Relational hyperevent models for polyadic interaction networks

Jürgen Lerner
University of Konstanz, Germany
juergen.lerner@uni-konstanz.de

Alessandro Lomi
University of the Italian Switzerland, Lugano, CH.
alessandro.lomi@usi.ch

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Abstract

Polyadic (one-to-many) social interaction happens when one sender addresses multiple receivers simultaneously. Currently available relational event models (REM) are not well suited to the analysis of polyadic interaction networks because they specify event rates for sets of receivers as functions of dyadic covariates associated with the sender and one receiver at a time. Relational hyperevent models (RHEM) alleviate this problem by specifying event rates as functions of hyperedge covariates associated with the sender and the entire set of receivers. In this article we demonstrate the potential benefits of RHEMs for the analysis of polyadic social interaction. We define and implement practically relevant effects that are not available for REMs but may be incorporated in empirical specifications of RHEM. In a reanalysis of the canonical Enron email data, we illustrate how RHEMs effectively (i) reveal evidence of polyadic dependencies in empirical data, (ii) improve the fit over comparable dyadic specifications of REMs, and (iii) better identify the set of recipients actually receiving the same email message from sets of potential recipients who could have received the same email message, but did not.

1 Introduction

Data generated by technology-mediated communication typically come in sequences of time stamped interaction events involving two or more actors simultaneously. This kind of one-to-many (or “multicast”) interaction is common. For instance, email users can send messages to any number of receivers (Zhou et al. 2007; Perry and Wolfe 2013). In this paper we call a social interaction process polyadic when it is characterized by events in which one “sender” addresses multiple “receivers” simultaneously.

Data produced by time-stamped polyadic interaction processes are not unique to social network data produced by technology-mediated interaction (e.g., email messaging) or social media (e.g., twitter). Polyadic interaction data are encountered frequently in empirical research across the social sciences. Examples include scientific papers citing several references (Radicchi et al. 2012), courts judgments citing multiple legal precedents (Fowler et al. 2007), patents approved by the patent office citing multiple prior patents (Verspagen 2007; Kuhn et al. 2020), and infected persons transmitting a virus to several others simultaneously through group contact (Colizza et al. 2007; Hancean et al. 2021). Face-to-face conversations where one speaker addresses multiple alters simultaneously also illustrate the generality of polyadic interaction (Gibson 2005). Polyadic interaction may be directed or undirected. Examples of undirected polyadic interaction networks include meetings attended by multiple participants (Freeman 2003; Lerner et al. 2021), coauthors jointly publishing a paper (Newman 2004), coordination in task-oriented teams (Ahmadpoor and Jones 2019; Guimerà et al. 2002; Leenders et al. 2016), countries agreeing to sign a multilateral treaty (Hollway and Koskinen 2016; Simmons and Hopkins 2005), and class action lawsuits where the plaintiff is a group of people simultaneously bringing a suit to one defendant (Bronstein and Fiss 2002).
Despite their obvious diversity, these empirical examples share two defining features. First, interaction among the agents takes the form of time stamped relational events – rather than relational states such as “being a friend of,” “seeking advice from,” or “regularly communicate with.” Second, the stream of observed events is not generated only by dyadic interaction, but involves multiple network nodes interacting at given points in time. In recent years, the availability of social interaction data sharing these features has increased significantly due to the diffusion of computer-mediated communication and collaborative technologies, the availability of large-scale databases, and the diffusion of automated data collection technologies (Lazer et al. 2009). However, the availability of statistical models capable of analyzing these data by accounting for their constitutive features has remained limited.

Relational event models (REMs) (Butts 2008; Brandes et al. 2009; Perry and Wolfe 2013; Lerner et al. 2013) provide the most promising framework for the analysis of sequences of social interaction events observed in (near) continuous time (Bianchi and Lomi 2022; Vu et al. 2017). Typical REM for networks of time stamped interaction events specify point processes whose intensities are functions of dyadic covariates. For instance, the intensity of events from sender $i$ to receiver $j$ might depend on the age or gender of $i$ and $j$, on their age difference, on the frequency of previous events from $i$ to $j$ or from $j$ to $i$, or on previous events to or from common third actors. Extant research recognizes the problem posed by polyadic interaction and multicast communication. The solution that is typically offered involves specifying intensities for events in which a sender $i$ sends a message to a set of receivers $J = \{j_1, \ldots, j_k\}$ (Perry and Wolfe 2013). However, these intensities are still modeled as functions of dyadic covariates $x(i, j)$, $j \in J$, considering the sender and only one receiver at a time. In this way, available models for relational events assume that the multiple dyads simultaneously produced by a single polyadic interaction are either independent (Perry and Wolfe 2013), or pertain to fictional “collective” receivers (Butts 2008), such as the whole “team” as a receiver of broadcast messages. While empirically useful, neither of these solutions is fully satisfactory.

Recently proposed relational hyperevent models (RHEMs) (Lerner et al. 2019, 2021) generalize REMs by specifying the rate of interaction events from sender $i$ to receiver set $J$ as a function of hyperedge covariates $x(i, J)$ – being a function of the sender and the entire set of receivers – that cannot necessarily be decomposed into dyadic covariates. For an example illustrating the difference between dyadic covariates and hyperedge covariates, we refer to Fig. 1. Generalizing edges in graphs that connect exactly two nodes, hyperedges in a hypergraph can connect any number of nodes (Berge 1989). A relational hyperevent is a time stamped event indexed by a hyperedge (Lerner et al. 2021).

While the conceptual transition from REM to RHEM may be intuitive, its analytical and empirical implications need more complete and rigorous articulation. In this paper, our goal is to improve our current understanding by specifying appropriate models for polyadic social interaction processes and then evaluating empirical differences with respect to comparable dyadic specifications.

A model specified strictly in terms of dyadic covariates potentially misses higher-order dependencies that are typically present in network data produced by polyadic interaction. In turn, this might potentially yield misleading estimates of network effects. In this paper we take the view that higher-order dependencies – when present – should not merely be considered as an inconvenient feature of the data to be, in the best cases, controlled away. Rather, we argue that such higher-order effects provide a unique opportunity for improving our understanding of the structure and dynamics of social interaction processes, and for developing and testing innovative theories of social interaction.

In this paper we intend to demonstrate how differences between REM and RHEM may be directly relevant for empirical studies analyzing polyadic social interaction processes. We define and discuss practically relevant network effects that can be expressed in RHEMs, but not in dyadic specifications of the event rate afforded by currently available REMs. In the empirical part of the paper, we use the canonical Enron email data (Zhou et al. 2007; Perry and Wolfe 2013) to address the following orienting questions about the relation between REM and RHEM.

RQ1 How strong is the evidence of higher-order effects in empirical data produced by polyadic social interaction processes?

RQ2 Is evidence of dyadic effects affected by the failure to control for higher-order dependencies?
RQ3: Do RHEMs fit polyadic interaction data better than REM based on dyadic specifications of the event rate?

RQ4: Are RHEMs better able to discriminate between the sets of receivers of actually observed messages and the associated sets of potential receivers who could have received a common message, but did not?

We demonstrate how the computation of hyperedge covariates for RHEMs may be performed with an extension of the open source software eventnet to the analysis of hyperevents. Specifically, the analysis of the Enron email data with RHEM is explained in a step-by-step tutorial from the data preprocessing and the computation of covariates over to model estimation in R. Thus, the analysis reported in the empirical part of this paper is fully reproducible. The software may be adopted in – and adapted to future empirical studies involving polyadic social interaction processes.

2 Background

2.1 REM based on dyadic covariates

We start by recalling the point process models for directed interaction networks, following closely the argument and the notation proposed by Perry and Wolfe (2013). The basic relational event model developed by Perry and Wolfe (2013) involves dyadic events with a single sender and a single receiver. We start our discussion from their model for the more general case of relational events defined over receiver sets of arbitrary size. As Perry and Wolfe (2013) observe, their theoretical results on the consistency of the maximum partial likelihood inference also apply to the case of polyadic (“multicast”) events. It may be worth repeating that the difference between RHEM and the model from Perry and Wolfe (2013) does not lie in the basic modeling framework, but in generalizing dyadic covariates to hyperedge covariates.

Let $I$ be a finite set of senders and $J$ be a finite set of receivers, not necessarily disjoint from $I$. We denote elements from $A = I \cup J$ as actors. For a sender $i \in I$ and a point in time $t > 0$, let $J_t(i) \subseteq J$ denote the set of actors that could potentially receive an interaction from $i$ at $t$. If $A = I = J$, then it is often the case that $J_t(i) = I \setminus \{i\}$, implying that a sender can send interaction events to everyone but herself – that is, loops are excluded. For a sender $i$ and receiver $j$, let $x_t(i, j)$ be a $p$-dimensional vector of covariates and $\beta_0$ a $p$-dimensional vector of unknown parameters. For a sender $i$ and a positive integer $L$ (giving the receiver set size), the baseline intensity is denoted by $\lambda_r(i, L)$.

Perry and Wolfe (2013) define a model for counting processes on $\mathbb{R}_+ \times I \times \mathcal{P}(J)$ where the intensity on $(i, J)$, with $i \in I$ and $J \subseteq J$, is modeled as

$$
\lambda_t(i, J) = \lambda_r(i, |J|) \exp \left\{ \beta_0^T \sum_{j \in J} x_t(i, j) \right\} \prod_{j \in J} 1_{\{j \in J_t(i)\}}. \tag{1}
$$

Intuitively, $\lambda_t(i, J) \Delta t$ is the expected number of events that $i$ sends to the receiver set $J$ in the time interval $[t, t + \Delta t)$. The intensity $\lambda_r(i, J)$ is assumed to be the baseline intensity $\lambda_r(i, |J|)$ multiplied with the relative rate, $\exp \left\{ \beta_0^T \sum_{j \in J} x_t(i, j) \right\}$. Thus, the parameter vector $\beta_0$ controls which covariates $\{x_t(i, j); j \in J\}$ make $J$ a more or less likely receiver set for interaction events sent by $i$. The covariates in $x_t(i, j)$ can depend on actor characteristics, such as the age of $j$ or the age difference of $i$ and $j$, but they can also depend on the history of the process. For instance, the covariates $x_t(i, j)$ can include a count of the number of past events that $i$ has sent to $j$, before $t$.

Importantly, the model from Perry and Wolfe (2013), summarized in (1), assumes that the covariate vectors $\{x_t(i, j)\}$ for $j \in J$ increase or decrease the rate of events from sender $i$ to the receiver set $J$ independently of each other. Equivalent to (1), we can model the intensity $\lambda_t(i, J)$ as the baseline intensity

https://github.com/juergenlerner/eventnet/wiki/Directed-RHEM-for-multicast-interaction-(tutorial)
Let $\lambda_t(i, J)$ multiplied with the relative rate $\exp \{ \beta_t^T x_t(i, J) \}$, where the covariates $x_t(i, J)$ of sender $i$ and receiver set $J$ are assumed to have the specific form:

$$x_t(i, J) = \sum_{j \in J} x_t(i, j).$$

The equation above implies that in the model from Perry and Wolfe (2013) the suitability of $j$ as a receiver of interaction sent by $i$ is assumed to be independent of the other receivers $j' \in J \setminus \{j\}$ of the same interaction. Generalizing the model from Perry and Wolfe (2013), RHEM allow covariates $x_t(i, J)$ that do not necessarily decompose into a sum of dyadic covariates of the form $\sum_{j \in J} x_t(i, j)$; compare Sect. 2 below.

To illustrate the limitations of dyadic covariates, we present the following stylized example; compare Fig. 1 below. Assume that Alice frequently sends emails jointly addressing her work colleagues Bob and Charles. Assume, further, that she frequently sends emails to Deborah and Eve who are in her same sports team. Assume, finally, that Alice has never send any email jointly addressing some of her work colleagues and some of her team mates (because, for example, she prefers to keep her work and leisure activities distinct). A model specifying event rates purely based on dyadic covariates would predict that an email from Alice to the receiver set $\{\text{Bob, Charles}\}$ is as likely as an email from Alice to $\{\text{Bob, Deborah}\}$. Thus, dyadic specifications of the event rate, assess the suitability of individual receivers without considering other receivers of the same interaction. In contrast, a model based on hyperedge covariates could recognize that the number of previous events from Alice to the receiver set $\{\text{Bob, Charles}\}$ is high, but the number of previous events from Alice to the receiver set $\{\text{Bob, Deborah}\}$ is zero – and in turn can predict a higher intensity for the former than for the latter. As we shall see, covariates that may be specified in RHEMs are not restricted to receiver sets of size two.

Let $(t_1, i_1, J_1), \ldots, (t_n, i_n, J_n)$ be the observed sequence of polyadic interactions where $(t, i, J)$ indicates that at time $t$ sender $i$ interacts with receiver set $J$. The model from Perry and Wolfe (2013), given in (1), leads to the log partial likelihood at $t$ evaluated at $\beta \in \mathbb{R}^p$:

$$\log L_t(\beta) = \sum_{t_m \leq t} \left( \beta^T \sum_{j \in J_m} x_{t_m}(i_m, j) - \log \left[ \sum_{j \in \binom{J_m\setminus\{i_m\}}{J_m-1}} \exp \left\{ \beta^T \sum_{j \in J} x_{t_m}(i_m, j) \right\} \right] \right),$$

where for a set $X$ and an integer $L$, we write $\binom{X}{L} = \{X' \subseteq X ; |X'| = L\}$ for the set of all subsets of size $L$.

There is a slight abuse of notation in using the same symbol $x_t$ for covariates taking values on pairs of senders and receivers and for covariates taking values on pairs of senders and receiver sets. However, we believe that this does not cause any confusion and that it makes the similarities and differences between the model from Perry and Wolfe (2013) and RHEM more transparent.
Perry and Wolfe (2013) propose an approximation for the log partial likelihood where the sum over all size-$|J_m|$ subsets of $J_m(i_m)$ in (3) is replaced by a sum over all receivers in $J_m(i_m)$ and develop methods to estimate and correct the approximation error. This approximation, which affords a large gain in computational speed, does not apply if receiver set covariates are no longer given as a sum of dyadic covariates. Instead we propose to speed up model estimation by case-control sampling (Borgan et al. 1995; Keogh and Cox 2014); compare Sect. 3.2 below.

2.2 Related work

A number of alternative approaches have been proposed to adapt REM to events with several receivers (or several senders). Butts (2008) proposes to create “virtual” nodes to represent sets of receivers (or senders). This approach can be appropriate for specifically chosen subsets (such as the whole “team” as a receiver of broadcast messages), but representing all possible subsets by virtual nodes becomes quickly unfeasible. Kim et al. (2018) propose the hyperedge event model for multicast events. Their model specifies dyadic intensities, associated with one sender and one receiver, as a function of dyadic covariates. These intensities then stochastically determine the sender of the next event and subsequently the receiver set of an interaction by that sender. Their framework does not allow for hyperedge covariates as in RHEM. Mulder and Hoff (2021) define a latent variable model for multicast interaction. However, in their model the mean suitability score of a receiver for messages initiated by a given sender is still a dyadic function, dependent on the sender and one receiver at a time.

Relational hyperevent models for undirected hyperevents (e.g., meeting events) have been proposed in Lerner et al. (2021). Earlier, RHEM have been mentioned by Lerner et al. (2019) who also defined RHEM for directed hyperevents – but did not analyze directed RHEM in any of their empirical examples. Directed RHEM have been applied to modeling contact elicitation networks by infected persons in Hancean et al. (2021).

3 RHEM and hyperedge covariates for multicast social interaction

3.1 RHEM

Building on the model proposed by Perry and Wolfe (2013), the definition of RHEM for directed polyadic interaction can be obtained by substituting $x_t(i, J)$ for $\sum_{j \in J} x_t(i, j)$ in (1) and (3). More precisely, given the notation from Sect. 2.1, RHEM define a model for counting processes on $\mathbb{R}_+ \times I \times \mathcal{P}(J)$ where the intensity on $(i, J)$, with $i \in I$ and $J \subseteq J$, is modeled as

$$\lambda_t(i, J) = \sum_{|J'|} \lambda_t(i, |J|) \exp \left\{ \beta \sum_{j \in J'} x_t(i, j) \right\} \mathbf{1}\{J \subseteq J_t(i)\} . \quad (4)$$

The covariates $x_t(i, J)$ do not necessarily decompose into a sum of dyadic covariates and may depend on exogenous actor-level characteristics or they can depend on the history of the process. Examples for the former include the average age of receivers in $J$, the average absolute age difference between the sender $i$ and the receivers in $J$, or the average absolute age difference between pairs of receivers in $J$. Examples for the latter (i.e., history-dependent) covariates include the number of past interactions that $i$ has sent to receiver set $J$:

$$\# \{\text{interaction } (t', i, J) \text{ with } t' < t \} ,$$

the average number of past interactions that actors $j \in J$ have received from $i$:

$$\frac{\sum_{j \in J} \# \{\text{interaction } (t', i, J') \text{ with } t' < t \text{ and } j \in J'\}}{|J|} ,$$

or the average number of past interactions that pairs of actors $\{j, j'\} \subseteq J$ have jointly received from $i$:

$$\frac{\sum_{\{j, j'\} \in \binom{J}{2}} \# \{\text{interaction } (t', i, J') \text{ with } t' < t \text{ and } j, j' \in J'\}}{\binom{|J|}{2}} .$$
The number of non-events per event, let \( \tilde{k} \) obtained via case control sampling (Borgan et al. 1995; Keogh and Cox 2014). For a given positive integer, suggested to reduce excessive computational runtime by replacing the risk set with a sampled risk set in intractable, computational runtime for all but the smallest covariates in Perry and Wolfe (2013). Having to sum over all subsets would lead to excessive, or even intractable, computational runtime for all but the smallest covariates in Perry and Wolfe (2013). Having to sum over all subsets would lead to excessive, or even intractable, computational runtime for all but the smallest set.

The log partial likelihood (5) can no longer be approximated by replacing the sum over all sets sampled without replacement uniformly and independently at random from \( \mathcal{R} \subseteq \binom{\mathcal{J}_m(i_m)}{|\mathcal{J}_m|} \) be a set of subsets of \( \mathcal{J}_m(i_m) \) that is sampled uniformly at random from:

\[
\left\{ \mathcal{R} \subseteq \binom{\mathcal{J}_m(i_m)}{|\mathcal{J}_m|}; J_m \in \mathcal{R} \land |\mathcal{R}| = k + 1 \right\}.
\]

In other words, \( \mathcal{R}_m(\mathcal{J}_m(i_m), J_m, k) \) contains the observed receiver set \( J_m \) plus \( k \) sets sampled without replacement uniformly and independently at random from \( \{ J \subseteq \mathcal{J}_m(i_m); |J| = |J_m| \land J \neq J_m \} \). We obtain the sampled log partial likelihood function:

\[
\log \tilde{L}_t(\beta) = \sum_{i_m \leq t} \left( \beta^T x_{i_m}(i_m, J_m) - \log \sum_{J \in \tilde{R}_m(\mathcal{J}_m(i_m), J_m, k)} \exp \{ \beta^T x_{i_m}(i_m, J) \} \right),
\]

and we estimate RHEM parameters by maximizing (6).

### 3.3 Hyperedge covariates

Since hyperedge covariates \( x_t(i, J) \) are defined on the entire set of receivers, we obtain a large number of possible structurally different covariates. While our list is certainly far from exhausting the possibilities, we define in the following a collection of hyperedge covariates that are practically relevant in the empirical analysis of multicast interaction networks. We define two types of covariates: “attribute effects”, dependent on actor-level attributes and “network effects,” dependent on the history of the process, that is, on previously observed events.

#### 3.3.1 Covariates dependent on actor-level attributes

Suppose that available data include information on one or several numeric actor-level attributes \( z: \mathcal{A} \rightarrow \mathbb{R} \). Then, in general, RHEM covariates \( x_t(i, J) \) based on such actor (“node-specific”) characteristics may be obtained either by computing summaries of the values in the receiver set such as mean\( \{ z(j) : j \in J \} \), or by computing summaries of the receivers’ values in relation to the sender’s value \( z(i) \), such as mean\( \{ |z(j) - z(i)| : j \in J \} \). Covariates that are only functions of the sender \( i \) would lead to a non-identifiable parameter, since their effect would be absorbed by the baseline rate \( \tilde{\lambda}_t(i, |J|) \).
Concrete examples of covariates are discussed below, where we also discuss the possibility of defining covariates based on categorical attributes. In this article we consider only time-invariant attributes. However, the covariates discussed below extend directly to time-varying actor-level attributes by considering values at the given time $t$.

**Receiver set average.** The receiver average of attribute $z$ is defined to be

$$\text{receiver.avg.}z_i, J = \frac{\sum_{j \in J} z(j)}{|J|}.$$ 

For example, in the empirical analysis we apply this definition to two binary attributes, *gender* and *seniority*, both of which are coded by the values $\{0, 1\}$ (*female* = 1 and *senior* = 1). The receiver set average, thus, gives the fraction of females (or seniors, respectively) in the receiver set.

In our empirical study we also apply this definition to a categorical attribute *department* taking values in $\{\text{Trading, Legal, Other}\}$ by recoding this information in two binary attributes, “trading” and “legal”, taking the value one if the actor works in the corresponding department. Thus, the resulting receiver set average is the fraction of receivers working in the trading department (or legal department, respectively). In general, a categorical attribute with $C$ different values can be coded in $C - 1$ binary attributes, with the remaining value as the base category.

**Sender-receiver homophily.** We assess homophily between senders and receivers by averaging the absolute difference between the attribute value of the sender $i$ and the values of the receivers $j \in J$:

$$\text{sender.receiver.abs.diff.}z_i, J = \frac{\sum_{j \in J} |z(i) - z(j)|}{|J|}.$$ 

For example, for the attribute *gender* we obtain the fraction of receivers having the opposite gender than the sender. If senders have a tendency to interact with similar receivers, then a higher value of $\text{sender.receiver.abs.diff.}z_i, J$ would make an interaction from $i$ to $J$ less likely. Thus, in presence of homophily we would expect a negative parameter.

For a categorical attribute $z$, we assess sender-receiver homophily via the fraction of receivers that have a different value than the source:

$$\text{sender.receiver.cat.diff.}z_i, J = \frac{\sum_{j \in J} 1\{z(i) \neq z(j)\}}{|J|}.$$ 

Since $\text{sender.receiver.cat.diff.}$ measures differences in the attribute value, a negative associated parameter would point to homophily.

**Receiver set homophily.** We assess homophily in the receiver set by averaging the absolute difference of the attribute value between receiver pairs:

$$\text{receiver.pair.abs.diff.}z_i, J = \sum_{\{j, j'\} \in \binom{J}{2}} \frac{|z(j') - z(j)|}{\binom{|J|}{2}}.$$ 

If senders have a tendency to interact with homogeneous receiver sets (irrespective of their own attribute value), then a higher value of $\text{receiver.pair.abs.diff.}z_i, J$ would make an interaction from $i$ to $J$ less likely, so that we would expect a negative associated parameter.

For a categorical attribute $z$, we assess receiver set homophily via the fraction of receiver pairs that have a different attribute value:

$$\text{receiver.pair.cat.diff.}z_i, J = \frac{\sum_{\{j, j'\} \in \binom{J}{2}} 1\{z(j') \neq z(j)\}}{\binom{|J|}{2}}.$$
Again, a negative parameter would suggest a tendency to send interaction to homogeneous sets of participants.

Note that receiver set homophily is structurally different from sender-receiver homophily (Snijders and Lomi 2019). Theoretically it might be the case that senders have neither a preference, nor a reluctance to send interaction events to receivers of their own gender. Yet, in the same data it is possible that typical receiver sets are either mostly composed of females or mostly composed of males. In such a hypothetical scenario, we would not find evidence for sender-receiver homophily but we would find evidence for receiver set homophily.

### 3.3.2 Network effects

In addition to covariates based on actor-level attributes, we define several covariates expressing dependence of the intensity $\lambda_t(i, J)$ on the history of the process. Given an observed sequence of events generated by polyadic interaction $E = \{(t_1, i_1, J_1), \ldots, (t_n, i_n, J_n)\}$, the value of these covariates at time $t$ is computed as a function of earlier events $E_{<t} = \{(t_m, i_m, J_m) \in E : t_m < t\}$.

Similar to previous work on REM and RHEM (Brandes et al. 2009, Lerner et al. 2013, Amati et al. 2019, Lerner et al. 2021), we let the effect of past events decay over time. More precisely, for a given half life period $T_{1/2} > 0$, the weight of a past event $(t_m, i_m, J_m)$ at current time $t > t_m$ is defined to be $w(t - t_m) = \exp\left(- \frac{(t - t_m) \log 2}{T_{1/2}}\right)$. While alternatives exist to exponential decay of past events, Schecter and Quintana (2021) found it to be generally adequate. The objective of our paper is to clarify the benefit of hyperedge covariates in comparison to dyadic covariates. We consider issues related to the effect of elapsed time of past events as orthogonal to the objective of the current paper.

**Exact repetition.** An empirically plausible effect would capture relational "inertia," or the tendency of actors to continue to do what they did in the past. In fact, in many cases it may be important to verify the presence of more complex mechanisms over and above the simple repetition of past behavior. In polyadic interaction networks this inertial behavioral tendency would lead to future events that repeat the sender and the entire receiver set of past events. Such an effect can be captured by the covariate:

$$\text{repetition}_t(i, J) = \sum_{(t_m, i_m, J_m) \in E_{<t}} w(t - t_m) \cdot 1(i_m = i \land J_m = J) .$$

Thus, we sum the current weight over all previous events that have the same sender and exactly the same receiver set. Exact repetition captures effects in which the same sender repeatedly addresses the same receivers, for instance, email communication with fixed receiver lists ("mailing lists").

**Unordered exact repetition.** Besides exact repetition, there can be situations of interaction within a stable group of actors with turn-taking among the senders (Gibson 2005). Such an effect can be captured by the covariate

$$\text{unordered.repetition}_t(i, J) = \sum_{(t_m, i_m, J_m) \in E_{<t}} w(t - t_m) \cdot 1(i_m \cup J_m = \{i\} \cup J) .$$

In contrast to (ordered) exact repetition, the unordered repetition effect allows that different actors take on the role of sender – as long as the union of the sender with the receiver set remains constant. A typical example of this behavior is email communication using a "reply-to-all" functionality: a receiver of a previous message sends a message to the previous sender and to all other receivers. See Fig. 2 for an illustration of unordered exact repetition. In contrast to unordered exact repetition, exact repetition would require that the future event in Fig. 2 is exactly on the hyperedge $(A, \{B, C, D, E\})$, that is, the same sender $A$ sends another interaction to the same receiver set $\{B, C, D, E\}$.
Figure 2: Stylized example illustrating unordered exact repetition. *Left*: history of a past event \( e_1 = (t_1, A, \{B, C, D, E\}) \) displayed as a gray-shaded area; dashed lines connect the sender to the receivers. *Right*: a candidate hyperedge \( h = (C, \{A, B, D, E\}) \) for a future hyperevent. The past event \( e_1 \) increases the value of unordered exact repetition on \( h \) at time \( t > t_1 \). In communication networks, unordered exact repetition could point to turn-taking among a stable set of conversation participants.

**Partial receiver set repetition.** The two (ordered and unordered) exact repetition covariates defined above still give an incomplete picture of stability in multicast interaction events, since they require that sets of actors involved in past and current events have to be identical. A possible event that mostly, but not exactly, repeats the receiver set of a past event – for instance, if some new receivers are added or if some previous receivers are removed – is treated identically to a possible event with a completely disjoint receiver set. To quantify partial repetition, we define a parametric family of covariates capturing to what extent subsets of a possible receiver set have jointly received past events.

To shorten notation, we define the *hyperedge indegree* of a set of receivers \( J' \subseteq J \) by considering past events that have been jointly received by all members of \( J' \) – possibly together with varying other receivers outside of \( J' \):

\[
hy.deg_{t}^{(in)}(J') = \sum_{(t_m, J_m) \in E_{<t}} w(t - t_m) \cdot 1(J' \subseteq J_m) .
\]

Partial receiver set repetition (or subset repetition among the receivers) is parametrized by a positive integer \( p \), giving the cardinality of the subsets that repeatedly receive joint messages:

\[
receiver.sub.rep_{t}^{(p)}(i, J) = \sum_{J \in \binom{J}{p}} \frac{hy.deg_{t}^{(in)}(J')}{\binom{|J'|}{p}} .
\]

For \( p = 1 \) we obtain the average indegree of individual receivers \( j \in J \) by considering past interactions received by \( j \), downweighted by the elapsed time. For \( p = 2 \) we consider past interactions that have been jointly received by pairs \( \{j, j'\} \subseteq J \), and so on.

Partial receiver set repetition – and the related sender-specific partial receiver set repetition, defined below – are illustrated in Fig. 3. In the notation of that figure, if we ignore the decay over time and if \( e_1 \) is the entire history, we get for a point in time \( t > t_1 \) the following values:

\[
\begin{align*}
receiver.sub.rep_{t}^{(1)}(h) &= 3/4 \\
receiver.sub.rep_{t}^{(2)}(h) &= 3/6 \\
receiver.sub.rep_{t}^{(3)}(h) &= 1/4 .
\end{align*}
\]

Partial receiver set repetition of order \( p > 3 \) is zero in this example.

**Sender-specific partial receiver set repetition.** Partial receiver set repetition defined above does not consider whether past interactions jointly received by \( J' \subseteq J \) originated from the same sender \( i \). Thus, these
Figure 3: Stylized example illustrating (sender-specific) partial receiver set repetition. Left: history of a past event $e_1 = (t_1, A, \{B, C, D, E\})$ displayed as a gray-shaded area; dashed lines connect the sender to the receivers. Right: a candidate hyperedge $h = (A, \{C, D, E, F\})$ for a future hyperevent. Among the four receivers in $h$, three have individually received the past event $e_1$. Among the six pairs of receivers in $h$, three have jointly received the past event $e_1$. Among the four triples of receivers in $h$, one has jointly received the past event $e_1$. The past event $e_1$ increases the value of sender-specific partial receiver set repetition of order $p = 1$, $2$, $3$ on $h$ at $t > t_1$. If the sender of $h$ was another actor $G$, instead of $A$, then the past event would still increase the value of partial receiver set repetition, but it would not increase the value of the sender-specific variant.

covariates capture partial repetition of receivers by any sender. To distinguish partial receiver set repetition by different senders, we define the sender-specific hyperedge degree by:

$$hy.deg_t(i, J) = \sum_{(t_m, i_m, J_m) \in E < t} w(t - t_m) \cdot 1(i = i_m \land J' \subseteq J_m).$$

Sender-specific partial receiver set repetition (or sender-specific subset repetition in the receiver set) is parametrized by a positive integer $p$, giving the cardinality of the subset that repeatedly receives joint messages from the given sender $i$.

$$sender.receiver.sub.rep^{(p)}_t(i, J) = \sum_{J' \in \binom{J}{p}} hy.deg_t(i, J') \cdot \binom{|J| - 1}{p}.$$ 

For $p = 1$ we obtain the average weight of past interactions that individual actors $j \in J$ received from the given sender $i$, where the average is taken over all those receivers. For $p = 2$ we consider past interactions that have been sent by $i$ and that have been jointly received by pairs $\{j, j'\} \subseteq J$, and so on.

In the example illustrated in Fig. 3 we obtain the same values for sender-specific partial receiver set repetition as for partial receiver set repetition, since the candidate hyperedge for a future event $h = (A, \{C, D, E, F\})$ has the same sender as the past event $e_1$. However, another candidate hyperedge $h' = (G, \{C, D, E, F\})$ would have the same value than $h$ in the partial receiver set repetition covariates but would get the value zero in the sender-specific variants.

(Sender-specific) partial receiver set repetition may induce a clustering in the set of actors, implying subsets of receivers that are likely to jointly receive the same interactions. Partial receiver set repetition implies a “global” clustering applying to all interactions, irrespective of their sender. The sender-specific variants allow for different clustering of receivers that vary with the sender.

Addressing senders and their receivers. Yet another network effect in polyadic social interaction network arises if actors send interactions to a sender of a past interaction together with a subset of the receivers of that past interaction. This is, for instance, a frequent pattern in scientific citation network where a paper $P$ cites another paper $P'$ and some of the references of $P'$. This pattern can be captured by the following family of covariates, parametrized by a positive integer $p$, giving the number of the repeated receivers of the previous event:

$$interaction.among.receivers^{(p)}_t(i, J) = \sum_{j \in J, J' \in \binom{J \setminus \{i\}}{p}} hy.deg_t(j, J') \cdot \binom{|J| - 1}{p}.$$
Interaction among receivers is illustrated in Fig. 4. In the notation of that figure, if we ignore the decay over time and if $e_1$ is the entire history, we get for a point in time $t > t_1$ the following values:

$$interaction.among.receivers^{1}(t) = \frac{2}{4 \cdot 3}$$

$$interaction.among.receivers^{2}(t) = \frac{1}{4 \cdot 3} .$$

Reciprocation and generalized reciprocation. Actors often have the tendency to reply to interactions they received in the past. Reciprocation is captured by considering interactions that the sender $i$ of the current interaction has received from actors $j \in J$:

$$reciprocation_t(i, J) = \sum_{j \in J} hy.deg_t(j, \{i\})/|J| .$$

The effect generalized reciprocation, arises when senders of past interactions receive future interactions from any actor in the network – not necessarily from the receivers of the past interactions. We define the outdegree of an actor $i' \in A$ by considering past events that have been send by $i'$. (Note that in contrast to the hyperedge indegree, the outdegree cannot be defined for a set of more than one actor, since every interaction has only one sender.)

$$deg_t^{(out)}(i') = \sum_{(t_m, i_m, j_m) \in E_{<t}} w(t - t_m) \cdot 1(i' = i_m) .$$

Generalized reciprocation is defined by

$$generalized.reciprocation_t(i, J) = \sum_{j \in J} deg_t^{(out)}(j)/|J| .$$

Reciprocation and generalized reciprocation are illustrated in Fig. 5. In the notation of that figure, if we ignore the decay over time and if $e_1, e_2$ is the entire history, we get for a point in time $t > t_1, t_2$ the following values:

$$reciprocation_t(h) = \frac{1}{3}$$

$$generalized.reciprocation_t(h) = \frac{2}{3} .$$
Figure 5: Stylized example illustrating reciprocation and generalized reciprocation. Left: history of two past events \( e_1 = (t_1, A, \{D, E, F\}) \) and \( e_2 = (t_2, B, \{A, C\}) \) displayed as gray-shaded areas; dashed lines connect the sender to the receivers. Right: a candidate hyperedge \( h = (D, \{A, B, C\}) \) for a future hyperevent. The past event \( e_1 \) increases the value of the covariate \( \text{reciprocation}_t(h) \) at \( t > t_1, t_2 \), since \( e_1 \) has been sent by \( A \), a receiver of \( h \), among others to \( D \), the sender of \( h \). The past event \( e_2 \) does not increase \( \text{reciprocation}_t(h) \) – but it increases \( \text{generalized.reciprocation}_t(h) \), since \( e_2 \) has been send by \( B \), a receiver of \( h \).

**Triadic effects.** Interactions might further depend on past interactions that the sender and the receivers had with common third actors. By varying the directions of the past interactions to or from the sender and the receivers we obtain four variants of triadic closure. In the summations below, \( a \) iterates over all actors \( A \setminus \{i, j\} \).

\[
\begin{align*}
\text{transitive.closure}_t(i, J) &= \sum_{j \in J, a \neq i, j} \min \{\text{hy.deg}_t(i, \{a\}), \text{hy.deg}_t(a, \{j\})\} \bigg/ |J| \\
\text{cyclic.closure}_t(i, J) &= \sum_{j \in J, a \neq i, j} \min \{\text{hy.deg}_t(a, \{i\}), \text{hy.deg}_t(j, \{a\})\} \bigg/ |J| \\
\text{shared.sender}_t(i, J) &= \sum_{j \in J, a \neq i, j} \min \{\text{hy.deg}_t(a, \{i\}), \text{hy.deg}_t(a, \{j\})\} \bigg/ |J| \\
\text{shared.receiver}_t(i, J) &= \sum_{j \in J, a \neq i, j} \min \{\text{hy.deg}_t(i, \{a\}), \text{hy.deg}_t(j, \{a\})\} \bigg/ |J|.
\end{align*}
\]

Transitive closure and cyclic closure are illustrated in Fig. 6. In the notation of that figure, if we ignore the decay over time and if \( e_1, e_2 \) is the entire history, for a point in time \( t > t_1, t_2 \) we obtain the following values:

\[
\begin{align*}
\text{transitive.closure}_t(h) &= 2/2 \\
\text{cyclic.closure}_t(h) &= 1/2.
\end{align*}
\]

The \( \text{shared.sender} \) covariate is illustrated in Fig. 7. In the notation of that figure, if we ignore the decay over time and if \( e_1, e_2 \) is the entire history, for a point in time \( t > t_1, t_2 \) we obtain the following value:

\[
\text{shared.sender}_t(h) = 2/2.
\]

The \( \text{shared.receiver} \) covariate is illustrated in Fig. 8. In the notation of that figure, if we ignore the decay over time and if \( e_1, e_2 \) is the entire history, for a point in time \( t > t_1, t_2 \) we obtain the following value:

\[
\text{shared.receiver}_t(h) = 1/2.
\]

### 3.3.3 Dyadic covariates vs. hyperedge covariates

As mentioned, we denote a hyperedge covariate \( x_t(i, J) \) as a **dyadic covariate** if it admits a decomposition of the form \( x_t(i, J) = C(|J|) \cdot \sum_{j \in J} x_t(i, j) \), compare [2], where \( C(|J|) \) is a multiplicative constant depending
Figure 6: Stylized example illustrating transitive closure and cyclic closure. Left: history of two past events $e_1 = (t_1, A, \{B, C\})$ and $e_2 = (t_2, C, \{D, E\})$ displayed as gray-shaded areas; dashed lines connect the sender to the receivers. Right: two candidate hyperedges $h = (A, \{D, E\})$ and $h' = (E, \{A, F\})$ for future hyperevents. The past events $e_1, e_2$ increase the value of the covariate $\text{transitive.closure}_t(h) > t_1, t_2$, since $h$ transitively closes two paths: from $A$ over $C$ to $D$ and from $A$ over $C$ to $E$. The past events $e_1, e_2$ increase the value of the covariate $\text{cyclic.closure}_t(h') > t_1, t_2$, since $h'$ closes a cycle from $A$ to $C$ to $E$ to $A$.

Figure 7: Stylized example illustrating the shared.sender covariate. Left: history of two past events $e_1 = (t_1, C, \{A, B\})$ and $e_2 = (t_2, C, \{D, E\})$ displayed as gray-shaded areas; dashed lines connect the sender to the receivers. Right: a candidate hyperedge $h = (A, \{D, E\})$ for a future hyperevent. The past events $e_1, e_2$ increase the value of the covariate $\text{shared.sender}_t(h)$ at $t > t_1, t_2$, since the sender of $h$, $A$ received a past event from $C$ and two receivers of $h$, $D$ and $E$ also received a past event from the same sender $C$.

Figure 8: Stylized example illustrating the shared.receiver covariate. Left: history of two past events $e_1 = (t_1, A, \{B, C\})$ and $e_2 = (t_2, E, \{D, C\})$ displayed as gray-shaded areas; dashed lines connect the sender to the receivers. Right: a candidate hyperedge $h = (A, \{D, E\})$ for a future hyperevent. The past events $e_1, e_2$ increase the value of the covariate $\text{shared.receiver}_t(h)$ at $t > t_1, t_2$, since the sender of $h$, $A$ has sent a past event to the receiver $C$ and one receiver of $h$, $E$ has also sent a past event to the same receiver $C$. 

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only on the size of the receiver set $J$. Below we discuss, which of the hyperedge covariates defined above are dyadic covariates.

Among the covariates based on actor-level attributes, the receiver set average and sender-receiver homophily are dyadic covariates. In contrast, receiver set homophily is not a dyadic covariate since the terms in its definition jointly consider pairs of receivers.

Neither exact repetition nor unordered exact repetition are dyadic covariates in a general sense since the terms in their definition jointly consider an unbounded number of receivers. Partial receiver set repetition of order $p$ is a dyadic covariate for $p = 1$ but not for any $p > 1$. The same holds for sender-specific receiver set repetition of order $p$. Interaction among receivers of order $p$ is not a dyadic covariate for all $p \geq 1$. Note that the definition of this covariate considers subsets of the receiver set $J$ containing $p$ previous receivers and one previous sender. Reciprocation, generalized reciprocation, and all triadic closure covariates are dyadic. For the latter, note that in the definition of these covariates we sum over single receivers $j \in J$, rather than over larger subsets of $J$.

4 Empirical analysis

We fit RHEM to empirical data produced by polyadic social interaction networks with two main goals in mind. First, we want to illustrate typical findings on RHEM effects, and their interpretation in multicast interaction networks. Second, we intend to address the four research questions on the relation between RHEM and REM specified via dyadic covariates. Specifically, we want to determine: (RQ1) the strength of empirical evidence in support of higher-order dependencies in data produced by polyadic social processes; (RQ2) the implications of controlling for higher-order dependencies for the assessment of dyadic effects; (RQ3) the extent to which RHEMs fit empirical data better than REMs; and (RQ4) the ability of RHEMs better to distinguish the receiver sets of actually observed emails, from the associated sets of actors who could have jointly received the same email, but did not.

4.1 Empirical setting and data

We demonstrate the empirical value of the model in an analysis of the Enron email data – a collection of corporate emails exchanged by employees of Enron Corporation that was made public after the company filed for bankruptcy in December 2001. For additional information on the history of the data, we refer interested readers to Zhou et al. (2007). Prior empirical analyses of the email corpus may be found in Diesner et al. (2005) and in Keila and Skillicorn (2005). To facilitate comparability, we analyze the subset of the data cleaned and processed by Zhou et al. (2007) that has also been used in the empirical example reported in Perry and Wolfe (2013) and that is available at https://github.com/patperry/interaction-proc/tree/master/data/enron. Analysis with the eventnet software requires conversion of these data into a different format. The conversion steps are explained in https://github.com/juergenlerner/eventnet/tree/master/data/enron where the converted data are also available for download. As mentioned before, the entire analysis reported in this paper is detailed in a step-by-step tutorial; see Sect. 5.7.

In the empirical section that follows we refer to this subset as the “Enron data” or just the “data.” The data comprises 21,635 emails (treated as hyperevents) among 156 Enron employees. Additionally, we use the actor attributes, gender (female = 1, male = 0), seniority (senior = 1, junior = 0), and the categorical attribute department (taking the values “Legal”, “Trading”, and “Other”). From the department attribute, we derive two binary attributes legal and trading that take the value one if the employee works in the corresponding department and zero otherwise. None of the actor attributes changes over time and there are no missing values.

The observed emails have exactly one sender and between one and 57 receivers. About 30.7% of the emails have more than one receiver and the average number of receivers is 1.77. The receiver set size distribution is further detailed in Table 1. We analyze all of these emails, that is, we do not discard emails with many receivers.
The time values of the emails in the data correspond to seconds – but time resolution is by the minute since all given time values are divisible by 60. There are 20,994 emails (about 97%) that have a unique time stamp, there are 305 time points at which exactly two different emails have been sent, nine time points are shared by three emails, and there is one time point at which four different emails have been sent. We order simultaneous emails arbitrarily. Given the high level of time resolution and the relatively low number of simultaneous hyperevents (i.e., tied event times), we believe that this decision is unlikely to affect our results in any meaningful way.

4.2 Model specification and selection

The core objectives of our analysis are to understand higher order effects in polyadic interaction networks, and to answer the four core research questions on the relation between RHEM and REM specified with dyadic covariates. We estimate two types of models, a conventional REM (“dyadic model”) and a RHEM (“polyadic model”).

The dyadic model includes the covariates receiver.avg.z for the actor-level attributes gender, seniority, trading, and legal, the covariates sender.receiver.abs.diff.z for gender and seniority, and the covariate sender.receiver.cat.diff.z for the attribute department. The dyadic model also includes the network effects partial receiver set repetition and sender-specific partial receiver set repetition of order $p = 1$, reciprocation$_t$, generalized.reciprocation, and all four triadic closure effects.

The RHEM includes all covariates of the dyadic model and in addition the covariates receiver.pair.abs.diff.z for gender and seniority, and the covariate receiver.pair.cat.diff.z for the attribute department. Moreover, the RHEM includes repetition$_t$, unordered.repetition$_t$, as well as the effects receiver.sub.rep$_t^{(p)}$, sender.receiver.sub.rep$_t^{(p)}$, and interaction.among.receivers$_t^{(p)}$ for varying values of $p$ that have been determined in a preliminary analysis (see below). Additionally, we specify and estimate two further specifications of the dyadic REM and two of the polyadic RHEM. These restricted models include covariates based on actor-level attributes only, and network effects only, respectively.

Model estimation proceeds in two steps. In the first step we apply the eventnet software [Lerner and Lomi 2020; Lerner et al. 2021], available at https://github.com/juergenlerner/eventnet to sample $k$ non-event hyperevents associated with each observed hyperevent and to compute (a superset of) all covariates of observed events and sampled non-events. We set the number of non-events per event to $k = 100$. We initially compute the covariates receiver.sub.rep$_t^{(p)}$, sender.receiver.sub.rep$_t^{(p)}$, and interaction.among.receivers$_t^{(p)}$ for $p = 1, \ldots, 10$. All network effects are defined with the half life period $T_{1/2}$ set to one week. In the second step we estimate parameters of models specified by varying lists of covariates with the coxph function in the R package survival [Therneau and Grambsch 2013, Therneau 2015]. To assess the error caused by sampling non-events (rather than considering the full risk set), we repeat the sampling 100 times, recomputing the covariate values each time. This gives us, for each actual choice of the covariate vector $x_t$, 100 different sampled log partial likelihood functions $\ell_k$ and 100 potentially different estimated parameter vectors $\hat{\beta}$.

To determine reasonable values of $p$ for the three covariates receiver.sub.rep$_t^{(p)}$, sender.receiver.sub.rep$_t^{(p)}$, and interaction.among.receivers$_t^{(p)}$, we incrementally add these covariates for increasing values of $p$ and monitor robustness of parameter estimation over different samples, as well as model fit assessed by information criteria (AIC and BIC). To this end we use the term complexity of a covariate to denote the number of nodes that are jointly considered in its computation as subsets of previous event hypereedges. The covariate receiver.sub.rep$_t^{(p)}$ has complexity equal to $p$ and sender.receiver.sub.rep$_t^{(p)}$ and interaction.among.receivers$_t^{(p)}$ have complexity equal to $p + 1$. (Note that sender.receiver.sub.rep$_t^{(p)}(i, J)$ considers the sender $i$ together with $p$ receivers in $J$ and interaction.among.receivers$_t^{(p)}(i, J)$ considers $p + 1$ receivers in $J$, one

| $|J|$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | > 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| frequency: | 14,985 | 2,962 | 1,435 | 873 | 711 | 180 | 176 | 61 | 24 | 29 | 199 |

Table 1: Number of emails (bottom row) with given number of receivers (top row).
of which is a sender of a previous event and the $p$ others are receivers of the same previous event.) With this terminology, we incrementally add the covariates $\text{receiver.sub.rep}_t^{(p)}$, $\text{sender.receiver.sub.rep}_t^{(p)}$, and $\text{interaction.among.receivers}_t^{(p)}$ with growing complexity $2, 3, \ldots$.

For complexity equal to five we observe for the first time that some of the 100 repeated samples lead to non-convergence in parameter estimation. Concretely, for two out of 100 samples we obtain parameters that diverge during parameter estimation implemented in `coxph`. In contrast, for complexity up to four, parameter estimation converges for all 100 samples. Upon closer inspection of the distribution of covariates in samples that lead to non-convergence, we find that covariates of complexity five or higher are very sparse among the non-events, that is, a typical non-event takes very small values, most often equal to zero, on these covariates of complexity five or higher. For instance, the mean value of $\text{receiver.sub.rep}^{(5)}$ over the non-events is about $3 \cdot 10^{-5}$, the mean of $\text{sender.receiver.sub.rep}^{(4)}$ over the non-events is $7 \cdot 10^{-6}$, and that of $\text{interaction.among.receivers}^{(4)}$ is $1 \cdot 10^{-7}$. The respective mean values over the events are much higher: 0.16, 0.30, and 0.004, respectively. This implies that these covariates have an excessively strong effect on event probabilities which leads to divergence during parameter estimation. We therefore consider only effects of complexity up to four. Model fit measured in AIC and BIC consistently increases for growing complexity up to four (and even beyond four for those samples on which parameter estimation converges). We therefore decided to specify the most complex RHEM by including $\text{receiver.sub.rep}^{(p)}$ for $p = 1, 2, 3, 4$ and $\text{sender.receiver.sub.rep}^{(p)}$ and $\text{interaction.among.receivers}_t^{(p)}$ for $p = 1, 2, 3$. In the following we denote this model as “the RHEM”.

Among the covariates included in the RHEM, two pairs have correlation exceeding 0.9. These are the pairs ($\text{repetition}_t$, $\text{unordered.repetition}_t$) with a correlation of 0.93 and ($\text{receiver.sub.rep}_t^{(3)}$, $\text{sender.receiver.sub.rep}_t^{(3)}$) with a correlation equal to 0.98. To test how findings are affected by such high correlations, we additionally fit a reduced RHEM without $\text{unordered.repetition}_t$ and without $\text{sender.receiver.sub.rep}_t^{(3)}$. This reduced RHEM has a lower model fit than the RHEM including all effects. Moreover, the findings on all covariates assessing subset repetition in the receiver set (sender-specific or not) are not affected qualitatively: they are all consistently positive in the reduced RHEM and in the RHEM including all effects. However, the parameter of repetition switches its sign from significantly positive to significantly negative when we additionally include unordered repetition. We argue that this sign switch does not point to lack of robustness in our findings – but rather to a relevant effect in multicast communication networks that can be well explained (see the discussion below).

5 Results and discussion

5.1 Discussion of effects

Estimated parameters and standard errors of the three models specified with dyadic covariates are reported in Table 2 those of RHEM including dyadic and higher-order effects are reported in Table 3. Since covariates modeling network effects are skewed and have no natural unit, we first scale those covariates by the transformation $x \mapsto \log(1 + x)$ and then standardize them to mean zero and standard deviation equal to one.

In the following we discuss findings effect by effect across all models in which the respective covariate is included. Even though we fitted models to 100 different samples, the reported parameters have been estimated from one arbitrarily selected single sample. Indeed, we believe that this situation is more representative for empirical studies than results derived from repeated sampling. However, in our discussion we highlight findings related to some of the effects that are inconsistent across the 100 repeated samples from the risk sets, i.e., we draw attention on parameter signs or significance levels obtained from at least one sample that differ from those obtained in other samples (compare Sect. 5.2).

Effect of gender. The parameter of $\text{receiver.avg.female}$ is significantly positive in all four models including this effect. This means that female employees tend to receive emails at a higher rate than male – controlling for all other effects in the model. We note that there are 43 female and 113 male among the 156
employees so that receiver sets are still predominantly male. The parameter sign of receiver.avg.female is comparable across models. We also find sender-receiver homophily with respect to gender: the average absolute difference between the gender of the sender $i$ and that of the receivers $J$ has a negative effect on the event rate on the hyperedge $(i, J)$. For this effect we find that parameter values become much smaller in absolute value (i.e., closer to zero) when we additionally control for network effects. In the RHEM we find additional forms of gender-homophily in the receiver set: emails are more likely to be send to homogeneous sets of receivers (consisting mostly of female or mostly of male), irrespective of the gender of the sender. However, the finding on receiver.pair.abs.diff.gender is one of the few not consistent across all samples: it is not significant in one out of 100 samples and significantly negative at the 5% level in the remaining 99.

**Effect of seniority.** The positive parameter sign of receiver.avg.seniority suggests that senior employees receive emails at a higher rate than junior employees. Similarly to findings on the gender variable, we find seniority-homophily between a sender and her receivers as well as seniority-homophily within the receiver set in the RHEM. These parameters consistently approach zero when controlling for network effects. The latter finding suggests that failure to control for network effects could overestimate the effect size of actor-level covariates.

**Effect of department.** Employees working in the legal department have been found to be more likely to receive emails in three of the four models containing this effect. The parameter size decreases when controlling for network effects, and the parameter is not significant in the largest RHEM “att+net (rhem)”. Across the samples, this non-significance holds for 73 out of 100 samples; the parameter is significantly positive in the remaining 27 samples. Employees working in the trading department are less likely to receive emails and we find homophily with respect to department, both between senders and receivers and (in the RHEM) within the receiver set. All these parameters approach zero when controlling for network effects.

**Exact (ordered and unordered) repetition.** In the RHEM the parameter associated with unordered repetition is significantly positive and that of repetition is significantly negative. These two effects have to be interpreted together – as we illustrate building on the example from Fig. 2. Assume that at $t_1$
Table 3: Estimated parameters of RHEM.

|                           | att (rhem) | net (rhem) | att+net (rhem) |
|---------------------------|------------|------------|----------------|
| r.avg.female              | 0.28 (0.02)*** | 0.22 (0.02)*** |
| s.r.abs.diff.gender       | −0.50 (0.02)*** | −0.19 (0.02)*** |
| r.pair.abs.diff.gender    | −0.43 (0.04)*** | −0.19 (0.06)*** |
| r.avg.seniority           | 0.68 (0.02)*** | 0.28 (0.02)*** |
| s.r.abs.diff.seniority    | −0.94 (0.02)*** | −0.43 (0.02)*** |
| r.pair.abs.diff.seniority | −1.42 (0.04)*** | −0.79 (0.07)*** |
| r.avg.in.legal            | 1.23 (0.02)*** | 0.05 (0.03) |
| r.avg.in.trading          | −0.49 (0.03)*** | −0.08 (0.03)*** |
| s.r.cat.diff.dept         | −1.88 (0.02)*** | −0.76 (0.02)*** |
| r.pair.cat.diff.dept      | −2.22 (0.04)*** | −1.14 (0.07)*** |
| repetition                | −0.25 (0.01)*** | −0.22 (0.01)*** |
| undirected.repetition     | 0.41 (0.01)*** | 0.39 (0.01)*** |
| r.sub.rep.1               | 0.13 (0.02)*** | 0.10 (0.02)*** |
| r.sub.rep.2               | 0.14 (0.01)*** | 0.11 (0.01)*** |
| r.sub.rep.3               | 0.16 (0.02)*** | 0.14 (0.02)*** |
| r.sub.rep.4               | 0.43 (0.06)*** | 0.40 (0.06)*** |
| s.r.sub.rep.1             | 0.73 (0.01)*** | 0.68 (0.01)*** |
| s.r.sub.rep.2             | 0.54 (0.02)*** | 0.51 (0.02)*** |
| s.r.sub.rep.3             | 0.53 (0.09)*** | 0.41 (0.08)*** |
| reciprocation              | 0.06 (0.01)*** | 0.06 (0.01)*** |
| general.recip             | 0.02 (0.02) | 0.04 (0.02)*** |
| interact.receiver.1       | 0.17 (0.01)*** | 0.16 (0.01)*** |
| interact.receiver.2       | 0.17 (0.03)*** | 0.15 (0.03)*** |
| interact.receiver.3       | 1.42 (0.13)*** | 1.27 (0.13)*** |
| shared.sender             | 0.40 (0.02)*** | 0.35 (0.02)*** |
| shared.receiver           | 0.05 (0.02)*** | 0.01 (0.02)*** |
| transitive.closure        | −0.05 (0.02)** | −0.03 (0.02)*** |
| cyclic.closure            | −0.14 (0.01)*** | −0.12 (0.01)*** |

AIC 159,300.98 77,558.15 75,057.20
Num. events 21,635 21,635 21,635
Num. obs. 2,185,135 2,185,135 2,185,135

***p < 0.001, **p < 0.01, *p < 0.05
actor $A$ sends an email to $\{B, C, D, E\}$ and consider two alternative hyperedges $h = (C, \{A, B, D, E\})$ and $h' = (A, \{B, C, D, E\})$ at a later point in time $t > t_1$. The past email $e_1 = (t_1, A, \{B, C, D, E\})$ increases the value of the covariate $\text{repetition}_1(h')$ and it increases the value of the covariate $\text{unordered\_repetition}_1(h')$. The joint effect of these two covariates is positive (since unordered repetition has a larger parameter than the absolute value of the negative repetition parameter). Thus, the past event $e_2 = (t_1, A, \{B, C, D, E\})$ makes a repeated email from $A$ to $\{B, C, D, E\}$ at $t > t_1$ more likely. However, an event at $t$ on hyperedge $h$, that is, sent from $C$ to $\{A, B, D, E\}$ would be even more likely than an email from $A$ to $\{B, C, D, E\}$. This is because the past email $e_1 = (t_1, A, \{B, C, D, E\})$ increases the value of the covariate $\text{unordered\_repetition}_1(h)$ (having a positive effect) but it does not increase the value of $\text{repetition}_1(h)$ (which would have had a negative effect).

Thus, while conversations within fixed lists of actors (e.g., $A, B, C, D, E$ in the example from Fig. 2) are overrepresented, it is more likely that a future email within the same fixed set of actors has another sender than the preceding email. This points to a form of “turn-taking” (Gibson 2003) within fixed conversation groups. We note that it would not be possible to express these effects purely with dyadic covariates.

(Sender-specific) partial receiver set repetition. In the RHEM all covariates expressing subset repetition in the receiver set (with order $p = 1, 2, 3, 4$), as well as all covariates for sender-specific subset repetition in the receiver set (with order $p = 1, 2, 3$) have a significantly positive effect. This demonstrates that repetition from the same sender to the same receivers (possibly within a larger and varying set of yet other receivers) is not a purely dyadic phenomenon. Instead, if two (or three or four) actors have already jointly received an email, then they are more likely to do so again. This points to a clustering of the set of all possible receivers into subsets that are likely to receive the same emails. We find evidence for a “global” clustering (that is, one that applies to the average sender) and for a sender-specific clustering (so that one sender can structure the space of receivers in a different way than another sender).

(Sender-specific) receiver subset repetition of order one is a dyadic effect and thus also included in the dyadic model. We find the same parameter signs in both types of models – although the parameter of sender-specific receiver subset repetition of order one is smaller in the RHEM than in the dyadic model. Controlling for the effect of actor-level attributes consistently decreases the size of partial receiver set repetition. This may suggest that the model specified purely by network effects (i.e., ignoring the effect of actor-level attributes) can overestimate the effect size of partial receiver set repetition.

(Generalized) reciprocation. We find, in the dyadic models and in the largest RHEM, evidence for reciprocation and for generalized reciprocation. However, generalized reciprocation is not significant in the RHEM purely based on network effects, excluding attribute effects “net (rhem)”. The parameter values in the dyadic model are much larger than those in the RHEM.

Interaction among receivers. In the RHEM we find a positive effect for interaction among receivers of order $p = 1, 2, 3$. This suggests that actors are more likely to send emails to the sender of a previous email jointly with some of the receivers of the same previous email. This effect cannot be expressed with dyadic covariates, already for $p = 1$. This effect for $p = 3$ is one of the few effects that are inconsistent across samples. While it is significantly positive in 98 of the 100 samples, it is not significant in the remaining two.

Triadic effects. The triadic effects can be included in the dyadic model and in the RHEM. We find that $\text{shared\_sender}$ is significantly positive in all four models including it. This suggests that if two actors $A$ and $B$ both have received emails from a common “third” actor $C$, then $A$ becomes more likely to send an email to $B$ in the future. Having sent a past email to a common receiver ($\text{shared\_receiver}$) has a significantly positive effect in both models that do not control for attribute effects but is not significant in the models that do control for attribute effects. Transitive closure is significantly positive in the dyadic models, significantly negative in the RHEM purely based on network effects “net (rhem)” and not significant in “att+net (rhem)”.

More precisely, it is not significant in 83 out of 100 samples but significantly negative in the remaining 17. Finally, cyclic closure is significantly negative in all four models including this effect. A negative cyclic closure effect is consistent with a hierarchical interpretation of interaction, for instance, that interaction tends to
go from lower to higher status. Positive cyclic closure would contradict such a hierarchical interpretation (Lerner and Lomi 2017).

5.2 Variation across samples

We repeated the sampling from the risk set 100 times. Below we discuss which of the findings on the various covariates are qualitatively different across samples.

Most findings in the largest RHEM (“att+net (rhem)”) are qualitatively identical across all 100 samples. That is, for most covariates we could have chosen any of the 100 samples and would have obtained the same parameter signs and significance at the 5% level as reported in 3. Most of these effects on which the different samples yield consistent findings are significantly different from zero. However, one effect, namely shared.receiver is not significant in any of the 100 samples. Effects on which findings from at least one sample differs from the rest are the following. The effect receiver.pair.abs.diff.gender is significantly negative in 99 out of 100 samples. However, the remaining sample leads to a z-value equal to -1.736, that is, the parameter is not significant at the 5% level in this single sample. The effect receiver.avg.legal is not significantly different from zero in 73 out of 100 samples. In the remaining 27 samples the associated parameter is significantly positive. The effect interact.receiver has been found to be significantly positive in 98 out of 100 samples and not significant in the remaining two. The effect transitive.closure is not significant in 83 out of 100 samples but significantly negative in the remaining 17.

All findings in the largest dyadic model (“att+net (dyad)”) are qualitatively identical across all 100 samples. That is, for all dyadic covariates we could have chosen any of the 100 samples and would have gotten the same parameter signs and significance at the 5% level as reported in 3. All covariates except shared.receiver are consistently significant at the 5% level. The covariate shared.receiver is not significant in any of the 100 samples.

5.3 Evidence of higher-order effects

Overall, we find strong evidence of the presence of higher-order effects in the data. Covariates that go beyond purely dyadic effects have been found to be significantly predictive of future event distributions. As outlined above, we contend that such interdependence between different receivers should not be considered as an annoyance to be controlled away – but rather as an opportunity to develop and test additional theories about the structure of polyadic interaction networks. Some of the higher-order effects are highly relevant for empirical studies. Homophily in the receiver set, exact repetition and unordered exact repetition, and (sender-specific) partial receiver-set repetition of order \( p \geq 2 \) all have relevant implications for the structure and dynamics of polyadic interaction networks.

5.4 Differences between RHEM and dyadic models

In most cases, parameters of covariates that have been included in dyadic models and in RHEM are estimated to have the same sign. However, in most cases the absolute values of these parameters are considerably smaller in the RHEM than in the dyadic models. This may suggest that failure to control for higher-order effects could lead to an overestimation of effect sizes. In our empirical analysis the dyadic models suggested that transitive closure is positive and strongly significant, while RHEM found the same effect to be mostly not significant, or even negative.

5.5 Model fit

Regarding model fit, measured with information criteria, we find two patterns. First, network effects improve the model fit much more than attribute effects. However, the joint models including attribute and network effects have the best fit in the family of dyadic models and in RHEMs. Second, RHEM fit the data better than the corresponding dyadic models. We find that the RHEM purely specified with network effects “net (rhem)” already has a better fit than the full dyadic model “att+net (dyad)”.
Table 4: Within sample predictive performance.

|              | all | $|J| = 1$ | $|J| = 2$ | $|J| = 3$ | $|J| = 4$ | $|J| \geq 5$ |
|--------------|-----|---------|---------|---------|---------|-----------|
| num.emails   | 21,635 | 14,985 | 2,962  | 1,435  | 873   | 1,380   |
| RHEM # first | 13,129 | 7,292  | 2,382  | 1,302  | 832  | 1,321  |
| RHEM % first | 60.68 | 48.66  | 80.42  | 90.73  | 95.30  | 95.72 |
| RHEM avgrank | 3.76 | 4.95 | 1.91  | 0.68  | 0.25  | 0.24  |
| dyad # first | 12,580 | 7,169  | 2,142  | 1,228  | 786  | 1,255  |
| dyad % first | 58.15 | 47.84  | 72.32  | 85.57  | 90.03  | 90.94  |
| dyad avgrank | 4.11 | 5.06 | 2.66  | 1.36  | 1.27  | 1.55  |

5.6 Predictive performance

We further test how well models succeed in predicting events, that is, how well they are able to distinguish the one observed event from the 100 associated non-events sampled from the risk set. As an error measure we determine for each observed event the number of associated non-events whose predicted event rate exceeds the predicted event rate of the observed event. We then take the average over all events to assess the prediction error of the model. The best possible value is zero (implying that no non-event is considered more likely than the associated observed event), the worst possible value is 100 (meaning that all observed events are considered to be the least likely among all sampled alternatives), and random uniform guessing would lead to an expected average rank of 50. We perform two types of prediction experiments: “within sample” where we estimate models based on all 21,635 events and then evaluate the prediction error on the same data, and an “out-of-sample” scenario where we estimate model parameters based on a subset of “training data” comprising the first 90% of all events (19,471 emails) and then evaluate performance on the “test data” comprising the last 10% of the data (2,164 emails). We note that in the latter scenario, the predictions for the test data instances also take into account information from other test instances. Notably, the hyperedge covariates $x_t(i,J)$ are functions of earlier events ($t',i',J'$) with $t'<t$, even if $t'$ lies in the test interval. The difference is that in the latter scenario, model parameters are estimated without considering test instances. We assess predictive performance for the largest dyadic model “att+net (dyad)” and for the largest RHEM “att+net (rhem)”. Besides reporting the error and some other related measures on the entire (test) data, we also assess performance separately for emails with specific receiver set sizes to evaluate how relative performance gains differ between dyadic events (with a single receiver) and hyperevents with growing number of receivers.

Within sample predictive performance. Table 4 reports predictive performance in the within sample scenario. For both models we report, for all emails and for emails with given constraints on the receiver set size, the number and percentage of events whose predicted rate is not surpassed by any associated non-event and the average rank.

We find that the RHEM outperforms the dyadic model when considering all emails and also for any subset of emails (emails with number of receivers equal to 1, 2, 3, 4 and emails with five or more receivers). A striking result is that RHEM are even better at predicting dyadic events ($i,J$) where the receiver set has size $|J| = 1$. However, on those dyadic emails, RHEM improve dyadic models only by a small margin (e.g., average rank of 4.95, compared to 5.06). The relative gain implied by hyperedge covariates in addition to dyadic covariates increases with the size of the receiver set. For instance, for emails with five or more receivers the RHEM achieves an average rank of 0.24 compared with 1.55 for the dyadic model (i.e., an error that is more than six times as high). For both models, prediction of dyadic emails ($|J| = 1$) is the most difficult and predictive performance typically improves with growing receiver set size (however, average rank restricted to emails with at least five receivers gets worse for the dyadic model). An explanation for this pattern is that randomly selected actor sets of increasing size are more and more unlikely to be the receiver list of a common email and, thus, can be more easily recognized as non-events by models.
Table 5: Out-of-sample predictive performance.

|                | all | $|J| = 1$ | $|J| = 2$ | $|J| = 3$ | $|J| = 4$ | $|J| \geq 5$ |
|----------------|-----|---------|---------|---------|---------|-----------|
| num.emails (test) | 2,164 | 1,530 | 308 | 139 | 66 | 121 |
| RHEM # first      | 1,320 | 764 | 247 | 130 | 61 | 118 |
| RHEM % first      | 61.00 | 49.93 | 80.19 | 93.53 | 92.42 | 97.52 |
| RHEM avgrank      | 3.97 | 5.18 | 1.91 | 0.17 | 0.29 | 0.31 |
| dyad # first      | 1,251 | 738 | 237 | 112 | 57 | 107 |
| dyad % first      | 57.81 | 48.24 | 76.94 | 80.58 | 86.36 | 88.43 |
| dyad avgrank      | 4.44 | 5.51 | 2.49 | 1.35 | 0.45 | 1.54 |

Out-of-sample predictive performance. Table 5 reports predictive performance in the out-of-sample scenario. Again we find that the RHEM outperforms the dyadic model on all subsets of emails, even for dyadic emails, and that the relative gain typically increases with growing receiver set size. For this scenario, the RHEM achieves its best performance on emails with three receivers and the dyadic model on emails with four receivers.

5.7 Replicability

The analysis reported above is explained in a dedicated software tutorial[^1], from the data preprocessing and the computation of covariates over to model estimation in R. Thus, the analysis reported this paper is fully replicable and the tutorial points out some model variations that might be relevant in given application scenarios. The software may be adopted in – and adapted to future empirical studies involving polyadic social interaction processes.

6 Conclusion

The objectives of our paper are to introduce, analyze, and illustrate hyperedge covariates for polyadic interaction networks and to compare RHEM, including higher-order effects, with REM purely specified in terms of dyadic covariates. Answering our four research questions we obtain the following findings from our empirical analysis using the Enron email data. (RQ1) There is ample evidence for the presence of higher-order effects captured by hyperedge covariates that jointly consider receiver sets of size larger than one. (RQ2) Not controlling for such higher-order dependencies can affect estimated parameters of dyadic covariates. However, in our analysis, this change seems to affect mostly parameter sizes (and sometimes also their significance levels) but in most cases does not switch the direction of effects (i.e., parameter signs). (RQ3) RHEM consistently fit the data better than dyadic models, and (RQ4) RHEM consistently outperform dyadic models with respect to predictive performance.

The results we reported suggest that researchers interested in analyzing polyadic interaction networks should not restrict their models to dyadic specifications. From a high-level perspective, the step from models for multicast networks specified via dyadic covariates (Perry and Wolfe 2013) to RHEM is relatively intuitive. However, hyperedge covariates provide a much richer set of possible effects and considerably increase the fit of models for given empirical data.

These results strengthen our view that that higher-order effects should not be considered merely as an annoyance that has to be controlled for – but rather as an opportunity to develop and test additional theories about the structure and dynamics of social interaction and communication networks. Some of the effects that we have documented could not have been discovered or tested with models purely specified via dyadic covariates. Concretely, this includes findings on unordered repetition (pointing to turn-taking within a fixed list of conversation participants), partial receiver set repetition (related with a clustering of the actors into

[^1]: https://github.com/juergenlerner/eventnet/wiki/Directed-RHEM-for-multicast-interaction-(tutorial)
groups that are likely receivers of a joint message), and sender-specific partial receiver set repetition (pointing to patterns in which different senders cluster the actors into potentially different groups).

Our analysis found limits on the complexity of effects that can be included in RHEM. For instance, effects that consider five or more actors at a time could lead, in some samples, to non-convergence in parameter estimation. The reason for this is that effects of increasing complexity are increasingly sparse among the non-events, so that randomly selected hyperedges are too often equal to zero on such covariates. This finding is also due to the fact that average hyperedge sizes in the Enron email data are still rather small; 70% of the emails have exactly one receiver and the average number of receivers is 1.77. In other empirical settings, for instance, coauthorship networks (Newman 2004), hyperevents are typically much larger. For example, in the coauthorship networks considered in Lerner et al. (2019), the average number of authors per paper is close to eight and some events have size up to 100. As another example, in the meeting events extracted from contact diaries analyzed in Lerner et al. (2021), the average number of participants per meeting is 4.4. With larger hyperevents, it is possible to include subset repetition covariates of higher order.

Our primary goal in this paper was to establish hyperedge covariates and compare RHEM to dyadic models, rather than to elaborate parameter estimation techniques. To deal with the exponential size of the risk set, we proposed case-control sampling — which is readily available and a well established technique in the analysis of rare events (Borgan et al. 1995; Keogh and Cox 2014). Hyperevents could indeed be considered as rare events since a randomly selected subset of actors is very unlikely to experience even one common event. However, case-control sampling, in which we uniformly sample from the risk set, revealed some limitations, mostly due to the fact that some higher-order covariates are very sparse in the risk set. In turn, most sampled non-events did not provide any information on those covariates which increased the necessary sample size. A more efficient sampling scheme could be stratified sampling, where, for instance, one could include as strata non-events that are non-zero in some subset repetition covariates. Another possibility would be Markov-chain Monte Carlo (MCMC) methods which do not sample uniformly from the (stratified or unstratified) risk set but sample proportional to the conditional event probability of subsets. The additional cost of MCMC methods, however, is that sampling is no longer independent of model parameters.

We point out that, even if we do not include subset repetition covariates of order five or higher, our models do not preclude hyperevents of that size. The reason for this is that subset repetition covariates are nested within each other: whenever a past event implies a non-zero value in subset repetition of order \(p\), then it necessarily implies non-zero values in subset repetition of lower order \(p' < p\); compare the examples given in Sect. 3.3.2.

Besides improved parameter estimation techniques for RHEM, we consider the further development of other RHEM covariates as a promising direction for future work. We mentioned above that the collection of hyperedge covariates proposed in this paper is far from exhausting the possibilities. Developing further effects that are relevant for given application scenarios can further extend the potential of RHEM.

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