Quantum corrections to neutrino masses and mixing angles

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1 Introduction

Quark and lepton masses and mixing angles are free parameters of the Standard Model (SM). They are known with various degree of accuracy, from the precision direct measurements of charged lepton masses, a combination of experimental data and theoretical arguments for quark masses and relatively precise measurements of quark mixing angles and the phase parameterizing the violation of CP invariance in the SM. A new element in this picture are neutrino masses and mixing angles. There is at present strong experimental evidence for neutrino oscillations whose most obvious and most natural explanation is that neutrinos have non-zero masses and the neutrino mass eigenstates are different from the weak interaction eigenstates. Although far from having final interpretation, the present experimental data give interesting preliminary information about the neutrino mass sector and the forthcoming experiments are expected to resolve the remaining ambiguities.

On the theoretical side, the origin of the interactions giving rise to fermion masses is a problem that cannot be addressed in the framework of the SM. The physical scale (we shall call it $M$) of the, still unknown, theory of fermion masses is certainly above the electroweak scale and quite likely it may even be close to the GUT or Planck scales. Such a high scale is suggested, for instance, by the see-saw interpretation of the magnitude of the neutrino masses indicated by experiment. Ultimately, the theory will predict at the scale $M$ the running fermion mass parameters (and perhaps also other parameters) of the effective low energy theory, describing physics at energies $< M$.  

To relate the mass parameters of the effective low energy theory to the experimentally measured quantities one has to include quantum corrections that already do not depend on the specific theory of fermion masses but only on the low energy effective theory. Clearly, close to the electroweak scale this theory is the Standard Model (SM). One possibility is that the SM remains the correct effective theory up to the scale $M$. This is conceivable particularly if the scale $M$ is relatively low. Another possibility is that the SM needs to be embedded into a bigger effective theory already much below $M$. The latter case may, for instance, happen if the low energy supersymmetry is realized in Nature. In this review we discuss quantum corrections to neutrino masses both in the SM and in its supersymmetric extension, the Minimal Supersymmetric Standard Model (MSSM).

Quite generally one can distinguish two classes of quantum corrections that enter on a somewhat different footing. The first one is given by the RGEs describing the evolution

\footnote{The running mass parameters of the effective low energy theory can in principle be calculated in the underlying theory at any renormalization scale $Q$ by proper inclusion of the high energy threshold corrections. Choosing $Q \approx M$ one minimizes those corrections.}
of the fermion mass parameters in the SM or MSSM from the scale $M$ down to some scale close to the electroweak scale. Another source of quantum corrections are the so-called low energy threshold effects. Strictly speaking these are the corrections necessary to express measurable quantities like neutrino masses and mixing angles in terms of the running (renormalized) parameters of the effective theory Lagrangian. Formally, their inclusion renders the prediction for observables independent of the choice of the scale to which the RGEs are integrated. In the SM the threshold corrections are unambiguous and, if that scale is taken to be close to $M_Z$, they are small and can be neglected. Such corrections may, however, be very important in the MSSM since they depend also on the sparticle masses and couplings.

In Section 2 we review the neutrino masses and mixing and stress the differences with the quark sector. Contrary to the small mixing in the latter one, at least one neutrino mixing angle and, quite likely two, are close to maximal $[1]$. A maximal or bimaximal mixing would be quite natural for (approximately) degenerate masses, much larger than their differences $[4]$. Such mass patterns are very different from the hierarchical masses of the charged fermions but are consistent with experiment. Indeed, only mass squared differences can be inferred from the data and not the neutrino masses themselves.

The observed (mixing) and potential (mass pattern) differences between the neutrino and quark sectors provide strong motivation for studying quantum corrections to neutrino masses and mixing. It is well known that the corrections are small for the quarks, just because of their hierarchical masses and mixing. As we shall review in this paper, that remains true for neutrinos if their masses are hierarchical too. However, quantum corrections may give strong, qualitatively new, effects if the neutrino masses are not hierarchical.

The main parts of this review are Sections 4 and 5. In Section 4 we discuss quantum corrections described by the renormalization group evolution and in Section 5 the potential effects of the low energy threshold corrections are reviewed.

A brief overview of quantum corrections in the neutrino sector is given in Section 6.
2 Fermion masses and mixing

For quarks and charged leptons the Particle Data Group [3] gives the following values of the masses:

\[
\begin{align*}
    m_u &= 1.5 \pm 5.0 \text{ MeV}, \quad m_c = 1.15 \pm 1.35 \text{ GeV}, \quad m_t = 174.3 \pm 5.1 \text{ GeV}, \\
    m_d &= 3.0 \pm 9.0 \text{ MeV}, \quad m_s = 60 \pm 170 \text{ MeV}, \quad m_b = 4.0 \pm 4.4 \text{ GeV}, \\
    m_e &= 511 \text{ keV}, \quad m_\mu = 105.7 \text{ MeV}, \quad m_\tau = 1.777 \text{ GeV}
\end{align*}
\]

and of the quark mixing (absolute values of the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix):

\[
\begin{pmatrix}
    0.9742 - 0.9757 & 0.219 - 0.226 & 0.002 - 0.005 \\
    0.219 - 0.225 & 0.9743 - 0.9749 & 0.037 - 0.043 \\
    0.004 - 0.014 & 0.035 - 0.043 & 0.9990 - 0.9993
\end{pmatrix}
\]

The pattern of quark masses and mixing is clear: hierarchical masses and small (and also) hierarchical mixing: the larger the mass difference, the smaller the mixing angle. In the framework of the SM, fermion masses and mixing angles are used to fix the Yukawa matrices which are the actual free parameters of the theory. They cannot, however, be uniquely reconstructed from the experimental data: there is always the ambiguity of rotating the electroweak basis and all we can have at present is merely a phenomenological parametrization.

Neutrino masses and mixing are inferred from a number of oscillation experiments. The interpretation of neutrino oscillations in terms of massive neutrinos is the most natural one. To discuss this interpretation, let us first recall the oscillation pattern for two hypothetical neutrinos with masses \( m_1 \) and \( m_2 \) whose quantum fields are linear combinations of two quantum fields \( \nu_A \) and \( \nu_B \) that are weak eigenstates

\[
\begin{align*}
    \nu_1 &= \cos \theta \nu_A + \sin \theta \nu_B \\
    \nu_2 &= -\sin \theta \nu_A + \cos \theta \nu_B
\end{align*}
\]

The probabilities that in a given experiment a neutrino produced in the interaction of the charged lepton of flavour \( A \) with the \( W^\pm \) boson is detected at a distance \( L \) as the neutrino creating the charged lepton of flavour \( B, B \neq A, \) or \( A \) are

\[
\begin{align*}
    P^{2 \times 2}(\nu_A \to \nu_B) &= \sin^2 \theta \sin^2 \left( \frac{\Delta m^2 L_{\text{exp}}}{4E_{\text{exp}}} \right) \\
    P^{2 \times 2}(\nu_A \to \nu_A) &= 1 - P^{2 \times 2}(\nu_A \to \nu_B)
\end{align*}
\]

\footnote{For \( u, d \) and \( s \) quarks the quoted values are the running masses in the \( \overline{\text{MS}} \) scheme at \( Q = 2 \) GeV. The values given for the \( c \) and \( b \) quarks are the \( \overline{\text{MS}} \) running masses at \( Q = m_c \) and \( Q = m_b \), respectively. Finally \( m_t \) given here is the pole mass.}

\footnote{The formulae like (2.4) are usually derived in the framework of quantum mechanics. The proper...}
We see that the oscillation probability depends on \( \Delta m^2 \equiv m_2^2 - m_1^2 \), the distance \( L_{\text{exp}} \), the mixing angle \( \vartheta \) and the neutrino energy \( E_{\text{exp}} \). It is clear that for observing the oscillation pattern the factor \( \Delta m^2 L/4E \) should be of order \( \mathcal{O}(1) \). Hence, in general, longer oscillation distances are necessary to probe smaller mass squared differences. On the other hand, for \( \Delta m^2 L/4E \gg 1 \) the transition probability is \( P(\nu_A \rightarrow \nu_B) \approx \frac{1}{2} \sin^2 2\vartheta \), i.e. it is insensitive to \( \Delta m^2 \), because in realistic applications the expression (2.4) has to be averaged over some non-zero interval of the initial neutrino energies \( \Delta E \) [3].

We can now summarize the results of various experiments. They are usually interpreted in terms of the effective 2 × 2 parametrization (2.4). Historically, the first information came from the experiments measuring the flux of \( \nu_e \) neutrinos produced in the Sun (for references see e.g. the review [3]). A convincing evidence for the solar neutrino oscillation was provided by the Kamiokande and Superkamiokande experiments which established a strong (≈50\%) suppression of the flux of neutrinos from the nuclear reaction

\[ ^8B \rightarrow ^8Be^* + e^+ + \nu_e. \]

The most plausible explanation of those results is the transmutation of \( \nu_e \)'s produced in the core of the Sun into another type of neutrinos such as \( \nu_\mu \), \( \nu_\tau \) and/or the so-called sterile neutrino \( \nu_{\text{sterile}} \) which does not interact with the \( W^\pm \) or \( Z^0 \) bosons. Such a transmutation can occur either during their flight from the Sun to the Earth (the so-called vacuum oscillations (VO)) or through the resonant transition in the matter of the outer layers of the Sun (the so-called MSW effect [6]). As follows from the formulae (2.4) with \( A \equiv e \) and \( B \equiv \mu, \tau \) or \( s \), in the case of VO with \( L \approx 1.5 \times 10^8 \) km (the Sun-Earth mean distance) and for mean \( ^8B \) neutrino energy of order \( E \sim 10 \) MeV, the solar neutrino experiments are sensitive to \( \Delta m^2_{\text{sol}} \sim \mathcal{O}(10^{-11-10}) \) eV\(^2\). The deficit is then explained with \( \sin^2 2\vartheta_{\text{sol}} > 0.7 \). In the case of the MSW resonant conversion the formulae (2.4) for the transition and survival probabilities are replaced by more complicated expressions [3] which depend also on the electron and neutron number densities in the Sun. The observed \( \nu_e \) neutrino deficit can be then explained either for \( \Delta m^2_{\text{sol}} \sim \mathcal{O}(10^{-5}) \) eV\(^2\) and \( \sin^2 2\vartheta_{\text{sol}} \sim \mathcal{O}(10^{-3-2}) \) (the so-called SAMWS solution) or for \( \Delta m^2_{\text{sol}} \sim \mathcal{O}(10^{-4}) \) eV\(^2\) and \( \sin^2 2\vartheta_{\text{sol}} > 0.5 \) (the so-called LAMWS solution).

picture of neutrino oscillations is however the field theoretical one [4]. Due to the field mixing (2.3) there is in general a nonzero amplitude for emission (absorption) of any of the neutrino mass eigenstates \( \nu_a \) in the interaction of charged lepton of flavour \( A \) with the \( W \)-boson. The change of the neutrino flavour (inferred from the flavours of the charged leptons) is in this picture due to the coherent sum of Feynman diagrams describing exchanges of all virtual mass eigenstates of neutrinos between the emission and detection vertices. Taking properly into account the effects of wave packets describing initial and final states it can be shown that the formula (2.4) is in most cases a sufficient approximation to the full result, which automatically accounts for decoherence effects and depends on the overlap of the wave packets describing the initial and final states [4].

\[^4\text{Recall that } \frac{\Delta m^2 L}{4E} = 1.27 \times \frac{(\Delta m^2/1 \text{ eV}^2)(L/1 \text{ km})}{(E/1 \text{ GeV})}. \]
The deficit of neutrinos was also revealed by the measurements of the flux of $\nu_\mu$ and $\bar{\nu}_\mu$ neutrinos produced together with $\nu_e$'s in the Earth’s atmosphere by the cosmic rays. The results of the Superkamiokande experiment [7] are most easily explained by $\nu_\mu$ oscillation into another type of neutrinos. The oscillatory explanation is further supported by the zenith angle dependence of the $\nu_\mu$ and $\bar{\nu}_\mu$ flux deficit. For typical $\nu_\mu$ energies of order $\sim$GeV and $20 \text{ km} \lesssim L \lesssim 1.3 \times 10^4 \text{ km}$, the observed $\nu_\mu$ deficit is explained for $\Delta M^2_{\text{atm}} \approx 3.2 \times 10^{-3} \text{ eV}^2$ and $\sin^2 2\theta_{\text{atm}} > 0.82$. Also, the Superkamiokande data seem to favour $\nu_\mu \rightarrow \nu_\tau$ oscillations over $\nu_\mu \rightarrow \nu_{\text{sterile}}$ [8].

Neutrino oscillations are also intensively searched for in various reactor or accelerator based experiments (for review see [5, 9]). Except for the LSND experiment reporting [10] a positive signal for the $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$ oscillations, which would require $0.1 \text{ eV}^2 < \Delta M^2 < 1 \text{ eV}^2$ for the oscillatory explanation, the results of all other experiments are negative. Particularly strong constraint comes from the CHOOZ reactor experiment. It excludes disappearance of $\bar{\nu}_e$ with mean energies $\sim 3 \text{ MeV}$ at the distance $L \approx 1 \text{ km}$. In the $2 \times 2$ oscillation framework this translates into the limit

$$\sin^2 2\theta_{\text{react}} < 0.1 \quad \text{for} \quad \Delta M^2 > 10^{-3} \text{ eV}^2$$

(2.5)

Except for the unconfirmed LSND result, all experiments are consistent with oscillations between the three known neutrino flavours. Moreover, recent SNO measurement [11] of the total solar neutrino flux (i.e. the flux of $\nu_e$, $\nu_\mu$ and $\nu_\tau$) combined with the Superkamiokande data [12] strongly supports this assumption. Therefore, in the rest of the review we assume that the neutrino sector of the low energy effective theory consists of 3 active neutrinos only.

Within the $3 \times 3$ framework the solar, atmospheric and reactor data become interrelated and one can draw more definite conclusions. To make the discussion of the neutrino experiments more complete we recall the formula for the transition probabilities $P(\nu_A \rightarrow \nu_B)$ for the case of 3 neutrinos:

$$P(\nu_A \rightarrow \nu_B, L, E) = \left| \sum_a U_{Ba} e^{-i \frac{m^2_{\nu_a} L}{2E}} U^*_{Aa} \right|^2 = \sum_a |U_{Ba} U^*_{Aa}|^2 + \text{Re} \sum_a \sum_{b \neq a} U_{Ba} U^*_{Bb} U^*_{Aa} U_{Ab} e^{i \Delta m^2_{ab} L/4E}$$

(2.6)

where $\Delta m^2_{ab} \equiv m^2_{\nu_a} - m^2_{\nu_b}$. The complex matrix elements are defined in the basis in which charged lepton mass matrix is diagonal, by the decomposition of the $\nu_A$ neutrino fields belonging to $SU_L(2)$ doublets into the mass eigenstates field:

$$\nu_A = \sum_a U_{Aa} \nu_a.$$

(2.7)
For a real matrix $U$ the expression (2.6) can be written as

$$P(\nu_A \rightarrow \nu_B, L, E) = \delta_{AB} - 4 \sum_{a>b=1}^{3} U_{Aa} U_{Bb} U_{Ab} U_{Bb} \sin^2 \left( \frac{\Delta m_{ab}^2 L}{4E} \right).$$  \hspace{1cm} (2.8)

To simplify the notation we will denote: $\Delta m^2 \equiv \Delta m_{\text{sol}}^2 \equiv |m_2^2 - m_1^2|$, $\Delta M^2 \equiv \Delta M_{\text{atm}}^2 \equiv |m_3^2 - m_2^2|$. For $\Delta m^2 \ll \Delta M^2$ the measurements of the experiments listed above can now be summarized as follows:

$$P_{\text{sol}}(\nu_e \rightarrow \nu_e) = 1 - 2 \left( 1 - U_{13}^2 \right) U_{13}^2 - 4 U_{13}^2 U_{12}^2 \sin^2 \left( \frac{\Delta m_{21}^2 L_{\text{sol}}}{4E_{\text{sol}}} \right)$$

$$P_{\text{atm}}(\nu_\mu \rightarrow \nu_\mu) = 1 - 4 \left( 1 - U_{23}^2 \right) U_{23}^2 \sin^2 \left( \frac{\Delta M_{23}^2 L_{\text{atm}}}{4E_{\text{atm}}} \right)$$

$$P_{\text{react}}(\nu_e \rightarrow \nu_e) = 1 - 4 \left( 1 - U_{13}^2 \right) U_{13}^2 \sin^2 \left( \frac{\Delta m_{21}^2 L_{\text{react}}}{4E_{\text{react}}} \right)$$  \hspace{1cm} (2.9)

A convenient parametrization for the mixing matrix $U$ is (see also the next Section):

$$U_{\text{MNS}} = \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
    -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
    s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix} \times \Phi$$ \hspace{1cm} (2.10)

where $c_{ij} \equiv \cos \theta_{ij}$ ($s_{ij} \equiv \sin \theta_{ij}$) and $\Phi$ is a diagonal matrix $\Phi = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1)$ (or a similar diagonal matrix with two phases and $\Phi_{11} = 1$ or $\Phi_{22} = 1$). Our convention is such that for vanishing mixing angles $\theta_{ij}$ we have $\nu_1 = \nu_e$, $\nu_2 = \nu_\mu$ and $\nu_3 = \nu_\tau$, i.e. no ordering of the neutrino masses $m_{\nu_a}$ is assumed. CP is conserved if $\delta = 0 \mod \pi$ and $\alpha_{1,2} = 0 \mod \pi/2$. The relation to the effective $2 \times 2$ parametrization then reads

$$\sin^2 2\theta_{\text{react}} = \sin^2 2\theta_{13} \sin^2 2\theta_{\text{atm}} = \sin^2 2\theta_{23} \left( \cos^2 \theta_{13} + \frac{\sin^2 2\theta_{13}}{4\cos^2 23} \right)$$

$$\sin^2 2\theta_{\text{sol}} \approx \sin^2 2\theta_{12} \left( 1 - 2 \sin^2 \theta_{13} \right)$$  \hspace{1cm} (2.11)

In the limit of small $\theta_{13}$ angle imposed by the CHOOZ result (2.5) (non-negligible entries $U_{23}$ and $U_{33}$ and unitarity of $U$ exclude the possibility of $|\theta_{13}| \approx \pi/2$) we get

$$\sin^2 2\theta_{\text{atm}} \approx \sin^2 2\theta_{23} \sin^2 2\theta_{\text{sol}} \approx \sin^2 2\theta_{12}$$  \hspace{1cm} (2.12)

Finally we mention that the neutrino Majorana mass term of the form $\mathcal{L}_{\text{mass}} = -\frac{1}{2} m_\nu^{AB} \nu_A \nu_B + \text{H.c}$ violates the lepton number conservation and leads to the neutrino-less double beta decay. Non-observation of such decays implies [5]

$$|m_\nu^{ee}| < 0.35(0.27) \text{ eV \ at \ 90\% \ (68\%) \ C.L.},$$ \hspace{1cm} (2.13)
\[ |m_\nu c_{12}^c c_{13}^e e^{2i\alpha_1} + m_\nu s_{12}^c s_{13}^e e^{2i\alpha_2} + m_\nu s_{13}^e e^{2i\delta} | < 0.35(0.27) \text{eV at 90\% (68\%) C.L.} \]

Concluding this section, we stress that the pattern of neutrino mixing is distinctly different from the one observed in the quark sector. At least one mixing angle (\(\theta_{23}\)) is large and the recent data favour solutions with two large mixing angles [1]. The data indicate \(\Delta m^2 \ll \Delta M^2\) but, since the masses are not measured, several options for the values of the masses themselves are still possible:

1. \(\Delta M^2 \approx m_{\nu_1}^2 \gg m_{\nu_2}^{1(2)} \gg m_{\nu_3}^2\) (hierarchical)

2. \(\Delta M^2 \approx m_{\nu_1}^2, m_{\nu_2}^2 \gg m_{\nu_3}^2\) or \(\Delta M^2 \approx m_{\nu_1}^2, m_{\nu_2}^2 \gg \Delta m^2\) (partly degenerate)

3. \(m_{\nu_3}^2 \approx m_{\nu_1}^2 \approx m_{\nu_2}^2\) and all of order or larger than \(\Delta M^2\) (degenerate).

Special cases of the partly degenerate and degenerate patterns are equal masses, \(|m_{\nu_1}| = |m_{\nu_2}|\) or \(|m_{\nu_1}| = |m_{\nu_2}| = |m_{\nu_3}|\), respectively. In our language we refer to them as two-fold and three-fold degeneracies, to distinguish them from, more general, partly degenerate and degenerate pattern defined by ii) and iii). The last two possibilities are very different from the pattern known from the quark sector but almost maximal (bimaximal) mixing suggested by the experimental data makes them an interesting alternative to the hierarchical pattern. It is important to remember that in that respect neutrinos may be qualitatively different from quarks because, unlike the charged fermions, they can be Majorana particles. As we shall see, the magnitude and the importance of quantum corrections depends strongly on the assumed pattern of the masses.

### 3 Neutrino masses in the effective theory

We begin by recalling how the quark masses are incorporated in the SM (or the MSSM) Lagrangian. We shall use the Weyl spinor notation (see e.g. [16]), which is particularly convenient for Majorana particles. The relation to the standard Dirac notation is explained in the Appendix A. The original Lorentz and \(SU(3) \times SU(2) \times U(1)\) invariant Lagrangian contains the Yukawa interactions of the form

\[
\mathcal{L}_{\text{Yuk}} = -\epsilon_{ij} H_i u^A Y_u^{AB} q^B_j - H^*_j d^c C Y_d^{AB} q^B_i - H^*_j e^c A Y_e^{AB} l^B_j + \text{H.c.} \quad (3.1)
\]

where \(Y_u, Y_d, Y_e\) are a priori arbitrary complex Yukawa matrices, \(\epsilon_{12} = -\epsilon_{21} = -1\), the upper case letters enumerate the three generations of the matter fermions

\[
u \equiv \begin{pmatrix} u \\ d \end{pmatrix} \text{ and } l \equiv \begin{pmatrix} \nu \\ e \end{pmatrix} \quad (3.2)
\]
described by the left-handed Weyl spinors and $H$ is the SM Higgs doublet. In the MSSM there are two Higgs doublets, $H^{(u)}$ and $H^{(d)}$, and one has to replace: $H_i \rightarrow H_i^{(u)}$ and $H_j^{*} \rightarrow \epsilon_{ij}H_i^{(d)}$. Replacing the Higgs doublet $H$ (doublets $H^{(u)}$ and $H^{(d)}$ in the MSSM) by its vacuum expectation value leads to the following mass terms in the SM (MSSM) Lagrangian:

$$L_{\text{mass}} = -u^c A(v Y_u)^{AB} u^B - d^c A(v Y_d)^{AB} d^B - e^c A(v Y_e)^{AB} e^B + \text{H.c.} \quad (3.3)$$

(In the MSSM $v Y_u \rightarrow v_u Y_u$, $v Y_{d,e} \rightarrow -v_d Y_{d,e}$.) The fermion mass matrices are diagonalized by the unitary chiral rotations

$$u \rightarrow U_L u, \quad u^c \rightarrow u^c U_R^\dagger, \quad d \rightarrow D_L d, \quad d^c \rightarrow d^c D_R^\dagger, \quad e \rightarrow E_L e, \quad e^c \rightarrow e^c E_R^\dagger, \quad (3.4)$$

which give:

$$U_R^\dagger Y_u U_L = \text{diag}(y_u, y_c, y_t)$$
$$D_R^\dagger Y_d D_L = \text{diag}(y_d, y_s, y_b) \quad (3.5)$$
$$E_R^\dagger Y_e E_L = \text{diag}(y_e, y_\mu, y_\tau).$$

The only remnant of the nontrivial Yukawa matrices is then the CKM matrix

$$V_{\text{CKM}} \equiv V = U_L^\dagger D_L \quad (3.6)$$

appearing in the interactions of the charged quark currents with (massive) charged vector bosons $W_{\mu}^\pm$. It can be shown (see e.g. [14]) that the CKM matrix can be parameterized by 3 angles and one phase. The other $6-1=5$ phases can be absorbed in the redefinition of the quark fields.

In the SM defined as a renormalizable theory, with the neutrino and Higgs fields transforming as components of doublets of the $SU_L(2)$ gauge symmetry, neutrinos remain massless. Moreover, as a consequence of the gauge symmetry, renormalizability and the field content, the Lagrangian has two accidental global $U(1)$ symmetries that ensure baryon and lepton number conservation. In the absence of any mass terms for the neutrino fields $\nu^A$, one can always perform the rotation $\nu \rightarrow V_L \nu$ with $V_L = E_L$ so that the counterpart of the CKM matrix in the leptonic sector is trivial.

There are two easy possibilities for extending the particle content of the SM so that neutrino mass is generated by renormalizable interactions. One is to couple two lepton $SU_L(2)$ doublets to a $SU_L(2)$ Higgs triplet (singlets would violate the electric charge
conservation). This possibility is not particularly attractive for several reasons. The smallness of the neutrino masses would require either $v_{\text{triplet}} \ll v_{\text{doublet}}$ or the triplet couplings orders of magnitude smaller than the other Yukawa couplings. Moreover, the introduction of the triplet would make the parameter $\rho$ a free parameter of the theory (hence not calculable) with its own counterterm allowing to adjust its value at will. The other possibility is to introduce a number of new $SU_L(2) \times U_Y(1)$ singlet left-handed leptonic fields $\nu^c$ to the Lagrangian and couple them to the Higgs doublet in the same way as the quark fields $u^c$:

$$\Delta \mathcal{L}_{\text{Yuk}} = -\epsilon_{ij} H_i \nu^c K^{ji} A^j + \text{H.c.}$$

(3.7)

where $K = 1, \ldots, K_{\text{max}}$ and in principle $K_{\text{max}}$ could be arbitrary. The interaction (3.7) preserves the known structure of the SM and, if $K = 1, 2, 3$, the symmetry between quarks and leptons is restored. For singlet fields $\nu^c$ the Majorana mass term

$$\Delta \mathcal{L}_{\text{Maj}} = -\frac{1}{2} M_{\text{Maj}}^{KL} \nu^c K \nu^c L + \text{H.c.}$$

(3.8)

can also be added to the Lagrangian. In general both terms should be included in the theory defined by the $SU_L(2) \times U_Y(1)$ gauge symmetry and renormalizability. The Majorana mass term can be, however, eliminated by imposing the additional global $U(1)$ symmetry ensuring conservation of the lepton number $L$. The fields $\nu^c$ must then have $L = -1$, opposite to $L$ of the leptonic doublets $l_i$. Their complex conjugate $\overline{\nu}^c$ (see Appendix A) can be then interpreted as the right-handed neutrinos. Neutrinos are then Dirac particles like the other fermions. There are two reasons why Dirac masses are not so attractive. One is, again, the need for very small numerical values of the Yukawa couplings $Y_\nu$. The other is that the lepton number conservation has to be imposed as an additional global symmetry (remember that in the SM it is a consequence of the field content and of the renormalizability and not an additional assumption).

The presence of the Majorana mass term, which breaks the global $U(1)$ symmetry, inevitably makes neutrinos Majorana particles (in fact what allows to interpret in the SM the two helicity states described by the $\nu$ field as a particle and an anti-particle is just the lepton number!) Taking for simplicity three singlet neutrinos $\nu^c$ and replacing $H (H^u)$ in the MSSM) by its vacuum expectation value, the general form of the neutrino mass matrix (in the absence of Higgs triplets) is

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & m_D \\ m^T_D & M_{\text{Maj}} \end{pmatrix}$$

(3.9)

The formula (3.9) is easily generalized to an arbitrary number of singlet neutrinos $\nu^c$. 

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where \( m_D \equiv vY_\nu \) and \( M_{\text{Maj}} \) are \( 3 \times 3 \) matrices in the basis \((\nu, \nu^c)\). If both these matrices are diagonal with eigenvalues \( m_A \) and \( M_A \), respectively, the physical neutrinos are mixtures of \( \nu_A \) and \( \nu^c_A \) with the masses

\[
m_{\nu A \pm} = \frac{1}{2} \left( M_A \pm \sqrt{M_A^2 + 4m_A^2} \right).
\]

In general, each of the six leptonic fields \( \nu_A \) and \( \nu^c_K \) is a linear combination of the six Majorana neutrinos. There is the possibility of oscillations \( \nu \to \nu^c \). The \( \nu^c \) neutrinos are singlets of the \( SU_L(2) \times U(1) \) gauge group and are sterile neutrinos.\(^6\) The situation simplifies for \( M \gg v \) where \( M \) is the overall scale of the \( M_{\text{Maj}} \) entries. For simplicity, we identify it with the introduced earlier scale \( M \), although they could actually be different scales. In this limit there are two sets of mass eigenvalues. They are of the order \( \mathcal{O}(M) \) and \( \mathcal{O}(v^2/M) \), respectively, and three of the mass eigenstates are very heavy and decouple from the physics at the electroweak scale. This is the so-called see-saw mechanism.\(^7\)

It is instructive to repeat the decoupling procedure in the field theoretical language. The effects of the Born diagram, Fig. 1, present in the full theory valid above the scale \( M \) are, up to \( \mathcal{O}(1/M^2) \), reproduced in the effective theory, describing physics below the scale \( M \), by a non-renormalizable operator of dimension 5:

\[
\Delta L = -\frac{1}{4M} C^{AB} (\epsilon_{ij} H_i l_j^A) (\epsilon_{lk} H_l l_k^B) + \text{H.c.}
\]

in which

\[
M^{-1} C = Y_{\nu}^T (M_{\text{Maj}})^{-1} Y_{\nu}.
\]

The overall scale \( M \) of the entries of the matrix \( M_{\text{Maj}} \) has been factorized out from \( C \) to make it dimensionless. Thus, if the scale \( M \) is high enough, the effects of the Majorana masses can be discussed in the SM or MSSM supplemented by the operator \( (3.11) \). We should also stress that this operator is the only one of dimension 5 contributing to the neutrino Majorana mass matrix. Other possible contributions are of higher dimension \( \mathcal{O}(M^0) \). Thus, one may expect that in the effective SM or MSSM the neutrino masses are indeed described by the operator \( (3.11) \), even if its origin is different from the see-saw

---

\(^6\)Note that such oscillations exist only for Majorana neutrinos and are forbidden by \( L \) conservation for Dirac neutrinos i.e. the right-handed neutrinos \( \nu^c \) have correct \( L \) but wrong chirality to oscillate into \( \nu \).

\(^7\)However in some cases, one of the formally \( \mathcal{O}(M) \) mass eigenvalues can remain small as a result of some symmetries of the underlying theory of neutrino masses. The corresponding light state can play the role of a sterile neutrino.\(^8\)
mechanism. In particular, degenerate or partly degenerate patterns of neutrino masses require some interplay between the parameters entering the formula (3.12), but one can also imagine that such patterns originate from other mechanisms at the scale $M$.

\[
\nu^c \nu^c H H M_{\text{Maj}}
\]

Figure 1: Diagram generating the dimension 5 operator (3.11).

We end this section with the discussion of the parametrization of the neutrino mixing matrix. In the effective theory, after the electroweak symmetry breaking, the operator (3.11) is the source of the Majorana mass for the three active neutrinos $\nu$, $L_{\text{mass}} = -\frac{1}{2} \frac{v^2}{4M} C^{AB} \nu^A \nu^B + \text{H.c.}$ This mass matrix can be diagonalized by an additional unitary rotation $\nu^A \rightarrow U^{Aa} \nu_a$. Recall we work in the basis in which the leptonic Yukawa coupling is already diagonal and the matrix $C$ is assumed to be given in the same basis. We have:

\[
\frac{v^2}{4M} (U^T C U)_{ab} = \frac{v^2}{4M} C^{a} \delta_{ab} = m_{\nu_a} \delta_{ab}
\] (3.13)

The matrix $U$ will therefore appear in the couplings of neutrinos to $W^\pm$ bosons and charged Goldstone (and Higgs) bosons. Being unitary, the matrix $U$ depends on 3 angles and 6 phases and can be conveniently written as $U^{Aa} = e^{i \varphi_A} U^{Aa}_{\text{MNS}}$ where $U_{\text{MNS}}$ is given in eq. (2.10). Contrary to the quark case, if the Majorana masses $m_{\nu_a}$ in eq. (3.13) are to be real and positive, the only freedom that remains is the possibility to re-phase independently the three Dirac fields of the charged leptons $\psi_{e_A} \rightarrow \exp(i \varphi_A) \psi_{e_A}$ (i.e. $e_A \rightarrow \exp(i \varphi_A) e_A$ for the left-handed and $e^c_A \rightarrow \exp(-i \varphi_A) e^c_A$ for their right-handed components). It is then the MNS matrix (2.10) with three angles and three phases that enters the interactions of neutrinos with the SM particles. In the MSSM, however, the phases $\varphi_A$ can be eliminated (by appropriate rotations of the superpartner fields) only if the slepton mass matrices and charged lepton mass matrices are simultaneously diagonal. In the general case, the phases $\varphi_A$ appear in the neutrino-sneutrino-neutralino and neutrino-chargino-charged slepton vertices.

\[\text{In a general electroweak basis in which the leptonic Yukawa coupling is not necessarily diagonal the matrix } U \text{ is given by the product } U = E_L^T V_L \text{ where the rotations } e \rightarrow E_L e \text{ and } \nu \rightarrow V_L \nu \text{ diagonalize the mass matrices of charged leptons and neutrinos, respectively.}\]
It is sometimes convenient to work with complex neutrino masses \( m_{\nu_a} \) in eq. (3.13). The phases \( \alpha_{1,2} \) from eq. (2.10) are then absorbed into the masses: \( m_{\nu_{1,2}} = |m_{\nu_{1,2}}| \exp(-2i\alpha_{1,2}) \).

In particular, when CP is conserved the phases \( \alpha_{1,2} = 0, \pm \pi/2 \) reflecting different CP parities of different neutrinos, can be absorbed into the mass eigenvalues and make them real positive or negative. The MNS matrix is then real. Since only relative signs of the masses play a role, we will for definiteness fix our convention so that \( m_{\nu_3} \) is always positive. Such a parametrization simplifies greatly the qualitative analysis of the renormalization group equations discussed in the next section.

The value of the mass \( M \) is not known. One can expect that it is determined by a beyond the SM theory that provides the physical cut-off for the SM. The idea of Grand Unification and of the big desert would suggest very high value of \( M \). However there are also other models (large extra dimensions) in which \( M \) is \( \mathcal{O}(1 \text{ TeV}) \). In this review we assume that whatever the theory of neutrino masses is, the scale \( M \) is high enough to justify the description of the neutrino mass effects by the effective theory (SM or MSSM) with the single operator (3.11) added.

Finally, one should also mention that in the MSSM neutrino mass can originate from R-parity violating interactions [21]. Such models are usually based on low energy mechanisms for neutrino mass generation and will not be discussed in this review.

4 Quantum corrections from the renormalization group evolution

The neutrino mass parameters given at some scale \( M \) in the effective theory (SM or MSSM) supplemented by the operator (3.11), determine the measurable quantities after inclusion of quantum corrections. The first category of corrections are those described by the evolution of the effective theory parameters from the scale \( M \) down to some low energy scale close to the electroweak scale. They depend on arbitrary powers of the large logarithms of the ratio \( M/M_Z \) and are resummed to all orders of the perturbation expansion by means of the renormalization group equations (RGE). The second class of corrections are the so-called low energy threshold corrections. In the SM they connect the mass parameters of the SM Lagrangian renormalized at the scale \( M_Z \) to the fermion masses in the the effective theory obtained after the electroweak symmetry breaking (whose renormalizable part is QCD+QED) renormalized at the same scale \( M_Z \). In the MSSM, threshold corrections include also the superpartner contributions due to their mass splittings and/or flavour violation in slepton mass matrices. The relative magnitude of the two types of corrections will be discussed later on. In this section we assume that the RG corrections are
the dominant ones and, after deriving the relevant RG equations, we discuss the potential role played by these corrections for the neutrino masses and mixing.

4.1 RG equations for the CKM matrix

Before we derive the RGE for the neutrino masses and mixing we discuss the technically easier case of the evolution of the Yukawa matrices and the CKM matrix.

The renormalization group equations for the Yukawa couplings of the SM and the MSSM at one \cite{22,23} and two loop \cite{23,24} are well known. For convenience we reproduce the one loop RGE in Appendix B. They have a matrix structure which encodes the evolution of their eigenvalues and of all entries of the CKM mixing matrix. From the full RGE in the matrix form it is possible \cite{25,26} to derive the RGE for the eigenvalues of the Yukawa matrices and for the CKM matrix. It is often very instructive to discuss the evolution of those Lagrangian parameters that can be directly determined from the data. For instance, the experimentally known information can be unambiguously extrapolated to any high scale. Moreover, certain qualitative features of the Yukawa coupling pattern necessary to reproduce the experimental data can be easier to understand. The derivation goes as follows. One writes down auxiliary RGEs for the matrices $U_{L,R}, D_{L,R}$ defined in eq. (3.4) in the form:

$$
\frac{d}{dt} U_L = U_L \varepsilon^U_L, \quad \frac{d}{dt} U_R = U_R \varepsilon^U_R,
$$

$$
\frac{d}{dt} D_L = D_L \varepsilon^D_L, \quad \frac{d}{dt} D_R = D_R \varepsilon^D_R,
$$

(4.1)

where the matrices $\varepsilon^U_{L,R}, \varepsilon^D_{L,R}$ are antihermitean, in order to preserve unitarity of the matrices $D_{L,R}, U_{L,R}$, and $t = (1/16\pi^2) \ln(Q/M_Z)$. One then looks for the evolution of the $U_{L,R}$ and $D_{L,R}$ matrices such that Yukawa matrices remain diagonal during the evolution. Differentiating the four relations (obtained from eqs. (3.3))

$$
\text{diag}(y_u^2, y^2_e, y^2_t) = U_L^\dagger U_R = Y_u^\dagger Y_u
$$

$$
\text{diag}(y_d^2, y^2_s, y^2_b) = D_L^\dagger D_R = Y_d^\dagger Y_d
$$

(4.2)

and requiring the derivatives of the diagonal matrices on the rhs to be also diagonal matrices one gets the matrices $\varepsilon^U_{L,R}, \varepsilon^D_{L,R}$:

$$
(\varepsilon^U_L)_{JJ} = -u_{u,J} y^2_{u,J} + y^2_d \sum_K V^{JK} V^{IK} y^2_d, \quad (\varepsilon^U_L)_{JJ} = 0
$$

$$
(\varepsilon^D_L)_{KK} = -d_{u,K} y^2_{u,K} + y^2_d \sum_J y^2_d V^{JK} V^{JI}, \quad (\varepsilon^D_L)_{KK} = 0
$$

(4.3)
and similarly the matrices \( \varepsilon^U_R \) :

\[
(\varepsilon^U_R)_{JI} = -u_d \frac{2y_{u_J}y_{u_I}}{y_{u_J}^2 - y_{u_I}^2} \sum_K V^{JK}V^{IK*}y_{d_K}^2, \quad (\varepsilon^U_R)_{JJ} = 0
\]

\[
(\varepsilon^D_R)_{KL} = -d_u \frac{2y_{d_K}y_{d_L}}{y_{d_K}^2 - y_{d_L}^2} \sum_J y_{d_J}^2 V^{JK*}V^{JL}, \quad (\varepsilon^D_R)_{KK} = 0
\]

where \( u_d = d_u = -3/2 \) for the SM and \( u_d = d_u = 1 \) for the MSSM. The evolution of the CKM mixing matrix \( V \) is then given by the simple matrix equation

\[
\frac{d}{dt} V = -\varepsilon^U_L V + V\varepsilon^D_L.
\]

Because of the hierarchical pattern of quark masses the expressions (4.3) for \( \varepsilon^U,D_L \) can be simplified to

\[
(\varepsilon^D_L)_{KL} = -d_u y_{d_K}^2 V^{K3*}V^{3L} \quad \text{for} \quad K > L \quad \text{and} \quad (\varepsilon^D_L)_{KL} = -(\varepsilon^D_L)_{LK} \quad \text{for} \quad K < L
\]

\[
(\varepsilon^U_L)_{JI} = -u_d V^{J3}V^{I3*}y_{b_J}^2 \quad \text{for} \quad J > I \quad \text{and} \quad (\varepsilon^U_L)_{JI} = -(\varepsilon^U_L)_{IJ} \quad \text{for} \quad J < I.
\]

Taking next into account the hierarchy of the entries of the CKM matrix it is easy to find that, to a good approximation, the 2 \( \times \) 2 submatrix describing the mixing of the first two generations as well as the element \( V_{33} \) do not evolve, while the evolution of the remaining entries is universal and is given by [20, 27]:

\[
V(t) = V(0) \exp \left\{ -\int_0^t \left[ u_d y_{b_J}^2(t') + d_u y_{b_J}^2(t') \right] dt' \right\}
\]

where \( V = V^{31}, V^{32}, V^{13} \) or \( V^{32} \). It follows that to one loop the Jarlskog invariant \( J \equiv \text{Im}(V_{ud}V_{cs}V_{us}^*V_{cds}^*) \) does not change. By using e.g. the standard parametrization of the CKM matrix it is also easy to derive the RGE for the three mixing angles and the CP violating phase.\(^9\) For the SM and the MSSM this has been done up to two-loops in ref. [27].

### 4.2 RGEs for neutrino masses and mixing

The Majorana mass term for neutrinos arises from the dimension 5 operator (3.11). The RGE for \( C^{AB} \) in the SM has been correctly computed only recently in ref. [29] (previous

\(^9\)In the one loop approximation the running of the CKM matrix elements looks particularly simple in the Wolfenstein parametrization [28] (see e.g. [16]) in which only the parameter \( A \) changes with the scale.
calculations \cite{30,31,32} have errors) and in the MSSM in refs. \cite{31,32}:

\[
\frac{d}{dt} C = -KC - \kappa \left[ (Y_e^\dagger Y_e)^T C + C (Y_e^\dagger Y_e) \right] \tag{4.8}
\]

where in the SM \(\kappa = -3/2\) and \(K = -3g_2^2 + 2\text{Tr} \left( 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d + Y_e^\dagger Y_e \right) + 2\lambda\); the normalization of \(\lambda\) is fixed by the Higgs self interaction: \(L_{\text{self}} = -\frac{\lambda}{2} (H^\dagger H)^2\). In the MSSM, \(\kappa = +1\) and \(K = -6g_2^2 - 2g_1^2 + 2\text{Tr} \left( 3Y_u^\dagger Y_u \right)\). Since our initial conditions will be always given at the scale \(M\), we have written eq. (4.8) top-down, i.e. with \(16\pi^2 t = \ln(M/Q)\).

The equation (4.8) is valid in any flavour basis, in particular in the basis in which the Yukawa matrix \(Y_e\) is diagonal.\footnote{The MSSM RGEs given in ref. \cite{31} allow to treat also the case in which squarks and/or gluino are much heavier than sleptons, charginos and neutralinos so that the decoupling procedure \cite{33} can be employed; there are then four different operators which mix with each other below the squark/gluino threshold. Above it one has (in the notation of ref. \cite{31}) \(\epsilon_{11}^{ab} = 2\epsilon_{12}^{ab} = 2\epsilon_{21}^{ab} = \epsilon_3^{ab} = C^{ab}\) and the four equations of ref. \cite{31} merge into the one quoted here.}

In that basis, eq. (4.8) simplifies to

\[
\frac{d}{dt} C^{AB} = -KC^{AB} - \kappa \left[ y_{eA}^2 C^{AB} + C^{AB} y_{eB}^2 \right] \tag{4.9}
\]

where \(y_{eA}^2\) are the eigenvalues of the hermitean matrix \(Y_e^\dagger Y_e\). In this form it can be elegantly solved \cite{34}:

\[
C(t) = I_K J C(0) J \tag{4.10}
\]

where \(J = \text{diag}(I_e, I_\mu, I_\tau)\) and

\[
I_K = \exp \left( -\int_0^t K(t')dt' \right),
\]

\[
I_{eA} = \exp \left( -\kappa \int_0^t y_{eA}^2 (t')dt' \right). \tag{4.11}
\]

Note that any zero in the initial matrix \(C^{AB}\) is preserved by the RG evolution and that the phases of the \(C^{AB}\) entries do not evolve \cite{35}. We also note that, since \(I_K I_e^2 \approx 1\), the experimental constraint (2.14), with \(m_{\nu e}^e = (v^2/4M)C^{11}(0)\) and \(v^2 = v^2(M_Z)\), is renormalized negligibly.

Although the solution (4.10) to the RG equation for \(C^{AB}\) is simple, qualitative features of the running of the neutrino mass eigenvalues \(m_{\nu a}\) and of the MNS matrix are often masked by the diagonalization procedure. In that approach it has to be performed after

\footnote{This is because the matrix \(E_L\) defined in eq. (3.4) does not evolve; the matrices \(E_{L,R}^E\) analogous to the ones defined in eq. (4.1) are identically zero because below the scale \(M\), at which the operator (3.11) is generated, the RGE for the leptonic Yukawa coupling (3.11,3.2) do not depend on the couplings \(Y_\nu\), the only one that could change their matrix structure during the evolution.
the evolution. It is therefore useful to derive the RGE directly for the mass eigenvalues
and mixing angles. This is done by using a trick similar as for the CKM matrix. One
defines the auxiliary antihermitean matrix $\epsilon^\nu$ by the equation

$$\frac{d}{dt} U = U \epsilon^\nu$$

(4.12)

where $U$ satisfies eq. (3.13). Differentiating the equality $C^a \delta^{ab} = (U^T C U)^{ab}$ and requiring
that its rhs remains a diagonal matrix gives us the matrix $\epsilon^\nu$. The only difference is that
the resulting equation depends on $(\epsilon^\nu)^T$ instead of $(\epsilon^\nu)^\dagger$. Hence, one obtains two separate
equations for Re$(\epsilon^\nu)$ and Im$(\epsilon^\nu)$:

$$\text{Re}(\epsilon^\nu)^{ab} = \kappa A_{ab} \text{Re} \left( \sum_A U^{Aa} y_{e A}^2 U^{Ab} \right)$$

$$\text{Im}(\epsilon^\nu)^{ab} = \kappa (A_{ab})^{-1} \text{Im} \left( \sum_A U^{Aa} y_{e A}^2 U^{Ab} \right)$$

(4.13)

where

$$A_{ab} = \frac{C^a + C^b}{C^a - C^b} = \frac{m_{\nu_a} + m_{\nu_b}}{m_{\nu_a} - m_{\nu_b}}$$

(4.14)

and of course Re$(\epsilon^\nu)^{aa} = \text{Im}(\epsilon^\nu)^{aa} = 0$. For conserved CP ($U$ real), those equations have
been first derived in ref. [36]. For a general complex matrix $U$ they have been given in [37].

The formulae for $dU/dt$ written there is valid in a general electroweak basis; it reduces
to (4.13) after setting $P = Y^\dagger_e Y_e = P^\dagger$ and passing to the basis in which $P$ is diagonal.

Thus,

$$\frac{d}{dt} U^{Aa} = \kappa \sum_{b \neq a} U^{Ab} \left[ A_{ba} \text{Re} \left( \sum_B U^{Bb} y_{e B}^2 U^{Ba} \right) + \frac{i}{A_{ba}} \text{Im} \left( \sum_B U^{Bb} y_{e B}^2 U^{Ba} \right) \right]$$

(4.15)

$$\frac{d}{dt} C^a = - \left( K + 2 \sum_A y_{e A}^2 |U^{Aa}|^2 \right) C^a.$$  

(4.16)

Eq. (4.13) gives directly the running of the angles $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ [36] and the phases $\delta$,
$\alpha_a$, $\varphi_A$ parameterizing the matrix $U^{Aa} = e^{i\varphi_A} U^{Aa}_{\text{MNS}}$ [37]. In general, even if the phases $\varphi_A$
are zero at some scale they will be generated during the evolution. However, it is easy
to see that the differential equations for the parameters of the MNS matrix (2.10) do not
depend on the $\varphi_A$. Indeed, from eq. (4.13) it follows that $\epsilon^\nu$ does not depend on $\varphi_A$; furthermore, since $\dot{U}^{Aa} = e^{i\varphi_A} [\dot{U}^{Aa}_{\text{MNS}} + i \dot{\varphi}_A U^{Aa}_{\text{MNS}}]$, all factors $e^{i\varphi_A}$ cancel out in eq. (4.12).

If two of the three eigenvalues, say $m_{\nu_a}$ and $m_{\nu_b}$, are equal at some scale $t$ there is the
freedom in choosing the matrix $U(t)$, corresponding to the redefinition $U(t) \rightarrow \tilde{U}(t) =
\( U(t)R \) where \( R \) is a rotation in the \( ab \) plane. For the evolution, \( R \) has to be fixed by the condition

\[
\text{Re}\left[ \sum_A \bar{U}^{A a}_a y_{e A}^2 U^{A b}(t) \right] = 0
\]

(4.17)

so that \( \varepsilon'' \) is nonsingular. This is particularly important for considering mass patterns with exact degeneracy at the scale \( M \). The mixing matrix \( U \) is then ambiguous at that scale in the tree level approximation and becomes determined by quantum corrections, no matter how small. We shall illustrate this point in the next section. Note also that for conserved CP and neutrinos \( \nu_a \) and \( \nu_b \) of opposite CP parities such an ambiguity of rotations \( R \) does not exist. In such a case it is convenient to work with a real matrix \( U^{A a} \) and with the neutrino masses of opposite signs (see the comments at the end of Section 3). The RG equation (4.13) is then nonsingular because \( A_{ab} \), instead of being divergent, vanishes.

Eqs. (4.13) and (4.16) are very convenient for a qualitative discussion of the impact of the RG evolution directly on the neutrino masses and mixing angles. As we shall see, several physical effects are in this approach more transparent than in the approach based on evolving the matrix \( C^{AB} \).

### 4.3 Evolution of the neutrino masses

The effects of the RG quantum corrections on the neutrino mass eigenvalues are simple and, except for a few special cases, not very interesting. The solution to eq. (4.16) reads

\[
m_{\nu_a}(t) = I_K \exp \left( -2\kappa \int_0^t y_{e A}^2 |U^{A a}(t')|^2 dt' \right) m_{\nu_a}(0).
\]

Neglecting the small \( y_e \) and \( y_\mu \) Yukawa couplings\(^{12}\)

\[
m_{\nu_a}^2(t) = m_{\nu_a}^2(0) I_K^2 \exp \left( -4\kappa \int_0^t y_{r A}^2 |U_{3 a}|^2(t') dt' \right)
\]

(4.19)

Factors \( I_K \) and \( \kappa \) are different for the SM and the MSSM (\( \kappa = -3/2 \) and +1, respectively). We see that the possibility of some change in the mass pattern caused by the evolution resides solely in the differences in the mixing matrix elements \( U_{3 a} \) and their RG running.

In the MSSM where \( y_{r A}^2 \approx (\tan \beta/100)^2, \ t_Z \approx 0.12 \) for \( M = 10^{10} \) GeV and \( U_{3 a}^2 \) typically varies between 0 and 1/4 (except for \( U_{33}^2 \)), the exponent is at most of order of \( \epsilon \equiv y_{r A}^2 \log(M/M_Z)/16\pi^2 \approx \tan^2 \beta \times 10^{-5} < 2.5 \times 10^{-2} \) for \( \tan \beta < 50 \). In the SM, \( y_{r A}^2 \approx 10^{-4} \)

\(^{12}\)In the SM the Yukawa couplings are unambiguously determined at the electroweak scale by the charged lepton masses; in the MSSM \( y_{r A}^2_{\text{MSSM}} = (y_{r A}^2_{\text{SM}})/\cos \beta \) where \( \tan \beta \equiv v_u/v_d \) can vary from a few up to \( \sim 50 \).
and, consequently, $\epsilon \approx 10^{-5}$. We can then estimate the changes in the mass squared differences:

$$\Delta m_{ab}^2(t) = m_{\nu_a}^2(t) - m_{\nu_b}^2(t) = \Delta m_{ab}^2(0) - (\eta_a m_{\nu_a}^2(0) - \eta_b m_{\nu_b}^2(0)) \epsilon$$  \hspace{1cm} (4.20)

where we have neglected $I_K$ which is always close to 1, and the factors $\eta_{a(b)} > 0$ are typically in the range $0 - 2$, depending on the values of $U_{3a}$ factors and their evolution. Taking (for definiteness) $\Delta m_{ab}^2(0) = 0$, we see that the evolution of $\Delta m_{ab}^2(t)$ is limited by $m_{\nu_a}^2(0)\epsilon$ or $m_{\nu_b}^2(0)\epsilon$, i.e. by the value of the larger mass.

We see that the RG evolution cannot change the pattern of masses. It may still be of importance for precision tests of various models, particularly those with partly degenerate or degenerate patterns. We recall here that such patterns have (at least approximately) degenerate two and three neutrinos, respectively, at the scale $M$. Matching the experimental $\Delta m^2$ and $\Delta M^2$ needs then some fine-tuning between the initial values and the RG quantum corrections. That point can, however, be meaningfully discussed only for each concrete model.

The special cases are those with equal masses, $m_{\nu_1}^2 = m_{\nu_2}^2 \neq m_{\nu_3}^2$ or $m_{\nu_1}^2 = m_{\nu_2}^2 = m_{\nu_3}^2$ at the scale $M$. They are called in this review two-fold and three-fold degeneracies, respectively, to be distinguishable from the more general partly degenerate or degenerate patterns. The interesting question we shall discuss in Subsection 4.5 and Section 5 is whether only quantum corrections can then explain the observed mass squared differences.

In the next sections we shall discuss the RG quantum corrections to the neutrino mixing. We shall focus on interesting qualitative effects that depend only on the broad classification of the neutrino mass pattern, and do not depend on such details as whether the masses are equal or only approximately equal.

### 4.4 Mixing of two neutrinos

We first examine the mixing of two neutrinos, which was investigated in many papers \cite{32, 38, 39, 10, 11}. Strictly speaking, it could be physically relevant only if the atmospheric neutrino anomaly was due to the $\nu_{\mu}-\nu_{\tau}$ oscillations whereas the solar neutrino deficit resulted from the $\nu_{e}-\nu_{\text{sterile}}$ oscillations (at present strongly disfavoured by the SNO and Superkamiokande data). Moreover the two $2 \times 2$ neutrino systems would have to be completely independent due to some particular texture of the $U$ matrix. Nevertheless we will see, that the evolution of the $\theta_{23}$ angle in the $3 \times 3$ scenario is, in some cases very similar to the evolution of the mixing angle of two neutrinos only. It is therefore instructive to discuss the $2 \times 2$ evolution and to compare it later with the more realistic $3 \times 3$ mixing.
We begin with a real matrix $C^{AB}$ and hence, a real $U^{Aa}$:

$$U^{Aa} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}$$ (4.21)

where $s_\theta \equiv \sin \vartheta$, $c_\theta \equiv \cos \vartheta$. We shall consider the same or different CP parities of the two neutrinos $m_{\nu_1} m_{\nu_2} > 0$ or $m_{\nu_1} m_{\nu_2} < 0$, respectively. From eq. (4.15) one finds the following equations for the mixing angle

$$\frac{ds_\theta}{dt} = \kappa A_{21} (y_{e2}^2 - y_{e1}^2) s_\theta c_\theta$$

$$\frac{dc_\theta}{dt} = -\kappa A_{21} (y_{e2}^2 - y_{e1}^2) s_\theta^2 c_\theta,$$ (4.22)

where $\kappa = -3/2$ and $+1$ in the SM and the MSSM, respectively. Eqs. (4.22) give

$$\frac{d}{dt} \sin^2 2\vartheta = 2\kappa A_{21} (y_{e2}^2 - y_{e1}^2) \sin^2 2\vartheta \cos 2\vartheta$$ (4.23)

known from the literature\[32, 32, 38, 12, 13, 14\]. Eq. (1.23) has a trivial fixed point (FP) at $\sin^2 2\vartheta = 0$ but, contrary to the statements made in some papers, the maximal mixing $\sin^2 2\vartheta = 1$ is not its FP: Although, naively, $\sin^2 2\vartheta(t) \equiv 1$ solves eq. (1.23), it is easy to see that $s_\vartheta(t) = \pm 1/\sqrt{2}$ does not solve eqs. (4.22).\[4\]

It is clear from eq. (4.22) that, for fixed values of the Yukawa couplings $y_{eA}^2$ and for some fixed evolution “time” $t_Z \equiv (1/16\pi^2) \ln(M/M_Z)$, the evolution of the mixing angle depends on the factor $A_{21} = (m_{\nu_2} + m_{\nu_1})^2/\Delta m_{21}^2$. It is always small when the two neutrinos have opposite CP-parities i.e. when $m_{\nu_1} m_{\nu_2} < 0$ (as follows from eq. (1.18), $m_{\nu_a}$ cannot change sign during the evolution) or if the neutrino mass spectrum is hierarchical ($m_{\nu_1}^2 \ll m_{\nu_2}^2$ or $m_{\nu_1}^2 \ll m_{\nu_2}^2$). The evolution of the mixing angle is then negligible \[39, 34, 13, 34\]. For $m_{\nu_1} m_{\nu_2} > 0$ the evolution of $\vartheta$ can be significant, particularly if the neutrino masses are nearly degenerate so that $|A_{21}| \gg 1$ where $\epsilon \equiv (y_{e2}^2 - y_{e1}^2) t_Z$. It can be checked that for $|A_{21}| / \epsilon > 3$ the FP at $\sin^2 2\vartheta = 0$ is reached at the electroweak scale.

In view of the maximal $\nu_{\mu}-\nu_{\tau}$ mixing needed to explain the atmospheric neutrino data, some attention was paid to the possibility of increasing $\vartheta$ by the RG corrections from a small value at the $M$ scale to a (nearly) maximal ($|\vartheta| \approx \pi/4$) at the electroweak scale \[32, 38, 39, 40, 41\]. As we said earlier, the maximal mixing is not a FP of eqs. (1.22), so such an “explanation” of large mixing can merely be due to a coincidence of the running “time” and the initial values of the angles and the masses. The value of $|A_{21}| / \epsilon$ must be

\[13\] Its rhs is more frequently written in the equivalent form $2\kappa \frac{\epsilon}{\epsilon_0 + \epsilon_1} (y_{e2}^2 - y_{e1}^2) \sin^2 2\vartheta \cos 2\vartheta$.

\[14\] For example, for constant $\eta \equiv \kappa A_{21} (y_{e2}^2 - y_{e1}^2)$, eq. (1.23) is solved by $s_\theta^2(t) = s_0^2/(s_0^2 + c_0^2 \xi)$ where $\xi \equiv \exp(-2\eta t)$ and $s_0$ is the initial value of $s_\theta$. It is then straightforward to check that $\sin^2 2\vartheta(t) = \sin^2 2\vartheta_0 \xi/(s_0^2 + c_0^2 \xi)^2$ solves eq. (1.23) for any initial value $\sin^2 2\vartheta_0$. Thus, for $\sin^2 2\vartheta_0 = 1$, eq. (1.23) has two solutions: $\sin^2 2\vartheta(t) = 1$ and the one given here but only the latter satisfies the underlying eq. (1.22).
in the range such that the evolution is non-negligible, but not strong enough to reach the
FP. This can be most easily seen if we return to the solution (4.10) written in the form

\[
C^{AB}(t) = \begin{pmatrix} C^{11}(t) & C^{12}(t) \\ C^{12}(t) & C^{22}(t) \end{pmatrix} \propto \begin{pmatrix} C_0^{11} & C_0^{12}I_{21}(t) \\ C_0^{12}I_{21}(t) & C_0^{22}I_{21}(t) \end{pmatrix}
\]

(4.24)

where \( I_{21}(t) = I_{e_2}/I_{e_1} = \exp \left( -\kappa \int_0^t (y_{e_2}^2 - y_{e_1}^2)(t')dt' \right) \). One then has

\[
\sin^2 2\vartheta(t) = \frac{4 [C^{12}(t)]^2}{[C^{11}(t) - C^{22}(t)]^2 + 4[C^{12}(t)]^2}
\]

(4.25)

It is obvious that \( \sin^2 2\vartheta(t) \approx 1 \) is obtained whenever \( C^{11}(t) \approx C^{22}(t) \). Since \( C^{22}(t) \)
evolves differently from \( C^{11}(t) \) (as \( y_{e_2}^2 \gg y_{e_1}^2 \)) it is relatively easy to devise the situation
in which, at the initial scale \( M \), \( \sin^2 2\vartheta_0 \) is small (this requires \( |C_0^{11} - C_0^{22}| \gg 2|C_0^{12}| \))
and the evolution is such that at some lower scale \( C^{11}(t) = C^{22}(t) \) holds (this obviously
requires \( C_0^{22}C_0^{11} > 0 \)). With the judicious choice of the \( C^{AB}(0) \) matrix elements and \( \epsilon \equiv (y_{e_2}^2 - y_{e_1}^2)t_Z \) it is possible to obtain \( \sin^2 2\vartheta \approx 1 \) at the electroweak scale \([40, 41]\).
Expressing \( C_0^{11} \) and \( C_0^{22} \) in terms of the neutrino masses and mixing angle at the initial
scale \( M \), the relevant condition \( C_0^{11} = C_0^{22}I_{21}^2(t_Z) \) reads \([11]\)

\[
(m_{\nu_1}^0c_{\vartheta_0} + m_{\nu_2}^0s_{\vartheta_0}) = (m_{\nu_1}^0s_{\vartheta_0} + m_{\nu_2}^0c_{\vartheta_0})I_{21}(t_Z)
\]

(4.26)

where \( s_{\vartheta_0} \equiv \sin \vartheta(0) \) and \( m_{\nu_a}^0 \propto C^a(0) \). It is clear that for \( |m_{\nu_2}^0| \approx |m_{\nu_1}^0| \) and \( s_{\vartheta_0} \approx 0 \)
(or \( c_{\vartheta_0} \approx 0 \)) satisfying the condition \([4.26]\) requires \( m_{\nu_1}m_{\nu_2} > 0 \). On the other hand,
for \( m_{\nu_1}m_{\nu_2} < 0 \) the product \( C_0^{11}C_0^{22} \) can be positive only if \( m_{\nu_2}^2 < m_{\nu_1}^2 \) or \( m_{\nu_2}^2 > m_{\nu_1}^2 \),
which leads to a hierarchy \( C_0^{11} \gg C_0^{22} \) or \( C_0^{11} \ll C_0^{22} \). Getting \( C^{22}(t) = C^{11}(t) \) requires
then \( I_{21} \ll 1 \) or \( \gg 1 \). Thus, for \( m_{\nu_1}m_{\nu_2} < 0 \) the evolution parameter \( \epsilon \) must be large,
too large to be accommodated in realistic theories. This is in agreement with our earlier
observation that for opposite CP parities of the two neutrinos and/or their hierarchical
masses the RG evolution is very weak.

Of course, since \( \sin^2 2\vartheta = 1 \) is not the FP of the RGE, \( \sin^2 2\vartheta = 1 \) can hold only at
one particular scale.

Similar strategy can be also applied to analyze the general complex \( C^{AB} \). Once the
parameters of the \( U^{Aa} \) matrix

\[
U = \begin{pmatrix} e^{i\varphi_1} & 0 & c_{\vartheta} & -s_{\vartheta} \\ 0 & e^{i\varphi_2} & s_{\vartheta} & c_{\vartheta} \\ 1 & 0 & e^{i\alpha_2} \end{pmatrix}
\]

(4.27)

are expressed explicitly in terms of the \( C^{AB} \) entries, the solution \([1.10]\) allows to obtain
analytic formula for \( \sin^2 2\vartheta(t) \) and to study its behaviour. The relevant formulae have been
given in ref. \([16]\) (see also \([17]\)). As could be expected, for fixed initial values of \( \sin^2 2\vartheta \) and
the neutrino masses $|m_{\nu_1}|$ and $|m_{\nu_2}|$, the $\sin^2 2\theta$ obtained by the RG evolution interpolates smoothly between its value obtained for $\alpha_2 = 0$ (i.e. $m_{\nu_1}m_{\nu_2} > 0$) and $\alpha_2 = \pm \pi/2$ (i.e. with $m_{\nu_1}m_{\nu_2} < 0$) \cite{18}.

### 4.5 Mixing of three neutrinos and fixed points

For conserved CP, it is straightforward to derive from (4.13) the equations for the three independent mixing parameters $s_{12}$, $s_{23}$ and $s_{13}$ \cite{30,31}. Neglecting $y_e$ and $y_{\mu}$ Yukawa couplings we get, both in the SM and in the MSSM:

\begin{align}
\dot{s}_{12} &= -c_{12}(s_{12}s_{23} - c_{12}c_{23}s_{13})(-c_{12}s_{23} - s_{12}c_{23}s_{13})\kappa A_{21}y_\tau^2 \\
&\quad - s_{12}c_{12}c_{23}s_{13}(s_{12}s_{23} - c_{12}c_{23}s_{13})\kappa A_{31}y_\tau^2 + c_{12}^2c_{23}s_{13}(-c_{12}s_{23} - s_{12}c_{23}s_{13})\kappa A_{32}y_\tau^2
\end{align}

\begin{align}
\dot{s}_{23} &= s_{12}c_{23}^2(s_{12}s_{23} - c_{12}c_{23}s_{13})\kappa A_{31}y_\tau^2 \\
&\quad - c_{12}c_{23}^2(-c_{12}s_{23} - s_{12}c_{23}s_{13})\kappa A_{32}y_\tau^2
\end{align}

\begin{align}
\dot{s}_{13} &= -c_{12}c_{23}s_{13}^2(s_{12}s_{23} - c_{12}c_{23}s_{13})\kappa A_{31}y_\tau^2 \\
&\quad - s_{12}c_{23}c_{13}^2(-c_{12}s_{23} - s_{12}c_{23}s_{13})\kappa A_{32}y_\tau^2.
\end{align}

The evolution of the mixing angles can be classified into several universal types of behaviour, depending on the magnitude of the factors $A_{ab}$ in eqs. (4.28)-(4.30). We note that, neglecting the small effects of mass evolution, all possible mass configurations in patterns $i$-$iii$ listed at the end of Section 2 (except for the case of neutrinos (approximately) degenerate in mass and all having the same CP parity - to be discussed later) give one of the following four structures:

a) $A_{31} \approx A_{32}$ and $|A_{31}| \approx |A_{21}| \approx 1$

b) $A_{31} \approx A_{32}$ and $|A_{21}| \gg |A_{31}|, |A_{21}| \approx 1$

c) $A_{32} \approx A_{21} \approx 0, |A_{31}| \gg 1$

d) $A_{31} \approx A_{21} \approx 0, |A_{32}| \gg 1$

For hierarchical masses and for partly degenerate structure with opposite CP parities of the (almost) degenerate neutrinos all $A_{ab}$ are $O(1)$. For partly degenerate pattern with same CP parities or for degenerate pattern, at most one of them is large. In the first case, it follows from eqs. (4.28)-(4.30) and the value of $\epsilon$ ranging from $10^{-5}$ in the
SM to $2.5 \times 10^{-2}$ in the MSSM with $\tan \beta \approx 50$ that the evolution of the angles is very weak [34, 54, 36, 37]. On the other hand, for one of the $A_{ab}$ factors sufficiently big so that $|A_{ab}| > 1$, the angles evolve to an infrared quasi-fixed point (FP) [36, 37]. It is clear from eqs. (4.28)-(4.30) and from the parametrization (2.10) of the MNS matrix $U$, that depending on which $A_{ab}$ is large, the fixed points are either at $U_{31} = 0$ or $U_{32} = 0$.

Before discussing the approach to those fixed points in more detail, we can already now summarize several qualitative conclusions.

It is interesting to notice that in both fixed points we get the same relation between the mixing angles

$$
\sin^2 2\theta_{12} = \frac{s_{13}^2 \sin^2 2\theta_{23}}{(s_{23}^2 c_{13} + s_{13}^2)^2} \quad \text{with} \quad s_{23}^2 = \frac{1}{2} \left( 1 \pm \sqrt{1 - \sin^2 2\theta_{23}} \right).
$$

(4.31)

Thus, contrary to the $2 \times 2$ case, quantum corrections can give now an interesting, non-trivial fixed point relation. It is particularly interesting in the context of the present experimental indications for small $\theta_{13}$ angle from CHOOZ and maximal atmospheric and solar mixing ($\sin^2 2\theta_{23} \approx 1$ and $\sin^2 2\theta_{12} \approx 1$). The relation (4.31) is inconsistent with such a pattern of mixing. We stress that quantum corrections summarized in RGEs eqs. (4.28)-(4.30), if large, always give (4.31). Thus, if the presently most likely pattern of mixing is confirmed experimentally, all mass patterns generating large quantum corrections through RGE are ruled out!

Special cases easy to consider (still before a detailed study of the approach to the fixed points) are exact degeneracies at the scale $M$: $m_{\nu_1} = m_{\nu_2} \neq m_{\nu_3}$ or $m_{\nu_1} = m_{\nu_2} = m_{\nu_3}$. As follows from the discussion surrounding eq. (4.17), for the same CP parities, $m_{\nu_1} = m_{\nu_2}$ or $m_{\nu_1} = m_{\nu_3}$ or $m_{\nu_2} = m_{\nu_3}$ the angles must satisfy the FP relation (4.31) already at the scale $M$. If this is ruled out by experiment, then the two-fold degeneracy needs $m_{\nu_1} = -m_{\nu_2}$ and the three-fold degeneracy is ruled out. This conclusion holds (both in the SM and MSSM) under the assumption that the RGE corrections are the dominant ones (see next section for other possibilities). For $m_{\nu_1} = -m_{\nu_2}$ and $m_{\nu_1}^2 \gg m_{\nu_2}^2$, with $|m_{\nu_1}^2 - m_{\nu_2}^2| \approx \Delta M^2$, we can ask if quantum corrections can explain $\Delta m^2 = |m_{\nu_2}^2 - m_{\nu_1}^2|$. The answer to this question is positive in the MSSM [19] (in the SM see [50]) and will be discussed in more detail in Section 5. Another point worth a discussion is what if experiment will eventually be consistent with the FP relation (4.31). Two-fold degeneracy is easy: we need $m_{\nu_1} = m_{\nu_2}$, $|m_{\nu_3}^2 - m_{\nu_2}^2| \approx \Delta M^2$ and (as discussed in Section 5) quantum corrections can explain $\Delta m^2$. With three-fold degeneracy, we necessarily have at least one pair of neutrinos with the same CP parities but the question is whether we can explain both $\Delta m^2$ and $\Delta M^2$ by the discussed here class of quantum corrections. The answer is negative [51, 52, 53]. This can be explained as follows: from eq. (4.19) we would need
implies sin $2\theta_{12} \approx 1$ and the second sin $2\theta_{23} \approx 1$ ($|U_{33}| - |U_{32}|$ can be at most $\approx 1/2$). This is however incompatible with the relation (4.31) which should be satisfied! So, even if experiment was consistent with (4.31), the three-fold degenerate mass spectrum would still be unacceptable since $\Delta m^2$ and $\Delta m^2$ could not be explained by quantum corrections (always under the assumption about the dominance of the RG corrections).

Let us now discuss the approach to the fixed point and concentrate first on the structure b) which can be realized for partly degenerate or degenerate patterns, with same CP parities of $\nu_1$ and $\nu_2$. Since $A_{31} \approx A_{32}$ and $|A_{21}| \gg |A_{31}|$, equations (4.28-4.30) reduce to:

$$
\dot{s}_{12} = -c_{12}(s_{12}s_{23} - c_{12}c_{23}s_{13})(-c_{12}s_{23} - s_{12}c_{23}s_{13})A_{21}y^2_r - c_{12}s_{23}c_{23}s_{13}A_{32}y^2_r,
$$

$$
\dot{s}_{23} = s_{23}c^2_{23}A_{32}y^2_r,
$$

$$
\dot{s}_{13} = c_{23}^2s_{13}c_{13}A_{32}y^2_r.
$$

(4.32)

(we take MSSM, with $\kappa = 1$). The equation for $\dot{s}_{23}$ is the same as in the 2 $\times$ 2 scenario discussed in the previous subsection. The evolution of $\theta_{23}$ in the two cases is formally not identical as the evolution of the mass factor $A_{32}$ depends now also on the remaining mixing angles $\theta_{12}$ and $\theta_{13}$. Nevertheless, the qualitative behaviour of $s_{23}$ is similar because in most cases the scale dependence of $A_{32}$ can be neglected. Denoting

$$
\xi_r \equiv \exp \left( -\int_0^t 2A_{32}(t')y^2_r(t')dt' \right) \approx \exp (-2A_{32}(0)\epsilon)
$$

(4.33)

the solution for $s^2_{23}(t)$ reads

$$
s^2_{23}(t) = s^2_{23}(0)/\left[ s^2_{23}(0) + c^2_{23}(0)\xi_r \right]
$$

(4.34)

and yields

$$
\sin^2 2\theta_{23}(t) = \xi_r \sin^2 2\theta_{23}(0)/\left[ s^2_{23}(0) + c^2_{23}(0)\xi_r \right]^2.
$$

(4.35)

The solution for $s^2_{13}$ can also be given in a closed form:

$$
s^2_{13}(t) = s^2_{13}(0)/\left\{ s^2_{13}(0) + c^2_{13}(0)\left[ s^2_{23}(0) + c^2_{23}(0)\xi_r \right] \right\}.
$$

(4.36)

Thus, since $|A_{32}| \approx 1$, the evolution of both, $s_{23}$ and $s_{13}$ is very weak [34, 38, 39]. For example, the effect of the running for $\sin^2 2\theta_{\text{atm}}$ is a 2.5% change for extreme value of $\tan \beta \approx 50$.

For $|A_{21}| \gg 1$ (recall we consider $m^2_{\nu_{21}} > |\Delta m^2|$ and $m_{\nu_1}, m_{\nu_2} > 0$), the evolution is towards one of the two approximate fixed points of the RG equation for $s_{12}$. One can
easily check, for instance, by considering the equation for \(d\tan\theta_{12}/dt\), that for \(A_{21} > 0\) (i.e. for \(\Delta m^2 > 0\)) the point \(U_{31} = 0\) is the UV fixed point and \(U_{32} = 0\) is the IR fixed point. For \(A_{21} < 0\) (i.e. for \(\Delta m^2 < 0\)) the situation is reversed. It is also interesting to notice that in the limit \(s_{13} = 0\) we can follow analytically the approach to the fixed points. In this approximation

\[
s_{12} = s_{12}(t) = s_{12}(0)/\left\{s_{12}^2(0) + c_{12}^2(0) \left[c_{23}^2(0) + s_{23}^2(0)\xi^{-1}\right]^{-A_{21}/A_{32}}\right\},
\]

and the solution for \(s_{12}^2\) is of the form (4.34), with \(s_{23}(c_{23}) \rightarrow s_{12}(c_{12})\) and

\[
\xi_t \rightarrow \xi'_t = \exp\left(-\int_0^t 2s_{23}^2(t')A_{21}(t')y_{\tau}^2(t')dt'\right).
\]

For \(A_{21} > 0\), in the top-down running, the factor \(\xi' \rightarrow 0\) exponentially with decreasing the scale \(Q \rightarrow M_Z\) i.e. for growing \(t \propto A_{21}y_{\tau}\log(M/Q)\) and, consequently, we obtain \(s_{12}(t) = \pm 1\) (depending on its initial sign) and approach IR fixed point at \(U_{32} = 0\). Changing the signs of the right hand sides of eqs. (4.32) one can see that in the bottom-up evolution we approach \(s_{12}^2(t) \approx 0\) exponentially, i.e. the UV fixed point at \(U_{31} = 0\). For \(A_{21} < 0\) we get the reversed situation, in accord with our general expectations.

It is interesting to estimate the values of \(A_{21}\) and \(\tan \beta\), for which the approach to the IR fixed points is seen. In ref. [36] we estimated that for approaching the fixed point during the evolution in the range \((M, M_Z)\) with \(M \approx 10^{10}\) GeV one needs \(A_{21} \epsilon(M) > 3\), i.e. for \(\tan \beta = 20\) one needs \(m_{\nu_1} \approx m_{\nu_2} \gtrsim 10^{-4}\) eV for \(\Delta m^2 \sim 10^{-10}\) eV\(^2\) and \(m_{\nu_1} \approx m_{\nu_2} \gtrsim 0.01\) eV for \(\Delta m^2 \gtrsim 10^{-6}\) eV\(^2\). The qualitative change from a negligible evolution to the FP behaviour at the electroweak scale is abrupt and occurs in the small range \(0.5 \lesssim |A_{ab}\epsilon| \lesssim 3\).

Finally we note that, from the point of view of the initial conditions at the scale \(M\), the UV fixed point looks not realistic as the neglected muon Yukawa coupling \(y_{\mu}\) quickly destabilizes it during the evolution. We conclude that for hierarchical and partly degenerate mass patterns the evolution of the mixing angles is either very mild or shows (for \(|A_{21}| \epsilon \gtrsim 3\)) a fixed point behaviour.

The evolution of mixing angles in the degenerate case, \(m_{\nu_3}^2 \approx m_{\nu_2}^2 \approx m_{\nu_1}^2 \sim \mathcal{O}(\Delta M^2)\) or larger, partly falls into the same classes of behaviour. Indeed, as long as \(A_{31} \approx A_{32}\) with \(|A_{31}| \lesssim \mathcal{O}(1)\) and \(|A_{21}| \gg 1\), the angles evolve according to the same equations (4.32). One can easily identify the mass patterns of the degenerate case that fall into

\[
s_{12}(t) = s_{12}(0)/\left\{s_{12}^2(0) + c_{12}^2(0) \left[c_{23}^2(0) + s_{23}^2(0)\xi^{-1}\right]^{-A_{21}/A_{32}}\right\}.
\]
this category: the necessary condition is that $m_{\nu_1}$ and $m_{\nu_2}$ are of the same sign. For the evolutions of $s_{12}$ we then closely follow the two possibilities, depending on the sign of $A_{21}$, discussed for the partial degeneracy with $m_{\nu_1}m_{\nu_2} > 0$. We simply note that larger values of $|A_{21}|$ are generic for the present case and the approach to the fixed points is faster. However, the evolution of $s_{23}$ and $s_{13}$ are guaranteed to be mild only if $m_{\nu_1}$ and $m_{\nu_2}$ are negative. For positive $m_{\nu_1}$ and $m_{\nu_2}$, i.e. when all masses have the same CP parity, we can have $|A_{32}\epsilon| \gtrsim 3$ for $\tan \beta > 40$ (due to the bound (2.14) $|A_{32}\epsilon| \gtrsim 3$ cannot be realized for $\tan \beta < 40$). The angle $\theta_{23}$ behaves then as the angle $\vartheta$ of the $2\times2$ mixing discussed in Section 4.3 and, according to the solution (4.34), $s_{23}$ is exponentially focused to the stable FP $s_{23}(t) = 0$ or $s_{23}(t) = \pm 1$, depending on the sign of $A_{32}$, and on the direction of the evolution. The angle $s_{13}$ behaves in a similar way except that, as follows from eq. (4.36), it does not reach the value $s_{13}^2 = 1$ when $s_{23} \rightarrow 1$ (due to the presence of the factor $c_{23}^2$ in its RGE, the evolution of $s_{13}$ is then “frozen”). We conclude that in the regime in which the approach to the fixed points is relevant, the pattern with approximately degenerate neutrino masses and the same all three CP parities is not acceptable.

As we said earlier, the approach to the FP behaviour is abrupt as a function of $|A_{31}\epsilon| \approx |A_{32}\epsilon|$. However, in the small transition region of the values of $A_{31}$ and $A_{32}$, in agreement with our discussion in the previous subsection, it is possible to chose the initial condition for $s_{23}$ so to get $\sin^22\theta_{23} = 1$, $s_{13} \approx 0$ at the electroweak scale. This was exploited in ref. [55] as a possible mean to obtain maximal atmospheric neutrino mixing from the initially small $\sin^22\theta_{23}$. Since $|A_{21}\epsilon|$ is always much larger than $|A_{32}\epsilon|$, the evolution of $s_{12}$ is then such that the FP at $U_{32} = 0$ or $U_{31} = 0$ is quickly reached. Thus, a realistic solution, with maximal $\theta_{23}$ and $s_{13} \approx 0$, has $\sin^22\theta_{12} \approx 0$ and, moreover, the scheme cannot work unless there is an extreme fine tuning of the initial parameters [37].

The remaining degenerate mass patterns can be classified according to the relations $A_{32} \approx A_{21} \approx 0$ or $A_{31} \approx A_{21} \approx 0$. Consider first $A_{21} \approx A_{32} \approx 0$, i.e. $m_{\nu_1} \approx -m_{\nu_2} \approx m_{\nu_3}$. The equations for the evolution of the mixing angles can be approximated as

$$
\dot{s}_{12} = -s_{12}c_{12}c_{23}s_{13}(s_{12}s_{23} - c_{12}c_{23}s_{13})A_{31}y_{\tau}^2, \\
\dot{s}_{23} = s_{12}c_{23}^2(s_{12}s_{23} - c_{12}c_{23}s_{13})A_{31}y_{\tau}^2, \\
\dot{s}_{13} = -c_{12}c_{23}c_{13}^2(s_{12}s_{23} - c_{12}c_{23}s_{13})A_{31}y_{\tau}^2.
$$

These equations exhibit IR quasi-fixed point behaviour for $A_{31}\epsilon \ll -1$, corresponding to $U_{31} = 0$. As before, at the fixed point the angles satisfy the relation $s_{13} = \tan \theta_{12} \tan \theta_{23}$.

Since $\dot{s}_{12}$ is proportional to $s_{12}$ and suppressed by $s_{13}$, the running of $s_{12}$ is weak. The IR fixed point is reached due to strong running of $s_{23}$ and $s_{13}$. For $A_{31}\epsilon \gg 1$, $s_{23}(M_Z)$ is strongly focused at $\pm 1$. Thus, the mass and $\tan \beta$ configurations leading to $A_{31}\epsilon \gg 1$ are unacceptable. For $A_{31} \approx A_{21} \approx 0$ and $A_{32} < 0$ we get IR fixed point in $U_{32} = 0$. 

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The RG running of the mixing of three neutrinos for a complex matrix $U$, i.e. non-zero phase $\delta$ and $\alpha_1, \alpha_2 \neq 0$ modulo $\pi/2$ with all masses $m_{\nu_a}$ positive by definition, has been also investigated in the literature $^{37}$. For initially degenerate $\nu_a$ and $\nu_b$ the mixing pattern is determined by the condition $\text{Re} \left( U^{3a} U^{3b*} \right) \equiv \text{Re} \left( U^{3a}_{\text{MNS}} U^{3b*}_{\text{MNS}} \right) = 0$ playing in the general complex case the role of the FP condition. Qualitatively, our conclusions remain unchanged also in this case. For example, starting with $m_{\nu_1}^2 = m_{\nu_2}^2$, $\sin^2 2\theta_{23} = 1$, $s_{13} \approx 0$ and arbitrary phases $\alpha_1, \alpha_2$ at the scale $M$, the RG evolution leads to the relation $^{37}$

$$\sin^2 2\theta_{12}(M_Z) = \sin^2 2\theta_{12}(M) \sin^2(\alpha_1 - \alpha_2) + O(s_{13}^2) \quad (4.40)$$

We see therefore that maximal solar mixing can again be obtained only if the change of $\theta_{12}$ during the running is negligible, i.e. for $\alpha_1 - \alpha_2 \approx \pi/2$ at the scale $M$.

The running of the masses is always given by eq. (4.18) and is similar as for a real matrix $U$. The conclusion that the triple degeneracy of neutrinos at the scale $M$, $|m_{\nu_1}| = |m_{\nu_2}| = |m_{\nu_3}|$, cannot lead to an acceptable pattern of masses and mixing also remains valid.

In summary, with all $|A_{ab}\epsilon| \lesssim 0.5$ the evolution of the mixing is negligible. For the mass configurations such that at least one $|A_{ab}| \gg 1$ and $|A_{ab}\epsilon| \gtrsim 3$ the infrared fixed points are reached during the evolution, independently of further details of the mass matrices. However, only for $|A_{21}\epsilon| \gtrsim 3$ and $|A_{31}|, |A_{32}| \lesssim 1$, or for $A_{31}(A_{32}) \epsilon \ll -3$ and $A_{31}(A_{32}) \approx A_{21} \approx 0$ the evolution is consistent with a large atmospheric mixing angle at low energy. The mass configurations $m_{\nu_1} \approx m_{\nu_2} \approx m_{\nu_3}$ and $\pm m_{\nu_1} \approx \mp m_{\nu_2} \approx m_{\nu_3}$ with $A_{31}(A_{32}) > 0$ also lead to the infrared fixed points but at the same time the atmospheric mixing angle converges to zero. We also note that for $|A_{21}\epsilon| \gtrsim 3$ only $s_{12}$ runs to assure the fixed point relation, so the initial values for $s_{23}$ and $s_{13}$ have to be close to their experimental values already at the scale $M$. For $A_{31}\epsilon \lesssim -3$ or $A_{32}\epsilon \lesssim -3$, $s_{23}$ and $s_{13}$ evolve strongly and the evolution of $s_{12}$ is weak. The IR fixed point relation (4.31) is always one equation for three angles. Insisting on a large atmospheric mixing angle, it correlates small (as follows from CHOOZ) $\theta_{13}$ angle with a small solar mixing angle. The fixed point solution makes the low energy angles dependent on only two, instead of in general three, initial conditions at the scale $M$. We conclude that quantum corrections encoded in the RG running of mixing angles may have dramatic impact on their physical values if the mass pattern is partial degeneracy or degeneracy. If the bimaximal mixing solution was confirmed, all the mass patterns leading to the FP would be ruled out, unless the low energy threshold corrections change the results.
5 Low energy threshold corrections

5.1 Threshold corrections in the SM

The Wilson coefficient $C^{AB}$ of the dimension 5 $\Delta L = 2$ operator (3.11) is a renormalized parameter of the effective theory Lagrangian. Integrating its RGEs from the high scale $M$ down to some scale $Q \approx M_Z$ allows to resum potentially large corrections involving $\ln(M/Q)$ to all orders of the perturbation expansion. However, since the low energy scale $Q$ is not a priori determined by any physical requirement (apart from the condition $Q \approx M_Z$), the neutrino masses and mixing angles computed in the tree-level approximation from the Wilson coefficient $C^{AB}(Q)$ do depend (albeit weakly) on the actual choice of $Q$. This dependence can be removed by computing masses and mixing angles in the one-loop approximation in the MS scheme and with $Q$ as the renormalization scale.

In general, to compute the physical neutrino masses and mixing one has first to perform the RG evolution of the entire matrix $U$ and neutrino masses from the scale $M$ down to the scale $M_Z$, include threshold corrections and subsequently rediagonalize the resulting mass matrix $m^{1\text{-loop}}_{ab}$ by an additional unitary matrix $U'$. The physical matrix $U_{MNS}$ (whose elements are probed in neutrino experiments) is then given as $e^{i\varphi_A} U_{MNS}^{Aa} = (U \cdot U')^{Aa}$ where $U$ is the matrix obtained from the RG evolution.

In the basis in which neutrino masses are diagonal, we have

$$m^{1\text{-loop}}_{ab} = m_{\nu_a} \delta^{ab} + m_{\nu_a} I^{ab} + m_{\nu_b} I^{ba} \quad (5.1)$$

where

$$I^{ab} = \sum_{A,B} U^{Aa} I^{th}_{AB} U^{Bb} \quad (5.2)$$

In the SM one finds [56]

$$I^{th}_{AB} = \frac{\delta^{AB}}{16\pi^2} \frac{g_2^2 m_{\nu_A}^2}{2 M_W^2} \left[ \frac{11}{8} - \frac{3}{2} \ln \frac{M_W}{Q} + \mathcal{O} (x_A \ln x_A) \right] + \ldots \quad (5.3)$$

where $x_A \equiv m_{\nu_A}^2 / M_W^2$. Note that since $g_2^2 m_{\nu_A}^2 / 2M_W^2 = y_{\nu_A}^2$, the coefficient of $\ln(1/Q)$ agrees with the coefficient $\kappa$ of the $y_{\nu_A}^2 C^{AB} + C^{AB} y_{\nu_B}^2$ term in the RGE (4.8) for the SM. This confirms the correctness of the recent re-derivation [20] of the SM RGE. The remaining corrections indicated by dots in (5.3) affect only the overall scale of the neutrino masses and therefore are not interesting in view of the unspecified magnitude of the mass $M$ in eq. (3.13).

As could be expected, in the SM the nontrivial part (5.3) of the low energy threshold corrections that changes the structure of the mass matrix, and hence of the matrix $U^{Aa}$, is
small and always negligible compared to the quantum effects described by the RG running, which are enhanced by large logarithm of the ratio $M/M_Z$. The correction (5.3) can be most easily taken into account by stopping the RG evolution of the Wilson coefficient $C^{AB}$ at the scale $Q = M_W e^{-11/12}$.

5.2 Threshold corrections in the MSSM

In the supersymmetric model, apart from stabilizing the results obtained from the RG analysis, the low energy threshold corrections can be as or more important than the RG evolution and may have very important physical consequences for the neutrino masses and mixing angles [61, 60, 58]. Before we discuss some physical examples, it is worthwhile to adapt our calculational procedure to the new situation, so that the RG evolution effects and the threshold corrections can be treated on equal footing. Although the general procedure outlined earlier is in principle correct irrespectively of the relative magnitude of the RG and threshold effects, in practice it can mask simple qualitative features if the rediagonalization due to threshold corrections is not a small perturbation.

We observe that, to a very good accuracy, in the solution (4.10) to the RGE the factors $I_{eA}$ (4.11) can be approximated by

$$I_{eA} \approx 1 - \kappa \int_0^{t_Z} y_{eA}^2 dt' \equiv 1 - I_{rg}^A. \quad (5.4)$$

With this approximation (which can fail only for unrealistically large values of $\tan \beta$) all quantum effects of the physics below the scale $M$, the RG running as well as the low energy threshold corrections, can be described by a single formula (5.1). The $m_{\nu_a}$ and the matrix $U$ are now the neutrino mass eigenvalues and the neutrino mixing matrix, respectively, at the scale $M$, and the factors $I^{ab}$ are given by eq. (5.2) with

$$I_{AB}^{th} \rightarrow I_{AB}^{th} - \delta^{AB} I_{A}^{\delta} \equiv I_{AB}. \quad (5.5)$$

Again, the case of a real matrix $U$ is particularly easy because then the formula (5.1) can be written as

$$m^{1-loop}_{ab} = m_{\nu_a} \delta^{ab} + (m_{\nu_a} + m_{\nu_b}) I^{ab} \equiv m_{\nu_a} \delta^{ab} + \Delta m_{ab} \quad (5.6)$$

with the right hand side symmetric and real i.e. Hermitean. One can then use the formal perturbation calculus of quantum mechanics (see e.g. [57]) to find corrections to neutrino masses and mixing angles. Of course, if the threshold corrections are absent\footnote{If the threshold corrections are universal, i.e. if $I_{AB}^{th} = I^{th} \delta^{AB}$, they can be absorbed into the overall scale of neutrino masses and do not influence neither the mixing angles nor the ratio $\Delta M^2/\Delta m^2$.} one has

$I_{AB} \approx \delta^{AB} I_{A}$ with $|I_{e}| \gg |I_{\mu}| \gg |I_{\tau}|$. 

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Several physically interesting situations can be discussed. In the following we shall mainly focus on the scenarios with equal three or at least two masses at the scale $M$. Although the discussion of the mixing angles does not depend on whether the masses are equal or only approximately equal, the former case is much more constrained for the masses themselves and therefore more interesting.

### 5.2.1 Three-fold degeneracy and flavour diagonal corrections

In the MSSM the complete formulae for $I_{AB}^{\text{th}}$ are lengthy due to additional contributions of $H^\pm$ and chargino/charged slepton and neutralino/sneutrino loops [56] but we do not need them for our discussion. First of all, it is interesting to investigate whether it is possible to generate the right mass squared differences starting from degenerate neutrinos at the scale $M$: $|m_{\nu_1}| = |m_{\nu_2}| = |m_{\nu_3}|$ (we again allow for different CP parities of the neutrinos).

From the arguments presented in Section 4 it follows that this is impossible if the factors $I_{AB}$ are dominated by the RG effects, i.e. when $I_{AB} \approx \delta^{AB} I_A$ with $|I_\tau| \gg |I_\mu|, |I_e|$. The degeneracy Ansatz is however potentially interesting as it might help to understand the bimaximal mixing. Also, the neutrino masses of order of few eV are necessary if the neutrinos are to play a role of the hot component of the Dark Matter.\footnote{The hot component of the Dark Matter was previously needed to account for formation of largest scale structures in the Universe. At present, in view of the observational evidence for a significant contributions of the cosmological constant (or another form of dark energy as e.g. quintessence) to the energy density of the Universe, the hot component of the Dark Matter seems no longer necessary (but is not excluded).}

It is therefore interesting to see if some other corrections can split the mass squares. We start with the flavour diagonal threshold corrections: $I_{AB} = \delta^{AB} I_A$ and consider first the mass pattern $m_{\nu_1} = -m_{\nu_2} = m_{\nu_3} \equiv m_\nu$. The formula (5.6) now takes the form

$$m_{\nu}^{1-\text{loop}}(ab) = m_\nu \begin{pmatrix} 1 + 2U_{A1}^2 I_A & 0 & 2U_{A1} U_{A3} I_A \\ 0 & -1 - 2U_{A2}^2 I_A & 0 \\ 2U_{A1} U_{A3} I_A & 0 & 1 + 2U_{A3}^2 I_A \end{pmatrix}$$

The corrections to neutrino masses can be calculated perturbatively. Because $m_{\nu_1} = m_{\nu_3}$, one applies here the perturbation calculus to the case with degeneracy of the unperturbed “energy levels” [57]. Instead of solving the “secular” equation for the corrected eigenvalues, it is however better to exploit the freedom of an arbitrary rotation in the plane (13): $U \rightarrow U R_{13}$. This freedom can be used to diagonalize the perturbation by requiring

$$\sum_A U_{A1} U_{A3} I_A = 0$$

which fixes the matrix $U$ after taking into account the perturbation. The correction $\Delta m_{ab}$ to the zeroth order neutrino mass matrix becomes then diagonal and the neutrino masses
are given by

$$|m_{\nu_3}| = m_\nu \left(1 + 2U_{2\alpha}^2 I_A\right).$$

(5.9)

Let us now assume that $I_e \neq 0$, $I_\mu = I_\tau = 0$. The condition (5.8) reduces then to $U_{11}U_{13} = 0$. The experimentally viable solution is $U_{13} = 0$, i.e. $s_{13} = 0$. Note that this relation replaces the FP relation (1.31) of Section 4. For the neutrino masses one finds

$$\Delta m_{21}^2 = -4m_\nu^2 \cos^2 2\theta_{12} I_e, \quad \Delta m_{32}^2 = -4m_\nu^2 s_{12}^2 I_e.$$

(5.10)

It is therefore possible to generate $\Delta M^2 \gg \Delta m^2$ provided the solar mixing is nearly maximal. The same result is obtained starting from $-m_{\nu_1} = m_{\nu_2} = m_{\nu_3} \equiv m_\nu$. Thus, if the underlying theory gives one of these two degeneracies together with the bimaximal mixing pattern, the low energy threshold corrections can be responsible for assuring $s_{13} = 0$ and generating the correct mass squared differences.

Of course it is unrealistic to expect only one correction $I_A$ to be nonzero. Note however, that the results for the mixing angles and the ratio $\Delta M^2/\Delta m^2$ are not altered by shifting all corrections $I_A$ by an overall additive constant: $I_A \rightarrow \tilde{I}_A \equiv I_A - I$. Indeed, using unitarity of the matrix $U$, the off-diagonal entries in the rhs of eq. (5.6) can be always written as $\sum A U_{Ah} U_{Ab} \tilde{I}_A$ ($a \neq b$) with arbitrary $I$. For the diagonal entries instead one has $1 + 2 \sum_A U_{Ah}^2 I_A = 1 + 2I + 2 \sum_A U_{Ah}^2 \tilde{I}_A \approx (1 + 2I)(1 + 2 \sum_A U_{Ah}^2 \tilde{I}_A)$. Hence, up to higher order terms, eq. (5.6) can be rewritten with $\tilde{I}^{ab} = \sum A U_{Ah} U_{Ab} \tilde{I}_A$ and $m_\nu$ replaced by $\tilde{m}_\nu \equiv (1 + 2I) m_\nu$. Thus, $I_\mu = I_\tau \neq 0$ is equivalent to $I_\mu = I_\tau = 0$.

It follows that $|I_e| \gg |I_\tau|$, $|I_\mu|$, equivalent to $|\tilde{I}_e| \gg |\tilde{I}_\tau| \neq 0$, $\tilde{I}_\mu = 0$, should give only small correction to the result (5.11) and $s_{13} = 0$. For $m_{\nu_1} = -m_{\nu_2} = m_{\nu_3}$, solving eq. (5.8) (with $I_e = \tilde{I}_e$, $I_\mu = 0$ and $I_\tau = \tilde{I}_\tau$) one finds (5.8) that

$$s_{13} = -\frac{s_{12}}{c_{12}} s_{23} c_{23} r + O(r^2)$$

$$\Delta m_{32}^2 \approx -4\tilde{m}_\nu^2 \tilde{I}_e s_{12}^2$$

$$\Delta m_{21}^2 \approx -4\tilde{m}_\nu^2 \tilde{I}_e \left[\cos 2\theta_{12} (1 - s_{23}^2 r) + (1 + 2c_{12}^2) s_{12}^2\right]$$

(5.11)

where $r \equiv \tilde{I}_\tau/\tilde{I}_e$. Thus, obtaining small but non-zero angle $\theta_{13}$ is also possible. To obtain the experimentally favoured value of $\Delta M^2$ one must have $|\tilde{m}_\nu^2 \tilde{I}_e| \approx 1.6 \times 10^{-3}$ eV$^2$ i.e. $|I_e - I_\mu| \sim 10^{-3}$ and $I_\mu \approx I_\tau$ for $m_\nu$ in the eV range. In the MSSM the required hierarchy $|I_e| \gg |I_\tau|$, $I_\tau \approx I_\mu$ can arise from the low-energy threshold corrections only as a result of non-universality of the left-handed charged slepton masses (and sneutrino masses).\footnote{In the $\tilde{W}^\pm$ loop approximation to $I_{AB}^{\text{th}}$, it was estimated in ref. (58) that in order to get $\tilde{I}_e \sim 10^{-3}$ and positive one needs $M_{\tilde{e}_L} \approx 1.7 M_{\tilde{\mu}_L}$ (and $M_{\tilde{\mu}_L} \approx M_{\tilde{e}_L}$). However, from the full expression for $I_{AB}^{\text{th}}$ it follows (50) that $\tilde{I}_e \sim 10^{-3}$ and positive can be achieved only for relatively light first chargino ($m_{C_1} \lesssim 300$ GeV) and $M_{\tilde{\mu}_L} \approx (1.2 - 1.6) M_{\tilde{\mu}_L}$. For $M_{\tilde{e}_L} \approx (1.2 - 1.6) M_{\tilde{\mu}_L}$, which can be realized in the inverted hierarchy models (59), one gets $\tilde{I}_e \sim -10^{-3}$ i.e. positive $\Delta m_{21}^2$ for (almost) maximal solar mixing.}
Moreover, for exactly maximal solar mixing, obtaining $\Delta m^2$ appropriate for the LAMSW solution requires $s_{13}^2 \approx 10^{-2(2-3)}$. This is consistent with the CHOOZ data and is realized for $r \lesssim 0.1$, i.e. $|I_\tau - I_\mu| \lesssim 10^{-4}$. For $\tan \beta \lesssim 3$ such a small difference between the corrections $I_\tau$ and $I_\mu$ can be due to the RG effects in $I^{\text{rg}}$ and does not require any mass splitting between $\mu_L$ and $\tau_L$. For larger values of $\tan \beta$, for which $I^{\text{rg}}_\tau - I^{\text{rg}}_\mu > 10^{-4}$, some conspiracy between $I^{\text{th}}_\tau - I^{\text{th}}_\mu$ due to $M_\tilde{\mu}_L < M_\tilde{\tau}_L$ and $I^{\text{rg}}_\tau - I^{\text{rg}}_\mu$ is required. The amount of the necessary fine tuning grows, of course, with $\tan \beta$. For the VO solution to the solar neutrino problem, with $\sin^2 2\theta_{12} = 1$, a more severe fine tuning is always necessary: $\Delta m^2 \sim O(10^{-10} \text{ eV}^2)$ requires $|I_\tau - I_\mu| \lesssim 10^{-7}$ which is always smaller than $I^{\text{rg}}_\tau - I^{\text{rg}}_\mu \gtrsim 10^{-5}$.

Thus obtaining the correct $\Delta m^2$ for this solution requires a cancellation at least to one part per hundred between the RG and threshold effects. For $\Delta m^2 \sim O(10^{-10} \text{ eV}^2)$ one gets $s_{13}^2 \lesssim 10^{-8}$. Similar results are obtained also for $-m_{\nu_1} = m_{\nu_2} = m_{\nu_3}$.

It can be checked [60] that, for the initially degenerate neutrino masses, the patterns $m_{\nu_1} = -m_{\nu_2} = m_{\nu_3}$ or $-m_{\nu_1} = m_{\nu_2} = m_{\nu_3}$ and $|I_e| \gg |I_\tau|$, $I_\tau \approx I_\mu$ are the only ones that can produce the required mass squared splittings with flavour diagonal threshold corrections. The necessary condition is that the underlying theory valid above the scale $M$ gives the bimaximal mixing in the basis in which eq. (5.8) is satisfied [58, 60].

5.2.2 Three-fold degeneracy and flavour non-diagonal corrections

Qualitatively new possibilities for mixing angles and for splitting initially degenerate neutrinos originate from nonzero off-diagonal elements of the corrections $I^{\text{th}}_{AB}$ in eq. (5.2) [61, 60]. This is possible in the MSSM if the slepton mass matrices are not diagonal in the flavour space in the basis in which the leptonic Yukawa couplings are diagonal. The amount of flavour mixing is then best quantified in terms of the so-called mass insertions [62] defined as the ratio of the off-diagonal (in flavour space) elements of the charged sleptons mass squared matrices to some average charged slepton mass squared. Current limits on such mass insertions following from the non-observation of the decays $\mu \rightarrow e\gamma$ etc. are not very stringent: Only the insertion $\delta_{LR}^{12}$ mixing left(right)-handed $\tilde{\mu}$ with right(left)-handed $\tilde{e}$ has to be smaller than $O(10^{-5})$. The insertions $\delta_{LR}^{13}$ and $\delta_{LR}^{23}$ causing similar, chirality changing, $\tilde{e} \leftrightarrow \tilde{\tau}$ and $\tilde{\mu} \leftrightarrow \tilde{\tau}$ transitions are bounded by $\approx 0.5$ and $\approx 0.1$, respectively, for slepton masses $\sim 500$ GeV. Bounds on the chirality preserving insertions $\delta_{LL}^{AB}$, $\delta_{RR}^{AB}$ exist only for the $\tilde{e} \leftrightarrow \tilde{\mu}$ transition and are $\approx 0.2$ for $M_{\tilde{\ell}} \sim 500$ GeV. The other chirality preserving mass insertions are practically unrestricted.\footnote{Due to the underlying $SU_L(2)$ symmetry, mass insertions inducing transitions between left-handed sleptons of different generations are always accompanied by the insertions inducing similar transitions between sneutrinos.}
If the mass insertions are non-vanishing, there is a flavour non-diagonal contribution to $I_{AB}^{th}$. In general it takes the form

\[ I_{AB}^{th} \approx \frac{1}{16\pi^2} \sum_{X,Y=L,R} \delta_{XY}^A h_{XY}(M_{Iw}, m_{C_j}, m_{N_i}) \]  \hspace{1cm} (5.12)

where $h_{LL}$, $h_{RR}$ and $h_{LR}$ are some functions of the chargino and neutralino masses $m_{C_j}$, $m_{N_i}$ and of some average mass $M_{Iw}$ of charged sleptons. The largest values of $I_{AB}^{th}$ can be obtained for relatively light charginos, heavy sleptons ($\sim 1$ TeV) and $M_2/\mu \approx -1$ and reach $h_{LL}/16\pi^2 \approx \text{few} \times 10^{-3}$ ($h_{RR}$ and $h_{LR}$ are always smaller) \[56\]. For comparable chargino and slepton masses one has $h_{LL}/16\pi^2 \approx 2 \times 10^{-4}$. In principle the mass insertion approximation should fail for $|\delta_{XY}^A| \leq 0.1$. In practice it works as an order of magnitude estimate even for $|\delta_{XY}^A| \leq 1$ (the error is then of order 25%).

In the presence of non-zero mass insertions and for $m_{\nu_a} = m_{\nu_b} (-m_{\nu_c})$ the condition for vanishing of the appropriate off-diagonal entry of the correction to the zeroth order mass matrix reads

\[ \sum_{A,B} U_{Aa} U_{Bb} I_{AB} = 0. \]  \hspace{1cm} (5.13)

Consider first the situation in which the single correction $I_{e\mu}^{th}$, $I_{e\tau}^{th}$ or $I_{\mu\tau}^{th}$ dominates over all other corrections. The condition \[5.13\] gives then relations between the mixing angles that are different from \[4.31\] and are listed in Table \[4\] \[\{1\}. Only three of the nine possibilities are compatible with the bimaximal mixing: dominant $I_{\mu\tau}^{th}$ for $m_{\nu_1} = m_{\nu_3}$ or $m_{\nu_2} = m_{\nu_3}$ and dominant $I_{e\tau}^{th}$ for $m_{\nu_1} = m_{\nu_2}$. For initially degenerate neutrinos, the latter combination gives wrong relation $\Delta m^2 \approx 2\Delta M^2$. (Other six combinations leading through \[5.13\] to $\sin^2 2\theta_{12} \sim \sin^2 2\theta_{13}$ also give bad relations $\Delta m^2 \approx \Delta M^2$ or $\Delta M^2 = 0$.) The former two are however interesting giving $\Delta m^2 \approx -4m^2_\nu \cos 2\theta_{12} \sin 2\theta_{23} I_{\mu\tau}^{th}$ and $\Delta M^2 \approx 4m^2_\nu (1 + c^2_{12}) \sin 2\theta_{23} I_{\mu\tau}^{th}$. Obtaining $\Delta M^2 \approx 3.2 \times 10^{-3}$ eV$^2$ is therefore possible but only for $m_\nu \gtrsim 1$ eV and $\delta_{LL}^3 \gtrsim 0.5$ i.e. for rather large flavour mixing in the slepton mass matrices.

Similarly as for a non-zero $\tilde I_e$ correction, the solar mass squared difference $\Delta m^2 \ll \Delta M^2$ can be generated either by an appropriately tuned departure of the angle $|\theta_{12}|$ from maximal value $\pi/4$ or by another, hierarchically smaller, correction $I \neq 0$. It has been demonstrated \[\{10\}, that including on the top of the dominant $I_{\mu\tau}^{th}$ correction a hierarchically smaller perturbation in the form of either $\tilde I_\mu$, $\tilde I_\tau$ ($\tilde I_e$ does not work) or $I_{e\mu}^{th}$, $I_{e\tau}^{th}$ allows to split $m_{\nu_1} = -m_{\nu_2}$ even for exactly bimaximal mixing.\footnote{We note \[\{60\], that with a large $I_e$ diagonal perturbation discussed in the Subsection 5.2.1, $\Delta m^2 \ll \Delta M^2$ (and $0 \neq s^2_{13} \ll 1$) can be also induced by $I_{e\mu}^{th}$ or $I_{e\tau}^{th}$ instead of $I = \tilde I_e$ (small $I_{\mu\tau}^{th} \neq 0$ would..."
Table 1: Relations of the FP type between the mixing angles for dominant correction $I_{AB}^{th}$

| $I_{e\mu}^{th}$ | $I_{e\tau}^{th}$ | $I_{\mu\tau}^{th}$ |
|-----------------|-----------------|-----------------|
| $m_{\nu_1} = m_{\nu_2}$ | $s_{13} = \cot 2\theta_{12} \cot \theta_{23}$ | $s_{13} = \cot 2\theta_{12} \tan 2\theta_{23}$ |
| $m_{\nu_1} = m_{\nu_2}$ | $s_{13} = \cot 2\theta_{12} \tan 2\theta_{23}$ | $s_{13} = \cot 2\theta_{12} \tan 2\theta_{23}$ |
| $m_{\nu_2} = m_{\nu_3}$ | $s_{13} = \cot 2\theta_{12} \tan 2\theta_{23}$ | $s_{13} = \cot 2\theta_{12} \tan 2\theta_{23}$ |

In all those cases of two hierarchically different corrections the important difference between the two alternatives: perturbation by a diagonal correction $\tilde{I}_A$ or perturbation by an off-diagonal correction $I_{AB}^{th}$ is that, for exactly bimaximal mixing, in the former case $\Delta m^2 \propto r^2$, whereas in the latter $\Delta m^2 \propto r$ only ($s_{13} \sim r$ in all cases), where $r \ll 1$ is the ratio of the smaller to larger correction as in (5.11) \[60\]. Therefore, similarly as for $|\tilde{I}_e| \gg |\tilde{I}_r|$, obtaining $\Delta m^2$ appropriate for the VO solution with $I_{\mu\tau}^{th}$ dominance and hierarchically smaller $I_{e\tau}$ or $I_{e\mu}$ would require some tuning of slepton masses to cancel too large a contribution of $I_{\mu\tau}^{th}$ to $I_r$.

Finally, we remind the reader that the relations listed in Table 1 remain approximately valid when the equalities of the masses are relaxed and replaced by the corresponding approximate degeneracies. The listed relations play then the role of the FP relations discussed in Section 4. They are satisfied at the electroweak scale irrespectively of the initial values of the angles. The observed $\Delta m^2$ and $\Delta M^2$ can be obtained by adjusting the initial values of only approximately equal masses. The role of the threshold corrections $I_{AB}^{th}$ is then the same as the role of the RG corrections. If large enough, they give one of the “fixed points” relations of Table 1.

5.2.3 Two-fold degeneracy and threshold corrections

We can also discuss the effect of threshold corrections in the case of the two-fold degeneracy $m_{\nu_4} = m_{\nu_2} \equiv m_{\nu} \gg m_{\nu_3}$ or $m_{\nu_4} = m_{\nu_2} \ll m_{\nu_3}$. For $m_{\nu_1} = -m_{\nu_2}$ the $\Delta m_{12}$ off-diagonal entry in eq. (5.6) automatically vanishes but the correction matrix $\Delta m_{ab}$ as the whole needs not be diagonal. In the first order of the perturbation calculus \[54\] the neutrino masses are then given by eq. (5.9) and receive also further corrections of order $O((\Delta m_{ab})^2/\max(|m_{\nu_3}|, |m_{\nu}|))$. The mixing angles also receive corrections of order not split $m_{\nu_1}$ and $m_{\nu_2}$ for $\sin^2 2\theta_{12} = 1$. This leads to $s_{13} \approx -r s_{23} (c_{23}) (r = r_{\mu\tau}^{th}/r_e)$ and $\Delta m^2 \approx 4m_{\nu}^2 \sin 2\theta_{12} c_{23} s_{23} r_{\mu\tau}^{th}$. Obtaining $\Delta m^2 \sim O(10^{-4} \text{ eV}^2)$ requires, for $m_{\nu} \sim 1 \text{ eV}$ the correction $I_{\mu\tau}^{th} \sim 10^{-4(1-5)}$ which is possible in the MSSM and gives $r \sim 10^{-4(1-2)}$ i.e. acceptable $s_{13}$. 

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\( O (\Delta m_{ab}/\max(|m_{\nu_1}|, |m_{\nu_2}|)) \), i.e. small if the hierarchy of neutrino masses is large. If only flavour diagonal threshold corrections are present the solar mass squared splitting is given by\(^{21}\)

\[
\Delta m_{21}^2 \approx 4 m_{\nu}^2 \left[ \left( U_{32}^2 - U_{31}^2 \right) \tilde{I}_\tau + \left( U_{12}^2 - U_{11}^2 \right) \tilde{I}_\tau \right]. \tag{5.14}
\]

The interesting aspect of this situation is that the (generically) dominant first order contribution to \( \Delta m^2 \) proportional to \( \tilde{I}_\tau = \tilde{I}_{\tau}^{\text{rg}} + \tilde{I}_{\tau}^{\text{th}} \) vanishes for \( U_{32}^2 = U_{31}^2 \), i.e. for \( s_{13} = \pm (s_{23}/c_{23})(c_{12} \pm s_{12})/(c_{12} \mp s_{12}) \), a special case of which is the bimaximal mixing solution with \( s_{13} = 0 \) (also the second term in eq. (5.14) then vanishes). This has been discussed in refs. \[^{49, 48, 36}\] in connection with the possibility of realizing the VO solution to the solar neutrino problem, within the inversely hierarchical pattern \( m_{\nu_3}^2 \ll m_{\nu}^2 \sim 3 \times 10^{-3} \) eV\(^2\). Note that for generic mixing angles the RG corrections would always give too large a \( \Delta m^2 \).

Since the second order correction \((\propto \tilde{I}_\tau^2)\) to \( \Delta m^2 \) is proportional to \( m_{\nu_3} \), the right \( \Delta m^2 \) for the VO solution can be obtained by appropriately tuning \( m_{\nu_3} \approx 0 \) and/or due to the threshold effects.

As discussed in \[^{36}\], with \( m_{\nu_1} = -m_{\nu_2}, m_{\nu_3}^2 \ll m_{\nu_1} \sim 3 \times 10^{-3} \) eV\(^2\), the correct \( \Delta m^2 \) for the LAMSW (or SAMSW) solutions can be obtained from the RG running. From eq. (5.14) we get

\[
\Delta m^2 \approx 4 m_{\nu}^2 \left| s_{23}^2 \cos 2\theta_{12} + s_{13} \sin 2\theta_{23} \sin 2\theta_{12} \right| \tilde{I}_{\tau}^{\text{rg}} \tag{5.15}
\]

and for the LAMSW solution, with \( |s_{13}| \approx 0.1, \Delta m^2 \sim O(10^{-5}) \) eV\(^2\) can be obtained for \( \tan \beta \approx 30 \) (\( \tan \beta \approx 10 \) for the SAMSW solution). For smaller values of \( \tan \beta \) one has to investigate potentially larger threshold corrections, which give \[^{30}\]

\[
\begin{align*}
\Delta m^2 &\approx 4 m_{\nu}^2 \left| \sin 2\theta_{12} (c_{23} I_{\mu e}^{\text{th}} - s_{23} I_{e\tau}^{\text{th}}) \right| \quad \text{LAMSW} \\
\Delta m^2 &\approx 4 m_{\nu}^2 \left| \cos 2\theta_{12} (I_{e\mu}^{\text{th}} - \frac{1}{2} \sin 2\theta_{23} I_{\mu\tau}^{\text{th}}) \right| \quad \text{SAMSW}. \tag{5.16}
\end{align*}
\]

For \( \Delta m^2 \sim O(10^{-5}) \) eV\(^2\) one needs therefore corrections \( \tilde{I}_{\tau}^{\text{th}} \sim 10^{-3} - 10^{-2} \) which is definitely too big a value for \( I_{\mu e}^{\text{th}} \) in the MSSM and also impossible for \( I_{e\tau}^{\text{th}} \) and \( I_{\mu\tau}^{\text{th}} \) (at least within the validity of the mass insertion approximation, it would require \( |\delta_{e\mu\tau}^{\text{th}}| > 1 \)).

Our discussion here does not cover fully the potential role of quantum corrections for the two-fold degeneracy pattern. Several other possibilities do exist, depending on the chosen solar solution and we refer the reader to the literature \[^{11, 10, 58}\] for further details.

\(^{21}\)We choose to work with \( \tilde{I}_A \equiv I_A - I_{\mu} \).
6 Conclusions

For every theory of neutrino masses, for a meaningful comparison with experimental information, it is necessary to discuss quantum corrections. This is relatively easy if the effective low energy theory is the SM or its supersymmetric extension and the neutrino masses enter via the effective operator \((\mathcal{L}_{\text{eff}})\). In Sections 4 and 5 we reviewed the formalism for including quantum correction in those cases. Next we applied that formalism to several different classes of the neutrino mass sector at the scale \(M\), hypothetically given by the theory of neutrino masses. They split into two broad groups. The first one is characterized by small quantum corrections that may eventually be important for precision tests of the future theory. However, they are irrelevant at the level of present, qualitative considerations. Here belongs the hierarchical pattern and the partly degenerate pattern with opposite CP parities of the (almost) degenerate neutrinos. For those mass matrices, quantum corrections cannot substantially alter the structures present at the scale \(M\), so the agreement with experimental data has to be assured by the boundary values at \(M\).

Equal masses \(m_{\nu_1} = -m_{\nu_2}\), at the scale \(M\) are possible. Quantum corrections can explain the observed \(\Delta m^2\).

The other group consists of partly degenerate pattern with the same CP parities of the (almost) degenerate neutrinos and of the (approximately) degenerate structures. Large quantum corrections can originate either from the RG evolution or from low energy threshold corrections (in the MSSM). They never change qualitatively the mass eigenvalue pattern, although they may explain their observed splitting. However, large quantum corrections always lead to a “fixed point” relation for the mixing angles. The sufficient condition is that for at least one pair of neutrinos \((m_{\nu_a} + m_{\nu_b})/(m_{\nu_a} - m_{\nu_b}) \gg 1\). It is interesting that the transition from small (qualitatively irrelevant) quantum corrections to the “fixed point” behaviour is very abrupt. So, to a good approximation there are those two and only two physical situations.

A fixed point relation is always one equation for three angles (if CP is conserved) and makes their low energy values dependent on only two boundary conditions for the angles at the scale \(M\). If the dominant quantum corrections come from the RG evolution (i.e. originate from the large \(\tau\) Yukawa coupling) or also from several configurations of the low energy threshold corrections, the “fixed point” relation links small \(\theta_{13}\) angle (constrained by the CHOOZ experiment) to a small angle responsible for the mixing of solar neutrinos. If presently favoured bimaximal mixing was confirmed by future experimental data, all mass patterns leading to large RG corrections and the regions of the MSSM parameter space leading to the same “fixed point” relation would be ruled out.

There are, however, other “fixed point” relations, generated by some special sfermion
mass configurations and/or by flavour non-conserving effects in the slepton sector, that are consistent with bimaximal mixing and small $\theta_{13}$ angle. Such solutions are phenomenologically interesting as, at the same time, they explain the observed mass squared differences, as the effect of quantum corrections, with degenerate spectrum at the scale $M$. However, the simple Ansatz at the scale $M$ needs a deeper theoretical justification.

Quantum corrections do not explain the origin of the neutrino masses and do not replace its theory. Nevertheless, they are important piece of the overall picture. They will constrain strongly the acceptable mass structures once the experimental ambiguities are resolved.

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Appendix A

In Section 3, instead of the familiar Dirac bispinors we used the two-component Weyl spinor notation. It is particularly convenient for dealing with Majorana particles. Here we explain this notation shortly. More details can be found in modern textbooks on QFT (see e.g. [16, 54]).

In four dimensions, the Lorentz group or more precisely its covering group $SL(2, C)$ has two non-equivalent complex two-dimensional representations denoted as $(1/2, 0)$ and $(0, 1/2)$. The Grassmann fields (or fermionic field operators) transforming according to these representations are conventionally written as $\lambda_\alpha$ and $\bar{\chi}^\dagger$ and called left- and right-handed spinors, respectively. Since the complex conjugation of a left-handed spinor $\lambda_\alpha$

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22It has also been discussed in the literature [52, 53] that quantum corrections generated in the full theory above the scale $M$, can break the degeneracy already at the scale $M$ and lead to correct mass squared differences and mixing angles at the electroweak scale. For an interesting link between neutrino masses and flavour changing processes see ref. [63].

23Somewhat incorrectly; properly they should be called left- and right-chiral.
transforms as a right-handed one \( ((\lambda_\alpha)^* \sim \bar{\lambda}_\dot{\alpha}) \), the fermion content of any Lagrangian can be specified by listing only the left-handed spinors used for its construction.

If the two left-handed fields \( \lambda_\alpha \) and \( \chi_\beta \) transform as representations \( R \) and \( R^* \), respectively under the final unbroken symmetry group (global or local) of the theory, they can be combined to form a Dirac bispinor:

\[
\psi(\lambda) = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}_{\dot{\alpha}} \end{pmatrix}, \quad \bar{\psi}(\lambda) = \begin{pmatrix} \chi^\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.1}
\]

transforming as \( R \) and \( R^* \), respectively. The raising and lowering of Weyl spinor indices is done with the help of the antisymmetric tensors \( \epsilon^{\alpha\beta}, \epsilon^{\dot{\alpha}\dot{\beta}} \) and \( \epsilon_{\dot{\beta}\beta} \):

\[
\lambda^\alpha = \epsilon^{\alpha\beta} \lambda^\beta, \quad \bar{\lambda}_{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{\dot{\beta}}, \quad \bar{\lambda}_{\dot{\alpha}} = \bar{\lambda}_{\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}}. \tag{A.2}
\]

The two kinetic terms for \( \lambda \) and \( \chi \) can be then rewritten in the familiar form

\[
\mathcal{L}_{\text{kin}} = i \bar{\lambda} \gamma^\mu \partial_\mu \lambda + i \bar{\chi} \sigma^\mu \partial_\mu \lambda + \text{(total der)} = i \bar{\psi}(\lambda) \gamma^\mu \partial_\mu \psi(\lambda) \tag{A.3}
\]

where the Dirac matrices \( \gamma^\mu \) in the Weyl representation are constructed as

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma_{\mu\dot{\alpha}} \\ \bar{\sigma}_{\mu\alpha} & 0 \end{pmatrix} \tag{A.4}
\]

with \( \sigma^\mu \equiv (I, \sigma), \quad \bar{\sigma}^\mu \equiv (I, -\sigma) \) (\( \sigma \)'s are the Pauli matrices). For such a pair of Weyl fields also a Dirac mass term can be constructed

\[
\mathcal{L}_{\text{mass}} = -m \left( \lambda^\alpha \chi_\alpha + m \bar{\lambda}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} \right) = -m \bar{\psi}(\lambda) \psi(\lambda) \tag{A.5}
\]

If the field \( \lambda \) (\( \chi \)) has no left-handed partner transforming in the complex conjugate representation \( R^* \) (\( R \)), it is convenient to introduce chiral Dirac bispinors

\[
\psi_{(\lambda)L} = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}, \quad \psi_{(\lambda)R} = \begin{pmatrix} 0 \\ \bar{\chi}_{\dot{\alpha}} \end{pmatrix}. \tag{A.6}
\]

For chiral Dirac bispinors e.g. \( \psi_{(\lambda)L} = P_L \psi_{(\lambda)\bar{L}} \) etc., where \( P_L \equiv (1 - \gamma^5)/2 \). Note that (see eq.(A.1)),

\[
\bar{\psi}_{(\lambda)L} = \begin{pmatrix} 0 \\ \bar{\lambda}_{\dot{\alpha}} \end{pmatrix}, \quad \bar{\psi}_{(\lambda)R} = \begin{pmatrix} \chi^\alpha \\ 0 \end{pmatrix}. \tag{A.7}
\]

The typical Yukawa coupling of a scalar field \( \phi \) in the representation \( R_\phi \) and two left-handed Weyl spinors \( \lambda \) and \( \chi \) transforming as representations \( R_\lambda \) and \( R_\chi \), respectively (such that \( 1 \subset R_\phi \times R_\lambda \times R_\chi \)) can be written as (we omit the Clebsch-Gordan coefficients)

\[
\mathcal{L}_{\text{Yuk}} = -Y \phi \lambda \chi - Y^* \phi^\dagger \bar{\lambda} \bar{\chi}
= -Y \phi \psi_{(\lambda)\bar{R}} \bar{\psi}_{(\lambda)\bar{L}} - Y^* \phi^\dagger \psi_{(\chi)\bar{L}} \bar{\psi}_{(\chi)\bar{R}} \tag{A.8}
\]
The Yukawa part of the SM Lagrangian \( (3.1) \) and the SM mass terms \( (3.3) \) are the example of \( (A.8) \) and \( (A.5) \), respectively, with fields \( u^c, d^c \) and \( e^c \) playing the role of \( \chi \), and \( q \) and \( l \) (or \( u, d \) and \( e \)) playing the role of \( \lambda \).

Finally, Weyl spinor fields \( \lambda_\alpha \) which are singlets of all unbroken symmetries of the theory can form 4-component Majorana bispinors

\[
\psi_{(\lambda)\text{Maj}} = \begin{pmatrix} \lambda^\alpha \\ \bar{\lambda} \end{pmatrix}.
\]

Of course in this case \( \psi_{(\lambda)} = C \bar{\psi}_{(\lambda)}^T \equiv \psi_{(\lambda)}^c \) which means that the field is self-conjugate. One also has (up to raising or lowering indices)

\[
\lambda = \psi_{(\lambda)L} = \bar{\psi}_{(\lambda)R} \quad \text{and} \quad \bar{\lambda} = \psi_{(\lambda)R} = \bar{\psi}_{(\lambda)L}.
\]

For such a field a Majorana mass term can be formed

\[
\mathcal{L}_{\text{Maj}} = -\frac{1}{2} m (\lambda \lambda + \bar{\lambda} \bar{\lambda}) = -\frac{1}{2} m \left( \bar{\psi}_{(\lambda)R} \psi_{(\lambda)L} + \bar{\psi}_{(\lambda)L} \psi_{(\lambda)R} \right)
\]

\[
\equiv -\frac{1}{2} m \bar{\psi}_{(\lambda)} \psi_{(\lambda)} \equiv -\frac{1}{2} m \bar{\psi}_{(\lambda)}^T C \psi_{(\lambda)}
\]

where \( C \) is the charge conjugation matrix. The Majorana mass term \( (3.8) \) is precisely of this form with \( \nu^c \) playing the role of \( \lambda \) and the Yukawa coupling \( (3.7) \) is usually written as

\[
\Delta \mathcal{L}_{\text{Yuk}} = -\epsilon_{ij} H_i \overline{\psi}_{(\nu^c)L} Y_{\nu \psi_{(l)L}} - \epsilon_{ij} H_i^* \overline{\psi}_{(\lambda)L} Y_{\nu \psi_{(l)L}}^T
\]

### Appendix B

In this Appendix we recall the well known RGEs for Yukawa coupling matrices defined in eq. \( (3.1) \) in the SM and in the MSSM. In the SM they read [25, 24]:

\[
\frac{d}{dt} Y_u = Y_u \left[ -8 g_3^2 - \frac{9}{4} g_2^2 - \frac{17}{12} g_Y^2 + T + \frac{3}{2} (Y^\dagger_u Y_u - Y^\dagger_d Y_d) \right],
\]

\[
\frac{d}{dt} Y_d = Y_d \left[ -8 g_3^2 - \frac{9}{4} g_2^2 - \frac{5}{12} g_Y^2 + T + \frac{3}{2} (Y^\dagger_d Y_d - Y^\dagger_u Y_u) \right],
\]

\[
\frac{d}{dt} Y_e = Y_e \left[ -\frac{9}{4} g_2^2 - \frac{15}{4} g_Y^2 + T + \frac{3}{2} Y^\dagger_e Y_e \right],
\]

where

\[
t \equiv \frac{1}{16\pi^2} \ln \left( \frac{Q}{M_Z} \right),
\]

\[
T \equiv \text{Tr} \left( 3 Y^\dagger_u Y_u + 3 Y^\dagger_d Y_d + Y^\dagger_e Y_e \right)
\]

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and the gauge couplings $g_3$, $g_2$ and $g_Y$ evolve according to

$$\frac{d}{dt} g_i = b_i g_i^3 \quad i = 3, 2, Y$$

(B.4)

with $b_3 = -7$, $b_2 = -19/6$ and $b_Y = 41/6$.

In the MSSM one finds instead:

$$\frac{d}{dt} Y_u = Y_u \left[ -\frac{16}{3} g_2^2 - 3 g_2^2 - \frac{13}{9} g_Y^2 + \text{Tr} \left( 3 Y_u^\dagger Y_u + 3 Y_u^\dagger Y_u + Y_d^\dagger Y_d \right) \right],$$

$$\frac{d}{dt} Y_d = Y_d \left[ -\frac{16}{3} g_2^2 - 3 g_2^2 - \frac{7}{9} g_Y^2 + \text{Tr} \left( 3 Y_d^\dagger Y_d + Y_e^\dagger Y_e + 3 Y_d^\dagger Y_d + Y_u^\dagger Y_u \right) \right],$$

$$\frac{d}{dt} Y_e = Y_e \left[ -3 g_2^2 - 3 g_Y^2 + \text{Tr} \left( 3 Y_d^\dagger Y_d + Y_e^\dagger Y_e + 3 Y_e^\dagger Y_e \right) \right],$$

(B.5)

and the factors $b_i$ change to $b_3 = -3$, $b_2 = +1$ and $b_Y = 11$.

For completeness we give here also the RGEs above the scale $M$ i.e. for the theory whose set of fermion fields includes additional three $SU_L(2) \times U_Y(1)$ singlet neutrino fields $\nu^A$ ($A = 1, 2, 3$) and whose Lagrangian is identical to the one for the effective theory valid below $M$ except for the Yukawa interaction (B.7) and the Majorana mass terms (B.8).

If the theory above the $M$ scale extends the SM, then $T$ given in eq. (B.3) has to be replaced by [39, 35]

$$T \equiv \text{Tr} \left( 3 Y_u^\dagger Y_u + 3 Y_d^\dagger Y_d + Y_e^\dagger Y_e + Y_\nu^\dagger Y_\nu \right),$$

(B.6)

the last equation in (B.1) should be replaced by

$$\frac{d}{dt} Y_e = Y_e \left[ -\frac{9}{4} g_2^2 - \frac{15}{4} g_Y^2 + T + \frac{3}{2} \left( Y_e^\dagger Y_e - Y_\nu^\dagger Y_\nu \right) \right]$$

(B.7)

and the neutrino Yukawa matrix RGE reads:

$$\frac{d}{dt} Y_\nu = Y_\nu \left[ -\frac{9}{4} g_2^2 - \frac{3}{4} g_Y^2 + T - \frac{3}{2} \left( Y_e^\dagger Y_e - Y_\nu^\dagger Y_\nu \right) \right].$$

(B.8)

In addition, the Majorana mass matrix also runs [52]:

$$\frac{d}{dt} M^{KL}_{\text{Maj}} = M^{KL}_{\text{Maj}} \left( Y_\nu^\dagger Y_\nu \right)^{LJ} + \left( Y_\nu^\dagger Y_\nu \right)^{KJ} M^{LJ}_{\text{Maj}}.$$

(B.9)

If the low energy theory is the MSSM, then [39, 35] in the first equation of (B.3)

$$\text{Tr} \left( 3 Y_u^\dagger Y_u \right) \rightarrow \text{Tr} \left( 3 Y_u^\dagger Y_u + Y_\nu^\dagger Y_\nu \right),$$

(B.10)

the last equation of (B.3) is replaced by

$$\frac{d}{dt} Y_e = Y_e \left[ -3 g_2^2 - 3 g_Y^2 + \text{Tr} \left( 3 Y_d^\dagger Y_d + Y_e^\dagger Y_e + 3 Y_e^\dagger Y_e \right) \right]$$

(B.11)
and the neutrino Yukawa matrix RGE reads:

$$\frac{d}{dt} Y_\nu = Y_\nu \left[ -3g_2^2 - g_Y^2 + \text{Tr} \left( 3Y_u^\dagger Y_u + Y_\nu^\dagger Y_\nu \right) + 3Y_e^\dagger Y_e + Y_\nu^\dagger Y_\nu \right].$$  \hspace{1cm} (B.12)

Finally, the Majorana mass matrix running is dictated by

$$\frac{d}{dt} M_{\text{Maj}}^{KL} = 2M_{\text{Maj}}^{KJ} \left( Y_\nu Y_\nu^\dagger \right)^{LJ} + 2 \left( Y_\nu Y_\nu^\dagger \right)^{KJ} M_{\text{Maj}}^{IL}. \hspace{1cm} (B.13)$$

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