A weak energy identity and the length of necks for a Sacks-Uhlenbeck $\alpha$-harmonic map sequence

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Abstract

Assume that $M$ is a closed surface and $N$ is a compact Riemannian manifold without boundary. Let $u_\alpha : M \to N$ be the critical point of $E_\alpha$ with $E_\alpha(u_\alpha) < C$. Assume $u_0$ is the weak limit of $u_\alpha$ in $W^{1,2}(M, N)$ and $x_1$ is the only blow-up point in $B_\sigma(x_1) \subset M$ with $n_0$ bubbles. Then, on a local coordinate system on $B_\sigma(x_1)$ which origin is $x_1$, we can find sequences $x_\alpha \to 0$, $\lambda_\alpha \to 0$ ($i = 1, \cdots, n_0$) s.t. $u_\alpha(x_\alpha + \lambda_\alpha x) \to v^i$, where $v^i$ are harmonic maps from $S^2$ to $N$. We define

$$\mu_i = \liminf_{\alpha \to 1} (\lambda_\alpha^{2-2\alpha}).$$

We will prove that

$$\lim_{\alpha \to 1} E_\alpha(u_\alpha, B_\sigma(x_1)) = E(u_0, B_\sigma(x_1)) + |B_\sigma(x_1)| + \sum_{j=1}^{n_0} \mu_j^2 E(v^j).$$

Further, when $n_0 = 1$, we define

$$\nu_1 = \liminf_{\alpha \to 1} (\lambda_\alpha^{1-\nu_1}),$$

then we have:

- If $\nu_1 = 1$, then $u_0(B_\sigma(x_1)) \cup v^1(S^2)$ is connected;
- If $1 < \nu_1 < +\infty$, then $u_0(B_\sigma(x_1))$ and $v^1(S^2)$ are connected by a geodesic with length

$$L = \sqrt{\frac{E(v^1)}{\pi}} \log \nu_1.$$

- If $\nu_1 = +\infty$, the neck contains at least one geodesic with infinite length.

We also give an example of neck which shows the neck contains at least one geodesic of infinite length.

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1 Introduction

Let $(M, g)$ be a smooth closed Riemann surface, and $(N, h) \subset \mathbb{R}^K$ be an $n$-dimensional smooth compact Riemannian submanifold. We always assume that $N \hookrightarrow \mathbb{R}^K$ is an isometric embedding and has no boundary.

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Let $W^{1,2}(M,N)$ denote the Sobolev space of $W^{1,2}$ maps from $M$ into $N$. If $u \in W^{1,2}(M,N)$, locally, we define the energy density $e(u)$ of $u$ at $x \in M$ by

$$e(u)(x) = |\nabla g_{u}|^2 = g^{ij}(x)h_{\alpha\beta}(u(x))\frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$ 

It is easy to check that

$$e(u) = \text{Trace}_g u^*h,$$

where $u^*h$ is the pull-back of the metric tensor $h$. Usually, the energy $E(u)$ of $u$ is defined by

$$E(u) = \int_M e(u) dV_g,$$

and the critical points of $E$ are called harmonic maps. We know that a harmonic map $u$ satisfies the following equation:

$$\tau(u) = \Delta u + A(u)(\nabla u, \nabla u) = 0,$$

where $A$ is the second fundamental form of $N$ in $\mathbb{R}^K$.

It is not easy to find a harmonic map, since $E$ does not satisfy the Palais-Smale condition when the dimensions of domain manifold $\dim(M) \geq 2$. Eells and Sampson first employed the heat flow method to approach the existence problems of harmonic maps and successfully deformed a map from a closed manifold into a manifold with nonpositive sectional curvature into a homotopic harmonic map. Concretely, they considered the heat flow for harmonic maps (or the negative gradient flow of the energy functional $E(u)$):

$$\frac{\partial u}{\partial t} = \tau_g(u).$$

If we can establish the global existence of the above flow with respect to the time variable $t$, or roughly speaking, the flow flows to infinity smoothly, then we are able to find a sequence $u_k = u(x, t_k)$ s.t. $t_k \rightarrow +\infty$ and $u_k$ converges to a harmonic map (see [E-S]).

As $\dim(M) = 2$, it is well-known that the energy functional is of conformal invariance and harmonic maps for this case are of special importance and interest. In fact, mathematicians pay more attention to this case. To prove the existence of harmonic maps from a closed surface Sacks and Uhlenbeck in their pioneering paper [S-U] employed a perturbed energy functional which satisfies the Palais-Smale condition, hence defined the so called $\alpha$-harmonic map to approximate the harmonic map. More precisely, for every $u \in W^{1,2\alpha}(M,N)$ Sacks and Uhlenbeck defined the so called $\alpha$-energy $E_{\alpha}$ as

$$E_{\alpha}(u) = \int_M (1 + |\nabla u|^2)^\alpha dV_g,$$

which can be regarded as a perturbation of energy $E$, and considered the $\alpha$-harmonic maps, i.e. the critical points of $E_{\alpha}$ in $W^{1,2\alpha}(M,N)$, which satisfy the following equation:

$$\Delta_g u_\alpha + (\alpha - 1) \frac{\nabla_g |\nabla g u_\alpha|^2}{1 + |\nabla g u_\alpha|^2} \nabla_g u_\alpha + A(u_\alpha)(du_\alpha, du_\alpha) = 0.$$ 

If there is a subsequence $u_k = u_{\alpha_k}$ which converges smoothly as $\alpha_k \rightarrow 1$, $u_{\alpha_k}$ will converge to a harmonic map.
Later, Struwe used the heat flow method of Eells and Sampson to approach the existence problems for harmonic maps from a closed surface and he obtained almost the same results as in \[SU\]. Chang showed the same results as in \[ST\] for the case where the domain manifold is a compact surface with smooth boundary (see \[C\]).

However, for both cases, the blow-up might happen. That is to say, we are only sure that the convergence is smooth away from finitely many points (which are called blow-up points) to a smooth harmonic map \(u_0\), which might be a trivial map. Around a blow-up point \(p\), the energy will concentrate, i.e., we will have

\[
\lim_{r \to 0} \liminf_{k \to +\infty} \int_{B_r(p)} |\nabla u_k|^2 dV_g > 0.
\]

And then, we can find sequences \(\lim_{k \to +\infty} x^i_k \to p, \lim_{k \to +\infty} \lambda^i_k \to 0, i = 1, \cdots, n_0, \) s.t.

\[
u_k(x^i_k + \lambda^i_k x) \to w^i \text{ in } C^{\alpha}_{\text{loc}}(\mathbb{R}^2 \setminus A^i),
\]

where all \(w^i\) are non-trivial harmonic maps from \(S^2\) to \(N\), and \(A^i\) is a finite set.

Then two problems occur. One is that if we have the energy identity, i.e.

\[
\lim_{k \to +\infty} \int_{B_\sigma} |\nabla u_k|^2 dV_g = \int_{B_\sigma} |\nabla u_0|^2 dV_g + \sum_{i=1}^{n_0} E(w^i).
\]

The other one is what the neck is if it exists?

When \(u_k = u(x, t_k)\) is a subsequence of a heat flow for two dimensional harmonic maps, the above two problems are deeply studied. The energy identities have been proved by Qing \[Q\] (in the case \(N = S^n\)) and Ding-Tian \[D-T\] in the general case. In \[Lin-W\], Lin-Wang gave another proof of the energy identity. For the neck, Qing-Tian \[Q-T\] proved that there is no neck if the blow-up happened at infinite time (Ding \[D\] proved a more general case), and Topping \[T\] gave a surprising example of heat flow blowing up at finite time s.t. the weak limit is not continuous.

Unexpectedly, the energy identity for an \(\alpha\)-harmonic sequence with bounded energy is still open. Now, many people believe that the methods used to solve the identity for heat flow, or more generally a sequence with tension fields \(\tau\) bounded in \(L^2\), are not powerful enough to solve the energy identity for an \(\alpha\)-harmonic map sequence. The reason lies in the identity (2.3) in this paper. For a sequence with tension fields \(\tau\) bounded in \(L^2\), (2.3) becomes

\[
\int_{\partial B_r} |\partial_r u_k|^2 ds_0 - \frac{1}{2} \int_{\partial B_r} |\nabla_0 u_k|^2 ds_0 = O(\int_{B_r} |\tau(u_k)||\nabla u_k|dV_g) + O(1).
\]

then the right side of the above identity is bounded. However, in (2.3), a very bad term

\[
\frac{\alpha - 1}{r} \int_{B_r} (1 + |\nabla u_\alpha|^2)^{\alpha-1} |\nabla u_\alpha|^2 dV_g
\]

appears.

The known energy identities for some special \(\alpha\)-harmonic sequences are usually obtained by methods which are completely different with the one of \[D-T\]. Now we would like to mention the following cases.

If \(\{u_\alpha\}\) is a sequence of minimizing \(\alpha\)-harmonic map, i.e. every \(u_\alpha\) is the minimizer of \(E_\alpha\), which belongs to the same homotopic class, Chen and Tian \[C-T\] proved that the necks consist
of some geodesics of finite length, and moreover this implies no loss of energy in necks for the sequence (see also [D-K]).

Another important case is the energy identity for a minimax sequence. We let $M$ be a compact Riemann surface, $A$ be a parameter manifold. Let $h_0 : M \times A \to N$ be continuous. Assume $H$ be the class of all maps homotopic to $h_0$, and

$$\beta_\alpha(H) = \inf_{h \in H} \sup_{t \in A} E_\alpha(h(\cdot,t)).$$

We can deduce from Jost’s result [J] that there is at least one sequence $u_{\alpha_k}$ which attained $\beta_{\alpha_k}(H)$ satisfies the energy identity as $\alpha_k \to 1$ (Also see [C-M] and [L]).

In this paper, we will adopt some methods and techniques in [D-T] and [D] to discuss the energy identity for an $\alpha$-harmonic sequence, especially the necks between the bubbles. However, we can not give a final proof on the energy identity for such Sacks-Uhlenbeck sequence, instead, we only show a weaker energy identity and give some observation on this subject. On the other hand, we exploit the details of the necks. Precisely we provide a new method to show that the necks converge to geodesics and obtain the formula on the length of the geodesics.

Now, we assume that $u_\alpha$ is a sequence of $\alpha$-harmonic maps from $(M, g)$ to $(N, h)$ with

$$E_\alpha(u_\alpha) < \Theta.$$

Then, by the theory of Sacks and Uhlenbeck, we are able to assume that there exists a sequence $\alpha_k \to 1$, s.t. $u_{\alpha_k}$ converges to a harmonic map $u_0 : M \to N$ smoothly away from a finite many points $\{x_i\}$ as $\alpha_k \to 1$. We assume that there are $n_0$ bubbles at the point $x_1$. Then we are able to assume that there are $x^{i}_{\alpha_k} \to x_1$ and $\lambda^{i}_{\alpha_k} \to 0$ for $j = 1, \cdots, n_0$, such that

$$v^{i}_{\alpha_k} = u_{\alpha_k}(x^{i}_{\alpha_k} + \lambda^{i}_{\alpha_k} x)$$

converge in $C^{k}_{\text{loc}}(\mathbb{R}^2 \setminus \{p_1, p_2, \cdots, p_{n_0}\})$ to non-trivial harmonic maps

$$v^{i} : S^2 \to N.$$

Moreover, we assume that one of the following holds:

\begin{itemize}
  \item \textbf{H1.} For any fixed $R$, $B_{R \lambda^{i}_{\alpha_k}}(x^{i}_{\alpha_k}) \cap B_{R \lambda^{j}_{\alpha_k}}(x^{j}_{\alpha_k}) = \emptyset$ whenever $(\alpha_k - 1)$ are sufficiently small.
  \item \textbf{H2.} $\frac{\lambda^{i}_{\alpha_k}}{\lambda^{j}_{\alpha_k}} + \frac{\lambda^{j}_{\alpha_k}}{\lambda^{i}_{\alpha_k}} \to +\infty$ as $\alpha_k \to 1$.
\end{itemize}

\textbf{Remark 1.} One is easy to check that if $(\lambda^{i}_{\alpha_k}, x^{i}_{\alpha_k})$ and $(\lambda^{j}_{\alpha_k}, x^{j}_{\alpha_k})$ do not satisfy \textbf{H1} and \textbf{H2}, then we can find subsequences of $\lambda^{i}_{\alpha_k}$, $x^{i}_{\alpha_k}$, $\lambda^{j}_{\alpha_k}$, $x^{j}_{\alpha_k}$ s.t. $\frac{\lambda^{j}_{\alpha_k}}{\lambda^{i}_{\alpha_k}} \to \lambda \in (0, \infty)$ and $\frac{x^{j}_{\alpha_k} - x^{i}_{\alpha_k}}{\lambda^{i}_{\alpha_k}} \to a \in \mathbb{R}^2$. Since

$$u_{\alpha_k}(x^{i}_{\alpha_k} + \lambda^{i}_{\alpha_k} x) = u_{\alpha_k}(x^{j}_{\alpha_k} + \lambda^{j}_{\alpha_k} \left( \frac{x^{j}_{\alpha_k} - x^{i}_{\alpha_k}}{\lambda^{i}_{\alpha_k}} + \frac{\lambda^{i}_{\alpha_k}}{\lambda^{j}_{\alpha_k}} x \right)),$$

we have

$$v^{i}(x) = v^{j}(a + \lambda x),$$

and then $v^{i}$ and $v^{j}$ are in fact the same bubble.
Fixing an $R$, we have
\[
\int_{B_R \setminus (x_{\alpha_k}^j \cup (\cup_{i=1}^s B_{\lambda_{\alpha_k}^i}(x_{\alpha_k}^i + \lambda_{\alpha_k}^i p_i)))} |\nabla g u_{\alpha_k}|^{2\alpha_k} dV_g = (\lambda_{\alpha_k}^j)^{2-2\alpha} \int_{B_R \setminus (\cup_{i=1}^s B_{\lambda_{\alpha_k}^i}(p_i))} |\nabla g u_{\alpha_k}|^{2\alpha_k} dV_g.
\] (1.2)

Since
\[
\int_{B_R \setminus (\cup_{i=1}^s B_{\lambda_{\alpha_k}^i}(p_i))} |\nabla g u_{\alpha_k}|^{2\alpha_k} dV_g(x_{\alpha_k}^i + \lambda_{\alpha_k}^i x) \to \int_{B_R \setminus (\cup_{i=1}^s B_{\lambda_{\alpha_k}^i}(p_i))} |\nabla_0 v^j|^2 dx,
\]
and
\[
\lambda_{\alpha_k}^j < 1,
\]
we define
\[
\mu_j = \liminf_{\alpha \to 1} \left( \frac{\int_{B_R \setminus (x_{\alpha_k}^j \cup (\cup_{i=1}^s B_{\lambda_{\alpha_k}^i}(x_{\alpha_k}^i + \lambda_{\alpha_k}^i p_i)))} |\nabla g u_{\alpha_k}|^{2\alpha_k} dV_g}{\int_{B_R \setminus (\cup_{i=1}^s B_{\lambda_{\alpha_k}^i}(p_i))} |\nabla_0 v^j|^2 dx} \right) \leq \frac{\theta - |M| - E(u_0)}{\theta},
\]
where
\[
\theta = \inf\{E(u) : u \text{ is a nontrivial harmonic map from } S^2 \to N\}.
\]

Therefore, we know
\[
\mu_j \in \left[1, \frac{\theta - |M| - E(u_0)}{\theta}\right].
\]

The first task of this paper is to get the following weak energy identity:

**Theorem 1.1.** Let $M$ be a smooth closed Riemann surface and $N$ be a smooth compact Riemannian manifold without boundary. Assume that $u_{\alpha_k} \in C^\infty(M, N)$ ($\alpha_k \to 1$) is a sequence of $\alpha_k$-harmonic maps with uniformly bounded energy and $x_1$ be the only blow-up point of the sequence $\{u_{\alpha_k}\}$ in $B_\sigma(x_1) \subset M$. Then, passing to a subsequence, there exist $u_0 : M \to N$ which is a smooth harmonic map and finitely many bubbles $v_j : S^2 \to N$ such that $u_{\alpha_k} \rightharpoonup u_0$ weakly in $W^{1,2}(M, N)$ and in $C^\infty_{loc}(B_\sigma(x_1) \setminus \{x_1\}, N)$ and the following identity holds
\[
\lim_{k \to +\infty} E_{\alpha_k}(u_{\alpha_k}, B_\sigma(x_1)) = E(u_0, B_\sigma(x_1)) + |B_\sigma(x_1)| + \sum_{j=1}^{n_0} \mu_j^2 E(v^j),
\] (1.4)

where $\mu_j$ is defined by (1.3) and $n_0$ is the number of bubbles at $x_1$.

This theorem tells us that the energy identity holds true if and only if $\mu_j = 1$. It provides a new route to approach the problem whether the necks contain energy or not.

**Remark 2.** By Lemma 2.2 in section 2, $\mu_j = 1$ implies
\[
\lim_{k \to +\infty} E(u_{\alpha_k}, B_\sigma(x_1)) = E(u_0, B_\sigma(x_1)) + \sum_{j=1}^{n_0} E(v^j),
\] (1.5)
and reversely, by Lemma 2.2 and (2.4), (1.5) also implies \( \mu_j = 1 \).

It is our another purpose to study the behavior of the necks connecting bubbles. For this sake, we need to define

\[
\nu_j = \liminf_{\alpha \to 1} (\lambda_j^\alpha)^{-\sqrt{\alpha - 1}}.
\]

We will see that the above quantity play an important role in the discussion on the behavior of blowing up. Our main results are stated as follows:

**Theorem 1.2.** Let \( M \) be a smooth closed Riemann surface and \( N \) be a smooth closed Riemannian manifold and \( u_{\alpha_k} \in C^\infty(M, N) \) be a sequence of \( \alpha_k \)-harmonic maps with uniformly bounded energy and \( u_{\alpha_k} \) converges to a smooth harmonic map \( u_0 : M \to N \) in \( C^\infty_{loc}(B_\sigma(x_1) \setminus \{x_1\}, N) \) as \( \alpha_k \to 1 \). Assume there is only one bubble in \( B_\sigma(x_1) \subset M \) for \( \{u_{\alpha_k}\} \) and \( v_1 : S^2 \to N \) is the bubbling map. Let \( \nu_1 = \liminf_{\alpha \to 1} (\lambda_1^\alpha)^{-\sqrt{\alpha - 1}}. \) Then we have

1) when \( \nu_1 = 1 \), the set \( u_0(B_\sigma(x_1)) \cup v_1(S^2) \) is a connected subset of \( N \);

2) when \( \nu_1 \in (1, \infty) \), the set \( u_0(B_\sigma(x_1)) \) and \( v_1(S^2) \) are connected by a geodesic with Length

\[
L = \sqrt{\frac{E(v_1)}{\pi}} \log \nu_1;
\]

3) when \( \nu_1 = +\infty \), the neck contains at least an infinite length geodesic.

**Remark 3.** Although we state and prove Theorem 1.2 only for one bubble case, it is not difficult to follow the steps in section 3.2 to prove the general case. However, the general case is quite complicated, for example, if we have 2 bubbles:

\[
u_1^\alpha(x + x_1) \to v_1, \quad \nu_2^\alpha(x + x_1) \to v_2
\]

which satisfy: \( \lambda_1^\alpha / \lambda_2^\alpha \to 0 \) and \( \nu_1, \nu_2 < \infty \), then \( u_0(B_\sigma(x_1)) \), \( v_1(S^2) \) are connected by a geodesic with length

\[
L = \sqrt{\frac{E(v_1) + E(v_2)}{\pi}} \log \nu_2,
\]

and \( v_1(S^2) \), \( v_2(S^2) \) are connected by a geodesic with length

\[
L = \sqrt{\frac{E(v_1)}{\pi}} \log \frac{\nu_1}{\nu_2}.
\]

We should mention that after we completed the paper we found that Moore had proved that if a neck is of finite length \( L \) and \( \tilde{g} \geq 1 \) (the genus of \( M \)), then \( L = \sqrt{\frac{E(u)}{\pi}} \log \nu \) (note that in \( [M] \), \( E(u) \) is defined to be \( \frac{1}{2} \int_M |\nabla u|^2dV_{\tilde{g}} \)). However, the arguments to prove Theorem 1.2 in this paper is completely different from Moore’s proof. The key estimation of us is the Proposition 4.1 in section 4, which gives the details of the necks.

The Proposition 4.1 also provides a new method to prove that the necks consist of geodesics, which has been already proved by Chen and Tian \([C-T]\). In this paper, we will make use of the following curve

\[
\Gamma_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} u_\alpha(r, \theta) d\theta
\]
to approximate the necks. With the help of Proposition [4.1], one can easily calculate the second fundamental form of the approximation curve, and then to prove that the limiting curve satisfies the equation of geodesic in $N$.

We failed to find a sufficient condition s.t. $\nu^j < +\infty$, but we will show that there are indeed many cases that the necks contain at least one infinite length geodesic:

**Corollary 1.3.** Let $\alpha_k \to 1$, and $u_k : M \to N$ be a minimizer of $E_{\alpha_k}$ in the homotopic class containing $u_k$. We assume for any $i \neq j$, $u_i$ and $u_j$ are not in the same homotopic class. If

$$\sup_k E_{\alpha_k}(u_k) < +\infty,$$

then $u_k$ will blow up, and the neck contains at least one infinite length geodesic.

**Remark 3.** In the last section, by constructing a manifold $N$ we will give an example of a minimizing $\alpha$-harmonic map sequence, which satisfies the condition in the above corollary. This indicates that there exists a neck joining bubbles which is a geodesic of infinite length.

We conclude this introduction with showing the following proposition as a consequence of Theorem 1.1, which implies the result due to Chen-Tian that, if the necks consist of some geodesics of finite length, then the energy identity is true:

**Proposition 1.4.** The energy identity holds true for a subsequence of $u_\alpha$ if and only if

$$\liminf_{\alpha \to 1} \| \nabla u_\alpha \|_{C^0(M)}^{\alpha - 1} = 1. \quad (1.6)$$

The limit set of such subsequence has no neck if and only if

$$\liminf_{\alpha \to 1} \| \nabla u_\alpha \|_{\overline{C^0(M)}}^{\alpha - 1} = 1.$$

The bubbles in limit set of such subsequence are joined by some geodesics of finite length, if and only if

$$\liminf_{\alpha \to 1} \| \nabla u_\alpha \|_{\overline{C^0(M)}}^{\alpha - 1} < +\infty.$$

**Proof.** We only prove the first claim.

First, we prove (1.6) implies $\mu_j = 1$. We assume $v^j(x) = u_\alpha(x^j + \lambda^j_\alpha x)$ converges to $v^j$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{p_1, p_2, \cdots, p_s\})$. Then we have

$$(\lambda^j_\alpha)_{1-\alpha} = \frac{|\nabla u_\alpha(x^j + \lambda^j_\alpha x)|^{\alpha - 1}}{|\nabla v^j(x)|^{\alpha - 1}}$$

for any $x$ with $|\nabla v^j(x)| \neq 0$. Hence we get $\mu_j \leq 1$ and then $\mu_j = 1$.

Now, we will prove “$\mu_j = 1$ for all $j$” implies (1.6). Let $x_\alpha$ to be the point s.t. $|\nabla u_\alpha|(x_\alpha) = \max |\nabla u_\alpha|$, and

$$\lambda_\alpha = \frac{1}{|\nabla u_\alpha|(x_\alpha)}.$$

We set $v_\alpha(x) = u_\alpha(x + \lambda_\alpha x)$. One is easy to check that $v_\alpha$ will converge to a non-trivial harmonic map $v_0$ locally. By $\mathbf{H1}$ and $\mathbf{H2}$ we must find a $j$, s.t.

$$B_{R\lambda^j_\alpha}(x_\alpha^j) \cap B_{R\lambda_\alpha}(x_\alpha) \neq \emptyset, \quad \text{and} \quad \frac{1}{C} \lambda^j_\alpha < \lambda_\alpha < C \lambda^j_\alpha.$$
for some \( C > 0 \). Hence we get \( |\lambda_\alpha|^{\alpha-1} \to 1 \).

\[ \square \]

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\section{Preliminary}

In this section we intend to establish some integral formulas on \( \alpha \)-harmonic maps from a closed surfaces by the variations of domain. Of course, we need to choose some suitable variational vector fields on \( M \) which generate the transformations of \( M \). We will see that these integral relations will play an important role in the proofs of main theorems.

Note that the functional \( E_\alpha \) is not conformal invariant. For example, on an isothermal coordinate system around a point \( p \in M \), if we set the metric

\[ g = e^\varphi((dx)^2 + (dy)^2) \]

with \( p = (0,0) \), \( \varphi(0) = 0 \) and \( \tilde{u}_\alpha(x) = u_\alpha(\lambda x) \), then we will get

\[ \int_{B_\delta} (1 + |\nabla_g u_\alpha|^2)\alpha dV_g = \int_{B_{\delta/\lambda}} \lambda^{2-2\alpha}(\lambda^2 + |\nabla_g \tilde{u}_\alpha|^2)^\alpha dV_{g'}, \]

where \( g' = e^{\varphi(p+\lambda x)}((dx)^2 + (dy)^2) \). We also ought to note that an \( \alpha \)-harmonic map sequence \( u_\alpha \) may have several bubbles near a blowing up point, for example, there are sequences \( \lambda_\alpha^1, \lambda_\alpha^2 \), s.t.

\[ \frac{\lambda_\alpha^1}{\lambda_\alpha^2} \to 0, \quad \lambda_\alpha^2 \to 0, \]

as \( \alpha \to 1 \), and

\[ v_\alpha^1(x) = u_\alpha(\lambda_\alpha^1 x) \to v^1 \text{ in } C^k_{\text{loc}}(\mathbb{R}^2), \quad v_\alpha^2(x) = u_\alpha(\lambda_\alpha^2 x) \to v^2 \text{ in } C^k_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}), \]

where \( v^1, v^2 \) are non-trivial harmonic maps from \( S^2 \) to \( N \). For this case, we have

\[ v_\alpha^1(x) = v_\alpha^2 \left( \frac{\lambda_\alpha^1}{\lambda_\alpha^2} x \right), \]

i.e. \( v^1(x) \) is in fact a bubble for the sequence \( v_\alpha^2 \). Therefore, we need to consider the equation of \( v_\alpha^2 \), and one is easy to check that \( v_\alpha^2 \) is locally a critical point of the functional

\[ F(v) = \int_{B_\delta} ((\lambda_\alpha^2)^2 + |\nabla_v|^2)\alpha dV_{g_\alpha}, \]

where \( g_\alpha = e^{\varphi(\lambda_\alpha^2 x)}((dx)^2 + (dy)^2) \). For this reason, we need to consider a more general \( \alpha \)-energy which is of the following form:

\[ E_{\alpha, \epsilon_\alpha}(u) = \int_{B_\delta} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^\alpha dV_{g_\alpha}. \]

Let \( u_\alpha \) be the critical point of the above functional. Then, \( u_\alpha \) satisfies the following elliptic system which is also called the equation of \( \alpha \)-harmonic maps:

\[ \Delta_{g_\alpha} u_\alpha + (\alpha - 1) \frac{\nabla_{g_\alpha} |\nabla_g u_\alpha|^2}{\epsilon_\alpha + |\nabla_g u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) = 0. \tag{2.1} \]
Here we always assume that the sequence $\epsilon_\alpha$ ($\epsilon_\alpha \leq 1$) satisfies

$$\lim_{\alpha \to 1} \epsilon_\alpha^{q-1} > \beta_0 > 0. \quad (2.2)$$

It follows from (1.3) that this assumption is reasonable.

From now on, we consider $u_\alpha$ to a map sequence from $(B, g)$ to $(N, h)$ which satisfy equation (2.1). We assume that $g = e^{\varphi_\alpha}((dx_1^1)^2 + (dx_2^2)^2)$ with $\varphi_\alpha(0) = 0$ and $\varphi_\alpha \to \varphi$ smoothly. Moreover, we assume that $u_\alpha \to u_0$ in $C^{k,\beta}_{loc}(B \setminus \{0\})$.

Next, we recall the well-known $\epsilon$-regularity theorem due to Sacks-Uhlenbeck [S-U]:

**Theorem 2.1.** Let $u : B \to N$ satisfies equation (2.1) where $B \subset M$ is a ball with radius 1. There exists $\epsilon > 0$ and $\alpha_0 > 1$ such that if $E(u, B) < \epsilon$ and $1 \leq \alpha \leq \alpha_0$, then for all smaller $r < 1$, we have

$$\|\nabla u\|_{W^{1,p}(B_r)} \leq C(p, r)E(u, B),$$

here $B_r \subset B$ is a ball with radius $r$, $1 < p < \infty$.

We also have

**Lemma 2.2.** Let $u_\alpha$ be the critical point of $E_\alpha$ with $E_\alpha(u_\alpha) \leq \Theta$. We have

$$\beta_0 < \liminf_{\alpha \to 1} \|\epsilon_\alpha + |\nabla g u_\alpha|^2\|^{\alpha-1}_{C^0(B)} \leq \limsup_{\alpha \to 1} \|\epsilon_\alpha + |\nabla g u_\alpha|^2\|^{\alpha-1}_{C^0(B)} < \beta_1,$$

where $\beta_1$ is independent of $\alpha$.

**Proof.** Obviously, we only need to prove $\|\nabla g u_\alpha\|^{\alpha-1}_{C^0(B)} < C$. We assume that there is sequence $\alpha_k \to 1$, s.t. $\|\nabla g u_{\alpha_k}\|^{\alpha_k-1}_{C^0(B)} \to +\infty$ as $k \to +\infty$.

Let $|\nabla g u_{\alpha_k}|(x_{\alpha_k}) = \max\{|\nabla g u_{\alpha_k}|\}$, and $\lambda_k = \frac{1}{|\nabla g u_{\alpha_k}|}$, $v_k(x) = u_{\alpha_k}(x_{\alpha_k} + \lambda_k x)$. Then a subsequence of $\{v_{k_j}\}$ converges to a new nontrivial harmonic map from $S^2$ to $N$. Then by (1.3), we obtain the following $\lambda_{k_j}^{1-\alpha_{k_j}} < C$, which contradicts with the choice of $\alpha_k$. $\square$

### 2.1 Variational formula

Take an 1-parameter family of transformations $\{\phi_s\}$ which is generated by the vector field $X$. If we assume $X$ is supported in $B$, then we have

$$E_{\alpha, \epsilon_\alpha}(u \circ \phi_s) = \int_B (\epsilon_\alpha + |\nabla g(u \circ \phi_s)|^2)^{\alpha} dV_g$$

$$= \int_B (\epsilon_\alpha + \sum_\beta |d(u \circ \phi_s)(e_\beta(x))|^2)^{\alpha} dV_g(x)$$

$$= \int_B (\epsilon_\alpha + \sum_\beta |du(\phi_s e_\beta(x))|^2)^{\alpha} dV_g(x)$$

$$= \int_B (\epsilon_\alpha + \sum_\beta |du(\phi_s(\phi_s^{-1}(x))))|^2)^{\alpha} Jac(\phi_s^{-1}) dV_g,$$
where \( \{e_\alpha\} \) is a local orthonormal basis of \( TB \). Noting
\[
\frac{d}{ds}Jac(\phi_s^{-1})dV_g|_{s=0} = -\text{div}(X)dV_g,
\]
we have proved the formula
\[
dE_f(u_\alpha(u_\alpha(X))) = -\int_B (\epsilon_\alpha + |\nabla_g u|^2)^{\alpha} \text{div}(X)dV_g
\]
\[+ 2\alpha \sum_\beta \int_B (\epsilon_\alpha + |\nabla_g u|^2)^{\alpha-1}\langle du(\nabla_{e_\beta} X), du(e_\beta)\rangle dV_g.
\]
Now, we assume \( u_\alpha \) to be the critical point of \( E_\alpha \). For any vector field \( X \) on \( B \), we have
\[
-\int_B (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha} \text{div}X dV_g + 2\alpha \sum_\beta \int_B (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1}\langle du(\nabla_{e_\beta} X), du(e_\beta)\rangle dV_g = 0.
\]
Next, for \( 0 < t' < t \leq \rho \), we choose a vector field \( X \) with compact support in \( B_\rho \) by
\[
X = \eta(r) r \frac{\partial}{\partial r} = \eta(|x|) x^i \frac{\partial}{\partial x^i},
\]
where \( \eta \) is defined by
\[
\eta(r) = \begin{cases} 
1 & \text{if } r \leq t', \\
\frac{t-r}{t-t'} & \text{if } t' \leq r \leq t \\
0 & \text{if } r \geq t,
\end{cases}
\]
where \( r = \sqrt{(x^1)^2 + (x^2)^2} \). By a direct computation we obtain
\[
\text{div}(X) = 2\eta + r\eta' + r\eta \frac{\partial \varphi}{\partial r},
\]
and
\[
\nabla_{\frac{\partial}{\partial r}} X = \eta \frac{\partial}{\partial x^i} + \eta'(x^j) \frac{x^j}{r} \frac{\partial}{\partial x^1} + \frac{x^1}{r} \frac{\partial}{\partial x^2} + \eta x^1 \Gamma_1^2 \frac{\partial}{\partial x^1} + \eta x^2 \Gamma_1^2 \frac{\partial}{\partial x^2} + \eta x^2 \Gamma_2^1 \frac{\partial}{\partial x^1} + \eta x^2 \Gamma_2^1 \frac{\partial}{\partial x^2}.
\]
Then,
\[
\sum_\beta \langle du_\alpha(\nabla_{e_\beta} X), du_\alpha(e_\beta)\rangle dV_g = \langle du_\alpha(\nabla_{\frac{\partial}{\partial r}} X), du_\alpha(\frac{\partial}{\partial r})\rangle dx + \langle du_\alpha(\nabla_{\frac{\partial}{\partial x^1}} X), du_\alpha(\frac{\partial}{\partial x^1})\rangle dx
\]
\[= (\eta|\nabla_0 u_\alpha|^2 + \eta' r \frac{\partial u_\alpha}{\partial r})^2 + O(|x|)|\nabla_0 u_\alpha|^2 dx,
\]
where \( \nabla_0 \) is the Riemannian connection with respect to standard metric. Hence, we derive
\[
0 = (2\alpha - 2) \int_{B_t} \eta(\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1}|\nabla_0 u_\alpha|^2 dx
\]
\[+ \int_{B_t} O(|x|)(\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1}|\nabla_0 u_\alpha|^2 dx
\]
\[-2\epsilon_\alpha \int_{B_t} \eta(\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1}dV_g + \frac{\epsilon_\alpha}{t-t'} \int_{B_t \setminus B_{t'}} r(\epsilon_\alpha + |\nabla_g u|^2)^{\alpha-1} dV_g
\]
\[+ \frac{1}{t-t'} \int_{B_t \setminus B_{t'}} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1}|\nabla_0 u_\alpha|^2 r - 2\alpha r \frac{\partial u_\alpha}{\partial r}|^2 dx
\]
\[-\int_{B_t} \epsilon_\alpha (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1} r\eta \frac{\partial \varphi}{\partial r} dV_g.
\]
Letting $t' \to t$ in the above identity and using Lemma 2.2, we obtain the following
\[
\int_{\partial B_t} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} \frac{\partial u_\alpha}{\partial r}^2 ds_0 - \frac{1}{2\alpha} \int_{\partial B_t} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla_0 u_\alpha|^2 ds_0
\]
\[
= \frac{(\alpha - 1)}{\alpha t} \int_{B_t} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla_0 u_\alpha|^2 dx + O(t),
\]
where $ds_0$ is the volume element of $\partial B_t$ with respect to the Euclidean metric. We know that the metric $g$ can be written as $g = e^r(dr^2 + r^2d\theta^2)$ in the polar coordinate system. Set
\[
u_{\alpha, \theta} = \frac{1}{r} \frac{\partial u_\alpha}{\partial \theta}.
\]
Since $|\nabla_0 u_\alpha|^2 = |\frac{\partial u_\alpha}{\partial r}|^2 + |u_{\alpha, \theta}|^2$, we get from the above identity
\[
(1 - \frac{1}{2\alpha}) \int_{\partial B_t} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\frac{\partial u_\alpha}{\partial r}|^2 ds_0 - \frac{1}{2\alpha} \int_{\partial B_t} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |u_{\alpha, \theta}|^2 ds_0
\]
\[
= \frac{(\alpha - 1)}{\alpha t} \int_{B_t} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla_0 u_\alpha|^2 dx + O(t).
\]

2.2 Pohozaev identity

Denote $\Delta_0 = \frac{\partial^2}{\partial (x^i)^2} + \frac{\partial^2}{\partial (x^j)^2}$. By (2.1), we have the equation:
\[
\Delta_0 u_\alpha + (\alpha - 1) \frac{\nabla_0 |\nabla g u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla g u_\alpha|^2} + A(u_\alpha)(du_\alpha, du_\alpha) = 0.
\]

As in [Lin-W], we multiply the both sides of the above equation with $r \frac{\partial u_\alpha}{\partial r}$ to obtain
\[
\int_{B_t} r \frac{\partial u_\alpha}{\partial r} \Delta_0 u_\alpha dx = (\alpha - 1) \int_{B_t} \frac{\nabla_0 |\nabla g u_\alpha|^2 \nabla_0 u}{\epsilon_\alpha + |\nabla g u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx.
\]

It is easy to see
\[
\int_{B_t} r \frac{\partial u_\alpha}{\partial r} \Delta_0 u_\alpha dx = \int_{\partial B_t} r \frac{\partial u_\alpha}{\partial r} |\nabla_0 u_\alpha|^2 ds_0 - \int_{B_t} \nabla_0 (r \frac{\partial u_\alpha}{\partial r}) \nabla_0 u_\alpha dx.
\]

Since
\[
\int_{B_t} \nabla_0 (r \frac{\partial u_\alpha}{\partial r}) \nabla_0 u_\alpha dx = \int_{B_t} \nabla_0 (x^k \frac{\partial u_\alpha}{\partial x^k}) \nabla_0 u_\alpha dx
\]
\[
= \int_{B_t} |\nabla_0 u_\alpha|^2 dx + \int_0^t \int_{\partial B_t} |\nabla_0 u_\alpha|^2 r \frac{\partial (|\nabla_0 u_\alpha|^2)}{\partial r} r d\theta dr
\]
\[
= \int_{B_t} |\nabla_0 u_\alpha|^2 dx + \frac{1}{2} \int_{\partial B_t} |\nabla_0 u_\alpha|^2 t ds_0 - \int_{B_t} |\nabla_0 u_\alpha|^2 dx
\]
\[
= \frac{1}{2} \int_{\partial B_t} |\nabla_0 u_\alpha|^2 t ds_0,
\]
then, we have
\[
\int_{\partial B_t} (|\frac{\partial u_\alpha}{\partial r}|^2 - \frac{1}{2} |\nabla_0 u_\alpha|^2) ds_0 = \frac{\alpha - 1}{t} \int_{B_t} \frac{\nabla_0 |\nabla g u_\alpha|^2 \nabla_0 u_\alpha}{\epsilon_\alpha + |\nabla g u_\alpha|^2} r \frac{\partial u_\alpha}{\partial r} dx.
\]
Hence, it follows
\[
\int_{\partial B_t} \left( \frac{\partial u_\alpha}{\partial r} \right)^2 - |u_\alpha,\theta|^2 \, ds_0 = -\frac{2(\alpha - 1)}{t} \int_B \nabla_0 |\nabla g u_\alpha|^2 \nabla_0 u_\alpha \frac{\partial u_\alpha}{\partial r} \, dx.
\] (2.6)

Thus, we obtain two key variational identities (2.4) and (2.6) which will be used repeatedly in our following argument.

### 3 The proof of Theorem 1.1

In this section, we discuss the weak energy identity on a sequence of $\alpha$-harmonic maps. By following the idea of Ding and Tian in [D-T] we will apply (2.4) (2.5) to give the proof of Theorem 1.1.

Let $B_{2\sigma} = B_{2\sigma}(0)$ be a ball in $\mathbb{R}^2$ with the metric $g = e^{\varphi_\alpha(x)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$, where $\varphi \in C^\infty(B_{2\sigma})$ and $\varphi_\alpha(0) = 0$, and $\varphi_\alpha$ converges smoothly. We set $u_\alpha : B_{2\sigma} \to N$ be a map which satisfies equation (2.1). Clearly, (2.4), (2.5) and (2.6) hold.

We assume that for any $\alpha$
\[
E_{\alpha,\epsilon_\alpha}(u_\alpha, B_\sigma) < C_1,
\]
and 0 is the only blow-up point in $B_{2\sigma}$. Without loss of generality, we assume $u_\alpha \to u_0$ in $C^k_{loc}(B_\sigma \setminus \{0\})$, where $u_0$ is a harmonic map from $B_\sigma$ to $N$.

We can get the first bubble in the following way. Let $x_1^1 \in B_\delta$ s.t. $|\nabla u_\alpha(x_1^1)| = \max_{B_\delta} |\nabla u_\alpha|$, and $\lambda_1^1 = \max_{B_\delta} \frac{1}{|\nabla u_\alpha|}$. Then, without loss of generality, we may assume in $C^k_{loc}(\mathbb{R}^2)$

\[
u_\alpha(x_1^1 + \lambda_1^1 x) \to v^1.
\]

Now, we assume there exists another $n_0 - 1$ bubbles $v^2, \ldots, v^{n_0}$, and sequences $x_\alpha^i, \lambda_\alpha^i$ s.t.

\[
u_\alpha(x_\alpha^i + \lambda_\alpha^i x) \to v^i
\]
in $C^k_{loc}(\mathbb{R}^2 \setminus A^i)$, where $A^i$ are finite sets. Clearly, we may assume

\[
\lambda_\alpha^1 = \min_{i \in \{1, \ldots, n_0\}} \{\lambda_\alpha^i\}.
\]

Moreover, we assume that for any $i \neq j$, one of the H1 and H2 holds.

#### 3.1 The weak energy identity for the case of only one bubble

First we prove the Theorem 1.1 in the case of $n_0 = 1$, where $n_0$ is the number of the bubbles. The general case will be explained in the next subsection.

We denote $\lambda_\alpha = \lambda_\alpha^1$, $x_\alpha = x_\alpha^1$, and $v = v^1$. We define
\[
\Lambda_\alpha(R) = \int_{B_{R\lambda_\alpha}(x_\alpha)} |\nabla g u_\alpha|^{2\alpha} dV, \quad \Lambda = \lim_{R \to +\infty} \lim_{\alpha \to 1} \Lambda_\alpha(R).
\]
and
\[
\mu = \lim_{\alpha \to 1} \lambda_\alpha^{2-2\alpha}.
\]
By (1.3), we have \( \Lambda = \mu E(v) \). Moreover, we also have

\[
\lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{R\lambda \alpha}} (\epsilon_{\alpha} + |\nabla_{g} u_{\alpha}|^{2})^{\alpha - 1}|\nabla_{g} u_{\alpha}|^{2} dV_{g} = \lambda \mu \int_{\mathbb{R}^{2}} |\nabla_{0} v|^{2} dx = \Lambda.
\]

Furthermore, we claim that for any \( \epsilon > 0 \) there exist \( \delta_{1} \) and \( R \) such that, \( \forall \lambda \in (\frac{R\lambda}{2}, 4\delta_{1}) \), there holds

\[
\int_{B_{2\lambda \alpha \lambda}} |\nabla_{g} u_{\alpha}|^{2} dV_{g} \leq \epsilon.
\]

Suppose that the claim is false, then we may assume that there exist \( \alpha_{i} \to 1 \) and \( \lambda'_{i} \to 0 \) satisfying \( \frac{\lambda'_{i}}{\lambda_{\alpha_{i}}} \to +\infty \) such that

\[
\int_{B_{2\lambda'_{i} \alpha \lambda_{\alpha_{i}}}} |\nabla_{g} u_{\alpha_{i}}|^{2} dV_{g} \geq \epsilon.
\]

Denote \( v'_{\alpha_{i}}(x) = u_{\alpha_{i}}(\lambda'_{i}x + x_{\alpha_{i}}) \), we may assume \( v'_{\alpha_{i}} \to v' \) in \( C_{loc}^{0}(\mathbb{R}^{2} \setminus (\{0\} \cup A), N) \), where \( A \) is a finite set which does not contain 0. If \( A = \emptyset \) then it follows from (3.3) that \( v' \) is a nonconstant harmonic sphere which is different from \( v^{1} \). This contradicts the assumption \( n_{0} = 1 \). Next, if there exists \( x_{1} \in A \), then, by a similar argument with that we get \( v = v^{1} \), we can still obtain a sequence \( x_{i} \to x_{1} \), \( \lambda_{i} \to 0 \), s.t. \( v'(x_{i} + \lambda_{i}x) \) converges to a harmonic map \( v^{2} \). Hence we get \( u_{\alpha_{i}}(x_{\alpha_{i}} + \lambda_{i}(\lambda_{\alpha_{i}}x + x_{i})) \) converges to \( v^{2} \) strongly, and then \( v^{2} \) is the second harmonic map. This proves that the claim (3.2) must be true.

Set

\[
u^{*}_{\alpha} = \frac{1}{2\pi} \int_{0}^{2\pi} u_{\alpha}(x_{\alpha} + re^{i\theta}) d\theta.
\]

One is easy to check that, for any \( a < b \), the following inequality holds true

\[
\int_{B_{b} \setminus B_{a}(x_{\alpha})} \left| \frac{\partial u^{*}_{\alpha}}{\partial r} \right|^{2} dx = \int_{a}^{b} \int_{0}^{2\pi} \left| \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\partial u_{\alpha}}{\partial \theta} d\theta \right|^{2} d\theta dr \\
\leq \frac{1}{2\pi} \int_{a}^{b} \left( \int_{0}^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^{2} d\theta \right) dr \\
= \int_{a}^{b} \int_{0}^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^{2} d\theta dr = \int_{B_{b} \setminus B_{a}(x_{\alpha})} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^{2} dx.
\]

By applying (3.2) and Sacks-Uhlenbeck \( \epsilon \)-regularity theorem (Theorem 2.1), we have the following

**Lemma 3.1.** For any \( R\lambda \alpha < a < b < \delta_{1}, \) we have

\[
\int_{B_{b} \setminus B_{a}(x_{\alpha})} \left| \nabla_{g} u_{\alpha} \right|^{2} dV_{g} \leq C \int_{B_{b} \setminus B_{b}(x_{\alpha})} \left| \nabla_{g} u_{\alpha} \right|^{2} dV_{g}.
\]
and
\[ \int_{B_b \setminus B_a(x_\alpha)} |\nabla^2 u_\alpha| \cdot |u_\alpha - u^*_\alpha| dV_g \leq C \int_{B_{4b} \setminus B_{4a}(x_\alpha)} |\nabla g u_\alpha|^2 dV_g, \]
where \( C \) does not rely on \( \alpha \).

**Proof.** First, we prove the first inequality in the above lemma. We assume that \( 2^K a \in (b, 2b) \) and set
\[ D_i = B_{2^i a} \setminus B_{2^{i-1} a}(x_\alpha). \]

We rescale \( D_i \) to \( B_{2^i} \setminus B_1 \), and \( u_\alpha \) to \( \tilde{u}_\alpha \). By Theorem 2.1 (the \( \epsilon \)-regularity theory), we have on \( D_i \)
\[ |\nabla g u_\alpha| \leq \frac{1}{2^{i-1} a} |\nabla_0 \tilde{u}_\alpha|_{C^0(B_{2^i} \setminus B_1)} \leq \frac{C_1}{2^{i-1} a} \|\nabla_0 \tilde{u}_\alpha\|_{L^2(B_{4^i} \setminus B_{3^i/2})}. \]

Hence, it follows
\[ \|r \nabla g u_\alpha\|_{C^0(D_i)} \leq 2C_1 |\nabla g u_\alpha| \leq C_2 \|\nabla g u_\alpha\|_{L^2(D_i \cup D_{i+1} \cup D_{i-1})}. \]

Similarly, we have
\[ \|r^2 \nabla^2 g u_\alpha\|_{C^0(D_i)} \leq C'_2 \|\nabla g u_\alpha\|_{L^2(D_i \cup D_{i+1} \cup D_{i-1})}. \]

Then we have
\[ \int_{D_i} |\nabla^2 g u_\alpha| r |\nabla g u_\alpha| dV_g \leq C \int_{D_{i+1} \cup D_{i-1} \cup D_i} |\nabla g u_\alpha|^2 dV_g \int_{D_i} \frac{dV_g}{r^2} \leq C' \int_{D_{i+1} \cup D_{i-1} \cup D_i} |\nabla g u_\alpha|^2 dV_g. \]

Therefore, we get the first inequality in this Lemma. The proof of the second inequality goes to almost the same. \( \square \)

### 3.1.1 The estimate of \( \int_{B_b \setminus B_{R\lambda}(x_\alpha)} |u_{\alpha, \theta}|^2 dx \)

The goal of this subsection is to prove the following

**Lemma 3.2.** For \( \alpha \)-harmonic map sequence \( u_\alpha \) (\( \alpha \to 1 \)), there holds true
\[ \lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_b \setminus B_{R\lambda}(x_\alpha)} |u_{\alpha, \theta}|^2 dx = 0. \]

**Proof.** We adopt the technique of Sacks-Uhlenbeck [SU] and [L-W] to show the lemma. Using (3.2) we have
\[ |u^*_\alpha(r) - u_\alpha(r, \theta)| \leq \epsilon_1. \]

We compute
\[
\int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} |\nabla g u_\alpha|^2 dV_g \\
= \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \nabla g u_\alpha \nabla (u_\alpha - u_\alpha^*) dx + \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \nabla g u_\alpha \nabla u_\alpha^* dV_g \\
= -\int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \Delta g u_\alpha (u_\alpha - u_\alpha^*) dx + \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \nabla g u_\alpha \nabla (u_\alpha - u_\alpha^*) dx \\
+ \int_{\partial (B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha))} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 \tag{3.6}
\]

On the other hand, noting (3.4) we have
\[
\int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} \frac{\partial u_\alpha^*}{\partial r} dx \leq \sqrt{\int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 dx \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 dx} \leq \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 dx \tag{3.7}
\]

Hence, by using Lemma 3.1, 3.5, 3.7 and noting the following fact
\[
\frac{\nabla g |\nabla g u_\alpha|^2 |\nabla g u_\alpha|}{\epsilon_\alpha + |\nabla g u_\alpha|^2} \leq |\nabla^2 u_\alpha|,
\]
we can infer from (3.6)
\[
\int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} |\nabla_0 u_\alpha|^2 dx \leq \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 dx + 3C(\alpha - 1) \int_{B_\delta} |\nabla g u_\alpha|^2 dV_g \\
+ \int_{\partial B_\delta(x)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 \\
- \int_{\partial B_{R\lambda_\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 \\
+ \epsilon_1' \int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} |\nabla_0 u_\alpha|^2 dx,
\]
where \(\epsilon_1' = \epsilon_1 \|A\|_{L^\infty(M)}\).

Since \(|\nabla_0 u_\alpha|^2 = |\frac{\partial u_\alpha}{\partial r}|^2 + |u_{a,\theta}|^2\), we get
\[
\int_{B_\delta \setminus B_{R\lambda_\alpha}(x_\alpha)} |u_{a,\theta}|^2 dx \leq -\int_{\partial B_\delta(x)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 + \int_{\partial B_{R\lambda_\alpha}(x_\alpha)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 \\
+ C'((\alpha - 1) + \epsilon).
\]

Keeping (3.2) in mind, we have
\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{\partial B_{\delta}(x_0)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 = 0,
\]
and
\[
\lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{\partial B_{R\lambda_\alpha}(x_0)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 = 0.
\]
Hence, we can see the above inequality implies the conclusion of Lemma 3.2.

Immediately we infer from Lemma 2.2 and Lemma 3.2

**Corollary 3.3.** There holds true
\[
\lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{\delta} \setminus B_{R\lambda_\alpha}(x_0)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1} |u_{\alpha,\theta}|^2 dx = 0.
\]

### 3.1.2 The energy of the neck

We set
\[
F_\alpha(t) = \int_{B_{\lambda_\alpha}(x_0)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1} |\nabla_0 u_\alpha|^2 dx,
\]
\[
E_{r,\alpha}(t) = \int_{B_{\lambda_\alpha}(x_0) \setminus B_{\lambda_\alpha}(x_0)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1} |\frac{\partial u_\alpha}{\partial r}|^2 dx,
\]
and
\[
E_{\theta,\alpha}(t) = \int_{B_{\lambda_\alpha}(x_0) \setminus B_{\lambda_\alpha}(x_0)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1} |u_{\alpha,\theta}|^2 dx.
\]

By (2.3), for \(t \in [\epsilon, t_0]\), we have
\[
(1 - \frac{1}{2\alpha}) E_{r,\alpha}(t) - \frac{1}{2\alpha} E_{\theta,\alpha}(t) = \frac{\alpha - 1}{\alpha} \log \lambda_\alpha F_\alpha(t) + O(\lambda_\alpha^t \log \lambda_\alpha).
\]

Then
\[
(1 - \frac{1}{2\alpha}) E_{r,\alpha}(t) - \frac{1}{2\alpha} E_{\theta,\alpha}(t) = \frac{1}{2} \int_{t_0}^t \frac{1}{\alpha} \log \lambda_\alpha^{2(\alpha-1)} F_\alpha(t) + O(\lambda_\alpha^t \log \lambda_\alpha)|dt.
\]

It is easy to check that the sequences \{(1 - \frac{1}{2\alpha}) E_{r,\alpha}(t) - \frac{1}{2\alpha} E_{\theta,\alpha}(t)\} and \{F_\alpha(t)\} are compact in \(C^0([\epsilon, t_0])\) topology for any \(\epsilon > 0\). Therefore, there exist two functions \(F\) and \(E_r\) which belong to \(C^0([\epsilon, t_0])\) such that, as \(\alpha \to 1\),

\[
F_\alpha \to F, \quad E_{r,\alpha} \to E_r \quad \text{in} \quad C^0([\epsilon, t_0]).
\]

Hence, we infer from the above integration equality
\[
E_r(t) = -\log \mu \int_{t_0}^t F dt = -\log \mu \int_{t_0}^t (E_r(t) + F(t_0)) dt.
\]

This implies that \(E_r(t) \in C^1\) and
\[
E'_r = -\log \mu (E_r + F(t_0)).
\]
It follows
\[ E_r(t) = \mu^{t_0-t}F(t_0) - F(t_0). \]

Next, we prove that
\[ \lim_{t_0 \to 1} F(t_0) = \Lambda. \] (3.8)

Integrating (2.3) with respect to \( t \) on the interval \([R\lambda, \lambda_0^\alpha]\) we obtain
\[
F_\alpha(t_0) - \int_{B_{R\lambda \alpha}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla_0 u_\alpha|^2 \, dx \\
\leq C \int_{B_{\lambda_0^\alpha t_0} \setminus B_{R\lambda \alpha}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |u_{\alpha, \theta}|^2 \, dx \\
+ C \int_{R\lambda \alpha} \frac{\alpha - 1}{r} \, dr + C(\lambda_0^\alpha - R\lambda_\alpha).
\]

Noting the following holds true (from Corollary 3.3)
\[
\lim_{t_0 \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{\lambda_0^\alpha t_0} \setminus B_{R\lambda \alpha}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |u_{\alpha, \theta}|^2 \, dx = 0,
\]
and
\[
\lim_{t_0 \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{R\lambda \alpha} \frac{\alpha - 1}{r} \, dr = \lim_{t_0 \to 1} \frac{(1 - t_0)}{2} \log \mu = 0.
\]
Thus, (3.8) follows from the above inequality in view of (3.1).

On the other hand, we have
\[
\int_{B_{\lambda_0^\alpha t_0}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla g u_\alpha|^2 \, dV_g = E_{r, \alpha}(t) + E_{\theta, \alpha}(t) + F_\alpha(t_0).
\]

Noting Corollary 3.3 i.e. \( \lim_{\alpha \to 1} E_{\theta, \alpha}(t) = 0 \), we can deduce the following
\[
\lim_{\alpha \to 1} \int_{B_{\lambda_0^\alpha t_0}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla g u_\alpha|^2 \, dV_g = E_r(t) + F(t_0) = \mu^{t_0-t}F(t_0).
\]
Thus, we have shown the following

**Lemma 3.4.** For any \( t \in (0, 1) \) and \( \epsilon_\alpha > 0 \) with \( \lim_{\alpha \to 1} \epsilon_\alpha^{\alpha-1} \geq \beta_0 \), there holds true
\[
\lim_{\alpha \to 1} \int_{B_{\lambda_0^\alpha t_0}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla g u_\alpha|^2 \, dV_g = \mu^{1-t} \Lambda.
\]

**Proof of Theorem 1.1** Here we restrict us to the case of one bubble. By taking almost the same argument as we proved (3.8), we obtain
\[
\lim_{\delta \to 0} \lim_{t_0 \to 1} \lim_{\alpha \to 1} \int_{B_{\delta} \setminus B_{\lambda_0^\alpha t_0}(x_\alpha)} (\epsilon_\alpha + |\nabla g u_\alpha|^2)^{\alpha-1} |\nabla g u_\alpha|^2 \, dV_g = 0, \quad (3.9)
\]
which leads to
\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{B_{\delta}(x_{\alpha})} (\epsilon_{\alpha} + |\nabla g u_{\alpha}|^2)^{\alpha - 1} |\nabla g u_{\alpha}|^2 dV = \mu \Lambda = \mu^2 E(v).
\]
Obviously, this implies the required conclusion. So we have completed the proof of Theorem 1.1 in the case that \(n_0 = 1\).

### 3.2 The weak energy identity for the case of several bubbles

For the general case that \(n_0 > 1\), the proof can be completed by induction in \(n_0\), the number of bubbles.

We set
\[
\lambda_{\alpha}^i = \max_i \{|x_{\alpha}^i - x_{\alpha}^1| + |\lambda_{\alpha}^i|\}.
\]
Without loss of generality, we assume \(x_{\alpha}^1 \equiv 0\) and \(\lambda_{\alpha}^i\) is attained by the \(n_0\)-th bubble, i.e.
\[
\lambda_{\alpha}^i = |x_{\alpha}^{n_0}| + |\lambda_{\alpha}^n|.
\]

Let \(v_{\alpha}(x) = u_{\alpha}(\lambda_{\alpha}^i x)\). Then \(v_{\alpha}\) will converges to \(v_0\) except finite points. Since \(\lambda_{\alpha}^0\) and \(\lambda_{\alpha}^1\) satisfies \(H1\) or \(H2\), then we have \(\frac{|x_{\alpha}^0|}{\lambda_{\alpha}^0} \to +\infty\), or \(\frac{|x_{\alpha}^n|}{\lambda_{\alpha}^n} \to +\infty\), and therefore we have \(\frac{\lambda_{\alpha}^i}{\lambda_{\alpha}^0} \to +\infty\).

So, it is easy to check that 0 is a blowup point of the sequence \(\{v_{\alpha}\}\).

Similar to the proof of (3.2), we have for any \(\epsilon > 0\), there are \(\delta_1\) and \(R\) s.t.
\[
\int_{B_2 \setminus B_\delta(x_{\alpha})} |\nabla g u_{\alpha}|^2 dV \leq \epsilon, \quad \forall \lambda \in (R \lambda_{\alpha}^i, \delta_1).
\]
(3.10)

We set \(v_{\alpha}(x) = u_{\alpha}(\lambda_{\alpha}^i x)\) and assume \(v_{\alpha} \to v_0\). Then using the arguments in the above subsection (in this case \(F(t_0) \to \lim_{R \to +\infty} \lim_{\alpha \to 1} E_{\alpha}(v_{\alpha}, B_R)\) as \(t_0 \to 1\)), we have
\[
\lim_{\alpha \to 1} \int_{B_2(x_{\alpha})} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^2)^{\alpha - 1} |\nabla u_{\alpha}|^2 dV = \lim_{\alpha \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_R} (\lambda_{\alpha}^2 \epsilon_{\alpha} + |\nabla u_{\alpha}|^2)^{\alpha - 1} |\nabla u_{\alpha}|^2.
\]
Moreover, (3.10) implies that all the blowup points lie in \(B_R\) for some \(R > 0\).

The rest of the proof will be divided into two cases: i) \(v_0\) is a non-trivial harmonic map. ii) \(v_0\) is trivial.

In case i), \(v_0\) is a bubble, then we can assume \(v_0\) is in fact one of \(v_i\)'s for \(i \in \{2, \cdots, n_0\}\). We set \(v^\alpha_{\lambda\alpha}\) to be equivalent to \(v_0\), then \(\lim_{\alpha \to 1} (\lambda')^{2-2\alpha} = \mu_{\lambda\alpha}\), and \(E(v_{\lambda\alpha}) = E(v^\alpha_{\lambda\alpha})\). Since there is only \(n_0 - 1\) bubbles of the sequence \(\{v_{\lambda\alpha}\}\), by induction, we have
\[
\lim_{\alpha \to 1} \int_{B_R} (\lambda_{\alpha}^2 \epsilon_{\alpha} + |\nabla u_{\alpha}|^2)^{\alpha - 1} |\nabla u_{\alpha}|^2 dV = E(v_{\lambda\alpha}, B_R) + \sum_{i \neq \lambda\alpha} (\mu_i/\mu_{\lambda\alpha})^2 E(v^i).
\]
In case ii), one is easy to check that \(\frac{|x_{\alpha}^0|}{\lambda_{\alpha}^0} \to +\infty\). Then \(x_0 = \lim_{\alpha \to 1} \frac{x_{\alpha}^0}{\lambda_{\alpha}^0}\) which lies on \(\partial B_1\) is a blow-up point. Then there are at least two blowup points 0 and \(x_0\). So, at any blowup point of \(v_{\alpha}\), there are most \(n_0 - 1\) bubbles, and then we can use the induction. Hence, we will get
\[
\lim_{\alpha \to 1} \int_{B_R} (\lambda_{\alpha}^2 \epsilon_{\alpha} + |\nabla u_{\alpha}|^2)^{\alpha - 1} |\nabla u_{\alpha}|^2 dV = \sum_{i = 1}^{n_0} (\lim_{\alpha \to 1} (\lambda_{\alpha}^i)^{2-2\alpha})^2 E(v^i).
\]
Thus, we complete the proof of Theorem 1.1.
4 Description and further analysis of the necks

In this section, we always assume there is only one bubble on some small ball $B_{\delta}$.

4.1 The proof of Theorem 1.2 in the case $\nu = 1$

In this subsection, we assume $\nu = 1$. Then we have $\mu = 1$, and

$$
\lim_{\delta \to 0} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{\delta} \setminus B_{R\lambda}(x_\alpha)} |\nabla g u_\alpha|^2 dV_g = 0. \tag{4.1}
$$

We will use the arguments of Ding [1].

For simplicity, we assume $P = \log \delta - \log R\lambda$ is an integer. For any integer $k \in [1, P - 1]$, we set

$$
Q_k(t) = B_{2k+t\lambda}(x_\alpha) \setminus B_{2k-t\lambda}(x_\alpha),
$$

where $t + k \leq P$ and $k - t \geq 0$.

Using the same approximate method as in Section 3.1, we can conclude that on $Q_k(t)$ the following inequality holds

$$
\int_{Q_k(t)} |\nabla u_\alpha|^2 dx \leq \int_{Q_k(t)} A(u_\alpha)(du_\alpha, du_\alpha)(u_\alpha - u_\alpha^*) dx + C(\alpha - 1) \int_{Q_k(t+2)} |\nabla u_\alpha|^2 dx + \int_{\partial Q_k(t)} \frac{\partial u_\alpha}{\partial r}(u_\alpha - u_\alpha^*) ds_0 + \int_{Q_k(t)} |\nabla u_\alpha|^2 dx. \tag{4.2}
$$

Next, we will apply Pohozaev identity (2.5) to control the last term in the above inequality, i.e. $\int_{Q_k(t)} |\nabla u_\alpha|^2 dx$. For the sake of convenience, we set

$$
H(r) = -\int_{B_{\epsilon}(x_\alpha)} \frac{\nabla g \nabla u_\alpha}{\epsilon_\alpha + |\nabla g u_\alpha|^2} \frac{\partial u_\alpha}{\partial r} dV_g = -\int_{B_{\epsilon}(x_\alpha)} \frac{\nabla_0 \nabla g u_\alpha |\nabla g u_\alpha|^2}{\epsilon_\alpha + |\nabla g u_\alpha|^2} \frac{\partial u_\alpha}{\partial r} dx.
$$

Using Lemma 3.1, we have

$$
|H(r)| \leq \int_{B_{\epsilon}(x_\alpha)} |\nabla g u_\alpha|^2 \frac{\partial u_\alpha}{\partial r} dx + |H(R\lambda)| \leq C \int_{B_{\epsilon}(x_\alpha)} |\nabla u_\alpha|^2 dx + |H(R\lambda)| < C',
$$

where we use the fact

$$
\lim_{\alpha \to 1} |H(R\lambda)| \leq \lim_{\alpha \to 1} \int_{B_{R\lambda}(x_\alpha)} |\nabla g u_\alpha|^2 |\nabla g v_\alpha| dx = \int_{B_{R\lambda}} |\nabla_0 v_\alpha|^2 |\nabla g v_\alpha| dx < C(R).
$$

Therefore, combining these with (2.5) we obtain
\[
\int_{Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 dx - \frac{1}{2} \int_{Q_k(t)} |\nabla_0 u|^2 dx \leq C \int_{2^{-t} R_\lambda}^{|u| \frac{1}{r} dr} \leq C(\alpha - 1)t.
\]

It follows (4.2) and the above inequality
\[
(\frac{1}{2} - \epsilon_1) \int_{Q_k(t)} |\nabla_0 u|^2 dx \leq C(\alpha - 1)(t + 1) + \int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right| (u_\alpha - u_\alpha^*) ds_0.
\] (4.3)

On the other hand, we have
\[
\left| \int_{\partial Q_k(t)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 \right| \leq \sqrt{\int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 ds_0 \int_{\partial Q_k(t)} |u_\alpha - u_\alpha^*|^2 ds_0}
\]
\[
\leq \sqrt{\int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 ds_0 \int_{\partial Q_k(t)} |u_\alpha,0|^2 r^2 ds_0}
\]
\[
\leq \frac{1}{2} \left[ \int_{\partial Q_k(t)} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 r ds_0 + \int_{\partial Q_k(t)} |u_\alpha,0|^2 r ds_0 \right]
\]
\[
= \frac{1}{2} \int_{\partial Q_k(t)} r |\nabla u_\alpha|^2 ds_0
\]
\[
= 2^{t+k-1} R_\lambda \int_{\partial B_{2^{t+k} R_\lambda}(x_\alpha)} |\nabla_0 u_\alpha|^2 ds_0
\]
\[
- 2^{k-t-1} R_\lambda \int_{\partial B_{2^{k-t} R_\lambda}(x_\alpha)} |\nabla_0 u_\alpha|^2 ds_0.
\]

Let
\[
f_k(t) = \int_{Q_k(t)} |\nabla u_\alpha|^2 dx.
\]

From (4.4) we know
\[
\int_{\partial Q_k(t)} \frac{\partial u_\alpha}{\partial r} (u_\alpha - u_\alpha^*) ds_0 \leq \frac{1}{2} \log 2 f_k'(t).
\]

Hence, by combining (4.3) and the above inequality we have
\[
(1 - 2\epsilon_1) f_k(t) \leq \frac{1}{2} \log 2 f_k'(t) + C(\alpha - 1)(t + 1).
\]

Multiplying the two sides of the above inequality by \(2^{-(1-2\epsilon_1)t}\) and integrating we obtain
\[
f_k(1) \leq C 2^{-(1-2\epsilon_1)t_k} f_k(t_k) + C(\alpha - 1).
\]

It is easy to check that, if we set
\[
t_1 = L_k = \left\{ \begin{array}{ll} k & \text{if } 2k - 1 \leq P \\ P - k & \text{if } 2k - 1 > P \end{array} \right.
\]
then, we get
\[
\sqrt{E(u_\alpha, Q_k(1))} \leq C 2^{-aL_k} \sqrt{E(u_\alpha, B_\delta \setminus B_{R_\lambda}(x_\alpha))} + C \sqrt{\alpha - 1}
\]
for some positive \(a\) and \(C\).
By the standard $L^p$ estimate, we have
\[
\osc_{B_{2^{k+1}R\lambda\alpha}} B_{2^{-k}R\lambda\alpha}(x_{\alpha}) u_{\alpha} \leq C 2^{-aL_k} \sqrt{E(u_{\alpha}, B_{\delta} \setminus B_{R\lambda\alpha}(x_{\alpha}))} + C \sqrt{\alpha - 1}.
\]
These inequalities imply
\[
\osc_{B_\delta \setminus B_{R\lambda\alpha}}(x_{\alpha}) u_{\alpha} \leq C \sqrt{E(u_{\alpha}, B_\delta \setminus B_{R\lambda\alpha}(x_{\alpha}))} + C (R, \delta) \sqrt{\alpha - 1} + C \log \lambda_{\alpha} \sqrt{\alpha - 1}.
\]
Letting $\alpha \to 1$, and then $R \to +\infty$, $\delta \to 0$, we get
\[
\osc_{B_\delta \setminus B_{R\lambda\alpha}}(x_{\alpha}) u_{\alpha} \to 0.
\]
Thus we proved Theorem 1.2 in the case $\nu = 1$.

4.2 The details of the neck when $\nu > 1$

The goal of this section is to show the neck converges to a geodesic in $N$ and furthermore calculate the length of the geodesic.

For this sake, we will consider the behaviors of $u_{\alpha}$ on $\partial B_{2^{\lambda}t_{\alpha}} (x_{\alpha})$ with $t \in [t_2, t_1]$, where $0 < t_2 < t_1 < 1$. By the arguments in section 3.1.2, we can see easily that
\[
\int_{B_{2^{\lambda}t_{\alpha}} (x_{\alpha})} |\nabla g u_{\alpha}|^2 dV_g \to \mu^2 t E(v^1)
\]
in $C^0([t_2, t_1])$. Then, it is easy to yield
\[
\int_{B_{2^{\lambda}t_{\alpha}} \setminus B_{t_{\alpha}} (x_{\alpha})} |\nabla g u_{\alpha}|^2 dV_g \to 0
\]
in $C^0([t_2, t_1])$. Therefore, for any $t \in [t_2, t_1]$, we have
\[
\osc_{\partial B_{t_{\alpha}} (x_{\alpha})} u_{\alpha} \leq C \int_{B_{2^{\lambda}t_{\alpha}} \setminus B_{t_{\alpha}} (x_{\alpha})} |\nabla g u_{\alpha}|^2 dV_g \to 0,
\]
i.e. $u_{\alpha}|_{\partial B_{t_{\alpha}} (x_{\alpha})}$ will subconverge to a point belonging to $N$. Especially, we have that, as $\alpha \to 1$,
\[
u_{\alpha}(\partial B_{t_{\alpha}} (x_{\alpha})) \to y_1 \in N \text{ and } u_{\alpha}(\partial B_{t_2} (x_{\alpha})) \to y_2 \in N.
\]

For simplicity, we will use “$(r, \theta)$” to denote “$x_{\alpha} + r(\cos \theta, \sin \theta)$”. Now we can state the main results of this subsection as follows:

**Proposition 4.1.** When $\nu > 1$ and $0 < t_2 \leq t_{\alpha} \leq t_1 < 1$, we have, after passing to a subsequence,
\[
\lim_{\alpha \to 1} \frac{1}{\alpha - 1} \int_{B_{R\lambda\alpha} \setminus B_{R\lambda\alpha}(x_{\alpha})} |u_{\alpha, \theta}|^2 dx = 0
\]
for any $R > 0$, and

$$\frac{1}{\sqrt{\lambda_0^\alpha}} \left( u_\alpha(\lambda_0^\alpha r, \theta) - u_\alpha(\lambda_0^\alpha, 0) \right) \to \bar{a} \log r$$

strongly in $C^k(S^1 \times [\frac{1}{R}, R], \mathbb{R}^n)$, where $\theta$ is the angle parameter of the ball centered at $x_\alpha$, $\bar{a} \in T_y N$ is a vector in $\mathbb{R}^n$ with

$$|\bar{a}| = \mu \lim_{\alpha \to 1} t_{\bar{a}} \sqrt{E(v)}$$

and $y = \lim_{\alpha \to 1} u_\alpha(\lambda_0^\alpha, \theta)$.

To prove Proposition 4.1, we first prove the following

**Lemma 4.2.** When $\nu > 1$ and $0 < t_2 \leq t_\alpha \leq t_1 < 1$, we have

$$\lim_{\alpha \to 1} \frac{1}{\alpha - 1} \int_{B_{R \lambda_0^\alpha} \setminus B_{\frac{1}{2} \lambda_0^\alpha}(x_\alpha)} |u_{\alpha, \theta}|^2 dx < C$$

where $C$ does not depend on $R$.

**Proof.** We set

$$Q(t) = B_{2t \lambda_0^\alpha}(x_\alpha) \setminus B_{2 - t \lambda_0^\alpha}(x_\alpha).$$

Here we assume $2^t \leq \lambda_0^{-\varepsilon}$, where

$$\varepsilon < \min\{t_2, 1 - t_1\}.$$

Applying (2.5), we get from (4.2) the following

$$\left( \frac{1}{2} - \epsilon_1 \right) \int_{Q(t)} \left| \nabla_0 u_\alpha \right|^2 dx \leq \left( \alpha - 1 \right) \left( \int_{2 - t \lambda_0^\alpha}^{2t \lambda_0^\alpha} \frac{1}{r} H(r) dr \right) + C \int_{B_{2t \lambda_0^\alpha} \setminus B_{2 - t \lambda_0^\alpha}(x_\alpha)} |\nabla_0 u_\alpha|^2 dx - \int_{\partial Q(t)} \frac{\partial u_\alpha}{\partial r}(u_\alpha - u_\alpha^\circ)ds.$$  

For any $r \in [\lambda_0^{\alpha + \varepsilon}, \lambda_0^{-\varepsilon}]$, it is easy to check that

$$|H(r) - H(\lambda_0^\alpha)| \leq \int_{B_{\lambda_0^{\alpha - \varepsilon}} \setminus B_{\lambda_0^{\alpha + \varepsilon}}(x_\alpha)} \left| \nabla_0^2 u_\alpha \right| r \left| \frac{\partial u_\alpha}{\partial r} \right| dx.$$  

Using Lemma 3.1, we can get

$$\int_{B_{\lambda_0^{\alpha}} \setminus B_{\lambda_0^{\alpha + \varepsilon}}(x_\alpha)} \left| \nabla_0^2 u_\alpha \right||r \frac{\partial u_\alpha}{\partial r}| dx \leq C \int_{B_{2 \lambda_0^{\alpha - \varepsilon}} \setminus B_{\lambda_0^{\alpha + \varepsilon}}(x_\alpha)} \left| \nabla_0 u_\alpha \right|^2 dx.$$  

By integrating (2.3) we obtain

$$\left( 1 - \frac{1}{2\alpha} \right) \int_{\frac{1}{2} \lambda_0^{\alpha - \varepsilon}}^{2\lambda_0^{\alpha - \varepsilon}} ds \int_{\partial B_s(x_\alpha)} (\epsilon_\alpha + \left| \nabla g u_\alpha \right|^2)^{\alpha - 1} \left| \frac{\partial u_\alpha}{\partial r} \right|^2 ds_0$$

$$- \frac{1}{2\alpha} \int_{\frac{1}{2} \lambda_0^{\alpha + \varepsilon}}^{2\lambda_0^{\alpha + \varepsilon}} ds \int_{\partial B_s(x_\alpha)} (\epsilon_\alpha + \left| \nabla g u_\alpha \right|^2)^{\alpha - 1} \frac{1}{s^2} \left| \frac{\partial u_\alpha}{\partial \theta} \right|^2 ds_0$$

$$= \left( \frac{\alpha - 1}{\alpha s} \right) \int_{B_s(x_\alpha)} (\epsilon_\alpha + \left| \nabla g u_\alpha \right|^2)^{\alpha - 1} \left| \nabla_0 u_\alpha \right|^2 dx + O(\lambda_0^{2(t_\alpha - \varepsilon)}).$$
By Corollary 3.3 we know that the second term on the left hand side of the above inequality vanishes as \( \alpha \to 0 \). On the other hand, noting the fact \((\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1}\) is bounded, we have

\[
\int_{\frac{1}{2}\lambda_\alpha^{t_\alpha-\epsilon}}^{2\lambda_\alpha^{t_\alpha-\epsilon}} \left( \frac{(\alpha - 1)}{\alpha s} \int_{B_s(x_\alpha)} (\epsilon_\alpha + |\nabla_g u_\alpha|^2)^{\alpha-1} |\nabla_0 u_\alpha|^2 dx \right) ds \leq C \int_{\frac{1}{2}\lambda_\alpha^{t_\alpha-\epsilon}}^{2\lambda_\alpha^{t_\alpha-\epsilon}} \frac{(\alpha - 1)}{\alpha s} ds = \frac{C \epsilon}{\alpha} \log \lambda_\alpha^{2+2\alpha}.
\]

Noting the fact \( \lim_{\alpha \to 1} \epsilon_\alpha^{\alpha-1} = \beta_0 > 0 \), we can infer from (4.9) that, as \( \epsilon_\alpha \) is close to 1 enough and \( \lambda_\alpha \) is bounded, we have

\[
\int_{B_2\lambda_\alpha^{t_\alpha-\epsilon}\setminus B_{\frac{1}{2}\lambda_\alpha^{t_\alpha+\epsilon}}(x_\alpha)} |\nabla_0 u_\alpha|^2 dx \leq C \epsilon (\log \lambda_\alpha^{2-2\alpha} + 1) + O(\lambda_\alpha^{2(t_\alpha-\epsilon)}).
\]

So, when \( \alpha \) is close to 1 enough we can always choose \( \epsilon \) such that

\[
C \int_{B_2\lambda_\alpha^{t_\alpha-\epsilon}\setminus B_{\frac{1}{2}\lambda_\alpha^{t_\alpha+\epsilon}}(x_\alpha)} |\nabla_0 u_\alpha|^2 dx \leq C \epsilon (\log \mu + 1) < \epsilon_1.
\]

Hence, we have

\[
H(\lambda_\alpha^{t_\alpha}) - \epsilon_1 \leq H(\tau) \leq H(\lambda_\alpha^{t_\alpha}) + \epsilon_1. \tag{4.10}
\]

Let

\[
f(t) = \int_{Q(t)} |\nabla_g u_\alpha|^2 dV = \int_{Q(t)} |\nabla_0 u_\alpha|^2 dx.
\]

By using a similar estimate with (4.4) and (4.10), we infer that as \( \alpha \) is close to 1 enough there holds

\[
(1 - 2\epsilon_1)f(t) \leq \frac{1}{\log 2} f'(t) + (\alpha - 1)(at + \epsilon_1),
\]

where

\[
a = 4 \log 2 H(\lambda_\alpha^{t_\alpha}) + \epsilon_1.
\]

Then, it is easy to see

\[
(2^{-(1-2\epsilon_1)\tau} f)' \geq -(\alpha - 1)(at + \epsilon_1)2^{-(1-2\epsilon_1)\tau} \log 2.
\]

Hence, we get

\[
f(t) \leq 2^{-(1-2\epsilon_1)(\tau-t)} f(\tau) + \frac{\alpha - 1}{1 - 2\epsilon_1} \left( \epsilon_1 + at + \frac{a}{\log 2} - a\tau 2^{-(1-2\epsilon_1)(\tau-t)} - a \frac{2^{-(1-2\epsilon_1)(\tau-t)}}{(1 - 2\epsilon_1) \log 2} \right).
\]

Then, it follows

\[
f(k) \leq C_1(k) 2^{-(1-2\epsilon_1)\tau} f(\tau) + \frac{\alpha - 1}{1 - 2\epsilon_1} \left( \epsilon_1 + ak + \frac{a}{\log 2} + aC_2(k)a\tau 2^{-(1-2\epsilon_1)\tau} \right).
\]

Let \( 2^\tau = \lambda_\alpha^{t_\alpha-\epsilon} \). Then

\[
\int_{B_2\lambda_\alpha^{t_\alpha}\setminus B_{\frac{1}{2}\lambda_\alpha^{t_\alpha}}(x_\alpha)} |\nabla_0 u_\alpha|^2 dx \leq C(k) \lambda_\alpha^{\epsilon(1-2\epsilon_1)} + \frac{\alpha - 1}{1 - 2\epsilon_1} (H(\lambda_\alpha^{t_\alpha})) 4k \log 2 + \frac{a}{\log 2} + C(k) \lambda_\alpha^{\epsilon(1-2\epsilon_1) \log \lambda_\alpha}. \tag{4.11}
\]
On the other hand, by (2.6) and (4.10), we get
\[
\int_{B_{2^k \lambda_{\alpha}} \setminus B_{\frac{1}{2^k} \lambda_{\alpha}}(x_\alpha)} \left( |\frac{\partial u_{\alpha}}{\partial r}|^2 - |u_{\alpha}|^2 \right) dx \geq (\alpha - 1)4k \log 2 (H(\lambda_{\alpha}) - \epsilon_1).
\] (4.12)

Therefore, subtracting (4.11) by (4.12) we obtain
\[
2 \int_{B_{2^k \lambda_{\alpha}} \setminus B_{\frac{1}{2^k} \lambda_{\alpha}}(x_\alpha)} |u_{\alpha, \theta}|^2 dx \leq C(k) \lambda_{\alpha}^{(1-2\epsilon_1)} + \frac{(\alpha - 1)}{1 - 2\epsilon_1} (2\epsilon_1 H(\lambda_{\alpha})) 4k \log 2 + \frac{a}{\log 2} 
+ C(k) \lambda_{\alpha}^{(1-2\epsilon_1)} \log \lambda_{\alpha} + \epsilon_1 (\alpha - 1)4k \log 2.
\] (4.13)

Since \( \nu = \lim_{\alpha \to 1} \lambda_{\alpha}^{-\frac{\alpha-1}{\alpha}} > 1 \), it is easy to see that, for any \( m > 0 \),
\[
\lambda_{\alpha}^{(1-2\epsilon_1)} = o((\alpha - 1)^m).
\] (4.14)

Then, noting (4.14) and letting \( \epsilon_1 \to 0 \) in the above inequality (4.13), we get
\[
\lim_{\alpha \to 1} \frac{1}{\alpha - 1} \int_{B_{2^k \lambda_{\alpha}} \setminus B_{\frac{1}{2^k} \lambda_{\alpha}}(x_\alpha)} |u_{\alpha, \theta}|^2 dx \leq \frac{a'}{2 \log 2}
\]
where \( a' \) is a constant which does not depend on \( R \). Thus, we finish the proof of the lemma. \( \square \)

Now, we are in the position to give the proof of Proposition 4.1.

**Proof.** First we show (4.6). Since lemma 4.2 says
\[
\int_{B_{2^k \lambda_{\alpha}} \setminus B_{\frac{1}{2^k} \lambda_{\alpha}}(x_\alpha)} |u_{\alpha, \theta}|^2 dx = \int_{2^{-k} \lambda_{\alpha}} \frac{1}{\alpha - 1} r \left( \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 d\theta \right) dr \leq \frac{a'}{2 \log 2},
\]
for any \( \epsilon > 0 \), we can always find \( k_0 \) which is independent of \( \alpha \), s.t. there exist
\[ L_\alpha \in [2^{k_0}, 2^{2k_0}] \]
such that
\[
\frac{1}{\alpha - 1} \int_{\partial B_{L_\alpha \lambda_{\alpha}}(x_\alpha)} |u_{\alpha, \theta}|^2 r ds = \frac{1}{\alpha - 1} \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^2 (L_\alpha \lambda_{\alpha}, \theta) d\theta < \epsilon,
\]
and
\[
\frac{1}{\alpha - 1} \int_{\partial B_{\frac{1}{2^k} \lambda_{\alpha}}(x_\alpha)} |u_{\alpha, \theta}|^2 r ds = \frac{1}{\alpha - 1} \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^2 d\theta < \epsilon.
\]

Then
\[
\left| \int_{\partial Q(\log L_\alpha/\log 2)} \frac{\partial u_{\alpha}}{\partial r} (u_{\alpha} - u_{\alpha}^*) ds \right| \leq \sqrt{\int_{\partial Q(\log L_\alpha/\log 2)} r \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 ds \int_0^{2\pi} \left| \frac{\partial u_{\alpha}}{\partial \theta} \right|^2 d\theta} 
\leq \epsilon (\alpha - 1) \int_{\partial Q(\log L_\alpha/\log 2)} r \left| \frac{\partial u_{\alpha}}{\partial r} \right|^2 ds.
\]

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From Lemma 2.2 and (2.3), we get
\[
\int_{\partial Q} r \frac{\partial u_\alpha}{\partial r}^2 ds \leq C \int_{\partial Q} \frac{|u_\alpha|}{\log 2}^2 ds + C(\alpha - 1) + C\lambda_\alpha^t
\leq (C + \epsilon)(\alpha - 1).
\]

By (4.8) and (4.10), we get
\[
(1 - 2\epsilon_1) \int_{B_{\lambda^t_\alpha \lambda} \setminus B_{\lambda^t_\alpha \lambda} (x_\alpha)} |\nabla_0 u_\alpha|^2 dx \leq \epsilon_1 (\alpha - 1) + 2(\alpha - 1)(2H(\lambda^t_\alpha) \log L_\alpha + \epsilon). \quad (4.15)
\]
Noting (4.10) we can infer from (2.6)
\[
\frac{1}{\alpha - 1} \int_{B_{\lambda^t_\alpha \lambda} \setminus B_{\lambda^t_\alpha \lambda} (x_\alpha)} \left( \frac{\partial u_\alpha}{\partial r}^2 - |u_\alpha| \right) dx = \int_{\lambda^t_\alpha \lambda} \frac{2}{r} H(r) dr,
\]
which implies that
\[
\frac{1}{\alpha - 1} \int_{B_{\lambda^t_\alpha \lambda} \setminus B_{\lambda^t_\alpha \lambda} (x_\alpha)} \left( \frac{\partial u_\alpha}{\partial r}^2 - |u_\alpha| \right) dx \geq 4 \log L_\alpha (H(\lambda^t_\alpha) - \epsilon).
\]
Combining (4.15) with the above inequality we conclude the following inequality holds true as \( \alpha \) is close to 1 sufficiently
\[
\frac{1}{\alpha - 1} \int_{B_{\lambda^t_\alpha \lambda} \setminus B_{\lambda^t_\alpha \lambda} (x_\alpha)} |u_\alpha| \right) dx \leq \epsilon_1 + C\epsilon.
\]
Thus, we have shown (4.6).

Next, we turn to proving the remaining assertions of Proposition 4.1.
For \( \{t_\alpha\} \subset [t_2, t_1] \), we assume
\[
u_\alpha(x) \rightarrow y, \quad \alpha \rightarrow 1.
\]
As \( N \) is regarded as an embedded submanifold in \( \mathbb{R}^K \), for simplicity, we may assume \( y = 0 \in N \) and \( T_y N = \mathbb{R}^n \), where \( \mathbb{R}^K = \mathbb{R}^n \times \mathbb{R}^{K-n} \). We also let \( \lambda'_\alpha = \lambda^t_\alpha, x'_\alpha = (\lambda'_\alpha, 0) + x_\alpha \) and
\[
u'_\alpha(x) = u_\alpha(x), \quad v'_\alpha(x) = \frac{1}{\sqrt{\alpha - 1}} [u_\alpha(x) - u_\alpha(x'_\alpha)].
\]
By (4.11) and Theorem 2.1, we get
\[
\|\nabla u'_\alpha\|_{C^0(B_{2k} \setminus B_{2-k})} + \|\nabla^2 u'_\alpha\|_{C^0(B_{2k} \setminus B_{2-k})} < C(k)\sqrt{\alpha - 1},
\]
and then
\[
\|\nabla v_\alpha\|_{C^0(B_{2k} \setminus B_{2-k})} + \|\nabla^2 v_\alpha\|_{C^0(B_{2k} \setminus B_{2-k})} < C(k).
\]
Noting that \( v_\alpha(1, 0) = 0 \), we get
\[
\|v_\alpha\|_{C^0(B_{2k} \setminus B_{2-k})} < C'(k).
\]
Obviously, we have the equation:

$$\Delta_{0} v_{\alpha} + \sqrt{\alpha - 1} (A(y) + o(1)) (dv_{\alpha}, dv_{\alpha}) + (\alpha - 1) O(|\nabla^{2} v_{\alpha}|) = 0,$$

hence, the sequence

$$v_{\alpha} \rightarrow v_{0} \quad \text{in} \quad C^{k}_{\text{loc}}(R^{2} \setminus \{0\})$$

where $v_{0}$ satisfies

$$\Delta_{0} v_{0} = 0 \quad \text{with} \quad v_{0} = v_{0}(|x|).$$

Set

$$v = (a_{1}, a_{2}, \cdots, a_{n}, 0, \cdots, 0) \log r.$$

We deduce from (2.4) that

$$\int_{\partial B} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^{2})^{\alpha - 1} |\nabla^{0} v_{\alpha}|^{2} ds_{0} = \frac{2\alpha}{\frac{2\alpha - 1}{t} \int_{B_{t}} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^{2})^{\alpha - 1} |v_{\alpha, \theta}|^{2} ds_{0}} + \frac{2}{(2\alpha - 1) t} \int_{B_{t}} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^{2})^{\alpha - 1} |\nabla_{0} u_{\alpha}|^{2} dx.$$

Recalling that

$$F_{\alpha}(t) = \int_{B_{\lambda_{t}^{\alpha}}} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^{2})^{\alpha - 1} |\nabla_{0} u_{\alpha}|^{2} dx$$

and keeping (4.6) in our minds, we infer from the above identity and Lemma 2.2 that, as $\alpha \rightarrow 1$,

$$\int_{B_{2\lambda_{t}^{\alpha}} \setminus B_{\lambda_{t}^{\alpha}}} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^{2})^{\alpha - 1} |\nabla_{0} v_{\alpha}|^{2} dx = \frac{2\alpha}{2\alpha - 1} \int_{B_{\lambda_{t}^{\alpha}}} \frac{1}{t} F_{\alpha}(\log \lambda_{t}^{\alpha}, t) dt + o(1)$$

$$= \frac{2\alpha}{2\alpha - 1} \log 2 F_{\alpha}(t_{\alpha}) + o(1)$$

$$\rightarrow 2 \log 2 F(\lim_{\alpha \rightarrow 1} t_{\alpha}).$$

On the other hand, we have that, as $\alpha \rightarrow 1$, there holds

$$\int_{B_{2\lambda_{\alpha}^{0}} \setminus B_{\lambda_{\alpha}^{0}}} (\epsilon_{\alpha} + |\nabla u_{\alpha}|^{2})^{\alpha - 1} |\nabla_{0} v_{\alpha}|^{2} dx = \int_{B_{2} \setminus B_{1}} \left( \epsilon_{\alpha} + |\nabla_{0} v_{\alpha}|^{2} \right) \alpha - 1 \frac{2\alpha - 1}{\lambda_{\alpha}^{2}} |\nabla_{0} v_{\alpha}|^{2} dx$$

$$\rightarrow 2 \pi \mu^{\alpha - 1} |a|^{2} \log 2.$$

Hence, we get

$$\lim_{\alpha \rightarrow 1} v_{\alpha} = (a_{1}, \cdots, a_{n}, 0, \cdots, 0) \log r$$

with

$$\sum_{i=1}^{m} a_{i}^{2} = \frac{\Lambda}{\pi} \mu^{1 - 2 \lim_{\alpha \rightarrow 1} t_{\alpha}}.$$

As $v : S^{2} \rightarrow N$ is the corresponding only bubble, then the above identity can be written as

$$|a|^{2} = \frac{E(v, S^{2})}{\pi} \mu^{2 - 2 \lim_{\alpha \rightarrow 1} t_{\alpha}}.$$

Thus, we complete the proof of Proposition 4.1. \square
Corollary 4.3. Let $\alpha_k$ be a sequence s.t.

$$E_{\alpha_k}(u_{\alpha_k}, B_{\lambda_{\alpha_k}}(x_\alpha)) \to \mu^{2-t}E(v)$$

in $C^0([t_2, t_1])$ with respect to $C^0$-norm. If $\nu > 1$, then

$$\int_{\lambda_{\alpha_k}}^{2\lambda_{\alpha_k}} \frac{1}{\sqrt{\alpha - 1}} \frac{\partial u_{\alpha_k}}{\partial r} |dr| \to \log 2 \mu^{1-t} \frac{E(v)}{\pi}$$

in $C^0([t_2, t_1])$, and

$$\frac{1}{\sqrt{\alpha - 1}} (r \frac{\partial u_{\alpha_k}}{\partial r})(\lambda_{\alpha_k}^t, \theta) \to \mu^{1-t} \frac{E(v)}{\pi}$$

in $C^0([t_2, t_1] \times S^1)$.

Proof. We need only to prove the first claim, since the proof of the second claim is similar. If the first claim was not true, then we assumed that there was a subsequence $\alpha_{k_i}, t_i \to t_0$ s.t.

$$\left| \int_{\lambda_{\alpha_{k_i}}}^{2\lambda_{\alpha_{k_i}}} \frac{1}{\sqrt{\alpha - 1}} \frac{\partial u_{\alpha_{k_i}}}{\partial r} |dr| - \log 2 \mu^{1-t_0} \frac{E(v)}{\pi} \right| \geq \epsilon > 0.$$

On the other hand, from the above arguments on Proposition 4.1 we know that, after passing to a subsequence, there holds

$$\frac{u_{\alpha_{k_i}}(\lambda_{\alpha_{k_i}} x) - u_{\alpha_{k_i}}(\lambda_{\alpha_{k_i}}^l, 0)}{\sqrt{\alpha - 1}} \to \bar{a} \log r,$$

with $|\bar{a}| = \left| \mu^{1-t_0} \sqrt{\frac{E(v)}{\pi}} \right|$. Hence we derive the following

$$\lim_{i \to +\infty} \int_{\lambda_{\alpha_{k_i}}}^{2\lambda_{\alpha_{k_i}}} \frac{1}{\sqrt{\alpha - 1}} \frac{\partial u_{\alpha_{k_i}}}{\partial r} |dr| = |\bar{a}| \int_1^2 \frac{1}{r} dr = \log 2 \mu^{1-t_0} \sqrt{\frac{E(v)}{\pi}}.$$

This is a contradiction. \hfill $\square$

4.3 The proof Theorem 1.2 in the case $\nu > 1$

First, we need to show the necks for the $\alpha$-harmonic map sequence converge to some geodesics in $N$ which join the bubbles. For this goal, we denote the curve

$$\frac{1}{2\pi} \int_0^{2\pi} u_\alpha(r, \theta) d\theta : [\lambda_{\alpha}^l, \lambda_{\alpha}^t] \rightarrow \mathbb{R}^n$$

by $\Gamma_\alpha$. For simplicity, we set

$$\omega_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} u_\alpha(r, \theta) d\theta.$$

First, we claim that if $\Gamma_\alpha \to \Gamma$, then $\Gamma$ must lies on $N$. This is a direct corollary of (4.5). Next, we will prove a subsequence of $\Gamma_\alpha$ will converges locally to a geodesic of $N$ and then give the formula of length of $\Gamma$. 

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By computation we have
\[
\ddot{\omega}_\alpha = \frac{1}{2\pi} \int_0^{2\pi} \dddot{u}_\alpha(r, \theta) d\theta \\
= \frac{1}{2\pi} \int_0^{2\pi} (\dddot{u}_\alpha(r, \theta) + \frac{u_{\alpha, \theta \theta}}{r^2}) d\theta \\
= \frac{1}{2\pi} \int_0^{2\pi} \Delta_0 u_\alpha d\theta - \frac{1}{2\pi} \int_0^{2\pi} \dddot{u}_\alpha d\theta \\
= -\frac{1}{2\pi} \int_0^{2\pi} A(u_\alpha)(du_\alpha, du_\alpha) - \frac{\alpha - 1}{2\pi} \int_0^{2\pi} \nabla_0 [\nabla_g u_\alpha]^2 \nabla_0 u_\alpha d\theta - \frac{\dot{\omega}_\alpha}{r} 
\]

where we have used the fact
\[
\int_0^{2\pi} u_{\theta \theta}(r, \theta) d\theta = 0.
\]

Let
\[
G_\alpha = -\ddot{\omega}_\alpha - \frac{\dot{\omega}_\alpha}{r}.
\]

Denote the induced metric of $\Gamma_\alpha$ in $\mathbb{R}^K$ by $h_\alpha$, and let $A_{\Gamma_\alpha}$ be the second fundamental form of $\Gamma_\alpha$ in $\mathbb{R}^K$.

Given $\lambda_\alpha^{t_0} \in [\lambda_\alpha^{t_1}, \lambda_\alpha^{t_2}]$. As before, we always have
\[
u \alpha - 1 \| \lambda_\alpha^{t_0} \rangle 
\overline{\alpha} - 1
\int_0^{2\pi} A(u_\alpha)(du_\alpha, du_\alpha) d\theta + \frac{\alpha - 1}{2\pi} \int_0^{2\pi} \nabla_0 [\nabla_g u_\alpha]^2 \nabla_0 u_\alpha d\theta
\]
\[
= \frac{\alpha - 1}{\lambda_\alpha^{2t_0}} \left( \frac{1}{2\pi} \int_0^{2\pi} A(y)(\ddot{a}, \ddot{a}) d\theta + o(1) \right) + (\alpha - 1) \int_0^{2\pi} O(\| \nabla_0^2 u_\alpha \|) d\theta
\]
\[
= \frac{\alpha - 1}{\lambda_\alpha^{2t_0}} \left( A(y)(\ddot{a}, \ddot{a}) + o(1) + O(\sqrt{\alpha - 1}) \right)
\]
\[
= \frac{\alpha - 1}{\lambda_\alpha^{2t_0}} \left( A(y)(\ddot{a}, \ddot{a}) + o(1) \right).
\]

In the above identity we have made use of the fact $\nu > 1$ which implies that for any $m > 0$
\[
\lambda_\alpha^{2t_0} = o((\alpha - 1)^m).
\]

Noting that $\langle A(y)(\ddot{a}, \ddot{a}) \rangle = 0$, we get
\[
-A_{\Gamma_\alpha}(d\omega_\alpha, d\omega_\alpha) = -A_{\Gamma_\alpha}(\ddot{\omega}_\alpha, \ddot{\omega}_\alpha) + -\frac{\dot{\omega}_\alpha}{|\dot{\omega}_\alpha|^2} \dddot{\omega}_\alpha = -A_{\Gamma_\alpha}(\ddot{\omega}_\alpha, \dddot{\omega}_\alpha) + \frac{G_\alpha - 1}{|\dot{\omega}_\alpha|^2} \dddot{\omega}_\alpha
\]
\[
= -\frac{\alpha - 1}{\lambda_\alpha^{2t_0}} \left( A(y)(\ddot{a}, \ddot{a}) + o(1) \right) + \frac{G_\alpha}{|\dot{\omega}_\alpha|^2} \dddot{\omega}_\alpha.
\]

\[
\text{(4.17)}
\]
Hence, we get
\[ \| A_{\alpha} \|^2_{h_\alpha}(\lambda_{\alpha}^t) < C. \]

Similar to the proof of Corollary 4.3, we have, after passing to a subsequence,
\[ \| A_{\alpha} \|^2_{h_\alpha}(\lambda_{\alpha}^t) < C \]
for any \( t \in [t_2, t_1] \).

Now, we fix \( y \in N \), and let \( s \) to be the arc length parameter of \( \omega_\alpha(t) \) with \( s(\lambda_{\alpha}^t) = 0 \). We assume \( \omega_\alpha(\lambda_{\alpha}^t) \rightarrow y \) as \( \alpha \rightarrow 1 \). It is well-known that \( \| A_{\alpha} \|^2_{h_\alpha}(\lambda_{\alpha}^t) \) does not depend on the choice of parameter, and
\[ \frac{d^2\omega_\alpha}{ds^2} = -A_{\alpha}(\omega_\alpha)(\frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds}), \]
then \( \omega_\alpha(s) \) will converges locally to \( \omega(s) \) in \( C^1 \), where \( s \) is still the arc length parameter. This implies that \( \Gamma_\alpha|_{[\lambda_{\alpha}^{t_1}, \lambda_{\alpha}^{t_2}]} \) converges locally to a curve \( \Gamma \) locally. We claim that
\[ A_{\alpha}(\omega_\alpha)(\frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds}) \rightarrow A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}) \]
strongly in \( C^0([0, s_1], \mathbb{R}^n) \) for sufficiently small \( s_1 \). If this was not true, then for any small \( s_1 \) we could find a subsequence of \( \{u_\alpha\} \), still denoted by \( \{u_\alpha\} \), such that
\[ s'_\alpha = \int_{\lambda_{\alpha}^{t_\alpha}}^{\lambda_{\alpha}^{t_\alpha}} |\dot{\omega}_\alpha|dr \rightarrow s' \in (0, s_1) \]
s.t.
\[ \left| A_{\alpha}(\omega_\alpha)(\frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds}) - A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}) \right|_{s=s'_\alpha} > \epsilon. \]
To apply Proposition 4.1 we must ensure that \( t'_\alpha \in [t_2, t_1] \). For simplicity, we may assume \( \lambda_{\alpha}^{t_\alpha} = 2^P \lambda_{\alpha}^{t_2} \) where \( P \) is an integer. Then, applying Corollary 4.3 we have
\[ \int_{2^1 \lambda_{\alpha}^{t_\alpha}}^{2^1 \lambda_{\alpha}^{t_\alpha}} |\dot{\omega}_\alpha|dr = \sqrt{\alpha - 1} \mu^{-i(t_\alpha-i\log \lambda_{\alpha}^1)} \left( \log 2 \sqrt{\frac{E(v)}{\pi}} + o_\alpha(1) \right). \]
Therefore, as \( \alpha \) is close to 1 enough,
\[ \int_{\lambda_{\alpha}^{t_2}}^{\lambda_{\alpha}^{t_\alpha}} |\dot{\omega}_\alpha|dr = \sum_{i=0}^{P-1} \int_{2^i \lambda_{\alpha}^{t_\alpha}}^{2^1 \lambda_{\alpha}^{t_\alpha}} |\dot{\omega}_\alpha|dr \geq P \sqrt{\alpha - 1} \left( \sqrt{\frac{E(v)}{\pi}} \log 2 + o_\alpha(1) \right) \geq C(t_\alpha - t_2) \log \lambda_{\alpha}^{-\sqrt{\alpha-1}} \geq C \frac{t_2}{2} \log \nu > 0. \]
Therefore, we may always choose \( s_1 \) to be very small, for example \( s_1 < C \frac{t_2}{2} \log \nu \), such that \( t'_\alpha \in [\frac{t_2}{2}, t_1] \). Then, as before there holds
\[ \frac{u_\alpha(\lambda_{\alpha}^{t'_\alpha} r, \theta) - u_\alpha(\lambda_{\alpha}^{t'_\alpha} r, 0)}{\sqrt{\alpha - 1}} \to a^2 \log r. \]
Obviously,
\[
\frac{\dot{\omega}_\alpha(\lambda'_\alpha)}{|\dot{\omega}_\alpha(\lambda'_\alpha)|} = \frac{d\omega_\alpha}{ds}(s'_\alpha) \to \frac{d\omega}{ds}(s').
\]
Applying (4.16) and (4.17), we get that, after passing a subsequence the following holds
\[
A_{\Gamma_\alpha}(\omega_\alpha)(\frac{d\omega_\alpha}{ds}, \frac{d\omega_\alpha}{ds})\big|_{s=s'_\alpha} = \frac{1}{|\dot{\omega}_\alpha(\lambda'_\alpha)|^2} A_{\Gamma_\alpha}(\omega_\alpha)(\dot{\omega}_\alpha, \dot{\omega}_\alpha)\big|_{r=\lambda'_\alpha} \to A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds})\big|_{s=s'},
\]
which contradicts the choice of \(s'_\alpha\). So, we infer that
\[
\frac{d\omega}{ds}(s) - \frac{d\omega}{ds}(0) = - \int_0^s A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}) ds
\]
holds near \(s = 0\). This shows \(\omega\) is smooth near 0 and satisfies
\[
d^2 \omega \frac{ds^2}{ds} = - A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}).
\]
Therefore, we obtain finally
\[
\nabla^N \frac{d\omega}{ds} = d^2 \omega \frac{ds^2}{ds} + A(\omega)(\frac{d\omega}{ds}, \frac{d\omega}{ds}) = 0,
\]
which means that \(\Gamma\) is a geodesic.

Next, we calculate the length of the geodesic \(\Gamma\). For simplicity, we assume \(\lambda'^2_\alpha = 2^P \lambda'^3_\alpha\) for some integer \(P\). Then we have
\[
P = \frac{t_2 - t_1}{\log 2} \log \lambda_\alpha.
\]
When \(\nu = +\infty\), by Corollary 4.3, we have
\[
L(\Gamma_\alpha|_{B_{2k+1}\lambda'^1_\alpha, x_\alpha}) \geq \sqrt{\alpha - 1} \left( \sqrt{\frac{E(v)}{\pi}} \log 2 + o(1) \right).
\]
Then
\[
L(\Gamma_\alpha) \geq CP \sqrt{\alpha - 1} \geq C \log \lambda^{-\sqrt{\alpha - 1}} \to +\infty.
\]
This implies
\[
L(\Gamma) = +\infty.
\]
Now, we assume \(\nu < +\infty\). By Corollary 4.3
\[
L(\Gamma_\alpha|_{B_{2k+1}\lambda'^1_\alpha, x_\alpha}) = \sqrt{\alpha - 1} \left( \sqrt{\frac{E(v)}{\pi}} \log 2 + o(1) \right),
\]
where \(o(1) \to 0\) as \(\alpha \to 1\) uniformly. Hence
\[
L(\Gamma) = \lim_{\alpha \to 1} \sqrt{\alpha - 1} \sqrt{\frac{E(v)}{\pi}} \log 2 = (t_1 - t_2) \frac{E(v)}{\pi} \log \nu.
\]
Now, it is easy to see that to complete the proof of Theorem 1.2 we only need to prove the following:

\[
osc_{B_{\delta} \setminus B_{R\lambda}}(x_{\alpha}) u_{\alpha} \to 0, \quad \text{as} \quad \alpha \to 1, \quad \text{then} \quad R \to +\infty \quad \text{and} \quad t \to 1 \tag{4.18}
\]

and

\[
osc_{B_{\delta} \setminus B_{\lambda_{i}^{t}}} (x_{\alpha}) u_{\alpha} \to 0, \quad \text{as} \quad \alpha \to 1, \quad \text{then} \quad \delta \to 0 \quad \text{and} \quad t \to 0. \tag{4.19}
\]

Since \( \nu < +\infty \) implies \( \mu = 1 \), from Theorem 1.1 we know

\[
\lim_{t \to 1} \lim_{R \to +\infty} \lim_{\alpha \to 1} \int_{B_{\delta} \setminus B_{R\lambda}} |\nabla u_{\alpha}|^2 = 0.
\]

Therefore, we can use the same method as in Subsection 4.1 (we replace \( \delta \) with \( \lambda_{i}^{t} \)) to deduce

\[
osc_{B_{\lambda_{i}^{t}} \setminus B_{R\lambda}} u_{\alpha} \leq C \sqrt{E(u_{\alpha}, B_{\lambda_{i}^{t}} \setminus B_{R\lambda}(x_{\alpha}))} + C(1 - t) \log \nu + C\sqrt{\alpha - 1},
\]

then (4.18) follows. Similarly, we can prove (4.19). Hence, we derive the length formula of the geodesic \( \Gamma \)

\[
L = \sqrt{\frac{E(\nu)}{\pi}} \log \nu.
\]

Thus, we finish the proof of Theorem 1.2. \( \square \)

Next, we want to give the proof of Corollary 1.3. However, to prove the corollary we only need to prove the following proposition:

**Proposition 4.4.** If \( \nu < +\infty \), then when \( (\alpha - 1) \) is sufficiently small, all the \( u_{\alpha} \) are in the same homotopy class.

**Proof.** When \( \alpha_{i} - 1 \) and \( \alpha_{j} - 1 \) are sufficiently small, we have \( ||u_{i} - u_{j}||_{C^{0}} \leq i(N) \) where \( i(N) \) is the injective radius of \( N \). Hence, by using exponential map we know that \( u_{\alpha_{i}} \) and \( u_{\alpha_{j}} \) are homotopic in \( M \setminus B_{\delta}, B_{\lambda_{i}^{t_{1}}} \setminus B_{\lambda_{j}^{t_{2}}} \) and \( B_{R\lambda} \) respectively.

Let \( p = u_{0}(0) \) and \( q = v(+\infty) \). By (4.18), we know that \( u_{i}(B_{\delta}(x_{\alpha}) \setminus B_{\lambda_{i}^{t_{2}}}(x_{\alpha})) \) is contained in a simply connected ball centered at \( p \) when \( \alpha \) is close to 1 enough, \( \delta \) and \( t_{1} \) are small enough. Similarly, by (4.19) we also have \( u_{j}(B_{\lambda_{j}^{t_{2}}} \setminus B_{R\lambda}(x_{\alpha})) \) is contained in a small simply connected ball in \( N \) with center \( q \) when \( \alpha - 1, \delta \) and \( 1 - t_{1} \) are sufficiently small. Hence \( u_{i} \) and \( u_{j} \) are homotopic in \( B_{\delta} \setminus B_{\lambda_{i}^{t_{1}}} \) and \( B_{\lambda_{i}^{t_{1}}} \setminus B_{R\lambda} \) respectively. So \( u_{i} \) and \( u_{j} \) are homotopic. \( \square \)

5 Some comments and an example

In this paper we only consider the case \( u_{\alpha} \) is an \( \alpha \)-harmonic maps when the conformal structure of \( M \) is fixed. Naturally, one will ask the following problems (i) what could we say in the case \( u_{\alpha} \) is an \( \alpha \)-harmonic maps and the conformal structure of \( M \) varies with \( \alpha \), (ii) whether the methods in this paper can be extended to a class of variational problem which is more general than \( \alpha \)-energy or not. In a forthcoming paper we will further develop some tools to discuss some issues which relate to the above problems.

On the other hand, one want to know whether one can give an example to show there is a neck joining the bubbles in the limit of an \( \alpha \)-harmonic map sequence is of infinite length or not.
However, if we can construct a manifold $N$ and find a minimizing $\alpha$-harmonic map sequence which satisfies the condition of Corollary 1.3 then the corollary tells us that indeed there exists a necks in the limit which if of infinite length. By modifying the example of Duzaar and Kuwert (see page 304 of [D-K]) we can construct such example as following.

Example. Let $\mathbb{Z}^3$ act on $\mathbb{R}^3$ by $\tau_\kappa(x, y, z) = (x+4k_1, y+4k_2, z+4k_3)$, where $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$. Consider

$$\tilde{X} = \mathbb{R}^3 \setminus \cup_{\kappa} \tau_\kappa(B_1(0))$$

and $X$ is the quotient of $\tilde{X}$. Then $X$ is a compact manifold with boundary. Topologically, $X$ is $T^3$ minus a small ball.

Let $\Phi$ be a conformal map from $\mathbb{R}^2$ to $\partial B_1(0)$, s.t. $\Phi(x) = (1, 0, 0)$ when $|x| > 2$ and $\Phi(x) = (-1, 0, 0)$ when $|x| < 1$, and $deg(\Phi) = 1$ if we consider $\Phi$ be a map from $S^2$ to $S^2$. Moreover, we let $\gamma_k : [0, 1] \to \tilde{X}$ be a curve connect $(4k-1, 0, 0)$ and $(1, 0, 0)$. We define

$$v_k = \begin{cases} 
\Phi(x), & |x| \geq \delta \\
\gamma_k \left( \log \frac{r}{\log |\Phi|} \right), & R\epsilon < |x| < \delta \\
\tau_{(k,0,0)}(\Phi(\tau)), & |x|/\epsilon \leq R
\end{cases}$$

We denote $\pi$ to be the projection from $\tilde{X}$ to $X$, then $\pi(v_k) \in \pi_2(X)$. We have

$$\int_{B_2 \setminus B_{R\epsilon}} |\nabla v_k|^2 = 2\pi \int_{R\epsilon}^{\delta} \frac{\partial \gamma_k}{\partial r}^2 r dr < c \frac{\|\gamma\|^2_{L^\infty}}{(-\log R\epsilon + \log \delta)^2} \int_{R\epsilon}^\delta \frac{dr}{r} = c \frac{\|\gamma\|^2_{L^\infty}}{\log \delta - \log R\epsilon}$$

$$\int_{R^2 \setminus B_\delta} |\nabla v_k|^2 \leq E(\Phi), \quad \text{and} \quad \int_{B_{R\epsilon}} |\nabla v_k|^2 \leq E(\Phi).$$

So, we can find suitable $R$ and $\epsilon$, s.t.

$$E(\pi(u_k)) = E(v_k) \leq 2E(\Phi) + 1.$$

We claim that $[\pi(v_k)]$ are different homotopy classes. Assuming this is not true, we can find a continuous map

$$H(x, t) : S^2 \times [0, 1] \to X$$

s.t.

$$H(x, 0) = \pi(v_i) \quad \text{and} \quad H(x, 1) = \pi(v_j).$$

Since $S^2 \times [0, 1]$ is simply connected, we are able to lift $H$ to $\tilde{H}$ which is a map from $S^2 \times [0, 1] \to \tilde{X}$ with $\tilde{H}(x, 0) = v_i$. We assume that $\tilde{H}(x, 1) = \tau_\kappa(v_{i'})$. Hence $[v_i] = [\tau_\kappa(v_{i'})]$. Therefore

$$[\partial B_1(0) + \partial \tau_{(i,0,0)}(B_1(0))] = [\partial \tau_\kappa(B_1(0)) + \partial \tau_{(j,0,0)} \tau_\kappa(B_1(0))] \quad \text{in} \quad \pi_2(\tilde{X}),$$

where $\pi_2(\tilde{X})$ is the second homotopy group of $\tilde{X}$. However, it is easy to check that $\pi_1(\tilde{X}) = \{1\}$, then by Hurewicz Theorem, the above identity is not true.

Now, we proceed to construct $N$. Let $f$ be a homeomorphism from $X$ to $Y = X$. We consider the quotient space of $X \cup Y$, obtained by glueing every point $x \in \partial X$ with $f(x) \in \partial Y$.
together. In this way, we get a closed compact manifold $N$ and a projection $\phi : N \to X$. One is easy to check that $\pi(v_k)$ can be also considered as a map from $S^2$ to $N$ with $E(\pi(v_k)) < C$. We claim that $[\pi(v_k)]$ are some different homotopic classes with each other in $\pi_2(N)$. Assuming it is not true. Then, we can find a continuous map $H(x, t) : S^2 \times [0, 1] \to N$ such that $H(x, 0) = \pi(v_i)$ and $H(x, 1) = \pi(v_j)$. Hence, $\phi(H(x, t))$ is just a homotopic map of $\pi(v_i)$ and $\pi(v_j)$ in $X$. A contradiction.

Finally, we would like to ask the following problems:

**Problem 1.** Suppose all $\alpha$-harmonic maps $u_\alpha$ belong to the same homotopic class and satisfy the energy identity as $\alpha \to 1$. Do the necks consist of some geodesics of finite length?

**Problem 2.** Could we find a sequence $\alpha_k \to 1$, and $\alpha_k$ harmonic maps $u_{\alpha_k}$, s.t. 1) the Morse index tends to infinite; 2) $\sup_k E_{\alpha_k}(u_{\alpha_k}) < \infty$; 3) for any $i \neq j$, $u_{\alpha_i}$ and $u_{\alpha_j}$ are not homotopic to each other.

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