Deformation stability of $p$-SKT and $p$-HS manifolds

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Abstract
We study the notions of $p$-Hermitian-symplectic and $p$-pluriclosed compact complex manifolds, which are defined as generalisations for an arbitrary positive integer $p$ not exceeding the complex dimension of the manifold of the standard notions of Hermitian-symplectic and SKT manifolds that correspond to the case $p = 1$. We then notice that these two notions are equivalent on $\partial\overline{\partial}$-manifolds and go on to prove that in (smooth) complex analytic families of $\partial\overline{\partial}$-manifolds, the properties of being $p$-Hermitian-symplectic and $p$-pluriclosed are deformation-open. Concerning closedness results, we prove that the cones $A_p$, resp. $C_p$, of Aeppli cohomology classes of strictly weakly positive $(p, p)$-forms $\Omega$ that are $p$-pluriclosed, resp. $p$-Hermitian-symplectic, must be equal on the limit fibre if they are equal on the other fibres and if some rather weak $\partial\overline{\partial}$-type assumptions are made on the other fibres.

Keywords Deformations of complex structures · Positivity · $p$-SKT manifold · $p$-HS manifold

Mathematics Subject Classification 32G05 · 32Q57 · 53C55

1 Introduction
Let $\mathcal{X}$ be a complex manifold and let $\Delta$ be an open ball containing the origin in $\mathbb{C}^m$ for some $m \in \mathbb{N}^*$. Recall the following standard notion: a complex analytic family of compact complex manifolds is a proper holomorphic submersion $\pi : \mathcal{X} \to \Delta$. This means that the following conditions are satisfied:

- For each $t \in \Delta$, $X_t = \pi^{-1}(t)$ is a compact complex connected submanifold of $\mathcal{X}$.
- The rank of the Jacobian matrix of $\pi$ is equal to $n$ at every point of $\mathcal{X}$.

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There is a locally finite open covering \( \{ U_j : j = 1, 2, \ldots \} \) of \( X \) and complex-valued holomorphic functions \( \xi_j^\nu(p) \), defined on \( U_j \), such that for each \( t \) the set
\[
\{ p \in U_j \rightarrow (\xi_j^1(p), \ldots, \xi_j^n(p)) : U_j \cap \pi^{-1}(t) \neq \emptyset \}
\]
is a system of local holomorphic coordinates of \( X_t \).

By a result of Ehresmann [25, Theorem 9.3], all the fibres \( X_t \) are \( C^\infty \)-diffeomorphic to a fixed \( C^\infty \)-manifold \( X \). Therefore, the holomorphic family \( (X_t)_{t \in \Delta} \) of compact complex manifolds can be viewed as a single \( C^\infty \)-manifold \( X \) endowed with a \( C^\infty \) family of complex structures \( (J_t)_{t \in \Delta} \).

The purpose of this paper is to relax the notion of \( p \)-Kählerianity introduced by Alessandrini and Andreatta in [2] for compact complex manifolds to two notions that offer analogues at the level of \((p, p)\)-forms (satisfying a mild strict positivity assumption) for the classical notions of Hermitian-symplectic, respectively SKT (or pluriclosed), manifolds that correspond to the case \( p = 1 \). Specifically, a smooth strictly weakly positive \((p, p)\)-form \( \Omega \) (cf. Definition 2.1) will be said to be \( p \)-Hermitian-symplectic \((p\text{-HS})\), respectively \( p \)-SKT \((p\text{-SKT})\), if \( \Omega \) is the \((p, p)\)-type component of a real \( d \)-closed \((2p)\)-form, respectively if \( \partial \overline{\partial} \Omega = 0 \). Compact complex manifolds carrying such forms will be termed \( p \)-Hermitian-symplectic \((p\text{-HS})\) manifolds, respectively \( p \)-SKT \((p\text{-SKT})\) manifolds, see Definition 2.2. As was pointed to us by the referee, these notions were already introduced in [1] where it was observed that they coincide on \( \partial \overline{\partial} \)-manifolds—unfortunately we were unaware of this when writing the first version of this paper. Nevertheless, our approach seems new in that it uses the Frölicher spectral sequence.

Our first main result (cf. Theorems 4.3 and 4.5 for precise statements) places these two notions in the context of deformations of compact structures.

**Theorem 1.1** The properties of being \( p \)-Hermitian-symplectic and \( p \)-SKT \( \partial \overline{\partial} \)-manifolds are open under holomorphic deformations of the complex structure.

Recall that the property of \( p \)-Kählerianity of [2,4] is not deformation-open (cf. [3]).

In the second part of the paper, we will investigate some deformation-closedness issues. To this end, we introduce the following positivity cones in the Aeppli cohomology of bidegree \((p, p)\) on any compact complex \( n \)-dimensional manifold \( X \):

\[
A_p(X) = \{ [\Omega]_A : \Omega \text{ is strictly weakly positive such that } \partial \overline{\partial} \Omega = 0 \},
\]
\[
C_p(X) = \{ [\Omega]_A : \Omega \text{ is strictly weakly positive such that } \Omega \text{ is } p\text{-HS} \}.
\]

The inclusion \( C_p(X) \subset A_p(X) \subset H^p,^p_A(X, \mathbb{R}) \subset H^p,^p_A(X, \mathbb{C}) \) always holds trivially.

Recall that (a smooth (1, 1)-form positive definite) \( \omega \) is called Gauduchon if \( \partial \overline{\partial} \omega^{n-1} = 0 \) and strongly Gauduchon (sG) if \( \partial \omega^{n-1} \in \text{Im} \overline{\partial} \). The Gauduchon cone of \( X \) introduced in [17],

\[
\mathcal{G}_X := \{ \omega^{n-1, n-1} \text{ is a Gauduchon metric on } X \},
\]
is an open convex cone in $H^{n-1, n-1}_A(X, \mathbb{R})$. It is known that Gauduchon metric always exists [12], thus $\mathcal{S}_X \neq \emptyset$.

We consider the following canonical linear map:

$$T : H^{n-1, n-1}_A(X, \mathbb{C}) \to H^{n-1, n-1}_{\partial}(X, \mathbb{C})$$

It is easy to check that this map is well defined. The strongly Gauduchon cone is defined as (cf. [17])

$$\mathcal{S}_G X := \mathcal{S}_X \cap \ker T \subset \mathcal{S}_X \subset H^{n-1, n-1}_A(X, \mathbb{C}). \quad (2)$$

If no sG metric exists, then $\mathcal{S}_G X = \emptyset$.

**Definition 1.2** ([19, Definition 1.2]) Let $X$ be a compact complex manifold of dimension $n$. $X$ is called an sGG manifold if the strongly Gauduchon cone of $X$ coincides with the Gauduchon cone of $X$, i.e., if $\mathcal{S}_G X = \mathcal{S}_X$.

It is proved in [19] that, in complex dimension 3, the deformation limit $X_0$ of sGG manifolds $X_t$, $t \neq 0$, is not sGG in general. Thus it is important to identify which extra conditions on the holomorphic family guarantee the equality $A_p(X_0) = \mathcal{C}_p(X_0)$ in its central limit. We show (cf. Proposition 5.8) the following behaviour of these cones under holomorphic deformations of the complex structure.

**Proposition 1.3** Let $(X_t)_{t \in \Delta}$ be a holomorphic family of compact complex manifolds. If $A_p(X_t) = \mathcal{C}_p(X_t)$ for all $t \in \Delta \setminus \{0\}$, if some special cases of the $\partial \bar{\partial}$-lemma (that we call the hypotheses $\widehat{H}_1, \ldots, \widehat{H}_{p+1}$) are satisfied for all $t \neq 0$ and if the cohomology numbers

$$h^{p+k-1, p-k+1}_A(t) := \dim H^{p+k-1, p-k+1}_A(X_t, \mathbb{C}),$$
$$h^{p+k, p-k+1}_B(t) := \dim H^{p+k, p-k+1}_B(X_t, \mathbb{C}),$$
$$h^{p+k, p-k+1}_{\partial}(t) := \dim H^{p+k, p-k+1}_{\partial}(X_t, \mathbb{C})$$

are independent of $t \in \Delta$ for all $k \in \{1, \ldots, p+1\}$, then $A_p(X_0) = \mathcal{C}_p(X_0)$.

We hope that these results will contribute to a possible resolution of Barlet’s **Properness Conjecture** for the Barlet space of relative cycles associated with a holomorphic family of class $\mathcal{C}$ manifolds. Before presenting this conjecture let us recall that a compact complex manifold $X$ is said to be in the (Fujiki) class $\mathcal{C}$ if there exists a proper holomorphic bimeromorphic map (i.e. a modification) $\mu : \widetilde{X} \to X$ from a compact Kähler manifold $\widetilde{X}$ to $X$ (see [24]).

Fujiki introduced the class $\mathcal{C}$ of manifolds, in [11], as meromorphic images of compact Kähler manifolds. This definition was proved to be equivalent to the above-given definition by Varouchas in [24].
Denote by $\mathcal{C}^p(X/\Delta)$ the relative Barlet space of effective analytic $p$-cycles contained in the fibres $X_t$, and let

$$\mathcal{C}(X/\Delta) = \bigcup_{0 \leq p \leq n} \mathcal{C}^p(X/\Delta).$$

**Conjecture 1.4** ([15], one of the versions of Barlet’s Conjecture) Let $\pi : \mathcal{X} \to \Delta$ be a complex analytic family of compact complex manifolds such that the fibre $X_t$ is a class $C$ manifold for every $t \in \Delta$. Then the irreducible components of the relative Barlet space $\mathcal{C}(X/\Delta)$ of cycles on $X$ are proper over $\Delta$ in the following sense. Consider the holomorphic map

$$P : \mathcal{C}(X/\Delta) \to \Delta$$

$$Z_t \mapsto P(Z_t) = t$$

mapping every divisor $Z_t \subset X_t$ contained in some fibre $X_t$ to the base point $t \in \Delta$. The map $P$ has the property that its restrictions to the irreducible components of $\mathcal{C}(X/\Delta)$ are proper.

In the end, we prove (cf. Corollary 5.11) that on a fixed compact complex manifold $X$ satisfying some assumptions weaker than the $\partial\bar{\partial}$-lemma, the natural map

$$H^k_{DR}(X, \mathbb{C}) \oplus H^{2n-k}_{DR}(X, \mathbb{C}) \to \bigoplus_{p+q=k} H^p_A(X, \mathbb{C}) \oplus \bigoplus_{p+q=2n-k} H^p,q_A(X, \mathbb{C})$$

is an isomorphism.

## 2 Preliminaries

In this section, we recall some background that will be needed in the sequel and introduce our first definitions. Let $X$ be a compact complex manifold of complex dimension $n$.

The following positivity notion is standard, though a different name has been used in the literature, and goes back to Lelong. We spell out the details of the definition in order to dispel the confusion over the English and French meanings of “positivity”.

**Definition 2.1** (see e.g. [8, Chapter III] and [4, Definition 1.1]) (i) Let $V$ be a complex vector space of dimension $n$ and let $V^*$ be its dual. Fix any integer $1 \leq p \leq n-1$. A $(p, p)$-form $\alpha \in \Lambda^p.p V^*$ is said to be strictly weakly positive if for all linearly independent $\tau_1, \ldots, \tau_q \in V^*$, with $q = n - p$, the $(n, n)$-form

$$\alpha \wedge i\tau_1 \wedge \overline{\tau}_1 \wedge \cdots \wedge i\tau_q \wedge \overline{\tau}_q$$

is positive (i.e. $> 0$) whenever $\alpha \wedge i\tau_j \wedge \overline{\tau}_j \neq 0$ for all $j \in \{1, \ldots, q\}$.

(ii) Let $X$ be a complex manifold of dimension $n$. A smooth $(p, p)$-form $\alpha \in C^\infty_p(X, \mathbb{C})$ on $X$ is said to be strictly weakly positive if, for every point $x \in X$,
\(\alpha(x) \in \Lambda^{p,p} T^* X\) is a strictly weakly positive \((p, p)\)-form on the holomorphic tangent space \(T_x X\) to \(X\) at \(x\). (These forms are called transverse in [4].)

It is standard that a \((p, p)\)-form \(\alpha \in \Lambda^{p,p} V^*\) on a vector space \(V\) is strictly weakly positive if and only if its restriction \(\alpha|_E\) to every \(p\)-dimensional vector subspace \(E \subset V\) is a positive (i.e. \(> 0\)) volume form on \(E\). (See e.g. [8, Chapter III].)

Consequently, a smooth \((p, p)\)-form \(\alpha \in C_{p,p}^\infty(X, \mathbb{C})\) on a manifold \(X\) is strictly weakly positive if and only if, for every coordinate patch \(U \subset X\) and every \(p\)-dimensional complex submanifold \(Y \subset U\), its restriction \(\alpha|_Y\) is a positive (i.e. \(> 0\)) volume form on \(Y\). (See e.g. [4].)

It is also standard that, for \(p = 1\) and \(p = n - 1\), the notion of strict weak positivity coincides with the usual notion of positive definiteness for \((1, 1)\) and \((n - 1, n - 1)\)-forms (= the positivity of all the eigenvalues of the coefficient matrix). Moreover, all strictly weakly positive \((p, p)\)-forms \(\alpha\) are always real, in the sense that \(\alpha = \overline{\alpha}\). (See e.g. [8, Chapter III].)

Now, we define \(p\)-Hermitian-symplectic \((p\text{-HS})\) and \(p\text{-SKT}\) forms and manifolds as generalisations of the notions of \(p\text{-Kähler}\) forms and manifolds introduced in [2] and further studied in [4]. These two notions (i.e. \(p\text{-HS}\) and \(p\text{-SKT}\) forms) were already introduced in [1] under the names of \(p\text{-symplectic}\) and \(p\text{-pluriclosed}\) forms.

**Definition 2.2** Let \(X\) be a compact complex manifold of complex dimension \(n\) and let \(\Omega\) be a \(C^\infty\) strictly weakly positive \((p, p)\)-form on \(X\).

- \(\Omega\) is said to be a \(p\text{-Hermitian-symplectic form}\) if there exist forms \(\alpha^{i,2p-i} \in C^\infty_{i,2p-i}(X, \mathbb{C}), i \in \{0, \ldots, p - 1\}\), such that \(d\left(\sum_{i=0}^{p-1} \alpha^{i,2p-i} + \sum_{i=0}^{p-1} \overline{\alpha^{i,2p-i}}\right) = 0\).
- \(\Omega\) is said to be a \(p\text{-SKT form}\) if \(\partial \overline{\partial} \Omega = 0\).
- \(X\) is said to be a \(p\text{-SKT manifold}\) (resp. a \(p\text{-HS manifold}\)) if there exists a \(p\text{-SKT}\) (resp. a \(p\text{-HS}\)) strictly weakly positive \((p, p)\)-form on \(X\).
- ([4]) A compact complex manifold \(X\) is said to be \(p\text{-Kähler}\) if it supports a \(d\)-closed strictly weakly positive \((p, p)\)-form \(\Omega\). Such an \(\Omega\) is called \(p\text{-Kähler form}\).

Finally, recall the following standard definitions. For all \(p, q = 0, \ldots, n\), the **Bott–Chern cohomology group** of \(X\) of type \((p, q)\) is defined as

\[
H^{p,q}_{\text{BC}}(X, \mathbb{C}) = \frac{\ker\{\partial: C_{p,q}^\infty(X) \to C_{p+1,q}^\infty(X)\} \cap \ker\{\overline{\partial}: C_{p,q}^\infty(X) \to C_{p,q+1}^\infty(X)\}}{\operatorname{Im}\{\partial \overline{\partial}: C_{p-1,q-1}^\infty(X, \mathbb{C}) \to C_{p,q}^\infty(X, \mathbb{C})\}},
\]

while the **Aeppli cohomology group** of type \((p, q)\) is defined as

\[
H^{p,q}_A(X, \mathbb{C}) = \frac{\ker\{\partial \overline{\partial}: C_{p,q}^\infty(X, \mathbb{C}) \to C_{p+1,q+1}^\infty(X, \mathbb{C})\}}{\operatorname{Im}\{\partial: C_{p-1,q}^\infty(X) \to C_{p,q}^\infty(X)\} + \operatorname{Im}\{\overline{\partial}: C_{p,q-1}^\infty(X) \to C_{p,q}^\infty(X)\}}.
\]
3 Deformations of pluriclosed manifolds

Let us now recall some basic notions.

Definition 3.1 Let $\omega > 0$ be a Hermitian metric on a complex manifold $X$.

- $\omega$ is said to be pluriclosed (or SKT) if $\partial \bar{\partial} \omega = 0$.
- $\omega$ is called Hermitian-symplectic (H-S for simplicity) (cf. [23, Definition]) if there exists $\alpha^{0,2} \in C_{0,2}^\infty(X, \mathbb{C})$ such that
  \[
  d(\alpha^{0,2} + \omega + \alpha^{0,2}) = 0.
  \]
  This is equivalent to the existence of a $C^\infty$ real 2-form $\Omega$ on $X$ such that $d\Omega = 0$ and $\Omega^{1,1} = \omega > 0$.
- $X$ is called an SKT manifold (resp. H-S manifold) if there exists an SKT (resp. an H-S) metric on $X$.
- $X$ is called a $\partial \bar{\partial}$-manifold (this term was introduced in [16]) if the $\partial \bar{\partial}$-lemma is satisfied on $X$. As explained in the introduction of [7], on a compact complex manifold, there exist at least two versions of $\partial \bar{\partial}$-lemma. The first (appeared in [9]) applies to a smooth form $u$ of any degree that is not necessarily of pure type with $u \in \ker \partial \cap \ker \bar{\partial}$ and satisfying the following implication:
  \[
  u \in \text{Im} \, d \implies u \in \text{Im} \, (\partial \bar{\partial}).
  \]
  The second version of $\partial \bar{\partial}$-lemma (appeared in [16]) applies to a pure type $d$-closed form $u$ (of any bidegree) on $X$ satisfying the following equivalences:
  \[
  u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \bar{\partial}\text{-exact} \iff u \text{ is } \partial \bar{\partial}\text{-exact}.
  \]
  It is proved in [1,17] that on every $\partial \bar{\partial}$-manifold, the notion of SKT metric and the notion of Hermitian-symplectic metric are equivalent.

Lemma 3.2 Let $X$ be a $\partial \bar{\partial}$-manifold. For any Hermitian metric $\omega$ on $X$, the following equivalence holds:

\[
\omega \text{ is SKT} \iff \omega \text{ is Hermitian-symplectic}.
\]

Proof $(\iff)$ This implication holds on any compact complex manifold. Suppose that there exists $\alpha^{0,2} \in C_{0,2}^\infty(X, \mathbb{C})$ such that $d(\alpha^{0,2} + \omega + \alpha^{0,2}) = 0$. This is equivalent to

\[
\begin{align*}
\partial \alpha^{0,2} &= 0, \\
\bar{\partial} \alpha^{0,2} + \bar{\partial} \omega &= 0.
\end{align*}
\]

Applying $\partial$ to the above identity we get $\partial \bar{\partial} \omega = 0$.

$(\Rightarrow)$ Suppose that $\partial \bar{\partial} \omega = 0$ and $X$ is a $\partial \bar{\partial}$-manifold, then $\partial \omega \in \ker \bar{\partial}$. Meanwhile $\partial \omega$ is a $(2,1)$-form which is $d$-closed and $\partial$-exact, hence by the $\partial \bar{\partial}$-lemma $\partial \omega$ must be
\( \overline{\partial} \)-exact. This means that there exists \( \alpha^{2,0} \in C^\infty_{2,0}(X, \mathbb{C}) \) such that \( \partial \omega = - \overline{\partial} \alpha^{2,0} = - \overline{\partial} \alpha^{0,2} \) with \( \alpha^{0,2} := \alpha^{2,0} \). We get

\[
d(\alpha^{0,2} + \omega + \overline{\alpha^{0,2}}) = \partial \alpha^{0,2} + \overline{\partial} \alpha^{0,2} + \partial \omega + \overline{\partial} \omega + \overline{\partial} \alpha^{0,2} + \overline{\partial} \alpha^{0,2} = (\partial \omega + \overline{\partial} \alpha^{0,2}) + (\partial \omega + \overline{\partial} \alpha^{0,2}) + \overline{\partial} \alpha^{0,2} + \overline{\partial} \alpha^{0,2}
\]

\[
= \overline{\partial} \alpha^{0,2} + \overline{\partial} \alpha^{0,2}.
\]

It remains to show that \( \partial \alpha^{2,0} = 0 \). Indeed, we have \( \overline{\partial} \partial \alpha^{2,0} = - \overline{\partial} \partial \alpha^{2,0} = \partial^2 \omega = 0 \). So \( \partial \alpha^{2,0} \) is a \( d \)-closed and \( \overline{\partial} \)-exact \((3, 0)\)-form, therefore by the \( \partial \overline{\partial} \)-lemma it is \( \overline{\partial} \)-exact. For bidegree reasons, this means that \( \partial \alpha^{2,0} = 0 \). This proves that \( \omega \) is H-S. \( \square \)

In the rest of this section, we will show that the property of a manifold being Hermitian-symplectic is open under holomorphic deformations.

**Theorem 3.3** Let \( (X_t)_{t \in \Delta} \) be a holomorphic family of compact complex manifolds over an open disc \( \Delta \subset \mathbb{C} \) containing the origin. If there exists an H-S metric \( \omega_0 \) on \( X_0 \), then after possibly shrinking \( \Delta \) about 0, there exists a \( C^\infty \) family \( (\omega_t)_{t \in \Delta} \) of H-S metrics on the fibres \( (X_t)_{t \in \Delta} \) whose term corresponding to \( t = 0 \) is the original \( \omega_0 \).

**Proof** Suppose that there exists an H-S metric \( \omega_0 \) on \( X_0 \). Then, there exists \( \alpha_{0,2} \in C^\infty_{0,2}(X_0, \mathbb{C}) \) such that

\[
d(\alpha_{0,2}^0 + \omega_0 + \overline{\alpha_{0,2}^0}) = 0.
\]

Putting \( \Omega := \alpha_{0,2}^0 + \omega_0 + \overline{\alpha_{0,2}^0} \), we get \( \Omega \) a \( d \)-closed real 2-form on \( X_0 \). By Ehresmann, \( \Omega \) does not depend on the complex structure of the fibre. Let \( (\Omega_t^{1,1})_{t \in \Delta} \) be the \( C^\infty \) family of components of \( \Omega \) of \( J_1 \)-type \((1, 1)\), namely \( \Omega_t^{1,1} = (\Omega_t^1)^1 \), so \( (\Omega_t^{1,1})_{t \in \Delta} \) vary smoothly with \( t \). Thus, by the continuity of the family \( (\Omega_t^{1,1})_{t \in \Delta} \), the strict positivity of \( \Omega_t^{1,1} \) implies the strict positivity of \( \Omega_t^{1,1} \) for all \( t \in \Delta \) sufficiently close to 0. So, we obtain an H-S metric \( \omega_t \) on \( X_t \) for every \( t \in \Delta \) close to 0. \( \square \)

Now, recall the following theorem of Wu [26] according to which the \( \partial \overline{\partial} \)-property of compact complex manifolds is open under holomorphic deformations of the complex structure:

\[
X_0 \text{ is a } \partial \overline{\partial} \text{-manifold} \implies X_t \text{ is a } \partial \overline{\partial} \text{-manifold for } t \in \Delta, \ t \sim 0.
\]

As a consequence of Wu’s theorem and of our Theorem 3.3, we get the following result.

**Corollary 3.4** Let \( (X_t)_{t \in \Delta} \) be a holomorphic family of compact complex manifolds over an open disc \( \Delta \subset \mathbb{C} \) containing the origin. If \( X_0 \) is an SKT \( \partial \overline{\partial} \)-manifold, then \( X_t \) is an SKT \( \partial \overline{\partial} \)-manifold for all \( t \in \Delta \) sufficiently close to 0. Moreover, if \( X_0 \) is an SKT \( \partial \overline{\partial} \)-manifold, any SKT metric \( \omega_0 \) on \( X_0 \) deforms in a \( C^\infty \) way to a family \( (\omega_t)_t \) of SKT metrics on the nearby fibres \( (X_t)_t \).
4 Deformations of $p$-HS and $p$-SKT manifolds

In this section, we will prove that the property of a compact complex $\partial\bar{\partial}$-manifold to
carry a $p$-SKT form is deformation-open.

The following result, stating that the notions of $p$-HS and $p$-SKT metrics coincide
on a compact complex $\partial\bar{\partial}$-manifold, was already proven in [1, Corollary 3.4] (in a
different way).

Lemma 4.1 Let $X$ be a $\partial\bar{\partial}$-manifold with $\dim\mathbb{C} X = n$. Fix $p \in \{1, \ldots, n - 1\}$. If
there exists a $p$-SKT form $\omega$ on $X$, then $\omega$ is a $p$-Hermitian-symplectic form.

Proof Suppose there exists a $p$-pluriclosed form $\omega$ on $X$. Namely, there exists a $C^\infty$
strictly weakly positive real $(p, p)$-form $\omega$ such that $\partial\bar{\partial}\omega = 0$. Notice that $\partial\omega$ is a $d$-closed $(p + 1, p)$-form. By the $\partial\bar{\partial}$-lemma, $\partial\omega$ is $\bar{\partial}$-exact, that is there exists
$
\alpha^{p+1,p-1} \in C^\infty_{p+1,p-1}(X, \mathbb{C})
$
such that $\partial\omega = -\bar{\partial}\alpha^{p+1,p-1}$. Note that $\partial\alpha^{p+1,p-1}$
is $d$-closed, so by the $\partial\bar{\partial}$-lemma there exists $\alpha^{p+2,p-2} \in C^\infty_{p+2,p-2}(X, \mathbb{C})$ such that
$\partial\alpha^{p+1,p-1} = -\bar{\partial}\alpha^{p+2,p-2}$.

We continue until we find a form $\alpha^{2p,0} \in C^\infty_{2p,0}(X, \mathbb{C})$ such that $\partial\alpha^{2p-1,1} =
-\bar{\partial}\alpha^{2p,0}$. Thus, $\partial\alpha^{2p,0}$ is $d$-closed hence by the $\partial\bar{\partial}$-lemma it must also be $\bar{\partial}$-exact.
However, the only $\bar{\partial}$-exact $(2p, 0)$-form is zero, thus $\partial\alpha^{2p,0} = 0$. As a consequence, we have

\[
d\left( \sum_{i=0}^{p-1} \alpha^{i,2p-i} + \omega + \sum_{i=0}^{p-1} \alpha^{i,2p-i} \right) 
= \partial\alpha^{0,2p} + \bar{\partial}\alpha^{0,2p} + \partial\bar{\partial}\alpha^{0,2p} + \bar{\partial}\alpha^{1,2p-1} + \partial\alpha^{1,2p-1} + \bar{\partial}\alpha^{1,2p-1} 
+ \partial\alpha^{1,2p-1} + \bar{\partial}\alpha^{1,2p-1} + \cdots + \partial\alpha^{p-3,p+3} + \bar{\partial}\alpha^{p-3,p+3} 
+ \partial\alpha^{p-3,p+3} + \bar{\partial}\alpha^{p-3,p+3} + \partial\alpha^{p-2,p+2} + \bar{\partial}\alpha^{p-2,p+2} 
+ \partial\alpha^{p-2,p+2} + \bar{\partial}\alpha^{p-2,p+2} + \partial\alpha^{p-1,p+1} + \partial\bar{\partial}\alpha^{p+1,p-1} 
= 0.
\]

Thus, $\omega$ is a $p$-HS form. \qed

A key observation for us will be the following deformation-openness property of the
strict weak positivity for $(p, p)$-forms.

Lemma 4.2 Let $(X_t)_{t \in \Delta}$ be a holomorphic family of compact complex $n$-dimensional
manifolds over an open disc $\Delta \subset \mathbb{C}$ containing the origin. Fix any $p \in \{1, \ldots, n - 1\}$. Let $(\Omega_t)_{t \in \Delta}$ be a $C^\infty$ family of real $(2p)$-forms on the fibres $X_t$ such that every $\Omega_t$
is of type $(p, p)$ for the complex structure $J_t$ of $X_t$. If $\Omega_0$ is strictly weakly positive
for the complex structure $J_0$ of $X_0$, then $\Omega_t$ is strictly weakly positive for the complex
structure $J_t$ of $X_t$ for all $t \in \Delta$ sufficiently close to $0 \in \Delta$. \[ Springer \]
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Proof Let $X = \bigcup_{t \in \Delta} X_t$ be the total space of the family and let $\mathcal{X} = U^{(1)} \cup \cdots \cup U^{(N)}$ be an open covering by coordinate patches of $\mathcal{X}$ such that $X_0 \subset U^{(1)} \cup \cdots \cup U^{(N)}$. For every $v = 1, \ldots, N$, let $(z_1^{(v)}, \ldots, z_m^{(v)}, t)$ be a local holomorphic coordinate system on $U^{(v)} \subset \mathcal{X}$, where $t$ is a holomorphic coordinate on a neighbourhood of $0$ in $\Delta$. Put $U_t^{(v)} := U^{(v)} \cap X_t \subset X_t$ and $z_j^{(v)}(t) := z_j^{(v)}|_{X_t}$.

Now, the strict weak positivity for $(p, p)$-forms on a complex manifold is a point-wise, hence local, property. For a fixed $t' \in \Delta$ and a fixed index $v \in \{1, \ldots, N\}$, let $\tau_j(t'), \ldots, \tau_q(t')$ (where $q = n - p$) be linearly independent $C^\infty J_t^p(1, 0)$-forms on $U_t^{(v)} \subset X_t$. We extend the forms $\tau_j(t')$ to $C^\infty (1, 0)$-forms $\tau_1, \ldots, \tau_q$ on $U^{(v)} \subset \mathcal{X}$. (This is always possible for forms defined on a coordinate patch. For example, we can extend their coefficients by requiring them to remain constant in the $t$-direction.) For every $t \in \Delta$, the restrictions $\tau_1(t), \ldots, \tau_q(t)$ of the forms $\tau_1, \ldots, \tau_q$ to $U_t^{(v)} \subset X_t$ are of type $(1, 0)$ for $J_t$. They are also linearly independent if $t$ lies in a sufficiently small neighbourhood (depending on the forms $\tau_1(t'), \ldots, \tau_q(t')$) of $0$. Since $\Omega_0$ is strictly weakly positive for $J_0$, we have

$$\Omega_0 \wedge i \tau_1(0) \wedge \overline{\tau_1}(0) \wedge \cdots \wedge i \tau_{n-p}(0) \wedge \overline{\tau_{n-p}}(0) > 0 \text{ on } U_0^{(v)} \subset X_0.$$  

By continuity, we still have $\Omega_t \wedge i \tau_1(t) \wedge \overline{\tau_1}(t) \wedge \cdots \wedge i \tau_{n-p}(t) \wedge \overline{\tau_{n-p}}(t) > 0$ for all $t$ lying in a sufficiently small neighbourhood (depending on the forms $\tau_1(t'), \ldots, \tau_q(t')$) of $0$.

Meanwhile, the strict weak positivity at a point has to be tested on all the $p$-dimensional complex vector subspaces of an $n$-dimensional space. These subspaces form a compact complex manifold, the Grassmannian $G_{p,n}$. Thanks to the compactness of this Grassmannian and of the fibres, there exists a uniform open neighbourhood $\Delta_0$ of $0$ in $\Delta$ such that $\Omega_t$ is strictly weakly positive for the complex structure $J_t$ of $X_t$ for all $t \in \Delta_0$. \hfill $\square$

The first main result of this section is the following.

Theorem 4.3 Let $X$ be a compact complex manifold of dimension $n$. Fix $p \in \{1, \ldots, n-1\}$. If there exists a $p$-HS form $\omega_0$ on $X_0$, then there exists a $p$-HS form $\omega_t$ on $X_t$ for $t \in \Delta$ sufficiently close to $0$.

Proof Assume that there exists a $p$-HS form $\omega_0$ on $X_0$. This means that there exist $\alpha^{i,2p-i} \in C^\infty_{i,2p-i}(X_0, \mathbb{C})$ for $i = 0, \ldots, 2p$ such that $d(\sum_{i=0}^{p-1} \alpha^{i,2p-i} + \omega_0 + \sum_{i=0}^{p-1} \overline{\alpha^{i,2p-i}}) = 0$.

Put $\Omega := \sum_{i=0}^{p-1} \alpha^{i,2p-i} + \omega + \sum_{i=0}^{p-1} \overline{\alpha^{i,2p-i}}$. This is a $d$-closed real $(2p)$-form on the $C^\infty$-manifold $X$ underlying the fibres $X_t$. Let $(\Omega_t^{p,p})_{t \in \Delta}$ be the $C^\infty$ family of components of $\Omega$ of $J_t$-type $(p, p)$. The forms $\Omega_t^{p,p}$ vary smoothly with $t \in \Delta$.

Thanks to Lemma 4.2, the weak strict positivity of $\Omega_0^{p,p} = \omega_0$ implies the weak strict positivity of $\Omega_t^{p,p}$ for all $t \in \Delta$ sufficiently close to $0$. Thus, $\omega_t := \Omega_t^{p,p}$ is a $p$-HS form on $X_t$ for all $t \in \Delta$ sufficiently close to $0$. \hfill $\square$

As a consequence we get
Conclusion 4.4 If $X_0$ is a $p$-SKT $\bar{\partial} \bar{\partial}$-manifold, then $X_0$ is a $p$-HS manifold. Therefore, $X_t$ is a $p$-HS manifold, hence also a $p$-SKT manifold, for $t \in \Delta$ sufficiently close to 0. On the other hand, by [26], the $\partial \bar{\partial}$-lemma property is open under holomorphic deformations.

So, we have proved the following result.

Theorem 4.5 Let $\pi : \tilde{X} \to \Delta$ be a holomorphic family of compact complex $n$-dimensional manifolds. Fix $p \in \{1, \ldots, n-1\}$. If $X_0$ is a $p$-SKT $\bar{\partial} \bar{\partial}$-manifold, then $X_t$ is a $p$-SKT $\partial \bar{\partial}$-manifold for $t \in \Delta$ sufficiently close to 0.

5 Deformation limits of positive cones

We consider the Frölicher spectral sequence $(E_r, d_r)$. It relates the Dolbeault cohomology groups as invariants of the complex structure to the De Rham cohomology groups as topological invariants. For more details, see e.g. [18] or [7].

Let us start by defining the notions of $E_k$-closedness and $\bar{E}_k$-closedness.

Definition 5.1 A smooth $(r, s)$-form $\alpha^{r, s}$ is called $E_k$-closed (resp. $\bar{E}_k$-closed) if $\bar{\partial} \alpha^{r, s} = 0$, $\partial \alpha^{r, s} = \bar{\partial} \alpha^{r+1, s-1}$, $\ldots$, $\partial \alpha^{r+k-2, s-k+2} = \bar{\partial} \alpha^{r+k-1, s-k+1}$ (resp. $\bar{\partial} \alpha^{r, s} = 0$, $\partial \alpha^{r, s} = \partial \alpha^{r-1, s+1}$, $\ldots$, $\partial \alpha^{r-k+2, s+k-2} = \partial \alpha^{r-k+1, s+k-1}$) with $\alpha^{r-l,s+l} \in C_{r-l, s+l}(X, \mathbb{C})$ and $\alpha^{r+l,s-l} \in \bar{C}_{r+l, s-l}(X, \mathbb{C})$ for $l \in \{0, \ldots, k-1\}$.

Recall that a $C^\infty$ strictly weakly positive $(p, p)$-form $\Omega$ is $p$-Hermitian-symplectic ($p$-HS) if and only if there exist $\alpha^{i,2p-i} \in C^\infty_{i,2p-i}(X, \mathbb{C})$ for $i = 0, \ldots, 2p$ such that

$$d \left( \sum_{i=0}^{p-1} \alpha^{i,2p-i} + \Omega + \sum_{i=0}^{p-1} \alpha^{i,2p-i} \right) = 0.$$ 

Therefore,

$\Omega$ is $p$-HS $\iff$ $\partial \alpha^{0,2p} = 0$, $\partial \alpha^{2p-1,1} + \bar{\partial} \alpha^{2p,0} = 0$, $\ldots$, $\partial \alpha^{p+1,p-1} + \bar{\partial} \alpha^{p+2,p-2} = 0$, $\partial \Omega + \bar{\partial} \alpha^{p+1,p-1} = 0$ $\iff$ $\alpha^{2p,0}$ is $E_{p+1}$-closed and $d_{p+1}(\{\alpha^{2p,0}\}_{E_{p+1}}) = \{\partial \Omega\}_{E_{p+1}}$ $\iff$ $\alpha^{0,2p}$ is $E_{p+1}$-closed and $d_{p+1}(\{\alpha^{0,2p}\}_{E_{p+1}}) = \{\partial \Omega\}_{E_{p+1}}$.

So, given a strictly weakly positive $(p, p)$-form $\Omega$, we have

$\Omega$ is $p$-HS $\iff$ $\partial \Omega$ is $E_{p+1}$-closed and $\{\partial \Omega\}_{E_{p+1}} \in \text{Im} d_{p+1}$ $\iff$ $\partial \Omega$ is $E_{p+2}$-exact.

This is equivalent to the property

$$(E_k) : \bar{\partial} \alpha^{p+k,p-k} = -\partial \alpha^{p+k-1,p-k+1} \quad \text{for all } k = 1, \ldots, p.$$
We now introduce our main objects of study in this section.

**Definition 5.2** Let $X$ be a compact $n$-dimensional complex manifold. Let $p \in \{1, \ldots, n\}$. The cones $\mathcal{A}_p(X)$ and $\mathcal{C}_p(X)$ are defined as

$$\mathcal{A}_p(X) = \{ [\Omega]_A : \Omega \text{ is strictly weakly positive such that } \partial \bar{\partial} \Omega = 0 \}$$

(3)

$$\subset H^{p,p}_\mathcal{A}(X, \mathbb{R}) \subset H^{p,p}_\mathcal{A}(X, \mathbb{C}),$$

$$\mathcal{C}_p(X) = \{ [\Omega]_A : \Omega \text{ is strictly weakly positive such that } \Omega \text{ is } p\text{-HS} \}$$

(4)

$$\subset H^{p,p}_\mathcal{A}(X, \mathbb{R}) \subset H^{p,p}_\mathcal{A}(X, \mathbb{C}).$$

Note that $\mathcal{C}_p(X) \subset \mathcal{A}_p(X)$.

**Lemma 5.3** The subsets $\mathcal{C}_p(X) \subset \mathcal{A}_p(X)$ are open convex cones in $H^{p,p}_\mathcal{A}(X, \mathbb{R})$.

**Proof** To see that $\mathcal{A}_p(X)$ is a convex cone, let $[\Omega]_A, [\tilde{\Omega}]_A \in \mathcal{A}_p(X)$ and $\lambda > 0$. Then $\Omega + \lambda \tilde{\Omega}$ remains strictly weakly positive and $\partial \bar{\partial} (\Omega + \lambda \tilde{\Omega}) = 0$, i.e., $[\Omega + \lambda \tilde{\Omega}]_A \in \mathcal{A}_p(X)$.

To prove openness for $\mathcal{A}_p(X)$, let $[\Omega]_A \in \mathcal{A}_p(X)$ and $[\gamma]_A \in H^{p,p}_\mathcal{A}(X, \mathbb{C})$ be arbitrary. We will prove that $[\Omega]_A + \varepsilon [\gamma]_A \in \mathcal{A}_p(X)$ for all sufficiently small $\varepsilon > 0$.

Pick arbitrary representatives $\Omega$ and $\gamma$ of their respective Aeppli cohomology classes such that $\Omega$ is strictly weakly positive. Notice that $\partial \bar{\partial} (\Omega + \varepsilon \gamma) = 0$ for every $\varepsilon$ since $\Omega$ and $\gamma$ represent Aeppli classes.

It remains to prove that $\Omega + \varepsilon \gamma$ is strictly weakly positive for all sufficiently small $\varepsilon > 0$. Since this positivity is a pointwise property, we can reason locally. Let $x_0 \in X$ be an arbitrary point.

By hypothesis, $(\Omega \wedge i\alpha_1 \wedge \bar{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})(x_0) > 0$ for every choice of locally defined, linearly independent, smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_{n-p}$. Equivalently, the restriction of $\Omega(x_0)$ to the $p$-dimensional vector subspace $E_0 = E_{x_0} = \ker \alpha_1(x_0) \cap \cdots \cap \ker \alpha_{n-p}(x_0) \subset T_{x_0}^{1,0} X$ is a positive (i.e., $> 0$) volume form on $E_0 = E_{x_0}$. Now, the quantity $(\Omega + \varepsilon \gamma)(x)|_{E_x}$ depends in a $C^\infty$ way on the triple $(\varepsilon, x, E)$. Therefore, by continuity, there exist a constant $\varepsilon_0 > 0$ and open neighbourhoods $U_{x_0}$ of $x_0$ in $X$ and $\mathcal{U}_{E_0}$ of $E_0$ in the Grassmannian $G_{p,n}$ of $p$-dimensional vector subspaces of $\mathbb{C}^n$, such that

$$(\Omega + \varepsilon \gamma)|_{E_x} > 0 \quad \text{for all} \quad (\varepsilon, x, E) \in (0, \varepsilon_0) \times U_{x_0} \times \mathcal{U}_{E_0}.$$ 

Since both $X$ and the Grassmannian $G_{p,n}$ are compact manifolds, we conclude that there exists a uniform constant $\varepsilon_0 > 0$ such that the $(p, p)$-form $\Omega + \varepsilon \gamma$ is strictly weakly positive for all $0 < \varepsilon < \varepsilon_0$.

Therefore, the cone $\mathcal{A}_p(X)$ is open. The same arguments apply to $\mathcal{C}_p(X)$. $\square$

We now show that the cones $\mathcal{A}_p(X)$ and $\mathcal{C}_p(X)$ are equal if the hypotheses ($H_k$) spelt out in (5) below for $k = 1, \ldots, p + 1$ are satisfied. These hypotheses are a collection of special cases of the $\partial \bar{\partial}$-lemma in a few selected bidegrees. Recall that neither the property of $p$-Kählerianity nor being a balanced manifold are open under
small deformations. It is proved that being a balanced manifold is deformation-open under a weaker condition than the $\partial \bar{\partial}$-lemma (called the $(n - 1, n)$-th mild $\partial \bar{\partial}$-lemma in [20], $(n - 1, n)$-th weak $\partial \bar{\partial}$-lemma in [10] and $(n - 1, n)$-th strong $\partial \bar{\partial}$-lemma in [6]) as well as the property of $p$-Kählerianity under the $(p, p + 1)$-th mild $\partial \bar{\partial}$-lemma in [21]. Thus, it is not necessary to assume the validity of the $\partial \bar{\partial}$-lemma in full generality.

**Proposition 5.4** Let $X$ be a compact complex manifold with $\dim \subset X = n$. For a fixed $p \in \{1, \ldots, n - 1\}$ and a fixed $k \in \{1, \ldots, p + 1\}$, let us consider the following hypothesis $(H_k)$:

$$(H_k): \text{ for all } \Gamma \in C_{p+k,p-k+1}^\infty(X, \mathbb{C}) \text{ such that } d\Gamma = 0,$$

$$\Gamma \in \text{Im } \partial \Rightarrow \Gamma \in \text{Im } \partial .$$

(5)

(i) If the hypotheses $(H_1), \ldots, (H_{p+1})$ are satisfied, then $A_p(X) = C_p(X)$.

(ii) If $A_p(X) = C_p(X)$, the hypothesis $(H_1)$ holds.

**Proof** (i) Assume that the hypotheses $(H_1), \ldots, (H_{p+1})$ are satisfied. Let $[\Omega]_A \in A_p(X)$. Since $\partial \bar{\partial} \Omega = 0$, $\partial \Omega$ is a $d$-closed and $\partial$-exact $(p + 1, p)$-form and by the hypothesis $(H_1)$ there exists $\alpha^{p+1,p-1} \in C_{p+1,p-1}^\infty(X, \mathbb{C})$ that solves the equation $(E_1)$: $\partial \alpha^{p+1,p-1} = - \partial \Omega$. Let $\alpha^{p+1,p-1} \in C_{p+1,p-1}^\infty(X, \mathbb{C})$ be an arbitrary solution of the equation $(E_1)$. On the other hand, we have $\partial \alpha^{p+1,p-1} \in C_{p+2,p-1}^\infty(X, \mathbb{C})$. This implies that $\partial \alpha^{p+1,p-1}$ is a $d$-closed and $\partial$-exact $(p + 2, p - 1)$-form. Hence, under the $(H_2)$ assumption we have $\partial \alpha^{p+1,p-1} \in \text{Im } \partial$, so there exists $\alpha^{p+2,p-2} \in C_{p+1,p-1}^\infty(X, \mathbb{C})$ that solves $(E_2)$, i.e., $\partial \alpha^{p+1,p-1} = - \partial \alpha^{p+2,p-2}$.

We follow the same process until we obtain, under the hypothesis $(H_p)$, that the equation $(E_p)$ has a solution $\alpha^{2p,0}$ such that $\partial \alpha^{2p,0} = - \partial \alpha^{2p-1,1}$. This implies that $\partial \partial \alpha^{2p,0} = 0$, so $\partial \alpha^{2p,0}$ is a $(2p + 1, 0)$-form that is $d$-closed and $\partial$-exact. This implies that $\partial \alpha^{2p,0} \in \text{Im } \partial$, hence $\partial \alpha^{2p,0} = 0$.

Therefore $d \left( \sum_{i=0}^{p-1} \alpha^{i,2p-i} + \Omega + \sum_{i=0}^{p-1} \alpha^{i,2p-i} \right) = 0$, i.e., $[\Omega]_A \in C_p(X)$. It follows that $A_p(X) = C_p(X)$.

(ii) Consider the following linear map:

$$H^p_{\partial_{\bar{\partial}}} \left( X, \mathbb{C} \right) \to H^{p+1,p}_{\partial_{\bar{\partial}}} \left( X, \mathbb{C} \right),$$

$$[\Omega]_A \mapsto [\partial \Omega]_{\bar{\partial}} .$$

Let us show that this map is well defined. Let $[\Omega]_A \in H^p_{\partial_{\bar{\partial}}}(X, \mathbb{C})$, hence $\partial \bar{\partial} \Omega = 0$, i.e., $\partial (\partial \Omega) = 0$, thus $\partial \Omega$ defines a class $[\partial \Omega]_{\bar{\partial}} \in H^{p,p}_{\partial_{\bar{\partial}}}(X, \mathbb{C})$. If $[\Omega]_A = [\Omega_2]_A$, then there exist a $(p - 1, p)$-form $u$ and a $(p, p - 1)$-form $v$ such that $\Omega_1 - \Omega_2 = \partial u + \bar{\partial} v$. Thus $\partial (\Omega_1 - \Omega_2) = \partial \bar{\partial} v = \bar{\partial} (- \partial v) \in \text{Im } \partial$. This proves that $[\partial \Omega_1]_{\bar{\partial}} = [\partial \Omega_2]_{\bar{\partial}}$. Consequently the map $T$ is well defined.

Now, we want to prove that the map $T$ vanishing identically is equivalent to the hypothesis $(H_1)$. Indeed, let $\Gamma \in C^\infty_{p+1,p}(X, \mathbb{C})$ be such that $\Gamma = \partial \Omega \in \text{ker } d$, so
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\[ \overline{\partial} \Gamma = \overline{\partial} \partial \Omega = - \partial \overline{\partial} \Omega. \] But $\Gamma$ is a $d$-closed form of pure type $(p+1, p)$, i.e., $\partial \Gamma = 0$ and $\overline{\partial} \Gamma = 0$, then $\partial \overline{\partial} \Omega = 0$.

On the other hand, $T \equiv 0$ means that $T ([\Omega]_A) = [\partial \Omega]_\overline{\partial} = 0 \in H^{p+1, p}_\overline{\partial} (X, \mathbb{C})$, for every $[\Omega]_A \in H^{p, p}_A (X, \mathbb{C})$. Hence $\partial \Omega \in \text{Im} \overline{\partial}$.

Conversely, assume that for all $\Gamma \in C^\infty_{p, p+1} (X, \mathbb{C})$ such that $d \Gamma = 0$, we have the implication $\Gamma \in \text{Im} \partial \Rightarrow \Gamma \in \text{Im} \overline{\partial}$. Let $[\Omega]_A \in H^{p, p}_A (X, \mathbb{C})$. Then $\Gamma = \partial \Omega$ implies that $\Gamma = \partial \Omega \in \text{Im} \overline{\partial}$, so $[\partial \Omega]_\overline{\partial} = 0 = T ([\Omega]_A)$. This proves $T \equiv 0$.

Now, since $A_p(X) = \mathcal{C}_p(X)$, we have

\[ A_p(X) \cap \ker T = \{ [\Omega]_A : \Omega \text{ is strictly weakly positive such that } \partial \overline{\partial} \Omega = 0 \] and $\partial \Omega \in \text{Im} \overline{\partial} \} \supset \mathcal{C}_p(X)\]

So $\mathcal{C}_p(X) \subset A_p(X) \cap \ker T \subset A_p(X)$. Since $A_p(X) = \mathcal{C}_p(X)$, this implies that $A_p(X) \cap \ker T = A_p(X)$.

Hence $\ker T = H^p_{A, p} (X, \mathbb{C})$, which is equivalent to the $(H_1)$ assumption. \( \square \)

We shall now consider, for $k = 1, \ldots, p+1$, the linear map

\[ H^{p+k-1, p-k+1}_A (X, \mathbb{C}) \xrightarrow{\widehat{T}_k} H^{p+k, p-k+1}_B (X, \mathbb{C}) \]

\[ [\Omega]_A \rightarrow [\partial \Omega]_B. \]

This map is well defined. Indeed. Suppose that $\partial \overline{\partial} \Omega = 0$, i.e., $\overline{\partial} (\partial \Omega) = 0$, so $\partial \Omega \in \ker \overline{\partial}$ this is equivalent to $\partial \Omega \in \ker \partial \cap \ker \overline{\partial}$. Moreover, if $\Omega = \partial u + \overline{\partial} v$ with $u$ and $v$ are $(p+k-2, p-k+1)$-form and $(p+k-1, p-k)$-form respectively, hence $\partial \Omega = \partial \overline{\partial} v \in \text{Im} \partial \overline{\partial}$. Thus the map $\widehat{T}_k$ is well defined.

Consider the linear map

\[ I^{p+k, p-k+1} : H^{p+k, p-k+1}_B (X, \mathbb{C}) \rightarrow H^{p+k, p-k+1}_\partial (X, \mathbb{C}) \]

\[ [\Gamma]_B \rightarrow [\Gamma]_\partial. \]

We have $\partial \Gamma = 0$, and $\overline{\partial} \Gamma = 0$. To show that the map $I^{p+k, p-k+1}$ is well defined, we still have to show that the definition is independent of the choice of representative of the class $[\Gamma]_B$. In other words, if $\Gamma = \partial \overline{\partial} u$ with $u$ a $(p+k-1, p-k)$-form, then $\Gamma \in \text{Im} \partial$. This is obvious, so the map $I^{p+k, p-k+1}$ is well defined.

We will infer the following

\textbf{Remark 5.5} The map $\widehat{T}_k$ vanishes identically if and only if $\partial \Omega \in \text{Im} \partial \overline{\partial}$ for all $\Omega \in C^\infty_{p+k-1, p-k+1} (X, \mathbb{C}) \cap \ker \partial \overline{\partial}$. This is equivalent to

\[ (\widehat{H}_k) : \quad \text{for all } \Gamma \in C^\infty_{p+k, p-k+1} (X, \mathbb{C}) \text{ such that } d \Gamma = 0, \]

we have $\Gamma \in \text{Im} \partial \Rightarrow \Gamma \in \text{Im} \partial \overline{\partial}$. \( (6) \)

This is further equivalent to $I^{p+k, p-k+1}$ is injective, hence to $\ker I^{p+k, p-k+1} = \{0\}$. \( \square \) Springer
Consider the diagram

\[
\begin{array}{ccc}
H^p_{\partial_{\log}, p-k+1}(X, \mathbb{C}) & \xrightarrow{\widehat{T}_k} & H^{p+k}_{p-k+1}(X, \mathbb{C}) \\
g & & \downarrow I^{p+k,p-k+1} \\
H^p_{\partial}, p-k+1(X, \mathbb{C})
\end{array}
\]

where \( g = I^{p+k,p-k+1} \circ \widehat{T}_k \). We have \( \ker I^{p+k,p-k+1} = \{ \{\Gamma\} : [\Gamma] = 0 \} \) and \( \text{Im} \, \widehat{T}_k = \{ [\partial \Omega]_{BC} : \Omega \in \ker \partial \partial \subset C^{\infty}_{p+k-1,p-k+1}(X, \mathbb{C}) \} \subset \ker I^{p+k,p-k+1} \).

Conversely, if \([\Gamma]_{BC} \in \ker I^{p+k,p-k+1}\), then there exists a \((p+k-1, p-k+1)\)-form \(\Omega\) such that \(\Gamma = \partial \Omega\). Since \([\Gamma]_{BC} \in H^{p+k}_{BC}(X, \mathbb{C})\), then \(0 = \partial \partial \Gamma = \partial \partial \Omega\). This implies that \([\Gamma]_{BC} = [\partial \Omega]_{BC} = \widehat{T}_k([\Omega]_{A})\). Therefore we always have

\[
\ker I^{p+k,p-k+1} = \text{Im} \, \widehat{T}_k.
\]

Furthermore, the hypothesis \((\widehat{T}_k)\) (cf. 6) is satisfied if and only if \(\ker I^{p+k,p-k+1} = \text{Im} \, \widehat{T}_k = \{0\}\).

We can now prove the following result.

**Theorem 5.6** Let \((X_t)_{t \in \Delta}\) be a holomorphic family of compact complex manifolds with \(\dim_{\mathbb{C}} X_t = n\) and \((\omega_t)_{t \in \Delta}\) a smooth family of Hermitian metrics on \((X_t)_{t \in \Delta}\) for \(t \in \Delta\). If for all \(t\) sufficiently close to 0 we have

\[
\begin{align*}
h^p_{A, p-k+1}(0) &= h^p_{A, p-k+1}(t) := \dim_{\mathbb{C}} H^p_{A, p-k+1}(X_t, \mathbb{C}), \\
h^p_{BC, p-k+1}(0) &= h^p_{BC, p-k+1}(t) := \dim_{\mathbb{C}} H^p_{BC, p-k+1}(X_t, \mathbb{C}), \\
h^p_{\partial, p-k+1}(0) &= h^p_{\partial, p-k+1}(t) := \dim_{\mathbb{C}} H^p_{\partial, p-k+1}(X_t, \mathbb{C}),
\end{align*}
\]

then

\[
\begin{align*}
\Delta \ni t &\implies \mathcal{H}^p_{A, p-k+1}(X_t, \mathbb{C}), \\
\Delta \ni t &\implies \mathcal{H}^p_{BC, p-k+1}(X_t, \mathbb{C}), \\
\Delta \ni t &\implies \mathcal{H}^p_{\partial, p-k+1}(X_t, \mathbb{C})
\end{align*}
\]

are \(C^\infty\)-vector bundles. Moreover, the linear maps

\[
\begin{array}{ccc}
H^p_{A, p-k+1}(X_t, \mathbb{C}) & \xrightarrow{\widehat{T}_k(t)} & H^p_{BC, p-k+1}(X_t, \mathbb{C}) \\
gk(t) & & \downarrow I^{p+k,p-k+1}(t) \\
H^p_{\partial, p-k+1}(t) & & \end{array}
\]

vary in a \(C^\infty\) way with \(t \in \Delta\), where \(g_k = I^{p+k,p-k+1} \circ \widehat{T}_k\). 

\(\square\) Springer
Proof The Laplace-type operators

\[
\begin{align*}
\Delta_{A,t} &= \partial_t \partial_t^* + \overline{\partial_t \partial_t^*} + \overline{\partial_t^* \partial_t} + \partial_t \overline{\partial_t^* \partial_t} + \partial_t \partial_t^* \partial_t^* + \overline{\partial_t^* \partial_t} \partial_t^* + \overline{\partial_t \partial_t^*} \partial_t^* + \overline{\partial_t^* \partial_t} \partial_t^* + \partial_t \partial_t \partial_t (\text{cf. [22]}) \\
\Delta_{BC,t} &= \partial_t \partial_t^* + \overline{\partial_t \partial_t^*} + \overline{\partial_t^* \partial_t} + \partial_t \overline{\partial_t^* \partial_t} + \partial_t \partial_t^* \partial_t^* + \overline{\partial_t^* \partial_t} \partial_t^* + \overline{\partial_t \partial_t^*} \partial_t^* + \overline{\partial_t^* \partial_t} \partial_t^* + \partial_t \partial_t \partial_t (\text{cf. [14]}) \\
\Delta'_t &= \partial_t \partial_t^* + \partial_t^* \partial_t
\end{align*}
\]

are elliptic (cf. [14,22]), so we have the following Hodge isomorphisms:

\[
\begin{align*}
H_{A}^{p+k-1,p-k+1}(X_t, \mathbb{C}) &\simeq \mathcal{H}_{\Delta_{A,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}), \\
H_{BC}^{p+k,p-k+1}(X_t, \mathbb{C}) &\simeq \mathcal{H}_{\Delta_{BC,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}), \\
H_{\partial_t}^{p+k,p-k+1}(X_t, \mathbb{C}) &\simeq \mathcal{H}_{\Delta'_t}^{p+k-1,p-k+1}(X_t, \mathbb{C}),
\end{align*}
\]

where \( \mathcal{H}_{\Delta_{BC,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}) = \ker \Delta_{BC,t} \) and \( \mathcal{H}_{\Delta_{A,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}) = \ker \Delta_{A,t} \) stand for the Bott–Chern and Aeppli harmonic spaces (spaces of Bott–Chern and Aeppli harmonic forms) respectively and \( \mathcal{H}_{\Delta'_t}^{p+k-1,p-k+1}(X_t, \mathbb{C}) = \ker \Delta'_t \).

On the other hand, since \( h_{A}^{p+k-1,p-k+1}(0) = h_{BC}^{p+k-1,p-k+1}(t) = h_{BC}^{p+k,p-k+1}(t) \) and \( \partial_{t}^{h_{BC}}(0) = h_{BC}^{p+k,p-k+1}(t) \) for all \( t \) sufficiently close to 0, by Kodaira–Spencer (cf. [13, Theorem 7.4]) we have

\[
\begin{align*}
&\begin{array}{c}
\xrightarrow{t} \mathcal{H}_{A}^{p+k-1,p-k+1}(X_t, \mathbb{C}) \simeq \mathcal{H}_{\Delta_{A,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}), \\
\xrightarrow{t} \mathcal{H}_{BC}^{p+k,p-k+1}(X_t, \mathbb{C}) \simeq \mathcal{H}_{\Delta_{BC,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}), \\
\xrightarrow{t} \mathcal{H}_{\partial_{t}}^{p+k,p-k+1}(X_t, \mathbb{C}) \simeq \mathcal{H}_{\Delta'_t}^{p+k-1,p-k+1}(X_t, \mathbb{C})
\end{array}
\end{align*}
\]

are \( C^\infty \)-vector bundles. Recall that the linear maps \( \widehat{T}_k(t) \) and \( I_{h_{BC}}^{p+k,p-k+1}(t) \) are well defined as shown above. Let \( h_t \) (resp. \( F_t \)) be the orthogonal projection of \( C_{p+k-1,p-k+1}(X_t) \) (respectively \( C_{p+k-1,p-k+1}(X_t) \)) onto \( \mathcal{H}_{A}^{p+k-1,p-k+1}(X_t, \mathbb{C}) \) (respectively \( \mathcal{H}_{BC}^{p+k-1,p-k+1}(X_t, \mathbb{C}) \)).

\[
\begin{align*}
C_{p+k-1,p-k+1}(X_t) &\xrightarrow{h_t} C_{p+k-1,p-k+1}(X_t) \\
H_{A}^{p+k-1,p-k+1}(X_t, \mathbb{C}) &\xrightarrow{\mathcal{H}_{\Delta_{A,t}}} \mathcal{H}_{\Delta_{A,t}}^{p+k-1,p-k+1}(X_t, \mathbb{C}) \xrightarrow{\widehat{T}_k(t)} \mathcal{H}_{BC}^{p+k-1,p-k+1}(X_t, \mathbb{C}) \xrightarrow{F_t} H_{BC}^{p+k-1,p-k+1}(X_t, \mathbb{C})
\end{align*}
\]

We have \( F_t \circ f_t = \widehat{T}_k(t) \circ h_t \), \( f_t \) is \( C^\infty \) and by Kodaira–Spencer as in [13], \( F_t \) and \( h_t \) vary smoothly with \( t \in \Delta \). Therefore \( \widehat{T}_k(t) \) varies smoothly with \( t \). Let \( P_t \) the orthogonal projection of \( C_{p+k-1,p-k+1}(X_t) \) onto \( \mathcal{H}_{\Delta'_t}^{p+k-1,p-k+1}(X_t, \mathbb{C}) \).
We have \( P_t = I_{p+k,p-k+1}(t) \circ F_t \) and by Kodaira–Spencer as in [13], \( P_t \) and \( F_t \) are \( C^\infty \) with \( t \in \Delta \), then \( I_{p+k,p-k+1}(t) \) is also \( C^\infty \) with \( t \in \Delta \). Consequently \( g(t) \) varies smoothly with \( t \in \Delta \).

One obtains sections \( \tilde{T}_k = (\tilde{T}_k(t))_{t \in \Delta} \in C^\infty(\Delta, \text{End}(\mathcal{H}_{\Delta A}, \mathcal{H}_{\Delta BC})) \) and \( I_{p+k,p-k+1} = (I_{p+k,p-k+1}(t))_{t \in \Delta} \in C^\infty(\Delta, \text{End}(\mathcal{H}_{BC}, \mathcal{H}_{\partial})) \).

As a consequence, we obtain the following conclusion on the deformation limit of the \( X_t \)'s that satisfy the \((\mathcal{H}_k)\) assumption.

**Corollary 5.7** Let \((X_t)_{t \in \Delta}\) be a holomorphic family of compact complex manifolds and \((\omega_t)_{t \in \Delta}\) a smooth family of metrics on \((X_t)_{t \in \Delta}\). Assume that

\[
\begin{align*}
\frac{h_A^{p+k-1,p-k+1}(0)}{h_A^{p+k-1,p-k+1}(t)} = \frac{h_{BC}^{p+k,p-k+1}(0)}{h_{BC}^{p+k,p-k+1}(t)}, \\
\frac{h_{\partial}^{p+k,p-k+1}(0)}{h_{\partial}^{p+k,p-k+1}(t)},
\end{align*}
\]

for all \( t \sim 0 \).

If \( X_t \) satisfies the hypothesis \((\mathcal{H}_k)\), for all \( t \in \Delta \setminus \{0\} \), then \( X_0 \) also satisfies the hypothesis \((\mathcal{H}_k)\).

**Proof** Recall that \( X_t \) satisfying \((\mathcal{H}_k)\) is equivalent to the map \( \tilde{T}_k(t) \) vanishing identically. Moreover, if \( \tilde{T}_k(t) \equiv 0 \), for all \( t \in \Delta \setminus \{0\} \), then by continuity of \( \tilde{T}_k(t) \), \( \tilde{T}_k(0) \equiv 0 \) which is equivalent to \( X_0 \) satisfying the \((\mathcal{H}_k)\) assumption.

Hence we obtain the following result.

**Proposition 5.8** Fix \( p \in \{1, \ldots, n-1\} \) and let \( k \in \{1, \ldots, p+1\} \) be such that \( p+k \leq n \).

(i) For all \( k \in \{1, \ldots, p+1\} \), \((\mathcal{H}_k) \Rightarrow (H_k)\).

(ii) \((\mathcal{H}_1) + \cdots + (\mathcal{H}_{p+1}) \Rightarrow (H_1) + \cdots + (H_{p+1}) \Rightarrow \mathcal{A}_p(X) = \mathcal{C}_p(X)\).

(iii) Suppose that \( \mathcal{A}_p(X_t) = \mathcal{C}_p(X_t) \) (cf. (3), (4)) for all \( t \in \Delta \setminus \{0\} \). Then

\[
\forall k \in \{1, \ldots, p+1\} \quad (\mathcal{H}_k) + \cdots + (\mathcal{H}_{p+1}) \quad \forall t \in \Delta \setminus \{0\} \\
\frac{h_A^{p+k-1,p-k+1}(0)}{h_A^{p+k-1,p-k+1}(t)} = \frac{h_{BC}^{p+k,p-k+1}(0)}{h_{BC}^{p+k,p-k+1}(t)} = \frac{h_{\partial}^{p+k,p-k+1}(0)}{h_{\partial}^{p+k,p-k+1}(t)},
\]

\( \Rightarrow \mathcal{A}_p(X_0) = \mathcal{C}_p(X_0) \).
Remark 5.9 In the case where \( n = 3 \) and \( p = 2, k \in \{1, 2, 3\} \) and \( 2 + k \leq 3 \) we must have \( k = 1 \). So, \( A_2(X) = \{[\Omega]_A : \Omega > 0 \} \) is such that \( \partial \bar{\partial} \Omega = 0 \) is \( \mathcal{S}_X \) Gauduchon cone (1), \( C_2(X) = \{[\Omega]_A : \Omega > 0 \} \) and there exists \( \alpha^{1,3} \in C^\infty_{1,3}(X, \mathbb{C}) \) such that 
\[
d(\alpha^{1,3} + \Omega + \bar{\alpha}^{1,3}) = 0 \}
\( = \mathcal{S}_X \) sG cone (2) and the following equivalence holds:
\[
A_2(X) = C_2(X) \iff X \text{ is sGG, i.e., } \mathcal{S}_X = \mathcal{S}_X.
\]

Proposition 5.10 Let \( p, q \in \{1, \ldots, n\} \) be fixed. Suppose that the implication
\[
u \in \text{Im } \partial \implies \nu \in \text{Im } \partial \bar{\partial}
\]
holds for all \( d \)-closed forms of types \((p, q), (q, p), (p + 1, q), \) and \((q + 1, p)\). Then, there exists a canonical injective linear map
\[
H^{p,q}_A(X, \mathbb{C}) \leftrightarrow H^{p+q}_{DR}(X, \mathbb{C})
\]
\[
[\alpha]_A \mapsto [\alpha]_{DR}
\]
where \( \alpha \) is any \( d \)-closed representative of the class \([\alpha]_A \) (such an \( \alpha \) exists due to the hypothesis).

Proof The map
\[
H^{p,q}_A(X, \mathbb{C}) \to H^{p+q}_{DR}(X, \mathbb{C})
\]
\[
[\alpha]_A \mapsto [\alpha]_{DR}
\]
is well defined since if \( \partial \bar{\partial} \alpha = 0 \), then by the hypothesis \( \text{Im } \partial \cap \ker \bar{\partial} = \text{Im } \partial \bar{\partial} \) on \((p + 1, q)\)-forms, there exists a \((p, q - 1)\)-form \( v \) such that \( \partial \alpha = -\bar{\partial}v \). On the other hand, \( \bar{\alpha} \) is a \((q, p)\)-form and \( \partial \bar{\alpha} \) is a \( \partial \)-exact \((q + 1, p)\)-form. Hence, by assumption, \( \partial \bar{\alpha} \) is \( \bar{\partial} \)-exact. Then, by conjugation, \( \bar{\partial} \alpha \) still is \( \bar{\partial} \)-exact, i.e., there exists some \((p - 1, q)\)-form \( u \) such that \( \bar{\partial} \alpha = \bar{\partial} u \). This implies that \( d(\alpha + \bar{\partial} u + \bar{\partial} v) = 0 \), it means that every Aeppli cohomology class contains a \( d \)-closed representative.

Let \( \alpha \) be a \((p, q)\)-form such that \( d \alpha = 0 \) and \( \alpha = \bar{\partial} u + \bar{\partial} v \). Then \( \partial \alpha = 0 = \bar{\partial} v \). Note that \( \partial \bar{\partial} \) is a \( d \)-closed \( \partial \)-exact \((p, q)\)-form, so \( \partial \bar{\partial} \in \text{Im } \partial \bar{\partial} \). Thus \( \partial \bar{\partial} \in \text{Im } d \). Meanwhile, \( \bar{\partial} v \) is a \( d \)-closed \( \bar{\partial} \)-exact \((p, q)\)-form, so \( \bar{\partial} v \in \text{Im } \bar{\partial} \), then \( \bar{\partial} v \in \text{Im } d \). Therefore \( \alpha \in \text{Im } d \), thus the map above is well defined.

Injectivity: for all \( \alpha \in C^\infty_{p,q}(X, \mathbb{C}) \) such that \( \partial \bar{\partial} \alpha = 0 \) and \( \alpha \in \text{Im } d \), we have \( \alpha \in \text{Im } \partial + \text{Im } \bar{\partial} \). Hence the map is injective once it is well defined.

We can now infer the following

Corollary 5.11 (i) Fix \( k \in \{0, 1, \ldots, 2n\} \), suppose that the implication
\[
u \in \text{Im } \partial \implies \nu \in \text{Im } \partial \bar{\partial}
\]
holds for all $d$-closed forms of types $(p, q), (q, p), (p + 1, q),$ and $(q + 1, p)$ for all $p, q$ such that $p + q = k.$ Then there exists a canonical injection

$$\bigoplus_{p+q=k} H^{p,q}_A(X, \mathbb{C}) \hookrightarrow H^{k}_{	ext{DR}}(X, \mathbb{C}).$$

(ii) Fix $k \in \{0, 1, \ldots, 2n\}.$ Suppose that $(\ast_k)$ holds, where

$$(\ast_k): \text{the implication } u \in \text{Im } \partial \Rightarrow u \in \text{Im } \partial \partial \text{ holds for all } d\text{-closed forms of types } (p, q), (q, p), (p + 1, q), \text{ and } (q + 1, p) \text{ for all } p, q \text{ such that } p + q = k \text{ or } p + q = 2n - k.$$ (7)

Then there exists a canonical injection

$$\bigoplus_{p+q=k} H^{p,q}_A(X, \mathbb{C}) \oplus \bigoplus_{p+q=2n-k} H^{p,q}_A(X, \mathbb{C}) \hookrightarrow H^{k}_{\text{DR}}(X, \mathbb{C}) \oplus H^2_{\text{DR}}(X, \mathbb{C}).$$

Due to Angella–Tomassini [5], we have

$$2b_k \leq \sum_{p+q=k} h^{p,q}_A + \sum_{p+q=2n-k} h^{p,q}_A$$

this map is an isomorphism and $2b_k = \sum_{p+q=k} h^{p,q}_A + \sum_{p+q=2n-k} h^{p,q}_A.$

Another immediate consequence is the following degeneration at $E_1$ of the Frölicher spectral sequence.

**Corollary 5.12** If $(\ast_k)$ (cf. (7)) is satisfied for all $k \in \{0, 1, \ldots, 2n\}$, then the $\partial \partial$-lemma holds. Thus $E_1(X) = E_{\infty}(X)$ (i.e. the Frölicher spectral sequence degenerates at $E_1$).

In the end, the above results (cf. Corollary 5.11) lead to the following

**Proposition 5.13** It is clear that $(\ast_{2p})$ (cf. (7)) implies $\tilde{H}_k$ (cf. (6)) for all $k \in \{1, \ldots, p + 1\}.$ If $X_0$ satisfies $(\ast_{2p})$ and $(\ast_{2p+1}),$ then

$$\begin{cases}
\text{for all } k \in \{1, \ldots, p + 1\} \\
h^{p+k-1,p-k+1}_A(0) = h^{p+k-1,p-k+1}_A(t), \\
h^{p+k,p-k+1}_A(0) = h^{p+k,p-k+1}_A(t), \quad \text{for all } t \sim 0, \\
h^{p+k,p-k+1}_\partial(0) = h^{p+k,p-k+1}_\partial(t),
\end{cases}$$

**Proof** Suppose that $(\ast_{2p})$ holds on $X_0.$ It follows from Corollary 5.11(ii) that

$$2b_{2p} = \sum_{p+q=2p} h^{p,q}_A + \sum_{p+q=2n-2p} h^{p,q}_A.$$
Since the De Rham cohomology group does not depend on \( t \in \Delta \) and by the upper-semicontinuity of \( t \mapsto h^r_A(t) \), one obtains

\[
\sum_{r+s=2p} h^r_A(0) + \sum_{r+s=2n-2p} h^r_A(0) = \sum_{r+s=2p} h^r_A(t) + \sum_{r+s=2n-2p} h^r_A(t)
\]

for all \( t \in \Delta, t \sim 0 \). Then

\[
h^r_A(0) = h^r_A(t) \quad \text{for all } t \in \Delta, t \sim 0 \quad \text{with} \quad r+s=2p \quad \text{and} \quad r+s=2n-2p.
\]

By taking \( r = p+k-1 \) and \( s = p-k+1 \), we have \( r+s=2p \). Hence

\[
h^{p+k-1,p-k+1}_A(0) = h^{p+k-1,p-k+1}_A(t) \quad \text{for all } t \in \Delta, t \sim 0.
\]

Similarly, we can show that the condition \((**_{2p+1})\) on \( X_0 \) implies that

\[
h^r_A(0) = h^r_A(t) \quad \text{for all } t \in \Delta, t \sim 0 \quad \text{with} \quad r+s=2p+1, r+s=2n-(2p+1).
\]

By duality, we get

\[
h^r_{BC}(0) = h^r_{BC}(t) \quad \text{for all } t \in \Delta, t \sim 0 \quad \text{with} \quad r+s=2p+1, r+s=2n-(2p+1).
\]

This means that for \( r = p+k \) and \( s = p-k+1 \), we have

\[
h^{p+k,p-k+1}_{BC}(0) = h^{p+k,p-k+1}_{BC}(t) \quad \text{for all } t \in \Delta, t \sim 0.
\]

We consider the following map:

\[
R: H^{p+k,p-k+1}_{BC}(X_0, \mathbb{C}) \to H^{p+k,p-k+1}_{\partial}(X_0, \mathbb{C}) \quad \quad [\alpha]_{BC} \mapsto [\alpha]_{\partial}.
\]

It is clear that \( R \) is well defined.

Assume that for any \( d \)-closed \( (p+k, p-k+1) \)-form \( \alpha \), we have \( [\alpha]_{\partial} = 0 \in H^{p+k,p-k+1}_{\partial}(X_0, \mathbb{C}) \), i.e., \( \alpha \) is \( \partial \)-exact. Then, by the \((**_{2p+1})\) assumption, \( \alpha \) is \( \partial \bar{\partial} \)-exact. This proves that the map \( R \) is injective.

To prove that \( R \) is surjective, let \( [\alpha]_{\partial} \in H^{p+k,p-k+1}_{\partial}(X_0, \mathbb{C}) \). We need to prove the existence of a \((p+k-1, p-k+1)\)-form \( u \) such that \( \bar{\partial}(\alpha + \partial u) = 0 \). It remains to show that \( \bar{\partial} \alpha \) is \( \partial \bar{\partial} \)-exact. Notice that a \( d \)-closed form \( \partial \bar{\partial} \) is \( \partial \)-exact, then by the \((**_{2p+1})\) assumption it is \( \partial \bar{\partial} \)-exact. By conjugation, \( \partial \alpha \) is also \( \partial \bar{\partial} \)-exact, i.e., there exists a \((p+k-1, p-k+1)\)-form \( u \) such that \( \partial \alpha = \partial \bar{\partial} u \). Consequently,

\[
[\alpha]_{\partial} = [\alpha + \partial u]_{\partial} = R([\alpha + \partial u]_{\partial}).
\]
The surjectivity statement follows. As a result, we get

\[ h^p_k(0) = h^p_k(0). \]

Since the Bott–Chern numbers are constant in a neighbourhood of \( X_0 \),

\[ h^p_k(0) = h^p_k(t) \quad \text{for all} \quad t \in \Delta, \ t \sim 0. \]

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