Reduction of Coupling Parameters
and Duality

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Abstract

The general method of the reduction in the number of coupling parameters is discussed. Using renormalization group invariance, theories with several independent couplings are related to a set of theories with a single coupling parameter. The reduced theories may have particular symmetries, or they may not be related to any known symmetry. The method is more general than the imposition of invariance properties. Usually, there are only a few reduced theories with an asymptotic power series expansion corresponding to a renormalizable Lagrangian. There also exist ‘general’ solutions containing non-integer powers and sometimes logarithmic factors. As an example for the use of the reduction method, the dual magnetic theories associated with certain supersymmetric gauge theories are discussed. They have a superpotential with a Yukawa coupling parameter. This parameter is expressed as a function of the gauge coupling. Given some standard conditions, a unique, isolated power series solution of the reduction equations is obtained. After reparametrization, the Yukawa coupling is proportional to the square of the gauge coupling.

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coupling parameter. The coefficient is given explicitly in terms of the numbers of colors and flavors. ‘General’ solutions with non-integer powers are also discussed. A brief list is given of other applications of the reduction method.
1. Introduction.

The method of reduction in the number of coupling parameters [1, 2, 3, 4, 5], [6, 7, 8, 9, 10], [11], [12] has found many theoretical and phenomenological applications. It is a very general method, based essentially upon the requirement of renormalization group invariance of the original multi-parameter theory, as well as the related reduced theories with fewer couplings. Combining the renormalization group equations of original and reduced theories, we obtain a set of reduction equations. These are differential equations for the removed couplings considered as functions of the remaining parameters. They are necessary and sufficient for the independence of the reduced theories from the normalization mass. We consider massless theories, or mass independent renormalization schemes, so that no mass parameters appear in the coefficient functions of the renormalization group equations. This can be arranged, provided the original coefficient functions have a well defined zero-mass limit [12].

In this paper, we discuss only reductions to a single coupling, which covers most cases of interest. Usually, we can choose one of the original couplings as the remaining parameter. The multi-parameter theory is assumed to be renormalizable with an asymptotic power series expansion in the weak coupling limit. However the reduced theories, as obtained from the reduction equations, may well not all have such expansions in the remaining coupling. Non-integer powers and logarithms can appear, often with undetermined coefficients. Such general solutions do not correspond to conventional renormalized power series expansions associated with a Lagrangian. But they are still well defined in view of their embedding in the renormalized multi-parameter theory. Nevertheless, it is the relatively small number of uniquely determined power series solutions of the reduction equations, which is of primary interest. Depending upon the character of the system considered, there may be additional requirements which further reduce the number of these solutions. Although we consider renormalizable theories, with appropriate assumptions, the reduction method can also be applied in cases where the original theory is non-renormalizable.

Regular reparametrization is a very useful tool in connection with the reduction method. For theories with two or more coupling parameters, it is not possible to reduce the $\beta$-function expansions to polynomials. However, in the reductions to one coupling, we can usually remove all but the first term in the power series...
solutions of the reduction equation with determined coefficients. The $\beta$-functions of the corresponding reduced theories remain however infinite series. As seen from many examples, these reparametrizations lead to frames which are very natural for the reduced theories.

The imposition of a symmetry on the multi-parameter theory is a conventional way of relating the coupling parameters. If there appear no anomalies, we get a renormalizable theory with fewer parameters so as to implement the symmetry. These situations are all included in the reduction scheme, but our method is more general, leading also to unique power series solutions which exhibit no particular symmetry. This situation is illustrated by an example we have included. An $SU(2)$ gauge theory with matter fields in the adjoint representation. Besides the gauge coupling, there are three additional couplings. With only the gauge coupling remaining after the reduction, we get two acceptable power solutions. One of the reduced theories is an $N = 2$ supersymmetric gauge theory, while the other solution leads to a theory with no particular symmetry.

The main example presented in this article is connected with duality [13, 14, 15, 16]. We consider $N = 1$ supersymmetric QCD (SQCD) and the corresponding dual theory, magnetic SQCD. The primary interest is in the phase structure of the physical system described by these theories. Essential aspects of this phase structure were first obtained on the basis of supercovergence relations and BRST methods [17, 18, 19], and more recently with the help of duality [13, 15]. We exhibit the quantitative agreement of both approaches [20, 21]. While duality is formulated only in connection with supersymmetry, the supercovergence arguments can be used also for QCD and similar theories [18, 19, 22]. Of particular interest is the transition point at $N_F = \frac{3}{2}N_C$ for SQCD [17, 13], where $N_F$ and $N_C$ are the numbers of flavors and colors respectively. It is the lower end of the conformal window. For smaller values of $N_F$, the quanta of free, electric SQCD are confined, the system is described by free magnetic excitations of the dual theory (for $N_C > 4$), and eventually by mesons and baryons. (The corresponding transition point for QCD is given by $N_F = \frac{13}{4}N_C$).

As the original theory, SQCD has only the gauge coupling $g_e$. The dual theory is constructed on the basis of the anomaly matching conditions [13, 13]. It involves the two coupling parameters $g_m$ and $\lambda_1$, where $\lambda_1$ is a Yukawa coupling associated with a superpotential. This potential is required by duality, mainly since theories,
which are dual to each other, must have the same global symmetries.

At first, we apply the reduction method to the magnetic theory in the conformal window $\frac{2}{3}N_C < N_F < 3N_C$. We find two power series solutions. After reparametrization, one solution is given by $\lambda_1(g_m^2) = g_m^2 f(N_C, N_F)$, with $f$ being a known function of the numbers of colors and flavors for SQCD. The other solution is $\lambda_1(g_m^2) \equiv 0$. Since the latter removes the superpotential, it is excluded, and we are left with a unique single power solution. This solution implies a theory with a single gauge coupling $g_m$, and renormalized perturbation expansions which are power series in $g_m^2$. It is the appropriate dual of SQCD. There are ‘general’ solutions, but they all approach the excluded power solution $\lambda_1(g^2) \equiv 0$. With one exception, they involve non-integer powers of $g_m^2$. The reduction can be extended to the ‘free electric region’ $N_F > 3N_C$, and to the ‘free magnetic region’ $N_C + 2 < N_F < \frac{2}{3}N_C$, $(N_C > 4)$. The results are similar, and discussed in detail in [23]. In the free magnetic case, we deal however with the approach to a trivial infrared fixed-point.

Possible connections of the reduction results with features of brane dynamics remain to be considered. Internal fluctuations of branes may be of relevance for the field theory properties obtained here.

2. Reduction Equations

We consider renormalizable quantum field theories with several coupling parameters. It is assumed that there is a mass-independent renormalization scheme, so that no mass parameters occur in the coefficient functions of the renormalization group equations. Let $\lambda, \lambda_1, \ldots, \lambda_n$ be $n + 1$ dimensionless coupling parameters of the theory. One can reduce this system in various ways, but we want to consider the parameter $\lambda$ as the primary coupling, and express the remaining $n$ couplings as functions of $\lambda$:

$$\lambda_k = \lambda_k(\lambda), \quad k = 1, \ldots, n .$$

(1)

It is assumed, that these functions $\lambda_k(\lambda)$ are independent of the renormalization mass $\kappa$, which can always be arranged.

The Green’s functions $G(k_i, k^2, \lambda, \lambda_1, \ldots, \lambda_n)$ of the original multi-parameter version of the theory satisfy the usual renormalization group equations with the coefficient functions $\beta, \beta_k$, and the anomalous dimension $\gamma_G$, which depend upon the $n + 1$ coupling parameters. The corresponding Green’s functions of the reduced
theory are given by
\[ G(k, \kappa^2, \lambda) = G \left( k, \kappa^2, \lambda, \lambda_1(\lambda), \ldots, \lambda_n(\lambda) \right) . \]  
(2)

Renormalization group invariance requires that they satisfy the equation
\[ \left( \kappa^2 \frac{\partial}{\partial \kappa^2} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \gamma_G(\lambda) \right) G(k, \kappa^2, \lambda) = 0 , \]  
(3)

where \( \beta(\lambda) \) and \( \gamma_G(\lambda) \) are given by the corresponding original coefficients with the insertions \( \lambda_k = \lambda_k(\lambda), \quad k = 1, \ldots, n \). Comparison of Eq.(3) with the original multi-parameter renormalization group equation implies then
\[ \beta(\lambda) \frac{d\lambda_k(\lambda)}{d\lambda} = \beta_k(\lambda), \quad k = 1, \ldots, n \]  
(4)

These are the Reduction Equations, which are necessary and sufficient for the validity of Eq.(3).

It is of interest to briefly consider the relationship between the reduction method as described above, and the equations for the effective coupling functions \( \overline{\lambda}(u), \overline{\lambda}_k(u) \), where \( u \) is the dimensionless scaling parameter \( u = k^2/\kappa^2 \). These functions satisfy the equations
\[ u \frac{d \overline{\lambda}}{du} = \beta(\overline{\lambda}, \overline{\lambda}_1, \ldots, \overline{\lambda}_n) , \]
\[ u \frac{d \overline{\lambda}_k}{du} = \beta_k(\overline{\lambda}, \overline{\lambda}_1, \ldots, \overline{\lambda}_n) . \]  
(5)

With \( \overline{\lambda}(u) \) being an analytic function, we can choose a point where \( (d\overline{\lambda}(u)/du) \neq 0 \) and introduce \( \overline{\lambda}(u) \) as a new variable Eqs.(5,6). The result is again the reduction equations (4).

With effective couplings, we study the multi-parameter theory at different mass scales. In the reduction method, we consider the set of different field theories with one coupling parameter (or a reduced number), which can be obtained from a given multi-parameter theory as solutions of the reduction equations. The elements of this set are labeled by the free parameters of the solution, and all are considered at the same fixed mass scale. With some natural assumptions the number of theories in this set is usually smaller than the number of original coupling parameters, and the different theories have characteristic physical and mathematical features. This is best seen in examples, some of which we discuss below. It must be remembered,
that the origin of the coupling parameter space is a singular point, so that the Picard-Lindeloef theorem about the uniqueness of solutions at regular points does not apply.

As described so far, the reduction scheme is very general, but in practice we usually know the \( \beta \)-functions only as asymptotic expansions in the small coupling limit. Within the framework of renormalized perturbation theory, we restrict ourselves here to expansions of the form

\[
\beta(\lambda, \lambda_1, \ldots, \lambda_n) = \beta_0 \lambda^2 + (\beta_1 \lambda^3 + \beta_{1k} \lambda_k \lambda^2 + \beta_{1kk'} \lambda_k \lambda_{k'} \lambda) \\
+ \sum_{n=4}^{\infty} \sum_{m=0}^{n-1} \beta_{n-2,k_1,\ldots,k_m} \lambda_{k_1} \cdots \lambda_{k_m} \lambda^{n-m},
\]

\[
\beta_k(\lambda, \lambda_1, \ldots, \lambda_n) = (c_k^{(0)} \lambda^2 + c_{k,k'}^{(0)} \lambda_{k'} \lambda + c_{k,k',k''}^{(0)} \lambda_{k'} \lambda_{k''} \lambda_{k''}) \\
+ \sum_{n=3}^{\infty} \sum_{m=0}^{n-2} c_{k,k_1,\ldots,k_m}^{(n-2)} \lambda_{k_1} \cdots \lambda_{k_m} \lambda^{n-m}.
\]  

In writing the expansions (6), we have assumed that the primary coupling \( \lambda \) is chosen such that \( \beta(0, \lambda_1, \ldots, \lambda_n) = 0 \).

With the original \( \beta \)-functions given as asymptotic power series expansions, we will consider in the following solutions \( \lambda_k(\lambda) \) of the reduction equations, which are also of the form of asymptotic expansions. Of special interest are solutions which are power series expansions. But in general, non-integer powers as well as logarithmic terms are possible.

### 3. Power Series Solutions

Let us first consider solutions of the reduction equations (4) which are asymptotic power series expansions. Then the Green’s functions \( G(k_i, \kappa^2; \lambda) \) of the reduced theory have power series expansions in \( \lambda \) and are associated with a corresponding renormalizable Lagrangian. It is reasonable to write

\[
\lambda_k(\lambda) = \lambda f_k(\lambda), \quad k = 1, \ldots, n,
\]

where the functions \( f_k(\lambda) \) are bounded for \( \lambda \to 0 \) so that \( \lambda_k(0) = 0 \). According to the reduction equations, if we had \( \lambda_k(0) \neq 0 \), the vanishing of \( \beta(0, \lambda_1(0), \ldots, \lambda_n(0)) \) would imply that also \( \beta_k(0, \lambda_1(0), \ldots, \lambda_n(0)) \) vanishes, which is too strong a restriction and not fulfilled by Eq.(6). In terms of the functions \( f_k(\lambda) \) the reduction equations are
of the form
\[ \beta \left( \lambda \frac{df_k}{d\lambda} + f_k \right) = \beta_k , \] (8)
where we have introduced the \( \beta \)-functions
\[ \beta(\lambda) = \beta(\lambda, \lambda f_1, \ldots, \lambda f_n) = \sum_{n=0}^{\infty} \beta_n(\lambda) \lambda^{n+2} , \] (9)
\[ \beta_k(\lambda) = \beta_k(\lambda, \lambda f_1, \ldots, \lambda f_n) = \sum_{n=0}^{\infty} \beta_k^{(n)}(\lambda) \lambda^{n+2} . \] (10)

Here the argument \( f \) stands for \( f_1(\lambda), \ldots, f_n(\lambda) \). The coefficients are easily obtained from Eqs.(6). For example, the one-loop terms are given by
\[ \beta_0(f) = \beta_0, \quad \beta_k^{(0)}(f) = c_k^{(0)} + c_{kk'}^{(0)} f_{k'} + c_{kk'k''}^{(0)} f_k f_{k''} . \] (11)

For the functions \( f_k(\lambda) \), we write the expansions
\[ f_k(\lambda) = f_k^0 + \sum_{m=1}^{\infty} \chi_k^{(m)}(\lambda) \lambda^m , \] (12)
and insert them, together with the series (9) and (10), into the reduction equations. At the one-loop level, there result then the relations
\[ \beta_k(f^0) - f_k^0 \beta_0 = 0 , \] (13)
or in explicit form using Eq.(11),
\[ c_k^{(0)} + (c_{kk'}^{(0)} - \beta_0 \delta_{kk'}) f_k^0 + c_{kk'k''}^{(0)} f_k^0 f_{k''} = 0 . \] (14)

These are the fundamental formulae for the reduction.

Given a solution \( f_k^0 \) of the quadratic equations (14), we obtain for the expansion coefficients \( \chi_k^{(m)} \) the relations
\[ \left( M_{kk'}(f^0) - m \beta_0 \delta_{kk'} \right) \chi_k^{(m)} = \left( \beta_m(f^0) f_k^0 - \beta_k^{(m)}(f^0) \right) + X_k^{(m)} , \] (15)
where \( m = 1, 2, \ldots, k = 1, \ldots, n \). The matrix \( M(f^0) \) is given by
\[ M_{kk'}(f^0) = c_{kk'}^{(0)} + 2c_{kk'k''}^{(0)} f_k^0 f_{k''} - \delta_{kk'} \beta_0 . \] (16)

The rest term \( X^{(m)} \) depends only upon the coefficients \( \chi^{(1)}, \ldots, \chi^{(m-1)} \), and upon the \( \beta \)-function coefficients in (9) and (10), evaluated at \( f_k = f_k^0 \), for order \( m - 1 \) and lower. They vanish for \( \chi^{(1)} = \ldots = \chi^{(m-1)} = 0 \).
We see that the one-loop criteria
\[
\text{det} \left( M_{k'k}(f^0) - m\beta_0\delta_{k'k} \right) \neq 0 \quad \text{for} \quad m = 1, 2, \ldots
\]  
(17)
are sufficient to insure that all coefficients $\chi^{(m)}$ in the expansion (12) are determined.

Then the reduced theory has a renormalized power series expansion in $\lambda$. All possible solutions of this kind are determined by the one-loop equation (13) for $f_k^0$.

With the coefficients $\chi^{(m)}$ fixed, we can use regular reparametrization transformations in order to remove all but the first term in the expansion (12) of the functions $f_k(\lambda)$. These reparametrization transformations are of the form
\[
\begin{align*}
\lambda' &= \lambda'(\lambda, \lambda_1, \ldots, \lambda_n) = \lambda + a^{(20)}\lambda^2 + a^{(11)}_k\lambda + \cdots , \\
\lambda_k' &= \lambda_k'(\lambda, \lambda_1, \ldots, \lambda_n) = \lambda_k + b^{(20)}_{kk'k''}\lambda_k\lambda_{k'}\lambda_{k''} + b^{(11)}_{kk'}\lambda_k\lambda + \cdots .
\end{align*}
\]  
(18)
They leave invariant the one-loop quantities
\[
\begin{align*}
f_k^0, \beta_0(f^0), \beta_k^{(0)}(f^0), M_{kk'}(f^0).
\end{align*}
\]  
(19)
Given the condition (17), we then have a frame where
\[
\begin{align*}
\lambda_k(\lambda) &= \lambda f_k^0.
\end{align*}
\]  
(20)
This result is valid to all orders of the asymptotic expansion and determined by one-loop information. With the expressions (20), the $\beta$-function expansions (9) and (10) of the reduced theory have constant coefficients $\beta_m(f^0)$, $\beta_k^{(m)}(f^0)$, but they are generally not polynomials. They satisfy the relations
\[
\begin{align*}
\beta_k^{(m)}(f^0) - f_k^0\beta_m(f^0) &= 0.
\end{align*}
\]  
(21)
for all values of $m$. Only the relations for $m = 0$ are reparametrization invariant. They are the fundamental formulae (13).

So far, we have implicitly assumed that $f_k^0 \neq 0$. But it is straightforward to include the cases where $f_k^0 = 0$. They are of particular interest for supersymmetric theories. Suppose we have a solution of the reduction equations with the asymptotic expansion
\[
\begin{align*}
f_k(\lambda) &= \chi_k^{(N)}\lambda^N + \sum_{m=N+1}^{\infty} \chi_k^{(m)}\lambda^m,
\end{align*}
\]  
(22)
where $N \geq 1$ and $\chi_k^{(N)} \neq 1$. Then coefficients appearing in this equation are again determined except for the first one, which is invariant. Hence, using regular reparametrization, there is a frame where
\[
f_k(\lambda) = \chi_k^{(N)} \lambda^N.
\] (23)

We have considered here only expansions at the origin in the space of coupling parameters. However, one can use the method also in connection with any non-trivial fixed point of the theory.

2. General Solutions

At first, let us briefly consider the case where the determinant appearing in Eq.(17) vanishes. Suppose there is a positive eigenvalue of the matrix $\beta_0^{-1} M(f^0)$ for some $m = N \leq 1$, $\beta_0 \neq 0$. Then the asymptotic power series must be supplemented by terms of the form $\lambda^m (\log \lambda)^p$, with $m \leq N$ and $1 < p < \sigma(N)$. After reparametrization, we obtain then an expansion of the form
\[
f_k(\lambda) = f_k^0 + \chi_k^{(N,1)} \lambda^N \log \lambda + \chi_k^{(N)} \lambda^N + \ldots ,
\] (24)

All parameters in Eq.(24) are determined except the vector $\chi_k^{(N)}$, which contains as many free parameters as the degeneracy of the eigenvalue. Even though the theory considered here can have logarithmic terms in the asymptotic expansion, it is ‘renormalized’ in view of it’s embedding into the original, renormalized multi-parameter theory. In special cases it may happen that the coefficients of the logarithmic terms vanish, as in the example of the massless Wess-Zumino model.

We now return to systems with non-vanishing determinant for all values of $m$. In addition to the power series solutions described before, there can be general solutions of the reduction equations, which approach the latter asymptotically. In order to describe a characteristic case, we assume that $\beta_0 \neq 0$ and that the matrix $\beta_0^{-1} M(f^0)$ has one positive eigenvalue $\eta$ which is non-integer, with all others being negative. Then the reduction equations (4) have solutions of the form
\[
f_k(\lambda) = f_k^0 + \sum_{a,b} \chi_k^{(a\eta+b)} \lambda^{a\eta+b} + \sum_m \chi_k^{(m)} \lambda^m
\] (25)

with $a = 1, 2, \ldots , b = 0, 1, \ldots , a\eta + b = $ non-integer. After reparametrization, powers with $m < \eta$ are removed, and we have
\[
f_k(\lambda) = f_k^0 + \chi_k^{(\eta)} \lambda^\eta + \ldots
\] (26)
In this expansion all coefficients are determined except $\chi_k^{(n)}$, which may contain up to $r$ arbitrary parameters if the eigenvalue $\eta$ is $r$-fold degenerate:

$$\chi_k^{(n)} = C_1 \xi_k^{(1)} + \ldots + C_r \xi_k^{(r)},$$

(27)

where the $\xi_k^{(i)}$ are the eigenvectors.

The results described above can be generalized to situations with several positive, non-integer eigenvalues. In special cases, where the matrix also has a zero eigenvalue, logarithmic factors may appear.

So far, we have assumed that $\beta_0 \neq 0$, and obtained general solutions which approach the power series solution (20) asymptotically with a power law as indicated in Eq.(26). The situation is quite different if $\beta_0 = 0$. Then the matrix $M$ is given by

$$M_{kk'}(f^0) = \left( \frac{\partial^2 \beta_0^{(0)}}{\partial f_{k'}} \right)_0$$

(28)

and we find that the general solutions and the power series solutions differ asymptotically by terms which vanish exponentially. We refer to [2] for more details.

Besides the general solutions, which approach the power series solutions asymptotically, there can be others which move away in the limit $\lambda \to 0$. These are not calculable unless the $\beta$-functions are known more explicitly. However, we can get information about the existence or non-existence of such solutions on the basis of the linear part of the reduction differential equations (4). We find that the theorems of Lyapunov [24], with generalizations by Malkin [25], are applicable here [26]. We refer to [5] for some more discussion, and to [27] for an application. Generally, it turns out that a power series solution (20) is asymptotically stable if there are no negative eigenvalues of the matrix $\beta_0^{-1} M(f^0)$ (or the matrix $\beta_N^{-1} M(f^0)$ in the case of the solution (23)). A solution is unstable if there is at least one negative eigenvalue.

4. Gauge Theory

It should be most helpful to discuss briefly an example. We use a gauge theory with one Dirac field, one scalar and one pseudoscalar field, all in the adjoint representation of SU(2) [4]. Besides the usual gauge couplings, the direct interaction part of the Lagrangian is given by

$$\mathcal{L}_{\text{dir.int.}} = i \sqrt{\lambda_1} \epsilon^{abc} \bar{\psi}^a (A^b + i \gamma_5 B^b) \psi^c - \frac{1}{4} \lambda_2 (A^a A^a + B^a B^a)^2 + \frac{1}{4} \lambda_3 (A^a A^b + B^a B^b)^2$$

(29)
Writing $\lambda = g^2$, where $g$ is the gauge coupling, and $\lambda_k = \lambda f_k$, with $k=1,2,3$, the one-loop $\beta$-function coefficients of this theory are given by

\[
\begin{align*}
(16\pi^2)\beta_{g^0} &= -4 \\
(16\pi^2)\beta_{f_1}^0 &= 8f_1^2 - 12f_1 \\
(16\pi^2)\beta_{f_2}^0 &= 3f_3^2 - 12f_3f_2 + 14f_2^2 + 8f_1f_2 - 8f_1^2 - 12f_2 + 3 \\
(16\pi^2)\beta_{f_3}^0 &= -9f_3^2 + 12f_3f_2 + 8f_3f_1 - 12f_3 - 3.
\end{align*}
\]

(30)

The algebraic reduction equations (14) have four real solutions, which are given by

\[
\begin{align*}
&f_1^0 = 1, \quad f_2^0 = 1, \quad f_3^0 = 1 \\
&f_1^0 = 1, \quad f_2^0 = \frac{9}{\sqrt{105}}, \quad f_3^0 = \frac{7}{\sqrt{105}},
\end{align*}
\]

(31)

and two others with reversed signs of $f_2^0$ and $f_3^0$, so that the classical potential approaches negative infinity with increasing magnitude of the scalar fields. These latter solutions will not be considered further. We note that the Yukawa coupling is required for the consistency of the reduction.

The eigenvalues of the matrix $\beta_{g^0}^{-1}M(f^0)$ are respectively

\[
\left(-2, -3, +\frac{1}{2}\right)
\]

and

\[
\left(-2, -\frac{3}{4} \frac{25 + \sqrt{343}}{\sqrt{105}}, -\frac{3}{4} \frac{25 - \sqrt{343}}{\sqrt{105}}\right) = (-2, -3.189..., -0.470...).
\]

(33)

There are no positive integers appearing in the equations (32) or (33). Hence the coefficients of the power series solutions are determined and can be removed by reparametrization, except for the invariant first term. With $\lambda = g^2$ as the primary coupling, $g$ being the gauge coupling, these solutions are

\[
\begin{align*}
(a) & \quad \lambda_1 = \lambda_2 = \lambda_3 = g^2, \\
(b) & \quad \lambda_1 = g^2, \quad \lambda_2 = \frac{9}{\sqrt{105}} g^2, \quad \lambda_3 = \frac{7}{\sqrt{105}} g^2,
\end{align*}
\]

(34) (35)

which corresponds to an $N = 2$ extended SUSY Yang-Mills theory, and

which is not associated with any known symmetry, at least in four dimensions. Both theories are ‘minimally’ coupled gauge theories with matter fields. The eigenvalues
of the matrix $\beta_{g0}^{-1} M(f^0)$, given in Eqs.(32),(33), are all negative with the exception of the third one for the N=2 supersymmetric theory. In this case we have a general solution corresponding to Eq.(26) with $\eta = +\frac{1}{2}$, and with the coefficient given by $\chi^{(\frac{1}{2})} = (0, C, 3C)$, where $C$ is an arbitrary parameter. The theory with $C \neq 0$ corresponds to one with hard breaking of SUSY. It has an asymptotic power series in $g$ and not in $g^2$, as is the case for the invariant theory.

As we see from Eqs.(32) and (33), both power series solutions have some negative eigenvalues of the matrix $\beta_{g0}^{-1} M(f^0)$, and are therefore unstable. Not all nearby solutions approach them asymptotically.

From the present example, and many others, we realize that the special frame, where the power series solutions of the reduction equations are of the simple form (20), is a natural frame as far as the reduced one-parameter theories are concerned. The $\beta$-functions of the reduced theories are still power series and are not reduced to polynomials.

5. Dual SQCD

As the main application of the reduction method, we consider here the reduction of multi-parameter theories appearing in connection with duality. As a particular example, we discuss the dual magnetic theory associated with SQCD [13, 16]. While SQCD, as the ‘electric’ theory, has the gauge coupling $g_e$ as the only coupling parameter, the dual ‘magnetic’ theory has two parameters: the magnetic gauge coupling $g_m$ and a Yukawa coupling $\lambda_1$, which measures the strength of the interaction of color-singlet superfields with the magnetic quark superfields. It is our aim to discuss the reduced theories where the Yukawa coupling is expressed in terms of the gauge coupling.

For SQCD the gauge group is $SU(N_C)$ with N=1 supersymmetry. There are $N_F$ quark superfields $Q_i$ and their antifields $\tilde{Q}^i$, $i = 1, 2, ..., N_F$ in the fundamental representation. For completeness and later reference, we give here the $\beta$-function coefficients for the electric SQCD theory:

$$\beta_e(g_e^2) = \beta_{e0} g_e^4 + \beta_{e1} g_e^6 + \cdots,$$

with

$$\beta_{e0} = (16\pi^2)^{-1} \left(-3N_C + N_F\right)$$
\[ \beta_{e1} = (16\pi^2)^{-2} \left( 2N_C(-3N_C + N_F) + 4N_F \frac{N_C^2 - 1}{2N_C} \right). \]  

(37)

The corresponding dual magnetic theory is constructed mainly on the basis of the anomaly matching conditions \[ \text{[13, 15, 28]} \]. It involves the gauge group \( G^d = SU(N^d_C) \) with \( N^d_C = N_F - N_C \). Here \( N_F \) is the number of quark superfields \( q_i, \tilde{q}^i, i = 1, 2, ..., N_F \) in the fundamental representation of \( G^d \). Because both theories must have the same global symmetries, the number of flavors \( N_F \) should be the same for SQCD and it’s dual. As we have mentioned, duality requires a non-vanishing Yukawa coupling in the form of a superpotential

\[ W = \sqrt{\lambda_1} M^i_j q_i \tilde{q}^j. \]  

(38)

The \( N^2_F \) gauge singlet superfields \( M^i_j \) are independent and cannot be constructed from \( q \) and \( \tilde{q} \). The superpotential not only provides for the coupling of the \( M \) superfield, but also removes a global \( U(1) \) symmetry acting on \( M \), which would have no counterpart in the electric theory.

In the following, we will be dealing essentially only with the magnetic theory. For convenience, we therefore write \( g \) in place of \( g_m \) for the corresponding gauge coupling. We also omit the subscript \( m \) for the \( \beta \)-function coefficients. Then the \( \beta \)-function expansions of the magnetic theory are

\[ \begin{align*}
\beta(g^2, \lambda_1) &= \beta_0 g^4 + (\beta_1 g^6 + \beta_{11} g^4 \lambda_1) + \cdots \\
\beta_1(g^2, \lambda_1) &= c_1^{(0)} g^2 \lambda_1 + c_{11}^{(0)} \lambda_1^2 + \cdots .
\end{align*} \]  

(39)

The coefficients are given by \[ \text{[21, 23, 29, 30]} \]

\[ \begin{align*}
\beta_0 &= (16\pi^2)^{-1}(3N_C - 2N_F) \\
\beta_1 &= (16\pi^2)^{-2} \left( 2(N_F - N_C)(3N_C - 2N_F) + 4N_F \frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)} \right) \\
\beta_{11} &= (16\pi^2)^{-2} \left( -2N_F^2 \right) \\
c_1^{(0)} &= (16\pi^2)^{-1} \left( -4 \frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)} \right) \\
c_{11}^{(0)} &= (16\pi^2)^{-1} \left( 3N_F - N_C \right).
\end{align*} \]  

(40)

Already at the one-loop level, we see some important features from Eqs.(37),(40). In the interval

\[ \frac{3}{2} N_C < N_F < 3N_C, \]  

(41)
both theories are asymptotically free at large momenta, in particular the magnetic theory for $N_F > \frac{3}{2}N_C$. For $N_F > 3N_C$, the electric theory is not asymptotically free in the UV but in the IR, where the magnetic version remains strongly coupled. Hence we expect that the original electric excitations are present in the ‘physical’ state space. The situation is reversed for $N_F < \frac{3}{2}N_C$, where the electric quanta are confined, and the elementary magnetic excitations describe the system, at least for $N_F > N_C + 2$ where the dual theory exists which is the ‘free magnetic region’. This is the duality picture as proposed by Seiberg, with both theories describing the same physical system.

In the conformal window given in Eq.(41), the electric as well as the magnetic theory are in an interacting non-Abelian Coulomb phase, and it is indicated that they both have non-trivial conformal fixed points at zeroes of the exact $\beta$-functions. At these fixed points the theories are actually equivalent. Near an endpoint of the window, in the infrared limit, one theory may be in a weak coupling situation, and the other, dual theory in a strong coupling regime. Since both theories represent the same system, we can describe the strongly coupled field theory by the weakly coupled dual. The free excitations of the latter may be considered as composites of those of the former theory.

Within the framework of this duality picture, the system undergoes an important phase transition at the point $N_F = \frac{3}{2}N_C$. As has already been mentioned above, below this point the elementary electric quanta are confined in the sense that they are not elements of the physical state space. In the original electric theory, the transition at $N_F = \frac{3}{2}N_C$ is not apparent from the $\beta$-function coefficients, in contrast to the phase change at $N_F = 3N_C$, where $\beta_{e0} = 0$. But in the duality picture, we have $\beta_0 = 0$ at $N_F = \frac{3}{2}N_C$ for the magnetic theory, and this is the indication for the phase transition of the system.

Many years ago, we have obtained the phase transition of SQCD at $N_F = \frac{3}{2}N_C$ by using a rather different method [17]. It involves analyticity and superconvergence of the gauge field propagator, as well as the BRST-cohomology in order to define the physical state space of the theory [18, 21]. The superconvergence relations, where they exist, are exact. They connect long and short distance information, and are not valid in perturbation theory [31, 32, 33].

The asymptotic form of the gauge field propagator is governed by the ratio $\gamma_{00}/\beta_0$, where $\gamma_{00}$ is the anomalous dimension of the gauge field (not the superfield)
at the fixed point $\alpha = 0$. Here $\alpha \geq 0$ is the conventional gauge parameter. Because this parameter is effectively a function of the momentum scale, it tends to a fixed point asymptotically. For example, in general covariant gauges, the discontinuity of the structure function has the asymptotic form

$$ -k^2 \rho(k^2, \kappa^2, g, \alpha) \simeq C(g^2, \alpha) \left( -\beta_0 \ln \frac{k^2}{\kappa^2} \right)^{-\gamma_{00}/\beta_0} ,$$

which is independent of $\alpha$ with the possible exception of the coefficient.

For the discussion of confinement using the BRST cohomology or the quark-antiquark potential, it is most convenient to work in the Landau gauge, where the superconvergence relation is of the form

$$ \int_{-0}^{\infty} dk^2 \rho(k^2, \kappa^2, g, 0) = 0, \quad (43)$$

provided $\gamma_{00}/\beta_0 > 0$. For general gauges $\alpha \geq 0$, the relation is the same except that the right hand side is given by $\alpha/\alpha_0$, were $\alpha_0 = -\frac{\gamma_{00}}{\beta_0}$, with $\gamma_0(\alpha) = \gamma_{00} + \alpha \gamma_{01}$.

In contrast to the duality arguments, the superconvergence method is applicable to non-supersymmetric theories like QCD, where the interval corresponding to the window (53) for SQCD is given by

$$ \frac{13}{4} N_C < N_F < \frac{22}{4} N_C, \quad (44)$$

For $N_F < \frac{13}{4} N_C$ for QCD and for $N_F < \frac{3}{2} N_C$ for SQCD, our arguments show that the transverse gauge field excitations are not elements of the physical state space and hence confined. With some further arguments one can extend this result to quark fields.

For SQCD and similar theories, the connection between duality and superconvergence results is quantitative. For electric and magnetic SQCD, we have the anomalous dimensions

$$ \gamma_{e00} = (16\pi^2)^{-1}(-\frac{3}{2}N_C + N_F)$$

$$ \gamma_{m00} = (16\pi^2)^{-1}\frac{1}{2}(-3N_C + N_F), \quad (45)$$

and with the $\beta$-function coefficients from Eqs.(37),(40), we obtain the relations

$$ \beta_{m0}(N_F) = -2\gamma_{e00}(N_F)$$

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\[ \beta_{e0}(N_F) = -2\gamma_{m00}(N_F) , \quad (46) \]

where the argument \( N_F \) on both sides refers to matter fields with different quantum numbers corresponding to electric and magnetic gauge groups. We have restored the subscript \( m \) for these duality relations. We see that \( \gamma_{e00}(N_F) \) changes sign at the same point \( N_F = \frac{3}{2}N_C \) as \( \beta_{m0}(N_F) \), and the ratio \( \gamma_{e00}(N_F)/\beta_{e0}(N_F) \) is positive below this point, indicating superconvergence and confinement as discussed before.

The exact relations (46) are an indication, that the anomalous dimension coefficients of the gauge fields at the fixed point \( \alpha = 0 \) may have a more fundamental significance, similar to the one-loop \( \beta \)-function coefficients.

Our discussion about the relation of superconvergence and duality results can be extended to similar supersymmetric gauge theories with other gauge groups [21, 34, 35]. The results are analogous. However, in the presence of matter superfields in the adjoint representation [36], the problem is more complicated. There the construction of dual theories requires a superpotential already for the original electric theory, and a corresponding reduction of couplings would be called for. Also the application of the superconvergence arguments is not straightforward. These cases deserve further study.

Duality in general superconformal theories has been discussed in [37], and for softly broken SQCD in [38].

6. Reduced Dual SQCD

The magnetic theory dual to SQCD contains two parameters, the gauge coupling \( g \) and the Yukawa coupling \( \lambda_1 \). We now want to apply the reduction method described in the previous sections and express the coupling parameter \( \lambda_1 \) as a function of \( g^2 \). With Eq.(7) we write

\[ \lambda_1(g^2) = g^2 f_1(g^2) , \quad \text{with} \quad f_1(g^2) = f^0 + \sum_{l=1}^{\infty} \chi^{(l)} g^{2l} . \quad (47) \]

The essential one-loop reduction equation is then

\[ \beta_0 f^0 = \left( c_{11}^{(0)} f^0 + c_1^{(0)} \right) f^0 . \quad (48) \]

There are two solutions:

\[ f^0 = f_{01} = \frac{\beta_0 - c_1^{(0)}}{c_{11}^{(0)}} \quad \text{and} \quad f^0 = f_{00} = 0 , \quad (49) \]
where $f_{01}$ is a function of $N_C$ and $N_F$, and is given by

$$f_{01}(N_C, N_F) = \frac{N_C (N_F - N_C - 2/N_C)}{(N_F - N_C)(3N_F - N_C)},$$

(50)

using the explicit expressions (40) for the coefficients. Here and in the following, we do not consider possible additional terms which vanish exponentially or faster \cite{footnote}. The criteria for the unique definition of the coefficients $\chi^{(l)}$ in the expansion (47) are given by

$$\left(M(f^0) - l\beta_0\right) \neq 0 \text{ for } l = 1, 2, \ldots$$

(51)

with

$$M(f^0) = c_1^{(0)} + 2c_{11}^{(0)} f^0 - \beta_0.$$  

(52)

Upon substitution of the solutions (49) and the explicit form of the coefficients from Eqs.(40), we find

$$M(f_{01}) - l\beta_0 = -\beta_0(\xi + l)$$

$$M(f_{00}) - l\beta_0 = +\beta_0(\xi - l),$$

(53)

with $\beta_0$ from Eq.(40) and $\xi$ as a function of $N_C$ and $N_F$ given by

$$\xi(N_C, N_F) = \frac{N_C (N_F - N_C - 2/N_C)}{(N_F - N_C)(2N_F - 3N_C)}.$$  

(54)

The equations for the coefficients $\chi^{(l)}$ are of the general form given in Eqs.(15). For $l + 1$ loops, they are simply

$$\left(M(f^0) - l\beta_0\right) \chi^{(l)} = \left(\beta_l(f^0)f^0 - \beta^{(l)}(f^0)\right) + X^{(l)},$$

(55)

where $l = 1, 2, \ldots$, and where $f^0$ is to be replaced by the solutions $f_{01}$ or $f_{00}$ respectively. The $\beta$-function coefficients are as in Eq.(11) with appropriate substitutions.

In the following, we consider characteristic intervals in $N_F$ separately, and concentrate on the conformal window.

We have already discussed the window $\frac{3}{2}N_C < N_F < 3N_C$, where both SQCD and dual SQCD are asymptotically free at small distances. Considering first the solution $f_{01}(N_C, N_F)$ as given in Eq.(50), we see that it is positive in the window, as is the function $\xi(N_C, N_F)$. Since also $\beta_0 < 0$, the coefficients in Eq.(55) do not
vanish. Consequently the expansion coefficients $\chi^{(l)}$ are uniquely determined and can be removed by a regular reparametrization transformation. We are left with the explicit solution

$$\lambda_1(g^2) = g^2 f_{01}(N_C, N_F), \quad (56)$$

with $f_{01}$ given by Eq.(50). The $\beta$-functions of the reduced theory, as defined by the solution (56), are now simply given by Eqs.(9) and (10)) with the argument $f$ of the coefficient functions replaced by $f_{01}(N_C, N_F)$, so that they are constants:

$$\beta(g^2) = \beta(g^2, g^2 f_{01}) = \sum_{l=0}^{\infty} \beta_l(f_{01})(g^2)^{l+2}, \quad \beta_1(g^2) = f_{01} \beta_1(g^2). \quad (57)$$

The second relation follows from the reduction equation (4) with Eq.(56). The coefficient $\beta_0$ is as given in Eq.(40), and for $\beta_1(f_{01})$ we obtain explicitly [21, 23]

$$(16\pi^2)^2 \beta_1(f_{01}) = 2(N_F - N_C)(3N_C - 2N_F) + 4N_F \frac{(N_F - N_C)^2 - 1}{2(N_F - N_C)} - 4N_F^2 \frac{N_C(N_F - N_C - 2/N_C)}{2(N_F - N_C)(3N_F - N_C)}. \quad (58)$$

These relations are used later in connection with the infrared fixed point of dual SQCD in the conformal window near $N_F = \frac{3}{2}N_C$. We must note here, that for the expansion (57), in addition to $\beta_0$, the two-loop coefficient $\beta_1(f_{01})$ is reparametrization invariant. This result follows because $f_{01}$ satisfies the reduction equation (48).

It remains to consider the second solution presented in Eq.(49), with $f^0 = f_{00} = 0$. In this case the second expression in Eq.(53) is relevant for the determination of the higher coefficients in the expansion of $f_1(g^2)$. There could be a zero if $\xi(N_C, N_F)$ is a positive integer in the window. Generally however, this is not the case (at least for $N_C < 16$), with the characteristic exception of $N_C = 3, N_F = 5$, where $\xi(3, 5) = 2$ and the magnetic gauge group is $SU(2)$. Ignoring this case, we have again the situation that all coefficients $\chi^{(l)}$ are determined and can be removed by regular reparametrization. Then the second power series solution of the reduction equations is given by

$$\lambda_1(g^2) \equiv 0, \quad (59)$$

and leads to a theory without superpotential. As we have discussed earlier, this situation is not acceptable for the dual magnetic theory.
Returning to the exceptional case with the magnetic gauge group $SU(2)$, we find that, after reparametrization, it leads to a solution of the form

$$\lambda_1(g^2) = A g^6 + \chi^{(3)} g^8 + \cdots,$$

where the coefficient $A$ is undetermined, and the higher ones are fixed once $A$ is given. They vanish if $A = 0$. We do not discuss this case here any further.

Finally, we briefly consider possible ‘general’ solutions of the reduction equations. It turns out that for dual SQCD there are no such solutions which asymptotically approach the relevant polynomial solution $\lambda_1(g^2) = g^2 f_{01}$ given in Eq.(56). The only general solution we obtain is associated with the excluded polynomial solution $\lambda_1(g^2) \equiv 0$. It is given by

$$\lambda_1(g^2) = A(g^2)^{1+\xi} + \cdots,$$

where $A$ is again an undetermined parameter with properties analogous to those discussed above for Eq.(60). As we have pointed out, the exponent $\xi$, as given in Eq.(54), is positive and generally non-integer in the limit. The only exception is for $N_C = 3, N_F = 5$, in which case we are back to the exceptional solution (60) discussed above.

We see that, within the set of solutions of the reduction equations for magnetic SQCD, the power series $\lambda_1(g^2) = g^2 f_{01}$ is the unique choice for duality. Ignoring the isolated $SU(2)$ case, the second power series solution $\lambda_1(g^2) \equiv 0$ is excluded. The general solution (61), which is associated with it, leads to asymptotic expansions of Green’s functions involving non-integer powers. This is not consistent with a conventional, renormalizable Lagrangian formulation. Since there are no general solutions approaching the power solution $\lambda_1(g^2) = g^2 f_{01}$, the latter is isolated or unstable.

From the one- and two-loop expressions for the $\beta$-functions of the electric and the reduced magnetic theories given in Eqs.(37) and (40), we can obtain some information about non-trivial infrared fixed points in the conformal window [39, 40]. These expansions are useful as long as the fixed points occur for values of $N_C, N_F$ near the appropriate endpoint of the window. We find [23]

$$\beta_m(g^*^2) = 0 \quad \text{for} \quad \frac{g^*^2}{16\pi^2} = \frac{7}{3} \frac{N_F - \frac{3}{2} N_C}{N_F^2/4 - 1} + \cdots,$$
and

$$β_e(g_e^2) = 0 \quad \text{for} \quad \frac{g_e^2}{16\pi^2} = \frac{3N_C - N_F}{6(N_C^2 - 1)} + \cdots,$$

(63)

for sufficiently small and positive values of $3N_C - N_F$ and $N_F - \frac{3}{2}N_C$ respectively. Larger values of $N_C$ may be needed in order to have a useful approximation. Higher order terms have been calculated and may be found in [29].

With the reduced dual theory depending only upon the magnetic gauge coupling, it is straightforward to obtain the critical exponent $γ_m = γ_m(N_C, N_F)$ near the lower end of the window at $N_F = \frac{3}{2}N_C$ [29]. This exponent is relevant for describing the rate at which a given charge approaches the infrared fixed point. With Eqs.(40), (58) and (62), the lowest order term is given by

$$γ_m = \left(\frac{dB_m(g^2)}{dg^2}\right)_{g^2 = g_m^2} = \frac{14}{3} \left(\frac{3N_F - \frac{3}{2}N_C}{\frac{3}{2}N_C^2 - 1}\right)^2 + \cdots,$$

(64)

where we have written $g$ in place of $g_m$ as before. For the electric theory in the window near $N_F = 3N_C$, the corresponding expression is

$$γ_e = \left(\frac{dB_e(g_e^2)}{dg_e^2}\right)_{g_e^2 = g_e^2} = \frac{1}{6} \left(\frac{3N_C - N_F}{N_C^2 - 1}\right)^2 + \cdots.$$

(65)

In both cases we refer to [29] for the next order.

In this report we consider mainly the reduction of dual magnetic SQCD in the conformal window. A detailed discussion of the situation in the free magnetic phase $N_C + 2 \leq N_F < \frac{3}{2}N_C$ may be found in [23]. This interval is non-empty for $N_F > 4$. The electric theory is UV-free and the magnetic theory IR-free. At low energies, it is the latter which describes the spectrum. Because of the lack of UV-asymptotic freedom, one may be concerned that the magnetic theory may not exist as a strictly local field theory. However, it can be considered as a long distance limit of an appropriate brane construction in superstring theory, which can also confirm duality [41]. Except for special cases involving again $SU(2)$ as the magnetic gauge group, the unique power series solution (56) remains the appropriate choice also for this phase. It correspond here to the approach to the trivial infrared fixed point. Below $N_F = N_C + 2$ there is no dual magnetic gauge theory, and the spectrum should contain massless baryons and mesons associated with gauge invariant fields.

In the free electric phase for $N_F > 3N_C$, the magnetic theory remains UV-free, and the results of the reduction method are the same as in the conformal window.
7. Conclusions

In the application to Duality, we see that the reduction method is most helpful in bringing out characteristic features of theories with superpotentials. In the case of the dual of SQCD, we get an essentially unique solution of the reduction equations, which corresponds to a renormalizable Lagrangian theory with an asymptotic power series expansion in the remaining gauge coupling. This dual magnetic theory is asymptotically free. It is UV-free in the conformal window and above, and IR-free in the free magnetic region below the window. In this latter region, it describes the low energy excitations. These can be considered as composites of the free quanta of the electric theory, which is strongly coupled there.

As we have mentioned before, dual theories can be obtained as appropriate limits of brane systems [41]. In these brane constructions, duality corresponds essentially to a reparametrization of the quantum moduli space of vacua of a given brane structure. It is of interest to find out how the reduction solutions are related to special features of these constructions, in particular as far as the unique power solution (56) is concerned.

Besides the use of the reduction method in connection with duality, which we have described in this article, there are many other theoretical as well as phenomenological applications. Examples of applications in more phenomenological situations are discussed in this volume by J. Kubo [42].

Without detailed discussions, we mention here only a few applications:

* Construction of gauge theories with “minimal” coupling of Yang-Mills and matter fields [1].
* Proof of conformal invariance (finiteness) for $N = 1$ SUSY gauge theories with vanishing lowest order $\beta$-function on the basis of one-loop information [13, 14].
* Reduction of the infinite number of coupling parameters appearing in the light-cone quantization method [15].
* Reduction in an effective field theory formulation of quantum gravity and in effective scalar field theory [16].

We see that the reduction method can be used also within the framework of non-renormalizable theories, where the number of couplings is infinite a priori.
* Applications of reduction to the standard model (non-SUSY) give values for the top-quark mass which are too small, indicating the need for more matter fields [47].

* Gauge-Yukawa unifications within the framework of SUSY GUT’s. Successful calculations of top-quark and bottom-quark masses within the framework of finite and non-finite theories [9, 48, 42].

* Reduction and soft symmetry breaking parameters. In softly broken \( N = 1 \) SUSY theories with gauge-Yukawa reduction, one finds all order renormalization group invariant sum rules for soft scalar masses [49, 50, 42]. There are interesting agreements with results from superstring based models.

There are other problems where the reduction scheme is a helpful and often an important tool [51].

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