Dissipative spinodal decomposition in systems with nonreciprocal effective interactions

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Abstract. Reciprocity of action and reaction, well known as the third Newton's law, can be violated in open complex systems of different nature. We analyze the dynamic behavior of the binary system with variable nonreciprocal interactions. We demonstrate that the resulting state in these systems is determined by interplay between dissipation and energy release (due to the interaction nonreciprocity), allowing bi-stability and corresponding dissipative phase transitions. The results of the Article shed light to a role of interactions nonreciprocity, intrinsic for different classes of open dynamical systems, from chemical physics and materials science to multi-agent systems and financial markets.

1. Introduction

Open systems of various kinds are intrinsic for physics, chemistry, materials sciences, biology, and even sciences which deal with multi-agent systems, including neuron nets, artificial intelligence, and financial markets. Owing to its interdisciplinarity, understanding of principal mechanisms and drivers responsible for collective phenomena in both inanimate and live systems, like self-organization, synchronization, dissipative phase transitions and criticality, is important for a wide range of problems in statistical physics and a lot of related areas. In particular, it is valuable for material science in tasks related to the creation of new materials with unique properties for example for photonics problems [1–6].

Interactions between individual particles in open or non-equilibrium complex systems can be non-conservative, that is related with a visible violation of the third Newton's law, one of the basic cornerstones of equilibrium mechanics: The forces of action and reaction have an equal magnitude and the opposite directions. The brake of the action-reaction symmetry results in fundamentally changed dynamics of the system[7–9].
Nonreciprocal interactions between particles can be determined by their internal state or by local environment. Figure 1 illustrates the “nonreciprocal World” of such systems. As the simplest examples, one can mention colloidal suspensions and complex plasmas Fig. 1(a) with interaction driven by flows [7, 10–12], non-equilibrium fluctuations [13], optical beams [14], environment flows [10–12], diffusiophoresis [15–18], shadows and wakes in complex plasmas [8, 19–25]. Systems of self-moving particles, whose dynamics is determined by the local environment [26–29], are next in complexity. Owing to nonreciprocity of interactions, the systems exhibit complex collective behavior, for example in bacteria systems [30–32], Fig. 1(b). Nonreciprocity plays an important role in behavior of large groups of animals, despite the “interaction” has a completely different nature. The most clearly they manifest themselves in the formation of swarms, flocks [33–37] or other groups [38–40], shown sketchily in Figs. 1(c-f). Moreover, non-reciprocity is an important feature of humans’ behavior in crowds [41], pedestrian dynamics [42–44] or panic behavior [45], as schematically shown in Fig. 1(g). Additionally to fundamental significance for statistical physics, nonreciprocity appears in such practically-important systems as distributed self-organized networks of mobile agents (for instance, sensors) [35, 46, 47], automated delivery of payloads [35], traffic management of unmanned vehicles [48]. In spite of generalized coordinates and forces in these systems have different nature and specificity, their dynamics is well described by equation of Langevin’s type, that means that collective phenomena can occur in similar manner.

In present Article, we study an influence of nonreciprocal interactions on statistical mechanics and collective phenomena in dynamical systems. We show that radial-dependent nonreciprocity leads to heating of the system until it reaches a stationary state determined by competition between this energy source and dissipation. If nonreciprocity parameter changes with the distance between particles, a set of dynamical equilibrium points arises. As a result arise conditions for dissipative phase transitions of the first kind, which have never been studied in such systems yet, to our knowledge.

Figure 1. The ‘nonreciprocal World’: Examples of systems with broken the third Newton’s law, schematically depicted in the center. (a) Complex plasmas and colloidal suspensions in flows; (b) bacterial colonia, (c) birds flocks, (d) swarms of insects, (e) school of fish, (f) various systems of victims and predators, (g) human crowds, (h) networks, financial markets, epidemic spreading, neuron systems etc.
2. Methods

To analyze the behavior of the system with nonreciprocal interactions we performed molecular dynamic (MD) simulation, which is a standard tool for many-particle modeling [8–10, 49–53]. We consider a two-dimensional system consisting of \( N = 2 \times 10^4 \) particles separated into two equal subsystems 1 and 2. The motion of the particles corresponds to Langevin dynamics:

\[
m\ddot{\mathbf{r}}_i = \mathbf{F}_i - m\tau^{-1}\dot{\mathbf{r}}_i + \sqrt{2mT_{th}/\tau}\mathbf{R}_i(t),
\]

(1)

where \( i \) is particle index, \( \mathbf{F}_i \) is the total interaction force acting on the \( i \)-th particle, \( \tau \) and \( T_{th} \) are the damping time and the thermostat temperature respectively and \( \mathbf{R}_i(t) \) is the stochastic force satisfying the conditions:

\[
\langle \mathbf{R}_i(t) \rangle = 0, \quad \langle R_{\alpha\beta}(t)R_{\gamma\beta}(t') \rangle = \delta_{\alpha\gamma}\delta_{\beta\delta}(t - t'),
\]

(2)

where \( i, j \) are the indices of different particles, \( \alpha \) and \( \beta \) are the spatial projection indices, and \( \delta(t) \) denotes the Dirac delta function. Force \( \mathbf{F}_i \) of interparticle interaction is determined in accordance with expression:

\[
\mathbf{F}_i = -\sum_{j\neq i} (1 + \Delta_{s's}(r)) \frac{\partial \varphi(|\mathbf{r}_j - \mathbf{r}_i|)}{\partial \mathbf{r}_i}
\]

(3)

where \( \varphi(r) \) potential energy of reciprocal part of interaction, \( \Delta_{s's}(r) \) is function characterizing nonreciprocity, \( s' \) and \( s \) are indexes of \( j \) and \( i \) particles subsystems respectively i.e., they can take on values 1 or 2 depending on belonging of particle to correspond subsystem. We chose following explicit forms of \( \varphi(r) \) and \( \Delta_{s's}(r) \) function:

\[
\varphi(r) = \begin{cases} 
\epsilon(1 - r/r_0)^2/2, & r < r_0; \\
0, & r \geq r_0;
\end{cases}
\]

(4)

\[
\Delta_{s's}(r) = (1 - r/r_0)\text{sign}(s'-s)
\]

(5)

where \( r_0 \) and \( \epsilon \) are spatial scale and magnitude of interaction. All simulations by molecular dynamic method were performed in LAMMPS simulation package [54–55]. The numbers of time steps in different simulations were ranged from \( 4 \times 10^6 \) to \( 2 \times 10^8 \) with constant time step value \( \Delta t = 5 \times 10^{-3}\sqrt{mr_0^2/\epsilon} \). All results presented in units of \( r_0, \epsilon, \sqrt{mr_0^2/\epsilon} \) for length, energy, and time values respectively. The simulations and analysis were performed at different damping times \( \tau \) and thermostat temperatures \( T_{th} \).

In addition to direct MD simulation, we implement the balance approach to analyze steady states [9], which based on comparing the powers of the energy source and dissipation in the system. In the steady state, energy release compensated by dissipation:

\[
N_1P_1 + N_2P_2 - (N_1 + N_2)P_{NR} = 0,
\]

(6)

where \( N_{1,2} \) are the number of particles in the subsystems, \( P_{1,2} = 2\tau^{-1}(T_{1,2} - T_{th}) \) is the energy dissipation power per particle in subsystems 1 and 2, \( P_{NR} \) is the power of energy release per particle due to the nonreciprocity of interactions. In the simulations, we observed that in a steady state both subsystems have nearly Maxwellian velocity distributions with effective temperatures \( T_1 \) and \( T_2 \) of the subsystems. Moreover, the ratio of kinetic energy \( K_1 \) of subsystem 1 to kinetic energy \( K_2 \) of subsystem 2 is weakly influenced by the thermostat (if \( \min\{T_1, T_2\} > T_{th} \)) and mainly determined by the average kinetic energy of the system \( K \) [9].

To calculate \( P_{NR}(K) \), we firstly determine the dependence \( K_1/K_2 \) on the average energy \( K \) at \( \tau \to \infty \). After what we performed simulations with separate thermostating of subsystems
Figure 2. Results for a binary system with the radially-dependent nonreciprocity of interactions. (a) Time evolution of the average kinetic energy $K(t)$ in the system (at different values of the relaxation time $\tau$) from the initial states with low and high energies, shown by solid and dashed lines, respectively. (b) Comparison of the energy powers caused by nonreciprocity $P_{NR}$ and interaction with thermostat $P_D$. $P_{NR}$ is shown by the black dashed line, while the colored solid lines are $P_D$ at different relaxation time $\tau$. (c) The inset demonstrates the ratio of kinetic energy $K_1$ of subsystem 1 to kinetic energy $K_2$ of subsystem 2 at different average kinetic energy $K$ in the system. (d) Average kinetic energy $K$ in steady states obtained at different relaxation time $\tau$. Blue dashed and red solid lines correspond to the MD calculations and balance approach. Thermostat temperature is $T_{th} = 10^{-2}$, the relaxation times $\tau$ are notified in the legend in panel (d), labeled by letters A-H and colored in the same manner over the figures (a), (b), and (d).

at the temperatures $T_1$ and $T_2$ such that $T_1/T_2 = K_1/K_2$ where ratio $K_1/K_2$ determined by $K$ in accordance with the result of the previous step. Finally, the energy source power can be calculated in MD as:

$$P_{NR} = \langle \mathbf{v} \cdot \mathbf{F} \rangle.$$  \hspace{1cm} (7)

where $\mathbf{v}$ is the velocity of the system particle and $\mathbf{F}$ is force acting on it, $\langle ... \rangle$ denote averaging over the ensemble.

3. Results

In Fig. 2 are summarized results obtained at thermostat temperature $T_{th} = 10^{-2}$. Evolution of the average kinetic energy $K(t)$ starting from ‘cold’ and ‘hot’ initial states at different relaxation time $\tau$ of the thermostat is shown in Fig. 2(a) by solid and dotted lines respectively. The purple curve $A$, corresponding to the non-dissipative case $\tau \to \infty$ ($\gamma = 0$). Notice that exist the region of finite $\tau$ where the steady state becomes depending on the initial state as is shown by lines D–G in Fig. 2(a), i.e. there is bistability. As result, the hysteresis loop are formed on the dependence of asymptotic kinetic energy values from $\tau$ as it seen in Fig. 2(d).

The time evolution of the energy $K(t)$ is determined by competition between the powers of the energy release $P_{NR}$ due to nonreciprocity and dissipation $P_D$. The dependencies of power $P_{NR}$ (dashed black line) and $P_D$ (lines colored accordingly to $\tau$) from average kinetic energy $K$ are presented at Fig. 2(b). Steady states are arise where $P_{NR} = P_D$ and $P_D$ increase faster when $P_{NR}$ with $K$ increasing. At large and low $\tau$ values $P_D$ and $P_{NR}$ curves intersect only once forming the stable states. However, at intermediate values of $\tau$ (cases E and F) the points of tangency arise, as shown in Fig. 2(b). In the range of $\tau$ between these two cases of tangency, the curves $P_D$ and $P_{NR}$ intersect in three points, two of which are stable and the middle one is
Figure 3. Dissipative phase transitions in the system with nonreciprocal interaction
(a) Dependencies of the steady state energy $K$ on the relaxation time $\tau$ at different temperatures of thermostat $T_{th}$. The dissipative “spinodal” is shown by the dashed red line with critical point (CP) in the vertex, while the dashed black line is the averaged kinetic energy at activated and non-activated branches of the spinodal. The legend provides the temperatures of thermostat $T_{th}$, the lines are colored and marked by letters A-F accordingly. (b) Dependencies of the entropy production rate $\dot{S}$ on the relaxation time $\tau$ at different temperatures of thermostat $T_{th}$. The dissipative spinodal are justified by area of bi-stability

The evolution $K(t)$ at different $\tau$ and discussed hysteresis behavior of steady states are summarized in Fig. 2(d), where the blue symbols are MD results, blue dashed line is their interpolation. The solid red line is given by the power balance, while the points B-H mark the steady states shown in Fig. 2(b). Note that the balance method yields a remarkable accuracy of the determined asymptotic steady states.

The observed behavior of steady states can be interpreted as a dissipative phase transition, that provides a remarkable analogy between the nonreciprocal systems and, e.g., classical van-der-Waals theory of spinodal decomposition. This analogy is illustrated in Fig. 3(a), where the steady states in $\tau - K$ plane are shown at different $T_{th}$. With an increase in the thermostat temperature $T_{th}$, energy dissipation and the width of the hysteresis loop decrease, and dynamic decomposition vanishes at value $T_{CP}$, which plays a role of critical thermostat temperature. Lines corresponding to different $T_{th}$ play a role of dissipative isotherms in $\tau - K$ coordinates, presented in Fig. 3(a). The limit points at each isotherm form a line of dissipative spinodal with the vertex in the dissipative critical point (CP).

Inspired by this analogy of the observed dissipative phase transitions with Van-der-Waals theory, we analyzed the dissipative spinodal in vicinity of the critical point and observed that it is well described by the power law (see Fig. 4 for details):

$$K_+ - K_- \propto (\tau - \tau_{CP})^\beta,$$

where $\tau_{CP}$ is the critical value of the relaxation time, $K_\pm$ are the kinetic energies of activated and non-activated steady states denoted by $\pm$ indices, respectively. The parameters of the critical point are determined by particular form of the nonreciprocal interaction, similarly to the theory of phase transitions.

Apart from kinetic energy, the hysteresis dissipative behavior manifests in entropy production rate, $\dot{S}$. Since the powers of energy release per particle in the subsystems are $Q_{1,2} =
Figure 4. Near-critical behavior of energy. Energy decomposition $\Delta K$ on the dissipative spinodal near the critical value of relaxation time $\tau_{CP}$. Blue symbols are results of MD simulations, orange lines are fits by Eq. 8, the vertical orange dashed line indicates position of $\tau_{CP}$.

\[2\tau^{-1}(T_{1,2} - T_{th}) + P_{NR},\] for the rate $\dot{S} = \dot{Q}_1/T_1 + \dot{Q}_2/T_2$ we have

\[\frac{dS}{dt} = 4\tau^{-1} + (P_{12} - 4\tau^{-1}T_{th}) \left( \frac{1}{T_1} + \frac{1}{T_2} \right). \tag{9}\]

Dependencies $\dot{S}(\tau)$ are shown in Fig. 3(b) for the same temperatures $T_{th}$ as in Fig. 3(a). One can see that the dependencies justify the hysteresis behavior, existence of bistability area, dissipative phase transitions and critical point. The activated and inactivated states correspond to the high or low rate of entropy production. This also provides analogy with high-entropy (strongly disordered, gas) and low-entropy (fluid, more ordered) states. One can see in Fig. 3(b) that the discrepancy between $\dot{S}(\tau)$ given by the balance approach and MD results increases significantly at small $\tau$, while the energy $K$ in the system is close to the temperature of Langevin thermostat $T_{th}$. In this case, the kinetic energy of the ‘cold’ subsystem becomes smaller than $T_{th}$, deviations from Maxwellian velocity distribution in the subsystem increase, that enhances the error of $P_{NR}$ given by the balance approach.

4. Conclusions

In conclusion, we have studied the dynamics of systems with nonreciprocal effective interactions between particles. Shown that in case of radially-dependent nonreciprocity there are areas of bistability, i.e. exist the specific range of $\tau$ values (at low thermostat temperatures) where the average kinetic energy of a steady state depends on the initial state and hysteresis loop is observed. At the same time, in limits of low and high values of $\tau$ bistability is absent. Moreover, the range of $\tau$ values corresponding to bistability decrease with thermostat temperature $T_{th}$ increasing and vanish at the critical value. As a result, in the $\tau$–$K$ plane formed the dissipative spinodal line with the dissipative critical point. Such complex behavior is explained by the complicated dependence of powers of the energy source and dissipation on the average kinetic energy of the system.
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References
[1] Yakovlev E V, Konarova K A, Zaytsev K I, Kryuchkov N P, Koshelev I K, Zotov A K, Shelestov D A, Tokstoguzov V L, Kurlov V N, Ivlev A V and Yurchenko S O 2017 Scientific Reports 7 13727 URL https://www.nature.com/articles/s41598-017-14001-y
[2] Yurchenko S O, Zaytsev K I, Gorbunov E A, Yakovlev E V, Zotov A K, Masalov V M, Emelchenko G A and Gorelik V S 2017 Journal of Physics D: Applied Physics 50 055105 URL http://stacks.iop.org/0022-3727/50/i=5/a=055105
[3] Zaytsev K I and Yurchenko S O 2014 Applied Physics Letters 105 051902 URL http://scitation.aip.org/content/aip/journal/apl/105/5/10.1063/1.4852383
[4] Zaytsev K I, Katyba G M, Yakovlev E V, Zotov A K, Masalov V M, Emelchenko G A and Gorelik V S 2014 Journal of Applied Physics 115 213505 URL http://scitation.aip.org/content/aip/journal/jap/115/21/10.1063/1.4880299
[5] Niu R and Palberg T 2018 Soft Matter 14(18) 3435–3442 URL http://dx.doi.org/10.1039/C8SM00256H
[6] Niu R, Özgüç E C, Müller H, Reinmüller A, Botin D, Löwen H and Palberg T 2017 Phys. Chem. Chem. Phys. 19(4) 3104–3114 URL http://dx.doi.org/10.1039/C6CP07231C
[7] Hayashi K and ichi Sasa S 2006 Journal of Physics: Condensed Matter 18 2825 URL http://stacks.iop.org/0953-8984/18/i=10/a=008
[8] Ivlev A V, Bartnick J, Heinen M, Du C R, Nosenko V and Löwen H 2015 Phys. Rev. X 5(1) 011035 URL https://link.aps.org/doi/10.1103/PhysRevX.5.011035
[9] Kryuchkov N P, Ivlev A V and Yurchenko S O (to be published)
[10] Dzubiella J, Löwen H and Likos C N 2003 Phys. Rev. Lett. 91(24) 248301 URL https://link.aps.org/doi/10.1103/PhysRevLett.91.248301
[11] Sriram I and Furst E M 2012 Soft Matter 8(12) 3335–3341 URL http://dx.doi.org/10.1039/C2SM06784F
[12] Mejia-Monasterio C and Oshanin G 2011 Soft Matter 7(3) 993–1000 URL http://dx.doi.org/10.1039/C0SM00465K
[13] Buenzli P R and Soto R 2008 Phys. Rev. E 78(2) 020102 URL https://link.aps.org/doi/10.1103/PhysRevE.78.020102
[14] Dholakia K and Zemánek P 2010 Rev. Mod. Phys. 82(2) 1767–1791 URL https://link.aps.org/doi/10.1103/RevModPhys.82.1767
[15] Sabass B and Seifert U 2010 Phys. Rev. Lett. 105(21) 218103 URL https://link.aps.org/doi/10.1103/PhysRevLett.105.218103
[16] Soto R and Golestanian R 2014 Phys. Rev. Lett. 112(6) 068301 URL https://link.aps.org/doi/10.1103/PhysRevLett.112.068301
[17] Keh H J 2016 Current Opinion in Colloid & Interface Science 24 13 – 22 ISSN 1359-0294 URL http://www.sciencedirect.com/science/article/pii/S1359029416300553
[18] Bartnick J, Heinen M, Ivlev A V and Löwen H 2016 Journal of Physics: Condensed Matter 28 025102 URL http://stacks.iop.org/0953-8984/28/i=2/a=025102
[19] Tsytovich V N 1997 Phys. Usp. 40 53–94 URL http://ufn.ru/en/articles/1997/1/1/
[20] Chaudhuri M, Ivlev A V, Krhapak S A, Thomas H M and Morfill G E 2011 Soft Matter 7(4) 1287–1298 URL http://dx.doi.org/10.1039/C0SM00813C
[21] Morfill G E and Ivlev A V 2009 Rev. Mod. Phys. 81(4) 1353–1404 URL https://link.aps.org/doi/10.1103/RevModPhys.81.1353
[22] Vaulina O S, Lisina I I and Lisin E A 2015 EPL (Europhysics Letters) 111 50003 URL http://stacks.iop.org/0295-5075/111/i=5/a=50003
[23] Yurchenko S O, Yakovlev E V, Couédel L, Kryuchkov N P, Lipaev A M, Naumkin V N, Kislov A Y, Ovcharov P V, Zaytsev K I, Vorob’ev E V, Morfill G E and Ivlev A V 2017 Phys. Rev. E 96(4) 043201 URL https://link.aps.org/doi/10.1103/PhysRevE.96.043201
[24] Bartnick J, Kaiser A, Löwen H and Ivlev A V 2016 The Journal of Chemical Physics 144 224901 URL https://link.aps.org/doi/10.1063/1.4952252
[25] Kryuchkov N P, Yakovlev E V, Gorbunov E A, Couédel L, Lipaev A M and Yurchenko S O 2018 Phys. Rev. Lett. 121(7) 075003 URL https://link.aps.org/doi/10.1103/PhysRevLett.121.075003
[26] Vicsek T, Czirók A, Ben-Jacob E, Cohen I and Shochet O 1995 Phys. Rev. Lett. 75(6) 1226–1229 URL https://link.aps.org/doi/10.1103/PhysRevLett.75.1226
[27] Mahapatra P S, Kulkarni A, Mathew S, Panchagnula M V and Vedantam S 2017 Phys. Rev. E 95(6) 062610 URL https://link.aps.org/doi/10.1103/PhysRevE.95.062610
[28] Bricard A, Caussin J B, Desreumaux N, Dauchot O and Bartolo D 2013 Nature 503 95–98 ISSN 0028-0836 letter URL http://dx.doi.org/10.1038/nature12673
[29] Nagai K H, Sumino Y, Montagne R, Aranson I S and Chaté H 2015 Phys. Rev. Lett. 114(16) 168001 URL https://link.aps.org/doi/10.1103/PhysRevLett.114.168001
[30] Dombrowski C, Cisneros L, Chatkaew S, Goldstein R E and Kessler J O 2004 Phys. Rev. Lett. 93(9) 098103 URL https://link.aps.org/doi/10.1103/PhysRevLett.93.098103
[31] Zhang H P, Be'er A, Florin E L and Swinney H L 2010 Proceedings of the National Academy of Sciences 107 13626–13630 URL http://www.pnas.org/content/107/31/13626.abstract
[32] Marchetti M C, Joanny J F, Ramaswamy S, Liverpool T B, Prost J, Rao M and Simha R A 2013 Rev. Mod. Phys. 85(3) 1143–1189 URL https://link.aps.org/doi/10.1103/RevModPhys.85.1143
[33] Hildenbrandt H, Carere C and Hemelrijk C 2010 Behavioral Ecology 21 1349–1359 URL http://dx.doi.org/10.1093/beheco/arq149
[34] Levine H, Rappel W J and Cohen I 2000 Phys. Rev. E 63(1) 017101 URL https://link.aps.org/doi/10.1103/PhysRevE.63.017101
[35] Olfati-Saber R 2006 IEEE Transactions on Automatic Control 51 401–420 ISSN 0018-9286
[36] Nagy M, Akos Z, Biro D and Vicsek T 2010 Nature 464 890–893 ISSN 0028-0836 URL http://dx.doi.org/10.1038/nature08891
[37] Ballerini M, Cabibbo N, Candelier R, Cavagna A, Cisbani E, Giardina I, Lecomte V, Orlandi A, Parisi G, Procaccini A, Viale M and Zdravkovic V 2008 Proceedings of the National Academy of Sciences 105 1232–1237 URL http://www.pnas.org/content/105/4/1232.abstract
[38] Ferdinandy B, Ozogány K and Vicsek T 2017 Physica A: Statistical Mechanics and its Applications 479 467 – 477 ISSN 0378-4371 URL http://www.sciencedirect.com/science/article/pii/S0378437117302777
[39] Couzin I D, Krause J, Frank S R and Levin S A 2005 Nature 433 513–516 ISSN 0028-0836 URL http://dx.doi.org/10.1038/nature03236
[40] Lihoreau M, Charleston M A, Senior A M, Clissold F J, Raubenheimer D, Simpson S J and Buhl J 2017 Philosophical Transactions of the Royal Society of London B: Biological Sciences 372 ISSN 0962-8436 URL http://rstb.royalsocietypublishing.org/content/372/1727/20160238
[41] Helbing D and Molnár P 1995 Phys. Rev. E 51(5) 4282–4286 URL https://link.aps.org/doi/10.1103/PhysRevE.51.4282
[42] Moussaïd M, Helbing D and Theraulaz G 2011 Proceedings of the National Academy of Sciences 108 6884–6888 URL http://www.pnas.org/content/108/17/6884.abstract
[43] Karamouzas I, Skinner B and Guy S J 2014 Phys. Rev. Lett. 113(23) 238701 URL https://link.aps.org/doi/10.1103/PhysRevLett.113.238701
[44] Helbing D, Farkas I and Vicsek T 2000 Nature 407 478–480 ISSN 0028-0836 URL http://dx.doi.org/10.1038/35035023
[45] Cortés J and Bullo F 2005 SIAM Journal on Control and Optimization 44 1543–1574 URL https://doi.org/10.1137/S0363012903428652
[46] Akyildiz I F, Su W, Sankarasubramaniam Y and Cayirci E 2002 IEEE Communications Magazine 40 102–114 ISSN 0163-6804 URL http://ieeexplore.ieee.org/document/1024422
[47] Dong X, Yu B, Shi Z and Zhong Y 2015 IEEE Transactions on Control Systems Technology 23 340–348 ISSN 1063-6536 URL http://ieeexplore.ieee.org/document/6798711/
[48] Ovcharov P V, Kryuchkov N P, Zaytsev K I and Yurchenko S O 2017 The Journal of Physical Chemistry C 121 26860–26868 URL https://link.aps.org/doi/10.1021/acs.jpcc.7b09317
[49] Khrapak S A, Kryuchkov N P and Yurchenko S O 2018 Phys. Rev. E 97(2) 022616 URL https://link.aps.org/doi/10.1103/PhysRevE.97.022616
[50] Kryuchkov N P, Yurchenko S O, Fomin Y D, Tsioik E N and Ryzhov V N 2018 Soft Matter 14(11) 2152–2162 URL http://dx.doi.org/10.1039/C7SM02429X
[52] Yurchenko S O, Komarov K A, Kryuchkov N P, Zaytsev K I and Brazhkin V V 2018 The Journal of Chemical Physics 148 134508 URL https://doi.org/10.1063/1.5022969

[53] Kryuchkov N P, Khrapak S A and Yurchenko S O 2017 The Journal of Chemical Physics 146 134702 URL https://doi.org/10.1063/1.4979325

[54] http://lammps.sandia.gov URL http://lammps.sandia.gov

[55] Plimpton S 1995 Journal of Computational Physics 117 1 – 19 ISSN 0021-9991 URL http://www.sciencedirect.com/science/article/pii/S002199918571039X