When $G^2$ is a König-Egerváry graph?

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Abstract

The square of a graph $G$ is the graph $G^2$ with the same vertex set as in $G$, and an edge of $G^2$ is joining two distinct vertices, whenever the distance between them in $G$ is at most 2. $G$ is a square-stable graph if it enjoys the property $\alpha(G) = \alpha(G^2)$, where $\alpha(G)$ is the size of a maximum stable set in $G$.

In this paper we show that $G^2$ is a König-Egerváry graph if and only if $G$ is a square-stable König-Egerváry graph.

Keywords: Square of a graph; Perfect matching; Maximum stable set.

1 Introduction

All the graphs considered in this paper are finite, undirected, loopless and without multiple edges. For such a graph $G = (V, E)$ we denote its vertex set by $V = V(G)$ and its edge set by $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$.

The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \bigcup \{N(v) : v \in A\}$, for $A \subseteq V$. If $|N(v)| = |\{w\}| = 1$, then $v$ is a leaf and $vw$ is a pendant edge of $G$.

By $C_n$, $K_n$, $P_n$ we denote the chordless cycle on $n \geq 4$ vertices, the complete graph on $n \geq 1$ vertices, and respectively the chordless path on $n \geq 3$ vertices.

A stable set of maximum size will be referred as to a stability system of $G$. The stability number of $G$, denoted by $\alpha(G)$, is the cardinality of a stability system in $G$. Let $\Omega(G)$ denotes $\{S : S$ is a stability system of $G\}$. 
A matching is a set of non-incident edges of $G$; a matching of maximum cardinality $\mu(G)$ is a maximum matching, and a matching covering all the vertices of $G$ is called a perfect matching. $G$ is a König-Egerváry graph provided $\alpha(G) + \mu(G) = |V(G)|$, [1], [11].

If $S$ is an independent set of a graph $G$ and $H = G[V - S]$, then we write $G = S \ast H$.

Clearly, any graph admits such representations.

**Theorem 1.1** [5] If $G$ is a graph, then the following assertions are equivalent:

(i) $G$ is a König-Egerváry graph;

(ii) $G = S \ast H$, where $S \in \Omega(G)$ and $|S| \geq \mu(G) = |V(H)|$;

(iii) $G = S \ast H$, where $S$ is an independent set with $|S| \geq |V(H)|$ and $(S, V(H))$ contains a matching $M$ of size $|V(H)|$.

$G$ is well-covered if it has no isolated vertices and if every maximal stable set of $G$ is also a maximum stable set, i.e., it is in $\Omega(G)$ [8]. $G$ is called very well-covered [2], provided $G$ is well-covered and $|V(G)| = 2\alpha(G)$. Some interrelations between well-covered and König-Egerváry graphs were studied in [3], [4].

The distance between two vertices $v, w \in V(G)$ is denoted by $dist_G(v, w)$, or $dist(v, w)$ if no ambiguity. $G^2$ denotes the second power of graph $G$, i.e., the graph with the same vertex set $V$ and an edge is joining distinct vertices $v, w \in V$ whenever $dist_G(v, w) \leq 2$. Clearly, any stable set of $G^2$ is stable in $G$, as well, while the converse is not generally true. Therefore, we may assert that $1 \leq \alpha(G^2) \leq \alpha(G)$. Let notice that the both bounds are sharp. For instance, if:

- $G$ is not a complete graph and $dist(a, b) \leq 2$ holds for any $a, b \in V(G)$, then $\alpha(G) \geq 2 > 1 = \alpha(G^2)$; e.g., for the $n$-star graph $G = K_{1,n}$, with $n \geq 2$, we have $\alpha(G) = n > \alpha(G^2) = 1$;

- $G = P_4$, then $\alpha(G) = \alpha(G^2) = 2$.

The graphs $G$ for which the upper bound of the above inequality is achieved, i.e., $\alpha(G) = \alpha(G^2)$, are called square-stable; e.g., the graph from Figure 1.

![Figure 1: A square-stable graph $G$ and its $G^2$.](image)

**Theorem 1.2** [6] The graph $G$ is square-stable if and only if there is some $S \in \Omega(G)$ such that $dist_G(a, b) \geq 3$ holds for all distinct $a, b \in S$.

In this paper we prove that $G^2$ is a König-Egerváry graph if and only if $G$ is a square-stable König-Egerváry graph. In particular, we deduce that the square of the tree $T$ is a König-Egerváry graph if and only if $T$ is well-covered.
It is quite evident that \( G \) and \( G^2 \) are simultaneously connected or disconnected. Thus in the rest of the paper all the graphs are connected.

**Lemma 2.1** If \( G \) is a square-stable graph with 2 vertices at least, then \( \alpha(G) \leq \mu(G) \).

**Proof.** According to Theorem 1.2 there exists a maximum stable set \( S = \{ v_i : 1 \leq i \leq \alpha(G) \} \) in \( G \) such that \( \text{dist}_G(a, b) \geq 3 \) for all pairwise distinct \( a, b \in S \). It follows that for every \( i \in \{1, 2, ..., \alpha(G) - 1\} \) there is a shortest path in \( G \), of length 3 at least, connecting \( v_i \) to \( v_{\alpha(G)} \), say \( v_i, w_i, ..., w_i, v_{\alpha(G)} \) (see Figure 2).

![Figure 2](image)

Figure 2: \( S = \{ v_1, v_i, ..., v_{\alpha(G)} \} \in \Omega(G) \) and \( M = \{ v_1w_1, ..., v_iw_i, ..., v_{\alpha(G)}w_1 \} \) is a matching in \( G \).

All the vertices \( w_i, 1 \leq i \leq \alpha(G) - 1 \) and \( w^1 \) are pairwise distinct, i.e.,

\[
w_i \neq w^1, 1 \leq i \leq \alpha(G) - 1,
\]

because, otherwise, there will be a pair of vertices in \( S \) at distance 2, in contradiction with the hypothesis on \( S \). Hence we deduce that

\[
M = \{ v_iw_i : 1 \leq i \leq \alpha(G) - 1 \} \cup \{ v_{\alpha(G)}w_1 \}
\]

is a matching in \( G \) that saturates all the vertices of \( S \in \Omega(G) \). Consequently, we obtain \( \alpha(G) = |S| = |M| \leq \mu(G) \). 

**Remark 2.2** The vertex \( w^1 \) in the proof of Lemma 2.1 may be a common vertex for more shortest paths connecting various \( v_i \) to \( v_{\alpha(G)} \) (see Figure 3).

![Figure 3](image)

Figure 3: \( G \) has \( \alpha(G) = \alpha(G^2) = 3 = |\{ v_1w_1, v_2w_2, v_3u \}| < \mu(G), \) where \( w^1 = w^2 = u. \)

The graph \( G \) in Figure 1 is square-stable and has \( \mu(G) = \mu(G^2) = 2 \), while the square-stable graph \( G \) from Figure 1 satisfies \( \mu(G) < \mu(G^2) \). Notice that, in the both examples, neither \( G \) nor \( G^2 \) is a König-Egerváry graph.
Proposition 2.3 Let $G^2$ be a König-Egerváry graph with 2 vertices at least. Then the following assertions are equivalent:

(i) $\alpha(G) = \alpha(G^2)$;
(ii) $\mu(G) = \mu(G^2)$;
(iii) $G$ is a König-Egerváry graph with a perfect matching.

Proof. The following inequalities are true for every graph $G$:

$$\mu(G) \leq \mu(G^2) \leq \alpha(G^2) \leq \alpha(G).$$

Since $G^2$ is a König-Egerváry graph, $\mu(G^2) \leq \alpha(G^2)$. Consequently, we get

$$(i) \Rightarrow (ii),(iii) \text{ If } G \text{ is square-stable, then these inequalities together with Lemma 2.1 give }$$

$$\mu(G) = \mu(G^2) = \alpha(G^2) = \alpha(G).$$

Moreover, we infer that

$$|V(G)| = \mu(G^2) + \alpha(G^2) = \mu(G) + \alpha(G),$$

which means that $G$ is a König-Egerváry graph. In addition, $G$ has a perfect matching, because $\mu(G) = \alpha(G)$.

$$(iii) \Rightarrow (i) \text{ If } G \text{ is a König-Egerváry graph with a perfect matching, then }$$

$$\mu(G) + \alpha(G) = |V(G)| = \mu(G^2) + \alpha(G^2) \text{ and } \mu(G) = \mu(G^2).$$

Thus, we deduce that $\alpha(G) = \alpha(G^2)$, i.e., $G$ is a square-stable graph.

$$(ii) \Rightarrow (i) \text{ If } \mu(G) = \mu(G^2), \text{ then it follows that }$$

$$|V(G)| = \alpha(G^2) + \mu(G^2) \leq \alpha(G) + \mu(G) = \alpha(G) + \mu(G) \leq |V(G)|,$$

which assures that $\alpha(G) = \alpha(G^2)$, i.e., $G$ is a square-stable graph. \[\Box\]

Remark 2.4 There are König-Egerváry graphs, whose squares are not König-Egerváry graphs; e.g., every even chordless cycle.

Remark 2.5 There are non-König-Egerváry graphs, whose squares are not König-Egerváry graphs; e.g., every odd chordless cycle.
Theorem 2.6 If $G^2$ is a König-Egerváry graph, then $G$ is a square-stable König-Egerváry graph with a perfect matching.

Proof. Since $G^2$ is a König-Egerváry graph, Theorem 1.1 ensures that $G^2 = S * H$, where $S \in \Omega(G^2)$, $\mu(G^2) = |V(H)|$ and every maximum matching of $G^2$ is contained in $(S, V(H))$.

Let $S = \{ s_j : 1 \leq j \leq \alpha(G^2) \} \in \Omega(G^2)$ and $V(H) = \{ h_k : 1 \leq k \leq |V(G)| - \alpha(G^2) \}$.

Claim 1. Every $h \in V(H)$ is joined, by an edge from $G$, to at most one vertex of $S$.

Otherwise, if some $h \in V(H)$ has two neighbors $s_i, s_j \in S$ such that $hs_i, hs_j \in E(G)$, then $s_i s_j \in E(G^2)$, in contradiction to the fact that $S$ is independent.

Claim 2. $S_G(H) = S_{G^2}(H)$, where

$S_G(H) = \{ s \in S : (\exists) hs \in E(G), h \in V(H) \}$, and

$S_{G^2}(H) = \{ s \in S : (\exists) hs \in E(G^2), h \in V(H) \}$.

Since $E(G) \subseteq E(G^2)$, we get that $S_G(H) \subseteq S_{G^2}(H)$. Assume that there is some $s \in S_G(H) - S_{G^2}(H)$. Hence, it follows that there is some $h_j s \in E(G^2) - E(G)$.

Consequently, in $G$ must exist some path on two edges from $s$ to $h_j$, and because $S$ is stable, it follows that there is some $h_k \in V(H)$, such that $h_k h_j, h_k s \in E(G)$ and this contradicts the fact that $s \in S_G(H) - S_{G^2}(H)$.

Claim 3. There is a maximum matching in $G^2$ containing only edges from $G$.

Combining Claim 1 and Claim 2, it follows that every $h \in V(H)$ is joined, by an edge from $G$, to exactly one vertex of $S$, say $s(h)$, because, otherwise, we get $S_G(H) \neq S_{G^2}(H)$. Now, the set $M = \{ hs(h) : h \in V(H) \}$ is a matching both in $G$ and in $G^2$. Moreover, by Theorem 1.1 $M$ is a maximum matching in $G^2$, because $|M| = |V(H)|$. Consequently, we deduce that $|M| \leq \mu(G) \leq \mu(G^2) = |M|$, which implies $\mu(G) = \mu(G^2)$.

According to Proposition 2.3 it follows that $G$ is a square stable König-Egerváry graph having a perfect matching.

Notice that the converse of Theorem 2.6 is not generally true; e.g., $G = C_{2n}, n \geq 2$.

Now we are ready to formulate the main finding of the paper.

Theorem 2.7 For a graph $G$ of order $n \geq 2$ the following assertions are equivalent:

(i) $G^2$ is a König-Egerváry graph;

(ii) $G$ is a square-stable König-Egerváry graph;

(iii) $G$ has a perfect matching consisting of pendant edges;

(iv) $G$ is very well-covered with exactly $\alpha(G)$ leaves.

Proof. The implication (i) $\implies$ (ii) follows from Theorem 2.6. The proof of the implication (ii) $\implies$ (i) is in the following series of inequalities:

$|V(G)| = \alpha(G) + \mu(G) = \alpha(G^2) + \mu(G) \leq \alpha(G^2) + \mu(G^2) \leq |V(G^2)| = |V(G)|$.

All the equivalences between (ii), (iii) and (iv) have been proved in 7. □

It was shown in 11 that a tree having at least two vertices is well-covered if and only if it has a perfect matching consisting of pendant edges. It was also mentioned there that every well-covered tree of order at least two is very well-covered as well. Combining these observations with Theorem 2.7 we obtain the following.

Corollary 2.8 The square of a tree is a König-Egerváry graph if and only if the tree is well-covered.
3 Conclusions

Recall that $\theta(G)$ is the clique covering number of $G$, i.e., the minimum number of cliques whose union covers $V(G)$; $i(G) = \min\{|S| : S \text{ is a maximal stable set in } G\}$, and $\gamma(G) = \min\{|D| : D \text{ is a minimal domination set in } G\}$. In general, it can be shown that the graph invariants mentioned above are related by the following inequalities:

$$\alpha(G^2) \leq \theta(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \theta(G),$$

which turn out to be equalities, when $\alpha(G^2) = \alpha(G)$ or $\theta(G^2) = \theta(G)$ [9].

It seems interesting to find out some other graph operations and invariants such that interrelations between them may lead to König-Egerváry graphs.

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