A practical approach to the problem of the missing imaginary part of the handbag diagram in the confined Bethe-Salpeter framework

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In the confined Bethe-Salpeter (BS) methods, the quark propagator is an entire function. Hence, the imaginary part of the handbag diagram disappears, leading to a problem of the vanishing parton distribution function (PDF). In contrast, the direct calculation of the light-cone (LC) momentum distribution does give a non-vanishing result even in the confined BS framework. We consider their precise relation and difference, and propose to use the latter as a practical approach to this problem. Our formalism is general enough to be applied to various effective models.

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The parton distribution function (PDF) is defined through the factorization procedure in the perturbative QCD (pQCD) as a low energy-scale quantity which cannot be calculated within the framework of pQCD itself. It provides us with a unique tool to extract the information on the quark-gluon structure of hadrons directly based on experiments. In order to calculate PDF theoretically, the information of the hadronic wave function plays the essential role. Hence, it is necessary to resort to various low-energy nonperturbative methods, such as low energy effective models, the light-front quantization, lattice QCD, etc.

The Bethe-Salpeter (BS) method is one such method, which makes it possible to evaluate a number of hadronic low-energy observables in a relativistically covariant manner. The BS equation has been applied to mesons which consist of mainly a quark and an antiquark. The BS description of baryons as the relativistic three quark system has been developed in the last decade, utilizing the Faddeev method for the quantum three body problem. The Faddeev method transforms the three-quark BS equation into the relativistic Faddeev equation, which justifies the use of the quark-diagonal BS equation as its approximation.

In the BS method, the effect of the color confinement can be taken into account through the entire function nature of the nonperturbative quark propagator. Thereby, many of the shortcomings can be improved such as the unphysical threshold for decay into colored particles, description of higher resonances, etc. However, there is actually a serious drawback in calculating PDF. Since it is essentially the wave function squared, the PDF is realized as the Bjorken limit of its imaginary part. In most of the low energy effective models, this procedure has been adopted providing reasonable answers. In the confined BS framework, due to the entire nature of the propagator, the imaginary part of the handbag diagram disappears, leading to a serious problem of the vanishing PDF.

A straightforward solution of this problem may be found by considering the intermediate “hadronic loops”, which can provide imaginary parts due to the color-singlet nature of hadrons. If we adopt this as the solution, we have to consider the full forward Compton amplitude instead of the handbag diagram. Remember that, in order to calculate the full Compton amplitude, one has to adopt the recently developed “gauge method”, which determines the unique set of Feynman diagrams associated with the particular choice of BS interaction kernel. (Other choice of Feynman diagrams lead to the violation of Ward identity.) It follows that, in order to obtain the hadronic loop, one has to solve a highly complicated BS equation. This is of course formidable. Its numerical solution is practically impossible to be obtained. Moreover, the construction of the BS equation itself is extremely non-trivial. The transparent relation to the parton picture will be lost as well.

In this way, one has to figure out another practical approach to PDF in the confined BS framework. Actually, there is another method for PDF by resorting to the direct calculation of the light-cone (LC) momentum distribution. Since it is essentially the wave function squared with the plus component of the LC momentum of one of the partons fixed, it is not expected to vanish unless the wave function itself vanishes. This procedure has been mainly adopted in the light-front framework, where the LC momentum distribution is naturally expressed as the equal light-front time correlator. On the other hand, in the effective models (including BS method), this procedure has been scarcely adopted except for a few authors.

The latter method is conceptually sound in the sense that the Bjorken limit should not be considered within the framework of the low energy effective theories. The product of the currents should be factorized at the level of pQCD into the product of the high energy scale quantity (the coefficient functions)

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and low scale quantity (the composite operators, i.e., the LC momentum distributions). The BS method should be used to calculate the matrix element of the low energy quantities. Then, the structure function $F_2(x, Q^2)$ is expressed for large $Q^2$ as $F_2(x, Q^2) \approx x \left(\frac{1}{2} u(x, Q^2) + \frac{1}{2} \bar{u}(x, Q^2) + \frac{1}{2} s(x, Q^2) + \frac{1}{2} \bar{s}(x, Q^2) + \cdots\right)$ with the $Q^2$-evoluted LC momentum distributions $u(x, Q^2), \bar{u}(x, Q^2), s(x, Q^2), \bar{s}(x, Q^2)$, etc. Fortunately, in most of the cases so far, these two methods do not result in a serious inconsistency. Especially, if the propagator satisfies the scaling property \[20\], both methods lead to exactly the same result. This seems to be the reason why the latter has been scarcely adopted in the effective models. However, the results of these two methods do not agree at all in the confined BS framework.

We begin by applying the inverse Mellin transformation analytically to the matrix elements of the twist-two operators. The aim of doing so is twofold. The first aim is to ensure that the direct calculation of the LC momentum distribution gives the equivalent result with the one in the moment space. As is well-known, pQCD establishes the equivalence between PDF in the moment space representation and PDF in the $x$-space representation, where the equivalence is provided by the Mellin transformation. The second aim is to introduce the quantities such as the “forward Compton-like amplitude” and the “handbag-like diagram”, which will be used to consider the precise differences between the two methods for PDF. We will argue their physical meaning later.

We consider the spin averaged matrix element of the twist-two operators as follows:

$$A_n \equiv \frac{1}{2} \sum_{s=\pm} \frac{\langle N(p, s)\bar{\psi}(\gamma^+ (i\partial^\perp)^{n-1}\psi)N(p, s)\rangle}{(p^+)^n},$$  \hspace{1cm} (1)

where $\psi$ and $\bar{\psi}$ denote the quark fields, and $|N(p, s)\rangle$ denotes the state vector for the nucleon with momentum $p$ and helicity $s$. The light-cone variables are defined as $p^\pm \equiv \frac{1}{\sqrt{2}}(p^0 \pm p^3)$, etc. We adopt the covariant normalization $\langle N(p, s)|N(p', s')\rangle = 2E_N(p)(2\pi)^3\delta^{(3)}(p' - p')\delta_{ss'}$ with $E_N(p) \equiv \sqrt{m_N^2 + p^2}$. The actual calculation of the matrix elements of these bilinear operators in the BS framework amounts roughly to the diagram depicted in Fig. 1. The cross “×” indicates the point where we insert the following vertex as

$$\gamma^+(k^+)^{n-1}/(p^+)^n.$$  \hspace{1cm} (2)

(For more details, see \[18, 19\], where the matrix element of the conserved current is discussed based on the gauge method \[18, 19, 20\], which may work as explicit examples. Ref. \[2\] is also helpful.)

We apply analytically the inverse Mellin transformation to $A_n$. For this purpose, we introduce an analytic function $T(x)$ for $x \in C$ in the following way:

$$T(x) \equiv \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{A_n}{x^n}. \hspace{1cm} (3)$$

Since $T(x)$ is the analogue of the forward Compton amplitude $T_{\mu\nu}(x, Q^2)$ in pQCD, we refer to $T(x)$ as “forward Compton-like amplitude”. We will argue its physical meaning later. To proceed, we have to make two assumptions on $T(x)$. (1) The power series in Eq. (3) converges, if $|x| > 1$. (2) The singularity structure of $T(x)$ is the cut along the segment $[-1,1]$. Due to the assumption (1), in the region $|x| > 1$, $T(x)$ defines an analytic function, which can be analytically continued to the inside as much as possible. The most important property of the forward Compton-like amplitude $T(x)$ is the following identity:

$$A_n = \int_C dx\ x^{n-1}T(x), \hspace{1cm} (4)$$

where the contour $C$ is a sufficiently large circle centered at the origin as depicted in Fig. 2. This identity follows from the residue theorem. Due to the assumption (2), we can deform the contour $C$ to the thin contour $C'$ as depicted in Fig. 2. We are thus lead to the following identity:

$$A_n = \int_{-1}^{1} dx\ x^{n-1}H(x), \hspace{1cm} (5)$$

where $H(x)$ is the discontinuity of $T(x)$ along the cut, i.e.,

$$H(x) \equiv \text{disc} \left( T(x) \right) = T(x - ie) - T(x + ie). \hspace{1cm} (6)$$
We define the quark and the anti-quark distribution functions for \( x \) \((0 \leq x \leq 1)\) as
\[
q(x) \equiv H(x), \quad \bar{q}(x) \equiv -H(-x),
\]
respectively \[24\]. Now, Eq. (6) reads
\[
A_n = \int_0^1 dx x^{n-1} \left( q(x) + (-1)^n \bar{q}(x) \right).
\]
From the relation to the moments, this identity shows that \( q(x) \) and \( \bar{q}(x) \) really work as PDF. We note that \( H(x) \) is directly expressed in the canonical operator representation in the following way \[20\]:
\[
H(x) = \frac{1}{2} \sum_{s = \pm} \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{i xp^+ z^-} \langle N(p, s) | \bar{\psi}(0) \gamma^+ \psi(z^-) | N(p, s) \rangle,
\]
where \( \psi(z^-) \) is a shorthand notation for \( \psi(z^+) = 0, \bar{z}^-, z_- = 0 \). Indeed, by inserting Eq. (9) into Eq. (6), we obtain Eq. (10).

To see explicitly that \( q(x) \) and \( \bar{q}(x) \) in Eq. (7) calculate the LC momentum distribution, we complete the summation in Eq. (10) analytically. Since the cross symbol in Fig. 4 represents the vertex \( \gamma^+(k^+) n_1/(p^+) n \), the summation in Eq. (10) reduces to the summation of these vertices as
\[
V(x) = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \frac{\gamma^+(k^+) n_1}{x^n (p^+) n} = \frac{1}{2\pi i} \frac{\gamma^+}{xp^+ - k^+} = \frac{1}{2\pi i} \frac{\gamma^+}{xp^+ - k^+}.
\]

The diagram, which corresponds to the direct calculation of \( T(x) \), is depicted in Fig. 3 where the dotted line represents \( V(x) \). Since this diagram is the analogue of the handbag diagram, we refer to it as “handbag-like diagram”. In order to calculate \( H(x) \), all we have to do is to replace \( V(x) \) by the following quantity:
\[
V(x - i\epsilon) - V(x + i\epsilon) = \frac{\gamma^+}{2\pi i} \left( \frac{1}{xp^+ - k^+ - i\epsilon} - \frac{1}{xp^+ - k^+ + i\epsilon} \right) = \gamma^+ \delta(xp^+ - k^+),
\]
where we used the identity: \( \delta(x) = \frac{1}{2\pi i} \left( \frac{1}{x - \frac{i\epsilon}{2}} - \frac{1}{x + \frac{i\epsilon}{2}} \right) \).

Since Eq. (11) counts the “number” of each parton which carries the LC momentum \( xp^+ \), \( q(x) \) and \( \bar{q}(x) \) in Eq. (7) calculate directly the LC momentum distribution. By construction, it is clear that the quantity which is equivalent to the twist-two moments under the Mellin transformation is the PDF obtained as the LC momentum distribution rather than the PDF obtained from the handbag diagram.

We consider the physical meaning of the handbag-like diagram and the precise differences between the two methods. We may think of the handbag-like diagram as the “proper Bjorken limit” of the handbag diagram. Indeed, the LC momentum distribution is obtained as the imaginary part of this “proper Bjorken limit” of the handbag diagram, while, in the original method, the imaginary part of the handbag diagram in the straightforward Bjorken limit is considered. (We emphasize again that these two methods would agree, if the propagator of the struck quark could satisfy the scaling property \[20\].) We can find the precise difference between the two methods in the difference between these two diagrams. In the handbag-like diagram, the free propagator is used for the struck quark, while, in the handbag diagram, the nonperturbative propagator is used. Intuitively, since the extremely large momentum is transferred to the struck quark, there is no reason to believe in the effective nonperturbative propagator, which is constructed to reproduce the low energy properties. Furthermore, since the origin of the propagator of the struck quark in our formulation is the derivatives in the twist-two operators, it should not be the non-trivial one.

Several comments are in order. First, the nontrivial propagator should be used except for the struck quark. In Ref. \[14\], the first attempt to calculate PDF for pion in the confined BS framework is performed, where the free propagator is used not only for the struck parton but also for the spectator parton. Their calculation is different from our formalism in this way. Second, by construction, our LC momentum distribution naturally satisfies the various sum rules, i.e., the fermion number sum rule, the momentum sum rule, and so on. (See Eq. (5). Its moments calculate the matrix elements of various conserved currents.) The third comment is concerning the calculability of the LC momentum distribution. Usually,
to evaluate the LC momentum distribution, one insert \( \gamma^+ \delta(xp^+ - k^+) \) as the vertex in the diagram like Fig. [1] In some cases, it is not easy to perform this procedure straightforwardly. (For instance, when the integration momentum becomes complex due to the Wick rotation, the meaning of the delta function with the complex variable is highly nontrivial.) In such cases, the technique of the inverse Mellin transformation can extend the calculability of the LC momentum distribution [25]. As is often the case, the calculation in the moment representation is much easier than the one in the \( x \)-representation. The last comment is concerning the gluon. Once the gluon is given, the analyticity of the LC momentum distribution [25], as is often argued, its practical implementation is formidable. Another solution may be provided by the non-perturbative quark propagator replaced by the non-perturbative one [12].

\[ A^+ = 0 \]

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