Extremely Correlated Fermi Liquid theory of the $t$-$J$ model in 2 dimensions:

Low Energy properties

B. Sriram Shastry and Peizhi Mai

Physics Department, University of California, Santa Cruz, Ca 95064

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Abstract

Low energy properties of the metallic state of the 2-dimensional $t$-$J$ model are presented at various densities and temperatures for second neighbor hopping $t'$, with signs that are negative or positive corresponding to hole or electron doping. The calculation employs a closed set of equations for the Greens functions obtained from the extremely correlated Fermi liquid theory. These equations, when used in $d = \infty$ reproduce most of the known low energies features of the $U = \infty$ Hubbard model. In 2-dimensions we are able to study the variations due to the superexchange $J$. The resulting Dyson self energy is found to be momentum dependent as expected. The density and temperature dependent quasiparticle weight, decay rate and the peak spectral heights over the Brillouin zone are calculated. We also calculate the resistivity, Hall conductivity and cotangent of the Hall angle in experimentally relevant units. These display significant thermal sensitivity for density $n \gtrsim 0.8$, signifying an effective Fermi-liquid temperature scale which is two or three orders of magnitude below the bare bandwidth. Flipping the sign of the hopping $t'$, i.e. studying hole versus electron doping, is found to induce a change in curvature of the temperature dependent resistivity from convex to concave at low temperatures. Our results provide a natural route for understanding the observed difference in the temperature dependent resistivity of strongly correlated electron-doped and hole-doped matter.
I. INTRODUCTION

The $t$-$J$ model in 2-dimensions (2-d) has been argued to be of fundamental importance for understanding strongly correlated matter, including the high $T_c$ superconductors$^{1,2}$. Due to the difficulties inherent in a strong coupling problem, very few techniques are available for extracting its low temperature physics. Towards this end we have recently developed the extremely correlated Fermi liquid (ECFL) theory$^{3,4}$. It is an analytical method for treating very strong correlations of lattice Fermions, employing Schwinger’s technique of functional differential equations together with several important added ingredients. While further details can be found in Ref. (3,4), a brief summary of the main idea behind the ECFL theory seems appropriate. We consider the Hubbard model with a large interaction $U \to \infty$, and hence the name of the theory. A well known expansion in the inverse powers of $U$ leads to the $t$-$J$ model (defined below$^2$). Taking the infinite $U$ limit forces one to abandon the conventional Feynman diagram based perturbation theory in $U$, and to make a fresh start. The ECFL theory starts with the graded Lie-algebra of the Gutzwiller projected, i.e. infinite-$U$ limit Fermi operators Eqs. (2,3). This leads to an exact functional differential equation for the Greens functions, known as the Schwinger equation of motion Eq. (18 or 22). In this equation, a parameter $\lambda$ is introduced; $\lambda$ is bounded in the range $\in [0,1]$ and represents the evolution from the free Fermi limit. We then use a systematic expansion in the parameter $\lambda$, for solving the Schwinger equations perturbatively in $\lambda$. In this scheme we start with the uncorrelated Fermi gas at $\lambda = 0$ and end up at the fully correlated projected Fermion problem at $\lambda = 1$. The scheme thus represents a generalization of the usual perturbation theory for canonical Fermionic models, in order to handle a non-canonical Fermionic problem such as the $t$-$J$ model. The context of interacting Bosons provides a useful parallel. In the well known problem of representing spin $S$ variables in terms of canonical Bosons, one uses the expansion parameter $\frac{1}{2S}$ with a similar range $\frac{1}{2S} \in [0,1]$. We may think of $\lambda$ as being analogous to the parameter $\frac{1}{2S}$ as shown in Ref. (4). The introduction of the parameter $\lambda$ and the $\lambda$-expansion scheme thus enabled are among the main technical advances introduced in the ECFL theory.

This approach has been recently benchmarked$^{5-7}$ against the numerically exact results from the single impurity Anderson model, and the $d = \infty$ Hubbard model from Dynamical Mean Field Theory (DMFT)$^{9,10}$. These tests provide quantitative support to our general
scheme described below, especially for low energy response. Our scheme has no specific limitation to \( d = \infty \), and is expected to be reasonably accurate in any dimension \( d > 1 \), including 2 dimensions, a case of great experimental importance due to the High Tc cuprate materials. It is applied here to probe the metallic state of the \( t-J \) model in 2-d. We present results for the electron self energy, the spectral functions, the resistivity the Hall constant and the Hall angle at various temperatures and electron density \( n = N/N_s \) (number of electrons per site). We also use the notation of hole density \( \delta = (1 - n) \) instead of \( n \), following the convention used in several experimental studies of doped Mott systems.

We explore various values of the parameters of the \( t-J \) model, including the second neighbor hopping, which turns out to play a very important role in determining the effective Fermi Liquid (FL) temperature scale. We investigate the resistivity due to mutual collisions of electrons at low temperatures, and its dependence on the parameters of the model. We pay special attention to the resistivity since this easily measured -but notoriously hard to calculate object, reveals the lowest energy scale physics of charge excitations in metallic systems, and therefore is of central importance.

II. METHODS

In this section we summarize the equations used in the present calculation, together with the arguments leading to them- further details may be found in earlier papers on this theory\textsuperscript{3,5–7}. In Sec. (II A) the model is defined and the exact Schwinger-Dyson equations of motion are written out. In Sec. (II B) the \( \lambda \) parameter is introduced and the exact factorization of the Greens function into an auxiliary Greens function and a caparison function are noted. In Sec. (II C) we summarize the shift identities of the \( t-J \) model. The shift transformation is a simple and yet important invariance of the \( t-J \) model leading to important constraints on possible approximations. Within the \( \lambda \) expansion, this invariance obligates the introduction of a second chemical potential \( u_0 \), which is then treated as a Lagrange multiplier to be fixed through sum-rules. In Sec. (II D) we collect the equations of the second order theory. In Sec. (II E) we summarize the rationale for a high energy cutoff of the equations given in Sec. (II D).
A. The \( t-J \) model preliminaries

The \( t-J \) model is a two component Fermi system on a lattice, defined on the restricted subspace of three local states, obtained by excluding all doubly occupied configurations. The allowed states at a single site are \(|a\rangle\) with \( a = 0, \uparrow, \downarrow\), and the double occupancy state \(|\uparrow\downarrow\rangle\) is removed by the (Gutzwiller) projection operator \( P_G = \prod_i (1 - n_i \uparrow n_i \downarrow) \). We use the Hubbard operators \( X^a_{i,b} = |a\rangle\langle b| \), which are expressible in terms of the usual Fermions \( C_{i\sigma}, C^\dagger_{i\sigma} \) and the Gutzwiller projector \( P_G \) as:

\[
X^\sigma_0 i = P_G C^\dagger_{i\sigma} P_G; \quad X^\sigma_0 i = P_G C_{i\sigma} P_G; \quad X^\sigma\sigma'_i = P_G C^\dagger_{i\sigma} C_{i\sigma'} P_G.
\]

These obey the anti-commutation relations

\[
\{X^0_{i\sigma}, X^0_{j\sigma}\} = \delta_{i,j} \left( \delta_{\sigma_i,\sigma_j} - \sigma_i \sigma_j X^0_{i\bar{\sigma},j\bar{\sigma}} \right)
\]

and the commutators

\[
[X^0_{i\bar{\sigma}}, X^0_{j\sigma}] = \delta_{ij} \delta_{\sigma_i,\sigma_j} X^0_{i\sigma}, \quad [X^0_{i\sigma}, X^0_{j\sigma}] = -\delta_{ij} \delta_{\sigma_i,\sigma_j} X^0_{i\sigma}.
\]

The Hamiltonian of the general \( t-J \) model \( H_{t,J} \) is

\[
H_{t,J} = H_t + H_J,
\]

\[
H_t = -\sum_{ij} t_{ij} X^0_{i\sigma} X^0_{j\sigma} - \mu \sum_i X^\sigma_i; \quad H_J = \frac{1}{2} \sum_{ij} J_{ij} \left( \vec{S}_i \cdot \vec{S}_j - \frac{1}{4} X^\sigma_i X^\sigma_{j^\prime} \right),
\]

where we sum over repeated spin indices. Here \( \mu \) is the chemical potential and the spin is given in terms of the Fermions and the Pauli matrices \( \vec{\tau} \) as usual \( \vec{S} = \frac{1}{2} X^\sigma_i \vec{\tau} \sigma X^0_i \). We will restrict in the following to nearest neighbor exchange \( J \), and first (t) and second neighbor (t') hopping on a square lattice.

For the purpose of computing the Green’s functions we add Schwinger sources to the Hamiltonian; the commuting (Bosonic) potential \( V \) couples to the charge as well as spin density. These sources serve to generate compact Schwinger equations of motion (EOM), and are set to zero at the end. The zero source equations are usually termed as the Schwinger-Dyson equations. In that limit we recover spatial and temporal translation invariance of the Greens function. Explicitly we write

\[
\hat{A}_S = \sum_i \int_0^\beta \hat{A}_S(i,\tau) d\tau; \quad \hat{A}_S(i,\tau) = V^\sigma\sigma_{i\sigma}(\tau) X^\sigma_0(i,\tau),
\]

\[
4
\]
and all time dependences are as in $Q(\tau) = e^{\tau H_{tJ}}Qe^{-\tau H_{tJ}}$. The generating functional of Green’s functions of the $t$-$J$ model is

$$Z[\mathcal{V}] \equiv \text{Tr}_{tJ} e^{-\beta H_{tJ}}T_{\tau}\left(e^{-\hat{A}_{S}}\right).$$

(6)

it reduces to the standard partition function on turning off the indicated source terms. The Green’s functions for positive times $0 \leq \tau_j \leq \beta$, are defined as usual:

$$G_{\sigma\sigma'}(i\tau_i, f\tau_f) = -\langle T_{\tau}\left(e^{-\hat{A}_{S}}X_i^{0\sigma}(\tau_i)X_f^{\sigma'0}(\tau_f)\right)\rangle.$$  

(7)

where for an arbitrary $Q$ we define

$$\langle Q \rangle \equiv \frac{1}{Z} \text{Tr}_{tJ} e^{-\beta H_{tJ}}T_{\tau}\left(e^{-\hat{A}_{S}}Q\right).$$

(8)

We note that $n_\sigma$, the number of particles per site, is determined from the number sum rule:

$$n_\sigma = G_{\sigma\sigma}(i\tau^-, i\tau),$$

(9)

and $\mu$ the chemical potential is fixed by this constraint. By taking the time derivative of Eq. (7) we see that the Green’s function satisfies the EOM

$$\partial_{\tau} G_{\sigma_{i}\sigma_{f}}(i, f) = -\delta(\tau_i - \tau_f)\delta_{if}(1 - \gamma_{\sigma_{i}\sigma_{f}}(i\tau_i)) - \langle T_{\tau}\left(e^{-\hat{A}_{S}}[H_{tJ} + \hat{A}_{S}(i, \tau_i), X_i^{0\sigma_i}(\tau_i)] X_f^{\sigma'0}(\tau_f)\right)\rangle$$

(10)

where the local Green’s function is defined as

$$\gamma_{\sigma_{a}\sigma_{b}}(i\tau) = \sigma_{a}\sigma_{b}G_{\bar{\sigma}_{a}\bar{\sigma}_{b}}(i\tau^-, i\tau),$$

(11)

with the notation

$$\bar{\sigma}_i = -\sigma_i.$$  

(12)

Using the Hamiltonian Eq. (4) and canonical relations Eqs. (2, 3) we find

$$[H_{tJ}, X_i^{0\sigma_i}] = \sum_j t_{ij}X_j^{0\sigma_i} + \mu X_i^{0\sigma_i} - \sum_{j\sigma_j} t_{ij}(\sigma_i\sigma_j)X_i^{\sigma_i\sigma_j}X_j^{\sigma'0} + \frac{1}{2} \sum_{j\neq i} J_{ij}(\sigma_i\sigma_j)X_i^{\sigma_i\sigma_j}X_j^{\sigma'0},$$

(13)

and

$$[\hat{A}_{S}(i\tau_i), X_i^{0\sigma_i}] = -\mathcal{V}_i^{\sigma_i\sigma_j}X_i^{0\sigma_j}.$$  

(14)
Substituting into Eq. (10) and using the free Fermi gas Green’s function:

\[
g_{0,\sigma_i,\sigma_j}^{-1}(i\tau_i, j\tau_j) = \left\{ \delta_{\sigma_i,\sigma_j} [\delta_{ij}(\mu - \partial_i) + t_{ij}] - \delta_{ij}V_{i}^{\sigma_i,\sigma_j}(\tau_i) \right\} \delta(\tau_i - \tau_j),
\]

we obtain

\[
g_{0,\sigma_i,\sigma_j}^{-1}(i\tau_i, j\tau_j)G_{\sigma_i,\sigma_j}(j\tau_j, f\tau_f) = \delta(\tau_i - \tau_f)\delta_{ij}(1 - \gamma_{\sigma_i,\sigma_j}(i\tau_i)) - \sum_{j^\sigma_j} t_{ij}(\sigma_i,\sigma_j) \left\langle T_\tau \left( X_i^{\sigma_i,\sigma_j}(\tau_i)X_j^{0,\sigma_j}(\tau_i)X_f^{0,\sigma_f}(\tau_f) \right) \right\rangle + \frac{1}{2} \sum_{k,\sigma_j} J_{ik}(\sigma_i,\sigma_j) \left\langle T_\tau \left( X_k^{\sigma_i,\sigma_j}(\tau_i)X_i^{0,\sigma_j}(\tau_i)X_f^{0,\sigma_f}(\tau_f) \right) \right\rangle.
\]

(16)

We next “reduce” the higher order Green’s function to a lower one using the identity (valid for any operator \(Q\)):

\[
\left\langle T_\tau X_i^{\sigma_i,\sigma'}(\tau)Q \right\rangle = \left\langle T_\tau X_i^{\sigma_i,\sigma'}(\tau) \right\rangle \left\langle T_\tau Q \right\rangle - \frac{\delta}{\delta V_i^{\sigma_i,\sigma'}(\tau)} \left\langle T_\tau Q \right\rangle,
\]

(17)

and rearranging terms we obtain the fundamental Schwinger EOM:

\[
\left( g_{0,\sigma_i,\sigma_j}^{-1}(i\tau_i, j\tau_j) - \hat{X}_{\sigma_i,\sigma_j}(i\tau_i, j\tau_j) - Y_{1,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) \right) \times G_{\sigma_i,\sigma_j}(j\tau_j, f\tau_f) = \delta_{ij}\delta(\tau_i - \tau_f) \left( \delta_{\sigma_i,\sigma_j} - \gamma_{\sigma_i,\sigma_j}(i\tau_i) \right),
\]

(18)

where we defined the functional derivative operator at site \(i\) and time \(\tau_i\)

\[
D_{\sigma_i,\sigma_j}(i\tau_i) = \sigma_i\sigma_j \frac{\delta}{\delta V_i^{\sigma_i,\sigma_j}(\tau_i)},
\]

(19)

the composite derivative operator

\[
\hat{X}_{\sigma_i,\sigma_j}(i\tau_i, j\tau_j) = \delta(\tau_i - \tau_j) \times \left( -t_{ij}D_{\sigma_i,\sigma_j}(i\tau_i) + \delta_{ij} \sum_k \frac{1}{2} J_{ik}D_{\sigma_i,\sigma_j}(k\tau_i) \right),
\]

(20)

and corresponding \(Y_1\) as

\[
Y_{1,\sigma_i,\sigma_j}(i\tau_i, j\tau_j) = -\delta(\tau_i - \tau_j) \times \left( -t_{ij}\gamma_{\sigma_i,\sigma_j}(i\tau_i) + \delta_{ij} \sum_k \frac{1}{2} J_{ik}\gamma_{\sigma_i,\sigma_j}(k\tau_i) \right).
\]

(21)

By considering the spin, space and time variables as generalized matrix indices, we can symbolically write Eq. (18) as

\[
\left( g_{0}^{-1} - \hat{X} - Y_1 \right) \cdot G = \delta (1 - \gamma).
\]

(22)
B. The $\lambda$ expansion and the auxiliary Greens function

The main task is to compute solutions of the Schwinger-Dyson equation, i.e. the functional differential equation Eq. (18) or Eq. (22). If symmetry-breaking, such as magnetism or superconductivity is ignored, then a liquid state ensues, where we would like the solution to connect continuously with the Fermi gas. For this purpose we seek guidance from standard Feynman-Dyson perturbation theory for canonical models. The repulsive Hubbard model is an ideal example, where the corresponding Schwinger-Dyson equation can be schematically written as:

\[
(g_0^{-1} - U\delta/\delta V - UG) \cdot G = \delta \cdot 1. \tag{23}
\]

Comparing with Eq. (22), we see that the left-hand sides are of the same form, but the right-hand sides differ, in Eq. (22) the local Greens function $\gamma$ multiplies the delta function. In turn this extra term originates from the second (non canonical) term in the anti-commutator in Eq. (2), and is therefore the signature term of extremely strong correlations.

Within the Schwinger viewpoint of Eq. (23), we can view the skeleton graph perturbation theory (Feynman-Dyson) as an iterative scheme in $U$, i.e. using the \(n\)th order results to generate the \((n+1)\)th order terms by functional differentiation. In the ECFL theory the iterative scheme used is defined by generalizing Eq. (22) to

\[
(g_0^{-1} - \lambda\hat{X} - \lambda Y_1) \cdot G = \delta \cdot (1 - \lambda\gamma). \tag{24}
\]

The explicit solutions in the ECFL theory start from this basic equation. More explicitly, in Eq. (24) the exact Eq. (18) is generalized to include the $\lambda$ parameter\(^{38}\) by scaling $\hat{X}_{\sigma_i\sigma_j}, Y_{\sigma_i\sigma_j}, \gamma_{\sigma_i\sigma_j} \rightarrow \lambda\hat{X}_{\sigma_i\sigma_j}, \lambda Y_{\sigma_i\sigma_j}, \lambda\gamma_{\sigma_i\sigma_j}$. The starting point for the iteration is $\lambda = 0$, corresponding to the Fermi gas. As we iterate towards $\lambda = 1$, Eq. (24) reduces to the exact equation Eq. (22). The Gutzwiller projection is fully effective only at the end point of the iterative scheme $\lambda = 1$, while for intermediate values of $\lambda$, we have only a partial reduction of the number of doubly occupied sites. The role of $U$ in Eq. (23) is roughly similar, at $U = 0$ we have the Fermi gas, which evolves into an interacting theory with increasing $U$, giving us the Feynman-Dyson perturbation theory. The range of $\lambda$ ($\in [0, 1]$) in Eq. (24) is bounded above, as opposed to that of $U \in [0, \infty]$ in Eq. (23). Therefore the ECFL theory avoids dealing with a major headache of the canonical theory whenever a coupling constant becomes
large. Recall that realistic interactions in correlated matter usually involve a large coupling parameter $U$. For this purpose one is forced to make hard-to-control approximations, such as summing specific classes of diagrams in different parameter ranges. The introduction of $\lambda$ into the ECFL equations opens the possibility that a low order calculation might suffice to give accurate results at low excitation energies. This possibility is in-fact realized for important strong coupling problems as shown earlier Ref. (5).

We found in Ref. (3) that an efficient method for proceeding with the iterative scheme is to first perform a factorization of the Greens function into two parts. The first is an auxiliary Greens function $g$ satisfying a canonical equation, thus admitting a Dysonian expansion with its attendant advantage of summing a geometric series with every added term of the denominator. There remain some terms that cannot be pushed into the denominator, these are collected together as the caparison function $\tilde{\mu}$. In the matrix notation used above we first decompose the Greens function as:

$$\mathcal{G} = g \tilde{\mu},$$

(25)

this implies a product in the $\vec{k}, \omega$ domain as written below in Eq. (32). The differential operator $X$ in equation Eq. (24) is distributed over the two factors of Eq. (25) using the Leibniz product rule, as

$$X.g.\tilde{\mu} = \overline{X}g.\tilde{\mu} + X.g.\tilde{\mu}$$

(26)

where the contraction symbol $\overline{X}a$ indicates the term being differentiated by the functional derivative terms in $X$, while the matrix indices follow the dots. Using $g^{-1}.g = 1$ Eq. (24) is now written as

$$\left(g_{0}^{-1} - \lambda \overline{X}.g.g^{-1} - \lambda Y_{1}\right).g.\tilde{\mu} = \delta (1 - \lambda \gamma) + \lambda \overline{X}.g.\tilde{\mu}.$$  

(27)

This equation factors exactly into two equations upon insisting that $g$ has a canonical structure:

$$\left(g_{0}^{-1} - \lambda \overline{X}.g.g^{-1} - \lambda Y_{1}\right) = g^{-1}$$

(28)

and

$$\tilde{\mu} = \delta (1 - \lambda \gamma) + \lambda \overline{X}.g.\tilde{\mu}.$$  

(29)
We can then use $g \cdot g^{-1} = 1$ to simplify the term $X g \cdot g^{-1} = -X g \cdot g^{-1}$, giving rise to a Dyson self-energy expressed in terms of a Dyson vertex functions as usual. The idea then is to iterate the pair of Equations (28,29) jointly in $\lambda$. Details of the skeleton expansion nature can be found in Ref. (3,5,6). The main point to note is that while $g^{-1}, \mu$ in Eq. (28) and Eq. (29) are expanded in powers of $\lambda$, the function $g$ is kept unexpanded as a basis term (or “atom”) of the skeleton expansion, temporarily ignoring its relationship as the inverse of $g^{-1}$. The equal time value of the variable $\gamma$ in Eq. (11) is taken from the exact sum-rule for $\mathcal{G}$ in Eq. (9). The initial values at $\lambda = 0$ are $g = g_0$ and $\mu = 1$, and we must remember to use the product form Eq. (25) to determine the local Greens function $\gamma$ in Eq. (11). We should note that when the source is turned off $V \rightarrow 0$ we recover space and time translation invariance so that Eq. (25) is simply $\mathcal{G}(\vec{k}, i\omega_j) = g(\vec{k}, i\omega_j), \mu(\vec{k}, i\omega_j)$, with the Matsubara frequency $\omega_j = (2j+1)\pi\beta$. At low $T$, the leading singularities of $\mathcal{G}$ are co-located with those of $g$ provided the caparison function $\mu$ is sufficiently smooth- this situation is realized in all studies done so far.

C. The shift identities and second chemical potential $u_0$

Before proceeding with the iterative scheme, it is important to discuss a simple but crucial symmetry of the $t$-$J$ model- the shift invariance, first noted in Ref. (11). In an exact treatment shifting $t_{ij} \rightarrow t_{ij} + c_t \delta_{ij}$ with $c_t$ arbitrary, is easily seen to be innocuous, it merely adds to Eq. (4) a term $-c_t \sum_{\sigma} \hat{N}_{\sigma}$ whereby the center of gravity of the band is displaced. (Here $\hat{N}_{\sigma}$ is the number operator for electrons with spin $\sigma$.) However in situations such as the $\lambda$ expansion, the Gutzwiller constraint is released at intermediate values, here it has the effect of adding terms derivable from a local (i.e. Hubbard type) interaction term. To see this consider the fundamental commutator term $[H_{tJ}, X_i^{0\sigma}]$ in Eq. (13), here under the shift $t_{ij} \rightarrow t_{ij} + c_t \delta_{ij}$, the third term gives rise to an extra term $c_t X_i^{\sigma, \alpha} X_i^{0\alpha}$. This term vanishes only in a Gutzwiller projected state, the equations of motion by themselves do not eliminate it. Its appearance is tantamount to adding a Hubbard like term $\frac{c_t}{2} \sum_{i} X_i^{\sigma} X_i^{\bar{\sigma}}$ to the Hamiltonian $H_{tJ}$. As argued in Ref. (11) we would like the equations of motion for the Greens functions to be explicitly invariant under the above shift of $t_{ij}$ to each order in $\lambda$. Enforcing this shift invariance to each order in the $\lambda$ expansion plays an important “watchdog” role on the $\lambda$ expansion.
An efficient method to do so is to explicitly introduce an extra Lagrange multiplier $u_0$ through a term $\lambda u_0 \sum_i N_{it} N_{ti}$ to the Hamiltonian Eq. (4). This amounts to replacing $t_{ij} \rightarrow t_{ij} + \delta_{ij} u_0$ in all terms other than in the bare propagator $g_0$. The $u_0$ term makes no difference when $\lambda$ is set at unity in the exact series, since double occupancy is excluded. In practice, we set $\lambda = 1$ in equations that are truncated at various orders of $\lambda$, and the magnitude of $u_0$ is fixed through a second constraint. We thus have two variables to fix, namely $u_0$ and $\mu$. We also have two constraints, the number sum-rules $n_\sigma = g_{\sigma\sigma}(i\tau^-, i\tau^-)$, and $n_\sigma = G_{\sigma\sigma}(i\tau^-, i\tau^-)$(Eq. (9)). In the absence of a magnetic field the number densities $n_\sigma$ reduce as $n_\sigma \rightarrow \frac{n}{2}$, where $n$ is the number of particles per site.

After turning off the sources, in the momentum-frequency space we can further introducing two self energies $\Psi(k, i\omega_j)$, and $\Phi(k, i\omega_j)$ with

$$\tilde{\mu}(k, i\omega_j) = 1 - \lambda \frac{n}{2} + \lambda \Psi(k, i\omega_j)$$
$$g^{-1}(k, i\omega_j) = g^{(-1)}_{0}(k, i\omega_j) + \lambda \left( \frac{n}{2} \varepsilon_k + \frac{n}{4} J_0 \right) - \lambda \Phi(k, i\omega_j).$$

(30) (31)

Here $\varepsilon_k$ and $J_k$ are the Fourier transforms of $-t_{ij}$ and $J_{ij}$. In the right hand side of Eq. (31), the second and third terms arise respectively from the equal-time limit of $\lambda Y_1$ and $\lambda \sqrt{g} g^{-1}$ in Eq. (28) respectively. The two self energies $\Phi, \Psi$ are explicitly $\lambda$ dependent, they vanish at infinite frequency for any $\lambda$. Thus we write

$$G(k, i\omega_j) = g(k, i\omega_j) \times \tilde{\mu}(k, i\omega_j).$$

(32)

The auxiliary Greens function satisfies a second sum-rule that is identical to Eq. (9), both may written in the Fourier domain:

$$(k_B T) \sum_{k, \omega} e^{i\omega \theta_+} G_{\sigma\sigma}(k, i\omega_j) = n_\sigma; \text{ for both } G = G \text{ and } g.$$  

(33)

Eq. (25) can now be written explicitly in the non-Dysonian form proposed in Ref. (3)

$$G(k, i\omega_j) = \frac{1 - \lambda \frac{n}{2} + \lambda \Psi(k, i\omega_j)}{g^{(-1)}_{0}(k, i\omega_j) + \lambda \frac{n}{2} \varepsilon_k + \lambda \frac{n}{4} J_0 - \lambda \Phi(k, i\omega_j)}.$$  

(34)

We observe that simple Fermi liquid type self energies $\Psi$ and $\Phi$ can, in the combination above, lead to highly asymmetric (in frequency) Dyson self energies. Finally we note that our calculations are performed in terms of spectral function obtainable from analytic
continuation of the Matsubara frequencies into the upper complex half plane of frequencies:

\[ \rho_G(\vec{k}, \omega) = -\frac{1}{\pi} \Im m \mathcal{G}(\vec{k}, i\omega_j \rightarrow \omega + i0^+) , \]

\[ \mathcal{G}(\vec{k}, i\omega_j) = \int \frac{\rho_G(\vec{k}, \omega)}{i\omega_j - \omega} \tag{35} \]

and similarly defined spectral functions for variables \(g, \Phi, \Psi\) etc.

D. Summary of equations to second order in \(\lambda\)

In the following, we use the minimal second order equations\(^5\)\(^7\) obtained by expanding Eq. (28) and Eq. (29) to second order in \(\lambda\). The calculation is straightforward and a systematic notation is detailed in Ref. (6), which is followed here. We use the abbreviation\(^8\)

\[ k \equiv (\vec{k}, i\omega_n) , \]

and also redefine \(\Phi(k) = \chi(k) + \varepsilon_k \Psi(k)\), keeping in mind that one set of terms in \(\Phi\) have an external common factor of \(\varepsilon_k\) multiplied by all terms in \(\Psi\). We next collect the answers below in terms of the two self energies \(\chi, \Psi\)

\[ g^{-1}(k) = i\omega_n + \mu - \varepsilon_k + \lambda \frac{1}{4} n J_0 - \varepsilon_k (-\lambda \frac{n}{2} + \lambda \Psi) - \lambda \chi(k) . \tag{36} \]

We now expand \(\Psi\) and \(\chi\) from Eq. (28) and Eq. (29) in powers of \(\lambda\). To the lowest two orders we find \(\Psi = \lambda \Psi_{[1]} + O(\lambda^2)\) and \(\chi = \chi[0] + \lambda \chi[1] + O(\lambda^2)\), where \(\chi[0] = -\sum_p g_p (\varepsilon_p + \frac{\mu}{2} J_{k-p})\).

The next step is to introduce \(u_0\) explicitly: we write \(\varepsilon_k \rightarrow \varepsilon'_k = \varepsilon_k - \frac{u_0}{2}\) in every occurrence of \(\varepsilon_k\), except in the bare propagation term (the term with an underbrace) in Eq. (36).

\[ g^{-1}(k) = i\omega_n + \mu + \lambda \frac{1}{4} n J_0 - \frac{1}{2} u_0 - \tilde{\mu}(k) \varepsilon'_k - \lambda \chi[0](k) - \lambda^2 \chi[1](k) . \tag{37} \]

Note that the shift with \(u_0\) also applies to the term \(\chi[0]\), it now reads \(\chi[0] = -\sum_p g_p (\varepsilon'_p + \frac{\mu}{2} J_{k-p})\). We note the expressions for \(\chi[1], \Psi[1]\) from Ref. (6) Eq. (65-67):

\[ \chi[1](k) = -\sum_{pq} \left( \varepsilon'_p + \varepsilon'_q + \frac{1}{2} (J_{k-p} + J_{k-q}) \right) \times \left( \varepsilon'_{p+q-k} + J_{q-k} \right) g(p) g(q) g(p + q - k) , \tag{38} \]

\[ \Psi[1](k) = -\sum_{pq} (\varepsilon'_p + \varepsilon'_q + J_{k-p}) g(p) g(q) g(p + q - k) \tag{39} . \]
We now set $\lambda = 1$ and record the final equations:

$$
\mathbf{g}^{-1}(k) = i\omega_n + \left( \mu + \frac{1}{4} n(J_0 - u_0) - \frac{1}{2} u_0 + \sum_p g_p \varepsilon_p + \frac{J_k}{2} \sum_p g_p \cos p_x \right) - \tilde{\mu}(k) \varepsilon'_k - \chi_{|1|}(k),
$$

(40)

$$
\tilde{\mu}(k) = 1 - \frac{n}{2} + \Psi_{|1|}(k),
$$

(41)

where we used a nearest neighbor $J_{ij}$ and cubic symmetry in the simplifications. We can verify that the above expressions obey the shift invariance: if we shift $\varepsilon_k \rightarrow \varepsilon_k + c_0$, the arbitrary constant $c_0$ can be absorbed by shifting $\mu \rightarrow \mu + c_0$ and $u_0 \rightarrow u_0 + 2c_0$, and is thus immaterial. The band energy is given explicitly as $\varepsilon_k = -2t(\cos(k_x a_0) + \cos(k_y a_0)) - 4t' \cos(k_x a_0) \cos(k_y a_0)$, where $t$ and $t'$ are the first and second neighbor hopping amplitudes.

E. High energy cutoff scheme

The self consistent solution of the second order equations of Eqs. (38,39,40,41) plus the number sum-rules, can be found numerically by discretizing the momentum and frequency variables on a suitable grid. This procedure can be carried out in a straightforward way for low $T \lesssim t$ and high hole densities $\delta \gtrsim 0.3$ (low particle densities $n \lesssim 0.7$). At lower hole densities or at high temperature $T \gg t$, the equations run into convergence problems. The origin of this problem is the formation of weak and featureless tails of the spectral functions extending to quite high energies. These tails are known to be artificial, since they do not occur in the exact numerical solutions where available. Thus the second order theory seems insufficient in the regime of low hole densities $\delta \lesssim 0.2$, where much of the current interest lies. A technically rigorous resolution of the problem of weak tails seems possible. However it requires the non-trivial calculation of higher order terms in the $\lambda$ expansion. Such higher order terms oscillate in sign and hence cancellations at high energies are expected.

In view of the substantial magnitude of the program of summing the $\lambda$ series to high orders, it seems worthwhile to investigate simpler and physically motivated approximations for improving the lowest order scheme. It turns out that there are a few interesting alternatives in this direction. In Ref. (5) we showed one convenient way to handle the high energy tail problem practically, through the introduction of a high energy cutoff. The choice of an objective cutoff was rationalized by considering two physically different limits, that of high particle density $n \rightarrow 1$ and the simpler high temperature limit, where related tails are found.
The cutoff is chosen using the analytically available high T limit results and then applied to all densities and T.

The cutoff scheme of Ref. (5) is not rigorous, but enables us to extract meaningful results for low energy excitations from the second order \( \lambda \) equations, out to fairly low hole densities \( \delta \lesssim 0.2 \). It is benchmarked in the case of \( d = \infty \), where the cutoff scheme quantitatively reproduces the spectral weights in the most interesting regime of low energies \( |\omega| \ll t \), while erring somewhat at energies above the scale of quarter bandwidth. In Ref. (5,7) the resulting physical quantities such as resistivity are shown to be in good correspondence to the exact results from DMFT. In view of this success we use a similar cutoff scheme for 2-d below, with the expectation that the physics of the low energy excitations is captured. In the present 2-d case we employ a single (re)-normalization the spectral function for each \( \vec{k} \) as

\[
\hat{\rho}_g(\vec{k}, \omega) = \frac{1}{N_k \omega} W_T(\omega - \varepsilon_k) \rho_g(\vec{k}, \omega),
\]

where \( W_T \) is a smooth window (even) function shown in Fig. (3) Ref. (5). It is centered on the bare band energy and has width \( 4D \), where \( 2D \) is the bandwidth (\( \sim 8t \) in this case). The constant \( N_k \) is fixed by the normalization condition \( \int \hat{\rho}_g(\vec{k}, \omega) d\omega = 1 \). In the present case of 2-d we can impose this cutoff window at each \( \vec{k} \) individually, so that only \( \vec{k} \) states very far from the chemical potential are affected.

The two chemical potentials \( \mu \) and \( u_0 \) are determined through the number sum rules written in terms of the Fermi function \( f(\omega) = (1 + e^{\beta \omega})^{-1} \) and the spectral functions:

\[
\sum_k \int \hat{\rho}_g(k, \omega) f(\omega) d\omega = \frac{n}{2} = \sum_k \int \rho_G(k, \omega) f(\omega) d\omega. \tag{43}
\]

The set of equations Eqs. (38,39,40,41, 42, 43) constitute the final set of equations to be computed. These are valid in any dimension, and reduce to the ones benchmarked in \( d = \infty \) after setting \( J \to 0 \).

After analytically continuing \( i\omega_n \to \omega + i0^+ \) we determine the spectral function of the interacting electron spectral function \( \rho_G(\vec{k}, \omega) = \frac{1}{\pi} \Im G(\vec{k}, \omega) \). The set of Equations (1-5) was solved iteratively on \( L \times L \) lattices with \( L = 19, 37, 61 \) and a frequency grid with \( N_\omega = 2^{14}, 2^{16} \) points. Other details are essentially the same as in our recent study of the \( d = \infty, J \to 0 \) case in Ref. (5,7).
FIG. 1: Hole density $\delta$, and $t'/t$ variation of the nodal $Z(k_F)$ at $T = 63$K. $t'/t$ is marked at the top. Decreasing $t'$ has a similar effect to decreasing $\delta$.

III. RESULTS

§ Band Parameters: The $t$-$J$ model is studied on the square lattice with hopping parameters $t$ and $t'$ for first and second neighbors. The hopping parameter $t > 0$, while $t'$ is varied between $-0.4t$ and $0.4t$, thereby changing the Fermi surface (FS) from hole-like to electron-like. Parameters relevant to cuprate High Tc materials are summarized in $^2,14,15$. Following $^2$ we assume $t \sim 0.45$ eV, giving a bandwidth $\sim 3.6$ eV.

§ Single-particle spectrum: The quasiparticle energy $E(\vec{k})$ and quasiparticle weight $Z(\vec{k})$ are found from $G$ as usual$^5$. In Fig. (1) we display the hole density $\delta$ and $t'$ dependence of the low temperature $Z(k_F)$, along the nodal (i.e. $\langle 11 \rangle$) direction. The typical magnitudes of $Z$ are comparable or lower than those reported in $d = \infty^5$. A new and important feature is the strong sensitivity of $Z(k_F)$ to the sign and magnitude of $t'/t$. Both decreasing $t'$ (at fixed $\delta$) and decreasing $\delta$ (at fixed $t'$) reduce $Z$. This feature is basic to understanding our main results. We next study the decay rate of the electrons

$$\Gamma(\vec{k}) = -Z(\vec{k}) \times \Im m\Sigma(\vec{k}, E(\vec{k})), \tag{44}$$

found as the half-width at half-maximum of the spectral function $\rho_G(\vec{k}, \omega)$ at fixed $\vec{k}$. We display the $T$ variation of $\Gamma$ and $-\Im m\Sigma$ at the Fermi surface for three representative values of $t'/t$ in Fig. (2). Both variables display considerable variation with modest change of $T$. The case of $t' > 0$ shows a distinct quadratic $T$ dependence, but for $t' \leq 0$ we note the strong reduction, or absence, of such a quadratic dependence. Below we note a closely parallel $T$ and $t'$ dependence of the resistivity.
In Fig. (3) we display the photoemission accessible peak heights of the spectral function \{t \ast \rho_G(\vec{k}, \omega)\}_{max} over the BZ at three representative values of \(t'/t\), at three temperatures \(T = 63, 210, 334\)K. The peaks track the non-interacting FS, changing from hole-like in Panels (a,b,c) to strongly electron-like in Panels (g,h,i). Several features are noteworthy. The peaks are higher in the nodal relative to the anti-nodal direction at low T. We observe the high sensitivity to warming, in going from \(T = 63\)K to \(T = 334\)K a small (~ 0.7%) change in \(T\) relative to the bandwidth causes a five to fifteen-fold drop in the spectral peaks at the Fermi surface. This is correlated to the thermal variation of \(-\Im m \Sigma\) at the same set of \(t'\), shown in the inset of Fig. (2), since the intensity at \(k_F\) is essentially the inverse of this object. Meanwhile the background spectral weight rises rapidly in all cases, to a roughly similar magnitude. The figure shows that at low \(T\) the curve with \(t' > 0\) has much higher peaks than \(t' \leq 0\), giving the impression of weaker correlations. However the drop on warming is the largest in this case, which signifies another facet of strong correlations.

§ Resistivity: We now study the behavior of the resistivity from electron-electron scattering. We use the popular bubble approximation, factoring the current correlator as \(\langle J(t)J(0)\rangle \sim \sum_k v_k^2 G^2(k)\), where the bare current vertex is the velocity \(\hbar v_k^\alpha = \frac{\partial\varepsilon_k}{\partial k_\alpha}\). In tight binding theory \(v_k^\alpha\) oscillates in sign, resulting in a vanishing average over the Brillouin zone. This oscillation is expected to reduce magnitude of the vertex corrections\(^{16}\). For a 3-d metal having well separated sheets in the c direction (\(c_0\) the separation of the sheets), with
FIG. 3: $\delta = 0.15$. The peak height of the spectral function $A(\vec{k}, \omega)$ over the Brillouin zone at various $t'$ and $T$. At low temperatures the steep increase of the peak heights in going from Panels $(a) \rightarrow (d) \rightarrow (g)$ illustrates the almost FL nature of $(g)$ $t' > 0$, relative to $(a)$ with $t' < 0$. The complementary view of variation with $T$ in going from Panels $(a) \rightarrow (b) \rightarrow (c)$ etc illustrates the dramatic thermal sensitivity in all cases. Recalling that our bandwidth is $\sim 3.6$ eV, we observe that the tiny 0.35% variation of temperature relative to the bandwidth, in warming from 63K to 210K drops the peak height by a factor between 5 and 10, followed by a more gentle fall to 334K. This extraordinary thermal sensitivity is characteristic of our solution of the $t$-$J$ model, it is also reflected in other variables discussed here such as the resistivity.
each sheet represented by the 2-d $t$-$J$ model, the DC resistivity $\rho_{xx}$ can be written in terms of the electron spectral function as follows. We define a dimensionless resistivity $\bar{\rho}_{xx}$ whose inverse is given by

$$\bar{\sigma}_{xx} = \langle \Upsilon(\vec{k})(\hbar v_F^2/a_0^2) \rangle_k,$$

where $\langle A \rangle_k \equiv \frac{1}{N_s} \sum_{\vec{k}} A(\vec{k})$, while the momentum resolved relaxation scale is:

$$\Upsilon(\vec{k}) = (2\pi)^2 \int_{-\infty}^{\infty} d\omega (-\partial f/\partial \omega) \rho^2_G(\vec{k},\omega),$$

and $f \equiv 1/(1 + \exp \beta \omega)$ is the Fermi function. This object resembles the spectral peaks in Fig. (3), losing height and broadening rapidly with $T$. The physical 3-d resistivity is given by $\rho_{xx} = \rho_0 \times \bar{\rho}_{xx}$, where $\rho_0 \equiv c_0 h/e^2 (\sim 1.71 m\Omega cm)$ serves as the scale of resistivity$^{17}$, and using the measured values of the lattice constants we can express our results in absolute units. For understanding the magnitude of the inelastic scattering it can be useful to convert the resistivity into the dimensionless parameter $\langle k_F \rangle \lambda_m$ of an effective 2-d continuum theory, where $\lambda_m$ is the mean-free-path and where $\langle k_F \rangle$ is an (angle averaged) effective Fermi momentum. We can use a relation argued for in Ref. (36,37)

$$\langle k_F \rangle \lambda_m = \frac{hc_0}{e^2 \rho_{xx}} = \frac{1}{\bar{\rho}_{xx}},$$

In Ref. (36,37) the authors note that in a metallic system this parameter is expected to be greater than unity, and its least value is $\langle k_F \rangle \lambda_m = 1$ for the case of unitary (impurity) scattering. Thus we expect that $\rho_{xx} \leq \rho_0$, i.e. $\bar{\rho}_{xx} \leq 1$ in a good metal. The Ioffe-Regel-Mott resistivity scale used in Ref. (5,7,9) provides a similar measure for quantifying the magnitudes of the resistivity found in strongly correlated metals. However we should keep in mind that both estimates suffer from ambiguities in defining a precise threshold value of resistivity, since factors of 2 (or of $2\pi$) cannot be ruled out in Eq. (47).

Fig. (4) shows the temperature dependence of the resistance at three densities, and their strong variation with $t'/t$. $J$ is taken as 900K here, varying $J$ between 0 and 1500K makes almost no difference at these temperatures. We see that the scale of the resistivity for $t' \leq 0$ exceeds the (approximately estimated) unitarity value 1.71$m\Omega$cm already at modest $T$, indicating very strong inelastic scattering. On the other hand $t' > 0$ shows a considerably smaller resistivity at most densities.

In all curves we see that the curvature changes from positive (for $t' \geq 0$) to negative (for $t' < 0$) at say 150K. To understand the role of $t'/t$ we note that the resistivity in
FIG. 4: Panels (a,b,c) show the resistivity at three densities. We expect the very low-T region is cutoff by superconductivity. Panel (b) (Inset) is the local approximation from Eq. (48). It illustrates the $t'/t$ dependence of $\rho_{\text{Local}}$ from self energies, further enhanced by velocity factors in the full curve. The Fermi liquid $\rho \propto T^2$ regime is shrunk (enlarged) as $t'/t \to -0.4$ (+0.4). For a fixed $T$ the curvature changes from positive to negative as $t'/t$ varies upwards in each panel, and also as $n$ increases across the panels - resembling the experimental findings of Ref. (18–22). Note that the scale of the resistivity exceeds the approximately estimated unitarity value $1.71 \mu\Omega\text{cm}$ at modest $T$ for $t' \leq 0$, indicating very strong inelastic scattering. In the displayed range, the case $t' > 0$ shows a somewhat smaller resistivity, and crosses $1.71 \mu\Omega\text{cm}$ only at the lowest hole density $\delta = 0.12$.

Eq. (45) depends on $t'/t$ through the velocity $v_k^x$, in addition to a dependence through the self energies Eqs. (39,38). To gauge their relative importance it is useful to examine a local
approximation of Eq. (45) where the two functions are averaged separately over momentum:

$$\bar{\sigma}_{xx}^{\text{local}} = \langle \Upsilon(\vec{k}) \rangle_k \times \langle (\hbar v_k^x)^2/a_0^2 \rangle_k.$$  (48)

The velocity squared average is independent of the sign of $t'$, therefore the local approximation, shown in the inset of Panel (b), probes only the dependence through Eqs. (39,38).

Comparing the inset and main figure in Panel (b), we see that at $t' = 0.4t$ both resistivity curves display a positive curvature. At $t' = -0.4t$ we see that $\rho^{\text{Local}}$ is essentially linear in $T$, while $\rho_{xx}$ shows a negative curvature. The behavior of $-\Sigma''$ in the inset of Fig. (2) qualitatively resembles the resistivity. The difference is actually related to the velocity factors, which are very different effect between $t' < 0$ and $t' > 0$. These cause the integrals to have very different thermal variation.

§Hall response: Within the bubble scheme, we may also calculate the Hall conductivity$^{10,23–25}$ as $\sigma_{xy} = -2\pi^2/\rho_0 \times (\Phi/\Phi_0) \times \bar{\sigma}_{xy}$, the dimensionless conductivity:

$$\bar{\sigma}_{xy} = \frac{4\pi^2}{3} \int_{-\infty}^{\infty} d\omega (-\partial f/\partial \omega) \langle \rho_G^3(k,\omega) \eta(k) \rangle_k,$$  (49)

and $\eta(k) = \frac{\hbar^2}{a_0^5} \{ (v_k^x)^2 \frac{\partial^2 \varepsilon_k}{\partial k_x^2} - (v_k^x v_k^y) \frac{\partial^2 \varepsilon_k}{\partial k_x \partial k_y} \}$. Here $\Phi = B a_0^2$ is the flux$^{17}$ and $\Phi_0 = \hbar c/(2|e|)$ is the flux quantum. In terms of these we can compute the Hall constant $R_H$ and Hall angle $\Theta_H$ from

$$c R_H = -\frac{4\pi^2 v_0}{|e|} \bar{\sigma}_{xy} \times \bar{\sigma}_{xx}^2,$$  (50)

$$\cot(\Theta_H) = -\frac{1}{2\pi^2} \frac{\bar{\sigma}_{xx}}{\bar{\sigma}_{xy}} \times \frac{\Phi_0}{\Phi},$$  (51)

with $v_0 = (a_0^2 c_0)^{17,26}$.

In Fig. (5) we display the computed Hall variables. In Panel (a) $\tan \Theta_H$ is shown for two values of $t'/t$ displaying hole-like and electron-like behavior. A decrease in hole density reduces the magnitude in either case. In Panel (b) we display the computed $\cot(\Theta_H)$ versus $T^2$ with three values of $t'$ giving an electron-like FS. We note that $\cot \Theta_H$ is approximately linear with $T^2$ and strongly affected by the magnitude of $t'$. The two distinct $\cot(\Theta_H) \propto T^2$ regimes seen in Fig. (5-b) are also seen in many experiments, the crossover to a strange metal corresponding to the bending temperature$^{29}$. Our results for the hole-like case $t' < 0$ at low $T$ are less robust due to the small magnitude of $\sigma_{xy}$ and errors from the oscillating sign of $\eta$ in Eq. (49). In Panel (c) we show the Hall constant $R_H$ at three densities
for representative values of $t'/t$. Its sign is electron-like for $t' > 0$ and hole-like for $t' \leq 0$, tracking the change in topology of the Fermi surface in Fig. (2). The magnitude of $R_H$ is substantially affected by changing $t'$. This is a strong correlation effect, and discourages envisaging any simple relationship between the Hall number and hole density.

IV. DISCUSSION

Using the recently developed second order equations of the ECFL theory in Ref. (5), we have presented results for the 2-d $t$-$J$ model at low and intermediate temperatures. In keeping with our recent findings for the $d = \infty$ solution of the same equations, we note that the quasiparticle weight $Z(k_F)$ is non-zero, but remarkably small. This fragile Fermi liquid therefore has an extremely low effective Fermi temperature, above which it displays characteristics of a Gutzwiller correlated strange metal, as listed in Ref. (5,7), including a resistivity that is linear in $T$.

By varying $t'$, the second neighbor hopping at a fixed $t$ and $J$, we found in Fig. (1) a remarkable variation of the quasiparticle weight $Z(k_F)$ that is characteristic of the 2-d square lattice, with no simple analog in $d = \infty$. We found $t' < 0$ leads to a considerable reduction in its magnitude, while $t' > 0$ leads to a larger value and thus a more robust Fermi liquid. A direct calculation of the single particle spectral width $\Gamma = -Z\Sigma''$ confirms this observation in Fig. (2), and when studied as a function of the temperature, shows a much larger magnitude, and hence broader spectral lines.

Our two striking results concern the spectral heights over the Brillouin zone, and the resistivity as a function of $T$ at various densities and $t'$. The spectral height is the peak value of $\rho_G(\vec{k}, \omega)$ scanned over $\omega$, and equals the inverse of the least magnitude of $\Im m \Sigma(\vec{k}, \omega)$. In Fig. (3) we present both the $T$ evolution (going horizontally) and the $t'$ evolution (going vertically) of this important object visible in ARPES. We note that $t' < 0$ model with a very small $\Gamma$ also displays a rapid loss of coherence on warming. The quasiparticle peaks drop rapidly, while the valleys, representing the background spectral weight in photoemission, catch up with the peaks in magnitude. A similar variation happens for $t' = 0$ but the drop of the peak heights is more pronounced. The case of $t' > 0$ has the largest drop of peak heights, while its effective Fermi temperature is the largest of the three cases. It follows that the electron doped case has a more robust Fermi liquid appearance for $T$ lower than
its Fermi scale. Our study provides absolute scale values for these observable heights, and it should be interesting to study these experimentally for comparison. Towards that objective we note that $t' > 0$ maps to the electron doped High Tc superconductors, while $t' \leq 0$ maps to the hole doped cases, as we may also deduce from the shapes of the Fermi surfaces in the above figure.

The other striking results concerns the resistivity. We are able to calculate the longitudinal resistivity $\rho_{xx}$ on a doubly absolute scale, both the magnitude of $\rho_{xx}$ and that of $T$ are given in physical units by using reasonable values for the basic parameters of the $t$-$J$ model and the lattice constants Fig. (4). We find essentially the experimentally observed scales for both axes, and there is room for further adjustments of bare scales if needed. The main finding is that as $\delta$ is varied towards half filling, the regime of linear resistivity increases in the hole-like cases ($t' \leq 0$) and the quadratic dependence regime shrinks to very low $T$ scales, falling below the known superconducting transition temperatures. The other important finding is that the concavity (convexity) of resistance versus $T$, usually taken to denote a (non) Fermi liquid behavior, requires an enlarged viewpoint; we find that the sign of $t'$ flips the two cases. As an example, the case $t' \leq 0$ has a pronounced convex regime at low $T$. This could be naively ascribed to a non-Fermi liquid behavior, but in reality is a crossover range to the strange metal regime.

We also present results in Fig. (5) for the Hall constant and the Hall angle. These are calculated using simple versions of the Kubo formula, found by neglecting the vertex corrections, in the same spirit as the longitudinal resistivity. It must be kept in mind that the vertex corrections are likely to be more significant for the Hall response, since there are two vertices involved- the second one from the magnetic field derivative of the Greens function. Additionally there are no exact results in literature on correlated matter for the Hall constant to benchmark the ECFL results. For these reasons one might place lesser confidence in the quantitative aspect of the Hall results as compared to the longitudinal transport functions. We find that the Hall angle changes sign with $t'$. The cot($\Theta_H$) is found to be roughly linear with $T^2$, in agreement with the experimental situation. Interestingly the magnitude of the computed results is also roughly in accord with the experiments. While these results are encouraging, we believe that further work is needed to unravel what we appears to be a knee in the cot($\Theta_H$) versus $T^2$ curve, and also to better estimate the density dependence of the Hall constant.
V. CONCLUSIONS:

In this work, we used a scheme from the ECFL theory where the second order $\lambda$ expansion terms are supplemented with a high energy cutoff. This scheme has been benchmarked in $d = \infty$ against DMFT$^{5,7}$ for computing transport and other low energy excitations, giving good agreement with exact numerical results. As detailed in Ref. (5) the magnitude of the quasiparticle weight $Z$ is somewhat lower in this scheme as compared to the exact DMFT values for hole density $\delta \lesssim 0.8$ resulting in a larger magnitude of the resistivity as well. In this work the same formalism has been applied to the 2-d $t$-$J$ model. It is possible that the close agreement found in the $d = \infty$ case might not survive to the same extent for $d = 2$. Hence we might expect to find further refinements of the absolute values of the quasiparticle weight and resistivity to emerge from further studies of the $\lambda$ expansion. However it seems likely that the crucial variation of resistivity and Hall constant with the magnitude and sign of $t'$ found here will persist in more exact future results. Hence it seems that we can draw some useful conclusions already regarding the difference between hole and electron doping.

We have shown a range of results for the 2-d $t$-$J$ model, obtained by varying different parameters within our scheme. It is interesting that the magnitudes of various transport variables, presented here in physical units$^{30}$, are roughly on the scale of reported measurements$^{18–22,31–33}$. Although it is not our primary aim here to produce exact fits, we note that the agreement can be improved in many cases with suitable changes of the bare (band) parameters.

In the range of parameters considered here, a metallic state has been posited, and therefore the role of the exchange $J$ is limited; we find very little variation of the transport quantities with a change in $J$. The transport parameter variation with density seems very similar to that found in $d = \infty$ in Ref. (5,7,9) where a large variety of Gutzwiller correlated metallic states were shown to arise$^{7}$, with their origin in the $U = \infty$ or Gutzwiller correlation rather than with $J$. The added feature in $d = 2$ is the important role played by $t'$, as stressed here. We expect magnetic, superconducting and possibly other broken symmetry states at the lowest $T$ and $\delta$ to arise, largely due to the effect of $J$. Further work is necessary to find reliable calculational schemes for these broken symmetry states.

A few broad conclusions suggest themselves. The parameter $t'/t$ plays a key role in determining the low-energy scales. In Fig. (1) we see that the quasiparticle weight $Z$ has a
large variation with \( t' \). The origin of this sensitivity lies in the self energies in Eqs. (39,38), where combinations of the band energies \( \varepsilon_k \) play the role of an effective interaction. Varying \( t'/t \) therefore changes the self-energies strongly, in contrast to the usual weak change via the band parameters in Eq. (37).

Our main findings are as follows. (I) The spectral functions are highly sensitive to thermal variation; in Fig. (3) we observe a five to fifteen fold drop in intensity with a variation of \( k_B T \) about 1/100th the bandwidth \( \sim 3.6 \text{ eV} \). This is in severe conflict with expectations from conventional theories of metals. (II) We note from Fig. (4) that with \( t' \leq 0 \), a Fermi liquid (FL) resistivity \( \rho \propto T^2 \) is seen only at very low \( T \). The very low \( T \) (FL) regime is followed by a “strange metal” regime, also at low \( T \), where we find a \( \rho \text{ vs } T \) curve, with zero or negative curvature. This regime parallels the Gutzwiller-correlated strange metal regime reported in \( d = \infty^7 \), the negative curvature making it even stranger. (III) For the electron-doped case \( t' > 0 \), Fig. (4) shows that the curvature is positive and the Fermi liquid regime extends to higher temperatures.

It is significant that the ECFL theory captures the diametrically opposite resistivity behaviors of hole doped\(^{18-20} \) and electron doped materials\(^{21,22} \) within the same scheme, only differing in the sign of \( t'/t \). The resistivity curvature mapping of Ref. (18) can also be viewed in terms of a variation of this ratio and the temperature, as in Fig. (4). In conclusion this work provides a sharp picture of the difference made by the second neighbor hopping \( t' \) in the presence of Gutzwiller correlations. It also yields quantitative results for several famously hard to compute variables in correlated matter, that in rough agreement with a variety of experiments.

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39 Similarly we note that shifting J_{ij} → J_{ij} + c_f δ_{ij} with arbitrary c_f also adds a similar unphysical local interaction term, as discussed in greater detail in Ref. (11).
FIG. 5: Panel (a) $\tan \Theta_H$ for $B = 10T$ at three densities vs $T$. The set with $t' = -0.4t$ have a hole-like FS while the set with $t' = 0$ an electron-like FS. In both cases we see a rapid fall-off with $T$, and a decreasing magnitude with $\delta$. Panel(b) for $B = 10T$ at $\delta = 0.15$ shows $|\cot(\Theta_H)|$ for three values of $t'/t$. It is approximately linear with $T^2$ over the range, in fact it is linear on both sides of a bend$^{20,22,27–29,31–33}$. Panel(c) gives the $T$ dependent $R_H$ for three densities, each with four values of $t'/t$. The sign change resembles the change seen in experiments$^{34}$. 

\begin{itemize}
  \item Panel (a) $\tan \Theta_H$, $\delta$ marked above
  \item Panel (b) $|\cot(\Theta_H)|$, $\delta = 0.15$. $t'/t$ marked above
  \item Panel (c) $R_H$ full,dotted,dashed lines are at $\delta = 0.18, 0.15, 0.12$. $t'/t$ marked above
\end{itemize}