Improvement on Brook’s theorem for 3K₁-free Graphs
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Abstract: Problem of finding an optimal upper bound for the chromatic no. of a 3K₁-free graph is still open and pretty hard. Here we prove that for a 3K₁-free graph G with Δ(G) ≥ 8, χ(G) ≤ max{Δ-1, 4}. We also prove that if G is 3K₁-free, ω = 4 and Δ(G) ≥ 7, then χ(G) ≤ Δ-1. This implies that Borodin & Kostochka Conjecture is true for 3K₁-free graphs as a corollary.

Introduction:
In [1], [2], [3], [4] chromatic bounds for graphs are considered especially in relation with ω and Δ. Gyárfás [5] and Kim [6] show that the optimal χ-binding function for the class of 3K₁-free graphs has order ω²/log(ω). If we forbid additional induced subgraphs, the order of the optimal χ-binding function drops below ω²/log(ω). In 1941, Brooks’ theorem stated that for any connected undirected graph G with maximum degree Δ, the chromatic number of G is at most Δ unless G is a complete graph or an odd cycle, in which case the chromatic number is Δ + 1 [5]. In 1977, Borodin & Kostochka conjectured that if Δ(G) ≥ 9, then χ(G) ≤ max{ω, Δ-1} [6]. In 1999, Reed proved the conjecture for Δ ≥ 10¹⁴ [7]. Also D. W. Cranston and L. Rabern [8] proved it for claw-free graphs. Here we prove that if a graph G is 3K₁-free and Δ(G) ≥ 8, then χ(G) ≤ max{Δ-1, 4}. For a 3K₁-free graph with ω = 4 we prove a stronger result that if Δ(G) ≥ 7, then χ(G) ≤ Δ-1. These results prove Borodin & Kostochka conjecture for 3K₁-free graphs as a corollary.

Notation: For a graph G, V(G), E(G), Δ, ω, χ denote the vertex set, edge set, maximum degree, size of a maximum clique, chromatic number of G resp. For u ∈ V(G), N(u) = {v ∈ V(G) / uv ∈ E(G)}, and \( \overline{N}(u) = N(u) \cup \{u\} \). If S ⊆ V, then <S> denotes the subgraph of G induced by S. If C is some coloring of G and if a vertex u of G is colored i in C, then u is called a i-vertex. Also if P is a path in G s.t. vertices on P are alternately colored say i and j, then P is called an i-j path. All graphs considered henceforth are simple. We consider here simple and undirected graphs. For terms which are not defined herein we refer to Bondy and Murty [9].

Main Result 1: Let G be 3K₁-free, if ω = 4 and Δ ≥ 7, then χ ≤ Δ-1.
Proof: Let if possible G be a smallest 3K₁-free graph with Δ ≥ 7 and χ > Δ-1. Then clearly as G ≠ C₂₀+₁ or K_{|V(G)|}, χ ≥ Δ > ω. Let u ∈ V(G). Then G-u ≠ K_{|V(G)|} (else χ = ω). If Δ(G-u) ≥ 7, then by minimality χ(G-u) ≤ max{ω(G-u), Δ(G-u)-1}. Clearly if ω(G-u) ≤ Δ(G-u)-1, then χ(G-u) = Δ(G-u)-1 ≤ Δ-1 and otherwise χ(G-u) = ω(G-u) ≤ ω < Δ. In any case χ(G-u) ≤ Δ-1. Also if Δ(G-u) < 7, then as G-u ≠ C₂₀+₁ (else as G is 3K₁-free, G-u ~ C₅), by Brook’s Theorem χ(G-u) ≤ Δ (G-u) < 7 ≤ Δ. Thus always χ(G-u) ≤ Δ-1 and in fact, χ(G-u) = Δ-1 and deg v ≥ Δ-1 ∨ v ∈ V(G).

Let Q ⊆ V(G) be s.t. <Q> is a maximum clique in G. Let u ∈ Q be s.t. deg u = max v ∈ Q deg v. Let S = \{1,..., Δ\} be a Δ-coloring of G s.t. u is colored Δ and vertices in Q are colored 1,..., |Q|-1. Every vertex v of N(u) with a unique color say i has at least one j-vertex j ≠ i (else color v by j and u by i). Also as G is 3K₁-free, <V(G)- \( \overline{N}(u) \)> is complete and hence by maximality |Q| ≥ |V(G)- \( \overline{N}(u) \)|

Case 1: deg u = Δ-1 for every maximal clique <Q> in G and a vertex u of maximum degree in Q.
Then deg v ≥ Δ-1 ∀ v ∈ Q. As G is 3K₁-free and deg u ≥ 6, |Q| ≥ 4. Let A, C, D ∈ Q be colored 1, 2, 3 resp. Clearly every j-vertex v of Q has a unique i-vertex in N(v) where i ≠ j (else N(v) has color i
missing in \( \overline{N(v)} \). Color \( v \) by \( i, u \) by \( j \). As \( \Delta > \omega \), \( |N(u)\cdot Q| \geq 1 \). W.l.g. let \( B \in N(u)\cdot Q \) be colored \( \omega \) and \( A \in Q \) be s.t. \( AB \not\in E(G) \). Let \( R \) be a component containing \( A \) s.t. vertices in \( R \) are colored either 1 or \( \omega \). Then \( B \in R \) (else alter colors in \( R \) and color \( u \) by 1). Let \( P = \{A, E, F, B\} \) be a 1-\( \omega \) path in \( R \). \( F \) is the only 1-vertex of \( B \) and \( FV \not\in E(G) \) for any \( V \in Q \). Then \( F \) has a \( r \)-vertex adjacent in \( V(G)\cdot \overline{N(u)} \) \( \forall r, 2 \leq r \leq |Q|-1 \) (else color \( F \) by \( r \), \( B \) by 1 and \( u \) by \( \omega \)) . By I, \( E \) is adjacent to all these \( |Q| \cdot 2 \) vertices and \( |V(G)\cdot \overline{N(u)}| \geq |Q| \). Hence by I, \( |V(G)\cdot \overline{N(u)}| \geq |Q| \) or \( B \) is non-adjacent to all vertices of \( Q \Rightarrow E \) is adjacent to all vertices of \( Q \) and by assumption \( \Delta - 1 = \deg E \geq 2(|Q| - 1) = 2(\Delta - 1) \Rightarrow \Delta = 1 \), a contradiction.

**Case 2:** Let \( \deg u = \Delta \).

Let \( A, B, C \in Q \) be colored 1, 2, 3.

First let \( N(u)\cdot Q \) have a repeat color and \( D, E \in N(u)\cdot Q \) be colored 4 and 5, 6 resp. W.l.g. let \( DB \not\in E(G) \Rightarrow BE \in E(G) \) and w.l.g. let \( EA \not\in E(G) \Rightarrow AD \in E(G) \). Now \( D \) has a 2-vertex in \( V(G)\cdot \overline{N(u)} \) (else color \( D \) by 2, A by 4, \( u \) by 1). Similarly \( E \) has a 1-vertex in \( V(G)\cdot \overline{N(u)} \).

Also as \( <Q> \) is a maximum clique in \( G \), \( F \in G \) is non-adjacent to some vertex in \( Q \) and \( V(G)\cdot \overline{N(u)} \) has a 5-vertex and a 6-vertex. Then by I, \( |V(G)\cdot \overline{N(u)}| = |Q| \) and \( V(G)\cdot \overline{N(u)} \) has no 3-vertex. Clearly \( N(C) = N(u) \) (else \( G \) is \( (\Delta - 1) \)-colorable). But then as \( G \) is \( 3K_1 \)-free, \( \omega(A,B,D,E,F,G) \geq 3 \) and \( \omega(G) \geq 5 \), a contradiction.

Next w.l.g. let \( D, E, F, G \in N(u)\cdot Q \) be colored 1, 4, 5, 6 resp. Now \( \omega(E,F,G) \geq 2 \). W.l.g. let \( EF \in E(G) \). Again each of \( E, F, G \) is non-adjacent to \( B \) or \( C \) (else if say \( EB, EC \in E(G) \), then by replacing \( Q \) with \( Q-A+E \) we get the earlier case). Thus \( |V(G)\cdot \overline{N(u)}| \geq 5 \), a contradiction.

This proves the result.

**Main Result 2:** If \( G \) is \( 3K_1 \)-free and \( \Delta \geq 8 \), then \( \chi \leq \max \{\omega, \Delta - 1\} \).

Proof: As before let \( G \) be a smallest \( 3K_1 \)-free graph with \( \Delta \geq 8 \) and \( \chi > \max \{\omega, \Delta - 1\} \). As before we have \( \chi(G-u) = \Delta - 1 \) and \( \deg u \geq \Delta - 1 \ \forall \ v \in V(G) \). Let \( |Q| \geq 5 \).

Let \( Q \subseteq V(G) \) be s.t. \( <Q> \) is a maximum clique in \( G \). Let \( u \in Q \) be s.t. \( \deg u = \max_{v \in Q} \deg v \). Let \( \{1, 2, \ldots, \Delta\} \) be a \( \Delta \)-coloring of \( G \) s.t. \( u \) is colored \( \Delta \) and vertices in \( Q \) are colored 1, 2, \ldots, \( |Q| - 1 \). Every vertex \( v \) of \( N(u) \) with a unique color say \( i \) has at least one \( j \)-vertex \( j \neq i \) (else color \( v \) by \( j \) and \( u \) by \( i \)). Also as \( G \) is \( 3K_1 \)-free, \( |V(G)\cdot \overline{N(u)}| \) is complete and hence by maximality \( |Q| \geq |V(G)\cdot \overline{N(u)}| \)

**Case 1:** \( \deg u = \Delta - 1 \) for every maximal clique \( <Q> \) in \( G \) and a vertex \( u \) of maximum degree in \( Q \).

Same proof as in the **Main Result 1** holds good.

**Case 2:** \( \deg u = \Delta \geq 8 \)

**Case 2.1:** \( N(u)\cdot Q \) has a repeat color

Let \( A, B \in N(u)\cdot Q \) be colored 5. W.l.g. let \( \exists D \) colored 2 in \( Q \) s.t. \( AD \not\in E(G) \). Then \( DB \in E(G) \).

Also let \( \exists C \) colored 1 in \( Q \) s.t. \( BC \not\in E(G) \). Then \( CA \in E(G) \).

Let \( E, F \in Q \) be colored 3, 4 resp.

**Case 2.1.1:** \( \exists \) a vertex \( X \in Q \) s.t. \( XA, XB \in E(G) \). W.l.g. let \( X = E \).
**Case 2.1.1:** \( \exists \) a vertex \( Y \in Q \cdot C \) s.t. \( YA \in E(G) \) and \( YB \not\in E(G) \). W.l.g. let \( Y = F \). As \( D \) has at the most one repeat color in \( \overline{N(D)} \), w.l.g. let \( C \) be the only 1-vertex of \( D \). Now let \( G \) be the 2-vertex of \( A \) in \( V(G) \cdot \overline{N(u)} \) (else color \( A \) by 2, \( C \) by 5, \( u \) by 1) and similarly \( H \) be the 1-vertex, 4-vertex of \( B \) in \( V(G) \cdot \overline{N(u)} \) resply. Now \( AH \in E(G) \) (else color \( A \) by 1, \( C \) by 5, \( D \) by 1, \( u \) by 2) \( \Rightarrow \) \( F \) is the only 4-vertex of \( A \) \( \Rightarrow DJ \in E(G) \) (else color \( A \) by 4, \( F \) by 5, \( D \) by 4, \( u \) by 2). Thus \( E \) is the only 3-vertex of \( A \) and \( D \). Also as \( E \) has two 5-vetices \( D \) is its only 2-vertex. Color \( E \) by 2, \( D \) by 3, \( A \) by 3, \( C \) by 5, \( u \) by 1, a contradiction.

**Case 2.1.2:** \( \forall V \in Q \cdot \{C, D\}, VA, VB \in E(G) \).

As before let \( G \) be the 2-vertex of \( A \) and \( H \) be the 1-vertex of \( B \). As \( E, F \) have two 5-vetices each, \( C, D \) are the only 1, 2 vertices of \( E \) and \( F \). Also w.l.g. let \( E \) be the only 3-vertex of \( A \) \( \Rightarrow DJ \in E(G) \) where \( J \) is the 3-vertex in \( V(G) \cdot \overline{N(u)} \) (else color \( E \) by 1, \( C \) by 5, \( A \) by 3, \( D \) by 3, \( u \) by 2). Again \( AK \in E(G) \) where \( K \) is the 4-vertex in \( V(G) \cdot \overline{N(u)} \) (else color \( F \) by 1, \( C \) by 5, \( A \) by 4, \( D \) by 4, \( u \) by 2). Thus \( C \) (\( A \)) is the only 1 (5) vertex of \( A \) (\( C \)) and \( C \) is the only 1-vertex of \( D \). Color \( C \) by 5, \( A \) by 1, \( D \) by 1, \( u \) by 2, a contradiction.

**Case 2.1.2.a:** Every \( V \in Q \) is adjacent to only one of \( A \) or \( B \).

**Case 2.1.2.1:** \( \exists \) two vertices in \( Q \cdot C \) say \( E, F \) adjacent to say \( A \).

W.l.g. let \( C, E \) be the unique 1, 3 vertices of \( A \) and \( E \) be the unique 3-vertex of \( D \). Color \( E \) by 5, \( A \) by 3, \( D \) by 3, \( u \) by 2, a contradiction.

**Case 2.1.2.2:** No two vertices in \( Q \cdot C \) (\( Q \cdot D \)) are adjacent to \( A (B) \).

Then \( |Q| = 5 \) (else we get Case 2.1.2.1). W.l.g. let \( EA, FB \in E(G) \) and \( C, D \) be the only 1, 2 vertex of \( A, B \) resply \( \Rightarrow C (D) \) has another 2 (1) vertex in \( V(G) \cdot \overline{N(u)} \) and \( A (B) \) has another 3 (4) vertex in \( V(G) \cdot \overline{N(u)} \) and \( |V(G) \cdot \overline{N(u)}| \geq 4 \). Also as \( \Delta \geq 8 \), and \( |Q| = 5 \), \( |N(u) \cdot Q \cdot \{A, B\}| \geq 2 \). Clearly \( |V(G) \cdot \overline{N(u)}| \geq 6 \), contrary to \( I \).

**Case 2.2:** \( A \in Q \) and \( B \in N(u) \cdot Q \) have a common color 1.

Now every vertex say \( V \in R = N(u) \cdot Q \cdot B \) is non-adjacent to some vertex of \( Q \cdot u \cdot A \) (replace \( Q \) by \( Q \cdot A + V \) to get Case 2.1). Let \( C, D, E \in Q \) have colors 2, 3, 4 resply.

**Claim:** \( |Q| \geq 6 \)

If \( |Q| = 5 \), then as \( \Delta \geq 8 \), \( |R| \geq 4 \) and \( w < R \geq 2 \). W.l.g. let \( FG \in E(G) \) and \( FC, GD \not\in E(G) \). Let \( F, G, H \in R \) be colored 5, 6, 7 resply. \( \Rightarrow V(G) \cdot \overline{N(u)} \) has a 2-vertex, 3-vertex, 5-vertex, 6-vertex and a 7-vertex. Hence by \( I \) \( E \) is the unique 4-vertex in \( V(G) \). As \( E \) has at the most one repeat color in \( N(E) \), w.l.g. let \( F \) be the only 5-vertex of \( E \). Color \( F \) by 4, \( E \) by 5, \( C \) by 4, \( u \) by 2, a contradiction.

This proves the Claim.

Let \( L \in Q \) be colored 6 and \( F \in R \) be colored 5.

**Case 2.2.1:** \( \exists \) a vertex in \( R \) non-adjacent to two vertices in \( Q \cdot A \).

W.l.g. let \( FC, FD \not\in E(G) \). Then \( \exists \) a 2-5 path \( P = \{C, G, H, F\} \). Similarly let \( R = \{D, G, J, F\} \) 3-5 path. As \( G \) has two 2-vertices and 3-vertices, \( G \) has a unique 4-vertex (else some color \( r \) is missing in \( \overline{N(G)} \), color \( G \) by \( r \), \( C \) by 5, \( u \) by 2). Then \( GE \not\in E(G) \) (else color \( E \) by 2, \( C \) by 5, \( G \) by 4, \( u \) by 4), \( F \) is the only 5-vertex of \( E \) and let \( K \in V(G) \cdot \overline{N(u)} \) be the unique 4-vertex of \( G \).
If $|Q| \geq 7$, then let $M$ be the 7-vertex in $Q$. Now as before $G$ has a unique 6-vertex, 7-vertex and $LG$, $MG \notin E(G)$. Also as $F$ has at most one repeat color in $\overline{N(F)}$ w.l.g. let $E, L$ be the only 4-vertex, 6-vertex of $F$. Again $C$ has either a unique 4-vertex or 6-vertex. W.l.g. $E$ be the only 4-vertex of $C$. Color $E$ by 5, $F$ by 4, $C$ by 4, $u$ by 2, a contradiction.

Hence let $|Q| = 6$. W.l.g. let $E$ be the only 4-vertex of $F$. If $E$ is the only 4-vertex of $C$ ($D$), then color $E$ by 5, $F$ by 4, $C$ ($D$) by 4, $u$ by 2 (3), a contradiction. Hence $CK, DK \in E(G)$ where $K$ is the 4-vertex in $V(G) - \overline{N(u)}$. Then $L$ is the only 6-vertex of $C, D$. If $L$ is the only 6-vertex of $F$, then color $L$ by 5, $F$ by 6, $C$ by 6, $u$ by 2, a contradiction. Hence $F$ has another 6-vertex $M$ in $V(G) - \overline{N(u)}$. As $\Delta \geq 8$, and $|Q| = 6$, $\exists N, P \in N(u) - Q$ colored 7, 8 resp. By $I$, either $N$ or $P$ is the unique 7-vertex or 8-vertex in $V(G)$. W.l.g. let $N$ be the unique 7-vertex in $V(G)$. As $N$ has a unique 2-vertex or 5-vertex, w.l.g. let $C$ be its unique 2-vertex. Color $N$ by 2, $C$ by 7, $F$ by 7, $u$ by 5, a contradiction.

Case 2.2.2: Every vertex of $R$ is non-adjacent to exactly one vertex in $Q$-$A$.

Let $FC \notin E(G)$. As $F$ has at most one color repeated in $\overline{N(F)}$ w.l.g. let $D, E$ be the unique 3-vertex and 4-vertex of $F$. Again w.l.g. let $D$ be the unique 3-vertex of $C$. Now $D$ has either a unique 2-vertex or 5-vertex. W.l.g. let $C$ be the unique 2-vertex of $D$. Color $D$ by 2, $C$ by 3, $F$ by 3, $u$ by 5, a contradiction.

This proves the theorem.

Corollary: Borodin and Kostochka conjecture is true for $3K_1$-free graphs.

References

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