Branched Coverings and Interacting Matrix Strings in Two Dimensions

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Abstract

We construct the lattice gauge theory of the group $G_N$, the semidirect product of the permutation group $S_N$ with $U(1)^N$, on an arbitrary Riemann surface. This theory describes the branched coverings of a two-dimensional target surface by strings carrying a $U(1)$ gauge field on the world sheet. These are the non-supersymmetric Matrix Strings that arise in the unitary gauge quantization of a generalized two-dimensional Yang-Mills theory. By classifying the irreducible representations of $G_N$, we give the most general formulation of the lattice gauge theory of $G_N$, which includes arbitrary branching points on the world sheet and describes the splitting and joining of strings.

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1 Introduction

The relation between two dimensional Yang-Mills theories in the large $N_c$ limit and two dimensional string theories was established by Gross and Taylor in a series of papers [1, 2, 3]. They showed that the coefficients of the $1/N_c$ expansion of the YM partition function, with gauge group $SU(N_c)$ on a Riemann surface $\mathcal{M}$ of area $A$ and arbitrary genus $G$ count the number of string configurations without folds, namely the number of branched coverings of the surface. As a matter of fact two dimensional YM does not give exactly a pure theory of coverings for two reasons: the presence of two distinct chiral sectors, which are weakly coupled by pointlike tubes and correspond to the two possible orientations of the world sheet, and the presence, for $G > 1$, of the so called $\Omega^{-1}$ points, whose geometric meaning was eventually clarified in [4, 5].

A lattice gauge theory of complex $N_c \times N_c$ matrices that describes, in the large $N_c$ limit, a pure theory of branched coverings with just one chiral sector and without $\Omega^{-1}$ points was later discovered in [6, 7]. In fact, when expanded in powers of $q = e^{-A}$, the partition function of this complex matrix model can be regarded as the generating function of the number of branched coverings $Z_{G,N}$ that wrap around $\mathcal{M}$ $N$ times. Actually for finite $N_c$ only the first $N_c$ coefficient of the expansion correctly reproduce the corresponding $Z_{G,N}$ and the generating function is only reproduced in the large $N_c$ limit. A similar property holds for two dimensional YM theories [6].

For fixed $N$, the number of branched coverings $Z_{G,N}$ is itself the partition function of a lattice gauge theory [2, 6, 7] whose gauge group is the symmetric group $S_N$ and whose plaquette action is the standard heat-kernel action for $S_N$:

$$ e^{-S_{pl}(P,A_{pl})} = \frac{1}{(N!)^2} \sum_r d_r \text{ch}_r(P) e^{A_{pl} g_r} , $$

where $P \in S_N$ and $r$ labels the representations of $S_N$ of characters ch$_r$ and dimensions $d_r$; $A_{pl}$ is the area of the plaquette and the arbitrary coefficients $g_r$ carry all the information about the density of all different types of branch points. The relation between the lattice matrix theory of [7] and the $S_N$ gauge theory is a direct consequence of Frobenius formula, as shown in [6, 7].

A completely different way of obtaining a string theory from generalized two dimensional YM theories was proposed in [9, 10]. The mechanism is analogue to the one through which string configurations emerge in "Matrix string theory" [11] and it is based on the quantization of a generalized YM theory in a unitary gauge. More precisely, one starts from the generalized action

$$ S_{gen} = \int_{\mathcal{M}} d\mu V(B) - i \text{tr} \int_{\mathcal{M}} (dA - iA \wedge A)B , $$


\footnote{Generalized YM theories have been discussed in [12, 13].}
where the gauge group is $U(N)$ and $V(B)$ is an arbitrary gauge invariant potential of the $N \times N$ hermitian matrix $B$. One chooses the unitary gauge in which the matrix $B$ is diagonal. This gauge choice is incomplete, as it leaves a residual $U(1)^N$ gauge invariance. It is also affected by Gribov ambiguities as at each point of the space-time manifold the gauge is determined up to an arbitrary permutation of the eigenvalues of $B$. A global smooth diagonalization is then in general not possible because, as one goes round a loop non contractible to a point, the eigenvalues of $B$ may be subjected to a permutation. This gives origin to twisted sectors, which are in one-to-one correspondence with the homomorphisms $\Pi_1(x, \mathcal{M}) \to S_N$ of the homotopy group of $\mathcal{M}$ onto the symmetric group $S_N$, namely with the unbranched $N$-coverings of $\mathcal{M}$.

In four space-time dimensions unitary gauges lead to divergences when two eigenvalues coincide, and the theory is apparently non-renormalizable \cite{24} in these gauges. In two dimensions these divergences still occur, but they are exactly cancelled in flat space-time. In fact there is an exact cancellation, due to a fermionic symmetry, between the contributions of the non-diagonal components of the gauge field and the ghost-antighost system \cite{25}. This fermionic symmetry is however anomalous in presence of space-time curvature, and it appears that the only consistent way to eliminate the resulting divergences is to make use of the arbitrariness of the potential $V(B)$ in \cite{26} and introduce in it a term, which depends on the curvature, that cancels exactly the anomaly. With this prescription the theory becomes completely abelian, consisting of $N$ $U(1)$ gauge theories, whose field strengths are the eigenvalues of $B$, and which are only coupled, in each twisted sector, by the boundary conditions induced by the homomorphism $\Pi_1(x, \mathcal{M}) \to S_N$. The original generalized YM theory, with $U(N)$ gauge group, is then described by a string theory (unbranched coverings) that covers $\mathcal{M}$ $N$ times, and has a $U(1)$ gauge theory on its world sheet.

As shown in \cite{10}, this theory can be formulated as a lattice gauge theory whose gauge group $\mathcal{G}_N$ is the semi-direct product of $S_N$ and $U(1)^N$ defined by the multiplication rule

$$(P, \varphi) (Q, \theta) = (PQ, \varphi + P\theta),$$

where $(P, \varphi)$ can be represented as the $N \times N$ matrix

$$(P, \varphi)_{ij} = e^{i\varphi_i \delta_j} \delta_{iP(j)}$$

and $P$ is an element of $S_N$. The twisted sectors considered in \cite{10} did not include the possibility of branch points. Correspondingly, the plaquette action in the $\mathcal{G}_N$ lattice gauge theory had the form

$$e^{-S_{pl}} = \delta(P_{pl}) \sum_{n_i} \exp \left\{ \sum_{i=1}^{N} \left( in_i \varphi_i^{(pl)} - A_{pl} v(n_i) \right) \right\},$$

\footnote{We denote here the number of colours by $N$ as in this case it coincides with the number of times the world sheet of the string wraps round the target space. This marks an important difference between this approach and the one of \cite{26} or \cite{25}.}
where \( v(n_i) \) is the remaining part of \( V(B) \) after the anomaly cancellation. The plaquette action for the \( S_N \) subgroup in (3) is just a delta function \( \delta(P_{\mu}) \); this denotes the absence of branch points.

The problem of introducing twisted sectors that allow branch points, so that the strings carrying the \( U(1) \) field can join and split, will be addressed in this paper. From the point of view of the lattice gauge theory of \( G_N \) this is equivalent to writing the most general heat kernel action for \( G_N \) (as it is done in eq. (1) for the group \( S_N \)). This entails finding an explicit expression for the characters of the irreducible representations of \( G_N \) in terms of the characters of \( S_N \) and \( U(1)^N \). From the point of view of the (generalized) Yang-Mills theory, the string interactions should originate from non-perturbative effects, in analogy to what happens in the supersymmetric case of \( N = 8 \) super Yang–Mills theory. In this case the string theory that arises in the diagonal gauge is the Matrix String of [11], which is believed to be equivalent to the Type IIB string theory in the light-cone frame. The string interactions [14, 15, 16, 17, 18] arise precisely because there are classical instantonic configurations [19, 20, 21] which lead to a structure of eigenvalues describing a branched covering of the target spacetime manifold. The string coupling constant \( g_s \) is, consistently with this picture, proportional to \( 1/g_{YM}^2 \).

In our case it is not yet clear how to explicitly derive analogous instantonic configurations for the generalized Yang-Mills theory; nevertheless we can investigate the interacting theory, i.e. the theory of branched coverings, irrespectively of its dynamical origin.

2 \( N \)-coverings as \( S_N \) gauge theory

In this Section we will show that the partition function of \( N \)-coverings of a general Riemann surface can be expressed as the partition function of a lattice gauge theory, defined on a cell decomposition of the surface, with the symmetric group \( S_N \) as the gauge group.

A covering of a target Riemann surface \( \Sigma_T \) is essentially a smooth map

\[
    f : \Sigma_W \to \Sigma_T
\]

from a covering surface \( \Sigma_W \) (the “world-sheet”) to \( \Sigma_T \) (the “target”) such that each point of \( \Sigma_T \) has exactly \( N \) counterimages in \( \Sigma_W \). Each of these counterimages is said to belong to a different sheet of the covering surface. Consider now a path \( \gamma \) in \( \Sigma_T \), from a point \( P_1 \) to a point \( P_2 \). It is natural to define the \( N \) liftings \( \tilde{\gamma}_i \) of \( \gamma \) as the paths on \( \Sigma_W \) such that

\[
    f \circ \tilde{\gamma}_i = \gamma .
\]

If the liftings of all closed, homotopically trivial paths are closed, the covering is said unbranched. Otherwise it has branch points: these are points in \( \Sigma_T \) such that

\[
    h \in \mathbb{C} \setminus \frac{1}{g_{YM}^2} .
\]

Here the field which are diagonalized in the Yang-Mills strong coupling limit are the eight scalar superpartners of the gauge field.
Figure 1: Dual lattice: the solid lines are the links of the original lattice and the dashed lines the links of the dual lattice. The plaquettes $p_1$ and $p_2$ are joined by a dual link to which a permutation $Q \in S_N$ is associated: $Q$ dictates how to glue together copies of $p_1$ to copies of $p_2$ to construct the covering surface.

a closed loop around them, when lifted, goes from one sheet of the coverings to a different one. Therefore each branch point is associated to a non trivial permutation of the $N$ sheets of the coverings.

Consider now a cell decomposition of the target surface $\Sigma_T$, made of $N_0$ sites, $N_1$ links and $N_2$ plaquettes, so that the Euler formula gives the genus $G$ as

$$N_0 - N_1 + N_2 = 2 - 2G.$$  

To construct a branched $N$-covering of this discretized target surface, consider $N$ copies of each plaquette $p$ of $\Sigma_T$: these will be the plaquettes of $\Sigma_W$. Consider now two neighboring plaquettes $p_1$ and $p_2$ in $\Sigma_T$, let $l$ be their common link, and glue each of the $N$ copies of $p_1$ to one of the $N$ copies of $p_2$ along $l$ according to a permutation $P \in S_N$. Repeat the procedure for all links to construct the discretized version of the covering surface $\Sigma_W$.

In this way one associates a permutation $P \in S_N$ to each link of $\Sigma_T$. It is easier to visualize $P$ as associated to an oriented link of the dual lattice, that is the one in which sites and plaquettes are exchanged, as shown in Fig. [1]. A dual link goes from a plaquette $p_1$ to a plaquette $p_2$ and its associated permutation dictates how to glue copies of $p_1$ to copies of $p_2$.

A closed path around a site $s$ of $\Sigma_T$ will be represented in the discretized version as a plaquette of the dual lattice. If the ordered product of the permutations around the plaquette is not the identical permutation, then $s$ is a branch point of our covering: the liftings of the closed path to $\Sigma_W$ are not all closed, as some of them
Figure 2: Left: One-plaquette discretization of the torus. The solid lines are the original lattice links exiting from the site $s$. The dashed lines are the dual links, to which the permutations $P$ and $Q$ are associated. A closed path around the site $s$ gives rise to the permutation $PQP^{-1}Q^{-1}$, so that if this is not the identical permutation then $s$ is a branch point. Right: Integration over internal (dual) links.

start on a sheet and end on a different one. Therefore in our discretization the branch points of the coverings are localized on the lattice sites.

Now we want to construct a model in which all possible branched $N$-coverings of a given (discretized) target surface are counted with Boltzmann weights depending on their branch point structure. The previous discussion suggests that this model can be written as a lattice gauge theory with $S_N$ as the gauge group.

The possible types of branch points are in one-to-one correspondence with the conjugacy classes of permutations, that is with the partitions of $N$. To each conjugacy class we want to assign an arbitrary, positive Boltzmann weight. Therefore we must construct a lattice gauge theory of the symmetric group, defined on the discretized target surface: the Boltzmann weight of a configuration will be

$$\prod_s w_s(P_s),$$

where the product is extended to all the sites, $P_s \in S_N$ is the ordered product of the permutations on the dual links around the site $s$, that is around the dual plaquette corresponding to the site $s$ and the weights $w_s(P_s)$ vary in general from site to site and depend on the conjugacy class of $P_s$ only, namely:

$$w_s(P) = w_s(QPQ^{-1}) \quad \forall P, Q \in S_N.$$
The partition function of our model is then
\[ Z_N = \left( \frac{1}{N!} \right)^{N_2} \sum_{\{P\}} \prod_s w_s(P_s) , \] (11)

where the sum is extended to all the configurations, that is to all ways of assigning a permutation to each link, and the product to all sites. The normalization takes into account the arbitrary relabelings of the sheets on top of each plaquette.

This partition function can be computed by first expanding the class function \( w_s(P) \) in characters of the symmetric group:
\[ w_s(P) = \frac{1}{N!} \sum_r d_r \tilde{w}_s(r) \text{ch}_r(P) \] (12)
\[ \tilde{w}_s(r) = \sum_{P \in S_N} \frac{\text{ch}_r(P)}{d_r} w_s(P) \] (13)

We then follow the standard method used for solving two-dimensional lattice gauge theory, first introduced in Ref. [22, 23]. That is, we use the orthogonality and completeness properties of the characters (already implicitly used in deriving (12) and (13)), which in the case of the symmetric group read:
\[ \sum_r \text{ch}_r(Q)s \text{ch}_r(P) = \sum_R \delta(\text{PR}Q)\text{ch}_R(P) \] (14)
\[ \frac{1}{N!} \sum_{P \in S_N} \text{ch}_r(P_1 P) \text{ch}_r(P^{-1} P_2) = \delta_{rr} \frac{\text{ch}_r(P_1 P_2)}{d_r} \] (15)
\[ \frac{1}{N!} \sum_{P \in S_N} \text{ch}_r(P P_1 P^{-1} P_2) = \frac{\text{ch}_r(P_1) \text{ch}_r(P_2)}{d_r} \] (16)

These properties can be used to integrate over all the internal dual links of the discretized surface. Suppose for example we want to integrate over \( Q \) as shown in Fig. 2(right). Using the character expansion Eq. (12) and the orthogonality property Eq. (15), we find
\[ \sum_Q w_{s_2}(P_1 P_2 Q^{-1})w_{s_1}(Q P_3 P_4) = \frac{1}{N!} \sum_r d_r \tilde{w}_{s_1}(r) \tilde{w}_{s_2}(r) \text{ch}_r(P_1 P_2 P_3 P_4) \] (17)

In this way one can integrate over all the internal links, and end up with an effective one-site model, whose form depends on the topology of the target surface. Let us consider first the partition function on the disk: if \( P \) is the permutation around the boundary we get
\[ Z_{N, \text{disk}}(P) = (N!)^{-2} \sum_r d_r \text{ch}_r(P) \prod_{s=1}^{N_0} \tilde{w}_s(r) \] (18)
From this expression one can calculate the partition function for a surface without boundaries and arbitrary genus $G$. This is done by representing the surface as a polygon with sides suitably identified:

$$Z_{N,G} = \sum_{P_1, Q_1, \ldots, P_G, Q_G} Z_{N, \text{disk}} \left( P_1 Q_1 P_1^{-1} Q_1^{-1} \cdots P_G Q_G P_G^{-1} Q_G^{-1} \right).$$  \hspace{1cm} (19)

At this point one can use Eqs. (16,17) to perform the sum over $P_1, Q_1, \ldots, P_G, Q_G$ and obtain the final form of the partition function:

$$Z_{N,G} = (N!)^{2G-2} \sum_r d_r^{2-2G} \prod_{s=1}^{N_0} \tilde{w}_s(r)$$ \hspace{1cm} (20)

This expression allows one to calculate the number of coverings with any prescribed branch point number and structure. As an example, let us consider the limiting case of unbranched $N$-coverings: this is obtained by choosing

$$w_s(P) = \delta(P) \quad \forall s$$ \hspace{1cm} (21)

In this case eq. (13) gives simply

$$\tilde{w}_s(r) = 1$$ \hspace{1cm} (22)

so that

$$Z_{N,G} = \sum_r \left( \frac{d_r}{N!} \right)^{2-2G}$$ \hspace{1cm} (23)

and in particular for the torus we obtain the well-known result

$$Z_{N,1} = \sum_r 1 = p(N),$$ \hspace{1cm} (24)

where $p(N)$ is the number of partitions of $N$.

The previous example can be generalized to coverings with any branch point structure by associating to each site $s$ a permutation $Q_s$ and choosing the Boltzmann weight $w_s(P)$ of the form:

$$w_s(P) = \delta(P, Q_s) = \frac{1}{N!} \sum_{R \in S_N} \delta(P R Q_s R^{-1}) ,$$ \hspace{1cm} (25)

which means that $w_s(P)$ is different from zero only if $P$ is in the same conjugacy class as $Q_s$. From (23) and the orthogonality of the characters we get:

$$\tilde{w}_s(r) = \frac{\text{ch}_r(Q_s)}{d_r} .$$ \hspace{1cm} (26)

\[^4\text{This partition function has been studied already in the literature, for instance in connection with the interpretation of two dimensional Yang-Mills theories as string theories.} \]
By inserting (26) into (18) and (20) we obtain:

\[
Z_{N,\text{disk}}(P, \{Q_s\}) = \frac{1}{N!^2} \sum_r d_r \text{ch}_r(P) \prod_s \left( \frac{\text{ch}_r(Q_s)}{d_r} \right)
\]

(27)

and

\[
Z_{N,G}(\{Q_s\}) = \sum \left( \frac{d_r}{N!} \right)^{2-2G} \prod_s \left( \frac{\text{ch}_r(Q_s)}{d_r} \right).
\]

(28)

The r.h.s. of both eq. (27) and eq. (28) depend only on the equivalence classes of \(Q_s\) and ultimately on how many branch points of each equivalence class are present on the surface. Let us then denote by \(\hat{Q}\) the equivalence classes, and by \(p_{\hat{Q}}\) the number of branch points in the equivalence class \(\hat{Q}\); then eqs. (27) and (28) can be rewritten as:

\[
Z_{N,\text{disk}}(P, p_{\hat{Q}}) = \frac{1}{N!^2} \sum_r d_r \text{ch}_r(P) \prod_{\hat{Q} \neq \hat{1}} \left( \frac{\text{ch}_r(\hat{Q})}{d_r} \right)^{p_{\hat{Q}}};
\]

(29)

\[
Z_{N,G}(p_{\hat{Q}}) = N!^{2G-2} \sum_r d_r^{2-2G-p} \prod_{\hat{Q} \neq \hat{1}} \left( \frac{\text{ch}_r(\hat{Q})}{d_r} \right)^{p_{\hat{Q}}},
\]

(30)

where \(p = \sum_{\hat{Q} \neq \hat{1}} p_{\hat{Q}}\) is the total number of branch points. In eqs. (29) and (30) the numbers \(p_{\hat{Q}}\) of branched points of a given type are kept fixed. Suppose now to consider the numbers \(p_{\hat{Q}}\) as additional degrees of freedom, and to let them vary. This leads us to consider a partition function, that in a broad sense we can call grand-canonical, given by

\[
Z_{N,\text{disk}}(P, \mathcal{A}) = \sum \{p_{\hat{Q}}\} \mathcal{A}^p \prod_{\hat{Q} \neq \hat{1}} \frac{\sigma_{\hat{Q}} g_{\hat{Q}}}{p_{\hat{Q}}!} Z_{N,\text{disk}}(P, p_{\hat{Q}}),
\]

(31)

where \(\mathcal{A}\) is the area of the disk, \(\sigma_{\hat{Q}}\) the number of permutations in the conjugacy class \(\hat{Q}\), and \(g_{\hat{Q}}\) a Boltzmann weight, referred to the unit area, attached to a branch point characterized by a permutation \(Q \in \hat{Q}\). The factor \(p_{\hat{Q}}!\) accounts for the fact that branch points in the same conjugacy class are indistinguishable. The sum is over all the \(p_{\hat{Q}}\) with \(\hat{Q} \neq \hat{1}\). It can be done explicitly and gives

\[
Z_{N,\text{disk}}(P, \mathcal{A}) = \frac{1}{N!^2} \sum_r d_r \text{ch}_r(P) e^{\mathcal{A} g_r},
\]

(32)

with

\[
g_r = \sum_{Q \neq 1} g_Q \frac{\text{ch}_r(Q)}{d_r},
\]

(33)

where of course \(g_Q = g_{\hat{Q}}\) if \(Q \in \hat{Q}\). In eq. (33) the sum over the conjugacy classes \(\hat{Q}\) has been replaced by the sum over the group elements, using the fact that according
to the definition of $\sigma_{\tilde{Q}}$ we have $\sum_{\tilde{Q}} \sigma_{\tilde{Q}} \equiv \sum_{\tilde{Q}}$. The quantity $g_r$ in (32) can be thought of as an arbitrary function of the representation $r$.

The partition function $Z_{N,\text{disk}}(P, A)$ can be used as the building block of a different lattice gauge theory of the group $S_N$. Consider once again a cell decomposition of a Riemann surface of genus $G$, and associate an element of $S_N$ to each link. Let us now associate to each plaquette $\alpha$ of area $A_\alpha$ in our cell decomposition a Boltzmann weight given by $Z_{N,\text{disk}}(P_\alpha, A_\alpha)$, where $P_\alpha$ is the ordered product of the link elements around $\alpha$. Unlike the original theory defined in (11), this theory has no branch points on its sites, but a dense distribution of branch points (characterized by $g_r$ or equivalently by $g_{\tilde{Q}}$) on each plaquette. The integration over the link variables can be performed again by means of the orthogonality properties of the characters, and the result for a Riemann surface of genus $G$ is

$$Z_{N,G}(A) = \sum_r \left( \frac{d_r}{N!} \right)^{2-2G} e^{A g_r} .$$

We can regard the theory introduced at the beginning of this section, with the branch points at the sites of the cell decomposition as a “microscopic” theory, and the one defined by the partition function (34) as a “continuum” limit. The theory characterized by the plaquette action (32) has the same structure as a generalized two dimensional YM theory, but with $S_N$ as gauge group. As a special case we can consider the one where all branch points are quadratic. This is obtained from (33) by choosing

$$g_r = g \xi_r^2 \equiv g \frac{N(N-1)}{2} \frac{\text{ch}(2^1)}{d_r} ,$$

where $2^1$ donotes the conjugacy class consisting of just one exchange. $\xi_r^2$ can be expressed in terms of the lengths $m_\alpha$ and $n_\beta$ of the rows and the columns of the Young tableau labeling the representation $r$ as $\xi_r^2 = 1/2(\sum_{\alpha} m_\alpha^2 - \sum_{\beta} n_\beta^2)$. It is related to the quadratic Casimir of the representation of a unitary group corresponding to the same Young tableau.

The statistics of branched coverings with a finite number $p$ of branched points, as summarized in eqs (29) and (30), can also be viewed in a slightly different way. Consider a disk with $p$ holes, and let the holonomies on the internal and external boundaries be respectively $Q_s$ and $P$. A cell decomposition of this surface is shown in Fig. 3 and an element of $S_N$ can be associated to each link of the cell decomposition. Assuming that there is no branch point on the surface we associate to each plaquette $\alpha$ the Boltzmann weight

$$w_\alpha = \delta(P_\alpha) ,$$

where $P_\alpha$ is the ordered product of the elements of $S_N$ associated to the links around the plaquette $\alpha$. There is a second type of plaquettes in the cell decomposition that we are considering, namely the plaquettes that correspond to the boundaries of the holes. We can choose the holonomies on these boundaries to be
Figure 3: Cell decomposition of a disk with $p$ holes ($p = 4$ in this case).

fixed and given by $Q_s$ as in eq. (27). This is equivalent to associate the Boltzmann weight $w_s(P_s) = \delta(P_s, Q_s)$ introduced in eq. (27) to the corresponding plaquettes. We have in this way constructed an $S_N$ lattice gauge theory with two types of plaquettes, the ones obtained by the cell decomposition of the surface, and the ones corresponding to the holes, which are in fact nothing else but blown-up branched points. Clearly, the partition function of such lattice theory is again given by (29).

Instead, if we associate to all boundary plaquettes the same Boltzmann weight $w_s(P_s) = \sum R d_R \chi_R(P_s)$ and we sum over the number of boundaries we reproduce the "continuum" partition function (34).

This construction is not obviously limited to the gauge group $S_N$. Consider for instance YM theory on a surface of genus $G$ with $p$ boundaries, with a cell decomposition analogous to the one of the disk in Fig. 3. We associate to the internal plaquettes $\alpha$ a Boltzmann weight

$$w_\alpha(g_\alpha) = \sum_R d_R \chi_R(g_\alpha), \quad (37)$$

where $g_\alpha$ is the ordered product of the link variables around the plaquette. This defines on the surface a topological (BF) theory. To each of the $p$ plaquettes forming the boundaries we associate instead a Boltzmann weight

$$w_s(g_s) = \sum_R d_R C_R A \chi R(g_s), \quad (38)$$
3 Matrix strings of generalized YM2, coverings, and the lattice gauge theory of \( G_N \)

As already discussed in the introduction, the quantization in the unitary gauge of a generalized U(N) Yang-Mills theory is described \([9, 10]\) by a theory of coverings of the space–time manifold, \( i.e. \) a string theory. This theory is however not the pure theory of coverings that we described in the previous section as a lattice gauge theory of the permutation group \( S_N \). In fact a U(1) gauge theory is defined on the world-sheet of the string. This is the result of the residual gauge freedom after choosing the unitary gauge in which the auxiliary field \( B \) is diagonal.

We have already shown in \([10]\) that in the case of unbranched coverings a string theory with U(1) gauge fields on the world sheet can be described as a lattice gauge theory in which the gauge group is the semidirect product \( G_N \) of \( S_N \) and U(1)

The elements of this group, which we denote by \((P, \phi)\), can be represented by U(N) matrices of the special form

\[
(P, \phi)_{ij} = e^{i\phi_i} \delta_{i,P(j)},
\]

where \( P \in S_N \).

In order to understand how the lattice gauge theory of the group \( G_N \) originates, let us go back to Fig. 1, at the beginning of Sec. 2, and consider again \( N \) copies of each plaquette \( p \). Although we are interested here in introducing on the world sheet a gauge group U(1), we consider the case of an arbitrary gauge group \( G \) to make the construction completely general. Let us introduce on each plaquette \( p \) a matter field \( \Psi_\alpha^i(p) \) that transform under a given representation of \( G \). In \( \Psi_\alpha^i(p) \) the index \( i = 1, 2, \ldots, N \) labels the copies of the plaquette and \( \alpha \) is the index of the representation of \( G \). A gauge transformation consists in a relabeling of the sheets induced by a permutation \( P \) and, on each sheet \( i \), in the gauge transformation induced by \( g_i \in G \), namely:

\[
\Psi_\alpha^i(p) \rightarrow (P, g)\Psi_\alpha^i(p) = D_\alpha^\beta(g_i)\Psi_\beta^i(p_{P^{-1}(i)}),
\]

where \( D_\alpha^\beta(g_i) \) denote the matrix elements of \( g_i \) in the given representation. From eq. \([41]\) the composition rule of two elements of the gauge group can be easily derived and found to define the semidirect product of \( S_N \) and \( G^N \):

\[
(P, g)(Q, h) = (PQ, g \cdot Ph) \quad \text{with} \quad (g \cdot Ph)_i = g_i h_{P^{-1}(i)}.
\]

From \([41]\) one also obtains:

\[
(P, g)^{-1} = (P^{-1}, P^{-1}g^{-1}).
\]

Consider now the dual lattice. The matter fields sit on the sites \( s \) of the dual lattice, while on the links we have to define gauge connections, given by group elements,
which are needed to define covariant differences, the discrete analogue of covariant derivatives. Non trivial holonomies are associated to the plaquettes of the dual lattice, namely to the sites of the original lattice, and are given by the ordered product along the plaquette of the group elements on the links. Let \((P_s, g(s))\) be such product relative to a plaquette \(s\). A gauge transformation on the plaquette variable \((P_s, g(s))\) is given by:

\[
(P_s, g(s)) \rightarrow (Q, h)(P_s, g(s))(Q, h)^{-1} = (QP_s Q^{-1}, h \cdot Qg \cdot QPQ^{-1}h^{-1}).
\] (43)

For the \(S_N\) part, this gauge transformation amounts, as in the model of the previous section, to a relabeling of the sheets and leaves unchanged the decomposition into cycles of \(P_s\), which describes the branching structure of the point \(s\). As for the \(G\) gauge transformations, they can be derived from (43) by setting \(Q = 1\), and read

\[
g_i(s) \rightarrow h_i \cdot g_i(s) \cdot h_{P^{-1}(i)}^{-1}.
\] (44)

Clearly, unless \(P^{-1}(i) = i\), the transformation given in eq.(44) is not a gauge transformation on \(g_i(s)\). This is due to the fact that the lifting of a closed loop around the point \(s\) on the target space \(\Sigma_T\) is in general not closed. The closed loops on the world sheet \(\Sigma_W\) are given by the cycles of \(P_s\), and correspondingly the gauge covariant loop variables are the products \(g_{i_1}(s) \cdot g_{i_2}(s) \cdots g_{i_k}(s)\) where \((i_1, i_2, \ldots, i_k)\) is a cycle of \(P_s\).

We have shown that the theory of branched \(N\)-coverings with a \(G\)-gauge theory of on the world sheet is equivalent to a lattice gauge theory where the gauge group is the semi-direct product of \(S_N\) and \(G_N\). In order to write the partition function of this theory, we need to find the irreducible representations of such group. Although this can be done for an arbitrary group \(G\), we shall restrict ourselves in the rest of this section to the case \(G = U(1)\). The extension to arbitrary groups, although cumbersome from the point of view of notations, is rather straightforward.

### 3.1 Representation theory of \(G_N\)

Consider now the group \(G_N\), the semidirect product of \(S_N\) and \(U(1)^N\), whose elements we denote by \((P, \phi)\) where \(P\) is an element of \(S_N\) and \(\phi\) stands for the set of invariant angles \(\phi_i \quad i = 1, 2, \ldots, N\) that characterize an element of \(U(1)^N\). The product of two generic elements of the group can be obtained from (39) or directly from (41) by putting \(g_i = e^{ih_i}:

\[
(P, \phi)(Q, \theta) = (PQ, \phi + P\theta),
\] (45)

where \((P\theta)_i = \theta_{P^{-1}(i)}\). With this product law, the inverse of an element is

\[
(P, \phi)^{-1} = (P^{-1}, -P^{-1}\phi)
\] (46)

and the expression of a conjugated element is

\[
(Q, \theta)(P, \phi)(Q, \theta)^{-1} = (QPQ^{-1}, \theta + Q\phi - QPQ^{-1}\theta).
\] (47)
The structure of the group $\mathcal{G}_N$ is similar to that of the Poincaré group, the $U(1)^N$ elements playing the role of the translations and the permutations the role of the Lorentz rotations. To describe the irreducible representations of $\mathcal{G}_N$ we can thus follow Wigner's method of induced representations usually employed for the Poincaré group.

We begin by choosing an irreducible representation $n \equiv \{n^i\}$ of $U(1)^N$, with $i = 1, \ldots, N$ and $n^i \in \mathbb{Z}$. This is nothing but the assignment of quantized momenta in all of the $N$ compact directions: the $U(1)^N$ element $\phi$ is represented by the phase $\exp(i \sum n^i \phi_i)$. The chosen unordered $N$-ple of momenta is invariant under the action of the permutation group; however, this is not the case for the specific ordering of them which defines the representation $n$. We use again the notation

$$P n = \{n^{p-1(i)}\}$$

(48)

to denote a permutation of the momenta. In the analogy with the Poincaré group, the unordered $N$-ple of the $n^i$’s is the invariant that plays the role of the squared mass.

Given the representation $n$, Wigner’s idea is to construct the representations of the semidirect product group in terms of irreducible representations of the so-called little group $\mathcal{L}_n \subset S_N$, which contains those permutations that preserve $n$. We can split any permutation $P \in S_N$ as

$$P = \pi p : \left\{ \begin{array}{l} p \in \mathcal{L}_n, \\ \pi \in S_N/\mathcal{L}_n. \end{array} \right.$$  

(49)

While $pn = n$, the elements $\pi$ in the coset $S_N/\mathcal{L}_n$ act non-trivially, mapping $n$ into $\pi n$, with

$$(\pi n)^i = n^{\pi^{-1}(i)} ,$$

(50)

in accordance with Eq. (15). Notice that the coset class $\pi$ is of course defined only up to little group transformations. We can single out a specific representative $\hat{\pi}$ in this class by requiring, for instance, that

$$\hat{\pi}(i) < \hat{\pi}(j) \text{ if } i < j \text{ with } n^i = n^j .$$

(51)

This removes any ambiguity from the decomposition Eq. (49); where not otherwise indicated, we assume such a choice in what follows.

The little group is determined, up to isomorphisms, only by the structure of the set $\{n^i\}$ defining the representation. The construction is the following: let us denote by $N_n$ with $\sum_a N_a = N$ the number of times a given momentum $n$ appears in the set $\{n_i\}$. In other words there are $N_n$ values of $i$ for which $n_i = n$. Then the little group consists of the direct product

$$\bigotimes_n S_{N_n}$$

(52)
of the symmetric groups $S_{N_i}$ acting on the subsets of indices $i$ for which $n_i = n$. In the Poincaré group there are only two different little group structures corresponding to the mass squared being bigger than or equal to zero. In our case we have many more possibilities, specified by the possible degeneracies amongst the set of $n_i$’s.

The explicit expression $L_n$ in Eq. (49) of the little group Eq. (52) as a subgroup of $S_N$ depends on the specific ordering of the $n_i$’s that defines the representation $n$. If we modify the ordering $n$ to $\pi n$, we have an isomorphic expression

$$L_{\pi n} = \pi L_n \pi^{-1}.$$  \hfill (53)

As it follows from Eq. (52), an irreducible representation $r$ of the little group $L_n$ is a tensor product of irreducible representations $r_a$ of the symmetric groups $S_{N_a}$, for $a = 1, \ldots M$. The irreducible representations $r_a$, which are in one-to-one correspondence with the Young tableaux of $N_a$ boxes, and their characters $\operatorname{ch}_{r_a}$ have been discussed in the previous Section. We denote the matrix elements of $p \in L_n$ in the representation $r$ by

$$[D_r(p)]_{\alpha}^{\beta} = \bigotimes_{a=1}^{M} [D_{r_a}(p_a)]_{\alpha_a}^{\beta_a},$$  \hfill (54)

where $p_a$ is the component of $p$ in the $a$-th factor, $S_{N_a}$, of the little group. Correspondingly, the characters $\operatorname{Ch}_r(p)$ in this representation are given by

$$\operatorname{Ch}_r(p) = \prod_{a=1}^{M} \operatorname{ch}_{r_a}(p_a).$$  \hfill (55)

Consider now the action of a permutation $P$ on a state with momenta $n$ and which transforms in the irreducible representation $r$ of the little group. Denoting such a state as $|\alpha\rangle^{(n)}$, we have

$$P|\alpha\rangle^{(n)} = [D_r(p)]_{\alpha}^{\beta} |\beta\rangle^{(\pi n)},$$  \hfill (56)

having used the decomposition Eq. (49) with a specific choice of coset representative, e.g., the one in Eq. (51). We see that the states at fixed $n$ do not span a representation by themselves, and we are forced to form a single representation including all of the possible reorderings of the $n_i$’s, parametrized by the classes of $S_N/L_n$. This is analogous to the obvious fact that a Lorentz rotation $\Lambda$ on a state of momentum $p^\mu$ produces a state of rotated momentum $\Lambda^\mu \nu p^\nu$.

Eq. (56) suggests that we can represent, for each fixed $U(1)^N$ representation $n$, the entire group $G_N$ on a finite-dimensional space spanned by state vectors $|\sigma;\alpha\rangle$, with $\sigma \in S_N/L_n$ and $\alpha$ in the carrier space of an irreducible representation $r$ of the little group $L_n$, as above. The matrix representing an element $(P,\phi) \in G_N$ can then be written, using the composite index $A = (\sigma;\alpha)$, as

$$[D_{n,r}(P,\phi)]_A^{B} = e^{i \sum (\sigma n)^i \phi_i} [D_n(P)]^\tau_{\sigma} [D_r(\sigma^{-1}P^\tau)]^{\beta}_{\alpha}.$$  \hfill (57)
where

\[ [D_n(P)]^\tau_\sigma = \delta(P^{-1}\sigma n, \tau n) \]  

(58)

The \([D_n(P)]^\tau_\sigma\) factor simply states that we have non-zero matrix elements in the space of \(\sigma, \tau\) indices, i.e., in \(S_N/\mathcal{L}_n\), \(\sigma n\) (momenta of the state we are acting on) to \(\tau n\). This being the case, we see that \(\sigma^{-1}P\tau n = n\), i.e., for any choice of representatives of the coset classes, \(\sigma^{-1}P\tau\) is an element of the little group and can rightly appear in the last factor in Eq. (57). However, which specific element \(\hat{\sigma}^{-1}P\hat{\tau}\) one gets depends on the choice of representatives, so to fully describe the representation matrices we have to make a choice, e.g., the one of Eq. (51), as indicated in Eq. (57). Different choices of coset representatives lead to equivalent representations.

It is easy to verify that the matrices \(D_{n,r}\) defined as in Eq. (57) provide indeed a representation of the group \(G_N\), i.e., that

\[ [D_{n,r}(P, \phi)]^B_A [D_{n,r}(Q, \theta)]^C_B = [D_{n,r}(PQ, \phi + P\theta)]^C_A . \]  

(59)

One can furthermore show that the representations \(D_{n,r}\) are irreducible, by employing Schur’s lemma: if one assumes that a matrix \(F^B_A\) commutes with all the elements \(D_{n,r}(P, \phi)\), one finds that \(F\) has to be proportional to the identity matrix.

The representations \(D_{n_{\mu,r}}\), with \(\mu \in S_N/\mathcal{L}_n\), obtained by considering a different reordering of the \(n^i\)’s, can be shown to be equivalent to the representations \(D_{n,r}\).

Summarizing, the set of inequivalent irreducible representations of the group \(G_N\) is described by the possible unordered \(N\)-ples of momenta \(n^i\) and the irreducible representations of the corresponding little group \(\mathcal{L}_n\).

The characters in these representations we denote by \(CH_{n,r}\) and are given, as it follows from Eq. (57), by

\[ CH_{n,r}(P, \phi) = \sum_A [D_{n,r}(P, \phi)]^A_F = \sum_\sigma e^{i\sum_i (\sigma n)^i \phi_i} \delta(P^{-1}\sigma n, \sigma n) Ch_r(\sigma^{-1}P\sigma) . \]  

(60)

The dimension of the representation \(D_{n,r}\), which we denote by \(d_{n,r}\), is given by \(CH_{n,r}(1, 0)\), i.e., by

\[ d_{n,r} = \sum_\sigma d_\sigma = |S_N/\mathcal{L}_n| d_\sigma = \frac{N!}{\prod_n N_n!} d_\sigma , \]  

(61)

where \(d_\sigma\) is the dimension of the chosen irreducible representation \(r\) of the little group.

An extreme case is the one in which all the \(n^i\)’s are different; in this case the little group is trivial and has only the trivial representation \((r = 0)\); all representations \(D_{n,r=0}\) have then dimension \(N!\), and are given by:

\[ [D_{n,r=0}(P, \phi)]^R_Q = e^{i\sum_i (Qn)^i \phi_i} \delta(P^{-1}Q, R) , \]  

(62)
where $Q$ and $R$ are indices of $S_N/L_n$ which coincides in this case with $S_{N}$. At the other end we have the case where all the $n^i$’s are equal, the little group coincides with $S_N$, and the only coset class is the identity one. All representations $D_{n,r}$ have the same dimensionality $d_r$ as the chosen representation of the little group, and coincide, up to the $U(1)^N$ phase factor, with the representations of $S_N$.

The characters given in Eq. (60) satisfy the usual orthogonality and completeness relations, which we will need to construct and solve the lattice theory. One has the “fusion” rule

$$
\int D\phi \frac{1}{N!} \sum_{P \in S_N} \text{CH}_{n,r}( (Q_1, \theta_1)(P, \phi)^{-1}) \text{CH}_{n',r'} ((P, \phi)(Q_2, \theta_2))
$$

$$
= \delta_{n,n'} \delta_{r,r'} \frac{\text{CH}_{n,r}( (Q_1, \theta_1)(Q_2, \theta_2))}{d_{n,r}},
$$

which in the particular case $Q_1 = Q_2 = 1$ and $\theta_1 = \theta_2 = 0$ becomes the orthogonality relation

$$
\int D\phi \frac{1}{N!} \sum_{P \in S_N} \text{CH}_{n,r}( (\phi)^{-1}) \text{CH}_{n',r'} ((P, \phi) = \delta_{n,n'} \delta_{r,r'} .
$$

In both equations the measure on $U(1)^N$ is defined as $D\phi \equiv \prod_i d\phi_i/2\pi$. Besides we have the following “fission” property:

$$
\int D\phi \frac{1}{N!} \sum_{P \in S_N} \text{CH}_{n,r}( (P, \phi)(Q_1, \theta_1)(P, \phi)^{-1}(Q_2, \theta_2))
$$

$$
= \frac{1}{d_{n,r}} \text{CH}_{n,r}(Q_1, \theta_1) \text{CH}_{n,r}(Q_2, \theta_2)
$$

and the completeness relation

$$
\frac{1}{N!} \sum_{n} \sum_{r} \text{CH}_{n,r}(Q, \theta) \text{CH}_{n,r}(P, \phi) = \delta_{G_N}[(P, \phi), (Q, \theta)].
$$

The $G_N$-invariant delta-function appearing in the l.h.s above is explicitly given by

$$
\delta_{G_N}[(P, \phi), (Q, \theta)] = \frac{1}{N!} \sum_{R \in S_N} \int D\psi \delta((P, \phi)(R, \psi)(Q, \theta)(R, \psi)^{-1})
$$

$$
= \sum_{R} \delta(PRQR^{-1}) \frac{1}{(2\pi)^N} \prod_{l=1}^{N} \prod_{A=1}^{r_l} 2\pi \delta(\sum_{\alpha=0}^{l-1} \phi_{l,A,\alpha} + \sum_{\alpha=0}^{l-1} (R\theta)_{l,A,\alpha}),
$$

where we decomposed $P$ into $r_l$ cycles of length $l$ ($l = 1, \ldots, N$) and replaced the index $i = 1, \ldots, N$ with the multiindex

$$
(l, A, \alpha), \quad l = 1, \ldots, N, \quad A = 1, \ldots, r_l, \quad \alpha = 0, \ldots, l - 1 .
$$

\footnote{Notice that if one omits in Eq. (62) the $U(1)^N$ phase factors, one obtains the regular representation of $S_N$ which, however, is reducible.}
Notice that the $G_N$-invariant delta-function depends only on the sums of the angles $\phi$ belonging to the same cycles of $P$ and on the sums of the angles $\theta$ belonging to the same cycles of $R^{-1}PR$, that is of $Q$. A particular case of (66) is obtained by putting $Q = 1$ and $\theta_i = 0$, and reads:

$$\frac{1}{N!} \sum_n \sum_r d_{n,r} \chi_{n,r}(P,\phi) = \delta(P) (2\pi)^N \delta(\phi_1) \ldots \delta(\phi_N) .$$

(69)

All these formulas can be checked by direct although somewhat cumbersome calculations.

### 3.2 Branched coverings endowed with a $U(1)^N$ flux and the lattice gauge theory of $G_N$

We are now ready to generalize the results of Section 2 and study a theory of branched $N$-coverings of a Riemann surface $\Sigma_T$, endowed with a $U(1)$ gauge theory on the world sheet $\Sigma_W$. According to the discussion at the beginning of the present section this will be described by a lattice gauge theory of $G_N$, namely by a theory where an element $(P,\phi) \in G_N$ is associated to each link in the dual lattice. A specific covering is determined now by its branch-point structure and by the values of the $U(1)^N$ fluxes. To each plaquette $s$ of the dual lattice (that is, to each site of the original lattice) we associate a weight $W_s(P_s,\phi_s)$, where $(P_s,\phi_s)$ is the ordered product of the elements of $G_N$ associated to the links around the plaquette $s$. Such weights must be class functions, that is they depend on the conjugacy class of $(P_s,\phi_s)$ only. The partition function is obtained summing over all configurations:

$$Z_{G,N} = \left( \frac{1}{N!} \right)^N \sum_{(P,\phi)} \prod_s W_s(P_s,\phi_s)$$

(70)

(the normalization is as in Eq. (11)). We can expand the weight $W_s(P,\phi)$ into characters of $G_N$,

$$W_s(P,\phi) = \frac{1}{N!} \sum_{n,r} d_{n,r} \tilde{W}_s(n,r) \chi_{n,r}(P,\phi) ,$$

(71)

where, according to Eq. (64), we have

$$\tilde{W}_s(n,r) = \sum_P \int D\phi \frac{\chi_{n,r}((P,\phi)^{-1})}{d_{n,r}} W_s(P,\phi) .$$

(72)

Using the character expansion, it is possible to integrate over all internal links of the dual lattice to obtain finally, just as in Section 2, the partition function on a disk:

$$Z_{\text{disk},N}(P,\phi) = (N!)^{-2} \sum_{n,r} d_{n,r} \prod_{s=1}^{N_0} \tilde{W}_s(n,r) \chi_{n,r}(P,\phi) ,$$

(73)
where \((P, \phi)\) is the group element associated to the boundary of the disk, and from this the partition function for a closed surface of genus \(G\):

\[
Z_{G,N} = \sum_{n,r} \left( \frac{d_{n,r}}{N!} \right)^{2-2G} \prod_{s=1}^{N_0} \tilde{W}_s(n, r). \tag{74}
\]

In [10] we considered the case of unbranched coverings, and we showed that it corresponds to the following choice of weights:

\[
\forall s : \quad W_s(P, \phi) = \delta(P) \sum_{n_i} e^{\sum_{n_i} \phi_i - A_s \sum_i v(n_i)}, \tag{75}
\]

where \(A_s\) is the area of the dual plaquette \(s\). The lattice gauge theory corresponding to this plaquette action was shown to be equivalent to the generalized U(N) YM theory based on a potential \(V(F)\) if the finite potential \(\sum_i v(n_i)\) is obtained from \(V(\{n_i\})\) by subtracting the logarithmic divergences that arise when the genus \(G\) of the target manifold is different from 1. The \(n_i\)'s correspond to the quantized eigenvalues of the diagonal components of \(F\). With this choice, we get from Eq. (72)

\[
\tilde{W}_s(n, r) = e^{-A_s \sum_i v(n_i)}, \tag{76}
\]

that is, the \(\tilde{W}\)'s have no dependence on the little group representation \(r\). The partition function for a closed surface is then simply

\[
Z_{G,N} = \sum_{n,r} \left( \frac{d_{n,r}}{N!} \right)^{2-2G} e^{-A \sum_i v(n_i)}, \tag{77}
\]

where \(A = \sum_s A_s\) is the total area of the target surface. In [11], the grand-canonical partition function \(Z_G(q) \equiv \sum_N Z_{G,N} q^N\) was investigated by directly enumerating the coverings and associating to each connected component a generalized U(1) partition function \(\sum_{n \in \mathbb{Z}} e^{-A v(n)}\). A closed expression was given in the case of the torus, and a list of the first few terms (lowest values of \(N\)) in its \(q\) expansion was given in the appendix of [11] also for other values of \(G\). A closed expression for the grand canonical partition function \(Z_G(q)\), generalizing the one given in [11] for the torus, can be obtained from (77) by using eq. (61) and the relations \(\sum_i v(n_i) = \sum_n N_n v(n)\) and \(N = \sum_n N_n\). We find that the infinite sums factorize in a product over \(n\), namely:

\[
Z_G(q) \equiv \sum_N Z_{G,N} q^N = \prod_n Z_{G}(e^{-A v(n)} q), \tag{78}
\]

where \(Z_G(q)\) in the grand-canonical partition function for the unbranched coverings, that is:

\[
Z_G(q) = \sum_N Z_{N,G} q^N, \tag{79}
\]
and $Z_{N,G}$ is given in (23). In Appendix A, the torus grand-canonical function $Z_G(q)$ will be described in more detail to appreciate the relation between the treatment of [10] and the more general one given here.

Let us go back to the general case of an arbitrary branching structure. We follow the same pattern as in Section 2 and associate to each dual plaquette $s$ an element $(Q_s, \theta_s)$ of $G_N$. This is done by assigning to it the weight

$$W_s(P, \phi) = \delta_{G_N}((Q_s, \theta_s), (P, \phi))$$

(80)

where the $G_N$-invariant delta function is the one defined in (67). The permutation $Q_s$ gives the branching structure at the site $s$ of the original lattice, while the invariant angles $\theta_{s,i}$ determine the $U(1)$ holonomies. Notice however that closed loops on the world sheet around the site $s$ are in one to one correspondence to the cycles of the permutation $Q_s$. The corresponding $U(1)$ gauge invariant holonomies are given by the angles $\theta_{s,\{l,A\}} = \sum_{i \in \{l,A\}} \theta_{s,i}$ where $\{l,A\}$ denotes the $A$-th cycle of length $l$ in $Q_s$. These are the only angles that appear, according to (67), in the definition of the covariant delta function and are the only angles which are left invariant, according to the general discussion at the beginning of the present section, under conjugacy transformations of $G_N$. From Eq. (80) and the orthogonality of characters we obtain

$$\tilde{W}_s(n, r) = \frac{\text{CH}_{n,r}(Q_s, \theta_s)}{d_{n,r}}.$$  

(81)

The partition function on a disk and on a closed surface of genus $G$ can be obtained by inserting (81) in (73) and (74) leading to

$$Z_{\text{disk},N}((P, \phi), \{(Q_s, \theta_s)\}) = (N!)^{-2} \sum_{n,r} d_{n,r}^{N_0} \prod_{s=1}^{N_0} \left( \frac{\text{CH}_{n,r}(Q_s, \theta_s)}{d_{n,r}} \right) \text{CH}_{n,r}(P, \phi)$$  

(82)

and

$$Z_{G,N}(\{(Q_s, \theta_s)\}) = (N!)^{2G-2} \sum_{n,r} d_{n,r}^{2G-2} \prod_{s=1}^{N_0} \left( \frac{\text{CH}_{n,r}(Q_s, \theta_s)}{d_{n,r}} \right).$$  

(83)

The continuum limit can be taken on Eq.s (82) and (83) following the same track as in Section 2. Let us assume that each branch point with holonomy $(Q_s, \theta_s)$ appears with a coupling $A g[(Q_s, \theta_s)]$, where $A$ is the area of the surface and $g[(Q_s, \theta_s)]$ is a class function of $G_N$. Let us then define $g_{n,r}$ as the coefficients of the expansion of $g[(Q_s, \theta_s)]$ in characters of $G_N$:

$$g[(Q_s, \theta_s)] = \frac{1}{N!} \sum_{n,r} d_{n,r} g_{n,r} \text{CH}_{n,r}(Q_s, \theta_s).$$  

(84)

If we multiply the partition functions in (82) and (83) by $\prod_s A g[(Q_s, \theta_s)]$, integrate over $(Q_s, \theta_s)$ and sum over the number $N_0$ of branch points (with a $1/N_0!$ factor) we obtain

$$Z_{\text{disk},N}((P, \phi), A) = (N!)^{-2} \sum_{n,r} d_{n,r} \text{CH}_{n,r}(P, \phi) e^{A g_{n,r}}$$  

(85)

(19)
and
\[ Z_{G,N}(\mathcal{A}) = (N!)^{2G-2} \sum_{\mathbf{n},\mathbf{r}} q_{\mathbf{n},\mathbf{r}}^{2-2G} e^{A g_{n,r}} . \] (86)

The partition function given in (86) is very general, as it corresponds to arbitrary weights for the different types of branch points, and to arbitrary U(1) holonomies associated to each type of branch point. Consider now a more specific case, where in all sites \( s \) there is either no branch point or one corresponding to a single exchange. This means that the coupling \( g[(Q, \theta)] \) consists of two terms:
\[ g[(Q, \theta)] = g_0[(Q, \theta)] + \lambda g_1[(Q, \theta)] , \] (87)
where \( \lambda \) is a free parameter. The functions \( g_0[(Q, \theta)] \) and \( g_1[(Q, \theta)] \) are the most general class functions of \( G_N \) with support, respectively, in \( Q = 1 \) and in \( Q \) consisting of a single exchange. Their explicit expression is:
\[ g_0[(Q, \theta)] = \delta(Q) \sum_{\{n_i\}} e^{\sum_i n_i \theta_i} v(n_1, n_2, ... n_N) , \] (88)
\[ g_1[(Q, \theta)] = \sum_R \delta(RP_{12}R^{-1}Q) \sum_{\{n_i\}} \delta_{n_1,n_2} f(n_1; n_3, ..., n_N) e^{\sum_i n_i \theta_R(i)} , \] (89)

where \( P_{12} \) is a permutation consisting of the exchange of the labels 1 and 2, \( v(n_1, n_2, ... n_N) \) and \( f(n_1; n_3, ..., n_N) \) are arbitrary functions, which are symmetric respectively under permutations of all the \( n_i \)’s and of the \( n_i \)’s with \( i = 3, 4, ... N \). We consider now the case where the \( f(n_1; n_3, ..., n_N) = 1 \), and \( v(n_1, n_2, ... n_N) = \sum_i v(n_i) \). The former condition implies that there is no U(1) holonomy attached to the quadratic branch points or, in other words, that the U(1) electromagnetic field is not localized on the branch points, but distributed on the world sheet. The condition \( v(n_1, n_2, ... n_N) = \sum_i v(n_i) \) is equivalent to the statement that the U(1) gauge action is local on the world sheet, namely that what happens on one sheet has no effect on the others. With this choice the coefficients \( g_{n,r} \) can be easily calculated, leading to:
\[ g_{n,r} = \sum_n \left( N_n v(n) + \lambda \xi^r_{2(n)} \right) , \] (90)

where as before \( N_n \) is the number of times the integer \( n \) appears in \( \mathbf{n} \), \( r_{\{n\}} \) is the representation of \( S_{N_n} \) in \( r \), and \( \xi^r_{2} \) is defined in (35). We can now insert (90) into (86) and consider the grand canonical partition function, defined as in the case without branch points (see Eq. (78)). It is not difficult to verify that with the choice (90) for \( g_{n,r} \), the grand canonical partition function factorizes in an infinite product over \( n \), namely
\[ Z_G(\mathcal{A}, q) \equiv \sum_{N} Z_{G,N}(\mathcal{A}) q^N = \prod_n Z_G(e^{-A v(n)} q, \lambda A) . \] (91)
Here $Z_G(q, A)$ is the grand canonical partition function for coverings with quadratic branch points, namely:

$$Z_G(q, A) \equiv \sum_N Z_{G,N}(A) q^N = \sum_N \sum_{r|S_N} \left( \frac{d_r}{N!} \right)^{2-2G} e^{4\xi q} q^N$$  \hspace{1cm} (92)$$

where the second sum at the r.h.s. is over the representations $r$ of $S_N$.

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**A Unbranched coverings**

In this appendix, we make more explicit some points of the discussion, given in the main text after Eq. (77), of the grand-canonical partition functions describing the particular case of unbranched covers.

The partition functions for coverings with a U(1) on their world-sheet were already discussed, in the unbranched case, in [10]. In that paper a procedure was given to construct the grand-canonical partition function $Z_G(q) \equiv \sum_N Z_{G,N} q^N$. One starts from the grand-canonical partition function $Z_G(q)$ for the pure coverings, and considers the corresponding free energy $F_G(q)$. Each connected world-sheet covering $k$ times the target is then weighed by a factor of $z(kA)$, where $z(A) \equiv \sum_{n \in \mathbb{Z}} \exp(-A v(n))$ is the generalized U(1) partition function. This produces the free energy $F_G(q)$ for our theory, which can then be exponentiated to obtain the partition function $Z_G(q)$.

In the case of the torus, $G = 1$, the free energy counting connected coverings is given by $F_1(q) = \sum_p \sum_{m|p} (1/m) q^p$ (here $m|p$ means "$m$ divides $p"$), so that an explicit expression for $Z_1(q)$ is

$$Z_1(q) = \exp \left( \sum_p \sum_{m|p} \frac{1}{m} q^p z(pA) \right).$$  \hspace{1cm} (93)$$

The partition functions $Z_{1,N}$ at fixed $N$ are then straightforwardly obtained by expanding Eq. (93) to the desired order (see the Appendix of [10] for the first few values of $N$). For instance, for $N = 2$ and $N = 3$ one gets

$$Z_{1,2} = \frac{1}{2} \left( z^2(A) + 3z(2A) \right),$$  \hspace{1cm} (94)$$

$$Z_{1,3} = \frac{1}{6} \left( z^3(A) + 9z(2A) + 8z(3A) \right).$$  \hspace{1cm} (95)$$
On the other hand, it was noticed in [9, 10] that the partition function Eq. (93) can be re-expressed as an infinite product:

\[
Z_1(q) = \prod_{n \in \mathbb{Z}} \prod_{k=1}^{\infty} \frac{1}{1 - q^k e^{-kA v(n)}} = \prod_{n \in \mathbb{Z}} \left( \sum_{N=0}^{\infty} p(s) q^s e^{-sA v(n)} \right),
\]

(96)
p(s) being the number of partitions of s. This coincides with our general formula Eq. (78), which for \(G = 1\) reduces to

\[
Z_1(q) = \prod_{n} \left( \sum_{s} Z_{s,1} q^s e^{-sA v(n)} \right),
\]

(97)
since for the pure coverings of the torus one has \(Z_{s,1} = p(s)\), as given in Eq. (24).

The equality between the expressions Eq. (93) and Eq. (96) of the grand-canonical partition function summarizes the rearrangements by which, for any fixed \(N\), one reconstructs from the sum over the momenta \(n\) appearing in Eq. (77) the independent sums over integers which appear in the expansion of Eq. (93). Such an expansion contains in fact products of \(U(1)\) partition functions \(z(kA)\). For instance, let us consider the case \(N = 2\). The inequivalent set of momenta \(n\) are: \(i)\) \(n_1 < n_2\), and \(ii)\) \(n_1 = n_2\). In the case \(i)\), the little group is trivial and so it only has one representation; in case \(ii)\) the little group is \(S_2\), which admits two irreducible representations. We have therefore

\[
Z_{1,2} = \frac{1}{2} \sum_{n_1 \neq n_2} e^{-A(v(n_1)+v(n_2))} + 2 \sum_{n_1} e^{-2A v(n_1)}
\]

\[
= \sum_{n_1, n_2} e^{-A(v(n_1)+v(n_2))} + (2 - \frac{1}{2}) \sum_{n_1} e^{-2A v(n_1)} = \frac{1}{2} z^2(A) + \frac{3}{2} z(2A),
\]

in agreement with Eq. (24). One can easily repeat the same check for \(N = 3\) and (with increasing effort) higher \(N\)’s.

When the target space has genus \(G > 1\), in [10] the expressions of \(Z_{G,N}\) could be worked out case by case, but a closed expression could not be exhibited. The reason is that no closed form is in fact known for the free energy \(F_G(q)\) of connected coverings in this case. Here, attacking the problem by the point of view of the \(G_N\) lattice gauge theory, we have obtained in Eq. (77) such a closed expression.

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