Cramér–Rao Bound Analog of Bayes’ Rule

The estimation of multiple parameters is a common task in signal processing. The Cramér–Rao bound (CRB) sets a statistical lower limit on the resulting errors when estimating parameters from a set of random observations. It can be understood as a fundamental measure of parameter uncertainty [1], [2]. As a general example, suppose \( \theta \) denotes the vector of sought parameters and that the random observation model can be written as

\[
y = x_\theta + w,
\]

where \( x_\theta \) is a function or signal parameterized by \( \theta \) and \( w \) is a zero-mean Gaussian noise vector. Then the CRB for \( \theta \) has the following notable properties:

1) For a fixed \( \theta \), the CRB for \( \theta \) decreases as the dimension of \( y \) increases.
2) For a fixed \( y \), if additional parameters \( \tilde{\theta} \) are estimated, then the CRB for \( \theta \) increases as the dimension of \( \tilde{\theta} \) increases.
3) If adding a set of observations \( \tilde{y} \) requires estimating additional parameters \( \tilde{\theta} \), then the CRB for \( \theta \) decreases as the dimension of \( \tilde{y} \) increases, provided the dimension of \( \tilde{\theta} \) does not exceed that of \( \tilde{y} \) [3]. This property implies both 1) and 2) above.
4) Among all possible distributions of \( w \) with a fixed covariance matrix, the CRB for \( \theta \) attains its maximum when \( w \) is Gaussian, i.e., the Gaussian scenario is the “worst case” for estimating \( \theta \) [4]–[6].

In this lecture note, we show a general property of the CRB that quantifies the interdependencies between the parameters in \( \theta \). The presented result is valid for more general models than (1) and also generalizes the result in [7] to vector parameters. It will be illustrated via two examples.

RELEVANCE
In probability theory, the chain rule and Bayes’ rule are useful tools to analyze the statistical interdependence between multiple random variables and to derive tractable expressions for their distributions. In this lecture note, we provide analogs of the chain rule and Bayes’ rule for the CRB associated with multiple parameters. The results are particularly useful when estimating parameters of interest in the presence of nuisance parameters.

PREREQUISITIES
The reader needs basic knowledge about linear algebra, elementary probability theory, and statistical signal processing.

PRELIMINARIES
We will consider a general scenario in which we observe an \( n \times 1 \) random vector \( y \). Its probability density function (pdf) \( p(y; \theta) \) is parameterized by a \( k \times 1 \) deterministic vector \( \theta \). The goal is to estimate \( \theta \), or subvectors of \( \theta \), given \( y \).

Let \( l(\theta) \triangleq \ln p(y; \theta) \) denote the log-likelihood function, and let \( \hat{\theta} \) be any unbiased estimator. Then the mean square error (MSE) matrix \( P_\theta \triangleq E[(\theta - \hat{\theta})(\theta - \hat{\theta})^H] \) is bounded from below by the inverse of the Fisher information matrix \( J_\theta \triangleq -E[\partial^2 l(\theta)] \), where \( \partial^2 l(\theta) \) denotes the second-order differential or Laplacian operator with respect to \( \theta \). That is, \( P_\theta \geq J_\theta^{-1} \), assuming from hereon that \( J_\theta \) is nonsingular. This is the Cramér–Rao inequality [2], [8], [9].

The determinant of the MSE matrix, \( |P_\theta| \), is a scalar measure of the error magnitude. For unbiased estimators, \( |P_\theta| \) equals the “generalized variance” of errors [10]. By defining \( \text{CRB}(\theta) \triangleq |J_\theta^{-1}| \) the generalized error variance is bounded by \( |P_\theta| \geq \text{CRB}(\theta) \).

In the following, we are interested in subvectors or elements of \( \theta \). Letting \( \theta = [\alpha^T \beta]^T \), we can write the Fisher information matrix in block form,

\[
J_\theta = -E \left[ \frac{\partial^2 l(\alpha, \beta)}{\partial \alpha} \frac{\partial^2 l(\alpha, \beta)}{\partial \beta} \right] = J_{\alpha \alpha} J_{\beta \beta}^{-1} J_{\beta \alpha} J_{\alpha \beta}^{-1}.
\]

MAIN RESULT
Let \( a \) and \( b \) be two random vectors. Two useful rules in probability theory are the chain rule

\[
p(a, b) = p(a | b)p(b)
\]

and Bayes’ rule

\[
p(a) = \frac{p(b | a)p(a)}{p(b)}.
\]

Now consider two parameter vectors \( \alpha \) and \( \beta \). When both are unknown, their joint CRB bound is given by

\[
\text{CRB}(\alpha, \beta) = \begin{bmatrix} J_{\alpha \alpha} & J_{\alpha \beta} \end{bmatrix}^{-1}.
\]

The bound for \( \alpha \) with known \( \beta \) is simply

\[
\text{CRB}(\alpha | \beta) = |J_\alpha^{-1}|,
\]

and the bound for \( \alpha \) with unknown \( \beta \) is

\[
\text{CRB}(\alpha) = |(J_\alpha - J_{\alpha \beta} J_\beta^{-1} J_{\beta \alpha})^{-1}|.
\]

[Equation (7) follows by evaluating the inverse in (5) and extracting the upper-left block corresponding to \( \alpha \).]
By applying the Schur determinant formula [8], [11]
\[
\begin{vmatrix}
J_{a} & J_{ab} \\
J_{ba} & J_{b}
\end{vmatrix} = |J_{a}| |J_{b} - J_{ba}J_{a}^{-1}J_{ab}| = |J_{a}||J_{b} - J_{ba} J_{a}^{-1}J_{ab}|
\]
along with \(|J_{a}^{-1}||J_{b}|^{-1}\), to (5)–(7), we can now state the CRB analogs of the chain rule (3),
\[
\text{CRB}(\alpha, \beta) = \text{CRB}(\alpha | \beta) \text{CRB}(\beta) \tag{8}
\]
and of Bayes’ rule (4),
\[
\text{CRB}(\alpha) = \frac{\text{CRB}(\beta)}{\text{CRB}(\beta | \alpha)} \text{CRB}(\alpha | \beta). \tag{9}
\]
The results are, of course, symmetric, i.e., one can interchange \(\alpha\) and \(\beta\).

From (8) we see that the joint error bound for \(\alpha\) and \(\beta\) equals the error bound for \(\alpha\), when \(\beta\) is known, multiplied by the error bound for \(\beta\). More interestingly, (9) tells us that the error bound for \(\alpha\) is equal to the bound for \(\beta\) when \(\beta\) is known, multiplied by a factor, viz. \(\text{CRB}(\beta)/\text{CRB}(\beta | \alpha) \geq 1\), that quantifies the influence of \(\beta\) on one’s ability to estimate \(\alpha\).

**REMARK 1**
The rules can be applied to cases with any number of additional parameters, besides \(\alpha\) and \(\beta\). Consider, for instance, the case of \(\alpha\), \(\beta\), and \(\gamma\), where \(\gamma\) is an unknown nuisance parameter. Then applying the chain rule twice yields
\[
\text{CRB}(\alpha, \beta, \gamma) = \text{CRB}(\gamma | \alpha, \beta) \text{CRB}(\alpha, \beta)/\text{CRB}(\beta) = \text{CRB}(\gamma | \alpha, \beta) \text{CRB}(\beta | \alpha) \text{CRB}(\alpha), \tag{10}
\]
where the factors without \(\gamma\) signify that the nuisance parameter is unknown. Combining the two expressions in (10) yields the analog of Bayes’ rule (9) for any number of additional parameters.

The joint error bound for a set of parameters \(\alpha_{1}, \alpha_{2}, \alpha_{3}, \ldots\) can be similarly decomposed by a recursive application of the chain rule to analyze their interdependency and its impact on estimation.

**REMARK 2**
The CRB analog of Bayes’ rule (9) generalizes the result in [7], which concerns only scalar parameters \(\alpha\) and \(\beta\) amid a vector of nuisance parameters \(\gamma\). Our proof of (9) is also more direct than in [7].

**REMARK 3**
These results are also applicable to the posterior, or Bayesian, CRB (PCRB), in which \(\theta\) is modeled as a random variable with a prior distribution. The PCRB is valid for the entire class of estimators \(\hat{\theta}\), whether biased or not [2]. The posterior Cramér–Rao inequality is then \(P_{\theta} \geq J_{\theta}\), where \(J_{\theta} = -E[\partial^{2} \ln p(y, \theta)]\) is the Bayesian Fisher information matrix, \(p(y, \theta)\) is the joint pdf and the expectation is with respect to this pdf. Letting \(\theta = [\alpha^{T} \beta^{T}]^{T}\), the matrix can be partitioned correspondingly,
\[
J_{\theta} = \begin{bmatrix} J_{a} & J_{ab} \\
J_{ba} & J_{b}
\end{bmatrix},
\]
and thereby the results (8)–(10) can be applied to the PCRB as well.

**EXAMPLES**
Next, we illustrate via two examples how a decomposition like (9) can be used for analysis. The examples show that, by quantifying the impact of nuisance parameters, it is possible to study the tradeoff between the gain of obtaining them through independent side information versus estimating them jointly with the parameters of interest.

**LINEAR MIXED MODEL**
Consider a linear model
\[
y = Ax + Bz + w \in \mathbb{R}^{n},
\]
where \(w\) is Gaussian noise with covariance matrix \(\mathbf{I}\), and \(x \in \mathbb{R}^{k}\) and \(z \in \mathbb{R}^{k}\) are unknown parameters. The matrices are known and of rank \((\mathbf{A} \mathbf{B}) = k_{x} + k_{z} < n\), which implies that the parameters \(x\) and \(z\) are embedded into two distinct range spaces, \(\mathcal{R}(\mathbf{A})\) and \(\mathcal{R}(\mathbf{B})\), respectively. Here \(\mathcal{R}(\mathbf{A})\) denotes the linear subspace spanned by the columns of \(\mathbf{A}\). Under these conditions the joint Fisher information matrix equals [9]
\[
\begin{bmatrix} J_{x} & J_{zx} & J_{zx} \\
J_{zx} & J_{z} & J_{z} \\
J_{zx} & J_{z} & J_{z}
\end{bmatrix} = \frac{1}{\nu} \begin{bmatrix} A^{T} A & A^{T} B & 0 \\
B^{T} A & B^{T} B & 0 \\
0 & 0 & \frac{n_{2}}{2\nu}
\end{bmatrix}.
\]
From this expression, we see that the bound for \(\nu\) is independent of that for \(x\) and \(z\). That is, \(\text{CRB}(x, z, \nu) = \text{CRB}(x, z, \nu)\) CRB(\nu). This is a CRB analog of the independence for random variables. Furthermore, we obtain \(\text{CRB}(x | z) = |J_{x}^{-1}| = |v(B^{T} B)^{-1}| = \nu^{-1}|B^{T} B|^{-1}\) and \(\text{CRB}(z) = |(J_{z} - J_{zx} J_{zx}^{-1})| = |v(B^{T} B - B^{T} A (A^{T} A)^{-1} A^{T} B)^{-1}| = \nu^{-1}|B^{T} \Pi_{z} B|^{-1}\), where \(\Pi_{z}\) is the projector onto the orthogonal complement of \(\mathcal{R}(\mathbf{A})\).

The increase in the error bound for \(x\) due to the lack of information about \(z\) can now be quantified using (9)
\[
\text{CRB}(x) = \frac{|B^{T} B|}{|B^{T} \Pi_{z} B|} \text{CRB}(x | z) \tag{11}
\]
where the factor \(|B^{T} \Pi_{z} B|^{-1}\) measures the alignment of \(\mathcal{R}(\mathbf{A})\) and \(\mathcal{R}(\mathbf{B})\). When the range spaces are orthogonal we have that \(|B^{T} \Pi_{z} B| = |B^{T} B|\), and by (11) the bound for \(x\) is unaffected by one’s ignorance about \(z\). In scenarios where it is possible to obtain \(z\) through additional side-information or calibration instead of estimation, the cost can be weighed against the reduction of the error bound for \(x\) by the given factor \(|B^{T} \Pi_{z} B|^{-1} / |B^{T} B|\).

This example has illustrated the interdependencies between the unknown parameters \(x\), \(z\), and \(\nu\). Next we consider an example where the unknown parameters become asymptotically independent as the number of samples \(n\) grows large.

**SINE-WAVE FITTING**
Sine-wave fitting is a problem that arises in system testing, e.g., of waveform recorders, and IEEE Standard 1057 formalizes procedures to do so (see [12] and references therein).

Consider \(n\) uniform samples of a sinusoid in noise
\[
y(k) = \alpha \sin(\omega k + \phi) + C + w(k),
\]
where \(w(k)\) is a Gaussian white noise process with variance \(\nu\) and \(k = 0, \ldots, n - 1\). The amplitude \(\alpha\) and phase \(\phi\) of the sinusoidal signal, along with the offset \(C\), are of interest. In certain cases, the frequency \(\omega\) of the test signal may be obtained separately from the estimation of \(\alpha, \phi\) and \(C\). For simplicity, we first consider an alternative parameterization of the sinusoid: \(\alpha \sin(\omega k + \phi) = A \cos(\omega k) + B \sin(\omega k)\), where \(A = \alpha \sin(\phi)\) and \(B = \alpha \cos(\phi)\). The parameters are \(\theta = [\alpha B C \omega \nu]^{T}\).
As shown in [12], the Fisher information matrix can be decomposed into $J_\alpha = J_\alpha + J_\theta$, where $J_\alpha$, shown in the box at the bottom of the page, contains the dominant terms and $J_\theta$ contains the remainder, so that $J_\alpha = J_\alpha^\dagger$ for large $n$. Using this approximation we now analyze the bounds for $A$, $B$, and $C$ by application of (9).

First, let $\theta^\dagger = [ABC \mathbf{v}]^\top$ be the parameter vector without $\alpha$. Then

$$CRB(\alpha) = \left| J_\omega - J_{\omega \alpha} J_{\alpha \omega}^{-1} J_{\alpha \omega} \right|^{-1}$$

$$= 2 \alpha \left( \frac{A^2 + B^2}{n^2} \right)$$

$$= \frac{2}{n^2} \left( A^2 + B^2 \right)$$

where the bounds for $B$ and $C$ are derived in a similar manner as for $A$. This shows that the bound for the offset $C$ becomes independent of the knowledge of the frequency $\omega$ as $n$ increases, while the bounds for $A$ and $B$ are inflated by factors ranging between one and four due to one’s ignorance about $\omega$.

When considering the original parameterization $\theta^\dagger = [\alpha \phi C \mathbf{v}]^\top$ there exists an invertible relation, $\theta^\dagger = \theta^\dagger(\alpha)$. Therefore we have that $J_{\alpha \theta} = \frac{\partial \theta^\dagger}{\partial \theta^\dagger}$, where $\theta^\dagger$ denotes the first-order differential or gradient with respect to $\theta$ and

$$\partial_\phi \theta^\dagger(\alpha) = \begin{bmatrix} \sin \phi & \cos \phi & 0 \\ -\sec^2 \phi & \sec \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exploiting the approximation $J_{\alpha \theta} = J_{\alpha \theta}^\dagger$ once again, one obtains [12]

$$CRB(\sigma) \equiv CRB(\sigma | \alpha)$$

$$CRB(\phi) \equiv 4CRB(\phi | \alpha)$$

This shows that, in large samples, the error bound for the amplitude $\sigma$ also becomes independent of knowledge about the frequency $\omega$, whereas not knowing $\omega$ inflates the bound for the phase $\phi$ by a factor of four.

For large data records, the cost of pre-calibrating the frequency can be weighed against a reduction of the error bound for the phase, while the error bounds for the amplitude and offset will not be improved.

**WHAT WE HAVE LEARNED**

An analog of Bayes’ rule for the CRB has been derived. This analogous rule enables a formalized decomposition and quantification of the mutual dependencies between multiple unknown parameters.

The use of the rule was illustrated in two estimation problems.

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