Abstract. We investigate intersections of geodesic lines in $\mathbb{H}^2$ and in an associated tree $T$, proving the following result. Let $M$ be a punctured hyperbolic torus and let $\gamma$ be a closed geodesic in $M$. Any edge of any triangle formed by distinct geodesic lines in the preimage of $\gamma$ in $\mathbb{H}^2$ is shorter than $\gamma$. However, a similar result does not hold in the tree $T$. Let $W$ be a reduced and cyclically reduced word in $\pi_1(M) = \langle x, y \rangle$. We construct several examples of triangles in $T$ formed by distinct axes in $T$ stabilized by conjugates of $W$ such that an edge in those triangles is longer than $L(W)$. We also prove that if $W$ overlaps two of its conjugates in such a way that the overlaps cover all of $W$ and the overlaps do not intersect, then there exists a decomposition $W = BC^kI, k > 0$, with $B$ a terminal subword of $C$ and $I$ an initial subword of $C$.

1. Introduction

The study of curves on surfaces is a classical subject going back to the origins of topology, [1]. Of particular interest are closed geodesics which can be investigated by looking at their lifts in covering spaces of the surface, [2], [3], [4], [6], [7], [9]. In this paper we consider hyperbolic surfaces and study the intersections of geodesic lines in $\mathbb{H}^2$, [5]. In general, the patterns of such intersections are very complicated, so we restrict ourselves to three geodesic lines in $\mathbb{H}^2$ which are lifts of the same closed geodesic in a punctured hyperbolic surface. For the sake of clarity we choose the surface to be a punctured torus.

An important tool in studying geodesic lines in $\mathbb{H}^2$ is the tree $T$ in $\mathbb{H}^2$, defined as follows, cf. [5], pp.111-112.

Let $M$ be a hyperbolic punctured torus and let $x_0$ and $y_0$ be disjoint infinite geodesic arcs on $M$ such that $M$ cut along $x_0 \cup y_0$ is an open two-dimensional disk $D$. There exist closed geodesics $x$ and $y$ in $M$ such that $x \cap x_0 =$ point, $x \cap y_0 = \emptyset$, $y \cap x_0 = \emptyset$, and $y \cap y_0 =$ point, which generate the fundamental group of $M$. Note that the fundamental group of $M$ is a free group of rank two, $\pi_1(M) = \langle x, y \rangle$.

The universal cover of $M$ is the hyperbolic plane $\mathbb{H}^2$, so $M = \pi_1(M) \setminus \mathbb{H}^2$. Let $\tilde{D}$ be a lift of the disc $D$ to $\mathbb{H}^2$. Note that $\mathbb{H}^2$ is tiled by the translates of the closure of $\tilde{D}$ by $\pi_1(M)$. Let $T$ be the graph in $\mathbb{H}^2$ dual to this tiling, i.e. the vertices of $T$ are located one in each translate of $\tilde{D}$, and each edge of $T$ connects two vertices of $T$ in adjacent copies of $\tilde{D}$, so each edge intersects one lift of either $x_0$ or $y_0$ once. As $\mathbb{H}^2$ is simply connected, $T$ is a tree. Note that $T$ is the Cayley graph of $\pi_1(M) = \langle x, y \rangle$.

Define the distance $d_T(v, u)$ between two vertices of $T$ to be the number of edges in a shortest path in $T$ connecting $v$ and $u$.

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Any element \( f \) of \( \pi_1(M) \) acts on \( T \) leaving invariant a unique line, called the axis of \( f \), which contains all vertices \( v \) with minimum \( d_T(v, f(v)) \). That minimum is called the translation length of \( f \), and is equal to the length of the word \( W \) in \( \pi_1(M) = \langle x, y \rangle \) obtained from \( f \) by reduction and cyclic reduction. Denote the length of the word \( W \) in \( \pi_1(M) = \langle x, y \rangle \) by \( L(W) \).

We prove the following result in Section 2.

**Theorem 1.** Let \( M \) be a punctured hyperbolic torus and let \( \gamma \) be a closed geodesic in \( M \). Any edge of any triangle formed by distinct geodesic lines in the preimage of \( \gamma \) in \( \mathbb{H}^2 \) is shorter than \( \gamma \).

However, a similar result does not hold in the tree \( T \). In Section 3 we construct several examples of triangles in \( T \) formed by distinct axes in \( T \) stabilized by conjugates of \( W \), such that an edge in those triangles is longer than \( L(W) \).

In Section 4 we determine the general form of a reduced and cyclically reduced word \( W \) in \( \pi_1(M) = \langle x, y \rangle \) which overlaps two of its conjugates in such a way that the overlaps cover all of \( W \), proving the following result.

**Theorem 2.** Let \( W \) be a reduced and cyclically reduced word in \( \pi_1(M) = \langle x, y \rangle \) which overlaps two of its conjugates in such a way that the overlaps cover all of \( W \) and the overlaps do not intersect. Then there exists a decomposition \( W = BC^kI, k > 0 \), with \( B \) a terminal subword of \( C \) and \( I \) an initial subword of \( C \).

## 2. Triangles in \( \mathbb{H}^2 \)

We use the notation from the previous section.

**Lemma 1.** Let \( f \) be an element in \( \pi_1(M) = \langle x, y \rangle \) and let \( W \) be its reduced and cyclically reduced conjugate. Consider two axes in the tree \( T \) stabilized by \( f \) and its conjugate \( f' \in \pi_1(M) \). If such axes intersect in an interval labeled with a word \( W_0 \) such that \( L(W_0) = L(W) - 1 \) then they coincide.

**Proof.** WLOG \( W_0 \) is an initial subword of \( W \), hence WLOG there exists a decomposition \( W = W_0x \), where \( x \) is a generator of \( \pi_1(M) = \langle x, y \rangle \). Let \( W' \) be a reduced and cyclically reduced conjugate of \( f' \) containing \( W_0 \). Then the abelianization of \( W \) implies that either \( W' = xW_0 \) or \( W' = W_0x = W \). In both cases the axes coincide. \( \square \)

**Proof of Theorem 1**

Assume to the contrary that there exists a triangle in \( \mathbb{H}^2 \) formed by geodesic lines \( l, m, \) and \( n \), which are distinct lifts of the geodesic \( \gamma \), such that the length of the side lying in \( l \) is longer than \( \gamma \). Note that \( l \) is stabilized by some element \( f \) in \( \pi_1(F) \) which acts as a hyperbolic isometry of \( \mathbb{H}^2 \). Let \( P \) be the intersection of \( l \) and \( n \), and let \( X \) be the intersection of \( l \) and \( m \). The length of \( \gamma \) is equal to the length of the segment \( Pf(P) \) which is equal to the length of the segment \( f(P)f^2(P) \).

Consider two cases.

**Case 1.** The side \( PX \) of the triangle formed by lines \( l, m, \) and \( n \) is shorter than the segment \( Pf^2(P) \).

See Figure 1.
By assumption, the side $PX$ is longer than $\gamma$, so the segment $Xf^2(P)$ is shorter than the segment $PX$. Consider the geodesics $f(n)$ and $f^2(n)$. As $f$ is an isometry, the geodesics $n, f(n)$, and $f^2(n)$ make the same angle with $l$. Then as $Xf^2(P)$ is shorter than $PX$, the angle between $n$ and $l$ is equal to the angle between $f^2(n)$ and $l$, and the opposite angles between $m$ and $l$ are equal, it follows that $m$ and $f^2(n)$ intersect, as shown in Figure 1.

Let $T$ be the tree in $\mathbb{H}^2$ defined above and let $W$ be a reduced and cyclically reduced word conjugate to $f$ in $\pi_1(F)$. The geodesic lines $l, m, n$ are transversal to the lifts of the geodesics $x_0$ and $y_0$ in $\mathbb{H}^2$. Consider the intersections of the lifts of the geodesics $x_0$ and $y_0$ with lines $l, m, n$. Choose a projection $s: \mathbb{H}^2 \to T$ which respects the action of $\pi_1(F)$ on $\mathbb{H}^2$. It can be arranged that the restriction of $s$ to each component of the lift of $\gamma$ in $\mathbb{H}^2$ is monotone, so $s$ maps each component of the lift of $\gamma$ onto a geodesic in $T$.

Let $b$ lifts of $x_0$ and $y_0$ intersect both $l$ and $n$ to the left of the point $P$ and let $a$ lifts of $x_0$ and $y_0$ intersect both $l$ and $n$ to the right of the point $P$. Then there are $a + b$ lifts of $x_0$ and $y_0$ crossing $l$ and $n$, hence the length of the intersection $s(l) \cap s(n)$ is $a + b$. Lemma 1 implies that $a + b < L(W) - 1$. By a similar argument, the number $c$ of the lifts of $x_0$ and $y_0$ intersecting both $l$ and $m$ is also less than $L(W) - 1$. As $f$ is an isometry, there are $b$ lifts of $x_0$ and $y_0$ crossing $l$ and $f^2(n)$ to the left of $f^2(P)$. Then the total number of the lifts of $x_0$ and $y_0$ crossing $l$ between the points $P$ and $f^2(P)$ is $a + b + c$, which is strictly less than $2L(W)$. However by construction, the number of the lifts of $x_0$ and $y_0$ crossing $l$ between the points $P$ and $f^2(P)$ should be equal to $2L(W)$. This contradiction completes the proof of Theorem 1 in Case 1.
Case 2. The side $PX$ of the triangle formed by lines $l, m,$ and $n$ is longer or equal than the segment $Pf^2(P)$.
See Figure 2.

Let $a$ lifts of $x_0$ and $y_0$ intersect both $l$ and $n$ to the right of the point $P$. Then the length of the intersection $s(l) \cap s(n)$ is not shorter than $a$, hence Lemma 1 implies that $a < L(W) - 1$. Let $c$ be the number of the lifts of $x_0$ and $y_0$ intersecting both $l$ and $m$ to the left of the point $f^2(P)$. Then the length of the intersection $s(l) \cap s(m)$ is not shorter than $c$, hence Lemma 1 implies that $c < L(W) - 1$. Therefore the total number of the lifts of $x_0$ and $y_0$ crossing $l$ between the points $P$ and $f^2(P)$ is $a + c$, which is strictly less than $2L(W)$. However by construction, the number of the lifts of $x_0$ and $y_0$ crossing $l$ between the points $P$ and $f^2(P)$ should be equal to $2L(W)$.
This contradiction completes the proof of Theorem 1 in Case 2. $\Box$

The author would like to thank Max Neumann-Coto for sharing his ideas about Theorem 1.

3. Triangles in the Tree $T$

Consider again the tree $T$ defined above. As was mentioned already, $T$ can be considered to be the Cayley graph of the free group $\pi_1(M) = \langle x, y \rangle$. Let $W$ be a reduced and cyclically reduced word in $\{x, y, x^{-1}, y^{-1}\}$. Consider three distinct axes in $T$ stabilized by the word $W$ and two of its conjugates $f_1$ and $f_2$. Call the axes $\lambda, \lambda_1,$ and $\lambda_2$. Let $\tilde{W}$ denote the bi-infinite product of the word $W$. Note that all the axes $\lambda, \lambda_1,$ and $\lambda_2$ are labeled by the bi-infinite word $\tilde{W}$. 
Choose a copy of the word $W$ in $\lambda$. We will work with that chosen copy. Assume that the axes intersect in such a way that $\lambda_1 \cap \lambda$ and $\lambda_2 \cap \lambda$ cover the word $W$ in $\lambda$. Note that the intervals $\lambda_1 \cap \lambda, \lambda_2 \cap \lambda,$ and $\lambda_1 \cap \lambda_2$ form a tripod in the tree $T$ which is a degenerate triangle. Denote the label of the interval $\lambda \cap \lambda_1$ by $U$ and the label of the interval $\lambda \cap \lambda_2$ by $V$. The four examples below show that, in contrast with Theorem 1, $W, \lambda_1,$ and $\lambda_2$ can be chosen in such a way that $L(U \cup V) \geq L(W)$.

Note that Lemma 1 implies that $L(U) \leq L(W) - 2$ and $L(V) \leq L(W) - 2$, so $L(U \cup V) \leq 2L(W) - 4$.

Let $\mu_i, i = 1, 2$ be a subinterval of $\lambda_i$ containing $\lambda_i \cap \lambda$ such that its label $W_i$ is a reduced and cyclically reduced conjugate of $f$. Then $W_1$ contains $U$ and $W_2$ contains $V$.

**Example 1.** See Figure 3.

Let $W = xyxyx, W_1 = x^{-1}Wx = yxyx^2,$ and $W_2 = xWx^{-1} = x^2yxy$. Then $U = W_1 \cap W = xyx$ and $V = W_2 \cap W = xyx$, so that $U \cap V = x$. Then $(W_1 \cup W_2) \cap W = U \cup V = W$.

![Image of Figure 3](image-url)

**Example 2.** See Figure 4.

Let $W = xy^2x y^2x, W_1 = y^{-1}x^{-1}Wxy = yxy^2x^2y,$ and $W_2 = xWx^{-1} = x^2y^2xy^2$. Then $U = W_1 \cap W = xy^2x$ and $V = W_2 \cap W = xy^2x$, so that $U \cap V = x$. Then $(W_1 \cup W_2) \cap W = U \cup V = W$. 
Example 3. See Figure 5.

Let $W = yxy^2xy^2x$, $W_1 = y^{-1}Wy = xy^2xy^2y$, and $W_2 = y^2xWx^{-1}y^{-2} = y^2xyxy^2x$. Then $U = W_1 \cap W = yxy^2xy$ and $V = W_2 \cap \tilde{W} = yxy$, so that $U \cap V = \text{point}$. Then $(W_1 \cup W_2) \cap \tilde{W} = (U \cup V) \cap \tilde{W} = Wy$. 

Figure 4
Example 4. See Figure 6.
Let $W = yxyxyxyxyxyx$, $W_1 = x^{-1}y^{-1}x^{-1}y^{-1}Wyxxy = yxyxyxyxyxyx$, and let $W_2 = xWx^{-1} = xyxyxyxyxyxyxy$. Then $U = W_1 \cap W = yxyxyxyxyxyx$ and $V = W_2 \cap \tilde{W} = yxyxy$, so $U \cap V = \text{point}$.
Hence $(W_1 \cup W_2) \cap \tilde{W} = (U \cup V) \cap \tilde{W} = Wyxxy$.
4. CONJUGATE WORDS IN A FREE GROUP

Let $W$ and $\tilde{W}$ be as in the previous section. Note that $\tilde{W}$ has a Z-shift.

Lemma 2. Assume that there exists an initial subword $U$ of $W$ such that $\tilde{W}$ contains a nonequivalent (i.e., not obtained by the Z-shift) copy of $U$. Call it $U_2$. If $U$ and $U_2$ overlap in such a way that the beginning of $U_2$ lies in $U$ and $L(U \cap U_2) > 0$, then there exist decompositions $U = BC^k$ and $U \cup U_2 = BC^{k+1}$ with $k > 0$ such that $B$ is a terminal subword of $C$. If $U \cup U_2$ contains $W$, then there exists a decomposition $W = BC^kI$, $k > 0$, where $B$ is a terminal subword of $C$ and $I$ is an initial subword of $C$. If $U \cup U_2 = W$, then there exists a decomposition $W = BC^k$, $k > 1$, where $B$ is a terminal subword of $C$. 

Figure 6
Proof. Let $P$ be the overlap of $U$ and $U_2$. Then there exists a decomposition $U_2 = PC$, and $L(U_2) = k \cdot L(C) + n$ with $k > 0$. As $U = U_2$, it follows that $U = BC^k$, where $B$ is a terminal subword of $C$, and $L(B) = n$, see Figure 7. Then $U \cup U_2 = BC^{k+1}$. If $U \cup U_2$ contains $W$, then $W$ is a proper initial subword of $BC^{k+1}$ containing $U$. Hence $W = BC^k I, k > 0$, where $B$ is a terminal subword of $C$ and $I$ is an initial subword of $C$. If $U \cup U_2 = W$, then $I$ is trivial and there exists a decomposition $W = BC^k, k > 1$, where $B$ is a terminal subword of $C$. \hfill $\square$

Remark 1. Note that $U \cup U_2$ might be a proper subword of $W$. In that case we do not have much information about $W$.

Proof of Theorem 2

Let $W_1, W_2, U$ and $V$ be as in the previous section. Assume that $U \cap V =$ point and $U \cup V = W$, hence $U$ is a proper initial subword of $W$ and $V$ is a proper terminal subword of $W$. Assume WLOG that $L(U) > L(V)$, (the case $L(U) = L(V)$ is considered separately at the end of the section). Then $L(U) > \frac{1}{2}L(W)$.

As the axes are generated by conjugate elements, there exist non-equivalent (i.e. not obtained by the $Z$-shift on $\tilde{W}$) copies of the words $U$ and $V$ in $\tilde{W}$. As $L(U) > \frac{1}{2}L(W)$ there exists a non-equivalent copy of $U$ in $\tilde{W}$ whose beginning is contained in $U$. Call that copy $U_2$. Also there exists a non-equivalent copy of $V$ in $\tilde{W}$ whose beginning is contained in $W$. Call that copy $V_2$. 

Figure 7
As $U$ and $U_2$ satisfy the conditions of Lemma 2, there exist decompositions $U = BC^k$ and $U \cup U_2 = BC^{k+1}$ with $k > 0$ such that $B$ is a terminal subword of $C$. If $W = U \cup V \subseteq U \cup U_2$, then Theorem 2 follows from Lemma 2. Hence we need to rule out the case $U \cup U_2 \subset U \cup V$.

Assume that $U \cup U_2 \subset U \cup V$. It follows that $V$ and, hence $V_2$, begin with $C$. Note that either $V_2 \cap V \neq \emptyset$ or $V_2 \subset U$.

Consider 4 cases.

**Case 1.**
$V_2 \subset U$ and the beginning $C$ of $V_2$ is "standard" in $U$, i.e. it is one of the $k$ copies of $C$ defined by the decomposition $U = BC^k$. See Figure 8.

![Figure 8](image)

It follows that $V_2 = V = C^lD$, where $l > 0$ and $D$ is an initial subword of $C$. So the word $U_2$ in $\tilde{W}$ is followed by the word $C^{l-1}D$. Note that the word $U_2$ in $\tilde{W}$ corresponds to the word $U$ in $W_1$, so in the word $W_1$ the word $U$ is followed by a copy of the word $C^{l-1}D$, call it $V'$. However, the word $U$ in $W$ is followed by a copy of the word $C$. If the word $V'$ is non-trivial, it should have non-trivial intersection with that copy of the word $C$ in $W$, so the intersection of $W_1$ with $W$ should be longer than $U$. This contradiction implies that $l = 1$ and $D$ is trivial, hence $V_2 = V = C$. Note that the word $V$ in $\tilde{W}$ corresponds to the word $V_2$ in $W_2$. As the word $V_2$ in $W$ is preceded by the word $B$, the word $V$ in $W_2$ is preceded by...
a copy of the word $B$, hence the intersection of $W_2$ and $\tilde{W}$ should be longer than $V$. This contradiction shows that Case 1 cannot happen.

**Case 2.**

$V_2 \subset U$ and the beginning $C$ of $V_2$ is "non-standard" in $U$. See Figure 9.

![Figure 9](image)

Note that there exist decompositions $C = C_1C_2 = C_2C_1$, see Figure 9 and Figure 10.
As $C_1$ and $C_2$ commute, they are powers of some $C_0$.\footnote{p.10} It follows that $C = C_0^m, m > 1$ and $U = BC^k = BC_0^{km}$. As $B$ is a terminal subword of $C$, it follows that $B = B_0 C_0^l, l \geq 0$, where $B_0$ is a terminal subword of $C_0$ which might be empty. So $U = BC^k = B_0 C_0^{m+k+1}$ and $V_2 = V = C_0^n D_0$, where $l > 0, n > 0$ and $D_0$ is an initial subword of $C_0$. Note that $C_0$ is standard in both $U$ and $V$. Hence we can apply the same argument as in Case 1, demonstrating that Case 2 cannot happen.

**Case 3.**
$V \cap V_2 \neq \emptyset$ and the initial $C$ of $V_2$ is in $U$.
If the beginning $C$ of $V_2$ is "standard" in $U$, we can use the same argument as in Case 1, to obtain a contradiction.
If the beginning $C$ of $V_2$ is "non-standard" in $U$, we can use the same argument as in Case 2, to obtain a contradiction.
Therefore Case 3 is impossible.

**Case 4.**
$V \cap V_2 \neq \emptyset$ and the initial $C$ of $V_2$ intersects $V$.
We can use the same argument as in Case 2 to obtain a contradiction, so Case 4 is also impossible.

Therefore $U \cup V \subset U \cup U_2$, proving Theorem 2 when $L(U) > L(V)$. 

**Figure 10**

As $C_1$ and $C_2$ commute, they are powers of some $C_0$. It follows that $C = C_0^m, m > 1$ and $U = BC^k = BC_0^{km}$. As $B$ is a terminal subword of $C$, it follows that $B = B_0 C_0^l, l \geq 0$, where $B_0$ is a terminal subword of $C_0$ which might be empty. So $U = BC^k = B_0 C_0^{m+k+1}$ and $V_2 = V = C_0^n D_0$, where $l > 0, n > 0$ and $D_0$ is an initial subword of $C_0$. Note that $C_0$ is standard in both $U$ and $V$. Hence we can apply the same argument as in Case 1, demonstrating that Case 2 cannot happen.
Now consider the case when \( L(U) = L(V) = \frac{1}{2}L(W) \).
If \( U_2 = V \), then \( W = U^2 \) and the axes \( \lambda, \lambda_1, \text{and} \lambda_2 \) (defined in the previous section) coincide, contradicting their choice to be distinct.
If \( U_2 \neq V \) then \( U \) and \( U_2 \) have a non-trivial intersection. If the beginning of \( U_2 \) is contained in \( U \) then Lemma 2 implies that there exists a decomposition \( U \cup U_2 = BC^{k+1} \), where \( B, C, \text{and} k \) are defined above. If the beginning of \( U \) is contained in \( U_2 \) we can reduce this case to the previous one by considering the words \( W_0 = U^{-1}V^{-1}, U^{-1}, \text{and} V^{-1} \) instead of \( W, U, \text{and} V \).
So \( U \cup U_2 \subset W \subseteq U \cup V \). It follows again that \( V \) and, hence \( V_2 \), begin with \( C \).
Consider the word \( V_2 \). If \( V_2 = U \), then \( W = V^2 \) and the axes \( \lambda, \lambda_1, \text{and} \lambda_2 \) (defined in the previous section) coincide, contradicting their choice to be distinct.
Otherwise, \( V \cap V_2 \neq \emptyset \). Consider two cases.

**Case 5.**
\( V \cap V_2 \neq \emptyset \) and the initial \( C \) of \( V_2 \) is in \( U \).
If the beginning \( C \) of \( V_2 \) is "standard" in \( U \) then, as in Case 1(above), \( V = V_2 = C \).
As \( L(V) = L(U) \), it follows that \( U = C \), hence \( W = C^2 \) and the axes \( \lambda, \lambda_1, \text{and} \lambda_2 \) (defined in the previous section) coincide, contradicting their choice to be distinct.
If the beginning \( C \) of \( V_2 \) is "non-standard" in \( U \) then, as in Case 2(above), \( C = C^m \). Hence as in Case 2(above), it follows that \( V = V_2 = C^m = C \). We got a contradiction with the assumptions that \( V_2 \subset U \) and \( V \cap V_2 \neq \emptyset \). Therefore this case is impossible.

**Case 6.**
\( V \cap V_2 \neq \emptyset \) and the initial \( C \) of \( V_2 \) intersects \( V \).
As in Case 2(above) it follows that \( V = V_2 = C \), hence as \( L(V) = L(U) \) it follows that \( W = U^2 = C^2 \) and the axes \( \lambda, \lambda_1, \text{and} \lambda_2 \) (defined in the previous section) coincide, contradicting their choice to be distinct.
Therefore in the special case when \( L(U) = L(V) = \frac{1}{2}L(W) \) the axes \( \lambda, \lambda_1, \text{and} \lambda_2 \) (defined in the previous section) coincide, contradicting their choice to be distinct.
That contradiction completes the proof of Theorem 2.

**Remark 2.** Note that there exists a decomposition \( C = IT \), where \( T \) is a terminal subword of \( C \). If \( T = B \), then \( W \) is a conjugate of \( C^{k+1} \), so the axes \( \lambda, \lambda_1, \text{and} \lambda_2 \) (defined in the previous section) coincide, contradicting their choice to be distinct.
What can be said about \( W \) if \( T \neq B \)?

The following conjecture was formulated by Max Neumann-Coto.

**Conjecture** Assume that \( W \) overlaps two of its conjugates in such a way that the overlaps cover all of \( W \) and the overlaps do not intersect. Then \( W = DC^k \), where \( C \) is non-trivial and \( k > 1 \) and the conjugates have the form \( C^rDC^{k-r} \) and \( C^sDC^{k-s} \).

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