A square root of Hurwitz numbers

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Abstract
We show that a generating function of spin Hurwitz numbers analogous to (disconnected) double Hurwitz numbers is a tau function of the two-component BKP (2-BKP) hierarchy such that it is a square root of a tau function of the two-component KP (2-KP) hierarchy defined by related Hurwitz numbers.

1 Introduction
The Gromov-Witten theory of Kähler surfaces (with smooth canonical divisor) is entirely determined by the GW theory of spin curves (see [15, 10, 18]). The (dimension zero) sum formula for spin curves (Theorem 1.1 of [16]) indicates that the spin Hurwitz theory is to the GW theory of spin curves as the Hurwitz theory is to the GW theory of curves. So it would be interesting to find some connections between the following theories:

\[ \text{GW theory of curves} \quad \xrightarrow{\text{GW/H}} \quad \text{Hurwitz theory} \]

\[ \text{GW theory of spin curves} \quad \xrightarrow{\text{spin Hurwitz theory}} \]

In the top arrow is the celebrated GW/H correspondence developed in [21, 22]. This paper aims to find a connection in the right arrow and also to address a question on a correspondence in the bottom arrow analogous to the GW/H correspondence.

In [20], A. Okounkov showed that a generating function of (disconnected) double Hurwitz numbers is a tau function of the 2-Toda lattice hierarchy. The key idea is to write the generating function in terms of Schur functions that are special tau functions of the KP hierarchy.

Following the same idea, we will compare the generating functions of Hurwitz numbers and spin Hurwitz numbers defined below. The Hurwitz generating function can be written via Schur functions and the spin Hurwitz generating function via Schur Q-functions that are tau functions of the BKP hierarchy.

Given partitions \( \mu, \nu, \eta_2 = (2,1^{d-2}) \) and \( \eta_3 = (3,1^{d-3}) \) of \( d > 0 \) and integers \( r_2, r_3 \geq 0 \), the Hurwitz number of \( \mathbb{P}^1 \) is the weighted sum

\[ H^0_d(\mu, \nu, \eta_2^{r_2}, \eta_3^{r_3}) = \sum_f \frac{1}{|\text{Aut}(f)|} \]
of possibly disconnected ramified covers $f$ of $\mathbb{P}^1$ with ramification profiles $\mu$ over $0 \in \mathbb{P}^1$, $\nu$ over $\infty \in \mathbb{P}^1$ and $\eta_2$ and $\eta_3$ over $r_2$ and $r_3$ other fixed points of $\mathbb{P}^1$. Here $\eta_2$ denotes the simple ramification.

The spin Hurwitz numbers of the spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$ also count ramified covers of $\mathbb{P}^1$, but with only odd ramifications and with sign induced from $\mathcal{O}(-1)$.

Specifically, suppose $\rho$ and $\sigma$ are odd partitions of $d$ (i.e., all parts in $\rho$ and $\sigma$ are odd) and consider possibly disconnected ramified covers $f : C \to \mathbb{P}^1$ with ramification profiles $\rho$ over $0 \in \mathbb{P}^1$, $\sigma$ over $\infty \in \mathbb{P}^1$ and $\eta_3$ over other $r$ fixed points of $\mathbb{P}^1$. The domain Euler characteristic $\chi(C)$ is related to the partition lengths $\ell(\rho)$ and $\ell(\sigma)$ and the number $r$ by the Riemann-Hurwitz formula

$$\chi(C) = \ell(\rho) + \ell(\sigma) - 2r. \quad (1.3)$$

As the ramification divisor $R_f$ of $f : C \to \mathbb{P}^1$ is even, the twisted pull-back bundle

$$N_f = f^* \mathcal{O}(-1) \otimes \mathcal{O}_C(\frac{1}{2}R_f)$$

is a square root of the canonical bundle of $C$ (or a theta characteristic on $C$). Given odd partitions $\rho, \sigma$ of $d$ and the number $r$, the spin Hurwitz number of $(\mathbb{P}^1, \mathcal{O}(-1))$ is defined to be

$$H^0_d(\rho, \sigma, \eta_3^r) = \sum_{f} \frac{(-1)^{h(N_f)}}{|\text{Aut}(f)|}, \quad (1.4)$$

where the superscript $+$ denotes the parity of the spin curve $(\mathbb{P}^1, \mathcal{O}(-1))$.

Let $p = (p_1, p_2, \cdots)$ and $p' = (p'_1, p'_2, \cdots)$ be two sets of variables where $p_n$ and $p'_n$ are power-sum symmetric functions. For a partition $\mu = (\mu_1, \mu_2, \cdots)$, let $p_\mu = \prod p_{\mu_i}$. Let $P(d)$ be the set of partitions of $d$. Now introduce a generating function of the Hurwitz numbers (1.2) by

$$\Phi(p, p', b, q) = \sum_{d} q^d \sum_{\mu, \nu \in P(d)} p_\mu p'_\nu \sum_{s=0}^\infty \frac{b^s}{s!} \sum_{r_1+r_2+r_3=s} \frac{s!}{r_1! r_2! r_3!} \left( \frac{d^2 + d}{2} \right)^{r_1} H^0_d(\mu, \nu, \eta_2^{r_2}, \eta_3^{r_3}).$$

Under the restriction, $p_2 = p_4 = \cdots = 0$ and $p'_2 = p'_4 = \cdots = 0$, the function $\Phi(p, p', b, q)$ specializes to the function $\Phi_B(p_B, p'_B, b, q)$ where

$$p_B = (p_1, 0, p_3, 0, \cdots) \quad \text{and} \quad p'_B = (p'_1, 0, p'_3, 0, \cdots).$$

Let $OP(d)$ denote the set of odd partitions of $d$ and introduce a generating function of the spin Hurwitz numbers (1.4) by

$$\Phi_B(p_B, p'_B, b, q) = \sum_{d} q^d \sum_{\rho, \sigma \in OP(d)} p_\rho p'_\sigma \sum_{s=0}^\infty \frac{b^s}{s!} \sum_{r=0}^s \binom{s}{r} d^{2(s-r)} 2^{-\frac{(s+1)}{2}} H^0_d(\rho, \sigma, \eta_3^r),$$

where $\chi(C)$ is the domain Euler characteristic in (1.3).

Theorem 1.1. The function $\Phi(p, p', b, q)$ is a tau function of the two-component KP (2-KP) hierarchy and the function $\Phi_B(p_B, p'_B, b, q)$ is a tau function of the two-component BKP (2-BKP) hierarchy such that

$$\Phi_B^2(p_B, p'_B, b, q) = \Phi(p_B, p'_B, b, q). \quad (1.5)$$
This theorem is proved in Section 4. Its proof is based on the reduction of the KP hierarchy to the BKP hierarchy. Both hierarchies are formulated by a single tau function such that a square of a BKP tau function is a KP tau function (cf. Proposition 4 of [2] and Proposition 1 of [28]).

A discussion on a conjectural spin curve analog of the GW/H correspondence is presented in Section 5.

2 Symmetric functions

With transition matrices between linear bases of algebras relevant to our case, we express the Hurwitz generating function $\Phi$ via Schur functions and shifted symmetric power sums and the spin Hurwitz generating function $\Phi_B$ via Schur Q-functions and odd power-sum symmetric functions.

2.1 Central characters of symmetric group

Irreducible representations and conjugacy classes of the symmetric group $S(d)$ on $d$ letters are indexed by partitions of $d$. Let $C_\mu$ denote the conjugacy class indexed by $\mu \in P(d)$. The order of the centralizer of any element of $C_\mu$ is

$$z_\mu = \prod_k \mu(k)! k^{\mu(k)},$$

(2.1)

where $\mu(k)$ is the number of parts in $\mu$ equal to $k$.

Let $\pi^\lambda$ be the irreducible representation indexed by $\lambda \in P(d)$ and $\chi^\lambda$ be its character. The class sum $\sum_{g \in C_\mu} g$ acts on $\pi^\lambda$ as multiplication by constant. This constant is the central character

$$f^\lambda_\mu = \frac{|C_\mu|}{\dim \pi^\lambda} \chi^\lambda_\mu,$$

(2.2)

where $|C_\mu| = d! / z_\mu$ and $\chi^\lambda_\mu$ is the value of the character $\chi^\lambda$ on any element of $C_\mu$. The character formula for the Hurwitz number (1.2) is given as

$$H^0_d(\mu, \nu, \eta_2^r, \eta_3^r) = \sum_{\lambda \in P(d)} \frac{\chi^\lambda_\mu}{z_\mu} \frac{\chi^\lambda_\nu}{z_\nu} \left( f^\lambda_{\eta_2} \right)^{r_2} \left( f^\lambda_{\eta_3} \right)^{r_3}.$$

(2.3)

2.1.2 The algebra of shifted symmetric functions

The algebra of shifted symmetric functions $\Lambda^*$ is generated by the shifted symmetric power sums $p_n$ defined by

$$p_n(\lambda) = \sum_{i=1}^\infty \left( (\lambda_i - i + \frac{1}{2})^n - (-i + \frac{1}{2})^n \right).$$

From the central characters of the symmetric group, Kerov and Olshanski [9] obtained shifted symmetric functions as follows. For any partition $\mu = (\mu_1, \cdots, \mu_r)$, let $|\mu| = \sum \mu_i$ and set $\mu \cup (1^k) = (\mu_1, \cdots, \mu_r, 1, \cdots, 1) \in P(|\mu| + k)$. 

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Define a function \( f_\mu \) on the set of partitions by
\[
f_\mu(\lambda) = \left( \frac{\mu(1) + k}{\mu(1)} \right)^{\lambda_{(1)}} f_{\mu(1)}^{\lambda_{(1)}} \quad \text{if} \quad k = |\lambda| - |\mu| \geq 0
\]
and \( f_\mu(\lambda) = 0 \) if \( |\lambda| < |\mu| \). Then
\[
f_\mu = \prod_{\mu_i} p_\mu + \cdots,
\]
where \( p_\mu = \prod P_{\mu_i} \) and the dots denote the lower degree terms (see [9] and also [21]). This implies \( \{ f_\mu \} \) and \( \{ p_\mu \} \) are mutually triangular linear bases of \( \Lambda^* \). Observe that for \( \eta_3 = (3, 1^{d-3}) \) in the character formula (2.3),
\[
f_{(3)}(\lambda) = f_{\eta_3}^\lambda.
\]

The lemma below is a consequence of the Wassermann formula [27] (see also [6]). For a formal series \( A(t) \), let
\[
[t^k] \{ A(t) \} = \text{the coefficient of } t^k \text{ in } A(t).
\]

**Lemma 2.1.** Let \( f_2 = f_{(2)} \) and \( f_3 = f_{(3)} \). We have:

- (a) \( f_2 = \frac{1}{2} p_2 \).
- (b) \( f_3 = \frac{1}{3} p_3 - \frac{1}{2} p_1^2 + \frac{5}{4} p_1 \).

**Proof.** (a) is a well-known fact (cf. [20]), which also follows by the same argument as for (b). Consider the function \( \mathbf{p}_d^{\#} \) on the set of partitions defined by
\[
\mathbf{p}_d^{\#}(\lambda) = \begin{cases} 
\frac{d^{|\lambda|} \chi_{(3, 1^{d-3})}^\lambda}{\dim \pi^\lambda}, & d := |\lambda| \geq 3 \\
0, & d < 3,
\end{cases}
\]
where \( d^3 = d(d-1)(d-2) \). Form (3.2) of [6], one has
\[
\mathbf{p}_d^{\#} = [t^0] \left\{ -\frac{1}{3} \prod_{j=1}^3 \left( 1 - (j - \frac{1}{2}) t \right) \exp \left( \sum_{j=1}^{\infty} \frac{p_j}{j} (1 - (1 - 3t)^{-j}) \right) \right\}
\]
\[
= p_3 - \frac{3}{2} p_1^2 + \frac{5}{4} p_1.
\]
This together with (2.2) proves (b). \( \square \)

### 2.1.3 Schur functions

The Schur functions \( s_\lambda \) and the monomials \( p_\mu \) are related by
\[
s_\lambda(p) = \sum_{\mu \in P(d)} \frac{\chi_\mu^\lambda}{\mu} p_\mu,
\]
where \( \lambda \in P(d) \) (see page 114 of [17]). Now by (2.3) and (2.4) one can write the generating function \( \Phi(p, p', b, q) \) of Hurwitz numbers as
\[
\Phi(p, p', b, q) = \sum_d \sum_{\lambda \in P(d)} q^d e^{b(d+d') + f_2(\lambda) + f_3(\lambda)} s_\lambda(p)s_{\lambda}(p').
\]

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2.2

2.2.1 Central characters of Sergeev group

The Sergeev group \( C(d) \) is the semidirect product \( \text{Cliff}(d) \rtimes S(d) \), where \( \text{Cliff}(d) \) is the Clifford group generated by \( \xi_1, \cdots, \xi_d \) and a central element \( \epsilon \) subject to the relations
\[
\xi_i^2 = 1, \quad \epsilon^2 = 1, \quad \xi_i \xi_j = \epsilon \xi_j \xi_i \quad (i \neq j)
\]
and the symmetric group \( S(d) \) acts on \( \text{Cliff}(d) \) by permuting the \( \xi_i \)'s.

Let \( \text{SP}(d) \) denote the set of strict partitions \( \lambda = (\lambda_1 > \lambda_2 > \cdots) \) of \( d \). A spin \( C(d) \)-supermodule is a supermodule over the group algebra \( C[C(d)] \) on which \( \epsilon \) acts as multiplication by \( -1 \) such that (i) the irreducible spin \( C(d) \)-supermodules \( V_\lambda \) are indexed by strict partitions \( \lambda \in \text{SP}(d) \), (ii) the character \( \zeta_\lambda \) of \( V_\lambda \) is determined by the values \( \zeta_\lambda^\rho \) on the conjugacy classes \( C_\rho \) indexed by odd partitions \( \rho \in \text{OP}(d) \), and (iii) for the conjugacy class \( C_\rho \) indexed by \( \rho \in \text{OP}(d) \),
\[
|C_\rho| = \frac{|C(d)|}{2^{\ell(\rho)+1} z_\rho} = \frac{2^{|\rho|-\ell(\rho)}|\rho|! z_\rho}{z_\rho},
\]
where \( z_\rho \) is introduced in (2.1) (see [7, 24] and also [4]). The class sum of \( C_\rho \) acts on \( V_\lambda \) as multiplication by a constant, which is the central character of \( V_\lambda \) given by
\[
f_\lambda^\rho = \frac{|C_\rho|}{\dim V_\lambda} \zeta_\lambda^\rho.
\] (2.6)

For \( \lambda \in \text{SP}(d) \), let
\[
\delta(\lambda) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even} \\ 1, & \text{if } \ell(\lambda) \text{ is odd} \end{cases}
\]
From the Gunningham formula [3, 14], one has the character formula for the spin Hurwitz number (1.4):
\[
H^{\rho,\sigma,\eta}_d = \sum_{\lambda \in \text{SP}(d)} 2^{-\delta(\lambda)} \frac{\zeta_\lambda^\rho \zeta_\lambda^\sigma}{z_\rho z_\sigma} (f_\lambda^\eta)^r.
\] (2.7)

2.2.2 The algebra of supersymmetric functions

The algebra of supersymmetric functions \( \Gamma \) is generated by \( p_1, p_3, \cdots \). In the same manner as for \( f_\mu \), one can obtain supersymmetric functions \( f_\rho \) from the central characters of the Sergeev group. For any odd partition \( \rho \), let
\[
f_\rho(\lambda) = \begin{cases} \rho(1)+k \rho(1) f_\rho^{\lambda,(k)} & \text{if } k = |\lambda| - |\rho| \geq 0 \\ 0 & \text{if } |\rho| > |\lambda| \end{cases}
\]
and \( f_\rho(\lambda) = 0 \) if \( |\rho| > |\lambda| \). Then
\[
f_\rho = \frac{1}{z_\rho} p_\rho + \cdots,
\]
where the dots denote the lower degree terms (see Section 6 of [4]). It follows that \( \{ f_\rho \} \) and \( \{ p_\rho : \rho \text{ is odd} \} \) are mutually triangular linear bases of \( \Gamma \).

The lemma below follows from Proposition 3.4 of [5].
Lemma 2.2. Let \( f_3 = f_{(3)} \). We have
\[
f_3 = \frac{1}{4}p_3 - p_1^2 + \frac{2}{3}p_1.
\]

Proof. Consider the function \( p^\#_3 \) on the set of strict partitions defined by
\[
p^\#_3(\lambda) = \begin{cases} 
   d^{13} \frac{X^\lambda_{(3,1^{d-3})}}{X^\lambda_{(1^d)}}, & d := |\lambda| \geq 3, \\
   0, & d < 3,
\end{cases}
\]
where \( X^\lambda_{(3)} \) corresponds to \( X^\lambda_{(1)}(-1) \) in [17] such that
\[
X^\lambda_{(3,1^{d-3})} = 2^{\frac{\ell(\lambda) - \delta(\lambda)}{2} - 2(d-2)} \zeta(\lambda) \dim V^\lambda
\]
(see Proposition 3.3 of [4]). By (3.5) of [5], one has:
\[
p^\#_3 = \left\{ \frac{1}{6} \right\} \left( \frac{1}{2} (1 - \frac{1}{2} t)^2 \prod_{j=1}^\infty (1 - j t) \exp \left( \sum_{j=1}^\infty \frac{2p_{2j-1}t^{2j-1}}{2j-1} (1 - (1 - 3t)^{-(2j-1)}) \right) \right) = p_3 - 3p_1^2 + 2p_1.
\]
This together with (2.6) shows the lemma.

2.2.3 Schur Q-functions

The algebra of super symmetric functions \( \Gamma \) has another linear basis formed by Schur Q-functions \( \{ Q_\lambda : \lambda \) is strict \}. For \( \lambda \in \text{SP}(d) \),
\[
Q_\lambda(p_B) = \sum_{\rho \in \text{OP}(d)} 2^{\frac{\ell(\lambda) - \delta(\lambda)}{2} - 2d} \frac{\zeta^\rho}{z_\rho} p_\rho
\]
(see Corollary 3 of [7]). Thus, for \( \frac{1}{2} p_B = \left( \frac{1}{2} p_1, 0, \frac{1}{2} p_3, \cdots \right) \),
\[
Q_\lambda(\frac{1}{2} p_B) = \sum_{\rho \in \text{OP}(d)} 2^{\frac{\ell(\lambda) - \delta(\lambda) - 2d(-1)}} \frac{\zeta^\rho}{z_\rho} p_\rho. \tag{2.8}
\]
By (2.7) and (2.8), one can write the generating function \( \Phi_B(p_B, p'_B, b, q) \) of spin Hurwitz numbers as
\[
\Phi_B(p_B, p'_B, b, q) = \sum_d \sum_{\lambda \in \text{SP}(d)} q^{d} e^{b(d^2 + f_3(\lambda))} 2^{-\ell(\lambda)} Q_\lambda(\frac{1}{2} p_B) Q_\lambda(\frac{1}{2} p'_B). \tag{2.9}
\]

3 The operator formalism

The fermion operator formalism is a handy tool for handling various generating functions. We employ the formalism to express the generating functions (2.5) and (2.9) as vacuum expectations. For more detailed discussions of the formalism, we refer to Chapter 14 of [8], Appendix of [19] and Section 2 of [21].
3.1 The infinite wedge space

Let $V$ be a vector space with basis $\{ k \}$ indexed by the half-integers $k \in \mathbb{Z} + \frac{1}{2}$. The infinite wedge space (or fermion Fock space) is the vector space

$$\Lambda^\infty V$$

with basis $\{ v_S \}$ where $S = (s_1 > s_2 > s_3 > \cdots)$ with $s_i - s_{i+1} = 1$ for $i \gg 0$ and

$$v_S = s_1 \wedge s_2 \wedge s_3 \wedge \cdots.$$

The half-infinite wedge space, $\Lambda^\infty_0 V$, is the subspace generated by vectors $v_\lambda$ indexed by partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$. These are defined by

$$v_\lambda = \lambda_1 - \frac{1}{2} \wedge \lambda_2 - \frac{3}{2} \wedge \cdots \wedge \lambda_i - i + \frac{1}{2} \wedge \cdots.$$

The vector indexed by the empty partition $\emptyset = (0, 0, \cdots)$ is the vacuum vector

$$v_\emptyset = -\frac{1}{2} \wedge -\frac{3}{2} \wedge -\frac{5}{2} \wedge \cdots.$$

Let $(\cdot, \cdot)$ be the inner product on $\Lambda^\infty V$ for which $\{ v_S \}$ is an orthonormal basis. The vacuum expectation of an operator $A$ on $\Lambda^\infty V$ is defined as

$$\langle A \rangle := (A v_\emptyset, v_\emptyset).$$

3.1.2 Charged fermions

For each $k \in \mathbb{Z} + \frac{1}{2}$, the operator $\psi_k$ on $\Lambda^\infty V$ and its adjoint $\psi_k^*$ are defined by

$$\psi_k v = k \wedge v, \quad \psi_k^* v = \iota_k v.$$

These are charged fermions and satisfy the canonical anti-commutative relations:

$$\psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k,\ell}, \quad \psi_k \psi_\ell + \psi_\ell \psi_k = \psi_k^* \psi_\ell^* + \psi_\ell^* \psi_k^* = 0. \quad (3.1)$$

Infinite sums of the quadratics $\psi_j \psi_j^*$ make sense as operators on $\Lambda^\infty V$ if we write them in terms of normal ordering defined by

$$: \psi_k \psi_\ell^* : \xrightarrow{\text{def}} \psi_k \psi_\ell^* - \langle \psi_k \psi_\ell^* \rangle = \begin{cases} \psi_k \psi_\ell^*, & \ell > 0, \\ -\psi_\ell^* \psi_k, & \ell < 0. \end{cases}$$

3.1.3 Operators related to shifted symmetric functions

For any integer $n \geq 0$, define

$$E_n = \sum_{k \in \mathbb{Z} + \frac{1}{2}} k^n : \psi_k \psi_k^* :.$$

One of the salient features of the half-integer infinite wedge lies in the relation between the operators $E_n$ and the shifted symmetric power sums $p_n$:

$$E_n v_\lambda = p_n(\lambda) v_\lambda. \quad (3.2)$$
The operators $E_1$ and $E_0$ are the energy and charge operator. As $p_1(\lambda) = |\lambda|$, $E_1 v_{\lambda} = |\lambda| v_{\lambda}$. On the other hand, $E_0 v_{\lambda} = 0$ and hence the half-infinite wedge space is the 0-eigenspace of $E_0$ (cf. Appendix of [19]).

To express the generating function (2.5) of Hurwitz numbers as a vacuum expectation, we consider the following operator

$$F = \frac{1}{3}E_3 + \frac{1}{2}E_2 + \frac{11}{12}E_1.$$ 

By Lemma 2.1 and (3.2), for every vector $v_{\lambda}$ with $|\lambda| = d$ one has

$$F v_{\lambda} = \left( \frac{1}{2} (d + d^2) + f_2(\lambda) + f_3(\lambda) \right) v_{\lambda}.$$ (3.3)

As shown in [20], it is convenient to use the operator $q^{E_1} = e^{E_1 \ln q}$ because it can be written as

$$q^{E_1} = \sum_d P_d q^d,$$

where $P_d$ is the orthogonal projection onto the $d$-eigenspace of $E_1$.

### 3.1.4 The Boson-Fermion correspondence

Introduce a set $t = (t_1, t_2, \cdots)$ of Miwa variables $t_n = p_n/n$ and let $\alpha^*(t)$ denote the adjoint of the operator

$$\alpha(t) = \sum_{n > 0} t_n \sum_{k \in \mathbb{Z} + \frac{1}{2}} : \psi_{k-n} \psi_k^* :.$$ 

Then from the remarkable Boson-Fermion correspondence (cf. §14.10 of [8]), one has

$$e^{\alpha^*(t)} v_0 = \sum_{\lambda} s_\lambda(p) v_{\lambda},$$

where the sum is over all partitions $\lambda$. This together with (2.5) and (3.3) gives

$$\Phi(p, p', b, q) = \left\langle e^{\alpha(t)} q^{E_1} e^{bF} e^{\alpha^*(t')} \right\rangle,$$ (3.4)

where $t' = (t'_1, t'_2, \cdots)$ is another set of Miwa variables $t'_n = p'_n/n$.

### 3.2 Neutral fermions

There is an involution on the charged fermions defined by

$$w(\psi_k) = (-1)^{k+\frac{1}{2}} \psi_{k-1}^*, \quad w(\psi_k^*) = (-1)^{k+\frac{1}{2}} \psi_{-k-1}.$$ 

The neutral fermions are defined as $\pm 1$-eigenvectors of the involution.

**Definition 3.1.** For $m \in \mathbb{Z}$ and $i = \sqrt{-1}$, let

$$\phi_m = \frac{1}{\sqrt{2}} \left( \psi_{m-\frac{1}{2}} + (-1)^m \psi_{m+\frac{1}{2}}^* \right), \quad \hat{\phi}_m = \frac{i}{\sqrt{2}} \left( \psi_{m-\frac{1}{2}} - (-1)^m \psi_{m+\frac{1}{2}}^* \right).$$
By the canonical anti-commutative relations (3.1) for charged fermions, the neutral fermions also satisfy the canonical anti-commutative relations:

\[ \phi_n \phi_m + \phi_m \phi_n = \hat{\phi}_m \phi_n + \hat{\phi}_n \phi_m = (-1)^m \delta_{m,-n}, \quad \phi_m \hat{\phi}_n + \hat{\phi}_n \phi_m = 0. \tag{3.5} \]

With the relations (3.1), the charged fermions \( \psi_i \) and \( \psi_j^* \) generate a Clifford algebra, denoted \( Cl \). The neutral fermions \( \phi_m \) and \( \hat{\phi}_m \) respectively generate isomorphic subalgebras. The isomorphism is given by the involution on the Clifford algebra \( Cl \) induced from the map \( \phi_m \leftrightarrow \hat{\phi}_m \), which we denote by

\[ X \mapsto \hat{X}. \tag{3.6} \]

Let \( Cl_B \) be the subalgebra of \( Cl \) generated by \( \phi_m \)'s. There is a decomposition

\[ Cl_B = Cl_B^0 \oplus Cl_B^1, \]

where \( Cl_B^p \) is spanned by all products of the form \( \phi_{m_1} \cdots \phi_{m_s} \) with \( s \equiv p \pmod{2} \).

Recall that the half-infinite wedge space has an orthonormal basis \( \{ v_\lambda \} \). Its neutral analog is the subspace spanned by vectors \( v^B_\lambda \) indexed by strict partitions \( \lambda = (\lambda_1 > \cdots > \lambda_\ell) \), which are defined by

\[ v^B_\lambda = \begin{cases} \phi_{\lambda_1} \cdots \phi_{\lambda_\ell} v_\emptyset & \text{if } \ell \text{ is even}, \\
\sqrt{2} \phi_{\lambda_1} \cdots \phi_{\lambda_\ell} \phi_0 v_\emptyset & \text{if } \ell \text{ is odd}. \end{cases} \]

As \( \phi_m^* = (-1)^m \phi_{-m} \) and \( \phi_0^2 = \frac{1}{2}, \{ v^B_\lambda \} \) is an orthonormal basis of the subspace (3.7).

### 3.2.2 Operators related to supersymmetric functions

For any odd integer \( n \geq 1 \), define a neutral analog of the operator \( E_n \) by

\[ E_n^B = \sum_{m > 0} (-1)^m m^n \phi_m \phi_{-m}. \]

From the canonical anti-commutative relations (3.5), it is easy to see that

\[ (-1)^m \phi_m \phi_{-m} v^B_\lambda = \begin{cases} v^B_\lambda & \text{if } m = \lambda_j \text{ for some } j, \\
0 & \text{otherwise}. \end{cases} \]

This implies \( v^B_\lambda \) is an eigenvector of \( E_n^B \) with eigenvalue \( p_n(\lambda) \), that is,

\[ E_n^B v^B_\lambda = p_n(\lambda) v^B_\lambda. \tag{3.8} \]

As \( p_1(\lambda) = |\lambda| \), the operator \( E_1^B \) plays the same role as the energy operator \( E_1 \).

The following lemma together with (3.2) and (3.8) makes a connection between supersymmetric functions and shifted symmetric functions via neutral and charged fermions.

**Lemma 3.2.** For any odd integer \( n \geq 1 \),

\[ E_n^B + \hat{E}_n^B = \sum_{i=0}^{n} \binom{n}{i} \frac{E_{n-i}}{2^i}, \]

where the operator \( \hat{E}_n^B \) is defined by the isomorphism (3.6) in an obvious way.
To express the generating function (2.9) of spin Hurwitz numbers as a vacuum expectation, consider the following operator

\[ F_B = \frac{1}{3} E_3^B + \frac{2}{3} E_1^B. \]

It follows from Lemma 2.2 that for every vector \( v_\lambda^B \) with \( |\lambda| = d \),

\[ F_B v_\lambda^B = (d^2 + f_3(\lambda)) v_\lambda^B. \]  

(3.9)

### 3.2.3 The neutral Boson-Fermion correspondence

Let \( \beta^*(t) \) be the adjoint of the operator

\[ \beta(t) = \frac{1}{2} \sum_{n \geq 0} t_{2n+1} \sum_{m \in \mathbb{Z}} (-1)^{m+1} \phi_m \phi_{-m-2n-1}. \]

Then from the neutral Boson-Fermion correspondence, one has

\[ e^{\beta^*(t)} v_\emptyset = \sum_\lambda 2^{-\frac{u(\lambda)}{2} Q_\lambda(\frac{1}{2} p_B)} v_\lambda^B, \]

where the sums are over all strict partitions \( \lambda \) (see [28]). This together with (2.9) and (3.9) yields:

\[ \Phi_B(p_B, p_B', b, q) = \left\langle e^{\beta(t)} q e^{bF} e^{\beta^*(t')} \right\rangle = \left\langle e^{\beta(t)} q e^{bF} e^{\beta^*(t')} \right\rangle, \]  

(3.10)

where the second vacuum expectation is given by the isomorphism (3.6).

### 4 Square root

Integrable hierarchies of KP type (including the 2-Toda lattice hierarchy) have fermionic forms of Hirota equations and fermionic formulas for tau functions (see [1] and also Appendix of [23]). We apply those in the proof of Theorem 1.1 below.

**Proof of Theorem 1.1:** The operators \( q e^{E_{1}} e^{bF} \) and \( q e^{E_{1}} e^{bF} \) satisfy the following fermionic forms of Hirota equations:

\[ \left[ q e^{E_{1}} e^{bF} \otimes q e^{E_{1}} e^{bF}, \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k \otimes \psi_k^* \right] = 0, \]  

(4.1)

\[ \left[ q e^{E_{1}} e^{bF} \otimes q e^{E_{1}} e^{bF}, \sum_{m \in \mathbb{Z}} (-1)^{m} \phi_m \otimes \phi_{-m} \right] = 0. \]  

(4.2)

One can obtain these commutation relations from: for \( n \geq 0 \),

\[ E_{n} \psi_{k} = \psi_{k}(E_{n} + k^{n}), \quad E_{n} \psi_{k}^{*} = \psi_{k}^{*}(E_{n} - k^{n}), \]

and for odd \( n \geq 1 \),

\[ E_{n}^{B} \phi_{\pm m} = \phi_{\pm m}(E_{n}^{B} \pm m^{n}). \]

It follows from (4.1) that the sequence

\[ \tau_{n}(t, t') = \left( e^{\alpha(t)} q e^{E_{1}} e^{bF} e^{\alpha^*(t')} v_{n}, v_{n} \right) \quad (n \in \mathbb{Z}) \]
is a sequence of tau functions of the 2-Toda lattice hierarchy where $v_n$ is the vacuum vector in the $n$-eigenspace of the charge operator $E_0$ defined by

$$v_n = n - \frac{1}{2} \wedge n - \frac{3}{2} \wedge n - \frac{5}{2} \wedge \cdots$$

(see Appendix of [19]). The Hurwitz generating function $\Phi(p, p', b, q) = \tau_0(t, t')$ by (3.4) and $\tau_0(t, t')$ is a tau function of the 2-KP hierarchy (see Theorem 1.12 of [26]).

It also follows from (4.2) that the vacuum expectation in (3.10),

$$\Phi_B(p_B, p'_B, b, q) = e^{\beta(t) q E_B} e^{b F_B} e^{\beta^*(t')} \langle \rangle,$$

is a tau function of the 2-BKP hierarchy in time variables $t$ and $t'$ (see [25]).

Now it remains to prove (1.5). By (3.4) we have

$$\Phi(p_B, p'_B, b, q) = e^{\alpha_B(t) q E_B} e^{b F_B} e^{\alpha^*_B(t')} \langle \rangle,$$

where $\alpha_B(t) = \alpha(t_1, 0, t_3, 0, \cdots)$. By definition, one has

$$\beta(t) + \tilde{\beta}(t) = \alpha_B(t).$$

Since $\Lambda_0^\infty V$ is the 0-eigenspace of the charge operator $E_0$, Lemma 3.2 shows that

$$(E_1^B + \hat{E}_1^B)|_{\Lambda_0^\infty V} = E_1, \quad (F^B + \hat{F}^B)|_{\Lambda_0^\infty V} = F.$$

Therefore noting $\langle Z \rangle \langle \hat{W} \rangle = \langle Z \hat{W} \rangle$ for $Z, W \in \mathcal{C} t_B$ (see Lemma 1 of [28]), we conclude

$$\Phi_B^2(p_B, p'_B, b, q) = e^{\beta(t) + \tilde{\beta}(t) q E_B + \hat{E}_B} e^{b F_B + \hat{F}_B} e^{\beta^*(t) + \tilde{\beta}^*(t')} \langle \rangle = \Phi(p_B, p'_B, b, q).$$

This completes the proof of Theorem 1.1.

5 Remark

In [21], the GW/H correspondence was defined by the two linear bases $\{p_\mu\}$ and $\{f_\mu\}$ of the algebra of shifted symmetric functions $\Lambda^*$. Let $P_Q$ denote the vector space over $Q$ with basis the set of partitions (i.e., every vector in $P_Q$ is a formal linear combination of partitions). As $\{f_\mu\}$ is a linear basis of $\Lambda^*$, there is a linear isomorphism $\varphi : P_Q \rightarrow \Lambda^*$

given by $\mu \mapsto f_\mu$. Using this isomorphism, one can extend the character formula for Hurwitz numbers of the curve of genus $h$ to the following multilinear form on $P_Q$:

$$H_d^h(v_1, \cdots, v_n) = \sum_{\lambda \in \text{P}(d)} \left( \frac{\dim \pi_\lambda}{d!} \right)^{2-2h} \prod_{i=1}^n \varphi(v_i)(\lambda).$$

The GW/H correspondence is then the following equality:

$$\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \rangle_d^h = H_d^h \left( \frac{P_{k_1+1}}{(k_1+1)!}, \cdots, \frac{P_{k_n+1}}{(k_n+1)!} \right),$$
where the left-hand side is the degree $d$ descendent GW invariants of the curve of genus $h$, $\omega$ is the Poincaré dual of the point class and $\bullet$ denotes the disconnected theory.

One may ask whether interplays between the following functions yield connections between theories in (1.1):

\[
\begin{array}{c}
\{p_\rho\} & \xrightarrow{\text{GW/H}} & \{f_\rho\} \\
\downarrow & & \downarrow \\
\{p_\rho\} & \xleftarrow{\text{Theorem 1.1}} & \{f_\rho\}
\end{array}
\]

We will use the two linear bases \( \{p_\rho : \rho \text{ is odd}\} \) and \( \{f_\rho\}\) of the algebra of supersymmetric functions $\Gamma$ to describe a conjectural spin curve analog of the GW/H correspondence. Let $\text{OP}_\mathbb{Q}$ denote the vector space over $\mathbb{Q}$ with basis the set of odd partitions. There is a linear isomorphism $\varphi_B : \text{OP}_\mathbb{Q} \rightarrow \Gamma$ given by

\[
\varphi_B(\rho) = 2^{\frac{|\rho| - |\rho|}{2}} f_\rho.
\]

Using this isomorphism, one can also extend the Gunningham formula [3, 14] for spin Hurwitz numbers of the spin curve of genus $h$ and parity $p$ to the following multilinear form on $\text{OP}_\mathbb{Q}$:

\[
H^{h,p}_d(w_1, \ldots, w_n) = 2^{(d+1)(1-h)} \sum_{\lambda \in \text{SP}(d)} (-1)^{p\delta(\lambda)} \left( 2^{1 - \frac{1}{2} \frac{\dim V^\lambda}{|C(d)|}} \right)^{2 - 2h} \prod_{i=1}^n \varphi_B(w_i)(\lambda).
\]

Then the Maulik-Pandharipande formulae ((8) and (9) of [18]), which were proved in [11, 13, 12], show that for degree $d = 1, 2$, the descendent GW invariants of the spin curve of genus $h$ and parity $p$ are given by

\[
\left\langle \tau_{k_1}(\omega) \cdots \tau_{k_n}(\omega) \right\rangle_d^{h,p} = H^{h,p}_d \left( \varphi_B^{-1} \left( \frac{(-1)^{k_1} k_1!}{2k_1(2k_1 + 1)!} p_{2k_1 + 1} \right), \cdots, \varphi_B^{-1} \left( \frac{(-1)^{k_n} k_n!}{2k_n(2k_n + 1)!} p_{2k_n + 1} \right) \right).
\]

**Question 5.1.** Does the equality (5.1) hold for $d \geq 3$?

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