On the maximum number of edges of non-flowerable coin graphs

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Abstract

For \( n \in \mathbb{N} \) and \( 3 \leq k \leq n \) we compute the exact value of \( E_k(n) \), the maximum number of edges of a simple planar graph on \( n \) vertices where each vertex bounds an \( \ell \)-gon where \( \ell \geq k \). The lower bound of \( E_k(n) \) is obtained by explicit construction, and the matching upper bound is obtained by using Integer Programming (IP). We then use this result to conjecture the maximum number of edges of a non-flowerable coin graph on \( n \) vertices. A flower is a coin graph representation of the wheel graph. A collection of coins or discs in the Euclidean plane is non-flowerable if no flower can be formed by coins from the collection.

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1 Introduction

In this article, we will prove a result which gives the maximum number of edges in a plane graph on \( n \) vertices, where each vertex bounds some \( \ell \)-gon for \( \ell \geq k \). We will find the exact upper bound using integer programming and the matching lower bound by construction. This problem arose from an investigation of extremal coin graphs with multiple radii that satisfy certain conditions we will discuss in Section 3. Here a coin graph is a graph whose vertices can be represented as closed, non-overlapping disks in the Euclidean plane such that two vertices are adjacent if and only if their corresponding disks intersect at their boundaries, i.e. they touch.

Coin graphs are ubiquitous in the discrete geometry literature especially since the most general ones (with no restrictions on the radii of the coins) are, by a well-known theorem of Thurston [1], precisely the planar graphs. One of the best known extremal problems of coin graphs is perhaps one posed by Erdős [4] in 1946 and again by Reutter [5] in 1972: for a given natural number \( n \), determine the maximum number of edges a coin graph can have if all the coins have the same radius (called unit coin graphs.) This problem has an unusually nice solution, due to Harborth [6] from 1974, who showed that the maximum number of edges is given by \( T(n) := \lfloor 3n - \sqrt{12n-3} \rfloor \). This problem can be generalized in many ways, as suggested in a recent excellent survey of open research problems in discrete geometry [7, p. 222]. For instance it can be generalized to (i) graphs

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1This theorem can also be attributed to Koebe [2] and Andreev [3]. Koebe’s original proof covered only the case of fully-triangulated planar graphs. Thurston reduced the proof to the previous theorem of Andreev. Thurston’s proof is of the more general case of all planar graphs.
embedded in other surfaces such as the sphere or to graphs embedded in \( n \)-dimensional Euclidean space for \( n \geq 3 \), where the definition of a coin graph is modified appropriately to an \( n \)-dimensional sphere graph, or (ii) by adding the constraint that no three vertices of the graph can be collinear forces the maximum degree of any vertex to be 5, leading to a different upper bound, or (iii) by defining a similar class of graphs by connecting two vertices if and only if their distance \( d \) satisfies \( 1 \leq d \leq 1 + \epsilon \) for some given small \( \epsilon > 0 \). This structure can be pictured as a unit coin graph using elastic disks that can stretch some small amount. It is conjectured that for small \( \epsilon \) (less than 0.15 times the defined unit distance) the maximum number of edges is still \( T(n) \) as in the case of the unit coin graph [7]. Finally, (iv) Swanepoel [8] recently conjectured that the largest number of edges in a coin graph with no triangular faces is given by \( \lfloor 2n - 2\sqrt{n} \rfloor \). All these slight modifications are still open problems. Another natural generalization of the unit coin graph problem above, that is not discussed in [7], is to allow coins of more than one possible radius. Brightwell and Scheinerman [9] explored integral representations of coin graphs, where the radii of the coins can take arbitrary positive integer values.

The results of this article were inspired by an extremal problem in the same vein as mentioned above, namely to determine the exact maximum number of edges in a coin graph on \( n \) coins where their radii is such that no wheel graph can by formed by them. Here a wheel graph is formed from a simple cycle by connecting one additional central vertex to each of the vertices of the cycle. A physical interpretation of this is to have a collection of \( n \) coins on the table and their sizes are such that it is impossible to completely “surround” one coin with other coins such that they all touch. This means that the underlying plane graph of the coin graph has no vertex that only borders triangular faces. The article is organized as follows: in Section 2 we first introduce our notation and terminology. Then for given \( n \) and \( k \) we compute the tight upper bound of the maximum number of edges a simple plane graph on \( n \) vertices can have, if every vertex borders some \( \ell \)-gon where \( \ell \geq k \). The lower bound is obtained by direct construction, whereas the matching upper bound is obtained with Integer Programming (IP). Unlike many IP problems, the one we obtain is simple enough to be able to solve completely. In Section 3 we give an upper bound for the maximum number of edges a coin graph on \( n \) vertices with no induced wheel graphs, and conjecture that this bound is indeed tight.

## 2 The general tight upper bound

**Notation and terminology** The set \( \{1, 2, 3, \ldots\} \) of natural numbers will be denoted by \( \mathbb{N} \). The set of real numbers is \( \mathbb{R} \) and the Euclidean plane \( \mathbb{R}^2 \), the Cartesian product of two copies of \( \mathbb{R} \). The complex number field is \( \mathbb{C} \). Unless otherwise stated, all graphs in this article will be finite, simple and undirected. The cycle graph on \( n \) vertices and \( n \) edges will be denoted by \( C_n \) and the wheel graph on \( k + 1 \) vertices will be denoted by \( W_k \).

Our main objective in this section is to prove the following theorem.

**Theorem 2.1** Let \( k, n \in \mathbb{N} \) with \( n \geq k \geq 4 \). The maximum number \( E_k(n) \) of edges of a plane graph on \( n \) vertices, where each vertex bounds some \( \ell \)-gon for \( \ell \geq k \), is given by

\[
E_k(n) = T_k(n) := \left\lfloor \frac{(2k + 3)n}{k} - 6 \right\rfloor - \alpha
\]
where
\[ \alpha = \begin{cases} 
0 & \text{if } n \equiv k - 1 \pmod{k} \\
2 - \frac{\beta}{k} & \text{if } n \equiv k - 2 \pmod{k} \\
\frac{3\beta}{k} & \text{if } n \equiv \beta \pmod{k} \text{ for } 0 \leq \beta \leq k - 3.
\end{cases} \]

We will show that \( T_k(n) \) is both an upper bound and a lower bound for \( E_k(n) \). We start with the easier case and show that \( T_k(n) \) is lower bound using an explicit construction, and we consider each of the three cases, \( n \equiv k - 1, k - 2, \beta \pmod{k} \) where \( 0 \leq \beta \leq k - 3 \), separately, since each case has a unique construction. We then conclude the section with the more involved case and prove that the matching lower bound \( T_k(n) \) is also an upper bound.

**The lower bound** Write \( n = kj + \beta \) where \( 0 \leq \beta \leq k - 1 \). Form \( j - 1 \) disjoint copies of \( C_k \) and one copy of \( C_{k+\beta} \) in the plane, no cycle containing another cycle, consisting of \( n \) edges altogether. We need \( 3(j - 1) \) edges to connect the cycles into one connected component such that (i) the infinite face is bounded by a simple \( n \)-cycle and (ii) the internal faces of this \( n \)-cycle other than the \( C_k \)s and the \( C_{k+\beta} \) are triangular. Then we add \( n - 3 \) edges to fully triangulate the infinite face. The total number of edges thus obtained is \( e(n, j) := n + 3(j - 1) + (n - 3) \). Consider now the various cases for \( \beta \).

**First case:** \( \beta = k - 1 \). In this case \( C_{k+\beta} = C_{2k-1} \) and two additional edges can be added to the interior of the cycle \( C_{2k-1} \) to create 3 regions, 2 bounded by \( k \)-gons and one by a triangle such that every vertex is bounded by a \( k \)-gon. Add these additional edges between appropriate vertices of the cycle \( C_{2k-1} \). The total number of edges is then given by
\[ e(n, j) + 2 = 2n + 3j - 4 = \frac{(2k+3)n}{k} - 6 - \left(1 - \frac{3}{k}\right) = \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor. \]

**Second case:** \( \beta = k - 2 \). Here \( C_{k+\beta} = C_{2k-2} \) and one additional edge can be added to the interior of the cycle \( C_{2k-2} \) to create 2 regions bounded by \( k \)-gons. Add this additional edge between appropriate vertices of the cycle \( C_{2k-2} \). The total number of edges is now given by
\[ e(n, j) + 1 = 2n + 3j - 5 = 2n + 3 \left(1 + \frac{2}{k}\right) - 6 - 2 = \frac{(2k+3)n}{k} - 6 - \left(2 - \frac{6}{k}\right). \]

Since for any real numbers \( x, y \) with \( x - y \) a positive integer we have \( x - y = \lfloor x \rfloor - \lfloor y \rfloor \), then this last expression equals \( \lfloor \frac{(2k+3)n}{k} - 6 \rfloor - \lfloor 2 - \frac{6}{k} \rfloor \).

**Third case:** \( 0 \leq \beta \leq k - 3 \). Here the total number of edges is given by
\[ e(n, j) = 2n + 3j - 6 = \frac{(2k+3)n}{k} - 6 - \frac{3\beta}{k} = \left\lfloor \frac{(2k+3)n}{k} - 6 \right\rfloor - \left\lfloor \frac{3\beta}{k} \right\rfloor, \]
the last step just as in the previous case when \( \beta = k - 2 \). These three cases show that the mentioned bound \( T_k(n) \) can always be reached.

**The upper bound** We will derive the matching upper bound using Integer Programming. Unlike most integer programs, it turns out that our specific one in this case will be simple enough to be able to spot a general pattern to solve it exactly.
Assume we have a plane graph $G$ on $n$ vertices with the property mentioned in the theorem. The number of edges is $m$ and the number of faces is $f$. Form a new graph $G'$ by adding a vertex inside each $\ell$-gon, where $\ell \geq k$ and connect that vertex to all the vertices bounding the $\ell$-gon. Let $n'$, $m'$, and $f'$ be the number of vertices, edges, and faces of $G'$. Note that $G'$ is planar and fully triangulated. For $i \in \{3, \ldots, k-1\}$, let $f_i$ denote the number of $i$-sided faces of $G$ and $f_k$ be the number of all $\ell$-sided faces where $\ell \geq k$. Then $f = f_3 + f_4 + \cdots + f_{k-1} + f_k$. By assumption we have $n' = n + f_4 + \cdots + f_{k-1} + f_k$ and $m' = 3n' - 6$, by Euler’s formula.

Let $d$ be the sum of the degrees of all the vertices that were added above, so $d$ also equals the number of edges added to $G$ to obtain $G'$. Hence $m' = m + d = 3(n + f_4 + \cdots + f_k) - 6$, so $m = 3n - 6 - (d - 3(f_4 + \cdots + f_{k-1} + f_k))$. Note that $d = d_4 + d_5 + \cdots + d_{k-1} + d_k$ where for each $i \in \{4, \ldots, k-1\}$, $d_i$ is the sum of the degrees of the vertices of degree $i$ added to $G$ and $d_k$ is the sum of degrees of vertices of degree greater than or equal to $k$ added to $G$. Therefore we have $d_i = i f_i$ for each $i \in \{4, \ldots, k-1\}$ and so $d = 4 f_4 + \cdots + (k-1) f_{k-1} + d_k$ and hence

$$m = 3n - 6 - (f_4 + 2 f_5 + \cdots + (k-3) f_{k-1} + d_k - 3 f_k).$$

Note that $m$ is maximized if $f_4 + 2 f_5 + \cdots + (k-3) f_{k-1} + d_k - 3 f_k$ is minimized. Since the conditions are (1) $n \leq d_k$, (2) $f_i \geq 0$ for $i \in \{4, \ldots, k\}$, and (3) $k f_k \leq d_k$, we can simplify this optimization problem by setting $f_i = 0$ for $i = 4, \ldots, k-1$ and the problem reduces to minimizing the value of $d_k - 3 f_k$ over nonnegative integers, given the constraints $d_k \geq n$ and $k f_k \leq d_k$.

**Lemma 2.2** If $k, n \in \mathbb{N}$ and $n \geq k \geq 4$ and $\mu(n, k) := \min\{x - 3y : x, y \in \mathbb{N} \cup \{0\}, x \geq n, ky \leq x\}$, then

$$\mu(n, k) = n + \gamma - 3 \left\lfloor \frac{n + \gamma}{k} \right\rfloor$$

where $\gamma = \begin{cases} 1 & \text{if } n \equiv k - 1 \pmod{k} \\ 2 & \text{if } n \equiv k - 2 \pmod{k} \\ 0 & \text{otherwise}. \end{cases}$

**Proof.** Drawing the vector $(1, -3)$ and the lines $x = n$ and $ky = x$ in the Euclidean plane $\mathbb{R}^2$, we can spot the solution to our Integer Program $\mu(n, k)$, since the function $x - 3y = (1, -3) \cdot (x, y)$, a dot product of two vectors, will obtain its minimum value at $x = n$ and $y = \left\lfloor \frac{n}{k} \right\rfloor$ in the case of $n \equiv i \pmod{k}$ where $i = 0, 1, \ldots, k-3$, and at $x = k \left\lceil \frac{n}{k} \right\rceil$ and $y = \left\lfloor \frac{n}{k} \right\rfloor = \left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n}{k} \right\rfloor$ otherwise. The Figures 1 and 2 illuminate this general pattern, which here remains the same for all other values of $n$ and $k$. Using the above definition of $\gamma$ in the lemma, we can write $x = n + \gamma$ as the $x$-value that will always minimize the function. Then we have $y = \left\lfloor \frac{x + \gamma}{k} \right\rfloor$ as the $y$-value that will always minimize the function. \hfill \square

Continuing to obtain the upper bound of $m$ from (11), we have by Lemma 2.2 that $d_k - 3 f_k$ is minimized when $d_k = n + \gamma$ and $f_k = \left\lfloor \frac{n+\gamma}{k} \right\rfloor$ and hence we have

$$m = 3n - 6 - (d_k - 3 f_k) \leq 3n - 6 - n - \gamma + 3 \left\lfloor \frac{n + \gamma}{k} \right\rfloor = 2n - 6 + 3 \left\lfloor \frac{n + \gamma}{k} \right\rfloor - \gamma.$$

If $n \equiv k - 1 \pmod{k}$, then $\gamma = 1$ and

$$m \leq 2n - 6 + 3 \left\lfloor \frac{n + \gamma}{k} \right\rfloor - 1 = \frac{(2k + 3)n}{k} - 6 - \left(1 - \frac{3}{k}\right) = \left\lfloor \frac{(2k + 3)n}{k} \right\rfloor - 6 + \left\lfloor 2 - \frac{6}{k}\right\rfloor.$$

If $n \equiv k - 2 \pmod{k}$ then $\gamma = 2$ and

$$m \leq 2n - 6 + 3 \left\lfloor \frac{n + 2}{k} \right\rfloor - 2 = \frac{(2k + 3)n}{k} - 6 + \left(2 - \frac{6}{k}\right) = \left\lfloor \frac{(2k + 3)n}{k} \right\rfloor - 6 + 2 - \frac{6}{k}.$$
Figure 1: When $k = 6$ and $n = 8$, the function is minimized at $x = 8, y = 1$.

Figure 2: When $k = 4$ and $n = 7$, the function is minimized at $x = 8, y = 2$. 
If \( n \equiv \beta \) where \( \beta \in \{0, 1, \ldots, k-3\} \) then \( \gamma = 0 \) and
\[
m \leq 2n - 6 + 3 \left\lfloor \frac{n}{k} \right\rfloor = 2n - 6 + \left( \frac{n - \beta}{k} \right) = \frac{(2k + 3)n}{k} - 6 - \left( \frac{3\beta}{k} \right) = \left\lfloor \frac{(2k + 3)n}{k} - 6 \right\rfloor - \left\lfloor \frac{3\beta}{k} \right\rfloor.
\]

The above three cases show that \( m \leq T_k(n) \), the matching lower bound. This proves Theorem 2.1 that \( E_k(n) = T_k(n) \).

In the especially interesting case when \( k = 4 \), the discrepancy term \( \alpha \in \{0, \left\lfloor \frac{2}{k} - \frac{6}{k} \right\rfloor, \left\lfloor \frac{3}{k} \beta \right\rfloor \} \) for \( 0 \leq \beta \leq k - 3 \), will be 0 in all cases, and hence we obtain the following:

**Corollary 2.3** The maximum number of edges \( E_4(n) \) of a plane graph on \( n \) vertices, where each vertex bounds some \( \ell \)-gon for \( \ell \geq 4 \), is given by
\[
E_4(n) = \left\lfloor \frac{11}{4} n - 6 \right\rfloor.
\]

### 3 Non-flowerable coins

A coin graph representation of the wheel graph is called a flower. A coin graph with no flowers is non-flowered and a collection \( C \) of coins is non-flowerable if no flower can be formed by coins from \( C \). This terminology is consistent with that found in [10]. Note that it is not necessary for a non-flowerable collection to contain coins of distinct radii, but it cannot contain seven or more coins of equal radii, since seven coins with the same radius can form a regular hexagonal flower.

**Definition 3.1** For \( n \in \mathbb{N} \) denote by \( \overline{NF}(n) \) the set of all non-flowerable collections of \( n \) coins. For each \( C \in \overline{NF}(n) \) let \( NF(C) \) denote the maximum number of edges of a coin graph formed from coins in \( C \). Finally let
\[
NF(n) = \max(\{NF(C) : C \in \overline{NF}(n)\})
\]

Note that every non-flowered coin graph must have each coin bounded by an \( \ell \)-gon where \( \ell \geq 4 \). Hence, by Corollary 2.3 we obtain the following corollary.

**Corollary 3.2** For \( n \in \mathbb{N} \) we have
\[
NF(n) \leq E_4(n) = \left\lfloor \frac{11}{4} n - 6 \right\rfloor.
\]

Whether \( NF(n) = E_4(n) \) or not is unknown to us as of writing this article.

**Conjecture 3.3** For \( n \in \mathbb{N} \) we have \( NF(n) = E_4(n) = \left\lfloor \frac{11}{4} n - 6 \right\rfloor \).

**Remark:** Given \( n \in \mathbb{N} \). By Thurston’s theorem [1] we can obtain a coin graph representation of each of the planar graphs constructed for the lower bound of Theorem 2.1 in Section 2. By construction it is guaranteed that it will be non-flowered. However, we do not know if the underlying collection of coins used in this representation is non-flowerable, since some flower could be formed by a subset of them. We do suspect that each such coin graph representation of the graphs formed for the lower bound in Theorem 2.1 can be represented by a non-flowerable collection of coins: Recall that the map \( \mathbb{C} \to \mathbb{C} \) given by \( z \mapsto 1/z \) is an inversion about the unit circle centered at origin. It is known fact in plane geometry that every inversion of the complex plane maps a coin
graph to another coin graph with the same underlying planar graph. However, the radii of coins have all changed. We suspect that a proof of Conjecture 3.3 can be obtained by inverting a carefully chosen embedding of a coin graph on \( n \) coin with the maximum number \( E_4(n) \) of edges, resulting in a representation using non-flowerable collection of coins. However, as far as our investigation goes, we will stop here for the moment.

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