Quasi-Minimal, Pseudo-Minimal Systems and Dense Orbits

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Abstract

Minimal sets are an important aspect in the theory of dynamical systems. A new phenomenon are the quasi-minimal sets which appear for example for the Cherry-flow and only few is known about those sets. We say that a dynamical system (group action) \((X, T)\) is pseudo-minimal if there exists an open set \(U\) such that for all \(x \in U\) the orbit \(O(x) = \{xt \mid t \in T\}\) is dense. One could also say that the subset of the points whose orbit is dense has a non empty interior or the union of the dense orbits is an open set. Pseudo-minimal systems appear on surfaces of high genus and can be constructed from parabolic billiard systems. This article is about this new aspect in the theory of dynamical systems. We are interested in giving conditions for a pseudo-minimal system to be minimal, but we also show that some systems (often near chaotic behavior) are not pseudo-minimal. These results can be applied to pseudo-minimal sets. These conditions can be of algebraical or topological type. We also develop techniques to construct such systems and counterexamples.

1 Basic Facts

In the complete paper all systems are flows or homeomorphisms, so the time space is \(T = \mathbb{R}, \mathbb{Z}\).

**Definition 1.1.** A system \((X, T)\) on a topological space \(X\) is said to be pseudo-minimal (we just simple write p.m.), if there exists a non empty open set \(U \subset X\) such that each orbit \(O(x)\) is dense in \(X\) for each \(x \in U\).

If the system is a flow or an homeomorphism, then a system on \(X\) is said to be positively pseudo-minimal (p.p.m.), if there exists a non empty open set \(U\) such that each semi-orbit \(O_+(x)\) is dense in \(X\) for each \(x \in U\). In the same way negatively pseudo-minimal (n.p.m.) is defined.

If the system is p.m. but not minimal, then it is called strictly p.m..

An invariant closed subset \(P \subset X\) of a dynamical system \((X, T)\) is called a pseudo-minimal set if the system restricted to \(P\) is p.m..

Note that we can therefore define p.p.m. for a continuous map, but in most cases this map is already minimal. There is always a maximal open dense set containing only the dense orbits if the system is p.m., which we denote with \(X_M\). Therefore \((X_M, T)\) is a minimal system on an open (maybe not compact)
Let $X_{M,+}$ denote the subset containing all $x \in X_M$ such that $O_+(x)$ is dense in $X$. Similarly we define $X_{M,-}$. Therefore a p.m. system is p.p.m. iff the interior of $X_{M,+}$ is non empty. A stronger form of pseudo-minimal will be discussed in section 4.

While we are working with p.m. systems, there is often the following assumption which we refer to as (A): If $f : X \times \mathbb{R} \to X$ is a flow, then $X$ is a $C^\infty$ mfd. of dimension $> 1$ and the flow is $C^1$. If $g$ is a homeomorphism then $X$ is a locally compact metric space with a countable basis and perfect, i.e. without isolated points.

Note that under assumption (A), $O_+(x)$ is dense iff $\omega(x)$ is dense. The next lemma is trivial but useful.

**Lemma 1.2.** Let $(X,T)$ be a p.m. system. If $x \in X_M$, then $\omega(x)$ is $X$ or $\omega(x) \cap X_M = \emptyset$.

Proof: Assume that $y \in \omega(x) \cap X_M$, then $O(y) \subset \omega(x)$, so $\omega(x)$ is dense.\□

Indeed p.m. is not the same as minimal by regarding a flow on $S^1$. It is also well-known that there are discrete locally compact group actions on $S^1$ such that every orbit is dense except one point. It is known that for a 1-dim. mfd., minimality and topological transitivity is the same for discrete flows. We will construct real and discrete p.m. system on mfd.’s which admit no real nor discrete minimal flows. We will discuss this later on.

Here is another useful lemma:

**Lemma 1.3.** Let $(X,T)$ be a topologically transitive system that satisfies (A) then there is a $y$ in $X$ such that $O_+(y)$ lies dense in $X$ and a $z$ in $X$ such that $O_-(z)$ lies dense in $X$. Moreover if $O(x)$ is dense then either $O_+(x)$ or $O_-(x)$ is dense.

We consider only the case where the flow is real, since the other case is similar. We only need to show the existence of $y$. Regard the following sets for $r > 0$:

$$N_r := \{x | \exists n_k \to +\infty \text{ with } f(B,n_k) \cap B \neq \emptyset \text{ and } B := B_r(x)\}$$

$$M_r := \{x | \exists m_k \to -\infty \text{ with } f(B,m_k) \cap B \neq \emptyset \text{ and } B := B_r(x)\}.$$

Since $f$ is a flow, it follows that $N_r = M_r$. At first we show that for all open sets $U$ and all $N > 0$, a $t > N$ exists with $f(U,t) \cap U \neq \emptyset$.

Assume that $O(x)$ is a dense orbit. Given $x_0 := f(x,t_0) \in U$ and $B_0(x_0) \subset U$, \{f(x,s)\}_{|s| > M}$ is dense in $X$ for all $M > 0$ since $f$ satisfies (A), so there exists an $|t_1| > |t_0| + N$ with $x_1 := f(x_0,t_1 - t_0) = f(x,t_1) \in U$ and $d(x_1,x_0) < r_0$.

By repeating this argument the result follows, since $x_0$ is in $N_{r_0}$ or $M_{r_0}$, but these sets are the same.

Now we show that for all open sets $U,V \subset X$ and all $M > 0$, a $t > M$ exists with $f(U,t) \cap V \neq \emptyset$, so that the lemma is follows with the proof of [1.4.2] in the book [Ka]. For an open set $Q$, define $Q_t := f(Q \cap V,-t)$, therefore $f(Q,t) = Q \cap V$.

Choose an open set $Q \subset V$ with $Q_{t_0} \subset U$, and by the preceding result a $t_1$
exists with \( t_1 > -t_0 + M \), such that \( W := f(Q_{t_0}, t_1) \cap Q_{t_0} \neq \emptyset \). One has \( t_1 + t_0 > M \) and

\[
f(U, t_1 + t_0) \cap V \supset f(Q_{t_0}, t_1 + t_0) \cap V \supset f(W, t_0) \cap V \neq \emptyset
\]

since \( f(W, t_0) \subset Q \subset V \) and \( f(Q_{t_0}, t_1) \supset W \).

If \( O(x) \) is dense, then either \( O_+(x) \) or \( O_-(x) \) contains an open set, hence one of the sets is dense.

\[ \square \]

It is not difficult to show that all the subsets of the points whose positive semi-orbit is dense, form a generic set. Hence the subset of the points whose each semi-orbit is dense form a generic set. The following lemma is based on the theorem [10.29] in the book [G].

**Lemma 1.4.** Let \( X \) be a topological space and \( f : X \to X \) continuous. Let \( A \subset U \) be given, where \( U \) is open and the complement of a compact set \( K \) and \( A \subset f^{-1}(A) \). At least one of the following statements are valid:

1. There is a closed set \( E \) with \( A \subset E \notin U \) and \( f(E) \subset E \).
2. There is an open set \( V \) with \( A \subset V \subset U \) and \( f^{-1}(V) \subset V \).

If 1. holds, \( E \) is of the form \( E = \bigcap_{n=1}^{\infty} f^{-n}(U) \).

Proof: Let \( E = \bigcap_{n=1}^{\infty} f^{-n}(U) \). \( E \) is closed and \( f(E) \subset E \). Since \( A \subset f^{-n}(A) \subset f^{-n}(U) \), we conclude \( A \subset E \). If \( E \notin U \) then 1. is true. If \( E \subset U \), choose an \( m \) with \( \bigcap_{n=1}^{m} f^{-n}(U) \subset U \).

This \( m \) exists: Consider the set \( E_i := K \cap \bigcap_{n=1}^{i} f^{-n}(U) \). Since \( E_i \) is closed (also as a subset of \( K \) ), \( E_{i+1} \subset E_i \),

\[
\bigcap_{n=1}^{\infty} E_i = E \cap K = \emptyset,
\]

and \( K \) is compact, there exists an \( m \) with \( E_m = \emptyset \), therefore \( \bigcap_{n=1}^{m} f^{-n}(U) \subset U \).

Let us set \( V := \bigcap_{n=1}^{m} f^{-n}(U) \), clearly we have

\[
A \subset V \subset U
\]

and since \( A \subset f^{-n}(A) \subset f^{-n}(U) \), \( V = U \cap \bigcap_{n=1}^{m} f^{-n}(U) \), we get \( f^{-1}(V) \subset V \).

\[ \square \]

If we compactify a space, then we only use the following:

If \( X \) is a \( T_1 \)-space, then we can construct a compact space \( X_\infty \) which is containing \( X \). The procedure is the same as the Alexandrov one-point compactification, but we take all sets which are closed and compact to construct this space.

Every homeomorphism \( f : X \to X \) can be extended to \( f : X_\infty \to X_\infty \) by setting \( f(\infty) = \infty \).

If \( X \) is not compact, then \( \{\infty\} \) is closed, but if \( X \) is compact, then \( \{\infty\} \) is open. Every compact set in \( X \) is compact in \( X_\infty \).

Next we prove two important theorems for this article with two different technics. These technics will be repeated in other proofs. However these theorems have been proved many times, maybe the first time in [Hom] (1953).
Theorem 1.5. Let \((X, f)\) be a p.p.m. which satisfies (A), then \(X\) is compact and \((X, f)\) is minimal. In particular for every p.p.m. continuous map \(g : X \to X\) every orbit is positively dense and \(X\) is compact.

Proof: Assume \(X\) is not compact, let \(Y\) be the compactification and \(f : Y \to Y\) the extension. We set \(A = \{\infty\}\) and \(U = K^c \cup \{\infty\}\) as in lemma [1.4] above, where \(K\) is compact in \(X_M\) and contains an open set \(Q\). If 1. holds as in lemma [1.4], we get \(E \not\subset U\). \(f\) is p.p.m., \(K \subset X_M\) and \(f(E) \subset E\), so we get \(X \subset \bar{E} \subset E\). Therefore \(Y = E\) and \(U = Y\), contradicting the fact that \(K\) contains an open set.

If 2. holds as in lemma [1.4], then \(f^{-1}(V) \subset V\) for an open set \(V\). If \(\bar{V} \neq X\) then for \(y \in V \cap X_{M,+}\) there exists a \(k\) such that \(f^k(y) \notin V\), but \(f^{k+1}(y) \in V\). This a contradiction, so \(\bar{V} = Y\), which as well is a contradiction. Therefore \(X\) is compact. To show that every orbit is dense, we define

\[
W := \{y \mid \exists k \in \mathbb{Z} : y \in f^k(x)\},
\]

if \(O_+(x)\) is not dense. The same arguments for \((X, f)\) holds for \((X - W, f)\). So \(X - W\) is compact which is a contradiction, since \(X - W\) is not compact, so every orbit is dense. \(\square\)

Theorem 1.6. If \((X, T)\) is a p.p.m. flow that satisfies (A), then \(X\) is compact and the flow is minimal.

Proof: For \(x \in X_M\), choose a ball \(B_1 := B(x, \delta) \subset X_M\) and a smaller ball \(B_2\) whose closure is compact in \(B_1\). Since every point in \(X_M\) is positively recurrent and its positive semi-orbit is dense, we conclude that for all \(y \in \overline{B_2}\), there exists a small ball \(N_y\) such that \(N_y t_y \subset B_1\) for some \(t_y > 0\). Choose a finite subcover \(\{N_y\}\) and let \(t_0\) be the max of \(\{t_y\}\). We conclude that \(O_+(x) \subset \overline{B_2[0, t_0]} \subset X_M\). This fact implies that \(X\) is compact and every orbit is dense. \(\square\)

If one replaces p.p.m. by p.m., then the lemma is false for flows and as we will see even for diffeomorphisms. The simplest counterexample for a flow is mentioned above. In the up coming sections we construct counterexamples by taking the product of p.m. systems. From the next theorem it follows that if a product of systems is p.m., then at least one factor is minimal. We only state it for discrete systems, but the proof is analogous for the case \(T = \mathbb{R}\).

Theorem 1.7. Given two dynamical systems \((X, f)\) and \((Y, g)\) such that \(X, Y\) satisfy (A). If \(f \times g : X \times Y \to X \times Y\) is p.m., then both systems are p.m. and at least one system is a minimal system on a compact space. Moreover if \(f \times f : X \times X \to X \times X\) is p.m., then \(X\) consists of one point.

Proof: Set \(X \times Y = Z\) and \(h = f \times g\) and suppose that \(h\) is p.m.. It is easy to see that \(f\) and \(g\) are p.m.. Given \((x_0, x_1) \in Z_{M,+}\) then choose an \(\epsilon\) such that \(U := B(x_0, \epsilon) \times B(x_1, \epsilon) \subset Z_M\). W.l.o.g. we can suppose that \(x_0 \in X_{M,+} - X_{M,-}\), otherwise apply theorem [1.5]. Hence \(B(x_1, \epsilon)\) lies in \(Y_{M,+}\), thus \(Y\) is compact and \(g\) minimal.

If \(f\) is p.m., then set \(X \times X = Z\) and \(F = f \times f\). Note that \(X\) is compact and \(f\) is minimal. Given \((x_0, x_1) \in Z_{M,+}\), w.l.o.g. we can choose an
must contain \( (x_0, f^m(x_0)) \in Z_{M, +} \). Choose a sequence \( m_k \to \infty \) such that \( F^{mk}(x_0, f^m(x_0)) \to (y, y) \), hence \( f(y) = y \), so \( f = Id \) and \( X \) is a point. \( \square \)

Finally we answer the question if the iteration of a p.m. homeomorphism is p.m.. An homeomorphism is called totally minimal if every iteration is minimal.

**Proposition 1.8.** If \( (X, f) \) is p.m., \( X \) satisfies (A) and \( X_M \) is connected, then \( (X_M, f) \) is totally minimal.

We first make a definition and prove a lemma. Given a prime number \( p > 1 \), we denote the elements of \( \mathbb{Z}_p \) with \([q]\).

For \( i \in \{0, 1, \ldots, p - 1\} \) and \( x \in X_M \); we set

\[
A^{\pm}_{[i]}(x) := \{ y \in X_M | \exists \{n_k\}_k \in \mathbb{Z} \text{ with } pn_k + i \to \pm \infty \text{ and } f^{pn_k + i} \to y \}.
\]

It is easy to see that \( A^{+}_{[i]}(x) \) and \( A^{-}_{[i]}(x) \) are closed subsets of \( X_M \).

**Lemma 1.9.**

1. If \( x \in X_{m, \pm} \) then \( \bigcup_i A^{\pm}_{[i]}(x) = X_M \), \( f(A^{\pm}_{[i]}(x)) = A^{\pm}_{[i+1]}(x) \) and the interior of all \( A^{\pm}_{[i]} \) are non empty.

2. Given \( x \in X_M \) and \( y \in A^{+}_{[i_0]}(x) \), then for all \( i \) it holds that:

\[
A^{+}_{[i]}(y) \subset A^{+}_{[i_0 + i]}(x).
\]

Proof: 1) The first two results are trivial. From Baire’s theorem we conclude that at least one \( A^{\pm}_{[i_0]}(x) \) has a non empty interior. From the second fact we conclude this holds for all, since \( f \) is a homeomorphism.

2) Choose, for \( z \in A^{\pm}_{[i]}(y) \), a sequence \( b_k = pm_k + i \) as in the definition, such that \( f^{b_k}(y) \to z \), and a sequence \( a_k = pm_k + i_0 \) as in the definition such that \( f^{a_k}(x) \to y \). By choosing \( a_k \) growing fast, it follows that

\[
0 < c_k = a_k + b_k = p(n_k + m_k) + i + i_0 \to \infty \text{ and } f^{c_k}(x) \to z. \quad \square
\]

Proof of the proposition: We prove that \( (X_M, f^p) \) is minimal for every prime number, then it follows that \( (X_M, f^n) \) is minimal, where \( n \) is a product of prime numbers.

Let \( x \in X_{M, +} \) be given. \( X_M \) is connected, by lemma [1.9.1] there exists an \( y \in A^{+}_{[i_0]}(x) \cap A^{+}_{[i_1]}(x) \).

By lemma [1.9.2] we get for all \( k \) and \( i \in \{i_0, i_1\} \) that

\[
A^{\pm}_{[k]}(y) \subset A^{\pm}_{[k + i]}(x).
\]

From this fact and lemma [1.9.1] we conclude that

\[
A^{+}_{[0]}(x) \cap A^{+}_{[1]}(x)
\]

must contain \( A^{+}_{[i_0]}(y) \), hence an open set, since \( y \in X_M \). W.l.o.g. suppose that \( y \in X_{M, +} \) and take an \( k_0 \) such that \( x \in A^{+}_{[k_0]}(y) \). Again we conclude that for all \( n \) and \( i \in \{i_0, i_1\} \) we have:

\[
A^{+}_{[i]}(x) \subset A^{+}_{[k_0 + n]}(y) \subset A^{+}_{[k_0 + n + i]}(x).
\]
We suppose that $[c_0] := [k_0 + i_0] \neq 0$ (otherwise take $i_1$) and conclude, since $p$ is a prime number, that for all $n$

$$A_{[n]}^+(x) = A_{[n+1]}^+(x) = A_{[0]}^+(x),$$

which means the orbit of $x$ under $f^p$ is dense. \hfill □

Let us generalize the theorem.

**Theorem 1.10.** If $(X, f)$ is p.m., $X$ satisfies (A) and $X$ is locally connected, then there is a finite number of connected components $\{C_i\}_{i \in I}$ and there exists an $k$ such that $f^k(C_i) = C_i$ for all $i$. In particular, $k$ is the number of connected components $\{C_i\}_{i \in I}$ and for every prime number $p > k$, $(X_M, f^p)$ is minimal.

Proof: Let $\{C_i\}_{i \in I}$ be the connected components. Given $x \in X_{M,+} \cap C_0$, let $j(i)$ denote the unique element of $I$ such that $f^i(x) \in C_{j(i)}$. We conclude $f^i(C_0) = C_{j(i)}$, since $f$ is an homeomorphism. The orbit of $x$ is dense, so there exist two $i_0 < i_1$ such that $f^{i_0}(C_0) = f^{i_1}(C_0)$, and $f^{i_1-i_0}(C_0) = C_0$. So there can be only finitely many components and $k$ must be the number of connected components $\{C_i\}_{i \in I}$.

The proof of proposition [1.8] works well in this case if $p$ is bigger than the number of connected components, since then there must exist an

$$y \in A_{[i_0]}^+(x) \cap A_{[i_1]}^+(x),$$

for every $x \in X_{m,+}$. \hfill □

## 2 Topological Dynamics

In this section, we study the set of dense orbits and how topological properties interfere with p.m. systems. Two of them are the classes of equicontinuous systems and expansive systems. The class of expansive appears in chaotic systems. At least p.m. systems seems to behave nearly like minimal systems on manifolds. Structure makes p.m. systems minimal, but p.m. systems must not be arbitrary chaotic, as we will see. First we want to characterize the points where only one semi-orbit is dense.

**Definition 2.1.** Let $(X, T)$ be a p.m. system, which satisfies (A). Let

$$\mathcal{A} := \{x \in X_M | \alpha(x) \text{ is not dense in } X\}$$

and

$$\mathcal{W} := \{x \in X_M | \omega(x) \text{ is not dense in } X\}.$$  

**Lemma 2.2.** Let $(X, T)$ be a p.m. system that satisfies (A) such that for all $x \in \mathcal{W}$ we have $\omega(x) \neq \emptyset$, and denote with $T_-$ all negative numbers of $T$. Let $N$ be an isolated closed subset of $X^c_M$ such that $NT_- \subset N$ and suppose there exists an open set $U$ with $\overline{U}$ compact, containing $N$, then

$$N \cap \bigcup_{x \in \mathcal{W}} \omega(x) \neq \emptyset.$$


Proof: Suppose this is not the case and define
\[ M := \bigcup_{x \in W} \omega(x). \]
Note that \( M \cap \Omega = \emptyset \), since
\[ \bigcup_{x \in W} \omega(x) \subset X^c_M. \]

We first prove the discrete case \((X, f)\). Take very small neighbourhoods \( U \) of \( \Omega \) and \( V \) of \( M \) such that \( f^{-1}(U) \cap V = \emptyset \) and \( U \cap V = \emptyset \). Now, as in lemma [1.4], consider \( E = \bigcap_{n=1}^{\infty} f^{-n}(\overline{U}) \). Note that \( E \cap V = \emptyset \). If \( E \subset U \), then we get a contradiction as in the proofs above, so take \( x \in E \cap U^c \), but note that for \( x \in U^c \) we have \( \omega(x) \cap M \neq \emptyset \), which contradicts \( f(E) \subset E \) and \( E \cap V = \emptyset \). This proves the discrete case.

The non discrete case is similarly. Let \( T \) denote the group action. First take again \( U \) and \( V \) so small such that \( \overline{U}([-2, 2]) \cap V = \emptyset \) and set \( f^n(x) = x(n) \). Again set \( E = \bigcap_{n=1}^{\infty} f^{-n}(\overline{U}) \). If \( E \subset U \), then we get an open set \( W \) with \( f^{-1}(W) \subset W \) and \( N \subset W \subset U \). We conclude that for a point \( x \in W \), \( x(n) \) lies in \( W \) after the time \( n = -1 \), so \( O_+(x) \subset \overline{U}([-0, 1]) \), and \( x \) can not be dense. If \( x \in E \cap U^c \), then the orbit \( x(t) \) lies always in \( E \) for discrete time \( t \in \mathbb{N} \), hence by using the fact \( E \subset \overline{U}([-1,0]) \), we conclude
\[ O_+(x) \cap V \subset E([0,1]) \cap V \subset \overline{U}([-2, 2]) \cap V = \emptyset, \]
but for every \( x \in E \cap U^c \) we have \( \omega(x) \cap M \neq \emptyset \). □

The next lemma of this section shows that the sets \( X_{m,+} \) and \( X_{m,-} \) must besituated in \( X_M \) around closed invariant sets in a complicated way.

**Lemma 2.3.** Let \( X \) be a manifold and \((X, f)\) a discrete strictly p.m. homeomorphism. If there is a closed set \( U \) such that \( \partial U \subset X_{M,+} \) and \( U \) is the closure of an open connected set then \( U \) does not contain a closed connected invariant set.

Proof: We transfer a similar argument from [K2] which will remind the reader of the previous proofs. It make use of the so-called Birkhoff construction. First note that if \( U = X \), then the boundary is an invariant set, hence \( f \) is not p.m., so we can assume that \( U \neq X \) holds. Suppose \( U \) contains a closed invariant set \( A \) and \( \partial U \subset X_{M,-} \), otherwise \( f^{-1} \). We set \( D \) the be the interior of \( U \). We define domains by induction:

1. \( D_0 = D \),
2. \( D_{n+1} \) is the connected component of \( A \) in \( f(D_n) \cap D_0 \).
We set $K = \bigcap_{n \in \mathbb{N}} D_n$. $K$ is closed, $K \subset f(K)$, contains $A$ and is contained in $U$. Let $C$ denote the boundary of $U$. Assume $K \cap C = \emptyset$, then choose an $n_0$ such that $D_{n_0+1} \cap C = \emptyset$, so

$$\partial D_{n_0+1} \subset \partial f(D_{n_0})$$

and hence

$$D_{n_0+1} = f(D_{n_0}).$$

We will conclude

$$\partial D_{n_0+2} \subset \partial f(D_{n_0+1}) = \partial f^2(D_{n_0}),$$

and hence $D_{n_0+2} = f^2(D_{n_0})$. Let us prove this in detail.

Keep in mind that the sequence $D_n$ is a decreasing sequence of sets, i.e.:

$$D_{n+1} \subset D_n.$$

We prove this by induction. This is trivial for the case $n = 0$ trivial. Given now $x \in D_{n+1}$, take a path $\gamma$ from $A$ to $x$ which is contained in $D_{n+1}$. The path $\beta := f^{-1}(\gamma)$ must lie in

$$D_n \cap f^{-1}(D_0) \subset D_{n-1} \cap f^{-1}(D_0).$$

Hence $\gamma$ must be in $f(D_{n-1}) \cap D_0$ and is connecting $x$ and $A$, therefore $x$ lies in $D_n$ by the definition of $D_n$.

We have by construction

$$\partial D_{n+2} \subset \partial f(D_{n+1}) \cup \partial D_0$$

but $D_{n+2} \subset D_{n+1}$ and $\overline{D_{n+1} \cap \partial D_0} = \emptyset$ and hence $\partial D_{n+2} \cap \partial D_0 = \emptyset$. This proves $D_{n+2} = f^2(D_{n_0})$.

Iterating this argumentation we conclude $f^i(D_{n_0}) = D_{n_0+i} \subset D_{n_0}$ and therefore $X_{M,+}$ is empty, which is a contradiction. Thus $K \cap C \neq \emptyset$ and therefore take $y \in X_{M,-} \cap K$. We conclude that $U = K = X$, since $O_-(y) \subset f^{-1}(K) \subset K$, contradicting $U \neq X$. \qed

Lemma 2.4. A p.m. homeomorphism on a connected 1-dim. mfd. is minimal. In particular every homeomorphism on a connected 1-dim. mfd. $X$ which is topologically transitive, is minimal and $X \cong S^1$.

We omit the proof, since this is well-known. In dimension 2 the following is true (and as we will see, only in dimension 1 and 2):

Proposition 2.5. Every non-singular p.m. $C^0$ flow $(X, T)$ on a 2-dimensional compact manifold is minimal.

Proof: From theorem [7.1.4] of [Ar] we conclude that our flow is topologically conjugate to a $C^1$ flow. We know that every minimal set $A$ must neither contain a periodic orbit (closed trajectory) nor a fixpoint, since no topologically transitive $C^1$ flow on a 2-dimensional closed manifold admits a periodic orbit (see chapter 2.1.6 in [Arv]). Since $A \subset X^c_M$, we conclude that $A$ is a nontrivial
minimal set in the sense of definition [6.4.1] of [Ar], since it is nowhere dense and does neither contain a periodic orbit (closed trajectory) nor a fixpoint. Again from theorem [7.1.4] of [Ar] we conclude that \((X, T)\) is topologically conjugate to a \(C^\infty\) flow. From theorem of Schwartz [14.3.1] in [Ka] we conclude that \(A\) contains a periodic orbit, if the flow is not minimal. Thus the flow is minimal. □

**Definition 2.6.** A system \((X, T)\) is called equicontinuous (regular), if for all \(\epsilon > 0\) there exists an \(\delta(\epsilon) > 0\) such that for all \(x, y\) with \(d(x, y) \leq \delta\) we have \(d(xt, yt) < \epsilon\) for all \(t \in T\).

A point is positively (resp. negatively) equicontinuous if for all \(\epsilon > 0\) there exists an \(\delta(\epsilon) > 0\) such that for all \(y\) with \(d(x, y) \leq \delta\) we have \(d(xt, yt) < \epsilon\) for all \(t \in T_+\) (resp. \(t \in T_-\)).

A point is called equicontinuous if the point is positively and negatively equicontinuous.

Equicontinuous minimal system and transitive systems with equicontinuous points are well studied. The following is true.

**Theorem 2.7.** If \((X, T)\) is a p.m. system which satisfies (A) and \(X\) is compact then the following conditions are equivalent:

1. \((X_M, T)\) is pointwise equicontinuous.
2. There exists one point which is equicontinuous.
3. \((X, T)\) is an equicontinuous minimal system.

Proof: (3) → (1): This is trivial.

(1) → (2): This is trivial.

(2) → (3): We first prove the case \(T = \mathbb{Z}\). Given a point \(x\) that is positively equicontinuous, by theorem [3.1] in [Kol] every point is positively recurrent, hence \((X, T)\) is p.m.p. and therefore minimal by theorem [1.5].

The case \(T = \mathbb{R}\): Let \(\pi : X \times \mathbb{R} \to X\) denote our system. Choose an \(s\) such that \(x \to \pi_s(x)\) is p.m. (see theorem [4.9]) and apply the results above. □

Note that in (A) there is dimension assumption hidden (dimension > 1) and theorem [2.7.] does not hold in the case dimension 1.Indeed, take for example a p.m. flow on \(S^1\) having exactly one fixpoint, then the flow \((X_M, T)\) is pointwise equicontinuous, but our flow is not minimal nor equicontinuous.

Now we want to show that an expansive homeomorphism \(f : X \to X\) under certain conditions is never p.m.. Mañe proved that an expansive minimal homeomorphism \(f : X \to X\) on a compact space exists only on zero-dimensional spaces [Ma]. Therefore there are no expansive minimal homeomorphisms on manifolds. There are many motivations to research on expansive systems and one of them is that expansiveness appears in many chaotic systems and therefore describes some kind of chaos. In [Ko] it is shown that expansive homeomorphism on compact surfaces are pseudo-Anosov (hence the periodic points are dense) and on the contrary there are examples of topological transitive expansive homeomorphisms on manifolds, so one could ask: does there exists p.m.
expansive homeomorphisms on manifolds? The answer is no, but our proof is not based on the proof of the result of Mañé, since he shows that there is a periodic point if the dimension of the set is not zero. However Mañé sketched in [Ma] an idea how to prove that a power of an expansive map is not minimal, but he did not prove the details. We are following this idea of Mañé and filling the gaps with results appeared years later after the original paper of Mañé!

**Definition 2.8.** A homeomorphism $f : X \to X$ on a metric space $(X,d)$ is called expansive if there is a constant $e > 0$ such that $\sup_{n \in \mathbb{Z}} \{d(f^n(x), f^n(y))\} \leq e$ implies that $x = y$.

**Theorem 2.9.** Suppose $(X,d)$ is a non trivial, locally connected and compact metric space and $(X,f)$ is expansive, then $f$ is not p.m..

**Corollary 2.10.** Let $X$ be a manifold and $K$ be an invariant hyperbolic set (i.e. $f(K) = K$) for a $C^1$ diffeomorphism $f : X \to X$. If $K$ is locally connected, then $K$ is not a pseudo-minimal set or $K$ is the orbit of a periodic point.

Proof: $f$ is expansive on $K$. □

Note that in general, the periodic orbits do not have to be dense in hyperbolic set. It is easy to construct a p.m. expansive homeomorphism on a compact metric space that is not minimal (take a totally disconnected set on the unit interval). One might conjecture that expansive p.m. flows do not exist on manifolds, but even for the case that the flow is minimal the problem is unsolved. At least Keynes and Sears could answer in [Ke] this question for a certain class of minimal flows.

Note that expansiveness does not depend on the metric if the space is compact. Given $\epsilon > 0$ and $x \in X$, we can define the local stable set $W^s_\epsilon(x)$ and the local unstable set $W^u_\epsilon(x)$ by

$$W^s_\epsilon(x) = \{y \in X \mid d(f^i(x), f^i(y)) \leq \epsilon, \forall \ i \geq 0\},$$

$$W^u_\epsilon(x) = \{y \in X \mid d(f^i(x), f^i(y)) \leq \epsilon, \forall \ i \leq 0\}.$$  

Here are some fundamental theorems.

**Theorem 2.11** (Reddy). If $f : X \to X$ is an expansive homeomorphism on a compact metric space $(X,d)$, then there are constants $a > 0$, $b > 0$, $0 < \gamma < 1$ and a compatible metric $D$ such that for all $\epsilon \leq a$ we have

$$y \in W^s_\epsilon(x) \to D(f^i(x), f^i(y)) \leq b\gamma^i D(x,y) \quad \forall \ i \geq 0$$

$$y \in W^u_\epsilon(x) \to D(f^i(x), f^i(y)) \leq b\gamma^i D(x,y) \quad \forall \ i \leq 0$$

Proof: See [Re] □

Let us define, for $\sigma = \{u, s\}$ $C_\sigma^\sigma(x)$, the connected component of $x$ in $W^\sigma_\epsilon(x)$ and $S_\delta(x) := \{y \mid d(x, y) = \delta\}$. Note that $C_\sigma^\sigma(x) \subset B(x, \epsilon)$. The following holds:
Theorem 2.12 (Hiriade). If $f : X \to X$ is an expansive homeomorphism on a non trivial, connected, locally connected and compact metric space $(X, d)$ then for every $\epsilon > 0$ there is an $\delta(\epsilon) > 0$ such that

$$C^\alpha_\epsilon(x) \cap S_{\delta(\epsilon)}(x) \neq \emptyset.$$ 

Proof: See proposition C in [Hi].

It is easy to see that in theorem [2.12] for all $\epsilon \leq \delta(\epsilon)$

$$C^\alpha_\epsilon(x) \cap S_\epsilon(x) \neq \emptyset$$

holds.

Before proving our theorem we need two useful lemmas.

Lemma 2.13. Suppose $(X, D)$ is a non trivial, connected, locally connected and compact metric space and $(X, f)$ is an expansive p.m. homeomorphism, where $D$ is the metric from theorem [2.11]. Moreover $a, b$ and $\gamma$ are the constants from theorem [2.11]. Then there exist constants $\epsilon > 0$, $c > 0$, $r > 0$ and a prime integer $p$ such that the following hold:

1. $Y := X - B(X^c_M, 2c)$ contains an $z$ such that $B(z, r) \subset Y$
2. $0 < \epsilon < a$ and $4\epsilon < r$
3. $4c < \epsilon$, $8c < \frac{\epsilon}{16}$ and $4c < \delta(\epsilon)$ where $\delta(\epsilon)$ is taken from theorem [2.12]
4. $f^p$ is minimal on $X_M$
5. If $\Gamma$ is a connected compact subset of $W^s_\epsilon(x)$ such that $x \in \Gamma$ and $c \leq Diam(\Gamma) \leq 2c$, then $f^{-p}(\Gamma) \subset W^s_\epsilon(f^{-p}(x))$ and $\text{Diam}(f^{-p}(\Gamma)) \geq 6c$.

Proof: The proof is easier than its formulation. Choose first $0 < \epsilon < a$ so small that 1 and 2 is satisfied for some $r > 0$. Choose now a large prime integer $p$ such that 4 holds (use theorem [1.11]) and moreover such that for $g, h \in X$ with $g \in W^s_\epsilon(h)$ we have $D(f^p(g), f^p(h)) \leq \frac{D(g, h)}{12}$ (use [2.11]). Choose now a small $c > 0$ such that 3 holds and moreover, if $Q$ is any set with $\text{Diam}(Q) \leq 2c$, then $D(f^i(g), f^i(h)) < \frac{\epsilon}{2}$ for all $-p \leq i \leq 0$ and $g, h \in Q$ (use compactness). If $\Gamma$ is a connected compact subset of $W^s_\epsilon(x)$ such that $x \in \Gamma$ and $c \leq \text{Diam}(\Gamma) \leq 2c$, then $f^{-p}(\Gamma) \subset W^s_\epsilon(f^{-p}(x))$. This follows from theorem [2.11], since we have by construction $D(f^i(g), f^i(h)) < \frac{\epsilon}{2}$ for all $p \geq i \geq 0$ and $g, h \in f^{-p}(\Gamma)$. Moreover we have $\text{Diam}(\Gamma) < 2c$ and therefore for all $i \geq 0$ and $l \in \Gamma$

$$D(f^i(l), f^i(x)) \leq b\gamma^i D(l, x) < b\gamma^i 2c \leq \frac{\epsilon}{4},$$

hence $D(f^i(g), f^i(h)) < \frac{\epsilon}{2}$ for all $i \geq 0$ and $g, h \in \Gamma$, thus $f^{-p}(\Gamma) \subset W^s_\epsilon(f^{-p}(x))$. If $\text{Diam}(f^{-p}(\Gamma)) < 6c$ then $\text{Diam}(\Gamma) < \frac{\epsilon}{2}$, so the lemma is proved. \qed

Lemma 2.14. Given a non trivial, connected, locally connected and compact metric space $(X, D)$, the numbers $c, \epsilon$ as in lemma [2.13] and a connected compact subset $\Gamma$ of $W^s_\epsilon(x)$ with $x \in \Gamma$ and $\text{Diam}(\Gamma) \geq 6c$, then there are two points $\alpha, \beta \in \Gamma$ and two compact connected sets $\alpha \in \Gamma_\alpha, \beta \in \Gamma_\beta$ such that the following holds:
1. $\Gamma_\alpha, \Gamma_\beta \subset \Gamma$

2. $\inf\{d(g, h) \mid g \in \Gamma_\alpha, h \in \Gamma_\beta\} > \frac{c}{2}$

3. $\Gamma_\alpha \subset W^s_\epsilon(\alpha)$ and $\Gamma_\beta \subset W^s_\epsilon(\beta)$

4. $c \leq \text{Diam}(\Gamma_{\alpha, \beta}) \leq 2c$

Proof: Since $\Gamma$ is compact and connected, choose two points $\alpha, \beta \in \Gamma$ such that $d(\alpha, \beta) = 4c$. Let $\Gamma_\alpha$ be the connected component of $\alpha$ in $B(\alpha, c) \cap \Gamma$ and define $\Gamma_\beta$ analogously. It is clear that 1, 2 holds and 4 follows from $c \leq \delta(\epsilon)$. 3 follows from the triangle inequality as in the proof of [2.13]. □

Proof of the theorem: First we know that there are at most finitely many connected components, since $X$ is locally connected and $f$ is p.m.. W.l.o.g $f$ is strictly p.m., since if $f$ is minimal, its dimension is zero, but a non trivial, connected, locally connected and compact metric space has not dimension zero (Theorem in [Mane]). By applying lemma [2.13] we choose an $x$ in $B := B(z, r)$ such that $W := W^s_\epsilon(x)$ lies in $B$ and there is an open set $U$ with $\text{Diam}U < \frac{c}{2}$ with $W \cap U = \emptyset$. Moreover we can assume that $O_+(x)$ is not dense hence $O_+(y)$ is not dense for all $y \in W$ and so the backward orbit of $y$ under $f^p$ is dense for all $y \in W$, otherwise we choose $\epsilon$ smaller. Choose by applying lemma [2.12] and [2.13, 3] a compact connected set $\Gamma$ in $W$ containing $x$ with $c \leq \text{Diam}(\Gamma) \leq 2c$. Indeed, $C'_\epsilon$ intersects $S_\delta(x)$ for all $\delta \leq \delta(\epsilon)$. But $4c < \delta(\epsilon)$ and therefore the connected component of $x$ (denote by $\Gamma$) in $C'_\epsilon \cap B(x, c)$ satisfies $c \leq \text{Diam}(\Gamma) \leq 2c$. We know from lemma [2.13, 5] that $f^{-p}(\Gamma) \subset W^s_\frac{c}{2}(f^{-p}(x))$ and $\text{Diam}(f^{-p}(\Gamma)) \geq 6c$, so we apply lemma [2.13] to $f^{-p}(\Gamma)$ to get a set $\Gamma_1$ with $\Gamma_1 \subset f^{-p}(\Gamma)$, $x_1 \in \Gamma_1 \subset W^s_\epsilon(x_1)$ $c \leq \text{Diam}(\Gamma) \leq 2c$ and $\Gamma_1 \cap U = \emptyset$. Again we conclude that $f^{-p}(\Gamma_1) \subset W^s_\frac{c}{2}(f^{-p}(x))$ and $\text{Diam}(f^{-p}(\Gamma_1)) \geq 6c$.

Repeating this construction, we get a sequence of compact connected sets $\Gamma_i$ with $\Gamma_0 = \Gamma, \Gamma_i \cap U = \emptyset$ and $f^p(\Gamma_{i+1}) \subset \Gamma_i$. Take a point $w$ in $\bigcap_i f^{Ip}(\Gamma_i) \subset W$. Then we have that $O_{-f^p}(w)$ does not intersect $U$ hence $f$ is not p.m. □

One could ask why the proof does not work in the case where the map is only transitive: We need pseudo-minimality to have $W := W^s_\epsilon(x) \subset X_M$.

Expansivity and the shadowing-property appears in hyperbolic dynamics, thus we see that the p.m. homeomorphisms are far away from being Anosov diffeomorphisms. A p.m. homeomorphism can not have the shadowing-property on certain space, but we skip the proof.

### 3 Algebraic Dynamics

This section is about automorphisms and affine transformations which are p.m.. A pseudo-minimal system with "something extra" should be minimal. The "something extra" in this section is that the system is an automorphism on a topological group. In this section every group is an abelian locally compact metric group and $\tau$ denotes a bicontinuous ($\tau$ and its inverse are continuous) automorphism. Our main theorem is based on well-known facts, though some of them are non-trivial.
Theorem 3.1. Let \((G, \circ)\) be a Lie group, \(\tau\) an automorphism and \(T = \tau \circ a\) be an affine transformation.

1. If \((G, \tau)\) is p.m., then \(G\) is trivial.

2. If \(G\) is abelian and \((G, T)\) is p.m., then \(G\) is a compact torus and \(T\) is minimal.

We first prove [3.1.1].

Lemma 3.2. If \((G, \tau)\) is p.m. system and there exists a normal subgroup \(H\) such that \(\tau(H) = H\), then \(H\) induces a p.m. system \((G/H, \tau)\).

Proof: Let \(\pi\) denote the projection. The projection is open, so \(V := \pi(G_M)\) is open. Every orbit starting in \(V\) is dense, hence \((G/H, \tau)\) is a p.m. system.

Lemma 3.3. If \((G, \tau)\) is p.m., then \(G\) is compact and connected.

Proof: Aoki proved in [A] that an automorphism having a dense orbit on a locally compact metric group \(X\) implies that \(X\) is compact. Therefore \(G\) is compact. We claim that \(G\) is connected. If \(G\) is compact and not connected, take the closed component \(C\) of \(e\). The induced automorphism \(\tau : G/C \to G/C\) is again p.m.. This automorphism must be ergodic by [R], since there is a dense orbit and therefore there must exists a dense set of periodic orbits by [Y], since \(\tau\) is ergodic, \(G/C\) is totally disconnected and compact. Thus \(G/C\) is finite and therefore \(G\) is the finite union of disjoint compact connected sets, but \(C\) is \(\tau\)-invariant, so the system can not be p.m..

Proof of theorem [3.1.1] : W.l.o.g. \(G\) is compact and connected.

First case: \(G\) is abelian:
It follows from proposition [2.16] from [Arv] that \(G\) is a torus. It is well-known, that for every automorphism on \(G\) the periodic orbits are dense, hence \(G\) is a finite set, therefore \(G\) is trivial.

General case: We prove the lemma by induction on the dimension. It holds for \(\dim G = 1\), since in this case \(G\) is abelian. Given now a non abelian Lie group of \(\dim G = n\). \(\tau\) is ergodic by [R], since there is a dense orbit and therefore from lemma 1 in [K] we conclude that \(G\) is nilpotent and the center \(Z\) of \(G\) has positive dimension. The induced automorphism on the lower dimensional Lie group \(G/Z\) is p.m. and therefore is \(G = Z\), so \(G\) is abelian.

Remark 3.4. In particular one could show more: If \(G\) is a locally compact metric group and \((G, \tau)\) is a pseudo-minimal system, then \(G\) might be trivial.

Here is a idea that could work. \(G\) is compact and connected by lemma [3.2].
Suppose that \(G\) is no Lie group, hence \(G\) has small subgroups (i.e. in every neighbourhood of \(\{e\}\) there is a non trivial subgroup). If we could find a \(\tau\)-invariant normal subgroup \(N\) such that \(G/N\) is a Lie group and \(N \neq G\) (this is the hard part and one has to argue as in [K]), then we have a contradiction. Since \((G/N, \tau)\) is a p.m. automorphism we have \(G = N\). So \(G\) is trivial.
Now we prove the last part. The prove is based on the paper [Ho]. Let \( T(x) = a + \tau(x) \) and \( \beta(x) = \tau(x) - x \). The following theorem can be found in [Ho] as theorem 4.

**Theorem 3.5.** Let \( T(x) = a + \tau(x) \) be an affine transformation on an abelian compact connected group \( X \) such that \( \beta(x) = \tau(x) - x \) is limit nilpotent, then \( T(x) \) is totally minimal, iff the closure of \( \{na\}_{n \in \mathbb{Z}} \) and \( \beta(X) \) together generate \( X \).

A homeomorphism \( \alpha \) is nilpotent if \( \alpha^n(X) = \{0\} \) for some \( n \), and limit nilpotent if \( \bigcap_{n \geq 0} \alpha^n(X) = \{0\} \).

**Lemma 3.6.** If \( T(x) = a + \tau(x) \) is an affine transformation on a compact connected group \( X \) with a dense orbit, then the closure of \( \{na\}_{n \in \mathbb{Z}} \) and \( \beta(X) \) together generate \( X \).

Proof: Let \( Y \) denote the group generated by the closure of \( \{na\}_{n \in \mathbb{Z}} \) and \( \beta(X) \). Choose a non-constant character \( \alpha \) which annihilates \( Y \). Therefore

\[
\alpha(T(x)) = \alpha(a)\alpha(x)\alpha(\beta(x)) = \alpha(x),
\]

hence \( \alpha \) is constant equal to 1, so \( Y = X \). \( \square \)

**Lemma 3.7.** \( T \) is conjugate to \( \tau \) via a translation \( \phi(x) = x + b \) iff \( \beta(b) = -a \).

Proof: Let \( Y \) denote the group generated by the closure of \( \{na\}_{n \in \mathbb{Z}} \) and \( \beta(X) \). Choose a non-constant character \( \alpha \) which annihilates \( Y \). Therefore

\[
\alpha(T(x)) = \alpha(a)\alpha(x)\alpha(\beta(x)) = \alpha(x),
\]

hence \( \alpha \) is constant equal to 1, so \( Y = X \).

Now we are in the case that \( X \) is a torus and \( T(x) = a + \tau(x) \) is an affine transformation on \( X \).

**Lemma 3.8.** If \( T(x) = a + \tau(x) \) is a p.m. affine transformation on a finite-dimensional torus \( X \), then \( a \notin \beta(X) \) or \( X = \{0\} \).

This follows directly from the lemma above. Otherwise \( T \) is topologically conjugate to the automorphism \( \tau \). Therefore \( \tau \) is p.m., hence \( X = \{0\} \) by theorem [3.1,1]. \( \square \)

**Lemma 3.9.** \( \{\beta^n(X)\}_n \) is a decreasing sequence of compact connected subgroups and there exists an \( n_0 \) such that \( \beta^{n_0}(X) = \{0\} \), i.e. \( \beta \) is nilpotent.

Proof: Since \( \beta(X) \subset X \), we have \( \beta^2(X) \subset \beta(X) \subset X \) and so on. \( \beta \) is continuous, \( X \) is compact and connected, therefore \( \beta^n(X) \) is compact and connected.

Let \( p_i \) denote the projection of \( X \) on \( S^1 \). We have that \( p_i \circ \beta^n(X) \) is a compact and connected subgroup of \( S^1 \), hence \( p_i \circ \beta^n(X) \) is trivial or \( S^1 \). So \( \beta^n(X) \) is a
torus.
The next part is more difficult. If \( n_0 \) does not exist then there is an \( m \) such that \( \beta^m(X) = H = \beta(H) \) for all \( n \geq m \), where \( H \) is a subtorus. Let us define an affine transformation \( S : H \to H \) by

\[
S(\beta^m(x)) = \beta^m(a) + \tau(\beta^m(x)) = \beta^m(T(x)).
\]

\( S \) is well-defined, continuous and the last equation holds, since \( \beta \) and \( \tau \) commute. Let us define \( S^{-1} : H \to H \) by

\[
S^{-1}(\beta^m(x)) = \beta^m(\tau^{-1}(-a)) + \tau^{-1}(\beta^m(x)) = \beta^m(T^{-1}(x)).
\]

Again \( S^{-1} \) is well-defined, continuous and the last equation holds, since \( \beta \) and \( \tau^{-1} \) commute and \( T^{-1}(x) = \tau^{-1}(-a) + \tau^{-1}(x) \). It is clear that \( S^{-1} \) is the inverse of \( S \).

We show that \( S \) is p.m. on \( H \). If \( x \) is in \( X_M \), then the orbit of \( \beta^m(x) \) is dense under \( S \). \( \beta^m(X_M) \) contains an open set, since \( \beta^m : X \to H \) as an homomorphism between Lie groups is \( C^\infty \) we can conclude from the theorem of Sard that \( X_M \) contains at least one regular point since \( \beta^m : X \to H \) is onto. Hence \( \beta^m(X_M) \) contains an open subset of \( H \). Thus \( S \) is p.m. and therefore by lemma [3.8] \( \beta^{1+m}(a) \notin \beta(H) = H \) or \( H \) is trivial. This shows \( H \) is trivial since \( \beta^{1+m}(a) \in H \). \hfill \square

Proof of the theorem [3.1, 2] : If \( G \) is a torus then \( \beta \) is nilpotent, so we conclude from lemma [3.5] and [3.6] that \( T \) is totally minimal. Now let \( T(x) = a + \tau(x) \) be an affine transformation on an abelian Lie group \( G \). Dani proved in [D] that a topologically transitive affine transformation on a connected Lie Group \( G \) implies that \( G \) is compact. The connected component \( X_0 \) containing 0 is a connected Lie group. Choose the minimal \( k \) such that \( T^k(X_0) = X_0 \). We conclude that \( S = T^k \) is p.m. on \( X_0 \) and hence by the result of [D] we conclude that \( X_0 \) is compact, thus \( X_0 \) is compact connected abelian Lie group and therefore \( S \) is a torus and therefore \( S \) is minimal. We know that a minimal system on a compact manifold is positively minimal, hence every point of \( X \) is positively recurrent under \( T \), thus \( T \) is p.p.m. by lemma [1.2] and therefore minimal by theorem [1.5]. \hfill \square

4 Constructing Pseudo-Minimal Flows by Blowing up Points and Counterexamples

It is well-known that the only compact surface which admits a minimal flow is the torus. We want to show an easy method to construct strictly p.m. systems on other manifolds. There exists a beautiful method to construct those flows on orientable surfaces (see [Ka], 14.4.a), but we construct these systems by blowing up points. Let \( X \) be a manifold of dimension strictly higher than 1 which admits a p.m. \( C^1 \) flow.

Choose a p.m. flow on \( X \) and fix a point \( x \in X_{M,+} \cap X_{M,-} \) then choose a positive function, which is only zero on this point and multiply it with the
induced vectorfield of the flow. The new flow is a strictly p.m. flow and one can repeat this construction. One could also cut off the point and then construct a p.m. flow on $X - \{x\}$.

If one has a p.m. $C^1$ flow on an open bounded polygon $P$ in $\mathbb{R}^n$, then one could choose a function, which is only zero on the topological boundary and otherwise strictly positive, and multiply it with the induced vectorfield of the p.m. flow $\pi$ such that the new flow is completely integrable. So we get a real p.m. flow being the identity on the boundary. Choosing equivalence relations generating manifolds which are trivial on the interior, we get a large class of p.m. flows on distinct mfd’s.

However there are no examples of p.m. $C^1$ flows on an open bounded polygon $P$ in $\mathbb{R}^n$, therefore let us modify the technic to construct p.m. flows. It is based upon blowing up a point of a manifold. Assume $X$ is a mfd. with a p.m. $C^1$ flow. We are only working locally, so we assume that the manifold is $\mathbb{R}^n$, $0 \in X_{M,+} \cap X_{M,-}$ and the flow is trivial. Multiply the induced vectorfield $V$ with a suitable function which is only zero in 0. Cut off 0 and blow up the hole to $B := B(0,1)$ and expand the vectorfield on $B^c$ by setting it zero on the boundary to get a new vectorfield $W$.

Note that if the vectorfield $V$ is $C^r$, the new vectorfield $W$ can be chosen $C^r$. Take an equivalence relation $\sim$ being trivial on the interior, so $X/\sim$ is a mfd. without boundary and therefore having a p.m. flow on $X/\sim$.

For example, one could take for the points on the boundary $x \sim y$ iff $x = y$ or $x = -y$. $Y := B^c/\sim$ generates a mfd. with a p.m. flow. Note that in this case $Y = X\sharp\mathbb{R}P^n$ where $\sharp$ denotes the connected sum. This means that, for example, we have p.m. flows on $T^n\sharp\mathbb{R}P^n \cdots \sharp\mathbb{R}P^n$.

Thus we get p.m. flows on $T^2\sharp\mathbb{R}P^2 \cdots \sharp\mathbb{R}P^2$ which does not admit minimal flows. One could also construct real p.m. flows on $T^n\sharp T^n \cdots \sharp T^n$.

We want to state this as a theorem:

**Theorem 4.1.** If a manifold $X$ admits a p.m.$C^1$ flow, then $Y\sharp X - F$ also admits a real p.m. flow, where $F$ is a finite subset of $X$ and $Y$ is a manifold generated as above.

It must be pointed out that, if one has a p.m. $C^1$ flow on $\mathbb{R}^n$, then on all mfd.’s of dimension $n$, since we can embed $\mathbb{R}^n$ densely in every manifold. Therefore it is very interesting in the view of pseudo-minimality, if there are minimal $C^1$ flows on the spheres.

The real flows on submanifolds of the Klein bottle, $S^2$ and $P\mathbb{R}^2$ are well studied, so we do not spend much time on it. It is well-known (compare with [Ar]) that flows on compact surfaces, which are closed subsets of $S^2$, $P\mathbb{R}^2$ or the Klein bottle, do not admit a dense orbit since the Poincaré-Bendixon theorem holds. Markley proved it for the Klein bottle in [M]. Therefore it is more interesting to ask whether there exist p.m. flows on the other surfaces. This has been answered in the paper [L] but the authors have not stated the fact that the real topologically transitive flows they have constructed are p.m.. In [L], p.m. flows of the torus are “embedded” in other surfaces while on the contrary we “blow up” the singularities of a p.m. flow on the torus.
Proposition 4.2. On all compact orientable surfaces $X$ of genus $g(X) > 0$ and unorientable surfaces of genus $g(X) > 2$ there exist real p.m. flows.

Proof: We prove only the case where the manifolds are closed, since we can change the flow to get a p.m. flow with some singularities, and then blow them up to boundary components. Above we constructed p.m. flows on all orientable closed surfaces (even with boundary) of genus $g(X) > 0$. Note that the genus raises by adding projective planes to the torus, and on these manifolds there exist p.m. flows by the results above. □

Now we want to construct p.m. diffeomorphisms on a large class of compact surfaces and higher dimensional manifolds. It is well-known (compare [Bl]) that the only closed connected surfaces which admit minimal homeomorphisms are the Klein bottle and the torus. In particular, this fact is proven for p.p.m. maps, hence also for (positively) minimal homeomorphisms.

We will see that for every finite set $F$ of $T^2$ there exists a diffeomorphism that is minimal on $T^2 - F$. On the contrary there is a theorem of Le Calvez and Yoccoz [C], that states that there is no minimal homeomorphism of the the space $S^2 - F$, where $F$ is a finite subset. This shows, that the theorem of Le Calvez and Yoccoz does not hold for arbitrary closed surfaces. Moreover we construct a strictly p.m. flow without fixed points.

At first, we construct explicitly a p.m. diffeomorphism, although we will see that there is a general way to construct them.

Proposition 4.3. There is a strictly p.m. $C^\infty$ diffeomorphism on $T^n$ where $n > 1$ and $X^c_M$ is a point.

We will construct other strictly p.m. systems on mfd.’s, but if one understand this proof, one understands the construction of the other diffeomorphisms. The proof is similarly to the proof of proposition [1.4.1] from [Ka].

We first state some easy lemmas.

Lemma 4.4. A continuous open map $f$ of a locally compact separable metric space is topologically transitive, iff there are no two disjoint open non empty $f$-invariant sets.

Proof: For a proof see corollary [1.4.3] in [Ka]. □

Let $f = \pi(\cdot, s)$ be an induced discrete system of a flow $\pi$. Then let us denote $O_{f,+}(x)$ the positive semi-orbit under $f$ and $O(x)$ the orbit of $\pi$.

Lemma 4.5. Let $f = \pi(\cdot, s)$ be an induced discrete system of a flow $\pi$ on a metric space that complies (A), then the following holds:

1. If $O_{f,+}(x)$ is dense, then $O_{f,+}(\pi(x,t))$ is dense for all $t$.

2. If $O_{f,+}(x)$ is dense and for $y$ one have $O_{f,+}(y) \cap O(x)$, then $O_{f,+}(y)$ is dense.

Proof: i) The claim follows from

$O_{f,+}(\pi(x,t)) = \{\pi(\pi(x,t), sn)\}_{n \in \mathbb{N}_0} = \{\pi(\pi(x,sn),t)\}_{n \in \mathbb{N}_0} = \pi(O_{f,+}(x),t)$. 

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ii) If \( z \in \overline{O_+(y)} \cap O(x) \), then from (i) we conclude
\[
X = \overline{O_+(z)} \subset \overline{O_+(y)}.
\]
\(\square\)

We now construct a vectorfield \( V \) that induces our strictly p.m. diffeomorphism. The picture shows the preimage of our constructed sets under the natural projection. Let \( W \) denote the constant vectorfield \( W(x) = (\gamma_1, \cdots, \gamma_n) \) on \( \mathbb{R}^n \), where \( \{\gamma_1, \cdots, \gamma_n\} \) are rationally independent, i.e.
\[
\sum_i k_i \gamma_i \notin \mathbb{Z} \quad \forall \quad (k_1, \cdots, k_n) \in \mathbb{Z}^n - \{0\}.
\]
Now choose a positive \( C^\infty \) function \( \phi \) that is constant equal to 1 outside a small neighbourhood of 0, bounded above by 1 and is only zero in 0. Now set
\[
V(x) = \left( \prod_{(k_1, \cdots, k_n) \in \mathbb{Z}^n} \phi(x + (k_1, \cdots, k_n)) \right) W(x).
\]
The flow induced by \( W \) is denoted with \( \tilde{\pi} \) and the flow of \( V \) is denoted with \( \tilde{\rho} \). The vectorfields are \( \mathbb{Z}^n \)-invariant and induce therefore flows on the torus, which we write as \( \pi \) and \( \rho \).
Now choose a small positive \( s \in \mathbb{Q} \) and define \( f(x) := \rho(x, s) \) (\( \tilde{f}(x) := \tilde{\rho}(x, s) \)). If \( s \) is chosen small, there are two open connected sets \( Q_1 \subset Q_2 \) such that the vectorfields \( W \) and \( V \) coincide on \( Q_2 \) and \( f(Q_1) \subset Q_2 \). Moreover we have that \( \tilde{f}(x) = x + s(\gamma_1, \cdots, \gamma_n) \) on \( Q_2 \) and \( Q_1 \) contains a rectangle \( R \) such that every orbit segment of our flow in \( R \) has a length larger than \( s \). \( O_1 \) and \( O_2 \) are the only orbits of the flow \( \rho \) which are only dense in one direction. Take a look at the picture.
Lemma 4.6. \( f \) is topologically transitive.

This part of the proof is based on the proof of [1.4.1] in [Ka]. Let \( U, V \) be two disjoint open non empty \( f \)-invariant sets. Let \( \chi \) denote the characteristic function of \( U \). \( \chi \) is \( f \)-invariant. Take the Fourier expansion

\[
\chi(x_1, \cdots, x_n) = \sum_{(k_1, \cdots, k_n) \in \mathbb{Z}^n} \chi(k_1, \cdots, k_n) \exp(2\pi i \sum_{j=1}^{n} k_j x_j).
\]

Since \( s \) is chosen small, we find an open set \( U_0 \) in \( U \cap Q \), hence for \( x \in U_0 \)

\[
\chi(x) = \chi(\tilde{f}(x)) = \chi(x + s(\gamma_1, \cdots, \gamma_n)) = \sum_{(k_1, \cdots, k_n) \in \mathbb{Z}^n} \chi(k_1, \cdots, k_n) \exp(2\pi i \sum_{j=1}^{n} k_j x_j) \exp(2\pi i \sum_{j=1}^{n} sk_j \gamma_j).
\]

From the uniqueness of the Fourier expansion we conclude

\[
\chi(k_1, \cdots, k_n) = \chi(k_1, \cdots, k_n) \exp(2\pi i \sum_{j=1}^{n} sk_j \gamma_j).
\]

\( U \) has Lebesgue measure \( \lambda(U) \in (0,1) \), thus we conclude that \( \chi(k_1, \cdots, k_n) \neq 0 \) for some \( (k_1, \cdots, k_n) \) and therefore \( \exp(2\pi i \sum_{j=1}^{n} sk_j \gamma_j) = 1 \), thus \( \sum_{j=1}^{n} sk_j \gamma_j \in \mathbb{Z} \),

which contradicts the fact that \( \{\gamma_1, \cdots, \gamma_n\} \) are rationally independent. Hence \( f \) is topologically transitive by lemma [4.4]. \(\square\)

Proof of the proposition: Given an \( x_0 \) such that \( O_{f,+}(x_0) \) is dense and an \( x_1 \) such that \( O_{f,-}(x_1) \) is dense (those exist by lemma [1.3]), it is sufficient to show that for every point \( x \in T^n - \{0 \cup O_2\} \), its positive semi orbit is dense, since the other case is similar.

The orbit of \( x \) under \( \pi \) is positively dense, therefore we can find subsegments \( r_i \) of \( \pi(x, [0, \infty)) \) in \( R \) that are converging to a subsegment \( r \) of \( \pi(x_0, [0, \infty)) \) in \( R \). The length of the orbit segments are larger than \( s \), hence there is an accumulation point \( y \) of the set \( O_{f,+}(x) \) on \( r \), but lemma [4.5] then already implies that \( O_{f,+}(x) \) is dense. \(\square\)

In fact we could construct, for every given finite set \( F \), a strictly p.m. \( C^\infty \) diffeomorphisms on \( T^n \), where \( n > 1 \) and \( X^c_M = F \). Comparing construction of p.m. flows in theorem [4.1], with the same method we could construct, for every given compact surface \( X \) that admits a p.m. flow and any given \( m \geq g(X) - 1 \) (\( g(X) \) is the genus), a strictly p.m. \( C^\infty \) diffeomorphism on \( X \) induced from a flow such that \( X^c_M \) has \( m \) components.

We show that a p.m. flow without fixed points does not have to be minimal. Thus theorem [4.2] does not hold in general.

Proposition 4.7. For every \( n > 2 \) there are strictly p.m. \( C^\infty \) flows on \( T^n \) without fixed points, and the orbits that are not dense are closed and isolated.
Proof: We have proved in the proof of proposition [4.3] that for a small $s_0$, every $s \in S := (0, s_0) \cap \mathbb{Q}$ induces a strictly p.m. diffeomorphism $f_s(x) = \rho(x, s)$. Thus for a dense set $D$ in $\mathbb{Q}$, every induced diffeomorphism $x \rightarrow \rho(x, s)$ is strictly p.m., since $X_M$ is connected, and one can therefore apply theorem [1.10]. In particular let $D = \mathbb{Q} - \{0\}$. Take such a flow $\rho$ on $T^{n-1}$ and the flow $\phi : (x, t) \rightarrow x \exp(2\pi it)$ on $S^1$. The product flow $\pi$ is strictly p.m. and without fixpoints.

Let us show this: Fix a point $x_0 \in T^{n-1} \times S^1$. The closure of the orbit $O(x_0)$ of the induced discrete system for $(s, s) \in D \times \mathbb{Q}$ is in $T^{n-1} \times \mathbb{R}$, where $A$ is the orbit $O_{g_s}(x)$ of the map $g_s : x \rightarrow x \exp(2\pi is)$, since the map $g$ is periodic on $A$ and $f_s$ is totally minimal on $T^{n-1}_M$. A can be chosen arbitrary dense by chosing $s$ small, hence the orbit of $x_0$ under $\pi$ is dense if $x_0 \in T^{n-1} \times S^1$, otherwise it is periodic. The proof is similar, if one wants to have more closed orbits. □

If one combines both methods of the last two theorems, one gets the following theorem.

**Theorem 4.8.** For any given minimal translation $A : T^m \rightarrow T^m$ and every $l \geq 0, n \geq 2 + m$, there is a strictly p.m. $C^\infty$ diffeomorphism on $T^n$ without fixed points and every component $C$ of $X^e_M$ is isolated and a torus of dim$(C) = m$. $X^e_M$ has exactly $l$ components, and the diffeomorphism restricted on every component of $X^e_M$ is $C^\infty$ topologically conjugate to $A : T^m \rightarrow T^m$.

Proof: Start with a linear flow $A : T^m \rightarrow T^m$ and take a product with a strictly p.m. system such that the new flow is p.m.. □

The last theorem of this section generalizes proposition [4.3].

**Theorem 4.9.** Given a p.m. $C^1$ flow $(X, \pi)$ on a Riemannian mfd. $X$ of dimension $> 1$ such that the induced vectorfield of the flow is bounded then

$$D := \{ s \in \mathbb{R} | \pi(\cdot, s) \text{ is minimal on } X_M \} = \{ s \in \mathbb{R} | \pi(\cdot, s) \text{ is topologically transitive} \}$$

is a dense $G_\delta$-set. In particular, if $x \in X_{M,+}$ and $s \in D$, then the positive semi orbit of $x$ under $\pi(\cdot, s)$ is dense.

The proof is based on the theorem 6 from [O]:

**Theorem 4.10.** Given a real topologically transitive flow $(X, \pi)$ on separable metric space ,and suppose there is no "isolated streamline". For all values of $s \in \mathbb{R}$, except a set of first category, the homeomorphisms $\pi(\cdot, s)$ are topologically transitive.

The fact that there is no "isolated streamline" is used to show that for any given number $s_0 > 0$ and open sets $U, V$ there exists an open set $W \subset V$ such that $W[-s_0, s_0]$ is not dense in $U$. This clearly holds for all $C^1$ flows on a mfd. $X$ of dimension $> 1$.

Proof of the theorem: The set

$$D = \{ s \in \mathbb{R} | \pi(\cdot, s) \text{ is topologically transitive} \}$$
is a dense $G_δ$-set. Take an $s \in D$ and choose a positive integer $m$ such that $\frac{s}{m}$ is very small (note $\frac{s}{m} \in D$). Define $g(x) := \pi(x, \frac{s}{m})$ and take an $x_0$ such that $O_{g,+}(x_0)$ is dense. We show that for every orbit $x \in X_{M,+} O_{g,+}(x)$ is dense. As in the proof of proposition [4.3] we choose a very small flowbox $U$ in $X_{M}$ and a rectangle $R$ in $U$. If $\frac{s}{m}$ is chosen small enough, then since the induced vectorfield is bounded, we conclude as in [4.3], that a point of the orbit $x_0(−\infty, \infty)$ is an accumulation point of the orbit $O_{g,+}(x)$, hence the orbit is dense. Since $X_{M}$ is connected, $g^m(x) = \pi(x, s)$ is p.m. by theorem [1.10]. □

There already exist definitions for weaker forms of minimality. For example, the definition of quasi-minimality exists. Unfortunately there are different definitions of quasi-minimal flows. If one defines a system quasi-minimal, if $(X_{M})^c$ is at most a finite set (as in [Gu]), then p.m. systems do not need to be quasi-minimal, while on the contrary p.m. systems are always quasi-minimal in the sense of exercise 2 in chapter 6 of [S]. In [S], a flow is quasi-minimal if there is a point $x$ such that its orbit is dense and $x$ is positively and negatively recurrent. We use the definition of quasi-minimality used in [K] and [Ha]:

**Definition 4.11.** A flow on a surface (possible with boundary) is said to be quasi-minimal, if it has finitely many fixed points and every semi-orbit other than a fixed point or a separatrix of a saddle is dense.

The following is true:

**Theorem 4.12.** Transitivity, topological mixing and quasi-minimality are equivalent for flows on compact surfaces that have finitely many fixed points and preserve a measure positive on open sets.

Proof: See corollary 8.4.3. in [Ha] □

**Theorem 4.13.** On every compact orientable surface $X$ of genus $g > 1$, there exists a quasi-minimal flow that preserves a measure positive on open sets. Moreover, $X_{M}^c$ is a point.

Proof: See corollary 8.4.3. in [Ha] □

**Corollary 4.14.** Given a compact orientable surfaces $X$ of genus $g > 1$ and a non empty finite subset $F$, there exists a quasi-minimal flow (diffeomorphism) that preserves a positive measure on open sets and $X_{M}^c = F$.

**Theorem 4.15.** For any area-preserving flow on a compact orientable surface of genus $g > 1$, there are at most $g$ ergodic nonatomic invariant measures. Furthermore, for any $k$ with $1 \leq k \leq g$, there exists a quasiminimal area-preserving flow that has exactly $k$ nonatomic ergodic invariant measures.

Proof: See theorem 8.4.5 in [Ha]. □

Thus we see that pseudo-minimal area-preserving flows do not have to be minimal and hence we conclude with the technics of the previous results that pseudo-minimal area-preserving diffeomorphism exists:
Corollary 4.16. Let $X$ be compact orientable surface of genus $g > 1$ and $1 \leq k \leq g$. Then there exists a quasi-minimal area-preserving diffeomorphism that has $k$ nonatomic invariant measures and $|X^c_M|$ is finite.

It is not difficult to construct an area-preserving $C^\infty$ p.m. flow on a 3-manifold without fixed points, whose orbits, which are not dense, are closed or asymptotic to closed orbits.

5 Questions and Conjectures

Indeed many authors researched on systems like pseudo-minimal systems but still most of the results in this paper are new and here is a list of some interesting questions concerning pseudo-minimal systems.

Are flows and diffeomorphism on manifolds in the $C^r$ topology generically not pseudo-minimal?

Does every compact manifold of strictly higher dimension than 2 admit a p.m. flow?

Does every compact manifold admit a p.m. diffeomorphism? (Maybe the method of fast-conjugating could produce some p.m. systems.)

Which compact surfaces admit pseudo-minimal or quasi-minimal diffeomorphism?

Does a p.m. geodesic flow exists?

At least the authors conjecture that on all three dimensional manifolds, there exists a p.m. flow. Maybe one could use the fact that from $S = S^1 \times D^2$, one can construct $S^3$.

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