SYMMETRY IN $n$-BODY PROBLEM VIA GROUP REPRESENTATIONS

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Abstract. We introduce an algebraic method to study local stability in the Newtonian $n$-body problem when certain symmetries are present. We use representation theory of groups to simplify the calculations of certain eigenvalue problems. The method should be applicable in many cases, we give two main examples here: the square central configurations with four equal masses, and the equilateral triangular configurations with three equal masses plus an additional mass of arbitrary size at the center. We explicitly found the eigenvalues of certain $8 \times 8$ Hessians in these examples, with only some simple calculations of traces. We also studied the local stability properties of corresponding relative equilibria in the four-body problems.

Keywords: the Newtonian $n$-body problem, symmetry, representation theory, eigenvalues

1. Introduction

In this article, we consider the Newtonian $n$-body problem with certain symmetries. We study the local stability problems of certain relative equilibria corresponding to symmetric central configurations. The idea is very classical: the representation theory of finite groups can be a very useful and effective tool to simplify the calculation of eigenvalue problems associated with central configurations and relative equilibria. This old technique provide us with some new perspectives in celestial mechanics and related fields, and help us to get more information that is difficult to obtain otherwise. There are many interesting scenerios where this technique would apply, we start with two very specific examples.

A relative equilibrium in the $n$-body problem is where the $n$ point masses keep a fixed relative position during their motions. Only a special set of configurations of the $n$ bodies can maintain such relative equilibria, these are called central configurations. Central configurations are well-studied objects (cf. [2, 3, 8, 10, 11, 12, 15]), dating back to Euler and Lagrange, a simple question whether the number of central configurations is finite, for any given set of $n$ positive masses, remain a major open problem in mathematics (cf. Smale [12], Albouy and Kaloshin[1], Hampton & Moeckel [4]).

In this paper, we consider two examples, with different symmetries, in the planar four-body problem. The first example is a simple central configuration where we place four equal masses on corners of a square. The second example is where three masses are placed at vertices of an equilateral triangle, with the fourth particle, of arbitrary mass, placed at the center. The associated eigenvalue problems are for some $8 \times 8$ matrices, they are not directly solvable analytically. Moreover, the second example involves an arbitrary parameter, the mass at the center, which complicates any analytical or numerical solutions. In both cases, we are able to give a simple analytical solutions, by using represtation theory of associate symmetry groups (dihedral group $D_4$ and permutation group $S_3$).
Interestingly, Palmore\cite{9,10} showed that a degeneracy arose in the 4-body central configurations. The existence of degeneracies in relative equilibria answers several questions raised by Smale. In this work, by giving an analytical formula, we find exactly where and how Palmore’s degeneracy occurs.

We obtained the complete local stability analysis for corresponding relative equilibria for above two examples.

2. PRELIMINARIES

In this section, we introduce some basic concepts of the central configurations as well as main ideas of the representation theory of finite groups.

2.1. Planar Central Configurations. Let $q_i \in \mathbb{R}^2$ be the position of the particle $m_i, i = 1, 2, \ldots, n$. If we have a constant $\lambda$ such that

$$\lambda m_i q_i = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|q_i - q_j|^3} (q_j - q_i)$$

for all $1 \leq i \leq n$, then the $n$ particles are said to form a (planar) central configuration. A planar central configuration remains a central configuration after a rotation in $\mathbb{R}^2$ and a scalar multiplication (cf. Xia \cite{15}). Rotation and scaling generate an equivalent classes of central configurations.

It turns out that a central configuration is a critical point of the function $\sqrt{IU}$, where

$$I = \frac{1}{2} \sum_{i=1}^{n} m_i |q_i|^2,$$

$$U = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|q_i - q_j|}.$$  

The function $I$ is the moment of inertia of the $n$-body system, while $U$ is the potential function. Equivalently, central configurations can be described as critical points of the function $U$ on the ellipsoid \cite{15}

$$S = \{q = (q_1, \ldots, q_n) \in \mathbb{R}^{2n} | I = 1\}.$$  

2.2. Hamiltonian of the $n$ Body Problem. Let $p_i = m_i \dot{q}_i$, then the $n$-body problem has a Hamiltonian formulation with the Hamiltonian\cite{7}

$$H = K - U,$$

where

$$K = \sum_{i=1}^{n} \frac{|p_i|^2}{2m_i}.$$  

And the equations of motion are

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m_i},$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = \frac{\partial U}{\partial q_i}.$$  

There are some special solutions where the $n$ particles move on concentric circles with uniform angular velocity\cite{6}. Let the center of circle be at the origin, then the solutions have the form

$$q_i^* = e^{\omega Jt} a_i,$$

$$p_i^* = -m_i \omega J e^{\omega Jt} a_i,$$
where \(a_i\) is constant vector,

\[
J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

and

\[
\exp(-\omega J t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix},
\]

and \(\omega\) is a positive constant that satisfies

\[
(1) \quad \omega^2 a_i + \sum_{j=1}^{n} \frac{m_j(a_j - a_i)}{|a_j - a_i|^3} = 0.
\]

Solutions \(a_1, \ldots, a_n\) to the above equation are positions for the \(n\) particles respectively, they are precisely the central configuration. Under a rotating coordinate system with constant frequency \(\omega\), these solutions will be fixed points, and therefore called \textit{relative equilibria}. Let \(q_i = \exp(-\omega J t)x_i\) and \(p_i = \exp(-\omega J t)y_i\). The Hamiltonian, under uniform rotating coordinates, becomes

\[
H = \sum_{i=1}^{n} \left( \frac{|y_i|^2}{2m_i} - \omega x_i^T J y_i \right) - \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|x_i - x_j|}.
\]

The equations of motion are

\[
(2) \quad \dot{x}_i = \omega J x_i + \frac{y_i}{m_i},
\]

\[
\dot{y}_i = \omega J y_i + \sum_{j=1}^{n} \frac{m_j(x_j - x_i)}{|x_j - x_i|^3}.
\]

The equations can be written as a second-order equation by eliminating \(y_i\):

\[
(3) \quad \ddot{x}_i = 2\omega J \dot{x}_i + \omega^2 x_i + \frac{1}{m_i} \sum_{j=1}^{n} \frac{m_j(x_j - x_i)}{|x_j - x_i|^3}.
\]

2.3. Groups and Symmetry. Let \(G\) be a group. The following are useful examples in our applications:

- \(S_i, i = 2, 3, 4, \ldots\), the symmetric, or permutation, group of degree \(i\).
- \(D_i, i = 2, 3, 4, \ldots\), dihedral finite groups of degree \(i\). (The symmetry groups of the planar regular \(i\)-gon.)

2.4. The Representation of Finite Groups. We will review some basic notations, concepts and results in the group representation theory. Let \(GL(n, \mathbb{C})\) be the group of \(n \times n\) (complex) non-singular matrices. A representation of \(G\) of degree \(n\) is a homomorphism \(\mathcal{D}\) from \(G\) to \(GL(n, \mathbb{C})\).

For any two representations \(\mathcal{D}_1\) and \(\mathcal{D}_2\) of \(G\), respectively of degree \(n_1\) and \(n_2\), the direct sum \(\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2\) can be defined as another representation of degree \(n_1 + n_2\). For each \(A \in G\), \(\mathcal{D}(A)\) is the \(n_1 + n_2\) invertible matrix

\[
\mathcal{D}(A) = \begin{pmatrix} \mathcal{D}_1(A) & 0 \\ 0 & \mathcal{D}_2(A) \end{pmatrix}.
\]

Similarly, for representations \(\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_k\) of \(G\) respectively of degree \(n_1, n_2, \ldots, n_k\), we can define a representation \(\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_k\) of degree \(n_1 + n_2 + \cdots + n_k\).
We say two representations \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) of degree \( n \) of \( G \) are equivalent if there is an invertible \( n \times n \) matrix \( P \) such that for any \( A \in G \), \( \mathcal{D}_2(A) = P \mathcal{D}_1(A)P^{-1} \). A representation is reducible if it is equivalent to a direct sum of other representations. Otherwise, we say that this representation is irreducible.

Let \( \mathcal{D} \) be a representation of a group \( G \) of degree \( n \). A character \( \chi \) is a complex-valued function defined on group \( G \). For each \( A \in G \), let \( \chi(A) = \text{Tr}(\mathcal{D}(A)) \). Equivalent representations have the same character. Most remarkably, representations which have the same character are necessarily equivalent\[13\].

The (Hermitian) inner product of characters can be defined as follows
\[
(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{A \in G} \overline{\chi_1(A)} \chi_2(A),
\]
where \( G \) is a finite group and \(|G|\) is the number of elements in \( G \). With this inner product, for \( \chi_1, \chi_2, \ldots, \chi_k \) which are characters of distinct (pairwise non-equivalent) irreducible representations, we have\[13\]
\[
(\chi_i, \chi_j) = \delta_{ij}, \ i, j = 1, 2, \ldots, k.
\]

Suppose the finite group \( G \) has \( h \) number of conjugacy classes. Then the characters of representations of \( G \) as vectors span \( \mathbb{C}^h \). The following theorem\[13\] is a classic result in the representation theory of finite groups.

**Theorem 1.** For any finite group \( G \) with \( h \) conjugacy classes, there are \( h \) non-equivalent irreducible representations. Let \( \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_h \) be irreducible representations of group \( G \). Let \( \chi_1, \ldots, \chi_h \) be the characters of these representations respectively. Then any representation \( \mathcal{D} \) of \( G \) with character \( \chi \) is equivalent to
\[
n_1 \mathcal{D}_1 \oplus n_2 \mathcal{D}_2 \oplus \cdots \oplus n_h \mathcal{D}_h
\]
where
\[
n_i = (\chi, \chi_i) \in \mathbb{Z}, \ i = 1, \ldots, h.
\]

The above theorem provides an easy way to decompose any representation \( G \) into a direct sum of irreducible group representations.

2.5. **The Dihedral Group \( D_4 \).** The dihedral group \( D_4 = \langle a, r \mid a^4 = r^2 = (ra)^2 = e \rangle \) has 8 elements. It is the symmetry group for a square. We put 4 identical particles at vertices and number them with 1, 2, 3, 4. Each element in \( D_4 \) acts as a permutation of the 4 particles \( \{1, 2, 3, 4\} \).

The number of conjugacy classes for the group \( D_4 \) is 5 which are \( \{e\}, \{a, a^3\}, \{a^2\}, \{r, a^2r\}, \{ar, a^3r\} \).

By Theorem \[1\] the group \( D_4 \) also has 5 irreducible representations. The characters for the irreducible group representation with \( D_4 \) are listed in Table \[2\]. The degrees of \( \chi_1, \chi_2, \chi_3, \chi_4, \chi_5 \) are 1, 1, 1, 1, 2 respectively.

2.6. **The Symmetric Group \( S_3 \).** The group \( S_3 \) has 6 elements and is the symmetry group of equilateral triangles. The elements of \( S_3 \) can be written as matrices in the following
\[
I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]
\[
R^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad TR = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad TR^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]
| $D_4$ | Action |
|-------|--------|
| $e$   | Identity |
| $a$   | Rotating by $\frac{\pi}{2}$ |
| $a^2$ | Rotating by $\pi$ |
| $a^3$ | Rotating by $\frac{3\pi}{2}$ |
| $r$   | Reflection |
| $ar$  | Reflection then Rotating by $\frac{\pi}{2}$ |
| $a^2r$| Reflection then rotating by $\pi$ |
| $a^3r$| Reflection then rotating by $\frac{3\pi}{2}$ |

Table 1. The actions in $D_4$

| $A/\chi$ | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ |
|----------|----------|----------|----------|----------|----------|
| $e$ | 1 | 1 | 1 | 1 | 2 |
| $a, a^3$ | 1 | 1 | -1 | -1 | 0 |
| $a^2$ | 1 | 1 | 1 | 1 | -2 |
| $r, a^2r$ | 1 | -1 | 1 | -1 | 0 |
| $ar, a^3r$ | 1 | -1 | -1 | 1 | 0 |

Table 2. The irreducible character table for $D_4$

This is a group representation for $S_3$ with degree 3. For $S_3$ the conjugacy class are $\{I\}, \{R, R^2\}, \{T, TR, TR^2\}$. So there are three irreducible representations. First, for any element $A$ in $S_3$, $D_1(A) = 1$. This is the trivial representation with degree 1. Second, for any $A \in S_3$, let $D_2(A) = \text{det}(A)$. It is also a representations of degree 1. Third, there is a representation of degree 2. It is

\[
D_3(I) = \bar{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_3(T) = \bar{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
D_3(R) = \bar{R} = R\left(\frac{2\pi}{3}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},
\]

\[
D_3(R^2) = \bar{R}^2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},
\]

\[
D_3(TR) = \bar{T} \bar{R} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.
\]

It can be proved that $D_1, D_2$ and $D_3$ enumerates all the irreducible representations of $S_3$. The character table for $S_3$ with $D_1, D_2$ and $D_3$ is presented as follows

3. APPLICATION

In this section, we apply the representation theory of finite groups to study central configurations with symmetries in the Newtonian $n$-body problem.
3.1. The Technique from the Representation Theory. Suppose $G$ is a finite group. Its element $A$ can act on some vector space as a linear operator. Assume there is another linear operator $H$ defined on the same vector space. We say that $H$ is invariant under the group $G$ if $HA = AH$ for each $A$ in $G$. Consider the eigenspace $V_\lambda$ for $H$ with $\phi \in V_\lambda$, we have

$$H\phi = \lambda \phi \Rightarrow AH\phi = \lambda A\phi \Rightarrow H(A\phi) = \lambda A\phi.$$

Then $A\phi$ is also an eigenvector for $H$ with the same eigenvalue. Denote a basis for the eigenspace $V_\lambda$ by $\phi_1, \phi_2, \ldots, \phi_k$, it follows

$$A\phi_j = \sum_{i=1}^{k} \mathcal{D}_{ij}(A)\phi_i, \quad j = 1, 2, \ldots, k.$$

Let $\mathcal{D}(A)$ be the matrix with entries $\mathcal{D}_{ij}(A)$. And if $A, B \in G$,

$$BA\phi_m = \sum_{j=1}^{k} \mathcal{D}_{jm}(A)B\phi_j$$

$$= \sum_{j=1}^{k} \mathcal{D}_{jm}(A) \sum_{i=1}^{k} \mathcal{D}_{ij}(B)\phi_i$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{k} \mathcal{D}_{ij}(B)\mathcal{D}_{jm}(A)\phi_i, \quad m = 1, 2, \ldots, k$$

$$\Rightarrow BA\phi_m = \sum_{i=1}^{k} \sum_{j=1}^{k} \mathcal{D}_{ij}(B)\mathcal{D}_{jm}(A)\phi_i$$

$$\Rightarrow \mathcal{D}(BA) = \mathcal{D}(B)\mathcal{D}(A).$$

Therefore $\mathcal{D}$ is a group representation of $G$.

Suppose the $n \times n$ matrix $H$ is the Hessian for a smooth function of $n$ variables. It is a symmetric matrix and has $n$ independent eigenvectors which form a basis of $\mathbb{R}^n$. In this case, there is an invertible matrix $P$ (which acts as the change of coordinates) such that $PAP^{-1} = \mathcal{D}(A)$ for all $A \in G$ and $PHP^{-1} = H'$, where

$$H' = \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_n
\end{pmatrix}.$$
The transformation $A$ does not intermix eigenvectors. We group the eigenvectors with the same eigenvalue together. In the new basis the transformation will be represented by a matrix of the form

$$
D(A) = \begin{pmatrix}
\mathcal{D}_1(A) & 0 & \ldots & 0 \\
0 & \mathcal{D}_2(A) & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \mathcal{D}_k(A)
\end{pmatrix}
$$

with $\deg(\mathcal{D}_1(A)) + \deg(\mathcal{D}_2(A)) + \cdots + \deg(\mathcal{D}_k(A)) = n$. Then $H'D(A)$ is

$$
\begin{pmatrix}
\lambda_1 \mathcal{D}_1(A) & 0 & \ldots & 0 \\
0 & \lambda_2 \mathcal{D}_2(A) & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_k \mathcal{D}_k(A)
\end{pmatrix}.
$$

Moreover $\mathcal{D}$ is equivalent to $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_k$. For each $i = 1, 2, \ldots, k$, it is a general rule that the constituents $\mathcal{D}_i$ associated with the eigenspaces are already irreducible\[5\] in the $n$-body problem. Since $G$ is a finite group, we can find the characters of its distinct irreducible representations. It allows us to decompose the group representation into irreducible components. Then $\mathcal{D}$ is equivalent to the irreducible decomposition of the group representation. And $\mathcal{D}_i$ is equivalent to some irreducible group representation. We can get the trace of $\mathcal{D}_i(A)$, $i = 1, 2, \ldots, k$. Since $Tr(HA) = Tr(PHAp^{-1}) = Tr(H'D(A))$, it can deduce some equations with eigenvalues. By solving these equations we may get information about eigenvalues.

3.2. The Square Configuration. For the 4-body problem in $\mathbb{R}^2$ which four equal masses, the central configuration are critical points of the function $\sqrt{IU}$, where

$$I = \sum_{i=1}^{4} \frac{1}{2}(x_i^2 + y_i^2),$$

$$U = \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}.$$

Then squares centered at the origin are a central configuration as shown in Figure[1].

Let $z \in \mathbb{R}^8$ be the vector $z = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$. Then a configuration for the square is $z_0 = (2, 0, 0, 2, -2, 0, 0, -2)$.
Figure 1. The Square Configuration

Clearly $z_0$ is a critical point for $\sqrt{TU}$, and the Hessian $\frac{\partial^2 (\sqrt{TU})}{\partial z^2}$ at $z_0$ is

$$
H_1 = \begin{pmatrix}
\frac{3\sqrt{2}}{16} + \frac{5}{32} & 0 & -\frac{\sqrt{2}}{8} & \frac{3\sqrt{2}}{16} - \frac{1}{32} & 0 & -\frac{\sqrt{2}}{16} & \frac{3}{32} \\
0 & \frac{3\sqrt{2}}{8} + \frac{1}{16} & \frac{3\sqrt{2}}{16} - \frac{\sqrt{2}}{16} & 0 & \frac{1}{16} & -\frac{3\sqrt{2}}{16} & -\frac{\sqrt{2}}{16} \\
-\frac{\sqrt{2}}{8} & \frac{3\sqrt{2}}{16} + \frac{5}{32} & \frac{3\sqrt{2}}{8} & 0 & -\frac{\sqrt{2}}{16} & -\frac{3\sqrt{2}}{16} & 0 \\
\frac{3\sqrt{2}}{16} - \frac{1}{32} & 0 & -\frac{\sqrt{2}}{16} & \frac{3\sqrt{2}}{8} + \frac{1}{16} & \frac{3\sqrt{2}}{16} - \frac{\sqrt{2}}{16} & 0 & -\frac{3\sqrt{2}}{16} \\
0 & \frac{1}{16} & -\frac{\sqrt{2}}{16} & -\frac{\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{8} + \frac{1}{16} & \frac{3\sqrt{2}}{16} - \frac{\sqrt{2}}{16} \\
-\frac{\sqrt{2}}{16} & -\frac{3\sqrt{2}}{16} & \frac{1}{16} & 0 & -\frac{\sqrt{2}}{16} & -\frac{3\sqrt{2}}{16} & 0 \\
\frac{3}{32} & -\frac{\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} - \frac{1}{32} & 0 & -\frac{3\sqrt{2}}{16} & 0 \\
-\frac{\sqrt{2}}{16} & 0 & \frac{3\sqrt{2}}{16} - \frac{1}{32} & 0 & -\frac{3\sqrt{2}}{16} & 0 & 0
\end{pmatrix}
$$

We now use the dihedral group $D_4$ to find the eigenvalues of $H_1$. As shown in Figure 1, the square has four equal mass particles at vertices. Every element in $D_4$ act in the square by permuting the particles.

For example, take $a \in D_4$ that rotates squares though $\frac{\pi}{2}$, we have

$$
\begin{pmatrix}
x'_1 \\
y'_1
\end{pmatrix} = R\left(\frac{\pi}{2}\right) \begin{pmatrix}x_1 \\
y_1\end{pmatrix},
\begin{pmatrix}x'_2 \\
y'_2\end{pmatrix} = R\left(\frac{\pi}{2}\right) \begin{pmatrix}x_2 \\
y_2\end{pmatrix},
\begin{pmatrix}x'_3 \\
y'_3\end{pmatrix} = R\left(\frac{\pi}{2}\right) \begin{pmatrix}x_3 \\
y_3\end{pmatrix},
\begin{pmatrix}x'_4 \\
y'_4\end{pmatrix} = R\left(\frac{\pi}{2}\right) \begin{pmatrix}x_4 \\
y_4\end{pmatrix}.
\end{array}
$$

After rotating by $\frac{\pi}{2}$, the central configuration shown in Figure 1 becomes as Figure 2.

With the vector $z = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$, we have

$$
z' = \begin{pmatrix}
0 & 0 & 0 & R(\frac{\pi}{2}) \\
R(\frac{\pi}{2}) & 0 & 0 & 0 \\
0 & R(\frac{\pi}{2}) & 0 & 0 \\
0 & 0 & R(\frac{\pi}{2}) & 0
\end{pmatrix} z,
$$

where $z'$ is the vector after the transformation. Hence, action $a$ can be represented by the $8 \times 8$ matrix.
Similarly $a^2, r, ar \in D_4$ are respectively rotating $\pi$, reflection and rotating $\pi/2$ after reflection. They have representations as follows:

\[
D(a^2) = \begin{pmatrix}
0 & 0 & 0 & R(\pi/2) \\
0 & 0 & 0 & 0 \\
0 & R(\pi/2) & 0 & 0 \\
0 & 0 & R(\pi/2) & 0
\end{pmatrix}, \quad \text{where} \quad R(\pi/2) = \begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix},
\]

\[
D(r) = \begin{pmatrix}
0 & 0 & F \\
0 & F & 0 \\
F & 0 & 0 \\
0 & 0 & F
\end{pmatrix}, \quad \text{where} \quad F = \begin{pmatrix} -1 & 0 \\
0 & 1 \end{pmatrix},
\]

\[
D(ar) = \begin{pmatrix}
0 & 0 & 0 & G \\
0 & 0 & G & 0 \\
0 & G & 0 & 0 \\
G & 0 & 0 & 0
\end{pmatrix}, \quad \text{where} \quad G = R(\pi/2)F = \begin{pmatrix} 0 & -1 \\
-1 & 0 \end{pmatrix}.
\]

Since the character of $D$ is constant on conjugacy classes, it is easily calculated.

\[
\chi(e) = 8, \quad \chi(a) = 0,
\]

\[
\chi(a^2) = 0, \quad \chi(r) = 0, \quad \chi(ar) = 0.
\]

According to Theorem 1 we have

\[
n_1 = (\chi, \chi_1) = \frac{1}{8} \times 8 = 1, \quad n_2 = (\chi, \chi_2) = 1,
\]

\[
n_3 = (\chi, \chi_3) = 1, \quad n_4 = (\chi, \chi_4) = 1, \quad n_5 = (\chi, \chi_5) = 2.
\]
Hence \( \chi = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5 \), and \( D \) is equivalent to \( D_1 \oplus D_2 \oplus D_3 \oplus D_4 \oplus 2D_5 \).

The Hessian \( H_1 \) is invariant in group \( D_4 \) under the symmetry, i.e., \( D(A)H_1 = H_1 D(A) \) for every \( A \in D_4 \). This implies that the action \( D(A) \) for any \( A \in D_4 \) doesn’t intermix these eigenspaces for \( H_1 \). Assume eigenvectors with the same eigenvalue are grouped together. Change the coordinate system and take eigenvectors for \( H_1 \) as new coordinates. Then, under the new coordinates, the transformation \( D \) is equivalent to \( D' \) which is represented by

\[
D'(A) = \begin{pmatrix}
D_1(A) & 0 & 0 & 0 & 0 & 0 \\
0 & D_2(A) & 0 & 0 & 0 & 0 \\
0 & 0 & D_3(A) & 0 & 0 & 0 \\
0 & 0 & 0 & D_4(A) & 0 & 0 \\
0 & 0 & 0 & 0 & D_5(A) & 0 \\
0 & 0 & 0 & 0 & 0 & D_5(A)
\end{pmatrix}.
\]

With the degree of \( D_1, D_2, D_3, D_4, D_5 \) respectively is 1, 1, 1, 1, 2, the transformation \( H'_1 \) induced by \( H_1 \) has the form

\[
H'_1 = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_6
\end{pmatrix}.
\]

The trace of \( H_1 D(A) \) is therefore equivalent to the trace of

\[
\begin{pmatrix}
\lambda_1 D_1(A) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 D_2(A) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 D_3(A) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 D_4(A) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_5 D_5(A) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_6 D_6(A)
\end{pmatrix},
\]

i.e.,

\[
(4) \quad \text{Tr}(H_1 D(A)) = \lambda_1 \chi_1(A) + \lambda_2 \chi_2(A) + \lambda_3 \chi_3(A) + \lambda_4 \chi_4(A) + \lambda_5 \chi_5(A) + \lambda_6 \chi_5(A).
\]
On the other hand, the trace of $H_1 \mathcal{D}(A)$ can be calculated in the original coordinate. Then we have the following equations

\[
\begin{align*}
\text{Tr}(H_1 \mathcal{D}(e)) &= \text{Tr}(H_1) = \frac{9\sqrt{2}}{4} + \frac{7}{8} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2(\lambda_5 + \lambda_6), \\
\text{Tr}(H_1 \mathcal{D}(a)) &= -\frac{3}{8} - \frac{3\sqrt{2}}{4} = \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4, \\
\text{Tr}(H_1 \mathcal{D}(a^2)) &= -\frac{1}{8} - \frac{3\sqrt{2}}{4} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - 2(\lambda_5 + \lambda_6), \\
\text{Tr}(H_1 \mathcal{D}(r)) &= \frac{3}{8} - \frac{3\sqrt{2}}{4} = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \\
\text{Tr}(H_1 \mathcal{D}(ar)) &= -\frac{3}{8} + \frac{3\sqrt{2}}{4} = \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4.
\end{align*}
\]

Solving these five equations, we obtain

\[
\begin{align*}
\lambda_1 &= \lambda_2 = 0, & \lambda_3 &= \frac{3}{8}, & \lambda_4 &= \frac{3\sqrt{2}}{4}, & \lambda_5 + \lambda_6 &= \frac{3\sqrt{2}}{4} + \frac{1}{4}.
\end{align*}
\]

And observe that

\[
\begin{align*}
\frac{11}{8} \left( \frac{3\sqrt{2}}{4} + 1 \right) (\lambda_5 + 1)^2 (\lambda_5' + 1)^2 &= \det(H + I)
= \frac{340505}{32768} + \frac{963897\sqrt{2}}{131072},
\end{align*}
\]

\[
\Rightarrow (\lambda_5 + 1)(\lambda_5' + 1) = \frac{97}{64} + \frac{27\sqrt{2}}{32},
\]

we get

\[
\begin{align*}
\lambda_1 &= \lambda_2 = 0, & \lambda_3 &= \frac{3}{8}, & \lambda_4 &= \frac{3\sqrt{2}}{4}, \\
\lambda_5 &= \frac{\sqrt{2}}{4} + \frac{1}{8}, & \lambda_6 &= \frac{\sqrt{2}}{2} + \frac{1}{8}.
\end{align*}
\]

We point out that two zero eigenvalues are expected, as they correspond to rotation and scaling invariance of $\sqrt{U}$.

### 3.3. The Center + Equilateral Triangle Configuration

Next we consider the central configuration with three particle with mass $1$ at the vertices of an equilateral triangle and a fourth particle with mass $m$ at the origin, as shown in Figure 3. Clearly, this is a central configuration.

Let

\[
I = \frac{1}{2} \left( \sum_{i=1}^{3} (x_i^2 + y_i^2) + m(x_4^2 + y_4^2) \right),
\]

\[
U = \sum_{1 \leq i < j \leq 3} \frac{1}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} + m \sum_{i=1}^{3} \frac{1}{\sqrt{(x_i - x_4)^2 + (y_i - y_4)^2}}.
\]

Let $z \in \mathbb{R}^8$ be the vector $z = (x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)$. Then the equilateral triangular can be described by

\[
z_0 = (1, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0, 0)
\]
It is a critical point of the function $\sqrt{1}U$. The Hessian $H_2$ for $\frac{\partial^2(\sqrt{1}U)}{\partial z^2}$ at $z_0$ is:

$$
\begin{pmatrix}
\frac{5}{6} + 2\sqrt{3}m & 0 & \frac{1}{12} + \sqrt{3}m & -\frac{\sqrt{3}m}{2} & \frac{1}{12} + \sqrt{3}m & \frac{\sqrt{3}m}{2} + \frac{3m}{2} & -2\sqrt{3}m & 0 \\
0 & \frac{5}{6} + \sqrt{3}m & -\frac{\sqrt{3}m}{2} & \frac{1}{12} + \sqrt{3}m & -\frac{\sqrt{3}m}{2} & \frac{3m}{2} & 0 & \sqrt{3}m \\
\frac{1}{12} + \sqrt{3}m & \frac{5}{6} & -\frac{\sqrt{3}m}{2} & \frac{1}{12} + \sqrt{3}m & -\frac{\sqrt{3}m}{2} & \frac{3m}{2} & 0 & \sqrt{3}m \\
-\frac{1}{12} + \sqrt{3}m & \frac{5}{6} + \sqrt{3}m & -\frac{\sqrt{3}m}{2} & \frac{1}{12} + \sqrt{3}m & -\frac{\sqrt{3}m}{2} & \frac{3m}{2} + \frac{3m}{2} & \frac{\sqrt{3}}{2} & \frac{9m}{2} & \frac{9m}{2} & \frac{9m}{2} + \sqrt{3}(2m^2 + 3m) + m \\
0 & -\frac{\sqrt{3}m}{2} & \frac{1}{12} & \frac{5\sqrt{3}m}{4} & \frac{9m}{4} & \frac{9m}{4} & -\frac{5\sqrt{3}m}{4} & 0 & \sqrt{3}(2m^2 + 3m) + m \\
0 & -\frac{\sqrt{3}m}{2} & \frac{1}{12} & \frac{5\sqrt{3}m}{4} & \frac{9m}{4} & \frac{9m}{4} & -\frac{5\sqrt{3}m}{4} & 0 & \sqrt{3}(2m^2 + 3m) + m
\end{pmatrix}
$$

The group $S_3$ is a symmetry group for the central configuration. An action $A$ in $S_3$ displaces particles by linear transformations $D(A)$. For instance, the action $R$ that rotates the entire configuration by $\frac{2\pi}{3}$, can be described by

$$
\begin{align*}
\begin{pmatrix}
x_1' \\
y_1'
\end{pmatrix} &= R\left(\frac{2\pi}{3}\right) \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}, \\
\begin{pmatrix}
x_2' \\
y_2'
\end{pmatrix} &= R\left(\frac{2\pi}{3}\right) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \\
\begin{pmatrix}
x_3' \\
y_3'
\end{pmatrix} &= R\left(\frac{2\pi}{3}\right) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \\
\begin{pmatrix}
x_4' \\
y_4'
\end{pmatrix} &= R\left(\frac{2\pi}{3}\right) \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}.
\end{align*}
$$

It induces a representation for $R$,

$$
D(R) = \begin{pmatrix}
0 & 0 & \bar{R} & 0 \\
\bar{R} & 0 & 0 & 0 \\
0 & \bar{R} & 0 & 0 \\
0 & 0 & 0 & \bar{R}
\end{pmatrix}, \quad \text{where} \quad \bar{R} = R\left(\frac{2\pi}{3}\right) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.
$$

Similarly, we have

$$
D(T) = \begin{pmatrix}
\bar{T} & 0 & 0 & 0 \\
0 & \bar{T} & 0 & 0 \\
0 & 0 & \bar{T} & 0 \\
0 & 0 & 0 & \bar{T}
\end{pmatrix}, \quad \text{where} \quad \bar{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.
$$

**Figure 3.** The center + equilateral triangle solution for the 4 body problem

Therefore, $\mathcal{D}$ is a representation of $S_3$ with degree 8. The character of $\mathcal{D}$ can be calculated with
\[ \chi(I) = 8, \quad \chi(R) = -1, \quad \chi(T) = 0. \]
Referring the character table for $S_3$, we have Table 5.

| $A$ | $\chi$ | $\chi_1$ | $\chi_2$ | $\chi_3$ |
|-----|--------|----------|----------|----------|
| $I$  | 8      | 1        | 1        | 2        |
| $R, R^2$ | -1 | 1        | 1        | -1       |
| $T, TR, TR^2$ | 0  | 1        | -1       | 0        |

Table 5. The character of $\mathcal{D}$ and character table for $S_3$

According to Theorem 1, it implies that
\[ n_1 = (\chi, \chi_1) = \frac{1}{6} \times (8 \times 1 - 2 \times 1 \times 1 + 3 \times 0 \times 1) = 1, \]
\[ n_2 = (\chi, \chi_2) = 1, \quad n_3 = (\chi, \chi_3) = 3 \]

Then $\chi = \chi_1 + \chi_2 + 3\chi_3$. Therefore $\mathcal{D}$ is equivalent to $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus 3\mathcal{D}_3$. It can be verified that $H_2 \mathcal{D}(A) = \mathcal{D}(A)H_2$ for all $A \in S_3$. Since $H_2$ is invariant under the symmetry group $S_3$, the action by $\mathcal{D}(A)$ does not change the eigenspaces for each eigenvalue of $H_2$. Choosing eigenvectors for $H_2$ as a basis in $\mathbb{R}^8$, there is an invertible matrix $P$ such that $P \mathcal{D}(A)P^{-1} = \mathcal{D}(A)'$, where $\mathcal{D}(A)'$ is equivalent to
\[
\begin{pmatrix}
\mathcal{D}_1(A) & 0 & 0 & 0 & 0 \\
0 & \mathcal{D}_2(A) & 0 & 0 & 0 \\
0 & 0 & \mathcal{D}_3(A) & 0 & 0 \\
0 & 0 & 0 & \mathcal{D}_3(A) & 0 \\
0 & 0 & 0 & 0 & \mathcal{D}_3(A)
\end{pmatrix},
\]
and degrees of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are respectively 1, 1, 2. Under the new coordinates $H_2$ is given by
\[
H'_2 = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}.
\]

Therefore the trace of $H_2 \mathcal{D}(A)$ is equal to the trace of
\[
\begin{pmatrix}
\lambda_1 \mathcal{D}_1(A) & 0 & 0 & 0 & 0 \\
0 & \lambda_2 \mathcal{D}_2(A) & 0 & 0 & 0 \\
0 & 0 & \lambda_3 \mathcal{D}_3(A) & 0 & 0 \\
0 & 0 & 0 & \lambda_4 \mathcal{D}_3(A) & 0 \\
0 & 0 & 0 & 0 & \lambda_5 \mathcal{D}_3(A)
\end{pmatrix}.
\]
We obtain an equation:
\[ (5) \quad Tr(H_2 \mathcal{D}(A)) = \lambda_1 \chi_1(A) + \lambda_2 \chi_2(A) + \lambda_3 \chi_3(A) + \lambda_4 \chi_3(A) + \lambda_5 \chi_3(A). \]
For $A = I$, \[
\lambda_1 + \lambda_2 + 2(\lambda_3 + \lambda_4 + \lambda_5) = Tr(H_2) = 2\sqrt{3}m^2 + (9\sqrt{3} + 2)m + 5.
\]
For $A = R$, \[
\lambda_1 + \lambda_2 - (\lambda_3 + \lambda_4 + \lambda_5) = Tr(H_2 \mathcal{D}(R)) = -\frac{1}{2}(2\sqrt{3}m^2 + (9\sqrt{3} + 2)m + 5).
\]
For $A = T$, \[
\lambda_1 - \lambda_2 + 0(\lambda_3 + \lambda_4 + \lambda_5) = Tr(H_2 \mathcal{D}(T)) = 0.
\]
Solving these equations, we have \[
\lambda_1 = \lambda_2 = 0, \quad \lambda_3 + \lambda_4 + \lambda_5 = \frac{1}{2}(2\sqrt{3}m^2 + (9\sqrt{3} + 2)m + 5).
\]
Consider the characteristic polynomial \[
f(\lambda) = |\lambda I_8 - H| = \lambda^8 + a_1 \lambda^7 + \cdots + a_8.
\]
Since $\lambda_1 = \lambda_2 = 0$, we have \[
\lambda_3^2 + \lambda_4^2 + \lambda_5^2 + 4\lambda_3 \lambda_4 + 4\lambda_3 \lambda_5 + 4\lambda_4 \lambda_5 = a_2,
\]
\[
\lambda_3^2 \lambda_4^2 \lambda_5^2 = a_6.
\]
Then \[
\lambda_3 \lambda_4 + \lambda_3 \lambda_5 + \lambda_4 \lambda_5 = \frac{1}{2}(a_2 - (\lambda_3 + \lambda_4 + \lambda_5)^2), \quad \lambda_3 \lambda_4 \lambda_5 = \sqrt{a_6}.
\]

Figure 4. Eigenvalues with the parameter $m$
The expression of $\lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5$ and $\lambda_3\lambda_4\lambda_5$ can be determined with the parameter $m$. Let

\[
f_1(m) = \lambda_3 + \lambda_4 + \lambda_5
\]

\[
= \frac{1}{2}(2\sqrt{3}m^2 + (9\sqrt{3} + 2)m + 5),
\]

\[
f_2(m) = \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5
\]

\[
= \frac{1}{2}(18m^3 - 18m^2 + 11\sqrt{3}m^2 + 15\sqrt{3}m + 5m + 3),
\]

\[
f_3(m) = \lambda_3\lambda_4\lambda_5
\]

\[
= -\frac{27\sqrt{3}m^3}{2} + \frac{21\sqrt{3}m^2}{4} - \frac{27m^2}{4} + \frac{3m}{2} + \frac{9\sqrt{3}m}{4} + \frac{45m^3}{4}.
\]

This implies that $\lambda_1$, $\lambda_2$ and $\lambda_3$ are three roots of the following cubic equation:

\[
\lambda^3 - f_1(m)\lambda^2 + f_2(m)\lambda - f_3(m) = 0,
\]

Solving about equation, we obtain explicitly $\lambda_3, \lambda_4, \lambda_5$ as functions of $m$. We sketch the graph of the solutions in Figure 4.

As we remarked before, two zero eigenvalues are expected, these correspond to rotation and scaling invariance of the function $\sqrt{I}U$. A central configuration is said to be degenerate if there are more than two zero eigenvalues. This indeed happens when $f_3 = \lambda_1\lambda_2\lambda_3 = 0$. We conclude that for $m^* = \frac{2\sqrt{3} + 9}{18\sqrt{3} - 15}$, the Hessian have one additional zero eigenvalue. This implies that there exists a degenerate central configuration the 4-body problem, as discovered in Palmore’s work [9]. See also the last section of Xia [14].

4. Stability of relative equilibria

In this section, we study the local orbital structure of these relative equilibria corresponding to the types of central configurations we have discussed.

4.1. Triangle + center. The first case, the configuration with equilateral triangle plus center. As discussed in section 3.3, the 4-body central configuration is

\[
z_0 = (1, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0).
\]

The corresponding relative equilibrium solution is

\[
q_1 = \left(\cos\omega t \sin\omega t\right), \quad q_2 = \left(\cos(\omega t + \frac{2\pi}{3}) \sin(\omega t + \frac{2\pi}{3})\right),
\]

\[
q_3 = \left(\cos(\omega t + \frac{4\pi}{3}) \sin(\omega t + \frac{4\pi}{3})\right), \quad q_4 = \left(0 \right),
\]

where $\omega^2 = \frac{\sqrt{3}}{4} + m$ according to equation (1). Under uniform rotating coordinates [9], they become

\[
x_1 = \left(\frac{1}{1} \right), \quad x_2 = \left(-\frac{1}{\sqrt{3}} \right),
\]

\[
x_3 = \left(\frac{1}{\sqrt{3}} \right), \quad x_4 = \left(0 \right).
\]
To study the stability of this relative equilibrium, we need to linearize the equation of motion around the orbit. Recall the equation of motion \([3]\),

\[
x_i = 2\omega J \dot{x}_i + \omega^2 x_i + \frac{1}{m_i} \sum_{j=1}^{n} \frac{m_j m_j (x_j - x_i)}{|x_j - x_i|^3}.
\]

All terms are linear except

\[
\sum_{j=1, j \neq i}^{n} \frac{m_j m_j (x_j - x_i)}{|x_j - x_i|^3},
\]

which is the gradient of potential function \(U\). To linearize, we compute the Hessian of \(U\) at

\[
z_0 = (1, 0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2} - \frac{\sqrt{3}}{2}, 0, 0).
\]

With help of Maple, denoting the Hessian as \(H_3\), we have

\[
\begin{pmatrix}
\frac{5\sqrt{3}}{18} + 2m & 0 & -\frac{5\sqrt{3}}{36} & \frac{1}{4} & -\frac{5\sqrt{3}}{36} & -\frac{1}{4} & -2m & 0 \\
0 & -\frac{\sqrt{3}}{18} - m & \frac{3\sqrt{3}}{18} & -\frac{1}{4} & \frac{3\sqrt{3}}{36} & -\frac{3\sqrt{3}}{18} & 0 & m \\
-\frac{5\sqrt{3}}{36} & \frac{1}{4} & \frac{3\sqrt{3}}{36} & -\frac{1}{4} & -\frac{5\sqrt{3}}{36} & \frac{3\sqrt{3}}{36} & 0 & -\frac{m}{2} \\
\frac{1}{4} & \frac{3\sqrt{3}}{36} & -\frac{1}{4} & -\frac{3\sqrt{3}}{36} & 2\frac{\sqrt{3}}{36} + \frac{5m}{4} & 0 & -2\frac{\sqrt{3}}{9} & \frac{3\sqrt{3}}{4} \\
-\frac{5\sqrt{3}}{36} & -\frac{1}{4} & \frac{3\sqrt{3}}{9} & 0 & \frac{\sqrt{3}}{36} - \frac{m}{2} & \frac{1}{2} + \frac{3\sqrt{3}}{4} & \frac{m}{4} & -\frac{3\sqrt{3}}{4} \\
-2m & 0 & \frac{m}{4} & \frac{3\sqrt{3}}{4} & -3\frac{\sqrt{3}}{36} + \frac{5m}{4} & 0 & -3\frac{\sqrt{3}}{4} & \frac{3m}{2} \\
0 & m & \frac{3\sqrt{3}}{4} & -\frac{5m}{4} & -3\frac{\sqrt{3}}{36} + \frac{5m}{4} & 0 & 0 & \frac{3m}{2}
\end{pmatrix}
\]

For any element \(A\) in \(S_3\), we still have \(H_3\) is invariant under linear transformation \(\mathcal{D}(A)\), i.e., \(H_3 \mathcal{D}(A) = \mathcal{D}(A) H_3\). Choosing eigenvectors for \(H_3\) as new coordinates, we have equation \([4]\). Then For \(A = I\),

\[
\lambda_1 + \lambda_2 + 2(\lambda_3 + \lambda_4 + \lambda_5) = Tr(H_3) = \frac{2\sqrt{3}}{3} + 6m.
\]

For \(A = R\),

\[
\lambda_1 + \lambda_2 - (\lambda_3 + \lambda_4 + \lambda_5) = Tr(H_3 \mathcal{D}(R)) = \frac{\sqrt{3}}{6} - \frac{3m}{2}.
\]

For \(A = T\)

\[
\lambda_1 - \lambda_2 + 0(\lambda_3 + \lambda_4 + \lambda_5) = Tr(H_3 \mathcal{D}(T)) = \sqrt{3} + 3m.
\]

Solving these equations, it gives

\[
\lambda_1 = \frac{2\sqrt{3}}{3} + 2m, \quad \lambda_2 = -\frac{\sqrt{3}}{3} - m,
\]

\[
\lambda_3 + \lambda_4 + \lambda_5 = \frac{\sqrt{3}}{6} + \frac{5m}{2}.
\]

By calculating the characteristic polynomial of \(H_3\), there must be at least two zero eigenvalue. We assume \(\lambda_3 = 0\). With the determinant \((H_3 + I)\), we have another equation

\[
(\lambda_4 + 1)(\lambda_5 + 1) = \frac{3}{18}(2\sqrt{3}m - 48m^2 + \sqrt{3} + 15m + 6).
\]
Then we obtain
\[
\lambda_4 = \frac{\sqrt{3}}{12} + \frac{5m}{4} + \frac{\sqrt{3 - 18\sqrt{3}m + 1377m^2}}{12},
\]
\[
\lambda_5 = \frac{\sqrt{3}}{12} + \frac{5m}{4} - \frac{\sqrt{3 - 18\sqrt{3}m + 1377m^2}}{12}.
\]

Linearize equation (3) at \(z_0\) and choose eigenvectors of \(H_3\) as a basis in \(R^8\), it becomes

\[
(6) \ddot{x}_i = 2\omega J \dot{x}_i + (\omega^2 I_2 + \frac{1}{m_i} L_i) x_i,
\]

where
\[
L_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_3 \end{pmatrix},
\]
\[
L_3 = \begin{pmatrix} \lambda_4 & 0 \\ 0 & \lambda_4 \end{pmatrix}, \quad L_4 = \begin{pmatrix} \lambda_5 & 0 \\ 0 & \lambda_5 \end{pmatrix},
\]
\[
m_1 = m_2 = m_3 = 1, \quad m_4 = m.
\]

For \(i = 1, 2, 3, 4\), equation (6) is a system of second order differential equations. Let \(y'_i = \dot{x}_i, z_i = (x_i, y'_i)\), Equation (6) can be written as

\[
\dot{z}_i = B_i z_i, \quad i = 1, 2, 3, 4,
\]

where
\[
B_i = \begin{pmatrix} 0 & \omega^2 I_2 \frac{1}{m_i} L_i & I_2 \\ \omega J \end{pmatrix}.
\]

We obtain all eigenvalues of the second-order differential equations (6).

For \(B_1\),
\[
\lambda'_1 = \lambda'_2 = 0, \quad \lambda'_3 = \frac{\sqrt{3\sqrt{3} + 9m}}{3} i, \quad \lambda'_4 = -\frac{\sqrt{3\sqrt{3} + 9m}}{3} i.
\]

For \(B_2\),
\[
\lambda'_5 = \lambda'_6 = -\frac{\sqrt{3\sqrt{3} + 9m}}{3} i, \quad \lambda'_7 = \lambda'_8 = \frac{\sqrt{3\sqrt{3} + 9m}}{3} i.
\]

For \(B_3\),
\[
\lambda'_9 = \frac{\sqrt{3\sqrt{3} + 9m}}{3} i + \frac{\sqrt{3\sqrt{3} - 18m + 1377m^2}}{6} + 3\sqrt{3} + 45m,
\]
\[
\lambda'_{10} = \frac{\sqrt{3\sqrt{3} + 9m}}{3} i - \frac{\sqrt{3\sqrt{3} - 18m + 1377m^2}}{6} + 3\sqrt{3} + 45m,
\]
\[
\lambda'_{11} = -\frac{\sqrt{3\sqrt{3} + 9m}}{3} i - \frac{\sqrt{3\sqrt{3} - 18m + 1377m^2}}{6} + 3\sqrt{3} + 45m,
\]
\[
\lambda'_{12} = -\frac{\sqrt{3\sqrt{3} + 9m}}{3} i - \frac{\sqrt{3\sqrt{3} - 18m + 1377m^2}}{6} + 3\sqrt{3} + 45m.
\]

For \(B_4\),
\[
\lambda'_{13} = \frac{1}{6} (2\sqrt{3\sqrt{3} + 9m} + \sqrt{3\sqrt{m} - 3\sqrt{3 - 18\sqrt{3}m + 1377m^2}} + 45),
\]
\[
\lambda'_{14} = \frac{1}{6} (2\sqrt{3\sqrt{3} + 9m} - \sqrt{3\sqrt{m} - 3\sqrt{3 - 18\sqrt{3}m + 1377m^2}} + 45),
\]
\[ \lambda'_{15} = \frac{1}{6}(-2\sqrt{3}\sqrt{3} + 9m + \sqrt{3}\frac{3\sqrt{3} - 18\sqrt{3}m + 1377m^2}{m} + 45), \]
\[ \lambda'_{16} = \frac{1}{6}(-2\sqrt{3}\sqrt{3} + 9m - \sqrt{3}\frac{3\sqrt{3} - 18\sqrt{3}m + 1377m^2}{m} + 45). \]

For all \( m > 0 \), we have \( 3 - 18m + 1377m^2 > 0 \). Therefore \( \lambda'_9, \lambda'_{10}, \lambda'_{11}, \lambda'_{12} \) all have nonzero real parts. We conclude that the relative equilibrium is unstable for all values of \( m \). [7]

![Figure 5. f with the parameter m](image)

To classify the local orbital structure, let

\[ f = \frac{3\sqrt{3} - 3\sqrt{3} - 18\sqrt{3}m + 1377m^2}{m} + 45. \]

As shown in Figure 5, \( f \) becomes negative when \( m \) is large enough. Eigenvalues \( \lambda'_{13}, \lambda'_{14}, \lambda'_{15}, \lambda'_{16} \) are pure imaginary number. When \( m \) is large enough, the eigenvalues \( \lambda'_{13}, \lambda'_{14}, \lambda'_{15}, \lambda'_{16} \) become pure imaginary numbers. We conclude that the number of elliptic direction changes from 6 to 10 when \( m \) is large enough.

4.2. Square configuration. The second case is the square configuration. As investigated in section 3.2, the 4-body central configuration is

\[ z_0 = (2, 0, 0, 2, -2, 0, 0, -2). \]

The corresponding relative equilibrium solution is

\[ q'_1 = \begin{pmatrix} 2\cos\omega't \\ 2\sin\omega't \end{pmatrix}, \quad q'_2 = \begin{pmatrix} 2\cos(\omega't + \frac{\pi}{2}) \\ 2\sin(\omega't + \frac{\pi}{2}) \end{pmatrix}. \]
\[ q'_3 = \begin{pmatrix} 2 \cos(\omega' t + \pi) \\ 2 \sin(\omega' t + \pi) \end{pmatrix}, \quad q'_4 = \begin{pmatrix} 2 \cos(\omega' t + \frac{3\pi}{2}) \\ 2 \sin(\omega' t + \frac{3\pi}{2}) \end{pmatrix}, \]

with \( \omega'^2 = \frac{\sqrt{3}}{16} + \frac{1}{32} \) according to equation (1). Under uniform rotating coordinates \( H \), they become

\[ x'_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad x'_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \]

\[ x'_3 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad x'_4 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}. \]

Similarly, we linearize the equation of motion (3) around the orbit. We compute the Hessian of \( U \) at

\[ z_0 = (2, 0, 0, 2, -2, 0, 0, -2). \]

With help of Maple, denoting the Hessian as \( H_4 \), we have

\[
\begin{pmatrix}
\sqrt{\frac{7}{32}} + \frac{1}{32} & 0 & -\sqrt{\frac{7}{64}} & -\frac{3\sqrt{2}}{64} & -\frac{1}{32} & 0 & -\sqrt{\frac{7}{64}} & -\frac{3\sqrt{2}}{64} \\
0 & \sqrt{\frac{7}{32}} - \frac{1}{64} & \frac{3\sqrt{2}}{64} & -\frac{1}{64} & 0 & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \\
-\frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & 0 & -\frac{3\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} \\
-\frac{1}{64} & -\frac{3\sqrt{2}}{64} & -\frac{1}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & 0 & -\frac{3\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} \\
\frac{1}{64} & -\frac{3\sqrt{2}}{64} & -\frac{1}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & 0 & -\frac{3\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} \\
-\frac{3\sqrt{2}}{64} & -\frac{1}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} & -\frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} \\
-\frac{1}{64} & \frac{1}{64} & -\frac{3\sqrt{2}}{64} & -\frac{1}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} \\
-\frac{3\sqrt{2}}{64} & -\frac{1}{64} & \frac{1}{64} & -\frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64} & \frac{3\sqrt{2}}{64}
\end{pmatrix}
\]

For any \( A \) in \( D_4 \), we have \( H_4 \) is invariant under linear transformation \( \mathcal{D}(A) \), i.e., \( H_4 \mathcal{D}(A) = \mathcal{D}(A)H_4 \). Let eigenvectors for \( H_4 \) as new coordinates, we have equation \( 4 \). Then For \( A = e \),

\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2(\lambda_5 + \lambda_6) = \frac{\sqrt{7}}{4} + \frac{1}{16}. \]

For \( A = a \),

\[ \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0. \]

For \( A = a^2 \),

\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - 2(\lambda_5 + \lambda_6) = \frac{1}{16}. \]

For \( A = r \),

\[ \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = \frac{3}{16}. \]

For \( A = ar \),

\[ \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 = -\frac{3\sqrt{2}}{8}. \]

Solving these equations, we have

\[ \lambda_1 = \frac{1}{16} + \frac{\sqrt{2}}{8}, \quad \lambda_2 = -\frac{1}{32} - \frac{\sqrt{2}}{16}, \quad \lambda_3 = \frac{1}{16} - \frac{\sqrt{2}}{16}, \quad \lambda_4 = -\frac{1}{32} + \frac{\sqrt{2}}{8}, \quad \lambda_5 + \lambda_6 = \frac{\sqrt{2}}{16}. \]
By computing the character polynomial of $H_4$, we find it exists two zero eigenvalue. Assuming $\lambda_3 = 0$, then $\lambda_6 = \frac{\sqrt{2}}{16}$. Linearize equation (3) at $z_0$ and choose eigenvectors of $H_4$ as a basis in $R^8$, we have

$$\ddot{x}_i = 2\omega J \dot{x}_i + (\omega^2 I_2 + L'_i)x_i,$$

where

$$L'_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad L'_2 = \begin{pmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{pmatrix},$$

$$L'_3 = \begin{pmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \end{pmatrix}, \quad L'_4 = \begin{pmatrix} \lambda_6 & 0 \\ 0 & \lambda_5 \end{pmatrix}.$$

It is easy to calculate eigenvalues for the second-order differential equations.

For $i = 1$,

$$\lambda'_1 = \lambda'_2 = 0, \quad \lambda'_3 = \frac{\sqrt{2} + 4\sqrt{2}}{8}i, \quad \lambda'_4 = -\frac{\sqrt{2} + 4\sqrt{2}}{8}i.$$

For $i = 2$,

$$\lambda'_5 = \frac{\sqrt{68\sqrt{2} - 9i - 2\sqrt{2} - 1}}{8}, \quad \lambda'_6 = -\frac{\sqrt{68\sqrt{2} - 9i - 2\sqrt{2} - 1}}{8},$$

$$\lambda'_7 = \frac{\sqrt{-68\sqrt{2} - 9i - 2\sqrt{2} - 1}}{8}, \quad \lambda'_8 = -\frac{\sqrt{-68\sqrt{2} - 9i - 2\sqrt{2} - 1}}{8}.$$

For $i = 3$,

$$\lambda'_9 = \lambda'_{10} = \frac{\sqrt{2} + 4\sqrt{2}}{8}i, \quad \lambda'_{11} = \lambda'_{12} = -\frac{\sqrt{2} + 4\sqrt{2}}{8}i.$$

For $i = 4$,

$$\lambda'_{13} = -\frac{2i}{4} + \frac{\sqrt{2} + 4\sqrt{2}}{8}i, \quad \lambda'_{14} = \frac{2i}{4} + \frac{\sqrt{2} + 4\sqrt{2}}{8}i,$$

$$\lambda'_{15} = -\frac{2i}{4} - \frac{\sqrt{2} + 4\sqrt{2}}{8}i, \quad \lambda'_{16} = \frac{2i}{4} - \frac{\sqrt{2} + 4\sqrt{2}}{8}i.$$

Therefore, $\lambda'_5, \lambda'_6, \lambda'_7, \lambda'_8, \lambda'_9, \lambda'_{13}, \lambda'_{14}, \lambda'_{15}, \lambda'_{16}$ all have nonzero real parts, we conclude that the square configuration is unstable. Eigenvalues $\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4, \lambda'_{10}, \lambda'_{11}, \lambda'_{12}$ are pure imaginary numbers. This implies that the elliptic direction has dimension 6.

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