QUATERNIONIC k-VECTOR FIELDS ON QUATERNIONIC KÄHLER MANIFOLDS

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ABSTRACT. In this paper, we define a differential operator as a modified Dirac operator. Using the operator, we introduce a quaternionic k-vector field on a quaternionic Kähler manifold and show that any quaternionic k-vector field corresponds to a holomorphic k-vector field on the twistor space. We calculate the dimension of the space of quaternionic k-vector fields on $\mathbb{H}P^n$.

1. INTRODUCTION

Let $(M, g)$ be a quaternionic Kähler manifold, that is, a $4n$-dimensional Riemannian manifold whose holonomy group is reduced to a subgroup of $\text{Sp}(n) \cdot \text{Sp}(1)$. Let $E$ and $H$ denote the associated bundles with the canonical representations of $\text{Sp}(n) \cdot \text{Sp}(1)$ on $\mathbb{C}^{2n}$ and $\mathbb{C}^2$, respectively. Then $TM \otimes \mathbb{C} = E \otimes_{\mathbb{C}} H$. The complex vector bundles $E, H$ admit symplectic structures and anti-$\mathbb{C}$-linear maps which square to $-\text{id}$. The Levi-Civita connection of $(M, g)$ is decomposed into connections of $E$ and $H$. They induce the covariant derivative $\nabla : \Gamma(\wedge^k E \otimes S^k H) \to \Gamma(\wedge^k E \otimes S^k H \otimes E^* \otimes H^*)$ where $\Gamma(\wedge^k E \otimes S^k H)$ means the space of smooth sections of $\wedge^k E \otimes S^k H$. The bundle $\wedge^k E \otimes S^k H \otimes E^* \otimes H^*$ is isomorphic to $\wedge^k E \otimes E^* \otimes S^k H \otimes H$ by the symplectic structure of $H$. Moreover, $S^k H \otimes H \cong S^{k+1} H \oplus S^{k-1} H$ by the Clebsch-Gordan decomposition. Thus, the covariant derivative $\nabla$ is regarded as

$$\nabla : \Gamma(\wedge^k E \otimes S^k H) \to \Gamma(\wedge^k E \otimes E^* \otimes S^{k+1} H) \oplus \Gamma(\wedge^k E \otimes E^* \otimes S^{k-1} H).$$

Dirac operator $\mathfrak{D}_{\wedge^k E}$ is defined by the $\wedge^k E \otimes E^* \otimes S^{k+1} H$-part of $\nabla$ (c.f. [3]) :

$$\mathfrak{D}_{\wedge^k E} : \Gamma(\wedge^k E \otimes S^k H) \to \Gamma(\wedge^k E \otimes E^* \otimes S^{k+1} H)$$

The trace of $(\otimes^k E) \otimes E^*$ induces the map $\text{tr} : \wedge^k E \otimes E^* \to \wedge^{k-1} E$ by the restriction of $\otimes^k E \otimes E^*$ to $\wedge^k E \otimes E^*$. Let $(\wedge^k E \otimes E^*)_0$ denote the kernel of $\text{tr} : \wedge^k E \otimes E^* \to \wedge^{k-1} E$. The bundle $\wedge^k E \otimes E^*$ is decomposed into $(\wedge^k E \otimes E^*)_0$ and $(\wedge^{k-1} E) \otimes \text{id}_E$. We define an operator

$$\mathfrak{D}_{\wedge^k E}^0 : \Gamma(\wedge^k E \otimes S^k H) \to \Gamma((\wedge^k E \otimes E^*)_0 \otimes S^{k+1} H)$$

as the $(\wedge^k E \otimes E^*)_0$-part of $\mathfrak{D}_{\wedge^k E}$. A section of $\wedge^k E \otimes S^k H$ is a k-vector field since $\wedge^k E \otimes S^k H$ is the subbundle of $\wedge^k TM \otimes \mathbb{C}$.

Definition 1.1. A section $X$ of $\wedge^k E \otimes S^k H$ is a quaternionic k-vector field on $M$ if $\mathfrak{D}_{\wedge^k E}^0(X) = 0$ for $1 \leq k \leq 2n - 1$ and $\mathfrak{D}_{\wedge^{2m-1} E} \circ \text{tr} \circ \mathfrak{D}_{\wedge^{2n} E}(X) = 0$ for $k = 2n$.

The twistor space $Z$ of $M$ is given by the projective bundle of the dual bundle $H^*$. It has a natural complex structure and a real structure. It is known that a section $X$ of $\wedge^k E \otimes S^k H$ with $\mathfrak{D}_{\wedge^k E}(X) = 0$ is lifted to a holomorphic k-vector field on $Z$. However, any holomorphic k-vector field on $Z$ does not correspond to such a k-vector field on $M$. The following is the main theorem :

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**Theorem 1.2.** Let \( Q(\wedge^k E \otimes S^k H) \) be the sheaf of quaternionic k-vector fields on \( M \) and \( O(\wedge^k T^{1,0} Z) \) that of holomorphic k-vector fields on \( Z \). Then, \( H^0(Q(\wedge^k E \otimes S^k H)) \) is isomorphic to \( H^0(O(\wedge^k T^{1,0} Z)) \).

The space \( \Gamma(\wedge^k E \otimes S^k H) \) admits a real structure \( \tau \). Any \( \tau \)-invariant section of \( \wedge^k E \otimes S^k H \) is a real k-vector field on \( M \). On the other hand, the real structure of \( Z \) induces the real structure \( \bar{\tau} \) on the space of holomorphic k-vector fields. We obtain the following theorem:

**Theorem 1.3.** The space \( H^0(Q(\wedge^k E \otimes S^k H))^\tau \) of quaternionic real k-vector fields on \( M \) is isomorphic to the space \( H^0(O(\wedge^k T^{1,0} Z))^\bar{\tau} \) of \( \bar{\tau} \)-invariant holomorphic k-vector fields on \( Z \).

This paper is organized as follows. In Section 2, we prepare some fundamental facts about quaternionic vector spaces and quaternionic Kähler manifolds. In Section 3 and 4, we see some theorems in twistor theory from the view point of the principal bundle of \( H^n \) and the connection. In Section 5, we introduce quaternionic sections on quaternionic Kähler manifolds as sections of \( \wedge^k E \otimes S^m H \) satisfying some differential equations. In the final section, we provide the definition of quaternionic k-vector fields. We show Theorem 1.2 and Theorem 1.3 (see Theorem 6.12 and Theorem 6.14). As an example, we consider the \( n \)-dimensional quaternionic projective space \( \mathbb{H}P^n \) and compute the dimension of the space of quaternionic k-vector fields.

### 2. Preliminaries

#### 2.1. The \( n \)-dimensional quaternionic vector space \( \mathbb{H}^n \)

Let \( \mathbb{H} \) be Hamilton’s quaternionic number field \( \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R} \). We define endomorphisms \( i, j, k \) of \( \mathbb{H}^n \) as the right action of \( i, j, k \) on \( \mathbb{H}^n \), respectively. The product \( \text{Sp}(n) \times \text{Sp}(1) \) acts on \( \mathbb{H}^n \) by \( (A, q) \cdot \xi = A\xi q \) for \( (A, q) \in \text{Sp}(n) \times \text{Sp}(1) \) and \( \xi \in \mathbb{H}^n \). Since \((-1, -1)\) acts as the identity, the quotient \( \text{Sp}(n) \cdot \text{Sp}(1) = \text{Sp}(n) \times \text{Sp}(1)/\mathbb{Z}_2 \) acts on \( \mathbb{H}^n \). The action of \( \text{Sp}(n) \cdot \text{Sp}(1) \) on \( \mathbb{H}^n \) preserves the subspace \( \langle i, j, k \rangle \) of \( \text{End}_g(\mathbb{H}^n) \). Under the identification \( \mathbb{H}^n \cong \mathbb{R}^{4n} \), we denote by \( I_n, J_n, K_n \) the endomorphisms of \( \mathbb{R}^{4n} \) corresponding to \( i, j, k \), respectively. Then \( \text{Sp}(n) \cdot \text{Sp}(1) \) is identified with a subgroup of \( SO(4n) \) which preserves \( \langle I_n, J_n, K_n \rangle \). We denote by \( Q_0 \) the subspace \( \langle I_n, J_n, K_n \rangle \). In particular, \( \text{Sp}(n) \) is considered as a subgroup of \( SO(4n) \) preserving each \( I_n, J_n, K_n \).

#### 2.2. Quaternionic Kähler manifolds

Let \((M, g)\) be a Riemannian manifold of dimension \( 4n \). A subbundle \( Q \) of \( \text{End}(TM) \) is called an almost quaternionic Hermitian structure if there exists a local basis \( I, J, K \) of \( Q \) such that \( I^2 = J^2 = K^2 = -\text{id} \) and \( K = IJ \). A pair \((Q, g)\) is an almost quaternionic Hermitian structure if any section \( \varphi \) of \( Q \) satisfies \( g(\varphi X, Y) + g(X, \varphi Y) = 0 \) for \( X, Y \in TM \). For \( n \geq 2 \), if the Levi-Civita connection \( \nabla \) preserves \( Q \), then \((Q, g)\) is called a quaternionic Kähler structure, and \((M, Q, g)\) a quaternionic Kähler manifold. A Riemannian manifold is a quaternionic Kähler manifold if and only if the holonomy group is reduced to a subgroup of \( \text{Sp}(n) \cdot \text{Sp}(1) \). Alekseevskii [4] shows that a quaternionic Kähler manifold is Einstein and the curvature of \( Q \) is described by the scalar curvature (we also refer to [1, 8]). For \( n = 1 \), since \( \text{Sp}(1) \cdot \text{Sp}(1) = SO(4) \), a manifold satisfying the above condition is just an oriented Riemannian manifold. A 4-dimensional oriented Riemannian manifold \( M \) is said to be a quaternionic Kähler manifold if it is Einstein and self-dual. The scalar curvature of a quaternionic Kähler manifold \( M \) vanishes if and only if the holonomy group is reduced to \( \text{Sp}(n) \), that is, \( M \) is a hyperkähler manifold.

The frame bundle of a quaternionic Kähler manifold \( M \) is reduced to a principal \( \text{Sp}(n) \cdot \text{Sp}(1) \)-bundle \( F \). The Levi-Civita connection \( \nabla \) induces a connection of \( F \). The bundle \( F \)
is lifted to a principal $\text{Sp}(n) \times \text{Sp}(1)$-bundle $\tilde{F}$, locally. Given a representation $V$ of $\text{Sp}(n) \times \text{Sp}(1)$, we construct an associated bundle $\tilde{F} \times_{\text{Sp}(n) \times \text{Sp}(1)} V$ with the induced connection. The bundle is globally defined on $M$ if either $\tilde{F}$ exists globally or $(-1, -1) \in \text{Sp}(n) \times \text{Sp}(1)$ acts as the identity in the representation. The obstruction to the global existence of $\tilde{F}$ is provided by a cohomology class $\varepsilon \in H^2(M, \mathbb{Z}_2)$ introduced by Marchiafava-Romani [6]. In $n$ is odd, the class $\varepsilon$ is given by the second Stiefel-Whitney class $w_2$.

The symplectic group $\text{Sp}(n)$ acts on the right $\mathbb{H}$-module $\mathbb{H}^n$ by $A \xi = A_\xi$ for $A \in \text{Sp}(n)$ and $\xi \in \mathbb{H}$. On the other hand, $\text{Sp}(1)$ has an action on the left $\mathbb{H}$-module $\mathbb{H}$ by $q \xi$ for $q \in \text{Sp}(1)$ and $\xi \in \mathbb{H}$. Let $E$, $H$ denote the associated bundles with the representations $\text{Sp}(n)$, $\text{Sp}(1)$ on $\mathbb{H}^n$, $\mathbb{H}$, respectively. Then $E$ is the right $\mathbb{H}$-module bundle and $H$ is the left $\mathbb{H}$-module bundle. The dual representations of $\text{Sp}(n)$ and $\text{Sp}(1)$ induce the left $\mathbb{H}$-module bundle $E^*$ and the right $\mathbb{H}$-module bundle $H^*$. Then

$$TM = E \otimes_{\mathbb{H}} H, \quad T^* M = H^* \otimes_{\mathbb{H}} E^*.$$ 

The $\mathbb{H}$-bundles $E$, $H$ are regarded as the $\mathbb{C}$-vector bundles with anti $\mathbb{C}$-linear maps $J_E$, $J_H$ satisfying $J_E^2 = -\text{id}_E$, $J_H^2 = -\text{id}_H$. Then there exist symplectic structures $\omega_E$, $\omega_H$ on $E$, $H$ which are compatible with $J_E$, $J_H$, respectively. As the same manner, $E^*$ and $H^*$ are $\mathbb{C}$-vector bundles with anti $\mathbb{C}$-linear maps $J_{E^*}$, $J_{H^*}$. Then $J_E^* (\alpha) = -\alpha \circ J_E$, $J_H^* (\beta) = -\beta \circ J_H$ for $\alpha \in E^*$, $\beta \in H^*$. The correspondences $\xi \mapsto \omega_E(\cdot, \xi)$, $u \mapsto \omega_H(\cdot, u)$ provide the $\mathbb{C}$-isomorphisms $E \cong E^*$, $H \cong H^*$, which are denoted by $\omega_E^*$, $\omega_H^*$. The tensor space $(\otimes^p E) \otimes (\otimes^q H) \otimes (\otimes^r H^*) \otimes (\otimes^s E^*)$ is globally defined if $p + p' + q + q' + s$ is even.

Then $J_{E^*} \otimes J_{H^*}^2 \otimes J_{H^*}^r \otimes J_{E^*}^p$ is a real structure on the tensor space. In particular, $J_{E^*} \otimes J_V$ and $J_{H^*} \otimes J_{H^*}^r$, $J_{H^*} \otimes J_{E^*}$ are real structures on $E \otimes \mathbb{C} H$ and $H^* \otimes \mathbb{C} E^*$. The real forms of $J_{E^*} \otimes J_H$ and $J_{H^*} \otimes J_{E^*}$ are $TM$ and $T^* M$, respectively. Hence

$$TM \otimes \mathbb{C} = E \otimes \mathbb{C} H, \quad T^* M \otimes \mathbb{C} = H^* \otimes \mathbb{C} E^*.$$ 

The tensor product $\omega_E \otimes \omega_H$ is the complexification of the Riemannian metric $g$. The technique is called $EH$-formalism, which was introduced by Salamon in [8].

2.3. The twistor space. The quaternionic structure $Q$ is an associated bundle with the representation of the action of $\text{Sp}(n) \cdot \text{Sp}(1)$ on $Q_0$. The representation is reduced to that of $\text{Sp}(1)$ on the subspace $\langle i, j, k \rangle$ of $\text{End}_{\mathbb{H}}(\mathbb{H})$, where $i, j, k$ act on a left $\mathbb{H}$-module $\mathbb{H}$ by the right multiplication. Hence, $Q$ is considered as a subbundle of the real vector bundle $\text{End}_{\mathbb{H}}(H)$. We identify $\text{End}_{\mathbb{H}}(H)$ with the real form of $\text{End}_{\mathbb{C}}(H) = H \otimes \mathbb{C} H^*$ with respect to $J_H \otimes J_{H^*}$. Then $Q$ is contained in $\text{End}_{\mathbb{C}}(H)$. Let $u$ be an $\mathbb{H}$-frame of $H$. We define local sections $I, J, K$ of $\text{End}_{\mathbb{H}}(H)$ as $I(hu) = hiu$, $J(hu) = hju$, $K(hu) = hku$ for any $h \in \mathbb{H}$. Then $\{I, J, K\}$ is a local basis of $Q$ and represented by elements

$$I = i(u \otimes u^* - ju \otimes (ju)^*), \quad J = ju \otimes u^* - u \otimes (ju)^*, \quad K = i(ju \otimes u^* + u \otimes (ju)^*) \ (1)$$

of $\text{End}_{\mathbb{C}}(H)$ for the $\mathbb{C}$-frame $\{u, ju\}$ of $H$. Let $Z$ be a sphere bundle

$$Z = \{aI + bJ + cK \in Q \mid a^2 + b^2 + c^2 = 1\}$$

over $M$. Let $f : Z \to M$ denote the projection. The bundle $Z$ is called a twistor space of the quaternionic Kähler manifold $M$. Let $I'$ be an element of $Z$. We set $x$ as the point $f(I')$ of $M$. There exists an element $(u')^* \in H_x^*$ such that $I' = i(u' \otimes (u')^* - ju' \otimes (ju')^*)$. The element $I'$ is extended to a complex structure of $T_x M$ by the decomposition $TM \otimes \mathbb{C} = E \otimes H$. It follows from $(ju')^* = -(u')^* j$ that

$$\wedge^{1,0} T_x^* M = E_x^* \otimes \langle (u')^* \rangle_\mathbb{C}, \quad \wedge^{0,1} T_x^* M = E_x^* \otimes \langle (u')^* j \rangle_\mathbb{C}.$$ 

Thus any element of $Z$ defines a complex structure of the tangent space of $M$. Let $p : \text{P}(H^*) \to M$ be a frame bundle of $H^*$, whose fiber consists of right $\mathbb{H}$-bases of $H^*$. 
Then \( P(H^*) \) is a principal \( GL(1, \mathbb{H}) \)-bundle by the right action. The twistor space \( Z \) is regarded as the quotient space \( P(H^*)/GL(1, \mathbb{C}) \). We denote by \( \pi : P(H^*) \to Z \) the quotient map and regard \( P(H^*) \) as a principal \( GL(1, \mathbb{C}) \)-bundle over \( Z \). By the definition, the twistor space \( Z \) is a \( CP^1 \)-bundle over \( M \).

### 3. The Principal Bundle \( P(H^*) \)

Let \((M, g)\) be a quaternionic Kähler manifold of dimension \(4n\). The frame bundle of \( H^* \) is the principal \( GL(1, \mathbb{H}) \)-bundle \( p : P(H^*) \to M \). Let \( \mathcal{A}^0(\wedge^k E \otimes S^m H) \) denote the sheaf of \( \wedge^k E \otimes S^m H \)-valued smooth \( q \)-forms on \( M \).

#### 3.1. Lift of \( \mathcal{A}^0(\wedge^k E \otimes S^m H) \) to \( P(H^*) \).

In this section, the bundles \( H \) and \( H^* \) are regarded as bundles of the left \( \mathbb{C} \)-module and the right \( \mathbb{C} \)-module, respectively. We denote the complex representation \( \rho \) of \( GL(1, \mathbb{H}) \) on \( \mathbb{H} \) by \( \rho(a)h = ah \) for \( a \in GL(1, \mathbb{H}) \) and \( h \in \mathbb{H} \). The dual representation \( \rho^* \) of \( \rho \) is given by \( \rho^*(a)h = ha^{-1} \) for \( a \in GL(1, \mathbb{H}) \) and \( h \in \mathbb{H} \). Then \( H \) is the associated bundle \( P(H^*) \times_{\rho^*} \mathbb{H} \) with the representation \( \rho^* \). Let \( S^m \mathbb{H} \) be the \( \mathbb{C} \)-vector space \( S^m \mathbb{C}^2 \). We denote by \( s^m \rho^* \) the representation of \( GL(1, \mathbb{H}) \) on the \( m \)-th symmetric tensor \( S^m \mathbb{H} \) of the \( \mathbb{C} \)-vector space \( \mathbb{H} \) induced by \( \rho^* \). The bundle \( S^m H \) over \( M \) is given by the associated bundle \( S^m H = P(H^*) \times_{s^m \rho^*} S^m \mathbb{H} \) with the representation \( s^m \rho^* \). In the case \( m = 0 \), \( \rho^0 = id_\mathbb{C} \) and \( S^0 H = \mathbb{C} \).

The point \( u^* \in P(H^*) \) induces the point \( u \) of \( P(H) \) by taking the dual \( \mathbb{H} \)-basis of \( H \) at \( p(u^*) \). The \( \mathbb{H} \)-basis \( u \) provides the \( \mathbb{C} \)-basis \( \{u, ju\} \) of the \( \mathbb{C} \)-vector bundle \( H \). Thus, any element \( u \in P(H) \) is regarded as a \( \mathbb{C} \)-isomorphism \( u : \mathbb{H} \to H_{p(u)} \), and it induces a \( \mathbb{C} \)-isomorphism \( S^m \mathbb{H} \to S^m H_{p(u)} \), which is denoted by \( u \) for simplicity. For a section \( \xi \) of \( S^m H \to M \), a section \( \xi \) of the trivial bundle \( S^m \mathbb{H} \to P(H^*) \) is given by \( \xi_u = u^{-1}(\xi(p(u))) \) at \( u^* \in P(H^*) \). Then \((Ra)^* \xi = (s^m \rho^*)(a^{-1}) \xi \) for \( a \in GL(1, \mathbb{H}) \). We write \( \rho_m(a) \) as \( \rho_m(a) = (s^m \rho^*)(a^{-1}) \) for \( a \in GL(1, \mathbb{H}) \). Conversely, any section \( \xi \) of \( S^m H \to M \) is induced by a section \( \tilde{\xi} \) of \( S^m \mathbb{H} \to P(H^*) \) such that \((Ra)^* \tilde{\xi} = \rho_m(a) \tilde{\xi} \) for any \( a \in GL(1, \mathbb{H}) \).

Let \( \mathcal{A}^q(S^m \mathbb{H}) \) and \( \mathcal{A}^0_{P(H^*)}(S^m \mathbb{H}) \) be the sheaf of \( S^m \mathbb{H} \)-valued smooth \( q \)-forms on \( M \) and \( P(H^*) \), respectively. We consider the pull-back \( p^* \mathcal{A}^q(S^m \mathbb{H}) \) as the subsheaf of \( \mathcal{A}^q_{P(H^*)}(S^m \mathbb{H}) \). If \( \tilde{\xi} \) is an element of the inverse image \( p^{-1}p_* (p^* \mathcal{A}^q(S^m \mathbb{H})) \) of the direct image of \( p^* \mathcal{A}^q(S^m \mathbb{H}) \), then \((Ra)^* \tilde{\xi} \) is in \( p^{-1}p_*(p^* \mathcal{A}^q(S^m \mathbb{H})) \) for any \( a \in GL(1, \mathbb{H}) \). We define a sheaf \( \mathcal{A}^q_m(S^m \mathbb{H}) \) by

\[
\mathcal{A}^q_m(S^m \mathbb{H}) = \{ \tilde{\xi} \in p^{-1}p_* (p^* \mathcal{A}^q(S^m \mathbb{H})) \mid (Ra)^* \tilde{\xi} = \rho_m(a) \tilde{\xi}, \forall a \in GL(1, \mathbb{H}) \}.
\]

The sheaf \( \mathcal{A}^q_0(S^0 \mathbb{H}) \) is just that of pull-back of smooth \( q \)-forms on \( M \) by \( p \). We denote \( \mathcal{A}^q_0(S^0 \mathbb{H}) \) by \( \mathcal{A}^q \) for simplicity. In particular, \( \mathcal{A}^0 \) is the sheaf of smooth functions on \( P(H^*) \) which are constant along each fiber of \( p \). Then

\[
\mathcal{A}^q_m(S^m \mathbb{H}) = \mathcal{A}^q \otimes_{\mathcal{A}^0} \mathcal{A}^q_m(S^m \mathbb{H}).
\]

An element \( \xi \in \mathcal{A}^q(S^m H) \) induces \( \xi \in \mathcal{A}^q_m(S^m \mathbb{H}) \) by

\[
\tilde{\xi}_u^* = u^{-1}(p^* \xi)_u^*.
\]

at each point \( u^* \in P(H^*) \). The Levi-Civita connection of \( M \) induces a connection \( \nabla \) on \( H \), and the covariant exterior derivative \( d^\nabla : \mathcal{A}^q(S^m H) \to \mathcal{A}^{q+1}(S^m H) \). The tangent space \( T_{u^*} P(H^*) \) is decomposed into the horizontal space \( \mathcal{H}_{u^*} \) and the vertical space \( \mathcal{V}_{u^*} \) at \( u^* \in P(H^*) \). We define \( d_{\mathcal{H}} : \mathcal{A}^q_m(S^m \mathbb{H}) \to \mathcal{A}^{q+1}_m(S^m \mathbb{H}) \) by the exterior derivative restricted to the horizontal \( \mathcal{H} \). The sheaf \( \mathcal{A}^q(S^m H) \) is isomorphic to \( \mathcal{A}^q_m(S^m \mathbb{H}) \) by the correspondence \( \xi \mapsto \tilde{\xi} \). Moreover, \( d^\nabla \xi = d_{\mathcal{H}} \tilde{\xi} \) for any \( \xi \in \mathcal{A}^q(S^m H) \) (c.f. Chapter II, §5 in \[5\]).
We fix a point $u_0^q$ of $P(H^*)$. The complex coordinate $(z,w)$ of the fiber is given by $u_0^q(z+jw)$. A function $f$ on $P(H^*)$ is a polynomial of degree $(m-i,i)$ along fiber if $f(u_0^q(z+jw))$ is a polynomial of $z, w, \bar{z}, \bar{w}$ of degree $m$ such that $(R_c)^* f = c^{m-i} \circ f$ for $c \in \text{GL}(1, \mathbb{C})$. We denote by $\mathcal{A}_0^{m}(m-i,i)$ the sheaf of elements of $p^{-1} p_* \mathcal{A}_0^{0}(P(H^*)|\mathbb{C})$ which are polynomials of degree $(m-i,i)$ along fiber on $P(H^*)$. We also define a sheaf $\mathcal{A}_0^{m}(m-i,i)$ as

$$\mathcal{A}_0^{m}(m-i,i) = \mathcal{A}_0^q \otimes \mathcal{A}_0^{0}(m-i,i).$$

Let $a_1 a_2 \cdots a_m$ denote the symmetrization $\frac{1}{m!} \sum_{\sigma \in S_m} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(m)}$ of $a_1 \otimes \cdots \otimes a_m \in \otimes^m \mathbb{H}$ where $S_m$ is the symmetric group of degree $m$. The set $\{1^m, 1^m-1^1, 1^2^1^2, \ldots , j^m\}$ is a $\mathbb{C}$-basis of the vector space $S^m \mathbb{H}$. An $S^m \mathbb{H}$-valued $q$-form $\xi$ on $P(H^*)$ is given by

$$\tilde{\xi} = \tilde{\xi}_0 1^m + \tilde{\xi}_1 1^{m-1}j + \tilde{\xi}_2 1^{m-2}j^2 + \cdots + \tilde{\xi}_m j^m$$

for $q$-forms $\tilde{\xi}_0, \ldots, \tilde{\xi}_m$ on $P(H^*)$.

**Proposition 3.1.** If $\tilde{\xi}$ is in $\mathcal{A}_0^{m}(S^m \mathbb{H})$, then $\tilde{\xi}_0 \in \mathcal{A}_0^{m}(m,0,0)$. Conversely, for $\tilde{\xi}_0 \in \mathcal{A}_0^{m}(m,0,0)$, there exists a unique section $\tilde{\xi} \in \mathcal{A}_0^{m}(S^m \mathbb{H})$ in which the coefficient of $1^m$ is $\tilde{\xi}_0$.

**Proof.** It suffices to show the case $q = 0$. Let $\tilde{\xi}$ be an $S^m \mathbb{H}$-valued function on $P(H^*)$. We fix a point $u_0^q$ of $P(H^*)$. The complex coordinate $(z,w)$ of the fiber is given by $u_0^q(z+jw)$. If we take $a = z + jw$, then there exists

$$\begin{pmatrix}
(1a)^m \\
(1a)^{m-1}ja \\
\vdots \\
(ja)^m
\end{pmatrix} =
\begin{pmatrix}
p_{00} & p_{01} & \cdots & p_{0m} \\
p_{10} & p_{11} & \cdots & p_{1m} \\
p_{m0} & p_{1m} & \cdots & p_{mm}
\end{pmatrix}
\begin{pmatrix}
1^m \\
1^{m-1}j \\
\vdots \\
j^m
\end{pmatrix}
$$

where $p_{ij}$ is a polynomial of $z, w, \bar{z}, \bar{w}$ of degree $m$. Each component $p_{ij}$ is a polynomial of degree $(m-i', i')$ for $i, i' = 0, 1, \ldots, m$. In particular, $p_{i0} = (-1)^{i} z^{m-i} w^i$ for $i = 0, 1, \ldots, m$.

It follows from $(R_\alpha)^* \xi = \rho_\beta(a) \xi$ that $\tilde{\xi}_0(u_0^q(a)) = \sum_{i=0}^{m} \tilde{\xi}_i(u_0^q)^{m} p_{ij} = \sum_{i=0}^{m} (-1)^{i} \xi_i(u_0^q)^{m} z^{m-i} w^i$. Conversely, $\tilde{\xi}_0(u_0^q) = \sum_{i=0}^{m} \xi_i(a) c_i z^{m-i} w^i$ for complex number $c_0, c_1, \ldots, c_m$. We define an $S^m \mathbb{H}$-valued function $\xi$ on $P(H^*)$ as $\tilde{\xi}_0(a) = \sum_{i=0}^{m} (-1)^i c_i a^{m-i} (ja)^i$ for any $a \in \text{GL}(1, \mathbb{H})$. Then $\tilde{\xi}$ is in $\mathcal{A}_0^{m}(S^m \mathbb{H})$. By the definition, the coefficient of $1^m$ in $\tilde{\xi}_0^q$ is $\sum_{i=0}^{m} (-1)^i c_i p_{i0} = \sum_{i=0}^{m} c_i z^{m-i} w^i = \xi_0$, and hence it completes the proof.

The coefficient $\tilde{\xi}_i$ of $\tilde{\xi}$ is in $\mathcal{A}_0^{m}(m-i,i)$. **Proposition 3.1** implies the following

**Corollary 3.2.** The sheaf $\mathcal{A}_0^{q}(S^m H)$ is isomorphic to $\mathcal{A}_0^{m}(m,0,0)$ by the correspondence $\xi \mapsto \tilde{\xi}_0$. Moreover, $(d\overline{\nabla})_0 = d_R \tilde{\xi}_0$ for any $\xi \in \mathcal{A}_0^{q}(S^m H)$. 

We extend the above argument of $\mathcal{A}_0^{q}(S^m H)$ to $\mathcal{A}_0^{q}(\wedge^k E \otimes S^m H)$ as follows. Let $\mathcal{A}_0^{q}(\wedge^k E)$ denote the sheaf of pull-back of $\wedge^k E$-valued smooth $q$-forms on $M$ by $p$. We define $\mathcal{A}_0^{m}(\wedge^k E \otimes S^m H)$ and $\mathcal{A}_0^{q}(m-i,i)(\wedge^k E)$ as the sheaves

$$\mathcal{A}_0^{m}(\wedge^k E \otimes S^m H) = \mathcal{A}_0^{q}(\wedge^k E) \otimes \mathcal{A}_0^{0}(S^m H)$$

and

$$\mathcal{A}_0^{q}(m-i,i)(\wedge^k E) = \mathcal{A}_0^{q}(\wedge^k E) \otimes \mathcal{A}_0^{0}(m-i,i).$$

Any element $\tilde{\xi}$ of $\mathcal{A}_0^{m}(\wedge^k E \otimes S^m H)$ is written by

$$\tilde{\xi} = \tilde{\xi}_0 1^m + \tilde{\xi}_1 1^{m-1}j + \tilde{\xi}_2 1^{m-2}j^2 + \cdots + \tilde{\xi}_m j^m$$

(2)
for \( p^{-1}(\wedge^k E) \)-valued 1-forms \( \tilde{\xi}_0, \ldots, \tilde{\xi}_m \). The Levi-Civita connection induces connections of \( E, H \) and the covariant exterior derivative \( d^\nabla : \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \to \mathcal{A}^{k+1}(\wedge^k E \otimes \mathbb{H}) \).

Let \( \mathcal{H} \) be the horizontal subbundle of \( TP(H^*) \). By the same argument in Proposition 3.1 and Corollary 3.2, we obtain

**Corollary 3.3.** (i) \( \mathcal{A}^q_m(\wedge^k E \otimes \mathbb{H}) \cong \mathcal{A}_{(m,0)}^q(\wedge^k E) \) by \( \tilde{\xi} \mapsto \tilde{\xi}_0 \). Moreover, \( d_\mathcal{H} \tilde{\xi}_0 = d_\mathcal{H} \tilde{\xi}_0 \) for any \( \tilde{\xi} \in \mathcal{A}^q_m(\wedge^k E \otimes \mathbb{H}) \).

(ii) \( \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \cong \mathcal{A}_{(m,0)}^q(\wedge^k E) \) by \( \xi \mapsto \tilde{\xi}_0 \). Moreover, \( d_\mathcal{H} \xi_0 = d_\mathcal{H} \tilde{\xi}_0 \) for any \( \xi \in \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \).

For \( \xi \in \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \), the element \( \tilde{\xi}_0 \in \mathcal{A}_{(m,0)}^q(\wedge^k E) \) is said to be a lift to \( P(H^*) \).

### 3.2. Real structures on \( P(H^*) \)

We define an anti-\( \mathbb{C} \)-linear map \( \tau : \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \to \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \) by

\[
\tau(\xi) = \sum_i (J^k_E \otimes J^{\overline{k}}_E)(v_i) \otimes \overline{\alpha}^i
\]

for \( \xi = \sum v_i \otimes \alpha^i \) where \( \{v_i\} \) is a frame of \( \wedge^k E \otimes \mathbb{H} \) and \( \alpha^i \) is a \( q \)-form. We denote by \( \mathcal{A}^q(\wedge^k E \otimes \mathbb{H})^\tau \) the sheaf of \( \tau \)-invariant elements of \( \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \). The map \( \tau \) is a real structure of \( \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \) if \( k + m \) is even. Especially, in the case \( k = m = 1 \), \( \tau \) is the complex conjugate on \( \mathcal{A}^q(TM \otimes \mathbb{C}) \).

Let \( \mathcal{A}^q_P(H^*) \) denote the inverse image \( \mathcal{A}^q_P(H^*)(p^{-1}(\wedge^k E)) \) of the direct image of the sheaf \( \mathcal{A}^q_P(H^*)(p^{-1}(\wedge^k E)) \). If we take a frame \( \{e_i\} \) of \( \wedge^k E \), then any element \( \beta \) of \( \mathcal{A}^q_P(H^*)(\wedge^k E) \) is written by \( \beta = \sum_i e_i \otimes \alpha^i \) for some \( q \)-forms \( \alpha^i \). An endomorphism \( \bar{\tau} \) of \( \mathcal{A}^q_P(H^*)(\wedge^k E) \) is defined as \( \bar{\tau}(\beta) = \sum_i J^k_E r^i j^i \otimes \overline{\alpha}^i \). We also denote \( \bar{\tau}(\beta) \) by \( J^k_E r^i j^i \).

Moreover, we extend the endomorphism \( \bar{\tau} \) to that of \( \mathcal{A}^q_P(H^*)(\wedge^k E \otimes \mathbb{H}) \) by

\[
\bar{\tau}(\beta \otimes 1^{m-i}j^i) = J^k_E r^i j^i \otimes 1^{m-i}j^i
\]

for \( \beta \otimes 1^{m-i}j^i \in \mathcal{A}^q_P(H^*)(\wedge^k E \otimes \mathbb{H}) \). The map \( \bar{\tau} \) is anti-\( \mathbb{C} \)-linear and \( \bar{\tau}^2 = (-1)^k R_{-1}^{-1} \).

**Proposition 3.4.** The map \( \bar{\tau} \) defines endomorphisms of \( \mathcal{A}^q_m(\wedge^k E \otimes \mathbb{H}) \) and \( \mathcal{A}^q_{(m-i,i)}(\wedge^k E) \) such that \( \bar{\tau}(\tilde{\xi}) = \tau(\xi) \) and \( \bar{\tau}(\tilde{\xi}_i) = \tau(\xi)_i \) for \( \xi \in \mathcal{A}^q(\wedge^k E \otimes \mathbb{H}) \). Moreover, \( \bar{\tau} \) induces real structures on \( \mathcal{A}^q_m(\wedge^k E \otimes \mathbb{H}) \) and \( \mathcal{A}^q_{(m-i,i)}(\wedge^k E) \) if \( k + m \) is even.

**Proof.** Let \( \tilde{\xi} \) be an element of \( \mathcal{A}^q_m(\mathbb{H}) \). Then \( R_{z^j w}^z R_{w}^{z} \tilde{\xi} = R_{z^j w}^z (R_{z}^{z} \tilde{\xi}) = R_{z^j w}^{z} R_{z}^{z} \tilde{\xi} = R_{z^j}^{z} R_{z}^{z} \tilde{\xi} = \rho_m(z + jw) \xi = \rho_m(z + jw) R_{z}^{z} \tilde{\xi} \). Hence \( \bar{\tau}(\tilde{\xi}) \) is in \( \mathcal{A}^q_m(\mathbb{H}) \). Let \( f \) be a polynomial of degree \( (m-i,i) \) along fiber. Since \( R_{j}^{z} f(z^j w) = f (u^j (z^j w)) = f(u^j (-\bar{w} + j\bar{z})) \), \( \bar{\tau}(f) \) is a polynomial of \( z, w, \bar{z}, \bar{w} \) of degree \( m \). Furthermore, \( (R_{c})^* \bar{\tau}(f) = (R_{c})^* R_{j}^{z} f = R_{j}^{z} R_{c}^{c} f = c^{m-i} c^{i} f = c^{m-i} c^{i} \bar{\tau}(f) \) for \( c \in \text{GL}(1, \mathbb{C}) \). Hence \( \bar{\tau}(f) \) is in \( \mathcal{A}^q_{(m-i,i)} \). By the bundle map \( J^k_E : \wedge^k E \to \wedge^k E, \bar{\tau} \) is extended to endomorphisms of \( \mathcal{A}^q_m(\wedge^k E \otimes \mathbb{H}) \) and \( \mathcal{A}^q_{(m-i,i)}(\wedge^k E) \). For \( \tilde{\xi} = \sum_i \tilde{\xi}_i \otimes 1^{m-i}j^i \),

\[
\bar{\tau}(\tilde{\xi}) = \sum_i J^k_E \tilde{\xi}_i \otimes \rho_m(j) 1^{m-i}j^i = \sum_i J^k_E \tilde{\xi}_i \otimes (1^{m-i}j^i).
\]

On the other hand, \( \xi \) is given by \( \xi = \sum_i \xi_i \otimes u^{m-i}(ju)^i \) for a frame \( u \) of \( H \). Then \( \tau(\xi) = \sum_i J^k_E \xi_i \otimes J_H^m(u^{m-i}(ju)^i) = \sum_i J^k_E \xi_i \otimes (-1)^i (ju)^{m-i} u^i \). Thus \( \tau(\xi) = \bar{\tau}(\tilde{\xi}) \). By the definition, \( \bar{\tau}(\tilde{\xi})_i = J^k_E \tilde{\xi}_i \otimes \bar{\tau}(\tilde{\xi})_i \). Hence \( \bar{\tau}(\tilde{\xi}_i) = \tau(\xi)_i \). It follows from \( \bar{\tau}^2 = (-1)^k R_{-1}^{-1} \).
that \( \tau^2 = (-1)^{k+m} \text{id} \) on \( \tilde{A}_{m0}^q(\wedge^k E \otimes \mathbb{S}^m \mathbb{H}) \) and \( \tilde{A}_{(m-i,i)}^q(\wedge^k E) \). If \( k + m \) is even, the anti-C-linear \( \tilde{\eta} \) is real structures on \( \tilde{A}_{m0}^q(\wedge^k E \otimes \mathbb{S}^m \mathbb{H}) \) and \( \tilde{A}_{(m-i,i)}^q(\wedge^k E) \). It completes the proof. \( \square \)

We also obtain

**Lemma 3.5.** Let \( \tilde{\xi} \) be an element of \( \tilde{A}_{m0}^q(\wedge^k E \otimes \mathbb{S}^m \mathbb{H}) \). Under the representation \( \mathbb{Z} \), \( \tilde{\xi} \) is \( \tilde{\eta} \)-invariant if and only if \( \tilde{\xi}_i \) is \( \tilde{\eta} \)-invariant for each \( i \), and \( \tilde{\xi}_i = (-1)^{m-i} J_E \tilde{\xi}_{m-i} \). \( \square \)

Let \( \tilde{A}_{m0}^q(\wedge^k E \otimes \mathbb{S}^m \mathbb{H}) \tilde{\eta} \) and \( \tilde{A}_{(m,0)}^q(\wedge^k E) \tilde{\eta} \) denote the sheaves of \( \tilde{\eta} \)-invariant elements of \( \tilde{A}_{m0}^q(\wedge^k E \otimes \mathbb{S}^m \mathbb{H}) \) and \( \tilde{A}_{(m,0)}^q(\wedge^k E) \), respectively. Corollary 3.3 and Proposition 3.4 imply the following corollary:

**Corollary 3.6.** \( \tilde{A}_{m}^q(\wedge^k E \otimes \mathbb{S}^m H)^\tilde{\eta} \cong \tilde{A}_{m0}^q(\wedge^k E \otimes \mathbb{S}^m \mathbb{H}) \tilde{\eta} \cong \tilde{A}_{(m,0)}^q(\wedge^k E) \tilde{\eta} \) by \( \xi \mapsto \tilde{\xi} \mapsto \tilde{\xi}_0 \). \( \square \)

3.3. **Canonical 1-form on** \( P(H^*) \). We define a \( p^{-1}(E) \otimes \mathbb{H} \)-valued 1-form \( \tilde{\theta} \) on \( P(H^*) \) as

\[
\tilde{\theta}_{u^*}(v) = u^{-1}(p_*(v))
\]

for \( v \in T_u P(H^*) \) at \( u^* \). The 1-form \( \tilde{\theta} \) is called the canonical 1-form on \( P(H^*) \). The identity map \( \text{id}_{TM} \) of \( TM \) is regarded as an element \( \theta \) of \( A^1(E \otimes H) \). The canonical form \( \tilde{\theta} \in \tilde{A}_1^q(E \otimes \mathbb{H}) \) is the lift of \( \theta \in \tilde{A}_1^q(E \otimes H) \). We define \( p^{-1}(E) \)-valued 1-forms \( \tilde{\theta}_0 \) and \( \tilde{\theta}_1 \) on \( P(H^*) \) as

\[
\tilde{\theta} = \tilde{\theta}_0 + \tilde{\theta}_1 j.
\]

Then \( \tilde{\theta}_0 \in \tilde{A}_1^q((1,0)) \) and \( \tilde{\theta}_1 \in \tilde{A}_1^q((0,1)) \). Since \( \theta = \text{id}_{TM} \) is \( \tau \)-invariant, the lift \( \tilde{\theta} \) is \( \tilde{\eta} \)-invariant. Lemma 3.5 implies that \( \tilde{\theta}_0 \) and \( \tilde{\theta}_1 \) are \( \tilde{\eta} \)-invariant, and \( \tilde{\theta}_1 = J_E \tilde{\theta}_0 \).

Let \( A \) denote the connection form of \( P(H^*) \). We consider \( H^* \) as the right \( \mathbb{C} \)-bundle, that is, the Lie algebra \( \mathfrak{gl}(1, \mathbb{H}) = \mathbb{H} \) is identified with \( \mathbb{C} + j \mathbb{C} \). Then \( A \) is written by

\[
A = \eta_0 + j \eta_1
\]

for complex valued 1-forms \( \eta_0, \eta_1 \) on \( P(H^*) \). The connection form \( A \) is \( \tilde{\eta} \)-invariant since \( R_j^2 A = ad(j)(A) = \tilde{A} \). It yields that \( \tilde{\eta}_0 \) and \( \tilde{\eta}_1 \) are \( \tilde{\eta} \)-invariant.

**Proposition 3.7.** \( dE \tilde{\theta}_0 = -\tilde{\theta}_0 \wedge \eta_0 - \eta_1 \wedge \tilde{\theta}_1, \ dE \tilde{\theta}_1 = -\tilde{\theta}_0 \wedge \tilde{\eta}_1 - \tilde{\theta}_1 \wedge \tilde{\eta}_0 \).

**Proof.** The element \( d_{\tilde{H}} \tilde{\theta} \in \tilde{A}_2^q(E \otimes \mathbb{H}) \) corresponds to the torsion \( d^\nabla \theta \in A^2(E \otimes H) \). It yields that \( d_{\tilde{H}} \tilde{\theta} = 0 \). It follows from \( d_{\tilde{H}} \tilde{\theta} = dE \tilde{\theta} + \tilde{\theta} \wedge A \) that \( dE \tilde{\theta} = -\tilde{\theta} \wedge A \). We obtain

\[
dE \tilde{\theta} = -(\tilde{\theta}_0 + \tilde{\theta}_1 j) \wedge (\eta_0 + j \eta_1) = -\tilde{\theta}_0 \wedge \eta_0 + \tilde{\theta}_1 \wedge \eta_1 - (\tilde{\theta}_0 \wedge \tilde{\eta}_1 + \tilde{\theta}_1 \wedge \tilde{\eta}_0) j.
\]

Hence we finish the proof. \( \square \)

Let \( s_H^2 \) denote the symmetrization \( \otimes^2 H \rightarrow S^2 H \). We define an \( S^2 H \)-valued 2-form \( \omega \) on \( M \) as

\[
\omega = \omega_E \otimes s_H^2.
\]

Then the lift \( \tilde{\omega} \in \tilde{A}_2^q(S^2 \mathbb{H}) \) is decomposed by

\[
\tilde{\omega} = \tilde{\omega}_0 1 \cdot 1 + \tilde{\omega}_1 1 \cdot j + \tilde{\omega}_2 j \cdot j
\]

for \( \tilde{\omega}_0 \in \tilde{A}_2^q(2,0), \tilde{\omega}_1 \in \tilde{A}_2^q(1,1) \) and \( \tilde{\omega}_2 \in \tilde{A}_2^q(0,2) \). The \( S^2 H \)-valued 2-form \( \omega \) is \( \tau \)-invariant since \( \tau(\omega_E \otimes s_H^2) = J_{H,1}^2(\omega_E) \otimes (J_E^2 \otimes \tilde{J}_{H*}^2)(s_H^2) = \omega_E \otimes s_H^2 \). It follows from Lemma 3.5.
that \(\tilde{\omega}_0, \tilde{\omega}_1\) and \(\tilde{\omega}_2\) are \(\tau\)-invariant, \(\tilde{\omega}_2 = \overline{\tilde{\omega}_0}\) and \(\tilde{\omega}_1 = -\overline{\tilde{\omega}_1}\). Since \(p^*\omega_E \otimes s^2(\theta_{u^*}(v), \theta_{u^*}(w)) \) for \(v, w \in T_{u^*} P(H^*)\) at \(u^*\),

\[
\omega_0 = \omega_E(\tilde{\theta}_0, \tilde{\theta}_0), \quad \tilde{\omega}_1 = \omega_E(\tilde{\theta}_0, \tilde{\theta}_1) + \omega_E(\tilde{\theta}_1, \tilde{\theta}_0), \quad \tilde{\omega}_2 = \omega_E(\tilde{\theta}_1, \tilde{\theta}_1).
\]

3.4. Curvature on \(P(H^*)\). Let \(\{I, J, K\}\) be a local basis of \(Q\) for a \(\mathbb{C}\)-frame \(\{u, ju\}\) of \(H\) as in (1). The dual bundle \(Q^*\) of \(Q\) is contained in the real form of \(\text{End}_\mathbb{C}(H^*) = H^* \otimes \mathbb{C} H\) with respect to \(JH^* \otimes JH\). The Killing form \(B_0\) on \(sp(1)\) induces a symmetric bi-linear form \(B\) on \(Q^*\).

**Lemma 3.8.** \(B = -2(I \otimes I + J \otimes J + K \otimes K)\).

**Proof.** Let \(I^*, J^*, K^*\) be the dual of endomorphisms \(I, J, K\), respectively. The dual basis \(\{(I^*)^\vee, (J^*)^\vee, (K^*)^\vee\}\) of \(\{I^*, J^*, K^*\}\) satisfies \((I^*)^\vee = -\frac{1}{2}I, (J^*)^\vee = -\frac{1}{2}J\) and \((K^*)^\vee = -\frac{1}{2}K\). The Killing form \(B_0\) is given by \(B_0 = -8(i^\vee \otimes i^\vee + j^\vee \otimes j^\vee + k^\vee \otimes k^\vee)\) on \(sp(1) = \langle i, j, k \rangle\). Hence \(B = -8((I^*)^\vee \otimes (I^*)^\vee + (J^*)^\vee \otimes (J^*)^\vee + (K^*)^\vee \otimes (K^*)^\vee) = -2(I \otimes I + J \otimes J + K \otimes K)\). \(\Box\)

The endomorphisms \(I, J, K\) of \(H\) induce almost complex structures on \(M\), locally. We define local 2-forms \(\omega_I, \omega_J, \omega_K\) and \(\omega_{K^*}\) on \(M\) by \(\omega_I(X, Y) = g(I'X, Y), \omega_J(X, Y) = g(J'X, Y)\) and \(\omega_{K^*}(X, Y) = g(K'X, Y)\) for \(X, Y \in TM\). They are written by

\[
\omega_I = \omega_E \otimes \frac{i}{|u|^2}(u^* \otimes (ju)^* + (ju)^* \otimes u^*), \quad \omega_J = \omega_E \otimes \frac{1}{|u|^2}(u^* \otimes u^* + (ju)^* \otimes (ju)^*),
\]

\[
\omega_K = \omega_E \otimes \frac{-i}{|u|^2}(u^* \otimes u^* - (ju)^* \otimes (ju)^*)
\]

for the \(\mathbb{C}\)-frame \(\{u, ju\}\) of \(H\). We define \(B^\sharp\) by a global \(Q\)-valued 2-form \(B^\sharp = -2(I \otimes \omega_I + J \otimes \omega_J + K \otimes \omega_K)\) on \(M\).

Let \(\Omega\) be the curvature form of \(P(H^*)\). We define a function \(r\) on \(P(H^*)\) by

\[
r(u^*) = |u^*|\]

for \(u^* \in P(H^*)\), where \(|\cdot|\) means the norm of \(H^*\).

**Proposition 3.9.** Let \(t\) be the scalar curvature of \(M\). Then \(\Omega = -2c_n tr^{-2}(\tilde{\omega}_1 + 2j\tilde{\omega}_0)\) where \(c_n\) is a positive number depending only on \(n\).

**Proof.** The curvature \(R_H\) of \(H\) is given by \(c_n t B^\sharp\) for a positive number \(c_n\) depending on \(n\) (c.f. [1], [2]). Then \(R_{H^*} = -c_n t (B^\sharp)^*\) where \((B^\sharp)^* = -2(I \otimes \omega_I + J^* \otimes \omega_J + K^* \otimes \omega_K)\). Hence \(R_{H^*} = 2c_n t(I' \otimes \omega_I + J^* \otimes \omega_J + K^* \otimes \omega_K)\). The curvature form \(\Omega\) of \(H^*\) is given by the \(sp(1)\)-valued 2-form

\[
\Omega = 2c_n t(i \otimes \omega_I + j \otimes \omega_J + k \otimes \omega_K).
\]

We take \(v, w \in T_{u^*} P(H^*)\). Then \(p_* (v) = e_0 \otimes u + e_1 \otimes ju\) and \(p_* (w) = e'_0 \otimes u + e'_1 \otimes ju\) for \(e_i, e'_i \in E_{P(u^*)}\). An easy calculation shows that

\[
p^* \omega_I (v, w) = i|u^*|^{-2}(\omega_E(e_0, e'_1) + \omega_E(e_1, e'_0)) = i|u^*|^{-2}(\tilde{\omega}_1)_{u^*}(v, w),
\]

\[
p^* \omega_J (v, w) = i p^* \omega_K (v, w) = 2|u^*|^{-2}\omega_E(e_0, e'_0) = -2|u^*|^{-2}(\tilde{\omega}_0)_{u^*}(v, w)
\]

at \(u^*\). Thus we obtain

\[
i \omega_I = -r^{-2} \tilde{\omega}_1, \quad \omega_J - i \omega_K = -2r^{-2} \tilde{\omega}_0
\]

on \(P(H^*)\). Hence \(\Omega = 2c_n t(i \omega_I + j \omega_J + k \omega_K) = -2c_n tr^{-2}(\tilde{\omega}_1 + 2j\tilde{\omega}_0)\). \(\Box\)

From now on, we set \(c = 2c_n t\). Then \(\Omega = -cr^{-2}(\tilde{\omega}_1 + 2j\tilde{\omega}_0)\). By \(dA = \Omega - A \wedge A\), we obtain

**Proposition 3.10.** \(\eta_0 = -cr^{-2} \tilde{\omega}_1 - \eta_1 \wedge \overline{\eta}_1, \quad \eta_1 = -2cr^{-2} \tilde{\omega}_0 + \eta_0 \wedge \eta_1 + \eta_1 \wedge \overline{\eta}_0\). \(\Box\)
3.5. **Complex structure on** $P(H^*)$. We identify each fiber of $p : P(H^*) \to M$ with $\mathbb{C}^2 \setminus \{0\}$ by $\mathbb{H} = \mathbb{C} + j\mathbb{C} \simeq \mathbb{C}^2$. Then the fiber admits a complex structure. Let $\mathring{\cal V}$ be a vector bundle consisting of tangent vectors to fibers. We denote by $i_{\mathring{\cal V}}$ the complex structure of $\mathring{\cal V}$. The horizontal $\mathring{\cal H} \otimes \mathbb{C}$ is isomorphic to $TM \otimes \mathbb{C} = E \otimes H$ by $dp$. The element $I$ as in (1) induces a complex structure of $\mathring{\cal H}$ such that $\mathring{\cal H}^{1,0} \simeq E \otimes u$ and $\mathring{\cal H}^{0,1} \simeq E \otimes ju$ at $u^*$, denoted also by $I$. We define an almost complex structure $\mathring{I}$ on $P(H^*)$ by

$$\mathring{I} = I + i_{\mathring{\cal V}}$$

under the decomposition $TP(H^*) = \mathring{\cal H} \oplus \mathring{\cal V}$. Then $\eta_0$ and $\eta_1$ are $(1,0)$-forms. Actually, we fix a point $x$ of $M$ and take an unitary frame $u_0^*$ of $P(H^*)$ with $\nabla u_0^* = 0$ at $p(u_0^*)$. At $u^* = u_0^*(z + jw)$,

$$\eta_0 = r^{-2}(\bar{z}dz + \bar{w}dw), \quad \eta_1 = r^{-2}(-wdz + zdw).$$

By the definition of $\bar{\theta}$, the $p^{-1}(E)$-valued 1-forms $\bar{\theta}_0$ and $\bar{\theta}_1$ are $(1,0)$ and $(0,1)$-forms, respectively.

**Proposition 3.11.** (c.f. Theorem 4.1 in [2], Theorem 4.1 in [8]) The almost complex structure $\mathring{I}$ is integrable.

**Proof.** Taking an open set $U$ of $P(H^*)$ and a local basis $\{e_i\}$ of $E$, then we define local 1-forms $\alpha^i, \beta^i$ by $\bar{\theta}_0 = \sum_{i=1}^{2n} e_i \otimes \alpha^i$ and $\bar{\theta}_1 = \sum_{i=1}^{2n} e_i \otimes \beta^i$. Let $D$ denote the distribution $\langle \alpha^1, \alpha^2, \cdots, \alpha^{2n}, \eta_0, \eta_1 \rangle$ on $U$. Then $\Lambda^{1,0}TP(H^*) = D$ on $U$. It follows from Proposition 3.10 the $(2,0)$-form $\bar{\omega}_0$ and the $(1,1)$-form $\bar{\omega}_1$ that $d\eta_0, d\eta_1 \in D \wedge A^1$ where $A^1$ is the sheaf of differential 1-forms on $P(H^*)$. Proposition 3.7 implies $d\alpha^i \in D \wedge A^1$. It turns out that $dD \subset D \wedge A^1$, and hence the almost complex structure is integrable. \hfill $\Box$

**Remark 3.12.** We take a torsion free connection $\nabla$ of $TP(H^*)$ preserving $\mathring{I}$. Let $F$ be a holomorphic vector bundle on $P(H^*)$ and $\nabla_F$ a $(1,0)$-connection $\nabla_F : F \to F \otimes T^*$ of $F$. We consider the connection $\nabla_{F \otimes \Lambda^q}$ of $F \otimes \Lambda^q$ as the map $F \otimes \Lambda^q \to F \otimes \Lambda^q \otimes T^*$. Then the covariant exterior derivative $d_{\nabla_F}$ is given by $(-1)^q \Lambda \circ \nabla_{F \otimes \Lambda^q}$. The operator $\partial_F : F \otimes \Lambda^{q,0} \to F \otimes \Lambda^{q+1}$ satisfies $\partial_F = (-1)^q \Lambda \circ \nabla_{F \otimes \Lambda^q}$. If $\partial_F \alpha = \sum_{i,j} \beta_i \wedge \gamma_j$ for $\beta_j \in \Lambda^{q,0}, \gamma_i \in \Lambda^{0,1}$, then $\nabla_{F \otimes \Lambda^q} \alpha = (-1)^q \sum_{i,j} \beta_i \wedge \gamma_j$.

**Lemma 3.13.**

$$\nabla^{0,1}_0 \eta_0 = cr^{-2} \omega_E(\bar{\theta}_0, \bar{\theta}_1) + \eta_1 \otimes \pi_1,$$

$$\nabla^{0,1}_1 \eta_1 = -\eta_1 \otimes \eta_0,$$

$$\nabla^{0,1}_{E \otimes \Lambda^q} \bar{\theta}_0 = \eta_1 \otimes \bar{\theta}_1.$$

**Proof.** Proposition 3.10 and 3.11 imply that

$$\partial \eta_0 = -cr^{-2} \omega_1 - \eta_1 \wedge \pi_1, \quad \partial \eta_1 = \eta_1 \wedge \pi_0, \quad \partial_E \bar{\theta}_0 = -\eta_1 \wedge \bar{\theta}_1.$$

By Remark 3.12, we obtain the formula in this Lemma. \hfill $\Box$

From now on, we write $\nabla^{0,1}_{E \otimes \Lambda^q} \bar{\theta}_0$ instead of $\nabla^{0,1}_{E \otimes \Lambda^q} \bar{\theta}_0$ for simplicity. We define a $p^{-1}(\Lambda^k E)$-valued $(k,0)$-form $\tilde{\theta}_0^k$ by the $k$-th wedge $\sum_{i_1, \cdots, i_k=1}^{2n} e_{i_1} \wedge \cdots \wedge e_{i_k} \otimes \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}$ of $\bar{\theta}_0 = \sum_{i=1}^{2n} e_i \otimes \alpha_i$. Lemma 3.13 implies the following:
Proposition 3.14.
\[ \nabla^0,1\tilde{\theta}_k = k\tilde{\theta}_0^{k-1} \wedge \eta_1 \wedge E \tilde{\theta}_1 \\
\nabla^0,1(\tilde{\theta}_0^{-1} \wedge \eta_0) = -(k-1)\tilde{\theta}_0^{-k-2} \wedge \eta_0 \wedge \eta_1 \wedge E \tilde{\theta}_1 + \tilde{\theta}_0^{-k-1} \wedge (c r^{-2} \omega_E(\tilde{\theta}_0, \tilde{\theta}_1) + \eta_1 \otimes \bar{\eta}_1), \\
\nabla^0,1(\tilde{\theta}_0^{-1} \wedge \eta_1) = -\tilde{\theta}_0^{-k-1} \wedge \eta_1 \otimes \bar{\eta}_0, \\
\nabla^0,1(\tilde{\theta}_0^{-2} \wedge \eta_0 \wedge \eta_1) = -\tilde{\theta}_0^{-k-2} \wedge \eta_1 \wedge (cr^{-2} \omega_E(\tilde{\theta}_0, \tilde{\theta}_1) - \eta_0 \otimes \bar{\eta}_0). \]

\[ \square \]

3.6. Holomorphic symplectic structure and hyperkähler structure on \( P(H^*) \).

We recall that the function \( r \) on \( P(H^*) \) is given by \( r(u^*) = |u^*|^2_{p(u^*)} \) at \( u^* \in P(H^*) \).

Lemma 3.15. \( dr^2 = r^2(\eta_0 + \bar{\eta}_0), \quad \bar{\partial} r^2 = r^2 \bar{\eta}_0 \)

Proof. The symplectic form \( \omega_{H^*} \) of \( H^* \) is the element of \( \mathcal{A}^0(\wedge^2 H) \). Then \( (\omega_{H^*}) = r^2 1 \wedge j \) on \( P(H^*) \), and \( d\bar{\theta}(\omega_{H^*}) = 0 \) since \( \nabla \omega_{H^*} = 0 \). It follows from \( d\bar{\theta}(\omega_{H^*}) = d(\omega_{H^*}) - (\eta_0 + \bar{\eta}_0) \wedge \bar{\theta} \) that \( dr^2 = r^2(\eta_0 + \bar{\eta}_0) \). Immediately, it implies that \( \bar{\partial} r^2 = r^2 \bar{\eta}_0 \). \[ \square \]

The complex manifold \( P(H^*) \) has a holomorphic symplectic structure if the scalar curvature \( t \) of \( M \) is not zero as follows:

Proposition 3.16. The form \( r^2 \eta_1 \) is a holomorphic \((1,0)\)-form on \( P(H^*) \). If \( t \neq 0 \), then \( d(r^2 \eta_1) \) is a holomorphic symplectic form on \( P(H^*) \).

Proof. Proposition 3.10 and Lemma 3.15 imply that \( d(r^2 \eta_1) = 2(-c \bar{\omega}_0 + r^2 \eta_0 \wedge \eta_1) \). It turns out that \( d(r^2 \eta_1) \) is a \((2,0)\)-form, and so \( r^2 \eta_1 \) is a holomorphic \((1,0)\)-form on \( P(H^*) \). The 2-form \( \eta_0 \wedge \eta_1 \) is non-degenerate on \( \mathcal{V}^1,0 \). If \( t \neq 0 \), then \( c \bar{\omega}_0 \) is non-degenerate on \( \mathcal{V}^1,0 \). Hence, the 2-form \( d(r^2 \eta_1) \) is a non-degenerate holomorphic \((2,0)\)-form on \( P(H^*) \). \[ \square \]

The quaternionic vector space \( \mathbb{H} \) is regarded as \( \mathbb{C}_j + k \mathbb{C}_j \) where \( \mathbb{C}_j \) means a complex vector space with respect to \( j \). Then a complex structure \( j_{\bar{\mathcal{V}}} \) of \( \mathcal{V} \) is induced by the decomposition of \( \mathbb{H} \) as the same manner as \( i_{\bar{\mathcal{V}}} \). Considering the decomposition \( \mathbb{H} = \mathbb{C}_k + i \mathbb{C}_k \), where \( \mathbb{C}_k \) is a complex vector space with respect to \( k \), then we also obtain a complex structure \( k_{\bar{\mathcal{V}}} \) of \( \mathcal{V} \). The endomorphisms \( J, K \) of \( H \) defined by \( (1) \) also induce complex structures of \( \mathcal{H} \), which denote by \( J, K \), respectively. Then \( \bar{J} = J + i \bar{\mathcal{V}}, \bar{K} = K + k_{\bar{\mathcal{V}}} \) are complex structures on \( P(H^*) \) by the same argument of Proposition 3.11. We remark that \( \bar{I} \bar{J} = -\bar{K} \). Hence \( (\bar{I}, \bar{J}, -\bar{K}) \) is a hyperkähler structure on \( P(H^*) \). We define symmetric 2-forms \( g_{\bar{\mathcal{V}}} \) and \( \bar{g} \) by

\[ g_{\bar{\mathcal{V}}} = \eta_0 \wedge \bar{\eta}_0 + \bar{\eta}_0 \wedge \eta_0 + \eta_1 \wedge \bar{\eta}_1 + \bar{\eta}_1 \wedge \eta_1 \]

and

\[ \bar{g} = r^2(c p^* g + g_{\bar{\mathcal{V}}}) \]

on \( P(H^*) \).

Proposition 3.17. If \( t > 0 \), then \( (\bar{g}, \bar{I}, \bar{J}, -\bar{K}) \) is a hyperkähler structure on \( P(H^*) \) such that \( -id(r^2 \eta_0), d(r^2 \eta_1) \) are Kähler forms with respect to \( \bar{I}, \bar{J}, -\bar{K} \), respectively.

Proof. It follows from Proposition 3.10, Lemma 3.15 and the equation \( (3) \) that

\[ -id(r^2 \eta_0) = r^2(c \omega_I + i(\eta_0 \wedge \bar{\eta}_0 + \eta_1 \wedge \bar{\eta}_1)), \]

\[ d(r^2 \eta_1) = r^2(c \omega_J - i \omega_K) + 2 \eta_0 \wedge \eta_1. \]

It is easy to see that \( i(\eta_0 \wedge \bar{\eta}_0 + \eta_1 \wedge \bar{\eta}_1) = g_{\bar{\mathcal{V}}} (i_{\bar{\mathcal{V}}} \cdot \cdot) \). Hence \( -id(r^2 \eta_0) \) is a Kähler form with respect to \( (\bar{g}, \bar{I}) \). We fix \( u^*_0 \in P(H^*) \) and provide a coordinate \((z, w)\) of fiber such that \( u^* = u_0^*(z + jw) \). Then \( j \) maps \((z, w)\) to \((-w, \bar{z})\). Since \( dz \circ j_{\bar{\mathcal{V}}} = -dw \)
and $dw \circ j_\eta = d\bar{z}$, we obtain $\eta_0 \circ j_\eta = -\overline{\eta}$ and $\eta_1 \circ j_\eta = \overline{\eta}_0$. By the same manner, $\eta_0 \circ k_\eta = i\eta_1$ and $\eta_1 \circ k_\eta = -i\eta_0$. It yields that $g_\eta(j_\eta \cdot, \cdot) = \eta_0 \wedge \eta_1 + \overline{\eta}_0 \wedge \overline{\eta}_1$ and $g_\eta(k_\eta \cdot, \cdot) = i(\eta_0 \wedge \eta_1 - \overline{\eta}_0 \wedge \overline{\eta}_1)$. Thus $g_\eta(j_\eta \cdot, \cdot) - ig_\eta(k_\eta \cdot, \cdot) = 2\eta_0 \wedge \eta_1$. It turns out that $d(r^2\eta)_\text{Re} = d(r^2\eta)_\text{Im}$ are Kähler forms with respect to $\J$ and $-K$, respectively. Hence we finish the proof. 

The hyperkähler structure $(\overline{g}, \overline{I}, \overline{J}, -\overline{K})$ induces that on $P(H^*)/\mathbb{Z}_2$. This coincides the hyperkähler structure constructed by Swann [9].

### 3.7. Holomorphic $k$-vector fields on $P(H^*)$.

Let $\overline{H}_{1,0}^1$ and $\overline{V}_{1,0}^1$ be the vector bundles of horizontal $(1,0)$-vectors and vertical $(1,0)$-vectors on $P(H^*)$. Then $\overline{H}_{1,0}^1 = \text{Ker} \eta_0 \cap \text{Ker} \eta_1$. There exist fundamental vector fields $\hat{1}, \hat{i}, \hat{j}, \hat{k}$ associated with the elements $1, i, j, k$ of Lie algebra $gl(1, \mathbb{H}) = \mathbb{H}$. We define the complex vector fields $v_0$ and $v_1$ as $v_0 = \frac{1}{2}(\hat{1} - i\hat{i})$ and $v_1 = \frac{1}{2}(\hat{j} - i\hat{k})$. Then $\{v_0, v_1\}$ is the dual basis of $\{\eta_0, \eta_1\}$. The two vector fields $v_0$ and $v_1$ span the space $\overline{V}_{1,0}^1$. Let $X'$ be a $(1, 0)$-vector field on $P(H^*)$. Then $X'$ is decomposed into

$$X' = X'_h + f_0 v_0 + f_1 v_1$$

for a horizontal vector field $X'_h$ and functions $f_0, f_1$ on $P(H^*)$.

**Lemma 3.18.** The $(1, 0)$-vector field $X'$ is holomorphic if and only if

1. $\overline{\partial}(\overline{\theta}_0(X'_h)) - f_1 \overline{\theta}_1 = 0$,
2. $\overline{\partial}f_0 = cr^{-2}\omega_E(\overline{\theta}_0(X'_h), \overline{\theta}_1) + f_1 \overline{\eta}_1$ under the decomposition $[7]$. 

**Proof.** The vector field $X'$ is holomorphic if and only if $\nabla V_{0,1} X' = 0$. The equation is equal to $\theta_0(\nabla V_{0,1} X') = 0$, $\eta_0(\nabla V_{0,1} X') = 0$ and $\eta_1(\nabla V_{0,1} X') = 0$. The first equation induces the third one since $\overline{\partial}(\theta_0(X'_h)) = \eta_1(\nabla V_{0,1} X') \wedge \overline{\theta}_1 + \theta_0(\Omega^{0,2}_{FP(H^*)}(X')) = \eta_1(\nabla V_{0,1} X') \wedge \overline{\theta}_1$ and the map $\wedge \overline{\theta}_1 : \wedge^1 \to p^{-1}(E) \otimes \wedge^1$ is injective. Lemma 3.13 implies that $\overline{\theta}_0(\nabla V_{0,1} X') = \overline{\partial}(\theta_0(X'_h)) - f_1 \overline{\theta}_1$ and $\eta_0(\nabla V_{0,1} X') = \overline{\partial}f_0 - cr^{-2}\omega_E(\overline{\theta}_0(X'_h), \overline{\theta}_1) - f_1 \overline{\eta}_1$. It turns out that $\nabla V_{0,1} X' = 0$ is equivalent to the conditions (i) and (ii). 

Let $k$ be an integer which is greater than 1. Any $(k, 0)$-vector field $X'$ is decomposed into

$$X' = X'_h + Y_0 \wedge v_0 + Y_1 \wedge v_1 + Z_0 \wedge v_0 \wedge v_1$$

for $X'_h \in \wedge^k \overline{H}_{1,0}^1$ and $Y_0, Y_1 \in \wedge^{k-1} \overline{H}_{1,0}^1$ and $Z_0 \in \wedge^{k-2} \overline{H}_{1,0}^1$.

**Lemma 3.19.** For $2 \leq k \leq 2n$, the $(k, 0)$-vector field $X'$ is holomorphic if and only if

1. $\overline{\partial}(\theta_0(X'_h)) - k^2 \overline{\partial}^{k-1}_0(Y_1) \wedge E \overline{\theta}_1 = 0$,
2. $k^2 \overline{\partial}(\theta_0^{-k-2}(Z_0)) + k^2(2-1)^2 \overline{\partial}^{k-2}_0(Z_0) \wedge E \overline{\theta}_1 - cr^{-2}\omega_E(\overline{\theta}_0(X'_h), \overline{\theta}_1) = 0$,
3. $\overline{\partial}(\theta_0^{-k-1}(Y_1)) + \theta_0^{k-1}(Y_1) \wedge E \overline{\theta}_0 = 0$,
4. $(k-1)^2 \overline{\partial}(\theta_0^{-k-2}(Z_0)) + (k-1)^2 \overline{\theta}_0^{-k-2}(Z_0) \wedge E \overline{\theta}_0 - cr^{-2}\omega_E(\overline{\theta}_0^{-k-1}(Y_1), \overline{\theta}_1) = 0$

under the decomposition $[8]$. Especially, in the case $k \neq 2n$, $X'$ is holomorphic if and only if the equations (i), (ii), (iv) hold.

**Proof.** The equation $\nabla V_{0,1} X' = 0$ is equal to $\overline{\partial}_0(\nabla V_{0,1} X') = 0$, $\overline{\partial}_0^{k-1} \wedge \eta_0(\nabla V_{0,1} X') = 0$, $\overline{\partial}_0^{k-2} \wedge \eta_0 \wedge \eta_1(\nabla V_{0,1} X') = 0$ and $\overline{\partial}_0^{k-2} \wedge \eta_0 \wedge \eta_1(\nabla V_{0,1} X') = 0$. Proposition 3.14 implies that
\[ \partial_0^k(\nabla^{0,1}X') = \bar{\partial}_0^k(X'_h) - k^2\partial_0^{k-1}(Y_1) \land E \bar{\theta}_1 \text{ and } \partial_0^{k-1} \land \eta_0(\nabla^{0,1}X') = k(\bar{\partial}_0^{k-1}(Y_1)) + \partial_0^{k-1}(Y_1) \land \eta_0 \]. By \( \omega_E(\partial_0^{k-1}(Y_1), \bar{\theta}_1) = k(\partial_0^{k-1} \land \omega_E(\theta_0, \theta_1))(X') \), we obtain
\[ \partial_0^{k-1} \land \eta_0(\nabla^{0,1}X') = k\bar{\partial}(\partial_0^{k-1}(Y_0)) + ((k-1)\partial_0^{k-2} \land \eta_0 \land \eta_1 \land E \bar{\theta}_1 - \partial_0^{k-1} \land (cr^{-2}\omega_E(\theta_0, \theta_1) + \eta_0 \land \eta_1))(X') \]
\[ = k\bar{\partial}(\partial_0^{k-1}(Y_0)) + (k-1)^2\partial_0^{k-2}(Z_0) \land E \bar{\theta}_1 - \frac{1}{k}c^{-2}\omega_E(\partial_0^{k}(X'_h), \bar{\theta}_1) - k\partial_0^{k-1}(Y_1) \land \eta_1. \]
and
\[ \partial_0^{k-2} \land \eta_0 \land \eta_1(\nabla^{0,1}X') = k(k-1)\bar{\partial}(\partial_0^{k-2}(Z_0)) - kcr^{-2}\partial_0^{k-2} \land \omega_E(\theta_0, \bar{\theta}_1)(Y_1) + k(k-1)\partial_0^{k-2}(Z_0) \land \eta_0 \]
\[ = k(k-1)\bar{\partial}(\partial_0^{k-2}(Z_0)) - kcr^{-2}\omega_E(\partial_0^{k-1}(Y_1), \bar{\theta}_1) + k(k-1)\partial_0^{k-2}(Z_0) \land \eta_0. \]
Therefore, \( \nabla^{0,1}X' = 0 \) if and only if \((i), (ii), (iii), (iv)\) hold. Now \( \bar{\partial}(\nabla^{0,1}X') = (-1)^{k+1}(\partial_0^{k-1} \land \eta_1)(\nabla^{0,1}X') \land \theta_1 \). The map \( \bar{\theta}_1 : p^{-1}(\land^{k-1}E) \rightarrow p^{-1}(\land^kE) \land \eta_0 \) is injective in the case \( 1 \leq k \leq 2n - 1 \). Hence \( \bar{\partial}(\nabla^{0,1}X') = 0 \) implies that \( \theta_1(\nabla^{0,1}X') = 0 \) for \( 1 \leq k \leq 2n - 1 \). It turns out that \( \nabla^{0,1}X' = 0 \) is equivalent to equations \((i), (ii), (iii) \) and \((iv)\) in the case \( 1 \leq k \leq 2n - 1 \).

From now on, we extend the decomposition \((\ref{9})\) to the case \( k = 1 \) as \( Z_0 = 0 \).

**Proposition 3.20.** Let \( k \) be an integer with \( 1 \leq k \leq 2n - 1 \). If \( X'_h \in \land^k\bar{\mathcal{H}}^{1,0} \) and \( Y_1 \in \land^{k-1}\bar{\mathcal{H}}^{1,0} \) satisfy
\[ (i) \quad \bar{\partial}(\partial_0^{k}(X'_h)) - k^2\partial_0^{k-1}(Y_1) \land E \bar{\theta}_1 = 0, \]
then there exist \( Y_0 \) and \( Z_0 \) locally such that \( X' \) is holomorphic. If \( X'_h \in \land^{2n}\bar{\mathcal{H}}^{1,0} \) and \( Y_1 \in \land^{2n-1}\bar{\mathcal{H}}^{1,0} \) satisfy \((i)\) and
\[ (iii) \quad \bar{\partial}(\partial_0^{2n-1}(Y_1)) + \partial_0^{2n-1}(Y_1) \land \eta_0 = 0, \]
then there exist \( Y_0 \) and \( Z_0 \) locally such that \( X' \) is holomorphic.

**Proof.** We assume that \( X'_h \in \land^k\bar{\mathcal{H}}^{1,0} \) and \( Y_1 \in \land^{k-1}\bar{\mathcal{H}}^{1,0} \) satisfy \((i)\) for \( 1 \leq k \leq 2n - 1 \), \((i)\) and \((iii)\) for \( k = 2n \). It follows from Proposition \[3.7\] and Lemma \[3.13\] that \( \bar{\partial}(r^2\bar{\theta}_1) = 0 \). By taking the derivative \( \bar{\partial} \) on \((i)\), we obtain \( \bar{\partial}(r^2\bar{\theta}_0^{2n-1}(Y_1)) \land \bar{\theta}_1 = 0 \). Since the wedge \( \land \bar{\theta}_1 : p^{-1}(\land^kE) \rightarrow p^{-1}(\land^{k-1}E) \land T^*P(H^*) \) is injective for \( 1 \leq k \leq 2n - 1 \), \( \bar{\partial}(r^2\bar{\theta}_0^{2n-1}(Y_1)) = 0 \). The equation \((iii)\) is equal to \( \bar{\partial}(r^2\bar{\theta}_0^{2n-1}(Y_1)) = 0 \). Hence \( \bar{\partial}(r^2\bar{\theta}_0^{k-1}(Y_1)) = 0 \) for \( 1 \leq k \leq 2n \).

It is easy to see that the condition \((iv)\) in Lemma \[3.19\] is equivalent to
\[ \bar{\partial}(r^2\bar{\theta}_0^{k-1}(Y_1)) = c\omega_E(r^2\bar{\partial}_0^{k-1}(Y_1), r^{-2}\bar{\theta}_1). \]
(6)

The derivative \( \bar{\partial} \) on the right hand side of \((6)\) vanishes since \( \bar{\partial}(r^2\bar{\theta}_0^{k-1}(Y_1)) = 0 \) and \( \bar{\partial}(r^{-2}\bar{\theta}_1) = 0 \). By Dolbeaux’s lemma, there exists an element \( \zeta \in A_0^0\mathcal{P}(H^*)((\land^{k-2}E)) \) such that \( \bar{\partial}\zeta = c\omega_E(\partial_0^{k-1}(Y_1), r^{-2}\bar{\theta}_1) \). The \((k-2)\)-th wedge \( \partial_0^{k-2} \) is an isomorphism from \( A_0^0\mathcal{P}(H^*)((\land^{k-2}E)) \) to \( A_0^0\mathcal{P}(H^*)((\land^{k-2}E)) \). Hence there exists \( Z_0 \in A_0^{0}\mathcal{P}(H^*)((\land^{k-2}H^{1,0})) \) satisfying \((6)\), and \((iv)\). In order to find a solution \( Y_0 \) of the equation \((ii)\) in Lemma \[3.19\] we consider the cases \( k \neq 1 \) and \( k = 1 \). In the case \( k \neq 1 \), we write \((ii)\) as
\[ \bar{\partial}(\bar{\partial}_0^{k-1}(Y_0)) = k^2c\omega_E(\partial_0^{k}(X'_h), r^{-2}\bar{\theta}_1) + r^2\partial_0^{k-1}(Y_1) \land r^{-2}\eta_1 - (k-1)^2r^2\partial_0^{k-2}(Z_0) \land E r^{-2}\bar{\theta}_1. \]
(7)
The derivative $\bar{\partial}$ on the right hand side of (7) is provided by
\[
cr^{-2}\{(ω_E(\tilde{θ}_0^{-1}(Y_1) ∧_E \tilde{θ}_1, \tilde{θ}_1))-2\tilde{θ}_0^{-1}(Y_1) ∧ E_0(\tilde{θ}_1, \tilde{θ}_1)-ω_E(\tilde{θ}_0^{-1}(Y_1), \tilde{θ}_1) ∧ \tilde{θ}_1\}.\tag{8}
\]
We take a point $u^*$ of $P(H^*)$ and denote by $e$ the element $\tilde{θ}_0^{-1}(Y_1)u^*$. The $(0, 1)$-form $\tilde{θ}_1$ is given by $id_E ∗ u^*j$ at the point $u^*$. Then (8) is written as an element
\[
∧_{E^*}(ω_E(e ∧_E id_E, id_E)−2e ∗ω_E(id_E, id_E)−ω_E(e, id_E) ∧_{E^*} id_E)
\]
of $\wedge^{k−1}E ⊗ \wedge^2E^*$ by the basis $cr^{-2}u^*j ∗ u^*j$. However, the element (9) vanishes by the calculation. Therefore (8) vanishes at each point $u^* ∈ P(H^*)$. It turns out that the derivative $\bar{\partial}$ on the right hand side of (7) is zero for $k ≠ 1$. In the case $k = 1$, by the same argument, the derivative $\bar{θ}$ on the right hand side of (ii) in Lemma 3.18 is reduced to $\wedge_E(ω_E(id_E, id_E)−2ω_E(id_E, id_E)$ which vanishes. Hence, there exists $Y_0 ∈ \mathcal{A}^{0}_{P(H^*)}(\wedge^{k−1}H)$ such that (7) and (ii) hold for any $1 ≤ k ≤ 2n$. It completes the proof. □

Lemma 3.19 and Proposition 3.20 induce the following

**Theorem 3.21.** Horizontal $k$ and $(k−1)$-vector fields $X'_h, Y_1$ satisfy for $1 ≤ k ≤ 2n − 1$, (i) $\bar{\partial}(\tilde{θ}_0^k(X'_h))−k^2\tilde{θ}_0^{k−1}(Y_1) ∧ E \tilde{θ}_1 = 0$ and for $k = 2n$, (i) and
\[
\bar{\partial}(\tilde{θ}_0^{2n−1}(v^2Y_1)) = 0
\]
if and only if the $(k, 0)$-vector field $X'_h + Y_0 ∧ v_0 + Y_1 ∧ v_1 + Z_0 ∧ v_0 ∧ v_1$ is holomorphic for local horizontal $(k − 1)$ and $(k − 2)$-vector fields $Y_0, Z_0$ on $P(H^*)$. □

4. The twistor space $Z$

The complex structure $\tilde{I}$ on $P(H^*)$ induces a complex structure $\hat{I}$ on $Z$ since the action of $\text{GL}(1, \mathbb{C})$ on $P(H^*)$ is holomorphic. Each fiber of the projection $f : Z → M$ is isomorphic to $\mathbb{C}P^1$. We denote by $l$ a line bundle over $Z$ which is the hyperplane bundle on each fiber of $f$. If $H$ is not global, then $l$ is not also. However, $l^2$ is globally defined.

4.1. Lift of $\mathcal{A}^q(\wedge^kE ⊗ S^mH)$ to $Z$. Let $\mathcal{A}^q_Z(l^m)$ be a sheaf of smooth section of the $m$-th tensor product $l^m$ over $Z$. We define a sheaf $\widehat{\mathcal{A}}^q(l^m)$ by
\[
\widehat{\mathcal{A}}^q(l^m) = \{ζ ∈ f^{-1}f_*((\mathcal{A}^q_Z(l^m)) \mid ζ : \text{holomorphic along each fiber of } f}\).
\]
In the case $m = 0$, $\widehat{\mathcal{A}}^q(l^0)$ is just the sheaf of functions on $Z$ which are constant along each fiber of $f$. We write $\widehat{\mathcal{A}}^q(l^0)$ as $\widehat{\mathcal{A}}^q$ for simplicity. Let $\widehat{\mathcal{A}}^q(\wedge^kE)$ denote the sheaf of pull-back of $\wedge^kE$-valued $q$-forms on $M$ by $f$. We define a sheaf $\widehat{\mathcal{A}}^q(\wedge^kE ⊗ l^m)$ as
\[
\widehat{\mathcal{A}}^q(\wedge^kE ⊗ l^m) = \widehat{\mathcal{A}}^q(\wedge^kE) ⊗ \widehat{\mathcal{A}}^q(l^m).
\]
Since a polynomial of degree $m$ on $\mathbb{C}^2\setminus\{0\}$ induces a holomorphic section of the $m$-th tensor product $l^m$, any element $\tilde{ξ}_0$ of $\widehat{\mathcal{A}}^q_{(m, 0)}$ defines an element of $\widehat{\mathcal{A}}^q(l^m)$, which we denote by $\tilde{ξ}$. Such an element $\tilde{ξ}$ is called a lift of $ξ$ to $Z$. The correspondence $\tilde{ξ}_0 ⇔ \tilde{ξ}$ provides the isomorphism
\[
\widehat{\mathcal{A}}^q_{(m, 0)}(\wedge^kE) ≅ \widehat{\mathcal{A}}^q(\wedge^kE ⊗ l^m).
\]
It follows from Corollary 3.3 that

**Corollary 4.1.** $\mathcal{A}^q(\wedge^kE ⊗ S^mH) ≅ \widehat{\mathcal{A}}^q(\wedge^kE ⊗ l^m)$ by the correspondence $ξ → \tilde{ξ}$. □
4.2. Real structures on $Z$. A differential $q$-form $\alpha$ on $P(H^*)$ is called of $GL(1, \mathbb{C})$-order $m$ if \((R_c)\alpha = e^m\alpha\) for any $c \in GL(1, \mathbb{C})$. Let $A^q_{P(H^*)}(\wedge^k E)$ denote a subsheaf of $A^q_{P(H^*)}(\wedge^k E)$ whose elements are of $GL(1, \mathbb{C})$-order $m$. The anti-$\mathbb{C}$ linear endomorphism $\bar{\tau}$ of $A^q_{P(H^*)}(\wedge^k E)$ as in \[3.3\] induces that of $A^q_{P(H^*)}(\wedge^k E)$. If $k + m$ is even, $\bar{\tau}$ is a real structure of $A^q_{P(H^*)}(\wedge^k E)$.

Let $A^q_{Z}(f^{-1}(\wedge^k E) \otimes l^m)$ be a sheaf of $f^{-1}(\wedge^k E) \otimes l^m$-valued differential $q$-form on $Z$. We denote $A^q_{Z}(\wedge^k E \otimes l^m)$ the sheaf $f^{-1} f_* A^q_{Z}(f^{-1}(\wedge^k E) \otimes l^m)$. The sheaf $A^q_{Z}(\wedge^k E \otimes l^m)$ is isomorphic to $A^q_{P(H^*)}(\wedge^k E)$. We define $\hat{\tau}$ as the endomorphism of $A^q_{Z}(\wedge^k E \otimes l^m)$ induced by $\bar{\tau}$. The right action $R_j$ of $j$ on $P(H^*)$ induces an anti-holomorphic involution of $Z$, and we denote it by $R_{(j)} : Z \rightarrow Z$. The map $R_{(j)}$ is the antipodal map of each fiber $\mathbb{C}P^1$ of $f$. The anti-$\mathbb{C}$ linear endomorphism $\hat{\tau}$ of $A^q_{Z}(\wedge^k E \otimes l^m)$ is given by $\hat{\tau}(\beta_Z) = J^E_{R_{(j)}^{-1}} R_{(j)} \beta_Z \hat{\tau}$ for $\beta_Z \in A^q_{Z}(\wedge^k E \otimes l^m)$. It follows from Proposition \[3.4\] and the isomorphism \[10\] that

**Proposition 4.2.** The map $\hat{\tau}$ defines an endomorphism of $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$ such that $\hat{\tau}(\hat{\xi}) = \bar{\tau}(\hat{\xi})$ for $\xi \in A^q(\wedge^k E \otimes S^m H)$. Moreover, $\hat{\tau}$ is a real structure on $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$ if $k + m$ is even.

Let $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)_{\hat{\tau}}$ denote the sheaf of $\hat{\tau}$-invariant elements of $\hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)$. Corollary \[3.6\] and Proposition \[4.2\] imply the following corollary:

**Corollary 4.3.** $A^q(\wedge^k E \otimes S^m H)^{\tau} \cong \hat{\mathcal{A}}^q(\wedge^k E \otimes l^m)_{\hat{\tau}}$ by $\xi \mapsto \hat{\xi}$.

4.3. Canonical 1-form on $Z$. The principal $GL(1, \mathbb{C})$-bundle $\pi : P(H^*) \rightarrow Z$ is regarded as the frame bundle of $l^*$. An $l^m$-valued differential $(q, q')$-form on $Z$ is induced by a differential $(q, q')$-form on $P(H^*)$ of $GL(1, \mathbb{C})$-order $m$ which is annihilate to vectors along each fiber of $\pi$. We define $\hat{\theta}_0$ and $\hat{\theta}_1$ as the $f^{-1}(E) \otimes l$-valued $(1, 0)$-form and the $f^{-1}(E) \otimes l^{-1}$-valued $(0, 1)$-form on $Z$ induced by $\theta_0$ and $r^{-2} \theta_1$, respectively. Let $\eta$ and $\tilde{\omega}$ be the $l^2$-valued $(1, 0)$-form and the $l^2$-valued $(2, 0)$-form on $Z$ induced by $r^2 \eta_1$ and $\tilde{\omega}_0$, respectively. The forms $\tilde{\theta}_0$ and $\tilde{\omega}$ are the lift (as in the section \[1.1\]) of $\theta_0, \theta_1 \in \mathbb{H}^1(E \otimes H)$ and $\omega = \tilde{\omega}_0 \otimes s^2 H \in \mathbb{H}^2(S^2 H)$ to $Z$, respectively. The forms $\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\omega}$ are $\hat{\tau}$-invariant since $\tilde{\theta}_0, \tilde{\theta}_1, \tilde{\omega}$ and $\tau$ are $\hat{\tau}$-invariant.

The line bundle $l$ admits a connection with the connection form $\eta_0$ on the frame bundle $P(H^*)$. Let $d^l$ be the covariant exterior derivative. Then $d^l$ corresponds to the restriction of $d^E$ to the horizontal ker $\eta_0$. By Proposition \[3.7\] and \[3.10\] we obtain

**Proposition 4.4.** $d^l \hat{\theta}_0 = -\eta \wedge \hat{\theta}_1$, $d^l \eta = -2c \tilde{\omega}$.

It implies that $\eta$ is a holomorphic 1-form valued with $l^2$.

**Proposition 4.5.** (c.f. Theorem 4.3 in \[8\]) If the scalar curvature $t$ is not zero, then $\eta$ is a holomorphic contact form on $Z$ such that $l^2$ is the contact bundle.

**Proof.** If $t \neq 0$, then $c \neq 0$, and $(d^l \eta)^n \wedge \eta = (-2c)^n \tilde{\omega}^n \wedge \eta \neq 0$.

The symmetric 2-tensor $\eta_1 \otimes \overline{\eta}_1 + \overline{\eta}_1 \otimes \eta_1$ on $P(H^*)$ is GL(1, $\mathbb{C}$)-invariant and annihilated by tangent vectors to the fiber of $\pi : P(H^*) \rightarrow Z$. Let $g_\hat{\tau}$ be a real symmetric 2-form on $Z$ such that $\pi^* g_\hat{\tau} = \eta_1 \otimes \overline{\eta}_1 + \overline{\eta}_1 \otimes \eta_1$. We define a real symmetric 2-form $\hat{g}$ by $\hat{g} = cf^* g + g_\hat{\tau}$ on $Z$. 

Proposition 4.6. (c.f. Theorem 6.1 in [S]) If $t$ is positive, then $(\hat{g}, \hat{I})$ is a Kähler-Einstein structure on $Z$ with positive scalar curvature.

Proof. It follows from $\pi^*(\hat{g}(\hat{I}, \cdot)) = -i\partial\bar{\partial}\log r^2$ that $(\hat{g}, \hat{I})$ is a Kähler structure on $Z$. Proposition 4.5 implies that the canonical bundle $K_Z$ of $Z$ is isomorphic to $\mathbb{L}^{2(n+1)}$. The Ricci form $\rho_{\hat{g}}$ with respect to $\hat{g}$ is given by the 2-form $-2(n+1)i\partial\bar{\partial}H_0$ on $P(H^*)$. Hence, $\hat{g}$ is a Kähler-Einstein metric on $Z$ with positive scalar curvature. □

As in the above proof, $\hat{g}$ is the Fubini-Study metric on the fiber $\mathbb{C}P^1$ of $f : Z \to M$.

Let $\nabla$ be a torsion free connection on $Z$ such that $\nabla^0,1 = \partial$. It follows from Proposition 4.4 that $\nabla^0,1 \hat{\theta}_0 = \eta \otimes \hat{\theta}_1$ and $\nabla^0,1 \eta = 0$. We define a $f^{-1}(\wedge^k E) \otimes l^k$-valued $(k, 0)$-form $\hat{\theta}_0^k$ by the $k$-th wedge of $\hat{\theta}_0$. Then

Proposition 4.7. $\nabla^0,1 \hat{\theta}_0^k = k\hat{\theta}_0^{k-1} \wedge \eta \wedge_E \hat{\theta}_1$, $\nabla^0,1 (\hat{\theta}_0^{k-1} \wedge \eta) = 0$. □

4.4. Holomorphic $k$-vector fields on $Z$. The horizontal bundle $\hat{\mathcal{H}}$ induces a bundle $\hat{\mathcal{H}}$ over the twistor space $Z$ since $\hat{\mathcal{H}}$ is invariant under the action of $GL(1, \mathbb{C})$. Let $\hat{\mathcal{V}}$ be the bundle of tangent vectors of each fiber of $f : Z \to M$. The tangent bundle $TZ \otimes \mathbb{C}$ is decomposed into $TZ \otimes \mathbb{C} = \hat{\mathcal{H}} \oplus \hat{\mathcal{V}}$. The bundle $\hat{\mathcal{H}}$ is isomorphic to the pull back bundle $f^{-1}(TM \otimes \mathbb{C})$. We call a section of $\wedge^k \hat{\mathcal{H}}$ a horizontal $k$-vector field on $Z$. Let $\hat{\mathcal{H}}^{1,0}$ be a vector bundle of horizontal $(1, 0)$-vectors on $Z$. The bundle $\hat{\mathcal{H}}^{1,0}$ is a holomorphic subbundle of $T^{1,0}Z$ since $\hat{\mathcal{H}}^{1,0}$ is the kernel of the holomorphic form $\eta$. The 1-form $\eta$ provides the map from $\wedge^k \hat{\mathcal{H}}^{1,0}Z$ to $l^2 \otimes \wedge^{k-1}T^{1,0}Z$, which we also denote by $\eta$. Then $\wedge^k \hat{\mathcal{H}}^{1,0}$ is the kernel of the map $\eta$. We denote by $\nu$ the $l^{-2}$-valued $(1, 0)$-vector field on $Z$ induced by the vector field $r^{-2}v_1$ on $P(H^*)$. The $l^{-2}$-valued vector field $\nu$ is regarded as the dual of $\eta$ since $\eta(\nu) = 1$. We remark that $\nu$ is not holomorphic on whole $Z$ but holomorphic along each fiber. Let $X'$ be a $(k, 0)$-vector field on $Z$. Then $X'$ is given by

$$X' = X_h' + Y \wedge \nu$$

for $X_h' \in \wedge^k \hat{\mathcal{H}}^{1,0}$ and $Y \in l^2 \otimes \wedge^{k-1} \hat{\mathcal{H}}^{1,0}$.

Theorem 4.8. For $1 \leq k \leq 2n - 1$, the $(k, 0)$-vector field $X_h' + Y \wedge v$ is holomorphic if and only if

$$\partial^1(\hat{\theta}_0^k(X_h')) - k^2 \hat{\theta}_0^{k-1}(Y) \wedge_E \hat{\theta}_1 = 0.$$  

The $(2n, 0)$-vector field $X_h' + Y \wedge v$ is holomorphic if and only if

$$\partial^1(\hat{\theta}_0^{2n}(X_h')) - 4n^2 \hat{\theta}_0^{2n-1}(Y) \wedge_E \hat{\theta}_1 = 0,$$

$$\partial^1(\hat{\theta}_0^{2n-1}(Y)) = 0.$$  

Proof. The $(k, 0)$-vector field $X'$ is holomorphic if and only if $\nabla^0,1 X' = 0$. The equation is equal to $\hat{\theta}_0^k(\nabla^0,1 X') = 0$ and $(\hat{\theta}_0^{k-1} \wedge \eta)(\nabla^0,1 X') = 0$. Proposition 4.7 implies that $\hat{\theta}_0^k(\nabla^0,1 X') = \partial^1(\hat{\theta}_0^k(X_h')) - k^2 \hat{\theta}_0^{k-1}(Y) \wedge E \hat{\theta}_1$. Now $\nabla^0,1(\hat{\theta}_0^k(\nabla^0,1 X')) = (-1)^{k+1}(\hat{\theta}_0^{k-1} \wedge \eta)(\nabla^0,1 X') \wedge \hat{\theta}_1$. In the case $1 \leq k \leq 2n - 1$, the map $\wedge \hat{\theta}_1 : f^{-1}(\wedge^{k-1} E) \otimes l^{k+1} \otimes l^{0,1} \to f^{-1}(\wedge^k E) \otimes l^k \otimes l^{0,2}$ is injective, and the equation $(\hat{\theta}_0^{k-1} \wedge \eta)(\nabla^0,1 X') = 0$ follows from $\hat{\theta}_0^k(\nabla^0,1 X') = 0$. In the case $k = 2n$, $(\hat{\theta}_0^{2n-1} \wedge \eta)(\nabla^0,1 X') = 2n \partial^1(\hat{\theta}_0^{2n-1}(Y))$. Hence we finish the proof. □

5. Quaternionic sections

In this section, we provide a definition of a quaternionic section of $\wedge^k E \otimes S^m H$. We show that the lifts of the quaternionic section satisfy some $\partial$-equations on $P(H^*)$ and $Z$. 
5.1. Quaternionic sections of $\wedge^k E \otimes S^m H$. The Levi-Civita connection of $(M, g)$ induces the covariant derivative $\nabla : \Gamma(\wedge^k E \otimes S^m H) \to \Gamma(\wedge^k E \otimes S^{m+1} H \otimes E^* \otimes H^*)$ where $\Gamma(\wedge^k E \otimes S^m H)$ means the space of smooth sections of $\wedge^k E \otimes S^m H$. The space $\wedge^k E \otimes S^m H \otimes E^* \otimes H^*$ is isomorphic to $\wedge^k E \otimes E^* \otimes S^m H \otimes H$ by the isomorphism $\omega_H^2 : H^* \to H$. Moreover, $S^m H \otimes H \cong S^{m+1} H \oplus S^{m-1} H$ by the Clebsch-Gordan decomposition. Thus, the covariant derivative $\nabla$ is regarded as

$$\nabla : \Gamma(\wedge^k E \otimes S^m H) \to \Gamma(\wedge^k E \otimes E^* \otimes S^{m+1} H) \oplus \Gamma(\wedge^k E \otimes E^* \otimes S^{m-1} H).$$

Dirac operator is defined as the $\wedge^k E \otimes E^* \otimes S^{m+1} H$-part of $\nabla$ (c.f. [3])

$$\mathcal{D}_{\wedge^k E} : \Gamma(\wedge^k E \otimes S^m H) \to \Gamma(\wedge^k E \otimes E^* \otimes S^{m+1} H)$$

Let $k$ be a positive integer. By the restriction of $\otimes^k E \otimes E^*$ to $\wedge^k E \otimes E^*$, the trace of $(\otimes^k E) \otimes E^*$ induces the map $\text{tr} : \wedge^k E \otimes E^* \to \wedge^{k-1} E$. Let $(\wedge^k E \otimes E^*)_0$ denote the kernel of $\text{tr} : \wedge^k E \otimes E^* \to \wedge^{k-1} E$. We rescale the trace map as $\frac{1}{2n-k+1} \text{tr}$, and also denote it by the same notation $\text{tr}$. Then the map $\text{tr} : \wedge^k E \otimes E^* \to \wedge^{k-1} E$ have a right inverse $\alpha \mapsto \alpha \wedge \text{id}_E$ where $\alpha \wedge \text{id}_E$ is the image of $\alpha \otimes \text{id}_E$ by the anti-symmetrization $\wedge^{k-1} E \otimes E^* \otimes E^* \to \wedge^{k-1} E \otimes E^*$. Hence, the bundle $\wedge^k E \otimes E^*$ is decomposed into $(\wedge^k E \otimes E^*)_0$ and $(\wedge^{k-1} E) \otimes \text{id}_E$.

$$\wedge^k E \otimes E^* = (\wedge^k E \otimes E^*)_0 \oplus (\wedge^{k-1} E) \otimes \text{id}_E.$$ We define an operator

$$\mathcal{D}_{\wedge^k E}^0 : \Gamma((\wedge^k E \otimes E^*)_0 \otimes S^{m+1} H)$$

as the $(\wedge^k E \otimes E^*)_0$-part of $\mathcal{D}_{\wedge^k E}$.

**Definition 5.1.** Let $m$ be a non-negative integer. A section $X$ of $\wedge^k E \otimes S^m H$ is quaternionic if $\mathcal{D}_{\wedge^k E}^0(X) = 0$ for $1 \leq k \leq 2n-1$ and $\mathcal{D}_{\wedge^{2n-1} E \circ \text{tr} \circ \mathcal{D}_{\wedge^{2n} E}}(X) = 0$ for $k = 2n$.

Any section $X$ of $\wedge^{2n} E \otimes S^m H$ satisfies $\mathcal{D}_{\wedge^{2n} E}^0(X) = 0$ since $(\wedge^{2n} E \otimes E^*)_0 = \{0\}$.

Let $\tau$ be an anti-$\mathbb{C}$-linear endomorphism of $\Gamma((\otimes^k E) \otimes (\otimes^k E^*) \otimes (\otimes^m H) \otimes (\otimes^{m'} H^*) \otimes \wedge^9 T^* M)$ as

$$\tau(\xi) = (J_E^k \otimes J_{E'}^k \otimes J_H^m \otimes J_{H'}^{m'})(v) \otimes \alpha^i$$

for $\xi = \sum v_i \otimes \alpha^i$ where $\{v_i\}$ is a frame of $(\otimes^k E) \otimes (\otimes^k E^*) \otimes (\otimes^m H) \otimes (\otimes^{m'} H^*)$ and $\alpha^i$ is a q-form. If $k+k'+m+m'$ is even, then $\tau$ is a real structure. The covariant derivative $\nabla : \Gamma(\wedge^k E \otimes S^m H) \to \Gamma(\wedge^k E \otimes S^m H \otimes T^* M)$ satisfies $\tau \circ \nabla = \nabla \circ \tau$ since the connections of $E$ and $H$ preserve $J_E$ and $J_H$, respectively. The maps $\omega_H^2 : \Gamma(E^* \otimes H^*) \to \Gamma(E^* \otimes H)$ and $s_H^{m+1} : \Gamma(\wedge^k E \otimes E^* \otimes S^m H \otimes H) \to \Gamma(\wedge^k E \otimes E^* \otimes S^{m+1} H)$ are commutative with $\tau$.

Since $\mathcal{D}_{\wedge^k E}$ is provided by $s_H^{m+1} \circ \omega_H^2 \circ \nabla$, the operator $\mathcal{D}_{\wedge^k E}$ is also commutative with $\tau$. The trace map $\text{tr} : \wedge^k E \otimes E^* \to \wedge^{k-1} E$ satisfies $\text{tr} \circ (J_E^k \otimes J_{E'}^k) = J_E^{k-1} \circ \text{tr}$. Hence, the operators $\mathcal{D}_{\wedge^k E}^0$ and $\mathcal{D}_{\wedge^{2n-1} E \circ \text{tr} \circ \mathcal{D}_{\wedge^{2n} E}}$ are commutative with $\tau$.

5.2. Lift of quaternionic sections to $P(H^*)$. The map $\omega_H^2 : H^* \to H$ induces the isomorphism from $\Lambda^1 = A^0(E^* \otimes H^*)$ to $A^0(E^* \otimes H)$, and denote it also by $\omega_H^2$. On $P(H^*)$, $\omega_H^2$ provides a map from $A^0_{P(H^*)}(\tilde{H}^*)$ to $A^0_{P(H^*)}(E^* \otimes H)$ since $\tilde{H}^* \cong p^{-1}(E^* \otimes H^*)_{u^*}$ at $u^* \in P(H^*)$. The sheaf $A^0_{P(H^*)}(E^* \otimes H)$ is isomorphic to $A^0_{P(H^*)}(E^* \otimes \mathbb{H})$ by considering a point $u^* \in P(H^*)$ as a frame of $H^*$. Moreover, taking a coefficient of $1 \in \mathbb{H}$, we have a map $A^0_{P(H^*)}(E^* \otimes \mathbb{H}) \to A^0_{P(H^*)}(E^*)$. Therefore, we obtain a map $A^0_{P(H^*)}(\tilde{H}^*) \to A^0_{P(H^*)}(E^*)$ and denote it by $\omega_{\tilde{H}^*}^2$. The $(1,0)$ and $(0,1)$-subspaces $(\tilde{H}^*)^{1,0}$ and $(\tilde{H}^*)^{0,1}$ of $\tilde{H}^*$ are given by $E^* \otimes u^*$ and $E^* \otimes u^*j$ at $u^* \in P(H^*)$. Since $\omega_{\tilde{H}^*}^2(u^*) = -r^2(-u^*j)^* = \omega_{\tilde{H}^*}^2(u^*) = -r^2(-u^*j)^*$.
\(-r^2ju\) and \(\omega_H^2(\mathbf{u}^*) = r^2u\), the map \(\tilde{\omega}_H^2: \mathcal{A}_{P(H^*)}^0(\mathcal{H}^*) \to \mathcal{A}_{P(H^*)}^0(E^*)\) has the kernel \(\mathcal{A}_{P(H^*)}^0(\mathcal{H}^{*,1,0})\) and it is injective on \(\mathcal{A}_{P(H^*)}^0(\mathcal{H}^{*,0,1})\). The restriction of \(\tilde{\omega}_H^2\) to \(\mathcal{A}^1\) is the map \(\tilde{\omega}_H^2: \mathcal{A}^1 \to \mathcal{A}_{P(H^*)}^0(E^*)\). Then \(\tilde{\omega}_H^2\) is regarded as a lift of \(\omega_H^2: \mathcal{A}^1 \to \mathcal{A}_{P(H^*)}^0(E^* \otimes H)\) to \(P(H^*)\). By the tensor product of the lift and \(\tilde{\mathcal{A}}^0_{(m,0)}\), we obtain the map \(\tilde{\omega}_H^2: \tilde{\mathcal{A}}^1_{(m,0)} \to \tilde{\mathcal{A}}^0_{(m,0)} \otimes \tilde{\mathcal{A}}^0_{(1,0)} \otimes \tilde{\mathcal{A}}^0_{(m,0)}\) which corresponds to the map \(\tilde{\omega}_H^2: \mathcal{A}^1(S^mH) \to \mathcal{A}^0(E^* \otimes H \otimes S^mH)\). It follows from \(\tilde{\mathcal{A}}^0_{(1,0)} \otimes \tilde{\mathcal{A}}^0_{(m,0)} = \tilde{\mathcal{A}}^0_{(m+1,0)}\) that there exists a commutative diagram

\[
\begin{array}{cccc}
\tilde{\mathcal{A}}^1_{(m,0)} & \xrightarrow{\tilde{\omega}_H^2} & \tilde{\mathcal{A}}^0_{(1,0)} \otimes \tilde{\mathcal{A}}^0_{(m,0)} & = \tilde{\mathcal{A}}^0_{(m+1,0)}(E^*) \\
\uparrow & & \uparrow & \\
\mathcal{A}^1(S^mH) & \xrightarrow{\omega_H^2} & \mathcal{A}^0(E^* \otimes H \otimes S^mH) & \xrightarrow{s_{H}^{m+1}\omega_H^2} \mathcal{A}^0(E^* \otimes S^{m+1}H).
\end{array}
\]

By extending the above diagram to \(\wedge^kE\)-valued 1-forms, we have

\[
\begin{array}{cccc}
\tilde{\mathcal{A}}^1_{(m,0)}(\wedge^kE) & \xrightarrow{\tilde{\omega}_H^2} & \tilde{\mathcal{A}}^0_{(1,0)}(\wedge^kE \otimes \wedge^*E) & = \tilde{\mathcal{A}}^0_{(m+1,0)}(\wedge^kE \otimes \wedge^*E) \\
\uparrow & & \uparrow & \\
\mathcal{A}^1(\wedge^kE \otimes S^mH) & \xrightarrow{s_{H}^{m+1}\omega_H^2} \mathcal{A}^0(\wedge^kE \otimes S^{m+1}H).
\end{array}
\]

The \(\mathcal{H}^*\)-part \(d_{\mathcal{H}}: \mathcal{A}^0_{P(H^*)}(\wedge^kE) \to \mathcal{A}^0_{P(H^*)}(\mathcal{H}^* \otimes \wedge^kE)\) of the exterior derivative induces \(d_{\tilde{\mathcal{H}}}: \tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE) \to \tilde{\mathcal{A}}^1_{(m,0)}(\wedge^kE)\) on the subsheaf \(\tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE)\) of \(\mathcal{A}_{P(H^*)}^0(\wedge^kE)\). By Corollary 3.3, we obtain a commutative diagram

\[
\begin{array}{cccc}
\tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE) & \xrightarrow{d_{\tilde{\mathcal{H}}}} & \tilde{\mathcal{A}}^1_{(m,0)}(\wedge^kE) & \xrightarrow{\tilde{\omega}_H^2} \tilde{\mathcal{A}}^0_{(m+1,0)}(\wedge^kE \otimes \wedge^*E) \\
\uparrow & & \uparrow & \\
\mathcal{A}^0(\wedge^kE \otimes S^mH) & \xrightarrow{s_{H}^{m+1}\omega_H^2} \mathcal{A}^1(\wedge^kE \otimes S^mH) & \xrightarrow{s_{H}^{m+1}\omega_H^2} \mathcal{A}^0(\wedge^kE \otimes \wedge^*E \otimes S^{m+1}H).
\end{array}
\]

The operator \(d_{\tilde{\mathcal{H}}}: \tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE) \to \tilde{\mathcal{A}}^1_{(m,0)}(\wedge^kE)\) is decomposed into two operators \(\partial_{\mathcal{H}}: \tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE) \to \mathcal{A}^0_{P(H^*)}(\mathcal{H}^* \otimes \wedge^kE)\) and \(\tilde{\partial}_{\mathcal{H}}: \tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE) \to \mathcal{A}^0_{P(H^*)}(\mathcal{H}^{*,1,0} \otimes \wedge^kE)\) by the decomposition \(\mathcal{H}^* = (\mathcal{H}^*)^* \otimes (\mathcal{H}^*)^{0,1}\). The operator \(\tilde{\partial}_{\mathcal{H}}\) coincides with \(\tilde{\partial}\) since an element of \(\tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE)\) is holomorphic along each fiber.

**Proposition 5.2.** \(\tilde{\mathcal{D}}_{\wedge^kE}(\xi) = \tilde{\omega}_H^2(\tilde{\partial}\xi_0)\) for \(\xi \in \mathcal{A}^0(\wedge^kE \otimes S^mH)\).

**Proof.** It follows from \(\mathcal{D}_{\wedge^kE} = s_{H}^{m+1} \circ \omega_H^2 \circ \nabla\) and the diagram (11) that \(\tilde{\mathcal{D}}_{\wedge^kE}(\xi)_0 = \tilde{\omega}_H^2(d_{\tilde{\mathcal{H}}}\xi_0)\) for \(\xi \in \mathcal{A}^0(\wedge^kE \otimes S^mH)\). Since the kernel of \(\tilde{\omega}_H^2\) is \(\mathcal{A}^0_{P(H^*)}(\mathcal{H}^{*,1,0} \otimes \wedge^kE)\), \(\tilde{\omega}_H^2(d_{\tilde{\mathcal{H}}}\xi_0) = \tilde{\omega}_H^2(\tilde{\partial}\xi_0)\). Hence \(\tilde{\mathcal{D}}_{\wedge^kE}(\xi)_0 = \tilde{\omega}_H^2(\tilde{\partial}\xi_0)\). \(\square\)

We denote by \(\tilde{\mathcal{O}}_m(\wedge^kE)\) the kernel of \(\tilde{\partial}\) on \(\tilde{\mathcal{A}}^0_{(m,0)}(\wedge^kE)\). By Proposition 5.2, the injectivity of \(\tilde{\omega}_H^2\) on \(\mathcal{A}^0_{P(H^*)}(\mathcal{H}^{*,1,0} \otimes \wedge^kE)\), we obtain an isomorphism

\[
\text{Ker} \mathcal{D}_{\wedge^kE} \cong \tilde{\mathcal{O}}_m(\wedge^kE)
\]

by \(\xi \mapsto \tilde{\xi}_0\). We extend Proposition 5.2 to the following :

**Proposition 5.3.** \(\tilde{\mathcal{D}}_{\wedge^kE}(\xi \wedge \wedge E \cdot \text{id}_{E})_0 = \tilde{\omega}_H^2(\tilde{\partial}\xi_0 - \tilde{\xi}_0 \wedge E \cdot \tilde{\partial}^2_0)\) for \(\xi \in \mathcal{A}^0(\wedge^kE \otimes S^mH)\) and \(\xi \in \mathcal{A}^0(\wedge^{k-1}E \otimes S^{m+1}H)\).
Proposition 5.5 and the injectivity of $\hat{\omega}_{H^*}$. Each fiber of $f$ and extend the map to $\tilde{\omega}_{H^*}(\tilde{\theta}_1) = \omega_{H^*}(\theta_1) = r^2\tilde{id}_E$ since $\omega_{H^*} (u^*(-j)) = r^2u$. Hence we obtain

$$\tilde{\omega}_{H^*}(\tilde{\zeta} \wedge E r^{-2}\tilde{\theta}_1) = \tilde{\zeta} \wedge E \tilde{id}_E = (\zeta \wedge E id_E)_{0}. \quad (12)$$

The equation (12) and Proposition 5.2 imply the equation of this Proposition.

Proposition 5.4. Let $\xi$ and $\zeta$ be elements of $A^0(\wedge^k E \otimes S^m H)$ and $A^0(\wedge^{k-1} E \otimes S^{m+1} H)$, respectively. The element $\xi$ is quaternionic and $\zeta = \text{tr}O_{\wedge^k E}(\xi)$ if and only if $\tilde{\omega}_{H^*}$, $\text{tr}(\xi \wedge E id_E)$ is quaternionic and $\tilde{\omega}_{H^*}$ is injective on $(\tilde{\omega}_{H^*})_{0,1}$. $O_{\wedge^k E} \zeta - \xi \wedge E id_E = 0$ for $k = 2n$. By Proposition 5.2 and the injectivity of $\tilde{\omega}_{H^*}$, on $(\tilde{\omega}_{H^*})_{0,1}$, $O_{\wedge^k E} \zeta - \xi \wedge E id_E = 0$ is equal to $\tilde{\omega}_{H^*} = 0$ for $k = 2n$.

Proposition 5.5. Let $\xi \in A^0(\wedge^k E \otimes S^m H)$ and $\xi \in A^0(\wedge^{k-1} E \otimes S^{m+1} H)$ be elements of $A^0(\wedge^k E \otimes S^m H)$ and $A^0(\wedge^{k-1} E \otimes S^{m+1} H)$, respectively. The element $\xi$ is quaternionic and $\tilde{\omega}_{H^*} \zeta = \tilde{\omega}_{H^*} \xi \zeta$. By Proposition 5.2 and the injectivity of $\tilde{\omega}_{H^*}$, on $(\tilde{\omega}_{H^*})_{0,1}$, $O_{\wedge^k E} \zeta - \xi \wedge E id_E = 0$ is equal to $\tilde{\omega}_{H^*} = 0$ for $k = 2n$.

5.3. Lift of quaternionic sections to $Z$. The sheaf $A^0_{P(H^*)}(\tilde{H}^*)$ is considered as that of $\tilde{\omega}_{H^*}$ maps an element of $A^0_{P(H^*)}(\tilde{H}^*)$ of $\text{GL}(1,\mathbb{C})$-order $m$ to an element of $A^0_{P(H^*)}((E^*)^{m+1})$ on $Z$. Since the kernel of $\tilde{\omega}_{H^*}$, $O_{\wedge^k E} \zeta - \xi \wedge E id_E = 0$ for $k = 2n$. By Proposition 5.2 and the injectivity of $\tilde{\omega}_{H^*}$, on $(\tilde{\omega}_{H^*})_{0,1}$, $O_{\wedge^k E} \zeta - \xi \wedge E id_E = 0$ is equal to $\tilde{\omega}_{H^*} = 0$ for $k = 2n$. Then there exists a commutative diagram

$$\begin{array}{ccc}
\tilde{\omega}_{H^*} \zeta & \rightarrow & \tilde{\omega}_{H^*} \xi \\
\uparrow & & \uparrow \\
\tilde{\omega}_{H^*} \zeta & \rightarrow & \tilde{\omega}_{H^*} \xi
\end{array}$$

The operator $\tilde{\omega}_{H^*}$ is decomposed into two operators $\tilde{\omega}_{H^*} = \tilde{\omega}_{H^*} \zeta \wedge E id_E = \tilde{\omega}_{H^*} \xi$ for $\xi \in A^0(\wedge^k E \otimes S^m H)$. We denote by $O(\wedge^k E \otimes l^m)$ the kernel of $\bar{\partial}$ on $\tilde{\omega}_{H^*} \zeta \otimes E id_E = \tilde{\omega}_{H^*} \xi$. Then $O(\wedge^k E \otimes l^m)$ is a subsheaf of $A^0(\wedge^k E \otimes l^m)$, whose elements are holomorphic sections of $f^{-1}(\wedge^k E \otimes l^m)$. Proposition 5.5 and the injectivity of $\tilde{\omega}_{H^*}$ on $(\tilde{\omega}_{H^*})_{0,1}$ induce an isomorphism

$$\text{Ker} O_{\wedge^k E} \cong \tilde{\omega}_{H^*} \zeta \wedge E id_E = \tilde{\omega}_{H^*} \xi \zeta \wedge E \tilde{\theta}_1. \quad (13)$$

by $\xi \mapsto \tilde{\xi}$. We extend Proposition 5.5 to the following proposition:

Proposition 5.6. $\tilde{\omega}_{H^*} \zeta \wedge E id_E = \tilde{\omega}_{H^*} (\bar{\partial} \xi - \zeta \wedge E \tilde{\theta}_1).$
Proof. The terms $\tilde{\omega}^\sharp_{H^*}(\tilde{\gamma_0} \wedge E \tau^{-1} \tilde{\theta}_1)$ and $(\zeta \wedge E \text{id}_E)\tilde{\theta}_1$ in (12) correspond to $\tilde{\omega}^\sharp_{H^*}(\tilde{\zeta} \wedge E \tilde{\theta}_1)$ and $\zeta \wedge E \text{id}_E$, respectively. Then $\tilde{\omega}^\sharp_{H^*}(\tilde{\zeta} \wedge E \tilde{\theta}_1) = \zeta \wedge E \text{id}_E$. The equation and Proposition 5.5 induce this Proposition. \qed

As the proof of Proposition 5.4, Proposition 5.5 and 5.6 imply the following:

Proposition 5.7. Let $\xi$ and $\zeta$ be elements of $A^0(\wedge^k E \otimes S^m H)$ and $A^0(\wedge^{k-1} E \otimes S^{m+1} H)$, respectively. The element $\xi$ is quaternionic and $\zeta = \text{tr} \circ D_{\wedge^k E}(\xi)$ if and only if $\tilde{\partial}^k \xi - \tilde{\zeta} \wedge E \tilde{\theta}_1 = 0$ for $1 \leq k \leq 2n - 1$, $\tilde{\partial}^k \xi - \tilde{\zeta} \wedge E \tilde{\theta}_1 = 0$ and $\tilde{\partial}^k \zeta = 0$ for $k = 2n$. \qed

6. Quaternionic $k$-vector fields

In this section, we consider a section of $\wedge^k E \otimes S^m H$ in the case $m = k$. Such a section is a $k$-vector field on $M$ since $\wedge^k E \otimes S^k H$ is the subbundle of $\wedge^k TM \otimes \mathbb{C}$. We call a quaternionic section of $\wedge^k E \otimes S^k H$ as a quaternionic $k$-vector field on $M$. We prove that any quaternionic $k$-vector field corresponds to a holomorphic $(k,0)$-vector field on $Z$.

6.1. Definition of quaternionic $k$-vector fields.

Definition 6.1. A section $X$ of $\wedge^k E \otimes S^k H$ is called a quaternionic $k$-vector field on $M$ if $D^{0}_{\wedge^k E}(X) = 0$ for $1 \leq k \leq 2n - 1$ and $D^{0}_{\wedge^{k-1} E \otimes \text{tr} \circ D_{\wedge^k E}(X)} = 0$ for $k = 2n$. If a quaternionic $k$-vector field $X$ is $\tau$-invariant, then $X$ is said to be a quaternionic real $k$-vector field on $M$.

This definition is also valid in quaternionic manifolds. We denote by $Q(\wedge^k E \otimes S^k H)$ the sheaf of quaternionic $k$-vector fields on $M$. The sheaf $Q(\wedge^k E \otimes S^k H)^\tau$ of $\tau$-invariant elements of $Q(\wedge^k E \otimes S^k H)$ is the sheaf of quaternionic real $k$-vector fields on $M$.

Proposition 6.2. A 1-vector field $X$ on $M$ is quaternionic if and only if $X$ preserves the quaternionic structure $Q$, that is, $L_X Q \subset Q$ where $L_X$ is the Lie derivative with respect to $X$.

Proof. Let $\nabla$ be the Levi-Civita connection on $M$. We define an endomorphism $[\nabla X, I]$ for $I \in Q$ as $\nabla X \circ I - I \circ \nabla X$. Since $L_X I = \nabla_X I - [\nabla X, I]$, the condition $L_X Q \subset Q$ is equivalent to $[\nabla X, Q] \subset Q$. Under the decomposition $\text{End}(TM) = (E \otimes E^*)_0 \otimes Q + E \otimes E^* \otimes \text{id}_H$, $[\nabla X, Q] \subset Q$ is equivalent that $\nabla X$ has no component of $(E \otimes E^*)_0 \otimes Q$. This condition is written by $D^0_{E}(X) = 0$ since $(E \otimes E^*)_0 \otimes Q \cong (E \otimes E^*)_0 \otimes S^2 H$ by $\omega^2_{H^*}$. \qed

6.2. Horizontal lift of $k$-vector fields to $P(H^*)$. Let $\widetilde{A}_0(\wedge^k \widetilde{H})$ denote the sheaf of $\text{GL}(1, \mathbb{H})$-invariant horizontal $k$-vector fields on $P(H^*)$. The isomorphism $A^0(\wedge^k TM) \cong A^0(\wedge^k \widetilde{H})$ is given by taking the horizontal lift $\widetilde{X}_h$ of a $k$-vector field $X \in A^0(\wedge^k TM)$. From now on, we denote by $\wedge^k E \widetilde{S}^k H$ the tensor space $\wedge^k E \otimes S^k H$ for simplicity. We define $\wedge^k E \widetilde{S}^k H$ as a subbundle of $\wedge^k \widetilde{H}$ corresponding to $\wedge^k E \widetilde{S}^k H$. Let $\widetilde{A}_0(\wedge^k E \widetilde{S}^k H)$ be a subsheaf of $\widetilde{A}_0(\wedge^k \widetilde{H})$ which consists of the horizontal lift of elements of $A^0(\wedge^k E \widetilde{S}^k H)$. Then $A^0(\wedge^k E \widetilde{S}^k H) \cong \widetilde{A}_0(\wedge^k E \widetilde{S}^k H)$ by $X \mapsto \widetilde{X}_h$. The $k$-th wedge $\widetilde{\theta}^k_0$ provides a bundle map $\widetilde{\theta}^k_0 : \wedge^k \widetilde{H} \to p^{-1}(\wedge^k E)$ and an isomorphism

$$\widetilde{A}_0(\wedge^k E \widetilde{S}^k H) \cong \widetilde{A}^0_{(k,0)}(\wedge^k E)$$

such that $\widetilde{\theta}^k_0(\widetilde{X}_h) = \widetilde{X}_0$ for $X \in A^0(\wedge^k E \widetilde{S}^k H)$.

Lemma 6.3. Let $\widetilde{X}_h$ be an element of $\widetilde{A}_0(\wedge^k E \widetilde{S}^k H)$. The $(k,0)$-part $\widetilde{X}^{k,0}_h$ is $\text{GL}(1, \mathbb{C})$-invariant and holomorphic along each fiber of $p : P(H^*) \to M$. 

Proof. The horizontal $k$-vector field $\widetilde{X}_h$ is $GL(1, \mathbb{C})$-invariant. Since the action of $GL(1, \mathbb{C})$ on $P(H^*)$ is holomorphic, the $(k, 0)$-part $\widetilde{X}_h^{k, 0}$ is also $GL(1, \mathbb{C})$-invariant. The vector field $\widetilde{X}_h^{k, 0}$ is holomorphic along each fiber if and only if $\nabla^V_v \widetilde{X}_h^{k, 0}$ vanishes for any $v \in A^0(\widetilde{V})$. We remark that the bundle $H^1$ is not a holomorphic subbundle of $T^{1,0}P(H^*)$. The condition $\nabla^V_v \widetilde{X}_h^{k, 0} = 0$ is equal that $\nabla^V_v \widetilde{X}_h^{k, 0}$ is annihilated by $\tilde{\theta}^k_0, \tilde{\theta}^{k-1}_0 \wedge \eta_0, \tilde{\theta}^{k-1}_0 \wedge \eta_1$ and $\tilde{\theta}^{k-2}_0 \wedge \eta_0 \wedge \eta_1$. By Proposition 3.14, $\tilde{\theta}^{k-1}_0 \wedge \eta_0 (\nabla^V_v \widetilde{X}_h^{k, 0})$, $\tilde{\theta}^{k-1}_0 \wedge \eta_1 (\nabla^V_v \widetilde{X}_h^{k, 0})$ and $\tilde{\theta}^{k-2}_0 \wedge \eta_0 \wedge \eta_1 (\nabla^V_v \widetilde{X}_h^{k, 0})$ vanish. Furthermore, $\tilde{\theta}^k_0 (\nabla^V_v \widetilde{X}_h^{k, 0}) = \partial_v (\tilde{\theta}^k_0 (\widetilde{X}_h^{k, 0})) = \partial_v (\tilde{\theta}^k_0 (\widetilde{X}_h)) = \tilde{\theta}^k_0 (\nabla^V_v \widetilde{X}_h) = 0$ for any $v \in A^0(\widetilde{V})$. Hence $\widetilde{X}_h^{k, 0}$ is holomorphic along each fiber. □

We denote by $\widetilde{A}_0(\wedge^k \widetilde{H}^{1,0})$ the sheaf of horizontal $(k, 0)$-vector fields which are $GL(1, \mathbb{C})$-invariant and holomorphic along each fiber of $p : P(H^*) \rightarrow M$.

Proposition 6.4. The isomorphism $A^0(\wedge^k ES^k H) \cong \widetilde{A}_0(\wedge^k \widetilde{H}^{1,0})$ is given by $X \mapsto \widetilde{X}_h^{0}$.

Proof. Let $X'$ be a horizontal $(k, 0)$-vector field on $P(H^*)$. As in the proof of Lemma 6.3, $\tilde{\theta}^k_0 (\nabla^V_v X') = \partial_v (\tilde{\theta}^k_0 (X'))$ for any $v \in A^0(\widetilde{V})$. Moreover,

\[ R^c_v (\tilde{R}^c_0 (X')) = (R^c_v (\tilde{R}^c_0 ((R_{c-1})* X'))) = c^k \tilde{\theta}^k_0 (\tilde{R}_{c-1} (X')) \tag{15} \]

for any $c \in GL(1, \mathbb{C})$. Thus the bundle isomorphism $\tilde{\theta}^k_0 : \wedge^k \widetilde{H}^{1,0} \cong p^{-1} (\wedge^k E)$ induces an isomorphism

\[ \widetilde{A}_0(\wedge^k \widetilde{H}^{1,0}) \cong \widetilde{A}^0_{(k, 0)}(\wedge^k E). \tag{16} \]

By Lemma 6.3, we have a map $\widetilde{A}_0(\wedge^k ES^k H) \rightarrow \widetilde{A}_0(\wedge^k \widetilde{H}^{1,0})$. This map is the composition of two isomorphisms in (14) and (16), and it is isomorphic. Hence $A^0(\wedge^k ES^k H) \cong \widetilde{A}_0(\wedge^k \widetilde{H}^{1,0})$. It follows from the isomorphism (13) that $\widetilde{X}_0 = \tilde{\theta}^k_0 (\widetilde{X}_h) = \tilde{\theta}^k_0 (\widetilde{X}_h^{k, 0})$ for $X \in A^0(\wedge^k ES^k H)$. □

Under the irreducible decomposition of $\wedge^k TM$, the horizontal lift of the components except for $\wedge^k ES^k H$ vanish by $\tilde{\theta}^k_0$. Hence, Proposition 6.4 implies the following:

Corollary 6.5. Let $X$ be an element of $A^0(\wedge^k TM)$. The $(k, 0)$-part $\widetilde{X}_h^{k, 0}$ of the horizontal lift $\widetilde{X}_h$ is $GL(1, \mathbb{C})$-invariant and holomorphic along each fiber of $p : P(H^*) \rightarrow M$. □

6.3. Holomorphic lift of quaternionic $k$-vector fields to $P(H^*)$. A horizontal $(k, 0)$-vector field $X'$ on $P(H^*)$ is called of $GL(1, \mathbb{C})$-order $m$ if $(R_{c-1})* X' = c^m X'$ for any $c \in GL(1, \mathbb{C})$. We define $\widetilde{A}_m(\wedge^k \widetilde{H}^{1,0})$ as the sheaf of horizontal $(k, 0)$-vector fields which are of $GL(1, \mathbb{C})$-order $m$ and holomorphic along each fiber of $p : P(H^*) \rightarrow M$. By the equation (15), we obtain an isomorphism

\[ \widetilde{A}_m(\wedge^k \widetilde{H}^{1,0}) \cong \widetilde{A}^0_{(k, m, 0)}(\wedge^k E) \tag{17} \]

given by $X' \mapsto \tilde{\theta}^k_0 (X')$. Let $\xi$ be an element of $A^0(\wedge^k ES^{k+m} H)$. The lift $\tilde{\xi}_0$ of $\xi$ to $P(H^*)$ is in $\widetilde{A}^0_{(k, m, 0)}(\wedge^k E)$. By the isomorphism (17), there exists a unique element $\tilde{\xi}_0$ of $\widetilde{A}_m(\wedge^k \widetilde{H}^{1,0})$ such that $\tilde{\theta}^k_0 (\tilde{\xi}_0) = \tilde{\xi}_0$.

Using the isomorphism (17), we have

\[ A^0(\wedge^k ES^{k+m} H) \cong \widetilde{A}_m(\wedge^k \widetilde{H}^{1,0}) \tag{18} \]

by $\xi \mapsto \tilde{\xi}_0$. In the case $m = 0$, by considering $\xi$ as the $k$-vector field $X$ on $M$, the isomorphism $A^0(\wedge^k ES^k H) \cong \widetilde{A}_0(\wedge^k \widetilde{H}^{1,0})$ is given by $X \mapsto \widetilde{X}_h^{k, 0}$. 

Proposition 6.6. Let $X$ and $\zeta$ be elements of $\mathcal{A}^0(\wedge^k ES^k H)$ and $\mathcal{A}^0(\wedge^k E S^{k+1} H)$, respectively. The $k$-vector field $X$ is quaternionic and $\zeta = \text{tr} \circ \mathcal{D} \wedge E(X)$ if and only if there exist $Y_0 \in \mathcal{A}^0_{P(H)}(\wedge^{k-1} \mathcal{H}^{1,0})$ and $Z_0 \in \mathcal{A}^0_{P(H)}(\wedge^{k-2} \mathcal{H}^{1,0})$ such that the $(k,0)$-vector field $\tilde{X}_h^{k,0} + Y_0 \wedge v_0 + Y_1 \wedge v_1 + Z_0 \wedge v_0 \wedge v_1$ is holomorphic for $Y_1 = \frac{1}{\sqrt{\eta}} \tilde{Y}_\zeta$.

Proof. Proposition 6.4 implies that $\tilde{X}_0 = \tilde{\theta}_0^k(\tilde{X}_h^{k,0})$, and $\tilde{\zeta}_0 = \tilde{\theta}_0^{k-1}(\tilde{Y}_\zeta)$. Setting $Y_1 = \frac{1}{\sqrt{\eta}} \tilde{Y}_\zeta$, then we obtain $\tilde{\zeta}_0 = k^2 \tilde{\theta}_0^{k-1}(r^2 Y_1)$. It follows from Proposition 7.4 that $X$ is quaternionic and $\zeta = \text{tr} \circ \mathcal{D} \wedge E(X)$ if and only if $\tilde{\partial} \tilde{X}_0 - \tilde{\zeta}_0 \wedge E r^{-2} \tilde{\theta}_1 = 0$ for $1 \leq k \leq 2n - 1$, $\tilde{\partial} \tilde{X}_0 - \tilde{\zeta}_0 \wedge E r^{-2} \tilde{\theta}_1 = 0$ and $\tilde{\partial} \tilde{X}_0 = 0$ for $k = 2n$. The condition is equivalent to $\tilde{\partial} \tilde{\theta}_0^k(\tilde{X}_h^{k,0}) - k^2 \tilde{\theta}_0^{k-1}(Y_1) \wedge E \tilde{\theta}_1 = 0$ for $1 \leq k \leq 2n - 1$, $\tilde{\partial} \tilde{\theta}_0^k(\tilde{X}_h^{k,0}) - k^2 \tilde{\theta}_0^{k-1}(Y_1) \wedge E \tilde{\theta}_1 = 0$ and $\tilde{\partial} \tilde{\theta}_0^{k-1}(r^2 Y_1) = 0$ for $k = 2n$. It is equal that there exist $Y_0 \in \mathcal{A}^0_{P(H)}(\wedge^{k-1} \mathcal{H}^{1,0})$, $Z_0 \in \mathcal{A}^0_{P(H)}(\wedge^{k-2} \mathcal{H}^{1,0})$ such that $\tilde{X}_h^{k,0} + Y_0 \wedge v_0 + Y_1 \wedge v_1 + Z_0 \wedge v_0 \wedge v_1$ is holomorphic by Theorem 3.21.

6.4. Horizontal lift of $k$-vector fields to $Z$. We denote by $\tilde{\mathcal{A}}_0(\wedge^k \mathcal{H})$ the sheaf of horizontal smooth $k$-vector fields which are constant along each fiber of $f : Z \to M$. For $X \in \mathcal{A}^0(\wedge^k TM)$, the horizontal lift $\tilde{X}_h$ is an element of $\tilde{\mathcal{A}}_0(\wedge^k \mathcal{H})$. Conversely, $\tilde{\mathcal{A}}_0(\wedge^k \mathcal{H})$ consists of such elements. Hence, we obtain an isomorphism $\mathcal{A}^0(\wedge^k TM) \cong \tilde{\mathcal{A}}_0(\wedge^k \mathcal{H})$ by $X \mapsto \tilde{X}_h$. We denote by $\tilde{\mathcal{A}}(\wedge^k \mathcal{H}^{1,0})$ the sheaf of horizontal $(k,0)$-vector fields which are holomorphic along each fiber of $f$. The vector field $\tilde{X}_h$ and the $(k,0)$-part $\tilde{X}_h^{k,0}$ correspond to $\tilde{X}_h^{k,0}$, respectively. Proposition 6.4 induces the following:

Proposition 6.7. The isomorphism $\mathcal{A}^0(\wedge^k ES^k H) \cong \tilde{\mathcal{A}}(\wedge^k \mathcal{H}^{1,0})$ is given by $X \mapsto \tilde{X}_h^{k,0}$. Moreover, $\tilde{X} = \tilde{\theta}_0^k(\tilde{X}_h^{k,0})$ for $X \in \mathcal{A}^0(\wedge^k ES^k H)$.

Corollary 6.8. Let $X$ be an element of $\mathcal{A}^0(\wedge^k TM)$. The $(k,0)$-part $\tilde{X}_h^{k,0}$ of the horizontal lift $\tilde{X}_h$ is holomorphic along each fiber of $f : Z \to M$.

We consider the holomorphic bundle $\wedge^k \mathcal{H}^{1,0} \otimes l^m$ for a non-negative integer $m$. Let $\tilde{\mathcal{A}}(\wedge^k \mathcal{H}^{1,0} \otimes l^m)$ be a subsheaf of $\wedge^k E \otimes l^m$-valued horizontal smooth $(k,0)$-vector fields which are holomorphic along each fiber of $f : Z \to M$. Let $\tilde{\mathcal{O}}(\wedge^k \mathcal{H}^{1,0} \otimes l^m)$ denote the subsheaf of $\tilde{\mathcal{A}}(\wedge^k \mathcal{H}^{1,0} \otimes l^m)$ of holomorphic $l^m$-valued horizontal $(k,0)$-vector fields. By the definition of $l$, we obtain the isomorphism

$$\tilde{\mathcal{A}}(\wedge^k \mathcal{H}^{1,0} \otimes l^m) \cong \tilde{\mathcal{A}}_m(\wedge^k \mathcal{H}^{1,0}).$$

The isomorphism $\tilde{\theta}_0^k : \wedge^k \mathcal{H}^{1,0} \to f^{-1}(\wedge^k E) \otimes l^k$ is extended to $\tilde{\theta}_0^k \otimes (\text{id})^m : \wedge^k \mathcal{H}^{1,0} \otimes l^m \to f^{-1}(\wedge^k E) \otimes l^{k+m}$.

Lemma 6.9. The map $\tilde{\theta}_0^k \otimes (\text{id})^m$ induces the isomorphisms $\tilde{\mathcal{A}}(\wedge^k \mathcal{H}^{1,0} \otimes l^m) \cong \tilde{\mathcal{A}}_m(\wedge^k E \otimes l^{k+m})$ and $\tilde{\mathcal{O}}(\wedge^k \mathcal{H}^{1,0} \otimes l^m) \cong \tilde{\mathcal{O}}(\wedge^k E \otimes l^{k+m})$.

Proof. Let $X''$ be an $l^m$-valued horizontal $(k,0)$-vector field. The covariant derivative $\nabla^{0,1} X''$ is a section of $\wedge^k \mathcal{H}^{1,0} \otimes l \otimes \wedge^{0,1} T^* \mathcal{O}$ since $\mathcal{H}^{1,0}$ is a holomorphic subbundle of $T^{1,0} Z$. It follows from Proposition 4.7 that $\tilde{\partial} \tilde{\theta}_0^k \otimes (\text{id})^m(X'') = \tilde{\theta}_0^k \otimes (\text{id})^m(\nabla^{0,1} X'')$. Thus $\tilde{\partial} \tilde{\theta}_0^k \otimes (\text{id})^m(X'') = 0$ if and only if $\nabla^{0,1} X'' = 0$. Hence, we obtain the two isomorphisms in this proposition.
We denote $\hat{\partial}_0 \otimes (\text{id})^m$ by $\hat{\partial}_0^m$ for short. Let $\xi$ be an element of $A^0(\wedge^k ES^{k+m} H)$. The lift $\tilde{\xi}$ of $\xi$ to $\tilde{Z}$ is in $\tilde{A}^0(\wedge^k E \otimes l^{k+m})$. Then there exists a unique element $\tilde{Y}_\xi$ of $\tilde{A}(\wedge^k \tilde{H}^{1,0} \otimes l^m)$ such that

$$\hat{\partial}_0^m(\tilde{Y}_\xi) = \tilde{\xi}$$

by Lemma [6.9] The isomorphisms in (18) and (19) yield

$$A^0(\wedge^k ES^{k+m} H) \cong \tilde{A}(\wedge^k \tilde{H}^{1,0} \otimes l^m)$$

by $\xi \mapsto \tilde{Y}_\xi$. In the case $m = 0$, by considering $\xi$ as a $k$-vector field $X$ in $M$, the isomorphism $A^0(\wedge^k ES^k H) \cong \tilde{A}(\wedge^k \tilde{H}^{1,0})$ is given by $X \mapsto \tilde{X}_h^k$. We consider the operator

$$\mathcal{D}_{\wedge^k E} : A^0(\wedge^k ES^{k+m} H) \to A^0(\wedge^k E \otimes E^* \otimes S^{k+m+1} H).$$

It follows from the isomorphism (13) that $\text{Ker} \mathcal{D}_{\wedge^k E} \cong \tilde{O}(\wedge^k E \otimes l^{k+m})$ by $\xi \mapsto \tilde{\xi}$. By Lemma [6.9] we obtain the following isomorphism:

**Corollary 6.10.** Ker $\mathcal{D}_{\wedge^k E} \cong \tilde{O}(\wedge^k \tilde{H}^{1,0} \otimes l^m)$ by $\xi \mapsto \tilde{Y}_\xi$. \( \square \)

6.5. Holomorphic lift of quaternionic $k$-vector fields to $Z$.

**Proposition 6.11.** Let $X$ and $\zeta$ be elements of $A^0(\wedge^k ES^k H)$ and $A^0(\wedge^k \times ES^{k+1} H)$, respectively. The $k$-vector field $X$ is quaternionic and $\zeta = \text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$ if and only if the $(k,0)$-vector field $\tilde{X}_h^k + Y \wedge v$ is holomorphic for $Y = k^{-2}\tilde{Y}_\zeta$.

**Proof.** It follows from Proposition 6.7 that $\tilde{X} = \hat{\partial}_0^k(\tilde{X}_h^k)$. We set $Y = k^{-2}\tilde{Y}_\zeta$. Then $\tilde{\zeta} = k^2\hat{\partial}_0^{-1}(Y)$ by [20]. Proposition 5.7 implies that $X$ is quaternionic and $\zeta = \text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$ if and only if $\hat{\partial} \tilde{X} - \tilde{\zeta} \wedge E \hat{\theta}_1 = 0$ for $1 \leq k \leq 2n - 1$, $\hat{\partial} \tilde{X} - \tilde{\zeta} \wedge E \hat{\theta}_1 = 0$ and $\hat{\partial} \tilde{\zeta} = 0$ for $k = 2n$. They are written by $\hat{\partial}(\hat{\partial}_0^k(\tilde{X}_h^k)) - k^2\hat{\partial}_0^{-1}(Y) \wedge E \hat{\theta}_1 = 0$ for $1 \leq k \leq 2n - 1$, $\hat{\partial}(\hat{\partial}_0^k(\tilde{X}_h^k)) - k^2\hat{\partial}_0^{-1}(Y) \wedge E \hat{\theta}_1 = 0$ and $\hat{\partial}(\hat{\partial}_0^{-1}(Y)) = 0$ for $k = 2n$. The condition is equal that $\tilde{X}_h^k + Y \wedge v$ is holomorphic for any $k$ by Theorem 4.8. \( \square \)

Let $\tilde{O}(\wedge^k T^{1,0} Z)$ be a sheaf of holomorphic $(k,0)$-vector fields defined in the pull-back of open sets on $M$ by $f : Z \to M$.

**Theorem 6.12.** An isomorphism $Q(\wedge^k ES^k H) \cong \tilde{O}(\wedge^k T^{1,0} Z)$ is given by $X \mapsto \tilde{X}_h^k + Y \wedge v$ where $Y$ is defined by $k^{-2}\tilde{Y}_\text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$. Moreover, $H^0(Q(\wedge^k ES^k H))$ is isomorphic to the space $H^0(\tilde{O}(\wedge^k T^{1,0} Z))$ of holomorphic $k$-vector fields on $Z$ by the correspondence.

**Proof.** By Proposition 5.11 we obtain a map $Q(\wedge^k ES^k H) \to \tilde{O}(\wedge^k T^{1,0} Z)$ by $X \mapsto \tilde{X}_h^k + Y \wedge v$. Conversely, any element $X'$ of $\tilde{O}(\wedge^k T^{1,0} Z)$ is written by $X' = X_h^k + Y \wedge v$ for $X_h^k \in \tilde{A}(\wedge^k \tilde{H}^{1,0})$, $Y \in \tilde{A}(\wedge^k \tilde{H}^{1,0} \otimes l^2)$. The isomorphism (21) implies that there exist $X \in A^0(\wedge^k ES^k H)$ and $\zeta \in A^0(\wedge^k \times ES^{k+1} H)$ such that $X_h^k = \tilde{X}_h^k$ and $Y = k^{-2}\tilde{Y}_\zeta$. Since $X' = \tilde{X}_h^k + Y \wedge v$ is holomorphic, $X$ is quaternionic and $\zeta = \text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$ by Proposition 6.11. Thus we obtain the isomorphism $Q(\wedge^k ES^k H) \cong \tilde{O}(\wedge^k T^{1,0} Z)$ by $X \mapsto \tilde{X}_h^k + Y \wedge v$. If $X$ is a global section of $Q(\wedge^k ES^k H)$, then $\zeta = \text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$ and $Y = k^{-2}\tilde{Y}_\zeta$ are also global. Hence $H^0(Q(\wedge^k ES^k H))$ is isomorphic to $H^0(\tilde{O}(\wedge^k T^{1,0} Z)) = H^0(\tilde{O}(\wedge^k T^{1,0} Z))$ by $X \mapsto \tilde{X}_h^k + Y \wedge v$. It completes the proof. \( \square \)

6.6. Holomorphic lift of quaternionic real $k$-vector fields to $Z$. We denote by $\tilde{A}^0_{\mathcal{P}(H^*)}(\wedge^k TP(H^*))$ the sheaf $p^{-1}_{\mathcal{P}(H^*)}A^0_{\mathcal{P}(H^*)}(\wedge^k TP(H^*))$ on $P(H^*)$. An endomorphism $\tilde{\tau}$ of $\tilde{A}^0_{\mathcal{P}(H^*)}(\wedge^k TP(H^*))$ is defined as

$$\tilde{\tau}(X') = (\mathcal{R}_{-})_{\ast}X'$$
for $X' \in \tilde{A}_m^0(\wedge^k T^*M)$). The action of $j$ preserves the horizontal space $\tilde{H}$ and it is anti-holomorphic. Thus $\tilde{\tau}$ induces endomorphisms of $\tilde{A}_0(\wedge^k \tilde{H})$ and $\tilde{A}_0(\wedge^k \tilde{H}^1,0)$. In fact, $\tilde{\tau}$ is the complex conjugate on $\tilde{A}_0(\wedge^k \tilde{H})$. The endomorphism $\tau = J_E \otimes J_H$ on $TM$ is extended to $\wedge^k TM$, and it is also the complex conjugate on $\mathcal{A}^0(\wedge^k TM)$. Thus $\mathcal{A}^0(\wedge^k TM)^\tau \cong \tilde{A}_0(\wedge^k \tilde{H})^\tilde{\tau}$ is given by taking the horizontal lift of real $k$-vector fields of $M$. Since $\tilde{\tau}$ preserves the horizontal $(k,0)$-vector space $\wedge^k \tilde{H}^{1,0}$, $\tilde{A}_0(\wedge^k \tilde{H})^\tilde{\tau} \to \tilde{A}_0(\wedge^k \tilde{H}^{1,0})^\tilde{\tau}$ by $\tilde{X}_h \mapsto \tilde{X}^{k,0}_h$. Hence we have the isomorphism $\mathcal{A}^0(\wedge^k ES^{k}H)^\tau \cong \tilde{A}_0(\wedge^k \tilde{H}^{1,0})^\tilde{\tau}$ by $X \mapsto \tilde{X}^{k,0}_h$.

The map $\tilde{\tau}$ induces an endomorphism of $\tilde{A}_m(\wedge^k \tilde{H}^{1,0})$ since

$$(R_{c,-1})_*(R_{c,-1})_sX' = (R_{c,-1})_s(R_{c,-1})_sX' = (R_{c,-1})_s\tilde{c}^mX' = \tilde{c}^m(R_{c,-1})_sX'$$

for $X' \in \tilde{A}_m(\wedge^k \tilde{H}^{1,0})$. Moreover,

$$\tilde{\tau}(\tilde{\theta}_0^k(X')) = J_E R^*_c(\tilde{\theta}_0^k(X')) = J_E R^*_c(\tilde{\theta}_0^k)((R_{c,-1})_sX') = \tilde{\tau}(\tilde{\theta}_0^k(\tau(X'))) = \tilde{\theta}_0^k(\tilde{\tau}(X'))$$

for $X' \in \tilde{A}_m(\wedge^k \tilde{H}^{1,0})$. It yields that $\tilde{A}_m(\wedge^k \tilde{H}^{1,0})^\tilde{\tau} \cong \mathcal{A}^{(k+m,0)}_0(\wedge^k E)^\tilde{\tau}$ by $X' \mapsto \tilde{\theta}_0^k(X')$. By Corollary 3.6 we have an $\mathcal{R}$-isomorphism

$$\mathcal{A}^0(\wedge^k ES^{k+m}H)^\tau \cong \tilde{A}_m(\wedge^k \tilde{H}^{1,0})^\tilde{\tau}$$

by $\xi \mapsto \tilde{Y}_\xi$. In the case $m = 0$, the isomorphism $\mathcal{A}^0(\wedge^k ES^{k}H)^\tau \cong \tilde{A}_0(\wedge^k \tilde{H}^{1,0})^\tilde{\tau}$ is given by $X \mapsto \tilde{X}^{k,0}_h$.

We denote by $\tilde{A}^{0}_Z(\wedge^k TZ)$ the sheaf $f^{-1}f_*A^0_Z(\wedge^k TZ)$ on $Z$. An endomorphism $\tilde{\tau}$ of $\tilde{A}^{0}_Z(\wedge^k TZ)$ is defined by

$$\tilde{\tau}(X') = (R_{c,[\xi]})_sX'$$

for $X' \in \tilde{A}^{0}_Z(\wedge^k TZ)$. Then $\tilde{\tau}$ induces endomorphisms of $\tilde{A}_0(\wedge^k \tilde{H})$ and $\tilde{A}(\wedge^k \tilde{H}^{1,0})$. Under isomorphisms $\tilde{A}_0(\wedge^k \tilde{H}) \cong \tilde{A}_0(\wedge^k \tilde{H})$ and $\tilde{A}(\wedge^k \tilde{H}^{1,0}) \cong \tilde{A}_0(\wedge^k \tilde{H}^{1,0})$, the elements $\tilde{\tau}(\tilde{X}_h), \tilde{\tau}(\tilde{\tilde{X}}^{k,0}_h)$ correspond to $\tau(\tilde{X}_h), \tau(\tilde{\tilde{X}}^{k,0}_h)$ for $X \in \mathcal{A}^0(\wedge^k E^{k}H)$, respectively. Therefore, $\mathcal{A}^0(\wedge^k ES^{k}H)^\tau \cong \tilde{A}(\wedge^k \tilde{H}^{1,0})^\tau$ by $X \mapsto \tilde{X}^{k,0}_h$. Moreover, $\tilde{\tau}$ induces a bundle map of $l^m$, and it is extended to an endomorphism of $\tilde{A}(\wedge^k \tilde{H}^{1,0} \otimes l^m)$. Then $\tilde{A}_m(\wedge^k \tilde{H}^{1,0})^\tilde{\tau} \cong \tilde{A}(\wedge^k \tilde{H}^{1,0} \otimes l^m)^\tilde{\tau}$. Using the isomorphism (22), we obtain an $\mathcal{R}$-isomorphism

$$\mathcal{A}^0(\wedge^k ES^{k+m}H)^\tau \cong \tilde{A}(\wedge^k \tilde{H}^{1,0} \otimes l^m)^\tilde{\tau}$$

by $\xi \mapsto \tilde{Y}_\xi$. In the case $m = 0$, the isomorphism $\mathcal{A}^0(\wedge^k ES^{k}H)^\tau \cong \tilde{A}(\wedge^k \tilde{H}^{1,0})^\tilde{\tau}$ is given by $X \mapsto \tilde{X}^{k,0}_h$. Moreover,

$$(\ker \mathcal{D}_{\wedge^k E})^\tau \cong \tilde{D}(\wedge^k \tilde{H}^{1,0} \otimes l^m)^\tilde{\tau}$$

under the correspondence. Corresponding to Proposition 6.11 we obtain the following:

**Proposition 6.13.** Let $X$ and $\zeta$ be elements of $\mathcal{A}^0(\wedge^k ES^{k}H)$ and $\mathcal{A}^0(\wedge^k -ES^{k+1}H)$, respectively. The $k$-vector field $X$ is quaternionic and real, and $\zeta = \text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$ if and only if the $(k,0)$-vector field $\tilde{X}^{k,0}_h + Y \wedge v$ is holomorphic and $\tilde{\tau}$-invariant for $Y = k^{-2}Y_\zeta$.

**Proof.** Let $X$ be a quaternionic $k$-vector field on $M$ and $\zeta$ the element $\text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$. It suffices to show that $X$ is real if and only if $\tilde{X}^{k,0}_h + Y \wedge v$ is $\tilde{\tau}$-invariant for $Y = k^{-2}Y_\zeta$. If $X$ is real, then $\zeta = \text{tr} \circ \mathcal{D}_{\wedge^k E}(X)$ is $\tau$-invariant. By the isomorphism (22), $\tilde{X}^{k,0}_h$ and $Y = k^{-2}Y_\zeta$ are $\tilde{\tau}$-invariant. Now we have $\tilde{\tau}(v) = v$ since $\tilde{\tau}(v_1) = v_1$. Thus $\tilde{X}^{k,0}_h + Y \wedge v$ is also $\tilde{\tau}$-invariant. Conversely, we assume that $\tilde{X}^{k,0}_h + Y \wedge v$ is $\tilde{\tau}$-invariant. Then $\tilde{\tau}(\tilde{X}^{k,0}_h) = \tilde{\tilde{X}}^{k,0}_h$ since $\tilde{\tau}$ preserves the decomposition $\wedge^k T^{1,0}Z = \wedge^k \tilde{H}^{1,0} \otimes (\wedge^k -\tilde{H}^{1,0}) \wedge \tilde{V}^{1,0}$. It follows from (22) that $X$ is $\tau$-invariant, that is, real. Hence we finish the proof. □
Proposition 6.13 implies

**Theorem 6.14.** An $\mathbb{R}$-isomorphism $Q(\wedge k E S^k H)^{\tau} \cong \hat{O}(\wedge k T^{1,0} Z)^{\tau}$ is given by $X \mapsto \hat{X}^{k,0}_k + Y \wedge v$ where $Y$ is defined by $k^{-2} \hat{V}_{10D}(X)$. Moreover, $H^0(Q(\wedge k E S^k H)^{\tau} \cong H^0(\hat{O}(\wedge k T^{1,0} Z))^{\tau}$ by the correspondence.

6.7. **Example.** Let $M$ be the $n$-dimensional quaternionic projective space $\mathbb{H}P^n$. Then $P(H^*) = \mathbb{C}^{2n+2}\{0\}$ as a complex manifold. The twistor space $Z$ is the $(2n+1)$-dimensional complex projective space $\mathbb{C}P^{2n+1}$. We consider the space $H^0(Q(\wedge k E S^k H))$ of quaternionic $k$-vector fields on $\mathbb{H}P^n$. We denote by $(z_0, z_1, \ldots, z_{2n+1})$ the standard coordinate of $\mathbb{C}^{2n+2}$. The vector field $v_0$ associated with the action of $GL(1, \mathbb{C})$ on $P(H^*)$ is written by

$$v_0 = z_0 \frac{\partial}{\partial z_0} + z_1 \frac{\partial}{\partial z_1} + \cdots + z_{2n+1} \frac{\partial}{\partial z_{2n+1}}$$

on $\mathbb{C}^{2n+2}\{0\}$. Let $\tilde{V}_k$ denote the space of $GL(1, \mathbb{C})$-invariant holomorphic $k$-vector fields on $\mathbb{C}^{2n+2}\{0\}$. Then

$$\tilde{V}_k = \left\{ \sum a_{i_1 \ldots i_k j_1 \ldots j_k} z_{i_1} \ldots z_{i_k} \frac{\partial}{\partial z_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{j_k}} \mid a_{i j k} \in \mathbb{C} \right\}$$

We regard the coefficient $(a_{i_1 \ldots i_k j_1 \ldots j_k})$ as an element of $\otimes^k \mathbb{C}^{2n+2} \otimes \otimes^k (\mathbb{C}^{2n+2})^*$. Then $(a_{i_1 \ldots i_k j_1 \ldots j_k})$ is in $S^k \mathbb{C}^{2n+2} \otimes \wedge^k (\mathbb{C}^{2n+2})^*$. We define $S^k \otimes \wedge^k$ as the projection from $\otimes^k \mathfrak{gl}(2n + 2, \mathbb{C}) \cong \otimes^k \mathbb{C}^{2n+2} \otimes \otimes^k (\mathbb{C}^{2n+2})^*$ to $S^k \mathbb{C}^{2n+2} \otimes \wedge^k (\mathbb{C}^{2n+2})^*$. Then $\tilde{V}_k \cong S^k \otimes \wedge^k \otimes^k \mathfrak{gl}(2n + 2, \mathbb{C})$. The space $H^0(\hat{O}(\wedge k T^{1,0} Z))$ of holomorphic $k$-vector fields on $\mathbb{C}P^{2n+1}$ is identified with the quotient space $\tilde{V}_k/\tilde{V}_{k-1} \wedge v_0$, where $\tilde{V}_{k-1} \wedge v_0$ is the subspace of $\tilde{V}_k$ consisting of $a_{i_1 \ldots i_{k-1} j_1 \ldots j_{k-1}} z_{i_1} \ldots z_{i_{k-1}} \frac{\partial}{\partial z_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{j_{k-1}}} \wedge v_0$ (c.f. §5.1 in [7]):

$$H^0(\hat{O}(\wedge k T^{1,0} Z)) \cong \tilde{V}_k/\tilde{V}_{k-1} \wedge v_0$$

The space $\tilde{V}_{k-1} \wedge v_0$ is isomorphic to $S^k \otimes \wedge^k (\otimes^{k-1} \mathfrak{gl}(2n + 2, \mathbb{C}) \otimes \operatorname{Id})$. Theorem 6.12 implies that

$$H^0(Q(\wedge k E S^k H)) \cong S^k \otimes \wedge^k (\otimes^k \mathfrak{gl}(2n + 2, \mathbb{C})) / S^k \otimes \wedge^k (\otimes^{k-1} \mathfrak{gl}(2n + 2, \mathbb{C}) \otimes \operatorname{Id})$$

and

$$\dim_{\mathbb{C}} H^0(Q(\wedge k E S^k H)) = \sum_{i=0}^k (-1)^{k+i} 2n+i+1 C_{i} 2n+2 C_{i}.$$

The real structure $\tau$ is associated with the action of $j$ on $\mathbb{H}^{n+1}\{0\} \cong \mathbb{C}^{2n+2}\{0\}$. Since $\tilde{V}_k \cong S^k \otimes \wedge^k (\otimes^k \mathfrak{gl}(n+1, \mathbb{H}))$,

$$H^0(Q(\wedge^k E S^k H))^\tau \cong S^k \otimes \wedge^k (\otimes^k \mathfrak{gl}(n+1, \mathbb{H})) / S^k \otimes \wedge^k (\otimes^{k-1} \mathfrak{gl}(n+1, \mathbb{H}) \otimes \operatorname{Id})$$

by Theorem 6.14.

Especially, in the case $\mathbb{H}P^1 \cong S^4$,

$$H^0(Q(EH))^\tau \cong \mathfrak{gl}(2, \mathbb{H})/\operatorname{Id} \cong \mathfrak{sl}(2, \mathbb{H}),$$

$$H^0(Q(\wedge^2 E S^2 H))^\tau \cong S^2 \otimes \wedge^2 (\otimes^2 \mathfrak{gl}(2, \mathbb{H})) / S^2 \otimes \wedge^2 (\mathfrak{gl}(2, \mathbb{H}) \otimes \operatorname{Id})$$

and $\dim_{\mathbb{R}} H^0(Q(EH))^\tau = 15, \dim_{\mathbb{R}} H^0(Q(\wedge^2 E S^2 H))^\tau = 45$.

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