Arithmetic homology and an integral version of Kato’s conjecture

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Summary. We define an integral Borel-Moore homology theory over finite fields, called arithmetic homology, and an integral version of Kato homology. Both types of groups are expected to be finitely generated, and sit in a long exact sequence with higher Chow groups of zero-cycles.

1 Introduction

For a separated scheme $X$ of finite type over a field $k$, let $\mathcal{Z}^c(0)(X)$ be Bloch’s complex of relative zero-cycles, generated in degree $-i$ by cycles of dimension $i$ on $X \times \Delta^i$ in good position. The higher Chow groups $CH_0(X, i)$ are defined as the homology $H_i(\mathcal{Z}^c(0)(X))$ of $\mathcal{Z}^c(0)(X)$. Varying $X$, we obtain a complex of etale sheaves $\mathcal{Z}^c(0)$ and can consider its etale hypercohomology $H_i(X_{\text{et}}, \mathcal{Z}^c(0))$. If $k$ is algebraically closed, then it is a consequence of the Beilinson-Lichtenbaum conjecture that $CH_0(X, i) \cong H_i(X_{\text{et}}, \mathcal{Z}^c(0))$ [9, Thm.3.1], but in general, these groups are not isomorphic. Over a finite field, Kato [19] defined for each $m$ a complex with homology $H^K_i(X, \mathbb{Z}/m)$ and conjectured that if $X$ is smooth, proper and connected, then $H^K_i(X, \mathbb{Z}/m)$ vanishes for $i > 0$, and $H^K_0(X, \mathbb{Z}/m) \cong \mathbb{Z}/m$. Jannsen and Saito [15] observed that Kato homology measures the difference between the finite coefficient versions $CH_0(X, i, \mathbb{Z}/m)$ and $H_i(X_{\text{et}}, \mathcal{Z}^c/\mathbb{Z}/m(0))$, and proved Kato’s conjecture assuming resolution of singularities [16].

In this paper, we construct Borel-Moore homology groups $H^c_i(X_{\text{ar}}, \mathbb{Z})$, which are a substitute for the (pathological) etale higher Chow groups $H_i(X_{\text{et}}, \mathcal{Z}^c(0))$, and define an integral version of Kato’s complex whose homology groups $H^K_i(X, \mathbb{Z})$ measure the difference between $CH_0(X, i)$ and $H^c_i(X_{\text{ar}}, \mathbb{Z})$. The analog of Kato’s conjecture is that if $X$ is smooth, proper and connected, then $H^K_i(X, \mathbb{Z})$ vanishes for $i > 0$, and $H^K_0(X, \mathbb{Z}) = \mathbb{Z}$.

The Borel-Moore homology theory, which we call arithmetic homology, is constructed by applying the Weil-etale formalism of Lichtenbaum to $\mathcal{Z}^c(0)$:

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Given a scheme $X$ over the finite field $\mathbb{F}_q$, $H^*_G(X_{\text{ar}}, \mathbb{Z})$ is the $-i$th cohomology group of the complex $R\Gamma_G R\Gamma(X_{\text{et}}, \mathbb{Z}^c(0))[1]$, where $X = X \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$, and $G$ is the Weil group of $\mathbb{F}_q$ acting on $\overline{\mathbb{F}_q}$. These groups are expected to be finitely generated. More precisely, we have

**Theorem 1.1** Let $X$ be smooth and proper. Assume resolution of singularities up to the dimension of $X$ and the Beilinson-Lichtenbaum conjecture (see below). Then the following statements are equivalent:

a) $CH_0(X, i)_\mathbb{Q} = 0$ for all $i > 0$.

b) There is an isomorphism $CH_0(X, i) \cong H^i_{\text{ar}}(X, \mathbb{Z})$ of finitely generated abelian groups for all $i \geq 0$.

c) There are short exact sequences for all $i$,

$$0 \to CH_0(\overline{X}, i + 1)_G \to CH_0(X, i) \to CH_0(\overline{X}, i)^G \to 0.$$

The vanishing of the Chow groups is a special case of Parshin’s conjecture $K^i_s(X) = 0$ for $i > 0$, and in particular it follows from Tate’s conjecture together with Beilinson’s conjecture that rational and homological equivalence agree up to torsion [5] over finite fields, or from finite dimensionality of smooth and projective schemes over $\mathbb{F}_q$ in the sense of Kimura-O’Sullivan [10].

The integral analog of Kato homology comes into play when comparing higher Chow groups with arithmetic homology, for not necessarily smooth or proper schemes over $\mathbb{F}_q$: Consider the complex

$$\cdots \to \bigoplus_{x \in X(i)} K^M_s(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q})_G \to \bigoplus_{x \in X(1)} (k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q})^X_0 \to \cdots \to \oplus_{x \in X(0)} \mathbb{Z},$$

where $X(i)$ denote the points of $X$ of dimension $s$, and $K^M_s(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q})$ is the group of Frobenius coinvariants of the Milnor $K$-group of the finite product of fields $k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. The maps in the complex are induced by boundary maps in localization sequences of higher Chow groups via the identification $K^M_s(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}) \cong H^s(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}, \mathbb{Z}(s))$. The group $H^K_i(X, \mathbb{Z})$ is the $i$th homology of this complex. Assuming the Beilinson-Lichtenbaum conjecture, we have a long exact sequence

$$\cdots \to H^K_i(X, \mathbb{Z}) \xrightarrow{m} H^K_i(X, \mathbb{Z}) \to H^K_i(X, \mathbb{Z}/m) \to \cdots,$$

which justifies calling the groups $H^K_i(X, \mathbb{Z})$ an integral version of Kato homology. As an analog of Kato’s conjecture, we propose the following

**Conjecture 1.2** If $X$ is smooth, proper and connected, then $H^K_i(X, \mathbb{Z}) = 0$ for $i > 0$ and $H^K_0(X, \mathbb{Z}) \cong \mathbb{Z}$.

It is easy to use the known results on the (torsion) Kato conjecture to show that the conjecture is true in degree 0, and that $H^K_i(X, \mathbb{Z}) \cong CH_0(X, i)_\mathbb{Q}$ for smooth and proper $X$ and $i = 1, 2$. Regarding the conjecture and the relationship between higher Chow groups and arithmetic homology, we show
Theorem 1.3 Assuming resolution of singularities and the Beilinson-Lichtenbaum conjecture, the following statements are equivalent:

a) For every smooth and proper $X$ over $\mathbb{F}_q$, $CH_0(X, i) = 0$ for $i > 0$.

b) Conjecture [1.2] holds, and for every separated scheme of finite type $X$ over $\mathbb{F}_q$, there is a long exact sequence of finitely generated groups

$$\cdots \to CH_0(X, i) \to H^{i+1}_{c}(X_{ar}, \mathbb{Z}) \to H^{i+1}_K(X, \mathbb{Z}) \to CH_0(X, i-1) \to \cdots.$$ 

The exact sequence of the Theorem exists unconditionally in degrees $i \leq 1$.

We show that Conjecture 1.2 holds for curves, i.e. there is an exact sequence

$$0 \to (k(C) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q)^{\chi} \to \bigoplus_{x \in C^{(0)}} \mathbb{Z} \to \mathbb{Z} \to 0.$$ 

(1)

for smooth and proper curves $C$. Since the sequence comparing Weil-etale to etale cohomology [6] gives a short exact sequence

$$0 \to (k(C) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q)^{\chi} \to k(C)_{\bar{\mathbb{Q}}} \to Br(k(C) \to 0),$$

we obtain $(k(C) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q)^{\chi} \otimes \mathbb{Q}/\mathbb{Z} \cong Br(k(C)$, and (1) is an integral version of the classical short exact sequence for the Brauer group of $k(C)$. For an arbitrary curve $C$ over $\mathbb{F}_q$, we obtain $H_0^0(C_{ar}, \mathbb{Z}) \cong H_0^K(C, \mathbb{Z})$, there is a short exact sequence

$$0 \to CH_0(C) \to H_1^c(C_{ar}, \mathbb{Z}) \to H_1^K(C, \mathbb{Z}) \to 0,$$

and $CH_0(C, i) \cong H_{i+1}^c(C_{ar}, \mathbb{Z})$ for $i \geq 1$. The latter groups are finitely generated for $i = 1$ and zero for $i > 1$. For curves, Kato homology is easy to calculate recursively; for example if $C$ is proper with dual graph $\Gamma_C$, then $H_i^K(C, \mathbb{Z}) = H_i(\Gamma_C, \mathbb{Z})$.

Arithmetic homology can be applied to study abelian class field theory of proper schemes. The main observation is that the group $H_1^c(X_{ar}, \mathbb{Z})$ (which is conjecturally finitely generated) becomes isomorphic to the abelianized fundamental group $\pi_1(X)^{ab}$ after profinite completion, and the reciprocity map factors as

$$CH_0(X) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q \to \pi_1(X)^{ab}.$$ 

Notation: For an abelian group $A$, $A^\wedge = \lim A/n$ is the profinite completion, and $A^* = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ the Pontrjagin dual. All schemes over a field are separated and of finite type; from section 3 we fix a finite field as the base field.

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2 Higher Chow groups

For a scheme $X$ over a field $k$, Bloch’s higher Chow complex $z_n(X, \ast)$ is defined as follows [1]. In degree $-i$, it is the free abelian group $z_n(X, i)$ generated by cycles of dimension $n + i$ on $X \times_k \Delta^i$ which meet all faces properly. The differentials are given by taking the alternating sum of intersection with face maps. Higher Chow groups $CH_n(X, i)$ are defined as the homology of this complex. We let $Z^c(n)_X = z_n(-, -)[2n]$ be the complex of etale sheaves on $X$ with $z_n(-, -2n-i)$ in degree $i$. For a proper map $f : X \to Y$ we have a push-forward $f_*Z^c(n)_X \to Z^c(n)_Y$, and for a flat, equidimensional map $f : X \to Y$ of relative dimension $d$, we have a pull-back $f^*Z^c(n)_Y \to Z^c(n+d)_X[2d]$. If $X$ is smooth of dimension $d$, then there is a quasi-isomorphism of complexes of Zariski sheaves $Z^c(n) \cong Z(d-n)[2d]$, where the right hand side is the motivic complex of Voevodsky [28]. For a finitely generated field $F$ over $k$, we define $CH_n(F, i) = \text{colim}_U CH_n(U, i)$, where the colimit runs through $U$ of finite type over $k$ with field of functions $F$. If $F$ has transcendence degree $d$ over $k$, then $CH_n(F, i) \cong H^{2d-2n-i}(F, Z(d-n))$, where the right hand side is motivic cohomology, and vanishes for $i < d-n$, and agrees with $K_n^{d-n}(F)$ for $i = d-n$. As a formal consequence of localization for higher Chow groups, one obtains an isomorphism

$$CH_n(X, i-2n) = H_i(X_{\text{Zar}}, Z^c(n)) =: H^c_i(X_{\text{Zar}}, Z(n)) \quad (2)$$

and spectral sequences

$$E_1^{s,t} = \oplus_{x \in X_{(s)}} H^{-s-1}(k(x), Z(s-n)) \Rightarrow H^{c,s+t}_i(X_{\text{Zar}}, Z(n)). \quad (3)$$

In particular, $H^c_i(X_{\text{Zar}}, Z(n)) = 0$ for $i < n$. For an abelian group $A$, we define

$$H^c_i(X_{\text{et}}, A(n)) = H^{-i}R\Gamma(X_{\text{et}}, A \otimes Z^c(n)). \quad (4)$$

If $A = \mathbb{Q}$, then $H^c_i(X_{\text{et}}, \mathbb{Q}(n)) = H^c_i(X_{\text{Zar}}, \mathbb{Q}(n)) = CH_n(X, i-2n)\mathbb{Q}$. In [2], we proved that for every integer $m$ and every scheme $f : X \to k$ over a perfect field $k$, there is a quasi-isomorphism

$$Rf^!Z/m \cong Z^c/m(0). \quad (5)$$

Here $Rf^!$ is the extraordinary inverse image of SGA 4 XVIII. Hence the etale homology groups $H^c_i(X_{\text{et}}, Z/m(0))$ with coefficients $Z^c/m(0)$ agree with usual etale homology groups $H_i(X_{\text{et}}, Z/m) := H^{-i}(X_{\text{et}}, Rf^!Z/m)$ of Laumon [21]. For a finitely generated field $F$ over $k$, we define $H^c_i(F_{\text{et}}, A(n)) = \text{colim}_U H^c_i(U_{\text{et}}, A(n))$.

The Beilinson-Lichtenbaum conjecture over $k$ in homological weight $n$ says that if $X$ is a smooth scheme of dimension $d$ over $k$, then the canonical map

$$CH_n(X, i-2n) = H^c_i(X_{\text{Zar}}, Z(n)) \to H^c_i(X_{\text{et}}, Z(n))$$
is an isomorphism for \( i \geq n+d -1 \). By considering cohomological dimension, it then must be an isomorphism for all \( i \) if \( k \) is algebraically closed and \( n \leq 0 \). As a special case, the conjecture implies that \( H^{d-n+1}(F_{et}, \mathbb{Z}(d-n)) = 0 \) for fields \( F \) of transcendence degree \( d \) over \( k \); this statement is often called "Hilbert’s Theorem 90". Over an algebraically closed field, Suslin [25] shows that the groups \( CH_0(X, i, \mathbb{Z}/m) \) and \( H^e_0(X_{et}, \mathbb{Z}/m(0)) \) are isomorphic for char \( k \mid m \), but it is not clear that the isomorphism is induced by the canonical change of topology map. In [11], it is shown that the canonical map is an isomorphism if \( m \) is a power of the characteristic. In [9], we use Suslin’s theorem to show the following result.

**Proposition 2.1** Let \( k \) be a perfect field and \( n \leq 0 \).

a) (Localization) For a closed embedding \( i : Z \to X \), the canonical map \( \mathbb{Z}_c^e(n) \to R^i\mathbb{Z}_X^e(n) \) is a quasi-isomorphism.

b) (Homotopy formula) If \( p : X \times \mathbb{A}^m \to X \) is the projection, then we have a quasi-isomorphism of complexes of etale sheaves \( Rp_*\mathbb{Z}_X^e(n) \cong \mathbb{Z}_X^e(n-m) \).

c) (Niveau spectral sequence) There is a spectral sequence

\[
E_{s,t}^1 = \bigoplus_{x \in X(n)} H^{s-t}(k(x)_{et}, \mathbb{Z}(s-n)) \Rightarrow H^{e}_{s+t}(X_{et}, \mathbb{Z}(n)).
\]

In particular, \( H^e_{s}(X_{et}, \mathbb{Z}(n)) = 0 \) for \( i < n \).

The last statement follows because \( s+t \leq n \) implies that \( s \leq n \) or \( t < 0 \), both of which imply that \( E_1^{s,t} \) vanishes. Note that by the homotopy formula, the Beilinson-Lichtenbaum conjecture in homological weight \( n \leq 0 \) implies the Beilinson-Lichtenbaum conjecture in all weights less than \( n \).

### 3 Arithmetic homology with compact support

We fix a finite field \( \mathbb{F}_q \) with Galois group \( \hat{G} = \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \) and let \( G \subset \hat{G} \) be the Weil group of \( \mathbb{F}_q \), i.e. the subgroup of \( \hat{G} \) generated by the Frobenius endomorphism \( \varphi \). Given a separated scheme of finite type \( X \) over \( \mathbb{F}_q \), let \( \overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q \). For an abelian group \( A \), we define arithmetic homology groups with compact support and coefficients in \( A \) as the homology groups of the complex \( R^i\Gamma \Gamma(X_{et}, A \otimes \mathbb{Z}^e(n))[1] \), where \( G \) acts on \( \overline{\mathbb{F}}_q \), so that

\[
H^e_i(X_{et}, A(n)) = H^{1-i}R^i\Gamma \Gamma(X_{et}, A \otimes \mathbb{Z}^e(n)).
\]

The Leray spectral sequence for composition of functors degenerates into short exact sequences

\[
0 \to H^e_i(\overline{X}_{et}, \mathbb{Z}(n))_G \to H^e_i(X_{et}, \mathbb{Z}(n)) \to H^e_{i-1}((\overline{X}_{et}, \mathbb{Z}(n)))^G \to 0.
\]

In particular, \( H^e_i(X_{et}, \mathbb{Z}(n)) = 0 \) for \( i < n \leq 0 \) by Proposition 2.1. If \( X \) is smooth of dimension \( d \), then the quasi-isomorphism \( \mathbb{Z}^e(n) \cong \mathbb{Z}(d-n)[2d] \) implies
\[ H^c_i(X, n) \cong H^{2d+1-i}(X, \mathbb{Z}(d-n)), \quad (8) \]

where the right hand side are the Weil-etale cohomology groups of \[22, 6\]. (For arbitrary \(X\) of finite type over \(\mathbb{F}_q\), one has to replace etale cohomology by eh-cohomology and Weil-etale cohomology by arithmetic cohomology of \[7\] to obtain well-behaved cohomology groups). Recall from \[6, \text{Thm.7.1}\] that for every smooth \(X\) over \(\mathbb{F}_q\), there is a long exact sequence

\[ \cdots \to H^i(\text{et}, \mathbb{Z}(p)) \to H^i(X, \mathbb{Z}(p)) \to H^{i-1}(\text{et}, \mathbb{Q}(p)) \to \cdots. \quad (9) \]

The same proof shows

**Theorem 3.1** There are long exact sequences

\[ \cdots \to H^c_{i-1}(\text{et}, Z(n)) \to H^c_i(X, Z(n)) \]
\[ \to CH_n(X, i-2n)_{\mathbb{Q}} \to H^c_{i-2}(\text{et}, Z(n)) \to \cdots, \]

hence for every integer \(m \geq 1\) an isomorphism

\[ H^c_i(X, m(n)) \cong H^c_{i+1}(X, \mathbb{Z}/m(n)). \]

With rational coefficients, the long exact sequence splits, and

\[ H^c_i(X, \mathbb{Q}(n)) \cong CH_n(X, i-2n)_{\mathbb{Q}} \oplus CH_n(X, i-2n-1)_{\mathbb{Q}}. \]

\(\square\)

From now on, we assume that \(n \leq 0\). Then for a closed subscheme \(Z\) of \(X\) with open complement \(U\) we have by Proposition 2.1 localization sequences

\[ \cdots \to H^c_i(Z, Z(n)) \to H^c_i(X, Z(n)) \to H^c_i(U, Z(n)) \to \cdots \quad (10) \]

**Corollary 3.2** Let \(n \leq 0\).

a) *(Projective bundle formula)* Let \(P^n_X \to X\) be a projective bundle of relative dimension \(r\). Then

\[ H^c_i(P^n \times X, Z(n)) \cong \oplus_{j=0}^r H^c_{i-2j}(X, Z(n-j)). \]

b) *(Homotopy formula)* For every \(r \geq 0\),

\[ H^c_i(\mathbb{A}^r \times X, Z(n)) \cong H^c_{i-2r}(X, Z(n-r)). \]

**Proof.** The homotopy formula follows from Prop. 2.1 via (7), and the projective bundle formula can be derived from this using localization and induction. \(\square\)

The Beilinson-Lichtenbaum conjecture over \(\mathbb{F}_q\) implies that, for \(n \leq 0\),

\[ R\Gamma(X, A \otimes \mathbb{F}(n)) \] has the explicit representative
Arithmetic cohomology of fields

Assuming the Beilinson-Lichtenbaum conjecture in homological
Lemma 3.4
Corollary 3.3
Example. Let \( P \) be a connected, zero-dimensional scheme over \( \mathbb{F}_q \). Then \( CH_0(\mathbb{F}_q) \) for \( i = 0 \) and zero otherwise, where \( c \) is the number of connected components of \( P \). Since the Galois group permutes the connected components of \( P \), (11) implies \( H^c_0(\mathbb{F}_q, \mathbb{Z}) \cong \mathbb{Z} \) for \( i = 0, 1 \), and zero otherwise.

3.1 Arithmetic cohomology of fields

For a finitely generated field \( k \) over \( \mathbb{F}_q \), \( k \otimes_{\mathbb{F}_q} \mathbb{F}_q \) is a finite product of algebraic extensions of \( k \), finitely generated over \( \mathbb{F}_q \), on which \( G \) acts. We define arithmetic cohomology \( H^i(k_{\mathbb{Q}_l}, A(n)) \) as the cohomology of \( R\Gamma_R(G, (k \otimes_{\mathbb{F}_q} \mathbb{F}_q)_{et}, A \otimes \mathbb{Z}(n)) \). There exist sequences (9) comparing arithmetic cohomology to étale cohomology. If the transcendence degree of \( k \) over \( \mathbb{F}_q \) is \( d \), then we can write \( k \) as the colimit of smooth schemes of dimension \( d \) over \( \mathbb{F}_q \), \( k = \text{colim} \ U \), and \( H^i(k_{\mathbb{Q}_l}, \mathbb{Z}(d)) \cong \text{colim} \ H^i(U_{\mathbb{Q}_l}, \mathbb{Z}(d)) \cong \text{colim} \ H^c_{2d+1-i}(U_{\mathbb{Q}_l}, \mathbb{Z}(0)) \) by (5). As a formal consequence of localization (10), we obtain

\begin{equation}
E^1_{s,t} = \bigoplus_{x \in X_{et}} H^{s-t}(k_{\mathbb{Q}_l}, \mathbb{Z}(s-n)) \Rightarrow H^c_{s+t+1}(k_{\mathbb{Q}_l}, \mathbb{Z}(s)).
\end{equation}

Lemma 3.4 Assuming the Beilinson-Lichtenbaum conjecture in homological weight \( d - s \), then for every field \( k \) with \( d = \text{trdeg}_{\mathbb{F}_q} k \) we have isomorphisms

\begin{align*}
H^{s+1}(k_{\mathbb{Q}_l}, \mathbb{Z}(s)) &\cong K^M_s(k \otimes_{\mathbb{F}_q} \mathbb{F}_q)G; \\
H^{s+2}(k_{\mathbb{Q}_l}, \mathbb{Z}(s)) &\cong H^{s+1}(k \otimes_{\mathbb{F}_q} \mathbb{F}_q, \mathbb{Q}/\mathbb{Z}(s))^G.
\end{align*}

If \( s \geq d \), then the latter group vanishes, and we obtain an exact sequence

\begin{equation}
0 \to K^M_s(k \otimes_{\mathbb{F}_q} \mathbb{F}_q)_G \to K^M_s(k)_G \xrightarrow{\delta} H^{s+1}(k_{\mathbb{Q}_l}, \mathbb{Q}/\mathbb{Z}(s)) \to 0,
\end{equation}

where \( \delta \) is the composition

\begin{equation}
K^M_s(k)_G \to K^M_s(k)_G \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{sym}} H^s(k_{\mathbb{Q}_l}, \mathbb{Q}/\mathbb{Z}(s)) \xrightarrow{e} H^{s+1}(k_{\mathbb{Q}_l}, \mathbb{Q}/\mathbb{Z}(s)),
\end{equation}

with \( \text{sym} \) the symbol map, and \( e \in H^1((\mathbb{F}_q)_{et}, \hat{\mathbb{Z}}) = \text{Hom}_{cont}(\hat{G}, \hat{\mathbb{Z}}) \) the isomorphism sending the Frobenius endomorphism to 1.
Proof. The first statements follow from the Leray spectral sequence

\[ 0 \to H^{i-1}((k \otimes \overline{F}_q)_\text{et}, \mathbb{Z}(s)) \to H^i(k_{\text{ar}}, \mathbb{Z}(s)) \to H^i((k \otimes \overline{F}_q)_\text{et}, \mathbb{Z}(s))^G \to 0 \]

and Hilbert’s Theorem 90. The exact sequence is (9), using the identifications

\[ H^s(k_{\text{et}}, \mathbb{Q}(s)) = K^M_s(k) \quad \text{and} \quad H^{s+2}(k_{\text{et}}, \mathbb{Z}(s)) \cong H^{s+1}(k_{\text{et}}, \mathbb{Q}/\mathbb{Z}(s)). \]

The description of \( \delta \) in (9) is given in [6, Thm.7.1a].

The argument of the Lemma for \( s = 0, 1 \) gives, for any \( k \), exact sequences

\[ 0 \to \mathbb{Z} \to \mathbb{Q} \to \text{Hom}(G_k, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_G(\text{Gal}(k \otimes \overline{F}_q), \mathbb{Q}/\mathbb{Z}) \to 0 \]

\[ 0 \to (k \otimes \overline{F}_q)^+_G \to (k^+)_Q \to \text{Br}(k) \to \text{Br}(k \otimes \overline{F}_q)^G \to 0. \] (14)

Lemma 3.5 Under the hypothesis of the previous Lemma, there are short exact sequences

\[ 0 \to H^{s+1}(k_{\text{ar}}, \mathbb{Z}(s)) \to H^s(k, \mathbb{Q}(s)) \to H^{s+1}(k_{\text{et}}, \mathbb{Q}/\mathbb{Z}(s)) \to 0; \]

and

\[ 0 \to H^{s+1}(k_{\text{et}}, \mathbb{Z}(s)) \xrightarrow{\times m} H^{s+1}(k_{\text{et}}, \mathbb{Z}(s)) \to H^{s+1}(k_{\text{et}}, \mathbb{Z}/m(s)) \to 0. \]

Proof. The first short exact sequence is (13). The long exact coefficient sequence for arithmetic cohomology gives the second short exact sequence, because the boundary map factors

\[ H^s(k_{\text{et}}, \mathbb{Z}/m(s)) \xrightarrow{\delta} H^{s+1}(k_{\text{et}}, \mathbb{Z}(s)) = 0 \]

\[ H^s(k_{\text{et}}, \mathbb{Z}/m(s)) \xrightarrow{\delta} H^{s+1}(k_{\text{et}}, \mathbb{Z}(s)), \]

the upper right group is trivial by Hilbert’s theorem 90. 

4 Finite generation

We fix a finite field \( \overline{F}_q \).

Conjecture \( P_0(X) \): For the smooth and proper scheme \( X \) over \( \overline{F}_q \), the group \( CH_0(X, i) \) is torsion for all \( i > 0 \).

This is a special case of Parshin’s conjecture, because if \( X \) has dimension \( d \), then

\[ CH_0(X, i)_Q = H^{2d-i}(X, \mathbb{Q}(d)) = K_i(X)_Q^{(d)}. \]

If Tate’s conjecture holds and rational equivalence and homological equivalence agree up to torsion for all \( X \), then Conjecture \( P_0(X) \) holds for all \( X \) [7].
Proposition 4.1 For a smooth and proper scheme $X$, the following statements are equivalent:

a) Conjecture $P_0(X)$.

b) The groups $H_i^c(X_{\text{ar}}, \mathbb{Z})$ are finitely generated for all $i$.

c) There are isomorphisms $H_i^c(X_{\text{ar}}, \mathbb{Z}) \otimes \mathbb{Z}_l \xrightarrow{\sim} \lim_{t \to 1} H_{i-1}(X_{\text{et}}, \mathbb{Z}/l^t)$ for all $i$, and all $l$.

d) There are isomorphisms $H_i^c(X_{\text{ar}}, \mathbb{Z}) \otimes \mathbb{Z}_l \xrightarrow{\sim} \lim_{t \to 1} H_{i-1}(X_{\text{et}}, \mathbb{Z}/l^t)$ for all $i$, and some $l$.

Proof. a) $\Rightarrow$ b): If $X$ is smooth and proper, then $H_i^c(X_{\text{ar}}, \mathbb{Z}) \cong H^{2d+1-i}(X_{\text{ar}}, \mathbb{Z}(d))$ by (3). By hypothesis and (9), for $j \leq 2d$,

$$H^j(X_{\text{ar}}, \mathbb{Z}(d)) \cong H^j(X_{\text{et}}, \mathbb{Z}(d)) \cong H^{j-1}(X_{\text{et}}, \mathbb{Q}/\mathbb{Z}(d)),$$

which is finite for $j < 2d$ by the Weil-conjectures and Gabber [4]. By (20), the group $H^{2d}(X_{\text{ar}}, \mathbb{Z}(d)) \cong H^{2d}(X_{\text{et}}, \mathbb{Z}(d))$ agrees with $CH_0(X)$, and is finitely generated. Finally, $H^{2d+1}(X_{\text{ar}}, \mathbb{Z}(d)) \cong \mathbb{Z}$ and $H^i(X_{\text{ar}}, \mathbb{Z}(d)) = 0$ for $i > 2d + 1$ by [4] Thm.7.5.

b) $\Rightarrow$ c): Finite generation implies $H_i^c(X_{\text{ar}}, \mathbb{Z}) \otimes \mathbb{Z}_l \cong \lim_{t \to 1} H_i^c(X_{\text{ar}}, \mathbb{Z}/l^t) \cong \lim_{t \to 1} H_{i-1}(X_{\text{et}}, \mathbb{Z}/l^t)$ by Theorem 3.1 and (5).

d) $\Rightarrow$ a): The group $\lim_{t \to 1} H_{i-1}(X_{\text{et}}, \mathbb{Z}/l^t) = H^{2d-i}(X_{\text{et}}, \mathbb{Z}(d))$ is torsion for $i > 1$ by the Weil-conjectures, and a) follows via Theorem 3.1. \qed

Proposition 4.2 The following statements are equivalent:

a) Conjecture $P_0(X)$ for all smooth and proper $X$.

b) The groups $H_i^c(X_{\text{ar}}, \mathbb{Z})$ are finitely generated for all $i$ and all $X$.

c) There are isomorphisms $H_i^c(X_{\text{ar}}, \mathbb{Z}) \otimes \mathbb{Z}_l \xrightarrow{\sim} \lim_{t \to 1} H_{i-1}(X_{\text{et}}, \mathbb{Z}/l^t)$ for all $i$, all $X$, and all $l$.

d) There are isomorphisms $H_i^c(X_{\text{ar}}, \mathbb{Z}) \otimes \mathbb{Z}_l \xrightarrow{\sim} \lim_{t \to 1} H_{i-1}(X_{\text{et}}, \mathbb{Z}/l^t)$ for all $i$, all $X$, and some $l$.

Proof. Following the proof of Prop. 4.1 it suffices to show that finite generation of $H_i^c(X_{\text{ar}}, \mathbb{Z})$ for smooth and proper $X$ implies finite generation for all $X$. By localization, induction on the number of irreducible components, and induction on the dimension, we reduce to the case that $X$ is irreducible, and see that finite generation for $X$ and any of its open subschemes $V$ are equivalent. Let $X' \to X$ be an alteration such that $X'$ is an open subscheme of a smooth and proper scheme $T$. We can assume that there is an open subset $U \subseteq X$ such that $U' = U \times_X X' \to U$ is Galois with group $A$. Since $H_i^c(T_{\text{ar}}, \mathbb{Z})$ is finitely generated, so is $H_i^c(U'_{\text{ar}}, \mathbb{Z})$. But there is a spectral sequence

$$E^2_{s,t} = H_s(A, H_t^c(U_{\text{ar}}', \mathbb{Z}(n))) \Rightarrow H^{s+t}_{\text{et}}(U_{\text{ar}}, \mathbb{Z}(n)),$$

and group homology of a finite group with finitely generated coefficients is finite. So $H_i^c(U_{\text{ar}}, \mathbb{Z})$, and hence $H_i^c(X_{\text{ar}}, \mathbb{Z})$ is finitely generated. \qed
Proposition 4.3 Let $X$ be connected. If $X$ is proper, then the degree map induces an isomorphism $H^0_\delta(X_{ar}, \mathbb{Z}) \cong \mathbb{Z}$, and if $X$ is not proper, then $H^0_\delta(X_{ar}, \mathbb{Z}) = 0$.

Proof. By [5] and [6] Thm.7.5, we have $H^0_\delta(X_{ar}, \mathbb{Z}) \cong H^{2d+1}(X_W, \mathbb{Z}(d)) \cong \mathbb{Z}$ for $X$ smooth and proper of dimension $d$. In the general case, we can assume that $X$ is irreducible by induction on the number of irreducible components. By induction on the dimension, we obtain $H^0_\delta(U', \mathbb{Z}) = 0$ for the complement $U'$ of a non-empty closed subscheme in a smooth and proper scheme $X'$. With the argument of Prop. 4.2 this implies that there is an open subscheme $U$ of $X$ with connected complement such that $H^0_\delta(U_{ar}, \mathbb{Z}) = 0$. But then by induction on the dimension, we get that $\mathbb{Z} = H^0_\delta((X-U)_{ar}, \mathbb{Z}) \to H^0_\delta(X_{ar}, \mathbb{Z}) \to H^0_\delta((\mathbb{F}_q)_{ar}, \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. □

Consider the complex
\[
\cdots \to \bigoplus_{x \in X(a)} H^s(k(x), \mathbb{Q}(s)) \to \cdots \to \bigoplus_{x \in X(0)} H^0(k(x), \mathbb{Q}(0)),
\]
(15)

arising as the $E_1$-terms and $d_1$-differentials in theiveau spectral sequence \[4\]. Let $\tilde{H}_i(X, \mathbb{Q}(0)) = E^2_{i,0}(X)$ be the homology of this complex. Then in [10], we used a theorem of Jannsen [14] to show that Conjecture $P_0(X)$ has two independent faces:

Proposition 4.4 Assume resolution of singularities. Then conjecture $P_0(X)$ for all smooth and proper $X$ is equivalent to the following statements for all smooth and proper $X$:

a) $CH_0(X, i)_\mathbb{Q} = 0$ for $i > \dim X$.

b) $H_i(X, \mathbb{Q}(0))$ for all $i > 0$ and $H_0(X, \mathbb{Q}(0)) = \mathbb{Q}$.

The first statement is equivalent to $H^i(k, \mathbb{Q}(d)) = 0$ for all fields $k$ of transcendence degree $d$ over $\mathbb{F}_q$ and all $i \neq d$, or to $CH_0(X, i)_\mathbb{Q} \cong H_i(X, \mathbb{Q}(0))$ for all $X$ and all $i$. The second statement is equivalent to the vanishing of $\tilde{H}_d(X, \mathbb{Q}(0)) = \tilde{H}_{d-1}(X, \mathbb{Q}(0)) = 0$ for $d = \dim X > 1$ [10].

4.1 Special values of zeta-functions

We can use Borel-Moore arithmetic homology to give formulas for special values of zeta-functions at non-positive integers $n$. There are other versions in Kahn [13] Thm. 72] (up to powers of $p$), and [7] Conj. 1.4] (assuming resolution of singularities). Let $\chi(H^*_e(X_{ar}, \mathbb{Z}(n)), e)$ be the Euler characteristic of the complex
\[
\cdots \to H^e_{i+1}(X_{ar}, \mathbb{Z}(n)) \xrightarrow{\cap e} H^e_i(X_{ar}, \mathbb{Z}(n)) \xrightarrow{\cap e} H^e_{i-1}(X_{ar}, \mathbb{Z}(n)) \to \cdots.
\]
(16)
The cap product is induced by the group cohomology product with a generator $e$ of $H^1(G, \mathbb{Z}) \cong \mathbb{Z}$.
Theorem 4.5 Under Conjecture $P_0(X)$ for all smooth and projective $X$, the following statements hold for all $n \leq 0$ and all $X$ of finite type over $\mathbb{F}_q$.

a) The groups $H^i_\star(X_{\text{ar}}, \mathbb{Z}(n))$ are finitely generated, and
$$\text{ord}_{s=n} \zeta(X, s) = \sum_i (-1)^i \cdot \text{rank} H^i_\star(X_{\text{ar}}, \mathbb{Z}(n)) =: \rho_n.$$  

b) For $s \to n$,
$$\zeta(X, s) \sim (1 - q^{n-s})^{\rho_n} \cdot \chi(H^\star_\text{et}(X_{\text{ar}}, \mathbb{Z}(n)), e)^{-1}.$$  

Proof. By the homotopy formula, we can assume that $n = 0$. Using Prop. 4.2 and the argument of [17, Thms. 1.6, 2.21] we can reduce to the case where $X$ is smooth, projective, and connected. In this case $\text{ord}_{s=0} \zeta(X, s) = \rho_0 = -1$, and by Prop. 4.1b),
$$H^i_\star(X_{\text{ar}}, \mathbb{Z}) \otimes \mathbb{Z} \cong H^{2d+1-i}(X_W, \mathbb{Z}(d)) \otimes \mathbb{Z} \cong H^{2d+1-i}(X_{\text{et}}, \mathbb{Z}(d))$$
is the usual étale cohomology. By Poincare-duality,
$$H^{2d+1-i}(X_{\text{et}}, \mathbb{Z}(d)) \cong \lim_{\to} (H^i(X_{\text{et}}, \mathbb{Z}/m)^\ast) \cong H^i(X_{\text{et}}, \mathbb{Q}/\mathbb{Z})^\ast,$$
and since $D^\ast$ is torsion free for every divisible group $D$, we obtain
$$H^{2d+1-i}(X_{\text{et}}, \mathbb{Z}(d))^\ast \cong ((H^i(X_{\text{et}}, \mathbb{Q}/\mathbb{Z})/\text{Div})^\ast)^\ast)_\text{tor} \cong (H^{i+1}(X_{\text{et}}, \mathbb{Z})^\ast)^\ast.$$  

It is easy to see that the map $H^{2d}(X_{\text{et}}, \mathbb{Z}(d))/\text{tor} \cong H^{2d+1}(X_{\text{et}}, \mathbb{Z}(d))/\text{tor}$ is an isomorphism. Since all other groups in (16) are finite, we obtain
$$\chi(H^i_\star(X_{\text{ar}}, \mathbb{Z}), e)^{-1} = \prod_i |H^i_\star(X_{\text{ar}}, \mathbb{Z})^\ast|^{-1} = \prod_i |H^{i+1}(X_{\text{et}}, \mathbb{Z})^\ast|^{-1} = \chi(H^\ast(X_W, \mathbb{Z}), e).$$  

The latter is equal to the limit of $\zeta(X, s)(1 - q^{-s})^{-1}$ for $s \to 0$ by [6, Thm.9.1].

5 An integral version of Kato’s conjecture

Throughout this section we assume the validity of the Beilinson-Lichtenbaum conjecture in weight 0. By Jannsen-Saito-Sato [17, Thms. 1.6, 2.21], the complex of $E_r^\ast$-terms and differentials in (9)
$$\cdots \to \oplus_{x \in X(\ast)} H^{s+1}(k(x)_{\text{et}}, \mathbb{Z}/m(s)) \to \cdots \to \oplus_{x \in X(0)} H^1(k(x)_{\text{et}}, \mathbb{Z}/m(0)) \to 0$$  
is isomorphic to the complex defined by Kato in [19].
**Conjecture 5.1 (Kato)** For a smooth, proper, and connected scheme X over a finite field, the homology groups $H^K_i(X, \mathbb{Z}/m)$ of (17) satisfy

$$H^K_i(X, \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & i = 0; \\ 0 & i \neq 0. \end{cases}$$

Comparing the niveau spectral sequence (3) and (6), one sees that there is a long exact sequence

$$\cdots \to CH_0(X, i, \mathbb{Z}/m) \to H_i(X_{et}, \mathbb{Z}/m) \to H_{i+1}^K(X, \mathbb{Z}/m) \to \cdots. \quad (18)$$

Indeed, with finite coefficients, the terms $E_{1s,t}^{s,t}$ in (3) and (6) agree for $t \geq 0$ by the Beilinson-Lichtenbaum conjecture, the term $E_{1s,t}^{s,t}$ in (3) vanish for $t < 0$ by definition, the terms $E_{1s,t}^{s,t}$ in (6) vanish for $t < -1$ by considering cohomological dimension, and the remaining terms $E_{1s,-1}^{s,-1}$ in (6) are given by the complex (17).

### 5.1 The theorem of Jannsen and Saito

**Theorem 5.2 (Jannsen, Saito [14, 16])** If X is smooth, proper and connected, and resolution of singularities holds for schemes up to dimension $\min \{i, \dim X\}$, then $H^K_0(X, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ and $H^K_i(X, \mathbb{Q}/\mathbb{Z}) = 0$ for $i > 0$.

The case $1 \leq i \leq 3$ was previously treated by Colliot-Thélène [2] and Suwa [26]. We can now prove Theorem 1.1 of the introduction:

**Proof.** (Thm. 1.1) a) $\Rightarrow$ b): By $P_0(X)$ and Thm. 3.1, the isomorphism holds rationally. With torsion coefficients, it follows from Thms. 3.1, 5.2 and (18). Hence we get the isomorphism by comparing coefficient sequences. The groups are finitely generated by Prop. 4.1.

b) $\Rightarrow$ c) is (11), and c) $\Rightarrow$ a) because $CH_0(X, i)_G \otimes \mathbb{Q} \cong CH_0(X, i)_G \otimes CH_0(X, i+1, \mathbb{Q}/\mathbb{Z}) \cong \lim CH_0(X, i, \mathbb{Z}/l) \otimes \mathbb{Q}$ by a transfer argument. 

**Proposition 5.3** Assuming resolution of singularities, the following statements are equivalent:

a) $P_0(X)$ for all smooth and proper X.

b) $CH_0(X, i)$ is finitely generated for all X and all $i$.

c) The canonical map $CH_0(X, i) \otimes \mathbb{Z}_l \to \lim CH_0(X, i, \mathbb{Z}/l')$ is an isomorphism for all i, all l, and all X.

**Proof.** a) $\Rightarrow$ b): By the standard argument using localization, it suffices to show finite generation for smooth and proper X. In this case, for $i > 0$, by $P_0(X)$ and Theorem 5.2

$CH_0(X, i) \cong \text{tor} CH_0(X, i) \leftarrow CH_0(X, i + 1, \mathbb{Q}/\mathbb{Z})$

$\xrightarrow{\sim} H_{i+1}(X_{et}, \mathbb{Q}/\mathbb{Z}) = H^{2d-i-1}(X_{et}, \mathbb{Q}/\mathbb{Z}(d))$
and this is finite by [11] and the Weil-conjectures. Finiteness of $CH_0(X)$ is a result of class field theory [20].

b) $\Rightarrow$ c) is clear, and c) $\Rightarrow$ a) follows from

$$CH_0(X,i) \otimes \mathbb{Z}_l \xrightarrow{\sim} \lim CH_0(X,i,\mathbb{Z}/l^r) \xrightarrow{\sim} \lim H_i(X_{et},\mathbb{Z}/l^r) = \lim H^{2d-i}(X_{et},\mathbb{Z}/l^r(d)),$$

because the right hand group is torsion for smooth and proper $X$ and for $i > 0$ by the Weil-conjectures.

5.2 The complex

Definition 5.4 For a separated scheme of finite type $X$ over $\mathbb{F}_q$, we define integral Kato-homology $H^K_i(X,\mathbb{Z})$ to be the homology of the complex $C^K(X)$ of $E_1^{n-1}$-terms and differentials in

$$\cdots \rightarrow \bigoplus_{x \in X_k} H^{n+1}(k(x)_{ar},\mathbb{Z}(s)) \rightarrow \cdots \rightarrow \bigoplus_{x \in X_{(0)}} H^1(k(x)_{ar},\mathbb{Z}(0)) \rightarrow 0. \tag{19}$$

By Lemma 3.4, the complex (19) can be rewritten as

$$\cdots \rightarrow \bigoplus_{x \in X_s} K^M(k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)^G \rightarrow \cdots \rightarrow \bigoplus_{x \in X_{(0)}} (k(x) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)^G \rightarrow \bigoplus_{x \in X_{(0)}} \mathbb{Z}. \tag{20}$$

An inspection of the spectral sequence (12) shows that there is a map of complexes $C^K(X) \rightarrow H_0^0(X,\mathbb{Z})$. In particular, if $X$ is proper and connected, then we obtain from Proposition 4.3 an augmentation $C^K(X) \rightarrow \mathbb{Z}$. A closed embedding $Z \rightarrow X$ with open complement $U$ gives rise to a short exact sequence of complexes (20), hence a long exact sequence

$$\cdots \rightarrow H^K_i(Z,\mathbb{Z}) \rightarrow H^K_i(X,\mathbb{Z}) \rightarrow H^K_i(U,\mathbb{Z}) \rightarrow \cdots.$$

The connection to Kato homology with finite coefficients is given as follows.

Proposition 5.5 There are long exact sequences

$$\cdots \rightarrow H^K_i(X,\mathbb{Z}) \xrightarrow{\times m} H^K_i(X,\mathbb{Z}) \rightarrow H^K_i(X,\mathbb{Z}/m) \rightarrow \cdots, \tag{21}$$

and

$$\cdots \rightarrow H^K_i(X,\mathbb{Z}) \rightarrow H_i(X,\mathbb{Q}(0)) \rightarrow H^K_i(X,\mathbb{Q}/\mathbb{Z}) \rightarrow \cdots. \tag{22}$$

Proof. By Lemma 5.5, there are short exact sequence of complexes

$$0 \rightarrow \bigoplus_{x \in X_{(s)}} H^{n+1}(k(x)_{ar},\mathbb{Z}(*)) \xrightarrow{\times m} \bigoplus_{x \in X_{(s)}} H^{n+1}(k(x)_{ar},\mathbb{Z}(*)) \rightarrow \bigoplus_{x \in X_{(s)}} H^{n+1}(k(x)_{et},\mathbb{Z}/m(\ast)) \rightarrow 0.$$
and

\[ 0 \to \bigoplus_{x \in X_{(\cdot)}} H^{i+1}(k(x)_{\text{ar}}, \mathbb{Z}(\ast)) \to \bigoplus_{x \in X_{(\cdot)}} H^i(k(x), \mathbb{Q}(\ast)) \]
\[ \to \bigoplus_{x \in X_{(\cdot)}} H^{i+1}(k(x)_{\text{et}}, \mathbb{Q}/\mathbb{Z}(\ast)) \to 0. \]

\[ \]
Remark. Taking the colimit over smooth schemes with function field $k$, the same argument shows that the statement in Prop. 4.4(a) implies that for a field $k$ of transcendence degree $d$ over $\mathbb{F}_q$, we have

$$H^i(k_{\text{ar}}, \mathbb{Z}(d)) \cong \begin{cases} H^i(k, \mathbb{Z}(d)) & i \leq d; \\
K^d_i(k, \mathbb{Z}) & i = d + 1; \\
0 & i > d + 1. \end{cases}$$

5.3 The conjecture

The following is an integral version of Kato’s conjecture.

**Conjecture 5.8** If $X$ is smooth, proper and connected, then $H^K_i(X, \mathbb{Z}) = 0$ for $i > 0$, and the augmentation map induces an isomorphism $H^0_K(X, \mathbb{Z}) \cong \mathbb{Z}$.

**Proposition 5.9** Conjecture 5.8 is equivalent to the conjunction of Conjecture 5.1 and the statement in Proposition 4.4(b).

**Proof.** If Conjecture 5.8 holds, then for smooth and proper $X$, the groups $H^i_K(X, \mathbb{Z}/m)$ vanish for $i > 0$ by (21), and then $\tilde{H}_i(X, \mathbb{Q}(0)) = 0$ for $i > 0$ and $\tilde{H}_0(X, \mathbb{Q}(0)) = \mathbb{Q}$ by (22). The converse follows by (22). $\square$

Note that Propositions 5.9 and 4.4 together with Theorem 5.7 imply Theorem 1.3.

Assuming resolution of singularities, Jannsen [14, Thm.5.9] defines weight-homology $H^W_i(X, \mathbb{Z})$ of $X$ as the homology of the complex

$$\cdots \rightarrow \mathbb{Z}^{\pi_0(X_1)} \rightarrow \mathbb{Z}^{\pi_0(X_0)} \rightarrow \cdots$$

where

$$\cdots \rightarrow M(X_1) \rightarrow M(X_0)$$

is the weight complex $W(X)$ of Gillet-Soulé [13], see the discussion in [10]. By definition for $X$ smooth and proper, $H^W_0(X, \mathbb{Z}) = \mathbb{Z}$, and $H^W_i(X, \mathbb{Z}) = 0$ for $i > 0$. If follows from the properties of weight complexes that $H^W_i(-, \mathbb{Z})$ has the localization property. We define weight homology of a field $k$ of finite type over $\mathbb{F}_q$ as $H^W_i(k, \mathbb{Z}) := \text{colim} H^W_i(U, \mathbb{Z})$, where $U$ runs through the filtered system of schemes over $\mathbb{F}_q$ with function field $k$. By Jannsen [14, Thm.5.9], $H^W_i(k, \mathbb{Z}) = 0$ for $i \neq \text{trdeg}_q k$, hence the niveau spectral sequence shows that the weight homology $H^W_i(X, \mathbb{Z})$ of $X$ is the homology of the complex

$$\cdots \rightarrow \oplus_{x \in X_{(1)}} H^W_s(k(x), \mathbb{Z}) \rightarrow \cdots \rightarrow \oplus_{x \in X_{(0)}} H^W_0(k(x), \mathbb{Z}). \quad (23)$$

**Lemma 5.10** For any scheme $X$ over $\mathbb{F}_q$, we have a canonical isomorphism

$$H^K_i(W(X), \mathbb{Z}) \xrightarrow{\sim} H^K_i(X, \mathbb{Z}),$$

compatible with localization triangles.
Proof. If $X$ is proper, then $W(X)$ is chain homotopy equivalent to a hyperenvelope $X$ of $X$, and the augmentation $W(X) \cong X \to X$ induces the indicated quasi-isomorphism by the localization property of Kato homology by the argument of [12]. For an arbitrary scheme $U$, we choose a compactification $X$ and complement $Z$, and obtain a commutative diagram

$$
\begin{array}{ccc}
C^K(W(Z)) & \longrightarrow & C^K(W(X)) \\
\downarrow & & \downarrow \\
C^K(Z) & \longrightarrow & C^K(X)
\end{array}
$$

The rows are distinguished triangles by the localization property of Kato-homology and property d) of weight complexes, respectively.

The augmentation maps $C^K(X_i) \to \mathbb{Z}^{\pi_0(X_i)}$ induce a natural homomorphism

$$H^K_i(X, Z) \cong H^K_i(W(X), Z) \to H^K_i(X, Z).$$

Taking the colimit over smooth $U$ with function field $k$, we obtain a map

$$H^{d+1}(k_{ar}, \mathbb{Z}(d)) \cong H^K_d(k, \mathbb{Z}) \to H^K_d(k, \mathbb{Z}).$$

**Proposition 5.11** Assume resolution of singularities. Then the following statements are equivalent:

a) Conjecture 5.8.

b) The canonical map $H^{d+1}(k_{ar}, \mathbb{Z}(d)) \to H^K_d(k, \mathbb{Z})$ is an isomorphism for all fields $k$ with $\text{trdeg} k/\mathbb{F}_q = d$.

c) The canonical map $H^K_i(X, Z) \to H^K_i(X, Z)$ is an isomorphism for all schemes $X$ and all $i$.

Proof. a) $\implies$ b): We proceed by induction on $d$, the case $d = 0$ being trivial. Choose a smooth and proper model $X$ for $k$ and compare the exact sequences [19] and [23]

$$\begin{array}{ccc}
H^{d+1}(k_{ar}, \mathbb{Z}(d)) & \longrightarrow & \bigoplus_{x \in X_{(d-1)}} H^d(k(x)_{ar}, \mathbb{Z}(d-1)) \\
\downarrow & & \downarrow \\
H^K_d(k, \mathbb{Z}) & \longrightarrow & \bigoplus_{x \in X_{(d-1)}} H^K_d(k(x), \mathbb{Z})
\end{array}$$

The upper sequence is exact by hypothesis, and the lower sequence is exact by Jannsen’s theorem, because $H^K_i(X, Z) = 0$ for $i > 0$.

b) $\implies$ c) follows by comparing [19] and [23], and c) $\implies$ a) is trivial.

**Corollary 5.12** Assume resolution of singularities. Then Conjecture $P_0(X)$ holds for all smooth and proper $X$ if and only if there is an exact sequence of finitely generated groups
Arithmetic homology and an integral version of Kato's conjecture

\[ \cdots \to CH_0(X, i) \to H_{i+1}^{\text{ar}}(X, \mathbb{Z}) \to H_{i+1}^W(X, \mathbb{Z}) \to \cdots \]

for all \( X \).

**Proof.** If \( P_0(X) \) holds for all \( X \), then we get the exact sequence from Theorem 5.7, and Prop. 4.4, 5.9 and 5.11. The finite generation statement follows because weight homology is finitely generated by construction. Conversely, the exact sequence gives \( CH_0(X, i) \cong H_{i+1}^{\text{ar}}(X, \mathbb{Z}) \) for smooth and proper \( X \) and \( i \geq 0 \), and hence \( P_0(X) \) by comparing to Thm. 3.1. \( \square \)

### 6 Curves

**Theorem 6.1** Conjecture 5.8 holds for all smooth and proper curves over \( \mathbb{F}_q \).

**Proof.** Consider the following map of short exact sequences

\[
\begin{align*}
H^2(k(C)_{\text{ar}}, \mathbb{Z}(1)) & \longrightarrow H^1(k(C), \mathbb{Q}(1)) \longrightarrow Br(k(C)) \\
\bigoplus_{x \in C_{(0)}} \mathbb{Z} & \longrightarrow \bigoplus_{x \in C_{(0)}} \mathbb{Q} \longrightarrow \bigoplus_{x \in C_{(0)}} \mathbb{Q}/\mathbb{Z}.
\end{align*}
\]

The upper row is (14), since \( Br(k \otimes \mathbb{F}_q, \mathbb{F}_q) \) vanishes by Tsen’s theorem. The snake Lemma gives a short exact sequence of kernels and cokernels

\[ 0 \to H^1_k(C, \mathbb{Z}) \to K_1(C)_\mathbb{Q} \to Br(C) \to H^1_0(C, \mathbb{Z}) \to K_0(C)_\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0, \]

and the claim follows because \( K_1(C)_\mathbb{Q} = 0 = Br(C) \). \( \square \)

The above theorem can be derived without using the vanishing of the Brauer group by analyzing the Frobenius coinvariants of the localization sequence

\[ 0 \to \Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}})^{\times} \to (k(C) \otimes \mathbb{F}_q, \mathbb{F}_q)^{\times} \to \bigoplus_{x \in C_{(0)}} \mathbb{Z} \to \text{Pic} \tilde{C} \to 0. \]

It is amusing to observe that

\[ 0 \to H^2(k(C)_{\text{ar}}, \mathbb{Z}(1)) \to \bigoplus_{x \in C_{(0)}} \mathbb{Z} \to \mathbb{Z} \to 0 \]

is an integral version of the classical short exact sequence

\[ 0 \to Br(k(C)) \to \bigoplus_{x \in C_{(0)}} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0. \]

Indeed, the upper row of (24) shows that \( H^2(k(C)_{\text{ar}}, \mathbb{Z}(1)) \otimes \mathbb{Q}/\mathbb{Z} \cong Br(k(C)) \).
Theorem 6.2 Let $C$ be any curve. Then $H_0^K(C_{ar}, \mathbb{Z}) \cong H_0^K(C, \mathbb{Z})$, there is a short exact sequence

$$0 \to CH_0(C) \to H_1^K(C_{ar}, \mathbb{Z}) \to H_1^K(C, \mathbb{Z}) \to 0,$$

an isomorphism of finitely generated groups $CH_0(C, 1) \cong H_2^K(C_{ar}, \mathbb{Z})$, and $CH_0(C, i) = H_{i+1}^K(C_{ar}, \mathbb{Z}) = 0$ for $i > 1$.

The Pointrjagin dual of the short exact sequence is \cite{20} Prop. 1.

Proof. The proof of Thm. \cite{5.7} works if one restricts oneself to schemes of dimension at most 1, in which case the Beilinson-Lichtenbaum conjecture and $P_0(X)$ is known. Similarly, finite generation follows by Prop. \cite{4.2} and \cite{5.3}. 

For a curve $C$, integral Kato homology can be calculated combinatorially. We first assume that $C$ is connected and proper. Let $C' = \coprod_{i \in I} C_i'$ be the decomposition of the normalization $C'$ of $C$ into irreducible components, $S$ the set of singular points of $C$, and $S' = S \times_C C'$. Consider the dual graph $\Gamma'$ of $C$ \cite{24}. It is a bipartite graph, with vertices the set $S$ of singular points of $C$ on the one hand, and the set of irreducible components $I$ of $C'$ on the other hand. For each point $s \in S'$, there is an edge connecting the image of $s$ in $S$ with the $i \in I$ such that $s \in C_i$. Comparing the exact sequence calculating the homology of $\Gamma$ to the sequence

$$0 \to H_1^K(C, \mathbb{Z}) \to H_0^K(S', \mathbb{Z}) \to H_0^K(S, \mathbb{Z}) \oplus H_0^K(C', \mathbb{Z}) \to H_0^K(C, \mathbb{Z}) \to 0,$$

we get

Proposition 6.3 If $C$ is a proper curve, then $H_1^K(C, \mathbb{Z}) \cong H_1(\Gamma, \mathbb{Z})$. In particular, $H_0^K(C, \mathbb{Z}) = \mathbb{Z}^{\gamma_0(C)}$, and $H_1^K(C, \mathbb{Z}) = \mathbb{Z}^{\gamma_0(S') - \gamma_0(S) - \gamma_0(C') + \gamma_0(C)}$.

If $C$ has only ordinary double points as singularities, then the homology of $\Gamma$ agrees with the homology of the graph $\Gamma'$ used in \cite{24}. Indeed, the graph $\Gamma'$ is gotten from $\Gamma$ by erasing all vertices corresponding to points in $S$. Since over every point of $s \in S$ there are exactly two point of $x, y \in S'$, the two edges corresponding to $x$ and $y$ combine to give an edge labeled $s$ in $\Gamma'$. Thus $\Gamma$ and $\Gamma'$ are homeomorphic as topological spaces.

If $C$ is not proper, let $\bar{C}$ be a proper curve containing $C$. Then the exact sequence

$$0 \to H_1^K(\bar{C}, \mathbb{Z}) \to H_1^K(C, \mathbb{Z}) \to \bigoplus_{p \in \bar{C} - C} \mathbb{Z} \to H_0^K(\bar{C}, \mathbb{Z}) \to H_0^K(C, \mathbb{Z}) \to 0$$

shows that $H_0^K(C, \mathbb{Z}) = 0$ and $H_1^K(C, \mathbb{Z}) = H_1^K(\bar{C}, \mathbb{Z}) \oplus \mathbb{Z}^{\gamma_0(C - C) - 1}$.

7 Class field theory

In this section, we assume that $X$ is proper and connected over $\mathbb{F}_q$. 

\newpage
Proposition 7.1 We have

\[ H_1^c(X_{ar}, \mathbb{Z})^\wedge \cong \pi_1^{ab}(X). \]

In particular, if \( H_1^c(X_{ar}, \mathbb{Z}) \) is finitely generated, then there is a short exact sequence

\[ 0 \to (\text{CH}_0(\overline{X}, 1)_G)^\wedge \to \pi_1^{ab}(X) \to (\text{CH}_0(\overline{X})^G)^\wedge \to 0. \]

Proof. By Prop. 4.3, \( H_0^c(X_{ar}, \mathbb{Z}) \) is torsion free, hence by Thm. 3.1

\[ H_1^c(X_{ar}, \mathbb{Z})/m \cong H_1^c(X_{ar}, \mathbb{Z}/m) \cong H_0(X_{et}, \mathbb{Z}/m). \]

By [9], the latter is isomorphic to \( H^1(X_{et}, \mathbb{Z}/m)^* = \pi_1^{ab}(X)/m \). The Proposition follows by taking inverse limits.

The results of this section could be formulated independently of Conjecture \( P_0(X) \) using the dual of arithmetic cohomology of [7]:

Proposition 7.2 We have a natural inclusion

\[ H^{-1}R\text{Hom}_{Ab}(R\Gamma(X_{ar}, \mathbb{Z}), \mathbb{Z}) \to \pi_1^{ab}(X) \]

which induces an isomorphism on completions. There are short exact sequences

\[ 0 \to (H^2(X_{ar}, \mathbb{Z}))_{tor}^* \to \pi_1(X)^{ab} \to \text{Hom}(H^1(X_{ar}, \mathbb{Z}), \mathbb{Z})^\wedge \to 0. \]

For normal \( X \), under resolution of singularities, \( (H^2(X_{ar}, \mathbb{Z}))_{tor}^* \) is isomorphic to the geometric part of the abelianized fundamental group \( \pi_1^{ab}(X)^0 \).

Proof. Consider the reduction mod \( m \) map

\[ H^{-1}R\text{Hom}_{Ab}(R\Gamma(X_{ar}, \mathbb{Z}), \mathbb{Z})/m \to H^{-1}(R\text{Hom}_{Ab}(R\Gamma(X_{et}, \mathbb{Z}), \mathbb{Z}) \otimes \mathbb{Z}/m) \]

\[ \to H^{-1}(R\text{Hom}_{Ab}(R\Gamma(X_{et}, \mathbb{Z}/m), \mathbb{Z}/m)) \cong \text{Hom}(H^1(X_{ar}, \mathbb{Z}/m), \mathbb{Q}/\mathbb{Z}). \]

Taking the limit over \( m \), we obtain a map

\[ H^{-1}R\text{Hom}_{Ab}(R\Gamma(X_{ar}, \mathbb{Z}), \mathbb{Z}) \to \lim \text{Hom}(H^1(X_{ar}, \mathbb{Z}/m), \mathbb{Q}/\mathbb{Z}) \]

\[ = \text{Hom}(H^1(X_{ar}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(H^1(X_{et}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = \pi_1(X)^{ab}. \]

Since the first term is finitely generated, it injects into its completion. The short exact sequence is the Leray spectral sequence for \( R\text{Hom}_{Ab}(-, \mathbb{Z}) \) (using that \( \text{Ext}^1(-, \mathbb{Z}) \cong (A_{tor})^* \) for finitely generated \( A \)). For normal \( X \), \( H^1(\overline{X}_{eh}, \mathbb{Z}) = 0 \) under resolution of singularities [7], hence \( H^1(X_{ar}, \mathbb{Z}) \cong H^0(X_{eh}, \mathbb{Z})_G \cong \mathbb{Z}. \) \( \square \)
Let $X$ be smooth, projective and geometrically connected. The isomorphism of class field theory of Bloch and Kato-Saito [20], factors through arithmetic homology

$$\theta_X : CH_0(X)_{\text{tor}} \to H^i_1(X_{\text{ar}}, Z)_{\text{tor}} \cong \pi_1^{ab}(X)_{\text{tor}}.$$ 

Let $T$ be the finite group $\text{colim} \text{Hom}_{GS/k}(\mu_m, \text{NS}_X)$, where $\text{Hom}_{GS/k}(-,-)$ denotes the group of homomorphisms of groups schemes over $k$, and recall that conjecturally $CH_0(X, 1)_Q = 0$.

**Proposition 7.3** There are short exact sequences

$$0 \to CH_0(\bar{X}, 1)_G \to H^i_1(X_{\text{ar}}, Z) \to \text{Alb}(\bar{X}) \oplus Z \to 0,$$

$$0 \to CH_0(X) \to H^i_1(X_{\text{ar}}, Z) \to CH_0(X, 1)_Q \to 0,$$

and

$$0 \to T^* \to CH_0(\bar{X}, 1)_G \to CH_0(X, 1)_Q \to 0.$$

**Proof.** The first exact sequence is (11), together with $CH_0(\bar{X})_G \cong (\text{Alb}(\bar{X}) \oplus Z)_G = \text{Alb}(X) \oplus Z$. The next sequence follows from Cor. [5.6] and Thm. [5.7]. Indeed, $H^1_2(X, Z)$ is uniquely divisible, hence its image in the finitely generated group $CH_0(X)$ must be trivial. The last sequence follows by considering the following diagram (the upper row is [20, Prop. 9(2)]).

Consider a cartesian diagram

$$Z' \longrightarrow X'$$

$$\downarrow \quad p \downarrow$$

$$Z \longrightarrow X;$$

such that $i$ is a closed embedding, $p$ is proper, and $p$ induces an isomorphism $X' - Z' \cong X - Z$.

**Proposition 7.4** If $X'$ and $Z$ are smooth, then there is an exact sequence

$$\ker \left( H^i_1(Z', Z) \to H^i_1(X', Z) \oplus H^i_1(Z, Z) \right) \to CH_0(X) \xrightarrow{2} H^i_1(X_{\text{ar}}, Z).$$

The first term agrees with $H^i_1(Z', Z)$ if $CH_0(X', 1)_Q \cong CH_0(Z, 1)_Q = 0.$
Proof. Consider the commutative diagram with exact rows and exact columns given by Theorem 5.7:

\[
\begin{array}{ccccccccc}
CH_0(Z') & \longrightarrow & CH_0(X') \oplus CH_0(Z) & \longrightarrow & CH_0(X) & \rightarrow & 0 \\
\theta_{Z'} & & \theta_Z & & \theta_X & & \\
H_1^c(Z'_ar, Z) & \longrightarrow & H_1^c(X'_ar, Z) \oplus H_1^c(Z_ar, Z) & \longrightarrow & H_1^c(X_ar, Z) & \downarrow & \\
& & & & & & \\
H_1^K(Z', Z) & \longrightarrow & H_1^K(X', Z) \oplus H_1^K(Z, Z) & \longrightarrow & H_1^K(X, Z) & \downarrow & \\
& & & & & & \\
0 & & 0 & & 0 & & 
\end{array}
\]

Using the injectivity of \(\theta_{X'}\) and \(\theta_Z\), the first statement follows by a diagram chase. The last claim follows from Corollary 5.6. \(\square\)

Theorem 5.7 and Proposition 7.3 can be applied to study the kernel of the reciprocity map. For example, we can recover some results of Asakura, Matsumi and Sato [23]. Let \(X\) be a normal surface with one singular point \(P\). Let \(S\) be a resolution of singularities \(X\) which is an isomorphism away from \(P\), and let \(D = P \times_X S\) be the exceptional fiber.

**Proposition 7.5** Let \(\Gamma\) be the dual graph of \(D\), and assume that \(CH_0(S, 1)_Q = 0\). Then there are short exact sequences

\[
H_1(\Gamma, Z) \xrightarrow{\delta_X} CH_0(X) \xrightarrow{\theta_X} H_1^c(X_ar, Z),
\]

and

\[
0 \rightarrow ker(CH_0(S) \rightarrow CH_0(X)) \rightarrow ker(H_1^c(S_ar, Z) \rightarrow H_1^c(X_ar, Z))
\]

\[
\rightarrow ker \theta_X \rightarrow coker(CH_0(D) \xrightarrow{deg} Z) \rightarrow 0.
\]

**Proof.** The first statement follows from Props. 6.3 and 7.4. To prove the second statement, consider the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & CH_0(S) & \xrightarrow{\theta_S} & H_1^c(S_ar, Z) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ker \theta_X & \longrightarrow & CH_0(X) & \xrightarrow{\theta_X} & H_1^c(X_ar, Z) & \longrightarrow & 0.
\end{array}
\]  

(25)

The upper row is exact by Prop. 7.3. Since \(coker(CH_0(S) \rightarrow CH_0(X)) = coker(CH_0(D) \rightarrow CH_0(P))\), it suffices to show that \(H_1^c(S_ar, Z) \rightarrow H_1^c(X_ar, Z)\) is surjective. Since \(X\) is normal, \(D\) is connected, hence \(H_1^c(D_ar, Z) \cong H_1^c(P_ar, Z)\), and we obtain an exact sequence
Thus it suffices to show that \( H_1^c(D_{ar}, \mathbb{Z}) \rightarrow H_1^c(P_{ar}, \mathbb{Z}) \oplus H_1^c(S_{ar}, \mathbb{Z}) \rightarrow H_1^c(X_{ar}, \mathbb{Z}) \rightarrow 0. \)

According to [23, Lemma 1.2, Prop. B.6], \( \text{coker}(CH_0(D) \rightarrow \mathbb{Z}) \cong \mathbb{Z}/m, \)
where \( m \) is the greatest common divisor of the degree of the field extension \( \Gamma(C_i, \mathcal{O}_{C_i})/\Gamma(D, \mathcal{O}_D) \). Here \( C_i \) runs through the irreducible components of \( D \).

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