We study conditions for which the mapping torus of a 6-manifold endowed with an SU(3)-structure is a locally conformal calibrated $G_2$-manifold, that is, a 7-manifold endowed with a $G_2$-structure $\varphi$ such that $d\varphi = -\theta \wedge \varphi$ for a closed non-vanishing 1-form $\theta$. Moreover, we show that if $(M, \varphi)$ is a compact locally conformal calibrated $G_2$-manifold with $L_{\theta^\#} \varphi = 0$, where $\theta^\#$ is the dual of $\theta$ with respect to the Riemannian metric $g_\varphi$ induced by $\varphi$, then $M$ is a fiber bundle over $S^1$ with a coupled SU(3)-manifold as fiber.

1. Introduction

A $G_2$-structure on a 7-manifold $M$ can be characterized by the existence of a globally defined 3-form $\varphi$, called the fundamental 3-form, which can be written at each point as
\begin{equation}
\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},
\end{equation}
with respect to some local coframe $\{e^1, \ldots, e^7\}$ on $M$. A $G_2$-structure $\varphi$ induces a Riemannian metric $g_\varphi$ and a volume form $dV_{g_\varphi}$ on $M$ given by
\begin{equation}
g_\varphi(X, Y) dV_{g_\varphi} = \frac{1}{6} i_X \varphi \wedge i_Y \varphi \wedge \varphi,
\end{equation}
for any pair of vector fields $X, Y$ on $M$.

The classes of $G_2$-structures can be described in the terms of the exterior derivatives of the fundamental 3-form $\varphi$ and the 4-form $* \varphi$ [5, 8], where $*$ is the Hodge operator defined from $g_\varphi$ and $dV_{g_\varphi}$. In this paper, we focus our attention on the class of locally conformal calibrated $G_2$-structures, which are characterized by the condition
\begin{equation}
d\varphi = -\theta \wedge \varphi,
\end{equation}
for a closed non-vanishing 1-form $\theta$ also known as the Lee form of the $G_2$-structure. We refer to a manifold endowed with such a structure as a locally conformal calibrated $G_2$-manifold.

A differential complex for locally conformal calibrated $G_2$-manifolds was introduced in [9], where such manifolds were characterized as the ones endowed with a $G_2$-structure $\varphi$ for which the space of differential forms annihilated by $\varphi$ becomes a differential subcomplex of the de Rham’s complex.

Locally conformal calibrated $G_2$-structures $\varphi$ whose underlying Riemannian metric $g_\varphi$ is Einstein were studied in [10], where it was shown that in the compact case the scalar curvature of $g_\varphi$ can not be positive. Moreover, in contrast to a result obtained in the compact homogeneous case, a non-compact example of homogeneous manifold endowed with a locally conformal calibrated $G_2$-structure whose associated Riemannian metric is...
Einstein and non Ricci-flat was given. The homogeneous Einstein metric is a rank-one extension of a Ricci soliton induced on the 3-dimensional complex Heisenberg group by a left-invariant coupled SU(3)-structure \((\omega, \psi_+\rangle\), i.e., such that \(d\omega = c\psi_+\), with \(c \in \mathbb{R} - \{0\}\) (see Definition 2.2 for details on coupled SU(3)-structures).

In the general case, it is not difficult to show that the product of a coupled SU(3)-manifold by \(S^1\) admits a natural locally conformal calibrated \(G_2\)-structure.

In [12], a characterization of compact locally conformal parallel \(G_2\)-manifolds as fiber bundles over \(S^1\) with compact nearly Kähler fiber was obtained (see also [21]).

Banyaga in [2] showed that special types of exact locally conformal symplectic manifolds are fibered over \(S^1\) with each fiber carrying a contact form. In this context, exact means that the locally conformal symplectic structure \(\Omega\) is \(d\theta\)-exact, i.e., \(\Omega = d\alpha + \theta \wedge \alpha\), for a 1-form \(\alpha\) and a closed 1-form \(\theta\).

Exact locally conformal symplectic structures are precisely the structures called of the first kind in Vaisman’s paper [20], where he showed that a manifold endowed with such a structure is a 2-contact manifold and has a vertical 2-dimensional foliation. Moreover, if this foliation is regular, then the manifold is a \(T^2\)-principal bundle over a symplectic manifold.

More in general, by [11] Proposition 3.3], every compact manifold admitting a generalized contact pair of type \((k, 0)\) fibers over the circle with fiber a contact manifold and the monodromy acting by a contactomorphism. Note that a contact pair \((\alpha, \beta)\) of type \((k, 0)\) induces a locally conformal symmetric form defined by \(d\alpha + \alpha \wedge \beta\). Conversely, every mapping torus of a contactomorphism admits a generalized contact pair of type \((k, 0)\) and an induced locally conformal symmetric form.

A theorem by Tischler [19] asserts that the existence of a non-vanishing closed 1-form on a compact manifold \(M\) is equivalent to the condition that \(M\) is a mapping torus. In the last years, mapping tori of certain diffeomorphisms have been very useful to study some geometric structures. For example, Li in [14] proved that any compact cosymplectic manifold is the mapping torus of a symplectomorphism. Furthermore, formality of mapping tori was studied in [3].

A natural problem is then to find a characterization of \(d\theta\)-exact locally conformal calibrated \(G_2\)-manifolds with Lee form \(\theta\) and to see under which conditions a locally conformal calibrated \(G_2\)-manifold is a mapping torus of a special type of SU(3)-manifold or, more in general, it is fibered over \(S^1\) with fiber endowed with a special type of SU(3)-structure.

In Section 3 we will show that if \(N\) is a six-dimensional, compact, connected coupled SU(3)-manifold, and \(\nu: N \rightarrow N\) is a diffeomorphism preserving the SU(3)-structure of \(N\), then the mapping torus \(N_\nu\) of \(\nu\) admits a locally conformal calibrated \(G_2\)-structure. Moreover, in the same section, we will show that a result of this kind also holds for a compact nearly Kähler SU(3)-manifold, which is a particular case of coupled SU(3)-manifold. In detail, we will prove that the mapping torus of a nearly Kähler SU(3)-manifold with respect to a diffeomorphism preserving the nearly Kähler structure is endowed with a locally conformal parallel \(G_2\)-structure.

In a similar way as in the paper [2] on locally conformal symplectic manifolds, in Section 4 we find some characterizations for locally conformal calibrated \(G_2\)-structures \(\varphi\) which are \(d\theta\)-exact, that is, such that \(\varphi = d\phi = d\omega + \theta \wedge \omega\), where \(\theta\) is the Lee form of \(\varphi\) and \(\omega\) is a 2-form on \(M\). In fact, for a locally conformal calibrated \(G_2\)-manifold \((M, \varphi)\),
we prove in Proposition 4.3 that if $X$ is the $g_\varphi$-dual vector field of $\theta$ and $\omega$ is the 2-form given by $\omega = i_X \varphi$, then $X$ is an infinitesimal automorphism of $\varphi$ ($L_X \varphi = 0$) if and only if $\theta(X) \varphi = d\omega$.

Section 5 is devoted to the construction of Lie algebras admitting a locally conformal calibrated $G_2$-structure from six-dimensional Lie algebras endowed with a coupled SU(3)-structure. As a consequence, new examples of compact manifolds with a locally conformal calibrated $G_2$-structure are constructed.

Finally, in the last section we obtain a characterization for compact locally conformal calibrated $G_2$-manifolds $(M, \varphi)$ under the assumption $L_\theta \# \varphi = 0$, where $\theta$ is the dual of the Lee form $\theta$ with respect to the Riemannian metric $g_\varphi$ induced by $\varphi$. More precisely, in Theorem 6.4 we show that $M$ is a fiber bundle over $S^1$ such that each fiber is equipped with a coupled SU(3)-structure.

2. Preliminaries

Given a seven-dimensional manifold $M$ endowed with a $G_2$-structure $\varphi$, let $g_\varphi$ and $dV_{g_\varphi}$ denote respectively the Riemannian metric and the volume form on $M$ induced by $\varphi$ via the relation (2).

A manifold endowed with a $G_2$-structure $\varphi$ is said to be locally conformal calibrated $G_2$-manifold if

$$d\varphi = -\theta \wedge \varphi,$$

where $\theta$ is a non-vanishing 1-form which has to be closed. $\theta$ is called the Lee form associated to the locally conformal calibrated $G_2$-structure $\varphi$ and can be defined as

$$\theta = \frac{1}{4} \ast (\ast d\varphi \wedge \varphi),$$

where $\ast$ is the Hodge star operator defined from the metric $g_\varphi$ and the volume form $dV_{g_\varphi}$.

If the Lee form vanishes, then the 3-form $\varphi$ is closed and the $G_2$-structure is called calibrated.

A $G_2$-structure $\varphi$ is said to be locally conformal parallel if

$$d\varphi = -\theta \wedge \varphi, \quad d \ast \varphi = -\frac{4}{3} \theta \wedge \ast \varphi.$$

By [11, Theorem 3.1], given a compact manifold $M$ admitting a $G_2$-structure $\varphi$, there exists a unique (up to homothety) conformal $G_2$-structure $e^M \varphi$ such that the corresponding Lee form is coclosed. A $G_2$-structure with co-closed Lee form is also called a Gauduchon $G_2$-structure.

Given a closed 1-form $\tau$ on $M$, we say that a $G_2$-structure $\varphi$ is $d_\tau$-exact if there exists a 2-form $\alpha \in \Lambda^2(M)$ such that $\varphi = d\alpha + \tau \wedge \alpha$. In this case, a simple computation shows that the $G_2$-structure is locally conformal calibrated with Lee form $\tau$. The converse is not true in general.

Remark 2.1. Given a locally conformal calibrated $G_2$-structure $\varphi$, we can consider the class

$$[\varphi] = \{ f \varphi : f \in C^\infty(M) \text{ and } f > 0 \}$$
of locally conformal calibrated $G_2$-structures which are conformally equivalent to $\varphi$. If $d\varphi = -\theta \wedge \varphi$, then $d(f \varphi) = (d(ln f) - \theta) \wedge f \varphi$ and $\varphi$ is $d\theta$-exact if and only if $f \varphi$ is $d(ln f) - \theta$-exact. Thus, being $d\theta$-exact is a conformal property for locally conformal calibrated $G_2$-structures.

Recall that an SU(3)-structure $(\omega, \psi_+)$ on a 6-manifold $N$ is said to be half-flat if both $\omega^2 = \omega \wedge \omega$ and $\psi_+$ are closed. By [1N], if a half-flat SU(3)-structure on a six-dimensional connected manifold $N$ is such that $d\omega = f \psi_+$ for some non-vanishing function $f \in C^\infty(N)$, then $f$ has to be a constant function. This motivates the following

**Definition 2.2.** Let $N$ be a six-dimensional connected manifold and let $(\omega, \psi_+)$ be an SU(3)-structure on it. We say that $(\omega, \psi_+)$ is a coupled SU(3)-structure if $d\omega = c \psi_+$ for some nonzero real constant $c$ called the coupled constant.

Given an SU(3)-structure $(\omega, \psi_+)$, denote by $J$ the almost complex structure induced by $\psi_+$, by $\psi_- = J \psi_+$ and by $h(\cdot, \cdot) = \omega(\cdot, J\cdot)$ the induced Riemannian metric. By [17], since an SU(3)-structure on a six-dimensional manifold $N$ is characterized by a pair of stable, compatible ad normalized forms $(\omega, \psi_+) \in \Lambda^2(N) \times \Lambda^3(N)$ inducing a Riemannian metric and since the construction of $J, \psi_-$ and $h$ is invariant, a diffeomorphism preserving the forms $\omega$ and $\psi_+$ is actually an automorphism of the SU(3)-structure itself and, in particular, an isometry. As a special case of this, if we have a coupled SU(3)-structure with $d\omega = c \psi_+$ for some nonzero constant $c$, then every diffeomorphism of $N$ preserving $\omega$ is an automorphism of the SU(3)-structure and an isometry.

**Remark 2.3.** Note that if $(\omega, \psi_+)$ is a nearly Kähler SU(3)-structure ($d\omega = 3\psi_+, d\psi_- = -2\omega^2$), then it is in particular a coupled SU(3)-structure. Moreover, if $(\omega, \psi_+)$ is a coupled SU(3)-structure with $d\omega = c \psi_+$, then the pair $\hat{\omega} := k^2 \omega, \hat{\psi}_+ := k^3 \psi_+$, where $k$ is a nonzero real constant, is still a coupled SU(3)-structure with $d\hat{\omega} = \frac{c}{k^2} \hat{\psi}_+$. As a consequence, it is always possible to find a coupled structure having coupled constant $c = 1$. Changing the constant $c$ of a coupled SU(3)-structure in the way just described, the almost complex structure $J$ is preserved, while the Riemannian metric $h$ is rescaled by $k^2$.

The following proposition shows a first link between coupled SU(3)-structures and locally conformal calibrated $G_2$-structures.

**Proposition 2.4.** Let $N$ be a six-dimensional connected manifold endowed with a coupled SU(3)-structure $(\omega, \psi_+)$ with coupled constant $c$, then the cylinder $(N \times \mathbb{R}, h + dr^2)$ over $N$ has a locally conformal calibrated $G_2$-structure given by $\varphi = \omega \wedge dr + \psi_+$ and such that $g_\varphi = h + dr^2$. Moreover, if $c \neq 3$, also the cone over $N$, $(N \times (0, +\infty), r^2 h + dr^2)$, is endowed with a locally conformal calibrated $G_2$-structure $\varphi = r^2 \omega \wedge dr + r^3 \psi_+$ whose associated metric $g_\varphi$ is the cone’s one.

### 3. Mapping Torus

Let $N$ be a compact manifold and $\nu : N \to N$ a diffeomorphism. The mapping torus $N_\nu$ of $\nu$ is the quotient space of $N \times [0, 1]$ in which any point $(p, 0)$ is identified with $(\nu(p), 1)$. $N_\nu$ is naturally a smooth manifold, since it is the quotient of $N \times \mathbb{R}$ by the infinite cyclic group generated by the diffeomorphism $(p, t) \mapsto (\nu(p), t + 1)$. The natural map $\pi : N_\nu \to S^1$
defined by \( \pi(p, t) = e^{2\pi it} \) is the projection of a locally trivial fiber bundle (here we think \( S^1 \) as the interval \([0, 1]\) with identified end points). Thus, any \( \nu \)-invariant form \( \alpha \) on \( N \)
defines a form \( \bar{\alpha} \) on \( N_\nu \) since the pullback of \( \alpha \) to \( N \times \mathbb{R} \) is invariant by the diffeomorphism \((p, t) \mapsto (\nu(p), t + 1)\). For the same reason, the 1-form \( dt \) on \( \mathbb{R} \), where \( t \) is the coordinate on \( \mathbb{R} \), induces a closed 1-form \( \eta \) on \( N_\nu \). Moreover, on \( N_\nu \) we have a distinguished vector field \( \xi \) induced by the vector field \( \frac{d}{dt} \). This vector field is such that \( \eta(\xi) = 1 \).

In the case of a compact manifold endowed with a coupled SU(3)-structure \((\omega, \psi_+)\) we can prove the following

**Proposition 3.1.** Let \( N \) be a six-dimensional, compact, connected manifold endowed with a coupled SU(3)-structure \((\omega, \psi_+)\) with \( d\omega = c\psi_+ \) and let \( \nu : N \to N \) be a diffeomorphism such that \( \nu^* \omega = \omega \). Then the mapping torus \( N_\nu \) admits a locally conformal calibrated \( G_2 \)-structure \( \varphi \). Moreover, \( \mathcal{L}_\xi \varphi = 0 \).

**Proof.** We have the following situation

\[
\begin{array}{cccc}
N & \xrightarrow{p_1} & N \times [0, 1] & \xrightarrow{q} & N_\nu \\
\downarrow p_2 & & \downarrow \pi & & \\
[0, 1] & \xrightarrow{\Pi} & S^1 \\
\end{array}
\]

where \( p_1 \) and \( p_2 \) are the projections from \( N \times [0, 1] \) on the first and on the second factor respectively, \( q \) is the quotient map, \( \pi \) is the fibration map and \( \Pi(t) = e^{2\pi it} \).

Let us observe that \( p_1^* (\omega) \in \Lambda^2(N \times [0, 1]) \) and \( p_1^* (\tilde{\psi}_+) \in \Lambda^3(N \times [0, 1]) \). Then, since \( \nu^* \omega = \omega \) and (as a consequence) \( \nu^* \tilde{\psi}_+ = \tilde{\psi}_+ \), we can glue up these pullbacks and obtain a 2-form \( \tilde{\omega} \in \Lambda^2(N_\nu) \) and 3-forms \( \tilde{\psi}_+ \in \Lambda^3(N_\nu) \) satisfying the same relations that hold on \( N \).

In particular, since \( d(p_1^* \omega) = c p_1^* \tilde{\psi}_+ \), we have

\[
d\tilde{\omega} = c \tilde{\psi}_+, \quad d\tilde{\omega}^2 = 0.
\]

Using the closed 1-form \( \eta \) defined on \( N_\nu \) we have that

\[
\varphi = \eta \wedge \tilde{\omega} + \tilde{\psi}_+
\]
is a 3-form on \( N_\nu \) defining a \( G_2 \)-structure on it. Moreover,

\[
d\varphi = d\eta \wedge \tilde{\omega} - \eta \wedge d\tilde{\omega} + d\tilde{\psi}_+ = -c\eta \wedge \tilde{\psi}_+ = -c\eta \wedge \varphi,
\]

that is, \( \varphi \) is locally conformal calibrated with Lee form \( \theta = c\eta \).

Since both \( \tilde{\omega} \) and \( \tilde{\psi}_+ \) derive from differential forms defined on \( N \), we have \( i_\xi \tilde{\omega} = 0 \) and \( i_\xi \tilde{\psi}_+ = 0 \). From these conditions it follows that \( i_\xi \varphi = \tilde{\omega} \) and, consequently, \( \mathcal{L}_\xi \varphi = 0 \) \( \square \).

**Remark 3.2.** As we mentioned before, if \((\omega, \psi_+)\) is a pair of compatible, normalized, stable forms on \( N \), then the remaining tensors appearing in the definition of an SU(3)-structure, namely the almost complex structure \( J \), the 3-form \( \psi_- = J\psi_+ \) and the Riemannian metric \( h \), can be completely determined from \((\omega, \psi_+)\). Thus, the diffeomorphism \( \nu \) preserves not only the 2-form \( \omega \) and the 3-form \( \tilde{\psi}_+ \) on \( N \) but also the Riemannian metric \( h \) of the SU(3)-structure \((\omega, \psi_+)\). Hence, (globally) the metric \( g_\varphi \) on \( N_\nu \) is \( g_\varphi = h + \eta^2 \) and we have that \( \xi \) is the vector field dual to \( \eta \) with respect to \( g_\varphi \).
Example 3.3. The previous proposition can be applied to compact nilmanifolds admitting a coupled SU(3)-structure. Nilpotent Lie algebras admitting a coupled SU(3)-structure were classified in [10]. One of these is the Iwasawa Lie algebra, i.e., the Lie algebra of the complex Heisenberg group of complex dimension 3

\[
G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}, \ z_i \in \mathbb{C}, \ i = 1, 2, 3 \right\}.
\]

This complex Lie group admits a co-compact discrete subgroup \( \Gamma \) which is defined as the subgroup of \( G \) for which \( z_i \) are Gaussian integers. Consider the automorphism \( \nu : G \to G, \)

\[
\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & z_1 & -iz_3 \\ 0 & 1 & -iz_2 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then, if we denote by

\[
e^1 + ie^2 = dz_1, \quad e^3 + ie^4 = dz_2, \quad e^5 + ie^6 = -dz_3 + z_1 \wedge dz_2,
\]

we have

\[
\nu^*(e^1) = e^1, \quad \nu^*(e^2) = e^2, \quad \nu^*(e^3) = e^4, \quad \nu^*(e^4) = -e^3, \quad \nu^*(e^5) = e^6, \quad \nu^*(e^6) = -e^5.
\]

The nilmanifold \( N = \Gamma \backslash G \) admits the coupled SU(3)-structure \( (\omega, \psi_+) \) defined by

\[
\omega = e^{12} + e^{34} - e^{56}, \quad \psi_+ = e^{136} - e^{145} - e^{235} - e^{246}.
\]

Since \( \nu^* \omega = \omega \), we can apply Proposition 3.1 and the mapping torus \( N_\nu \) admits a locally conformal calibrated \( G_2 \)-structure.

Remark 3.4. Since by Remark 2.3 we can always suppose that the coupled constant is \( c = 1 \), the form \( \varphi = \eta \wedge \tilde{\omega} + \tilde{\psi}_+ \) is \( d_{\eta} \)-exact with \( \varphi = d_{\eta} \tilde{\omega} \).

As we observed in Remark 2.3, a special case of coupled SU(3)-structures is given by the nearly Kähler SU(3)-structures. In this case, we can prove what follows.

Proposition 3.5. Let \( N \) be a six-dimensional, compact, connected manifold endowed with a nearly Kähler SU(3)-structure \( (\omega, \psi_+) \) and let \( \nu : N \to N \) be a diffeomorphism such that \( \nu^* \omega = \omega \). Then the mapping torus \( N_\nu \) admits a locally conformal parallel \( G_2 \)-structure.

Proof. As in Proposition 3.1, we can define the differential forms \( \tilde{\omega} \in \Lambda^2(\nu_\nu) \) and \( \tilde{\psi}_\pm \in \Lambda^3(\nu_\nu) \), which in this case satisfy the relations

\[
d\tilde{\omega} = 3\tilde{\psi}_+, \quad d\tilde{\psi}_- = -2\tilde{\omega}^2.
\]

The positive 3-form

\[
\varphi = \eta \wedge \tilde{\omega} + \tilde{\psi}_+
\]

defines a \( G_2 \)-structure on \( N_\nu \) with Hodge dual

\[
\ast \varphi = \tilde{\psi}_- \wedge \eta + \frac{1}{2} \tilde{\omega}^2.
\]
It follows from computations that
\begin{align*}
d\varphi &= 3(-\eta) \wedge \varphi, \\
d \ast \varphi &= 4(\eta) \wedge \ast \varphi.
\end{align*}
Therefore, \( \varphi \) is a locally conformal parallel \( G_2 \)-structure defined on \( N_\nu \). \hfill \Box

**Example 3.6.** Consider the six-dimensional compact manifold \( S^3 \times S^3 \). As a Lie group it is \( \text{SU}(2) \times \text{SU}(2) \) and its Lie algebra is \( \text{su}(2) \oplus \text{su}(2) \). Let \( \{e_1, e_2, e_3\} \) denote the standard basis for the first copy of \( \text{su}(2) \), let \( \{e_4, e_5, e_6\} \) denote it for the second one and let \( \{e^1, e^2, e^3\} \) and \( \{e^4, e^5, e^6\} \) denote their dual bases. The structure equations of \( \text{su}(2) \oplus \text{su}(2) \) are:

\begin{align*}
de e^1 &= e^{23}, \quad de e^2 = e^{31}, \quad de e^3 = e^{12}, \\
de e^4 &= e^{56}, \quad de e^5 = e^{64}, \quad de e^6 = e^{45}.
\end{align*}

On \( \text{su}(2) \oplus \text{su}(2) \) we have a pair of stable, compatible, normalized forms
\begin{align*}
\omega &= -\frac{\sqrt{3}}{36} e^{14} + e^{25} + e^{36}, \\
\psi^+ &= \frac{\sqrt{3}}{36} (-e^{234} + e^{156} + e^{135} - e^{246} - e^{126} + e^{345}),
\end{align*}

defining an \( \text{SU}(3) \)-structure which is nearly Kähler and induces a left-invariant nearly Kähler \( \text{SU}(3) \)-structure on \( S^3 \times S^3 \).

Let \( \nu : S^3 \times S^3 \to S^3 \times S^3 \) be the diffeomorphism such that
\[
\nu^* e^1 = e^1, \quad \nu^* e^2 = e^3, \quad \nu^* e^3 = -e^2, \quad \nu^* e^4 = e^4, \quad \nu^* e^5 = e^6, \quad \nu^* e^6 = -e^5.
\]
\( \nu \) preserves \( \omega \), therefore the mapping torus \((S^3 \times S^3)_\nu\) is endowed with a locally conformal parallel \( G_2 \)-structure by the previous proposition.

A characterization of compact locally conformal parallel \( G_2 \)-manifolds as fiber bundles over \( S^1 \) with compact nearly Kähler fiber was obtained in \cite{12} (see also \cite{21}). It was also shown there that for compact seven-dimensional locally conformal parallel \( G_2 \)-manifolds \((M, \varphi)\) with co-closed Lee form \( \theta \), the Lee flow preserves the Gauduchon \( G_2 \)-structure, i.e., \( \mathcal{L}_\theta \# \varphi = 0 \), where \( \theta^\# \) is the dual of \( \theta \) with respect to \( g_\varphi \). In the next section, we will characterize the locally conformal calibrated \( G_2 \)-structures such that \( \mathcal{L}_\theta \# \varphi = 0 \).

**4. Exact locally conformal calibrated \( G_2 \)-manifolds**

In a similar way as in the paper \cite{2} on locally conformal symplectic manifolds, we can find some characterizations for \( d_\theta \)-exact locally conformal calibrated \( G_2 \)-structures \( \varphi \) with Lee form \( \theta \).

We recall that \( X \in \mathfrak{X}(M) \) is a **conformal infinitesimal automorphism** of \( \varphi \) if an only if there exists a smooth function \( \rho_X \) on \( M \) such that \( \mathcal{L}_X \varphi = \rho_X \varphi \). If \( \rho_X \equiv 0 \), then \( X \) is a **conformal automorphism** of \( \varphi \). We start by proving the following

**Lemma 4.1.** Let \((M, \varphi)\) be a locally conformal calibrated \( G_2 \)-manifold with Lee form \( \theta \). A vector field \( X \in \mathfrak{X}(M) \) is a conformal infinitesimal automorphism of \( \varphi \) if and only if there exists a smooth function \( f_X \in C^\infty(M) \) such that \( d_\theta \omega = f_X \varphi \), where \( \omega = i_X \varphi \). Moreover, if \( M \) is connected, \( f_X \) is constant.
Proof. Let us compute the expression of the Lie derivative of $\varphi$ with respect to $X$

$$L_X \varphi = d(i_X \varphi) + i_X (d \varphi)$$

$$= d\omega + i_X (-\theta \wedge \varphi)$$

$$= d\omega - \theta(X) \varphi + \theta \wedge (i_X \varphi)$$

$$= d\omega + \theta \wedge \omega - \theta(X) \varphi$$

$$= d\theta \omega - \theta(X) \varphi,$$

where $\omega = i_X \varphi$. Therefore, $X$ is a conformal infinitesimal automorphism of $\varphi$ with $L_X \varphi = \rho_X \varphi$ if and only if $d\theta \omega = f_X \varphi$, where $f_X$ is a smooth real valued function on $M$ such that $f_X = \rho_X + \theta(X)$.

Suppose now that $M$ is connected and let $X$ be a conformal infinitesimal automorphism of $\varphi$, as we have just shown $d\theta \omega = f_X \varphi$ for some $f_X \in C^\infty(M)$. We have

$$0 = d\theta(d\theta \omega)$$

$$= d\theta(f_X \varphi)$$

$$= d(f_X \varphi) + \theta \wedge (f_X \varphi)$$

$$= df_X \wedge \varphi + f_X d\varphi + f_X \theta \wedge \varphi$$

$$= df_X \wedge \varphi + f_X d\varphi - f_X d\varphi$$

$$= df_X \wedge \varphi.$$

Since the linear mapping $\wedge \varphi : \Lambda^1(M) \to \Lambda^4(M)$ is injective, we obtain that $df_X = 0$ and from this the assertion follows. □

Remark 4.2. It is worth emphasizing here that if $X$ is a conformal infinitesimal automorphism of $\varphi$ with $f_X$ a nonzero constant, then $\varphi$ is $d\theta$-exact. Indeed

$$\varphi = \frac{1}{f_X} d\theta \omega = d\theta \left( \frac{\omega}{f_X} \right).$$

Recall the result contained in [15]:

Lemma 4.3 ([15]). Let $M$ be a seven-dimensional, compact manifold. Then for any $G_2$-structure $\varphi$ on $M$, any vector field $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ it holds

$$\int_M L_X \varphi \wedge * f \varphi = -3 \int_M df \wedge * X^\varphi.$$

From this Lemma with $f \equiv 1$ and $X$ conformal infinitesimal automorphism of $\varphi$ with $L_X \varphi = \rho_X \varphi$ we have

$$\int_M \rho_X dV_{g_\varphi} = 0.$$

Thus, thinking at the proof of Lemma 4.1 we get

$$\int_M \theta(X) dV_{g_\varphi} = \int_M f_X dV_{g_\varphi} = f_X \text{Vol}(M),$$

that is, the Riemannian integral of the function $\theta(X)$ over $M$ is constant.

In conclusion, we can prove the following characterization for a $d\theta$-exact locally conformal calibrated $G_2$-structure.
Proposition 4.4. Let \((M, \varphi)\) be a connected locally conformal calibrated \(G_2\)-manifold with Lee form \(\theta\). Let \(X\) be the \(g_\varphi\)-dual vector field of \(\theta\), i.e., \(\theta(\cdot) = g_\varphi(X, \cdot)\), and define the 2-form \(\omega := i_X \varphi\). Then \(\mathcal{L}_X \varphi = 0\) if and only if \(\theta(X) \varphi = d\theta\omega\). Moreover, if \(\mathcal{L}_X \varphi = 0\), then \(\theta(X) = |X|^2\) is a nonzero constant.

Proof. We have
\[
\mathcal{L}_X \varphi = d(i_X \varphi) + i_X d\varphi = d\omega + i_X (\theta \wedge \varphi) = d\omega - \theta(X) \varphi + \theta \wedge \omega.
\]
Therefore, \(\mathcal{L}_X \varphi = 0\) if and only if \(\theta(X) \varphi = d\theta\omega\).

If \(\mathcal{L}_X \varphi = 0\), from Lemma 4.1 we have that \(\theta(X) = |X|^2\) is a nonzero constant, since \(\theta(X) \varphi = d\theta\omega\) and \(X = \theta#\), where the map \(\cdot#: \Lambda^1(M) \to \mathfrak{X}(M)\) is an isomorphism. \(\square\)

5. Locally conformal calibrated \(G_2\) Lie algebras

In this section, we show that locally conformal calibrated \(G_2\)-structures defined on seven-dimensional Lie algebras are closely related to coupled SU(3)-structures on six-dimensional Lie algebras. This generalizes the result proved in [16] for calibrated \(G_2\)-structures on seven-dimensional Lie algebras from symplectic half-flat Lie algebras. First, we need to recall some definitions and results about SU(3)- and \(G_2\)-structures on Lie algebras.

Let \(g\) be a seven-dimensional Lie algebra. A \(G_2\)-structure on \(g\) is a 3-form \(\varphi\) on \(g\) which can be written as in (1) with respect to some basis \(\{e_1, \ldots, e_7\}\) of the dual space \(g^*\) of \(g\). \(\varphi\) is said to be a locally conformal calibrated \(G_2\)-structure on \(g\) if
\[
d\varphi = -\theta \wedge \varphi,
\]
for some closed 1-form \(\theta\) on \(g\), where \(d\) denotes the Chevalley-Eilenberg differential on \(g^*\).

We say that a six-dimensional Lie algebra \(h\) has an SU(3)-structure if there exists a pair \((\omega, \psi^+\)) of forms on \(h\), where \(\omega\) is a 2-form and \(\psi^\#\) a 3-form, which can be expressed as
\[
\omega = e^{12} + e^{34} + e^{56}, \quad \psi^+ = e^{135} - e^{146} - e^{236} - e^{245}
\]
with respect to some basis \(\{e^1, \ldots, e^6\}\) of the dual space \(h^*\). If \(\{e^1, \ldots, e^6\}\) is such a basis, the dual basis \(\{e_1, \ldots, e_6\}\) of \(h\) is called SU(3)-basis. An SU(3)-structure \((\omega, \psi^+)\) on \(h\) is said to be a coupled SU(3)-structure if
\[
d\omega = c\psi^+
\]
for some nonzero real constant \(c\), where \(d\) is the Chevalley-Eilenberg differential on \(h^*\).

If \(h\) is a six-dimensional Lie algebra and \(D \in \text{Der}(h)\) is a derivation of \(h\), then the vector space
\[
g = h \oplus D \mathbb{R} \xi
\]
is a Lie algebra with the Lie bracket given by
\[
[U, V] = [U, V]|_h, \quad [\xi, U] = D(U),
\]
for any \(U, V \in h\).

It is well known that there exists a real representation of the 3 \(\times 3\) complex matrices via
\[
\rho: \mathfrak{gl}(3, \mathbb{C}) \to \mathfrak{gl}(6, \mathbb{R}).
\]
More in detail, if $A \in \mathfrak{gl}(3, \mathbb{C})$, then $\rho(A)$ is the matrix $(B_{ij})_{i,j=1}^3$ with

$$B_{ij} = \begin{pmatrix} \text{Re}A_{ij} & \text{Im}A_{ij} \\ -\text{Im}A_{ij} & \text{Re}A_{ij} \end{pmatrix},$$

where $A_{ij}$ is the $(i,j)$ component of $A$.

Now, suppose that $(\omega, \psi_+)$ is a coupled SU(3)-structure on a six-dimensional Lie algebra $\mathfrak{h}$, and let $D$ be a derivation of $\mathfrak{h}$ such that $D = \rho(A)$, where $A \in \mathfrak{sl}(3, \mathbb{C})$. Then, the matrix representation of $D$ with respect to an SU(3)-basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{h}$ is

$$D = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
  -a_{12} & a_{11} & -a_{14} & a_{13} & -a_{16} & a_{15} \\
  a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
  -a_{32} & a_{31} & -a_{34} & a_{33} & -a_{36} & a_{35} \\
  a_{51} & a_{52} & a_{53} & a_{54} & -a_{11} - a_{33} & -a_{12} - a_{34} \\
  -a_{52} & a_{51} & -a_{54} & a_{53} & a_{12} + a_{34} & -a_{11} - a_{33}
\end{pmatrix},$$

where $a_{ij} \in \mathbb{R}$.

**Proposition 5.1.** Let $(\omega, \psi_+)$ be a coupled SU(3)-structure on a Lie algebra $\mathfrak{h}$ of dimension 6 and let $D = \rho(A)$, $A \in \mathfrak{sl}(3, \mathbb{C})$, be a derivation of $\mathfrak{h}$ whose matrix representation with respect to an SU(3)-basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{h}$ is as in (4). Then, the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus D \mathbb{R}\xi,$$

with the Lie bracket given by (3), has a locally conformal calibrated $G_2$-structure.

**Proof.** We define the $G_2$ form $\varphi$ on $\mathfrak{g} = \mathfrak{h} \oplus D \mathbb{R}\xi$ by

$$\varphi = \omega \wedge \eta + \psi_+, \quad (5)$$

where $\eta$ is the 1-form on $\mathfrak{g}$ such that $\eta(X) = 0$ for all $X \in \mathfrak{h}$ and $\eta(\xi) = 1$. We will see that

$$d\varphi = -c\eta \wedge \varphi,$$

where $c$ is the coupled constant of the coupled SU(3)-structure on $\mathfrak{h}$.

Suppose that $X, Y, Z, U \in \mathfrak{h}$. Then, it is clear that $(d\omega \wedge \eta)(X, Y, Z, U) = 0$, and

$$d\varphi(X, Y, Z, U) = d\psi_+(X, Y, Z, U) = \hat{d}\psi_+(X, Y, Z, U) = 0,$$

since $\psi_+$ is $\hat{d}$-closed and $d\eta = 0$.

Let us consider $X, Y, Z \in \mathfrak{h}$. Using (5), we have

$$d\varphi(X, Y, Z, \xi) = -\varphi([X, Y], Z, \xi) + \varphi([X, Z], Y, \xi) - \varphi([X, \xi], Y, Z) - \varphi([Y, Z], X, \xi) + \varphi([Y, \xi], X, Z) - \varphi([Z, \xi], X, Y) = -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) - \psi_+([X, \xi], Y, Z) + \psi_+([Y, \xi], X, Z) - \psi_+([Z, \xi], X, Y) = d\omega(X, Y, Z) + \psi_+(D(X), Y, Z) + \psi_+(X, D(Y), Z) + \psi_+(X, Y, D(Z)).$$
Taking into account the expressions of $D$ and $\psi_+$ in terms of the SU(3)-basis $\{e_1, \ldots, e_6\}$, it is easy to check that

$$\psi_+(D(e_i), e_j, e_k) + \psi_+(e_i, D(e_j), e_k) + \psi_+(e_i, e_j, D(e_k)) = 0,$$

for every triple $(e_i, e_j, e_k)$ of elements of the SU(3)-basis. Therefore,

$$d\varphi(X, Y, Z, \xi) = d\omega(X, Y, Z) = \hat{d}\omega(X, Y, Z) = c\psi_+(X, Y, Z).$$

Using (5) again, we get

$$d\varphi(X, Y, Z, \xi) = - (c\eta \wedge \varphi)(X, Y, Z, \xi),$$

which completes the proof that the 3-form $\varphi$ given by (5) defines a locally conformal calibrated $G_2$-structure on $\mathfrak{g}$. □

As an application of the previous proposition, we describe two examples of non-isomorphic solvable Lie algebras endowed with a locally conformal calibrated $G_2$-structure. They are obtained considering two different derivations on the Iwasawa Lie algebra.

**Example 5.2.** Consider the six-dimensional Iwasawa Lie algebra $\mathfrak{n}$ and let $\{e_1, \ldots, e_6\}$ denote an SU(3)-basis for it. With respect to the dual basis $\{e_1, \ldots, e_6\}$, the structure equations of $\mathfrak{n}$ are

$$(0, 0, 0, 0, e^{14} + e^{23}, e^{13} - e^{24})$$

and the pair

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

defines a coupled SU(3)-structure on $\mathfrak{n}$ with $\hat{d}\omega = -\psi_+$.

Let $D$ be the derivation of $\mathfrak{n}$ defined as follows

$$De_1 = -e_3, \quad De_2 = -e_4, \quad De_3 = e_1, \quad De_4 = e_2, \quad De_5 = 0, \quad De_6 = 0.$$ 

The Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus_D \mathbb{R} e_7$ has the following structure equations with respect to the basis $\{e^1, \ldots, e^6, e^7\}$ of $\mathfrak{s}^*$

$$(e^{37}, e^{47}, -e^{17}, -e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0).$$

By Proposition 5.1, the 3-form

$$\varphi = \omega \wedge e^7 + \psi_+$$

defines then a locally conformal calibrated $G_2$-structure on $\mathfrak{s}$ with Lee form $\theta = -e^7$.

Let $S$ denote the simply connected solvable Lie group with Lie algebra $\mathfrak{s}$, let $N$ denote the simply connected nilpotent Lie group such that $Lie(N) = \mathfrak{n}$ and let $e \in N$ denote the identity element. Observe that $S = \mathbb{R} \ltimes_\mu N$, where $\mu$ is the unique smooth action of $\mathbb{R}$ on $N$ such that $\mu(t)e = \exp(tD)$, for any $t \in \mathbb{R}$, and where $\exp$ denotes the map $\exp : \text{Der}(\mathfrak{n}) \to \text{Aut}(\mathfrak{n})$. Hence, $S$ is almost nilpotent in the sense of [13].
Now, in order to show a lattice of \( S \) we proceed as follows. The SU(3)-basis \( \{e_1, \ldots, e_6\} \) we considered is a rational basis for \( n \) and with respect to this basis we have

\[
\exp(tD) = \begin{bmatrix}
\cos(t) & 0 & \sin(t) & 0 & 0 & 0 \\
0 & \cos(t) & 0 & \sin(t) & 0 & 0 \\
-\sin(t) & 0 & \cos(t) & 0 & 0 & 0 \\
0 & -\sin(t) & 0 & \cos(t) & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

In particular, \( \exp(\pi D) \) is an integer matrix. Therefore, \( \exp(N(\mathbb{Z}\langle e_1, \ldots, e_6 \rangle)) \) is a lattice of \( N \) preserved by \( \mu(\pi) \) and, consequently,

\[
\Gamma = \pi\mathbb{Z} \ltimes \mu \exp(N(\mathbb{Z}\langle e_1, \ldots, e_6 \rangle))
\]

is a lattice in \( S \) (see [4]). Thus, the compact quotient \( \Gamma \setminus S \) is a compact solvmanifold endowed with an invariant locally conformal calibrated \( G_2 \)-structure \( \varphi \). Moreover, if \( X = -e_7 \) denotes the \( g_\varphi \)-dual vector field of the Lee form \( \theta = -e^7 \), then \( i_X \varphi = -\omega \), \( \mathcal{L}_X \varphi = 0 \) and \( \varphi = d_\theta(-\omega) \), as we expected from Proposition [4.4].

**Example 5.3.** Let us consider the coupled SU(3)-structure \( (\omega, \psi_+) \) on the Iwasawa Lie algebra \( n \) described in the previous example and the derivation \( D \in \text{Der}(n) \) given by

\[
De_1 = 2e_3, \quad De_2 = 2e_4, \quad De_3 = e_1, \quad De_4 = e_2, \quad De_5 = 0, \quad De_6 = 0,
\]

with respect to the SU(3)-basis \( \{e_1, \ldots, e_6\} \) of \( n \). Then, the Lie algebra \( q = n \oplus D \mathbb{R}e_7 \) has the following structure equations with respect to the basis \( \{e^1, \ldots, e^6, e^7\} \) of \( q^* \):

\[
\{e^{37}, e^{47}, 2e^{17}, 2e^{27}, e^{14} + e^{23}, e^{13} - e^{24}, 0\}.
\]

The 3-form

\[
\varphi = \omega \wedge e^7 + \psi_+
\]

defines a locally conformal calibrated \( G_2 \)-structure on \( q \) with Lee form \( \theta = -e^7 \) by Proposition [5.1].

In this case, it is easy to check that if \( X = -e_7 \) denotes the \( g_\varphi \)-dual vector field of \( \theta \), then \( \mathcal{L}_X \varphi \neq 0 \) and, accordingly with Proposition [4.4] \( \varphi \neq d_\theta(i_X \varphi) \). However, \( \varphi \) is \( d_\theta \)-exact. Indeed, \( \varphi = d_\theta \gamma \), where

\[
\gamma = \frac{5}{7}e^{12} - \frac{3}{7}e^{14} + \frac{3}{7}e^{23} - \frac{1}{7}e^{34} - e^{56}.
\]

As in the previous example, we have an almost nilpotent Lie group \( Q = \mathbb{R} \ltimes_{\mu} N \), where \( Q \) is the simply connected Lie group with solvable Lie algebra \( q \) and \( \mu \) is the unique smooth action of \( \mathbb{R} \) on \( N \) such that \( \mu(t)e_i = \exp(tD) \), for any \( t \in \mathbb{R} \). With respect to the rational basis \( \{X_1, \ldots, X_6\} \) of \( n \) given by \( X_1 = -\frac{1}{\sqrt{2}}e_2 + e_4, X_2 = -\frac{1}{\sqrt{2}}e_1 + e_3, X_3 = \sqrt{2}e_1 + e_3, X_4 = \)
\[ \frac{1}{\sqrt{2}} e_2 + e_4, \quad X_5 = \sqrt{2} e_5, \quad X_6 = \sqrt{2} e_6, \]
the matrix associated to \( \exp(\sqrt{2}D) \) is integer. More in detail, we have
\[ \exp(\sqrt{2}D) = \text{diag}(-2, -2, 2, 2, 0, 0). \]
Thus, \( \exp^{N}(\mathbb{Z}\langle X_1, \ldots, X_6 \rangle) \) is a lattice of \( N \) preserved by \( \mu(\sqrt{2}) \) and, as a consequence,
\[ \Gamma = \sqrt{2}\mathbb{Z} \ltimes \mu^{N}(\mathbb{Z}\langle X_1, \ldots, X_6 \rangle) \]
is a lattice in \( Q \). The compact quotient \( \Gamma \backslash Q \) is then a compact solvmanifold endowed with an invariant locally conformal calibrated \( G_2 \)-structure \( \varphi \).

6. Characterization as fiber bundles over \( S^1 \)
Let us now consider a seven-dimensional compact manifold \( M \) endowed with a locally conformal calibrated \( G_2 \)-structure \( \varphi \). We will show that if \( L^{X\varphi} = 0 \), where \( X = \theta^{\#} \) is the \( g_{\varphi} \)-dual of the Lee form \( \theta \), then \( M \) is fibered over \( S^1 \) and each fiber is endowed with a coupled \( \text{SU}(3) \)-structure given by the restriction of \( (\omega = i_X \varphi, \psi_+ = d\omega) \) to the fiber.

We begin recalling the following result, which we will use later.

**Proposition 6.1** ([7]). Let \( V \) be a seven-dimensional real vector space and \( \varphi \in \Lambda^3(V^*) \) a stable 3-form which induces the inner product \( g_{\varphi} \) on \( V \). Moreover, let \( n \in V \) be a unit vector with \( g_{\varphi}(n, n) = 1 \) and let \( W := \langle n \rangle^\perp \) denote the \( g_{\varphi} \)-orthogonal complement of \( \mathbb{R}n \).

Then the pair \((\omega, \psi_+ \rangle \) defined by
\[ \omega = (i_n \varphi)|_W, \quad \psi_+ = \varphi|_W \]
is a pair of compatible, normalized, stable forms. The inner product \( h \) induced by this pair on \( W \) satisfies \( h = g_{\varphi}|_W \) and the stabilizer is \( \text{SU}(3) \).

The next two lemmas will be useful to prove one part of the main theorem of this section.

**Lemma 6.2.** Let \((M, g)\) be a Riemannian manifold and consider two differential forms \( \theta \in \Lambda^1(M), \omega \in \Lambda^2(M) \). Then
\[ |\theta \wedge \omega|^2 = 3|\theta|^2|\omega|^2 - 6|u|^2, \]
where \(|\cdot|\) is the pointwise norm induced by \( g \) and \( u \in \Lambda^1(M) \) is defined locally as \( u = u_i dx^i \), \( u_i = g^{ik} \theta_i \omega_{ki} \). From this follows
\[ |\theta \wedge \omega|^2 \leq 3|\theta|^2|\omega|^2. \]

Moreover, with respect to the \( L^2 \) norm \( \|\cdot\| \) induced by the \( L^2 \) inner product on \( k \)-forms \( \langle \alpha, \beta \rangle = \int_M \alpha \wedge \ast \beta = \int_M g(\alpha, \beta) \ast 1 \), we have
\[ \|\theta \wedge \omega\|^2 \leq 3 \int_M |\theta|^2|\omega|^2 \ast 1. \]

**Proof.** Let us recall some definitions in order to clarify the conventions we use. Given a \( k \)-covariant tensor \( \eta \), we define the antisymmetrization of \( \eta \) as
\[ \text{Alt}(\eta) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} |\sigma| \eta^\sigma, \]
where \( |\sigma| \) is the sign of the permutation \( \sigma \) and given any \( k \) vectors \( X_{i_1}, \ldots, X_{i_k} \in \mathfrak{X}(M) \) we have \( \eta^\sigma(X_{i_1}, \ldots, X_{i_k}) = \eta(X_{\sigma(i_1)}, \ldots, X_{\sigma(i_k)}) \). Given the differential forms \( \alpha \in \Lambda^r(M), \beta \in \Lambda^s(M) \), we define the wedge product \( \wedge : \Lambda^r(M) \times \Lambda^s(M) \rightarrow \Lambda^{r+s}(M) \) as

\[
\alpha \wedge \beta = \frac{(r+s)!}{r!s!} \text{Alt} (\alpha \otimes \beta).
\]

In local coordinates we then have

\[
(\theta \wedge \omega)_{ijk} = \theta_{i} \omega_{jk} - \theta_{j} \omega_{ik} + \theta_{k} \omega_{ij}.
\]

We can now start with our computations:

\[
|\theta \wedge \omega|^2 = (\theta \wedge \omega)_{i j k} g^{i a} g^{j b} g^{k c} (\theta \wedge \omega)_{a b c} = 3 \theta_{i} \omega_{j k} g^{i a} g^{j b} g^{k c} \theta_{a} \omega_{b c} - 6 \theta_{i} \omega_{j k} g^{i a} g^{j b} g^{k c} \theta_{a} \omega_{b c} \theta_{j} \omega_{i k} + 3 \theta_{i} \omega_{j k} g^{i a} \theta_{a} g^{j b} g^{k c} \theta_{b} \omega_{i j k} = 3 |\theta|^2 |\omega|^2 - 6 u_{a} g^{a c} u_{j} g^{i b} g^{k c} \theta_{j} \omega_{i k} = 3 |\theta|^2 |\omega|^2 - 6 |u|^2.
\]

\[\square\]

For manifolds endowed with a \( G_2 \)-structure we can prove the following

**Lemma 6.3.** Let \( M \) be a manifold endowed with a \( G_2 \)-structure \( \varphi \). Consider a vector field \( X \in \mathfrak{X}(M) \) and define the 2-form \( \omega := i_X \varphi \). Then

\[|\omega|^2 = 3 |X|^2.\]

**Proof.** Using the identity \( \varphi \wedge (i_X \varphi) = 2 \ast (i_X \varphi) \), which reads \( \varphi \wedge \omega = 2 \ast \omega \) in our case, we have

\[|\omega|^2 \ast 1 = \omega \wedge \ast \omega = \frac{1}{2} \omega \wedge \varphi \wedge \omega = \frac{1}{2} (i_X \varphi) \wedge (i_X \varphi) \wedge \varphi = 3 |X|^2 \ast 1.\]

\[\square\]

We can now prove the main result of this section.

**Theorem 6.4.** Let \( M \) be a connected, compact, seven-dimensional manifold endowed with a locally conformal calibrated \( G_2 \)-structure \( \varphi \) such that \( \mathcal{L}_X \varphi = 0 \), where \( X \) is the \( g_{\varphi} \)-dual vector field of the Lee form \( \theta \). Then

1. \( M \) is fibered over \( S^1 \) and each fiber is endowed with a coupled \( SU(3) \)-structure given by the restriction of \( (\omega = i_X \varphi, \psi_+ = d\omega) \) to the fiber.
2. \( M \) has a locally conformal calibrated \( G_2 \)-structure \( \hat{\varphi} \) such that \( d\hat{\varphi} = -\hat{\theta} \wedge \varphi \), where \( \hat{\theta} \) is a 1-form with integral periods.

**Proof.** (1) First of all, observe that since the closed 1-form \( \theta \) is nowhere vanishing, we can consider the foliation \( \mathcal{F}_\theta \) generated by the integrable distribution \( ker(\theta) \). We prove now that the pair \( (\omega = i_X \varphi, \psi_+ = d\omega) \) defines a coupled \( SU(3) \)-structure when restricted to each
defines a coupled SU(3)-structure on $T$. We can approximate the 1-form $\theta$ defined closed 1-form $\theta$ have that the pair $(\theta, \omega, d\omega)$ is normalized since it is nowhere zero, then $\theta$ is close to $X$. Let $F$ be a leaf of the foliation $\mathcal{F}_\theta$, then for any $p \in L$ we have

$$T_pL = \{ Y_p \in T_pM \mid \theta_p(Y_p) = 0 \} \subset T_pM$$

and since $\theta(\cdot) = g_\varphi(X, \cdot)$, we also have

$$\ker(\theta) = \langle X \rangle^\perp.$$  

Thus, $T_pL = \langle X_p \rangle^\perp$ is a six-dimensional subspace of $T_pM$. Moreover, we can suppose that $X$ is normalized since it is nowhere zero, then $\theta(X) = |X|^2 = 1$ and by Proposition 6.1 we have that the pair $((i_{X_p}\varphi)|_{T_pL}, \varphi|_{T_pL})$ defines an SU(3)-structure on $T_pL$. Now,

$$(i_{X_p}\varphi)|_{T_pL} = \omega_p|_{T_pL}$$

and for any choice of tangent vectors $Q_p, Y_p, Z_p \in T_pL$ we have

$$\varphi_p(Q_p, Y_p, Z_p) = (d\omega_p + \theta_p \wedge \omega_p)(Q_p, Y_p, Z_p) = (d\omega_p)(Q_p, Y_p, Z_p),$$

since $\theta_p$ evaluated on any vector of $T_pL$ is zero. We then obtain that

$$\varphi|_{T_pL} = d\omega|_{T_pL}.$$

Summarizing, the pair $(\omega, d\omega)$ defines a coupled SU(3)-structure when restricted to each leaf $L$ of the foliation.

Let us now observe that $M$ is fibered over $S^1$, since it is compact and there is a globally defined closed 1-form $\theta$ which is nowhere vanishing. Using the same argument as in [19], we can approximate the 1-form $\theta$ by $q \hat{\theta}$ for some integer $q$ and some 1-form $\hat{\theta}$ with integral periods. Indeed, since $H^1(M, \mathbb{Z})$ is isomorphic to the set of homotopy classes of maps into the circle, Tischler showed that there exists a smooth map $\pi : M \to S^1$ such that $\hat{\theta} = \pi^*(ds)$, where $ds$ is the length form on $S^1$. The form $q \hat{\theta}$ (and therefore $\hat{\theta}$) has no zeroes since it is close to $\theta$. Thus $\pi^*(ds)$ has no zeros and then $\pi$ is a submersion. Since $M$ is compact, $\pi : M \to S^1$ is a fibration. The fibers are defined by the equation $\hat{\theta} = 0 = q \hat{\theta}$, which is close to the equation $\theta = 0$. Thus the tangent spaces to the fibers are close to the tangent spaces to the leaves.

We can now show that the restriction of $\omega$ and $d\omega$ to the fibers of $\pi$ defines a coupled SU(3)-structure. Let $p$ be a point in a fiber $F$ of $\pi$ and let $L$ be the leaf of the foliation $\mathcal{F}_\theta$ such that $p \in L$. Using normal coordinates in $M$ with respect to the Riemannian metric $g_\varphi$, we get a diffeomorphism $\Phi$ from a neighborhood $D_L$ of $p$ in the leaf $L$ to a neighborhood $D_F$ of $p$ in the fiber $F$ such that $\Phi(p) = p$ and $\Phi_\ast p$ is arbitrary close to the identity. Since $(\omega, d\omega)$ defines a coupled SU(3)-structure when restricted to the leaf $L$, there exists an orthonormal basis $\{e^1, \ldots, e^6\}$ of $T_pL^\ast$ such that

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi^+ = e^{135} - e^{146} - e^{236} - e^{245}.$$

Considering the basis $\{f^1 = \Phi^\ast e^1, \ldots, f^6 = \Phi^\ast e^6\}$ of $T_pF^\ast$, we have that $(\Phi^\ast(\omega), \Phi^\ast(d\omega))$ defines a coupled SU(3)-structure on $T_pF$. 
(2) Note that from Lemma 6.3 and using the fact that $X$ is normalized we obtain
\[ |\omega|^2 = 3|X|^2 = 3. \]

Define the 3-form $\hat{\varphi} := d\omega + q\hat{\theta} \wedge \omega$, it is a positive 3-form. Indeed, using Lemma 6.2 and the previous observation we have
\[
|\hat{\varphi} - \varphi|^2 = |(q\hat{\theta} - \theta) \wedge \omega|^2 \\
\leq 3|q\hat{\theta} - \theta|^2 |\omega|^2 \\
= 9|q\hat{\theta} - \theta|^2.
\]

Then
\[
\|\hat{\varphi} - \varphi\|^2 = \int_M |\hat{\varphi} - \varphi|^2 \ast 1 \leq 9 \int_M |q\hat{\theta} - \theta|^2 \ast 1 = 9\|q\hat{\theta} - \theta\|^2.
\]

Observe that $\|\hat{\varphi} - \varphi\|$ is arbitrary small since $\|q\hat{\theta} - \theta\|$ is, therefore the 3-form $\hat{\varphi}$ is positive since it lies in an arbitrary small neighborhood of the positive 3-form $\varphi$ and being a positive 3-form is an open condition. Therefore, since $d\hat{\varphi} = -q\hat{\theta} \wedge \hat{\varphi}$, the 3-form $\hat{\varphi}$ defines a new locally conformal calibrated $G_2$-structure with associated Lee form $q\hat{\theta}$, which is a 1-form with integral periods. □

Remark 6.5. By the previous theorem we have that $M = N_\nu$ is the mapping torus of a six-dimensional manifold, but the diffeomorphism $\nu$ in general does not preserve the coupled SU(3)-structure on the fiber.

Note that the compact locally conformal calibrated $G_2$-manifold $M = \Gamma\backslash G$ obtained in Example 5.2 satisfies the assumptions of previous theorem. Indeed, it admits a locally conformal calibrated $G_2$-structure $\varphi$ such that $\mathcal{L}_X\varphi = 0$, where $X$ is the $g_\varphi$-dual vector field of the Lee form $\theta$. Therefore, $M$ is fibered over $S^1$ and each fiber is endowed with a coupled SU(3)-structure given by the restriction of $(-\omega, d(-\omega))$ to the fiber.

Acknowledgments. This work has been partially supported by (Spanish) MINECO Project MTM2011-28326-C02-02, Project UPV/EHU ref. UFI11/52 and by (Italian) Project PRIN “Varietà reali e complesse: geometria, topologia e analisi armonica”, Project FIRB “Geometria Differenziale e Teoria Geometrica delle Funzioni”, and by GNSAGA of INdAM.

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