ONE-DIMENSIONAL SYMMETRY OF POSITIVE BOUNDED SOLUTIONS TO THE QUADRATIC NONLINEAR SCHRÖDINGER EQUATION IN THE HALF-SPACE UP TO DIMENSION 5

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ABSTRACT. We are concerned with the half-space Dirichlet problem

\[
\begin{aligned}
-\Delta v + v &= |v|^{p-1}v \quad \text{in } \mathbb{R}_+^N, \\
v &= c \quad \text{on } \partial \mathbb{R}_+^N, \\
\lim_{x_N \to \infty} v(x', x_N) &= 0 \text{ uniformly in } x' \in \mathbb{R}^{N-1},
\end{aligned}
\]

where \(\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}\) for some \(N \geq 2\), and \(p > 1, c > 0\) are constants. It was shown recently by Fernandez and Weth [Math. Ann. (2021)] that there exists an explicit number \(c_p \in (1, \sqrt{e})\), depending only on \(p\), such that for \(0 < c < c_p\) there are infinitely many bounded positive solutions, whereas, for \(c > c_p\) there are no bounded positive solutions. They also posed as an interesting open question whether the one-dimensional solution is the unique bounded positive solution in the case where \(c = c_p\). If \(N = 2, 3\), we recently showed this one-dimensional symmetry property in [Partial Differ. Equ. Appl. (2021)] by adapting some ideas from the proof of De Giorgi’s conjecture in low dimensions. Here, we focus on the case \(p = 2\) and prove this uniqueness property in dimensions \(2 \leq N \leq 5\). Our approach is completely different and relies on showing that a suitable auxiliary function, inspired by a Lyapunov-Schmidt type decomposition of the solution, is a nonnegative super-solution to the quadratic Lane-Emden-Fowler equation in \(\mathbb{R}^{N-1}\), for which classical Liouville type results are available.

1. Introduction

Recently in [8], the authors studied the half-space Dirichlet problem

\[
\begin{aligned}
-\Delta v + v &= |v|^{p-1}v \quad \text{in } \mathbb{R}_+^N, \\
v &= c \quad \text{on } \partial \mathbb{R}_+^N, \\
\lim_{x_N \to \infty} v(x', x_N) &= 0 \text{ uniformly in } x' \in \mathbb{R}^{N-1},
\end{aligned}
\]

where \(\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}\) for some \(N \geq 1\), and \(p > 1, c > 0\) are constants. We note that \(u(x, t) = e^{it}v(x)\) is a standing wave solution to the focusing nonlinear Schrödinger equation with the odd power nonlinearity and exponent \(p\). Let us clarify here that throughout this paper solutions will be understood in the classical sense (i.e. at least of class \(C^2\) and continuous up to the boundary).

If \(N = 1\), then the corresponding ODE has a unique positive even solution that decays to zero at infinity, it is given explicitly by the following formula

\[
t \to w_0(t) = c_p \left[ \cosh \left( \frac{p - 1}{2} t \right) \right]^{-\frac{p}{p-1}} \quad \text{with } c_p = \left( \frac{p + 1}{2} \right)^{\frac{1}{p-1}} = w_0(0) = \sup_{t \in \mathbb{R}} w_0(t). \tag{1.2}
\]

Still for \(N = 1\), it was shown in the aforementioned reference that if \(0 < c < c_p\) then (1.1) possesses exactly two positive solutions given by

\[
t \to w_0(t + t_{c,p}) \quad \text{and} \quad t \to w_0(t - t_{c,p})
\]
with
\[ t_{c,p} = \frac{2}{p-1} \ln \left( \sqrt{\frac{p+1}{2e^{c-1}}} + \sqrt{\frac{p+1}{2e^{c-1}} - 1} \right); \]
if \( c = c_p \) then \( w_0 \) is the unique positive solution; if \( c > c_p \) then there are no positive solutions. We note in passing that the above solutions play an important role in a class of boundary layer problems (see [12]).

If \( N \geq 2, \ p > 1, \) and \( 0 < c < c_p, \) using variational methods, it was shown in the same reference that (1.1) admits at least three positive bounded solutions that are geometrically distinct in the sense that they are not translates of each other in the \( x' \) direction. In particular, under the further restriction that \( p+1 \) is smaller than the critical Sobolev exponent in \( \mathbb{R}^N, \ N \geq 2, \) then (1.1) admits a positive bounded solution of the form
\[ x \to w_0(x_N + t_{c,p}) + u(x) \text{ with } u \in H^1_0(\mathbb{R}^N_+) \setminus \{0\} \text{ nonnegative.} \]

On the other hand, if \( c > c_p, \ p > 1, \) it was shown therein that (1.1) has no bounded positive solutions. This was accomplished by means of the famous sliding method [2].

Still in the same reference, it was posed as an interesting open question whether the function \( x \to w_0(x_N) \) is the unique bounded positive solution to (1.1) in the case \( c = c_p. \) In this regard, as we explained in [14], we note that the aforementioned sliding argument can be applied even in this case to establish that
\[ w_0(x_N) < v(x), \ x \in \mathbb{R}^N_+ \text{ or } w_0 \equiv v. \] (1.3)

If \( N = 2, 3, \) we were able to exclude the first scenario in (1.3) by adapting some ideas from the proof of the famous De Giorgi conjecture in the plane (see [3, 9]). More precisely, a key observation in the proof was that the convexity of the nonlinearity and (1.3) imply that, in the first case of (1.3), the difference \( v - w_0 \) would be a positive super-solution to the linearized problem on \( w_0 \) (see also [5, Ch. 1]). This brought the problem closer in spirit to that of the one-dimensional symmetry of bounded, stable solutions to semilinear elliptic equations (a stronger version of De Giorgi’s conjecture, see [4, 7]). The fact that we were dealing with the half-space and not the full space also created some technical difficulties in applying the approach of the aforementioned references. We point out that we were able to gain one more dimension, compared to the aforementioned works, owing to the exponential decay of \( w_0' \) as \( x_N \to \infty, \) which belongs to the kernel of the linearized operator on \( w_0. \)

In the current work, we will restrict ourselves to the case \( p = 2 \) and prove the following:

**Theorem 1.1.** If \( 2 \leq N \leq 5 \) and \( p = 2, \) then the only positive bounded solution of (1.1) with \( c = c_2 \) is \( v(x) = w_0(x_N), \) where \( c_2 \) and \( w_0 \) are as in (1.2).

Our main observation behind the proof is that the nonnegative auxiliary function
\[ u(x') = \int_0^\infty (v(x', x_N) - w_0(x_N)) (-w_0'(x_N)) \, dx_N, \ x' \in \mathbb{R}^{N-1}, \]
is a nonnegative super-solution to the quadratic Lane-Emden-Fowler equation \( \Delta u + u^2 = 0 \) (after a suitable re-scaling). We point out that this property is valid for all \( N \geq 2. \) Then, restricting ourselves to \( 2 \leq N \leq 5, \) we can apply a well known Liouville type result to conclude (see Appendix A below). We note in passing that the above auxiliary function \( u \) corresponds to the projection of the difference \( v - w_0 \) on \( w_0' \) which is in the kernel of the linearized problem on \( w_0. \) Lastly, let us remark that an approach of this nature has
been frequently applied in various parabolic problems for obtaining Liouville type results for ancient or eternal solutions (see for instance \cite{11}).

The proof of Theorem 1.1 will be given in the following section. Finally, in Appendix A we recall a well known Liouville type theorem concerning the nonexistence of positive super-solutions to the Lane-Emden-Fowler equation in the whole space.

### 2. Proof of Theorem 1.1

**Proof.** As we have already explained, the relation (1.3) is valid. We wish to show that the second alternative is the one which holds. To this end, let us argue by contradiction and suppose that

\[ w_0(x_N) < v(x', x_N), \quad (x', x_N) \in \mathbb{R}_+^N. \quad (2.1) \]

As we know from \cite{14}, the difference

\[ \varphi(x', x_N) = v(x', x_N) - w_0(x_N), \quad (x', x_N) \in \mathbb{R}_+^N, \quad (2.2) \]

furnishes a positive super-solution to the linearization of (1.1) on \( w_0 \). The main advantage of dealing with the case \( p = 2 \) is that the calculations involved for this purpose are completely transparent. Indeed, using that both \( v \) and \( w_0 \) satisfy the same PDE in (1.1), we find that

\[
\Delta(v - w_0) = v - v^2 - w_0 + w_0^2 \\
= v - w_0 - (v + w_0)(v - w_0) \\
= v - w_0 - (v - w_0 + 2w_0)(v - w_0) \\
= v - w_0 - 2w_0(v - w_0) - (v - w_0)^2.
\]

Summarizing, recalling the definition of \( \varphi \) from (2.2), we have

\[
\Delta \varphi = (1 - 2w_0)\varphi - \varphi^2, \quad \varphi > 0 \text{ in } \mathbb{R}_+^N; \quad \varphi = 0 \text{ on } \partial \mathbb{R}_+^N. \quad (2.3)
\]

Let us now consider the positive auxiliary function

\[
u(x') = \int_0^\infty \varphi(x', x_N)Z(x_N)dx_N, \quad x' \in \mathbb{R}^{N-1}, \quad (2.4)\]

where \( \varphi \) is as in (2.2), and

\[ Z(x_N) \equiv -w_0'(x_N). \quad (2.5)\]

For future reference, we observe that differentiation of the ODE satisfied by \( w_0 \) (i.e. (1.1) with \( p = 2 \)) yields

\[ -Z'' + (1 - 2w_0(x_N)) Z = 0, \quad x_N > 0. \quad (2.6)\]

We also note that

\[ Z(0) = 0, \quad Z > 0 \text{ in } (0, \infty) \text{ and } Z \text{ decays exponentially fast as } x_N \to \infty. \quad (2.7)\]
We compute that
\[
\Delta_{x'}u(x') = \int_0^\infty \Delta_{x'}\varphi(x', x_N)Z(x_N)dx_N
= \int_0^\infty (-\varphi_{x_Nx_N} + \Delta \varphi) Zdx_N
\]
using (2.3):
\[
= \int_0^\infty [-\varphi_{x_Nx_N} + (1 - 2w_0)\varphi - \varphi^2] Zdx_N
\]
integrating by parts:
\[
= \int_0^\infty \varphi [-Z_{x_Nx_N} + (1 - 2w_0)Z] dx_N - \int_0^\infty \varphi^2 Zdx_N
+ \varphi_{x_N}(x', 0)Z(0) - \varphi(x', 0)Z'(0)
\]
via (2.3), (2.6), (2.7):
\[
= -\int_0^\infty \varphi^2 Zdx_N.
\]
On the other hand, thanks to the Cauchy-Schwarz inequality and recalling the definition of Z from (2.5) (keeping in mind also (2.7)), we have
\[
u^2(x') = \left( \int_0^\infty \varphi Zdx_N \right)^2 \leq \int_0^\infty \varphi^2 Zdx_N \int_0^\infty Zdx_N = c_2 \int_0^\infty \varphi^2 Zdx_N.
\]
Hence, by combining (2.8) and (2.9), we arrive at
\[-\Delta_{x'}u \geq \frac{1}{c_2} u^2, \ u > 0 \text{ in } \mathbb{R}^{N-1}.
\]
In view of Theorem A.1 in Appendix A, after a simple rescaling in the above inequality, we can arrive at a contradiction and complete the proof, provided that we restrict $N \geq 2$ so that
\[
2 \leq p_{sg}(N - 1) \iff 2 \leq \frac{N - 1}{N - 3} \iff 2 \leq N \leq 5.
\]

Appendix A. A Liouville type theorem

For the reader’s convenience, we state below a well known result due to [10] (see also [1] and [13, Ch. I] for simpler proofs and extensions), which we used in the proof of Theorem 1.1.

Theorem A.1. Let $1 < p \leq p_{sg}(n)$, where
\[
p_{sg}(n) = \begin{cases} 
\infty & \text{if } n \leq 2, \\
\frac{n}{n-2} & \text{if } n > 2.
\end{cases}
\]
Then, the inequality
\[-\Delta u \geq u^p, \ x \in \mathbb{R}^n,
\]
does not possess any positive classical solution.
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