ON REAL TROPICAL BASES AND REAL TROPICAL DISCRIMINANTS

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ABSTRACT. We explore the concept of real tropical basis of an ideal in the field of real Puiseux series. We show explicit tropical bases of zero-dimensional real radical ideals, linear ideals and hypersurfaces coming from combinatorial patchworking. But we also show that there exist real radical ideals that do not admit a tropical basis. As an application, we show how to compute the set of singular points of a real tropical hypersurface, i.e. we compute the real tropical discriminant.

1. Introduction

In this article, we try to develop an analog of the tropical algebra applied to the case of real varieties. Our point of view is to define real tropical varieties as non-archimedean amoebas plus signs. They are the restriction to the reals of the complex tropical curves of G. Mikhalkin in [13] and a natural extension of combinatorial patchworking. Different approaches to real tropical geometry appear, for instance, in the study of logarithmic limits of semialgebraic sets in [1] or the study of initial real radical ideals in [18]. Another approach with nice combinatorial properties is the study of the positive part of a tropical variety [15].

We show that, contrary to the complex case, Kapranov's theorem or, more generally, the fundamental theorem of tropical geometry does not hold in the real case and there may not be tropical bases. cf. [4], [7], [11], [14], [16].

However, for sufficiently simple yet interesting varieties, we can compute real tropical basis. We include here real radical zero dimensional ideals, linear varieties and hypersurfaces constructed using combinatorial patchworking. As an application, we are able to describe the singular locus of a real tropical hypersurface using an analogue of the techniques introduced in [6].

The presentation is done over the field of real Puiseux series $\mathbb{K}$. In principle, we could work over any real closed field $\mathbb{F}$ provided with a nontrivial valuation and most of our results can be translated to this more general setting without trouble. However, some results depending on the residue field of $\mathbb{F}$ being archimedean or not, see for instance Proposition 3.8 and Example 3.9.

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The paper is structured as follows. In Section 2 we introduce the notation and basic definitions in real tropical geometry. In Section 3 we show the main results concerning the existence and computation of real tropical basis. Finally, in Section 4 we apply our results to the computation of the singular locus of a real tropical hypersurface.

2. Preliminaries

Let us introduce some notation. Let $\mathbb{K}$ be the valued field of real Puiseux series,

$$\mathbb{K} = \bigcup_{n \geq 1} \mathbb{R}((t^{1/n})) \quad v : \mathbb{K}^* \to \mathbb{Q} \subseteq \mathbb{R}.$$ 

Every element is a power series with real coefficients and rational exponents with bounded denominator

$$p = \sum_{i \geq r_0} a_i t^{i/n}, \quad a_{r_0} \neq 0$$

The valuation $v(p)$ of a nonzero power series $p$ is the least exponent $r_0$ appearing in the development of the series and the principal coefficient is $a_{r_0}$. This principal coefficient is the residue of the series $pt^{-v(p)}$ in the residue field $\mathbb{K}$. $\mathbb{K}$ is an ordered field with the order given by the relation $p > 0$ is positive if and only if its principal coefficient $a_{r_0}$ is positive. The valuation is compatible with the order in the sense that if $0 < p < q$ then $v(q) \leq v(p)$. Moreover, since $\mathbb{K}[i]$ is the algebraically closed fields of Puiseux series, $\mathbb{K}$ is a real closed field.

Definition 2.1. The real tropicalization is the valuation taking into account the sign of the series:

$$\text{trop} : \mathbb{K}^* \to \mathbb{T}\mathbb{R} = \{1, -1\} \times \mathbb{R}$$

where $s(x)$ is the sign function $s : \mathbb{K}^* \to \{1, -1\}$, $s(x) = 1$ if $x > 0$ and $s(x) = -1$ if $x < 0$. We will sometimes denote by $a^+ = (1, a)$ and $a^- = (-1, a) \in \mathbb{T}\mathbb{R}$. If $x = (p, a) \in \mathbb{T}\mathbb{R}$, then $p = s(x) \in \{1, -1\}$ is the sign of $x$ and $a = |x| \in \mathbb{R}$ is the modulus of $x$. If $a = ((s_1, a_1), \ldots, (s_n, a_n)) \in \mathbb{T}\mathbb{R}^n$, we denote by $s(a) = (s_1, \ldots, s_n) \in \{1, -1\}^n$, $|a| = (a_1, \ldots, a_n) \in \mathbb{R}^n$, the sign and modulus taken component-wise.

Remark 2.2. Note that, while $\mathbb{T}\mathbb{R}$ is a group with the tropical multiplication $(i, a) \odot (j, b) = (i \cdot j, a + b)$, it is not a tropical semiring, since there is not a reasonable definition of addition for $a^+ \oplus a^-$. 

We now define our main geometric objects, real tropical varieties, as the image of an algebraic set under the trop map.

Definition 2.3. Let $V \subseteq (\mathbb{K}^*)^n$ be a real variety in the torus. The real tropicalization of $V$ is the closure (in $\mathbb{R}^n$) of the image $\text{trop}(V) \subseteq \mathbb{T}\mathbb{R}^n$ of the real points of $V$ under the tropicalization map applied component-wise.
If \( V = V(I) \) and \( I \) is generated by polynomials in \( \mathbb{R}[x_1, \ldots, x_n] \), then we say that we are dealing with the constant coefficient case.

This definition is related with taking the non-archimedean amoeba and co-amoeba of the set of real points of a real variety (cf. the complex tropical curves presented in [13] section 6).

In the algebraically closed case, the tropicalization of a variety \( V \) can be computed using Kapranov’s theorem, also known as the fundamental theorem of tropical geometry. The real case seems more involved. The tropicalization in the constant coefficient case has been studied in [18] but there is a difference in the tropicalization in the non constant coefficient case we are studying here that has its root in Pólya’s Theorem [9] (See also [8] for a more general result).

**Theorem 2.4** ([9]). Let \( F \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous polynomial that is positive in the set \( \{ x \in \mathbb{R}^n | x_i \geq 0, x_1 + \ldots + x_n \neq 0 \} \), then for \( N \) sufficiently large \( F \cdot (x_1 + \ldots + x_n)^N \) has only positive coefficients.

However, it is well known that this theorem does not hold if the ground field is not archimedean (See Example 3.6). This translates to the fact that tropicalization of polynomials is not so useful in the non-constant coefficient case.

We now introduce a different way of defining tropical varieties in terms of a real tropical polynomial. The motivation of this alternative approach is...
to get rid of algebraic polynomials and deal only with tropical polynomials, this is a common approach in the tropical case over complex Puiseux series.

**Definition 2.5.** A **real polynomial** \( f \in \mathbb{TR}[w_1, \ldots, w_n] \) is just a formal sum of tropical monomials of the form \( f = \oplus_{\ell \in A} a_{\ell} w^\ell \in \mathbb{TR}[w_1, \ldots, w_n] \), \( A \subseteq \mathbb{N}^n \). Every real tropical polynomial defines a piecewise affine function \( f : \mathbb{TR}^n \to \mathbb{R} \). \( f(p) = \min \{|a_\ell| + \langle |p|, \ell \rangle \mid \ell \in A\} \).

We define the **real tropical hypersurface** defined by \( f \), \( \mathcal{T}_R(f) \) by \( p = (p_1, \ldots, p_n) \in \mathcal{T}_R(f) \) if there are two monomials \( i \neq j \) such that

\[
\begin{align*}
\sum_{\ell} s(a_i) s(p_\ell)^{i_\ell} &= 1, \\
\sum_{\ell} s(a_j) s(p_\ell)^{j_\ell} &= -1
\end{align*}
\]

and

\[
\sum_{\ell} |a_i| + \langle |p|, \ell \rangle = |a_j| + \langle |p|, \ell \rangle \leq |a_k| + \langle |p|, \ell \rangle
\]

for all \( k \neq i, j \).

That is, if the minimum of \( f(p) \) is attained at two different monomial \( i, j \) where the evaluation at the point \( p \) yields two different signs.

**Example 2.6.** Let \( f = 0^+ \oplus 1^+ w^0 \oplus 0^+ w^2 \oplus 0^+ w^3 \oplus 2^- w^4 \). We have associated the piecewise-linear map \( p \mapsto \min \{0,1 + p,0 + 2p,1 + 3p,2 + 4p\} \). The minimum of this piecewise-linear is attained at least twice for \( p = 0, -1 \). The value and sign attained in the candidates of roots are:

| \( a \) | \( 0 \) | \( 1 + p \) | \( 0 + 2p \) | \( 1 + 3p \) | \( 2 + 4p \) |
|-----|-----|-----|-----|-----|-----|
| \( 0^+ \) | \( 0^+ \) | \( 1^+ \) | \( 0^+ \) | \( 1^+ \) | \( 2^- \) |
| \( 0^- \) | \( 0^+ \) | \( 1^- \) | \( 0^+ \) | \( 1^- \) | \( 2^- \) |
| \( -1^+ \) | \( 0^+ \) | \( 0^- \) | \( -2^+ \) | \( -2^- \) | \( -2^- \) |
| \( -1^+ \) | \( 0^+ \) | \( 0^- \) | \( -2^- \) | \( -2^- \) | \( -2^- \) |

Neither \( 0^+ \) nor \( 0^- \) are real tropical roots of \( f \), because the minimum is attained at \( 0^+ \) so we do not have two different signs. On the other hand, \( -1^+ \) and \( -1^- \) are both real tropical roots, since the minimum is attained at different signs \( 2^+ \) and \( 2^- \). Hence \( \mathcal{T}_R(f) = \{-1^+, -1^-\} \).

In Figure 1 we show the four tropical quadrants of the conic given by the polynomial \( f = 1^+ \oplus 0^+ \alpha \oplus 0^- \alpha \oplus 0^- \beta \oplus 0^+ v \alpha \oplus 0^- v \beta \). Note that we are working with the min and that the subdivision induced by \( f \) is not a triangulation, so the picture is different from the usual ones in patchworking.

**Definition 2.7.** If \( F = \sum_{\ell \in A} a_\ell x^\ell \in \mathbb{K}[x_1, \ldots, x_n] \) is a real polynomial, write

\[
trop(F) = f = \bigoplus_{\ell \in A} (s(a_\ell),v(a_\ell)) w^\ell.
\]

By abuse of notation, we will write

\[
\mathcal{T}_R(F) = \mathcal{T}_R(f) \subseteq \mathbb{TR}^n
\]

Clearly \( \mathcal{T}(\mathcal{V}(F)) \subseteq \mathcal{T}(F) \) (Example 3.2), but contrary to the algebraically closed set, the inclusion may be strict. Next lemma shows further discrepancies between the real and usual tropical setting.
Lemma 2.8. Let $F$, $G \in \mathbb{K}[x_1, \ldots, x_n]$, then
\[ \mathcal{T}_\mathbb{R}(F \cdot G) \subseteq \mathcal{T}_\mathbb{R}(F) \cup \mathcal{T}_\mathbb{R}(G) \]
and the inclusion may be strict.

Proof. Let $p \in \mathbb{T}^{\mathbb{R}^n}$ such that $p \notin (\mathcal{T}_\mathbb{R}(F) \cup \mathcal{T}_\mathbb{R}(G))$. Without lost of generality, assume that $p = (0^+, \ldots, 0^+)$. Let $A_1$ (resp. $A_2$) be the monomials of $F$ (resp $G$) where the minimum of trop$(F)$ (resp. trop$(G)$) is attained at $p$. Then, the tropicalization of the monomials of $A_1$ attain the same sign at $p$ and so do the monomials in $A_2$. We may also suppose that all these monomials have positive coefficients. Then, the monomials of $FG$ where the minimum is attained at $p$ are all of the form $ca_1a_2$, with $c \in \mathbb{K}$, $c > 0$, $a_1 \in A_1$, $a_2 \in A_2$. It follows that all monomials yield the same sign at $p$ and $p \notin \mathcal{T}_\mathbb{R}(FG)$.

To see that the inclusion may be strict, let $F = x^2y + x^2 - xy - x + y + 1$, $G = xy^2 - xy + y^2 + x - y + 1$. Then $FG = x^3y^3 + x^3 + y^3 + 1$. So $p = (0^+, 0^+)$ belongs to both $\mathcal{T}_\mathbb{R}(F)$ and $\mathcal{T}_\mathbb{R}(G)$, but not to $\mathcal{T}_\mathbb{R}(FG)$. \(\square\)

We end this section with an elementary result concerning the number of real roots that an univariate polynomial admit.

Definition 2.9. Let $f = \oplus_{a \in A} a_\ell \omega^\ell \in \mathbb{T}^{\mathbb{R}[w]}$ be an univariate real tropical polynomial. Let $f' = \oplus_{a \in A} a_\ell \omega^\ell \in \mathbb{T}[w]$ be the usual tropical polynomial obtained from $f$ by forgetting signs.

Let $p \in \mathcal{T} \mathbb{R}$ with $|p|$ a tropical root of $f'$. Let $i_1 < \ldots < i_r$ be the monomials where $|a_\ell| + \langle |p|, \ell \rangle$ attains its minimum. Take the sequence of signs
\[ S_p = (s(a_{i_1})s(p)^{i_1}, \ldots, s(a_{i_n})s(p)^{i_n}) \]
then, the complex multiplicity of $|p|$ in $f'$ is $i_n - i_1$ and the real multiplicity of $p$ in $f$ is the number of changes of signs in $S_p$.

Lemma 2.10. Let $f = \oplus_{a \in A} a_\ell \omega^\ell \in \mathbb{T}^{\mathbb{R}[w]}$ be an univariate real tropical polynomial of degree $n$. Let $f' = \oplus_{a \in A} a_\ell \omega^\ell \in \mathbb{T}[w]$ be the (usual) tropical polynomial obtained from $f$ by forgetting the signs. Let $p \in \mathbb{T}$ be a (usual) tropical root of $f'$ of (usual) multiplicity $m$. Let $m^+$ (resp. $m^-$) be the real tropical multiplicity of $p^+$ (resp. $p^-$) as a real tropical root of $f$. Then $m \leq m^+ + m^-$ and the difference $m - m^+ - m^-$ is an even number.

Proof. Both multiplicities only depend on the support and signs of the coefficients where the minimum is attained when evaluating $f$ at $p$. Hence, we may assume, without loss of generality, that all coefficients of $f$ and the tropical root have modulus 0. consider the polynomial
\[ F = \sum_{\ell \in A} s(a_\ell) \cdot t^{\ell_2^2} x^\ell \in \mathbb{R}[t][x]. \]
By combinatorial patchworking \([19]\), for any sufficiently small evaluation $t_0$ of $t$, $0 < t_0 << 1$, $F(t = t_0)$ will have exactly $m^+$ positive roots and $m^-$
negative roots, $m$ equals the number of nonzero roots of $F$, so $m^+ + m^- \leq m$ and its difference is the number of non-real roots of $F$ is even.

\[ \square \]

**Example 2.11.** Let $f = 0^+ \oplus 1^+ w \oplus 0^- w^2 \oplus 1^+ w^3 \oplus 2^- w^4$ be the real tropical polynomial from Example 2.6. Let $f'$ be the usual tropical roots of $f$ and $-1$ with multiplicity 2 each root. Now, for the root zero, we know that $f$ does not admit a real tropical root. So for $|p| = 0$ we have that $m_+ = m_- = 0$, $m = 2$ and $m - m_+ - m_- = 2$. On the other hand, the sequence of signs $S_1 = (1, 1, -1)$ so $m^+ = 1$, $S_1^- = (1, -1, 1)$ and $m^- = 1$, $m - m_+ - m_- = 0$ is also even.

3. Real Tropical Basis

We start with the definition of real tropical basis, cf. [1]

**Definition 3.1.** Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$. A **real tropical basis** of $I$ is a finite set $\{F_1, \ldots, F_r\}$ that generates $I$ such that $\text{trop}(V(I)) = \cap_{r=1}^r T_R(F_i)$.

**Example 3.2.** Let $F = x^2 - x + 1$, $V(F) = \emptyset$, there are no real solutions. But $\text{trop}(F) = f = 0^+ w^2 \oplus 0^- w \oplus 0^+$ and $T_R(f) = \{0^+\}$. This means that the generator $\{x^2 - x + 1\}$ is not a real tropical basis of the ideal $(x^2 - x + 1)$. Let $x^3 + 1 = (x^2 - x + 1)(x + 1)$, $T_R(0^+ w^3 \oplus 0^+) = \{0^+\}$, hence $T_R(f) \cap T_R(0^+ w^3 \oplus 0^+) = \emptyset$ and $I = (x^2 - x + 1, x^3 + 1)$ is a real tropical basis of $I$. Note that, in this case, $I$ is not real radical.

The first result is that real-radical zero-dimensional ideals admit a tropical basis.

**Definition 3.3.** Let $I \subseteq \mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$. The **real radical** of $I$, $\sqrt{\mathbb{R}} I$ is the ideal

$$\sqrt{\mathbb{R}} I = \{ F \in \mathbb{K}[x] | \exists G_1, \ldots, G_r \in \mathbb{K}[x], \exists m > 0, F^{2m} + \sum_{i=1}^r G_i^2 \in I \}.$$  

By the real theorem of zeros [3],

$$\sqrt{\mathbb{R}} I = \{ F \in \mathbb{K}[x_1, \ldots, x_n] | \forall a \in V_k(I), F(a) = 0 \}$$

An ideal $I$ is **real radical** if $I = \sqrt{\mathbb{R}} I$.

**Theorem 3.4.** Let $I$ be a real radical zero dimensional ideal, with $V(I) \subseteq (\mathbb{K}^*)^n$ then $I$ admits a tropical basis.

**Proof.** Let $V = V(I) = \{ p_1, \ldots, p_r \} \subseteq (\mathbb{K}^*)^n$. Let $p_i = (p_{i1}, \ldots, p_{in})$ and $\text{trop}(p_i) = a_i = (a_{i1}, \ldots, a_{in})$, $1 \leq i \leq n$. So $\text{trop}(V) = \{ a_1, \ldots, a_r \}$. Let $F_j$ be the squarefree part of $\prod_{i=1}^n (x_j - p_{ij})$, since $I$ is real radical $F_j \in I, 1 \leq j \leq n$. The tropical roots of $\text{trop}(F_j)$ are precisely $a_{1j}, \ldots, a_{nj}$ and the number of tropical roots of each sign and valuation $a_{ij}$ can be recovered using Lemma 2.10 and taking into account that all the roots of $F_j$ are real. The set of tropical polynomials $\{ \text{trop}(F_j), 1 \leq j \leq n \}$ describe a finite set $S$ of points containing $\text{trop}(V)$.
Let $L = b_1 x_1 + \ldots + b_n x_n$ be a linear function with integer coefficients such that $L$ is injective in the set of modulus of $S$, $|S| = \{|a| \mid a \in S\}$. Let $F_0$ be the squarefree part of the numerator of the polynomial $\prod_{i=1}^{n}(x^{b_1}_i \ldots x^{b_n}_n - p^{b_1}_1 \ldots p^{b_n}_n)$. Since this is a polynomial in $I$ whose tropicalization is injective in $S$, it allows us to discriminate which modulus of points of $S$ belong to $trop(V)$ or not.

Finally, suppose that $c = (c_1, \ldots, c_n)$ is a tropical point in $S \cap T_{\mathbb{R}}(F_0)$ but not in $trop(V(I))$. This can only happen if there is a point in $a \in trop(V)$ with $b \neq a$, $|b| = |a|$ and for every index $j$ there is a point in $trop(V)$ with $j$-th coordinate $b_j$.

For every point $a_i \in trop(V)$ with $|a_i| = |c|$, there is a coordinate $h(i)$ with $s(a_{i,h(i)}) \neq s(c_{h(i)})$. Let $G_c$ be the squarefree part of

\[ \prod_{i, a_i = |c|} (x_{h(i)} - p_{i,h(i)}) \times \prod_{i, |a_i| \neq |c|} (x^{b_1}_i \ldots x^{b_n}_n - p^{b_1}_1 \ldots p^{b_n}_n) \in I \]

This polynomial vanishes by construction on every element of $trop(V)$, but it is not a real tropical root of $trop(G_c)$, because, for each factor of $G_c$ in $\mathbb{R}$, $c$ either has the wrong modulus or the wrong distribution of signs. By Lemma 2.8, $c \notin T_{\mathbb{R}}(G_c)$.

If $H$ is any generator set of $I$, then $H \cup \{F_j, 1 \leq j \leq n\} \cup \{F_0\} \cup \{G_c \mid c \in S - trop(V)\}$ is a real tropical basis of $I$. □

Example 3.5. Let $V = \{(−1, 2), (2, 3), (−3, −t), (1, −4), (2t, 4t)\} \subseteq (\mathbb{K})^2$, $trop(V) = \{(0−, 0+), (0+, 0+), (0+, 1−), (0+, 0−), (1+, 1+)\}$. Let $I = \mathcal{I}(V)$ be the ideal of $V$. We are computing a real tropical basis of $I$. First, define

\[
F_1(x) = x^5 + (-2t + 1) x^4 + (-2t - 7) x^3 + (14t - 1) x^2 + (2t + 6) x - 12t,
\]

\[
F_2(y) = y^5 + (-3t - 1) y^4 + (-4t^2 + 3t - 14) y^3 + (4t^2 + 42t + 24) y^2 + (56t^2 - 72t) y - 96t^2.
\]

$trop(F_1)$ and $trop(F_2)$ define the finite set

\[ S = \{(0+, 0+), (0+, 1+), (0+, 1−), (0+, 0−), (0−, 0+), (0−, 1+), (0−, 1−), (0−, 0−), (1+, 0+), (1+, 1+), (1+, 1−), (1+, 0−)\} \]

$|S| = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Note that $(1, 0) \notin |trop(V)|$. Let $L = x + 2y$ by a linear function injective in $|S|$. Then

\[ F_0 = (xy^2 - 18) \cdot (xy^2 - 16) \cdot (xy^2 + 4) \cdot (xy^2 + 3t^2) \cdot (xy^2 - 32t^3) \]

In usual tropical geometry $\{F_1, F_2, F_0\}$ would suffice to provide a tropical basis, See [10], [17]. But these polynomials are not enough in the real case. The elements of $S$ that are also in $T_{\mathbb{R}}(F_0)$ are $\{(0+, 0+), (0+, 0−), (0−, 0+), (0−, 1+), (0−, 1−), (0−, 0−), (1+, 1+), (1+, 1−), (1+, 0+)\}$. We have to compute polynomials to discard the points in $S$ not in $trop(V)$. Following the theorem: $G_{(0−, 0−)} = (x - 2)(y - 2)(xy^2 - 32t^3)(xy^2 + 3t^2)$, and $(0−, 0−) \notin T_{\mathbb{R}}(G_{(0+, 0+)})$. Define also $G_{(0−, 1+)} = (y + t)(xy^2 + 4)(xy^2 - 18)(xy^2 - 16)(xy^2 -$
Example 3.6. Let \( F = x^2 - (2 + t)x + 1 \). The discriminant of \( F \) is \( t^2 + 4t > 0 \), so there are no real roots of \( F \) and \( \sqrt[3]{F} = 1 \). However, \( I = \langle F \rangle \) has no real tropical basis. To see that, \( \{0^+\} \) is a root of \( \text{trop}(x^2 - (2 + t)x + 1) = 0^+x^2 \oplus 0^-x \oplus 0^+ \). Now \( v(F(1)) = v(-t) = 1 > 0 \), \( 1 \) is a real root of \( x^2 - 2x + 1 \). If \( G \in \mathbb{K}[x] \), assume without loss of generality that the minimum of the valuation of the coefficients of \( G \) is zero. Then \( 1 \) will be a root of the residue polynomial of \( F \cdot G \). This means that \( 0^+ \) will also be a tropical root of \( \text{trop}(FG) \) and \( \bigcap_{H \in I} \mathcal{T}_\mathbb{R}(H) = \{0^+\} \).

The reader may think that the failure here is due to the fact that the residue polynomial \( x^2 - 2x + 1 \) obtained by substituting \( t = 0 \) has a double root. This is true for the field of real Puiseux series but it may fail over more complicated fields.

Definition 3.7. Let \( F = \sum_{\ell \in A} p_{\ell} x^\ell \in \mathbb{K}[x_1, \ldots, x_n] \) be a polynomial, let \( w \in \mathbb{R}^n \). Let \( i_1, \ldots, i_r \) be the monomials where \( \text{trop}(F) \) attains its minimum at \( w \). We define the residue polynomial of \( F \) with respect to \( w \) to the polynomial

\[
F_w = \sum_{j=1}^r Pc(p_{i_j}) x^{i_j} \in \mathbb{R}[x_1, \ldots, x_n],
\]

where \( Pc(p_{i_j}) \in \mathbb{R} \) is the principal coefficient of the power series \( p_{i_j} \).

Proposition 3.8. Let \( F \in \mathbb{K}[x] \) be an univariate polynomial such that, for every real root \( p \in \mathbb{K}^\ast \) we have that \( F_{\text{trop}(p)} \) has only simple roots, then \( F \) admits a tropical basis.

Proof. Let \( \{p_1, \ldots, p_r\} = \tau_{\mathbb{R}}(F) \) be the real tropical roots of \( \text{trop}(F) \). If \( p_i \notin \text{trop}(\{F = 0\}) \) we have to find a certificate of this fact. Without loss of generality, we may assume that \( p_i = 0^+ \). \( F_0 \in \mathbb{R}[x] \) is an univariate polynomial that has no positive real root. From Theorem 2.4 there exists a natural \( N \) such that \( F_0 \cdot (1 + x)^N \) has all its coefficients positive. It follows that \( 0^+ \notin \tau_{\mathbb{R}}(F \cdot (1 + x)^N) \). Iterating over every element in \( \tau_{\mathbb{R}}(F) \) we can construct a real tropical basis of \( F \). \( \square \)

We now show that Proposition 3.8 cannot be extended to any real closed valued field.

Example 3.9. Let \( \mathbb{F} = \mathbb{K}\{\{s\}\} \) be the field of Puiseux series is \( s \) whose coefficients are Puiseux series in \( t \) over the reals. Take again \( F = x^2 - (2 + t)x + 1\mathbb{F}[x] \). Now, \( \text{trop}(F) \) still has a tropical root \( 0^+ \) but \( F \) has no root in \( \mathbb{F} \). By the same argument as in Example 3.6 \( F \) does not admit a tropical basis. Note that in this new context \( F = F_0 \) has no multiple root.
Let us consider now other ideals that also admit tropical basis. We can revisit combinatorial patchworking (See [19] for the details) from the point of view of tropical basis.

**Theorem 3.10.** Let \( F \in \mathbb{K}[x_1, \ldots, x_n] \) be a real polynomial and let \( f = \text{trop}(F) \) be the corresponding tropical polynomial. \( f \) defines a mixed subdivision in \( \text{Supp}(F) \) by duality. Assume that this subdivision is a triangulation that contains all the monomials in \( \text{Supp}(F) \) as vertices. Then \( \{F\} \) is a real tropical basis of \( (F) \). That is \( \mathcal{T}_\mathbb{R}(F) = \text{trop}(V(F)) \).

**Proof.** We are going to prove that if \( p \in \mathcal{T}_\mathbb{R}(F) \cap \mathbb{Q}^n \) then \( p \in \text{trop}(V(F)) \). Without loss of generality, we may assume that \( p = (0^+, \ldots, 0^+) \).

Let \( S = \text{Supp}(F) \), the set of monomials of \( S \) where \( f \) attains its minimum at \( p \) is a simplex \( C \) in \( S \) that is not a point. Since \( p \in \mathcal{T}_\mathbb{R}(f) \), then there is a positive monomial \( a \in C \) and a negative monomial \( b \in C \). Let \( L = r_0 + r_1 x_1 + \ldots + r_n x_n \) be an affine function with integer coefficients such that for all monomials \( c \in S - \{a, b\} \), \( L(a) < L(c) < L(b) \). Let \( c_0 x^a, -c_0 x^b \) be the corresponding monomials of \( F \). Consider the points of the form \( (s_1^r, \ldots, s_n^r) \). If \( 0 < s_- << 1 \) is a valuation zero positive element of \( \mathbb{K} \) small enough, then \( F(s_1^r, \ldots, s_n^r) \sim c_0 s^{L(a)} > 0 \). If \( 1 << s_+ \) is a valuation zero element big enough, \( F(s_1^r, \ldots, s_n^r) \sim -c_0 s^{L(b)} < 0 \). Hence, by the intermediate value theorem, the polynomial \( F(s_1^r, \ldots, s_n^r) \) has a root \( s_0 \) in the interval \([s_-, s_+]\), and hence \( s_0 \) is of valuation zero. By construction, \((s_0^r, \ldots, s_0^r)\) is a zero of \( F \) with tropicalization \((0^+, \ldots, 0^+)\). \( \quad \square \)

**Corollary 3.11.** Real vector hyperplanes have tropical basis. More precisely, let \( H = V(a_1 x_1 + \ldots + a_n x_n + a_0) \in \mathbb{K}[x_1, \ldots, x_n] \) be a hyperplane in \((\mathbb{K}^*)^n\). Then \( \text{trop}(H) = \mathcal{T}(\text{trop}(a_1)w_1 \oplus \ldots \oplus \text{trop}(a_n)w_n \oplus \text{trop}(a_0)) \).

**Proof.** We are in the hypothesis of Theorem 3.10 since \( \text{Supp}(H) \) is a simplex. \( \quad \square \)

The last goal of this section is to prove that linear ideals have a tropical basis (cf. [2] for the algebraically closed case). We start with a Lemma that states that there are infinite tropical basis.

**Lemma 3.12.** Let \( V \subseteq (\mathbb{K}^*)^n \) be a linear space, then
\[
\text{trop}(V) = \bigcap_{H \in \mathcal{H}} \text{trop}(H),
\]
where \( \mathcal{H} \) is the set of hyperplanes of \((\mathbb{K}^*)^n\).

**Proof.** Clearly \( \text{trop}(V) \subseteq \bigcap_{H \in \mathcal{H}} \text{trop}(H) \). We prove the equality by induction in the dimension \( n \) of the ambient space. For \( n = 2 \), \( V \) can only be a line, so the result holds by Corollary 3.11. Assume it is true for ambient dimension up to \( n - 1 \). We prove that the result is true in ambient dimension \( n \) by induction in the codimension of \( V \). If \( \text{codim}(V) = 1 \), then \( V \) is a hyperplane and this is again Corollary 3.11. Assume now that the result
is true for all codimension $r$ spaces and that $V$ has codimension $r + 1$. If $V \subseteq \{x_i = 0\}$ for some $i$ then $\text{trop}(V) = \text{trop}(\{x_i = 0\}) = \emptyset$. Let $p = (p_1, \ldots, p_n) \in \bigcap_{V \subseteq H} \text{trop}(H)$. Without loss of generality $p = (0^+, \ldots, 0^+)$. If $e_i \in V$ is the $i$-th vector of the canonical basis, the $i$-th coordinate of the points of $V$ may take any value and for all $V \subseteq H$, the $i$-th coefficient of the defining linear equation of $H$ is zero. It follows that we can project along the $i$-th coordinate and $p \in \text{trop}(V)$ iff $\pi_i(p) \in \text{trop}(\pi_i(V))$ and clearly $\pi_i(p) \in \bigcap_{V \subseteq H} \text{trop}(\pi_i(V)) = \bigcap_{V \subseteq H} \text{trop}(H)$ where $H$ is any hyperplane in dimension $n - 1$ containing $\pi_i(V)$ and we are done by induction.

Hence, we may assume that $e_i \notin V$ for $1 \leq i \leq n$. Consider the linear spaces $V^+ < e_i >, 1 \leq i \leq n$. Then $p \in \bigcap_{V^+ < e_i > \subseteq H} \text{trop}(H) = \text{trop}(V^+ < e_i >)$ by induction hypothesis in the codimension of $V$. Hence, for each $i, 1 \leq i \leq n$ there is a point $a_i = (a_{i1}, \ldots, a_{in}) \in V$ such that for all $i \neq j, a_{ij} > 0$ and $v(a_{ij}) = 0$. We can write the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{trop}(A) = \begin{pmatrix} q_1 & 0^+ & \cdots & 0^+ \\ 0^+ & q_2 & \cdots & 0^+ \\ 0^+ & 0^+ & \cdots & q_n \end{pmatrix}$$

Since $\text{codim}(V) > 1$, $\text{rank}(A) \leq n - 2$ and $A$ is singular, so it cannot happen that for all $i \mid q_i \mid < 0$. Assume that $\mid q_1 \mid \geq 0$. Since the minor $A_{11}$ is also singular, there must be another element $\mid q_1 \mid \geq 0$. Assume $\mid q_2 \mid \geq 0$. If for some $i$ is $\mid q_i \mid > 0$ let $\lambda$ be an element of valuation zero such that its residue coefficient $b$ is $0 < b << 1$ then $\lambda a_j + a_i, j \in 1, 2, j \neq i, \text{trop}(\lambda a_j + a_i) = p$. Thus, we may assume that $\mid q_i \mid \leq 0$ for all $i$. But again, since $A$ is singular, $\mid q_i \mid = 0$ for all $i$. If for some $i$ is $a_{ii} > 0$ we are done and $a_i$ is a lift of $p$. Last, we have the case that $q_i = 0^-$ for all $i$. In this case, we perform Gauss reduction on the matrix $A$. The coefficients we have to multiply $a_1$ in order to make zeros below the first column are $0 < \lambda_i, v(\lambda_i) = 0$. For all the elements $a_{ij}, i \neq 1 a_{ij} + \lambda_i a_{1j}$ is a positive element of valuation $0$. On the other hand $v(a_{ii} + \lambda a_{1i}) \geq 0$.

Now, if for some $i \neq 1, v(a_{ii} + \lambda_i a_{1i}) > 0$, let $k \notin \{1, i\}$, let $0 < \eta$ with, $v(\eta) = 0, 0 < Pc(\eta) << 1$ then $b = \eta a_k + (a_i + \lambda_i a_1)$ is a vector $b \in V$ such that $b_i > 0$ and $v(b_i) = 0$ and we are done.

Next case is if for some $i \neq 1, a_{ii} + \lambda_i a_{1i}$ is a positive element of valuation $0$, then for $0 < \mu$ of valuation $0, 1 << Pc(\mu), -a_1 + \mu(a_i + \lambda_i a_1)$ is the desired point.

Finally, if for all $i, a_{ii} + \lambda_i a_{1i}$ is a negative element of valuation $0$, then we perform Gauss reduction on the second column and repeat the process. Since $\text{rank}(A) \leq n - 2$, after a finite number of steps, we must arrive to a
matrix (after reordering the indices) of the form

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1k} & a_{1,k+1} & \ldots & a_{1n} \\
0 & b_{22} & \ldots & b_{2k} & b_{2,k+1} & \ldots & b_{2n} \\
\vdots & & & & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & b_{k-1,k-1} & b_{k-1,k} & \ldots & b_{k-1,n} \\
0 & 0 & \ldots & 0 & b_{k,k} & \ldots & b_{k,n} \\
0 & 0 & \ldots & 0 & b_{n,k} & \ldots & b_{nn}
\end{pmatrix}
\]

with real tropicalization

\[
\begin{pmatrix}
0^- & 0^+ & \ldots & 0^+ & 0^+ & \ldots & 0^+ \\
\infty & 0^- & \ldots & 0^+ & 0^+ & \ldots & 0^+ \\
\infty & \infty & \ldots & 0^- & 0^+ & \ldots & 0^+ \\
\infty & \infty & \ldots & \infty & c^+_{k,k} & \ldots & c^+_{k,n} \\
\infty & \infty & \ldots & \infty & c^+_{n,k} & \ldots & c^+_{nn}
\end{pmatrix}
\]

Such that \( k = \dim(V) \leq n - 2; b_{kk} \neq 0; b_{ij} = 0 \) if \( i > j < k; \) \( \text{trop}(b_{ij}) = 0^+, \) if \( i < j \) or \( i > j \geq k; \) \( \text{trop}(b_{ii}) = 0^- \) for \( i < k; \) vectors \( b_{k+1}, \ldots, b_n \) are all multiple of \( b_k. \) Hence, it must happen that \( \text{trop}(b_{kk}) = 0^+. \) Then, for \( \eta_2, \ldots, \eta_k \) of valuation zero and positive such that \( 1 \ll Pc(\eta_2) \ll Pc(\eta_3) \ll \ldots \ll Pc(\eta_k) \) the vector \(-b_1 - \eta_2b_2 - \ldots - \eta_{k-1}b_{k-1} + \eta_kb_k\) is a lift of \( p \) in \( V. \)

\[\square\]

**Theorem 3.13.** Let \( I \) be the ideal of an affine space \( V \) in \((\mathbb{K}^*)\). The affine polynomials in \( I \) with minimal support form a real tropical basis of \( I. \)

**Proof.** For each minimal support there is only one linear polynomial in \( I \) up to multiplication by a constant and they generate \( I. \) Call \( \{F_1, \ldots, F_r\} \) this generator set of \( I. \) Clearly, \( \text{trop}(V) \subseteq \cap_{i=1}^r \mathcal{T}_\mathbb{R}(F_i). \) To see the equality, let \( p \notin \text{trop}(V). \) Homogenizing if necessary and performing a monomial change of coordinates, we may assume, without loss of generality that \( I \) is homogeneous and that \( p = (0^+, \ldots, 0^+). \) By Lemma 3.12 there is a linear function \( F \in V \) such that \( p \notin \mathcal{T}_\mathbb{R}(F). \) It suffices to check that we can take \( F \) with minimal support. If \( F \) has no minimal support, then there exists another linear \( G \in I \) whose support is contained in the support of \( F. \) Multiply \( F \) and \( G \) by appropriate constants so that the minimum valuation of the coefficients of \( F \) and of \( G \) is zero. By substituting \( G \) with a linear combination of \( F \) and \( G \), we may also assume that the residue polynomials at \( p \) of \( F \) and \( G \) are linearly independent forms in \( \mathbb{R}[x_1, \ldots, x_n]. \) Let \( S_F \) (resp. \( S_G \)) be the variables where \( F \) (resp. \( G \)) attains the minimum at \( p. \) If there is a monomial \( g_mx_m \) in \( S_G \) that is not in \( S_F, \) we can use \( G \) to make zero the monomial \( x_m \) in \( F \) without modifying the residue polynomial of \( F \) at \( p \) and proceed recursively.
Assume then that \( S_G \subseteq S_F \), after reordering, we may assume that \( S_F = \{x_i, \ldots, x_n\} \). Let \( a_i \) (resp. \( b_i \)) be the principal coefficient \( F \) (resp. \( G \)) at the monomial \( x_i \). Let \( i \) be an index in \( S_F \) such that \(|b_i/a_i|\) is maximum of all the indices. If \( b_i > 0 \) substitute \( G \) by \(-G\) so that \( b_i < 0 \). Then, the polynomial \( H = F \cdot (-bi/ai) + G \), \( \text{Supp}(H) \subseteq \text{Supp}(F) \) and the residue polynomial of \( H \) at \( p \) has only positive coefficients. We proceed recursively until we arrive to a linear polynomial \( H' \) with minimal support in \( I \) such that \( p \notin \mathcal{T}_R(H') \).

We finish this section showing a real radical ideal that has no tropical basis. This dissonance has been studied in [18], but the example showed there is not real radical.

**Example 3.14.** Consider the cubic defined by:

\[
F = x^3 + y^3 - x^2 y - xy^2 + 2x^2 + 2y^2 + 4xy - 8x - 8y + 8
\]

The only point with positive coefficients is the singular point \((1,1)\) of valuation \((0,0)\).

The corresponding tropical polynomial is

\[
f = 0^+ v^3 \oplus 0^+ w^3 \oplus 0^- v^2 \oplus 0^- v w^2 \oplus 0^+ v^2 \oplus 0^+ w^2 \oplus 0^+ v w \oplus 0^- v \oplus 0^- w \oplus 0^+
\]

In the positive tropical quadrant the tropical polynomial defines a line-like tropical curve.

Positive part of a tropical singular cubic (left) and the set of points in every tropicalization of a polynomial in \( (F) \) (right).

We claim that the curve \( C = \{F = 0\} \) has no real tropical basis. Since the only positive point of the curve is the singular point \((1,1)\), then the only tropical positive point is the tropical point \((0^+, 0^+)\). The ideal of \( C \) is \((F)\) and this is a real radical ideal. We are showing that, for every \( a < 0 \), \((a^+, a^+) \in \mathcal{T}_R(FG)\) for every \( G \neq 0, G \in \mathbb{K}[x, y] \).

Let \( G \) be any real polynomial and \( a < 0 \). Let \( G_a \) be the residue polynomial of \( G \) at the point \((a, a)\). Then, \((FG)_{(a,a)} = (x^3 + y^3 - 2xy^2 - 2x^2y)G_a = (x+y)(x-y)^2G_a \). Since \((1,1)\) is a root of this polynomial, there must be two monomials in \((FG)_{(a,a)}\) with different sign where \((a, a)\) attains its minimum. That is \((a^+, a^+) \in \mathcal{T}_R(FG)\).

On the other hand, let us take a look at the points of the form \((a^+, 0^+)\) and \((0^+, a^+)\) with \( a > 0 \). The monomials where the minimum of \((0, a)\) is attained are \( F_{(0,a)} = x^3 + 2x^2 - 8x + 8 \). This polynomial have no positive root, hence, by Pólya’s theorem \( F_{(0,a)}(1 + x)^n \) has only positive monomials.

\[
(F(1 + x)^{11})_{(0,a)} = x^{14} + 13x^{13} + 69x^{12} + 195x^{11} + 308x^{10} + 242x^9 + 66x^8 + 198x^7 + 825x^6 + 1441x^5 + 1441x^4 + 903x^3 + 354x^2 + 80x + 8
\]
Hence, in the tropical positive quadrant
\[ \bigcap_{G} T_{\mathbb{R}}(FG) = \{(a^+, a^+) | a \leq 0\} \neq \text{trop}(C) = \{(0^+, 0^+)\}\]

4. THE REAL TROPICAL DISCRIMINANT

In this section we extend the study of tropical singularities and discriminants \([5, 6, 12]\) to the real case.

Consider the algebraic closed field \(\mathbb{K}[i]\). Let \(A \subseteq \mathbb{Z}^n\). \(|A| = d\) such that \(<A> = \mathbb{Z}^n\). Consider the Laurent polynomial ring \(\mathbb{K}[i][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\).

In this ring we have the \(\mathbb{K}[i]\)-linear space of polynomials with support \(A\), \(F = \sum_{i \in A} a_ix^i\). We may identify any such polynomial with the point \((a_i | i \in A) \in \mathbb{K}[i]^d\). Consider the incidence variety
\[ H = \{(F, u) \in (\mathbb{K}[i]^*)^d \times (\mathbb{K}[i]^*)^n | F \text{ is singular at } u\}\]

of singular hypersurfaces of support \(A\) and singular points. This is a \(\mathbb{Q}\)-defined variety. Consider the projections \(\pi_1: (\mathbb{K}[i]^*)^d \times (\mathbb{K}[i]^*)^n \rightarrow (\mathbb{K}[i]^*)^d\) and \(\pi_2: (\mathbb{K}[i]^*)^d \times (\mathbb{K}[i]^*)^n \rightarrow (\mathbb{K}[i]^*)^n\). \(\pi_1(H)\) is the \(A\)-discriminant variety of hypersurfaces with support \(A\) and a singular point, while \(\pi_2(H) = (\mathbb{K}^*)^n\), and the fiber over any point is a linear space isomorphic to \(\pi_{A}^{-1}(1, \ldots, 1) = \{F | F \text{ is singular at } (1, \ldots, 1)\}\). Our aim is to describe the real tropicalization of \(\pi_1(H)\).

**Definition 4.1.** Let \(f \in T_{\mathbb{R}}[x_1, \ldots, x_n]\) be a tropical polynomial with signs. Let \(p = (p_1, \ldots, p_n) \in T_{\mathbb{R}}(f)\). We say that \(p\) is a singular point of \(f\) if there exists a pair \(F \in \mathbb{K}[x_1, \ldots, x_n]\) and \(P \in \mathbb{K}^n\) such that \(P\) is a singular point of \(\{F = 0\}\), \(\text{trop}(F) = f\) and \(\text{trop}(P) = p\).

The definition of tropical Euler derivative \([6]\) can be easily extended to tropical polynomials with signs.

**Definition 4.2.** Let \(f = \bigoplus_{i \in A} a_{i}w^\ell \in T_{\mathbb{R}}[x_1, \ldots, x_n]\) and \(<A> = \mathbb{Z}^n\) be a tropical polynomial with signs. Let \(L = b_0 + b_1w_1 + \ldots + b_nw_n\) be an affine function with integer coefficients. The **Euler derivative** of \(f\) with respect to \(L\) is
\[ \bigoplus_{\ell \in A \atop L(\ell) \neq 0} s(L(\ell))a_{i}w^\ell.\]

We eliminate all the monomials in the support where \(L\) vanish and swap signs of the monomials where \(L\) is negative. If \(F \in \mathbb{K}[x_1, \ldots, x_n]\) is any polynomial such that \(\text{trop}(F) = f\) then
\[ \frac{\partial f}{\partial L} = T \left( \frac{\partial F}{\partial L} \right) = T \left( b_0F + b_1x_1 \frac{\partial F}{\partial x_1} + \ldots + b_nx_n \frac{\partial F}{\partial x_n} \right)\]

**Example 4.3.** Let \(f = 0^+ \oplus 0^+w_1 \oplus 0^+w_2 \oplus 0^+w_1^2 \oplus 0^+w_1w_2 \oplus 0^+w_2^2\). Then \(\frac{\partial f}{\partial w_{1-w_2}} = 0^+w_1 \oplus 0^-w_2 \oplus 0^+w_2^2 \oplus 0^-w_2^2\).

We have the following consequence of Theorem \([6,13]\)
Theorem 4.4. Let $f$ be a tropical polynomial with signs. The set of tropical singularities of $\mathcal{T}_R(f)$ is
\[
\bigcap_L \mathcal{T}_R \left( \frac{\partial f}{\partial L} \right)
\]

Proof. If $p$ is a singularity of $\mathcal{T}_R(f)$, then let $P \in (\mathbb{K}^*)^n$ and $F \in \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial, with $\text{trop}(P) = p$, $\text{trop}(F) = f$, and $P$ is in the singular locus of $\{F = 0\}$. Then, for any $L$, $\frac{\partial F}{\partial L}(P) = 0$. So $p \in \mathcal{T}_R(\frac{\partial f}{\partial L})$.

Conversely, let $p \in \bigcap_L \mathcal{T}_R \left( \frac{\partial f}{\partial L} \right)$. Without loss of generality, assume that $p = (0^+, \ldots, 0^+)$. The set tropical polynomials with signs having a singularity at $p$ and support $A$ is a linear space that is the tropicalization of the linear system $H_1$ of polynomials with support $A$ in $\mathbb{K}[x_1, \ldots, x_n]$ having a singularity at $P_1 = (1, \ldots, 1)$. Let $F = \sum_{\ell \in A} a_{\ell} x^\ell$ be the generic polynomial with indeterminates coefficients and support $A$. The linear system $H_1$ is generated by $F(P_1), (x_1 \frac{\partial F}{\partial x_1}(P_1)), \ldots, (x_n \frac{\partial F}{\partial x_n}(P_1))$. By Lemma 3.12
\[
\text{trop}(H_1) = \bigcap_L \mathcal{T}_f \left( \frac{\partial F}{\partial L} \right) = \bigcap_L \mathcal{T}_f \left( \frac{\partial f}{\partial L} \right)
\]

Hence, there exists $F \in H_1$ with a singularity at $(1, \ldots, 1)$ and $p$ is a singularity of $f$. \hfill \Box

Definition 4.5 (cf. [2, 5, 12, 6]). Let $p$ be a point in $\mathcal{T}_R(f)$. We define the flag of $f$ with respect to $p$ as the flag of subsets $F(p)$ of $A$ defined inductively by: $F_{-1} = \emptyset \subseteq F_0(p) \subseteq F_1(p) \subseteq \ldots \subseteq F_r(p)$, $\dim < F_r(p) \geq n$, and for any $\ell$: $F_{\ell+1}(p) - F_\ell(p)$ is the subset of $A_{\ell} < F_\ell(p) >$ where the tropical polynomial $\oplus_{j \in A_{\ell} < F_\ell(p)} a_j w^j$ attains its minimum at $p$. The weight class of the flag $F(p)$ are all the points $p' \in \mathcal{T}_R(f)$ for which $F(p') = F(p')$. If $p$ and $p'$ are in the same weight class and $s(p) = s(p')$ then $p$ is singular if and only if $p'$ is singular.

Let $f \in \mathcal{T}(w_1, \ldots, w_d)$ and $p = (p_1, \ldots, p_d) \in \mathbb{R}$ be a singular point of the usual tropical variety defined by $f$ forgetting signs (refer to [6]. We want to know if $p^+ = (p_1^+, \ldots, p_d^+)$ is a singular point of $\mathcal{T}_R(f)$. Let $F_{-1} = \emptyset \subseteq F_0(p) \subseteq F_1(p) \subseteq \ldots \subseteq F_r(p)$ be the weight class of $p$. For any $F_i$, write $F_i = F_i^+ \cup F_i^-$, where $F_i^+$ (resp. $F_i^-$) are the monomials whose corresponding tropical coefficient is positive (resp. negative).

Definition 4.6. Let $A, B$ disjoint sets. Let $L$ be a hyperplane, we say that $L$ separates $A$ and $B$ if $L$ does not contain $A \cup B$ and the sets are contained in different closed halfspaces defined by $L$. That is $(A \cup B) \not\subseteq L$, $A \subseteq L_{\geq 0}$ and $B \subseteq L_{\leq 0}$ (or $A \subseteq L_{\leq 0}$ and $B \subseteq L_{\geq 0}$).

Note, that, for example, if $A \subseteq L$ and $B \subseteq L_{\geq 0}$ then $L$ separates $A$ and $B$. Also, if $A = \emptyset$, any $L$ not containing $B$ separates $A$ and $B$.

Theorem 4.7. With the previous notation, let $p \in \mathcal{T}^n$ be a point with all its coordinates positive. Then, $p$ is a singular point of $\mathcal{T}_R(f)$ if and only
if for all $i = 0, \ldots, r - 1$ and for all hyperplane $L$ such that $F_{i-1}(p) \subseteq L$, $F_i(p) \not\subseteq L$, $L$ does not separate $F_i^+$ and $F_i^-$. 

Proof. Let $L$ be a hyperplane such that $F_{i-1}(p) \subseteq L$ and $L$ separates $F_i^+$ and $F_i^-$. Then, the set of monomials where $\frac{\partial f}{\partial L}$ attains its minimum at $p$ is $F_i(p)$. But, since $L$ separates $F_i^+$ and $F_i^-$, all the monomials in $F_i(p)$ yield the same sign, so $p \notin T_R(\frac{\partial f}{\partial L})$ and $p$ is not singular.

Assume now that $p$ is a singular point. Then, there exists a $L$ such that $p \notin T_R(\frac{\partial f}{\partial L})$. The monomials where the minimum of $\frac{\partial f}{\partial L}$ is attained at $p$ is one of the sets of the flag $F_i(p)$. Since $p$ is not in the tropical variety defined by this derivative, it follows that $L$ separates $F_i^+$ and $F_i^-$. □

Example 4.8. Consider the real tropical cubic

$$f = 0^+ v^3 \oplus 0^+ w^3 \oplus 0^- v^2 w \oplus 0^- w^2 \oplus 0^+ v^2 \oplus 0^+ w^2 \oplus 0^+ vw \oplus 0^- v \oplus 0^- w \oplus 0^+$$

The weight classes in the positive orthant are: $\{(0^+,0^+)\}$, $\{(a^+,a^+)\} | a < 0 \}$, $\{(0^+,a^+)\} | a > 0 \}$, $\{(a^+,0^+)\} | a > 0 \}$. The points in the weight class $\{(a^+,a^+)\} | a < 0 \}$ are not singular, since they do not belong to the real tropical variety defined by $\frac{\partial f}{\partial v}$. The points of the form $\{(0^+,a^+)\} | a > 0 \}$ are all singular, $F_o^+ = \{1, v^2, v^3\}$, $F_o^- = \{v\}$. No $L$ separates $F_o^+$ and $F_o^-$ and $(0^+,a^+) \in \mathbb{T}^2$. Similarly $\{(a^+,0^+)\} | a > 0 \}$ are all singular points. Finally the point $(0^+,0^+)$ is also singular, as shown in Example 3.14.

Example 4.9. Let $p = (0^+,0^+) \in \mathbb{T}^2$. We would like to describe the real tropical curves with a singularity at $p$. We only describe its maximal cones. Such a description in the complex case appears in [1] so we just discuss which are the valid distribution of signs on the coefficient of the tropical polynomial $f$. Up to symmetry and swap of signs the (maximal) combinatorial types of polynomials with a singularity at $p$ are.

(1) $F_0(p)$ is a two-dimensional circuit (four points, each three affinely independent). Then, the positive and negative monomials of the circuit cannot be separated by a line.

(2) $F_0(p)$ is a one-dimensional circuit (three collinear points) contained in a line $L$. The vertices of the circuit are of the same sign and the interior monomial is of the opposite sign. In this case, $F_1$ consists on two monomials, we distinguish two cases:

(a) The monomials of $F_1$ lie on different halfspaces with respect to $L$. In this case, the two monomials have the same sign.

(b) Both monomials of $F_1$ lie on the same halfspace with respect to $L$. In this case, the two monomials have different signs.
Figure 2. Possible distribution of signs on a circuit of dimension 2.

Figure 3. Possible distribution of signs around a circuit of dimension 1. The monomials in $\mathcal{F}_1$ are the white circles.

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