Optimal Bounds for Convergence of Expected Spectral Distributions to the Semi-Circular Law

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Abstract

Let $X = (X_{jk})_{j,k=1}^n$ denote a Hermitian random matrix with entries $X_{jk}$, which are independent for $1 \leq j \leq k \leq n$. We consider the rate of convergence of the empirical spectral distribution function of the matrix $X$ to the semi-circular law assuming that $\mathbb{E}X_{jk} = 0$, $\mathbb{E}X^2_{jk} = 1$ and that

$$\sup_{n \geq 1} \sup_{1 \leq j,k \leq n} \mathbb{E}|X_{jk}|^4 =: \mu_4 < \infty,$$

and

$$\sup_{1 \leq j,k \leq n} |X_{jk}| \leq D_0 n^{\frac{1}{4}}.$$

By means of a recursion argument it is shown that the Kolmogorov distance between the expected spectral distribution of the Wigner matrix $W = \frac{1}{\sqrt{n}}X$ and the semicircular law is of order $O(n^{-\frac{1}{2}})$.

1 Introduction

Consider a family $X = \{X_{jk}\}$, $1 \leq j \leq k \leq n$, of independent real random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $n \geq 1$. Assume that $X_{jk} = X_{kj}$, for $1 \leq k < j \leq n$, and introduce the symmetric

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1Research supported by SFB 701 “Spectral Structures and Topological Methods in Mathematics” University of Bielefeld. Research supported by grants RFBR N 14-01-00500 and by Program of Fundamental Research Ural Division of RAS, Project N 12-P-1-1013
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matrices

\[ W = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}. \]

The matrix \( W \) has a random spectrum \( \{\lambda_1, \ldots, \lambda_n\} \) and an associated spectral distribution function \( F_n(x) = \frac{1}{n} \text{card} \{j \leq n : \lambda_j \leq x\}, \ x \in \mathbb{R} \). Averaging over the random values \( X_{ij}(\omega) \), define the expected (non-random) empirical distribution functions \( F_n(x) = \mathbb{E} F_n(x) \). Let \( G(x) \) denote the semi-circular distribution function with density \( g(x) = G'(x) = \frac{1}{2\pi} \sqrt{4 - x^2} I_{[-2,2]}(x) \), where \( I_{[a,b]}(x) \) denotes the indicator function of the interval \( [a,b] \). The rate of convergence to the semi-circular law has been studied by several authors. We proved in [13] that the Kolmogorov distance between \( F_n(x) \) and the distribution function \( G(x) \), \( \Delta^*_n := \sup_x |F_n(x) - G(x)| \) is of order \( O_P(n^{-\frac{1}{2}}) \) (i.e. \( n^\frac{1}{2} \Delta^*_n \) is bounded in probability). Bai [11] and Girko [8] showed that \( \Delta_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac{1}{2}}) \). Bobkov, Götze and Tikhomirov [4] proved that \( \Lambda_n \) and \( \mathbb{E} \Lambda^*_n \) have order \( O(n^{-\frac{2}{3}}) \) assuming a Poincaré inequality for the distribution of the matrix elements. For the Gaussian Unitary Ensemble respectively for the Gaussian Orthogonal Ensemble, see [12] respectively [21], it has been shown that \( \Delta_n = O(n^{-1}) \). Denote by \( \gamma_{n1} \leq \ldots \leq \gamma_{nn} \), the quantiles of \( G \), i.e. \( G(\gamma_{nj}) = \frac{j}{n} \), and introduce the notation \( \log n := \log \log n \). Erdös, Yau and Yin [6] showed, for matrices with elements \( X_{jk} \) which have a uniformly sub exponential decay, i.e.

\[
\Pr\{|X_{jk}| > t\} \leq A \exp\{-t^\kappa\},
\]

for some \( \kappa > 0 \), \( A > 0 \) and for any \( t \geq 1 \), the following result

\[
\Pr\left\{ \exists j : |\lambda_j - \gamma_{nj}| \geq (\log n)^{C \log n} \left[ \min(j, N - j + 1) \right]^{-\frac{1}{3}} n^{-\frac{2}{3}} \right\} \leq C \exp\{-C (\log n)^{C \log n} \},
\]

for \( n \) large enough. It is straightforward to check that this bound implies that

\[
\Pr\left\{ \sup_x |F_n(x) - G(x)| \leq C n^{-1} (\log n)^{C \log n} \right\} \geq 1 - C \exp\{-C (\log n)^{C \log n} \}.
\]

From the last inequality it is follows that \( \mathbb{E} \Lambda^*_n \leq C n^{-1} (\log n)^{C \log n} \). Similar results were obtained in [20], Theorem 32, assuming additionally that the distributions of the entries of matrices have vanishing third moment.
In this paper we derive the optimal bound for the rate of convergence of the expected spectral distribution to the semi-circular law. Using arguments similar to those used in [17] we provide a self-contained proof based on recursion methods developed in the papers of Gärtzel and Tikhomirov [13], [9] and [22]. It follows from the results of Gustavsson [14] that the best possible bound in the Gaussian case for the rate of convergence in probability is $O(n^{-1} \sqrt{\log n})$. The best possible bound for $\Delta_n$ is of order $O(n^{-1})$. For Gaussian matrices such bounds were obtained in [12] and [21]. Our setup includes the case that the distributions of $X_{jk} = X_{jk}^{(n)}$ may depend on $n$. In the following we shall investigate the rate of convergence of expected spectral distribution function $F_n(x) = \mathbb{E}F_n(x)$ to the semi-circular distribution function by estimating the quantity $\Delta_n$. The main result of this paper is the following

**Theorem 1.1.** Let $\mathbb{E}X_{jk} = 0$, $\mathbb{E}X_{jk}^2 = 1$. Assume that

$$\sup_{n \geq 1} \sup_{1 \leq j,k \leq n} \mathbb{E}|X_{jk}|^4 =: \mu_4 < \infty. \quad (1.4)$$

Assume as well that there exists a constant $D_0$ such that for all $n \geq 1$

$$\sup_{1 \leq j,k \leq n} |X_{jk}| \leq D_0 n^{\frac{4}{7}}. \quad (1.5)$$

Then, there exists a positive constant $C = C(D_0, \mu_4)$ depending on $D_0$ and $\mu_4$ only such that

$$\Delta_n = \sup_x |F_n(x) - G(x)| \leq C n^{-1}. \quad (1.6)$$

**Corollary 1.1.** Let $\mathbb{E}X_{jk} = 0$, $\mathbb{E}X_{jk}^2 = 1$. Assume that

$$\sup_{n \geq 1} \sup_{1 \leq j,k \leq n} \mathbb{E}|X_{jk}|^8 =: \mu_8 < \infty. \quad (1.7)$$

Then, there exists a positive constant $C = C(\mu_8)$ depending on $\mu_8$ only such that

$$\Delta_n \leq C n^{-1}. \quad (1.8)$$

**Remark 1.2.** Note that the bound (1.6) in Theorem 1.1 and the bound (1.8) in the Corollary 1.1 are not improvable and coincide with the corresponding bounds in the Gaussian case.
We state here as well the results for the Stieltjes transform of the expected spectral distribution of the matrix $W$. Let $R$ denote the resolvent matrix of the matrix $W$,

$$R := R(z) = (W - zI)^{-1}. \quad (1.9)$$

Here and in what follows $I$ denotes the unit matrix of corresponding dimension. For any distribution function $F(x)$ we define the Stieltjes transform $s_F(z)$, for $z = u + iv$ with $v > 0$, via formula

$$s_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x). \quad (1.10)$$

Denote by $m_n(z)$ the Stieltjes transform of the distribution function $F_n(x)$. It is a well-known fact that

$$m_n(z) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\lambda_j - z} = \frac{1}{n} \text{Tr} R. \quad (1.11)$$

By $s(z)$ we denote the Stieltjes transform of the semi-circular law,

$$s(z) = \frac{-z + \sqrt{z^2 - 4}}{2}.$$ 

The Stieltjes transform of the semi-circular distribution satisfies the equation

$$s^2(z) + zs(z) + 1 = 0 \quad (1.12)$$

(see, for example, equality (4.20) in [13]).

Introduce for $z = u + iv$ and a positive constant $A_0 > 0$

$$v_0 := A_0 n^{-1}, \quad \text{and} \quad \gamma := \gamma(z) := |2 - |u||. \quad (1.13)$$

For any $0 < \varepsilon < \frac{1}{2}$, and $A_0 > 0$, define a region $G = G(A_0, n \varepsilon) \subset \mathbb{C}_+$, by

$$G := \{ z = u + iv \in \mathbb{C}_+: -2 + \varepsilon \leq u \leq 2 - \varepsilon, \, v \geq v_0/\sqrt{\gamma(z)} \}. \quad (1.14)$$

Let $a > 0$ be a positive number such that

$$\frac{1}{\pi} \int_{|u| \leq a} \frac{1}{u^2 + 1} du = \frac{3}{4}. \quad (1.15)$$

We prove the following result.
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**Theorem 1.3.** Let $\frac{1}{2} > \varepsilon > 0$ be a sequence of positive numbers in (1.14) such that

$$\varepsilon^3 \frac{2}{\nu_0 a} = 2^\nu_0 a. \tag{1.16}$$

Assuming the conditions of Theorem 1.1, there exists a positive constant $C = C(D_0, A_0, \mu_4)$ depending on $D$, $A_0$ and $\mu_4$ only, such that, for $z \in \mathbb{G}$,

$$|\mathbb{E}m_n(z) - s(z)| \leq \frac{C}{n\nu_4^3} + \frac{C}{n^2\nu_4^2 |z^2 - 4|^{1/4}}. \tag{1.17}$$

### 1.1 Sketch of the Proof

1. We start with an estimate of the Kolmogorov-distance to the Wigner distribution via an integral over the difference of the corresponding Stieltjes transforms along a contour in the upper half-plane using a smoothing inequality and Cauchy’s formula developed by the authors in [11]. The resulting bound involves an integral over a segment at a fixed distance, say $V = 4$, from the real axis and a segment $u + iA_0 n^{-1} (2 - |u|)^{-1/2}$, $u \leq x$ at a distance of order $n^{-1}$ but avoiding to come close to the endpoints $\pm 2$ of the support. These segments are part of the boundary of an $n$-dependent region $\mathbb{G}$ where bounds of Stieltjes transforms are needed. Since the Stieltjes-transform and the diagonal elements $R_{jj}(z)$ of the resolvent of the Wigner-matrix $W$ are uniformly bounded on the segment with $\text{Im} z = V$ by $1/V$ (see Section 3.1) proving a bound of order $O(n^{-1})$ for the latter segment near the $x$-axis is the essential problem.

2. In order to investigate this crucial part of the error we start with the 2nd resolvent or self-consistency equation for the expected Stieltjes transform resp. the quantities $R_{jj}(z)$ of $W$ (see (3.3) below) based on the difference of the resolvent of $W^{(j)}$ ($j$th row and column removed) and $W$. For the equivalent representation for the difference of Stieltjes transforms (see (3.6)) we have to show an error bound of order $O((n\nu)^{1/4} |z^2 - 4|^{-1/4})$ for $z \in \mathbb{G}$.

To prove this bound we use a recursive version of this representations as in (5.49). Obviously bounds for $\mathbb{E}|R_{jj}|^p$ for $z = u + iv$ close to the real line are needed for the sufficiently large $p$, $(p = O(\log n))$, which follow once the error terms $\varepsilon_j$ are small in the region $\mathbb{G}$. But proving that $\mathbb{E}|\varepsilon_j|^p$ is small requires in turn again bounds of $\mathbb{E}|R_{jj}|^{2p}$.

An approach suggested recently in [17] turns out to be very fruitful in dealing with this recursion problem. Assuming that $\mathbb{E}|R_{jj}|^{2p} \leq C_0^{2p}$ for some $z = u + iv$, we can show that $\mathbb{E}|R_{jj}|^p \leq C_0^p$ with $z = u + iv/s_0$ with some fixed scale factor $s_0 > 1$. This allows us to prove by induction a bound of type $\mathbb{E}|R_{jj}|^q \leq C_0^q$ for some fixed $q$ (independent of $n$) and $z = u + iv$ with...
$v \geq Cn^{-1}$ starting with $\mathbb{E}|R_{jj}|^p \leq C_0^p$ for $p = s_0$ and $z = u + iv$ for fixed $v = 4$, say. The latter assumption can be easily verified.

Note that one of the errors, that is $\varepsilon_{j2}$, in (3.3) is a quadratic form in independent random variables. Thus, in case that $W$ has entries with exponential or even sub-Gaussian tails, inequalities for quadratic forms of independent random variables, like [11], Lemma 3.8, or [17], Proposition A.1 could be applied.

Assuming eight moments in Corollary 1.1 or four moments and a truncation condition in Theorem 1.1 only, we can’t use these strong tail estimates for quadratic forms anymore. Our solution is an recursive application of Burkholder’s inequality for the $p$th moment resulting in a bound involving moments of order $p/2$ of another quadratic form in independent variables in each step. This is the crucial part of the moment recursion for $R_{jj}$ described above. Details of this procedure are described in Sections 5.1 and 5.2.

3. In Section 6 we prove a bound for the error $\Lambda_n := \mathbb{E}m_n(z) - s(z)$ of the form $(nv)^{-\frac{1}{4}} + (nv)^{-\frac{5}{2}}|z^2 - 4|^{-\frac{1}{4}}$ which suffices to prove the rate $O(n^{-1})$ in Theorem 1.1. Here we use a series of martingale-type decompositions to evaluate the expectation $\mathbb{E}m_n(z)$ combined with the bound $\mathbb{E}|\Lambda_n|^2 \leq C(nv)^{-2}$ of Lemma 7.24 in the Appendix which is again based on a recursive inequality for $\mathbb{E}|\Lambda_n|^2$ in (7.164). A direct application of this bound to estimate the error terms $\varepsilon_{j4}$ would result in a less precise bound of order $O(n^{-1} \log n)$ in Theorem 1.1. Bounds of such type will be shown for the Kolmogorov distance of the random spectral distribution to Wigner’s law in a separate paper. For the expectation we provide sharper bounds in Section 6.2 involving $m'_n(z)$.

4. The necessary auxiliary bounds for all these steps are collected in the Appendix.

2 Bounds for the Kolmogorov Distance of Spectral Distributions via Stieltjes Transforms

To bound the error $\Delta_n$ we shall use an approach developed in previous work of the authors, see [13].

We modify the bound of the Kolmogorov distance between an arbitrary distribution function and the semi-circular distribution function via their Stieltjes transforms obtained in [13] Lemma 2.1. For $x \in [-2, 2]$ define $\gamma(x) := 2 - |x|$. Given $\frac{1}{2} > \varepsilon > 0$ introduce the interval $\mathbb{J}_\varepsilon = \{x \in [-2, 2] : \gamma(x) \geq \varepsilon\}$ and $\mathbb{J}'_\varepsilon = \mathbb{J}_\varepsilon / 2$. For a distribution function $F$ denote by $S_F(z)$ its Stieltjes transform.
Proposition 2.1. Let $v > 0$ and $a > 0$ and $\frac{1}{2} > \varepsilon > 0$ be positive numbers such that
\[
\frac{1}{\pi} \int_{|u| \leq a} \frac{1}{u^2 + 1} du = \frac{3}{4} =: \beta, \tag{2.1}
\]
and
\[2va \leq \varepsilon^2. \tag{2.2}\]

If $G$ denotes the distribution function of the standard semi-circular law, and $F$ is any distribution function, there exist some absolute constants $C_1$ and $C_2$ such that
\[
\Delta(F, G) := \sup_x |F(x) - G(x)|
\leq 2 \sup_{x \in J'_\varepsilon} \left| \int_{-\infty}^x (S_F(u + i \frac{v'}{\sqrt{\gamma}}) - S_G(u + i \frac{v'}{\sqrt{\gamma}})) du \right| + C_1v + C_2\varepsilon^2.
\]

Remark 2.2. For any $x \in J'_\varepsilon$ we have $\gamma = \gamma(x) \geq \varepsilon$ and according to condition (2.2), $\frac{av}{\sqrt{\gamma}} \leq \frac{\varepsilon}{2}$.

For a proof of this Proposition see [11], Proposition 2.1.

Lemma 2.1. Under the conditions of Proposition 2.1, for any $V > v$ and $0 < v' \leq \frac{3\varepsilon^2}{2a}$ and $v' = v/\sqrt{\gamma}, \gamma = 2 - |x|, x \in J'_\varepsilon$ as above, the following inequality holds
\[
\sup_{x \in J'_\varepsilon} \left| \int_{-\infty}^x \left( \text{Im}(S_F(u + iv') - S_G(u + iv')) \right) du \right|
\leq \int_{-\infty}^V |S_F(u + iV) - S_G(u + iV)| du
\+ \sup_{x \in J'_\varepsilon} \left| \int_{x}^{V} (S_F(x + iu) - S_G(x + iu)) du \right|.
\]

Proof. Let $x \in J'_\varepsilon$ be fixed. Let $\gamma = \gamma(x)$. Put $z = u + iv'$. Since $v' = v/\sqrt{\gamma} \leq \frac{\varepsilon}{2a},$ see (2.2), we may assume without loss of generality that $v' \leq 4$ for $x \in J'_\varepsilon$. Since the functions $S_F(z)$ and $S_G(z)$ are analytic in the upper half-plane, it is enough to use Cauchy’s theorem. We can write for $x \in J'_\varepsilon$
\[
\int_{-\infty}^x \text{Im}(S_F(z) - S_G(z)) du = \text{Im} \lim_{L \to \infty} \int_{-L}^L (S_F(u + iv') - S_G(u + iv')) du.
\]
By Cauchy’s integral formula, we have
\[
\int_{-L}^{x} (S_F(z) - S_G(z))du = \int_{-L}^{x} (S_F(u + iV) - S_G(u + iV))du \\
+ \int_{v'}^{V} (S_F(-L + iu) - S_G(-L + iu))du \\
- \int_{v'}^{V} (S_F(x + iu) - S_G(x + iu))du.
\]

Denote by $\xi$ (resp. $\eta$) a random variable with distribution function $F(x)$ (resp. $G(x)$). Then we have
\[
|S_F(-L + iu)| = \left| \mathbb{E} \frac{1}{\xi + L - iu} \right| \leq v'^{-1} \Pr\{|\xi| > L/2\} + \frac{2}{L},
\]
for any $v' \leq u \leq V$. Similarly,
\[
|S_G(-L + iu)| \leq v'^{-1} \Pr\{|\eta| > L/2\} + \frac{2}{L}.
\]
These inequalities imply that
\[
\left| \int_{v'}^{V} (S_F(-L + iu) - S_G(-L + iu))du \right| \to 0 \quad \text{as} \quad L \to \infty,
\]
which completes the proof. \hfill \qed

Combining the results of Proposition 2.1 and Lemma 2.1 we get

**Corollary 2.2.** Under the conditions of Proposition 2.1 the following inequality holds
\[
\Delta(F, G) \leq 2 \int_{-\infty}^{\infty} |S_F(u + iV) - S_G(u + iV)|du + C_1v_0 + C_2 \varepsilon^{\frac{2}{3}} \\
+ 2 \sup_{x \in J'} \int_{v'}^{V} |S_F(x + iu) - S_G(x + iu)|du,
\]
where $v' = \frac{v_0}{\sqrt{\gamma}}$ with $\gamma = 2 - |x|$ and $C_1, C_2 > 0$ denote absolute constants.

### 3 Proof of Theorem 1.1

**Proof.** We shall apply Corollary 2.2 to prove the Theorem 1.1. We choose $V = 4$ and $v_0$ as defined in (1.13) and use the quantity $\varepsilon = (2av_0)^{\frac{3}{2}}$. 
3.1 Estimation of the First Integral in (2.2) for $V = 4$

Denote by $T = \{1, \ldots, n\}$. In the following we shall systematically use for any $n \times n$ matrix $X$ together with its resolvent $R$, its Stieltjes transform $m_n$ etc. the corresponding quantities $X^{(A)}$, $R^{(A)}$ and $m_n^{(A)}$ for the corresponding sub matrix with entries $X_{jk}, j, k \notin A, A \subset T = \{1, \ldots, n\}$. Observe that

$$m_n^{(A)}(z) = \frac{1}{n} \sum_{j \in T_A} \frac{1}{\lambda^{(A)} - z}. \quad (3.1)$$

Let $T_A = T \setminus A$. By $\mathfrak{M}^{(j)}$ we denote the $\sigma$-algebra generated by $X_{lk}$ with $l, k \in T_j$. If $A = \emptyset$ we shall omit the set $A$ as exponent index.

We shall use the representation

$$R_{jj} = \frac{1}{-z + \frac{1}{\sqrt{n}}X_{jj} - \frac{1}{n}\sum_{k,l \in T_j}X_{jk}X_{jl}R^{(j)}_{kl}}, \quad (3.2)$$

(see, for example, equality (4.6) in [13]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j R_{jj}, \quad (3.3)$$

where $\varepsilon := \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$ with

$$\varepsilon_1 := \frac{1}{\sqrt{n}}X_{jj}, \quad \varepsilon_2 := \frac{1}{n}\sum_{k \neq l \in T_j}X_{jk}X_{jl}R^{(j)}_{kl},$$

$$\varepsilon_3 := \frac{1}{n}\sum_{k \in T_j} (X_{jk}^2 - 1)R^{(j)}_{kk}, \quad \varepsilon_4 := \frac{1}{n}(\text{Tr} R^{(j)} - \text{Tr} R). \quad (3.4)$$

Let

$$\Lambda_n := \Lambda_n(z) := m_n(z) - s(z) = \frac{1}{n}\text{Tr} R - s(z). \quad (3.5)$$

Summing equality (3.3) in $j = 1, \ldots, n$ and solving with respect $\Lambda_n$, we get

$$\Lambda_n = m_n(z) - s(z) = \frac{T_n}{z + m_n(z) + s(z)}, \quad (3.6)$$

where

$$T_n = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j R_{jj}. \quad (3.7)$$
Obvious bounds like $|z+s(z)| \geq 1$, $|\lambda_j-z|^{-1} \leq v^{-1}$, max$\{|R^{(3)}_{jj}(z)|, |m_n^{(3)}(z)|\} \leq v^{-1}$, imply that for $V = 4$ and for any $J \subset T$,

$$|m_n^{(3)}(z)| \leq \frac{1}{4} \leq \frac{1}{2}|z+s(z)|,$$

$$|s(z)-m_n^{(3)}(z)| \leq \frac{1}{2} \leq \frac{1}{2}|z+s(z)|, \text{ a.s.}$$

(3.8)

and therefore,

$$|z+m_n^{(3)}(z)+s(z)| \geq \frac{1}{2}|z+s(z)|, \quad |z+m_n^{(3)}(z)| \geq \frac{1}{2}|s(z)+z|. \quad (3.9)$$

Using equality (3.6), we may write

$$E\Lambda_n = \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_j R_{jj}}{z+m_n(z)+s(z)}$$

$$= \frac{4}{n} \sum_{\nu=1}^{4} \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}s(z)}{z+m_n(z)+s(z)} + \sum_{\nu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}(R_{jj}-s(z))}{z+m_n(z)+s(z)}.$$  

(3.10)

We use that $E\{\varepsilon_{j\nu}\Re^{(j)}\} = 0$, for $\nu = 1, 2, 3$ and obtain

$$\frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}s(z)}{z+m_n(z)+s(z)} = -E \frac{\varepsilon_{j\nu}^4s(z)}{(z+m_n^{(j)}(z)+s(z))(z+m_n(z)+s(z))}.$$  

(3.11)

Thus, according to inequalities (3.9), Lemmas 7.8, 7.9, 7.10, 7.12 in the Appendix and equation (1.12), we obtain

$$|\frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}s(z)}{z+m_n(z)+s(z)}| \leq 4|s(z)|^3 \frac{1}{n} \sum_{j=1}^{n} E|\varepsilon_{j4}\varepsilon_{j\nu}| \leq \frac{C|s(z)|^2}{n^2}, \quad (3.12)$$

where $C$ depends on $\mu_4$ only. For $\nu = 4$, Lemma 7.12 in the Appendix, inequality (3.9) and relation (1.12) yield

$$\frac{1}{n} \sum_{j=1}^{n} \frac{|s(z)|\varepsilon_{j4}}{|z+m_n(z)+s(z)|} \leq \frac{C}{n}|s(z)|^2 \quad (3.13)$$
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with some absolute constant $C$. Furthermore, applying the Cauchy – Schwartz inequality and inequality (3.9) and relation (1.12), we get

$$\left| \sum_{\nu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}(R_{jj} - s(z))}{z + m_n(z) + s(z)} \right| \leq C |s(z)| \sum_{\nu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} E^{\frac{1}{2}} |\varepsilon_{j\nu}|^2 E^{\frac{1}{2}} |R_{jj} - s(z)|^2.$$  

(3.14)

We may rewrite the representation (3.3) using $\Lambda_n = m_n(z) - s(z)$ and (1.12) as (compare (3.12))

$$R_{jj} = s(z) - s(z)\varepsilon_j R_{jj} - s(z)\Lambda_n R_{jj}. \quad (3.15)$$

Applying representations (3.6) and (3.15) together with (3.9) and $|R_{jj}| \leq \frac{1}{4}$, we obtain

$$E |R_{jj}(z) - s(z)|^2 \leq C |s(z)|^2 \sum_{l=1}^{n} E |\varepsilon_l|^2. \quad (3.16)$$

Combining inequalities (3.14) and (3.16), we get

$$\left| \sum_{\nu=1}^{4} \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}(R_{jj} - s(z))}{z + m_n(z) + s(z)} \right| \leq C |s(z)|^2 \sum_{\nu=1}^{4} \frac{1}{n} \sum_{j=1}^{n} E |\varepsilon_{j\nu}|^2. \quad (3.17)$$

Applying now Lemmas 7.8, 7.9, 7.10 and 7.12 we get

$$\left| \sum_{\nu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu}(R_{jj} - s(z))}{z + m_n(z) + s(z)} \right| \leq C |s(z)|^2 \frac{1}{n}. \quad (3.18)$$

Inequality (3.13) and (3.18) together imply

$$|E\Lambda_n| \leq C \frac{|s(z)|^2}{n}. \quad (3.19)$$

Consider now the integral

$$Int(V) = \int_{-\infty}^{\infty} |Em_n(u + iV) - s(u + iV)| du \quad (3.20)$$

for $V = 4$. Using inequality (3.19), we have

$$|Int(V)| \leq C \frac{1}{n} \int_{-\infty}^{\infty} |s(u + Vi)|^2 du. \quad (3.21)$$
Finally, we note that
\[ \int_{-\infty}^{\infty} |s(z)|^2 dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(x-u)^2 + V^2} du dF_n(x) \leq \frac{\pi}{V}. \] (3.22)

Therefore,
\[ \int_{-\infty}^{\infty} |E_{m_n}(u + iV) - s(u + iV)| du \leq \frac{C}{n}. \] (3.23)

### 3.2 Estimation of the Second Integral in (2.3)

To finish the proof of Theorem 1.1 we need to bound the second integral in (2.3) for \( z \in G \) and \( v_0 = A_0 n^{-1} \), where \( \varepsilon = (2av_0)^{\frac{3}{4}} \) is defined in such a way that condition (2.2) holds. We shall use the results of Theorem 1.3. According to these results we have, for \( z \in G \),
\[ |E_{m_n}(z) - s(z)| \leq \frac{C}{nv_0^{\frac{3}{4}}} + \frac{C}{n^{\frac{3}{2}}v_0^{\frac{3}{4}}|z^2 - 4|^{\frac{1}{4}}} . \] (3.24)

We have
\[ \int_{v_0/\sqrt{\gamma}}^{V} |E(m_n(x + iv) - s(x + iv))| dv \leq \frac{1}{n} \int_{v_0/\sqrt{\gamma}}^{V} \frac{dv}{v^{\frac{3}{4}}} + \frac{1}{n\sqrt{n\gamma^{\frac{3}{4}}}} \int_{v_0/\sqrt{\gamma}}^{V} \frac{dv}{v^{\frac{3}{4}}} . \] (3.25)

After integrating we get
\[ \int_{v_0/\sqrt{\gamma}}^{V} |E(m_n(x + iv) - s(x + iv))| dv \leq \frac{C}{n} + \frac{C\gamma^{\frac{3}{4}}}{n^{\frac{1}{2}}v_0^{\frac{3}{4}}} \leq \frac{C}{n} . \] (3.26)

Inequalities (3.23) and (3.26) complete the proof of Theorem 1.1. Thus Theorem 1.1 is proved. \( \square \)

### 4 The proof of Corollary 1.1

To prove the Corollary 1.1 we consider truncated random variables \( \hat{X}_{jl} \) defined by
\[ \hat{X}_{jl} := X_{jl} 1_{\{|X_{jl}| \leq cn^{\frac{3}{4}}\}} . \] (4.1)

Let \( \hat{F}_n(x) \) denote the empirical spectral distribution function of the matrix \( \hat{W} = \frac{1}{\sqrt{n}}(\hat{X}_{jl}) \).
Lemma 4.1. Assuming the conditions of Theorem 1.1 there exists a constant $C > 0$ depending on $\mu_8$ only such that

$$
\mathbb{E}\{\sup_x |F_n(z) - \tilde{F}_n(x)|\} \leq \frac{C}{n}.
$$

Proof. We shall use the rank inequality of Bai. See [3], Theorem A.43, p. 503. According this inequality

$$
\mathbb{E}\{\sup_x |F_n(x) - \tilde{F}_n(x)|\} \leq \frac{1}{n} \mathbb{E}\{\text{rank}(X - \tilde{X})\}.
$$

Observing that the rank of a matrix is not larger than numbers of its non-zero entries, we may write

$$
\mathbb{E}\{\sup_x |F_n(x) - \tilde{F}_n(x)|\} \leq \frac{1}{n} \sum_{j,k=1}^n \mathbb{E}\{I\{|X_{jk}| \geq C n^{\frac{1}{4}}\}\} \leq \frac{1}{n^3} \sum_{j,k=1}^n \mathbb{E}|X_{jk}|^8 \leq \frac{C\mu_8}{n}.
$$

Thus, the Lemma is proved.

Remark 4.1. Note that

$$
|\tilde{X}_{jl}| \leq D_1 n^{\frac{1}{4}}, \quad \mathbb{E}\tilde{X}_{jl} = 0 \text{ and } \mathbb{E}\tilde{X}_{jk}^2 = 1,
$$

for some absolute constant $D_1$. That means that the matrix $\tilde{W}$ satisfies the conditions of Theorem 1.3.

Lemma 4.2. There exists some absolute constant $C$ depending on $\mu_8$ such that

$$
\mathbb{E}|\tilde{m}_n(z) - \bar{m}_n(z)| \leq \frac{C}{n^2 \nu^2}.
$$

Proof. Note that

$$
\bar{m}_n(z) = \frac{1}{n} \text{Tr} (\tilde{W} - z I)^{-1} =: \frac{1}{n} \text{Tr} \tilde{R}, \quad \tilde{m}_n(z) = \frac{1}{n} \text{Tr} (\tilde{W} - z I)^{-1} =: \frac{1}{n} \text{Tr} \tilde{R}.
$$
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Therefore,
\[
\tilde{m}_n(z) - \hat{m}_n(z) = \frac{1}{n} \text{Tr}(\tilde{R} - \hat{R}) = \frac{1}{n} \text{Tr}(\tilde{W} - \hat{W})\tilde{R}\hat{R}.
\] (4.7)

Using the simple inequalities \(|\text{Tr} AB| \leq \|A\|_2\|B\|_2\) and \(\|AB\|_2 \leq \|A\|\|B\|_2\), we get
\[
E|\tilde{m}_n(z) - \hat{m}_n(z)| \leq n^{-1} E^\frac{3}{2} \|\tilde{R}\|_2^2 \|\hat{R}\|_2^2 E^\frac{1}{2} \|\tilde{W} - \hat{W}\|_2^2.
\] (4.8)

Furthermore, we note that,
\[
\tilde{W} - \hat{W} = \frac{1}{\sqrt{n}}((1 - \sigma_{jk}) \tilde{X}_{jk}),
\] (4.9)
and
\[
\|\tilde{W} - \hat{W}\|_2 \leq \max_{1 \leq j,k \leq n} \{1 - \sigma_{jk}\} \|\tilde{W}\|_2.
\] (4.10)

Since
\[
0 < 1 - \sigma_{jk} \leq 1 - \sigma_{jk}^2 \leq Cn^{-\frac{3}{2}} \mu_8,
\] (4.11)
therefore
\[
E\|\tilde{W} - \hat{W}\|_2^2 \leq C\mu_8^2 n^{-2}.
\] (4.12)

Applying Lemma 7.6, inequality (7.28), in the Appendix and inequality (4.12), we obtain
\[
E|\tilde{m}_n(z) - \hat{m}_n(z)| \leq Cn^{-\frac{3}{2}} v^{-\frac{3}{2}} \left(\frac{1}{n} \sum_{j=1}^{n} E|\tilde{R}_{jj}|\right)^{\frac{1}{2}}.
\] (4.13)

According Remark 4.1, we may apply Corollary 5.14 in Section 5 with \(q = 1\) to prove the claim. Thus, Lemma 4.2 is proved.

Denote by \(\tilde{m}_n(z)\) the Stieltjes transform of the empirical distribution function of the matrix \(\tilde{W}\) and let \(\hat{m}_n(z)\) denote the Stieltjes transform of the matrix \(\hat{W}\).

**Lemma 4.3.** For some absolute constant \(C > 0\) we have
\[
E|\tilde{m}_n(z) - \hat{m}_n(z)| \leq \frac{C\mu_8}{n^{\frac{3}{2}} v^2}.
\] (4.14)
Proof. Similar to (1.7), we write

$$\tilde{m}_n(z) - \hat{m}_n(z) = \frac{1}{n} \text{Tr}(\tilde{R} - \hat{R}) = \frac{1}{n} \text{Tr}(\tilde{W} - \hat{W})\hat{R}\hat{R}. \quad (4.15)$$

This yields

$$\mathbb{E}|\tilde{m}_n(z) - \hat{m}_n(z)| \leq n^{-1}\mathbb{E}\|\hat{R}\|\|\hat{R}\|_2\|\mathbb{E}\hat{W}\|_2. \quad (4.16)$$

Furthermore, we note that, by definition (4.1) and condition (1.7), we have

$$|\mathbb{E}\hat{X}_{jk}| \leq Cn^{-\frac{1}{2}}\mu^{\frac{1}{2}}. \quad (4.17)$$

Applying Lemma 7.6, inequality (7.28), in the Appendix and inequality (4.17), we obtain using \(\|\hat{R}\| \leq v^{-1}\),

$$\mathbb{E}|\tilde{m}_n(z) - \hat{m}_n(z)| \leq n^{-\frac{1}{2}}v^{-\frac{1}{2}}\mathbb{E}\|\hat{R}\|\|\hat{R}\|_2|\tilde{m}_n(z)|. \quad (4.18)$$

By Lemma 4.2,

$$\mathbb{E}|\tilde{m}_n(z)| \leq \mathbb{E}|\hat{m}_n(z)| + C, \quad (4.19)$$

for some constant \(C\) depending on \(\mu_8\) and \(A_0\). According to Corollary 5.14 in Section 5.2 with \(q = 1\)

$$\mathbb{E}|\tilde{m}_n(z)| \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}\|\tilde{R}_{jj}\| \leq C, \quad (4.20)$$

with a constant \(C\) depending on \(\mu_4, D_0\). Using these inequalities, we get

$$\mathbb{E}|\tilde{m}_n(z) - \hat{m}_n(z)| \leq \frac{C\mu_8}{n^{\frac{1}{2}}v^{\frac{1}{2}}} \leq \frac{C\mu_8}{n^{\frac{1}{2}}v^{\frac{1}{2}}}. \quad (4.21)$$

Thus Lemma 4.3 is proved.

**Corollary 4.4.** Assuming the conditions of Corollary 1.1, we have for \(z \in \mathbb{G}\),

$$|\mathbb{E}\hat{m}_n(z) - s(z)| \leq \frac{C}{(nv)^{\frac{1}{2}}} + \frac{C}{n^{\frac{1}{2}}v^{\frac{1}{2}}} \sqrt{\gamma}. \quad (4.22)$$

Proof. The proof immediately follows from the inequality

$$|\mathbb{E}\hat{m}_n(z) - s(z)| \leq |\mathbb{E}(\tilde{m}_n(z) - \hat{m}_n(z))| + |\mathbb{E}\tilde{m}_n(z) - s(z)|, \quad (4.23)$$

Lemmas 4.2 and 4.3 and Theorem 1.3.

The proof of Corollary 1.1 follows now from Lemma 4.1 Corollary 2.2 ineqality (3.23) and inequality

$$\sup_{x \in \mathbb{H} \sqrt{\gamma}v_0} \mathbb{E}|\tilde{m}_n(x + iv) - s(x + iv)|dv \leq \frac{C}{n}. \quad (4.24)$$
5 Resolvent Matrices and Quadratic Forms

The crucial problem in the proof of Theorem 1.3 is the following bound for any \( z \in \mathbb{G} \)
\[
E|R_{jz}|^p \leq C^p,
\]
(5.1)
for \( j = 1, \ldots, n \) and some absolute constant \( C > 0 \). To prove this bound we use an approach similar to the proof of Lemma 3.4 in [17]. In order to arrive at our goal we need additional bounds of quadratic forms of type
\[
E\left| \frac{1}{n} \sum_{l \neq k} X_{jl} X_{jk} R_{kl}^{(j)} \right|^p \leq \left( \frac{C p}{\sqrt{nV}} \right)^p .
\]
(5.2)
To prove this bound we recurrently use Rosenthal’s and Burkholder’s inequalities.

5.1 The Key Lemma

In this Section we provide auxiliary lemmas needed for the proof of Theorem 1.1.

For any \( J \subset T \) introduce \( T_J = T \setminus J \). We introduce the quantity, for some \( J \subset T \),
\[
B_p^{(J)} := \left[ \frac{1}{n} \sum_{q \in T_J} \left( \sum_{r \in T_J} |R_{qr}^{(j)}|^2 \right)^p \right].
\]
By Lemma 7.6, inequality (7.29) in the Appendix, we have
\[
EB_p^{(J)} \leq v^{-p} \frac{1}{n} \sum_{q \in T_J} E|R_{qq}^{(j)}|^p .
\]
(5.3)
Furthermore, introduce the quantities
\[
Q_{\nu_1}^{(I,k)} = \sum_{r \in T_k} a_{rr}^{(j,k,\nu+1)},
\]
(5.4)
\[
Q_{\nu_2}^{(I,k)} = \sum_{r \in T_k} (X_{kr}^2 - 1) a_{rr}^{(j,k,\nu+1)},
\]
(5.5)
\[
Q_{\nu_3}^{(I,k)} = \sum_{r \neq q \in T_k} X_{kr} X_{kq} a_{qr}^{(j,k,\nu+1)},
\]
(5.6)
where, $a_{qr}^{(j,k,0)}$ are defined recursively via

\[
a_{qr}^{(j,k,0)} = \frac{1}{\sqrt{n}} R_{qr}^{(j,k)},
\]

\[
a_{qr}^{(j,k,\nu+1)} = \sum_{l \in \{\max\{q,r\}+1,\ldots,n\} \cap \mathcal{T}_{j,k}} a_{rl}^{(j,k,\nu)} a_{lq}^{(j,k,\nu)}, \quad \text{for } \nu = 0, \ldots, L. \tag{5.5}
\]

Using these notations we have

\[
Q_{\nu}^{(j,k)} = Q_{\nu_1}^{(j,k)} + Q_{\nu_2}^{(j,k)} + Q_{\nu_3}^{(j,k)}. \tag{5.6}
\]

**Lemma 5.1.** Under the conditions of Theorem 1.1 we have

\[
\sum_{r \in \mathcal{T}_{j,k}} |a_{qr}^{(j,k,\nu+1)}|^2 \leq \left( \sum_{l \in \mathcal{T}_{j,k}} |a_{lr}^{(j,k,\nu)}|^2 \right) \left( \sum_{l \in \mathcal{T}_{j,k}} |a_{ql}^{(j,k,\nu)}|^2 \right). \tag{5.7}
\]

Moreover,

\[
\sum_{q,r \in \mathcal{T}_{j,k}} |a_{qr}^{(j,k,\nu+1)}|^2 \leq \left( \sum_{q,r \in \mathcal{T}_{j,k}} |a_{qr}^{(j,k,\nu)}|^2 \right)^2. \tag{5.8}
\]

**Proof.** We apply Hölder’s inequality and obtain

\[
|a_{qr}^{(j,k,\nu+1)}|^2 \leq \sum_{l \in \mathcal{T}_{j,k}} |a_{lr}^{(j,k,\nu)}|^2 \sum_{l \in \mathcal{T}_{j,k}} |a_{ql}^{(j,k,\nu)}|^2. \tag{5.9}
\]

Summing in $q$ and $r$, (5.7) and (5.8) follow. \qed

**Corollary 5.2.** Under the conditions of Theorem 1.1 we have

\[
\sum_{q,r \in \mathcal{T}_{j,k}} |a_{qr}^{(j,k,\nu)}|^2 \leq \left( \text{Im} m_n^{(j)}(z) + \frac{1}{nv} \right)^{2\nu} \tag{5.10}
\]

and

\[
\sum_{r \in \mathcal{T}_{j,k}} |a_{qr}^{(j,k,\nu)}|^2 \leq \left( \text{Im} m_n^{(j)}(z) + \frac{1}{nv} \right)^{2\nu-1} n^{-1} v^{-1} \text{Im} R_{qq}^{(j,k)}. \tag{5.11}
\]
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Proof. By definition of $a^{(J,k,0)}_{qr}$, see (5.5), applying Lemma 7.6 equality (7.28), in the Appendix, we get

$$\sum_{q,r \in T_{J,k}} |a^{(J,k,0)}_{qr}|^2 \leq \frac{1}{n} \sum_{q,r \in T_{J,k}} |R^{(J,k)}_{qr}|^2 \leq \left( \text{Im} m_{n}^{(J)}(z) + \frac{1}{nv} \right) v^{-1}, \quad (5.12)$$

and by definition (5.5),

$$\sum_{r \in T_{J,k}} |a^{(J,k,0)}_{qr}|^2 \leq \frac{1}{n} \sum_{r \in T_{J,k}} |R^{(J,k)}_{qr}|^2. \quad (5.13)$$

The general case follows now by induction in $\nu$, Lemma 5.1, and inequality (7.28), Lemma 7.6 in the Appendix.

Corollary 5.3. Under the conditions of Theorem 1.1 we have

$$a^{(J,k,\nu+1)}_{rr} \leq \left( \left( \text{Im} m_{n}^{(J)}(z) + \frac{1}{nv} \right) v^{-1} \right)^{2^{\nu}-1} n^{-1} v^{-1} \text{Im} R^{(J,k)}_{rr}. \quad (5.14)$$

Proof. The result immediately follows from the definition of $a^{(k,\nu)}_{rr}$ and Corollary 5.2.

In what follows we shall use the notations

$$\Psi^{(J)} = \text{Im} m_{n}^{(J)}(z) + \frac{1}{nv}, \quad (A^{(J)}_{\nu,p})^2 = E(\Psi^{(J)})(2^{\nu-1}2p), \quad T^{(J,k)}_{\nu,p} = E|Q^{(J,k)}_{\nu}|^p,$$

$$A^{(J)}_{p} := 1 + E^{\frac{1}{p}}|\Psi^{(J)}|^{4p}. \quad (5.15)$$

Let $s_0$ denote some fixed number (for instance $s_0 = 2^8$). Let $A_1$ be a constant (to be chosen later) and $0 < v_1 \leq 4$ a constant such that $v_0 = A_0 n^{-1} \leq v_1$ for all $n \geq 1$.

Lemma 5.4. Assuming the conditions of Theorem 1.1 and for $p \leq A_1(nv)^{-\frac{1}{4}}$

$$E|R^{(J)}_{jj}|^p \leq C_0^p, \quad \text{for } v \geq v_1, \quad \text{for all } j = 1, \ldots, n, \quad (5.16)$$

we have for $v \geq v_1/s_0$ and $p \leq A_1(nv)^{-\frac{1}{4}}$, and $k \in T_J$

$$E(Q^{(J,k)}_0)^p \leq 6\left( \frac{C_0 p}{\sqrt{2}} \right)^{2p} v^{-p} A^{(J)}_{p}. \quad (5.17)$$
Proof. Using the representation (5.6) and the triangle inequality, we get
\[ E|Q_{\nu}^{(j,k)}|^p \leq 3^p \left( E|Q_{\nu}^{(j,k)}|^p + E|Q_{\nu}^{(j,k)}|^p + E|Q_{\nu}^{(j,k)}|^p \right). \]  
\[ (5.18) \]

Let \( \mathcal{M}(A) \) denote the \( \sigma \)-algebra generated by r.v.’s \( X_{j,l} \) for \( j, l \in \mathbb{T}_\lambda \), for any set \( A \). Conditioning on \( \mathcal{M}(A) \) (\( A = \mathbb{J} \cup \{ k \} \)) and applying Rosenthal’s inequality (see Lemma 7.1), we get
\[ E|Q_{\nu}^{(j,k)}|^p \leq C_1 p^p \left( E\left( \sum_{r \in \mathbb{T}_{j,k}} |a_{rr}^{(j,k,v+1)}|^2 \right)^{\frac{p}{2}} + \sum_{r \in \mathbb{T}_{j,k}} E|a_{rr}^{(j,k,v+1)}|^p E|X_{kr}|^{2p} \right). \]  
\[ (5.19) \]

where \( C_1 \) denotes the absolute constant in Rosenthal’s inequality. By Remark 4.1, we get
\[ E|Q_{\nu}^{(j,k)}|^p \leq C_1 p^p \left( E\left( \sum_{r \in \mathbb{T}_{j,k}} |a_{rr}^{(j,k,v+1)}|^2 \right)^{\frac{p}{2}} + n \sum_{r \in \mathbb{T}_{j,k}} E|a_{rr}^{(j,k,v+1)}|^p \right). \]  
\[ (5.20) \]

Analogously conditioning on \( \mathcal{M}(A) \) and applying Burkholder’s inequality (see Lemma 7.3), we get
\[ E|Q_{\nu}^{(j,k)}|^p \leq C_2 p^p \left( E\left( \sum_{q \in \mathbb{T}_{j,k} \cap \{ 1, \ldots, r-1 \}} \sum_{q \in \mathbb{T}_{j,k} \cap \{ 1, \ldots, r-1 \}} X_{kr} a_{rq}^{(j,k,v+1)} \right)^{\frac{p}{2}} + \sum_{q=1}^{n-1} \sum_{r=1}^{q-1} \sum_{q=1}^{n-1} \sum_{r=1}^{q-1} E|a_{rq}^{(j,k,v+1)}|^p E|X_{kq}|^p \right). \]  
\[ (5.21) \]

where \( C_2 \) denotes the absolute constant in Burkholder’s inequality. Conditioning again on \( \mathcal{M}(A) \) and applying Rosenthal’s inequality, we obtain
\[ E\left( \sum_{r \in \mathbb{T}_{j,k}} X_{kr} a_{rq}^{(j,k,v+1)} \right)^p \leq C_1 p^p \left( E\left( \sum_{r=1}^{q-1} |a_{rq}^{(j,k,v+1)}|^2 \right)^{\frac{p}{2}} \right) + \sum_{r \in \mathbb{T}_{j,k}} E|a_{rr}^{(j,k,v+1)}|^p E|X_{kr}|^p. \]  
\[ (5.22) \]

Combining inequalities (5.21) and (5.22), we get
\[ E|Q_{\nu}^{(j,k)}|^p \leq C_2 p^p E|Q_{\nu}^{(j,k)}|^p + C_1 p^2 E\left( \sum_{r \in \mathbb{T}_{j,k}} |a_{rq}^{(j,k,v+1)}|^2 \right)^{\frac{p}{2}} E|X_{kq}|^p \]
\[ + C_1 p^2 \sum_{q \in \mathbb{T}_{j,k}} \sum_{r \in \mathbb{T}_{j,k}} E|a_{rq}^{(j,k,v+1)}|^p E|X_{kq}|^p E|X_{kr}|^p. \]  
\[ (5.23) \]
Using Remark \[4.1\] this implies
\[
E|Q_{ν1}^{(j,k)}|^p ≤ C_2^p E|Q_{ν1}^{(j,k)}|^p + C_1^p C_2^p 2^p n^q \frac{1}{n} \sum_{q \in \mathbb{J}_{j,k}} \sum_{r \in \mathbb{T}_{j,k}} E|a_{qr}^{(j,k,ν+1)}|^p.
\]

Using the definition \[5.4\] of \(Q_{ν1}^{(j,k)}\) and the definition \[5.5\] of coefficients \(a_{rr}^{(j,k,ν+1)}\), it is straightforward to check that
\[
E|Q_{ν1}^{(j,k)}|^p ≤ E \left[ \sum_{q,r \in \mathbb{T}_{j,k}} |a_{qr}^{(j,k,ν)}|^2 \right].
\]  

Combining \[5.20\], \[5.24\] and \[5.25\], we get by \[5.18\]
\[
3^{-p} E|Q_{ν1}^{(j,k)}|^p ≤ C_2^p E|Q_{ν1}^{(j,k)}|^p + C_1^p C_2^p 2^p n^q \frac{1}{n} \sum_{q \in \mathbb{J}_{j,k}} \sum_{r \in \mathbb{T}_{j,k}} E|a_{qr}^{(j,k,ν+1)}|^p
\]
\[ + C_1^p C_2^p E \left[ \sum_{q,r \in \mathbb{T}_{j,k}} |a_{qr}^{(j,k,ν)}|^2 \right]^p
\]
\[ + C_1^p C_2^p p \left( E \left[ \sum_{q,r \in \mathbb{T}_{j,k}} |a_{rr}^{(j,k,ν+1)}|^2 \right]^2 + n^q \frac{1}{n} \sum_{r \in \mathbb{T}_{j,k}} E|a_{rr}^{(j,k,ν+1)}|^p \right). \]

Applying now Lemma \[5.1\] and Corollaries \[5.2\] and \[5.3\] we obtain
\[
E|Q_{ν1}^{(j,k)}|^p ≤ C_3^p E|Q_{ν1}^{(j,k)}|^p + C_1^p E(\text{Im} m_n(z) + \frac{1}{nν})^{(2ν-1)} p(2ν-1)p
\]
\[ + C_3^p p^{2p} n^{-2} E(\text{Im} m_n^{(2)}(z) + \frac{1}{nν})^{(2ν-1)} p(2ν-1)p \left( \frac{1}{n} \sum_{q \in \mathbb{T}_{j,k}} |R_{qq}^{(j,k)}|^2 \right)
\]
\[ + C_3^p p^{2p} n^{-2} E(\text{Im} m_n^{(2)}(z) + \frac{1}{nν})^{(2ν-1)} p \left( \frac{1}{n} \sum_{q \in \mathbb{T}_{j,k}} |R_{qq}^{(j,k)}|^p \right),
\]

where \(C_3 = 3C_1 C_2\). Applying Cauchy–Schwartz inequality, we may rewrite
the last inequality in the form
\[
E|Q_{p}(J,k)|^p \leq C_3^pE|Q_{p+1}(J,k)|^p + C_3^pE(\text{Im}m_{n}(3)(z) + \frac{1}{nv^p})^{(2^p-1)p}v^{-(2^p-1)p}p
\]
\[
+ C_3^p2^p n^{-\frac{4}{p}}E(\text{Im}m_{n}(3)(z) + \frac{1}{nv^p})^{(2^p-1)p}E\left(\frac{1}{n} \sum_{q \in T_{J,k}} |R_{qq}(J,k)|^{2p}\right)
\]
\[
+ C_3^p2^p n^{-\frac{2}{p}}E(\text{Im}m_{n}(3)(z) + \frac{1}{nv^p})^{(2^p-1)2p}E\left(\frac{1}{n} \sum_{q \in T_{J,k}} |R_{qq}(J,k)|^{2p}\right).
\]
(5.28)

Introduce the notation
\[
\Gamma_p(z) := \frac{1}{n} \sum_{q \in T_{J,k}} |R_{qq}(J,k)|^{2p}.
\]
(5.29)

We rewrite the inequality (5.28) using \(\Gamma_p(z)\) and the notations of (5.15) as follows
\[
T_{\nu,p}^{(J,k)} \leq (C_3^p)^p T_{\nu+1,p/2}^{(J,k)} + C_3^p A_{\nu,p}^{(J)} v^{-(2^p-1)p}
\]
\[
+ (C_3^p)^p \left(v^{-2^p p n^{-\frac{4}{p}} A_{\nu,p/2}^{(J)} \frac{1}{n} \Gamma_p(z)} + v^{-2^p p n^{-\frac{4}{p}} A_{\nu,p}^{(J)} \Gamma_p(z)}\right).
\]
(5.30)

Note that
\[
A_{0,p}^{(J)} = 1, \quad A_{\nu,p}^{(J)} \leq \sqrt{1 + E(\Psi^{(J)})^{2p}} \leq 1 + E\left(\frac{1}{n}\right)^{\frac{1}{4p}}(\Psi^{(J)})^{2p},
\]
(5.31)

where \(\Psi^{(J)} = \text{Im}m_{n}(3)(z) + \frac{1}{nv^p}\). Furthermore,
\[
\Gamma_p/\nu^{2p} \leq \frac{1}{p}. \quad (5.32)
\]

Without loss of generality we may assume \(p = 2^L\) and \(\nu = 0, \ldots, L\). We may write
\[
T_{0,p}^{(J,k)} \leq (C_3^p)^p T_{1,p/2}^{(J,k)} + C_3^p + (C_3^p)^2 p v^{-p} \left(n^{-\frac{4}{p}} \frac{1}{n} \Gamma_p(z) + n^{-\frac{4}{p}} \Gamma_p(z)\right).
\]
(5.33)
By induction we get

\[ T_{0,p}^{(J,k)} \leq \prod_{\nu=0}^{L} (C_3p/2^\nu)^{p/2^\nu} T_{L,1}^{(J,k)} + A_p^{(J)} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} (C_3p/2^\nu)^{p/2^\nu} \right) v^{-(2^l-1)p/2^l} \]

\[ + A_p^{(J)} v^{-p} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} (C_3p/2^\nu)^{p/2^\nu} \right) (n^{-p} \Gamma_p^2)^{\frac{1}{2^l+1}} \]

\[ + A_p^{(J)} L \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} (C_3p/2^\nu)^{p/2^\nu} \right) (n^{-p} \Gamma_p^2)^{\frac{1}{2^l}}. \]  

(5.34)

It is straightforward to check that

\[ \sum_{\nu=1}^{l-1} \frac{\nu}{2^\nu} = 2(1 - \frac{2l+1}{2^l}). \]  

(5.35)

Note that, for \( l \geq 1, \)

\[ \prod_{\nu=0}^{l-1} \left( C_3(p/2^\nu) \right)^{p/2^\nu} = \frac{(C_3p)^{2p(1-2^{-l})}}{2^{2p(1-2^{-l})}} = 2^{4p \frac{1}{2^l}} \left( \frac{C_3p}{2} \right)^{2p(1-2^{-l})}. \]  

(5.36)

Applying this relation, we get

\[ A_p^{(J)} \sum_{l=0}^{L} \left( \prod_{\nu=0}^{l-1} (C_3p/2^\nu)^{p/2^\nu} \right) v^{-(2^l-1)p/2^l} \leq A_p^{(J)} \left( \frac{C_3p}{2} \right)^{2p} v^{-p} \sum_{l=0}^{L-1} 2^{4p \frac{1}{2^l}} (\frac{4v}{C_3^2p^{2l}})^{\frac{1}{2^l}}. \]  

(5.37)

Note that for \( l \geq 0, \frac{1}{2^l} \leq \frac{1}{2} \) and recall that \( p = 2^L. \) Using this observation, we get

\[ A_p^{(J)} \sum_{l=0}^{L} \left( \prod_{\nu=0}^{l-1} (C_3p/2^\nu)^{p/2^\nu} \right) v^{-(2^l-1)p/2^l} \leq A_p^{(J)} \left( \frac{C_3p}{2} \right)^{2p} v^{-p} 2^{2p} \sum_{l=0}^{L-1} \left( \frac{4v}{C_3^2p^{2l}} \right)^{2^l}. \]  

(5.38)

This implies that for \( \frac{4v}{C_3^2p^{2l}} \leq \frac{1}{2}, \)

\[ A_p^{(J)} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} (C_3p/2^\nu)^{p/2^\nu} \right) v^{-(2^l-1)p/2^l} \leq (C_3p)^{2p} A_p^{(J)} v^{-p}. \]  

(5.39)
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Furthermore, by definition of $T_{p,q}$, we have

$$T_{L,1}^{(j,k)} = \mathbb{E}Q_{L}^{(j,k)} \leq \mathbb{E} \sum_{q,r \in \mathbb{I}_{k}} (a_{q,r}^{(j,k)})^2. \quad (5.40)$$

Applying Corollary 5.2 and Hölder’s inequality, we get

$$T_{L,1}^{(j,k)} \leq E(v^{-1}\Psi^{(j)})^p \leq v^{-p}A_p^{(j)}. \quad (5.41)$$

By condition (5.19), we have

$$\Gamma_p := \Gamma_p(u + iv) \leq s_0^2C_0^{2p}. \quad (5.42)$$

Using this inequality, we get,

$$A_p^{(j)}v^{-p} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} \frac{(C_{3p}/2^\nu)^{p/2^\nu}}{\frac{2^{p/2}C_{3p}}{2}} \right) n^{-\frac{2p}{2^{p+1}}} \Gamma_p^{\frac{1}{p^{\nu}}} \leq A_p^{(j)}v^{-p} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} \frac{(C_{3p}/2^\nu)^{p/2^\nu}}{\frac{2^{p/2}C_{3p}}{2}} \right) (s_0^4C_0^4n^{-1})^{\frac{p}{2^{p+1}}}. \quad (5.43)$$

Applying relation (5.36), we obtain

$$A_p^{(j)}v^{-p} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} \frac{(C_{3p}/2^\nu)^{p/2^\nu}}{\frac{2^{p/2}C_{3p}}{2}} \right) n^{-\frac{2p}{2^{p+1}}} \Gamma_p^{\frac{1}{p^{\nu}}} \leq \left( \frac{C_{3p}}{2} \right)^{2p} A_p^{(j)}v^{-p} \sum_{l=1}^{L} 2^{p/2^{p+1}} \left( \frac{C_{3p}}{2} \right)^{-\frac{2p}{2^{p+1}}} (s_0^4C_0^4n^{-1})^{\frac{p}{2^{p+1}}} \leq \left( \frac{C_{3p}}{2} \right)^{2p} A_p^{(j)}v^{-p} \sum_{l=1}^{L} 2^{p/2^{p+1}} \left( \frac{C_{3p}}{2} \right)^{-\frac{2p}{2^{p+1}}} (s_0^4C_0^4n^{-1})^{\frac{p}{2^{p+1}}} \leq \left( \frac{C_{3p}}{2} \right)^{2p} A_p^{(j)}v^{-p} \sum_{l=1}^{L} \left( \frac{s_0C_0}{C_{3p}^{n+1}} \right)^{2^{L-l+1}}. \quad (5.44)$$

Without loss of generality we may assume that $C_3 \geq 2(C_0s_0)$. Then we get

$$A_p^{(j)}v^{-p} \sum_{l=0}^{L} \left( \prod_{\nu=0}^{l-1} \frac{(C_{3p}/2^\nu)^{p/2^\nu}}{\frac{2^{p/2}C_{3p}}{2}} \right) n^{-\frac{2p}{2^{p+1}}} \Gamma_p^{\frac{1}{p^{\nu}}} \leq (C_{3p})^{2p} A_p^{(j)}v^{-p}. \quad (5.45)$$

Analogously we get

$$A_p^{(j)}v^{-p} \sum_{l=1}^{L} \left( \prod_{\nu=0}^{l-1} \frac{(C_{3p}/2^\nu)^{p/2^\nu}}{\frac{2^{p/2}C_{3p}}{2}} \right) n^{-\frac{2p}{2^{p+1}}} \Gamma_p^{\frac{1}{p^{\nu}}} \leq (C_{3p})^{2p} v^{-p} A_p^{(j)}. \quad (5.46)$$
Combining inequalities (5.34), (5.41), (5.43), (5.46), we finally arrive at
\[ T_{0,p}^{(j,k)} \leq 6(C_3p)^{2p}v^{-p}A_{p}^{(j)}. \] (5.47)

Thus, Lemma 5.4 is proved. \( \square \)

### 5.2 Diagonal Entries of the Resolvent Matrix

We shall use the representation, for any \( j \in \mathbb{T}_J, \)
\[ R_{jj}^{(j)} = \frac{1}{-z + \frac{1}{\sqrt{n}}X_{jj} - \frac{1}{n}\sum_{k,l \in \mathbb{T}_J,j} X_{jk}X_{jl}R_{kl}^{(j,j)}}, \]
(see, for example, equality (4.6) in [13]). Recall the following relations (compare (3.3), (3.15))
\[ R_{jj}^{(j)} = -\frac{1}{z + m_n^{(j)}(z)} + \frac{1}{z + m_n^{(j)}(z)}\varepsilon_jR_{jj}^{(j)}, \] (5.48)
or
\[ R_{jj}^{(j)} = s(z) - s(z)\Lambda_n^{(j)}R_{jj}^{(j)} - s(z)\varepsilon_jR_{jj}^{(j)}, \] (5.49)
where \( \varepsilon_j^{(j)} := \varepsilon_{j_1}^{(j)} + \varepsilon_{j_2}^{(j)} + \varepsilon_{j_3}^{(j)} + \varepsilon_{j_4}^{(j)} \) with
\[ \varepsilon_{j_1}^{(j)} := \frac{1}{\sqrt{n}}X_{jj}, \varepsilon_{j_2}^{(j)} := \frac{1}{n}\sum_{k \neq j \in \mathbb{T}_J} X_{jk}X_{jl}R_{kl}^{(j,j)}, \varepsilon_{j_3}^{(j)} := \frac{1}{n}\sum_{k \in \mathbb{T}_J} (X_{jk}^2 - 1)R_{kk}^{(j,j)}, \]
\[ \varepsilon_{j_4}^{(j)} := \frac{1}{n}(\text{Tr}\, R^{(j,j)} - \text{Tr}\, R^{(j)}), \Lambda_n^{(j)} := m_n^{(j)}(z) - s(z) = \frac{1}{n}\text{Tr}\, R^{(j)} - s(z). \] (5.50)

Since \( |s(z)| \leq 1, \) the representation (5.49) yields, for any \( p \geq 1, \)
\[ |R_{jj}^{(j)}|^p \leq 3^p + 3^p|\varepsilon_j^{(j)}|^p|R_{jj}^{(j)}|^p + 3^p|\Lambda_n^{(j)}|^p|R_{jj}^{(j)}|^p. \] (5.51)

Applying the Cauchy – Schwartz inequality, we get
\[ E|R_{jj}^{(j)}|^p \leq 3^p + 3^pE\frac{1}{2}|\varepsilon_j^{(j)}|^2E\frac{1}{2}|R_{jj}^{(j)}|^2 + 3^pE\frac{1}{2}|\Lambda_n^{(j)}|^2E\frac{1}{2}|R_{jj}^{(j)}|^2. \] (5.52)

We shall investigate now the behavior of \( E|\varepsilon_j^{(j)}|^{2p} \) and \( E|\Lambda_n^{(j)}|^{2p}. \) First we note,
\[ E|\varepsilon_j^{(j)}|^{2p} \leq 4^{2p} \sum_{\nu=1}^{4} E|\varepsilon_{j\nu}^{(j)}|^{2p}. \] (5.53)
Lemma 5.5. Assuming the conditions of Theorem 1.1 we have, for any \( p \geq 1 \), and for any \( z = u + iv \in \mathbb{C}_+ \),
\[
\mathbb{E}|\varepsilon_{j1}|^{2p} \leq \frac{\mu_4}{n^{\frac{p}{2} + 1}}.
\] (5.54)

Proof. The proof follows immediately from the definition of \( \varepsilon_{j1} \) and condition (1.4).

Lemma 5.6. Assuming the conditions of Theorem 1.3 we have, for any \( p \geq 1 \), and for any \( z = u + iv \in \mathbb{C}_+ \),
\[
\mathbb{E}|\varepsilon_{j4}|^{2p} \leq \frac{1}{n^{2p}v^{2p}}.
\] (5.55)

Proof. For a proof of this Lemma see [13], Lemma 4.1.

Let \( A_1 > 0 \) and \( 0 \leq v_1 \leq 4 \) be a fixed.

Lemma 5.7. Assuming the conditions of Theorem 1.1 and assuming for all \( J \subset T \) with \( |J| \leq L \) and all \( l \in T_J \)
\[
\mathbb{E}|R^{(j)}_{jl}|^q \leq C_0^q, \text{ for } 1 \leq q \leq A_1(nv)^{\frac{1}{4}} \text{ and for } v \geq v_1,
\] (5.56)
we have, for all \( v \geq v_1/s_0 \), and for all \( J \subset T \) with \( |J| \leq L - 1 \),
\[
\mathbb{E}|\varepsilon_{j3}|^{2p} \leq (C_1p)^{2p}n^{-p}s_0^{2p}C_0^{4p}, \text{ for } 1 \leq p \leq A_1(nv)^{\frac{1}{4}}.
\] (5.57)

Proof. Recall that \( s_0 = 2^4 \) and note that if \( p \leq A_1(nv)^{\frac{1}{4}} \) for \( v \geq v_1/s_0 \) then \( q = 2p \leq A_1(nv)^{\frac{1}{2}} \) for \( v \geq v_1 \). Let \( v' := vs_0 \). If \( v \geq v_1/s_0 \) then \( v' \geq v_1 \). We have
\[
q = 2p \leq 2A_1(nv)^{\frac{1}{4}} = 2A_1(nv's_0^{-1})^{\frac{1}{4}} = A_1(nv')^{\frac{1}{2}}.
\] (5.58)
We apply now Rosenthal’s inequality for the moments of sums of independent random variables and get
\[
\mathbb{E}|\varepsilon_{j3}|^{2p} \leq (C_1p)^{2p}n^{-2p}\left(\mathbb{E}\left(\sum_{l \in T_{j,l}} |R^{(j,l)}_{jl}|^2\right)^p + \mathbb{E}|X_{j1}|^{4p}\sum_{l \in T_{j,l}} \mathbb{E}|R^{(j,l)}_{jl}|^{2p}\right).
\] (5.59)
According to inequality (5.58) we may apply Lemma 7.7 and condition (5.56) for \( q = 2p \). We get, for \( v \geq v_1/s_0 \),
\[
\mathbb{E}|\varepsilon_{j3}|^{2p} \leq (C_1p)^{2p}n^{-p}s_0^{2p}C_0^{2p}.
\] (5.60)
We use as well that by the conditions of Theorem 1.1 \( \mathbb{E}|X_{jl}|^{4p} \leq D_0^{4p-1}n^{p-1}\mu_4 \), and by Jensen’s inequality, \( \left(\frac{1}{n} \sum_{l \in T_J} |R^{(j,l)}_{jl}|^2\right)^p \leq \frac{1}{n} \sum_{l \in T_J} |R^{(j,l)}_{jl}|^{2p} \). Thus, Lemma 5.7 is proved.
Lemma 5.8. Assuming the conditions of Theorem 1.1, condition (5.56), for \( v \geq v_1 \) and \( p \leq A_1(nv)^{1/4} \), we have, for any \( v \geq v_1/s_0 \) and \( p \leq A_1(nv)^{1/4} \),

\[
E|\epsilon_{j2}^{(j)}|^{2p} \leq 6(C_3p)^{4p}n^{-p}v^{-p}A_p^{(j)} + 2(C_3p)^{4p}n^{-p}v^{-p}(C_0s_0)^p. \tag{5.61}
\]

Proof. We apply Burkholder’s inequality for quadratic forms. See Lemma 7.3 in the Appendix. We obtain

\[
E|\epsilon_{j2}^{(j)}|^{2p} \leq (C_1p)^{2p}n^{-2p}E\left( \sum_{l \in T_{j,j}} \left( \sum_{r \in T_{j,j} \cap \{1, \ldots, l-1\}} X_{jr}R_{lr}^{(j)} \right)^2 \right)^p + \max_{j,k} E|X_{jk}|^{2p} \sum_{l \in T_{j,j}} E \left( \sum_{r \in T_{j,j} \cap \{1, \ldots, l-1\}} X_{jr}R_{lr}^{(j)} \right)^{2p}. \tag{5.62}
\]

Using now the quantity \( Q_0^{(j,j)} \) for the first term and Rosenthal’s inequality and condition (1.3) for the second term, we obtain with Lemma 7.6 inequality (7.29), in the Appendix and \( C_3 = C_1C_2 \)

\[
E|\epsilon_{j2}^{(j)}|^{2p} \leq (C_2p)^{2p}n^{-p}E|Q_0^{(j,j)}|^{p} + (C_3p)^{4p}n^{-p}v^{-p} \frac{1}{n} \sum_{l \in T_{j,j}, r \in T_{j,j} \cap \{1, \ldots, l-1\}} E|R_{lr}^{(j,j)}|^{2p} \leq (C_2p)^{2p}n^{-p}E|Q_0^{(j,j)}|^{p} + (C_3p)^{4p}n^{-p}v^{-p} \frac{1}{n} \sum_{l \in T_{j,j}} E|R_{lr}^{(j,j)}|^{p} + (C_3p)^{4p}n^{-p}v^{-p} \frac{1}{n^2} \sum_{l \in T_{j,j}} E|R_{lr}^{(j,j)}|^{p}. \tag{5.63}
\]

By Lemma 7.7 and condition (5.56), we get

\[
E|\epsilon_{j2}^{(j)}|^{2p} \leq (C_3p)^{2p}n^{-p}E|Q_0^{(j,j)}|^{p} + 2(C_3p)^{3p}n^{-p}v^{-p}(C_0s_0)^p. \tag{5.64}
\]

Applying now Lemma 5.4 we get the claim. Thus, Lemma 5.8 is proved. \( \square \)

Recall that

\[
\Lambda_{n}^{(j)} = \frac{1}{n} \text{Tr} R^{(j)} - s(z), \quad \text{and} \quad T_{n}^{(j)}(z) = \frac{1}{n} \sum_{j \in T_{j,j}} \epsilon_{j}^{(j)} R_{jj}^{(j)}. \tag{5.65}
\]
**Lemma 5.9.** Assuming the conditions of Theorem 1.1 we have

\[ |Λ_n^{(j)}| ≤ C(\sqrt{|T_n^{(j)}(z)|} + \frac{\sqrt{|J|}}{\sqrt{n}}). \]  

(5.66)

**Proof.** See e.g. inequality (2.10) in [17]. For completeness we include short proof here. Obviously

\[ Λ_n^{(j)}(z) = s(z)(T_n^{(j)}(z) - \frac{|J|}{n}) \]  

(5.67)

Solving this equation w.r.t. \( Λ_n(z) \), we get

\[ Λ_n(z) = \frac{-\sqrt{z^2 - 4} + \sqrt{z^2 - 4 - 4\tilde{T}_n(z)}}{2}, \]  

(5.68)

where

\[ \tilde{T}_n(z) = s(z)(T_n^{(j)}(z) - \frac{|J|}{n}). \]  

(5.69)

We take here the branch of \( \sqrt{z} \) such that \( \text{Im} \sqrt{z} ≥ 0 \). Since for any \( a, b \in \mathbb{C} \)

\[ |\sqrt{a + b} - \sqrt{a}| ≤ C\frac{|b|}{\sqrt{|a| + |b|}}, \]  

we get

\[ |Λ_n^{(j)}(z)| ≤ C(\sqrt{|T_n^{(j)}(z)|} + \frac{\sqrt{|J|}}{\sqrt{n}}). \]  

(5.70)

Thus, Lemma 5.9 is proved.

**Lemma 5.10.** Assuming the conditions of Theorem 1.1 and condition (5.56), we obtain, for \( |J| ≤ Cn^2 \)

\[ E|Λ_n^{(j)}|^{2p} ≤ \frac{Cp}{n^{\frac{p}{2}}} + \left( \frac{μ_4}{n^{\frac{p}{2}+1}} + \frac{1}{n^{2p_un_{2p_u}}} + (C_1p)^{2p}n^{-p}s_0^{2p}C_0^{4p} \right. \]

\[ + 6(C_3p)^{4p}n^{-p}v^{-p}A_p^{(j)} + 2(C_3p)^{4p}n^{-p}v^{-p}(C_0s_0)p \]  

(5.71)

**Proof.** By Lemma 5.8 we have

\[ E|Λ_n^{(j)}|^{2p} ≤ C^pE|T_n^{(j)}(z)|^p + \frac{|J|^{\frac{p}{2}}}{n^{\frac{p}{2}}} ≤ C^pE|T_n^{(j)}(z)|^p + \frac{C}{n^{\frac{p}{2}}}. \]  

(5.72)
Furthermore,
\[ E|T_n^{(j)}(z)|^p \leq \left( \frac{1}{n} \sum_{j \in \mathcal{T}} E|\varepsilon_j^{(j)}|^{2p} \right)^{\frac{1}{2}} \left( \frac{1}{n} \sum_{j \in \mathcal{T}} E|R_{jj}^{(j)}|^{2p} \right)^{\frac{1}{2}}. \tag{5.73} \]

Lemmas 5.5–5.8 together with Lemma 7.7 imply
\[ \frac{1}{n} \sum_{j \in \mathcal{T}} E|\varepsilon_j^{(j)}|^{2p} \leq 4^{2p-1} \left( \frac{\mu_4}{n^4 + 1} + \frac{1}{n^{2p-2}} + (C_1 p)^{2p} n^{-p} s_0^2 C_0^{4p} \right) \]
\[ + 6(C_3 p)^{4p} n^{-p} v^{-p} A_p^{(j)} + 2(C_3 p)^{4p} n^{-p} v^{-p}(C_0 s_0)^p \].
\[ \tag{5.74} \]

Thus, Lemma 5.10 is proved.

**Lemma 5.11.** Assuming the conditions of Theorem 1.1 and condition (5.56), there exists an absolute constant \( C_4 \) such that, for \( p \leq A_1(n v)^{\frac{1}{2}} \) and \( v \geq v_1 / s_0 \), we have, uniformly in \( J \subset \mathcal{T} \) such that \( |J| \leq C n^\frac{1}{2} \),
\[ A_p^{(j)} \leq C_4^p. \tag{5.75} \]

**Proof.** We start from the obvious inequality, using \( |s(z)| \leq 1 \),
\[ E|\Psi_j^{(j)}|^{2p} \leq 3^{2p} (1 + (n v)^{-p} + E|\Lambda_n^{(j)}(z)|^{2p}). \tag{5.76} \]

Furthermore, applying Lemma 5.10 we get
\[ E|\Psi_j^{(j)}|^{2p} \leq 3^{2p} \left( 1 + (n v)^{-p} + \left( \frac{\mu_4}{n^4 + 1} + \frac{1}{n^{2p-2}} + (C_1 p)^{2p} n^{-p} s_0^2 C_0^{4p} \right) \right. \]
\[ + 6(C_3 p)^{4p} n^{-p} v^{-p} A_p^{(j)} + 2(C_3 p)^{4p} n^{-p} v^{-p}(C_0 s_0)^p \left[ \frac{1}{2} (C_0 s_0)^p \right]. \tag{5.77} \]

By definition,
\[ A_p^{(j)} \leq 1 + E^\frac{1}{2}|\Psi_j^{(j)}|^{2p}. \tag{5.78} \]

Inequalities (5.78) and (5.77) together imply
\[ A_p^{(j)} \leq 1 + 3^p \left( 1 + (n v)^{-p} + (C_0 s_0)^p \left( \frac{1}{2 n^4 v^2} + \frac{1}{n^2 v^2} + (C_1 p)^{2p} n^{-p} s_0^2 C_0^{4p} \right) \right. \]
\[ + 3(C_3 p)^{4p} n^{-p} v^{-p} A_p^{(j)} + 2(C_3 p)^{4p} n^{-p} v^{-p}(C_0 s_0)^p \left[ \frac{1}{2} (C_0 s_0)^p \right]. \tag{5.79} \]
Let \( C' = s_0 \max \{ 9, C_2^2, C_0^2 C_1^2, 3C_3 C_0^2, C_3 C_0^2 \} \). Using Lemma 7.4 with \( x = (A_p^{(3)})^{1/4}, t = 4, r = 1, \) we get
\[
A_p^{(3)} \leq C^p \left( 1 + (nv)^{-\frac{7}{4}} + \mu_4 \frac{1}{n^2 v^2} + \frac{1}{n^2 v^2} \right.
+ p^\frac{5}{4} n^{-\frac{7}{4}} v^{-\frac{3}{2}} + p^2 n^{-\frac{7}{4}} v^{-\frac{3}{2}} + \left. p^\frac{5}{4} n^{-\frac{7}{4}} v^{-\frac{3}{2}} \right). \tag{5.80}
\]
For \( p \leq A_1 (nv)^{\frac{1}{4}} \), we get, for \( z \in \mathbb{C} \),
\[
A_p^{(3)} \leq C_4^p, \tag{5.81}
\]
where \( C_4 \) is some absolute constant. We may take \( C_4 = 2C' \). \qed

**Corollary 5.12.** Assuming the conditions of Theorem 1.1 and condition (5.56), we have, for \( v \geq v_1 / s_0 \), and for any \( J \subset T \) such that \( |J| \leq n \sqrt{v} \),
\[
\mathbf{E} |\Lambda_n^{(J)}|^{2p} \leq C_0^{2p} \left( \frac{4^p \mu_4^\frac{1}{4} s_0^\frac{1}{4}}{n^p v^p} + \frac{s_0^\frac{1}{p}}{n^p v^p} + C_0^{p^2} \right) \tag{5.82}
\]
where
\[
C_5 := 4C_2^2 s_0^4 + 6\mu_4 C_4 + 2^{\frac{1}{2}} C_3^3 s_0^3. \tag{5.83}
\]

**Proof.** Without loss of generality we may assume that \( C_0 > 1 \). The bound (5.82) follows now from Lemmas 5.10 and 5.11. \qed

**Lemma 5.13.** Assuming the conditions of Theorem 1.1 and condition (5.56) for \( J \subset T \) such that \( |J| \leq L \leq n \sqrt{v} \), there exist positive constant \( A_0, C_0, A_1 \) depending on \( \mu_4, D_0 \) only, such that we have, for \( p \leq A_1 (nv)^{\frac{1}{4}} \) and \( v \geq v_1 / s_0 \) uniformly in \( J \) and \( v_1 \)
\[
\mathbf{E} |R_{jj}^{(3)}|^{p} \leq C_0^{p} \tag{5.84}
\]
with \( |J| \leq L - 1 \).

**Proof.** According to inequality (5.52), we have
\[
\mathbf{E} |R_{jj}^{(3)}|^{p} \leq 4^p (1 + (\mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(J)}|^{2p} + \mathbf{E}^{\frac{1}{2}} |\varepsilon_j^{(j)}|^{2p}) \mathbf{E}^{\frac{1}{2}} |R_{jj}^{(3)}|^{2p}). \tag{5.85}
\]
Applying condition (5.56), we get
\[
\mathbf{E} |R_{jj}^{(3)}|^{p} \leq 4^p (1 + (\mathbf{E}^{\frac{1}{2}} |\Lambda_n^{(J)}|^{2p} + \mathbf{E}^{\frac{1}{2}} |\varepsilon_j^{(j)}|^{2p} + \ldots + \mathbf{E}^{\frac{1}{2}} |\varepsilon_j^{(j)}|^{2p}) s_0^p C_0^p). \tag{5.86}
\]
Combining results of Lemmas 5.5–5.8 and Corollary 5.12 we obtain

\[ E|R_{jj}^{(3)}|^p \leq 5^p \left( 1 + s_0^p C_0^{2p} \left( \frac{4^p \mu_4 s_0^p}{n^p v^n} + \frac{s_0^p}{n^p v^n} + \frac{C_0^p p^{2p}}{n^p v^n} \right) \right) \]

\[ + s_0^p C_0^{3p} \left( \frac{4^p \mu_4}{n^p v^n} + \frac{s_0^p}{n^p v^n} + \frac{C_0^p p^{2p}}{n^p v^n} \right). \]

(5.87)

We may rewrite the last inequality as follows

\[ E|R_{jj}^{(3)}|^p \leq C_0^p \left( \frac{5^p}{C_0^p} + \frac{\tilde{C}_1}{(nv)^{\frac{p}{8}}} + \frac{\tilde{C}_2 p^4}{(nv)^{\frac{p}{8}}} + \frac{\tilde{C}_3 p^4}{(nv)^{\frac{p}{2}}} + \frac{\tilde{C}_4}{(nv)^p} \right), \]

(5.88)

where

\[ \tilde{C}_1 = 5^8 s_0^{12} C_0^8 \mu_4^\frac{1}{3}; \]

\[ \tilde{C}_2 = 5^4 s_0^4 C_0^4 \mu_4^4 (1 + 2C_0^4 \mu_4^4); \]

\[ \tilde{C}_3 = 5^2 s_0^2 C_0^2 (s_0 + C_0^2); \]

\[ \tilde{C}_4 = 5 C_0 s_0^2. \]

(5.89)

Note that for

\[ A_0 \geq 2^8 A_1^4 \max\{\tilde{C}_1, \ldots, \tilde{C}_4\} \]

(5.90)

and \( C_0 \geq 25 \), we obtain that

\[ E|R_{jj}^{(4)}|^p \leq C_0^p. \]

(5.91)

Thus Lemma 5.13 is proved.

**Corollary 5.14.** Assuming the conditions of Theorem 1.1 we have, for \( p \leq 8 \) and \( v \geq v_0 = A_0 n^{-1} \) there exist a constant \( C_0 > 0 \) depending on \( \mu_4 \) and \( D_0 \) only such that for all \( 1 \leq j \leq n \) and all \( z \in \mathbb{G} \)

\[ E|R_{jj}|^p \leq C_0^p, \]

(5.92)

and

\[ E \left| \frac{1}{z + m_n(z)} \right|^p \leq C_0^p. \]

(5.93)
Proof. Let $L = [−\log s_0 v_0] + 1$. Note that $s_0 L ≤ v_0$ and $A_1 \frac{n_1}{s_0} ≥ A_1 (nv_0)^{\frac{1}{4}}$.

We may choose $C_0 = 25$ and $A_0, A_1$ such that (5.90) holds and

$$A_1 (nv)^{\frac{1}{4}} ≥ 8.$$  \hspace{1cm} (5.94)

Then, for $v = 1$, and for any $p ≥ 1$, for any set $J ⊂ T$ such that $|J| ≤ L − 1$. After repeated application of Lemma 5.13 (with (5.95) as assumption valid for $v ≥ 1/s_0$) we arrive at the conclusion that the inequality (5.95) holds for $v ≥ 1/s_0$, $p ≤ A_1 n_1^{\frac{1}{4}}/s_0^{\frac{1}{2}}$ and all $J ⊂ T$ such that $|J| ≤ L − 2$. Continuing this iteration inequality (5.95) finally holds for $v ≥ A_0 n^{-1}$, $p ≤ 8$ and $J = ∅$.

The proof of inequality of (5.93) is similar: We have by (1.12)

$$\frac{1}{|z + m_n(z)|} ≤ \frac{1}{|s(z) + z|} + \frac{|\Lambda_n|}{|z + m_n(z)||z + s(z)|} ≤ |s(z)| \left(1 + \frac{|\Lambda_n|}{|z + m_n(z)|}\right).$$  \hspace{1cm} (5.96)

Furthermore, using that $|m_n'(z)| ≤ \frac{1}{n} \text{Tr} R^2 ≤ \frac{1}{n} \sum_{j=1}^{n} E|R_{jj}|^2$ and Lemma 7.6 inequality (7.28) in the Appendix, we get

$$|d \frac{1}{dz} \log(z + m_n(z))| \leq \frac{1 + m_n'(z)}{|z + m_n(z)|} \leq \frac{1 + v + \text{Im} m_n(z)}{v |z + m_n(z)|} \leq \frac{1}{v}.$$  \hspace{1cm} (5.97)

By integration, this implies that (see the proof of Lemma 7.7)

$$\frac{1}{|(u + iv/s_0) + m_n(u + iv/s_0)|} ≤ \frac{s_0}{|(u + iv) + m_n(u + iv)|}.$$  \hspace{1cm} (5.98)

Inequality (5.96) and the Cauchy–Schwartz inequality together imply

$$E \frac{1}{|z + m_n(z)|^p} ≤ 2^p |s(z)|^p \left(1 + E \frac{1}{|\Lambda_n|^{2p} E \frac{1}{|z + m_n(z)|^{2p}}}ight).$$  \hspace{1cm} (5.99)

Applying inequality (5.98), we obtain

$$E \frac{1}{|z + m_n(z)|^p} ≤ 2^p |s(z)|^p \left(1 + E \frac{1}{|\Lambda_n|^{2p} s_0^p C_0^p} \right).$$  \hspace{1cm} (5.100)
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Using Corollary [5.12] we get, for \( v \geq 1/s_0 \)

\[
E \frac{1}{|z + m_n(z)|^p} \leq 2^p |s(z)|^p \left( 1 + \left( \frac{4^p \mu_4 s_0^p}{n^2 v^2} + \frac{s_0^p}{n^2 v^2} + \frac{C_0^p p^p}{n^2 v^2} \right) s_0^p C_0^{2p} \right).
\]

(5.101)

Thus inequality (5.93) holds for \( v \geq 1/s_0 \) as well. Repeating this argument inductively with \( A_0, A_1, \ldots, C \) satisfying (5.90) for the regions \( v \geq s_0^{-\nu} \), for \( \nu = 1, \ldots, L \) and \( z \in G \), we get the claim. Thus, Corollary 5.14 is proved.

6 Proof of Theorem 1.3

We return now to the representation (5.49) which implies that

\[
s_n(z) = \frac{1}{n} \sum_{j=1}^{n} E R_{jj} = s(z) + E A_n = s(z) + E \frac{T_n(z)}{z + s(z) + m_n(z)}.
\]

(6.1)

We may continue the last equality as follows

\[
s_n(z) = s(z) + E \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj} + E \frac{\hat{T}_n(z)}{z + s(z) + m_n(z)},
\]

(6.2)

where

\[
\hat{T}_n = \sum_{\nu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j\nu} R_{jj}.
\]

Note that the definition of \( \varepsilon_{j4} \) in (5.49) and equality (7.81) together imply

\[
\frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj} = \frac{1}{n} \text{Tr} R \frac{1}{R} = \frac{1}{n} \frac{dm_n(z)}{dz}.
\]

(6.3)

Thus we may rewrite (6.2) as

\[
s_n(z) = s(z) - \frac{1}{n} E \frac{m_n(z)}{z + s(z) + m_n(z)} + E \frac{\hat{T}_n(z)}{z + s(z) + m_n(z)}.
\]

(6.4)

Denote by

\[
\tilde{T} = E \frac{\hat{T}_n(z)}{z + s(z) + m_n(z)}.
\]

(6.5)
6.1 Estimation of $\mathcal{T}$

We represent $\mathcal{T}$

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2,$$  \hspace{1cm} (6.6)

where

$$\mathcal{T}_1 = -\frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{3} \mathbb{E} \frac{\varepsilon_{j\nu}}{z + m_n^{(j)}(z) + s(z)},$$

$$\mathcal{T}_2 = \frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{3} \mathbb{E} \frac{\varepsilon_{j\nu} (R_{jj} + \frac{1}{z + m_n^{(j)}(z)})}{z + m_n(z) + s(z)}. \hspace{1cm} (6.7)$$

6.1.1 Estimation of $\mathcal{T}_1$

We may decompose $\mathcal{T}_1$ as

$$\mathcal{T}_1 = \mathcal{T}_{11} + \mathcal{T}_{12},$$ \hspace{1cm} (6.8)

where

$$\mathcal{T}_{11} = -\frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{3} \mathbb{E} \frac{\varepsilon_{j\nu}}{z + m_n^{(j)}(z) + s(z)},$$

$$\mathcal{T}_{12} = -\frac{1}{n} \sum_{j=1}^{n} \sum_{\nu=1}^{3} \mathbb{E} \frac{\varepsilon_{j\nu}\varepsilon_{j4} \frac{1}{z + m_n^{(j)}(z)}}{(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))}. \hspace{1cm} (6.9)$$

It is easy to see that, for $\nu = 1, 2, 3$, by conditional expectation

$$\mathcal{T}_{11} = 0.$$ \hspace{1cm} (6.10)

Applying the Cauchy–Schwartz inequality, we get

$$\mathbb{E} \left| \frac{\varepsilon_{j\nu}\varepsilon_{j4} \frac{1}{z + m_n^{(j)}(z)}}{(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))} \right|^2 \leq \mathbb{E} \left| \frac{\varepsilon_{j\nu}}{z + m_n^{(j)}(z) + s(z)} \right|^2 \mathbb{E} \left| \frac{\varepsilon_{j4}}{z + m_n(z) + s(z)} \right|^2.$$ \hspace{1cm} (6.11)
Applying the Cauchy – Schwartz inequality again, we get
\[
\mathbb{E} \frac{\varepsilon_{j\nu}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))}^2 \leq \mathbb{E}_\frac{1}{|z + m_n^{(j)}(z) + s(z)|^4} \mathbb{E} \frac{1}{|z + m_n^{(j)}(z)|^4}.
\] (6.12)

Inequalities (6.11), (6.12), Corollaries 7.23 and 5.14 together imply
\[
\mathbb{E} \frac{\varepsilon_{j\nu} |\varepsilon_{j\nu}|^4}{(z + m_n^{(j)}(z) + s(z)) (z + m_n^{(j)}(z) + s(z))} \leq \frac{C}{nv} \mathbb{E} \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z) + s(z)|^4}.
\] (6.13)

By Lemmas 7.14 and 7.5 we have for \( \nu = 1 \)
\[
\mathbb{E} \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z) + s(z)|^4} \leq \frac{C}{\sqrt{nv}|z^2 - 4|}.
\] (6.14)

By Corollary 7.17, inequality (7.73) with \( \alpha = 0 \) and \( \beta = 4 \) in the Appendix we have for \( \nu = 2, 3 \)
\[
\mathbb{E} \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z) + s(z)|^4} \leq \frac{C}{\sqrt{nv}|z^2 - 4|}. \] (6.15)

with some constant \( C > 0 \) depending on \( \mu_4 \) and \( D_0 \) only. Using that, by Lemma 7.13 for \( z \in \mathbb{G} \),
\[
\frac{1}{\sqrt{nv}|z^2 - 4|} \leq \frac{\sqrt{v}}{\sqrt{nv}|z^2 - 4|} \leq \frac{C}{\sqrt{nv}|z^2 - 4|}. \] (6.16)

we get from (6.13), (6.14) and (6.15) that for \( z \in \mathbb{G} \),
\[
|\mathbb{I}_1| \leq \frac{C}{(nv)^{3/2}|z^2 - 4|^{1/4}} \] (6.17)

with some constant \( C > 0 \) depending on \( \mu_4 \) and \( D_0 \) only.

### 6.1.2 Estimation of \( \mathbb{I}_2 \)

Using the representation (5.49), we write
\[
\mathbb{I}_2 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu}^2 R_{jj}}{(z + m_n^{(j)}(z))(z + s(z) + m_n(z))}. \] (6.18)
Furthermore we note that
\[ \tilde{\varepsilon}_j^2 = \varepsilon_{j2}^2 + \eta_j, \]  
(6.19)
where
\[ \eta_j = (\varepsilon_{j1} + \varepsilon_{j3})^2 + 2(\varepsilon_{j1} + \varepsilon_{j3})\varepsilon_{j2}. \]  
(6.20)

We now decompose \( \mathfrak{T}_2 \) as follows
\[ \mathfrak{T}_2 = \mathfrak{T}_{21} + \mathfrak{T}_{22} + \mathfrak{T}_{23}, \]  
(6.21)
where
\[ \mathfrak{T}_{21} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j2}^2 R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n(z))}, \]
\[ \mathfrak{T}_{22} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\eta_j R_{jj}}{(z + m^{(j)}(z))(z + s(z) + m_n^{(j)}(z))}, \]
\[ \mathfrak{T}_{23} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\eta_j R_{jj} \varepsilon_{j4}}{(z + m^{(j)}(z))(z + s(z) + m_n^{(j)}(z))(z + s(z) + m_n(z))}. \]  
(6.22)

Applying the Cauchy – Schwartz inequality, we obtain
\[ |\mathfrak{T}_{22}| \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{|\eta_j|^2}{|z + m^{(j)}(z)|^2 |z + s(z) + m_n^{(j)}(z)|^2} \mathbb{E} \frac{|\varepsilon_{j4}|^2}{|R_{jj}|^2}. \]  
(6.23)

In what follows we denote by \( C \) a generic constant depending on \( \mu_4 \) and \( D_0 \) only. We note that
\[ \mathbb{E} \{ |\eta_j|^2 \mathfrak{R}^{(j)} \} \leq C (\mathbb{E} \{ |\varepsilon_{j1}|^4 \mathfrak{R}^{(j)} \} + \mathbb{E} \{ |\varepsilon_{j3}|^4 \mathfrak{R}^{(j)} \}) \]
\[ + C \left( \mathbb{E} \{ |\varepsilon_{j1}|^4 \mathfrak{R}^{(j)} \} + \mathbb{E} \{ |\varepsilon_{j3}|^4 \mathfrak{R}^{(j)} \} \right)^{\frac{1}{2}} \mathbb{E} \frac{1}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^4 \mathfrak{I} \{ |\varepsilon_{j2}|^4 \mathfrak{R}^{(j)} \}. \]  
(6.24)

Using Lemmas 7.14, 7.15, 7.16 we get
\[ \mathbb{E} \{ |\eta_j|^2 \mathfrak{R}^{(j)} \} \leq \frac{C}{n^2} + \frac{C}{n^2} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^4 \]
\[ + \left( \frac{C}{n} + \frac{C}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^4 \right) \frac{1}{n} \mathbb{E} \text{Im} m_n^{(j)}(z). \]  
(6.25)
This inequality and Lemma 7.5 together imply
\[
\begin{align*}
\mathbb{E}^{|\frac{1}{2}} |\eta_j|^2 & \leq \frac{C}{n\sqrt{|z^2 - 4|}} \frac{1}{|z + m_n^j(z)|^2} \\
+ \frac{C}{n\sqrt{|z^2 - 4|}} \left( \frac{1}{n} \sum_{t \in T_j} \mathbb{E} |R_{ij^t}|^4 \right)^{\frac{1}{4}} \mathbb{E}^{|\frac{1}{4}} \frac{1}{|z + m_n^j(z)|^4} \\
+ \frac{C}{nv^2|z^2 - 4|^{\frac{1}{4}}} \left( 1 + \left( \frac{1}{n} \sum_{t \in T_j} \mathbb{E} |R_{ij^t}|^4 \right)^{\frac{1}{4}} \right) \mathbb{E}^{|\frac{1}{4}} \frac{1}{|z + m_n^j(z)|^4}
\end{align*}
\]
(6.26)

Inequality (6.26) and Corollary 5.14 together imply
\[
|\Sigma_{22}| \leq \frac{C}{n\sqrt{|z^2 - 4|}} + \frac{C}{n\sqrt{|z^2 - 4|^{\frac{1}{4}}}}.
\]  
(6.27)

Applying Lemma (7.13) for \( z \in G \), we get
\[
|\Sigma_{22}| \leq \frac{C}{n\sqrt{|z^2 - 4|^{\frac{1}{4}}}} \leq \frac{C}{nv^{\frac{3}{4}}}.
\]  
(6.28)

By Hölder’s inequality, we have
\[
|\Sigma_{23}| \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}^{|\frac{1}{2}} |\eta_j|^2 \frac{1}{|z + m_n^j(z)|^2} |z + s(z) + m_n^j(z)|^2 \\
\times \mathbb{E}^{|\frac{1}{4}} \frac{|\varepsilon_j|^4}{|z + s(z) + m_n(z)|^4} \mathbb{E}^{|\frac{1}{4}} |R_{ij^t}|^4.
\]  
(6.29)

Using now Lemmas 7.22, 7.13 and Corollary 5.14 we may write, for \( z \in G \),
\[
|\Sigma_{23}| \leq \frac{C}{n\sqrt{|z^2 - 4|^{\frac{1}{4}}}} \leq \frac{C}{nv^{\frac{3}{4}}}.
\]  
(6.30)

We continue now with \( \Sigma_{21} \). We represent it in the form
\[
\Sigma_{21} = H_1 + H_2,
\]  
(6.31)

where
\[
H_1 = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_j^2}{(z + m_n^j(z))^2(z + s(z) + m_n(z))},
\]
\[
H_2 = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_j^2(R_{ij} + \frac{1}{z + m_n^j})}{(z + m_n^j(z))(z + s(z) + m_n(z))}.
\]  
(6.32)
Furthermore, using the representation
\[ R_{jj} = -\frac{1}{z + m_n^{(j)}(z)} + \frac{1}{z + m_n^{(j)}(z)}(\varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3})R_{jj} \] (6.33)
(compare with (3.3)), we bound \( H_2 \) in the following way
\[ |H_2| \leq H_{21} + H_{22} + H_{23}, \] (6.34)
where
\[ H_{21} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{4|\varepsilon_{j1}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n(z)|}, \]
\[ H_{22} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{2|\varepsilon_{j2}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n(z)|}, \]
\[ H_{23} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{2|\varepsilon_{j3}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n(z)|}. \] (6.35)

Using inequality (7.104) in the Appendix and Hölder inequality, we get, for \( \nu = 1, 2, 3 \)
\[ H_{2\nu} \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{4|\varepsilon_{j\nu}|^3|R_{jj}|}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)|} + \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{4|\varepsilon_{j\nu}|^3|R_{jj}|^4}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)||z + s(z) + m_n(z)|} \leq C \frac{n}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\varepsilon_{j\nu}|^4}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)|^2} \leq C \frac{n^2}{n^2}. \] (6.36)

Applying Corollary 7.17 with \( \beta = \frac{4}{3} \) and \( \alpha = \frac{8}{3} \), we obtain, for \( z \in \mathbb{G} \), and for \( \nu = 1, 2, 3 \)
\[ \mathbb{E} \frac{|\varepsilon_{j2}|^4}{|z + m^{(j)}(z)|^2|z + s(z) + m_n^{(j)}(z)|^2} \leq C \frac{(nv)^{\frac{4}{3}}}{(nv)^{\frac{8}{3}}}. \] (6.37)

This yields together with Corollary 5.14 and inequality (6.36)
\[ H_2 \leq C \frac{(nv)^{\frac{4}{3}}}{(nv)^{\frac{8}{3}}}. \] (6.38)
Consider now $H_1$. Using the equality
\[
1 \over z + m_n(z) + s(z) = 1 \over z + 2s(z) - \Lambda_n(z) \over (z + 2s(z))(z + m_n(z) + s(z)) \tag{6.39}
\]
and
\[
\Lambda_n = \Lambda_n^{(j)} + \varepsilon_{j4}, \tag{6.40}
\]
we represent it in the form
\[
H_1 = H_{11} + H_{12} + H_{13}, \tag{6.41}
\]
where
\[
H_{11} = -\frac{1}{(z + s(z))^2} n \sum_{j=1}^{n} \mathbb{E} \varepsilon_{j2}^2 \over z + s(z) + m_n(z),
\]
\[
= -s^2(z) - \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \varepsilon_{j2}^2 \over z + s(z) + m_n(z),
\]
\[
H_{12} = -\frac{1}{(z + s(z))^2} n \sum_{j=1}^{n} \mathbb{E} \varepsilon_{j2}^2 \Lambda_n^{(j)} \over (z + m_n^{(j)}(z))^2(z + s(z) + m_n(z))
\]
\[
H_{13} = -\frac{1}{(z + s(z))^2} n \sum_{j=1}^{n} \mathbb{E} \varepsilon_{j2}^2 \Lambda_n^{(j)} \over (z + m_n^{(j)}(z))(z + s(z) + m_n(z)). \tag{6.42}
\]
In order to apply conditional independence, we write
\[
H_{11} = H_{111} + H_{112}, \tag{6.43}
\]
where
\[
H_{111} = -s^2(z) - \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \varepsilon_{j2}^2 \over z + m_n^{(j)}(z) + s(z),
\]
\[
H_{112} = s^2(z) - \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \varepsilon_{j2}^2 \varepsilon_{j4} \over (z + s(z) + m_n(z))(z + m_n^{(j)}(z) + s(z)). \tag{6.44}
\]
It is straightforward to check that
\[
\mathbb{E} \{ \varepsilon_{j2}^2 | \mathcal{M}^{(j)} \} = \frac{1}{n^2} \text{Tr} \left( R^{(j)} \right)^2 - \frac{1}{n^2} \sum_{i \in T_j} \left( R_{ii}^{(j)} \right)^2. \tag{6.45}
\]
Using equality (6.3) for \( m'_n(z) \) and the corresponding relation for \( m'(j)_n(z) \), we may write

\[ H_{111} = L_1 + L_2 + L_3 + L_4, \]

where

\[ L_1 = s^2(z) \frac{1}{n} \mathbb{E} \frac{m'_n(z)}{z + m_n(z) + s(z)}, \]

\[ L_2 = s^2(z) \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{1}{n} \sum_{t \in T_j} (R_{ij}^j)^2 \frac{1}{z + m_{n}(j)(z) + s(z)}, \]

\[ L_3 = s^2(z) \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{1}{n} ((m_n(j)(z))' - m'_n(z)) \frac{1}{z + m_n(j)(z) + s(z)}, \]

\[ L_4 = s^2(z) \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{1}{n} ((m_n(j)(z))' - m'_n(z)) \varepsilon_j \frac{1}{(z + m_n(z) + s(z))(z + m_{n}(j)(z) + s(z))}. \]

Using Lemmas 7.5, 7.18, 7.26, and Corollary 5.14 it is straightforward to check that

\[ |L_2| \leq \frac{C}{n \sqrt{|z^2 - 4|}}, \]

\[ |L_3| \leq \frac{C}{n^2 v^2 \sqrt{|z^2 - 4|}}, \]

\[ |L_4| \leq \frac{C}{n^3 v^3 |z^2 - 4|}. \]

Applying inequality (7.104), we may write

\[ |H_{12}| \leq \frac{C}{n} \sum_{j=1}^{n} \mathbb{E} \| \varepsilon_j \|^2 |\Lambda_n(j)| \frac{|\Lambda_n(j)|}{|z + m_n(j)(z)||z + m_{n}(j)(z) + s(z)|}. \]

Conditioning on \( m(j) \) and applying Lemma 7.15, Lemma 7.5, inequality (7.22), Corollary 5.14, and equality (6.40), we get

\[ |H_{12}| \leq \frac{C}{nv} \sum_{j=1}^{n} \mathbb{E} \frac{|\Lambda_n(j)|}{|z + m_n(j)(z)|} \leq \frac{C}{nv} \mathbb{E} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \]

By Lemma 7.24, we get

\[ |H_{12}| \leq \frac{C}{n^2 v^2}. \]
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Similar we get

\[ |H_{13}| \leq \frac{C}{n^2 v^2}. \]  

(6.54)

We rewrite now the equations (6.2) and (6.4) as follows, using the remainder term \( T_3 \), which is bounded by means of inequalities (6.17), (6.28), (6.30), (6.38).

\[ \mathbf{E} \Lambda_n(z) = \mathbf{E} m_n(z) - s(z) = -\left(1 - s^2(z)\right) \frac{m_n'(z)}{n} \frac{z + m_n(z) + s(z)}{z} + \mathcal{T}_3, \]  

(6.55)

where

\[ |\mathcal{T}_3| \leq \frac{C}{n \sqrt{v} \sqrt{|z^2 - 4|}} + \frac{C}{n^2 v^2 |z^2 - 4|^{1/4}}. \]  

(6.56)

Note that

\[ 1 - s^2(z) = s(z) \sqrt{z^2 - 4}. \]  

(6.57)

In (6.55) we estimate now the remaining quantity

\[ \mathcal{T}_4 = -\frac{s(z) \sqrt{z^2 - 4}}{n} \mathbf{E} \frac{m_n'(z)}{z + m_n(z) + s(z)}. \]  

(6.58)

**6.2 Estimation of \( \mathcal{T}_4 \)**

Using that \( \Lambda_n = m_n(z) - s(z) \) we rewrite \( \mathcal{T}_4 \) as

\[ \mathcal{T}_4 = \mathcal{T}_{41} + \mathcal{T}_{42} + \mathcal{T}_{43}, \]  

(6.59)

where

\[ \mathcal{T}_{41} = -\frac{s(z)s'(z)}{n}, \]

\[ \mathcal{T}_{42} = \frac{s(z) \sqrt{z^2 - 4}}{n} \mathbf{E} \frac{m_n'(z) - s'(z)}{z + m_n(z) + s(z)}, \]

\[ \mathcal{T}_{43} = \frac{s(z)}{n} \mathbf{E} \left( m_n'(z) - s'(z) \right) \Lambda_n \frac{z + m_n(z) + s(z)}{z}. \]  

(6.60)

**6.2.1 Estimation of \( \mathcal{T}_{42} \)**

First we investigate \( m_n'(z) \). The following equality holds

\[ m_n'(z) = \frac{1}{n} \text{Tr } R^2 = -\frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj} = \frac{s(z)}{n} \sum_{j=1}^{n} \varepsilon_{j4} + D_1, \]  

(6.61)
where

\[ D_1 = \sum_{j=1}^{n} \varepsilon_{jj}(R_{jj} - s(z)). \] (6.62)

Using equality (7.81), we may write

\[ m'_n(z) = \frac{s^2(z)}{n} \sum_{j=1}^{n} (1 + \frac{1}{n} \sum_{l,k \in T_j} X_{jl}X_{jk}[(R^{(j)})_l^2]_{lk}) + D_1. \] (6.63)

Denote by

\[ \beta_{j1} = \frac{1}{n} \sum_{l \in T_j} [(R^{(j)})_l^2]_{ll} - \frac{1}{n} \sum_{l=1}^{n} [(R)_l^2]_{ll} = \frac{1}{n} \sum_{l \in T_j} [(R^{(j)})_l^2]_{ll} - m'_n(z), \]

\[ = \frac{1}{n} \frac{d}{dz} (\text{Tr} R - \text{Tr} R^{(j)}), \]

\[ \beta_{j2} = \frac{1}{n} \sum_{l \in T_j} (X_{jl}^2 - 1)[(R^{(j)})_l^2]_{ll}, \]

\[ \beta_{j3} = \frac{1}{n} \sum_{l \neq k \in T_j} X_{jl}X_{jk}[(R^{(j)})_l^2]_{lk}. \] (6.64)

Using these notation we may write

\[ m'_n(z) = \frac{s^2(z)}{n} (1 + m'_n(z)) + \frac{s^2(z)}{n} \sum_{j=1}^{n} (\beta_{j1} + \beta_{j2} + \beta_{j3}) + D_1. \] (6.65)

Solving this equation with respect to \( m'_n(z) \) we obtain

\[ m'_n(z) = \frac{s^2(z)}{1 - s^2(z)} + \frac{1}{1 - s^2(z)} (D_1 + D_2), \] (6.66)

where

\[ D_2 = \frac{s^2(z)}{n} \sum_{j=1}^{n} (\beta_{j1} + \beta_{j2} + \beta_{j3}). \] (6.67)

Note that for the semi-circular law

\[ \frac{s^2(z)}{1 - s^2(z)} = \frac{s^2(z)}{1 + \frac{s(z)}{z + 2s(z)}} = -\frac{s(z)}{z + 2s(z)} = s'(z). \] (6.68)
Applying this relation we rewrite equality (6.66) as

\[ m'_n(z) - s'(z) = \frac{1}{s(z)(z + 2s(z))}(D_1 + D_2). \]  

(6.69)

Using the last equality, we may represent \( \mathfrak{T}_{42} \) now as follows

\[ \mathfrak{T}_{42} = \mathfrak{T}_{421} + \mathfrak{T}_{422}, \]  

(6.70)

where

\[ \mathfrak{T}_{421} = \frac{1}{n} E \frac{D_1}{z + m_n(z) + s(z)}, \]

\[ \mathfrak{T}_{422} = \frac{1}{n} E \frac{D_2}{z + m_n(z) + s(z)}. \]

(6.71)

Recall that, by (6.62),

\[ \mathfrak{T}_{421} = \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_j (R_{jj} - s(z))}{(z + s(z) + m_n(z))}. \]  

(6.72)

Applying the Cauchy – Schwartz inequality, we get for \( z \in G \),

\[ |\mathfrak{T}_{421}| \leq \frac{1}{n} \sum_{j=1}^{n} E |R_{jj} - s(z)|^2 E \left| \frac{\varepsilon_j}{z + s(z) + m_n(z)} \right|^2. \]  

(6.73)

Using Corollary 7.23 and Corollary 5.14 we get

\[ |\mathfrak{T}_{421}| \leq \frac{C}{n^{\frac{3}{2}} v^{\frac{3}{2}}}. \]  

(6.74)

6.2.2 Estimation of \( \mathfrak{T}_{422} \)

We represent now \( \mathfrak{T}_{421} \) in the form

\[ \mathfrak{T}_{422} = \mathfrak{T}_{51} + \mathfrak{T}_{52} + \mathfrak{T}_{53}, \]  

(6.75)

where

\[ \mathfrak{T}_{5\nu} = \frac{1}{n^2} \sum_{j=1}^{n} E \frac{\beta_{j\nu}}{z + m_n(z) + s(z)}, \quad \text{for} \quad \nu = 1, 2, 3. \]  

(6.76)

At first we investigate \( \mathfrak{T}_{53} \). Note that, by Lemma 7.26

\[ |\beta_{j1}| \leq \frac{C}{nv^2}. \]  

(6.77)
Therefore, for $z \in \mathbb{G}$, using Lemma 7.5, inequality (7.23), and Lemma 7.13,

$$|\Sigma_{51}| \leq \frac{C}{n^2 v^2 \sqrt{|z^2 - 4|}} \leq \frac{C}{n^2 v^2 |z^2 - 4|^\frac{1}{4}}.$$  \hspace{1cm} (6.78)

Furthermore, we consider the quantity $\Sigma_{5\nu}$, for $\nu = 2, 3$. Applying the Cauchy-Schwartz inequality and inequality (7.104) in the Appendix as well, we get

$$|\Sigma_{5\nu}| \leq \frac{C}{n^2} \sum_{j=1}^{n} \mathbb{E} \frac{|\beta_{j\nu}|^2}{|z + m_n^j(z) + s(z)|^2}. \hspace{1cm} (6.79)$$

By Lemma 7.25 together with Lemma 7.5 in the Appendix, we obtain

$$\mathbb{E} \frac{|\beta_{j\nu}|^2}{|z + m_n^j(z) + s(z)|^2} \leq \frac{C}{n^\frac{1}{2} v^\frac{1}{2} |z^2 - 4|^\frac{1}{4}}. \hspace{1cm} (6.80)$$

This implies that

$$|\Sigma_{5\nu}| \leq \frac{C}{n^\frac{1}{2} v^\frac{1}{2} |z^2 - 4|^\frac{1}{4}}. \hspace{1cm} (6.81)$$

Inequalities (6.78) and (6.81) yield

$$|\Sigma_{422}| \leq \frac{C}{n^\frac{1}{2} v^\frac{1}{2} |z^2 - 4|^\frac{1}{4}}. \hspace{1cm} (6.82)$$

Combining (6.74) and (6.83), we get, for $z \in \mathbb{G}$,

$$|\Sigma_{42}| \leq \frac{C}{n^\frac{1}{2} v^\frac{1}{2} |z^2 - 4|^\frac{1}{4}}. \hspace{1cm} (6.83)$$

### 6.2.3 Estimation of $\Sigma_{43}$

Recall that

$$\Sigma_{43} = \frac{s(z)}{n} \mathbb{E} \frac{(m_n'(z) - s'(z)) \Lambda_n}{z + m_n(z) + s(z)}. \hspace{1cm} (6.84)$$

Applying equality (6.69), we obtain

$$\Sigma_{43} = \Sigma_{431} + \Sigma_{432}. \hspace{1cm} (6.85)$$
where

\[ \mathfrak{T}_{431} = \frac{1}{n(z + 2s(z))} E \frac{D_1 \Lambda_n}{z + m_n(z) + s(z)}, \quad (6.86) \]

\[ \mathfrak{T}_{432} = \frac{1}{n(z + 2s(z))} E \frac{D_2 \Lambda_n}{z + m_n(z) + s(z)}. \quad (6.87) \]

Applying the Cauchy–Schwarz inequality, we get

\[ |\mathfrak{T}_{431}| \leq \frac{1}{n(z + 2s(z))} E^\frac{1}{2} |D_1|^2 |z + m_n(z) + s(z)|^2 E^\frac{1}{2} |\Lambda_n|^2. \quad (6.88) \]

By definition of \( D_1 \) and Lemma 7.24, we get

\[ |\mathfrak{T}_{431}| \leq \frac{4}{n^2 v^2 |z^2 - 4|^\frac{3}{2}} \sum_{j=1}^{n} E^\frac{1}{4} |\varepsilon_j|^4 |z + m_n(z) + s(z)|^4 E^\frac{1}{4} |R_{jj} - s(z)|^4. \quad (6.89) \]

Applying now Corollary 5.14 and Lemma 7.22, we get

\[ |\mathfrak{T}_{431}| \leq \frac{4}{n^3 v^2 |z^2 - 4|^\frac{3}{2}}. \quad (6.90) \]

For \( z \in \mathbb{G} \) this yields

\[ |\mathfrak{T}_{431}| \leq \frac{4}{n^2 v^2 |z^2 - 4|^\frac{1}{2}}. \quad (6.91) \]

Applying again the Cauchy–Schwarz inequality, we get for \( \mathfrak{T}_{432} \) accordingly

\[ |\mathfrak{T}_{432}| \leq \frac{C}{n |z^2 - 4|^\frac{3}{2}} E^\frac{1}{2} |D_2|^2 E^\frac{1}{2} |\Lambda_n|^2. \quad (6.92) \]

By Lemma 7.24, we have

\[ |\mathfrak{T}_{432}| \leq \frac{C}{n^2 v^2 |z^2 - 4|^\frac{3}{2}} E^\frac{1}{2} |D_2|^2. \quad (6.93) \]

By definition of \( D_2 \),

\[ E|D_2|^2 \leq \frac{1}{n} \sum_{j=1}^{n} (E|\beta_{j1}|^2 + E|\beta_{j2}|^2 + E|\beta_{j3}|^2). \quad (6.94) \]

Applying Lemmas 7.25 with \( \nu = 2, 3 \), and 7.26, we get

\[ E|D_2|^2 \leq \frac{C}{n^2 v^4} + \frac{C}{nv^3}. \quad (6.95) \]
Inequalities (6.93) and (6.95) together imply, for $z \in \mathbb{G}$,

$$|\mathcal{F}_{432}| \leq \frac{C}{n^3 v^3 |z^2 - 4|^\frac{1}{2}} + \frac{C}{n^\frac{2}{3} v^\frac{4}{3} |z^2 - 4|^\frac{1}{2}} \leq \frac{C}{n^\frac{4}{3} v^\frac{4}{3} |z^2 - 4|^\frac{1}{2}}.$$  

(6.96)

Finally we observe that

$$s'(z) = -\frac{s(z)}{\sqrt{z^2 - 4}}$$  

(6.97)

and, therefore

$$|\mathcal{F}_{41}| \leq \frac{C}{n |z^2 - 4|^\frac{1}{2}}.$$  

(6.98)

For $z \in \mathbb{G}$ we may rewrite it

$$|\mathcal{F}_{41}| \leq \frac{C}{n \sqrt{v}}.$$  

(6.99)

Combining now relations (6.55), (6.41), (6.38), (6.78), (6.83), (6.99), we get for $z \in \mathbb{G}$,

$$|E \Lambda_n| \leq \frac{C}{nv^\frac{1}{3}} + \frac{C}{n^\frac{2}{3} v^\frac{4}{3} |z^2 - 4|^\frac{1}{2}}.$$  

(6.100)

The last inequality completes the proof of Theorem 1.3.

### 7 Appendix

#### 7.1 Rosenthal’s and Burkholder’s inequalities

In this subsection we state the Rosenthal and Burkholder inequalities, starting with Rosenthal’s inequality. Let $\xi_1, \ldots, \xi_n$ be independent random variables with $E \xi_j = 0$, $E \xi_j^2 = 1$ and for $p \geq 1$ $E|\xi_j|^p \leq \mu_p$ for $j = 1, \ldots, n$.

**Lemma 7.1.** (Rosenthal’s inequality)

There exists an absolute constant $C_1$ such that

$$E \left| \sum_{j=1}^{n} a_j \xi_j \right|^p \leq C_1^p p^p \left( \sum_{j=1}^{p} |a_j|^2 \right)^\frac{p}{2} + \mu_p \sum_{j=1}^{p} |a_j|^p.$$  

(7.1)

**Proof.** For a proof see [19] and [16].

Let $\xi_1, \ldots, \xi_n$ be a martingale-difference with respect to the $\sigma$-algebras $\mathcal{M}_j = \sigma(\xi_1, \ldots, \xi_{j-1})$. Assume that $E \xi_j^2 = 1$ and $E|\xi_j|^p < \infty$. 

Lemma 7.2. (Burkholder’s inequality) There exist an absolute constant $C_2$ such that
\[
E|\sum_{j=1}^{n} \xi_j|^p \leq C_2^p (\frac{n}{p} E(\sum_{k=1}^{n} E(\xi_k^2 | \mathcal{M}_{k-1}))^{\frac{p}{2}} + \sum_{k=1}^{p} E|\xi_k|^p).
\] (7.2)

Proof. For a proof of this inequality see [5] and [15].

We rewrite the Burkholder inequality for quadratic forms in independent random variables. Let $\zeta_1, \ldots, \zeta_n$ be independent random variables such that $E\zeta_j = 0$, $E|\zeta_j|^2 = 1$ and $E|\zeta_j|^p \leq \mu_p$. Let $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$.

Consider the quadratic form
\[
Q = \sum_{1 \leq j \neq k \leq n} a_{jk} \zeta_j \zeta_k.
\] (7.3)

Lemma 7.3. There exists an absolute constant $C_2$ such that
\[
E|Q|^p \leq C_2^p \left( E(\sum_{j=2}^{n} (\sum_{k=1}^{j-1} a_{jk} \zeta_k)^2)^{\frac{p}{2}} + \mu_p \sum_{j=2}^{n} E(\sum_{k=1}^{j-1} a_{jk} \zeta_k)^p \right).
\] (7.4)

Proof. Introduce the random variables, for $j = 2, \ldots, n$,
\[
\xi_j = \zeta_j \sum_{k=1}^{j-1} a_{jk} \zeta_k.
\] (7.5)

It is straightforward to check that
\[
E\{\xi_j | \mathcal{M}_{j-1}\} = 0,
\] (7.6)

and that $\xi_j$ are $\mathcal{M}_j$ measurable. That means that $\xi_1, \ldots, \xi_n$ are martingale-differences. We may write
\[
Q = 2 \sum_{j=2}^{n} \xi_j.
\] (7.7)

Applying now Lemma 7.2 and using
\[
E\{|\xi_j|^2 | \mathcal{M}_{j-1}\} = \left( \sum_{k=1}^{j-1} a_{jk} \zeta_k \right)^2 E\zeta_j^2,
\]
\[
E|\xi_j|^p = E|\zeta_j|^p E|\sum_{k=1}^{j-1} a_{jk} \zeta_j|^p,
\] (7.8)

we get the claim. Thus, Lemma 7.3 is proved.
We prove as well the following simple Lemma

**Lemma 7.4.** Let \( t > r \geq 1 \) and \( a, b > 0 \). Any \( x > 0 \) satisfying the inequality

\[
x^t \leq a + bx^r
\]

is explicitly bounded as follows

\[
x^t \leq ea + \left( \frac{2t - r}{t - r} \right) \frac{x}{t} b^r.
\]

**Proof.** First assume that \( x \leq a \frac{t}{r} \). Then inequality (7.10) holds. If \( x \geq a \frac{t}{r} \), then according to inequality (7.9)

\[
x^t - r \leq a \frac{t}{r} + b,
\]

or

\[
x^t \leq (a \frac{t}{r} + b)^r.
\]

Using that for any \( \alpha > 0 \) and \( a > 0, b > 0 \)

\[
(a + b)\alpha \leq (a + \frac{a}{\alpha})\alpha + (b + ab)\alpha \leq ea\alpha + (1 + \alpha)\alpha b\alpha,
\]

we get the claim.

In what follows we prove several lemmas about the resolvent matrices. Recall the equation, for \( j \in \mathbb{T}_J \), (compare with (3.3))

\[
R^{(j)}_{jj} = -\frac{1}{z + m_n^{(j)}(z)} + \frac{1}{z + m_n^{(j)}(z)} \varepsilon^{(j)}_{j} R^{(j)}_{jj},
\]

where

\[
\varepsilon^{(j)}_{j1} = \frac{X_{jj}}{\sqrt{n}}, \quad \varepsilon^{(j)}_{j2} = \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} X_{jl} X_{jk} R^{(j)}_{kl},
\]

\[
\varepsilon^{(j)}_{j3} = \frac{1}{n} \sum_{l \in \mathbb{T}_j} (X_{jl}^2 - 1) R^{(j,j)}_{ll}, \quad \varepsilon^{(j)}_{j4} = m_n^{(j)}(z) - m_n^{(j,j)}(z).
\]

Summing these equations for \( j \in \mathbb{T}_J \), we get

\[
m_n^{(j)}(z) = -\frac{n - |J|}{n(z + m_n^{(j)}(z))} + \frac{T_n^{(j)}}{z + m_n^{(j)}(z)},
\]
where
\[ T_n^{(J)} = \frac{1}{n} \sum_{j=1}^{n} e_j^{(J)} R_{jj}^{(J)}. \] (7.17)

Note that
\[ \frac{1}{z + m_n^{(J)}(z)} = \frac{1}{z + s(z)} - \frac{m_n^{(J)}(z) - s(z)}{(s(z) + z)(z + m_n^{(J)}(z))} = -s(z) + \frac{s(z)\Lambda_n^{(J)}(z)}{z + m_n^{(J)}(z)}, \] (7.18)
where
\[ \Lambda_n^{(J)} = \Lambda_n^{(J)}(z) = m_n^{(J)}(z) - s(z). \] (7.19)

Equalities (7.16) and (7.18) together imply
\[ \Lambda_n^{(J)} = -s(z)\Lambda_n^{(J)} + \frac{T_n^{(J)}}{z + m_n^{(J)}(z)} + \frac{|J|}{n(z + m_n^{(J)}(z))}. \] (7.20)

Solving this with respect to \( \Lambda_n^{(J)} \), we get
\[ \Lambda_n^{(J)} = \frac{T_n^{(J)}}{z + m_n^{(J)}(z) + s(z)} + \frac{|J|}{n(z + m_n^{(J)}(z) + s(z))}. \] (7.21)

**Lemma 7.5.** Assuming the conditions of Theorem 1.1, there exists an absolute constant \( c_0 > 0 \) such that for \( J \subset \mathbb{T} \),
\[ |z + m_n^{(J)}(z) + s(z)| \geq \text{Im} m_n^{(J)}(z), \] (7.22)
moreover, for any \( z \in \mathbb{G} \)
\[ |z + m_n^{(J)}(z) + s(z)| \geq c_0 \sqrt{|z^2 - 4|}. \] (7.23)

**Proof.** Firstly note that \( \text{Im} (z + s(z)) \geq 0 \) and \( \text{Im} m_n^{(J)}(z) \geq 0 \). Therefore
\[ |z + m_n^{(J)}(z) + s(z)| \geq \text{Im} (z + m_n^{(J)}(z) + s(z)) \geq \text{Im} m_n^{(J)}(z). \] (7.24)
Furthermore,
\[ |z + m_n^{(J)}(z) + s(z)| \geq \text{Im} (z + s(z)) = \frac{1}{2} \text{Im} (z + \sqrt{z^2 - 4}) \geq \frac{1}{2} \text{Im} \sqrt{z^2 - 4}. \] (7.25)

Note that for \( z \in \mathbb{G} \)
\[ \text{Re} (z^2 - 4) < 0. \] (7.26)
Therefore,
\[ \text{Im} \sqrt{z^2 - 4} \geq \frac{\sqrt{2}}{2} \sqrt{|z^2 - 4|}. \] (7.27)
Thus Lemma 7.5 is proved. □
Lemma 7.6. For any $z = u + iv$ with $v > 0$ and for any $J \subset \mathbb{T}$, we have

$$\frac{1}{n} \sum_{l,k \in T_J} |R_{kl}^{(j)}|^2 \leq v^{-1} \text{Im} m_n^{(j)}(z).$$  (7.28)

For any $l \in T_J$

$$\sum_{k \in T_J} |R_{kl}^{(j)}|^2 \leq v^{-1} \text{Im} R_{ll}^{(j)}.$$  (7.29)

and

$$\sum_{k \in T_J} |[(R^{(j)})^2]_{kl}|^2 \leq v^{-3} \text{Im} m_n^{(j)}(z).$$  (7.30)

Moreover, for any $J \subset T$ and for any $l \in T_J$ we have

$$\frac{1}{n} \sum_{l \in T_J} |[(R^{(j)})^2]_{ll}|^2 \leq v^{-3} \text{Im} m_n^{(j)}(z),$$  (7.31)

and, for any $p \geq 1$

$$\frac{1}{n} \sum_{l \in T_J} |[(R^{(j)})^2]_{ll}|^p \leq v^{-p} \frac{1}{n} \sum_{l \in T_J} \text{Im}^p R_{ll}^{(j)}.$$  (7.32)

Finally,

$$\frac{1}{n} \sum_{l,k \in T_J} |[(R^{(j)})^2]_{lk}|^2 \leq v^{-3} \text{Im} m_n^{(j)}(z),$$  (7.33)

and

$$\frac{1}{n} \sum_{l,k \in T_J} |[(R^{(j)})^2]_{lk}|^{2p} \leq v^{-3p} \frac{1}{n} \sum_{l \in T_J} \text{Im}^p R_{ll}^{(j)},$$  (7.34)

We have as well

$$\frac{1}{n^2} \sum_{l,k \in T_J} |[(R^{(j)})^2]_{lk}|^{2p} \leq v^{-2p} \left( \frac{1}{n} \sum_{l \in T_J} \text{Im}^p R_{ll}^{(j)} \right)^2.$$  (7.35)

Proof. For $l \in \mathbb{T}_J$ let us denote by $\lambda_l^{(j)}$ for $l \in \mathbb{T}_J$ eigenvalues of matrix $W^{(j)}$. Then we may write (compare (7.40))

$$\frac{1}{n} \sum_{l,k \in T_J} |R_{kl}^{(j)}|^2 \leq \frac{1}{n} \sum_{l \in T_J} \frac{1}{|\lambda_l^{(j)} - z|^2}.$$  (7.36)
Note that, for any $x \in \mathbb{R}$

$$\text{Im} \frac{1}{x - z} = \frac{v}{|x - z|^2}.$$  \hfill (7.37)

We may write

$$\frac{1}{|\lambda_l^{(j)} - z|^2} = v^{-1} \text{Im} \frac{1}{\lambda_l^{(j)} - z}$$  \hfill (7.38)

and

$$\frac{1}{n} \sum_{l,k \in T_J} |R_{lk}^{(j)}|^2 \leq v^{-1} \text{Im} \left( \frac{1}{n} \sum_{l \in T_J} \frac{1}{\lambda_l^{(j)} - z} \right) = v^{-1} \text{Im} m_n^{(j)}(z).$$  \hfill (7.39)

So, inequality (7.28) is proved. Let denote now by $u_l^{(j)} = (u_{lk}^{(j)})_{k \in T_J}$ the eigenvector of matrix $W^{(j)}$ corresponding to the eigenvalue $\lambda_l^{(j)}$. Using this notation we may write

$$R_{lk}^{(j)} = \sum_{q \in T_J} \frac{1}{\lambda_q^{(j)} - z} u_{lq}^{(j)} u_{kq}^{(j)}.$$  \hfill (7.40)

It is straightforward to check the following inequality

$$\sum_{k \in T_J} |R_{kl}^{(j)}|^2 \leq \sum_{q \in T_J} \frac{1}{|\lambda_q^{(j)} - z|^2} |u_{lq}^{(j)}|^2$$

$$= v^{-1} \text{Im} \left( \sum_{q \in T_J} \frac{1}{\lambda_q^{(j)} - z} |u_{lq}^{(j)}|^2 \right) = v^{-1} \text{Im} R_{ll}^{(j)}.$$  \hfill (7.41)

Thus, inequality (7.29) is proved. Similar we get

$$\sum_{k \in T_J} [(R^{(j)})^2]_{kl} \leq \sum_{q \in T_J} \frac{1}{|\lambda_q^{(j)} - z|^4} |u_{lq}^{(j)}|^2 \leq v^{-3} \text{Im} R_{ll}^{(j)}.$$  \hfill (7.42)

This proves inequality (7.30). To prove inequality (7.31) we observe that

$$|(R^{(j)})^2|_l \leq \sum_{k \in T_J} |R_{lk}^{(j)}|^2.$$  \hfill (7.43)

This inequality implies

$$\frac{1}{n} \sum_{l \in T_J} [(R^{(j)})^2]_l \leq \frac{1}{n} \sum_{l \in T_J} \left( \sum_{k \in T_J} |R_{lk}^{(j)}|^2 \right)^2.$$  \hfill (7.44)
Applying now inequality (7.29), we get

\[
\frac{1}{n} \sum_{l \in T_J} |(R^{(j)})^2|_{ll}|^2 \leq v^{-2} \frac{1}{n} \sum_{l \in T_J} \text{Im}^2 R^{(j)}_{ll}.
\] (7.45)

This leads (using \(|R^{(j)}_{ll}| \leq v^{-1}\)) to the following bound

\[
\frac{1}{n} \sum_{l \in T_J} |(R^{(j)})^2|_{ll}|^2 \leq v^{-3} \frac{1}{n} \sum_{l \in T_J} \text{Im} R^{(j)}_{ll} = v^{-3} \text{Im} m^{(j)}_n(z).
\] (7.46)

Thus inequality (7.31) is proved. Furthermore, applying inequality (7.43), we may write

\[
\frac{1}{n} \sum_{l \in T_J} |(R^{(j)})^2|_{ll}|^4 \leq \frac{1}{n} \sum_{l \in T_J} \left( \sum_{k \in T_J} |R^{(j)}_{lk}|^4 \right).
\] (7.47)

Applying (7.29), this inequality yields

\[
\frac{1}{n} \sum_{l \in T_J} |(R^{(j)})^2|_{ll}|^4 \leq v^{-4} \frac{1}{n} \sum_{l \in T_J} \text{Im}^4 R^{(j)}_{ll}.
\] (7.48)

The last inequality proves inequality (7.32). Note that

\[
\frac{1}{n} \sum_{l,k \in T_J} |(R^{(j)})^2|_{lk}|^2 \leq \frac{1}{n} \text{Tr} |R^{(j)}|^4 = \frac{1}{n} \sum_{l \in T_J} \frac{1}{|\lambda^{(j)}_l - z|^4}
\leq v^{-3} \text{Im} \frac{1}{n} \sum_{l \in T_J} \frac{1}{|\lambda^{(j)}_l - z|} = v^{-3} \text{Im} m^{(j)}_n(z).
\] (7.49)

Thus, inequality (7.33) is proved. To finish we note that

\[
\frac{1}{n} \sum_{l,k \in T_J} |(R^{(j)})^2|_{lk}|^4 \leq \frac{1}{n} \sum_{l \in T_J} \left( \sum_{k \in T_J} |(R^{(j)})^2|_{lk}|^2 \right)^2.
\] (7.50)

Applying inequality (7.30), we get

\[
\frac{1}{n} \sum_{l,k \in T_J} |(R^{(j)})^2|_{lk}|^4 \leq v^{-6} \frac{1}{n} \sum_{l \in T_J} (\text{Im} R^{(j)}_{ll})^2.
\] (7.51)

To prove inequality (7.35), we note

\[
|((R^{(j)})^2|_{lk}|^2 \leq (\sum_{q \in T_j} |R^{(j)}_{lq}|^2)(\sum_{q \in T_j} |R^{(j)}_{kq}|^2).
\] (7.52)
This inequality implies
\[
\frac{1}{n^2} \sum_{l,k \in \mathbb{T}_J} |(R^{(J)})^2_{lk}|^{2p} \leq \left( \frac{1}{n} \sum_{l,k \in \mathbb{T}_J} \left( \sum_{q \in \mathbb{T}_J} |R^{(J)}_{lj}|^2 \right)^p \right)^2.
\] (7.53)

Applying inequality (7.28), we get the claim. Thus, Lemma 7.6 is proved. \(\square\)

**Lemma 7.7.** For any \( s \geq 1 \), and for any \( z = u + iv \) and for any \( J \subset \mathbb{T} \), and \( j \in \mathbb{T}_J \),
\[
|R^{(J)}_{jj}(u + iv/s)| \leq s|R^{(J)}_{jj}(u + iv)|.
\] (7.54)

**Proof.** See Lemma 3.4 in [17]. For the readers convenience we include the short argument here. Note that, for any \( j \in \mathbb{T}_J \),
\[
\left| \frac{d}{dv} \log R^{(J)}_{jj}(u + iv) \right| \leq \frac{1}{|R^{(J)}_{jj}(u + iv)|} \left| \frac{d}{dv} R^{(J)}_{jj}(u + iv) \right|.
\] (7.55)

Furthermore,
\[
\frac{d}{dv} R^{(J)}_{jj}(u + iv) = [(R^{(J)})^2]|_{jj}(u + iv) \quad (7.56)
\]
and
\[
|[(R^{(J)})^2]|_{jj}(u + iv)| \leq v^{-1} \text{Im} R^{(J)}_{jj}. \quad (7.57)
\]

From here it follows that
\[
\left| \frac{d}{dv} \log R^{(J)}_{jj}(u + iv) \right| \leq v^{-1}. \quad (7.58)
\]

We may write now
\[
|\log R^{(J)}_{jj}(u + iv) - \log R^{(J)}_{jj}(u + iv/s)| \leq \int_{v/s}^v \frac{du}{u} = \log s. \quad (7.59)
\]

The last inequality yields the claim. Thus Lemma 7.7 follows. \(\square\)

**Lemma 7.8.** Assuming the conditions of Theorem 1.1, we get
\[
E|\varepsilon_{j1}|^2 \leq \frac{C}{n}.
\]

**Proof.** The proof follows immediately from the definition of \( \varepsilon_{j1} \) and conditions of Theorem 1.1. \(\square\)
7.1.1  Some Auxiliary Bounds for Resolvent Matrices for $\Im z = 4$

We need the bound for the $\varepsilon_{j\nu}$, and $\eta_j$ for $V = 4$.

**Lemma 7.9.** Assuming the conditions of Theorem 1.1, we get
\[
\mathbb{E}|\varepsilon_{j2}|^2 \leq \frac{C}{n}.
\]

*Proof.* Conditioning on $\mathfrak{M}^{(j)}$, we get
\[
\mathbb{E}|\varepsilon_{j2}|^2 \leq n^{-2} \sum_{k,l \in T_j} \mathbb{E}|R_{kl}^{(j)}|^2.
\]

(7.60)

Applying now Lemma 7.6, inequality (7.29), we get with $\Im R_{kl}^{(j)} \leq \frac{1}{4}$,
\[
\mathbb{E}|\varepsilon_{j2}|^2 \leq \frac{1}{4} n^{-2} \sum_{l \in T_j} \Im R_{ll}^{(j)} \leq \frac{1}{16n}.
\]

Thus Lemma 7.9 is proved.

**Lemma 7.10.** Assuming the conditions of Theorem 1.1, we get
\[
\mathbb{E}|\varepsilon_{j3}|^2 \leq \frac{C}{n}.
\]

*Proof.* Conditioning on $\mathfrak{M}^{(j)}$, we obtain as above
\[
\mathbb{E}|\varepsilon_{j3}|^2 \leq \frac{\mu_4}{n^2} \sum_{l \in T_j} |R_{ll}^{(j)}|^2 \leq \frac{\mu_4}{16n}.
\]

Thus Lemma 7.10 is proved.

**Lemma 7.11.** Assuming the conditions of Theorem 1.1, we get
\[
\mathbb{E}|\eta_j|^2 \leq \frac{C}{n}.
\]

*Proof.* The proof is similar to proof of Lemma 7.9. We need to use that $|(R_{kl}^{(j)})^2|_{kl} \leq V^{-2} = \frac{1}{16}$ and $\sum_{l \in T_j} |(R_{kl}^{(j)})^2|_{kl}^2 \leq V^{-4}$.

**Lemma 7.12.** Assuming the conditions of Theorem 1.1, we get
\[
\mathbb{E}|\varepsilon_{j4}|^q \leq \frac{C^q}{n^q},
\]

$k$, \[\]
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Proof. The result follows immediately from the bound

$$|\varepsilon_{j4}| \leq \frac{1}{nv}, \ a. \ s.$$ 

See for instance [10], Lemma 3.3.

\[\] 7.2 Some auxiliary bounds for resolvent matrices for \( z \in G \)

Introduce now the region

\[ G := \{ z = u + iv \in \mathbb{C}^+ : u \in \mathbb{J}_\varepsilon, v \geq v_0/\sqrt{\gamma} \}, \quad \text{where} \quad v_0 = A_0 n^{-1}, \quad (7.61) \]

\[ \mathbb{J}_\varepsilon = [-2 + \varepsilon, 2 - \varepsilon], \quad \varepsilon := c_1 n^{-\frac{2}{3}}, \quad \gamma = \gamma(u) = \min\{2 - u, 2 + u\}. \]

In the next lemma we state some useful inequalities for the region \( G \).

Lemma 7.13. For any \( z \in G \) we have

$$|z^2 - 4| \geq 2 \max\{\gamma, v\}, \quad nv\sqrt{|z^2 - 4|} \geq 2A_0. \quad (7.62)$$

Proof. We observe that

$$|z^2 - 4| = |z - 2||z + 2| \geq 2\sqrt{\gamma^2 + v^2}. \quad (7.63)$$

This inequality proves the Lemma.

Lemma 7.14. Assuming the conditions of Theorem 1.1 there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

$$E\{|\varepsilon_{j1}|^4|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n^2}. \quad (7.64)$$

Proof. The result follows immediately from the definition of \( \varepsilon_{j1} \).

Lemma 7.15. Assuming the conditions of Theorem 1.1 there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

$$E\{|\varepsilon_{j2}|^2|\mathfrak{M}^{(j)}\} \leq \frac{C}{nv} \text{Im} m_n^{(j)}(z), \quad (7.65)$$

and

$$E\{|\varepsilon_{j2}|^4|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4^2}{n^2v^2} \text{Im}^2 m_n^{(j)}(z). \quad (7.66)$$
Proof. Note that r.v.’s $X_{jl}$ for $l \in T_j$ are independent of $\mathfrak{M}^{(j)}$ and that for $l, k \in T_j$ $R_{lk}^{(j)}$ are measurable with respect to $\mathfrak{M}^{(j)}$. This implies that $\varepsilon_{j2}$ is a quadratic form with coefficients $R_{lk}^{(j)}$ independent of $X_{jl}$. Thus its variance and fourth moment are easily available.

$$E\{[\varepsilon_{j2}]^2|\mathfrak{M}^{(j)}\} = \frac{1}{n^2} \sum_{l \neq k \in T_j} |R_{lk}^{(j)}|^2 \leq \frac{1}{n^2} \text{Tr} |R^{(j)}|^2. \quad (7.67)$$

Here we use the notation $|A|^2 = AA^*$ for any matrix $A$. Applying Lemma 7.6, inequality (7.28), we get equality (7.65).

Furthermore, direct calculations show that

$$E\{[\varepsilon_{j2}]^4|\mathfrak{M}^{(j)}\} \leq C\mu_2 n^2 \left( \frac{1}{n} \sum_{l \neq k \in T_j} |R_{lk}^{(j)}|^2 \right)^2 \leq C\mu_2^2 \left( \frac{1}{n} \sum_{l \neq k \in T_j} |R_{lk}^{(j)}|^2 \right)^2. \quad (7.68)$$

Here again we used Lemma 7.6, inequality (7.28). Thus Lemma 7.15 is proved.

Lemma 7.16. Assuming the conditions of Theorem 1.1 there exists an absolute constant $C > 0$ such that for any $j = 1, \ldots, n$,

$$E\{[\varepsilon_{j3}]^2|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n} \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^2, \quad (7.69)$$

and

$$E\{[\varepsilon_{j3}]^4|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n^2} \left( \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^2 \right)^2 + \frac{C\mu_4}{n^2} \left( \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^4 \right). \quad (7.70)$$

Proof. The first inequality is easy. To prove the second, we apply Rosenthal’s inequality. We obtain

$$E\{[\varepsilon_{j3}]^4|\mathfrak{M}^{(j)}\} \leq \frac{C\mu_4}{n^2} \left( \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^2 \right)^2 + \frac{C\mu_8}{n^3} \left( \frac{1}{n} \sum_{l \in T_j} |R_{ll}^{(j)}|^4 \right). \quad (7.71)$$

Using $|X_{jl}| \leq Cn^{\frac{1}{2}}$ we get $\mu_8 \leq Cn\mu_4$ and the claim. Thus Lemma 7.16 is proved.
Corollary 7.17. Assuming the conditions of Theorem 1.1, there exists an absolute constant $C > 0$, depending on $\mu_4$ and $D_0$ only, such that for any $j = 1, \ldots, n$, $\nu = 1, 2, 3$, $z \in \mathbb{G}$, and $0 \leq \alpha \leq \frac{1}{2}A_1(n\nu)^{\frac{1}{2}}$ and $\beta \geq 1$

\[
E \frac{|\varepsilon_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{C}{nv|z^2 - 4|^{\frac{\beta - 1}{2}}} \tag{7.72}
\]

and for $\beta \geq 2$

\[
E \frac{|\varepsilon_{j\nu}|^4}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{C}{n^2v^2|z^2 - 4|^{\frac{\beta - 1}{2}}} \tag{7.73}
\]

We have as well, for $\nu = 2, 3$, and $\beta \geq 4$

\[
E \frac{|\varepsilon_{j\nu}|^8}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{C}{n^4v^4|z^2 - 4|^{\frac{\beta - 4}{2}}} \tag{7.74}
\]

Proof. For $\nu = 1$, by Lemma 7.15, we have

\[
E \frac{|\varepsilon_{j\nu}|^2}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{1}{nv|z^2 - 4|^{\frac{\beta - 1}{2}}} E|X_{jj}|^4 E \frac{1}{|z + m_n^{(j)}(z)|^{\alpha}}. \tag{7.75}
\]

Applying now Corollary 5.14 and Lemma 7.13, we get the claim. The proof of the second inequality for $\nu = 1$ is similar. For $\nu = 2$ we apply Lemmas 7.14 and 7.15, inequality (7.66), and obtain, using inequality (7.22),

\[
E \frac{|\varepsilon_{j2}|^2}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{1}{nv|z^2 - 4|^{\frac{\beta - 1}{2}}} E \frac{\text{Im}m_n^{(j)}(z)}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{C}{nv} \frac{1}{|z + m_n^{(j)}(z)|^{\alpha}}. \tag{7.76}
\]

Similar, using Lemma 7.15, inequality (7.66), and inequality (7.22) we get

\[
E \frac{|\varepsilon_{j2}|^4}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|^\alpha} \leq \frac{C}{n^2v^2|z^2 - 4|^{\frac{\beta - 4}{2}}} E \frac{\text{Im}^2m_n^{(j)}(z)}{|z + m_n^{(j)}(z) + s(z)|^2|z + m_n^{(j)}(z)|^\alpha} \leq \frac{C}{n^2v^2|z^2 - 4|^{\frac{\beta - 4}{2}}} \frac{1}{|z + m_n^{(j)}(z)|^{\alpha}}. \tag{7.77}
\]
Applying Corollary 5.14, we get the claim. For \( \nu = 3 \), we apply Lemma 7.10, inequalities (7.69) and (7.70) and Lemma 7.5. We get

\[
E \left| \varepsilon_{j3} \right|^2 \leq \frac{C}{n \sqrt{|z^2 - 4|}} \frac{1}{|z + m_n^{(j)}(z)|^\alpha} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2 \right), \tag{7.78}
\]

and

\[
E \left| \varepsilon_{j3} \right|^4 \leq \frac{C}{n^2 |z^2 - 4|} \frac{1}{|z + m_n^{(j)}(z)|^\alpha} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^4 \right). \tag{7.79}
\]

Using now the Cauchy – Schwartz inequality and Corollary 5.14, we get the claim.

**Lemma 7.18.** Assuming the conditions of Theorem 1.1, there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

\[
|\varepsilon_{j4}| \leq \frac{C}{nv} \quad \text{a.s.} \tag{7.80}
\]

**Proof.** This inequality follows from

\[
\text{Tr} \text{ } R - \text{Tr} \text{ } R^{(j)} = (1 + \frac{1}{n} \sum_{l,k \in \mathbb{T}_j} X_{jl}X_{jk}[(R^{(j)})^2]_{kl})R_{jj} = R_{jj}^{-1} dR_{jj} dz, \tag{7.81}
\]

which may be obtained using the Schur complement formula. See, for instance [10], Lemma 3.3.

**Corollary 7.19.** Assuming the conditions of Theorem 1.1, there exists an absolute constant \( C > 0 \) such that for any \( j = 1, \ldots, n \),

\[
E|\varepsilon_{j4}|^4 \leq \frac{C}{n^2 v^2}. \tag{7.82}
\]

**Proof.** By definition of \( \varepsilon_j \) (see 3.3), we have

\[
E|\varepsilon_{j4}|^4 \leq 4^4(E|\varepsilon_{j1}|^4 + \cdots + E|\varepsilon_{j4}|^4). \tag{7.83}
\]

Applying now Lemmas 7.14, 7.15 (inequality (7.66)), 7.16 (inequality (7.70)) and 7.18 and taking expectation where needed (7.82) follows. Thus the Corollary is proved.
Introduce quantities

\[ \eta_{j1} = \frac{1}{n} \sum_{l \in T_j} [(R^{(j)})^2]_{ll} = \frac{1}{n} \text{Tr} (R^{(j)})^2, \]

\[ \eta_{j2} = \frac{1}{n} \sum_{l \neq k \in T_j} X_{jl} X_{jk} [(R^{(j)})^2]_{lk}, \]

\[ \eta_{j3} = \frac{1}{n} \sum_{l \in T_j} (X_{jl}^2 - 1) [(R^{(j)})^2]_{ll}. \]

(7.84)

**Lemma 7.20.** Assuming the conditions of Theorem 1.1, we have, for \( z \in G \), for any \( j = 1, \ldots, n \)

\[ E\{|\eta_{j3}|^4 | \mathfrak{M}^{(j)}\} \leq C n^{-2} v^{-6} \text{Im}^2 m_n^{(j)} (z) + C n^{-2} v^{-4} \mu_4 \frac{1}{n} \sum_{l \in T_j} \text{Im}^4 R^{(j)}_{ll}, \]  

(7.85)

\[ E\{|\eta_{j3}|^8 | \mathfrak{M}^{(j)}\} \leq \frac{C \mu_4^4}{n^4 v^{12}} \text{Im}^4 m_n^{(j)} (z) + \frac{C \mu_4^4}{n^4 v^{8}} \frac{1}{n} \sum_{l \in T_j} \text{Im}^8 R^{(j)}_{ll}. \]  

(7.86)

**Proof.** Direct calculation shows

\[ E\{|\eta_{j3}|^4 | \mathfrak{M}^{(j)}\} \leq \frac{C \mu_4^4}{n^2} \left( \sum_{l \in T_j} \text{Im}^4 m_n^{(j)} (z) + \mu_8 \frac{1}{n} \sum_{l \in T_j} |(R^{(j)})^2|_{ll}^4 \right). \]

(7.87)

Applying Lemma 7.6, the inequality (7.32) and the inequality \( \mu_8 \leq C n \mu_4 \), we get

\[ E\{|\eta_{j3}|^4 | \mathfrak{M}^{(j)}\} \leq C n^{-2} v^{-6} \text{Im}^2 m_n^{(j)} (z) + C n^{-2} \mu_4 \frac{1}{n} \sum_{l \in T_j} \text{Im}^4 R^{(j)}_{ll}. \]  

(7.88)

Thus, inequality (7.85) is proved. Consider now the 8th moment of \( \eta_{j3} \). Applying Rosenthal’s inequality, we get

\[ E\{|\eta_{j3}|^8 | \mathfrak{M}^{(j)}\} \leq \frac{C \mu_4^4}{n^4 v^{12}} \left( \sum_{l \in T_j} \text{Im}^4 m_n^{(j)} (z) + \mu_8 \frac{1}{n} \sum_{l \in T_j} |(R^{(j)})^2|_{ll}^4 \right). \]  

(7.89)

Using now Lemma 7.6, inequalities (7.31) and (7.32), and that \( \mu_{16} \leq C n^3 \mu_4 \), we obtain

\[ E\{|\eta_{j3}|^8 | \mathfrak{M}^{(j)}\} \leq \frac{C \mu_4^4}{n^4 v^{12}} \text{Im}^4 m_n^{(j)} (z) + \frac{C \mu_4^4}{n^4 v^{8}} \frac{1}{n} \sum_{l \in T_j} \text{Im}^8 R^{(j)}_{ll}. \]  

(7.90)

Thus, inequality (7.86) is proved. \qed
**Lemma 7.21.** Assuming the conditions of Theorem 7.1, we have, for \( z \in \mathbb{G} \), for any \( j = 1, \ldots, n \)

\[
E\{|\eta_j|^2|\mathfrak{m}(j)\} \leq \frac{C|\text{Im}m_n(j)(z)|^2}{n^2 v^6} + \frac{C\mu_2^2}{n^3 v^6} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2,
\]

(7.91)

Furthermore, we have

\[
E\{|\eta_j|^8|\mathfrak{m}(j)\} \leq \frac{C}{n^4 v^{12}} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |R_{lk}^{(j)}|^2 \right)^2 + \frac{C\mu_4^2}{n^4 v^{8}} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} |\text{Im}m_n(j)(z)|^4 \right)^2 + \frac{1}{n^6 v^{12}} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} |(\text{Im}R_{ll}^{(j)})|^2 \right)^2 (\text{Im}m_n(j)(z))^2.
\]

(7.92)

**Proof.** Direct calculation shows that

\[
E\{|\eta_j|^4|\mathfrak{m}(j)\} \leq \frac{C}{n^2} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^2|^2 \right)^2 + \frac{C\mu_2^2}{n^4} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^4|.
\]

(7.93)

Applying inequalities (7.33) and (7.34) of Lemma 7.6 we get

\[
E\{|\eta_j|^4|\mathfrak{m}(j)\} \leq \frac{C|\text{Im}m_n(j)(z)|^2}{n^2 v^6} + \frac{C\mu_2^2}{n^3 v^6} \sum_{l \in \mathbb{T}_j} |R_{ll}^{(j)}|^2.
\]

(7.94)

Furthermore, we have

\[
E\{|\eta_j|^8|\mathfrak{m}(j)\} \leq \frac{C}{n^4 v^{12}} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^2|^2 \right)^4 + \frac{C\mu_4^2}{n^4 v^{8}} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^4|^8 + \frac{C\mu_6^2}{n^6} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^6|^2 \right) \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^2|^2 \right) + \frac{C\mu_4^2}{n^6} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^4|^2 \right) + \frac{C\mu_2^2}{n^6} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^2|^4 \right)^2 \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} |(R_{lk}^{(j)})^2|^2 \right)^2.
\]

(7.95)
Note that
\[ \mu_6 \leq C \sqrt{n} \mu_4, \quad \mu_8 \leq C n^2 \mu_4. \] (7.96)

Using this relations, we get
\[
E \{|\eta_j|^8|\mathfrak{M}^{(j)}\} \leq \frac{C}{n^4} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} \| (R^{(j)})^2_{kl} \| \right)^4 + \frac{C \mu_4^2}{n^6} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} \| (R^{(j)})^2_{kl} \| \right)^8 
+ \frac{C \mu_4^2}{n^6} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} \| (R^{(j)})^2_{kl} \| \right)^6 \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} \| (R^{(j)})^2_{kl} \| \right)^2 
+ \frac{C \mu_4^2}{n^6} \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} \| (R^{(j)})^2_{kl} \| \right)^4 \left( \frac{1}{n} \sum_{l \neq k \in \mathbb{T}_j} \| (R^{(j)})^2_{kl} \| \right)^2.
\] (7.97)

Applying now Lemma 7.6, we obtain
\[
E \{|\eta_j|^8|\mathfrak{M}^{(j)}\} \leq \frac{C}{n^4 v^2} \left( \text{Im} m^{(j)}_n(z) \right)^4 + \frac{C \mu_4^2}{n^4 v^8} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} \text{Im} R^{(j)}_{ll} \right)^4 
+ \frac{C \mu_4^2}{n^8 v^2} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} \text{Im}^3 R^{(j)}_{ll} \right) \left( \text{Im} m^{(j)}_n(z) \right) + \frac{C \mu_4^2}{n^6 v^2} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} \text{Im} R^{(j)}_{ll} \right)^2 
+ \frac{C \mu_4^2}{n^6} \left( \frac{1}{n} \sum_{l \in \mathbb{T}_j} \text{Im} R^{(j)}_{ll} \right) \left( \text{Im} m^{(j)}_n(z) \right)^2.
\] (7.98)

Thus, the Lemma is proved.

**Lemma 7.22.** Assuming the conditions of Theorem 1.1, we have
\[
E \left| \frac{\varepsilon_{j4}}{|z + s(z) + m_n(z)|^4} \right| \leq \frac{C}{n^4 v^4}.
\] (7.99)

**Proof.** Using the representations (7.81)-(7.84) we have
\[
\varepsilon_{j4} = \frac{1}{n} \left( 1 + \frac{1}{n} \sum_{l,k \in \mathbb{T}_j} X_{jl} X_{k} [(R^{(j)})^2]_{lk} R_{jj} \right) = \frac{1}{n} (1 + \eta_{j1} + \eta_{j2} + \eta_{j3}) R_{jj}.
\] (7.100)
Applying the Cauchy–Schwarz inequality, we get
\[
\mathbb{E}\left|\varepsilon_j 4 \left(\frac{1}{\pi} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right) \right|^4 \leq C \left(1 + \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right) \right)^8 \mathbb{E}\varepsilon_j^8 |R_{jj}|^8.
\]
(7.101)

Using Corollary 5.14 we may write
\[
\mathbb{E}\left|\varepsilon_j 4 \left(\frac{1}{\pi} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right) \right|^4 \leq C \left(1 + \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right) \right)^8 \mathbb{E}\varepsilon_j^8 \left|\eta_j 1 \right|^8 + \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \mathbb{E}\varepsilon_j^8 \left|\eta_j 2 \right|^8 + \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \mathbb{E}\varepsilon_j^8 \left|\eta_j 3 \right|^8.
\]
(7.102)

Observe that,
\[
\frac{1}{|z + s(z) + m_n(z)|} \leq \frac{1}{|z + s(z) + m_n(j)(z)|} (1 + \frac{|\varepsilon_j 4|}{|z + s(z) + m_n(z)|}).
\]
(7.103)

Therefore, by Lemmas 7.18, 7.3, and 7.13 for \(z \in \mathcal{G}\),
\[
\frac{1}{|z + s(z) + m_n(z)|} \leq C \frac{1}{|z + s(z) + m_n(j)(z)|}.
\]
(7.104)

Using (7.104) we get by definition of \(\eta_j 1\), Lemma 7.6 and inequality (7.28)
\[
\mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \leq C \mathbb{E}\varepsilon_j^8 \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(j)(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \leq v^{-4}.
\]
(7.105)

Furthermore, applying inequality (7.104) again, we obtain
\[
\mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \leq C \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(j)(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8.
\]
(7.106)

Conditioning with respect to \(\mathcal{M}^{(j)}\) and applying Lemma 7.20, we obtain
\[
\mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \leq C \mathbb{E}\varepsilon_j 4 \left(\frac{1}{|z + s(z) + m_n(j)(z)|} \sum_{l,k\in T_j} X_{jl} X_{jk} |(R^{(j)})^2|_{lk} \right)^8 \left(\frac{1}{n^4 v^{12}} (\text{Im} m_n(j)(z))^4 + \frac{\mu_4}{n^4 v^8} \frac{1}{n^4 v^{12}} \sum_{l\in T_j} (\text{Im} R_{ll}^{(j)})^8 \right).
\]
(7.107)
Using Lemma 7.5 inequality (7.22), together with Corollary 5.14 we get

\[ E \left| \frac{\eta_j}{z + m_n(z) + s(z)} \right|^8 \leq \frac{C}{n^2 v^6 |z^2 - 4|} + \frac{C \mu_1^2}{n^2 v^4 |z^2 - 4|^2}. \]  

(7.108)

Applying inequality (7.104) and conditioning with respect to \( \mathfrak{M}_n^{(j)} \) and applying Lemma 7.21 we get

\[ E \left| \frac{\eta_j}{z + m_n(z) + s(z)} \right|^8 \leq E \left( \frac{1}{|z + s(z) + m_n^{(j)}(z)|^8} \left( \frac{C}{n^4 v^8} (\text{Im} m_n^{(j)}(z))^4 \right. \right. \]
\[ \left. \left. + \frac{C \mu_1^2}{n^4 v^8} \left( \frac{1}{n} \sum_{l \in \mathcal{T}_j} (\text{Im} R_{ll}^{(j)})^4 \right)^2 \right. \right. \]
\[ \left. \left. + \frac{C \mu_1^2}{n^5 v^{12}} \left( \frac{1}{n} \sum_{l \in \mathcal{T}_j} (\text{Im} R_{ll}^{(j)})^3 \right) (\text{Im} m_n^{(j)}(z)) \right) \right. \]
\[ \left. \left. + \frac{C \mu_1^4}{n^6 v^{12}} \left( \frac{1}{n} \sum_{l \in \mathcal{T}_j} (\text{Im} R_{ll}^{(j)})^2 \right)^2 \right) \right. \]
\[ \left. + \frac{C \mu_1^2}{n^6 v^{12}} \left( \frac{1}{n} \sum_{l \in \mathcal{T}_j} (\text{Im} R_{ll}^{(j)})^2 \right) (\text{Im} m_n^{(j)}(z))^2 \right). \]  

(7.109)

Using that \(|z + m_n^{(j)}(z) + s(z)| \geq \text{Im} m_n^{(j)}(z)| \) together with Lemma 7.6 we arrive at

\[ E \left| \frac{\eta_j}{z + m_n(z) + s(z)} \right|^8 \leq \frac{C}{n^2 v^6 |z^2 - 4|} + \frac{C}{n^2 v^4 |z^2 - 4|^2} \]
\[ + \frac{C}{n^3 v^6 |z^2 - 4|^2} + \frac{C}{n^3 v^6 |z^2 - 4|^2} \]  

(7.110)

Summarizing we may write now, for \( z \in \mathcal{G} \),

\[ E \left| z + s(z) + m_n(z) \right|^4 \leq \frac{C}{n^4 v^4} + \frac{C}{n^6 v^6 |z^2 - 4|} + \frac{C}{n^6 v^4 |z^2 - 4|^2} \]
\[ + \frac{C}{n^7 v^6 |z^2 - 4|^2} + \frac{C}{n^7 v^6 |z^2 - 4|^2}. \]  

(7.111)

For \( z \in \mathcal{G} \), see (7.61) and Lemma 7.13 this inequality may be simplified by means of the following bounds (with \( v_0 = A_0 n^{-1} \))

\[ n^2 v^2 |z^2 - 4|^2 \geq n^2 v_0^2 \gamma^{-1 + \frac{1}{4}} \geq C \sqrt{n} \gamma^{\frac{3}{4}} \geq C, \]
\[ n^2 v^2 |z^2 - 4| \geq C, \quad n^3 v^2 |z^2 - 4|^2 \geq C, \]
\[ n^2 |z^2 - 4|^2 \geq C. \]  

(7.112)
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Using these relations, we obtain
\[ E \left| \frac{|\epsilon_j|^4}{|z + s(z) + m_n(z)|^4} \right| \leq \frac{C}{n^4 v^4}. \] (7.113)

Thus Lemma 7.22 is proved. \(\square\)

**Corollary 7.23.** Assuming the conditions of Theorem 1.1, we have
\[ E \left| \frac{|\epsilon_j|^2}{|z + s(z) + m_n(z)|^2} \right| \leq \frac{C}{n^2 v^2}. \] (7.114)

**Proof.** The result follows immediately from Lemma 7.22 and Jensen’s inequality. \(\square\)

**Lemma 7.24.** Assuming the conditions of Theorem 1.1, we have, for \(z \in \mathbb{C}\),
\[ E|\Lambda_n|^2 \leq \frac{C}{n^2 v^2}. \] (7.115)

**Proof.** We write
\[ E|\Lambda_n|^2 = E\Lambda_n \overline{\Lambda}_n = E \frac{T_n}{z + m_n(z) + s(z)} \overline{\Lambda}_n \]
\[ = \sum_{\nu=1}^4 E \frac{T_{n\nu}}{z + m_n(z) + s(z)} \overline{\Lambda}_n, \] (7.116)

where
\[ T_{n\nu} := \frac{1}{n} \sum_{j=1}^n \epsilon_{j\nu} R_{jj}, \text{ for } \nu = 1, \ldots, 4. \] (7.117)

First we observe that by (7.81)
\[ |T_{n4}| = \frac{1}{n} |m_n'(z)| \leq \frac{1}{nv} \text{Im} m_n(z). \] (7.118)

Hence \(|z + m_n(j)(z) + s(z)| \geq \text{Im} m_n(j)(z)\) and Jensen’s inequality yields
\[ |E \frac{T_{n4}}{z + m_n(z) + s(z)} \overline{\Lambda}_n| \leq \frac{1}{nv} E|\Lambda_n|^2. \] (7.119)

Furthermore, we represent \(T_{n1}\) as follows
\[ T_{n1} = T_{n11} + T_{n12}, \] (7.120)
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\[ T_{n11} = -\frac{1}{n} \sum_{j=1}^{n} \varepsilon_j \frac{1}{z + m_n(z)}, \]
\[ T_{n12} = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j (R_{jj} + \frac{1}{z + m_n(z)}). \]  
(7.121)

Using these notations we may write
\[ V_1 := E T_{n11} z + s_n(z) \Lambda_n = -E \left( \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j \right) \frac{(z + m_n(z))}{(z + m_n(z))(z + s_n(z) + m_n(z))} \Lambda_n. \]  
(7.122)

Applying the Cauchy – Schwartz inequality twice and using the definition of \( \varepsilon_j \) (see (3.3)), we get by Lemma 7.5
\[ |V_1| \leq \frac{1}{n |z^2 - 4|^{\frac{1}{2}}} E \left| \sum_{j=1}^{n} X_{jj} \right|^4 E \left| \frac{1}{z + m_n(z)} \right|^2 E \left| \Lambda_n \right|^2. \]  
(7.123)

By Rosenthal’s inequality, we have, for \( z \in \mathbb{C} \)
\[ |V_1| \leq \frac{C}{n |z^2 - 4|^{\frac{1}{2}}} \left| \sum_{j=1}^{n} X_{jj} \right|^4 \left| \frac{1}{z + m_n(z)} \right|^2 \left| \Lambda_n \right|^2 \leq \frac{1}{nv} \left| \Lambda_n \right|^2. \]  
(7.124)

Using (3.3) we rewrite \( T_{n12} \), obtaining
\[ V_2 := E T_{n12} z + m_n(z) + s_n(z) \Lambda_n = \frac{1}{n \sqrt{n}} \sum_{j=1}^{n} E \left( \varepsilon_{j} R_{jj} \right) \frac{X_{jj} \varepsilon_j}{(z + m_n(z))(z + s_n(z) + m_n(z))} \Lambda_n. \]  
(7.125)

By the Cauchy – Schwartz inequality, using the definition of \( \varepsilon_j \) (see representation 3.3), we obtain
\[ |V_2| \leq \frac{1}{\sqrt{n}} \sum_{\nu=1}^{4} \left| \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j \nu} X_{jj} R_{jj} \right|^2 \frac{|X_{jj} \varepsilon_j|^2}{|z + m_n(z) + s_n(z)|^2 |z + m_n(z)|^2} \left| \Lambda_n \right|^2 =: \sum_{\nu=1}^{4} V_{2\nu}. \]  
(7.126)

For \( \nu = 1 \), we have
\[ V_{21} \leq \frac{1}{n} E \left| \sum_{j=1}^{n} X_{jj} R_{jj} \right|^2 \frac{|X_{jj} \varepsilon_j|^2}{|z + m_n(z) + s_n(z)|^2 |z + m_n(z)|^2} \left| \Lambda_n \right|^2. \]  
(7.127)
Applying the Cauchy – Schwartz inequality twice and Lemma 7.5, we arrive at

\[ V_{21} \leq \frac{1}{n\sqrt{|z^2 - 4|}} E^\frac{1}{2} \left( \frac{1}{n} \sum_{j=1}^{n} X_{jj}^4 \right)^2 E^\frac{1}{2} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj}|^2 \right)^4 E^\frac{1}{2} \frac{1}{|z + m_n(z)|^8} E^\frac{1}{2} |\Lambda_n|^2. \]  

(7.128)

Observe that

\[ E \left( \frac{1}{n} \sum_{j=1}^{n} X_{jj}^4 \right)^2 = \left( \frac{1}{n} \sum_{j=1}^{n} E X_{jj}^4 \right)^2 + E \left( \frac{1}{n} \sum_{j=1}^{n} (X_{jj}^4 - E X_{jj}^4) \right)^2 \]

\[ \leq \mu_4^2 + \frac{2}{n^2} \sum_{j=1}^{n} E |X_{jj}|^8 \leq (\mu_4 + D_4^4) \mu_4 \leq C. \]  

(7.129)

The last inequality, inequality (7.128), Corollary 5.14 and Lemma 7.13 together imply

\[ V_{21} \leq \frac{C}{n\sqrt{|z^2 - 4|}} E^\frac{1}{2} |\Lambda_n|^2 \leq \frac{C}{nv} E^\frac{1}{2} |\Lambda_n|^2. \]  

(7.130)

Furthermore, for \( \nu = 4 \), by Lemma 7.18 we have

\[ V_{24} \leq \frac{1}{nv\sqrt{n}} E^\frac{1}{2} \frac{1}{n} \sum_{j=1}^{n} |X_{jj}|^2 |R_{jj}|^2 \frac{1}{|z + m_n(z) + s(z)|^2 |z + m_n(z)|^2} E^\frac{1}{2} |\Lambda_n|^2. \]  

(7.131)

Applying the Cauchy – Schwartz inequality and Lemma 7.5, we get

\[ V_{24} \leq \frac{1}{nv\sqrt{n}} E^\frac{1}{2} \left( \frac{1}{n} \sum_{j=1}^{n} X_{jj}^4 \right)^2 E^\frac{1}{2} \left( \frac{1}{n} \sum_{j=1}^{n} |R_{jj}|^2 \right)^4 E^\frac{1}{2} \frac{1}{|z + m_n(z)|^8} E^\frac{1}{2} |\Lambda_n|^2. \]  

(7.132)

Similar to inequality (7.130), applying Lemma 7.13 inequality (7.129) and Corollary 5.14 we get

\[ V_{24} \leq \frac{C'}{nv} E^\frac{1}{2} |\Lambda_n|^2. \]  

(7.133)

By Hölder’s inequality, we have for \( \nu = 2, 3 \),

\[ V_{2\nu} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} E^\frac{1}{2} \frac{|\varepsilon_j\nu|^4 |X_{jj}|^4}{|z + m_n(z) + s(z)|^4} E^\frac{1}{2} \frac{1}{|z + m_n(z)|^8} E^\frac{1}{2} |R_{jj}|^8 E^\frac{1}{2} |\Lambda_n|^2. \]  

(7.134)
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Note that for \( \nu = 2, 3 \), r.v. \( X_{jj} \) doesn’t depend on \( \varepsilon_{j\nu} \) and on \( \sigma \)-algebra \( \mathcal{M}^{(j)} \). Using inequality (7.104) for \( z \in \mathcal{G} \), we get

\[
E \frac{\varepsilon_{j\nu}^4 |X_{jj}|^4}{|z + m_n(z) + s(z)|^4} \leq C E \frac{\varepsilon_{j\nu}^4 |X_{jj}|^4}{|z + m_n^{(j)}(z) + s(z)|^4} \leq C \mu_4 E \frac{\varepsilon_{j\nu}^4}{|z + m_n^{(j)}(z) + s(z)|^4}
\]

(7.135)

Applying now Lemmas 7.15, 7.16 and 7.13 arrive at

\[
V_{2\nu} \leq \frac{C}{n^\nu} |\Lambda_n|^2, \quad \text{for } \nu = 2, 3.
\]

(7.136)

Inequalities (7.130), (7.133), (7.136) together imply

\[
V_2 \leq \frac{C}{n^\nu} |\Lambda_n|^2.
\]

(7.137)

Consider now the quantity

\[
Y_{\nu} := E \frac{T_{\nu\nu} z + m_n(z) + s(z)}{|z + m_n(z) + s(z)|^4} \Lambda_n,
\]

(7.138)

for \( \nu = 2, 3 \). We represent it as follows

\[
Y_{\nu} = Y_{\nu 1} + Y_{\nu 2},
\]

(7.139)

where

\[
Y_{\nu 1} = -\frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu} \Lambda_n}{(z + m_n^{(j)}(z))(z + m_n(z) + s(z))},
\]

\[
Y_{\nu 2} = \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu} (R_{jj} + \frac{1}{z + m_n^{(j)}(z)}) \Lambda_n}{z + m_n(z) + s(z)}.
\]

(7.140)

By the representation (5.49), which is similar to (3.3) we have

\[
Y_{\nu 2} = \frac{1}{n} \sum_{\mu=1}^{3} \frac{1}{n} \sum_{j=1}^{n} E \frac{\varepsilon_{j\nu} \varepsilon_{j\mu} \Lambda_n R_{jj}}{(z + m_n(z) + s(z))(z + m_n^{(j)}(z))}.
\]

(7.141)

Using inequality (7.104), we may write, for \( z \in \mathcal{G} \)

\[
|Y_{\nu 2}| \leq \frac{C}{n} \sum_{\mu=1}^{3} \sum_{j=1}^{n} E \frac{|\varepsilon_{j\mu}| |\varepsilon_{j\nu}| |\Lambda_n||R_{jj}|}{|z + m_n^{(j)}(z) + s(z)||z + m_n^{(j)}(z)|}.
\]

(7.142)
Applying the Cauchy – Schwartz inequality and the inequality \( ab \leq \frac{1}{2}(a^2 + b^2) \), we get

\[
|Y_{\nu 2}| \leq \sum_{\mu=1}^{3} \frac{C}{n} \sum_{j=1}^{n} \mathbb{E}^* \left| \frac{\varepsilon_{j\mu}}{z + m^{(j)}_n(z)} |R_{jj}|^2 |R_{jj}|^2 \right| \mathbb{E}^* |\Lambda_n|^2
\]

\[
\leq \sum_{\mu=2}^{3} \frac{C}{n} \sum_{j=1}^{n} \mathbb{E}^* \left| \frac{\varepsilon_{j\mu}}{z + m^{(j)}_n(z)} |R_{jj}|^2 |R_{jj}|^2 \right| \mathbb{E}^* |\Lambda_n|^2
\]

\[
+ \frac{C}{n} \sum_{j=1}^{n} \mathbb{E}^* \left| \frac{\varepsilon_{j\mu}}{z + m^{(j)}_n(z)} |R_{jj}|^2 |R_{jj}|^2 \right| \mathbb{E}^* |\Lambda_n|^2
\]

\[
\leq \sum_{\mu=2}^{3} \frac{C}{n} \sum_{j=1}^{n} \mathbb{E}^* \left| \frac{\varepsilon_{j\mu}}{z + m^{(j)}_n(z)} |R_{jj}|^2 |R_{jj}|^2 \right| \mathbb{E}^* |\Lambda_n|^2 \mathbb{E}^* |R_{jj}|^4
\]

\[
+ \frac{C}{n} \sum_{j=1}^{n} \mathbb{E}^* \left| \frac{\varepsilon_{j\mu}}{z + m^{(j)}_n(z)} |R_{jj}|^2 |R_{jj}|^2 \right| \mathbb{E}^* |\Lambda_n|^2 \mathbb{E}^* |R_{jj}|^4.
\]

(7.143)

Using Corollary 7.17 with \( \alpha = 2 \), we arrive at

\[
|Y_{\nu 2}| \leq \frac{C}{n^\beta} \mathbb{E}^* |\Lambda_n|^2.
\]

(7.144)

In order to estimate \( Y_{\nu 1} \) we introduce now the quantity

\[
\Lambda^{(j)}_n = \frac{1}{n} \text{Tr} \mathbf{R}^{(j)} - s(z) + \frac{s(z)}{n}.
\]

(7.145)

Note that

\[
\Lambda_n - \Lambda_n^{(j,1)} = \frac{1}{n} \text{Tr} \mathbf{R} - \frac{1}{n} \text{Tr} \mathbf{R}^{(j)} - \frac{s(z)}{n}
\]

\[
= \frac{R_{jj} - s(z)}{n} + \frac{1}{n^2} \sum_{l,k \in T_j} X_{jl}X_{jk}[\{R^{(j)}]^2]_{kl} = \delta_{nj}.
\]

(7.146)

We represent \( Y_{\nu 1} \) in the form

\[
Y_{\nu 1} = Z_{\nu 1} + Z_{\nu 2} + Z_{\nu 3}.
\]

(7.147)
where
\[
Z_{\nu 1} = -\frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \Lambda_n^{(j1)}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))},
\]
\[
Z_{\nu 2} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \delta_{nj}}{(z + m_n^{(j)}(z))(z + m_n(z) + s(z))},
\]
\[
Z_{\nu 3} = \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{\varepsilon_{j\nu} \Lambda_n^{(j1)} \delta_{nj}}{(z + m_n^{(j)}(z))(z + m_n^{(j)}(z) + s(z))(z + m_n(z) + s(z))}.
\] (7.148)

First, note that by conditionally independence
\[
Z_{\nu 1} = 0.
\] (7.149)

Furthermore, applying Hölder’s inequality, we get
\[
|Z_{\nu 3}| \leq \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \frac{4}{|z + m_n^{(j)}(z)|^4 |z + m_n^{(j)}(z) + s(z)|^4} \times \mathbb{E} \frac{4}{|z + m_n(z) + s(z)|^4} \mathbb{E} |\Lambda_n^{(j1)}|^2. \] (7.150)

Using Corollary 7.17 with \( \alpha = 4 \) and Lemmas 7.22 and 7.5 we obtain
\[
|Z_{\nu 3}| \leq \frac{C}{(nv)^{\frac{3}{2}} |z^2 - 4|} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} |\Lambda_n^{(j1)}|^2. \] (7.151)

Applying now Corollary (5.14), we get
\[
|Z_{\nu 3}| \leq \frac{C}{(nv)^{\frac{3}{2}} |z^2 - 4|} \mathbb{E} |\Lambda_n|^2 + \frac{C}{(nv)^{\frac{3}{2}} |z^2 - 4|} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} |\delta_{nj}|^2. \] (7.152)

For \( z \in \mathbb{G} \) we may rewrite this bound using Lemma 7.13
\[
|Z_{\nu 3}| \leq \frac{C}{nv} \mathbb{E} |\Lambda_n|^2 + \frac{C}{nv} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} |\delta_{nj}|^2. \] (7.153)

By definition of \( \delta_{nj} \), see (7.146), we have
\[
\mathbb{E} |\delta_{nj}|^2 \leq C \left( \frac{1}{n^2} \mathbb{E} |R_{jj} - s(z)|^2 + \mathbb{E} \frac{1}{n^2} \sum_{l,k \in T_j} X_{jl} X_{jk} [(R^{(j)})^2]_{kl} \right) |\Lambda_n^{(j1)}|^4. \] (7.154)
By representation (3.3), we have
\[ E|R_{jj} - s(z)|^2 \leq E|\Lambda_n|^2 + E^\frac{2}{3} |\varepsilon_j|^4 E^\frac{2}{3} |R_{jj}|^4. \] (7.155)

By Corollaries 5.14 and 7.19 we get
\[ E|R_{jj} - s(z)|^2 \leq E|\Lambda_n|^2 + \frac{C}{nv}. \] (7.156)

By Lemmas 7.21, inequality (7.91) and Corollary 5.14, we have
\[ E\left|\sum_{l,k \in T_j} X_{jl} X_{jk} (R^{(j)})^2_{kl}\right|^4 \leq \frac{1}{n} E|\eta_{j3}|^4 \leq \frac{C}{n^6 v^6}. \] (7.157)

Inequalities (7.156) and (7.157) together imply
\[ E|\delta_{nj}|^2 \leq \frac{1}{n^2} E|\Lambda_n|^2 + \frac{C}{n^8 v} + \frac{C}{n^6 v^6}. \] (7.158)

Therefore, for \( z \in \mathbb{G} \),
\[ |Z_{v3}| \leq \frac{C}{nv} E\frac{1}{n} |\Lambda_n|^2 + \frac{C}{n^4 v^4} + \frac{1}{n^2 v} E\frac{1}{n} |\Lambda_n|^2 \leq \frac{C}{nv} E\frac{1}{n} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \] (7.159)

To bound \( Z_{v2} \) we first apply inequality (7.104) and obtain
\[ |Z_{v2}| \leq \frac{1}{n} \sum_{j=1}^n E|\varepsilon_{j\nu}| \frac{|\delta_{nj}|}{|z + m_n^{(j)}(z)||z + m_n^{(j)}(z) + s(z)|}. \] (7.160)

Applying now Hölder’s inequality, we get
\[ |Z_{v2}| \leq \frac{C}{n} \sum_{j=1}^n E\frac{1}{n} |\varepsilon_{j\nu}|^2 \frac{|\delta_{nj}|^2}{|z + m_n^{(j)}(z)||z + m_n^{(j)}(z) + s(z)|^2}. \] (7.161)

Therefore, by Corollary 7.17
\[ |Z_{v2}| \leq \frac{C}{nv |z| z^2 - 4} \frac{1}{n} \sum_{j=1}^n E\frac{1}{n} |\delta_{nj}|^2. \] (7.162)

The last inequality together with Lemma 7.13 and inequality (7.158), imply
\[ |Z_{v2}| \leq \frac{C}{nv |z| z^2 - 4} E\frac{1}{n} |\Lambda_n|^2 + \frac{C}{n^2 v} + \frac{C}{n^2 v^2} \leq \frac{C}{nv} E\frac{1}{n} |\Lambda_n|^2 + \frac{C}{n^2 v^2}. \] (7.163)
Combining now inequalities (7.119), (7.124), (7.137), (7.159), (7.163), we get
\[
E|\Lambda_n|^2 \leq C \frac{1}{n\nu} E^{\frac{1}{2}} |\Lambda_n|^2 + C \frac{1}{n^2\nu^2}.
\] (7.164)

Applying Lemma 7.4 with \(t = 2, r = 1\) completes the proof of Lemma 7.24. Thus Lemma 7.24 is proved.

We relabel \(\eta_{2j}, \eta_{3j}\) and introduce the following quantity
\[
\beta_{j1} = \frac{1}{n} \sum_{l \in T_j} [(R^{(j)})^2]_{ll} - \frac{1}{n} \sum_{l=1}^{n} [(R)^2]_{ll},
\]
\[
\beta_{j2} = \frac{1}{n} \sum_{l \neq k \in T_j} X_{jl}X_{jk}[(R^{(j)})^2]_{lk} = \eta_{j2},
\]
\[
\beta_{j3} = \frac{1}{n} \sum_{l \in T_j} (X^2_{jl} - 1)[(R^{(j)})^2]_{ll} = \eta_{j3}.
\] (7.165)

**Lemma 7.25.** Assuming the conditions of Theorem 1.1, we have, for \(\nu = 2, 3\),
\[
E\{\|\beta_{j\nu}\|^2 |\mathcal{M}^{(j)}\} \leq C \frac{1}{n\nu^3} \text{Im} m_n^{(j)}(z).
\] (7.166)

**Proof.** We recall that by \(C\) we denote the generic constant depending on \(\mu_4\) and \(D_0\) only. By definition of \(\beta_{j\nu}\) for \(\nu = 2, 3\), conditioning on \(\mathcal{M}^{(j)}\), we get
\[
E\{\|\beta_{j2}\|^2 |\mathcal{M}^{(j)}\} \leq C \frac{1}{n^2} \sum_{l \neq k \in T_j} ||(R^{(j)})^2|_{kl}||^2 \leq C \frac{1}{n^2} \sum_{l,k \in T_j} ||(R^{(j)})^2|_{kl}||^2,
\]
\[
E\{\|\beta_{j3}\|^2 |\mathcal{M}^{(j)}\} \leq C \frac{1}{n^2} \sum_{l \in T_j} ||(R^{(j)})^2|_{ll}||^2 \leq C \frac{1}{n^2} \sum_{l,k \in T_j} ||(R^{(j)})^2|_{kl}||^2. \] (7.167)

Applying Lemma 7.6 we get the claim. Thus Lemma 7.25 is proved.

**Lemma 7.26.** Assuming the conditions of Theorem 1.1, we have, for \(j = 1, \ldots, n\),
\[
E|\beta_{j1}| \leq C \frac{1}{n\nu^2}.
\] (7.168)
Rate of Convergence to the Semi-Circular Law

Proof. Let $F_n^{(j)}(x)$ denote empirical spectral distribution function of matrix $W^{(j)}$. According to interlacing eigenvalues Theorem (see [18], Theorem 4.38) we have
\[
\sup_x |F_n(x) - F_n^{(j)}(x)| \leq \frac{C}{n}.
\] (7.169)
Furthermore, we represent
\[
\beta_{j1} = \int_\infty^{-\infty} \frac{1}{(x-z)^2} d(F_n(x) - F_n^{(j)}(x)) + \frac{1}{n(n-1)} \text{Tr} \left( R^{(j)} \right)^2.
\] (7.170)
Integrating by parts, we get the claim. Thus Lemma [7.26] is proved.

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