SIMPLE ROOTS OF DEFORMED PREPROJECTIVE ALGEBRAS

LIEVEN LE BRUYN

For Idun Reiten on her 60th birthday.

Abstract. In [1] W. Crawley-Boevey gave a description of the set \( \Sigma_\lambda \) consisting of the dimension vectors of simple representations of the deformed preprojective algebra \( \Pi_\lambda \). In this note we present alternative descriptions of \( \Sigma_\lambda \).

1. Reduction to \( \Pi_0 \)

Recall that a quiver \( \vec{Q} \) is a finite directed graph on a set of vertices \( Q_v = \{v_1, \ldots, v_k\} \), having a finite set of arrows \( Q_a = \{a_1, \ldots, a_l\} \) where we allow both multiple arrows between vertices and loops in vertices. The Euler form of \( \vec{Q} \) is the bilinear form on \( \mathbb{Z}^k \) determined by the integral \( k \times k \) matrix having as its \((i, j)\)-entry \( \chi_{ij} = \delta_{ij} - \# \{\text{arrows from } v_i \text{ to } v_j\} \). The double quiver \( \bar{Q} \) of the quiver \( \vec{Q} \) is the quiver obtained by adjoining to every arrow \( a \in Q_a \) an arrow \( a^* \) in the opposite direction. The path algebra \( \mathbb{C}\bar{Q} \) has as \( \mathbb{C} \)-basis the set of all oriented paths \( p = a_{i_1} \cdots a_{i_u} \) of length \( u \geq 1 \) together with the vertex-idempotents \( e_i \) considered as paths of length zero. Multiplication in \( \mathbb{C}\bar{Q} \) is induced by concatenation (on the left) of paths. For rational numbers \( \lambda_i \), the deformed preprojective algebra is the quotient algebra

\[
\Pi_\lambda = \Pi_\lambda(\bar{Q}) = \frac{\mathbb{C}\bar{Q}}{(\sum_{a \in Q_a} [a, a^*] - \sum_{v_i \in Q_v} \lambda_i e_i)}
\]

The (difficult) problem of describing the set \( \Sigma_\lambda \) of all dimension vectors of simple representations of \( \Pi_\lambda \) was solved by W. Crawley-Boevey in [1]. He proved that for \( \alpha \) a positive root of \( \bar{Q} \), \( \alpha \in \Sigma_\lambda \) if and only if

\[
p(\alpha) > p(\beta_1) + \ldots + p(\beta_r)
\]

for every decomposition \( \alpha = \beta_1 + \ldots + \beta_r \) with \( r \geq 2 \) all \( \beta_i \) positive roots of \( \bar{Q} \) such that \( \lambda.\beta_i = 0 \) and where \( p(\beta) = 1 - \chi(\beta, \beta) \).

For a given dimension vector \( \alpha = (a_1, \ldots, a_k) \in \mathbb{N}^k \) one defines the affine scheme \( \text{rep}_\alpha \Pi_\lambda \) of \( \alpha \)-dimensional representations of \( \Pi_\lambda \). There is a natural action of the basechange group \( GL(\alpha) = \prod_{i=1}^k GL_{a_i} \) on this scheme and the corresponding quotient morphism

\[
\text{rep}_\alpha \Pi_\lambda \xrightarrow{\sim} \text{iss}_\alpha \Pi_\lambda
\]

sends a representation \( V \) to the isomorphism class of the direct sum of its Jordan-Hölder factors. Let \( \xi \) be a geometric point of \( \text{iss}_\alpha \Pi_\lambda \), then \( \xi \) determines the isomorphism class of a semisimple \( \alpha \)-dimensional representation say with decomposition

\[
M_\xi = S_{1 e_1}^{\xi e_1} \oplus \ldots \oplus S_{l e_l}^{\xi e_l}
\]
with the $S_i$ distinct simple representations of $\Pi_\lambda$ with dimension vector $\beta_i$, which occurs in $M_\xi$ with multiplicity $e_i$. We say that $\xi$ is of representation type $\tau = (\epsilon_1, \beta_1; \ldots; \epsilon_l, \beta_l)$. Construct a graph $G_B$ depending on the set of simple dimension vectors $B = \{\beta_1, \ldots, \beta_l\}$ having $l$ vertices $\{w_1, \ldots, x_l\}$ having $2\rho(\beta_i) = 2(1 - \chi(\beta_i, \beta))$ loops in vertex $w_i$ and $-\chi(\beta_i, \beta_j) - \chi(\beta_j, \beta_i)$ edges between $w_i$ and $w_j$.

Let $Q_B$ be the (double) quiver obtained from $G_B$ by replacing each solid edge by a pair of directed arrows with opposite ordering. In [3 §4] W. Crawley-Boevey proved that there is an étale isomorphism between a neighborhood of $\xi$ in $\text{iss}_\alpha \Pi_\lambda$ and a neighborhood of the trivial representation $\overline{0}$ in $\text{iss}_\alpha$, $\Pi_0(\overline{Q})$ where $\alpha_\tau = (\epsilon_1, \ldots, e_l)$ determined by the multiplicities of the simple factors of $M_\xi$.

The arguments in [3 §4] actually prove that there is a $GL(\alpha)$-equivariant étale isomorphism between a neighborhood of the orbit of $\xi$ in $\text{rep}_\alpha \Pi_\lambda(\overline{Q})$ and a neighborhood of the orbit of $(1, 0)$ in the principal fiber bundle

$$GL(\alpha) \times^{GL(\alpha_\tau)} \text{rep}_\alpha \Pi_0(\overline{Q})$$

Using the description of $\Sigma_\lambda$ it was proved in [1] that $\text{iss}_\alpha \Pi_\lambda$ is irreducible whenever $\alpha \in \Sigma$.

In this note we will give two alternative descriptions of the set $\Sigma_\lambda$ stressing the fundamental role of the extended Dynkin quivers in the study of deformed preprojective algebras. Both descriptions rely on the above irreducibility result so they do not give a short proof of Crawley-Boevey’s result unless an independent proof of irreducibility of $\text{iss}_\alpha \Pi_\lambda$ for all $\alpha \in \Sigma_\lambda$ is found. In the statement of the results we have therefore separated the parts that depend on the irreducibility statement.

**Proposition 1.1.** Let $\xi$ be a geometric point of $\text{iss}_\alpha \Pi_\lambda$ of representation type $\tau = (\epsilon_1, \beta_1; \ldots; \epsilon_l, \beta_l)$. The following are equivalent

1. Any neighborhood of $\xi$ in $\text{iss}_\alpha \Pi_\lambda$ contains a point of representation type $(1, \alpha)$ (whence, in particular, $\alpha \in \Sigma_\lambda$).
2. $\alpha_\tau = (\epsilon_1, \ldots, e_l)$ is the dimension vector of a simple representation of $\Pi_0(\overline{Q})$.
3. Any neighborhood of $\overline{0}$ in $\text{iss}_\alpha$, $\Pi_0(\overline{Q})$ contains a point of representation type $(1, \alpha_\tau)$ (whence, in particular, $\alpha_\tau$ is the dimension vector of a simple representation of $\Pi_0(\overline{Q})$).

If moreover $\text{iss}_\alpha \Pi_\lambda$ is irreducible these statements are equivalent to

- $\bullet \ \alpha \in \Sigma_\lambda$.

**Proof.** By comparing the stabilizer subgroups of the closed orbits determined by corresponding points under the étale isomorphism it follows that (1) $\Leftrightarrow$ (3) and clearly (3) $\Rightarrow$ (2). Because the equations of $\Pi_0(\overline{Q})$ are homogeneous there is a $\mathbb{C}^*$-action on $\text{rep}_\alpha$, $\Pi_0(\overline{Q})$ (multiplying all matrices by $t \in \mathbb{C}^*$). The limit point $t \to 0$ of any representation is the trivial representation. Starting from a simple representation $V$, any neighborhood of $\overline{0}$ contains a point determined by $tV$ for suitable $t$ proving (2) $\Rightarrow$ (3). To prove that $\bullet \Rightarrow$ (1) observe that the set of all points of representation type $(1, \alpha)$ form an open subset of $\text{iss}_\alpha \Pi_\lambda$ (follows from the étale local description), whence if $\text{iss}_\alpha \Pi_\lambda$ is irreducible this set is dense. $\square$

This result allows us to describe $\Sigma_\lambda$ inductively if we can determine the sets of simple dimension vectors for preprojective algebras. The induction starts off by
taking the positive roots $\alpha$ for $\bar{Q}$ minimal w.r.t. $\lambda.\alpha = 0$. It follows from the easier part of $\Pi$ that these $\alpha \in \Sigma_\lambda$.

2. Genetic description of $\Sigma_0$

In this section we start with the quiver $\bar{Q}$ and will give an inductive procedure to determine $\Sigma_0$, the set of simple dimension vectors of $\Pi_0 = \Pi(\bar{Q})$.

Assume we have constructed a set $B = \{\beta_1, \ldots, \beta_l\}$ with $\beta_i \in \Sigma_0$ (we can take $\beta = \beta_i = \beta_j$ for $i \neq j$ provided $p(\beta) > 0$). We want to determine the minimal linear combinations

$$\alpha = e_1\beta_1 + \ldots + e_l\beta_l$$

such that $\alpha \in \Sigma_0$. We will do this in terms of the graph $G_B$ constructed in the previous section and the dimension vector $\alpha_\tau = (e_1, \ldots, e_l)$.

The tame settings are the couples $(D, \delta)$ where $D$ is an extended Dynkin diagram and $\delta$ the corresponding imaginary root. The list of tame settings is given in figure 1.

We say that a tame setting $(D, \delta)$ is contained in $(\bar{Q}, \alpha)$ if $D$ is a subgraph of $G_B$ and if $\delta \leq \alpha_\tau$.

Recall from $[4]$ that all polynomial invariants of quivers are generated by taking traces along oriented cycles in the quiver. As a consequence, the coordinate algebra $\mathbb{C}[\text{iss}_\alpha, \Pi_0] = \mathbb{C}[\text{rep}_\alpha, \Pi_0]^{GL(\alpha)}$ is generated by traces in the quiver $\bar{Q}$. Note that non-trivial invariants exist whenever $\alpha \in \Sigma_0$ and $\alpha$ is not a real root of $\bar{Q}$. The crucial ingredient in our descriptions is the following technical result.

**Proposition 2.1.** For $\alpha \in \Sigma_0$, if $\alpha$ is not a real root of $\bar{Q}$ and $\bar{Q}$ has only loops at vertices where $\alpha$ is one, then there is a non-loop tame setting $(\tilde{D}, \delta)$ contained in $(\bar{Q}, \alpha)$.

**Proof.** Assume $\bar{Q}$ is a counterexample with a minimal number of vertices. There are at most two directed arrows between two vertices ($(\tilde{A}_1, (1, 1))$ is not contained)
so we can define the graph $G$ replacing a pair of directed arrows by a solid edge. Then, $G$ is a tree \((\bar{A}_m,(1,\ldots,1))\) is not contained).

We claim that the component of $\alpha$ for every internal (not a leaf) vertex is at least two. Assume $v$ in internal and has dimension one, then any non-zero trace $tr(c)$ along a circuit in $\Gamma$ passing through $v$ (which must be the case by minimality of the counterexample) can be decomposed as

$$0 \neq tr(c) = tr(t_1)tr(t_2)\ldots tr(t_m)$$

where $t_i$ is part of the circuit along a subtree rooted at $v$. But then $tr(t_i) \neq 0$ when evaluated at representations of the preprojective algebra of the corresponding subtree, contradicting minimality of the counterexample.

Hence, $G$ is a binary tree \(((\bar{D}_4,(2,1,1,1))\) is not contained) and even a star with at most three arms \(((\bar{D}_m,(2,\ldots,2,1,1,1,1))\) is not contained). If $G$ does not contain $\bar{E}_i$ for $6 \leq i \leq 8$ as subgraph, then $Q$ is a Dynkin quiver and one knows that in this case there are no nontrivial invariants, a contradiction.

If $\delta_i$ is the vertex-simple concentrated in vertex $v$, we claim that

$$\chi(\alpha,\delta_i) + \chi(\delta_i,\alpha) \leq 0$$

for every vertex $v$. Indeed, it follows from [2] that for any non-isomorphic simple $\Pi_0$-representations $V$ and $W$ of dimension vectors $\beta$ and $\gamma$ we have

$$dim \ Ext^1_{\Pi_0}(V,W) = -\chi(\beta,\gamma) - \chi(\gamma,\beta)$$

Therefore, twice the dimension of $\alpha$ at $v$ is smaller or equal to the sum of the dimensions of $\alpha$ in the two (maximum three) neighboring vertices. Fill up the arm of $G$ corresponding to the longest arm of $\bar{E}_i$ with dimensions starting with 1 at the leaf and proceeding by the rule that twice the dimension is equal to the sum of the neighboring dimensions, then we obtain a dimension vector $\beta$ such that

$$\delta_i \leq \beta \leq \alpha$$

where $\delta_i$ is the imaginary root of $\bar{E}_i$, a contradiction. $\square$

**Theorem 2.2.** With notations as above, we have

1. $\alpha = e_1\beta_1 + \ldots + e_l\beta_l \in \Sigma_0$ whenever $\delta = (e_1,\ldots,e_l)$ is the imaginary root of an extended Dynkin subgraph $D$ of $G_B$.

2. If moreover $\text{iss}_\alpha \Pi_0$ is irreducible for all $\alpha \in \Sigma_0$, the set $\Sigma_0$ is obtained by iterating the procedure in (1) starting from the set of all real roots of $\bar{Q}$.

**Proof.** (1) : There is a point $\xi \in \text{iss}_\alpha \Pi_0$ determined by a semi-simple representation $M_\xi$ of representation type $\tau = (e_1,\beta_1;\ldots;e_l,\beta_l)$. A neighborhood of $\xi$ is étale isomorphic to a neighborhood of $\bar{0}$ in $\text{iss}_\delta \Pi_0(\bar{Q}_B)$. It is well known that $\text{iss}_\delta \Pi_0(D)$ contains points of representation type $(1,\delta)$ whence $\delta$ is a dimension vector of a simple representation of $\Pi_0(\bar{Q}_B)$ (take a simple of $\Pi_0(D)$ and add zero matrices for the remaining arrows). By proposition [1] it follows that $\alpha \in \Sigma_0$.

(2) : Let $\alpha \in \Sigma_0$ and take a decomposition (representation type)

$$\alpha = d_1\beta_1 + \ldots + d_l\beta_l$$

with all $\beta_i \in \Sigma_0$, $\beta_i < \alpha$ and $d = \sum d_i$ minimal. Note that we can take all $d_i = 1$ whenever $p(\beta_i) > 0$ (as then there are infinitely many non-isomorphic simples of dimension vector $\beta_i$). As a consequence $G_B$ only has loops at vertices where $\alpha_\tau$ is equal to one and $\alpha_\tau$ is a simple root for $\Pi_0(G_B)$ (here we used irreducibility of
iss_α \Pi_0 \text{ in order to apply proposition 1.1. By proposition 2.1 there is a non-loop tame subsetting } (D, \delta) \text{ contained in } (G_B, \alpha_\tau) \text{ and if } \delta = (e_1, \ldots, e_l) \text{ then we have a decomposition }

\alpha = (d_1 - e_1)\beta_1 + \ldots + (d_l - e_l)\beta_l + 1. (\delta, \beta)

which has strictly smaller total number of multiplicities unless } \alpha = \delta, \beta. \text{ Induction on the total dimension finishes the proof.}

3. Another description of } \Sigma_\lambda

In this section we reformulate the previous arguments in a more manageable statement.

Take a non-trivial representation type } \tau = (d_1, \beta_1; \ldots; d_l, \beta_l) \text{ of } \alpha \text{ with all } \beta_i \in \Sigma_\lambda. \text{ Let } \tau' \text{ be the representation type obtained from } \tau \text{ by replacing each } (d_i, \beta_i) \text{ by } (1, \beta_i; \ldots; 1, \beta_i) \text{ whenever } p(\beta_i) > 1 \text{ (see the proof of theorem 2.2) and let } B' \text{ be the corresponding set of simple root (some occurring more than once).}

**Theorem 3.1.** The following are equivalent

1. } \alpha \in \Sigma_\lambda \text{ and } \text{iss}_\alpha \Pi_\lambda \text{ is irreducible.}

2. For all non-trivial representation types } \tau \text{ of } \alpha \text{ there is a non-loop tame setting contained in } (G_B', \alpha_{\tau'}).

**Proof.** (2) ⇒ (1) : We claim that } (1, \alpha) \text{ is the unique maximal representation type in the ordering of inclusion in Zariski-closures. Assume not and let } \tau \text{ be another maximal type, then } \tau = \tau' \text{ and by proposition 2.1 there is a tame setting contained in } (G_B, \alpha_{\tau'}) \text{ but then there are non-loop polynomial invariants, whence } \tau \text{ is not maximal.}

(1) ⇒ (2) : Follows from proposition 1.1 and proposition 2.1

Hence, the dimension vectors obtained from the genetic construction of theorem 2.2 are exactly those } \alpha \in \Sigma_0 \text{ such that } \text{iss}_\alpha \Pi_0 \text{ is irreducible.

**Acknowledgement :** I thank W. Crawley-Boevey for drawing my attention to the circular argument used in the first version.

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Universiteit Antwerpen (UIA), B-2610 Antwerp (Belgium)

E-mail address: lebruyn@uia.ua.ac.be

URL: http://win-www.uia.ac.be/u/lebruyn/