Abstract

Many studies have been conducted on sensitivity analysis in DEA which focus on changing the value of data or reallocation of inputs. In this article, some models have been presented which can be used for allocating one or more new inputs between the DMUs in a way that some of the inefficient units have been modified to efficient units. These models have been offered for both crisp inputs and fuzzy inputs. The presented model allocates the new inputs between the DMUs in a way that the distance of new inputs from the fair allocation is minimized by applying the Chebyshev’s norm. The mentioned models are utilized in four numerical examples and the related results are reported. To the present, the related studies have been focused on changing the value of the old inputs, however, this article paid particular attention to allocating the new input(s) between the DMUs.

Keywords: Allocation, Data Envelopment Analysis, Efficiency, Fair Allocation, Fuzzy Source

1. Introduction

Sensitivity analysis of Data Envelopment Analysis models is very important. Other issues like modification in the number of items have not been considered. For example in many studies we have seen the fixed costs imposed to DMUs. The imposed fixed costs to DMUs can be considered as a new input which may keep fixed the relative efficiency measurement or change it. Anyhow the allocation of new costs to DMUs should be fair. For more information on this issue refer to 1-8. This issue has been tried to be investigated in this article and present the models which can be used for allocating one or more sources between the units and consider its impact on the efficiency status of other units. Another model is also introduced which can be used for dividing one source between the DMUs in a way that the most number of inefficient DMUs can become efficient.

Through the employment of fuzzy logic we can mathematically formulate a lot of concepts, variables and the vague and imprecise systems, providing a base for making decisions under conditions of risk and uncertainty. A lot of studies have been conducted in this area 9-14. Since in real problem the primary data and information are not precise but qualitative, ordinal, and interval, in this article we consider allocating a fuzzy type fixed cost to DMUs. Moreover, it is supposed that all inputs and outputs are fuzzy and allocating the new cost should be in a way that it increases the rate of relative efficiency of DMUs to a special possible percentage. The allocating amount to each DMUs would be fuzzy.

This paper is organized in the following way. This paper is organized in the following way. After proving the introduction in section one, DEA and Fuzzy DEA models and related concepts is presented in Section 2. Then the influence of adding new sources on efficiency statues of DMUs is shown in section 3. Section 4 gives a method for allocating a new fuzzy source for all DMUs, and some examples are solved in section 5. Finally the results have been synthesized and presented.

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2. Preliminary

2.1 Some of Definitions and Models of DEA

Let there is \( n \) decision making units \( DMU_j (j = 1, \ldots, n) \) that convert \( m \) inputs \( x_i (i = 1, \ldots, m) \) into \( s \) outputs \( y_j (r = 1, \ldots, s) \) and \( DMU_o \) is an under evaluation DMU. Let \( x = (x_1, x_2, \ldots, x_m) \) and \( y = (y_1, \ldots, y_s) \) be an optimal vector of output and input weights, respectively for \( DMU_o \). The original CCR model in order to maximize the relative efficiency score of \( DMU_o \) is as follows:

\[
\begin{align*}
\text{Max} & \quad \sum_{r=1}^{s} u_r y_{ro} \\
\text{S.t} & \quad \sum_{i=1}^{m} v_i x_{io} \\
& \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 1, \quad j = 1, \ldots, n \\
& \quad u_r \geq \varepsilon, \quad r = 1, \ldots, s \\
& \quad v_i \geq \varepsilon, \quad i = 1, \ldots, m
\end{align*}
\]

\( DMU_o \) is efficient if and only if \( \sum_{r=1}^{s} u_r y_{ro} = 1 \) in the Model (2) and otherwise, is inefficient.

Consider \( DMU_1 \) with input \( X_1 \) and output \( Y_1 \) and \( DMU_2 \) with input \( x \) and output \( Y_2 \). \( DMU_2 \) is dominated by \( DMU_1 \), if \( (x, Y_2) \leq (x, Y_1) \) and the inequality must be strict in at least one component.

2.2 Fuzzy Theory

A Fuzzy Number is a set of ordered pairs as \( \tilde{A}(x) = \{(x, \mu_{\tilde{A}}(x))\} \) in which \( \mu_{\tilde{A}} \) is called membership function of this fuzzy number. A Fuzzy Number \( \tilde{A} \) is called L-R Fuzzy Number if there exist functions \( L \) (for Left), \( R \) (for Right), and scalars \( a > 0, \beta > 0 \), such that

\[ \mu_{\tilde{A}}(x) = \begin{cases} 
L \left( \frac{m-x}{a} \right) & x \leq m \\
R \left( \frac{x-m}{\beta} \right) & x \geq m 
\end{cases} \]

where \( L \) and \( R \) are non-Archimedean element.

The top of model which is called fractional model that its equivalent linear form is used as follows:

\[
\begin{align*}
\text{Max} & \quad \sum_{r=1}^{s} u_r y_{ro} \\
\text{S.t} & \quad \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 1, \quad j = 1, \ldots, n \\
& \quad u_r \geq \varepsilon, \quad r = 1, \ldots, s \\
& \quad v_i \geq \varepsilon, \quad i = 1, \ldots, m
\end{align*}
\]

A fuzzy set \( \tilde{A} = (x^l, x^m, x^u) \) is a generalized Left Right Fuzzy Numbers (LRFN) of Dubois and Prade if its membership function satisfies the following:

\[ \mu_{\tilde{A}}(x) = \begin{cases} 
L \left( \frac{m-x}{m_1 - l} \right) & 1 \leq x \leq m_1 \\
1 & m_1 \leq x \leq m_2 \\
R \left( \frac{x-m_2}{u-m_2} \right) & m_2 \leq x \leq u \\
0 & \text{otherwise}
\end{cases} \]

where \( L \) and \( R \) are strictly decreasing functions defined on \( [0, 1] \) and satisfying the conditions:

\[ L(x) = R(x) = 1, x < 0 \]
\[ L(x) = R(x) = 0, x \leq 1 \]

For \( m_1 = m_2 \), we have the classical definition of (LRFN) of Dubois and Prade. Trapezoidal Fuzzy Numbers (TRFN) are special cases of Generalized Left right Fuzzy Numbers (GLRFN) with \( L(x) = R(x) = 1 - x \).
2.3 DEA and Fuzzy DEA Models

In customary data envelopment analysis, the values of inputs and outputs with the exact values are determined by experts. However in imprecise environments the experts’ premise of precise data is very unreal. So, the experts decide to consider DEA and its models in fuzzy environments\(^ {16-17}\). Suppose \( n \) DMUs with \( m \) inputs and \( s \) outputs are evaluated. Consider the vectors related to inputs and outputs of \( DMU_j \) for \( j = 1, \ldots, n \) are respectively in the form of \( \tilde{X}_j = (\tilde{x}_{ij}, \ldots \tilde{x}_{in}) \) and \( \tilde{Y}_j = (\tilde{y}_{ij}, \ldots \tilde{y}_{jn}) \).

Furthermore suppose \( \tilde{x}_j \) and \( \tilde{y}_j \) are a triangular fuzzy number as \( \tilde{x}_j = (x^l_j, x^m_j, x^u_j) \) and \( \tilde{y}_j = (y^l_j, y^m_j, y^u_j) \). Model (3) is CCR multiplier form for the fuzzy inputs and outputs.

\[
\begin{align*}
\text{Max} & \quad \sum_{r=1}^{s} u_r \tilde{y}_{ro} \\
\text{s.t} & \quad \sum_{i=1}^{m} v_i \tilde{x}_{io} = 1 \quad i = 1, \ldots, m \\
& \quad \sum_{r=1}^{s} u_r \tilde{y}_{ro} - \sum_{i=1}^{m} v_i \tilde{x}_{io} \leq 0 \quad j = 1, \ldots, n \\
& \quad v_i \leq 0 \quad i = 1, \ldots, m \\
& \quad u_r \geq 0 \quad r = 1, \ldots, s
\end{align*}
\]

The above model can be changed to interval model by using the \( a \)-cut\(^ \text{a} \).

The following model is CCR model in output oriented for the fuzzy inputs and outputs.

\[
\begin{align*}
\text{Max} & \quad \varphi_o \\
\text{s.t} & \quad \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \leq \tilde{x}_{io} \quad i = 1, \ldots, m \\
& \quad \sum_{j=1}^{n} \lambda_j \tilde{y}_{ij} \geq \varphi_o \tilde{y}_{ro} \quad r = 1, \ldots, s \\
& \quad \lambda_j \geq 0 \quad j = 1, \ldots, n
\end{align*}
\]

Definition 1 (Efficiency in fuzzy)

\( DMU_o \) is called CCR efficient if in Model (4), \( \varphi_o = 1 \) and in each optimal solution, all the constraints (except \( \lambda_j \geq 0 \)) will become equal. If not \( DMU_o \) is CCR inefficient.

3. Influence of Adding New Sources on Efficiency Statutes of DMUs

Many studies have been conducted on sensitivity analysis in DEA by Change in data of inputs and outputs in order to preserve the two categories of efficient and inefficient DMUs. However managers involve the problems that need modification in number of data. For example consider, some bank units are evaluating the types annual costs of bank as input. Afterward, it was decided to re-evaluate the units by adding one of the indexes. This leads to solve a linear programming problems in which if \( n \) is big, a lot of complicated calculation will be involved. So, this issue made us to pay attention to modification in the numbers of the data in this study. The fixed costs have been seen in a lot of problems in which DEA can be used in them. These costs are the new inputs which can be allocated equitably between the DMUs. This allocation has been done in different ways.

This article presents the models which can be used for allocating one or more new sources between the DMUs and considers its impact on the efficiency status of other DMUs.
3.1 Suggested Model for Allocating a New Source

In this section, first, a new source was allocated to DMUs and its impact on the efficiency of other DMUs was considered. Adding sources are equivalent to adding some new variables in Model (1). So the feasible space in at least one upper dimension will be discussed. Thereupon, the value of objective function which shows the efficiency measure of DMU will not worsen. As DMU, with m input and s output is efficient and its efficiency measure equals to one, it still preserves its efficiency. However, the inefficient DMUs may become efficient. So, we pay attention to the inefficient DMUs.

Suppose the DMU is inefficient. We have k unit from a new source and we are going to divide it among the inefficient DMUs in a way that the inefficient DMU become efficient. The portion of DMU from this new source equals to . The fair allocation of this input for DMU is presented is for this reason.

\[
\text{Min } \text{Max } \left\{ a_1 - K \sum_{j=1}^{m} x_{ij}, a_2 - K \sum_{j=1}^{m} x_{i2}, \ldots, a_n - K \sum_{j=1}^{m} x_{in} \right\}
\]

S.t \[ \sum_{r=1}^{m} u_r y_{rj} \leq 1 \quad j = 1, \ldots, n \]
\[ \sum_{j=1}^{n} v_i x_{ij} + \sum_{j=1}^{n} v_{m+1} a_j = 1 \]
\[ \sum_{j=1}^{n} a_j = K \]
\[ u_r \geq \varepsilon \quad r = 1, \ldots, s \]
\[ v_i \geq \varepsilon \quad i = 1, \ldots, m, m+1 \]
\[ a_j \geq 0 \quad j = 1, \ldots, n \]

(6)

In above model, \(v_{m+1}\) is the corresponding weight for the new source and \(\sum_{j=1}^{m} K a_j\) is related to the new source limitation. On one hand, the purpose is to minimize the distance of the intended allocation from the fair allocation among the DMUs. For this reason, this distance is minimized with the Chebyshev’s norm.

**Theorem 1.** Model (6) is feasible.

**Proof.** The following solution is a feasible solution for Model (6).

\[
v_j = \varepsilon i = 1, \ldots, m; \quad v_{m+1} = 1
\]
\[
u_r = M \epsilon r = 1, \ldots, s
\]
\[
a_j = \frac{\left( M \sum_{r=1}^{i} y_{ro} - \sum_{i=1}^{m} x_{io} \right)}{m+1} \quad j \neq 0
\]

Where M is a very large positive number.

With regarding to this fact that \(v_{m+1} \neq 0\), Model (6) can be written as follows:

\[
\text{Min } \text{Max } \left\{ a_1 - K \sum_{j=1}^{m} x_{ij}, a_2 - K \sum_{j=1}^{m} x_{i2}, \ldots, a_n - K \sum_{j=1}^{m} x_{in} \right\}
\]

S.t \[ \sum_{r=1}^{m} u_r y_{rj} - \sum_{j=1}^{m} v_j x_{ij} - a_j \leq 0 \quad j = 1, \ldots, n \]
\[ \sum_{r=1}^{m} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - a_o = 0 \]
\[ \sum_{j=1}^{n} a_j = K \]
\[ u_r \geq \varepsilon \quad r = 1, \ldots, s \]
\[ v_i \geq \varepsilon \quad i = 1, \ldots, m, m+1 \]
\[ a_j \geq 0 \quad j = 1, \ldots, n \]

**Theorem 2.** Suppose \(a_j^*\) for \(j = 1, \ldots, n\) is gained from solving Model (7). Then allocating \(a_j^*\) from the source \(m+1\) to DMU for \(j = 1, \ldots, n\), make the DMU efficient.
Proof. Suppose \((v^*_r, \ldots, v^*_{m1}, u^*_r, \ldots, u^*_{m1}, a^*_j, \ldots, a^*_n)\) is the optimal solution for Model (7) then we can say:

\[
\sum_{r=1}^{s} \bar{\nu}_{m+1} u^*_r y^*_r
\]

and also

\[
\sum_{i=1}^{n} a^*_j = K
\]

We have

\[
\sum_{j=1}^{n} a^*_j = K
\]

Therefore \((\nu^*_{m1}, \ldots, \nu^*_{m1}, v^*_{m1}, \nu^*_{m1}, u^*_{m1}, \ldots, v^*_{m1}, a^*_{j}, \ldots, a^*_n)\) is a feasible solution for Model (7).

Now, if we consider \(v^*_r = \bar{v}_{m+1} u^*_r, \bar{u}_{r} = \bar{v}_{m+1} u^*_r\) then \((\nu^*_r, \ldots, \nu^*_{m1}, \nu^*_{m1}, \nu^*_{m1}, u^*_r, \ldots, u^*_{m1})\) will be the feasible solution with the objective function one for the following model.

Max \(\sum_{r=1}^{s} u^*_r y^*_r + \sum_{i=1}^{m} v^*_{i} x^*_{i} + v^*_{m1} a^*_o\)

S.t. \(\sum_{r=1}^{s} u^*_r y^*_r + \sum_{m=1}^{m} v^*_{i} x^*_{i} + v^*_{m1} a^*_o \leq 1\)

\(u^*_r \geq \varepsilon, r = 1, \ldots, s\)

\(v^*_i \geq \varepsilon, i = 1, \ldots, m, m + 1\)

It means \(DMU_o\) becomes efficient in presence of new source.

Theorem 3. Suppose \(DMU_o\) is inefficient and \(a^*_j\) for \(j = 1, \ldots, n\) is the optimal solution of model (7). If \(a^*_j \leq a^*_o\) and \(a^*_j \geq a^*_o\) for \(j \neq 0\) such that \(\sum_{j=1}^{n} a^*_j = K\) then by allocating \(a^*_j\) to \(DMU_j\) \(j = 1, \ldots, n\), \(DMU_o\) will become efficient.

Proof. With regarding the constraints of the Model (7), it is clear that \(a^*_j\) for \(j = 1, \ldots, n\) can make the \(DMU_o\) efficient. Now it can be proved that if \(a^*_j \geq a^*_o\) for \(j \neq 0\) and \(0 \leq a^*_j \leq a^*_o\), then \(DMU_o\) will become efficient. Suppose the inefficient \(DMU_o\) is still preserved its inefficiency. So \(DMU_o\) is dominated by a member belong to \(T_i\) in this form \(\sum_{j=0}^{m} w_j (x_{ij}, \ldots, x_{mj}, a^*_j, -y^*_j\ldots-y^*_n)^T\). So we have the following inequality that is strict in at least one component.

\[
\sum_{j=1}^{n} a^*_j \left(\sum_{j=1}^{m} w_j (x_{ij}, \ldots, x_{mj}, a^*_j, -y^*_j\ldots-y^*_n)^T\right)
\]

In one hand based on the assumption of the Theorem we have the subsequent inequalities:

\[
\left(\sum_{j=1}^{m} w_j (x_{ij}, \ldots, x_{mj}, a^*_j, -y^*_j\ldots-y^*_n)^T\right)
\]

By utilizing (8),(9),(10) it can be concluded the following inequality that is strict in at least one component.

\[
\sum_{j=1}^{n} a^*_j \left(\left(\sum_{j=1}^{m} w_j (x_{ij}, \ldots, x_{mj}, a^*_j, -y^*_j\ldots-y^*_n)^T\right)\right)
\]

That is in contradiction with making \(DMU_o\) efficient through allocating \(a^*_j\) for \(j = 1, \ldots, n\).

3.2 Presented Model for Allocating Multiple New Sources

Suppose there exit \(h\) new sources that \(K_1, K_2, \ldots, K_h\) units of them are on hand. We want to divide these sources between DMUs in a way that the inefficient \(DMU_o\) will become efficient. Suppose \(a^*_{w_j}\) is the portion of \(DMU^*_{j}\) from the source \(w\) for \(j = 1, \ldots, n\) and \(w = 1, \ldots, h\). So Model (6) has been generalized as follows:
Influence of Allocating One New Fuzzy Source on DMUs Efficiency

\[ \text{Min Max} \left\{ \begin{array}{c} \frac{a_{ij} - K_i}{\sum_{j=1}^{n} X_{ij}} \quad \ldots \quad \frac{a_{jn} - K_n}{\sum_{j=1}^{n} X_{jn}} \quad \ldots \quad , \end{array} \right. \]

\[ \right. \left\{ \begin{array}{c} a_{ii} - K_h \quad \sum_{j=1}^{m} X_{ij} \quad \ldots \quad a_{in} - K_n \quad \sum_{j=1}^{m} X_{jn} \quad \ldots \quad , \end{array} \right. \]

(12)

\[ \text{S.t.} \quad \sum_{j=1}^{n} u_{j} y_{ij} \leq 1 \quad j = 1, \ldots, n \]

\[ \sum_{r=1}^{m} u_{i} y_{ri} = 1 \]

\[ \sum_{i=1}^{m} v_{i} x_{ij} + v_{m+1} a_{ij} + \ldots + v_{m+h} a_{ijh} \]

Those \( v_{m+1}, \ldots, v_{m+h} \) are the corresponding weights with the added sources. This model is a nonlinear programming and can be solved through the weighting method.

**Theorem 4.** Model (12) is feasible.

**Proof.** Proof is similar to theorem 1.

**Theorem 5.** Suppose DMU \( a \) is inefficient and \( a_{nj}^* \) for \( j = 1, \ldots, n \) and \( w = 1, \ldots, h \) is the optimal solution of Model (11). If \( a_{wo} \leq a_{wo}^* \) and \( a_{nj} \geq a_{nj}^* \) for \( j = 0 \) such that \( \sum_{j=1}^{n} a_{nj} = K_w \)

for \( w = 1, \ldots, h \), then by allocating \( a_{nj} \) from source \( w \) to DMU \( j \), DMU \( a \) will become efficient.

**Proof.** Proof is similar to theorem 6.

### 3.3 Getting the Most Number of Efficient DMUs by Allocating a New Source

In this section we are going to divide a new source between DMUs in a way that the number of efficient DMUs is maximized. Suppose \( K \) units of this source are on hand. Consider the portion of \( DMU_j \) for \( j = 1, \ldots, n \) from this source equals \( a_j \). So the following model is presented:

\[ \text{Max} \quad \sum_{p \in N,E} \gamma_p \]

\[ \text{S.t.} \quad \sum_{r=1}^{m} v_{i} x_{ij} + v_{m+1} a_{ij} + \ldots + v_{m+h} a_{ijh} \leq 1 \quad j = 1, \ldots, n \]

(13)

Where \( N.E \) is the index set of inefficient DMUs, \( M \) is a very large positive number and \( V_{i}^p, U_{r}^p \) are the corresponding weights \( i \) (th) input and \( r \) (th) output in evaluating \( DMU_j \). Also \( V_{m+1}^p \) is the corresponding weight with the new source in evaluating \( DMU_j \).

**Theorem 6.** Model (13) is feasible.

**Proof.** The following solution is a feasible solution for Model (13).

\[ \gamma_p = 0 \quad p \in N.E \]

\[ u_{r}^p \geq e \quad p \in N.E; r = 1, \ldots, s \]

\[ v_{i}^p \geq e \quad p \in N.E; i = 1, \ldots, m + 1 \]

\[ \gamma_p \in \{0,1\} \quad p \in N.E \]

\[ a_j \geq 0 \quad j = 1, \ldots, n \]

\[ \frac{K}{n} \quad j = 1, \ldots, n. \]

**Theorem 7.** The allocation is derived from solving Model (13), make the maximum number of DMUs efficient.

**Proof.** With regarding to this fact that \( \gamma_p (p \in N.E) \) is a binary variable two status have been considered.

The first status \( \gamma_p = 1 \)

In this case, the following relations have been derived from the first and second set of constraints in Model (13):
with regard to the constraint \( \sum_{j=1}^{m} v^p_i x_{ij} + v^p_{m+1} a_p \geq -M(1-\gamma_p) \) will be redundant. With regard to this fact that, Model (13) is a maximization model it is preferred \( \gamma_p \) is chosen 1. So maximizing \( \sum_{p \in N.E} \gamma_p \) is equivalent to maximizing the number of efficient DMUs.

**Theorem 8.** The Model (13) can be modified into the following integer linear programming model:

Max \( \sum_{p \in N.E} \gamma_p \) \hspace{1cm} (15)

S.t. \( \sum_{i=1}^{m} u^p_i y_{ij} - \sum_{i=1}^{m} v^p_i x_{ij} - a_j \leq 0 \) \hspace{0.5cm} \( p \in N.E; j = 1, \ldots, n \)

\( \sum_{r=1}^{s} u^p_r y_{rp} - \sum_{r=1}^{s} v^p_r x_{rp} - a_p \geq -M(1-\gamma_p) \) \hspace{0.5cm} \( p \in N.E \)

\( \sum_{j=1}^{n} a_j = K \)

\( u^p_i \geq \epsilon \) \hspace{1cm} \( p \in N.E; r = 1, \ldots, s \)

\( v^p_i \geq \epsilon \) \hspace{1cm} \( p \in N.E; i = 1, \ldots, m + 1 \)

\( \gamma_p \in \{0,1\} \) \hspace{1cm} \( p \in N.E \)

\( a_j \geq 0 \) \hspace{1cm} \( j = 1, \ldots, n \)

**Proof.** With regard to \( v_{m+1} \neq 0 \) for \( p \in N.E \) the Model (13) can be written as follows:

\[
\sum_{r=1}^{s} u^p_r y_{rp} - \sum_{r=1}^{s} v^p_r x_{rp} - a_p \geq -M(1-\gamma_p) \]

This is equivalent to maximizing the number of efficient DMUs.

The second status (\( \gamma_p = 0 \))

In this status the constraint \( \sum_{j=1}^{n} v^p_i x_{ij} + v^p_{m+1} a_p \geq -M(1-\gamma_p) \) will change as follows:

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} u^p_i y_{ij} - \sum_{i=1}^{m} v^p_i x_{ij} - a_j \leq 0 \]

So above constraint can be written as follows:

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} u^p_i y_{ij} - \sum_{i=1}^{m} v^p_i x_{ij} - a_p \geq -M(1-\gamma_p) \]

The second status (\( \gamma_p = 0 \))

In this status by attention to \( \sum_{i=1}^{m} v^p_i x_{ip} + a_p > 0 \) we will have:

\[
\left( \sum_{i=1}^{m} v^p_i x_{ip} + a_p \right) \left( -M(1-\gamma_p) \right) = -M
\]

So the constraint \( \sum_{r=1}^{s} u^p_r y_{rp} - \sum_{r=1}^{s} v^p_r x_{rp} - a_p \geq -M(1-\gamma_p) \) will become as follows:
\[ \sum_{r=1}^{n} u_r^p y_{rp} - \sum_{i=1}^{m} v_i^p x_{ip} - a_p \geq -M = -M (1 - \gamma_p) \]

Therefore in any two statuses Model (15) can be written as Model (14).

4. Suggested Method for Allocating a New Fuzzy Source

In this section the impacts of allocating one new fuzzy source to DMUs will be considered. As it was mentioned in section 3.1, we pay attention to Inefficient DMUs. Suppose \( DMU_o \) with \( m \) fuzzy input and \( s \) fuzzy output whit Model (3) is considered inefficient. Now, we are going to consider the impacts of one adding new (fuzzy) source. Suppose \( \tilde{K} = (K^1, K^m, K^u) \) units from the new source it exist. The purpose is to allocate this source between the units in a way that \( DMU_o \) will become efficient. Consider the portion of \( DMU \) from this source equals \( \tilde{a}_j = (a_j^1, a_j^m, a_j^u) \) for \( j = 1, \ldots, n \) and \( \tilde{r}_j = (r_j^1, r_j^m, r_j^u) \) is equal to:

\[ r_j^1 = K^1 \times \frac{\sum_{q=1}^{m} \sum_{i=1}^{n} x_{iqj}}{\sum_{q=1}^{n} \sum_{i=1}^{m} x_{iqj}} \quad j = 1, \ldots, n \]
\[ r_j^m = K^m \times \frac{\sum_{q=1}^{m} \sum_{i=1}^{n} x_{iqj}}{\sum_{q=1}^{n} \sum_{i=1}^{m} x_{iqj}} \quad j = 1, \ldots, n \]
\[ r_j^u = K^u \times \frac{\sum_{q=1}^{m} \sum_{i=1}^{n} x_{iqj}}{\sum_{q=1}^{n} \sum_{i=1}^{m} x_{iqj}} \quad j = 1, \ldots, n \]

Model (16) allocates the new source between the DMUs in a way that \( DMU_o \) becomes efficient and the intended allocation distance \( \tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n) \) from the fair allocation \( \tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_n) \) is minimized by applying the Chebyshev’s norm.

Min Max \[ \{ \| \tilde{a}_1 - \tilde{r}_1 \|, \ldots, \| \tilde{a}_n - \tilde{r}_n \| \} \]

S.t. \[ \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} + v_{m+1} \tilde{a}_j \leq \tilde{I} \quad j = 1, \ldots, n \]
\[ \sum_{r=1}^{s} \tilde{u}_r \tilde{y}_{ro} - \sum_{i=1}^{m} v_i \tilde{x}_{io} + v_{m+1} \tilde{a}_o \leq \tilde{I} \]

\[ \tilde{a}_j = \frac{\tilde{K}}{n-1} - \frac{M \sum_{r=1}^{s} \tilde{y}_{ro} - \sum_{i=1}^{m} \tilde{x}_{io}}{n-1} \quad j \neq 0. \]

Where \( M \) is a very large positive number.
Model (17) can be changed to a deterministic Model (18).

\[
\begin{align*}
\text{Min Max} \\
\left\{ a^l_i - r^l_i, a^m_i - r^m_i, a^u_i - r^u_i \right\} \\
\text{S.t.} \sum_{r=1}^{s} u_r y^l_j - \sum_{i=1}^{m} v_i x^l_{ij} - a^l_j \leq 1 & \quad j = 1, \ldots, n \\
\sum_{r=1}^{s} u_r y^m_j - \sum_{i=1}^{m} v_i x^m_{ij} - a^m_j \leq 1 & \quad j = 1, \ldots, n \\
\sum_{r=1}^{s} u_r y^u_j - \sum_{i=1}^{m} v_i x^u_{ij} - a^u_j \leq 1 & \quad j = 1, \ldots, n
\end{align*}
\]

5. Example

5.1 Example 1

Now the presented models in this paper are used for the data of Table 1\textsuperscript{3} related to twelve DMUs with two inputs and outputs. Evaluating these DMUs through Model (1) has been revealed that DMUs A, B, and C are efficient.

Suppose we want to divide a new source (the third input) between these DMUs in a way that a specific inefficient DMU will become efficient. Suppose \( K = 120 \). We solve Model (7) for all inefficient DMUs and the related results are depicted in Table 2.

5.2 Example 2

Consider again the data of example 1. We want to divide two new sources between these DMUS in a way that a specific inefficient DMU will become efficient. Suppose \( K_1 = 120 \) and \( K_2 = 240 \). Like the previous example, Model (12) is solved for all inefficient DMUs and related results are presented in Table 3.

5.3 Example 3

Consider the related data to seven DMUs with two inputs and one output have been presented in Table 4. By solving Model (1) for all of these DMUs, DMU\(_6\) and DMU\(_7\) are evaluated inefficient. We want to divide a new source between these DMUs in a way that the most

| Hospital | Inputs | Nurse | Outpatient | Inpatient |
|----------|--------|-------|------------|-----------|
| A        | 20     | 151   | 100        | 90        |
| B        | 19     | 131   | 150        | 50        |
| C        | 25     | 160   | 160        | 55        |
| D        | 27     | 168   | 180        | 72        |
| E        | 22     | 158   | 94         | 66        |
| F        | 55     | 255   | 230        | 90        |
| G        | 33     | 235   | 220        | 88        |
| H        | 31     | 206   | 152        | 80        |
| I        | 30     | 244   | 190        | 100       |
| J        | 50     | 268   | 250        | 100       |
| K        | 53     | 306   | 260        | 147       |
| L        | 38     | 284   | 250        | 120       |

In above model, each constraint with fuzzy parameters in Model (17) is changed in to three constraints with deterministic parameters.
### Table 2. The result of solving Model (7) for inefficient DMUs

|   | C     | E     | F     | G     | H     | I     | J     | K     | L     |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $a_1$ | 6.776 | 7.906 | 6.590 | 7.007 | 7.904 | 7.580 | 6.678 | 7.255 | 7.309 |
| $a_2$ | 6.529 | 6.800 | 7.198 | 6.616 | 6.800 | 6.731 | 6.881 | 6.317 | 6.460 |
| $a_3$ | 7.011 | 7.329 | 7.708 | 8.031 | 7.330 | 6.937 | 7.378 | 7.134 | 7.081 |
| $a_4$ | 8.347 | 8.874 | 9.017 | 8.435 | 8.874 | 8.550 | 8.700 | 8.225 | 8.279 |
| $a_5$ | 6.80 | 6.281 | 6.141 | 6.723 | 6.283 | 7.944 | 6.458 | 6.932 | 6.879 |
| $a_6$ | 12.666 | 11.536 | 11.394 | 12.435 | 11.538 | 11.862 | 11.712 | 12.873 | 12.927 |
| $a_7$ | 11.298 | 10.816 | 11.003 | 10.279 | 10.816 | 11.500 | 11.650 | 11.176 | 11.230 |
| $a_8$ | 9.114 | 8.588 | 8.444 | 10.132 | 8.586 | 8.912 | 9.698 | 9.325 | 9.182 |
| $a_9$ | 11.540 | 10.757 | 10.238 | 10.522 | 10.756 | 10.406 | 11.893 | 10.731 | 10.678 |
| $a_{10}$ | 13.319 | 12.277 | 12.476 | 12.300 | 12.277 | 12.185 | 12.035 | 13.197 | 12.456 |
| $a_{11}$ | 14.045 | 15.353 | 15.645 | 13.958 | 15.352 | 14.315 | 14.129 | 14.166 | 14.908 |
| $a_{12}$ | 12.549 | 13.489 | 14.150 | 13.568 | 13.488 | 13.084 | 12.794 | 12.672 | 12.617 |

### Table 3. The result of solving Model (12) for inefficient DMUs

|   | C     | E     | F     | G     | H     | I     | J     | K     | L     |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $a_1$ | 7.107 | 8.070 | 6.485 | 6.706 | 8.156 | 7.838 | 6.554 | 7.440 | 7.520 |
| $a_2$ | 6.618 | 6.923 | 7.632 | 6.797 | 6.984 | 6.884 | 7.220 | 6.458 | 6.672 |
| $a_3$ | 6.922 | 7.312 | 7.827 | 6.984 | 7.304 | 6.967 | 7.317 | 6.950 | 6.949 |
| $a_4$ | 8.436 | 9.040 | 9.451 | 8.616 | 9.126 | 8.808 | 9.039 | 8.410 | 8.490 |
| $a_5$ | 7.580 | 6.117 | 5.902 | 7.741 | 6.032 | 8.114 | 6.119 | 6.748 | 6.667 |
| $a_6$ | 11.975 | 11.372 | 10.960 | 11.796 | 11.286 | 11.604 | 12.682 | 12.080 | 11.922 |
| $a_7$ | 10.278 | 10.807 | 11.029 | 10.979 | 10.798 | 10.307 | 10.523 | 11.110 | 10.224 |
| $a_8$ | 9.025 | 8.424 | 8.011 | 8.845 | 8.335 | 8.653 | 8.795 | 9.857 | 10.188 |
| $a_9$ | 10.520 | 10.697 | 9.874 | 10.341 | 10.657 | 10.148 | 11.227 | 11.353 | 11.270 |
| $a_{10}$ | 13.158 | 12.186 | 12.747 | 13.337 | 12.137 | 11.927 | 11.695 | 13.131 | 13.020 |
| $a_{11}$ | 14.815 | 15.495 | 15.503 | 14.995 | 15.563 | 14.901 | 14.662 | 13.982 | 14.677 |
| $a_{12}$ | 13.570 | 13.562 | 14.584 | 13.749 | 13.629 | 13.854 | 14.172 | 12.487 | 12.406 |
| $b_1$ | 13.269 | 14.981 | 12.254 | 13.089 | 15.067 | 14.750 | 14.980 | 14.351 | 14.432 |
| $b_2$ | 12.680 | 12.726 | 13.695 | 12.880 | 13.370 | 13.052 | 13.283 | 12.348 | 12.734 |
| $b_3$ | 14.399 | 13.797 | 13.386 | 14.221 | 13.711 | 14.029 | 15.912 | 15.233 | 14.347 |
| $b_4$ | 16.318 | 16.613 | 17.332 | 16.497 | 17.007 | 16.690 | 16.920 | 16.291 | 16.372 |
| $b_5$ | 13.996 | 13.392 | 12.982 | 13.817 | 13.307 | 13.624 | 13.844 | 14.829 | 14.992 |
| $b_6$ | 24.505 | 23.902 | 23.490 | 24.325 | 23.815 | 24.133 | 23.902 | 24.532 | 25.318 |
| $b_7$ | 21.942 | 22.499 | 23.233 | 20.929 | 22.908 | 22.591 | 20.507 | 21.137 | 21.056 |
| $b_8$ | 19.436 | 18.001 | 17.590 | 19.463 | 17.914 | 18.888 | 18.001 | 18.631 | 19.417 |
| $b_9$ | 22.427 | 22.740 | 23.718 | 22.633 | 23.393 | 21.222 | 20.992 | 22.427 | 22.408 |
| $b_{10}$ | 25.983 | 24.548 | 24.137 | 26.190 | 24.462 | 25.971 | 24.548 | 25.178 | 25.098 |
| $b_{11}$ | 28.466 | 29.618 | 30.589 | 29.036 | 30.264 | 29.941 | 29.927 | 28.492 | 28.412 |
| $b_{12}$ | 26.584 | 27.187 | 27.598 | 26.764 | 24.786 | 25.102 | 27.187 | 26.557 | 25.420 |
number of DMUs will become efficient. Also suppose K = 5, ε = 0.001. We solve Model (13) and obtained allocation makes both of inefficient $DMU_6$ and $DMU_7$ efficient as follows:

$$a_1 = 0.7144571 \quad a_2 = 0.7145071 \quad a_3 = 0.7145571$$
$$a_4 = 0.7144571 \quad a_5 = 0.7143571 \quad a_6 = 0.7138571$$
$$a_7 = 0.7138071$$

5.4 Example 4

Consider Table 5. This table is related to 5 DMUs with 2 triangular fuzzy inputs and 1 triangular fuzzy output. This table is extracted from [18].

By evaluating these DMUs by Model (3), it is revealed that units A, C, D and E are inefficient. Let $k = (k^1, k^2, k^u) = (22.8, 30.5, 38.2)$ as a new source for allocating among DMUs. The Fair allocation of this new source is presented in Table 6.

Furthermore, the allocation which is derived by solving Model (18) for each of the inefficient DMUs is presented in Tables 7, 8, 9 and 10.

### Table 4. DMUs data related to example 3

| $DMU_1$ | $DMU_2$ | $DMU_3$ | $DMU_4$ | $DMU_5$ | $DMU_6$ | $DMU_7$ |
|---------|---------|---------|---------|---------|---------|---------|
| Input1  | 0.7     | 0.55    | 0.4     | 0.3     | 0.2     | 1       |
| Input2  | 1       | 1       | 0.45    | 0.3     | 0.1     | 0.45    |

### Table 5. Data of DMUs (extracted from [13])

| DMU | Input1 | Input2 | Output |
|-----|--------|--------|--------|
| A   | (1,3,5)| (1,2,3)| (0.5,1,1.5) |
| B   | (1.5,2,2.5) | (2,2,5,3) | (2,4,3,3.6) |
| C   | (2.5,3,3.5) | (3,4,4,4,6) | (3,4,5) |
| D   | (4.5,5,5,5) | (1,2,3) | (0.5,1,1.5) |
| E   | (2.5,3,3,5) | (3,4,4,4,6) | (0.5,0,75,1) |

### Table 6. Fair allocation of new source [13]

| DMU | Fair allocation |
|-----|-----------------|
| A   | (2,5,8)         |
| B   | (3,5,4,5,5,5)   |
| C   | (5,9,7,8,1)     |
| D   | (5,5,7,8,5)     |
| E   | (5,9,7,8,1)     |

### Table 7. The result of solving Model (18) for $DMU_o = A$

| DMU | Allocation        |
|-----|-------------------|
| A   | (0.3923069, 1.784613, 3.176919) |
| B   | (5.683079, 7.353849, 9.024619)  |
| C   | (7.353848, 10.13846, 12.92308)  |
| D   | (0.6769188, 2.176919, 3.676919) |
| E   | (8.693847, 9.046155, 9.398462)  |

### Table 8. The result of solving Model (18) for $DMU_o = C$

| DMU | Allocation        |
|-----|-------------------|
| A   | (1.099999, 4.099999, 7.099999) |
| B   | (3.800001, 5.000001, 6.200001) |
| C   | (4.999999, 7.000000, 9.000001) |
| D   | (6.100000, 6.499999, 7.599999) |
| E   | (6.800001, 7.900001, 8.300000) |

### Table 9. The result of solving Model (18) for $DMU_o = D$

| DMU | Allocation        |
|-----|-------------------|
| A   | (1.861538, 5.930769, 9.406155) |
| B   | (8.707697, 8.707697, 9.301542) |
| C   | (11.10770, 11.10770, 13.30770) |
| D   | (0.4307650, 1.861534, 3.292303) |
| E   | (0.6923030, 2.892303, 2.892303) |

### Table 10. The result of solving Model (18) for $DMU_o = E$

| DMU | Allocation        |
|-----|-------------------|
| A   | (0.5166660, 2.033331, 3.549996) |
| B   | (6.280003, 8.100004, 9.920004)  |
| C   | (8.100002, 11.13334, 14.16667) |
| D   | (7.386667, 7.958333, 8.530000)  |
| E   | (0.5166621, 1.274995, 2.033328) |

6. Conclusion

In this article the models have been presented which can be used for allocating one or more new sources between the DMUs in a way that a specific inefficient DMU will become efficient. Another model is also introduced which can be used for dividing a new source between the DMUs in a way that the most number of inefficient DMUs can
become efficient. Since in real problem the primary data and information are not precise, in this article allocating a fuzzy type fixed cost to DMUs has been investigated. Finally the mentioned models have been utilized in some sets of DMUs and the results have been presented.

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