On AZ-style identity

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February 17, 2022

Abstract

The AZ identity is a generalization of the LYM-inequality. In this paper, we will give a generalization of the AZ identity.

KEYWORDS: LYM-inequality, AZ-identity
MATHEMATICS SUBJECT CLASSIFICATION: 05D05

1 Introduction

Let \([n] = \{1, 2, \ldots, n\}\), \(\Omega_n\) be the family of all subsets of \([n]\), and \(\emptyset\) be the empty set. Let \(\emptyset \neq \mathcal{F} \subseteq \Omega_n\). If \(A \subseteq B\) for all \(A, B \in \mathcal{F}\) with \(A \neq B\), then \(\mathcal{F}\) is called a Sperner family or antichain. For any antichain \(\mathcal{F}\), the following inequality holds:

\[
\sum_{X \in \mathcal{F}} \frac{1}{n \binom{n}{|X|}} \leq 1.
\]  (1)

The inequality (1) is called the LYM-inequality (Lubell, Yamamoto, Meshalkin) (see [5, Chapter 13]). Many generalizations of the LYM-inequality have been obtained (see [4, 6, 7, 9]). In particular, Ahlswede and Zhang [3] discovered an identity (see equation (2)) in which the LYM-inequality is a consequence of it.

Let \(\mathcal{G}_n\) be the family of all \(\mathcal{F}\) such that \(\emptyset \neq \mathcal{F} \subseteq \Omega_n\). For every \(\mathcal{F} \in \mathcal{G}_n\), the set

\[
D_n(\mathcal{F}) = \{ Y \subseteq [n] : Y \subseteq F \text{ for some } F \in \mathcal{F}\},
\]

is called the downset, while the set

\[
U_n(\mathcal{F}) = \{ Y \subseteq [n] : Y \supseteq F \text{ for some } F \in \mathcal{F}\},
\]

is called the upset. For each \(X \subseteq [n]\), we set

\[
Z_{\mathcal{F}}(X) = \left\{ \begin{array}{ll}
\emptyset & \text{if } X \notin U_n(\mathcal{F}), \\
\bigcap_{F \in \mathcal{F}, F \subseteq X} F & \text{otherwise}.
\end{array} \right.
\]

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Theorem 1.1. [3] For any $\mathcal{F} \in \mathcal{G}_n$ with $\emptyset \notin \mathcal{F}$,
\[
\sum_{X \in U_n(\mathcal{F})} \frac{|Z_{\mathcal{F}}(X)|}{|X| \binom{n}{|X|}} = 1. \tag{2}
\]

Equation (2) is called the AZ-identity. Note that when $\mathcal{F}$ is an antichain, $Z_{\mathcal{F}}(F) = F$ for all $F \in \mathcal{F}$. So equation (2) becomes
\[
\sum_{F \in \mathcal{F}} \frac{1}{\binom{|F|}{n}} + \sum_{X \in U_n(\mathcal{F}) \setminus \mathcal{F}} \frac{|Z_{\mathcal{F}}(X)|}{|X| \binom{n}{|X|}} = 1,
\]

and as the second term on the left is non-negative, we obtain inequality (1).

Later, Ahlswede and Cai discovered an identity for two set systems.

Theorem 1.2. [1] Let $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$ be elements in $\mathcal{G}_n$. Suppose that $A_i \neq \emptyset$ for all $i$, and $A_j \subseteq B_k$ if and only if $j = k$. Then
\[
\sum_{i=1}^{q} \frac{1}{\binom{n-|B_i|+|A_i|}{|A_i|}} + \sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} \frac{|Z_{\mathcal{A}}(X)|}{|X| \binom{n}{|X|}} = 1. \tag{3}
\]

Ahlswede and Cai [2] also discovered AZ type of identities of several other posets. For the duality of equations (2) and (3), we refer the readers to [8,10].

Recently, Thu discovered the following generalizations of equations (2) and (3).

Theorem 1.3. [12] Let $m$ be an integer, and $\mathcal{F} \in \mathcal{G}_n$ with $\emptyset \notin \mathcal{F}$. If $|F| + m > 0$ for all $F \in \mathcal{F}$, then
\[
\sum_{X \in U_n(\mathcal{F})} \frac{|Z_{\mathcal{F}}(X)| + m}{|X| + m \binom{n+m}{|X|+m}} = 1. \tag{4}
\]

Theorem 1.4. [12] Let $m$ be an integer, and $\mathcal{A} = \{A_1, A_2, \ldots, A_q\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_q\}$ be elements in $\mathcal{G}_n$. Suppose that $A_i \neq \emptyset$ for all $i$, and $A_j \subseteq B_k$ if and only if $j = k$. If $|A| + m > 0$ for all $A \in \mathcal{A}$, then
\[
\sum_{i=1}^{q} \frac{1}{\binom{n+m-|B_i|+|A_i|}{|A_i|+m}} + \sum_{X \in U_n(\mathcal{A}) \setminus D_n(\mathcal{B})} \frac{|Z_{\mathcal{A}}(X)| + m}{|X| + m \binom{n+m}{|X|+m}} = 1. \tag{5}
\]

In this paper, we will give generalizations of equations (4) and (5) (see Theorem 2.4 and Theorem 2.7).
2 Main theorems

Let us denote the set of real numbers by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. Let $a, m \in \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that $ak + m \neq 0$ for $k = l, l + 1, \ldots, n$. We set

$$g_{a,m}(n,l) = \frac{(n - l)!a^{n-l}}{\prod_{k=l}^{n}(ak + m)}.$$ 

Lemma 2.1. Suppose $l < n$. If $ak + m \neq 0$ for $k = l, l + 1, \ldots, n$, then

$$g_{a,m}(n,l) + g_{a,m}(n,l + 1) = g_{a,m}(n - 1, l).$$

Proof. Note that

$$g_{a,m}(n,l) + g_{a,m}(n,l + 1) = \frac{(n - l)!a^{n-l}}{\prod_{k=l}^{n}(ak + m)} + \frac{(n - l - 1)!a^{n-l-1}}{\prod_{k=l+1}^{n}(ak + m)}$$

$$= \frac{\prod_{k=l}^{n}(ak + m) + (al + m)(n - l - 1)!a^{n-l-1}}{\prod_{k=l}^{n}(ak + m)}$$

$$= \frac{n(n - l - 1)!a^{n-l} + (n - l - 1)!a^{n-l-1}}{\prod_{k=l}^{n}(ak + m)}$$

$$= \frac{(n - l - 1)!a^{n-l-1}}{\prod_{k=l}^{n-1}(ak + m)}$$

$$= g_{a,m}(n - 1, l).$$

The following lemma can be verified easily.

 Lemma 2.2. Suppose that $ak + m \neq 0$ for $k = l, l + 1, \ldots, n$.

(a) If $a = 1$ and $m$ is an integer, then

$$g_{1,m}(n,l) = \frac{1}{(l + m)\binom{n+m}{l+m}}.$$ 

(b) If $a = 1$ and $m = 0$, then

$$g_{1,0}(n,l) = \frac{1}{(l)\binom{n}{l}}.$$ 

We shall need the following lemma (see equation (3) of [11], or Lemma 2 of [8]).

Lemma 2.3. Let $\emptyset \notin A \in \mathcal{G}_n$ and $\emptyset \notin B \in \mathcal{G}_n$. Set

$$A \lor B = \{A \cup B : A \in A, B \in B\}.$$ 

Then for each $\emptyset \neq X \subseteq [n]$,

$$|Z_{A \lor B}(X)| = |Z_A(X)| + |Z_B(X)| - |Z_{A \lor B}(X)|.$$ 

\qed
Case 2. Suppose $\mathcal{A} = \{A_1, \ldots, A_p\}$, $q \geq 2$. Assume that the theorem holds for all $q'$ with $1 \leq q' < q$. Let $\mathcal{B} = \{A_1, \ldots, A_{q-1}\}$ and $\mathcal{C} = \{A_q\}$. Then $\mathcal{B} \cup \mathcal{C} = \{A_1, \ldots, A_{q-1} \cup A_q\}$, $U_n(\mathcal{A}) = U_n(\mathcal{B}) \cup U_n(\mathcal{C})$ and $U_n(\mathcal{B} \cup \mathcal{C}) = U_n(\mathcal{B}) \cap U_n(\mathcal{C})$. By Lemma 2.3

$$\lvert Z_A(X) \rvert = \lvert Z_B(X) \rvert + \lvert Z_C(X) \rvert - \lvert Z_{B \cup C}(X) \rvert.$$
So if \( X \in U_n(B) \setminus U_n(C) \), then \( |Z_A(X)| = |Z_B(X)| \); if \( X \in U_n(C) \setminus U_n(B) \), then \( |Z_A(X)| = |Z_C(X)| \), and if \( X \in U_n(B) \cap U_n(C) \), then \( |Z_A(X)| = |Z_B(X)| + |Z_C(X)| - |Z_{B\lor C}(X)| \).

Therefore

\[
\sum_{X \in U_n(A)} (a|Z_A(X)| + m) g_{a,m}(n, |X|)
\]

\[
= \sum_{X \in U_n(B) \setminus U_n(C)} (a|Z_B(X)| + m) g_{a,m}(n, |X|)
+ \sum_{X \in U_n(C) \setminus U_n(B)} (a|Z_C(X)| + m) g_{a,m}(n, |X|)
+ \sum_{X \in U_n(B \lor C)} (a(|Z_B(X)| + |Z_C(X)| - |Z_{B\lor C}(X)|) + m) g_{a,m}(n, |X|)
\]

\[
= \sum_{X \in U_n(B)} (a|Z_B(X)| + m) g_{a,m}(n, |X|)
+ \sum_{X \in U_n(C)} (a|Z_C(X)| + m) g_{a,m}(n, |X|)
- \sum_{X \in U_n(B \lor C)} (a|Z_{B\lor C}(X)| + m) g_{a,m}(n, |X|),
\]

and by induction,

\[
\sum_{X \in U_n(A)} (a|Z_A(X)| + m) g_{a,m}(n, |X|) = 1 + 1 - 1 = 1.
\]

\[\square\]

Note that by Lemma 2.2, equations (2) and (4) are consequence of Theorem 2.1.

We shall need the following lemma (see Lemma 4 of [12]).

**Lemma 2.5.** Let \( A_1, A_2, B_1, B_2 \in G_n \) and \( \emptyset \notin A_1 \cup A_2 \cup B_1 \cup B_2 \). Suppose that \( U_n(A_1) \cap D_n(B_2) = \emptyset = U_n(A_2) \cap D_n(B_1) \). Let \( A = A_1 \cup A_2 \) and \( B = B_1 \cup B_2 \). If \( F \) is a non-zero function defined on \( U_n(A) \), then

\[
\sum_{X \in U_n(A) \setminus D_n(B)} \frac{a|Z_A(X)| + m}{F(X)} = \sum_{X \in U_n(A_1) \setminus D_n(B_1)} \frac{a|Z_{A_1}(X)| + m}{F(X)}
+ \sum_{X \in U_n(A_2) \setminus D_n(B_2)} \frac{a|Z_{A_2}(X)| + m}{F(X)}
- \sum_{X \in U_n(A_1 \lor A_2)} \frac{a|Z_{A_1 \lor A_2}(X)| + m}{F(X)}.
\]

\[\square\]

In fact Lemma 2.5 can be proved easily by noting that

\[
U_n(A) \setminus D_n(B) = (U_n(A_1) \setminus (D_n(B_1) \cup U_n(A_2)))
\cup (U_n(A_2) \setminus (D_n(B_2) \cup U_n(A_1))) \cup U_n(A_1 \lor A_2),
\]

and by applying Lemma 2.3.
Lemma 2.6. Let \( a, m \in \mathbb{R} \) and \( n \in \mathbb{N} \). Let \( A, B \) be non-empty subsets of \([n]\). If \( A \subseteq B \), and \( ak + m \neq 0 \) for all \( |A| \leq k \leq n \), then

\[
\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = g_{a,m}(n - |B| + |A|, |A|).
\]

Proof. We may assume that \( A = \{1, 2, \ldots, r_1\} \) and \( B = \{1, 2, \ldots, r_1, r_1 + 1, \ldots, r_2\} \). We shall prove by induction on \( p = r_2 - r_1 \).

Suppose \( p = 0 \), i.e., \( A = B \). Then

\[
\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = g_{a,m}(n, |A|).
\]

Suppose \( p > 1 \). Assume that the lemma holds for \( p' < p \).

Note that \( A \nsubseteq B \) and \( r_2 \notin A \). Set \( B' = B \setminus \{r_2\} \). Then \( A \subseteq B' \), and by Lemma 2.1

\[
\sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = \sum_{A \subseteq X \subseteq B'} g_{a,m}(n, |X|) + \sum_{A \subseteq X \subseteq B'} g_{a,m}(n, |X \cup \{r_2\}|) = \sum_{A \subseteq X \subseteq B'} (g_{a,m}(n, |X|) + g_{a,m}(n, |X| + 1)) = \sum_{A \subseteq X \subseteq B'} g_{a,m}(n - 1, |X|).
\]

By induction \( \sum_{A \subseteq X \subseteq B'} g_{a,m}(n - 1, |X|) = g_{a,m}(n - 1 - |B'| + |A|, |A|) = g_{a,m}(n - |B| + |A|, |A|) \). Hence \( \sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = g_{a,m}(n - |B| + |A|, |A|) \). \( \square \)

Theorem 2.7. Let \( a, m \in \mathbb{R} \) and \( n \in \mathbb{N} \). Let \( A = \{A_1, A_2, \ldots, A_q\} \) and \( B = \{B_1, B_2, \ldots, B_q\} \) be elements in \( G_n \). Suppose that \( A_i \neq \varnothing \) for all \( i \), and \( A_j \subseteq B_k \) if and only if \( j = k \). If \( ak + m \neq 0 \) for all \( \min \{A \in A \mid A \subseteq B \} \), then

\[
\sum_{i=1}^{q} (a|A_i| + m)g_{a,m}(n - |B_i| + |A_i|, |A_i|) + \sum_{X \in U_n(A) \setminus D_n(B)} (a|Z_A(X)| + m) g_{a,m}(n, |X|) = 1. \tag{7}
\]

Proof. Case 1. Suppose \( q = 1 \). Then \( A = \{A_1\} \), \( B = \{B_1\} \), \( \varnothing \neq A_1 \subseteq B_1 \), and \( a|A_1| + m \neq 0 \). Furthermore if \( X \in U_n(A) \), then \( Z_A(X) = A_1 \). By Theorem 2.3

\[
\sum_{X \in U_n(A) \cap D_n(B)} (a|Z_A(X)| + m) g_{a,m}(n, |X|) + \sum_{X \in U_n(A) \setminus D_n(B)} (a|Z_A(X)| + m) g_{a,m}(n, |X|) = 1.
\]

Note that by Lemma 2.6

\[
\sum_{X \in U_n(A) \cap D_n(B)} (a|Z_A(X)| + m) g_{a,m}(n, |X|) = (a|A_1| + m) \sum_{A \subseteq X \subseteq B} g_{a,m}(n, |X|) = (a|A_1| + m) g_{a,m}(n - |B_1| + |A_1|, |A_1|).
\]

Hence the theorem holds.
Case 2. Suppose $q > 1$. Assume that the theorem holds for all $q'$ with $1 \leq q' < q$. Let

$$A_1 = \{A_1, \ldots, A_{q-1}\}, \quad A_2 = \{A_q\},$$
$$B_1 = \{B_1, \ldots, B_{q-1}\}, \quad B_2 = \{B_q\}.$$

Note that $U_n(A_1) \cap D_n(B_2) = \emptyset = U_n(A_2) \cap D_n(B_1)$. By Lemma 2.5 and induction,

$$\sum_{X \in U_n(A_1) \setminus D_n(B)} \frac{a|Z_A(X)| + m}{F(X)} = \left(1 - \sum_{i=1}^{q-1} (a|A_i| + m)g_{a,m}(n - |B_i| + |A_i|, |A_i|)\right) + (1 - (a|A_q| + m)g_{a,m}(n - |B_q| + |A_q|, |A_q|) ) - \sum_{X \in U_n(A_1 \lor A_2)} (a|Z_{A_1 \lor A_2}(X)| + m)g_{a,m}(n, |X|).$$

Note that by Theorem 2.4, the \(\sum_{X \in U_n(A_1 \lor A_2)} (a|Z_{A_1 \lor A_2}(X)| + m)g_{a,m}(n, |X|) = 1\). Hence the theorem holds.

Note that by Lemma 2.2, equations (3) and (5) are consequence of Theorem 2.7.

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