THE MAX NOETHER FUNDAMENTAL THEOREM IS COMBINATORIAL

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Abstract. In the present paper we give a reformulation of the Noether Fundamental Theorem for the special case where the three curves involved have the same degree. In this reformulation, the local Noether’s Conditions are weakened. To do so we introduce the concept of Abstract Curve Combinatorics (ACC) which will be, in the context of plane curves, the analogue of matroids for hyperplane arrangements.

1. Introduction

In 1873 Noether stated his celebrated Fundamental Theorem [8], sometimes referred to as the “AF + BG” Theorem. This theorem brings together the geometric and algebraic conditions plane projective algebraic curves should satisfy when belonging to a pencil. The following statement can be found in [3].

Theorem 1.1 (Max Noether’s Fundamental Theorem). Let $F, G, H$ be homogeneous reduced polynomials in three variables defining projective algebraic curves $V(F), V(G),$ and $V(H).$ Assume $V(F)$ and $V(G)$ have no common components. Then there is an equation $H = AF + BG$ (with $A, B$ forms of degrees $\deg(H) - \deg(F)$ and $\deg(H) - \deg(G)$ respectively) if and only if $H_P \in (F_P, G_P) \subset \mathscr{O}_P(\mathbb{P}^2)$ for any $P \in V(F) \cap V(G)$.

Here we denote by $F_P$ the germ of $F$ at $P$, by $(F_P, G_P)$ the local ideal generated by the germs $F_P$ and $G_P$, and by $V(F) \subset \mathbb{P}^2$ the set of zeroes of $F$. The local conditions on the equations $F, G, H$ are called Noether conditions.

This theorem was originally attacked both from geometric and algebraic points of view ([9, 3]) and it has been recently generalized to the non-reduced case by Fulton [4].

Most of the efforts to understand and rewrite Noether’s Fundamental Theorem have been focused on finding conditions that are equivalent to the Noether conditions in particular instances like transversality of branches, ordinary singularities, etc.

Our purpose here is to concentrate on the case where $\deg F = \deg G = \deg H$ and to weaken the Noether’s conditions so as to have strictly weaker local conditions that can still provide the equivalence of the result. Note that the Noether Fundamental Theorem is a combination of a global condition (the existence of the curves $F, G, H$) and local conditions. Our weakened local conditions combined with the global condition result in this equivalence.

The weakened local conditions can be briefly described as follows: We say $F$ satisfies the combinatorial conditions with respect to $G$ and $H$ if for any point $P \in V(F) \cap V(GH)$ and any local branch $\delta$ of $F$ at $P$ then $\mu_P(\delta, G) = \mu_P(\delta, H)$, where $\mu_P$ denotes the multiplicity of intersection of branches at $P$. Also we say that $F, G, H$ satisfy the conditions for a combinatorial pencil if each equation satisfies the combinatorial conditions with respect to the other two

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equations. We also introduce the concept of a \textit{primitive combinatorial pencil} which corresponds with the geometric idea that the fibers of the map over \( \mathbb{P}^1 \) induced by the pencil after resolution of indeterminacy are connected. In Section 3 we prove that any combinatorial pencil can be refined to a primitive combinatorial pencil.

The global condition can be rewritten as follows: If \( \deg F = \deg G = \deg H \), then the condition \( H = AF + BG \) simply means that \( H \) belongs to the pencil generated by \( F \) and \( G \), or simply that \( F, G, H \) belong to a pencil.

The main result is the following.

**Theorem 1.2.** Let \( F, G, H \) be projective plane curves of the same degree. Assume \( F \) and \( G \) have no common components. If \( F, G, H \) belong to a primitive combinatorial pencil then they belong to a pencil.

To end this introduction we present two examples aimed to clarify the sharpness of these combinatorial conditions. The first one points out that the combinatorial conditions are indeed weaker than the Noether conditions and the second one suggests that the conditions cannot be weakened.

**Example 1.3.** This first example shows that the (local) Noether Conditions are stronger that the combinatorial condition described above. Consider the germs \( f = x^3, g = y^2 \) and \( h = y^2 + (x+y)^3 \) in \( \mathcal{O}_P(\mathbb{P}^2) = \mathbb{C}[x, y], P = [0 : 0 : 1] \). It is obvious that they satisfy the combinatorial conditions at \( P \) since \( \mu_P(f, g) = \mu_P(f, h) = \mu_P(g, h) = 6 \). However, \( h \notin (f,g) \) since \( h = y^2 + x^3 + y^3 + 3xy^2 + 3x^2y \), where \( h_1 = y^2 + x^3 + y^3 + 3xy^2 \in (x^3, y^2) \), but \( h_2 = x^2y \notin (x^3, y^2) \), and \( h = h_1 + h_2 \).

**Example 1.4.** This second example shows that the combinatorial conditions have to be stated for each branch, as opposed to each irreducible component. Consider \( F = ZY^2 - ZX^2 + X^3, G = (X + Y)^3, \) and \( H = (X - Y)^3 \), three cubics. Note that \( V(F) \cap V(G) = V(F) \cap V(H) = V(G) \cap V(H) = \{P = [0 : 0 : 1]\} \) and \( \mu_P(F, G) = \mu_P(F, H) = \mu_P(G, H) = 9 \). However, \( F, G, H \) are not in a pencil. Note that the combinatorial conditions are not satisfied, since \( F \) is not locally irreducible at \( P \) and the two branches \( \delta_1 \) and \( \delta_2 \) satisfy \( \mu_P(\delta_1, G) = \mu_P(\delta_2, H) = 6 \), and \( \mu_P(\delta_1, H) = \mu_P(\delta_2, G) = 3 \).

2. Settings

2.1. Abstract Curve Combinatorics.

**Definition 2.1.** An \textit{Abstract Curve Combinatorics} (ACC for short) is a sextuplet \( W := (r, S, \Delta, \partial, \phi, \mu) \), where

1. \( r, S, \) and \( \Delta \) are finite sets,
2. \( \partial : \Delta \to S \) and \( \phi : \Delta \to r \) are surjective maps,
3. \( \mu : \text{SP}^2(\Delta) \to \mathbb{N}, \) where \( \text{SP}^2(\Delta) \) is the symmetric product of \( \Delta \), such that \( \mu(\delta_1, \delta_2) > 0 \) if and only if \( \partial(\delta_1) = \partial(\delta_2) \) and \( \phi(\delta_1) \neq \phi(\delta_2) \).

For simplicity, we denote \( \Delta_P := \partial^{-1}(P), P \in S \).

We say that two ACC’s are equivalent if there are bijections preserving the corresponding maps.

**Remark 2.2.** Note that any projective curve \( C \subset \mathbb{P}^2 \) determines naturally an ACC \( W_C := (r, S, \Delta, \partial, \phi, \mu), \) (which will be referred to as the \textit{Weak Combinatorial Type of} \( C \)) as follows:
(i) The set $r$ is the set of irreducible components of $C$,
(ii) The set $S := \text{Sing}(C)$, is the set of singular points of $C$,
(iii) $\Delta := \bigcup_{P \in S} \{\Delta_P\}$ where $\Delta_P$ is the set of local branches of $C$ at $P \in S$, $\partial(\delta) := P$ if $\delta \in \Delta_P$, and $\phi$ assigns to each local branch the global irreducible component that contains it.
(iv) $\mu(\delta_1, \delta_2)$ is defined as the multiplicity of intersection between $\delta_1$ and $\delta_2$ (when $\partial(\delta_1) = \partial(\delta_2)$ and $\phi(\delta_1) \neq \phi(\delta_2)$) and as zero otherwise.

In accordance with this motivation, given an ACC $W = (r, S, \Delta, \partial, \phi, \mu)$, we will refer to the elements of $r$ (resp. $S$, and $\Delta$) as irreducible components, (points, and branches). Also $\mu$ will be referred to as the intersection multiplicity of two branches.

2.2. Bézout Condition and degrees. Consider $W$ an ACC and define $d_{i,j} := \sum_{\phi(\delta) = i} \mu(\delta_1, \delta_2)$, for any $i, j \in r, i \neq j$.

Definition 2.3. $W$ satisfies the Bézout Condition if $d_{i,j}d_{j,k}$ is independent of $j, k \in r$. In that case, one can define

$$d_i := + \sqrt{\frac{d_{i,j}d_{j,k}}{d_{j,k}}}.$$ 

and will be referred to as the degree of $i$.

Note that the Weak Combinatorial Type of a plane projective curve satisfies the Bézout Condition and $d_i$ coincides with the algebraic degree of the irreducible component $i$.

2.3. Combinatorial Pencils. Let $W$ be an ACC satisfying the Bézout Condition.

Definition 2.4. We say that $W$ contains a combinatorial pencil if there exist $\bar{m} := (m_i)_{i \in r}$ a list of integers and $\mathcal{F} = \{F_1, \ldots, F_k\}$, $k \geq 3$ a partition of $r$ such that:

(1) $\sum_{i \in F_j} m_i d_i$ is independent of $j \in \{1, \ldots, k\}$, (such constant will be denoted by $d_\mathcal{F}$) and
(2) for any $P \in S$ one of the following two conditions is satisfied:
(a) either $\phi(\Delta_P) \subset F_i$ for a certain $i = 1, \ldots, k$,
(b) or $\phi(\Delta_P) \not\subset F_i$, in which case for each $\delta \in \Delta_P$, the natural number

$$\sum_{\phi(\delta') \in F_j} m_{\phi(\delta')} \mu(\delta, \delta')$$

is independent of $j$ (as long as $\phi(\delta) \not\in F_j$). Such a constant will be denoted by $k_s$.

The points $P \in S$ satisfying (2) will be called the base points of the combinatorial pencil and each $F_i \in \mathcal{F}$ will be called a fiber. The integer $m_i$ will be called the multiplicity of the $i$-th component and the members of the partition $\mathcal{F}$ are the members of the pencil.

We also say that three curves $F$, $G$ and $H$ belong to a combinatorial pencil if ($\{F, G, H\}$, $\bar{m}$) is a combinatorial pencil, where $\bar{m}$ is the list of multiplicities of the components of $C = F \cup G \cup H$.

Our purpose will be to investigate under what circumstances three curves belonging to a combinatorial pencil, also belong to a pencil, that is $H = AF + BG$ for some $A, B \in \mathbb{C}^\ast$. Note that this is not true in general as one can simply see with line arrangements. Consider $F = XY$, $G = X^2 - Y^2$, and $H = X^2 - 4Y^2$. It is obvious that $F$, $G$, and $H$ belong to a combinatorial pencil, but not to a pencil. The geometrical reason behind this phenomenon is that the resolution of the rational map $[X : Y : Z] \mapsto [F : G]$ does not have connected fibers.
Definition 3.3. A sequence of \((\nu, \mu)\) of multiplicities of the local branches at a point of the total transform \(\hat{V} \leftarrow \Delta\) is called primitive if:

1. \(\nu \leq 1\) if \(\delta \notin \Delta^0\), where \(\nu_0\) is the multiplicity associated with \(\delta\) at a \(\sigma\)-process, and

2. \(\mu(\delta_1, \delta_2) = \mu(\delta_1, \delta_2) - \nu_{\delta_1}\nu_{\delta_2}\), if \(\delta_1, \delta_2 \in \Delta^P\) and \(\phi(\delta_1) \neq \phi(\delta_2)\), (the intersection multiplicity of two branches after blow up decreases by the product of the multiplicities of the branches),

(6) \(\mu\) extends \(\mu\) outside \(SP^2(\Delta_P)\).

Remark 3.2. Let \(W\) be the weak combinatorial type of a curve \(\mathcal{C}\) in a rational surface \(V\) and let \(\hat{V} \leftarrow \hat{\mathcal{C}}\) be a blow-up of \(V\) at a singular point \(P\) of \(\mathcal{C}\). Note that then the weak combinatorial type of the total transform \(\hat{\mathcal{C}}\) is obtained by a \(\sigma\)-process at \(P\) from \(\mathcal{C}\) by using as \(\tilde{\nu}\) the list of multiplicities of the local branches at \(P\).

This way one can extend the concept of resolution to general ACC’s.

Definition 3.3. A sequence of \(\sigma\)-processes \(W = W_0 \leftarrow W_1 \leftarrow \ldots \leftarrow W_n\) of ACC’s \(W_k := (r^{(k)}, S^{(k)}, \partial^{(k)}, \Delta^{(k)}, \phi^{(k)}, \mu^{(k)}), k = 0, 1, \ldots, n\) is called a resolution of \(W\) if:

1. \(\nu_\delta \leq 1\) if \(\delta \notin \Delta^{(0)}\), where \(\nu_\delta\) is the multiplicity associated with \(\delta\) at a \(\sigma\)-process, and
(2) (Normal-crossing condition) \( \#\Delta_P^{(n)} = 2 \) for any \( P \in S^{(n)} \) and \( \mu^{(n)} \) only takes values in \( \{0, 1\} \).

An ACC is called solvable if there exists a resolution.

Remark 3.4. Note that the ACC obtained from a curve in \( \mathbb{P}^2 \) admits a (combinatorial) resolution given by any (geometric) resolution of its singularities, that is, every weak combinatorial type is solvable. Such a resolution will be called a geometric resolution of \( W \). Note that weak combinatorial types might admit non-geometric resolutions as well.

3.2. Admissibility conditions. Let \( W \) be an ACC and let \( \{v_i\}_{i \in r} \) be a list of vectors in \( \mathbb{K}^k \). For any \( \delta \in \Delta \) define

\[
v_{\delta} := \sum_{j \in r} \mu(\delta, j) v_j,
\]

where \( \mu(\delta, j) := \sum_{\delta' \in \phi^{-1}(j)} \mu(\delta, \delta') \). Note that, by Definition 2.1.(3), the only branches that contribute to \( v_{\delta} \) are those in \( \Delta_P \).

We say that \( \{v_i\}_{i \in r} \) satisfies the admissibility conditions for \( W \) if:

\[
(2) \{v_{\phi(\delta)}, v_{\delta}\} \text{ are linearly dependent for all } \delta \in \Delta.
\]

We will often denote this by saying \( v_{\phi(\delta)} \parallel v_{\delta} \) (note that one of the vectors might be zero).

Definition 3.5. A list of vectors \( v_W := \{v_i\}_{i \in r} \) in \( \mathbb{K}^k \) satisfying the admissibility conditions (2) for \( W \) and spanning \( \mathbb{K}^k \) is called a \( k \)-admissible family for \( W \).

One has the following result:

Proposition 3.6. If \( (F, \bar{m}) \) is a combinatorial pencil of \((k + 1)\)-fibers of \( W \), then there exists a \( k \)-admissible family for \( W \).

Proof. Let us consider \( F = \{F_0, F_1, \ldots, F_k\} \) and define the following family of vectors \( v_i, i \in r \):

\[
v_i := \begin{cases} m_i e_j & \text{if } i \in F_j, j \neq 0 \\ -m_i (e_1 + \ldots + e_k) & \text{if } i \in F_0.
\end{cases}
\]

Under these conditions note that if \( P \in S \) is not a base point, then condition (2) is immediately satisfied since all the vectors involved are linearly dependent. Now, if \( P \in S \) is a base point, then the condition (2b) in Definition 2.4 above implies that \( k_{\delta}v_i + v_{\delta} = 0 \) and hence condition (2) is also true. \( \square \)

Definition 3.7. The \( k \)-admissible family for \( W \) associated with the combinatorial pencil \( (F, \bar{m}) \) as in Proposition 3.6 will be referred to as the admissible family of \( W \) associated with \( (F, \bar{m}) \).

We need the following result from linear algebra.

Lemma 3.8. Suppose \( v_1, \ldots, v_r \in \mathbb{K}^k \) are vectors such that

\[
v_1 \parallel \sum_{j=1}^r a_{1,j} v_j, \quad v_2 \parallel \sum_{j=1}^r a_{2,j} v_j, \quad \ldots \quad v_{r-1} \parallel \sum_{j=1}^r a_{r-1,j} v_j,
\]

then
where \( a_{i,j} = a_{j,i}. \) Then

\[
v_r \|( \sum_{j=1}^{r} a_{r,j} v_j ),
\]

**Proof.** Note that if \( v_r = 0 \) or \( v_1 = v_2 = \cdots = v_{r-1} = 0, \) then the result is immediate. Also, if any of the vectors \( \{ v_1 \}_{i=1,\ldots,r-1} \) are trivial, say \( v_1 = 0, \) it is enough to solve the same problem on the remaining vectors, since the coefficients \( a_{1,i} = a_{i,1} \) either multiply the vector \( v_1 \) (and hence they have no contribution) or multiply \( v_i \) in the condition \( v_r \|( \sum_{j=1}^{r} a_{1,j} v_j ), \) which is trivially satisfied since \( v_1 = 0. \)

Therefore, we will assume that all the vectors \( v_i \) \((i = 1, \ldots, r)\) are non-zero. Note that if \( v \neq 0, \) then \( v \| w, \) implies the existence of \( \lambda \in \mathbb{K} \) such that \( \lambda v + w = 0. \) Hence, in our case, there exist \( \lambda_i \in \mathbb{K} \) such that

\[
\lambda_i v_i + \sum_{j=1}^{r} a_{i,j} v_j = 0
\]

for \( i = 1, \ldots, r - 1. \) Consider an \( r \times r \) symmetric matrix \( A := (a_{i,j}) \) where the first \( r-1 \) rows are given by the coefficients of the equations \( (3) \) over the variables \( v_i, \) and the last row is given by the coefficients of \( \sum_{j=1}^{r-1} a_{r,j} v_j \) over the same variables. Also consider \( V := (v_{i,j}) \) an \( r \times k \) matrix whose \( i \)-th row is given by the coefficients of \( v_i. \) By \( (3) \) one has that

\[
AV = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
b_1 & b_2 & \ldots & b_{k-1} & b_k
\end{pmatrix},
\]

and hence

\[
V^t AV = \bar{b}^t v_r,
\]

where \( \bar{b} = (b_1, b_2, \ldots, b_{k-1}, b_k) \) (row notation) is the vector of coordinates of \( \sum_{j=1}^{r-1} a_{r,j} v_j. \) Since \( V^t AV = \bar{b}^t v_r \) is symmetric, one obtains that \( v_r \| \sum_{j=1}^{r-1} a_{r,j} v_j \) and therefore \( v_r \| \sum_{j=1}^{r} a_{r,j} v_j. \)

In other words, the admissibility conditions for each point are redundant.

In what follows we look into how the admissibility conditions change under a \( \sigma \)-process. Consider \( v_W := (v_i)_{i \in \mathbb{F}} \) a \( k \)-admissible family of vectors for \( W, \) and \( \tilde{W} \) a \( \sigma \)-process of \( W \) at \( P \) associated with the multiplicity list \( \tilde{v} := (v_{\delta})_{\delta \in \Delta_P}. \)

One has the following.

**Proposition 3.9.** Let \( W \) be an ACC and \( \tilde{W} \) a \( \sigma \)-process of \( W. \) Then any \( k \)-admissible family \( v_W \) on \( W \) induces a \( k \)-admissible family \( v_{\tilde{W}} \) on \( \tilde{W} \) and vice versa.

**Proof.** Let \( v_W \) be an admissible family on \( W. \) Then \( v_{\tilde{W}} = (\tilde{v}_i)_{i \in \mathbb{F}} \) is defined as follows: \( \tilde{v}_i = v_i \) if \( i \in \mathbb{F}. \) The new vector associated with the exceptional divisor \( \tilde{v}_E \) is defined as:

\[
\tilde{v}_E := \sum_{\delta \in \Delta_P} v_{\delta} v_{\phi(\delta')}.
\]

It remains to verify that the new admissible conditions are satisfied. In order to avoid ambiguity, all the new vectors in \( \tilde{W} \) will be denoted as \( \tilde{v}. \) First we fix an infinitely near point \( \tilde{P}, \) a branch \( \delta \in \Delta_P, \) and assume \( i := \phi(\delta). \) We have two cases:
connected if there is a sequence of components $A_1, A_2, \ldots, A_n = B$, a sequence of branches $\delta_i, \delta'_i \in \Delta, (i = 1, \ldots, n)$ such that $\phi(\delta_i) = A_{i-1}, \phi(\delta'_i) = A_i$, satisfying $\mu(\delta_i, \delta'_i) \neq 0$. Therefore the concept of connected components of an ACC can be defined.

One has the following interesting result.

**Lemma 3.13.** After resolution and after removing the trivial divisors, different fibers of a combinatorial pencil belong to different connected components.

**Proof.** Let $(\mathcal{C}, m)$ be a combinatorial pencil in $\mathcal{C}$ and let $v_W$ be its associated $k$-admissible family. Consider $W \leftarrow \hat{W}$ a resolution of singularities of $\mathcal{C}$. By Proposition 3.9 the associated family of vectors $v_{\hat{W}}$ also satisfies the admissibility conditions shown in (2). Since $\hat{W}$ corresponds to the combinatorics of a normal crossing divisor the condition at each normal crossing of two divisors, say $E$ and $E'$, means that either $v_E = kv_{E'}, k \in \mathbb{K}^*$, or $v_E = 0$, or $v_{E'} = 0$. In other words, the
vectors associated with $\tilde{W}$ are either multiples of the original vectors of $W$ or zero. Moreover, after removing the trivial divisors, components from different fibers are disconnected. 

3.4. Intersection matrix. Let $W$ be a solvable ACC satisfying the Bézout conditions (Definition 2.3). Consider $W = W_0 \leftarrow W_1 \leftarrow \cdots \leftarrow W_n$ a resolution of $W$, and $v_W := (v_i)_{i \in E}$ a $k$-admissible family of vectors. Denote by $P_\ell$ the point blown-up at each step $W_\ell \leftarrow W_{\ell+1}$ with multiplicity list $\hat{\mu}(\ell) := (\hat{\mu}(\ell))_{\delta \in \Delta_\ell}$, and define by $P$ the collection of such points. As seen in Proposition 3.9, associated with $v_W$ there are $k$-admissible families $v_W_\ell := (v_i)_{i \in E(\ell)}$ of $W_\ell$. Define $\tilde{r}^{(n)} := \{i \in r^{(n)} | v_i \neq 0\}$. Consider the following incidence matrix associated with the resolution $W_n$ of $W$ and with the admissible family $v_W$:

$$J := (a_{ij})_{i \in \tilde{r}^{(n)}, j \in P},$$

where

$$a_{ij} := \begin{cases} v^{(\ell)}_i & \text{if } j = P_\ell, i \in r^{(\ell)} \\ -1 & \text{if } j = P_\ell, i = E_{\ell+1} \\ 0 & \text{otherwise,} \end{cases}$$

(recall that $v_i := \sum_{\delta \in \Delta_\ell} \nu_\delta$ and note that $|P| = n$). Also define the degree matrix $D := d^T \hat{d}$, where $\hat{d} := (d_i)_{i \in \tilde{r}^{(n)}}$, and

$$d_i := \begin{cases} d_i & \text{if } i \in r^{(0)} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we combine both matrices in order to define $Q := D - JJ^T$. Also, for convenience, if $W$ is already normal crossing $\tilde{r}$, then we set $JJ^T = 0$.

Note that $Q$ is a square matrix of order $|P|$. 

Proposition 3.14. If $W$ satisfies the Bézout Conditions and $W \leftarrow W_n = \tilde{W} = (\tilde{r}, \tilde{S}, \tilde{\Delta}, \tilde{\phi}, \tilde{\mu})$ is a resolution of $W$, then the matrix $Q := (q_{ij})_{i,j \in \tilde{r}}$ satisfies the following:

1. $q_{ij} := \tilde{\mu}(i,j) := \sum_{\delta' \in \phi^{-1}(i)} \tilde{\mu}(\delta, \delta')$ if $i \neq j$ (intuitively, the number of points at which the components $i$ and $j$ intersect in $\tilde{W}$). In particular $q_{ij} \geq 0$.

2. $q_{ii} = d^T_1 - \sum_{\ell \in L_1} (v^{(\ell)}_i)^2$, where $L_1 := \{\ell \mid i \in \phi^{(\ell)}(\Delta^{(\ell)}_{P_1})\}.$

Moreover, if the resolution is geometric, then $Q$ is the intersection matrix of the non-trivial divisors of $\tilde{r}$ in $\tilde{W}$.

Proof. By definition of $Q$ one has that

$$q_{ij} = d_i d_j - \sum_{\ell \in L_{ij}} v^{(\ell)}_i v^{(\ell)}_j,$$

where $L_{ij} := \{\ell \mid i,j \in \phi^{(\ell)}(\Delta^{(\ell)}_{P_1})\}$. This implies (2).

By definition, one has $\mu^{(\ell)}(i,j) = v^{(\ell+1)}_i v^{(\ell+1)}_j + \mu^{(\ell+1)}(i,j)$. Also, if $W$ satisfies the Bézout Conditions then $d_i d_j = \mu(i,j)$. Hence, $d_i d_j = \sum_{\ell \in L_{ij}} v^{(\ell)}_i v^{(\ell)}_j + \mu(i,j)$ and (1) follows. Finally, since $\tilde{W}$ is a resolution, $q_{i,j}$ is exactly the number of points at which the components $i$ and $j$ intersect in $\tilde{W}$.

The second part is a consequence of the Noether formula for the multiplicity of intersection of branches after resolution. 

□
Example 3.15. Consider the combinatorics corresponding to the arrangement of four curves: a smooth conic, two tangent lines to the conic, and the line joining the tangency points. We will order them and the exceptional divisors of the expected resolution as in Figure 1. The only non-trivial case corresponds to when $E_3$ and $E_4$ are the only dicritical divisors. The corresponding matrices follow:

$$J := \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix}, D := \begin{pmatrix}
4 & 2 & 2 & 2 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \text{ and } Q := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}.$$

Definition 3.16. Let $\mathcal{P}_1 := (F_1, \bar{m}_1)$ and $\mathcal{P}_2 := (F_2, \bar{m}_2)$ be combinatorial pencils of $W$. We say $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ if the fibers of $\mathcal{P}_2$ are contained in the fibers of $\mathcal{P}_1$.

Definition 3.17. Let $W$ be a solvable ACC and let $W \xrightarrow{\pi} \hat{W}$ be a resolution, we say a combinatorial pencil $\mathcal{P} = (F, \bar{m})$ is primitive w.r.t. $\pi$ if each two components in the same fiber of $F$ can be connected in $\hat{W}$ by non-trivial divisors and the multiplicities of the components are coprime, that is, $\gcd(\bar{m}) = 1$.

If $W$ is a weak curve combinatorics, then we simply call a combinatorial pencil primitive if it is primitive w.r.t. a geometric resolution.

Note that the combinatorial notion of primitive is again combinatorial, since the resolution is given again combinatorially.

Theorem 3.18. Let $W$ be a solvable ACC satisfying Bézout Conditions and let $W \xrightarrow{\pi} \hat{W}$ be a resolution, then any combinatorial pencil in $W$ admits a primitive refinement w.r.t $\pi$.

Proof. According to Proposition 3.6 any combinatorial pencil admits a $k$-admissible family $v_W := \{v_i\}_{i \in \Gamma}$. Take a resolution $W = W_0 \leftarrow W_1 \leftarrow \ldots \leftarrow W_n = \hat{W}$ and consider the resulting admissible family $v_{\hat{W}}$.

By Lemma 3.13 there exist trivial divisors. After removing them, one can construct the matrices $J$, $D$ and $Q$ as above. Let $A$ be the matrix whose columns are the non-zero vectors of $v_{\hat{W}}$. It is easy to check that Bézout’s Theorem implies $D \cdot A^t = 0$, and, by construction, $J^t \cdot A^t = 0$. So the rows of $A$ are in the kernel of both $Q$ and $D$. Note that the kernel of $JJ^t$ coincides with the kernel of $J^t$.
After a suitable ordering of the elements of $\tilde{r}^{(n)}$, one can assume that the matrix $Q$ decomposes into a direct sum of irreducible boxes $Q_\lambda$, $\lambda = 1, \ldots, \kappa$. This induces a partition $\tilde{F} := \{ \tilde{F}_1, \ldots, \tilde{F}_n \}$ (with $\tilde{F}_\lambda \subset \tilde{r}^{(n)}$) of the components of the combinatorics, and hence, in the columns of $A$. Each submatrix $Q_\lambda$ is symmetric, and by Proposition 3.14, it has non-negative entries outside the diagonal. Hence, using the Vinberg classification of matrices (see [6, Thm. 4.3]) on $-Q_\lambda$, one can ensure that $Q_\lambda$ is of one of the following types:

(Fin) $\det(Q_\lambda) \neq 0$; there exists a vector $v$ with positive entries such that $Q_\lambda v$ has negative entries.

(Aff) corank $(Q_\lambda) = 1$, and its kernel is generated by a vector with only positive entries.

(Ind) There exists a vector $u$ with only positive entries such that $Q_\lambda u$ has only positive entries. Here (Fin), (Aff), and (Ind) stand for Finite, Affine, and Indefinite types respectively. We have seen that $Q$ has a nontrivial kernel, so the $Q_\lambda$ cannot be an (Fin)-matrix. If one of the $Q_\lambda$, say $Q_1$, is an (Ind)-matrix, one can consider a vector $u_1$ with positive entries such that $Q_1 u_1$ has only positive entries. For the rest of the $Q_\lambda$, one can find vectors $u_\lambda$ with only negative entries such that $Q_\lambda u_\lambda$ has only zero entries (if $Q_\lambda$ is an (Aff)-matrix) or only negative entries (if $Q_\lambda$ is an (Ind)-matrix). By multiplying the $u_\lambda$ by suitable positive numbers, one can reconstruct a vector $u$ such that $Du = 0$.

Now denoting by $(\cdot, \cdot)$ the standard scalar product:

$$0 \geq -(J^t u, J^t u) = (Qu, u) - (Du, u) = (Qu, u) = (Q_1 u_1, u_1) + \sum_{i \geq 2} (Q_\lambda u_\lambda, u_\lambda) \geq (Q_1 u_1, u_1) > 0,$$

which leads to contradiction. So we can conclude that all the $Q_\lambda$ are (Aff)-matrices.

Note that, by Proposition 3.14, the partition $\tilde{F}$ induced by the boxes $Q_\lambda$ is equal to the partition given by the connected components in $\tilde{r}^{(n)}$.

Note that a vector is in the kernel of $Q$ if and only if it is made up of vectors that are in the kernel of the $Q_\lambda$’s. In particular, the kernel of $Q$ has dimension equal to the number of irreducible boxes.

From now on, $K_Q$ will denote the kernel of $Q$, and $K_D$ will denote the kernel of $D$, the degree matrix. Let $u_\lambda$ be a positive vector that generates the kernel of the box $Q_\lambda$, and $\tilde{u}_\lambda$ the vector of $K^{r_\lambda}$, $r_\lambda := \#(r^{(n)})$ obtained from $u_\lambda$ by completing the entries corresponding to the other boxes with zeroes. As mentioned above, $\{\tilde{u}_\lambda\}_{\lambda=1,\ldots,\kappa}$ is a basis of $K_Q$. Since $D = \check{d} \cdot \check{d}$, thus $K_D = \ker \check{d}$, and thus $\text{codim } K_D = 1$. Also, since $K_Q$ has a set of non-negative vectors as a basis, one has that $K_Q \not\subset K_D$, and thus $\dim K_Q \cap K_D = \dim K_Q - 1$. By Bézout’s Theorem $\check{d} := \check{d} \cdot \check{d}_\lambda$ is independent of $\lambda$ and hence $\{\tilde{u}_\lambda - \tilde{u}_\kappa\}_{\lambda=1,\ldots,\kappa-1}$ is a basis of $K_Q \cap K_D$.

Consider $N$, the matrix whose rows are the family of vectors $\{\tilde{u}_\lambda - \tilde{u}_\kappa\}_{\lambda=1,\ldots,\kappa-1}$. As we have seen before, the rows of $A$ must be a linear combination of the rows of $N$.

Let us construct the family of vectors $w := \{w_i\}_{i \in r^{(n)}}$ as follows:

- $w_i = 0$ if $i \in r^{(n)}$ is a trivial divisor.
- Otherwise, $w_i$ is equal to the corresponding column of $N$.

**Lemma 3.19.** The set of vectors $w$ is a $(\kappa - 2)$-admissible family for $\hat{W}$.

**Proof.** Since $\hat{W}$ is a normal crossing ACC, the admissibility conditions are verified if whenever two components, say $i$ and $j$ intersect, then $w_i || w_j$. 
If one of them, say $i$, is a trivial divisor, then there is nothing to prove, since $w_i = 0$. If neither of them are trivial divisors, then $i, j \in F_\lambda$ for some $\lambda$ in the partition $\mathcal{F}$. Therefore $v_i$ (and also $v_j$) must be proportional to $e_\lambda$ (the obvious vector of the canonical basis) when $\lambda \neq \kappa$ and to $(-1, \ldots, -1)$ if $\lambda = \kappa$. Thus the result follows.

All this shows that rank $N = \dim(K_Q \cap K_D)$, and the columns of $N$ are multiples of the following $\kappa$ vectors: the $\kappa - 1$ vectors $e_\lambda$ of the canonical basis of $\mathbb{Z}^{\kappa - 1}$ and the vector $e_\kappa := (-1, \ldots, -1)$. There is essentially only one linear relation among them, which is $\sum_{\lambda = 1}^\kappa e_\lambda = 0$. Moreover, if $i \in F_\lambda$, then $v_i = m_i \cdot e_\lambda$, where $m_i$ is a positive rational number. Note that, without loss of generality, after multiplication by a natural number (for instance the least common multiple of the denominators) one can assume that the $m_i$'s are integer numbers.

Let us show that $\mathcal{F} := \mathcal{F} \cap r$ and $\tilde{m} := (m_i)_{i \in r}$ defines a combinatorial pencil of $\kappa$ fibers. In order to do so we need to check both properties in Definition 2.4. Let $P \in S$ not satisfying Property 2.4(2a) and let $\tilde{m} := (m_i)_{i \in r}$ defines a combinatorial pencil of $\kappa$ fibers. In order to do so we need to check both properties in Definition 2.4. Let $P \in S$ not satisfying Property 2.4(2a) and let $\tilde{m} := (m_i)_{i \in r}$ defines a combinatorial pencil of $\kappa$ fibers. In order to do so we need to check both properties in Definition 2.4.
component containing \( \gamma \). Note that,
\[
\begin{cases}
\mu_Q(h_1, \alpha, F + \beta, G) > \mu_Q(h_1, F) = \mu_Q(h_1, G) & \text{if } Q = P \\
\mu_Q(h_1, \alpha, F + \beta, G) \geq \mu_Q(h_1, F) = \mu_Q(h_1, G) & \text{otherwise.}
\end{cases}
\]
Therefore, by Bézout, \( d \cdot \deg h_1 = \sum_{Q \in S} \mu_Q(h_1, \alpha, F + \beta, G) \geq \sum_{Q \in S} \mu_Q(h_1, F) = d \cdot \deg h_1 \), which implies that \( \alpha, F + \beta, G \) is a multiple of \( h_1 \). Consider now a resolution of the pencil, obtained by blowing up the base points \( \pi : X \to \mathbb{P}^2 \). The morphism \( f : X \to \mathbb{P}^1 \) is now well defined on the rational surface \( X \) and if \( P \notin S \) then \( f(\pi^{-1}(P)) := [F(P) : G(P)] \). By hypothesis, the preimage of \( V(H) \) outside the dicritical divisors is connected and hence \( f \) is constant on the strict transform of \( V(H) \), for any \( Q \in \pi^{-1}(V(H)) \) one has that \( f(Q) = [-\beta : \alpha] \in \mathbb{P}^1 \). On the other hand we know that for any \( h_1 \) and \( h_2 \) components of \( H \) one has that \( \alpha, F + \beta, G = h_1 u_1 \) and \( \alpha, F + \beta, G = h_2 u_2 \).

Consider now \( P_1 \) (resp. \( P_2 \)) a regular point of \( V(h_1) \setminus S \) (resp. \( V(h_2) \setminus S \)) and \( Q_i := \pi^{-1}(P_i) \). By the remarks in the previous paragraph, \( f(Q_i) := [F(P_i) : G(P_i)] = [-\beta : \alpha] \). Therefore \( [-\beta_1 : \alpha] = [-\beta_2 : \alpha] = [-\beta : \alpha] \), and hence \( \alpha, F + \beta, G = H' \), where \( H' = h_1^{\mu_1}, h_2^{\mu_2}, \ldots, h_r^{\mu_r} \). We will denote this by \( H' \subseteq (F, G) \). Let us denote by \( Q_1 \) (resp. \( Q_2 \)) the intersection matrix associated with a resolution of \( F \cdot G \cdot H \) (resp. \( F' \cdot G' \cdot H' \cdot K \)) that dominates both. According to Proposition 3.5, \( Q_1 \) and \( Q_2 \) have the following form
\[
Q_1 = \begin{bmatrix} Q_F & 0 & 0 \\ 0 & Q_G & 0 \\ 0 & 0 & Q_H \end{bmatrix}, \quad Q_2 = \begin{bmatrix} Q_F & 0 & 0 & 0 \\ 0 & Q_G & 0 & 0 \\ 0 & 0 & Q_H & M \\ 0 & 0 & M & Q_K \end{bmatrix}
\]
where \( Q_H = Q_{H'} \). Note that the box \( Q_H \) has a kernel of dimension 1 and hence \( M = 0 \), or else, the box \( \tilde{Q} := \begin{bmatrix} Q_H & M \\ M & Q_K \end{bmatrix} \) would have a vector of type \((v_H, 0) \in \ker \tilde{Q} \), which contradicts \( \tilde{Q} \) being of Affine type. Therefore we can assume that \( M = 0 \), \( H' = (H'')^q, H = (H'')^p \), and
\[
Q_2 = \begin{bmatrix} Q_F & 0 & 0 & 0 \\ 0 & Q_G & 0 & 0 \\ 0 & 0 & Q_H & 0 \\ 0 & 0 & 0 & Q_K \end{bmatrix}
\]
which implies that the preimage of \( H'' \) and \( K \) by the resolution \( \pi \) are disconnected outside the dicritical divisors. By the algebraic Stein Factorization Theorem one can find a refinement of the pencil \((F, G, H', K')\) into a pencil \( (\tilde{F}, \tilde{G}, \tilde{H}) \). Since the original pencil is primitive, one has that \( \tilde{F}^{n_F} = F, \tilde{G}^{n_G} = G, \text{ and } \tilde{H}^{n_{H''}} = H' = (H'')^q \), that is, there exists \( n_{H''} = \frac{np}{q} \in \mathbb{N} \) such that \( \tilde{H}^{n_{H''}} = H' \). By the hypothesis on degrees, this implies that \( n_F = n_G = n_{H''} \). Since \( \gcd(n_F, n_G, n_{H''}) = 1 \) by hypothesis, then \( n_F = n_G = n_{H''} = 1 \). Thus, \( H = H' \) and therefore \( \alpha, F + \beta, G = H \).

\[ \square \]

As an immediate corollary one has the following.
Corollary 4.2. Let \( F, G, H \) be three homogeneous polynomials of the same degree in three variables such that their zero sets \( V(F), V(G), \) and \( V(H) \) are three irreducible curves with no common components. If \( V(F) \cap V(G) = V(F) \cap V(H) = V(G) \cap V(H) = \{ P \} \) and \( F, G, \) and \( H \) are locally irreducible at \( P \), then \( H = \alpha F + \beta G \).

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