WEAK AMENABILITY FOR DYNAMICAL SYSTEMS

ANDREW MCKEE

Abstract. Using the recently developed notion of a Herz–Schur multiplier of a $C^*$-dynamical system we introduce weak amenability of $C^*$- and $W^*$-dynamical systems. As a special case we recover Haagerup’s characterisation of weak amenability of a discrete group. We also consider a generalisation of the Fourier algebra to crossed products and study its multipliers.

1. Introduction

Among the many characterisations of amenability of a locally compact group $G$ is Leptin’s Theorem [14]: $G$ is amenable if and only if the Fourier algebra of $G$ has a bounded approximate identity. The idea to weaken the latter condition, by requiring the approximate identity to be bounded in a different norm, goes back to Haagerup [9]. Following this, Cowling–Haagerup [5] formally defined weak amenability, explored some equivalent conditions, and introduced the Cowling–Haagerup (or weak amenability) constant. This constant has been computed for a large number of groups — see Brown–Ozawa [4, Theorem 12.3.8] and the references given by Knudby [13]. An overview of the literature surrounding weak amenability can be found in the thesis of Knudby [13, Section 5].

Weak amenability is an example of a property defined in terms of functions on a group which can be characterised by an approximation property of the group von Neumann algebra and/or group $C^*$-algebra (see Brown–Ozawa [4, Chapter 12] for several examples of such properties); the aim of this paper is to extend this idea to crossed products. A $C^*$-algebra $A$ is said to have the completely bounded approximation property (CBAP) if there exists a net $(T_\gamma)$ of finite rank completely bounded maps on $A$ such that $T_\gamma \to \text{id}_A$ in the point-norm topology and $\sup_\gamma \|T_\gamma\|_{\text{cb}} = C < \infty$. The infimum of all such constants $C$ is denoted $\Lambda_{\text{cb}}(A)$. Similarly, a von Neumann algebra $M$ is said to have the weak* completely bounded approximation property (weak* CBAP) if there exists a net $(R_\gamma)$ of ultraweakly continuous, finite rank, completely bounded maps on $M$ such that $R_\gamma \to \text{id}_M$ in the point-weak* topology and $\sup_\gamma \|R_\gamma\|_{\text{cb}} = C < \infty$; again, the infimum of all such constants $C$ is denoted $\Lambda_{\text{cb}}(M)$. A locally compact group $G$ is called weakly amenable

\textit{2010 Mathematics Subject Classification.} Primary: 46L55, Secondary: 46L05.

\textit{Key words and phrases.} Schur multiplier; $C^*$-crossed products; approximation properties; weak amenability.
if there exists a net of compactly supported Herz–Schur multipliers on $G$, uniformly bounded in the Herz–Schur multiplier norm, converging uniformly to 1 on compact sets. Haagerup [9] proved that a discrete group is weakly amenable if and only if the reduced group $C^*$-algebra has the completely bounded approximation property, if and only if the group von Neumann algebra has the weak* completely bounded approximation property.

In this paper we define weak amenability of $C^*$- and $W^*$-dynamical systems and characterise a weakly amenable system in terms of the completely bounded approximation property of the corresponding crossed product. The results in this direction, Theorems 4.3 and 4.6, may be seen as a generalisation of Haagerup’s result above. Haagerup and Kraus [10, Section 3] have studied $W^*$-dynamical systems under the assumption that $G$ is weakly amenable; Proposition 4.8 was motivated by their Theorem 3.2(b) and Remark 3.10.

In Section 2 we review the definitions and results surrounding the notion of a Herz–Schur multiplier of a $C^*$-dynamical system. Section 3 is motivated by the description of Herz–Schur multipliers as completely bounded multipliers of the Fourier algebra; we introduce a predual for (the enveloping von Neumann algebra of) the reduced crossed product, consisting of vector-valued functions on the group, and describe the completely bounded multipliers of this space as certain Herz–Schur multipliers of the associated dynamical system. In Section 4 we define weak amenability of $C^*$- and $W^*$-dynamical systems, and characterise in terms of the completely bounded approximation property of the associated crossed product.

2. Preliminaries

In this section we review the definitions and results of [15] required later, as well as establishing notation. Throughout, $G$ will denote a second-countable, locally compact, topological group, with modular function $\Delta$, endowed with left Haar measure $m$; integration on $G$, with respect to $m$, over the variable $s$, is simply denoted $ds$. Let $\lambda^G$ denote the left regular representation of $G$ on $L^2(G)$ given by

$$\lambda^G_t(\xi)(s) := \xi(t^{-1}s), \quad s, t \in G, \quad \xi \in L^2(G).$$

The same symbol will be used to denote the associated representation of $L^1(G)$ on $L^2(G)$, given by

$$\lambda^G(f) := \int_G f(s)\lambda^G_s \, ds, \quad f \in L^1(G).$$

The reduced group $C^*$-algebra $C^*_r(G)$ and group von Neumann algebra $\text{vN}(G)$ of $G$ are, respectively, the closure of $\lambda^G(L^1(G))$ in the norm and weak* topology of $B(L^2(G))$; we also have $\text{vN}(G) = \{\lambda^G_s : s \in G\}''$. Let $A$ be a unital, separable, $C^*$-algebra, which unless otherwise stated will be considered as a $C^*$-subalgebra of $B(H_A)$, where $H_A$ denotes the Hilbert space of the universal representation of $A$. Let $\alpha : G \to \text{Aut}(A)$ be a group
homomorphism which is continuous in the point-norm topology, \textit{i.e.} for all \( a \in A \) the map \( s \mapsto \alpha_s(a) \) is continuous from \( G \) to \( A \); in short, consider a \( C^* \)-dynamical system \((A, G, \alpha)\). The space \( L^1(G, A) \) of all Bochner-integrable functions from \( G \) to \( A \) becomes a Banach \(*\)-algebra with the product \( \times \) defined by

\[
(f \times g)(t) := \int_G f(s) \alpha_s(g(s^{-1} t)) \, ds, \quad f, g \in L^1(G, A), \quad t \in G,
\]

involution \(*\) defined by

\[
f^*(t) := \Delta(t)^{-1} \alpha_t(f(t^{-1})^*), \quad f \in L^1(G, A), \quad t \in G,
\]

and \( L^1 \)-norm \( ||f||_1 := \int_G ||f(s)|| \, ds \). These definitions also give a \(*\)-algebra structure on \( C_c(G, A) \), which is a dense \(*\)-subalgebra of \( L^1(G, A) \). For a thorough introduction to \( L^1(G, A) \) see Williams [23, Appendix B].

Define a representation of \( A \) on \( L^2(G, \mathcal{H}_A) \) by

\[
\pi : A \to \mathcal{B}(L^2(G, \mathcal{H}_A)); \quad (\pi(a)\xi)(t) := \alpha_t^{-1}(a)(\xi(t)),
\]

for all \( a \in A, \ t \in G, \ \xi \in L^2(G, \mathcal{H}_A) \). If we define

\[
\lambda : G \to \mathcal{B}(L^2(G, \mathcal{H}_A)); \quad (\lambda_t\xi)(s) := \xi(t^{-1} s),
\]

for all \( s, t \in G, \ \xi \in L^2(G, \mathcal{H}_A) \), then \( \lambda \) is a continuous unitary representation of \( G \) and it is easy to check that

\[
\pi(\alpha_t(a)) = \lambda_t \pi(a) \lambda_t^*, \quad a \in A, \ t \in G.
\]

The pair \((\pi, \lambda)\) is therefore a \textit{covariant representation} of \((A, G, \alpha)\). Thus we obtain a representation \( \pi \times \lambda : L^1(G, A) \to \mathcal{B}(L^2(G, \mathcal{H}_A)) \) given by

\[
\pi \times \lambda(f) := \int_G \pi(f(s)) \lambda_s \, ds, \quad f \in L^1(G, A).
\]

The \textit{reduced crossed product} of \( A \) by \( G \) is defined as the closure of \((\pi \times \lambda)(L^1(G, A))\) in the operator norm of \( \mathcal{B}(L^2(G, \mathcal{H}_A)) \), and denoted by \( A \rtimes_{\alpha,r} G \). More on this construction can be found in Pedersen [16, Chapter 7] and Williams [23].

In [15] the present author, with Todorov and Turowska, introduced and studied Herz–Schur multipliers of a \( C^* \)-dynamical system, extending the classical notion of a Herz–Schur multiplier (see de Cannière–Haaagerup [6]). We now recall the definitions and results needed here; the classical definitions of Herz–Schur (and Schur) multipliers are the special case \( A = \mathbb{C} \) of the definitions below. A bounded function \( F : G \to \mathcal{B}(A) \) will be called \textit{pointwise-measurable} if, for every \( a \in A \), the map \( s \mapsto F(s)(a) \) is a weakly-measurable function from \( G \) to \( A \). For each \( f \in L^1(G, A) \) define \( F : f(s) := F(s)(f(s)) \) \((s \in G)\). If \( F \) is bounded and pointwise-measurable then \( F \cdot f \) is weakly measurable and \( \|F \cdot f\|_1 \leq \sup_{s \in G} \|F(s)\| \|f\|_1 \), so \( F \cdot f \in L^1(G, A) \) for every \( f \in L^1(G, A) \).
**Definition 2.1.** A bounded, pointwise-measurable, function \( F : G \to \mathcal{CB}(A) \) will be called a Herz–Schur \((A, G, \alpha)\)-multiplier if the map
\[
S_F : (\pi \times \lambda)(L^1(G, A)) \to (\pi \times \lambda)(L^1(G, A)); \quad S_F((\pi \times \lambda)(f)) := (\pi \times \lambda)(F \cdot f)
\]
is completely bounded; if this is the case then \( S_F \) has a unique extension to a completely bounded map on \( A \rtimes_{\alpha, r} G \). The set of all Herz–Schur \((A, G, \alpha)\)-multipliers is an algebra with respect to the obvious operations; we denote it by \( \mathcal{S}(A, G, \alpha) \) and endow it with the norm \( \|F\|_{\text{HS}} := \|S_F\|_{\text{cb}} \).

It will be necessary to consider covariant representations of \((A, G, \alpha)\) defined differently to the pair \((\lambda, \pi)\) above. We first introduce notation to account for the Hilbert space where \( A \) is represented, then consider representations involving the weak* topology. If \((\theta, \mathcal{H}_\theta)\) is a faithful representation of \( A \) then we can define a covariant pair \((\pi^\theta, \lambda^\theta)\) as follows:
\[
(5) \quad \pi^\theta : A \to \mathcal{B}(L^2(G, \mathcal{H}_\theta)); \quad (\pi^\theta(a)\xi)(t) := \theta(\alpha_{r^{-1}}(a))(\xi(t)),
\]
and
\[
(6) \quad \lambda^\theta : G \to \mathcal{B}(L^2(G, \mathcal{H}_\theta)); \quad (\lambda^\theta_s\xi)(s) := \xi(t^{-1}s)
\]
for all \( a \in A, s, t \in G, \xi \in L^2(G, \mathcal{H}_\theta) \). Define \( A \rtimes_{\alpha, \theta} G := (\pi^\theta \times \lambda^\theta)(A \rtimes_{\alpha, r} G) \).

Since the closure of \((\pi^\theta \times \lambda^\theta)(L^1(G, A))\) is isomorphic to \( A \rtimes_{\alpha, r} G \) (see e.g. Pedersen [16, Theorem 7.7.5]) it follows that \( F \) is a Herz–Schur \((A, G, \alpha)\)-multiplier if and only if the map
\[
S^\theta_F : (\pi^\theta \times \lambda^\theta)(f) \mapsto (\pi^\theta \times \lambda^\theta)(F \cdot f), \quad f \in L^1(G, A),
\]
is completely bounded, so Herz–Schur \((A, G, \alpha)\)-multipliers can be defined using any faithful representation of \( A \) [15, Remark 3.2(ii)]. Let \( \alpha^\theta : G \to \text{Aut}(\theta(A)) \) be given by \( \alpha^\theta_t(\theta(a)) := \theta(\alpha_t(a)) \) \((t \in G, a \in A)\); note that if \( \alpha \) is continuous in the point-norm topology then so is \( \alpha^\theta \). We say \( \alpha \) is a \( \theta \)-action if \( \alpha^\theta \) extends to a weak*-continuous automorphism of \( \theta(A)^\vee \) such that the map \( t \mapsto \alpha^\theta_t(x) \) is weak*-continuous for each \( x \in \theta(A)^\vee \).

Let \( \lambda^\theta \) be as above and define
\[
\pi : \theta(A) \to \mathcal{B}(L^2(G, \mathcal{H}_\theta)); \quad \left( \pi(\theta(a))\xi \right)(t) := \alpha^\theta_{t^{-1}}(a)(\xi(t)),
\]
for all \( a \in A, t \in G, \xi \in L^2(G, \mathcal{H}_\theta) \). Then \((\pi, \lambda^\theta)\) is a covariant pair, so can be used to define \( \theta(A) \rtimes_{\alpha^\theta, \theta} G \) and we have
\[
\left( \pi(\theta(a))\xi \right)(t) = \alpha^\theta_{t^{-1}}(\theta(a))(\xi(t)) = \theta(\alpha_{r^{-1}}(a))(\xi(t)) = (\pi^\theta(a)\xi)(t)
\]
for all \( a \in A, t \in G, \xi \in L^2(G, \mathcal{H}_\theta) \). It follows that \( A \rtimes_{\alpha, \theta} G = \theta(A) \rtimes_{\alpha^\theta, \theta} G \).

We will need to work with \( \overline{A \rtimes_{\alpha, \theta} G}^{\omega^*} \), which we denote by \( A \rtimes_{\alpha, \theta} G \).

Let \( M \) be a von Neumann algebra on a Hilbert space \( \mathcal{H} \), and \( \beta : G \to \text{Aut}(M) \) a group homomorphism which is continuous in the point-weak* topology; then the triple \((M, G, \beta)\) is called a \( W^* \)-dynamical system. If we define a normal representation \( \pi \) of \( M \) on \( L^2(G, \mathcal{H}) \), analogously to \([3]\), and \( \lambda \) as in \([4]\), then we again obtain a covariant pair of representations \((\pi, \lambda)\).
of \((M,G,\beta)\). The \textit{(von Neumann) crossed product} of \((M,G,\beta)\), denoted \(M \rtimes^\text{\text{\(\beta\)}} G\), is the von Neumann algebra generated by \(\pi(M)\) and \(\lambda(G)\) in \(L^2(G,H)\). See Takesaki [22, Chapter X] for more on this construction.

Classically, \(u : G \to \mathbb{C}\) is called a Herz–Schur multiplier if \(u\) is a completely bounded multiplier of the Fourier algebra of \(G\) (the Fourier algebra of \(G\), \(A(G)\), will be defined in Section 3) \(i.e.\) \(uv \in A(G)\) for all \(v \in A(G)\) and the map

\[
m_u : A(G) \to A(G); \quad m_u(v) := uv, \quad v \in A(G),
\]

is completely bounded; the space of such functions is denoted \(\text{M}^{\text{cb}} A(G)\).

Bożejko–Fendler [3] discuss several equivalent definitions of Herz–Schur multipliers, including: Herz–Schur multipliers on \(G\) coincide with the completely bounded multipliers of \(vN(G)\). One can further show that if \(u\) is a Herz–Schur multiplier of \(G\) then \(m_u^* : vN(G) \to vN(G)\) leaves \(C^*_r(G)\) invariant. In defining Herz–Schur \((A,G,\alpha)\)-multipliers we took the ‘reverse’ approach, defining first a map on \(A \rtimes_{\alpha,\theta} G\). If the dynamical system in question is \((\mathbb{C},G,1)\) then the corresponding crossed product is precisely \(C^*_r(G)\), so (identifying \(CB(\mathbb{C})\) with \(\mathbb{C}\)) we have that \(u\) is a Herz–Schur \((\mathbb{C},G,1)\)-multiplier if and only if \(u\) is a Herz–Schur multiplier. The goal of Section 3 is to introduce a space for a \(C^*\)-dynamical system \((A,G,\alpha)\) which generalises the Fourier algebra of a locally compact group, and identify Herz–Schur \((A,G,\alpha)\)-multipliers with the completely bounded ‘multipliers’ of this space. Unlike the classical case it is not clear if the map \(S_F\) corresponding to \(F \in \mathcal{G}(A,G,\alpha)\) extends to the weak*-closure of \(A \rtimes_{\alpha,\theta} G\), so we make the following definition.

**Definition 2.2.** Let \((\theta,H_\theta)\) be a faithful representation of \(A\). A bounded function \(F : G \to \mathcal{B}(A)\) will be called a \(\theta\)-multiplier of \((A,G,\alpha)\) if the map

\[
S_F^\theta : \pi^\theta(a)\lambda^\theta_t \mapsto \pi^\theta(F(t)(a))\lambda^\theta_t, \quad a \in A, \quad t \in G,
\]

has an extension to a bounded weak*-continuous map on \(A \rtimes_{\alpha,\theta}^w G\). We say \(F\) is a Herz–Schur \(\theta\)-multiplier if \(S_F^\theta\) extends to a completely bounded, weak*-continuous map on \(A \rtimes_{\alpha,\theta}^w G\).

Note that [15, Remark 3.4] shows that Herz–Schur \(\theta\)-multipliers of \((A,G,\alpha)\) act in the same way as Herz–Schur \((A,G,\alpha)\)-multipliers, when viewed through a weak*-continuous functional. To simplify notation I will often omit the superscript \(\theta\) from the multiplication map \(S_F\) associated to a Herz–Schur \((A,G,\alpha)\)-multiplier; it will be clear from the presence/absence of \(\theta\) elsewhere in the notation where \(S_F\) is acting.

Let \(\Gamma\) be another locally compact group. Then we define

\[
\alpha^\Gamma : \Gamma \times G \to \text{Aut}(A); \quad \alpha^\Gamma_{(\gamma,t)} := \alpha_t,
\]

and

\[
(\pi^\theta)^\Gamma : A \to \mathcal{B}(L^2(\Gamma \times G,H_\theta)); \quad (\pi^\theta)^\Gamma(a)\xi(\gamma,t) := \alpha^\Gamma_{(\gamma,t^{-1})}(a)\xi(\gamma,t),
\]
for all $\gamma \in \Gamma$, $t \in G$, $a \in A$, $\xi \in L^2(\Gamma \times G, \mathcal{H}_\theta)$. Note that if we identify $L^2(\Gamma \times G, \mathcal{H}_\theta)$ with $L^2(\Gamma) \otimes L^2(G, \mathcal{H}_\theta)$ in the obvious way then $(\pi^\theta)^\Gamma = L^2(\Gamma) \otimes \pi^\theta$. If $\lambda$ is the left regular representation of $\Gamma \times G$ on $L^2(\Gamma \times G, \mathcal{H}_\theta)$ (so $\lambda_{(s,t)} = \lambda^\Gamma_s \otimes \lambda^G_t$) then $((\pi^\theta)^\Gamma, \lambda)$ is a covariant representation of the $C^*$-dynamical system $(A, \Gamma \times G, \alpha^\Gamma)$ and $A \rtimes_{\alpha^\Gamma, \theta}^w (\Gamma \times G)$ can be identified with $\nu N(\Gamma) \widehat{\otimes} A \rtimes_{\alpha^G}^w G$ [15, Proposition 3.19]. We have the following characterisation [15, Proposition 3.19] in the spirit of de Cannière–Haagerup [6, Theorem 1.6].

**Proposition 2.3.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system, $F : G \to CB(A)$, and $(\theta, \mathcal{H}_\theta)$ a faithful representation of $A$. The following are equivalent:

i. $F$ is a Herz–Schur $\theta$-multiplier of $(A, G, \alpha)$;

ii. for any second-countable locally compact group $\Gamma$, $F^\Gamma$ is a $\theta$-multiplier of $(A, \Gamma \times G, \alpha^\Gamma)$;

iii. $F^{SU(2)}$ is a $\theta$-multiplier of $(A, SU(2) \times G, \alpha^{SU(2)})$.

In parallel with Herz–Schur $(A, G, \alpha)$-multipliers we have also introduced a more general version of Schur multipliers [15, Section 2]. I will recall the basics and give the results which we require.\footnote{In [15] some of these definitions and results are given in a slightly more general setting not required here.} Let $A$ be a $C^*$-algebra and assume $A \subseteq \mathcal{B}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$. Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces (in the sequel we will only need the case $X = Y = G$). To any $k \in L^2(Y \times X, A)$ one can associate an element $T_k \in \mathcal{B}(L^2(X, \mathcal{H}), L^2(Y, \mathcal{H}))$, with $\|T_k\| \leq \|k\|_2$, by

$$(T_k \xi)(y) := \int_X k(y, x) (\xi(x)) \, d\mu(x), \quad y \in Y, \; \xi \in L^2(X, \mathcal{H}).$$

The linear space of all such operators is denoted by $\mathcal{S}_2(Y \times X, A)$ and is norm dense in $\mathcal{K}(L^2(X), L^2(Y)) \otimes_{\min} A$. If $\varphi : X \times Y \to CB(A)$ is a bounded, pointwise-measurable, function we define $\varphi \cdot k \in L^2(Y \times X, A)$ by

$$\varphi \cdot k(y, x) := \varphi(x, y)(k(y, x)), \quad (y, x) \in Y \times X.$$ 

Let $S_{\varphi}$ denote the map on $\mathcal{S}_2(Y \times X, A)$ given by

$$S_{\varphi}(T_k) := T_{\varphi \cdot k}, \quad k \in L^2(Y \times X, A).$$

If $\varphi : X \times Y \to CB(A)$ is bounded and pointwise-measurable and $(\theta, \mathcal{H}_\theta)$ is a faithful representation of $A$ on a separable Hilbert space then we define $\varphi_\theta : X \times Y \to CB(\theta(A))$ by $\varphi_\theta(x, y)(\theta(a)) := \theta(\varphi(x, y)(a)) (a \in A, \; (x, y) \in X \times Y)$; one then obtains a map $S_{\varphi_\theta}$ on $\mathcal{S}_2(Y \times X, \theta(A))$ as $S_{\varphi}$ above. It is not difficult to show that if $\theta_1$ and $\theta_2$ are two faithful representations of $A$ on separable Hilbert spaces then $S_{\varphi_\theta_1}$ is completely bounded if and only if $S_{\varphi_\theta_2}$ is completely bounded, and in this case $\|S_{\varphi_\theta_1}\|_{cb} = \|S_{\varphi_\theta_2}\|_{cb}$ [15, Proposition 2.3]. Thus the definition below does not depend on the separable Hilbert space on which $A$ acts.
Definition 2.4. A bounded, pointwise-measurable, function \( \varphi : X \times Y \to \mathcal{CB}(A) \) will be called a Schur \( A \)-multiplier if \( S\varphi \) is a completely bounded map on \( S_2(Y \times X, A) \) We denote the space of such functions by \( \mathcal{S}_0(X, Y; A) \) and endow it with the norm \( \| \varphi \|_{\mathcal{S}} := \| S\varphi \|_{cb} \). Let \( \theta, H_\theta \) be a faithful representation of \( A \) on a separable Hilbert space. We say \( \varphi \) is a Schur \( \theta \)-multiplier of \( A \) if \( S\varphi \theta \) extends to a completely bounded, weak*-continuous, map on \( \mathcal{B}(L^2(X), L^2(Y)) \otimes \theta(A)' \).

When working with Schur \( A \)-multipliers it is convenient to assume that \( A \subseteq B(H) \) for some separable Hilbert space \( H \), removing the need for the subscripts denoting the representation in the above discussion. Unfortunately we have no such luxury for Schur \( \theta \)-multipliers as we do not know if the existence of a weak* extension is independent of the representation of \( A \). We have characterised Schur \( A \)-multipliers in the following theorem [15, Theorem 2.6].

Theorem 2.5. Let \( A \subseteq B(H) \) be a C*-algebra and \( \varphi : X \times Y \to \mathcal{CB}(A) \) be a bounded, pointwise-measurable, function. The following are equivalent:

i. \( \varphi \) is a Schur \( A \)-multiplier;

ii. there exist a separable Hilbert space \( H_\rho \), a non-degenerate representation \( \rho : A \to B(H_\rho) \), \( V \in L^\infty(X, B(H, H_\rho)) \), and \( W \in L^\infty(Y, B(H, H_\rho)) \), such that

\[
\varphi(x, y)(a) = W(y)^* \rho(a) V(x), \quad a \in A,
\]

for almost all \( (x, y) \in X \times Y \).

When the above conditions hold we may choose \( V \) and \( W \) so that \( \| \varphi \|_{\mathcal{S}} = \operatorname{esssup}_{x \in X} \| V(x) \| \operatorname{esssup}_{y \in Y} \| W(y) \| \).

Given a function \( F : G \to \mathcal{CB}(A) \), we define \( \mathcal{N}(F) : G \times G \to \mathcal{CB}(A) \) by

\[
\mathcal{N}(F)(s, t)(a) = \alpha_{t^{-1}} \left( F(ts^{-1}) \left( \alpha_t(a) \right) \right), \quad s, t \in G, \ a \in A.
\]

Note that if \( F \) is pointwise-measurable then so is \( \mathcal{N}(F) \). The following result [15, Theorem 3.5] relates Schur \( A \)-multipliers and Herz–Schur \( (A, G, \alpha) \)-multipliers, generalising the classical transference theorem; see e.g. Bożejko–Fendler [3].

Theorem 2.6. Let \( (A, G, \alpha) \) be a C*-dynamical system and let \( F : G \to \mathcal{CB}(A) \) be a bounded, pointwise-measurable, function. The following are equivalent:

i. \( F \) is a Herz–Schur \( (A, G, \alpha) \)-multiplier;

ii. \( \mathcal{N}(F) \) is a Schur \( A \)-multiplier.

Moreover, if the above conditions hold then \( \| F \|_{\mathcal{HS}} = \| \mathcal{N}(F) \|_{\mathcal{S}} \).

The next result shows that classical Herz–Schur multipliers are Herz–Schur multipliers of any C*-dynamical system.
Lemma 2.7. Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system, and assume \(A \subseteq \mathcal{B}(\mathcal{H})\) for some separable Hilbert space \(\mathcal{H}\). Let \(u : G \rightarrow \mathbb{C}\) be a bounded, continuous, function. Define

\[F_u : G \rightarrow \mathcal{CB}(A); \quad F_u(t)(a) := u(t)a, \quad t \in G, \ a \in A.\]

The following are equivalent:

i. \(u \in \mathcal{M}^{cb} A(G)\);
ii. \(F_u\) is a Herz–Schur \((A, G, \alpha)\)-multiplier;
iii. \(\mathcal{N}(F_u)\) is a Schur \(A\)-multiplier.

If the above conditions hold then \(\|u\|_{\mathcal{M}^{cb}} = \|F_u\|_{\mathcal{HS}} = \|\mathcal{N}(F_u)\|_{\mathcal{E}}\), and \(F_u\) is a Herz–Schur \(\theta\)-multiplier for every faithful representation \((\theta, \mathcal{H})\) of \(A\) on a separable Hilbert space.

Proof. That conditions (i)–(iii) are equivalent follows from Proposition 4.1 and Corollary 3.6 of [15]. It remains to show the equality of norms. It follows from the proof of [15, Proposition 4.1] that for any \(C^*\)-dynamical system \(\|u\|_{\mathcal{M}^{cb}} \leq \text{esssup}_{s \in G} \|V(s)\| \text{esssup}_{t \in G} \|W(t)\| = \|\mathcal{N}(F_u)\|_{\mathcal{E}} = \|F_u\|_{\mathcal{HS}}\)

where \(V\) and \(W\) are the maps associated to the Schur \(A\)-multiplier \(\mathcal{N}(F_u)\) in Theorem 2.5, chosen to satisfy the first equality. For the converse, since \(G\) is second-countable there exist \(\xi, \eta : G \rightarrow \ell^2\) be such that \(u(ts^{-1}) = \langle \xi(s), \eta(t) \rangle\) [11]. The proof of [15, Proposition 4.1] shows that \(\mathcal{N}(F_u)\) is a Schur \(A\)-multiplier, represented as

\[\mathcal{N}(F_u)(s, t)(a) = W(t)^*\rho(a)V(s), \quad s, t \in G, \ a \in A,\]

where \(\rho\) is the countable ampliation of the identity representation of \(A \subseteq \mathcal{B}(\mathcal{H})\), \(V(s) := (\xi_i(s)I_{\mathcal{H}})_{i \in \mathbb{N}}\) \((s \in G)\), \(W(t) = (\eta_i(t)I_{\mathcal{H}})_{i \in \mathbb{N}}\) \((t \in G)\). For any \(s \in G\) we have

\[\|V(s)\|^2 = \|V^*(s)V(s)\| = \sum_{i \in \mathbb{N}} \xi_i(s)\xi_i(s) = \|\xi(s)\|^2;\]

and similarly \(\|W(t)\| = \|\eta(t)\|\) for all \(t \in G\). It follows that \(\|F_u\|_{\mathcal{HS}} = \|\mathcal{N}(F_u)\|_{\mathcal{E}} \leq \|u\|_{\mathcal{M}^{cb}}\). \(\square\)

To close this section we record the definition and main result on weak amenability of a discrete group for reference. Weak amenability was formally defined by Cowling–Haagerup [5], though the result below was proved before this by Haagerup [9]; a concise summary of the argument is given by Brown–Ozawa [4, Theorem 12.3.10].

Definition 2.8. A locally compact group \(G\) is called weakly amenable if there exists a net \((\varphi_i)_{i \in \mathbb{I}} \subseteq \mathcal{M}^{ab} A(G) \cap C_c(G)\) such that \(\varphi_i \rightarrow 1\) uniformly on compact sets and \(\sup_{i \in \mathbb{I}} \|\varphi_i\|_{\mathcal{M}^{ab}} \leq C\), where \(\|\varphi\|_{\mathcal{M}^{ab}}\) denotes the norm of \(\varphi\) as a Herz–Schur multiplier. The infimum of all such \(C\) is called the Cowling–Haagerup constant of \(G\) and denoted \(\Lambda_{cb}(G)\).
If $G$ is not weakly amenable we set $\Lambda_{cb}(G) = \infty$.

**Remark 2.9.** There are several equivalent ways to define weak amenability. Each of the following is equivalent to the above definition of weak amenability of $G$:

- there is a net $(\varphi_i) \subseteq \text{M}^{cb}A(G) \cap C_c(G)$ such that $\|\varphi_i u - u\|_{A(G)} \to 0$ for all $u \in A(G)$, and $\sup_i \|\varphi_i\|_{\text{M}^{cb}} \leq C$;
- there is a net $(\varphi_i) \subseteq A(G)$ such that $\varphi_i \to 1$ uniformly on compact sets and $\sup_i \|\varphi_i\|_{\text{M}^{cb}} \leq C$;
- there is a net $(\varphi_i) \subseteq A(G)$ such that $\|\varphi_i u - u\|_{A(G)} \to 0$ for all $u \in A(G)$, and $\sup_i \|\varphi_i\|_{\text{M}^{cb}} \leq C$.

The fact that uniform convergence on compacta can be replaced with pointwise convergence in $A(G)$ follows from an averaging trick given by Cowling–Haagerup [5, Proposition 1.1] (the same trick had been used by Haagerup in a work which has recently been published [9]).

**Theorem 2.10.** Let $G$ be a discrete group. The following are equivalent:

i. $G$ is weakly amenable;

ii. $C^*_r(G)$ has the completely bounded approximation property;

iii. $vN(G)$ has the weak* completely bounded approximation property.

Moreover, if the conditions hold then $\Lambda_{cb}(G) = \Lambda_{cb}(C^*_r(G)) = \Lambda_{cb}(vN(G))$.

#### 3. Fourier space of a crossed product

In this section we develop a space for the crossed product which is analogous to the Fourier algebra in the setting of group $C^*$-algebras and von Neumann algebras, and study the multipliers of this space. To motivate this discussion and fix notation let us first recall some facts about the Fourier algebra of a locally compact group $G$. The Fourier algebra of $G$, introduced by Eymard [7], denoted $A(G)$, is the space of coefficients of the left regular representation; that is, the space of functions $u : G \to \mathbb{C}$ of the form

$$u(t) = \langle \lambda^G_t \xi, \eta \rangle, \quad t \in G, \ \xi, \eta \in L^2(G).$$

The linear space defined in this way becomes an algebra under pointwise multiplication, and turns out to be the predual of the group von Neumann algebra $vN(G)$; the duality is given by

$$\langle \lambda^G_s, u \rangle = u(s), \quad u \in A(G), \ s \in G.$$ 

Bożejko–Fendler [3] proved that the space $M^{cb}A(G)$ is isometrically isomorphic to the space of Herz–Schur multipliers of $G$, so they are treated as the same space.

Recall that $A$ denotes a unital $C^*$-algebra and $\alpha : G \to \text{Aut}(A)$ is a point-norm continuous homomorphism. The following definition is adapted from Pedersen [16, 7.7.4].
Definition 3.1. Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system and let \((\theta, \mathcal{H}_\theta)\) be a faithful representation of \(A\). Let \(\tilde{u} \in (A \rtimes_{\alpha, \theta} G)^*\) be a functional of the form

\begin{equation}
\tilde{u}(T) = \sum_n \langle T\xi_n, \eta_n \rangle, \quad T \in A \rtimes_{\alpha, \theta} G,
\end{equation}

where \(\xi_n, \eta_n \in L^2(G, \mathcal{H}_\theta)\) satisfy \(\sum_n \|\xi_n\|^2 < \infty, \sum_n \|\eta_n\|^2 < \infty\). The set of such functionals forms a linear space which can be identified with \(\pi(\mathcal{A}^* \rtimes_{\alpha, \theta} G)^*\). To each such \(\tilde{u}\) we associate the function \(u : G \to A^*\) defined by

\begin{equation}
u(t)(a) := \tilde{u}(\pi^\theta(a)\lambda^\theta_i), \quad a \in A, \; t \in G.
\end{equation}

The set of all functions from \(G\) to \(A^*\) associated to functionals of the form of \(\tilde{u}\) is a linear space (with the obvious operations), which we again identify with the predual of \((A \rtimes_{\alpha, \theta} G)^*\). To each such \(\tilde{u}\) we associate the function \(u : G \to A^*\) defined by

\begin{equation}
\|u\|_A := \|\tilde{u}\|
\end{equation}

where the right side means the norm of \(\tilde{u}\) as a member of the dual space of \((A \rtimes_{\alpha, \theta} G)^*\). This defines a norm on \(\mathcal{A}^\theta(A, G, \alpha)\) since \(u \in \mathcal{A}^\theta(A, G, \alpha)\) is the zero map if and only if the associated functional \(\tilde{u}\) is the zero functional. The resulting space is called the Fourier space of \((A, G, \alpha)\) and denoted \(\mathcal{A}^\theta(A, G, \alpha)\).

In the case of the system \((\mathbb{C}, G, 1)\) the only representation \(\theta\) of \(\mathbb{C}\) is trivial, \(\pi^\theta\) also becomes trivial, and we can identify \(\lambda^\theta\) with \(\lambda^G\); thus the above definition gives the predual of \((\mathbb{C} \rtimes_{1, \tau} G)^* \cong \nu^\mathbb{N}(G)\), so the space defined may be identified with \(A(G)\). Definition 3.1 also works unchanged for a \(W^*\)-dynamical system \((M, G, \beta)\); in this case the definition identifies the predual of the von Neumann algebra \(M \rtimes^\beta_{\nu^\mathbb{N}} G\) with the space of functions \(u : G \to M_\pi\) of the form \(\mathbb{R}\) [21]. Next we show that, as for \(A(G)\), the compactly supported functions are dense in the Fourier space of a dynamical system; the proof is from Fujita [8, Lemma 3.4].

Remark 3.2. Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system and \((\theta, \mathcal{H}_\theta)\) a faithful representation of \(A\). The compactly supported functions form a dense subset of \(\mathcal{A}^\theta(A, G, \alpha)\). The same holds for a \(W^*\)-dynamical system.

Proof. Let \(u \in \mathcal{A}^\theta(A, G, \alpha)\) and suppose first that the associated functional on \(A \rtimes_{\alpha, \theta} G\) is of the form \(u(T) = \langle T\xi, \eta \rangle\), with \(\xi, \eta\) elementary tensors in \(C_c(G) \otimes \mathcal{H}_\theta\); in this case it is clear that \(u\) has compact support. Now if \(\xi, \eta\) are arbitrary elements of \(L^2(G) \otimes \mathcal{H}_\theta\) we can approximate them by \(\xi_i, \eta_i\) respectively, where \(\xi_i, \eta_i\) are finite sums of elementary tensors in \(C_c(G) \otimes \mathcal{H}_\theta\). Let \(\tilde{u}_i\) denote the associated vector functional. Then

\begin{align*}
| \langle T, \tilde{u} \rangle - \langle T, \tilde{u}_i \rangle | &= | \langle T\xi, \eta \rangle - \langle T\xi_i, \eta_i \rangle | \\
&\leq \|T\| (\|\xi - \xi_i\|\|\eta\| + \|\eta - \eta_i\|\|\xi_i\|),
\end{align*}

where \(\|\xi - \xi_i\| \to 0\) as \(i \to \infty\) since \(\xi_i\) are finite sums of elementary tensors.
which implies \( \| \tilde{u} - \tilde{u}_i \| \to 0 \). The first part of the proof implies the function \( u_i \in A^\theta(A, G, \alpha) \) associated to \( \tilde{u}_i \) is compactly supported. To complete the proof note that for arbitrary \( u \in A^\theta(A, G, \alpha) \) the associated functional can be approximated by finite sums of the \( \tilde{u}_i \).

The proof for \( W^* \)-dynamical systems is identical. \( \square \)

It appears that the space \( A^\theta(A, G, \alpha) \) was first defined for \( W^* \)-dynamical systems and their crossed products by Takai [21]. Pedersen defined the Fourier space of a \( C^* \)-dynamical system \((A, G, \alpha)\), which is the predual of the enveloping von Neumann algebra of the reduced \( C^* \)-crossed product \( A \rtimes_{\alpha, r} G \), i.e. the predual of \((A \rtimes_{\alpha, r} G)^''\). Note that in the case of a \( W^* \)-dynamical system Fujita [8] introduces a Banach algebra structure on \( A^\theta(A, G, \alpha) \), but we do not pursue this here.

We now define multipliers of the Fourier space of a \( C^* \)-dynamical system, and study the relationship with Herz–Schur multipliers of the system. The results in this section are essentially predual versions of some results in [15, Section 3].

**Definition 3.3.** A bounded function \( F : G \to B(A) \) is called a multiplier of \( A^\theta(A, G, \alpha) \) if there is a bounded map \( s_F : A^\theta(A, G, \alpha) \to A^\theta(A, G, \alpha) \)

such that

\[(s_Fu)(t)(a) = u(t)(F(t)(a)), \quad u \in A^\theta(A, G, \alpha), \quad t \in G, \quad a \in A.\]

The norm of a multiplier \( F \) is defined by \( \| F \|_M := \| s_F^* \| \). If moreover \( F \) maps into \( \mathcal{C}B(A) \) and \( s_F^* \) is completely bounded then \( F \) is called a completely bounded multiplier of \( A^\theta(A, G, \alpha) \). In this case the completely bounded multiplier norm of \( F \) is defined \( \| F \|_{M_{cb}} := \| s_F^* \|_{cb} \). The spaces of bounded and completely bounded multipliers of \( A^\theta(A, G, \alpha) \) are denoted \( M^A \) and \( M_{cb}^A \) respectively.

In what follows I will use the definitions and notation used in Proposition 2.3.

**Lemma 3.4.** Let \( F : G \to \mathcal{B}(A) \) be a bounded, pointwise-measurable, function, and \((\theta, H_\theta)\) be a faithful representation of \( A \). The following are equivalent:

i. \( F \) is a multiplier of \( A^\theta(A, G, \alpha) \);

ii. there is an ultraweakly continuous bounded operator \( S_F \) on \((A \rtimes_{\alpha, \theta} G)^''\) such that \( S_F(\pi^\theta(a)\lambda^\theta_t) = \pi(\theta(F(t)(a))\lambda^\theta_t) \) for all \( a \in A, \ t \in G \).

Moreover, if either condition holds then \( \| F \|_M = \| S_F \| \). Finally, \( F \) is a completely bounded multiplier of \( A^\theta(A, G, \alpha) \) if and only if the map \( S_F \) of (ii) is completely bounded, and in this case \( \| F \|_{M_{cb}} = \| S_F \|_{cb} \).
Proof. If $F$ is a multiplier of $A^\vartheta(A,G,\alpha)$ then $S_F := s_F^*$ is the required map because for any $u \in A^\vartheta(A,G,\alpha)$

$$\langle S_F(\pi^\vartheta(a)\lambda_t^\vartheta), u \rangle = \langle \pi^\vartheta(a)\lambda_t^\vartheta, su \rangle = u(t)(F(t)(a)) = \langle \pi^\vartheta(F(t)(a))\lambda_t^\vartheta, u \rangle.$$ 

Conversely, given $u \in A^\vartheta(A,G,\alpha)$, the function

$$\pi^\vartheta(a)\lambda_t^\vartheta \mapsto \langle S_F(\pi^\vartheta(a)\lambda_t^\vartheta), u \rangle$$

extends to an ultraweakly continuous linear functional on $(A \rtimes_{\alpha,\vartheta} G)^\vartheta$. Therefore, there is $F u \in A^\vartheta(A,G,\alpha)$ with $\|F u\| \leq \|u\|_{A^\vartheta} \|S_F\|$, such that

$$\langle \pi^\vartheta(a)\lambda_t^\vartheta, F u \rangle = \langle S_F(\pi^\vartheta(a)\lambda_t^\vartheta), u \rangle.$$ 

It follows that the map $u \mapsto F u$ is continuous, and

$$(F u)(t)(a) = \langle \pi^\vartheta(a)\lambda_t^\vartheta, F u \rangle = \langle S_F(\pi^\vartheta(a)\lambda_t^\vartheta), u \rangle = u(t)(F(t)(a)),$$

for all $t \in G$, $a \in A$, so $F$ is a multiplier of $A^\vartheta(A,G,\alpha)$ with $s_F u := F u$ for all $u \in A^\vartheta(A,G,\alpha)$. Finally, $\|F\|_M = \|s_F^*\| = \|S_F\|$ by definition. The statements about completely bounded multipliers follow similarly. \qed

Since the ultraweak topology on $(A \rtimes_{\alpha,\vartheta} G)^\vartheta$ is the relative ultraweak topology from $B(L^2(G) \otimes \mathcal{H}_\vartheta)$ we consider the map $S_F$ of the previous lemma to be a weak*-continuous map on $A \rtimes_{\alpha,\vartheta} G$.

Lemma \[3.4\] suggests that (completely bounded) multipliers of the Fourier space of a $C^*$-dynamical system are connected to the (Herz–Schur) multipliers of the system. We will obtain this connection after generalising a result of de Cannière–Haagerup \[6, Theorem 1.6\]. The proof is based on their argument and the proof of Proposition \[2.3\].

**Proposition 3.5.** Let $F : G \to CB(A)$ be a multiplier of $A^\vartheta(A,G,\alpha)$ and let $(\vartheta, \mathcal{H}_\vartheta)$ be a faithful representation of $A$. The following are equivalent:

i. $F$ is a completely bounded multiplier of $A^\vartheta(A,G,\alpha)$;

ii. for any second-countable, locally compact, group $\Gamma$, $F^\Gamma$ is a multiplier of $A^\vartheta(A,\Gamma \times G, \alpha^\Gamma)$;

iii. $F^{SU(2)}$ is a multiplier of $A^{SU(2)}(A, SU(2) \times G, \alpha^{SU(2)}).

Moreover, when these conditions hold, $\|F\|_{M_{cb}} = \|F^{SU(2)}\|_M$.

**Proof.** (i)$\Rightarrow$(ii) If $F$ is a completely bounded multiplier of $A^\vartheta(A,G,\alpha)$ then $s_F^* = S_F : A \rtimes_{\alpha,\vartheta} G \to A \rtimes_{\alpha,\vartheta} G$ is completely bounded and weak*-continuous as in Lemma \[3.4\]. Now $A \rtimes_{\alpha,\vartheta} (\Gamma \times G) \cong vN(\Gamma) \otimes A \rtimes_{\alpha,\vartheta} G$ (see the proof of \[15, Proposition 3.15\]), in particular $(\pi^\vartheta)^\Gamma(a)\lambda_\gamma = \lambda_\gamma^\vartheta \otimes \pi^\vartheta(a)\lambda_\gamma^\vartheta$, so by de Cannière–Haagerup \[6, Lemma 1.5\] there is a weak*-continuous map $\tilde{S}_F$ on $vN(\Gamma) \overline{\otimes} A \rtimes_{\alpha,\vartheta} G$ such that $\tilde{S}_F(x \otimes y) = x \otimes S_F(y)$ and $\|\tilde{S}_F\| \leq \|S_F\|_{cb}$.
In particular, for all \( a \in A, \gamma \in \Gamma, \ t \in G \),
\[
\tilde{S}_F \left( (\pi^\theta)^\Gamma (a) \lambda_{(\gamma,t)} \right) =  \lambda_{\gamma}^T \otimes S_F (\pi^\theta(a) \lambda_t^\theta) = \lambda_{\gamma}^T \otimes \pi^\theta (F(t)(a)) \lambda_t^\theta
\]
\[
= (\pi^\theta)^\Gamma (F(t)(a)) \lambda_{(\gamma,t)}
\]
\[
= (\pi^\theta)^\Gamma (F^\Gamma (\gamma,t)(a)) \lambda_{(\gamma,t)}.
\]

It follows that \( S_{FT} := \tilde{S}_F \) satisfies Lemma 3.4(ii), so \( F^\Gamma \) is a multiplier of \( \mathcal{A}^0 (A,G \times G, \alpha^\Gamma) \).

(iii)⇒(i) By Lemma 3.4 there exists a weak*-continuous map \( S_{FSU(2)} \) on \( A \times_{\alpha_{SU(2)},\theta}^w (SU(2) \times G) \cong vN(SU(2)) \otimes A \times_{\alpha,\theta}^w G \), and it is easy to see that in this case \( S_{FSU(2)} = \id_{vN(SU(2))} \otimes S_F \). Since \( vN(SU(2)) \cong \bigoplus_{n \in \mathbb{N}} M_n \) the restriction of \( S_{FSU(2)} \) to each component in the direct summand of
\[
vN(SU(2)) \otimes A \times_{\alpha,\theta}^w G \cong \bigoplus_{n \in \mathbb{N}} (M_n \otimes A \times_{\alpha,\theta}^w G)
\]
implies that \( S_F \) is completely bounded, with \( \|S_F\|_{cb} \leq \|S_{FSU(2)}\|_M \).

Finally, from (i)⇒(ii) we have, for every locally compact group \( \Gamma \)
\[
\|F^\Gamma\|_M = \|S_{FT}\| = \|\tilde{S}_F\| \leq \|S_F\|_{cb} = \|F\|_{M_{cb}}.
\]

On the other hand, from (iii)⇒(i),
\[
\|F\|_{M_{cb}} = \|S_F\|_{cb} \leq \|S_{FSU(2)}\| = \|F^{SU(2)}\|_M.
\]
Hence \( \|F\|_{M_{cb}} = \|F^{SU(2)}\|_M \).

\[\Box\]

**Corollary 3.6.** The space of Herz–Schur \( \theta \)-multipliers of \((A,G,\alpha)\) coincides isometrically with the space of completely bounded multipliers of \( \mathcal{A}^0 (A,G,\alpha) \).

**Proof.** Lemma 3.4 implies that, for any locally compact group \( \Gamma \), \( F^\Gamma \) is a multiplier of \( \mathcal{A}^0 (A,\Gamma \times G, \alpha^\Gamma) \) if and only if \( F^\Gamma \) is a \( \theta \)-multiplier of \((A,\Gamma \times G, \alpha^\Gamma)\); thus condition (ii) of Proposition 2.3 is equivalent to condition (ii) of Proposition 3.5. Finally, by Lemma 3.4 and (the proof of) Proposition 2.3 we have
\[
\|F\|_{M_{cb}} = \|F^{SU(2)}\|_M = \|S_{FSU(2)}\| = \|S_F\|_{cb} = \|F\|_{HS}.
\]
\[\Box\]

In the next section we will use the description of Herz–Schur multipliers of a dynamical system as completely bounded multipliers of the Fourier space in studying weak amenability of the system.

**Remark 3.7.** Bédos and Conti [1, Section 4] have taken a Hilbert \( C^* \)-module approach to completely bounded multipliers of a discrete (twisted) \( C^* \)-dynamical system. It is easy to check that \( F : G \to CB(A) \) is a Herz–Schur \((A,G,\alpha)\)-multiplier if and only if \( T_F : G \times A \to A; T_F(t,a) := F(t)(a) \) \((t \in G, a \in A)\) is a completely bounded reduced multiplier of \((A,G,\alpha)\), in the sense of Bédos–Conti. The same authors have also introduced a version
of the Fourier–Stieltjes algebra for discrete (twisted) $C^*$-dynamical systems, again using Hilbert $C^*$-modules [2]. It is interesting to note that, for a $C^*$-dynamical system $(A, G, \alpha)$ with $A \subseteq B(H)$, it follows from Corollary A.10 and the above equivalence that the completely bounded reduced multipliers of Bédos–Conti which extend to the weak* closure of the reduced crossed product are completely bounded multipliers of the Fourier space $A^{\text{id}}(A, G, \alpha)$.

We close this section by considering a transformation group, which can be viewed as a $C^*$-dynamical system or a measured groupoid. Renault [18] has introduced the Fourier algebra of a measured groupoid and studied its multipliers; here we relate his perspective on multipliers of the Fourier algebra of a transformation group to the one given in this section (see also [15, Section 5.2]). We refer to Renault [17] for the necessary background on measured groupoids, in particular the transformation groups briefly outlined below. The calculations which show the groupoid $C^*$-algebra can be identified with a crossed product are given in [15, Section 5.2].

Let $G$ be a second-countable, locally compact, group acting on a locally compact Hausdorff space $X$ from the right, i.e. there is a jointly continuous map

$$X \times G \to X; \ (x, t) \mapsto xt, \ x \in X, \ t \in G,$$

such that $(xt)s = x(ts)$ $(x \in X, \ s,t \in G)$. The space $G := X \times G$ is a groupoid. The set $G^{(2)}$ of composable pairs is

$$G^{(2)} = \{(x, t), (y, s) \} \in G \times G : y = xt \},$$

with multiplication $G^{(2)} \to G$ given by $(x, t)(xt, s) := (x, ts)$. The domain and range maps are given by

$$d(x, t) := (x, t)^{-1}(x, t) = (xt, e), \ r(x, t) := (x, t)(x, t)^{-1} = (x, e),$$

for all $(x, t) \in G$; it follows that the unit space $G_0$ can be identified with $X$.

The space $C_c(G)$ is a Banach $*$-algebra when identified with a subalgebra of $C_c(G, C_0(X))$, with the $*$-algebra structure defined as in [1] and [2] except for the absence of the modular function in the definition of convolution. It is shown in [15, Section 5.2] that there is an injective $*$-homomorphism $\phi$ which identifies $C_c(G)$ with a subspace of $C_c(G, C_0(X))$ under the usual operations [1] and [2]. There is a distinguished representation of $C_c(G)$ on $L^2(G)$ called the regular representation and denoted Reg. The von Neumann algebra of $G$, denoted $\mathcal{vN}(G)$, is defined as $\mathcal{vN}(G) := (\text{Reg}(C_c(G)))''$. For $f \in C_0(X)$ define

$$M_f : L^2(X) \to L^2(X); \ M_f(x) := f(x)\xi(x), \ \xi \in L^2(X), \ x \in X,$$

to obtain a faithful representation $\theta : f \mapsto M_f$ of $C_0(X)$ on $L^2(X)$. For each $t \in G$ and $a \in C_0(X)$ define $\alpha_t(a)(x) := a(tx), \ (x \in X)$. Then $\alpha : G \to \text{Aut}(C_0(X)); \ t \mapsto \alpha_t$ is a homomorphism, continuous in the point-norm topology; thus $(C_0(X), G, \alpha)$ is a $C^*$-dynamical system. Associated to the representation $\theta$ of $C_0(X)$ is the covariant representation $(\pi^\theta, \lambda^\theta)$ of the
bounded. This characterisation, together with the C$^*$-view of $G$ system (5) and (6). It is shown in [15, Section 5.2] that $(\pi^\theta \rtimes \lambda^\theta) \circ \phi$ is unitarily equivalent to Reg.

Renault [18] defines the Fourier algebra of $\mathcal{G}$, $A(\mathcal{G})$, to be the space of coefficients of the regular representation of $\mathcal{G}$; we do not define this precisely because Renault shows in the same paper that $\varphi \in L^\infty(\mathcal{G})$ is a contractive multiplier of $A(\mathcal{G})$ if and only if the map

$$\text{Reg}(f) \mapsto \text{Reg}(\varphi f), \quad f \in C_c(\mathcal{G}),$$

where $(\varphi f)(x, t) := \varphi(x, t)f(x, t)$ $(x \in X, t \in G)$, defines a bounded linear map of norm at most 1 on $vN(\mathcal{G})$. Moreover $\varphi$ is a completely bounded multiplier of $A(\mathcal{G})$ if and only if the associated map on $vN(\mathcal{G})$ is completely bounded. This characterisation, together with the $C^*$-dynamical system view of $\mathcal{G}$ given above, imply the Proposition below. The same observation, given in terms of Herz–Schur $(C_0(X), G, \alpha)$-multipliers, was made in [15, Proposition 5.3]. Either of these can be derived from the other by applying Corollary 3.6.

**Proposition 3.8.** Let $G$ be a second-countable, locally compact, group acting on a locally compact Hausdorff space $X$ from the right, and let $\mathcal{G} = X \times G$ be the associated groupoid. Let $\theta : f \mapsto M_f$ denote the faithful representation of $C_0(X)$ on $L^2(X)$. Let $\varphi : X \times G \to \mathbb{C}$ be an element of $L^\infty(\mathcal{G})$, and define

$$F_\varphi : G \to CB(C_0(X)); \quad (F_\varphi(t)(a))(x) := \varphi(x, t)a(x),$$

for all $x \in X, t \in G, a \in C_0(X)$. The following are equivalent:

i. $\varphi$ is a completely bounded multiplier of $A(\mathcal{G})$;

ii. $F_\varphi$ is a completely bounded multiplier of $A^\theta(C_0(X), G, \alpha)$.

**Proof.** Consider $\pi^\theta \rtimes \lambda^\theta$ as a representation of $\phi(C_c(\mathcal{G}))$, and observe that

$$(F_\varphi \cdot (\phi(f))(t))(x) = \phi(\varphi f)(t)(x) \quad \text{for all } f \in C_c(\mathcal{G}), x \in X, t \in G.$$ 

The unitary equivalence of Reg and $\pi^\theta \rtimes \lambda^\theta \circ \phi$ stated above implies that the map $\text{Reg}(f) \mapsto \text{Reg}(\varphi f)$ is completely bounded if and only if the map $(\pi^\theta \rtimes \lambda^\theta)(\varphi f) \mapsto (\pi^\theta \rtimes \lambda^\theta)(F_\varphi \cdot (\phi(f)))$ is completely bounded; that is, $F_\varphi$ is a completely bounded multiplier of $A^\theta(C_0(X), G, \alpha)$. The result follows. \hfill \Box

## 4. Weak amenability

In this section we define weak amenability of a $C^*$-dynamical system; when the group is discrete we prove a generalisation of Theorem 2.10 (i)$\iff$(ii). We also define weak amenability of a $W^*$-dynamical system, and when the group is discrete prove a generalisation of Theorem 2.10 (i)$\iff$(iii). The weak* CBAP for crossed products of $W^*$-dynamical systems has been studied by Haagerup–Kraus [10, Section 3]; they showed that if $(M, G, \alpha)$ is a $W^*$-dynamical system with $G$ weakly amenable and $M$ having the weak* CBAP then it is not true in general that $M \rtimes^\alpha G$ has the weak* CBAP. We will give an example of an assumption under which this implication does hold. The CBAP for the reduced crossed product of a $C^*$-dynamical system has been studied by Sinclair–Smith [19] under the assumption that the group
is amenable; here we give some other conditions under which the reduced crossed product has the CBAP.

As before $A$ is a unital $C^*$-algebra, we assume $A \subseteq \mathcal{B}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$, and $(\theta, \mathcal{H}_\theta)$ is a faithful representation of $A$ on a separable Hilbert space. Moreover, $G$ will always denote a discrete group; we note that the second-countability of $G$ required in [15] is not necessary if $G$ is discrete. Denote by $\alpha : G \to \text{Aut}(A)$ a homomorphism, so that $(A, G, \alpha)$ is a $C^*$-dynamical system. Since $G$ is discrete there is a canonical conditional expectation $E^\theta : \theta(A) \rtimes_{\alpha, r} G \to \theta(A)$ (see Brown–Ozawa [4, Proposition 4.1.9]), which corresponds to taking the $(e, e)$-th entry of the operator matrix of an element of $\theta(A) \rtimes_{\alpha, r} G$ (written as a matrix over $A$ acting on $\ell^2(G) \otimes \mathcal{H}_\theta \iso \bigoplus_{g \in G} \mathcal{H}_\theta$). We denote by $E$ the completely positive map defined by

$$A \rtimes_{\alpha, \theta} G \iso \theta(A) \rtimes_{\alpha, r} G \to A; \sum_{t \in G} \pi^\theta(a_t) \lambda^\theta_t \mapsto a_e, \quad a_t \in A.$$ 

The triple $(M, G, \beta)$ will denote a (separable) $W^*$-dynamical system, i.e. $M$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}_M$, $G$ is again a discrete group, and $\beta : G \to \text{Aut}(M)$ a homomorphism. The symbol $E$ will also be used for the conditional expectation $M \rtimes_{\beta}^N G \to M$, defined similarly.

Our main questions are:

- For a $C^*$-dynamical system $(A, G, \alpha)$ what is a necessary and sufficient condition for $A \rtimes_{\alpha, r} G$ to have the completely bounded approximation property?
- For a $W^*$-dynamical system $(M, G, \beta)$ what is a necessary and sufficient condition for $M \rtimes_{\beta}^N G$ to have the weak* completely bounded approximation property?

Our approach to these problems is to consider certain Herz–Schur multipliers of the system in question. Since we have so far only considered Herz–Schur multipliers of a $C^*$-dynamical system we briefly describe a construction, mentioned by Fujita [8, page 56], which shows that Herz–Schur multipliers of a $W^*$-dynamical system are particular cases of the weak*-extendable multipliers of Definition 2.2. For the $W^*$-dynamical system $(M, G, \beta)$, where $M$ is a von Neumann algebra on the separable Hilbert space $\mathcal{H}_M$, consider the set

$$M_\beta := \{x \in M : t \mapsto \beta_t(x) \text{ is norm-continuous for all } t \in G\}.$$ 

Then $M_\beta$ is a $G$-invariant, weak*-dense $C^*$-subalgebra of $M$ containing the identity, and $(M_\beta, G, \beta)$ is a $C^*$-dynamical system, with $M_\beta$ faithfully represented on $\mathcal{B}(\mathcal{H}_M)$. The construction of the reduced crossed product $M_\beta \rtimes_{\beta, r} G$, using the faithful representation $\text{id} : M_\beta \to \mathcal{B}(\mathcal{H}_M)$, gives a weak*-dense $C^*$-subalgebra of $M \rtimes_{\beta}^N G$. It follows that $\mathcal{A}^{\text{id}}(M_\beta, G, \beta)$ can be identified with the predual of $M \rtimes_{\beta}^N G$, and that the Herz–Schur id-multipliers
of \((M_\beta, G, \beta)\) are completely bounded multipliers of \(\mathcal{A}^{id}(M_\beta, G, \beta)\) and the associated maps possess completely bounded, weak*-continuous, extensions to \(M \rtimes^\beta V G\).

For a \(C^*\)-algebra \(B\) let \(\mathcal{CB}_\sigma(B)\) be the space of completely bounded maps on \(B\) that extend to completely bounded, weak*-continuous, maps on \(B''\).

**Definition 4.1.** A \(C^*\)-dynamical system \((A, G, \alpha)\) will be called weakly amenable if there exists a net \((F_i)\) of finitely supported Herz–Schur \((A, G, \alpha)\)-multipliers such that \(F_i(t)\) is a finite rank completely bounded map on \(A\) for all \(t \in G\),

\[
F_i(t)(a) \xrightarrow{\text{cb}} a \quad \text{for all } t \in G, \ a \in A,
\]

and \(\sup \|F_i\|_{HS} = K < \infty\). The infimum of all such \(K\) is denoted by \(\Lambda_{cb}(A, G, \alpha)\).

A \(W^*\)-dynamical system \((M, G, \beta)\), with \(M\) acting on \(\mathcal{B}(H_M)\), will be called weakly amenable if there is a net \(F_i : G \rightarrow \mathcal{CB}_\sigma(M_\beta)\) of finitely supported Herz–Schur \(id\)-multipliers of \((M_\beta, G, \beta)\), such that \(F_i(t)\) extends to a finite rank completely bounded map on \(M\) for all \(t \in G\),

\[
F_i(t)(a) \xrightarrow{w^*} a \quad \text{for all } t \in G, \ a \in M,
\]

and \(\sup \|F_i\|_{HS} = K < \infty\).

Observe that if \(A = \mathbb{C}\) then the finite rank condition is always satisfied, so Definition 4.1 reduces to Definition 2.8.

**Remark 4.2.** If \((A, G, \alpha)\) is a weakly amenable \(C^*\)-dynamical system with \(A\) unital, such that \(A\) is faithfully represented on a separable Hilbert space \(\mathcal{H}\), and the maps \(F_i\) of Definition 4.1 satisfy

\[
F_i(t) \circ \alpha_r = \alpha_r \circ F_i(t), \quad r, t \in G,
\]

then \(G\) is weakly amenable.

**Proof.** Suppose \((A, G, \alpha)\) is weakly amenable and take a net \((F_i)\) of Herz–Schur \((A, G, \alpha)\)-multipliers satisfying the definition. Let \(\xi \in \mathcal{H}\) be a unit vector. Condition \((10)\) ensures that the map

\[
v_i : G \rightarrow \mathbb{C}; \ v_i(ts^{-1}) := \langle \mathcal{N}(F_i)(s, t)(1_A)\xi, \xi \rangle, \quad s, t \in G
\]

is well-defined. Let \(V_i\) and \(W_i\) be the maps associated to the Schur \(A\)-multiplier \(\mathcal{N}(F_i)\) in Theorem 2.5. Then

\[
v_i(ts^{-1}) = \langle \mathcal{N}(F_i)(s, t)(1_A)\xi, \xi \rangle = \langle V_i(s)\xi, W_i(t)\xi \rangle, \quad s, t \in G,
\]

Hence \(v_i : G \rightarrow \mathbb{C}\) is a Herz–Schur multiplier (see Bożejko–Fendler [3], these statements are part of the proof of Lemma 2.7 [15, Proposition 4.1] for a particular case where \((10)\) holds). Since \(F_i\) has finite support so does \(v_i\). We have

\[
\|v_i\|_{Mcb} \leq \text{esssup}_{s \in G} \|V_i(s)\| \text{esssup}_{t \in G} \|W_i(t)\| = \|\mathcal{N}(F_i)\|_{\mathcal{B}} = \|F_i\|_{HS}.
\]
Since
\[ v_t(ts^{-1}) = \langle \mathcal{N}(F_t)(s,t)(1_A)\xi, \xi \rangle = \langle F_t(ts^{-1})(1_A)\xi, \xi \rangle \]
\[ \to \langle 1_A\xi, \xi \rangle = 1, \]
\( G \) is weakly amenable. \( \square \)

We now prove the analogue of Theorem 2.10 for \( C^* \)-dynamical systems.

**Theorem 4.3.** Let \( G \) be a discrete group, \( A \) a unital \( C^* \)-algebra, \( (\theta, \mathcal{H}_\theta) \) a faithful representation of \( A \) on a separable Hilbert space, and \( (A, G, \alpha) \) a \( C^* \)-dynamical system. The following are equivalent:

i. \( (A, G, \alpha) \) is weakly amenable;

ii. \( A \rtimes_{\alpha, \theta} G \) has the completely bounded approximation property.

Moreover, if the conditions hold then \( \Lambda_{cb}(A, G, \alpha) = \Lambda_{cb}(A \rtimes_{\alpha, \theta} G) \).

**Proof.** Suppose that \( (F_t) \) is a net of Herz–Schur \( (A, G, \alpha) \)-multipliers satisfying weak amenability of the system. It follows immediately that the net \( (S_{F_t}) \) of corresponding maps on \( A \rtimes_{\alpha, \theta} G \) consists of completely bounded, finite rank, maps satisfying \( \sup \|S_{F_t}\|_{cb} \leq C < \infty \). It remains to show that \( \|S_{F_t}(T) - T\| \to 0 \) for all \( T \in A \rtimes_{\alpha, \theta} G \). For this, it suffices to show that \( \|S_{F_t}(\sum_t \pi^\theta(a_t)\lambda^\theta_t) - \sum_t \pi^\theta(a_t)\lambda^\theta_t\| \to 0 \) when the sums are finite. Indeed, for any \( T \in A \rtimes_{\alpha, \theta} G \) and \( \epsilon > 0 \), we can find \( a_t \in A \) with \( \|T - \sum_t \pi^\theta(a_t)\lambda^\theta_t\| < \epsilon \), where only a finite number of \( a_t \) are non-zero, so

\[
\|S_{F_t}(T) - T\| \leq \left\| S_{F_t}(T) - S_{F_t}\left( \sum_t \pi^\theta(a_t)\lambda^\theta_t \right) \right\| \\
+ \left\| S_{F_t}\left( \sum_t \pi^\theta(a_t)\lambda^\theta_t \right) - \sum_t \pi^\theta(a_t)\lambda^\theta_t \right\| + \left\| \sum_t \pi^\theta(a_t)\lambda^\theta_t - T \right\| \\
< C\epsilon + \left\| S_{F_t}\left( \sum_t \pi^\theta(a_t)\lambda^\theta_t \right) - \sum_t \pi^\theta(a_t)\lambda^\theta_t \right\| + \epsilon.
\]

Now
\[
\left\| S_{F_t}\left( \sum_t \pi^\theta(a_t)\lambda^\theta_t \right) - \sum_t \pi^\theta(a_t)\lambda^\theta_t \right\| = \left\| \sum_t \pi^\theta(F_t(t)(a_t))\lambda^\theta_t - \sum_t \pi^\theta(a_t)\lambda^\theta_t \right\| \\
\leq \sum_t \left\| \pi^\theta(F_t(t)(a_t) - a_t)\lambda^\theta_t \right\| \to 0
\]
as \( F_t(t)(a) \to a \) for all \( a \in A, \ t \in G \). It follows that \( \Lambda_{cb}(A \rtimes_{\alpha, \theta} G) \leq \Lambda_{cb}(A, G, \alpha) \).

For the converse we will use a similar idea to Haagerup’s proof of Theorem 2.10. First consider a finite rank, completely bounded, map \( \rho : A \rtimes_{\alpha, \theta} G \to A \rtimes_{\alpha, \theta} G \). Take \( T_1, \ldots, T_k \in A \rtimes_{\alpha, \theta} G \) which span ran \( \rho \), so
there are \( \phi_1, \ldots, \phi_k \in (A \rtimes_{\alpha, \theta} G)^* \) such that

\[
\rho = \sum_{j=1}^{k} \phi_j \otimes T_j,
\]

where \((\phi_j \otimes T_j)(T) = \phi_j(T)T_j \ (T \in A \rtimes_{\alpha, \theta} G)\). We note that, for a matrix \((x_{p,q}) \in M_n(A \rtimes_{\alpha, \theta} G), \]

\[
\left\| \left( \sum_{j=1}^{k} \phi_j \otimes T_j \right)^{(n)} \right\|_{cb} \leq \sum_{j=1}^{k} \| \phi_j^{(n)} \| \| (x_{p,q}) \| \| \left( T_j \right) \|
\]

where \( \text{diag}_n(T) \) denotes the diagonal \( n \times n \) matrix with each diagonal entry equal to \( T \). Thus

\[
(11) \quad \left\| \sum_{j=1}^{k} \phi_j \otimes T_j \right\|_{cb} \leq \sum_{j=1}^{k} \| \phi_j \| \| T_k \|.
\]

For each \( j \) and each \( n \in \mathbb{N} \) find \( a_{j,n}^i \in A \) and \( s_{j,n}^i \in G \) such that \( T_{j,n} := \sum_{i=1}^{k} \pi^{\theta}(a_{j,n}^i) s_{j,n}^i \) satisfies \( \| T_j - T_{j,n} \| < 1/(nk \max_j \| \phi_j \|) \). Define \( \rho_n := \sum_{j=1}^{k} \phi_j \otimes T_{j,n} \). Then

\[
\| \rho - \rho_n \|_{cb} = \left\| \left( \sum_{j=1}^{k} \phi_j \otimes T_j \right) - \left( \sum_{j=1}^{k} \phi_j \otimes T_{j,n} \right) \right\|_{cb}
\]

\[
(12) \quad \leq \sum_{j=1}^{k} \| \phi_j \| \| T_j - T_{j,n} \|_{cb}
\]

\[
\leq \sum_{j=1}^{k} \| \phi_j \| \| T_j - T_{j,n} \| < \frac{1}{n}.
\]

Now let \((\rho_\gamma)\) be a net of maps on \( A \rtimes_{\alpha, \theta} G \) satisfying the conditions of the CBAP. By the above procedure we obtain a net of maps \((\rho'_\gamma,n)\) on \( A \rtimes_{\alpha, \theta} G \) which are finite rank, with range in \( \text{span}\{\pi^{\theta}(a) \lambda^\theta_t : a \in A, \ t \in G\} \). It is easily checked that \( \rho'_\gamma,n \to \text{id} \) in point-norm, using the product directed set. As in (11) we have that each \( \rho'_\gamma,n \) is completely bounded; by (12) we have \( \| \rho_\gamma - \rho'_\gamma,n \|_{cb} < 1/n \) for all \( \gamma \) and all \( n \in \mathbb{N} \), so \( \| \rho'_\gamma,n \|_{cb} < \| \rho_\gamma \|_{cb} + 1/n \). Let
where \( C = \sup \|\rho\|_{cb} \) and define
\[
\rho_{\gamma,n} := \frac{C}{C + 1/n^2} \rho_{\gamma,n}^i,
\]
so that \((\rho_{\gamma,n})\) is a net satisfying the CBAP for \( A \rtimes_{\alpha,\theta} G \), uniformly bounded by \( C \), and with range in \( \text{span}\{\pi^\theta(a)\lambda_t^\theta : a \in A, t \in G\} \). Define \( F_{\gamma,n} : G \to \mathcal{CB}(A) \) by
\[
F_{\gamma,n}(t)(a) := \mathcal{E} \left( \rho_{\gamma,n} (\pi^\theta(a)\lambda_t^\theta) \lambda_{t^{-1}}^\theta \right), \quad a \in A, \ t \in G.
\]
It is easy to see that \( \text{supp} F_{\gamma,n} \subseteq \{s_{i,j}^r : 1 \leq i \leq k_n, 1 \leq j \leq k\} \). As \( \rho_{\gamma,n} \) is finite rank, with range spanned by finite sums of elements of the form \( \pi^\theta(a)\lambda_t^\theta \) \((a \in A, r \in G)\), it follows that each \( F_{\gamma,n}(t) \) is a finite rank map on \( A \), with \( \text{ran} F_{\gamma,n}(t) \subseteq \text{span}\{a \in A : \pi^\theta(a)\lambda_t^\theta \in \text{ran} \rho_{\gamma,n}\} \). Since \( \rho_{\gamma,n} \to \id \) in point-norm we have, for all \( t \in G, \ a \in A, \)
\[
F_{\gamma,n}(t)(a) = \left( \mathcal{E} \left( \rho_{\gamma,n} (\pi^\theta(a)\lambda_t^\theta) \lambda_{t^{-1}}^\theta \right) \right) \to \mathcal{E} \left( \pi^\theta(a)\lambda_{t^{-1}}^\theta \right) = a.
\]
It remains to show that each \( F_{\gamma,n} \) is a Herz–Schur \((A, G, \alpha)\)-multiplier and \( \|S_{F_{\gamma,n}}\|_{cb} = \|\rho_{\gamma,n}\|_{cb} \). Let \((e_l)_\lambda\) be a countable orthonormal basis for \( \mathcal{H}_\theta \),
\[
V : \ell^2(G) \otimes \mathcal{H}_\theta \to \ell^2(G) \otimes \ell^2(G) \otimes \mathcal{H}_\theta; \quad \delta_g \otimes e_l \mapsto \delta_g \otimes \delta_g \otimes e_l,
\]
where \( \{\delta_g : g \in G\} \) denotes the canonical orthonormal basis for \( \ell^2(G) \), and define a homomorphism
\[
\tau : A \rtimes_{\alpha,\theta} G \to C^*_r(G) \otimes_{\min} A \rtimes_{\alpha,\theta} G; \quad \pi^\theta(a)\lambda_t^\theta \mapsto \lambda_t^G \otimes \pi^\theta(a)\lambda_t^\theta,
\]
for all \( a \in A, \ t \in G \) (see Bédos–Conti [1, Lemma 4.1] for more on the coaction \( \tau \)). We claim
\[
S_{F_{\gamma,n}}(x) = V^* (\id \otimes \rho_{\gamma,n}) \tau(x) V, \quad x \in A \rtimes_{\alpha,\theta} G,
\]
which implies \( S_{F_{\gamma,n}} \) is completely bounded, with \( \|S_{F_{\gamma,n}}\|_{cb} = \|\rho_{\gamma,n}\|_{cb} \). To prove the claim we first assume \( \rho_{\gamma,n} \) has one-dimensional range generated by \( \pi^\theta(b)\lambda_l^\theta \) for some \( b \in A, \ r \in G \). Then, for \( x, y \in G, \ l, m \in A, \)
\[
\langle V^* (\id \otimes \rho_{\gamma,n}) \tau (\pi^\theta(a)\lambda_l^\theta) V (\delta_x \otimes e_m), \delta_y \otimes e_l \rangle
\]
\[
= \langle \lambda_l \otimes \rho_{\gamma,n} (\pi^\theta(a)\lambda_l^\theta) (\delta_x \otimes \delta_x \otimes e_m), \delta_y \otimes \delta_y \otimes e_l \rangle
\]
\[
= \langle \delta_{tx}, \delta_y \rangle \langle \rho_{\gamma,n} (\pi^\theta(a)\lambda_l^\theta) (\delta_x \otimes e_m), \delta_y \otimes e_l \rangle
\]
\[
= \langle \delta_{tx}, \delta_y \rangle \langle \pi^\theta(b)\lambda_l^\theta (\delta_x \otimes e_m), \delta_y \otimes e_l \rangle
\]
\[
= \langle \delta_{tx}, \delta_y \rangle \langle \pi^\theta(b)\lambda_l^\theta (\delta_x \otimes e_m)(y), e_l \rangle
\]
\[
= \langle \delta_{tx}, \delta_y \rangle \langle \alpha_{y^{-1}}(b)e_m, e_l \rangle \langle \delta_y, \delta_y \rangle.
\]
On the other hand,
\[
\left\langle S_{F, n} \left( \pi^{\theta}(a) \lambda_{t}^{\theta} \right) (\delta_{x} \otimes e_{m}), \delta_{y} \otimes e_{l} \right\rangle \\
= \left\langle \pi^{\theta} \left( F_{n} (t)(a) \right) \lambda_{t}^{\theta} (\delta_{x} \otimes e_{m}), \delta_{y} \otimes e_{l} \right\rangle \\
= \left\langle \pi^{\theta} \left( E \left( \rho_{1, n} \left( \pi^{\theta}(a) \lambda_{t}^{\theta} \right) \lambda_{t-1}^{\theta} \right) \right) \lambda_{t}^{\theta} (\delta_{x} \otimes e_{m}), \delta_{y} \otimes e_{l} \right\rangle \\
= \left\langle \pi^{\theta} \left( \pi^{\theta} (b) \lambda_{r t-1}^{\theta} \right) \right\rangle \lambda_{t}^{\theta} (\delta_{x} \otimes e_{m}), \delta_{y} \otimes e_{l} \right\rangle \\
= \langle \gamma, \delta_{l} \rangle \left( \pi^{\theta} (b) \lambda_{t}^{\theta} (\delta_{x} \otimes e_{m}), \delta_{y} \otimes e_{l} \right) \\
= \langle \gamma, \delta_{l} \rangle \langle \alpha_{y-1} (b) e_{m}, e_{l} \rangle \langle \delta_{t x}, \delta_{y} \rangle.
\]

It follows that \( V^{*}(\text{id} \otimes \rho_{1, n}) \pi^{\theta}(a) \lambda_{t}^{\theta} V = S_{F, n} \left( \pi^{\theta}(a) \lambda_{t}^{\theta} \right) \). By linearity and continuity we obtain \( (14) \) when \( \rho_{1, n} \) takes values in \( \text{span} \{ \pi^{\theta}(b_{i}) \lambda_{t}^{\theta} : i = 1, \ldots, k \} \). The equality \( \| S_{F, n} \|_{cb} = \| \rho_{1, n} \|_{cb} \) follows, so \( (F_{n}, \pi) \) is a net satisfying weak amenability of \( (A, G, \alpha) \). It also follows that \( \Lambda_{cb}(A, G, \alpha) \leq \Lambda_{cb}(A \rtimes_{\alpha, \theta} G) \). \( \Box \)

**Remark 4.4.** For degenerate \( C^{*} \)-dynamical systems the constant \( \Lambda_{cb} \) introduced in Definition 4.1 reduces to the familiar constants defined in Section 1. Indeed, if \( G \) is a discrete group such that the system \( (C, G, 1) \) is weakly amenable then \( G \) is weakly amenable by Remark 4.2 or Theorem 4.3 moreover, by Theorem 4.3,

\[
\Lambda_{cb}(C, G, 1) = \Lambda_{cb}(C \rtimes_{1, r} G) = \Lambda_{cb}(C_{r}^{*}(G)) = \Lambda_{cb}(G).
\]

Similarly, if the \( C^{*} \)-dynamical system \( (A, \{ e \}, 1) \) is weakly amenable then
\[
\Lambda_{cb}(A, \{ e \}, 1) = \Lambda_{cb}(A \rtimes_{1, r} \{ e \}) = \Lambda_{cb}(A).
\]

In fact, Sinclair–Smith [19, Theorem 3.4] have shown that for an amenable discrete group \( G \), \( \Lambda_{cb}(A \rtimes_{\alpha, r} G) = \Lambda_{cb}(A) \), so when \( (A, G, \alpha) \) is a discrete \( C^{*} \)-dynamical system with \( G \) amenable we have
\[
\Lambda_{cb}(A, G, \alpha) = \Lambda_{cb}(A \rtimes_{\alpha, r} G) = \Lambda_{cb}(A).
\]

We now turn to characterising weak amenability of \( W^{*} \)-dynamical systems.

**Lemma 4.5.** Let \((M, G, \beta)\) be a \( W^{*} \)-dynamical system, with \( G \) a discrete group, and \((F_{t})\) a net of Herz–Schur id-multipliers of the underlying \( C^{*} \)-dynamical system \((M_{\beta}, G, \beta)\). The following are equivalent:

i. \( F_{t}(t)(a) \overset{w^{*}}{\rightarrow} a \) for all \( t \in G \), \( a \in M \) (condition (14) above);

ii. \( s_{F} u \rightarrow u \) in \( A(M, G, \beta) \) for all \( u \in A(M, G, \beta) \).

**Proof.** Suppose condition (i) holds. By Remark 3.2 finitely supported functions are dense in \( A(M, G, \beta) \), so it suffices to prove the claim for singly supported \( u \in A(M, G, \beta) \). Suppose \( u \in A(M, G, \beta) \) is supported on \( \{ s \} \)
and \( u(t)(a) = \sum_{n=1}^{\infty} \langle \pi(a)\lambda_n, \eta_n \rangle \) (\( t \in G, \ a \in M \)) for some families satisfying \( \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty \) and \( \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty \). Since \( \lambda_n \) is an isometry it follows that the functional in \( \pi(M)_\ast \) given by \( \pi(a) \mapsto \sum_{n=1}^{\infty} \langle \pi(a)\lambda_n \xi_n, \eta_n \rangle \) has the same norm as \( u \); thus \( \|u(s)\| = \|u\|_A \). Since \( sF_i u \) is also supported on \( \{s\} \) we have

\[
\|sF_i u - u\|_A = \|u(s) \circ F_i(s) - u(s)\| = \sup_{\|a\| \leq 1} \left| u(s)(F_i(s)(a) - a) \right| \to 0.
\]

Condition (ii) follows.

For the converse suppose (ii) holds. Then, for any \( a \in A, \ t \in G \) and \( u \in \mathcal{A}(M, G, \beta) \),

\[
\left| \langle \pi(F_i(t)(a))\lambda_t - \pi(a)\lambda_t, u \rangle \right| = \left| \langle \pi(a)\lambda_t, sF_i u - \pi(a)\lambda_t, u \rangle \right| \to 0,
\]

so \( u(t)(F_i(t)(a)) \to u(t)(a) \). As \( u \) varies \( u(t) \) can take any value in \( M_\ast \); thus \( F_i(t)(a) \) converges to \( a \) in the weak* topology. \( \square \)

**Theorem 4.6.** Let \( G \) be a discrete group, \( M \subseteq \mathcal{B}(H_M) \) a von Neumann algebra acting on a separable Hilbert space, and \( (M, G, \beta) \) a \( W^\ast \)-dynamical system. Consider the conditions:

i. \( (M, G, \beta) \) is weakly amenable;

ii. \( M \rtimes_\beta G \) has the weak* completely bounded approximation property.

Then (i)\( \Rightarrow \) (ii). If \( G \) is weakly amenable then (i) and (ii) are equivalent.

**Proof.** (i)\( \Rightarrow \) (ii) Suppose that \((F_i)\) is a net of Herz–Schur id-multipliers of the underlying \( C^\ast \)-dynamical system \((M, G, \beta)\) satisfying Definition 4.1. Then the associated net of maps \((S_{F_i})\) on \( M \rtimes_\beta G \) are completely bounded, weak*-continuous, and finite rank. Finally, using the identification of \((M \rtimes_\beta G)_\ast\) with \( \mathcal{A}(M, G, \beta) \), we have for any \( u \in \mathcal{A}(M, G, \beta) \) and any \( T \in M \rtimes_\beta G \)

\[
\langle S_{F_i}T, u \rangle = \langle T, sF_i u \rangle \to \langle T, u \rangle
\]

by Lemma 4.5 so \( S_{F_i}T \) converges to \( T \) in the weak* topology.

(ii)\( \Rightarrow \) (i) For the converse suppose \( M \rtimes_\beta G \) has the weak* CBAP. Given a finite set \( E \subseteq G, \ \epsilon > 0 \), and a collection \( \Omega \subseteq M_\ast \), choose \( \rho : M \rtimes_\beta G \to M \rtimes_\beta G \) such that

\[
F : G \to \mathcal{CB}_\rho(M_\beta); \ F(t)(a) := \mathcal{E}(\rho(\pi(a)\lambda_t)\lambda_{t^{-1}}), \ a \in M, \ t \in G
\]

satisfies \( |\omega(a - F(t)(a))| < \epsilon \) for all \( a \in M, \ t \in E, \ \omega \in \Omega \). In this way we produce a net \((F_i)\), indexed by triples of the form \((E, \epsilon, \Omega)\), such that \( F_i(t)(a) \to a \) in the weak* topology. For each \( t \in G, \ F(t) \) defined above is a finite rank map on \( M \) as in the proof of Theorem 4.3; indeed, suppose \( \rho = \sum_{j=1}^{k} \phi_j \otimes T_j \), where \( \phi_j \) is a functional and \( T_j \in M \rtimes_\beta G \). Then

\[
F(t)(a) = \mathcal{E}(\rho(\pi(a)\lambda_t)\lambda_{t^{-1}}) = \sum_{j=1}^{k} \phi_j(\pi(a)\lambda_t)\mathcal{E}(T_j\lambda_{t^{-1}}),
\]
so that \( \mathcal{E}(T_j \Lambda_{j-1}) : j = 1, \ldots, k \) span \( \text{ran} \, F(t) \). The same calculation as in the proof of Theorem 4.3 shows that \( \|S_F\|_{cb} = \|\rho\|_{cb} \); in particular \( F \) is a Herz–Schur \( (M_\beta, G, \beta) \)-multiplier. Each \( S_F \) is a composition of weak*-continuous maps, so is weak*-extendable. We have that the net \((F_i)\) satisfies all the conditions of weak amenability of \((M, G, \beta)\) except that it may not be finitely supported. To correct this we use the assumption that \( G \) is weakly amenable. Let \((\varphi_j)\) be a net of functions on \( G \) satisfying Definition 2.8.

Define another net, indexed by the product directed set,

\[
F_{i,j} : G \to \mathcal{CB}_\sigma(M); \quad F_{i,j}(t)(a) := \varphi_j(t) F_i(t)(a), \quad t \in G, \ a \in M,
\]

which is a net of Herz–Schur id-multipliers of \((M_\beta, G, \beta)\), with \( S_{F_{i,j}} = S_{\varphi_j} \circ S_{F_i} \). From the properties of \( \varphi_j \) and \( F_i \) we have that each \( F_{i,j} \) is finitely supported, \( F_{i,j}(t) \) is finite rank for all \( t \in G \), and \( F_{i,j}(t)(a) \) converges to \( a \) in the weak* topology. Finally, \( \|F_{i,j}\|_{HS} = \|S_{F_{i,j}}\|_{cb} \leq \|S_{\varphi_i}\|_{cb}\|S_{F_i}\|_{cb} \), so the net is uniformly bounded.

**Remarks 4.7.** (i) In the proof of (ii)\( \Rightarrow \)(i) above we required weak amenability of \( G \); to see why this requirement arose let us return to the proof of Theorem 4.3. There we are able to approximate in norm the operators \( \rho_\gamma \), which implement the CBAP of \( \alpha \times _{a,b} G \), by operators \( \rho_{\gamma,n} \) with finite-dimensional range spanned by elements of the form \( \pi^\theta(a) \lambda_t^\theta \), such that \( \|\rho_{\gamma,n}\|_{cb} \) is closely related to \( \|\rho_\gamma\|_{cb} \); these estimates allowed us to identify the support and Herz–Schur norm of \( F_{\gamma,n} \). Such norm estimates are not available in the setting of Theorem 4.6, so the extra hypothesis seems to be required to use the techniques in this paper.

(ii) If in the above proof we make the stronger assumption that \( \Lambda_{cb}(G) = 1 \) then the net \((\varphi_{i,n})\) may be chosen such that \( \|S_{\varphi_{i,n}}\|_{cb} \) is uniformly bounded by 1. Therefore, with this assumption on \( G \), we obtain \( \Lambda_{cb}^N(M, G, \beta) \leq \Lambda_{cb}(M \rtimes_{\beta}^\gamma G) \), where \( \Lambda_{cb}^N \) is the natural weak amenability constant of a \( W^* \)-dynamical system. It follows that if \( \Lambda_{cb}(G) = 1 \) we have \( \Lambda_{cb}^N(M, G, \beta) = \Lambda_{cb}(M \rtimes_{\beta}^\gamma G) \). It would be interesting to have a characterisation of when these two weak amenability constants coincide.

Suppose that \((A, G, \alpha)\) is a \( C^* \)-dynamical system with \( G \) an amenable discrete group and \( A \) a nuclear \( C^* \)-algebra. It is well known (e.g. Brown–Ozawa [4, Theorem 4.2.6]) that this implies \( A \rtimes_{a,r} G \) is nuclear. It is natural to ask whether this fact persists for weak amenability and the CBAP: does the CBAP for \( A \) and weak amenability of \( G \) imply that \( A \rtimes_{a,r} G \) has the CBAP? Haagerup–Kraus give an example of a \( W^* \)-dynamical system showing that in general this is not true, which we reproduce here as a \( C^* \)-dynamical system. Both \( \text{SL}(2, \mathbb{Z}) \) and \( \mathbb{Z}^2 \) are weakly amenable, but their semidirect product \( \mathbb{Z}^2 \rtimes_{\mu} \text{SL}(2, \mathbb{Z}) \) is not [10, page 670] (\( \mu \) denotes the usual action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{Z}^2 \)). The unitary \( U \) on \( L^2(\mathbb{Z}^2 \times \text{SL}(2, \mathbb{Z})) \) given by

\[
(U \xi)(y, s) := \xi(\mu_s(y), s), \quad \xi \in L^2(\mathbb{Z}^2 \times \text{SL}(2, \mathbb{Z})), \ y \in \mathbb{Z}^2, \ s \in \text{SL}(2, \mathbb{Z}),
\]
implements a unitary equivalence between $C_\nu^r(\mathbb{Z}^2) \rtimes_{\nu,r} \text{SL}(2, \mathbb{Z})$ (acting on $L^2(\mathbb{Z}^2 \times \text{SL}(2, \mathbb{Z}))$) and $C_\nu^r(\mathbb{Z}^2 \rtimes_{\mu} \text{SL}(2, \mathbb{Z}))$ (acting on $L^2(\mathbb{Z}^2 \rtimes_{\mu} \text{SL}(2, \mathbb{Z}))$), where $\nu$ is the induced action on $C_\nu^r(\mathbb{Z}^2)$. It follows that the crossed product of a $C^*$-algebra with the CBAP by a weakly amenable group need not have the CBAP. Note that Sinclair–Smith [19] have shown that if $G$ is amenable and $A$ has the CBAP then $A \rtimes_{\alpha,r} G$ has the CBAP. To finish this paper we give an example of an additional assumption under which this implication can be recovered for weakly amenable groups.

**Proposition 4.8.** Let $G$ be a discrete group, $(A, G, \alpha)$ a $C^*$-dynamical system, and $(\theta, \mathcal{H}_\theta)$ a faithful representation of $A$ on a separable Hilbert space. The following are equivalent:

i. $G$ is weakly amenable, $A$ has the CBAP and the approximating maps $\phi_i : A \to A$ satisfy $\phi_i \circ \alpha_t = \alpha_t \circ \phi_i$ for all $t \in G$;

ii. $(A, G, \alpha)$ is weakly amenable and the approximating Herz–Schur $(A, G, \alpha)$-multipliers $F_i : G \to \text{CB}(A)$ satisfy $F_i(t)(\alpha_r(a)) = \alpha_r(F_i(t)(a))$ for all $r, t \in G$.

**Proof.** Suppose (i) holds. The condition on the maps $(\phi_i)$ implies that the map

$$\bar{\phi}_i : A \rtimes_{\alpha, \theta} G \to A \rtimes_{\alpha, \theta} G; \sum_t \pi^\theta(a_t) \lambda^\theta_t \mapsto \sum_t \pi^\theta(\phi_i(a_t)) \lambda^\theta_t, \quad a_t \in A, \ t \in G,$$

can be identified with the restriction of $I_{\ell^2(G)} \otimes \phi_i^\theta$ on $\mathcal{B}(\ell^2(G)) \otimes_{\min} \theta(A)$ to $A \rtimes_{\alpha, \theta} G$, where $\phi_i^\theta(\theta(a)) = \theta(\phi_i(a)) \ (a \in A)$. It follows from [6, Lemma 1.5] that $\bar{\phi}_i$ is completely bounded and $\|\bar{\phi}_i\|_{\text{cb}} \leq \|\phi_i\|_{\text{cb}}$. Let $(v_\gamma)$ be a net of scalar-valued functions on $G$ satisfying weak amenability of $G$ and let $S_{v_\gamma}$ be the completely bounded map on $A \rtimes_{\alpha, \theta} G$ associated to the (classical) Herz–Schur multiplier $v_\gamma$ as in Lemma [2.7]. Denote by $S_{\gamma,i}$ the composition $S_{v_\gamma} \circ \bar{\phi}_i$, which satisfies the CBAP for $A \rtimes_{\alpha, \theta} G$; indeed if $\sup_i \|\bar{\phi}_i\|_{\text{cb}} \leq C_1$ and $\sup_\gamma \|v_\gamma\|_{\text{Mcb}} \leq C_2$ then $\|S_{\gamma,i}\|_{\text{cb}} \leq C_1C_2$, each $S_{\gamma,i}$ is finite rank, and for any $T \in A \rtimes_{\alpha, \theta} G$

$$\|S_{\gamma,i}(T) - T\| \leq \|S_{v_\gamma}(\bar{\phi}_i(T)) - S_{v_\gamma}(T)\| + \|S_{v_\gamma}(T) - T\| \leq C_2\|\bar{\phi}_i(T) - T\| + \|S_{v_\gamma}(T) - T\| \to 0.$$

It follows from Theorem [4.3] that the system $(A, G, \alpha)$ is weakly amenable. To prove the covariance condition we first calculate the form of the Herz–Schur $(A, G, \alpha)$-multipliers defined in the proof of Theorem [4.3]

$$F_{\gamma,i}(t)(a) := \left(\mathcal{E}(S_{\gamma,i}(\pi^\theta(a)) \lambda^\theta_{t^{-1}})\right)$$

$$= \mathcal{E}(\pi^\theta(v_\gamma(t) \phi_i(a)))$$

$$= v_\gamma(t) \phi_i(a).$$

Thus, for any $r \in G$,

$$\alpha_r(F_{\gamma,i}(t)(a)) = v_\gamma(t) \alpha_r(\phi_i(a)) = v_\gamma(t) \phi_i(\alpha_r(a)) = F_{\gamma,i}(t)(\alpha_r(a)).$$
For the converse let \((F_i)\) be a net of Herz–Schur \((A, G, \alpha)\)-multipliers satisfying weak amenability of the system and the covariance condition. Weak amenability of \(G\) follows as in Remark 4.2. Define
\[
\phi_i : A \to A; \ a \mapsto \mathcal{E}\left(S_{F_i}(\pi^\theta(a))\right), \ a \in A,
\]
to obtain a net of maps easily seen to satisfy the CBAP for \(A\). Now calculate
\[
\phi_i(\alpha_t(a)) = \mathcal{E}\left(S_{F_i}(\pi^\theta(\alpha_t(a)))\right) = \mathcal{E}\left(\pi^\theta(F_i(e)(\alpha_t(a)))\right)
\]
\[
= \mathcal{E}\left(\alpha_t(F_i(e)(a))\right)
\]
\[
= \alpha_t(\phi_i(a)),
\]
as required. \(\square\)

**Acknowledgements.** My sincere thanks to my advisor Ivan Todorov for his guidance during this work. I would also like to thank the EPSRC for funding my PhD position.

**References**

[1] Erik Bédos and Roberto Conti, *Fourier series and twisted C\(^\ast\)*-crossed products*, J. Fourier Anal. Appl. **21** (2015), no. 1, 32–75. MR3302101

[2] ______________, *The Fourier–Stieltjes algebra of a C\(^\ast\)*-dynamical system*, Internat. J. Math. **27** (2016), no. 6, 1650050, 50. MR3516977

[3] Marek Bożejko and Gero Fendler, *Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group*, Boll. Un. Mat. Ital. A (6) **3** (1984), no. 2, 297–302. MR753889

[4] Nathanial P. Brown and Narutaka Ozawa, *C\(^\ast\)*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008. MR2391387

[5] Michael Cowling and Uffe Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. **96** (1989), no. 3, 507–549. MR996553

[6] Jean De Cannière and Uffe Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, American Journal of Mathematics **107** (1985), no. 2, 455–500.

[7] Pierre Eymard, *L’algèbre de Fourier d’un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236. MR0228628

[8] Masayuki Fujita, *Banach algebra structure in Fourier spaces and generalization of harmonic analysis on locally compact groups*, Journal of the Mathematical Society of Japan **31** (1979), no. 1, 53–67.

[9] Uffe Haagerup, *Group C\(^\ast\)*-algebras without the completely bounded approximation property*, J. Lie Theory **26** (2016), no. 3, 861–887. MR3476201

[10] Uffe Haagerup and Jon Kraus, *Approximation properties for group C\(^\ast\)*-algebras and group von Neumann algebras*, Transactions of the American Mathematical Society **344** (1994), no. 2, 667–699.

[11] Paul Jolissaint, *A characterization of completely bounded multipliers of Fourier algebras*, Colloq. Math. **63** (1992), no. 2, 311–313. MR1180643
[12] Richard V. Kadison and John R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II*, Graduate Studies in Mathematics, vol. 16, American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original. MR1468230

[13] Søren Knudby, *Approximation properties for groups and von Neumann algebras*, Ph.D. Thesis, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 København Ø, Denmark, 2014. Available at [http://www.math.ku.dk/noter/filer/phd14sk.pdf](http://www.math.ku.dk/noter/filer/phd14sk.pdf).

[14] Horst Leptin, *Sur l’algèbre de Fourier d’un groupe localement compact*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968).

[15] Andrew McKee, Ivan Todorov, and Lyudmila Turowska, *Herz–Schur multipliers of dynamical systems*, Preprint. arXiv:1608.01092 [math.OA].

[16] Gert K. Pedersen, *$C^*$-algebras and their automorphism groups*, London Mathematical Society Monographs, vol. 14, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. MR548006

[17] Jean Renault, *A groupoid approach to $C^*$-algebras*, Lecture Notes in Mathematics, vol. 793, Springer, Berlin, 1980. MR584266

[18] ____, *The Fourier algebra of a measured groupoid and its multipliers*, J. Funct. Anal. **145** (1997), no. 2, 455–490.

[19] A. M. Sinclair and R. R. Smith, *The completely bounded approximation property for discrete crossed products*, Indiana Univ. Math. J. **46** (1997), no. 4, 1311–1322. MR1631596

[20] Allan M. Sinclair and Roger R. Smith, *The Haagerup invariant for tensor products of operator spaces*, Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 1, 147–153. MR1373354

[21] Hiroshi Takai, *On a Fourier expansion in continuous crossed products*, Kyoto University. Research Institute for Mathematical Sciences. Publications **11** (1975/76), no. 3, 849–880.

[22] M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 125, Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry. 6.

[23] Dana P. Williams, *Crossed products of $C^*$-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007. MR2288954

Pure Mathematics Research Centre, Queen’s University Belfast, Belfast BT7 1NN, United Kingdom

E-mail address: amckee240@qub.ac.uk