NJL interaction derived from QCD: vector and axial-vector mesons

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In previous works effective non-local $SU(2) \times SU(2)$ NJL model was derived in the framework of the fundamental QCD. All the parameters of the model are expressed through QCD parameters: current light quark mass $m_0$ and average non-perturbative $\alpha_s$. The results for scalar and pseudo-scalar mesons are satisfactory agreement to existing data. In the present work the same model without introduction of any additional parameters is applied for a description of masses and strong decay widths of $\rho$- and $a_1$-mesons. The results for both scalar and vector sectors agree with data with only one adjusted parameter $m_0$, with account of average $\alpha_s \approx 0.415$, which is obtained in a previous work as well.

Keywords: compensation equation; effective interaction; fundamental QCD; low-energy meson physics.

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1 Introduction

For description of low-energy hadron physics phenomenological chiral quark Nambu – Jona-Lasinio model [1, 2, 3] (see also review [4] and references therein) is successfully applied for many years. Mass spectra of scalar, pseudo-scalar, vector and axial vector mesons and their low-energy interactions are satisfactorily described in this model [5, 6, 7]. In the simplest case of $SU(2) \times SU(2)$ chiral symmetry this model contains four arbitrary parameters: current mass $m_0$ of quark doublet $u, d$ (in approximation $m_u = m_d$), coupling constant $G_1$ of four-quark interaction of the scalar and pseudo-scalar quark currents in the chiral symmetric form, coupling constant $G_2$ of four quark interaction of vector and axial-vector currents, ultra-violet cut-off parameter $\Lambda$.

These four parameters allows to describe the pion, the $\sigma$-meson, vector mesons $\rho$ and $\omega$, axial-vector mesons $a_1$, masses and their main strong decays and weak pion decay constant $f_\pi$. However, only one of these parameters – $m_0$ – coincides with a parameter of fundamental QCD. So the very important problem is to express the rest parameters in terms of QCD parameters, e.g. $m_0$ and $\alpha_s$ (or $\Lambda_{QCD}$). But for a long time this problem was not solved in a sufficiently satisfactory form.

In recent works [8, 9] we have succeed in obtaining description of $SU(2) \times SU(2)$ NJL model using only QCD parameters. This becomes possible due to use of N.N. Bogoliubov compensation approach [10, 11] (application of the approach to QFT problems is described in work [12]). As a result a non-local version of NJL model was obtained with uniquely defined form-factor. Thus ultra-violet divergences disappear, therefore there is no need of introduction of parameter $\Lambda$. Constants $G_1$ and $G_2$ are expressed through $m_0$ and strong constant $\alpha_s$ in the non-perturbative region.

Remind, that application of these results to the sector of scalar and pseudo-scalar mesons leads to satisfactory description of $\pi$ and $\sigma$ masses, constant of weak pion decay $f_\pi$ and of strong $\sigma \to \pi \pi$ decay. Emphasize, that only parameters $m_0$ and $\alpha_s$ were used.

It is worth noting, that independent estimate of average non-perturbative value $\alpha_s$ was obtained in work [13] (see also [14]). The same Bogoliubov approach for a study of effective non-local three-gluon interaction results in existence of the stable solution for the definite form of non-perturbative contributions to running coupling $\alpha_s(q^2)$. This corresponds to average value for the running coupling in the non-perturbative region $\alpha_s = 0.415$ [14]. Taking into account this result only one parameter $m_0$ remains in our disposal. Note, that results of works [13, 14] lead to a consistent value of gluon condensate.

Here we use the version of of non-local NJL model obtained in [9] with the same parameters $m_0$ and $\alpha_s$ for calculation of masses and decay widths of vector and axial-vector mesons $\rho$ and $a_1$. Remind that we introduce no new parameters at all. A special attention will be paid to value $\alpha_s = 0.415$. 

\[2\]
2 Compensation equation for effective form-factor

In the same way as in work [9] we start from the standard Lagrangian of QCD with two light quarks and number of colours $N = 3$

$$L = \sum_{k=1}^{2} \left( \frac{i}{2} \left( \bar{\psi}_k \gamma^\mu \partial_\mu \psi_k - m_0 \bar{\psi}_k \psi_k - \partial_\mu \bar{\psi}_k \gamma_\mu \psi_k \right) + g_s \bar{\psi}_k \gamma^\mu t^a A_\mu^a \psi_k \right) - \frac{1}{4} \left( F_\mu^a F^a_\mu \right); \quad (1)$$

In accordance to the approach, application of which to such problems are described in details in work [12], we look for a non-trivial solution of a compensation equation, which is formulated on the basis of the Bogoliubov procedure add – subtract. Namely let us rewrite the initial expression (1) in the form

$$L = \frac{i}{2} \left( \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial^\mu \bar{\psi} \gamma^\mu \psi \right) - \frac{1}{4} F_\mu^a F^a_\mu - m_0 \bar{\psi} \psi + \frac{G_1}{2} \left( \bar{\psi} \tau^b \gamma_5 \psi \bar{\psi} \tau^b \gamma_5 \psi - \bar{\psi} \psi \bar{\psi} \psi \right) - \frac{G_2}{2} \left( \bar{\psi} \gamma^\mu \tau^b \gamma_5 \psi \bar{\psi} \gamma^\mu \tau^b \gamma_5 \psi + \bar{\psi} \gamma^\mu \gamma_5 \psi \bar{\psi} \tau^b \gamma_5 \gamma_\mu \psi \right) + \frac{G_3}{2} \left( \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma\mu \psi \right) + g_s \bar{\psi} \gamma^\mu t^a A_\mu^a \psi \right) - \frac{1}{4} \left( F_\mu^a F^a_\mu - F_0^a F^a_0 \right) - \frac{G_1}{2} \left( \bar{\psi} \tau^b \gamma_5 \psi \bar{\psi} \tau^b \gamma_5 \psi - \bar{\psi} \psi \bar{\psi} \psi \right) - \frac{G_2}{2} \left( \bar{\psi} \tau^b \gamma_5 \psi \bar{\psi} \tau^b \gamma_5 \psi + \bar{\psi} \tau^b \gamma_5 \gamma_\mu \psi \bar{\psi} \tau^b \gamma_5 \gamma_\mu \psi \right) - \frac{G_3}{2} \left( \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma\mu \psi \right). \quad (2)$$

Here $\psi$ is isotopic doublet, colour summation is performed inside each spinor bi-linear combination, $F_{0\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and e.g. notion $G_1 \cdot \bar{\psi} \psi \bar{\psi} \psi$ means non-local vertex in the momentum space

$$i (2\pi)^4 G_1 F_1 (p_1, p_2, p_3, p_4) \delta (p_1 + p_2 + p_3 + p_4), \quad (4)$$

where form-factor $F_1$ is introduced, which depends on incoming momenta. The Lagrangian contains contribution of both $G_2$ and $G_3$ which are connected correspondingly to isovector and isoscalar terms. In the present work we consider compensation equation only for isovector four-fermion terms.

Now we consider the first two lines of the Lagrangian (2) as new free Lagrangian $L_0$, and the two last ones as interaction Lagrangian $L_{int}$. Then compensation conditions (see again [12], [9] will consist in demand of full connected four-fermion vertices, following from Lagrangian $L_0$, to be zero. This demand gives a set of non-linear equations for form-factors $F_i$. These equations according to terminology of works [10] [11] are called compensation equations.
In a study of these equations the existence of a perturbative trivial solution (in our case \( G_i = 0 \)) is always evident, but a non-perturbative non-trivial solution may also exist. In the present work as well as in previous one we look for an adequate approach, the first non-perturbative approximation of which describes the main features of the problem. Improvement of a precision of results is to be achieved by corrections to the initial first approximation.

We follow works \([8], [9]\) in definition of the approximation.

1) In compensation equations we restrict ourselves by terms with loop numbers 0, 1, 2.
2) In compensation equations we perform a procedure of linearization over form-factor, which leads to linear integral equations. It means that in loop terms only one vertex contains the form-factor, while other vertices are considered to be point-like.
3) While evaluating diagrams with point-like vertices diverging integrals appear. Bearing in mind that as a result of the study we obtain form-factors decreasing at momentum infinity, we use an intermediate regularization by introducing UV cut-off \( \Lambda \) in the diverging integrals. It will be shown that results do not depend on the value of this cut-off.
4) We use a special approximation for integrals, which is connected with transfer of a quark mass from its propagator to the lower limit of momentum integration. Effectively this leads to introduction of IR cut-off at the lower limit of integration by Euclidean momentum squared \( q^2 \) at value \( m^2 \). To justify this prescription let us consider a typical integral to be encountered here and perform simple evaluations. Functions which we use here depend on variable of the form \( \alpha q^2 \), where \( \alpha \) is a parameter having \( 1/m^2 \) dimension.
5) We keep the first two terms of \( 1/N \) expansion in equations.
6) In case of vector vertices there are two Lorentz structures and thus we have generally speaking two form-factors instead of one in \([9]\). However the corresponding set of equations has no explicit solution similar to that of \([9]\) and so we proceed in the following way. In our approximation we impose simplified kinematic condition that left-side legs of diagrams have momenta \( p \) and \(-p\), while right-side ones have zero momenta. Now in addition to terms proportional to \( \gamma_e \times \gamma_v \), which we are interested in, terms of the form \( \hat{p} \times \hat{p} \) may be also present. Supposing, that the presence of a form-factor connected with the last structure gives small corrections we shall transform the initial equation (in diagram form see Fig. 1) to the scalar one contracting it with projector of the form

\[
\frac{1}{12} (\gamma_e - \frac{\hat{p}p_e}{p^2}).
\]

(5)

In the process of the study we have considered also equations obtained with use of projectors of more general form, namely

\[
\frac{1}{4(4 - d)} (\gamma_e - d \frac{\hat{p}p_e}{p^2});
\]

(6)

It becomes clear, that for values \( d \) between 1 and 2 the corresponding solutions lead to spread of physical values under interest in the range of \( 5 - 7\% \), that corresponds accuracy of the
method as a whole. So we take the formulated projection procedure as a component of the first approximation.

Now the demand of compensation of full connected four-fermion vertices proportional to $G_2$ multiplied by the vector form-factor leads us to the following equation (see Fig. 1)

$$G_2 F(p^2) + G_2^2 \left( \frac{65}{72} p^2 - \frac{7}{12} p^2 \ln \left( \frac{p^2}{\Lambda^2} \right) - \frac{5}{4} \Lambda^2 \right) +$$

$$\frac{G_3 G_2}{2 \pi^2} \left( -\frac{43}{72} p^2 + \frac{5}{12} p^2 \ln \left( \frac{p^2}{\Lambda^2} \right) + \frac{3}{4} \Lambda^2 \right) +$$

$$\frac{G_2^2 N}{32 \pi^4} \int_{m_0}^{\infty} F(k^2) \left(G_2^2 N \Lambda^2 - 4 \pi^2\right) d^4k + \frac{G_1^2}{\pi} \left( \frac{11}{288} p^2 - \frac{1}{16} \Lambda^2 - \frac{1}{48} p^2 \ln \left( \frac{p^2}{\Lambda^2} \right) \right) +$$

$$+ \frac{G_2^3 N}{2 \pi^6} \left( \frac{7}{36} \int_{m_0}^{\infty} \left( \frac{2(kp)^2}{p^2} + k^2 \right) (p-k)^2 \ln \left( \frac{(p-k)^2}{\Lambda^2} \right) F(k^2) d^4k \right) +$$

$$+ \int_{m_0}^{\infty} \left( -\frac{31}{108} \frac{(kp)^2}{p^2} - \frac{109}{864} k^2 \right) (p-k)^2 \frac{F(k^2) d^4k}{(k^2)^2} + \int_{m_0}^{\infty} \left( -\frac{1}{18} k^2 p^2 \ln \left( \frac{(p-k)^2}{\Lambda^2} \right) \right) +$$

$$+ \frac{3}{16} \left( \frac{2(kp)^2}{p^2} + k^2 \right) \Lambda^2 + \left( -\frac{5}{432} \right) \left( -k^2 + 3 (kp)^2 \right) \frac{F(k^2) d^4k}{(k^2)^2} +$$

$$+ \int_{m_0}^{\infty} \left( -\frac{1}{48} \frac{(kp)^2 - k^2 p^2}{(p-k)^4} m_0^4 - \frac{1}{96} m_0^4 \left( 7 \frac{k^2 p^2 + 8 (kp)^2}{p^2} \right) \right) \frac{F(k^2) d^4k}{(k^2)^2} +$$

$$+ \frac{G_2 G_3^2 N}{2 \pi^6} \left( \int_{m_0}^{\infty} \left( \frac{55}{576} \frac{(kp)^2}{p^2} + \frac{17}{2304} k^2 \right) (p-k)^2 \frac{F(k^2) d^4k}{(k^2)^2} \right) +$$

$$+ \int_{m_0}^{\infty} \left( \frac{17}{216} \frac{(kp)^2}{p^2} + \frac{1009}{13824} k^2 \right) (p-k)^2 \frac{F(k^2) d^4k}{(k^2)^2} + \int_{m_0}^{\infty} \left( \frac{11}{384} k^2 p^2 \ln \left( \frac{(p-k)^2}{\Lambda^2} \right) \right) +$$

$$+ \frac{49}{6816} \left( \frac{2(kp)^2}{p^2} + k^2 \right) \Lambda^2 + \frac{5}{6912} (31 (kp)^2 - 33 k^2 p^2) \frac{F(k^2) d^4k}{(k^2)^2} +$$

$$+ \int_{m_0}^{\infty} \left( \frac{1}{288} \frac{(kp)^2 - k^2 p^2}{(p-k)^4} m_0^4 + \frac{1}{384} m_0^4 \left( 7 \frac{k^2 p^2 + 8 (kp)^2}{p^2} \right) \right) \frac{F(k^2) d^4k}{(k^2)^2} +$$

$$+ \frac{G_2 G_1^2 N}{2 \pi^6} \left( -\frac{1}{288} \int_{m_0}^{\infty} \left( \frac{4(kp)^2}{p^2} - k^2 \right) \ln \left( \frac{(p-k)^2}{\Lambda^2} \right) \right) -$$

$$- \frac{1}{1728} \left( \frac{32(kp)^2}{p^2} + k^2 \right) (p-k)^2 \frac{F(k^2) d^4k}{(k^2)^2} +$$

$$+ \int_{m_0}^{\infty} \left( \frac{1}{864} - 5 + 6 \ln \left( \frac{(p-k)^2}{\Lambda^2} \right) \right) (k^2 p^2 - (kp)^2) +$$
We also use relation

\[ u_N \]

have to find a non-trivial solution we integrate by angular variables of four-dimensional space we proportional to \( N \) which is derived in the same works. After dividing by \( \Lambda^2 \). The lower limit of integration is defined by current quark mass \( m_0 \) corresponding to value \( u_0 = 1.925 \times 10^{-8} \), which is obtained in the course of consideration of scalar form-factor (see [8 [9]). We also use relation

\[ G_1 = \frac{6}{13} G_2; \]

which is derived in the same works. After dividing by \( G_2 \) that correspond to our intention to find a non-trivial solution we integrate by angular variables of four-dimensional space we have

\[
+ \frac{1}{96} \left( \frac{2(kp)^2}{p^2} + k^2 \right) \Lambda^2 \int F(k^2) \frac{d^4k}{(k^2)^2} + \\
+ \int_{m_0}^{\infty} \left( -\frac{1}{192} m_0^4 \left( \frac{5 k^2 p^2 - 2 k^2 p^2}{(p - k)^2 p^2} \right) + 1/48 \frac{m_0^4 \left( k^2 p^2 - k^2 p^2 \right)}{(p - k)^4} \right) F(k^2) \frac{d^4k}{(k^2)^2} = 0
\]

Here a conversion to Euclidean momentum space is performed, at one-loop level terms proportional to \( N \) and 1 are taken into account and for two loops respectively \( N^2 \) and \( N \). The lower limit of integration is defined by current quark mass \( m_0 \) corresponding to value \( u_0 = 1.925 \times 10^{-8} \), which is obtained in the course of consideration of scalar form-factor (see [8 [9]). We also use relation

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+ \int_{m_0}^{\infty} \left( -\frac{1}{192} m_0^4 \left( \frac{5 k^2 p^2 - 2 k^2 p^2}{(p - k)^2 p^2} \right) + 1/48 \frac{m_0^4 \left( k^2 p^2 - k^2 p^2 \right)}{(p - k)^4} \right) F(k^2) \frac{d^4k}{(k^2)^2} = 0
\]
in view of looking for solutions of Eq. (9) we apply the differential operator
\[
\frac{d^3}{dx^3} x \frac{d^2}{dx^2} x \frac{d^3}{dx^3} x^2 ,
\]
to this equation. As a result we obtain a differential equation, which with account of the following substitution
\[
z = \beta x^2 , \quad \beta = \frac{1}{26} \frac{N G_2 (12 G_2 - 7 G_3)}{24 \pi^4} ;
\]
reduces to the following form
\[
( z \frac{d}{dz} - b_1 ) ( z \frac{d}{dz} - b_2 ) ( z \frac{d}{dz} - b_3 ) ( z \frac{d}{dz} - b_4 ) ( z \frac{d}{dz} - b_5 ) ( z \frac{d}{dz} - b_6 ) x
\]
\[
\times (z \frac{d}{dz} - b_7) (z \frac{d}{dz} - b_8) F(z) = z \left( z \frac{d}{dz} - a_1 + 1 \right) \left( z \frac{d}{dz} - a_2 + 1 \right) F(z); \tag{11}
\]

i.e. it is Meijer equation of the eighth order. Solutions of the equation are represented in terms of the Meijer functions \[15\] with parameters \(b_i, a_i\), which we can calculate provided \(G_3\) is defined. We can naturally admit \(G_3 = G_2\) following rules of NJL model (see \[7\]). In what follows we present confirmation of this assumption. In this case we have

\[
\begin{align*}
&b_1 := 1.5; \quad b_2 = 1; \quad b_3 = 0.499991384; \quad b_4 = 0.500008866; \\
&b_5 = -1.45597130 \cdot 10^{-7}; \quad b_6 = 0; \quad b_7 = -0.50000003; \quad b_8 = -1.0000001; \\
&a_1 := -0.3944464; \quad a_2 := 1.9013991. \tag{12}
\end{align*}
\]

Values of parameters are calculated with account of value \(m_0\). To obtain a solution of the integral equation we choose four linearly independent solutions of Eq. (11) decreasing at infinity and form the following linear combination with coefficients \(C_i\)

\[
F(z) = C_1 G_{21}^{51} \left( z |^{a_1, a_2}_{b_5, b_4, b_3, b_2, b_1, b_8, b_7, b_6} \right) + C_2 G_{28}^{51} \left( z |^{a_1, a_2}_{b_6, b_5, b_3, b_2, b_1, b_8, b_7, b_4} \right) + C_3 G_{28}^{51} \left( z |^{a_1, a_2}_{b_7, b_4, b_3, b_2, b_1, b_8, b_6, b_5} \right) + C_4 G_{28}^{71} \left( z |^{a_1, a_2}_{b_8, b_6, b_5, b_4, b_3, b_2, b_1, b_7} \right). \tag{13}
\]

Coefficients \(C_i\) are fixed by boundary conditions, which are obtained in the same way as in work \[12\]

\[
3 \left( \frac{13}{96} G_2^2 + \frac{1}{192} G_1^2 + \frac{1}{384} G_3^2 - \frac{5}{96} G_2 G_3 \right) \frac{1}{\pi^4 \sqrt{\beta}} \int_{m_0^2}^{\infty} F(y) dy - \\
- \frac{7}{12} \frac{G_2}{\pi^2} - \frac{1}{48} \frac{G_1}{\pi^2 G_2} + \frac{5}{24} \frac{G_3}{\pi^2} = 0; \tag{14}
\]

\[
\int_{m_0^2}^{\infty} y F(y) dy = 0; \quad \int_{m_0^2}^{\infty} y^2 F(y) dy = 0; \quad \int_{m_0^2}^{\infty} y^3 F(y) dy = 0.
\]

As a result we have

\[
\begin{align*}
C_1 := 0.3330348455; & \quad C_2 := 6.254973002 \cdot 10^{-8}; \\
C_3 := 3.452159489 \cdot 10^{-8}; & \quad C_4 := 2.105889777 \cdot 10^{-15}. \tag{15}
\end{align*}
\]

Unlike of scalar case \[9\] we here do not force the form-factor value at lower integration limit to be unity. Using this condition one might try to define ratio of \(G_2\) and \(G_3\). However assuming equality of these constants we avoid solution of additional complicated transcendental equation, but we acquire a criterion of self-consistency of our approach as a whole, because calculations show, that changing this ratio in reasonable range we have satisfactory results for values of the form-factor at the normalization point. In our case we
have $F(u_0) = 0.96094$ and so we consider our assumption to be justified with reasonable accuracy. Admissible are values of ratio $\frac{G_2}{G_3} = \chi$ from 1 up to 1.2 as well. For the last value $F(u_0) = 1.098993576$. As a matter of fact to fix the ratio one should consider also equation for isoscalar vector terms. However this leads to a considerable complication of the procedure and so here we only noting, that preliminary estimates show that just for range $\chi = 1 - 1.2$ values of isoscalar vector form-factor differs from unity not more than by 10%. So admitting $\chi = 1$ we formulate the ground approximation bearing in mind necessity of further corrections.

3 Wave function of vector states

We have the non-trivial solution of the compensation equation and thus four-fermion terms are excluded from free Lagrangian. There is of course no compensation in interaction Lagrangian, which contains these terms with opposite sign. So we can study a problem of bound states with account of this four-fermion interaction. Bethe – Salpeter equation for vector case in the same approximation as above (see Fig. 2) has the following form. Remind that the first approximation corresponds to zero-mass states (in this approximation there is the same equation for vector and axial-vector).

$$
\Psi(y) = \frac{N}{\pi^4} \left( m^4 \left( \frac{3}{256} G_2^2 - \frac{G_1^2}{128} \right) \int_{m^2}^{\infty} \frac{1}{x} \Psi(y) dy + \right. \\
+ \left( \frac{G_2^2}{48} + \frac{G_1^2}{384} \right) \int_{m^2}^{\infty} \frac{y}{x^2} \Psi(y) dy + \left( \frac{23}{11520} G_2^2 + \frac{G_1^2}{5760} \right) \int_{m^2}^{\infty} \frac{y^3}{x^2} \Psi(y) dy + \\
+ \left( \frac{11}{128} G_2^2 + \frac{G_1^2}{192} \right) \ln(x) x \int_{m^2}^{\infty} \Psi(y) dy + \frac{5}{96} G_2^2 \ln(x) \int_{m^2}^{\infty} y \Psi(y) dy + \\
+ \left( \frac{139}{1152} G_2^2 + \frac{G_1^2}{192} \right) \int_{m^2}^{\infty} y \Psi(y) dy - \left( \frac{19}{4608} G_2^2 + \frac{G_1^2}{768} \right) \int_{m^2}^{\infty} \frac{y^2}{x} \Psi(y) dy + \\
+ \left( \frac{128}{384} G_2^2 + \frac{G_1^2}{192} \right) \frac{\Psi(y)}{y} y \int_{m^2}^{\infty} x \Psi(y) dy - \left( \frac{113}{1152} G_2^2 + \frac{G_1^2}{576} \right) \frac{\Psi(y)}{y} \int_{m^2}^{\infty} \frac{y^2}{x} \Psi(y) dy + \\
+ \left( \frac{G_1^2}{192} \right) x \int_{m^2}^{\infty} \ln(y) \Psi(y) dy + \left( \frac{11}{128} G_2^2 + \frac{G_1^2}{192} \right) x \int_{m^2}^{\infty} \Psi(y) dy + \\
+ \frac{5}{96} G_2^2 \frac{\Psi(y)}{y} \int_{m^2}^{\infty} \ln(y) \Psi(y) dy + \frac{N}{\pi^4} \left( \frac{5}{192} G_2^2 \int_{m^2}^{\infty} \frac{x}{y} \Psi(y) dy + \\
- \frac{5}{64} G_2^2 \int_{m^2}^{\infty} \Psi(y) dy \right) m^4 - \left( \frac{11}{128} G_2^2 + \frac{G_1^2}{192} \right) \ln(\Lambda^2) \int_{m^2}^{\infty} \Psi(y) dy + \\
- \frac{5}{64} G_2^2 \int_{m^2}^{\infty} \Psi(y) dy \right),
$$

(16)
where and coefficients \( b \) and one-meson exchange with corresponding constants \( \alpha \) define was obtained from stability condition for the effective potential. This procedure allows to with that of work \[9\] are presented in the summarizing table. For parameter \( m \) we use results of previous work \[9\] where it was obtained from stability condition for the effective potential. This procedure allows to define \( m \) corresponding to value of \( \alpha \). In Eq. (16) enters constituent mass \( m \), which correspond to values of \( \alpha \) in the range under study. We perform calculations for \( u = \beta m^4 \), which correspond to values of \( \alpha \) in the range under study. Values of \( \alpha \), which are slightly corrected in comparison with that of work \[9\] are presented in the summarizing table.

Differential equation now is the following

\[
\left( z \frac{d}{dz} - b_1 \right) \left( z \frac{d}{dz} - b_2 \right) \left( z \frac{d}{dz} - b_3 \right) \left( z \frac{d}{dz} - b_4 \right) \left( z \frac{d}{dz} - b_5 \right) \left( z \frac{d}{dz} - b_6 \right) \times
\]
\[
\times \left( z \frac{d}{dz} - b_7 \right) \left( z \frac{d}{dz} - b_8 \right) \Psi(z) = -z \left( z \frac{d}{dz} - a_1 + 1 \right) \left( z \frac{d}{dz} - a_2 + 1 \right) \Psi(z) ; \quad (17)
\]

where

\[
z = \beta x^2, \quad \beta = \frac{1}{2^6} \frac{N G_2 (12 G_2 - 7 G_3)}{24 \pi^4}; \quad \xi = \frac{G_1}{G_2};
\]

\[
a_1 = \frac{1}{80} \frac{59 \xi^2 + 6 - \sqrt{8281 \xi^4 + 708 \xi^2 + 36}}{\xi^2};
\]

\[
a_2 = \frac{1}{80} \frac{59 \xi^2 + 6 + \sqrt{8281 \xi^4 + 708 \xi^2 + 36}}{\xi^2}; \quad (18)
\]

and coefficients \( b_i \) are roots of the following equation ( \( G_3 = G_2 \))

\[
\left( -\frac{435}{5408} \frac{Nm^4 G_2^2 b^4}{\pi^4} + \frac{11539}{21632} \frac{Nm^4 G_2^2 b^5}{\pi^4} - \frac{28195}{21632} \frac{Nm^4 G_2^2 b^3}{\pi^4} + \frac{1217}{2704} \frac{Nm^4 G_2^2 b^2}{\pi^4} + \frac{2859}{5408} \frac{Nm^4 G_2^2 b}{\pi^4} \right) + \left( \frac{16}{\pi} \frac{b}{3} - \frac{38}{3} \frac{b^5}{\pi} + \frac{20}{3} \frac{b^4}{\pi} + \frac{110}{3} \frac{b^3}{\pi} + \frac{8}{3} \frac{b^2}{\pi} - \frac{148}{3} \frac{b}{\pi} \right) \left( \alpha_s - \frac{3 g_v^2}{8 \pi} \right) - 2 b^6 - 16 b^5 + 12 b^2 + 20 b^5 + b^8 - 4 b^7 - 11 b^4 = 0. \]
Solution of Eq. (17) decreasing at infinity has the following general form
\[
\Psi(z) = C_1 G_{28}^{41} \left( z \left| a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \right. \right) + C_2 G_{28}^{41} \left( z \left| a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \right. \right) + \\
C_3 G_{28}^{41} \left( z \left| a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \right. \right) + C_4 G_{28}^{61} \left( z \left| a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \right. \right) + \\
C_5 G_{28}^{61} \left( z \left| a_1, a_2, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 \right. \right) .
\] (20)

For \( u = 0.00030 \) values of parameters \( b_i \) read
\[
b_1 = 1.5; \quad b_2 = 1; \quad b_3 = 0.5; \quad b_4 = 0.763584407; \\
b_5 = 0.19323742; \quad b_6 = 0; \quad b_7 = -0.7956972342; \quad b_8 = -1.161124600 .
\] (21)

Parameters \( a_i \) are the same as before (12). Coefficients \( C_i \) are defined from the boundary conditions
\[
\Psi (m^2) = 1; \quad \int_{m^2}^{\infty} \Psi (y) dy = 0; \quad \int_{m^2}^{\infty} y \Psi (y) dy = 0; \\
\int_{m^2}^{\infty} y^2 \Psi (y) dy = 0; \quad \int_{m^2}^{\infty} y^3 \Psi (y) dy = 0; \] (22)

and value \( g_v \) is given by the iterative procedure being defined by normalization condition in one-loop approximation
\[
\frac{N g_v^2}{12 \pi^2} \int_{\tilde{u}}^{\infty} \frac{\Psi(z)^2 F(z)}{z} dz = 1; \quad \tilde{u} = \frac{\beta}{\beta_0} u; \quad \beta_0 = \frac{(G_1^2 + 6 G_1 G_2)}{16 \pi^4} .
\] (23)

Here we introduce into the integral form-factor \( F(z) \) which was obtained in the previous section in view to take into account decreasing of interaction for increasing momentum variable.

Ratio
\[
\frac{\beta}{\beta_0} = \frac{845}{754} .
\] (24)

gives coefficients for transitions to variable \( z \sim p^4 \) respectfully for vector and scalar sectors. Expression for \( \beta_0 \) is obtained in works [8], [9]

With parameters \( b_i \) (21) we have
\[
C_1 = 1.7465; \quad C_2 = 0.021266; \quad C_3 = 0.00107221; \quad C_4 = 0.00142116; \\
C_5 = -0.0000525341 ; \quad g_v = 5.00 .
\] (25)
4 Results and discussion

Now we proceed to calculation of observable parameters. Initial estimate of $\rho$-meson mass is given by expression [7]

$$M_0 = \frac{g_v}{\sqrt{G_2}}; \quad (26)$$

where $G_2$ is defined from relation (8) and $G_1$ is calculated according the method of work [9] for chosen value $u$. Then we introduce one-loop correction to mass squared, which is given by expression

$$\Delta(M^2_0) = -\frac{3 g_v^2}{8\pi^2\sqrt{\beta}} \int_{\tilde{u}}^\infty \frac{\Psi(z)^2 F(z)}{\sqrt{z}} dz; \quad (27)$$

so that

$$M_\rho = \sqrt{M^2_0 + \Delta(M^2_0)}. \quad (28)$$

For values of parameters presented above we have

$$M_0 = 974 MeV; \quad \Delta(M^2_0) = -325908 MeV^2; \quad M_\rho = 789 MeV. \quad (29)$$

We also estimate $a_1$-meson mass according to well known relation [7]

$$M^2_{a_1} = M^2_\rho + 6 m^2. \quad (30)$$

Coupling constant $g_{\rho \to 2\pi}$ of $\rho$-decay to two $\pi$-mesons we find with triangle diagram according to the following relation

$$g_{\rho \to 2\pi} = g_s^2 g_v \frac{3}{4\pi^2} \int_{\tilde{u}}^\infty \frac{\Psi_s(z)^2 \Psi \left( \frac{z}{\beta_0} \right) F(z)}{z} dz; \quad (31)$$

where $\Psi_s(z)$ is Bethe Salpeter wave function for scalar states and $g_s$ is scalar meson coupling according to definition in work [9].

The width of $\rho$ is the following

$$\Gamma_\rho = \frac{g_{\rho \to 2\pi}^2 (M^2_\rho - 4 m^2_\pi)^{3/2}}{24 \pi M^2_\rho}. \quad (32)$$

For $a_1$-meson width we consider two channels: $a_1 \to \rho \pi$ and $a_1 \to \sigma \pi$. The vertex for the first decay has the following form (we omit obvious isotopic factor $\epsilon_{abc}$)

$$V_{\mu\nu}(a_1 \to \rho \pi) = A_0 g_{\mu\nu} + A_2 p_\nu q_\mu; \quad (33)$$

$$A_0 = -\frac{N g_v^2 g_s m}{\pi^2} \int_{m^2}^\infty \frac{\Psi(y)^2 \Psi_s(y)}{y} dy; \quad A_2 = \frac{N g_v^2 g_s m}{2\pi^2} \int_{m^2}^\infty \frac{\Psi(y)^2 \Psi_s(y)}{y^2} dy;$$
where $p$, $\mu$ and $q$, $\nu$ are respectfully momentum and Lorentz index for $a_1$-meson and $\rho$-meson, $m$ is constituent quark mass.

The second decay is described by the following vertex (isotopic factor $\delta_{ab}$)

$$\nu^\mu (a_1 \rightarrow \sigma \pi) = g_{a_1 \rightarrow \sigma \pi} (q - k)_\mu; \quad (34)$$

$$g_{a_1 \rightarrow \sigma \pi} = \frac{Ng_\sigma g_\pi^2}{2\pi^2} \int_{m^2}^{\infty} \frac{\Psi(y) \Psi_s(y)^2}{y} dy;$$

where $k$ and $q$ are respectfully momentum of $\pi$ and $\sigma$ and $\mu$ is Lorentz index of $a_1$. Corresponding partial widths read

$$\Gamma(a_1 \rightarrow \rho \pi) = \frac{M_{a_1}^2 - M_\rho^2}{24\pi M_{a_1}^3} \left( A_0^2 \left( 2 + \frac{(M_{a_1}^2 + M_\rho^2)}{4M_{a_1}^2 M_\rho^2} \right) + A_0 A_2 \frac{(M_{a_1}^2 + M_\rho^2)(M_{a_1}^2 - M_\rho^2)^2}{4M_{a_1}^2 M_\rho^2} + A_2^2 \frac{(M_{a_1}^2 - M_\rho^2)^4}{16M_{a_1}^2 M_\rho^2} \right); \quad (35)$$

$$\Gamma(a_1 \rightarrow \sigma \pi) = \frac{g_{a_1 \rightarrow \sigma \pi}^2 (M_{a_1}^2 - M_\sigma^2)}{48\pi M_{a_1}^5}; \quad \Gamma_{a_1} = \Gamma(a_1 \rightarrow \rho \pi) + \Gamma(a_1 \rightarrow \sigma \pi).$$

Here we assume $m_\pi^2 << M_{a_1,\rho,\sigma}^2$.

Using all these expressions we calculate observable quantities for vector and axial-vector mesons. Note, that in calculation of widths of decays we substitute calculated masses of corresponding mesons. Results are presented in Table. We present there set of calculated parameters in dependence on average non-perturbative running coupling $\alpha_s$ in range 0.29 – 0.48. We normalize our calculations by the most precise parameter $f_\pi$. All other numbers in the Table are calculated. We take each column of the Table as a set of the corresponding parameters calculated starting from presented there parameters $\alpha_s$ and $m_0$. Following the general ideology of our approach we consider the last column with $\alpha_s = 0.415$ as the final result of our work, bearing in mind, that this value of average $\alpha_s$ is obtained in work [13]. In addition to values of parameters presented in this column we remind parameters of scalar sector, which are obtained in the course of performing of previous work [9]. For the same $\alpha_s = 0.415$ we have:

$$m_\pi = 134 \text{MeV}; \quad \langle \bar{q} q \rangle = -(230 \text{MeV})^3; \quad m_\sigma = 480 \text{MeV}; \quad \Gamma_\sigma = 560 \text{MeV}. \quad (36)$$

Thus the set of parameters seems to be in satisfactory agreement with data. The only parameter differing from corresponding experimental value by more than 12% is the mass of $a_1$. As a matter of fact a low value for $M_{a_1}$ is inherent to other NJL calculations (see, e.g. [7]). Presumably in this case there are some additional contributions to be taken into account.

The results being obtained here and in previous work [2] demonstrate that Bogoliubov compensation method leads to a reasonable form of effective non-local four-quark interaction of the NJL type. As a result this method allows to describe the light meson masses and
probabilities of their main decays. Emphasize, that advantages of this approach are absence of ultra-violet divergences in quark loop diagrams, inherent in the usual NJL model, and the presence of only one arbitrary parameter, namely current quark mass $m_0$. Note that in spite of this parameter being somewhat larger than its standard value we nevertheless obtain reasonable value for constituent quark mass $m \simeq 260\text{MeV}$ \cite{9}. Note, that value of $m_0$ is defined by compensation equation for scalar form-factor in works \cite{8,9}. Estimates show that presence of correction terms in this equation (e.g. the next terms of $1/N$ expansion) changes value of $m_0$ significantly, while the observable parameters change only slightly. A development of these considerations is the problem for future studies.

This method in future studies may also be applied to description of electro-weak properties of mesons (e.g. pion form-factor, pion polarizability), of $\pi - \pi$ scattering lengths etc. It is important problem to expand the approach for chiral $U(3) \times U(3)$ symmetry with inclusion of the strange quark $s$.

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|       | $u$       | 0.00015 | 0.00030 | 0.00045 | 0.00032 | exp/phen |
|-------|-----------|---------|---------|---------|---------|----------|
| $\alpha_s$ | 0.287 | 0.404 | 0.483 | 0.415 | 0.415 [14] |
| $f_\pi$ MeV | 93 | 93 | 93 | 93 | input |
| $g_s$ | 2.66 | 2.84 | 2.93 | 2.86 | – |
| $m_0$ MeV | 21.9 | 21.6 | 21.2 | 21.5 | 5 – 10 |
| $m$ MeV | 247 | 264 | 271 | 265 | $\approx$ 300 |
| $G_1^{1/2}$ MeV | 320 | 287 | 267 | 283 | – |
| $g_\rho$ | 4.30 | 5.00 | 5.52 | 5.11 | – |
| $M_\rho$ MeV | 713 | 785 | 830 | 791 | $771.1 \pm 0.9$ |
| $g_\rho \pi \pi$ | 4.29 | 4.41 | 4.36 | 4.44 | 4.26 |
| $\Gamma_\rho$ MeV | 136 | 166 | 175 | 170 | $149.2 \pm 0.7$ |
| $M_{a_1}$ MeV | 935 | 1017 | 1043 | 1018 | $1230 \pm 40$ |
| $\Gamma_{a_1}$ MeV | 268 | 330 | 312 | 334 | $250 – 600$ |
| $\Gamma(a_1 \rightarrow \sigma \pi)/\Gamma_{a_1}$ | 0.168 | 0.188 | 0.201 | 0.189 | $0.188 \pm 0.043$ [16] |
Figure captions

Fig. 1. Diagram representation of the compensation equation.

Fig. 2. Diagram representation of Bethe-Salpeter equation for vector bound state.
\[ GF(p) = G \]

Fig. 1.
Fig. 2.