N=2 Super Yang Mills Action and BRST Cohomology

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Abstract

The extended BRST cohomology of N=2 super Yang-Mills theory is discussed in the framework of Algebraic Renormalization. In particular, N=2 supersymmetric descent equations are derived from the cohomological analysis of linearized Slavnov-Taylor operator $B$. It is then shown that both off- and on-shell N=2 super Yang-Mills actions are related to a lower-dimensional gauge invariant field polynomial $Tr\phi^2$ by solving these descent equations. Moreover, it is found that these off- and on-shell solutions differ only by a $B$–exact term, which can be interpreted as a consequence of the fact that the cohomology of both cases are the same.

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1 Introduction and conclusions

One of the main reasons that supersymmetric quantum field theories have been extensively studied in recent years is that they display important finiteness properties due to the cancellations of ultraviolet divergences [1] that are first verified by using superfield formalism in the superspace [2]. In component field formalism, that is needed when calculations on nontrivial backgrounds are considered, these non-renormalization theorems are derived due to the fact that the non-renormalized interaction terms and/or the actions themselves can be written as multiple supervariations of one chirality of lower dimensional field monomials [3, 4, 5, 6]. The algebraic source of these results is related with the cohomological structure of supersymmetric models.

On the other hand, to construct the exact supersymmetric interaction terms and/or the actions by applying super-variations to lower dimensional field monomials\(^1\), one has to use off-shell formulation of the supersymmetry, i.e. supersymmetric field content of the theory should include auxiliary fields [5]. Otherwise, supersymmetry algebra realized without auxiliary fields (on-shell supersymmetry) closes modulo equations of motion terms and as a consequence the on-shell supervariations of these lower dimensional field monomials give different expressions than the original interaction terms and actions [5]. For instance, when both of the N=1 and N=2 super Yang-Mills (SYM) actions are derived by using pure on-shell supervariations of lower dimensional field monomials, the resulting expressions differ from the original one (that can be found by using off-shell variations) by equation of motion terms and there is not a non-trivial way to restore these missing terms [5].

The problem of above mentioned on-shell closure of the algebra can be overcome by extending the BRST transformations to include supersymmetry transformations [9, 10, 11] in the algebraic renormalization framework [13] that is structurally equivalent [14] to Batalin-Vilkovisky formalism [15]. (Note also that the extension of BRST transformations that includes arbitrary rigid symmetries is given in Ref. [12].) It is then possible to derive an off-shell nilpotent Slavnov-Taylor operator \( \mathcal{B} \) from an action functional that is extended by antifields (sources for extended BRST transformation of the fields) both for off- and on-shell supersymmetric cases. However, for on-shell supersymmetric case further extension of the action by adding some (none standard) quadratic terms in the anti-fields of fermions to the action is needed in order to get such a nilpotent operator [9, 10, 11]. The renormalization program then reduces to an algebraic discussion of the cohomology of a linearized Slavnov-Taylor operator defined on the space of integrated field polynomials. The counter terms and the possible anomalies are then the solutions of this cohomology problem with ghost number 0 and 1 respectively [13].

Therefore, since the supersymmetric actions also belong to the cohomology of Slavnov-Taylor operator \( \mathcal{B} \) with ghost number 0, in order to relate the above mentioned actions exactly to lower dimensional field monomials, it is natural to study a (extended) BRST cohomology problem in the algebraic renormalization framework by using the set of descent equations. In a recent paper [16] we were able to achieve this goal for both off- and on-shell N=1 SYM action.

\(^1\)For a similar approach of constructing N=1 globally and locally supersymmetric actions and also for the discussion of anomalies, see [7, 8].
and our aim in this paper is to extend the analysis of Ref. [16] to N=2 SYM theory.

Our motivation is twofold: first of all, it is well known that the perturbative beta function of the theory receives only one-loop contribution. This non-renormalization theorem has been proven algebraically in Ref. [17] by using the above mentioned set of descent equations for the twisted version of N=2 SYM theory [17]. The proof relies on a relationship between the lower dimensional gauge invariant field monomial $Tr\phi^2$ and the twisted action of N=2 SYM theory (see also Ref. [18]). Therefore, we find also interesting to study the original, untwisted N=2 theory. As it will be shown explicitly, solutions of the set of descent equations, that can be derived from the operator $B$ by using Wess-Zumino consistency condition, give also a similar relation between both off- and on-shell supersymmetric actions and the gauge invariant field monomial $Tr\phi^2$, where the scalar field $\phi$ is the lowest component of the N=2 vector multiplet. In other words, we show explicitly that both off- and on-shell actions of the theory can be constructed from $Tr\phi^2$. It is then straightforward to generalize the algebraic criterion for the non-renormalization theorem given for the twisted N=2 SYM [4] to both off- and on-shell supersymmetric (untwisted) N=2 SYM theory. Note that, this result is not surprising since the twisted Yang-Mills theory can be obtained from the original N=2 SYM by directly performing field redefinitions (i.e. without twisting) [20].

Our second motivation is to understand further the structure of the descent equations for globally supersymmetric gauge theories. The structure of these descent equations are quite different from the ones of nonsupersymmetric theories and the solutions of these equations are highly constrained due to supersymmetry. In a previous work [16], these descent equations are found for N=1 SYM by using the supersymmetric structure of the theory. Therefore, it is also useful to extend the method given in Ref. [16] in order to derive the complete set of descent equations for (untwisted) N=2 SYM theorem. The structure of these descent equations for N=2 SYM are similar with that of twisted SYM [4, 18, 21], as expected. However, it is worth mentioning that when these equations are compared with that of N=1 SYM case, it is seen that the structure and also the number of the descent equations are determined due to supersymmetry together with the corresponding R-symmetry of the theory. For instance, as it will be derived explicitly in this paper, the descent consists of five equations (i.e. four descendants) due to the $SU(2)_R$ symmetry of N=2 SYM theory whereas there was only three (i.e. two descendants) for N=1 case [16]. This fact may also have interesting outcomes when descent equations for N=4 SYM are considered since the internal R-symmetry group of the theory is $SU(4)$.

The organization and the results of the paper are as follows. In Sec.II, we review briefly the extension of BRST transformations to include N=2 supersymmetry and we introduce corresponding Slavnov-Taylor (ST) operator $B$ for both off- and on-shell N=2 supersymmetric Yang-Mills theory. In Sec.III, the descent equations, which arise from the cohomological analysis of $B$, are derived for N=2 SYM theory. In Sec.IV, by solving these equations for ghost number 0, we derive algebraic identities that relate the 2-dimensional gauge invariant field

\footnote{Note also that, in a recent paper [19] background field method (BFM) has been formulated for the twisted N=2 SYM. Our results may also be useful to formulate BFM for the original, untwisted case.}

\footnote{Recently, a systematic framework is proposed in order to solve the supersymmetric descent equations [21]. The approach given in Ref. [16] for N=1 SYM is also shown to be consistent with this framework [21].}
monomial $Tr \phi^2$ to both off and on-shell supersymmetric (extended) Yang-Mills actions. We also show that these two solutions of off- and on-shell supersymmetric cases differ from each other only by a $B$-exact term. This is a consequence of the fact that the BRST cohomology of both cases are the same. The missing terms that are found by climbing up with pure on-shell supervariations [5] are then restored with the help of this $B$-exact term.

2 N=2 SYM theory, extended BRST transformations and Slavnov-Taylor operator

In this work we study the formulation of N=2 SYM theory given in [22, 23] by using the supersymmetry conventions of [24]. We begin our analysis with the off-shell supersymmetric Yang-Mills theory in Wess-Zumino gauge. The action of the theory

$$S_{N=2} = \frac{1}{g^2} Tr \int d^4x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \bar{\lambda} i \slashed{D} \lambda_i + \phi D_\mu D^\mu \phi)$$

contains fields of N=2 vector multiplet $V = (A_\mu, \phi, \phi^\dagger, \lambda_i^\alpha, \bar{\lambda}_i^{\dot{\alpha}}, \bar{\lambda}_i^{\dot{\alpha}})$, where the gauge field $A_\mu$ and the scalar fields $\phi, \phi^\dagger$ are singlets, the Weyl spinors $\lambda_i^\alpha, \bar{\lambda}_i^{\dot{\alpha}}$ are doublets and the auxiliary field $\bar{D}$ is a triplet under the $SU(2)_R$ symmetry group [22, 23]. The $SU(2)_R$ indices of the spinors are raised and lowered due to

$$\lambda^i = \epsilon^{ij} \lambda_j, \quad \lambda_i = \lambda^j \epsilon_{ji}, \quad \bar{\lambda}_i = \epsilon_{ij} \bar{\lambda}_j, \quad \bar{\lambda}^i = \bar{\lambda}_j \epsilon^{ji}$$

where the antisymmetric tensor $\epsilon^{ij}$ is given as,

$$\epsilon_{12} = \epsilon^{12} = -\epsilon_{21} = -\epsilon^{21} = 1.$$ 

The action (1) is invariant under N=2 supersymmetry transformations,

$$\delta = \theta^i Q_i^\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$$

that obey the following off-shell algebra

$$\{ Q_i, \bar{Q}^j \} = -2i \delta^i_\beta \sigma^\mu D_\mu$$

$$\{ Q_\alpha, Q_\beta \} = -2i \sqrt{2} \epsilon_{ij} \epsilon_{\alpha\beta} \delta_\gamma(\phi)$$

$$\{ \bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}} \} = -2i \sqrt{2} \epsilon^{ij} \epsilon^{\dot{\alpha}\dot{\beta}} \delta_\gamma(\phi).$$

where $Q_\alpha$ and $\bar{Q}^{\dot{\alpha}}$ are chiral and antichiral part of the supersymmetry transformations, $\theta^i, \bar{\theta}_i$ are corresponding anti-commuting supersymmetry parameters, and $\delta_\gamma$ denotes field dependent gauge transformations with respect to its argument.

The extension of BRST transformations to include global symmetries is well known [9, 10, 11, 12] and is first given in Ref.[10] for N=2 supersymmetry. Following the standard procedure,
on the members of N=2 vector multiplet such an extended BRST generator $s$ can be defined as

$$s := s_0 - i\xi^i Q_i - i\bar{\xi}_i \bar{Q}^i$$

where $s_0$ is the ordinary BRST transformation and $\xi^{ia}$ and $\bar{\xi}_{ia}$ are the constant commuting chiral and antichiral SUSY ghosts respectively. Note that since $s_0$ carries ghost number and it is anti-commuting, the parameters of the global supersymmetry transformations are promoted to the status of constant ghosts and their Grassmann parity are changed so that the extended transformation $s$ is a homogeneous transformation.

The extended BRST transformation of the fields can now be written as,

$$sA_\mu = D_\mu c + \xi_i \sigma_\mu \bar{\lambda}^i + \bar{\xi}^j \bar{\sigma}_\mu \lambda_j$$

$$s\lambda_i = i\{c, \lambda_i\} - i\sigma^{\mu\nu} \xi_i F_{\mu\nu} + \xi_i[\phi, \phi^\dagger] - \sqrt{2}\sigma^{\mu\nu} \xi_i D_\mu \phi + \bar{\tau}^j \xi_j. \bar{D}$$

$$s\bar{\lambda}^i = i\{c, \bar{\lambda}^i\} - i\bar{\sigma}^{\mu\nu} \bar{\xi}^i F_{\mu\nu} - \bar{\xi}^i[\phi, \phi^\dagger] - \sqrt{2}\bar{\sigma}^{\mu\nu} \bar{\xi}^i D_\mu \phi - \bar{\xi}\bar{\tau}_i^j. D$$

$$s\phi = i\{c, \phi\} - i\sqrt{2}\xi_i \bar{\lambda}^i$$

$$s\phi^\dagger = i\{c, \phi^\dagger\} - i\sqrt{2}\bar{\xi}^i \lambda_i$$

$$s\bar{D} = i\{c, \bar{D}\} + i\bar{\tau}^j \xi_j \bar{D}\bar{\lambda}^i - \sqrt{2}\xi_i [\lambda_j, \phi^\dagger] - \sqrt{2}\bar{\xi}_j \bar{\lambda}^i$$

$$s\bar{c} = \frac{i}{2}\{c, \bar{c}\} - 2i\xi_i \sigma^\mu \bar{\xi}^i A_\mu - \sqrt{2}\xi_i \bar{\xi}^i \phi^\dagger - \sqrt{2}\bar{\xi}_i \xi^i \phi$$

where $c$ is the usual Faddeev-Popov ghost field and $\bar{\tau}$'s are Pauli spin matrices. Note that with the help of extra terms in $sc$, $s^2$ closes on translations,

$$s^2 = -2i\xi^i \sigma^\nu \xi^j \partial_\nu$$

in other words the complication that SUSY algebra is modified by field-dependent gauge transformations is solved. Note also that in order to get a nilpotent $s$, the definition (4) could be extended to include translations by introducing suitable translation ghosts (see for instance [10]). However, since our aim is to work with the integrated field polynomials the definition of $s$ given in (4) will cause no problems.

Since the action (1) is invariant under gauge transformations and supersymmetry, it is obviously invariant under extended BRST transformation $s$. The gauge fixing of the action (1) can be performed by adding an $s$-exact term [13], that is compatible with supersymmetry, since the extended BRST operator $s$ contains supersymmetry. We choose this term to be Landau type,

$$S_gf = - tr \int d^4 x s(\bar{c} \partial^\mu A_\mu)$$

where the fields $(\bar{c}, b)$ are the trivial pair that are introduced in the standard procedure of gauge fixing,

$$s\bar{c} = b \quad , \quad sb = -2i\xi^i \Sigma \partial_\nu \bar{c}.$$ 

In order to write Slavnov-Taylor (ST) identity from the gauge fixed action, the field content of the theory is extended to include antifields (sources) that couple to the corresponding $s$-transformations of the fields,

$$S_{quad} = tr \int d^4 x (A_\mu^* sA^\mu + \phi^* s\phi + \phi^\dagger s\phi^\dagger + \lambda^* s\lambda_i + \bar{\lambda}_i^* s\bar{\lambda}_i + \bar{D}^* s\bar{D} + c^* sc)$$
For off-shell supersymmetric case, there is no need to extend the action further. The total action is now given by,

$$ I = S_{N=2} + S_{gf} + S_{ext} $$

(16)

and satisfies the following ST identity,

$$ S(I) = tr \int d^4x (\frac{\delta I}{\delta \Phi_A} \frac{\delta I}{\delta \Phi_A^*} + s \bar{c} \frac{\partial I}{\partial \bar{c}} + s b \frac{\partial I}{\partial b}) $$

(17)

$$ = 2i \xi^i \sigma^\nu \bar{\xi}_i \Delta_\nu $$

(18)

where

$$ \Phi^A = \{ A_\mu, \lambda_{i\alpha}, \bar{\lambda}_{i\dot{\alpha}}, \phi, \phi^*, \vec{D}, c \} $$

$$ \Phi^{*A} = \{ A^*_\mu, \lambda^{*i\alpha}, \bar{\lambda}^{*i\dot{\alpha}}, \phi^*, \phi^{**}, \vec{D}^*, c^* \}. $$

Here, $\Delta_\nu$ is a classical breaking\(^5\),

$$ \Delta_\nu = tr \int d^4x ((-1)^A \Phi^{*A} \partial_\nu \Phi^A) $$

(19)

due to the fact that, the translation symmetry is not included in $s$. It is well known that this classical breaking has no effect on the renormalization of the theory, since it is linear in the fields \[13, 18\]. Note that with help of the extended BRST generator $s$, Ward identities for supersymmetry are transformed into a unique ST identity that includes all these symmetries and the identity (18) can be used to analyze the renormalization of N=2 SYM theory \[13, 10\].

The so called linearized ST operator $B$ \[13, 18\], that is the relevant object for cohomological analysis, can be obtained from (17) as,

$$ B_I = tr \int d^4x (\frac{\delta I}{\delta \Phi_A} \frac{\delta}{\delta \Phi_A^*} + s \bar{c} \frac{\partial I}{\partial \bar{c}} + s b \frac{\partial I}{\partial b}). $$

(20)

and it satisfies,

$$ B_I B_I = -2i \xi^i \sigma^\nu \bar{\xi}_i P_\nu $$

(21)

where $P_\nu = \int d^4x (\partial_\nu \Phi^{*A} \frac{\partial}{\partial \Phi^{*A}} + \partial_\nu \Phi_A \frac{\partial}{\partial \Phi_A})$ is a total derivative when it is acted on the space of integrated field polynomials and therefore $B$ can be considered as a nilpotent operator on this space.

The on-shell supersymmetric N=2 SYM theory is obtained, as usual, by eliminating the auxiliary field $\vec{D}$ with its equation of motion, that is $\vec{D} = 0$ for pure N=2 SYM theory. The on-shell action $\tilde{S}_{N=2} = S_{N=2|\vec{D}=0}$ is still invariant under on-shell supersymmetry transformations, but the supersymmetry algebra (3) is satisfied only when the equations of motion of the spinor fields are used. As a consequence, the extended BRST transformation that contains on-shell supersymmetry transformations

$$ \tilde{s} = s|_{\vec{D}=0} $$

satisfies

$$ \tilde{s}^2 = -2i \xi^i \sigma^\nu \bar{\xi}_i \partial_\nu $$

(modulo eq. of motion of $\lambda, \bar{\lambda}$).

\(^5\)\(-1)^A\) denotes the Grassman parity of the field $\Phi^A$. 

5
However, this complication that the algebra is modified by modulo terms involving equations of motion of spinor fields can be rectified by adding a quadratic term in the anti-fields to the extended action (16) and thus a Slavnov-Taylor operator that also squares to a boundary term can be obtained [9, 10, 11, 12]. For N=2 SYM it is found to be:

\[ S_{quad} = -\frac{1}{2} g^2 tr \int d^4 x (\bar{\sigma}^j_i (\xi_j \lambda^{*i} - \bar{\xi}_j \lambda^+_i) \bar{\sigma}^k_i (\xi_i \lambda^{-k} - \bar{\xi}_k \lambda^+_i)). \] (23)

The total on-shell classical action now reads,

\[ \bar{I} = S_{N=2}|_{D=0} + S_{gf} + S_{ext}|_{D=0} + S_{quad} \] (24)

and following the same steps corresponding ST identity and the linearized ST operator can be obtained as,

\[ \mathcal{S}(\bar{I}) = 2i \xi \sigma^\nu \bar{\xi} \bar{\Delta}^\nu \] (25)

\[ \mathcal{B}_{\bar{I}} = tr \int d^4 x \left( \frac{\delta \bar{I}}{\delta \bar{\Phi}^A} \frac{\delta}{\delta \bar{\Phi}^*_A} + \frac{\delta \bar{I}}{\delta \bar{\Phi}^*_A} \frac{\delta}{\delta \bar{\Phi}^A} + s c \frac{\partial}{\partial \bar{c}} + s b \frac{\partial}{\partial \bar{b}} \right) \] (26)

\[ \mathcal{B}_I \mathcal{B}_{\bar{I}} = -2i \xi \sigma^\nu \bar{\xi} \bar{\mathcal{P}}^\nu \] (27)

for \( \bar{\Phi}^A = \Phi^A|_{D=0}, \bar{\Phi}^*_A = \Phi^*_A|_{D=0}, \bar{\Delta}^\nu = \Delta^\nu|_{D=0}, \bar{\mathcal{P}}^\nu = \mathcal{P}^\nu|_{D=0}. \)

Note that, ST identity (25) and ST operator (27) given for on-shell case are quite similar with ones of off-shell case (18,21). This is due to the fact that the combination

\[ g^2 (\bar{\sigma}^j_i (\xi_j \lambda^{*i} - \bar{\xi}_j \lambda^+_i)) \]

exactly behaves like the auxiliary field \( \bar{D}, \)

\[ B_{\bar{I}} \lambda_i = \bar{s} \lambda_i + g^2 \bar{\sigma}^j_i (\xi_j \lambda^{*k} - \bar{\xi}_k \lambda^+_i) = B_{\bar{I}} \lambda_i|_{\bar{D}=\bar{r}_k} (\xi_i \lambda^{*k} - \bar{\xi}_k \lambda^+_i) \] (28)

\[ B_{\bar{I}} \bar{\lambda}^i = \bar{s} \bar{\lambda}^i - g^2 \xi_j \bar{\sigma}^j_i (\xi_i \lambda^{*k} - \bar{\xi}_k \lambda^+_i) = B_{\bar{I}} \bar{\lambda}^i|_{\bar{D}=\bar{r}_k} (\xi_i \lambda^{*k} - \bar{\xi}_k \lambda^+_i) \] (29)

\[ B_{\bar{I}} g^2 (\bar{\sigma}^j_i (\xi_j \lambda^{*i} - \bar{\xi}_i \lambda^+_j)) = B_{\bar{I}} \bar{D}|_{\bar{D}=\bar{r}_k} (\xi_i \lambda^{*k} - \bar{\xi}_k \lambda^+_i) \] (30)

It is worth underlining that, a similar relation between the auxiliary field and the certain combination of the supersymmetry ghosts and antifields of spinor fields also exists for N=1 SYM theory [8, 16]. It seems natural that, the quadratic terms in antifields of spinor fields that has to be added to the on-shell action in order to obtain a nilpotent operator, should be related to the auxiliary fields of SYM theories. Indeed, in BV formalism such combinations of anti-fields arise naturally when one eliminates the auxiliary fields using their 'generalized equations of motion' which are derived from the master action rather than from the classical action [3, 25]. It can be interesting to find out if this relation can be used to obtain an off-shell formulation of the supersymmetric theories where the auxiliary field content is not known, such as N=4 SYM theory.
3 Descent equations for N=2 SYM theory

As it is discussed in the previous section, when the linearized ST operator $\mathcal{B}$ is defined on the space of integrated polynomials of fields and antifields, it is nilpotent and it constitutes a cohomology problem on this functional space,

$$\mathcal{B} \int d^4x \mathcal{X} = 0$$

where the physically interesting solutions are the ones that can not be written as a $\mathcal{B}$-exact term,

$$\int d^4x \mathcal{X} \neq \mathcal{B} \int d^4x \mathcal{X}' .$$

One way of characterizing the cohomology classes of the operator $\mathcal{B}$ is to study the set of descent equations stemming from Wess-Zumino consistency condition. This framework is also the relevant one for our purposes since our aim is to relate the action of N=2 SYM with lower dimensional field polynomials.

To derive the set of descent equations we will generalize the strategy given in Ref.[16] to N=2 supersymmetric case. The first descent equation is obtained, as usual, by taking the local version of eq.(31):

$$\mathcal{B} \mathcal{X}^{(0)} = \bar{\xi}^i \sigma^{\mu\dot{\alpha}} \partial_\mu \mathcal{X}^{(1)}_{i\dot{\alpha}}$$

Here, the $\bar{\xi}^i \sigma^{\mu\dot{\alpha}}$ factor is assumed to appear due to supersymmetry algebra. To derive the second of the descent equations we apply $\mathcal{B}$ to Eq.(32)

$$\mathcal{B}^2 \mathcal{X}^{(0)} = -2i \bar{\xi}^i \sigma^\mu \xi_i \partial_\mu \mathcal{X}^{(0)} = \bar{\xi}^i \sigma^{\mu\dot{\alpha}} \partial_\mu \mathcal{B} \mathcal{X}^{(1)}_{i\dot{\alpha}}$$

that implies

$$\bar{\xi}^i \sigma^{\mu\dot{\alpha}} \partial_\mu (2i \xi_{i\alpha} \mathcal{X}^{(0)} + \mathcal{B} \mathcal{X}^{(1)}_{i\dot{\alpha}}) = 0 .$$

From the condition (33) the second descent equation can be found as,

$$\mathcal{B} \mathcal{X}^{(1)}_{i\dot{\alpha}} = -2i \xi_{i\alpha} \mathcal{X}^{(0)} + (\bar{\xi}^i \sigma^\mu)^{\beta} \partial_\mu \mathcal{X}^{(2)}_{i\alpha,j\beta}$$

where $\mathcal{X}^{(2)}_{i\alpha,j\beta}$ is a local polynomial antisymmetric in the pair of indices $(i\alpha)$ and $(j\beta)$, i.e. $\mathcal{X}^{(2)}_{i\alpha,j\beta} = -\mathcal{X}^{(2)}_{j\beta,i\alpha}$. Note that the antisymmetry of $\mathcal{X}^{(2)}$ follows from the fact that the term

$$\bar{\xi}^i \sigma^{\dot{\alpha}\mu} \partial_\mu \bar{\xi}^j \sigma^{\dot{\beta}\nu} \partial_\nu \mathcal{X}^{(2)}_{i\alpha,j\beta}$$

that can be added to condition (33), vanishes due to the commuting nature of the global supersymmetry ghosts only when $\mathcal{X}^{(2)}$ is taken to be antisymmetric in the pair of its indices.

\footnote{For the following discussion, $\mathcal{B}$ will stand for both $\mathcal{B}_I$ and $\mathcal{B}_J$.}
The rest of the descent equations can be found easily by iterating this procedure and the following set of descent equations for N=2 supersymmetry can be written:

\[
B\mathcal{X}^{(0)} = \bar{\xi}_{\alpha} \bar{\sigma}^{\mu \alpha} \partial_{\mu} \mathcal{X}^{(1)}_{\alpha}
\]

\[
B\mathcal{X}^{(1)}_{\alpha} = -2i\xi_{\alpha} \mathcal{X}^{(0)} + (\bar{\xi}^{i} \bar{\sigma}^{\mu})^{\beta} \partial_{\mu} \mathcal{X}^{(2)}_{\alpha, j\beta}
\]

\[
B\mathcal{X}^{(2)}_{\alpha, j\beta} = -2i\xi_{\alpha} \mathcal{X}^{(1)}_{j\beta} + 2i\xi_{\alpha} \mathcal{X}^{(1)}_{\alpha, j\beta} + (\bar{\xi}^{i} \bar{\sigma}^{\mu})^{\gamma} \partial_{\mu} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma}
\]

\[
B\mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma} = 2i\xi_{\alpha} \mathcal{X}^{(2)}_{j\beta, k\gamma} - 2i\xi_{\alpha} \mathcal{X}^{(2)}_{j\beta, k\gamma} - 2i\xi_{\alpha} \mathcal{X}^{(2)}_{\alpha, j\beta} + (\bar{\xi}^{i} \bar{\sigma}^{\mu})^{\lambda} \partial_{\mu} \mathcal{X}^{(4)}_{\alpha, j\beta, k\gamma, l\lambda}
\]

\[
B\mathcal{X}^{(4)}_{\alpha, j\beta, k\gamma, l\lambda} = -2i\xi_{\alpha} \mathcal{X}^{(3)}_{j\beta, k\gamma, l\lambda} - 2i\xi_{\alpha} \mathcal{X}^{(3)}_{j\beta, k\gamma, l\lambda} + 2i\xi_{\alpha} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma, l\lambda} + 2i\xi_{\alpha} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma, l\lambda}
\]

Here, the local polynomials \( \mathcal{X}^{(3)} \), \( \mathcal{X}^{(4)} \) are also totally antisymmetric in the pair of spinor and SU(2)-R indices (like when the pairs \( (i\alpha) \), \( (j\beta) \), ... are exchanged). The descent equations terminate at the fourth level, due to the fact that the pair of spinor and SU(2)-R indices can take only four distinct values, i.e. a totally antisymmetric \( \mathcal{X}^{(5)} \) is zero identically. Therefore, when these equations are compared with that of N=1 SYM [16], it is clear that the number of descent equations are related directly with the internal R-symmetry content of the theory.

The structure of descent equations (35-39) for N=2 SYM theory are quite different and complicated from the ones for standard gauge theories (see for instance Ref.s [13, 26]). Due to the assumption that the antichiral supersymmetry ghosts, \( \bar{\xi}^{i} \), appear explicitly in front of the derivatives on the R.H.S. of the descent equations, all the solutions \( \mathcal{X}^{(i)} \) carry the same ghost number and the possible solutions are highly constrained. This is due to the supersymmetric structure of the theory. Moreover, the last equation is not homogeneous. Nevertheless, the RHS of the last equation is homogeneous in \( \bar{\xi}^{i} \) and therefore, introduction of a filtration of the linearized ST operator with respect to chiral ghosts \( \xi^{i} \),

\[
\mathcal{N} = \xi^{i\alpha} \delta \frac{\delta}{\delta \xi^{i\alpha}} \quad ; \quad \mathcal{B} = \sum \mathcal{B}_{n} \quad , \quad [\mathcal{N}, \mathcal{B}_{n}] = n\mathcal{B}_{n}
\]

which leads to the algebra

\[
\mathcal{B}_{0}^{2} = 0 \quad (41)
\]

\[
\{\mathcal{B}_{0}, \mathcal{B}_{1}\} = -2i\xi^{i} \bar{\sigma}^{\mu} \bar{\xi} \partial_{\mu} \quad (42)
\]

\[
\{\mathcal{B}_{0}, \mathcal{B}_{2}\} + \mathcal{B}_{2}^{2} = 0 \quad (43)
\]

\[
\mathcal{B}_{2}^{2} = \{\mathcal{B}_{1}, \mathcal{B}_{2}\} = 0. \quad (44)
\]

is useful to find a solution.

Due to the filtration (40), the lowest descent equation (39) can be divided into two,

\[
\mathcal{B}_{0} \mathcal{X}^{(4)}_{\alpha, j\beta, k\gamma, l\lambda} = 0 \quad (45)
\]

\[
\xi^{i} \mathcal{Q}_{\alpha} \mathcal{X}^{(4)}_{\alpha, j\beta, k\gamma, l\lambda} = -2i\xi_{\alpha} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma, l\lambda} - 2i\xi_{\alpha} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma, l\lambda} + 2i\xi_{\alpha} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma, l\lambda} + 2i\xi_{\alpha} \mathcal{X}^{(3)}_{\alpha, j\beta, k\gamma, l\lambda} \quad (46)
\]

where we have defined the operator \( \mathcal{Q}_{\alpha} \) as

\[
\xi^{i} \mathcal{Q}_{\alpha} = \mathcal{B}_{1} + \mathcal{B}_{2}. \quad (47)
\]
Note that since the zeroth order operator $B_0$ in the filtration of $B$ is strictly nilpotent, it also constitutes a cohomology problem and the cohomology of the full operator $B$ is isomorphic to a subspace of the cohomology of the operator $B_0$. 

On the other hand, the equation (45,46) indicates that the operator $Q_\alpha$ can be used as a kind of climbing up operator. Indeed, whenever the descent equations can be divided into two like Eq.s(45,46), after some algebra it is found that the solutions of descent equations (35-39) are algebraically related to each other as,

$$Q^{i\alpha} Q^{j\beta} Q^{k\gamma} Q^{l\lambda} \mathcal{X}_{\alpha,j\beta,k\gamma,l\lambda}^{(4)} = (2i)^4 4! \mathcal{X}^{(0)}_{\alpha}$$
$$Q^{j\beta} Q^{k\gamma} Q^{l\lambda} \mathcal{X}_{\alpha,j\beta,k\gamma,l\lambda}^{(4)} = (2i)^3 3! \mathcal{X}^{(1)}_{\alpha}$$
$$Q^{k\gamma} Q^{l\lambda} \mathcal{X}_{\alpha,j\beta,k\gamma,l\lambda}^{(4)} = (2i)^2 2! \mathcal{X}^{(2)}_{\alpha,j\beta}$$
$$Q^{l\lambda} \mathcal{X}_{\alpha,j\beta,k\gamma,l\lambda}^{(4)} = (2i) 1! \mathcal{X}^{(3)}_{\alpha,j\beta,k\gamma}.$$ (48)

Therefore, once a explicit solution of the lowest descent equation is found, that belongs to the cohomology of the filtered operator $B_0$, the higher solutions in the descent can be obtained by applying the climbing up operator $Q_\alpha$ both for off- and on-shell supersymmetric cases. Note that the framework given above in order to solve the cohomology problem of the linearized ST operator $B$ is a direct generalization of the method presented in Ref.[16] for N=1 SYM theory and as expected, it is similiar with the ones given for the twisted N=2 SYM [11,18,21].

4 Construction of the N=2 SYM action

In algebraic renormalization framework the solutions of the cohomology problem (31) determined by the linearized ST operator $B$ for ghost numbers 0 and 1 gives the invariant counterterms that can be added to any order in the perturbation theory and the possible anomalies respectively. As a consequence the classical action also belongs to the cohomology of $B$ in the ghost sector zero and to study the descent equations for analyzing the cohomology of $B$ of a supersymmetric theory gives a natural framework to relate the corresponding action to the lower dimensional field polynomials.

Therefore, we are interested in a gauge invariant solution $\mathcal{X}^{(0)}$ of Eq.(31), which has the same quantum numbers with the classical Lagrangean of the N=2 SYM theory, i.e. a solution that has dimension four with vanishing ghost number and $SU(2)$-R charge and Grassmann even. The solutions of the lower descent equations are also constrained due to this requirement and as a consequence the gauge invariant solution $\mathcal{X}^{(4)}$ of the lowest descent equation (39) has dimension 2, and R-charge -4 (see table 1).

On the other hand, as discussed in the previous section, the solution $\mathcal{X}^{(4)}$ has to belong to the cohomology of the filtered operator $B_0$,

$$B_0 \mathcal{X}^{(4)} = 0$$
Table 1: Dimensions $d$, Grassmann parity $GP$, ghost number $Gh$ and R-weights.

| $d$ | $GP$ | $Gh$ | $R$ |
|-----|------|------|-----|
| 1   | 0    | 0    | 0   |
| 2   | 0    | 1    | -1  |
| 3   | 0    | 0    | 1   |
| 4   | 0    | 0    | 0   |

Note that this condition also implies that the solution $\mathcal{X}^{(4)}$ is gauge invariant. The only such gauge invariant field polynomial with correct quantum numbers is $\text{Tr} \phi^2$ for both off- and on-shell supersymmetric cases and the solution $\mathcal{X}^{(4)}$ with desired index structure to the descent equation (39) can be written as,

$$\mathcal{X}^{(4)}_{\alpha, \beta, k, \gamma, \lambda} = k E_{\alpha, \beta, k, \gamma, \lambda} \text{Tr}(\phi^2)$$

where $k$ is a constant and $E_{\alpha, \beta, k, \gamma, \lambda}$ is a totally anti-symmetric tensor in the pair of indices $(\alpha, (j \beta), (k \gamma), (i \lambda))$,

$$E_{\alpha, \beta, k, \gamma, \lambda} = \frac{2}{3} (\epsilon_{\alpha \beta \epsilon \gamma \lambda} (E_{ik} E_{jl} + E_{il} E_{jk}) - \epsilon_{\alpha \gamma \epsilon \beta \lambda} (E_{ij} E_{kl} + E_{il} E_{kj}) + \epsilon_{\alpha \lambda \epsilon \beta \gamma} (E_{ij} E_{kl} + E_{ik} E_{lj})).$$

It is worthwhile to remark that the field monomial $\text{Tr} \phi^2$ is a nontrivial element of the cohomology of operator $B_0$ only when $B_0$ is defined on the space of field polynomials that are analytic in constant supersymmetry ghosts. In other words, if the functional space is defined to be the polynomials of the fields that are not necessarily analytic in the constant ghosts $\bar{\xi}^i$, $\text{Tr} \phi^2$ can be written as an exact-$B_0$ term,

$$\text{Tr} \phi^2 = B_0 \text{Tr} \left( -\frac{\sqrt{2}}{2\bar{\xi}^i} c\phi + \frac{i\sqrt{2}}{12(\bar{\xi}^i \xi^i)^2} \{c, c\} c \right)$$

and therefore it belongs to the trivial cohomology of $B_0$. This fact has also been pointed out in Ref.s [4, 18], for twisted formulation of N=2 SYM that the twist of the N=2 theory can be interpreted as a topological theory only if the analyticity is lost in (scalar) SUSY ghosts. Moreover, in a recent paper a similar non-analyticity argument in constant supersymmetry ghosts is used to show that the topological Yang-Mills (TYM) [17] theory can be obtained by using field redefinitions i.e. as a change of variables (without twisting) [20]. Therefore, physical and topological interpretations of N=2 SYM are intertwined together due to the requirement of analyticity of supersymmetry ghosts [4, 18, 20].

It is straightforward to find the solution $\mathcal{X}^{(0)}$ from the lowest solution $\mathcal{X}^{(4)}$ (49) by using the lift given in (40) for both off-shell supersymmetric case,

$$\mathcal{X}^{(0)}_{\alpha, \beta, k, \gamma, \lambda} = k \text{Tr} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{i}{8} \epsilon^{\mu \nu \lambda \kappa} F_{\mu \nu} F_{\lambda \kappa} - i\lambda^i \bar{D} \bar{\lambda}_i + \phi D_{\mu} D^\mu \phi \right)$$

$$- \frac{i\sqrt{2}}{2} (\lambda_i [\lambda^i, \phi^\dagger] + \bar{\lambda}^i [\bar{\lambda}_i, \phi]) - \frac{1}{2} [\phi, \phi^\dagger]^2 + \frac{1}{2} \bar{D} \bar{D}$$

As noted in Ref.[4, 18], when perturbative calculations are considered one should obviously require analyticity of the parameters of a theory, the analyticity requirement for supersymmetry ghosts is mandatory for perturbative regime.
and for on-shell supersymmetric case

\[ \mathcal{X}^{(0)}_{on} = k \text{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{8} \epsilon^{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa} - \frac{i}{2} \lambda^i \psi_i^\dagger + \frac{1}{2} \phi D_\mu D^\mu \phi^\dagger - \frac{i\sqrt{2}}{4} \lambda^i [\bar{\lambda}_i, \phi] \right) + \mathcal{F}(\Phi^A, \Phi^*_A) \]

where \( \mathcal{F}(\Phi^A, \Phi^*_A) \) is a complicated polynomial of fields and anti-fields.

Note that in the above solutions the term \( \text{Tr} \epsilon^{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa} \) is a total derivative under the integral sign and it is seen easily that off-shell (extended) N=2 SYM action is directly related to the lower dimensional field monomial \( \text{Tr} \phi^2 \) as a consequence of the lift (48) for \( k = \frac{1}{g^2} \),

\[ -\frac{1}{2} g \frac{d}{dg} \tilde{I} = S_{N=2} = \frac{1}{g^2} \mathcal{Q}^4 \text{Tr} \int d^4 x \phi^2 \]

where we have used \( \mathcal{Q}^4 = \frac{1}{24} \epsilon_{\alpha\beta\gamma\delta} Q^{i\alpha} Q^{j\beta} Q^{k\gamma} Q^{l\delta} \) for notational simplicity.

The on-shell (extended) action can be obtained, by noting that the anti-field independent part of the solution differs from the original action by equations of motion terms due to on-shell supersymmetry. Since, \( \mathcal{B}\)-transformation of antifields includes the equation of motion of the corresponding field\(^8\),

\[ \mathcal{B}_I \Phi^*_A = \frac{\delta \tilde{I}}{\delta \Phi^A} = \frac{\delta S_{N=2}}{\delta \Phi^A} + \ldots \]

addition of a \( \mathcal{B}_I \) -exact term restores these missing terms and the on-shell (extended) action can be constructed as

\[ -\frac{1}{2} g \frac{d}{dg} \tilde{I} = S_{N=2|D=0} - S_{quad} = \frac{1}{g^2} \mathcal{Q}^4 \text{Tr} \int d^4 x \phi^2 + \frac{1}{2} \mathcal{B}_I \text{tr} \int d^4 x (\phi^* \phi - \lambda^i \lambda_i). \]

The above relations show that both off and on-shell supersymmetric SYM actions can be constructed from 2 dimensional field monomial \( \text{Tr} \phi^2 \). The difference between two cases is only a \( \mathcal{B} \)-exact term, which can be interpreted as a result of the theorem that local BRST cohomologies of two formulations of the same theory differing in auxiliary field content are the same \([27]\). Moreover, since \( \text{Tr} \phi^2 \) is the lowest component of the N=2 multiplet where the action belongs, the relations (53) and (54) imply that the above method of working the BRST cohomology through the descent equations, gives the relation between the action and the lowest component of the multiplet in an elegant way for both off and on-shell supersymmetric cases. A similar structure is also obtained for N=1 SYM theory \([16]\) and it should be straightforward to generalize this method to other supersymmetric theories.

\(^8\)This part of the operator \( \mathcal{B} \) is often called in the literature Koszul-Tate differential. See Ref. \([26]\) for its importance for BRST cohomological calculations.

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References

[1] J. Wess and B. Zumino, *Phys. Lett.* B 49 (1974) 52; J. Iliopoulos and B. Zumino, *Nucl. Phys.* B 79 (1974) 310.

[2] K. Fujikawa and W. Lang, *Nucl. Phys.* B 88 (1975) 61; M.T. Grisaru, W. Siegel and M. Rocek, *Nucl. Phys.* B 159 (1979) 429; M.T. Grisaru and W. Siegel, *Nucl. Phys.* B 201 (1982) 292.

[3] R. Flume and E. Kraus, *Nucl. Phys.* B 569 (2000) 625.

[4] A. Blasi, V. E. Lemes, N. Maggiore, S. P. Sorrella, A. Tanzini, O. S. Ventura and L. C. Vilar, *JHEP* 0005 (2000) 039; V. E. Lemes, M. S. Sarandy, S. P. Sorrella, O. S. Ventura and L. C. Vilar, *J. Phys. A* 34 (2001) 9485.

[5] K. Ulker, *Mod. Phys. Lett.A* 16 (2001) 881.

[6] E. Kraus and D. Stöckinger, *Phys. Rev.* D 64 (2001) 115012; E. Kraus, *Nucl. Phys.* B 620 (2002) 55.

[7] F. Brandt, *Nucl. Phys.* B 392 (1993) 428; F. Brandt, *Class. Quant. Grav.* 11 (1994) 849.

[8] F. Brandt, *Phys. Lett.* B 320 (1994) 57; F. Brandt, *Ann. Phys.* 259 (1997) 253; F. Brandt, *JHEP* 04 (2003) 035.

[9] P.L. White, *Class. Quant. Grav.* 9 (1992) 413; *Class. Quant. Grav.* 9 (1992) 1663.

[10] N. Maggiore, *Int. J. Mod. Phys.* A10 (1995) 3937; *Int. J. Mod. Phys.* A10 (1995) 3781.

[11] N. Maggiore, O. Piguet and S. Wolf, *Nucl. Phys.* B 458 (1996) 403; *Nucl. Phys.* B 476 (1996) 329.

[12] F. Brandt, M. Henneaux and A. Wilch, *Phys. Lett.* B 387 (1996) 320; *Nucl. Phys.* B 510 (1998) 640.

[13] O. Piguet and S.P. Sorrella, *Algebraic Renormalization*, Lecture Notes in Physics m28, Springer Verlag, Berlin Heidelberg,1995.

[14] S. Weinberg, *The Quantum Theory of Fields, Vol.II Modern Applications* Cambridge Univ. Press, 1996.

[15] I.A. Batalin and G.A Vilkovisky, *Phys. Lett.* B 69 (1983) 309; *Phys. Rev.* D 28 (1981) 2567; *Phys. Lett.* B 102 (1981) 27.

See also J. Gomis, J. Paris and S. Samuel *Phys. Rep.* 259 (1995) 1 for a self contained review.
[16] K. Ulker, *Mod. Phys. Lett.* A **17** (2002) 739.

[17] E. Witten, *Commun. Math. Phys.* **117** (1988) 353.

[18] F. Fucito, A. Tanzini, L. C. Vilar, O. S. Ventura, C. A. Sasaki and S. P. Sorella, Algebraic renormalization: Perturbative twisted considerations on topological Yang-Mills theory and on N = 2 supersymmetric gauge theories, Lectures given at the *First School on Field Theory and Gravitation*, Victoria, Esprinto, Brazil, 1997. hep-th/9707209.

[19] P. Grassi, T. Hurth and A. Quadri, *JHEP* **07** (2003) 008.

[20] K. Ulker, *N=2 Super Yang Mills Action as a BRST Exact Term, Topological Yang Mills and Instantons*, hep-th/0304154.

[21] V.E.R. Lemes, M. Picariello, M.S. Sarandy, S.P. Sorella, *J. Phys. A* **35** (2002) 3703.

[22] R. Grimm, M.F. Sohnius and J. Wess, *Nucl. Phys. B* **133** (1978) 275.

[23] M. Sohnius, *Phys. Rep.* **128** (1985) 39.

[24] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 1992, Princeton University Press, Princeton, NJ.

[25] M. Henneaux, *Phys. Lett.* B **238** (1990) 299;
A. Dresse, P. Gregoire, M. Henneaux, *Phys. Lett.* B **245** (1990) 192.

[26] G. Barnich, F. Brandt and M. Henneaux, *Phys. Rep.* **338** (2000) 439.

[27] G. Barnich, F. Brandt and M. Henneaux, *Commun. Math. Phys.* **174** (1995) 57.