A POSTERIORI ERROR ESTIMATORS FOR STABILIZED FINITE ELEMENT APPROXIMATIONS OF AN OPTIMAL CONTROL PROBLEM

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Abstract. We derive a posteriori error estimators for an optimal control problem governed by a convection–reaction–diffusion equation; control constraints are also considered. We consider a family of low–order stabilized finite element methods to approximate the solutions to the state and adjoint equations. We obtain a fully computable a posteriori error estimator for the optimal control problem. All the constants that appear in the upper bound for the error are fully specified. Therefore, the proposed estimator can be used as a stopping criterion in adaptive algorithms. We also obtain a robust a posteriori error estimator for when the error is measured in a norm that involves the dual norm of the convective derivative. Numerical examples, in two and three dimensions, are presented to illustrate the theory.

Key words. linear–quadratic optimal control problem; convection–reaction–diffusion equation; stabilized methods; fully computable a posteriori error estimator; robust a posteriori error estimator.

AMS subject classifications. 49K20, 49M25, 65K10, 65N15, 65N30, 65N50, 65Y20.

1. Introduction. The purpose of this work is to construct and analyze a posteriori error estimators for a control–constrained optimal control problem involving a convection–reaction–diffusion equation as state equation. To describe our problem, let $\Omega$ be an open and bounded polytopal domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, with Lipschitz boundary $\partial \Omega$. Given a desired state $y_\Omega \in L^2(\Omega)$, we define the cost functional

$$J(y, u) = \frac{1}{2} \| y - y_\Omega \|_{L^2(\Omega)}^2 + \frac{\vartheta}{2} \| u \|_{L^2(\Omega)}^2,$$

where $\vartheta > 0$ denotes the so–called regularization parameter. We shall be concerned with the following optimal control problem: Find

$$\min J(y, u)$$

subject to the convection–reaction–diffusion equation

$$- \nu \Delta y + b \cdot \nabla y + \kappa y = f + u \quad \text{in } \Omega,$$

$$y = 0 \quad \text{on } \partial \Omega,$$

and the control constraints

$$u \in U_{ad}, \quad U_{ad} := \{ v \in L^2(\Omega) : a \leq v(x) \leq b \text{ for almost every } x \in \Omega \}. (1.4)$$

Here, the bounds $a, b \in \mathbb{R}$ and are such that $a < b$ and $f \in L^2(\Omega)$. Assumptions on $\nu$, $b$ and $\kappa$ are deferred until later.
The a priori error analysis for standard finite element approximations of problem (1.2)–(1.4) has been well established; see [6, 18, 27, 30, 31, 51] and the references therein. This analysis strongly relies on the error estimates involved in the approximation of (1.3). However, it is well known that applying the standard finite element method to (1.3) produces poor results when convection-dominated regimes are considered. In order to overcome such a difficulty, in the last few decades a variety of finite element approaches, such as stabilized finite element methods, have been proposed in the literature. In this work, we will focus on low-order conforming stabilized schemes, for which [46] provides an extensive overview in the subject. In the context of optimal control, the numerical approximation of problem (1.2)–(1.4) relies additionally on the discretization of the so-called adjoint equation (see Section 2.12 in [52]). Since (1.2)–(1.4) is intrinsically non-linear and presents a crosswind phenomena, an efficient method for solving such a control problem has to properly treat the oscillatory behaviours that occur when approximating \( \bar{y} \) and its adjoint variable \( \bar{p} \) and resolve the boundary layers exhibited by the state and adjoint state. Failure to resolve boundary layers pollutes the numerical solution in the entire domain [28]. Different stabilized finite element methods have been proposed to solve (1.2)–(1.4); see [13, 14, 21, 38, 58, 60, 61]. However, considering only stabilized schemes is not sufficient to efficiently approximate the solution to (1.2)–(1.4); boundary or interior layers and possible geometric singularities need to be resolved. This motivates the design of stabilized adaptive finite element methods.

Adaptive procedures for obtaining finite element solutions are based on the so-called a posteriori error analysis, and it has a solid foundation for elliptic problems; see [4, 10, 11, 43, 55, 62]. In contrast, the a posteriori error analysis for finite element approximations of optimal control problems has not yet been fully understood. We refer the reader to [29, 37, 56] for contributions to the theory. An attempt to unify the available results in the literature is presented in [46], where the authors derive an important error equivalence that simplifies the a posteriori error analysis to, simply put, provide estimators for the state and adjoint equations. The analysis is based on the energy norm inherited by the state and adjoint equations. Recently, the authors of [49] provided a general framework that complements the one developed in [36], and measures the error in a norm that is motivated by the objective. The analysis relies on the convexity of \( \Omega \). Both approaches exploit the first-order optimality conditions to derive a posteriori error estimates. However, the derived error estimates involve several unknown constants in the analysis, in particular, in the upper bound for the error in terms of the proposed error estimators (see [36, Theorem 3.2] and [49, Theorem 3.3]). Hence, in real computations, it will be unclear whether over or under estimation of the error has occurred. In fact, in a practical setting, if the estimator is to be used as a stopping criterion, then the constant involved in the upper bound for the error must be fully computable. This motivates the design and analysis of fully computable error estimators [1, 22, 39, 45, 57] for our optimal control problem, which guarantee a genuine upper bound for the error in the sense that the value of the error estimator is greater than or equal to the value of the error; see Theorem 4.2.

One of the main aims of this work is to develop an a posteriori error estimator with the following features:

- to be fully computable, in order to have at hand a stopping criterion for the adaptive resolution;
- to be applicable to a wide variety of low-order stabilized methods, allowing different combinations of stabilization terms for the state and adjoint equa-
tions.

We follow the a posteriori error analysis from [2] in order to obtain a fully computable error estimator for the convection–reaction–diffusion equation (1.3). With this estimator at hand, we provide what we believe is the first fully computable error estimator for the optimal control problem (1.2). However, the proposed fully computable estimator is not robust; the constant involved in the lower bound for the error depends on $\kappa$. This motivates another aim of our work:

- to propose and analyze a robust a posteriori error estimator.

To accomplish this task we follow [50, 54, 55] and measure the error in a norm that involves the dual norm of the convective derivative.

ERROR ESTIMATORS FOR AN OPTIMAL CONTROL PROBLEM

2. Preliminaries. We shall use standard notation for Sobolev and Lebesgue spaces, norms, and inner products. For a bounded domain $G \subset \mathbb{R}^d$ ($d = 2, 3$): $L^2(G)$ denotes the space of square integrable functions over $G$, $H^1(G)$ is the usual Sobolev space and $H^1_0(G)$ denotes the subspace of $H^1(G)$ consisting of functions whose trace is zero on $\partial G$. Let $(\cdot, \cdot)_{L^2(G)}$ denote the inner product in $L^2(G)$. The norm of $L^2(G)$ is denoted by $\| \cdot \|_{L^2(G)}$. We use bold letters to denote the vector-valued counterparts of spaces, e.g. $L^2(G) = L^2(G)^d$.

Let $\mathcal{T} = \{K\}$ be a conforming partition of $\Omega$ into simplices $K$ in the sense of Ciarlet [19]. We denote by $\mathcal{T}$ a collection of conforming and shape regular meshes that are refinements of an initial mesh $\mathcal{R}_0$. For a fixed $\mathcal{T} \in \mathcal{T}$, let

- $\mathcal{F}$ denote the set of all element edges(2D)/faces(3D);
- $\mathcal{F}_i \subset \mathcal{F}$ denote the set of interior edges(2D)/faces(3D), $\mathcal{F}_{\partial \Omega} \subset \mathcal{F}$ denote the set of boundary edges(2D)/faces(3D);
- $\Omega_n = \{ K \in \mathcal{T} : x_n \in K \}$, the set of elements for which $x_n$ is a vertex;
- $\mathcal{F}_n = \{ \gamma \in \mathcal{F} : x_n \in \gamma \}$ denote the set of element edges(2D)/faces(3D) that have $x_n$ as a vertex.

For an element $K \in \mathcal{T}$, let

- $\mathcal{P}_n(K)$ denote the space of polynomials on $K$ of total degree at most $n$;
- $\mathcal{P}_n \subset \mathcal{F}$ denote the set containing the individual edges(2D)/faces(3D) of $K$;
- $\Omega_K := \{ K' \in \mathcal{T} : \mathcal{F}_K \cap \mathcal{F}_{K'} \neq \emptyset \}$;
- $\mathcal{V}_K$ index the set $\{ x_n \}$ of all the vertices of the element $K$;
- $|K|$ denote the area/volume of $K$;
- $h_K$ denote the diameter of $K$;
- $n^\times_K$ denote the unit exterior normal vector to the edge/face $\gamma \in \mathcal{F}_K$;
- $v_{|K}$ denote the restriction of $v$ to the element $K$;
- $\bar{v}_K$ denote the mean value of the function $v$ on $K$, i.e. $\bar{v}_K = \frac{1}{|K|} \langle v, 1 \rangle_{L^2(K)}$.

For an edge/face $\gamma \in \mathcal{F}$, let:

- $\mathcal{P}_n(\gamma)$ denote the space of polynomials on $\gamma$ of total degree at most $n$;
A posteriori error analysis for the state equation

We consider the following stationary convection–reaction–diffusion problem: Find \( y \) such that

\[
-\nu \Delta y + b \cdot \nabla y + \kappa y = q \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial \Omega. \tag{3.1}
\]

The weak formulation of the previous problems reads: Find \( y \in H^1_0(\Omega) \) such that

\[
B(y, v) = (q, v)_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega), \tag{3.2}
\]

where, for \( w, v \in H^1_0(\Omega) \), the bilinear form \( B \) is defined by

\[
B(w, v) := \nu(\nabla w, \nabla v)_{L^2(\Omega)} + (b \cdot \nabla w + \kappa w, v)_{L^2(\Omega)}. \tag{3.3}
\]

We assume that the data of problem \( 3.2 \) satisfy the following conditions:

- \( \nu \) and \( \kappa \) are real and positive constants;
- \( b \in W^{1,\infty}(\Omega) \) and is a solenoidal field, that is, \( \text{div } b = 0 \);
- \( q \in L^2(\Omega) \).

We define, for \( G = \Omega \) or \( G \in \mathcal{T} \), and \( v \in H^1(G) \), the norm

\[
\|v\|_G := \left( \nu \|\nabla v\|^2_{L^2(G)} + \kappa \|v\|^2_{L^2(G)} \right)^{1/2}. \tag{3.4}
\]

On the basis of \( A1-(A3) \), this definition implies that \( B(v, v) = \|v\|^2_G \) and that

\[
B(w, v) \leq (1 + (\kappa \nu)^{-1/2}) \|b\|_{L^\infty(\Omega)} \|v\|_G \|w\|_G \quad \text{for all } v, w \in H^1_0(\Omega). \]

Then, the Lax–Milgram Lemma immediately yields the well–posedness of problem \( 3.2 \) \cite{13, 25, 40}.

To approximate the solution to problem \( 3.2 \), we will consider stabilized finite element methods: Find \( y_{\mathcal{T}} \in \mathcal{V}(\mathcal{T}) \) such that

\[
B(y_{\mathcal{T}}, v_{\mathcal{T}}) + S(y_{\mathcal{T}}, q; v_{\mathcal{T}}) = (q, v_{\mathcal{T}})_{L^2(\Omega)} \quad \forall v_{\mathcal{T}} \in \mathcal{V}(\mathcal{T}), \tag{3.5}
\]

where \( \mathcal{V}(\mathcal{T}) \) denotes the space of continuous piecewise linear functions on \( \mathcal{T} \), i.e,

\[
\mathcal{V}(\mathcal{T}) := \{ v \in C^0(\overline{\Omega}) : v|_K \in P_1(K) \quad \forall K \in \mathcal{T} \quad \text{and } v|_{\partial \Omega} = 0 \}. \]
and $\mathcal{S}$ corresponds to a particular choice of a stabilization term; the election $\mathcal{S} = 0$ corresponds to the standard finite element method without stabilization. We note that $\mathcal{S}$ may contain contributions of the datum $q$. In the next subsection, we will be precise about the stabilized terms that are allowed in our analysis. Meanwhile, we will assume that problem (3.5) has a unique solution $y_\mathcal{S} \in \mathcal{V}(\mathcal{T})$.

In general, stabilized schemes add mesh-dependent terms to the standard Galerkin formulation of (3.1) with the aim of improving the stability of the numerical method in the regime where the layers are unresolved [46]. Recently, to improve the accuracy of the schemes, attention has shifted toward the development of a posteriori error estimators, which is the content of the following subsections.

3.2. Reliability: a fully computable upper bound. In order to construct a fully computable a posteriori error estimator, we follow [2, 5], where the a posteriori error analysis is based on two main ingredients: the construction of equilibrated error estimators, which is the content of the following subsections. For all $v \in H^1_0(\Omega)$. We now introduce boundary fluxes $g_{\gamma,K} \in P_1(\gamma)$ on elements $K \in \mathcal{T}$ and $\gamma \in \mathcal{F}_K$, which satisfy

- **Consistency**: $g_{\gamma,K} + g_{\gamma,K'} = 0$, if $\gamma \in \mathcal{F}_K \cap \mathcal{F}_{K'}$, $K, K' \in \mathcal{T}$, $K \neq K'$. (3.6)

We can then incorporate such fluxes into the error equation to see that

$$B(y - y_{\mathcal{T}}, v) = \sum_{K \in \mathcal{T}} \left( (\mathcal{R}_K, v)_{L^2(K)} + (\text{osc}_K, v)_{L^2(K)} + \sum_{\gamma \in \mathcal{F}_K} (\mathcal{R}_{\gamma,K}, v)_{L^2(\gamma)} \right),$$

(3.7)

where, for $K \in \mathcal{T}$ and $\gamma \in \mathcal{F}_K$,

$$\mathcal{R}_K := \Pi_K(q) - \Pi_K(b \cdot \nabla y_{\mathcal{T}}) - \kappa y_{\mathcal{T}|K}, \quad \mathcal{R}_{\gamma,K} := g_{\gamma,K} - \nu \nabla y_{\mathcal{T}|K} \cdot n_{\gamma}^K,$$

(3.8)

$$\text{osc}_K := q - \Pi_K(q) - b \cdot \nabla y_{\mathcal{T}|K} + \Pi_K(b \cdot \nabla y_{\mathcal{T}}).$$

(3.9)

In addition, for $K \in \mathcal{T}$, we introduce a smooth enough vector function $\sigma_K$, such that

$$\sigma_K \cdot n_{\gamma}^K = \mathcal{R}_{\gamma,K} \text{ on } \gamma, \text{ for all } \gamma \in \mathcal{F}_K,$$

(3.10)

and hence, integration by parts yields that

$$\left( \text{div } \sigma_K, v \right)_{L^2(K)} + (\sigma_K, \nabla v)_{L^2(K)} = \sum_{\gamma \in \mathcal{F}_K} (\mathcal{R}_{\gamma,K}, v)_{L^2(\gamma)}.$$

Consequently, we can rewrite and bound the error equation as

$$B(y - y_{\mathcal{T}}, v) \leq \sum_{K \in \mathcal{T}} \left( (\mathcal{R}_K + \text{div } \sigma_K, v)_{L^2(K)} + (\sigma_K, \nabla v)_{L^2(K)} + (\text{osc}_K, v)_{L^2(K)} \right)^{1/2} \|v\|_{\Omega},$$

(3.11)
where the error indicator $\tilde{\eta}_K$ is defined by

$$\tilde{\eta}_K := \frac{1}{\sqrt{h}} \| \mathcal{R}_K + \div \sigma_K \|_{L^2(K)} + \frac{1}{\sqrt{\nu}} \| \sigma_K \|_{L^2(K)} + C_{\text{osc},K} \| \text{osc}_K \|_{L^2(K)}.$$  \hspace{1cm} (3.12)

To obtain (3.11), we have used the Cauchy–Schwarz inequality and

$$\| (\text{osc}_K, v)_{L^2(K)} \| \leq C_{\text{osc},K} \| \text{osc}_K \|_{L^2(K)} \| v \|_K, \quad C_{\text{osc},K} = \min \left\{ \frac{h_K}{\pi}, \frac{1}{\sqrt{h_K}} \right\}.$$  

The latter holds because, by using (2.1) and the Poincaré inequality [12, 44], we have that

$$| (\text{osc}_K, v)_{L^2(K)} | = | (\text{osc}_K, v - \bar{v}_K)_{L^2(K)} | \leq \left( \frac{h_K}{\pi} \right) \| v \|_{L^2(K)}$$  

but also

$$| (\text{osc}_K, v)_{L^2(K)} | \leq \kappa^{-1/2} \| \text{osc}_K \|_{L^2(K)} \| v \|_K.$$  

Finally, taking $v = y - y_\mathcal{F}$ and using that $B(v, v) = \| v \|_{\Omega}^2$, we arrive at

$$\| y - y_\mathcal{F} \|_{\Omega} \leq \tilde{\eta}, \quad \tilde{\eta} := \left( \sum_{K \in \mathcal{F}} \frac{\eta_K^2}{\nu} \right)^{1/2}.$$  \hspace{1cm} (3.13)

**Remark 3.1 (fully computable upper bound).** The main advantage of the previous a posteriori error analysis is that it provides an upper bound for the error that is free of any unknown constants. Consequently, the error estimator $\tilde{\eta}$ can be confidently used as a stopping criterion for an adaptive mesh procedure.

We conclude this subsection by mentioning that the quality of the error estimation depends on the construction of the vector function $\sigma_K$ satisfying (3.10) and the equilibrated boundary fluxes. Before discussing such constructions, we introduce a family of stabilized schemes.

### 3.3. Stabilized schemes.

We now describe the stabilized finite element methods that will be the focus of our work. To do this, we write the stabilization term $S$, based on [24], in terms of local contributions coming from each element, namely,

$$S(y_\mathcal{F}, q; v_\mathcal{F}) = \sum_{K \in \mathcal{F}} S_K(y_\mathcal{F}, q; v_\mathcal{F}|K).$$

The local contributions, $S_K$, for the below mentioned stabilizations, are as follows:

*Streamline Upwind Petrov-Galerkin (SUPG)*: This stabilization technique was introduced by Brooks and Hughes [15]; see also [33, 41, 46]. The local contributions are

$$S_K(y_\mathcal{F}, q; v_\mathcal{F}|K) = \tau_K \left( b \cdot \nabla y_\mathcal{F} + \kappa y_\mathcal{F} - q, b \cdot \nabla v_\mathcal{F} \right)_{L^2(K)}.$$  \hspace{1cm} (3.14)

*Galerkin Least Squares (GLS)*: This method was introduced in [33]; see also [41, 46]. The local contributions are

$$S_K(y_\mathcal{F}, q; v_\mathcal{F}|K) = \tau_K \left( b \cdot \nabla y_\mathcal{F} + \kappa y_\mathcal{F} - q, b \cdot \nabla v_\mathcal{F} + \kappa v_\mathcal{F} \right)_{L^2(K)}.$$  \hspace{1cm} (3.15)

*Continuous Interior Penalty (CIP)*: This stabilization method was proposed by Douglas and Dupont [23]. Upon defining $[b \cdot \nabla y_\mathcal{F}]_\gamma|K := b \cdot \nabla (y_\mathcal{F}|K - y_\mathcal{F}|K)\gamma$ with $K\gamma$ being the element that shares $\gamma$ with $K$, the local contributions are

$$S_K(y_\mathcal{F}, q; v_\mathcal{F}|K) := \sum_{\gamma \in K \cap \mathcal{F}} \tau_\gamma \left( [b \cdot \nabla y_\mathcal{F}]_\gamma|K, b \cdot \nabla v_\mathcal{F}|K \right)_{L^2(\gamma)}.$$  \hspace{1cm} (3.16)
**Edge Stabilization (ES):** This technique was proposed by Burman and Hansbo in [17] (see also [16], [40]). Upon defining \( [\cdot] \) to be the usual jump on internal edges/faces and \( K \gamma \) to be the element that shares \( \gamma \) with \( K \), the local contributions are

\[
S_K(y_\gamma, q; v_\gamma|_K) := \sum_{\gamma \in K \cap F_i} \tau_\gamma (\| \nabla v_\gamma \cdot n_\gamma \|, \nabla v_\gamma|_K \cdot n^K_\gamma (h^K_\gamma + h^2_\gamma))_{L^2(\gamma)}. \tag{3.17}
\]

In all the previous schemes \( \tau_K \) and \( \tau_\gamma \) denote nonnegative stabilization parameters that can vary from one method to another.

### 3.4. Construction of the equilibrated fluxes.

We now describe a procedure for obtaining a set of equilibrated fluxes \( \{g_{\gamma,K} : \gamma \in F_K, K \in \mathcal{T} \} \) which satisfy (3.6) and

- **First order equilibration:** for all \( \lambda \in \mathbb{P}_1(K) \) and all \( K \in \mathcal{T} \),

\[
0 = (q, \lambda)_{L^2(K)} - B_K(y_\gamma, \lambda) + \sum_{\gamma \in F_K} (g_{\gamma,K}, \lambda)_{L^2(\gamma)} - S_K(y_\gamma, q; \lambda), \tag{3.18}
\]

where \( B_K(y_\gamma, \lambda) = \nu(\nabla y_\gamma, \nabla \lambda)_{L^2(K)} + (b \cdot \nabla y_\gamma + \kappa y_\gamma, \lambda)_{L^2(K)} \).

In addition, the equilibrated fluxes satisfy a property, namely (3.24), that will be used to prove that the local error indicators defined by (3.12) are locally efficient.

Let

\[
(J)_{\gamma,K} := \begin{cases} \frac{1}{2}(J_{\gamma,K} - J_{\gamma,K'}) & \text{if } \gamma \in F_K \cap F_{K'}, K \neq K', \\ J_{\gamma,K} & \text{if } \gamma \in F_K \cap F_{\partial \Omega}, \end{cases} \tag{3.19}
\]

where \( J_{\gamma,K} := \nu \nabla y_\gamma|_K \cdot n^K_\gamma \). For every \( i \in \mathcal{V} \), let \( \{\xi_{K,i} : K \in \Omega_i\} \) be a solution to the linear system of equations

\[
\frac{1}{2} \sum_{K' \in \Omega_K \cap \Omega_i} (\xi_{K,i} - \xi_{K',i}) + \sum_{\gamma \in F_K \cap F_{\partial \Omega}} \xi_{K,i} = \Delta_K(\lambda_i) \quad \forall \ K \in \Omega_i \tag{3.20}
\]

where

\[
\Delta_K(\lambda_i) = B_K(y_\gamma, \lambda_i) - (q, \lambda_i)_{L^2(K)} - \sum_{\gamma \in F_K} ((J)_{\gamma,K}, \lambda_i)_{L^2(\gamma)} + S_K(y_\gamma, q; \lambda_i|_K). \]

In terms of the solutions to these systems of equations, we define

\[
\hat{\mu}_{K,i} := \begin{cases} \frac{1}{2}(\xi_{K,i} - \xi_{K',i}) + ((J)_{\gamma,K} - (J)_{\gamma,K'})_{L^2(\gamma)} & \text{if } \gamma \in F_K \cap F_{K'}, K \neq K', \\ (\xi_{K,i} + ((J)_{\gamma,K} - (J)_{\gamma,K'})_{L^2(\gamma)} & \text{if } \gamma \in F_K \cap F_{\partial \Omega}, \end{cases} \tag{3.21}
\]

for \( i \in \mathcal{V}, K \in \Omega_i \) and \( \gamma \in \mathcal{F}_i \). Now, the terms \( S_K \) are defined in such a way that

\[
\sum_{K \in \Omega_i} \Delta_K(\lambda_i) = B(y_\gamma, \lambda_i) - (q, \lambda_i)_{L^2(\Omega_i)} - \sum_{K \in \Omega_i} \sum_{\gamma \in F_K} ((J)_{\gamma,K} - (J)_{\gamma,K'})_{L^2(\gamma)} + S(y_\gamma, q; \lambda_i) \]

for \( i \in \mathcal{V} \). Hence, we can conclude that

\[
\sum_{K \in \Omega_i} \Delta_K(\lambda_i) = 0 \quad \forall \ i \in \mathcal{V} : x_i \notin \partial \Omega \]

upon taking \( v_\gamma = \lambda_i \) in (5.5) and noticing that

\[
\sum_{K \in \Omega_i} \sum_{\gamma \in F_K} ((J)_{\gamma,K} - (J)_{\gamma,K'})_{L^2(\gamma)} = 0 \quad \forall \ i \in \mathcal{V} : x_i \notin \partial \Omega. \]

It then follows from [3, Lemma 5] (and its three dimensional analog which can be proved using the same arguments) that, for \( i \in \mathcal{V} \),
• if \( x_i \in \partial \Omega \), then (3.20) has a unique solution;
• if \( x_i \notin \partial \Omega \), then solutions to (3.20) exist and are of the form \( \{ \xi_{K,i} + c_i, K \in \Omega_i \} \), where \( c_i \) is any constant and \( \{ \xi_{K,i}, K \in \Omega_i \} \) is any solution to (3.20).

Consequently, the \( \mu_{K,i}^\gamma \) are uniquely defined by (3.21). This is due to the fact that, if \( x_i \notin \partial \Omega \), the solution to (3.20) only appears in (3.21) as \( \xi_{K,i} - \xi_{K',i} \) and so the nonuniqueness cancels out. Hence, for each \( i \in V \), the \( \mu_{K,i}^\gamma \), for \( K \in \Omega_i \) and \( \gamma \in F_i \), can be computed using (3.21) after obtaining a solution to (3.20).

For \( \gamma \in F_K \) and \( K \in \mathcal{S} \), we define

\[
g_{\gamma,K} = \frac{d}{|\gamma|} \sum_{j \in V_\gamma} \mu_{K,j}^\gamma \left( (d + 1)\lambda_j - 1 \right)
\]  

(3.22)

which is such that \( g_{\gamma,K} \in P_1(\gamma) \) and

\[
(g_{\gamma,K}, \lambda_i)_{L^2(\gamma)} = \mu_{K,i}^\gamma \text{ for all } i \in V_\gamma.
\]  

(3.23)

Now, let \( \gamma \in F_K \cap F_{K'} \), \( K, K' \in \mathcal{S} \), \( K \neq K' \). From (3.21) and (3.19) it can be seen that

\[
\mu_{K,j}^\gamma = \frac{1}{2} \left( \xi_{K,j} - \xi_{K',j} \right) + \left( (\langle J \rangle_{\gamma,K}, \lambda_j)_{L^2(\gamma)} \right)
\]

and, similarly,

\[
\mu_{K',j}^\gamma = \frac{1}{2} \left( \xi_{K',j} - \xi_{K,j} \right) + \left( (\langle J \rangle_{\gamma,K'}, \lambda_j)_{L^2(\gamma)} \right).
\]

Consequently, by (3.22) we have that

\[
g_{\gamma,K} + g_{\gamma,K'} = \frac{d}{|\gamma|} \sum_{j \in V_\gamma} \left( \mu_{K,j}^\gamma + \mu_{K',j}^\gamma \right) \left( (d + 1)\lambda_j - 1 \right) = 0
\]

and hence (3.3) is satisfied.

Now, let \( i \in V \) and \( K \in \Omega_i \). By (3.23) and (3.21) we have that

\[
\frac{1}{2} \sum_{K' \in \Omega_K \cap \Omega_i} (\xi_{K,i} - \xi_{K',i}) = \sum_{\gamma \in F_K \cap F_i : \gamma \notin F_{\partial i}} (g_{\gamma,K} - \langle J \rangle_{\gamma,K}, \lambda_i)_{L^2(\gamma)}
\]

and

\[
\sum_{\gamma \in F_K \cap F_i : \gamma \notin F_{\partial i}} \xi_{K,i} = \sum_{\gamma \in F_K \cap F_i : \gamma \notin F_{\partial i}} (g_{\gamma,K} - \langle J \rangle_{\gamma,K}, \lambda_i)_{L^2(\gamma)}.
\]

Consequently,

\[
\frac{1}{2} \sum_{K' \in \Omega_K \cap \Omega_i} (\xi_{K,i} - \xi_{K',i}) + \sum_{\gamma \in F_K \cap F_i : \gamma \notin F_{\partial i}} \xi_{K,i} = \sum_{\gamma \in F_K \cap F_i} (g_{\gamma,K} - \langle J \rangle_{\gamma,K}, \lambda_i)_{L^2(\gamma)}
\]

\[
= \sum_{\gamma \in F_K} (g_{\gamma,K} - \langle J \rangle_{\gamma,K}, \lambda_i)_{L^2(\gamma)}
\]
since \( \lambda_{i| \gamma} = 0 \) for \( \gamma \in F_K \setminus (F_K \cap F_I) \). Therefore, from (3.25) we can see that

\[
0 = (q, \lambda_i)_{L^2(K)} - B_K (y, \lambda_i) + \sum_{\gamma \in F_K} (g_{\gamma, K}, \lambda_i)_{L^2(\gamma)} - S_K (y, \lambda_i).
\]

The fact that (3.18) is satisfied then follows since \( (\cdot, \cdot)_{L^2(K)}, B_K (\cdot, \cdot), (\cdot, \cdot)_{L^2(\gamma)} \) and \( S_K (\cdot, \cdot, \cdot) \) are linear in their final arguments and \( \{\lambda_{i| K}, i \in V_K\} \) is a basis for \( P_1(K) \).

Furthermore, following the arguments in the proof of [4, Theorem 6.2], yields that

\[
\| g_{\gamma, K} - (J)_{\gamma, K} \|_{L^2(\gamma)} \leq C \frac{1}{\sqrt{|\gamma|}} \sum_{n \in V_{\gamma}} \sum_{K' \in \Omega_n} |\Delta_{K'}(\lambda_n)|
\]

and that

\[
|\Delta_{K'}(\lambda_n)| \leq \sqrt{|K'|} \| \mathscr{R}_{K'} \|_{L^2(K')} + \sum_{\gamma' \in F_{K'} \cap F_{\gamma}} \sqrt{|\gamma'|} \| \mathcal{J}_{\gamma'} \|_{L^2(\gamma')} + |\mathcal{S}_{K'}(y, \nu; \lambda_{n| K'})|.
\]

Consequently, for \( K \in T \) and \( \gamma \in F_K \), we have that

\[
\left( \frac{h}{p} \right)^{1/2} \| g_{\gamma, K} - (J)_{\gamma, K} \|_{L^2(\gamma)} \leq C \sum_{n \in V_{\gamma}} \sum_{K' \in \Omega_n} \left( \frac{h}{p} \right)^{1/2} \| \mathscr{R}_{K'} \|_{L^2(K')} + \sum_{\gamma' \in F_{K'} \cap F_{\gamma}} \left( \frac{h^{2-d}}{p} \right)^{1/2} \| \mathcal{S}_{K'}(y, \nu; \lambda_{n| K'}) \|ight) .
\tag{3.24}
\]

### 3.5. Construction of \( \sigma_K \)

In order to obtain a fully computable error estimator, we choose \( \sigma_K \in \mathbb{P}_2(K) \times \mathbb{P}_2(K) \) to be a solution to

\[
\begin{cases}
-\text{div } \sigma_K = p_K & \text{in } K, \\
\sigma_K \cdot n^K = p_{\gamma, K} & \text{on } \gamma \text{ for all } \gamma \in F_K,
\end{cases}
\tag{3.25}
\]

where

\[
p_K = \mathscr{R}_K - \overline{\mathscr{R}}_K - \frac{1}{|K|} \sum_{\gamma \in F_K} (\mathscr{R}_{\gamma, K}, 1)_{L^2(\gamma)} \in \mathbb{P}_1(K), \quad p_{\gamma, K} = \mathscr{R}_{\gamma, K} \in \mathbb{P}_1(\gamma)
\tag{3.26}
\]

where \( \mathscr{R}_K \) and \( \mathscr{R}_{\gamma, K} \) are defined in (3.18). Since the data of problem (3.25) satisfies a constant equilibration condition, that is,

\[
\sum_{\gamma \in F_K} (p_{\gamma, K}, 1)_{L^2(\gamma)} + (p_K, 1)_{L^2(K)} = 0,
\]

then [5, Theorem 6.3] provides an explicit formula for a solution to (3.25) that satisfies

\[
\|\sigma_K\|_{L^2(K)} \leq C \left( h_K^{1/2} \sum_{\gamma \in F_K} \|p_{\gamma, K}\|_{L^2(\gamma)} + h_K \|p_K\|_{L^2(K)} \right).
\tag{3.27}
\]

We note that (3.27) will also be satisfied by the \( \sigma_K \in \mathbb{P}_2(K) \times \mathbb{P}_2(K) \) which satisfies (3.25) and is such that \( \|\sigma_K\|_{L^2(K)} \) is minimized. Furthermore, once the discrete solution is obtained, the only problems that have to be solved in order to compute \( \sigma_K \) and \( g_{\gamma, K} \) are local problems of size at most \( \max\{\#\Omega_i : i \in V\} \). Hence, once the number of degrees of freedom is sufficiently large, the cost of obtaining the estimator should be inexpensive when compared with the cost of obtaining the solution to 3.5.
3.6. Final fully computable upper bound. Gathering all our findings of the previous sections, allows us to state the following reliability result.

**Theorem 3.2 (fully computable upper bounds).** Let \( y \in H^1_0(\Omega) \) and \( y_\mathcal{S} \in \mathcal{V}(\mathcal{S}) \) be the solutions to problems (3.2) and (3.3), respectively. Then, we have the following fully computable upper bound for the energy norm of the error:

\[
\|y - y_\mathcal{S}\|_\Omega \leq \eta = \left( \sum_{K \in \mathcal{S}} \eta_K^2 \right)^{1/2},
\]

where the error indicators \( \eta_K \) are defined by

\[
\eta_K := \frac{1}{\sqrt{|K|}} |S_K(y_\mathcal{S}, q; 1)| + \frac{1}{\sqrt{|V|}} \|\sigma_K\|_{L^2(K)} + C_{osc,K}\|osc_K\|_{L^2(K)}.
\]

**Proof.** Upon noticing that

\[
\sum_{\gamma \in \mathcal{F}_K} (\nabla y_\mathcal{S} |\nabla K \cdot n_{\gamma}^K, 1)_{L^2(\gamma)} = (\text{div} (\nabla y_\mathcal{S}), 1)_{L^2(K)} = 0
\]

and \( R_{\gamma,K} = g_{\gamma,K} - \nu \nabla y_\mathcal{S} |\nabla K \cdot n_{\gamma}^K \), it follows that

\[
\frac{1}{|K|} \sum_{\gamma \in \mathcal{F}_K} (R_{\gamma,K}, 1)_{L^2(\gamma)} = \frac{1}{|K|} \sum_{\gamma \in \mathcal{F}_K} (g_{\gamma,K}, 1)_{L^2(\gamma)}.
\]

Moreover, (2.1), the equilibration condition (3.18) and the definition of \( R_K \) yield that

\[
\sum_{\gamma \in \mathcal{F}_K} (g_{\gamma,K}, 1)_{L^2(\gamma)} = B_K(y_\mathcal{S}, 1) - (q, 1)_{L^2(K)} + S_K(y_\mathcal{S}, q; 1)
\]

\[
= -(R_{K}, 1)_{L^2(K)} + S_K(y_\mathcal{S}, q; 1).
\]

Combining this with (3.30) implies that

\[
\frac{1}{|K|} \sum_{\gamma \in \mathcal{F}_K} (R_{\gamma,K}, 1)_{L^2(\gamma)} = -R_{K} + \frac{1}{|K|} S_K(y_\mathcal{S}, q; 1).
\]

Consequently, we rewrite the norm of \( R_K + \text{div} \sigma_K \), as

\[
\|R_K + \text{div} \sigma_K\|_{L^2(K)} = \left\|R_K + \frac{1}{|K|} \sum_{\gamma \in \mathcal{F}_K} (R_{\gamma,K}, 1)_{L^2(\gamma)} \right\|_{L^2(K)}
\]

\[
= \left\|\frac{1}{|K|} S_K(y_\mathcal{S}, q; 1) \right\|_{L^2(K)} = \frac{1}{|K|^{1/2}} |S_K(y_\mathcal{S}, q; 1)|,
\]

which combined with (3.12), (3.19), and (3.29), yields the result claimed.

**Corollary 3.3.** Let \( y \in H^1_0(\Omega) \) and \( y_\mathcal{S} \in \mathcal{V}(\mathcal{S}) \) be the solutions to problems (3.2) and (3.3), respectively. Then, we have the following fully computable upper bound for the energy norm of the error:

\[
\|y - y_\mathcal{S}\|_\Omega \leq \left( \sum_{K \in \mathcal{S}} \left( \frac{\|\sigma_K\|_{L^2(K)}}{\sqrt{|V|}} + C_{osc,K}\|osc_K\|_{L^2(K)} \right)^2 \right)^{1/2}
\]
for SUPG, ES and CIP, and
\[ \| y - y_{\mathcal{T}} \|_{\Omega} \leq \left( \sum_{K \in \mathcal{T}} \left( \tau_K \sqrt{r} \| \mathcal{R}_K \|_{L^2(K)} + \frac{\| \mathcal{R}_K \|_{L^2(K)}}{\sqrt{r}} + C_{\text{osc}, \mathcal{K}} \| \text{osc}_K \|_{L^2(K)} \right)^2 \right)^{1/2} \]
for GLS.

Proof. The results follow from Theorem 3.2 and an inspection of the stabilization terms. The latter reveals that \( S_K (y_{\mathcal{T}}, q; 1) = 0 \) for SUPG, CIP and ES, and that \( |S_K (y_{\mathcal{T}}, q; 1)| = \tau_K \| (-\mathcal{R}_K, K)_{L^2(K)} \| = C_{\mathcal{K}} \sqrt{r} \| \mathcal{R}_K \|_{L^2(K)} \) for GLS. [2]

Remark 3.4 (extension of the theory). We are not assuming that \( S_K (y_{\mathcal{T}}, q; 1) = 0 \) as in Assumption 4.3 from [3]. This advantage allows us to consider, for example, the GLS scheme. From (3.20) and (3.31), it follows that \( -\text{div} \, \sigma_K = \mathcal{R}_K \) if \( S_K (y_{\mathcal{T}}, q; 1) \neq 0 \). If \( S_K (y_{\mathcal{T}}, q; 1) = 0 \) then \( -\text{div} \, \sigma_K \neq \mathcal{R}_K \) and the error indicator \( \eta_K \), in which the estimator \( \eta \) is defined in terms of, has an additional term.

3.7. Local efficiency. We now explore the local efficiency properties of the local indicator (3.29). From (3.27), we have that
\[ \| \sigma_K \|_{L^2(K)} \leq C \left( h_K^{1/2} \sum_{\gamma \in F_K} \| \mathcal{R}_{\gamma,K} \|_{L^2(\gamma)} + h_K \| \mathcal{R}_K \|_{L^2(K)} \right) \]
(3.33)
since \( p_{\gamma,K} = \mathcal{R}_{\gamma,K} \) and
\[ h_K \| p_K \|_{L^2(K)} \leq h_K \| \mathcal{R}_K - \mathcal{R}_K \|_{L^2(K)} + h_K |K|^{1/2} \sum_{\gamma \in F_K} \| \mathcal{R}_{\gamma,K} \|_{L^2(\gamma)} \]
\[ \leq \frac{h_K}{2} \| \mathcal{R}_K \|_{L^2(K)} + C h_K^{1/2} \sum_{\gamma \in F_K} \| \mathcal{R}_{\gamma,K} \|_{L^2(\gamma)} \]
because \( \| \mathcal{R}_K - \mathcal{R}_K \|_{L^2(K)} \leq \| \mathcal{R}_K \|_{L^2(K)} \). Now, we define
\[ [J_\gamma] := \left\{ \begin{array}{ll} & \frac{1}{2} (J_{\gamma,K} + J_{\gamma,K'}) \quad \text{if } \gamma \in F_K \cap F_{K'}, \ K \neq K', \\
& 0 \quad \text{if } \gamma \in F_K \cap F_{K'}, \ K = K' \end{array} \right. \]
where \( J_{\gamma,K} := \nu \nabla y_{\mathcal{T}} |_{K} \cdot n_{\gamma,K} \). This, combined with (3.28) and (3.31), allows us to state that \( \mathcal{R}_{\gamma,K} = g_{\gamma,K} - (J_{\gamma,K} - [J_\gamma]) \). Then, in view of (3.29), (3.33) provides the bound:
\[ \eta_K \leq C \left( h_K^{1/2} \| \mathcal{R}_K \|_{L^2(K)} + \sum_{\gamma \in F_K} \left( h_K^{1/2} \left( \| J_{\gamma,K} \|_{L^2(\gamma)} + \| g_{\gamma,K} - (J_{\gamma,K} - [J_\gamma]) \|_{L^2(\gamma)} \right) \right) \right) \]
\[ + C_{\text{osc}, \mathcal{K}} \| \text{osc}_K \|_{L^2(K)} + \frac{1}{\sqrt{|K|}} |S_K (y_{\mathcal{T}}, q; 1)|. \]
(3.34)
To prove local efficiency, the terms on the right hand side of (3.34) have to be bounded by the energy norm of the error \( y - y_{\mathcal{T}} \) plus data oscillation terms. To do this, we first note that we can rewrite the error equation (3.7), for any \( v \in H_0^1 (\Omega) \), as
\[ \sum_{K \in \mathcal{T}} (\mathcal{R}_K, v)_{L^2(K)} - 2 \sum_{\gamma \in F_1} ([J_\gamma], v)_{L^2(\gamma)} = \mathcal{B}(y - y_{\mathcal{T}}, v) - \sum_{K \in \mathcal{T}} (\text{osc}_K, v)_{L^2(K)}. \]
Applying standard bubble function arguments [4, 5] to this error equation yields that
\[ \frac{h_K}{\sqrt{r}} \| \mathcal{R}_K \|_{L^2(K)} \leq C \left( \mathcal{C}_K \| y - y_{\mathcal{T}} \|_K + \frac{h_K}{\sqrt{r}} \| \text{osc}_K \|_{L^2(K)} \right) \]
(3.35)
for $K \in \mathcal{T}$, and

$$
\left( \frac{h_K}{\nu} \right)^{1/2} \| J_\gamma \|_{L^2(\gamma)} \leq C \sum_{K' \in \Omega} \left( C_{K'} \| y - y_{\mathcal{B}} \|_{K'} + \frac{h_K}{\nu} \| \text{osc}_{K'} \|_{L^2(K')} \right)
$$

(3.36)

for $K \in \mathcal{T}$ and $\gamma \in \mathcal{F}_K$, where

$$
C_K := \max \left\{ 1, \frac{\| b \|_{L^\infty(K)} h_K}{\nu}, \sqrt{\frac{h_K}{\nu}} \right\}.
$$

(3.37)

Now, from (3.34), (3.35), (3.36), and (3.24) we have that

$$
\eta_K \leq C \sum_{n \in \mathcal{V}_K} \sum_{K' \in \Omega_n} \left( C_{K'} \| y - y_{\mathcal{B}} \|_{K'} + \frac{h_K}{\nu} \| \text{osc}_{K'} \|_{L^2(K')} \right)
$$

$$
+ \left( \frac{h_K^{2-d}}{\nu} \right)^{1/2} \| S_{K'}(y_{\mathcal{B}}, q; \lambda_n|K') \| + \frac{1}{\sqrt{\nu |K|}} |S_K(y_{\mathcal{B}}, q; 1)|.
$$

Consequently, to obtain the local efficiency of $\eta_K$ it remains to control the stabilization term $S_K$ in the previous inequality. We proceed to examine the local contribution of each method described in Section 3.3, namely, SUPG, GLS, ES, and CIP.

**Streamline Upwind Petrov–Galerkin (SUPG):** Clearly $S_K(y_{\mathcal{B}}, q; 1) = 0$. Moreover,

$$
\left( \frac{h_K^{2-d}}{\nu} \right)^{1/2} |S_K(y_{\mathcal{B}}, q; \lambda_n|K)| = \frac{h_K^{1-d/2}}{\nu} \tau_K \| (b \cdot \nabla y_{\mathcal{B}} + \nu y_{\mathcal{B}} - q, b \cdot \nabla \lambda_n)_{L^2(K)} \|
$$

$$
\leq C \tau_K \frac{\| b \|_{L^\infty(K)} h_K}{h_K} \left( \| \mathcal{R}_{K'} \|_{L^2(K)} + \| \text{osc}_{K} \|_{L^2(K)} \right).
$$

**Galerkin Least Square (GLS):** In view of (3.14), we have that

$$
\frac{1}{\sqrt{\nu |K|}} |S_K(y_{\mathcal{B}}, q, 1)| = \frac{1}{\sqrt{\nu |K|}} \tau_K K \sqrt{|K|} \frac{\| \mathcal{R}_{K} \|_{L^2(K)}}{\| \mathcal{R}_{K} \|_{L^2(K)}} = \tau_K \frac{\sqrt{\nu}}{h_K} \frac{h_K}{\nu} \| \mathcal{R}_{K} \|_{L^2(K)}
$$

and

$$
\left( \frac{h_K^{2-d}}{\nu} \right)^{1/2} |S_K(y_{\mathcal{B}}, q; \lambda_n|K)| = \frac{h_K^{1-d/2}}{\nu} \tau_K \left( \| (b \cdot \nabla y_{\mathcal{B}} + \nu y_{\mathcal{B}} - q, b \cdot \nabla \lambda_n + \nu \lambda_n)_{L^2(K)} \| \right)
$$

$$
\leq C \tau_K \max \left\{ \frac{\| b \|_{L^\infty(K)} h_K}{h_K}, K \right\} \frac{h_K^{1-d/2}}{\nu} \left( \| \mathcal{R}_{K} \|_{L^2(K)} + \| \text{osc}_{K} \|_{L^2(K)} \right).
$$

**Edge Stabilization (ES):** Clearly $S_K(y_{\mathcal{B}}, q; 1) = 0$. Moreover, (3.17) yields

$$
\left( \frac{h_K^{2-d}}{\nu} \right)^{1/2} |S_K(y_{\mathcal{B}}, q; \lambda_n|K)| \leq C \frac{h_K}{\nu} \sum_{\gamma \in \mathcal{F}_K \cap \mathcal{F}_I} \tau_\gamma \left( \frac{h_K}{\nu} \right)^{1/2} \| J_\gamma \|_{L^2(\gamma)}.
$$

**Continuous Interior Penalty (CIP):** Clearly $S_K(y_{\mathcal{B}}, q; 1) = 0$. Now, notice that $b = (n^K_\gamma \cdot b)n^K_\gamma + (t_\gamma \cdot b)t_\gamma$, where $t_\gamma$ is a unit tangent to $\gamma$. Hence, using the fact that $b \in W^\infty_1(\Omega)$, allows us to see that $\| b \cdot \nabla y_{\mathcal{B}} \|_{\gamma,K} = \frac{2}{\nu} (n^K_\gamma \cdot b) \| J_\gamma \|$ since

$$
\| b \cdot \nabla y_{\mathcal{B}} \|_{\gamma,K} = (n^K_\gamma \cdot b)n^K_\gamma \cdot \nabla (y_{\mathcal{B}}|K - y_{\mathcal{B}}|K) + (t_\gamma \cdot b)t_\gamma \cdot \nabla (y_{\mathcal{B}}|K - y_{\mathcal{B}}|K).
$$
Thus \( \| \mathbf{b} \cdot \nabla y_{\mathcal{F}} \|_{L^2(\gamma)} \leq \frac{2}{\nu} \| \mathbf{b} \|_{L^\infty(\gamma)} \| J_\gamma \|_{L^2(\gamma)} \). Consequently, (3.10) yields
\[
\left( \frac{b_{\text{max}}^2 - d}{\nu} \right)^{1/2} |S_K(y_{\mathcal{F}}, q; \lambda_{n[K]})| \leq C \sum_{\gamma \in \mathcal{F}_K \cap \mathcal{F}_I} \frac{\tau_{\gamma} \| b \|_{L^\infty(\gamma)}^2}{\nu K} \left( \frac{b_{\text{max}}}{\nu} \right)^{1/2} \| J_\gamma \|_{L^2(\gamma)}.
\]

By combining all of the previous results, we can finally summarize and bound the error indicator \( \eta_\gamma \) as described in the following theorem.

**Theorem 3.5.** Let \( y \in H^1_0(\Omega) \) and \( y_{\mathcal{F}} \in \mathcal{V}(\mathcal{F}) \) be the solutions to problems (3.2) and (3.3), respectively. Then, we have the following local lower bound for the energy norm of the error:
\[
\eta_\gamma \leq C \sum_{n \in \mathcal{V}_K} \sum_{K' \in \Omega_n} \left( C_{K'} \| y - y_{\mathcal{F}} \|_{L^2(K')} + \frac{h_{\text{osc}K'}}{\nu} \| b \|_{L^\infty(K')} \right)
\]
\[
+ C_{S_{K'}} \left( \sum_{K'' \in \Omega_{K'}} \left( C_{K''} \| y - y_{\mathcal{F}} \|_{L^2(K'')} + \frac{h_{\text{osc}K''}}{\nu} \| b \|_{L^\infty(K'')} \right) \right),
\]
where \( C_{K'} \) is defined as in (3.37) and
\[
C_{S_{K'}} = \begin{cases} 
\tau_K \| b \|_{L^\infty(K)} / h_K & \text{for SUPG}, \\
\frac{h_{\text{osc}K}}{\nu} \tau_{\gamma} & \text{for ES}, \\
\frac{\tau_K \| b \|_{L^\infty(K)}}{h_K} & \text{for GLS}, \\
\frac{\tau_{\gamma} \| b \|_{L^\infty(\gamma)}}{\nu h_K} & \text{for CIP}.
\end{cases}
\]

4. **Optimal control problem.** In this section, we analyze the optimal control problem (1.2)–(1.4). To approximate its solution, we propose a numerical method that is based on the stabilized schemes of Section 3.3. We derive fully computable a posteriori upper bounds for the error and prove local efficiency properties of the proposed error estimators.

Under the assumptions (A1)–(A3), the existence and uniqueness of an optimal pair \((\bar{y}, \bar{u}) \in H^1_0(\Omega) \times \mathcal{U}_{ad}\) satisfying (1.2)–(1.4) follows standard arguments [52]. An equivalent formulation can be obtained by introducing the so-called adjoint variable \( \bar{p} \). We then say that \((\bar{y}, \bar{p}, \bar{u}) \) is optimal if and only if it solves the nonlinear system
\[
\begin{cases} 
\bar{y} \in H^1_0(\Omega) : & B(\bar{y}, v) = (f + \bar{u}, v)_{L^2(\Omega)}, \quad \forall v \in H^1_0(\Omega), \\
\bar{p} \in H^1_0(\Omega) : & B^*(\bar{p}, w) = (\bar{y} - \bar{y}_0, w)_{L^2(\Omega)}, \quad \forall w \in H^1_0(\Omega), \\
\bar{u} \in \mathcal{U}_{ad} : & (\bar{p} + \nabla \bar{u}, \nabla \bar{u})_{L^2(\Omega)} \geq 0, \quad \forall \bar{u} \in \mathcal{U}_{ad},
\end{cases}
\]
which is necessary and sufficient for optimality. The form \( B \) is given in (3.3), and
\[
B^*(w, v) := \nu (\nabla w, \nabla v)_{L^2(\Omega)} + (\kappa w - \mathbf{b} \cdot \nabla w, v)_{L^2(\Omega)}.
\]
Finally, we recall the projection formula for the optimal control variable: the variational inequality in (4.1) can be equivalently written as (see [52] Chapter 2),
\[
\bar{u} = \Pi \left( -\frac{1}{\bar{p}} \right),
\]
where
\[
(\Pi (w)) (x) := \min \{ b, \max \{ a, w(x) \} \} \quad \text{for almost every } x \in \Omega
\]
with \( \mathbf{a} \) and \( \mathbf{b} \) being the control bounds that define the set \( U_{ad} \) in (1.4). We note that this operator is such that, for \( K \in \mathcal{T} \),

\[
\| \Pi(w_1) - \Pi(w_2) \|_{L^2(K)} \leq \| w_1 - w_2 \|_{L^2(K)} \quad \text{for all } w_1, w_2 \in H^1_0(\Omega). \tag{4.5}
\]

We now introduce a numerical technique to solve problem (1.2)–(1.4) that is based on the discretization of the optimality system (4.1), i.e., we consider the so-called optimize–then–discretize approach. The scheme incorporates stabilized terms into the standard Galerkin discretizations of the state and adjoint equations; no a priori relation between the stabilized terms is required. The stabilized scheme reads as follows: Find \((\bar{y}_\mathcal{T}, \bar{p}_\mathcal{T}, \bar{u}_\mathcal{T}) \in \mathcal{V}(\mathcal{T}) \times \mathcal{V}(\mathcal{T}) \times \mathcal{U}_{ad}(\mathcal{T})\) such that

\[
B(\bar{y}_\mathcal{T}, v_\mathcal{T}) + S(\bar{y}_\mathcal{T}, f + \bar{u}_\mathcal{T}; v_\mathcal{T}) = (f + \bar{u}_\mathcal{T}, v_\mathcal{T})_{L^2(\Omega)}, \quad \forall v_\mathcal{T} \in \mathcal{V}(\mathcal{T}),
\]

\[
B^*(\bar{p}_\mathcal{T}, w_\mathcal{T}) + S^*(\bar{p}_\mathcal{T}, \bar{y}_\mathcal{T} - y_\Omega; w_\mathcal{T}) = (\bar{y}_\mathcal{T} - y_\Omega, w_\mathcal{T})_{L^2(\Omega)}, \quad \forall w_\mathcal{T} \in \mathcal{V}(\mathcal{T}),
\]

\[
(\bar{p}_\mathcal{T} + \bar{v}_\mathcal{T}, u_\mathcal{T} - \bar{u}_\mathcal{T})_{L^2(\Omega)} \geq 0, \quad \forall u_\mathcal{T} \in \mathcal{U}_{ad}(\mathcal{T}),
\]

where \( \mathcal{U}_{ad}(\mathcal{T}) = \{ u_\mathcal{T} \in L^\infty(\Omega) : u_\mathcal{T}|_K \in \mathcal{P}_0(K) \forall K \in \mathcal{T} \} \cap \mathcal{U}_{ad}, \) and \( S \) and \( S^* \) are stabilization terms. The stabilization for the state equation \( S \) is defined in Section 3.3 for different stabilization methods. The stabilization for the adjoint equation reads:

\[
S^*(\bar{p}_\mathcal{T}, \bar{y}_\mathcal{T} - y_\Omega; w_\mathcal{T}|_K) = \sum_{K \in \mathcal{T}} S^*_K(\bar{p}_\mathcal{T}, \bar{y}_\mathcal{T} - y_\Omega; w_\mathcal{T}|_K),
\]

where the local terms \( S^*_K = S^*_K(\bar{p}_\mathcal{T}, \bar{y}_\mathcal{T} - y_\Omega; w_\mathcal{T}|_K) \), for the below mentioned stabilizations, are defined as follows:

- **SUPG:** \( S^*_K := \tau^*_K (-\mathbf{b} \cdot \nabla \bar{p}_\mathcal{T} + \kappa \bar{p}_\mathcal{T} - \bar{y}_\mathcal{T} + y_\Omega, -\mathbf{b} \cdot \nabla w_\mathcal{T})_{L^2(K)} \). \( \tag{4.7} \)

- **GLS:** \( S^*_K := \tau^*_K (-\mathbf{b} \cdot \nabla \bar{p}_\mathcal{T} + \kappa \bar{p}_\mathcal{T} - \bar{y}_\mathcal{T} + y_\Omega, -\mathbf{b} \cdot \nabla w_\mathcal{T} + \kappa w_\mathcal{T})_{L^2(K)} \). \( \tag{4.8} \)

- **CIP:** \( S^*_K := \sum_{\gamma \in \mathcal{F}_K \cap \mathcal{F}_I} \tau^*_\gamma (||\mathbf{b} \cdot \nabla \bar{p}_\mathcal{T}||_{L^2(\gamma)} ; \mathbf{b} \cdot \nabla w_\mathcal{T}|_K)_{L^2(\gamma)} \). \( \tag{4.9} \)

- **ES:** \( S^*_K := \sum_{\gamma \in \mathcal{F}_K \cap \mathcal{F}_I} \tau^*_\gamma (||\nabla \bar{p}_\mathcal{T} \cdot \mathbf{n}_{\gamma}|_{\gamma}|| ; \mathbf{n}_{\gamma} \cdot \mathbf{n}_{\gamma}^K(h^2_K + h^2_{K,\gamma}))_{L^2(\gamma)} \). \( \tag{4.10} \)

In all the above mentioned schemes \( \tau^*_K \) and \( \tau^*_\gamma \) denote nonnegative stabilization parameters that can vary from one method to another.

**Remark 4.1** (optimize–then–discretize). In general, there are two approaches to solve (1.2)–(1.4): optimize–then–discretize and discretize–then–optimize. The first approach is based on the discretization of the optimality system (4.1). In contrast, the second approach first discretizes the optimal control problem (1.2)–(1.4) and then deduces the discrete optimality conditions. In principle, these two approaches do not coincide: they could lead to different discrete problems. If \( S \) and \( S^* \) are based on the same scheme and \( S \) is symmetric, then both approaches lead to the same discrete system.

Before proceeding with the description of our solution technique for (1.2)–(1.4), let us comment on those advocated in the literature. Concerning the a priori theory,
to the best of our knowledge, the first work that analyzed a stabilized scheme is [20].
This work considers the SUPG method, elaborates on the fact that the approaches
optimize–then–discretize and discretize–then–optimize do not coincide and explores
their respective advantages; see [59] for an improvement on the theory. Later, local
projection stabilization (LPS) techniques were proposed in [13, 14]. These techniques
have the advantage that, due to the symmetry of the proposed stabilization term,
optimize–then–discretize and discretize–then–optimize coincide. The ES scheme has
also been employed to derive a discrete technique that approximates the solution to
(1.2)–(1.4) [32, 60]: optimize–then–discretize and discretize–then–optimize coincide.
We refer the reader to [61] for a survey that includes other discretization techniques
and an extensive list of references.

In contrast to this well–established theory, the a posteriori error analysis for
stabilized finite element discretizations of (1.2)–(1.4) is not as well developed. We refer
the reader to [21, 32, 36, 42, 59, 60] for a posteriori error estimators based on different
stabilized schemes: ES scheme, discontinuous Galerkin methods and the Lagrange
functional method. A common feature of all of the above–cited references is that
the upper bound for the error in terms of the estimator, when it is derived, involves
constants that are not known. This motivates the construction of fully computable a
posteriori error indicators.

4.1. A posteriori error analysis: reliability. We assume that the discrete
and nonlinear problem (4.6) has a unique solution. We then construct a posteriori error estimator associated with the optimal control variable

\[ \eta_{ct} := \left( \sum_{K \in \mathcal{T}} \eta_{ct,K}^2 \right)^{1/2} \]

where \( \eta_{ct,K} := \| \tilde{u}_T - \Pi(\frac{1}{2} \bar{p}_T) \|_{L^2(K)} \). (4.11)

Here, \( \Pi \) denotes the non–linear operator defined in (4.4).

We now construct the error estimators associated with the state and adjoint
optimal variables. To accomplish this task, we define \( \hat{y} \in H^1_0(\Omega) \) to be such that

\[ \mathcal{B}(\hat{y}, v) = (f + \tilde{u}_T, v) \quad \forall \ v \in H^1_0(\Omega), \] (4.12)

and \( \hat{p} \in H^1_0(\Omega) \) to be such that

\[ \mathcal{B}^*(\hat{p}, w) = (\hat{y}_T - y_\Omega, w) \quad \forall \ w \in H^1_0(\Omega). \] (4.13)

Performing the a posteriori error analysis presented in Section 3 to bound the error
between the solutions of (4.12) and the discretization of the state equation in (4.6),
and the error between the solutions of (4.13) and the discretization of the adjoint
equation from (4.6), we can conclude that

\[ \| \hat{y} - \bar{y}_T \|_{1, \Omega}^2 \leq \eta_{st}^2 \] and \( \| \hat{p} - \bar{p}_T \|_{1, \Omega}^2 \leq \eta_{ad}^2 \), (4.14)

where, for \( \varrho = st \) or \( \varrho = ad \), the error estimators \( \eta_{st} \) and \( \eta_{ad} \) are defined by

\[ \eta_{st}^2 := \sum_{K \in \mathcal{T}} \eta_{st,K}^2, \quad \eta_{st,K} := \frac{1}{\sqrt{|K|}} |S_K^\varrho(1)| + \frac{\| \sigma_K^\varrho_L^2(K) \|_{L^2(K)}}{\sqrt{\nu}} + C_{osc,K} \| osc_K^\varrho \|_{L^2(K)}. \] (4.15)
Here, for all \( K \in \mathcal{T} \), \( \sigma^0_K \in \mathbb{P}_2(K) \times \mathbb{P}_2(K) \) denotes the solution to
\[
\begin{aligned}
\left\{ \begin{array}{ll}
-\text{div} \, \sigma^0_K &= \mathcal{R}^0_K - \frac{1}{|K|} (\mathcal{R}^0_{\gamma,K})_{L^2(K)} - \frac{1}{|K|} \sum_{\gamma \in \mathcal{F}_K} (\mathcal{R}^0_{\gamma,K})_{L^2(\gamma)} & \text{in } K, \\
\sigma^0_K \cdot n^K &= \mathcal{R}^0_{\gamma,K} & \text{on } K \text{ for all } \gamma \in \mathcal{F}_K,
\end{array} \right.
\end{aligned}
\] (4.16)

which is such that \( \|\sigma^0_K\|_{L^2(K)} \) is minimized, with residuals and oscillation terms defined as
\[
\begin{aligned}
\mathcal{R}^{st}_{\gamma,K} &:= \Pi_K(f) + \bar{u}_{\gamma,K} - \Pi_K(b \cdot \nabla y_{\gamma}) - \kappa y_{\gamma,K}, \\
\mathcal{R}^{st}_{\gamma,K} &= g^{st}_{\gamma,K} - \nu \nabla y_{\gamma} \cdot n^K, \\
\text{osc}_K^{st} &= f - \Pi_K(f) - (b \cdot \nabla y_{\gamma,K} - \Pi_K(b \cdot \nabla y_{\gamma})),
\end{aligned}
\] (4.17)
and
\[
\begin{aligned}
\mathcal{R}^{ad}_{\gamma,K} &:= y_{\gamma,K} - \Pi_K(y_\Omega) + \Pi_K(b \cdot \nabla \bar{p}_{\gamma}) - \kappa \bar{p}_{\gamma,K}, \\
\mathcal{R}^{ad}_{\gamma,K} &= g^{ad}_{\gamma,K} - \nu \nabla \bar{p}_{\gamma} \cdot n^K, \\
\text{osc}^{ad}_K &= -(y_\Omega - \Pi_K(y_\Omega)) + b \cdot \nabla \bar{p}_{\gamma,K} - \Pi_K(b \cdot \nabla \bar{p}_{\gamma}).
\end{aligned}
\] (4.18)

The equilibrated boundary fluxes \( \{g^{0}_{\gamma,K}\} \) are constructed on the basis of the material presented in Section 3.4. First, they must satisfy the consistency property
\[
g^{0}_{\gamma,K} + g^{0}_{\gamma,K'} = 0, \quad \text{if } \gamma \in \mathcal{F}_K \cap \mathcal{F}_{K'}, \ K, K' \in \mathcal{T}, \ K \neq K'.
\] (4.19)

In addition, they must satisfy the first order equilibration condition that is
\[
0 = (f + \bar{u}_{\gamma}, \lambda)_{L^2(K)} - B_K(y_{\gamma}, \lambda) + \sum_{\gamma \in \mathcal{F}_K} (g^{st}_{\gamma,K}, \lambda)_{L^2(\gamma)} - S^t_K(\lambda),
\]
and
\[
0 = (\bar{y} - y_\Omega, \lambda)_{L^2(K)} - B^*_K(p_{\gamma}, \lambda) + \sum_{\gamma \in \mathcal{F}_K} (g^{ad}_{\gamma,K}, \lambda)_{L^2(\gamma)} - S^a_K(\lambda),
\]
for all \( \lambda \in \mathbb{P}_1(K) \) and all \( K \in \mathcal{T} \), and where \( B_K(y_{\gamma}, \lambda) = \nu(\nabla y_{\gamma} \cdot \nabla \lambda)_{L^2(K)} + (b \cdot \nabla y_{\gamma} + \kappa y_{\gamma,K}, \lambda)_{L^2(K)} \),
\( B^*_K(p_{\gamma}, \lambda) = \nu(\nabla \bar{p}_{\gamma} \cdot \nabla \lambda)_{L^2(K)} + (\kappa \bar{p}_{\gamma} - b \cdot \nabla \bar{p}_{\gamma}, \lambda)_{L^2(K)} \) and
\[
S^t_K(\lambda) := \begin{cases} 
S_K(\bar{y}, f + \bar{u}_{\gamma}; \lambda) & \text{for } \varrho = \text{st}, \\
S_K(\bar{p}_{\gamma}, \bar{y} - y_\Omega; \lambda) & \text{for } \varrho = \text{ad}.
\end{cases}
\] (4.20)

Finally, they must satisfy the corresponding analog of (3.24).

For \( G = \Omega \) or \( G \in \mathcal{T} \), we define
\[
\|(e_{\gamma}, e_{\varrho}, e_u)\|_G = \left( \|\bar{y} - y_{\gamma}\|_G^2 + \|\bar{p} - \bar{p}_{\gamma}\|^2_G + \|\bar{u} - \bar{u}_{\gamma}\|^2_{L^2(G)} \right)^{1/2}.
\]

We now present the analysis through which we obtain a fully computable upper bound for the total error for our optimal control problem.

**Theorem 4.2 (global reliability).** Let \( (y, p, \bar{u}) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \) be the solution to (3.1) and \( (y_{\gamma}, p_{\gamma}, \bar{u}_{\gamma}) \in \mathcal{V}(\mathcal{T}) \times \mathcal{V}(\mathcal{T}) \times \mathcal{U}_{ad}(\mathcal{T}) \) its numerical approximation obtained as the solution to (3.2), then
\[
\|(e_{\gamma}, e_{\varrho}, e_u)\|_\Omega \leq Y = \left( \sum_{K \in \mathcal{T}} Y^2_{\delta K} \right)^{1/2}
\] (4.21)
where
\[
\Upsilon^2_K := C_{st}\eta^2_{st,K} + C_{ad}\eta^2_{ad,K} + C_{ct}\eta^2_{ct,K},
\] (4.22)
with \(\eta^2_{st}\) and \(\eta^2_{ad}\) being given in (4.11), \(\eta^2_{ct}\) being defined in (4.11) and
\[
C_{st} := 2 + \frac{4}{k^3} + \frac{8}{\theta^2 k^6} (k^3 + 2k^2 + 4), \quad C_{ad} := 2 + \frac{4}{\theta^2 k^4} (k^3 + 2k^2 + 4), \quad C_{ct} := 2 + \frac{4}{\kappa^3} + \frac{8}{\theta^2 k^2} (k^3 + 2k^2 + 4).
\]

Proof. We proceed in four steps.

Step 1. The goal of this step is to control the error \(\bar{u} - \tilde{u}\). We define \(\tilde{u} = \Pi(-\frac{1}{T}\bar{\varphi}, \bar{\varphi})\), which can be equivalently characterized by
\[
(\bar{p} + \vartheta \tilde{u}, u - \bar{u})_{L^2(\Omega)} \geq 0 \quad \forall u \in \mathcal{U}_{ad}. \tag{4.23}
\]
With this definition at hand, we have that
\[
\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 \leq 2 \left(\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 + \|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2\right) = 2\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 + 2\eta^2_{ct}. \tag{4.24}
\]
Let us now focus on the first term on the right hand side of (4.24). The variational inequality of (4.11) with \(u = \bar{u}\) and (4.23) with \(u = \tilde{u}\) yield that
\[
(\bar{p} + \vartheta \tilde{u}, \tilde{u} - \bar{u})_{L^2(\Omega)} \geq 0 \quad \text{and} \quad (\bar{p} + \vartheta \tilde{u}, \tilde{u} - \bar{u})_{L^2(\Omega)} \geq 0,
\]
from which it follows that
\[
\vartheta \|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 \leq (\bar{p} - \tilde{p} + \tilde{u} - \bar{u})_{L^2(\Omega)}.
\]
To bound the right hand side of the above expression, we let \((\tilde{y}, \tilde{p})\) be such that
\[
\begin{align*}
\tilde{y} &\in H_0^1(\Omega) : \quad B(\tilde{y}, v) = (f + \bar{u}, v) \quad \forall v \in H_0^1(\Omega), \\
\tilde{p} &\in H_0^1(\Omega) : \quad B^*(\tilde{p}, w) = (\tilde{y} - \bar{y}, w) \quad \forall w \in H_0^1(\Omega).
\end{align*}
\]
We then have that
\[
\vartheta \|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 \leq (\bar{p} - \tilde{p} + \tilde{u} - \bar{u})_{L^2(\Omega)} + (\bar{p} - \tilde{p} - \tilde{u} - \bar{u})_{L^2(\Omega)} + (\bar{p} - \tilde{p} - \bar{u})_{L^2(\Omega)}
\]
upon using Cauchy–Schwarz and Young’s inequalities. Hence,
\[
\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 \leq \frac{2}{\vartheta}(\bar{p} - \tilde{p} + \tilde{u} - \bar{u})_{L^2(\Omega)} + \frac{2}{\vartheta} \|\bar{p} - \tilde{p}\|_{L^2(\Omega)}^2 + \frac{2}{\vartheta} \|\tilde{p} - \bar{p}\|_{L^2(\Omega)}^2. \tag{4.25}
\]
We now proceed to bound the term \((\bar{p} - \tilde{p} + \tilde{u} - \bar{u})_{L^2(\Omega)}\). To do this, we note that \(\tilde{y} - \bar{y}\) is such that \(B(\tilde{y} - \bar{y}, v) = (\tilde{u} - \bar{u}, v)_{L^2(\Omega)}\) for all \(v \in H_0^1(\Omega)\) and that \(\bar{p} - \tilde{p}\) solves \(B^*(\bar{p} - \tilde{p}, w) = (\tilde{y} - \bar{y}, w)\) for all \(w \in H_0^1(\Omega)\). Hence,
\[
(\bar{p} - \tilde{p} + \tilde{u} - \bar{u})_{L^2(\Omega)} = B(\tilde{y} - \bar{y}, \bar{p} - \tilde{p}) = B^*(\bar{p} - \tilde{p}, \tilde{y} - \bar{y})_{L^2(\Omega)} = -\|\tilde{y} - \bar{y}\|_{L^2(\Omega)}^2 \leq 0.
\]
This, in conjunction with (4.25) yields
\[
\|\bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 \leq \frac{2}{\vartheta^2} \|\bar{p} - \tilde{p}\|_{L^2(\Omega)}^2 + \frac{2}{\vartheta^2} \|\bar{p} - \tilde{p}\|_{L^2(\Omega)}^2 \leq \frac{2}{\vartheta^2} \|\bar{p} - \tilde{p}\|_{L^2(\Omega)}^2 + \frac{2}{\vartheta^2} \eta^2_{ad}. \tag{4.26}
\]
since \( \| \hat{p} - \hat{p}_\mathcal{T} \|_{L^2(\Omega)}^2 \leq \frac{4}{\kappa} \| \hat{p} - \hat{p}_\mathcal{T} \|_{L^2(\Omega)}^2 \leq \frac{4}{\kappa} \eta_{st}^2 \) because of (4.14). To bound \( \| \hat{p} - \hat{p} \|_{L^2(\Omega)}^2 \) we first note that \( B^*(\hat{p} - \hat{p}, w) = (\tilde{y} - \tilde{y}_\mathcal{T}, w)_{L^2(\Omega)} \) for all \( w \in H_0^1(\Omega) \). So, taking \( w = \hat{p} - \hat{p} \) and using the fact that \( \hat{b} \) is a solenoidal field allows us to conclude that

\[
\kappa \| \hat{p} - \hat{p} \|_{L^2(\Omega)}^2 \leq B^*(\hat{p} - \hat{p}, \hat{p} - \hat{p}) = (\tilde{y} - \tilde{y}_\mathcal{T}, \hat{p} - \hat{p})_{L^2(\Omega)} = (\tilde{y} - \tilde{y} - \tilde{y}_\mathcal{T}, \hat{p} - \hat{p})_{L^2(\Omega)}
\]

and hence,

\[
\| \hat{p} - \hat{p} \|_{L^2(\Omega)}^2 \leq \frac{1}{\kappa} \left( \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)} + \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \right) \| \hat{p} - \hat{p} \|_{L^2(\Omega)}.
\]

Consequently,

\[
\| \hat{p} - \hat{p} \|_{L^2(\Omega)}^2 \leq \frac{2}{\kappa} \left( \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}^2 + \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)}^2 \right) \leq \frac{2}{\kappa} \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}^2 + \frac{2}{\kappa} \eta_{st}^2,
\]

since \( \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \leq \frac{1}{\kappa} \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}^2 \leq \frac{1}{\kappa} \eta_{st}^2 \). This, in conjunction with (4.26), yields

\[
\| \tilde{u} - \tilde{u} \|_{L^2(\Omega)}^2 \leq \frac{2}{\kappa} \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}^2 + \frac{4}{\kappa^2} \eta_{st}^2 + \frac{2}{\kappa} \eta_{ad}^2.
\]

To bound \( \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}^2 \) we first note that \( B(\tilde{y} - \tilde{y}, v) = (\tilde{u} - \tilde{u}_\mathcal{T}, v)_{L^2(\Omega)} \) for all \( v \in H_0^1(\Omega) \). So, taking \( v = \tilde{y} - \tilde{y} \) allows us to conclude that

\[
\kappa \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)} \leq B(\tilde{y} - \tilde{y}, \tilde{y} - \tilde{y}) = (\tilde{u} - \tilde{u}_{\mathcal{T}}, \tilde{y} - \tilde{y})_{L^2(\Omega)} \leq \| \tilde{u} - \tilde{u}_{\mathcal{T}} \|_{L^2(\Omega)} \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}
\]

and hence, \( \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)} \leq \frac{1}{\kappa} \| \tilde{u} - \tilde{u}_{\mathcal{T}} \|_{L^2(\Omega)}^2 = \frac{1}{\kappa} \eta_{st}^2 \). Combining this with (4.27) implies that

\[
\| \tilde{u} - \tilde{u}_{\mathcal{T}} \|_{L^2(\Omega)}^2 \leq \frac{1}{\kappa} \eta_{st}^2 + \frac{4}{\kappa^2} \eta_{st}^2 + \frac{2}{\kappa} \eta_{ad}^2,
\]

which together with (4.24), allows us to conclude that

\[
\| \tilde{u} - \tilde{u}_{\mathcal{T}} \|_{L^2(\Omega)}^2 \leq \frac{8}{\kappa^2} \eta_{st}^2 + \frac{4}{\kappa^2} \eta_{ad}^2 + \frac{2}{\kappa} \eta_{ad}^2.
\]

**Step 2.** The goal of this step is to control the error \( \tilde{y} - \tilde{y}_\mathcal{T} \). Now, we have that

\[
\| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \leq 2 \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)} + \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \leq 2 \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)} + 2 \eta_{st}^2.
\]

Moreover, \( \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \leq \frac{1}{\kappa} \| \tilde{u} - \tilde{u}_\mathcal{T} \|_{L^2(\Omega)}^2 \), since

\[
\| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} = B(\tilde{y} - \tilde{y}, \tilde{y} - \tilde{y}) = (\tilde{u} - \tilde{u}_\mathcal{T}, \tilde{y} - \tilde{y})_{L^2(\Omega)} \leq \frac{1}{\sqrt{\kappa}} \| \tilde{u} - \tilde{u}_\mathcal{T} \|_{L^2(\Omega)} \| \tilde{y} - \tilde{y} \|_{L^2(\Omega)}.
\]

Therefore, upon combining this with (4.29) and (4.28), we can conclude that

\[
\| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \leq (2 + \frac{16}{\kappa^2}) \eta_{st}^2 + \frac{8}{\kappa^2} \eta_{ad}^2 + \left( \frac{4}{\kappa} + \frac{16}{\kappa} \right) \eta_{st}^2.
\]

**Step 3.** The goal of this step is to control the error \( \tilde{p} - \tilde{p}_\mathcal{T} \). Now, we have that

\[
\| \tilde{p} - \tilde{p}_\mathcal{T} \|_{L^2(\Omega)} \leq 2 \| \tilde{p} - \tilde{p} \|_{L^2(\Omega)} + \| \tilde{p} - \tilde{p}_\mathcal{T} \|_{L^2(\Omega)} \leq 2 \| \tilde{p} - \tilde{p} \|_{L^2(\Omega)} + 2 \eta_{ad}^2.
\]

Moreover, \( \| \tilde{p} - \tilde{p}_\mathcal{T} \|_{L^2(\Omega)} \leq \frac{1}{\kappa} \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)}^2 \), since

\[
\| \tilde{p} - \tilde{p}_\mathcal{T} \|_{L^2(\Omega)} = B^*(\tilde{p} - \tilde{p}, \tilde{p} - \tilde{p}) = (\tilde{y} - \tilde{y}_\mathcal{T}, \tilde{p} - \tilde{p})_{L^2(\Omega)} \leq \frac{1}{\kappa} \| \tilde{y} - \tilde{y}_\mathcal{T} \|_{L^2(\Omega)} \| \tilde{p} - \tilde{p} \|_{L^2(\Omega)}.
\]
Therefore, upon combining this with (4.21) and (4.30), we can conclude that

\[ \| \tilde{p} - \tilde{p}_{\gamma} \|^2_{L^2(\Omega)} \leq \left( \frac{1}{\kappa} + \frac{32}{\sigma^2 \kappa^2} \right) \eta_{st}^2 + \left( 2 + \frac{16}{\sigma^2 \kappa^2} \right) \eta_{ad}^2 + \left( \frac{8}{\kappa^3} + \frac{32}{\sigma^2 \kappa^2} \right) \eta_{ct}^2. \]  

(4.32)

**Step 4.** The result claimed follows upon gathering (4.28), (4.30) and (4.32). \( \square \)

**Remark 4.3** (fully computable a posteriori upper bound). The bound (4.21) is a genuine upper bound in the sense that the value of the estimator exceeds the value of the true error regardless of the coarseness of the mesh or the nature of the data of the problem. All constants appearing in the bound are fully specified.

**Remark 4.4** (Poisson problem). If we set \( \nu = 1, b = 0 \) and \( \kappa = 0 \), the state and adjoint equations become the Poisson problem. By examining the proof of Theorem 4.2, it can be seen that if we apply the Poincaré inequality \( \| v \|^2_{L^2(\Omega)} \leq C_P \| \nabla v \|^2_{L^2(\Omega)} \) for all \( v \in H^1_0(\Omega) \), instead of \( \| v \|^2_{L^2(\Omega)} \leq \kappa^{-1} \| v \|^2_{L^2(\Omega)} \), then (4.21) will hold if the constants \( C_{st}, C_{ad} \) and \( C_{ct} \) in (4.11) are taken to be

\[
\begin{align*}
C_{st} &= 2 + 4C_P + \frac{8}{\sqrt{2}} \left( C_P^3 + 2C_P^4 + 4C_P^5 \right), \\
C_{ad} &= 2 + \frac{4}{\sqrt{2}} \left( C_P + 2C_P^2 + 4C_P^3 \right), \\
C_{ct} &= 2 + 4C_P + 8C_P^3 + \frac{8}{\sqrt{2}} \left( C_P^2 + 2C_P^3 + 4C_P^4 \right).
\end{align*}
\]

The resulting bound is fully computable because an upper bound for \( C_P \) can be found, for instance, in [10].

### 4.2. Error estimator: efficiency

We first write the error equation for the state variable and its approximation, for all \( v \in H^1_0(\Omega) \), as

\[
\sum_{K \in \mathcal{T}} \langle \mathcal{R}^s_{K}, v \rangle_{L^2(K)} - 2 \sum_{\gamma \in \mathcal{F}_I} \langle \| J^s_{\gamma} \|, v \rangle_{L^2(\gamma)} = B(\bar{y} - \bar{y}_\gamma, v) - (\bar{u} - \bar{u}_\gamma, v)_{L^2(\Omega)} - \sum_{K \in \mathcal{T}} (osc^s_{K}, v)_{L^2(K)},
\]

(4.33)

with \( \| J^s_{\gamma} \| := \frac{1}{2} \left( J^s_{\gamma K} + J^s_{\gamma K'} \right) \) if \( \gamma \in \mathcal{F}_K \cap \mathcal{F}_{K'}, K \neq K' \) where \( J^s_{\gamma K} := \nu \nabla \bar{y}_\gamma |_K \cdot n_{\gamma} \).

Bubble function arguments then lead to

\[
\frac{h_K}{\sqrt{p}} \| \mathcal{R}^s_{K} \|_{L^2(\gamma)} \leq C \left( C_K \| \bar{y} - \bar{y}_\gamma \|_K + \frac{h_K}{\sqrt{p}} \left( \| osc^s_{K} \|_{L^2(K)} + \| \bar{u} - \bar{u}_\gamma \|_{L^2(K)} \right) \right),
\]

and

\[
\left( \frac{h_K}{\sqrt{p}} \right)^{1/2} \| J^s_{\gamma} \|_{L^2(\gamma)} \leq C \sum_{K \in \Omega_{\gamma}} \left( C_K \| \bar{y} - \bar{y}_\gamma \|_K, + \frac{h_K}{\sqrt{p}} \left( \| osc^s_{K} \|_{L^2(K')} + \| \bar{u} - \bar{u}_\gamma \|_{L^2(K')} \right) \right).
\]

Now, we write the error equation for the adjoint variable and its approximation as

\[
\sum_{K \in \mathcal{T}} \langle \mathcal{R}^{ad}_{K}, v \rangle_{L^2(K)} - 2 \sum_{\gamma \in \mathcal{F}_I} \langle \| J^{ad}_{\gamma} \|, v \rangle_{L^2(\gamma)} = B^*(\bar{p} - \bar{p}_\gamma, v) - (\bar{y} - \bar{y}_\gamma, v)_{L^2(\Omega)} - \sum_{K \in \mathcal{T}} (osc^{ad}_{K}, v)_{L^2(K)},
\]
for all $v \in H^1_0(\Omega)$, where, with $J^ad_{\gamma,K} := \nu \nabla \tilde{p} \cdot \nabla_{K} \mathbf{n}^K_\gamma$, $[J^ad_{\gamma}]$ is defined analogously to $[J^a_{\gamma}]$. By again using bubble function arguments, we can establish that

$$\frac{h_K}{\nu} \| J^ad_{\gamma,K} \|_{L^2(K)} \leq C \left( C_K \| \tilde{p} - \bar{p} \|_K + \frac{h_K}{\sqrt{\nu}} \left( \| \text{osc}^ad_{\gamma,K} \|_{L^2(K)} + \frac{1}{\sqrt{\nu}} \| \bar{y} - \bar{y}_{(K)} \|_K \right) \right),$$

and

$$\left( \frac{h_K}{\nu} \right)^{1/2} \| [J^ad_{\gamma}] \|_{L^2(\gamma)} \leq C \sum_{K' \in \mathcal{T}_h} \left( C_{K'} \| \tilde{p} - \bar{p} \|_{K'} + \frac{h_{K'}}{\sqrt{\nu}} \left( \| \text{osc}^ad_{\gamma,K'} \|_{L^2(K')} + \frac{1}{\sqrt{\nu}} \| \bar{y} - \bar{y}_{(K')} \|_{(K')} \right) \right).$$

Moreover, an application of the triangle inequality, (4.13) and (4.14), yield that

$$\eta_{et,K} = \| \bar{u}_{(\gamma)} - \Pi(-\frac{1}{\sigma} \tilde{p} \frac{\gamma}{s}) \|_{L^2(K)} \leq \| \bar{u} - \bar{u}_{(\gamma)} \|_{L^2(K)} + \| \Pi(-\frac{1}{\sigma} \tilde{p}) - \Pi(-\frac{1}{\sigma} \tilde{p} \frac{\gamma}{s}) \|_{L^2(K)}$$

$$\leq \| \bar{u} - \bar{u}_{(\gamma)} \|_{L^2(K)} + \frac{1}{\sqrt{\sigma}} \| \tilde{p} - \bar{p} \|_K$$

and hence

$$\eta_{et,K}^2 \leq 2 \left( \| \bar{u} - \bar{u}_{(\gamma)} \|_{L^2(K)}^2 + \frac{1}{\sqrt{\sigma}} \| \tilde{p} - \bar{p} \|_K^2 \right). \quad (4.34)$$

Gathering all of the previous results with (4.15) and following the analysis presented in Section 3.7 allows us to conclude the following result.

**Theorem 4.5 (local efficiency).** Let $(\bar{y}, \tilde{p}, \bar{u})$ be the solution to (4.1) and $(\bar{y}_{(\gamma)}, \bar{p}_{(\gamma)}, 
\bar{u}_{(\gamma)})$ be the solution to (4.10). In addition, let the stabilization parameters be such that, for all $K \in \mathcal{T}$ and for both the state and adjoint equations, $C_{S,K}$ can be bounded by a constant which is independent of the size of the elements in the mesh. Then,

$$\gamma_{K}^2 \leq C_{ef} \sum_{K' \in D_K} \left( \| (e_{y}, e_{\bar{p}}, e_{\bar{u}}) \|_{K'}^2 \right) + h_K \left( \| \text{osc}_{st}^2 \|_{L^2(K')} + \| \text{osc}_{ad}^2 \|_{L^2(K')} \right),$$

where the constant $C_{ef}$ depends on the physical parameters in (4.1) but is independent of the size of the elements in the mesh and $D_K = \{ K' \in \mathcal{T} : \mathcal{f}_{K'} \cap \mathcal{f}_{K''} \neq \emptyset, K'' \in \Omega_K \}$ with $\Omega_K = \{ K' \in \mathcal{T} : K' \cap K \neq \emptyset \}$.

**5. Robust a posteriori error estimation.** In this section we derive and analyze robust a posteriori error estimates for the approximation of the optimality system (4.1) obtained using the stabilized scheme (4.14). We immediately comment that by robustness we mean that the constants involved in the upper and lower bounds for the error are independent of the diffusion parameter $\nu$ and the vector field $\mathbf{b}$. Regarding the stabilization, in this section we only consider the following techniques: the streamline upwind Petrov–Galerkin method (SUPG) and the continuous interior penalty method (CIP).

The analysis is based on the results by Verfürth [54, 55] and Tobiska and Verfürth [56]. As in these works, we measure the error in a norm that adds, to the energy norm, the dual norm of the convective derivative. We also refer the reader to [2, 21, 24, 35, 47, 48] for different approaches.
5.1. The state and adjoint equations. We assume that the data of the problem (5.2) satisfy, in addition to the assumptions (A1)–(A3) of section 3.1, the following assumption:

(B1) \( 0 < \nu \ll 1 \).

This assumption emphasizes that, in this section, we are interested in the convection–dominated regime.

The presented analysis hinges on an appropriate choice of norm. We define

\[
\|v\|_R := \|v\|_\Omega + \|b \cdot \nabla v\|_\ast \quad \forall v \in H^1_0(\Omega),
\]

(5.1)

where \( \|v\|_\Omega \) is defined as in (3.4), and

\[
\|b \cdot \nabla v\|_\ast = \sup_{\phi \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle b \cdot \nabla v, \phi \rangle}{\|\phi\|_\Omega}.
\]

(5.2)

The term \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( H^1_0(\Omega) \) and \( H^{-1}(\Omega) \). In this setting, we have that

\[
\sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\mathcal{B}(w, v)}{\|v\|_\Omega} \geq \frac{1}{3}\|w\|_R \quad \forall w \in H^1_0(\Omega).
\]

(5.3)

We refer the reader to [54] [Lemma 3.1] and [55] [Proposition 4.17] for more details. We also have that

\[
\sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\mathcal{B}^*(w, v)}{\|v\|_\Omega} \geq \frac{1}{3}\|w\|_R \quad \forall w \in H^1_0(\Omega).
\]

(5.4)

Moreover, we note that

\[
\|v\|_{L^2(\Omega)}^2 \leq \frac{1}{\pi} \|v\|_\Omega^2 \leq \|v\|_R^2 \quad \forall v \in H^1_0(\Omega).
\]

(5.5)

We follow [50, section 2] and define, for \( \varrho = \text{st} \) or \( \varrho = \text{ad} \), the error estimator

\[
\mathcal{E}^2_{\varrho} := \sum_{K \in \mathcal{T}} \mathcal{E}^2_{\varrho, K},
\]

(5.6)

where the local error indicators are given by

\[
\mathcal{E}^2_{\varrho, K} := h_K^2 \|R^\varrho_K\|_{L^2(K)} + \sum_{\gamma \in \mathcal{F}_K \cap \mathcal{F}_I} \nu^{-1/2} h_{\gamma} \|\mathcal{J}^\varrho_{\gamma}\|_{L^2(\gamma)}.
\]

Here, \( h_\omega = \min \{ h_\omega^{-1/2}, h_K^{-1/2} \} \) for \( \omega = K \) or \( \omega = \gamma \), \( R^\varrho_K \) is defined as in (4.17) or (4.18), and the jump term \( \mathcal{J}^\varrho_{\gamma} \) is defined as in section 4.2. We also define, again for \( \varrho = \text{st} \) or \( \varrho = \text{ad} \), the global oscillation term

\[
\|\text{osc}^\varrho\|_{L^2(\Omega)} = \left( \sum_{K \in \mathcal{T}} h_K^2 \|\text{osc}^\varrho_K\|_{L^2(K)}^2 \right)^{1/2},
\]

(5.7)

where \( \text{osc}^\varrho_K \) is defined as in (4.17) or (4.18).
As stabilization techniques we will consider the SUPG and CIP methods; see (5.11), (5.10), (4.7) and (4.9). We assume that their stabilization parameters are such that (50)[ineq. (2.7) and (2.10)]:

\[ \|b\|_{L^\infty(K)} \tau_K \leq Ch_K \quad \text{and} \quad \|b\|_{L^\infty(K)} \tau^*_K \leq Ch_K \quad \forall K \in \mathcal{F} \quad (5.8) \]

and

\[ \tau_\gamma \leq Ch_\gamma^2 \quad \text{and} \quad \tau^*_\gamma \leq Ch_\gamma^2 \quad \forall \gamma \in \mathcal{F}_I, \quad (5.9) \]

where \( C \) is independent of the size of the elements in the mesh and the physical parameters. We also define \( \Theta(\xi) = \sum_{K \in \mathcal{F}} \Theta_K(\xi)^2 \), where

\[ \Theta_K(\xi) = \begin{cases} h_K \| (b - \Pi_K(b)) \cdot \nabla \xi \|_{L^2(K)} + h_K \| \nabla b \|_{L^\infty(K)} \| \nabla \xi \|_{L^2(K)}, & \text{CIP,} \\ 0, & \text{SUPG.} \end{cases} \]

The assumptions (5.8) and (5.9) on the stabilization parameters guarantee that the so-called consistency error can be bounded robustly by the residual estimator (5.10) and the oscillation term (5.7); see (50)[Lemma 2.3 and Lemma 2.6]. This is a key result in the derivation of the main result of (50) Theorem 2.8 (see also (54) Theorem 4.1). Let \( \tilde{y} \) be the solution to problem (4.12) and \( \bar{y}_\gamma \) be the solution to the first variational equation of the discrete optimality system (4.6). If (5.8) and (5.9) hold, then

\[ \frac{1}{\tau_\gamma} \mathcal{E}_{\text{st}} - \| \text{osc}^{\text{st}} \|_{L^2(\Omega)} \leq \| \tilde{y} - \bar{y}_\gamma \|_R \leq C_T \left( \mathcal{E}_{\text{st}}^2 + \| \text{osc}^{\text{st}} \|_{L^2(\Omega)}^2 + \Theta(\bar{y}_\gamma)^2 \right)^{\frac{1}{2}}. \quad (5.10) \]

We have a similar result for the numerical approximation of the adjoint equation. Let \( \tilde{p} \) be the solution to problem (1.13) and \( \bar{p}_\gamma \) be the solution to the second variational equation of the discrete optimality system (4.6). If (5.8) and (5.9) hold, then

\[ \frac{1}{\tau_p} \mathcal{E}_{\text{ad}} - \| \text{osc}^{\text{ad}} \|_{L^2(\Omega)} \leq \| \tilde{p} - \bar{p}_\gamma \|_R \leq C_T \left( \mathcal{E}_{\text{ad}}^2 + \| \text{osc}^{\text{ad}} \|_{L^2(\Omega)}^2 + \Theta(\bar{p}_\gamma)^2 \right)^{\frac{1}{2}}. \quad (5.11) \]

We immediately comment that the estimates (5.10) and (5.11) are robust in the sense that the constants \( C_T \), \( C_p \), \( D_\gamma \), and \( D_p \) are independent of \( \nu \) and \( b \).

5.2. The optimal control problem. We now derive a posteriori error estimates for the discretization of the optimal control problem (1.2)–(1.4) proposed in section 4. We now present a modification of the analysis elaborated in the proof of Theorem 4.2 in order to obtain an estimator, for the error \( \| \tilde{y} - \bar{y}_\gamma \|_R^2 + \| \tilde{p} - \bar{p}_\gamma \|_R^2 + \| \tilde{u} - \bar{u}_\gamma \|_{L^2(\Omega)}^2 \), that is robust with respect to \( \nu \) and \( b \) in the sense that the constants involved in the upper and lower bounds for the error are independent of \( \nu \) and \( b \).

THEOREM 5.1 (global reliability). Let \( (\tilde{y}, \tilde{p}, \tilde{u}) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \) be the solution to (1.1) and \( (\bar{y}_\gamma, \bar{p}_\gamma, \bar{u}_\gamma) \in V(\mathcal{F}) \times \mathcal{W}(\mathcal{F}) \times \mathcal{U}_{\text{ad}}(\mathcal{F}) \) be its numerical approximation obtained as the solution to (4.6). If the stabilization parameters are such that (5.8) and (5.9) hold, then

\[ \| \tilde{y} - \bar{y}_\gamma \|_R^2 + \| \tilde{p} - \bar{p}_\gamma \|_R^2 + \| \tilde{u} - \bar{u}_\gamma \|_{L^2(\Omega)}^2 \leq C_1 T_R^2 \quad (5.12) \]

where

\[ T_R^2 = \mathcal{E}_{\text{st}}^2 + \mathcal{E}_{\text{ad}}^2 + \eta_{\text{ct}}^2 + \| \text{osc}^{\text{st}} \|_{L^2(\Omega)}^2 + \| \text{osc}^{\text{ad}} \|_{L^2(\Omega)}^2 + \Theta(\bar{y}_\gamma)^2 + \Theta(\bar{p}_\gamma)^2 \quad (5.13) \]
and the positive constant $C_1$ is independent of the size of the elements in the mesh, $\nu$ and $b$.

**Proof.**  

**Step 1.** We begin the proof by recalling the estimate (4.24):

$$\|\bar{\nu} - \tilde{\nu}\|_{L^2(\Omega)}^2 \leq 2\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}^2 + 2\eta_{cl}^2, \quad \bar{\nu} = \Pi(-\frac{1}{\nu}\bar{\nu}),$$

where $\eta_{cl}$ is defined as in (4.11). We control the first term on the right hand side of the previous expression using the intermediate step in (4.20). We thus have that

$$\|\bar{\nu} - \tilde{\nu}\|_{L^2(\Omega)}^2 \leq \frac{2\nu}{\nu^2}\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}^2 + \frac{2\nu^3}{\nu^4}\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}^2$$

$$\leq \frac{2\nu}{\nu^2}\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}^2 + \frac{2C_2^2}{\nu^4}\left(\varepsilon_{ad}^2 + \|\text{osc}_{ad}\|_{L^2(\Omega)}^2 + \Theta(\bar{\nu})^2\right),$$

upon using (5.5) and (5.4). We now proceed to control the term $\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}$. To accomplish this task, we invoke (5.5) and (5.4) to obtain that

$$\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\kappa}}\|\bar{\nu} - \bar{\nu}\|_{\Omega} \leq \frac{3}{\sqrt{\kappa}} \sup_{v \in H^1_0(\Omega) \setminus \{0\}} B^*(\bar{\nu} - \bar{\nu}, v) \|v\|_{\Omega}.$$

We now utilize the problem that $\bar{\nu} - \bar{\nu}$ solves and (5.6) to arrive at

$$\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)} \leq \frac{3}{\sqrt{\kappa}} \sup_{v \in H^1_0(\Omega) \setminus \{0\}} (\bar{\nu} - \bar{\nu}, v)_{L^2(\Omega)} \|v\|_{\Omega} \leq \frac{3}{\kappa^{1/2}} \|\bar{\nu} - \bar{\nu}\|_{\Omega}. \tag{5.14}$$

To control the right hand side of the previous expression we invoke the triangle inequality and the a posteriori error estimate (5.10). This yields that

$$\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)} \leq \frac{9}{\nu^2}\|\bar{\nu} - \bar{\nu}\|_{\Omega} \leq \frac{18}{\nu^2}\|\bar{\nu} - \bar{\nu}\|_{\Omega} + \frac{18C_2^2}{\nu^4}\left(\varepsilon_{ad}^2 + \|\text{osc}_{ad}\|_{L^2(\Omega)}^2 + \Theta(\bar{\nu})^2\right).$$

The term $\|\bar{\nu} - \bar{\nu}\|_{\Omega}$ is controlled using similar arguments to the ones that lead to (5.14):

$$\|\bar{\nu} - \bar{\nu}\|_{\Omega} \leq \frac{3}{\sqrt{\kappa}} \sup_{v \in H^1_0(\Omega) \setminus \{0\}} B^*(\bar{\nu} - \bar{\nu}, v) \|v\|_{\Omega} \leq \frac{3}{\sqrt{\kappa}} \|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)} \leq \frac{3}{\sqrt{\kappa}} \eta_{cl},$$

where $\eta_{cl}$ is the estimator associated with the control variable that is defined in (4.11). By gathering our previous findings, we conclude that

$$\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}^2 \leq \left(2 + \frac{64}{\eta_{cl}}\right)\|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}^2 + \frac{4C_2^2}{\nu^2}\left(\varepsilon_{ad}^2 + \|\text{osc}_{ad}\|_{L^2(\Omega)}^2 + \Theta(\bar{\nu})^2\right)$$

$$+ \frac{72C_2^2}{\nu^4}\left(\varepsilon_{ad}^2 + \|\text{osc}_{ad}\|_{L^2(\Omega)}^2 + \Theta(\bar{\nu})^2\right). \tag{5.15}$$

**Step 2.** The goal of this step is to control the error $\|\bar{\nu} - \bar{\nu}\|_{\Omega}$. In fact, using standard inequalities and (5.11), we arrive at

$$\|\bar{\nu} - \bar{\nu}\|_{\Omega} \leq 2\|\bar{\nu} - \bar{\nu}\|_{\Omega} + 2C_2^2\left(\varepsilon_{ad}^2 + \|\text{osc}_{ad}\|_{L^2(\Omega)}^2 + \Theta(\bar{\nu})^2\right).$$

By using (5.3) and the problem that $\bar{\nu} - \bar{\nu}$ solves, we obtain that

$$\|\bar{\nu} - \bar{\nu}\|_{\Omega} \leq \sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{B^*(\bar{\nu} - \bar{\nu}, v)}{\|v\|_{\Omega}} \leq \frac{3}{\sqrt{\kappa}} \|\bar{\nu} - \bar{\nu}\|_{L^2(\Omega)}.$$
Therefore, invoking (5.15) we arrive at
\[
\|\bar{y} - \bar{y}_\sigma\|_R^2 \leq \frac{18}{\kappa} (2 + \frac{648}{\nu^2 \kappa^4}) \eta_{ct}^2 + \frac{72 C^2}{\nu^2 \kappa^2} \left( \mathcal{E}_{st}^2 + \|\text{osc}^d\|_{L^2(\Omega)}^2 + \Theta(\bar{p}, \sigma)^2 \right) \\
+ (2 + \frac{1296}{\nu^2 \kappa^2}) C^2 \left( \mathcal{E}_{ad}^2 + \|\text{osc}^d\|_{L^2(\Omega)}^2 + \Theta(\bar{p}, \sigma)^2 \right).
\] (5.16)

**Step 3.** In this step we bound \(\|\bar{p} - \bar{p}_\sigma\|_R\). We invoke similar arguments to the ones elaborated in step 2 and conclude that
\[
\|\bar{p} - \bar{p}_\sigma\|_R^2 \leq 2(\|\bar{p} - \bar{p}_R\|_R^2 + \|\bar{p} - \bar{p}_\sigma\|_R^2) \leq 2\|\bar{p} - \bar{p}_R\|_R^2 + 2C^2 \left( \mathcal{E}_{ad}^2 + \|\text{osc}^d\|_{L^2(\Omega)}^2 + \Theta(\bar{p}, \sigma)^2 \right)
\]
and that
\[
\|\bar{p} - \bar{p}_R\| \leq 3 \sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{B^*(\bar{p} - \bar{p}_R, v)}{\|v\|_\Omega} \leq \frac{3}{\kappa} \|\bar{y} - \bar{y}_\sigma\|_R.
\]

We thus use these estimates, and (5.10), to arrive at
\[
\|\bar{p} - \bar{p}_\sigma\|_R^2 \leq \frac{324}{\nu^2 \kappa^2} (2 + \frac{648}{\nu^2 \kappa^4}) \eta_{ct}^2 \\
+ (2 + \frac{1296}{\nu^2 \kappa^2}) C^2 \left( \mathcal{E}_{ad}^2 + \|\text{osc}^d\|_{L^2(\Omega)}^2 + \Theta(\bar{p}, \sigma)^2 \right) \\
+ \frac{18}{\kappa} (2 + \frac{1296}{\nu^2 \kappa^2}) C^2 \left( \mathcal{E}_{st}^2 + \|\text{osc}^e\|_{L^2(\Omega)}^2 + \Theta(\bar{y}, \sigma)^2 \right).
\] (5.17)

**Step 4.** Finally, combining the estimates (5.15), (5.10), and (5.17) allows us to arrive at (5.12).

We now provide an efficiency analysis.

**Theorem 5.2** (global efficiency). Let \((\bar{y}, \bar{p}, \bar{u}) \in H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)\) be the solution to (4.11) and \((\bar{y}_\sigma, \bar{p}_\sigma, \bar{u}_\sigma) \in \mathcal{V}(\mathcal{T}) \times \mathcal{V}(\mathcal{T}) \times \mathcal{U}_h(\mathcal{T})\) be its numerical approximation obtained as the solution to (4.10). If the stabilization parameters are such that (5.8) and (5.9) hold, then
\[
\mathcal{E}_{st}^2 + \mathcal{E}_{ad}^2 + \eta_{ct}^2 \leq C_2 \left( \|\bar{y} - \bar{y}_\sigma\|^2_2 + \|\bar{p} - \bar{p}_\sigma\|^2_2 + \|\bar{u} - \bar{u}_\sigma\|_{L^2(\Omega)}^2 + \|\text{osc}^e\|_{L^2(\Omega)}^2 + \|\text{osc}^d\|_{L^2(\Omega)}^2 \right),
\] (5.18)

where the positive constant \(C_2\) is independent of the size of the elements in the mesh, \(\nu\) and \(b\).

**Proof.** From (5.10) and (5.11) we have that
\[
\mathcal{E}_{st} \leq D_y \left( \|\bar{y} - \bar{y}_\sigma\|_R + \|\text{osc}^e\|_{L^2(\Omega)} \right) \leq D_y \left( \|\bar{y} - \bar{y}\|_R + \|\bar{y} - \bar{y}_\sigma\|_R + \|\text{osc}^e\|_{L^2(\Omega)} \right)
\]
and
\[
\mathcal{E}_{ad} \leq D_p \left( \|\bar{p} - \bar{p}_\sigma\|_R + \|\text{osc}^d\|_{L^2(\Omega)} \right) \leq D_p \left( \|\bar{p} - \bar{p}\|_R + \|\bar{p} - \bar{p}_\sigma\|_R + \|\text{osc}^d\|_{L^2(\Omega)} \right).
\]
Moreover, (5.3) and (5.4) imply that
\[
\|\bar{y} - \bar{y}\|_R \leq 3 \sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{B(\bar{y} - \bar{y}, v)}{\|v\|_\Omega} \text{ and } \|\bar{p} - \bar{p}\|_R \leq 3 \sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{B^*(\bar{p} - \bar{p}, v)}{\|v\|_\Omega}.
\]
Now, (4.11) and (4.12) yield that
\[ B(\bar{y} - \tilde{y}, \nu) = (\bar{u} - \tilde{u}, \nu)_L^2(\Omega) \leq \|\bar{u} - \tilde{u}\|_L^2(\Omega) \|\nu\|_L^2(\Omega) \leq \frac{1}{\sqrt{\kappa}}\|\bar{u} - \tilde{u}\|_L^2(\Omega) \|\nu\|_\Omega \]
and (4.11) and (4.13) yield that
\[ B^*(\bar{p} - \tilde{p}, \nu) = (\bar{y} - \tilde{y}, \nu)_L^2(\Omega) \leq \|\bar{y} - \tilde{y}\|_L^2(\Omega) \|\nu\|_L^2(\Omega) \leq \frac{1}{\kappa}\|\bar{y} - \tilde{y}\|_R \|\nu\|_\Omega. \]
Hence
\[
\varepsilon^2_{\text{st}} \leq 3D_y \left( \frac{9}{\kappa^2}\|\bar{u} - \tilde{u}\|_L^2(\Omega)^2 + \|\bar{y} - \tilde{y}\|_R^2 + \|\text{osc}_{\Omega}\|_L^2(\Omega)^2 \right) \tag{5.19}
\]
and
\[
\varepsilon^2_{\text{ad}} \leq 3D_p \left( \frac{9}{\kappa^2}\|\bar{y} - \tilde{y}\|_R^2 + \|\bar{p} - \tilde{p}\|_R^2 + \|\text{osc}_{\Omega}\|_L^2(\Omega)^2 \right). \tag{5.20}
\]
In addition, (4.31) leads to
\[
\eta^2_{\text{st}} \leq 2 \left( \|\bar{u} - \tilde{u}\|_L^2(\Omega)^2 + \frac{2}{\sqrt{\kappa}}\|\bar{p} - \tilde{p}\|_R^2 \right). \tag{5.21}
\]
The claimed result then follows upon combining (5.19), (5.20) and (5.21). □

Remark 5.3 (robustness). We remark that the a posteriori error estimator \( \Upsilon_\kappa \) is robust in the sense that the constants that appear in (5.12) and (5.18) are independent of \( \nu \) and \( b \). The estimator is not robust with respect to \( \kappa \) or \( \vartheta \). The dependence of the constant in (5.12) on \( \kappa \) and \( \vartheta \) can be seen from (5.19), (5.10), and (5.17). Similarly, the dependence of the constant in (5.18) on \( \kappa \) and \( \vartheta \) can be seen from (5.19), (5.20) and (5.21).

6. Numerical examples. In this section we show numerical examples that illustrate the performance of the error estimator. We wrote a code in C++ that implements the procedure described in Algorithm 1. The integrals involving the data \( y_\Omega \) and \( f \) were computed using quadrature formulas which are exact for polynomials of degree \( N \). We show results for \( N \in \{4, 19\} \) for \( d = 2 \) and \( N \in \{4, 14\} \) for \( d = 3 \). The error \( ||(e_y, e_p, e_\vartheta)||_\Omega \) was computed using a quadrature formula which is exact for polynomials of degree 19 for \( d = 2 \) and 14 for \( d = 3 \). The global linear systems were solved using the multifrontal massively parallel sparse direct solver (MUMPS) [18]. In order to construct exact solutions we fix the optimal state and adjoint state, and compute the optimal control and data \( y_\Omega \) and \( f \) using (4.4) and (4.1).

Algorithm 1: Adaptive Primal-Dual Active Set Algorithm.

| Step | Description |
|------|-------------|
| 1.   | Compute \((y_\Omega, p_\Omega, u_\Omega) \in V(\mathcal{T}) \times V(\mathcal{T}) \times \text{Lag}(\mathcal{T})\) that solves (4.11) using the active set strategy of (4.12). |
| 2.   | Compute the local error indicator \( \Upsilon_k \) given in (4.19) for each \( K \in \mathcal{T} \) and the error estimator \( \Upsilon \) given in (4.31). |
| 3.   | Mark an element \( K \in \mathcal{T} \) for refinement if \( \Upsilon_k^2 \geq \Upsilon^2/\#\mathcal{T} \). |
| 4.   | Refine the mesh \( \mathcal{T} \) using a longest edge bisection algorithm and return to step 1. |

Example 1: We set \( d = 2, a = -1, b = -0.1, \lambda = 1, \nu = 10^{-3}, b = (1, 0) \) and \( \kappa = 1 \). The exact optimal state and adjoint are given by, taking \( \varsigma := x_2(1 - x_2) \),
\[
\bar{y}(x_1, x_2) = \varsigma \left( x_1 + \frac{e^{-1} - e^{-\nu}}{e^{-\nu} - 1} \right), \quad \bar{p}(x_1, x_2) = \varsigma \left( 1 - x_1 + \frac{e^{-1} - e^{-\nu}}{e^{-\nu} - 1} \right). 
\]
Example 2: We set \(d = 3\), \(a = -0.01\), \(b = 0.01\), \(\lambda = 1\), \(\nu = 0.01\), \(\mathbf{b} = (3, 2, 1)\) and \(\kappa = 10\). The exact optimal state and adjoint are given by

\[
\bar{y}(x_1, x_2, x_3) = \prod_{i=1}^{3} x_i (1 - x_i), \quad \bar{p}(x_1, x_2, y_3) = \bar{y}(x_1, x_2, y_3) \tan^{-1} \left( \frac{x_1 - 0.5}{\nu} \right).
\]

In Figures 6.1 and 6.3, we present the performance of the adaptive procedure by showing the total error \(\| (e_y, e_p, e_b) \|_\Omega\) and error estimator \(\Upsilon\), as well as effectivity indices \(\Upsilon / \| (e_y, e_p, e_b) \|_\Omega\), for different combinations of stabilizations for state-adjoint equations. We use the following notation: SUPG–GLS corresponds to using a SUPG stabilization for the state equation and a GLS stabilization for the adjoint equation; SUPG–SUPG, SUPG–CIP and SUPG–ES are defined analogously. We took the stabilization parameters \(\tau_K\) where

\[
\tau_K = \begin{cases} 
\frac{h_K}{2 b L_{L^\infty(K)}}, & \text{if } P e_K > 1, \\
\frac{h_K}{2 b L_{L^\infty(K)}} + \frac{1}{12}, & \text{if } P e_K \leq 1,
\end{cases}
\]

for SUPG and GLS, \(\tau^*_S = 1/24\) for ES and \(\tau^*_S = h_K^2/12\) for CIP. The total number of degrees of freedom \(\text{Ndof} = 2 \dim(V(T)) + \#T\). In Figures 6.1 and 6.3, we observe that, once the mesh has been sufficiently refined, the experimental rates of convergence are optimal. Computationally, we observe that the estimator is never less than the error, the effectivity index never goes below 1; on the final meshes takes the numerical value stated in the plots. In Figures 6.2 and 6.4 we observe that the refinement is being concentrated around the boundary and interior layers, even though different values of \(N\) resulted in different adaptively refined meshes.

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Example 1 - Υ

Example 1: The error \( \|e_y, e_p, e_u\|_\Omega \), estimator \( \Upsilon \) and effectivity indices \( \Upsilon/\|e_y, e_p, e_u\|_\Omega \) obtained with \( N = 19 \) (top) and \( N = 4 \) (bottom).

Fig. 6.2. Example 1: The initial mesh (left) and the 25th adaptively refined meshes obtained with \( N = 19 \) (middle, \( \text{Ndof} = 156476 \)) and \( N = 4 \) (right, \( \text{Ndof} = 156508 \)).

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Example 2 - The error \( \|e_y, e_p, e_u\|_{\Omega} \), estimator \( \Upsilon \) and effectivity indices \( \Upsilon / \|e_y, e_p, e_u\|_{\Omega} \) obtained with \( N = 14 \) (top) and \( N = 4 \) (bottom).

Fig. 6.4. Example 2: The initial mesh (left) and the 25th adaptively refined meshes obtained with \( N = 14 \) (middle, \( \text{Ndo}f = 15261784 \)) and \( N = 4 \) (right, \( \text{Ndo}f = 13511617 \)).

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