DIMENSIONS OF GENERIC LOCAL ORBITS OF MULTIPARTITE QUANTUM SYSTEMS

DRAGOMIR Ž. ĐOKOVIĆ

ABSTRACT. We consider the action of the group of local unitary transformations, $U(m) \otimes U(n)$, on the set of (mixed) states $\mathcal{W}$ of the bipartite $(m \times n)$ quantum system. We prove that the generic $U(m) \otimes U(n)$-orbits in $\mathcal{W}$ have dimension $m^2 + n^2 - 2$. This problem was mentioned (and left open) by Kuś and Życzkowski in their paper Geometry of entangled states [3]. The proof can be extended to the case of arbitrary finite-dimensional multipartite quantum systems.

1. Motivation

We consider the action of the group of local unitary transformations, $U(m) \otimes U(n)$, on the set of (mixed) states $\mathcal{W}$ of the bipartite $(m \times n)$ quantum system. The Hilbert space of this system has the tensor product structure $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\mathcal{H}_A$ of dimension $m$ and $\mathcal{H}_B$ of dimension $n$. We assume that $n \geq m \geq 2$.

For the sake of convenience let us denote by $\text{Herm}(\mathcal{H})$ the real vector space of all hermitian operators on $\mathcal{H}$. The affine subspace of $\text{Herm}(\mathcal{H})$ consisting of operators of trace 1 will be denoted by $\text{Herm}_1(\mathcal{H})$. Note that this affine space has (real) dimension $m^2 n^2 - 1$.

The set $\mathcal{W}$ of (mixed) states of our quantum system is the set of all Hermitian positive semi-definite linear operators on $\mathcal{H}$ having trace 1. It is a compact convex set with non-empty interior, relative to the ambient real affine space $\text{Herm}_1(\mathcal{H})$. Let us recall that the action of the full (global) unitary group $U(mn)$ on the space $\text{Herm}(\mathcal{H})$ is given by $(U, W) \rightarrow UWU^\dagger$, where $U \in U(mn)$ and $W \in \text{Herm}(\mathcal{H})$. The action of $U(m) \otimes U(n)$ is given by

$$(U \otimes V, W) \rightarrow (U \otimes V) \cdot W \cdot (U^\dagger \otimes V^\dagger),$$

where $U \in U(m)$, $V \in U(n)$ and $W \in \text{Herm}(\mathcal{H})$.

The problem of computing the maximum dimension of $U(m) \otimes U(n)$-orbits in $\mathcal{W}$ was raised in a recent paper of Kuś and Życzkowski [3]. According to their expectation, this maximum dimension should be

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They carried out computations and verified this claim for some small dimensions.

The object of this note is to prove that the answer to the above question of Kuś and Życzkowski is positive for all dimensions.

2. Result and proof

We shall prove the following theorem.

**Theorem 2.1.** The maximum dimension of local unitary orbits in \( W \) is \( m^2 + n^2 - 2 \). The minimal isotropy subgroup for this action is exactly the center of \( U(m) \otimes U(n) \). The corresponding orbit is diffeomorphic to the product manifold \( SU(m)/Z_m \times SU(n)/Z_n \), where \( Z_m \) resp. \( Z_n \) is the center of \( SU(m) \) resp. \( SU(n) \).

*Proof.* The first and third assertions are consequences of the second. Thus we shall only prove the second one.

Since the center of \( U(m) \otimes U(n) \) acts trivially on \( \text{Herm}(\mathcal{H}) \), it suffices to exhibit a positive semi-definite operator \( W \in \text{Herm}(\mathcal{H}) \) whose isotropy subgroup (i.e., the stabilizer) in \( U(m) \otimes U(n) \) is precisely the center of this group. In fact we shall drop the condition that \( W \) be positive semi-definite because we can always add to \( W \) a positive scalar multiple of the identity operator to make it positive definite without affecting its isotropy subgroup.

Let \( e_1, \ldots, e_m \) be an orthonormal basis of \( \mathcal{H}_A \) and let \( P_1, \ldots, P_m \) be the corresponding orthogonal projectors, \( P_i = e_i e_i^\dagger \). Let us introduce an auxiliary vector \( v = \sum k e_k \). We now introduce our Hermitian operator \( W \) by setting

\[
(2.1) \quad W = \sum_{i=1}^m P_i \otimes X_i + vv^\dagger \otimes I,
\]

where \( I \) is the identity operator on \( \mathcal{H}_B \) and the \( X_i \)'s are traceless Hermitian operators on \( \mathcal{H}_B \) which are chosen so that their joint centralizer in \( U(n) \) is just the center of \( U(n) \), and the \( X_i \)'s and \( I \) are all linearly independent. (It is easy to choose such \( X_i \)'s, and there are infinitely many such choices.)

Assume that \( U \otimes V \in U(m) \otimes U(n) \) belongs to the isotropy subgroup of \( W \), i.e.,

\[
(U \otimes V) \cdot W \cdot (U^\dagger \otimes V^\dagger) = W.
\]

This can be rewritten as

\[
(2.2) \quad W = \sum_i U P_i U^\dagger \otimes V X_i V^\dagger + Uvv^\dagger U^\dagger \otimes I.
\]
Due to the linear independence of the \( X_i \)'s and \( I \) as well as the linear independence of the \( P_i \)'s and \( vv^\dagger \), we can apply a result from Linear Algebra (see [1, Chapitre II, §7, Proposition 17] and its first corollary) to deduce from Eqs. (2.1) and (2.2) that the operators \( P_i \) and the operator \( vv^\dagger \) span the same subspace as the operators \( UP_i U^\dagger \) and the operator \( Uvv^\dagger U^\dagger \).

Thus, for all \( i \)'s, we have

\[
UP_i U^\dagger = \sum_j \alpha_{ij} P_j + \beta_i vv^\dagger,
\]

and also

\[
Uvv^\dagger U^\dagger = \sum_i \gamma_i P_i + \delta vv^\dagger,
\]

for some real numbers \( \alpha_{ij} \), \( \beta_i \), \( \gamma_i \) and \( \delta \). By plugging these expressions into Eq. (2.2), we obtain

\[
W = \sum_i \left( \sum_j \alpha_{ij} P_j + \beta_i vv^\dagger \right) \otimes V X_i V^\dagger \\
+ \left( \sum_j \gamma_j P_j + \delta vv^\dagger \right) \otimes I \\
= \sum_j P_j \otimes \left( \gamma_j I + V \left( \sum_i \alpha_{ij} X_i \right) V^\dagger \right) \\
+ vv^\dagger \otimes \left( \delta I + V \left( \sum_i \beta_i X_i \right) V^\dagger \right).
\]

By comparing this expression with Eq. (2.1), we infer that

\[
\delta I + V \left( \sum_i \beta_i X_i \right) V^\dagger = I,
\]

i.e.,

\[
\sum_i \beta_i X_i + (\delta - 1) I = 0.
\]

As the \( X_i \)'s and \( I \) are linearly independent, we conclude that all \( \beta_i = 0 \) and \( \delta = 1 \).

From the above comparison we also obtain the equations

\[
\gamma_j I + V \left( \sum_i \alpha_{ij} X_i \right) V^\dagger = X_j.
\]

Since all \( X_i \)'s have trace 0, we deduce that all \( \gamma_j = 0 \).
Thus, for all $i$’s we have

$$UP_i U^\dagger = \sum_j \alpha_{ij} P_j.$$ 

For fixed $i$, the $\alpha_{ij}$ are the eigenvalues of $UP_i U^\dagger$, and so exactly one of them is 1 and all other are 0. Hence, there is a permutation $\sigma$ of \{1,2,\ldots,m\} such that

$$UP_i U^\dagger = P_{\sigma(i)}, \quad 1 \leq i \leq m.$$ 

Consequently,

$$U(e_i) = \xi_i e_{\sigma(i)}, \quad |\xi_i| = 1, \quad 1 \leq i \leq m.$$ 

We also have the equality

$$Uvv^\dagger U^\dagger = vv^\dagger$$

which implies that $Uv = \xi v$ for some complex number $\xi$ with $|\xi| = 1$. By using the definition of $v$, we obtain that

$$\sum_k k\xi_k e_{\sigma(k)} = \xi \sum_k k e_k.$$ 

This shows that $\sigma$ must be the identity permutation and that all $\xi_i = \xi$. Hence, $U$ belongs to the center of $U(m)$ and $UP_i U^\dagger = P_i$ for all $i$’s.

Eqs. (2.1) and (2.2) now give

$$\sum_i P_i \otimes (VX_i V^\dagger - X_i) = 0.$$ 

Since the $P_i$’s are linearly independent, it follows that $VX_i V^\dagger = X_i$ for all $i$. By our choice of the $X_i$’s, we conclude that $V$ belongs to the center of $U(n)$.

Hence, we have shown that $U \otimes V$ is in the center of $U(m) \otimes U(n)$. This concludes the proof of the theorem.

□

3. Comments

To illustrate the theorem, as the basic example one can take $m = n = 2$. In that case $m^2 + n^2 - 2 = 6$ which is discussed at length in [3], including the orbits of lower dimension.

In the general case, the orbits having the maximum orbit type (i.e., having the minimum isotropy subgroups, up to conjugacy) are known as orbits of principal type. It is well-known that the union of all orbits of principal type is an open dense set (see e.g. [2, Chapter IV, Theorem 3.1]). It is also common to refer to the orbits of principal type as the generic orbits.
We point out that there may exist also non-principal orbits having the maximal dimension (the same as the principal orbits). Such orbits are known as exceptional (see [2, p. 181]). They are non-trivial covering spaces of the principal orbit. For instance, in the case of two qubits \( (m = n = 2) \) there exist at least two types of exceptional orbits. For one of them the quotient group of the stabilizer modulo the center of \( U(2) \otimes U(2) \) is \( Z_2 \) and for the other it is \( Z_2 \times Z_2 \).

The proof of Theorem 2.1 given above can be extended to arbitrary multipartite systems. Let us formulate precisely this generalization.

**Theorem 3.1.** Let us consider the multipartite quantum system with \( k \) parties with the Hilbert space

\[
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_k.
\]

Let \( d_i = \dim \mathcal{H}_i, \) \( i = 1, 2, \ldots, k \) and assume that \( 2 \leq d_1 \leq d_2 \leq \cdots \leq d_k \). Then the maximum dimension of the local unitary orbits contained in the set \( \mathcal{W} \) of all mixed states of our quantum system is equal to

\[
d_1^2 + d_2^2 + \cdots + d_k^2 - k.
\]

The minimal isotropy subgroup coincides with the center of the group

\[
U(d_1) \otimes U(d_2) \otimes \cdots \otimes U(d_k)
\]

of local unitary transformations and the generic orbits in \( \mathcal{W} \) are diffeomorphic to the product manifold

\[
SU(d_1)/Z_{d_1} \times SU(d_2)/Z_{d_2} \times \cdots \times SU(d_k)/Z_{d_k},
\]

where \( Z_{d_i}, \) a cyclic group of order \( d_i, \) is the center of the special unitary group \( SU(d_i). \)

**References**

[1] N. Bourbaki, Algébre, Chapitres 1 à 3, Hermann, Paris, 1970.

[2] G. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, London, 1972.

[3] M. Kuś and K. Życzkowski, Geometry of entangled states, Phys. Rev. A 63, 032307 (2001).