Invariants and matrix elements of the quantum group $U_q[gl(n, \mathbb{C})]$ revisited

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Abstract
In a previous paper the generator matrix elements and (dual) vector reduced Wigner coefficients (RWCs) were evaluated via the polynomial identities satisfied by a certain matrix constructed from the $R$-matrix $R$ and its twisted counterpart $R_T = T \circ R$. Here we provide an alternative evaluation utilising the $R$-matrix $\tilde{R} = (R_T)^{-1}$. This provides a new direct derivation of the vector RWCs obtained indirectly in earlier work via a symmetry relation. This approach has the advantage that it generalises to the Lie superalgebra case, which will be investigated elsewhere.

Keywords: quantum groups, matrix elements, representation theory

1. Introduction

Quantum groups were introduced in the mid-1980s as deformations of the universal enveloping algebra of Lie algebras [1, 2], mainly due to their relevance to the quantum Yang–Baxter equation, a key ingredient to the quantum inverse scattering method [3]. Crucial to their applicability to such problems of physical interest, is a well-developed, and explicit, theory of representations. The key is to have access on demand to explicit expressions for quantities such as Wigner coefficients and matrix elements, ready to be used for applications such as the analysis of physically motivated quantum integrable systems.

Such a constructive approach was applied first to Lie algebras by Baird and Biedenharn [4] using pattern calculus methods, motivated by the seminal results of Gelfand and Tsetlin [5]. Then, using the characteristic identities originally developed systematically by Green and Bracken [6, 7], in a series of papers Gould presented a thorough derivation of various quantities such as formulae for reduced Wigner coefficients (isoscalar factors), Wigner coefficients, reduced matrix elements and matrix elements [8–13]. It is worth pointing out that for the

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case of simple Lie algebras, the form of the characteristic identities depend on the quadratic Casimir element.

The approach using characteristic identities was then further developed for quantum groups [14, 15]. In the case of Lie algebras, several identities are required to ultimately derive the relevant formulae—one for the characteristic matrix $A$, and another for the so-called adjoint characteristic matrix, $\overline{A}$. In the case of $gl(n)$ for instance, both matrices partition into a convenient block form that highlights the subalgebra embedding $gl(n - 1) \subset gl(n)$. For the quantum group case, the form of the characteristic identities do still depend on the Casimir element, which in turn is expressible in terms of the universal $R$-matrix. Furthermore, only one of the characteristic matrices partitions conveniently (depending on the choice of $R$-matrix), but this does not affect the calculation, as symmetry relations involving $q$-dimensions of the irreducible representations exist among the invariants associated to the two identities. These symmetry relations may be employed to streamline the derivation (see, for example, equation (38) in [14]) and bypass the issue of partitioning.

Along these lines, more recent work by the current authors have seen a development of characteristic identities utilised to produce analogous formulae for Lie superalgebras in the general linear case [16–20] and to a lesser extent, the orthosymplectic case [21]. A natural progression of this work would be to advance the case of quantum superalgebras. Unfortunately, for the case of quantum superalgebras, the partitioning problem remains manifest, but no such analogous symmetry relations exist (at least in such a straightforward manner) due to vanishing $q$-superdimensions.

In this technical article, we return to the case of quantum groups (i.e. without underlying $\mathbb{Z}_2$-grading), specifically $U_q[gl(n, \mathbb{C})]$, and make use of two distinct (but related) $R$-matrices, both of which lead to an appropriate block structure of each of the characteristic matrices, $A$ and a second, alternative characteristic matrix $\tilde{A}$ in place of the adjoint one. As such, we provide an alternative and direct derivation of the various formulae. Having established such a direct construction for the non-graded quantum case, we expect to garner insight into the derivation of formulae for the $\mathbb{Z}_2$-graded case.

The paper is organised as follows. In section 2 we review certain structural aspects of quasitriangular Hopf algebras, especially those aspects of the structure associated to the universal $R$-matrix. After a brief discussion on abstract tensor operators in this context, we make use of the preceding investigation surrounding the $R$-matrix to then apply the results to $U_q[gl(n, \mathbb{C})]$ in section 3, and to develop the pertinent characteristic identities. In section 4, the main construction is presented, where we provide an alternative characteristic matrix $\tilde{A}$ to the adjoint matrix, which partitions appropriately and allows us to then derive formulae for a variety of invariants (section 5) and matrix elements (section 6). We also include five appendices which help provide details of our calculations for the interested reader.

2. Quasi-triangular Hopf algebras

Let $H$ be a quasi-triangular Hopf algebra with universal $R$-matrix

$$R = \sum_i a_i \otimes b_i \equiv a_i \otimes b_i.$$  

Let $\Delta : H \rightarrow H \otimes H$ be the co-product on $H$, $\varepsilon : H \rightarrow \mathbb{C}$ the co-unit, and $\Delta^T = T \circ \Delta$ the opposite co-product, where $T : H \otimes H \rightarrow H \otimes H$ is the usual twist map. By definition $R$ satisfies
\( R\Delta(a) = \Delta^T(a)R, \ \forall a \in H. \)

and

\[
(\Delta \otimes \text{id})R = R_{13}R_{23}, \tag{1}
\]

\[
(\text{id} \otimes \Delta)R = R_{13}R_{12}. \tag{2}
\]

Let \( \iota : C \to H \) denote the unit map (so that \( \iota(1) = I \), the identity element on \( H \)). Applying \((\iota \circ \varepsilon \otimes \text{id} \otimes \text{id})\) to (1), then applying \( m \otimes \text{id} \) and making use of the fundamental properties of the Hopf structure, we deduce

\[
(\iota \circ \varepsilon \otimes \text{id})R = I \otimes I. \tag{3}
\]

In a similar way, using (2), we also have \((\text{id} \otimes \iota \circ \varepsilon)R = I \otimes I\). Similarly applying \((S \otimes \text{id} \otimes \text{id})\) or \((\text{id} \otimes S \otimes \text{id})\) to (1) and then applying \((m \otimes \text{id})\) gives

\[
I \otimes I = (S \otimes \text{id})R \cdot R = R \cdot (S \otimes \text{id})R,
\]

which shows explicitly that \( R \) must be invertible with

\[
R^{-1} = (S \otimes \text{id})R. \tag{3}
\]

Note that \( H \) is also quasi-triangular under the opposite structure with co-product \( \Delta^T \), antipode \( S^{-1} \) and \( R \)-matrix

\[
R^T = T(R) = b_i \otimes a_i.
\]

Thus (3) implies

\[
(R^T)^{-1} = (S^{-1} \otimes \text{id})R^T
\]

or

\[
R^{-1} = (\text{id} \otimes S^{-1})R, \tag{4}
\]

so that

\[
(\text{id} \otimes S)R^{-1} = R.
\]

Thus from (3) we arrive at

\[
(S \otimes S)R = (\text{id} \otimes S)R^{-1} = R. \tag{5}
\]

2.1. The \( u \)-operator

Following Drinfeld \cite{22} and Reshetikhin \cite{23}, we define

\[
u = \sum_i S(b_i)a_i \equiv S(b_i)a_i.
\]

Then in Sweedler’s \cite{24} notation,

\[
S(a^{(2)})ua^{(1)} = \varepsilon(a)\nu,
\]

with \( \Delta(a) = a^{(1)} \otimes a^{(2)} \). Alternatively, we may express this result as

\[
m(S \otimes \text{id})((I \otimes u)\Delta^T(a)) = \varepsilon(a)\nu, \ \forall a \in H.
\]

The result is true since
Moreover, we have the following result.

**Lemma 1.** Set $R^{-1} = c_i \otimes d_i$. Then

(i) $S^2(a)u = ua$, $\forall a \in H$,

(ii) $u$ is invertible with inverse $u^{-1} = S^{-1}(d_i)c_i$.

**Proof.**

(i) First note that the identity

$$(S \otimes S)\Delta^T(a) = \Delta(S(a))$$

in the Sweedler notation is

$$S(a_{(2)}) \otimes S(a_{(1)}) = S(a_{(1)}) \otimes S(a_{(2)}).$$

It follows, using (6) with $a$ replaced by $S(a)$, that

$$S^2(a)u = S^2(a_{(1)})uS(a_{(2)})a_{(3)}$$
$$= S(S(a_{(2)})uS(a_{(1)})a_{(3)})$$
$$= \varepsilon(S(a_{(1)})ua_{(2)}) = ua.$$

(ii) Explicitly

$$u \cdot u^{-1} = u \cdot S^{-1}(d_i)c_j$$
$$\overset{(i)}{=} S(d_j)uc_j$$
$$= S^{-1}(d_i)S(b_i)a_i c_j$$
$$= S(b_i)u a_i c_j$$
$$= m \circ (S \otimes \text{id})[b_i d_j \otimes a_i c_j]$$
$$= m \circ (S \otimes \text{id})[R^T \cdot (R^{-1})^T] = I \otimes I.$$

Thus lemma 1 implies

$$S^2(a) = uau^{-1}, \ \forall a \in H.$$  

That is, the automorphism $S^2$ is an inner automorphism.

The canonical element $u$ has an interesting connection with $R^T R$. Before investigating this connection, we present a technical lemma as follows.
Lemma 2. Define two maps $\varphi, \varphi : H^\otimes 4 \to H^\otimes 2$ by

$$\varphi(c_1 \otimes c_2 \otimes c_3 \otimes c_4) = S(c_3)c_1 \otimes S(c_4)c_2,$$

$$\varphi(c_1 \otimes c_2 \otimes c_3 \otimes c_4) = S^{-1}(c_3)c_1 \otimes S(c_4)c_2.$$  

Then we have

(i) $\varphi(R_{12}^TR_{23} c) = \varphi(c)$,

(ii) $\varphi(R_{13} c) = (u \otimes I)\varphi(c)$,

$\forall c \in H^\otimes 4$.

Proof.

(i) In obvious notation we have

$$\varphi(R_{12}^TR_{23} c) = \varphi((b_i \otimes a_i \otimes I \otimes I)(I \otimes a_i \otimes b_j \otimes I)(c_1 \otimes c_2 \otimes c_3 \otimes c_4))$$

$$= \varphi(b_i c_3 a_i c_1 \otimes b_j c_3 a_i c_1)$$

$$= S(b_i c_3) b_i c_3 \otimes S(c_4) a_i c_1$$

$$= (S(c_3) \otimes S(c_4))(S(b_i) b_i \otimes a_i a_j)(c_1 \otimes c_2)$$

$$= S(c_3)c_1 \otimes S(c_4)c_2 = \varphi(c).$$

since

$$S(b_i) b_i \otimes a_i a_j = (S \otimes \text{id})(S^{-1}(b_i) b_i \otimes a_i a_j) = (S \otimes \text{id})((R^T)^{-1}R^T) = I \otimes I.$$

(ii) $\varphi(R_{13} c) = \varphi((a_i \otimes I \otimes b_i \otimes I)(c_1 \otimes c_2 \otimes c_3 \otimes c_4))$

$$= \varphi(a_i c_1 \otimes c_2 \otimes b_i c_3 \otimes c_4)$$

$$= S(b_i c_3) a_i c_1 \otimes S(c_4) c_2$$

$$= S(c_3) S(b_i) a_i c_1 \otimes S(c_4) c_2$$

$$= S(c_3) u c_1 \otimes S(c_4) c_2$$

$$= (u \otimes I)(S^{-1}(c_3)c_1 \otimes S(c_4)c_2)$$

$$= (u \otimes I)\varphi(c).$$

Now we are in a position to prove the following.

Theorem 1.

$$\Delta(u) = (u \otimes u)(R^T R)^{-1}.$$ 

Proof.

$$\Delta(u) \cdot R^T R = \left[ (S \otimes S) \Delta^T(b_i) \right] \cdot \Delta(a_i) \cdot R^T R$$

$$= \left[ (S \otimes S) \Delta^T(b_i) \right] \cdot R^T R \cdot \Delta(a_i)$$

$$= \varphi(R_{12}^TR_{12}(\Delta \otimes \Delta^T)R).$$
with \( \varphi \) as defined in lemma 2. Using (1) and (2), we have

\[
\begin{align*}
(\Delta \otimes \Delta^T)R &= R_{13} R_{14} R_{23} R_{24} \\
&= R_{13} R_{23} R_{14} R_{24} \\
\Rightarrow \quad \Delta(u)R^T R &= \varphi (R_{12}^T R_{13} R_{23} R_{14} R_{24}) \\
&= \varphi (R_{12}^T R_{23} R_{13} R_{14} R_{24}),
\end{align*}
\]

where in the last step we have made use of the quantum Yang–Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

By repeated application of lemma 2, we have

\[
\begin{align*}
\Delta(u)R^T R &= \varphi (R_{13} R_{14} R_{24}) \\
&= (u \otimes I)^\tau (R_{12} R_{14} R_{24}) \\
&= (u \otimes I)(a_i \otimes S(b_k)S(b_j)b_ka_k) \\
&= (u \otimes I)(I \otimes S(b_k))(a_i a_j \otimes S(b_j)b_i)(I \otimes a_k) \\
&= (u \otimes I)(I \otimes S(b_k))[(id \otimes S)(a_i a_j \otimes S^{-1}(b_j)b_k)](1 \otimes a_k) \\
&= (u \otimes I)(I \otimes S(b_k))[(id \otimes S)(R^{-1} \cdot R)](1 \otimes a_k) \\
&= (u \otimes I)(I \otimes S(b_k)a_k) \\
&= (u \otimes I)(I \otimes u) \\
&= u \otimes u.
\end{align*}
\]

Thus

\[
R^{-1} = (S \otimes \text{id})R = S(a_i) \otimes b_i = c_i \otimes d_i,
\]

we may also write

\[
u^{-1} = S^{-1}(d_i)c_i = S^{-1}(b_i)S(a_i) = S^{-2}(b_i)a_i
\]

since \( R = (S \otimes S)R \). We also observe that

\[
\tilde{R} = (R^T)^{-1}
\]

is a universal \( R \)-matrix for \( H \). Below we shall utilise the corresponding \( u \)-operator, denoted \( \tilde{u} \). It is related to the \( u \)-operator above by the following.

**Lemma 3.** \( \tilde{u} = S(u^{-1}) \).

**Proof.** Set

\[
\tilde{R} = \tilde{a}_i \otimes \tilde{b}_i = (R^{-1})^T \\
= [(S \otimes \text{id})R]^T \\
= [(id \otimes S)R]^T \\
= b_i \otimes S(a_i).
\]
Therefore, for the $u$-operator, we obtain
\[ \tilde{u} = S(b_i)\tilde{a}_i = S^2(a_i)b_i = S[S^{-1}(b_i)S(a_i)] = S[S^{-2}(b_i)a_i] = S(u^{-1}). \]

\[ \square \]

**Corollary 1.** The element $S(u)u$ is central.

**Proof.** This follows from
\[ S^2(a) = uau^{-1} = \tilde{a}a\tilde{u}^{-1} \]
\[ \Rightarrow \tilde{u}^{-1}u = S(u)u \text{ is central.} \]

Thus we also have
\[ \Delta(\tilde{u}) = (\tilde{u} \otimes \tilde{u})(\tilde{R}^T\tilde{R})^{-1} = (\tilde{u} \otimes \tilde{u})(R^{-1}(R^T)^{-1})^{-1} = (\tilde{u} \otimes \tilde{u})(R^T R). \]

### 2.2. Tensor operators

Let $V$ be a finite dimensional irreducible $H$-module and $\pi$ the representation afforded by $V$. Given a basis $\{e_\alpha\}_{\alpha=1}^n$, $n = \dim V$, for $V$ we define an irreducible tensor operator of type $\pi$ to be a collection of operators
\[ T \equiv \{T_\alpha\}_{\alpha=1}^n \]
transforming according to the rule
\[ \text{Ada} \circ T_\alpha = \pi(a)_{\beta\alpha}T_\beta, \text{ where } \text{Ada} \circ T_\alpha \equiv a(1)T_\alpha S(a(2)). \] (7)

**Lemma 4.** $T \equiv \{T_\alpha\}_{\alpha=1}^n$ is an irreducible tensor operator of type $\pi$ iff
\[ aT_\alpha = T_\beta\pi(a(1))_{\beta\alpha}a(2). \] (8)

**Proof.** If $T$ is an irreducible tensor operator then
\[ aT_\alpha = a(1)T_\alpha S(a(2))a(3) \]
\[ \overset{(7)}{=} \pi(a(1))_{\beta\alpha}T_\beta a(2) \]
which is (8). Conversely, if (8) holds then
\[ \text{Ada} \circ T_\alpha = a(1)T_\alpha S(a(2)) \]
\[ \equiv T_\beta\pi(a(1))_{\beta\alpha}a(2)S(a(3)) \]
\[ = T_\beta\pi(a)_{\beta\alpha} \]

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which is (7).

Now consider the tensor operator \( T \) acting on a finite dimensional module \( W \). Then \( T \) determines a linear map
\[
T : V \otimes W \rightarrow X
\]
defined by
\[
T(e_\alpha \otimes w) = T_\alpha w, \quad \forall w \in W; \tag{9}
\]
where \( X \) is a finite dimensional \( H \)-module determined by the action of \( T \).

**Lemma 5.** \( T \) defined by (9) determines an intertwining operator (or \( H \)-module homomorphism), i.e.
\[
T\Delta(a) = aT.
\]

**Proof.**
\[
T\Delta(a)(e_\alpha \otimes w) = T(a_{(1)}e_\alpha \otimes a_{(2)}w)
= \pi(a_{(1)})\beta_\alpha T(e_\beta \otimes a_{(2)}w) \tag{9}
\leq \pi(a_{(1)})\beta_\alpha T_\beta a_{(2)}w
= aT_\alpha w
= aT(e_\alpha \otimes w).
\]

This shows that a tensor operator \( T \) acting on \( W \) is equivalent to an intertwining operator on \( V \otimes W \).

**Example.** \( R^\tau R \) commutes with the action of \( \Delta \). Hence we have a tensor operator defined by
\[
T_{\alpha\beta} = \pi(h_{\alpha j})_{\alpha\beta}a_{ibj}.
\]

To understand the nature of this tensor operator, of importance below, we need to consider some module constructions.

First, given finite dimensional \( H \)-modules \( V, W \), the space of linear maps from \( V \) to \( W \), denoted \( \ell(V, W) \), is also an \( H \)-module under the action defined by
\[
(a \circ f)(v) = a_{(1)}f(S(a_{(2)})v), \quad v \in V, f \in \ell(V, W). \tag{10}
\]

Note that \( f \) is an \( H \)-module homomorphism iff it is an invariant under this action, i.e.
\[
a_{(1)}f(S(a_{(2)})v) = \varepsilon(a)f(v).
\]

In the case \( W = \mathbb{C} \) is the trivial 1-dimensional module, the action (10) reduces to
\[
(a \circ f)(v) = \varepsilon(a_{(2)})f(S(a_{(1)})v),
\]
i.e.
\[
(a \circ f)(v) = f(S(a)v), \quad \forall f \in V^*.
\]

This shows that a dual module is also a module under the action determined by the antipode.
Proposition 1. Let $V, W$ be finite-dimensional $H$-modules. Then we have an $H$-module isomorphism

$$W \otimes V^* \cong \ell(V, W).$$

Proof. Following the standard argument, we define a linear map $\varphi : W \otimes V^* \to \ell(V, W)$ by

$$\varphi(w \otimes v^*)(u) = v^*(u)w, \ \forall u \in V, w \in W, v^* \in V^*.$$ 

Then $\varphi$ is well-defined and one-to-one. It is also onto since given any $f \in \ell(V, W)$ we may write (sum on $\alpha$)

$$f = \varphi(f(e_\alpha) \otimes e^*_\alpha)$$

where $\{e_\alpha\}$ is a basis for $V$ with dual basis $\{e^*_\alpha\}$ for $V^*$ defined by $e^*_\alpha(e_\beta) = \delta_{\alpha\beta}$. Indeed,

$$\varphi(f(e_\alpha) \otimes e^*_\alpha)(u) = e^*_\alpha(u)f(e_\alpha) = f(u), \ \forall u \in V.$$ 

It remains to show that $\varphi$ is an $H$-module homomorphism. To this end we have

$$\varphi(a \circ (w \otimes v^*))((u) = \varphi((a_{(1)}w \otimes a_{(2)}v^*))(u)$$

$$= (a_{(2)}v^*)(u)a_{(1)}w$$

$$= v^*(S(a_{(2)}u))a_{(1)}w$$

$$= a_{(1)}\varphi(w \otimes v^*)(S(a_{(2)}u))$$

$$= a \circ \varphi(w \otimes v^*))(u).$$

□

In the case that $W = V$, we arrive at the following.

Proposition 2. If $V$ is a finite dimensional irreducible $H$-module the identity module occurs exactly once in $V \otimes V^*$ and is spanned by the vector (notation as above)

$$\xi = e_\alpha \otimes e^*_\alpha,$$

with summation on the label $\alpha$.

Proof. Since $V \otimes V^* \cong \ell(V, V)$, Schur’s lemma implies the identity module occurs exactly once in $V \otimes V^*$ (all invariants in $\ell(V, V)$ are scalar multiples of the identity by classical arguments). Then

$$a \circ \xi = a_{(1)}e_\alpha \otimes a_{(2)}e^*_\alpha$$

$$= a_{(1)}e_\alpha \otimes (a_{(2)}e^*_\alpha, e^*_\beta e^*_\beta)$$

$$= \langle e^*_\alpha, S(a_{(2)}e^*_\beta)a_{(1)}e_\alpha \otimes e^*_\beta$$

$$= a_{(1)}S(a_{(2)})e^*_\beta \otimes e^*_\beta$$

$$= \varepsilon(a)\xi.$$

□
Note. The uniqueness of the identity representation in $V \otimes V^*$ also applies to $V^* \otimes V$ (see appendix B for details). In fact, using the $R$-matrix it is straightforward to show that the two spaces $V \otimes V^*$ and $V^* \otimes V$ are isomorphic.

2.3. Tensor operators from $R^T R$

Let $\pi$ be the representation afforded by a finite dimensional irreducible $H$-module $V$. Then $T \equiv (\pi \otimes \text{id})R^T R$ determines a tensor operator with components

$$T_{\alpha \beta} = (\zeta_{\alpha \beta} \otimes \text{id})R^T R,$$

where $\zeta_{\alpha \beta} : H \to \mathbb{C}$ is such that $\zeta_{\alpha \beta}(a) = \pi(a)_{\alpha \beta}$. Given another $H$-module $W$, the above components are defined by

$$(\pi \otimes \text{id})R^T R(e_{\beta} \otimes w) = e_{\alpha} \otimes T_{\alpha \beta}w$$

so

$$T_{\alpha \beta}w = (e_{\alpha}^* \otimes \text{id})R^T R(e_{\beta} \otimes w).$$

Proposition 3. The map $\varphi : V^* \otimes V \otimes W \longrightarrow W$ defined by

$$\varphi(e_{\alpha}^* \otimes e_{\beta} \otimes w) = T_{\alpha \beta}w$$

is an intertwining operator ($H$-module homomorphism).

Proof. We have, for any $a \in H$,

$$\varphi \cdot a(e_{\alpha}^* \otimes e_{\beta} \otimes w) = \varphi(a(1)e_{\alpha}^* \otimes a(2)e_{\beta} \otimes a(3)w)$$

$$= (a(1)e_{\alpha}^* \otimes \text{id})R^T R(a(2)e_{\beta} \otimes a(3)w)$$

$$= (a(1)e_{\alpha}^* \otimes \text{id})R^T R\Delta(a(2))(e_{\beta} \otimes w)$$

$$= (e_{\alpha}^* \otimes \text{id}) \cdot (S(a(1)) \otimes I)\Delta(a(2))R^T R(e_{\beta} \otimes w)$$

$$= (e_{\alpha}^* \otimes \text{id}) \cdot (I \otimes a)R^T R(e_{\beta} \otimes w)$$

$$= aT_{\alpha \beta}w$$

$$= a \cdot \varphi(e_{\alpha}^* \otimes e_{\beta} \otimes w).$$

In terms of components we have

$$aT_{\alpha \beta}w = \varphi(a(1)e_{\alpha}^* \otimes a(2)e_{\beta} \otimes a(3)w)$$

$$= \pi^*(a(1))_{\gamma \alpha} \pi(a(2))_{\delta \beta} \varphi(e_{\delta}^* \otimes e_{\delta} \otimes a(3)w)$$

$$= \pi^* \otimes \pi(\Delta(a(1)))_{\gamma \alpha \beta} T_{\gamma \delta} a(2)w$$

or

$$aT_{\alpha \beta} = \pi^* \otimes \pi(\Delta(a(1)))_{\gamma \alpha \beta} T_{\gamma \delta} a(2). \quad (11)$$

This shows explicitly that the entries of the matrix $(\pi \otimes \text{id})R^T R$ transform as a tensor operator of type $\pi^* \otimes \pi$. 

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Notes.

(1) If \( \{e_\alpha\} \) is a basis for \( V \), assumed above, then
\[
ae_\alpha = \pi(a)_{\beta\alpha} e_\beta \\
= (e_\beta^*, ae_\alpha) e_\beta
\]
\[
\Rightarrow \quad \pi(a)_{\beta\alpha} = (e_\beta^*, ae_\alpha),
\]
for \( a \in H \).

(2) If \( \{e_\alpha^*\} \) is the corresponding dual basis for \( V^* \), as above, then we have
\[
ae_\alpha^* = \pi^*(a)_{\beta\alpha} e_\beta^*.
\]
On the other hand,
\[
ae_\alpha^* = (e_\beta, ae_\alpha^*) e_\beta^* = (S(a)e_\beta, e_\alpha^*) e_\beta^* = \pi(S(a))_{\alpha\beta} e_\beta^*
\]
\[
\Rightarrow \quad \pi^*(a) = \pi'(S(a)), \quad \forall a \in H,
\]
which is the usual definition of dual representation adopted below.

3. Quantum groups

A well-known example of a quasi-triangular Hopf algebra is the quantum group \( U_q(L) \) associated with a simple Lie algebra \( L \) with co-unit and co-product given by
\[
\varepsilon(x) = 0, \quad x = e_i, f_i, h_i,
\]
\[
\Delta(x) = q^{h_i/2} \otimes x + x \otimes q^{-h_i/2}, \quad x = e_i, f_i, \quad \Delta(h_i) = h_i \otimes I + I \otimes h_i,
\]
and antipode
\[
S(a) = q^{-h_\rho} \gamma(a) q^{h_\rho}, \quad a \in U_q(L)
\]
where \( \gamma \) is the principal anti-homomorphism defined by
\[
\gamma(I) = I, \quad \gamma(x) = -x, \quad x = e_i, f_i, h_i.
\]
In this case the \( R \)-matrix is expressible
\[
R = q^{h_i \otimes h_j} \left( I + a_i \otimes b_j \right) \in U_q^{(-)}(L) \otimes U_q^{(+)}(L), \quad (12)
\]
with \( \{h_i | i = 1, \ldots, \ell\} \) a basis for the Cartan subalgebra of \( L \) with dual basis \( \{h^*_i | i = 1, \ldots, \ell\} \) under a given invariant bi-linear form \( (\, , \, ) \) unique up to scalar multiples on \( L \). Note here that \( \rho \) is half the sum of positive roots of \( L \), and \( h_\rho \) is defined with respect to \( (\, , \, ) \), for all \( \mu \) in the dual space of the Cartan subalgebra, as
\[
\mu(h_\rho) = (\mu, \rho).
\]
Algebraic relations satisfied by the generators are given, for example, in [25].
Observe that
\[
S^2(a) = q^{-2h_\rho} a q^{2h_\rho}, \quad a \in U_q(L)
\]
which implies that \( v = q^{2h_v}u \) is a central element with inverse (also central)
\[
v^{-1} = u^{-1} q^{-2h_v}.
\]
Moreover, since \( q^{h_{2h_v}} \) are group-like, we also have
\[
\Delta(v) = (v \otimes v)(R^TR)^{-1}
\]
or
\[
R^TR = (v \otimes v) \Delta(v^{-1}).
\]
We call \( U_q(L) \) a Ribbon Hopf algebra.

In terms of the \( R \)-matrix
\[
\tilde{R} = (R^T)^{-1}
\]
we have the corresponding \( u \)-operator
\[
\tilde{u} = S(u^{-1})
\]
and central element
\[
\tilde{v} = q^{2h_v} \tilde{u}
\]
\[
= q^{2h_v} S(u^{-1})
\]
\[
= S(u^{-1} q^{-2h_v})
\]
\[
= S(v^{-1}).
\]
In this case
\[
\Delta(\tilde{v}) = (\tilde{v} \otimes \tilde{v})(\tilde{R}^2 \tilde{R})^{-1}
\]
\[
= (\tilde{v} \otimes \tilde{v})(R^{-1}(R^T)^{-1})^{-1}
\]
\[
= (\tilde{v} \otimes \tilde{v})R^TR
\]
\[
= S(v^{-1}) \otimes S(v^{-1}) R^TR.
\]

3.1. Eigenvalues of \( v \)

Let \( D_+ \) denote the set of dominant integral weights. Let \( V(\Lambda) \) be a finite dimensional irreducible module with highest weight \( \Lambda \in D_+ \), and let \( e^\Lambda_- \) be the minimal weight vector. Then observe that with \( R \) as in (12),
\[
ue^\Lambda_- = q^{-h^\Lambda} e^\Lambda_- = q^{-(\Lambda \cdot \Lambda_\Lambda)} e^\Lambda_-
\]
\[
\Rightarrow ve^\Lambda_- = q^{2h_v} ue^\Lambda_- = q^{-(\Lambda \cdot \Lambda_{2\rho})} e^\Lambda_--(\Lambda \cdot \Lambda_{2\rho}) e^\Lambda_-
\]

Now observe that there is a unique Weyl group element \( \tau \) sending positive roots to negative roots so that
\[
\Lambda_- = \tau(\Lambda).
\]
It follows that
\[
(\Lambda_- , 2\rho) - (\Lambda_- , \Lambda_-) = (\Lambda_\tau (2\rho) - (\tau(\Lambda), \tau(\Lambda))
\]
\[
= -(\Lambda, 2\rho) - (\Lambda, \Lambda)
\]
\[
= -(\Lambda, \Lambda + 2\rho).
\]
Consequently, on $V(\Lambda)$ the invariant $v$ takes the constant value
\[ \chi_\Lambda(v) = q^{-(\Lambda,\Lambda+2\rho)}. \] (13)
On the other hand, $\tilde{v} = S(v^{-1})$ takes the eigenvalue
\[ \chi_\Lambda(\tilde{v}) = q^{(\Lambda,\Lambda+2\rho)} \] (14)
as may be verified directly by action on the highest weight vector.

**Note.** It follows that
\[ 1 - q^{-((\Lambda,\Lambda+2\rho) - (\Lambda,\Lambda+2\rho))} \rightarrow \frac{1}{2}(\Lambda, \Lambda + 2\rho) \]
in the limit $q \rightarrow 1$, so we may take
\[ C_q = 2 \left( \frac{1 - v}{q - q^{-1}} \right) \]
as a generalised quadratic invariant.

The properties of such invariants and their relationship to those considered below is worth further consideration. Note that
\[ \Delta(C_q) = \frac{2}{q - q^{-1}} \left( I \otimes I - \Delta(v) \right) = \frac{2}{q - q^{-1}} \left( I \otimes I - (v \otimes v)(R^T R)^{-1} \right). \]

### 3.2. Characteristic identities

Let $V(\Lambda)$ be a finite dimensional irreducible $U_q(L)$-module with highest weight $\Lambda \in D_+$ and $\pi_\Lambda$ the representation afforded by $V(\Lambda)$. Then we have the tensor matrix
\[ A = (q - q^{-1})^{-1} \pi_\Lambda \otimes \text{id}(I \otimes I - R^T R) \]
which reduces to the characteristic matrix for the semisimple Lie algebra $L$ previously considered by Green and Bracken [6, 7] and Gould [8, 9], in the limit $q \rightarrow 1$. We have seen that the entries $A_{\alpha,\beta}$ of the matrix $A$ transform as a tensor operator of type $\pi_\Lambda^* \otimes \pi_\Lambda$.

On the other hand, acting on a finite dimensional module $V(\mu)$, $A$ may be regarded as an intertwining operator on the tensor product module $V(\Lambda) \otimes V(\mu)$:
\[ A = (q - q^{-1})^{-1} \pi_\Lambda \otimes \pi_\mu(I \otimes I - R^T R) \]
\[ = (q - q^{-1})^{-1} \pi_\Lambda \otimes \pi_\mu(I \otimes I - (v \otimes v)\Delta(v^{-1})). \]
The module $V(\Lambda) \otimes V(\mu)$ is well known to be completely reducible into a direct sum of irreducible modules with highest weights $\mu + \lambda_i$ with $\lambda_i$ a weight occurring in $V(\Lambda)$. It follows that if $\lambda_1, \ldots, \lambda_k$ are the distinct weights in $V(\Lambda)$ then $A$ satisfies the polynomial identity
\[ \prod_{i=1}^{k} |A - a_i| = 0, \] (15)
where
\[ a_i = (q - q^{-1})^{-1} \left( 1 - q^{((\mu + \lambda_i,\mu + \lambda_i + 2\rho) - (\Lambda,\Lambda+2\rho) - (\mu,\mu+2\rho))} \right) \]
\[ = \frac{1 - q^{2\lambda_i}}{q - q^{-1}}. \]
where
\[
\alpha_i = \frac{1}{2} \left[ (\mu + \lambda_i, \mu + \lambda_i + 2\rho) - (\Lambda, \Lambda + 2\rho) - (\mu, \mu + 2\rho) \right] \\
= \frac{1}{2} \left[ (\lambda_i, \lambda_i + 2(\mu + \rho)) - (\Lambda, \Lambda + 2\rho) \right]
\]
are the classical characteristic roots. These are a quantum analogue of the characteristic identities considered by Green and Bracken [6, 7], Gould [8, 9] and others [26] for a simple Lie algebra \( L \).

3.3. Characteristic identities for \( U_q[gl(n, \mathbb{C})] \)

Recall that \( U_q(n) = U_q[gl(n, \mathbb{C})] \) is the quantum group with simple generators
\[
e_i = E_{i+1 i}, \quad f_i = E_{i+1 i}, \quad h_i = E_{ii} - E_{i+1 i+1}, \quad 1 \leq i \leq n.
\]
Here we consider the vector module \( V_0 \) of \( gl(n, \mathbb{C}) \) and \( U_q(n) \) which is undeformed and irreducible of highest weight \( \varepsilon_1 \). The corresponding representation is defined by
\[
\pi_0(E_{i+1 i}) = e_{i+1 i}, \quad \pi_0(E_{i+1 i}) = e_{i+1 i}, \quad \pi_0(E_{ii}) = e_{ii},
\]
where, as usual, \( e_{ij} \) is an elementary matrix.

In this case we have, from previous work of Jimbo [1] and others [27, 28],
\[
(\pi_0 \otimes \text{id})R = \sum_{i<j} e_{ij} \otimes \check{E}_{ij}, 
\]
\[
(\pi_0 \otimes \text{id})R^T = \sum_{i<j} e_{ij} \otimes \check{E}_{ji},
\]
where
\[
\check{E}_{ij} = \begin{cases} 
(q - q^{-1}) q^{(E_{ii} + E_{jj} - 1) E_{ij} - E_{ij}} & \text{for } i \neq j \\
q^{E_{ii}} & \text{for } i = j
\end{cases}
\]
with \( E_{ij} \) defined recursively according to
\[
E_{ij} = E_{ik} E_{kj} - q^{-1} E_{kj} E_{ik}, \quad i \leq k \leq j.
\]
This gives the characteristic matrix
\[
\check{A}_{ij} = (q - q^{-1}) \left\{ \delta_{ij} - (\pi_0)_{ij} \otimes \text{id}(R^T R) \right\}
\]
\[
= (q - q^{-1}) \left\{ \delta_{ij} - \sum_{k \neq i} \check{E}_{ik} \check{E}_{kj} \right\},
\]
where \( i \vee j = \max(i, j) \). This characteristic matrix is referred to as the adjoint matrix in [14].

**Note.** The operator (18) should not be confused with
\[
\check{E}_{ij} = \begin{cases} 
-(q - q^{-1}) q^{(E_{ii} + E_{jj} - 1) E_{ij} - E_{ij}} & \text{for } i \neq j \\
q^{-E_{ii}} & \text{for } i = j
\end{cases}
\]
with \( E_{ij} \) defined recursively by
appearing in [14].

In this case the antipode is given by

\[ S^\pm(e_i) = -q^{\pm h_i} e_i q^{\pm h_i} = -q^\mp 1 e_i, \quad S^\pm(f_i) = -q^{\pm 1} f_i, \quad S(E_{ii}) = -E_{ii}, \]

so the transformation properties of equation (11) reduce to

\[ q^{\pm h} \bar{A}_{kl} = \bar{A}_{kl} q^{\pm (e_i - e_j - e_k)}, \quad 1 \leq k, \ell \leq n, \]

\[ e_i \bar{A}_{kl} = q^{\frac{1}{2}(e_i - e_{i+1} - e_k - e_l)} \bar{A}_{kl} e_i + \left\{ \delta_{k,i+1} q^{-\frac{1}{2}(e_i - e_{i+1} - e_k)} \bar{A}_{li} - q^{-1} q^{\frac{1}{2}(e_i - e_{i+1} - e_k)} \delta_{l,i+1} \bar{A}_{il} \right\} q^{-h/2}, \]

\[ f_i \bar{A}_{kl} = q^{\frac{1}{2}(e_i - e_{i+1} - e_k - e_l)} \bar{A}_{kl} f_i + \left\{ \delta_{i,k+1} q^{\frac{1}{2}(e_i - e_{i+1} - e_k)} \bar{A}_{ki} - q \delta_{k,i+1} q^{-\frac{1}{2}(e_i - e_{i+1} - e_k)} \bar{A}_{il} \right\} q^{-h/2}, \]

or

\[ \left[ e_i, \bar{A}_{kl} \right]_{q^h} = \left\{ \delta_{k,i+1} q^{-\frac{1}{2}(e_i - e_{i+1} - e_k)} \bar{A}_{li} - q^{-1} q^{\frac{1}{2}(e_i - e_{i+1} - e_k)} \delta_{l,i+1} \bar{A}_{il} \right\} q^{-h/2}, \]

\[ \left[ f_i, \bar{A}_{kl} \right]_{q^h} = \left\{ \delta_{i,k+1} q^{\frac{1}{2}(e_i - e_{i+1} - e_k)} \bar{A}_{ki} - q \delta_{k,i+1} q^{-\frac{1}{2}(e_i - e_{i+1} - e_k)} \bar{A}_{il} \right\} q^{-h/2}, \]

where the \( q \)-bracket on the LHS is defined by

\[ [x, \bar{A}_{kl}]_{q^h} = x \bar{A}_{kl} - q^{\frac{1}{2}(e_i - e_{i+1} - e_k - e_l)} \bar{A}_{kl} x, \quad x = e_i, f_i. \]

We note here that \( e_i - e_{i+1} \) denotes the \( i \)th simple root.

### 3.4. Polynomial identities

In this case the identities (15) reduce to

\[ \prod_{r=1}^{n} (\bar{A} - \bar{a}_r) = 0, \quad (24) \]

where

\[ \bar{a}_r = \frac{1 - q^{2 \pi_r}}{q - q^{-1}}, \]

where

\[ \pi_r = \frac{1}{2} \left[ (\varepsilon_r, \varepsilon_r + 2(\Lambda + \rho)) - (\varepsilon_1, \varepsilon_1 + 2\rho) \right]. \]

In this case

\[ \rho = \frac{1}{2} \sum_{i=1}^{n} (n + 1 - 2i) \varepsilon_i \]

so that

\[ \pi_r = \frac{1}{2} \left[ 1 + 2\Lambda_r + (n + 1 - 2r) - 1 - (n + 1 - 2) \right] \]

\[ = \Lambda_r + 1 - r \]
which are the usual classical adjoint roots.

The above polynomial identities allow the construction of projection operators

\[ P_r = \prod_{\ell \neq r} \left( \frac{A - a_r}{\alpha_{\ell} - a_r} \right) \]

whose entries are polynomials in the generators which determine squares of Wigner coefficients (WCs). Of particular interest is the \((n, n)\) entry

\[ \varpi_r = (P_r)_{nn} \]  
(25)

which in fact determines an invariant of \(U_q(n - 1)\) whose eigenvalues are squares of reduced Wigner coefficients (RWCs). Specifically, following the notation of [15], on an irreducible \(U_q(n - 1)\)-module \(V(\Lambda_0)\) with highest weight \(\Lambda_0\) contained in \(V(\Lambda)\), the eigenvalue of the invariants (25) are given by

\[ \chi(\Lambda, \Lambda_0)(\varpi_r) = \left| \left\langle \Lambda \bigg| \Lambda_0 + \varepsilon_r \left| 0 \cdot \Lambda_0 \right. \right\rangle \right|^2. \]  
(26)

Our aim, following the approach of [14, 15], is to utilise the above characteristic identities (24) to evaluate the squared RWCs (26). Unfortunately this approach does not apply in this case since the matrix \(A\) of equation (18) does not have the correct block structure. Specifically, the first \(n - 1\) rows and columns do not give the corresponding adjoint matrix \(A\) for \(U_q(n - 1)\) which is a necessary requirement for the application of this approach. Here we present an alternative construction which remedies this deficiency.

4. An alternative construction

Here we utilise the alternative \(R\)-matrix \(\tilde{R} = (R^T)^{-1}\). Remarkably the \(\tilde{A}\) matrix constructed from this \(R\)-matrix has the required block structure, necessary to implement the approach of [14, 15], as we will see.

Using

\[ \tilde{R} = (R^T)^{-1} = [(S \otimes \text{id})R]^T = (\text{id} \otimes S)R^T, \]

\[ \tilde{R}^T = R^{-1} = (\text{id} \otimes S^{-1})R, \]

we have, making use of (16) and (17), the \(L\)-operators

\[ (\pi_0 \otimes \text{id})\tilde{R} = (\pi_0 \otimes \text{id})(\text{id} \otimes S)R^T = \sum_{i \leq j} e_{ij} \otimes S(\hat{E}_{ij}), \]

\[ (\pi_0 \otimes \text{id})\tilde{R}^T = (\pi_0 \otimes \text{id})(\text{id} \otimes S^{-1})R = \sum_{i \leq j} e_{ji} \otimes S^{-1}(\hat{E}_{ij}). \]

Therefore we arrive at the new characteristic matrix

\[ \tilde{A}_q = (q - q^{-1})^{-1} \left\{ \delta_{ij} - (\pi_0 \otimes \text{id})(\tilde{R}^T \tilde{R}) \right\} \]

where now

\[ (\pi_0 \otimes \text{id})(\tilde{R}^T \tilde{R}) = \sum_{i \leq j, k \leq \ell} e_{ji} e_{\ell k} \otimes S^{-1}(\hat{E}_{ij})S(\hat{E}_{\ell k}) \]

\[ = \sum_{k \leq \ell \neq j} e_{kj} \otimes S^{-1}(\hat{E}_{kj})S(\hat{E}_{\ell j}). \]
Thus we arrive at the alternative characteristic matrix

$$\tilde{A}_{ij} = (q - q^{-1})^{-1} \delta_{ij} - \sum_{k \leq i \wedge j} S^{-1}(\hat{E}_k) S(\hat{E}_j),$$

where $i \wedge j = \min(i, j)$. Note that the transformation properties (22) and (23) also hold for this matrix.

In this case the matrix $\tilde{A}$ may be written in invariant form as

$$\tilde{A} = (q - q^{-1})^{-1}(\pi_0 \otimes \text{id}) \left[ I \otimes I - (\tilde{v} \otimes \tilde{v}) \Delta(\tilde{v}^{-1}) \right]$$

so that, in view of equation (13), on an irreducible module $V(\Lambda)$ the matrix $\tilde{A}$ satisfies the polynomial identity

$$\prod_{r=1}^{n} (\tilde{A} - \tilde{a}_r) = 0,$$

where now $\tilde{a}_r = (q - q^{-1})^{-1}[1 - q^{-2\pi_r}]$ with $\pi_r = \Lambda_r + 1 - r$ the classical adjoint roots as before.

4.1. Partitioning of matrix $\tilde{A}$: $U_q(n) \supset U_q(n-1)$

We observe that the first $n - 1$ rows and columns of the matrix $\tilde{A}$ yields the corresponding $U_q(n-1)$ matrix, denoted $\tilde{A}_0$:

$$\begin{pmatrix}
\tilde{A}_0 \\
\vdots \\
\tilde{A}_n
\end{pmatrix}$$

Now observe that the elementary generators of $U_q(n-1)$ are given by $e_i, f_i, h_i, 1 \leq i < n - 1$ so the transformation law of equations (22) and (23) in the case $k = \ell = n$ gives

$$q^{E_i} \tilde{A}_{mn} = \tilde{A}_{mn} q^{E_i}, \quad 1 \leq i < n,$$

$$e_i \tilde{A}_{mn} - \tilde{A}_{mn} e_i = f_i \tilde{A}_{mn} - \tilde{A}_{mn} f_i = 0, \quad 1 \leq i < n - 1,$$

so that the entry $\tilde{A}_{mn}$ is a $U_q(n-1)$ invariant.

Now consider the entries

$$\phi_i \equiv \tilde{A}_{in}, \quad 1 \leq i < n$$

of the last column of the matrix $\tilde{A}$. Setting $\ell = n$ in equations (22) and (23) gives

$$q^{E_i} \phi_k = \phi_k q^{E_i} \delta_k = q^{E_i} \phi_k q^{E_i},$$

$$e_i \phi_k - q^{-\frac{1}{2}(\delta_k - \delta_{i+1})} \phi_k e_i = -q^{-1} \delta_k \phi_{i+1} q^{-h/2},$$

$$f_i \phi_k - q^{-\frac{1}{2}(\delta_k - \delta_{i+1})} \phi_k f_i = -q \delta_k \phi_{i+1} q^{-h/2}.$$
Similarly for the entries of the last row
\[ \psi_i \equiv \tilde{\Lambda}_n, \; 1 \leq i < n, \] (28)
we have, setting \( k = n \) into equations (22) and (23), the transformation law
\[ q^{\tilde{i}} \psi_\ell = q^{\delta_{\ell i}} \psi_\ell q^{\tilde{i}}, \]
\[ e_\ell \psi_\ell - q^{\frac{1}{2}(\varepsilon_i - \varepsilon_{i+1})} \psi_\ell e_i = \delta_{\ell i+1} \psi_i q^{-\hbar/2}, \]
\[ f_\ell \psi_\ell - q^{\frac{1}{2}(\varepsilon_i - \varepsilon_{i+1})} \psi_\ell f_i = \delta_{\ell i} \psi_i q^{-\hbar/2}. \]
Again, by comparison with appendix A, it follows that the operators (28) transform as a vector
operator of \( U_q(n - 1) \).

4.2. \( U_q(n) \) vector operators and shift components

The \( U_q(n) \) projection operators
\[ \tilde{P}_r = \prod_{\ell \neq r}^{n} \left( \frac{\tilde{\Lambda} - \tilde{a}_r}{\tilde{a}_r - \tilde{a}_\ell} \right) \] (29)
form an orthogonal set of idempotents adding up to the identity:
\[ \tilde{P}_r \tilde{P}_r = \delta_{\ell r} \tilde{P}_r, \sum_{r=1}^{n} \tilde{P}_r = I. \]

As an operator on the tensor product module \( V_0 \otimes V(\Lambda) \), \( \tilde{P}_r \) projects onto the submodule
\( V(\Lambda + \varepsilon_r) \subset V_0 \otimes V(\Lambda) \).
On the other hand, \( \tilde{P}_r \) may be viewed as an \( n \times n \) matrix with entries \( (\tilde{P}_r)_{ij} \) defined by
\[ \tilde{P}_r(e_i \otimes v) = e_j \otimes (\tilde{P}_r)_{ji} v, \; \forall v \in V(\Lambda). \]

Note. With this definition, powers of the matrix \( \tilde{A} \) are given recursively by
\[ (\tilde{A}^{m+1})_{ij} = (\tilde{A})_{ik} (\tilde{A}^m)_{kj}, \]
with summation on \( k \) from 1 to \( n \).

If \( \psi_i \) is a vector operator on \( V(\Lambda) \) then we have seen that \( \psi \) on \( V_0 \otimes V(\Lambda) \), defined by
\[ \psi(e_i \otimes v) = \psi_i v, \; \forall v \in V(\Lambda) \]
is an intertwining operator. So too is \( \psi[r] \) defined by
\[ \psi[r](e_i \otimes v) = \psi \tilde{P}_r(e_i \otimes v) \]
\[ = \psi(e_j \otimes (\tilde{P}_r)_{ji} v) \]
\[ = \psi((\tilde{P}_r)_{ji} v). \]
Thus
\[ \psi[r]_i \equiv \psi((\tilde{P}_r)_{ji}) \]
also gives rise to a vector operator which increases the highest weight \( \Lambda \) by \( \varepsilon_r \), i.e. affects the
shift \( \Lambda \to \Lambda + \varepsilon_r \). We call \( \psi[r] \), a shift vector operator.
By contrast, if $\phi$ is a dual vector operator on $V(\Lambda)$, $\tilde{P}_r$ projects out the shift components from the left (see appendix B):
\[ \phi[r]_i = (\tilde{P}_r)_i \phi_j \]
which affects the shifts $\Lambda \rightarrow \Lambda - \varepsilon_r$.

To project out the shift components of vector (respectively dual vector) operators from the left (respectively right), however, we also require the $U_q(n)$ vector matrix and its characteristic identity. These matrices are of particular importance for determining dual vector WCs and RWCs.

4.3. The vector matrix revisited
In what follows, $\pi_0$ denotes the (undeformed) ‘dual’ vector representation given by
\[
\pi_0(E_{i+1}) = \pi_0(e_i) = -e_{i+1}, \\
\pi_0(f_i) = -e_{i+1}, \\
\pi_0(E_{ii}) = -e_{ii}.
\]
Like $\pi_0$, this representation is undeformed and unitary (see appendix E) unlike the dual vector representation $\pi^*_0$. The dual vector representation is defined by
\[ \pi^*_0(a) = \pi_0^*(S(a)), \quad \forall a \in U_q(n), \]
whereas
\[ \overline{\pi}_0(a) = \pi_0(\gamma(a)), \]
with $\gamma$ the principal anti-automorphism. The two representations are related by
\[ \pi_0(a) = \pi^*_0(\gamma(a)) \]
\[ = \pi^*_0(S(q^{\rho} a q^{-h_r})) \]
\[ = \pi^*_0(q^{\rho} a q^{-h_r}) \]
\[ \Rightarrow \overline{\pi}_0(a)_{ij} = \pi^*_0(q^{\rho} a q^{-h_r})_{ij} = q^{(\rho, \varepsilon_i - \varepsilon_j)} \pi^*_0(a)_{ij} \] or
\[ \pi^*_0(\gamma(a))_{ij} = q^{(\rho, \varepsilon_i - \varepsilon_j)} \pi_0(a)_{ij}. \quad (30) \]
Now consider the dual vector $L$-operators:
\[ (\pi^*_0 \otimes \text{id})R = (\pi^*_0 \otimes \text{id})(S \otimes \text{id})R \]
\[ = (\pi^*_0 \otimes \text{id})(\text{id} \otimes S^{-1})R \]
\[ = \sum_{i \leq j} e_{ij} \otimes S^{-1}(\hat{E}_{ij}), \quad (16) \]
\[ (\pi^*_0 \otimes \text{id})R^T = (\pi^*_0 \otimes \text{id})(S \otimes \text{id})R^T \]
\[ = (\pi^*_0 \otimes \text{id})(\text{id} \otimes S^{-1})R^T \]
\[ = \sum_{i \leq j} e_{ji} \otimes S^{-1}(\hat{E}_{ji}), \quad (17) \]
It makes more sense, however, to consider the physically more meaningful \( L \)-operators arising from the (unitary) undeformed ‘dual’ representation \( \pi_0 \), referred to herein as the pseudo vector representation. In view of equation (30), this gives the following \( L \)-operators:

\[
(\pi_0 \otimes \text{id})R = \sum_{i \leq j} q^{(\rho,\varepsilon_j-\varepsilon_i)} e_j \otimes S^{-1}(\hat{E}_j),
\]

\[
(\pi_0 \otimes \text{id})R^T = \sum_{i \leq j} q^{(\rho,\varepsilon_i-\varepsilon_j)} e_j \otimes S^{-1}(\hat{E}_j).
\]

The corresponding matrix

\[
A = (q - q^{-1})^{-1} \{ I \otimes I - (\pi_0 \otimes \text{id})R^T R \}
\]

has entries

\[
A_{ij} = (q - q^{-1})^{-1} \left\{ \delta_{ij} - \sum_{k \leq i \leq j} q^{(\rho,\varepsilon_j-\varepsilon_i)} S^{-1}(\hat{E}_k)S^{-1}(\hat{E}_{kj}) \right\}.
\]

It follows that the entries of the matrix \( A \) transform according to

\[
aA_{ij} = (\pi_0^* \otimes \pi_0) \Delta(a(1)) \gamma_{ij} \delta_{A_1 A_2}.
\]

Notes.

(1) As seen in appendix C, this agrees with the matrix \( A \) of [14], although the conventions used are quite different.

(2) The dual pseudo-vector representation is given by

\[
\pi_0^* (a) = \pi_0^* (\gamma^*)^I = \pi_0 (S (\gamma (a))),
\]

where

\[
S (\gamma (a)) = q^{-h_\gamma} \gamma^2 (a) q^{h_\gamma} = q^{-h_\gamma} a q^{h_\gamma}.
\]

This implies that

\[
\pi_0^* (a) = \pi_0 (q^{-h_\gamma} a q^{h_\gamma}),
\]

so \( \pi_0^* \) is equivalent to the vector representation. Moreover,

\[
\pi_0^* (a)_{ij} = q^{(\rho,\varepsilon_j-\varepsilon_i)} \pi_0 (a)_{ij}.
\]

We have the following relations:

\[
q^{E_{\bar{a}}} A_{ij} = q^{(\varepsilon_i,\varepsilon_j)} A_{ij} q^{E_{\bar{a}}} = q^{(\varepsilon_i,\varepsilon_j)} A_{ij} q^{E_{\bar{a}}},
\]

\[
e_k A_{ij} = q^{(\varepsilon_i - \varepsilon_{i+1} - \varepsilon_j)} A_{ij} e_k + (\pi_0^* \otimes \pi_0) (q^{h_{1/2}} \otimes e_k + e_k \otimes q^{-h_{1/2}}) \gamma_{ij} \delta_{A_1 A_2} q^{-h_{1/2}}
\]

\[
\Rightarrow e_k A_{ij} - q^{(\varepsilon_i - \varepsilon_{i+1} - \varepsilon_j)} A_{ij} e_k
\]

\[
= q^{(\varepsilon_i - \varepsilon_{i+1} - \varepsilon_j)} (-e_{k+1}) \delta_{A_1 A_2} q^{-h_{1/2}} + q^{(\rho,\varepsilon_j - \varepsilon_i)} (e_{k+1}) \gamma q^{(\varepsilon_2 - \varepsilon_{i+1})} A_{ij} q^{h_{1/2}}
\]

\[
= -\delta_{ij} q^{(\varepsilon_i - \varepsilon_{i+1})} A_{ij} q^{-h_{1/2}} + \delta_{i+1} q^{(\varepsilon_{i+1} - \varepsilon_1)} A_{ij} q^{-h_{1/2}}
\]

and similarly
Thus the classical characteristic roots \[8\]. Thus

\[ f_k A_0 q = q^{(k - (k + 1, e_1, e_2, e_3))} A_0 f_k + (\pi_0 \otimes \pi_0) (q^{h_0 / 2} \otimes f_k + f_k \otimes q^{-h_0 / 2})_{\alpha, i} A_0 q^{-h_0 / 2} \]

\[ = f_k A_0 q - q^{(e_k - 2e_k - 1) / 2} A_0 f_k \]

\[ = q^{(e_k - 2e_k - 1) / 2} (e_k + 1 \otimes (q^{e_k - 1} + e_1, e_2, e_3)) A_0 q^{-h_0 / 2} \]

\[ \Rightarrow f_k A_0 q - q^{(e_k - 2e_k - 1) / 2} A_0 f_k \]

Acting on an irreducible \( U_q(n) \) module \( V(\Lambda) \), the matrix \( A \) may be regarded as an intertwining operator on \( V_0^* \otimes V(\Lambda) \), where it is understood that the representation afforded by \( V_0^* \) is actually \( \pi_0 \) (rather than \( \pi_0^* \)). In this case, \( A \) has eigenvalues

\[ \alpha_r = \frac{1 - q^{-2\alpha_r}}{q - q^{-1}}, \quad \alpha_r = \Lambda_r + n - r, \]

with \( \alpha_r \) the classical characteristic roots \[8\]. Thus \( A \) satisfies the polynomial identity

\[ \prod_{r=1}^{n} (A - \alpha_r) = 0. \]

The corresponding projection operators

\[ P_r = \prod_{\ell \neq r} \left( \frac{A - \alpha_{\ell}}{\alpha_r - \alpha_\ell} \right) \]

project \( V_0^* \otimes V(\Lambda) \) onto the submodule \( V(\Lambda - \varepsilon_r) \).

Note. Thus if \( \{ e_0^* \} \) is the basis for \( V_0^* \) dual to the standard basis of \( V_0 \), then we are defining

\[ a e_i^* = \pi_0(a) e_i^* \]

or

\[ \langle a_*, e_j \rangle = \langle e_i^*, \gamma(a) e_j \rangle. \]

Note, however, that if we set

\[ e_i^0 = q^{h_0} e_i^* \]

then

\[ a e_i^0 = q^{h_0} (q^{-h_0} a q^{h_0}) e_i^* \]

\[ = \pi_0(q^{-h_0} a q^{h_0}) e_i^0 = \pi_0^* (a) e_i^0. \]

Therefore, the dual vector representation is obtained from \( \pi_0 \) by a simple basis transformation.

4.4. Partitioning of matrix \( A \)

The first \( n - 1 \) rows and columns of the matrix \( A \) gives the matrix \( A_0 \) of \( U_q(n - 1) \). Setting \( i = j = n \) into equations \(34), (35) and (36) we obtain

\[ [e_k, A_{nm}] = [f_k, A_{nm}] = [q^{E_k}, A_{nm}] = 0 \]

so that \( A_{nm} \) is an invariant of \( U_q(n - 1) \).

Setting \( j = n \) into these equations, the entries of the last column, i.e.

\[ \psi_i = A_{in}, \quad 1 \leq i \leq n \]
satisfy
\[ q^{E_\mu} A_m = q^{(e_\mu, e_\nu)} A_m q^{E_\nu}, \]
i.e.
\[ q^{E_\mu} \overline{\psi}_i = q^{(e_\mu, e_\nu)} \overline{\psi}_i q^{E_\nu}, \]
\[ e_k \overline{\psi}_l = q^{(e_k - e_{k+1}, e_\nu)} \overline{\psi}_l = \delta_{k+1} q^{-1} \overline{\psi}_l q^{-h_\nu}/2, \]
\[ f_k \overline{\psi}_l = q^{(e_k - e_{k+1}, e_\nu)} \overline{\psi}_l = \delta_{k+1} q^{-1} \overline{\psi}_l q^{-h_\nu}/2, \]
and thus transform as a vector operator of rank \( \pi_0^* \) (i.e. a dual pseudo vector operator).

Setting \( i = n \) into the same equations, the entries of the last row, i.e.
\[ \overline{\phi}_i = A_m, \quad 1 \leq i < n \]
satisfy
\[ q^{E_\mu} \overline{\phi}_i = q^{(e_\mu, e_\nu)} \overline{\phi}_i q^{E_\nu}, \]
\[ e_k \overline{\phi}_l = q^{(e_k, e_{k+1}, e_\nu)} \overline{\phi}_l = -\delta_{k+1} q^{-1} \overline{\phi}_l q^{-h_\nu}/2, \]
\[ f_k \overline{\phi}_l = q^{(e_k, e_{k+1}, e_\nu)} \overline{\phi}_l = \delta_{k+1} q^{-1} \overline{\phi}_l q^{-h_\nu}/2, \]
and thus transform as a pseudo vector operator (i.e. as the undeformed dual vector representation \( \pi_0 \)).

Thus we have the following partitioning
\[ A = \begin{pmatrix} A_0 & A_m \\ A_m & A_{nn} \end{pmatrix}, \]
with \( A_0 \) the characteristic matrix of \( U_q(n-1) \), \( A_m \) a \( U_q(n-1) \) vector operator of rank \( \pi_0^* \), \( A_{nn} \) a \( U_q(n-1) \) pseudo vector operator, and \( A_{nn} \) a \( U_q(n-1) \) invariant.

Note. The above transformation laws also apply to the matrix powers of \( A, \tilde{A} \). In particular,
\[ (A^m)_m, (\tilde{A}^m)_m \]
determine \( U_q(n-1) \) invariants. One of our aims is to systematically determine the eigenvalues of these invariants.

5. On invariants and Wigner coefficients

In view of appendix B, lemma B.2, since the matrices \( A, \tilde{A} \) transform as tensor operators of type \( \pi_0^* \otimes \pi_0, \pi_0^* \otimes \pi_0 \) respectively, the following are invariants:
\[ C_m = \text{tr}(\pi_0 \otimes \text{id})(q^{2h_\nu} \otimes I)(I - R^TR)^m \]
\[ = \text{tr}(\pi_0(q^{2h_\nu})A^m) \]
\[ = \text{tr}_q(A^m) = \sum_{i=1}^{n} q^{-(2\rho, e_\nu)} A_{ii}^m, \]
\[ \tilde{C}_m \equiv \text{tr} \left( \pi_0(q^{2\rho}) \tilde{A}^m \right) \]
\[ = \sum_{i=1}^{n} q^{(2, \varepsilon_i)} \tilde{A}_i^m, \]
so \( C_m, \tilde{C}_m \) belong to the centre of \( U_q(n) \).

In particular, we have the first order invariants
\[ C_1 = \sum_{i=1}^{n} q^{-(2, \varepsilon_i)} A_{ii} = \sum_{i=1}^{n} q^{-(2, \varepsilon_i)} A_{ii}. \]
\[ \tilde{C}_1 = \sum_{i=1}^{n} q^{(2, \varepsilon_i)} \tilde{A}_i = \sum_{i=1}^{n} q^{(2, \varepsilon_i)} \tilde{A}_i. \]

On a maximal weight vector \( v_+ \in V(\Lambda) \) we observe that
\[ A_{ii} = (q - q^{-1})^{-1} \left\{ I - \sum_{k \leq i} S^{-1}(\hat{E}_{ik}) S^{-1}(\hat{E}_{ki}) \right\} \]
satisfies
\[ A_{ii} v_+ = \frac{1 - q^{-2}E_{ii}}{q - q^{-1}} v_+ = \frac{1 - q^{-2}(\Lambda, \varepsilon_i)}{q - q^{-1}} v_+. \]

Therefore the eigenvalue of \( C_1 \) on \( V(\Lambda) \) is given by
\[ \chi_{\Lambda}(C_1) = \sum_{i=1}^{n} q^{-(2, \varepsilon_i)} \frac{1 - q^{-2}(\Lambda, \varepsilon_i)}{q - q^{-1}} \]
\[ = \sum_{i=1}^{n} q^{-(\Lambda + 2, \varepsilon_i)} [(\Lambda, \varepsilon_i)]_q, \]
where we have used the \( q \)-number notation
\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \]

Similarly
\[ \tilde{A}_{ii} = (q - q^{-1})^{-1} \left\{ I - \sum_{k \leq i} S^{-1}(\hat{E}_{ik}) S(\hat{E}_{ki}) \right\} \]
and so on a minimal weight vector \( v_- \in V(\Lambda) \) of weight \( \Lambda_- \), we have
\[ \tilde{A}_{ii} v_- = (q - q^{-1})^{-1} \left\{ I - q^{-2}E_{ii} \right\} v_- = (q - q^{-1}) \left( 1 - q^{-2}(\Lambda_-, \varepsilon_i) \right) v_. \]

Therefore, on \( V(\Lambda) \) the invariant \( \tilde{C}_1 \) has eigenvalue
\[ \chi_{\Lambda}(\tilde{C}_1) = \sum_{i=1}^{n} q^{(2, \varepsilon_i)} \frac{1 - q^{-2}(\Lambda_-, \varepsilon_i)}{q - q^{-1}}. \]
Now if \( \tau \) is the unique Weyl group element sending positive roots to negative roots (and thus maximal weights to minimal weights), we have

\[
\chi_{\Lambda}(\tilde{C}_1) = \sum_{i=1}^{n} q^{(2\rho, \tau(\varepsilon_i))} \frac{1 - q^{-2(\Lambda - \tau(\varepsilon_i))}}{q - q^{-1}}
\]

\[
= \sum_{i=1}^{n} q^{(2\tau(\rho), \varepsilon_i)} \frac{1 - q^{-2(\Lambda, \varepsilon_i)}}{q - q^{-1}}
\]

\[
= \sum_{i=1}^{n} q^{(\Lambda + 2\rho, \varepsilon_i)} [(\Lambda, \varepsilon_i)]_q
\]

(37)

Thus \( C_1 \) and \( \tilde{C}_1 \) have the same eigenvalues.

Finally we have the spectral decompositions

\[
A^m = \sum_{r=1}^{n} a_r^m P_r, \quad \tilde{A}^m = \sum_{r=1}^{n} \tilde{a}_r^m \tilde{P}_r,
\]

\[
\Rightarrow C_m = \text{tr} (\pi_0 (q^{2\rho}) A^m) = \sum_{r=1}^{n} a_r^m \tau_q (P_r),
\]

\[
\tilde{C}_m = \text{tr} (\pi_0 (q^{2\rho}) \tilde{A}^m) = \sum_{r=1}^{n} \tilde{a}_r^m \tau_q (\tilde{P}_r).
\]

We have

\[
D_q [\Lambda] \tau_q (P_r) = \text{tr}_{U(n)} [(q^{2\rho} \otimes q^{2\rho}) P_r] = D_q [\Lambda - \varepsilon_r],
\]

\[
D_q [\Lambda] \tau_q (\tilde{P}_r) = \text{tr}_{U(n)} [(q^{2\rho} \otimes q^{2\rho}) \tilde{P}_r] = D_q [\Lambda + \varepsilon_r],
\]

where

\[
D_q [\Lambda] = \prod_{\alpha > 0} q^{(\Lambda + \rho, \alpha)} - q^{-(\Lambda + \rho, \alpha)}
\]

is the usual \( q \)-dimension. Thus we obtain

\[
\chi_{\Lambda}(C_m) = \sum_{r=1}^{n} a_r^m \prod_{\alpha > 0} [(\Lambda + \rho - \varepsilon_r, \alpha)]_q q^{[(\Lambda + \rho, \alpha)]_q},
\]

\[
\chi_{\Lambda}(\tilde{C}_m) = \sum_{r=1}^{n} \tilde{a}_r^m \prod_{\alpha > 0} [(\Lambda + \rho + \varepsilon_r, \alpha)]_q q^{[(\Lambda + \rho, \alpha)]_q}.
\]

Our aim, following [14, 15], is to utilise the characteristic identities for \( U_q(n) \) and \( U_q(n-1) \) to evaluate the \( U_q(n-1) \) invariants

\[
(A^m)_m = \sum_{k=1}^{n} a_k^m \omega_k, \quad (\tilde{A}^m)_m = \sum_{k=1}^{n} \tilde{a}_k^m \tilde{\omega}_k,
\]
where

\[ \omega_k = (P_k)_{\mu \nu}, \quad \tilde{\omega}_k = (\tilde{P}_k)_{\mu \nu}. \]

These latter invariants are of particular interest since their eigenvalues determine the squared RWCS

\[ \left\langle \begin{array}{c|c} \Lambda - \varepsilon_k & \Pi \varepsilon_k \Lambda \\ \hline \Lambda_0 & 0 \end{array} \right\rangle, \quad \left\langle \begin{array}{c|c} \Lambda_0 \varepsilon_k & \Pi \varepsilon_k \Lambda \\ \hline \Lambda_0 & 0 \end{array} \right\rangle \]

respectively, where \( \Lambda_0 \) is the highest weight of an irreducible \( U_q(n-1) \) submodule of \( V(\Lambda) \), and \( \tilde{\varepsilon} \) denotes the \( U_q(n-1) \) weight \((0, 0, \ldots, 0)\).

5.1. Evaluation of \( \omega_k, \tilde{\omega}_k \)

Using the \( U_q(n) \) characteristic identity we have

\[ (P_kA)_\mu = a_k(P_k)_\mu \]

which may be expanded to give

\[ a_k(P_k)_{\mu \nu} = \omega_k \tilde{\varphi}_\nu + (P_k)_{\mu \nu}(A_0)_{\mu \nu}, \]

with \( A_0 \) the \( U_q(n-1) \) characteristic matrix and \( \tilde{\varphi}_\nu = A_n (\text{pseudo vector operator}) \). Therefore we have

\[ \omega_k \tilde{\varphi}_\nu = (P_k)_{\mu \nu}(A_0)_{\mu \nu} - a_k \]

\[ \Rightarrow \sum \omega_k \tilde{\varphi}_{[r]}(a_k - a_{0 r})^{-1} = (P_k)_{\mu \nu} \]

where \( a_{0 r} \) are the \( U_q(n-1) \) characteristic roots.

Now using the shift property (see appendix B)

\[ \tilde{\varphi}_{[r], a_{0 r}} = (q^{-2}a_{0 r} + q^{-1})\tilde{\varphi}_{[r]}, \]

the above becomes

\[ \sum_r \omega_k (a_k - q^{-2}a_{0 r} - q^{-1})^{-1} \tilde{\varphi}_{[r]} = (P_k)_{\mu \nu}. \tag{38} \]

Using

\[ \sum_{k=1}^n (P_k)_{\mu \nu} = \delta_{\mu \nu} \]

we arrive at the equations

\[ \sum_{k=1}^n \omega_k (a_k - q^{-2}a_{0 r} - q^{-1})^{-1} = 0, \quad 1 \leq r < n, \]

\[ \sum_{k=1}^n \omega_k = 1. \]
Following [15], these equations uniquely determine the \( \omega_k \) and yield the solution

\[
\omega_k = \prod_{r=1}^{n-1} (a_k - q^{-2}a_{0r} - q^{-1}) \prod_{\ell \neq k} (a_k - a_\ell)^{-1}. \tag{39}
\]

Similarly applying the identity for the matrix \( \tilde{A} \) we have

\[
\tilde{a}_k (\tilde{P}_k)_{in} = (\tilde{A} \tilde{P}_k)_{in} = (\tilde{A})_{ij} (\tilde{P}_k)_{jm} + \phi_i \tilde{\omega}_k,
\]

where \( \phi_i = \tilde{A}_i \) (dual vector operator). Therefore

\[
\phi_i \tilde{\omega}_k = (\tilde{A}_i)_{0j} (\tilde{P}_k)_{jm},
\]

\[
\Rightarrow \sum_{r=1}^{n-1} (\tilde{a}_k - \tilde{a}_{0r})^{-1} \phi[r] \tilde{\omega}_k = (\tilde{P}_k)_{jm}.
\]

Here we note that, from appendix B,

\[
\tilde{a}_{0r} \phi[r]_i = \phi[r]_i (q^2 \tilde{a}_{0r} - q)
\]

\[
\Rightarrow \sum_{r=1}^{n-1} \phi[r]_i (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1} \tilde{\omega}_k = (\tilde{P}_k)_{jm}.
\]

From this we obtain the equations

\[
\sum_{k=1}^{n} (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1} \tilde{\omega}_k = 0, \quad 1 \leq r < n,
\]

\[
\sum_{k=1}^{n} \tilde{\omega}_k = 1,
\]

which yield the unique solution

\[
\tilde{\omega}_k = \prod_{r=1}^{n-1} (\tilde{a}_k - q^2 \tilde{a}_{0r} + q) \prod_{\ell \neq k} (\tilde{a}_k - \tilde{a}_\ell)^{-1}. \tag{41}
\]

Alternatively,

\[
(\tilde{P}_k \tilde{A})_{nj} = \tilde{a}_k (\tilde{P}_k)_{mj} = (\tilde{P}_k)_{nj} (\tilde{A}_0)_{ij} + \tilde{\omega}_k \psi_i,
\]

with \( \psi_i = \tilde{A}_i \) a vector operator. Therefore

\[
\tilde{\omega}_k \psi_i = (\tilde{P}_k)_{nj} (\tilde{a}_k - \tilde{A}_0)_{ij}
\]

\[
(\tilde{P}_k)_{nj} = \sum_{r=1}^{n-1} \tilde{\omega}_k \psi[r]_i (\tilde{a}_k - \tilde{a}_{0r})^{-1}
\]

\[
= \sum_{r=1}^{n-1} \tilde{\omega}_k (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1} \psi[r]_i
\]

where we have used (see appendix B)

\[
\psi[r]_i \tilde{a}_{0r} = (q^2 \tilde{a}_{0r} - q) \psi[r]_i.
\]

this yields exactly the same equation to solve for the \( \tilde{\omega}_k \) as above.
5.2. Reduced matrix elements

Below we adopt the following notation:

\[ \tilde{\phi}_i = \tilde{A}_i \text{— dual vector operator of } U_q(n-1) \text{ (matrix } \tilde{A} \text{ acts on left)}, \]

\[ \tilde{\psi}_i = \tilde{A}_i \text{— vector operator of } U_q(n-1) \text{ (matrix } \tilde{A} \text{ acts on right)}, \]

\[ \phi_i = A_i \text{— pseudo vector operator of } U_q(n-1) \text{ (matrix } A \text{ acts on right)}, \]

\[ \psi_i = A_i \text{— dual pseudo vector operator of } U_q(n-1) \text{ (matrix } A \text{ acts on left)}. \]

**Note.** With this notation, letting \( \rho_0 \) denote the half sum of positive roots from \( U_q(n-1) \), observe that

\[ \psi_i = q^{-(\rho_0, \varepsilon_i)} \tilde{\psi}_i \]

is a vector operator and

\[ \phi_i = q^{-(\rho_0, \varepsilon_i)} \tilde{\phi}_i \]

is a dual vector operator.

Now from equation (42) we have

\[ (\tilde{P}_k)_{ij} = \sum_{r=1}^{n-1} \tilde{\omega}_k (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1} \tilde{\psi}_r[i]. \]

Therefore we look for solutions \( \tilde{\gamma}_{ik} \) to equations

\[ \sum_{k=1}^{n} \tilde{\gamma}_{ik} \tilde{\omega}_k (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1} = \delta_{ik}, \quad (43) \]

\[ \sum_{k=1}^{n} \tilde{\gamma}_{ik} \tilde{\omega}_k = 0. \quad (44) \]

This yields the unique solution

\[ \tilde{\gamma}_{ik} = \gamma_r (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1}, \]

where

\[ \gamma_r = (-1)^{n-1} \prod_{k=1}^{n} (\tilde{a}_k - q^2 \tilde{a}_{0r} + q) \prod_{r' \neq r} (q^2 \tilde{a}_{0r} - q^2 \tilde{a}_{0r'}) \]

\[ = (-1)^{n-1} q^{4-n} \prod_{k=1}^{n} (q^{-1} \tilde{a}_k - q \tilde{a}_{0r} + 1) \prod_{r' \neq r} (\tilde{a}_{0r} - \tilde{a}_{0r'}). \quad (45) \]

These invariants have an interesting interpretation. From equations (43) and (44) above, we obtain

\[ \sum_{k=1}^{n} \tilde{\gamma}_{ik} (\tilde{P}_k)_{mi} = \sum_{k=1}^{n-1} \sum_{r=1}^{n} \tilde{\gamma}_{rk} \tilde{\omega}_k (\tilde{a}_k - q^2 \tilde{a}_{0r} + q)^{-1} \tilde{\psi}_r[f]_i \]

\[ = \tilde{\psi}_r[f]_i. \]

Therefore (summation on \( i \))
\( \tilde{\psi}[r], \tilde{\phi}[r]_i = \tilde{\psi}[r], \tilde{\phi}_i \)

\( = \tilde{\psi}[r]_i \tilde{A}_{in} \)

\( = \sum_{k=1}^{n} \tilde{\gamma}_r (P_k)_{in} \tilde{A}_{in} \)

\( = \sum_{k=1}^{n} \tilde{\gamma}_r (\tilde{a}_k \tilde{\omega}_k - \tilde{\omega}_k \tilde{A}_{in}) \)

\( = \tilde{\gamma}_r \sum_{k=1}^{n} (\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1} \tilde{a}_k \tilde{\omega}_k \)

\( = \tilde{\gamma}_r \sum_{k=1}^{n} (\tilde{\omega}_k + (q^2 \tilde{a}_0 - q)(\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1} \tilde{\omega}_k) \)

\( = \tilde{\gamma}_r. \)

Thus we can think of \( \tilde{\gamma}_r \) as determining the ‘length’ of the shift vector \( \tilde{\phi}[r]_i \).

This gives the projection property

\[ \tilde{\phi}[r]_i (\tilde{\gamma}_r)^{-1} \tilde{\psi}[r]_j = (\tilde{P}_r)_{ij}, \tag{46} \]

which generalises the projection property for normal vectors. We note that

\( \tilde{\gamma}_r \tilde{\psi}[r]_i = (-1)^{n-1} \prod_{k=1}^{n} (\tilde{a}_k - q^2 \tilde{a}_0 + q) \tilde{\psi}[r]_i \)

\( = \tilde{\psi}[r]_i \tilde{\mu}_r, \)

where

\( \tilde{\mu}_r = (-1)^{n-1} \frac{\prod_{k=1}^{n} (\tilde{a}_k - \tilde{a}_0)}{\prod_{\ell \neq r} (\tilde{a}_0 - q^2 \tilde{a}_0 + q)} \tag{47} \)

and where we have used (see appendix B)

\( (q^2 \tilde{a}_0 - q) \tilde{\psi}[r]_i = \tilde{\psi}[r]_i \tilde{a}_0, \)

\( \tilde{a}_0 \tilde{\psi}[r]_i = \tilde{\psi}[r]_i (q^{-2} \tilde{a}_0 + q^{-1}). \)

Thus equation (46) may be rearranged to give

\[ \tilde{\phi}[r]_i \tilde{\psi}[r]_j = \tilde{\mu}_r (\tilde{P}_r)_{ij}. \tag{48} \]

The invariants \( \tilde{\mu}_r \) are of interest since they determine the squared reduced matrix elements (RMEs) of the vector operator \( \tilde{\psi}[r]_i \).

**Note.** It is worth observing that

\( \tilde{\gamma}_r (\Lambda, \Lambda_0 + \varepsilon_r) = \tilde{\mu}_r (\Lambda, \Lambda_0) \)

which shows the relation between the two invariants.

Similarly using the identity for the matrix \( A \), we have from equation (38)

\( (P_k)_{mi} = \sum_{\ell=1}^{n-1} \omega_k (a_k - q^{-2} a_{0\ell} - q^{-1})^{-1} \tilde{\phi}[\ell]_j. \)
Now we consider the solution $\gamma_{rk}$ to
\[
\sum_{k=1}^{n} \gamma_{rk}\omega_k (a_k - q^{-2}a_{0r} - q^{-1})^{-1} = \delta_{r\ell},
\]
\[
\sum_{k=1}^{n} \gamma_{rk}\omega_k = 0,
\]
which yields the solution
\[
\gamma_{rk} = (a_k - q^{-2}a_{0r} - q^{-1})^{-1}\gamma_r.
\]
where
\[
\gamma_r = (-1)^{n-1} \prod_{k=1}^{n} (a_k - q^{-2}a_{0r} - q^{-1})
\]
\[
\prod_{\ell \neq r} (q^{-2}a_{0r} - q^{-2}a_{0\ell})
\]
\[
= (-1)^{n-1} q^{-n} \prod_{k=1}^{n} (a_kq - q^{-1}a_{0r} - 1).
\]

In this case we may write
\[
\overline{\phi}[r]_i = \sum_{k} \gamma_{rk}(P_k)_m
\]
so that, multiplying on the right by $\overline{\psi}_i = A_{in}$, we obtain (summation on $i$)
\[
\overline{\phi}[r]_i \overline{\psi}[r]_i = \sum_{k} \gamma_{rk}(P_k)_{mi}A_{in} = \gamma_r.
\]

We note that
\[
\gamma_r \overline{\phi}[r]_i = \overline{\phi}[r]_i \mu_r
\]
where now
\[
\mu_r (\Lambda, \Lambda_0) = \gamma_r (\Lambda, \Lambda_0 - \varepsilon_r)
\]
\[
= (-1)^{n-1} \prod_{k=1}^{n} \frac{(a_k - a_{0r})}{(a_{0r} - q^{-2}a_{0\ell} - q^{-1})}.
\]

Thus in this case we have the spectral decomposition
\[
\overline{\psi}[r]_i (\gamma_r)^{-1} \overline{\phi}[r]_i = (P_r)_{ij}
\]
(50)
or
\[
\overline{\psi}[r]_j \overline{\phi}[r]_j = \mu_r (P_r)_{ij}.
\]
(51)

5.3. Squared reduced Wigner coefficients

Using the identity satisfied by the matrix $\tilde{A}$ we have (see equation (40))
\[
\Lambda_{in}(\tilde{P}_k)_{mj} = (\tilde{a}_k - \Lambda_0)_{ie}(\tilde{P}_k)_{ij},
\]
or
\[
\tilde{\phi}_{ij}(\tilde{P}_k)_{mj} = (\tilde{a}_k - \Lambda_0)_{ie}(\tilde{P}_k)_{ij}.
\]
Therefore, multiplying on the left by the subalgebra projector \( \tilde{P}_0 \) (\( U_q(n - 1) \) analogue of \( \tilde{P}_k \)) gives
\[
\tilde{\varphi}[r]_i(\tilde{P}_k)_nj = (\tilde{a}_k - \tilde{a}_0)(\tilde{P}_0)_i(r)(\tilde{P}_k)_lj
\Rightarrow (\tilde{P}_0 \tilde{P}_k)_ij = (\tilde{a}_k - \tilde{a}_0)^{-1} \tilde{\varphi}[r]_i(\tilde{P}_k)_ij.
\]

Therefore from equation (42) we obtain
\[
(\tilde{P}_0 \tilde{P}_k \tilde{P}_0)_ij = (\tilde{a}_k - \tilde{a}_0)^{-1} \tilde{\varphi}[r]_i \tilde{\omega}_k(\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1} \tilde{\psi}[r]_j.
\]

Now using the fact that the expression
\[
\tilde{\omega}_k(\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1}
\]
is independent of \( \tilde{a}_0 \) (see formula (41)) we obtain
\[
(\tilde{P}_0 \tilde{P}_k \tilde{P}_0)_ij = \tilde{\omega}_k(\tilde{a}_k - \tilde{a}_0)^{-1}(\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1} \tilde{\varphi}[r]_i \tilde{\psi}[r]_j = (\tilde{a}_k - \tilde{a}_0)^{-1}(\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1} \tilde{\omega}_k \tilde{\mu}_i(\tilde{P}_0)_ij.
\]

Thus we arrive at
\[
(\tilde{P}_0 \tilde{P}_k \tilde{P}_0)_ij = \tilde{\omega}_k(\tilde{P}_0)_ij
\]

where
\[
\tilde{\omega}_k = (\tilde{a}_k - \tilde{a}_0)^{-1}(\tilde{a}_k - q^2 \tilde{a}_0 + q)^{-1} \tilde{\omega}_k \tilde{\mu}_i.
\]

These determine \( U_q(n - 1) \) invariant operators whose eigenvalues are the squared RWCs
\[
\begin{vmatrix}
\Lambda + \varepsilon_k & \varepsilon_j \\
\Lambda_0 + \varepsilon_k & \Lambda_0
\end{vmatrix}^2
\]
where \( \Lambda_0 \) is the highest weight of a \( U_q(n - 1) \) submodule of \( V(\Lambda) \).

Similarly we have
\[
A_0(\tilde{P}_k)_nj + (A_0)_i(\tilde{P}_k)_lj = a_k(\tilde{P}_k)_ij
\Rightarrow \tilde{\psi}[r]_i(\tilde{P}_k)_nj = (a_k - A_0)_i(\tilde{P}_k)_lj.
\]

Now multiplying on the left by \( U_q(n - 1) \) projector \( P_0 \) gives
\[
\tilde{\psi}[r]_i(\tilde{P}_k)_nj = (a_k - a_0)(P_0 \tilde{P}_k)_ij
\Rightarrow (P_0 \tilde{P}_k)_ij = (a_k - a_0)^{-1} \tilde{\psi}[r]_i(\tilde{P}_k)_nj.
\]

Therefore, from equation (38), we obtain
\[
(P_0 \tilde{P}_k \tilde{P}_0)_ij = (a_k - a_0)^{-1} \tilde{\psi}[r]_i \tilde{\omega}_k(a_k - q^{-2} a_0 - q^{-1})^{-1} \tilde{\psi}[r]_j = (a_k - a_0)^{-1}(a_k - q^{-2} a_0 - q^{-1})^{-1} \tilde{\omega}_k \tilde{\mu}_i(P_0)_ij.
\]

Therefore, in this case we obtain
\[
(P_0 \tilde{P}_k \tilde{P}_0)_ij = \tilde{\omega}_k(P_0)_ij,
\]
where
\[ \omega_{kr} = (a_k - a_0 r)^{-1} (a_k - q^{-2} a_0 r - q^{-1})^{-1} \omega_{kr}. \]

These invariants determine the squared pseudo vector RWCs
\[ \langle \Lambda - \varepsilon_k | \Omega_1 ; \Lambda_0 \rangle \]
where \( \varepsilon_1 = -\varepsilon_n \) is the highest weight of the pseudo vector module, and similarly for \( \varepsilon_0 = -\varepsilon_{n-1} \).

5.4. Evaluation of Wigner coefficients

Recall [15] that a (pseudo) vector Wigner coefficient (WC) is expressible as a product of RWCs for each quantum group in the canonical subalgebra chain
\[ U_q(n) \supset U_q(n-1) \supset \cdots \supset U_q(2) \supset U_q(1). \]

It thus suffices to evaluate \( U_q(n) : U_q(n-1) \) (pseudo) vector RWCs (also known as isoscalar factors).

For the vector RWCs, we have two types, namely
\[ \langle \Lambda + \varepsilon_k | \varepsilon_1 \Lambda_0 \rangle, \quad \langle \Lambda + \varepsilon_k | \varepsilon_01 \Lambda_0 \rangle \]
corresponding to the two irreducible \( U_q(n-1) \) submodules of the \( U_q(n) \) vector module \( V = V(\varepsilon_1) \). We have seen that the invariants \( \tilde{\omega}_k, \tilde{\omega}_{kr} \) determine the absolute value squared of these RWCs respectively, so it remains to determine the phases. The latter are determined by the phases \( \pm 1 \) of the corresponding classical RWCs obtained in the limit \( q \to 1 \). By this means, we obtain
\[ \langle \Lambda + \varepsilon_k | \varepsilon_1 \Lambda_0 \rangle = \tilde{\omega}_k^{1/2}, \]
\[ \langle \Lambda + \varepsilon_k | \varepsilon_01 \Lambda_0 \rangle = S(r-k)\tilde{\omega}_{kr}^{1/2} \]
where
\[ S(x) = \text{sgn}(x), \quad \text{with } S(0) = 1. \]

Similarly we obtain
\[ \langle \Lambda - \varepsilon_k | \varepsilon_1 \Lambda_0 \rangle = \omega_k^{1/2}, \]
\[ \langle \Lambda - \varepsilon_k | \varepsilon_01 \Lambda_0 \rangle = S(r-k)\tilde{\omega}_{kr}^{1/2} \]

Note. Here we adopt the convention that the square root is that with positive real part.
5.5. Alternative expressions

Following appendix D we may write
\[ a_k - q^{-2}a_{0r} - q^{-1} = q^{-(\alpha_k + \alpha_{0r} + 1)[\alpha_k - \alpha_{0r} - 1]} q, \]
\[ a_k - a_\ell = q^{-(\alpha_k + \alpha_\ell)[\alpha_k - \alpha_\ell]} q \]
\[ \Rightarrow \omega_k = \prod_{r=1}^{n-1} (a_k - q^{-2}a_{0r} - q^{-1}) \prod_{\ell \neq k} (a_k - a_\ell) \]
\[ = \prod_{r=1}^{n-1} q^{-(\alpha_k + \alpha_{0r} + 1)[\alpha_k - \alpha_{0r} - 1]} q \prod_{\ell \neq k} q^{-(\alpha_k + \alpha_\ell)[\alpha_k - \alpha_\ell]} q \]
\[ = q^\xi \prod_{r=1}^{n-1} [\alpha_k - \alpha_{0r} - 1] q, \]

where
\[ \xi_k = \sum_{\ell \neq k} \alpha_\ell - \sum_{r=1}^{n-1} (\alpha_{0r} + 1) \]
\[ = \sum_{\ell=1}^{n} \Lambda_\ell - \sum_{r=1}^{n-1} \Lambda_{0r} - \alpha_k. \]

Note. the first two terms on the right hand side of the previous expression actually determine the \( n \)th component of the weight of the semi-maximal state
\[ \begin{pmatrix} \Lambda \\ \Lambda_0 \\ \text{max} \end{pmatrix}, \]

which has weight \( \Lambda_0 + \theta(\Lambda, \Lambda_0) e_n \), where
\[ \theta(\Lambda, \Lambda_0) = \sum_{\ell=1}^{n} \Lambda_\ell - \sum_{r=1}^{n-1} \Lambda_{0r}. \]

Similarly, for \( \tilde{\omega}_k \) we have
\[ \tilde{\omega}_k = \prod_{r=1}^{n-1} (\tilde{a}_k - q^{2}\tilde{a}_{0r} + q) \prod_{\ell \neq k} (\tilde{a}_k - \tilde{a}_\ell)^{-1} \]
\[ = \prod_{r=1}^{n-1} q^{-\left(\alpha_k + \alpha_{0r} - 1\right)} q^{\left[\alpha_k - \alpha_{0r} + 1\right]} q \prod_{\ell \neq k} q^{-\left(\alpha_k + \alpha_\ell - 1\right)} q^{\left[\alpha_k - \alpha_\ell + 1\right]} q \]
\[ = q^{\tilde{\xi}} \prod_{r=1}^{n-1} \left[\alpha_k - \alpha_{0r} + 1\right] q \prod_{\ell \neq k} \left[\alpha_k - \alpha_\ell + 1\right] q \]

where
\[ \tilde{\xi}_k = \sum_{\ell \neq k} \tilde{\alpha}_\ell - \sum_{r=1}^{n-1} (\tilde{\alpha}_{0r} - 1) \]
\[ = \theta(\Lambda, \Lambda_0) - \tilde{\alpha}_k \]
\[ = \theta(\Lambda, \Lambda_0) - \alpha_k + n - 1 = \xi_k + n - 1. \]
Similarly, for the squared RMEs we have

$$\tilde{\mu}_r = (-1)^{n-1} \frac{\prod_{k=1}^{n}(\tilde{a}_k - \tilde{a}_0 \ell)}{\prod_{\ell \neq r}(\tilde{a}_0 \ell - q^2 \tilde{a}_0 \ell + q)}.$$

Now we use

$$\begin{align*}
\tilde{a}_k - \tilde{a}_0 \ell &= q^{-n(\tilde{\alpha}_0 + \tilde{\alpha}_\ell)}[\tilde{a}_k - \tilde{a}(\ell)] q, \\
\tilde{a}_0 \ell - q^2 \tilde{a}_0 \ell + q &= q^{-n(\tilde{\alpha}_0 + \tilde{\alpha}_\ell) - 1}[\tilde{a}_0 \ell - \tilde{a}_0 \ell + 1] q \\
\Rightarrow \tilde{\mu}_r &= (-1)^{n-1} q^{n \eta_r} \frac{\prod_{k=1}^{n}[\tilde{a}_k - \tilde{a}(\ell)] q}{\prod_{\ell \neq r}[\tilde{a}_0 \ell - \tilde{a}_0 \ell + 1] q},
\end{align*}$$

where

$$\eta_r = \sum_{\ell \neq r}(\tilde{a}_0 \ell + \tilde{a}_0 \ell - 1) - \sum_{k=1}^{n}(\tilde{a}_0 \ell + \tilde{a}_k)$$

$$= -2\tilde{a}_0 \ell + \sum_{\ell \neq r}(\tilde{a}_0 \ell - 1) - \sum_{k=1}^{n}\tilde{a}_k$$

$$= -3\tilde{a}_0 \ell + 1 + \sum_{\ell = 1}^{n-1}(\tilde{a}_0 \ell - \sum_{k=1}^{n}\tilde{A}_k)$$

$$= -3\tilde{a}_0 \ell + 1 - \theta(\tilde{A}, \tilde{A}_0). \quad (55)$$

Also,

$$\mu_r = (-1)^{n-1} \frac{\prod_{k=1}^{n}(a_k - a_0 \ell)}{\prod_{\ell \neq r}(a_0 \ell - q^{-2}a_0 \ell - q^{-1})}$$

$$= (-1)^{n-1} \frac{\prod_{k=1}^{n} q^{-(\alpha_0 + \alpha_\ell)}[a_k - a_0 \ell] q}{\prod_{\ell \neq r} q^{-(\alpha_0 + \alpha_\ell + 1)}[a_0 \ell - a_0 \ell - 1] q}$$

where we have used

$$a_0 \ell - q^{-2}a_0 \ell - q^{-1} = q^{-2}(q^2a_0 \ell - a_0 \ell - q)$$

$$= q^{-(\alpha_0 + \alpha_\ell + 1)}[a_0 \ell - a_0 \ell - 1] q,$$

$$a_k - a_0 \ell = q^{-(\alpha_0 + \alpha_\ell)}[a_k - a_0 \ell] q.$$

Therefore we obtain

$$\mu_r = (-1)^{n-1} q^{n \eta_r} \frac{\prod_{k=1}^{n}[a_k - a_0 \ell] q}{\prod_{\ell \neq r}[a_0 \ell - a_0 \ell - 1] q}$$

where

$$\eta_r = \sum_{\ell \neq r}(\alpha_0 \ell + \alpha_0 \ell + 1) - \sum_{k=1}^{n}(\alpha_0 \ell + \alpha_k)$$

$$= -3\alpha_0 \ell + 1 + \sum_{\ell = 1}^{n-1}(\alpha_0 \ell - \sum_{k=1}^{n}\alpha_k)$$

$$= -3\alpha_0 \ell + 1 - \theta(\alpha_0, \alpha).$$
Utilising the above formulae for $\omega_k$ and $\mu_k$, we have for the invariants

$$\omega_k = (a_k - a_0)^{-1}(a_k - q^{-2}a_0q^{-1})^{-1}\omega_k\mu_k$$

$$= (-1)^{n-1} \prod_{\ell \neq r} \frac{a_k - q^{-2}a_0q^{-1}}{a_0 - a_0} \prod_{p \neq k} a_p - a_0$$

$$= (-1)^{n-1} \prod_{\ell \neq r} \frac{a_k - q^{-2}a_0q^{-1}}{a_0 - q^{-2}a_0q^{-1}} \prod_{p \neq k} a_p - a_0$$

$$= \prod_{\ell \neq r} \frac{a_k - q^{-2}a_0q^{-1}}{a_0 - q^{-2}a_0q^{-1}} \prod_{p \neq k} (a_p - a_0)$$

$$= \prod_{\ell \neq r} q^{-\alpha_{0\ell}} \prod_{p \neq k} q^{-\alpha_{0p}} \frac{(\alpha_k - \alpha_0 - 1)_q}{(\alpha_0 - \alpha_0 - 1)_q} \prod_{p \neq k} \frac{a_p - a_0}{a_p - a_k}$$

$$= q^{n-\alpha_{0\ell}} \prod_{\ell \neq r} \frac{(\alpha_k - \alpha_0 - 1)_q}{(\alpha_0 - \alpha_0 - 1)_q} \prod_{p \neq k} \frac{a_p - a_0}{a_p - a_k}$$

Similarly,

$$\bar{\omega}_k = (\bar{a}_k - \bar{a}_0)^{-1}(\bar{a}_k - q^2\bar{a}_0q + q)^{-1}\bar{\omega}_k\bar{\mu}_k$$

$$= (-1)^{n-1} \prod_{\ell \neq r} \frac{\bar{a}_k - q^2\bar{a}_0q + q}{\bar{a}_0 - q^2\bar{a}_0q + q} \prod_{p \neq k} (\bar{a}_p - \bar{a}_0)$$

$$= \prod_{\ell \neq r} \frac{\bar{a}_k - q^2\bar{a}_0q + q}{\bar{a}_0 - q^2\bar{a}_0q + q} \prod_{p \neq k} \frac{(\bar{a}_p - \bar{a}_0)}{\bar{a}_p - \bar{a}_k}$$

$$= \prod_{\ell \neq r} q^{-\bar{\alpha}_{0\ell}} \prod_{p \neq k} q^{-\bar{\alpha}_{0p}} \frac{(\bar{\alpha}_k - \bar{\alpha}_0 - 1)_q}{(\bar{\alpha}_0 - \bar{\alpha}_0 - 1)_q} \prod_{p \neq k} \frac{\bar{a}_p - \bar{a}_0}{\bar{a}_p - \bar{a}_k}$$

$$= q^{n-\bar{\alpha}_{0\ell}} \prod_{\ell \neq r} \frac{(\bar{\alpha}_k - \bar{\alpha}_0 - 1)_q}{(\bar{\alpha}_0 - \bar{\alpha}_0 - 1)_q} \prod_{p \neq k} \frac{\bar{a}_p - \bar{a}_0}{\bar{a}_p - \bar{a}_k}$$

6. On matrix elements

6.1. Matrix elements of elementary operators $A_{m,m+1}$, $A_{m+1,m}$, $\tilde{A}_{m,m+1}$, $\tilde{A}_{m+1,m}$

Here we denote the characteristic roots of the canonical subalgebra $U_q(m)$ by

$$\tilde{a}_{k,m}, \quad a_{k,m}$$

so that

$$a_{k,m} = 1 - q^{-2\alpha_{k,m}}, \quad \tilde{a}_{k,m} = \frac{1 - q^{-2\alpha_{k,m}}}{q - q^{-1}}$$

with $\alpha_{k,m} = \Lambda_{k,m} + m - k$, $\tilde{\alpha}_{k,m} = \Lambda_{k,m} + 1 - k$. We also denote the $U_q(m)$ (dual) vector operators by
\[ \tilde{\phi}_{m,i} = \tilde{A}_{m+1,i}, \quad 1 \leq i \leq m, \]
\[ \tilde{\psi}_{m,i} = \tilde{A}_{m+1,i}, \quad 1 \leq i \leq m, \]

and we denote the \( U_q(m) \) analogues of the \( U_q(n-1) \) invariants \( \omega_k, \tilde{\omega}_k, \mu_r, \tilde{\mu}_r \) by \( \omega_{k,m}, \tilde{\omega}_{k,m}, \mu_{r,m}, \tilde{\mu}_{r,m} \) respectively. Thus with this notation we have
\[ \omega_k \equiv \omega_{k,m}, \quad \tilde{\omega}_k \equiv \tilde{\omega}_{k,m}, \quad \mu_r \equiv \mu_{r,m}, \quad \tilde{\mu}_r \equiv \tilde{\mu}_{r,m}. \]

Now let \( (\Lambda) \) be a Gelfand–Tsetlin pattern whose \( m \)th row is given by the \( U_q(m) \) highest weight \( (\Lambda_1,m, \Lambda_2,m, \ldots, \Lambda_{num}) \). Then the action of the elementary operator \( \tilde{\psi}_{m,m} = \tilde{A}_{m+1,m} \) on the corresponding Gelfand–Tsetlin state is given by
\[ \tilde{\psi}_{m,m}(\Lambda) = \sum_{r=1}^{m} \tilde{\psi}_m[r_m](\Lambda) \]
where
\[ \tilde{\psi}_m[r_m](\Lambda) = \tilde{M}_{r,m}(\Lambda + \varepsilon_{r,m}). \]

Choosing the positive real phases for these matrix elements, in agreement with the classical convention, we have
\[ \tilde{M}_{r,m} = (\langle \Lambda | \tilde{\psi}_m[r_m] \tilde{\psi}_m[r_m] | \Lambda \rangle)^{1/2} = (\langle \Lambda | \tilde{\phi}_m[r_m] \tilde{\phi}_m[r_m] | \Lambda \rangle)^{1/2} = (\langle \Lambda | \tilde{\phi}_{r,m} \tilde{\mu}_{r,m} | \Lambda \rangle)^{1/2}, \]
where, in the notation above,
\[ \tilde{\mu}_{r,m} \tilde{\mu}_{r,m} = (-1)^m \prod_{k=1}^{m+1} (\tilde{a}_{k,m+1} - \tilde{a}_{r,m}) \prod_{r=1}^{m-1} (\tilde{a}_{r,m} - q^n \tilde{a}_{r,m-1} + q) \]
\[ \prod_{r \neq r}^{m} (\tilde{a}_{r,m} - q^n \tilde{a}_{r,m} + q)(\tilde{a}_{r,m} - \tilde{a}_{r,m}). \]

**Note.** The above agrees with the expression for the elementary generator matrix elements in the classical limit \( q \to 1 \).

Similarly we consider the (dual/pseudo) vector operators
\[ \tilde{\phi}_{m,m} = \tilde{A}_{m,m+1}, \quad \tilde{\psi}_{m,m} = \tilde{A}_{m,m+1}, \quad 1 \leq i \leq m. \]

Then the action of \( \tilde{\phi}_{m,m} = \tilde{A}_{m+1,m} \) on a Gelfand–Tsetlin state is given by
\[ A_{m+1,m}(\Lambda) = \sum_{r=1}^{m} \tilde{\phi}_m[r_m](\Lambda), \]
where
\[ \tilde{\phi}_m[r_m](\Lambda) = \tilde{M}_{r,m}(\Lambda - \varepsilon_{r,m}). \]

Again, using positive real phases, we obtain
\[ \tilde{M}_{r,m} = (\langle \Lambda | \tilde{\phi}_m[r_m] \tilde{\phi}_m[r_m] | \Lambda \rangle)^{1/2} = (\langle \Lambda | \tilde{\phi}_m[r_m] \tilde{\phi}_m[r_m] | \Lambda \rangle)^{1/2} = (\langle \Lambda | \tilde{\phi}_{r,m} \tilde{\mu}_{r,m} | \Lambda \rangle)^{1/2}. \]
where
\[
\mu_{r,m}\omega_{r,m} = \frac{(-1)^m \prod_{k=1}^{m+1} (d_{k,m+1} - d_{r,m}) \prod_{\ell=1}^{m-1} (d_{r,m} - q^{-2}a_{\ell,m-1} - q^{-1})}{\prod_{\ell\neq r} (d_{r,m} - q^{-2}a_{\ell,m} - q^{-1}) (d_{r,m} - a_{\ell,m})}.
\]

Thus we obtain the following matrix element formulae
\[
\tilde{M}_{r,m} = q^{\frac{1}{2} \langle \tilde{v}_{r,m} + \tilde{\epsilon}_{r,m} \rangle} \left\{ \prod_{k=1}^{m+1} [\tilde{r}_{k,m+1} - \tilde{r}_{r,m}]_q \prod_{\ell=1}^{m-1} [\tilde{r}_{r,m} - \tilde{r}_{r,m} - 1]_q \right\}^{1/2},
\]
where, using notation extended from (54) and (55),
\[
\tilde{v}_{r,m} + \tilde{\epsilon}_{r,m} = \sum_{j=1}^{m} \Lambda_{j,m} - \sum_{\ell=1}^{m-1} \Lambda_{\ell,m-1} - \tilde{r}_{r,m} - 3\tilde{\epsilon}_{r,m} + 1 - \sum_{k=1}^{m+1} \Lambda_{k,m+1} + \sum_{j=1}^{m} \Lambda_{j,m}
= 1 - 4\tilde{r}_{r,m} + 2m \Lambda_{j,m} - \sum_{\ell=1}^{m-1} \Lambda_{\ell,m-1} - \sum_{k=1}^{m+1} \Lambda_{k,m+1}
= 1 - 4\tilde{r}_{r,m} + \nu (\nu, \varepsilon_m - \varepsilon_{m+1})
\]
where \(\nu\) is the weight of the (initial) Gelfand–Tsetlin state \(|\Lambda\rangle\)
\[
\Rightarrow \tilde{M}_{r,m} = q^{\frac{1}{2} \langle \nu + \rho - \varepsilon_m - \varepsilon_{m+1} \rangle - 2\nu_m} \left\{ \prod_{k=1}^{m+1} [\alpha_{k,m+1} - \alpha_{r,m}]_q \prod_{\ell=1}^{m} [\alpha_{r,m} - \alpha_{r,m} - 1]_q \right\}^{1/2}.
\]

Similarly we obtain
\[
\tilde{M}_{r,m} = q^{\frac{1}{2} \langle \nu + \rho - \varepsilon_m - \varepsilon_{m+1} \rangle - 2\nu_m} \left\{ \prod_{k=1}^{m+1} [\alpha_{k,m+1} - \alpha_{r,m}]_q \prod_{\ell=1}^{m} [\alpha_{r,m} - \alpha_{r,m} - 1]_q \right\}^{1/2}.
\]

**Note.** Here, \(\rho\) is the \(U_q(n)\) Weyl vector, so \(\langle \rho, \varepsilon_m - \varepsilon_{m+1} \rangle = 1\).

For the remaining elementary generators \(\tilde{A}_m \alpha_{m+1}, \tilde{A}_m \varepsilon_{m+1}\) we have, from appendix E,
\[
\tilde{A}_m \alpha_{m+1} = \tilde{\phi}_{m,m} = (\tilde{A}_m + 1)\tilde{\phi}_{m,m},
\]
so that on a Gelfand–Tsetlin state \(|\Lambda\rangle\) we have
\[
\tilde{A}_m \alpha_{m+1} |\Lambda\rangle = \sum_{r=1}^{m} \tilde{\phi}_{m,r} r |\Lambda\rangle,
\]
where
\[
\tilde{\phi}_{m,r} |\Lambda\rangle = \tilde{M}_{r,m} |\Lambda - \varepsilon_{r,m}\rangle,
\]
and
\[
\tilde{M}_{r,m} = \langle \Lambda - \varepsilon_{r,m} | \tilde{A}_m \alpha_{m+1} | \Lambda \rangle = \langle \Lambda | \tilde{A}_m | \Lambda - \varepsilon_{r,m} \rangle.
\]

From the above formulae for the matrix elements \(\tilde{M}_{r,m}\) of \(\tilde{A}_m \alpha_{m+1}\) \(m,\), we obtain
\[
\tilde{M}_{r,m} = q^{\frac{1}{2} \langle \nu + \rho - \varepsilon_m - \varepsilon_{m+1} \rangle - 2\nu_m} \left\{ \prod_{k=1}^{m+1} [\alpha_{k,m+1} - \alpha_{r,m}]_q \prod_{\ell=1}^{m} [\alpha_{r,m} - \alpha_{r,m} - 1]_q \right\}^{1/2}.
\]
Note. If $\nu$ is the weight of the initial Gelfand–Tsetlin state $|\Lambda\rangle$, then the weight of $|\Lambda - \varepsilon_{r,m}\rangle$ is actually $\nu - \varepsilon_m + \varepsilon_{m+1}$.

Similarly we have for the generators $A_{m \, m+1}$,

$$A_{m \, m+1}|\Lambda\rangle = \sum_{r=1}^{m} \overline{\psi}_m[r]_m|\Lambda\rangle,$$

where

$$\overline{\psi}_m[r]_m|\Lambda\rangle = M'_{r,m}|\Lambda + \varepsilon_{r,m}\rangle.$$

We obtain

$$M'_{r,m} = (|\Lambda + \varepsilon_{r,m}\rangle|A_{m \, m+1}|\Lambda\rangle) = (|\Lambda\rangle|A_{m+1 \, m}|\Lambda + \varepsilon_{r,m}\rangle) = M_{r,m}(\Lambda + \varepsilon_{r,m})$$

$$= q^{1/2} \frac{\prod_{i=1}^{m+1} (\alpha_{r,m+1} - \alpha_{r,m} - 1)}{q^{m+1} \prod_{j=1}^{m+1} (\alpha_{r,m} - \alpha_{r,m+1} + 1)} \frac{1}{2}.$$  

6.2. Matrix elements of quantum group generators

From the transformation properties (24) (set $k = \ell = m + 1$, $i = m$) we have

$$[e_m, \tilde{A}_{m+1 \, m+1}] = q^{-\frac{1}{2} (\varepsilon_m - \varepsilon_{m+1} - \varepsilon_{m+1})} \tilde{A}_{m+1 \, m+1} q^{-\frac{1}{2} \hbar} = q^2 A_{m+1 \, m+1} q^{-\frac{1}{2} \hbar}.$$

On the other hand, note that the first order invariant $\tilde{C}_1 = \tilde{C}_{1,m+1}$ for $U_q(m + 1)$ is expressible

$$\tilde{C}_{1,m+1} = q \tilde{C}_{1,m} + q^{-m} A_{m+1 \, m+1} \Rightarrow \tilde{A}_{m+1 \, m+1} = q^m [\tilde{C}_{1,m+1} - q \tilde{C}_{1,m}],$$

where $\tilde{C}_{1,m}$ denotes the first order invariant of $U_q(m)$. Since $\tilde{C}_{1,m+1}$ commutes with $e_m$ this gives

$$q^{m+1} \tilde{C}_{1,m} [e_m] = q^2 A_{m+1 \, m+1} q^{-\frac{1}{2} \hbar}$$

or

$$[\tilde{C}_{1,m}, e_m] = q^{-m+\frac{1}{2}} \tilde{A}_{m+1 \, m+1} q^{-\frac{1}{2} \hbar}.$$

It follows that the non-zero matrix elements of $e_m$ are of the form

$$N_{r,m} = \langle (\Lambda + \varepsilon_{r,m})|e_m|\Lambda\rangle$$

and are related to the matrix elements

$$M_{r,m} = \langle (\Lambda + \varepsilon_{r,m})|\tilde{A}_{m+1 \, m+1}|\Lambda\rangle$$

by

$$q^{-m+\frac{1}{2}} \langle \Lambda + \varepsilon_{r,m}\rangle M_{r,m} = \langle (\Lambda + \varepsilon_{r,m})|\tilde{C}_{1,m} e_m - e_m \tilde{C}_{1,m}|\Lambda\rangle = \langle \tilde{C}_{1,m} (\Lambda + \varepsilon_{r,m}) - \tilde{C}_{1,m}\Lambda\rangle N_{r,m}.$$
Now from equation (37) we have
\[ \tilde{C}_{1,m}(L + \varepsilon_r m) - \tilde{C}_{1,m}(L) = \sum_{i=1}^{m} \left( q^{-(L+2\rho_1,\varepsilon_i)}[(L + \varepsilon_r m)]_q - q^{-(L+2\rho_1,\varepsilon_i)}[(L,\varepsilon_i)]_q \right) \]
\[ = q^{-(L+2\rho_1,\varepsilon_i)}[(L + \varepsilon_r m)]_q - q^{-(L+2\rho_1,\varepsilon_i)}[(L,\varepsilon_i)]_q \]
\[ = q^{-2(L+\rho_1,\varepsilon_i) - 1} = q^{-2\ell_m - m} \]
\[ \Rightarrow q^{-(m+\frac{1}{2})\langle \nu,\varepsilon_m - \varepsilon_{m+1} \rangle} \tilde{M}_{r,m} = q^{-2\ell_m - m} N_{r,m} \]
or
\[ N_{r,m} = q^{2\ell_m - \frac{1}{2} \langle \nu,\varepsilon_m - \varepsilon_{m+1} \rangle} \tilde{M}_{r,m} \]
where, again, \( \nu \) is the weight of the (initial) Gelfand–Tsetlin state \((L)\). By comparison with equation (56) this gives
\[ N_{r,m} = \left\{ \frac{\prod_{k=1}^{m+1} [\alpha_{k,m+1} - \alpha_{r,m}]_q \prod_{i=1}^{m-1} [\alpha_{r,m} - \alpha_{i,m-1} + 1]_q}{\prod_{i=1}^{m} [\alpha_{r,m} - \alpha_{i,m} + 1]_q [\alpha_{r,m} - \alpha_{\ell_m}]} \right\}^{1/2}. \]

Similarly from equation (24) with \( i = m, k = \ell = m + 1 \) we obtain
\[ [f_m, \tilde{A}_{m+1,m+1}] = -q^{3/2} \tilde{A}_{m+1,m+1} q^{-h_m/2} \]
\[ \Rightarrow -q^{3/2} \tilde{A}_{m+1,m+1} q^{-h_m/2} = [f_m, \tilde{A}_{m+1,m+1}] = -q^{m+1} [f_m, \tilde{C}_{1,m}], \]
or
\[ [\tilde{C}_{1,m}, f_m] = -q^{-m+\frac{1}{2}} \tilde{A}_{m+1,m+1} q^{-h_m/2}. \]
Then the non-zero matrix elements of \( f_m \) are of the form
\[ N_{r,m} = \langle (L - \varepsilon_r m) | f_m | (L) \rangle \]
and are related to the matrix elements
\[ \tilde{M}'_{r,m} = \langle (L - \varepsilon_r m) | \tilde{A}_{m+1} | (L) \rangle \]
by
\[ [\tilde{C}_{1,m}(L - \varepsilon_r m) - \tilde{C}_{1,m}(L)] N_{r,m} = -q^{m+\frac{1}{2} \langle \nu,\varepsilon_m - \varepsilon_{m+1} \rangle} \tilde{M}'_{r,m} \]
where \( \nu \) is the weight of the Gelfand–Tsetlin state \((L)\).

Now we note that
\[ \tilde{C}_{1,m}(L - \varepsilon_r m) - \tilde{C}_{1,m}(L) = q^{-(L+2\rho_1,\varepsilon_r)}[(L - \varepsilon_r,\varepsilon_r)]_q - q^{-(L+2\rho_1,\varepsilon_r)}[(L,\varepsilon_r)]_q \]
\[ = q^{-2(L+\rho_1,\varepsilon_r) + 1} = -q^{-m+2\ell_m + 2} \]
\[ \Rightarrow N_{r,m} = q^{2\ell_m - \frac{1}{2} \langle \nu,\varepsilon_m - \varepsilon_{m+1} \rangle} \tilde{M}'_{r,m} \]
\[ = \left\{ \frac{\prod_{k=1}^{m+1} [\alpha_{k,m+1} - \alpha_{r,m}]_q \prod_{i=1}^{m-1} [\alpha_{r,m} - \alpha_{i,m-1} + 1]_q}{\prod_{i=1}^{m} [\alpha_{r,m} - \alpha_{i,m} + 1]_q [\alpha_{r,m} - \alpha_{\ell_m}]} \right\}^{1/2}. \]
Note. In the above, \( \nu \) denotes the weight of the initial Gelfand–Tsetlin state \(|(\Lambda)|\) as before. Also recall that \(-\frac{1}{2} - \frac{1}{2}(\nu, \varepsilon_m - \varepsilon_{m+1}) = -\frac{1}{2}(\nu + \varepsilon_m - \varepsilon_{m+1})
\). Finally, in terms of characteristic roots,

\[
\alpha_{k,m} = \overline{\alpha}_{k,m} + m - 1 \implies \overline{\alpha}_{k,m} = \alpha_{k,m} - m + 1,
\]

we obtain the following matrix element formulae for the simple generators:

\[
\mathcal{N}_{r,m} = \left\{ \prod_{k=1}^{m+1} \left[ (\alpha_{k,m+1} - \alpha_{r,m} - 1) q \prod_{k=1}^{m} \left[ (\alpha_{r,m} - \alpha_{k,m} - 1) q \right] \right] \right\}^{1/2},
\]

\[
\overline{\mathcal{N}}_{r,m} = \left\{ \prod_{k=1}^{m+1} \left[ (\alpha_{k,m+1} - \alpha_{r,m}) q \prod_{k=1}^{m} \left[ (\alpha_{r,m} - \alpha_{k,m} - 1) q \right] \right] \right\}^{1/2}.
\]

These expressions agree with the previous results of Gould et al [14, 15].

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Appendix A. Vector and contragredient vector operators

Following equation (8) we call a collection of components \( \psi \equiv \{ \psi_i \}_{i=1}^n \) a vector operator if the following transformation law holds:

\[
\alpha \psi_i = \pi_0(a_{(1)}) \psi_j a_{(2)}, \quad a \in U_q(n),
\]

where \( \pi_0 \) is the defining (vector) representation (which is undeformed). In terms of quantum group generators, this transformation law is expressible

\[
q^{E_{k,0}} \psi_i = q^{h_{0} \psi_i} q^{E_{k,0}},
\]

\[
e_k \psi_i = \pi_0(q^{h_{k}/2} \psi_j e_k + \pi_0(E_{k+1}) \psi_j q^{-h_{k}/2}) = q^{1/(e_k - e_{k+1} + 1)} \psi_i e_k + \delta_{k,1} \psi_i q^{-h_{k}/2},
\]

\[
f_k \psi_i = \pi_0(q^{h_{k}/2} \psi_j f_k + \pi_0(E_{k+1}) \psi_j q^{-h_{k}/2}) = q^{1/(e_k - e_{k+1} + 1)} \psi_i f_k + \delta_{k,1} \psi_i q^{-h_{k}/2}.
\]

In terms of the \( q \)-bracket

\[
[x_k, \psi_i]_{q} = x_k \psi_i - q^{1/(e_k - e_{k+1} + 1)} \psi_i x_k, \quad x_k = e_k f_k,
\]

the last two relations are expressible

\[
[e_k, \psi_i]_{q} = \delta_{k,k+1} \psi_i q^{-h_{k}/2},
\]

\[
[f_k, \psi_i]_{q} = \delta_{k,k+1} \psi_i q^{-h_{k}/2}.
\]

Dually, we call a collection of operators \( \phi \equiv \{ \phi_i \}_{i=1}^n \) a dual vector operator if the following transformation law holds:

\[
\alpha \phi_i = \pi_0^*(a_{(1)}) \phi_j a_{(2)}, \quad a \in U_q(n),
\]

(A.1)
where \( \pi^*_0 \) is the dual vector representation defined by
\[
\pi^*_0(a)_{ij} = \langle e_j, a e^*_i \rangle = \langle S(a)e_j, e^*_i \rangle = \pi_0(S(a))_{ij}.
\]

Note. If \( a \vec{\phi}_i \) transforms as \( \pi_0 \), called a 
**pseudo vector operator**. Thus
\[
a^* a \vec{\phi}_i = \pi_0(a_{(1)})_{ij} a \vec{\phi}_j a_{(2)}, \ a \in U_q(n).
\]

In terms of elementary generators, the above transformation law \((A.1)\) is expressible
\[
q^{E_a} \vec{\phi}_i = q^{-\delta_{ki}} \phi_i q^{E_a},
\]
\[
e_k \phi_i = \pi^*_0(q^{\rho_{(2)} / 2})_{ji} \phi_j e_k - q^{-1} \pi_0(E_k)_{ij} \phi_j q^{-h_k / 2}
\]
\[
\Rightarrow e_k \phi_i = q^{-1/2(\epsilon_k - \epsilon_{k+1})} \phi_i e_k = -q^{-1} \delta_{ik} \phi_{k+1} q^{-h_k / 2}
\]
and similarly
\[
f_k \phi_i - q^{-1/2(\epsilon_k - \epsilon_{k+1})} \phi_i f_k = -q^{\delta_{ik} q^{-1} q^{-h_k / 2}}.
\]

It is also worth noting that a collection of components \( \vec{\psi}_i \) transforms as a tensor operator of rank \( \pi^* \) (dual pseudo vector operator) if and only if it satisfies
\[
a^* a \vec{\psi}_i = \pi^*_0(a_{(1)})_{ij} \vec{\psi}_j a_{(2)}.
\]

In terms of elementary generators we have
\[
e_k \vec{\psi}_i = q^{1/2(\epsilon_k - \epsilon_{k+1})} \vec{\psi}_j e_k + \pi^*_0(q^{\rho_{(2)} / 2})_{ji} \vec{\psi}_j q^{-h_k / 2}
\]
\[
\Rightarrow e_k \vec{\psi}_i = q^{1/2(\epsilon_k - \epsilon_{k+1})} \vec{\psi}_j e_k = \delta_{ik} q^{-1/2} q^{-h_k / 2},
\]
and similarly
\[
f_k \vec{\psi}_i = q^{1/2(\epsilon_k - \epsilon_{k+1})} \vec{\psi}_j f_k = \pi^*_0(q^{\rho_{(2)} / 2})_{ji} \vec{\psi}_j q^{-h_k / 2}
\]
\[
= \delta_{ik} q q^{-1} q^{-h_k / 2}
\]

with
\[
q^{E_a} \vec{\psi}_i = q^{\epsilon_a} \vec{\psi}_j q^{E_a}.
\]

Note. If \( \vec{\psi}_i \) is a vector operator then
\[
\vec{\psi}_i = q^{(\rho_{(2)})} \vec{\psi}_i
\]
transforms as \( \pi^* \) and conversely, i.e. if \( \vec{\psi}_i \) transforms as \( \pi^* \) then
\[
\vec{\psi}_i = q^{-1} \rho_{(2)} \vec{\psi}_i,
\]
transforms as a vector operator.

**Appendix B. Shift components**

Let \( \{ \phi_i \} \) be a dual vector operator of \( U_q(n) \) which, acting on an irreducible module \( V(\Lambda) \), determines an intertwining operator \( \phi \):
\[
\phi(e_i \otimes v) = \phi v, \ v \in V(\Lambda).
\]
As for vector operators, \( \phi \) may be resolved into shift components
\[ \phi_r = \sum_{r=1}^{n} \phi[r] \]

where

\[ \phi[r](V^* \otimes V(\Lambda)) \cong V(\Lambda - \varepsilon_r). \]

Here we show that the projection operators \( \tilde{P}_r \) of equation (29) project out the above shift components from the left. We first need (see proposition 2)

**Lemma B.1.** The identity module occurs exactly once in \( V \otimes V^* \) and is spanned by the vector \( \xi = e_\alpha \otimes e_\alpha^* \) (sum on \( \alpha \)), where \( \{e_\alpha\} \) is a basis for \( V \) with corresponding dual basis \( \{e_\alpha^*\} \) for \( V^* \) (regarded as a module under the usual action determined by the antipode \( S \)).

**Proof.**

\[
a \xi = a_1 e_\alpha \otimes a_2 e_\alpha^* \\
= a_1 e_\alpha \otimes (a_2 e_\alpha^* \otimes e_\beta) \\
= a_1 e_\alpha \otimes (a_2 e_\alpha^* \otimes e_\beta = a_1 S(a_2) e_\beta \otimes e_\alpha^* = \varepsilon(a) \xi.
\]

Thus \( \tilde{P}_r \) clearly projects out the shift components of a vector operator \( \psi = \{\psi_i\} \) from the right since

\[
\psi[r]_i \nu = \psi[r](e_i \otimes \nu) = \psi(\tilde{P}_r)(e_i \otimes \nu) = \psi(\tilde{P}_r)_i \nu,
\]

Now let \( \tilde{P}_r \) be the projection operators of equation (29) and \( \phi[l] \) be the shift components of \( \phi \). We observe that

\[
(id \otimes \phi)(\xi \otimes \nu) = e_i \otimes \phi(e_i^* \otimes \nu), \quad \nu \in V(\Lambda) \\
= e_i \otimes \phi \nu
\]
gives an isomorphic copy of \( V(\Lambda) \). Now

\[
\tilde{P}_r (id \otimes \phi[l])(\xi \otimes \nu) = \tilde{P}_r (e_i \otimes \phi[l](e_i^* \otimes \nu)) \subseteq V(\Lambda + \varepsilon_r - \varepsilon_l) \cap V(\Lambda)
\]

\[ \Rightarrow \tilde{P}_r (id \otimes \phi[l])(\xi \otimes V(\Lambda)) = (0), \quad r \neq l. \]

Therefore we have

\[
e_i \otimes \phi[r]_i \nu = e_i \otimes \phi[r](e_i^* \otimes \nu) \\
= \tilde{P}_r (id \otimes \phi)(\xi \otimes \nu) \\
= \tilde{P}_r (e_i \otimes \phi(e_i^* \otimes \nu)) \\
= e_i \otimes (\tilde{P}_r)_i \phi \nu \\
\Rightarrow \phi[r]_i = (\tilde{P}_r)_i \phi \nu.
\]

Thus \( \tilde{P}_r \) projects out shift components of dual vector operators from the left. However, \( \tilde{P}_r \) clearly projects out the shift components of a vector operator \( \psi = \{\psi_1\} \) from the right since

\[
\psi[r]_i \nu = \psi[r](e_i \otimes \nu) \\
= \psi(\tilde{P}_r)(e_i \otimes \nu) \\
= \psi(\tilde{P}_r)(\tilde{P}_r)_i \nu \\
= \psi(\tilde{P}_r)_i \nu.
\]
\[ \psi[r] = \psi_j(\tilde{P}_r) \mu. \]

However, to project out these shift components from the left we need.

**Lemma B.2.** The identity module occurs exactly once in \( V_0^* \otimes V_0 \) and is spanned by

\[ \eta = e_i^* \otimes q^{2h_\nu} e_i = q^{\chi(\rho, \nu)} e_i^* \otimes e_i \]

(sum on \( i \)).

**Proof.**

\[
\begin{align*}
\alpha\eta = a_1 e_i^* \otimes a_2 q^{2h_\nu} e_i \\
= \langle a_1 e_i^*, e_j \rangle e_i^* \otimes a_2 q^{2h_\nu} e_i \\
= e_i^* \otimes a_2 q^{2h_\nu} e_i (e_i^*, S(a_1) e_j) \\
= e_i^* \otimes a_2 q^{2h_\nu} S(a_1) e_j \\
= e_i^* \otimes a_2 S^{-1}(a_1) q^{2h_\nu} e_j = \varepsilon(a)\eta.
\end{align*}
\]

\[ \blacksquare \]

**Note.** This result can be regarded as a particular case of lemma B.1 if we note that the transformed basis \( q^{2h_\nu} e_i \) gives a basis for the representation \( \pi^{**} \).

Now let \( P_r \) on \( V^* \otimes V(\Lambda) \) be the projection operator arising from the matrix \( A \), so here we regard \( V^* \) as a module under the action defined by \( \pi_0 \), i.e.

\[ a e_i^* = \pi_0(a) e_i^*, \]

or

\[ \langle a e_i^*, e_j \rangle = \langle e_i^*, \gamma(a) e_j \rangle. \]

Under this action the invariant of lemma B.2 becomes

\[ \overline{\eta} = e_i^* \otimes q^{h_\nu} e_i = q^{\chi(\rho, \nu)} e_i^* \otimes e_i = q^{-h_\nu} e_i^* \otimes e_i \]

since

\[
\begin{align*}
a\overline{\eta} = a_1 e_i^* \otimes a_2 q^{h_\nu} e_i \\
= \langle e_i^*, \gamma(a_1) e_i \rangle e_i^* \otimes a_2 q^{h_\nu} e_i \\
= e_i^* \otimes a_2 q^{h_\nu} \gamma(a_1) e_j \\
= e_i^* \otimes a_2 S^{-1}(a_1) q^{h_\nu} e_j \\
= \varepsilon(a)\overline{\eta}.
\end{align*}
\]

Now let \( \psi = \{ \psi_i \} \) be a vector operator acting on \( V(\Lambda) \). Following our derivation above we have for \( v \in V(\Lambda) \)
\[ q^{-h_r} e_i^* \otimes \psi_i[r]v = q^{-h_r} e_i^* \otimes \psi_i[r](e_i \otimes v) \]
\[ = (\text{id} \otimes \psi_i[r]) (\Pi \otimes v) \]
\[ = P_r (\text{id} \otimes \psi_i[r]) (\Pi \otimes v) \]
\[ = P_r (q^{-h_r} e_i^* \otimes \psi_i(r \otimes v)) \]
\[ = q^{(\rho, \varepsilon_i)} e_i^* \otimes (P_r)_{ji} \psi_i(e_i \otimes v) \]
\[ = e_i^* \otimes (P_r)_{ji} q^{(\rho, \varepsilon_i)} \psi_i v \]
\[ = e_i^* \otimes (P_r)_{ji} q^{(\rho, \varepsilon_i)} \psi_j v \]
\[ \Rightarrow q^{(\rho, \varepsilon_i)} \psi_i[r] = (P_r)_{ji} q^{(\rho, \varepsilon_i)} \psi_j. \]

**Note.** \( \psi_i = q^{(\rho, \varepsilon_i)} \) defines a dual pseudo vector operator, i.e. transforming as \( \pi_0^* \). Then we have
\[ \overline{\psi_i} = (P_r)_{ji} \psi_j, \]
thus showing that \( P_r \) projects out the shift components of such tensor operators from the left.

On the other hand, if \( \phi_i \) is a pseudo vector operator, then for \( v \in V(\Lambda) \)
\[ \overline{\phi_i} v = \overline{\phi_i}(e_i^* \otimes v) \]
\[ = \overline{\phi_i} P_r (e_i^* \otimes v) \]
\[ = \overline{\phi_i} (e_i^* \otimes (P_r)_{ji} v) \]
\[ = \overline{\phi_i} (P_r)_{ji} v \]
\[ \Rightarrow \overline{\phi_i} v = \overline{\phi_i} (P_r)_{ji} v, \]
showing that \( P_r \) projects out the shift components of \( \overline{\phi} \) from the right.

**B.1. Characteristic root shifts**

Let \( \psi \) be a vector (or dual pseudo vector) operator and \( \phi \) a dual vector (or pseudo vector) operator. Then the shift components affect the following shifts on the classical roots:
\[ \alpha_r \phi_i[r] = \phi_i[r](\alpha_r - 1), \]
\[ \overline{\alpha}_r \phi_i[r] = \phi_i[r](\overline{\alpha}_r - 1). \]
\[ \alpha_r \psi_i[r] = \psi_i[r](\alpha_r + 1), \]
\[ \overline{\alpha}_r \psi_i[r] = \psi_i[r](\overline{\alpha}_r + 1). \]

Using
\[ \frac{1 - q^{-2\alpha_r - 2}}{q - q^{-1}} = q^{-2} \frac{1 - q^{-2\alpha_r}}{q - q^{-1}} + \frac{1 - q^{-2}}{q - q^{-1}} = q^{-2} a_r + q^{-1}, \]
\[ \frac{1 - q^{-2\alpha_r + 2}}{q - q^{-1}} = q^{2} \frac{1 - q^{-2\alpha_r}}{q - q^{-1}} + \frac{1 - q^{-2}}{q - q^{-1}} = q^{2} a_r - q, \]
(and similarly for \( \overline{\alpha}_r \) and \( \overline{a}_r \)) we have the following shifts
\[ a_i \psi[r] = \psi[r] (q^{-2} a_i + q^{-1}), \]
\[ \tilde{a}_i \psi[r] = \psi[r] (q^{-2} \tilde{a}_i + q^{-1}), \]
\[ a_i \phi[r] = \phi[r] (q^2 a_i - q), \]
\[ \tilde{a}_i \phi[r] = \phi[r] (q^2 \tilde{a}_i - q), \]

or equivalently

\[ \psi[r] a_i = (q^2 a_i - q) \psi[r], \]
\[ \psi[r] \tilde{a}_i = (q^2 \tilde{a}_i - q) \psi[r], \]
\[ \phi[r] a_i = (q^{-2} a_i + q^{-1}) \phi[r], \]
\[ \phi[r] \tilde{a}_i = (q^{-2} \tilde{a}_i + q^{-1}) \phi[r]. \]

**Appendix C**

Here we show that the operators \( \tilde{E}_{ij} \) of equation (18) are related to those of equation (21), i.e. \( \tilde{E}_{ij} \), via action of the antipode.

First we recall the following definitions, with \( i \leq k \leq j \):

\[ E_{ij} = E_{ik} E_{kj} - q^{-1} E_{kj} E_{ik}, \]
\[ E'_{ij} = E'_{ik} E'_{kj} - q E'_{kj} E'_{ik}. \]

Now observe that

\[ S^{-1} (E_{i+1}) = -q E_{i+1} = -q E'_{i+1} \]

\[ \Rightarrow S^{-1} (E_{i+1}) = S^{-1} (E_{i+1} E_{i+1} E_{i+1} - q^{-1} E_{i+1} E_{i+1} E_{i+1}) \]
\[ = q^2 E'_{i+1} E'_{i+1} E'_{i+1} - q E'_{i+1} E'_{i+1} E'_{i+1} \]
\[ = -q (E'_{i+1} E'_{i+1} E'_{i+1} - q^{-1} E_{i+1} E_{i+1} E_{i+1}) \]
\[ = -q E_{i+1}. \]

More generally, it follows by induction that

\[ S^{-1} (E_{ij}) = -q E'_{ij}, \quad i < j. \]

In the case \( i > j \), we observe first that

\[ S^{-1} (E_{i+1}) = -q^{-1} E_{i+1} = -q^{-1} E'_{i+1}. \]

Then

\[ S^{-1} = S^{-1} (E_{i+2} E_{i+1} E_{i+1} - q^{-1} E_{i+1} E_{i+2} E_{i+1}) \]
\[ = q^{-2} E_{i+1} E_{i+1} E_{i+1} - q^{-3} E_{i+2} E_{i+1} E_{i+1} \]
\[ = -q^{-3} (E'_{i+2} E_{i+1} E_{i+1} - q E_{i+1} E_{i+2} E_{i+1}) \]
\[ = -q^{-3} E'_{i+2}. \]

More generally we observe that
\( S^{-1}(E_{ij}) = -q^{(2p,\varepsilon_i-\varepsilon_j)} E_{ij}', \quad i < j, \)

i.e.

\[
S^{-1}(E_{ij}) = -q \begin{cases} 
E_{ij}', & i < j \\
q^{(2p,\varepsilon_i-\varepsilon_j)} E_{ij}', & i > j.
\end{cases}
\]

Thus

\[
S^{-1}(E_{ij}) = \begin{cases} 
(q - q^{-1}) q^{-\frac{1}{2}(E_{ui} + E_{uj} - 1)} S^{-1}(E_{ij}), & i \neq j \\
q^{-E_{ui}}, & i = j
\end{cases}
\]

\[
\begin{cases} 
-(q - q^{-1}) q^{-\frac{1}{2}(E_{ui} + E_{uj} - 1)} E_{ij}', & i < j \\
q^{-E_{ui}}, & i = j
\end{cases}
\]

\[
\begin{cases} 
-(q - q^{-1}) q^{-\frac{1}{2}(E_{ui} + E_{uj} - 1)} q^{(2p,\varepsilon_i-\varepsilon_j)} E_{ij}', & i > j
\end{cases}
\]

\[
\begin{cases} 
\tilde{E}_{ij}, & i \leq j \\
q^{(2p,\varepsilon_i-\varepsilon_j)} \tilde{E}_{ij}, & i > j.
\end{cases}
\]

Thus from equation (30) we obtain

\[
(\pi_0^* \otimes \text{id}) R = \sum_{i \leq j} e_{ij} \otimes \tilde{E}_{ij}
\]

\[
(\pi_0^* \otimes \text{id}) R^T = \sum_{i \leq j} q^{(2p,\varepsilon_i-\varepsilon_j)} e_{ji} \otimes \tilde{E}_{ji}.
\]

As we have seen, the dual vector representation utilised in [14] is here denoted \( \pi_0 \) (the pseudo vector representation) and is defined by

\[
\pi_0(a) = \pi_0^*(\gamma(a)) = \pi_0^*(q^{a q^{-a}}, \quad a \in U_q(n),
\]

so (see equation (30))

\[
\pi_0(a)_{ij} = q^{(\rho,\varepsilon_i-\varepsilon_j)} \pi_0^*(a)_{ij}.
\]

This gives the following expression for the \( L \)-operators of equations (31) and (32) in terms of the \( \tilde{E}_{ij} \):

\[
(\pi_0 \otimes \text{id}) R = \sum_{i \leq j} q^{(\rho,\varepsilon_i-\varepsilon_j)} e_{ij} \otimes \tilde{E}_{ij},
\]

\[
(\pi_0 \otimes \text{id}) R^T = \sum_{i \leq j} q^{(\rho,\varepsilon_i-\varepsilon_j)} e_{ji} \otimes \tilde{E}_{ji}.
\]

This provides an alternative description of the \( L \)-operators and agrees with [14].

**Appendix D. Characteristic roots**

We have
\[ \tilde{a}_k = \frac{1 - q^{-2\pi_k}}{q - q^{-1}}, \quad \overline{a}_k = \Lambda_k + 1 - k, \]
\[ a_k = \frac{1 - q^{-2\alpha_k}}{q - q^{-1}}, \quad \alpha_k - \Lambda_k + n - k = \overline{\alpha}_k + n - 1.\]

Therefore
\[ \tilde{a}_k - q^2\tilde{a}_0 = \frac{1 - q^{-2\pi_k} - q^2(1 - q^{-2\pi_0})}{q - q^{-1}} = \frac{q^{2-2\pi_0} - q^{-2\pi_0}}{q - q^{-1}} + \frac{q(q^{-1} - q)}{q - q^{-1}} \]
\[ \Rightarrow \tilde{a}_k - q^2\tilde{a}_0 + q = \frac{q^{2-2\pi_0} - q^{-2\pi_0}}{q - q^{-1}} \]
\[ = \frac{q^{2-2(\alpha_0 + 2 - n)} - q^{-2(\alpha_0 + 1 - n)}}{q - q^{-1}} \]
\[ = q^{2(1-n)} \frac{q^{2\alpha_0} - q^{-2\alpha_0}}{q - q^{-1}} \]
\[ = q^{2(1-n)} (a_k - a_{0\nu}). \]

Therefore we have the following identity:
\[ \tilde{a}_k - q^2\tilde{a}_0 + q = q^{2(n-1)} (a_k - a_{0\nu}). \]

Also,
\[ \tilde{a}_k - a_{\ell} = \frac{q^{-2\pi_{\ell}} - q^{-2\pi_k}}{q - q^{-1}} = q^{-(\pi_{\ell} + \pi_k)} \frac{q^{\pi_{\ell} - \pi_k} - q^{\pi_k - \pi_\ell}}{q - q^{-1}} \]
\[ \Rightarrow \tilde{a}_k - a_{\ell} = q^{-(\pi_{\ell} + \pi_k)} [\tilde{a}_k - \tilde{a}_{\ell}] \]
and similarly
\[ a_k - a_{\ell} = \frac{q^{-2\alpha_{\ell}} - q^{-2\alpha_k}}{q - q^{-1}} = q^{-(\alpha_{\ell} + \alpha_k)} [\alpha_k - \alpha_{\ell}] \]
\[ a_k - a_{0\nu} = \frac{q^{-2\alpha_0} - q^{-2\alpha_k}}{q - q^{-1}} = q^{-(\alpha_0 + \alpha_k)} [\alpha_k - \alpha_{0\nu}] \]
and so on. Also observe that
\[ \alpha_k - \alpha_{\ell} = \overline{\alpha}_k - \overline{\alpha}_{\ell} \]
so that
\[ \tilde{a}_k - a_{\ell} = q^{-(\pi_{\ell} + \pi_k)} [\alpha_k - \alpha_{\ell}] \]
\[ = q^{2(n-1)} (a_k - a_{\ell}) \]
\[ \Rightarrow a_k = q^{-\alpha_{\ell}} \left( \frac{q^{\alpha_k} - q^{-\alpha_k}}{q - q^{-1}} \right) = q^{-\alpha_{\ell}} [\alpha_k]_q, \]
\[ \tilde{a}_k = q^{-\pi_k} [\tilde{a}_k]_q. \]
The above formulae allow for alternative expressions for the invariants $\omega_k$, $\tilde{\omega}_k$, $\mu_r$, $\tilde{\mu}_r$ and so on.

We similarly have

$$\tilde{a}_k - q^2 \tilde{a}_r + q = q^{2(n-1)}(a_k - q^2 a_r + q)$$

so

$$\tilde{a}_0\ell - q^2 \tilde{a}_0r + q = q^{2(n-2)}(a_0\ell - q^2 a_0r + q).$$

We also note that

$$(\Lambda + \rho, \varepsilon_r - \varepsilon_k) = \alpha_r - \alpha_k = \tilde{\alpha}_r - \tilde{\alpha}_k,$$

$$(\Lambda + \rho \pm \varepsilon_r, \varepsilon_r - \varepsilon_k) = \alpha_r - \alpha_k \pm 1 = \tilde{\alpha}_r - \tilde{\alpha}_k \pm 1,$$

and

$$[\alpha_r - \alpha_\ell - 1]_q = q^{\alpha_r + 2\alpha_\ell - 1}(q^2\alpha_r - a_\ell - q)$$

$$= [\alpha_r - \alpha_\ell - 1]_q$$

$$= q^{2\alpha_\ell - \alpha_r - 1}(q^2\tilde{a}_r - \tilde{a}_\ell - q),$$

$$[\alpha_r - \alpha_\ell + 1]_q = q^{\alpha_r + 2\alpha_\ell - 1}(a_r - q^2 a_\ell + q)$$

$$= [\alpha_r - \alpha_\ell + 1]_q$$

$$= q^{2\alpha_\ell + \alpha_r - 1}(\tilde{a}_r - q^2 \tilde{a}_\ell + q),$$

$$[\alpha_r - \alpha_\ell]_q = q^{\alpha_r + \alpha_\ell}(a_r - a_\ell)$$

$$= [\alpha_r - \alpha_\ell]_q$$

$$= q^{\alpha_r + \alpha_\ell}(\tilde{a}_r - \tilde{a}_\ell).$$

Furthermore, this implies

$$\frac{D_q[\Lambda \pm \varepsilon_r]}{D_q[\Lambda]} = \prod_{\ell \neq r} \frac{[\alpha_r - \alpha_\ell \pm 1]_q}{[\alpha_r - \alpha_\ell]_q}$$

so that

$$\frac{D_q[\Lambda + \varepsilon_r]}{D_q[\Lambda]} = \prod_{\ell \neq r} q^{\alpha_r + \alpha_\ell - 1}(a_r - q^2 a_\ell + q)$$

$$= \prod_{\ell \neq r} q^{-1} a_r - q a_\ell + 1$$

$$= \prod_{\ell \neq r} q^{-1} \tilde{a}_r - q \tilde{a}_\ell + 1,$$

$$\frac{D_q[\Lambda - \varepsilon_r]}{D_q[\Lambda]} = \prod_{\ell \neq r} q^{\alpha_r + \alpha_\ell - 1}(q^2\alpha_r - a_\ell - q)$$

$$= \prod_{\ell \neq r} q a_r - q^{-1} a_\ell - 1$$

$$= \prod_{\ell \neq r} q \tilde{a}_r - q^{-1} \tilde{a}_\ell - 1.$$

In summary:
Finally we also note that
\[ a_k - q^2a_0 - q^{-1} = \frac{1 - q^{-2\alpha_k}}{q^{-1} - q^{-1}} - q^{-1} \]
\[ = \frac{q^{-2 - 2\alpha_k} - q^{-2\alpha_k}}{q - q^{-1}} = q^{-(\alpha_k + \alpha_0 + 1)}[a_k - \alpha_0 - 1]_q. \]

and similarly
\[ \tilde{a}_k - q^2\tilde{a}_0 + q = q^{-(\alpha_k + \alpha_0 - 1)}[\tilde{a}_k - \alpha_0 + 1]_q. \]

**Appendix E. Hopf * structure (real q > 0)**

Observe that we have a conjugation (or star) operation defined by
\[ e_i^\dagger = f_i, \quad f_i^\dagger = e_i, \quad h_i^\dagger = h_i, \]
which extends to an (anti-linear) anti-homomorphism on all of \( U_q(n) \) and thus defines a conjugation (or star) operation of all \( U_q(n) \). Recall the defining properties, \( \forall a, b \in U_q(n), \alpha, \beta \in \mathbb{C} \):

(i) \((a^\dagger)^\dagger = a, \)
(ii) \((ab)^\dagger = b^\dagger a^\dagger, \)
(iii) \((\alpha a + \beta b)^\dagger = \overline{\alpha} a^\dagger + \overline{\beta} b^\dagger. \)

With this operation, \( U_q(n) \) is a Hopf * algebra for generic real \( q > 0 \), i.e. \( \Delta, \varepsilon \) determine *-algebra homomorphisms so
\[ \Delta(a)^\dagger = \Delta(a^\dagger), \quad \varepsilon(a^\dagger) = \overline{\varepsilon(a)}, \quad \forall a \in U_q(n). \]

It is easily seen that for any \( a \in U_q(n), \)
\[ S(a)^\dagger = S^{-1}(a^\dagger) \]
(uniqueness of the antipode) - observe that \( \tilde{S}(a) = S^{-1}(a)^\dagger \) is also an antipode. Moreover for the \( R \)-matrix we have
\[ R^\dagger = R^T, \quad (R^T)^\dagger = R \]
where \( \dagger \) on the left hand side is the naturally induced conjugation operation on \( U_q(n) \otimes U_q(n) \).

Following the standard approach, we have
**Definition.** A finite dimensional $U_q(n)$-module $V$ is called *unitary* if it can be equipped with an inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle a^\dagger v, w \rangle = \langle v, aw \rangle, \; \forall a \in U_q(n), \; v, w \in V.$$ 

**Note.** If $\pi$ is the representation afforded by $V$ then this is equivalent to

$$\pi(a^\dagger) = \pi(a)^\dagger.$$ 

That is,

$$\pi(a)^\dagger = \pi(a^\dagger).$$

It is well known [29] that all finite dimensional irreducible $U_q(n)$-modules are equivalent to unitary ones. In particular the (undeformed) vector module is unitary. However, care needs to be taken since the dual vector module $V^*$ defined in the usual way by

$$\langle av^*, w \rangle = \langle v^*, S(a)w \rangle, \; v^* \in V^*, \; w \in V$$

is *not* unitary (only equivalent to a unitary one). Indeed

$$\pi_0(a)^\dagger = \pi_0(S(a))^\dagger = \pi_0(S(a)^\dagger) = \pi_0(S(q^{2h_r}a^\dagger q^{-2h_r})) = \pi_0(q^{2h_r}a^\dagger q^{-2h_r}).$$

However the pseudo vector representation $\pi_0$ is unitary. Hence when discussing Wigner coefficients, matrix elements, etc, it is important to work with unitary modules. Hence we assume that the representation afforded by $V^*$ is given by $\pi_0$ rather than $\pi_0^\dagger$.

**E.1. Fundamental matrices**

Since

$$R^\dagger = R^T, \quad (R^T)^T = R$$

it follows that

$$\tilde{A} = (q - q^{-1})^{-1}(\pi_0 \otimes \text{id})(I - \tilde{R}^T \tilde{R})$$

is self-adjoint, since $\pi_0$ is unitary. Thus

$$\tilde{A}^\dagger = \tilde{A}.$$ 

Writing

$$\tilde{A} = \sum_{ij} e_{ij} \otimes \tilde{A}_{ij}$$

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\[ \Rightarrow \sum_{ij} e_{ij} \otimes \tilde{A}_{ij} = \tilde{A} \]

\[ = \tilde{A}^\dagger = \sum_{j,i} e_{ji} \otimes (\tilde{A}_{ij})^\dagger \]

\[ \Rightarrow (\tilde{A}_{ij})^\dagger = \tilde{A}_{ji} \].

Similarly, since \( \pi_0 \) is unitary,

\[(A_{ij})^\dagger = A_{ji}.\]

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