Nonet meson properties in Nambu–Jona-Lasinio model with dimensional regularization at finite temperature and chemical potential

T. Inagaki  
Information Media Center, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8521, Japan

D. Kimura  
Faculty of Education, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8524, Japan

H. Kohyama  
Department of Physics, Chung-Yuan Christian University, Chung-Li 32023, Taiwan

A. Kvinikhidze  
A. Razmadze Mathematical Institute of Georgian Academy of Sciences,  
M. Alexidze Str. 1, 380093 Tbilisi, Georgia

(Dated: October 27, 2011)

The nonet meson properties are studied in the Nambu–Jona-Lasinio model at finite temperature and chemical potential using dimensional regularization. This study leads to the reasonable description which is mainly similar to one obtained in the model with the cutoff regularization. However, remarkable differences between the two regularizations are observed in the behavior of the chiral phase transition at finite chemical potential.

PACS numbers: 11.10.Wx, 11.30.Qc, 12.39.-x

I. INTRODUCTION

Mesons are composed of quarks, interacting with each other through the exchange of gluons. Under the low temperature and density conditions, quarks are confined in hadrons and therefore are not observed as free particles, which makes their investigation challenging. The phenomenon of confinement is closely related to the dynamical chiral symmetry breaking. While the chiral symmetry broken in vacuum is expected to be restored at high temperature and density, it is also expected that the properties of mesons are affected by this chiral phase transition. Thus the investigation of meson properties at finite temperature and density is an important part of hadron physics.

Such investigation unfortunately cannot be pursued directly in the framework of quantum chromodynamics (QCD), the ultimate theory of strong interaction, due to its large coupling constant at low energy scale and thereby the necessity of dealing non-perturbatively with its notoriously complicated structure. The investigations are usually performed by using the effective models which share the same symmetry properties with QCD such as the Nambu–Jona-Lasinio (NJL) model [1], the linear sigma model [2], or by using the discretized version of QCD, i.e., the lattice QCD [3].

We employ, in this paper, the NJL model to study the nonet meson properties (for reviews on the model, see e.g., [4–7]). It is not renormalizable therefore one has to introduce the regularization procedure to handle the divergences appearing in loop integrals. There are several regularization schemes: the three- and four-momentum cutoff regularization, the dimensional regularization, the Pauli-Villars methods, and so on. We shall use the dimensional regularization (DR) in this paper. In the DR method the divergent fermion loop integrals which are the functions of the space-time dimensions are regularized by applying an analytic continuation in the dimensions variable. Many works are devoted to the studies based on it (see e.g., [8–12]).

We believe that a regularization procedure is an important dynamical part of the NJL model, it gives interaction between quarks the size and shapes it, which otherwise is point-like in the traditionally used leading order approximation of the $1/N_c$ expansion. That is why it is important to chose a proper regularization. Dimensional and cutoff regularizations can be considered as corresponding to the same size but different shape of the interaction. The fact that some physical results depend on the regularization way implies that the corresponding physics is sensitive (probes) not only to the size of the interaction but to its shape as well. If so the regularization dependent properties would be determined by higher energy (shorter distances) than those which do not depend on the regularization way.

The purpose of this paper is to study the nonet meson properties such as masses and decay constants in the three-flavor NJL model with the DR at finite temperature $T$ and chemical potential $\mu$. To this end one has to firstly determine the model parameters so that the model describes the physical ingredients properly. This has been done in the paper [13], where the present authors have found that the various parameter sets reproduce the meson properties nicely at $T = 0$ and $\mu = 0$. As our goal is to study meson properties at finite $T$ and $\mu$, the paper is the straightforward extension of the previous work [13].

This paper is organized as follows: In Sec. [11] the three-
flavor NJL model and its parameters are introduced. In Sec. III we discuss the constituent quark masses and chiral condensates which are derived through solving the gap equations. The results on the meson properties and topological susceptibility are shown in Sec. IV and V. Sec. VI is devoted to the discussion of the critical temperature. Summary and discussions are given in Sec. VII. The required calculations of the chiral condensates and meson properties are aligned in App. A and B. The fitted parameters are shown in App. C.

II. NJL MODEL WITH $U_A(1)$ ANOMALY

A. The model

The Lagrangian of our three-flavor NJL model is

$$\mathcal{L}_{\text{NJL}} = \sum_{i,j} \bar{q}_i (i\partial - \hat{m}) q_j + \mathcal{L}_4 + \mathcal{L}_6,$$  \hspace{1cm} (1)

where

$$\mathcal{L}_4 = G \sum_{a=0}^{2} \left[ \left( \sum_{i,j} \bar{q}_i \lambda_a q_j \right)^2 + \left( \sum_{i,j} \bar{q}_i i\gamma_5 \lambda_a q_j \right)^2 \right], \hspace{1cm} (2)$$

$$\mathcal{L}_6 = -K \left[ \det \bar{q}_i (1 - \gamma_5) q_j + \text{h.c.} \right]. \hspace{1cm} (3)$$

Here the subscripts $i,j$ represent the flavor indices, $\hat{m}$ is the current quark mass matrix diag$(m_u, m_d, m_s)$, and $\lambda_a$ are the Gell-Mann matrices in flavor space with $\lambda_0 = \sqrt{2/3} \cdot 1$. $\mathcal{L}_4$ and $\mathcal{L}_6$ are the four- and six-fermion interactions with the effective coupling constants $G$ and $K$. $\mathcal{L}_6$ is introduced to break the $U_A(1)$ symmetry which is not realized in the real world. The six-quark vertex in the determinant form was firstly discussed by Kobayashi and Maskawa [14] and later derived by ‘t Hooft as an instanton-induced quark interaction [15], so this vertex is called Kobayashi-Maskawa-‘t Hooft term.

To study the thermal system we evaluate the thermodynamic potential $\Omega = -\ln Z/V \beta$, with the partition function $Z$, the volume of the system $V$ and the inverse temperature $\beta(=1/T)$. Here we use the imaginary time formalism to treat the system at finite $T$ and $\mu$. Applying the mean-field approximation in the DR scheme, we obtain the following thermodynamic potential:

$$\Omega = 2G(\phi_u^2 + \phi_d^2 + \phi_s^2) - 4K \phi_u \phi_d \phi_s$$

$$- \frac{D^{1/2}}{2} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \left[ E_u + E_d + E_s \right]$$

$$- \frac{D^{1/2}}{2} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \sum_{i,\pm} \ln \left[ 1 + e^{-\beta E_i^{\pm}} \right], \hspace{1cm} (4)$$

where $\phi_i \equiv (\bar{i}i)$ represent the chiral condensates and $N_c(=3)$ is the number of colors. $E_i = \sqrt{p^2 + m_i^2}$ is the energy of the quasi-particle, and $E_i^{\pm} = E_i \pm \mu$, with the constituent quark masses $m_i^\pm$ and a chemical potential $\mu$. In this scheme we regularize the divergent integrals by performing the analytic continuation of the space-time dimension $D$ to a value less than four, then the integral can be written as

$$\int \frac{d^{D-1}p}{(2\pi)^{D-1}} = \frac{2(4\pi)^{-(D-1)/2}}{\Gamma[1 - (D - 1)/2]} M_0^{D-4} \int_0^\infty dp \, p^{D-2}, \hspace{1cm} (5)$$

where $M_0$ is a renormalization scale. One can switch to cutoff regularization through the replacement

$$\int \frac{d^{D-1}p}{(2\pi)^{D-1}} \to \frac{1}{2\pi^2} \int_0^\Lambda dp \, p^2. \hspace{1cm} (6)$$

The chiral condensates $\phi_i$, which are the order parameters of the model are evaluated through solving the gap equations, $\partial \Omega/\partial \phi_i = 0$. After some algebra, one arrive at the following self-consistent equations,

$$m_i^* = m_i - 4G \phi_i + 2K \delta_{ij} \phi_j, \hspace{1cm} (i \neq j \neq k) \hspace{1cm} (7)$$

where

$$\phi_i = \frac{1}{\beta} \sum_{n=0}^{\infty} \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \text{tr} S^i(p), \hspace{1cm} (8)$$

$$S^i(p) = \frac{1}{p \cdot \gamma - (\omega_n - i\mu)\gamma_4 + m_i^* - ie^\dagger} \hspace{1cm} (9)$$

with $\omega_n = (\pi/\beta)(2n + 1)$. ($n = 0, \pm 1, \pm 2, \cdots$). The tr stands for the trace in spinor and color indices. The explicit form of the integral is presented in the App. A.

To evaluate meson properties, one needs to carry out further calculations as discussed in [13] where only the case of $T, \mu = 0$ is presented. We carry out the corresponding calculations for the case of finite $T$ and $\mu$ in the App. B.

B. Model parameters

In the DR, the three-flavor NJL model has seven parameters:

- current quark masses $m_u, m_d, m_s$
- four-point coupling $G$
- six-point coupling $K$
- dimensions $D$
- renormalization scale $M_0$.

Since the mass difference between $m_u$ and $m_d$ is small as compared to one between $m_s$ and $m_u$ (or $m_d$), we take the isospin limit, $m_d = m_u$, and test several values, $m_u = 3, 4, 5, 5.5, 6$ MeV. To fix the other parameters, we usually employ the following physical observables [16]:

$$m_\pi = 138 \text{MeV}, \hspace{0.5cm} f_\pi = 92 \text{MeV}, \hspace{0.5cm} m_K = 495 \text{MeV}, \hspace{0.5cm} m_{\eta'} = 958 \text{MeV}. \hspace{1cm} (9)$$

One more physical quantity is needed to fix all seven above listed parameters. Possible candidates are the alternative meson properties such as the $\eta$ meson mass, $m_\eta$, topological susceptibility, $\chi$, kaon decay constant, $f_K$, etc. We choose the following three cases:
Case $m_\eta, m_\pi, m_K, f_\pi, m_{d'}, m_\eta = 548\text{MeV}$
Case $\chi_{170} m_\pi, m_K, f_\pi, m_{d'}, \chi^{1/4} = 170\text{MeV}$
Case $\chi_{179} m_\pi, m_K, f_\pi, m_{d'}, \chi^{1/4} = 179\text{MeV}$

As mentioned in the introduction, the parameters are fixed at $T, \mu = 0$ in [13]; we show the complete sets of them in the App. C. We do not consider the $T$ and $\mu$ dependence of these parameters in this paper. In App. C we also show the sets in the three-momentum sharp cutoff case to make comparison of the two different regularization schemes. In the cutoff regularization the six parameters, $m_u, m_d(= m_u), m_s, G, K$ and cutoff scale $\Lambda$, are fixed by using the four physical observables and choosing several values of $m_u$.

III. CONSTITUENT QUARK MASSES

Before going through the meson properties, let us discuss the results for the constituent quark masses $m^*_i$. This is important because the constituent quark masses determine the scale of the system, and they play a crucial role in the studies of the transition temperature and chemical potential.

A. $m^*$ in the DR scheme

![Graph 1](image1)

**FIG. 1:** $m^*_u$ in the Case $m_\eta$ with $m_u = 3\text{MeV}$. (a) finite $T$ and $\mu = 0$. (b) finite $\mu$ and $T = 10\text{MeV}$.

![Graph 2](image2)

**FIG. 2:** $m^*_u$ in the Case $m_\eta$ with various $m_u$. (a) finite $T$ and $\mu = 0$. (b) finite $\mu$ and $T = 10\text{MeV}$.

In Fig. 1 we display the numerical results for the constituent quark masses which are obtained by solving the gap equations Eq. (7) as explained in the previous section. We note that the absolute values of $m^*_u$ and $m^*_s$ gradually decrease with increasing temperature, while at low temperature they do not depend on the chemical potential up to its critical value where they suddenly fall. This is a clear signal of the first order phase transition.

Similar pictures correspond to the different values of $m_u$ as it is shown in Fig. 2 where the dependence of $m^*_u$ on $T$ and $\mu$ is given in the Case $m_\eta$ for fixed values of $m_u = 3, 4, 5, 5.5$ and $6\text{MeV}$. These pictures again indicate the crossover transition for low $\mu$ and the first order phase transition for low $T$. Thus the behavior of the first order and crossover transitions holds for the different values of $m_u$ in the Case $m_\eta$.

In Fig. 4 there are shown $m^*_u$ for different values of $m_u$ as the functions of $T$ or $\mu$ in the Case $m_\eta$. According Eq. (7), due to the large contribution of $m_u$, the absolute value of $m^*_u$ is larger than that of $m^*_s$. We see that the qualitative behavior of $m^*_u$ as a function of $T$ (at fixed low $\mu$) is similar to that of $m^*_s$. Yet this is not the case with the $\mu$ dependence of $m^*_u$ at fixed low $T$ where there appear two gaps (see lower panel). The first gap comes from the effect of $m_\eta^*$, the second one comes from that of $m_\eta^*$. We noted that the critical chemical potential is less than the constituent quark mass at $T = 0$ in the DR for
FIG. 3: $m_u^*$ in the Case $m_u$ with various $m_u$. (a) finite $T$ and $\mu = 0$. (b) finite $\mu$ and $T = 10\text{MeV}$.

$2 \leq D < 3$ at the chiral limit $\eta$.

Other three parameter sets, namely Cases $\chi_{170}$, $\chi_{179}$ and $\chi_{179}^{HD}$, lead qualitatively to similar results. However, it is worth mentioning that the Case $\chi_{179}^{HD}$, although reproduces proper values for the meson properties discussed in [13], shows quite different quantitative behavior. The constituent quark masses are one order of magnitude larger than the ones in the other parameter sets. This leads to unphysical results for the chiral phase transition: the transition temperature becomes of the order of 1GeV which is much larger than the physically expected value. This may indicate that the parameters of the Case $\chi_{179}^{HD}$ are not suitable to describe the chiral dynamics in this model. To explicitly compare these parameter sets, we align the numerical results for the constituent quark mass, $m_u^*$, in Tab. I. There one can clearly see that the values in the Case $\chi_{179}^{HD}$ are considerably larger than the ones in the other parameter sets. Since the model predictions in the Cases $\chi_{170}$ and $\chi_{179}$ are almost the same due to the similar values of $m_u^*$ and the model parameters, we shall only show the results in the Case $\chi_{170}$. In what follows, therefore, we will focus on the Cases $m_u^{LD}$, $m_u$ and $\chi_{170}$.

TABLE I: Constituent quark mass $m_u^*$ [MeV].

| $m_u$ | $m_u^{LD}$ | $m_u$ | $\chi_{170}$ | $\chi_{179}$ | $\chi_{179}^{HD}$ |
|-------|------------|-------|---------------|---------------|-------------------|
| 3.0   | −467       | −597  | −453          | −455          | −1942             |
| 4.0   | −460       | −623  | −439          | −442          | −2260             |
| 5.0   | −453       | −662  | −423          | −427          | −          |
| 5.5   | −679       | −415  | −420          | −          |
| 6.0   | −696       | −408  | −412          | −          |

FIG. 4: $m_u^*$ and $m_s^*$ in the case of cutoff for $m_u = 5.5\text{MeV}$. (a) finite $T$ and $\mu = 0$. (b) finite $\mu$ and $T = 10\text{MeV}$.

B. $m^*$ in the cutoff regularization scheme

Here we present the results obtained in the widely used sharp cutoff scheme to make comparison with our results of the DR.

The numerical results for the constituent quark masses $m_u^*$ and $m_s^*$ for the case $m_u = 5.5\text{MeV}$ are shown in Fig. 4. It is found that the $T$ and $\mu$ dependences are similar to the DR case. Fig. 5 shows qualitative difference between the cases of $m_u = 5.5\text{MeV}$, and, $m_u = 3, 4, 5\text{MeV}$. $\mu$ dependence of the constituent quark mass $m_u^*$ has discontinuity in the case of $m_u = 5.5\text{MeV}$, while $m_s^*$ decreases continuously in the cases of $m_u = 3, 4, 5\text{MeV}$. Thus, we have the phase transition of the first order in the case of $m_u = 5.5\text{MeV}$ and crossover in the other cases. This is a critical qualitative difference between the dimensional and cutoff regularizations.

Fig. 6 displays the corresponding results for $m_s^*$. We
FIG. 5: $m_\pi^*$ in the case of cutoff. (a) finite $T$ and $\mu = 0$. (b) finite $\mu$ and $T = 10\text{MeV}$.

note that the $T$ dependence of $m_\pi^*$ resembles each other in the two regularizations, which is seen in Figs. 3(a) and 3(b). However, as seen from Figs. 3(b) and 3(b), the $\mu$ dependence of $m_\pi^*$ is different. The two gaps in the cutoff regularization are smaller and smoother than those in the dimensional scheme.

IV. MESON PROPERTIES

In this section, we will show the meson properties, $\{m_\pi, f_\pi, m_K, m_\eta, m_{\eta'}\}$, in the two regularizations. We present the Cases $m_\eta$ and $\chi_{170}$ with $m_u = 5.5\text{MeV}$, and $m_u^\text{LD}$ with $m_u = 3\text{MeV}$. The reason of these choices is that the case with $m_u = 5.5\text{MeV}$ is frequently used value in the cutoff regularization [6, 17]. As the Case $m_u^\text{LD}$ does not have solutions at $m_u = 5.5\text{MeV}$, so we select the $m_u = 3\text{MeV}$ case in which the critical temperature of the chiral phase transition is close to empirical data (see, Sec. VI).

A. Meson properties in DR

Here the numerical results of the meson properties are presented for the Cases $\chi_{170}$, $m_\eta$ and $m_\eta^\text{LD}$ with some discussion on their behavior.

Fig. 6 shows the results of the meson properties for (a) finite $T$ and $\mu = 0$, and (b) finite $\mu$ and $T = 10\text{MeV}$ in the Case $\chi_{170}$ with $m_u = 5.5\text{MeV}$. As one can observe from the upper panel, $m_\pi$ and $m_\eta$ stay almost constant until they cross $2|m_u^*|$, then suddenly jump. In the similar manner, $m_K$ is nearly constant below the value $|m_\pi^* + m_\eta^*|$, and it enhances when $m_K$ becomes $|m_\pi^* + m_\eta^*|$. $f_\pi$ decreases monotonously and becomes negligibly small after $m_\pi$ exceeds $2|m_u^*|$. At low $T$ the dependence of meson properties on $\mu$ has sharp discontinuities as seen in Fig. 7(b), where the masses $m_\pi$, $m_K$, $m_\eta$ suddenly get larger and $m_{\eta'}$, $f_\pi$ falls off. This sharp discontinuities come from the discontinuous change of the chiral condensate, which corresponds to the strong first order phase transition. On the other hand in the upper panel the meson masses also show jumps as functions of $T$, although the chiral condensates change continuously with $T$. This comes from the singular behavior which appears particularly in the lower dimensions $D < 4$. Below, we will discuss the origin of these discontinuities through the analysis of the solution for the pion state.

In Fig. 8 the function $F_\pi(k^0) = \text{Re}[1 - 2K_3\Pi_\pi(k^0)]$ is plotted which determines the pion mass through the condition $F_\pi(k^0) = 0$ (see App. III). Note that all the lines diverge at $k^0 = 2|m_u^*|$. Due to this divergent behavior of $F_\pi(k^0)$, there is a jump in the dependence of the pion mass on $T$. Namely, we find the unique solution around
finite three candidates for the solution at $k_0 = 0$.

We select the larger value, $k_0 \simeq 330$ MeV, because the pion should acquire a larger mass at high $T$ where the chiral symmetry is expected to be restored. This is the reason why the discontinuous change of $m_\pi$ at some $T$, is not originated from the sudden change of the chiral condensate $\phi_u$. In App. [B1] we give the technical details on the divergence in $F_\pi$.

In Figs. [9] and [10], we show the results for the Case $m_\eta$ and $m_\eta^{LD}$. We observe the similar pictures in these two cases; smooth dependence on $T$ at zero $\mu$, and similar discontinuities in the dependence on $\mu$ at low $T$. As is seen in Figs. [9] b) and [10] b), there are two discontinuities. The first discontinuity comes from the effect of the gap for $m_\eta^*$, and the second discontinuity comes from the one for $m_\eta$. These gaps are confirmed in Figs. [2] b) and [3] b). It is interesting to note that $m_\eta^*$ survives at high temperature region in Fig. [9] a). This behavior may be attributed to the fact that $m_\eta^*$ is smaller than the threshold $2|m_\eta^*|$ at $T = 0$.

We have also calculated meson properties for the cases of different $m_u$, and found that the differences between various cases are rather nominal, therefore we showed the results for the case of a particular $m_u$.

B. Meson properties in cutoff regularization

Here we present the results for the meson properties in the cutoff case.

Fig. [11] shows the dependence of the meson properties on $T$ and $\mu$. The results in the two regularizations look similar to each other. The most noticeable difference is in that the curves are rather smooth in the cutoff case.
There are no discontinuities for low \( \mu \) in Fig. 11(a), because Eq. (11) does not include divergence for \( D = 4 \), and the solution on the real axis survives at high temperature region. In Fig. 11(b) the gap at \( \mu \approx 340 \text{MeV} \) is smaller than the gap of the DR case. This can be understood from the fact that the gap of \( m_\eta^* \) in the cutoff scheme is smaller than the one in the DR scheme (see, Figs. 2(b), 5(b)).

V. TOPOLOGICAL SUSCEPTIBILITY

The topological susceptibility \( \chi \) is an important quantity because it is intimately related to the chiral and \( U_\chi(1) \) symmetry of QCD. We show the results of \( \chi \) in the dimensional and cutoff regularizations in this section.

Figs. 12 and 13 display the curves for the topological susceptibility \( \chi \) as the functions of \( T \) or \( \mu \) which resemble the curves for \( m_\eta^* \) in Figs. 2 and 5 respectively. Thus we confirm that the topological susceptibility is influenced indeed by the chiral symmetry breaking.

VI. CRITICAL TEMPERATURE

We have seen that the chiral condensate and the topological susceptibility always show discontinuous changes at low temperature in the DR, which indicates the first order phase transition. The tendency remains in all cases with different parameter sets. On the other hand, the order of transition is different for the different parameter sets in the cutoff case as confirmed in Fig. 5(b) and 13(b). This may indicate that the transition pattern crucially depends on the regularization procedures. Therefore it is interesting to study the values of the critical temperature \( T_c \) for various cases both in dimensional and cutoff regularizations. Here we focus on \( T_c \) at \( \mu = 0 \), and make the comparison among the two regularization and the lattice QCD simulation.

There are several ways to define the critical temperatures. In this paper, we shall employ two definitions given by the maxima of

\[
\frac{\partial \phi_u}{\partial T}, \quad \text{and} \quad -\frac{\partial \chi^{1/4}}{\partial T}.
\]

The resulting critical temperatures will be denoted by, \( T_c^{(\phi)} \) and \( T_c^{(\chi)} \), respectively.

In Tab. 11 the numerical results for the critical temperature are aligned for various parameter sets. In the Case \( m_\Lambda^{LD} \) with \( m_u = 4 \) and 5MeV, we can not find a physically meaningful behavior for the phase transition; the absolute value of \( m_\eta^* \) becomes larger with increasing \( T \). That is why these cases are marked by “none” in the table.

Note that the values of \( T_c \) in the DR except for the
FIG. 12: Case $m_\eta$: Topological susceptibility $\chi^{1/4}$. (a) finite $T$ and $\mu = 0$. (b) finite $\mu$ and $T = 10\text{MeV}$.

TABLE II: Critical temperature $T_c^{(\phi)}$.

| $m_u$ | $m_\eta^{LD}$ | $m_\eta$ | $\chi_{170}$ | Cutoff |
|-------|---------------|----------|---------------|--------|
| 3.0   | 184          | 304      | 253           | 146    |
| 4.0   | none         | 336      | 249           | 155    |
| 5.0   | none         | 374      | 243           | 170    |
| 5.5   | –            | 395      | 240           | 184    |
| 6.0   | –            | 419      | 237           | –      |

Case $m_\eta^{LD}$ are larger than ones in the cutoff case. The Case $m_\eta^{LD}$ at $m_u = 3\text{MeV}$ leads almost to the same value of $T_c$ as in the cutoff case at $m_u = 5.5\text{MeV}$ which is comparable with the result in the lattice QCD simulation, $150 - 200\text{MeV}$ \[18\]. We see that $T_c$ becomes larger with increasing $m_u$ in the Case $m_\eta$ while $T_c$ does not change drastically with $m_u$ in the Case $\chi_{170}$. This can be explained by the values of $m_\eta^*$.

VII. SUMMARY AND DISCUSSIONS

In this paper we studied the nonet meson characteristics, $m_\pi$, $f_\pi$, $m_K$, $m_\eta$, $m_\eta'$, and $\chi$ at finite temperature and chemical potential in the NJL model with DR. Many of meson properties show reasonable behavior, which is similar to one obtained in the model with the cutoff regularization.

We examined the behavior of the constituent quark masses, $m_u^*$ and $m_s^*$, in the two regularizations. $T$ dependence of the constituent quark masses at low $\mu$ does not depend essentially on the way of regularization. However, $\mu$ dependence of the constituent quark masses at low $T$ depends on the regularization way around the critical chemical potential. In the DR, the phase transition at low $T$ is always of the first order for various choices of $m_u$. This tendency is consistent with the current consensus about the transition to be of the first order in the chiral limit \[12\]. On the other hand in the cutoff case, the gap at the critical chemical potential becomes smaller with decreasing value of $m_u$, and it eventually vanishes resulting in crossover transition (see, Fig. 5). This happens due to the fact that the larger cutoff $\Lambda$ diminishes the
coupling strength which leads to the smoother change of the order parameters. Decreasing the coupling strength weakens nonperturbative effects. This explains the effect of the UV-cutoff which is also seen in the work [20] as the effect of the cutoff in the temporal direction.

Using the obtained values of the constituent quark masses, we evaluated meson properties at finite $T$ and/or $\mu$ in the two regularizations. $T$ dependence of the meson masses at low $\mu$ are similar to each other except around $m_\mu \simeq |m_\mu^* + m_\mu^t|$. Since the singularity of the pion self-energy $\Pi_\pi(k^2)$ appears at $k^2 = (m_\pi^* + m_\pi^t)^2$ for $D < 4$, the behavior of meson masses becomes discontinuous in the DR. $\mu$ dependence of the meson masses at low $T$ also does not essentially depend on the way of regularization except at the region of the critical chemical potential. Since the gap of $m_\eta^*$ in the DR case is larger than one in the cutoff case, the gap of the meson mass becomes large. The behavior of $f_\pi$ as the functions of $T$ or $\mu$ in the DR case resembles one in the cutoff case. We also evaluated the topological susceptibility $\chi$ as the functions of $T$ or $\mu$ in the two regularizations. The behavior of $\chi$ is similar to one of $m_\eta^*$.

In the DR, we examined the three parameter set cases: $\chi_{170}$, $m_\eta$ and $m_\eta^{LD}$. Parameters of each set are fixed by fitting physical quantities at $T, \mu = 0$ [13]. Most of the properties concerning meson masses and decay constants are similar to each other in the three cases. However, the large difference is seen in the values of the critical temperature and of the critical chemical potential. One of the sources of this difference may be the strength of couplings, i.e., increasing the values of the coupling strength decreases the values of the critical temperature and the critical chemical potential. We actually calculated the critical temperatures, $T^{(c)}_{\pi, K}$ and $T^{(c)}_{\pi, \eta}$. In the Case $m_\eta$ at $m_\mu = 3$MeV, the critical temperature is almost the same as in the cutoff case at $m_\mu = 5.5$MeV which is comparable to the result in the lattice QCD simulation.

The lessons we learned in this paper are twofold: (i) similarities in the predicted meson properties and (ii) remarkable differences seen in the order of the chiral phase transition at finite chemical potential. Regarding the point (i), we have been convinced that the NJL model predictions do not depend drastically on whether one uses the dimensional or cutoff regularization procedure. This may be the regularization independent aspect of the NJL model, which is intriguing problem to be studied in more detail in future. As to the point (ii), the results of the paper suggest that the critical point in the phase diagram on $T - \mu$ plane shows different behavior. We think that it is interesting to study the behavior of the chiral phase transition and the phase diagrams of the model in more detail, especially in context of how the location of the critical point depends on the parameters in the different regularization schemes.

Acknowledgments

HK is supported by the grant NSC-99-2811-M-033-017 from National Science Council (NSC) of Taiwan.

Appendix A: Chiral condensate

By taking the trace with respect to the Dirac spinor indices in Eq. (3), we get the chiral condensate

$$\phi_i = tr \cdot m_i^\alpha \int \frac{dD-1}{(2\pi)^{D-1}} T \sum_{n=-\infty}^{\infty} \frac{-1}{\omega^2 + E^2_i},$$

(A1)

where $tr = N_c \cdot 2^{D/2}$. After performing the frequency summation, we arrive at the expression

$$\phi_i = tr \cdot m_i^\alpha \int \frac{dD-1}{(2\pi)^{D-1}} \frac{-1}{2E_i} \left[ 1 - \sum_{\pm} f(E_i^\pm) \right],$$

(A2)

with the Fermi-Dirac distribution $f(E)$ given by

$$f(E_i^\pm) = \frac{1}{1 + e^{\beta E_i^\pm}}.$$  

(A3)

Appendix B: Meson properties

Here we evaluate meson masses, decay constants and topological susceptibility. More detailed, self-contained calculations are presented in the review paper [5].

1. $\pi$ and $K$ masses

Meson masses are obtained through examining the pole structure of their propagators. Using the random-phase approximation and the $1/N_c$ expansion, we get the meson propagators of the form [5, 6]

$$\Delta_P(k^2) = \frac{2K_\alpha}{1 - 2K_\alpha \Pi_P(k^2)} + O(N_c^{-1}),$$

(B1)

where the index $P(= \pi, K)$ denotes the meson species and $\alpha$ labels the isospin channel. The pole position is determined by the equation

$$1 - 2K_\alpha \Pi_P(x^2) = 0$$

(B2)

where $x^2$ is regarded as the square of a meson mass, i.e., $m_\pi^2$ or $m_K^2$.

The effective couplings, $K_\alpha$, of mesons become

$$K_\pi \equiv G - \frac{1}{2} K \phi_s, \quad \text{for } \pi^0,$$

(B3)

$$K_0 \equiv G - \frac{1}{2} K \phi_d, \quad \text{for } K^0, \bar{K}^0.$$  

(B4)
The self-energy, $\Pi_P$, of the propagator is given by
\[
\Pi_P(k^2)\delta_{\alpha\beta} = \int \frac{d^Dp}{i(2\pi)^D} \text{tr} \left[ \gamma_5 T_\alpha S^I (p + k/2) \gamma_5 T_\beta S^J (p - k/2) \right],
\]  
(B5)
where the trace runs over flavor, spinor and color indices. The flavor $SU(3)$ matrices $T_\alpha$ are: $T_3 = \lambda_3$ for $\pi^0$, $T_6 = (\lambda_6 + i\lambda_7)/\sqrt{2}$ for $K^0$ and $T_\alpha^0$ for $\bar{K}^0$. Then the self-energy for $\pi^0$, $K^0$ and $\bar{K}^0$ can be written as
\[
\Pi_\pi(k^2 = m_\pi^2) = 2I^{\pi}_{uu}(k^2 = m_\pi^2),
\]
(B6)
\[
\Pi_K(k^2 = m_K^2) = 2I^{\bar{K}}_{uu}(k^2 = m_K^2),
\]
(B7)
where $\Pi_\pi^{ij}(k^2)$ is defined by
\[
\Pi_\pi^{ij}(k^2) = \int \frac{d^Dp}{i(2\pi)^D} \text{tr} \left[ \gamma_5 S^I (p + k/2) \gamma_5 S^J (p - k/2) \right] = \frac{1}{2} \left( \frac{\phi_i}{m_i^2} + \frac{\phi_j}{m_j^2} \right) + \frac{1}{2} \left( k^2 - (m_i^* - m_j^*)^2 \right),
\]
(B8)
with
\[
I_{ij} = \int \frac{d^Dp}{i(2\pi)^D} \frac{\text{tr}1}{(p^2 - m_i^*)^2} \left( 2E_i - 1 \sum_{s=\pm} S^{js}_{ij} f(E_i^s) \right)
\]
(B9)
The integral above becomes
\[
I_{ij} = \int \frac{d^{D-1}p}{i(2\pi)^{D-1}} \frac{\text{tr}1}{D_{ij}^0} \left\{ \sum_{s=\pm} \frac{1}{2E_i} \left[ 1 - \sum_{s=\pm} S^{js}_{ij} f(E_i^s) \right] \right\}
\]
(B10)
where $D_{ij}^\pm = (E_i \pm E_j)^2 - k^2$ and $S_{ij} = m_i^* - m_j^* - k^2$.

In practice we first solve the gap equations to get the chiral condensates $\phi_i$ and constituent quark masses $m_i^*$, then we scan the solution of Eq. (B12) in evaluating $m_\pi$ and $m_K$. In the actual numerical calculations, we choose the rest frame $k^\mu = (k^0, \mathbf{0})$.

It should be noted that $I_{ij}$ as a function of $k^0$ ($k = 0$) has a singularity. Since the pion self-energy $\Pi_\pi$ consists of the function $I_{uu}$, it is divergent at the singularity. $\Pi_\pi(k^2)$ has a cut in the region $k^0 \geq 2|m_\pi^*|$ on the real axis. One should move the other Riemann sheet to calculate $\Pi_\pi(k^2)$ at the cut. The vacuum part $I_{uu}^{(v)}$ is written as
\[
I_{uu}^{(v)} = \frac{N_c}{(2\pi)^D/2} \Gamma(2 - \frac{D}{2}) \int_0^1 dx L_{uu}^{D/2-2}(k^2),
\]
(B11)
where
\[
L_{uu}(k^2) = m_u^* - k^2x(1 - x).
\]
(B12)
The function $I_{uu}(k^2)$ is divergent at $k^2 = 4m_u^* - k^2 < 4$. Then $\Pi_\pi(k^2)$ has the singularity at $k^0 = 2|m_\pi^*|$ which is confirmed in Fig. [S] where $D = 2.47$ (Case $\chi_{170}$ with $m_u = 5.5$MeV). The divergence appears at the resonance position. It does not mean the existence of a bound state except for $k^0 = 2|m_\pi^*|$. In Fig. [S] three solutions are observed for $F_\pi = 0$ at $T = 271$ and 279MeV. Two of the solutions are located near the singularity at $k^0 = 2|m_\pi^*|$. We take the solution with the larger $k^0$ to express the pion mass. Thus the meson masses jump as functions of $T$ near the critical temperature.

As is shown in Fig. [S], the larger solution disappears for $T \gtrsim 279$MeV. In this case we determine the meson mass as $k^0$ at the maximum of $F_\pi$. On the contrary, the reliable solution always exists and the smaller two solutions are degenerate above the critical chemical potential at $T = 10$MeV.

2. $\pi$ and $K$ decay constants

Here, we display the equations determining $f_\pi$, the pion and kaon decay constants, with $P(=\pi, K)$.

\[
if_\pi f_P \delta_{\alpha\beta} = -M_0^{4-D} \int \frac{d^Dp}{(2\pi)^D} \text{tr} \left[ \gamma_\mu \gamma_5 T_\alpha \frac{\sqrt{2}}{2} g_{\nu\nu} (0) \gamma_5 T_\beta S^\nu \right],
\]
(B13)
where the renormalization scale $M_0$ is introduced so that the decay constant has a correct mass dimension, $\text{dim}(f_\pi) = 1$. The trace is with respect to the flavor, spinor and color indices. The meson-to-quark-quark coupling, $g_{\nu\nu}$, is defined by
\[
g_{\nu\nu}(k^2)^{-2} = M_0^{4-D} \frac{\partial \Pi_P(k^2)}{\partial k^2}.
\]
(B14)
Eq. (B13) can be rewritten in the following form
\[
f_\pi^\pm = -M_0^{4-D} g_{\nu\nu} I_\pi^\pm / k^2,
\]
(B15)
where $I_\pi = m_u^* I_{uu}(k^2)k_\mu$ and
\[
I_K = \text{tr}1 \int \frac{d^Dp}{i(2\pi)^D} \frac{(m_\pi^* - m_\pi^*)p_{\mu} - m_u^*k_\mu}{(p^2 - m_u^*)^2 [(p - k)^2 - m_u^*]^2}.
\]
By inserting the values of the constituent quark masses $m_u^*$ and the renormalization scale $M_0$, one can easily evaluate the decay constants.

3. $\eta$ and $\eta'$ mesons

The equation which determines $\eta$ meson masses becomes $2 \times 2$ matrix due to the well known $U_A(1)$ anomaly. In the leading order of the $1/N_c$ expansion, the propagator of the $\eta - \eta'$ system becomes
\[
\Delta^+(k^2) = 2K^+ [1 - 2K^+ \Pi(k^2)]^{-1},
\]
(B16)
where $K^+$ and $\Pi$ are given by

$$K^+ = \begin{pmatrix} K_{00} & K_{08} \\ K_{80} & K_{88} \end{pmatrix}, \quad (B17)$$

$$\Pi = \begin{pmatrix} \Pi_{00} & \Pi_{08} \\ \Pi_{80} & \Pi_{88} \end{pmatrix}, \quad (B18)$$

with

$$K_{00} = G + \frac{1}{3} K(\phi_s + 2\phi_u),$$
$$K_{88} = G + \frac{1}{6} K(\phi_s - 4\phi_u),$$
$$K_{08} = K_{80} = \frac{\sqrt{2}}{6} K(\phi_s - \phi_u),$$

and

$$\Pi_{00}(k^2) = \frac{2}{3} \left[ 2\Pi_{uu}^{00}(k^2) + \Pi_{ss}^{00}(k^2) \right],$$
$$\Pi_{88}(k^2) = \frac{2}{3} \left[ \Pi_{uu}^{00}(k^2) + 2\Pi_{ss}^{00}(k^2) \right],$$
$$\Pi_{08}(k^2) = \Pi_{80}(k^2) = \frac{2\sqrt{2}}{3} \left[ \Pi_{uu}^{00}(k^2) - \Pi_{ss}^{00}(k^2) \right].$$

The condition which determines the $\eta$ and $\eta'$ masses is,

$$\text{det}[1 - 2K^+ \Pi(x^2)] = 0. \quad (B19)$$

Using the constituent quark masses derived from the gap equations, we search for the appropriate solutions for $m_\eta$ and $m_{\eta'}$.

### 4. Topological susceptibility

The topological susceptibility $\chi$ is given by the correlation function between the topological charge densities $Q(x)$ at different points,

$$\chi = \int d^4x \langle 0 | TQ(x)Q(0) | 0 \rangle_{\text{connected}}, \quad (B20)$$

here the charge density is defined as

$$Q(x) \equiv \frac{g^2}{32\pi^2} F_{\mu \nu}^a F^{a \mu \nu} = 2K \text{Im} \left[ \text{det} \bar{q}(1 - \gamma_5)q \right], \quad (B21)$$

with $g$, the strong coupling constant of QCD, and $F_{\mu \nu}^a$, the gluon field strength. The explicit form of $\chi$ is given below [21],

$$\chi = -\frac{4K^2}{M_0^2} \phi_{\mu} \phi_{\mu} \left[ \phi_u \phi_s \left( \frac{2\phi_s}{m_u^2} + \frac{\phi_u}{m_u^2} \right) \right]$$
$$- \left\{ \frac{1}{\sqrt{6}} (2\phi_s + \phi_u) \left( \Pi_{00}(0), \Pi_{08}(0) \right) \right\} \Delta^+(0)$$
$$+ \left\{ \frac{1}{\sqrt{6}} (\phi_s - \phi_u) \left( \Pi_{08}(0), \Pi_{88}(0) \right) \right\} \Delta^+(0)$$
$$\times \left\{ \frac{1}{\sqrt{3}} (2\phi_s + \phi_u) \left( \Pi_{00}(0), \Pi_{08}(0) \right) \right\}$$
$$+ \left\{ \frac{1}{\sqrt{3}} (\phi_s - \phi_u) \left( \Pi_{08}(0), \Pi_{88}(0) \right) \right\}. \quad (B22)$$

After solving the gap equations, one can numerically calculate this quantity by inserting the values of $\phi_i$ and $m_i^*$.

### Appendix C: Parameter sets

Here we present the parameter sets obtained in [13]. In the Case $m_\eta$ and $\chi_{170}$, there exist two sets of parameters, and we distinguish between these cases by using the superscript LD (lower dimension) and HD (higher dimension).

In Tab. [IX] we also present the parameter sets of the Case Cutoff. In the cutoff regularization, the number of
TABLE VII: Case $\chi^{179}$

| $m_u$ | $m_s$ | $G$       | $K$       | $M_0$ | $D$  |
|-------|-------|-----------|-----------|-------|------|
| 3.0   | 78.0  | $-0.0170$ | $2.70 \times 10^{-7}$ | 120   | 2.28 |
| 4.0   | 106   | $-0.0144$ | $2.52 \times 10^{-7}$ | 115   | 2.36 |
| 5.0   | 136   | $-0.0120$ | $2.11 \times 10^{-7}$ | 111   | 2.44 |
| 5.5   | 152   | $-0.0109$ | $1.90 \times 10^{-7}$ | 110   | 2.47 |
| 6.0   | 168   | $-0.00905$| $1.70 \times 10^{-7}$ | 108   | 2.50 |

TABLE VIII: Case $\chi^{179}$

| $m_u$ | $m_s$ | $G$       | $K$       | $M_0$ | $D$  |
|-------|-------|-----------|-----------|-------|------|
| 3.0   | 74.8  | $-4.24 \times 10^{-5}$ | $1.38 \times 10^{-13}$ | 2.91  | 3.22 |
| 4.0   | 100   | $-6.91 \times 10^{-8}$ | $3.60 \times 10^{-19}$ | $1.15 \times 10^{-25}$ | 3.90 |

TABLE IX: Case Cutoff.

| $m_u$ | $m_s$ | $G\Lambda^2$ | $K\Lambda^5$ | $\Lambda$ |
|-------|-------|---------------|--------------|-----------|
| 3.0   | 89.5  | 1.55          | 8.34         | 960       |
| 4.0   | 110   | 1.60          | 8.38         | 797       |
| 5.0   | 128   | 1.71          | 8.77         | 682       |
| 5.5   | 136   | 1.81          | 9.17         | 630       |
| 5.87  | 139   | 2.09          | 10.1         | 580       |

parameters is five, and these are fixed by using the four physical observables $m_\pi$, $f_\pi$, $m_K$, $m_{\eta'}$ for several values of $m_u$.