**Abstract**

We present a lattice investigation of the heavy-light meson decay constants using Wilson light quarks and NRQCD heavy quarks, partially including the effects of dynamical sea quarks. We calculate the pseudoscalar and vector decay constants over a wide range in heavy quark mass and are able to perform a detailed analysis of heavy quark symmetry. We find consistency between the extrapolation of the NRQCD results and the static case, as expected. We find the slope of the decay constants with $1/M$ is significantly larger than naive expectations and the results of previous lattice calculations. For the first time we extract the non-perturbative coefficients of the slope arising from the $O(1/M)$ heavy quark interactions separately and show the kinetic energy of the heavy quark is dominant and responsible for the large slope. In addition, we find that significant systematic errors remain in the decay constant extracted around the $B$ meson mass due to truncating the NRQCD series at $O(1/M)$. We estimate the higher order contributions to $f_B$ are approximately 20%; roughly the same size as the systematic errors introduced by using the Wilson action for light quarks.

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1 Introduction

The decay constant of the $B$ meson, $f_B$, is of fundamental importance to tests of the Standard Model because it is needed in the determination of the CKM matrix. While one of the simplest weak matrix elements to study, it is not very well known and dominates the uncertainty in extracting $V_{td}$, one of the smallest CKM elements, from $B - \bar{B}$ mixing. The best chance of an experimental measurement of $f_B$ lies in the decay $B^+ \rightarrow \tau^+ + \nu$. However, this decay has yet to be observed, and $f_B$ appears in combination with $V_{ub}$, which is only known to within a factor of 2.

Theoretical calculations using for example QCD sum rules within the framework of Heavy Quark Effective Theory (HQET) [1,2] and lattice field theory have been actively pursued over the last few years with the current world average standing at $f_B = 200 \pm 20\%$ MeV [3,4]. As a first principles approach, lattice simulations offer ultimately the most reliable calculation of $f_B$. The initial problem of simulating a $b$ quark with Compton wavelength of the order of the typical lattice spacing presently achievable was first solved by using the static approximation [5,17], where the heavy quark is treated as a static colour source within the heavy-light meson. Simulations using conventional relativistic methods around the $D$ meson showed the corrections to the static approximation in this region, and hence also at the $B$ meson, are much larger than naively expected. This has contributed to the sizable uncertainty remaining in $f_B$, shown above. The determination of $f_B$ directly at $M_B$ on the lattice is required, and this is now possible with the development of Non Relativistic QCD (NRQCD) [7,8]. Using this approach the error in the decay constant can be significantly reduced with the computational power currently available. Initial calculations have already been performed by Hashimoto [9] and Davies [10].

This analysis forms the second stage of a project applying NRQCD to heavy-light systems and studying heavy quark symmetry at the $B$ meson. Our previous papers have dealt with the $B$ meson spectrum in both the quenched approximation [11] and partially including the effects of sea quarks [12]. We now present our results for the decay constants of the heavy-light mesons from dynamical configurations. Note that the combination $f_B^2 B_B$, where $B_B$ is the mixing parameter for the $B$ system, is the experimentally relevant quantity, and thus a calculation of $B_B$ is also needed.

The paper is organised as follows. In section 2 we describe the simulation
details including the systematic errors associated with the calculation. The meson operators required to compute the decay constant to $O(1/M)$ are set out in section 3 and we discuss computing the decay constant consistently to this order in terms of the evolution equation used. Next, we examine the predictions of heavy quark symmetry for the behaviour of the decay constant in the heavy quark mass limit. The contributions from the interactions of the heavy quark at $O(1/M)$ are analysed and combinations of decay constants which isolate these individual contributions are discussed. Section 4 illustrates our results and fitting analysis with a subset of the data. The heavy quark mass dependence of the decay constants are investigated in section 5.2; the individual non-perturbative $O(1/M)$ coefficients of the slope of the decay constant are extracted. Finally we present our conclusions in section 6.

2 Simulation Details

The simulations were performed using 100 $16^3 \times 32$ gauge configurations at $\beta = 5.6$ with two flavours of staggered dynamical sea quarks with a bare quark mass of $am_{\text{sea}} = 0.01$. These configurations were generously made available by the HEMCGC collaboration; more details can be found in [13]. We fixed the configurations to the Coulomb gauge. The light quark propagators were generated using the Wilson fermion action, without an $O(a)$ improvement term, at two values of the hopping parameter, $\kappa = 0.1585$ and 0.1600. The former corresponds to a quark mass close to strange, where $\kappa_s = 0.1577$ from $M_\phi$, while 0.1600 is somewhat lighter, with $\kappa_c = 0.1610$.

With only two values of light quark mass it is not possible to perform a trustworthy chiral extrapolation. In addition, the systematic errors associated with the light quarks, discussed below, affect the determination of the bare strange quark mass; $\kappa_s$ extracted from pseudoscalar mesons, using the lowest order chiral mass dependence to find the ‘experimental mass’ of the pure $s\bar{s}$ pseudoscalar, differs from $\kappa_s$ obtained from the vector meson $\phi$. Thus, most of the results will be presented for $\kappa_l = 0.1585$.

In this simulation we truncate the NRQCD series at $O(1/M_0)$, where $M_0$ is the bare heavy quark mass, and the action takes the form:

$$S = Q^\dagger (D_t + H_0 + \delta H) Q$$  \hspace{1cm} (1)
where
\[ H_0 = -\frac{\Delta^{(2)}}{2M_0} \quad \text{and} \quad \delta H = -c_B \frac{\sigma \cdot B}{2M_0}. \]

Tadpole improvement of the gauge links is used throughout i.e. they are divided by \( u_0 \), where we use \( u_0 = 0.867 \) measured from the plaquette, and the hyperfine coefficient is given the tree-level value \( c_B = 1 \). We use the standard Clover-leaf operator for the B field. The heavy quark propagators were computed using the evolution equation \[ 14 \]:

\[
G_1 = \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n \delta_{x,0}
\]

\[
G_{t+1} = \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n (1 - a\delta H) G_t \quad (t > 1)
\]

where \( n \) is the stabilising parameter. In the static limit this reduces to

\[
G_{t+1} - U_4^\dagger G_t = \delta_{x,0}.
\]

We generated heavy quark propagators at 11 values of \((aM_0,n)\) corresponding to \((0.8,5), (1.0,4), (1.2,3), (1.7,2), (2.0,2), (2.5,2), (3.0,2), (3.5,1), (4.0,1), (7.0,1)\) and \((10.0,1)\), and the static limit. This roughly corresponds to a range of meson masses from \( M_B/3 \) to \( 4M_B \) and is sufficient for a reasonable investigation of heavy quark symmetry. However, it is not possible to simulate the \( D \) meson on this lattice using NRQCD; a larger lattice spacing is required, \( \beta_{n_f=2} \lesssim 5.5 \) or \( \beta_{n_f=0} \lesssim 5.85 \).

Details of the construction of meson operators, the smearing functions used and our fitting analysis can be found in \[ 12 \]. As discussed in this reference there are large systematic errors associated with this simulation arising from the use of Wilson light fermions. The emphasis of our analysis will be on determining the heavy quark mass dependence of the decay constants and extracting the nonperturbative coefficients of the heavy quark expansion to \( O(1/M) \). The systematic error arising from the light quark in these \( O(\Lambda_{QCD}^2) \) coefficients is naively estimated to be a relative error of \( O(\Lambda_{QCD}a) \sim 20\% \) for this ensemble (where we take \( a\Lambda_{QCD} = a\Lambda_V = 0.185 \) for these configurations). The truncation of the NRQCD series and the use of the tree-level value for the hyperfine coefficient introduce absolute errors of \( O(\Lambda_{QCD}(\Lambda_{QCD}/M)^2) \) and \( O(\alpha_S\Lambda_{QCD}(\Lambda_{QCD}/M)) \) respectively. Both correspond to approximately 1% errors in the value and 10% errors in the slope of the decay constant at

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The coefficients of the quadratic term in the heavy quark expansion cannot be correctly determined in this analysis. The size of the systematic errors induced by the light quark action are indicated by the large uncertainty in the lattice spacing; we use $a^{-1} = 1.8 - 2.4$ GeV. This gives $aM_0 = 2.4 - 1.7$ as corresponding to the bare $b$ quark mass. We are currently repeating our analysis using tadpole-improved Clover light fermions \[15\].

### 3 Meson Operators

The pseudoscalar decay constant is defined by

$$\langle 0 | A_0 | PS \rangle = f_{PS} M_{PS},$$

(5)

where $A_0 = \bar{q} \gamma_5 \gamma_0 h$ at tree level and $q$ and $h$ represent 4-component light and heavy quark fields respectively. In this convention the experimental value of $f_\pi$ is 131 MeV. Similarly, the vector decay constant is given by

$$\langle 0 | V_i | V_i \rangle = \epsilon_i f_V M_V,$$

(6)

where $V_i = \bar{q} \gamma_i h$ at tree level. The calculation of these matrix elements, and thus decay constants, to $O(1/M)$ in NRQCD requires several operators. At tree level these can be obtained by relating $h$ of full QCD to the 2-component NRQCD fields, $Q$ and $\tilde{Q}$ for heavy quarks and heavy antiquarks respectively, using the Foldy-Wouthuysen transformation. To $O(1/M)$,

$$h = e^{-iS(0)} \left( \frac{Q}{\tilde{Q}^i} \right) \simeq (1 - iS(0)) \left( \frac{Q}{\tilde{Q}^i} \right), \quad S(0) = -\frac{i\gamma \cdot \vec{D}}{2M_0}$$

(7)

With the convention that our meson states be built out of a heavy quark and a light antiquark, the $\tilde{Q}$ field will not contribute to the above matrix elements through $O(1/M)$, and we obtain

$$\langle 0 | q^{(4)} h | P \rangle_{QCD} = -\langle 0 | \gamma_{34} \Gamma^{(2)} | P \rangle_{NRQCD} + \langle 0 | \gamma_{12} \Gamma^{(2)} \left( \frac{i\sigma \cdot \vec{D}}{2M_0} \right) Q \langle P \rangle_{NRQCD}$$

(8)

\[3\]This is a more convenient definition than the more commonly used $\langle 0 | \bar{q} \gamma_i h | V_i \rangle = \epsilon_i M_V^2 / f_V$. 

4
where $\Gamma^{(4)} = \gamma_5 \gamma_0$ and $\Gamma^{(2)} = I$ for the axial current, and $\Gamma^{(4)} = \gamma_i$ and $\Gamma^{(2)} = i\sigma_i$ for the vector current. $q_{12}$ and $q_{34}$ denote the upper and lower two components of $q$ respectively. Our Euclidean space $\gamma$ matrices are chosen as,

$$
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & i\tilde{\sigma} \\ -i\tilde{\sigma} & 0 \end{pmatrix}.
$$

(9)

Beyond tree level all operators with the same quantum numbers can mix under renormalisation. Thus at $O(\alpha/M)$ the basis of operators for the axial current, is given by

$$
\mathcal{O}_1 = -q_{34}^\dagger Q \quad \mathcal{O}_2 = q_{12}^\dagger i\sigma \cdot \vec{D} \frac{1}{2M_0} Q \quad \mathcal{O}_3 = q_{12}^\dagger -i\tilde{D} \cdot \sigma \frac{1}{2M_0} Q
$$

(10)

Using translation invariance, the zero momentum currents involving $\mathcal{O}_2$ and $\mathcal{O}_3$ are identical on the lattice and comparing with equation 8 it is sufficient to calculate matrix elements of the tree level operators $\mathcal{O}_1$ and $\mathcal{O}_2$. Similarly, the basis of operators for the vector current,

$$
\mathcal{O}_1 = -i\sigma_i q_{34}^\dagger Q \quad \mathcal{O}_2 = q_{12}^\dagger \sigma_i \frac{-\sigma \cdot \vec{D}}{2M_0} Q \quad \mathcal{O}_3 = q_{12}^\dagger -\frac{\tilde{D} \cdot \sigma_i}{2M_0} Q
$$

(11)

$$
\mathcal{O}_4 = q_{12}^\dagger \frac{-\tilde{D}_i}{2M_0} Q \quad \mathcal{O}_5 = q_{12}^\dagger \frac{\tilde{D}_i}{2M_0} Q
$$

(12)

is reduced, for the numerical computation, to $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_2 - \mathcal{O}_4$, where $\sigma$ matrix properties have also been used. We computed the pseudoscalar and vector matrix elements corresponding to this minimal set of operators by inserting a smeared interpolating field with the same quantum numbers as $\mathcal{O}_1$ at the source and all combinations of local currents at the sink for each meson.

The perturbative calculation of the renormalisation factors needed to relate the matrix elements in NRQCD to the currents in full QCD has not yet been completed [16]. Thus, we use the static renormalisation factors [17] and are restricted to the tree level $O(1/M)$ corrections; we omit $\mathcal{O}_4$ for the vector meson. Since this correction is of $O(\alpha_s\Lambda_{QCD}(\Lambda_{QCD}/M))$ it is roughly the same size as that induced by unknown perturbative matching corrections in

\footnote{In the limit of zero light quark mass.}
the coefficient $c_B$, which are not included in the analysis. The renormalisation factors in NRQCD depend on the heavy quark mass and hence using the static value introduces an $O(\alpha_S) \sim 20\%$ error into the $O(1/M)$ coefficients extracted from our results.

It is possible that the power divergences which appear in NRQCD may lead to significant non-perturbative contributions to the renormalisation factors [18]. No clear indication of ‘renormalon’ effects has been found in other quantities calculated perturbatively in NRQCD [14], however, we intend to compute the renormalisation factors non-perturbatively to check this.

Strictly, our analysis is not fully consistent to $O(1/M)$. Note that the evolution equation given in equation (3) induces an $O(\Lambda_{QCD}/M)$ systematic error in the currents. This is because it does not apply the relativistic correction operator $(1 - \delta H)$ on the initial timeslice. Since all subsequent timeslices include this operator the transfer matrix is consistent to $O(1/M)$. Thus, it is only in the amplitude of the meson correlator, i.e. the decay constant, that is missing this $O(a\Lambda_{QCD}/M)$ contribution. A better but computationally more costly evolution equation is given by

$$G_{t+1} = (1 - \frac{a\delta H}{2}) \left(1 - \frac{aH_0}{2n}\right)^n U_4^\dagger \left(1 - \frac{aH_0}{2n}\right)^n (1 - \frac{a\delta H}{2}) G_t$$

for all timeslices. Since our use of the Wilson action for light quarks also induces errors of this order of magnitude, we used equation (3) in this project. In a comparison of these evolution equations using Clover fermions for the light quarks [15] we found the insertion of $(1 - \delta H)$ on the source timeslice introduces $\sim 4\%$ correction to $\langle 0 | O_1 | PS \rangle_{NRQCD}$ and $\sim 3\%$ correction to $\langle 0 | O_2 | PS \rangle_{NRQCD}$ around $M_b$. Considering the systematic errors inherent in this calculation and our statistical accuracy of $\sim 2\%$, the omission of this correction will not significantly affect our findings.

4 Heavy Quark Symmetry

In the heavy quark limit the decay constant can be parameterised in terms of an expansion in $1/M$:

$$f\sqrt{\bar{M}} = (f\sqrt{\bar{M}})^\infty \left(1 + \frac{c_p}{\bar{M}} + O\left(\frac{1}{\bar{M}^2}\right)\right).$$

(14)
The coefficient $c_P$ is determined by nonperturbative contributions arising from the hyperfine interaction ($G_{\text{hyp}}$) and the kinetic energy of the heavy quark ($G_{\text{kin}}$), which appear in the NRQCD action, and the corrections to the current arising from the Foldy-Wouthuysen transformation. From first order perturbation theory in $1/M$ about the static limit:

$$c_P = G_{\text{kin}} + 2d_M G_{\text{hyp}} + d_M G_{\text{corr}}/6 \quad (15)$$

where $d_M = 3$ and $-1$ for pseudoscalar and vector mesons respectively. The integer factors in this equation are introduced for convenience when comparing with HQET.

$$\langle 0 | q^\dagger \Gamma^{(2)} Q | P \rangle_\infty G_{\text{kin}} = \langle 0 | \int dy T \{ q^\dagger \Gamma^{(2)} Q(0), Q^\dagger (-\vec{D}^2/2) Q(y) \} | P \rangle_\infty , \quad (16)$$

$$\langle 0 | q^\dagger \Gamma^{(2)} Q | P \rangle_\infty 2d_M G_{\text{hyp}} = \langle 0 | \int dy T \{ q^\dagger \Gamma^{(2)} Q(0), Q^\dagger (-c_B \vec{\sigma} \cdot \vec{B}/2) Q(y) \} | P \rangle_\infty , \quad (17)$$

$$\langle 0 | q^\dagger \Gamma^{(2)} Q | P \rangle_\infty d_M G_{\text{corr}}/6 = \langle 0 | q^\dagger \Gamma^{(2)} \frac{\vec{\sigma} \cdot \vec{D}}{2} Q | P \rangle_\infty , \quad (18)$$

where $|P\rangle_\infty$ represents the meson in the limit of infinite heavy quark mass, and equations (16)-(18) are tree level expressions. Note that the kinetic and hyperfine terms contribute to the decay constant through the correction to the meson wavefunction.

Equation (16) suggests that the individual contribution from each $O(1/M)$ term can be obtained separately by taking appropriate combinations of the pseudoscalar and vector decay constants, $(f \sqrt{M})_{\text{PS}}$ and $(f \sqrt{M})_{\text{V}}$ respectively. For clarity below, we denote

$$(f \sqrt{M})^{\text{uncorr}}_{\text{PS}} = \frac{1}{\sqrt{M_{\text{PS}}}} \langle 0 | \mathcal{O}^{\text{PS}}_1 | PS \rangle_{\text{NRQCD}} \quad (19)$$

$$(f \sqrt{M})_{\text{PS}} = \frac{1}{\sqrt{M_{\text{PS}}}} \langle 0 | \mathcal{O}^{\text{PS}}_2 | PS \rangle_{\text{NRQCD}} \quad (20)$$

$$(f \sqrt{M})^{\text{tot}}_{\text{PS}} = (f \sqrt{M})^{\text{uncorr}}_{\text{PS}} + \delta(f \sqrt{M})_{\text{PS}} \quad (21)$$
\[
(f \sqrt{M})_{PS}^{uncorr} / (f \sqrt{M})_{V}^{uncorr} = 1 + c''_P/M, \quad c''_P = 8G_{hyp} = 8G_{hyp} + 2G_{corr}/3.
\]

Table 1: Expansions about the static limit to \(O(1/M)\) of combinations of the pseudoscalar and vector decay constant, with and without the current corrections.

with similar quantities defined for the vector particle by replacing PS with V. We also denote the spin-average of the decay constants,

\[
(f \sqrt{M}) = ((f \sqrt{M})_{PS}^{uncorr} + 3(f \sqrt{M})_{V}^{uncorr})/4.
\]  

(22)

Table 1 displays our notations for the expansions in \(1/M\) and the corresponding coefficients of \((f \sqrt{M})\) and the ratio of the pseudoscalar to vector decay constants, with and without the current correction. It is easy to see from this that both \(G_{kin}\) and \(G_{hyp}\) can be extracted in a simple way from \(c'_P\) and \(c''_P\) respectively; \(G_{corr}\) must be obtained from the combination \(c''_P - c'_P\).

Since we compute \(\langle 0 | O_2 | P \rangle\) separately, however, \(G_{corr}\) can be more accurately found by examining the current correction in the infinite heavy quark mass limit:

\[
G_{corr} = \frac{[2M_0 \langle 0 | O_2 | P \rangle]_{\infty}}{\langle 0 | O_1 | P \rangle_{\infty}} = \lim_{M_0 \to \infty} \frac{2M_0 \delta(f \sqrt{M})_{PS}^{uncorr}}{(f \sqrt{M})_{PS}^{uncorr}}.
\]  

(23)

HQET provides an analogous decomposition of these coefficients [19]. However, since HQET constructs an effective theory in terms of the heavy quark pole mass, as opposed to the bare quark mass in NRQCD, only physical combinations of \(G\)'s, i.e. \(c_P, c''_P\) and \(c'_P = G_{kin}\), can be compared. Note that while these coefficients are expected to be of \(O(\Lambda_{QCD})\) \(\sim 200\)–\(500\) MeV (and thus the corrections to the static limit at \(M_B O(\Lambda_{QCD}/M_B) \sim 10\%\)); the unphysical quantities \(G_{hyp}\) and \(G_{corr}\) need not be of this order.

In HQET \(G_{corr}\) is related to the meson binding energy, \(G_{corr} = -\bar{\Lambda} [19].\) This follows from the observation

\[
\langle 0 | i\partial_{\alpha}(\bar{q} \Gamma(4) h) | P \rangle_{\infty} = (M_M - M_Q)\langle 0 | \bar{q} \Gamma(4) h | P \rangle_{\infty},
\]  

(24)
= \bar{\Lambda} v_\alpha \langle 0 | \bar{q} \Gamma^{(4)} h | P \rangle_\infty, \quad (25)

where \( v_\alpha \) is the velocity of the heavy quark, and using the equations of motion of the heavy and light quark. Note that in NRQCD the R.H.S. of equation (25) becomes \( E_{\text{sim}}^\infty v_\alpha \langle 0 | \bar{q} \Gamma^{(4)} h | P \rangle_\infty \) and following the arguments of [19] the analogous expression in NRQCD can be obtained:

\[ G_{\text{corr}} = -E_{\text{sim}}^\infty. \quad (26) \]

At finite heavy quark mass this relation is broken by terms of \( O(1/M) \). Similarly, on the lattice, discretisation modifies (26) and the operators in equation (23) must be renormalised.

5 Results

5.1 Fitting results for the decay constants

To begin with we extract the decay constant without the correction to the current, i.e. from \( \langle 0 | O_1 | P \rangle \). We performed a single exponential vector fit to \( C(1, l_{O_1}) \) and \( C(1, 1) \) for the pseudoscalar and vector mesons. \( C(1, l_{O_1}) \) denotes a correlation function constructed at the source from an interpolating field with appropriate quantum numbers for the vector or pseudoscalar state and the heavy quark smeared with a ground S-state smearing function (described in [12]); at the sink the local current \( O_i \) is inserted. In the limit of large values of \( t \)

\[
C(1, l_{O_1}) = Z_{l_{O_1}} Z_1 e^{-E_{\text{sim}}^t},
\]

\[
C(1, 1) = Z_1^2 e^{-E_{\text{sim}}t}. \quad (28)
\]

By taking a bootstrap ratio of the amplitudes of these correlators the decay constant can be obtained:

\[
\sqrt{2} Z_{l_{O_1}} = \langle 0 | O_1 | P \rangle_{\text{NRQCD}} / \sqrt{M} = (f \sqrt{M})^{uncorr}. \quad (29)
\]

To illustrate our results we use the data at \( aM_0 = 1.0 \) and \( \kappa_l = 0.1585 \). The effective masses of the pseudoscalar \( C(1, l_{O_1}) \) and \( C(1, 1) \) correlators, presented in figure [4], show a clear signal out to approximately timeslice 20. A one exponential \( (n_{\text{exp}} = 1) \) simultaneous fit to both correlators that
satisfies the fit criteria $Q > 0.1$ is possible from an initial timeslice ($t_{\text{min}}$) of 4 out to timeslice 20 ($t_{\text{max}}$). The variation of $E_{\text{sim}}$ with the fit range is $\sim 2\sigma$ for $t_{\text{min}} > 4$ and this is taken to be the fitting error in our analysis. A final fitting range of 7 – 20 is chosen, for which $Q$ is judged to be roughly stable to increases in $t_{\text{min}}$.

To estimate the remaining excited state contribution to the fitting parameters extracted the results are compared in table 2 with a multi-exponential analysis using a vector and matrix of smeared correlators [12]. The agreement, to within the fitting error, of the energies and amplitudes provides confidence that we have isolated the ground state. Since the overlaps of both $C(1, \ell_{O_1})$ and $C(1, 1)$ with the first excited state are small, $n_{\text{exp}} = 2$ fits could not be performed successfully to these correlators.

| Data | $n_{\text{exp}}$ | fit range | $Q$ | $aE_{\text{sim}}^{PS}$ | $a^3Z_1Z_1$ | $a^{3/2}Z_1$ |
|------|-----------------|-----------|-----|------------------------|-------------|-------------|
| 11,11 | 1 | 7-20 | 0.7 | 0.487(2) | 1.09(2) | 10.1(1) |
| 11,21 | 2 | 3-20 | 0.3 | 0.484(3) | 1.02(3) | - |
| 11,12,21,22 | 2 | 3-20 | 0.4 | 0.485(2) | - | 10.0(1) |

Table 2: The fit parameters, ground state energies and amplitudes extracted from vector and matrix fits to the pseudoscalar meson correlators (without the current correction) for $aM_0 = 1.0$ and $\kappa_l = 0.1585$.

In the static case, it is more difficult to obtain a satisfactory fit. From figure 2 the signal for the $C(1, 1)$ dies away as $C(1, \ell_{O_1})$ appears to be reaching a plateau. While a $n_{\text{exp}} = 1$ fit to both correlators is possible from $t_{\text{min}} = 4$ for $Q > 0.1$, figure 2 shows the ground state energy is not stable to changes in the fitting range. In addition, it was not possible to bootstrap the fits for $t_{\text{min}} < 9$. Thus, we choose 9 – 11 for which $E_{\text{sim}}$ is consistent with the results from different fitting ranges. Agreement is found when comparing with the energy and amplitudes obtained from fits to other combinations of correlators, detailed in table 3. However, a better tuned ground state smearing function would enable the fit parameters to be extracted with greater confidence. As noted in reference [12], even for the heaviest finite heavy quark

\(^5\text{Defined as the one-standard deviation confidence level.}\)
Figure 1: The effective masses of the pseudoscalar $C(1, l_{\sigma_1})$ (circles) and $C(1, 1)$ (diamonds) correlators, and the corresponding ground state energy extracted as a function of $t_{\text{min}}$ from a $n_{\text{exp}} = 1$ vector fit for $aM_0 = 1.0$ and $\kappa_l = 0.1585$. The effective mass of the $C(1, l_{\sigma_1})$ correlator is offset by +0.2. Also shown are the values of $Q$ for the vector fit; $E_{s_{\text{in}}}$ is only presented for those values of $t_{\text{min}}$ which satisfy $Q > 0.1$. $t_{\text{max}}$ is fixed at 20.
mass value, $M_0 = 10.0$ shown in figure 3. NRQCD does not suffer from the same \textit{signal/noise} problems.

| Data | $n_{exp}$ | fit range | $Q$  | $aE_{\text{sim}}^\infty$ | $a^3Z_lZ_l$ | $a^{3/2}Z_l$ |
|------|-----------|------------|------|--------------------------|-------------|-------------|
| 11,11 | 1         | 9-11       | 0.65 | 0.529(13)                | 2.07(20)    | 10.7(5)     |
| 11,21 | 1         | 8-15       | 0.2  | 0.530(9)                 | 2.1(1)      | -           |
| 11    | 1         | 3-11       | 0.6  | 0.527(6)                 | -           | 10.4(2)     |

Table 3: The fit parameters, ground state energies and amplitudes extracted from vector fits to the pseudoscalar meson correlators for $aM_0 = \infty$ and $\kappa_l = 0.1585$.

The heavy quark is expected to be almost a spectator in the heavy-light meson and so the wavefunction changes very little with $M_0$ or the spin content of the meson. We find it is sufficient to use the same fitting range, 7–20, for the pseudoscalar mesons for all finite values of the heavy quark mass. This fitting range is also used for the vector mesons, apart from $M_0 = 0.8–1.2$ where it was necessary to use 9–20. The corresponding values of $E_{\text{sim}}$ and the pseudoscalar and vector decay constants (without the current correction) are given in table 4 for $\kappa_l = 0.1585$. The pseudoscalar meson mass, $M_{PS}$, also shown in the table, is calculated using the mass shifts, $M_{PS} - E_{\text{sim}}$, given in reference [12]. Repeating the analysis for the results at $\kappa_l = 0.1600$, we found 7–20 is the optimal fitting range for all $M_0$.

The current correction is extracted separately by taking the jackknife ratio of the pseudoscalar $C(1, l_{O_2})$ and $C(1, l_{O_1})$ correlators. In the limit of large times, the ratio tends to a constant:

$$\frac{C(1, l_{O_2})}{C(1, l_{O_1})} = \frac{\delta(f\sqrt{M})_{PS}}{(f\sqrt{M})_{PS}^{\text{uncorr}}}. \quad (30)$$

This ratio is shown in figure 4 for $aM_0 = 1.0$ and $\kappa_l = 0.1585$. While a plateau seems to be reached around timeslice 5, $Q$ does not stabilise until $t_{\text{min}} = 12$. Thus, we choose a fitting range of 12–20; this was found to be optimal for all $M_0$. In addition, picking an earlier $t_{\text{min}}$ leads to difficulties when investigating the heavy quark mass dependence of the current correction. The statistical uncertainty becomes so small that terms above $O(1/M^4)$ in the
Figure 2: The effective masses of the pseudoscalar $C(1, l_{O_1})$ (circles) and $C(1, 1)$ (diamonds) correlators, and the corresponding ground state energy extracted as a function of $t_{min}$ from a $n_{exp} = 1$ vector fit for $aM_0 = \infty$ and $\kappa_l = 0.1585$. The effective mass of the $C(1, l_{O_1})$ correlator is offset by +0.2. Also shown are the values of $Q$ for the vector fit. $t_{max}$ is fixed at 11.
heavy quark expansion of this quantity are required, and it becomes difficult
to extract the coefficients reliably. Using the previous results for \((f\sqrt{M})_{PS}\)
we obtain the values of \(\delta(f\sqrt{M})_{PS}\) shown in table 4 for \(\kappa_l = 0.1585\).
It is clear that as \(\delta(f\sqrt{M})_{PS}\) is a \(\sim 15\%)\) correction to \((f\sqrt{M})_{PS}^{uncorr}\) around \(M_b\) (\(\sim 2.0\)), it is essential to include the current correction when calculating \(f_B\)
and investigating its corresponding heavy quark mass dependence, especially
when other sources of error from light quarks etc are under control.

As stated in equations 14 and 15, group theory relates the spin dependent
corrections to the slopes of the vector and pseudoscalar decay constants in
the continuum. In terms of the current correction:

\[
\left|\frac{\delta(f\sqrt{M})_{PS}/(f\sqrt{M})_{PS}^{uncorr}}{\delta(f\sqrt{M})_V/(f\sqrt{M})_V^{uncorr}}\right| = 3. \tag{31}
\]

On the lattice this relation only holds in the limit of infinite statistics. Figure 5 shows \(\left|\frac{C(1,l_0_2)/C(1,l_0_1)_{PS}}{C(1,l_0_2)/C(1,l_0_1)_V}\right|\) for \(aM_0 = 1.0\) and \(\kappa_l = 0.1585\). In the limit
of large times this ratio of correlators tends to the R.H.S. of equation 31.
We find equation 31 is satisfied to within \(\sim 1\sim 2\sigma\) and thus, we can further
reduce the number of matrix elements that need to be calculated by using

\[-\frac{1}{3}[\delta(f\sqrt{M})_{PS}/(f\sqrt{M})_{PS}^{uncorr}](f\sqrt{M})_V^{uncorr} \] for the vector current correction.

Figure 3: The effective masses of the pseudoscalar \(C(1,l_0_1)\) (circles) and
\(C(1,1)\) (diamonds) meson correlators for \(aM_0 = 10.0\) and \(\kappa_l = 0.1585\).
Figure 4: The jackknife ratio of the pseudoscalar $C(1, l_{O_2})$ and $C(1, l_{O_1})$ correlators and the corresponding amplitude, $\delta(f\sqrt{M})_{PS}/(f\sqrt{M})_{uncorr}$, as a function of $t_{\text{min}}$ obtained from a constant fit for $aM_0 = 1.0$ and $\kappa_l = 0.1585$. The values of $Q$ for the fits are also shown. $t_{\text{max}}$ is fixed at 20.
Table 4: The pseudoscalar ground state energy, meson mass, decay constant and current correction, and the vector decay constant for all $aM_0$ and $\kappa_l = 0.1585$. Note that the renormalisation factors are not included.

We compare $E_{PS}^{P_S}$ with $2M_0 \langle \langle 0 | O_2^{PS} | PS \rangle \rangle_{NRQCD} / \langle \langle 0 | O_1^{PS} | PS \rangle \rangle_{NRQCD}$ in table 5. The 20% $\sim O(\alpha_s)$ disagreement between these two quantities is approximately independent of heavy quark mass, and hence is most likely to be due to the omission of the lattice renormalisation factors rather than $O(1/M)$ terms. Note that both quantities are roughly a factor of 2 larger than $\bar{\Lambda}$ i.e. quite different from the analogous quantities in HQET.

The spin-average of the vector and pseudoscalar decay constants, as well as the ratio, with and without the current correction were computed. The results are given in table 6. In order to investigate the heavy quark mass dependence of these quantities, detailed in the next section, 100 bootstrap ensembles were generated for each correlated fit. The bootstrap averages of the fit parameters are consistent with the corresponding unbooted values given in the tables. In the static case we obtain,

$$aE_{sym}^{\infty} = 0.559(37) \quad a^3Z_1Z_1 = 2.82(88) \quad a^{3/2}Z_1 = 12.4(17).$$

These values are consistent, but with much larger errors than those obtained from unbooted data. This underscores the signal/noise problems and the
Figure 5: The jackknife ratio of $C(1,l_0)/C(1,l_1)$ for the vector and pseudoscalar mesons for $aM_0 = 1.0$ and $\kappa_l = 0.1585$.

rough nature of our determinations of $(f\sqrt{M})^{\text{static}}$. The corresponding estimate of $(f\sqrt{M})^{\text{stat}} = 0.4(1)$ is so uncertain as not to be useful. Since the purpose of our static result is to compare with the more accurate extrapolation to the infinite mass limit of the NRQCD results, a rough value is sufficient and we use a value consistent with the unbooted fits, $(f\sqrt{M})^{\text{stat}} = 0.341(37)$, which has smaller error bars.

While the one loop correction to the decay constant is not included in this analysis, the size of the vector matrix element $\langle 0 \mid O_1 \mid V \rangle_{\text{NRQCD}}$ is shown in figure 3 as a percentage of $\langle 0 \mid O_1 \mid V \rangle_{\text{NRQCD}}$ for $aM_0 = 1.0$ and $\kappa_l = 0.1585$. This is roughly the same size as the tree level matrix element,

$$\delta(f\sqrt{M})_V/(f\sqrt{M})_{V,\text{uncorr}} = -\frac{1}{3}\delta(f\sqrt{M})_{PS}/(f\sqrt{M})_{PS,\text{uncorr}} \sim 0.28/3.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (33)

Since the one loop contributions should be suppressed by a factor of $O(\alpha_s) \sim 20\%$, the contribution to the vector decay constant is around 2\% of $\langle 0 \mid O_1 \mid V \rangle_{\text{NRQCD}}$. This is of order $\sim 2\sigma$ in the statistical errors of $(f\sqrt{M})_{V,\text{uncorr}}$ at this mass. The one loop correction to the pseudoscalar decay constant is similarly small.
$$aM_0 \quad aE_{\text{sim}} \quad -2M_0a\frac{\delta(f\sqrt{M})_{PS}}{(f\sqrt{M})_{PS}^{\text{uncorr}}}$$

| $aM_0$ | $aE_{\text{sim}}$ | $-2M_0a\frac{\delta(f\sqrt{M})_{PS}}{(f\sqrt{M})_{PS}^{\text{uncorr}}}$ |
|-------|-----------------|-------------------------------------------------|
| 0.8   | 0.475(3)        | 0.537(2)                                        |
| 1.0   | 0.488(3)        | 0.554(3)                                        |
| 1.2   | 0.497(3)        | 0.567(3)                                        |
| 1.7   | 0.510(3)        | 0.589(3)                                        |
| 2.0   | 0.514(3)        | 0.600(3)                                        |
| 2.5   | 0.519(4)        | 0.612(4)                                        |
| 3.0   | 0.522(3)        | 0.621(4)                                        |
| 3.5   | 0.524(3)        | 0.627(4)                                        |
| 4.0   | 0.525(3)        | 0.631(5)                                        |
| 7.0   | 0.528(4)        | 0.649(6)                                        |
| 10.0  | 0.527(4)        | 0.657(8)                                        |

Table 5: A comparison of the pseudoscalar ground S-state energy and $\langle 0|{\mathcal O}_2|PS\rangle_{NRQCD}/\langle 0|{\mathcal O}_1|PS\rangle_{NRQCD}$ for all $aM_0$ and $\kappa_l = 0.1585$. Note that the renormalisation factors are not included.

Figure 6: The jackknife ratio of $C(1, l_{\mathcal O_4})/C(1, l_{\mathcal O_1})$ for the vector meson at $aM_0 = 1.0$ and $\kappa_l = 0.1585$. 
| $aM_0$ | $a^{3/2}(f\sqrt{M})$ | $(f\sqrt{M})_{PS}^{uncorr}$ | $(f\sqrt{M})_{PS}^{tot}$ |
|-------|---------------------|------------------------|------------------------|
| 0.8   | 0.172(5)            | 1.070(18)              | 0.641(11)              |
| 1.0   | 0.182(5)            | 1.065(19)              | 0.706(13)              |
| 1.2   | 0.190(5)            | 1.059(18)              | 0.751(13)              |
| 1.7   | 0.208(3)            | 1.047(7)               | 0.819(6)               |
| 2.0   | 0.217(5)            | 1.042(7)               | 0.845(6)               |
| 2.5   | 0.229(5)            | 1.037(7)               | 0.875(5)               |
| 3.0   | 0.241(5)            | 1.032(7)               | 0.895(5)               |
| 3.5   | 0.246(5)            | 1.028(6)               | 0.910(5)               |
| 4.0   | 0.253(5)            | 1.025(7)               | 0.920(6)               |
| 7.0   | 0.282(6)            | 1.013(5)               | 0.952(4)               |
| 10.0  | 0.298(8)            | 1.007(4)               | 0.964(5)               |

Table 6: The spin-averaged decay constant, and the ratio of the pseudoscalar to vector decay constant, both with and without the current correction for all $aM_0$ and $\kappa_l = 0.1585$. Note that the renormalisation factors are not included.

5.2 Heavy quark mass dependence of the decay constants

The decay constants have been computed over a wide range of heavy quark mass and this is ideal for investigating heavy quark symmetry. Figure 7 presents the results for $(f\sqrt{M})_{PS}^{tot}$ as a function of $1/M_{PS}$ for $\kappa_l = 0.1585$. The static result is also included in the plot. A correlated fit to the data was performed using the functional form:

$$f\sqrt{M} = C_0 + \frac{C_1}{M} + \frac{C_2}{M^2} + \frac{C_3}{M^3}$$

(34)

The fitting range was varied keeping the starting point fixed at the heaviest data point, $aM_0 = 10.0$. Beginning with a linear function, the fitting procedure was repeated adding quadratic and cubic terms. In general, in order to have confidence in the value for $C_1$ a quadratic fit is required. Note that in reference [12] we found when performing correlated fits to the results as a function of $1/M$ the values of $n$ used in the evolution equation for $aM_0 = 3.5$
and 4.0 are too low; the results for these data points are not included in any of the fits.

Table 7 summarises the coefficients extracted from various fits. A ‘good fit’ is defined in the same way as for the propagator fits: $Q > 0.1$. The fitting range for which $Q$ falls below this cut-off is taken to signify when higher terms can be resolved. Thus, we do not present fits including an additional term over a smaller fitting range. The decay constant is only consistent with a linear dependence on $1/M_{PS}$ for the first three data points, for which $1/M_{PS} \gtrsim 0.3$. The quadratic term is needed for $1/M_{PS} \sim 0$, i.e. there are significant $O(1/M^2)$ contributions to the decay constant in the region of the $B$ meson ($1/(aM_{PS}) \sim 0.4$). Since only the $O(1/M)$ contributions are correctly included in this analysis it is necessary to go to the next order in the NRQCD expansion to calculate $f_B$ accurately. A fit to all the data points is possible including the cubic term.

A quadratic fit to the heaviest six data points is shown in figure 7. The extrapolation of the NRQCD results at finite heavy quark mass is clearly consistent with the static data point. Since the static theory is the heavy quark mass limit of NRQCD this is expected and we assume this is not spoilt.

Figure 7: $a^{3/2}(f \sqrt{M})_{PS}^{tot}$ vs $1/(aM_{PS})$ for $\kappa_l = 0.1585$. The solid line indicates a quadratic fit to the heaviest six points and $1/(aM_B) \sim 0.4$. The static data point, shown as a burst, is not included in the fit. Note that the renormalisation factors are not included.
Table 7: The coefficients extracted from fits to $a^{3/2}(f\sqrt{M})_{PS}^{tot}$ as a function of $1/(aM_{PS})$ for $aM_0 = 1.0$ and $\kappa_l = 0.1585$. Note that the renormalisation factors are not included. The fit range 1\text{-}3 denotes a fit to the data at $aM_0 = 10.0, 7.0$ and 3.0. The data at $aM_0 = 4.0$ and 3.5 are not included in the fit.

| Order | fit range | n.d.o.f. | Q   | $a^{3/2}C_0$ | $a^{5/2}C_1$ | $a^{7/2}C_2$ | $a^{9/2}C_3$ |
|-------|-----------|----------|-----|--------------|--------------|--------------|--------------|
| 1     | 1-3       | 1        | 0.3 | 0.324(6)     | -0.371(22)   | -            | -            |
|       | 1-4       | 2        | 0.0 | 0.317(6)     | -0.329(15)   | -            | -            |
| 2     | 1-4       | 1        | 0.7 | 0.342(10)    | -0.561(71)   | 0.458(137)   | -            |
|       | 1-5       | 2        | 0.7 | 0.338(10)    | -0.525(55)   | 0.379(75)    | -            |
|       | 1-6       | 3        | 0.7 | 0.335(9)     | -0.486(40)   | 0.304(60)    | -            |
|       | 1-7       | 4        | 0.7 | 0.332(7)     | -0.456(23)   | 0.254(26)    | -            |
| 3     | 1-8       | 4        | 0.8 | 0.346(9)     | -0.623(52)   | 0.740(119)   | -0.425(104) |
|       | 1-9       | 5        | 0.9 | 0.346(8)     | -0.614(45)   | 0.716(100)   | -0.406(69)  |

by the inclusion of the renormalisation factors. A striking feature of the plot is the very large slope and the corresponding large deviations from the static limit of $\sim 50\%$ around $1/M_B$.

Note that with only the last two data points requiring a cubic fit, the coefficient $C_1$ is not as well determined as for the quadratic fit. Hence, we only consider the quadratic fits and find $a^{3/2}C_0 = 0.34(1)$. Similarly, $a^{5/2}C_1 \sim -0.46(5)$. Thus, we arrive at $ac_P = aC_1/C_0 \sim -1.35(15)$, much larger than the naive expectation of $O(a\Lambda_{QCD}) \sim -0.2$. Converting to physical units, $c_P \sim -2.8(5)$ GeV, using $a^{-1} = 1.8-2.4$. We found the slope parameters do not depend significantly on the light quark mass.

This slope is significantly larger than that found in previous lattice simulations which found $c_P \sim -0.8$ GeV \cite{20} and $-1.14$ GeV \cite{21}. We believe the discrepancy is due to previous results being obtained by extrapolation from around the $D$ meson with too small a range of masses when using Wilson or Clover heavy quarks; consistency between the static limit and results in this region has not been shown. In addition, our results indicate the deviations from the static limit may be so large at $M_D$ that a heavy quark expansion is not valid in this region.

Onogi et al, using the Fermilab approach to an order equivalent to $O(1/M)$
in NRQCD, also find a slope around $-1 \text{ GeV}$ \cite{22}; data points around the $B$ and $D$ meson are used. The pseudoscalar renormalisation factors for this method have not yet been calculated and a proper comparison of the extrapolation of the results with the static limit cannot yet be made. Consistency should be found between this method and NRQCD, and it is important to confirm the slope in the large mass region, $M_{PS} \gg M_B$.

To examine the large slope of the decay constant further we obtained the individual contributions to the pseudoscalar and vector matrix elements from each of the $O(1/M)$ interactions. Figure 8 shows $\langle f\sqrt{M} \rangle$, $\langle f\sqrt{M} \rangle_{PS}^{uncorr}$ and $\langle f\sqrt{M} \rangle_{PS}^{corr}$ as a function of $1/M_{PS}$. From section 4 the slope of the spin-average decay constant depends only on the kinetic energy of the heavy quark, while the other two quantities include contributions from the spin dependent interactions. It is clear from the large mass region where the slope is linear that $G_{kin}$ is much greater than $G_{hyp}$ or $G_{corr}$, and is the source of the large slope. Qualitatively, the spin dependent terms are expected to be suppressed since they break spin as well as flavour symmetry. Performing correlated fits to $\langle f\sqrt{M} \rangle$, detailed in table 8, we find $aG_{kin} = C_1/C_0 \sim -1.26(15)$ or $-2.6(5) \text{ GeV}$.

This result agrees with Hashimoto \cite{9}, who calculates the pseudoscalar decay constant at $\beta = 6.0$ in the quenched approximation and finds the coefficient of the slope at $O(1/M)$ roughly $-2.4(1.1) \text{ GeV}$. Since the correction to the current is omitted in his calculation, this coefficient corresponds to $G_{kin} + 6G_{hyp}$.

Note that even though only $O(1/M)$ terms are included in the NRQCD action we see quadratic and cubic behaviour in the decay constant in figure 7; this is due to higher powers of the existing $O(1/M)$ terms. These higher order powers are dominated by powers of the kinetic energy terms as is clear from figure 8, which shows that the spin-dependent and spin-independent slopes of $f\sqrt{M}$ show little difference even at relatively small values of $M$. Also, the missing $O(1/M^2)$ terms in the NRQCD action are all spin-dependent so they cannot contribute to the spin-independent slope at $O(1/M^2)$. We would therefore expect that our spin-independent slope was correct even through $O(1/M^2)$ except that there are spin-independent matrix element corrections at this order. We do not know the size of these corrections, but it is unlikely that they would be larger than the spin-independent $1/M^2$ terms that are present in our calculation already. Later we will allow for this by taking a
rather pessimistic 100% error in our $1/M^2$ coefficient.

To isolate the effects of $G_{hyp}$ and $G_{corr}$, consider the ratio of the pseudoscalar and vector decay constants, with and without the current corrections, shown in figure 8. Both data sets are consistent with 1 in the static limit, as expected. Note that without the current corrections the ratio, which is determined solely by the hyperfine interaction, is greater than 1 indicating $G_{hyp}$ is positive. We find $ac''_P \sim +0.11(1)$ or $0.23(4)$ GeV in physical units, and hence $aG_{hyp} = c''_P/8 \sim 0.014(1)$ or 0.029(5) GeV. Including $\delta(f\sqrt{M})$, the slope is a physical quantity and $ac'''_P = -0.40(4)$ or $-0.85(15)$ GeV.

By subtracting $c''_P$ and $c'''_P$, we estimate $aG_{corr} \sim -0.77(8)$. However, the correction to the current can be extracted directly and more accurately as the intercept of $2M_0\delta(f\sqrt{M})_{PS}/(f\sqrt{M})_{PS}^{uncorr}$ shown in figure 10. From table 8, we obtain consistent results, $aG_{corr} = -0.675(7)$ or $-1.42(20)$ GeV. As in the finite heavy quark mass case, $G_{corr}$ agrees with $-E_{sim}^\infty = -0.528(5)$ to within approximately $O(\alpha_s)$.

A comparison can be made with the predictions of QCD sum rules. The results obtained by Neubert [19] using HQET, shown in table 9, are in good
Table 8: The coefficients extracted from fits to combinations of the pseudoscalar and vector decay constants as a function of $1/(aM_{PS})$ for $\kappa_l = 0.1585$. The errors include the variation in the coefficients using different orders in the fit function.

| Expression                                      | $a^{3/2}C_0$   | $a^{5/2}C_1$   |
|-------------------------------------------------|----------------|----------------|
| $a^{3/2}(f\sqrt{M})$                           | 0.335(7)       | -0.43(5)       |
| $(f\sqrt{M})_{PS}^{uncorr}$                    | 0.996(5)       | 0.11(1)        |
| $(f\sqrt{M})_{PS}^{tot}$                        | 1.009(5)       | -0.40(4)       |
| $2M_0a^{3/2}(f\sqrt{M})_{PS}^{uncorr}$          | $aC_0$         | $a^2C_1$       |
| $2M_0a^{3/2}(f\sqrt{M})_{PS}^{uncorr}$          | -0.675(7)      | 0.19(1)        |

Figure 9: The ratio of the pseudoscalar to vector decay constant both with (crosses) and without (pluses) the current correction as a function of $1/(aM_{PS})$ for $\kappa_l = 0.1585$. The solid lines indicate a linear fit to $((f\sqrt{M})_{PS}/(f\sqrt{M})_V)^{uncorr}$ and a quadratic fit to $((f\sqrt{M})_{PS}/(f\sqrt{M})_V)^{tot}$, where all data points are included in the fits. The error in the extrapolation to the static limit is indicated for both fits. Note that the renormalisation factors are not included.
Figure 10: $2M_0 a\delta (f \sqrt{M})_{PS}/(f \sqrt{M})_{PS}^{uncorr}$ vs $1/(aM_{PS})$ for $\kappa_l = 0.1585$. The solid line indicates a linear fit to the six heaviest data points. Note that the renormalisation factors are not included.

agreement with our analysis. As noted previously, the unphysical parameters $G_{hyp}$ and $G_{corr}$ differ; from table 4 the former differs in sign as well as in magnitude with our result. It remains unclear, however, why the slope, and thus $G_{kin}$, is much larger than naively expected; a similar analysis of the spectrum [12] showed deviations from the static limit in agreement with naive expectations. The decay constant may be unique in this respect, however, it is clear this should not be assumed, particularly for quantities such as semi-leptonic form factors of $B$ mesons which are not protected by Luke’s theorem.

By chirally extrapolating the results and interpolating to $1/(aM_B)$, we find $f_B \sim 126\text{–}166$ MeV. The static renormalisation factor, $Z^{stat}_{A} = 0.73$ [17] [1], is used, where we guess $aq_{Z_A}^2 = 2.0$ at $aM_0 = \infty$. To estimate the error in $f_B$ due to the truncation of the NRQCD series we assume a 100% error in the quadratic coefficients of the heavy quark mass expansion. Thus, we find a $\sim 20\%$ uncertainty in $f_B$, which is comparable to the uncertainties arising from the light quark sector. This is likely to be an overestimate;

\footnote{This renormalisation constant corresponds to $\tilde{Z}Z_{cont}$ in the notation used in this reference. Also, the anomalous dimension factor is set to 1.}
Table 9: A comparison in GeV of the decay constant slope parameters and coefficients extracted in this simulation with those computed by Neubert [19] using HQET and QCD sum rules. The errors are dominated by the uncertainty in $a^{-1}$.

|     | Sum Rules | NRQCD     |
|-----|-----------|-----------|
| $c_p$ | -2.9(5)   | -2.8(5)   |
| $c_p'$ | -0.9(1)   | -0.85(15) |
| $G_{kin}$ | -2.3(4)   | -2.6(5)   |
| $G_{hyp}$ | -0.07(3)  | 0.029(5)  |
| $G_{corr}$ | -0.50(7)  | -1.42(20) |

since the contributions from the kinetic energy term in the NRQCD action dominate the slope, the corrections to the matrix element at $O(1/M^2)$ are unlikely to be larger than the quadratic contributions already present. Similarly, from the size of $G_{kin}$ the next order spin-independent terms in the action, $-p^4/8M^3$, are expected to lead to larger contributions to $f_B$ than naively predicted, but this is unlikely to be larger than the lower order contributions. The $O(\alpha_S) \sim 20\%$ error in the slope introduced by using the static renormalisation factor is less significant. In the static limit, we obtain $f_{PS}^{stat} = 237-295$ MeV at $\kappa_l = 0.1585$ which is close to $\kappa_s$.

Our determination of $f_B$ is consistent with the current world average of 200(40) MeV [3]. Our analysis is being performed in conjunction with a simulation using $\beta = 6.0$ quenched configurations [24]. With such large systematic errors it is not possible to discern any difference between the partial inclusion of dynamical quarks in this analysis and quenched results. However, the decay constant is related to the wavefunction at the origin, $f_B\sqrt{M_B} = \sqrt{2}|\psi(0)|$, and thus it is expected to be sensitive to short distance physics and the quenched approximation. Booth [23] has estimated the effect of quenching to be $10-15\%$ in $f_B$ and $\sim 5\%$ in $f_{B_s}/f_B$ using chiral perturbation theory coupled with HQET. However, the prediction is fairly dependent on the light quark mass. In the future we aim to reduce the systematic errors in both our quenched and dynamical simulations by using tadpole improved clover light fermions and including higher orders terms in the NRQCD expansion. Thus, we expect to isolate and remove the full effects.
of quenching.

With an experimental measurement of mixing also in the $\bar{B}_s^0-B_s^0$ system only the ratio $f_{B_s}B_{B_s}/f_BB_B$ is required to extract $|V_{ts}/V_{td}|$. Some cancellation of systematic errors is expected for this ratio, and hence it should be determined more accurately. Table 10 presents the results for $(f\sqrt{M})_s/(f\sqrt{M})_d^{tot}$, where $s$ and $d$ denote results for a light quark mass extrapolated to the strange quark mass and the chiral limit respectively. The expected $O(m_s/M)$ dependence on the heavy quark mass is not resolved in our results and the ratio is independent of $M_0$. The error indicated is purely statistical. There is an additional $\sim 1\sigma$ error due to the uncertainty in $\kappa_s$.

| $aM_0$ | 0.8 | 1.0 | 1.2 | 1.7 | 2.0 | 2.5 |
|--------|-----|-----|-----|-----|-----|-----|
| $(f\sqrt{M})_s^{tot}/(f\sqrt{M})_d^{tot}$ | 1.26(3) | 1.26(3) | 1.25(3) | 1.25(3) | 1.25(3) | 1.26(4) |
| $aM_0$ | 3.0 | 3.5 | 4.0 | 7.0 | 10.0 | $\infty$ |
| $(f\sqrt{M})_s^{tot}/(f\sqrt{M})_d^{tot}$ | 1.26(4) | 1.26(4) | 1.25(4) | 1.24(5) | 1.25(3) | - |

Table 10: The ratio of the pseudoscalar decay constant for $\kappa_l = \kappa_s$ and $\kappa_l = \kappa_c$ for all $aM_0$, where $\kappa_s = 0.1577$ is obtained using the ratio $M_\phi/M_\rho$. The errors shown are purely statistical.

6 Conclusions

We presented a lattice study, partially including the effects of dynamical quarks, of the heavy-light pseudoscalar and vector decay constants using Wilson light quarks and NRQCD heavy quarks. All $O(1/M)$ terms in the NRQCD action and matrix elements were included; an $O(a\Lambda_{QCD}/M)$ wave-function renormalization induced by the evolution equation is the same order as the light quark discretization errors. We find consistency between the extrapolation of the NRQCD result and the static case, as expected, although the static result is not very well determined. There are significant $O(1/M^2)$ corrections to $f\sqrt{M}$ around $M_B$, and this leads to a large systematic error in extracting $f_B$; the use of Wilson light quarks introduces a systematic error of roughly the same magnitude. It is necessary to go to the next order in
NRQCD and use an improved light quark action in order to reliably extract $f_B$.

We found the slope of the decay constants with $1/M$ to be significantly larger than that found in previous lattice simulations. We believe this is because we are closer to the static limit. In particular, for the first time we obtained the three $O(1/M)$ contributions to the linear component of the slope separately. We found that $G_{\text{kin}}$ dominates and thus is responsible for the large slope. Good agreement is found between our results for $G_{\text{kin}}$ and physical combinations of $G_{\text{hyp}}$ and $G_{\text{corr}}$ and a prediction of QCD sum rules. The results show that the kinetic energy of the heavy quark gives rise to contributions to $f\sqrt{M}$ much greater than $O(\Lambda_{\text{QCD}})$, contrary to naive expectations.



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