Submodular Optimization under Noise

Avinatan Hassidim
Bar Ilan University
avinatan@cs.biu.ac.il

Yaron Singer
Harvard University
yaron@seas.harvard.edu

Abstract

We consider the problem of maximizing a monotone submodular function under noise, which to the best of our knowledge has not been studied in the past. There has been a great deal of work on optimization of submodular functions under various constraints, with many algorithms that provide desirable approximation guarantees. However, in many applications we do not have access to the submodular function we aim to optimize, but rather to some erroneous or noisy version of it. This raises the question of whether provable guarantees are obtainable in presence of error and noise. We provide initial answers, by focusing on the question of maximizing a monotone submodular function under a cardinality constraint when given access to a noisy oracle of the function. We show that:

- For a cardinality constraint \( k \geq 2 \), there is an approximation algorithm whose approximation ratio is arbitrarily close to \( 1 - 1/e \);
- For \( k = 1 \) there is an approximation algorithm whose approximation ratio is arbitrarily close to \( 1/2 \) in expectation. No randomized algorithm can obtain an approximation ratio better than \( 1/2 + o(1) \) in expectation;
- If the noise is adversarial, no non-trivial approximation guarantee can be obtained.
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1 Introduction

In this paper we study the effects of error and noise on submodular optimization. A function $f : 2^N \rightarrow \mathbb{R}_+$ defined on a ground set $N$ of size $n$ is submodular if for any $S, T \subseteq N$:

$$f(S \cup T) \leq f(S) + f(T) - f(S \cap T).$$

Equivalently, submodularity can be defined as a natural diminishing returns property: for any $S \subseteq T \subseteq N$ and $a \in N \setminus T$ the function is submodular if:

$$f_S(a) \geq f_T(a),$$

where $f_A(B) = f(A \cup B) - f(A)$ for any $A, B \subseteq N$. In general, submodular functions may require a representation that is exponential in the size of the ground set, in which case the natural assumption is that we are given access to a value oracle which given a set $S$ returns $f(S)$. It is well known that submodular functions admit desirable approximation guarantees and are heavily used in applications such as market design, data mining, and machine learning (see related work section for further discussion). Before defining error and noise formally, we can consider the following example.

Example: maximizing coverage with error. In the maximum-coverage problem we are given a family of sets that cover a universe of items, and the goal is to select a fixed number of sets whose union is maximal. This classic problem is an example of maximizing a monotone (i.e. $S \subseteq T \implies f(S) \leq f(T)$) submodular function under a cardinality constraint. It is well known that the celebrated greedy algorithm which iteratively adds the set which includes the largest number of items that have not yet been covered provides a $1 - 1/e$ approximation guarantee [66] and that this guarantee is optimal unless P=NP [29]. But imagine that the algorithm cannot estimate the underlying coverage of the sets exactly. Instead, the algorithm queries an oracle that has some error. For a concrete example, consider the instance illustrated in Figure 1. If we apply the greedy algorithm with the oracle described in the example, it will begin by selecting a set that covers an element in $A$, and rather than continuing to select sets from $B$, the algorithm will select elements from $A$ until it terminates. The approximation ratio in this case would be linear in the size of the input. In this example, for every set queried, the oracle is guaranteed to return a value that is at least $2/3$ of the underlying objective value, which may at a first glance seem like an excessive error. Note however that for any $\epsilon > 0$ the example can be replicated by planting not 2 but $1/\epsilon$ items in $A$, and still the greedy algorithm could only guarantee an $O(1/k)$-approximation when choosing $k$ elements.

The greedy algorithm is heavily used in the theory and applications of submodular optimization, which can make its lack of robustness to small errors somewhat alarming. Since submodular functions can be exponentially representative, it may be reasonable to assume that there are cases where one faces some error in their evaluation. In market design where submodular functions often model agents’ valuations for goods, it seems reasonable to assume that agents do not precisely know their valuations. Even with compact representation, evaluation of a submodular function may be prone to error. In machine learning and data mining the submodular objective functions are often learned from data, and may be subject to error. One recent line of work in machine learning theory, for example, seeks to learn submodular functions by observing samples and returning some surrogate function which is a constant-factor approximation of the submodular function [40, 7, 6, 3, 35, 36, 25, 26, 34, 37, 5]. Optimizing the function using the output of these learning
Figure 1: An illustration of an instance of max-cover for which the greedy algorithm fails with access to an oracle with error. In the instance above there is one family of sets \( A \) depicted on the left where all sets cover the same two items, and another family of disjoint sets \( B \) that each cover a single unique item. Consider an oracle which evaluates sets as follows. For any combination of sets the oracle evaluates the cardinality of the union of the subsets exactly, except for a few special cases: For \( S = A \cup b \quad \forall A \subseteq A, b \in B \) the oracle returns \( \tilde{f}(S) = 2 \), and for \( S \subseteq A \) the oracle returns \( \tilde{f}(S) = 2 + \delta \) for some arbitrarily small \( \delta > 0 \). With access to this oracle, the greedy algorithm will only select sets in \( A \) which may be as bad as linear in the size of the input.

algorithms implies optimizing an erroneous version of the original function. In general, as we consider optimization over large data sets, it seems plausible that there are domains in which the objective is not known precisely.

Can we retain desirable approximation guarantees in the presence of error?

The fact that the greedy algorithm fails does not yet exclude the possibility that a variant of the greedy algorithm or perhaps some other algorithm can overcome small errors. To discuss this rigorously we need a formal model of optimization that accounts for error and noise.

1.1 Separation between error and noise

For a given function \( f : 2^N \to \mathbb{R} \) and some \( \epsilon > 0 \) we say that an oracle \( \tilde{f} : 2^N \to \mathbb{R} \) is \( \epsilon \)-erroneous if for every set \( S \subseteq N \), it respects:

\[
(1 - \epsilon)f(S) \leq \tilde{f}(S) \leq (1 + \epsilon)f(S)
\]

Stated in these terms, in the max-coverage example we tricked the greedy algorithm with a 1/3-erroneous oracle. The same consequences apply to an \( \epsilon \)-erroneous oracle for any \( \epsilon > 0 \) by planting \( 1/\epsilon \) items in \( A \), as mentioned above. Notice that given access to an \( \epsilon \)-erroneous oracle, one can trivially approximate the solution well in exponential time by evaluating all possible subsets and return the best solution. Is there a polynomial-time algorithm that can obtain desirable approximation guarantees for maximizing a monotone submodular function under a cardinality constraint given access to \( \epsilon \)-erroneous oracles? Unfortunately, in Section 6 we exclude this possibility.

**Theorem.** No randomized algorithm can obtain an approximation strictly better than \( O(n^{-1/2+\delta}) \) to maximizing monotone submodular functions under a cardinality constraint using \( e^{n^\delta} / n \) queries to an \( \epsilon \)-erroneous oracle, for any fixed \( \epsilon, \delta < 1/2 \), with high probability.

The above statement implies that even near-optimal approximations of submodular functions do not provide us with guarantees suitable for optimization. In particular, even constant-factor estimations of submodular functions from learning and sketching do not suffice.
The seven stages of grief. Once we pass shock and denial, we can turn to bargaining. To bargain with the above impossibility result we can consider two main relaxations. The first would be to consider stricter classes of problems. It is trivial to show for example, that additive functions where $f(S) = \sum_{a \in S} f(a)$ can be optimized well when given access to $\epsilon$-erroneous oracles: since each element can be evaluated within a factor of $1 \pm \epsilon$ of its true value, choosing the elements with largest values using the erroneous oracle would be at most $(1 - 2\epsilon)$ away from optimal. Unfortunately, it seems like there are not many interesting classes of submodular functions that enjoy these properties. In fact, our impossibility result applies to very simple affine functions, and even coverage functions like the one in the example above. An alternative relaxation is to consider error models that are not necessarily adversarial. Since we cannot handle small adversarial errors, it may be beneficial to consider the case in which errors adhere to some universal structure.

Optimization under noise. Note that we can equivalently say that $\tilde{f} : 2^N \to \mathbb{R}$ is $\epsilon$-erroneous if for every $S \subseteq N$ we have that $\tilde{f}(S) = \xi_S f(S)$ for some $\xi_S \in [1 - \epsilon, 1 + \epsilon]$. The lower bound stated above applies to the case in which the error multipliers $\xi_S$ are adversarially chosen. A natural question is whether some relaxation of the adversarial error model can lead to possibility results.

**Definition.** For a function $f : 2^N \to \mathbb{R}$ we say that $\tilde{f} : 2^N \to \mathbb{R}$ is a **noisy oracle** if there exists some distribution $\mathcal{D}$ s.t. $\tilde{f}(S) = \xi_S f(S)$ where $\xi_S$ is independently drawn from $\mathcal{D}$ for every $S \subseteq N$.

Note that a noisy oracle as defined above is consistent: for any $S \subseteq N$ the noisy oracle returns the same answer regardless of how many times it is queried. When the noisy oracle is inconsistent, mild conditions on the noise distribution allow the noise to essentially vanish after logarithmically-many queries, reducing the problem to standard submodular optimization (see e.g. [46]). Note also that in the definition we do not impose any restrictions on the distribution. In particular, it is not required that $\xi_S \in [1 - \epsilon, 1 + \epsilon]$. In Section 7 we show simple examples for which the greedy algorithm fails, even when the noise distribution and uniformly distributed in $[1 - \epsilon, 1 + \epsilon]$.

Noise distributions. Naturally, if the distribution always returns 0, any algorithm with access to a noisy oracle is helpless. We will therefore be interested in defining a broad class of distributions that avoids such trivialities which is general enough to contain natural distributions. In this paper we define a class which we call generalized exponential tail distributions. This class includes Gaussian and Exponential distributions, as well as uniform distributions under a mild assumption (the probability of seeing the supremum is not zero). The class is formally defined as follows.

**Definition.** A noise distribution $\mathcal{D}$ has a **generalized exponential tail** if there exists some $x_0$ such that for $x > x_0$ the probability density function $\rho_{\mathcal{D}}(x) = e^{-g(x)}$, where $g(x) = \sum_i a_i x^{\alpha_i}$. We do not assume that all the $\alpha_i$’s are integers, but only that $\alpha_0 \geq \alpha_1 \geq \ldots$, and that $\alpha_0 \geq 1$.

An additional requirement that we have on the distribution is that it is bounded away from zero. In particular, we assume that the likelihood of obtaining values that are smaller than some polynomial in $n$ is polynomially small. Notice that this definition depends on the $n$, but we only use it to state upper bounds. Any impossibility result we state is independent of $n$. We further discuss this class of distributions in Section 7. From here on, we will assume every distribution $\mathcal{D}$ has a generalized exponential tail bounded away from zero.
1.2 Main results

Our main result is that for the problem of optimizing a monotone submodular function under a cardinality constraint, near-optimal approximations are achievable under noise.

**Theorem.** For any generalized exponential tail distribution which is bounded away from zero, there exists $n_0$ such that for any monotone submodular function $f$ defined on a ground set $n > n_0$, there is a polynomial-time algorithm with access to a noisy oracle of $f$, which optimizes $f$ under a cardinality constraint $k > 2$, whose approximation ratio is w.h.p arbitrarily close to $1 - 1/e$.

This theorem is a summary of three different results, each obtained for different regime of $k$ as a function of $n$. More specifically for any fixed $\epsilon > 0$ we show:

- **$1 - 1/e - \epsilon$ guarantee for large $k$:** We say that $k$ is large when $k \in \Omega(\log \log n/e^2)$. For $k$ that is sufficiently larger than $\log \log n/e^2$ we give a deterministic algorithm which obtains a $(1 - 1/e - \epsilon)$ approximation guarantee with high probability over the noise distribution.

- **$1 - 1/e - \epsilon$ guarantee for small $k$:** We say that $k$ is small when $k \in O(\log \log n)$ and also $k \in \Omega(1/\epsilon)$. For values of $k$ in $O(\log \log n)$ the problem is surprisingly harder. We develop a different algorithm for this case which is deterministic and achieves the coveted $(1 - 1/e - \epsilon)$ guarantee, with high probability over the noise distribution. This algorithm can be generalized to a randomized algorithm which would then obtain a $1 - 1/e - \epsilon$ approximation for any $k \in \Omega(1/\epsilon)$ but this guarantee only holds in expectation over the randomization of the algorithm.

- **Guarantees for very small $k$:** We say that $k$ is very small when it is a constant smaller than $1/\epsilon$. The algorithm for small $k$ depends on $k$ being sufficiently larger than $1/\epsilon$ to obtain its $1 - 1/e - \epsilon$ guarantee. When $k$ is a small constant, a variation on the algorithm for $k \in \Omega(1/\epsilon)$ produces a randomized algorithm whose approximation ratio is $1 - 1/k - \epsilon$ which holds with high probability over the randomization of the algorithm and the noise distribution. Note that this gives a $1 - 1/e - \epsilon$ for any $k > 2$, and $1/2 - \epsilon$ approximation for $k = 2$. For $k = 1$ this implies a randomized $1/4 - \epsilon$ approximation: run the algorithm for $k = 2$ and select an element at random. We also show a $k/(k + 1)$ algorithm whose approximation ratio holds in expectation over the randomization of the algorithm. This achieves $1 - 1/e$ even for $k = 2$. For $k = 1$ no randomized algorithm can obtain an approximation ratio better than $1/2 + O(1/\sqrt{n})$ and $(2k - 1)/2k + O(1/\sqrt{n})$ for general $k$.

1.3 Overview of techniques

In the absence of noise, the greedy algorithm obtains a $1 - 1/e$ approximation guarantee by iteratively choosing the element $a$ which maximizes $f(S \cup a)$ where $S$ is the set chosen in previous iterations. In the presence of noise, the greedy algorithm can no longer obtain a constant factor guarantee. This is true even in cases where the noise is uniformly distributed in $[1 - \epsilon, 1 + \epsilon]$ or when it is a Gaussian or Exponential (see Section 7).

At a high level, in this paper we modify the greedy algorithm so that instead of selecting elements that maximize the noisy version of the function, it selects elements that maximize noisy versions of surrogate functions that we construct from the noisy oracle. We refer to this technique as smoothing. The challenge then reduces from dealing with noise to constructing meaningful surrogates through which we can obtain desirable guarantees to the original function we aim to optimize.
1.3.1 Smoothing

Intuitively, the idea of smoothing allows us to identify elements whose marginal contribution is large. To get some intuition behind this idea, consider evaluating a continuous function with access to a noisy oracle. One reasonable approach to obtain the value of a certain point is by sampling noisy valuations of polynomially-many points around it and use their average as an estimation for the true value. In this paper we mimic this approach by selecting a family of sets $\mathcal{H}$ and for every pair of sets $A,B \subseteq N$ we average the values of $\tilde{f}(A \cup X)$ and $f(B \cup X)$ for all $X \in \mathcal{H}$, and select the set whose averaged noisy contribution is maximal. In continuous optimization one can sample arbitrarily close points to a point of interest and apply concentration bounds. In our setting, the combinatorial nature of the problem does not enable this. In Section 2 we provide general tools for bounding effects of noise that are used by the different algorithms. Each one of the algorithms uses a different smoothing technique that employs these bounds in different ways.

1.4 A Bird’s-eye view on the algorithms

At a high level, we present two different approaches for optimization under noise, each relying on a different type of smoothing. When $k$ is large, we can afford to take “dummy” elements without hurting the approximation ratio too much. This lets us choose elements with high probability, and with a reasonable runtime of $\tilde{O}(nk)$, compared to the greedy algorithm’s $O(nk)$. When $k$ is small, we have to use elements which we will not take into the final set for smoothing. This creates a difficulty, since it’s possible that the elements we entered have large values. We overcome this by adding $O(1/e)$ elements at a time (and then “sacrificing” one of them does not hurt the approximation ratio by much), but this has runtime costs, and hurts the success probability. If we want the second method to succeed with success probability $1 - o(1)$ we can afford at most $k = o(\sqrt{\log n})$, and hence it is important that SMOOTH-GREEDY succeeds even on relatively small values of $k$.

1.4.1 Algorithm for Large $k$

Intuitively, when $k$ is sufficiently large we would like to discard a small number of elements that could be used as the family of sets for smoothing, and hope that the smooth values we obtain are sufficiently close to the true values of the function. Intuitively, if $k$ is sufficiently large, discarding a small set should not severely hurt the approximation ratio. This is the mindset behind the approach for designing an algorithm for large values of $k$.

The main algorithm for large $k$ is Slick-Greedy which we describe in Section 3. The idea is that this algorithm is competitive against a restricted optimal solution that is forced to choose some small set of arbitrary elements. Once we committed to such a set, we can use it for smoothing. To do this, the algorithm uses the simpler SMOOTH-GREEDY algorithm.

The smooth greedy algorithm. The SMOOTH-GREEDY algorithm chooses a set $H$ of $\ell \in O(\log \log n)$ arbitrary elements when $k \in O(\log \log n)$ and $\ell \in O(\log n)$ elements when $k$ is sufficiently larger than $\log n$. Then, it runs the greedy algorithm for $k - \ell$ steps, starting with $S = \emptyset$ adding elements one at a time. However, instead of evaluating $f(S \cup a)$ to choose the next element
to add to $S$, it uses the smooth values:
\[
\frac{1}{2\ell} \sum_{H' \subseteq H} \tilde{f}(S \cup a \cup H')
\]
In the end of the procedure, the output is $S \cup H$.

**Noise elimination via smoothing.** Since the noise distribution has a mean and a variance, and since it is independent of $n$ (our main theorem says that for every noise distribution there exists some $n_0$ such that if there are $n > n_0$ elements the approximation succeeds), then averaging over $2^{O(\log \log n)} = \text{poly} \log n$ samples of the distribution is enough to give us concentration bounds that hold with $1 - 1/\text{poly}(n)$ probability. Thus, we can union bound the failure of these bounds\(^1\) and focus on analyzing Smooth-Greedy assuming it queries the ideal submodular function\(^2\).

**Analysis of the smooth greedy algorithm.** If we were “lucky” and at some point $f_S(H) = 0$ we are done, since from this point onwards we are essentially computing the “correct” marginal contribution of every element to $S$, and running the standard greedy algorithm. In the beginning, this is clearly not the case. However, letting $S$ denote the set of elements selected at the beginning of some iteration, we show that in every iteration we add an element $a$ whose marginal contribution to $S$ is arbitrarily close to $\max_b f_{S \cup H}(b)$. We then show that the resulting set $S$ together with the smoothing set $H$ give a $1 - 1/e$ approximation against the optimal solution that aims to optimize $f_H$ with $k - \ell$ elements\(^3\). This is an important property that is then used by Slick-Greedy. This also gives a simple algorithm whose approximation ratio turns out to be arbitrarily close to $((e - 1)/(2e - 1))$ against the true optimal solution.

**Slick Greedy: boosting the smooth greedy algorithm.** In principle, Smooth-Greedy does not provide the $1 - 1/e$ guarantee since it may use a smoothing set which encompasses a large-valued fraction of the optimal solution. In this case, the optimal solution evaluated on $f_H$ may be insignificant to the optimal solution, and the constant factor approximation guarantee crucially depends on including $H$ in the solution. The main idea behind Slick-Greedy is to select a large yet constant number of smoothing sets $H_1, \ldots, H_c$, run multiple iterations of a variant of Smooth-Greedy and choose the best solution. In this variant, in each iteration we set one of the smoothing sets $H_j$ aside and run Smooth-Greedy that is initialized with $S = \cup_{i \neq j} H_i$ and $H_j$ as its smoothing set. Intuitively, the idea is that one of the smoothing sets has relatively small value to the rest, and Smooth-Greedy obtains a $1 - 1/e - \epsilon$ approximation to the restricted optimal solution that is forced to take $\cup_{i \neq j} H_i$, which in itself is close to the (unrestricted) optimal solution.

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\(^1\)There is an oversimplification here. In general, we want the smoothing to have two properties. First, we need to have an accuracy which is at least proportional to $1/k$. This means that when $k = O(\log n)$, but when $k$ is much larger than $\log n$ we already use $\ell = O(\log n)$ elements. In addition, we need to have a union bound over a polynomial number of evaluations, and hence always need at least $\ell = O(\log \log n)$ and cannot hope to scale this argument down.

\(^2\)There is a subtle issue here if most of the contribution of $\sum_{H' \subseteq H} \tilde{f}(S \cup H' \cup a)$ comes from a small number of elements. We gloss over this here, but in the paper we show a weaker (and correct) concentration bound which lower bounds the maximal set, and upper bounds the bad ones.

\(^3\)Note that $f_H$ is a very different function from $f$. In particular, it could be that $f(H)$ is huge, and $f_H(S) = 0$ for every $S$. 

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1.4.2 Algorithm for small $k$

When $k \approx \log \log n$, we cannot afford to choose arbitrary elements and include them in the solution. Instead, we design the sampled mean method (SM-GREEDY) as follows. Initialize with $S = \emptyset$. Given that SM-GREEDY already chose some $S$, it goes over all sets $C$ of size $|C| = c$ for some $c = O(1/\epsilon)$, computes the score of $C$, defined as $\phi(C) = \frac{1}{|C||n-|C|-|S||} \sum_{i \in C} \sum_{j \notin S \cup C} \tilde{f}(S \cup C \setminus \{i\} \cup \{j\})$, and adds to $S$ a random term in the set with the highest score (this term is called a representative of the set).

**Analysis of SM-Greedy.** As we did with Smooth Greedy, the first step in the analysis is to show that we can ignore the noise, as we are averaging over $O(n)$ samples every time. The argument here is a little more subtle than for SLICK-GREEDY. Letting $A^*$ denote the set with the maximal marginal contribution at some iteration, we show that the score of $A^*$ is at least $1 - 1/c$ of the marginal contribution of $A^*$. Since we choose a set $A$ with the highest score, it must be that the score of $A$ is at least $1 - 1/c$ of the marginal contribution of $A^*$. Since we are choosing a random element from $A$, the expected contribution we get is $1 - 1/c$ of the marginal contribution of $A^*$.

**From expectation to high probability.** Improving the success probability to $1 - o(1)$ requires us to first find a set $A$ with maximal score as before, but then to choose a different representative for $A$ which we add to $S$: instead of choosing a random term we use the term that the oracle assigned the highest value to. Formally, we add to $S$ the set $A \setminus \{i^*\} \cup \{j^*\}$, where $i^*, j^*$ are defined as $\arg \max_{i \in A, j \notin S \cup A} \tilde{f}(S \cup A \setminus \{i\} \cup \{j\})$. When making this choice we are no longer averaging over the effect of the noise, so we need a new type of analysis. Let $\text{good}$ be the set of pairs $i \in A, j \notin S \cup A$, where a pair $i, j$ is in $\text{good}$ if $f(S \cup A \setminus \{i\} \cup \{j\})$ is very close to $\phi(A)$. Since $\phi(A)$ is very close to $f(S \cup A^*)$, no single term in the score can contribute too much, and we have that $|\text{good}| = O(n)$. Let $\text{bad}$ be the set of pairs $i \in A, j \notin S \cup A$, where a pair $i, j$ is in $\text{bad}$ if $f(S \cup A \setminus \{i\} \cup \{j\})$ is very close to $\phi(A)$. Clearly $|\text{bad}| < n|C|$. We utilize the generalized exponential tail of the noise distribution. The difficulty is that the noise distribution may not be monotone. Imagine for example that with probability to show that with probability $1 - 1/\sqrt{\log n}$, we have:

$$\max_{i,j \in \text{good}} \tilde{f}(S \cup A \setminus \{i\} \cup \{j\}) > \max_{i,j \in \text{bad}} \tilde{f}(S \cup A \setminus \{i\} \cup \{j\})$$

This shows that a bad element did not win\(^4\), and hence the contribution of the chosen term $(i^*, j^*)$ is not too far from $\phi(A)$. Note that it is not necessarily true that they are in $\text{good}$\(^5\).

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\(^4\)The challenge in proving the equation is that we need use the effect of the noise on the pairs in $\text{good}$. Imagine for example that $|\text{good}| = n/10$, $|\text{bad}| = n$, and the noise distribution would give us $1$ with probability $1 - 1/2n$ and $2$ otherwise. This would make the claim false, and a bad element is likely to win. The “problematic” distribution we just described depends on $n$, and hence is ruled out by the condition that for every distribution there exist $n_0$ such that the theorem holds when $n > n_0$. Showing that this bad case can not happen just because the tail is unbounded is difficult.

\(^5\)Another direction that we tried is a black box reduction, saying that if a greedy algorithm for submodular maximization succeeds in expectation, it will also succeed w.h.p., since it has several “independent” trials to succeed in. We did not manage to pull the proof, but conjecture that the original SM-GREEDY succeeds w.h.p. and we are just not smart enough.
1.4.3 Algorithms for very small $k$

When $k < 1/e$ we cannot add $1/e$ elements at a time as required by SM-GREEDY. In this case, we can iterate over all sets of size $k$, and compute their score, adding a representative of the set with the highest score. Exact analysis of this technique gives an approximation ratio of $1 - 1/k - \epsilon$. Since we show that no algorithm (regardless of runtime considerations) can obtain an expected approximation ratio of $k/k + 1 + \epsilon$ this is almost tight for large (albeit constant) $k$. For $k = 1$, we get that the best we can hope for is an approximation ratio of 2, which is worse than what we get for large $k$. We also have an approximation algorithm which gives an expected value of $1/4$, by finding the set of size 2 with maximal score, choosing a random term in the score and choosing a random item in the pair. This essentially equivalent to flipping a coin and choosing either a random element or the element with the largest average value (taken together with every other element).

1.5 Paper organization

Following the discussion on more related work, we begin by laying out definitions and proving concentration bounds of smoothing in Section 2. These bounds are later used in the analysis of the algorithms in subsequent sections. In Section 3 we describe and analyze the SLICK-GREEDY algorithm. In Section 4 we describe and analyze the SM-GREEDY algorithm. We describe the variants for very small $k$ and give information-theoretic lower bounds on constant $k$ in Section 5. In Section 6 we give a lower bound on adversarial noise. We conclude with discussion of the model in Section 7. To optimize for readability we defer many proofs to the appendix. In general, we defer many of the smoothing arguments to the appendix, to help focus on the underlying algorithmic ideas and mitigate the noise around them.

1.6 More related work

**Submodular optimization.** Maximizing monotone submodular functions under cardinality and matroid constraints is heavily studied. The seminal works of [64, 38] show that the greedy algorithm gives a factor of $1 - 1/e$ for maximizing a submodular function under a cardinality constraint and a factor $1/2$ approximation for matroid constraints. For max-cover which is a special case of maximizing a submodular function under a cardinality constraint, Feige shows that no poly-time algorithm can obtain an approximation better than $1-1/e$ unless P=NP [29]. Vondrak presented the continuous greedy algorithm which gives a $1 - 1/e$ ratio for maximizing a monotone submodular function under matroid constraints [74]. This is optimal, also in the value oracle model [63, 48, 65]. It is interesting to note that with a demand oracle the approximation ratio is strictly better than $1 - 1/e$ [33]. When the function is not monotone, constant factor approximation algorithms are known to be obtainable as well [30, 57, 12, 13]. In general, in the past decade there has been a development in the theory of submodular optimization, through concave relaxations [1, 17], the multilinear relaxation [16, 74, 18], and general rounding technique frameworks [76]. In this paper, the techniques we develop arise from first principles: we only rely on basic properties of submodular functions, concentration bounds, and the algorithms are variants of the standard greedy algorithm.

**Submodular optimization in game theory.** Submodular functions have been studied in game theory almost fifty years ago [72]. In mechanism design submodular functions are used to model
agents’ valuations [58] and have been extensively studied in the context of combinatorial auctions (e.g. [22, 23, 21, 63, 14, 20, 67, 27, 24]). Maximizing submodular functions under cardinality constraints have been studied in the context of combinatorial public projects [68, 71, 15, 62] where the focus is on showing the computational hardness associated with not knowing agents valuations and having to resort to incentive compatible algorithms. Our adversarial lower bound implies that if agents err in their valuations, optimization may be hard, regardless of incentive constraints.

**Submodular optimization in machine learning.** In the past decade submodular optimization has become a central tool in machine learning and data mining (see surveys [52, 53, 9]). Problems include identifying influencers in social networks [46, 70] sensor placement [59, 41], learning in data streams [73, 43, 55, 4], information summarization [60, 61], adaptive learning [42], vision [45, 44, 50], and general inference methods [51, 44, 19]. In many cases the submodular function is learned from data, and our work aims to address the case in which there is potential for noise in the model.

**Combinatorial optimization under noise.** Combinatorial optimization with noisy inputs can be largely studied through consistent (independent noisy answers when querying the oracle twice) and inconsistent oracles. For inconsistent oracles, it usually suffices to repeat every query $O(\log n)$ times, and eliminate the noise. To the best of our knowledge, submodular optimization has been studied under noise only in instances where the oracle is inconsistent or equivalently small enough so that it does not affect the optimization [46, 54]. One line of work is studies methods for reducing the number of samples required for optimization (see e.g. [32, 8]), primarily for sorting and finding elements. On the other hand, if two identical queries to the oracle always yield the same result, the noise can not be averaged out so easily, and one needs to settle for approximate solutions, which has been studied in the context of tournaments and rankings [47, 11, 2].

**Convex optimization under noise.** Maximizing functions under noise is also an important topic in convex optimization. The analogue of our model here is one where there is a zeroth-order noisy oracle to a convex function. As discussed in the paper, the question of polynomial-time algorithms for noisy convex optimization is straightforward and the work in this area largely aims at improving the convergence rate [28, 39, 49, 56, 69].
2 Combinatorial Smoothing

In this section we illustrate a general framework we call *combinatorial smoothing* that we will use in the subsequent sections. Intuitively, combinatorial smoothing mitigates the effects of noise and enables finding elements whose marginal contribution is large.

**Some intuition.** Recall from our discussion in Section 1.3 that implementing the greedy algorithm requires identifying arg max \( f(S \cup a) \) for a given set \( S \) of elements selected by the algorithm in previous iterations. Thus, if for some \( a, b \in N \) we can compare \( S \cup a \) and \( S \cup b \) and decide whether \( f(S \cup a) > f(S \cup b) \) or vice versa, we can implement the greedy algorithm. Put differently, viewing a set as a point on the hypercube, given two points in \( \{0, 1\}^n \) we need to be able to tell which one has the larger true value, using a noisy oracle. In a world of continuous optimization, a reasonable approach to estimate the true value of a point in \( [0, 1]^n \) with access to a noisy oracle is to take a small neighborhood around the point, sample values of points in its neighborhood, and average their values. Taking polynomially-many samples allows concentration bounds to kick in, and using a small enough diameter can often guarantee that the averaged value is a reasonable estimate of the point’s true value. Surprisingly, the spirit of this simple idea can used in submodular optimization.

**Smoothing neighborhood.** For a given subset \( A \subseteq N \) a smoothing function is a method which assigns a family of sets \( \mathcal{H}(A) \) called the smoothing neighborhood. The smoothing function will be used to create a smoothing neighborhood for a small set \( A \). This set \( A \) whose marginal contribution we aim to evaluate, is essentially a candidate for a greedy algorithm. In the application in Section 3 the set \( A \) will simply be a single element, whereas in Section 4 the set \( A \) will be of size \( O(1/\epsilon) \).

**Definition 2.1.** For a given function \( f : 2^N \rightarrow \mathbb{R}, A, S \subseteq N, \) and smoothing neighborhood \( \mathcal{H}(A) \):

- \( F_S(A) := E_{X \in \mathcal{H}(A)}[f_S(X)] \) (called the smooth marginal contribution of \( A \)),
- \( F(S \cup A) := E_{X \in \mathcal{H}(A)}[f(S \cup X)] \) (called the smooth value of \( S \cup A \))
- \( \tilde{F}(S \cup A) := E_{X \in \mathcal{H}(A)}[\tilde{f}(S \cup X)] \) (called the noisy smooth value of \( S \cup A \)).

The idea behind combinatorial smoothing is to select a smoothing neighborhood which includes sets whose value is in some sense close to the value of the set \( A \) whose marginal contribution we wish to evaluate. Intuitively, when the sets are indeed close, by averaging the values of the sets in \( \mathcal{H}(A) \) we can mitigate the effects of noise and produce meaningful statistics (see Figure 2).

2.1 Smoothing arguments

In our model, the algorithm may only have access \( \tilde{F}(S \cup A) \). Ideally, given a set \( S \) and a smoothing neighborhood \( \mathcal{H}(A) \) we would have liked to apply concentration bounds and show that the noisy smooth value is arbitrarily close to the non-noisy smooth value, i.e. \( F(S \cup A) \approx \tilde{F}(S \cup A) \) or:

\[
\sum_{i \in \mathcal{H}(A)} f(S \cup X_i) \approx \sum_{i \in \mathcal{H}(A)} \xi_i f(S \cup X_i)
\]
Figure 2: An illustration of smoothing. For every element in the ground set we associate an index \( i \in [n] \) and define the submodular function as \( f(S) = \sqrt{\sum_{i \in S} i/2} - c \) for a constant \( c > 0 \). The blue dot depicts the true value of the element \( a \) associated with the index \( i = 400 \) and the red dot depicts the true value of the element \( b \) associated with the index \( j = 900 \). The light blue and light red dots depict the noisy function values of elements associated with indices \( i \) in the range \( |i - 400| \leq 100 \) and \( |i - 900| \leq 100 \). For \( S = \emptyset \), and smoothing neighborhoods \( \mathcal{H}(a) = \{i : |i - a| \leq 100\} \) and \( \mathcal{H}(b) = \{i : |i - b| \leq 100\} \) we depict \( \tilde{F}(S \cup a) \) and \( \tilde{F}(S \cup b) \) as the blue and red triangles, respectively. Intuitively, an algorithm which needs to decide whether \( a \) (blue point) is larger than \( b \) (red point) will decide by comparing \( \tilde{F}(S \cup a) \) (blue triangle) and \( \tilde{F}(S \cup b) \) (red triangle).

If the values \( \{f(S \cup X_i)\}_{i=1}^{\mathcal{H}(A)} \) were arbitrarily close, we could simply apply a concentration bound by taking the value of any one of the sets, say \( S \cup X_j \), and for \( v_j = f(S \cup X_j) \), since all the values are close, we would be guaranteed that:

\[
\sum_{i \in \mathcal{H}(A)} \xi_i f(S \cup X_i) \approx \sum_{i \in \mathcal{H}(A)} \xi_i f(S \cup X_j) = v_j \cdot \sum_{i \in \mathcal{H}(A)} \xi_i
\]

In continuous optimization this is usually the case when averaging over an arbitrarily small ball around the point of interest, and concentration bounds apply. The problem in our case is that due to the combinatorial nature of the problem, the values of the sets in the smoothing neighborhood may take on very different values. For this reason we cannot simply apply concentration bounds. The purpose of this section is to provide machinery that overcomes this difficulty. The main ideas can be summarized as follows:
1. In general, there may be cases in which we cannot perform smoothing well and cannot get the noisy smooth values to be similar to the true smooth values. We therefore define a more modest, yet sufficient goal. Since our algorithms essentially try to replace the step of adding the element $a \in \arg\max_b f(S \cup b)$ in the greedy algorithm with $a' \in \arg\max_b F(S \cup b)$, it suffices to guarantee that for the set $A$ which maximizes the noisy smooth values, that set also well approximates the (non-noisy) smooth values. More precisely our goal is to show that if for an arbitrarily small $\delta > 0$ we have that $A \in \arg\max_B \tilde{F}(S \cup B)$ then $F(S \cup A) \geq (1 - \delta) \max_B F(S \cup B)$;

2. To show that $A \in \arg\max \tilde{F}(S \cup A)$ implies $F(S \cup A) \geq (1 - \delta) \max_B F(S \cup B)$ for an arbitrarily small $\delta > 0$, we prove two bounds. Lemma 2.4 lower bounds the noisy smooth contribution of a set in terms of its (true) smooth contribution. Lemma 2.5 upper bounds the smooth noisy contribution of any element against its smooth contribution. The key difference between these lemmas is that Lemma 2.4 lower bounds the value in terms the variation of the smoothing neighborhood. The variation of the neighborhood is the ratio between the set with largest value and that with lowest value in the neighborhood. Intuitively, for elements with large values the variation of the neighborhood is bounded, and thus we can show that the noisy smooth value of these elements is nearly as high as their true smooth values.

3. Together, these lemmas are used in subsequent sections to show that an element with the largest noisy smooth marginal contribution is an arbitrarily good approximation to the element with the largest (non-noisy) smooth marginal contribution. This is achieved by showing that the lower bound on the smooth value of an element with large (non-noisy) smooth marginal contribution beats the upper bound on the smooth (non-noisy) value of an element with slightly smaller smooth contribution.

As discussed in the introduction, in the interest of readability we defer the proofs related to smoothing (and hence the proofs from this section) to the appendix.

The first lemma gives us tail bounds for exponential tail distributions. We later use these tail bounds in concentration bounds we use in the smoothing procedures.

**Lemma 2.2.** Let $\omega_{\text{max}}$ and $\omega_{\text{min}}$ be the upper and lower bounds on the value of the noise multiplier in any of the calls made by a polynomial-time algorithm. For any $\delta > 0$, we have that:

- $\Pr[\omega_{\text{max}} < t^\delta] > 1 - e^{-\Omega(t^\delta / \ln t)}$
- $\Pr[\omega_{\text{min}} > t^{-\delta}] > 1 - e^{-\Omega(t^\delta / \ln t)}$

The definition below of the variation of the neighborhood quantifies the ratio between the largest possible value and the smallest possible value achieved by a set in the neighborhood.

**Definition 2.3.** For given sets $A, S \subseteq N$, the variation of the neighborhood denoted $v_S(H(A))$ is:

$$v_S(H(A)) = \frac{\max_{T \in H(A)} f_S(T)}{\min_{T \in H(A)} f_S(T)}.$$

The following lemma gives a lower bound on the noisy smooth value in terms of the (non-noisy) smooth value and the variation. Intuitively, when an element has large value its variation is bounded, and the lemma implies that its noisy smooth value is close to its smooth value. Essentially, when the variation is bounded $\tilde{F}(S) \approx (1 - \lambda)(1 - \epsilon)F(S)$ for $\lambda$ and $\epsilon$ that vanish as $n$ grows large.
Lemma 2.4. Let \( f : 2^N \to \mathbb{R}, A, S \subseteq N, \omega = \max_{A_i \in \mathcal{H}(A)} \xi_{A_i}, \) and \( \mu \) be the mean of the noise distribution. For \( \epsilon = \min \{1, 2v_S(\mathcal{H}) \cdot |\mathcal{H}(A)|^{-1/4}\} \) for any \( \lambda < 1 \) w.p \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \) we have:

\[
\tilde{F}(S \cup A) > (1 - \lambda)\mu \cdot (f(S) + (1 - \epsilon) \cdot F_S(A)).
\]

The next lemma gives us an upper bound on the noisy smooth value. The bound shows that for sufficiently large \( t \) (the size of the smoothing neighborhood, which always depends on \( n \)), for small \( \lambda > 0 \) we have that \( \tilde{F}(S) \approx (1 + \lambda)F(S) + 3t^{-1/4} \alpha_{\text{max}}. \) In our applications of smoothing \( \alpha_{\text{max}} \leq \text{OPT}, \) and \( t \) is large. Since we use this upper bound to compare against elements whose value is at least some bounded factor of \( \text{OPT}, \) the dependency of the additive term on \( \alpha_{\text{max}} \) will be insignificant.

Lemma 2.5. Let \( f : 2^N \to \mathbb{R}, A, S \subseteq N, \omega = \max_{A_i \in \mathcal{H}(A)} \xi_{A_i}, \alpha_{\text{max}} = \max_{A_i \in \mathcal{H}(A)} f_S(A_i) \) and \( \mu \) be the mean of the noise distribution. For \( \epsilon = 3t^{-1/4} \alpha_{\text{max}} \) we have that for any \( \lambda < 1 \) with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):

\[
\tilde{F}(S \cup A) < (1 + \lambda)\mu \cdot (f(S) + F_S(A) + \epsilon).
\]
3 Optimization for Large \( k \)

In this section we describe the Slick-Greedy algorithm whose approximation is arbitrarily close to 1 − 1/e for sufficiently large \( k \). The algorithm is deterministic and can be applied when the cardinality constraint \( k \) is in \( \Omega(\log \log n) \). In particular, given a desired degree of accuracy \( \epsilon > 0 \), our guarantees apply when \( k \geq 3168 \log \log n / \epsilon^2 \). We first describe and analyze the performance of the Smooth-Greedy algorithm which is used as a subroutine by the Slick-Greedy algorithm. We then describe and analyze the Slick-Greedy algorithm which is the main result of this section.

3.1 The smooth greedy algorithm

We begin by describing the smoothing neighborhood used by the algorithm. We select an arbitrary set \( H \) and for a given element \( a \), the smoothing neighborhood is simply \( H = \{ H' \subseteq H : H' \cup a \} \). Throughout the rest of this section we assume that \( H \) is an arbitrary set of size \( \ell \), where \( \ell \) depends on \( k \). In the case where \( k \geq 2400 \log n \) we will assign \( \ell = 25 \log n \), and when \( k < 2400 \log n \) we will have \( \ell = 33 \log \log n \). We will use \( t = 2^\ell \) to denote the number of subsets of \( H \) and \( k' = k - \ell \). The precise choice for \( \ell \) will become clear later in this section. Intuitively, \( \ell \) is on the one hand small enough so that if \( k \) is sufficiently large we can afford to sacrifice \( \ell \) elements for smoothing the noise, and on the other hand \( \ell \) is large enough so that taking all its subsets gives us a large smoothing neighborhood which enables applying concentration bounds. Denoting \( H^{(1)}, \ldots, H^{(t)} \) as all the subsets of \( H \), the smooth contribution and smooth marginal contribution are:

\[
F(S \cup a) = \frac{1}{t} \sum_{i=1}^{t} f \left( H^{(i)} \cup (S \cup a) \right)
\]

\[
F_S(a) = \frac{1}{t} \sum_{i=1}^{t} f_S(H^{(i)} \cup a)
\]

3.1.1 The algorithm

The smooth greedy algorithm is a variant of the standard greedy algorithm which replaces the procedure of finding argmax_{a \in N} f(S \cup a) with its smooth analogue. The algorithm receives an arbitrary set of elements \( H \) of size \( \ell \) and at every stage adds the element \( a \notin S \cup H \) to the solution \( S \) for which the smooth noisy contribution is largest.

Algorithm 1 Smooth-Greedy

**Input:** budget \( k \), set \( H \)

1. \( S \leftarrow \emptyset \)
2. **while** \( |S| < k - |H| \) **do**
3. \( S \leftarrow S \cup \arg \max_{a \notin S \cup H} \tilde{F}(S \cup a) \)
4. **end while**
5. **return** \( S \)

\(^6\)W.l.o.g. we assume that \( k < n - 25 \log n \) as for sufficiently large \( n \) this then implies that \( k \geq (1 - \epsilon)n \) and by submodularity optimizing with \( k' = n - 25 \log n \) suffices to get the \( 1 - 1/e - \epsilon \) guarantee for any fixed \( \epsilon > 0 \).
At a high level, the idea behind the analysis is to compare the performance of the solution returned by the algorithm against an optimal solution which ignores the value of $H$ and any of its partial substitutes. That is, we compare the true value of the solution against the optimal solution evaluated on $f_H$ where $f_H(S) = f(H \cup S) - f(H)$. To gain some intuition behind this idea in the analysis we encourage the reader to consult Figure 3. Essentially, we will show that at every step Smooth-Greedy selects an element whose marginal contribution is larger than that of the optimal solution evaluated on $f_H$. We will use $\text{OPT}$ to denote the value of the optimal solution with $k$ elements evaluated on $f$ and $\text{OPT}_H$ to denote the value of the optimal solution with $k'$ elements evaluated on $f_H$.

One of the artifacts of smoothing is that our comparisons are not precise. That is, when we select an element that maximizes $\tilde{F}(S \cup a)$, our smoothing guarantee will be that this element respects $F_S(a) \geq (1 - \delta) \max_{b \notin S \cup H} F_S(b)$ for $\delta > 0$ that depends on $\epsilon$ and $k$. This can be guaranteed only for an iteration where there is at least a single element not yet selected (and not in $H$) whose marginal contribution is at least $\epsilon/k$ fraction of $\text{OPT}_H$. We call such iterations $\epsilon$-relevant.

**Definition 3.1.** Let $O \in \arg \max_{T:|T| \leq k} f_H(T)$ and $S$ be the current set of elements in some iteration of Smooth-Greedy. The iteration is $\epsilon$-relevant if $f_{H \cup S}(O) \geq \epsilon \cdot \text{OPT}_H$.

We will analyze Smooth-Greedy in the case where the iterations are $\epsilon$-relevant as it allows applying the smoothing arguments. In the analysis we will then ignore iterations that are not $\epsilon$-relevant at the expense of a negligible loss in the approximation guarantee. The main steps in the analysis of Smooth-Greedy are:

1. In Lemma 3.2 we show that when the iterations are $\epsilon$-relevant, the element selected in each iteration by the algorithm is with high probability an arbitrarily good approximation to the (non-noisy) smooth marginal contribution $F_S(a)$. To do so we need claims 3.3, 3.4 and 3.5;

2. Next, in Claim 3.7 we show that the element $a$ whose smooth marginal contribution $F_S(a)$ is maximal has true marginal contribution $f_S(a)$ that is roughly a $k'$th fraction of the marginal contribution of the optimal solution over $f_H$;

3. Finally, in Lemma 3.8 we apply a standard inductive argument to show that the fact that the algorithm selects an element with large marginal contribution in each step results in an approximation arbitrarily close to $1 - 1/e$ to $\text{OPT}_H$ (not $\text{OPT}$). In Corollary 3.9 we show that the bound against $\text{OPT}_H$ can already be used to give a constant factor approximation to $\text{OPT}$. To get arbitrarily close to $1 - 1/e$ we boost Smooth-Greedy as described in Section 3.2.

### 3.1.2 Smoothing guarantees

As this stage our goal is to prove Lemma 3.2. This lemma shows that at every step as Smooth-Greedy adds the element that maximizes the noisy contribution $\arg \max_{a \notin S \cup H} \tilde{F}(S \cup a)$, that element nearly maximizes the (non-noisy) smooth marginal contribution $F_S$, with high probability.

**Lemma 3.2.** For any fixed $\epsilon > 0$, assume $\text{OPT}_H \geq \text{OPT}/3$ and consider an $\epsilon$-relevant iteration of Smooth-Greedy where $S$ is the set of elements selected in previous iterations and
Figure 3: An illustration of Claim 3.3 applied on a coverage function. The set of all elements $N$ and $A, B, H \subset N$ are depicted as circles that illustrate the area of the universe they cover. Claim 3.3 essentially says that if we select $A$ rather than $B$ this means that the total area $A$ covers (white and grey) must be larger than the white-only (i.e. universe not covered by $H$) of $B$. Stated in these terms, we use this claim to analyze the performance of Smooth-Greedy evaluated on the white and grey area against the optimal solution evaluated on the white-only area.

$$a \in \arg\max_{b \in S \cup H} F(S \cup b)$$ is the element selected in that iteration. Then, for sufficiently large $n$, for $\delta = \epsilon^2/4k$ we have that with probability at least $1 - 1/n^4$:

$$F_S(a) \geq (1 - \delta) \max_{b \notin S \cup H} F_S(b).$$

To prove the above lemma we will need claims 3.3, 3.4, and 3.5.

**Claim 3.3.** If $F_S(a) \geq F_S(b)$ then $f_S(a) \geq f_{S \cup H}(b)$.

**Proof.** Assume for purpose of contradiction that $f_S(a) < f_{S \cup H}(b)$. Since $f$ is a submodular function, the function $f_S(T) = f(S \cup T) - f(S)$ is also submodular (and hence also subadditive). Therefore for every $H' \subseteq H$:

$$f_S(H' \cup a) \leq f_S(H') + f_S(a) \quad \text{(subadditivity of } f_S)$$

$$< f_S(H') + f_{S \cup H}(b) \quad \text{(by assumption)}$$

$$\leq f_S(H' \cup b) \quad \text{(submodularity of } f_S)$$

Notice however, that this contradicts our assumption:

$$F_S(a) = \frac{1}{t} \sum_{H' \subset H} f_S(H' \cup a) < \frac{1}{t} \sum_{H' \subset H} f_S(H' \cup b) = F_S(b). \qed$$

The following claim bounds the variation of the smoothing neighborhood of the element we selected. This is a necessary property for applying the smoothing arguments in a meaningful way.

**Claim 3.4.** Let $\epsilon > 0$ and assume $\OPT_H \geq \OPT/3$. For an $\epsilon$-relevant iteration of Smooth-Greedy, let $S$ be the set of elements selected in previous iterations. If $a^* \in \arg\max_{a \notin S \cup H} F_S(a)$ then $v_S(H(a^*)) \leq 3k/\epsilon$.

**Claim 3.5.** For a fixed $\epsilon > 0$ assume $t \geq \left(\frac{110k \cdot \log n}{\epsilon \delta}\right)^8$, $\OPT_H \geq \OPT/3$. For an $\epsilon$-relevant iteration of Smooth-Greedy, let $S$ be the set of elements selected in previous iterations and $a \in \arg\max_{b \notin S \cup H} F(S \cup b)$ be the element selected at that stage. Then, w.p. at least $1 - 1/n^4$:

$$F_S(a) \geq (1 - \delta) \max_{b \notin S \cup H} F_S(b).$$
The proof of Claim 3.5 uses the smoothing arguments from the previous section together with claims 3.3 and 3.4. The proofs of 3.4 and 3.5 as well as proof of Lemma 3.2 can be found in Appendix B. The proof of Lemma 3.2 simply verifies that the number of elements in the smoothing set respects the condition in the statement of Claim 3.5.

### 3.1.3 Approximation guarantee

Lemma 3.2 lets us forget about noise, at least for the remainder of the analysis of Smooth-Greedy. We can now focus on the consequences of selecting an element \( a \) which (up to factor \( 1 - \delta \)) maximizes the smooth contribution \( F_S \) at every stage of the algorithm, rather than the true marginal contribution \( f_S(a) \).

**Claim 3.6.** For any \( \epsilon > 0 \), let \( \delta \leq \epsilon^2/4k \). Suppose that the iteration is \( \epsilon \)-relevant, that \( OPT_H \geq OPT/3 \), and let \( o^* \in \text{argmax}_O f_{H \cup S}(O) \) where \( O \in \text{argmax}_{T:|T| \leq k'} f_H(T) \). If \( F_S(a) \geq (1 - \delta) F_S(o^*) \), then:

\[
F_S(a) \geq (1 - \epsilon) F_{H \cup S}(a).
\]

The main principle is similar to the proof of Claim 3.3. In this version we have a slightly weaker condition since \( F_S(a) \) is not greater than \( F_H(o^*) \) but rather \( (1 - \delta) F(o^*) \). We use a different approach to show that under the stated conditions of this claim, \( \delta \) is sufficiently small for the statement to hold. The proof can be found in Appendix B.

**Claim 3.7.** For any fixed \( \epsilon > 0 \), consider an \( \epsilon \)-relevant iteration of Smooth-Greedy, let \( S \) be the set of elements selected in previous iterations, and assume \( OPT_H \geq OPT/3 \). Let \( a \in \text{argmax}_{b \notin S \cup H} \tilde{F}(S \cup b) \) be the element selected at that iteration. Then, with prob. \( > 1 - 1/n^4 \):

\[
F_S(a) \geq (1 - \epsilon) \left[ \frac{1}{k'} (OPT_H - f(S)) \right].
\]

**Proof.** Let \( O \in \text{argmax}_{T:|T| \leq k'} f_H(T) \), and \( o^* \in \text{argmax}_{O \in O} f_{H \cup S}(o) \). From Lemma 3.2 we know that with probability \( 1 - 1/n^4 \) we have \( F_S(a) \geq (1 - \delta) F_S(o^*) \) for \( \delta = \epsilon^2/4k \). Claim 3.6 implies that:

\[
F_S(a) \geq (1 - \epsilon) F_{H \cup S}(o^*)
\]

From subadditivity \( F_{H \cup S}(o^*) \geq F_{H \cup S}(O)/k' \) and thus:

\[
F_S(a) \geq (1 - \epsilon) F_{H \cup S}(o^*) \geq \left( \frac{1 - \epsilon}{k'} \right) F_{H \cup S}(O) \geq \left( \frac{1 - \epsilon}{k'} \right) \left( f_H(O) - f(S) \right).
\]

Having proved Claim 3.7 we can now state the main lemma of this subsection.

**Lemma 3.8.** Let \( S \) be the set returned by Smooth-Greedy and \( H \) its smoothing set. Then, for any fixed \( \epsilon > 0 \) when \( k \geq 3\ell/\epsilon \) with probability of at least \( 1 - 1/n^3 \) we have that:

\[
f(S \cup H) \geq (1 - 1/\epsilon - \epsilon/3) OPT_H.
\]

A simple generalization of Lemma 3.8 provides us the guarantee we need to analyze the Slick-Greedy algorithm in Section 3.2.
provides the approximation guarantee. Otherwise, \( \OPT_H \geq \OPT / 3 \) and we can apply Claim 3.7 using a standard inductive argument to show that \( S \) alone provides the approximation. The subtle yet crucial aspect of the proof is that the inductive argument is applied to analyze the quality of the solution against the optimal solution for \( f_H \) and not against the optimal solution on \( f \). The proof is in Appendix B.

Before continuing to the next subsection, it is worth noting that by applying Lemma 3.8 one can show that Smooth-Greedy provides a constant (but not optimal) approximation guarantee.

**Corollary 3.9.** Let \( S \) be the set returned by Smooth-Greedy and \( H \) be its smoothing set. For any fixed \( \epsilon > 0 \) and \( k > 3\ell/\epsilon \), we have that with probability at least \( 1 - 1/n^3 \):

\[
f(S \cup H) > \left( \frac{e - 1}{2e - 1 - \epsilon} - 2\epsilon \right) \OPT
\]

**Proof.** Let \( O_H \in \arg\max_{|T| \leq k} f_H(T) \). From Lemma 3.8, with probability at least \( 1 - 1/n^3 \):

\[
f(S \cup H) > \left( 1 - \frac{1}{e} - \frac{\epsilon}{3} \right) f(O_H)
\]  

Let \( O' \in \arg\max_{|T| \leq k - |H|} f(T) \). From submodularity and the fact that \( k \geq 3\ell/\epsilon > |H|/\epsilon \) we get that \( (1 - \epsilon)\OPT \leq f(O') \). Putting everything together:

\[
(1 - \epsilon)\OPT \leq f(O') \quad \text{submodularity of } f
\]

\[
\leq f(O_H \cup H) \quad \text{monotonicity of } f
\]

\[
\leq f(O_H) + f(H) \quad \text{subadditivity of } f
\]

\[
\leq \left( \frac{e}{e - 1 - \epsilon} \right) f(S \cup H) + f(H) \quad \text{by (1)}
\]

\[
\leq \left( \frac{2e - 1 - \epsilon}{e - 1 - \epsilon} \right) f(S \cup H). \quad \text{monotonicity of } f
\]

Therefore \( f(S \cup H) > \left( \frac{e - 1}{2e - 1 - \epsilon} - 2\epsilon \right) \OPT \) as required.

### 3.2 The slick greedy algorithm: optimal approximation for sufficiently large \( k \)

The reason Smooth-Greedy alone cannot obtain an arbitrarily close approximation to \( 1 - 1/e \) is due to the fact that a substantial portion of the optimal solution’s value may be attributed to \( H \). This is because we essentially bound a solution \( S \cup H \) against \( f(O_H) + f(H) \). This would be resolved if we had a way to guarantee that the contribution of the smoothing set we use is small. The idea behind Slick-Greedy is to obtain this type of guarantee. Intuitively, by running a large albeit constant number of instances of Smooth-Greedy with different smoothing sets, and selecting the “best” solution we can ensure the contribution of the smoothing set is relatively minor.

#### 3.2.1 The algorithm

We can now describe the Slick-Greedy algorithm which is the main result of this section. The idea is to boost the smooth greedy algorithm. To do so, given a constant \( \epsilon > 0 \) we set \( \delta = \epsilon/6 \). We
generate sets $H_1, \ldots, H_{1/\delta}$, each of size $\ell$ s.t. $H_i \cap H_j = \emptyset$ for every $i, j \in [1/\delta]$. We then run the Smooth-Greedy algorithm $1/\delta$ times, where in each iteration $j$ we initialize Smooth-Greedy with $R_j = \cup_{i \neq j} H_i$ and use $H_j$ to generate the smoothing neighborhood. By initializing the Smooth-Greedy with $R_j$ we mean that the first iteration starts with $S = R_j$ rather than $S = \emptyset$ and the algorithm terminates after $k - |R_j| - |H_j|$ steps. We denote this as Smooth-Greedy$(k, R_j, H_j)$. For each such iteration $j \in [1/\delta]$ we keep the solution returned by the algorithm $S_j$ and the set $H_j$. We then compare the solutions $T_j = S_j \cup H_j$ to the best set $T_i$ we’ve seen so far through a procedure we call Smooth-Cmpare described below. The Smooth-Cmpare procedure begins with a solution $T_i$ and compares it to another solution $T_j$ by generating a set $H_{ij}$ s.t. $H_{ij} \cap (T_j \cup T_i) = \emptyset$ and $|H_{ij}| = \ell$. If $T_i$ wins, the procedure returns $T_i$ and otherwise returns $T_j$. The Slick-Greedy then returns the set $T_i$ that survived the Smooth-Cmpare tournament.

\begin{algorithm}
\textbf{Algorithm 2 Slick-Greedy}
\begin{algorithmic}[1]
\State \textbf{Input:} budget $k$
\State 1: Select $\ell/\delta$ elements in $N$ and partition them into disjoint sets of equal size $H_1, \ldots, H_{1/\delta}$
\State 2: $T_i \leftarrow \emptyset$
\For{$j \in [1/\delta]$}
\State 4: $R_j \leftarrow \cup_{i \neq j} H_i$
\State 5: $T_j \leftarrow \text{Smooth-Greedy}(k, R_j, H_j) \cup H_j$
\State 6: $H_{ij} \leftarrow$ arbitrary set of $\ell$ elements disjoint from $T_i \cup T_j$
\State 7: $T_i \leftarrow \text{Smooth-Cmpare}([T_i, T_j], H_{ij})$
\EndFor
\State 9: \textbf{return} $T_i$
\end{algorithmic}
\end{algorithm}

Consider the smoothing sets $H_1, \ldots, H_{1/\delta}$. Let $H_l$ be the smoothing set whose marginal contribution to the others is minimal, i.e. $H_l \in \arg\min_{i \in [1/\delta]} f_{R_l}(H_i)$. Notice that from submodularity we are guaranteed that $f_{R_l}(H_l) \leq \delta f(R_l \cup H_l)$. In this case, the fact that the marginal contribution of $H_l$ to the rest of the smoothing sets $R_l$ is small, together with the fact that the solution is initialized with $R_l$, enables the tight analysis. The two main steps are:

1. We first describe and analyze the Smooth-Cmpare procedure. We prove that with high probability the set $T_i$ that wins the Smooth-Cmpare tournament (i.e. the set $T_i$ eventually returned by Slick-Greedy) satisfies $f(T_i) \geq (1 - \epsilon/3) \max_{j \in [1/\delta]} f(T_j)$. In particular, $f(T_i) \geq (1 - \epsilon/3) f(T_l)$;

2. We can then apply the guarantee of Smooth-Greedy on $T_i$, which gives us that with high probability $f(T_i \cup H_l)$ approximates $\text{OPT}$ to a factor arbitrarily close to $(1 - 1/e)$. Intuitively this happens since the marginal contribution of $H_l$ to the rest of the smoothing sets is small, and since the solution to Smooth-Greedy is initialized with $R_l = \cup_i H_i \setminus H_l$ losing the value of $H_l$ is negligible. Since Smooth-Cmpare returns $T_i$ that respects $f(T_i) \geq (1 - \epsilon/3) f(T_l)$, this essentially concludes the proof.

### 3.2.2 The smooth comparison procedure

We can now describe the Smooth-Cmpare procedure we use in the algorithm. For a given set $H_{ij} \subseteq N$ of size $\ell$ and two sets $T_i, T_j \subseteq N \setminus H_{ij}$, we compare $\hat{f}(T_i \cup H_{ij}^t)$ with $\hat{f}(T_j \cup H_{ij}^t)$ for all $H_{ij}^t \subset H_{ij}$. We select $T_i$ if in the majority of the comparisons with $H_{ij}^t \subset H_{ij}$ (breaking ties
lexicographically) we have that \( \tilde{f}(T_i \cup H_{ij}') \geq \tilde{f}(T_j \cup H_{ij}') \), and otherwise we select \( T_j \). A formal description is added below.

Algorithm 3 Smooth-Compare

Input: \( T_i, T_j, H_{ij} \subseteq N \setminus (T_i \cup T_j) \),

1. Compare \( \tilde{f}(T_i \cup H_{ij}') \) with \( \tilde{f}(T_j \cup H_{ij}') \) for all \( H_{ij}' \subseteq H_{ij} \)
2. if \( T_i \) won the majority of comparisons return \( T_i \) otherwise return \( T_j \)

Lemma 3.10. Assume \( k \geq 96\ell /e^2 \). Let \( T_i \) be the set that won the Smooth-Compare tournament. Then, with probability at least \( 1 - 1/n^2 \):

\[
\tilde{f}(T) \geq (1 - \epsilon/3) \max_{j \in [1/\delta]} f(T_j).
\]

The proof of this lemma has two parts.

1. First we show that if a set \( T_i \) has moderately larger value than another set \( T_j \) (more specifically, if the gap is \( 1 - \epsilon\delta/3 \)) then it must be that \( f(T_i \cup H_{ij}') \) is also larger than \( f(T_j \cup H_{ij}') \), when \( H_{ij}' \subseteq H_{ij} \) (Claim B.1). At a high level, this is due to the fact that elements in \( H_{ij}' \) are candidates for Smooth-Greedy and the fact that they are not selected indicates that their marginal contribution to \( T_j = S_j \cup H_j \) is low. Thus, elements in \( H_{ij}' \) cannot add much value, and since \( |H_{ij}| \ll k \) adding subsets of \( H_{ij} \) does not distort the comparison by much.

2. The next step (Claim B.2) then shows that if for every subset \( H_{ij}' \) we have that \( f(T_i \cup H_{ij}') \geq f(T_j \cup H_{ij}') \) then with high probability \( T_i \) wins the comparison in Smooth-Compare.

Using these two parts we then conclude that since we are running the Smooth-Compare tournament between 1/\( \delta \) sets, the winner is an \( (1 - \epsilon\delta/3)^{1/\delta} \geq (1 - \epsilon/3) \) approximation to the competing set with the highest value. The proofs can be found in Appendix B.

3.2.3 Approximation guarantee of slick greedy

Definition 3.11. Given two disjoint sets \( H \) and \( R \), let \( \text{OPT}_{H,R} = f(H \cup R \cup O_{H,R}) - f_R(H) \) where:

\[
O_{H,R} \in \arg\max_{T:|T| \leq k - |H \cup R|} f(H \cup R \cup T) - f_R(H).
\]

Notice that when \( R = \emptyset \) we have that \( O_{H,R} = O_H \in \arg\max_{T:|T| \leq k - |H|} f_H(T) \) as defined in the previous subsection. In that sense, the value of \( O_{H,R} \) is that of the optimal solution evaluated on \( f_H \) when initialized with \( R \). In the same way we show that smooth greedy obtains a \( 1 - 1/e - \epsilon/3 \) approximation to \( \text{OPT}_H \) in Lemma 3.8 one can show that when Smooth-Greedy is initialized with \( R \) it obtains the same approximation guarantee against \( \text{OPT}_{H,R} \). Details are in Appendix B.

Lemma 3.12. Let \( S \) be the set returned by Smooth-Greedy that is initialized with a set \( R \subseteq N \) and has \( H \) as its smoothing set of size \( \ell \), which is disjoint from \( R \) and \( S \). Then, for any fixed \( \epsilon > 0 \) when \( k \geq 3|H \cup R|/\epsilon \) with probability of at least \( 1 - 1/n^3 \) we have that:

\[
f(R \cup S \cup H) \geq (1 - 1/e - \epsilon/3) \text{OPT}_{H,R}.
\]
As discussed above, we will instantiate this definition with \( R = R_t \) and \( H = H_t \) defined as follows: for any \( i \in [1/\delta] \) we will define \( R_i = \bigcup_{j \neq i} H_j \) and use the index \( l \) to denote the smoothing set in \( \{H_i\}_{i=1}^{1/\delta} \) which has the least marginal contribution to the rest, i.e. \( H_l = \arg\min_{i \in [1/\delta]} f_{R_l}(H_i) \). We will first show that the iteration of SLICK-GREEDY finds a solution arbitrarily close to \( 1 - 1/e \) for sufficiently large \( k \).

**Lemma 3.13.** Let \( S_l \) be the set returned by SMOOTH-GREEDY that is initialized with \( R_l \) and \( H_l \) its smoothing set. Then, for any fixed \( \epsilon > 0 \) when \( k \geq 36\ell/e^2 \) with probability of at least \( 1 - 1/n^3 \) we have that:

\[
 f(S_l \cup H_l) \geq (1 - 1/e - 2\epsilon/3)OPT
\]

**Proof.** To ease notation, let \( R = R_t, H = H_t \), and \( O = O_t \) where \( O_t \) is the solution which maximizes \( f(H \cup R \cup T) - f_R(T) \) over all subsets \( T \) of size at most \( k - |H \cup R| \). Let \( \beta = |H \cup R|/k \). Notice that by submodularity we have that:

\[
 f(H \cup R \cup O) \geq \left( 1 - \frac{|H \cup R|}{k} \right) \text{OPT} = (1 - \beta)\text{OPT}
\]  

(2)

Notice also that by the minimality of \( H = H_t \) and submodularity we have that \( f_{R_l}(H) \leq \delta f(H \cup R) \). Recall also that \( \delta = \epsilon/6 \) and notice that whenever \( k \geq \ell/\delta^2 = 36\ell/e^2 \) we have that \( \beta \leq \delta \) and hence \( \beta + \delta < \epsilon/3 \). Therefore, by application of Lemma 3.12 we get that with probability \( 1 - 1/n^3 \):

\[
 f(S \cup R \cup H) \geq \left( 1 - \frac{1}{e} - \frac{\epsilon}{3} \right) \text{OPT}_{H,R} \quad \text{by Lemma 3.12}
\]

\[
 = \left( 1 - \frac{1}{e} - \frac{\epsilon}{3} \right) (f(H \cup R \cup O) - f_R(H)) \quad \text{by definition}
\]

\[
 \geq \left( 1 - \frac{1}{e} - \frac{\epsilon}{3} \right) (f(H \cup R \cup O) - \delta \cdot f(H \cup R)) \quad \text{by monotonicity of f}
\]

\[
 \geq \left( 1 - \frac{1}{e} - \frac{\epsilon}{3} - \delta \right) \cdot f(H \cup R \cup O) \quad \text{by Lemma 3.12}
\]

\[
 \geq \left( 1 - \frac{1}{e} - \frac{\epsilon}{3} - \delta \right) (1 - \beta)\text{OPT} \quad \text{by (2)}
\]

\[
 \geq \left( 1 - \frac{1}{e} - \frac{2\epsilon}{3} \right) \text{OPT}. \quad \beta + \delta < \epsilon/3 \quad \square
\]

Finally, putting everything together, we can prove the main result of this section.

**Theorem 3.1.** Let \( f : 2^N \rightarrow \mathbb{R} \) be a monotone submodular function. For any fixed \( \epsilon > 0 \), when \( k \geq 3168\log \log n/e^2 \), then given access to noisy oracle whose noise is an exponentially decaying tail distribution, the SMOOTH-GREEDY algorithm returns a set which is a \((1 - 1/e - \epsilon)\) approximation to \( \max_{S: |S| \leq k} f(S) \), with probability at least \( 1 - 1/n \).

**Proof.** From Lemma 3.13 we know that when \( k \geq 36\ell/e^2 \) with probability at least \( 1 - 1/n^3 \) SMOOTH-GREEDY initialized with \( R_t \) outputs \( T_l = S_l \cup H_l \) which is a \((1 - 1/e - 2\epsilon/3)\) approximation to \( \text{OPT} \). When \( 3168\log \log n/e^2 \leq k \leq 2400\log \log n/e^2 \) we use \( \ell = 33\log \log n \) and then \( k \geq 96\ell/e^2 \). In the case
that $k \geq 2400 \log n / \epsilon^2$ we have that $\ell = 25 \log n$ and in this case too $k \geq 96 \ell / \epsilon^2$. Therefore, the conditions for Lemma 3.10 hold, and we know that with probability $1 - 1/n^2$ the set $T_i$ returned by Slick-Greedy respects $f(T_i) \geq f(T_l)$. Applying a union bound on these events we get that with probability at least $1 - 1/n$:

\[
\begin{align*}
    f(T_i) &\geq (1 - \epsilon/3) f(T_l) & \text{by Lemma 3.10} \\
    &\geq (1 - \epsilon/3) (1 - 1/e - 2\epsilon/3) \text{OPT} & \text{by Lemma 3.13} \\
    &\geq (1 - 1/e - \epsilon) \text{OPT}. & \text{hallelujah.}
\end{align*}
\]
4 Optimization for Small \( n \)

When \( k \) is small we cannot use the smoothing technique from the previous section, since it requires including the smoothing set of size \( \Theta(\log \log n) \) in the solution. In this section we describe the \textit{sampling mean method} which can be applied to \( k \in O(\log \log n) \). This method leads to an approximation arbitrarily close to \( 1 - 1/e \) for any \( k \in \Omega(1/\epsilon) \). The smoothing idea in this section is different from that of the previous section. The smoothing method we use makes it easy to obtain a good approximation that holds in expectation. The main technical challenge is the transition from a guarantee that holds in expectation to one that holds with high probability. This difficulty is what limits the algorithm to be applicable only when \( k \) ranges between \( \Omega(1/\epsilon) \) and \( O(\log \log n) \).

4.1 Combinatorial averaging

Similar to the Slick-Greedy algorithm, the sampled-mean method is based on averaging sets to find elements whose marginal contribution is high, which can then be greedily added to the solution. Unlike the Slick-Greedy algorithm however, the idea here is to select smoothing sets that are intuitively close to the elements whose marginal contribution we estimate. Borrowing again from continuous optimization, the idea in this section is to define a small ball around a point. For this purpose, unlike the previous algorithm, we will add sets of constant size \( c \in O(1/\epsilon) \) to the solution each time, which can be viewed as \( c \)-dimensional points. Unlike the previous section, for a given set we will generate multiple smoothed values and average them. We call these values the \textit{mean contribution} of a set and we will define this concept shortly.

**Smoothing.** Consider some arbitrary ordering on the elements s.t. \( N = \{a_1, a_2, \ldots, a_n\} \). For a given set \( A \subseteq N \), let \( A_{-i} \) denote \( A \setminus \{a_i\} \). Throughout the rest of this section we will work on sets of size \( c \), where \( c \) is a constant that depends on \( \epsilon \) and will be fixed later. For given sets \( S \) and \( A \), for every \( i \), the smoothing neighborhood of \( A_{-i} \) is \( \mathcal{H}(A_{-i}) = \{A_{-i} \cup \{a_j\} : a_j \in N \setminus (S \cup A)\} \). Thus:

\[
F(S \cup A_{-i}) = \frac{1}{n - c - |S|} \sum_{j \notin S \cup A} f(S \cup A_{-i} \cup a_j)
\]

\[
F_S(A_{-i}) = \frac{1}{n - c - |S|} \sum_{j \notin S \cup A} f_S(A_{-i} \cup a_j)
\]

And similarly, \( \tilde{F}(S \cup A_{-i}) = \frac{1}{n - c - |S|} \sum_{j \notin S \cup A} \tilde{f}(S \cup A_{-i} \cup a_j) \). For a given set \( A \) and every given \( a_i \in A \) we will apply smoothing arguments on each \( A_{-i} \) to show that the noisy smooth value is close to the true smooth value. We will compare between candidate solutions to be selected by the greedy algorithm using the \textit{mean smooth value} which is simply an average of the smooth values of \( A_{-i} \), for all \( i \in [c] \). Intuitively, this mean value well approximates the true marginal contribution.

**Definition 4.1.** Suppose that we’ve already committed to a set \( S \). For a set \( A \) of fixed size \( c \), \( a_i \in A \) and \( a_j \in N \setminus (S \cup A) \) we define \( A_{ij}(S) := (A \setminus \{a_i\}) \cup \{a_j\} \). When \( S \) is clear from context, we will use the shorthand \( A_{ij} \). The mean smooth value, noisy mean smooth value and marginal smooth value of a set \( A \) to an existing set \( S \) are, respectively:

- \( \phi(S \cup A) := \frac{1}{c} \sum_{i \in A} F(S \cup A_{-i}) = \frac{1}{c(n - c - |S|)} \sum_{i \in A} \sum_{j \notin S \cup A} f(S \cup A_{ij}) \)
\[
\bar{\phi}(S \cup A) := \frac{1}{c} \sum_{i \in A} \bar{F}(S \cup A - i) = \frac{1}{c(n - c - |S|)} \sum_{i \in A} \sum_{j \notin A \cup S} \bar{f}(S \cup A) \\
\phi_S(A) := \frac{1}{c} \sum_{i \in A} F_S(A - i) = \frac{1}{c(n - c - |S|)} \sum_{i \in A} \sum_{j \notin A \cup S} f_S(A_{ij})
\]

4.2 The sampled mean algorithm

The SM-Greedy begins with the empty set \( S \) and at every iteration considers subsets of size \( c \in O(1/\epsilon) \) to add to \( S \). At every iteration, the algorithm first takes the set \( A \) which maximizes the noisy mean smooth value. After taking \( A \) the algorithm then considers all possible subsets \( A_{ij} \) and takes the set whose noisy value is largest. We describe the algorithm formally below.

\begin{algorithm}
\begin{algorithmic}[1]
\State \( S \leftarrow \emptyset \)
\While{\( |S| < c \cdot \lfloor k/c \rfloor \)}
\State \( A \leftarrow \arg \max_{|A| = c} \bar{\phi}(S \cup A) \)
\State \( S \leftarrow S \cup \arg \max_{i \in A, j \in N \setminus (S \cup A)} f(S \cup A) \)
\EndWhile
\State \text{return } S
\end{algorithmic}
\end{algorithm}

At a high level, the major steps in the analysis can be described as follows.

1. We begin by giving guarantees on smoothing. In Lemma 4.2 we motivate the idea behind the mean contribution. We show that when \( c = 1/\epsilon \) the mean contribution is a \( 1 - \epsilon \) approximation to \( f_S(A^*) \) where \( A^* \in \arg \max_{A:|A| = c} f_S(A) \). Using Claim 4.3, in Lemma 4.5 we then show that the set selected in each iteration is with high probability an arbitrarily good approximation to the set with maximal (non-noisy) smooth mean contribution;

2. Once done with smoothing arguments, in Lemma 4.6 we prove that if the marginal contribution \( f_S(\hat{A}) \) of the set \( \hat{A} \) we select at every iteration is close to the mean smooth marginal contribution \( \phi_S(A) \) we obtain an approximation arbitrarily close to \( 1 - 1/e \). This suffices for an approximation guarantee that holds in expectation;

3. The last step is Lemma 4.13 which is the most technically challenging part of this section. We show that taking \( A \in \arg \max_{i,j} f(S \cup A_{ij}) \) in line 4 of the algorithm gives us, with high probability, that the marginal contribution of the set we select is close to its smooth marginal contribution. We can therefore invoke Lemma 4.6 and obtain the approximation guarantee.

4.3 Smoothing guarantees

We begin by showing that the marginal smooth value of the best subset can well approximate its marginal contribution.

**Lemma 4.2.** For a fixed set \( S \subset N \), let \( A^* \in \arg \max_{A:|A| = c} f_S(A) \). Then:

\[
\left( 1 - \frac{1}{c} \right) f_S(A^*) \leq \phi_S(A^*) \leq f_S(A^*)
\]
Proof. By the maximality of $A^*$ we have that $f(A^*) \geq f(A^*_{ij})$ for any $i, j$ since $A^*_{ij}$ is generated by replacing $a_i \in A^*$ with $a_j \in N \setminus A^*$. Therefore, the average of all $A_{ij}$’s is upper bounded by $f_S(A^*)$.

For the lower bound, consider some arbitrary ordering on the elements $a_1, \ldots, a_c \in A^*$. From the diminishing returns property of submodular functions we have that for any $i \in [c]$:

$$f_{S \cup A^*_{i}}(a_i) = f(S \cup A^*_{i} \cup a_i) - f(S \cup A^*_{i}) \leq f(S \cup \{a_1, \ldots, a_i\}) - f(S \cup \{a_1, \ldots, a_{i-1}\})$$

Thus:

$$\sum_{i=1}^{c} f_{S \cup A^*_{i}}(a_i) \leq \sum_{i=1}^{c} (f(S \cup \{a_1, \ldots, a_i\}) - f(S \cup \{a_1, \ldots, a_{i-1}\})) = f_S(A^*)$$

Therefore we get:

$$\phi_S(A^*) = \frac{1}{c(n - c - |S|)} \sum_{j=1}^{n-c-|S|} \sum_{i=1}^{c} f_S(A^*_{ij})$$

$$\geq \frac{1}{c} \sum_{i=1}^{c} f_S(A^*_{i})$$

$$= \frac{1}{c} \sum_{i=1}^{c} \left( f_S(A^*_{i} \cup a_i) - f_{S \cup A^*_{i}}(a_i) \right)$$

$$= \frac{1}{c} \sum_{i=1}^{c} f_S(A^*) - \frac{1}{c} \sum_{i=1}^{c} f_{S \cup A^*_{i}}(a_i)$$

$$\geq f_S(A^*) - \frac{1}{c} f_S(A^*) \quad \text{by (3)}$$

$$= \left( 1 - \frac{1}{c} \right) f_S(A^*) \quad \square$$

We would now like to employ the smoothing arguments from Section 2. We will show that for the set $A$ selected by the SM-Greedy algorithm in line 3, $\phi(A)$ is a good approximation to $\phi_S(A^*)$, where $A^* \in \text{argmax}_{B: |B|=c} f_S(B)$. From Lemma 4.2 above, this implies that $\phi_S(A)$ well approximates $f_S(A^*)$. To do so, we use the next claim which essentially relies on bounding the variation of the smoothing neighborhoods $\mathcal{H}(A^*_{-i})$, for almost all sets $A^*_{-i}$. To bound the average variation of the sets $\{A^*_{-i}\}_{i=1}^{c}$ we argue that at most one set $A^*_{-i}$ will be s.t. $f_S(A^*_{-i}) < f_S(A^*)/2$. We defer the proof to Appendix C.

Claim 4.3. Let $A^* \in \text{argmax}_{B: |B|=c} f_S(B)$, $c \geq 4/\epsilon$. Then:

$$\frac{1}{c} \sum_{i=1}^{c} \max \left\{ 0, 1 - 2v_S(\mathcal{H}(A^*_{-i})) \cdot t^{-1/4} \right\} f_S(A^*_{-i}) \geq (1 - \epsilon) f_S(A^*)$$

Similarly to the definition of $\epsilon$-relevant iterations from the previous section, we introduce the definition of $\epsilon$-significant iterations to employ smoothing arguments in the next lemma.

Definition 4.4. Let $O \in \text{argmax}_{T: |T| \leq k} f(T)$. An iteration of SM-Greedy is $\epsilon$-significant if for the given set $S$ selected before the iteration we have that $f_S(O) \geq \epsilon f(O)$.
The following lemma gives us the smoothing guarantee we need to work with smooth values rather than noisy ones. The proof uses Claim 4.3 to bound the variation as well as Lemma 4.2. The proof is deferred to Appendix C.

**Lemma 4.5.** Let \( A \in \arg\max_{B:|B|=c} \tilde{\phi}(S \cup B) \) where \( c \geq 16/\epsilon \), and assume that the iteration is \( \epsilon/4 \)-significant. Then, with probability at least \( 1 - e^{-\Omega(t^{1/10})} \) we have that:

\[
\phi_S(A) \geq (1 - \epsilon) \max_{B:|B|=c} \phi_S(B)
\]

### 4.4 Approximation guarantee

We begin by giving an approximation guarantee, assuming that the marginal contribution of the set of size \( c \) we select at every iteration is close to its mean smooth marginal contribution. This condition suffices for an approximation guarantee that holds in expectation. We will soon after discuss how to obtain a guarantee that holds with high probability.

**Lemma 4.6.** Assume that at every \( \delta/4 \)-significant iteration of SM-Greedy when the set selected at previous iterations is \( S \) and the set selected is \( A \) we have that: \( f_S(A) \geq (1 - \delta) \max_{A:|A|=c} \phi_S(A) \), for \( \delta > 0 \). Assume that \( k > c/\delta \geq 16 \). Let \( \bar{S} \) be the set of elements selected in all the iterations of the algorithm SM-Greedy. Then, with probability \( 1 - 1/n^2 \):

\[
f(\bar{S}) = (1 - 1/e - 5\delta)OPT
\]

The proof follows by applying Lemma 4.5 together with the assumption that \( f_S(A) \geq (1 - \delta) \max_{A:|A|=c} \phi_S(A) \). There is some bookkeeping to keep track of the fact that in each iteration we choose a subset of size \( c \), and the minor loss in approximation due to smoothing. The rest follows by a standard inductive argument. The full proof is in Appendix C.

#### 4.4.1 From expectation to high probability

Lemma 4.5 gives us a bound on the performance of the mean marginal contribution (measured by \( \phi \)), but not the actual marginal contribution, which is determined by the set we actually choose to add to the solution. Note that adding the set which maximizes the mean smooth value can easily lead to an arbitrarily bad approximation \(^7\). Choosing a random \( A_{ij} \) would have given us the correct expected marginal contribution, but then the result would not be with high probability of success. It remains to show that after we selected the set \( A \) which maximizes \( \phi_S(A) \), choosing the set which maximizes \( \tilde{f}(S \cup A_{ij}) \) is a good approximation of \( \phi_S(A) \), with high probability.

**High-level overview to show high probability guarantee.** Let \( A^* \) be the set of the largest marginal contribution and \( A \) be the set selected by the algorithm at an iteration. That is, \( A^* \in \arg\max_{B:|B|=c} f_S(B) \) and \( A \in \arg\max_{B:|B|=c} \tilde{\phi}(B) \). We will define two kinds of sets in \( \{A_{ij}\}_{i \in [c], j \notin S \cup A} \), called **good** and **bad**. The good sets are those whose true marginal contribution

\(^7\) As an example, consider an instance with \( n - 1 \) complementary elements \( M \) for whom for any \( S \subseteq M \) the function evaluates to \( f(S) = \alpha \) for some arbitrarily large value \( \alpha \), and an additional subset of elements \( A \) s.t. \( f(A) = \epsilon \) and for any \( S \subseteq M \) we have \( f(S \cup A) = M + \epsilon \), for some arbitrarily small \( \epsilon > 0 \). The sampled mean of \( A \) is maximal, while its value is arbitrarily small.
is at most $1 - 2\epsilon$ from $f_S(A^*)$ and the bad sets are those whose marginal contribution is at least $1 - 3\epsilon$ from $f_S(A^*)$. Our goal would be to prove that with high probability no bad set can be returned by the algorithm. To do so, we will prove that with high probability a good set $A'$ beats a bad set $A''$ when comparing $f(S \cup A')$ with $f(S \cup A'')$. This will be done using the following steps:

1. After defining good and bad sets, in Claim 4.8 we show that for $A \in \arg\max_{B:|B|=c} \phi(B)$, at least half of the sets in $\{A_{ij}\}_{i \in A, j \notin S \cup A}$ are good, and at most half are bad;

2. Next, we define two thresholds: $M_g$ and $M_b$. Intuitively, $M_g$ is the value of the noise multiplier that one of the good sets should obtain, and $M_b$ is an upper on the value of the noise multiplier that any one of the bad sets can obtain;

3. We will then show in Lemma 4.12 that $M_g \geq (1 - \gamma)M_b$, for any $\gamma = \Theta(1/\log \log n)$ to be fixed later. Notice that this then implies that the noisy value of a good set is then larger than the noisy value of a bad set. This lemma is quite technical, and it is where we fully leverage the property of generalized exponential tail distribution and the fact that $k \in O(\log \log n)$;

4. Putting everything together, the fact that a bad set loses to a good set w.h.p. implies that the value of the set we end up selecting must at least be as high as that of a bad set, i.e. $(1-3\epsilon)f_S(A^*)$. This essentially enables us to then apply Lemma 4.6 and obtain the guarantee.

**Definition 4.7.** For a given set $S$, let $A^* \in \arg\max_{B:|B|=c} f_S(B)$, $A \in \arg\max_{B:|B|=c} \tilde{\phi}(S \cup B)$, and $A = \{A_{ij}\}_{i \in A, j \notin A}$. For a fixed $\epsilon > 0$:

- $A_{ij} \in A$ is $\epsilon$-good if $f_S(A_{ij}) \geq (1 - 2\epsilon)f_S(A^*)$; let $\epsilon$-good($A$) denote all $\epsilon$-good $A_{ij} \in A$;
- $A_{ij} \in A$ is $\epsilon$-bad if $f_S(A_{ij}) \leq (1 - 3\epsilon)f_S(A^*)$; let $\epsilon$-bad($A$) denote all $\epsilon$-bad $A_{ij} \in A$.

**Claim 4.8.** For a set $S \subseteq N$ let $A \in \arg\max_{B:|B|=c} \tilde{\phi}(S \cup B)$ and assume the iteration is $\epsilon/8$-significant and that $c \geq \epsilon/2$. Then with probability at least $1 - 1/n^{10}$:

- $|\epsilon$-good($A$)| $\geq \frac{c(n-c-|S|)}{2}$;
- $|\epsilon$-bad($A$)| $\leq \frac{c(n-c-|S|)}{2}$.

**Proof.** Since the sets $A_{ij}$ are distinct both $\epsilon$-good($A$) and $\epsilon$-bad($A$) contain no repetitions and we can argue about their size. To lower bound the size of $\epsilon$-good($A$), let $A^* \in \arg\max_{A:|A|=c} f_S(A)$. When the iteration is $\epsilon/8$-significant, from Lemma 4.5 we know that with exponentially high probability:

$$\phi_S(A) \geq (1 - \epsilon/2)\phi_S(A^*)$$

When $c \geq 2/\epsilon$, from Lemma, we know that:

$$\phi_S(A^*) \geq (1 - \epsilon/2)f_S(A^*)$$

Denoting $m = c(n - c - |S|)$, we get with exponentially high probability:

$$\phi_S(A) = \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{c} f_S(A_{ij}) \geq (1 - \epsilon)f_S(A^*)$$  \hspace{1cm} (4)
In addition, due to the maximality of $A^*$ we have that $f_S(A_{ij}) \leq f_S(A^*)$ for every $i, j$. Therefore:
\[
\sum_{j=1}^{m} \sum_{i=1}^{c} f_S(A_{ij}) \leq |\epsilon\text{-good}(A)| \cdot f_S(A^*) + (m - |\epsilon\text{-good}(A)|) \cdot (1 - 2\epsilon) f_S(A^*)
\]
(5)

Putting (4) and (5) together we get that for sufficiently large $n$, with probability at least $1 - 1/n^{10}$:
\[
m(1 - \epsilon) f_S(A^*) \leq (|\epsilon\text{-good}(A)| + (m - |\epsilon\text{-good}(A)|))(1 - 2\epsilon) f_S(A^*)
\]
Rearranging and using $m = c(n - c - |S|)$ we get that $|\text{good}| \geq c(n - c - |S|)/2$. Since there are a total of $c(n - c - |S|)$ it follows that $|\epsilon\text{-bad}(A)| \leq c(n - c - |S|)$ as required.

**Definition 4.9.** Let $\rho_D(x)$ denote the probability density function of $D$. For a set $S : |S| \leq O(\log n)$, $c > 0$, $\gamma > 0$, we define $M_g$ and $M_b$ as:
- $\int_{M_b}^\infty \rho_D(x)dx = \frac{2}{c(n-c-|S|) \log n}$;
- $\int_{M_g}^\infty \rho_D(x)dx = \frac{2 \log n}{c(n-c-|S|)}$.

The following claim immediately follows from the definition, yet it is still useful to specify explicitly. The claim considers $c(n - c - |S|)/2$ samples since this is an upper and lower bound on $|\epsilon\text{-good}(A)|$ and $|\epsilon\text{-bad}(A)|$. Therefore the claim gives us the likelihood that the largest noise multiplier of $\epsilon\text{-bad}(A)$ does not exceed $M_b$ and that at least one set from $\epsilon\text{-good}(A)$ exceeds $M_g$.

**Claim 4.10.** For a fixed set $S$ and $A \in \arg\max_{B:\|B\|=c} \phi(S \cup B)$, let $m = c(n-c-|S|)$ and consider $m/2$ independent samples from the noise distribution. Then:

- $\Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \leq M_b] > \left(1 - \frac{2}{\log n}\right)$;
- $\Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \geq M_g] > 1 - 2/n$.

**Proof.** For a single sample $\xi$ from $D$, we have that:
\[
\Pr[\xi \leq M_b] = 1 - \frac{2}{m \log n}
\]
If we take $m/2$ independent samples $\xi_1, \ldots, \xi_{m/2}$, the probability they are all bounded by $M_b$ is:
\[
\Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \leq M_b] \geq \left(1 - \frac{2}{m \log n}\right)^{\frac{m}{2}} > \left(1 - \frac{2}{\log n}\right)
\]
In the case of $M_g$, the probability that a single sample $\xi$ taken from $D$ is at most $M_g$ is equal to:
\[
\Pr[\xi \leq M_g] = 1 - \frac{2 \log n}{m}
\]
If we take independent samples $\xi_1, \ldots, \xi_{m/2}$, the probability they are all bounded by $M_g$ is:
\[
\Pr[\max\{\xi_1, \ldots, \xi_{c(n-c-|S|)}\} \leq M_g] = \left(1 - \frac{2 \log n}{m}\right)^{\frac{m}{2}} < \frac{2}{2 \log n} = \frac{2}{n}
\]
And accordingly the probability that at least one of these samples is greater than $M_g$ is:
\[
\Pr[\max\{\xi_1, \ldots, \xi_{m/2}\} \geq M_g] > 1 - 2/n.
\]
The Lemma 4.12 below relates \( M_g \) and \( M_b \) assuming that \( D \) has a generalized exponential tail. This lemma makes the result applicable for Exponential and Gaussian distributions, and it fully leverages the fact that \( k \in O(\log \log n) \). The lemma is quite technical, and we first prove the much simpler case where the distribution is bounded, which applies to the case of uniform distributions.

**Lemma 4.11.** Assume \( D \) has a generalized exponential tail and that \( D \) is bounded. Let \( x_0 \) be the infimum s.t. \( D = 1 \) and assume \( \rho_D(x_0) \neq 0 \). Then: \((1 - \gamma)M_b \leq M_g, \forall \gamma \in \Omega(1/\log \log n)\).

**Proof.** Since \( x_0 \) is the infimum number such that \( D = 1 \), we have that \( M_b \leq x_0 \). Let \( x_g = (1 - \gamma)x_0 \). Since we know that \( x_0 \) is an infimum and \( \rho_D(x_0) \neq 0 \) it must be that \( D(x_g) = 1 - \delta \) for some \( \delta = \Omega(1/\log \log n) \). But in this case, if we take \( n > \log n/\gamma \delta \) we get that \( M_g > x_g \), and hence \( M_g \geq (1 - \gamma)M_b \). □

**Lemma 4.12.** If \( D \) has a generalized exponential tail then \((1 - \gamma)M_b \leq M_g, \forall \gamma \in \Omega(1/\log \log n)\).

**Proof.** The proof follows three stages:

1. We use properties of \( D \) to argue upper and lower bounds for \( \rho_D(x) \);
2. We show an upper bound \( M \) on \( M_b \);
3. We show that integrating a lower bound of \( \rho_D(X) \) from \((1 - \gamma)M \) to \( \infty \), yields a probability mass at least \( \frac{\log n}{\gamma c(n-c-|S|)} \). Now suppose for contradiction that \( M_g < (1 - \gamma)M_b \), we would get that \( \int_{M_g}^{\infty} \rho_D(x) \) is strictly greater than \( \frac{\log n}{\gamma c(n-c-|S|)} \), which contradicts the definition of \( M_g \).

We now elaborate each on stage. Recall that by definition of \( D \) for \( x \geq x_0 \), we have that \( \rho_D(x) = e^{-g(x)} \), where \( g(x) = \sum_i a_i\alpha_i \) and that we do not assume that all the \( \alpha_i \)'s are integers, but only that \( \alpha_0 \geq \alpha_1 \geq \ldots \), and that \( \alpha_0 \geq 1 \). We do not assume anything on the other \( \alpha_i \) values.

For the first stage we will show that for every \( g(x) \), there exists \( n_0 \) such that for any \( n > n_0 \) and \( x \geq \left( \frac{\log n}{2a_0} \right)^{1/\alpha_0} \) we have that for \( \beta = \gamma / 100 < 1/100 \):

\[
(1 + \beta)a_0x^{\alpha_0-1}e^{-(1+\beta)a_0x^{\alpha_0}} \leq \rho_D(x) \leq (1 - \beta)a_0x^{\alpha_0-1}e^{-(1-\beta)a_0x^{\alpha_0}}
\]

We explain both directions of the inequality. To see \( a_0x^{\alpha_0-1}(1 + \beta)e^{-(1+\beta)a_0x^{\alpha_0}} \leq \rho_D(x) \) we first show:

\[
e^{-(1+\beta/2)a_0x^{\alpha_0}} \leq \rho_D(x)
\]

This holds since for sufficiently large \( n \), we have that:

\[
x \geq \left( \frac{\log n}{2a_0} \right)^{1/\alpha_0} \geq \left( 2 \frac{\sum_{i=1}^n |a_i|}{\beta a_0} \right)^{\alpha_0-\alpha_1}
\]

So the term \( \frac{\beta}{2} \) dominates the rest of the terms. We now show that:

\[
e^{-(1+\beta/2)a_0x^{\alpha_0}} \geq a_0x^{\alpha_0-1}(1 + \beta)e^{-(1+\beta)a_0x^{\alpha_0}}
\]

This is equivalent to:

\[
e^{\beta a_0/2x^{\alpha_0}} \geq a_0x^{\alpha_0-1}(1 + \beta)
\]
Which hold for $x = \log \log^3 n$ and large enough $n$.

The other side of the inequality is proved in a similar way. We want to show that:

$$\rho_D(x) \leq (1 - \beta)a_0x^{\alpha_0-1}e^{-(1-\beta)a_0x^{\alpha_0}}$$

Clearly for $x > \log \log^3 n$ we have that $(1 - \beta)a_0x^{\alpha_0-1} > 1$. Hence we just need to show that:

$$\rho_D(x) \leq e^{-(1-\beta)a_0x^{\alpha_0}}$$

But this holds for sufficiently large $n$ s.t.:

$$x \geq \frac{(\log n)^{1/\alpha_0}}{2a_0} \geq \left(\frac{\sum_{i=1}^{\beta} |a_i|}{\beta a_0}\right)^{\alpha_0-\alpha_1}$$

We now proceed to the second stage, and compute an upper bound on $M_b$. Note that if

$$\int_{M_b}^{\infty} \rho_D(x) = \int_{M}^{\infty} g(x)$$

and for every $x \geq M$ we have $\rho_D(x) \leq g(x)$ then it must be that $M \geq M_b$. Applying this to our setting, we bound $\rho_D(x) \leq (1 - \beta)a_0x^{\alpha_0-1}e^{-(1-\beta)a_0x^{\alpha_0}}$ to get:

$$\frac{1}{c(n-c-|S|)\log n} = \int_{M}^{\infty} (1 - \beta)a_0x^{\alpha_0-1}e^{-(1-\beta)a_0x^{\alpha_0}}$$

Taking the logarithm of both sides, we get:

$$-(1 - \beta)a_0M^{\alpha_0} = \log \frac{1}{c(n-c-|S|)\log n}$$

$$= -\log(c(n-c-|S|)\log n)$$

Multiplying by $-1$, dividing by $(1 - \beta)a_0$ and taking the $1/\alpha_0$ root we get:

$$M = \left(\frac{\log(c(n-c-|S|)\log n)}{(1 - \beta)a_0}\right)^{\alpha_0}$$

Note that $(1 - \gamma)M > \left(\frac{\log n}{2a_0}\right)^{1/\alpha_0}$ and hence our bounds on $\rho_D(x)$ hold for this regime.

We move to the third stage, and bound $\int_{(1-\gamma)M}^{\infty} \rho_D(x)$ from below. If we show that: $\int_{(1-\gamma)M}^{\infty} \rho_D(x)$ is greater than $\frac{\log n}{\gamma c(n-c-|S|)}$, this implies that $M_g \geq (1 - \gamma)M$, as $M_g$ is defined as the value such that when we integrate $\rho_D(x)$ from $M_g$ to $\infty$ we get exactly $\frac{\log n}{\gamma c(n-c-|S|)}$. We show:

$$\int_{(1-\gamma)M}^{\infty} \rho_D(x) \geq (1 + \beta)a_0\alpha_0x^{\alpha_0-1}e^{-(1+\beta)a_0x^{\alpha_0}}$$

$$= -e^{-(1+\beta)a_0x^{\alpha_0}}\int_{(1-\gamma)M}^{\infty} \rho_D(x)$$

$$= e^{-(1+\beta)a_0((1-\gamma)M)^{\alpha_0}}$$

$$= e^{-(1+\beta)a_0M^{\alpha_0}(1-\gamma)^{\alpha_0}}$$

$$\geq e^{-(1+\beta)a_0M^{\alpha_0}(1-\gamma)}$$
However $a_0 M^{a_0} = \left( \frac{\log(c(n-c-|S|) \log n)}{(1-\beta)} \right)$. Since $\beta < 0.1$ we have that $\frac{1+\beta}{1-\beta} < 1 + 3\beta$. Substituting both expressions we get:

$$e^{-(1+\beta)a_0 M^{a_0}(1-\gamma)} \geq e^{-(1+3\beta)(1-\gamma) \log(c(n-c-|S|) \log n)}$$

$$= \left( \frac{1}{c(n-c-|S|) \log n} \right)^{(1-\gamma)(1+3\beta)}$$

$$\geq \left( \frac{1}{c(n-c-|S|) \log n} \right)^{(1-\gamma)/2}$$

Where we used that $\beta = \gamma / 100$ and hence $(1-\gamma)(1+3\beta) < 1 - \gamma / 2$. We now need to compare this to $\frac{\gamma}{\gamma c(n-c-|S|)}$. To do this, note that:

$$\left( \frac{1}{c(n-c-|S|) \log n} \right)^{(1-\gamma)/2} \geq \frac{1}{(c(n-c-|S|))^{1-\gamma/2 \log n}}$$

$$\geq \frac{2^{\sqrt{\log n}}}{\gamma c(n-c-|S|) \log n}$$

Where $n$ is large enough that $\frac{\gamma}{2} \log(n-c-|S|) > \sqrt{\log n}$. This completes the proof, since $M_g \geq (1-\gamma) M \geq (1-\gamma) M_b$ as required.

**Lemma 4.13.** Consider an $\epsilon/8$-significant iteration of SM-Greedy with a set $S : S \in O(\log \log n)$, and let $\hat{A} = \arg\max_{i \in A, j \in N \setminus (S \cup A)} \hat{f}(S \cup A_{ij})$, where $A = \arg\max \phi(S \cup A)$ and $c \geq 16/\epsilon$. For every $\gamma = \Omega(1/\log \log n)$, with probability at least $1 - 3/\log n$ we have that: $f_S(\hat{A}) \geq (1 - 3\epsilon) \phi_S(A)$.

**Proof.** We will use the above claims to argue that with probability at least $1 - 4/\log n$ the noisy mean value of any set in $\epsilon$-bad$(A)$ is smaller than the largest noisy mean value of a set in $\epsilon$-good$(A)$. Since a bad set is defined as a set $B$ for which $f(B) \leq (1-3\epsilon)f_S(A^*)$ this implies that the set returned by the algorithm has value at least $(1-3\epsilon)f_S(A^*)$. Since for any set $A : |A| = c$ we have that $f_S(A^*)$ is an upper bound on $\phi_S(A)$ it will complete the proof.

We will condition on the event that $|\epsilon$-good$(A)| \geq c(n-c-|S|)/2$ which happens with probability at least $1 - 1/n^{10}$ from Claim 4.8. Under this assumption, from Claim 4.10 we know that with probability at least $1 - 2/n$ at least one of the noise multipliers of sets in $\epsilon$-good$(A)$ has value at least $M_g$, and from Lemma 4.12 we know that $M_g \geq (1-\gamma) M_b$ for any $\gamma \in \Theta(1/\log \log n)$. Thus:

$$\max_{A_{ij} \in \epsilon$-good$(A)} \tilde{f}(S \cup A_{ij}) = \max_{A_{ij} \in \epsilon$-good$(A)} \xi_{A_{ij}} (f(S) + f_S(A_{ij}))$$

$$\geq M_g \cdot (f(S) + (1-2\epsilon)f_S(A^*)) \geq (1-\gamma) M_b \cdot (f(S) + (1-2\epsilon)f_S(A^*))$$

Let $B \in \arg\max_{C \in \epsilon$-bad$(A)} \tilde{f}(S \cup C)$. From Claim 4.10 we know that with probability at least $1 - 2/\log n$ all noise multipliers of sets in $\epsilon$-bad$(A)$ are at most $M_b$. Thus:

$$\tilde{f}(S \cup B) = \max_{A_{ij} \in \epsilon$-bad$(A)} \tilde{f}(S \cup A_{ij}) = \max_{A_{ij} \in \epsilon$-bad$(A)} \xi_{A_{ij}} f(S \cup A_{ij}) \leq M_b \cdot (f(S) + (1-3\epsilon)f_S(A^*))$$

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Let $d$ be some constant such that $|S| \leq d \log \log n$. Note that the iteration is $\epsilon$-significant, and therefore due to the maximality of $A^*$ and since $f(S) \leq \text{OPT}$ and the optimal solution has at most $d \cdot \log \log n$ elements we have that:

$$f_S(A^*) \geq \frac{\epsilon}{d \log \log n} f(S).$$

Putting it all together and conditioning on all events we have with probability at least $1 - 3/\log n$:

$$\tilde{f}(S \cup \hat{A}) - \tilde{f}(S \cup B) \geq (1 - \gamma)M_b \cdot (f(S) + (1 - 2\epsilon) f_S(A^*)) - M_b \cdot (f(S) + (1 - 3\epsilon) f_S(A^*))$$

$$\geq M_b (\epsilon f_S(A^*) - \gamma ((1 - 2\epsilon) f_S(A^*) + f(S)))$$

$$> M_b \cdot f_S(A^*) \left( \epsilon - \gamma \left( 1 + \frac{d \cdot \log \log n}{\epsilon} \right) \right)$$

Since Lemma 4.12 applies to any $\gamma \in \Theta(1/\log \log n)$, we know that for any constant $d$ there is a large enough value of $n$ such that $\gamma (1 + d \log \log n/\epsilon) < \epsilon$. Therefore the difference is strictly positive, implying that a bad set will not be selected by the algorithm which concludes our proof.

**Theorem 4.14.** For any monotone submodular function and any $\epsilon > 0$, with probability $1 - 4/\log n$ there is a $(1 - 1/\epsilon - \epsilon)$ approximation for $\text{max}_{S:|S|\leq k} f(S)$ for super-constant $k = O(\log \log n)$, given access to a noisy oracle whose distribution has a generalized exponential tail.

**Proof.** Let $\delta = \epsilon/5$ and set $c \geq 16/\delta$. At any given $\delta/8$-significant iteration of SM-Greedy from Lemma 4.13 we know that with probability at least $1 - 3/\log n$ we have that $f(\hat{A}) \geq 1 - \delta \phi_S(A)$, where $A \in \text{argmax}_{B:|B| = c}$. We can then apply Lemma 4.6 which implies that with probability at least $1 - 4/\log n$ we have a $1 - 1/\epsilon - 5\delta = 1 - 1/\epsilon - \epsilon$ approximation.

Somewhat counter-intuitively, when $k$ is constant the optimization problem becomes strictly harder. In the following section we show lower bounds for small values of $k$ and when $k = 1$ that no algorithm can obtain an expected approximation ratio better than $1/2 + o(1)$.

**Corollary 4.15.** For any monotone submodular function and any $\epsilon > 0$, given access to a noisy oracle whose distribution has a generalized exponential tail, there is a $(1 - 1/k - \epsilon)$ approximation for $\text{max}_{S:|S| \leq k} f(S)$ for $k \in \Omega(1/\epsilon)$ with probability $1 - 3/\log n$.

**Proof.** Enumerate over all possible sets of size $k$ and output $\hat{A} = \text{argmax} \tilde{f}(A_{ij})$ where $A = \text{argmax}_{B:|B| = k} \tilde{\phi}(B)$. Let $A^* \in \text{argmax}_{B:|B| = k} f(B)$. Lemma 4.5 implies that w.h.p. for sufficiently large $n$ we have that: $\phi(A) \geq (1 - \epsilon/2) \phi(A^*)$ and from Lemma 4.2 this implies that

$$\phi(A) \geq (1 - \epsilon/2)(1 - 1/k)f_S(A^*).$$

Lemma 4.13 gives that $f(\hat{A})$ is an $1 - \epsilon/2$ approximation to $\phi(A)$ with probability $1 - 3/\log n$. 

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5 Optimization for Very Small \( k \)

The smoothing guarantee from the previous actually depends on \( c \in O(1/\epsilon) \) which may not apply to small values of \( k \). The dependency on \( \epsilon \) originates in Claim 4.3 where we bound on the variation of \( c - 1 \) sets \( A_{-i} \), and thus the result depends on \( c \geq 4/\epsilon \). For small constants we propose a slightly different algorithm than the ones we used before, which is simple and follows the same principles.

**Smoothing.** The smoothing here is straightforward. For every set \( A \) consider the smoothing neighborhood \( \mathcal{H}(A) = \{A \cup x : x \notin A\} \), \( F(A) = \mathbb{E}_{X \in \mathcal{H}(A)}[f(X)] \) and \( \tilde{F}(A) = \mathbb{E}_{X \in \mathcal{H}(A)}[\tilde{f}(X)] \).

**Lemma 5.1.** Let \( \hat{A} \in \arg\max_{B : |B| = k} \tilde{F}(B) \). Then, for any fixed \( \epsilon > 0 \) w.p. \( 1 - e^{-\Omega(\epsilon^2(n-k))} \):

\[
F(A) \geq (1 - \epsilon) \max_{B : |B| = k} F(B)
\]

**Approximation guarantee in expectation.** The algorithm will simply select the set \( \hat{A} \) to be a random set of \( k \) elements from a random set of \( \mathcal{H}(A) \) where \( A \in \arg\max_{B : |B| = k} \tilde{F}(B) \).

**Algorithm 5** EXP-SMALL-GREEDY

**Input:** budget \( k \)

1. \( A \leftarrow \arg\max_{B : |B| = k} \tilde{F}(B) \)
2. \( x \leftarrow \) select random element from \( N \setminus A \)
3. \( \hat{A} \leftarrow \text{random set of size } k \text{ from } A \cup x \)
4. return \( \hat{A} \)

**Theorem 5.2.** For any constant \( k \) and any fixed \( \epsilon > 0 \) there is a \( (k/(k+1) - \epsilon) \)-approximation algorithm that holds in expectation.

**Proof.** From Lemma 4.2 we know that \( f(\hat{A}) \geq (k/(k+1)) F(A) \). Let \( A^* = \arg\max_{B : |B| = k} f(B) \). From monotonicity we know that \( f(A^*) \leq F(A^*) \). Applying Lemma 5.1 we get that for the set \( F(A) \geq (1 - \epsilon) F(A^*) \). Hence:

\[
f(\hat{A}) \geq \left( \frac{k}{k+1} \right) F(A) \geq (1 - \epsilon) \left( \frac{k}{k+1} \right) F(A^*) \geq (1 - \epsilon) \left( \frac{k}{k+1} \right) f(A^*) \geq \left( \frac{k}{k+1} - \epsilon \right) \text{OPT}
\]

**High probability.** To obtain a result that holds with high probability we will do a modest variation on the algorithm above. The algorithm will enumerate all possible subsets of size \( k-1 \), and then select the set \( \hat{A} \in \arg\max_{B : |B| = k-1} \tilde{F}(B) \). The algorithm will then select \( \hat{A} \in \arg\max_{X \in \mathcal{H}(A)} \tilde{f}(X) \).

**Algorithm 6** HP-SMALL-GREEDY

**Input:** budget \( k \)

1. \( \hat{A} \leftarrow \arg\max_{B : |B| = k-1} \tilde{F}(B) \)
2. \( \hat{A} \leftarrow \arg\max_{x \in N \setminus A} \tilde{f}(A \cup x) \)
3. return \( \hat{A} \)

The analysis of the algorithm is similar to the high-probability proof from Section 4.
**Theorem 5.3.** For any constant $k$ and any fixed $\epsilon > 0$ there is a $(1 - 1/k - \epsilon)$-approximation algorithm that holds with probability at least $1 - 6/\log n$.

**Proof.** Let $A \in \arg\max_{B:|B|=k-1} F(B)$, and let $A^* \in \arg\max_{B:|B|=k-1} f(B)$. Since $A^*$ is the optimal solution over $k - 1$ elements, from submodularity we know that $f(A^*) \geq (1 - 1/k)OPT$. What now remains to show is that $\hat{A} \in \arg\max_{x \in X \setminus A} \hat{f}(A \cup x)$ is a $(1 - \epsilon)$ approximation to $F(A)$. To do so recall the definitions of good and bad sets from the previous section. Let $\delta$ be an arbitrary constant. Suppose that a set $X$ is in $\delta$-good$(A)$ if $f(X) \geq (1 - 2\delta)f(A^*)$ and in $\delta$-bad$(A)$ if $f(X) \leq (1 - 3\delta)f(A^*)$. We will show that the set selected has value at least as high as that of a bad set, i.e. $(1 - 3\delta)f(A^*)$ which will complete the proof.

We first show that with probability at least $1 - 6/\log n$ the noise multiplier of some good set is at least $M_g$ and of a bad set is at most $M_b$. To do so we will first argue about the size of $\delta$-good$(A)$ and $\delta$-bad$(A)$. From Lemma 5.1 and the maximality of $A$ we know that with exponentially high probability $F(A) \geq (1 - \delta)F(A^*)$. Therefore for $m = n - k$:

$$F(A) = \frac{1}{m} \sum_{x \notin A} f(A \cup x) \geq (1 - \delta) \frac{1}{m} \sum_{x \notin A^*} f(A^* \cup x) \geq (1 - \delta)f(A^*)$$

Due to the maximality of $A^*$ and submodularity we know that $f(A \cup x) \leq 2f(A^*)$ for all $x \notin A$:

$$\sum_{x \notin A} f(A \cup x) \leq |\delta$-good$(A)|2f(A^*) + (m - |\delta$-good$(A)|)(1 - 2\delta)f(A^*)$$

Putting the these bounds on $F(A)$ together and rearranging we get that:

$$|\delta$-good$(A)| \geq \frac{\delta \cdot m}{1 + 2\epsilon} \geq \frac{\delta m}{3}$$

Therefore, for sufficiently large $n$ the likelihood of at least one set achieving value at least $M_g$ is:

$$\Pr[\max\{\xi_1, \ldots, \xi_{\delta m/3}\} \geq M_g] \geq 1 - \left(1 - \frac{2\log n}{m}\right)^{\frac{\delta m}{3}} \geq 1 - \frac{2}{n^{\delta/3}} \geq 1 - \frac{1}{\log n}$$

To bound $\delta$-bad$(A)$ we will simply note that it is trivial that $\delta$-bad$(A) < m$. Thus, the likelihood that all noise multipliers of bad sets are bounded from above by $M_b$ is:

$$\Pr[\max\{\xi_1, \ldots, \xi_m\} \leq M_b] \geq \left(1 - \frac{2}{m \log n}\right)^m > \left(1 - \frac{4}{\log n}\right)$$

Thus, by a union bound and conditioning on the event in Lemma 5.1 we get that $M_b$ is an upper bound on the value of the noise multiplier of bad sets and $M_g$ is with lower bound on the value of the noise multiplier of a good stem all with probability at least $1 - 6/\log n$. From Lemma 4.12 we know that for any $\gamma \in \Theta(1/\log \log n)$ we have that $M_g \geq (1 - \gamma)M_b$. Thus:

$$\max_{X \in \delta$-good$(A)} \hat{f}(X) = \max_{X \in \delta$-good$(A)} \xi_X f(X) \geq M_g \cdot (1 - 2\delta)f(A^*) \geq (1 - \gamma)M_b \cdot (1 - 2\delta)f(A^*)$$

Let $B \in \arg\max_{C \in \delta$-bad$} \tilde{f}(S \cup C)$. From Claim 4.10 we know that with probability at least $1 - 2/\log n$ all noise multipliers of sets in $\epsilon$-bad$(A)$ are at most $M_b$. Thus:

$$\tilde{f}(S \cup B) = \max_{X \in \delta$-bad$} \hat{f}(X) = \max_{X \in \epsilon$-bad$(A)} \xi_X f(X) \leq M_b \cdot (1 - 3\delta)f(X)$$

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Putting it all together we have with probability at least $1 - 6/\log n$:

$$\tilde{f}(\hat{A}) - \tilde{f}(B) \geq M_0 f(A^*) \cdot ((1 - \gamma)(1 - 2\delta) - (1 - 3\delta)) > M_0 f(A^*) (\delta - \gamma)$$

Since Lemma 4.12 applies to any $\gamma \in \Theta(1/\log \log n)$, and $\delta$ is fixed it applies to $\gamma < \delta$ and the difference is positive. Since $\delta = \epsilon/6$ this completes our proof.

5.1 Information theoretic lower bounds for constant $k$

Surprisingly, even for $k = 1$ no algorithm can obtain an approximation better than $1/2$, which proves a separation between large and small $k$.\(^8\) The following is a tight bound for $k = 1$.

**Claim 5.4.** There exists a submodular function and noise distribution for which w.h.p. no randomized algorithm with a noisy oracle can obtain an approximation better than $1/2 + O(1/\sqrt{n})$ for $\max_{a \in N} f(a)$.

**Proof.** We will construct two functions that are identical except that one function attributes a value of 2 for a special element $x^*$ and 1 for all other elements, whereas the other is assigns a value of 1 for each element. In addition, these functions will be bounded from above by 2 so that the only queries that give any information are those of singletons. More formally, consider the functions $f_1(S) = \min\{|S|, 2\}$ and $f_2(S) = \min\{g(S), 2\}$ where $g : 2^N \to \mathbb{R}$ is defined for some $x^* \in N$ as:

$$g(S) = \begin{cases} 2, & \text{if } S = x^* \\ |S|, & \text{otherwise} \end{cases}$$

The noise distribution will return 2 with probability $1/\sqrt{n}$ and 1 otherwise.

We claim that no algorithm can distinguish between the two functions with success probability greater than $1/2 + O(1/\sqrt{n})$. For all sets with two or more elements, both functions return 2, and so no information is gained when querying such sets. Hence, the only information the algorithm has to work with is the number of 1, 2, and 4 values observed on singletons. If it sees the value 4 on such a set, it can conclude that the underlying function is $f_2$. This happens with probability $1/\sqrt{n}$.

Conditioned on the event that the value 4 is not realized, the only input that the algorithm has is the number of 1s and 2s it sees. The optimal policy is to choose a threshold, such if a number of 2s observed is or above this threshold, the algorithm returns $f_2$ and otherwise it reruns $f_1$. In this case, the optimal threshold is $\sqrt{n} + 1$.

The probability that $f_2$ has at most $\sqrt{n}$ twos is $1/2 - 1/\sqrt{n}$, and so is the probability that $f_1$ has at least $\sqrt{n} + 1$ twos, and hence the advantage over a random guess is $O(1/\sqrt{n})$ again.

An algorithm which approximates the maximal set on $f_2$ with ratio better than $1/2 + \omega(1/\sqrt{n})$ can be used to distinguish the two functions with advantage $\omega(1/\sqrt{n})$. Having ruled this out, the best approximation one can get is $1/2 + O(1/\sqrt{n})$ as required.

We generalize the construction to general $k$. The lower for general $k$ behaves like $2k/(2k - 1)$, where our upper bound is $(k - 1)/k$.

---

\(^8\)We note that if the algorithm is not allowed to query the oracle on sets of size bigger than $k$, Claim 5.5 can be extended to show an approximation ratio of $O(n)$, so choosing a random element is almost the best possible course of action.
Claim 5.5. There exists a submodular function and noise distribution for which w.h.p. no randomized algorithm with a noisy oracle can obtain an approximation better than $(2k - 1)/2k + O(1/\sqrt{n})$ for the optimal set of size $k$.

Proof. Consider the function:

$$f_1(S) = \begin{cases} 2|S|, & \text{if } |S| < k \\ 2k - 1, & \text{if } |S| = k \\ 2k, & \text{if } |S| > k \end{cases}$$

And the function $f_2$, which is dependent on the identity of some random set of size $k$, denoted $S^*$:

$$f_2(S; S^*) = \begin{cases} 2|S|, & \text{if } |S| < k \\ 2k - 1, & \text{if } |S| = k, S \neq S^* \\ 2k, & \text{if } S = S^* \\ 2k, & \text{if } |S| > k \end{cases}$$

Both functions are submodular.

The noise distribution will return $2k/(2k - 1)$ with probability $n^{-1/2}$ and 1 otherwise. Again we claim that no algorithm can distinguish between the functions with probability greater than $1/2$. Indeed, since $f_1, f_2$ are identical on sets of size different than $k$, and their value only depends on the set size, querying these sets doesn’t help the algorithm (the oracle calls on these sets can be simulated). As for sets of size $k$, the algorithm will see a mix of $2k - 1$, $2k$, and at most one value of $4k^2/(k - 1)$. If the algorithm sees the value $4k^2/(k - 1)$ then it was given access to $f_2$. However, the algorithm will see this value only with probability $1/\sqrt{n}$. Conditioning on not seeing this value, the best policy the algorithm can adopt is to guess $f_2$ if the number of $2k$ values is at least $1 + \binom{n}{k}/\sqrt{n}$, and guess $f_1$ otherwise. The probability of success with this test is $1/2 + O(1/\sqrt{n})$ (regardless of whether the underlying function is $f_1$ or $f - 2$). Any algorithm which would approximate the best set of size $k$ to an expected ratio better than $(2k - 1)/2k + \omega(1/\sqrt{n})$ could be used to distinguish between the function with an advantage greater than $1/\sqrt{n}$, and this puts a bound of $(2k - 1)/2k + O(1/\sqrt{n})$ on the expected approximation ratio. ∎
6 Impossibility for Adversarial Noise

In this section we show that there are very simple submodular functions for which no randomized algorithm with access to an \( \epsilon \)-erroneous oracle can obtain a reasonable approximation guarantee with a subexponential number of queries to the oracle. Intuitively, the main idea behind this result is to show that a noisy oracle can make it difficult to distinguish between two functions whose values can be very far from one another. The functions we use here are similar to those used to prove information theoretic lower bounds for submodular optimization and learning [63, 68, 31, 7, 75].

Theorem 6.1. No randomized algorithm can obtain an approximation strictly better than \( O(n^{-1/2+\delta}) \) to maximizing monotone submodular functions under a cardinality constraint using \( e^{\delta^2}/n \) queries to an \( \epsilon \)-erroneous oracle, for any fixed \( \epsilon, \delta < 1/2 \).

Proof. We will consider the problem of \( \max_{S:|S| \leq k} f(S) \) where \( k = n^{1/2+\delta} \). Let \( X \subseteq N \) be a random set constructed by including every element from \( N \) with probability \( n^{-1/2+\delta} \). We will use this set to construct two functions that are close in expectation but whose maxima have a large gap, and show that access to a noisy oracle implies distinguishing between these two functions. The functions are:

\[
\begin{align*}
\bullet \ f_1(S) &= \min \left\{ |S \cap X| \cdot n^{1/2} + \frac{n^{1/2+\delta}}{\epsilon}, |S| \cdot n^{1+\delta} \right\} \\
\bullet \ f_2(S) &= \min \left\{ |S| \cdot n^{\delta} + \frac{n^{1/2+\delta}}{\epsilon}, |S| \cdot n^{1+\delta} \right\}
\end{align*}
\]

By the Chernoff bound we know that \( |X| \geq n^{1/2+\delta}/2 \) with probability \( 1 - e^{-\Omega(n^{1/2+\delta})} \). Conditioned on this event we have that \( \max_{S:|S| \leq k} f_1(S) = f_1(X) \in O(n^{1+\delta}) \) whereas \( f_2 \) is symmetric and \( \max_{S:|S| \leq k} f_2(S) \in O(n^{1/2+2\delta}) \). Thus, an inability to distinguish between these two functions implies there is no approximation algorithm with approximation better than \( O(n^{-1/2+\delta}) \). We define the erroneous oracle as follows. If the function is \( f_2 \), its oracle returns the exact same value as \( f_2 \) for any given set. Otherwise, the function is \( f_1 \) and its erroneous oracle is defined as:

\[
\tilde{f}(S) = \begin{cases} 
  f_2(S), & \text{if } (1 - \epsilon)f_1(S) \leq f_2(S) \leq (1 + \epsilon)f_1(S) \\
  f_1(S), & \text{otherwise}
\end{cases}
\]

Notice that this oracle is \( \epsilon \)-erroneous, by definition.

Suppose now that the set \( X \) is unknown to the algorithm, and the objective is \( \max_{S:|S| \leq k} f_1(S) \). We will first show that no deterministic algorithm that uses a single query to the erroneous oracle \( \tilde{f} \) can distinguish between \( f_1 \) and \( f_2 \), with exponentially high probability (equivalently, we will show that a single query the algorithm cannot find a set \( S \) for which \( f_1(S) < (1 - \epsilon)f_2(S) \) or \( f_1(S) > (1 + \epsilon)f_2(S) \) with exponentially high probability). For a single query algorithm, we can imagine that the set \( X \) is chosen after the algorithm chooses which query to invoke, and compute the success probability over the choice of \( X \). In this case, all the elements are symmetric, and the function value is only determined by the size of the set that the single-query algorithm queries.

In case the query is a set \( S \) of cardinality smaller or equal to \( n^{1/2} \), by the Chernoff bound we have
that $|S \cap X| \leq (1 + \beta)n^\delta$ for any $\beta < 1$ with probability at least $1 - e^{-\Omega(\beta^2 n^\delta)}$. Thus:

$$\frac{n^{1/2 + \delta}}{\epsilon} \leq f_1(S) \leq (1 + \beta + \frac{1}{\epsilon})n^{1/2 + \delta}$$

$$\frac{n^{1/2 + \delta}}{\epsilon} \leq f_2(S) \leq (1 + \frac{1}{\epsilon})n^{1/2 + \delta}$$

It is easy to verify that for $\beta < \epsilon/(1 - \epsilon)$: $(1 - \epsilon)f_1(S) \leq f_2(S) \leq (1 + \epsilon)f_1(S)$. Thus, for any query of size less or equal to $n^{1/2}$ the likelihood of the oracle returning $f_1$ is $1 - e^{-\Omega(n^\delta)}$.

In case the oracle queries a set of size greater than $n^{1/2}$ then again by the Chernoff bound, for any $\beta < 1$ we have that with probability at least $1 - e^{-\Omega(\beta n^{1/2})}$:

$$(1 - \beta)\frac{|S|}{n^{1/2 - \delta}} \leq |S \cap X| \leq (1 + \beta)\frac{|S|}{n^{1/2 - \delta}}$$

For $\beta \leq \epsilon/(1 - \epsilon)$, this implies that:

$$(1 - \epsilon)f_1(S) \leq f_2(S) \leq (1 + \epsilon)f_1(S)$$

Therefore, for any fixed $\epsilon \in (0, 1)$, the algorithm cannot distinguish between $f_1$ and $f_2$ with probability $1 - e^{-\Omega(n^\delta)}$ by querying the erroneous oracle with a set larger than $n^{1/2}$. To conclude, by a union bound we get that with probability $1 - e^{-\Omega(n^\delta)}$ no algorithm can distinguish between $f_1$ and $f_2$ using a single query to the erroneous oracle, and the ratio between their maxima is $O(n^{1/2 - \delta})$.

To complete the proof, suppose we had an algorithm running in time $e^{n^\delta}/n$ which can approximate the value of a submodular function, given access to an $\epsilon$-erroneous oracle with approximation ratio strictly better than $O(n^{-1/2 + \delta})$ which succeeds with probability $2/3$. This would let us solve the following decision problem: "Given access to an $\epsilon$-erroneous oracle for either $f_1$ or $f_2$, determine which function is being queried." To solve the decision problem, given access to an erroneous oracle of unknown function, we would use the hypothetical approximation algorithm to estimate the value of the maximal set of size $n^{1/2 + \delta}$. If this value is strictly more than $n^{1/2 + 2\delta}$, the function is $f_1$ (since $f_1(X) = O(n^{1+\delta})$), and otherwise it is $f_2$.

The reduction allows us to show that distinguishing between the functions in time $e^{n^\delta}/n$ and success probability $2/3$ is impossible. For purpose of contradiction, suppose that there is a (randomized) algorithm for the decision problem, and let $p$ denote the probability that it outputs $f_2$ if it sees an oracle which is fully consistent with $f_2$. To succeed with probability $2/3$, it must be the case that whenever the algorithm gets $f_1$ as an input, it finds a set $S$ for which the noisy oracle returns $f_1(S)$ with probability at least $2/3 - p/2 \geq 1/6$. Whenever it finds such a set, the algorithm is done, since it can compute $f_2(S)$ without calling the oracle, and hence it knows that $f_1$ was chosen in the decision problem.

In this case, we know that the algorithm makes up to $e^{n^\delta}/n$ queries, until it sees a set for which it gets $f_1(S)$. But this means that there is an algorithm with success probability at least $O(n/6e^{n^\delta})$ that makes a single query. This algorithm guesses some index $i < e^{n^\delta}/n$, and simulates the original algorithm for $i - 1$ steps (by feeding it with $f_2$ without using the oracle), and then using the oracle in step $i$. If the algorithm guesses $i$ to be the first index in which the exponential time algorithm sees $f_1(S)$, then the single query algorithm would succeed. Hence, since we showed that no single query (randomized) algorithm can find a set $S$ such that $f_1(S) < (1 - \epsilon)f_2(S)$ or $f_1(S) > (1 + \epsilon)f_2(S)$ with just one query this concludes the proof. \qed
Somewhat surprisingly, the above theorem suggests that a good approximation to a submodular function does not suffice to obtain reasonable approximation guarantees. In particular, guarantees from learning or sketching where the goal is to approximate a submodular function up to constant factors may not necessarily be meaningful for optimization. It is important to note that for some classes of submodular functions such as additive functions \( f(S) = \sum_{a \in S} f(a) \), we can obtain algorithms that are robust to adversarial noise. A very interesting open question is to characterize the class of submodular functions that are robust to adversarial noise.
7 Modeling Optimization under Noise

In this section we discuss some of the modeling choices made in the paper. A theoretical model inevitably introduces tradeoffs, and the abstractions we choose are never a perfect fit for any reality. In this section we justify some of the modeling decisions we made, and discuss their consequences.

7.1 Alternative models of noise

As we have shown in Section 6, it is impossible to obtain non trivial guarantees in the presence of adversarial error, and hence it makes sense to focus on probabilistic noise. We briefly discuss the limitations and advantages of other variants of this model.

Inconsistent oracles. If we can sample the oracle multiple times on the same point and get independent results then we can just query the oracle $O(\log n)$ times, obtain an arbitrarily good approximation to the marginal contributions of elements to sets. Given a $1 - \epsilon$ approximate oracle to the marginal contributions, it is easy to show that one can obtain a $1 - \epsilon$ approximation via the standard greedy algorithm. While there are interesting questions that arise when given inconsistent oracles such as sample complexity (e.g. [10]) the question of whether polynomial-time algorithms can obtain optimal approximations is largely solved.

Degradation of information. We have written the paper as if the algorithm gains no additional information for querying a point twice. The generalization to a case where the algorithm gets more information each time but there is a degradation of information is simple: whenever the algorithms we presented here want to query a point just query it multiple times, and feed the expected value of the point given all the information one has to the algorithm. Hence it makes sense to focus on the extreme case where only the first query is helpful, as common in the literature of noisy optimization (e.g. [11])

Impossibility result for correlated distributions. Following the discussion above, the assumption that multiple samples on the same point do not give any extra information seems like the next logical step. Having taken the first step showing algorithms for the i.i.d. model, a natural question is whether the assumption that the noise is i.i.d on different sets is a good one.

**Theorem 7.1.** Even for unit demand functions there are simple correlated distributions for which no algorithm can achieve an approximation strictly better than $1/n$.

**Proof sketch.** Consider a unit demand function $f(S) = \max_{a \in S} f(a)$ which operates on a ground set with $n$ elements. There are $n - 1$ regular elements, and one special element $s$. The value of $f$ on any regular element is 1, but $f(s) = M$ for some large $M$. The noise distribution is such that it returns 1 on sets which do not contain $s$, and $1/M$ on sets that contain $s$. The best one can do in this case is to choose a random element without using the oracle at all. 

It is possible to add various assumptions that will mitigate bad examples like the one above (e.g. requiring that the correlation between different elements would be chosen “at random”, trying to
achieve $1 / \log n$ approximations etc.), but many of them also have similar bad examples. Moreover, the definitions of natural assumption are inevitably subjective and debatable, especially without motivating concrete applications. Hence, we decided that a reasonable first step would be to study submodular maximization in the uniform noise model, and left correlated cases for future research.

**Impossibility result for multiple distributions.** One can consider having multiple noise distributions which act on different sets. A noise distribution can be assigned to a set either in adversarial manner, or at random. If sets are assigned to noise distributions in an adversarial manner, it is possible to construct the bad example of the correlated case with just two noise distributions. If sets are assigned to a noise distribution in an i.i.d manner, this reduces to the i.i.d case when there is a single distribution.

### 7.2 Properties of our noise distribution

One of the requirements we had was that the noise distribution could potentially be Gaussian, Exponential, or any bounded distribution. Moreover, we wanted the algorithm to be oblivious to the specific noise distribution, and only rely on properties of the distribution in the analysis. As discussed in the Introduction, to obtain meaningful results, we introduce the class of *generalized exponential tail* distributions, formally defined as follows.

**Definition.** A noise distribution $\mathcal{D}$ has a *generalized exponential tail* if there exists some $x_0$ such that for $x > x_0$ the probability density function $\rho_\mathcal{D}(x) = e^{-g(x)}$, where $g(x) = \sum_i a_i x^{\alpha_i}$. We do not assume that all the $\alpha_i$’s are integers, but only that $\alpha_0 \geq \alpha_1 \geq \ldots$, and that $\alpha_0 \geq 1$.

Note that in particular if $\mathcal{D}$ is a Gaussian or an Exponential distribution, it belongs to the family of generalized exponential tail distributions. For $\alpha_i < 1$ which implies that a generalized exponential tail also includes cases where the probability density function denoted $\rho_\mathcal{D}$ respects $\rho_\mathcal{D}(x) = \rho_\mathcal{D}(x_0)e^{-g'(x-x_0)}$. This is because we can simply add $\rho_\mathcal{D}(x_0)$ to $g$ using $\alpha_i = 0$ for some $i$, and do a coordinate change moving from $g'(x-x_0)$ to an equivalent $g(x)$. Technically, the definition of a generalized exponential tail doesn’t allow a bounded distribution. However, all our results also hold for bounded distributions, assuming that if $x_0$ is the supremum of the distribution then $\rho_\mathcal{D}(x_0) \neq 0$ which then includes the uniform distribution. In fact the results hold even if $\rho_\mathcal{D}(x_0) = 0$, as long as the decay to 0 is slower then doubly exponential.

**The noise distribution is independent of $n$.** One important property that we keep using is that the distribution is independent of $n$ (formally we show that for every specific distribution there exists $n_0$ such that if $n > n_0$ some approximation guarantee holds). This requirement is necessary. Indeed, suppose that one was trying to maximize some submodular function bounded between 1 and $M$ (which can be a function of $n$). Suppose that we take a noise distribution with a finite support: $1, M, M^2, \ldots M^{2^n}$. The noise distribution is chooses a value in its support uniformly at random. Then no approximation is possible (regardless of runtime). A similar bad example occurs when the noise distribution is 1 with probability $1/2^n$ and 0 otherwise. Again, one can remove these specific examples with assumptions, but other bad example occur, and hence we do not allow any explicit dependency on $n$. 

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Making close sets have close values. Another type of criticism is that the distribution generating the noise is quite permissive, potentially making two sets whose true values are close have dramatically different values under noise. It seems reasonable that in many applications this shouldn’t be the case. For such applications, one can consider bounded distributions and in particular distributions whose values are in \([1 - \epsilon, 1 + \epsilon]\). Even for such degenerate cases, the classic greedy algorithm fails and we are not aware of simpler approaches than those presented in this paper. We do note however, that when the distribution is bounded, the running time of the algorithms can be significantly improved, and in Section 4 the analysis can be simplified. We highlight this and give the proof for this special in Section 4 as well.

7.3 Maximizing almost submodular functions

In this paper our goal is to obtain near optimal guarantees as defined on the original function that was distorted through noise. That is, we assume that there is an underlying submodular function which we aim to optimize, and we only get to observe noisy samples of it. An alternative direction would be to consider the problem of optimizing functions that are approximately submodular. In this case, the quality of the solution would be evaluated against \(\tilde{f}\) and not on \(f\). If we assume that the function is an adversarial modification of a submodular function, our lower bound from Section 6 for erroneous oracles essentially shows that nothing can be done. If we assume that a noisy process altered a submodular function, then there are trivial impossibility results. Suppose that the initial submodular function is the constant function that gives 1 to every set. If we apply (e.g.) Gaussian noise to it, then the optimal algorithm is just to try random sets and hope for the best, and no polynomial time algorithm can achieve a constant factor approximation.

7.4 Greedy fails with random noise

In practice, the greedy algorithm is often used although we know the data may be noisy. Hence, a different direction for research could be to analyze the effect of noise on the existing greedy algorithm. Unfortunately, it turns out that the greedy algorithm fails even on very simple examples.

**Theorem 7.2.** Given a noise distribution that is either uniformly distributed in \([1 - \epsilon, 1 + \epsilon]\) for any \(\epsilon > 0\), a Gaussian, or an Exponential, the greedy algorithm cannot obtain a constant factor approximation ratio even in the case of maximizing additive functions under a cardinality constraint.

**Proof sketch.** Consider an additive function, which has two types of elements: \(k = \sqrt{n}\) good elements, each worth \(n^{1/4}\), and \(n - k\) bad elements, each worth 1. Suppose that the noise is uniform in \([1 - \epsilon, 1 + \epsilon]\). Then after taking \(k^{2/3}\) good elements greedy is much more likely to take bad elements, which leads to an approximation ratio of \(O(1/n^{1/6})\). Similar examples hold for Gaussian and Exponential noise. \(\square\)
8 Acknowledgements

A.H. was supported by ISF 1241/12; Y.S. was supported by NSF grant CCF-1301976, CAREER CCF-1452961, a Google Faculty Research Award, and a Facebook Faculty Gift. We are deeply indebted to Lior Seeman, who has carefully read previous versions of the manuscript and made multiple invaluable suggestions.
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A Missing Proofs: Combinatorial Smoothing

Smoothing arguments

Lemma. Let $\omega_{\text{max}}$ and $\omega_{\text{min}}$ be the upper and lower bounds on the value of the noise multiplier in any of the calls made by a polynomial-time algorithm. For any $\delta > 0$, we have that:

- $\Pr[\omega_{\text{max}} < t^\delta] > 1 - e^{-\Omega(t^\delta/\ln t)}$
- $\Pr[\omega_{\text{min}} > t^{-\delta}] > 1 - e^{-\Omega(t^\delta/\ln t)}$

Proof. As $n$ tends to infinity, this is lemma trivial for any noise distribution which is bounded, or has finite support. If the noise distribution is unbounded, we know that its tail is subexponential. Thus, at any given sample the probability of seeing the value $n^\delta$ is at most $e^{-\Omega(n^\delta)}$ where the constant in the big $O$ notation depends on the magnitude of the tail. Iterating this a polynomial number of times gives the bound. The proof of the lower bound is equivalent.

Lemma. Let $f : 2^N \rightarrow \mathbb{R}$, $A, S \subset N$, $\omega = \max_{A_i \in \mathcal{H}(A)} \xi_{A_i}$, and $\mu$ be the mean of the noise distribution. For $\epsilon = \min \{1, 2v_S(\mathcal{H}) \cdot |\mathcal{H}(A)|^{-1/4}\}$ with probability $1 - e^{-\Omega(\frac{\lambda^2t}{\omega}1/4)}$ we have:

$$\tilde{F}(S \cup A) > (1 - \lambda)\mu \cdot (f(S) + (1 - \epsilon) \cdot F_S(A)).$$

Proof. Let $A_1, \ldots, A_t$ be the sets in $\mathcal{H}(A)$ and let $\alpha_1, \ldots, \alpha_t$ denote the corresponding marginal contributions and $\xi_1, \ldots, \xi_t$ denote their noise multipliers. In these terms the noisy smooth value is:

$$\tilde{F}(S \cup A) = \frac{1}{t} \sum_{i=1}^{t} \xi_i(f(S) + \alpha_i) = \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) + \frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i. \quad (6)$$

Let $\omega$ be the upper bound on the value of the noise multiplier. Applying the Chernoff bound, we get that for any $\lambda < 1$ with probability at least $1 - e^{-\Omega(\frac{\lambda^2t}{\omega})}$:

$$\frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) \geq (1 - \lambda)\mu f(S).$$

To complete the proof we need to argue about concentration of the second term in (6). To do so, in our analysis we will consider a fine discretization of $\{\alpha_i\}_{i \in [t]}$ and apply concentration bounds on each discretized value. Define $\alpha_{\text{max}} = \max_{i \in [t]} \alpha_i$ and $\alpha_{\text{min}} = \min_{i \in [t]} \alpha_i$. We can divide the set of values $\{\alpha_i\}_{i \in [t]}$ to $t^{1/4}$ bins $\text{BIN}_1, \ldots, \text{BIN}_{t^{1/4}}$, where a value $\alpha_i$ is placed in the bin $\text{BIN}_q$ if

$$(q - 1) \cdot \alpha_{\text{max}}^{t^{-1/4}} \leq \alpha_i \leq q \cdot \alpha_{\text{max}}^{t^{-1/4}}$$

Say a bin is dense if it contains at least $t^{1/4}$ values and sparse otherwise. Consider some dense bin $\text{BIN}_q$ and let $\alpha_{\text{min}(q)} = \min_{i \in \text{BIN}_q} \alpha_i$ and $\alpha_{\text{max}(q)} = \max_{i \in \text{BIN}_q} \alpha_i$. Since every bin is of width $\alpha_{\text{max}} \cdot t^{-1/4}$ we know that:

$$\alpha_{\text{min}(q)} \geq \alpha_{\text{max}(q)} - \alpha_{\text{max}} \cdot t^{-1/4}$$
Applying concentration bounds as above, we get that
\[ \sum_{i \in \text{bin}_q} \xi_i \geq (1 - \lambda) \mu \cdot |\text{bin}_q| \]
with probability at least \( 1 - e^{-\Omega(\lambda^2 t^{1/4} / \omega)} \) for any \( \lambda < 1 \). Thus, with this probability:
\[
\sum_{i \in \text{bin}_q} \xi_i \alpha_i \geq \sum_{i \in \text{bin}_q} \xi_i \alpha_{\min(q)} \\
\geq (1 - \lambda) \mu \cdot |\text{bin}_q| \cdot \alpha_{\min(q)} \\
\geq (1 - \lambda) \mu \cdot |\text{bin}_q| \cdot \left( \max \left\{ 0, \alpha_{\max(q)} - \alpha_{\max} \cdot t^{-1/4} \right\} \right) \\
> (1 - \lambda) \mu \cdot |\text{bin}_q| \cdot \left( \max \left\{ 0, 1 - \frac{\alpha_{\max}}{\alpha_{\min}} \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)} \\
\geq (1 - \lambda) \mu \cdot |\text{bin}_q| \cdot \left( \max \left\{ 0, 1 - \alpha_{\max} \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)} \\
= (1 - \lambda) \mu \cdot |\text{bin}_q| \cdot \left( \max \left\{ 0, 1 - v_S(H(A)) \cdot t^{-1/4} \right\} \right) \alpha_{\max(q)}
\]

Taking a union bound over all (at most \( t^{1/4} \)) dense bins, we get that with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4} / \omega)} \):
\[
\sum_{i \in \text{dense}} \xi_i \alpha_i \geq (1 - \lambda) \mu \cdot \left( 1 - \max \left\{ 0, v_S(H(A)) \cdot t^{-1/4} \right\} \right) \sum_{\text{BIN}_q \in \text{dense}} |\text{BIN}_q| \cdot \alpha_{\max(q)} \\
\geq (1 - \lambda) \mu \cdot \left( \max \left\{ 0, 1 - v_S(H(A)) \cdot t^{-1/4} \right\} \right) \sum_{i \in \text{dense}} \alpha_i. \quad (7)
\]

Let \( \alpha = \frac{1}{t} \sum_{i=1}^{t} \alpha_i \). Since we have less than \( t^{1/4} \) elements in a sparse bin, and in total \( t^{1/4} \) bins, the number of elements in sparse bins is at most \( t^{1/2} \). We can use this to effectively lower bound the values in sparse bins in terms of \( \alpha \):
\[
\sum_{i \in \text{dense}} \alpha_i = \sum_{i=1}^{t} \alpha_i - \sum_{i \in \text{sparse}} \alpha_i \\
\geq \max \left\{ 0, \sum_{i=1}^{t} \alpha_i - t^{1/2} \alpha_{\max} \right\} \\
\geq \max \left\{ 0, t \alpha - t^{1/2} \alpha_{\max} \right\} \\
> \max \left\{ 0, t \cdot \left( 1 - \frac{\alpha_{\max}}{\alpha_{\min}} \cdot t^{-1/2} \right) \alpha \right\} \\
= \max \left\{ 0, t \cdot \left( 1 - v_S(H) \cdot t^{-1/2} \right) \alpha \right\} \quad (8)
\]
Putting (7) and (8) we get that for any positive \( \lambda < 1 \), with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):
\[
\tilde{F}_S(A) = \frac{1}{t} \sum_{i=1}^{t} \xi_i \cdot \alpha_i
\geq \frac{1}{t} \sum_{i \in \text{dense}} \xi_i \cdot \alpha_i
\geq (1 - \lambda) \mu \cdot (\max \left\{ 0, 1 - v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\}) \cdot \frac{1}{t} \sum_{i \in \text{dense}} \alpha_i
\geq (1 - \lambda) \mu \cdot (\max \left\{ 0, 1 - v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\}) \cdot (\max \left\{ 0, 1 - v_S(\mathcal{H}(A)) \cdot t^{-1/2} \right\}) \alpha
> (1 - \lambda) \mu \cdot (\max \left\{ 0, 1 - 2v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\}) \alpha
= (1 - \lambda) \mu \cdot (\max \left\{ 0, 1 - 2v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\}) F_S(A)
\]
Taking a union bound we get that for any positive \( \lambda < 1 \) with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):
\[
\tilde{F}(S \cup A) = \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) + \frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i
\geq (1 - \lambda) \mu \cdot \left( f(S) + \max \left\{ 0, 1 - 2v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} \cdot F_S(A) \right)
= (1 - \lambda) \mu \cdot \left( f(S) + (1 - \min \left\{ 1, 2v_S(\mathcal{H}(A)) \cdot t^{-1/4} \right\} F_S(A) \right).
\]

**Lemma.** Let \( f : 2^N \to \mathbb{R} \), \( A, S \subseteq N \), \( \omega = \max_{A_i \in \mathcal{H}(A)} \xi_{A_i} \), \( \alpha_{\max} = \max_{A_i \in \mathcal{H}(A)} f_S(A_i) \) and \( \mu \) be the mean of the noise distribution. For \( \epsilon = 3t^{-1/4} \alpha_{\max} \) we have that for any \( \lambda < 1 \) with probability \( 1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)} \):
\[
\tilde{F}(S \cup A) < (1 + \lambda) \mu \cdot (f(S) + F_S(A) + \epsilon).
\]

**Proof.** As in the proof of Lemma 2.4 let \( A_1, \ldots, A_t \) denote the sets in \( \mathcal{H}(A) \), and for each set \( A_i \) we will again use \( \alpha_i \) to denote the marginal value \( f_S(A_i) \) and \( \xi_i \) to denote the noise multiplier \( \xi_{S \cup \{A_i\}} \).
\[
\tilde{F}(S \cup A) = \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) + \frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i.
\] (9)
As before, we will focus on showing concentration on the second term. Define \( \alpha_{\max} = \max_i \alpha_i \) and \( \alpha_{\min} = \min_i \alpha_i \). To apply concentration bounds on the second term, we again partition the values of \( \{\alpha_i\}_{i \in [t]} \) to bins of width \( \alpha_{\max} \cdot t^{-1/4} \) and call a bin dense if it has at least \( t^{1/4} \) values and sparse otherwise. Using this terminology:
\[
\sum_{i=1}^{t} \xi_i \alpha_i = \sum_{i \in \text{dense}} \xi_i \alpha_i + \sum_{i \in \text{sparse}} \xi_i \alpha_i.
\]
Let \( \text{BIN}_\ell \) be the dense bin whose elements have the largest values. Consider the \( t^{1/4}/2 \) largest values in \( \text{BIN}_\ell \) and call the set of indices associated with these values \( L \). We have:
\[
\sum_{i=1}^{t} \xi_i \alpha_i = \sum_{i \in \text{dense} \setminus L} \xi_i \alpha_i + \sum_{i \in L \cup \text{sparse}} \xi_i \alpha_i
\]
The set $L \cup_{sparse}$ is of size at least $t^{1/4}/2$ and at most $t^{1/4}/2 + t^{1/2}$. This is because $L$ is of size exactly $t^{1/4}/2$ and there are at most $t^{1/2}$ values in bins that are sparse since there are $t^{1/4}$ bins and a bin that has at least $t^{1/4}$ is already considered dense. Thus, when $\omega$ is an upper bound on the value of the noise multiplier, from Chernoff, for any $\lambda < 1$ with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\sum_{i \in L \cup_{sparse}} \xi_i \alpha_i \leq \sum_{i \in L \cup_{sparse}} \xi_i \alpha_{\max} \\
< (1 + \lambda) \mu \cdot |L \cup_{sparse}| \cdot \alpha_{\max} \\
\leq (1 + \lambda) \mu \cdot \left(\frac{t^{1/4}}{2} + t^{1/2}\right) \alpha_{\max} \\
< (1 + \lambda) \mu \cdot 2t^{1/2} \alpha_{\max}
$$

We will now use the same logic as in the proof of Lemma 2.4 to apply concentration bounds on the values in the dense bins. For a dense bin $\text{BIN}_q$, let $\alpha_{\max(q)}$ and $\alpha_{\min(q)}$ be the maximal and minimal values in the bin, respectively. As in Lemma 2.4, for any $\lambda < 1$ with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\sum_{i \in \text{BIN}_q} \xi_i \alpha_i \leq \sum_{i \in \text{BIN}_q} \xi_i \cdot \alpha_{\max(q)} \\
\leq (1 + \lambda) \mu \cdot \alpha_{\max(q)} \cdot |\text{BIN}_q| \\
\leq (1 + \lambda) \mu \cdot \left(\alpha_{\min(q)} + \alpha_{\max} \cdot t^{-1/4}\right) \cdot |\text{BIN}_q| \\
\leq (1 + \lambda) \mu \cdot \left(|\text{BIN}_q| \cdot \alpha_{\min(q)} + |\text{BIN}_q| \alpha_{\max} \cdot t^{-1/4}\right)
$$

Applying a union bound we get with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\sum_{i \in \text{dense} \setminus L} \xi_i \alpha_i < \sum_{q} (1 + \lambda) \mu \cdot \left(|\text{BIN}_q| \cdot \alpha_{\min(q)} + |\text{BIN}_q| \alpha_{\max} \cdot t^{-1/4}\right) \\
< (1 + \lambda) \mu \cdot t \left(\alpha + t^{-1/4} \alpha_{\max}\right)
$$

Together we have:

$$
\frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i = \frac{1}{t} \left(\sum_{i \in \text{dense} \setminus L} \xi_i \alpha_i + \sum_{i \in L \cup_{sparse}} \xi_i \alpha_i\right) \\
< (1 + \lambda) \mu \cdot \left(\alpha + t^{-1/4} \alpha_{\max} + 2t^{-1/2} \alpha_{\max}\right) \\
< (1 + \lambda) \mu \cdot \left(\alpha + 3t^{-1/4} \alpha_{\max}\right) \\
< (1 + \lambda) \mu \cdot \left(F_S(A) + 3t^{-1/4} \alpha_{\max}\right)
$$

By a union bound we get that with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$
\bar{F}(S \cup A) = \frac{1}{t} \sum_{i=1}^{t} \xi_i f(S) + \frac{1}{t} \sum_{i=1}^{t} \xi_i \alpha_i \leq (1 + \lambda) \mu \cdot \left(f(S) + F_S(A) + 3t^{-1/4} \alpha_{\max}\right). \quad \square
$$
\section*{B Missing Proofs: Optimization for Large \( k \)}

\textbf{Smoothing guarantees}

\textbf{Lemma (3.2).} For any fixed \( \epsilon > 0 \), assume \( \OPT_H \geq \OPT/3 \) and consider an \( \epsilon \)-relevant iteration of Smooth-Greedy where \( S \) is the set of elements selected in previous iterations and \( a \in \arg\max_{b \in S \cup H} F(S \cup b) \) is the element selected in that iteration. Then, for sufficiently large \( n \), for \( \delta = \epsilon^2/4k \) we have that with probability at least \( 1 - 1/n^4 \):

\[
F_S(a) \geq (1 - \delta) \max_{b \notin S \cup H} F_S(b).
\]

To prove the above lemma we will need claims 3.3, 3.4 and 3.5. After proving 3.5 the proof will follow by verifying that the number of sets in the smoothing set is sufficient to obtain the desired approximation \((1-\delta)\). For convenience we restate Claim 3.3. The proof is given in the main body of the paper.

\textbf{Claim (3.3).} If \( F_S(a) \geq F_S(b) \) then \( f_S(a) \geq f_{S \cup H}(b) \).

\textbf{Claim (3.4).} Let \( \epsilon > 0 \) and assume \( \OPT_H \geq \OPT/3 \). For an \( \epsilon \)-relevant iteration of Smooth-Greedy, let \( S \) be the set of elements selected in previous iterations. If \( a^* \in \arg\max_{a \notin S \cup H} F_S(a) \) then \( v_S(H(a^*)) \leq 3k/\epsilon \).

\textbf{Proof.} Let \( O \in \arg\max_{T:|T| \leq k'} f_H(T) \), and let \( o^* \in \arg\max_{o \in O} f_{H \cup S}(o) \). By the maximality of \( a^* \) we have that \( F_S(a^*) \geq F_S(o^*) \), and thus by Claim 3.3 we get \( f_S(a^*) \geq f_{S \cup H}(o^*) \). Since the iteration is \( \epsilon \)-relevant we have that \( f_{S \cup H}(O) \geq \epsilon \cdot \OPT_H \) and from subadditivity we get:

\[
f_{S \cup H}(o^*) \geq \frac{\epsilon \cdot \OPT_H}{k'} \geq \frac{\epsilon \cdot \OPT_H}{k}
\]

From \( f_S(a^*) \geq f_{S \cup H}(o^*) \) and \( f_{S \cup H}(o^*) \geq \epsilon \cdot \OPT_H / k \), and monotonicity of \( f \) we get:

\[
\min_{H' \subseteq H} f_S(H' \cup a^*) \geq f_S(a^*) \geq \frac{\epsilon \cdot \OPT_H}{k}
\]

and since every set in \( H(a^*) \) is of size at most \( k \) we know that \( \max_{H' \subseteq H} f_S(H' \cup a^*) \leq \OPT \). Together with the fact that \( \OPT \leq 3\OPT_H \) we get:

\[
v_S(H(a^*)) = \frac{\max_{H' \subseteq H} f_S(H' \cup a^*)}{\min_{H' \subseteq H} f_S(H' \cup a^*)} \leq \frac{\OPT}{\OPT_H} \cdot \frac{k}{\epsilon} \leq \frac{3k}{\epsilon}. \quad \square
\]

We can now show (assuming \( \OPT_H \) is sufficiently large) that in \( \epsilon \)-relevant iterations the value of the element which maximizes the noisy smooth value is comparable to that of the (non-noisy) smooth value, with high probability.

\textbf{Claim (3.5).} For a fixed \( \epsilon > 0 \) assume \( t \geq \left( \frac{110k \log n}{\epsilon^2} \right)^8 \), \( \OPT_H \geq \OPT/3 \). For an \( \epsilon \)-relevant iteration of Smooth-Greedy, let \( S \) be the set of elements selected in previous iterations and \( a \in \arg\max_{b \in S \cup H} F(S \cup b) \) be the element selected at that stage. Then, w.p. at least \( 1 - 1/n^4 \):

\[
F_S(a) \geq (1 - \delta) \max_{b \notin S \cup H} F_S(b).
\]
Proof. Let \( a^* \) be the element which maximizes the smooth marginal contribution:

\[
a^* \in \arg \max_{b \notin S \cup H} F_S(a)
\]

We will show that for any element \( b \) whose smooth marginal contribution is more than \((1 - \delta)\) away from the smooth marginal contribution of \( a^* \), the noisy smooth marginal contribution of \( a^* \) must be larger than that of \( b \), w.h.p. That is, for any \( b \notin S \cup H \) for which \( F_S(b) < (1 - \delta)F_S(a^*) \) we get that \( \bar{F}(S \cup b) < \bar{F}(S \cup a^*) \) with probability at least \( \Omega(1 - 1/n^5) \). The result will then follow by taking a union bound over a comparison of \( \bar{F}(a^*) \) with all possible elements whose smooth value is at most \((1 - \delta)F(a^*)\). We will show that \( a^* \) likely beats \( b \) by lower bounding \( \bar{F}(S \cup a^*) \) and upper bounding \( \bar{F}(S \cup b) \) using the smoothing arguments from the previous section.

- **Lower bound on** \( \bar{F}(S \cup a^*) \): First, from Claim 3.4 we know that \( v_S(H(a^*)) \leq 3k/\epsilon \). Together with Lemma 2.4 we get that \( \forall \lambda < 1 \) with probability \( 1 - e^{-\Omega(\lambda^2t^{1/4}/\omega)} \):

\[
\bar{F}(S \cup a^*) > (1 - \lambda)\mu \cdot \left( f(S) + \left( 1 - \frac{6k}{\epsilon} \cdot t^{-1/4} \right) \cdot F_S(a^*) \right)
\]  

(10)

- **Upper bound on** \( \bar{F}(S \cup b) \): Letting \( \beta_{\max} \) denote the set with the highest marginal contribution in \( H(b) \), from Lemma 2.5, we get that \( \forall \lambda < 1 \) with probability \( 1 - e^{-\Omega(\lambda^2t^{1/4}/\omega)} \):

\[
\bar{F}(S \cup b) < (1 + \lambda)\mu \cdot \left( f(S) + F_S(b) + 3t^{-1/4} \beta_{\max} \right)
\]  

(11)

We’ll express this inequality in terms of \( f(S) \) and \( F_S(a^*) \) as well. First, since all sets in \( H(b) \) are of size at most \( k \) we also know that \( \beta_{\max} \leq \text{OPT} \). Thus:

\[
3t^{-1/4} \beta_{\max} \leq 3t^{-1/4} \cdot \text{OPT}
\]  

(12)

We will now bound \( \text{OPT} \) in terms of \( F_S(a^*) \). Since every set in \( H(a^*) \) includes \( a^* \), from monotonicity we get that \( F_S(a^*) \geq f_S(a^*) \). Let \( O_H = \arg\max_{T:|T| \leq k} f_H(T) \) and \( o^* \in \arg\max_{o \in O_H} f_{S \cup H}(o) \). Due to the maximality of \( a^* \) we have that \( F_S(a^*) \geq F_S(o^*) \) and by Claim 3.3 we know that \( F_S(a^*) \geq f_{S \cup H}(o^*) \). Since the iteration is \( \epsilon \)-relevant and \( \text{OPT}_H \geq \text{OPT}/3 \) we get:

\[
F_S(a^*) \geq f_S(a^*) \geq f_{S \cup H}(o^*) \geq \frac{f_{S \cup H}(O_H)}{k} \geq \frac{\epsilon \cdot \text{OPT}_H}{k} \geq \frac{\epsilon \cdot \text{OPT}}{3k}
\]  

(13)

Putting (13) together with (12) we get:

\[
3t^{-1/4} \beta_{\max} \leq \frac{k}{\epsilon} \cdot 9t^{-1/4} \cdot F_S(a^*)
\]

(14)

Plugging into (11) and using the assumption that \( F_S(b) < (1 - \delta)F_S(a^*) \) we get:

\[
\bar{F}(S \cup b) < (1 + \lambda)\mu \cdot \left( f(S) + F_S(b) + \left( 9t^{-1/4} \cdot \frac{k}{\epsilon} \right) F_S(a^*) \right)
\]  

(15)

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Putting (10) together with (15) we get that $\forall \lambda < 1$ with probability at least $1 - 2e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:  

$$\bar{F}(S \cup a^*) - \bar{F}(S \cup b) > \mu \cdot \left( F_S(a^*) \left[ (1 - \lambda) \left( 1 - \frac{6k}{\epsilon} t^{-1/4} \right) - (1 + \lambda) \left( \frac{9k}{\epsilon} t^{-1/4} + (1 - \delta) \right) \right] - 2\lambda f(S) \right)$$

$$\geq \mu \cdot \left( F_S(a^*) \left[ (1 - \lambda) \left( 1 - \frac{6k}{\epsilon} t^{-1/4} \right) - (1 + \lambda) \left( \frac{9k}{\epsilon} t^{-1/4} + (1 - \delta) \right) \right] - 2\lambda \text{OPT} \right)$$

$$> \mu \cdot \left( F_S(a^*) \left[ (1 - \lambda) \left( 1 - \frac{6k}{\epsilon} t^{-1/4} \right) - (1 + \lambda) \left( \frac{9k}{\epsilon} t^{-1/4} + (1 - \delta) \right) \right] - 2\lambda \frac{3k}{\epsilon} F_S(a^*) \right)$$

$$= \mu \cdot \left( F_S(a^*) \left[ (1 - \lambda) \left( 1 - \frac{6k}{\epsilon} t^{-1/4} \right) - (1 + \lambda) \left( \frac{9k}{\epsilon} t^{-1/4} + (1 - \delta) \right) \right] - 2\lambda \frac{3k}{\epsilon} \right)$$

$$= \mu \cdot \left( F_S(a^*) \left[ (1 - \lambda) \left( 1 - \frac{6k}{\epsilon} t^{-1/4} \right) - (1 + \lambda) \left( \frac{9k}{\epsilon} t^{-1/4} + (1 - \delta) \right) \right] - 2\lambda \frac{3k}{\epsilon} \right)$$

Thus, the third condition is satisfied when:

1. $\frac{k}{\epsilon} \cdot 15 t^{-1/4} \leq \frac{\delta}{2}$, and
2. $10\lambda \leq \frac{\delta}{2}$, and
3. $1 - 2 \exp\left( \frac{-\lambda^2 t^{1/4}}{\omega} \right) \in \Omega(1 - 1/n^5)$. 

The first condition holds when $t \geq (30k/\epsilon \delta)^4$; the second condition holds when $\lambda = \epsilon \delta / 20k$. For $\omega = 6 \log n$ and $\lambda = \epsilon \delta / 20k$, the third condition is satisfied when:

$$\frac{(\epsilon \delta)^2 t^{1/4}}{20^2 k^2 \omega} = \frac{(\epsilon \delta)^2 t^{1/4}}{20^2 k^2 6 \log n} \geq 5 \log n$$

rearranging:

$$t \geq 12000^4 \left( \frac{k \log n}{\epsilon \delta} \right)^8$$

Thus, since $t$ in the lemma statement respects:

$$t \geq \left( \frac{110k \log n}{\epsilon \delta} \right)^8 > 12000^4 \left( \frac{k \log n}{\epsilon \delta} \right)^8$$

we have that the first, second, and third conditions are met conditioned on $\omega \leq 6 \log n$. That is, we have that the difference is positive with probability $1 - 2 \exp\left( \frac{-\lambda^2 t^{1/4}}{\omega} \right) \geq 1 - 2/n^5$, conditioned on $\omega \leq 6 \log n$. From lemma 2.2 we know that the probability of $\omega > 6 \log n$ is smaller than $1/n^5$ for sufficiently large $n$. Therefore, by taking a union bound on the probability of the event in which the difference is negative and the probability that $\omega > 6 \log n$, both occurring with probability smaller than $2/n^5$ we have that the probability of the difference being positive is at least $1 - 4/n^5 \in \Omega(1 - 1/n^5)$, as required. 

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Proof of Lemma 3.2. By Claim 3.5, when $\delta = \epsilon^2/4k$ for any fixed $\epsilon > 0$ we need to verify that for sufficiently large $n$:

$$t > \left( \frac{110k \log n}{\epsilon \delta} \right)^8 = \left( \frac{440k^2 \log n}{\epsilon^3} \right)^8$$

In the case where $k \geq \log n$ we use $\ell = 25 \log n$ and thus $t = 2^\ell = n^{25}$ and the above inequality holds. When $k < \log n$ we use $\ell = 33 \log \log n$ and thus $t = \log 33 n$ and the above inequality holds in this case as well. We therefore have the result with probability at least $1 - 1/n^4$.\qed

Approximation guarantee

Claim (3.6). For any $\epsilon > 0$, let $\delta \leq \epsilon^2/4k$. Suppose that the iteration is $\epsilon$-relevant, that $\text{OPT}_H \geq \text{OPT}/3$, and let $o^* \in \arg\max f_{H \cup S}(O)$ where $O \in \arg\max_{T:|T| \leq k'} f_H(T)$. If $F_S(a) \geq (1 - \delta)F_S(o^*)$. Then:

$$f_S(a) \geq (1 - \epsilon)f_{H \cup S}(o^*).$$

Proof. First, we upper bound $F_S(a)$:

$$F_S(a) = \frac{1}{t} \sum_{H' \subseteq H} f_S(H' \cup a) \quad \text{by definition of } F_S$$

$$= \frac{1}{t} \sum_{H' \subseteq H} (f_S(H') + f_{S \cup H'}(a)) \quad \text{by submodularity of } f$$

$$\leq \frac{1}{t} \sum_{H' \subseteq H} (f_S(H') + f_S(a)) \quad \text{by submodularity of } f$$

$$= f_S(a) + \frac{1}{t} \sum_{H' \subseteq H} f_S(H') \quad \text{by submodularity of } f$$

Next, we lower bound $(1 - \delta)F_S(o^*)$:

$$(1 - \delta)F_S(o^*) = (1 - \delta)^2 \frac{1}{t} \sum_{H' \subseteq H} f_S(H' \cup o^*) \quad \text{by definition of } F_S$$

$$= (1 - \delta)^2 \frac{1}{t} \sum_{H' \subseteq H} (f_S(H') + f_{S \cup H'}(o^*)) \quad \text{by submodularity of } f$$

$$\geq (1 - \delta) \frac{1}{t} \sum_{H' \subseteq H} (f_S(H') + f_{S \cup H}(o^*)) \quad \text{by submodularity of } f$$

$$= (1 - \delta)f_S(o^*) - \frac{1}{t} \sum_{H' \subseteq H} f_S(H') + \frac{1}{t} \sum_{H' \subseteq H} f_S(H') \quad t = 2^{|H|}$$

Note that we could have used smaller values of $\ell$ to achieve the desired bound. The reason we exaggerate the values of $\ell$ is to be consistent with the analysis of Slick-Greedy which necessitates these slightly larger values of $\ell$.\footnote{Note that we could have used smaller values of $\ell$ to achieve the desired bound. The reason we exaggerate the values of $\ell$ is to be consistent with the analysis of Slick-Greedy which necessitates these slightly larger values of $\ell$.}
Since $F_S(a) \geq (1 - \delta)F_S(o^*)$ this implies that:

$$f_S(a) \geq (1 - \delta)f_{H \cup S}(o^*) - \frac{1}{t} \sum_{H' \subseteq H} f_S(H')$$

$$\geq (1 - \delta)f_{H \cup S}(o^*) - \frac{1}{t} \sum_{H' \subseteq H} f_S(H) \quad \text{monotonicity of } f$$

$$\geq (1 - \delta)f_{H \cup S}(o^*) - \frac{1}{t} \sum_{H' \subseteq H} f_S(H) \quad t = |H'|$$

$$\geq (1 - \delta)f_{H \cup S}(o^*) - \delta \text{OPT}$$

$$\geq (1 - \delta)f_{H \cup S}(o^*) - 3\delta \text{OPTH}$$

$$\geq (1 - \delta)f_{H \cup S}(o^*) - 3\delta \cdot \frac{k}{\epsilon} f_{H \cup S}(o^*)$$

$$= \left(1 - \delta \left(1 + \frac{3k}{\epsilon}\right)\right) f_{H \cup S}(o^*)$$

$$\geq \left(1 - \delta \left(\frac{4k}{\epsilon}\right)\right) f_{H \cup S}(o^*)$$

$$= (1 - \epsilon/c) f_{H \cup S}(o^*). \quad \delta \leq \epsilon^2/4k$$

**Lemma (3.8).** Let $S$ be the set returned by Smooth-Greedy and $H$ its smoothing set. Then, for any fixed $\epsilon > 0$ when $k \geq 3\ell/\epsilon$ with probability of at least $1 - 1/n^3$ we have that:

$$f(S \cup H) \geq (1 - 1/e - \epsilon/3) \text{OPT}_H.$$  

**Proof.** In case $\text{OPT}_H < \text{OPT}/3$ then $H$ alone provides a $1 - 1/e - \epsilon/3$ approximation. To see this, let $O \in \text{argmax}_{T:|T| \leq k} f(T)$ and $O' \in \text{argmax}_{T:|T| \leq k'} f(T)$, and $O_H \in \text{argmax}_{T:|T| \leq k'} f_H(T).$ We get:

$$(1 - \epsilon/3) f(O) \leq f(O') \quad k' = k - \ell \text{ and } k \geq 3\ell/\epsilon$$

$$\leq f(H \cup O') \quad \text{monotonicity}$$

$$= f(H) + f_H(O')$$

$$\leq f(H) + f_H(O_H) \quad \text{optimality of } O_H$$

$$< f(H) + 1/3 f(O) \quad \text{OPT}_H < 1/3 \text{OPT}$$

Thus:

$$f(H) \geq \left(1 - \frac{1}{3} - \frac{\epsilon}{3}\right) f(O) \geq \left(1 - \frac{1}{3} - \frac{\epsilon}{3}\right) \text{OPT} \geq \left(1 - \frac{1}{3} - \frac{\epsilon}{3}\right) \text{OPT}_H$$

In case $\text{OPT}_H \geq \text{OPT}/3$ we can apply a standard induction argument on Claim 3.7 to show that $S$ alone provides a $1 - 1/e - \epsilon/3$ approximation. We will use the following notation. At every iteration $i \in [k']$ of the while loop in the algorithm, we will use $a_i$ to denote the element that was added in that step, and $S_i := \{a_1, \ldots, a_i\}$. For $\gamma = \epsilon/6$, let $\hat{k} \leq k'$ be the last $\gamma$-relevant iteration, i.e. the last iteration $i$ for which $f_{H \cup S}(O_H) \geq \gamma \cdot \text{OPT}_H$. We can invoke Claim 3.7 whose guarantee holds with probability $1 - 1/n^4$ and take a union bound on all $\hat{k} < n$ iterations. Let $\hat{O} \subseteq O_H$ be the subset of $\hat{k}$ elements whose value (in terms of $f_H$) is largest. Since our smoothing guarantees depend on the assumption that $f_{H \cup S}(O_H) \geq \gamma \cdot \text{OPT}_H$, we will analyze the algorithm against $\hat{O}$ whose total value is $(1 - \gamma)\text{OPT}_H$. Claim 3.7 states that for $\gamma$-relevant iterations, at every stage $i \in [\hat{k}]$:

$$f(S_{i+1}) - f(S_i) \geq (1 - \gamma) \left[\frac{1}{k} \left(f_H(\hat{O}) - f(S_i)\right)\right]. \quad (16)$$
We will show that at every stage $i \in [k]$:

$$f(S_i) \geq (1 - \gamma) \left(1 - \left(1 - \frac{1}{k}\right)^i\right) f_H(\mathcal{O}).$$

The proof is by a standard induction on $i$. For $i = 1$ we have that $S_i = \{a_1\}$ and invoking Claim 3.7 with $S = \emptyset$ we get that $f(a_1) \geq (1 - \gamma) \frac{1}{k} f_H(\mathcal{O})$. Therefore:

$$f(S_1) = f(a_1) \geq (1 - \gamma) \frac{1}{k} f_H(\mathcal{O}) = (1 - \gamma) \left(1 - \left(1 - \frac{1}{k}\right)^1\right) f_H(\mathcal{O}).$$

We can now assume the claim holds for $i = l < k$ and show that it holds for $i = l + 1$:

$$f(S_{l+1}) \geq (1 - \gamma) \left(\frac{1}{k} f_H(\mathcal{O}) - f(S_l)\right) + f(S_l) \quad \text{By (16)}$$

$$\geq (1 - \gamma) \left(\frac{1}{k} f_H(\mathcal{O})\right) + (1 - \gamma) \left(1 - \left(1 - \frac{1}{k}\right)\right) f_H(\mathcal{O}) \quad \text{inductive hypothesis}$$

$$= (1 - \gamma) \left(1 - \left(1 - \frac{1}{k}\right)^{l+1}\right) f_H(\mathcal{O})$$

Note that for any $l > 1$ we have that $(1 - 1/l)^l \leq 1/e$, and thus:

$$f(S) \geq f(S_k) \quad \text{monotonicity}$$

$$\geq (1 - 1/e - \gamma) f_H(\mathcal{O}) \quad \text{by the induction}$$

$$\geq (1 - 1/e - 2\gamma)\text{OPT}_H \quad f(\mathcal{O}) \geq (1 - \delta)\text{OPT}_H$$

$$= (1 - 1/e - \epsilon/3)\text{OPT}_H. \quad \gamma = \epsilon/6$$

Thus, from monotonicity $f(S \cup H) \geq \max\{f(S), f(H)\} \geq (1 - 1/e - \epsilon/3)\text{OPT}$.

\[\square\]

The slick greedy algorithm: optimal approximation for sufficiently large $k$

**Lemma (3.10).** Assume $k \geq 96\ell^2/e^2$. Let $T_i$ be the set that won the Smooth-Compare tournament. Then, with probability at least $1 - 1/n^2$:

$$f(T) \geq (1 - \epsilon/3) \max_{j \in [1/\delta]} f(T_j).$$

The proof of the lemma uses the two claims below.

**Claim B.1.** Let $T_i = S_i \cup H_i$ and $T_j = S_j \cup H_j$ be two sets that are compared by Smooth-Compare, and suppose that $f(T_i) \geq (1 + 2\beta)f(T_j)$ where $\beta = |H_{ij}|/k''$ and $k'' = k - \ell/\delta$. Then, for any set $H'_{ij} \subseteq H_{ij}$ with probability at least $1 - 1/n^3$:

$$f(T_i \cup H'_{ij}) \geq f(T_j \cup H'_{ij}).$$
Proof. Recall that \( H_{ij} \cap (T_i \cup T_j) = \emptyset \). Let \( H'_{ij} \subseteq H_{ij} \). Due to submodularity and the fact that every element in \( H'_{ij} \) was a candidate for selection by the Smooth-Greedy and wasn’t selected, this implies that the marginal contribution of \( H'_{ij} \) to \( S_i \cup H_i \) is bounded from above by \( 2\beta \cdot f(S_i) \) with exponentially high probability. To see this, let \( R_i \) be the set used for initialization and let \( a \in S_i \setminus R_i \) be the last element chosen by the smooth greedy procedure, and \( H'_{ij} \subseteq H_{ij} \). Since \( T_i \cap H_{ij} = \emptyset \) we know that every element \( h \in H'_{ij} \subseteq H_{ij} \) was not chosen by the smooth greedy procedure since it is not in \( S_i \subseteq T_i \), and from Claim 3.3 and Lemma 3.2 we know that with probability at least \( 1 - 1/n^3 \):

\[
f_{S_j \setminus \{a\}}(a) \geq \frac{1}{2} \max_{h \in H} f(S_j \setminus \{a\}) \cup H_j(h)
\]

From the above argument and submodularity, applying a union bound over \( O(\log n) \) elements of \( H_{ij} \) we get that with probability at least \( 1 - 1/n^3 \):

\[
f_{T_j}(H'_{ij}) = f_{S_j \cup H_j}(H'_{ij}) \leq |H'_{ij}| \cdot \max_{h \in H'_{ij}} f(S_j \cup H_j(h)) \leq 2|H'_{ij}| \cdot f_{S_j \setminus \{a\}}(a) \leq 2\beta \cdot f(S_j)\]

Notice that showing this suffices since then we get:

\[
f(T_j \cup H'_{ij}) = f(T_j) + f_{T_j}(H'_{ij}) \leq (1 + 2\beta) f(T_j) < f(T_i) \leq f(T_i \cup H'_{ij}).
\]

Claim B.2. For \( k \geq 96\ell/\epsilon^2 \) suppose that \( f(T_i) \geq (1 + \epsilon\delta/3) f(T_j) \). Then, with probability at least \( 1 - 2/n^3 \) we have that \( f(T_i) \) wins in the smooth comparison procedure.

Proof. Let \( \beta = |H_{ij}|/k'' \) where \( k'' = k - (|H_{ij}|+|R_i|) \). Since we assume that \( k \geq 96\ell \) and \( \delta = \epsilon/6 \) this implies that \( 2\beta < \epsilon^2/45 \). We therefore have:

\[
f(T_i) > \left(1 + \frac{\epsilon\delta}{3}\right) f(T_j) \geq \left(1 + \frac{\epsilon^2}{18}\right) f(T_j) > \left(1 + \frac{\epsilon^2}{45}\right)^2 f(T_j) > (1 + 2\beta)^2 f(T_j)
\]

From Claim B.1 this implies that for any \( H'_{ij} \subseteq H_{ij} \) we have that with probability at least \( 1 - 1/n^3 \):

\[
f(T_j \cup H'_{ij}) \leq (1 + 2\beta) f(T_j \cup H'_{ij})
\]

We will condition on this event as well as the event that the maximal value obtained throughout the iterations of the algorithm is \( \nu_{\max} \) and minimal value is \( \nu_{\min} \), and that \( \nu_{\max}/\nu_{\min} \leq n^\tau \) for some
constant $\tau > 0$.

\[
\Pr \left[ \tilde{f}(T_i \cup H'_{ij}) \geq \tilde{f}(T_j \cup H'_{ij}) \bigg| f(T_i) \geq \left( 1 + \frac{\epsilon \delta}{3} \right) f(T_j) \right] = \Pr \left[ \xi_i f(T_i \cup H'_{ij}) \geq \xi_j f(T_j \cup H'_{ij}) \bigg| f(T_i) \geq \left( 1 + \frac{\epsilon \delta}{3} \right) f(T_j) \right] > \Pr \left[ (1 + 2\beta) \cdot \frac{\xi_i}{\xi_j} \geq 1 \right] \geq \frac{1}{2} + \frac{1}{2 \log_{1+2\beta}(\frac{\nu_{\max}}{\nu_{\min}})}
\]

The last inequality follows from a discretization argument: Consider the $m \in O(\log n)$ intervals, where the $i$'th interval is $[\nu_{\min}(1+2\beta)^i, \nu_{\min}(1+2\beta)^{i+1}]$, and $i$ ranges from 0 to $\log_{1+2\beta}(\frac{\nu_{\max}}{\nu_{\min}})$. Due to symmetry of $\xi_i$ and $\xi_j$, the likelihood of $\xi_j$ falling in the same or higher interval than $\xi_j$ is:

\[
\sum_{i=1}^{m} \frac{i}{m^2} = \frac{1}{2} + \frac{1}{2m} = \frac{1}{2} + \frac{1}{2 \log_{1+2\beta}(\frac{\nu_{\max}}{\nu_{\min}})} = \frac{1}{2} + \frac{1}{2 \tau \log_{1+2\beta} n}
\]

Applying a Chernoff bound, for any constants $\epsilon, \delta > 0$, s.t. $\epsilon \delta / 8 > 1 + 2\beta$, and $\nu_{\max} / \nu_{\min} \leq n^\tau$ for some constant $\tau > 0$, we get that $T_i$ is chosen with probability at least $1 - \exp(-\Omega(n/\log(n)))$, conditioned on $\nu_{\max} / \nu_{\min} < n^\tau$ which by Lemma 2.2 occurs with probability $1 - \exp(-\Omega(n^\alpha))$ for some constant $\alpha > 0$. For sufficiently large $n$, $T_i$ therefore wins with probability at least $1 - 1/n^3$.

**Proof of Lemma 3.10.** Since Smooth-Compare is called to make $1/\delta$ comparisons, and we proved that $\forall i, j \in [1/\delta]$ the call Smooth-Compare($\{T_i, T_j\}, H_{ij}$) returns $T_i$ as long as $f(T_i) \geq (1 - \epsilon \delta / 3) f(T_j)$. We get:

\[
f(T_i) \geq (1 - \epsilon \delta / 3)^{1/\delta} \max_{j \in [1/\delta]} f(T_j) \geq (1 - \epsilon / 3) \max_{j \in [1/\delta]} f(T_j).
\]

**Approximation guarantee for slick greedy**

**Lemma (3.12).** Let $S$ be the set returned by Smooth-Greedy that is initialized with a set $R \subseteq N$ and has $H$ as its smoothing set of size $\ell$, which is disjoint from $R$ and $S$. Then, for any fixed $\epsilon > 0$ when $k \geq \max \{ 3|H \cup R| / \epsilon, \ell \}$ with probability of at least $1 - 1/n^3$ we have that:

\[
f(R \cup S \cup H) \geq (1 - 1/e - \epsilon / 3) OPT_{H,R}.
\]

The proof follows by defining the monotone submodular function $g(S) = f_R(S)$ and applying the same arguments we have on $f$ in the previous section. In this case we define the value of the optimal solution to be $OPT[g] = \max_{T:|T| \leq k''} g(T)$ where $k'' = k - \max \{ 3|H \cup R| / \epsilon, \ell \}$. We can then also define $O := \arg \max_{T:|T| \leq k''} g_H(T)$, and $OPT[g]_H = g_H(O)$. An $\epsilon$-relevant iteration is one in which $g_{H \cup S}(O) \geq \epsilon OPT[g]_H$.

We will first show that for every element $a$ selected in the smooth greedy procedure (after the initialization of $R$) we have that $g_S(a) \geq (1 - \gamma) g_{H \cup H}(a^*)$ where $a^*$ is the element from the optimal solution on $g$ whose marginal contribution is highest and $\gamma > 0$ is any small constant. This will simply be a consequence of the smoothing arguments from the previous subsection. We will then show that this allows us to show that $g(S \cup H) \geq (1 - 1/e - \epsilon / 3)OPT[g]$. This, as we will soon show, suffices to prove the lemma. First, we show that $g_S(a) \geq (1 - \gamma) g_{S \cup H}(a^*)$. 63
Claim B.3. For a given set \( R \subset N \), let \( g(T) = f_R(T) \). For any fixed \( \gamma > 0 \) consider a \( \gamma \)-relevant iteration of Smooth-Greedy initialized with some set \( R \) using smoothing set \( H \neq R \), assume \( \text{OPT}[G]_H \geq \text{OPT}[G]/3 \) and let \( S \) be the set of elements selected before the iteration. If \( a \in \arg\max_b F(R \cup S \cup b) \) then with probability at least \( 1 - 1/n^4 \):

\[
g_S(a) \geq (1 - \gamma)g_{H \cup S}(o^*)
\]

Proof. Let \( G \) denote the smooth value function of \( g \), i.e. \( G(S \cup a) = \frac{1}{t} \sum_{H' \subset H} g(S \cup H' \cup a) \), where \( g(S) = f_R(S) \). The proof follows four simple steps. The arguments of each step are either direct applications of previous lemmas, or use the same analysis. Let \( \delta = \gamma^2/4k \) and \( \alpha = \gamma\delta/3k \). We show:

1. \( \tilde{F}(R \cup S \cup a) \geq \tilde{F}(R \cup S \cup o^*) \implies F_{R \cup S}(a) \geq (1 - \alpha)F_{R \cup S}(o^*) \)
2. \( F_{R \cup S}(a) \geq (1 - \alpha)F_{R \cup S}(o^*) \implies G(S \cup a) \geq (1 - \alpha)G(S \cup o^*) \)
3. \( G(S \cup a) \geq (1 - \alpha)G(S \cup o^*) \implies G_S(a) \geq (1 - \delta)G_S(o^*) \)
4. \( G_S(a) \geq (1 - \delta)G_S(o^*) \implies g_S(a) \geq (1 - \gamma)g_{H \cup S}(o^*) \)

The above arguments can be justified as follows:

1. To see that \( \tilde{F}(R \cup T \cup a) \geq \tilde{F}(R \cup T \cup o^*) \) implies \( F_{R \cup T}(a) \geq (1 - \alpha)F_{R \cup T}(o^*) \), we can invoke Claim 3.5 on \( S = R \cup T \). To do so, since \( \alpha = \gamma^3/24k^2 \) for sufficiently large \( n \) we need to verify:

\[
t > \left( \frac{110k \log n}{\gamma \alpha} \right)^8 = \left( \frac{2640k^3 \log n}{\gamma^3} \right)^8
\]

In the case where \( k \geq 2400 \log n \) we use \( \ell = 25 \log n \) and thus \( t = 2^\ell = n^{25} \) and the above inequality holds. When \( k < 2400 \log n \) we use \( \ell = 33 \log n \) and thus \( t = \log n^{33} \) and the above inequality holds in this case as well. We therefore have the result with probability at least \( 1 - 1/n^4 \).

2. Assuming that \( F_{R \cup S}(a) \geq (1 - \alpha)F_{R \cup S}(o^*) \) we will show that \( G(S \cup a) \geq (1 - \alpha)G(S \cup o^*) \):

\[
F_{R \cup S}(a) \geq (1 - \alpha)F_{R \cup S}(o^*) \implies \sum_{H' \subset H} f_{R \cup S}(H' \cup a) \geq (1 - \alpha) \sum_{H' \subset H} f_{R \cup S}(H' \cup o^*) \implies \sum_{H' \subset H} (f(R \cup S \cup H' \cup a) - f(R \cup S)) \geq (1 - \alpha) \sum_{H' \subset H} (f(R \cup S \cup H' \cup o^*) - f(R \cup S)) \implies \sum_{H' \subset H} (f(R \cup S \cup H' \cup a) - f(R)) \geq (1 - \alpha) \sum_{H' \subset H} (f(R \cup S \cup H' \cup o^*) - f(R)) \implies \sum_{H' \subset H} f_R(S \cup H' \cup a) \geq (1 - \alpha) \sum_{H' \subset H} f_R(S \cup H' \cup o^*) \implies \sum_{H' \subset H} g(S \cup H' \cup a) \geq (1 - \alpha) \sum_{H' \subset H} g(S \cup H' \cup o^*) \implies G(S \cup a) \geq (1 - \alpha)G(S \cup o^*)
\]
3. \( G(S \cup a) \geq (1 - \alpha)G(S \cup o^*) \implies G_S(a) \geq (1 - \delta)G_S(o^*) \): We first argue \( G_S(o^*) \geq \frac{\gamma \cdot \text{OPT}[G]}{3k''} \):

\[
G_S(o^*) = \frac{1}{t} \sum_{H' \subseteq H} (g(S \cup o^* \cup H') - g(S)) \geq \frac{1}{t} \sum_{H' \subseteq H} (g(S \cup o^* \cup H') - g(S \cup H')) \quad \text{monotonicity of } g
\]

\[
\geq \frac{1}{t} \sum_{H' \subseteq H} (g(S \cup o^* \cup H) - g(S \cup H)) \quad \text{submodularity of } g
\]

\[
= g(S \cup o^* \cup H) - g(S \cup H) = g_S(a)
\]

\[
\geq \frac{\gamma}{k''} \cdot \text{OPT}[G]_H \quad \gamma\text{-relevant iteration}
\]

\[
\frac{1}{t} \sum_{H' \subseteq H} (g(S \cup o^* \cup H') - g(S \cup H')) \geq \frac{1}{t} \sum_{H' \subseteq H} (g(S \cup o^* \cup H) - g(S \cup H)) = g(S \cup o^* \cup H) - g(S \cup H) \geq \frac{\gamma}{k''} \cdot \text{OPT}[G]
\]

Now, in a similar fashion to Claim 3.6:

\[
G_S(a) = \sum_{H' \subseteq H} g_S(a \cup H') \geq \sum_{H' \subseteq H} (g(S \cup a \cup H') - g(S)) = G(S \cup a) - G(S) \geq (1 - \alpha) (G(S \cup o^*) - G(S)) - \alpha G(S) \geq (1 - \alpha) (G(S \cup o^*) - G(S)) - \alpha \text{OPT}[G]
\]

\[
\geq (1 - \alpha) (G(S \cup o^*) - G(S)) - \alpha \frac{3k''}{\gamma} \cdot G_S(o^*) \quad \text{previous inequality}
\]

\[
= \left(1 - \alpha \left(1 - \frac{3k''}{\gamma}\right)\right) G_S(o^*) \quad \alpha = \frac{\epsilon \delta}{3k} \text{ and } k \geq k'' + 1
\]

4. \( G_S(a) \geq (1 - \delta)G_S(o^*) \implies g_S(a) \geq (1 - \gamma)g_{H \cup S}(o^*) \): by direct application of Claim 3.6

**Proof of Lemma 3.12.** From Claim B.3 we have that \( g_S(a) \geq (1 - \gamma)g_{S \cup H}(o^*) \) we can then apply Lemma 3.8 on \( g \) exactly in the same manner that we have on \( f \). By Lemma 3.8 the fact that \( g_S(a) \geq (1 - \gamma)g_{S \cup H}(o^*) \) and \( \gamma = \epsilon/6 \) implies:

\[
g(S \cup H) \geq (1 - 1/e - \epsilon/3) \text{OPT}[G]_H
\]
But this implies our claim:

\[
\begin{align*}
f(R \cup S \cup H) - f(R) &= f_R(S \cup H) \\
&= g(S \cup H) \\
&\geq (1 - 1/e - \epsilon)g_H(O) \\
&\geq (1 - 1/e - \epsilon)(g(O \cup H) - g(H)) \\
&\geq (1 - 1/e - \epsilon)(f_R(O \cup H) - f_R(H)) \\
&\geq (1 - 1/e - \epsilon)(f(R \cup O \cup H) - f(R) - f_R(H)) \\
&\geq (1 - 1/e - \epsilon)(f(R \cup O \cup H) - f_R(H)) - (1 - 1/e - \epsilon)f(R)
\end{align*}
\]

and we therefore get that \( f(R \cup S \cup H) \geq (1 - 1/e - \epsilon)(f(R \cup O \cup H) - f_R(H)) \).
C  Missing Proofs: Optimization for Small $k$

Smoothing guarantees

For convenience we restate Lemma 4.2 since it is used in the following proofs. The proof of
Lemma 4.2 is in the main body of the paper.

Lemma (4.2). For a fixed set $S \subset N$, let $A^* \in \arg\max_{|A| = c} f_S(A)$. Then:

$$\left(1 - \frac{1}{c}\right) f_S(A^*) \leq \phi_S(A^*) \leq f_S(A^*)$$

Claim (4.3). Let $A^* \in \arg\max_{B : |B| = c} f_S(B)$, $c \geq 4/\epsilon$. Then:

$$\frac{1}{c} \sum_{i=1}^{c} \max \left\{0, 1 - 2v_S(H(A^*_{-i})) \cdot t^{-1/4}\right\} F_S(A^*_{-i}) \geq (1 - \epsilon) f_S(A^*)$$

Proof. To bound the average variation of the sets $\{A^*_{-i}\}_{i=1}^{c}$ we argue that at most one set $A^*_{-i}$ will
be s.t. $f_S(A^*_{-i}) < f_S(A^*)/2$. To see this, assume for purpose of contradiction there are $A^*_{-i}$ and $A^*_{-j}$ for which
$f_S(A^*_{-i}) \leq f_S(A^*_{-j}) < f_S(A^*)/2$, then since $A^* = A^*_{-i} \cup A^*_{-j}$ we get a contradiction:

$$f_S(A^*) = f_S(A^*_{-i} \cup A^*_{-j}) \leq f_S(A^*_{-i}) + f_S(A^*_{-j}) < 2 \cdot \frac{f_S(A^*)}{2} = f_S(A^*)$$

We therefore have at least $c - 1$ sets such that each $A^*_{-i}$ in this set respects $f_S(A^*_{-i}) \geq f_S(A^*)/2$. Call these sets bounded. For any such bounded set $A^*_{-i}$, since $A^*_{-i} \subset A^*_j$ for any $j \in N \setminus (S \cup A^*)$, monotonicity implies:

$$\min_{A^*_j \in H(A^*_i)} f_S(A^*_j) \geq \frac{f_S(A^*)}{2}$$

For a given set $A^*_{-i}$, note that for every $j$, every set $A_{ij} \in H(A^*_i)$ respects $f_S(A^*_{ij}) \leq f_S(A^*)$ due to
the maximality of $A^*$. Thus for any bounded set $A^*_j$:

$$v_S(H(A^*_i)) = \frac{\max_{A^*_j \in H(A^*_i)} f_S(A^*_{ij})}{\min_{A^*_j \in H(A^*_i)} f_S(A^*_{ij})} \leq \frac{f_S(A^*)}{f_S(A^*)/2} = 2$$

Let $l$ be the index of the set $A^*_{-i}$ with the lowest value $f_S(A^*_{-i})$. Our discussion above implies that
this is the only set whose variation may not be bounded from above by 2. Assume $n$ sufficiently
large s.t. $t \geq 2^{12}/\epsilon^4$. We therefore get:
\[
\frac{1}{c} \sum_{i=1}^{c} \left( \max \{0, 1 - 2vS(\mathcal{H}(A_{\omega,i}))t^{-\frac{1}{2}} \} \right) F_S(A_{\omega,i}^*) \geq \frac{1}{c} \sum_{i \neq l} \left( \max \{0, 1 - 2vS(\mathcal{H}(A_{\omega,i}))t^{-\frac{1}{2}} \} \right) F_S(A_{\omega,l}^*)
\]

(18)

\[
\geq \frac{1}{c} \sum_{i \neq l} \left( 1 - 4t^{-\frac{1}{2}} \right) F_S(A_{\omega,i}^*)
\]

(19)

\[
\geq \frac{1}{c} \sum_{i \neq l} \left( 1 - 4t^{-\frac{1}{2}} \right) f_s(A_{\omega,i}^*)
\]

(20)

\[
\geq \left( 1 - 4t^{-\frac{1}{2}} \right) \frac{1}{c} \left( \sum_{i=1}^{c} f_s(A_{\omega,i}^*) - f_s(A_{\omega,l}^*) \right)
\]

(21)

\[
\geq \left( 1 - 4t^{-\frac{1}{2}} \right) \frac{1}{c} \left( (c-1)f_s(A^*) - f_s(A_{\omega,l}^*) \right)
\]

(22)

\[
\geq \left( 1 - 4t^{-\frac{1}{2}} \right) \frac{1}{c} \left( (c-1)f_s(A^*) - f_s(A^*) \right)
\]

(23)

\[
\geq \left( 1 - 4t^{-\frac{1}{2}} \right) \left( \frac{c-2}{c} \right) f_s(A^*)
\]

(24)

\[
\geq \left( \frac{c-2}{c} - 4t^{-\frac{1}{2}} \right) f_s(A^*)
\]

(25)

\[
\geq (1 - \epsilon) f_s(A^*)
\]

(26)

The inequality (19) is justified by the bound we established on bounded sets; (20) is due to monotonicity of \( f_S \), since \( F_S(A_{\omega,i}^*) \) is an average of the marginal contribution over all possible \( A_{\omega,i}^* \), which is a superset of \( A_{\omega,i}^* \); (22) is due to an argument in the proof of Lemma 4.2; (23) is due to the optimality of \( A^* \); (26) is due to the assumption on the parameters in the statement of the claim. \( \square \)

**Lemma (4.5).** Let \( A \in \text{argmax}_{B:|B|=c} \phi(S \cup B) \) where \( c \geq 16/\epsilon \), and assume that the iteration is \( \epsilon/4 \)-significant. Then, with probability at least \( 1 - \epsilon e^{-\Omega(t^{1/10})} \) we have that:

\[
\phi_S(A) \geq (1 - \epsilon) \max_{B:|B|=c} \phi_S(B)
\]

**Proof.** Let \( A^* = \text{argmax}_{A:|A|=c} f_S(A) \) and let \( B : |B| = c \) be such that \( \phi_S(B) < (1 - \epsilon)\phi_S(A^*) \). Similar the proof from the previous section we will apply the smoothing arguments and show that with high probability \( \hat{\phi}(S \cup A^*) \geq \hat{\phi}(S \cup B) \). By taking a union bound over all possible \( O(n^c) \) sets \( B \) we will then conclude that the set whose smooth noisy contribution is largest must have smooth contribution at least factor \( (1 - \epsilon) \) from that of \( A^* \), with high probability.

We will denote \( \epsilon_1 = \epsilon \) and \( \epsilon_2 = \epsilon/4 \). Notice that the conditions of Claim 4.3 are met with \( \epsilon_2 \) and that the iteration is \( \epsilon_2 \)-significant, which from submodularity implies \( f_S(A^*) \geq \epsilon_2 \cdot f(S)/k \).

For a set \( B_i \in B \), using Lemma 2.5, for \( t = n - c - |S| \), when \( \omega \) denotes the highest realized value
of a noise multiplier, we know that for $\lambda \in [0, 1)$ with probability $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{\phi}(S \cup B) = \frac{1}{c} \sum_{i} \tilde{F}(S \cup B_{-i})$$

$$< \frac{1}{c} \sum_{i} (1 + \lambda) \mu \cdot \left( f(S) + F_S(B_{-i}) + 3t^{-1/4} \max_{B_{ij} \in \mathcal{H}(B_{-i})} f_S(B_{ij}) \right)$$

$$\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} \max_{B_{ij} \in \bigcup_{i \in [t]} \mathcal{H}(B_{-i})} f_S(B_{ij}) + \frac{1}{c} \sum_{i=1}^{c} F_S(B_{-i}) \right)$$

$$\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + \frac{1}{c} \sum_{i=1}^{c} F_S(B_{-i}) \right)$$

$$\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + \phi(S \cup B) \right)$$

$$\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + (1 - \epsilon_1) \phi(S \cup A^*) \right)$$

$$\leq (1 + \lambda) \mu \cdot \left( f(S) + 3t^{-1/4} f_S(A^*) + (1 - \epsilon_1) f_S(A^*) \right)$$

$$= (1 + \lambda) \mu \cdot \left( f(S) + f_S(A^*) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) \right)$$

We now need to argue that $\phi(S \cup A^*)$ is sufficiently large to beat $\phi(S \cup B)$. Assuming $n$ is sufficiently large s.t. $t \geq 2^{20}/\epsilon^4$, from lemma 2.4 and 4.3 we know that for $\lambda \in [0, 1)$ w.p. $1 - e^{-\Omega(\lambda^2 t^{1/4}/\omega)}$:

$$\tilde{\phi}(S \cup A^*) = \frac{1}{c} \sum_{i=1}^{c} \tilde{F}(S \cup A^*)$$

$$> (1 - \lambda) \mu \cdot \left( f(S) + \frac{1}{c} \sum_{i=1}^{c} \left( 1 - 2\nu(A^*) \right) \cdot t^{-1/4} \right) \cdot f_S(A^*)$$

$$> (1 - \lambda) \mu \cdot (f(S) + (1 - \epsilon_2) f_S(A^*))$$

We therefore get that:

$$\tilde{\phi}(S \cup A^*) - \tilde{\phi}(S \cup B) \geq \mu \left( (1 - \lambda) \cdot (f(S) + (1 - \epsilon_2) f_S(A^*)) - (1 + \lambda) \cdot \left( f(S) + f_S(A^*) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) \right) \right)$$

$$\geq \mu \left( (1 - \lambda)(1 - \epsilon_2) f_S(A^*) - 2\lambda f(S) - (1 + \lambda) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) f_S(A^*) \right)$$

$$\geq \mu \left( (1 - \lambda)(1 - \epsilon_2) f_S(A^*) - \frac{2\lambda k}{\epsilon_2} f_S(A^*) - (1 + \lambda) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) f_S(A^*) \right)$$

$$\geq \mu \cdot f_S(A^*) \left( (1 - \lambda)(1 - \epsilon_2) - \frac{2\lambda k}{\epsilon_2} - (1 + \lambda) \left( 3t^{-1/4} + (1 - \epsilon_1) \right) \right)$$

$$\geq \mu \cdot f_S(A^*) \left( (1 - \lambda)(1 - \epsilon_2) - \frac{2\lambda k}{\epsilon_2} - (1 + \lambda) (\epsilon_4 + (1 - \epsilon_1)) \right)$$

$$\geq \mu \cdot f_S(A^*) \left( 1 - \lambda - \epsilon_2 - \frac{2\lambda k}{\epsilon_2} - \epsilon_2 - \lambda \epsilon_2 - 1 - \lambda + \epsilon_1 \right)$$

$$> \mu \cdot f_S(A^*) \left( \epsilon_1 - 3\epsilon_2 - \lambda \left( \frac{2k}{\epsilon_2} \right) \right)$$

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For any $\lambda \leq e^2/2k$ the difference above is strictly positive. Conditioning on $\omega$ being bounded from above by $t^{1/5}$ which happens with probability $1 - e^{-\Omega(t^{1/5}/\log t)}$, since $k \in O(\log \log n)$ we that the result holds with probability at least $1 - e^{-\Omega(t^{1/10})}$.

### Approximation guarantee

**Lemma (4.6).** Assume that at every $\delta/4$-significant iteration of SM-GREEDY when the set selected at previous iterations is $S$ and the set selected is $A$ we have that: $f_S(A) \geq (1 - \delta) \max_{A:|A|=c} \phi_S(A)$, for $\delta > 0$. Assume that $k > c/\epsilon$, $c \geq 16/\delta$. Let $S$ be the set of elements selected in all the iterations of the algorithm SM-GREEDY. Then, with probability $\geq 1 - 1/n^2$:

$$f(S) = (1 - 1/e - 5\delta)OPT$$

**Proof.** We condition on the high probability event in Lemma 4.5 and that every one of the $k$ iterations produces $f_S(A) \geq (1 - \delta) \max_{A:|A|=c} \phi_S(A)$. Since there are less than $k$ iterations in which an set of elements $A$ is added to the solution, by a union bound the result will hold with probability at least $1 - 1/n^2$. We assume that $n$ is sufficiently large s.t. $t \geq 2^{20}/\delta^4$.

First, we rely on the smoothing argument which assumes that the iterations are $\delta$-significant. Therefore, we will compare against $(1 - \delta)$ of the optimal value; let $k$ be the last $\delta$-significant iteration and $O$ be the subset of size $k$ of the optimal solution whose value is largest. By submodularity:

$$f(O) \geq (1 - \delta)OPT$$  \hspace{1cm} (27)

Second, we argue that optimizing over sets of size $c$ rather than singletons is inconsequential when $k > c/\epsilon$. To be convinced, notice that when the algorithm selects $c$ elements in every iteration the total number of elements selected will be $k' > k - c$. Let $O' \in \arg \max_{T:|T|\leq k'} f(T)$. As in previous arguments, from submodularity we have that: $(1 - c/k)f(O) \leq f(O')$. Since $k > c/\epsilon$ we have that:

$$f(O') > (1 - \delta)f(O) > (1 - 2\delta)OPT$$  \hspace{1cm} (28)

We will henceforth analyze the algorithm against $O'$. In a similar manner to the analysis of the greedy algorithm which selects singletons at every stage $i \in [k]$, we can analyze the greedy algorithm which selects sets of size $c$ at every stage $i \in [k'/c]$. To ease notation assume $|k'/c| = k'/c$.

For a given stage of the algorithm, assume the set $S$ has been previously selected and that a set $A$ is being added into the solution. Let $O^* = \arg \max_{T\subseteq O':|O|=c} f_S(O')$ and $A^* = \arg \max_{B:|B|=c} f_S(B)$.

$$f_S(A) \geq (1 - \delta) \phi_S(A)$$  \hspace{1cm} assumption in the statement

$$> (1 - 2\delta) \phi_S(A^*)$$  \hspace{1cm} Lemma 4.5 applied with $\epsilon = \delta$

$$> (1 - 3\delta) f_S(A^*)$$  \hspace{1cm} Lemma 4.2 and $c \geq 1/\delta$

$$> (1 - 3\delta) f_S(O^*)$$  \hspace{1cm} maximality of $A^*$

$$> (1 - 3\delta) \frac{c}{k'} \cdot f_S(O')$$  \hspace{1cm} subadditivity.

$$= (1 - 3\delta) \frac{c}{k'} \cdot (f(O' \cup S) - f(S))$$

$$\geq (1 - 3\delta) \frac{c}{k'} \cdot (f(O') - f(S))$$

An inductive argument similar to that in proof of Lemma 3.8 from the previous section can show $f(S) \geq (1 - 1/e - 3\delta) f(O')$. Since we lose $2\delta$ from (28) this concludes our proof.
D Missing Proofs: Optimization for Very Small $k$

Lemma D.1. Let $A \in \arg\max_{B:|B|=k} \tilde{F}(B)$. Then, for any fixed $\epsilon > 0$ w.p. $1 - e^{-\Omega(\epsilon^2(n-k))}$:

$$F(A) \geq (1 - \epsilon) \max_{B:|B|=k} F(B)$$

Proof. The proof follows the same arguments as the ones from the previous sections. Let $A^* = \arg\max_{B:|B|=k} F(B)$. We will show that w.h.p. no set $B$ for which $F(B) < (1 - \epsilon)F(A^*)$ beats $A$. The size of the smoothing set is $t = n - k$, and $\omega$ is an upper bound on the value of the noise multiplier.

Note that the optimality of $A^*$ and submodularity imply that $f(A^* \cup x) \leq 2f(A^*)$, for all $x \in N \setminus A^*$. Hence from monotonicity the variation is bounded by 2:

$$v(A^*) = \max_{x \in N \setminus A} f(A^* \cup x) - \min_{x \in N \setminus A} f(A^* \cup x) \leq 2$$

We can therefore apply Lemma 2.5 and get that with probability at least $1 - e^{\Omega(\lambda^2t^4/\omega)}$:

$$\tilde{F}(A^*) \geq (1 - \lambda)\mu \left(1 - 4t^{-1/4}F(A^*)\right)$$

To upper bound $\tilde{F}(B)$ for a set $B$ s.t. $F(B) < (1 - \epsilon)F(A^*)$, note that the value of largest set in the smoothing neighborhood is $\max_{x \in N \setminus B} f(B \cup x) \leq 2f(A^*)$. Hence, from Lemma 2.4 we get that with probability at least $1 - e^{\Omega(\lambda^2t^4/\omega)}$:

$$F(B) \leq (1 + \lambda)\mu \left(F(B) + 6t^{-1/4}F(A^*)\right)$$

Therefore when $n$ is sufficiently large s.t. $t^{-1/4} \leq \epsilon/100$ and $\lambda < 1$ we get that:

$$F(A^*) - F(B) \geq (1 - \lambda)\mu(1 - 4t^{-1/4})F(A^*) - (1 + \lambda)\mu \left(F(B) + 6t^{-1/4}F(A^*)\right)$$

$$\geq \mu \left((1 - \lambda)(1 - \frac{4\epsilon}{100})F(A^*) - (1 + \lambda)(1 - \epsilon)F(A^*) - (1 + \lambda)\frac{6\epsilon}{100}F(A^*)\right)$$

$$\geq \mu \left((1 - \lambda)(1 - \frac{4\epsilon}{100})F(A^*) - (1 + \lambda)(1 - \epsilon)F(A^*) - (1 + \lambda)\frac{6\epsilon}{100}F(A^*)\right)$$

$$> \mu \cdot F(A^*) (\epsilon - 2\lambda - \epsilon/5)$$

Using $\lambda < \epsilon/10$ the above inequality is strictly positive. Conditioning on the event of $\omega$ being sufficiently small completes the proof. \qed