A sufficient and necessary condition for superdense coding of quantum states

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Abstract

Recently, Harrow et al. [Phys. Rev. Lett. 92, 187901 (2004)] gave a method for preparing an arbitrary quantum state with high success probability by physically transmitting some qubits, and by consuming a maximally entangled state, together with exhausting some shared random bits. In this paper, we discover that some states are impossible to be perfectly prepared by Alice and Bob initially sharing those entangled states that are superposed by the ground states, as the states to be prepared. In particular, we present a sufficient and necessary condition for the states being enabled to be exactly prepared with probability one, in terms of the initial entangled states (maybe nonmaximally) superposed by the ground states. In contrast, if the initially shared entanglement is maximal, then the probabilities for preparing these quantum states are smaller than one. Furthermore, the lower bound on the probability for preparing some states are derived.

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1. Introduction

Entanglement is one of the most intriguing phenomena in quantum information theory and plays a pivotal role in quantum information processing [1,2], including superdense coding [3], quantum teleportation [4], remote state preparation [5], quantum algorithms [6], and quantum cryptograph [7]. Superdense coding, originally introduced by Bennett and Wieser [3] is the surprising utilization of entanglement to enhance the capacity of a quantum communication channel. That is, by making use of shared entanglement, it is possible to communicate two classical bits by physically transmitting only one qubit [3]. In a more general fashion, if one shares $\log_2 d$ ebits of entanglement, then one can extract $2 \log_2 d$ classical bits of information by sending a $d$-level quantum system (a qudit). The relationship between quantum teleportation and superdense coding was investigated by Werner and the others [8].

To date, superdense coding has been generalized in different manners (for example, see [9] and the references therein). There are mainly two scenarios: one is concerning communication between multiparties [10] and the other is regarding nonmaximally entangled states initially shared by Alice and Bob [11-15]. Hausladen et al. [12] showed that if Alice and Bob share a nonmaximally entangled state then the capacity of dense coding scheme is not $2 \log_2 d$ but equal to $S(\rho_{AB}) + \log_2 d$ bits of information in the asymptotic limit, where $S(\rho_{AB})$ is the entropy of entanglement of the shared state, and satisfies $0 \leq S(\rho_{AB}) \leq \log_2 d$. Therefore, we cannot deterministically send $2 \log_2 d$ bits using nonmaximally entangled states. Indeed, Mozes et al. [14] have dealt with the relationship between the entanglement of a given nonmaximally entangled state and the maximum number of alphabets which can be perfectly transmitted in a deterministic fashion. However, if the scheme is allowed to work in a probabilistic manner, then it should be possible to send $2 \log_2 d$ bits of information by sharing a nonmaximally entangled state [15]. Furthermore, it was shown that, by initially sharing some W-states [16], superdense coding and teleportation can also be perfectly performed [17].

Recently, another scheme, called superdense coding of quantum states was proposed by Harrow, Hayden, and Leung [18]. (Furthermore, Abeyesinghe et al. [19] dealt with preparing entangled states with minimal cost of entanglement and quantum communication.) That is,
if the sender knows the identity of the state to be sent, then two qubits can be communicated with a certain probability by physically transmitting one qubit and consuming one bit of entanglement [18]. Superdense coding of quantum states is analogous to remote state preparation [5] but the classical communication is now replaced by quantum communication. To be precise, the purpose of superdense coding of quantum states is to prepare a quantum state in Bob’s system or “sharing” a state that is entangled between Alice and Bob’s systems, for which Alice and Bob initially share a maximally entangled state, and Alice first performs a physically operation on her party with a certain success probability and then sends it to Bob. Furthermore, Harrow et al. [18] presented a protocol succeeding with high probability for communicating a 2l-qubit quantum state but some shared random bits are necessarily consumed, besides transmitting \( l + o(l) \) qubits and consuming \( l \) ebits of entanglement.

However, if the shared randomness is not required, Hayden, Leung and Winter [20] proposed a different protocol of superdense coding of quantum states that can always successfully perform the physical process, but may not guarantee the result to be exact. Rather, the protocol may result in an approximate outcome with high fidelity.

A natural question is that if Alice and Bob initially share nonmaximally entangled states then how about the success probability for preparing a quantum state; or, to prepare a quantum state, could we fix on an appropriately partially entangled state firstly shared by Alice and Bob, leading to the optimal success probability? As we know, due to the Schmidt Decomposition Theorem [1], any bipartite quantum state \( |\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i\rangle_A |j\rangle_B \) (where \( \{ |i\rangle_A |j\rangle_B \}_{1 \leq i,j \leq d} \) is an orthonormal basis for \( \mathbb{C}^d \otimes \mathbb{C}^d \)) can be written in the form \( \sum_{j} r_{j} |e_{j}\rangle |f_{j}\rangle \) with \( r_{j} \geq 0 \), where \( \{ |e_{j}\rangle \} \) and \( \{ |f_{j}\rangle \} \) are two orthonormal bases of systems A and B, respectively, so, it is evident that, by sharing this state they can exactly prepare this state \( |\psi\rangle \) with probability one. As we know, the ground states are in general easier to be prepared. However, \( \{ |e_{j}\rangle \} \) and \( \{ |f_{j}\rangle \} \) may not equal to the ground states \( \{ |i\rangle_A \} \) and \( \{ |j\rangle_B \} \), respectively, so, we here ask that, if the initial entangled states (maybe nonmaximally) are superposed by the ground states (i.e., \( \{ |e_{j}\rangle \} = \{ |j\rangle_A \} \) and \( \{ |f_{j}\rangle \} = \{ |j\rangle_B \} \)), as the states to be prepared, then how about the superdense coding of quantum states? The main goal of this paper is to clarify this question in detail.
The remainder of the paper is structured as follows. In Section 2, we recall an exact probabilistic protocol of superdense coding of quantum states, in which Alice and Bob initially share a maximally entangled state, and Alice implements a transformation on her party with a certain success probability and then sends it to Bob. Section 3 is the main part and we discover that some states are impossible to be perfectly prepared by Alice and Bob initially sharing those entangled states that are superposed by the ground states, as the states to be prepared. In particular, we present a sufficient and necessary condition for the states being enabled to be exactly prepared with probability one. Furthermore, the lower bound on the probability for preparing some states are derived. Finally some remarks are made in Section 4.

2. A probabilistic protocol for preparing quantum states

In this section, we recall a probabilistic protocol for preparing quantum states which was dealt with by Harrow et al. [18].

Suppose we want to prepare a $d^2$-dimensional state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i_A|j_B\rangle$ in Bob’s system, by sending $\log_2 d$ qubits and consuming $\log_2 d$ ebits of shared entanglement, where $\{|i_A|j_B\rangle\}_{1 \leq i,j \leq d}$ is an orthonormal basis for $\mathbb{C}^d \otimes \mathbb{C}^d$. The procedure can be described as follows. Alice and Bob initially share $\log_2 d$ ebits, or equivalently the maximally entangled state

$$ |\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A |i\rangle_B. \quad (1) $$

Alice performs a physical operation $X$ on her party and then sends it to Bob, which may result in the state $|\psi\rangle$ to be prepared with a certain success probability. We can represent it by Equation (2):

$$ (X \otimes I)|\Phi_d\rangle = |\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i\rangle_A |j\rangle_B, \quad (2) $$

where $I$ denotes the identity operator.

Nevertheless, $X$ may not be unitary, so the above scheme for successfully preparing fixed $|\psi\rangle$ depends on the successful application of $X$. One method to carry out $X$ is by the
generalized measurement \( \rho \rightarrow \sum_k E_k \rho E_k^\dagger \) with Kraus operators \([21,22]\):

\[
E_0 = \frac{X}{\|X\|}, \quad E_1 = \sqrt{I - E_0^\dagger E_0},
\]

where the operator norm \( \|X\| \) of \( X \), is taken to be the square norm, i.e., the square root of the largest eigenvalue of \( X^\dagger X \). When the measurement outcome is 0, \( X \) is successfully performed, and the success probability \( P_s \) is then

\[
P_s = Tr E_0^\dagger E_0 \frac{I}{d} = \frac{Tr(X^\dagger X)}{d\|X^\dagger\|\|X\|}.
\]

From Equations (1, 2) we know that \( X|j\rangle_A = \sum_{d} x_{i,j} |i\rangle_A \) for \( j = 1, 2, \ldots, d \). Therefore, we have

\[
Tr(X^\dagger X) = \sum_{j=1}^{d} \langle j|X^\dagger X|j\rangle
\]

\[
= \sum_{j=1}^{d} \sum_{i_1,i_2=1}^{d} x_{i_1,j}^* x_{i_2,j} \langle i_1|i_2\rangle
\]

\[
= \sum_{j,i=1}^{d} |x_{i,j}|^2
\]

\[
= d
\]

where the last equality results from \( \langle \psi|\psi\rangle = 1; \{|j\rangle : j = 1, 2, \ldots, d\} \) is an orthonormal basis of system \( A \), as above.

Due to \( Tr(X^\dagger X) = d \), and \( \|X^\dagger\|\|X\| = \|X^\dagger X\| \), we further have

\[
P_s = \frac{1}{\|X^\dagger X\|}.
\]

Clearly, if \( |\psi\rangle \) is a maximally entangled state, i.e., \( |\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_A |i\rangle_B \), then \( P_s = 1 \); if \( |\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i\rangle_A |j\rangle_B \) with \( \sum_{i=1}^{d} x_{i,j_1}^* x_{i,j_2} = 0 \) for any \( j_1 \neq j_2 \), then \( X^\dagger X = \text{diag}(a_1, a_2, \ldots, a_d) \) where \( a_i = \sum_{j=1}^{d} |x_{i,j}|^2 \), and, consequently,

\[
P_s = \frac{1}{\text{max}(a_1, a_2, \ldots, a_d)}.
\]

From equation (6) it follows that when \( \text{max}(a_1, a_2, \ldots, a_d) > 1 \), \( P_s < 1 \). Therefore, we consider that it is possible to increase the probability \( P_s \) by changing the maximally entangled state \( |\Phi_d\rangle \) initially shared by Alice and Bob. Indeed, we will show that, in terms of the state
$|\psi\rangle$ to be prepared, the state $|\Phi_d\rangle$ shared by the two parties can be, instead, partially entangled to lead to the success probability one. This is in contrast to superdense coding, in which partially (nonmaximally) entangled state will decrease the bits of information communicated between Alice and Bob [12,14].

### 3. Superdense coding of quantum states with partially entangled states

Motivated by the issue addressed above, in this section we try to answer it by deriving appropriate partially entangled states, for preparing some fixed states with perfect success.

Let $\{|i\rangle_A|j\rangle_B\}_{1\leq i,j \leq d}$ be a given orthonormal basis for $\mathbb{C}^d \otimes \mathbb{C}^d$. Suppose state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i\rangle_A |j\rangle_B$ to be prepared, where

$$\sum_{i,j} |x_{i,j}|^2 = d. \quad (7)$$

By means of the Schmidt Decomposition Theorem [1], there are orthonormal bases $\{|e_i\rangle_A : i = 1, 2, \ldots, d\}$ and $\{|f_i\rangle_B : i = 1, 2, \ldots, d\}$ of systems $A$ and $B$, respectively, such that

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i\rangle_A |j\rangle_B = \sum_{i=1}^{d} \lambda_i |e_i\rangle_A |f_i\rangle_B \quad (8)$$

for some $\lambda_i \geq 0$, $i = 1, 2, \ldots, d$, with $\sum_{i=1}^{d} \lambda_i^2 = 1$. Therefore, if Alice and Bob initially share state $\sum_{i=1}^{d} \lambda_i |e_i\rangle_A |f_i\rangle_B$, then they can clearly prepare the desired state $|\psi\rangle$ succeeding with probability one.

In general, the ground states are easier to be prepared. Therefore, we naturally ask if the entangled states initially shared by Alice and Bob are superposed by the ground states, as the state $|\psi\rangle$ to be prepared, i.e., the initial entangled states have the form $\sum_{i=1}^{d} \mu_i |g_i\rangle_A |h_i\rangle_B$ with $\sum_{i=1}^{d} |\mu_i|^2 = 1$, where

$$\{|g_i\rangle_A : i = 1, 2, \ldots, d\} = \{|i\rangle_A : i = 1, 2, \ldots, d\}, \quad (9)$$

$$\{|h_i\rangle_B : i = 1, 2, \ldots, d\} = \{|i\rangle_B : i = 1, 2, \ldots, d\}, \quad (10)$$
then whether $|\psi\rangle$ can be exactly prepared with probability one by sharing some appropriate states $\sum_{i=1}^{d} \mu_i |g_i\rangle_A |h_i\rangle_B$ (equations (9,10) are required) between Alice and Bob? Now, we give a proposition to verify that this may not be true.

**Proposition 1.** If the state $|\psi\rangle = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle_A |j\rangle_B$ is to be prepared, then with any entangled state of the form $\sum_{i=1}^{d} \mu_i |g_i\rangle_A |h_i\rangle_B$ (equations (9,10) are required) initially shared by Alice and Bob, the protocol of superdense coding of quantum states described above can not perfectly prepare $|\psi\rangle$.

**Proof.** We present a proof by contradiction. If the protocol of superdense coding of quantum states could exactly prepare $|\psi\rangle$ with probability one, then there exists unitary transformation $U_A$ on system $A$ such that

$$(U_A \otimes I) \sum_{i=1}^{d} \mu_i |g_i\rangle_A |h_i\rangle_B = |\psi\rangle = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle_A |j\rangle_B$$

(11)

for some $\mu_i$ with $\sum_{i=1}^{d} |\mu_i|^2 = 1$ and equations (9,10) holding. Suppose that

$$U_A |g_i\rangle_A = \sum_{j=1}^{d} a_{j,i} |j\rangle_A, \quad i = 1, 2, \ldots, d.$$  

(12)

Then the unitarity of $U_A$ results in

$$\sum_{j=1}^{d} a_{j,i_1} a_{j,i_2}^* = \begin{cases} 1, & i_1 = i_2, \\ 0, & i_1 \neq i_2. \end{cases}$$

(13)

With equation (12) we have

$$(U_A \otimes I) \sum_{i=1}^{d} \mu_i |g_i\rangle_A |h_i\rangle_B = \sum_{i,j=1}^{d} \mu_i a_{j,i} |j\rangle_A |h_i\rangle_B,$$

(14)

which together with equation (11) results in

$$\mu_i a_{j,i} = \frac{1}{\sqrt{d}}, \quad i, j = 1, 2, \ldots, d.$$  

(15)

Thus,

$$a_{1,k} = a_{2,k} = \ldots = a_{d,k} = \frac{1}{\sqrt{d}} e^{i \theta_k}$$

(16)

for some real numbers $\theta_k$, $k = 1, 2, \ldots, d$. Therefore, for any $i_1 \neq i_2$,

$$\sum_{j=1}^{d} a_{j,i_1} a_{j,i_2}^* = e^{i(\theta_{i_1} - \theta_{i_2})} \neq 0,$$

(17)
a contradiction to equation (13). The proposition has been verified.

As well, Proposition 1 clearly implies that the state $|\psi\rangle = \frac{1}{d} \sum_{i,j=1}^{d} |i\rangle_{A} |j\rangle_{B}$ can not be prepared with probability one by initially sharing the maximally entangled state $\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_{A} |i\rangle_{B}$ between Alice and Bob.

However, if the state $|\psi\rangle$ to be prepared satisfies a certain condition, we can still choose an appropriate initial state having the same orthonormal vectors as those in $|\psi\rangle$, and by sharing this state exactly prepare $|\psi\rangle$. This is further described by the following theorem.

**Theorem 1.** Let $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i\rangle_{A} |j\rangle_{B}$ be the state to be prepared. Then $|\psi\rangle$ can be exactly prepared with probability one by initially sharing the entangled state $|\Phi_{d}\rangle = \sum_{i=1}^{d} c_{i} |i\rangle_{A} |i\rangle_{B}$ between Alice and Bob for some $c_{i}$ with

$$\sum_{i=1}^{d} |c_{i}|^{2} = 1,$$

if and only if

$$\sum_{i=1}^{d} x_{i,j_{1}}^{*} x_{i,j_{2}} = 0,$$

for any $j_{1} \neq j_{2}$.

**Proof.** (If): To prepare $|\psi\rangle$, Alice performs a transformation $Y = \sum_{i,j} y_{i,j} |i\rangle_{A} \langle j|$ on her half in $|\Phi_{d}\rangle$ and then sends it to Bob’s system. Thus

$$(Y \otimes I)|\Phi_{d}\rangle = |\psi\rangle,$$

that is,

$$\sum_{i,j} c_{j} y_{i,j} |i\rangle_{A} \langle j|_{B} = \frac{1}{\sqrt{d}} \sum_{i,j} x_{i,j} |i\rangle_{A} \langle j|_{B}.$$

Therefore,

$$c_{j} y_{i,j} = \frac{x_{i,j}}{\sqrt{d}}$$

for any $i, j$. We know that the transformation $Y$ can be successfully implemented with certain probability $P_{s}$ in terms of Kraus operators $E_{0} = \frac{Y}{\|Y\|}$, $E_{1} = \sqrt{I - E_{0}^{\dagger} E_{0}}$. Therefore,

$$P_{s} = Tr E_{0}^{\dagger} E_{0} \frac{I}{d} = \frac{Tr(Y^{\dagger} Y)}{d \|Y^{\dagger}\| \|Y\|}.$$
Next, in order to show that $P_s$ can arrive at one, it suffices to construct appropriate $y_{i,j}$ and $c_j$ such that $Y$ is unitary and equations (18,22) hold. First we know that $Y$ is unitary if and only if

$$
\sum_{i=1}^{d} y_{i,j_1}^* y_{i,j_2} = \begin{cases} 1, & j_1 = j_2, \\
0, & j_1 \neq j_2. 
\end{cases} \quad (24)
$$

We take $c_j$ satisfying:

$$|c_j|^2 = \frac{1}{d} \sum_{i=1}^{d} |x_{i,j}|^2, \quad j = 1, 2, \ldots, d. \quad (25)$$

Clearly, if $|c_j|^2 = 0$, then $x_{i,j} = 0$ for $i = 1, 2, \ldots, d$. Furthermore we take $y_{i,j}$ in terms of the following:

$$y_{i,j} = \begin{cases} \frac{x_{i,j}}{\sqrt{|c_j|}}, & \text{if } c_j \neq 0, \\
\frac{1}{\sqrt{d}}, & \text{otherwise.} \end{cases} \quad (26)$$

Now, in terms of equations (19,25,26) and $\langle \psi | \psi \rangle = 1$, it is straightforward to check that these $c_j$ and $y_{i,j}$ determined satisfy equations (18,22). The unitarity of $Y$ results in $\text{Tr}(Y^\dagger Y) = d$ and $\|Y^\dagger\| = \|Y\| = 1$. Thus, by equation (23) $P_s = 1$ for the constructed transformation $Y$.

(Only if): The known conditions say that there exists transformation $Y = \sum_{i,j} y_{i,j}|i\rangle\langle j|$ such that equations (18,22) hold and $P_s = 1$. From $P_s = 1$ it follows that

$$\text{Tr}(Y^\dagger Y) = d\|Y^\dagger\|\|Y\| = d\|Y\|^2. \quad (27)$$

Suppose that $\lambda_i \geq 0$ ($i = 1, 2, \ldots, d$) are the eigenvalues of $Y^\dagger Y$. Then $\text{Tr}(Y^\dagger Y) = \sum_{i=1}^{d} \lambda_i$, and $\|Y\|^2 = \max(\lambda_1, \lambda_2, \ldots, \lambda_d)$. If there exist two different eigenvalues of $Y^\dagger Y$, then

$$\text{Tr}(Y^\dagger Y) = \sum_{i=1}^{d} \lambda_i < d \max(\lambda_1, \lambda_2, \ldots, \lambda_d) = d\|Y^\dagger\|\|Y\|. \quad (28)$$

Consequently, $P_s < 1$, a contradiction to $P_s = 1$. Therefore, we have $\lambda_1 = \lambda_2 = \ldots = \lambda_d = \lambda > 0$ for some $\lambda > 0$. Thus, $Y^\dagger Y = \sum_{i=1}^{d} \lambda |e_i\rangle\langle e_i|$ for some orthonormal base $\{|e_i\rangle\}$, which implies that $\frac{Y^\dagger Y}{\lambda} = I$ ($I$ denotes the identity operator). This also shows that $\frac{Y}{\sqrt{\lambda}}$ is a unitary operator. Therefore, for any $j_1 \neq j_2$,

$$\sum_{i=1}^{d} y_{i,j_1}^* y_{i,j_2} = \langle j_1 | Y^\dagger Y | j_2 \rangle = 0. \quad (29)$$

From equations (22,29) it follows directly that equation (19) holds. Therefore, we have completed the proof.
Remark. In the above Theorem 1, state $|\Phi_d⟩ = \sum_{i=1}^{d} c_i |i⟩_A |i⟩_B$ can be generalized to the more generic form

$$|\Phi_d⟩ = \sum_{i=1}^{d} c_i |g_i⟩_A |h_i⟩_B$$

(30)

where $\{|g_i⟩_A : i = 1, 2, \ldots, d\} = \{|i⟩_A : i = 1, 2, \ldots, d\}$, and $\{|h_i⟩_B : i = 1, 2, \ldots, d\} = \{|i⟩_B : i = 1, 2, \ldots, d\}$. Therefore, there exist permutations $\Pi_A(i) = g_i$ and $\Pi_B(i) = h_i$ for $i = 1, 2, \ldots, d$. The proof of the theorem with this change is analogous, only by changing $Y$ to $\sum_{i,j} y_{i,j} |g_i⟩⟨g_j|$, by changing the left side of equation (21) to $\sum_{i,j} c_j y_{i,j} |g_i⟩_A |h_j⟩_B$, and in places, by changing $x_{i,j}$ to $x_{\Pi_A(i), \Pi_B(j)}$.

A straightforward corollary from Theorem 1 is as follows.

**Corollary 1.** Let $|ψ⟩ = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i⟩|j⟩$ be the state to be prepared. If equation (19) holds, i.e., there exist $j_1 \neq j_2$ such that $\sum_{i=1}^{d} x_{i,j_1}^* x_{i,j_2} \neq 0$, then for any state $|Φ⟩ = \sum_{i=1}^{d} c_i |i⟩_A |i⟩_B$ with $\sum_{i=1}^{d} |c_i|^2 = 1$, initially shared by Alice and Bob, the success probability for preparing $|ψ⟩$ is strictly smaller than one.

Naturally, we may ask how about the lower bound on the success probability for superdense coding if the condition described by equation (19) does not hold. Next we consider the case of which equation (19) does not hold only for arbitrarily given two $j_1$ and $j_2$, and for the others, equation (19) is still preserved.

**Proposition 2.** Let $|ψ⟩ = \frac{1}{\sqrt{d}} \sum_{i,j=1}^{d} x_{i,j} |i⟩|j⟩$ be the state to be prepared. If for $\{j_1, j_2\} = \{k_1, k_2\}$, equation (19) does not hold, but for the other cases, equation (19) is preserved, then by initially sharing $|Φ⟩ = \sum_{i=1}^{d} c_i |i⟩_A |i⟩_B$ between Alice and Bob, where $\sum_{i=1}^{d} |c_i|^2 = 1$, the maximum success probability $P_s^{(m)}$ for preparing $|ψ⟩$ satisfies

$$1 > P_s^{(m)} \geq \left(1 + \frac{1}{d |c_{k_1} c_{k_2}|} \left|\sum_{i=1}^{d} x_{i,k_1}^* x_{i,k_2}\right|^{-1}\right)^{-1}. \quad (31)$$

**Proof.** First, $P_s^{(m)} < 1$ follows directly from Theorem 1. Next we prove the other inequality. We take $c_j$ as equation (25), i.e., $|c_j|^2 = \frac{1}{d} \sum_{i=1}^{d} |x_{i,j}|^2$. As above, let $Y = \sum_{i,j=1}^{d} y_{i,j} |i⟩⟨i|$ be the transformation on system $A$ performed by Alice. Then equation (22)
holds, i.e., for any \( i, j \), \( c_j y_{i,j} = \frac{x_{i,j}}{\sqrt{d}} \). By taking

\[
y_{i,j} = \begin{cases} \frac{x_{i,j}}{\sqrt{d}c_j}, & \text{if } c_j \neq 0, \\ \frac{1}{\sqrt{d}}, & \text{otherwise}, \end{cases}
\]

we then have \( Tr(Y^\dagger Y) = d \), and

\[
Y^\dagger Y = \sum_{i,j_1,j_2=1}^{d} y_{i,j_1,j_2}^* y_{i,j_1} |j_1\rangle \langle j_2|
\]

\[
= \sum_{j=1}^{d} |j\rangle \langle j| + \sum_{i=1}^{d} \frac{x_{i,k_1}^* x_{i,k_2}}{dc_{k_1}^* c_{k_2}} |k_1\rangle \langle k_2| + \sum_{i=1}^{d} \frac{x_{i,k_2}^* x_{i,k_1}}{dc_{k_2}^* c_{k_1}} |k_2\rangle \langle k_1|.
\]

Then we can determine that the eigenvalues of \( Y^\dagger Y \) are 1, and \( 1 \pm \frac{1}{d|c_{k_1} c_{k_2}|} \left| \sum_{i=1}^{d} x_{i,k_1}^* x_{i,k_2} \right| \). Therefore, by virtue of equation (23) we obtain

\[
P_s^{(m)} \geq \frac{Tr(Y^\dagger Y)}{d\|Y\|\|Y^\dagger\|} = \left( 1 + \frac{1}{d|c_{k_1} c_{k_2}|} \left| \sum_{i=1}^{d} x_{i,k_1}^* x_{i,k_2} \right| \right)^{-1},
\]

the lower bound as desired.

Especially, if \( \sum_{i=1}^{d} x_{i,k_1}^* x_{i,k_2} = 0 \), then the above bound described by inequality (31) reduces to 1, complying with Theorem 1.

### 4. Concluding Remarks

Superdense coding of quantum states, first proposed by Harrow, Hayden, and Leung [16], describes that if the sender knows the identity of the state to be sent, then two qubits can be communicated with a certain probability by physically transmitting one qubit and consuming one bit of entanglement. The objective of this protocol is to prepare a quantum state in Bob’s system or “sharing” a state that is entangled between Alice and Bob’s systems, for which Alice and Bob initially share a (maximally) entangled state, and Alice first performs a physically operation on her party with a certain success probability and then sends it to Bob. Furthermore, Harrow et al. [18] presented a protocol succeeding with high probability for communicating a 2\( l \)-qubit quantum state but some shared random bits are necessarily consumed, besides transmitting \( l + o(l) \) qubits and consuming \( l \) ebits of entanglement. Notably, if the shared randomness is not required, Hayden, Leung and Winter [20] proposed a different protocol of superdense coding of quantum states that can always successfully...
perform the physical process, but may not guarantee the result to be exact. Rather, the protocol may result in an approximate outcome with high fidelity.

In this paper, we discovered that some states are impossible to be perfectly prepared if Alice and Bob initially share the entangled states that are superposed by the ground states, as the states to be prepared. Particularly, we gave a sufficient and necessary condition for the states being enabled to be exactly prepared with probability one, by initially sharing these entangled states (maybe not maximally) between Alice and Bob. Furthermore, the lower bound on the probability for preparing some states was derived. Thus, this is another profile regarding superdense coding of quantum states. Also, in a way, this partially makes up the existing outcomes [18,20].

As well, for exactly preparing some quantum states, we determined some partially entangled states initially shared by Alice and Bob that result in the optimal success probability one. However, if, instead, the initial entanglement shared by the two parties is maximal, then the success probabilities for preparing these states may be smaller than one, a different phenomenon from superdense coding [3,12,14].

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