The Fourier transform approach to Hyers-Ulam stability of differential equation of second order

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Abstract. The use of Fourier transform has increased in the light of recent events in different application. Fourier transform is also seen as the easiest and an effective way among the other transformation. In line with this, the research deals with the Hyers-Ulam stability of second order differential equation using Fourier transform. The study aims at deriving a generalized Hyers-Ulam results for second order differential equations with constant co-efficient

\[ H''(v) + aH'(v) + bH(v) = r(t) \]

with the help of Fourier Transform.

Key Words: Hyers-Ulam stability, Generalized Hyers-Ulam stability, Linear Differential Equation, Fourier Transform

1. Introduction

The Fourier transform is a kind of integral transform. The Integral transform were pioneered by Swiss mathematician Leonhard Euler(1707-1783) for the second order differential equation problem [1]. The most applied integral transform is Fourier transform. Fourier transform was first used by French mathematician Jean Baptiste Joseph Fourier in 1807. The Fourier transform is consisting frequency value of function [2].

The ordinary and partial differential equation describe the quantities vary with respect to time. Its a difficult task to solve a complex equation. The most used method to solve such problem is Fourier transform. In this paper, we apply Fourier Transform method to prove Hyers-Ulam stability of Second order linear differential equation. The concept of Hyers-Ulam stability is provide an existence and unique solution of differential equation. Its history starts from middle of the 19th century. In 1940, Ulam [3] asked the following question in a seminar at Wisconsin University about stability of group homomorphism

"Let \( H_1 \) and \( H_2 \) be a group with a metric \( \rho \). For every \( \epsilon > 0 \), there exist a \( \delta > 0 \) such that \( g : H_1 \rightarrow H_2 \) satisfies \( \rho(g(uv), g(u)g(v)) < \delta, \forall u, v \in H_1 \). Then under what condition we can find a homomorphism \( h : H_1 \rightarrow H_2 \) exist with \( \rho(g(u), h(u)) < \epsilon, \forall u \in H_1.\)"

In 1941, Hyers [4] gave the positive answer to Ulam’s question in case of \( H_1 \) and \( H_2 \) are assumed to be Banach space. In view of Hyers result, there is no reason for the cauchy difference \( g(u + v) - g(u) - g(v) \) to be bounded. Towards this point, Rassias [5] was generalized Hyers concept. After that, this finding is named as Hyers-Ulam Rassias satbility. In 1993, Obloza [6]
created the Hyers-Ulam stability of linear differential equation. After some years, Alsina and Ger [7] developed stability of differential equations in way of Hyers-Ulam. The stability results of differential equation in various direction have been obtained by Miura and Takahasi [8]. Jung initiated the application of these concepts to integral equations via a fixed point method. In 2016, Jung [11] established the Hyers-Ulam stability of second order differential equation by direct method.

By motivation of above result, the aim of this paper is to investigate the Hyers-Ulam stability of second order differential Equation with constant co-efficient.

2. Preliminaries

In this section, we give some definition and property to prepare our main results.

**Definition 2.1 (Fourier Transform)** If a function \( \mathcal{H}(v) \) is piecewise continuous and is absolutely integrable in \( \mathbb{R} \), then the Fourier transform of \( \mathcal{H}(v) \) is defined as

\[
F(\mathcal{H}(v)) = \tilde{\mathcal{H}}(\eta) = \int_{-\infty}^{\infty} \mathcal{H}(\eta)e^{i\eta v} \, dv
\]  

The Dirichlet’s Conditions of a single valued function \( \mathcal{H}(u) \) is

(i) The integral value of \( \mathcal{H}(v) \) is finite.

(ii) \( \mathcal{H}(v) \) has finite number of discontinuous and extrema.

**Definition 2.2 (Inverse Fourier Transform)** If \( \mathcal{H}(v) \) satisfies Dirichlet’s conditions in every finite interval and absolutely convergent, then

\[
\mathcal{H}(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\mathcal{H}}(\eta)e^{i\eta v} \, d\eta
\]  

**Definition 2.3 (convolution)** The function \((\mathcal{H}_1 * \mathcal{H}_2)(v) = \int_{-\infty}^{\infty} \mathcal{H}_1(\eta)\mathcal{H}_2(v-\eta) \, d\eta \) is called the convolution of the function \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) defined on \((-\infty, \infty)\)

Now, we recall some properties of Fourier transform which is closely related to the solution process.

(i) If \( F(\mathcal{H}_1(v)) = F(\mathcal{H}_2(v)) \) then \( \mathcal{H}_1(v) = \mathcal{H}_2(v) \), \( \forall v \in \mathbb{R} \) (one-to-one)

(ii) \( F(\mathcal{H}^n(v)) = (-i\eta)^n \tilde{\mathcal{H}}(\eta) \) (differentation)

(iii) \( F(\mathcal{H}_1(v) * \mathcal{H}_2(v)) = F(\mathcal{H}_1(v)) F(\mathcal{H}_2(v)) \) (convolution)

**Definition 2.4** The differential equation

\[
\mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v) = 0
\]  

is said to be Hyers-Ulam stable if the mapping \( \mathcal{H}:(-\infty, \infty) \rightarrow \mathbb{F} \) satisfies the inequality \(|\mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v)| \leq \epsilon \), there exist a solution \( \mathcal{M}:(-\infty, \infty) \rightarrow \mathbb{F} \) of differential Eq.\( \text{(2.3)} \) with

\[
|\mathcal{H}(v) - \mathcal{M}(v)| \leq K(\epsilon), \quad \forall v \in \mathbb{R}
\]

where, \( \epsilon > 0 \) and \( K \) is a positive constant.

Throughout this paper \( \mathbb{F} \) is represent as Real or complex domain.
Definition 2.5 we that the differential equation

\[ H''(v) + aH'(v) + bH(v) = r(v) \]  

is Hyers-Ulam stable if \( H:(-\infty, \infty) \to \mathbb{F} \) is a continuously differentiable function satisfies \( |H''(v) + aH'(v) + bH(v) - r(t)| \leq \epsilon \), then for every \( \epsilon > 0 \), \( K > 0 \) and there exist a solution \( M: (-\infty, \infty) \to \mathbb{F} \) of differential Eq.(2.4) with

\[ |H(v) - M(v)| \leq K\epsilon, \quad \forall v \in \mathbb{R} \]

Moreover, if \( \epsilon \) and \( K(\epsilon) \) are replace by continuous functions \( \phi(v) \) and \( \chi(v) \), then we say that Eq.(2.3) and Eq.(2.4) is said to be generalized (or) Rassias Hyers-Ulam stability.

3. Stability Results for Homogeneous Linear Equation

In the section, we are going to verify that the approximate solution is near the exact solution of linear homogeneous differential Eq.(2.3).

Theorem 3.1 For every \( \epsilon > 0 \) and there exist a positive constant \( K \) such that function \( H:(-\infty, \infty) \to \mathbb{F} \) satisfies the inequality

\[ |H''(v) + aH'(v) + bH(v)| \leq \epsilon, \quad \forall v \in \mathbb{R} \]  

Then there exist a solution \( M: (-\infty, \infty) \to \mathbb{F} \) of differential Eq.(2.3) such that

\[ |H(v) - M(v)| \leq K\epsilon, \quad \forall v \in \mathbb{R} \]

proof: Define a function \( q: (-\infty, \infty) \to \mathbb{F} \) such that

\[ q(v) = H''(v) + aH'(v) + bH(v), \forall v \in \mathbb{R} \]  

Suppose that \( H \) be a continuously differentiable function satisfies the inequality (3.1). We have, \( |q(v)| \leq \epsilon \). Now, taking Fourier transform of Eq.(3.2)

\[ F(q(v)) = F(H''(v)) + aF(H'(v) + bH(v)) \]

\[ \tilde{Q}(\eta) = ((-i\eta)^2 F(H(v)) - i\eta F(H'(v)) + bF(H(v))) \]

\[ \tilde{H}(\eta) = \frac{\tilde{Q}(\eta)}{(-i\eta)^2 + a(-i\eta) + b} \]  

Let \( u_1, u_2 \) are the distinct roots of the characteristic equation

\[ (-i\eta)^2 + a(-i\eta) + b = 0 \]

then we have,

\[ (-i\eta)^2 + a(-i\eta) + b = (i\eta - u_1)(i\eta - u_2) \]

Thus

\[ F(H(v)) = \tilde{H}(\eta) = \frac{\tilde{Q}(\eta)}{(i\eta - u_1)(i\eta - u_2)} \]  

We now set \( \tilde{P}(\eta) = F(p(v)) = \frac{1}{(i\eta - u_1)(i\eta - u_2)} \). By inverse Fourier transform

\[ p(v) = \mathcal{F}^{-1} \left( \frac{1}{(i\eta - u_1)(i\eta - u_2)} \right) \]  

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Define a function $\mathcal{M} : (-\infty, \infty) \to \mathbb{F}$ such that $\mathcal{H}(v) = c_1 e^{-u_1 v} + c_2 e^{-u_2 v}$, where $c_1$ and $c_2$ are constant. We obtain,

$$F(\mathcal{H}(v)) = \tilde{\mathcal{H}}(\eta) = \int_{-\infty}^{\infty} (c_1 e^{-u_1 v} + c_2 e^{-u_2 v}) e^{i\eta v} dv = 0 \quad (3.6)$$

By taken account of property of Fourier transform and In view of Eq. (3.6), it's holds that

$$F(\mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v)) = ((-i\eta)^2 + a(-i\eta) + b) \tilde{\mathcal{H}}(\eta) = 0$$

Since Fourier transform satisfies one-to-one property. We get, $\mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v) = 0$ which yields that $\mathcal{H}$ is a solution of differential Eq. (2.3). Further, from Eq. (3.4) and (3.6) and convolution property of Fourier transform

$$F(\mathcal{H}(v) - \mathcal{M}(v)) = \tilde{\mathcal{H}}(\eta) - \tilde{\mathcal{M}}(\eta) = \frac{\tilde{Q}(\eta)}{(i\eta - u_1)(i\eta - u_2)} = \tilde{Q}(\eta) \tilde{P}(\eta) = F(q(v)) F(p(v)) = F(q(v) * p(v))$$

Hence, $\mathcal{H}(v) - \mathcal{M}(v) = q(v) * p(v)$. Now taking modulus,

$$|\mathcal{H}(v) - \mathcal{M}(v)| = |q(v) * p(v)| = \left| \int_{-\infty}^{\infty} q(\xi)p(v - \xi)d\xi \right| \leq \int_{-\infty}^{\infty} |q(\xi)p(v - \xi)| d\xi$$

We obtain,

$$|\mathcal{H}(v) - \mathcal{M}(v)| \leq \epsilon \int_{-\infty}^{\infty} p(v - \xi)d\xi \leq K \epsilon$$

where $K = \int_{-\infty}^{\infty} p(v - \xi)d\xi$ for any value of $v$. Clearly, this implies that the homogeneous linear differential Eq. (2.3) has Hyers-Ulam stability.

The main result of Hyers-Ulam Rassias stability of Eq. (2.3) is given in the following theorem. Define a function

$$\chi(v) = \int_{-\infty}^{\infty} G(\xi)p(v - \xi)d\xi$$

**Theorem 3.2** Let $a$ be a scalar in $\mathbb{F}$ and for every $\epsilon > 0$, there exist a positive constant $K$ and function $G : (-\infty, \infty) \to \mathbb{R}_+$ such that $\mathcal{H} : (-\infty, \infty) \to \mathbb{F}$ satisfies the inequality

$$|\mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v)| \leq G(v), \forall v \in \mathbb{R} \quad (3.7)$$

Then there exist a solution $\mathcal{M} : (-\infty, \infty) \to \mathbb{F}$ of differential Eq. (2.3) such that

$$|\mathcal{H}(v) - \mathcal{M}(v)| \leq KG(v), \forall v \in \mathbb{R}$$

**proof:** Assume that $\mathcal{H}(v)$ satisfies the inequality (3.7) and define a function $q : (-\infty, \infty) \to \mathbb{F}$ such that

$$q(v) = \mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v) \quad (3.8)$$
We have, $|q(v)| \leq G(v)$. The Fourier transform of Eq.(3.8) is

$$F(q(v)) = F(H''(v)) + aF(H'(v) + bH(v))$$

$$\tilde{Q}(\eta) = ((-i\eta)^2) F(H(v)) - i\eta F(H(v)) + bF(H(v))$$

$$\tilde{H}(\eta) = \frac{\tilde{Q}(\eta)}{(-i\eta)^2 + a(-i\eta) + b}$$  \quad (3.9)

Let $u_1, u_2$ are the distinct roots of the characteristic equation $(-i\eta)^2 + a(-i\eta) + b = 0$. We have,

$$(-i\eta)^2 + a(-i\eta) + b = (i\eta - u_1)(i\eta - u_2)$$

Thus,

$$F(H(v)) = \tilde{H}(\eta) = \frac{\tilde{Q}(\eta)}{(i\eta - u_1)(i\eta - u_2)}$$  \quad (3.10)

We now set $\tilde{P}(\eta) = F(p(v)) = \frac{1}{(i\eta - u_1)(i\eta - u_2)}$. By inverse Fourier transform

$$p(v) = F^{-1}\left(\frac{1}{(i\eta - u_1)(i\eta - u_2)}\right)$$  \quad (3.11)

Define $M: (-\infty, \infty) \to \mathbb{F}$ such that $M(v) = c_1e^{-u_1v} + c_2e^{-u_2v}$, where $c_1$ and $c_2$ are constant. Then,

$$F(M(v)) = \tilde{M}(\eta) = \int_{-\infty}^{\infty} (c_1e^{-u_1v} + c_2e^{-u_2v})e^{i\eta v}dv = 0$$  \quad (3.12)

By differentiation property of Fourier transform and In view of Eq.(3.12),

$$F(M''(v) + aM'(v) + bM(v)) = ((-i\eta)^2 + a(-i\eta) + b)\tilde{M}(\eta) = 0$$

Since Fourier transform is one to one operator. We get, $M''(v) + aM'(v) + bM(v) = 0$ Hence, $M(v)$ is a solution of differential Eq.(2.3). Further, it follows from Eq.(3.10) and (3.12) and convolution property of Fourier transform

$$F(H(v) - M(v)) = \tilde{H}(\eta) - \tilde{M}(\eta)$$

$$= \frac{\tilde{Q}(\eta)}{(i\eta - u_1)(i\eta - u_2)}$$

$$= \tilde{Q}(\eta)\tilde{P}(\eta)$$

$$= F(q(v))F(p(v))$$

$$= F(q(v)*p(v))$$

We obtain,

$$|H(v) - M(v)| = |q(v) * p(v)| = \left|\int_{-\infty}^{\infty} q(\xi)p(v - \xi)d\xi\right| \leq \int_{-\infty}^{\infty} |q(\xi)p(v - \xi)|d\xi$$

Therefore,

$$|H(v) - M(v)| \leq \int_{-\infty}^{\infty} G(\xi)p(v - \xi)d\xi = \chi(v)$$  \quad (3.13)

Hence, there exist a solution $M(v)$ of differential Eq.(2.3) such that the inequality (3.13) is always true $\forall v \in \mathbb{R}$. This completes the proof.
4. Stability Results for Non-Homogeneous Linear Equation

In this section, we are going to investigate stability of Non-homogeneous linear differential Eq.(2.4) of order two.

**Theorem 4.1** Let $a$ is a scalar in $\mathbb{F}$ and for every $\epsilon > 0$, there exist $K > 0$ such that $\mathcal{H}:(-\infty, \infty) \to \mathbb{F}$ satisfies the property

\[
|\mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v) - r(v)| \leq \epsilon, \forall v \in \mathbb{R}
\] (4.1)

Then there exist a solution $\mathcal{N}:(-\infty, \infty) \to \mathbb{F}$ of differential Eq.(2.4) such that

\[
|\mathcal{H}(v) - \mathcal{N}(v)| \leq K\epsilon, \forall v \in \mathbb{R}
\]

**proof:** Define a function $q:(-\infty, \infty) \to \mathbb{F}$ such that

\[
q(v) = \mathcal{H}''(v) + a\mathcal{H}'(v) + b\mathcal{H}(v) - r(v), \forall t \in \mathbb{R}
\] (4.2)

Suppose that $|q(v)| \leq \epsilon$. By definition of Fourier transformation and from Eq.(4.2)

\[
F(q(v)) = F(\mathcal{H}''(v)) + aF(\mathcal{H}'(v) + b\mathcal{H}(v)) - f(r(v))
\]

\[
\tilde{Q}() = ((-i\eta)^2) F(\mathcal{H}(v)) - i\eta f(\mathcal{H}(v)) + bF(\mathcal{H}(v)) - \tilde{R}(\eta)
\]

\[
\tilde{\mathcal{H}}(\eta) = \frac{\tilde{Q}(\eta) + \tilde{R}(\eta)}{(-i\eta)^2 + a(-i\eta) + b}
\] (4.3)

Let $u_1, u_2$ are the two distinct roots of the characteristic equation

\[
(-i\eta)^2 + a(-i\eta) + b = 0
\]

then we have,

\[
(-i\eta)^2 + a(-i\eta) + b = (i\eta - u_1)(i\eta - u_2)
\]

Thus

\[
F(\mathcal{H}(v)) = \tilde{\mathcal{H}}(\eta) = \frac{\tilde{Q}(\eta) + \tilde{R}(\eta)}{(i\eta - u_1)(i\eta - u_2)}
\] (4.4)

We now set $\tilde{P}(\eta) = F(p(v)) = \frac{1}{(i\eta - u_1)(i\eta - u_2)}$. By inverse Fourier transform

\[
p(v) = F^{-1}\left(\frac{1}{[i\eta - u_1](i\eta - u_2)}\right)
\] (4.5)

Let us define a function $z:(-\infty, \infty) \to \mathbb{F}$ such that $\mathcal{N}(t) = c_1e^{-u_1t} + c_2e^{-u_2t} + r(v) * p(v)$, where $c_1$ and $c_2$ are constant. we obtain,

\[
F(\mathcal{N}(v)) = \tilde{\mathcal{N}}(\eta) = \int_{-\infty}^{\infty}(c_1e^{-u_1v} + c_2e^{-u_2v})e^{iv\eta} dv + \tilde{R}(\eta)\tilde{P}(\eta) = \tilde{R}(\eta)\tilde{P}(\eta)
\] (4.6)

By taking differentiation property of Fourier transform and In view of Eq.(4.6)

\[
F\left(\mathcal{N}''(v) + a\mathcal{N}'(v) + b\mathcal{N}(v)\right) = ((-i\eta)^2 + a(-i\eta) + b)Z(\eta) = R(\eta)
\]
Since Fourier transform is one to one operator. We get, \( N''(v) + aN'(v) + bN(v) = r(v) \)
Hence, \( z(t) \) is a solution of differential Eq.(2.3). Further, it follows from Eq.(4.4) and (4.6) and convolution property of Fourier transform
\[
F(H(v) - N(v)) = \tilde{N}(\eta) - \tilde{N}(\eta) \\
= \frac{\tilde{Q}(\eta) + \tilde{R}(\eta)}{(i\eta - u_1)(i\eta - u_2)} - \tilde{R}(\eta)\tilde{P}(\eta) \\
= \frac{\tilde{Q}(\eta)\tilde{P}(\eta)}{\tilde{Q}(\eta)\tilde{P}(\eta)} \\
= F(q(v))F(p(v)) \\
= F(q(v) * p(v))
\]
we have, \( H(v) - N(v) = q(v) * p(v) \). Now taking modulus on both sides, we have
\[
|H(v) - N(v)| = |q(v) * p(v)| = \left| \int_{-\infty}^{\infty} q(\xi)p(v - \xi)d\xi \right| \leq \int_{-\infty}^{\infty} |q(\xi)p(v - \xi)|d\xi
\]
We obtain,
\[
|H(v) - N(v)| \leq \epsilon \int_{-\infty}^{\infty} p(v - \xi)d\xi \leq K\epsilon
\]
where \( K = \int_{-\infty}^{\infty} p(v - \xi)d\xi \) for any value of \( v \). The proof is completed.

Theorem 4.2 for ever \( H: (-\infty, \infty) \rightarrow \mathbb{F} \) satisfies
\[
|H''(v) + aH'(v) + bH(v) - r(v)| \leq \phi(v), v \in \mathbb{R}
\]  
(4.7)
there exist a solution \( J: (-\infty, \infty) \rightarrow \mathbb{F} \) of differential Eq.(2.4) such that
\[
|H(v) - J(v)| \leq \chi(v), \quad \forall v \in \mathbb{R}
\]
proof: Let \( H: C^1(\mathbb{R}, \mathbb{F}) \) satisfies (4.7) and define
\[
q(v) = H''(v) + aH'(v) + bH(v) - r(v)
\]  
(4.8)
We have, \( |q(v)| \leq \epsilon \). By definition and property of Fourier transform and from Eq.(4.8)
\[
F(q(v)) = F(H''(v)) + aF(H'(v)) + bF(H(v)) - f(r(v)) \\
\tilde{Q}(\eta) = \left( (-i\eta)^2 \tilde{H}(v) - i\eta aF(H(v)) + bF(H(v)) - \tilde{R}(\eta) \right) \\
\tilde{H}(\eta) = \frac{\tilde{Q}(\eta)\tilde{R}(\eta)}{(i\eta - u_1)(i\eta - u_2)}
\]  
(4.9)
The function \( \tilde{H}(\eta) \) represents Fourier transformation of \( H(v) \). Let \( u_1, u_2 \) are the two roots of the characteristic equation
\[
(-i\eta)^2 + a(-i\eta) + b = 0
\]
then we have,
\[
(-i\eta)^2 + a(-i\eta) + b = (i\eta - u_1)(i\eta - u_2)
\]
Thus
\[
F(H(v)) = \tilde{H}(\eta) = \frac{\tilde{Q}(\eta)\tilde{R}(\eta)}{(i\eta - u_1)(i\eta - u_2)}
\]  
(4.10)
We now set \( \tilde{P}(\eta) = F(p(v)) = \frac{1}{(i\eta-u_1)(i\eta-u_2)} \). By inverse Fourier transform

\[
p(v) = F^{-1}\left(\frac{1}{(i\eta-u_1)(i\eta-u_2)}\right)
\]

(4.11)

define a function \( \mathcal{N} : (\mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{F} \) such that \( \mathcal{N}(v) = c_1e^{-u_1v} + c_2e^{-u_2v} + r(v) \). Taking Fourier transform to \( \mathcal{H}(v) \), we obtain

\[
F(\mathcal{N}(v)) = \tilde{\mathcal{N}}(\eta) = \int_{-\infty}^{\infty}(c_1e^{-u_1v} + c_2e^{-u_2v})e^{i\eta v}dv + \tilde{R}(\eta)\tilde{P}(\eta) = \tilde{R}(\eta)\tilde{P}(\eta)
\]

(4.12)

By differentiation property of Fourier transform and In view of Eq.(4.6)

\[
F(\mathcal{N}''(v) + a\mathcal{N}'(v) + b\mathcal{N}(v)) = ((-i\eta)^2 + a(-i\eta) + b)\tilde{\mathcal{N}}(\eta) = \tilde{R}(\eta)
\]

Since Fourier transform is one to one operator. We get, \( \mathcal{N}''(v) + a\mathcal{N}'(v) + b\mathcal{N}(v) = r(v) \)

Hence, \( \mathcal{N}(v) \) is a solution of differential Eq.(2.4). Further, it follows from Eq.(4.10) and (4.12) and convolution property of Fourier transform

\[
F(\mathcal{H}(v) - \mathcal{N}(v)) = \tilde{\mathcal{H}}(\eta) - \tilde{\mathcal{N}}(\eta) = \frac{\tilde{Q}(\eta) + \tilde{R}(\eta)}{(i\eta-u_1)(i\eta-u_2)} - \tilde{R}(\eta)\tilde{P}(\eta) = \tilde{Q}(\eta)\tilde{P}(\eta) = F(q(v))F(p(v)) = F(q(v)*p(v))
\]

Hence, \( \mathcal{H}(v) - \mathcal{N}(v) = q(v)*p(v) \). Now taking modulus on both sides, we have

\[
|\mathcal{H}(v) - \mathcal{N}(v)| = |q(v)*p(v)| = \left|\int_{-\infty}^{\infty}q(\xi)p(v-\xi)d\xi\right| \leq \int_{-\infty}^{\infty}|q(\xi)p(v-\xi)|d\xi
\]

We obtain,

\[
|\mathcal{H}(v) - \mathcal{N}(v)| \leq \int_{-\infty}^{\infty}\phi(\xi)p(v-\xi)d\xi = \chi(\xi)
\]

Hence, the non-homogeneous linear differential Eq.(2.4) has Hyers-Ulam Rassias stability.

5. Conclusions
This research has made an attempt to analysis the Hyers-Ulam stability of differential equation of second order using Fourier transform method. This paper has also proved Generalized Hyers-Ulam stability of differential equation with constant co-efficients of order two. Hence, the study satisfied the stability in sense of "Hyers-Ulam".

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