NON-RELATIVISTIC RADIATION MEDIATED SHOCK BREAKOUTS. II. BOLOMETRIC PROPERTIES OF SUPERNOVA SHOCK BREAKOUT

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ABSTRACT

Exact bolometric light curves of supernova shock breakouts are derived based on the universal, non-relativistic, planar breakout solutions, assuming spherical symmetry, constant Thomson scattering opacity, \(\kappa\), and angular intensity corresponding to the steady-state planar limit. These approximations are accurate for progenitors with a scale height much smaller than the radius. The light curves are insensitive to the density profile and are determined by the progenitor radius \(R\), and the breakout velocity and density, \(v_0\) and \(\rho_0\), respectively, and \(\kappa\). The total breakout energy, \(E_{\text{BO}}\), and the maximal ejecta velocity, \(v_{\text{max}}\), are shown to be \(E_{\text{BO}} = 8.0\pi R^2\kappa^{-1} c v_0\) and \(v_{\text{max}} = 2.0v_0\), respectively, to an accuracy of about 10\%. The calculated light curves are valid up to the time of transition to spherical expansion, \(t_{\text{sph}} \approx R/4v_0\). Approximate analytic expressions for the light curves are provided for breakouts in which the shock crossing time at breakout, \(t_0 = c/\kappa\rho_0 v_0^2\), is \(\ll R/c\) (valid for \(R < 10^{14}\) cm). Modifications of the flux angular intensity distribution and differences in shock arrival times to the surface, \(\Delta\text{asym}\), due to moderately asymmetric explosions, affect the early light curve but do not affect \(v_{\text{max}}\) and \(E_{\text{BO}}\). For \(4v_0 \ll c\), valid for large (red supergiant) progenitors, \(L \propto r^{-1/3}\) at max(\(\Delta\text{asym}\), \(R/c\)) \(< t < t_{\text{sph}}\) and \(R\) may be accurately estimated from \(R \approx 2 \times 10^{13}(L/10^{43}\text{erg s}^{-1})^{2/3}(t/1\text{hr})^{1/15}\).  

Key words: radiation mechanisms: non-thermal – shock waves – supernovae: general – X-rays: bursts

Color figures

1. INTRODUCTION

During a core collapse supernova (SN) explosion, a strong radiation mediated shock (RMS) traverses the exploding stars’ mantle/envelope. Once the shock reaches the surface of the star, a burst of high energy radiation (UV to \(\gamma\)-rays) is expected to be emitted (Colgate 1974; Falk 1978; Klein & Chevalier 1978; Ensmann & Burrows 1992; Matzner & McKee 1999; Blinnikov et al. 2000; Katz et al. 2010; Piro et al. 2010; Nakar & Sari 2010). The observed properties of this breakout were derived in earlier analyses using either analytic order-of-magnitude estimates (e.g., Matzner & McKee 1999; Katz et al. 2010; Piro et al. 2010; Nakar & Sari 2010) or numerical calculations for particular progenitors (e.g., Ensmann & Burrows 1992; Blinnikov et al. 2000; Utrobin 2007; Tominaga et al. 2009, 2011; Tolstov 2010; Dessart & Hillier 2010; Kasen et al. 2011). Here we provide an accurate description of the time dependent radiation emission following shock breakout for a general progenitor without an optically thick wind.

In the first paper of this series (Sapi\(r\) et al. 2011, hereafter Paper I) we have solved the problem of a non-steady planar RMS breaking out from a surface with a power-law density profile, \(\rho \propto x^n\), in the approximation of diffusion with constant opacity (a similar solution for exponential atmospheres, valid only for the early part of the planar breakout, was given in Lasher & Chan 1979). In this paper we use the results of the planar calculation to derive the observed bolometric properties of SN shock breakout bursts, taking into account limb darkening. In a third paper (N. Sapi\(r\) et al. 2011, in preparation) we calculate the temperature profiles and spectral properties of the burst assuming local Compton equilibrium and photon generation by bremsstrahlung (see Section 5 of paper I).

The derivation of exact light curves for a general progenitor is possible thanks to the universality of the planar breakout solutions (Paper I). Radiation escapes the shock front, producing the observed breakout burst, when the optical depth \(\tau\) of the plasma lying ahead of it is equal to the optical depth of the shock transition layer, \(\tau_{\text{sh}} = c/v_{\text{sh}}\) (Weaver 1976). We denote the shock velocity and (pre-shock) density at this point by \(v_0\) and \(\rho_0\), respectively (see Section 2 for exact definitions). Measuring length, time, and mass in units of \(x_0 = c/\kappa\rho_0 v_0\), \(t_0 = x_0/v_0\) and \(m_0 = c/\kappa v_0\), respectively, the RMS breakout solution is universal and depends only on the density power law index \(n\). It was numerically found that the luminosity depends weakly on the density structure. In fact, throughout most of the emission, the luminosity changes by less than 25\% for \(n\) in the range 1–10. The fact that the planar luminosity curve changes so little for a wide range of density power law indexes suggests that the results are insensitive to the decreasing density structure, and are applicable to profiles which are not power laws.

The results presented in this paper, based on the non-relativistic planar breakout solution, have corrections of the order of \(v_0/c\). Estimating these and higher-order corrections require a relativistic calculation of a planar shock breakout, and are beyond the scope of this paper. We do estimate one obvious first-order correction resulting from the transformation of the comoving calculated flux to the observer frame in Appendix B. An additional complication occurs at shock velocities \(v/c \gtrsim 0.3\), where the post shock temperatures reach 50 keV (Weaver 1976; Katz et al. 2010) and electron–positron pairs are created and increase the opacity of the material. Our discussion is applicable to smaller velocities/lower temperatures. Order-of-magnitude estimates for relativistic shock breakouts can be
obtained by using the steady-state solution of a relativistic shock (Budnik et al. 2010; Katz et al. 2010).

This paper is organized as follows. The planar non-relativistic RMS breakout solution is briefly described in Section 2. The application of the planar solution to an SN breakout is discussed in Section 3. The planar breakout parameters $v_0$ and $\rho_0$ are expressed in terms of progenitor properties and explosion energy, and complications due to spherical expansion, asymmetry, and relativistic corrections are discussed. Numerically calibrated analytic expressions for the total breakout energy, $E_{BO}$, the asymptotic velocity of the fastest moving ejecta, $v_{max}$, and the luminosity at late times, $t > \max(\Delta t_{\text{asym}}, R/c)$, are given in Section 4. These are all insensitive to small deviations from spherical symmetry with differences in the shock arrival times $\Delta t_{\text{asym}}$ and from the constant flux angular intensity distribution. In Section 5, the observed bolometric light curve during breakout is calculated, assuming that the shock arrives at the surface simultaneously and taking into account light travel time effects. Approximate analytic expressions for the light curve are derived. Some of the first-order corrections in $v/c$ to the burst properties are analyzed in Appendix B. Our results are compared to those of previous studies in Section 6. The main results and conclusions are summarized in Section 7. Appendices A–C provide details omitted from the manuscript.

2. PLANAR NON-RELATIVISTIC RMS BREAKOUT

The problem of a non-relativistic RMS breaking out from the surface of a planar decreasing density profile was solved in Paper I (see also Lasher & Chan 1979). The analysis is preformed by neglecting the thermal energy of the matter and by approximating the radiation energy transport by diffusion with constant opacity $\kappa$.

The initial density profile is assumed to be a power law, $\rho \propto \chi^n$, where $\chi$ is the distance from the surface. The initial density and the asymptotic shock velocity at large optical depth are parameterized by

$$\rho(\tau) = \rho_0 (v_0 \tau/c)^{\frac{1}{n}}$$

(1)

and

$$v(\tau) \rightarrow v_0 (v_0 \tau/c)^{-\frac{1}{n}}$$

(2)

respectively, where $\tau = \kappa \int \rho d\chi$ is measured with respect to the surface and where $\beta_\nu$ is a function of $n$, which can be obtained numerically by solving the pure hydrodynamic shock evolution (Sakurai 1960). The values of $\rho_0$ and $v_0$ are the density and velocity at the point $\tau = c/v_{th}$ that would have been obtained in a pure hydrodynamic shock propagation.

The limits $n \rightarrow 0$ and $n \rightarrow \infty$ are well defined and correspond to a homogeneous density distribution and an exponential profile (Hayes 1968),

$$\rho \propto \rho_0 \exp(\kappa \rho_0 \Delta x), \quad \Delta x = x(\rho) - x(\rho_0)$$

(3)

respectively. The value of $\beta_\nu$ in Equation (2) decreases monotonically from 0.207 to 0.176 for $0 < n < \infty$.

As mentioned in Section 1, measuring length, time, and mass in units of $x_0 = c/\kappa \rho_0 v_0$, $t_0 = x_0/v_0$ and $m_0 = x_0^3 = c/\kappa v_0$, respectively, the RMS breakout solution is universal, i.e., it depends on the dimensionless parameter $n$ alone. In particular, the planar emitted energy flux, i.e., the luminosity per unit area, can be expressed as

$$L(t) = L_0 \tilde{E} \left( \frac{t - t_{\text{ref}}}{t_0}, n \right)$$

(4)

where $L_0 = \rho_0 v_0^3$ and $\tilde{E} \equiv E/L_0$ is given in Table 3 of Paper I for $n$ in the range 1–10. The shock crossing time at breakout is defined as

$$t_0 = \frac{c}{\kappa \rho_0 v_0^2}$$

(5)

The reference time $t_{\text{ref}}$ corresponds to $t_{\text{peak}}$ in Paper I, the time at which the luminosity per unit area $\tilde{E}$ peaks in the planar solution. Since the observed SN light curves do not peak at the same time (due to light travel time effects) we have replaced $t_{\text{peak}}$ with $t_{\text{ref}}$ to avoid confusion.

As can be seen in Table 3 of Paper I, the luminosity depends weakly on the density structure. In fact, throughout most of the emission, the luminosity changes by less than 25% for $n$ in the range 1–10. The fact that the planar luminosity curve changes so little for a wide range of density power law indexes suggests that the result is insensitive to the decreasing density structure, and likely represents also profiles which deviate from a power-law structure.

For convenience, we include a surface area of $4\pi R^2$ in the expressions below. The luminosity, $L(t) = 4\pi R^2 \tilde{E}(t)$, may be written as

$$L(t) = L_0 \tilde{E} \left( \frac{t - t_{\text{ref}}}{t_0}, n \right)$$

(6)

where $L_0$ is the breakout luminosity defined by

$$L_0 \equiv 4\pi R^2 \rho_0 v_0^3$$

(7)

At late times, $t \gg t_0$, the luminosity follows $L(t) \propto t^{-4/3}$ (Piro et al. 2010; Nakar & Sari 2010). In Paper I it was found that the exact solutions for the luminosity and its integral, $E(t) = \int_0^t L(t')dt'$, are well approximated by

$$L(t) \approx L_\infty \left( \frac{t - t_{\text{ref}}}{t_0} \right)^{-4/3}$$

$$E(t) \approx E_\infty \left[ 1 - \left( \frac{t - t_{\text{ref}}}{t_0} \right)^{-1/3} \right]$$

(8)

where

$$E_\infty = 2.0 \times 4\pi R^2 v_0^2 c$$

$$L_\infty = 0.33 \times 4\pi R^2 \rho_0 v_0^3$$

$$a_t = (3L_\infty/t_0/E_\infty)^{3} = 0.125$$

(9)

Equations (8) and (9) describe the emitted flux to an accuracy of better than 30% (10%) in $L(t)$ ($E(t)$) for $1 < n < 10$ and $1 < (t - t_{\text{ref}})/t_0 < 100$. Individual fits to different values of $n$ allow higher accuracy. Note that $E_\infty$ is the total energy emitted in the planar approximation.

Finally, in the non-relativistic approximation in planar geometry an exact relation exists between the velocity of the outermost mass element and the emitted luminosity (Lasher & Chan 1979; Sapir et al. 2011),

$$v(t) = \frac{\kappa}{c} \int_{t_{\text{ref}}}^{t} L(t')dt' = \frac{\kappa E(t)}{4\pi R^2 c}$$

(10)

In particular, the asymptotic value of the velocity of the surface is

$$v_{\infty} = \frac{\kappa E_\infty}{4\pi R^2 c} = 2.0 v_0$$

(11)

For a constant density profile, $n = 0$, the decline is steeper, $L(t) \propto t^{-9/8}$ (Sapir et al. 2011).
Equation (10) simply states that photons that hit a given particle transfer all their momentum to the particle on average. It holds for any elastic scattering which has forward/backward symmetry, regardless of whether the diffusion approximation is valid or not.

3. APPLICATION OF THE PLANAR SOLUTION TO AN SN BREAKOUT

In this section the application of the solution of shock breakout in the planar approximation to supernova shock breakouts is discussed. In Section 3.1 the relation between the progenitor effects that need to be taken into account in calculating the light parameters and the breakout parameters is briefly reviewed. The limits of the planar approximation are discussed in Section 3.3.

3.1. Breakout Parameters

The relation between the parameters at breakout and the physical parameters of the SN explosion were given in numerous publications (e.g., Matzner & McKee 1999; Katz et al. 2010; Nakar & Sari 2010). In particular, a complete set of such relations is given in Appendix A of Nakar & Sari (2010). For convenience, we reproduce the relation for \( v_0 \) and \( \rho_0 \) below. As explained in Section 2, these quantities completely define the planar problem.

We use the approximate relation for the evolution of the shock velocity throughout the star (Equation (19) in Matzner & McKee 1999) to set

\[
v_0 \approx 1.0v_s [\tilde{\rho}/\rho_0]^{0.19},
\]

with

\[
v_s = (E_{in}/M_{ej})^{1/2}, \quad \tilde{\rho} = M_{ej}/(4\pi R^3/3).
\]

Here, \( M_{ej} \) is the mass of the ejecta and \( E_{in} \) is its energy (note that \( \tilde{\rho} \) is different from \( \rho_s = M_{ej}R^{-3} \) by a factor of \( 4\pi/3 \)). Note that the scaling in Equations (12) and (13) is the same as for the planar Sakurai solution, given by Equations (1) and (2), which is valid close to the edge of the star, with \( \beta_s \) replaced by its approximate value 0.19 which is not sensitive to the density profile (see discussion after Equation (3)). We further use the density parameterization \( \rho(x) = f_\rho \tilde{\rho}(x/R)^n \), where \( f_\rho \) is a dimensionless parameter of the order of unity (see Calzavara & Matzner 2004, Appendix A for detailed estimates; note that \( \rho_1/\rho_s \) defined by the authors is related to \( f_\rho \) defined here by \( f_\rho = 4\pi/3(\rho_1/\rho_s) \)). Solving for \( \tau = \beta_s^{-1} \), the following relations are obtained for \( n = 3 \) (appropriate for a blue supergiant (BSG)) and for \( n = 3/2 \) (appropriate for a red supergiant (RSG)):

\[
v_0/v_s = 13M_{10}^{0.16}v_{s,8.5}^{0.16}R_{12}^{0.32}\kappa_0^{0.16}f_\rho^{-0.05} \quad \text{(BSG)}
\]

\[
= 4.5M_{10}^{0.13}v_{s,8.5}^{0.13}R_{12}^{0.26}\kappa_0^{0.13}f_\rho^{-0.09} \quad \text{(RSG)},
\]

\[
\rho_0 = 7 \times 10^{-9}M_{10}^{0.13}v_{s,8.5}^{-0.87}R_{12}^{1.26}\kappa_0^{-0.87}f_\rho^{0.29} \text{ g cm}^{-3} \quad \text{(BSG)}
\]

\[
= 2 \times 10^{-9}M_{10}^{0.32}v_{s,8.5}^{-0.08}R_{13}^{1.64}\kappa_0^{-0.68}f_\rho^{0.45} \text{ g cm}^{-3} \quad \text{(RSG)},
\]

where \( M_{ej} = 10M_{10}M_{O}, R = 10^{12}R_{12} \) cm = \( 10^{13}R_{13} \) cm, and \( v_s = 3,000 v_{s,8.5} \) km s\(^{-1}\).

The results are also applicable to shock breakouts from Type Ia supernovae as long as the shock does not reach relativistic velocities; however, the bursts are much weaker and shorter due to their small sizes and are thus much harder to detect (Piro et al. 2010).

3.2. Applicability of the Planar Solution to an SN Breakout

Several effects must be taken into account when using the planar solution to describe the observed breakout burst.

Light travel time smearing. A distant observer sees the breakout emission coming from different locations on the surface at different times due to the finite light travel time. This can be accounted for by appropriately “smearing” the instantaneous luminosity on a timescale \( t_{smean} \sim R/c \), as described in Section 5.

Relativistic corrections. At high velocities, \( \beta_0 \equiv v_0/c \gtrsim 0.1 \), relativistic effects may introduce corrections of the order of tens of percents to the observed luminosity. One aspect which is easily accounted for are first-order corrections to the relation between the calculated comoving fluxes and the observed fluxes. These corrections can be accounted for by Lorentz transforming the quantities in the comoving frame to the laboratory frame and finding the retarded time \( t_{obs} \) of the emission of photons arriving at the observer at time \( t_{obs} \). This is discussed in Appendix B.

3.3. Limitations

The applicability of the planar solution is limited by the following complications.

Spherical expansion. As the outer mass elements expand, their optical depth decreases like \( \tau \propto r^{-2} \), where \( r(m, t) \) is the radius to which the mass element moved. During the first, planar stage of the expansion, the optical depth and velocities of each mass shell are constant with time and radiation escapes only the outer \( \tau \sim c/v_0 \) shell where the diffusion time equals the local expansion time. The planar approximation breaks once the optical depth of the outermost elements drops significantly. Given that the outermost mass elements move with a velocity \( v \approx 2v_0 \), the optical depth of the outermost elements drops by a factor of \( \sim 2 \)

\[
t_{sph} \sim \frac{\sqrt{2} - 1}{2} \frac{R}{v_0} \sim R/(4v_0)
\]

\[
\sim 60M_{10}^{0.16}v_{s,8.5}^{-1.16}\kappa_0^{1.32}f_\rho^{-0.16}f_\rho^{0.05}\text{ s (BSG)}
\]

\[
\sim 0.5M_{10}^{-0.13}v_{s,8.5}^{-1.13}R_{13}^{1.26}\kappa_0^{0.13}f_\rho^{0.09}\text{ hr (RSG)}. \quad (16)
\]

The use of the planar solution is limited to times \( t \ll t_{sph} \).

For the planar solution to be applicable, it is required that \( t_{sph} \gg t_0 \) (this is equivalent to \( R \gg x_0 \)). Using Equations (14)–(16) we have

\[
t_0/t_{sph} \sim 0.01M_{10}^{0.29}v_{s,8.5}^{-0.29}\kappa_0^{0.58}f_\rho^{-0.29} \quad \text{(BSG)}
\]

\[
\sim 0.01M_{10}^{0.45}v_{s,8.5}^{-0.45}\kappa_0^{0.9}f_\rho^{-0.37} \quad \text{(RSG)}, \quad (17)
\]

implying that \( t_0 \ll t_{sph} \) for practically all progenitors.

We emphasize that our results are not applicable for progenitors with optically thick winds.

Non-spherically symmetric explosions. It is likely that SN explosions are not spherically symmetric. The use of the planar solution for asymmetric explosions may be limited due to several effects.

1. The break out velocities may be different at different locations on the surface.
2. The shock may reach the surface at oblique angles.
3. The shock arrival time to the surface may depend on location.

The treatment of the first two effects is beyond the scope of this paper. If the shock arrives to the surface at a large angle, the planar symmetry does not hold and our solution is not applicable. As the shock propagates through the star the anisotropy is expected to be smoothed, and it is reasonable to expect that there is a wide range of parameters for which the velocities are approximately uniform and the obliqueness is small. We note that even if the velocities are different, our solution can be applied locally, with the local speed of the shock velocity, at any location where the shock arrives at small obliqueness.

The third problem can, in principle, be considered within the context of the planar solution. It amounts to an appropriate location where the shock arrives at small obliqueness.

As the shock propagates through the star the anisotropy is expected to be smoothed, and it is reasonable to expect that this paper. If the shock arrives to the surface at a large angle, the planar symmetry does not hold and our solution is not applicable.

3.3. The shock arrival time to the surface may depend on location.

\[ t \sim 2R/v_s = 18R_{13}v_{8.5}^{-1/3} \text{hr} \gg t_{\text{ph}}. \]  

At earlier times, the accumulated energy emitted in the spherical phase, \( \Delta E \), relative to the breakout energy, is roughly given by

\[ \frac{\Delta E}{E_{\text{BO}}} = \left( \frac{t}{t_{\text{BO}}} \right)^{1-\alpha_{\text{BO}}} . \]  

The difference in shock arrival time to the surface due to asymmetry is always much shorter than \( R/v_s \), implying that the breakout energy can be accurately integrated, even if the time difference in arrival times is significant.

4.2. Maximal Velocity

The velocity of the fastest moving ejecta can be obtained from the planar relation Equation (10). This velocity can be probed by other observations, including the spectrum of the breakout and radio observations of the collisionless shock propagating through the circumstellar medium (e.g., Chevalier & Fransson 2006; Waxman et al. 2007). As long as the planar approximation is valid, the velocity of the surface is proportional to the emitted energy and is given by Equation (10). Once the radius changes considerably, the acceleration declines sharply, \( \dot{v} \propto L/R^2 \), and the velocity does not increase significantly any more (see also Matzner & McKee 1999). The resulting velocity of the fastest part of the ejecta is thus related to the breakout energy

\[ v_{\text{max}} = \frac{\kappa}{4\pi R^2c} E_{\text{BO}}, \]  

and is given by

\[ v_{\text{max}} = 2.0v_0 \left[ 1 - \left( \frac{t_{\text{ph}}}{a t_0} \right)^{-1/3} \right] . \]  

A rough estimate of the additional acceleration beyond the transition to spherical expansion can be obtained by approximating \( L(t > t_{\text{ph}}) = \text{const} = L_{\text{sph}} \) and \( R \propto t \). In this limit, the acceleration drops like \( t^{-2} \) beyond \( t_{\text{ph}} \), and the total additional velocity \( \Delta v_{\text{max}} \) is roughly given by (using Equation (18))

\[ \frac{\Delta v_{\text{max}}}{v_{\text{max}}} \sim \frac{t_{\text{ph}}}{t_{\text{BO}}} \frac{L_{\text{sph}}}{E_{\text{BO}}} \sim \frac{1}{3} \left( \frac{t_{\text{ph}}}{a t_0} \right)^{-1/3} , \]  

implying a correction of a few percent for all progenitors considered.

4.3. Asymptotic Luminosity

In the regime \( t_{\text{smear}} \ll t \ll t_{\text{sph}} \), where \( t_{\text{smear}} = \max(\Delta t_{\text{asym}}, R/c) \), the luminosity is given by Equation (8) and can be expressed as

\[ L(t) = L(\infty)(t/t_0)^{-4/3} \]

\[ = \frac{4}{3.0} \pi R^2 \kappa c^{-4/3} \left( \frac{v_0}{\rho_0} \right)^{1/3} \left( \frac{\rho_0}{\kappa_{0.4}} \right)^{4/3} \left( \frac{v_{0.9}}{\rho_{0.9}} \right)^{-1/3} t_{\text{hr}}^{-4/3} \text{erg s}^{-1}. \]  

where \( v_{0.9} = 10^9 v_0 \text{ cm s}^{-1} \), \( \rho_{0.9} = 10^{-9} \rho_0 \text{ g cm}^{-3} \) and \( t = t_{\text{hr}} \) hr. The weak dependence on the parameters \( \rho_0 \) and \( \kappa_0 \).
v_0 indicates that, if detected, this power-law tail can be used for an accurate determination of the stellar radius.

We emphasize that since t_{\text{meas}} \gg R/c and t_{\text{ph}} \sim R/4v_0, the time interval t_{\text{meas}} \ll t_{\text{ph}} exists only for small velocities satisfying v_0/c \ll 1/4. This condition is met only for large RSG progenitors. Given that v_0/\rho_0 \propto R^{1.38} for n = 3/2, it is useful to rewrite the relation (27) as

\[ R_{\text{late}-L} = \left( \frac{3L}{4\pi} \right)^{2/5} \int_{0}^{8/15} \left( \frac{\rho_0 R^{3/2}}{v_0} \right)^{2/15} = 1.8 \times 10^{13} \left( \frac{R}{L_43} \right)^{2/5} \left( \frac{h}{4h} \right)^{1/15} \left( \frac{\rho_0}{0.09} \right)^{3/2} \left( \frac{v_0}{0.9} \right)^{18/15} \kappa_{0.4}^{8/15} \kappa_{0.4}^{8/15} \right. \]

where \( L = 10^{43} L_43 \) erg s^{-1} and the factor in the last line is close to unity for RSG parameters. Using Equations (14) and (15),

\[ \left( \frac{\rho_0 - 0.9 R^{3/2}}{v_0} \right) = 1.1 M_{10}^{0.22} \nu_{1.5}^{-0.15} \kappa_{0.4}^{0.15} / \kappa_{0.4}^{0.07} \, (\text{RSG}) \]  

Note that to actually infer the progenitor radius from observations using Equation (28), (at least) two measurements at two different times are required. The power-law scaling (27) can be self consistently checked with (at least) three observations.

5. LIGHT CURVE

In this section, the observed bolometric light curve is calculated assuming that the breakout is strictly spherically symmetric (i.e., assuming negligible difference in shock arrival times to the surface at different locations). In Section 5.1 exact expressions for the observed luminosity are given. In Section 5.2, an approximate analytic expression (Equation (36)) is derived assuming \( ct_0/R \ll 1 \), which is valid for all progenitors with the exception of the largest RSGs. Approximate expressions for the peak luminosity and the luminosity at the time \( R/c \) are given. The results of this section are summarized in Section 5.3.

5.1. Exact Light Curve

Due to the difference in arrival times of photons originating from different positions on the surface of the star, the actual luminosity \( L_{\text{obs}}(t) \), that a distant observer would measure, is related to the instantaneous luminosity \( L(t) \) by

\[ L_{\text{obs}}(t) = \int_{0}^{1} h(\mu) L(t - R(1 - \mu)/c) \mu d\mu, \]  

where \( h(\mu) = 2\pi I(\mu)_{\nu=0}/\lambda \) is the angular distribution of the radiation intensity at the surface normalized so that \( \int_{0}^{1} h(\mu) \mu d\mu = 1 \).

The precise value of \( h(\mu) \) requires the solution of radiation transport up to the surface. For the non-relativistic breakouts considered here, the transport equations can be solved in the steady-state approximation with the flux given by the diffusion solution. For the considered case of Thomson scattering, \( h(\mu) \) was obtained analytically by Chandrasekhar (1950) and can be fit by a linear relation,

\[ h(\mu) \approx a_1 + b_1 \mu, \quad a_1/2 + b_1/3 = 1, \]  

with

\[ a_1 = 0.85, \quad b_1 = 1.725, \]  

to a good approximation (better than 3%; see Appendix C). For comparison, Equation (31) with \( a_1 = 2, b_1 = 0 \) represents a black body surface (isotropic emission), while \( a_1 = 1, b_1 = 3/2 \) represents isotropic scattering in the Eddington approximation. The resulting light curves for a density power law index \( n = 3 \) and for different values of \( ct_0/R \) are plotted in Figures 1 and 2.
5.2. Approximate Light Curve for $R/c \gg t_0$

For most progenitors, the smearing timescale $R/c$ is much larger than $t_0$. In fact, using Equations (14) and (15) we have

$$c_{\text{to}}/R = 0.02 M_t^{-0.45} v_{\infty,8.5}^{-1.45} R_9^{0.9} \kappa_4^{-0.45} f_\rho^{-0.18} \quad \text{(BSG)}$$

$$= 0.05 M_t^{-0.58} v_{\infty,8.5}^{-1.58} R_9^{1.16} \kappa_4^{-0.58} f_\rho^{-0.28} \quad \text{(RSG)}.$$  \(33\)

As can be seen, except for very large RSGs with $R \sim 10^{14}$ cm, $c_{\text{to}}/R$ may be assumed to be small.

In this case, the burst timescale is $R/c$ and the typical luminosity is $E_{\infty}c/R$. It is useful to describe $L_{\text{obs}}$ as a function of

$$s \equiv \frac{c(t-t_{\text{ref}})}{R}. \quad (34)$$

Consider first the formal limit $t_{\text{to}}c/R \to 0$. In this limit $L(t) \to E_{\infty}\delta(t-t_{\text{ref}})$ and the observed luminosity (Equation (30)) goes to

$$L_{\text{obs}}(t) \underset{t_{\text{to}}c/R \to 0}{\longrightarrow} \frac{E_{\infty}c}{R} (1-s) h(1-s) \cdot (0 < s < 1). \quad (35)$$

In Figure 1, the light curves are shown as a function of $s$ (blue solid lines) while the limit of Equation (35) is shown for comparison (dashed red). As can be seen, the observed luminosity converges slowly with $c_{\text{to}}/R$ to the limiting value. This is due to the luminosity tale of $L \propto t^{-4/3}$ at late times, the integral of which converges slowly. This can be taken into account by the approximation of Equation (8). Using Equations (31), (30), and (8) we find

$$L_{\text{obs}}(t) = \frac{c E_{\infty}}{R} s_c^{1/3} \frac{1}{3} \int_0^{\text{min}(1,s-s_c)} (1-s')[a_1 + b_1(1-s')] ds' \times (s-s')^{-4/3} ds', \quad (36)$$

where

$$s_c = c a_1 t_0 / R. \quad (37)$$

An explicit expression for the integral in Equation (36) is given in Equation (A2) and the resulting observed luminosities are shown in Figure 2 (dash-dotted magenta lines, values of $E_{\infty}/E_0 = 2.03$ and $a_1 = 0.1$ fitted for the case $n = 3$ considered were used). As can be seen in the figure, this is an excellent approximation to the calculated observed luminosity.

A few properties of the light curve can be derived from Equation (36). At very large $s \gg 1$ Equation (36) reduces to Equation (27) as required. Given that $s_c \ll 1$, the value of $L_{\text{obs}}$ at $s = 1 = (t_{\text{to}} + R/c)$ is

$$L_{\text{obs}}(t = t_{\text{to}} + R/c) = \frac{c E_{\infty}}{R} s_c^{1/3} (a_1/2 + b_1/5), \quad (38)$$

where for $a_1 = 0.85(2)$ we have $(a_1/2 + b_1/5) = 0.77(1)$. Comparing to Equation (27) (and using the relation between $a_1$, $E_{\infty}$, and $L_{\infty}$ in Equation (9)), we see that at $t = R/c$ the luminosity drops to a value about three times larger than the extrapolation of the asymptotic luminosity (Equation (27)) to this time.

As shown in Appendix A, the peak observed luminosity is, to a good approximation, given by (Equation (A5)):

$$L_{\text{peak}} = (a_1 + b_1) \frac{E_{\infty}c}{R} \left[1 - \left(\frac{c t_0}{R}\right)^{1/4}\right]. \quad (39)$$

5.3. Light-curve Calculation Summary

The observed light curve can be calculated using Equations (30) and (31), with $L(t)$ given by (6) and tabulated in Sapir et al. (2011, Table 3). The following parameters completely determine the light curve: breakout luminosity $L_0$, breakout shock crossing time $t_0$, time of peak planar flux $t_{\text{ref}}$, progenitor radius $R$ and density power law index $n$. As shown in Sapir et al. (2011), the luminosity depends weakly on $n$. The parameters $L_0$ and $t_0$ are related to the breakout density, velocity, and opacity through Equations (7) and (5). The resulting light curves for the case $n = 3$ are shown in Figure 2.

For most progenitors $R/c \gg t_0$ and the observed light curve can be calculated using Equation (36) with $s$ given by Equation (34) and $a_1 = 0.85, b_1 = 1.725$ given by Equation (32). An explicit algebraic expression is given in Equation (A2). In this case the light curve is determined by the following parameters: breakout energy $E_{\infty}$, time of peak planar flux $t_{\text{ref}}$, progenitor radius $R$ and a dimensionless parameter $s_c$. The parameters $E_{\infty}$ and $s_c$ are related to the breakout density, velocity, and opacity (and $R$) through Equations (37), (5), and (9).

The peak luminosity in this case can be approximated by Equation (39). At $t = t_{\text{ref}} + R/c$ the luminosity drops by a factor $\sim (0.1c t_0/R)^{1/3}$ compared to the peak, to a value which is approximately three times higher than the extrapolation (to $t = t_{\text{ref}} + R/c$) of the asymptotic luminosity given by Equation (27).

In the extreme limit $t_0 \to 0$ a simple approximation for the light curve (for $t < R/c$) is given by (35), which depends on two parameters only, $E_{\infty}$ and $R$.

6. COMPARISON TO PREVIOUS STUDIES

In Section 6.1 we compare our results with those of the numerical calculations of Ensmann & Burrows (1992) for 1987A-like (BSG) progenitors. In Section 6.2 we compare our calibrated analytic results with the order-of-magnitude estimates of Matzner & McKee (1999) and Nakar & Sari (2010).

6.1. Comparison to Ensmann & Burrows (1992)

In Figure 3 we compare our results with the numerical light curves of Ensmann & Burrows (1992). As can be seen, the numerical light curves are in excellent agreement with our analytic light curves, for an appropriate choice of $\rho_0$ and $v_0$. The fitted values are $v_0 = 16,500 \, \text{km} \, \text{s}^{-1}, \rho_0 = 1.6 \times 10^{-9} \, \text{g} \, \text{cm}^{-3}$ and $v_0 = 25,000 \, \text{km} \, \text{s}^{-1}, \rho_0 = 1.2 \times 10^{-9} \, \text{g} \, \text{cm}^{-3}$ for the progenitor models “500full1” and “500full2”, respectively (the opacity is assumed to be $\kappa = 0.33 \, \text{cm}^2 \, \text{g}^{-1}$, appropriate to a mixture of ionized hydrogen and helium with mass fractions $X = 0.67$ and $Y = 0.33$).
Figure 3. Comparison of our light curves with those obtained by the numerical calculations of Ensman & Burrows (1992) for two progenitor models (“500full1” and “500full2”). Our results are shown in color (blue and red for source frame and observed luminosities, respectively), overlaid on the original Figure 2 of Ensman & Burrows (1992). For the comparison we have used ε + p = 2.5c/¯c and b(β) = 2, as used by Ensman & Burrows (1992), although these values do not yield an accurate description of the light curves (see the text).

(43x369)As can be seen in Figure 3, the planar solution agrees with the spherical calculation at the latest times presented (which are less than t_{gh}) up to a factor of ~1.5. The maximal velocities predicted by Equation (25) are 28,000 km s^{-1} and 43,000 km s^{-1}, in good agreement (up to ~10%) with the maximal velocities of 30,000 km s^{-1} and 48,000 km s^{-1} obtained in Ensman & Burrows (1992).

6.2. Comparison with Previous Analytical Work

A quantitative comparison of the analytic results presented here (calibrated using the numerical results of Sapir et al. 2011) to those of the order-of-magnitude estimates made in the literature is complicated since the breakout parameters $v_0$ and $\rho_0$ were not well defined in earlier analyses. We remind the reader that we define $\rho_0$ and $v_0$ to be the upstream density and the shock velocity that would be obtained in a pure hydrodynamic calculation (ignoring radiation diffusion) at the point where $\tau = c/v_0$ (see Section 2). For illustration purposes we compare a few of our results with order-of-magnitude estimates made by Matzner & McKee (1999) and Nakar & Sari (2010) in which $\rho_0$ and $v_0$ were (vaguely) defined in a similar way.

1. The numerical values of the order-of-magnitude estimates of the power-law decline of the luminosity, $L \propto r^{-4/3}$, made by Nakar & Sari (2010) can be summarized as $L(t) \sim 4\pi R^2 \rho^{-4/3} (v_0/\rho_0)^{1/3} c^{4/3} t^{-4/3}$ (using the values for $\rho_0$ and $v_0$ given in their appendix). This result is in rough agreement with our Equation (27), with their expressions overestimating the luminosity by a factor of about 3.

2. Matzner & McKee (1999) estimate that the maximal velocity obtained by the fastest ejecta is about twice the breakout shock velocity $v_0$. Coincidently (as far as we can tell), this turns out to be accurate (see Equation 25).

3. The numerical values for the breakout energy of RSGs and BSGs in Matzner & McKee (1999) are related to the numerical values of the maximal velocities by $E_{BO} = 4\pi R^2 c v_{\text{max}}/k$ and $E_{BO} = 1.7 \times 4\pi R^2 c v_{\text{max}}/k$, respectively. The RSG relation is accurate while in the BSG relation the emitted energy is overestimated by a (modest) factor of about 1.7 (see Equation (24)).

7. SUMMARY AND DISCUSSION

We have derived exact bolometric light curves of supernova shock breakouts using the universal planar breakout solutions (Sapir et al. 2011), assuming spherical symmetry, constant Thomson scattering opacity, $\kappa = (Z/A) \sigma_T/m_p$, and angular intensity corresponding to the constant flux limit. The light curves are insensitive to the form of the density profile. This was demonstrated by calculating the emission for power-law profiles $\rho \propto x^n$ with resulting luminosities changing by <30% for a broad range of power-law indexes, 1 < n < 10 (Sapir et al. 2011).

The breakout emission properties are determined by four dimensional parameters: the progenitor radius $R$, the breakout velocity and density, $v_0$ and $\rho_0$, respectively, and $\kappa$. $v_0$ and $\rho_0$ are the shock velocity and (pre-shock) density at the point where $\tau_\phi = c/v_\phi$ is reached in the pure hydrodynamic (neglecting radiation diffusion) solution (see Section 2 for exact definitions). The relations between the SN parameters, the ejecta mass $M_{ej}$ and bulk velocity $v_\text{ej}$, and the breakout parameters, $\rho_0$ and $v_0$, are given in Equations (14) and (15) (see also Matzner & McKee 1999; Katz et al. 2010; Nakar & Sari 2010).

The application of the planar solution to SN breakouts was discussed in Section 3. The planar approximation is applicable provided that the shock crossing time at breakout, $t_\phi = c/(\kappa \rho v_\phi)$, is much smaller than the time for transition to spherical expansion, $t_{gh} \sim R/(4v_\phi)$ (Equation (16); Piro et al. 2010; Nakar & Sari 2010). This is valid for practically all progenitors, see Equation (17). At $t > t_{gh}$ the expansion is no longer planar, and the planar approximation no longer holds. Our results are not applicable for progenitors with optically thick winds.

Deviation from spherical symmetry may affect breakout light curves due to several effects: the breakout velocity may be different at different surface locations, the shock may reach the surface at oblique angles, and the shock arrival time at the surface may depend on location. In this paper we have assumed that the shock reaches the surface with approximately the same velocity everywhere and parallel to the surface. Differences in shock arrival times to the surface, $\Delta t_{\text{asym}}$, due to moderately asymmetric explosions, affect the early light curve (e.g., Calzavara & Matzner 2004), at $t < \Delta t_{\text{asym}}$, but do not affect the maximal velocity of the ejecta and the total emitted energy, $v_{\text{max}}$ and $E_{BO}$.

Analytic expressions for $v_{\text{max}}$, $E_{BO}$ and the late time luminosity were derived in Section 4. The total energy of the breakout burst is approximately (Equation (18))

$$E_{BO} = 8\pi R^2 \kappa^{-1} v_0 c = 1.9 \times 10^{57} \text{ erg} \frac{R^2}{13} \rho_0 v_0 k_{0.1}^{-1},$$

where $v_0 = 10^9 \text{ cm s}^{-1}$. We have shown that integrating the luminosity to times greater than $t_{gh}$ affects the total energy considerably only at very late times, $t \gtrsim R/v_\phi$ (Equation (21)). The maximal velocity of the ejecta is directly related to the emitted energy by (Equation (24); see also Lasher & Chan 1979; Sapir et al. 2011)

$$v_{\text{max}} = \kappa E_{BO}/(4\pi R^2 c) = 2.0 v_0.$$
We note that these results are not sensitive to deviations from the steady-state flux angular intensity distribution we have used. For $4v_0 \ll c$, valid for large RSG progenitors, there is a significant separation between $R/c$ and $t_{\text{ph}}$. In this case, the luminosity at max($\Delta t_{\text{sym}}$, $R/c$) $< t < t_{\text{ph}}$ is approximately given by (Equation (27))

$$L(t) = (4/3) \pi R^2 (v_0/\rho_0)^{1/3} (c/\kappa)^{1/3} t^{-4/3}.$$  \hfill (42)

The strong dependence of the asymptotic luminosity on $R$ and weak dependence on $\rho_0$ and $v_0$ allows one to accurately determine the progenitor radius of RSG breakouts (Equation (28)),

$$R = 2 \times 10^{13} L_{45}^{2/5} (8/15)^{1/5} \text{ cm}.$$ \hfill (43)

The bolometric light curve, assuming negligible spread in shock arrival times, are calculated in Section 5. A proper calculation of the effects of finite light travel time requires knowledge of the angular dependence of the intensity. Fortunately, the problem of radiation transport in an optically thick medium with opacity dominated by Thomson scattering was solved in closed form (Chandrasekhar 1950; results summarized in Appendix C). Exact light curves can be calculated using Equations (30) and (31) and the planar luminosity functions, $\mathcal{L}$, given in Sapir et al. (2011). Some examples are shown in Figures 1 and 2. For cases where $c t_0 \ll R$, applicable in all progenitors except for the largest RSGs (see Equation (33)), the planar luminosity $\mathcal{L}$ can be approximated by a power law, Equation (8), allowing the derivation of an analytical expression for the light curve, given in Equations (36) and (A2). The analytic expression is compared to the exact calculation (both without relativistic corrections) in Figures 1 and 2. In this case, the peak luminosity is approximately given by

$$L_{\text{obs, peak}} = 2.5 (E_{\text{BOC}}/R)[1 - (c t_0/R)^{1/4}],$$ \hfill (44)

where

$$E_{\text{BOC}}/R = 5.6 \times 10^{54} R_{13} v_{50}^{-1} \text{ erg s}^{-1}$$ \hfill (45)

is the typical peak luminosity. In addition, we have shown that at $t = R/c$ the normalized luminosity $L t^{4/3}$ is about three times larger than its asymptotic value given by Equation (27). A short summary which explains how to use the different expressions to obtain the light curves is provided in Section 5.3.

The transformation of the rest frame intensity to the lab frame and the value of the retarded time introduces corrections of the order of $\beta^4$. These are calculated in Appendix B. For the velocities considered, $\beta_0 \lesssim 0.3$, the only considerable correction is to the early light curve and is given by Equation (B19). We note that other corrections of the order of $\beta^4$ are not excluded.

The results of this paper are compared to previous results in Section 6. The calculated bolometric light curve is shown to be in excellent agreement with the numerical calculation of Ensmann & Burrows (1992), see Figure 3. The order-of-magnitude estimates given by Matzner & McKee (1999) and by Nakar & Sari (2010) for the emitted energy, maximal velocity, and asymptotic luminosity agree to within factors of a few with our exact analytic expressions.

The properties of the breakout emission depend strongly on the radius of the progenitor $R$ and on the breakout shock velocity $v_0$, depend weakly on the value of the density at breakout $\rho_0$, and are insensitive to the density structure. Breakout observations therefore allow one to accurately determine $R$ and $v_0$. These quantities are directly related to other observables: the breakout shock velocity $v_0$ is roughly proportional to the ejecta velocity $v_e$, which is probed by SN observations. The maximal velocity of the ejecta (about twice the breakout shock velocity) can be constrained by the radio and X-ray emission produced by the interaction of the ejecta with the circumstellar medium (e.g., Waxman et al. 2007; Soderberg et al. 2008). Both parameters affect the subsequent spherical expansion (cooling envelope) phase of the emission (e.g., Chevalier 1992; Rabinik & Waxman 2011; Nakar & Sari 2010).

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APPENDIX A

LIGHT TRAVEL TIME AVERAGING

The integral

$$g(s; s_c) = s_c^{1/3} \int_0^{\min(1, s-s_c)} (1 - s') [a_I + b_I (1 - s')] (s - s')^{-4/3} ds'$$ \hfill (A1)

appearing in Equation (36) is explicitly given by

$$g(s; s_c) = \begin{cases} \frac{3}{2} s_c^{1/3} \left[ -a_I (s - 1)^{2/3} + \frac{5}{2} b_I (s - 1)^{5/3} - \left[ a_I \left( \frac{2}{3} - s \right) + 2 b_I \left( \frac{1}{3} - s + \frac{3}{5} s^2 \right) \right] s^{-1/3} \right] & s > 1 + s_c \\ a_I (1 - s - \frac{1}{2} s_c) + b_I (1 + (s_c - 2) s + s^2 - s_c \left( 1 + \frac{3}{5} s_c \right)) + (s_c/s)^{1/3} \left[ -a_I - b_I + \frac{3}{2} (a_I + 2 b_I) s - \frac{9}{5} b_I s^2 \right] & s_c < s \leq 1 + s_c \\ 0 & s \leq s_c \end{cases}$$ \hfill (A2)

To lowest order in $s_c$, the peak of $g(s; s_c)$ is reached at

$$s_{\text{peak}} = \left( \frac{a_I + b_I}{3 (a_I + 2 b_I)} \right)^{3/4} s_c^{1/4}.$$ \hfill (A3)
with a peak value of
\[
 g_{\text{peak}} = (a_I + b_I) \left[ 1 - (3^{-3/4} + 3^{1/4}) \left( \frac{a_I + 2b_I}{a_I + b_I} \right)^{1/4} s_c^{1/4} \right].
\] (A4)

Using Equation (37), the fact that for \(0 < a_I < 2\) we have \(1 < \left[ (a_I + 2b_I)/(a_I + b_I) \right]^{1/4} < 1.2\), and the numerical value \((3^{-3/4} + 3^{1/4})s_c^{1/4} = 1.0\) (using \(a_t = 0.125\), see Equation (9)), we conclude that to a good approximation
\[
 L_{\text{obs, peak}} = \frac{cE_{\infty}}{R} (a_I + b_I) \left[ 1 - \left( \frac{c_0}{R} \right)^{1/4} \right].
\] (A5)

APPENDIX B
FIRST-ORDER CORRECTIONS DUE TO THE TRANSFORMATION BETWEEN THE COMOVING FRAME AND THE OBSERVER FRAME

In this section the first-order corrections in \(\beta\) due to the transformation between the comoving frame and the observer frame (lab frame) are calculated. Note that the expression for the asymptotic luminosity (27) is valid only for small velocities \(v_0 \ll c/4\), for which the first-order corrections are negligible. We estimate the first-order corrections in \(\beta_0 = v_0/c\) to the value of the breakout energy, Equation (18), in Appendix B.1, and the corrections to the light curve in Appendix B.2.

B.1. First-order Corrections to the Breakout Energy

The rate of energy escaping from the surface of the expanding envelope as measured in the laboratory frame, \(dE_{\text{lab}}/dt\), is given by
\[
 \frac{dE_{\text{lab}}}{dt} = 4\pi r^2 (L_{\text{lab}} - ve_{\text{lab}}),
\] (B1)
where \(r(t)\) is the radius of the surface and \(L_{\text{lab}}\) and \(e_{\text{lab}}\) are the radiation flux and energy density at the surface in the laboratory frame.

The values of \(L_{\text{lab}}\) and \(e_{\text{lab}}\) can be expressed using the comoving frame flux \(\mathcal{L}\), energy density \(e\) and pressure \(p\) by
\[
 L_{\text{lab}} = \mathcal{L} + v(p + e) + O(\beta^2),
\]
\[
 e_{\text{lab}} = e + 2\beta c^{-1} \mathcal{L} + O(\beta^2).
\] (B2)

The emitted energy per unit time to first order in \(\beta\) is thus
\[
 \frac{dE_{\text{lab}}}{dt} = 4\pi r^2 \left( 1 + \beta \frac{pc}{\mathcal{L}} \bigg|_{t=0} \right) L.
\] (B3)

The term \((pc/\mathcal{L})|_{t=0}\) depends on the angular distribution of the radiation intensity at the surface, which depends on the transport properties of the medium. For Thomson scattering in the constant flux limit, the value is \((pc/\mathcal{L})|_{t=0} = 0.71\) (Chandrasekhar 1950; see also Appendix C). For comparison, for a black body surface (isotropic emission), the value is \((pc/\mathcal{L})|_{t=0} = 2/3\) and for isotropic scattering in the Eddington approximation it is \((pc/\mathcal{L})|_{t=0} = 17/24\).

Assuming that \((pc/\mathcal{L})|_{t=0}\) is constant over time and neglecting the change in the surface radius during the emission, Equation (B3) can be analytically integrated over time. To do this note that \(4\pi R^2 \int \mathcal{L}(t') dt' = E(t)\) and that
\[
 4\pi R^2 \int \mathcal{L}(t') v(t') dt' = \frac{c}{k} \int \frac{dE}{dt'} Edt' = \frac{1}{2} E(t) v(t),
\] (B4)
where Equation (10) was used. Using Equations (B4) the integration of Equation (B3) to infinity results in
\[
 E_{\text{lab, \infty}} = E_{\infty} \left( 1 + \frac{1}{2} \frac{pc}{\mathcal{L}} |_{t=0} \beta_{\infty} \right),
\] (B5)
where \(\beta_{\infty} = v_{\infty}/c\) is the asymptotic value of \(\beta\) in the planar approximation (approximately unchanged by spherical geometry; see Section 4.3). Using the approximation of Equation (B4), and adopting \((pc/\mathcal{L})|_{t=0} = 0.7\), we have
\[
 E_{\text{lab, \infty}} = E_{\infty} (1 + 0.7\beta_0) = E_{\infty} (1 + 0.35\beta_{\infty}).
\] (B6)

Linearly adding this correction and the small term \((t_{\text{sph}}/(a_t t_0))^{-1/3}\) in Equation (18) we obtain
\[
 E_{\text{BO}} = E_{\infty} \left[ 1 - \left( \frac{t_{\text{sph}}}{a_t t_0} \right)^{-1/3} + 0.7\beta_0 \right].
\] (B7)
B.2. First-order Correction to the Observed Light Curve

First-order corrections to the light curve arise from corrections to the intensity, direction and value of the retarded time. At times \( t \gtrsim R/c \), the surface of the star moved a distance of \((v/c)R\), implying that there are corrections of the order of \( v/c \) due to the spherical nature of the expansion. An estimate of this correction is beyond the scope of this paper. Here we focus on the first-order corrections at early times \( t \ll R/c \) (including the peak observed luminosity). At these early times the angle between the emitting region radius and the direction towards the observer is small, \( \mu - 1 \ll 1 \), allowing a simple derivation of the first-order correction.

Consider a spherically symmetric moving surface with radius \( r(t) \) emitting radiation with lab frame intensity \( I(t; \Omega) \) which is axisymmetric with respect to the surface normal. Consider an observer located in the direction \( \Omega_{\text{obs}} \) at a very large distance. The position on the sphere is parameterized by the angle \( \theta \) between \( \Omega \) and \( \Omega_{\text{obs}} \). Since the intensity \( I \) is assumed to be the same at any position on the sphere and to be axisymmetric with respect to the normal we have \( I(t, \Omega_{\text{obs}}) = I(t, \theta) \). The luminosity inferred by the observer is given by

\[ L_{\text{obs}}(t_{\text{obs}}) = 4\pi \int 2\pi ada I(t; \theta), \tag{B8} \]

where

\[ a = r \sin \theta \tag{B9} \]

and the lab (retarded) time \( t \) is related to the observer time \( t_{\text{obs}} \) through

\[ R + ct - r \cos \theta = ct_{\text{obs}}. \tag{B10} \]

The integral (B8) should be evaluated at constant \( t_{\text{obs}} \), with \( t \) and \( \theta \) functions of \( a \) through the relations (B9) and (B10), in the regime \( \beta < \cos \theta < 1 \) (it is assumed that there are no photons with \( \cos \theta < \beta \), which are coming from outside of the surface).

It is useful to solve for \( \mu = \cos \theta \)

\[ \mu = \frac{R - c(t_{\text{obs}} - t)}{r}, \tag{B11} \]

where \( t' = t_{\text{obs}} - t \). Using (B10), we find

\[ cdt = rd\mu + \mu vdt \Rightarrow rd\mu = c(1 - \beta \mu)dt \tag{B12} \]

and

\[ ada = r \sin^2 \theta vdt + r^2 \sin \theta \cos \theta d\theta \]

\[ = c r [(1 - \mu^2)\beta - \mu(1 - \beta \mu)]dt = c r (\beta - \mu)dt. \tag{B13} \]

Equation (B8) can be written as

\[ L_{\text{obs}}(t_{\text{obs}}) = 4\pi \int dt cr(\mu - \beta)2\pi I(t; \mu) \tag{B14} \]

with \( \mu \) given by (B11).

The lab frame \( \mu \) is related to the surface frame \( \mu, \mu_{\text{com}}, \) by

\[ \mu_{\text{com}} = \frac{\mu - \beta}{1 - \beta \mu}. \tag{B15} \]

while the lab frame intensity \( I \) is related to the surface frame \( I_{\text{com}} \) by

\[ I(t, \mu) = [\gamma(1 + \beta \mu_{\text{com}})]^3 I_{\text{com}}(t, \mu_{\text{com}}), \tag{B16} \]

where \( \gamma = (1 - \beta^2)^{1/2} \) is the surface Lorentz factor.

Assuming that the angular dependence of the comoving intensity is time independent, \( I_{\text{com}}(t, \mu) = h(\mu) f_{\text{com}}(t)/(2\pi) \), Equation (B14) can be written as

\[ L_{\text{obs}}(t_{\text{obs}}) = \int dt \frac{c}{r}(\mu - \beta)\gamma^3(1 + \beta \mu_{\text{com}})^3 h(\mu_{\text{com}}) L_{\text{com}}(t), \tag{B17} \]

where \( \mu \) and \( \mu_{\text{com}} \) are given in Equations (B11) and (B15).

Consider next the first-order corrections in the parameters \( \beta \) and \( (1 - \mu) \), neglecting the difference between \( r \) and \( R \). Note that in this approximation we can set \( \beta \mu \rightarrow \beta, \gamma \rightarrow 1, \mu_{\text{com}} \rightarrow \mu \) and

\[ L_{\text{obs}}(t_{\text{obs}}) = \frac{c}{R} \int dt \mu h(\mu)(1 + 2\beta) L_{\text{com}} \tag{B18} \]

with \( \mu = 1 - c(t_{\text{obs}} - t)/R \).

Using \( \int_{-\infty}^{t} L(t)u(t)dt \approx \tfrac{1}{2} v_{\infty} E_{\infty} \approx 2 v_{0} E_{\infty} \) (see (B4)), we find that in the limit \( t_{0}/c \ll 1 \), the correction at early times \( t_{\text{obs}} \ll R/c \) is

\[ L_{\text{obs}} = L_{\text{obs}, 0}(1 + 2\beta_{0}). \tag{B19} \]
APPENDIX C
STEADY-STATE RADIATION TRANSPORT WITH THOMSON SCATTERING

The steady-state problem of radiation transfer in a semi-infinite medium with Thomson scattering was analytically solved by Chandrasekhar (1950). Assuming a constant flux $\mathcal{L}$ coming from inside the medium and zero incident radiation, the intensities at the surface, $I_l(\mu)$ and $I_r(\mu)$, polarized in and perpendicular to the meridian plane, respectively, were found in closed form. The polarized intensities are given by

$$I_l = \frac{q}{\sqrt{2}} \frac{3}{8\pi} \mathcal{L} H_l(\mu),$$

$$I_r = \frac{1}{\sqrt{2}} \frac{3}{8\pi} \mathcal{L} H_r(\mu)(\mu + c),$$

(C1)

where $H_i (i = l, r)$ are the solutions to the integral equations

$$H_i(\mu) = 1 + \mu H_i(\mu) \int_0^1 \frac{a_i(1 - \mu')^2}{\mu + \mu'} H_i(\mu')d\mu'$$

(C2)

with $a_l = 3/4$ and $a_r = 3/8$, while $q$ and $c$ are the solutions to the equations

$$q^2 = 2(1 - c^2),$$

$$q H_l(1) = (1 + c)H_r(1).$$

(C3)

The resulting intensities were numerically calculated by Chandrasekhar and his secretary, Mrs. Frances H. Breen, to five decimal points for the 20 values of $\mu = 0 : 0.05 : 1$ using pen and paper (Table XXIV in Chandrasekhar 1950, note that $F = \mathcal{L}/\pi$). The resulting total intensity, $I = I_l + I_r$, can be fit by the linear relation

$$h_{CB}(\mu) = \frac{2\pi I}{\mathcal{L}} \approx 0.85 + 1.725\mu$$

(C4)

to an accuracy better than 3% for all $0 < \mu < 1$ (by assumption, $I(\mu) = 0$ for $\mu < 0$).

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