GENERALIZED KRONECKER FORMULA FOR
BERNOULLI NUMBERS AND SELF-INTERSECTIONS OF CURVES ON A SURFACE

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Abstract. We present a new explicit formula for the \(m\)-th Bernoulli number \(B_m\), which involves two integer parameters \(a\) and \(n\) with \(0 \leq a \leq m \leq n\). If we set \(a = 0\) and \(n = m\), then the formula reduces to the celebrated Kronecker formula for \(B_m\). We give two proofs of our formula. One is analytic and uses a certain function in two variables. The other is algebraic and is motivated by a topological consideration of self-intersections of curves on an oriented surface.

1. Introduction

The Bernoulli numbers \(B_m (m \geq 0)\) are defined by the generating function

\[
\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m.
\]

We have: \(B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, \ldots\), and \(B_m = 0\) for all odd \(m \geq 3\). A large number of identities involving the Bernoulli numbers has been known \cite{2} \cite{3} \cite{10} \cite{11}. Most of them give relationships between \(B_m\) and \(B_i (0 \leq i < m)\). These identities provide various ways to compute \(B_m\) recursively from the \(B_i\)'s for \(0 \leq i < m\).

Contrary to the above recursive approach, the following formula of Kronecker gives a direct method for computing \(B_m\).

**Theorem 1** (Kronecker \cite{8}, see also \cite{2} \cite{4} \cite{10} \cite{11}). For any integer \(m \geq 2\), it holds that

\[
B_m = \sum_{k=1}^{m+1} \frac{(-1)^{k+1}}{k} \binom{m+1}{k} \sum_{i=1}^{k-1} i^m.
\]

In this article we generalize the formula (1) to a formula with two parameters:
Theorem 2. Let \( m, n, a \) be integers satisfying \( 0 \leq a \leq m \leq n \). Then it holds that

\[
B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right].
\]

Here \( \delta_{a,m} \) is the Kronecker delta. Furthermore, if \( m \geq 2 \), it holds that

\[
B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{i=1}^{k-1} i^a (k-i)^{m-a}.
\]

It is clear that, in the case \( a = 0 \) and \( n = m \), the formula (3) reduces to the Kronecker formula (1).

We give two proofs of Theorem 2. In §2, we introduce a two-variable function \( g(x, y) \) and compute its series expansion in two different ways. This leads to a proof of Theorem 2. In §3, we construct a certain continuous map \( \hat{\mu} : \mathbb{Q}[[Z]] \to \mathbb{Q}[[X, Y]] \) between the rings of formal power series. A key observation is that \( \hat{\mu}(Z) \) is expressed in terms of the Bernoulli numbers, and this leads to another proof of Theorem 2.

The map \( \hat{\mu} \) is motivated by an operation \( \mu \) to a curve on an oriented surface. This operation was introduced in [7] inspired by a construction of Turaev [12], and, among other things, it computes self-intersections of curves. In §4 we first recall the operation \( \mu \) from [7]. Then we obtain an exact formula for \( \mu \) (Theorem 3) based on the results in §3. The Bernoulli numbers have already appeared in the tensorial description of the homotopy intersection form on an oriented surface [9]. Our formula provides yet another evidence for a close connection between topology of surfaces and Bernoulli numbers.

2. The First Proof

Let \( f(x, y) \) and \( g(x, y) \) be functions in variables \( x \) and \( y \) defined by

\[
f(x, y) := \int_x^y (e^t - 1)^{n+1} dt, \quad \text{and} \quad g(x, y) := \frac{f(x, y)}{e^y - x - 1}.
\]

We will examine the coefficient of \( x^a y^{m-a} \) in the series expansion of \( g(x, y) \).

First we compute \( f(x, y) \) as follows:

\[
f(x, y) = \int_x^y (e^t - 1)^{n+1} dt
= \int_x^y \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kt} dt
= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} (e^{ky} - e^{kx}) + (-1)^{n+1}(y - x).
\]
Since
\[ \frac{e^{ky} - e^{kx}}{e^{y-x} - 1} = \frac{e^{kx}(e^{k(y-x)} - 1)}{e^{y-x} - 1} = \sum_{i=1}^{k-1} e^{ix} e^{(k-i)y} + e^{kx}, \]
we can compute \( g(x, y) \) as follows:
\[
g(x, y) = \frac{f(x, y)}{e^{y-x} - 1} = (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left( \frac{e^{ky} - e^{kx}}{e^{y-x} - 1} \right) + (-1)^{n+1} \frac{y-x}{e^{y-x} - 1}
\]
\[
= (-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} e^{ix} e^{(k-i)y} + e^{kx} \right]
\]
\[
+ (-1)^{n+1} \sum_{b=0}^{\infty} \frac{B_b}{b!} (y-x)^b.
\]

Then using the identities:
\[
e^{ix} e^{(k-i)y} = \sum_{b,c=0}^{\infty} \frac{i^b(k-i)^c}{b!c!} x^b y^c \quad \text{and} \quad e^{kx} = \sum_{b=0}^{\infty} \frac{k^b}{b!} x^b,
\]
we see that the coefficient of \( x^a y^{m-a} \) in \( g(x, y) \) is given by
\[
(-1)^{n+1} \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} \frac{a^i (k-i)^{m-a}}{a! (m-a)!} + \delta_{a,m} \frac{k^m}{m!} \right]
\]
\[
+ (-1)^{n+1+a} \frac{B_m}{m!} \binom{m}{a}.
\]

This is equal to \((((-1)^{n+1+a}/m!)\binom{m}{a}) \times \) times
\[
(-1)^a \sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} i^a (k-i)^{m-a} + \delta_{a,m} k^m \right] + B_m.
\]

(4)

Secondly, we expand \( g(x, y) \) in a different way. Put \( g_1(x, y) = f(x, y)/(y-x) \). Then we have
\[
g(x, y) = \frac{f(x, y)}{y-x} \frac{y-x}{e^{y-x} - 1} = g_1(x, y) \sum_{b=0}^{\infty} \frac{B_b}{b!} (y-x)^b.
\]

Writing \((e^t - 1)^{n+1} = \sum_{i \geq n+1} a_i t^i\), we have
\[
f(x, y) = \int_x^y (e^t - 1)^{n+1} dt = \sum_{i \geq n+1} \frac{a_i}{i+1} (y^{i+1} - x^{i+1}).
\]
Thus the series expansion of $g_1(x, y)$ has all terms of degree $\geq n + 1$, so does that of $g(x, y)$. In particular, the coefficient of $x^a y^{m-a}$ in this expansion is zero. Therefore, the expression (3) is zero, and we obtain the formula (2).

Finally, we can derive the formula (3) in Theorem 2 from the formula (2) by applying the following lemma. Although it might be well known, we give its proof for the sake of completeness.

**Lemma 1.** Let $m, n$ be integers satisfying $0 \leq m \leq n$. Then it holds that

$$\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m = \left\{ \begin{array}{ll} 0 & \text{if } m \geq 1, \\ -1 & \text{if } m = 0. \end{array} \right.$$ 

**Proof.** Set $f(x) := (e^x - 1)^{n+1}$. Since $m \leq n$, the coefficient of $x^m$ in the series expansion of $f(x)$ is zero.

On the other hand, we compute

$$f(x) = \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} e^{kx}$$

$$= (-1)^{n+1} \left[ \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} e^{kx} + 1 \right]$$

$$= (-1)^{n+1} \left[ \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \sum_{a=0}^{\infty} \frac{k^a}{a!} x^a + 1 \right].$$

Since the coefficient of $x^m$ in the last expression is equal to

$$\left\{ \begin{array}{ll} \frac{(-1)^{n+1}}{m!} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} k^m & \text{if } m \geq 1, \\ (-1)^{n+1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} + 1 & \text{if } m = 0, \end{array} \right.$$ 

the assertion follows. \hfill \Box

This completes the proof of Theorem 2.

3. **The second proof**

First of all, we describe a preliminary construction.

Let $\mathbb{Q}[[Z]]$ (resp. $\mathbb{Q}[[X,Y]]$) be the ring of formal power series in an indeterminate $Z$ (resp. in indeterminates $X$ and $Y$). For a non-negative integer $p$, let $F_p^Z$ (resp. $F_p^{X,Y}$) be the set of formal power series in $\mathbb{Q}[[Z]]$ (resp. $\mathbb{Q}[[X,Y]]$) which has only terms of (total) degree $\geq p$. We have natural isomorphisms $\mathbb{Q}[[Z]] \cong \varprojlim_p \mathbb{Q}[[Z]]/F_p^Z$ and $\mathbb{Q}[[X,Y]] \cong \varprojlim_p \mathbb{Q}[[X,Y]]/F_p^{X,Y}$.
Therefore, we have \( \hat{\mu} \) of the map \( f \) for any Laurent polynomial. It is sufficient to prove that 

\[
\hat{\mu}(z^k) = \begin{cases} 
- \sum_{i=1}^{k} e^{iX} e^{(k-i)Y} & (k > 0) \\
0 & (k = 0) \\
\sum_{i=0}^{k-1} e^{-iX} e^{(k+i)Y} & (k < 0) 
\end{cases}
\]

From the definition of \( \hat{\mu} \) it is easy to see that 

\((e^{-X} e^{Y} - 1)\hat{\mu}(z^k) = e^{kX} - e^{kY}, \quad k \in \mathbb{Z}.
\)

Therefore, we have 

\[
(\mu^{-1}(k) e^{jY} - 1)\hat{\mu}(f(z)) = f(e^{X}) - f(e^{Y})
\]

for any Laurent polynomial \( f(z) \in \mathbb{Q}[z, z^{-1}] \). Consider 

\[
\Phi(X, Y) := \sum_{i=0}^{\infty} \frac{B_i}{i!} (-X + Y)^i.
\]

Then we have \((\mu^{-1}(k) e^{jY} - 1)\Phi(X, Y) = -X + Y\). Multiplying \( \Phi(X, Y) \) to the both sides of (5), we have 

\[
(\Phi(X, Y) e^{jY} - 1)\hat{\mu}(f(z)) = (f(e^{X}) - f(e^{Y}) \Phi(X, Y)
\]

for any \( f(z) \in \mathbb{Q}[z, z^{-1}] \).

**Lemma 2.** There is a unique continuous extension \( \hat{\mu}: \mathbb{Q}[[z]] \rightarrow \mathbb{Q}[[X, Y]] \) of the map \( \hat{\mu} \) in (5).

**Proof.** It is sufficient to prove that \( \hat{\mu}(p) \subset F_{-1}^{X,Y} \) for any \( p \geq 1 \). Suppose \( f(z) \in I \). Then \( f(e^{X}) \) and \( f(e^{Y}) \) lie in \( F_{p}^{X,Y} \). This means that the right hand side of (7) is an element of \( F_{-1}^{X,Y} \). Therefore, \( \hat{\mu}(f(z)) \in F_{-1}^{X,Y} \).

Now for each \( k \geq 1 \) we can put \( f(z) = (\log z)^{k} = Z^{k} \) in (7), and we obtain 

\[
(\Phi(X, Y) e^{jY} - 1)\hat{\mu}(Z^{k}) = (X^{k} - Y^{k})\Phi(X, Y).
\]

This shows that \( \hat{\mu}(Z^{k}) \in F_{k-1}^{X,Y} \). Setting \( k = 1 \), we have 

\[
\hat{\mu}(Z) = -\Phi(X, Y) = -\sum_{i=0}^{\infty} \frac{B_i}{i!} \sum_{j=0}^{i} (-1)^{j} {i \choose j} X^{j} Y^{i-j}.
\]
The second proof of Theorem 2. We will give another proof to the formula (2) alone. In what follows, ≡ means an equality in \( \mathbb{Q}[[X,Y]] \) modulo \( F_{n+1}^{X,Y} \).

For \( k = 1, \ldots, n+1 \), we have

\[
(9) \quad \hat{\mu}(z^k) = \hat{\mu}(e^kZ) = \sum_{i=1}^{k+1} \frac{k^i}{i!} \hat{\mu}(Z^i) = \sum_{i=1}^{n+1} \frac{k^i}{i!} \hat{\mu}(Z^i).
\]

Consider the square matrix \( D = (D_{ki})_{k,i} \) of order \( n+1 \), where \( D_{ki} = \frac{k^i}{i!} \). Then \( D \) is invertible, and the inverse matrix of \( D \) has the first row \((a_1, \ldots, a_{n+1})\), where

\[
a_k = \frac{(-1)^{k+1}}{k} \binom{n+1}{k}
\]

(see also Lemma 1). From (9) we have

\[
(10) \quad \hat{\mu}(Z) \equiv \sum_{k=1}^{n+1} a_k \hat{\mu}(z^k) = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \hat{\mu}(z^k).
\]

Furthermore, for \( k = 1, \ldots, n+1 \), from (5) we have

\[
(11) \quad \hat{\mu}(z^k) = -\sum_{i=1}^{\infty} \sum_{a,b=0}^\infty \frac{i^a(k-i)^b}{a!b!} X^a Y^b - \sum_{a=0}^\infty \frac{k^a}{a!} X^a.
\]

By (10) and (11), the coefficient of \( X^a Y^{m-a} \) in \( \hat{\mu}(Z) \) is

\[
\sum_{k=1}^{n+1} \frac{(-1)^k}{k} \binom{n+1}{k} \left[ \sum_{i=1}^{k-1} \frac{i^a(k-i)^{m-a}}{a!(m-a)!} + \delta_{m,a} \frac{k^m}{m!} \right].
\]

On the other hand, by (8), this coincides with

\[
(-1)^{a+1} \frac{B_m}{m!} \binom{m}{a} = (-1)^{a+1} \frac{B_m}{a!(m-a)!} B_m.
\]

This completes the proof.

\[\square\]

4. A Topological Background for the Second Proof

Let \( S \) be a compact connected oriented surface with \( \partial S \neq \emptyset \). Fix a basepoint \( \ast \in \partial S \) and set \( \pi_1(S) := \pi_1(S, \ast) \). We denote by \( \hat{\pi}(S) \) the set of free homotopy classes of oriented loops on \( S \). For any \( p \in S \), we denote by \( | |: \pi_1(S, p) \to \hat{\pi}(S) \) the forgetful map of the basepoint.

We recall the operation \( \mu: \mathbb{Q}\pi_1(S) \to \mathbb{Q}\pi_1(S) \otimes (\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1) \), which has been introduced in [7] inspired by a construction of Turaev [12]. Here, 1 is the class of a constant loop. Let \( \gamma: [0,1] \to S \) be an immersed based loop. We arrange so that the pair of tangent vectors \((\dot{\gamma}(0), \dot{\gamma}(1))\) is a positive basis of the tangent space \( T_{\gamma}S \), and that the self-intersections of \( \gamma \) (except for the base point \( \ast \)) lie in the interior \( \text{Int}(S) \) and consist of transverse double points.
Let $\Gamma$ be the set of double points of $\gamma$. For $p \in \Gamma$ we denote $\gamma^{-1}(p) = \{t^p_1, t^p_2\}$, so that $0 < t^p_1 < t^p_2 < 1$. We define

\[
\mu(\gamma) := - \sum_{p \in \Gamma} \epsilon(\dot{\gamma}(t^p_1), \dot{\gamma}(t^p_2)) (\gamma_{0t^p_1} \gamma_{t^p_2}) \otimes |\gamma_{t^p_1 t^p_2}| \in \mathbb{Q}_{\pi_1} \otimes (\mathbb{Q}_{\hat{\pi}(S)}/\mathbb{Q}1).
\]

Here,

- the sign $\epsilon(\dot{\gamma}(t^p_1), \dot{\gamma}(t^p_2))$ is $+1$ if the pair $(\dot{\gamma}(t^p_1), \dot{\gamma}(t^p_2))$ is a positive basis of $T_p S$, and is $-1$ otherwise,
- the based loop $\gamma_{0t^p_1} \gamma_{t^p_2}$ is the conjunction of the paths $\gamma |_{[0, t^p_1]}$ and $\gamma |_{[t^p_1, 1]}$.
- the element $\gamma_{t^p_1 t^p_2} \in \pi_1(S, p)$ is the restriction of $\gamma$ to $[t^p_1, t^p_2]$ and we understand that $|\gamma_{t^p_1 t^p_2}| = 0$ if the loop $\gamma_{t^p_1 t^p_2}$ is homotopic to a constant loop.

We remark that the alternating part of $(| \otimes 1) \mu(\gamma)$ is exactly the Turaev cobracket \[13\] of the free loop $|\gamma|$.

We observe that if $\gamma$ is simple under the condition that the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of $T_* S$, then for any integer $k \in \mathbb{Z}$,

\[
\mu(\gamma^k) = \begin{cases} 
- \sum_{i=1}^{k-1} \gamma^i \otimes |\gamma^{k-i}| & (k > 0) \\
0 & (k = 0) \\
\sum_{i=0}^{k-1} \gamma^{-i} \otimes |\gamma^{k+i}| & (k < 0).
\end{cases}
\] (12)

See Figure 1. The definition of $\hat{\mu}$ in \[5\] is motivated by this formula.

In \[7\], it was shown that the map $\mu$ extends to a map between completions $\mu: \mathbb{Q}_{\pi_1}(S) \to \mathbb{Q}_{\pi_1}(S) \otimes \mathbb{Q}_{\hat{\pi}(S)}$. Here $\mathbb{Q}_{\pi_1}(S)$ and $\mathbb{Q}_{\hat{\pi}(S)}$ are the completions of the group ring $\mathbb{Q}_{\pi_1}(S)$ and the Goldman-Turaev Lie bialgebra $\mathbb{Q}_{\hat{\pi}(S)}/\mathbb{Q}1$, respectively, with respect to the augmentation ideal of $\mathbb{Q}_{\pi_1}(S)$. See \[4\]. Then we can consider $\log \gamma = \sum_{i=1}^{\infty} ((-1)^{i+1}/i)(\gamma - 1)^i \in \mathbb{Q}_{\pi_1}(S)$. 

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**Figure 1.** computation of $\mu(\gamma^k)$ for simple $\gamma$ ($k = 4$)
Theorem 3. Let $\gamma \in \pi$ be represented by a simple loop, and assume the pair $(\dot{\gamma}(0), \dot{\gamma}(1))$ is a positive basis of the tangent space $T_*S$. Then we have

$$\mu(\log \gamma) = \frac{1}{2} \hat{1} \otimes |\log \gamma| - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \sum_{p=0}^{2k-1} (-1)^p \binom{2k}{p} (\log \gamma)^p \hat{1} \otimes (\log \gamma)^{2k-p}.$$ 

Proof. We identify the ring $\mathbb{Q}[[X,Y]]$ with the complete tensor product $\mathbb{Q}[[Z]] \hat{\otimes} \mathbb{Q}[[Z]]$ by the map $X \mapsto Z \hat{\otimes} 1$ and $Y \mapsto 1 \hat{\otimes} Z$. Then the computation (8) implies

$$\hat{\mu}(\log z) = -1 \hat{\otimes} 1 - \frac{1}{2}(\log z) \hat{\otimes} 1 + \frac{1}{2} \hat{1} \otimes (\log z)^2.$$

(13) 

Since the curve $\gamma$ satisfies the formula (12) and we agree that $|1| = 0$, the theorem follows from (13). 

As an application of Theorem 3, the second-named author gives an explicit tensorial description of the Turaev cobracket on any genus 0 compact surface with respect to the standard group-like expansion [5]. It seems to suggest a certain connection between the operation $\mu$, or equivalently, the Turaev cobracket, and the Kashiwara-Vergne problem in the formulation by Alekseev-Torossian [1].

References

[1] Alekseev, A. and Torossian, C.: The Kashiwara-Vergne conjecture and Drinfeld’s associators, Ann. of Math. 175 (2012), 415–463.
[2] Dilcher, K., Skula, L., and Slavutskii, I.: Bernoulli numbers Bibliography (1713–1990), Queen’s Papers in Pure and Applied Mathematics, Kingston, Ontario (1987).
[3] Gould, H.: Explicit formulas for Bernoulli numbers, Amer. Math. Monthly 79 (1972), 44–51.
[4] Higgins, J.: Double series for the Bernoulli and Euler numbers, J. London Math. Soc. (2) 2 (1970), 722–726.
[5] Kawazumi, K.: A tensorial description of the Turaev cobracket on genus 0 compact surfaces, in preparation.
[6] Kawazumi, N and Kuno, Y.: Groupoid-theoretical methods in the mapping class groups of surfaces, preprint, arXiv:1109.6479v3 (2011).
[7] Kawazumi, K. and Kuno, Y.: Intersections of curves on surfaces and their applications to mapping class groups, to appear in Ann. Inst. Fourier, available at arXiv:1112.3841v3.
[8] Kronecker, L.: Ueber die Bernoullischen Zahlen, J. Reine Angew. Math. 94 (1883), 268–269.
[9] Massuyeau, G. and Turaev, V.: Fox pairings and generalized Dehn twists, Ann. Inst. Fourier 63 (2013), 2403–2456.
[10] Nielsen, N.: Traité élémentaire des nombres de Bernoulli, Paris: Gauthier-Villars et Cie (1923).
[11] Saalschütz, L.: Vorlesungen über die Bernoullischen Zahlen, ihren Zusammenhang mit den Secanten-Coefficienten und ihre wichtigeren Anwendungen, Berlin: Julius Springer (1893).
[12] Turaev, V.: Intersections of loops in two-dimensional manifolds, Math. USSR Sbornik 35 (1979), 229–250.

[13] Turaev V.: Skein quantization of Poisson algebras of loops on surfaces, Ann. sci. École Norm. Sup. (4) 24 (1991), 635–704.

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