

One-loop corrections to a scalar field during inflation

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Submitted to: JCAP

PACS numbers: 98.80.-k, 98.80.Cq, 11.10.Hi

Abstract. The leading quantum correction to the power spectrum of a gravitationally-coupled light scalar field is calculated, assuming that it is generated during a phase of single-field, slow-roll inflation.

Keywords: Inflation, Cosmological perturbation theory, Physics of the early universe, Quantum field theory in curved spacetime.
1. Introduction

Over the last several decades, our theories of the early universe have been promoted from an area of speculation to a field of intense scientific study. The most important extension of our knowledge concerns the nature of the primordial curvature perturbation $\zeta$, which is widely believed to have seeded temperature variations in the cosmic microwave background (CMB). It is now understood that $\zeta$ must have had a spectrum which was close to scale invariance on the scales probed by the CMB [2, 3, 4].

Many theories have been proposed to explain how a set of primordial perturbations with an almost scale-invariant spectrum could have been generated in the early universe. The most widely-studied candidate is the suggestion that an era of inflation may have taken place at high energy [5, 6, 7, 8, 9, 10, 11], where ‘inflation’ is defined to be any epoch in which the scale factor accelerates, $\ddot{a} > 0$. Under these circumstances, local regions of the universe are exponentially driven to spatial flatness, homogeneity and isotropy [12], and each light bosonic field acquires a spectrum of perturbations generated by amplification of quantum-mechanical vacuum fluctuations [13, 14, 15, 16]. This spectrum is close to scale-invariance when the universe inflates at a rate $\dot{a}/a$ which is almost constant. The curvature perturbation observed in the CMB is taken to be a model-dependent mix of these fluctuations, giving rise to anisotropies in the temperature which are compatible with observation. Therefore, inflation apparently provides a natural framework in which one can understand both the large-scale regularity of the universe and its small-scale irregularity. For this reason, among others, it has become the dominant paradigm in which to model the very early universe.

Inflation is not a single model, but rather a whole collection of scenarios which fit into the above framework. The only necessary ingredients are: (i) a specification of the field content, which allows a division into ‘light’ and ‘heavy’ fields; (ii) a background evolution $a(t)$ which gives rise to $\dot{a} > 0$ with the Hubble parameter $H \equiv \dot{a}/a$ slowly varying; and (iii) a rule for generating $\zeta$ from the light bosonic fields.

This prescription is extremely general and implies that very many models (perhaps with wildly different and mutually incompatible microphysics) may simultaneously be compatible with the observational data, since they will make equivalent predictions for the spectrum of $\zeta$. In view of this redundancy, we must expect that it will be difficult to learn about the microscopic physics which was operative during the very early universe. In particular, it will almost certainly be insufficient simply to study the spectrum of $\zeta$. In order to distinguish between wildly different models of the early universe it is necessary to find another source of data.

Any detailed model of the inflationary era does not merely predict the spectrum

† There are two commonly encountered perturbations in the literature. The first of these is the comoving curvature perturbation, written $\mathcal{R}$, which is proportional to the laplacian of the Ricci curvature of comoving spatial slices. On the other hand, the uniform density curvature perturbation $\zeta$ is proportional to the laplacian of the Ricci curvature on spatial slices of uniform density. $\mathcal{R}$ and $\zeta$ are equivalent up to a sign convention on superhorizon scales [11].
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of $\zeta$; it also implies a subtle but calculable network of correlations between the higher-order moments. These moments collectively measure the so-called non-gaussianity of $\zeta$ and arise from self-interactions among the quanta of the effective $\zeta$-field in the early universe. Such self-interactions are mandatory in any realistic theory of inflationary physics: since gravity is non-linear and couples universally to matter, there will always be interactions mediated by gravitational effects, quite apart from whatever interactions are explicitly postulated among the matter fields.

Non-gaussianities from self-interactions of $\zeta$ quanta have been investigated very extensively over the last few years, with the hope that observations of such effects may be able to discriminate between different models for physics in the early universe. Non-gaussianities from self-interactions of $\zeta$ quanta have been investigated very extensively over the last few years, with the hope that observations of such effects may be able to discriminate between different models for physics in the early universe. However, much more is possible. Self-interactions do not only imply non-gaussian statistics in the three- and higher $n$-point correlations functions: they also imply quantum corrections to all correlation functions, and in particular the power spectrum or two-point function. It is possible that such corrections may be large in their own right, demanding that they be taken into account in accurate analyses of the observational data, as recently suggested by Sloth [46, 47]. However, regardless of their exact magnitude (provided they are detectable), by searching for signatures of such quantum corrections in the power spectrum and correlating the results with predictions for non-gaussian statistics in the higher $n$-point functions we obtain a more sensitive test of physics during inflation. This gives rise to the hope that it will eventually be possible to place restrictions on the effective field theory which was operative during inflation, although at present that hope is still somewhat distant.

A second powerful motivation for studying loop corrections is a simple point of principle. The tree-level formula for the spectrum of $\zeta$ is widely used to make predictions for the amplitude and scale-dependence of fluctuations generated in a very large class of early universe scenarios. Before deciding what degree of credence we should attach to any of these predictions, it is necessary to thoroughly investigate whether the tree-level amplitude is a genuine approximation to the full quantum result.

In this paper, the prospects for detecting quantum corrections to the power spectrum are assessed in the simplest model of inflationary physics, that of inflation with a single scalar $\phi$ and arbitrary potential $V(\phi)$. In view of the importance of accurate comparisons with the precision measurements which are becoming available, this issue has attracted considerable previous attention. Early work by Mukhanov, Abramo & Brandenberger [48, 49] and Abramo & Woodard [50] demonstrated that significant effects were possible. Sloth [46, 47] estimated the one-loop correction to the power spectrum of field fluctuations, finding a significant cumulative effect (as large as 70% in some models) which could affect the precision determination of cosmological parameters from CMB experiments. This effect enhanced the one-loop correction by essentially a factor $N$, the number of e-folds between horizon exit of the mode under consideration and the end of inflation. A similar estimate has been made by Weinberg [51], who found corrections which scale like powers of $N$. Boyanovsky, de Vega &
Sanchez [52, 53] made an earlier estimate but did not take into account the coupling of field fluctuations to the metric and obtained an enhancement which scaled like $N^2$. Most recently, Bilandžić & Prokopec [54] reconsidered the issue in a different framework, finding an enhancement of quantum corrections scaling like $N$.

This paper attempts to readdresses these issues using a formalism similar to that applied by Sloth [46, 47]. However, in complete contrast to all previous analyses, the estimate is divided into two parts, the first of which appears in the present paper. First, the one-loop correction to the power spectrum of field fluctuations is computed soon after horizon exit. This correction is not observable by itself; it must be combined with other correlators of the fields in a second step to yield the one-loop correction to the power spectrum of the observable perturbation $\zeta$ long after horizon exit. The correct combination can be computed using the $\delta N$ formula [55, 56, 19, 57, 23]. This two-step process has several advantages. We shall see that the loop correction is generally afflicted by divergences at late times and on large scales. The $\delta N$ formalism naturally resums these late-time divergences into time evolution, whereas the divergences on large scales can be controlled by performing the calculation within a finite box. (It has recently been shown by Byrnes et al. [45] that these divergences can be resummed into spatial variation on large scales.) This is what we would naturally expect from our experience with the renormalization group in Minkowski space quantum field theory.

The present paper is concerned with the mostly technical issue of computing the loop correction for the power spectrum of the field fluctuations. This calculation involves the application of standard methods from quantum field theory, adapted to the case of an expanding spacetime. On the other hand, the resummation of field correlators into $\zeta$ correlators is an essentially classical calculation using the $\delta N$ formula. For clarity, this calculation will be presented separately elsewhere [58].

In §2 the background evolution and perturbation theory of the single scalar field are briefly described. The perturbations are characterized (as in more complex cases) by cubic and higher self-interactions which involve the time derivative of the perturbation, a fact which has important consequences for the calculation of quantum corrections. These corrections are introduced in §3. In §3.1 a path-integral expression for a general one-loop, single-vertex correction to the power spectrum is given in the Schwinger formalism, and in §3.2 the question of deriving a correct path integral expression for theories with derivative interactions is considered. In such cases the correct path-integral formula is well-known to contain a ‘ghost’ field, whose quanta do not appear in physical states but which circulate in the loops which give rise to quantum corrections. The Feynman rules for this theory are written down in §3.3. In §4 the assembled formalism is used to compute the leading radiative correction to the one-point function of the field. This is of interest in its own right, but also provides a simple setting in which some subtle features of the calculational machinery can be resolved. The one-loop correction to the two-point function is computed in §5.

§2 is introductory and merely serves to fix notation. The reader who is mostly interested in the computation of the two-point function $\langle \delta \phi(k_1) \delta \phi(k_2) \rangle$ may wish to
dispense with §3.1 and §3.2. These sections are largely dominated by the question of setting up a correct formalism in which the one-loop correction may be computed.

Units are chosen throughout such that $\hbar = c = M_P = 1$, where $M_P^{-2} = 8\pi G$ is the reduced Planck mass. The metric convention is $(-,+,+,+)$, and the unperturbed background is written in cosmic time $t$ as

$$ds^2 = -dt^2 + a^2(t) \, dx^2. \quad (1)$$

It is frequently more convenient to employ a conformally rescaled time variable, defined by $\eta = \int_0^t dt'/a(t')$. Indices labelling spacetime coordinates are chosen from the beginning of the Latin alphabet $(a,b,\ldots)$; indices labelling purely spatial coordinates are chosen from the middle of the alphabet $(i,j,\ldots)$. The different species of light bosonic fields are labelled with Greek indices $(\alpha,\beta,\ldots)$.

2. Inflation from a single scalar field

2.1. The background evolution

The simplest realistic microphysical model capable of supporting an inflationary epoch consists of Einstein gravity coupled to a single scalar field $\phi$ with potential $V(\phi)$, which can be taken to be arbitrary except that it must support inflation for some values of $\phi$. The field $\phi$ is known as the inflaton. The combined action for this system is

$$S = -\frac{1}{2} \int dt \, d^3x \sqrt{-g} \left\{ R - \nabla^a \phi \nabla_a \phi - 2V(\phi) \right\}, \quad (2)$$

where $g = \det g_{ab}$ is the metric determinant and $R$ is the spacetime Ricci curvature. The background field $\phi$ is taken to be spatially homogeneous and the background metric is parametrized by the scale factor $a$, given in (1). The evolution of $a$ is determined by the Friedmann equation

$$3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (3)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and $\dot{\phi}$ obeys the homogeneous Klein–Gordon equation

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0, \quad (4)$$

where a prime $'$ denotes a derivative with respect to $\phi$. The condition that inflation occurs, $\ddot{a} > 0$, can be rewritten as $aH^2(1 - \epsilon) > 0$, where $\epsilon$ satisfies

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \quad (5)$$

Using Eqs. (3)–(4) one can show that an equivalent definition is $\epsilon \equiv \dot{\phi}^2/2H^2$. Thus $\epsilon$ is manifestly positive. The condition that inflation occurs is $\epsilon < 1$.

When $\epsilon$ obeys the stronger condition $\epsilon \ll 1$, the rate of change of $\phi$ is negligible in comparison with the expansion rate $H$. In this case one says that the field is in the slow-roll régime. Although slow-roll is not mandatory for inflation to occur, the near scale-invariance of the power spectrum imprinted on the scales which are observed
in the CMB suggests that the field was indeed slowly rolling if the CMB perturbation has an inflationary origin. When $\epsilon \ll 1$ applies, one can develop a useful perturbation expansion in $\epsilon$, known as the slow-roll approximation. In this paper, we compute all effects to leading order in $\epsilon$.

2.2. Scalar perturbations

Now consider the possibility of small spatially-dependent perturbations in the inflaton, $\phi = \phi_0 + \delta \phi(t, x)$, where $\phi_0$ is the homogeneous background evolution and $\delta \phi$ obeys the smallness condition $|\delta \phi| \ll |\phi_0|$. Since $\phi$ dominates the energy density of the universe by assumption, any perturbation in $\phi$ will lead to a perturbation in the metric. These perturbations can be parametrized by a scalar $N$ (the lapse), a spatial vector $N^i$ (the shift), and a spatial metric $h_{ij}$,

$$ds^2 = -N(t, x)^2 dt^2 + h_{ij}(x) \left\{ dx^i + N_i(t, x) dt \right\} \left\{ dx^j + N^j(t, x) dt \right\}.$$  \hspace{1cm} (6)

Because of general coordinate invariance, not all choices of $\{N, N^i, h_{ij}\}$ lead to different configurations of the gravitational field. This redundancy is removed by fixing a gauge, which requires a choice of slicing into spatial hypersurfaces accompanied by a prescription for threading these spatial slices together. We will choose to work in the spatially flat gauge, where $h_{ij}$ is given by its background value $h_{ij} = a^2(t)\delta_{ij}$. Having done so, the metric functions $N$ and $N^i$ are completely determined in terms of $\delta \phi$ by the constraint part of the Einstein equations.

These constraint equations can be obtained by inserting the metric (6) in the Einstein–scalar action (2). One obtains

$$S = -\frac{1}{2} \int dt d^3x \sqrt{h} \left\{ N (\nabla^i \phi \nabla_i \phi + 2V) - \frac{1}{N} (E^i_j E_{ij} - E^2 + \pi^2) \right\},$$  \hspace{1cm} (7)

where $E_{ij} = \frac{1}{2} h_{ij} - \nabla_i N_j$ is the ‘gravitational momentum’ associated with $h_{ij}$, $\nabla_i$ is the spatial covariant derivative compatible with $h_{ij}$, and $\pi = \dot{\phi} - N^j \nabla_j \phi$ is the field momentum. The equations of motion for the lapse and shift follow by varying $S$ with respect to $N$ and $N^i$ respectively, and do not involve time derivatives. Therefore they are not evolution equations but constraints and can be solved algebraically: $N$ and $N^i$ are not propagating fields. Once $N$ and $N^i$ are known they may be substituted in (7) to obtain a reduced action which depends only on $\delta \phi$.

The $N$ constraint is

$$\nabla^i \phi \nabla_i \phi + 2V + \frac{1}{N^2} (E^i_j E_{ij} - E^2 + \pi^2) = 0$$  \hspace{1cm} (8)

and the $N^i$ constraint is

$$\nabla_i \left\{ \frac{1}{N} (E^j_j - E \delta^j_j) \right\} = \frac{\pi}{N} \nabla_j \phi.$$  \hspace{1cm} (9)

† We are adopting a useful convention used throughout this paper in which repeated spatial indices in complementary raised and lowered positions are contracted with the spatial metric $h_{ij}$, whereas a pair of repeated indices which both appear in the lowered position are contracted with the Euclidean metric $\delta_{ij}$. Thus, $a^i b_i = \sum_{i,j} h^{ij} a_i b_i$, whereas $a_i b_i = \sum_i a_i b_i$. Spacetime indices always appear in complementary raised and lowered pairs, and are contracted with the spacetime metric $g_{ab}$.
One solves Eqs. (8)–(9) order by order in $\delta \phi$. We write

$$N = 1 + \sum_{m=1}^{\infty} \alpha_m, \quad \text{and} \quad N_i = \nabla_i \left( \sum_{m=1}^{\infty} \vartheta_m \right) + \sum_{m=1}^{\infty} \beta_{mi}$$

(10)

where $\alpha_m$, $\vartheta_m$ and $\beta_{mi}$ are all $m$th order in $\delta \phi$ and the $\beta_{mi}$ are chosen to be divergenceless, so that $\nabla^i \beta_{mi}$ for all $m$. The expressions necessary to compute $S$ to third order in $\delta \phi$ were given by Maldacena in the comoving slicing [18] and rewritten in the flat slicing for multiple fields in Ref. [21]. Working to one order greater, the expressions necessary to compute $S$ to fourth order were obtained in the flat slicing by Sloth [46, 47] (in an approximation where all vector modes were absent), and given in complete generality in Ref. [59].

We work to leading order in the slow-roll approximation. At first order in $\delta \phi$ the leading terms are $o(\epsilon^{1/2})$,

$$\alpha_1 = \frac{1}{2H} \dot{\phi} \delta \phi, \quad \partial^2 \vartheta_1 = -\frac{a^2}{2H} \dot{\phi} \delta \phi \quad \text{and} \quad \beta_{ii} = 0.$$  

(11)

At second order in $\delta \phi$ the leading terms are $o(\epsilon^0)$

$$\alpha_2 = \frac{1}{2H} \partial^2 \Sigma, \quad \frac{4H}{a^2} \partial^2 \vartheta_2 = -\frac{1}{a^2} \partial_i \delta \phi \partial_i \delta \phi - \delta \phi \dot{\phi} - 12H^2 \alpha_2,$$  

(12)

$$\frac{1}{2a^2} \partial^4 \beta_{2i} = \delta^{rs} (\partial_i \Sigma_{rs} - \partial_r (\Sigma_{si})),$$  

(13)

where bracketed indices (· · ·) are symmetrized with total weight unity and $\Sigma_{rs}$ is defined by

$$\Sigma_{rs} \equiv \partial_r \dot{\phi} \partial_s \delta \phi + \delta \phi \partial_r \partial_s \delta \phi,$$  

(14)

with $\Sigma = \text{tr} \Sigma_{rs}$ its trace in the Euclidean metric. Eqs. (11)–(13) can be inserted into the action, Eq. (7), after which one obtains an expansion of $S$ in powers of $\delta \phi$. The first non-trivial term is quadratic. At $o(\epsilon^0)$ it is equal to

$$S_2 = \frac{1}{2} \int dt \, d^3x \left( \dot{\delta \phi}^2 - \frac{1}{a^2} \left( \partial \delta \phi \right)^2 \right)$$  

(15)

There is a cubic interaction whose leading term enters at $o(\epsilon^{1/2})$ [21], which can be written

$$S_3 = \int dt \, d^3x \, a^3 \left\{ \frac{\dot{\phi}}{2H} \delta \phi \partial_j \partial^2 \delta \phi \partial_j \delta \phi - \frac{\dot{\phi}}{4H} \delta \phi \left[ \delta \phi^2 + \frac{1}{a^2} \left( \partial \delta \phi \right)^2 \right] \right\}.$$  

(16)

The quartic term has leading terms of order $o(\epsilon^0)$ [59]. These terms correspond to

$$S_4 = \int dt \, d^3x \left\{ -\frac{1}{4a^2} \beta_{2j} \partial^2 \beta_{2j} - \frac{a^3}{4H} \partial^2 \Sigma \left[ \delta \phi^2 + \frac{1}{a^2} \left( \partial \delta \phi \right)^2 \right] - \frac{3}{4} a^3 (\partial^2 \Sigma)^2 - a \delta \phi \beta_{2j} \partial_j \delta \phi \right\}.$$  

(17)

The free field action $S_2$ and the interactions $\{ S_3, S_4 \}$ as written here are all accompanied by terms of higher-order in slow-roll parameters, which we neglect.
2.3. Expectation values

A class of especially important observables in this theory are expectation values of products of $n$ factors of the perturbation $\delta\phi$, taken at a common time $t_*$ but at distinct spatial coordinates $\{x_1, \ldots, x_n\}$. However, it is often more convenient to work with momentum space expectation values which are obtained by taking Fourier transforms with respect to the $x_i$, giving $k$-space correlators which are functions of $\{k_1, \ldots, k_n\}$.

At tree level the one-point expectation value vanishes, $\langle \delta\phi(k) \rangle = 0$, since $\delta\phi$ is by definition a perturbation in the comoving region under consideration. However, the gravitational background is time dependent since the scale factor $a(t)$ varies with $t$, and therefore the vacuum state of the theory is changing continuously. This effect leads to the gravitational production of inflaton particles \[60\]. Therefore we must expect a non-zero one-point function to be generated radiatively, reflecting the emergence of $\phi$ quanta from the vacuum. This issue is discussed in more detail in §4 below.

The two-point expectation value defines the power spectrum $P(k)$,

$$\langle \delta\phi(k_1)\delta\phi(k_2) \rangle_\ast = (2\pi)^3 \delta(k_1 + k_2)P_\ast(k_1).$$

The subscript ‘$\ast$’ denotes evaluation at the time when the $k$-mode under consideration left the horizon. At tree-level, $P_\ast(k) = H_\ast^2/2k^3$. It is often useful to work instead with the so-called dimensionless power spectrum, which is related to $P(k)$ by the rule $P(k) = k^3P_\ast(k)/2\pi^2$.

The three-point expectation value defines the bispectrum, $B(k_1, k_2, k_3)$, and the four-point expectation value defines the trispectrum, $T(k_1, k_2, k_3, k_4)$,

$$\langle \delta\phi(k_1)\delta\phi(k_2)\delta\phi(k_3) \rangle_\ast = (2\pi)^3 \delta(\sum_i k_i)B_\ast(k_1, k_2, k_3),$$

and

$$\langle \delta\phi(k_1)\delta\phi(k_2)\delta\phi(k_3)\delta\phi(k_4) \rangle_\ast = (2\pi)^3 \delta(\sum_i k_i)T_\ast(k_1, k_2, k_3, k_4).$$

Typically, $B$ and $T$ at tree-level are proportional to $P_\ast^2$ and $P_\ast^3$ respectively, multiplied by a momentum-dependent form-factor.

3. Quantum corrections

3.1. Loop corrections from Schwinger integrals

The appropriate formalism for computing expectation values in a quantum field theory was outlined by Schwinger \[61\]. Consider the vacuum expectation value of any observable $O$, observed at some time $t_*$, taken in some theory with light scalar fields $\{\phi^\alpha\}$. By inserting a complete set of states at any time $t_\sharp > t_*$, this expectation value can be written

$$\langle \Omega|O|\Omega \rangle_\ast = \int[d\phi^\alpha]\langle \Omega|\phi^\alpha_\sharp = \varphi^\alpha\rangle\langle\phi^\alpha_\sharp = \varphi^\alpha|O|\Omega \rangle_\ast,$$
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where $|\Omega\rangle$ is the vacuum state at $t \to -\infty$, the subscript ‘∗’ indicates that the fields in the expectation value are evaluated at $t_∗$, and $\phi_2$ denotes $\phi$ evaluated at $t_2$. The integral $\int [d\phi^α]$ is taken over all three-dimensional field configurations of all fields in the theory at $t_2$. Each factor in the product of transition amplitudes on the right-hand side of (21) can be expressed using the conventional Feynman path integral formula [62, 63, 64], $\langle \phi_2 = \varphi | O | \Omega \rangle = \int [d\phi^α]|O\rangle \Omega \exp iS$, where $S$ is the action functional and the integral is taken over all field configurations which begin in the state $|\Omega\rangle$ and end in the state $|\phi_2 = \varphi\rangle$, denoted by the limits attached to $[d\phi]$.

The interacting vacuum

In order to evaluate such integrals by the usual Feynman diagram expansion it is necessary to remove these restrictions on the range of integration, so that we integrate over all $\phi$ unrestrictedly. We follow the analysis of Weinberg [51]. To remove the restriction that the field must begin in the vacuum state, one can integrate over all $\phi$ obeying an arbitrary boundary condition at $t \to -\infty$, after multiplying the integrand by the vacuum wavefunctional, $\Psi[\psi] = \langle \phi(t \to -\infty) = \psi | \Omega \rangle$. This has the desired effect of restricting the integral to field configurations which begin in the correct vacuum. The exact expression for $\Psi$ depends on what we assume about $|\Omega\rangle$, but because the theory is supposed to be free as $t \to -\infty$ it must be a gaussian in the fields [62, 51]. Therefore we assume

$$\Psi[\psi] \propto \exp \left\{ -\frac{1}{2} \int \frac{d^3q d^3r}{(2\pi)^3} \delta(q + r)\Omega(q)\psi(q)\psi(r) \right\} \equiv \exp \left\{ -\frac{1}{2} (\psi, \Omega \psi) \right\}, \tag{22}$$

for some weight functional $\Omega(q)$, where $(\psi, \Omega \psi)$ is a convenient abbreviation for the integral. The expectation value (21) can therefore be written [51]

$$\langle O | \Omega \rangle = \int [d\phi^α] \left\{ \int [d\phi^β_\pm ] \exp \left( iS[\phi^β_\pm ] - \frac{1}{2} \sum_\beta (\psi^\beta_\pm , \Omega \psi^\beta_\pm ) \right) \right\}^\dagger \left\{ \int [d\phi^γ_\pm ] O \exp \left( iS[\phi^γ_\pm ] - \frac{1}{2} \sum_\gamma (\psi^\gamma_\pm , \Omega \psi^\gamma_\pm ) \right) \right\} \tag{23}$$

where ‘†’ denotes Hermitian conjugation and the fields in the two path integrals have been differentiated by the addition of subscripts ‘+’ and ‘−’. The integral over final field configurations means that we can drop the restriction on the fields $\phi^\pm$ at $t_2$, provided we guarantee that the + and − fields for each species share a common value at this time. This can be accommodated by inserting a $\delta$-function into the integrand which constrains the fields to agree [51]

$$\prod_\alpha \delta \left\{ \phi^\alpha_+(t_2) - \phi^\alpha_-(t_2) \right\} \propto \lim_{\varepsilon \to 0} \exp \left\{ -\frac{1}{2\varepsilon} \sum_\alpha \left[ \phi^\alpha_+(t_2) - \phi^\alpha_-(t_2) \right]^2 \right\}. \tag{24}$$

Since the action is real by assumption, the effect of the Hermitian conjugation in (23) is only to flip the sign of the $iS$ term. Eq. (23) can therefore be written as an unrestricted
path integral over the fields \( \{ \phi_+, \phi_- \} \) with a quadratic term which is given by
\[
\exp \left\{ \frac{i}{2} \int \frac{dt \, dq \, ds \, dr}{(2\pi)^3} \sum_\alpha \left( \begin{array}{c}
\phi_+^\alpha(t, q) \\
\phi_-^\alpha(s, r)
\end{array} \right)^T K_{ts}^\alpha(q, r) \left( \begin{array}{c}
\phi_+^\alpha(t, q) \\
\phi_-^\alpha(s, r)
\end{array} \right) \right\},
\]  
(25)
where we have assumed that there are no linear couplings among the fields, \( T \) denotes a transpose, and \( K_{ts}^\alpha(q, r) \) is the \((2 \times 2)\) kernel
\[
K_{ts}^\alpha(q, r) = \delta(q + r) \left( \begin{array}{cc}
\Delta^\alpha - \frac{2}{i\epsilon} \delta_{s\tau} \delta_{t\tau} - \frac{1}{i} \delta_{s\infty} \delta_{t\infty} \Omega & \frac{2}{i\epsilon} \delta_{s\tau} \delta_{t\tau} - \frac{1}{i} \delta_{s\infty} \delta_{t\infty} \Omega \\
\frac{2}{i\epsilon} \delta_{s\tau} \delta_{t\tau} - \frac{1}{i} \delta_{s\infty} \delta_{t\infty} \Omega & -\Delta^\alpha - \frac{2}{i\epsilon} \delta_{s\tau} \delta_{t\tau} - \frac{1}{i} \delta_{s\infty} \delta_{t\infty} \Omega
\end{array} \right).
\]  
(26)
In Eq. (26), \( \Delta \) is a differential kernel which to leading order in the slow-roll approximation can be taken to be given by the laplacian of exact de Sitter space,
\[
\Delta_{ts}(q, r) = \frac{\partial}{\partial t} \frac{\partial}{\partial s} \{ a^3 \delta_{ts} \} + a(q \cdot r) \delta_{ts};
\]  
(27)
\( \delta_{ts} \) is an abbreviation for the \( \delta \) function \( \delta(t - s) \); \( \delta_{t\tau} \) is the \( \delta \)-function \( \delta(t - t_\tau) \); and \( \delta_{t\infty} \) is the \( \delta \)-function \( \delta(t + \infty) \). The field propagator matrix for any given species is found by inverting (25),
\[
\int ds^3 r \, K_{ts}(q, r) \left( \begin{array}{cc}
G_{++} & G_{+-} \\
G_{-+} & G_{--}
\end{array} \right)_{(s, r), (u, w)} = i(2\pi)^3 \delta_{tw} \delta(q - u) \left( \begin{array}{cc}
1 & 0 \\
0 & -1
\end{array} \right).
\]  
(28)
Eq. (28) splits into coupled equations for \( G_{++}, G_{--}, G_{-+} \) and \( G_{+-} \), but because the doublets \( (G_{++}, G_{+-}) \) and \( (G_{--}, G_{-+}) \) are to be regarded as forming complex conjugate pairs half of these equations are related to the other half by complex conjugation. The \( G_{++} \) equation reads
\[
\frac{\partial^2}{\partial t^2} G_{++}^{tw} + 3H \frac{\partial}{\partial t} G_{++}^{tw} + \frac{q^2}{a^2} G_{++}^{tw} - \frac{i}{a^3} \delta_{t\infty} G_{++}^{tw} \Omega(q) = -\frac{1}{a^3} \delta_{tw}.
\]  
(29)
\( G_{--} \) obeys the homogeneous version of (29), whereas \( G_{-+} \) obeys the complex conjugate of (29) and \( G_{+-} \) its homogeneous complex conjugate. In addition, Eq. (28) gives \( G_{+-} \) and \( G_{--} \) the boundary conditions
\[
\delta_{t\tau} G_{++}^{tw} = \delta_{t\tau} G_{--}^{tw} \quad \text{and} \quad \delta_{t\tau} G_{-+}^{tw} = \delta_{t\tau} G_{+-}^{tw}.
\]  
(30)
Consider any solution to the \emph{homogeneous} version of (29). Any such solution is a function of the single variable \( t \), which after changing to conformal time \( \eta \) can be written in the form \( G \equiv \zeta/a \). [The dependence of the mixed propagators on a second time argument enters only through the boundary conditions (30).] The mode function \( \zeta \) must obey
\[
\zeta'' + \left\{ q^2 - (2 - \epsilon)(aH)^2 \right\} \zeta - i \delta_{t\infty} \frac{\Omega(q)}{a} \frac{\zeta}{a} \bigg|_{\eta\to-\infty} = 0.
\]  
(31)
Eq. (31) is equivalent to the condition \( \zeta'' + \left\{ q^2 - (2 - \epsilon)(aH)^2 \right\} \zeta = 0 \) almost everywhere, together with the boundary condition
\[
\frac{\zeta(\eta)}{a^2(\eta)} \to 0 \quad \text{as} \quad \eta \to -\infty.
\]  
(32)
One-loop corrections to a scalar field during inflation

In virtue of the $q$-dependence of $\Omega(q)$, Eq. (31) can be thought of as a modification of the momentum term $q^2$ to include an imaginary component at early times, where $\eta \to -\infty$, causing $\zeta$ to vanish in obedience with (32), or alternatively a small evolution of $\eta$ into imaginary time for large $|\eta|$. This prescription was used in Refs. [18, 21, 59] to compute tree-level correlation functions of the $\{\delta \phi^\alpha\}$ in the interacting vacuum.

One can now construct an explicit solution for $G_{++}$, which satisfies (in conformal time with arguments $\eta$ and $\tau$)

$$G_{++}^{\eta \tau}(r, u) = (2\pi)^3 \delta(r + u) \times \left\{ \begin{array}{ll} \xi(\eta, \tau) & \text{if } \eta < \tau \\ \xi^*(\eta, \tau) & \text{if } \tau < \eta \end{array} \right.,$$

where ‘*’ denotes complex conjugation and $\xi(\eta, \tau)$ is defined by

$$\xi(\eta, \tau) \equiv \frac{W^{-1}(\zeta, \zeta)}{a(\eta)a(\tau)} \xi^*(\tau)\zeta(\eta),$$

in which $W(f, g)$ is the Wronskian $W(f, g) \equiv fg' - gf'$. The propagator $G_{--}$ is obtained by complex conjugation of Eq. (33); the mixed propagator $G_{+-}$ is obtained from a homogeneous equation and therefore is smooth at $\eta = \tau$. The boundary condition (30) implies that it must satisfy

$$G_{+-}^{\eta \tau}(r, u) = (2\pi)^3 \delta(r + u)\xi(\eta, \tau)$$

and $G_{--}$ is given by its complex conjugate. [Note that there is no ambiguity in deciding which propagator should be assigned to a pair $\langle \delta \phi_+ \delta \phi_- \rangle$, because the mode $\zeta$ is always assigned to the argument of the + field, and $\zeta^*$ is always assigned to the argument of the − field.]

The above analysis was carried out for a single field, but where more than one species of light field is present similar results apply, with a mode function $\zeta^\alpha$ for each species which obeys a vacuum boundary condition similar to (32). The propagators which connect two fields of different species $\alpha$ and $\beta$ then obey analogues of (33) and (35) with the function $\xi$ replaced by a matrix $\xi^{\alpha \beta}$. If the fields do not couple linearly, then it follows that $\xi^{\alpha \beta} = \delta^{\alpha \beta} \xi$.

One-vertex, one-loop amplitudes In the remainder of this paper, we shall be concerned with computing expectation values in which of a set of $n$ external fields $\{\phi(k_n)\}$, observed at some time $\eta_*$ and carrying momenta $\{k_n\}$, are paired with a single $(n + 2)$-valent internal vertex with coupling constant $g$. Applying Schwinger’s formula shows that the term in such an expectation value of leading order in $g$ is given by

$$i(2\pi)^3 \int \frac{d^3q_1 \cdots d^3q_n}{(2\pi)^{3(n+2)}} \delta(\sum_{i=1}^n q_i) \int_{-\infty}^{\eta_*} d\eta \, g M,$$

where $M$ is defined by

$$M \equiv \left\langle \phi_+^\alpha(k_1) \cdots \phi_+^\beta(k_n) \phi_+^\gamma(q_1) \cdots \phi_+^\delta(q_n) \phi_+^\alpha(q_{n+1}) \phi_+^\beta(q_{n+2}) \right\rangle$$

$$- \left\langle \phi_+^\alpha(k_1) \cdots \phi_+^\beta(k_n) \phi_+^\gamma(q_1) \cdots \phi_+^\delta(q_n) \phi_+^\alpha(q_{n+1}) \phi_+^\gamma(q_{n+2}) \right\rangle,$$

(37)
where Greek indices label the species of fields, which are here allowed to run over field
derivatives as well as the fields themselves. The issue of obtaining a correct path integral
for theories with derivative interactions will be taken up again in the next section, but
causes no difficulties in the present context. Any such amplitude is automatically of
one-loop order, because the two field operators left over after all external fields have
been paired with \( n \) of the vertex fields must contract amongst themselves, leaving a
single unconstrained integral over momentum.

The time integral in (36) has been carried to some late time \( \eta^{\#} \) which satisfies
\( \eta^{\#} > \eta^{\ast} \). Using Eqs. (33)–(35) and their complex conjugates in Eq. (36), it follows that
the expectation value can be written

\[
i(2\pi)^3 \delta(k_1 + \cdots + k_n) \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\eta^{\#}} d\eta \ \xi_{k_1}^{\alpha\gamma}(\eta, \eta^{\ast}) \cdots \xi_{k_n}^{\beta\delta}(\eta, \eta^{\ast}) \xi_{q}^{\rho\sigma}(\eta, \eta)
\]

\[
- \ i(2\pi)^3 \delta(k_1 + \cdots + k_n) \int \frac{d^3q}{(2\pi)^3} \int_{-\infty}^{\eta^{\#}} d\eta \ \xi_{k_1}^{\alpha\gamma*}(\eta, \eta^{\ast}) \cdots \xi_{k_n}^{\beta\delta*}(\eta, \eta^{\ast}) \xi_{q}^{\rho\sigma*}(\eta, \eta)
\]

+ permutations,

\[ (38) \]

in which the second term is the complex conjugate of the first, and all permutations
likewise assemble into complex conjugate pairs. Observe that the internal term \( \xi_{q}(\eta, \eta) \)
does not spoil the complex conjugation property, since it follows from (34) that for
general propagating fields it is real when evaluated with equal arguments.\[†\]

The part of the integration over times between \( \eta^{\ast} \) and \( \eta^{\#} \) has cancelled out, since
in this region Eq. (35) is given by the same expression as Eq. (33), whereas in the
region \( \eta < \eta^{\ast} \) it is given by its complex conjugate. Note that for interactions which
contain more than a single vertex and a single loop the process of deriving expressions
such as (38) becomes increasingly cumbersome. For such interactions, the diagrammatic
operator formalism recently elaborated by Musso [65] is likely to prove superior.

3.2. Theories with derivative interactions

An important feature of the interactions (16) and (17) is that they include time
derivatives of the perturbation, \( \delta \dot{\phi} \) [66]. This means that the lagrangian can not be
written in the canonical form \( L(\delta \phi, \delta \dot{\phi}) = \frac{1}{2} \delta \dot{\phi} \Delta \delta \dot{\phi} + V(\delta \phi) \) (where the operator \( \Delta \) is
field independent), in which there is only a quadratic dependence on \( \delta \dot{\phi} \). As a result the
textbook construction of the path integral formula based on \( L \) does not work.

In the standard construction one identifies a momentum, \( \pi \), canonically conjugate
to \( \delta \phi \) and writes the lagrangian as a Legendre transformation of the hamiltonian function
\( H \),

\[
L(\delta \phi, \delta \dot{\phi}) = \pi \delta \dot{\phi} - H(\delta \phi, \pi).
\]

In the quantum theory \( \delta \phi \) and \( \pi \) cannot be specified simultaneously. Since it is \( H \)
that generates time evolution, when one constructs the path integral one naturally
arrives at a functional integration that involves independent integrals over \( \delta \phi \) and \( \pi \).

† We will find a small refinement to this statement in §3.3 below.
If $L$ depends at most quadratically on $\delta \dot{\phi}$ then $H$ depends at most quadratically on $\pi$ and the momentum integral can be performed immediately. This has the effect of setting the value of $\pi$ equal to the one stipulated by Hamilton’s equations and results in the standard lagrangian path integral formula \[22\]. However, when $H$ has a more complicated dependence on $\pi$ the momentum integral must be treated more carefully.

The properties of lagrangians with derivative interactions have been studied extensively in the context of the non-linear $\sigma$-model. (See, eg., Coleman \[67\]; a path integral treatment is given in Ref. \[62\], whereas the canonical approach was followed in Ref. \[68\].) After inspection of Eqs. (16)–(17) it is clear that no term contains as many as four time derivatives, although there are terms containing one, two or three. Let us parametrize a general action for a field $\theta$ with arbitrary interactions containing as many as three time derivatives in form

$$S = (2\pi)^3 \int d\eta \left( \frac{1}{2} \gamma_{\alpha\beta} \dot{\theta}^\alpha \dot{\theta}^\beta - \frac{1}{2} \delta_{\alpha\beta} \partial \theta^\alpha \partial \theta^\beta - V(\theta) + \lambda_\alpha \dot{\theta}^\alpha + \frac{1}{3} \omega_{\alpha\beta\gamma} \dot{\theta}^\alpha \dot{\theta}^\beta \dot{\theta}^\gamma \right). \tag{40}$$

In order to keep this and subsequent expressions manageable, Eq. (40) has been written in an abbreviated ‘de Witt’ notation, where contraction over indices implies not only a summation over species, but also an integration over momentum variables with measure $d^3k/(2\pi)^3$. With these conventions the function $\hat{\delta}_{\alpha\beta} = \delta(k_\alpha + k_\beta)$ is a pseudo-metric on $k$-space which is numerically identical to its index-raised counterpart, $\hat{\delta}^{\alpha\beta}$. Note that we have taken any interactions involving two factors of $\dot{\theta}$ to be included in $\gamma_{\alpha\beta}$ with the kinetic term. Without loss of generality $\gamma_{\alpha\beta}$ and $\omega_{\alpha\beta\gamma}$ can be taken to be symmetric under exchange of their indices. We assume that $\gamma_{\alpha\beta}$ is invertible, with inverse $\gamma^{\alpha\beta}$.

The momentum conjugate $\dot{\theta}^\alpha$ is $\pi_\alpha$,

$$\pi_\alpha = \frac{\delta S}{\delta \theta^\alpha} = (2\pi)^3 \left( \gamma_{\alpha\beta} + \lambda_\alpha + \omega_{\alpha\beta\gamma} \dot{\theta}^\beta \dot{\theta}^\gamma \right) \tag{41}$$

where we have used the assumed symmetry under index exchange to simplify this expression. In order to apply this formalism to the cubic and quartic interactions (16) and (17) it is only necessary to compute to $O(\theta^4)$, where we assume that $\pi \sim \theta$ in order of magnitude. Since there are no three-derivative interactions in $S_3$, this implies that $\omega = O(\theta)$ and it will be sufficient to work to leading order in $\omega$. To this order, the hamiltonian can be written

$$H = \frac{1}{2} \frac{1}{(2\pi)^3} \gamma_{\alpha\beta \pi_\alpha \pi_\beta} + \frac{1}{2} (2\pi)^3 \delta_{\alpha\beta \partial \theta^\alpha \partial \theta^\beta} + \frac{1}{2} (2\pi)^3 \gamma_{\alpha \beta \lambda_\lambda} + (2\pi)^3 V$$

$$- \frac{1}{3} \frac{1}{(2\pi)^6} \omega_{\alpha\beta\gamma} \gamma^{\alpha\rho \gamma\beta\sigma \gamma\pi \pi_\sigma \pi_\tau} - \gamma_{\alpha \beta \pi_\alpha \pi_\lambda}. \tag{42}$$

This hamiltonian can be used to construct a path integral for $\theta$, giving

$$\int [d\theta^\alpha d\pi_\beta] \exp \left\{ i \int d\eta \left( \pi_\alpha \dot{\theta}^\alpha - H \right) \right\}. \tag{43}$$

† The prefix pseudo- is applied because with our conventions, the object obtained by mixing indices, $\hat{\delta}_{\alpha\beta} \hat{\delta}^{\beta\gamma}$, is not the identity operator $\delta(k_\alpha - k_\gamma)$, but instead is proportional to it.
The fields $\dot{\theta}^\alpha$ and $\pi_\beta$ are now independent, so we are free to redefine the momentum variable

$$\pi_\alpha \mapsto (2\pi)^3 \left\{ \pi_\alpha + \chi_\alpha \right\}$$

(44)

with $\chi_\alpha$ chosen to eliminate the term in Eq. (43) which is linear in $\pi_\alpha$,

$$\chi_\alpha \equiv \gamma_{\alpha\beta} \dot{\theta}^\beta + \lambda_\alpha + \omega_{\alpha\beta\gamma} \dot{\theta}^\beta \dot{\theta}^\gamma.$$  

(45)

Having done so, one may rearrange terms to find a simplified path integral expression

$$\int [d\theta^\alpha d\pi_\beta] \exp \left\{ i(\mathcal{S}_\theta + \mathcal{S}_{gh}) \right\},$$

(46)

where $\mathcal{S}_\theta$ is the original $\theta$ action (40) with all derivative interactions in their original form, and $\mathcal{S}_{gh}$ is an effective action for the ‘ghost’ field $\pi$,

$$\mathcal{S}_{gh} = (2\pi)^3 \int \left( -\frac{1}{2} \gamma^{\alpha\beta} \pi_\alpha \pi_\beta + \omega_{\alpha\beta\gamma} \gamma^{\beta\gamma} \theta^\alpha \pi_\sigma \pi_\tau + \frac{1}{3} \omega_{\alpha\beta\gamma} \gamma^{\beta\gamma} \gamma^{\sigma\tau} \pi_\rho \pi_\sigma \pi_\tau \right)$$

(47)

The quanta associated with $\pi$ do not appear in physical states, although they couple to $\theta$ and so affect its expectation values when loop corrections are taken into account. This explains why it has been permissible to ignore such ghosts in previous tree-level calculations; the $\pi$ integral makes no contribution to tree-level expectation values. Note that unlike the more familiar Fadeev–Popov ghost, the $\pi$ field is a spacetime scalar and not a spin-1/2 fermion.

Eq. (47) is not yet in a form suitable for perturbative calculations. In particular the ghost kinetic term involves the inverse $\gamma^{\alpha\beta}$. This will be a complicated object even for relatively simple choices of $\gamma_{\alpha\beta}$, but it is pointless to compute beyond $O(\theta^4)$ since the action to which we wish to apply this formalism [namely Eqs. (15)–(17)] was truncated at this point. For a canonically normalised scalar field in an almost-de Sitter spacetime $\gamma_{\alpha\beta}$ can be written

$$\gamma_{\alpha\beta} = a^2 \delta(k_\alpha + k_\beta) + 2\Gamma_1(k_\alpha, k_\beta) + 2\Gamma_2(k_\alpha, k_\beta) + \cdots$$

(48)

where the $\Gamma_m$ are taken to be $o(\theta^m)$, the factor of two has been inserted for future convenience, and ‘$\cdots$’ denotes higher order terms which have been omitted.

The identity operator with our conventions is $(2\pi)^3 \delta(k_\alpha - k_\beta)$. The inverse $\gamma^{\alpha\beta}$ can be written

$$\gamma^{\alpha\beta} = (2\pi)^6 \left( \frac{1}{a^2} \delta(k_\alpha + k_\beta) + \psi_1(k_\alpha, k_\beta) + \psi_2(k_\alpha, k_\beta) \right),$$

(49)

where the $\psi_m$ are taken to be $o(\theta^m)$. One can verify that the normalization in Eq. (49) is correct, since when (48) and (49) are contracted together the $o(\theta^6)$ term becomes

$$\int \frac{d^3k_\beta}{(2\pi)^3} a^2 \delta(k_\alpha + k_\beta) \cdot (2\pi)^3 \frac{1}{a^2} \delta(k_\beta + k_\gamma) = (2\pi)^3 \delta(k_\alpha - k_\gamma) = \delta_\gamma.$$  

(50)

The $o(\theta)$ equation implies that $\psi_1$ satisfies

$$\psi_1(k_\alpha, k_\beta) \equiv -\frac{2}{a^2} \Gamma_1(-k_\alpha, -k_\beta),$$

(51)
whereas the $o(\theta^2)$ equation implies that $\psi_2$ satisfies

$$\psi_2(k, k_\beta) \equiv -\frac{2}{a^4} \Gamma_2(-k, -k_\beta) - \frac{4}{a^4} \int d^3q \, \Gamma_1(-k, q) \Gamma_1(-q, -k_\beta). \quad (52)$$

The ghost action can therefore be written

$$S_{gh} = (2\pi)^9 \int d\eta \left\{ -\frac{1}{2a^2} \hat{\delta}^{\alpha\beta} \pi_\alpha \pi_\beta - \psi_2^{\alpha\beta} \pi_\alpha \pi_\beta - \psi_2^{\alpha\beta} \pi_\alpha \pi_\beta + \frac{(2\pi)^6}{a^4} \omega_\alpha \omega_\beta \hat{\delta}^{\beta\sigma} \hat{\delta}^{\gamma\tau} \pi_\sigma \pi_\tau \right. \\
+ \left. \frac{(2\pi)^{12}}{3a^6} \omega_\alpha \omega_\beta \hat{\delta}^{\alpha\rho} \hat{\delta}^{\beta\sigma} \hat{\delta}^{\gamma\tau} \pi_\rho \pi_\sigma \pi_\tau \right\} \quad (53)$$

The first term is $o(\theta^2)$ and can be taken as the free-field part of the ghost action, whereas the remainder is $O(\theta^3)$ and can be taken as the interaction term. In this form, the ghost action is suitable for perturbative evaluation.

### 3.3. Feynman rules for the interacting scalar/ghost theory

We are now in a position to write down the Feynman rules for the $\delta\phi$ theory, including the effect of the ghost field. In this section we will not be obliged to carry out any of the complicated manipulations which characterized §3.2, and so we will revert to a notation in which momentum labels and integrals are written explicitly.

The propagators for the pure $\delta\phi$ theory were written down in §3.1. The free part of the ghost action can be inverted immediately to find the ghost propagator. For the $+$ fields this gives

$$\langle \pi_+ (\eta_1, k_1) \pi_+ (\eta_2, k_2) \rangle = -\frac{i}{(2\pi)^3} a(\eta_1) a(\eta_2) \delta(\eta_1 - \eta_2) \delta(k_1 + k_2). \quad (54)$$

from which the $--$ propagator can be obtained by complex conjugation. Eq. (54) is the propagator for a so-called static ultra-local field [69, 63]. Its $k$ and $\eta$ dependence is constrained by the appearance of $\delta$-functions, so the ghost does not propagate: its purpose is to provide corrections to the vertices of the theory which account for the presence of coincident time derivatives there. At one-loop, we do not require the mixed propagator: the ghost field only appears in loops, and at one-loop the only mixed contractions involve pairings of external fields with internal fields, and these cannot occur for the ghost.

Eq. (54) entails a small refinement of the one-loop, single-vertex formula, Eq. (38). The above discussion shows that the ghost propagator always appears in the role of the propagator evaluated at equal argument in (38), which we assumed to be real in §3.1. This is no longer true for the ghost. However, because the $++$ and $--$ propagators remain complex conjugates, this does not spoil the general rule derived in that section, that to obtain the full expectation value (containing both $+$ and $-$ fields) we can compute with only the $+$ field and then take twice the real part of the resulting expression.
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It remains to identify the interaction terms $V$, $\lambda$, $\Gamma_1$, $\Gamma_2$ and $\omega$. Reading these off from Eqs. (16)–(17) we obtain

$$V = a^2 \frac{\dot{\phi}}{4H} \int \frac{d^3 q_1 d^3 q_2 d^3 q_3}{(2\pi)^9} \delta^3 \left( \sum_{i=1}^3 q_i \right) \left\{ \prod_{j=1}^3 \delta \phi(q_j) \right\} (q_2 \cdot q_3)$$

$$\lambda(q_1) = \frac{a}{4H} \int \frac{d^3 q_2 d^3 q_3 d^3 q_1}{(2\pi)^9} \delta^4 \left( \sum_{i=1}^4 q_i \right) \left\{ \prod_{j=2}^4 \delta \phi(q_j) \right\} (q_2 \cdot q_3) \frac{\sigma(q_1, q_4)}{q_{14}^2},$$

$$\Gamma_1(q_1, q_2) = -a^2 \frac{\dot{\phi}}{4H} \int \frac{d^3 q_3}{(2\pi)^9} \delta^3 \left( \sum_{i=1}^3 q_i \right) \delta \phi(q_3) \left( 1 + 2 \frac{\sigma(q_2, q_3)}{q_1^2} \right)$$

$$\Gamma_2(q_1, q_2) = -a^2 \int \frac{d^3 q_3 d^3 q_4}{(2\pi)^9} \delta^4 \left( \sum_{i=1}^3 q_i \right) \left\{ \prod_{j=3}^4 \delta \phi(q_j) \right\} \times \left( \frac{z(q_1, q_3) \cdot z(q_2, q_4)}{q_{13}^2 q_{24}^2} + \frac{3 \sigma(q_1, q_3) \sigma(q_2, q_4)}{4 q_{13}^2 q_{24}^2} - \frac{2 q_4 \cdot z(q_2, q_3)}{q_{23}^4} \right)$$

$$\omega(q_1, q_2, q_3) = -\frac{a}{4H} \int \frac{d^3 q_4}{(2\pi)^9} \delta^4 \left( \sum_{i=1}^3 q_i \right) \delta \phi(q_4) \frac{\sigma(q_1, q_4)}{q_{14}^2},$$

where $q_{ij} = q_i + q_j$. The functions $\sigma$ and $z$ are defined by

$$\sigma(a, b) \equiv a \cdot b + b^2$$

and

$$z(a, b) = \sigma(a, b)a - \sigma(b, a)b.$$

These are the momentum-space counterparts of Eqs. (13)–(14). Note that as written, Eqs. (57)–(58) for $\Gamma$ and Eq. (59) for $\omega$ are not symmetric under exchange of their arguments. For $\Gamma$ this is actually immaterial, since (40) and (53) show that it always appears in a symmetric contraction. On the other hand $\omega$ appears only once in an asymmetric contraction, $\omega_{\alpha\beta\gamma} \dot{\theta}^\alpha \delta^\beta \dot{\delta}^\gamma \pi^\sigma \pi^\tau$. To avoid an unnecessarily tripling of the length of (59) we leave it in asymmetric form, carrying out an explicit symmetrization when computing amplitudes involving the asymmetric vertex.

Diagrammatic representation Eqs. (55)–(59) lead to a rather complicated diagrammatic formalism in which the vertices produce a number of related terms, depending on the number of derivatives which apply to the lines entering the vertex. In order to keep track of these related contributions it is useful to introduce a refinement of the Feynman rules in which the lines of scalar propagators to which derivatives are applied are decorated with a dot.

For the pure $\delta \phi$ vertices, the resulting diagrams are depicted in Fig. [1]. For the mixed $\delta \phi$/ghost vertices, the resulting diagrams are shown in Fig. [2].
Figure 1. Pure $\delta \phi$ vertices. A dot on a scalar line entering a vertex shows that a time derivative is applied to the field at the point of interaction. In terms of Eq. (40), the diagrams correspond to the vertices produced by (a) the potential $V$; (b) the $\lambda$ vertex; (c) the $\Gamma_1$ vertex; (d) the $\Gamma_2$ vertex; and (e) the $\omega$ vertex.

Figure 2. Scalar/ghost vertices. Solid lines represent that scalar field $\delta \phi$, whereas the dashed lines represent the ghost. A dot on a scalar line entering a vertex shows that a time derivative is applied to the field at the point of interaction. Time derivatives are never applied to ghost fields. In terms of Eq. (53) the diagrams correspond to the vertices produced by (a) the $\psi_1 \pi^2$ interaction; (b) the $\psi_2 \pi^2$ interaction; (c) the $\omega \dot{\theta} \pi \pi$ interaction; and (d) the $\omega \pi^3$ interaction.

Figure 3. Instability of the vacuum due to condensation. $\delta \phi$ particles emerge from the vacuum (represented by the hatched condensate) in a zero momentum state. The accumulation of such particles changes the homogeneous classical field configuration associated with the vacuum.

4. The one-point function

In [23], we observed that at tree-level, the one point function of $\delta \phi$ is zero, $\langle \delta \phi(k) \rangle = 0$. This is not merely a question of convention; if the one-point function was not zero then so-called ‘tadpole’ diagrams such as Fig. (3) would mean that $\delta \phi$ quanta would emerge from the vacuum. Conservation of momentum forces such particles to appear in a zero-momentum or ‘condensate’ state, and the accumulation of such particles causes the classical field configuration to change. Any such instability implies that a perturbation theory based on the original unstable vacuum state could not give meaningful answers. The problem can be cured be ensuring that the vacuum which we take as the basis of our perturbation theory is stable, at least at tree-level.

In an inflationary universe, the emergence of $\delta \phi$ quanta from the vacuum is exploited
to produce small density fluctuations on superhorizon scales. Therefore we may expect to encounter some symptoms of vacuum instability when quantum corrections are taken into account. These symptoms manifest themselves as a radiatively generated one-point function,

$$\langle \delta \phi(k) \rangle = (2\pi)^3 \delta(k) O,$$

where $O \neq 0$ is a dimensionless quantity. Although it is not the observable in which we are principally interested, the present section is devoted to a calculation of $O$. This is important for two reasons. The first is that it provides a consistency check on $\delta N$ calculations [21, 59, 23, 34, 70, 44, 45] which typically assume $O = 0$, even beyond tree level. The second is that it allows us to develop some aspects of the calculational formalism in a simpler setting than the computation of the two-point function.

### 4.1. Ghost diagrams

Consider the one-point function associated with some wavenumber $k$. We aim to compute this at the time $\eta^*$ when $k$ crosses the horizon, which is roughly defined by the condition $-k\eta^* = 1$.

We deal first with the diagrams which contain a ghost loop. There is only one such diagram, which arises from the $\psi_1 \pi^2$ coupling,

$$\langle \delta \phi(k) \rangle \subseteq \langle \ldots \rangle.$$

This diagram makes a contribution to $\langle \delta \phi(k) \rangle$ equal to

$$\langle \delta \phi(k) \rangle \supseteq - (2\pi)^3 \delta(k) \int_{-\infty}^{\eta^*} \frac{d^3 q}{(2\pi)^3} \frac{H_* H}{2k^3} \delta(0)(1 - ik\eta)e^{ik\eta} \frac{\dot{\phi}}{4H} \left(1 + \frac{2\sigma(-q,k)}{q^2}\right) + \text{complex conjugate},$$

where the symbol ‘$\supseteq$’ indicates that $\langle \delta \phi(k) \rangle$ contains the indicated contribution (among others), and in deference to the vacuum prescription outlined in §3.1 we should deform the contour of the $\eta$ integral to include some evolution in imaginary time for large $|\eta|$. In this region the exponential factor is strongly decaying [cf. (32)], so there is very little contribution to the integral from very early times; it receives its dominant part from times around horizon crossing, where $\eta \sim -1/k$. We may therefore approximate the slowly varying factors $H_* H$ and $\dot{\phi}/H$ by their values at the time of horizon crossing, which are equal to $H_*^2$ and $\dot{\phi}_*/H_*$ respectively. In this simple example the $\eta$ integral and the integral over the internal momentum $q$ factorize, leaving a final result

$$\langle \delta \phi(k) \rangle \supseteq -(2\pi)^3 \delta(k) P_*(k) \frac{\dot{\phi}_*}{4H_*} \int_{-\infty}^{\eta^*} d\eta \delta(0)(1 - ik\eta)e^{ik\eta} \int \frac{d^3 q}{(2\pi)^3} \left(1 + \frac{2\sigma(-q,k)}{q^2}\right) + \text{c.c.},$$

where $P_*(k)$ is the tree-level power spectrum evaluated at $\eta_*$. The object $\delta(0)$ is the $\eta$ delta-function evaluated at zero argument, and is badly divergent. In the present case, however, this is not material. The $\eta$ integral can be rotated to imaginary time, leaving a result which is purely imaginary. Hence, although divergent, this diagram makes no contribution to the one-point function.
4.2. Pure $\delta \phi$ diagrams

Now consider the pure $\delta \phi$ diagrams.

The vacuum prescription and renormalization In order to assess the contribution that such diagrams make to $O$, it is convenient to adopt the approximation made above that early times make almost no contribution to the $\eta$ integral, so that slowly varying quantities such as $H$ and $\dot{\phi}/H$ can be evaluated at $\eta_*$. A generic pure-$\delta \phi$ contribution to $O$ will then take the form

$$O \supset i P_s(k) \frac{\dot{\phi}_*}{4H_*} \int_{-\infty}^{\eta_*} d\eta \int \frac{d^3 q}{(2\pi)^3} e^{i k \eta \Sigma} + \text{complex conjugate}, \quad (66)$$

where $\Sigma$ is a $k$- and $q$-dependent quantity which is to be calculated. In evaluating $\Sigma$ we will encounter instances where a $\delta \phi$ propagator begins and ends at the same vertex, giving it coincident time arguments. We will choose to set such a propagator equal to

$$\langle \delta \phi(q_1, \eta) \delta \phi(q_2, \eta) \rangle = \frac{H^2}{2q^3} (1 + q^2 \eta^2), \quad (67)$$

where the exponential factors $e^{i q \eta} e^{-i q \eta}$ have cancelled among themselves.

In view of the discussion of the vacuum boundary conditions discussed in §3.1, one may question whether (67) is the correct choice, or whether it should be modified to read

$$\langle \delta \phi(q_1, \eta) \delta \phi(q_2, \eta) \rangle \rightarrow \frac{H^2}{2q^3} |1 - i q \eta|^2 e^{-2q \text{Im}(\eta)}, \quad \text{where } \text{Im}(\eta) > 0. \quad (68)$$

This would have the very desirable effect of decoupling our prediction for $\langle \delta \phi(k) \rangle$ from the deep ultra-violet régime where $q \to \infty$; in this limit, Eq. (67) is divergent, whereas the modified propagator (68) is strongly decaying due to the exponential factor. Therefore, our choice of (67) apparently raises the possibility of unphysical divergences owing to an incorrect treatment of the vacuum. On the other hand, Eq. (68) has the undesirable feature that it leads to a non-holomorphic integrand. This means that it would be necessary to rescind the possibility of contour rotation in evaluating the $\eta$ integral. A loop amplitude computed using (68) would therefore depend sensitively upon the entirely arbitrary value we assign to $\text{Im}(\eta)$.

This can be understood as follows. At tree-level the integral is holomorphic, meaning that all contours of integration which start from $|\eta| = \infty$ with $\text{Im}(\eta) > 0$ (strictly) and end at $\eta = \eta_*$ are equivalent. Hence, at tree-level the vacuum prescription (32) gives completely unambiguous results which are independent of the deep ultra-violet region. However, the non-holomorphic nature of the integrand at loop level means it is not sufficient to deal with ultra-violet divergences at higher-order. Such divergences can be accommodated more effectively by applying a method of regularization which separates the divergent ultra-violet contribution. This regularized divergence can be subtracted by conventional methods of renormalization.

† It makes no difference if we allow $\eta$ to develop a small imaginary component in the prefactor $|1 - i q \eta|^2$, since the exponential term is so strongly decaying for large $|\eta|$. 
This leaves open the question of suitable regularizations. Since the Einstein action is supposed to be an effective theory of gravity for energies less than the Planck scale $M_P \approx 10^{18}$ GeV, and inflation is usually supposed to occur at energies at least a few orders of magnitude less than $M_P$, one might imagine applying a cut-off on the loop momenta of order the Planck scale. However, this in itself is ambiguous since the Planck scale, unlike the speed of light $c$, is not a Lorentz invariant and varies between locally inertial frames. In particular, the comoving Planck scale at a given instant $\eta$ is given by $a(\eta)M_P$. Therefore a momentum cut-off of this form entangles the $\eta$ and $q$ integrations. For this reason it is preferable to use a method of regularization, such as dimensional regularization, which does not depend on the use of a cut-off.

Unfortunately, the integrals which we will encounter, especially in the computation of the two-point function (to be considered in §5 below) do not lend themselves to evaluation by dimensional regularization (see, for example, the example calculation given in Ref. [51]). In the present paper we will compute expectation values using a hard momentum cutoff both in the infra-red and ultra-violet. However, in the case of the one-point function where dimensional regularization can be usefully applied, it can be checked that when the ultra-violet region has been discarded the two methods yield comparable predictions for the leading infra-red divergences (up to numerical factors and logarithms of the IR cutoff).

This is sufficient for the purposes of the present paper, since it is the behaviour in the infra-red rather than the ultra-violet which is of principal interest in a cosmological context. Divergences in the ultra-violet come from the behaviour of the fields at high energies and small scales. Such small scale modes exist far inside the horizon, where the equivalence principle suggests that flat spacetime quantum field theory is expected to be a good approximation. On the other hand, the infra-red behaviour comes from low energies and large scales, where the field modes are well outside the horizon. On such scales, flat spacetime field theory is a very poor approximation and we are obliged to take account of the gravitational background.

This does not preclude the appearance of new ultra-violet divergences in our expectation values. Indeed, many of the integrals we shall encounter do contain ultra-violet divergences, of which the ultra-violet divergent quantity $\delta(0)$ which appears in Eq. (65) is an example. Such divergences do not interfere with our ability to perform accelerator or laboratory particle physics experiments on earth, which are characterized by time- and length-scales that are small compared to the expansion time-scale and horizon length-scale of the universe. On such small scales the ultra-violet divergences we shall encounter (none of which are present in the $\delta\phi$ theory in Minkowski space) are presumably subdominant with respect to divergences from the pure matter theory and therefore do not interfere with our ability to perform terrestrial experiments, or with the success of the principle of equivalence.

In what follows we shall generally assume that the ultra-violet divergences can be correctly subtracted, and focus our attention on the infra-red behaviour.
Zero-derivative interactions  We now return to the $\delta \phi$ diagrams. It is simplest to classify these diagrams according to the number of derivatives applied to propagators entering the vertex.

There is a zero-derivative interaction from the vertex in Fig. (I)(a),

$$\langle \delta \phi(k) \rangle_s \supset \bigcirc.$$  (69)

This term makes a contribution to $\Sigma$ which equals

$$\Sigma \supset -\left( \frac{1}{2q\eta^2} - \frac{k \cdot q}{q^3\eta^2} \right) (1 - i k \eta) (1 + q^2 \eta^2).$$  (70)

The term involving $k \cdot q$ is not rotationally invariant and disappears in the integral over $q$. Let us introduce an ultra-violet cut-off $k \Lambda$, where $\Lambda$ is a dimensionless number, and a comparable infra-red cut-off $km$ for some dimensionless quantity $m$. Evaluating the $q$ and $\eta$ integrals gives

$$O_s \supset P_s \frac{\dot{\phi}_s}{4H_s} \left( -\frac{k^3}{4\pi^2} \left[ \Lambda^4 - m^4 + \Lambda^2 - m^2 \right] \right).$$  (71)

Two derivatives, both derivatives on internal leg The next class of diagrams contain two derivative operators, and divide naturally into two sorts: those where the derivatives are applied to both ends of the internal loop, and those where one derivative is applied to the loop but the other is applied to the external leg.

The first sort give rise to diagrams of the form

$$\langle \delta \phi(k) \rangle_s \supset \bigcirc;$$  (72)

such diagrams contribute an amount to $\Sigma$ corresponding to

$$\Sigma \supset -\frac{q}{2} (1 - i k \eta) \left( 1 + 2 \frac{\sigma(q, -k)}{q^2} \right).$$  (73)

Evaluating the integrals by the method described above, one arrives at

$$O_s \supset P_s(k) \frac{\dot{\phi}_s}{4H_s} \left( -\frac{k^3}{4\pi^2} \left[ \frac{1}{2} \Lambda^4 - \frac{1}{2} m^4 + 2\Lambda^2 - 2m^2 \right] \right).$$  (74)

Two derivatives, single derivative on internal leg The final class of diagrams contain a single derivative on the internal line, and apply the remaining derivative to the external leg. These diagrams are of the form

$$\langle \delta \phi(k) \rangle_s \supset \bigcirc$$  (75)

and contribute to $\Sigma$ according to the rule

$$\Sigma \supset -\frac{k^2}{2q} (1 - i k \eta) \left( 1 + 2 \frac{\sigma(-k, q)}{q^2} \right).$$  (76)

This class of diagrams makes a contribution to $O_s$ which equals

$$O_s \supset P_s(k) \frac{\dot{\phi}_s}{4H_s} \left( -\frac{k^3}{4\pi^2} \left[ \frac{4}{3} \Lambda^3 - \frac{4}{3} m^3 + 2\Lambda^2 - 2m^2 \right] \right).$$  (77)
4.3. Infra-red behaviour

We now collect terms in Eqs. (71), (74) and (77). The ultra-violet divergences will be cancelled by renormalization, but without a definite prescription for the necessary counterterms it is not possible to assign a value to the finite remainder. Therefore, the residue of the UV divergences will be manifest in a renormalization-scheme dependent number $\alpha$. Accordingly, the one-point amplitude behaves in the infra-red like

$$\langle \delta\phi(k) \rangle_{*,m} \sim (2\pi)^3 \delta(k) \frac{P_*}{8 H_*} \left( \alpha + \frac{3}{2} m^4 + \frac{4}{3} m^3 + 5 m^2 \right).$$

(78)

We have computed $O$ as if any value of $k$ were allowed, whereas in reality momentum conservation restricts us to the zero-momentum case $k = 0$. In doing so, we are tacitly assuming that the correct way to regularize $O$ is to compute for finite $k$ and then study the limit $k \to 0$. This leads to Eq. (78) for vanishingly small $m$. The finite constant $\alpha$ will in general be time dependent. Although the present argument cannot capture its time dependence, Eq. (78) does show that it is free of infra-red divergences. Indeed, (78) shows that $O$ is insensitive to the box size in the limit of a large box, but varies as the box is reduced. This is in accordance with general expectations [71]. It follows that when working in a sufficiently large box $O$ can be eliminated by a finite renormalization, leaving $O = 0$ as usually assumed.

5. The two-point function

We now turn to the central purpose of this paper, the computation of the leading loop correction to the two-point function $\langle \delta\phi(k_1)\delta\phi(k_2) \rangle_*$.

It is first necessary to decide which classes of diagrams are to be included in the computation. In general, the one-loop correction to the two-point function of the $\delta\phi$ will be given by a sum of diagrams, the leading terms of which are of the form

$$\text{loop correction} \supset \begin{array}{c}
\text{loop} \\
\text{diagram} \\
\text{other terms}
\end{array} + \begin{array}{c}
\text{loop} \\
\text{diagram} \\
\text{other terms}
\end{array} + \cdots.$$

Eqs. (16)–(17) show that the first term is $O(1)$ in slow-roll, whereas the second term is $O(\epsilon)$. Therefore, provided $\epsilon \ll 1$ and there are no large logarithms which can compensate for the smallness of $\epsilon$, the expectation value will be dominated by the lowest-order part of the first diagram. This is opposite to the case considered by Sloth [46, 47], where a large logarithm was used to compensate for the smallness of $\epsilon$. In this régime the first diagram will be dominated by its subleading slow-roll part, although in Refs. [46, 47] the second diagram was still ignored. We will compute the contribution of the leading part of the first term only, and return to the issue of when this is a good approximation in Ref. [58].
5.1. Ghost diagrams

The relevant vertices here come from the $\psi_2 \pi \pi$ and $\omega \delta \phi \pi \pi$ terms in the ghost action. There is no contribution from the $\omega \pi \pi \pi$ interaction because this involves three ghost fields, which must appear in loops, and at one-loop order there is always one ghost field which is left unpaired. Therefore this term can be disregarded, although it would play a role in a two- or higher-loop calculation. To determine the $\psi_2$ contribution explicitly, consider Eq. (52) which gives $\psi_2$ in terms of the known functions $\Gamma_1$ and $\Gamma_2$. We are computing to leading order in slow-roll, so the term involving $\Gamma_1^2$ can be discarded, because Eq. (57) shows that it is proportional to the slow-roll parameter $\epsilon \sim \dot{\phi}^2/H^2$, whereas the leading terms in the fourth-order interaction are $o(\epsilon^0)$.

The relevant ghost diagrams are

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle * \supset \frac{-i \delta}{2 \pi} + \frac{i \delta}{2 \pi}.$$ (79)

Both these diagrams are purely imaginary and cancel between the $++$ and $--$ propagators in exactly the same manner described in §4 for the computation of the one-point function.

The ghost diagrams have therefore entirely cancelled out in both the one- and two-point functions. This leads to expressions which agree with those reported in Refs. [46, 47]. However, one should not immediately conclude that the ghost diagrams always sum to zero. Although this issue deserves more detailed attention, at two-loop order and above one can presumably expect the factors of $i$ to combine to give non-vanishing contributions. This will apparently occur whenever there are an even number of ghost propagators in the diagram.

5.2. Pure $\delta \phi$ diagrams

As in the one-point calculation, it is convenient to classify the pure $\delta \phi$ diagrams according to the number of derivatives they contain.

**Single derivative** There are no zero-derivative interactions, because the gravitational interactions responsible for generating the vertices in Fig. 1 make no contribution to the potential at $o(\delta \phi^4)$, and the cubic contribution which is generated would make a contribution to the loop correction which is subleading in slow-roll.

The first non-trivial term contains a single derivative, which can be applied to an internal or external line,

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle * \supset \frac{-i \delta}{2 \pi} + \frac{i \delta}{2 \pi}.$$ (80)

As in the case of the one-point function, it is useful to parametrize the contribution each diagram makes to $\langle \delta \phi(k_1) \delta \phi(k_2) \rangle *$ in terms of a function $\Pi$, which is defined by

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle * = \frac{1}{2} \left( P_s(k) \right)^2 \int d\eta \int \frac{d^3q}{(2\pi)^3} \, e^{2ik\eta} \Pi + \text{complex conjugate},$$ (81)
where \( P_\ast(k)^2 \) is the square of the tree-level power spectrum, and the quantity \( \Pi \) (to be calculated in this section) depends on the external momenta \( \{k_1, k_2\} \) and the loop momentum \( q \).

The class of diagrams where the derivative is applied to the external leg makes a contribution to \( \Pi \) which corresponds to

\[
\Pi \supset (1 - i \kappa \eta)(1 + q^2 \eta^2) \left( \frac{k^2}{4q^3}(q \cdot k_2) \frac{\sigma(-k_1, q)}{|k_1 + q|^2} + \frac{k^2 \sigma(-k_1, k_2)}{8q^2 k_{12}^2} \right) + [k_1 \leftrightarrow k_2],
\]

(82)

where \([k_1 \leftrightarrow k_2]\) denotes the same term with \( k_1 \) and \( k_2 \) interchanged. The ratio \( \sigma(-k_1, k_2)/k_{12}^2 \) is obviously singular when the momentum conservation condition \( k_1 + k_2 = 0 \) is enforced and must be treated carefully to avoid an unwanted divergence. Consider the non-singular quotient \( \sigma(a, b)/|a + b|^2 \) where \( a \) and \( b \) approach \( k \) and \(-k\) respectively,

\[
\lim_{\epsilon, \delta \to 0} \frac{\sigma(k + \delta, -k + \epsilon)}{|\epsilon + \delta|^2} = \lim_{\epsilon, \delta \to 0} \frac{k \cdot (\delta + \epsilon) + \epsilon^2 + \delta \cdot \epsilon}{\delta^2 + \epsilon^2 + 2\delta \cdot \epsilon}
\]

(83)

This is not symmetric between \( \delta \) and \( \epsilon \), because \( \sigma \) is not a symmetric function of its arguments; as a result, the limits do not commute. Moreover, as \( \delta \) and \( \epsilon \) approach zero the numerator of (83) vanishes linearly, as fast as \( O(\epsilon, \delta) \), whereas the denominator is vanishing quadratically, like \( O(\epsilon^2, \delta^2) \). Therefore (82) is naively divergent. In fact, the value of (83) depends on what is assumed about \( k \cdot \delta \) and \( k \cdot \epsilon \); if we demand that the limit is approached along a sequence of vectors of magnitude \( k = |k| \) then it follows that \( |k + \delta| = |\epsilon + \delta| = k \) and therefore

\[
k \cdot \epsilon = \frac{\epsilon^2}{2} \quad \text{and} \quad k \cdot \delta = -\frac{\delta^2}{2}.
\]

(84)

With this choice, Eq. (83) evaluates to \( 1/2 \) and the limits become commuting. This prescription was used implicitly in §4.2 of Ref. [59], but there does not seem to be any compelling reason to demand the limit is approached along such a specific sequence of vectors. Fortunately a catastrophic divergence is averted, since Eq. (82) requires symmetrization over the exchange \( k_1 \leftrightarrow k_2 \). The problematic term \( k \cdot (\delta + \epsilon) \) is antisymmetric under this exchange and cancels out of the expectation value (82), leaving a finite limit. The result of this procedure gives the same answer as if we had adopted Eq. (84), which can be regarded as an \emph{a posteriori} justification for the analysis presented in Ref. [59].

After performing the symmetrization over \( k_1 \) and \( k_2 \) and integrating over \( q \) and \( \eta \), this class of diagrams contribute to the two-point function a quantity

\[
\langle \delta \phi(k_1) \delta \phi(k_2) \rangle_\ast \supset (2\pi)^3 \delta(k_1 + k_2) P_\ast(k)^2 \left( -\frac{3}{16} k^3 \ln k - \frac{1}{120} k^3 + \cdots \right)
\]

(85)

where ‘\( \cdots \)’ denotes ultra-violet divergent terms which have been omitted, together with terms which vanish in the limit \( m \to 0 \).
Now consider the diagrams in which the derivative is applied to the internal line. Such diagrams contribute to $\Pi$ according to

$$\Pi \supseteq (1 - i k \eta)^2 (1 - i q \eta) \left( \frac{k_1 \cdot q \sigma(-q, -k_2)}{4q |k_2 + q|^2} + \frac{k^2 \sigma(q, -q)}{8q |q - q|} \right) + [k_1 \Leftarrow k_2], \quad (86)$$

This class of diagrams contains a similar ill-defined ratio, $\sigma(q, -q)/|q - q|$. Consider Eq. (83) again, with $k$ replaced by $q$. Although there is no longer any injunction to symmetrize over $q \Leftarrow -q$, the non-rotationally-invariant part $q \cdot (\delta + \epsilon)$ will vanish underneath the integral and does not give rise to any divergence. In order to assign a definite value to the remaining limit, we must assume something about $\delta$ and $\epsilon$. Since $q$ is merely a variable of integration and can be freely replaced by $-q$, we assume that $\sigma(q, -q)$ is to be regularized by taking its symmetric part. With this prescription, the ratio $\sigma(q, -q)/|q - q|$ evaluates to $1/2$.

Symmetrizing over $k_1$ and $k_2$ and omitting ultra-violet divergent terms together with any terms which vanish in the limit $m \to 0$, we find

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle_* \supseteq (2\pi)^3 \delta(k_1 + k_2) \frac{P_s(k)^2}{\pi^2} \left( \frac{1}{8} k^3 \ln k - \frac{3}{20} k^3 + \cdots \right) \quad (87)$$

**Two derivatives** We have now exhausted all diagrams with only a single derivative. The next set of diagrams all involve two derivatives and break naturally into three sets: the first class includes all diagrams with the derivatives applied to both external legs of the two-point function; the second set includes all diagrams where where one derivative is applied to an external leg while other applies to the internal propagator; and the third set includes all diagrams with both derivatives applied to the internal propagator:

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle_* \supseteq \begin{array}{c} \text{Diagram} \end{array} \quad (88)$$

Consider first the set of diagrams with both derivatives on the external legs. We obtain

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle_* \supseteq \begin{array}{c} \text{Diagram} \end{array} \quad \frac{-k^4}{2q^2} (1 + q^2 \eta^2) Q, \quad (89)$$

where $Q$ is the quantity

$$Q \equiv \frac{z(-k_1, q) \cdot z(-k_2, q)}{|q - k_1|^4 |q + k_2|^2} + \frac{3}{4} \frac{\sigma(-k_1, q) \sigma(-k_1, -q)}{|q - k_1|^2 |q + k_2|^2} + \frac{2}{20} \frac{q \cdot z(-k_2, q)}{|q - k_2|^2} + [k_1 \Leftarrow k_2]. \quad (90)$$

Unlike the previous examples, none of the ratios which appear in $Q$ are ill-defined. However, this result can still be significantly simplified using the symmetry properties of $z$ and $\sigma$. In particular, we observe that

$$\sigma(-a, -b) = \sigma(a, b) \quad \text{and} \quad z(-a, -b) = z(b, a). \quad (91)$$
These identities can be used together with the obvious antisymmetry of $z \ [z(a, b) = -z(b, a)]$. After performing the symmetrization over $k_1$ and $k_2$, $P$ can be reduced to the simpler form

$$Q = -2 \frac{z(q, k)^2}{|q + k|^6} + \frac{3 \sigma(k, q)^2}{2 |q + k|^4} + 4 \frac{q \cdot z(q, k)}{|q + k|^4}. \quad (92)$$

In this expression, $k$ can be taken to be either $k_1$ or $k_2$; after integration, the result depends only on the magnitude $k$ and not its orientation, and we obtain

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq (2\pi)^3 \delta(k_1 + k_2) \frac{P_+(k)^2}{\pi^2} \left( \frac{15}{16} k^3 \ln k + k^3 + \cdots \right). \quad (93)$$

The set of diagrams with one derivative on an external leg and one derivative on the internal propagator are the most complicated. To evaluate them, we write

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq (2\pi)^3 \delta(k_1 + k_2) \frac{P_+(k)^2}{\pi^2} \left( 3k^3 \ln k - \frac{5}{2} k^3 + \cdots \right). \quad (96)$$

After integration, one obtains

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq (2\pi)^3 \delta(k_1 + k_2) \frac{P_+(k)^2}{\pi^2} \left( -2 \frac{z(q, k)^2}{|q + k|^6} + \frac{3 \sigma(q, k)^2}{2 |q + k|^4} + 4 \frac{k \cdot z(q, k)}{|q + k|^4} \right), \quad (97)$$

which does not require regularization. After integration, one obtains

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq (2\pi)^3 \delta(k_1 + k_2) \frac{P_+(k)^2}{\pi^2} \left( - \frac{15}{16} k^3 \ln k + \frac{1}{4} k^3 + \cdots \right). \quad (98)$$

**Three derivatives** The only remaining class of diagrams are those containing three derivatives at the vertex. These diagrams break into two groups: those in which one end of the internal propagator is free of a derivative, and those in which an external leg is free of a derivative:

$$\langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq \frac{Q}{2} + \frac{Q}{2}. \quad (99)$$
Both types give rise to comparatively simple expressions. For the first we obtain

\[ \Pi \geq \frac{k^4}{8q} \eta^2 (1 - i q \eta) \left( 4 \frac{\sigma(k, q)}{|q + k|^2} + 1 \right); \]  

after integration this class of diagrams give contributions totalling

\[ \langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq (2\pi)^3 \delta(k_1 + k_2) \frac{P_*(k)^2}{\pi^2} \left( -\frac{1}{24} k^3 \ln k + \frac{1}{18} k^3 + \cdots \right). \]  

On the other hand, for the second type of diagram we obtain

\[ \Pi \geq \frac{k^2 q}{8 \eta^2} (1 - i k \eta) \left( 4 \frac{\sigma(q, k)}{|q + k|^2} + 1 \right). \]  

After integration, we find

\[ \langle \delta \phi(k_1) \delta \phi(k_2) \rangle \geq (2\pi)^3 \delta(k_1 + k_2) \frac{P_*(k)^2}{\pi^2} \left( \frac{1}{48} k^3 \ln k - \frac{1}{90} k^3 + \cdots \right). \]  

### 5.3. Infra-red behaviour

Having obtained the relevant contributions to the two-point function, given by Eqs. (85), (87), (93), (98), (101) and (103), we may collect these quantities to obtain an estimate of the total loop correction. It can be written

\[ \langle \delta \phi(k_1) \delta \phi(k_2) \rangle \sim (2\pi)^3 \delta(k_1 + k_2) P_*(k) P_* \left( \frac{35}{6} \ln k - \frac{491}{180} + \hat{\beta} \right) \]

\[ \simeq (2\pi)^3 \delta(k_1 + k_2) P_*(k) P_*(5.83 \ln k + \beta), \]  

where \( \hat{\beta} \) is an unknown renormalization-scheme dependent quantity left over from cancellation of the ultra-violet divergences. In the last equality this has been combined with the constant term to give an overall unknown constant \( \beta \). The coefficient of the logarithm, however, is scheme-independent [51].

### 6. Discussion

In this paper, we have computed estimates for the infra-red behaviour of the leading radiative corrections to the one- and two-point expectation values of the inflaton field perturbation during a phase of single-field, slow-roll inflation. After suitable ultra-violet renormalization, the loop correction to the one-point function was found to be given on large scales by an unknown renormalization-scheme dependent quantity \( \alpha \), which was free of infra-red divergences, whereas the loop correction to the two point function gave a correction to the power spectrum of the form

\[ P_*^{\text{loop}} = P_* \left\{ 1 + P_* \left( \frac{35}{6} \ln k + \beta \right) \right\}. \]  

where \( \beta \) is a similar scheme-dependent constant. This is not really a limitation, since \( \alpha \) and \( \beta \) would be negligible in comparison with \( \ln k \) on large scales. Although the amplitude of the \( \delta \phi \) power spectrum itself is not observable, the amplitude of \( \zeta \) is known
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accurately to be of order $10^{-10}$. At tree-level, the two are related via the approximate relation $P_\zeta \sim P_\star / \epsilon$, where $\epsilon$ is the slow-roll parameter introduced in Eq. (5). Since $\epsilon$ is expected to be of order $10^{-2}$ or less, we can conservatively suppose that $P_\star \lesssim 10^{-10}$.

The loop correction given by Eq. (105) is therefore extremely small for moderate values of $k$.

This does not allow us to conclude that loop corrections are too small to be observable in the CMB, because it is the loop corrections in $\zeta$ rather than the $\delta \phi$ themselves which are accessible to experiment. Therefore, the prediction (105) needs to be translated into a prediction for $P_\zeta^{\text{loop}}$ before a final determination concerning the magnitude of loop corrections can be made. This calculation will be presented elsewhere [58]. However, it is already clear from Eq. (105) that quantum effects do not greatly disturb the magnitude of the fluctuations imprinted in $\delta \phi$ as successive $k$-modes pass outside the horizon. Any large loop correction will have to come from superhorizon scales, where the fields are effectively in a classical régime.

Eq. (105) is entirely consistent with previous estimates which have been made in the literature. In particular, Weinberg has estimated a correction to $P_\star$ from matter loops in a multi-field theory [51] which has the same functional form as (105). Sloth [46, 47] has given a similar estimate, based on the same action given in Eqs. (25)–(17), but evaluated several tens of e-folds after horizon crossing when large infra-red divergences can compensate for a suppression in powers of slow-roll parameters; in this limit a different set of terms extracted from Eq. (17) dominate the loop correction. This loop correction is proportional to $\langle \delta \phi^2 \rangle \sim P_\star \ln(k)$ for a flat spectrum, which reproduces the logarithmic $k$-dependence described by (105).

Acknowledgments

I acknowledge support from PPARC under grant PPA/G/S/2003/00076. I would like to thank M. Sloth, D. Lyth, K. Malik, J. Lidsey and C. Byrnes for useful conversations, and especially F. Vernizzi for lengthy conversations and correspondence which have helped clarify my understanding. I would like to acknowledge the hospitality of the Abdus Salam Institute for Theoretical Physics, Trieste, and the Department of Physics, University of Cardiff, where portions of the work outlined in this paper were carried out.

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