Relation between the separable and one-boson-exchange potential for the covariant Bethe-Salpeter equation

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Abstract

We investigate the relation between the rank I separable potential for the covariant Bethe-Salpeter equation and the one-boson-exchange potential. After several trials of the parameter choices, it turns out that it is not always possible to reproduce the phase-shifts calculated from a single term of the one-boson-exchange potential especially of the \( \sigma \)-exchange term, separately by the rank I separable potential. Instead, it is shown that the separable potential is useful to parameterize the total nucleon-nucleon interaction.

1 Introduction

Relativistic approaches of the nuclear physics becomes important for high momentum phenomena, in particular for those associated with spin observables [1]. Furthermore, the phenomena where pions appear as in many nuclear physics phenomena, the relativistic treatment is essential, since the fundamental chiral symmetry is related to the relativistic nature of particles [2].

The starting point of the relativistically covariant theory is the Bethe-Salpeter equation for the two nucleon system [3]. In order to overcome a difficulty of solving the integral equation, a separable interaction is often employed, primarily as a mathematical manipulation [4]. Ignoring a possible dependence on the total momentum \( P = p_1 + p_2 = p'_1 + p'_2 \), where \( p_1 \) and \( p_2 \) are the momenta for the initial two nucleons, while \( p'_1 \) and \( p'_2 \) for the final state ones, the interaction is given as a function of the relative momenta \( p = (p_1 - p_2)/2 \), \( p' = (p'_1 - p'_2)/2 \),

\[
V_{\text{sep}}(p', p) = \lambda g(p') g(p). \tag{1}
\]

This rank I separable potential is non-local and is very much different from the widely used one-boson-exchange potential (OBEP) which is, to the leading order, given as a function of the momentum transfer \( q = p' - p \) and is local:

\[
V_{\text{OBEP}}(p', p) = V(q). \tag{2}
\]

In their form of Eq. (1) and (2), they can not be equivalent, but, instead, higher rank separable interactions may be used to generate the OBEP when infinitely many terms are introduced. Practically, a finite rank (usually up to rank three) potential is used with a finite number of parameters determined in the phase shift analysis [5].

In this paper, we investigate whether the parameters of the separable potential may be related to those of the OBEP, since the latter is considered to be physically more fundamental, at least for longer range part of the NN interaction. By doing this, we
expect that the separable potential can be understood with physics ground, not just a mathematically convenient tool.

In section 2, we briefly formulate the Bethe-Salpeter equation and provide an analytic solution when a rank I separable potential is used for a single channel problem of \( l = 0 \). The rank I separable potential and OBEP are then related in the long wave length approximation. In section 3, we compare the phase shifts calculated from the two potentials. Final section is devoted to conclusions of the present work.

## 2 The BS equation with a separable interaction

Let us consider a single channel equation for \( ^1S_0 \). This is sufficient for our present qualitative discussions. After angular integration, the BS equation is given by [6]

\[
T(p', p; s) = V(p', p) + \frac{i}{4\pi^3} \int dk_0 k^2 dk \frac{V(p', k) T(k, p; s)}{\left( \frac{s^2}{2} - e_k + i\epsilon \right)^2 - k_0^2},
\]

where \( T(p', p; s) \) is the \( T \)-matrix, \( V(p', p) \) the interaction kernel, \( s = (p_1 + p_2)^2 \) and \( e_k = \sqrt{k^2 + M_N^2} \) with \( M_N \) being the mass of the nucleon [6]. The momentum variables are for four momentum, e.g., \( p = (p_0, \vec{p}) \), etc. In Eq. (3), we have considered the equation in the center of mass system. The rank I separable ansatz assumes to write the interaction

\[
V_{\text{sep}}(p', p) = \lambda g(p') g(p)
\]

with a coupling constant \( \lambda \) and a function \( g(p) \) as a scalar function of \( p \) and \( p' \). The \( T \)-matrix is then obtained in the separable form as

\[
T(p', p; s) = \tau(s) g(p') g(p), \quad \tau(s) = \frac{1}{\lambda + h(s)},
\]

where

\[
h(s) = -\frac{i}{4\pi^3} \int dk_0 k^2 dk \frac{g(k)^2}{\left( \frac{s^2}{2} - e_k + i\epsilon \right)^2 - k_0^2}.
\]

Phase shifts are then given by the relation

\[
T(p', p; s) = -\frac{16\pi}{\sqrt{s} \sqrt{s - 4M_N^2}} e^{i\delta} \sin \delta,
\]

where the relative momenta are \( p = (0, \vec{p}) \), \( p' = (0, \vec{p}') \) for on-shell nucleons. Finally the phase shift \( \delta(s) \) can be presented by

\[
\cot \delta(s) = -\frac{\lambda^{-1} + \text{Re}(h(s))}{\text{Im}(h(s))}.
\]

Now important terms of the OBEP can be written as

\[
V_\sigma(q) = -g_\sigma^2 \frac{1}{-q^2 + m_\sigma^2} \left( \frac{\Lambda_\sigma^2 - m_\sigma^2}{\Lambda_\sigma^2 - q^2} \right)^2, \quad V_\omega(q) = g_\omega^2 \frac{1}{-q^2 + m_\omega^2} \left( \frac{\Lambda_\omega^2 - m_\omega^2}{\Lambda_\omega^2 - q^2} \right)^2.
\]
\[ V_\pi(q) = \frac{g_\pi^2}{4M_N^2 - q^2 + m_\pi^2} \left( \frac{\Lambda_\pi^2 - m_\pi^2}{\Lambda_\pi^2 - q^2} \right)^2, \quad V_\rho(q) = \frac{g_\rho^2}{2M_N^2 - q^2 + m_\rho^2} \left( \frac{\Lambda_\rho^2 - m_\rho^2}{\Lambda_\rho^2 - q^2} \right)^2, \]

where \( q = p - p' = (0, \vec{p} - \vec{p}') \). Here the masses \( m_\alpha \), the coupling constants \( g_\alpha \) and the cutoff parameters \( \Lambda_\alpha (\alpha = \sigma, \omega, \pi, \rho) \) are given in Ref. [7], and are summarized in Table 1.

In Eqs. (9) and (10) we picked up the dominant piece of the one boson exchange potential. The higher order terms are proportional to the initial and final relative momenta \( p \) and \( p' \). For the \( \rho \)-exchange potential, we use only the tensor coupling term, where the correction from the vector term is about 5%. The coupling strength given in Table 1 (in the second row) produces only the \( f \)-coupling in Ref. [7].

For the parameterization of the separable potential, we assume the Yukawa function for \( g(p) \) with the same mass parameter \( m \) as in the OBEP, \( g(p) = 1/(p^2 - m_b^2) \). Then we try to impose that \( V_{\text{sep}} \) equals \( V_b \) in the long wave length limit [6]:

\[ V_{\text{sep}}(0, 0) = V_b(0, 0), \]

which determines the strength \( \lambda \). In this way, we have a separable potential approximately related to the OBEP

\[ V_{\text{sep}}(p', p) = \lambda_b \frac{1}{p'^2 - m_b^2} \frac{1}{p^2 - m_b^2}, \]

where \( \lambda_b \) are given by

\[ \lambda_\sigma = -g_\sigma^2 m_\sigma^2 \left( 1 - \frac{m_\sigma^2}{\Lambda_\sigma^2} \right)^2, \quad \lambda_\omega = g_\omega^2 m_\omega^2 \left( 1 - \frac{m_\omega^2}{\Lambda_\omega^2} \right)^2, \]

\[ \lambda_\pi = -g_\pi^2 m_\pi^4 \left( 1 - \frac{m_\pi^2}{\Lambda_\pi^2} \right)^2, \quad \lambda_\rho = -g_\rho^2 m_\rho^4 \left( 1 - \frac{m_\rho^2}{\Lambda_\rho^2} \right)^2. \]

In the case of \( \pi \) and \( \rho \), we excluded \( q^2 \) dependence in the numerator of the Eqs. (15), otherwise we can not determine the \( \lambda \) parameter. It corresponds to excluding the \( \delta \)-function term in the \( r \) space, namely for the Eq. (11) we have used

\[ V_\pi(q) = -\frac{g_\pi^2}{4M_N^2 - q^2 + m_\pi^2} \left( \frac{\Lambda_\pi^2 - m_\pi^2}{\Lambda_\pi^2 - q^2} \right)^2, \quad V_\rho(q) = -\frac{g_\rho^2}{2M_N^2 - q^2 + m_\rho^2} \left( \frac{\Lambda_\rho^2 - m_\rho^2}{\Lambda_\rho^2 - q^2} \right)^2. \]

The numerical values of the \( \lambda \)'s are also given in the Table 1 (last column).

3 Comparison of phase shifts

We have calculated phase shifts using the BS Eqs. (3) – (6), which are compared with those obtained from the OBEP. Since we make the comparison at relatively low energy region, it is sufficient to solve the Schrödinger equation. Here we compare various phase shifts calculated by using a single term corresponding to \( \sigma \)-, \( \omega \)-, \( \pi \)- or \( \rho \)-exchange potentials. The resulting phase shifts are shown in Figs. 1. For later use, we show the one boson exchange potential of the \( ^1S_0 \) channel as a function of \( r \) in Figs. 2, where various terms of the OBEP are shown separately. The thick solid line in Fig. 2(a) is the total potential including the \( \sigma \), \( \omega \), \( \pi \) and \( \rho \) exchange potentials.

Now we discuss the phase shifts calculated from each meson exchange potential.

\( ^1 \)The relations shown here differ from those of Ref. [8], where extra factor of 4\( \pi^2 \) was erroneously included.
Fig. 1(a) shows the phase shifts calculated from the potentials of the $\sigma$ channel as functions of $T_{\text{lab}}$, the kinetic energy in the laboratory frame. Here $T_{\text{lab}}$ is related to $s$ by

$$T_{\text{lab}} = \frac{s - 4M_N^2}{2M_N}. \quad (16)$$

The thick solid line represents the phase shift for the separable potential, and the thin solid line for the OBEP. As shown in Fig. 1(a), both phase shifts start from 180 degrees, indicating a strong attraction as accommodating one bound state. Indeed, as shown in Fig. 2(a), the depth of the $\sigma$-exchange potential of the OBEP reaches about 200 MeV at 0.75 fm. The strong attraction of the OBEP causes the raising behavior at $T_{\text{lab}}=0$, which turns to decreasing at $T_{\text{lab}}=20$ MeV. On the other hand, the separable potential can not be that strongly attractive. This can be checked by analyzing the scattering matrix of Eq. (5), which will be discussed later. In fact, the phase shift approaches the upper limit as indicated by the dashed line in Fig. 1(a) in the limit $\lambda \to -\infty$. Furthermore we have checked that it is not possible to reproduce the strong attraction of the $\sigma$-exchange potential, whatever $m_b$ value of the separable potential we choose. The fact that the separable potential can not be too strong has been discussed previously [11].

Fig. 1(b) shows the phase shifts calculated from the potentials of the $\omega$ channel as functions of $T_{\text{lab}}$. The phase shifts calculated from the two interactions (Separable and OBEP) show repulsive nature as it starts from 0 degree and decreases as the energy increases. As shown in Fig. 2(a), the repulsive force of the $\omega$-exchange potential is very strong. As shown in Fig. 1(b), the result of the separable potential resembles that of OBEP. However, that strong repulsion of the OBEP can not be reproduced by the separable potential, which is similar to the case of the $\sigma$-exchange potential. Once again, as shown by the dashed line in Fig. 1(b) there is a lower bound of the phase shift of the separable potential in the limit $\lambda \to \infty$. However if we allow $m_b$ to change, it is possible to reproduce the phase shift of the $\omega$-exchange potential by using a parameter set of, for instance, $m_b = 630$ MeV and $\lambda = 81.5 \times 10^6$ MeV$^2$. To make $m_b$ small corresponds to the increase of repulsion. At this point, we recognize that the physical meaning of the mass parameter $m_b$ in the separable potential is different from that in the OBEP.

Fig. 1(c) shows the phase shifts calculated from the potentials of the $\pi$ channel as functions of $T_{\text{lab}}$. The phase shifts calculated from the two interactions look very different. Fig. 2(b) shows that $\pi$-exchange potential is attractive at long distances $r > 1$ fm and repulsive at middle and short distances $r < 1$ fm. Therefore the phase shift calculated from OBEP starts from 0 degree, raising at first, then turns to decrease at $T_{\text{lab}} \sim 5$ MeV and becomes repulsive at $T_{\text{lab}} \sim 20$ MeV. Due to the form factor, the one-pion-term of the OBEP here is written as a sum of the long range attraction and the short range repulsion. On the other hand, the phase shift of the separable interaction is weakly attractive. Because there is only one term in the rank I separable potential, the contributions of attraction and repulsion can not be reproduced simultaneously. In particular, the coupling strength $\lambda_{\pi}$ determined from the relation (14) at low momentum region $(p = p' = 0)$ is too attractive, which therefore can not reproduce the repulsive behavior at higher $T_{\text{lab}}$. However, one can fit the repulsive behavior at higher energies by changing the range parameter $m_{\pi}$ and the coupling constant $\lambda$. For instance if we choose $m_{\pi} = \Lambda_{\pi}=1300$ MeV and $\lambda = 135 \times 10^6$ MeV$^2$, we can reproduce the phase shift at around $T_{\text{lab}} \approx 200$ MeV as indicated by the dashed line of Fig. 1(c).
Fig. 1-(d) shows the phase shifts calculated from the potentials of the $\rho$ channel as functions of $T_{lab}$. The phase shifts calculated from the two interactions look very different. With $\lambda = -37.2 \times 10^6$ MeV$^2$ which is determined by Eq. (14), the result of the separable potential is too attractive, such that it generates one bound state and the phase shift starts from 180 degrees. In contrast, as shown in Fig. 2-(b) the $\rho$-exchange piece of the OBEP is attractive at long distances $r > \sim 0.6$ fm and repulsive at middle and short distances $r < \sim 0.6$ fm. The attractive interaction here, however, is not very large due to the cancellation by the repulsive component. Therefore the phase shift calculated from OBEP starts raising from 0 degree, turns to decrease at $T_{lab} \sim 40$ MeV, and change into repulsion at $T_{lab} \sim 160$ MeV. Just as in the $\pi$ case, we can re-fit the strength of the separable potential $\lambda_{\rho}$. By reducing the strength by about factor 8, $\lambda = -4.52 \times 10^6$ MeV$^2$, we obtain the phase shift of the separable potential as shown by the dashed line of Fig. 1-(d), which looks rather close to the result of OBEP.

These results show that it is difficult to reproduce the phase shifts of each terms of the one-boson-exchange potential separately by the rank I separable potential when we use the parameters determined in the long wave length limit. As explained above in detail, the separable potential can not be stronger than a certain strength both for attractive and repulsive cases if we do not change the $m_b$ parameter.

The fact that the separable potential can not be stronger than a certain strength may be understood from Eq. (5) where the factor $1/\lambda$ vanishes in the limit $|\lambda| \to \infty$. Interestingly, in this limit there is no distinction between attractive ($\lambda \to -\infty$) and repulsive ($\lambda \to +\infty$) interactions. In order to find the maximum strength of the separable potential, we plot in Fig. 4 the real and imaginary parts of $h(s)$ from which we can calculate the phase shift by using Eq. (8). The result is shown in Fig. 4. The real part monotonically decreases from 0.276 GeV$^{-2}$, while the imaginary part starts from 0, reaches the maximum value at some $s$ and turns to decrease monotonically. This behavior resembles what is familiar in the non-relativistic scattering theory where the phase shift varies from 0 to 180 degrees when there is one bound state. In a relativistic theory, however, a naive argument in the non-relativistic theory can not be applied, since in the large $s$ region particle production may occur and the discussion within a fixed particle number can not be applied. In the present separable potential model, the treatment will break down at and beyond $s = 4(M_N+m_b)^2$ where an unphysical pole of mass $m_b$ appears. In our calculation of the phase shift, in order to determine the initial value $\delta(T_{lab} = 0)$, for the attractive interaction we increased $\lambda$ gradually from a small value and verified that there is a jump from $\delta(T_{lab} = 0) = 0$ to $\delta(T_{lab} = 0) = 180$ degrees at certain strength of $\lambda$ only once. Therefore, we conclude that the maximum strength of the separable potential in our method is what allows one bound state for an attractive interaction. Similarly for the repulsive case, it is also possible to show that there is the maximum strength of the interaction if $m_b$ is fixed.

Now turning to the full result of the $^1S_0$ channel, as indicated in Fig. 3, the separable potential can reproduce rather well the result of the total nuclear force of OBEP when we take $\lambda = -0.294 \times 10^6$ MeV$^2$ and $m_b = 224$ MeV [6]. The very strong attractive and repulsive forces of the $\sigma$- and $\omega$-exchange potentials are largely canceled, yielding a rather mild nuclear force. This is the reason that the separable potential for nuclear reaction have been successful.
Figure 1: Phase shifts of the $^1S_0$ channel calculated from the separable potentials (thick solid lines) and those from the OBEP (thin solid lines). Dashed lines (a) and (b) represent the upper limit and the lower limit of the phase shift calculated from the separable potential with $|\lambda| \rightarrow \infty$. The dashed line of (c) represents the phase shift fitted to the phase shift of the OBEP around $T_{lab} \approx 200$MeV. The dashed line of (d) represents the phase shift calculated by the separable potential which fits the phase shift of the OBEP around $T_{lab} \approx 0$ MeV.
Figure 2: The separable contributions to the Bonn potential of the $^1S_0$ channel from the various meson exchange terms as denoted by the labels. (a): Thick solid, solid and thin dashed lines are for the total nuclear force, $\sigma$- and $\omega$-exchange potentials, respectively. (b): Dashed and dotted lines are $\pi$- and $\rho$-exchange potentials of the $^1S_0$ channel, respectively.

Figure 3: The phase shifts of the $^1S_0$ channel calculated from the best fitted separable potential as a function of the kinetic energy in the laboratory frame as compared with the experimental data (data are taken using SAID program [http://gwdac.phys.gwu.edu/]).
Figure 4: The real and imaginary part of $h(s)$ when $m_b = 500$(MeV) as a function of the total momentum square.
Table 1: Parameters of the OBEP from the Bonn potential \[7\].

|   | $m_b$(MeV) | $g^2/4\pi$ | $\Lambda_b$(MeV) | $\lambda$(MeV$^2$) |
|---|------------|------------|------------------|------------------|
| $\sigma$ | 550       | 7.78       | 2000             | $-25.3 \times 10^6$ |
| $\rho$  | 769       | 34.77      | 1300             | $-37.2 \times 10^6$ |
| $\omega$ | 783       | 20.0       | 1500             | $81.5 \times 10^6$  |
| $\pi$   | 138       | 14.9       | 1300             | $-0.0189 \times 10^6$ |

4 Summary

We have studied the relation between the rank I separable potential for the covariant Bethe-Salpeter equation and the one-boson-exchange potential (OBEP). Individual channels of $\sigma$, $\omega$, $\pi$- and $\rho$-exchanges were investigated separately. As a result, it turned out that the rank I separable potential could not reproduce the phase shift calculated from each component of the OBEP when we use the parameters determined in the long wave length limit. As for the $\sigma$ channel, where the potential is strongly attractive, we could not reproduce the phase shift of OBEP even if we take the limit $\lambda \to -\infty$ and we change $m_b$ parameter. Similarly as for the $\omega$ channel with strong attraction, the separable potential could not reproduce again the phase shift of OBEP even in the limit $\lambda \to \infty$. However we could reproduce the strong repulsion, if we change the $m_b$ parameter. These observations imply that the physical meaning of the mass parameters in the separable potential and OBEP are different. The mass parameter of the OBEP represents the interaction range of a local potential, while that of the separable potential could mimic, for instance, the range of a non-local interaction. The non-locality of the nuclear force is related to the structure of the nucleon at short ranges $r < \sim 0.5$ fm \[9\]. Concerning the $\pi$ and $\rho$ channels, where the potential consists of attraction at long distances and repulsion at short distances, the rank I separable potential could not reproduce the mixed nature of the interaction, although the interaction strengths are not as strong as the $\sigma$ and $\omega$ channels. Despite the above fact, the rank I separable potential can reproduce the experimental data of the $^1S_0$ phase shift up to the energy $T_{lab} > 200$ MeV where a mild attractive interaction dominates.

These results show that the rank I separable potential is not suited to the description of very strong attraction. For instance, phase shifts calculated from the separable potential can not become larger than 180 degrees, no matter how large the attraction coupling constant takes. Rather, the separable potential can describe relatively mild attraction and all repulsion. In the realistic nuclear force, such a mild strength is obtained by the sum of the strongly attractive $\sigma$-exchange and the strongly repulsive $\omega$-exchange potentials.

In this work, we have shown that the separable potential works well for the two nucleon system if parameters are chosen suitably, although the decomposition into components of physical OBEP does not make sense. In a sense, the different nature of the two potential should have been expected. The main purpose of the present paper was to see whether it is possible to make physical meaning of the separable potential in comparison with the OBEP by using a simple parameterization of one term of rank I. In order to perform a good description of phenomena in the covariant Bethe-Salpeter formalism, we can introduce a higher rank form. Such a work is now in progress, where the use of the improved rank one ansatz and of higher rank interactions are tested \[10\].
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