resp. \( \text{mes} \left( \Phi_j \zeta (U_j \cap A) \right) = 0 \), \( j = 1, 2 \)).

A Jordan-measurable set \( A \subset \partial D \) is said to be of positive length if it is not of zero length.

Suppose that \( D \) is Jordan-curve-like at a point \( \zeta \in \partial D \). We define the concept of angular approach regions at \( \zeta \) as follows. For any \( 0 < \alpha < \pi/2 \), the Stolz region or angular approach region \( A_\alpha (\zeta) \) is given by:

**Case 1:** \( \zeta \) is of type 1.

\[
A_\alpha (\zeta) := \Phi^{-1} \left\{ t \in E : \left| \arg \left( \frac{\Phi(\zeta) - t}{\Phi(\zeta)} \right) \right| < \alpha \right\},
\]

where \( \arg : \mathbb{C} \rightarrow (-\pi, \pi] \) is as usual the argument function.

**Case 2:** \( \zeta \) is of type 2.

\[
A_\alpha (\zeta) := \bigcup_{j=1,2} \Phi_j^{-1} \left\{ t \in E : \left| \arg \left( \frac{\Phi_j(\zeta) - t}{\Phi_j(\zeta)} \right) \right| < \alpha \right\}.
\]

Geometrically, \( A_\alpha (\zeta) \) is the intersection of \( D \) with one or two “cones” of aperture \( 2\alpha \) and vertex \( \zeta \) according to the type of \( \zeta \).

Let \( \zeta \in \partial D \) a point at which \( D \) is Jordan-curve-like and let \( U \) be an open neighborhood of \( \zeta \). We say that a function \( f \) defined on \( U \cap D \) admits the angular limit \( \lambda \) at \( \zeta \) if

\[
\lim_{z \in A_\alpha (\zeta), z \to \zeta} f(z) = \lambda,
\]

for all \( 0 < \alpha < \pi/2 \).

Let \( A \subset \partial D \) be a Jordan-measurable set and \( f : D \rightarrow \mathbb{C}, g : A \rightarrow \mathbb{C} \) two functions. Then \( f \) is said to have the angular limit \( g(a) \) for Jordan a.e. \( a \in A \), if the set

\[
\{ a \in A : f \text{ does not admit the angular limit } g(a) \text{ at } a \}
\]

is of zero length. For simplicity, in the future we only write “a.e.” instead of “Jordan a.e.”.

We conclude this subsection with a simple example which may clarify the above definitions. Let \( G \) be the open square in \( \mathbb{C} \) whose four vertices are \( 1 + i, -1 + i, -1 - i, \) and \( 1 - i \). Define the domain

\[
D := G \setminus \left[ -\frac{1}{2}, \frac{1}{2} \right].
\]

Then \( D \) is Jordan-curve-like on \( \partial G \cup (-\frac{1}{2}, \frac{1}{2}) \). Every point of \( \partial G \) is of type 1 and every point of \( (-\frac{1}{2}, \frac{1}{2}) \) is of type 2.

2.2. **Harmonic measure.** Let \( X \) be a complex manifold of dimension 1, let \( D \) be an open subset of \( X \) and let \( A \subset \partial D \). Consider the characteristic function

\[
1_{\partial D \setminus A}(\zeta) := \begin{cases} 1, & \zeta \in \partial D \setminus A, \\ 0, & \zeta \in A. \end{cases}
\]
Then the harmonic measure of the set $\partial D \setminus A$ (denoted by $\omega(\cdot, A, D)$) is the Perron solution of the generalized Dirichlet problem with boundary data $1_{\partial D \setminus A}$. In other words, one has

$$\omega(\cdot, A, D) := \sup_{u \in \mathcal{U}} u,$$

where $\mathcal{U} = \mathcal{U}(A, D)$ denotes the family of all subharmonic functions $u$ on $D$ such that $\limsup_{D \ni z \to \zeta} u(z) \leq 1_{\partial D \setminus A}(\zeta)$ for each $\zeta \in \partial D$.

It is well-known (see, for example, the book of Ransford [13] for the case $X := \mathbb{C}$) that $\omega(\cdot, A, D)$ is harmonic on $D$.

For a point $\zeta \in \partial D$ at which $D$ is a Jordan-curve-like, we say that it is a locally regular point relative to $A$ if

$$\lim_{z \to \zeta, z \in A} \alpha(\zeta) \omega(z, A \cap U, D \cap U) = 0$$

for any $0 < \alpha < \frac{\pi}{2}$ and any open neighborhood $U$ of $\zeta$. Obviously, $\zeta \in \overline{A}$. If, moreover, $\zeta \in A$, then $\zeta$ is said to be a locally regular point of $A$. The set of all locally regular points relative to $A$ is denoted by $A^*$. Observe that, in general, $A^* \not\subset A$, $A \not\subset A^*$.

As an immediate consequence of the Subordination Principle for the harmonic measure (see Corollary 4.3.9 in [13]), one gets

$$\lim_{z \to \zeta, z \in A^*} \omega(z, A, D) = 0, \quad \zeta \in A^*, \ 0 < \alpha < \frac{\pi}{2}. \tag{2.1}$$

We extend the function $\omega(\cdot, A, D)$ to $D \cup A^*$ by simply setting

$$\omega(z, A, D) := 0, \quad z \in A^*.$$

Geometric properties of the harmonic measure will be discussed in Section 4 below.

2.3. Cross and separate holomorphicity. Let $X$, $Y$ be two complex manifolds of dimension 1, let $D \subset X$, $G \subset Y$ be two open sets, let $A$ (resp. $B$) be a subset of $\partial D$ (resp. $\partial G$) such that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and that $A$ and $B$ are of positive length. We define a 2-fold cross $W$, its regular part $W^*$, its interior $W^\circ$, as

$$W := X(A, B; D, G) := ((D \cup A) \times B) \cup (A \times (B \cup G)),$$

$$W^* := X(A^*, B^*; D, G),$$

$$W^\circ := X^\circ(A, B; D, G) := (A \times G) \cup (D \times B).$$

Moreover, put

$$\omega(z, w) := \omega(z, A^*, D) + \omega(w, B^*, G), \quad (z, w) \in (D \cup A^*) \times (G \cup B^*).$$

It is clear that $\omega|_{D \times G}$ is harmonic.

For a 2-fold cross $W := X(A, B; D, G)$ define its wedge

$$\widehat{W} = \widehat{X}(A, B; D, G) := \{(z, w) \in (D \cup A^*) \times (G \cup B^*) : \omega(z, w) < 1\}.$$
Then the set of all interior points of the wedge $\hat{W}$ is given by

$$\hat{W}^\circ := \hat{X}(A, B; D, G) := \{ (z, w) \in D \times G : \omega(z, w) < 1 \}.$$

In particular, if $A$ (resp. $B$) is an open set of $\partial D$ (resp. $\partial G$), one has $A \times B \subset A^* \times B^*$ and $W \subset W^* \subset \hat{W}$.

We say that a function $f : W \rightarrow \mathbb{C}$ is separately holomorphic on $W^\circ$ and write $f \in \mathcal{O}_s(W^\circ)$, if for any $a \in A$ (resp. $b \in B$) the function $f(a, \cdot)|_G$ (resp. $f(\cdot, b)|_D$) is holomorphic on $G$ (resp. on $D$).

We say that a function $f : W \rightarrow \mathbb{C}$ (resp. $f : A \times B \rightarrow \mathbb{C}$) is separately continuous on $W$ (resp. on $A \times B$) and write $f \in \mathcal{C}_s(W)$ (resp. $f \in \mathcal{C}_s(A \times B)$), if it is continuous with respect to any variable when the remaining variable is fixed.

In the remaining part of this subsection we introduce two notions. As in Subsection 2.1 we like to point out that these notions are intrinsic, i.e., they do not depend on any choice we made in their definitions.

We say that a function $f : A \times B \rightarrow \mathbb{C}$ is Jordan-measurable on $A \times B$, if for every point $\zeta \in A$ with type $n$ (resp. $\eta \in B$ with type $m$) there is an open neighborhood $U$ of $\zeta$ (resp. $V$ of $\eta$) such that $U \cap D = \bigcup_{1 \leq j \leq n} U_j$ (resp. $V \cap G = \bigcup_{1 \leq k \leq m} V_k$) with Jordan domains $U_j$, $V_k$, and conformal mappings $\Phi_j$ (resp. $\Psi_k$) from $U_j$ (resp. $V_k$) onto $E$ which extends homeomorphically from $\overline{U_j}$ (resp. $\overline{V_k}$) onto $E$ such that $f(\Phi_j^{-1}(\cdot), \Psi_k^{-1}(\cdot)) : \Phi_j(U_j \cap A) \times \Psi_k(V_k \cap B) \rightarrow \mathbb{C}$ is Lebesgue measurable.

Two Jordan-measurable functions $f, g : A \times B \rightarrow \mathbb{C}$ are said to be equal a.e. on $A \times B$, if for every point $\zeta \in A$ with type $n$ (resp. $\eta \in B$ with type $m$), the functions

$$f(\Phi_j^{-1}(\cdot), \Psi_k^{-1}(\cdot)), g(\Phi_j^{-1}(\cdot), \Psi_k^{-1}(\cdot)) : \Phi_j(U_j \cap A) \times \Psi_k(V_k \cap B) \rightarrow \mathbb{C}$$

are equal a.e. (we keep the previous notation).

We say that a function $f : \hat{W}^\circ \rightarrow \mathbb{C}$ admits an angular limit $\lambda \in \mathbb{C}$ at $(a, b) \in \hat{W}$ if the following limit holds:

**Case 1:** $a \in D$ and $b \in G$ :

$$\lim_{z \rightarrow a, w \rightarrow b} f(z, w) = \lambda;$$

**Case 2:** $a \in A^*$ and $b \in G$ :

$$\lim_{z \rightarrow a, \ z \in A_0(a), \ w \rightarrow b} f(z, w) = \lambda, \quad 0 < \alpha < \frac{\pi}{2};$$

**Case 3:** $a \in D$ and $b \in B^*$ :

$$\lim_{z \rightarrow a, \ w \rightarrow b, \ w \in A_0(b)} f(z, w) = \lambda, \quad 0 < \alpha < \frac{\pi}{2};$$

**Case 4:** $a \in A^*$ and $b \in B^*$ :

$$\lim_{z \rightarrow a, \ z \in A_0(a), \ w \rightarrow b, \ w \in A_0(b)} f(z, w) = \lambda, \quad 0 < \alpha < \frac{\pi}{2}.$$
2.4. Motivations for our work. We are now able to formulate what, in the sequel, we quote as the classical version of the boundary cross theorem.

**Theorem 1.** (Gonchar \[5, 6\]) Let $D, G \subset \mathbb{C}$ be Jordan domains and $A$ (resp. $B$) a nonempty open set of the boundary $\partial D$ (resp. $\partial G$). Then, for any function $f \in C(W) \cap O_s(W^o)$, there is a unique function $\hat{f} \in C(\hat{W}) \cap O(\hat{W}^o)$ such that $\hat{f} = f$ on $W$. Moreover, if $|f|_W < \infty$ then

$$|\hat{f}(z, w)| \leq |f|_{A \times B}^{1-\omega(z, w)} |f|_W^{\omega(z, w)}, \quad (z, w) \in \hat{W},$$

where $W, W^o, \text{ and } \hat{W}$ denote the 2-fold cross, its interior and its wedge, respectively, associated to $A, B, D, G$.

Theorem 1 admits various generalizations. The following result is announced by Gonchar in \[5\].

**Theorem 2.** Let $D, G \subset \mathbb{C}$ be Jordan domains and $A$ (resp. $B$) a nonempty open and rectifiable set of the boundary $\partial D$ (resp. $\partial G$). Let $f$ be a function defined on the 2-fold cross $W$ with the following properties:

(i) $f|_W^o \in C(W^o) \cap O_s(W^o)$;
(ii) $f$ is locally bounded on $W$;
(iii) for any $a \in A$ (resp. $b \in B$), the holomorphic function $f(a, \cdot)|_G$ (resp. $f(\cdot, b)|_D$) has the angular limit $f_1(a, b)$ at $b$ for a.e. $b \in B$ (resp. $f_2(a, b)$ at $a$ for a.e. $a \in A$) and $f_1 = f_2 = f$ a.e. on $A \times B$.

1) Then there is a unique function $\hat{f} \in O(\hat{W}^o)$ such that

$$\lim_{(z, w) \to (\zeta, \eta)} \hat{f}(z, w) = f(\zeta, \eta), \quad (\zeta, \eta) \in W^o.$$

2) If, moreover, $|f|_W < \infty$, then

$$|\hat{f}(z, w)| \leq |f|_{A \times B}^{1-\omega(z, w)} |f|_W^{\omega(z, w)}, \quad (z, w) \in \hat{W}^o.$$

3) If, moreover, $f$ is continuous at a point $(a, b) \in A \times B$, then

$$\lim_{(z, w) \to (a, b)} \hat{f}(z, w) = f(a, b).$$

On the other hand, the following result due to Drużkowski \[2\] gives a different flavor.

**Theorem 3.** Let $D, G \subset \mathbb{C}$ be Jordan domains and $A$ (resp. $B$) a nonempty open connected set of the boundary $\partial D$ (resp. $\partial G$). Let $f$ be a function defined on $W$ with the following properties:

(i) $f \in C_s(W) \cap O_s(W^o)$;
(ii) $f$ is locally bounded on $W$;
(iii) $f|_{A \times B}$ is continuous on $A \times B$.

Then all conclusions of Theorem 1 still hold.
Observe that all these theorems require the following very strong hypothesis: $D$ and $G$ are Jordan domains in $\mathbb{C}$ and $A \times B$ is an open set of $\partial D \times \partial G$. Moreover, the assumptions on the boundedness and continuity of $f$ are rather restrictive.

A natural question is whether Theorems 1–3 are still true if $D, G$ are open sets in complex manifolds of dimension 1 and the $A$ (resp. $B$) is not necessarily an open set of $\partial D$ (resp. $\partial G$). In addition, if one drops the hypothesis on the local boundedness and the continuity of $f$, can one obtain a holomorphic extension of $f$ and what are its properties? These matters seem to be of interest, especially when one seeks to generalize Theorems 1–3 to higher dimensions.

The present paper is motivated by these questions. Our first purpose is to generalize Gonchar’s theorems to a very general situation, where $A, B$ are almost general open subsets of complex manifolds of dimension 1 and where the boundary sets $A, B$ are almost general subsets of $\partial D, \partial G$. Our second goal is to establish, in this general context, an extension theorem analogous to Drużkowski’s theorem with a minimum of hypotheses on $f$.

3. Statement of the main results and outline of the proofs

We are now ready to state our main result.

**Theorem A.** Let $X, Y$ be two complex manifolds of dimension 1, let $D \subset X, G \subset Y$ be two open sets and $A$ (resp. $B$) a subset of $\partial D$ (resp. $\partial G$) such that $D$ (resp. $G$) is Jordan-curve-like on $A$ (resp. $B$) and that $A$ and $B$ are of positive length. Let $f : W \rightarrow \mathbb{C}$ be such that:

(i) $f$ is locally bounded on $W$ and $f \in O_s(W^o)$;
(ii) $f|_{A \times B}$ is Jordan-measurable;
(iii) for any $a \in A$ (resp. $b \in B$), the holomorphic function $f(a, \cdot)|_G$ (resp. $f(\cdot, b)|_D$) has the angular limit $f_1(a,b)$ at $b$ for a.e. $b \in B$ (resp. $f_2(a,b)$ at $a$ for a.e. $a \in A$) and $f_1 = f_2 = f$ a.e. on $A \times B$.

Then there exists a unique function $\hat{f} \in \mathcal{O}(\hat{W}^o)$ with the following property:

1) there are subsets $\hat{A} \subset A \cap A^*$ and $\hat{B} \subset B \cap B^*$ such that the sets $A \setminus \hat{A}$ and $B \setminus \hat{B}$ are of zero length\(^1\) and $\hat{f}$ admits the angular limit $f(\zeta, \eta)$ at every point $(\zeta, \eta) \in X^o(\hat{A}, \hat{B}; D, G)$.

In addition, $\hat{f}$ enjoys the following properties:

2) If $|f|_W < \infty$, then

$$|\hat{f}(z, w)| \leq |f|_{A \times B}^{1-\omega(z,w)} |f|_W^{\omega(z,w)}, \quad (z, w) \in \hat{W}^o.$$  

3) For any $(a_0, w_0) \in A^* \times G$ (resp. $(z_0, b_0) \in D \times B^*$) if

$$\lim_{(z, w) \rightarrow (a_0, w_0)} f(z, w)(=: \lambda) \quad \text{(resp.} \quad \lim_{(z, w) \rightarrow (z_0, b_0)} f(z, w)(=: \lambda) \text{exists)},$$

then $\hat{f}$ admits the angular limit $\lambda$ at $(a_0, w_0)$ (resp. at $(z_0, b_0)$).

4) For any $(a_0, b_0) \in A^* \times B^*$, if

$$\lim_{(a, b) \rightarrow (a_0, b_0)} f(a, b)(=: \lambda) \text{ exists},$$

then $\hat{f}$ admits the angular limit $\lambda$ at $(a_0, b_0)$.

\(^1\)Under this condition it follows from Part 1) of Theorem below that $\hat{A} \subset \hat{A}^*$ and $\hat{B} \subset \hat{B}^*$.\)
5) If \( f|_{A \times B} \) can be extended to a continuous function defined on \( A^* \times B^* \), then \( f \) can be extended to a unique continuous function (still denoted by) \( f \) defined on \( W^* := X(A^*, B^*; D, G) \) and \( \hat{f} \) admits the angular limit \( f(\zeta, \eta) \) at every \( (\zeta, \eta) \in W^* \) and \( f_1 = f_2 = f \) on \( (A \cap A^*) \times (B \cap B^*) \).

Theorem A has an immediate consequence.

**Corollary A'.** We keep the hypotheses and the notation of Theorem A. Suppose in addition that \( f \in \mathcal{C}(W^*) \). Then the function \( \hat{f} \in \mathcal{O}(\hat{W}^*) \) provided by Theorem A admits the angular limit \( f(\zeta, \eta) \) at every point \( (\zeta, \eta) \in ((A \cap A^*) \times G) \cup (D \times (B \cap B^*)) \).

It is worthy to note that Theorem A and Corollary A' generalize, in some sense, Theorems 1–3.

Now we drop the hypothesis on local boundedness and continuity of \( f \). Then the examples of Družkowski in [2] (see Section 10 below) show that, without these conditions, the extended function \( \hat{f} \) (if it does exist) is, in general, not continuous on \( \hat{W} \). However, our second main result gives a partially positive answer to this question.

**Theorem B.** Let \( X, Y \) be two complex manifolds of dimension 1, let \( D \subset X \), \( G \subset Y \) be two open sets, let \( A \) (resp. \( B \)) be a subset of \( \partial D \) (resp. \( \partial G \)) such that \( D \) (resp. \( G \)) is Jordan-curve-like on \( A \) (resp. \( B \)) and that \( A \) and \( B \) are of positive length. Let \( f : W \to \mathbb{C} \) satisfy the following properties:

(i) \( f|_{A \times B} \in \mathcal{C}_s(A \times B) \) and \( f \in \mathcal{O}_s(W^*) \);

(ii) for any \( a \in A \) (resp. \( b \in B \)), the function \( f(a, \cdot) \) (resp. \( f(\cdot, b) \)) is locally bounded on \( G \cup B \) (resp. \( D \cup A \)) and the (holomorphic) restriction function \( f(a, \cdot)|_G \) (resp. \( f(\cdot, b)|_D \)) has the angular limit \( f(a, b) \) at \( b \) for every \( b \in B \) (resp. at \( a \) for every \( a \in A \)).

Then there are subsets \( \hat{A} \subset A \cap A^* \) and \( \hat{B} \subset B \cap B^* \), and a unique function \( \hat{f} \in \mathcal{O}(\hat{W}^*) \) with the following properties:

1) the sets \( A \setminus \hat{A} \) and \( B \setminus \hat{B} \) are of zero length;

2) \( \hat{f} \) admits the angular limit \( f(\zeta, \eta) \) at every point \( (\zeta, \eta) \in X(\hat{A}, \hat{B}; D, G) \).

Observe that if \( f \in \mathcal{C}_s(W) \cap \mathcal{O}_s(W^*) \), then conditions (i)–(ii) above are fulfilled. Although our results have been stated only for the case of a 2-fold cross, they can be formulated for the general case of an \( N \)-fold cross with \( N \geq 2 \) (see also [9] [10]).

Now we present some ideas how to prove Theorems A and B.

Our method consists of two steps. In the first step we suppose that \( D \) and \( G \) are Jordan domains in \( \mathbb{C} \). In the second one we treat the general case. The key technique here is to use level sets of the harmonic measure. More precisely, we exhaust \( D \) (resp. \( G \)) by the level sets of the harmonic measure \( \omega(\cdot, A, D) \) (resp. \( \omega(\cdot, B, G) \)), i.e. by \( D_\delta := \{z \in D : \omega(z, A, D) < 1 - \delta \} \) (resp. \( G_\delta := \{w \in G : \omega(w, B, G) < 1 - \delta \} \)) for \( 0 < \delta < 1 \).

In order to carry out the first step, we improve Gonchar’s method [5] [9] and make intensive use of Carleman’s formula and of geometric properties of the level sets of harmonic measures.
The main ingredient for the second step is a mixed cross type theorem (see also \[10\]) valid for measurable boundary sets in the context of complex manifolds of dimension 1. We prove that theorem using a recent work of Zeriahi (see \[15\]) and the classical method of doubly orthogonal bases of Bergman type.

In the second step we apply this mixed cross type theorems in order to prove Theorems A and B with $D$ (resp. $G$) replaced by $D_\delta$ (resp. $G_\delta$). Then we construct the solution for the original open sets $D$ and $G$ by means of a gluing procedure (see also \[9\]).

4. Properties of the harmonic measure and its level sets

In this section $X$ is a complex manifold of dimension 1, $D \subset X$ an open set, and $A$ a nonempty Jordan-measurable subset of $\partial D$. Observe that then $\partial D$ is non-polar.

Let $\mathcal{P}_D$ be the generalized Poisson integral of $D$. If, in addition, $A$ is a Borel set, then, by Theorem 4.3.3 of \[13\], the harmonic measure of $\partial D \setminus A$ is given by

$$\omega(\cdot, A, D) = \mathcal{P}_D[1_{\partial D \setminus A}].$$

The following elementary lemma will be useful.

**Lemma 4.1.** Let $E$ be the unit disc and $A$ a measurable subset of $\partial E$.

1) Let $u$ be a subharmonic function defined on $E$ with $u \leq 1$ and let $\alpha \in (0, \frac{\pi}{2})$ be such that

$$\limsup_{z \to \zeta, z \in A_\alpha(\zeta)} u(z) \leq 0 \quad \text{for a.e. } \zeta \in A.$$

Then $u \leq \omega(\cdot, A, E)$ on $E$.

2) For all density points $\zeta$ of $A$,

$$\lim_{z \to \zeta, z \in A_\alpha(\zeta)} \omega(z, A, E) = 0, \quad 0 < \alpha < \frac{\pi}{2}.$$

In particular, all density points of $A$ are contained in $A^*$.

3) For all interior points $\zeta$ of $A$,

$$\lim_{z \to \zeta} \omega(z, A, E) = 0.$$

**Proof.** It follows almost immediately from the explicit formula for $\mathcal{P}_E$. \qed

**Proposition 4.2.** (Maximum Principle) Let $u \in \mathcal{SH}(D)$ be such that $u$ is bounded from the above and

$$\limsup_{z \to \zeta} u(z) \leq 0, \quad \zeta \in \partial D \setminus A,$$

$$\limsup_{z \to \zeta, z \in A_\alpha(\zeta)} u(z) \leq 0, \quad \zeta \in A, \quad 0 < \alpha < \frac{\pi}{2}.$$

Then $u \leq 0$ on $D$.

**Proof.** Suppose that $u < M$ for some $M$. Let $\zeta_0$ be an arbitrary point of $A$. Fix a Jordan domain $U$ such that $U \subset D$ and $\partial U \cap \partial D$ is a closed arc which is a
neighborhood of $\zeta_0$ in $\partial D$. Let $B$ be an open arc in $\partial U \cap \partial D$ which contains $\zeta_0$. Part 1) and Part 3) of Lemma 4.1 applied to $u|_U$ yield that
\[
\limsup_{z \to \zeta, z \in U} u(z) \leq M \cdot \limsup_{z \to \zeta, z \in U} \omega(z, B, U) = 0, \quad \zeta \in B.
\]
Since $\zeta_0 \in B$ and $\zeta_0$ is an arbitrary point of $A$, we deduce that
\[
\limsup_{z \to \zeta, z \in D} u(z) \leq 0, \quad \zeta \in A.
\]
Combining this with the hypothesis, the desired conclusion follows from the classical Maximum Principle (see Theorem 2.3.1 in [13]).

In the sequel we formulate some important stability property of the harmonic measure. Let $\phi : \partial D \to \mathbb{R}$ be a bounded function. The associated Perron function $H_{D,A} : D \to \mathbb{R}$ is defined by
\[
(4.2) \quad H_{D,A}[\phi] := \sup_{u \in \widehat{U}} u,
\]
where $\widehat{U} = \widehat{U}(\phi, A, D)$ denotes the family of all subharmonic functions $u$ on $D$ such that
\[
\limsup_{z \to \zeta, z \in \partial D \setminus A} u(z) \leq \phi(\zeta), \quad \zeta \in \partial D \setminus A,
\]
\[
\limsup_{z \to \zeta, z \in A, 0 < \alpha < \pi/2} u(z) \leq \phi(\zeta), \quad \zeta \in A.
\]
In the sequel, $\widehat{U}(A, D)$ will stand for $\widehat{U}(1_{\partial D \setminus A}, A, D)$.

Using the above proposition, the corresponding results in Sections 4.1 and 4.2 of [13] with respect to $H_{D,A}$ (instead of $H_D$) are still valid making the obviously necessary changes. In particular, we have the following (see Corollary 4.2.6 in [13]):

**Proposition 4.3.** Let $D$ be an open subset of $X$, $A$ a nonempty Jordan-measurable subset of $\partial D$, and $\phi : \partial D \to \mathbb{R}$ a bounded function which is continuous nearly everywhere on $\partial D$. Then there exists a unique bounded harmonic function $h$ on $D$ such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for nearly everywhere $\zeta \in \partial D$. Moreover, $h = H_D[\phi] = H_{D,A}[\phi]$.

In virtue of this result, Theorem 4.3.3 in [13] is still valid in the context of $H_{D,A}$. More precisely,

**Proposition 4.4.** Let $D$ be an open subset of $X$, $A$ a nonempty Jordan-measurable subset of $\partial D$, and $\phi : \partial D \to \mathbb{R}$ a bounded Borel function. Then $H_D[\phi] = H_{D,A}[\phi] = \mathcal{P}_D[\phi]$.

In the special case $X := \mathbb{C}$ we can say even more.

**Proposition 4.5.** Let $D$ be a proper open subset of $\mathbb{C}$. Let $A$ be a nonempty Borel subset of $\partial D$ such that $D$ is Jordan-curve-like on $A$ and $A$ is of zero length. Then $\mathcal{P}_D[1_A] \equiv 0$ on $D$.\footnote{A property is said to hold nearly everywhere on $\partial D$ if it holds everywhere on $\partial D \setminus \mathcal{N}$ for some Borel polar set $\mathcal{N}$.}
Proof. Suppose without loss of generality that $D$ is Jordan-curve-like on the interval $[0, 1] \subset \partial D$ and that $A$ is a Borel subset of $[0, 1]$ with $\text{mes}(A) = 0$. Since $D \subset \mathbb{C}\setminus[0, 1]$, it follows from the Subordination Principle that

$$P_D[1_A] \leq P_{\mathbb{C}\setminus[0,1]}[1_A] \quad \text{on } D.$$ 

Therefore, it suffices to show that $P_{\mathbb{C}\setminus[0,1]}[1_A] \equiv 0$ on $\mathbb{C}\setminus[0,1]$. To this end consider the conformal mapping $\Phi(z) := \sqrt{\frac{1}{z} - 1}$ which maps $\mathbb{C} \cup \{\infty\} \setminus [0,1]$ onto $\mathbb{H} := \{z \in \mathbb{C} : \text{Im } z > 0\}$. It is not difficult to show that

$$P_{\mathbb{C}\setminus[0,1]}[1_A] = P_{\mathbb{H}}[1_{\Phi(A)}] \circ \Phi^{-1} \equiv 0.$$

This concludes the proof. \qed

Now we arrive at one of the main results of the section

**Theorem 4.6.** Let $D$ be an open subset of $X$, $A$ a nonempty Jordan-measurable subset of $\partial D$, and $\mathcal{N}$ a Jordan-measurable subset of $\partial D$ which is of zero length.

1) Then $A^*$ is a Borel set and $(A^*)^* = A^*$ and $(A \setminus \mathcal{N})^* = A^*$ and $A \setminus A^*$ is of zero length.

2) If $A$ is a Borel set then $\omega(z, A, D) = H_{D,A}[1_{\partial D\setminus A}]$ for $z \in D$. In particular,

$$\omega(z, A^*, D) = H_{D,A^*}[1_{\partial D\setminus A^*}] = H_{D,(A\setminus\mathcal{N})\setminus\mathcal{N}}[1_{\partial D\setminus((A\setminus\mathcal{N})\setminus\mathcal{N})}] = \omega(z, A, D),$$

$z \in D$.

3) If $X = \mathbb{C}$ then $\omega(z, A, D) = \omega(z, A \setminus \mathcal{N}, D) = \omega(z, A^*, D)$.

Proof. Part 1) can be checked using the definition and Lemma 4.5.

Part 2) is an immediate consequence of Proposition 4.3 and Part 1).

Now we turn to Part 3). Choose two Borel sets $A_1$, $A_2$ so that $A_1 \subset A \setminus \mathcal{N}$ and $A \subset A_2 \subset \partial D$ and $A_2 \setminus A_1$ is of zero length. Then we conclude by the Subordination Principle and Proposition 4.3 that

$$\omega(z, A_2, D) \leq \omega(z, A, D) \leq \omega(z, A \setminus \mathcal{N}, D) \leq \omega(z, A_1, D) = \omega(z, A_2, D), \quad z \in D.$$

This proves the first identity.

Since $A^*$ is, by Part 1), a Borel set, Part 2) gives that

$$\omega(z, A^*, D) = H_{D,A^*}[1_{\partial D\setminus A^*}].$$

Consequently, $\omega(z, A, D) \leq \omega(z, A^*, D)$, $z \in D$. On the other hand, let $B$ be a Borel set such that $B \subset A \cap A^*$ and $A \setminus B$ is of zero length. Then

$$\omega(\cdot, A, D) = \omega(\cdot, B, D) = H_{D,B}[1_{\partial D\setminus B}] \geq H_{D,A^*}[1_{\partial D\setminus A^*}] = \omega(\cdot, A^*, D) \quad \text{on } D.$$

Combining the above estimates, the proof of the last identity in Part 3) follows. \qed

**Proposition 4.7.** Let $D$ be an open subset of $X$ and $A$ a nonempty Jordan-measurable subset of $\partial D$. Let $(D_k)_{k=1}^\infty$ be a sequence of open subsets $D_k$ of $D$ and $(A_k)_{k=1}^\infty$ a sequence of Jordan-measurable subset of $\partial D$ which are also subsets of $A$ such that

(i) $D_k \subset D_{k+1}$ and $\bigcup_{k=1}^\infty D_k = D$;
(ii) $A_k \subset A_{k+1}$ and $A_k \subset \partial D \cap \partial D_k$ and $D_k$ is Jordan-curve-like on $A_k$ and $\bigcup_{k=1}^\infty A_k = A$;
(iii) for any point $\zeta \in A$ there is an open neighborhood $V = V_\zeta$ of $\zeta$ in $\mathbb{C}$ such that $V \cap D = V \cap D_k$ for some $k$.

Then 
$$\omega(z, A^*, D) = \lim_{k \to \infty} \omega(z, A^*_k, D_k), \quad z \in D.$$ 

**Proof.** Using the Subordination Principle it is easy to see that the sequence $(\omega(\cdot, A^*_k, D_k))_k^{\infty}$ is decreasing and the following limit 
$$u := \lim_{k \to \infty} \omega(\cdot, A^*_k, D_k)$$ 
exists and defines a subharmonic function in $D$. By the Subordination Principle again, we have $u \geq \omega(\cdot, A^*, D)$. Therefore, it remains to establish the converse inequality. In virtue of (i)–(iii), we conclude that 
$$\sup_{0 < \alpha < \frac{\pi}{2}} \limsup_{z \to \zeta, z \in A_\alpha(\zeta)} u = 0, \quad \zeta \in B,$$ 
where $B := \bigcup_{k=1}^{\infty} A_k$.

On the other hand, since $(A \cap A^*) \setminus B \subset \bigcup_{k=1}^{\infty} (A_k \setminus A^*_k)$, Part 1) of Theorem 4.6 implies that $(A \cap A^*) \setminus B$ is of zero length. Consequently, we deduce from (4.3) and Part 2) of Theorem 4.6 that 
$$u(z) \leq \omega(z, A^*, D), \quad z \in D.$$ 
This completes the proof. 

Next, we introduce a notion which will be relevant for our further study.

**Definition 4.8.** Let $D, G \subset X$ be two open sets such that $G \subset D$ and let $\zeta$ be a point in $\partial D$ such that $D$ is Jordan-curve-like at $\zeta$. Then the point $\zeta$ is said to be an end-point of $G$ in $D$ if, for every $0 < \alpha < \frac{\pi}{2}$, there is an open neighborhood $U = U_\alpha$ of $\zeta$ such that $U \cap A^*_\alpha(\zeta) \subset G$. The set of all end-points of $G$ in $D$ is denoted by $G^D$.

It is worthy to remark that the above definition is intrinsic.

The remaining part of this section is devoted to the study of level sets of the harmonic measure. We begin with the following important properties of these sets.

**Theorem 4.9.** Let $D \subset X$ be an open set and $A$ a Jordan-measurable set of $\partial D$ such that $A$ is of positive length. Then, for any $0 < \epsilon < 1$, the “$\epsilon$-level set” 
$$D_\epsilon := \{ z \in D : \omega(z, A^*, D) < 1 - \epsilon \}$$ 
enjoys the following properties:

(i) Let $G_1, G_2$ be arbitrary distinct connected components of $D_\epsilon$, then $G_1^D \cap G_2^D = \emptyset$.

(ii) For any point $\zeta \in A^*$, there is exactly one connected component $G$ of $D_\epsilon$ such that $\zeta \in G^D$.

(iii) $G^D \cap A$ is Jordan-measurable (on $\partial D$) and of positive length for every connected component $G$ of $D_\epsilon$.

**Proof.** To prove (i), suppose, in order to reach a contradiction, that $G_1^D \cap G_2^D \neq \emptyset$. Fix a point $\zeta_0 \in G_1^D \cap G_2^D$. Then, for every $0 < \alpha < \frac{\pi}{2}$, there is an open neighborhood $U_\alpha$ of $\zeta_0$ such that $A_\alpha(\zeta_0) \cap U_\alpha \subset G_1 \cap G_2$. This implies that $G_1 \cap G_2 \neq \emptyset$. Hence,
$G_1 = G_2$, which contradicts the hypothesis that $G_1 \neq G_2$. The proof of (i) is complete.

Next, we turn to the proof of (ii). Fix a $\zeta_0 \in A^*$. In virtue of assertion (i), it suffices to show the existence of a connected component $G$ of $D_\epsilon$ such that $\zeta_0 \in G^D$. Since $\zeta \in A^*$, for every $0 < \alpha < \frac{\pi}{2}$, there is an open neighborhood $U_\alpha$ of $\zeta_0$ such that

\begin{equation}
A_\alpha(\zeta_0) \cap U_\alpha \subset D_\epsilon.
\end{equation}

Fix an arbitrary $0 < \alpha_0 < \frac{\pi}{2}$, and let $G$ be the connected component of $D_\epsilon$ containing $A_{\alpha_0}(\zeta_0) \cap U_{\alpha_0}$. Since $A_{\alpha_0}(\zeta_0) \cap U_{\alpha_0}$ is not empty, we deduce from (4.4) that $G$ also contains $A_\alpha(\zeta_0) \cap U_\alpha$ for every $0 < \alpha < \frac{\pi}{2}$. Hence $\zeta_0 \in G^D$. The proof of (ii) is finished.

Finally, we prove (iii). First, we may find a sequence $(U_k)_{k=1}^\infty$ of open sets of $X$ such that $U_k \cap D$ is either a Jordan domain or the disjoint union of two Jordan domains and $A \subset \bigcup_{k=1}^\infty \partial(U_k \cap D)$. Since $A$ is Jordan-measurable, we see that in order to prove the Jordan-measurability of $G^D \cap A$, it is sufficient to check that $G^D \cap \partial(D \cap U_k)$ is Jordan-measurable for every $k \geq 1$. To prove the latter assertion, fix an $k_0 \geq 1$ and let $U := U_{k_0}$. Let $\Phi$ be a conformal mapping from $D \cap U$ onto $E$ which extends to a homeomorphic mapping (still denoted by) $\Phi$ from $\overline{D \cap U}$ onto $\overline{E}$. It is clear that for any $\zeta \in \partial(D \cap U)$, $\zeta \in G^D$ if and only if $\Phi(\zeta) \in [\Phi(G \cap U)]^E$. We shall prove, in the sequel, that $[\Phi(G \cap U)]^E$ is a Borel subset of $\partial E$. Taking this for granted, then $G^D \cap \partial(D \cap U)$ is also a Borel set. Consequently, $G^D \cap A$ is Jordan-measurable.

To check that $[\Phi(G \cap U)]^E$ is a Borel set, put

\begin{equation}
A_{n,m}(\eta) := \left\{ w \in E \cap A_{(1-\frac{1}{m})^{\frac{2}{3}}}(\eta) : |w - \eta| < \frac{1}{m} \right\}, \quad n, m \geq 1, \quad \eta \in \partial E.
\end{equation}

For any $n, m, p \geq 1$, let

\begin{equation}
T_{nmp} := \left\{ \eta \in \partial E : A_{n,m}(\eta) \subset \Phi(G \cap U) \quad \text{and} \quad \omega(\Phi^{-1}(w), A_*, D) \leq 1 - \epsilon - \frac{1}{p}, \quad \forall w \in A_{n,m}(\eta) \right\}.
\end{equation}

We observe the following geometric fact:

Let $\eta_0 \in \partial E$ and $(\eta_q)_{q=1}^\infty \subset \partial E$ such that $\lim_{q \to \infty} \eta_q = \eta_0$. Then

\[ A_{n,m}(\eta_0) \subset \bigcup_{q=1}^\infty A_{n,m}(\eta_q). \]

The proof of this fact follows immediately from the geometric shape of the cone $A_{n,m}(\eta)$ given in (4.3).
Let \((\eta_q)_{q=1}^{\infty} \subset T_{nmp}\) such that \(\lim_{q \to \infty} \eta_q = \eta_0 \in \partial E\). Using the above geometric fact, we see that \(A_{n,m}(\eta_0) \subset \Phi(G \cap U)\). This, combined with (4.6) and the continuity of \(\omega(\Phi^{-1}(\cdot), A, D)|_E\), implies that \(\eta_0 \in T_{nmp}\). Hence, the set \(T_{nmp}\) is closed. Clearly, we have

\[
[\Phi(G \cap U)]^E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{\infty} T_{nmp}
\]

It follows immediately from this identity that \([\Phi(G \cap U)]^E\) is a Borel set. Consequently, as was already discussed before, \(G^D \cap A\) is Jordan-measurable.

To finish assertion (iii), it remains to prove that \(G^E \cap A\) is of positive length. Suppose, in order to reach a contradiction, that \(G^E \cap A\) is of zero length. Consider the following function

\[
u(z) := \begin{cases} \omega(z, A^*, D), & z \in D \setminus G \\ 1 - \epsilon, & z \in G \end{cases} \]

Then clearly \(u \in SH(D)\) and \(u \leq 1\). In virtue of assertions (i) and (ii) and the definition of locally regular points, we have that

\[
\sup_{0<\alpha<\frac{\pi}{2}} \limsup_{z \to \zeta, z \in A_\alpha(\zeta)} u(z) = \sup_{0<\alpha<\frac{\pi}{2}} \limsup_{z \to \zeta, z \in A_\alpha(\zeta)} \omega(z, A^*, E) = 0, \quad \zeta \in (A \cap A^*) \setminus (G^D \cap A).
\]

Consequently, using the notation in (4.2), we conclude that

\[
u \in \tilde{U}((A \cap A^*) \setminus N, D),
\]

where \(N := G^D \cap A\). Since, by our above assumption, \(N\) is of zero length, it follows from Theorem (4.10) that \(u \leq \omega(\cdot, A^*, D)\). But on the other hand, one has \(\omega(z, A^*, D) < 1 - \epsilon = u(z)\) for \(z \in G\). This leads to the desired contradiction. Hence, the proof of (iii) is finished. \(\square\)

**Theorem 4.10.** Let \(D \subset X\) be an open set and \(A\) a Jordan-measurable set of \(\partial D\) such that \(A\) is of positive length. For any \(0 \leq \epsilon < 1\), let \(D_\epsilon := \{z \in D : \omega(z, A^*, D) < 1 - \epsilon\}\).

1) For any Jordan-measurable subset \(N \subset \partial D\) of zero length, let

\[
U_\epsilon(A, N, D) := \left\{ u \in SH(D) : u \leq 1 \text{ and } \sup_{0<\alpha<\frac{\pi}{2}} \limsup_{z \to \zeta, z \in A_\alpha(\zeta)} u(z) \leq 0, \quad \zeta \in (A \cap A^*) \setminus N \right\}.
\]

Then \(U_\epsilon(A, N, D) = U_\epsilon(A, \emptyset, D)\).

2) Define the “harmonic measure of the \(\epsilon\)-level set” \(\omega_\epsilon(\cdot, A, D)\) as

\[
\omega_\epsilon(z, A, D) := \begin{cases} \sup_{u \in U_\epsilon(A, \emptyset, D)} u(z), & z \in D_\epsilon \\ 0, & z \in A^* \end{cases}.
\]

Then

\[
\omega_\epsilon(z, A, D) = \frac{\omega(z, A^*, D)}{1 - \epsilon}, \quad z \in D_\epsilon \cup A^*.
\]
Proof. Clearly, by definition, \( U_e(A, \emptyset, D) \subset U_e(A, \mathcal{N}, D) \). To prove the converse inclusion, fix an arbitrary \( u \in U_e(A, \mathcal{N}, D) \). Consider the following function

\[
\hat{u}(z) := \begin{cases} 
\max \{(1 - \epsilon)u(z), \omega(z, A^*, D)\}, & z \in D_e \\
\omega(z, A^*, D), & z \in D \setminus D_e.
\end{cases}
\]

Then \( \hat{u} \in \mathcal{SH}(D) \) and \( \hat{u} \leq 1 \). Moreover, in virtue of (ii) of Theorem \ref{thm:angular}, we have that \( A^* \subset (D_e)^D \). Consequently, for every \( \zeta \in (A \cap A^*) \setminus \mathcal{N} \),

\[
\tag{4.7} \sup_{0 < \alpha < \frac{\pi}{2}} \limsup_{z \to \zeta, \ z \in A_\alpha(\zeta)} \hat{u}(z) 
\leq \max \left\{ \sup_{0 < \alpha < \frac{\pi}{2}} \limsup_{z \to \zeta, \ z \in A_\alpha(\zeta)} u(z), \sup_{0 < \alpha < \frac{\pi}{2}} \limsup_{z \to \zeta, \ z \in A_\alpha(\zeta)} \omega(z, A, D) \right\}.
\]

Observe that the first term in the latter line of (4.7) is equal to 0 because \( u \in U_e(A, \mathcal{N}, D) \). In addition, the second term in the latter line of (4.7) is also equal to 0. Hence, \( \hat{u} \in \hat{U}((A \cap A^*) \setminus \mathcal{N}, D) \). Consequently, by Theorem \ref{thm:angular} \( \hat{u} \leq \omega(\cdot, A^*, D) \).

In particular, one has

\[
\tag{4.8} u(z) \leq \frac{\omega(z, A^*, D)}{1 - \epsilon}, \quad z \in D, \ u \in U_e(A, \mathcal{N}, D).
\]

On the other hand, it is clear that \( \frac{\omega(z, A^*, D)}{1 - \epsilon} \in U_e(A, \emptyset, D) \subset U_e(A, \mathcal{N}, D) \). This, combined with (4.8), implies the desired conclusions of Part 1) and Part 2). \( \square \)

An immediate consequence of Theorem \ref{thm:angular} is the following Two-Constant Theorem for level sets.

**Corollary 4.11.** Let \( D \subset X \) be an open set and \( A, \mathcal{N} \) two Jordan-measurable subsets of \( \partial D \) such that \( A \) is of positive length and \( \mathcal{N} \) is of zero length. Let \( 0 \leq \epsilon < 1 \) and put \( D_\epsilon := \{ z \in D : \omega(z, A^*, D) < 1 - \epsilon \} \). If \( u \in \mathcal{SH}(D_\epsilon) \) satisfies \( u \leq M \) on \( D_\epsilon \) and \( \sup_{0 < \alpha < \frac{\pi}{2}} \limsup_{z \to \zeta, \ z \in A_\alpha(\zeta)} u(z) \leq m, \ \zeta \in (A \cap A^*) \setminus \mathcal{N} \), then

\[
u(z) \leq m(1 - \omega(\cdot, (A, D))) + M \cdot \omega(\cdot, D).
\]

5. **Boundary behaviour of the Gonchar–Carleman operator**

Before recalling the Gonchar–Carleman operator and investigating its boundary behavior, we first introduce the following notion and study its properties.

5.1. **Angular Jordan domains.** Let \( E \) be the unit disc. We begin with the

**Definition 5.1.** For every closed subset \( F \) of \( \partial E \) and any real number \( h \) such that \( \text{mes}(F) > 0 \) and \( \sup_{x, y \in F} |x - y| < h < 1 - \frac{\sqrt{2}}{2} \), the open set

\[
\Omega = \Omega(F, h) := \bigcup_{\zeta \in F} \{ z \in A_\frac{\pi}{2}(\zeta) : |z| > 1 - h \}
\]

is called the angular Jordan domain with base \( F \) and height \( h \).

Now we give a list of properties of such angular Jordan domains.
Proposition 5.2. Let $\Omega = \Omega(F, h)$ be an angular Jordan domain.

1) Then there exist exactly two points $\zeta_1, \zeta_2 \in F$ such that $|\zeta_1 - \zeta_2| = \sup_{x,y \in F} |x-y|$ and $F \subset [\zeta_1, \zeta_2]$, where $[\zeta_1, \zeta_2]$ is the (small) closed arc of $\partial E$ which is oriented in the positive sense and which starts from $\zeta_1$ and ends at $\zeta_2$.

2) Write the open set $[\zeta_1, \zeta_2] \setminus F$ as the union of disjoint open arcs

$$[\zeta_1, \zeta_2] \setminus F = \bigcup_{j \in J} (a_j, b_j),$$

where $(a_j, b_j)$ is the (small) open arc of $\partial E$ which goes from $a_j$ to $b_j$ and which is oriented in the positive sense, and the index set $J$ is finite or countable.

For $j \in J$, we construct the isosceles triangle with the three vertices $a_j, b_j$ and $c_j$ such that the base of the isosceles triangle is the segment connecting $a_j$ to $b_j$, and $c_j$ satisfies

$$\arg\left(\frac{c_j - a_j}{a_j}\right) = \frac{3\pi}{4} \quad \text{and} \quad \arg\left(\frac{c_j - b_j}{b_j}\right) = -\frac{3\pi}{4}.$$ 

Let $[a_j, c_j]$ (resp. $[c_j, b_j]$) denote the segment connecting $a_j$ to $c_j$ (resp. the segment connecting $c_j$ to $b_j$). Put

$$F_0 := F \cup \bigcup_{j \in J} ([a_j c_j] \cup [c_j b_j]).$$

Then $F_0$ is a rectifiable Jordan curve starting from $\zeta_1$ and ending at $\zeta_2$.

3) Let $\eta_1$ (resp. $\eta_2$) be the unique point in the circle $\partial B(0, 1 - h)$ such that

$$\arg\left(\frac{\eta_1 - \zeta_1}{\zeta_1}\right) = -\frac{3\pi}{4} \quad \text{(resp.} \quad \arg\left(\frac{\eta_2 - \zeta_2}{\zeta_2}\right) = \frac{3\pi}{4})$$

and that $|\eta_1 - \zeta_1|$ (resp. $|\eta_2 - \zeta_2|$) is minimal. Let $F_1$ (resp. $F_2$) denote the segment connecting $\eta_1$ to $\zeta_1$ (resp. the segment connecting $\zeta_2$ to $\eta_2$). Let $F_3$ be the (small) closed arc of the circle $\partial B(0, 1 - h)$ which starts from $\eta_2$ and ends at $\eta_1$ and which is oriented in the negative sense.

Then $\Omega$ is a rectifiable Jordan domain and its boundary $\Gamma$ consists of the rectifiable Jordan curve $F_0$, two segments $F_1, F_2$ and the closed arc $F_3$.

4) For every $\epsilon \in (0, \frac{h}{2})$ define the dilatation $\tau_\epsilon : E \rightarrow E$ as follows

$$\tau_\epsilon(z) := (1 - \epsilon)z, \quad z \in E.$$

Put

$$\Omega_\epsilon := \tau_\epsilon(\Omega) \setminus \overline{B(0, (1 + \epsilon)(1 - h))}.$$ 

Then $\Omega_\epsilon$ is a rectifiable Jordan domain and its boundary $\Gamma_\epsilon$ consists of the rectifiable Jordan curve $F_{0\epsilon} := \tau_\epsilon(F_0)$, a sub-segment $F_{1\epsilon}$ of $\tau_\epsilon(F_1)$, a sub-segment $F_{2\epsilon}$ of $\tau_\epsilon(F_2)$, and a closed arc $F_{3\epsilon}$ of $\partial B(0, (1 + \epsilon)(1 - h))$.

5) Consider the projection $\tau : E \setminus \{0\} \rightarrow \partial E$ given by $\tau(z) := \frac{1}{|z|}, \; z \in E \setminus \{0\}$. For every $\epsilon \in (0, \frac{h}{4})$ notice that $F_{0\epsilon} \cup F_{1\epsilon} \cup F_{2\epsilon} = \Gamma_\epsilon \setminus \partial B(0, (1 + \epsilon)(1 - h))$. Then the two maps

$$F_{0\epsilon} \cup F_{1\epsilon} \cup F_{2\epsilon} \ni \zeta \mapsto \tau(\zeta) \in \partial E,$$

$$F_{3\epsilon} \ni \zeta \mapsto \tau(\zeta) \in \partial E,$$
are one-to-one. In addition, for any linearly measurable subset $A$ of $\Gamma_\epsilon$,
\[ \text{mes}(A) \leq 10 \cdot \text{mes}(\tau(A)). \]

6) $\Omega_\epsilon \searrow \Omega$ as $\epsilon \searrow 0$.

7) For any closed Jordan curve $\mathcal{C}$ contained in $\Omega$ there is an $\epsilon > 0$ such that $\mathcal{C} \subset \Omega_\epsilon$.

8) $\text{mes}(F \setminus \Omega^\epsilon) = 0$.

Proof. All assertions are quite simple using an elementary geometric argument. Therefore, we leave the details of their proofs to the reader. However, we will give the proof that $\Omega$ is a domain. This proof will clarify Definition 5.1.

In virtue of the condition on $F$ and $h$ given in Definition 5.1, we see that
\[ \{ z \in A_\pi^\prime(\zeta) : |z| > 1 - h \} \setminus \{ z \in A_\pi^\prime(\eta) : |z| > 1 - h \} \neq \emptyset, \]
\[ \forall \zeta, \eta \in \partial E : |\zeta - \eta| < h < 1 - \sqrt{2}/2. \]

Hence, $\Omega$ is a domain.

Theorem 5.3. Let $X$ be a complex manifold of dimension 1, $D \subset X$ an open set and $A$ a Jordan measurable subset of $\partial D$ such that $A$ is of positive length. Then, for any $0 \leq \epsilon < 1$ and any connected component $G$ of $D_\epsilon := \{ z \in D : \omega(z, A^*, D) < 1 - \epsilon \}$, there is an open set $U \subset X$, a conformal mapping $\Phi : E \rightarrow X$, and an angular Jordan domain $\Omega = \Omega(F, h)$ such that

(i) $U \cap D$ is either a Jordan domain or the disjoint union of two Jordan domains;
(ii) $\Phi$ maps $E$ conformally onto one connected component of $U \cap D$ (notice that, in virtue of (i), $U \cap D$ has at most two connected components);
(iii) $\Phi(F) \subset A \cap A^* \cap G^D$ and $\Phi(\Omega) \subset G$.

Proof. We have already shown in the proof of (iii) of Theorem 4.9 that there is a sequence $(U_k)_{k=1}^\infty$ of open sets of $X$ such that $U_k \cap D$ is either a Jordan domain or the disjoint union of two Jordan domains, and $A \subset \bigcup_{k=1}^\infty \partial(U_k \cap D)$, and $A \cap A^* \cap G^D$ is of positive length. Consequently, there is an index $k_0$ such that
\[ (5.1) \quad \left( A \cap A^* \cap G^D \cap \partial(D \cap U) \right) \text{ is of positive length,} \]
where $U := U_{k_0}$. Suppose without loss of generality that $U \cap D$ is a Jordan domain. The remaining case where $U \cap D$ is the disjoint union of two Jordan domains may be proved in the same way. Let $\Phi$ be a conformal mapping from $E$ onto $D \cap U$. By Carathéodory Theorem (see [4]), $\Phi$ extends to a homeomorphic map (still denoted by) $\Phi$ from $\overline{E}$ onto $\overline{D \cap U}$. Hence, (i) and (iii) are satisfied.

On the other hand, it follows from (5.1) that
\[ (5.2) \quad \text{mes} \left( \Phi^{-1} \left( A \cap A^* \cap G^D \cap \partial(D \cap U) \right) \right) > 0. \]

For any $m \geq 1$, let
\[ (5.3) \quad A_m := \{ \eta \in \partial E : A_{2m}(\eta) \subset \Phi^{-1}(G) \}, \]
where $A_{2m}(\eta)$ is given by formula (3.5).

Using the Geometric fact just after (4.6), we see that $A_m$ is closed. On the other hand, it is clear that

$$\Phi^{-1}(A \cap A^* \cap GD \cap \partial(D \cap U)) \subset \bigcup_{m=1}^{\infty} A_m.$$ 

Therefore, in virtue of (5.2), there is an index $m_0$ such that

$$\text{mes}\left(A_{m_0} \cap \Phi^{-1}(A \cap A^* \cap GD \cap \partial(D \cap U))\right) > 0.$$ 

Put $h := \frac{1}{2m_0}$. By the latter estimate one may find a closed set $F$ contained in $A_{m_0} \cap \Phi^{-1}(A \cap A^* \cap GD \cap \partial(D \cap U))$ such that $\text{mes}(F) > 0$ and $\sup_{x,y \in F} |x - y| < h$.

Since $h = \frac{1}{2m_0}$, a geometric argument shows that

$$\{z \in \mathcal{A}_+^{\#}(\zeta) : |z| > 1 - h\} \subset \mathcal{A}_{2m_0}(\zeta), \quad \zeta \in \partial D.$$ 

This together with (5.3) implies that $\Omega = \Omega(F, h) \subset \Phi^{-1}(G)$. Hence, (iii) is verified. This completes the proof. \qed

In the sequel, the following uniqueness theorem will play a vital role.

**Theorem 5.4.** Let $X$ be a complex manifold of dimension 1, $D \subset X$ an open set, and $A, \mathcal{N}$ two Jordan-measurable subsets of $\partial D$ such that $A$ is of positive length and $\mathcal{N}$ is of zero length. Let $0 \leq \varepsilon < 1$ and $G$ a connected component of $D_\varepsilon := \{z \in D : \omega(z, A^*, D) < 1 - \varepsilon\}$. If $f \in \mathcal{O}(G)$ admits the angular limit 0 at every point of $(A \cap A^* \cap GD) \setminus \mathcal{N}$, then $f \equiv 0$.

**Proof.** Applying Theorem 5.3 we obtain an open set $U$ in $X$, a conformal mapping $\Phi$ from $E$ onto $D \cap U$ which extends homeomorphically to $\overline{E}$, and an angular Jordan domain $\Omega := \Omega(F, h)$ satisfying assertions (i)–(iii) listed in that theorem.

Consider the function $f \circ \Phi : \Omega \rightarrow \mathbb{C}$. By the hypothesis, $f \circ \Phi \in \mathcal{O}(\Omega)$ admits the angular limit 0 at a.e point in $F$. Since $\text{mes}(F) > 0$, Privalov’s Uniqueness Theorem (see [4]) gives that $f \circ \Phi \equiv 0$ on $\Omega$. Hence, $f \equiv 0$ on the subdomain $\Phi(\Omega)$ of $G$. This proves $f \equiv 0$. \qed

5.2. **Main result of the section.** Let $D, G \subset \mathbb{C}$ be open discs and let $A$ (resp. $B$) be a measurable subset of $\partial D$ (resp. $\partial G$) with $\text{mes}(A) > 0$ (resp. $\text{mes}(B) > 0$).

Let $f$ be a function defined on $W := \mathbb{X}(A, B; D, G)$ with the following properties:

(i) $f|_{A \times B}$ is measurable and there is a finite constant $C$ with $|f|_W < C$;

(ii) $f \in \mathcal{O}_s(W^0)$;

(iii) there exist two functions $f_1, f_2 : A \times B \rightarrow \mathbb{C}$ such that for any $a \in A$ (resp. $b \in B$), $f(a, \cdot)$ (resp. $f(\cdot, b)$) has the angular limit $f_1(a, b)$ at $b$ for a.e. $b \in B$ (resp. $f_2(a, b)$ at $a$ for a.e. $a \in A$), and $f_1 = f_2 = f$ a.e. on $A \times B$.

Let $\bar{\omega}(\cdot, A, D)$ (resp. $\bar{\omega}(\cdot, B, G)$) be the conjugate harmonic function of $\omega(\cdot, A, D)$ (resp. $\omega(\cdot, B, G)$) such that $\bar{\omega}(z_0, A, D) = 0$ (resp. $\bar{\omega}(w_0, B, G) = 0$) for a certain
fixed point \( z_0 \in D \) (resp. \( w_0 \in G \)). Thus we define the holomorphic functions
\[ g_1(z) := \omega(z, A, D) + i\tilde{\omega}(z, A, D), \]
\[ g_2(w) := \omega(w, B, G) + i\tilde{\omega}(w, B, G), \]
and
\[ g(z, w) := g_1(z) + g_2(w), \quad (z, w) \in D \times G. \]

Each function \( e^{-g_1} \) (resp. \( e^{-g_2} \)) is bounded on \( D \) (resp. on \( G \)). Therefore, in virtue of [4, p. 439], we may define \( e^{-g_1} \) (resp. \( e^{-g_2} \)) for a.e. \( a \in A \) (resp. \( b \in B \)) to be the angular boundary limit of \( e^{-g_1} \) at \( a \) (resp. \( e^{-g_2} \) at \( b \)).

In virtue of (i), for each positive integer \( N \), we define the Gonchar–Carleman operator as follows (5.4)
\[ K_N(z, w) = K_N[f](z, w) := \frac{1}{(2\pi i)^2} \int_{A \times B} e^{-N(g(a,b) - g(z,w))} \frac{f(a,b) \, da \, db}{(a-z)(b-w)}, \quad (z, w) \in D \times G. \]

We recall from Gonchar’s work in [6] that the following limit (5.5)
\[ K(z, w) = K[f](z, w) := \lim_{N \to \infty} K_N(z, w) \]
exists for all \((z, w) \in \hat{W}^o\), and its limit is uniform on compact subsets of \( \hat{W}^o \).

The boundary behavior of Gonchar–Carleman operator is described below.

**Theorem 5.5.** We keep the above hypothesis and notation. Let \( 0 < \delta < 1 \), \( w \in G \) be such that \( \omega(w, B, G) < \delta \), and let \( U \) be any connected component of
\[ D_\delta := \{ z \in D : \omega(z, A, D) < 1 - \delta \}. \]

Then there is an angular Jordan domain \( \Omega = \Omega(F, h) \) such that \( \Omega \subset U \), \( F \subset A \cap A^* \cap U^D \), and the Gonchar–Carleman operator \( K[f] \) (see formula (5.4)–(5.5) above) satisfies
\[ \lim_{z \to a, \ z \in A_\alpha(a)} K[f](z, w) = f(a, w), \quad 0 < \alpha < \frac{\pi}{2}, \]
for a.e. \( a \in F \).

The proof of this theorem will be given in Subsection 5.4 below.

5.3. **Preparatory results.** For the proof of Theorem 5.5 we need the following results.

In the sequel, for every function \( f \in L^1(\partial\Omega, |d\zeta|) \), let \( \mathcal{C}[f] \) denote the Cauchy integral
\[ \mathcal{C}[f](z) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta) \, d\zeta}{z - \zeta}, \quad z \in \Omega. \]

For a function \( F : \Omega \to \mathbb{C} \), the radial maximal function \( M_{\text{rad}}F : \partial\Omega \to [0, \infty) \) is defined by
\[ (M_{\text{rad}}F)(\zeta) := \sup_{0 \leq r < 1} |F(r\zeta)|, \quad \zeta \in \partial\Omega. \]

Now we are able to state the following classical result (see Theorem 6.3.1 in Rudin’s book [14])
Theorem 5.6. (Korányi-Vági type theorem) There is a constant $C > 0$ such that
\[
\int_{\partial E} |M_{\text{rad}} C [f](\zeta)|^2 |d\zeta| \leq C \int_{\partial E} |f(\zeta)|^2 |d\zeta|
\]
for every $f \in L^2(\partial E, |d\zeta|)$.

We recall the definition of the Smirnov class $E^p$, $p > 0$, on rectifiable Jordan domains.

Definition 5.7. Let $p > 0$ and $\Omega$ a rectifiable Jordan domain. A function $f \in \mathcal{O}(\Omega)$ is said to belong to the Smirnov class $E^p(\Omega)$ if there exists a sequence of rectifiable closed Jordan curves $(C_n)_{n=1}^{\infty}$ in $\Omega$, tending to the boundary in the sense that $C_n$ eventually surrounds each compact subdomain of $\Omega$, such that
\[
\int_{C_n} |f(z)|^p |dz| \leq M < \infty, \quad n \geq 1.
\]

Next, we rephrase some facts concerning the Smirnov class $E^p$, $p > 0$ on rectifiable Jordan domains in the context of angular Jordan domains $\Omega(F, h)$.

Theorem 5.8. 1) Let $\Omega$ be a rectifiable Jordan domain. Then every $f \in E^p(\Omega)$ ($p > 0$) admits the angular limit $f^*$ a.e. on $\partial \Omega$.
2) Let $\Omega := \Omega(F, h)$ be an angular Jordan domain and let $\Gamma := \partial \Omega$. For any $0 < \epsilon < \frac{h}{4}$, let $\Gamma_{\epsilon}$ be the rectifiable closed Jordan curve defined in Part 4) of Proposition 5.2. Then $f \in E^p(\Omega)$ if $\sup_{0 < \epsilon < \frac{h}{4}} \int_{\Gamma_{\epsilon}} |f(z)|^p |dz| < \infty$. In addition, for an $f \in E^p(\Omega)$, $p > 0$, it holds that
\[
\int_{\Gamma_{\epsilon}} |f^*(z)|^p |dz| \leq \sup_{0 < \epsilon < \frac{h}{4}} \int_{\Gamma_{\epsilon}} |f(z)|^p |dz|.
\]
3) Every $f \in E^1(\Omega)$ has a Cauchy representation $f := \mathcal{C}[f^*]$. Conversely, if $g \in L^1(\partial E, |dz|)$ and
\[
\int_{\partial E} z^n g(z) dz = 0, \quad n = 0, 1, 2, \ldots,
\]
then $f := \mathcal{C}[g] \in E^1(\Omega)$ and $g$ coincides with $f^*$ a.e. on $\partial E$.

Proof. For the proof of Parts 1) and 3), see [4, p. 438–441]. Taking into account Parts 6) and 7) of Proposition 5.2. Part 2) also follows from the results in [4, p. 438–441]. Hence, the proof is complete.

5.4. Proof of Theorem 5.5. We fix $w_0 \in G$ and $0 < \delta_0 < \delta$ with $\omega(w_0, B, G) < \delta_0$ and an arbitrary connected component $U$ of $D_\delta := \{z \in D : \omega(z, A, D) < 1 - \delta\}$. Applying Theorem 5.3 we may find an angular Jordan domain $\Omega := \Omega(F, h) \subset U$ such that $F \subset A \cap A^* \cap U^D$. In the course of the proof, the letter $C$ will denote a positive constant that is not necessarily the same at each step.
Applying Carleman Theorem (see, for example, [1, p.2]), we have
\[
f(z, b) = \lim_{N \to \infty} \frac{1}{2\pi i} \int_A e^{-N(g_1(a)-g_1(z))} \frac{f(a, b)da}{a - z}, \quad z \in D, \ b \in B,
\]
\[
f(a, b) = \lim_{r \to 1^-} f(ra, b), \quad a \in \partial D, \ b \in B.
\]
Consequently, \( f \rvert_{\partial D \times B} \) is measurable. In addition, by (iii) this function is bounded. Therefore, for every \( N \in \mathbb{N} \) we are able to define the function \( K_{\infty,N}(\cdot, w_0) : \partial D \to \mathbb{C} \),
\[
K_{\infty,N}(a, w_0) := \frac{1}{2\pi i} \int_B e^{N(g_2(w_0) - g_2(b))} \frac{f(a, b)db}{b - w_0}, \quad a \in \partial D.
\]
Since, in virtue of (ii)–(iii), \( f(a, \cdot) \in \mathcal{O}(G) \) and \(|f(a, \cdot)|_G < C\) for \( a \in A \), it follows from Carleman Theorem that
\[
\lim_{N \to \infty} K_{\infty,N}(a, w_0) = f(a, w_0), \quad a \in A,
\]
and the above convergence is uniform with respect to \( a \in A \).

On the other hand, by (5.6) we see that \( K_{\infty,N}(\cdot, w_0) \) is measurable and bounded. In addition, for any \( n = 0, 1, 2, \ldots \), taking (ii) into account, we have that
\[
\int_{\partial D} K_{\infty,N}(a, w_0) a^n da = \frac{1}{2\pi i} \int_B \left( \int_{\partial D} f(a, b) a^n da \right) e^{N(g_2(w_0) - g_2(b))} \frac{db}{b - w_0} = 0,
\]
where the first equality follows from an application of Fubini’s Theorem and the second one from an application of Part 3) of Theorem 5.8 to \( f(\cdot, b), \ b \in B \). Consequently, in virtue of Part 3) of Theorem 5.8 we can extend \( K_{\infty,N}(\cdot, w_0) \) to \( \mathbb{D} \) by setting
\[
K_{\infty,N}(z, w_0) := C[K_{\infty,N}(\cdot, w_0)](z) = \frac{1}{2\pi i} \int_{\partial D} \frac{K_{\infty,N}(a, w_0) da}{a - z}, \quad z \in D.
\]
Then the following identity holds
\[
\lim_{z \to a, \ z \in A_\alpha(a)} K_{\infty,N}(z, w_0) = K_{\infty,N}(a, w_0), \quad 0 < \alpha < \frac{\pi}{2},
\]
for a.e. \( a \in \partial D \).

Now we come back to the angular Jordan domain \( \Omega \). We keep the notation introduced in Proposition 5.2. For any \( 0 < \epsilon < \frac{b}{4} \) and any \( z \in \Gamma_\epsilon \), applying the Cauchy integral formula, we obtain
\[
K_{\infty,N}(z, w_0) - K_N(z, w_0) = \frac{1}{(2\pi i)^2} \int_{\partial D \setminus A} \int_B e^{N(g_1(z) - g_1(a) + N(g_2(w_0) - g_2(b))} \frac{f(a, b) da db}{(a - z)(b - w_0)}
\]
\[
= e^{N(g_1(z) - (1-\delta))} \int_{\partial D} \frac{p_N(a) da}{a - z}.
\]
Using the choice of $U$ and the hypothesis on $\delta$ and $\delta_0$, it can be checked that
\begin{equation}
|e^{N(\delta(\cdot)-1)}|_U \leq 1, \quad |p_N|_{\partial D} \leq Ce^{-N(\delta-\delta_0)}.
\end{equation}

Therefore, recalling the projection $\tau : E \setminus \{0\} \rightarrow \partial E$ (see Part 5) of Proposition 5.2, we estimate
\begin{equation}
(5.11) \quad \int_{\Gamma_e} |K_{\infty,N}(z, w_0) - K_N(z, w_0)|^2|dz| \leq C \int_{\Gamma_e} \left| \left| M_{\text{rad}} C[p_N] (\tau(z)) \right| \right|^2|dz|
\end{equation}

\begin{equation}
\leq 10C \int_{\tau(F_\delta \cup F_{3\delta} \cup F_{5\delta})} \left| M_{\text{rad}} C[p_N] (a) \right|^2|da| + 10C \int_{\tau(F_{3\delta})} \left| M_{\text{rad}} C[p_N] (a) \right|^2|da|
\end{equation}

\begin{equation}
\leq 20C \int_{\partial E} \left| M_{\text{rad}} C[p_N] (a) \right|^2|da| \leq C \int_{\partial E} \left| p_N (a) \right|^2|da| \leq Ce^{-N(\delta-\delta_0)}.
\end{equation}

Here the first estimate follows from (5.10)–(5.11) and the definition of the radial maximal function, the second and the third one are consequences of Part 5) of Proposition 5.2 and the fourth estimate holds by an application of Theorem 5.6 and the last one follows from (5.11).

On the other hand, for any $0 < \epsilon < \frac{h}{4}$,
\begin{equation}
(5.13) \quad \int_{\Gamma_e} |K_{N+1}(z, w_0) - K_N(z, w_0)|^2|dz|
\end{equation}

\begin{equation}
\leq 2 \int_{\Gamma_e} \left| A_N(z, w_0) \right|^2|dz| + 2 \int_{\Gamma_e} \left| B_N(z, w_0) \right|^2|dz| \leq Ce^{-N(\delta-\delta_0)},
\end{equation}

where $A_N$ and $B_N$ are given by formula (6) in [6] and the latter estimate follows from the same argument as in the proof of (5.10)–(5.12). We recall from (5.5) that
\begin{equation}
\lim_{N \rightarrow \infty} K_N(z, w_0) = K(z, w_0), \quad z \in \Gamma_e.
\end{equation}

This, combined with (5.12)–(5.13), implies that
\begin{equation}
(5.14) \quad \int_{\Gamma_e} |K_{\infty,N}(z, w_0) - K(z, w_0)|^2|dz| \leq C \cdot e^{-N(\delta-\delta_0)}, \quad 0 < \epsilon < \frac{h}{4}.
\end{equation}

Since we have already shown that $|K_{\infty,N}(\cdot, w_0)|_D < \infty$, in virtue of Part 2) of Theorem 5.8, we deduce from (5.14) that $K(\cdot, w_0)|_\Omega \in E^2(\Omega)$. For every $a \in \partial D$, let $K(a, w_0)$ denote the angular limit of $K(\cdot, w_0)|_\Omega$ at $a$ (if the limit exists). It follows from (5.14) and Part 2) of Theorem 5.8 that
\begin{equation}
\lim_{N \rightarrow \infty} \int_{\Gamma_e} |K_{\infty,N}(a, w_0) - K(a, w_0)|^2|da|
\end{equation}

\begin{equation}
\leq \sup_{0 < \epsilon < \frac{h}{4}} \int_{\Gamma_e} |K_{\infty,N}(z, w_0) - K(z, w_0)|^2|dz| \leq \lim_{N \rightarrow \infty} C \cdot e^{-N(\delta-\delta_0)} = 0.
\end{equation}
This, combined with (5.7) and Part 8) of Proposition 5.2, implies finally that
\[ K(a, w_0) = f(a, w_0), \quad \text{for a.e. } a \in F. \]
Hence, Theorem 5.5 has been proved. \[ \square \]

6. Proof of Theorem A for the case where \( D \) and \( G \) are Jordan domains

Using an exhaustion argument, a compactness argument and conformal mappings, the case where \( D \) and \( G \) are Jordan domains can be reduced to the following case:

We assume that \( D = G = E \), and \( |f|_W < 1 \). (\*)

Using hypotheses (i)–(iii) and (\*), we may apply Theorem 5.5 and obtain a function \( K[f] \in \mathcal{O}(\hat{W}^o) \). Consequently, we are able to define the desired extension function \( \hat{f} \) as follows
\[ \hat{f} := K[f]. \]

In this section we will use repeatedly Part 3) of Theorem 4.6
\[ \omega(\cdot, A, \Omega) = \omega(\cdot, A^*, \Omega), \]
where \( \Omega \subset \mathbb{C} \) is an open set and \( A \) is a Jordan measurable subset of \( \partial \Omega \).

The remaining part of the proof is divided into several steps.

**Step 1:** Proof of the estimate
\[ |\hat{f}|_{\hat{W}^o} \leq |f|_W. \]

**Proof of Step 1.** Let \((z_0, w_0)\) be an arbitrary point of \( \hat{W}^o \). Then we may find an \( \delta \in (0, 1) \) such that \( 0 < \omega(w_0, B, G) < \delta < 1 - \omega(z_0, A, D) \). Let \( U \) be the connected component of \( D_\delta := \{ z \in D : \omega(z, A, D) < 1 - \delta \} \) that contains \( z_0 \). By Theorem 5.3 we may find an angular Jordan domain \( \Omega := \Omega(F,h) \) contained in \( U \) such that \( F \subset A \cap A^* \cap U^D \). In addition, for every \( N \in \mathbb{N} \), applying Theorem 5.5 to the function \( f^N \), we obtain the function \( K[f^N] \in \mathcal{O}(\hat{W}^o) \) with the following property
\[ \lim_{z \to a, \, z \in A_\alpha(a)} K[f^N](z, w_0) = f(a, w_0)^N \]
\[ = \lim_{z \to a, \, z \in A_\alpha(a)} (K[f](z, w_0))^N, \quad 0 < \alpha < \frac{\pi}{2}, \]
for a.e. \( a \in F. \)
 Consequently, an application of Theorem 5.4 gives that
\[ K[f^N](z_0, w_0) = (K[f](z_0, w_0))^N, \quad N \in \mathbb{N}, \]
Since \((z_0, w_0) \in \hat{W}^o\) is arbitrarily chosen, it follows from the latter identity that
\[ (6.1) \quad K[f^N](z, w) = (K[f](z, w))^N, \quad N \in \mathbb{N}, \quad (z, w) \in \hat{W}^o. \]

Now we are able to conclude the proof in the same way as in [6, p. 23]. More precisely, taking into account (6.1), one gets that
\[ |\hat{f}^N(z, w)| \leq |K[f^N](z, w)| \leq \frac{C|f|^N_W}{(1 - |z|)(1 - |w|)(1 - e^{-(1-\omega(z,w))})}, \quad (z, w) \in \hat{W}^o. \]
Extracting the $N$th roots of both sides and letting $N$ tend to $\infty$, the desired estimate of Step 1 follows. □

**Step 2:** We shall prove that $\hat{f}$ is the unique function $O(\hat{W}^0)$ which verifies Property 1).

**Proof of Step 2.** First we show that the function $\hat{f}$ satisfies Property 1). Without loss of generality, it suffices to prove that there is a subset $\tilde{B}$ of $B \cap B^*$ such that $\text{mes}(\tilde{B}) = \text{mes}(B)$ and $\hat{f}$ admits the angular limit $f$ at every point of $D \times \tilde{B}$.

For any $a \in A$ put

$$B_a := \left\{ b \in B : f(a, \cdot) \text{ has an angular limit at } b \right\}.$$

By hypothesis (iii), we have $\text{mes}(B_a) = \text{mes}(B)$, $a \in A$. Consequently, applying Fubini's Theorem, we obtain that

$$\int_A \text{mes}(B_a) |da| = \text{mes}(A) \text{mes}(B) = \int_B \text{mes}\left(\left\{ a \in A : b \in B_a \right\}\right) |db|.$$

Hence,

$$\text{mes}\left(\left\{ a \in A : b \in B_a \right\}\right) = \text{mes}(A) \quad \text{for a.e. } b \in B.$$ (6.2)

The same reasoning also gives that

$$\text{mes}\left(\left\{ a \in A : f(a, b) = f_1(a, b) \right\}\right) = \text{mes}(A) \quad \text{for a.e. } b \in B.$$ (6.3)

Set

$$\tilde{B} := \left\{ b \in B \cap B^* : \text{mes}\left(\left\{ a \in A : b \in B_a \right\}\right) = \text{mes}(A) \right\}.$$

We deduce from (6.2)–(6.4) that

$$\text{mes}(\tilde{B}) = \text{mes}(B).$$ (6.5)

Fix an arbitrary point $b_0 \in \tilde{B}$ and let $(w_n)_{n=1}^{\infty}$ be an arbitrary sequence of $G$ such that $\lim_{n \to \infty} w_n = b_0$ and $w_n \in A_\alpha(b_0)$ for some fixed number $0 < \alpha < \frac{\pi}{2}$. Fix an arbitrary point $z_0$ of $D$ and let $(z_n)_{n=1}^{\infty}$ be an arbitrary sequence of $D$ such that $\lim_{n \to \infty} z_n = z_0$.

Clearly, we may find $0 < \delta_1 < 1$ such that

$$\sup_{n \in \mathbb{N}} \omega(z_n, A, D) < 1 - \delta_1.$$ (6.6)

Fix an $\delta_2$ such that $0 < \delta_2 < \delta_1$. Since $b_0$ is locally regular relative to $B$ and $\lim_{n \to \infty} w_n = b_0$ and $w_n \in A_\alpha(b_0)$, there is a sufficiently large number $N_0$ with

$$\omega(w_n, B, G) < \delta_2, \quad n > N_0.$$ (6.7)

Let $U$ be that connected component of the following open set

$$D_{\delta_1} := \{ z \in D : \omega(z, A, D) < 1 - \delta_1 \}$$

which contains $z_0$ (see (6.6)). Applying Theorem 5.3 we may find an angular Jordan domain $\Omega := \Omega(F, h)$ contained in $U$ such that $F \subset A \cap A^* \cap U^D$. Let $V$ be a rectifiable
Jordan domain with $\Omega \subset V \subset U$, $w_0 \in V$, and $V \cap U = \Omega \cap U$ for some neighborhood $U$ of the base $F$ of $\Omega$.

In virtue of (6.7) and of the fact that $V \subset U \subset D_\delta$, we obtain that
\begin{equation}
V \times \{w_n\} \subset \hat{W}^o, \quad n > N_0.
\end{equation}
Consequently, Theorem 5.5 yields that for any $n > N_0$,
\begin{equation}
f(a, w_n) = \lim_{z \to a, z \in A_\alpha(a)} \hat{f}(z, w_n), \quad 0 < \alpha < \frac{\pi}{2},
\end{equation}
for a.e. $a \in F$.

Next, for any $n > N_0$ let
\[
F_n := \left\{ a \in F : b_0 \in B_a \text{ and } f(a, w_n) = \lim_{z \to a, z \in A_\alpha(a)} \hat{f}(z, w_n) \right\},
\]
\[
F_0 := \bigcap_{n=N_0+1}^\infty F_n.
\]
It follows from (6.4), (6.9) and the fact that $b_0 \in \tilde{B}$ that $\text{mes}(F_n) = \text{mes}(F), \ n > N_0$. Hence
\begin{equation}
(6.10) \quad \text{mes}(F_0) = \text{mes}(F) > 0.
\end{equation}

In virtue of (6.8), consider the following holomorphic functions on $V$
\begin{equation}
(6.11) \quad h_n(t) := \hat{f}(t, w_n) \quad \text{and} \quad h_0(t) := f(t, b_0), \quad t \in V, \ n > N_0.
\end{equation}
Since we have already shown in Step I that $|h_n|_V \leq |f|_X < \infty, \ n > N_0$ or $n = 0$, applying Part 1) of Theorem 5.8 we may find a subset $\Delta$ of $F_0$ with $\text{mes}(\Delta) = \text{mes}(F_0) > 0$ such that $h_n, n > N_0$ (resp. $h_0$) admits the angular limit $\hat{f}_1(t, w_n)$ (resp. $f_1(t, b_0)$) at $t \in \Delta$. Observe that by (6.1) and the fact that $b_0 \in \tilde{B}$ we have that
\[
\lim_{n \to \infty} f_1(t, w_n) = f_1(t, b_0) = f(t, b_0) \quad \text{for a.e. } t \in \Delta.
\]
Using this and (6.11), we are able to apply Khinchin–Ostrowski Theorem (see [4, Theorem 4, p. 397]) to the sequence $(h_n)_{n=0}^\infty$. Consequently, one gets
\[
\lim_{n \to \infty} \hat{f}(z_n, w_n) = f(z_0, b_0).
\]
This shows that $\hat{f}$ admits the angular limit $f$ at every point of $D \times \tilde{B}$. Hence, $\hat{f}$ satisfies Property 1).

In order to complete Step 2 we need to show the uniqueness of $\hat{f}$. To do this, let $\hat{f} \in \mathcal{O}(\hat{W}^o)$ be a function with the following property: There is a subset $\tilde{A}$ (resp. $\tilde{B}$) of $A \cap A^*$ (resp. $b \cap B^*$) such that $\text{mes}(A \setminus \tilde{A}) = \text{mes}(B \setminus \tilde{B}) = 0$ and $\hat{f}$ admits the angular limit $f$ at every point of $(\tilde{A} \times G) \cup (D \times \tilde{B})$. Fix an arbitrary point $(z_0, w_0) \in \hat{W}^o$. Let $U$ be the connected component containing $z_0$ of the following open set
\[
\{ z \in D : \omega(z, A, D) < 1 - \omega(w_0, B, G) \}.
\]
We deduce from the property of \( \hat{f} \) and \( \hat{\dot{f}} \) that both holomorphic functions \( \hat{f}(\cdot, w_0)|_U \) and \( \hat{\dot{f}}(\cdot, w_0)|_U \) admit the angular limit \( f(\cdot, w_0) \) at every point of \( \tilde{A} \cap \tilde{A} \cap U^D \). Consequently, applying Theorem 5.4 yields that \( \hat{f}(\cdot, w_0) = \hat{\dot{f}}(\cdot, w_0) \) on \( U \). Hence, \( \hat{f}(z_0, w_0) = \hat{\dot{f}}(z_0, w_0) \). Since \((z_0, w_0) \in \hat{W}^o\) is arbitrary, the uniqueness of \( \hat{f} \) is established. This completes Step 2. \( \square \)

**Step 3: Proof of Part 2.**

**Proof of Step 3.** Fix \((z_0, w_0) \in \hat{W}^o\). For every \( b \in B \) we have

\[
|f(a, b)| \leq |f|_{A \times B}, \quad a \in A, \quad \text{and} \quad |f(z, b)| \leq |f|_W, \quad z \in D.
\]

Therefore, the Two-Constant Theorem (see Theorem 2.2 in [10]) implies that

\[
|f(z, b)| \leq |f|_{A \times B}^{1-\omega(z, A, D)} f_W^{\omega(z, A, D)}, \quad z \in D, \quad b \in B.
\]

Let \( \delta := \omega(z_0, A, D) \) and consider the \( \delta \)-level set

\[
G_\delta := \{w \in G : \omega(w, B, G) < 1 - \delta\}.
\]

Clearly, \( w_0 \in G_\delta \).

Recall from Step 2 that \( \tilde{B} \subset B \cap B^* \), \( \text{mes} \left((B \cap B^*) \setminus \tilde{B}\right) = 0 \), and

\[
f(z_0, b) = \lim_{w \to h, w \in A_\delta(b)} \hat{f}(z_0, w), \quad 0 < \alpha < \frac{\pi}{2}, \quad b \in \tilde{B}.
\]

Consider the following function \( h : G_\delta \cup \tilde{B} \to \mathbb{C} \) defined by

\[
h(t) := \begin{cases} \hat{f}(z_0, t), & t \in G_\delta \\ f(z_0, t), & t \in \tilde{B}. \end{cases}
\]

Clearly, \( h|_{G_\delta} \in O(G_\delta) \).

On the other hand, in virtue of (6.14) and the result of Step 1, we have

\[
h|_{G_\delta} \leq |f|_W^{\omega} \leq |f|_W < \infty.
\]

In addition, applying Corollary 4.11 and taking (6.13)–(6.14) into account yields

\[
|h(t)| \leq |h|_{B_\delta}^{1-\omega(t, A, D)} |h|_{G_\delta}^{\omega(t, A, D)}, \quad t \in G_\delta,
\]

where, by Theorem 4.10

\[
\omega(t, B, G) = \frac{\omega(t, B, G)}{1-\omega(z, A, D)}.
\]

This, combined with (6.12)–(6.15), implies that

\[
|\hat{f}(z_0, w_0)| = |h(w_0)| \leq |f|_{A \times B}^{1-\omega(z_0, A, D)-\omega(w_0, B, G)} f_W^{\omega(z_0, A, D)+\omega(w_0, B, G)}.
\]

Hence Part 2) for the point \((z_0, w_0)\) is proved. \( \square \)

**Step 4: Proof of Part 3.**

**Proof of Step 4.** Let \((a_0, w_0) \in A^* \times G\) be such that the following limit exists

\[
\lambda := \lim_{(a, w) \to (a_0, w_0), (a, w) \in A \times G} f(a, w).
\]
We like to show that \( \hat{f} \) admits the angular limit \( \lambda \) at \((a_0, w_0)\).

For any \( 0 < \epsilon < \frac{1}{2} \), we may find an open neighborhood \( A_{a_0} \) of \( a_0 \) in \( A \) and a positive number \( r > 0 \) such that \( \mathbb{B}(w_0, r) \subseteq G \) and
\[
(6.16) \quad |f(a, w) - \lambda| < \epsilon^2, \quad a \in A_{a_0}, \; |w - w_0| \leq r.
\]
Put
\[
(6.17) \quad \delta := \sup_{w \in \mathbb{B}(w_0, r)} \omega(w, B, G).
\]
Since \( a_0 \in A^* \), it is clear that \( \text{mes}(A_{a_0}) > 0 \). Next, consider the level set
\[
D_\delta := \{ z \in D : \omega(z, A_{a_0}, D) < 1 - \delta \}.
\]
In virtue of \( (6.17) \), we can define
\[
(6.18) \quad h(t, w) := \hat{f}(t, w) - \lambda, \quad t \in D_\delta, \; w \in \mathbb{B}(w_0, r).
\]
Clearly,
\[
(6.19) \quad |h|_{D_\delta} \leq 2|\hat{f}|_{\mathbb{B}} = 2|f|_W = 2.
\]
By \( (6.18) \) and using the result of Step 2, we know that for every \( w \in \mathbb{B}(w_0, r) \) the holomorphic function \( h(\cdot, w)|_{D_\delta} \) admits the angular limit \( f(a, w) - \lambda \) at \( a \) for \( a \in \tilde{A} \cap A_{a_0} \), where \( \tilde{A} \) is given in Step 2. Consequently, applying Corollary 4.11 and taking \( (6.16) \) and \( (6.19) \) into account, we see that
\[
|h(t, w)| < 2^{(1-\omega_\delta(t, A_{a_0}, D))}2^{\omega_\delta(t, A_{a_0}, D)}, \quad t \in D_\delta.
\]
Let \( 0 < \alpha < \frac{\pi}{2} \). In virtue of Theorem 4.10 and the hypothesis that \( a_0 \in A^* \), we deduce that \( \lim_{t \to a_0, \; t \in A_{a_0}(a_0)} \omega_\delta(t, A_{a_0}, D) = 0 \). Consequently, there is an \( r_\alpha > 0 \) such that
\[
|f(z, w) - \lambda| = |h(z, w)| < \epsilon, \quad z \in A_{a_0}(a_0) \cap \{|z - a_0| < r_\alpha\}, \; w \in \mathbb{B}(w_0, r).
\]
This completes the above assertion.

Similarly, we can prove that \( \hat{f} \) admits the angular limit
\[
\lim_{(z, b) \to (z_0, b_0), \; (z, b) \in D \times B} f(z, b)
\]
at any point \((z_0, b_0)\), if the latter limit exists. Hence the proof of Step 4 (i.e. Part 3)) is finished. \( \square \)

**Step 5: Proof of Part 4).**

**Proof of Step 5.** Let \((a_0, b_0) \in A^* \times B^* \) be such that the following limit exists
\[
\lambda := \lim_{(a, b) \to (a_0, b_0), \; (a, b) \in A \times B} f(a, b).
\]
We like to show that \( \hat{f} \) admits the angular limit \( \lambda \) at \((a_0, b_0)\).

Recall that \( |f|_X < 1 \), and fix an arbitrary \( 0 < \epsilon < \frac{1}{2} \). Since \((a_0, b_0) \in A^* \times B^* \), we may find an open neighborhood \( A_{a_0} \) of \( a_0 \) in \( A \) (resp. an open neighborhood \( B_{b_0} \) of \( b_0 \) in \( B \)) such that
\[
(6.20) \quad |f(a, b) - \lambda| < \epsilon^2, \quad a \in A_{a_0}, \; b \in B_{b_0}.
\]
It is clear that $\text{mes}(A_{a_0}) > 0$ and $\text{mes}(B_{b_0}) > 0$.

Consider the function

\begin{equation}
(6.21) \quad h(z, w) := f(z, w) - \lambda, \quad (z, w) \in \mathbb{X}(A_{a_0}, B_{b_0}; D, G).
\end{equation}

Clearly,

\begin{equation}
(6.22) \quad |h(z, w)| \leq 2, \quad (z, w) \in \mathbb{X}(A_{a_0}, B_{b_0}; D, G).
\end{equation}

Applying the results of Steps 1–3 to $h$, we obtain the function

\begin{equation}
(6.23) \quad \hat{h} := K[h] \quad \text{on} \quad \mathbb{X}(A_{a_0}, B_{b_0}; D, G).
\end{equation}

so that $\hat{h}$ admits the angular limit $h$ on $(A_{a_0} \times G) \cup (D \times B_{b_0})$, where $A_{a_0}, B_{b_0}$ are given by Step 2. Clearly,

\begin{equation}
\mathbb{X}(A_{a_0}, B_{b_0}; D, G) \subset \mathbb{X}(D, G).
\end{equation}

Consequently, arguing as in Step 1 and taking into account the above mentioned angular limit of $\hat{h}$, we conclude that

\begin{equation}
\hat{h} = \hat{f} - \lambda \quad \text{on} \quad \mathbb{X}(A_{a_0}, B_{b_0}; D, G).
\end{equation}

Consequently, applying Step 3 and taking into account (6.20)–(6.23) and the inequality $|f|_X < 1$, we see that

\begin{align*}
\left| \hat{f}(z, w) - \lambda \right| & = |\hat{h}(z, w)| \\
& \leq |h|_{A_{a_0} \times B_{b_0}}^{1-\omega(z, A_{a_0}, D) - \omega(w, B_{b_0}, G)}(2|f|_X)^{\omega(z, A_{a_0}, D) + \omega(w, B_{b_0}, G)} \\
& < \epsilon 2^{1-\omega(z, A_{a_0}, D) - \omega(w, B_{b_0}, G)} 2^{\omega(z, A_{a_0}, D) + \omega(w, B_{b_0}, G)}.
\end{align*}

Therefore, for all $(z, w) \in \mathbb{X}(A_{a_0}, B_{b_0}; D, G)$ satisfying

\begin{equation}
(6.24) \quad \omega(z, A_{a_0}, D) + \omega(w, B_{b_0}, G) < \frac{1}{3},
\end{equation}

we deduce from the latter estimate that

\begin{equation}
(6.25) \quad \left| \hat{f}(z, w) - \lambda \right| < \epsilon.
\end{equation}

Since $a_0$ (resp. $b_0$) is locally regular relative to $A_{a_0}$ (resp. $B_{b_0}$), there is an $r_\alpha > 0$ such that (6.24) is fulfilled for

\begin{equation}
(z, w) \in (A_{\alpha}(a_0) \cap \{|z - a_0| < r_\alpha\}) \times (A_{\alpha}(b_0) \cap \{|w - b_0| < r_\alpha\}).
\end{equation}

This, combined with (6.25), completes the proof. Hence Step 5 (i.e. Part 4) is finished.

**Step 6: Proof of Part 5.**

**Proof of Step 6.** In virtue of Step 5, we only need to show that $\hat{f}$ admits the angular limit $f$ on $(A^* \times G) \cup (D \times B^*)$. To do this let $(a_0, w_0) \in A^* \times G$ and choose an arbitrary $0 < \epsilon < 1$. Fix a compact subset $K$ of $B \cap B^*$ such that $\text{mes}(K) > 0$ and a sufficiently large $N$ such that

\begin{equation}
(6.26) \quad \epsilon^N(1-\omega(w_0, K, G))(2|f|_X)^{\omega(w_0, K, G)} < \frac{\epsilon}{2}.
\end{equation}
Using the hypothesis that \( f \) can be extended to a continuous function on \( A^* \times B^* \), we may find an open neighborhood \( A_{a_0} \) of \( a_0 \) in \( A^* \) such that
\[
|f(a, b) - f(a_0, b)| \leq \epsilon^N, \quad a \in A_{a_0} \cap A_{a_0}^*, \ b \in K.
\]
(6.27)

On the other hand,
\[
|f(a, w) - f(a_0, w_0)| \leq |f|_X < 2, \quad a \in A_{a_0} \cap A_{a_0}^*, \ w \in G.
\]
(6.28)

For \( a \in A_{a_0} \cap A_{a_0}^* \), applying the Two-Constant Theorem to the function \( f(a, \cdot) - f(a_0, \cdot) \in \mathcal{O}(G) \) and taking (6.26)–(6.28) into account, we deduce that
\[
|f(a, w_0) - f(a_0, w_0)| \leq \epsilon^{N(1 - \omega(w_0, K, G))} (2|f|_X)^{\omega(w_0, K, G)} < \frac{\epsilon}{2}.
\]
(6.29)

Since \( f(a, \cdot)|_G \) is a bounded holomorphic function for \( a \in A \), there is an open neighborhood \( V \) of \( w_0 \) such that
\[
|f(a, w) - f(a, w_0)| < \frac{\epsilon}{2}, \quad a \in A, \ w \in V.
\]

This, combined with (6.29), implies that
\[
|f(a, w) - f(a_0, w_0)| \leq |f(a, w_0) - f(a_0, w_0)| + |f(a, w) - f(a, w_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad a \in A_{a_0}, \ w \in V.
\]

Therefore, \( f \) is continuous at \( (a_0, w_0) \). Consequently, we conclude, by Step 4, that \( \hat{f} \) admits the angular limit \( f(a_0, w_0) \) at \( (a_0, w_0) \). Similarly, we may also show that \( \hat{f} \) admits the angular limit \( f(z_0, b_0) \) at every point \( (z_0, b_0) \in D \times B^* \). This completes the proof of the last step. \( \square \)

7. Preparatory results

We first develop some auxiliary results. This preparation will enable us to generalize the results of section 6 to the general case considered in Theorem A.

**Definition 7.1.** Let \( \Omega \) be a complex manifold of dimension 1 and \( A \subset \Omega \). Define
\[
\omega(\cdot, A, \Omega) := \sup \{ u : u \in \mathcal{SH}(\Omega), \ u \leq 1 \text{ on } \Omega, \ u \leq 0 \text{ on } A \}.
\]
The function \( \omega(\cdot, A, \Omega) \) is called the the harmonic measure of \( A \) relative to \( \Omega \). A point \( \zeta \in \overline{A} \cap \Omega \) is said to be a locally regular point relative to \( A \) if
\[
\lim_{z \to \zeta} \omega(z, A \cap U, \Omega \cap U) = 0
\]
for any open neighborhood \( U \) of \( \zeta \). If, moreover, \( \zeta \in A \), then \( \zeta \) is said to be a locally regular point of \( A \). The set of all locally regular points relative to \( A \) is denoted by \( A^* \). \( A \) is said to be locally regular if \( A = A^* \).

**Proposition 7.2.** Let \( X \) be a complex manifold of dimension 1, \( D \subset X \) an open set and \( A \subset \partial D \) a Jordan measurable subset of positive length. Let \( \{a_j\}_{j \in J} \) be a finite or countable subset of \( A \) with the following properties:

(i) For any \( j \in J \), there is an open neighborhood \( U_j \) of \( a_j \) such that \( D \cap U_j \) is either a Jordan domain or the disjoint union of two Jordan domains;
For any \(0 < \delta < \frac{1}{2}\), define
\[
U_{j,\delta} := \{z \in D \cap U_j : \omega(z, A^* \cap U_j, D \cap U_j) < \delta\}, \quad j \in J,
\]
\[
A_\delta := \bigcup_{j \in J} U_{j,\delta},
\]
\[
D_\delta := \{z \in D : \omega(z, A^*, D) < 1 - \delta\}.
\]

Then:
1) \(A \cap A^* \subset A^D_\delta\) and \(A_\delta \subset D_{1-\delta} \subset D_\delta\);
2) \(\omega(z, A^*, D) - \delta \leq \omega(z, A_\delta, D) \leq \omega(z, A^*, D), \quad z \in D\).

Proof. To prove Part 1), let \(a \in A \cap A^*\) and fix an \(j \in J\) such that \(a \in U_j\). Then
\[
\lim_{z \to a, \, z \in A_\alpha(a)} \omega(z, A^* \cap U_j, D \cap U_j) = 0, \quad 0 < \alpha < \frac{\pi}{2}.
\]
Consequently, for every \(0 < \alpha < \frac{\pi}{2}\), there is an open neighborhood \(V_\alpha \subset U_j\) of \(a\) such that
\[
\omega(z, A^* \cap U_j, D \cap U_j) < \delta, \quad z \in A_\alpha(a) \cap V_\alpha.
\]
This proves \(A \cap A^* \subset A^D_\delta\).

To prove the second assertion of Part 1), one applies the Subordination Principle and obtains for \(z \in U_{j,\delta}\),
\[
(7.1) \quad \omega(z, A^*, D) \leq \omega(z, A^* \cap U_j, D \cap U_j) < \delta < 1 - \delta.
\]
Hence, \(z \in D_{1-\delta}\). This implies that \(A_j \subset D_{1-\delta}\). In addition, since \(0 < \delta < \frac{1}{2}\), it follows that \(D_{1-\delta} \subset D_{1-\delta}\), Hence, Part 1) is proved.

We turn to Part 2). Since \(A_\delta\) is an open set and, by Part 1), \(A \cap A^* \subset A^D_\delta\), it follows from Definitions \([4.8, 7.7]\) that
\[
\omega(z, A_\delta, D) \leq \omega(z, A \cap A^*, D), \quad z \in D.
\]
Hence, in virtue of Theorem \([4.6]\) it follows that
\[
\omega(z, A_\delta, D) \leq \omega(z, A^*, D), \quad z \in D,
\]
which proves the second estimate of Part 2).

To complete Part 2), let \(z \in A_\delta\). Choose \(j \in J\) such that \(z \in U_{j,\delta}\). We deduce from (7.1) that \(\omega(z, A^*, D) - \delta \leq 0\). Hence,
\[
\omega(z, A^*, D) - \delta \leq 0, \quad z \in A_\delta.
\]
On the other hand, \(\omega(z, A^*, D) - \delta < 1, \quad z \in D\). Consequently, the first estimate of Part 2) follows. The proof of the lemma is finished. \(\square\)

The main ingredient in the proof of Theorem A is the following mixed cross theorem.
Theorem 7.3. Let $X$ and $Y$ be complex manifolds of dimension 1, $D \subset X$ and $\Omega \subset Y$ open subsets, and $A \subset D$ and $B \subset \partial \Omega$. Assume that $A = \bigcup_{k=1}^{\infty} A_k$ with $A_k$ locally regular compact subsets of $D$, $A_k \subset A_{k+1}$, $k \geq 1$. In addition, $B \subset \partial \Omega$ is a Jordan measurable subset of positive length. For $0 \leq \delta < 1$ put $G := \{w \in \Omega : \omega(w, B, \Omega) < 1 - \delta\}$. Let $W := X(A, B; D, G)$, $W^o := X^o(A, B; D, G)$, and (using the notation $\omega_\delta(\cdot, B, \Omega)$ of Theorem 4.10)

$W^o = \hat{X}^o(A, B; D, G) := \{(z, w) \in D \times G : \omega(z, A^*, D) + \omega_\delta(w, B, \Omega) < 1\}$.

Let $f : W \rightarrow \mathbb{C}$ be such that

1. $f \in \mathcal{O}_c(W^o)$;
2. $f$ is Jordan measurable and locally bounded on $W$;
3. for any $z \in A$,

$$\lim_{w \rightarrow \eta, w \in A_c(\eta)} f(z, w) = f(z, \eta), \quad \eta \in B, \quad 0 < \alpha < \frac{\pi}{2}.$$  

Then there is a unique function $\hat{f} \in \mathcal{O}(\hat{W}^o)$ such that $\hat{f} = f$ on $A \times G$ and

$$\lim_{z \rightarrow z_0, w \rightarrow \eta_0, w \in A_c(\eta_0)} \hat{f}(z, w) = f(z_0, \eta_0), \quad 0 < \alpha < \frac{\pi}{2},$$

for every $z_0 \in D$ and $\eta_0 \in B \cap B^*$. Moreover, $|\hat{f}|_{\hat{W}^o} \leq |f|_W$.

Proof. First one proves the existence and uniqueness of $\hat{f}$. Fix an $f : W \rightarrow \mathbb{C}$ which satisfies (i)–(iii) above.

Step 1: Reduction to the case where $D \subset X$ is an open hyperconvex set\(^3\) and $A$ is a locally regular compact subset of $D$ and $|f|_W < \infty$.

Since $X$ is countable at infinity, we may find an exhaustion sequence $(D_k)_{k=1}^{\infty}$ of relatively compact, hyperconvex open subsets $D_k$ of $D$ with $A_k \subset D_k \rightarrow D$ (for example, we can choose open subsets $D_k$ of $D$ with smooth boundary which contains $A_k$). Similarly, since $Y$ is countable at infinity, we may find a sequence $(\Omega_k)_{k=1}^{\infty}$ of open subsets of $\Omega$ and a sequence $(B_k)_{k=1}^{\infty}$ of Jordan measurable subsets of $B$ which satisfy the hypothesis of Proposition 4.10. Let $G_k := \{w \in \Omega_k : \omega(w, B_k, \Omega_k) < 1 - \delta\}$. Using a compactness argument, we see that $|f|_{X(A_k, B_k; D_k, G_k)} < \infty$.

By reduction assumption, for each $k$ there exists an $\hat{f}_k \in \mathcal{O}_c(\hat{X}^o(A_k, B_k; D_k, G_k))$ such that $\hat{f}_k$ admit the angular limit $f|_{X(A_k, B_k \cap B_k^*; D_k, G_k)}$ on $X(A_k, B_k \cap B_k^*; D_k, G_k)$.

We claim that $\hat{f}_{k+1} = \hat{f}_k$ on $\hat{X}^o(A_k, B_k; D_k, G_k)$. Indeed, fix an arbitrary $k_0 \geq 1$ and an arbitrary point $(z_0, w_0) \in \hat{X}^o(A_{k_0}, B_{k_0}; D_{k_0}, G_{k_0})$. Let $k \in \mathbb{N}$ such that $k \geq k_0$. Let $D$ be the connected component containing $z_0$ of the following open set

$$\{z \in D : \omega(z, A_{k_0}, D_{k_0}) < 1 - \omega_\delta(w_0, B_{k_0}, \Omega_k)\}.$$  

Observe that both functions $\hat{f}_{k_0}(\cdot, w_0)|_D$ and $\hat{f}_k(\cdot, w_0)|_D$ are holomorphic and

$$\hat{f}_k(z, w_0) = f_k(z, w_0) = \hat{f}_{k_0}(z, w_0), \quad z \in A_k \cap D.$$  

\(^3\)An open set $D \subset X$ is said to be hyperconvex if it admits an exhaustion function which is bounded subharmonic.
Since $A_k \cap D$ is non-polar, we deduce that $f_{k_0}(|, w_0)|_D = f_k(|, w_0)|_D$. Hence, $f_{k_0}(z_0, w_0) = f_k(z_0, w_0)$, which proves the above assertion.

On the other hand, by Proposition 4.7 one gets $\hat{X}^\circ(A_k, B_k; D_k, G_k)$ as $k \to \infty$. Therefore, we may glue $f_k$ together to obtain a function $\hat{f} \in O(\hat{X}^\circ)$ such that $\hat{f}$ admits the angular limit $f$ on $W$ and $\hat{f} = f$ on $A \times G$. The uniqueness of such an extension $\hat{f}$ can be proved using the argument given in the previous paragraph.

This completes Step I.

**Step II:** The case where $D \in X$ is an open hyperconvex set, $A$ is a locally regular compact subset of $D$, and $|f|_W < \infty$.

Suppose without loss of generality that $|f|_W < 1$. We will apply Théorème 3.3 in the work of Zeriahi [15] to the pair of condenser $(A, D)$. In the sequel, we will use the notation from this work.

Let $\mu_0 := \mu_{A,D}$ and $\mu_1$ a $B$-admissible Lebesgue measure of $D$. Let $H_1 := L^2(D, \mu_1)$, $H_0 :=$ the closure of $H_1|_A$ in $L^2(A, \mu_0)$, let $(b_j)_{j=1}^\infty \subset H_1$ be a system of doubly orthogonal bases in $H_1$ and $H_0$. Recall that $\|b_j\|_{H_0} = 1$. Putting $\gamma_j := \|b_j\|_{H_1}$, $j \in \mathbb{N}$, we have that

$$
\sum_{j=1}^\infty \gamma_j^{-\epsilon} < \infty, \quad \epsilon > 0.
$$

For any $w \in B$, we have $f(|, w) \in H_1$ and $f(|, w)|_A \in H_0$. Hence

$$
f(|, w) = \sum_{j=1}^\infty c_j(w) b_j,
$$

where

$$
c_j(w) = \frac{1}{\gamma_j} \int_D f(z, w) \overline{b_j(z)} d\mu_1(z) = \int_A f(z, w) \overline{b_j(z)} d\mu_0(z), \quad j \in \mathbb{N}.
$$

Taking the hypotheses (i)–(iii) into account and applying Lebesgue’s Dominated Convergence Theorem, we see that the formula

$$
\hat{c}_j(w) := \int_A f(z, w) \overline{b_j(z)} d\mu_0(z), \quad w \in G \cup B, \ j \in \mathbb{N};
$$

defines a bounded function which is holomorphic in $G$. Moreover, by (iii) and (7.4)–(7.5) it follows that

$$
\lim_{w \to \eta, \ w \in A_\alpha(\eta)} c_j(w) = \hat{c}_j(\eta) = c_j(\eta), \quad \eta \in B, \ 0 < \alpha < \frac{\pi}{2}.
$$

Using (7.4)–(7.6), we obtain the following estimates

$$
\limsup_{w \to \eta, \ w \in A_\alpha(\eta)} \frac{\log \gamma_j}{\log \gamma_j} \leq \frac{\log \sqrt{\mu_1(A)}}{\log \gamma_j}, \quad w \in G, \ j \in \mathbb{N},
$$

$$
\limsup_{w \to \eta, \ w \in A_\alpha(\eta)} \frac{\log \gamma_j}{\log \gamma_j} \leq \frac{\log \sqrt{\mu_1(D)}}{\log \gamma_j} - 1, \quad \eta \in B, \ 0 < \alpha < \frac{\pi}{2}, \ j \in \mathbb{N}.
$$
This shows that for any \( \epsilon > 0 \), there is a sufficiently large \( N \) such that for all \( j \geq N \),
\[
(7.7) \quad \frac{\log |\hat{c}_j|}{\log \gamma_j} \leq \omega_\delta(\cdot, B, \Omega) + \epsilon - 1 \quad \text{on } G.
\]

Take a compact set \( K \subseteq D \) and let \( 1 > \alpha = \alpha(K) > \max_{\Omega} \omega(\cdot, A, D) \). Choose an \( \epsilon = \epsilon(K) > 0 \) so small that \( \alpha + 2\epsilon < 1 \). Consider the open set
\[
G_K := \{ w \in G : \omega_\delta(\cdot, B, \Omega) < 1 - \alpha - 2\epsilon \}.
\]
By (7.1) there is a constant \( C'(K) \) such that
\[
(7.8) \quad |\hat{c}_j|_{G_j} \leq C'(K)\gamma_j^{\omega_\delta(\cdot, B, \Omega) + \epsilon - 1} \leq C'(K)\gamma_j^{-\alpha - \epsilon}, \quad j \geq 1.
\]
Now we wish to show that
\[
(7.9) \quad \sum_{j=1}^\infty \hat{c}_j(w)b_j(z)
\]
converges locally uniformly in \( \hat{W}^\alpha \). Indeed, by (7.2) and (7.8), we have that
\[
(7.10) \quad \sum_{j=1}^\infty |\hat{c}_j|_{G_k} |b_j|_K \leq \sum_{j=1}^\infty C'(K)\gamma_j^{-\alpha - \epsilon} C(K, \alpha) \gamma_j^\alpha \leq C'(K)C(K, \alpha) \sum_{j=1}^\infty \gamma_j^{-\epsilon} < \infty,
\]
which gives the normal convergence on \( K \times G_K \). Since the compact set \( K \) and \( \epsilon > 0 \) are arbitrary, the series in (7.9) converges uniformly on compact subsets of \( \hat{W}^\alpha \). Let \( \hat{f} \) denote this limit function in (7.9).

Fix \( z_0 \in D \) and \( \eta_0 \in B \cap B^* \). We choose a compact \( K_0 \subseteq D \) so that \( K_0 \) is a neighborhood of \( z_0 \). Let \( \epsilon_0 > 0 \).

In virtue of (7.10), there is an \( N_0 \) such that
\[
(7.11) \quad \sum_{j=N_0+1}^\infty |\hat{c}_j|_{G_{K_0}} |b_j|_{K_0} < \frac{\epsilon_0}{2}.
\]
On the other hand, in virtue of (7.8)–(7.9), we may find, for any \( 0 < \alpha < \frac{\pi}{2} \), an open neighborhood \( V_\alpha \) of \( \eta_0 \) such that
\[
\left| \sum_{j=1}^{N_0} \hat{c}_j(w)b_j(z) - \sum_{j=1}^{N_0} c_j(\eta_0)b_j(z) \right| < \frac{\epsilon_0}{2}, \quad z \in K_0, \ w \in A_\alpha(\eta_0) \cap V_\alpha.
\]
This, combined with (7.9) and (7.11), implies that
\[
\limsup_{z \to z_0, \ w \to \eta_0, \ w \in A_\alpha(\eta_0)} \left| \hat{f}(z, w) - f(z_0, \eta_0) \right| < \epsilon_0, \quad 0 < \alpha < \frac{\pi}{2}.
\]
Since \( \epsilon_0 > 0 \) and \( (z_0, \eta_0) \in D \times (B \cap B^* \) can be arbitrarily chosen, we conclude that
\[
\lim_{z \to z_0, \ w \to \eta_0, \ w \in A_\alpha(\eta_0)} \hat{f}(z, w) = f(z_0, \eta_0), \quad (z_0, \eta_0) \in D \times (B \cap B^*), \ 0 < \alpha < \frac{\pi}{2}.
\]

To complete Step II, it remains to show that \( \hat{f} = f \) on \( A \times G \). To do this, fix an arbitrary \( (z_0, w_0) \in A \times G \). Let \( G \) be the connected component of \( G \) containing \( w_0 \).
Recall that $G = \{ w \in \Omega : \omega(w, B, \Omega) < 1 - \delta \}$. Then observe that both functions $\hat{f}(z_0, \cdot)|_G$ and $f(z_0, \cdot)|_G$ admit the same angular limit $f$ on $B \cap \mathcal{G}^\Omega$. Consequently, applying Theorem 5.3 yields that $\hat{f}(z_0, \cdot)|_G = f(z_0, \cdot)|_G$. Hence, $\hat{f}(z_0, w_0) = f(z_0, w_0)$, which proves the above assertion.

This completes the proof of Step II.

It remains to prove the estimate $|\hat{f}|_{\tilde{W}} \leq |f|_W$. In order to reach a contradiction assume that there is a point $z^0 \in \tilde{W}$ such that $|\hat{f}(z^0)| > |f|_W$. Put $\alpha := \hat{f}(z^0)$ and consider the function

$$g(z) := \frac{1}{\hat{f}(z) - \alpha}, \quad z \in \tilde{W}.$$  

Using the above assumption, it can be checked that $g$ satisfies hypotheses (i)–(iii) of Theorem 7.3. Hence applying the first assertion of the theorem, there is exactly one function $\hat{g} \in \mathcal{O}(\tilde{W})$ with $\hat{g} = g$ on $A \times \tilde{G}$. Therefore, by (7.12) we have on $A \times \tilde{G}$:

$$g(f - \alpha) \equiv 1.$$  

Thus $\hat{g}(\hat{f} - \alpha) \equiv 1$ on $\hat{W}$. In particular,

$$0 = \hat{g}(z^0)(\hat{f}(z^0) - \alpha) = 1;$$  

a contradiction. Hence the inequality $|\hat{f}|_{\tilde{W}} \leq |f|_W$ is proved. \hfill \Box

Finally, we conclude this section with two uniqueness results.

**Proposition 7.4.** Let $X, Y$ be two complex manifolds of dimension 1, $D \subset X$, $G \subset Y$ two open sets and $A \subset \partial D, B \subset \partial G$ two Jordan measurable subsets of positive length. Let $\tilde{D} \subset X$ be an open set, $D \cap \tilde{D} \neq \emptyset$, and let $\tilde{A} \subset \partial \tilde{D}$ be a Jordan measurable subset of positive measure. Put

$$\tilde{W} := \hat{X}(A, B; D, G),$$  

$$\tilde{\tilde{W}} := \hat{X}(\tilde{A}, B; \tilde{D}, G).$$

Let $\hat{f} \in \mathcal{O}(\tilde{W})$, $\hat{f} \in \mathcal{O}(\tilde{\tilde{W}})$, and $z_0 \in D \cap \tilde{D}$ be such that both $\hat{f}$ and $\hat{f}$ admit the same angular limit at $(z_0, b)$ for a.e. $b \in B$. Then $\hat{f}(z, w) = \hat{f}(z, w)$ for every $(z, w) \in \tilde{W} \cap \tilde{\tilde{W}}$.

**Proof.** Fix an arbitrary $w_0 \in G$ such that $(z_0, w_0) \in \tilde{W} \cap \tilde{\tilde{W}}$. Choose $0 < \epsilon < 1$ so that

$$(z_0, w_0) \in D_{1-\epsilon} \times G_\epsilon \cap \tilde{D}_{1-\epsilon} \times G_\epsilon,$$

where we have used the notation of level sets introduced in Section 4. Applying Theorem 5.3 to $\hat{f}(z_0, \cdot)|_{C_\epsilon}$ and $\hat{f}(z_0, \cdot)|_{C_\epsilon}$, it follows that $\hat{f}(z_0, w_0) = \hat{f}(z_0, w_0)$.

Hence, the proof is finished. \hfill \Box

Now we are able to prove the uniqueness stated in Theorem A.

**Corollary 7.5.** We keep the hypotheses and the notation of Theorem A. Then there is at most one function $\hat{f} \in \mathcal{O}(\tilde{W})$ which satisfies Property 1) of Theorem A.

**Proof.** It follows immediately from Proposition 7.4. \hfill \Box
8. Proof of Theorem A

Recall that by Corollary \[7.5\] the function \( \hat{f} \) satisfying Part 1) is uniquely determined (if it exists). We only gives here the proof of Part 1). Using this part, we conclude the proof of Parts 2)–5) of Theorem A in exactly the same way as we did in Section 6 starting from Step 2 of that section. The proof is divided into two steps.

**Step 1:** Proof of Theorem A for the case where \( G \) is a Jordan domain.

**Proof of Step 1.** In virtue of Proposition \[7.2\] let \( \{a_j\}_{j \in J} \) be a finite or countable subset of \( A \) with the following properties:

- For any \( j \in J \), there is an open neighborhood \( U_j \) of \( a_j \) such that \( D \cap U_j \) is either a Jordan domain or the disjoint union of two Jordan domains (according to the type of \( a_j \));
- \( A \subset \bigcup_{j \in J} U_j \).

For any \( 0 < \delta < \frac{1}{2} \), define

\[
U_{j, \delta} := \{ z \in D \cap U_j : \omega(z, A^* \cap U_j, D \cap U_j) < \delta \}, \quad j \in J,
\]

\[
A_\delta := \bigcup_{j \in J} U_{j, \delta},
\]

\[
G_\delta := \{ w \in G : \omega(w, B, G) < 1 - \delta \}.
\]

Moreover, for every \( j \in J \) let

\[
W_j := \mathbb{X}(\partial(D \cap U_j) \cap A, B; D \cap U_j, G),
\]

\[
\hat{W}_j := \mathbb{X}^o(\partial(D \cap U_j) \cap A, B; D \cap U_j, G),
\]

\[
\check{f}_j := f|_{W_j}.
\]

Using the hypotheses on \( f \), we conclude that \( \check{f}_j, j \in J \), satisfies (i)–(iii) of Theorem A. Moreover, since \( G \) is a Jordan domain and \( D \cap U_j, j \in J \), is either a Jordan domain or the disjoint union of two Jordan domains, we are able to apply the result of Section 6 to \( \check{f}_j \). Consequently, we obtain, for \( j \in J \), a unique function \( \hat{f}_j \in \mathcal{O}\left(\check{W}_j^o\right) \), a subset \( A_j \) of \( \partial(D \cap U) \cap A \), a subset \( B_j \) of \( B \) such that

\[
A_j \subset A_j' ,
\]

\[
(\partial(D \cap U) \cap A) \setminus A_j \text{ and } B \setminus B_j \text{ is of zero length},
\]

\( \check{f}_j \) admits the angular limit \( f \) on \( ((\partial(D \cap U_j) \cap A_j) \times G) \cup (D \times B_j) \).

Put

\[
\tilde{A} := \bigcap_{j \in J} A_j \quad \text{and} \quad \tilde{B} := \bigcap_{j \in J} B_j,
\]

\[
W_\delta := \mathbb{X}\left( A_\delta, \tilde{B}; D, G_\delta \right),
\]

\[
\hat{W}_\delta := \mathbb{X}^o\left( A_\delta, \tilde{B}; D, G_\delta \right) .
\]
In virtue of Proposition 7.4 we are able to collect the family \( \left( \tilde{f}_j|_{U_{j,\delta} \times G_{\delta}} \right)_{j \in J} \) in order to obtain a function \( \tilde{f}_\delta \in O(A_\delta \times G_\delta) \).

Next, consider the function \( \tilde{f}_\delta : W_\delta \rightarrow \mathbb{C} \) given by

\[
\tilde{f}_\delta := \begin{cases} 
\tilde{f}_\delta, & \text{on } A_\delta \times G_\delta \\
 f, & \text{on } D \times (\tilde{B} \cap \tilde{B}^*)
\end{cases}
\]

In virtue of (8.1)–(8.4), we deduce that

\[
\text{In virtue of (8.1)–(8.4), we deduce that}
\]

\[
A \setminus \tilde{A} \text{ and } B \setminus \tilde{B} \text{ is of zero length,}
\]

and

\[
\lim_{z \to z_0, w \to b_0, w \in A_\alpha(b_0)} \tilde{f}_\delta(z, w) = f(z_0, b_0), \quad 0 < \alpha < \frac{\pi}{2}, \quad z_0 \in D, \quad b_0 \in \tilde{B} \cap \tilde{B}^*,
\]

\[
\lim_{z \to a_0, z \in A_\alpha(a_0), w \to w_0} \tilde{f}_\delta(z, w) = f(a_0, w_0), \quad 0 < \alpha < \frac{\pi}{2}, \quad a_0 \in \tilde{A}, \quad w_0 \in G_\delta.
\]

In virtue of (8.4)–(8.6), \( \tilde{f}_\delta \) satisfies the hypotheses (i)–(iii) of Theorem 7.3. Applying this theorem to \( \tilde{f}_\delta \), we obtain, for every \( 0 < \delta < \frac{1}{2} \), a function \( \hat{f}_\delta \in O\left(\hat{W}_\delta^o\right) \). In virtue of (8.8), we see that

\[
\hat{f}_\delta = \tilde{f}_\delta \quad \text{on } A_\delta \times G_\delta,
\]

\[
\lim_{z \to z_0, w \to b_0, w \in A_\alpha(b_0)} \hat{f}_\delta(z, w) = f(z_0, b_0), \quad 0 < \alpha < \frac{\pi}{2}, \quad z_0 \in D, \quad b_0 \in \tilde{B} \cap \tilde{B}^*,
\]

\[
\lim_{z \to a_0, z \in A_\alpha(a_0), w \to w_0} \hat{f}_\delta(z, w) = f(a_0, w_0), \quad 0 < \alpha < \frac{\pi}{2}, \quad a_0 \in \tilde{A}, \quad w_0 \in G_\delta.
\]

We are now in a position to define the desired extension function \( \hat{f} \). Indeed, one glues \( \left( \hat{f}_\delta \right)_{0 < \delta < \frac{1}{2}} \) together to obtain \( \hat{f} \) in the following way

\[
\hat{f} := \lim_{\delta \to 0} \hat{f}_\delta \quad \text{on } \hat{W}^o = \hat{X}^o(A, B; D, G).
\]

Now one has to check that the limit \( \hat{f} \) exists and possesses all the required properties. This will be an immediate consequence of the following

**Lemma 8.1.** For any point \( (z, w) \in \hat{W}^o \) put

\[
\delta_{(z, w)} := \frac{1 - \omega(z, A^*; D) - \omega(w, B^*; G)}{2}.
\]

Then \( \hat{f}(z, w) = \hat{f}_\delta(z, w) \) for all \( 0 < \delta \leq \delta_{(z, w)} \).
Proof of Lemma 8.1. Fix an arbitrary point \((z_0, w_0) \in \hat{\mathcal{X}}^0(A, B; D, G)\) and let \(\delta_0 := \delta(z_0, w_0)\). Let \(0 < \delta \leq \delta_0\). Then, \(\omega(w_0, B^*, G) < 1 - \delta_0\) and

\[
\omega(z_0, A_\delta, D) + \omega_{\delta_0}(w_0, B^*, G) \leq \omega(z_0, A^*, D) + \frac{\omega(w_0, B^*, G)}{1 - \delta_0} \leq \frac{\omega(z_0, A^*, D) + \omega(w_0, B^*, G)}{1 - \delta_0} < 1,
\]

where the latter estimate follows from formula (8.9). Consequently,

\[(8.10) \quad (z_0, w_0) \in \hat{\mathcal{X}}^0(A_\delta, B; D, G_{\delta_0}).\]

On the other hand, using Part 1) of Proposition 7.2, it is clear that

\[(8.11) \quad \hat{\mathcal{X}}^0(A_\delta, B; D, G_{\delta_0}) \subset \hat{\mathcal{X}}^0(A_\delta, B; D, G_{\delta_0}) \cap \hat{\mathcal{X}}^0(A_{\delta_0}, B; D, G_{\delta_0}).\]

Moreover, in virtue of (8.4) and (8.7), we have

\[(8.12) \quad \hat{f}_\delta = \tilde{f}_\delta = \hat{f}_{\delta_0} \quad \text{on} \quad A_\delta \times G_{\delta_0}.
\]

Next, let \(D\) be the connected component containing \(z_0\) of the following open set

\[\{z \in D : \omega(z, A_\delta, D) < 1 - \omega_{\delta_0}(w_0, B, G)\}\]

Observe that, in virtue of (8.10)–(8.11), both functions \(\hat{f}_\delta|_D\) and \(\hat{f}_{\delta_0}|_D\) are holomorphic and \(D \cap A_\delta\) is a nonempty open set. Therefore, we deduce from (8.12) that \(\hat{f}_\delta = \hat{f}_{\delta_0}\) on \(D\). Hence, \(\hat{f}_\delta(z_0, w_0) = \hat{f}_{\delta_0}(z_0, w_0)\), which completes the proof of the lemma.

We complete the proof (of Part 1)) as follows. An immediate consequence of Lemma 8.1 is that \(\hat{f} \in \mathcal{O}(\hat{W}^0)\). Next, we apply Lemma 8.1 and make use of (8.4)–(8.9) and of the fact that \(\hat{W}_\delta^0 \to \hat{W}^0\) as \(\delta \searrow 0\). Consequently, we conclude that \(\hat{f}\) satisfies the conclusion of Part 1). Hence, the proof of Step 1 is finished.

\[\text{Step 2: Proof of Theorem A for the general case.}\]

Proof of Step 2. We proceed using Step 1 in exactly the same way as we proved Step 1 using the result of Section 6. Hence, Step 2 is finished.

This completes the proof of Theorem A.

We conclude this section with the following remark. Using the above proof, one can also derive Gonchar’s Theorem (Theorem 1) from Druzkowski’s Theorem (Theorem 3). Indeed, in Step 1 above, let \(\{a_j\}_{j \in J}\) be finite or countable subset of \(A\) with the following properties:

- For any \(j \in J\), there is an open neighborhood \(U_j\) of \(a_j\) such that \(D \cap U_j\) is a Jordan domain and \(A \cap U_j\) is one open arc;
- \(A \subset \bigcup_{j \in J} U_j\).

Then we repeat Step 1 (\(B\) is only one open arc) and Step 2 (the general case) above using Druzkowski’s Theorem. Gonchar’s Theorem follows.
9. Proof of Theorem B

We will only give the proof of Theorem B for the case when $D$ and $G$ are the unit disc $E$. Since the general case can be proved using the scheme of Section 6 and 8, it is left to the interested reader. The proof is divided into the following two steps.

**Step 1:** Proof of Theorem B for the case when the slice functions $f(a, \cdot)|_G$ and $f(\cdot, b)|_D$ are bounded for every $a \in A$ and $b \in B$.

**Proof of Step 1.** For any $N \in \mathbb{N}$ let
\begin{equation}
A_N := \{ a \in A : |f(a, \cdot)|_G \leq N \} \quad \text{and} \quad B_N := \{ b \in B : |f(\cdot, b)|_D \leq N \}.
\end{equation}
Using the assumption of Step 1 and (9.1), we obtain
\begin{equation}
\lim_{N \to \infty} A_N \nearrow A \quad \text{and} \quad \lim_{N \to \infty} B_N \nearrow B \quad \text{as} \quad N \nearrow \infty.
\end{equation}
Now we would like to show that for every $N \in \mathbb{N}$,
\begin{equation}
A_N \quad \text{is a closed subset of} \quad A \quad \text{and} \quad f|_{A_N \times G} \in \mathcal{C}(A_N \times G),
\end{equation}
\begin{equation}
B_N \quad \text{is a closed subset of} \quad B \quad \text{and} \quad f|_{D \times B_N} \in \mathcal{C}(D \times B_N).
\end{equation}
To do this fix an arbitrary $N \in \mathbb{N}$ and let $(a_n)_{n=1}^\infty$ be a sequence in $A_N$ such that $a_n \to a_0 \in A_N$. Consequently, by hypothesis (i),
\begin{equation}
\lim_{n \to \infty} f(a_n, t) = f(a_0, t), \quad t \in B.
\end{equation}
On the other hand, it follows from the assumption $(a_n)_{n=1}^\infty \subset A_N$ and the hypothesis of Step 1 that
\[ |f(a_n, \cdot)|_G \leq N \quad \text{and} \quad |f(a_0, \cdot)|_G < \infty. \]
Combining this and (9.4), we are able to apply Khinchin–Ostrowski Theorem (see [4, Theorem 4, p. 397]) to the sequence $(f(a_n, \cdot)|_G)_{n=1}^\infty \subset \mathcal{O}(G)$. Consequently, this sequence converges uniformly on compact subsets of $G$ to $f(a_0, \cdot)$. This completes the proof of (9.3).

On the other hand, by hypothesis (ii), the holomorphic function $f(a, \cdot)$ admits the angular limit $f(a, b)$ at $b \in B$. Hence, it follows that $f|_{A_N \times B_N}$ is measurable. Moreover, by (9.1), $|f|_{\overline{X}(A_N, B_N; D, G)} \leq N$ for every $N \in \mathbb{N}$. In addition, in virtue of (9.2), there exists a sufficiently large integer $N_0$ such that $\text{mes}(A_N) > 0$ and $\text{mes}(B_N) > 0$ for $N \geq N_0$. Consequently, we are in a position to apply Theorem A to the function $f$ restricted to the cross $\overline{X}(A_N, B_N; D, G)$ for $N \geq N_0$. Therefore, we obtain a function $\hat{f}_N \in \mathcal{O}\left(\overline{X}(A_N, B_N; D, G)\right)$ and a subset $\tilde{A}_N$ (resp. $\tilde{B}_N$) of $A_N$ (resp. $B_N$) for $N \geq N_0$, such that
\begin{equation}
\text{mes}(A_N \setminus \tilde{A}_N) = \text{mes}(B_N \setminus \tilde{B}_N) = 0,
\end{equation}
\begin{equation}
\hat{f}_N \quad \text{admits the angular limit} \quad f \quad \text{on} \quad \left(\tilde{A}_N \times G\right) \cup \left(D \times \tilde{B}_N\right).
\end{equation}
Put
\begin{equation}
\tilde{A} := \bigcup_{N=N_0}^\infty \tilde{A}_N \quad \text{and} \quad \tilde{B} := \bigcup_{N=N_0}^\infty \tilde{B}_N.
\end{equation}
Applying (9.2), (9.5), and Corollary 7.5, we obtain
\[ \hat{f}_N = \hat{f}_{N+1} \quad \text{on} \quad \hat{X}^o \left( \hat{A}_N, \hat{B}_N; D, G \right), \quad N \geq N_0. \]

Therefore, we may glue the \( \hat{f}_N \) together to obtain the desired extension function \( \hat{f} \) as
\[ \hat{f} = \lim_{N \to \infty} \hat{f}_N \quad \text{on} \quad \hat{W}^o := \hat{X}^o \left( A, B; D, G \right). \]

Moreover, in virtue of (9.5)–(9.8), we get that 
\[ \text{mes} \left( A \setminus \hat{A} \right) = \text{mes} \left( B \setminus \hat{B} \right) = 0, \]
and
\[ \hat{f} \text{ admits the angular limit } f \text{ on } (\hat{A} \times G) \cup (D \times \hat{B}). \]

Next, for every \( N \geq N_0 \), in virtue of (9.2)–(9.3) and (9.5), one may find a sequence \( (F_{N,n})_{n=1}^\infty \) (resp. \( (H_{N,n})_{n=1}^\infty \)) of compact subsets of \( \partial D \) (resp. \( \partial G \)) such that
\[ F_{N,n} \subset F_{N,n+1} \subset A, \quad H_{N,n} \subset H_{N,n+1} \subset B, \]
\[ \text{mes} (F_{N,n}) > 0, \quad \text{mes} (H_{N,n}) > 0, \]
\[ \text{mes} \left( \hat{A}_N \setminus \bigcup_{n=1}^\infty F_{N,n} \right) = 0, \quad \text{mes} \left( \hat{B}_N \setminus \bigcup_{n=1}^\infty H_{N,n} \right) = 0. \]

Moreover, for any \( k \in \mathbb{N}, k \geq 1 \), and for any \( m \in \mathbb{N} \), put
\[ A_{Nnmk} := \left\{ a \in A_N : \left| f(a, \zeta) - f(a, \eta) \right| \leq \frac{1}{2k^2}, \quad \zeta, \eta \in H_{N,n} : \left| \zeta - \eta \right| < \frac{1}{m} \right\}, \]
\[ B_{Nnmk} := \left\{ b \in B_N : \left| f(\zeta, b) - f(\eta, b) \right| \leq \frac{1}{2k^2}, \quad \zeta, \eta \in F_{N,n} : \left| \zeta - \eta \right| < \frac{1}{m} \right\}. \]

Since, by hypothesis (i), \( f \in C_s (A \times B) \), we deduce from (9.10) and (9.11) that \( A_{Nnmk} \) (resp. \( B_{Nnmk} \)) is a closed subset of \( A_N \) (resp. \( B_N \)) and
\[ A_{Nnmk} \nearrow A_N \quad \text{and} \quad B_{Nnmk} \nearrow B_N \quad \text{as} \quad m \nearrow \infty, \quad k \geq 1. \]

Consequently, there is an \( m_0 := m_0(N, n, k) \) such that \( \text{mes} (A_{Nnmk} \cap F_{N,n}) > 0 \) and \( \text{mes} (B_{Nnmk} \cap H_{N,n}) > 0 \) for any \( m > m_0 \). Now we are in a position to apply Theorem A to the function \( f \) restricted on the cross \( \hat{X}^o \left( A_{Nnmk} \cap F_{N,n}, B_{Nnmk} \cap H_{N,n}; D, G \right) \).

Using (9.1)–(9.9) and Corollary 7.5 we obtain exactly the function \( \hat{f} \) restricted to \( \hat{X}^o \left( A_{Nnmk} \cap F_{N,n}, B_{Nnmk} \cap H_{N,n}; D, G \right) \). Let \(^4\)
\[ \hat{A}_{Nnmk} := (A_{Nnmk} \cap F_{M,n}) \cap (A_{Nnmk} \cap F_{N,n})^*, \]
\[ \hat{B}_{Nnmk} := (B_{Nnmk} \cap H_{M,n}) \cap (B_{Nnmk} \cap H_{N,n})^*. \]

\(^4\) Recall from Subsection 2.2 that for a boundary subset \( T, T^* \) denotes as usual the set of locally regular points relative to \( T \).
Taking (9.11)–(9.13) into account and arguing as in Step 5 of Section 6, we may show that
\[
\text{mes}\left(\tilde{A}_{Nnmk} \setminus F_{N,n}\right) = 0, \quad \text{mes}\left(\tilde{B}_{Nnmk} \setminus H_{N,n}\right) = 0,
\]
(9.14)
\[
\limsup_{(z,w) \to (a,b)} |\hat{f}(z, w) - f(a, b)| < \frac{1}{k}, \quad 0 < \alpha < \frac{\pi}{2},
\]
for every \((a, b) \in \tilde{A}_{Nnmk} \times \tilde{B}_{Nnmk}\). Now it suffices to put
\[
\tilde{A} := \bigcap_{k=1}^{\infty} \bigcup_{N=N_0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=m_0(N,n,k)}^{\infty} \tilde{A}_{Nnmk}\quad \text{and}\quad \tilde{B} := \bigcap_{k=1}^{\infty} \bigcup_{N=N_0}^{\infty} \bigcup_{n=1}^{\infty} \bigcup_{m=m_0(N,n,k)}^{\infty} \tilde{B}_{Nnmk}.
\]
Combining this and (9.14), (9.12), (9.9) and (9.2), we may check that all the conclusions of Theorem B are satisfied. Hence the proof is complete in this first step. \(\square\)

Step 2: The general case.

Proof of Step 2. We begin with the following

Definition 9.1. For a closed subset \(F\) of \(\partial E\) and an \(n \in \mathbb{N}\) with \(n > 1\), define the following open set
\[
\Delta = \Delta(F, n) := \bigcup_{\zeta \in F} \left\{ z \in A_\pi(\zeta) : |z| \geq 1 - \frac{1}{n} \right\} \cup \mathbb{B} \left(0, 1 - \frac{1}{n}\right).
\]

The reader should compare this definition with Definition 5.1. Below we give a list of properties of such open sets.

Proposition 9.2. Let \(F\) be a closed subset of \(\partial E\).
1) Let \(\Delta(F, n)\) be as in Definition 9.1, then \(\Delta(F, n)\) is a rectifiable Jordan domain and \(F \subset \partial \Delta(F, n)\).
2) \(\Delta(F, n) \nearrow E\) as \(n \nearrow \infty\).
3) Consider a locally bounded function \(f : E \cup F \to \mathbb{C}\). Then \(|f|_{\Delta(F,n)} < \infty\) for every \(n \in \mathbb{N}\) with \(n > 1\).
4) There holds the following equality
\[
\omega(z, F, E) = \lim_{n \to \infty} \omega(z, F, \Delta(F, n)), \quad z \in E.
\]

Proof of Proposition 9.2. Part 1) may be done as in the proof of Proposition 5.2.
Part 2) is an immediate consequence of Definition 9.1.
Part 3) follows immediately from the compactness of \(F\).
The proof of Proposition 5.7 still works in the context of Part 4) making the obviously necessary changes. This completes Part 4). \(\square\)

Now we are in a position to complete Step 2. Indeed, first suppose that both \(A\) and \(B\) are closed. Then consider the sequence of rectifiable Jordan domain \((D_n)_{n=2}^{\infty}\) and \((G_n)_{n=2}^{\infty}\) given by
\[
D_n := \Delta(A, n)\quad \text{and}\quad G_n := \Delta(B, n), \quad n \in \mathbb{N}, \ n > 1.
\]
For \( n \in \mathbb{N}, n > 1 \), let \( f_n := f|_{X(A,B,D_n,G_n)} \). In virtue of Proposition 9.2, we are able to apply the result of Step 1 to \( f_n \). Consequently, we obtain a function \( \hat{f}_n \in \hat{X}(A,B,D_n,G_n) \). Therefore, we may glue \( \hat{f}_n \) together in order to obtain the desired extension function \( \hat{f} \) as

\[
\hat{f} = \lim_{n \to \infty} \hat{f}_n \quad \text{on } \hat{W}^o = \hat{X}(A,B,D,G).
\]

Using Proposition 9.2, we can show that \( \hat{f} \) possesses all the assertions of Theorem B.

The case when \( A \) and \( B \) are only measurable is similar. It suffices to find a sequence \( (A_m)_{m=1}^{\infty} \) of subsets of \( A \) such that \( A_m \) is compact and \( \operatorname{mes}(A \setminus \bigcup_{m=1}^{\infty} A_m) = 0 \), and a similar sequence \( (B_m)_{m=1}^{\infty} \) for \( B \). Then we may apply the previous discussion to \( f|_{X(A_m,B_m,D,G)} \) in order to obtain a function \( \hat{f}_m \in \hat{X}(A_m,B_m,D,G) \), and define the desired extension function \( \hat{f} \) by \( \hat{f} := \lim_{m \to \infty} \hat{f}_m \) on \( \hat{W}^o \). This completes the proof in this last step. \( \square \)

10. Examples and Concluding remarks

The following examples of Drużkowski [2] show the optimality of Theorem A and B.

Consider \( D = G = E, A = B = \{ t \in \partial E : \Re t > 0 \} \), \( W := X(A,B,D,G) \), and \( T := (D \cup A) \times (G \cup B) \).

**Example 1.** Define a function \( h : T \to \mathbb{C} \) as follows

\[
h(z,w) := \begin{cases} 
\exp\left( -[\log (1-z) + \log (1-w)] \log \frac{2+zw}{3} \right), & z \neq 1, w \neq 1 \\
0, & z = 1 \text{ or } w = 1
\end{cases}
\]

where \( \log \) is the principal branch of logarithm.

Put \( f := h|_W \). As in [2] observe that \( f \) is measurable, \( f \in \mathcal{C}_s(W) \cap \mathcal{O}_s(W^o) \), \( |f|_W < \infty \), but \( f|_{A \times B} \) is not continuous at \((1,1)\). Since \( h|_{\hat{W}^o} \in \mathcal{O}(W^o) \), using the uniqueness established in Theorem A, we conclude that the solution \( \hat{f} \) provided by Theorem A and B satisfies \( \hat{f} = h|_{\hat{W}^o} \). In addition, we see that, for \( 0 < \alpha < \frac{\pi}{2} \), the angular limit of \( \hat{f} \) at \((1,1)\) does not exist. Thus the condition in assertion 3) of Theorem A is necessary. Moreover, the sets \( \hat{A}, \hat{B} \) given by Theorem B do depend on \( f \).

**Example 2.** Define a function \( h : T \to \mathbb{C} \) as follows

\[
h(z,w) := \begin{cases} 
\exp\left( -(z-\lambda) \log \frac{2+zw}{1-w} \right), & w \neq 1 \\
0, & w = 1
\end{cases}
\]

where \((z,w) \in T, 0 < \lambda \leq \frac{\sqrt{2}}{2}\).

Define \( f := h|_W \). Then \( \hat{f} = h|_{\hat{W}^o} \). As in [2] observe that \( f|_{A \times B} \) is continuous, \( f \in \mathcal{C}_s(W) \cap \mathcal{O}_s(W^o) \), but \( f \) is not locally bounded on \( W \).
In addition, for $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$, consider the functions $z_{\alpha, \lambda}, w_\alpha : [0, 1] \to \mathbb{C}$ given by

$$w_\alpha(t) := 1 + te^{i\left(\pi - \frac{\alpha}{10}\right)},$$

$$z_{\alpha, \lambda}(t) := \lambda + \left(\text{Re Log} \frac{3 + w_\alpha(t)}{1 - w_\alpha(t)}\right)^{-1} + i\lambda, \quad t \in [0, 1].$$

We may prove that there is an $t_{\alpha, \lambda} > 0$ and a neighborhood $U_{\alpha, \lambda}$ of $\lambda + i\lambda$ in $\mathbb{C}$ such that

$$(z_{\alpha, \lambda}(t), w_\alpha(t)) \in \begin{cases} 
\left(\left(\mathcal{A}_\alpha(\lambda + i\lambda) \cap U_{\alpha, \lambda}\right) \times \mathcal{A}_\alpha(1)\right) \cap \hat{W}^\circ, & 0 < t < t_{\alpha, \lambda}, \lambda = \frac{\sqrt{2}}{2} \\
\left(U_{\alpha, \lambda} \times \mathcal{A}_\alpha(1)\right) \cap \hat{W}^\circ, & 0 < t < t_{\alpha, \lambda}, 0 < \lambda < \frac{\sqrt{2}}{2}.
\end{cases}$$

In addition, it can be checked that

$$\lim_{t \to 0} (z_{\alpha, \lambda}(t), w_\alpha(t)) = (\lambda + i\lambda, 1) \quad \text{and} \quad \lim_{t \to 0} \left| \hat{f}(z_{\alpha, \lambda}(t), w_\alpha(t)) \right| = \infty.$$

This shows that the assumption of the local boundedness on $f$ is necessary in Theorem A.

Finally, we conclude the article by some remarks and open questions.

1. It may be proved that $\hat{W}^\circ$ provided by Theorem A is the maximal domain of holomorphic extension of the function $f$. We postpone the proof of this result to an ongoing work (see [12]).

2. Does Theorem A still hold if we omit the assumption (ii) “$f|_{A \times B}$ is Jordan-measurable”?

3. Does Theorem B still hold if we omit the assumption that $f|_{A \times B} \in C_s(A \times B)$?

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