Comments on Supersymmetric Yang-Mills Theory on a Noncommutative Torus

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D0-brane theory on a torus with a nonvanishing $B$ field is embedded into a string theory in the weak coupling limit. It is shown that the usual supersymmetric Yang-Mills theory on a noncommutative torus can not be the whole story. The Born-Infeld action survives the noncommutative torus limit.

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In a pair of interesting papers by Douglas and collaborators [1,2] (for further development see [3,4,5]), it was pointed out that the super Yang-Mills theory on a noncommutative torus is naturally related to compactification of matrix theory on a dual torus with a constant $C^{(3)}$ field. This field has a tangent index along the longitudinal direction as well as two indices along the compact torus. These two indices meet the minimal requirement of noncommutativeness. Thus the geometric correspondence to a noncommutative torus is quite natural, and the vertices arising from the noncommutativeness can be derived in string theory.

The field theory is superficially nonrenormalizable, since the action involves infinitely many high derivative terms. However these terms sum to an exponential form, and this becomes a damping factor at higher energies. It may well be that this theory is a well-defined quantum field theory on a two dimensional or three dimensional torus. Our aim in this note is not to attempt a front attack on this renormalization problem. We shall try to embed these theories into string theory, and ask whether there is a proper limit in which the theory is decoupled from string theory.

We shall argue that on a two dimensional torus, like the $C = 0$ case, the SYM can be embedded into a weakly coupled string theory. Actually, the string coupling constant for the fixed dimensionless Yang-Mills couplings tends to zero faster than in the case when $C = 0$. This indicates that the SYM on the noncommutative torus is indeed renormalizable. The fact that the decoupling works better than on a usual torus might have some root in an intrinsic property of SYM on a noncommutative torus: The nonlocal vertices have damping effects at high energies. The new result of our analysis is that in addition to the higher derivative terms originating from the noncommutative torus, there are higher derivative terms from the Born-Infeld action which are also important. Actually, when the compactification scales are smaller than the Planck scale, the Born-Infeld dominates.

Seiberg argued that the DLCQ of matrix theory on a finite light-like circle can be obtained by infinitely boosting a small space-like circle. The M theory on the space-like circle is a weakly coupled IIA string. Its dual on a two torus is again a weakly coupled string. In the three torus case, the dual string coupling is fixed. In this way one argues that indeed the 2D and 3D SYM decouple from the corresponding string theory in the limit $l_s \to 0$. Applying the infinite boost argument, we find the following: On a two torus, the string coupling constant still goes to zero faster than the $C = 0$ case. The Born-Infeld action survives in the decoupling limit.
Consider D0-branes on two torus $T^2$ of size $(R_1, R_2)$. Assume this torus be a rectangular one. A slanted torus introduces no new novelty. A constant $B_{12} = B$ is switched on. Our normalization for the $B$ field is such that it is dimensionless and the coupling to the string world sheet is

$$\int B_{\mu\nu} dx^\mu \wedge dx^\nu,$$

where we always use the coordinates with the period $2\pi$. Thus, on the two torus the metric is $ds^2 = (R_1 dx_1)^2 + (R_2 dx_2)^2$. The two complex moduli are given by

$$\tau = \frac{R_1}{R_2} i, \quad \rho = B + l_s^{-2} R_1 R_2 i. \quad (2)$$

Following Douglas and Hull, we switch to the T-dual torus obtained by performing T-duality along $R_2$. This amounts to exchanging the two moduli $\tau$ and $\rho$. In this dual picture, the new field $B' = 0$, and the two new radii are

$$R'_1 = \frac{1}{R_2} \sqrt{l_s^4 B^2 + R_1^2 R_2^2}, \quad R'_2 = \frac{l_s^2}{R_2}. \quad (3)$$

An original D0-brane is transformed to a D-string wrapped in the $x'_2$ direction. The new torus is a slanted one with the angle $\theta$ determined by $\sin \theta = R_1 R_2 / \sqrt{l_s^4 B^2 + R_1^2 R_2^2}$. The shortest string stretched between a D-string and its nearest image has a mass

$$M = l_s^{-2} R'_1 \sin \theta = l_s^{-2} R_1. \quad (4)$$

![Figure 1](image_url)

**Figure 1.** The shortest string stretched between a D-string and its image.

There is a displacement between the two ends in the new coordinate $x'_2$, $\delta x'_2 = 2\pi B l_s^2 / R_2$. This is the origin of the nonlocality of the field theory describing winding modes on the original two torus. In a vertex involving three open string fields there is an insertion of operator

$$\exp(i2\pi B (\partial_1^1 \partial_2^2 - \partial_1^2 \partial_2^1)), \quad (5)$$
where we used dual coordinates with period $2\pi$ which parametrize winding numbers. $\partial_i^a$ denotes $\partial_i$ acting on the $a$-th field.

Figure 2. An open string appears on the D-string as a dipole. Two dipoles join to form a larger dipole, this is the origin of nonlocality.

The stretched string described above can be interpreted as a winding mode stretched between a D0-brane and its nearest image in the $x_1$ direction, on the original torus. A more general formula for the mass for a winding open string is given by

$$M = l_s^{-2} \sqrt{(mR_1)^2 + (nR_2)^2}. \quad (6)$$

Ignoring the D0-brane and its image on which this string ends, it can be regarded as a closed string of winding numbers $m, n$. However, one should distinguish between this open string mode and the corresponding closed string mode. Indeed, the mass formula for the latter is

$$M = \frac{\sqrt{l_s^4 B^2 + R_1^4 R_2^4} l_s^{-2} \sqrt{(mR_1)^2 + (nR_2)^2}}{R_1 R_2}. \quad (7)$$

This mass is larger than the mass of the open string state with the same winding numbers. It depends on the background field $B$. The reason for the independence of the mass on $B$ of an open string is simple: The world sheet coupling $\int B$ vanishes for an open string upon imposing the Dirichlet boundary conditions. This fact meshes well with the mass formula we obtained on the dual torus $(x'_1, x'_2)$, eq.(4). On the other hand, the world sheet coupling of $B$ to a closed winding string contributes to the quantization of the spectrum. Although the spectrum of open string decouples from $B$, the interaction does not. The standard way of computing interaction is by insertions of the open string vertex operators on the boundary of the world sheet. Now an anomaly will arise from the jump of boundary conditions crossing a vertex operator.

To decouple these winding open string modes from the open string modes with oscillators as well as closed string modes with oscillators, the necessary condition is $R_i \ll l_s$ such that the energy of a stretched open string is much smaller than the string scale. One would expect that in this limit these modes are also decoupled from the closed string massless states. This is simply because the coupling between these two sets are not set by $g_s$, but by $\kappa = g_s l_s^4$. However, we shall use $g_s$ on the dual torus obtained by performing T-duality
along both directions \( x_i \). The new complex moduli associated to this dual torus are given by

\[
\tilde{\tau} = -\frac{1}{\tau}, \quad \tilde{\rho} = -\frac{1}{\rho}.
\]

The new torus is still a rectangular one with nonvanishing \( B \) field. The real parameters are read off from the above equations

\[
\Sigma_1 = \frac{l_s^2 R_2}{\sqrt{l_s^4 B^2 + R_1^2 R_2^2}}, \\
\Sigma_2 = \frac{l_s^2 R_1}{\sqrt{l_s^4 B^2 + R_1^2 R_2^2}}, \\
\tilde{B} = -\frac{l_s^4 B}{l_s^4 B^2 + R_1^2 R_2^2},
\]

Although the new radii depend on \( B \), the mass of a momentum mode is independent of \( B \). This is because it corresponds to a winding mode on the original torus. Another way to see this is by a direct computation. This time \( \tilde{B} \) is not decoupled in the world sheet action, since the relevant boundary condition is Neumann. In the low energy limit \( R_i/l_s \to 0 \), we can ignore the second term in \( l_s^4 B^2 + R_1^2 R_2^2 \) for a fixed \( B \). Thus, the new radii are given by \( \Sigma_1 = R_2/B, \Sigma_2 = R_1/B \). These are much different from the formulas when \( B = 0 \).

For a background \( B \sim 1 \), the size of the dual torus is the same order of the size of the original torus. In the limit we are interested in, both are very small. At this point it is interesting to note that it is impossible for an open string momentum mode to decay into closed string momentum modes. This is simply due to the fact that the energy scale of the former is \( R_i l_s^{-2} \), much smaller than the energy scale \( B/R_i \) of the latter.

Ignoring the high derivative terms introduced by noncommutativity of the dual torus, the low energy theory is a 2 + 1 SYM theory. This theory is well-understood and is the low energy world-volume theory of D2-branes. Naively, one has a paradox here. If the 2 + 1 Yang-Mills is the theory of D2-branes wrapped on the dual torus, why is not the spectrum determined by \( \Sigma_i \) which depend on \( B \), while is determined by the sizes \( l_s^2 / R_i \) as if there is no \( B \) field? Furthermore, what is the Yang-Mills coupling constant, is it still given by \( \tilde{g}_s l_s^{-1} \)? A careful analysis of the low energy action will help to answer these questions.

For convenience, we set \( l_s = 1 \). The low energy condition becomes \( R_i \ll 1 \). The standard Born-Infeld action reads

\[
S = -\frac{1}{g_s} \int d^3 x \det^{1/2} \left( G_{\mu\nu} + F_{\mu\nu} - B_{\mu\nu} \right),
\]

(9)
where for simplicity we dropped the tilde symbol. We also suppressed terms associated to scalars. If we expand the above action in the usual fashion, assuming $G_{\mu \nu}$ dominates other two terms in the determinant, we would obtain the usual low energy Yang-Mills action with $g_{YM}^2 = \tilde{g}_s l_s^{-1}$. In the case of interest, $G_{\mu \nu}$ are given by $\Sigma_i^2$ which are much smaller than the $B$ field, thus we shall expand the determinant in a different way. The determinant is
\[
\det^{1/2}(G_{\mu \nu} + F_{\mu \nu} - B_{\mu \nu}) = \alpha [1 - \frac{1}{\alpha^2}(2 \tilde{B} F_{12} - F_{12}^2 + \Sigma_1^2 F_{01}^2 + \Sigma_2^2 F_{02}^2)]^{1/2},
\]
where $\alpha^2 = \tilde{B}^2 + \Sigma_1^2 \Sigma_2^2$. We see that the terms $F_{0i}^2$ are weighted differently than $F_{12}$. The weights of the former are much smaller. As we shall see in a moment, for an on-shell field configuration, $F_{0i}^2 \sim \Sigma^2 F_{12}^2$. Thus, we can ignore higher orders in $F_{0i}^2$ in the expansion of the determinant, except for the quadratic ones which are needed for having interesting dynamics.

Thus, the appropriate low energy action is
\[
S = \frac{1}{2\tilde{g}_s \alpha} \int \left( \Sigma_2^2 F_{01}^2 + \Sigma_1^2 F_{02}^2 - 2\alpha^2 \left[1 + \frac{1}{\alpha^2}(F_{12}^2 - 2\tilde{B} F_{12})\right]^2 \right). \tag{10}
\]
Since $\alpha^2 \sim \tilde{B}^2 \sim 1$, naively one expects that the quadratic term in $F_{12}$ has a coefficient of order 1. This is incorrect, for the second term in the expansion of the square root almost cancels the first (we drop the linear term in $F_{12}$, it is a total derivative). So to the quadratic order,
\[
S = \frac{1}{2\tilde{g}_s \alpha} \int \left( \Sigma_2^2 F_{01}^2 + \Sigma_1^2 F_{02}^2 - \left(\frac{\Sigma_1 \Sigma_2}{\alpha}\right)^2 F_{12}^2 \right). \tag{11}
\]
An on-shell state will have, for instance $\Sigma_2^2 F_{01}^2 \sim (\Sigma_1 \Sigma_2 / \alpha)^2 F_{12}^2$. This justifies eq.\((11)\).

Now use $\alpha \sim \tilde{B} \sim 1/B$ and $\Sigma_i \sim \epsilon_{ij} R_j / B$, and do rescaling $x^i \to R_i x_i$, $A_i \to 1/R_i A_i$, the Yang-Mills action is put into the standard form $F_{\mu \nu}^2$ with the coupling $g_{YM}^2 = \tilde{g}_s B / (R_1 R_2)$. The fact that the Yang-Mills has the standard form in the new coordinates system implies that the dispersion relation of the spectrum is exactly the same as we argued for before, eq.\((6)\). A typical momentum mode has a mass $R_i$, much smaller than the scale $1/\Sigma_i$.

The Yang-Mills coupling is quite different from $\tilde{g}_s (l_s^{-1})$. Indeed, use the T-duality relation $\tilde{g}_s = g_s / B$, $g_{YM}^2 = g_s / (R_1 R_2)$, the same as one might expect from the T-duality relation for $B = 0$. Since the energy gap is $R_i$, the dimensionless couplings are $g_{YM}^2 / R_i = \tilde{g}_s B / (R_i)^3$. For intermediate such couplings, we require $\tilde{g}_s \sim (R_i)^3 / B \ll 1/B$. The string theory is weakly coupled. When $B = 0$, $g_{YM}^2 = \tilde{g}_s$ and the dimensionless couplings are given by
\( \tilde{g}_s / R_i \). For intermediate couplings we have \( \tilde{g}_s \sim R_i \ll 1 \). Compared to the case \( B \sim 1 \), \( \tilde{g}_s \) goes to zero more slowly.

It is important to realize that we recover the correct physics from the Born-Infeld action only when we expand the square root of the determinant to the second order. In the \( B = 0 \) case, the low energy physics is reproduced by taking only the first order terms. Indeed, there are higher order terms in \( F_{12} \) and these terms can not be ignored in the low energy physics. We now show that for a single quanta, the terms in the square root of eq. (10) are comparable to 1. Use the normalization in (10), a single quanta of energy \( \tilde{R} \) (assuming \( R_1 \sim R_2 \sim R \)) corresponds to the field strength \( F_{12}^2 \sim \tilde{g}_s B \tilde{R}^{-3} \). This gives

\[
\frac{1}{\alpha^2} F_{12}^2 = B^2 F_{12}^2 \sim B^3 (\tilde{g}_s R^{-3}), \quad \frac{\tilde{B}}{\alpha^2} F_{12} = BF_{12} = B^{3/2} (\tilde{g}_s R^{-3})^{1/2}.
\]

For the intermediate Yang-Mills coupling, \( \tilde{g}_s \sim R^3 / B \). So the above terms are comparable to 1 if \( B \sim 1 \), and we have to use the effective action (10), or better, the full Born-Infeld action (9).

It then appears that there is a large correction to the mass of a momentum mode in the gauge theory. We expect that this correction vanishes for some BPS states. A state with constant \( F_{12} \) represents a bound state of D0-branes and a D2-brane, and indeed the presence of \( \tilde{B} \) causes a correction to the mass formula, and the correction is large in the regime of noncommutative torus. We have not discussed the scalar fields. It is not hard to see that the dispersion relation is what was expected, and the correction of the Born-Infeld action to their kinetic energy is not large. For a non-BPS process involving \( F_{12} \), the interaction determined by the Born-Infeld action is large. It is larger than the kinetic energy, and also larger than the interaction energy caused by the higher derivative terms coming from the noncommutativeness. For \( B \sim 1 \), the latter is comparable to the kinetic energy.

Recently Sen and Seiberg proposed a systematical approach to matrix theory on a torus [6]. Seiberg’s argument involves boosting along a small spatial circle to get a near light-like circle of radius \( R \). The resulting matrix theory is the DLCQ version proposed by Susskind [7]. We now employ Seiberg’s boost argument to determine how important the Born-Infeld action is in matrix theory. Let the Planck mass scale be \( M_P \) and the radius of the light-like circle be \( R \). The corresponding scales on the small spatial circle are \( m_P \) and \( R_s \). Consider compactification on a two torus of radii \( R_i \). The corresponding radii in
the other theory with a spatial M circle are denoted by \( r_i \). The scales in the two theories are related through

\[
R_s m_P^2 = R M_P^2, \quad r_i m_P = R_i M_P. \tag{13}
\]

Consequently

\[
g_s = R_s^{3/4}(R M_P^2)^{3/4}, \quad l_s^2 = R_s^{1/2}(R M_P^2)^{-3/2},
\]

\[
r_i = \left( \frac{R_s}{R} \right)^{1/2} R_i. \tag{14}
\]

The matrix theory is obtained as the D0-brane theory in the limit \( R_s \to 0 \). Following [1], we switch on a \( C \) field \( C_{-12} \neq 0 \). This is the field in the DLCQ M theory. Let the corresponding field in the M theory with a small spatial circle be denoted by \( \tilde{C}_{-12} \). There must be a relation

\[
M_P^3 R R_1 R_2 C_{-12} = m_P^3 R_s r_1 r_2 \tilde{C}_{-12}. \tag{15}
\]

The \( B \) field is given by \( B = l_s^{-2} r_1 r_2 \tilde{C}_{-12} \). Using the above relation \( B = M_P^3 R R_1 R_2 C_{-12} \). We see that \( B \) is fixed in the limit \( R_s \to 0 \), and indeed there is a noncommutative torus.

The Yang-Mills coupling on the dual torus is, according to the previous analysis,

\[
g_{YM} = \tilde{g}_s l_s B / (r_1 r_2) = g_s l_s / (r_1 r_2) = R / (R_1 R_2). \]

It is fixed in the limit \( R_s \to 0 \). Note that \( \tilde{g}_s \) goes to zero as \( R_s \to 0 \), and the D2-brane theory is embedded into a weakly coupled string theory.

We have seen that the important terms in the Born-Infeld action for a momentum quanta are of the order given in (12). The \( F_{12}^2 \) term is estimated to be \( B^3 \tilde{g}_s l_s^3 r_i^{-3} = B^2 g_s l_s^3 r_i^{-3} \), where we restored a factor associated to \( l_s \). Using relations (14), this is \( B^2 (M_P R_i)^{-3} \). This is finite in the limit \( R_s \to 0 \), and is small only when the compactification scale \( R_i \) is much larger than the Planck scale. M theory compactified on a two torus correspond to IIA/IIB string theory. When both scales of the two torus are large, this is the strong coupling string limit. (One may employ the \( SL(2, Z) \) duality to go to a weak coupling limit.) If one of the circles, say \( R_1 \), is much smaller than the Planck length, this is a weakly coupled IIA string, and the Born-Infeld action can not be ignored.

Generalization to include multiple D0-branes poses a serious problem. The full non-abelian Born-Infeld action is not known, despite an interesting proposal [8]. In any case, inclusion of Born-Infeld adds to the problem of renormalization.

One way to get around the problem associated with the Born-Infeld action is to go to the large N limit and try to decouple momentum modes. The momentum modes represent longitudinal objects in matrix theory, and in the decompactification limit of the
longitudinal direction, these are much too heavy to play a role in the dynamics. In the large N limit, a toron, for instance, is described by a small $F_{12}$ proportional to $1/N$, and the Born-Infeld action is not important for such a small field strength. Thus a transverse membrane in matrix theory can be described by the usual matrix theory action. It must be noted that, however, the light-cone energy of the bound state of a transverse membrane and N D0-branes gets corrected by a constant term $B/R$. This is simply the statement that a transverse membrane in the background $B = RR_1R_2C$ induces a D0-brane charge $B$. This is achieved in [1] by adding a term $B \int F_{12}$ to the action. Here this term is a consequence of the expansion of the Born-Infeld action [9]. In the matrix string context, the momentum modes are important, but now in a twisted sector. $F_{12}$ can be made arbitrarily small again, if the length of the twisted sector is long enough. It is an interesting question whether the gauge field can be dualized to a scalar on a noncommutative torus. This dualization plays a crucial role in recovering the eighth scalar in the light-cone string theory [9].

It is not clear to us whether the $B$ moduli should play a role in the decompactification limit of the longitudinal direction, since to hold $B$ fixed, $C_{-12} \to 1/R$. There is no doubt that for finite $R$, one can always switch on a nonvanishing $C$ field. In such a case the inclusion of momentum modes of the gauge field necessitates the use of the Born-Infeld action. It is therefore desirable to know the whole action which includes all the relevant higher derivative terms. For the time being, we do not know how to do this. The best way to see a vertex associated with the noncommutative torus is to work with D-strings obtained after T-dualizing one circle. The best way to obtain the Born-Infeld action is to T-dualize both circles and work with a constant background of $F_{\mu\nu}$. Now, since the Born-Infeld involves vertices higher order in the open string field, one might follow Douglas and Hull to derive the general form of a differential operator such as (5). For example, consider a vertex involving four fields. Assume three fields representing three open strings join to form a forth open string. The differential operator is a product of three operators like (5), each is concerned with a pair of fields out of the three fields. In general, define the $\ast$ product for two fields as follows

$$\phi_1(x_1, x_2) \ast \phi_2(x_1, x_2) = e^{i2\pi B(\partial_1^1 \partial_2^2 - \partial_1^2 \partial_2^1)} \phi_1 \phi_2,$$

the three vertex can be written as

$$\int \phi_3(\phi_1 \ast \phi_2).$$
A four vertex can be written as

\[ \int \phi_4((\phi_1 \ast \phi_2) \ast \phi_3), \tag{18} \]

and another possible form

\[ \int (\phi_1 \ast \phi_2)(\phi_3 \ast \phi_4), \tag{19} \]

eq. The \ast product is associative, not commutative. This reflects the nature of open string interaction. Of course an actual vertex may contain additional differential operators. Higher order terms contained in the BI action are weighted by \( B \) as well as \( g_s \). It is difficult to see how these terms could arise in the D-string picture, since there is a slanted torus without a \( B \) background. One would say that the calculation of interaction is identical to the one on a rectangular torus without \( B \), except for insertions of differential operators (5). Is it possible that further terms are needed to cure some problem caused by nonlocality?

Finally, we do not see how to derive higher vertices from the matrix lagrangian, except those contained in the Yang-Mills action, using the procedure of [10]. On general grounds, the IMF Hamiltonian for a given background is not necessarily derivable from the IMF Hamiltonian of another background, due to the subtlety brought about by integrating out zero modes. Here might be a simple example demonstrating this point.

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