LARGE DEVIATION PRINCIPLE FOR FINITE-STATE MEAN FIELD
INTERACTING PARTICLE SYSTEMS

PAUL DUPUIS, KAVITA RAMANAN, AND WEI WU

ABSTRACT. We establish a large deviation principle for the empirical measure process associated
with a general class of finite-state mean field interacting particle systems with Lipschitz continuous
transition rates that satisfy a certain ergodicity condition. The approach is based on a variational
representation for functionals of a Poisson random measure. Under an appropriate strengthening
of the ergodicity condition, we also prove a locally uniform large deviation principle. The main
novelty is that more than one particle is allowed to change its state simultaneously, and so
a standard approach to the proof based on a change of measure with respect to a system of
independent particles is not possible. The result is shown to be applicable to a wide range of
models arising from statistical physics, queueing systems and communication networks. Along
the way, we establish a large deviation principle for a class of jump Markov processes on the
simplex, whose rates decay to zero as they approach the boundary of the domain. This result
may be of independent interest.

1. INTRODUCTION

Markovian particle systems on finite state spaces under mean field interactions arise in many dif-
ferent contexts. They appear as approximations of statistical physics models in higher dimensional
lattices (for various types of spin dynamics, see [27] and references therein), kinetic theory [21],
game theory [18] and as models of communication networks [1], [17], [19], [33]. The dynamics of
these particle systems have the following common features: a) particles are exchangeable, that is,
their joint distribution is invariant under permutation of their indices; b) at each time, multiple
particles in some finite subset can switch their states simultaneously; c) the interaction between
particles is global but weak, in the sense that the jump rate of each group of particles is a function
only of the initial and final configurations of that group of particles, and the empirical measure of
all particles. The precise dynamics of the Markovian $n$-particle system we consider are described
in Section 2.1.

Due to the exchangeability assumption, many essential features of the state of the particle system
can be captured by its empirical measure, which evolves as a jump Markov process on (a sublattice
of) the unit simplex. Under mild assumptions on the jump rates, standard results on jump Markov
processes (see [28]) show that the functional law of large numbers limit of the sequence of $n$-
particle empirical measures is the solution of a nonlinear ordinary differential equation (ODE) on

Date: January 26, 2016.

1991 Mathematics Subject Classification. Primary: 60F10, 60K35; Secondary: 60K25.

Key words and phrases. Large deviation principle, interacting particle systems, mean field limits, nonlinear Markov
process, McKean-Vlasov limits, rate function, locally uniform LDP, jump Markov processes, empirical measure,
Curie-Weiss model, network with alternate routing.

Division of Applied Mathematics, Brown University and New York University. This research was supported in part
by the Army Research Office (W911NF-12-1-0222).
the unit simplex. The ODE also characterizes the transition probabilities of a certain “nonlinear Markov process” that describes the limiting distribution of a typical particle in the system, as the number of particles goes to infinity [23], and is commonly referred to as the McKean-Vlasov limit. In this paper we consider the sample path large deviation properties of the sequence of empirical measure processes as the number of particles tends to infinity. In the case of interacting diffusion processes, such a large deviation principle (LDP) was first established by Dawson and Gartner in [22]. The sample path large deviation principle over finite time intervals has a number of applications, including the study of metastability properties via Freidlin-Wentzell theory [16] (see also [29] and [2] for the reversible case), and the study of the possible evolution of a Gibbs measure into a non-Gibbs measure under stochastic (e.g., spin-flip) dynamics (which is referred to as a Gibbs-non Gibbs transition in [15]).

Large deviation principles for jump Markov processes are known if the jump rates are Lipschitz continuous and uniformly bounded below away from zero (cf. [31]). In this case, the large deviation rate function admits an integral representation in terms of a so-called local rate function. However, the jump rates in our model do not satisfy this condition. Specifically, as the empirical measure approaches the boundary of the simplex, its jump rates along certain directions converge to zero. Nevertheless, we show that (under general conditions on the jump rates), the sequence of empirical measure processes satisfies a sample path LDP with the rate function having the standard integral representation. Under mild conditions, we also establish a “locally uniform” refinement [31], which characterizes the decay rate of the probabilities of hitting a convergent sequence of points. Such a result is of relevance only for discrete Markov processes (and not for diffusions) and does not follow immediately from the LDP. The locally uniform refinement is shown in [31,7] to be relevant for the study of stability properties of the nonlinear ODE that describes the law of large numbers (LLN) limit. All the main results of this paper are formulated for a more general class of jump Markov processes on the simplex whose rates diminish to zero at the boundary, and the interacting particle models are obtained as a special case.

Other works that have studied large deviations for jump Markov processes with vanishing rates include [32], [22], [25] and [4]. However, the results in [32] impose special conditions on the jump rates near the boundary, which do not apply to our model (see Appendix A of [35]). On the other hand, the methods used in [22] and [4] are adaptations of the argument used by Dawson and Gartner in [22], which crucially relies on the fact that the measure on path space induced by the interacting n-particle process is absolutely continuous with respect to that induced by n independent (non-interacting) particles, each evolving according to a time inhomogenous Markov process. This property does not hold when multiple particles jump simultaneously. Simultaneous jumps are a common feature of models used in many applications (see Example 2.7 and also [33] and [14, Chapter 8]).

The large deviation upper bound follows from general results in [12] (see Section 5). The subtlety arises in the proof of the large deviation lower bound. Our strategy for the proof is based on a variational representation for the n-particle empirical measure process and a perturbation argument near the boundary. The starting point of our variational representation is a representation formula for functionals of Poisson random measures [8], and an SDE representation of the empirical measure process in terms of a sequence of Poisson random measures. However, the state-dependent nature of the jump rates leads to a somewhat complicated variational problem. We use the special structure of the SDE to simplify the representation formula. The perturbation argument takes inspiration from [13], where an LDP was established for a discrete time one-dimensional Markov chain. Our model is higher dimensional, where the perturbation argument becomes substantially more intricate,
and geometry comes into play. The variational representation that we establish holds more generally for jump Markov processes with bounded jump rates, and could be useful for obtaining other asymptotics.

The outline of this paper is as follows. In Section 2 we set up the mean field interacting particle system, and describe a few examples in the literature that fit into the framework. In Section 3 we state the main results, namely a sample path LDP for a general class of weakly interacting particle systems (Theorem 3.6), its locally uniform refinement (Theorem 3.12) and an LDP for the corresponding sequence of stationary measures (Theorem 3.17). In Section 4 we show that our assumptions on the transition rates of the mean field interacting particle system imply that the jump rates of the associated empirical measure process satisfy certain useful properties, which are the only ones used in the proof of our results. As a consequence, our main results in fact apply to the larger class of jump Markov processes on the simplex whose jump rates possess these properties (see Remark 5.6 for a precise statement). Section 5 establishes the variational representation for the larger class of jump Markov processes on the simplex whose jump rates possess these properties. Section 6 establishes the variational representation for the larger class of jump Markov processes on the simplex whose jump rates possess these properties. Section 7 studies properties of the local rate function. Section 8 is devoted to the proof of the locally uniform LDP.

2. The Interacting Particle Systems

2.1. Model Description. In this work, we consider an \( n \)-particle system in which the state of each individual particle takes values in the finite set \( \mathcal{X} = \{1, 2, \ldots, d\} \). For each \( i = 1, \ldots, n \), let \( X^{i,n}(t) \) be the state of the \( i \)-th particle at time \( t \). For simplicity of notation, we assume that the sequence of processes \( X^n(\cdot) = \{X^n(t) = (X^{1,n}(t), \ldots, X^{n,n}(t)), t \geq 0\} \), \( n \in \mathbb{N} \), are defined on a common probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Each \( X^n(\cdot) \) evolves as a càdlàg, \( \mathcal{X}^n \)-valued jump Markov process. The associated empirical measure is denoted by

\[
\mu^n(t, \omega) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X^{i,n}(t, \omega)}, \quad t \geq 0, \omega \in \Omega,
\]

where \( \delta_x \) represents the Dirac mass at \( x \). In subsequent discussions, we often suppress the dependence of \( \mu^n \) on \( \omega \).

Let \( \mathcal{P}(\mathcal{X}) \) denote the space of probability measures on \( \mathcal{X} \). We identify \( \mathcal{P}(\mathcal{X}) \) with the simplex \( \mathcal{S} = \{x \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^{d} x_i = 1\} \) and endow \( \mathcal{S} \) with the topology induced from \( \mathbb{R}^d \), so that \( \mathcal{S} = \mathcal{P}(\mathcal{X}) \) is equipped with the Euclidean norm \( \|\cdot\| \). Define \( \mathcal{P}_n(\mathcal{X}) = \{\frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} : x \in \mathcal{X}^n\} \subset \mathcal{P}(\mathcal{X}) \). Then \( \mathcal{P}_n(\mathcal{X}) \) can be similarly identified with the lattice \( \mathcal{S}_n = \mathcal{S} \cap \frac{1}{n} \mathbb{Z}^d \), and clearly \( \mu^n(\cdot) = \{\mu^n(t), t \geq 0\} \) is an \( \mathcal{S}_n \)-valued stochastic process.

The possible transitions of \( X^n \) are as follows. It is assumed that there exists \( K \in \mathbb{N} \) such that at most \( K \) particles jump simultaneously. When \( K = 1 \), almost surely at most one particle can instantaneously change its state. For \( i, j \in \mathcal{X}, i \neq j \), and \( t \geq 0 \), the rate at which a particle changes its state from \( i \) to \( j \) at time \( t \) is assumed to be \( \Gamma_{ij}(\mu^n(t)) \), where \( \{\Gamma^n(x), x \in \mathcal{S}_n\} \) is a family of nonnegative \( d \times d \) matrices, and we set \( \Gamma^n_{ii}(x) = -\sum_{j=1, j \neq i}^{d} \Gamma^n_{ij}(x) \) for \( i = 1, \ldots, d \). For general \( K \in \mathbb{N} \) (in which case, we will always assume without loss of generality that \( n \geq K \)), for each \( k \in \{1, \ldots, K\} \), an ordered collection of \( k \) particles among all possible ordered \( k \)-tuples of the \( n \)-particle system can simultaneously change its configuration from \( i = (i_1, \ldots, i_k) \in \mathcal{X}^k \) to \( j = (j_1, \ldots, j_k) \in \mathcal{X}^k \), where \( i_l \neq j_l \), for \( l = 1, \ldots, k \).
Note that it is possible that multiple particles in the \( k \)-tuple may be in the same state. Let
\[
\mathcal{J} \equiv \bigcup_{k=1}^{K} \mathcal{J}^k,
\]
where for \( k = 1, \ldots, K \),
\[
(2.1) \quad \mathcal{J}^k \equiv \{(i,j) \in \mathcal{X}^k \times \mathcal{X}^k : i_l \neq j_l \text{ for } l = 1, \ldots, k\}
\]
is the collection of all possible pairs of initial and final configurations for an ordered \( k \)-tuple of particles. At time \( t \), the rate of a simultaneous transition of a \( k \)-tuple from \( i \in \mathcal{X}^k \) to \( j \in \mathcal{X}^k \) is given by \( \Gamma_{ij}^{k,n} (\mu^n (t)) \), where for each \( (i,j) \in \mathcal{J}^k \), \( \Gamma_{ij}^{k,n} \) is a function from \( \mathcal{S}_n \) to \([0, \infty)\). We also assume that the transition rate is independent of the ordering of the particles: if \( S_k \) denotes the group of permutations on \( \{1, \ldots, k\} \), then
\[
(2.2) \quad \Gamma_{ij}^{k,n} (x) = \Gamma_{i\sigma(j)}^{k,n} (x) , \text{ for any } n \in \mathbb{N}, k = 1, \ldots, K, x \in \mathcal{S}_n \text{ and } \sigma \in S_k.
\]

2.2. Dynamics of the Empirical Measure Process. If the initial configuration \( X^n (0) = (X_1^n (0), \ldots, X_n^n (0)) \) is exchangeable, then it is clear that at any time \( t \), the configuration \( X^n (t) \) of the \( n \)-particle system described above is also exchangeable, and thus essential features of its state at that time can be described by the empirical measure \( \mu^n (t) \). We now identify the generator \( \mathcal{L}_n \) of the empirical measure process \( \{\mu^n (t), t \geq 0\} \), which is an \( \mathcal{S}_n \)-valued càdlàg jump Markov process. Let \( \{e_i, i = 1, \ldots, d\} \) represent the standard basis of \( \mathbb{R}^d \). When \( K = 1 \), the possible jump directions of \( \mu^n (\cdot) \) lie in the set \( \mathcal{V}_1 \), where \( \mathcal{V}_1 \equiv \{e_j - e_i, (i,j) \in \mathcal{J}^1\} \). Moreover, the number of particles in state \( i \) when the empirical measure is equal to \( x \in \mathcal{S}_n \) is \( nx_i \). Hence, the jump rate of \( \mu^n (\cdot) \) in the direction \( \frac{1}{n} (e_j - e_i) \) is \( nx_i \Gamma_{ij}^{1,n} (x) \), and \( \mathcal{L}_n \) takes the form
\[
(2.3) \quad \mathcal{L}_n (f) (x) = n \sum_{(i,j) \in \mathcal{J}^1} x_i \Gamma_{ij}^{1,n} (x) \left[ f \left( x + \frac{1}{n} (e_j - e_i) \right) - f (x) \right]
\]
for any function \( f : \mathcal{S}_n \to \mathbb{R} \).

In the general case of simultaneous transitions with \( K \in \mathbb{N} \), for fixed \( 1 \leq k \leq K \), \( i = \{i_1, \ldots, i_k\} \in \mathcal{X}^k \), \( n \in \mathbb{N} \), \( n \geq K \), and \( x \in \mathcal{S}_n \), define \( A_k (n,i,x) \) to be the number of ordered \( k \)-tuples of particles with configuration \( i = \{i_1, \ldots, i_k\} \) when the empirical measure of the \( n \)-particle system is \( x \). In other words, \( A_k (n,i,x) \) is the number of ordered \( k \)-tuples \( \{r_1, \ldots, r_k\} \subset \{1, \ldots, n\} \) such that \( \{i'_1, \ldots, i'_k\} = \{i_1, \ldots, i_k\} \), so that the \( l \)-th particle in the \( k \)-tuple is in state \( i_l \), for some configuration \( \mathcal{V}' = \{i'_1, \ldots, i'_n\} \in \mathcal{X}^n \) of the \( n \)-particle system whose empirical measure is \( x \): \( \frac{1}{n} \sum_{l=1}^{n} \mathbb{I}_{i'_l = m} = x_m \) for \( m = 1, \ldots, d \). It is easily seen that this quantity depends on \( \mathcal{V}' \) (and hence, \( \mathcal{V} \)) only through the empirical measure \( x \) and takes the form
\[
(2.4) \quad A_k (n,i,x) = n^k \prod_{l=1}^{k} x_{i_l} + O (n^{k-1}),
\]
where the error term is non-zero precisely when the states \( \{i_l\}_{l=1}^{k} \) are not all distinct.

For \( k = 1, \ldots, K \) and \( i = \{i_1, \ldots, i_k\} \in \mathcal{X}^k \), denote \( e_i = \sum_{l=1}^{k} e_{i_l} \). Also, recall that \( \mathcal{J} \equiv \bigcup_{k=1}^{K} \mathcal{J}^k \) with \( \mathcal{J}^k \) defined by (2.1), and set
\[
\mathcal{V} \equiv \{e_j - e_i : (i,j) \in \mathcal{J}\}.
\]
We call \( v = e_j - e_i \) the jump direction associated with the transition \( (i,j) \in \mathcal{J} \). In what follows, \( |B| \) denotes the cardinality of a set \( B \).
Lemma 2.1. The generator of the Markov process $\mu^n(\cdot)$ is given by

\begin{equation}
\mathcal{L}_n(f)(x) = n \sum_{k=1}^{K} \sum_{(i,j) \in J^k} \alpha_{ij}^{k,n}(x) \left[ f \left( x + \frac{1}{n} e_j - \frac{1}{n} e_i \right) - f(x) \right]
\end{equation}

for any function $f : \mathcal{S}_n \mapsto \mathbb{R}$, with

\begin{equation}
\alpha_{ij}^{k,n}(x) \doteq \frac{1}{n(k!)} A_k(n, i, x) \Gamma_{ij}^{k,n}(x), \ x \in \mathcal{S}_n.
\end{equation}

Alternatively, the generator can be rewritten as

\begin{equation}
\mathcal{L}_n(f)(x) = n \sum_{v \in V} \lambda^n_v(x) \left[ f \left( x + \frac{1}{n} v \right) - f(x) \right],
\end{equation}

where

\begin{equation}
\lambda^n_v(x) \doteq \sum_{k=1}^{K} \sum_{e_j = e_i = v} \alpha_{ij}^{k,n}(x),
\end{equation}

with $J^k$ given by (2.4).

Proof. Fix $k \in \{1, \ldots, K\}$ and define an equivalence relation on $J^k$ as follows: for $(i_1, j_1), (i_2, j_2) \in J^k$, $(i_1, j_1) \sim (i_2, j_2)$ if and only if there exists $\sigma \in S_k$ such that $\sigma(i_1) = i_2$, $\sigma(j_1) = j_2$. Let $[i, j]$ denote the equivalence class containing $(i, j)$, let $[J^k]$ denote the collection of equivalence classes, and define $S_k[i, j] = \{\sigma \in S_k : \sigma(i) = i, \sigma(j) = j\}$. Since the particles are assumed indistinguishable, when $(i_1, j_1) \sim (i_2, j_2)$, the jump direction associated with $(i_1, j_1)$ coincides with that associated with $(i_2, j_2)$. Therefore, when the empirical measure of the $n$-particle system is $x \in \mathcal{S}_n$, given $(i, j) \in J^k$, the number of distinguishable ordered $k$-tuple transitions from configuration $i$ to $j$ is equal to $A_k(n, i, x) / |S_k[i, j]|$, where $A_k(n, i, x)$ satisfies (2.4). By the permutation symmetry (2.2), we can set $\Gamma_{ij}^{k,n}(\cdot) = \Gamma_{ij}^{k,n}(\cdot)$, and the generator of the Markov process $\{\mu^n(\cdot)\}$ is given by

\begin{equation}
\mathcal{L}_n(f)(x) = \sum_{k=1}^{K} \sum_{[i, j] \in [J^k]} \frac{A_k(n, i, x)}{|S_k[i, j]|} \Gamma_{ij}^{k,n}(x) \left[ f \left( x + \frac{1}{n} e_j - \frac{1}{n} e_i \right) - f(x) \right], \ x \in \mathcal{S}_n,
\end{equation}

for any function $f : \mathcal{S}_n \mapsto \mathbb{R}$. An alternative way to write the generator (2.9) is as a sum over $J^k$ rather than over $[J^k]$. Using (2.2) and noting that $|[i, j]| = |S_k[i, j]| = k! / |S_k[i, j]|$, we can rewrite (2.9) as in (2.6), since the sum in (2.9) for a given $(i, j)$ corresponds to $|[i, j]|$ summands in (2.6). Finally, (2.7) is a direct consequence of (2.9) and the definition of $\lambda^n_v(\cdot)$ in (2.8).

We will refer to $\lambda^n_v$ as the jump rate (of the empirical measure $\mu^n$) in the direction $v$.

2.3. The Law of Large Numbers Limit. We now describe the functional LLN limit for the sequence of jump Markov processes $\{\mu^n\}_{n \in \mathbb{N}}$ under a suitable assumption on the particle transition rates.

Assumption 2.2. For every $k = 1, \ldots, K$ and $(i, j) \in J^k$, there exists a Lipschitz continuous function $\Gamma_{ij}^k : \mathcal{S} \mapsto \mathbb{R}$ such that for every $x \in \mathcal{S}$ and sequence $x_n \in \mathcal{S}_n$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} x_n = x$,

\begin{equation}
\Gamma_{ij}^k(x) = \lim_{n \to \infty} n^{k-1} \Gamma_{ij}^{k,n}(x_n).
\end{equation}
Note that Assumption 2.2 implies that the transition rates are uniformly bounded:

$$R_0 \equiv \max_{k=1,...,K} \max_{(i,j) \in \mathcal{J}^k} \max_{x \in \mathcal{S}} \lambda_{ij}^k(x) < \infty,$$

and that the associated jump rates \( \{\lambda_v^k, v \in \mathcal{V}\} \) of the empirical measure process \( \mu^n(\cdot) \), given by (2.8), satisfy the following property.

**Property 2.3.** For every \( v \in \mathcal{V} \), there exists a Lipschitz continuous function \( \lambda_v : \mathcal{S} \to [0, \infty) \) such that given any sequence \( x_n \in \mathcal{S}_n, n \in \mathbb{N}, \) such that \( x_n \to x \in \mathcal{S} \) as \( n \to \infty \), \( \lambda_v^n(x_n) \to \lambda_v(x) \).

To see why this is true, for every \( k = 1, \ldots, K \) and \((i,j) \in \mathcal{J}^k\), define \( \alpha_{ij}^k(\cdot) \) by

$$\alpha_{ij}^k(x) \equiv \frac{1}{k!} \left( \prod_{l=1}^{k} x_i \right) \Gamma_{ij}^k(x), \quad x \in \mathcal{S}.$$

If Assumption 2.2 holds, then (2.10) and (2.11) together imply that \( \alpha_{ij}^k(x) = \lim_{n \to \infty} \alpha_{ij}^{k,n}(x) \) for \( x \in \mathcal{S} \). Together with (2.12), this shows that Property 2.3 is satisfied with

$$\lambda_v(x) \equiv \sum_{k=1}^{K} \sum_{\substack{(i,j) \in \mathcal{J}^k: \epsilon_j - \epsilon_i = v}} \alpha_{ij}^k(x), \quad x \in \mathcal{S},$$

for \( v \in \mathcal{V} \).

For future purposes, we also define

$$R_0 \equiv \sup_{v \in \mathcal{V}, x \in \mathcal{S}} \lambda_v(x) < \infty,$$

where \( R \) is finite because \( \mathcal{S} \) is compact and the rates \( \lambda_v(\cdot), v \in \mathcal{V} \), are continuous. Since the jump rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \) satisfy Property 2.3, the LLN limit for \( \{\mu^n\}_{n \in \mathbb{N}} \) follows from a general result due to [24] (see also [25].)

**Theorem 2.4.** Suppose that the sequence \( \{\lambda_v^n, v \in \mathcal{V}\} \) of jump rates associated with the sequence of empirical measure processes \( \{\mu^n(\cdot)\}_{n \in \mathbb{N}} \) satisfies Property 2.3 and let \( \lambda_v, v \in \mathcal{V}, \) be the associated limit jump rates defined in (2.13). Also, assume \( \mu^n(0) \) converges in probability to \( \mu_0 \in \mathcal{P}(\mathcal{X}) \) as \( n \) tends to infinity. Then \( \{\mu^n(\cdot)\}_{n \in \mathbb{N}} \) converges (uniformly on compact time intervals) in probability to \( \mu(\cdot) \), where \( \mu(\cdot) \) is the unique solution to the nonlinear Kolmogorov forward equation

$$\dot{\mu}(t) = \sum_{v \in \mathcal{V}} v \lambda_v(\mu(t)), \quad \mu(0) = \mu_0.$$

In particular, the above assertion holds when the sequence of transition rates \( \{\Gamma_{ij}^{k,n} : (i,j) \in \mathcal{J}^k, k = 1, \ldots, K\}_{n \in \mathbb{N}} \) satisfies Assumption 2.2 and \( \mu^n(0) \) converges in probability to \( \mu_0 \in \mathcal{P}(\mathcal{X}) \) as \( n \) tends to infinity.

Since properties of the LLN trajectory will be used in the large deviation proof, we present an alternative proof of Theorem 2.4 in Section 5.3. In the single jump case \( (K = 1) \), substituting (2.12) and (2.13) into (2.15) and rearranging terms, it is easy to see that the nonlinear ODE describing the LLN limit can be rewritten in the form

$$\dot{\mu}(t) = \mu(t) \Gamma(\mu(t)), \quad \mu(0) = \mu_0.$$
where $\Gamma(\cdot) = \Gamma^1(\cdot)$ is the transition rate matrix $\{\Gamma_{ij}(\cdot), i, j = 1, \ldots, d\}$. We now show that the LLN limit of the empirical measure of an interacting particle system with $K > 1$ can be viewed as the LLN limit of the empirical measure of a corresponding particle system with no simultaneous transitions (i.e., with $K = 1$).

**Remark 2.5.** Given a jump Markov process with generator $\{\tilde{\alpha}_{ij}(\cdot), \tilde{\alpha}_{ij}(\cdot) \geq 0\}$, consider the associated “single transition” interacting particle process, with transition rate matrix

$$
\Gamma^{{\text{eff}}}_{ij}(x) = \sum_{k=1}^{K} \sum_{(i,j) \in \mathcal{J}^k} \sum_{l=1}^{k} \tilde{\alpha}^{k,n}_{ij}(x) x_i \mathbb{I}_{\{i=i_l, j=j_l\}}, \quad x \in \mathcal{S}, (i, j) \in \mathcal{J}^1,
$$

for $i \neq j$ and $\Gamma^{{\text{eff}}}_{ii}(x) = -\sum_{j=1, j \neq i}^{d} \Gamma^{{\text{eff}}}_{ij}(x)$, $n \in \mathbb{N}$. In (2.17), for $i = (i_1, \ldots, i_k)$, if $x_{i_l} = 0$ for some $l \in \{1, \ldots, k\}$, then $\tilde{\alpha}^{k,n}_{ij}(x)/x_{i_l}$ is understood as the pointwise limit of $\alpha^{k,n}_{ij}(y)/y_{i_l}$ when $y$ lies in the relative interior of $\mathcal{S}$ and $y \to x$ in the Euclidean norm; the form of $\tilde{\alpha}^{k,n}_{ij}(\cdot)$ in (2.17) guarantees the existence of this pointwise limit. From Assumption 2.2 and 2.12, it is clear that for each $x \in \mathcal{S}$ and $i, j \in \mathcal{X}, i \neq j$, as $n \to \infty$, $\Gamma^{{\text{eff}}}_{ij}(x)$ converges to

$$
\Gamma^{{\text{eff}}}_{ij}(x) = \sum_{k=1}^{K} \sum_{(i,j) \in \mathcal{J}^k} \sum_{l=1}^{k} \prod_{r=1, r \neq l}^{k} x_i \Gamma^{k}_{ij}(x) \mathbb{I}_{\{i=i_l, j=j_l\}},
$$

where a product over an empty set is to be interpreted as 1. If, as usual, we set $\Gamma^{{\text{eff}}}_{ii} = -\sum_{j \neq i}^{d} \Gamma^{{\text{eff}}}_{ij}$, then it is easy to see that the LLN limit for the simultaneous transitions case, which has the form (2.15) with $\{\lambda_{i}(\cdot), v \in \mathcal{V}\}$ as in (2.13), coincides with the LLN limit for the single transition case in (2.16), but with the matrix $\Gamma$ replaced by the matrix $\Gamma^{{\text{eff}}}$. The superscript “eff” in (2.17) and (2.18) stands for “effective”, and is used to indicate that the $n$-particle system with simultaneous transitions and the corresponding single-transition particle system have the same LLN limit. However, it is important to note that the two systems have different dynamics and large deviation behavior (for instance, see Example 3.1.26 of [35]).

### 2.4. Examples

The particle systems that we describe naturally occur in a wide range of areas, including statistical mechanics (Curie-Weiss model), graphical models and algorithms, networks and queueing systems (rerouting, loss networks). We present two illustrative examples below.

**Example 2.6.** *The opinion dynamics or Curie-Weiss model [70].* This is a mean field model on a complete graph. As before, let $n$ be the number of particles or individuals and let $X^{i,n}(t) \in \mathcal{X} = \{-1, 1\}$ denote the opinion of the $i$th individual at time $t$, and let $\beta > 0$ be a parameter that measures the proclivity of an individual to change opinion. (Note that $d = |\mathcal{X}| = 2$, but we write $\mathcal{X} = \{-1, 1\}$ instead of $\mathcal{X} = \{1, 2\}$.) At time 0, each individual adopts an opinion $X^{i,n}(0)$ in $\mathcal{X}$ independently and uniformly at random. Each individual has an independent and identically distributed (iid) Poisson clock of rate 1. If the clock of individual $i$ rings at time $t$, he/she computes the opinion imbalance $M^{(i)} = \sum_{j \neq i}^{d} X^{j,n}(t^-)$, and changes opinion with probability

$$
P_{\text{flip}}(X^{i,n}(t)) = \begin{cases} 
\exp\left(-2\beta \left|M^{(i)}(t^-)\right|/n\right) & \text{if } M^{(i)}(t^-)X^{i,n}(t^-) > 0, \\
1 & \text{otherwise}.
\end{cases}
$$

The empirical measure process $\mu^n$ only takes jumps of the form $\mathcal{V} = \{e_j - e_i, i, j \in \{-1, 1\}\}$, with points in $\mathcal{S}$ denoted by $(x_{-1}, x_1)$. The particle transition rates satisfy Assumption 2.2 with $\Gamma = \Gamma^1$.
taking the form

\[
\begin{align*}
\Gamma_{1,-1}(x) &= \begin{cases} 
\exp(-2\beta(x_1-x_{-1})) & \text{if } x_1-x_{-1} > 0, \\
1 & \text{otherwise},
\end{cases} \\
\Gamma_{-1,1}(x) &= \begin{cases} 
\exp(-2\beta(x_{-1}-x_1)) & \text{if } x_{-1}-x_1 < 0, \\
1 & \text{otherwise},
\end{cases}
\end{align*}
\]

and, as usual, \( \Gamma_{1,1}(x) = -\Gamma_{1,-1}(x) \) and \( \Gamma_{-1,-1}(x) = -\Gamma_{-1,1}(x) \). A generalization of this example is the Curie-Weiss-Potts model with Glauber dynamics, the mixing time of which has interesting phase transition properties (see [20]).

Interacting particle systems with simultaneous transitions arise naturally as models of communication networks. We now provide one such example, from [17]. More examples can be found in [33], [27] and [20].

**Example 2.7.** Alternative rerouting networks [17]. Consider a network that consists of \( n \) links, each with finite capacity \( C \). Let \( \mathcal{X} = \{0, \ldots, C\} \) and let \( \mathcal{X}^{i,n}(t) \) denote the number of packets (or customers) using link \( i \) at time \( t \). Packets arrive at each link as a Poisson process with rate \( \gamma > 0 \). If a packet arrives at a link with a unit of spare capacity, then it is accepted to the link and occupies one unit of capacity for an exponentially distributed time with mean one. On the other hand, if a packet arrives at a link that is fully occupied, two other links are chosen uniformly at random from amongst the remaining \( n-1 \) links. If both chosen links have a unit of spare capacity available, the packet occupies one unit of capacity on each of the two links, for two independent, exponential clocks with mean one. Otherwise, the packet is lost. This model seeks to understand the impact of allowing alternative routes that occupy a greater number of resources on the performance of the network.

The empirical measure process \( \mu^n \) is a jump Markov process with jump rates summarized as follows: for any \( i, j \in \mathcal{X} \):

\[
\mu^n \rightarrow \begin{cases} 
\mu^n + \frac{1}{n} (e_{i+1} - e_i) & \text{at rate } n\gamma \mu^n_1, \quad 0 \leq i \leq C-1, \\
\mu^n + \frac{1}{n} (e_{i-1} - e_i) & \text{at rate } n\gamma \mu^n_1, \quad 1 \leq i \leq C, \\
\mu^n + \frac{1}{n} (e_{i+1} - e_i) + e_{j+1} - e_j & \text{at rate } 2\gamma \mu^n_1 \mu^n_1 \mu^n_1, \quad 0 \leq i \neq j \leq C-1, \\
\mu^n + \frac{2}{n} (e_{i+1} - e_i) & \text{at rate } 2\gamma \mu^n_1 \mu^n_1 \mu^n_1, \quad 0 \leq i \leq C-1.
\end{cases}
\]

The transition rates of the particle system satisfy Assumption 2.2 with \( K = 2 \), and

\[
\Gamma_{i,i+1}(x) = \gamma, \quad \Gamma_{i,i-1}(x) = 1, \quad \Gamma_{i,j}(i+1,j+1)(x) = \gamma x_c,
\]

and \( \Gamma_{i,j}^k = 0 \) for all other transitions \((i,j) \in \mathcal{J}^k, k=1,2 \). By (2.13), we can calculate the effective transition rate as

\[
\Gamma_{i,j}^{\text{eff}}(x) = \Gamma_{i,j}^1(x) + \sum_{\substack{i' \neq i, j' \neq j \\text{or } i' \neq j}} 2x_i \Gamma_{(i,i')(j,j')}^2(x) + x_i \Gamma_{(i,i')(j,j)}^2(x),
\]

which gives \( \Gamma_{i,i+1}^{\text{eff}}(x) = \gamma + 4 \sum_{i' \neq i} x_{i'} \lambda x_c + 2x_i \gamma x_c = \gamma (1 + 2 \lambda_c (2 - x_i)) \), \( \Gamma_{i,i-1}^{\text{eff}}(x) = \gamma \), and \( \Gamma_{i,j}^{\text{eff}}(x) = 0 \) for all other \((i,j) \in \mathcal{J}^1 \).
3. Main Results

Throughout the rest of the paper, we always assume, without explicit mention, that the transition rates associated with the sequence of $n$-particle systems satisfy the symmetry condition (2.2). We also assume that they satisfy Assumption 2.2 with associated limit transition rates $\{\Gamma^k_{ij}(\cdot), (i,j) \in J^k, k = 1, \ldots, K\}$. Then, as follows from Theorem 2.4, the corresponding sequence of empirical measure processes $\{\mu^n\}_{n \in \mathbb{N}}$ has a LLN limit $\mu(\cdot)$ whose evolution is governed by the limit jump rates $\{\lambda_v, v \in \mathcal{V}\}$, defined in (2.13). In practice one is often interested in estimating the tail probabilities $\mathbb{P}(\mu^n(\cdot) \in A)$ for certain sets of paths $A$ that do not contain the LLN limit. This can be studied in the framework of an LDP. First, in Section 3.1, we introduce additional assumptions on the limit transition rate functions, and then in Section 3.2, we state the sample path large deviation principle for the sequence $\{\mu^n\}_{n \in \mathbb{N}}$. Asymptotics of the tail probabilities at a given time $t$ will follow from the contraction principle. In Section 3.3, we introduce an additional condition that allows us to establish a locally uniform refinement to the LDP and in Section 3.4, we discuss the LDP for the associated sequence of invariant measures. As a by-product of our proof technique, we in fact establish these large deviation results for a larger class of sequences $\{\mu^n\}_{n \in \mathbb{N}}$ of jump Markov processes on the simplex. A precise statement of this more general result is given in Remark 8.6. For simplicity we assume from now on that $t \in [0,1]$, while all results in this paper can be established for $t$ in any compact time interval by the same argument.

3.1. Assumptions on the Limit Transition Rates. Below, we introduce three additional assumptions on the limit transition rates of the interacting particle system: a uniformity condition (Assumption 3.1), a type of ergodicity (Assumption 3.3), and a mild restriction on the type of simultaneous jumps allowed (Assumption 3.8). For $k \in \{1, \ldots, K\}$, denote

\begin{equation}
\mathfrak{M}^k_{ij} \doteq \inf_{x \in S} \Gamma^k_{ij}(x), \quad \text{for } (i,j) \in J^k,
\end{equation}

and let the set

\begin{equation}
J^k_+ \doteq \{(i,j) \in J^k : \mathfrak{M}^k_{ij} > 0\}
\end{equation}

denote the set of $k$-tuple transitions whose transition rates are uniformly bounded away from zero. Also, set

\begin{equation}
c_0 \doteq \min_{k=1,\ldots,K} \{\mathfrak{M}^k_{ij} : (i,j) \in J^k_+\}.
\end{equation}

The first assumption states that each transition rate function is either identically zero, or uniformly bounded below from zero on the simplex.

Assumption 3.1. For $k = 1, \ldots, K$ and $(i,j) \in J^k$, either $(i,j) \in J^k_+$ (equivalently, $\mathfrak{M}^k_{ij} > 0$) or $\Gamma^k_{ij}(x) = 0$ for every $x \in S$.

Note that, nevertheless, the limit jump rates $\lambda_v(\cdot), v \in \mathcal{V}$, of the associated sequence of empirical measure processes will not be bounded away from zero on the simplex. More precisely, for $v \in \mathcal{V}$, let $\mathcal{N}_v$ be the set of coordinates of $v$ that are strictly negative:

\begin{equation}
\mathcal{N}_v \doteq \{i \in \mathcal{X} : (v, e_i) < 0\}.
\end{equation}

Note that for every $v \in \mathcal{V}$, $v \neq 0$, the fact that $\sum_{i \in \mathcal{X}} v_i = 0$ implies $\mathcal{N}_v \neq \emptyset$. Now, we claim (and justify below) that $\lambda_v(x) \to 0$ whenever $x_i \to 0$ for any $i \in \mathcal{N}_v$. Indeed, the claim can be deduced from the form of $\lambda_v(\cdot)$ in (2.13), the fact that for any $k = 1, \ldots, K$ and $(i,j) \in J^k$, we have

\begin{equation}
e_j - e_i = v \quad \Rightarrow \quad \text{for every } i \in \mathcal{N}_v, \{\{l = 1, \ldots, k : i_l = i\} \geq |\langle v, e_i \rangle| \geq 1,
\end{equation}
and the property that \( a_l^k (x) \to 0 \) if \( x_l \to 0 \) for some \( l = 1, \ldots, k \), where the latter assertion follows from (2.12) and the uniform boundedness of \( \Gamma^k_{ij} \) on \( \mathcal{S} \), which is a consequence of the continuity of \( \Gamma^k_{ij} \) specified in Assumption 2.2.

Next, we impose a type of ergodicity property on the transition rates specified below.

**Definition 3.2.** For two states \( u, w \in \mathcal{X} \), \( w \) is said to be \( K \)-accessible from \( u \) if there exist \( M \in \{2, \ldots, d\} \) and a sequence of distinct states in \( \mathcal{X} \): \( u = u_1, u_2, \ldots, u_M = w \), such that for \( m = 1, \ldots, M - 1 \), the following three properties hold:

(i) there exist \( k_m \in \{1, \ldots, K\} \), \( (i_m, j_m) \in \mathcal{J}_{km}^\pm \), and \( l_m, l'_m \in \{1, \ldots, k_m\} \), such that \( u_m = i_m, l_m \) and \( u_{m+1} = j_m, l'_m \);

(ii) for \( l = 1, \ldots, k_m \), \( i_m, l \in \{u_1, \ldots, u_m\} \);

(iii) \( M_{m, i_m j_m}^{k_m} > 0 \), i.e., \( (i_m, j_m) \in \mathcal{J}_{km}^\pm \).

We say the family \( \{\Gamma^k_{ij} (\cdot), (i, j) \in \mathcal{J}, k = 1, \ldots, K\} \) is \( K \)-ergodic if for any \( u, w \in \mathcal{X} \), \( w \) is \( K \)-accessible from \( u \).

The \( K \)-ergodicity condition, roughly speaking, requires that one can reach any state \( w \in \mathcal{X} \) from any state \( u \in \mathcal{X} \) via a finite sequence of states, where each adjacent pair of states represents a state transition that can be effected by a simultaneous \( k \)-tuple transition with a strictly positive rate. Note that in general, the adjacent pair need not represent initial and final states of any one particle involved in the \( m \)th simultaneous transition; the latter is true only when \( l_m = l'_m \) in Definition 3.2(i), which in particular always holds when \( K = 1 \). Instead, the first state in the pair could be the initial state of one particle and the other state could be the final state of another particle involved in the simultaneous transition. However, as stipulated in property ii) above, \( K \)-ergodicity also requires that the initial states of all particles involved in the \( m \)th (simultaneous) transition must be a subset of the previous states \( u_1, \ldots, u_m \) in the sequence. The latter property, which is trivially satisfied when \( K = 1 \), ensures that at the \( m \)th stage “mass” is moved exclusively from the subset of states \( \{u_1, \ldots, u_m\} \) to \( u_{m+1} \), which helps in the construction of so-called communicating paths for the associated empirical measure process between different states on the simplex (see Definition 4.1 and Proposition 4.7).

**Assumption 3.3.** The family \( \{\Gamma^k_{ij} (\cdot), (i, j) \in \mathcal{J}, k = 1, \ldots, K\} \) is \( K \)-ergodic.

To provide further insight into the \( K \)-ergodicity property, we now state a simpler, and perhaps more intuitive, condition that (in the presence of Assumption 3.1) implies \( K \)-ergodicity. Recall that \( \Gamma^\text{eff} \) is the effective transition rate matrix introduced in (2.18).

**Assumption 3.4.** For every \( x \in \mathcal{S} \), the Markov process on \( \mathcal{X} \) with transition rate matrix \( \Gamma^\text{eff} (x) \) is ergodic.

**Lemma 3.5.** If the family \( \{\Gamma^k_{ij} (\cdot), (i, j) \in \mathcal{J}, k = 1, \ldots, K\} \) satisfies Assumptions 3.3 and 3.4 then it also satisfies Assumption 3.5 that is, it is \( K \)-ergodic.

**Proof.** Take any \( u, w \in \mathcal{X} \), \( u \neq w \). Since \( \Gamma^\text{eff} (e_u) \) is ergodic by Assumption 3.3 there exist \( M \in \{2, \ldots, d\} \) and a sequence of distinct states \( u = u_1, u_2, \ldots, u_M = w \) such that \( \Gamma^\text{eff} (e_{u_{m+1}} (e_u)) > 0 \), \( m = 1, \ldots, M - 1 \). By the definition of \( \Gamma^\text{eff} \) given in (2.18), this implies that for \( m = 1, \ldots, M - 1 \), there exist \( k_m \in \{1, \ldots, K\} \), \( (i_m, j_m) \in \mathcal{J}_{km}^\pm \) and \( l_m \in \{1, \ldots, k_m\} \) such that \( u_m = i_m, l_m \) and \( u_{m+1} = j_m, l_m \) and

\[
\prod_{m=1}^{k_m} \langle e_u, e_{i_m, r} \rangle \Gamma^k_{i_m j_m, e_u} > 0.
\]
Thus, $\Gamma$ of Example 3.7, first consider the case and Assumption 3.4 fails to hold. Therefore, Definition 3.2(ii) is satisfied with $\Gamma = \Gamma_1$ in (2.16). Thus, when $K = 1$ and Assumption 3.1 is satisfied, $K$-ergodicity, Assumption 3.3 and Assumption 3.1 below (which requires that $\Gamma^1(x)$ be ergodic for every $x \in S$) are all equivalent.

However, as the following example illustrates, when $K > 1$, $K$-ergodicity is strictly weaker than Assumption 3.4.

**Example 3.7.** Let $d = 4$, $K = 2$, and define the generator of the Markov process $\{\mu^n\}_{n \in \mathbb{N}}$ as in (2.3) with $a_{ij}^{k,n}$ defined as in (2.0), in terms of $\Gamma_1^{1,n}$ and $\Gamma_2^{2,n}$ given by

$$
\begin{align*}
\Gamma_1^{1,n}(x) &= c_1, \\
\Gamma_2^{2,n}(x) &= c_4,
\end{align*}
$$

with $c_i > 0$, $i = 1, \ldots, 6$, and $\Gamma_{ij}^{k,n} = 0$ for all other $(i,j) \in J^k$, $k = 1, 2$, $n \in \mathbb{N}$. Note that Assumption 2.2 trivially holds with $\Gamma_1^{1} = \Gamma_{ij}^{k,n}$ for $(i,j) \in J^k$, $k = 1, 2$, and Assumption 3.1 is also satisfied. Also, the associated limit jump rates $\{\lambda_v, v \in V\}$ defined in (2.13) take the form

$$
\begin{align*}
\lambda_{e_3-e_1}(x) &= x_1 c_1, \\
\lambda_{e_4-e_3}(x) &= x_3 c_3, \\
\lambda_{e_3+e_4-e_1-e_2}(x) &= x_1 x_2 c_5/2, \\
\lambda_{e_1-e_2}(x) &= x_2 c_2, \\
\lambda_{e_3-e_4}(x) &= x_4 c_4, \\
\lambda_{e_1+e_2-e_3-e_4}(x) &= x_3 x_4 c_6/2.
\end{align*}
$$

Furthermore, the effective transition rate matrix $\Gamma^{\text{eff}}$ defined in (2.15) takes the form

$$
\begin{align*}
\Gamma^{\text{eff}}(x) &= c_1, \\
\Gamma^{\text{eff}}(x) &= c_2, \\
\Gamma^{\text{eff}}(x) &= c_3, \\
\Gamma^{\text{eff}}(x) &= c_4.
\end{align*}
$$

Thus, $\Gamma^{\text{eff}}$ is not ergodic on the part of the boundary given by $\{x \in S : x_3 = x_4 = 0 \text{ or } x_1 = x_2 = 0\}$, and Assumption 3.4 fails to hold.

We now show that nevertheless, this particle system is 2-ergodic. To verify the 2-ergodicity of Example 3.7 first consider the case $u = 1$ and $w \in \{2, 3, 4\}$. If $w = 2$, then we can take $M = 2$, $k_1 = 1$ and $(i_1,j_1) = (1, 2)$. If $w = 3$, one might be tempted to set $M = 2$ again and use the simultaneous jump $(1, 2) \rightarrow (3, 4)$. However, this would violate property (ii) of Definition 3.2. Instead, we take $M = 3$, $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $k_1 = 1$, $(i_1,j_1) = (1, 2)$, $k_2 = 2$ and $(i_2,j_2) = ((1, 2), (3, 4))$. The case $w = 4$ is similar to the case $w = 3$, except that $u_3 = 4$. It is easy to check in each case that the sequence of states $\{u_m, m = 1, \ldots, M\}$ satisfy conditions i), ii), iii) of Definition 3.2. The symmetry of the problem allows one to deal with the case $u \in \{2, 3, 4\}$ in an analogous fashion (we omit the details) to show that the example is 2-ergodic.

As explained in the introduction and shown in Section 5 by applying a general result for jump-diffusion Markov processes that was obtained in [22], it is possible to establish a large deviation upper bound for jump Markov processes with generator (2.7) in the form of the integral of a so-called “local rate function”; see (3.8) and (3.7) below. The more delicate part of the sample path LDP is the proof of the large deviation lower bound. Since each jump rate $\lambda^\mu_v(\cdot)$ of $\mu^n$ tends to zero as $n$ approaches some part of the boundary of $S$, the local rate function can approach infinity, which makes the analysis difficult. The third assumption we require is a mild technical restriction
on the type of simultaneous particle transitions that are allowed, which allows us to overcome this difficulty. This assumption is used only to show that the LLN trajectory moves into the (relative) interior of the simplex sufficiently quickly (see Property 4.3 for a precise statement). However, as elaborated in Remark 8.3, our proof applies to the broader class of systems for which the LLN trajectory still possesses this property, even if Assumption 8.3 may fail to hold. Assumption 8.3 simply serves to identify a large class of systems for which the LLN trajectory can be a priori verified. We recall that \( \mathcal{J}_+ \) was defined in (4.2) and let \( J_+ \equiv \cup_{k=1}^{K} J^k_+ \).

**Assumption 3.8.** For every \( v \in \mathcal{V}\{0\} \) such that \( \lambda_v \) is not identically zero, at least one of the following two properties is true:

1. There exists \((i^*, j^*) \in \mathcal{J}_+ \) such that \( e_j^* - e_i^* = v \) and
   \[
   |\{l : i_l = j\}| = \begin{cases} |\{v, e_j\}| & \text{if } j \in \mathcal{N}_v, \\ 0 & \text{otherwise,} \end{cases}
   \]
   where we recall that \( \mathcal{N}_v = \{i \in \mathcal{X} : v_i < 0\} \),

2. There exist \( r_j \geq 1, j \in \mathcal{N}_v \), such that given any \((i, j) \in \mathcal{J}_+ \), we have \((i, j) \in \mathcal{J}_+^{k^*} \) where
   \[
   k^* = \sum_{j \in \mathcal{N}_v} r_j, \quad \text{and}
   \]
   \[
   r_j = |\{l = 1, \ldots, k^* : i_l = j\}|, \quad j \in \mathcal{N}_v.
   \]

To better understand what this assumption says, consider a particle system with \( d = 4 \) and suppose \( v = 2e_1 + e_3 - 2e_2 - e_4 \). There are many transitions \((i, j)\) that could lead to the jump direction \( v \), including, for example, (a) \((2, 2, 4) \mapsto (1, 1, 3)\); (b) \((2, 2, 4) \mapsto (1, 3, 1)\); (c) \((2, 2, 4, 4) \mapsto (4, 1, 3, 1)\); (d) \((2, 2, 4, 4) \mapsto (4, 3, 1, 1)\); (e) \((2, 2, 4, 1) \mapsto (1, 1, 1, 3)\). Here \( \mathcal{N}_v = \{2, 4\} \). First consider Assumption 3.8(1). For \( j = 2 \), the number of particles jumping from type 2 should be \(|\{v, e_2\}| = 2 \). Similarly, the number of particles of type 4 before the jump should be 1. In particular, transitions (a) or (b) above would meet this requirement, and a system in which all the above transitions [and their permuted versions, by virtue of (2.2)] have strictly positive rates would also satisfy the assumption. In contrast, a system in which the only transitions associated with \( v \) that have positive rate are of type (c) or (d) would not satisfy Assumption 3.8(1) because for such transitions the first inequality in (5.3) is strict for \( j = 4 \), thus violating the stipulated condition. However, this system would satisfy Assumption 3.8(2) since the vector of initial particle values for both these transitions have the same "type", namely containing a pair of 2’s and a pair of 4’s, whereas a system that has both transitions (a) and (c) would not satisfy the second condition in Assumption 3.8[although, as mentioned above, it would satisfy Assumption 3.8(1)]. An example of a particle system that would violate both conditions in Assumption 3.8 is one in which only transitions of type (e) have positive rate. In this case, \(|\{l : i_l = j\}|\) is non-zero for \( j = 1 \), which does not lie in \( \mathcal{N}_v = \{2, 4\} \) and, moreover, \((i, j) \in \mathcal{J}_+^{k^*}, \) but \( \sum_{i \in \mathcal{N}_v} |\{l : i_l = j\}| = 3 \). The technical problem with such a system is that one could have \( \lambda_v(x) \to 0 \) as one approaches parts of the boundary where \( x_i = 0 \) even though \( v_i > 0 \). In other words, the jump rates in a certain direction \( v \) could diminish to zero at certain points on the boundary when jumps in the direction \( v \) from such points would take the empirical measure back to the interior of the domain. The difficulty is that we rely on such jumps to move the LLN limit to the interior of the simplex quickly.

### 3.2. Large Deviation Principles

We now state our first large deviation result, which is the sample path LDP. To define the rate function, we need some notation. For \( x \in \mathcal{S} \), let

\[
\ell(x) \equiv \begin{cases} x \log x - x + 1 & x \geq 0, \\
\infty & x < 0, \end{cases}
\]

where we recall that \( \mathcal{S} \) was defined in (3.6) and let \( \mathcal{S} = \cup_{k=1}^{K} \mathcal{S}^k \).
be the local rate function associated with the standard Poisson process. Let \( \Delta^{d-1} = \{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0 \} \). Then for \( x \in \mathcal{S} \) and \( \beta \in \Delta^{d-1} \), we define

\[
L(x, \beta) = \inf_{\sum_{v \in V} q_{v} = \beta, v \in V} \sum_{v \in V} \lambda_{v}(x) \ell \left( \frac{q_{v}}{\lambda_{v}(x)} \right).
\]

For \( t \in [0,1] \) and an absolutely continuous function \( \gamma : [0,t] \to \mathcal{S} \), define

\[
I_t(\gamma) = \int_0^t L(\gamma(s), \dot{\gamma}(s)) \, ds,
\]

and in all the other cases, set \( I_t(\gamma) = \infty \). We write \( I(\gamma) \) to denote \( I_1(\gamma) \).

In what follows we equip \( D([0,1] : \mathcal{S}) \) with the Skorokhod \( J_1 \)-topology, and let \( B(D([0,1] : \mathcal{S})) \) be the associated Borel sets.

**Theorem 3.9.** Suppose the family \( \{ \Gamma^k_{ij}(x), x \in \mathcal{S}, (i,j) \in \mathcal{J}^k, k = 1, \ldots, K \} \) satisfies Assumptions 2.2, 3.1, 3.3 and 3.8. Also, assume that the initial conditions \( \{ \mu^n(0) \}_{n \in \mathbb{N}} \) are deterministic and satisfy \( \mu^n(0) \to \mu_0 \in \mathcal{P}(\mathcal{X}) \) as \( n \) tends to infinity. Then the associated sequence of empirical measure processes \( \{ \mu^n \}_{n \in \mathbb{N}} \) satisfies the sample path LDP with rate function \( I \). Specifically, for any measurable set \( A \in B(D([0,1] : \mathcal{S})) \), we have the large deviation upper bound

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu^n \in A) \leq - \inf \{ I(\gamma) : \gamma \in \mathcal{A}, \gamma(0) = \mu_0 \},
\]

and the large deviation lower bound

\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu^n \in A) \geq - \inf \{ I(\gamma) : \gamma \in \mathcal{A}^c, \gamma(0) = \mu_0 \}.
\]

Moreover, for any compact set \( K \subset \mathcal{S} \) and \( M < \infty \), the set

\[
\{ \gamma \in D([0,1] : \mathcal{S}) : I(\gamma) \leq M, \gamma(0) \in K \}
\]

is compact.

The proof of the upper bound (3.9) and the compactness of the set in (3.11), which only uses Property 2.3 (which is implied by Assumption 2.2) is given at the end of Section 6. The proof of the lower bound (3.10) is given at the end of Section 8.

Theorem 3.9 together with an application of the contraction principle (see, e.g., [34]), yields the following variational representation for the rate function of \( \{ \mu^n(t) \}_{n \in \mathbb{N}} \) for any \( t \in [0,1] \).

**Corollary 3.10.** Suppose the conditions of Theorem 3.9 hold. Then for each \( t \in [0,1] \), the sequence of random variables \( \{ \mu^n(t) \}_{n \in \mathbb{N}} \) satisfies an LDP with rate function

\[
J_t(\mu_0, x) = \inf \{ I_t(\gamma) : \gamma \in D([0,1] : \mathcal{S}), \gamma(0) = \mu_0, \gamma(t) = x \}.
\]

### 3.3. A locally uniform refinement

In applications, it is often useful to estimate the probability that \( \mu^n \) hits a specific point \( x_n \in \mathcal{S} \) at some given time, where \( x_n \to x \in \mathcal{S} \) as \( n \to \infty \). The ordinary LDP does not imply an asymptotic rate for this hitting probability since it applies only to fixed sets, and the “moving” set \( \{ x_n \} \) in the present case has empty interior. To obtain such a “locally uniform” result we need a strengthening of the \( K \)-ergodicity condition. Recall the single-transition rate matrix \( \Gamma^1 \).

**Assumption 3.11.** For every \( x \in \mathcal{S} \), the Markov process on \( \mathcal{X} \) with transition rate matrix \( \Gamma^1(x) \) is ergodic.
Note that Assumption 3.11 implies Assumption 3.4 and, thus, is stronger than Assumption 3.4, which in itself (in the presence of Assumption 3.1) is a strengthening of K-ergodicity (see Lemma 3.9). We now state the locally uniform LDP result, which is proved in Section 9.

**Theorem 3.12.** Suppose \( \{ \Gamma_i^k(x), x \in S, (i, j) \in J^k, k = 1, \ldots, K \} \) satisfies Assumptions 2.2, 3.1, 3.8, and 3.14, and let \( \{ \mu^n \}_{n \in \mathbb{N}} \) be the associated sequence of empirical measure processes. Also, assume the initial conditions \( \{ \mu^n(0) \}_{n \in \mathbb{N}} \) are deterministic, and \( \mu^n(0) \to \mu_0 \in \mathcal{P}(X) \) as \( n \) tends to infinity. Let \( \{ x_n \}_{n \in \mathbb{N}} \subset S_n, x \in S \), be such that \( x_n \to x \) as \( n \to \infty \). Then for any \( t \in [0, 1) \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} (\mu^n(t) = x_n) = -J_t (\mu_0, x),
\]

where \( J_t \) is as defined in (3.13).

For the \( n \)-particle systems we study, it is also natural to start with random initial conditions. Depending on the large deviation rate of the sequence of initial conditions, this gives rise to an additional cost in the rate function. The LDP for empirical measure processes with random initial conditions are stated in the following corollary.

**Corollary 3.13.** Suppose that Assumptions 2.2, 3.1, 3.8, and 3.14 are satisfied. Also, assume that the sequence of initial conditions \( \{ \mu^n(0) \}_{n \in \mathbb{N}} \) converges to \( \mu_0 \) in such a way that they satisfy an LDP with rate function \( J_0 (\cdot) \). Then the corresponding sequence of empirical measure processes \( \{ \mu^n \}_{n \in \mathbb{N}} \) satisfies the sample path LDP with rate function \( J_0 (\gamma (0)) + I (\gamma) \).

The proof of the corollary relies on the continuity of the following functional: given a bounded and continuous functional \( h \) on \( D ([0, 1] : S) \), define

\[
U(y) = \inf \{ I (\gamma) + h (\gamma) : \gamma \in D ([0, 1] : S), \gamma (0) = y \}, \quad y \in S.
\]

Then it follows from Lemma 3.2 that \( U \) is continuous.

**Proof of Corollary 3.13.** Given a bounded and continuous function \( h : D ([0, 1] : S) \to \mathbb{R} \), for any \( y \in S_n \) denote \( U^n(y) = -\frac{1}{n} \log \mathbb{E}_y [e^{-nh(\mu^n)}] \). Since \( U \) is continuous and \( \{ \mu^n \}_{n \in \mathbb{N}} \) satisfies an LDP (Theorem 3.12), the equivalence between the LDP and the Laplace principle \([11, \text{Theorems } 1.2.1 \text{ and } 1.2.3]\) implies that \( U^n \) converges to \( U \) uniformly on \( S \). In particular, this shows that if \( y_n \to y \) in \( S \), then \( U^n(y_n) \to U(y) \). Let \( \nu^n \) denote the law of \( \mu^n(0) \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mu^n(0)} \left[ e^{-nh(\mu^n)} \right] = \lim_{n \to \infty} -\frac{1}{n} \log \sum_{y_n \in S_n} e^{-nU^n(y_n)} \nu^n \{ y_n \} = \lim_{n \to \infty} -\frac{1}{n} \log \int e^{-n(U(y)+o(1))} \nu^n (dy) = \inf_{y \in S} \{ U(y) + J_0 (y) \} \leq \inf_{\gamma \in D([0, 1]:S)} \{ h (\gamma) + J_0 (\gamma (0)) + I (\gamma) \},
\]

where the third equality follows from the assumed LDP for deterministic initial conditions and the continuity of \( U \), and the fourth equality follows from the definition of \( U \) in (3.13). The conclusion of the corollary then follows from the equivalence between the LDP and the Laplace principle. \( \square \)

**Remark 3.14.** An example of initial conditions in the \( n \)-particle system that satisfy the assumptions of Corollary 3.13 is the case when particles are initially distributed as iid \( X \)-valued random variables, with common distribution \( \nu \). Then by Sanov’s theorem, \( J_0 (\mu_0) = R (\mu_0 || \nu) = \sum_{i=1}^d \mu_{0,i} \log \frac{\mu_{0,i}}{\nu_i} \).
Remark 3.15. The assumptions of the locally uniform case are used in the proof of Lemma 9.2 to establish that $U$ is continuous on $S$. Any set of conditions implying this continuity can also be used, and under the conditions of Theorem 5.9, $U$ is continuous on the interior of $S$, and hence the corollary holds if the distributions of initial conditions have support in a compact subset of the relative interior of $S$.

3.4. LDP for Invariant Measures. We now discuss some ramifications of the locally uniform LDP. In [16] a uniform (with respect to initial conditions) sample path LDP for small noise diffusions is used to study its metastability properties, including the mean exit time and most likely exit location from a given domain, and to establish an LDP for the sequence of invariant measures with the rate function given by the so-called quasipotential. The program of [16] was carried out for non-degenerate diffusions in $\mathbb{R}^d$; here we have a sequence of jump processes on lattice approximations of a compact set. However, we remark here that the same arguments carry through without essential change in the presence of a certain communication property, namely Property 4.16.i) in Section 4.6, which is shown to be implied by Assumptions 2.2, 3.1, 3.11 and 3.8 in Lemma 4.6 (see also [4] for details in the case of empirical measures arising from single-jump interacting particle systems, that is, systems with $K = 1$). In [16] extra conditions are assumed to guarantee that the process does not escape to infinity with significant probability; for our model, since the state space is compact, this is automatic.

When Assumption 3.11 is satisfied, for each $n \in \mathbb{N}$, all states in $S_n$ communicate under the dynamics of $\mu_n$, and hence there exists a unique invariant measure $\pi_n$ for this Markov process. In our setting, the quasipotential is defined by

$$V(x, y) = \inf \{I_t(\gamma) : \gamma \in D([0, t] : S), \gamma(0) = x, \gamma(t) = y, t < \infty\}$$ for $x, y \in S$.

For the results of [16] to carry over to our setting, we need the quasipotential to be continuous on its domain.

Lemma 3.16. If Assumption 2.2 and Assumption 3.11 hold, then $V(\cdot, \cdot)$ is jointly continuous in $S \times S$.

The proof of Lemma 3.16 is given in Section 9.1; it essentially follows from the property that for any $x, y \in S$ that are sufficiently close, one can construct a path that connects $x$ to $y$ with arbitrary small cost (see Lemma 9.1 for a precise statement). We now state the LDP for invariant measures in the case when the LLN limit $\mu(\cdot)$ has a unique fixed point. In view of (2.15), a fixed point of the LLN dynamics is a measure $\pi^* \in S$ with the property that $\sum_{v \in V} v\lambda_v(\pi^*) = 0$. Moreover, the fixed point is said to be globally attracting if for every $x_0 \in S$, the solution $\mu(\cdot)$ to (2.15) with initial condition $x_0$ satisfies $\mu(t) \rightarrow \pi^*$ as $t \rightarrow \infty$.

Theorem 3.17. Assume that $x_0$ is the unique fixed point of the LLN dynamics (2.13), and is globally attracting (in $S$). Also assume Assumptions 2.2, 3.1, 3.8 and 3.11 are satisfied. Then for any $n > 0$, there exists a unique invariant measure $\pi^n$ of the Markov process with generator (2.7). Moreover, the sequence $\{\pi^n\}_{n \in \mathbb{N}}$ satisfies an LDP with rate function $V(x_0, \cdot)$.

When the LLN limit (2.13) has multiple stable equilibria, following the same approach as in the case of non-degenerate diffusions (see [16], Chapter 6.4), a generalization of Theorem 3.17 can be obtained.
4. Properties of the Limit Jump Rates

In this section we establish certain important properties of the limit jump rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \) associated with interacting particle systems whose transition rates satisfy the assumptions introduced in the last section. First, in Section 4.1 we describe certain communication conditions (Property 4.3) that are required to avoid singularities in the large deviation analysis. In Sections 4.2 and 4.3 we show that these communication conditions are satisfied by the limit jump rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \) associated with any interacting particle system model that satisfies Assumptions 2.2, 3.1 and 3.3. Next, in Section 4.4 we show that Assumptions 2.2, 3.1 and 3.3 together imply certain estimates (Lemma 4.12) on the jump rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \). Then, in Section 4.5 we show that if the jump rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \) satisfy some of the estimates from Lemma 4.12 and the communication property (Property 4.3), then one can obtain a suitable upper bound on the time taken by the LLN path to hit a compact subset of the interior of the simplex \( \mathcal{S} \). The latter property plays a crucial role in the proof of the large deviation lower bound. In fact, as made precise in Remark 4.6, the above properties of \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \) are the only ones used to establish the LDP, and thus the conclusion of Theorem 3.9 in fact holds for the larger class of sequences of jump Markov processes that satisfy Property 2.3 and the above-stated properties. Finally, in Section 4.6 we establish a discrete version of the communication condition that is used (only) in the proof of the locally uniform LDP in Section 9. This is a technical section of the paper. Readers only interested in the LDP proof may want to skip Section 4.6.

4.1. Communication Conditions. In Definition 4.1, we first introduce the notion of a communicating path associated with the limit jump rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \).

Definition 4.1. Given rates \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \); for any \( x, y \in \mathcal{S} \) and \( t \in (0, 1] \), a communicating path on \([0, t]\) from \( x \) to \( y \) with constants \( c > 0, p < \infty \), and \( F \in \mathbb{N} \) is a piecewise linear function \( \phi : [0, t] \to \mathcal{S} \) that satisfies \( \phi(0) = x \), \( \phi(t) = y \) and the following two properties:

i). there exist \( \{v_m\}_{m=1}^{F} \subset \mathcal{V} \), \( 0 = t_0 < t_1 < \cdots < t_F = t \), \( \{U_m\}_{m=1}^{F} \subset \mathbb{R}_+ \), such that

\[
\dot{\phi}(s) = \sum_{m=1}^{F} U_m v_m \mathbb{1}_{[t_{m-1}, t_m)}(s), \quad \text{a.e. } s \in [0, t],
\]

ii). for \( m = 1, \ldots, F \),

\[
\lambda_{v_m}(\phi(s)) \geq c \left( \min_{i=1, \ldots, d} y_i \right)^p \text{ for } s \in [t_{m-1}, t_m).
\]

Remark 4.2. Definition 4.1 ii) implies that for \( y \) in a compact subset of \( \mathcal{S}^\circ \), \( \lambda_{v_m}(\phi(s)) \) is uniformly bounded from below. In fact, one can weaken Definition 4.1 ii) (and correspondingly, Property 4.3) in this way and the proof of the LDP lower bound still holds. Nevertheless, we choose to define a communicating path using the slightly stronger condition in Definition 4.1 because it is naturally satisfied by interacting particle systems with \( K \)-ergodic jump rates (see Definition 3.2), and it is analogous to the corresponding condition in Definition 4.4 of a strongly communicating path, which is used in the proof of the locally uniform LDP in Section 4.6.

Let \( AC([0, T] : \mathcal{S}) \) denote the absolutely continuous functions from \([0, T]\) to \( \mathcal{S} \). In what follows, given \( t > 0 \) and a path \( \phi \in AC([0, t] : \mathcal{S}) \), let

\[
\text{Len} (\phi) \doteq \int_{0}^{t} \| \dot{\phi}(s) \| \, ds
\]
denote the length of $\phi$. We now state the communication condition on the jump rates $\{\lambda_v(\cdot), v \in V\}$.

**Property 4.3.** The rates $\{\lambda_v(\cdot), v \in V\}$ are such that there exist constants $c > 0$, $C' < \infty$, $p < \infty$ and $F \in \mathbb{N}$, such that for every $x \in S$ and $y \in \text{int}(S)$, there exist $t \in (0, 1]$, and a communicating path $\phi$ on $[0, t]$ from $x$ to $y$ exists with the given $c, p, F$ such that

$$(4.3) \quad \text{Len}(\phi) \leq C'\|x - y\|.$$ 

For the locally uniform LDP, we need the following strengthening of the notion of a communicating path.

**Definition 4.4.** Given $x, y \in S$ and $t \in (0, 1]$, a piecewise linear function $\phi : [0, t] \to S$ is said to be a strongly communicating path on $[0, t]$ from $x$ to $y$ with constants $c > 0$, $p < \infty$, $F \in \mathbb{N}$, $c_1 > 0, p_1 < \infty$ if it is a communicating path on $[0, t]$ from $x$ to $y$ with constants $c, p, F$ and, in addition,

i). if $\phi$ has the representation (4.4), then for $m = 1, \ldots, F$,

$$(4.4) \quad \lambda_{v_m}(\phi(s)) \geq c_1 \left( \prod_{j \in N_{v_m}} \phi_j(s) \right)^{p_1}, s \in [t_{m-1}, t_m),$$

where, as defined in (4.4), $N_{v_m} = \{i \in X : (v_m, c_i) < 0\}$.

Note that when $y$ lies in $\partial S$, $\min_{y_1=1,\ldots,d}y_k = 0$, and therefore property (ii) in Definition 4.1 is trivially satisfied. In constrast, the polynomial lower bound in (4.4) imposes stronger requirements on the paths that are not automatically satisfied even when $y \in \partial S$.

**Remark 4.5.** There is some flexibility in the choice of $t$, $\{U_m\}_{m=1}^F$ and $\{t_m\}_{m=1}^F$ both in Definition 4.1 and Definition 4.2. By a reparametrization of the respective paths, we can always assume $t = 1$ and $U_m = U$ for every $m$. Moreover, if $t$ is allowed to take any values in $(0, \infty)$, we can always choose $U_m = 1$.

**4.2. A Preliminary Result.** Here, we show that Assumption 3.1 and Assumption 3.11 together imply a certain strong controllability property of the associated (limit) jump rates $\{\lambda_v(\cdot), v \in V\}$. This result is used both in the verification (under suitable assumptions) of the communication condition in Section 4.3 and of its strengthening (under more restrictive assumptions) in Section 4.6.

**Lemma 4.6.** Suppose the transition rates $\{\Gamma_{ij}, (i, j) \in J\}$ satisfy Assumption 3.1 and Assumption 3.11. Then the associated jump rates $\{\lambda_v(\cdot), v \in V\}$ defined in (2.13) have the property that there exist constants $C' < \infty$, $c, c_1 > 0$, $p, p_1 < \infty$ and $F \in \mathbb{N}$, such that for every $x, y \in S$ there exists a strongly communicating path $\phi$ from $x$ to $y$ with constants $c, c_1, p, p_1$ and $F$ such that, in addition, (4.3) is satisfied. Moreover, if the requirement that $t \in (0, 1]$ is dropped, the path $\phi$ can be chosen so that its derivatives all lie in the set $\mathcal{V}_1 = \{e_j - e_i, i, j \in X, i \neq j\}$. Furthermore, for any $n \in \mathbb{N}$, if $x, y \in S_n$, then there exists a strongly communicating path $\phi^{P_n}$ from $x$ to $y$ whose representation (4.1) satisfies $U_m = 1/n$, $\phi(t_m) \in S_n$ and $t_m - t_{m-1} \in \mathbb{N}$, for $m = 1, \ldots, F$.

**Proof.** We will prove the result by a recursive construction. We claim that for any $r \in \{2, \ldots, d\}$, there exists $C_r = C_r(d, K) < \infty$ such that for every $x, y \in S$, there exists $X_r \subset X$ with $|X_r| \geq r - 1$, $z(r) \in S$ with $z_i(r) = y_i$ for $i \in X_r$, $0 \leq t_{r-1} < \infty$ and a strongly communicating path $\phi(r)$ on $[0, t_{r-1}]$ from $x$ to $z(r)$ such that $\text{Len}(\phi(r)) \leq C_r \|x - y\|$ and all the derivatives of $\phi(r)$ lie in $\mathcal{V}_1$. 


The first two assertions of the lemma then follows on taking \( r = d \) because the fact that both \( z^{(r)} \) and \( y \) lie on the simplex implies that they are equal if and only if they agree on \( d - 1 \) coordinates.

We will prove the claim by induction. We first consider the case \( r = 2 \), which is easy. We assume without loss of generality that \( x_i \neq y_i \) for some \( i \in \mathcal{X} \), for otherwise the construction is trivial. Choose \( u_1 \in \mathcal{X} \) such that \( x_{u_1} \geq y_{u_1} \) (such a \( u_1 \) always exists because \( x, y \in \mathcal{S} \)), then set \( \mathcal{X}_1 = \{ u_1 \} \), \( t_1 = x_1 - y_1 \) and choose any state \( u_2 \in \mathcal{X} \setminus \{ u_1 \} \) such that \( m_{u_1, u_2} > 0 \) (the existence of \( u_2 \) is implied by Assumption 3.1 and Assumption 3.11). Define

\[
\phi^{(2)}(t) = x + (e_{u_2} - e_{u_1}) t, \quad t \in [0, t_1].
\]

Then clearly, \( \phi^{(2)}(0) = x, \ z^{(2)} = \phi^{(2)}(t_1) \in \mathcal{S}, \ z^{(2)} = y_1 \) and \( \text{Len}(\phi^{(2)}) = \sqrt{2}t_1 \leq \sqrt{2}\|x - y\| \).

Moreover, by (2.1.2) and (2.1.3), setting \( v_1 = e_{u_2} - e_{u_1} \), we have

\[
\lambda_{v_1}(x) = \sum_{k=1}^{K} \sum_{i \in J, i \neq v_1} \alpha_{ij}^k(x) \geq x_{u_1} \rho_{u_1, u_2}(x) \geq c_0 x_{u_1},
\]

where we have used the fact that \( m_{u_1, u_2} > 0 \) and \( c_0 \) is defined by (3.4). Thus, the lower bound (4.4) holds with \( \mathcal{N}_{v_1} = \{ u_1 \} \), \( p_1 = 1 \), and some \( c_1 = c_0 > 0 \). Thus, \( \phi^{(2)} \) is a path of the desired form.

Now, assume the claim holds for \( r = l < d \), and let \( \mathcal{X}_l, z^{(l)}, t_{l-1}, \) and \( \phi^{(l)} \) be the corresponding quantities in the claim. We now prove the claim for \( r = l + 1 \). By the induction hypothesis, we have \( \phi^{(l)}(t_{l-1}) = z^{(l)} \) and \( z^{(l)}_i = y_i \) for \( i \in \mathcal{X}_l \), and \( |\mathcal{X}_l| \geq l - 1 \). We can assume without loss of generality that \( \mathcal{X}_l = \{ i \in \mathcal{X} : z^{(l)}_i = y_i \} \) satisfies \( |\mathcal{X}_l| = l - 1 \), for otherwise the claim clearly also holds for \( r = l + 1 \). Under this assumption, we have \( \sum_{i \in \mathcal{X} \setminus \mathcal{X}_l} z^{(l)}_i = \sum_{i \in \mathcal{X} \setminus \mathcal{X}_l} y_i \) and there exists \( j \in \mathcal{X} \setminus \mathcal{X}_l \) such that \( z^{(l)}_j > y_j \). For notational simplicity, we assume without loss of generality that \( \mathcal{X}_l = \{ 1, \ldots, l - 1 \} \) and \( j = l \). Then \( z^{(l)}_l > y_l \), and to prove the claim we will move mass from state \( l \) to some state in \( \{ l + 1, \ldots, d \} \), without changing the mass in any state with a lower index. In other words, we will construct a path \( \psi \in \text{AC}([0, t_*] : \mathcal{S}) \) for some \( t_* < \infty \), such that \( \psi_i(t_*) = \psi_i(0) \) for every \( i = 1, \ldots, l - 1 \) and \( \psi_l(t_*) - \psi_l(0) = -(z^{(l)}_l - y_l) \). To do this, take any \( w \in \{ l + 1, \ldots, d \} \). By Assumption 3.1 and Assumption 3.11, there exist \( M \leq d \) and a sequence of distinct states \( u_0 = l, \ldots, u_M = w \), such that for \( 1 \leq m \leq M - 1 \), \( m_{u_m, u_{m+1}} > 0 \). Now, let \( M_* = \min \{ m \geq 1 : u_m \in \{ l + 1, \ldots, d \} \} \), and note that \( u_{M_*} \) is the first state in the sequence that lies outside \( \{ 1, \ldots, l \} \), and \( M_* \) is the number of steps it took to get there. Define \( t_* \equiv M_*(z^{(l)}_l - y_l) \), and let \( \psi \in \text{AC}([0, t_*] : \mathcal{S}) \) be defined by \( \psi(0) = z^{(l)} \) and

\[
\psi(t) = v_m \equiv e_{u_m} - e_{u_{m-1}}, \quad t \in ((m - 1)(z^{(l)}_l - y_l), M_*(z^{(l)}_l - y_l)), \ m = 1, \ldots, M_*.
\]

Since the states are distinct, \( M_* \leq d \), and we have

\[
\text{Len}(\psi) \leq \sqrt{2d} (z^{(l)}_l - y_l) \leq \sqrt{2d} (|z^{(l)}_l - y_l| + |z^{(l)}_l - x_l|) \leq \sqrt{2d} \left( \|x - y\| + \int_0^{t_{l-1}} \|\phi^{(l)}(s)\|ds \right) \leq \sqrt{2d} (1 + C_l) \|x - y\|,
\]

where the last inequality follows from the induction assumption for \( r = l \).
Define \( t_{l+1} = t_l + t_s \), and let \( \phi^{(l+1)} \in AC([0, t_{l+1}] : S) \) be the concatenation of \( \phi^{(l)} \) and \( \psi \). As we show below, \( \phi^{(l+1)} \) is a path of the desired form. Clearly, if \( z^{(l+1)} = \phi^{(l+1)}(t_{l+1}) \), then \( \mathcal{X}_{l+1} = \{ j \in \mathcal{X} : z_j^{(l+1)} = y_j \} = \{ 1, \ldots, l \} \). Further, since the derivatives of both \( \psi \) and \( \phi^{(l)} \) all lie in \( \mathcal{V}_1 \), the same holds true for \( \phi^{(l+1)} \). Now, given any \( v \in \mathcal{V}_1 \) of the form \( v = e_j - e_i \) for some \( i, j \in \mathcal{X}, \ i \neq j \), with \( \mathfrak{R}_{ij} > 0 \), as before, we have \( \lambda_v(x) \geq x_i \Gamma_{ij}^1(x) \geq c_0 x_i \), and therefore, for a.e. \( s \in [t_l, t_{l+1}] \), \( \lambda_v(\phi^{(l+1)}(s)) \geq c_0 \phi^{(l+1)}_i(s), \) where \( v = \phi^{(l+1)}(s) \) and \( \mathcal{N}_v = \{ i \} \), from which it is easy to see that \( \phi^{(l+1)} \) is a strongly communicating path. Furthermore, we note that

\[
\text{Len}(\phi^{(l+1)}) = \int_0^{t_{l+1}} \| \dot{\phi}^{(l+1)}(s) \| \, ds = \text{Len}(\psi) + \text{Len}(\phi^{(l)}) \\
\leq \left( \sqrt{2d} + \left( \sqrt{2d} + 1 \right) C_1 \right) \| x - y \|,
\]

which establishes (4.3), with \( \phi \) replaced by \( \phi^{(l+1)} \), and \( C' \) replaced by \( C_{l+1} = \sqrt{2d} + (\sqrt{2d} + 1)C_1 \). By induction, it follows that the claim holds for \( r = d \), thus completing the proof of the first two assertions of the lemma.

To prove the last assertion (which is only used in the proof of the locally uniform LDP), suppose we restrict to \( x, y \in S_n \) for some \( n \in \mathbb{N} \), and let \( \phi \) be the strongly communicating path constructed above. Then, it is easy to see from the construction that the lengths of the intervals on which \( \phi \) has constant derivative that all lie in \( N_n = \{ \tau/n : \tau = 1, 2, \ldots \} \) and thus, the value of \( \phi \) at the end of each such interval lies in \( S_n \). A simple time reparametrization (see Remark 4.5) then yields a path with the stated properties. This concludes the proof of the lemma. \( \square \)

### 4.3. Verification of the Communication Condition

This section is devoted to establishing the following result.

**Proposition 4.7.** Suppose the family \( \{ \Gamma_{ij}(\cdot), (i, j) \in \mathcal{J}^k, k = 1, \ldots, K \} \) satisfies Assumption 3.4 and Assumption 3.3. Then the associated jump rates \( \{ \lambda_v(\cdot), v \in \mathcal{V} \} \) defined via (2.13) satisfy Property 4.3.

The proof of the proposition consists of three steps. First, given \( x, y \in S, \ y \notin \partial S \), we show that Assumption 4.1 and \( K \)-ergodicity (Assumption 3.3) allow one to move from any point \( x \) on the boundary \( \partial S \) to some compact convex subset of \( \text{int}(S) \) containing \( y \) along a piecewise linear path, each of whose segments is parallel to a jump direction \( v \in \mathcal{V} \) whose rate is uniformly bounded below away from zero on that segment (Lemma 4.3). Then we show that we have stronger controllability within the compact subset, which allows us to move along any coordinate direction, again with rates that are uniformly bounded away from zero (Lemma 4.4). This property is then used in a straightforward manner to construct a communicating path in this compact subset (Lemma 4.11). The proof of the proposition is completed at the end of the section by concatenating the two paths constructed above.

We now define two compact subsets of \( \text{int}(S) \): for \( a \in [0, 1) \), define

\[
S^a = \{ x \in S : x_i \geq a, \ i = 1, \ldots, d \},
\]

and

\[
S^a = \{ x \in S : \text{dist}(x, \partial S) \geq a \} = \left\{ x \in S : \inf_{z \in \partial S} \| x - z \| \geq a \right\}.
\]
Note that for $x \in S^a$, $x_i \geq a/\sqrt{2}$ for $i = 1, \ldots, d$, and therefore $\tilde{S}^a \subset S^a \subset \tilde{S}^{a/\sqrt{2}}$. Thus, the two sets have similar properties, but it will be more convenient to use one or the other depending on the context.

**Lemma 4.8.** Suppose $K \geq 2$ and that the family $\{\Gamma_{ij}(), (i, j) \in J^k, k = 1, \ldots, K\}$ satisfies Assumption 3.7 and Assumption 3.3 and let $c_0$ be as defined in 3.3. Then for any $x \in S$ and $a \in [0, 1/((K + 1)^{-1})]$, there exist $z \in \tilde{S}^a$, $t_0 < \infty$ and a communicating path $\phi$ on $[0, t_0]$ from $x$ to $z$ with constants $c = c_0/K$, $p = d$ and $F \leq 2^d$ that also satisfies the following two properties:

1. for any $s \in [0, t_0]$ and $i \in \mathcal{X}$ such that $\phi_i(s) < 0$, the inequality $\phi_i(s) \geq a$ holds.
2. $\text{Len}(\phi) \leq C_x \text{dist}(x, \tilde{S}^a)$ for some $C_x < \infty$, which does not depend on $a$.

Furthermore, the family of paths can be chosen so that $C \equiv \sup_{x \in S} C_x < \infty$.

Before proving the lemma in general, we first illustrate the argument for Example 3.7, which has $d = 4$ and $K = 2$.

**Example 3.7 cont’d.** We assume that $x_1 \geq x_2 \geq x_3 \geq x_4$ (the other cases can be treated in an exactly analogous fashion). Note that then $x_1 \geq 1/4$. Fix $a \in [0, 1/108]$. If $x_i \geq a$ for all $i = 1, \ldots, 4$ then we can set $z = x$ and the null path $\phi$ is trivially a communicating path. Otherwise, we consider three mutually exclusive and exhaustive cases and discuss the construction of the path in each case. We set $(i_1, j_1) = (1, 2)$, and $(i_2, j_2) = ((1, 2), (3, 4))$.

**Case I.** $x_3 \geq a > x_4$. Take $\phi(s) = (e_{i_1} - e_{i_1})I_{(0, 2(a-x_4))}(s) + (e_{j_2} - e_{j_2})I_{(2(a-x_4), 3(a-x_4))}(s)$ and $\phi(0) = x$. Then $\phi$ clearly satisfies property i) of Definition 4.1 with $F = 2$. Moreover, note that for $s \in [0, 3(a - x_4)]$, $\phi_1(s) \geq \phi_1(0) - 2(a - x_4) - (a - x_4) \geq x_1 - 3a \geq a$, $\phi_2(s) \geq \phi_2(0) \geq a$, $\phi_3$ and $\phi_4$ are nondecreasing and $\phi_4(3(a - x_4)) = a$. Thus, we have $z = \phi(3(a - x_4)) \in \tilde{S}^a$. Moreover, $\lambda_v(\phi(s)) = c_1 \phi_1(s) \geq c_1 a$ for $v = e_{j_2} - e_{i_1}$, and $s \in [0, 2(a - x_4)]$ and $\lambda_v(\phi(s)) = \frac{c_0}{2} \phi_1(s) \phi_2(s) \geq \frac{c_0}{2} a^2$ for $v = e_{j_2} - e_{i_1}$ and $s \in [2(a - x_4), 3(a - x_4)]$. Thus $\phi$ also satisfies property (ii) of Definition 4.1 with $p = 2$ and $c = \min(c_1, c_0/2)$, and is thus a communicating path from $x$ to $z$. The only $i$ for which $\phi_i(s) < 0$ for any $s$ are $i = 1, 2$. However, we already verified that $\phi_i(s) \geq a$ for all $s \in [0, 3(a - x_4)]$ and $i = 1, 2, 3$. Moreover, it is clear that $\text{Len}(\phi) \leq \sqrt{2}(a - x_4) \leq \sqrt{2} \text{dist}(x, \tilde{S}^a)$, and thus $\phi$ also satisfies properties (1) and (2) of Lemma 4.8 with $C_x = \text{dist}(x, \tilde{S}^a)$ satisfying $\sup_{x \in S} C_x < \infty$.

**Case II.** $x_2 \geq a > x_3$. Set $\phi^{(1)}(s) = (e_{j_1} - e_{i_1})I_{(0, 2(a-x_3))}(s) + (e_{j_2} - e_{j_2})I_{(2(a-x_3), 3(a-x_3))}(s)$ and $\phi^{(1)}(0) = x$. As in Case 1, it is easy to verify that $\phi^{(1)}$ is a communicating path on $[0, 3(a - x_3)]$ from $x$ to $z^{(1)} = \phi^{(1)}(3(a - x_3))$ that satisfies property 1 of Lemma 4.8 and has $\text{Len}(\phi^{(1)}) \leq 4\sqrt{2}|x_3 - a| \leq 4\sqrt{2} \text{dist}(x, \tilde{S}^a)$. Moreover, we have $z_1^{(1)} \geq a$, $z_2^{(1)} \geq a$, $z_3^{(1)} = a$ and $a - x_4^{(1)} \leq a - x_4$, and hence, $\text{dist}(z^{(1)}, \tilde{S}^a) \leq \text{dist}(x, \tilde{S}^a)$. Then, using the construction in Case 1 (if $z_1^{(1)} > z_2^{(1)}$, and if not then the construction in Case 1 should be modified by setting $(i_1, j_1) = (2, 1)$, there exists a communicating path $\phi^{(2)}$ from $z^{(1)}$ to a point $z^{(2)} \in \tilde{S}^a$ that satisfies properties (1) and (2) of Lemma 4.8 and has $\text{Len}(\phi^{(2)}) \leq 4\sqrt{2} \text{dist}(z^{(1)}, \tilde{S}^a) \leq 4\sqrt{2} \text{dist}(x, \tilde{S}^a)$. The path $\phi$ obtained from concatenating $\phi^{(1)}$ and $\phi^{(2)}$ is then easily seen to satisfy the properties of the lemma.

**Case III.** $x_1 > a > x_2$. In this case, set $\phi^{(1)}(0) = x$, $\phi^{(1)}(s) = (e_{j_1} - e_{i_1})I_{(0, a-x_2)}(s)$. Then $\phi^{(1)}$ is a communicating path from $x$ to $z^{(1)} = \phi^{(1)}(a - x_2)$. It satisfies property (2) of Lemma 4.8 since $\text{Len}(\phi^{(1)}) \leq 2\sqrt{2}|a - x_2| \leq 2\sqrt{2} \text{dist}(x, \tilde{S}^a)$. Since $x_1 \geq 1/4$, $z_1^{(1)} > x_2 = a > x_3 \geq x_4$. Thus property 1 holds ($\phi_i(s) < 0$ only for $i = 1$), $\text{dist}(z^{(1)}, \tilde{S}^a) \leq \text{dist}(x, \tilde{S}^a)$, and $z^{(1)}$ satisfies the
conditions of Case II. So, the desired path can be obtained by concatenating $\phi^{(1)}$ with a path $\phi^{(2)}$ from $z^{(1)}$ to $\hat{S}^a$ constructed as in Case II.

The construction in the above example can be generalized into the following proof.

**Proof of Lemma 4.8.** If $a = 0$ or $x \in \hat{S}^a$, we can choose $z = x$ and there is nothing to prove. Therefore, we assume $a \in (0, 1/(K + 1)^{d-1}d)$ and $x \notin \hat{S}^a$, which in particular implies that $x \neq (1/d, \ldots, 1/d)$. Then, assume without loss of generality that $x_1 \geq x_2 \geq \cdots \geq x_d$, and let
\[(4.8) \quad N = N(x) = |\{l = 1, \ldots, d : x_l < a\}|.\]

We will prove the lemma by induction on the quantity $N$.

We first construct a family of paths that will be used in the inductive argument. Since Assumption 3.2 implies that the state $d$ is $K$-accessible from the state 1, there exist $M \in \{2, \ldots, d\}$ and a sequence of distinct states $1 = u_1, u_2, \ldots, u_M = d$, such that for $m = 1, \ldots, M - 1$, there exist $k_m \in \{1, \ldots, K\}$, $(l_m, m) \in J^k_m$, and $i_m, j_m \in \{1, \ldots, k_m\}$, such that $u_m = i_m, l_m$, $u_{m+1} = j_m, l'_m$, and $\mathcal{M}_{i_m, m} > 0$. Now, for any $m_0 \in \{1, \ldots, M\}$, we introduce the constants
\[(4.9) \quad c_{m, m_0} - c_{m-1, m_0} \geq K \sum_{r=m}^{m_0-1} (c_{r+1, m_0} - cr, m_0).\]

Next, fix $0 < h < a$, let $t_0 = t_0(m_0) = c_{m_0, m_0}h = (K + 1)^{m_0-2}h$, and on $[0, t_0]$, define the piecewise linear path $\phi$ (associated with $m_0$ and $h$) with initial condition $x$ as follows: $\phi(0) = x$, and
\[(4.10) \quad \dot{\phi}(s) = e_{j_m} - e_{i_m} \quad \text{for} \quad s \in (c_{m, m_0}h, c_{m+1, m_0}h), \quad m = 1, \ldots, m_0 - 1.\]

The proof proceeds via three main claims.

**Claim 1.** The path $\phi$ associated with $m_0$, $h$ and initial condition $x$ satisfies the following properties:

a) $\phi_t(t) > a$ for $t \in [0, t_0]$.

b) $\phi_{u_m}(t) \geq x_{u_m}$ for $t \in [0, t_0]$ and $m = 2, \ldots, m_0 - 1$.

c) $\phi_u$ is non-decreasing on $[0, t_0]$ for $u \in \{1, \ldots, d\} \setminus \{u_1, \ldots, u_{m_0-1}\}$.

d) $\phi_u(t_0) \geq x_u + h$ for every $u \in (j_{m_0-1}, k_{m_0-1}) \setminus \{u_1, \ldots, u_{m_0-1}\}$.

e) For $m = 1, \ldots, m_0 - 1$, if $v_m = e_{j_m} - e_{i_m}$, then for $s \in (0, t_0)$,
\[\lambda_v(\phi(s)) \geq c_0 \frac{(\min_{m=1, \ldots, m_0-1} a \wedge x_{u_m})^{k_m}}{k_m!},\]
where $c_0$ is defined by \[3.3\].

f) If $\min_{m=1, \ldots, m_0-1} x_{u_m} \geq a$ and $h \leq d(x, \hat{S}^a)$ then $\phi$ is a communicating path from $x$ to $\phi(t_0)$ with constants $c_0/K, d$ and $m_0$, and $\phi$ also satisfies properties (1) and (2) of the lemma with $C_x = \sqrt{2K(K + 1)^{m_0}}$.

**Proof of Claim 1.** We start with the proof of property a). Recall that the assumed ordering of the components of $x \in \mathcal{S}$ and the assumption that $x \neq (1/d, 1/d, \ldots, 1/d)$ implies that $x_1 > 1/d$, and that we also have the inequalities $(e_{i_m}, e_1) \leq K$, $h \leq a \leq 1/((K + 1)^{d-1}d) < 1/d$, and
\[ c_{m_0,m_0} = (K + 1)^{m_0-2} \leq (K + 1)^{d-2}. \] Substituting this into (4.10), we obtain for \( t \in [0, t_0] \),

\[
\phi_1(t) = x_1 + \int_0^t \dot{\phi}_1(s) \, ds \geq x_1 - \sum_{m=1}^{m_0-1} \langle e_{i_m}, e_1 \rangle (c_{m+1,m_0} - c_{m,m_0}) h \geq \frac{1}{d} - Kc_{m_0,m_0} \frac{1}{(K + 1)^{d-1}} d \\
\geq a.
\]

For the next property, note that for \( m \in \{1, \ldots, m_0 - 1\} \), \( \langle e_{i_m}, e_m \rangle \leq K \) for all \( r = 1, \ldots, m \). Recall that by Definition 3.2(ii), for any \( m = 1, \ldots, m_0 - 1 \) mass is only moved from indices \( \{u_1, \ldots, u_m\} \), in that the components \( i_m, l = 1, \ldots, k_m \), of \( i_m \) must be from this set. Since the \( u_m, m = 1, \ldots, m_0 - 1 \), are distinct, this means that if \( m > 1 \), then \( \langle e_{i_m}, e_m \rangle = 0 \) for \( m'<m \), which in turn implies that \( \phi_{u_m} \) is non-decreasing on \([0, c_{m,m_0} h]\). On the other hand, since \( u_m = j_{m-1}' \), and \( \langle e_{i_m}, e_m \rangle \leq K \) for all \( r, l \in [c_{m,m_0} h, t_0] \) we have

\[
\phi_{u_m}(t) \geq x_{u_m} + \langle e_{i_{m-1}'}, e_{u_m} \rangle (c_{m,m_0} - c_{m-1,m_0}) h - \sum_{r=m}^{m_0-1} \langle e_{i_r}, e_{u_m} \rangle (c_{r+1,m_0} - c_{r,m_0}) h \\
\geq x_{u_m} + (c_{m_0,m_0} - c_{m-1,m_0}) h - K \sum_{r=m}^{m_0-1} (c_{r+1,m_0} - c_{r,m_0}) h,
\]

which implies property b) due to (4.9).

Property c) is a simple consequence of (4.10) and the fact that \( \langle e_{i_m}, e_u \rangle > 0 \) only if \( u \in \{u_1, \ldots, u_m\} \) due to Definition 3.2(ii). The latter property also implies that for \( u \in \{j_{m_0-1}, l = 1, \ldots, k_{m_0}\} \backslash \{u_1, \ldots, u_{m_0-1}\} \), \( \langle e_{i_m}, e_u \rangle = 0 \) for \( m = 1, \ldots, m_0 - 1 \). For any such \( u \), clearly we also have \( \langle e_{i_{m_0-1}'}, e_u \rangle \geq 1 \) (where the strict inequality \( > \) holds if more than one particle transitions to state \( u \) during the simultaneous transition), and hence,

\[
\phi_u(t_0) = x_u + \sum_{m=1}^{m_0-1} (\langle e_{i_m}, e_u \rangle - \langle e_{i_m}, e_u \rangle) (c_{m+1,m_0} - c_{m,m_0}) h \\
\geq x_u + (c_{m_0,m_0} - c_{m_0-1,m_0}) h \\
= x_u + h,
\]

where the last inequality uses the identity \( c_{m_0,m_0} - c_{m_0-1,m_0} = 1 \). This establishes property d).

Next, for \( m = 1, \ldots, m_0 - 1 \), setting \( v_m = e_{i_m} - e_{i_{m-1}'}, \) \( (4.12) \), Definition 3.2(iii) and (5.3) show that for \( s \in [0, t_0] \),

\[
\lambda_{v_m}(\phi(s)) \geq \alpha_{i_m,j_m}^{k_m}(\phi(s)) = \frac{1}{k_m!} (\prod_{l=1}^{k_m} \phi_{i_m,l}(s)) \Gamma_{i_m,j_m}^{k_m}(\phi(s)) \geq \frac{c_0}{k_m!} \left( \min_{l=1, \ldots, k_m} \phi_{i_m,l}(s) \right)^{k_m},
\]

where \( c_0 > 0 \) due to Assumption 3.1. When combined with properties a) and b) and the fact that \( \{i_m, l = 1, \ldots, k_m\} \subset \{u_1, \ldots, u_{m_0-1}\} \), \( m = 1, \ldots, m_0 - 1 \), this proves property e). Furthermore, when \( m_{\min} = 1, \ldots, m_0 - 1 \) \( x_{u_m} \geq a \), \( (4.10) \), properties a), b), e) and the fact that \( k_m \leq K \leq d \) show that \( \phi \) is a communicating path with constants \( c_0/K! \), \( d \) and \( m_0 \), whereas properties a)–c) and the fact that \( f \) assumes \( \min_{m=1, \ldots, m_0-1} x_{u_m} \geq a \) show that property 1 of the lemma is satisfied. Lastly, (4.10) and the definition of \( t_0 \) directly imply that \( \text{Len}(\phi) \leq \sqrt{2K}(K + 1)^{m_0} h \), which shows that \( \phi \) satisfies property 2 of the lemma with \( C_x = \sqrt{2K}(K + 1)^{m_0} \) if \( h \leq \text{dist}(x, S^x) \). This completes the proof of property f) and hence, of Claim 1.
We now proceed with the induction argument. Recall the definition of $N(x)$ given in (4.8). As our induction hypothesis, we assume that there exists $N_0 \in \{1, \ldots, d\}$ and $C_{N_0} < \infty$ such that for every $x \in S$ with $N(x) \leq N_0$, there exist $y \in \hat{S}^a$, $t_0 > 0$, $C_x \leq C_{N_0}$ and a communicating path $\phi$ on $[0, t_0]$ from $x$ to $y$ with constants $c = c_0$, $p = d$ and $F \leq N_0d$ that satisfy properties (1) and (2) of the lemma.

**Claim 2.** The induction hypothesis holds with $N_0 = 1$.

**Proof of Claim 2.** Suppose $N = N(x) = N_0 = 1$, where recall that $N(x)$ is defined by (4.8). Then since $x_i \geq x_{i+1}$, $x_d$ is the only component such that $x_d < a$. Now, set $\hat{h} = a - x_d > 0$, $t_0 = c_M \hat{h}$ and let $\phi$, as constructed prior to Claim 1, be a path on $[0, t_0]$ associated with $\hat{h}$ and $\phi$ and with initial condition $x$. Also, define $y = \phi(t_0)$. Since the $\{u_m, m = 1, \ldots, M\}$ are distinct and $u_M = d$, $\{u_m, m = 1, \ldots, M - 1\} \subset \{1, \ldots, d - 1\}$ and thus, $\min_{m = 1, \ldots, M - 1} x_{u_m} \geq a$. Moreover, $d = u_M = j_{M-1} j_{d-1}$ and hence, property d) of Claim 1 shows that $\phi_d(t_0) \geq x_d + \hat{h} = a$.

The last two assertions, when combined with properties a) and b) of Claim 1, imply that $y \in \hat{S}^a$ and $\min_{m = 1, \ldots, M - 1} \inf_{t \in [0, t_0]} \phi_{u_m}(t) \geq \min_{m = 1, \ldots, M - 1} x_{u_m} \geq a$, thus verifying property 1 of the lemma. Since we also have $\hat{h} \leq d(x, \hat{S}^a)$, property f) of Claim 1 shows that Claim 2 holds with $C_1 = \sqrt{2K(K + 1)} \hat{h} \leq \sqrt{2K(K + 1)}d$.

**Claim 3.** If the induction hypothesis holds for some $N_0 \in \{1, \ldots, d - 1\}$, then it also holds for $N_0 + 1$.

**Proof of Claim 3.** Due to the induction hypothesis, it suffices to consider $x$ such that $N = N(x) = N_0 + 1$. To prove the claim, we will first construct a communicating path that goes from $x$ to some $\bar{y} \in S$ such that $N(\bar{y}) \leq N_0$ and then invoke the induction hypothesis to construct a communicating path from $\bar{y}$ to some $\bar{y} \in \hat{S}^a$. The assumed ordering of $x$ and the fact that $N(x) = N_0 + 1$ imply that $x_i < a$ if and only if $i \geq d - N_0$. Define $\bar{m} = \min\{m \in 1, \ldots, M : u_m \geq d - N_0\}$, which is well defined because $u_M = d$, and set $\bar{h} = a - x_{u_{\bar{m}}} > 0$ and $\bar{t}_0 = c_{\bar{m}} \bar{m} \bar{h}$. Now, let $\bar{\phi}$ be the path on $[0, \bar{t}_0]$ with initial condition $x$ as defined in (4.10), but with $\bar{m}$ and $\bar{h}$ taking the roles of $m_0$ and $h$, and set $\bar{y} = \bar{\phi}(\bar{t}_0)$. Then, by the choice of $\bar{m}$,

$$\min_{m = 1, \ldots, \bar{m} - 1} x_{u_m} \geq \min_{i = 1, \ldots, d - N_0 - 1} x_i \geq a. \tag{4.11}$$

Since, in addition, $\bar{h} \leq d(x, \hat{S}^a)$, property f) of Claim 1 shows that $\bar{\phi}$ is a communicating path from $x$ to $\bar{y}$ with constants $c_0, d$ and $\bar{m} \leq d$, which satisfies property 1 of the lemma and also

$$\text{Len}(\bar{\phi}) \leq C_1 \text{dist}(x, \hat{S}^a). \tag{4.12}$$

Recall that in the construction of the paths we have $u_{\bar{m} + 1} = j_{\bar{m}, t_0}$ for $t_0 \in \{1, \ldots, k_{\bar{m}}\}$. Thus property d) of Claim 1 shows that $\bar{y}_{u_{\bar{m}}} \geq x_{u_{\bar{m}}} + \bar{h} = a$, whereas (4.11) and properties a)–c) of Claim 1 show that $\bar{y}_i \geq a$ for $i \in 1, \ldots, d - N_0 - 1$. Indeed, for $i \in \{1, \ldots, d - N_0 - 1\} \setminus \{u_1, \ldots, u_{\bar{m} - 1}\}$, $\bar{y}_i = x_i \geq a$, and for $i \in \{u_1, \ldots, u_{\bar{m} - 1}\}$, properties a)–b) of Claim 1 implies $\bar{y}_i = \phi_i(\bar{t}_0) \geq \min\{a, x_i\} \geq a$.

This, in turn, implies, that $N(\bar{y}) \leq N - 1 = N_0$. Thus, applying the induction assumption, there exists a communicating path $\phi$ from $\bar{y}$ to $y \in \hat{S}^a$ with constants $c_0, d$ and $F \leq N_0d$ that satisfies property 1 of the lemma and for which

$$\text{Len}(\bar{\phi}) \leq C_{N_0} \text{dist}(\bar{y}, \hat{S}^a). \tag{4.13}$$

Let $\phi$ be the concatenation of $\bar{\phi}$ and $\bar{\phi}$. Then it is clear that $\phi$ is a communicating path from $x$ to $y \in \hat{S}^a$ with constants $c_0, d$ and $F \leq (N_0 + 1)d$, and also satisfies property 1 of the lemma. Moreover, combining (4.13) and (4.12) with the inequalities $\text{dist}(\bar{y}, \hat{S}^a) \leq ||y - x|| + \text{dist}(x, \hat{S}^a)$ and
\[ ||y - x|| \leq \text{Len}(\tilde{\phi}) \], it follows that
\[ \text{Len}(\phi) = \text{Len}(\tilde{\phi}) + \text{Len}(\tilde{\phi}) \leq [C_1 + C_{N_0}(1 + C_1)] \text{dist}(x, \tilde{S}^k). \]
Thus, the induction hypothesis is satisfied for \( N_0 + 1 \) with \( C_{N_0+1} = [C_1 + C_{N_0}(1 + C_1)] < \infty. \) By induction, the hypothesis holds for \( N_0 = d \), which proves the lemma.

We now establish a uniform controllability property within any compact subset of \( \text{int}(S) \).

**Lemma 4.9.** Suppose that the family \( \{\Gamma_{ij}(.), (i,j) \in \mathcal{F}^k, k = 1, \ldots, K\} \) satisfies Assumptions 3.4 and 3.3. Then for any \( u, w \in \mathcal{X}, u \neq w \), there exists a finite constant \( M = M_{u,w} < \infty \), and for \( m = 1, \ldots, M - 1 \), there exist \( k_m \in \{1, \ldots, K\} \), \( a_m \geq 0 \) and \( (i_m, j_m) \in \mathcal{F}^{k_m} \) such that \( 2M_{i_m,j_m} > 0 \) and
\[ e_w - e_u = \sum_{m=1}^{M-1} a_m (e_{j_m} - e_{i_m}). \]
We first verify this assertion for Example 4.7. Without loss of generality, we set \( w = 4 \) and \( u = 1 \). As before, we take \( (i_1, j_1) = (1, 2) \), and \( (i_2, j_2) = ((1, 2), (3, 4)) \). Then \( (e_1 - c_1) + (e_2 - c_2) = e_3 + e_4 - 2e_1 \). To cancel the term \( e_3 \), we further take \( (i_3, j_3) = (3, 4) \). Then \( \sum_{m=1}^{d} (e_{j_m} - e_{i_m}) = 2(e_4 - e_1) \), or \( e_4 - e_1 = \sum_{m=1}^{d} \frac{1}{2} (e_{j_m} - e_{i_m}) \).

**Proof of Lemma 4.9.** Fix \( u, w \in \mathcal{X}, u \neq w \). Due to the assumed \( K \)-ergodicity, \( w \) is \( K \)-accessible from \( u \) and hence, there exist \( M = M_{u,w} \in \{2, \ldots, d\} \) and a sequence of distinct states \( u = u_1, \ldots, u_M = w \), \( k_m \in \{1, \ldots, K\} \), \( (i_m, j_m) \in \mathcal{F}^{k_m} \) for \( m = 1, \ldots, M - 1 \), that satisfy the properties in Definition 3.2. If \( K = 1 \), then \( u = u_1, i_m = u_m \) and \( \sum_{m=1}^{M-1} (e_{j_m} - e_{i_m}) > 0 \) for \( m = 1, \ldots, M - 1 \), and also \( j_{M-1} = u_M = w \). Thus, we can simply take \( e_w - e_u = \sum_{m=1}^{M-1} (e_{u_{m+1}} - e_{u_m}) \).

Now, suppose \( K \geq 2 \). The proof of (4.14) is more subtle in this case, and consists of two main steps. In the first step, we show that for any sequence \( \{i_m, j_m\}_{m=1}^{M-1} \) as above, there exist nonnegative (and in fact strictly positive) coefficients \( \{b_m^{(u)}\}_{m=1}^{M-1} \) such that
\[ \sum_{m=1}^{M-1} b_m^{(u)} (e_{j_m} - e_{i_m}) = \sum_{i=1, i \neq u}^{d} c_i^{(u)} e_i - \left( \sum_{i=1, i \neq u}^{d} c_i^{(u)} \right) e_u, \]
where the constants \( c_i^{(u)} \doteq (\sum_{m=1}^{M-1} b_m^{(u)} (e_{j_m} - e_{i_m}), e_i), i \in \{1, \ldots, d\} \setminus \{u\} \) satisfy
\[ c_i^{(u)} \geq 0, i \in \{1, \ldots, d\} \setminus \{u, w\}, \text{ and } c_u^{(u)} > 0. \]
The second step shows that (4.14) can be deduced from the fact that a representation of the form (4.15)-(4.16) holds for all \( u \neq w \).

We now turn to the proof of (4.15)-(4.16). If \( M = 2 \), it is directly implied by Definition 3.2 and \( (c_1, c_2) \geq 1 \) that (4.15)-(4.16) holds with \( b_1^{(u)} = 1 \). Next, suppose \( M > 2 \). The construction is now a bit more involved. First, assume \( \{b_m^{(u)}\}_{m=1}^{M-1} \) is any strictly positive sequence. Then, since Definition 3.2 implies \( i_m, j_m \in \{u_1, \ldots, u_M\} \) for \( l = 1, \ldots, k_m, m = 1, \ldots, M - 1 \) and \( (c_{j_m-1}, c_{u_M}) \geq 1 \), we have
\[ c_i^{(u)} = \left( \sum_{m=1}^{M-1} b_m^{(u)} (e_{j_m} - e_{i_m}), e_i \right) = \left( \sum_{m=1}^{M-1} b_m^{(u)} e_{j_m}, e_i \right) \geq 0, \quad i \in \{1, \ldots, d\} \setminus \{u_m\}_{m=1}^{M-1}, \]
with strict inequality for \( i = u_M = w \), implying that \( c_w^{(u)} > 0 \). Next, we must argue that we can pick strictly positive \( b_m^{(u)}, m = 1, \ldots, M - 1 \), such that the corresponding \( \{c_i^{(u)}\} \) also satisfy \( c_i^{(u)} \geq 0 \).
for \(i = u_2, u_3, \ldots, u_{M-1}\). For \(m_0 \in \{2, \ldots, M-1\}\), recall from Definition 3.2(i) that \(\langle e_{i_m}, e_i \rangle \leq K\) for \(i \in \{1, \ldots, d\}\) and \(\langle e_{i_m}, e_{u_m} \rangle = 0\) if \(m \in \{1, \ldots, m_0-1\}\), and hence

\[
\begin{align*}
\sum_{m=1}^{M-1} b_m^{(u)} (e_{j_m} - e_{i_m}) \langle e_{j_{m_0}}, e_{u_{m_0}} \rangle & \geq \sum_{m=m_0}^{M-1} b_m^{(u)} (e_{j_{m_0}-1}, e_{u_{m_0}}) - \sum_{m=m_0}^{M-1} b_m^{(u)} (e_{i_m}, e_{u_{m_0}}) \\
& \geq \kappa_{m_0-1} k_{m_0-1} b_{m_0-1}^{(u)} - K \sum_{m=m_0}^{M-1} b_m^{(u)},
\end{align*}
\]

(4.18)

where \(\kappa_m \doteq \langle e_{j_m}, e_{u_{m+1}} \rangle / k_m\) for \(m = 1, \ldots, M - 2\). Defining \(\{b_m^{(u)}\}_{m=1}^{M-1}\) recursively according to

\[
\begin{align*}
\kappa_{M-1} b_1^{(u)} &= 1 \\
\kappa_{M-2} b_2^{(u)} &= K b_M^{(u)} \\
& \vdots \\
\kappa_{1} b_1^{(u)} &= K \left(b_{M-1}^{(u)} + b_{M-2}^{(u)} + \cdots + b_2^{(u)}\right),
\end{align*}
\]

(4.19)

it is clear that \(\{b_m^{(u)}\}_{m=1}^{M-1}\) are strictly positive, and \(c_{u_{m_0}} \geq 0\) for all \(m_0 \in \{2, \ldots, M-1\}\). Together with (4.17) this shows that (4.15) and (4.16) hold for this choice of \(\{b_m^{(u)}\}_{m=1}^{M-1}\).

We now proceed to the second step of the proof. To obtain (4.14) from (4.15) we will eliminate the terms involving \(e_i, i \neq u, w\), on the right hand side of (4.15) (see (4.15)). Now, we have assumed for each \(s \neq u, w\), that \(u\) is \(K\)-accessible from \(s\). Hence, applying the same argument as in Step 1, but with \(u\) replaced by \(s\), we obtain the existence of \(M_s < \infty\), a sequence of jumps \((\{i_{m}^{(u)}, j_{m}^{(s)}\})_{m=1}^{M_s-1}\) and strictly positive coefficients \(\{b_m^{(s)}\}_{m=1}^{M_s-1}\) and \(\{c_m^{(s)}\}_{m=1}^{d} \neq i, j \neq w\) with \(c_m^{(s)} > 0\) and

\[
\sum_{m=1}^{M_s-1} b_m^{(s)} (e_{j_m^{(s)}} - e_{i_m^{(s)}}) = \sum_{i=1, i \neq s}^{d} c_i^{(s)} e_i - \left(\sum_{i=1, i \neq s}^{d} c_i^{(s)}\right) e_s.
\]

(4.20)

Further, by a simple rescaling, it is clear that one can assume without loss of generality that \(\sum_{i=1, i \neq s}^{d} c_i^{(s)} = 1\), which in turn implies that

\[
\sum_{i=1, i \neq w, s}^{d} c_i^{(s)} < 1, \quad \text{for every } s = 1, \ldots, d, s \neq w.
\]

(4.21)

It suffices to find nonnegative \(\{\theta_s\}_{s=1, s \neq w}^{d}\) such that

\[
\sum_{s=1, s \neq w}^{d} \left(\theta_s \left(\sum_{i=1, i \neq s}^{d} c_i^{(s)} e_i - e_s\right)\right) = e_w - e_u,
\]

(4.22)

for then one can substitute (4.20) into (4.22) to obtain (4.14) with \(M \leq \sum_{s=1, s \neq w}^{d} (M_s - 1)\) and coefficients \(a_m\) of the form \(\theta_{s_m} b_m^{(s)}\). For notational simplicity, below we assume without loss of generality that \(w = d\). Then, using (4.20), it is clear that both sides of (4.22) are zero when \(a_m^{(u)} = 0\) and \(a_m^{(w)} = 0\) for all \(m \in \{1, \ldots, M_s - 1\}\). Thus \(\{\theta_s\}_{s=1}^{d-1}\) satisfies (4.22) if and only if the \(s\)th components of both sides of (4.22) are
equal for $i \in \{1, \ldots, d-1\}$, or in other words, if the following system of linear equations is satisfied:

\[
\begin{align*}
\theta_1 &= c_1^{(2)} \theta_2 + \cdots + c_1^{(d-1)} \theta_{d-1} \\
\theta_2 &= c_2^{(1)} \theta_1 + \cdots + c_2^{(d-1)} \theta_{d-1} \\
&\vdots \\
\theta_u &= c_u^{(1)} \theta_1 + \cdots + c_u^{(d)} \theta_{d-1} + 1 \\
&\vdots \\
\theta_{d-1} &= c_{d-1}^{(1)} \theta_1 + \cdots + c_{d-1}^{(d-2)} \theta_{d-2}.
\end{align*}
\]

This system can be expressed more concisely as $[I_{d-1} - C]\theta = \tilde{c}_u$, where $I_{d-1}$ is the $(d-1) \times (d-1)$-dimensional identity matrix, $\tilde{c}_u$ is the $(d-1)$-dimensional vector that has a one in the $u$th component and 0 elsewhere, and $C$ is the $(d-1) \times (d-1)$ matrix given by

\[
C = \begin{pmatrix}
0 & c_1^{(2)} & \cdots & c_1^{(d)} \\
c_2^{(1)} & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & c_{d-1}^{(d-1)} \\
c_{d-1}^{(1)} & \cdots & c_{d-1}^{(d-2)} & 0
\end{pmatrix}.
\]

Note that $C$ is a non-negative matrix with row sums strictly less than 1 due to (4.21). Thus, applying Lemma 4.10 below with $N = d-1$ and $y = \tilde{c}_u$, we conclude that the system of linear equations has a unique solution, which is also nonnegative. This completes the proof of the lemma.

**Lemma 4.10.** Suppose that $A = I_N - C$, where $I_N$ is an $N \times N$ identity matrix and $C = (c_{ij})_{i,j=1}^N$ for some $N \in \mathbb{N}$, such that $c_{ii} = 0$, $c_{ij} \geq 0$ and $\sum_{j=1}^N c_{ij} < 1$. Also, let $y \in [0, \infty)^N$. Then the system of linear equations $Ax = y$ has a unique nonnegative solution $\{x_i\}_{i=1}^N$.

**Proof.** The spectral radius of $C$ is less than 1 since its matrix norm is less than 1. Therefore $\det A > 0$, and $A^{-1}$ exists. The fact that $A^{-1}$ is a positive matrix follows from a general result in inverse positivity [3], Theorem 6.3.8. The nonnegativity of $x$ then follows from the nonnegativity of $y$. $\square$

**Lemma 4.11.** Suppose Assumption 3.1 and Assumption 3.3 hold. Then for any $a > 0$, there exist $c > 0, C' < \infty$ and $F \in \mathbb{N}$ such that for any $x, y \in \tilde{S}^a$, there exists a communicating path $\phi$ from $x$ to $y$ with constants $c, d$ and $F$ such that $\phi$ lies in $S^{a/2}$ and satisfies $\text{Len}(\phi) \leq C'|x-y|$.\[\text{Len}(\phi) \leq C'|x-y|].

**Proof.** Fix $a > 0$. Then we observe the following elementary fact: there exists $C_1 < \infty$, such that for every $x, y \in \mathcal{S}$, there exists a continuous piecewise linear path $\phi_0$ from $x$ to $y$ that lies in $\mathcal{S}$, uses only velocities in the directions $\{e_j - e_i : i, j \in \mathcal{X}\}$, and for which $\text{Len}(\phi_0) \leq C_1 \|x-y\|$. Since $\tilde{S}^a$ is similar to $\mathcal{S}$, a rescaling implies that for every $x, y \in \tilde{S}^a$, there is a continuous piecewise linear path $\phi_0$ from $x$ to $y$ that lies in $\tilde{S}^a$ and only uses velocities in the directions $\{e_j - e_i : i, j \in \mathcal{X}\}$, and for which $\text{Len}(\phi_0) \leq C_1 \|x-y\|$. Given such a path $\phi_0$ with $\phi_0(0) = x$, let $M < \infty$, and $0 = t_1 < \cdots < t_M$ and $\{w_m\}_{m=1}^M \in \mathcal{X}$ be such that

\[
\begin{align*}
\phi_0(t) = e_{w_m} - e_{u_m}, & \quad t \in (t_m, t_{m+1}), m = 1, \ldots, M-1,
\end{align*}
\]

where a uniform bound on $M$ can be assumed. We now use Lemma 4.9 to replicate these velocities using jumps with positive rates. For each $m = 1, \ldots, M-1$, there exist $M_m \in \mathbb{N}, \{(i_k^{(m)}, j_k^{(m)})\}_{k=1}^{M_m-1}$
Lemma 4.12. Suppose the family \( \{ \phi_{k}^{(m)} \}_{m=1}^{M-1} \) such that \( e_{w_{m}} - e_{u_{m}} = \sum_{r=1}^{M-1} a_{r}^{(m)} (e_{j_{r}}^{(m)} - e_{i_{r}}^{(m)}) \). With the appropriate partition of \((t_{m}, t_{m+1})\), we can construct a trajectory \( \phi \) that uses only these velocities and satisfies \( \phi(t_{m}) = \phi_{0}(t_{m}) \) and \( \phi(t_{m+1}) = \phi_{0}(t_{m+1}) \). If for any \( s \in (t_{m}, t_{m+1}) \) we have \( \| \phi(s) - \phi_{0}(s) \| > a/2 \), then we can partition \((t_{m}, t_{m+1})\) into an integral number \( K \) of smaller segments on which we replicate the velocity \( e_{w_{m}} - e_{u_{m}} \), and guarantee \( \phi(t_{m} + k[t_{m+1} - t_{m}]/K) = \phi_{0}(t_{m} + k[t_{m+1} - t_{m}]/K) \) for \( k = 1, \ldots, K \). For large enough \( K \) this implies \( \| \phi(s) - \phi_{0}(s) \| \leq a/2 \) for all \( t \in [0, T_{M}] \). Since \( X \) is finite there is a maximum velocity used in this process, and hence we can assume a uniform bound on \( K \) that depends only on \( a \), and also the existence of \( C < \infty \) such that \( \text{Len}(\phi) \leq C \text{Len}(\phi_{0}) \), and so can take \( C' = CC_{1} \).

The path so constructed satisfies \( \phi \in AC([0, T_{M}] : S^{a/2}) \). By (2.12), (2.13) and the fact that \( \mathcal{M}_{\text{int}} > 0 \) with \( c_{0} > 0 \) as in (3.3) we have that for all \( s \in [0, T_{M}] \), \( \lambda_{v}(\phi(s)) \geq \frac{1}{K} \left( \frac{a}{2} \right)^{d} c_{0} \). Since \( \min_{j=1,\ldots,d} y_{i} \geq a \), \( \phi \) is a communicating path on \([0, t_{M}]\) from \( x \) to \( y \) with constants \( c = c_{0}/2^{d}K! \), \( p = d \), and uniformly bounded \( F \).

We now complete the proof of Proposition 4.7.

Proof of Proposition 4.7. When \( K = 1 \), this follows from the stronger result proved in Lemma 4.6. Hence, suppose \( K \geq 2 \). Given \( x \in S \) and \( y \in \text{int}(S) \), let \( 0 < a < \min(1/(K+1)^{d-1}d), \min_{i=1,\ldots,d} y_{i} \).

By Lemma 4.7 there exist \( z \in \mathcal{S}^{a} \), a communicating path from \( x \) to \( z \) with constants independent of \( x \) and \( z \). We also have \( y \in \mathcal{S}^{a} \), and so by Lemma 4.11 there exists a communicating path from \( z \) to \( y \), with constants independent of \( z \) and \( y \). It is straightforward to see that the concatenation of these two paths is a communicating path from \( x \) to \( y \) with constants independent of \( x \) and \( y \). Moreover, it follows from the properties stated in Lemmas 4.8 and 4.11 that there exists \( C' < \infty \) such that the family of communicating paths thus constructed satisfies (3.3). This proves that Property 1.3 is satisfied. 

4.4. Estimates on the Jump Rates. In this section we derive certain estimates on the jump rates that are satisfied under our assumptions on the transition rates. These estimates are used in the subsequent proofs. We recall that \( \mathcal{N}_{v} = \{ i : v_{i} < 0 \} \).

Lemma 4.12. Suppose the family \( \{ \Gamma_{i,j}^{k}(\cdot) \in \mathcal{J}^{k}, k = 1, \ldots, K \} \) satisfies Assumption 2.2 and Assumption 3.1 and let \( \{ \lambda_{v}, v \in V \} \) be the associated jump rates. Then the following assertions hold.

1. There exists \( \bar{C} < \infty \) such that for every \( v \in V \),

\[
\lambda_{v}(x) \leq \bar{C} \prod_{i \in \mathcal{N}_{v}} x_{i}, \quad x \in S.
\]

2. There exists a continuous function \( \bar{C} : [0, \infty) \to [0, \infty) \) such that \( \bar{C}(r) \to 1 \) as \( r \to 0 \) such that for every \( v \in V \) and \( x, y \in S \),

\[
\frac{\lambda_{v}(x)}{\lambda_{v}(y)} \leq \bar{C}(||x - y||) \prod_{i \in X; y_{i} < x_{i}} \left( \frac{x_{i}}{y_{i}} \right)^{K}.
\]

3. For every \( v \in V \), either \( \lambda_{v} \equiv 0 \) or

\[
\lambda_{v}(x) \geq \frac{c_{0}}{K^{!}} \prod_{i=1}^{k} x_{i} \geq \frac{c_{0}}{K^{!}} \prod_{i=1}^{d} x_{i}^{K}, \quad x \in S,
\]
and hence, for every \( a > 0 \), \( \inf_{x \in S} \lambda_v(x) > 0 \) and \( \inf_{x \in S} \lambda_v(x) > 0 \), where \( \tilde{S}^a \) and \( S^a \) are defined in (4.6) and (4.7), respectively.

(4) If, in addition, Assumption 3.8 holds, then there exist \( \bar{c} > 0 \) such that for every \( v \in V \), either \( \lambda_v(\cdot) \equiv 0 \) or there exist \( r_i \geq 1 \), \( i \in N_v \), with \( \sum_{i \in N_v} r_i \leq K \) such that for every \( x \in S \) and \( y \in \text{int}(S) \),

\[
\frac{\lambda_v(x)}{\lambda_v(y)} \geq \bar{c} \prod_{i \in N_v} \left( \frac{x_i}{y_i} \right)^{r_i}.
\]

Proof. We start with the proof of the first property. For any \( v \in V \), for every \( k = 1, \ldots, K \) and \((i, j) \in J^k\) such that \( e_j - e_i = v \), (3.5) shows that each \( i \in N_v \) appears at least once in \( \{i_l, l = 1, \ldots, k\} \), and so \( \prod_{i=1}^{k} x_i \leq \prod_{i \in N_v} x_i \). Substituting this and the bound on the transition rates in (2.11) into the definition of \( \lambda_v \) in (2.13), it follows that the first property holds with \( \bar{C} \leq R \| J \| \).

Next, fix \( v \in V \) and note that by Assumption 3.1, we can rewrite \( \lambda_v \) from (2.13) as

\[
\lambda_v(x) = \sum_{k=1}^{K} \sum_{\substack{\{i,j\} \in J^k_v \colon e_j - e_i = v}} a_{ij}^k(x), \quad x \in S,
\]

where recall the definition of \( J^k_v \) from (3.2). Now, let \( k^* = k^*(x,y) \in \{1, \ldots, K\} \), \((i^*, j^*) = (i^*(x,y), j^*(x,y)) \in J^k_v \) be such that

\[
\frac{\alpha_{i^* j^*}^k(x)}{\alpha_{i^* j^*}^k(y)} = \max_{k=1, \ldots, K} \max_{\substack{\{i,j\} \in J^k \colon e_j - e_i = v}} \frac{\alpha_{ij}^k(x)}{\alpha_{ij}^k(y)}.
\]

Then, since for any finite index set \( I \) and numbers \( a_i, b_i > 0, i \in I \),

\[
\sum_{i \in I} a_i \leq \max_{i \in I} \frac{a_i}{b_i},
\]

from (4.24) and (2.12) it follows that

\[
\frac{\lambda_v(x)}{\lambda_v(y)} \leq \frac{\alpha_{i^* j^*}^k(x)}{\alpha_{i^* j^*}^k(y)} = \left( \prod_{l=1}^{k^*} \frac{x_{i^*_l}}{y_{i^*_l}} \right) \frac{\Gamma_{i^* j^*}^k(x)}{\Gamma_{i^* j^*}^k(y)}.
\]

Combining the lower bounds on \( \Gamma_{i^* j^*}^k \) in (3.3), and letting \( C_1 \) denote the maximum of the Lipschitz constants of \( \Gamma_{i^* j^*}^k \), \( (i,j) \in J^k, k = 1, \ldots, K \) (which is finite by Assumption 2.2), we obtain the inequality

\[
\frac{\Gamma_{i^* j^*}^k(x)}{\Gamma_{i^* j^*}^k(y)} = 1 + \frac{\Gamma_{i^* j^*}^k(x) - \Gamma_{i^* j^*}^k(y)}{\Gamma_{i^* j^*}^k(y)} \leq \left( 1 + \frac{C_1}{c_0} \| x - y \| \right).
\]

On the other hand, for any \( k = 1, \ldots, K \) and \((i, j) \in J^k\),

\[
\prod_{l=1}^{k} \frac{x_{i_l}}{y_{i_l}} \leq \prod_{l=1}^{k, x_{i_l} > y_{i_l}} \left( \frac{x_{i_l}}{y_{i_l}} \right) \leq \prod_{i \in I, x_i > y_i} \left( \frac{x_i}{y_i} \right)^K.
\]

Substituting this and (4.26) into (4.25), we see that the second property is satisfied by the function \( \bar{C}(r) = 1 + C_1 r / c_0, \ r \geq 0 \).
For the remaining two properties we can fix $v \in V$ and assume without loss of generality that $\lambda_v$ is not identically zero. Then, by the continuity of $\lambda_v(\cdot)$, which follows from Assumption 2.2, there exists $x \in \text{int}(S)$ such that $\lambda_v(x) > 0$. In turn, from the form of $\lambda_v$ in (1.24), it follows that there exists $k = 1, \ldots, K$ and $(i, j) \in J^k_j$ such that $v = c_j - e_i$ and $\Gamma_j(x) > 0$. The relations (2.13) and (2.14) and the definition of $c_0$ in (3.3) then show that

$$\lambda_v(x) \geq \alpha_{ij}(x) \geq \frac{c_0}{K^j} \prod_{i=1}^k x_{ii}, \quad x \in S,$$

which implies the inequality in property 3) because for each $i \in \{1, \ldots, d\}$, $x_i \in [0, 1]$ and $|i| = 1, \ldots, k : i_i = i$) $\leq k \leq K$. The bound $\inf_{x \in S^a} \lambda_v(x) > 0$ and $\inf_{x \in S^a} \lambda_v(x) > 0$ are an immediate consequence of the inequality, the definition of $S^a$ and the fact that $S^a \subseteq S^a/2$.

For the last property, first note that for any $x, y \in S$, using (4.24), (3.3), (2.11) and (2.12), we have

$$\lambda_v(x) \geq \alpha_{ij}(x) \geq \frac{c_0}{K^j} \prod_{i=1}^k x_{ii}, \quad x \in S,$$

If Assumption 3.8(1) holds then let $(i^*, j^*) \in J_+^k$ be as in the assumption, and define $r_j = r_j(v) = |\langle v, e_j \rangle|$ for $j \in N_v$. Then clearly $(i^*, j^*) \in J_+^{K^*}$, where $K^* = \sum_{j \in N_v} r_j$, and

$$\sum_{k=1}^{K^*} \sum_{(i, j) \in J_+^{K^*}} \left( \prod_{i=1}^{K^*} x_{ii} \right) \geq \prod_{i=1}^{K^*} x_{ii} = \prod_{j \in N_v} (x_j)^{r_j}.$$  

On the other hand, (3.3) along with the fact that $y \in [0, 1]^d$ implies that

$$\sum_{k=1}^{K^*} \sum_{(i, j) \in J_+^{K^*}} \left( \prod_{i=1}^{K^*} y_{ii} \right) \leq |J_+^k| \prod_{j \in N_v} (y_j)^{r_j}.$$  

Substituting the last two inequalities into (4.27), we see that (4.23) holds with $\bar{c} = c_0/R_0K^j|J_+|$, and $r_j, j \in N_v$, as specified above. On the other hand, if Assumption 3.8(2) is satisfied then let $r_j, j \in N_v$, and $k^* = \sum_{j \in N_v} r_j$ be as stated in the assumption. Notice that for all $(i, j) \in J_+^{K^*}$ such that $c_j - e_i = v$, the equality $\prod_{i=1}^{K^*} x_{ii} = \prod_{j \in N_v} (x_j)^{r_j}$ holds. Therefore, there is some constant $C' < \infty$, that only depends on $r_j, j \in N_v$, such that

$$\sum_{k=1}^{K^*} \sum_{(i, j) \in J_+^{K^*}} \left( \prod_{i=1}^{K^*} x_{ii} \right) = \sum_{(i, j) \in J_+^{K^*}} \left( \prod_{i=1}^{K^*} x_{ii} \right) = C' \prod_{j \in N_v} (x_j)^{r_j},$$

with the same equality also holding when $x$ is replaced by $y$. When substituted back into (4.27), this shows that (4.23) holds with $\bar{c} = c_0/R_0K^j$ and the given $r_j, j \in N_v$. This completes the proof of the lemma. \square

4.5. A property of the LLN trajectory. In this section, we show that the communication and growth conditions on the limit rates $\{\lambda_v(\cdot), v \in V\}$ imply that the associated LLN trajectory $\mu$ has the following property, which is crucially used in the proof of the large deviation lower bound.
Property 4.13. There exist constants \( b > 0 \) and \( D \in [1, \infty) \) such that for any \( x \in S \), the associated LLN path \( \mu \) that solves the ODE (4.15) and starts at \( x \) is such that for every \( i \in X \),
\[
\mu_i(t) \geq bt^D, \quad t \in [0,1].
\]

We now state the main result of this section.

Proposition 4.14. Suppose the family of jump rates \( \{\lambda_v(\cdot), v \in V\} \) satisfies Property 2.13, Property 4.3 and properties (1) and (4) of Lemma 4.12. Then Property 4.13 is also satisfied.

To provide insight into the proof of Proposition 4.14, we first show why the conclusion holds for the specific \( K \)-ergodic particle system with \( d = 4, K = 2 \) introduced in Example 3.7 when \( c_i = 1 \) for \( i = 1, \ldots, 6 \).

Example 3.7 cont’d. Let \( x = \mu(0) \), and assume without loss of generality that \( x_1 \geq x_2 \geq x_3 \geq x_4 \), and therefore that \( x_1 \geq 1/4 \). We start by establishing a basic inequality for \( \mu_i(t) \). Recall the form of \( \lambda_v(\cdot), v \in V \), given in Example 3.7, and note that the ODE (2.15) implies
\[
\dot{\mu}_i(t) = \sum_{v \in V} \langle v, e_i \rangle \lambda_v(\mu(t)) \geq -\mu_i(t) - \frac{1}{2} \mu_1(t) \mu_2(t) \geq -\frac{3}{2} \mu_i(t).
\]

Thus, for \( t \in [0,1], \mu_i(t) \geq x_1 e^{-\frac{3}{4}t} \geq b_1 \), where \( b_1 = e^{-\frac{3}{4}}/4 \). Then we have
\[
\dot{\mu}_2(t) = \sum_{v \in V} \langle v, e_j \rangle \lambda_v(\mu(t)) \geq \mu_1(t) - \mu_2(t) - \frac{1}{2} \mu_1(t) \mu_2(t) \geq b_1 - \frac{3}{2} \mu_2(t),
\]

which implies \( \mu_2(t) \geq \frac{2}{3} b_1 (1 - e^{-\frac{3}{4}t}) \geq b_2 t \) for some \( b_2 > 0 \) (for example, \( b_2 = b_1/2 \)). Substituting the last two bounds into the equations for \( i = 3, 4 \), we obtain
\[
\dot{\mu}_3(t) = \sum_{v \in V} \langle v, e_i \rangle \lambda_v(\mu(t)) \geq \frac{1}{2} \mu_1(t) \mu_2(t) - \mu_1(t) - \frac{1}{2} \mu_3(t) \mu_4(t) \geq \frac{b_1 b_2}{2} t - \frac{3}{2} \mu_3(t),
\]

and hence, for \( t \in [0,1], \mu_i(t) \geq b_3 t^2 \) for some \( b_3 > 0 \) (for example, \( b_3 = b_1 b_2 e^{-3/2}/4 \)). Thus, we have shown that Example 3.7, with the chosen parameters, satisfies Property 4.13.

The proof for the general case is more technical and is given below.

Proof of Proposition 4.14. Define \( \delta_1 = \max \{|\langle e_j, v \rangle| : j \in X, v \in V\} < \infty \), and let \( R_1 = \delta_1 \tilde{C} |V| \), where \( \tilde{C} \) is the constant in property (1) of Lemma 4.12. Then the fact that \( \mu \) solves the ODE (2.15) implies that for \( i \in X \) and \( t \in [0,1] \),
\[
\mu_i(t) = \sum_{v \in V} \langle v, e_i \rangle \lambda_v(\mu(t)) \geq \sum_{v \in V, \langle v, e_i \rangle > 0} \langle v, e_i \rangle \lambda_v(\mu(t)) + \sum_{v \in V, \langle v, e_i \rangle < 0} \langle v, e_i \rangle \lambda_v(\mu(t))
\]
\[
\geq \sum_{v \in V, \langle v, e_i \rangle > 0} \langle v, e_i \rangle \lambda_v(\mu(t)) - R_i \mu_i(t).
\]

Since each \( \lambda_v(\cdot) \) is nonnegative, (4.29) and the comparison principle for ODEs imply that
\[
\mu_i(t) \geq \mu_i(0) e^{-R_i t}, \quad t \in [0,1], i \in X.
\]

In order to use (4.29) to show that (4.28) holds (for suitable \( b > 0 \) and \( D < \infty \)), we first obtain a lower bound on \( \lambda_v(\mu(t)) \) by comparing it to \( \lambda_v(\phi(t)) \) for a suitable communicating path \( \phi \), and then apply the estimate (4.28). Define \( y = \left( \frac{1}{4}, \ldots, \frac{1}{4} \right) \), and note that by Property 4.3 and Remark
there exists a communicating path $\phi$ from $\mu(0) = x$ to $y$ on $[0, 1]$. Let $\phi$ admit a representation in terms of $F < \infty, \{t_m\}_{m=0}^{F}, \{v_m\}_{m=1}^{F}$ and $\{U_m\}_{m=1}^{F}$ as in (4.11) and for $m = 0, 1, \ldots, F$, denote $y^{(m)} = \phi(t_m)$. Then, applying the inequality in property $ii)$ of Definition (4.1) with $s = t_m$, $m = 1, \ldots, F$, and $y = \frac{1}{d} \sum_{i=1}^{d} \epsilon_i$, we see that there exists $c' \in (0, 1)$ such that
\begin{equation}
\lambda_{v_m} \left( y^{(m-1)} \right) \geq c', \quad m = 1, \ldots, F.
\end{equation}

Now define $X_0 = \emptyset$ and for $l = 1, \ldots, F$, define $X_l = \{ j \in X : \mu_j(0) < y_j^{(l)} \}$. Then, for $m = 1, \ldots, F$, by the continuity of $\lambda_{v_m}$ (Property (2.6)), there exists $\hat{y}^{(m-1)} \in \text{int}(S)$ sufficiently close to $y^{(m-1)}$ such that
\begin{equation}
\lambda_{v_m} \left( \hat{y}^{(m-1)} \right) \geq \frac{c'}{2}, \quad m = 1, \ldots, F,
\end{equation}
and
\begin{equation}
\{ j \in X : \mu_j(0) < \hat{y}^{(m-1)} \} = X_{m-1}.
\end{equation}

Now, fix $m \in \{1, \ldots, F\}$. Let $c > 0$ and $r_j = r_j(v_m), j \in \mathcal{N}_{v_m}$, be the constants in property (4) of Lemma (4.12). For $t \in [0, 1]$, we can first apply the estimate (4.23) with $x = \mu(t)$ and $y = \hat{y}^{(m-1)}$ and use (4.30), and then use (4.30) and $\sum_{j \in X} r_j \leq K$ to obtain
\begin{equation}
\lambda_{v_m} (\mu(t)) \geq \frac{c'}{2} \prod_{j \in \mathcal{N}_{v_m}} \left( \frac{\mu_j(t)}{\hat{y}^{(m-1)}} \right)^{r_j} \geq \tilde{c}_1 \prod_{j \in \mathcal{N}_{v_m}} \left( \frac{\mu_j(0)}{\hat{y}^{(m-1)}} \right)^{r_j},
\end{equation}
where $\tilde{c}_1 \equiv c' e^{-R_1 K^2} > 0$. Since (4.33) implies $\mu_j(0) \geq \hat{y}^{(m-1)}$ for $j \in X \setminus X_{m-1}$ and $\hat{y}^{(m-1)} \leq 1$ for all $j$ (and in particular $j \in X_{m-1}$), we can further simplify the last inequality to obtain
\begin{equation}
\lambda_{v_m} (\mu(t)) \geq \tilde{c}_1 \prod_{j \in X_{m-1} \cap \mathcal{N}_{v_m}} (\mu_j(t))^{r_j},
\end{equation}
where the product over an empty set is to be interpreted as 1.

We claim, and show below, that for every $m = 1, \ldots, F$, there exists $b^{(m)} \in (0, 1)$ such that for every $i \in X_m$, (4.28) holds with $b = b^{(m)}$ and $D = D(m) \equiv \sum_{i=0}^{m-1} K^i < \infty$. Setting $m = F$, this then proves (4.28) for all $i \in X_F$ with $b = b^{(F)}$ and $D = D(F)$. This suffices to complete the proof because for $i \in X \setminus X_F$, $\mu_i(0) \geq y_i^{(F)} = y_i = 1/d$ and so (4.30) implies that (4.28) holds with $b \equiv \frac{1}{d} e^{-R_1} > 0$ and $D = 0$.

We now use an inductive argument to prove the claim. Define
\begin{equation}
\delta_2 \equiv \min \{ \langle v, e_j \rangle : j \in X, v \in \mathcal{V}, \text{ s.t. } \langle v, e_j \rangle > 0 \} > 0.
\end{equation}
We first consider the case $m = 1$. For every $i \in X_1$, $\phi_i(0) = \mu_i(0) < y_i^{(1)} = \phi_i(t_1)$, which implies $\langle v_1, e_i \rangle > 0$ since $\phi_i(t) = U_1(v_1, e_i)$ for a.e. $t \in (0, t_1)$ due to the assumed representation (4.1) of $\phi$. Moreover, (4.31) and the fact that $X_0$ is the empty set, together imply $\lambda_{v_1}(\mu(t)) \geq \tilde{c}_1$ for $t \in [0, 1]$. Substituting this into (4.29), one sees that for $i \in X_1$,
\begin{equation}
\mu_i(t) \geq \langle v_1, e_i \rangle \lambda_{v_1}(\mu(t)) - R_1 \mu_i(t) > \delta_2 \tilde{c}_1 - R_1 \mu_i(t), \quad t \in [0, 1].
\end{equation}
By the comparison principle for ODEs, and the fact that $\mu_i(0) \geq 0$, we see that there exists some $b^{(1)} \in (0, 1)$ such that for $t \in [0, 1]$,
\begin{equation}
\mu_i(t) \geq \frac{\delta_2 \tilde{c}_1}{R_1} \left( 1 - e^{-R_1 t} \right) \geq b^{(1)} t,
\end{equation}
Since $D(1) = 1$, this shows that the claim holds for $m = 1$.

Next, assume that for some $m_0 \in \{1, \ldots, F - 1\}$, the claim holds for all $m \in \{1, \ldots, m_0\}$, and let $\hat{m} = m_0 + 1$. Fix $i \in X_\hat{m}$. Then, since $\phi_i(0) = \mu_i(0) < g_{i, \hat{m}} < \phi_i(t_{\hat{m}})$, it is clear from the representation \([4.34]\) for $\phi_i$ that there exists $m^* \in \{1, \ldots, \hat{m}\}$ such that $\langle v_{m^*}, e_i \rangle > 0$. If $m^* = 1$, this shows that \([4.35]\), and hence \([4.36]\), holds. Since $D(\hat{m}) \geq 1$, the claim holds for $m \equiv \hat{m}$ with $b^{(\hat{m})} = b^{(1)}$. On the other hand, if $m^* \in \{2, \ldots, \hat{m}\}$, then by \([4.34]\) and the induction hypothesis we have for $t \in [0, 1]$, \[\lambda_{v_{m^*}}(\mu(t)) \geq \bar{c}_1 \prod_{j \in \mathcal{X}_{m^*} - i} \langle \mu_j(t) \rangle_{\mathcal{A}_{m^*}^n} \geq \bar{c}_2 t^{KD(m^* - 1)},\]

with $\bar{c}_2 \geq \bar{c}_1 (l^{(m^* - 1)})^K > 0$, where we have used the fact that $\sum_{j \in \mathcal{N}_{m^*}^n} r_j \leq K$. Since $m \mapsto D(m)$ is increasing, this gives a lower bound of $\bar{c}_2 t^{KD(\hat{m} - 1)}$ on $\lambda_{v_{m^*}}(\mu(t))$. Substituting this into \([4.29]\) we see that for $t \in [0, 1]$,

$$
\dot{\mu}_i(t) \geq \langle v_{m^*}, e_i \rangle \lambda_{v_{m^*}}(\mu(t)) - R_i \mu_i(t) \\
\geq \bar{c}_3 t^{KD(\hat{m} - 1)} - R_i \mu_i(t),
$$

where $\bar{c}_3 = \delta_2 \bar{c}_2$, which we can assume without loss of generality to lie in $(0, 1)$. Note that the solution to the ODE $u(t) = \bar{c}_3 t^{KD(\hat{m} - 1)} - R_i u(t)$ with $u(0) \geq 0$ satisfies $u(t) \geq l^{(\hat{m})} t^{KD(\hat{m} - 1) + 1}$ with $l^{(\hat{m})} \equiv e^{-R_i} \bar{c}_3 (KD(\hat{m} - 1) + 1) \in (0, 1)$. Applying the comparison principle for ODEs and noting that $D(\hat{m}) = KD(\hat{m} - 1) + 1$, it follows that $\mu_i(t) \geq b^{(\hat{m})} t^{D(\hat{m})}$, which proves the claim for $\hat{m} = m_0 + 1$. This completes the proof of the proposition. \[\square\]

4.6. A discrete communication condition. We now show that under the slightly stronger assumption, Assumption \([4.11]\) on the transition rates of the interacting particle system, the empirical version of the strong communication condition that was introduced in Definition \([4.4]\). From Section 8. We first describe a discrete version of the strong communication condition that was introduced in Definition \([4.4]\).

Given $u, v \in \mathbb{N}$, we denote $[u, v] = \{u, u + 1, \ldots, v - 1\}$.

Definition 4.15. For any $x, y \in X_n$ and $T \in \mathbb{N}$, a discrete strongly communicating path of length $T$ from $x$ to $y$ with constants $c_1 > 0$, $p_1 < \infty$, and $F \in \mathbb{N}$ is a set of points $\{\phi_0, \phi_1, \ldots, \phi_F\} \subset X_n$ such that $\phi_0 = x$, $\phi_F = y$ and the following properties are satisfied.

i). There exist $\{v_m\}_{m=1}^F \subset \mathcal{V}$, and $0 = t_0 < t_1 < \cdots < t_F = T$, such that

$$
\phi_{s+1} - \phi_s = \frac{1}{n} v_m, \ s \in \{t_{s-1}, t_{s-1} + 1, \ldots, t_s - 1\}.
$$

ii). For all $n$ sufficiently large and $m = 1, \ldots, F$,

$$
\lambda_{v_m}^{(k)}(\phi_s) \geq c_1 \prod_{j \in \mathcal{N}_{v_m}} (\phi_s)_j, \ s \in \{t_{m-1}, t_{m-1} + 1, \ldots, t_m - 1\},
$$

where for $v \in \mathcal{V}$, $N_v$ is as defined in \([3.4]\).

Property 4.16. There exist constants $c, c_1 > 0, C', C'_1, p, p_1 < \infty$ and $\bar{F} \in \mathbb{N}$ with the following properties.

i). For any $x, y \in S$, there exist $t \in (0, 1]$, and a strongly communicating path $\phi$ that connects $x$ to $y$ on $[0, t]$ with constants $c, p, \bar{F}, c'_1, p_1$ such that the scaling property \([4.3]\) is satisfied.
\[ T_{x,y} \leq C_1 n \|x - y\|. \]

Clearly, Property \ref{prop:scaling} is a strengthening of the communication property stated in Property \ref{prop:comm}

**Proposition 4.17.** If \( \{\Gamma_{ij}(\cdot), (i,j) \in J^k, k = 1, \ldots, K \} \) satisfies Assumption \ref{assump:assumption1}, Assumption \ref{assump:assumption2}, and Assumption \ref{assump:assumption3} then the associated jump rates \( \{\lambda_{ij}(\cdot), v \in V\} \) satisfy Property \ref{prop:scaling} and hence, also Property \ref{prop:comm}.\hfill \box

**Proof.** The first part of Property \ref{prop:scaling}, namely the existence of a strongly communicating path, follows immediately from the first assertion of Lemma \ref{lem:main}. For the second part, we construct a discrete strongly communicating by discretizing the strongly communicating path constructed in Lemma \ref{lem:main}. Specifically, for \( n \in \mathbb{N} \), given \( x, y \in S_n \), given the strongly communicating path \( \phi_{P,n} \) on \([0, t_F]\) for some \( t_F \in \mathbb{N} \), which satisfies the additional properties stated in the last assertion of Lemma \ref{lem:main}, then it is easy to verify that if \( \phi_m = \phi_{P,n}(m) \) for \( m \in [0, t_F] \cap \mathbb{N} \), then \( \{\phi_m\} \) is a discrete strongly communicating path from \( x \) to \( y \) and the property \ref{eq:scaling} can be deduced from the corresponding property \ref{prop:comm} for communicating paths that was established in Lemma \ref{lem:main} \hfill \box

## 5. The Variational Representation Formula

### 5.1. Variational Representation for a Poisson Random Measure.\hfill \box

We first review the variational representation formula for a Poisson random measure stated in \cite[Theorem 2.1]{Variational}. For a locally compact Polish space \( S \), let \( B(S) \) denote the Borel sigma algebra and let \( M_F(S) \) denote the space of all measures \( \nu \) on \( (S, B(S)) \) satisfying \( \nu(K) < \infty \) for every compact \( K \subset S \). Letting \( C_c(S) \) denote the space of continuous functions with compact support, we equip \( M_F(S) \) with the weakest topology such that for every \( f \in C_c(S) \), the function \( \nu \mapsto \int_S f d\nu, \nu \in M_F(S) \), is continuous. Let \( Y \) be a locally compact Polish space, \( Y_T = [0, T] \times Y \), both equipped with the usual Euclidean topology, and let \( \mathcal{M} = M_F(Y_T) \). For some fixed measure \( \nu \in M_F(Y) \), let \( \nu_T = m_T \otimes \nu \), where \( m_T \) is Lebesgue measure on \([0, T] \). [In our use below we take \( Y = [0, \infty) \) and \( \nu \) to be Lebesgue measure.] For \( \theta \in [0, \infty) \), let \( \mathbb{P}_\theta \) denote the unique probability measure on \((\mathcal{M}, B(M))\) under which the canonical map \( \tilde{N} : \mathcal{M} \to \mathcal{N}, \tilde{N}(\omega) = \omega \), is a Poisson random measure with intensity measure \( \theta \nu_T \). Let \( \mathbb{E}_\theta \) denote expectation with respect to \( \mathbb{P}_\theta \). For notational convenience, we omit the dependence on \( \mathbb{P}_\theta \) and \( \mathbb{E}_\theta \) on the fixed measure \( \nu_T \).

We define a controlled Poisson random measure as follows. Let \( \mathcal{W} = Y \times [0, \infty) \) and \( \mathcal{W}_T = [0, T] \times \mathcal{W} = Y_T \times [0, \infty) \), both equipped with the Euclidean product topology. Let \( \mathcal{M} = M_F(\mathcal{W}_T) \) and let \( \bar{\mathcal{P}} \) be the unique probability measure on \((\mathcal{M}, B(M))\) under which the canonical map \( \tilde{\mathcal{N}} : \mathcal{M} \to \mathcal{N}, \tilde{\mathcal{N}}(\omega) = \omega \), is a Poisson random measure with intensity measure \( \tilde{\nu}_T = \nu_T \otimes m \), where \( m \) is Lebesgue measure on \([0, \infty) \). Let \( \bar{\mathbb{E}} \) denote expectation with respect to \( \bar{\mathbb{P}} \). Also, define

\[
\mathcal{G}_t = \sigma \left\{ \tilde{N} \left( (0, s] \times A \right) : 0 \leq s \leq t, A \in B(W) \right\},
\]

and let \( \mathcal{F}_t \) denote its completion under \( \bar{\mathbb{P}} \). We equip \((\mathcal{M}, B(M))\) with the filtration \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) and denote by \( \overline{\mathcal{P}} \) the corresponding predictable \( \sigma \)-field on \([0, T] \times \mathcal{M} \).

**Definition 5.1.** Let \( \bar{\mathcal{A}} \) be the class of \( (\bar{\mathcal{P}} \otimes B(Y)) \setminus B[0, \infty) \) measurable maps \( \varphi : [0, T] \times \mathcal{M} \times Y \to [0, \infty) \).
The role of $\varphi$ is to control the intensity of jumps at $(s, \omega, y)$ by thinning in the additional $r$-variable in (5.1) below. For $\varphi \in \tilde{A}$, define $N^\varphi : \tilde{M} \to M$ by
\begin{equation}
(5.1) \quad N^\varphi_\omega ((0, t] \times U) = \int_{(0,t] \times U} \mathbb{1}_{[0,\varphi(s,\omega,y)]}(r) \tilde{N}_\omega( dsdydr), \quad t \in [0, T], U \in \mathcal{B}(Y), \ \omega \in \tilde{M}.
\end{equation}

In what follows, we often suppress the dependence of $\varphi(t, \omega, y)$, $\tilde{N}_\omega$ and $N^\varphi_\omega$ on $\omega$. Under $\tilde{P}$, $N^\varphi$ is a controlled random measure on $\mathcal{Y}_T$ with $\varphi(s, y)$ determining the intensity for points at location $y$ and time $s$. With some abuse of notation, for $\theta \in [0, \infty)$ we will let $N^\theta$ be defined as in (5.1) with $\varphi(s, y) \equiv \theta$. Note that the law (on $\mathcal{M}$) of $N^\theta$ under $\tilde{P}$ coincides with the law of $N$ under $\mathcal{P}_\theta$.

Recall $\ell(\cdot)$ as defined in (4.6). For $\varphi \in \tilde{A}$ define the random variable $L_T(\varphi)$ by
\begin{equation}
(5.2) \quad L_T(\varphi)(\omega) = \int_{\mathcal{Y}_T} \ell(\varphi(t, \omega, y)) \nu_T( dtdy) = \left( \int_0^T \left( \int_{(0, \infty)} \ell(\varphi(t, \omega, y)) \nu(dy) \right) dt, \omega \in \tilde{M}. \right.
\end{equation}

**Definition 5.2.** Define $\tilde{A}_b$ to be the class of $(\tilde{P} \otimes \mathcal{B}(Y)) \setminus \mathcal{B}[0, \infty)$ measurable maps $\varphi$ such that for some $B < \infty$, $\varphi(t, \omega, y) \leq B$ for all $(t, \omega, y) \in [0, T] \times \mathcal{M} \times Y$.

In later sections we will set $T = 1$, and hence the dependence of $\tilde{A}$ and $\tilde{A}_b$ on $T$ can be omitted. Let $M_b(\mathcal{M})$ denote the space of bounded Borel measurable functions on $\mathcal{M}$. We then have the following representation formula for Poisson random measures.

**Theorem 5.3.** Let $F \in M_b(\mathcal{M})$. Then for any $\theta > 0$,
\begin{equation}
(5.3) \quad -\log \tilde{E}_\theta[\exp(-F(N))] = \inf_{\varphi \in \tilde{A}_b} \tilde{E}_\theta[F(L_T(\varphi) + F(N^\theta \varphi))].
\end{equation}

**Proof.** For $F \in M_b(\mathcal{M})$ and $\theta > 0$, it follows from Theorem 2.1 of [5] that
\begin{align*}
-\log \tilde{E}_\theta[\exp(-F(N))] & = -\log \tilde{E}[\exp(-F(N^\theta))] \\
& = \inf_{\varphi \in \tilde{A}} \tilde{E}[\theta L_T(\varphi) + F(N^\theta \varphi)].
\end{align*}

Moreover, Theorem 2.4 of [5] states that the above infimization can in fact be taken over the smaller class of controls, where for each control $\varphi$ there is $B \in (0, \infty)$ and compact $K \subseteq Y$ such that $1/B \leq \varphi(t, \omega, y) \leq B$ for all $(t, \omega, y) \in [0, T] \times \mathcal{M} \times K$ and $\varphi(t, \omega, x) = 1$ for all $(t, \omega, y) \in [0, T] \times \mathcal{M} \times K^c$. Since $\tilde{A}_b \subseteq \tilde{A}$ contains this class of controls, we obtain (5.3). \qed

### 5.2. Variational representation for the empirical measure process.

In this section we derive a variational representation formula for the empirical measure process $\mu^n$. We represent $\mu^n$ as a solution to a stochastic differential equation that is driven by finitely many iid Poisson random measures, and use thinning functions to obtain the desired jump rates. We then derive a variational representation formula for $\mu^n$, by viewing it as the image of a measurable mapping that acts on a collection of rescaled Poisson random measures.

Take $\nu = m$, so that $\nu_T = m_T \otimes m$. For $n \in \mathbb{N}$, let $\{N^n_v, v \in \mathcal{V}\}$ be a collection of iid Poisson random measures on $\mathcal{Y}_T$ with intensity measure $m \nu_T$. Then we have the following SDE representation for the empirical measure process: for $t \in [0, T]$,
\begin{equation}
(5.4) \quad \mu^n(t) = \mu^n(0) + \sum_{v \in \mathcal{V}} v \int_{[0, t]} \int_{\mathcal{Y}} \mathbb{1}_{[0, \lambda^n_v(\mu^n(s-))]}(y) \frac{1}{n} N^n_v(dsdy).
\end{equation}
The existence of a solution to (5.4) is justified by the following argument.

We set
\[ R \doteq \sup_{v \in V, x \in S_n, n \in \mathbb{N}} \lambda_v^n (x) < \infty, \]
where the finiteness of $R$ follows because Property 2.3 is satisfied and (2.14) holds. Let $M_{\text{atom}}$ denote the set of all $m = \{ m_v, v \in V \}$, where for each $v \in V$, $m_v$ is an atomic measure on $Y_T$, with the property that $m_v(\{ t \times [0, \tilde{R}] \}) > 0$ for only finitely many $t$. Recall from Section 2.1 that $S$ denotes the unit $(d-1)$-dimensional simplex and $S_n$ is the corresponding sublattice. Define $\Delta^{d-1} = \{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1 \}$ to be the $(d-1)$-dimensional hyperplane that contains $S$. Denote by $D ([0, T] : S)$ and $D \left( [0, T] : \Delta^{d-1} \right)$ respectively the space of càdlàg functions on $[0, T]$ that take values in $S$ and $\Delta^{d-1}$.

Let $\lambda^n = \{ \lambda^n_v, v \in V \}$. We also extend the definition of $\lambda^n$ to $\Delta^{d-1}$ by define it to be zero in $\Delta^{d-1} \setminus S_n$. Define $h_n : M_{\text{atom}} \times S \times \left( [0, \infty) \Delta^{d-1} \right)^{\otimes |V|} \to D \left( [0, T] : \Delta^{d-1} \right)$ as the mapping that takes $(m, \rho, \lambda^n) \in M_{\text{atom}} \times S \times \left( [0, \infty) \Delta^{d-1} \right)^{\otimes |V|}$ to the process $\eta \in D \left( [0, T] : \Delta^{d-1} \right)$ defined by
\[ \eta(t) \doteq \rho + \sum_{v \in V} v \int_{[0,t]} \int_S \mathbb{1}_{[0,\lambda_v^n (\eta(s^-))]} (y) m_v (dsdy), \quad t \in [0, T]. \]

In particular, when $m = \frac{1}{n} N^n$ the process $\eta$ lie in $S_n$.

The existence of a solution $\eta(\cdot)$ to (5.4) is easily verified by the following recursive construction. Set $t_0 = 0$, and define $\eta^0(t) \doteq \rho$ for $t \geq 0$. Assume as part of the recursive construction that for some $k \in \mathbb{N}_0$, a solution $\eta^k(\cdot)$ to (5.6) has been constructed on the interval $[0, t_k]$, and that $\eta^k(t) = \eta^k(t_k)$ for $t \geq t_k$. For any $t \in [t_k, T]$ and $v \in V$, let
\[ A_v(t) \doteq \{ (s, y) : s \in [t_k, t], y \in [0, \lambda_v^n (\eta^k(s))] \}, \]
and
\[ t_{k+1} = \inf \{ t > t_k \text{ such that for some } v \in V, m_v (A_v(t)) > 0 \} \wedge T, \]
where for $a, b \in \mathbb{R}$, $a \wedge b$ denotes the minimum of $a$ and $b$. We then define $\eta^{k+1} : [0, T] \to \mathbb{R}^d$ by setting
\[ \eta^{k+1}(t) \doteq \eta^k(t) \quad \text{for } t \in [0, t_{k+1}], \]
\[ \eta^{k+1}(t_{k+1}) \doteq \eta^k(t_{k+1}) + \sum_{v \in V} v \int_{[t_k, t_{k+1}]} \int_S \mathbb{1}_{[0,\lambda_v^n (\eta^k(s^-))]} (y) m_v (dsdy), \]
and $\eta^{k+1}(t) \doteq \eta^{k+1}(t_{k+1})$ for $t \in [t_{k+1}, T]$. Since $m_v$ has finitely many atoms on $[0, T] \times [0, \tilde{R}]$, the construction will produce a function defined on all of $[0, T]$ in $M < \infty$ steps, at which time we set $\eta(t) = \eta^M(t)$. Since $N \in M_{\text{atom}}$ for $P_n$-a.e. $\omega \in \mathcal{M}$, we can write
\[ \mu^n(t, \omega) = h_n \left( \frac{1}{n} N^n, \mu^n (0, \omega), \lambda^n \right) (t). \]

We now describe two classes of controls that will be used below. Recall that $\nu_T = \nu_T \otimes m$. Let $\{ N^n_v, v \in V \}$ be a collection of iid Poisson random measures on $W_T$ with intensity measure $n \nu_T$. We will apply the representation that is appropriate for these $|V|$ independent Poisson random measures. The underlying probability space is now the product space $(\mathcal{M}, \mathcal{B}(\mathcal{M}))^{\otimes |V|}$ (with an
abuse of notation we retain $\hat{\mathbb{P}}$ to denote the probability measure on this space). Let

$$G_t^{[V]} = \sigma \{ N^n((0, s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}(Y), v \in V \}$$

and let $\mathcal{F}_t^{[V]}$ denote its completion under $\hat{\mathbb{P}}$. Denote by $\hat{\mathcal{P}}^{[V]}$ the predictable $\sigma$-field on $[0, T] \times \mathcal{M}^{[V]}$ with the filtration $\{ \mathcal{F}_t^{[V]} : 0 \leq t \leq T \}$ on $(\hat{\mathcal{M}}, \mathcal{B}(\hat{\mathcal{M}}))^{[V]}$, and let $\hat{A}^{[V]}$ and $\hat{A}_b^{[V]}$ be defined analogously as was done for the case of a single Poisson random measure. Given $\varphi \in \hat{A}_b^{[V]}$ and $\mu^n(0) \in S$, we define the controlled jump Markov process $\hat{\mu}^n \in D([0, T] : S)$ to be the solution to the following SDE: for $t \in [0, T]$,

$$\hat{\mu}^n(t) = \mu^n(0) + \sum_{v \in V} v \int_{[0,t]} \int_Y I_{[0,\lambda^n_v(\hat{\mu}^n(s))]}(y) \int_{[0,\hat{\mu}^n(s)-y]} I_{[0,\varphi_v(s,w)]}(r) \frac{1}{n} N^n_v(dsdydr).$$

As described previously, $\varphi_v(s, y)$ will control the jump rate as a function of $(s, \omega, y)$. In particular, the overall jump rate for a jump of type $v$ is the product $\lambda^n_v(\hat{\mu}^n(s)) \varphi_v(s, y)$, so that $\varphi_v(s, y)$ perturbs the jump rate away from that of the original model, and the cumulative impact of the perturbation is found by integrating $y$ over $[0, \lambda^n_v(\hat{\mu}^n(s))]$. For $v \in V$, let $N^n_{v, \varphi}$ be defined as in (5.1), with $\varphi$ replaced by $\varphi_v$ and $\bar{N}$ replaced by $\bar{N}^n_v$, and let $N^{n,\varphi} = \{N^n_{v, \varphi}, v \in V\}$. For fixed $\varphi \in \hat{A}_b^{[V]}$, $N^{n,\varphi} \in \mathcal{M}_{atom}$ a.s. From the definition of $h_n(\cdot)$ and $N^{n,\varphi}$, it is clear that (5.8) is equivalent to the relation

$$\hat{\mu}^n = h_n \left( \frac{1}{n} N^{n,\varphi}, \frac{1}{n} \mu^n(0), \lambda^n \right).$$

Applying Theorem 5.3 and 5.7 with $F = G \circ h_n$, we obtain the following representation formula for $\mu^n$.

**Lemma 5.4.** For $G \in M_b(D([0, T] : S))$,

$$-\frac{1}{n} \log \mathbb{E} \left[ \exp(-nG(\mu^n)) \right] = \inf_{\varphi \in \hat{A}_b^{[V]}} \mathbb{E} \left[ \sum_{v \in V} L_T(\varphi_v) + G(\hat{\mu}^n) : \hat{\mu}^n = h_n \left( \frac{1}{n} N^{n,\varphi}, \frac{1}{n} \mu^n(0), \lambda^n \right) \right].$$

We now derive a simpler form of the variational representation formula than the one given in Lemma 5.3. The starting point for Lemma 5.3 is the representation given in [8] Theorem 2.1, which is general enough to cover situations where different points in $y \in Y$ correspond to different “types” of jumps. For our purposes this is in fact more general than we need, since all points in $\mathcal{Y}$ correspond to exactly the same type of jump, and all that is needed from the space $\mathcal{Y}$ is that it be big enough that arbitrary jump rates [such as $\lambda^n_{v_j}(\hat{\mu}^n(s))$] can be obtained by thinning. For example, we could have used $\mathcal{Y} = [0, \bar{R}]$ rather than $[0, \infty)$. The $y$’s in an interval such as $[0, \lambda^n_{v_j}(\hat{\mu}^n(s))]$ all play the same role, which is to indicate that a jump of type $v$ should occur. Hence one expects, and we will verify using Jensen’s inequality, that one can reformatulate the representation in terms of controls with no explicit $y$-dependence. Thus we will replace the $t$, $y$ and $v$ dependent controls $\hat{A}_b^{[V]}$ by controls $\hat{A}_b^{[V]}$ that only have $t$ and $v$ dependence, and rewrite the running cost as a function of the new controlled jump rates.

**Definition 5.5.** Define $\hat{A}_b^{[V]}$ to be the class of $\hat{\mathbb{P}} \mathcal{B}([0, \infty)^{[V]}$ measurable maps $\varphi : [0, T] \times \mathcal{M}^{[V]} \rightarrow [0, \infty)^{[V]}$ such that $\sup_{t \in [0, T], \omega \in \mathcal{M}^{[V]}, \varphi} \varphi_v(t, \omega) \leq B$ for some $B < \infty$. 

Define \( \Lambda^n : \mathcal{A}_b^{\otimes |V|} \times \mathcal{S} \rightarrow D \left( [0, T] : \Delta^{d-1} \right) \) by

\[
\Lambda^n (\tilde{\alpha}, \rho) (t) = \rho + \sum_{v \in V} v \int_{[0,t]} \int_\mathcal{Y} \mathbb{I} \{ \tilde{\alpha}_v (s-) \} (y) \frac{1}{n} N^n_v (dsdy).
\]  

(5.9)

\( \Lambda^n (\cdot, \rho) \) is well-defined for \( \tilde{\alpha} \in \mathcal{A}_b^{\otimes |V|} \).

We are now in a position to state the main variational representation formula, the proof of which is deferred to the Appendix. This representation appears to be the most appropriate one for finite state Markov chains, and expresses the variational functional as the sum of the expected cost for perturbing the jump rates, plus the expected value of the test function evaluated at the process whose dynamics follow the perturbed rates.

**Theorem 5.6.** Let \( F \in M_b \left( D \left( [0, T] : \Delta^{d-1} \right) \right) \). Then

\[
- \frac{1}{n} \log \mathbb{E} \left[ \exp \left( - nF (\mu^n) \right) \right]
\]

\[
= \inf_{\tilde{\alpha} \in \mathcal{A}_b^{\otimes |V|}} \mathbb{E} \left[ \sum_{v \in V} \int_0^T \xi^n_v (\tilde{\alpha}_v (t)) \ell \left( \frac{\tilde{\alpha}_v (t)}{\mu^n_v (\tilde{\alpha}_v (t))} \right) dt + F(\tilde{\mu}^n) : \tilde{\alpha}, \mu^n (0) \right].
\]

**Remark 5.7.** Since the integrand in the right hand side of the equation above is singular when \( \tilde{\mu}^n (t) \notin \mathcal{S} \) for some \( t \in [0, T] \), it is equivalent to infimize over a smaller class of control, namely, \( \tilde{\alpha} \in \mathcal{A}_b^{\otimes |V|} \) such that \( \tilde{\alpha}_v (t) = 0 \) when \( \tilde{\mu}^n (t) = x \in \partial \mathcal{S} \) and \( x + \frac{1}{n} v \) is taken outside \( \mathcal{S} \). By (5.7), the controlled process \( \tilde{\mu}^n \) will then lie in \( \mathcal{S} \). Therefore it suffices to prove Theorem 5.6 for \( F \in M_b \left( D \left( [0, T] : \mathcal{S} \right) \right) \).

5.3. **The Law of Large Numbers limit.** We next prove the law of large numbers limit stated in Theorem 2.4. First recall the law of large numbers result for scaled Poisson random measures: for any \( A \in \mathcal{B} \left( [0, T] \times [0, \infty) \right) \) such that \( m_T \otimes m (A) < \infty \), \( \frac{1}{n} N^n_v (A) \rightarrow m_T \otimes m (A) \) in probability, for any \( v \in V \). This implies that for any \( f \in C_c (\mathcal{S}) \), we have \( \int [0,T] \times [0,\infty) f (s, y) \frac{1}{n} N^n_v (dsdy) \rightarrow \int [0,T] \times [0,\infty) \tilde{f} (s, y) dsdy \). Rewrite (\[\tilde{\alpha}, \mu^n (0)\]) as

\[
\mu^n (t) = \rho_0 + \sum_{v \in V} v \int_{[0,t]} \xi^n_v (\mu^n (s-)) ds + M^n (t),
\]

where \( M^n (t) \doteq \sum_{v \in V} \int_{[0,t]} \xi^n_v (\mu^n (s-)) ds + M^n (t) \), is an \( \{ F^n_t \} \)-martingale. For any \( \varepsilon > 0 \), with \( \bar{R} \) defined as in (5.5), Doob’s maximal inequality gives

\[
P \left( \sup_{t \in [0,T]} \| M^n (t) \| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E} \left[ \| M^n (T) \|^2 \right]
\]

\[
= \frac{1}{\varepsilon^2} \mathbb{E} \left[ \left\| \sum_{v \in V} \int_0^T \int_{[0,\infty)} \mathbb{I} \{ \xi^n_v (\mu^n (s-)) \} (y) \frac{1}{n} N^n_v (dsdy) - ndsdy \right\|^2 \right]
\]

\[
\leq \frac{|V|}{\varepsilon^2} \mathbb{E} \left[ \int_0^T \frac{\bar{R}}{\int_0^T dsdx} \right],
\]
which tends to zero as $n \to \infty$. Let $\mu$ the unique solution $\mu$ of
\[
\mu(t) = \rho_0 + \sum_{v \in V} v \int_{[0,t]} \lambda_v (\mu(s-)) \, ds,
\]
which is the integral version of (2.15). Combining $\mathbb{P}(\sup_{t \in [0,T]} \|M^n(t)\| > \varepsilon) \to 0$ with the fact that $\lambda^n_v$ converges uniformly to $\lambda_v$ and that $\lambda_v$ is Lipschitz continuous (by Property 2.3), it follows from Gronwall’s inequality that $\mu^n \to \mu$ in probability (uniformly on $t \in [0,T]$). This completes the proof.

6. Proof of the LDP Upper Bound

A large deviation upper bound for a general class of sequences of Markov processes was obtained in [12]. We will apply the result of [12] to establish a large deviation upper bound for the sequence $\{\mu^n(\cdot)\}_{n \in \mathbb{N}}$, in which for each $n \in \mathbb{N}$, $\mu^n(\cdot)$ is a jump Markov process on $\mathcal{S}_n$ with generator $\mathcal{L}_n$ in (6.1) such that the associated sequence of rates $\{\lambda^n_v(\cdot), v \in V\}, n \in \mathbb{N}$, satisfy Property 2.3 for suitable Lipschitz continuous functions $\{\lambda_v(\cdot), v \in V\}$. Theorem 1.1 of [12] applies to Markov processes whose infinitesimal generator uses the limit jump rates:

(6.1) \[ \mathcal{L}^0_n (f) (x) = n \sum_{v \in V} \lambda_v (x) \left( f \left( x + \frac{1}{n} v \right) - f(x) \right). \]

However, as discussed below the uniform convergence of $\lambda^n_v(x)$ to $\lambda_v(x)$ implies that the large deviation properties the sequence of Markov processes with generators $\mathcal{L}^0_n$ and those generators 2.7 coincide.

To state the result from [12], for $x, \theta \in \mathbb{R}^d$, define
\[ H (x, \theta) \doteq \sum_{v \in V} \lambda_v (x) (\exp(\theta, v) - 1). \]

Note that $H$ is continuous. Let $L^0$ be its Legendre-Fenchel transform defined by
\[ L^0 (x, \beta) \doteq \sup_{\theta \in \mathbb{R}^d} [\langle \theta, \beta \rangle - H (x, \theta)]. \]

Also, for $t \in [0,1]$ define $L^0_t$ as in (3.8), but with $L$ replaced by $L^0$.

**Proposition 6.1.** For any compact set $\mathcal{K} \subset \mathcal{S}$ and $M < \infty$, the set
\[ \{ \gamma : L^0 (\gamma) \leq M, \gamma(0) \in \mathcal{K} \} \]
is compact. Assume the family of jump rates $\{\lambda_v(\cdot), v \in V\}$ satisfies Property 2.3. Also, assume that the initial conditions $\{\mu^n(0)\}_{n \in \mathbb{N}}$ are deterministic, and $\mu^n(0) \to \mu_0 \in \mathcal{P}(\mathcal{X})$ as $n$ tends to infinity. Let $\{Y^n\}_{n \in \mathbb{N}}$ be a sequence of Markov processes with generator $\mathcal{L}^0_n$, and $Y^n = \mu^n(0)$. Then $\{Y^n\}$ satisfies the large deviation upper bound with rate function $I^0$.

**Proof.** This result follows from Theorem 1.1 of [12] with (in the notation of [12]) $\varepsilon = 1/n$, $a(\cdot) = b(\cdot) = 0$, and $\mu_x(\cdot) = \mathbb{I}_{\{x \in \mathcal{S}\}} \sum_{v \in V} \lambda_v (x) \delta_v (\cdot)$.

We have introduced the function $L^0$ in (6.3) and the “local rate function” $L$ in (3.7), defined respectively in terms of a Legendre transform and the Poisson local rate function $\ell$. We now show that these functions are equal (see also Lemma 3.1 of [31]).
Proposition 6.2. Assume the family of jump rates \( \{ \lambda_v (\cdot), v \in \mathcal{V} \} \) satisfies Property P. For all \( x \in \mathcal{S}, \beta \in \Delta^{d-1} \),

\[
L^0 (x, \beta) = \inf_{q \in [0, \infty)^{\mathcal{V}} : \sum_{v \in \mathcal{V}} vq_v = \beta} \sum_{v \in \mathcal{V}} \lambda_v (x) \ell \left( \frac{q_v}{\lambda_v (x)} \right) = L (x, \beta).
\]

Moreover, \( I = I^0 \).

Proof. Defining \( h_{v,a} : \mathbb{R}^d \rightarrow \mathbb{R} \) by \( h_{v,a} (\theta) = a (\exp (\langle \theta, v \rangle) - 1) \) for \( v \in \mathbb{R}^d \) and \( a \in [0, \infty) \), we can write \( H (x, \theta) = \sum_{v \in \mathcal{V}} h_{v, \lambda_v (x)} (\theta) \). The Legendre-Fenchel transform of \( h_{v,a} \) can be computed explicitly as

\[
h^*_{v,a} (\beta) = \begin{cases} 
a \ell (y) & \text{if } \beta = a v, \\
\infty & \text{otherwise}. 
\end{cases}
\]

Since \( H \) is a finite sum of convex functions, we can apply a standard result in convex analysis to calculate its Legendre-Fenchel transform (see, e.g., Theorem D.4.2 of [11]):

\[
\left( \sum_{v \in \mathcal{V}} h_{v, \lambda_v (x)} \right)^* (\beta) = \inf \left\{ \sum_{v \in \mathcal{V}} h^*_{v, \lambda_v (x)} (\beta_v) : \sum_{v \in \mathcal{V}} \beta_v = \beta \right\}.
\]

Hence, (6.4) holds, which immediately implies \( I = I^0 \). \( \square \)

7. Properties of the Local Rate Function

In this section we establish useful properties of the proposed local rate function

\[
L (x, \beta) = \inf_{q \in [0, \infty)^{\mathcal{V}} : \sum_{v \in \mathcal{V}} vq_v = \beta} \sum_{v \in \mathcal{V}} \lambda_v (x) \ell \left( \frac{q_v}{\lambda_v (x)} \right), \ x \in \mathcal{S}, \beta \in \Delta^{d-1},
\]

first introduced in (3.7). The following observation will be useful in establishing properties of the function \( L \). Given a set of vectors \( \{ w_j, j = 1, \ldots, F \} \subset \mathbb{R}^d \), let the positive cone spanned by \( \{ w_j, j = 1, \ldots, F \} \) be denoted by

\[
\mathcal{C} (\{ w_j \}) = \left\{ w \in \mathbb{R}^d : \text{there exist } a_j \geq 0, j = 1, \ldots, F, \text{ with } w = \sum_{j=1}^F a_j w_j \right\}.
\]

Remark 7.1. Define

\[
\mathcal{V}_+ = \left\{ v \in \mathcal{V} : \text{for any } a > 0, \inf_{x \in \mathcal{S}^a} \lambda_v (x) > 0 \right\}.
\]
to be the set of directions for which the associated jump rates are bounded below away from zero on every compact subset of int(S). We claim that if \( \{ \lambda_v(x) \mid v \in V \} \) satisfies Property 2.3 Property 4.3 and property (3) of Lemma 4.12 then for every \( x \in \text{int}(S) \),
\[
C(\{ v \in V : \lambda_v(x) > 0 \}) = C(V_+) = \Delta^{d-1}.
\]
The first equality is a direct consequence of property (3) of Lemma 4.12. To show the second equality, it is clear that \( C(V_+) \subset \Delta^{d-1} \). To see why the reverse containment is true, given any \( w \in \Delta^{d-1} \), choose \( x, y \in \text{int}(S) \) such that \( y = x + rw \) for some \( r > 0 \). Then Property 4.3 implies that there exists \( t > 0 \) and a communicating path \( \phi \) on \([0, t]\) from \( x \) to \( y \). By Definition 4.11, this means that there exists \( F \in \mathbb{N} \) and \( v_m \in V, m = 1, \ldots, F \), such that \( w = \phi(t) - \phi(0) \) is a positive linear combination of the vectors \( v_m, m = 1, \ldots, F \). On the other hand, since \( y \in \text{int}(S) \), property ii of Definition 4.3 implies that for each \( m = 1, \ldots, F \), \( \lambda_{v_m} \) is not identically zero on the simplex. By property (3) of Lemma 4.12 this implies that \( v_m \in V_+ \) for every \( m \), which in turn implies \( w \in C(V_+) \). Since \( w \) is an arbitrary vector in \( \Delta^{d-1} \), this proves the claim.

**Lemma 7.2.** Assume that \( \{ \lambda_v(x) \mid v \in V \} \) satisfies Property 2.3 Property 4.3 and property (3) of Lemma 4.12. Then \( L \) is nonnegative and uniformly continuous on compact subsets of \( \text{int}(S) \times \Delta^{d-1} \), and for each \( x \in S \), \( L(x, \cdot) \) is strictly convex on its domain of finiteness.

**Proof.** \( L \) is nonnegative by definition 7.1 and for each \( x \in S \), relation 0.3 exhibits \( L(x, \cdot) \) as the Legendre-Fenchel transform of the smooth convex function \( H(x, \cdot) \) defined in 6.2. It follows from [30] Theorem 12.2 that \( L(x, \cdot) \) is strictly convex on its domain of finiteness. Since Property 2.3 holds, by Proposition 6.2 we have \( L = L^0 \). Due to property (3) of Lemma 4.12 we can replace the sum over \( v \in V \) in the expression 0.3 for \( L_0 \) by the sum over \( v \in V_+ \). According to Remark 4.1 under the assumptions of the lemma, the convex cone generated by \( \{ v \in V_+ \} \) is all of \( \Delta^{d-1} \). Since \( x \in \text{int}(S) \) implies that all elements of \( \{ \lambda_v(x), v \in V_+ \} \) are strictly positive, 0.4 implies \( L(x, \beta) < \infty \) for \( \beta \in \Delta^{d-1} \). Since Property 2.3 implies each \( \lambda_v(x), v \in V_+ \), is continuous, the joint continuity of \( L \) on \( \text{int}(S) \times \Delta^{d-1} \) follows also from 0.4 and positivity of \( \lambda_v(x), v \in V_+ \) for \( x \in \text{int}(S) \). This implies uniform continuity on compact subsets of \( \text{int}(S) \times \Delta^{d-1} \). □

The following elementary inequality can be proved using Legendre transforms.

**Lemma 7.3.** For \( r, q \in [0, \infty) \) we have \( r \ell \left( \frac{q}{r} \right) + r (e - 1) \geq q \).

We now study the asymptotic behavior of \( L \) in the second variable.

**Proposition 7.4.** Suppose \( \{ \lambda_v(x), v \in V \} \) satisfies the assumptions stated in Lemma 7.2. Given \( a > 0 \), there exist constants \( B = B(a) < \infty \) and \( C_2 = C_2(a, B), C_3 = C_3(a, B) \) such that
\[
L(x, \beta) \leq \begin{cases} 
C_2 \| \beta \| \log \| \beta \| & \text{if } x \in S^a \text{ and } \beta \in \Delta^{d-1}, \| \beta \| > B \\
C_3 & \text{if } x \in S^a \text{ and } \beta \in \Delta^{d-1}, \| \beta \| \leq B.
\end{cases}
\]
Moreover, for \( B < \infty \) sufficiently large, there exists \( c_1 = c_1(B) > 0 \) such that if \( x \in S \), then
\[
L(x, \beta) \geq c_1 \| \beta \| \log \| \beta \| \quad \text{for all } \beta \in \Delta^{d-1}, \| \beta \| > B.
\]
In particular, \( \beta \mapsto L(x, \beta) \) is superlinear, uniformly in \( x \).

**Proof.** Fix \( a > 0 \). For any \( B < \infty \), since \( \{ (x, \beta) \mid x \in S^a \times \Delta^{d-1} : \| \beta \| \leq B \} \) is a compact subset of \( \text{int}(S) \times \Delta^{d-1} \), the uniform boundedness of \( L \) on this set follows directly from the uniform continuity of \( L \) established in Lemma 7.2.

For the upper bound when \( \| \beta \| > B \), we first assume \( \| \beta \| = 1 \). By Remark 7.3 there exists a vector \( q = q(\beta) \in [0, \infty)^{|V|} \), such that \( \sum_{v \in V_+} v_q \beta = \beta, q_v > 0 \) for \( v \in V_+ \) and \( q_v = 0 \) for \( v \in V \setminus V_+ \).
Since $C(V_+^e) = \Delta^{d-1}$, we can assume $\max_{\| \beta \|=1} |q_\nu (\beta)|$ is finite. By scaling, it follows that there exists some constant $c_0 < \infty$, such that for any $\beta \in \Delta^{d-1}$, there exists a vector $q \in [0, \infty)^{|V|}$ such that $\sum_{v \in V_+^e} vq_\nu = \beta$, $\max_{\| \beta \|} |q_\nu| \leq c_0 \| \beta \|$, and $q_\nu = 0$ for $v \in V \setminus V_+^e$. It follows that for some $c_4 < \infty$,

$$L(x, \beta) \leq c_4 \sum_{v \in V} q_\nu \log \frac{q_\nu}{\lambda_\nu(x)} \leq C_2 \| \beta \| \log \| \beta \|$$

if $\| \beta \| \geq B$, for some $B = B(a)$ sufficiently large and all $x \in S^a$. This finishes the proof of the upper bound.

Now, consider the lower bound for $L$ on $\{ (x, \beta) \in S \times \Delta^{d-1} : \| \beta \| > B \}$. Since $L = L^0$ due to Proposition 6.2, by the definition (6.3) of $L^0$, we have for $t > 0, \theta = t \frac{\beta}{\| \beta \|}$, and $R < \infty$ defined as in (2.4),

$$L(x, \beta) \geq \langle \theta, \beta \rangle - H(x, \theta)$$

$$\geq t \| \beta \| - \sum_{v \in V} \lambda_\nu(x) \exp(\langle \theta, v \rangle)$$

$$\geq t \| \beta \| - R|V| \exp \left( \max_{v \in V} \| v \| t \right).$$

Substituting $t = \frac{1}{\max_{v \in V} \| v \|} \log \| \beta \|$, this implies

$$L(x, \beta) \geq \frac{1}{\max_{v \in V} \| v \|} \| \beta \| \log \| \beta \| - R|V| \| \beta \| \geq c_1 \| \beta \| \log \| \beta \|$$

for some constant $c_1 > 0$, provided $\| \beta \|$ is sufficiently large.

Recall that, given $\xi > 0$, $D([a,b] : S^\xi)$ denotes the space of càdlàg functions on $[a,b]$ taking values in $S^\xi$.

**Proposition 7.5.** Suppose $\{\lambda_\nu(\cdot), v \in V\}$ satisfies the assumptions stated in Lemma 7.2. Given $0 < a < b \leq 1$ and $\xi > 0$, suppose that $\gamma \in AC([a, b] : S^\xi)$ satisfies $\int_a^b L(\gamma(s), \dot{\gamma}(s)) ds < \infty$. Let $\{\dot{\gamma}_\delta\}_{\delta \in (0, 1)} \subset D([a, b] : S^\xi)$ be such that $\sup_{t \in [a,b]} \| \dot{\gamma}_\delta(t) - \dot{\gamma}(t) \| \to 0$ as $\delta \to 0$. Then for any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\xi, \varepsilon) > 0$ such that for $\delta < \delta_0$,

$$\left| \int_a^b L(\gamma(s), \dot{\gamma}(s)) ds - \int_a^b L(\dot{\gamma}_\delta(s), \dot{\gamma}(s)) ds \right| < \varepsilon.$$

**Proof.** Fix $\xi > 0$ and $0 < a < b \leq 1$. Let $A$ be the measurable set of points $s \in [a, b]$ for which $\dot{\gamma}(s)$ is well defined and lies in $\Delta^{d-1}$, so that $[a, b] \setminus A$ has zero Lebesgue measure. Let $C_2 < \infty, c_1 > 0$ and $B < \infty$ be chosen according to Proposition 7.4, so that $L(x, \beta) \leq C_2 \| \beta \| \log \| \beta \|$ if $x \in S^\xi/2$ and $\| \beta \| > B$, and $L(x, \beta) \geq c_1 \| \beta \| \log \| \beta \|$ if $x \in S$ and $\beta \in \Delta^{d-1}, \| \beta \| > B$. For $B \in (B, \infty)$ define $\tilde{A} = \{ s \in A : \| \dot{\gamma}(s) \| \leq B \}$. Assume $\delta_0 > 0$ is small enough that $\dot{\gamma}_\delta(s) \in S^\xi/2$ for all $\delta < \delta_0$ and $s \in [a, b]$. Then

$$\int_{[a,b] \setminus \tilde{A}} L(\dot{\gamma}_\delta(s), \dot{\gamma}(s)) ds \leq C_2 \int_{[a,b] \setminus \tilde{A}} \| \dot{\gamma}(s) \| \log \| \dot{\gamma}(s) \| ds \leq \frac{C_2}{c_1} \int_{[a,b] \setminus \tilde{A}} L(\gamma(s), \dot{\gamma}(s)) ds,$$

and since by assumption $s \mapsto L(\gamma(s), \dot{\gamma}(s))$ is integrable on $[a, b]$, for large enough $B \sim \infty$,

$$\int_{[a,b] \setminus \tilde{A}} L(\dot{\gamma}_\delta(s), \dot{\gamma}(s)) ds + \int_{[a,b] \setminus \tilde{A}} L(\gamma(s), \dot{\gamma}(s)) ds \leq \varepsilon/2.$$
On the other hand, since \( \int_a^b L(\gamma (s), \dot{\gamma} (s)) \, ds < \infty \), by dominated convergence and the continuity of \( L(\cdot, \beta) \) for fixed \( \beta \in \Delta^{d-1} \) established in Lemma 7.2, we have

\[
\int_A L(\gamma^\delta(s), \dot{\gamma}(s)) \, ds \to \int_A L(\gamma(s), \dot{\gamma}(s)) \, ds.
\]

Hence by choosing \( \delta_0 > 0 \) smaller if need be, (7.5) and (7.6) imply that for \( \delta \in (0, \delta_0) \)

\[
\left| \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds - \int_a^b L(\gamma^\delta(s), \dot{\gamma}(s)) \, ds \right| \leq \varepsilon.
\]

Lemma 7.6. Suppose \( \{\lambda_v(\cdot), v \in V\} \) satisfies the assumptions stated in Lemma 7.2. Suppose that \( \gamma \in AC([0, 1] : S) \) satisfies \( \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds < \infty \) for some \( 0 \leq a < b \leq 1 \). Then

\[
\|\gamma(t) - \gamma(a)\| \log \frac{1}{t-a} \to 0 \quad as \ t \downarrow a.
\]

Proof. Fix \( t \in (a, b) \) and let \( A \) be the measurable set of points \( s \in [a, b] \) be for which \( \dot{\gamma}(s) \) is well defined and lies in \( \Delta^{d-1} \). We claim, and show below, that \( \|\dot{\gamma}(\cdot)\| \log \|\dot{\gamma}(\cdot)\| \) is integrable on \( [a, b] \). By Proposition 7.4 there exists \( B \) sufficiently large and \( c_1(B) > 0 \) such that (7.4) holds. Therefore, defining \( A_1 = \{s \in A : \|\dot{\gamma}(s)\| \leq B\} \), we have

\[
\int_a^b \|\dot{\gamma}(s)\| \log \|\dot{\gamma}(s)\| \, ds = \int_A \|\dot{\gamma}(s)\| \log \|\dot{\gamma}(s)\| \, ds \leq \frac{1}{c_1} \int_{A \setminus A_1} L(\gamma(s), \dot{\gamma}(s)) \, ds + \int_{A_1} B \log B \, ds
\]

\[
\leq \frac{1}{c_1} \int_a^b L(\gamma(s), \dot{\gamma}(s)) \, ds + (B \log B)(b-a),
\]

where the last inequality uses the nonnegativity of \( L \). On the other hand, by Jensen’s inequality, for \( t \in (a, 1) \),

\[
\int_a^t \|\dot{\gamma}(s)\| \log \|\dot{\gamma}(s)\| \, ds \geq (t-a) \left\| \frac{\gamma(t) - \gamma(a)}{t-a} \right\| \log \left\| \frac{\gamma(t) - \gamma(a)}{t-a} \right\|
\]

\[
= \|\gamma(t) - \gamma(a)\| \log \left\| \frac{\gamma(t) - \gamma(a)}{t-a} \right\|.
\]

Now, since \( \|\dot{\gamma}(\cdot)\| \log \|\dot{\gamma}(\cdot)\| \) is integrable, the left-hand side of the last display goes to zero as \( t \downarrow a \). The lemma follows by observing that \( \|\gamma(t) - \gamma(a)\| \log \|\gamma(t) - \gamma(a)\| \) also goes to zero as \( t \downarrow a \). \( \square \)

Recall the definition of \( V_+ \), given in Remark 7.1. The following result is used in Lemma 5.1 which contains a perturbation argument used in proving the LDP lower bound.

Lemma 7.7. Suppose \( \{\lambda_v(\cdot), v \in V\} \) satisfies the assumptions stated in Lemma 7.2. Let \( c_0(\rho) \) be given such that \( c_0(\rho) \to 1 \) as \( \rho \to 0 \). Suppose that \( x \in S, \{x^\rho\}_{\rho > 0} \subset \text{int}(S) \), are such that \( \|x - x^\rho\| \to 0 \) as \( \rho \to 0 \), and for any \( \rho > 0 \) and \( v \in V_+ \), \( \lambda_v(x) / \lambda_v(x^\rho) \leq c_0(\rho) \). Then there exists \( c = c(\rho) \) that only depends on \( c_0(\rho) \) and \( \|x - x^\rho\| \), that satisfies \( c(\rho) \to 0 \) as \( \rho \to 0 \), and has the property that

\[
L(x^\rho, \beta) \leq (1 + c(\rho)) L(x, \beta) + c(\rho), \quad \beta \in \Delta^{d-1}.
\]
Proof. Fix $\beta \in \Delta^{d-1}$. We can assume without loss of generality that there exists $q \in [0, \infty)^{|V_+|}$ such that $\sum_{v \in V_+} v q_v = \beta$ because, if not, then $L(x, \beta)$ is infinite and (7.7) holds trivially. Now, we claim (and justify below) that to prove the lemma, it suffices to show that for every $\rho > 0$, there exists a function $c(\rho)$ (depending only on $\|x^0 - x\|$ and $c_0(\rho)$) such that for every $q \in [0, \infty)^{|V_+|}$ such that $\sum_{v \in V_+} v q_v = \beta$, 

$$
(7.8) \quad \sum_{v \in V_+} \lambda_v (x^p) \ell \left( \frac{q_v}{\lambda_v (x^p)} \right) \leq (1 + c(\rho)) \sum_{v \in V_+} \lambda_v (x) \ell \left( \frac{q_v}{\lambda_v (x)} \right) + c(\rho),
$$

and $c(\rho) \to 0$ as $\rho \to 0$. To see that the claim holds, recall the expression (6.4) for $L$ and note that the left-hand side of (7.7) is dominated by the left-hand side of (7.8). The right-hand side of (7.7) is the infimum of the right-hand side of (7.8) over all $q \in [0, \infty)^{|V_+|}$ such that $\sum_{v \in V_+} v q_v = \beta$, where we have used the fact that $\lambda_v (x) > 0$ then $v \in V_+$, which follows from property (3) of Lemma 4.12

We have the following relations, each line of which is explained below.

$$
\sum_{v \in V_+} \lambda_v (x^p) \ell \left( \frac{q_v}{\lambda_v (x^p)} \right) - \sum_{v \in V_+} \lambda_v (x) \ell \left( \frac{q_v}{\lambda_v (x)} \right) = \sum_{v \in V_+} q_v \log \lambda_v (x) - \sum_{v \in V_+} \left( \lambda_v (x^p) - \lambda_v (x) \right) \\
\leq \log c_0 (\rho) \sum_{v \in V_+} q_v + C_1 \|x^p - x\| \\
\leq \log c_0 (\rho) \sum_{v \in V_+} \left( \lambda_v (x) \ell \left( \frac{q_v}{\lambda_v (x)} \right) + \lambda_v (x) (e - 1) \right) + C_1 \|x^p - x\| \\
\leq \log c_0 (\rho) \sum_{v \in V_+} \lambda_v (x) \ell \left( \frac{q_v}{\lambda_v (x)} \right) + \bar{C}(\rho) .
$$

The equality follows from the expression (3.6) for $\ell$; the first inequality (with $C_1 < \infty$) due to the assumption of the lemma and the Lipschitz continuity of $\lambda_v$; the second inequality follows from Lemma 7.3 with $r = \lambda_v (x)$ and $q = q_v$; and the final inequality just uses the definition $\bar{C}(\rho) \doteq \log c_0(\rho) R |V| (e - 1) + C_1 \|x - x^p\|$. Then (7.8) holds with $c(\rho) = \max \{ \log c_0(\rho), \bar{C}(\rho) \}$. Since $c(\rho)$ depends only on $c_0(\rho)$ and $\|x - x^p\|$, and the assumptions of the lemma imply that $c(\rho) \to 0$ as $\rho \to 0$, this completes the proof. \hfill \Box

For $t \in [0, 1]$ and $c > 0$, define

$$
\gamma_c (s) \doteq \gamma (cs), \quad s \in [0, t],
$$

which is a time reparametrization of $\gamma$. The next result is used in the proof of the locally uniform LDP in Section 9. It states that given a path $\gamma$ with finite cost, the cost of the path depends continuously on the reparametrization of time.

**Proposition 7.8.** Suppose $\{ \lambda_v (\cdot), v \in V \}$ satisfies the assumptions stated in Lemma 7.2. For $t \in [0, 1)$, suppose $\gamma \in AC ([0, 1]; S)$ is such that $I_t (\gamma) < \infty$. Then the function $c \mapsto I_{t/c} (\gamma_c)$ is continuous at 1.
Proof. First note that for $c$ close to 1, $\gamma_c \in AC ([0, t/c] : S)$ and

$$I_{t/c} (\gamma_c) = \int_0^{t/c} L (\gamma (cs), \dot{c}(cs)) ds = \frac{1}{c} \int_0^{t} L (\gamma (r), \dot{c}(r)) dr.$$  

We now bound the integral of $\frac{1}{c}L (\gamma, c\dot{\gamma}) - L (\gamma, \dot{\gamma})$ over $[0, t]$. Recall the definition of $L$ in (3.7). Since $\gamma$ is absolutely continuous and $I_{t}(\gamma) < \infty$, $\dot{\gamma}(u)$ is well defined and $L(\gamma(u), \dot{\gamma}(u)) < \infty$ for almost every $u \in [0, t]$. Thus, for any such $u \in [0, t]$ and $\varepsilon > 0$, there exists $q \in [0, \infty)^{|\mathcal{V}|}$ such that $\sum_{v \in \mathcal{V}} vq_v = c\dot{\gamma}(u)$ and

$$\sum_{v \in \mathcal{V}} \lambda_v (\gamma (u)) \ell \left( \frac{q_v/c}{\lambda_v (\gamma (u))} \right) \leq L (\gamma (u), \dot{\gamma} (u)) + \varepsilon.$$  

On the other hand, using the expression (3.6) for $\ell$ we also have

$$\sum_{v \in \mathcal{V}} \lambda_v (\gamma (u)) \ell \left( \frac{q_v/c}{\lambda_v (\gamma (u))} \right) = \sum_{v \in \mathcal{V}} \left( \frac{q_v}{c} \log \frac{q_v/c}{\lambda_v (\gamma (u))} - q_v/c + \lambda_v (\gamma (u)) \right)$$

$$= \frac{1}{c} \sum_{v \in \mathcal{V}} \left( q_v \log \frac{q_v}{\lambda_v (\gamma (u))} - q_v + \lambda_v (\gamma (u)) \right) + \left( 1 - \frac{1}{c} \right) \sum_{v \in \mathcal{V}} q_v$$

$$\geq \frac{1}{c} L (\gamma (u), c\dot{\gamma} (u)) + \left( \frac{1}{c} \log \frac{1}{c} \right) \sum_{v \in \mathcal{V}} q_v + \left( 1 - \frac{1}{c} \right) \sum_{v \in \mathcal{V}} \lambda_v (\gamma (u)).$$

The last two relations imply that

$$(7.9) \quad \frac{1}{c} L (\gamma (u), c\dot{\gamma} (u)) - L (\gamma (u), \dot{\gamma} (u)) \leq \varepsilon - \left( \frac{1}{c} \log \frac{1}{c} \right) \sum_{v \in \mathcal{V}} q_v - \left( 1 - \frac{1}{c} \right) \sum_{v \in \mathcal{V}} \lambda_v (\gamma (u)).$$

Similarly, by taking $q \in [0, \infty)^{|\mathcal{V}|}$, such that $\sum_{v \in \mathcal{V}} vq_v = \dot{\gamma}(u)$ and

$$\sum_{v \in \mathcal{V}} \lambda_v (\gamma (u)) \ell \left( \frac{cq_v}{\lambda_v (\gamma (u))} \right) \leq L (\gamma (u), c\dot{\gamma} (u)) + c\varepsilon,$$

an analogous computation yields

$$(7.10) \quad \frac{1}{c} L (\gamma (u), c\dot{\gamma} (u)) - L (\gamma (u), \dot{\gamma} (u)) \geq (\log c) \sum_{v \in \mathcal{V}} q_v - \left( 1 - \frac{1}{c} \right) \sum_{v \in \mathcal{V}} \lambda_v (\gamma (u)) - \varepsilon.$$  

We now apply Lemma 7.3 with $r = \lambda_v (\gamma (u))$ and $q = q_v/c$, and the bound (2.14) for $\lambda_v$ to obtain

$$\frac{1}{c} \sum_{v \in \mathcal{V}} q_v \leq \sum_{v \in \mathcal{V}} \left( \lambda_v (\gamma (u)) \ell \left( \frac{q_v/c}{\lambda_v (\gamma (u))} \right) + \lambda_v (\gamma (u)) (e - 1) \right)$$

$$\leq L (\gamma (u), \dot{\gamma} (u)) + \varepsilon + R.$$
where \( R_1 \equiv |V|R(e - 1) < \infty \), with \( R \) being the bound in (2.14). Combining this with (7.10), we see that for \( c \) sufficiently close to 1,
\[
\left| \frac{1}{c} L(\gamma(u), c\dot{\gamma}(u)) - L(\gamma(u), \dot{\gamma}(u)) \right| \\
\leq M |V| \left| 1 - \frac{1}{c} \right| + \max \left\{ \log \frac{1}{c}, c \log c \right\} (L(\gamma(u), \dot{\gamma}(u)) + \varepsilon + M_1) + \varepsilon.
\]
Since this holds for almost every \( u \in [0, t] \), one can first integrate over \([0, t]\), then take \( c \to 1 \) and use the finiteness of \( I(\gamma) \) and finally send \( \varepsilon \to 0 \) to complete the proof. \( \square \)

8. Proof of the LDP lower bound

We now turn to the proof of the LDP lower bound, which we will establish for a somewhat larger class of jump Markov processes than the empirical measure processes. Again, we assume for each \( n \in \mathbb{N} \), \( \mu^n(\cdot) \) is a jump Markov process on \( S_n \) with generator \( \mathcal{L}_n \) in (2.47) such that the associated sequence of rates \( \{\lambda^n_v(\cdot)\}_{v \in V}, n \in \mathbb{N} \), satisfy Property (2.3) that is, converge uniformly to suitable Lipschitz continuous functions \( \{\lambda_v(\cdot)\}_{v \in V} \). Additional conditions imposed on \( \{\lambda_v(\cdot)\}_{v \in V} \) will be stated in the lemmas below. Recall that for notational convenience we assume the time interval is of the form \([0, 1]\). To prove the lower bound it suffices to show that for any fixed trajectory \( \gamma \in D([0, 1]: S) \), given any \( \varepsilon > 0 \) and \( \delta > 0 \) there exists \( \eta > 0 \) such that if \( \|\mu^n(0) - \gamma(0)\| < \eta \) for all \( n \) large enough,
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\|\mu^n - \gamma\|_{\infty} < \delta) \geq -I(\gamma) - \varepsilon.
\]

Without loss of generality we assume \( I(\gamma) < \infty \), which in particular implies that \( \gamma \in AC([0, 1], S) \).

One source of difficulty here is that the transition rates of \( \mu^n \) may tend to zero as \( \mu^n \) approaches the boundary of \( S \), which could lead to singularity of the local rate function. Our approach here adapts an idea from the study of a discrete time model in [13]. We first show that the singularity can be avoided except at \( t = 0 \), by slightly perturbing the original path, with arbitrarily small additional cost.

8.1. Perturbation argument. The idea of the perturbation argument is as follows. Recall the definition of \( S^a \) in (4.7). For any \( a > 0 \) fixed, by property (3) of Lemma 4.12 the rates \( \lambda_v(\cdot) \) are either identically zero or uniformly bounded below away from zero within \( S^a \). Therefore, a standard approximation argument can be used to establish the LDP in \( S^a \), uniformly with respect to the initial condition. When \( \gamma(0) = x \in S/S^a \), by using Proposition 4.13 one can construct a perturbed trajectory of \( \gamma \) that hits \( S^a \) in an arbitrarily short time as \( a \to 0 \), and in such a way that the difference in cost between \( \gamma \) and the perturbed trajectory can be made sufficiently small.

Lemma 8.1. Assume the family of jump rates \( \{\lambda_v(\cdot) : v \in V\} \) satisfies Property (2.3) Property (4.3) properties (2) and (3) of Lemma 4.12, and the associated LLN trajectory satisfies Property 4.13. Consider \( \gamma \in AC([0, 1]: S) \) such that \( I(\gamma) < \infty \). Then given any \( \varepsilon > 0 \), there exists \( \bar{b} > 0, D < \infty \) and a trajectory \( \psi \in AC([0, 1]: S) \) such that
i) \( \psi(0) = \gamma(0) \) and \( \|\psi - \gamma\|_{\infty} \leq \varepsilon \),
ii) \( \psi_i(t) \geq \bar{b} t^D \) for \( i = 1, ..., d \) and any \( t \in [0, 1] \),
iii) \( I(\psi) \leq I(\gamma) + \varepsilon \).
Proof. For $0 < \rho < 1$, define $\psi^\rho \equiv \rho \mu + (1 - \rho) \gamma$, where $\mu$ is the law of large numbers trajectory defined in (2.13) with $\mu (0) = \gamma (0)$. Let $C_d \equiv \max_{x, y \in \mathcal{S}} ||x - y||$ be the diameter of $\mathcal{S}$. Then we have

$$\psi (0) = \gamma (0)$$

and $||\psi^\rho - \gamma||_{\infty} = \rho ||\mu - \gamma||_{\infty} \leq C_d \rho$. By Property 4.13 of the LLN trajectory, there exist $b > 0$ and $D < \infty$ such that $\mu_i (t) \geq bt^D$ for $i = 1, \ldots, d$, which in turn implies the lower bound $\psi^\rho_i (t) \geq \rho \mu_i (t) \geq \rho bt^D$, $i = 1, \ldots, d$. Thus, for all $\rho < \varepsilon / C_d$, $\psi = \psi^\rho$ satisfies property (i) and property (ii) holds with $b \rho > 0$. It only remains to show that there exists some $\rho \in (0, \varepsilon / C_d)$ such that $\psi = \psi^\rho$ satisfies property (iii). We first show that there exists $c (\rho) < \infty$ which goes to zero as $\rho \to 0$, such that for almost every $t \in [0, 1]$,

$$L (\psi^\rho (t), \hat{\gamma} (t)) \leq (1 + c (\rho)) L (\gamma (t), \hat{\gamma} (t)) + c (\rho).$$

For $t \in [0, 1]$, property (ii) shows that $\psi^\rho (t) \in \text{int} (\mathcal{S})$, and we also have $\gamma_i (t) / \psi_i^\rho (t) \leq 1 / (1 - \rho)$ for every $i = 1, \ldots, d$. By property (2) of Lemma 4.12, there exists a function $\bar{C} : [0, \infty) \to [0, \infty)$ with $\bar{C} (r) \to 1$ as $r \to 0$ such that

$$\frac{\lambda_v (\gamma (t))}{\lambda_v (\psi^\rho (t))} \leq \bar{C} (||\psi^\rho (t) - \gamma (t)||_{\infty}) \prod_{i=1, \ldots, d: \gamma_i (t) > \psi_i^\rho (t)} \left( \frac{\gamma_i (t)}{\psi_i^\rho (t)} \right)^K \leq c_0 (\rho),$$

where

$$c_0 (\rho) \equiv \bar{C} (||\psi^\rho (t) - \gamma (t)||_{\infty}) \left( \frac{1}{1 - \rho} \right)^{K_d}.$$ As $\rho \to 0$, $c_0 (\rho) \to 1$ because $||\psi^\rho (t) - \gamma (t)||_{\infty} \to 0$. Thus, an application of Lemma 7.7 with $x = \gamma (t)$ and $\varphi = \psi^\rho (t)$ shows that (8.2) holds for suitable $c (\rho)$. Likewise, since $\mu_i (t) / \psi_i^\rho (t) \leq 1 / \rho$ for $i = 1, \ldots, d$, property (2) of Lemma 4.12 implies

$$\frac{\lambda_v (\mu (t))}{\lambda_v (\psi^\rho (t))} \leq \bar{C}_* \left( \frac{1}{\rho} \right)^{K_d},$$

where $\bar{C}_* \equiv \max_{r \in [0, C_d]} \bar{C} (r)$ is finite because $\bar{C}$ is continuous. Therefore, by the definition of $L$ in (3.1) and the fact that $\hat{\mu} (t) = \sum_{v \in \mathcal{V}} \nu \lambda_v (\mu (t))$, we have

$$L (\psi^\rho (t), \hat{\mu} (t)) \leq \sum_{v \in \mathcal{V}} \nu \lambda_v (\psi^\rho (t)) \ell \left( \frac{\lambda_v (\mu (t))}{\lambda_v (\psi^\rho (t))} \right)$$

$$= \sum_{v \in \mathcal{V}} \nu \lambda_v (\mu (t)) \log \left( \frac{\lambda_v (\mu (t))}{\lambda_v (\psi^\rho (t))} \right) + \nu \lambda_v (\psi^\rho (t)) - \nu \lambda_v (\mu (t))$$

$$\leq C_2 \left( \log \frac{1}{\rho} + 1 \right)$$

for some $C_2 < \infty$, where to obtain the last inequality we apply (8.3), use the Lipschitz continuity of $\lambda_v$ (Property 2.23) and the estimate $||\psi^\rho (t) - \mu (t)|| \leq C_d$.

Using the convexity and nonnegativity of $L (x, \cdot)$ stated in Proposition 7.2, along with relations (8.2) and (8.4), one has for almost every $t \in [0, 1]$,

$$L (\psi^\rho (t), \hat{\psi}^\rho (t)) \leq L (\psi^\rho (t), \hat{\gamma} (t)) + \rho L (\psi^\rho (t), \hat{\mu} (t))$$

$$\leq (1 + c (\rho)) L (\gamma (t), \hat{\gamma} (t)) + c_5 (\rho),$$

with $c_5 (\rho) \equiv C_2 \rho (\log 1 / \rho + 1) + c (\rho)$. Integrating both sides of the last inequality over $[0, 1]$, we get

$$I (\psi^\rho) \leq (1 + c (\rho)) I (\gamma) + c_5 (\rho).$$
Since \( c(\rho) \to 0 \) and \( c_\delta(\rho) \to 0 \) as \( \rho \to 0 \), property (iii) holds with \( \psi = \psi^\rho \) for all \( \rho > 0 \) sufficiently small.

In view of Lemma 5.1, it suffices to establish the lower bound \( \textsc{lbd} \) for paths \( \gamma \in \text{AC} (\{0,1\} \times \mathcal{S}) \) with \( I(\gamma) < \infty \) that satisfy the additional condition that

\[
\text{there are } b_0 > 0, \ D < \infty \text{ such that } \gamma_i(t) \geq b_0 t^D \text{ for all } i = 1, \ldots, d, \ t \in [0,1].
\]

8.2. Analysis for short times. We first state the main result of this subsection.

**Lemma 8.2.** Suppose \( \{\lambda_v(\cdot), v \in \mathcal{V}\} \) satisfies Property 2.3 and Property 4.3, and the sequence of deterministic initial conditions \( \{\mu^n(t)\}_{n \in \mathbb{N}} \) converges to \( \mu_0 \in \mathcal{S} \) as \( n \) tends to infinity, and let \( \epsilon > 0 \) and \( \delta > 0 \) be given. Then there exists \( \tau > 0 \) such that for any \( \sigma > 0 \), there is \( \eta = \eta(\sigma) > 0 \) such that \( \|\mu_0 - \gamma(0)\| \leq \eta \) implies

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P} \left( \|\mu^n_\tau(\cdot) - \gamma(\cdot)\| \leq \sigma, \sup_{t \in [0,\tau]} \|\mu^n(t) - \gamma(t)\| \leq \delta \right) \geq -\frac{\epsilon}{2}.
\]

We first present the idea behind the proof. Given \( \delta > 0 \), for \( \tau > 0 \) sufficiently small we use excursion bounds for jump Markov processes (Lemma 8.3 below) to establish a lower bound for the quantity

\[
\mathbb{P} \left( \sup_{t \in [0,\tau]} \|\mu^n(t) - \gamma(t)\| < \delta \right).
\]

The more difficult part is to obtain, for any \( 0 < \sigma < \delta \), a lower bound for \( \mathbb{P}(\|\mu^n(\tau) - \gamma(\tau)\| < \sigma) \) that is uniform in \( \mu^n(0) \) as long as \( \|\mu^n(0) - \gamma(0)\| \) is sufficiently small. For the latter, given \( \epsilon > 0 \), \( \tau \in (0,1] \), for any \( \sigma > 0 \), consider the penalty function \( g: \mathcal{S} \to \mathbb{R} \) defined by

\[
g(x) = \begin{cases} 0 & \text{if } ||x - \gamma(\tau)|| < \sigma, \\ 2\epsilon & \text{otherwise.} \end{cases}
\]

We then have \( g \in \mathcal{M}_b(\mathcal{S}) \) and

\[
\mathbb{P}(\|\mu^n(\tau) - \gamma(\tau)\| < \sigma) + e^{-2n\epsilon} \geq \mathbb{E} \left[ \exp(-ng(\mu^n(\tau))) \right].
\]

To lower bound the right-hand side of (8.7), we will use the variational representation formula from Theorem 5.6

\[
-\frac{1}{n} \log \mathbb{E} \left[ \exp(-ng(\mu^n(\tau))) \right]
\]

\[
= \inf_{\tilde{\alpha} \in \mathcal{A}_0^{\mathcal{S}}(\mathcal{V})} \mathbb{E} \left[ \sum_{v \in \mathcal{V}} \int_0^\tau \lambda^n_v(\tilde{\mu}^n(t)) \ell \left( \frac{\tilde{\alpha}_v(t)}{\lambda^n_v(\tilde{\mu}^n(t))} \right) \, dt + g(\tilde{\mu}^n(\tau)) : \tilde{\mu}^n = \Lambda^n(\tilde{\alpha}, \mu^n(0)) \right],
\]

with \( \mathcal{A}_0^{\mathcal{S}}(\mathcal{V}) \) and \( \Lambda^n \) defined as in Definition 5.3 and 5.4, respectively. Thus, to prove Lemma 5.2 we need to construct a suitable controlled process \( \tilde{\mu}^n \) that has “low cost” and is sufficiently close to \( \gamma(\tau) \) at time \( \tau \). We now provide the details of the proof.

**Proof of Lemma 5.2.** The idea is to argue that for large \( n \) and small \( \tau \), if \( \mu^n \) starts close to \( \gamma(0) \), then it stays close to a communicating path (see Definition 4.1) that connects \( \gamma(0) \) to \( \gamma(\tau) \), which lies in \( \text{int}(\mathcal{S}) \) due to (8.5). Since the jump rates (along the directions used to get from \( \gamma(0) \) to \( \gamma(\tau) \)) are bounded below away from zero along such a path, one obtains a nice upper bound for the cost. Specifically, by Property 4.3, Definition 4.1 and Remark 4.5 there exists a communicating
path $\phi \in C([0, \tau]: \mathcal{S})$, with $\phi(0) = \gamma(0)$ and $\dot{\phi}(\tau) = \gamma(\tau)$, and $F, U < \infty$, $\{v_m\}_{m=1}^F \subset \mathcal{V}$ and $0 = t_0 < t_1 < \cdots < t_F = \tau$, such that
\[
\frac{d}{dt} \phi(t) = \sum_{v \in \mathcal{V}} \tilde{\alpha}_v(t)v, \quad \text{a.e. } t \in [0, \tau],
\]
where
\[
(8.9) \quad \tilde{\alpha}_v(t) = \begin{cases} 
U[\tau_{m-1}, \tau_m](t) & \text{if } v = v_m, m = 1, \ldots, F, \\
0 & \text{if } v \notin \{v_m\}_{m=1}^F.
\end{cases}
\]
Also, by Definition 4.1 there exist $p, D < \infty$ and $c > 0$, such that
\[
(8.10) \quad \lambda_{v_m}(\phi(t)) \geq c \left( \min_{i=1, \ldots, d} \gamma_i(\tau) \right)^p \geq c_0 \tau^{DP}, \quad \text{if } t \in [t_{m-1}, t_m], \ m = 1, \ldots, F,
\]
where the second inequality uses (8.5) and $c_0 = cb_0 > 0$.

Now define $\mu^n = \Lambda^n(\alpha, \mu^n(0))$, where $\Lambda^n$ is as defined in (5.9). Property 2.3 and the LLN for Poisson random measures (see Section 5.3) imply that $\{\mu^n\}_{n \in \mathbb{N}}$ converges uniformly on $[0, \tau]$ in probability to $\bar{\mu}$, where $\bar{\mu}(0) = \mu_0$, and
\[
(8.11) \quad \frac{d}{dt} \bar{\mu}(t) = \sum_{v \in \mathcal{V}} \bar{\alpha}_v(t)v, \quad \text{a.e. } t \in [0, \tau].
\]
Since the trajectories $\phi$ and $\bar{\mu}$ satisfy the same (state-independent) ODE, we have $\|\bar{\mu}(t) - \phi(t)\| = \|\mu_0 - \gamma(0)\|$. Thus, by the Lipschitz continuity of $\lambda_v$ (Property 2.3) and (8.10), for any fixed $\tau$, there exists some $\eta_0(\tau) > 0$, such that for any $\eta \leq \eta_0(\tau)$, if $\|\mu_0 - \gamma(0)\| \leq \eta$ then
\[
(8.12) \quad \lambda_{v_m}(\bar{\mu}(t)) \geq \frac{c_0}{2} \tau^{DP}, \quad \text{for } t \in [t_{m-1}, t_m], \ m = 1, \ldots, F.
\]
We now bound the cost for the sequence of jump processes $\{\mu^n\}_{n \in \mathbb{N}}$ by making use of the bound (8.12) on its law of large numbers limit. Given the form of $\bar{\alpha}_v$ and $\ell$ in (8.9) and (3.6), respectively, we have
\[
(8.13) \quad \mathbb{E}\left[ \sum_{v \in \mathcal{V}} \int_0^\tau \lambda^n_v(\bar{\mu}(t)) \ell \left( \frac{\bar{\alpha}_v(t)}{\lambda^n_v(\bar{\mu}(t))} \right) dt + g(\bar{\mu}^n(\tau)) \right] = \mathbb{E}\left[ \sum_{m=1}^F \int_{t_{m-1}}^{t_m} \left( U \log \left( \frac{U}{\lambda_{v_m}(\bar{\mu}^n(t))} \right) - U + \sum_{v \in \mathcal{V}} \lambda^n_v(\bar{\mu}^n(t)) \right) dt + g(\bar{\mu}^n(\tau)) \right].
\]
Now fix $\tau > 0$ and $\eta < \min \{\eta_0(\tau), \sigma/2\}$. Then by (8.12), if $\|\mu_0 - \gamma(0)\| \leq \eta$ then for each $m = 1, \ldots, F$, on the interval $[t_{m-1}, t_m]$, $\lambda_{v_m}(\bar{\mu}(t))$ is uniformly bounded below away from zero. Since $\bar{\mu}^n$ converges in probability to $\bar{\mu}$, uniformly on $[0, \tau]$, and Property 2.3 holds, this implies that for each $m = 1, \ldots, F$ and $t \in [t_{m-1}, t_m]$, $\log \left( \lambda_{v_m}(\bar{\mu}^n(t)) \right)$ converges in probability to $\log(\lambda_{v_m}(\bar{\mu}(t)))$ uniformly for $t \in [t_{m-1}, t_m]$. Thus, taking the limit superior as $n \to \infty$ in (8.13), by the dominated
convergence theorem and the upper semicontinuity of $g$ defined in (8.6), we obtain

$$
\limsup_{n \to \infty} \mathbb{E} \left[ \sum_{v \in V} \int_0^\tau \lambda_v^n (\bar{\mu}^n(t)) \ell \left( \frac{\bar{\alpha}_v(t)}{\lambda_v^n (\bar{\mu}^n(t))} \right) dt + g(\bar{\mu}^n(\tau)) \right] 
\leq \mathbb{E} \left[ \sum_{v \in V} \int_{t_{m-1}}^{t_m} \left( U \log \left( \frac{U}{\lambda_v (\bar{\mu}(t))} \right) - U + \sum_{v \in V} \lambda_v (\bar{\mu}(t)) dt + g(\bar{\mu}(\tau)) \right) \right] 
\leq \tau \left( U \log U + U \log \left( \frac{C_0}{2} \tau^{D_p} \right) + |V|R \right),
$$

where the last inequality uses the lower bound in (8.12), the upper bound in (2.14), the identity $t_F = \tau$ and the fact that $g(\bar{\mu}(\tau)) = 0$ because $\|\bar{\mu}(\tau) - \gamma(\tau)\| = \|\bar{\mu}_0 - \gamma(0)\| \leq \eta < \sigma$. Choose $\tau > 0$ sufficiently small such that the last expression is less than $\varepsilon/4$. Observing that the control $\bar{\alpha}$ in (8.9) is a deterministic process that is uniformly bounded, and hence, lies in $A_0^\otimes V$, we can combine the last display with the representation formula (8.8): for all sufficiently large $n$ and sufficiently small $\eta$, $\|\bar{\mu}_0 - \gamma(0)\| < \eta$ implies

$$
- \frac{1}{n} \log \mathbb{E} [\exp(-ng(\mu^n(\tau)))] \leq \frac{\varepsilon}{2}.
$$

When combined with (8.7), this gives the lower bound

(8.14)

$$
\mathbb{P}(\|\mu^n(\tau) - \gamma(\tau)\| < \sigma) \geq e^{-n\varepsilon/2} - e^{-2n\varepsilon}.
$$

We will conclude the argument by establishing an upper bound on the probability of $\mu^n$ having a large excursion during the interval $[0, \tau]$. Given $\varepsilon > 0$, applying a standard martingale inequality (stated as Lemma 8.3 below), for sufficiently small $\tau$ we have

$$
\mathbb{P} \left( \sup_{t \in [0, \tau]} \|\mu^n(t) - \mu^n(0)\| > \frac{\delta}{3} \right) \leq 2d \exp (-n\varepsilon).
$$

On the other hand, since $\gamma$ is continuous, by taking $\tau$ smaller if necessary we can guarantee that $\sup_{t \in [0, \tau]} \|\gamma(t) - \gamma(0)\| \leq \delta/3$. It follows that for $\eta \in [0, \frac{\delta}{4}]$,

$$
\mathbb{P} \left( \sup_{t \in [0, \tau]} \|\mu^n(t) - \gamma(\tau)\| > \delta \right) \leq \mathbb{P} \left( \sup_{t \in [0, \tau]} \|\mu^n(t) - \mu^n(0)\| > \frac{\delta}{3} \right) \leq 2d \exp (-n\varepsilon).
$$

Combining this with the estimate (8.14) we arrive at the desired conclusion. \hfill \square

The following lemma is an adaptation of Lemma 2.3 in [14]. The lemma follows from bounds for certain exponential martingales.

**Lemma 8.3.** Let $\bar{C}_1 = \max_{v \in V} \|v\|, \bar{C}_2 = R |V| \bar{C}_1$, and for $\varrho > \bar{C}_2$ define $\bar{\ell}(\varrho) = \varrho \log (\varrho/\bar{C}_2) - 1)/\bar{C}_1$. Then $\bar{\ell}(\varrho)/\varrho \to \infty$ as $\varrho \to \infty$, and given any $\delta > 0$, for all $\tau \leq \delta/2\sqrt{d}\bar{C}_2$

$$
\mathbb{P} \left( \sup_{t \in [0, \tau]} \|\mu^n(t) - \mu^n(0)\| \geq \delta \right) \leq 2d \exp \left( -\frac{\delta}{2\sqrt{d}\tau} \right).
$$
8.3. Analysis for $t \in [\tau, 1]$. As shown in Section 8.1, to establish the large deviation lower bound, it suffices to establish the estimate (8.1) for $\gamma \in AC([0,1] : S)$ that satisfies $I(\gamma) < \infty$ and the bound (8.2). So for any $\tau > 0$ there exists $\xi > 0$ such that $\gamma(t)$ lies in $S^\xi$ for all $t \in [\tau, 1]$. Therefore, we now fix $\tau > 0$ and $\xi > 0$ and consider large deviations of $\mu^n$ in $[\tau, 1]$ from a path $\gamma \in AC([\tau, 1] : S^\xi)$.

For $\gamma \in S$ and $r > 0$, let $B(y, r)$ denote the open Euclidean ball centered at $y$ with radius $r$. For $\psi \in AC([\tau, 1] : S)$ with $\psi(0) = y$, we denote

$$P^y(\psi) \doteq \int_{\tau}^1 L \left( \psi(s), \dot{\psi}(s) \right) ds,$$

to emphasize the dependence on $y$ (though we omit the dependence on $\tau$). Given $y_n \in S$, let $P_{y_n}$ and $E_{y_n}$ denote the probability and expectation, respectively, conditioned on $\mu^n(\tau) = y_n$. Define the mapping $\Lambda^n_{\gamma} : A^\square_{\gamma} \times S \to D\left([\tau, 1] : D^{d-1}\right)$ by

$$\Lambda^n_{\gamma}(a, \rho)(t) = \rho + \sum_{v \in V} v \int_{[\tau, t]} \int_S \|\gamma(s)(x) - v\| n^{-1} N^n_{v}(dsdx),$$

for $a \in A^\square_{\gamma}$ and $\rho \in S$. We will prove the following uniform Laplace principle lower bound for $\{\mu^n(\cdot)\}_{n \in N}$ on $[\tau, 1]$, where we restrict to Lipschitz continuous test functions. By [11 Corollary 1.2.5], this implies the corresponding large deviation lower bound.

**Proposition 8.4.** Suppose the assumptions of Lemma 8.2 hold. Fix $\tau \in (0,1)$. Let $\xi > 0$ and $\gamma \in AC([\tau, 1] : S^\xi)$ be such that $\gamma(\tau) = y$ and $I^y(\gamma) < \infty$. Then there exists $\sigma > 0$ such that for any bounded and Lipschitz continuous functional $F$ on $D([\tau, 1] : S)$,

$$\liminf_{n \to \infty} \inf_{y_n \in B(y, \sigma)} \left( \frac{1}{n} \log E_{y_n} \left[ \exp(-nF(\mu^n)) \right] - G(y_n, F) \right) \geq 0,$$

where

$$G(y, F) \doteq \inf_{\psi \in AC([\tau, 1] : S^\xi)} \left[ I^y(\psi) + F(\psi) \right].$$

In particular, this implies the following uniform (with respect to initial conditions) large deviation lower bound: for any $\varepsilon > 0$ and $\delta > 0$, there exists $\sigma > 0$ such that for any sequence $\{y_n\}_{n \in N} \subset B(y, \sigma)$,

$$\liminf_{n \to \infty} \frac{1}{n} \log P_{y_n} \left( \sup_{t \in [\tau, 1]} \|\mu^n(t) - \gamma(t)\| < \delta \right) \geq -I^y(\gamma) - \frac{\varepsilon}{2}.$$

The proof of Proposition 8.4 relies on the following approximation argument. Fix $y \in S^\xi$ and a bounded and Lipschitz continuous functional $F$ on $D([\tau, 1] : S)$. By Proposition 1.2.7 of [11], to prove (8.15), it suffices to show that for any sequence $\{y_n\}_{n \in N}$ such that $\|y_n - y\| \to 0$ as $n \to \infty$,

$$\liminf_{n \to \infty} \frac{1}{n} \log E_{y_n} \left[ \exp(-nF(\mu^n)) \right] \geq G(y, F).$$

It suffices to show that for any $\varepsilon > 0$ and $\gamma_\varepsilon \in AC([\tau, 1] : S^\xi)$ such that $- (I^y(\gamma_\varepsilon) + F(\gamma_\varepsilon)) \geq G(y, F) - \varepsilon$, we have

$$\liminf_{n \to \infty} \frac{1}{n} \log E_{y_n} \left[ \exp(-nF(\mu^n)) \right] \geq -(I^y(\gamma_\varepsilon) + F(\gamma_\varepsilon)),$$
or, equivalently,

\(\limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{E}_{y_n} [\exp(-nF(\mu^n))] \leq I^\nu(\gamma) + F(\gamma).\)

Fix \(\varepsilon > 0\) and denote \(\gamma_\varepsilon\) simply by \(\gamma\). We now approximate \(\gamma\) by a piecewise linear path. Let 

\[\Delta = \frac{\xi_j}{j}\]

for some \(J \in \mathbb{N}\). For \(j = 0, 1, \ldots, J - 1\) let

\[a_j^\Delta = \frac{1}{\Delta} \int_{\tau + j\Delta}^{\tau + (j + 1)\Delta} \dot{\gamma} (s) \, ds.\]

Define

\[\dot{\gamma}^\Delta (t) = a_j^\Delta\]

if \(t \in (\tau + j\Delta, \tau + (j + 1)\Delta), \ j = 0, \ldots, J - 1,\]

and

\[\gamma^\Delta (t) = y + \int_{\tau}^{t} \dot{\gamma}^\Delta (s) \, ds \quad \text{for} \quad t \in [\tau, 1].\]

Then \(\gamma^\Delta\) is the piecewise linear interpolation of the continuous process \(\gamma\) with mesh size \(\Delta\). Note that for any \(\nu \in \mathcal{V}\), \(\lambda^n (\gamma^\Delta (\cdot))\) is continuous and uniformly bounded away from zero on \([\tau, 1]\).

The proof of \((8.15)\) thus relies on the following standard approximation result (we refer to Lemma 65 in Section 3.6.3 of [30] for a complete proof).

**Lemma 8.5.** Suppose \(\{\lambda_v (\cdot), v \in \mathcal{V}\}\) satisfies Property \((2.23)\) and Property \((4.1)\). Let \(\tau, \xi, y\) and \(\gamma\) be as in Proposition \((8.4)\) and define \(\gamma^\Delta\) as in \((8.21)\). Then for any \(\varepsilon > 0\), there exists \(\Delta (\varepsilon) > 0\), such that for any \(\Delta < \Delta (\varepsilon)\), and a.e. \(t \in [\tau, 1]\), there exists a piecewise constant vector \(q^\Delta (t) \in [0, \infty)^{\mathcal{V}}\) such that \(\sum_{v \in \mathcal{V}} v q^\Delta (t) = \gamma^\Delta (t)\), and

\[
\left(1\right) \int_{\tau}^{1} \sum_{v \in \mathcal{V}} \lambda_v (\gamma^\Delta (t)) \ell \left( \frac{q_v^\Delta (t)}{\lambda_v (\gamma^\Delta (t))} \right) \, dt \leq I^\nu (\gamma) + \varepsilon.
\]

We now complete the proof of Proposition \((8.4)\). By Lemma \((8.5)\) for any \(\varepsilon > 0\), there exists \(\Delta\) sufficiently small and a collection of piecewise constant functions \(\{q_v^\Delta (\cdot)\}_{v \in \mathcal{V}}\) on \([\tau, 1]\) that satisfy \((8.21)\). It follows directly from the LLN for Poisson random measures that as \(n \to \infty\),

\[\bar{\mu}^n = \Lambda^n (q^\Delta, y_n)\]

converges uniformly on \([\tau, 1]\) in probability to \(\gamma^\Delta\). Therefore, by the uniform continuity of \(\lambda_v (\cdot) \ell (q_v^\Delta (\cdot))\) on \(\mathcal{S}\) and the uniform convergence of \(\lambda^n_v (\cdot)\) to \(\lambda_v (\cdot)\) on \(\mathcal{S}\) by Property \((2.3)\),

\[
\Lambda^n_v (\bar{\mu}^n (\cdot)) \ell (q_v^\Delta (\cdot)) \bar{\lambda}_v^n (\bar{\mu}^n (\cdot)) \leq \lambda_v (\gamma^\Delta (\cdot)) \ell (q_v^\Delta (\cdot)) \lambda_v (\gamma^\Delta (\cdot)).
\]

Combining the variational representation formula (Theorem \((5.6)\) and the dominated convergence theorem, for any Lipschitz continuous functional \(F\) on \(D ([\tau, 1] : \mathcal{S})\), we have

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mathbb{E}_{y_n} [\exp(-nF(\mu^n))]
\]

\[
= \limsup_{n \to \infty} \inf_{\hat{\alpha} \in \Lambda^n_{\mathcal{V}}} \mathbb{P}_{y_n} \left[ \sum_{v \in \mathcal{V}} \int_{\tau}^{1} \lambda_v^n (\bar{\mu}^n (t)) \ell \left( \frac{\hat{\alpha}_v (t)}{\lambda_v^n (\bar{\mu}^n (t))} \right) \, dt + F(\bar{\mu}^n) : \bar{\mu}^n = \Lambda^n_v (\hat{\alpha}, y_n) \right]
\]

\[
\leq \limsup_{n \to \infty} \mathbb{P}_{y_n} \left[ \int_{\tau}^{1} \sum_{v \in \mathcal{V}} \lambda_v^n (\bar{\mu}^n (t)) \ell \left( \frac{q_v^\Delta (t)}{\lambda_v^n (\bar{\mu}^n (t))} \right) \, dt + F(\bar{\mu}^n) : \bar{\mu}^n = \Lambda^n_v (q^\Delta, y_n) \right]
\]

\[
= \int_{\tau}^{1} \sum_{v \in \mathcal{V}} \lambda_v (\gamma^\Delta (t)) \ell \left( \frac{q_v^\Delta (t)}{\lambda_v (\gamma^\Delta (t))} \right) \, dt + F(\gamma^\Delta)
\]

\[
\leq I^\nu (\gamma) + \varepsilon + F(\gamma^\Delta).
\]
Letting $\Delta \to 0$ gives the upper bound $J^\gamma (\gamma) + F(\gamma) + \varepsilon$, and since $\varepsilon > 0$ is arbitrary this gives (8.19).

We now have all the ingredients to complete the proof of the LDP lower bound.

**Proof of the lower bound (3.10) of Theorem 3.9** We start by showing that the assumptions on the transition rates $\{T^k_{ij}(\cdot), (ij) \in J^k, k = 1, \ldots, K\}$ imply all the required conditions on the jump rates $\{\lambda_v(\cdot), v \in V\}$ that are necessary to apply the results in Section 8. Indeed, Property 2.3 follows from Assumption 2.2. Lemma 4.12 shows that all four properties of the lemma follow from Assumption 5.1 and Assumption 5.5 and finally, since Assumption 5.3 also holds, Proposition 4.7 shows that the jump rates also satisfy Property 4.3. From the discussion at the beginning of Section 8 and Lemma 8.1 of Section 8.1, it follows that to prove the LDP lower bound (8.1) it suffices to establish (8.1) for $\gamma \in AC([0, 1]: S)$ that satisfies the lower bound (8.5). The latter lower bound guarantees that, even if $\gamma$ starts on the boundary of $S$, for any $\tau > 0$ it lies a strictly positive distance from that boundary, and thus after $\tau$, Proposition 8.4 can be applied to get a uniform lower bound for initial conditions close to $\gamma(\tau)$. Due to the Markov property, the proof is then completed by observing that Lemma 8.2 shows that, with an error that is vanishingly small as $\tau \to 0$, $\mu(\tau)$ can be brought into the required sufficiently small neighborhood of $\gamma(\tau)$, while staying close to $\gamma$ on $[0, \tau]$. \hfill \square

**Remark 8.6.** From the proof of the upper bound in Section 6 and the proof of the lower bound above, it is clear that the conclusions of Theorem 3.9 in fact holds for a more general class of jump Markov processes. Specifically, it holds for any sequence $\{\mu^n\}_{n \in \mathbb{N}}$, of jump Markov processes on $S$ with generators of the form (2.7), for which the associated sequence of jump rates $\{\lambda^n_v(\cdot), v \in V\}_{n \in \mathbb{N}}$ satisfies Property 2.3, Property 4.3 and the properties stated in Lemma 4.12. Moreover, the only place where Assumption 3.8 is used is in the proof of property (4) of Lemma 4.12, which in turn is only used in the proof of Property 4.13 of the LLN trajectory. Thus, to extend the results to situations where Assumption 3.8 fails to hold, it suffices to directly verify Property 4.13.

### 9. The Locally Uniform LDP

We now turn to the proof of Theorem 3.12. We assume throughout this section that the conditions (and conclusions) of Theorem 3.9 are satisfied, and below, only specify additional conditions that are imposed. Fix $t \in [0, 1]$. As shown in Corollary 3.10 one can express the rate function $J_t$ of $\{\mu^n(t)\}_{n \in \mathbb{N}}$ in terms of a variational problem. In what follows, fix $x \in S$ and $\{x_n\}_{n \in \mathbb{N}}$ such that $x_n \in S$ and $\|x_n - x\| \to 0$ as $n \to \infty$.

#### 9.1. Proof of the locally uniform LDP upper bound

Given any $\varepsilon > 0$, recall that $B(x, \varepsilon)$ denotes the open Euclidean ball centered at $x$ with radius $\varepsilon$, and that $\bar{B}(x, \varepsilon)$ denotes its closure. For $n$ sufficiently large such that $x_n \in \bar{B}(x, \varepsilon)$, by the LDP upper bound stated in Corollary 3.10

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu^n(t) = x_n) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\mu^n(t) \in \bar{B}(x, \varepsilon)) \leq -J^c_t(\mu_0, x),$$

where we define

$$J^c_t(\mu_0, x) \doteq \inf \left\{ J_t(\gamma) : \gamma \in D([0, 1]: S), \gamma(0) = \mu_0, \gamma(t) \in \bar{B}(x, \varepsilon) \right\}. \tag{9.1}$$

To prove the locally uniform LDP upper bound, it suffices to show that

$$\liminf_{\varepsilon \to 0} J^c_t(\mu_0, x) \geq J_t(\mu_0, x). \tag{9.2}$$
Lemma 9.1. Assume Property \([4.16]\) holds. Then there exists a function \(c : [0, \infty) \to [0, \infty)\) and \(C < \infty\) such that

i). \(c(\varepsilon) \to 0\) as \(\varepsilon \to 0\), and

ii). given any \(\varepsilon > 0\) and \(x, y \in S\) such that \(\|x - y\| < \varepsilon\), one can construct a path \(\gamma \in AC([0, \varepsilon] : S)\) such that \(\gamma(0) = x, \quad \gamma(\varepsilon) = y, \sup_{s \in [0, \varepsilon]} |\gamma(s) - x| \leq C\varepsilon\) and \(J_{\varepsilon}(x, y) \leq I_{\varepsilon}(\gamma) \leq c(\varepsilon)\), where \(I_{\varepsilon}, J_{\varepsilon}\) are defined in \((3.8), (3.12)\), respectively.

Before proving Lemma 9.1 we first describe how it can be used to prove Lemma 3.16. By Lemma 4.6, Assumption 3.1 and Assumption 3.11 (which are the conditions of Lemma 3.16) imply Property 4.16.i), thus the condition of Lemma 9.1 is satisfied.

**Proof of Lemma 3.16.** For any \(\varepsilon > 0\), take \(t \in (0, \infty)\) and \(\gamma \in AC([0, t] : S)\) such that \(\gamma(0) = x, \quad \gamma(t) = y, \quad \text{and} \quad I_{\varepsilon}(\gamma) \leq V(x, y) + \varepsilon/2\). Given \(\delta > 0\), and any \(y^\delta \in S\) such that \(\|y^\delta - y\| \leq \delta\), by Lemma 9.1 there exists a path \(\nu \in AC([0, \delta] : S)\) with \(\nu(0) = y, \quad \nu(\delta) = y^\delta\) with \(I_{\delta}(\nu) \leq c(\delta)\). Let \(\bar{\gamma}\) be the concatenation of \(\gamma\) and \(\nu\). Then we have

\[
V(x, y^\delta) \leq I_{t+\delta}(\bar{\gamma}) = I_{t}(\gamma) + I_{\delta}(\nu) \leq V(x, y) + \varepsilon/2 + c(\delta).
\]

It suffices to choose \(\delta\) such that \(c(\delta) \leq \varepsilon/2\). The reverse inequality and the joint continuity with respect to both variables can be proved using similar arguments.

A similar construction leads to the proof of the following lemma, which is used in the proof of Corollary 3.13. Notice Corollary 3.13 also assumes Assumption 3.1 and Assumption 3.11 which imply Property 4.16.i).

**Lemma 9.2.** Assume Property \([4.16]\) holds. Given a bounded and continuous function \(h\), the function

\[
U(y) = \inf \{ I(\gamma) + h(\gamma) : \gamma \in D([0, 1] : S), \gamma(0) = y \}
\]

is continuous on \(S\).

**Proof.** Fix \(\varepsilon > 0\). Take \(\gamma \in AC([0, 1] : S)\) such that \(I(\gamma) + h(\gamma) < U(y) + \varepsilon/3\). Given \(\delta > 0\) such that \(c(\delta) \leq \varepsilon/3\), and any \(y^\delta \in S\) such that \(\|y^\delta - y\| \leq \delta\), by Lemma 9.1 there exists a path \(\nu \in AC([0, \delta] : S)\) with \(\nu(0) = y, \nu(\delta) = y^\delta\) such that \(I_{\delta}(\nu) \leq c(\delta)\), and \(\sup_{s \in [0, \delta]} \|\nu(s) - y\| \leq C\delta\) for some \(C < \infty\). We now rescale \(\gamma\) to obtain a new path \(\gamma^\delta\): for \(c = (1 - \delta)^{-1}\), define \(\gamma^\delta \in AC([0, 1 - \delta] : S)\) by \(\gamma^\delta(s) = \gamma(cs)\). By Proposition 7.3 we can take \(\delta\) smaller if necessary such that \(I_{1-\delta}(\gamma^\delta) \leq I(\gamma) + \varepsilon/3\). Let \(\bar{\gamma}\) be the concatenation of \(\nu\) and \(\gamma^\delta\). Then \(\|\gamma - \bar{\gamma}\|_\infty \to 0\) as \(\delta \to 0\). Therefore, we have

\[
U(y^\delta) \leq h(\bar{\gamma}) + I(\bar{\gamma}) \leq U(y) + \varepsilon/3 + c(\delta) + h(\gamma) - h(\gamma) + \varepsilon/3.
\]

The other inequality is proved in the same way. Therefore,

\[
U(y^\delta) - U(y) \leq 2\varepsilon/3 + |h(\bar{\gamma}) - h(\gamma)|,
\]

by taking \(\delta\) sufficiently small, the right hand side is less than \(\varepsilon\).\(\square\)

Assuming Lemma 9.1 we next show \((3.22)\) and therefore complete the proof of the locally uniform LDP upper bound. For \(\delta > 0\), pick \(\gamma \in AC([0, 1] : S)\) such that \(\gamma(0) = \mu_0, \gamma(t) \in B(x, \varepsilon)\), and \(I_{\varepsilon}(\gamma) \leq J_{\varepsilon}(\mu_0, x) + \delta\). By Lemma 9.1 there exists a path \(\nu \in AC([0, \varepsilon] : S)\) with \(\nu(0) = \gamma(t), \nu(\varepsilon) = x\) with \(I_{\varepsilon}(\nu) \leq c(\varepsilon)\), where \(c(\varepsilon) \to 0\) as \(\varepsilon \to 0\). Let \(\bar{\gamma}\) be the concatenation of \(\gamma\) and \(\nu\). We now rescale \(\bar{\gamma}\) to obtain a new path: for \(c = (t + \varepsilon)/t\), define \(\bar{\gamma}_c \in AC([0, t] : S)\) by \(\bar{\gamma}_c(s) = \gamma(cs)\),
\( s \in [0,t] \). Then \( \tilde{\gamma}_c (0) = \mu_0, \ \tilde{\gamma}_c (t) = x \). Moreover, by Proposition 7.3 for \( \varepsilon \) sufficiently small, \( I_t (\tilde{\gamma}_c) \leq I_{t+\varepsilon} (\tilde{\gamma}) + \delta \), and by the construction above,

\[
J_t (\mu_0, x) \leq I_t (\tilde{\gamma}_c) \leq I_{t+\varepsilon} (\tilde{\gamma}) + \delta = I_t (\gamma) + I_{\varepsilon} (\nu) + \delta \leq I_{\varepsilon} (\mu_0, x) + 2\delta + c (\varepsilon).
\]

Taking the limit inferior as \( \varepsilon \to 0 \) and then sending \( \delta \to 0 \), (9.2) follows.

**Proof of Lemma 9.1.** By Property 4.16(i) and Remark 4.5, there exists a strongly communicating path \( \gamma \in AC ([0,\varepsilon] ; S) \) that satisfies \( \gamma (0) = x, \ \gamma (\varepsilon) = y \), and has constant speed \( U \leq c' \|x - y\| / \varepsilon \leq c' \). Precisely, there exist \( F < \infty \) and \( 0 = t_0 < t_1 < \cdots < t_F = 1 \), such that

\[
\dot{\gamma} (t) = \sum_{m=1}^{F} U_{\nu_m} \mathbb{I}_{[t_{m-1} \varepsilon, t_m \varepsilon]} (t) \text{ for a.e. } t \in [0,\varepsilon].
\]

Since \( I_{\varepsilon} (\gamma) = \sum_{m=1}^{F} (I_{t_m \varepsilon} (\gamma) - I_{t_{m-1} \varepsilon} (\gamma)) \), it suffices to bound each term from above.

Recall from (3.4) that for any \( m \in \mathbb{N}, (\varepsilon, \nu_m) < 0 \). Let \( b_1 = \min_{m_1 < \cdots < m_F} |\langle \varepsilon_j, \nu_m \rangle| > 0 \). Note that for \( s \in [t_{m-1} \varepsilon, t_m \varepsilon] \), and any \( m \in \mathbb{N}, \gamma_j (t_m \varepsilon - s) - \gamma_j (s) = \langle \varepsilon_j, \nu_m \rangle U (t_m \varepsilon - s) \), and thus \( \gamma_j (s) \geq b_1 U (t_m \varepsilon - s) \). Therefore, by Definition 4.4, there exist constants \( c_1 > 0, p_1 < \infty \), such that

\[
\lambda_{\nu_m} (\gamma (s)) \geq c_1 \left( \prod_{j \in \mathcal{N}_{\nu_m}} \gamma_j (s) \right)^{p_1} \geq c_1 U^\kappa (t_m \varepsilon - s)^\kappa,
\]

where \( \kappa = dp_1 < \infty \) and \( \tilde{c}_1 \doteq c_1 b_1^{dp_1} > 0 \). Thus, by taking \( q_{\nu_m} = U \), and \( q_{\nu} = 0 \) for \( \nu \neq \nu_m \) in the line below, we have

\[
L (\gamma (s), \dot{\gamma} (s)) = \inf_{q \in [0,\infty) \cap \mathcal{V} : \sum_{\nu \in \mathcal{V}} q_{\nu} = \gamma (s)} \sum_{\nu \in \mathcal{V}} \lambda_{\nu} (\gamma (s)) \ell \left( \frac{q_{\nu}}{\lambda_{\nu} (\gamma (s))} \right)
\]

\[
\leq \lambda_{\nu_m} (\gamma (s)) \ell \left( \frac{U}{\lambda_{\nu_m} (\gamma (s))} \right) + \sum_{\nu \in \mathcal{V} \setminus \{\nu_m\}} \lambda_{\nu} (\gamma (s))
\]

\[
\leq U \log \left( \frac{U}{c_1 U^\kappa (t_m \varepsilon - s)^\kappa} \right) - U + C_2
\]

\[
= - (\kappa - 1) U \log U - \kappa U \log (t_m \varepsilon - s) - U (1 + \log \tilde{c}_1) + C_2,
\]

with \( C_2 \doteq R |\mathcal{V}| < \infty \), where \( R \) is the bound in (2.14). Therefore,

\[
I_{t_m \varepsilon} (\gamma) - I_{t_{m-1} \varepsilon} (\gamma) = \int_{t_{m-1} \varepsilon}^{t_m \varepsilon} L (\gamma (s), \dot{\gamma} (s)) \, ds
\]

\[
\leq \int_{t_{m-1} \varepsilon}^{t_m \varepsilon} (- (\kappa - 1) U \log U - \kappa U \log (t_m \varepsilon - s) - U (1 + \log \tilde{c}_1) + C_2) \, ds
\]

\[
\leq - C_3 (U) \varepsilon \log \varepsilon + C_4 (U) \varepsilon
\]

for some constants \( C_3 (U), C_4 (U) \) such that \( \sup_{U \in [0,c']} (C_3 (U) \vee C_4 (U)) < \infty \). Summing over \( m \), we have \( J_\varepsilon (\mu_0, y) \leq J_\varepsilon (\gamma) \leq c (\varepsilon) \), where \( c (\varepsilon) = O (\varepsilon |\log \varepsilon|) \) as \( \varepsilon \to 0 \). \( \square \)
9.2. Proof of the Lower Bound. For the proof of the lower bound, take any \( \varepsilon > 0 \) small. Then by the Markov property for \( \{ \mu^n \} \), we have
\[
P_{\mu^n} (\mu^n (t) = x_n) \geq P_{\mu^0} (\mu^n (t - \varepsilon) \in B(x, \varepsilon)) \cdot \inf_{w_n \in B(x, \varepsilon) \cap S_n} P_{\mu^n} (\mu^n (\varepsilon) = x_n).
\]
The LDP lower bound in Corollary 3.10 implies
\[
\liminf_{n \to \infty} \frac{1}{n} \log P_{\mu^n} (\mu^n (t - \varepsilon) \in B(x, \varepsilon)) \geq -J^\varepsilon_{t - \varepsilon} (\mu_0, x),
\]
where \( J^\varepsilon_t \) is defined by (9.1). The proof of the lower bound will be complete if we can show both of the following:

i) \( \limsup_{\varepsilon \to 0} J^\varepsilon_{t - \varepsilon} (\mu_0, x) \leq J_t (\mu_0, x) \).

ii) The Local Communication Property: There exist a function \( c : [0, \infty) \to [0, \infty) \) that satisfies \( c(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) and is such that for all \( \varepsilon > 0 \) sufficiently small,
\[
\inf_{w_n \in B(x, \varepsilon) \cap S_n} P_{\mu^n} (\mu^n (\varepsilon) = x_n) \geq \exp(-nc(\varepsilon) + o(n)).
\]

To prove the first property, we will use Proposition 7.8. For any \( \delta > 0 \), take \( \gamma \in AC([0,1] : S) \) such that \( \gamma (t) = x \) and \( I_t (\gamma) \leq J_t (\mu_0, x) + \delta \). Take \( c = t / (t - \varepsilon) \) and consider the path \( \gamma_c \in AC([0,t - \varepsilon] : S) \), such that \( \gamma_c (s) = \gamma(cs) \), \( s \in [0, t] \). Then \( \gamma_c (0) = \mu_0 \), \( \gamma_c (t - \varepsilon) = x \). By Proposition 7.8 given \( \delta > 0 \), for \( \varepsilon \) sufficiently small, \( I_{t/c} (\gamma_c) \leq I_t (\gamma) + \delta \), and we have
\[
J^\varepsilon_{t - \varepsilon} (\mu_0, x) \leq I_{t - \varepsilon} (\gamma_c) = I_{t/c} (\gamma_c) \leq I_t (\gamma) + \delta \leq J_t (\mu_0, x) + 2\delta.
\]
The conclusion follows on taking first \( \varepsilon \to 0 \) and then \( \delta \to 0 \).

To prove the local communication property, we start with a direct evaluation of the hitting probability of jump Markov processes on a finite state space.

Lemma 9.3. Let \( \{ Y(t) \}_{t \geq 0} \) be a jump Markov process with finite state space \( \{ s_0, s_1, \ldots, s_N \} \). For \( i = 0, \ldots, N - 1 \), suppose that the jump rate from state \( s_i \) to \( s_{i+1} \) is \( b_{i+1,i} \), and the sum of jump rates from state \( s_i \) to all other states is bounded above by \( c < \infty \). If \( Y(0) = s_0 \), then
\[
P (Y(t) = s_N) \geq \frac{1}{N!} (\Pi_{i=1}^N b_i) t^N \exp(-ct).
\]

Proof. Let \( p(t) \) be the probability distribution of the process at time \( t \): \( p_i (t) = P (Y(t) = s_i) \). Then the Kolmogorov forward equation takes the form \( \dot{p} = Ap \), where \( A \) is the \( N \times N \) rate matrix for \( Y \). Let \( r \) be the unique solution to the system of linear ODEs given by
\[
\begin{cases}
\dot{r}_0 &= -cr_0, \\
\dot{r}_i &= b_ir_{i-1} - cr_i, & i = 1, \ldots, N, \\
r(0) &= e_{s_0}.
\end{cases}
\]
Solving this equation explicitly gives \( r_N (t) = \frac{1}{N!} (\Pi_{i=1}^N b_i) t^N \exp(-ct) \). Since \( r(0) = p(0) \), the comparison principle for ODEs shows that \( p_i (t) \geq r_i (t) \) for all \( i = 1, \ldots, N \), and the lemma is proved.

Proof of the local communication property. We will use Property 4.10 and Lemma 9.3. Fix some \( w_n \in B(x, \varepsilon) \cap S_n \), note that the probability of \( \mu^n (\varepsilon) = x_n \) is no less than the probability that \( \mu^n \) hitting \( x_n \) at \( \varepsilon \) by passing through the states of a given discrete strongly communicating path \( \phi \) that connects \( w_n \) and \( x_n \).
By Property 4.10 there exists $F < \infty$, $0 = t_0 \leq t_1 \leq \cdots \leq t_F = T$, \( \{v_m\}_{m=1}^F \), and constants $c_1 > 0$, $c_1', p_1 < \infty$, such that $\phi_0 = w_n$, $\phi_T = x_n$, and
$$
\phi_{s+1} - \phi_s = \frac{1}{n}v_m, \quad \text{for } s \in [t_{m-1}, t_m) \cap \mathbb{Z},
$$
with $T \leq c'n \|x_n - w_n\| \leq c'n \epsilon$. Also, for $s \in [t_{m-1}, t_m) \cap \mathbb{Z}$ and large $n$, $\lambda^n_{v_{mn}} (\phi_s) > c_1 (\prod_{j \in N_m} (\phi_s)_j)^{p_1}$.

Let $z^{(m)} = \phi_s (t_m)$. By the Markov property,
$$
P_{w_n} (\mu^n (\epsilon) = x_n) \geq \prod_{m=1}^F P_{z^{(m)}} (\mu^n (\left( \frac{t_{m+1} - t_m}{T} \right) \epsilon) = z^{(m+1)}),
$$
and it suffices to give a lower bound for each term in the product. This will be proved by comparison with another Markov process $Z^n$. Thus, without modifying the notation, we let $\mu^n (t)$ denote the process starting at $z^{(m)}$ and stopped when it first leaves the set of points $\{\phi_s : s \in [t_{m-1}, t_m) \cap \mathbb{Z}\}$.

For each $m$ and $t \in [0, (t_{m+1} - t_m) \epsilon / T)$, define $Z^n$ to be the jump Markov process with $Z^n (0) = z^{(m)}$, with the same set $\mathcal{V}$ of jump directions, and jump rates
$$
\tilde{\lambda}_v (x) = \begin{cases} 
nc_1 \left( \prod_{j \in N_{vm}} x_j \right)^{p_1} & \text{if } v = v_m, \\
nc_1 \left( \prod_{j \in N_{vm}} x_j \right)^{p_1} & \text{if } v \in \mathcal{V} \setminus \{v_m\}, 
\end{cases}
$$
as long as $Z^n$ stays in the set $\{\phi_s : s \in [t_{m-1}, t_m) \cap \mathbb{Z}\}$, and with the process stopped when it jumps off the line segment. Note that $\tilde{\lambda}_v (x)$ bounds $\lambda^n_{vm} (x)$ from below in the set, while $nR$ is an upper bound on all jump rates.

It follows by the comparison principle in Lemma 4.3 that $\mu^n$ has a higher probability to reach $x_{m+1}$ at time $(t_{m+1} - t_m) \epsilon / T$ than $Z^n$ does:
$$
P_{x_m} (\mu^n (\left( \frac{t_{m+1} - t_m}{T} \right) \epsilon) = x_{m+1}) \geq P_{x_m} (Z^n (\left( \frac{t_{m+1} - t_m}{T} \right) \epsilon) = x_{m+1}).
$$

Let $l = l (n, \epsilon) = t_{m+1} - t_m$. Then by Definition 4.13 $l \leq C_2 n \epsilon$ for some $C_2 < \infty$. The product of the jump rates of $Z^n$ along this segment satisfies
$$
\prod_{x : x \in \{\phi_s : s \in [t_{m-1}, t_m) \cap \mathbb{Z}\}} \left( nnc_1 \left( \prod_{j \in N_{vm}} x_j \right)^{p_1} \right) \geq \frac{c_4 \left( \frac{l!}(l^{(k_m)p_1}) \cdot \left( \frac{t_{m+1} - t_m}{T} \right) \epsilon \right)^d n R |\mathcal{V}| \left( \frac{t_{m+1} - t_m}{T} \epsilon \right)^d \exp \left( -nR |\mathcal{V}| \left( \frac{t_{m+1} - t_m}{T} \epsilon \right)^d \right)}{l!}
$$
where $\kappa_m = |N_m| \leq d$. The lower bound in the last inequality is achieved when $\{\phi_s : s \in [t_{m-1}, t_m) \cap \mathbb{Z}\}$ is a segment that ends at $x_{m+1} \in \partial \mathcal{S}$, and for all $j \in N_m$, $x_j = 1, \ldots, l$ along the segment. Then it follows from Lemma 9.3 that for $\epsilon > 0$ sufficiently small,
$$
P_{x_m} (\mu^n \left( \frac{t_{m+1} - t_m}{T} \epsilon \right) = x_{m+1}) \geq \frac{c_4 \left( \frac{l!}{l^{(k_m)p_1}} \right) \left( \frac{t_{m+1} - t_m}{T} \epsilon \right)^d n R |\mathcal{V}| \exp \left( -nR |\mathcal{V}| \left( \frac{t_{m+1} - t_m}{T} \epsilon \right)^d \right)}{l!} \epsilon^T \exp \left( -nR |\mathcal{V}| \epsilon \right).$$
To obtain the last inequality, we write \( x = (t_{m+1} - t_m)/T \), and use the fact that \( x \leq 1 \), and for \( \varepsilon < e^{-1} \), the function
\[
(x\varepsilon)^{xT} \exp(-nR|V|\varepsilon x) = \exp(xT \log(x\varepsilon) - nR|V|\varepsilon x)
\]
is decreasing for \( x \in (0, 1] \). Applying Stirling’s approximation and noticing \( T \leq c'n\varepsilon \), we have
\[
\frac{1}{n} \log \mathbb{P}_{x,w} \left( \mu^n \left( \frac{t_{m+1} - t_m}{T} \right) \varepsilon = x_{m+1} \right) \\
\geq c' \varepsilon \log \alpha_1 + (dp_1 - 1) \frac{1}{n} \log \frac{1}{n} - (dp_1 - 1) c' \varepsilon \log \varepsilon + c' \varepsilon \log \varepsilon - R|V|\varepsilon + o(\varepsilon) \\
\geq dp_1 c' \varepsilon \log \varepsilon + O(\varepsilon) + o(1),
\]
where \( o(1) \) tends to zero as \( n \to \infty \). Taking the product in \( m \), we conclude \( \frac{1}{n} \log \mathbb{P}_{w} \left( \mu^n (\varepsilon) = x_n \right) \geq -c(\varepsilon) + O(\varepsilon) + o(1) \) with \( c(\varepsilon) = O(\varepsilon \log \varepsilon) \), as desired. \( \square \)

### Appendix A. Proof of Theorem 5.6

We now present the proof of Theorem 5.6. Recall that \( h_n \) maps a controlled PRM into a controlled process, and is defined in (5.3). Recall also the definitions of \( \tilde{A}_{b}^{\otimes|V|} \) and \( \tilde{\mathcal{A}}_{b}^{\otimes|V|} \) in Definition 5.5 and Definition 5.2 respectively. The claim of Theorem 5.6 is essentially that the additional dependence of controls in \( \tilde{A}_{b}^{\otimes|V|} \) on the “type” of jump is not needed, and that the variational representation is valid with the simpler controls \( A_{b}^{\otimes|V|} \). We recall that the controls in \( \tilde{A}_{b}^{\otimes|V|} \) modulate the intensity of the driving PRM in an \( s, x \) and \( \omega \) dependent fashion, while the controls in \( A_{b}^{\otimes|V|} \) multiply the jump rates \( \lambda_{s}^{n} \) in an \( n \) and \( \omega \) dependent way.

The proof of Theorem 5.6 will follow from Lemma 5.4 and the results Corollary A.3 and Lemma 5.5 established below. For simplicity we assume \( T = 1 \). We start with two lemmas that elucidate the relation between elements of \( \tilde{A}_{b}^{\otimes|V|} \) and \( \tilde{\mathcal{A}}_{b}^{\otimes|V|} \).

**Lemma A.1.** There exists a map \( \Theta^{n} : \tilde{A}_{b}^{\otimes|V|} \to A_{b}^{\otimes|V|} \times D([0, 1] : S) \times \tilde{\mathcal{A}}_{b}^{\otimes|V|} \) that takes \( \varphi \in \tilde{A}_{b}^{\otimes|V|} \) into a triple \((\hat{\alpha}^{n}, \mu^{n}, \hat{\varphi}^{n})\), such that for any \( v \in V \), the following is true:

1. \( \hat{\alpha}^{n}(s) = \int_{0}^{\lambda_{s}^{n}(\mu^{n}(s))} \varphi_{v}(s, y) \, dy \),
2. \( \hat{\mu}^{n} = h_{n} \left( \frac{1}{n} \mathbb{N}^{n} \hat{\mu}^{n}, \mu^{n}(0), \lambda^{n} \right) \).
3. \( \hat{\varphi}^{n}(s, y) = \frac{\hat{\alpha}^{n}(s)}{\lambda_{s}^{n}(\mu^{n}(s))} \mathbb{I}_{[0, \lambda_{s}^{n}(\mu^{n}(s))]}(y) + \mathbb{I}_{[0, \lambda_{s}^{n}(\hat{\mu}^{n}(s))]}(y) \).

Note that given any control \( \varphi \in \tilde{A}_{b}^{\otimes|V|} \), this lemma identifies a structurally simpler control \( \hat{\varphi}^{n} \in \tilde{\mathcal{A}}_{b}^{\otimes|V|} \).

**Proof.** We prove the claim by a recursive construction.

1. To begin the recursion set \( t_{0} = 0 \), and given any \( \varphi \in \tilde{A}_{b}^{\otimes|V|} \), define for \( s \geq t_{0} \) and \( v \in V \),
   \[
   \hat{\mu}^{n, 0}(s) = \mu^{n}(0), \\
   \hat{\alpha}^{n, 0}(s) = \int_{0}^{\lambda_{s}^{n}(\hat{\mu}^{n, 0}(s))} \varphi_{v}(s, y) \, dy, \\
   \hat{\varphi}^{n, 0}(s, y) = \frac{\hat{\alpha}^{n, 0}(s)}{\lambda_{s}^{n}(\hat{\mu}^{n, 0}(s))} \mathbb{I}_{[0, \lambda_{s}^{n}(\hat{\mu}^{n, 0}(s))]}(y) + \mathbb{I}_{[0, \lambda_{s}^{n}(\hat{\mu}^{n, 0}(s))]}(y).
   \]
In other words, for $s > 0$ and $y$ inside the compact set $[0, \lambda^n_v(\mu^n_v(s))]$, we set $\hat{\varphi}^n_v(s, \cdot)$ to be the average of $\varphi_v(s, \cdot)$ over the set, while for $y$ in the complement we set $\hat{\varphi}^n_v = 1$. We see that by construction $\|\hat{\varphi}^n_v\|_\infty \leq \|\varphi_v\|_\infty \vee 1$.

2. Assume now that for some $k \in \mathbb{N}_0$, $t_k$ is well defined, $\{\hat{\varphi}^n_v(s), \hat{\alpha}^n_v(s), \{\mu^n_v(s)\}\}$ is well defined for $s \in [0, 1]$, and

$$\|\hat{\varphi}^n_v\|_{\infty, (t_k, \infty)} \doteq \sup_{(s, y) \in [t_k, \infty) \times \mathbb{R}} |\hat{\varphi}^n_v(s, y)| \leq \|\varphi_v\|_\infty \vee 1.$$  

For any $t \geq t_k$ and $v \in \mathcal{V}$, define

$$\hat{B}_{k,v} (t) = \{(s, y, r) : s \in [t_k, t], y \in \left[0, \lambda^n_v(\hat{\mu}^n_k(s))\right], r \in \left[0, \hat{\varphi}^n_v(s, y)\right]\}$$  

and

$$t_{k+1} = \inf \left\{ t > t_k \text{ such that for some } v \in \mathcal{V}, \hat{N}^n_v(\hat{B}_{k,v}(t)) > 0 \right\} \wedge 1.$$  

We define $\hat{\mu}^{n,k+1}$ on $[0, 1]$ by first setting $\hat{\mu}^{n,k+1}(s) = \hat{\mu}^{n,k}(s)$ for $s \in [0, t_{k+1})$. Then, at $t_{k+1}$, we update $\hat{\mu}^{n,k+1}$ by setting

$$\hat{\mu}^{n,k+1}(t_{k+1}) = \hat{\mu}^{n,k}(t_k) + \sum_{v \in \mathcal{V}} v \int_{[t_k, t_{k+1}]} \int_{\mathbb{R}} \int_{[0, \lambda^n_v(\hat{\mu}^{n,k}(s))]} (y) \int_{[0, \lambda^n_v(\hat{\varphi}^n_v(s, y))]} (r) \frac{1}{n} \hat{N}_v(dsdydr),$$  

and set $\hat{\mu}^{n,k+1}(s) = \hat{\mu}^{n,k+1}(t_{k+1})$ for $s \geq t_{k+1}$. Define

$$\hat{\alpha}^{n,k+1}_v(s) = \int_0^{\lambda^n_v(\hat{\mu}^{n,k+1}(s))} \varphi_v(s, y) dy,$$

$$\hat{\varphi}^{n,k+1}_v(s, y) = \lambda^n_v(\hat{\mu}^{n,k+1}(s)) \int_{[0, \lambda^n_v(\hat{\mu}^{n,k+1}(s))]} (y) + \int_{[0, \lambda^n_v(\hat{\varphi}^n_v(s, y))]} (y).$$

We also have $\|\hat{\varphi}^{n,k+1}_v\|_{\infty, (t_{k+1}, \infty)} \leq \|\varphi_v\|_\infty \vee 1$.

3. Recall $\hat{R}$ defined as in [X]. Since $\hat{N}^n_v$ has a.s. finitely many atoms on $[0, 1] \times [0, \hat{R}) \times [0, \|\varphi_v\|_\infty]$, the construction will produce functions defined on $[0, 1]$ in $L < \infty$ steps. Then set

$$\hat{\mu}^n(s) = \hat{\mu}^{n,L}(s), \hat{\alpha}^n_v(s) = \hat{\alpha}^{n,L}_v(s), \hat{\varphi}^n_v(s) = \hat{\varphi}^{n,L}_v(s), \text{ if } s \in [0, 1].$$

Then $\hat{\varphi}^n \in \hat{A}^\otimes |\mathcal{V}|$. Furthermore, by construction

$$\hat{\mu}^n = h_n\left(\frac{1}{n} N^n \hat{\varphi}^n, \mu^n(0), \lambda^n\right).$$

The next lemma shows that from controls in $\hat{A}^\otimes |\mathcal{V}|$ we can produce corresponding controls in $\hat{\mathcal{A}}^\otimes |\mathcal{V}|$.

Lemma A.2. There exists a map $\Xi^n : \hat{A}^\otimes |\mathcal{V}| \to D([0, 1] : \mathcal{S}) \times \hat{\mathcal{A}}^\otimes |\mathcal{V}|$ which takes $\hat{\alpha} \in \hat{A}^\otimes |\mathcal{V}|$ into a pair $(\hat{\mu}^n, \hat{\varphi})$, such that $\hat{\mu}^n = h(\frac{1}{n} N^n \hat{\varphi}, \mu^n(0), \lambda^n)$, where for $v \in \mathcal{V}$, $\hat{\varphi}_v(s, y) = \frac{x(v)(s)}{\lambda^n_v(\hat{\mu}^n(s))} \int_{[0, \lambda^n_v(\hat{\mu}^n(s))]} (y) + \int_{[0, \lambda^n_v(\hat{\varphi}_v(s, y))]} (y)$.
Proof. 1. Define \( t_0 = 0 \) and for any \( \bar{\alpha} \in \mathcal{A}_b^{[\alpha]} \), \( s \geq t_0 \) and \( v \in \mathcal{V} \), define
\[
\tilde{\mu}^n_0 (s) = \mu^n(0),
\]
\[
\tilde{\varphi}^0 (s, y) = \frac{\bar{\alpha}_v(s)}{\lambda^n_v(\tilde{\mu}^n_0(s))} [\mathbb{I}_{[0, \lambda^n_v(\tilde{\mu}^n_0(s))]}(y) + \mathbb{I}_{[\lambda^n_v(\tilde{\mu}^n_0(s)), \infty]}(y)].
\]

2. Assume now that for some \( k \in \mathbb{N} \), \( t_k \) is well defined, and that \( (\tilde{\mu}^{n,k}(s), \{\tilde{\varphi}^k_v(s)\}) \) is well defined for \( s \geq t_k \). For any \( t \geq t_k \), define
\[
\tilde{A}_{k,v}(t) = \{(s, y, r) : s \in [t_k, t], y \in [0, \lambda^n_v(\tilde{\mu}^{n,k}(s))], r \in [0, \tilde{\varphi}^k_v(s, y)]\}
\]
and
\[
t_{k+1} = \inf \{t > t_k \text{ such that for some } v \in \mathcal{V}, \tilde{N}^{n}_{v}(\tilde{A}_{k,v}(t)) > 0\} \wedge 1.
\]
We define \( \tilde{\mu}^{n,k+1} \) on \([0, 1]\) by first setting \( \tilde{\mu}^{n,k+1}(s) = \tilde{\mu}^{n,k}(s) \) for \( s \in [0, t_{k+1}) \). Then, at \( t_{k+1} \), we update \( \tilde{\mu}^{n,k+1} \) by
\[
\tilde{\mu}^{n,k+1}(t_{k+1}) = \tilde{\mu}^{n,k}(t_{k+1}) + \sum_{v \in \mathcal{V}} \int_{[t_k, t_{k+1}]} \int_{[0, \lambda^n_v(\tilde{\mu}^{n,k}(s))]} \mathbb{I}_{[0, \lambda^n_v(\tilde{\mu}^{n,k}(s))]}(y) \mathbb{I}_{[0, \tilde{\varphi}^k_v(s, y)]}(r) \frac{1}{n} \tilde{N}^{n}_{v}(dsdydr),
\]
and set \( \tilde{\mu}^{n,k+1}(s) = \tilde{\mu}^{n,k+1}(t_{k+1}) \) for \( s \geq t_{k+1} \). Define
\[
\tilde{\varphi}^{k+1}_v(s, y) = \frac{\bar{\alpha}_v(s)}{\lambda^n_v(\tilde{\mu}^{n,k+1}(s))} [\mathbb{I}_{[0, \lambda^n_v(\tilde{\mu}^{n,k+1}(s))]}(y) + \mathbb{I}_{[\lambda^n_v(\tilde{\mu}^{n,k+1}(s)), \infty]}(y)].
\]

3. Since \( \tilde{N}^{n}_{v} \) has a.s. finitely many atoms on \([0, 1] \times [0, \|\bar{\alpha}_v\|_{\infty}]\), the construction will produce functions defined on \([0, 1]\) in \( L < \infty \) steps. Then set
\[
\tilde{\mu}^n(s) = \tilde{\mu}^{n,L}(s), \tilde{\varphi}_v(s) = \tilde{\varphi}^L_v(s), s \in [0, 1].
\]
Note that
\[
\tilde{\varphi}_v(s, y) = \frac{\bar{\alpha}_v(s)}{\lambda^n_v(\tilde{\mu}^n(s))} [\mathbb{I}_{[0, \lambda^n_v(\tilde{\mu}^n(s))]}(y) + \mathbb{I}_{[\lambda^n_v(\tilde{\mu}^n(s)), \infty]}(y)]
\]
and \( \tilde{\mu}^n \) satisfies
\[
\tilde{\mu}^n = h_n \left( \frac{1}{n} N^n \tilde{\varphi}, \mu^n(0), \lambda^n \right).
\]

The next result is a corollary to the construction in Lemma [A.2]. Let \( \Sigma^n_1 : \mathcal{A}_b^{[\alpha]} \to D([0, 1] : \mathcal{S}) \) denote the first component of the map in Lemma [A.2].

**Corollary A.3.** For any given \( \{\tilde{\alpha}_v\} \in \mathcal{A}_b^{[\alpha]} \) and \( t \in [0, 1] \), \( \tilde{\Sigma}^{n}_1(\tilde{\alpha})(t) \) has the same distribution as \( \Lambda^n(\tilde{\alpha}, \mu^n(0))(t) \), where \( \Lambda^n \) is as defined in [A.3].

**Proof.** Recall that \( \tilde{\mu}^n = \tilde{\Sigma}^{n}_1(\tilde{\alpha}) \). We have \( \tilde{\Sigma}^{n}_1(\tilde{\alpha})(0) = \mu^n(0) \). Given \( s \in [0, 1] \), the total jump intensity of \( \tilde{\mu}^n(s) \) in the direction \( v \) is
\[
\int_{0}^{\lambda^n_v(\tilde{\mu}^n(s))} \tilde{\varphi}^n_v(s, y) dy = \int_{0}^{\lambda^n_v(\tilde{\mu}^n(s))} \frac{\bar{\alpha}_v(s)}{\lambda^n_v(\tilde{\mu}^n(s))} dy = \tilde{\alpha}_v(s)
\]
which is the same as that of \( \Lambda^n(\tilde{\alpha}, \mu^n(0))(s) \). \( \square \)
Lemma A.4. Let $\mathcal{A}_b$, $\tilde{\mathcal{A}}_b$ and $\bar{\mathcal{A}}$ be as defined in Definitions 6.7, 6.2 and 5.7 respectively, and let $L_1$ be defined as in (5.2) with $T = 1$. Then for $F \in \mathcal{M}_b(D([0, T] : \mathcal{S}))$,

$$\inf_{\varphi \in \tilde{\mathcal{A}}_b^{\otimes |V|}} \mathbb{E} \left[ \sum_{v \in V} L_1(\varphi_v) + F(\tilde{\mu}^n) : \tilde{\mu}^n = h_n \left( \frac{1}{n} N^{n\varphi}, \mu^n(0), \lambda^n \right) \right]$$

$$= \inf_{\varphi \in \mathcal{A}_b^{\otimes |V|}} \mathbb{E} \left[ \sum_{v \in V} L_1(\varphi_v) + F(\mu^n) : \mu^n = h_n \left( \frac{1}{n} N^{n\varphi}, \mu^n(0), \lambda^n \right) \right]$$

$$= \inf_{\bar{\alpha} \in \mathcal{A}_b^{\otimes |V|}} \mathbb{E} \left[ \sum_{v \in V} \int_0^1 \lambda^n_v(\tilde{\mu}^n(t)) \ell \left( \frac{\bar{\alpha}_v(t)}{\lambda^n_v(\tilde{\mu}^n(t))} \right) dt + F(\mu^n) : \mu^n = \Xi^n (\bar{\alpha}) \right],$$

where $\Xi^n$ is as defined in Lemma A.2.

Proof. The first equality is a consequence of Theorem 2.4 of [5]. To prove the rest of the claim, for $\bar{\alpha} \in \mathcal{A}_b^{\otimes |V|}$ fixed, let $(\tilde{\mu}^n, \bar{\alpha}) = \Xi^n (\bar{\alpha})$. Then by definition $\tilde{\varphi} \in \tilde{\mathcal{A}}_b^{\otimes |V|}$, and since $\nu$ in (5.2) is Lebesgue measure

$$L_1(\tilde{\varphi}_v) = \sum_{v \in V} \int_0^1 \int_0^1 \ell (\tilde{\varphi}_v(t, y)) dt \, dy = \sum_{v \in V} \int_0^1 \lambda^n_v(\tilde{\mu}^n(t)) \ell \left( \frac{\bar{\alpha}_v(t)}{\lambda^n_v(\tilde{\mu}^n(t))} \right) dt.$$

Now it follows from Lemma A.2 that

$$\inf_{\varphi \in \mathcal{A}_b^{\otimes |V|}} \mathbb{E} \left[ \sum_{v \in V} L_1(\varphi_v) + F(\mu^n) : \mu^n = h_n \left( \frac{1}{n} N^{n\varphi}, \mu^n(0), \lambda^n \right) \right]$$

$$\leq \mathbb{E} \left[ \sum_{v \in V} L_1(\tilde{\varphi}_v) + F \circ h_n \left( \frac{1}{n} N^{n\varphi}, \mu^n(0), \lambda^n \right) + F(\mu^n) \right]$$

$$= \mathbb{E} \left[ \sum_{v \in V} \int_0^1 \lambda^n_v(\bar{\mu}^n(t)) \ell \left( \frac{\bar{\alpha}_v(t)}{\lambda^n_v(\tilde{\mu}^n(t))} \right) dt + F(\mu^n) \right].$$

The reverse inequality is proved by a convexity argument. Recall the definition of $\Theta^n$ given in Lemma A.1. For given $\varphi \in \mathcal{A}_b^{\otimes |V|}$, let $(\bar{\alpha}, \bar{\mu}) = (\Theta^n_1 (\varphi), \Theta^n_2 (\varphi))$. Then $\bar{\alpha} \in \mathcal{A}_b^{\otimes |V|}$. By convexity of $\ell (\cdot)$ and Jensen’s inequality,

$$\int_0^1 \int_0^1 \ell (\varphi_v(s, y))\, dy \, ds$$

$$\leq \int_0^1 \int_0^1 \lambda^n_v(\bar{\mu}^n(s)) \ell \left( \frac{1}{\lambda^n_v(\tilde{\mu}^n(s))} \int_0^1 \lambda^n_v(\bar{\mu}^n(s)) \ell (\varphi_v(s, y))\, dy \right) \, ds$$

$$\leq \int_0^1 \lambda^n_v(\bar{\mu}^n(s)) \ell \left( \frac{1}{\lambda^n_v(\tilde{\mu}^n(s))} \int_0^1 \lambda^n_v(\tilde{\mu}^n(s)) \varphi_v(s, y)\, dy \right) \, ds$$

$$= \int_0^1 \lambda^n_v(\tilde{\mu}^n(s)) \ell \left( \frac{\bar{\alpha}_v(s)}{\lambda^n_v(\tilde{\mu}^n(s))} \right) \, ds.$$

Summing over $v \in V$, applying Lemma A.2 and infimizing over $\bar{\alpha} \in \mathcal{A}_b^{\otimes |V|}$ we obtain the desired result. \qed
References

[1] Nelson Antunes, Christine Fricker, Philippe Robert, and Danielle Tibi. Analysis of loss networks with routing. The Annals of Applied Probability, 16(4):2007–2026, 2006.
[2] J. Beltrán and C. Landim. Metastability of reversible finite state Markov processes. Stochastic Processes and their Applications, 121:1633–1677, 2011.
[3] Abraham Berman and Robert J Plemmons. Nonnegative Matrices. SIAM, 1979.
[4] Vivek Shripad Borkar and Rajesh Sundaresan. Asymptotics of the invariant measure in mean field models with jumps. In Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on, pages 1258–1263. IEEE, 2011.
[5] Amarjit Budhiraja, Jiang Chen, and Paul Dupuis. Large deviations for stochastic partial differential equations driven by a Poisson random measure. Stochastic Processes and their Applications, 123:523–560, 2013.
[6] Amarjit Budhiraja, Paul Dupuis, Markus Fischer, and Kavita Ramanan. Limits of relative entropies associated with weakly interacting particle systems. Electronic Journal of Probability, 20, 2015.
[7] Amarjit Budhiraja, Paul Dupuis, and Vassilios Maroulas. Large deviations for stochastic partial differential equations driven by a Poisson random measure. Markov Processes and Related Fields, 3(4):475–492, 1997.
[8] Carl Graham and Neil O'Connell. Large deviation asymptotics for occupancy problems. The Annals of Probability, 32(3B):2765–2818, 2004.
[9] Soummya Kar and Kavita Ramanan. Sample path large deviations for jump Markov processes revisited. preprint, 2015.
[10] Vassili N Kolokoltsov. Nonlinear Markov Processes and Kinetic Equations. Cambridge University Press London, 2010.
[11] David A Levin, Malwina J Luczak, and Yuval Peres. Glauber dynamics for the mean-field Ising model: cut-off, critical power law, and metastability. Probability theory and related fields, 146(1-2):223–265, 2010.
[12] Fabio Martinelli. Lectures on Glauber dynamics for discrete spin models. In Lectures on probability theory and statistics, pages 93–191. Springer, 1999.
[13] Karl Oelschlager. A martingale approach to the law of large numbers for weakly interacting stochastic processes. The Annals of Probability, 12(2):458–479, 1984.
[29] Enzo Olivieri and Maria Eulália Vares. *Large Deviations and Metastability*. Encyclopaedia of Mathematics and its Applications. Cambridge University Press, 2005.

[30] R Tyrrell Rockafellar. *Convex Analysis*. Number 28 in Princeton Landmarks in mathematics. Princeton university press, 1970.

[31] Adam Shwartz and Alan Weiss. *Large Deviations for Performance Analysis*. Chapman and Hall, 1995.

[32] Adam Shwartz and Alan Weiss. Large deviations with diminishing rates. *Mathematics of Operations Research*, 30(2):281–310, 2005.

[33] Danielle Tibi. Metastability in communication networks. *arXiv preprint arXiv:1002.0796*, 2010.

[34] SR Srinivasa Varadhan. *Large Deviations and Applications*. SIAM, 1984.

[35] Wei Wu. *Large Deviations and Quasipotential for Finite-state Mean Field Interacting Particle Systems*. PhD thesis, Brown University, Providence, RI, 2014.