THRESHOLD CORRECTIONS AND SYMMETRY ENHANCEMENT IN STRING COMPACTIFICATIONS

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ABSTRACT

We present the computation of threshold functions for Abelian orbifold compactifications. Specifically, starting from the massive, moduli-dependent string spectrum after compactification, we derive the threshold functions as target space duality invariant free energies (sum over massive string states). In particular we work out the dependence on the continuous Wilson line moduli fields. In addition we concentrate on the physically interesting effect that at certain critical points in the orbifold moduli spaces additional massless states appear in the string spectrum leading to logarithmic singularities in the threshold functions. We discuss this effect for the gauge coupling threshold corrections; here the appearance of additional massless states is directly related to the Higgs effect in string theory. In addition the singularities in the threshold functions are relevant for the loop corrections to the gravitational coupling constants.
1 Introduction

Four-dimensional string models possibly provide a consistent description of all known interactions including gravity. At energy scales small compared to the typical string scale $M_{\text{string}}$, which is directly related to the Planck mass $M_{\text{Planck}} = O(10^{19} \text{ GeV})$, it is convenient to use an effective Lagrangian for the light string modes $\phi_L$ where the effect of the heavy string modes $\phi_H$ is integrated out. Based on several motivations, the main area of research in this context is on four-dimensional string vacua with $N = 1$ space-time supersymmetry, implying that the effective string interactions are of the form of an $N = 1$ supergravity-matter action. Studying four-dimensional string vacua with space-time supersymmetry, it turns out that infinitely many of them are connected by continuous deformations of the underlying two-dimensional conformal field theory, parametrized by coupling constants called moduli, collectively denoted by $T_i$. At the level of the effective supergravity action, and neglecting non-perturbative effects, moduli are described by massless scalar fields with flat potential, whose vacuum expectation values parametrize the continuous deformations. The moduli play a very important role in the effective action, since various coupling constants among the matter fields like the tree-level Yukawa couplings [1, 2] or the loop corrections to the gauge and gravitational coupling constants [3]-[19] are moduli dependent functions. Specifically, the one-loop moduli dependence of the gauge and gravitational coupling constants arises from $\sigma$-model and Kähler anomalies [5, 7, 8, 10] and from integrating out the infinite tower of massive string modes with moduli dependent masses.

The moduli spaces $\mathcal{M}$ of four-dimensional string theories have a very interesting and rich structure. First, as it is true for many known compactification schemes of the ten-dimensional heterotic string, the underlying superconformal field theory is invariant under the target space duality symmetries (see [20]) which act on the the moduli as discrete reparametrizations. These target space duality transformations act in general non-trivially on the infinite massive spectrum $\phi_H$, in the sense that states with different quantum numbers (e.g. discrete internal momentum and winding numbers) are mapped onto each other. Consequently, integrating out the massive spectrum $\phi_H$ implies that certain low energy couplings are given in terms of automorphic functions of the corresponding duality group. This observation can provide very useful informations about the structure of the effective low-energy supergravity action, in particular when combined with some analyticity arguments of holomorphic coupling functions [21].

A second very interesting feature of the string moduli spaces comes from the fact that at certain critical points $P_c$ in the moduli spaces a finite number of additional massless
states may appear in the moduli-dependent string spectrum, which are otherwise massive at generic points in $\mathcal{M}$. We will call these states $\phi_H'$. Very often, at $P_c$ these fields correspond to additional holomorphic spin one currents on the world sheet. Then the gauge symmetry of the four-dimensional string is enlarged which is nothing else than the stringy version of the well known Higgs effect. The role of the Higgs fields is now taken by moduli fields. The field-theoretical formulation of the stringy Higgs effect, i.e. the correct coupling of the relevant moduli to the gauge bosons, was investigated in $[22, 23]$ for the case of the standard $\mathbb{Z}_3$ orbifold $[24, 25]$.

The appearance of additional massless fields at some critical points $P_c$ in $\mathcal{M}$ implies that the description of the string compactification by an low-energy effective action contains discontinuities, since near the critical points the fields $\phi_H'$ should be kept as light degrees of freedom and should not be integrated out from the spectrum. In other words, when integrating over all massive fields including $\phi_H'$ the effective action may acquire singularities at the critical points $P_c$. Let us make this more clear by considering as an example the one loop running of a gauge coupling constant in the low-energy field theory. The one-loop running coupling constant at a scale $p$ is given by (neglecting Kähler and $\sigma$-model anomalies)

$$\frac{1}{g^2(p^2)} = \frac{1}{g^2(M_{\text{string}}^2)} + \frac{b}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{|\Delta(T_i)|^2}{16\pi^2}, \quad (1.1)$$

where $b$ is the one-loop $\beta$-function coefficient of the light modes $\phi_L$ and $\Delta(T_i)$ is the moduli-dependent threshold correction of the heavy string modes. $\Delta(T_i)$ can be regarded as the suitably regularized free energy $[26, 27]$ of all massive modes, $\Delta(T_i) \propto \log \det M^2_{\phi_H}(T_i)$, where the $M_{\phi_H}(T_i)$’s are the moduli-dependent masses of the heavy modes. Clearly, if this sum contains also states $\phi_H'$ which become massless at $P_c$, $\Delta(T_i)$ possesses a singularity at $P_c$. As we will discuss in detail, the masses of $\phi_H'$ are generically of the form $M_{\phi_H'}(T_i) \propto (T_i - P_c)$ (here, $T_i$ is one specific ‘critical’ modulus). Thus we see that $\Delta(T_i)$ exhibits a logarithmic singularity of the form

$$\Delta \rightarrow n \log(T_i - P_c), \quad (1.2)$$

where $n$ accounts for the degeneracy of states which become massless at $P_c$. In order to get finite threshold corrections $\tilde{\Delta}(T_i)$ over the whole moduli space it is useful to separate in eq. (1.1) the contribution of the states $\phi_H'$ from the states $\phi_H$ which are always massive. Then eq. (1.1) can be written as

$$\frac{1}{g^2(p^2)} = \frac{1}{g^2(M_{\text{string}}^2)} + \frac{b}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{b'}{16\pi^2} \log |T_i - P_c|^2 + \frac{|\tilde{\Delta}(T_i)|^2}{16\pi^2}. \quad (1.3)$$
Here \( b' \) is the contribution of the states \( \phi'_H \) to the \( \beta \)-function coefficient, and \( \tilde{\Delta}(T_i) \) does not contain the states \( \phi'_H \). Thus the logarithmic singularity in the moduli \( T_i \) is nothing else than the threshold effect of \( \phi'_H \) with (intermediate) mass scale \( M_{\phi'_H}(T_i) \propto (T_i - P_c) \).

This discussion was entirely based on field theoretical arguments. As already discussed in [28], in string theory these threshold functions are given in terms of automorphic functions of the underlying target space duality group with the appropriate singularity structure. In fact, since in string theory there is generically an infinite number of states which may become massless at duality equivalent points in \( \mathcal{M} \) (at one particular point in \( \mathcal{M} \) only a finite number of states can become massless) the relevant automorphic functions possess singularities at an infinite number of points being related by discrete duality transformations. In this paper we will calculate explicitly these divergent threshold functions which are related to the discontinuities in the string spectrum and to the stringy Higgs effect. To be specific we concentrate on Abelian orbifold compactifications of the ten-dimensional heterotic string. We will discuss the dependence of the threshold corrections as functions of the moduli associated with the six-dimensional orbifold, denoted by \( T \) and \( U \), as well as of the so-called Wilson line moduli [29] which take values in the heterotic gauge group. The orbifold moduli \( T, U \) (metric and antisymmetric tensor) will be relevant for the discussion of the Higgs effect in the compactification sector; here the relevant automorphic functions will be given in terms of the absolute modular invariant function \( j \). On the other hand, the Wilson line moduli are responsible for the Higgs effect in the heterotic gauge group. This is of rather phenomenological importance since the Wilson line Higgs field may break some GUT gauge group to the gauge group of the standard model. Moreover it may be even possible to identify some of the Wilson line moduli with the supersymmetric standard model Higgs fields \( H_1 \) and \( H_2 \).

Our paper is organized as follows. In sections 2 and 3 we discuss the structure of the gauge groups in orbifold compactifications as a function of the various moduli fields. In section 4 we compute the masses of the generically massive fields. Some of these masses become zero at certain fixed points in the orbifold moduli spaces. Here we will use the results of our previous paper [30] where we have determined the moduli spaces plus the target space duality transformations in the presence of Wilson line moduli. In sections 5, 6 and 7 we will apply the results of section 4 to compute the target space free energies as (infinite) sums over the massive string states. Finally we will explain the relation of these free energies to the string threshold corrections. We will display the dependence of the threshold corrections in terms of \( T \) and \( U \) as well as in terms of the generic Wilson line moduli. A discussion of this is provided in section 8.
2 Gauge groups in orbifold compactifications with continuous Wilson lines

In this section we will start to discuss the moduli dependence of the gauge group of an orbifold compactification. We will concentrate on the so called gauge sector here, whereas the compactification sector will be studied in the next section. Our aim is to determine the unbroken gauge group in the presence of the most general continuous Wilson lines compatible with a given twist. For a certain class of twists the minimal and therefore generic gauge groups are easily determined, and we describe the method for determining them. The results are listed in tables given at the end of this paper. We also point out under which circumstances this method fails to give the minimal gauge groups, in which case it only yields a lower bound on these minimal gauge groups, that is not saturated. For these cases, in which a more detailed analysis is necessary, we outline how one has to proceed in order to determine the generic gauge groups.

2.1 Wilson lines in the Narain model

Since many properties of an orbifold model can be understood easily in terms of the underlying Narain model [31, 32], let us first recall how the gauge group of a toroidal compactification depends on the Wilson line moduli [33]. We will concentrate on the so called gauge sector which is generated by the sixteen extra left–moving worldsheet bosons $X(z) := X^I_L (z), \ I = 1, \ldots, 16$.

There are sixteen chiral conserved currents $\partial X^I_L (z)$ on the worldsheet. These can be combined with the right–moving ground state, which is an $N = 4$ space time vector supermultiplet, to give a $U(1)^{16} \ N = 4$ gauge theory. For special values of the moduli the Narain lattice $\Gamma = \Gamma_{22,6}$ contains vectors of the form

$$ P = (p_L; p_R) = (v, 0_6; 0_6) \in \Gamma, \ \ p_L^2 = v^2 = 2. \quad (2.1) $$

Then there are extra conserved currents $\exp(i \ v \cdot X(z))$, leading to vertex operators for massless charged gauge bosons with charges $v = (v^I)$, thus extending the gauge group to a rank 16 reductive non–abelian Lie group

$$ G^{(16)} = G^{(l)} \otimes U(1)^{16-l}. \quad (2.2) $$

where $G^{(l)}$ is semi–simple and has rank $l$.

As discussed in [33] these extended symmetries can be described in the following way. Every vector $P \in \Gamma$ of the Narain lattice can be written as

$$ P = q^I l_I + n^i \kappa_i + m_i k_i, \quad (2.3) $$

4
where the integers \( q^I, \ n^i \) and \( m_i \) are the charge, winding and momentum quantum numbers, \( I = 1, \ldots, 16, i = 1, \ldots, 6 \). The standard basis vectors are

\[
l_I = \left(e_I, -\frac{1}{2}(e_I \cdot A_i)e^i; -\frac{1}{2}(e_I \cdot A_i)e^i\right),
\]

\[
\mathbf{k}_i = \left(A_i, (4G_{ij} + D_{ij})\frac{1}{2}e^i; D_{ij}\frac{1}{2}e^j\right)
\]

and

\[
k^i = \left(0_{16}, \frac{1}{2}e^i; \frac{1}{2}e^i\right),
\]

where

\[
D_{ij} = 2(B_{ij} - G_{ij} - \frac{1}{4}A_i \cdot A_j).
\]

This basis is a function of the moduli

\[
G_{ij} = G_{ji} \in M(6, 6, \mathbb{R}), \quad B_{ij} = -B_{ji} \in M(6, 6, \mathbb{R}), \quad A_i \in \mathbb{R}^{16}
\]

of the Narain model, namely the metric and the axionic background field and the Wilson lines. \( e_I \) are basis vectors of a selfdual sixteen dimensional lattice \( \Gamma_{16} \) (the \( E_8 \otimes E_8 \) root lattice or the \( SO(32) \) root lattice extended by the spinor weights of one chirality), and the \( e^i \) are a basis of the dual \( \Lambda^* \) of the compactification lattice.

Vectors of the form (2.1) have quantum numbers such that

\[
q^I C_{IJ} q^J := q^T C^{(16)} q = 2,
\]

where \( C^{(16)} \) is the lattice metric of \( \Gamma_{16} \) and \( n^i m_i := n^T m = 0 \). The Wilson lines \( A_i \) must be chosen such that

\[
v \cdot A_i \in \mathbb{Z}
\]

where \( v = q^I e_I \). Then

\[
P = q^I l_I + (v \cdot A_i) k^i = (v, 0_6; 0_6)
\]

is a Narain vector with \( v^2 = q^T C^{(16)} q = 2 \). If one sets for example \( A_i = 0 \), then all roots of the lattice \( \Gamma_{16} \) are in the Narain lattice and therefore the generic gauge group \( U(1)^{16} \) is extended to \( E_8 \otimes E_8 \) or \( SO(32) \), depending on the choice of \( \Gamma_{16} \). Other solutions, which have as gauge groups all possible maximal rank regular reductive subgroups of \( E_8 \otimes E_8 \) and \( SO(32) \) were constructed in [33].

Finally note that a small deformation \( \delta A_i \) of the Wilson lines, if it destroys some of the conditions \( v \cdot A_i \in \mathbb{Z} \), acts on the lattice as a deformation

\[
(v, 0_6; 0_6) \rightarrow (v, w; w)
\]

with \( w = -\frac{1}{2}(v \cdot \delta A_i)e^i \), which makes the corresponding state acquire a mass

\[
\frac{\alpha'}{2} M^2 = w^2
\]

in a smooth way. This is a version of the stringy Higgs effect [23].

\[\text{\footnotesize More generally all possible extra massless states correspond to Narain vectors with quantum numbers satisfying } q^T C^{(16)} q + 2n^T m = 2. \text{ This is a consequence of the mass formula as we will recall in section 5. A second subclass of this set will be the subject of the next section.}\]
2.2 Definition of the orbifold

We can now proceed to extend this to the untwisted sector of an orbifold compactification. First of all one has to select a Narain lattice with some symmetry that can be moded out. Our reference lattice will be the one with vanishing Wilson lines, which therefore factorises as $\Gamma_{22;6} = \Gamma_{16} \oplus \Gamma_{6;6}$. For definiteness, $\Gamma_{16}$ will be the $E_8 \otimes E_8$ lattice. Then, we have to specify the twist action on $\Gamma_{16}$ and $\Gamma_{6;6}$. Whereas the twist action on $\Gamma_{6;6}$ is defined by choosing one of the 18 twists $\theta$ of the compactification lattice $\Lambda$ that lead to $N = 1$ space time supersymmetry \[35\], the twist on $\Gamma_{16}$ will be a Weyl twist (inner automorphism) $\theta'$ of $E_8 \otimes E_8$. The total twist $\Theta = (\theta', \theta, \theta)$ of the Narain lattice is constrained by world sheet modular invariance \[36\]. The level matching conditions worked out in \[36\] restrict the eigenvalues of $\Theta$. In this paper we will not present a classification of $\mathbb{Z}_N$ Weyl orbifolds. Instead we will take one of the $E_8$ as a hidden sector and assume that the Weyl twist in this sector has been chosen in such a way that it cancels the WS modular anomalies of the internal twist and of the Weyl twist in the first $E_8$. The orbifold model defined this way still has some Wilson line moduli left. In order to be compatible with the twist the Wilson lines must satisfy \[29, 37\]

$$\theta^I_J A_{Ij} = A_{Ii} \theta^i_j \quad (2.12)$$

where $A_{Ij}$ is a matrix containing the components of the Wilson lines and $\theta^I_J$, $\theta^i_j$ are the matrices of the gauge and of the internal twist with respect to the lattice bases $e_I$ and $e^i$ of $\Gamma_{16}$ and $\Lambda^*$ respectively. Wilson line moduli do exist if an eigenvalue appears both in the gauge twist and in the internal twist. More precisely, if a complex conjugated pair of eigenvalues (a real eigenvalue) appears $d$ times in the gauge twist and $d'$ times in the internal twist, this then leads to $2dd'$ ($dd'$) real moduli \[30, 37\].

2.3 Minimal gauge groups in the presence of Wilson lines

Let us now work out the gauge groups for Weyl twists of a $E_8$ in the presence of generic continuous Wilson lines. The basic idea is the following. All Weyl twists of $E_8$ are induced by twists that have a nontrivial action on some sublattice $\mathcal{N}$. This sublattice can be chosen to be the root lattice of a regular semi-simple subalgebra. The twist action on the sublattice can then be described by a so called Carter diagram, which can be thought of as a generalization of the well known Dynkin diagram \[38\]. In fact most Weyl twists of $E_8$ are induced by Coxeter twists of regular subalgebras and in this case the Carter diagram is identical with the Dynkin diagram of this subalgebra. To get all inequivalent Weyl twist one has to add a few twists of subalgebras, which are not Coxeter...
twists. These are then described by Carter diagrams that are not Dynkin diagrams. For more details on Carter diagrams and their relation to Weyl twists see [39] and [40].

Given the sublattice \( \mathcal{N} \) on which the twist acts non-trivially one has to look for the largest complementary sublattice \( \mathcal{I} \) on which it acts trivially. That means that \( \mathcal{I} \) is defined by

\[
E_8 \supset \mathcal{N} \oplus \mathcal{I} \tag{2.13}
\]

together with

\[
(E_8 \supset \mathcal{N} \oplus \mathcal{I} \text{ and } \mathcal{I}' \supset \mathcal{I}) \implies \mathcal{I}' = \mathcal{I}. \tag{2.14}
\]

If we now decompose \( E_8 \) into conjugacy classes with respect to \( \mathcal{N} \oplus \mathcal{I} \), a familiar procedure used in covariant lattice models, we get schematically that

\[
E_8 = (\mathcal{N}, 0) + (0, \mathcal{I}) + \sum_i (W_i(\mathcal{N}), W_i(\mathcal{I})) \tag{2.15}
\]

This means that there are three types of lattice vectors, namely those belonging to the sublattices \( \mathcal{N} \) and \( \mathcal{I} \) and those which have a non-vanishing projection onto both \( \mathcal{N} \) and \( \mathcal{I} \). For a review of lattice techniques see [41].

Using this decomposition the effect of switching on continuous Wilson lines becomes quite obvious. If we assume for the moment that that all eigenvalues of the gauge twist also appear in the internal twist, which is true for all \( \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6' \) and \( \mathbb{Z}_7 \) orbifolds, then the Wilson lines will take arbitrary values in \( \langle \mathcal{N} \rangle_{\mathbb{R}} \). Thus for any generic choice of the Wilson lines only states corresponding to lattice vectors in \( (0, \mathcal{I}) \) are massless. Note that these states are automatically twist invariant. Therefore the gauge group of the orbifold contains at least a semi-simple group corresponding to the roots of the lattice \( \mathcal{I} \). However, the rank of this group is not a priori guaranteed to be \( 8 - \dim(\mathcal{N}) \). This follows from the fact that the subgroup, to which \( E_8 \) is broken, need not to be semi-simple but only reductive, that is there can be \( U(1) \) factors around. On the other hand, when considering the action of the twist on the Cartan subalgebra, one knows that \( \dim(\mathcal{N}) \) Cartan generators are not invariant under the twist and are therefore projected out, whereas \( 8 - \dim(\mathcal{N}) \) are invariant under the twist. Therefore, the rank of the unbroken gauge group is \( 8 - \dim(\mathcal{N}) \) and the gauge group itself is given by

\[
G_\mathcal{I} \otimes U(1)^{8-\dim(\mathcal{N})-\text{rk}(G_\mathcal{I})} \tag{2.16}
\]

where \( G_\mathcal{I} \) is the semi-simple Lie group associated to the roots of the lattice \( \mathcal{I} \).

Decompositions of the form \( \mathcal{N} \oplus \mathcal{I} \) can easily be found using the formalism of extended Dynkin diagrams [42]. There are, however, cases where the decomposition is not unique. This is not in contradiction with \( \mathcal{I} \) being maximal, because \( \supset \) is a partial ordering.
relation, only. It is well known that it happens in a few number of cases that the twist
of the full algebra does not only depend on the isomorphic type of the subalgebra that is
twisted, but also on the precise embedding \[38\]. This is easily illustrated by the following
example, namely by decomposing $E_8$ into cosets with respect to $A_1^3$ and then taking the
$A_1^1$ Coxeter twist. From the decomposition one sees that, depending on the choice of
the $A_1$'s, one has that $\mathcal{I} = D_4$ or $\mathcal{I} = A_1^4$. These twists are known as $A_1^{1\text{I}}$ and $A_1^{4\text{I}}$,
respectively.

Fortunately, we can use the results of the classification of conjugacy classes of the Weyl
group, which is equivalent to the classification of Weyl twist modulo conjugation, for
determining the minimal gauge groups. In his work \[38\] Carter gives, for all twists, the
decompositions of the root system, which specify the group $G_I$. Then, according to
(2.16), all one has to do is to add some $U(1)$ factors, if necessary. We have listed all
the minimal gauge groups that can appear in the context of $N = 1$ supersymmetric $Z_N$
orbifold compactifications.

Note, however, that we have so far assumed that the Wilson lines are really allowed
to take values in all of $\langle \mathcal{N} \rangle_R$. But this is not the case if an eigenvalue of the gauge
twist does not appear in the internal twist. The simplest example for this is provided
by the $A_3$ Coxeter twist which has, as a lattice twist, the eigenvalues $\omega^i$, $i = 1, 2, 3$
where $\omega = \exp(2\pi i/4)$. This can be combined with a $Z_8$ twist of the internal space
with eigenvalues $\Omega^k$, $k = 1, 2, 3, 5, 6, 7$, where $\Omega = \exp(2\pi i/8)$. Since $\theta$ does not have
the eigenvalue $-1$, there are no Wilson lines taking values in the $-1$ eigenspace of $\theta'$.
Therefore, the minimal gauge group is larger then the expected $SO(10)$. A detailed
analysis shows that the unbroken gauge group is the non–simply laced group $SO(11)$.
Note that this is not in contradiction with the twist being defined through a Weyl twist of
$E_8$, because an inner automorphism of the full group may be an outer one of a subgroup.
Therefore, breaking to non–regular subgroups is possible, if Wilson lines are turned on.

In these cases, which include many of the $Z_6$ and $Z_8$, and most of the $Z_{12}$ and $Z_{12}'$
orbifolds, our list only provides a lower bound on the gauge group that is not saturated.
Note that the analysis to be performed in order to get the minimal gauge group is, in
these cases, the same as the one one has to use in order to get intermediate gauge groups,
that is gauge groups that are neither minimal nor maximal. A combination of counting
and embedding arguments as used in \[37\] is in many cases sufficient to determine the
gauge group.

The maximal gauge groups which appear for vanishing Wilson lines have been described
in \[10\]. Here the gauge group of the torus modes is $E_8$ and therefore all the three classes
of vectors in the decomposition are present. In order to determine the gauge group of the orbifold one has to form twist invariant combinations of the states corresponding to lattice vectors in \((\mathcal{N}, 0)\) and \((W_i(\mathcal{N}), W_i(\mathcal{Z}))\), because these vectors transform non-trivially under the twist. For convenience we have included the results of \([40]\) in our table.

The table is organized as follows. We list all twists that have order 2, 3, 4, 6, 7, 8 or 12 and can therefore appear in the context of \(N = 1\) orbifolds. We quote the conjugacy class of the twist and its name from \([38]\). So, a Coxeter twist in the subalgebra \(X\) is called \(X\), whereas the non–Coxeter twists, if they are needed, are called \(X(a_i)\). If the twist depends on the embedding of the subalgebra, then the inequivalent choices are labeled by \(X^{I}, X^{II}, \ldots\). Next, we list the order of the twist. An asterisk is put on those numbers, where the order of the Lie algebra twist is twice the order of the corresponding lattice twist, following \([40]\). We also list the non–trivial eigenvalues of the twist, because they specify the structure of the untwisted moduli space \([30]\). Actually, we list the corresponding powers of the \(N\)-th primitive root of unity where \(N\) is the order of the twist. The eigenvalues of the Coxeter twists have been calculated using their relation to the ranks of the inequivalent Casimir operators, whereas the eigenvalues of non–Coxeter twists are quoted from \([39]\). Then we give the minimal gauge groups, by taking the semi–simple part from \([38]\) and adding \(U(1)\)s if needed. Finally, we also quote the maximal gauge groups from \([40]\).

## 3 Extended gauge groups from the compactification sector

In this section we will discuss the enhancement of the gauge group occurring in the compactification (also called internal) sector of Narain compactifications and Narain orbifolds. We will also take into account the effect of continuous Wilson lines on this enhancement.

In the toroidal case, the generic gauge symmetries come from the twelve conserved chiral world sheet currents \(\partial X^i_L(z)\) and \(\bar{\partial} X^i_R(\bar{z})\) corresponding to the left- and right–moving parts of the six internal coordinates. Thus, the generic gauge group coming from the compactification sector is \(U(1)_L^6 \otimes U(1)_R^6\). The six gauge bosons of the \(U(1)_L^6\) are graviphotons, that is they are part of an \(N = 4\) gravitational supermultiplet, whereas the other six gauge bosons belong to \(N = 4\) vector multiplets.

In order to break the extended \(N = 4\) supersymmetry to \(N = 1\) one must break the \(U(1)_R^6\) completely. This is done by every twist that doesn’t have 1 as an eigenvalue. To preserve
the minimal $N = 1$ supersymmetry, the twist must be in the subgroup $SU(3) \subset SO(6)$ \cite{25}. This leads to the classification \cite{35} of $N = 1$ twists.

Since the twist acts in a left - right symmetric way the $U(1)^6_L$ is automatically also broken completely. There are, however, similarly to the gauge sector, special values of the moduli for which one has extra conserved currents. Let us discuss this for the untwisted model first. The additional massless gauge bosons are related to Narain vectors $P = q^l l_l + n^i k_i + m_i k_i$ with quantum numbers $q^l = 0$, $n^i m_i := n^T m = 1$. In order to extend the generic gauge group $U(1)^6_L$ to $G^{(l)} \otimes U(1)^{6-l}$, where $G^{(l)}$ is a rank $l \leq 6$ semi-simple simply laced Lie group with Cartan matrix $C_{ij}$, $i, j = 1, \ldots, l$, the moduli must satisfy the relations \cite{33}

$$A_i \cdot A_j + 4 G_{ij} = C_{ij} \quad (3.1)$$

and

$$4 B_{ij} = C_{ij} \text{ modulo 2.} \quad (3.2)$$

This implies that $D_{ij} \in \mathbb{Z}$ and moreover the vectors

$$P^{(i)} = (p^{(i)}_L ; p^{(i)}_R) = \mathbb{Z} - D_{ij} k_j = (A_i, 2e_i ; 0_6) \quad (3.3)$$

are in the Narain lattice. The vectors $e_i = G_{ij} e^j$ are basis vectors of the compactification lattice $\Lambda$. Since $C_{ij}$ is a Cartan matrix, $(p^{(i)}_L)^2 = 2$ and therefore the $p^{(i)}_L$ are a set of simple roots of $G^{(l)}$.

For vanishing Wilson lines the conditions (3.1), (3.2) reduce to the more special conditions for points of extended gauge symmetry that are known from \cite{34}. Note that an extended gauge group is compatible with, at least, small deformations of the Wilson lines, because their effect can be compensated by tuning the metric.

In the orbifold case, the moduli are further restricted by the condition of compatibility with a given twist. These conditions are, in the absence of discrete background fields, given by (2.12) and

$$D_{ij} \theta_{jk} = \theta_{ik} D_{jk} \quad (3.4)$$

where

$$D_{ij} = 2 \left( B_{ij} - G_{ij} - \frac{1}{4} A_i \cdot A_j \right) \quad (3.5)$$

and $\theta_{ij}^j$, $\theta_{ij}^i$ are the matrices of the internal twist with respect to lattice bases of the compactification lattice $\Lambda$ and its dual \cite{37}.

As an example, let us discuss the compactification on a 2-torus $T_2$. There are two rank two semi-simple simply laced Lie algebras, namely $A_1 \oplus A_1$ and $A_2$ with corresponding gauge groups $SU(2)^2$ and $SU(3)$. These enhanced gauge groups occur at special points
in the moduli space. Let us in the following first consider the case of vanishing Wilson lines. The special points of enhanced gauge symmetries are determined by equations (3.1) and (3.2). For the maximal enhancement of the gauge group to \( SU(2)^2 \) or \( SU(3) \), these equations do completely fix the moduli \( G_{ij} \) and \( B_{ij} \), up to discrete transformations. Introducing complex moduli \( T \) and \( U \),

\[
T = 2 \left( \sqrt{G} - iB_{12} \right), \quad U = \frac{1}{G_{11}} \left( \sqrt{G} - iG_{12} \right), \quad (3.6)
\]

where \( G_{ij}, B_{ij}, i, j = 1, 2 \) are the real moduli of the 2-torus, these conditions can be expressed as

\[
U = T = 1 \quad (3.7)
\]

and

\[
U = T = e^{i\pi/6} \quad (3.8)
\]

for a maximal enhancement to \( A_1 \oplus A_1 \) and \( A_2 \), respectively [23]. Note that in these equations we have restricted ourselves to the critical points of the standard fundamental domain. If one allows for an abelian factor, then one can also have the gauge group \( SU(2) \otimes U(1) \). Inspection of equations (3.1) and (3.2) shows that, in this case, not all of the moduli are fixed by these equations, but that two real moduli parameters are left unfixed. These two parameters can be taken to be one of the two radii of the \( T_2 \) as well as the angle between these two radii. In terms of the complex moduli \( T \) and \( U \), this means that the extended gauge group \( SU(2) \otimes U(1) \) occurs for points in moduli space for which \( U = T \). Note that this complex subspace \( U = T \) contains the two points (3.7) and (3.8) of maximally extended symmetry. For generic values \( U \neq T \), the toroidal gauge group is given by \( U(1)^2 \).

Let us now discuss the enhancement of the gauge group in the context of orbifold compactifications for which the underlying internal 6-torus decomposes into a \( T_6 = T_2 \oplus T_4 \). For concreteness, let us study the effect on this enhancement of a \( \mathbb{Z}_2 \) acting on the 2-torus \( T_2 \). This is the situation encountered in a \( \mathbb{Z}_4 \)-orbifold, for instance. The \( \mathbb{Z}_2 \) twist, given by the reflection \(-I_2\), doesn’t put any additional constraints on the four real moduli \( G_{ij} \) and \( B_{ij} \) and on their complex version \( U \) and \( T \). Therefore, the moduli space associated with the \( T_2 \) is the same as in the toroidal case. As is well known from one-dimensional compactifications, the enhanced \( (SU(2)) \) and the generic \( (U(1)) \) gauge groups get broken to \( SU(2) \rightarrow U(1) \) and \( U(1) \rightarrow \emptyset \) by the \( \mathbb{Z}_2 \)-twist, respectively. Thus, in the case of the \( \mathbb{Z}_2 \)-twist acting on the internal \( T_2 \), the gauge groups for the points \( U = T = 1 \), \( \exp(i\pi/6) \neq U = T \neq 1 \) and \( U \neq T \) in moduli space are given by \( U(1)^k \) with \( k = 2, 1, 0 \), respectively.
Something more interesting happens at the $SU(3)$ symmetric point $U = T = \exp(i\pi/6)$ in the orbifold case. The $\mathbb{Z}_2$-twist on $T_2$ can be decomposed into

$$-I_2 = W_1C^{-1}D$$

where $W_1$, $C$ and $D$ are the first Weyl reflection, the Coxeter twist and the diagram automorphism of $A_2$, respectively. Therefore the twist acts as an outer automorphism and breaks the $SU(3)$ to the maximal non-regular subgroup $SU(2)$. Thus, we have found a bosonic realization of the conformal embedding $SU(2)_{k=4} \subset SU(3)_{k=1}$ and expect that the $SU(2)$ is realized at the higher level $k = 4$ in order to have central charge $c = 2$. Indeed, a direct calculation of the OPE of the twist-invariant combinations of conserved currents shows that there is a $SU(2)$ current algebra at level $k = 4$.

Note that this phenomenon of rank reduction and simultaneous increase of the level is also quite generically present in the gauge sector because, as mentioned in the previous section, a Weyl twist of the $E_8$ will often act as an outer automorphism of a subalgebra left unbroken by Wilson lines. Take as an example the $SU(3)_3$ model obtained in [43] through switching on Wilson line moduli. By inspection of the vertex operators given in [43] one easily sees that the algebra of the corresponding deformed untwisted model is $SO(8)_1$, like in some of the models described in [37]. Therefore all these models should be bosonic realizations of the conformal embedding $SU(3)_3 \subset SO(8)_1$.

The points of extended gauge symmetry are fixed points under some transformation belonging to the modular group $SO(2,2,\mathbb{Z})$. There is a remarkable relation between the order of that transformation and the number of extra massless gauge bosons. Namely, we will now show, for a two-dimensional torus compactification and for its $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds, that

$$\text{order of fixed point} = (\text{order of twist}) \times (\text{number of extra gauge bosons}) \quad (3.10)$$

We will present here the case of the two-dimensional torus and its $\mathbb{Z}_2$ orbifold. The $\mathbb{Z}_3$ orbifold will be discussed at the end of this section.

The modular group of both the torus compactification and its $\mathbb{Z}_2$ orbifold is, in the absence of Wilson lines, the group $SO(2,2,\mathbb{Z})$. This group has four generators, which we can take to be $S, T, D_2, R$ as defined in [20]. $S$ and $T$ generate the subgroup $SL(2,\mathbb{Z}) \subset SO(2,2,\mathbb{Z})$, which is the subgroup of orientation preserving basis changes in $\Lambda$, whereas $R$ is the reflection of the first coordinate and $D_2$ is the factorized duality in the second coordinate. Note that there is a second $SL(2,\mathbb{Z})$ subgroup which commutes with the first one. It is generated by $S' = D_2SD_2$ and $T' = D_2TD_2$. Whereas $T'$ is the axionic shift symmetry, $S'$ is almost the full duality $D = D_1D_2$, namely $S' = SD$. Note that $D_1$, which
is the factorized duality transformation of the first coordinate, is not an independent
generator of the group because \( P := RS \) permutes the two coordinates and therefore
\( D_1 = PD_2 P \). See [20] for a detailed discussion. The explicit matrices given there specify
the linear action of the modular group \( SO(2,2,\mathbb{Z}) \) on the quantum numbers.

The group \( SO(2,2,\mathbb{Z}) \) acts non–faithfully and fractionally linear on the moduli. The
faithfull transformation group \( PSO(2,2,\mathbb{Z}) \) is known to be generated by the transforma-
tions [50]

\[
\tilde{S} : (U,T) \to \left( \frac{1}{U},T \right) \quad \tilde{T} : (U,T) \to (U + i,T) \quad (3.11)
\]

\[
\tilde{R} : (U,T) \to (U,T) \quad \tilde{M} : (U,T) \to (T,U) \quad (3.12)
\]

Note that as a consequence of our definition of the \( U \) modulus with \( G_{11} \) in the de-
nominator, we had to rearrange the generators as: \( \tilde{S} = S, \tilde{T} = STS, \tilde{R} = R \) and
\( \tilde{M} = RSD_2 \), in order to achieve that the transformations take their standard form. The
well known subgroups \( SL(2,\mathbb{Z})_U \) and \( SL(2,\mathbb{Z})_T \) are generated by \( \tilde{S}, \tilde{T} \) and \( \tilde{S}' = \tilde{M} \tilde{S} \tilde{M} \),
\( \tilde{T}' = \tilde{M} \tilde{T} \tilde{M} \) respectively.

We can now prove the statement given above. The extended gauge groups \( SU(2) \otimes U(1),
SU(2)^2 \) and \( SU(3) \) appear at the points \( U = T \neq 1, e^{i\pi/6}, U = T = 1 \) and \( U = T = e^{i\pi/6} \).
These are fixed points of the transformations \( \tilde{M}, \tilde{M} \tilde{S} \) and \( \tilde{M} \tilde{T} \tilde{S} \) which have the orders
2, 4 and 6 respectively. By formally taking the twist to be the identity here, we see
that equation (3.10) holds, because the orders of the fixed point transformations are
equal to the numbers of extra massless gauge bosons appearing at these points. This can
moreover be extended to the point at infinity, which is a fixed point of order \( \infty \) under
the translation \( \tilde{M} \tilde{T} \). Since the limit \( U,T \to \infty \) describes the decompactification of the
torus, infinitely many Kaluza–Klein states become massless there, as predicted by the
order of the fixed point [28].

In the \( \mathbb{Z}_2 \) orbifold the critical points are the same, but since one must form twist invariant
combinations the numbers of extra massless gauge bosons is divided by the order of the
twist. Therefore equation (3.10) holds as well. The case of the \( \mathbb{Z}_3 \) orbifold will be
discussed below.

Let us now switch on Wilson lines in the \( \mathbb{Z}_2 \) orbifold model discussed above. As already
argued above in terms of the real moduli, any extended gauge group can be preserved by
an appropriate tuning of the moduli of the compactification sector. All we have to add
here is the prescription of this tuning in terms of the complex moduli. We will do this
for the case of two complex Wilson line moduli \( B, C \) associated with the internal 2-torus
The $B, C$ moduli are

$$B = \frac{1}{G_{11}} \left( A_{11} \sqrt{G} - A_{21} G_{12} + A_{22} G_{11} + i(-A_{11} G_{12} + A_{12} G_{11} - A_{21} \sqrt{G}) \right)$$

$$C = \frac{1}{G_{11}} \left( A_{11} \sqrt{G} + A_{21} G_{12} - A_{22} G_{11} + i(-A_{11} G_{12} + A_{12} G_{11} + A_{21} \sqrt{G}) \right).$$  (3.13)

The $T$ modulus now reads

$$T = 2 \left( \sqrt{G} \left(1 + \frac{1}{4} A_i^\mu A_{\mu i} \right) - i(B_{12} + \frac{1}{4} A_i^\mu A_{\mu i} \frac{G_{12}}{G_{11}} - \frac{1}{4} A_i^\mu A_{\mu 2} \right) \right)$$  (3.14)

whereas the $U$ modulus is not modified. Here $A_{\mu i}$ denotes the $\mu$-th component of the $i$-th Wilson line with respect to an orthonormal frame, $\mu, i = 1, 2$.

States which become massless for generic $U = T$ (where the orbifold gauge group is $SU(2) \otimes U(1) \to U(1)$) stay massless when Wilson lines are turned on. Thus, no tuning of the complex moduli $U$ and $T$ is necessary. In order to have the maximally extended gauge group $SU(2)^2 \to U(1)^2$, the original condition $T = U = 1$ is, in the presence of Wilson lines, replaced by

$$T = U = \sqrt{1 + \frac{BC}{2}}$$  (3.15)

whereas in order to have $SU(3) \to SU(2)$, the new condition is given by

$$T = U = \frac{i}{2} + \sqrt{\frac{3}{4} + \frac{BC}{2}}.$$  (3.16)

This follows from the discussion of the zeros of the mass formula, which will be presented in section 6.

Consider, as another example, a $Z_3$ orbifold defined by the Coxeter twist of the $A_2$ root lattice. Here, the moduli have to be restricted in order to be compatible with the twist. In terms of complex moduli one has to set $U = \frac{1}{2}(\sqrt{3} + i) = e^{i\pi/6}$ and $C = 0$, whereas $T$ and $B =: \sqrt{3} A$ remain moduli. Clearly, the $SU(3)$ point ($U = T = e^{i\pi/6}$) is in the $Z_3$ moduli space, and at this special point in moduli space the toroidal gauge group $SU(3)$ is broken to $U(1)^2$. For generic $T$, the toroidal gauge group is $U(1)^2$, whereas there is no leftover gauge group in the orbifold case. Note again that the product of twist order and number of extra massless states in the orbifold model equals the order of the fixed point with respect to the modular group of the untwisted model, namely $3 \cdot 2 = 6$. No tuning of $T$ is required to preserve the $SU(3) \to U(1)^2$ gauge group in the presence of Wilson lines. The condition for having an extended gauge symmetry is that $T = e^{i\pi/6}$, whereas $B =: \sqrt{3} A$ is arbitrary.

Since the $U$ and the $C$ modulus are frozen to discrete values, the modular group of the $Z_3$ orbifold is smaller than the one of the untwisted model. In the case of vanishing Wilson
lines the modular group is obviously $SU(1, 1, \mathbb{Z}) \simeq SL(2, \mathbb{Z})_T$ with generators $\tilde{S}'$ and $\tilde{T}'$ as defined above. The point $T = e^{i\pi/6}$ of extended $U(1)^2$ symmetry is a fixed point under $\tilde{T}'\tilde{S}'$, which is a transformation of order 3. This reduction of the order, as compared to the modular group of the untwisted model, is due to the fact that the transformation $\tilde{M}$, which is of order 2, is not in the reduced modular group $SU(1, 1, \mathbb{Z})$.

4 Mass formulae for $SO(p + 2, 2)$ and $SU(m + 1, 1)$ cosets

In this chapter we show that in the case of a factorizing 2–torus $T_2$, the moduli dependent part of the mass formula for the untwisted sector of an $N = 1$ orbifold can be written as $|M|^2/Y$. $M$ is a holomorphic function of the moduli and depends on the quantum numbers. $Y$ is a real analytic function of the moduli, only, and is related to the Kähler potential by $K = -\log Y$.

4.1 Torus compactifications and the $SO(22, 6)$ coset

Let us first recall that the mass formula for the heterotic string compactified on a torus is

$$\frac{\alpha'}{2} M^2 = N_L + N_R + \frac{1}{2}(p_L^2 + p_R^2) - 1 \quad (4.1)$$

and that physical states must also satisfy the level matching condition

$$\frac{\alpha'}{2} M_L^2 := N_L + \frac{1}{2}p_L^2 - 1 \equiv N_R + \frac{1}{2}p_R^2 =: \frac{\alpha'}{2} M_R^2. \quad (4.2)$$

Here, we have absorbed the normal ordering constant of the NS sector into the definition of the right moving number operator and restricted ourselves to states surviving the GSO projection. Thus $N_R$ has an integer valued spectrum in both the NS and the R sector.

Substituting the second equation into the mass formula yields

$$\frac{\alpha'}{2} M^2 = p_R^2 + 2N_R. \quad (4.3)$$

Our aim is to make the moduli dependence of the mass explicit.

The first step is to express $p_R$ in terms of the quantum numbers $q^I, n^i, m_j$ and the real moduli $G_{ij}, B_{ij}, A_i$. This can be done by expanding the Narain vector $P = (p_L; p_R) \in \Gamma$ in terms of the lattice basis $l_I, k_i, k^j$ (2.4) - (2.7),

$$P = q^I l_I + n^i k_i + m_j k^j \quad (4.4)$$
and then projecting onto the right handed part by multiplying with the vectors $e^{(R)}_\mu = (0_{16}, 0_6, e_\mu)$, where the $e_\mu$ are an orthonormal basis of $\mathbb{R}^6$

$$\mathbf{p}_R = (q^i, n^i, m_j) \begin{pmatrix} l_I \cdot e^{(R)}_\mu \\ k_i \cdot e^{(R)}_\mu \\ k^j \cdot e^{(R)}_\mu \end{pmatrix} e^{(R)}_\mu.$$  (4.5)

The norm squared takes then the form

$$\mathbf{p}_R^2 = \mathbf{v}^T \Phi \Phi^T \mathbf{v}$$  (4.6)

where $\mathbf{v}$ is the vector of quantum numbers,

$$\mathbf{v}^T = (q^I, n^i, m_j) \in M(1, 28, \mathbb{Z}) \simeq \mathbb{Z}^{28}.$$  (4.7)

The matrix $\Phi$ contains the moduli

$$\Phi = \begin{pmatrix} l_I \cdot e^{(R)}_\mu \\ k_i \cdot e^{(R)}_\mu \\ k^j \cdot e^{(R)}_\mu \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -AE^* \\ DE^* \\ E^* \end{pmatrix} \in M(28, 6, \mathbb{R})$$  (4.8)

where

$$A = (A_{Ii}) = (e_I \cdot A_i) \in M(16, 6, \mathbb{R})$$  (4.9)

and

$$D = (D_{ij}) = 2 \left( B_{ij} - G_{ij} - \frac{1}{4} A_i \cdot A_j \right) \in M(6, 6, \mathbb{R})$$  (4.10)

are the moduli matrices and

$$E^* = (E^i_\nu) = (e^i \cdot e_\nu)$$  (4.11)

is a 6–bein whose appearance reflects the fact that the $e^{(R)}_\mu$ can be rotated by an $SO(6)$ transformation. $e^i$ are a basis of the dual $\Lambda^*$ of the compactification lattice $\Lambda$.

The matrix $\Phi$ satisfies the equation

$$\Phi^T H_{22,6} \Phi = -I_6,$$  (4.12)

where $H_{22,6}$ is the standard pseudo–euclidean lattice metric of $\Gamma = \Gamma_{22,6}$:

$$H_{22,6} = \begin{pmatrix} C_{(10)}^{-1} & 0 & 0 \\ 0 & 0 & I_6 \\ 0 & I_6 & 0 \end{pmatrix},$$  (4.13)
where $C_{(16)}$ is the Cartan matrix of $E_8 \otimes E_8$. By a coordinate transformation $\Phi \to \tilde{\Phi}$ equation (4.12) can be brought to the form
\[
\tilde{\Phi}^T \eta_{22,6} \tilde{\Phi} = -I_6,
\]
with the standard pseudo–euclidean metric of type $(+)^{22}(-)^6$
\[
\eta_{22,6} = \begin{pmatrix}
I_{22} & 0 \\
0 & -I_6
\end{pmatrix}.
\]

This is the standard form of the constraint equation which defines the coset space $SO(22,6)/(SO(22) \otimes SO(6))$ in terms of homogeneous coordinates $\tilde{\Phi}$ [14]. Therefore $\Phi$ is a modified homogenous coset coordinate. It has the advantage that not only the deformation group $SO(22,6)$ acts linearly on it, as usual for homogenous coordinates, but that the subgroup of modular transformations acts by integer valued matrices [30]. In the following the pseudo–euclidean lattice metric of an integer lattice $\Gamma_{m,n}$ and the standard metric of type $(+)^m(-)^n$ will be denoted by $H_{m,n}$ and $\eta_{m,n}$ respectively.

4.2 Orbifold compactifications and symmetric Kähler spaces

In the following subsections we will derive mass formulae and parametrizations for the untwisted moduli of orbifold compactifications. This will be done in four steps. In the first step we will recall how one can derive an explicit real parametrization of orbifold moduli spaces by solving the constraint equations, imposed by the compatibility requirement with a given twist, on the moduli $G_{ij}, B_{ij}, A_i$ of the Narain model. This solution can then be used in a second step to locally factorize the moduli space into spaces corresponding to distinct eigenvalues of the twist. The third step is then to find appropriate complex coordinates on each factor space which make explicit its Kähler structure. This can be done by using the relations between the the real moduli and homogenous coset coordinates. Finally, one can solve the constraint equations for an independent set of complex moduli. Plugging these results into the mass formula allows one to write its moduli dependent part as the ratio of a holomorphic and a real analytic piece, where the latter one is related to the Kähler potential.

4.2.1 Untwisted orbifold moduli

We will now implement the first step of the four described above. Let us recall that it was shown in [37], based on earlier results of [29, 45], that the continuous parts of the
background fields $G_{ij}, B_{ij}, A_i$ must satisfy the equations

$$D_{ij} \theta^j_k = \theta^j_i D_{jk}, \quad A_{ij} \theta^j_k = \theta^j_i A_{jk}$$

(4.16)

where $\theta^j_i, \theta^j_k$ and $\theta^j_i$ are the matrices of the internal twist $\theta$ and of the gauge twist $\theta'$ with respect to the lattice bases $e_i$, $e^j$ and $e_I$ of the lattices $\Lambda, \Lambda^* \text{ and } \Gamma_{16}$. Although $\theta$ and $\theta'$ are orthogonal maps their matrices with respect to non–orthonormal lattice bases are not. Thus, it is convenient to express equations (4.16) in terms of orthonormal bases $e_\mu$ and $e_M$ of $\mathbb{R}^6$ and $\mathbb{R}^{16}$, before solving them [45, 37]. The transformations between the lattice and orthonormal frames is given by n–bein matrices $E_i = (e_i \cdot e^j)$, $E = (E_{i\nu}) := (e_i \cdot e_\nu)$, $E = (E_{i\nu}) := (e_i \cdot e_\nu)$, $E = (E_{i\nu}) := (e_i \cdot e_\nu)$, etc. Note that orthonormal bases are "selfdual", $e_M = e^M$, $e_\mu = e^{\mu}$, whereas lattice bases are (generically) not: $e_I \neq e^I$, $e_i \neq e^i$. Therefore $E_{i\nu} = E^{i\mu} \neq E_{i\mu}$. A useful relation to be used later is $E^* = E^{T,-1}$.

One complication is that the metric moduli drop out of the $D$–matrix when written with respect to an orthonormal frame, because $e_\mu \cdot e_\nu = \delta_{\mu\nu}$ and therefore (see below for details of the transformation)

$$D_{\mu\nu} = 2 \left( B_{\mu\nu} - \delta_{\mu\nu} - \frac{1}{4} A_\mu \cdot A_\nu \right)$$

(4.17)

Instead, they are now contained in the 6–bein $E$ which is sensitive to deformations of $\Lambda$.

To study the effect of lattice deformations let us fix a reference lattice $\overline{\Lambda}$ and introduce a deformation map $S$ (or better a family of deformation maps) which maps it to $\Lambda$

$$S : \overline{\Lambda} \to \Lambda : \overline{e}_i \to e_i = S^j_i \overline{e}_j \implies \overline{G}_{ij} = \overline{e}_i \cdot \overline{e}_j \to G_{ij} = e_i \cdot e_j$$

(4.18)

This is compatible with the twist $\theta : \overline{e}_i \to \theta^j_i \overline{e}_j$ if

$$S^j_i \theta^j_k = \theta^j_i S^j_k.$$  

(4.19)

In order to transform equations (4.16) and (4.19) into the orthonormal frame we will have to work out some formulae. Consider therefore the deformation described in terms of the $e_\mu$

$$S : e_\mu \to e'_\mu = S^\nu_\mu e_\nu \implies e_\mu \cdot e_\nu = \delta_{\mu\nu} \to e'_\mu \cdot e'_\nu =: G_{\mu\nu}$$

(4.20)

Thus, a compactification on $\Lambda$ with background metric $\delta_{\mu\nu}$ can be reinterpreted as a compactification on a fixed lattice $\overline{\Lambda}$ in a deformed background $G_{\mu\nu}$. Noting that

$$S^\nu_\mu = e'_\mu \cdot e^\nu = e'_\mu \cdot e_\nu =: S^\nu_\mu \text{ and } S^\mu_\nu := e'^\mu_\nu \cdot e'^\nu = G^{\mu\nu} S_{\mu\nu}$$

(4.21)

and defining $S^* = (S^\nu_\mu)$, $S = (S^\nu_\mu)$ we have that $S^* = S^{T,-1}$.

Next we introduce a fixed (moduli independent) coordinate transformation which connects the orthonormal basis $e_\mu$ to the lattice basis $\overline{e}_i$ of the reference lattice $\overline{\Lambda}$:

$$\overline{e}_i = T^\mu_i e_\mu \implies T^\mu_i = e_i \cdot e^\mu$$

(4.22)
This matrix and its inverse \( T^i\mu = e^i_\mu \cdot e^j \) connect the deformation matrices by

\[
S^\mu_\nu = T^i_\mu S^j_i T^j_\nu
\]  

(4.23)

Therefore, the matrix \( T^i_\mu \) also connects the deformed orthonormal basis \( e^j_\mu \) to the deformed lattice basis \( e_i \):

\[
e_i = T^i_\mu e^j_\mu \Rightarrow T^i_\mu = e_i \cdot e^j = e_i \cdot e^j. 
\]  

(4.24)

The relations between the four \( n \)–bein matrices \( E^i_\mu, T^i_\mu, S^j_i, S^\mu_\nu \) and the four bases \( e_i, e^j, e^j_\mu, e^j_\mu \) are summarized by

\[
E^i_\nu = S^j_i T^j_\nu = T^i_\mu S^\mu_\nu \]  

(4.25)

which shows that \( S^j_i \) and \( S^\mu_\nu \) relate the undeformed lattice basis \( e_i \) and the orthonormal basis \( e^j \) to their images \( e^j_\mu, e^j_\mu \) under the deformation \( S \), whereas \( T \) describes a fixed coordinate transformation relating \( e_i \) to \( e^j \). \( E \) is a family of coordinate transformations relating the moving frame \( e_i \) to the fixed orthonormal frame \( e^j \). Equation (4.23) shows that \( E \) can be factorised into a moduli–dependent piece and a moduli–independent one in two different ways.

We can now use the \( n \)–bein matrices and transform the equations (4.16), (4.19) into the orthonormal bases. In order to do this we have to introduce the transformed moduli matrices

\[
D_{\mu\nu} = E^i_\mu D^j_i E^j_\nu = T^i_\mu \overline{D}^j_i T^j_\nu, \quad A_M^I = \overline{E}^I_M A_I^J E^J_\nu = \overline{E}^I_M \overline{A}^I_J T^j_\nu, \quad S^\nu_\mu = T^i_\mu S^j_i T^j_\nu 
\]  

(4.26)

and the transformed twist matrices

\[
\theta^\mu_\nu = E^i_\mu \theta^j_i E^j_\nu = T^i_\mu \theta^j_i T^j_\nu, \quad \theta^\mu_\nu = E^i_\mu \theta^j_i E^j_\nu = T^i_\mu \theta^j_i T^j_\nu, \quad \theta^N_M = \overline{E}^I_M \theta^J_I \overline{E}^J_N. 
\]  

(4.27)

Note that these relations are consistent thanks to (4.19). Since \( e^j_\mu, e^j_M \) are orthonormal bases it follows that

\[
\theta^\mu_\nu = \theta^\mu_\nu =: \theta_{\mu\nu} \quad \text{and} \quad \theta^N_M =: \theta_{MN} 
\]  

(4.28)

are orthogonal matrices.

Using the formulae derived above we find that equations (4.16), (4.19) are equivalent to

\[
\theta_{MN} A^{N\nu} = A_M^I \theta^\mu_\nu, \quad \theta_{\mu\nu} D_{\nu\rho} = D^{\mu\nu} \theta_{\nu\rho}, \quad \theta_{\mu\nu} S_{\nu\rho} = S_{\mu\nu} \theta_{\nu\rho}. 
\]  

(4.29)

The orthonormal bases can be chosen such that the twists \( \theta, \theta' \) take their real standard forms, with nonvanishing \( 2 \times 2 \) blocks along the diagonal. A slight modification will turn out to be useful in order to display the coset structure. Namely, we will group together
degenerate eigenvalues into bigger blocks. For definiteness let us consider the gauge twist \( \theta' \) and assume that it has complex eigenvalues \( e^{\pm i\psi_j} \) and real eigenvalues \(-1\) and \(1\) with multiplicities \(m_j, p\) and \(q\). Then there exists a basis \( e_M \) such that

\[
(\theta_{MN}) = \bigoplus_j R_j \oplus -I_p \oplus I_q \tag{4.30}
\]

with

\[
R_j = \begin{pmatrix}
\cos(\psi_j)I_{m_j} & -\sin(\psi_j)I_{m_j} \\
\sin(\psi_j)I_{m_j} & \cos(\psi_j)I_{m_j}
\end{pmatrix} \in O(2m_j, \mathbb{R}) \tag{4.31}
\]

The internal twist \( \theta \) can be brought into the same form with multiplicities \(n_j, r\) and \(s\). In these coordinates the equations (4.29) can be easily solved. The result for the Wilson lines matrix is

\[
(A_{\mu\nu}) = \bigoplus_j A^{(j)} \oplus A^{(-1)} \oplus A^{(+1)}, \tag{4.32}
\]

with

\[
A^{(j)} = \begin{pmatrix}
A^{(j)}_1 & A^{(j)}_2 \\
-A^{(j)}_2 & A^{(j)}_1
\end{pmatrix} \in M(2m_j, 2n_j, \mathbb{R}), \quad A^{(-1)} \in M(p, r, \mathbb{R}), \quad A^{(+1)} \in M(q, s, \mathbb{R}). \tag{4.33}
\]

and for the \( D \)-matrix the result is

\[
(D_{\mu\nu}) = \bigoplus_j D^{(j)} \oplus D^{(-1)} \oplus D^{(+1)}, \tag{4.34}
\]

with

\[
D^{(j)} = \begin{pmatrix}
D^{(j)}_1 & D^{(j)}_2 \\
-D^{(j)}_2 & D^{(j)}_1
\end{pmatrix} \in M(2n_j, 2n_j, \mathbb{R}), \quad D^{(-1)} \in M(r, r, \mathbb{R}), \quad D^{(+1)} \in M(s, s, \mathbb{R}). \tag{4.35}
\]

\( S \) has the same structure as \( D \). Note, however, that only the symmetric positive part (in the polar decomposition) of \( S \) is physically relevant, because the orthogonal part describes a pure rotation of \( \Lambda \). We use here that, when referring to an orthonormal frame, the matrix of the positive symmetric (orthogonal) part of an invertible map is a positive symmetric (orthogonal) matrix. The irrelevance of rotations will also be manifest by the fact that the masses only depend on the metric through

\[
G_{ij} = S^k_i \overrightarrow{G}_{kl} S^l_j = T^{\mu}_{i} G_{\mu\nu} T^{\nu}_{j} = E^\mu_{i} E^\mu_{j} \tag{4.36}
\]

or

\[
G_{\mu\nu} = S^\rho_{\mu} \delta_{\rho\sigma} S^\sigma_{\nu}. \tag{4.37}
\]
4.2.2 Factorization of the moduli space

We can now perform the second step, namely use the constrained form (4.32) - (4.35) of the moduli in order to factorize the untwisted orbifold moduli space in factors corresponding to the various distinct eigenvalues. In order to display its coset structure we must work with the homogenous coset coordinate. Therefore we start by replacing the lattice indices appearing in \( \Phi \) by orthonormal indices. This defines a new homogenous coordinate

\[
\hat{\Phi} (4.38)
\]

The new coset coordinate satisfies the coset equation

\[
\hat{\Phi}^T \hat{H}_{22,6} \hat{\Phi} = -I_6
\]

in which the pseudo–euclidean lattice metric \( H_{22,6} \) is replaced by

\[
\hat{H}_{22,6} = \begin{pmatrix}
I_{16} & 0 & 0 \\
0 & 0 & I_6 \\
0 & I_6 & 0
\end{pmatrix}
\]

because of the coordinate transformation. Inside the mass formula we absorb the transformation matrix into the component vector \( \hat{v} \)

\[
\hat{v}^T = \begin{pmatrix}
q^l, n^i, m_j \\
0 & T^i_\mu & 0 \\
0 & 0 & T^i_\nu
\end{pmatrix}
\]
such that

$$\mathbf{v}^T \phi = \hat{\mathbf{v}}^T \hat{\phi}. \quad (4.42)$$

Now we can use the block–diagonal form of the $A$, $D$ and $S$ matrices and permute the rows and columns of $\hat{\Phi}$ such that it becomes block–diagonal

$$\hat{\Phi} \rightarrow \bigoplus_j \hat{\phi}^{(j)} \oplus \hat{\phi}^{(-1)} \oplus \hat{\phi}^{(+1)} \quad (4.43)$$

with

$$\hat{\phi}^{(j)} = \frac{1}{2} \begin{pmatrix} -A^{(j)} \\ S^{(j)}D^{(j)} \\ (S^{(j)})^T \end{pmatrix} \in M(2m_j + 2n_j, 2n_j, \mathbb{R}) \quad (4.44)$$

and a similar expression for $\hat{\phi}^{(-1)} \in M(p + r, r, \mathbb{R})$ and $\hat{\phi}^{(+1)} \in M(q + s, s, \mathbb{R})$.

In the coset equation (4.39) this permutation results in replacing

$$\hat{H}_{22,6} \rightarrow \bigoplus_j \hat{H}_{2m_j + 2n_j, 2n_j} \oplus \hat{H}_{p+r, r} \oplus \hat{H}_{q+s, s}, \quad (4.45)$$

which makes manifest the factorization into factors corresponding to the distinct eigenvalues of the twist.

Again we can keep the mass formula form–invariant by absorbing this permutation into the components $\hat{\mathbf{v}}$. Note, however, that the underlying lattice will not have a corresponding decomposition, because a generic lattice vector has nonvanishing projections onto more than one eigenspace. Therefore the mass formula will not factorize for all states, but only for those having quantum numbers which live in only one of the eigenspaces.

Before we proceed to discuss the irreducible factors in the decomposition, let us recall from [23] that $N = 1$ space time supersymmetry requires $\theta$ to have no eigenvalue +1 and the eigenvalue −1 can only have multiplicities 0 or 2. Therefore $s = 0$ and either $r = 0$ or $r = 2$. This implies that $\hat{\phi}^{(+1)}$ does not appear. More generally, if some eigenvalue does only appear in $\theta$ ($\theta'$) but not in $\theta'$ ($\theta$) this gives rise to vanishing rows in the Wilson line matrix and therefore also in the rearranged $\hat{\Phi}$. They correspond to directions of the Narain lattice which possess no deformations.

4.2.3 Real eigenvalues and the $SO(p + 2, 2)$ coset

We will now study the subspace which corresponds to the real twist eigenvalue $-1$ and is parametrized by $\hat{\phi}^{-1}$. For the later discussion of modular symmetries it is convenient
to go one step back and to introduce a lattice basis in this sector. More precisely (see [30]) we can find a sublattice $\Gamma_{p+2,2}$ of the Narain lattice $\Gamma_{22,6}$ on which the twist $\Theta$ acts as $-I_{p+4}$. Note, however, that this lattice is only a sublattice but not a direct factor, which means that there is no decomposition $\Gamma_{22,6} = \Gamma_{p+2,2} \oplus \cdots$, but only a sublattice $\Gamma_{p+2,2} \oplus \cdots \subset \Gamma_{22,6}$. We will now consider those states that have non–vanishing quantum numbers lying in $\Gamma_{p+2,2}$ only.

The moduli dependent part of the mass formula for such states now takes the form

$$P_R^2 = v^T \phi \phi^T v. \quad (4.46)$$

with quantum numbers

$$v^T = (q^1, \ldots, q^p, n^1, n^2, m_1, m_2) \quad (4.47)$$

which are lattice coordinates for $\Gamma_{p+2,2}$. For vanishing Wilson lines $\Gamma_{p+2,2}$ factorizes as

$$\Gamma_{p+2,2} = \Gamma_p \oplus \Gamma_{2,2} \quad (4.48)$$

where $\Gamma_p$ is a sublattice but not a factor of the $E_8 \otimes E_8$ or $SO(32)$ lattice $\Gamma_{16}$. $\Gamma_{2,2}$ denotes the momentum/winding lattice corresponding to a two–dimensional sublattice $\Lambda^{(2)}$ of the compactification lattice $\Lambda$, which we assume to have a decomposition $\Lambda = \Lambda^{(2)} \oplus \Lambda^{(4)}$, as this happens for example for the $\mathbb{Z}_4$ twist. If Wilson lines are switched on, $\Gamma_{p+2,2}$ does not factorize any more but can still be described in terms of $\Gamma_p$ and $\Gamma_{2,2}$.

The matrix $\phi = \phi^{(-1)}$

$$\phi = \frac{1}{2} \begin{pmatrix} -A_{i\nu} E^i_{\nu} \\ D_{ik} E^k_{\nu} \\ E^i_{\nu} \end{pmatrix} \in M(p + 2, 2, \mathbb{R}) \quad (4.49)$$

is the analog of $\Phi$, because it satisfies the constraint equation of a $SO(p+2,2)$ coset with the standard metric replaced by the lattice metric $H_{p+2,2}$ of $\Gamma_{p+2,2}$.

$$\phi^T H_{p+2,2} \phi = -I_2 \quad (4.50)$$

where

$$H_{p+2,2} = \begin{pmatrix} C_{(p)}^{-1} & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix} \quad (4.51)$$
and $C_{(p)}$ is the lattice metric of $\Gamma_p$.

In order to introduce complex coordinates and to make explicit the Kähler structure of the $SO(p+2,2)$ coset, we will now transform equation (4.51) into its standard form. This can be done in two steps. The first step consists in converting the lattice indices $I, J, \ldots = 1, \ldots, p$ which refer to a lattice basis of $\Gamma_p$, into orthonormal indices $M, N, \ldots$ as well as converting the lattice indices $i, j, \ldots = 1, 2$, which refer to lattice bases of $\Lambda_2$ (as lower indices) and of $\Lambda_2^*$ (as upper indices), into orthonormal indices $\mu, \nu$. This is done in a similar way to the one discussed in the last subsection, and one arrives at the coordinate $\hat{\phi} = \hat{\phi}^{(-1)}$ introduced already there.

The second step for bringing equation (4.51) into its canonical form consists in replacing the metric $\hat{H}_{p+2,2}^T$ by the standard metric $\eta_{p+2,2}$. This is achieved by

$$\bar{\nu}^T \hat{\phi} = \bar{\nu}^T \phi$$

with

$$\bar{\phi} = \begin{pmatrix} I_p & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_2 & \frac{1}{\sqrt{2}} I_2 \\ 0 & \frac{1}{\sqrt{2}} I_2 & -\frac{1}{\sqrt{2}} I_2 \end{pmatrix} \hat{\phi}, \quad \bar{\nu}^T = \bar{\nu}^T = \begin{pmatrix} I_p & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} I_2 & \frac{1}{\sqrt{2}} I_2 \\ 0 & \frac{1}{\sqrt{2}} I_2 & -\frac{1}{\sqrt{2}} I_2 \end{pmatrix}$$ (4.53)

In terms of the real moduli we have that

$$\bar{\phi} = \frac{1}{2} \begin{pmatrix} -A \\ \frac{1}{\sqrt{2}} (SD + S^{T,-1}) \\ \frac{1}{\sqrt{2}} (SD - S^{T,-1}) \end{pmatrix}$$ (4.54)

with $A = A^{(-1)}$, etc.

The new homogenous coset coordinate $\bar{\phi}$ satisfies the standard coset relation

$$\bar{\phi}^T \eta_{p+2,2} \bar{\phi} = -I_2.$$ (4.55)

Recall that $\eta_{p+2,2}$ is the standard metric of type $(+)^{p+2}(-)^2$.

We can now introduce complex coset coordinates by

$$\bar{\phi} = \begin{pmatrix} \phi_{(1)}^1 & \phi_{(2)}^1 \\ \vdots & \vdots \\ \phi_{(1)}^{p+4} & \phi_{(2)}^{p+4} \end{pmatrix} \in M(p+4,\mathbb{R}) \quad \rightarrow \quad \phi_c = \begin{pmatrix} \phi_{(1)}^1 + i \phi_{(2)}^1 \\ \vdots \\ \phi_{(1)}^{p+4} + i \phi_{(2)}^{p+4} \end{pmatrix} \in M(p+4,\mathbb{C})$$ (4.56)
The complex coordinate $\phi_c$ satisfies the equations

$$\dot{\phi}_c^+ \eta_{p+2,2} \phi_c = -2, \quad \phi_c^+ \eta_{p+2,2} \phi_c = 0$$

and is therefore the standard complex homogenous coordinate on the $SO(p+2,2)$ coset [44]. Note that we can replace $\tilde{v}^T \tilde{\phi}$ by $\tilde{v}^T \phi_c$ in the mass formula, because

$$p_R^2 = \tilde{v}^T \tilde{\phi} \tilde{v} = \tilde{v}^T \phi_c \phi_c^+ \tilde{v} = |\tilde{v}^T \phi_c|^2$$

Thus the moduli dependent part of the mass is proportional to the norm squared of a complex number.

Our next step towards the derivation of the mass formula is to solve the complex constraint equations in terms of unconstrained complex coordinates and to make explicit the Kähler structure of the moduli space and the Kähler potential. This procedure is well known both in the mathematical [44] and in the physics [27] literature. We will use first the physicists approach and then explain the relation to the results in the mathematical literature.

**A bounded realization**

One way of solving the constraint equations is to first extract a scale factor $\sqrt{Y}$ from the coordinates, where $Y$ is a positive, real analytic functions of the moduli. It turns out that this function is closely related to the Kähler potential [27]. In terms of the new variables $y \in \mathbb{C}^{p+4}$

$$y = \sqrt{Y} \phi_c$$

the constraints read

$$\sum_{i=1}^{p+2} |y_i|^2 - |y_{p+3}|^2 - |y_{p+4}|^2 = -2Y$$

and

$$\sum_{i=1}^{p+2} y_i^2 - y_{p+3}^2 - y_{p+4}^2 = 0$$

One can now take the first $p + 2$ variables as the independent ones and express the other two in terms of them [27]

$$y_i = z_i, \quad i = 1, \ldots, p + 2, \quad y_{p+3} = \frac{1}{2} \left( 1 + \sum_{i=1}^{p+2} z_i^2 \right), \quad y_{p+4} = \frac{i}{2} \left( 1 - \sum_{i=1}^{p+2} z_i^2 \right)$$

This solves equation (4.60). Defining $z = (z_i) \in \mathbb{C}^{p+2}$ we see that equation (4.60) yields that

$$Y = Y_b = \frac{1}{4} \left( 1 + (\bar{z}^T \bar{z})(z^T z) - 2\bar{z}^T z \right).$$

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Solution (4.63) is $SO(p + 2)$ symmetric.

The domain defined by $Y > 0$ has two connected components, which is readily seen from $|z^T z| = 1 \Rightarrow Y = 0$. Choosing $|z^T z| < 1$ defines a bounded open domain in $C^{p+2}$, called a complex polydisc,

$$PD_{p+2} = \{ z \in C^{p+2} | |z^T z| < 1 \text{ and } 1 + (z^T z)(z^T z) - 2z^T z > 0 \}$$ (4.64)

which provides the standard bounded realization of the $SO(p + 2, 2)$ coset [44]. The domain possesses a Kähler metric, with Kähler potential [40]

$$K = -\log Y. \quad (4.65)$$

An unbounded realization

Another useful parametrization is defined by taking $y = (y_1, \ldots, y_{p+1}, y_{p+3})$ as independent variables. Setting [27, 30]

$$y_{p+2} = -i \left( 1 - \frac{1}{4} \left( - \sum_{i=1}^{p+1} y_i^2 + y_{p+3}^2 \right) \right), \quad y_{p+4} = i \left( 1 + \frac{1}{4} \left( - \sum_{i=1}^{p+1} y_i^2 + y_{p+3}^2 \right) \right) \quad (4.66)$$

solves the rescaled constraint equations with

$$Y = Y_u = \frac{1}{4} \left( (y_{p+3} + \overline{y}_{p+3})^2 - \sum_{i=1}^{p+1} (y_i + \overline{y}_i)^2 \right). \quad (4.67)$$

This solution is $SO(p + 1, 1)$ symmetric.

Again, $Y > 0$ has two connected components, because $y_{p+3} + \overline{y}_{p+3} \to 0$ implies $Y \to 0$. Taking for definiteness $y_{p+3} + \overline{y}_{p+3} > 0$ we get the unbounded open domain

$$L^{p+1,1}_+ + i \mathbb{R}^{p+2} := \{ y \in C^{p+2} | (y_{p+3} + \overline{y}_{p+3})^2 - \sum_{i=1}^{p+1} (y_i + \overline{y}_i)^2 > 0, \ y_{p+3} + \overline{y}_{p+3} > 0 \} \quad (4.68)$$

which differs by a factor of $i$ from the one used in the mathematical literature [44]. Note that the imaginary part of $y$ is unconstrained whereas the real part lives in the forward light cone of a $p + 2$ dimensional Minkowski space.

Using the holomorphic transformation between the two parametrizations given in [48] we know that

$$K = -\log Y_u \quad (4.69)$$

also is a Kähler potential.
Another unbounded realization

For applications one prefers to rearrange the complex moduli $y$ in terms of a $T$ and an $U$ modulus and additional complex Wilson line moduli $B_k, C_k$. Then, for vanishing Wilson lines, one is left with an $SO(2,2)$ coset which factorizes into two $SU(1,1)$ cosets parametrised by the $T$ and the $U$ modulus. To do this one has to set (generalizing the treatment of $SO(4,2)$ cosets [27, 30])

$$T = y_{p+1} + y_{p+3}, \quad 2U = y_{p+3} - y_{p+1}, \quad B_k = y_{2k-1} - iy_{2k}, \quad C_k = y_{2k-1} + iy_{2k}$$  \hspace{1cm} (4.70)

with $k = 1, \ldots, r$. If $p$ is even, then $r = \frac{p}{2}$. If $p$ is odd, $r = \frac{p-1}{2}$ and then there is one additional unpaired complex coordinate

$$A = y_p$$  \hspace{1cm} (4.71)

In terms of the new moduli, the $Y_u$ function now reads

$$Y_u = \frac{1}{2}(T + \overline{T})(U + \overline{U}) - \frac{1}{4} \sum_{k=1}^{r} (B_k + \overline{C_k})(C_k + \overline{B_k}) - \frac{1}{4}(A + \overline{A})^2$$  \hspace{1cm} (4.72)

For completeness, let us display how the $y_i$ look in terms of the new moduli

$$y_1 = \frac{1}{2}(B_1 + C_1), \quad y_2 = \frac{i}{2}(B_1 - C_1), \ldots, \text{ and } y_p = A, \text{ if } p \text{ is odd}$$

$$y_{p+1} = \frac{1}{2}(T - 2U), \quad y_{p+2} = -i \left(1 - \frac{1}{4}(2TU - \sum_k B_kC_k - A^2)\right)$$

$$y_{p+3} = \frac{1}{2}(T + 2U), \quad y_{p+4} = i \left(1 + \frac{1}{4}(2TU - \sum_k B_kC_k - A^2)\right)$$  \hspace{1cm} (4.73)

Let us now comment on several special cases. First, by setting all of the Wilson moduli to zero, $B_k = C_k = A = 0$, one obtains the Kähler potential for a $SO(2,2)$ coset, which factorizes into two $SU(1,1)$ cosets parametrised by the $U$ and the $T$ modulus, respectively. Again, $Y > 0$ has two connected components, and $U$ and $T$ have been defined in such a way, that the condition $y_{p+3} + \overline{y}_{p+3} > 0$ implies that they both have a positive real part, as usual.

Next, by inspection of the Kähler potential, one sees that, for a fixed value of $k$, $T, U, B_k, C_k$ parametrize a subspace which is a $SO(4,2)$ coset. Likewise $T, U, A$ parametrize a $SO(3,2)$ coset. Note also that by setting $B_k = C_k$ one can truncate a $SO(4,2)$ coset to a $SO(3,2)$ coset. Moreover, for even $p = 2m$ we can eliminate half of the moduli by setting $U + \overline{U} = r_1$ and $(\overline{B_k} + C_k)(B_k + \overline{C_k}) = r_2 \overline{A_k}A_k$, with real positive
constants $r_{1,2}$ and thus truncate the $SO(p+2,2)$ coset to a $SU(m+1,1)$ coset. This is obvious from the fact that $Y$ then takes the form

$$Y = \frac{1}{2} r_1 (T + \overline{T}) - \frac{1}{4} r_2 \sum_k \overline{A}_k A_k$$  \hspace{1cm} (4.74)$$

which leads to the Kähler potential of a $SU(m+1,1)$ coset.

The mass formula

Having found various parametrizations $y = \sqrt{Y} \phi_c$ with $y = y(z), y(y)$ or $y = y(T, U, B_k, C_K, A)$ of the coset, we can substitute each of them into the mass formula with the result that

$$p_R^2 = \frac{|\tilde{v}^T y|^2}{Y}. \hspace{1cm} (4.75)$$

Therefore the mass squared can be written as the ratio of the square of the norm of a chiral mass $\mathcal{M} = \tilde{v}^T y$, which is a holomorphic function of the moduli and contains the dependence on the quantum numbers, and of a real analytic positive function $Y$, which is related to a Kähler potential by $K = -\log Y$.

Let us recall that this expression is invariant under the group of target space modular transformations. To be precise we will consider here the modular group of the sublattice $\Gamma_{p+2,2}$ only. The connection of this group with the full modular group was discussed in the extended version of our paper [30]. A general lattice vector $\mathbf{P} \in \Gamma_{p+2,2}$ can be decomposed as

$$\mathbf{P} = v^A \mathcal{E}^A_M(m) e_M, \hspace{1cm} A, M = 1, \ldots, p+4. \hspace{1cm} (4.76)$$

Here $v = (v^A)$ denotes the set of all components with respect to a lattice basis of $\Gamma_{p+2,2}$ and $e_M$ an orthonormal basis. The $p+4$ bein $\mathcal{E}^A_M(m)$ connecting the two bases contains the dependence on the moduli $m = (T, U, B_k, C_K, A)$. Note that in [30] the full $p+4$ bein ($\mathcal{E}^A_M$) was denoted by $\phi$, a symbol that we now use for the right moving part ($\mathcal{E}^A_M$) of it. Note also that we discussed the full Narain lattice there, but these considerations also apply to sublattices. Now a modular transformation consists of acting with a matrix $\Omega^{-1} \in SO(p+2,2, \mathbb{Z})$ on the quantum numbers,

$$v \rightarrow \Omega^{-1} v, \hspace{1cm} (4.77)$$

while simultaneously transforming the moduli $m \rightarrow m'$ such that $\mathbf{P}$ is invariant [49, 50].

As we discussed in some detail in [30] the modular group can be interpreted as a discrete subgroup of the deformation group $SO(p+2,2)$ and therefore, the transformation of the moduli is given by the left action of this discrete subgroup on the $SO(p+2,2)$ coset.
Starting from this observation we then showed how the concrete transformation laws of the moduli can be worked out.

We can now combine this with the results of this section to deduce the general form of the transformation of $Y$ and $\mathcal{M}$. First we know that $Y$ only depends on the moduli, but not on the quantum numbers, and that it is related to the Kähler potential by $K = - \log Y$. Since the moduli are holomorphic coset coordinates, and modular transformations result from the left action of $SO(p + 2, 2)$ on its coset, which is a symmetric Kähler manifold, $K$ must transform by a Kähler transformation

$$ K \rightarrow K + F + \overline{F}, $$

where $F$ is a holomorphic function of the moduli. Therefore $Y \rightarrow e^{-F} Y$. But since by construction $\mathbf{P} = (\mathbf{p}_L; \mathbf{p}_R)$ and therefore $\mathbf{p}_R$ are invariant, this implies

$$ \mathcal{M} \rightarrow e^{-F} \mathcal{M} $$

**Examples: The $SO(4,2)$, $SO(3,2)$ and $SO(2,2)$ mass formulae**

Let us illustrate the formalism with a concrete example, namely a 2–torus with 2 two–component Wilson lines turned on. This leads to a $SO(4,2)$ coset. The reference compactification lattice is chosen to be the $A_1 \oplus A_1$ root lattice. We also have to chose a two–dimensional sublattice of the $E_8 \oplus E_8$ lattice and, again, we take it to be $A_1 \oplus A_1$. Let us work out, step by step, the transformations of the quantum numbers $v$. To implement the first step $v \rightarrow \hat{v}$, we need the matrices

$$ (\mathcal{E}_I^\mathcal{M}) = (T_i^\mu) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad (T_{ij}^a) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} $$

in order to convert the lattice basis of our reference lattice to an orthonormal one (with respect to the Euclidean scalar product)

$$ \hat{v}^T = (\hat{q}_1, \hat{q}_2, \hat{n}_1, \hat{n}_2, \hat{m}_1, \hat{m}_2) = (\sqrt{2}q_1, \sqrt{2}q_2, \sqrt{2}n_1, \sqrt{2}n_2, \frac{1}{\sqrt{2}}m_1, \frac{1}{\sqrt{2}}m_1) $$

Note that we have written all of the indices as lower ones for simplicity. In a second step, we have to switch to a basis which is also orthonormal with respect to the pseudo–euclidean metric. This gives

$$ \hat{v}^T = (\hat{q}_1, \hat{q}_2, \hat{n}_1, \hat{n}_2, \hat{m}_1, \hat{m}_2) = (\sqrt{2}q_1, \sqrt{2}q_2, n_1 + \frac{1}{2}m_1, n_2 + \frac{1}{2}m_2, n_1 - \frac{1}{2}m_1, n_2 - \frac{1}{2}m_2) $$

(4.82)
Next, let us use the solution (4.73) of the coset equations for the case $p = 2$ with $U, T, B, C$ as independent variables. Then the chiral mass is given by

$$M = \sum_{i=1}^{6} \tilde{v}_i y_i = -i \left( m_2 - im_1 U + in_1 T - n_2 (TU - \frac{1}{2} BC) \right) + \frac{i}{\sqrt{2}} q_1 (B + C) - \frac{1}{\sqrt{2}} q_2 (B - C) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) 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via

\[ P = \begin{pmatrix}
I_m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_2 & 0 \\
0 & I_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_2 \\
\end{pmatrix} \]  

one finds that \( \phi' \) has the form

\[ \phi' = \begin{pmatrix}
\phi_1' & \phi_2' \\
-\phi_2' & \phi_1' \\
\end{pmatrix} \]  

(4.90)

This effectively leads to the replacement \( \eta_{2m+2,2} \to \eta_{m+1,1} \oplus \eta_{m+1,1} \) in (4.87). Then, by introducing the complex coordinate

\[ \phi_c = \phi_1' + i \phi_2' \in M(m + 2, 1, C) \]  

(4.91)

one finds that \( \phi_c \) satisfies the relation

\[ \phi_c^+ \eta_{m+1,1} \phi_c = -1 \]  

(4.92)

characterizing the coset \( SU(m + 1, 1)/ (SU(m + 1) \otimes U(1)) \) [44]. Finally, one also has to introduce complex quantum numbers by

\[ v^i_{(c)} = v_i' + i v_{m+2+i}' \quad i = 1, \ldots, m + 2 \]  

(4.93)

in order to be able to rewrite the mass formula as

\[ P_R^2 = v'^T \phi' \phi'^T v' = |v_{(c)}^T \phi_c|^2 \]  

(4.94)

A bounded realization

Since the homogenous coset coordinate \( \phi_c \) is again a complex vector (and not a matrix) we can proceed as in the last subsection. First we introduce rescaled coordinates \( y_i = \sqrt{Y} \phi^i_c \), \( i = 1, \ldots, m + 2 \) and obtain the equation

\[ \sum_{i=1}^{m+1} \bar{y}_i y_i - \bar{y}_{m+2} y_{m+2} = -Y. \]  

(4.95)
Let us introduce unconstrained coordinates \( z = (z_i, z), i = 1, \ldots, m \) by
\[
y_i = z_i, \ i = 1, \ldots, m, \quad y_{m+1} = z, \quad y_{m+2} = 1.
\] (4.96)

This solves (4.92) with
\[
Y = Y_b = 1 - \mathbf{z}^T \mathbf{z}.
\] (4.97)

Note that \( Y > 0 \) implies that \( 1 - \mathbf{z}^T \mathbf{z} > 0 \). Therefore we have found a realization of the coset by the bounded open domain \([44]\)
\[
D_{m+1} = \{ z \in \mathbb{C}^{m+1} \mid 1 - \mathbf{z}^T \mathbf{z} > 0 \}
\] (4.98)
and the standard Kähler potential for this realization is \([44]\)
\[
K = -\log Y_b.
\] (4.99)

Comparing to standard projective coordinates (which are also known to provide a solution to the constraints \([44]\))
\[
Z_i := \frac{\phi_i^c}{\phi_{m+2}^c} = \frac{y_i}{y_{m+2}} = z_i, \ i = 1, \ldots, m, \quad Z_{m+1} := \frac{\phi_{m+1}^c}{\phi_{m+2}^c} = \frac{y_{m+1}}{y_{m+2}} = z
\] (4.100)
we see that those are identical to the \(z_i, z\).

An unbounded realization

Again, we would like to have another, unbounded representation in terms of a \( T \) modulus parametrizing a \( SU(1,1) \) coset and additional complex Wilson line moduli \( A_i, i = 1, \ldots, m \). The \( y_i = \sqrt{Y} \phi_i^c \) are now given by
\[
y_i = A_i, \ i = 1, \ldots, m, \quad y_{m+1} = \frac{1}{2}(T - 1), \quad y_{m+2} = \frac{1}{2}(T + 1).
\] (4.101)

This solves (4.92) with
\[
Y = Y_u = \frac{1}{2}(T + \overline{T}) - \sum_{i=1}^{m} A_i \overline{A}_i
\] (4.102)
Clearly we have found an unbounded realization, because the imaginary part of \( T \), for example, is not constrained at all, whereas the real part can be arbitrarily large.

There is a second way of connecting the bounded and the unbounded realization. Namely one can start with the bounded realization and then introduce \( T \) and \( A_i \) by the map
\[
z \to T = \frac{1 - z}{1 + z}, \quad z_i \to A_i := \frac{z_i}{1 + z}
\] (4.103)
For vanishing Wilson lines this reduces to the standard map from the open unit disc onto the right half plane

\[ D_1 = \{ z \in \mathbb{C} \, | \, |z| < 1 \} \rightarrow H = \{ T \in \mathbb{C} \, | \, T + \overline{T} > 0 \} . \]  

(4.104)

Substituting the transformation (4.103) into \( K' = -\log Y_u \) yields

\[ K' = -\log \left( 1 - \overline{\tau} z - \sum_{i=1}^{m} \overline{\tau}_i z_i \right) + \log \left( |1 + z|^2 \right) \]  

(4.105)

which differs from \( K = -\log Y_b \) by a Kähler transformation. Note that, when relating \( z, z_i \) and \( T, A_i \) by equating (4.101) and (4.96), this is equivalent to relating them by (4.103) modulo this Kähler transformation.

The mass formula

Again, the mass formula is given by the ratio of the square of the chiral mass and the function \( Y \)

\[ p_R^2 = \frac{|v^{T}_{(c)} y|^2}{Y} \]  

(4.106)

where \( y \) and \( Y \) are functions of the complex moduli \( T, A_i, i = 1, \ldots, m \).

Again this expression is invariant under the group of modular transformations. The relevant group \( SU(m+1,1,\mathbb{Z}) \) is a subgroup of the group \( SO(p+2,2,\mathbb{Z}) \), namely the normalizer with respect to the \( \mathbb{Z}_N \) group generated by the twist [49]. Recall that these groups explicitly depend on the reference lattice, so we did not make this explicit in our notation. For the same reasons as discussed before in the case of \( SO(p+2,2) \) cosets, \( Y \) and the chiral mass \( \mathcal{M} \) transform as

\[ Y \rightarrow e^{-F-\overline{F}} Y, \quad \mathcal{M} \rightarrow e^{-F} \mathcal{M}, \]  

(4.107)

where \( F \) is a holomorphic function of the moduli.

Example: The \( SU(2,1) \) and \( SU(1,1) \) mass formulae

We will again illustrate the general procedure with a concrete example, a two dimensional \( \mathbb{Z}_3 \) orbifold with one independent two–component Wilson line. This time we take both the reference compactification lattice and the sublattice of \( \Gamma_{16} \) to be \( A_2 \) root lattices. Both the internal and the gauge twist are taken to be the \( A_2 \) Coxeter twist.

Now the transformation matrices from the lattice to the orthonormal basis are given by

\[ (E^M_i) = (T^{\mu}_{i}) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}, \quad (E^i_{\mu}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{6} \sqrt{3} \\ 0 & \frac{1}{3} \sqrt{6} \end{pmatrix} \]  

(4.108)
The transformed quantum numbers are given by
\[
\hat{v}^T = (\hat{q}_1, \hat{q}_2, \hat{n}_1, \hat{n}_2, \hat{m}_1, \hat{m}_2) = \left( \sqrt{2} q_1 - \frac{1}{\sqrt{2}} q_2, \sqrt{2} n_1 - \frac{1}{\sqrt{2}} n_2, \frac{1}{\sqrt{2}} m_1 + \frac{1}{6} \sqrt{6} m_1 + \frac{1}{3} \sqrt{6} m_2 \right) 
\]

(4.109)

Diagonalizing the pseudo–euclidean lattice metric yields
\[
\tilde{v}^T = (\sqrt{2} q_1 - \frac{1}{\sqrt{2}} q_2, n_1 - \frac{1}{2} n_2 + \frac{1}{2} m_1 + \frac{1}{2} \sqrt{3} n_2 - \frac{1}{6} \sqrt{3} m_1 + \frac{1}{3} \sqrt{3} m_2) 
\]

Next, we have to reorder the components \( \tilde{v} \to v' \) and finally complexify them, \( v' \to v_c \).

Introducing the complex quantum numbers
\[
q_c = \sqrt{2} q_1 - \frac{1}{\sqrt{2}} q_2 + i \frac{1}{\sqrt{2}} q_2, \\
n_c = n_1 - \frac{1}{2} n_2 + \frac{1}{2} m_1 + i \frac{3}{2} (n_2 + \frac{1}{3} m_1 + \frac{2}{3} m_2), \\
m_c = n_1 - \frac{1}{2} n_2 - \frac{1}{2} m_1 + i \frac{3}{2} (n_2 - \frac{1}{3} m_1 - \frac{2}{3} m_2) 
\]

(4.111)

yields that
\[
v^T_{(c)} = (q_c, n_c, m_c) 
\]

(4.112)

Therefore, the chiral mass (setting \( y_1 = y \)) is given by
\[
\mathcal{M} = \sum_{i=1}^{3} v^i_{(c)} y_i = q_c y + n_c \frac{1}{2} (T - 1) + m_c \frac{1}{2} (T + 1) = q_c y + \frac{1}{2} (n_c + m_c) T + \frac{1}{2} (m_c - n_c) 
\]

(4.113)

giving raise to the mass formula
\[
p_R^2 = \frac{|q_c y + \frac{1}{2} (n_c + m_c) T + \frac{1}{2} (m_c - n_c)|^2}{\frac{1}{2} (T + T) - y y} 
\]

(4.114)

We now proceed to show that one can get the mass formula (4.114) of an \( SU(2, 1) \) coset by a suitable truncation of the one for the \( SO(4, 2) \) coset given in (4.84). To do this truncation correctly, one has to take two things into account. First, the lattice \( \Lambda \) must be proportional to the \( A_2 \) root lattice in order to have the \( A_2 \) Coxeter twist as a lattice automorphism. This freezes the \( U \) modulus to the value \( U = \frac{1}{2} (\sqrt{3} + i) \), while \( T \) is still arbitrary. Secondly, one has to choose inside the \( E_8 \oplus E_8 \) lattice a sublattice with the appropriate symmetry. Taking an \( A_2 \) sublattice amounts to setting
\[
(\hat{q}_1, \hat{q}_2) = (\sqrt{2} q_1 - \frac{1}{\sqrt{2}} q_2, \sqrt{3} q_2) 
\]

(4.115)
in (4.81). Also note that the Wilson line moduli have to be fixed to $B = \sqrt{3}A$, $C = 0$.

Specializing in this way, the chiral mass given in (4.83) turns into

$$M = \frac{\sqrt{3}}{2} q_c A + \frac{1}{2} (m_c + n_c) T + \frac{\sqrt{3}}{2} (m_c - n_c)$$

(4.116)

with the complex quantum numbers as defined above. The mass formula (4.84) then turns into

$$p_R^2 = \frac{|\sqrt{3} q_c A + \frac{1}{2} (m_c + n_c) T + \frac{\sqrt{3}}{2} (m_c - n_c)|^2}{\sqrt{3} (\frac{1}{2} (T + T^*) - \frac{\sqrt{3}}{4} A A)}$$

(4.117)

Equation (4.114) is then indeed obtained from (4.117) by rescaling $T$ by a factor $1/\sqrt{3}$ and by setting $y = \frac{4}{T}$.

And finally, when switching off the complex Wilson line $A$, $A = 0$, one arrives at the wellknown mass formula for a $SU(1,1)$-coset

$$M = \frac{i}{2} \left( (m_c + n_c) T + \sqrt{3} (m_c - n_c) \right)$$

(4.118)

5 Target space modular invariant orbits of massive untwisted states

Massive untwisted states play an important role in the context of 1-loop corrections to gauge and gravitational couplings [3]-[6], [8]-[19], [26, 27, 55] in $\mathbb{Z}_N$-orbifold models. These states give rise to moduli dependent threshold corrections, which are given in terms of automorphic functions of the modular group under consideration [5, 11, 26, 27]. We will thus focus on massive untwisted states in the following.

We will assume that the internal 6-torus factorises into $T_6 = T_2 \oplus T_4$ and that the lattice twist $\theta$ acts on the 2-torus $T_2$ as a $\mathbb{Z}_2$-twist. We will then focus on the $SO(p + 2, 2)$-coset space associated with the $T_2$ and discuss its mass formula. That is, we will consider those massive untwisted states which have non-vanishing quantum numbers $v^T = (q^1, ..., q^{p}, n^1, n^2, m_1, m_2)$ in the Narain sublattice $\Gamma_{p+2,2} \subset \Gamma_{22,6}$, as discussed in section 4.2.3. We will, in addition, also allow these massive untwisted states to carry non-vanishing quantum numbers in an orthogonal sublattice $\Gamma_{20-p,4}$ with $\Gamma_{p+2,2} \oplus \Gamma_{20-p,4} \subset \Gamma_{22,6}$.

Recall that the level matching condition for physical states in the heterotic string reads

$$p_L^2 - p_R^2 = 2(N_R + 1 - N_L) + P_R^2 - P_L^2$$

(5.1)
where \((p_L; p_R) \in \Gamma_{p+2,2}\) and \((P_L; P_R) \in \Gamma_{20-p,4}\). The mass formula for physical states can then be written as
\[
\frac{\alpha'}{2} M^2 = p_R^2 + P_R^2 + 2N_R
\] (5.2)
The untwisted states associated with the Narain sublattice \(\Gamma_{p+2,2}\) do, on the other hand, satisfy
\[
p_L^2 - p_R^2 = 2n^T m + q^T C q
\] (5.3)
where \(C\) denotes the lattice metric of the sublattice \(\Gamma_p\) of the \(E_8 \oplus E_8\) lattice \(\Gamma_{16}\), as explained in section 4.2.3. Then, equating (5.1) and (5.3) yields
\[
p_L^2 - p_R^2 = 2\left(N_R + \frac{1}{2} P_R^2 + 1 - N_L - \frac{1}{2} P_L^2\right) = 2n^T m + q^T C q
\] (5.4)
We have shown in section 4.2.3 that for a \(SO(p + 2, 2)\)-coset \(p_R^2\) can be written as
\[
p_R^2 = \frac{|M|^2}{Y}
\] (5.5)
where \(Y\) is related to the Kähler potential by \(K = -\log Y\), and where \(M\) is a holomorphic function of the complex coordinates for the \(SO(p + 2, 2)\)-coset. From the study of a few examples in the past [11, 26, 27], it is expected that a prominent role in threshold corrections to gauge and gravitational couplings is going to be played by those massive untwisted states which satisfy \(2N_R + P_R^2 = 0\), so that
\[
\frac{\alpha'}{2} M^2 = p_R^2 = \frac{|M|^2}{Y}
\] (5.6)
Thus, we will in the following only consider untwisted states for which \(2N_R + P_R^2 = 0\). Then, (5.4) turns into
\[
p_L^2 - \frac{\alpha'}{2} M^2 = 2(1 - N_L - \frac{1}{2} P_L^2) = 2n^T m + q^T C q
\] (5.7)
Note that, for any given \(2N_L + P_L^2\), the orbit \(2n^T m + q^T C q = 2(1 - N_L - \frac{1}{2} P_L^2)\) is invariant under modular \(SO(p + 2, 2, \mathbb{Z})\)-transformations. Let us now consider the following cases:
a) first, consider untwisted states for which \(N_L = 0, P_L^2 = 0\). This defines an orbit for which
\[
2n^T m + q^T C q = 2
\] (5.8)
This is the orbit which is relevant for the discussion of the stringy Higgs effect. It can namely happen [23] that at certain finite points in the fundamental region of the
moduli space certain (otherwise) massive states precisely become massless, $M^2 = 0$. Then, at these special points in moduli space one has that $p^2_L = 2$, and, hence, one has additional massless gauge bosons in the spectrum as well as additional massless scalar fields. These additional massless gauge bosons enhance the gauge group of the underlying two-dimensional $\mathbb{Z}_2$-orbifold. This enhancement can give rise to an additional $SU(2)$, as discussed in section 3. Thus, it is crucial to take into account orbit (5.8) when discussing 1-loop corrections to gauge couplings associated with the enhanced gauge group of the compactification sector of the orbifold. Similarly, orbit (5.8) must also be taken into account when discussing threshold corrections to gravitational couplings.

b) now consider untwisted states for which $2N_L + P^2_L = 2$. This defines an orbit for which $p^2_L = p^2_R$ and, hence,

$$2n^T m + q^T C q = 0 \tag{5.9}$$

An example is provided by setting $N_L = 0, P^2_L = 2$. Then, take $P^2_L = 2$ to be a vector in the root lattice of the gauge group $G'$ which is not affected by the moduli associated with $\Gamma_{p+2,2}$. For instance, $G'$ can be taken to be the $E_8'$ in the hidden sector left unbroken when turning on Wilson lines. Then, the orbit given in (5.9) is the relevant one for the discussion of threshold corrections for this type of gauge couplings in the presence of Wilson lines.

For untwisted states with quantum numbers $q = 0$, (5.9) obviously reduces down to $n^T m = 0$. This reduced orbit is not invariant under general modular $SO(p + 2, 2, \mathbb{Z})$-transformations anymore, but it is still invariant under modular transformations belonging to the subgroup $SL(2, \mathbb{Z})_T \otimes SL(2, \mathbb{Z})_U \subset SO(p + 2, 2, \mathbb{Z})$. This reduced orbit is the one which has been discussed quite extensively in the literature [26, 27, 11] in the context of $(2, 2)\mathbb{Z}_N$-orbifold theories. In this context, it is well known that there are no finite points in the fundamental region of the $(T, U)$-moduli space where (otherwise) massive states might become massless. Indeed, by setting $n_1 = r_1 s_2, n_2 = s_1 s_2, m_1 = -r_2 s_1$ and $m_2 = r_1 r_2$, it follows that $p^2_R = 2 \left(\frac{|r_1 + is_1 U|^2}{U + U}\right) \left(\frac{|r_2 + is_2 T|^2}{T + T}\right)$. This shows that it is only in the large radius limit, namely at $T = \infty$ with $U$ fixed (and vice-versa), that states (Kaluza-Klein states) become massless. This is in manifest contrast to what happens for the orbit discussed in a).

c) finally, consider untwisted states for which $2N_L + P^2_L \geq 4$. Inspection of (5.7) shows that there aren’t any points, finite or otherwise, in moduli space for which $M^2 = 0$, since then $p^2_L < 0$, which isn’t allowed.

Hence, it is only for the orbits a) and b) that it can happen that massive states become massless at some special points in the fundamental region of moduli space. Threshold
corrections to couplings should then exhibit a singular behaviour at precisely those points in moduli space. Thus, it appears that the interesting physics contained in threshold corrections is associated with orbits a) and b). This is then why we will be focussing on orbits a) and b) in the following.

It was shown \cite{11, 27} in the context of threshold corrections to the $E_6$ and $E_8'$ gauge couplings in $(2,2)\mathbb{Z}_N$-orbifold theories, that it is of relevance to consider the quantity $\sum_{\text{orbit}} \log M$. There, the relevant orbit is the reduced orbit discussed in case b), given by $nT \equiv m, q = 0$. Indeed, when suitably regularised \cite{27}, $\sum_{nT \equiv m, q = 0} \log M|_{\text{reg}} = \log (\eta^{-2}(T)\eta^{-2}(U))$, this quantity is precisely the object appearing in these threshold corrections. In the context of threshold corrections to gravitational couplings, on the other hand, one should a priori consider both the orbits a) and b). We will thus introduce the following quantities for cases a) and b), respectively

$$\Delta_0 = \frac{1}{L_0} \sum_{nT \equiv m, q = 0, C_q = 2} \log M$$

$$\Delta_1 = \frac{1}{L_1} \sum_{nT \equiv m, q = 0} \log M \quad (5.10)$$

It is implied in \cite{11, 10} that a regularisation procedure should exist for turning these formal expressions into meaningful ones. This regularisation procedure should be compatible with the transformation property of $\log M$ under modular $SO(p + 2, 2, \mathbb{Z})$-transformations. Thus, each of these $\Delta_i$, $i = 0, 1$, is expected to be expressed in terms of automorphic functions of the modular group $SO(p + 2, 2, \mathbb{Z})$. The constants $L_i$ are such as to ensure that, under the subgroup of $SL(2, \mathbb{Z})_{T,U}$-transformations, each of the $\exp \Delta_i$ has a modular weight of $-1$.

Now consider the case when all Wilson lines have been turned off. As will be discussed in detail in the next section, $\Delta_0$ should contain a term of the form

$$\Delta_0 \propto \log \{(j(T) - j(U))^r \eta(T)^{-2} \eta(U)^{-2}\} + \ldots \quad (5.11)$$

where $r$ denotes some non-zero integer and where $j(T)$ denotes the absolute modular invariant function. We will explain that such a term should arise when summing over all the states lying on the orbit $nT \equiv m, q = 0$. The singular behaviour of this term at the points in the fundamental region where $T = U$ reflects the fact that (otherwise) massive states become massless at these points in moduli space, as discussed in section 3. The role of the eta-terms in (5.11) is to ensure the correct transformation property of $\exp \Delta_0$ under $SL(2, \mathbb{Z})_{T,U}$-transformations.\footnote{As stated above, we have made the important assumption that the regularisation procedure used in}

\[38\]
\( \Delta_1 \), on the other hand, will contain a term of the form
\[
\Delta_1 \propto \log \{ \eta(T)^{-2} \eta(U)^{-2} \} + \ldots
\]  
(5.12)

As shown in [27], this term arises when summing over all massive states for which \( n^T m = 0, q = 0 \).

As discussed above, all the physically interesting information relevant to the threshold corrections to gravitational couplings should be contained in \( \Delta_0 \) and in \( \Delta_1 \). These gravitational threshold corrections should then, for the case of vanishing Wilson moduli, be proportional to
\[
\Delta_0 + \Delta_1 \propto \alpha \log(j(T) - j(U)) + \beta \log\{ \eta(T)^{-2} \eta(U)^{-2} \}
\]  
(5.13)

where the constants \( \alpha \) and \( \beta \) have to be determined from an appropriate string scattering amplitude calculation. We will, in the next sections, study \( \Delta_0 \) and \( \Delta_1 \) in detail. We will, in particular, also study the effect of non-vanishing Wilson moduli on (5.13). The last section will be devoted to a discussion of threshold corrections to gauge and gravitational couplings.

Note that the analogous result to (5.13) for a \( SU(m+1,1) \)-coset can be obtained by an appropriate truncation of the mass formula (5.9) for a \( SO(2m+2,2) \)-coset, as discussed in section 4.2.4.

6 Automorphic functions for the orbit \( 2n^T m + q^T C q = 2 \)

In this section we will consider massive untwisted states possessing \( N_L = 0, P_L^2 = 0 \). Then the associated modular invariant orbit is defined by
\[
2n^T m + q^T C q = 2
\]  
(6.1)

It can happen that certain massive states, characterised by a set of quantum numbers satisfying (6.1), become massless at special points in moduli space. Then, at these special points one has that \( p_L^2 = 2 \), and, hence, there are some additional massless states corresponding to massless gauge bosons. Thus, orbit (6.1) is of big relevance for the stringy Higgs-effect.

(5.10) respects both holomorphicity and modular covariance. Otherwise additional terms transforming inhomogenously under modular transformations might appear on the r.h.s. of (5.11). Such additional terms do not exhibit singular behaviour in the fundamental region, and so their appearance would not change the singular behaviour of \( \Delta_0 \).
We will in the following try to construct automorphic functions for the orbit (6.1). We will, for concreteness, first consider a $SO(4,2)$-coset associated with an internal 2-torus $T_2$. At the end of this section, by making use of the truncation procedure given in section 4.2.4, we will also discuss automorphic functions for a $SU(2,1)$-coset.

Consider the quantity $\Delta_0$ introduced in (5.10). Taking into account that the Cartan matrix $C$ for the Narain sublattice $\Gamma_{4,2}$ under consideration reads $C = 2 \text{diag}(1,1)$, yields (5.10) as

$$\Delta_0 \propto \sum_{n^T m + q^2 = 1} \log M$$

$$= \sum_{n^T m = 1, q = 0} \log M + \sum_{n^T m = 0, q^2 = 1} \log M + \sum_{n^T m = -1, q^2 = 2} \log M + \ldots$$

where we have rewritten the sum over all the states laying on the $SO(4,2,\mathbb{Z})$-invariant orbit $n^T m + q^2 = 1$ into a sum over $SL(2,\mathbb{Z})_{T,U}$-invariant orbits $n^T m = \text{constant}$. Then, (6.2) can be conveniently written as

$$\Delta_0 \propto \sum_{q^2 \geq 0} \Delta_{0,q^2} = \Delta_{0,0} + \Delta_{0,1} + \ldots$$

where

$$\Delta_{0,0} = \sum_{n^T m = 1, q = 0} \log M, \quad \Delta_{0,1} = \sum_{n^T m = 0, q^2 = 1} \log M$$

Each of the $\exp \Delta_{0,q^2}$ should carry a definite modular weight under $SL(2,\mathbb{Z})_{T,U}$-transformations. The complete $\exp \Delta_0$ should then have a modular weight of $-1$.

We begin by studying the case where the Wilson lines $B$ and $C$ have been switched off. We will later generalise our results to the case of non-vanishing Wilson lines.

As already mentioned, there is an enhancement of the gauge group of the model at some special points in the $(T,U)$-moduli space. The relevant untwisted states are the ones for which $q_1 = q_2 = 0, n^T m = 1$. For these states the mass formula (4.83) reduces to

$$M = m_2 - im_1 U + im_1 T - n_2 U T$$

Note that we have redefined the chiral mass $M$ by an irrelevant overall phase factor for convenience. Let us first discuss the issue of enhancement of the gauge group in the context of a two-dimensional toroidal model. We will then, in a second step, discuss it in the context of the two-dimensional $\mathbb{Z}_2$-orbifold. At generic values for the moduli $T$ and $U$ the gauge group of the two-dimensional torus $T_2$ under consideration is given by $G = U(1) \otimes U(1)$. At special points in the $(T,U)$-moduli space, namely for i) $T = U,$
Let us consider case i) first. Setting \( T = U \) turns (6.5) into
\[
\mathcal{M} = m_2 + i(n_1 - m_1)T - n_2 T^2
\]  
(6.6)
Then, the holomorphic mass (6.6) vanishes for the following 2 untwisted states
\[ m_1 = n_1 = \pm 1, \quad m_2 = n_2 = 0 \]  
(6.7)
Note that these states satisfy \( n^T m = 1 \). Thus, they are precisely the 2 states corresponding to the 2 additional gauge bosons which enhance the gauge group to \( G = SU(2) \otimes U(1) \) at the points where \( T = U \).

Next, consider case ii). Setting \( T = U = 1 \) turns (6.6) into
\[
\mathcal{M} = m_2 - n_2 + i(n_1 - m_1)
\]  
(6.8)
In addition to the 2 states (6.7) there are now 2 additional ones
\[ m_2 = n_2 = \pm 1, \quad m_1 = n_1 = 0 \]  
(6.9)
satisfying \( n^T m = 1 \), for which the holomorphic mass (6.8) vanishes. These 2 states correspond to the 2 additional massless gauge bosons which further enhance the gauge group from \( G = SU(2) \otimes U(1) \) to \( G = SU(2) \otimes SU(2) \).

Finally, consider case iii). Inserting \( T = U = e^{i\pi/6} \) into (6.6) yields
\[
\mathcal{M} = \frac{1}{2} \left( 2m_2 - n_2 - n_1 + m_1 + i\sqrt{3}(n_1 - m_1 - n_2) \right)
\]  
(6.10)
The holomorphic mass (6.10) vanishes not only for the states (6.7), but also for the following 4 additional states satisfying \( n^T m = 1 \)
\[
m_2 = n_2 = 1 : \quad m_1 = -1, \; n_1 = 0 \quad \text{or} \quad m_1 = 0, \; n_1 = 1
\]
\[
m_2 = n_2 = -1 : \quad m_1 = 0, \; n_1 = -1 \quad \text{or} \quad m_1 = 1, \; n_1 = 0
\]  
(6.11)
These 4 additional massless states correspond again to 4 additional massless gauge bosons which enhance the gauge group to \( G = SU(3) \). Thus, the ratio of additional massless gauge bosons for the cases i)-iii) is given by 1:2:3. Note that there is an infinite set of states with dual transformed quantum numbers, whose masses vanish at the transformed critical points.
The enhancement of the gauge group for the $\mathbb{Z}_2$-orbifold occurs at the same points in moduli space as in the toroidal case. The associated gauge groups are, however, smaller, since one has to project out all the non-twist invariant states. The resulting gauge groups are then, for the cases i)-iii), given by i) $U(1)$, ii) $U(1) \otimes U(1)$ iii) $SU(2)$ at level $k = 4$, as discussed in section 3. Thus, the ratio of additional massless gauge bosons is, as in the toroidal case, given by 1:2:3.

Now, what do the individual $\Delta_{0,q^2}$ given in (6.4) look like? First consider $\Delta_{0,0}$. It is defined in terms of a sum over all the states laying on the orbit $n^T m = 1, q = 0$. As shown above, this is precisely the orbit for which some of the massive states become massless at the special points i)-iii) in the fundamental region of the $(T,U)$-moduli space. Thus, at each of these special points $\Delta_{0,0}$ has to exhibit a logarithmic singularity. The order of this singularity should be determined by the number of states which become massless at these special points. As shown above, the ratio of additional massless states at the special points i)-iii) is 1:2:3. Thus, the order of the zeros of $\exp \Delta_{0,0}$ should be 1:2:3 for the points i)-iii). According to well-known theorems of modular functions \[51\], the order of the zeros and of the poles, together with the modular weights (possibly leading to non-trivial multiplier systems), determine a modular function in a unique way. Applying this to the above case yields that

$$\exp \Delta_{0,0} \propto (j(T) - j(U))^r \eta(T)^{-2} \eta(U)^{-2}$$

(6.12)

where $r$ denotes some non-zero integer. $j(T)$ denotes the absolute modular invariant function. The formal definition (6.4) of $\Delta_{0,0}$ implies that $\Delta_{0,0}$ has to carry a modular weight of $-1$ under $SL(2,\mathbb{Z})_{T,U}$-transformations. This explains the presence of the eta-terms in (6.12), which indeed ensure that $\exp \Delta_{0,0}$ transforms with modular weight of $-1$ under $SL(2,\mathbb{Z})_{T,U}$-transformations. Furthermore, $\exp \Delta_{0,0}$, as given in (6.12), obviously vanishes for $T = U$. For $T$ in the vicinity of $U$ one has that $j(T) - j(U) = j'(U)(T - U)$. For $T \sim U = 1$ one finds that $j(T) - j(U) = \frac{1}{2}j''(1)(T - 1)^2$, because of $j'(1) = 0$. And, finally, for $T \sim U = \rho$ one has that $j(T) - j(U) = \frac{1}{2}j''(\rho)(T - \rho)^3$, because $j'(\rho) = j''(\rho) = 0$. Thus, the ansatz $j(T) - j(U)$ does indeed reflect the ratio 1:2:3 in the order of the zeros of $\exp \Delta_{0,0}$. This observation is just a reflection of the fact that the order of the zeros of the $j$-function is determined by the order of the fixed points of the duality group; this order is proportional to the number of massless states at the fixed points, as proven in (3.10).

\[5\] In (6.12) we have ignored possible additional multiplicative terms of the type $\exp f(T,U)$ with $f(T,U)$ exhibiting regular behaviour inside the fundamental region. Such terms might be necessary in order to reproduce the correct asymptotic behaviour of $\Delta_{0,0}$ as $T,U \to \infty$. 

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Comparing (6.14) with (6.12), on the other hand, gives that
\[ \Delta_{0,0} = \sum_{n^T m = 1} \log(m_2 - im_1 U + in_1 T - n_2 UT) \]
\[ = \log \{ (j(T) - j(U))^r \eta(T)^{-2} \eta(U)^{-2} \} + \ldots \]  
(6.13)
for some non-zero integer \( r \). In (6.13) a regularisation of the infinite sum is implied. Thus, (6.13) yields a representation of \( j(T)^r \eta(T)^{-2} \) as a regularised sum (of chiral masses) over all the states on the orbit \( n^T m = 1, q = 0 \). It would be very interesting to prove this directly by explicitly performing the sum. To our knowledge such a representation is unknown in the literature. The dots in (6.13) stand for additional terms which are regular inside the fundamental region of the \( T,U \)-moduli space (see footnote \( \text{[3]} \)).

Let us now switch on the Wilson lines \( B \) and \( C \) and discuss the resulting modifications to (6.13). We will, unfortunately, have to restrict ourselves to the case where \( B \) and \( C \) are small, that is, we will work to lowest non-trivial order in \( B \) and \( C \), only. When switching on Wilson lines, \( \Delta_{0,0} \) becomes equal to
\[ \Delta_{0,0} = \sum_{n^T m = 1} \log \{ (j(T) - j(U))^r \eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} BC X(T,U) \} \]  
(6.14)
as can be seen from (6.13). Then, a suitable ansatz for the modification to (6.13) is, to lowest order in \( BC \), given by
\[ \Delta_{0,0} = \log \left( j(T) - j(U) + \frac{1}{2} BC X(T,U) \right)^r + \log \left( \eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} BC Y(T,U) \right) \]
\[ + \ldots \]  
(6.15)
where \( X(T,U) \) and \( Y(T,U) \) will be determined in the following. Let us first discuss the modification to the \( (j(T) - j(U)) \)-term given by \( X(T,U) \). Under \( SL(2,\mathbb{Z})_{U} \)-transformations \([17, 30]\)

\[ \begin{aligned} 
U &\to \frac{\alpha U - i\beta}{i\gamma U + \delta}, \quad T \to T - \frac{i\gamma}{2i\gamma U + \delta} \frac{BC}{\alpha\delta - \beta\gamma = 1} \\
B &\to \frac{B}{i\gamma U + \delta}, \quad C \to \frac{C}{i\gamma U + \delta} 
\end{aligned} \]  
(6.16)
it follows that, to lowest order in \( BC \),
\[ j(T) - j(U) \to j(T) - j(U) - \frac{i\gamma}{2i\gamma U + \delta} \frac{BC}{j'(T)} \]  
(6.17)
Then, if one requires that \( j(T) - j(U) + \frac{1}{2} BCX(T,U) \) should be invariant under \( SL(2,\mathbb{Z})_{T,U} \)-transformations (6.16), one obtains that \( X(T,U) \) has to transform as
\[ X(T,U) \to (i\gamma U + \delta)^2 X(T,U) + i\gamma (i\gamma U + \delta) j'(T) \]  
(6.18)
Similarly it follows that, under $SL(2,\mathbb{Z})_T$-transformations, $X(T,U)$ has to transform to lowest order in $BC$ as

$$X(T,U) \to (i\gamma T + \delta)^2 X(T,U) - i\gamma (i\gamma T + \delta) j'(U)$$

(6.19)

Note that, in addition, $X(T,U)$ has to vanish at $T = U$. Only then does $\Delta_{0,0}$ again become singular at $T = U$. This has to be the case because, as discussed in section 3, the enhancement of the gauge group still occurs at $T = U$ when Wilson lines are switched on.

Indeed, when inserting (6.7) into (6.14) it follows that $\Delta_{0,0}$ has a singularity at $T = U$. Similarly, when inserting (6.9) into (6.14) shows that another singularity of $\Delta_{0,0}$ occurs at $T = U = \sqrt{1 + \frac{BC}{2}}$. And finally, when inserting (6.11) into (6.14) yields yet another singularity of $\Delta_{0,0}$ at $T = U = i\frac{2}{4} + \sqrt{3 + \frac{BC}{2}}$, thus substantiating our claims in section 2.

All these requirements do not uniquely specify $X(T,U)$, however. The following $X(T,U)$ satisfies all the above requirements

$$X(T,U) = \partial_U \log \eta^2(U) j'(T) - \partial_T \log \eta^2(T) j'(U) + a (j(T) - j(U)) \eta^4(T) \eta^4(U) + O((BC)^2)$$

(6.20)

where $a$ is an arbitrary constant which cannot be determined alone by symmetry arguments. Note that the first term in (6.20) transforms inhomogenously as in (6.19), whereas the second term in (6.20) transforms homogenously with modular weight 2 under $SL(2,\mathbb{Z})_{T,U}$-transformations.

One can now try to determine the function $Y(T,U)$ in a similar manner [17] by demanding that $\eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} BC Y(T,U)$ should transform with modular weight of $-1$ under $SL(2,\mathbb{Z})_{T,U}$-transformations. Then, one has to require $Y(T,U)$ to transform as [17]

$$Y(T,U) \to (i\gamma U + \delta) Y(T,U) + i\gamma \eta^{-2}(U) \partial_T \eta^{-2}(T)$$

(6.21)

under $SL(2,\mathbb{Z})_U$-transformations as well as [17]

$$Y(T,U) \to (i\gamma T + \delta) Y(T,U) + i\gamma \eta^{-2}(T) \partial_U \eta^{-2}(U)$$

(6.22)

under $SL(2,\mathbb{Z})_T$-transformations. These transformation laws do, however, not uniquely specify $Y(T,U)$. The following $Y(T,U)$ transforms appropriately up to corrections of

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6 This is, strictly speaking, not quite correct. We have ignored the issue of field-independent phases $\phi = \phi(\alpha, \beta, \gamma, \delta)$, which arise when performing modular transformations. These phases, also called multiplier systems, show up in the transformation laws of the Wilson line moduli $B$ and $C$ as well as in the transformation law of the eta-function. Thus, in order to really construct the appropriate function $X(T,U)$, care needs to be taken of all the different multiplier systems appearing in the transformation laws.
order \((BC)^2\)

\[ Y(T, U) = -\eta(T)^{-2} \eta(U)^{-2} \partial_T \log \eta^2(T) \partial_U \log \eta^2(U) + b \eta^2(T) \eta^2(U) + \cdots \quad (6.23) \]

where \(b\) denotes an arbitrary constant which cannot be determined alone by symmetry arguments. Note that the first term \(\:\text{[17]}\) in (6.23) transforms inhomogeneously under \(SL(2, \mathbb{Z})_{T,U}\)-transformations, whereas the second term in (6.23) transforms homogeneously with modular weight 1. \(\:\text{[17]}\) Hence, it follows from (6.15) that the \(BC\)-corrected exp \(\Delta_{0,0}\) is given by

\[
\exp \Delta_{0,0} \propto (j(T) - j(U)) r \eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} BC r X(T, U)(j(T) - j(U))^{r-1} \eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} BC Y(T, U)(j(T) - j(U))^r \quad (6.24)
\]

with \(X(T, U)\) and \(Y(T, U)\) given by (6.20) and (6.23), respectively.

Next, let us discuss \(\Delta_{0,1}\) in the presence of Wilson lines. The sum now runs over all the states for which \(n^T m = 0, q^2 = 1\). This set of states can be divided into two subsets. The first subset consists of the 4 states \(n = m = 0, q^2 = 1\), whereas the second subset consists of all the states for which \((n, m) \neq (0, 0), q^2 = 1\). Then, it follows from (4.83) that

\[
\Delta_{0,1} = 2 \log(B + C) + 2 \log(B - C) + \sum_{(n,m)\neq(0,0),q^2=1} \log M + \text{const} \quad (6.25)
\]

The first two terms in (6.25) are the contribution arising from the 4 states \(n = m = 0, q^2 = 1\). When turning off the Wilson lines, these 4 states become massless, thus giving rise to logarithmic singularities in \(\Delta_{0,1}\), as required. The last term in (6.25), \(\sum_{(n,m)\neq(0,0),q^2=1} \log M\), can be written as

\[
\sum_{(n,m)\neq(0,0),q^2=1} \log M = \sum_{(n,m)\neq(0,0)} \log \{\hat{M} + \frac{i}{\sqrt{2}}(B + C)\} + \sum_{(n,m)\neq(0,0)} \log \{\hat{M} - \frac{i}{\sqrt{2}}(B + C)\} + \sum_{(n,m)\neq(0,0)} \log \{\hat{M} + \frac{1}{\sqrt{2}}(B - C)\} + \sum_{(n,m)\neq(0,0)} \log \{\hat{M} - \frac{1}{\sqrt{2}}(B - C)\} \quad (6.26)
\]

where

\[
\hat{M} = m_2 - im_1 U + im_1 T + n_2 (UT + \frac{1}{2} BC) \quad (6.27)
\]

\(^7\)We have, again, ignored the issue of multiplier systems appearing in the transformation laws given above.
Then, expanding (6.26) to lowest order in $B$ and $C$ yields that

$$
\sum_{(n,m)\neq(0,0), q^2=1} \log \mathcal{M} = 4 \left( \sum_{(n,m)\neq(0,0)} \log(m_2 - im_1 U + in_1 T - n_2 U T) \right) \\
+ \frac{1}{2} B C \sum_{(n,m)\neq(0,0)} \frac{n_2}{(m_2 - im_1 U + in_1 T - n_2 U T)} \\
+ 2 B C \sum_{(n,m)\neq(0,0)} \frac{1}{(m_2 - im_1 U + in_1 T - n_2 U T)^2} \tag{6.28}
$$

As we will show in the next section, the terms in the big bracket in (6.28) give, when suitably regularised, rise to

$$
\left( \sum_{(n,m)\neq(0,0)} \{ \log(m_2 - im_1 U + in_1 T - n_2 U T) + \frac{B C}{2} \frac{n_2}{(m_2 - im_1 U + in_1 T - n_2 U T)} \} \right)_{reg}
$$

$$
= \log \left( \eta(T)^{-2} \eta(U)^{-2} - \frac{1}{2} B C \eta(T)^{-2} \eta(U)^{-2} \partial_T \log \eta^2(T) \partial_U \log \eta^2(U) \right) \tag{6.29}
$$

Furthermore, (6.28) should not exhibit any singular behaviour at finite points in the fundamental region of moduli space, and it should transform appropriately under $SL(2, \mathbb{Z})_{T,U}$-transformations. Together with (6.29) this then restricts (6.28), to lowest order in $B$ and $C$, to be given by.

$$
\sum_{(n,m)\neq(0,0), q^2=1} \log \mathcal{M} = 4 \log \left( \eta(T)^{-2} \eta(U)^{-2} \right) \\
- \frac{1}{2} B C \eta(T)^{-2} \eta(U)^{-2} \partial_T \log \eta^2(T) \partial_U \log \eta^2(U) + c B C \eta(T)^4 \eta(U)^4 \tag{6.30}
$$

where $c$ is an arbitrary constant which cannot be fixed by symmetry considerations alone. Note that the last term in (6.30) is invariant under $SL(2, \mathbb{Z})_{T,U}$-transformations. Equation (6.30) can also be written as

$$
\sum_{(n,m)\neq(0,0), q^2=1} \log \mathcal{M} = 4 \log \left( \eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} B C Y(T,U) \right) \tag{6.31}
$$

with $Y$ given in (6.23). Inserting (6.31) into (6.25) yields that

$$
\Delta_{0,1} = p \log(B + C) + p \log(B - C) \\
+ t \log \left( \eta(T)^{-2} \eta(U)^{-2} + \frac{1}{2} B C Y(T,U) \right) + \text{const} \tag{6.32}
$$

8Holomorphic regularisation of the last term in (6.28) yields $\sum_{(n,m)\neq(0,0)} \frac{1}{(m_2 - im_1 U + in_1 T - n_2 U T)^2} \propto G_2(T)G_2(U)$. The holomorphically regularised Eisenstein function $G_2(T)$, however, transforms inhomogeneously under $SL(2, \mathbb{Z})_T$-transformations. We thus expect that the procedure given in (6.28) for computing $\Delta_{0,1}$ is too naive.

9Ignoring again complications due to the appearance of multiplier systems.
with appropriate constants $p$ and $t$. Note that each of the factors in $\exp \Delta_{0,1}$ transforms with modular weight of $-1$ under $SL(2,\mathbb{Z})_{T,U}$-transformations.

Inserting (6.15) and (6.32) into (6.2) yields that

$$\Delta_0 \propto r \log \left( j(T) - j(U) + \frac{1}{2} BC \left( \partial_U \log \eta^2(U) j'(T) - \partial_T \log \eta^2(T) j'(U) \right) \right)$$

$$+ \tilde{t} \log \left( \eta(T)^{-2} \eta(U)^{-2} - \frac{1}{2} BC \eta(T)^{-2} \eta(U)^{-2} \partial_T \log \eta^2(T) \partial_U \log \eta^2(U) \right)$$

$$+ \tilde{b} BC \eta^4(T) \eta^4(U) + p \log(B + C) + p \log(B - C) + \ldots$$

(6.33)

where, again, $p, r, \tilde{t}$ and $\tilde{b}$ denote some appropriate constants. Note that each of the terms in (6.33) is either invariant or the logarithm of a term transforming with weight of $-1$ under modular $SL(2,\mathbb{Z})$-transformations. Also note that, up to quadratic order in $B$ and $C$, the terms given in (6.33) are the only ones one can write down which can exhibit singular behaviour at finite points in the moduli space. At these points, otherwise massive states become massless, thus inducing this singular behaviour. The terms in (6.33) represent possible threshold corrections terms which one should find in calculations of the running of gauge and gravitational couplings in orbifolds with $\mathbb{Z}_2$-sectors. In that context, the coefficients $p, r, \tilde{t}$ and $\tilde{b}$ are related to beta-function coefficients as well as to Kähler and $\sigma$-model anomaly coefficients. This will be discussed in the last section of this paper. Finally, note that the terms given in (6.33) transform appropriately under modular $SL(2,\mathbb{Z})_{T,U}$-transformations, but not under additional transformations belonging to the full modular $SO(4,2,\mathbb{Z})$-group. The dots in (6.33) stand for additional contributions, which we haven’t computed, whose role is to restore the proper transformation behaviour of $\Delta_0$ under full $SO(4,2,\mathbb{Z})$-transformations.

Finally, let as look at the $SU(2,1)$-coset discussed in section 4.2.4 which was based on an $A_2$-root lattice. Applying the truncation $U = \frac{1}{\sqrt{3}}(\sqrt{3} + \imath), C = 0, B = \sqrt{3}A$ to (6.33) yields that

$$\Delta_0 \propto r \log j(T) + \tilde{t} \log \eta(T)^{-2} + \tilde{p} \log A + \text{const} + \ldots$$

(6.34)

where we have used that $j(U = \rho) = 0$. The dots in (6.34) stand for additional terms whose role is to restore the proper transformation behaviour of (6.34) under transformations belonging to the full modular $SU(2,1,\mathbb{Z})$-group.
7 Automorphic functions for the orbit $2n^T m + q^T C q = 0$

In this section we will consider massive untwisted states possessing $2N_L + P^2_T = 2$. Then, the associated modular invariant orbit is defined by

$$2n^T m + q^T C q = 0$$

(7.1)

Let us again look at the example of an $SO(4,2)$-coset, for concreteness. As in the previous section, by taking into account that the Cartan matrix $C$ for the Narain sublattice $\Gamma_{4,2}$ under consideration reads $C = 2\text{diag}(1,1)$, then yields (5.10) as

$$\Delta_1 \propto \sum_{n^T m + q^2 = 0} \log \mathcal{M} = \sum_{n^T m = 0, q = 0} \log \mathcal{M} + \sum_{n^T m = -1, q^2 = 1} \log \mathcal{M} + \ldots$$

(7.2)

where we have rewritten the sum over all the states laying on the $SO(4,2,\mathbb{Z})$-invariant orbit $n^T m + q^2 = 0$ into a sum over $SL(2,\mathbb{Z})_{T,U}$-invariant orbits $n^T m = \text{constant}$.

Let us consider the term $\sum_{n^T m = 0, q = 0} \log \mathcal{M}$ in (7.2). That is, let us consider massive untwisted states for which $q_1 = q_2 = 0, n^T m = 0$ and $(n, m) \neq 0$. For these states the mass formula (4.83) reduces to

$$\mathcal{M} = m_2 - im_1 U + i n_1 T + n_2 (-UT + \frac{1}{2} BC)$$

(7.3)

Expanding to lowest order in $B$ and $C$ yields

$$\sum_{n^T m = 0, q = 0} \log \mathcal{M} = \sum_{(n,m) \neq 0} \log(m_2 - im_1 U + in_1 T - n_2 UT)$$

$$+ \frac{1}{2} BC \sum_{(n,m) \neq 0} \frac{n_2}{m_2 - im_1 U + in_1 T - n_2 UT} + \mathcal{O}((BC)^2)$$

(7.4)

Introducing a set of integers $r_1, r_2, s_1$ and $s_2$ [27] subject to $n^T m = 0, (n, m) \neq 0$,

$$m_2 = r_1 r_2 , \quad n_2 = s_1 s_2 , \quad m_1 = -r_2 s_1 , \quad n_1 = r_1 s_2$$

(7.5)

allows one to rewrite (7.4) as

$$\sum_{n^T m = 0, q = 0} \log \mathcal{M} = \sum_{(r_1, s_1) \neq (0,0)} \log(r_1 + is_1 U) + \sum_{(r_2, s_2) \neq (0,0)} \log(r_2 + is_2 T)$$

$$+ \frac{1}{2} BC \left( \sum_{(r_1, s_1) \neq (0,0)} \frac{s_1}{r_1 + is_1 U} \right) \left( \sum_{(r_2, s_2) \neq (0,0)} \frac{s_2}{r_2 + is_2 T} \right)$$

$$+ \mathcal{O}((BC)^2)$$

(7.6)
and, hence, as

\[ \sum_{n^T m=0, q=0} \log \mathcal{M} = \sum_{(r_1, s_1) \neq (0,0)} \log(r_1 + is_1 U) + \sum_{(r_2, s_2) \neq (0,0)} \log(r_2 + is_2 T) \]

\[ - \frac{1}{2} BC \left( \partial_U \sum_{(r_1, s_1) \neq (0,0)} \log(r_1 + is_1 U) \right) \left( \partial_T \sum_{(r_2, s_2) \neq (0,0)} \log(r_2 + is_2 T) \right) \]

\[ + O((BC)^2) \quad (7.7) \]

Using that upon regularisation, \( \sum_{(r_1, s_1) \neq (0,0)} \log(r_1 + is_1 U) = \log \eta^{-2}(U) \) [20, 27], allows one to rewrite (7.7) as

\[ \sum_{n^T m=0, q=0} \log \mathcal{M} = \log (\eta^{-2}(U) \eta^{-2}(T)) - \frac{1}{2} BC \partial_U \log \eta^{-2}(U) \partial_T \log \eta^{-2}(T) \]

\[ + O((BC)^2) \quad (7.8) \]

This can also be written as

\[ \sum_{n^T m=0, q=0} \log \mathcal{M} = \log (\eta^{-2}(U) \eta^{-2}(T)(1 - \frac{1}{2} BC \partial_U \log \eta^2(U) \partial_T \log \eta^2(T))) \]

\[ + O((BC)^2) \quad (7.9) \]

Inserting (7.9) into (7.2) yields that

\[ \Delta_1 \propto \log (\eta^{-2}(U) \eta^{-2}(T)(1 - \frac{1}{2} BC \partial_U \log \eta^2(U) \partial_T \log \eta^2(T))) + \ldots \quad (7.10) \]

This result agrees with the one given in [17], which was obtained by requiring \( \exp \Delta_1 \) to transform with weight \(-1\) under \( SL(2, \mathbb{Z})_{T,U} \)-transformations [17]. Note that \( \Delta_1 \) also has to transform appropriately under additional \( SO(4,2, \mathbb{Z}) \)-transformations. The dots in (7.10) stand for additional contributions, which we haven’t computed, whose role is to restore the proper transformation behaviour of \( \Delta_1 \) under transformations belonging to the full modular \( SO(4,2, \mathbb{Z}) \)-group.

### 8 Threshold corrections

In this section, we will discuss 1-loop corrections to gauge and gravitational couplings in the context of \( (0,2)\mathbb{Z}_N \)-orbifold compactifications. We will begin by reviewing some well-known facts about effective gauge couplings in locally \( N = 1 \) supersymmetric effective quantum field theories (EQFT).

Consider first the case of a locally supersymmetric EQFT with gauge group \( G = \otimes G_a \) where the light charged particles are exactly massless, and where the massive charged
fields decouple at some scale, say $M_X$. Then, at energy scales $p^2 \ll M_X^2$, the 1-loop corrected low energy gauge couplings are given by [3, 4, 5, 8, 57]

$$\frac{1}{g_a^2(p^2)} = \frac{1}{g_a^2(M_X^2)} + \frac{b_a}{16\pi^2} \log \frac{M_X^2}{p^2} + \frac{\hat{\Delta}_a}{16\pi^2} \tag{8.1}$$

where $b_a$ is the coefficient of the 1-loop $N = 1 \beta$-function, $\beta_a = \frac{b_a g_a^3}{16\pi^2}$, computed from the massless charged spectrum of the theory. $b_a$ describes the running between $M_X^2$ and $p^2 \ll M_X^2$ and is given by $b_a = -3c(G_a) + \sum_C T_a(r_C)$. Here, $c(G_a)$ denotes the quadratic Casimir of the gauge group and the sum is over chiral matter superfields transforming under some representation $r$ of the gauge group $G_a$. $\hat{\Delta}_a$, on the other hand, determines the boundary conditions for the running gauge couplings at $M_X^2$ and is given by

$$\hat{\Delta}_a = \left( c_a K - 2 \sum_r T_a(r) \log \det g_r \right) + \Delta_a \tag{8.2}$$

The massive charged fields, which have been integrated out, contribute a finite threshold correction $\Delta_a$ to the low energy gauge coupling. There are, however, also contributions to $\hat{\Delta}_a$ from the massless modes in the theory. They too need to be taken into account when discussing effective couplings. These massless contributions arise due to non-vanishing Kähler and $\sigma$-model anomalies [3, 4, 8, 10, 57], present in generic supergravity-matter systems, and they are given by the term $c_a K - 2 \sum_r T_a(r) \log \det g_r$ in (8.2). Here, $c_a = -c(G_a) + \sum_C T_a(r_C)$, and $g_r$ denotes the $\sigma$-model metric of the massless subsector of the matter fields in the representation $r$.

Next consider the case where some gauge or matter particles do have small masses of the order $M_I \ll M_X$. In a regime where $M_I^2 \ll p^2 \ll M_X^2$, all interactions can be described in terms of a massless EQFT, whereas for $p^2 \ll M_I^2$ there is another EQFT given in terms of the truly massless fields, only [57]. We will assume that, at the threshold scale $M_I$, supersymmetry remains unbroken whereas the gauge group $G = \otimes G_a$ is spontaneously broken down to $\hat{G} = \hat{\otimes} G_{a,i}$. Let us consider one such factor $G_a$ and assume that it gets spontaneously broken down to $G_a \to \hat{G}_a = \hat{\otimes} \hat{G}_{a,i}$. Let us then discuss the running of the coupling $\hat{g}_{a,i}(p^2)$ of one such subgroup $\hat{G}_{a,i}$. In order to simplify the notation, we will, in the following, simply denote this subgroup $\hat{G}_{a,i}$ by $\hat{G}_a$ and its associated coupling constant $\hat{g}_{a,i}$ by $\hat{g}_a$. At low energies, $p^2 \ll M_I^2$, the effective gauge coupling $\hat{g}_a^2(p^2)$ is given as [55, 57]

$$\frac{1}{\hat{g}_a^2(p^2)} = \frac{1}{g_a^2(M_I^2)} + \frac{\hat{b}_a}{16\pi^2} \log \frac{M_I^2}{p^2} + \frac{1}{16\pi^2} \left( \hat{c}_a K - 2 \sum_{\hat{r}} \hat{T}_{a}(\hat{r}) \log \det g_{\hat{r}} \right) \tag{8.3}$$
where the coefficients \( \hat{b}_a = -3c(\hat{G}_a) + \sum \hat{T}_a(\hat{r}_C) \) and \( \hat{c}_a = -c(\hat{G}_a) + \sum \hat{T}_a(\hat{r}_C) \) are now determined only in terms of the truly massless fields transforming under \( \hat{G}_a \). Here, \( \hat{C} \) denotes a truly massless chiral superfield transforming under some representation \( \hat{r}_C \) of the unbroken gauge group \( \hat{G}_a \). Above the threshold \( M_I \), the running of the gauge coupling is determined by the gauge group \( G_a \). Hence, \( \hat{g}_a(M_I^2) \) is given by

\[
\frac{1}{\hat{g}_a^2(M_I)} = \frac{1}{g_a^2(M_X)} + \frac{b_a}{16\pi^2} \log \frac{M_X^2}{M_I^2} + \frac{\Delta_a}{16\pi^2} \tag{8.4}
\]

Here, \( b_a \) describes the running between \( M_X \) and \( M_I \) and is given by \( b_a = -3c(\hat{G}_a) + \sum \hat{T}_a(\hat{r}_C) \), where the sum runs over all the light chiral matter superfields charged under \( \hat{G}_a \). \( \Delta_a \) denotes the contribution from all the massive charged states which decouple at \( M_X \).

It is useful to note that the running of the gauge coupling between \( M_X \) and \( M_I \) can also be described in terms of the light fields which are charged under the gauge group \( \hat{G}_a \). Above the threshold \( M_I \) all of these light fields are effectively massless. Thus, they all contribute to the running and therefore, at least for regular embeddings, \( b_a \) can also be written as

\( b_a = -3c(\hat{G}_a) - 3\sum \hat{T}_a(\hat{r}_V) + \sum \hat{T}_a(\hat{r}_C) \). Here, \( \hat{r}_V \) and \( \hat{r}_C \) denote the representation of a light vector multiplet and of a light chiral multiplet, respectively. As an example, consider the breaking of \( G_a = SU(5) \) down to \( SU(3) \otimes SU(2) \otimes U(1) \). Decomposing the adjoint representation of \( SU(5) \) into representations of \( SU(3) \otimes SU(2) \), \( 24 = (8, 1) + (1, 3) + (1, 1) + (3, 2) + (3, 2) \), yields that \( c(SU(5)) = c(SU(3)) + 4\hat{T}_{SU(3)}(3) = 10 \). Decomposing \( 5 = (3, 1) + (1, 2) \) yields that \( T_{SU(5)}(5) = \hat{T}_{SU(3)}(3) = 1 \). More generally, \( \sum \hat{T}_{SU(5)}(r_C) = \sum \hat{\bar{T}}_{SU(3)}(r_C) \), and it follows then indeed that \( b_{SU(5)} = -3c(SU(3)) - 3\sum \hat{\bar{T}}_{SU(3)}(r_C) \).

We now turn to string theory and consider orbifolds with \( N = 2 \) spacetime sectors. The reason for this is as follows. Explicit string scattering amplitude calculations of threshold corrections to gauge \([4, 8, 11, 13, 7, 19]\) and gravitational \([13]\) couplings in the context of \((2, 2)\mathbb{Z}_N\)-orbifold compactifications show that a non-trivial moduli dependence of these thresholds only arises if the orbifold point twist group \( \mathcal{P} \) contains a subgroup \( \breve{\mathcal{P}} \) that, by itself, would produce an orbifold with \( N = 2 \) spacetime supersymmetry. Furthermore, if the underlying \( T_6 \) torus factorises into \( T_6 = T_2 \oplus T_4 \), where the \( T_2 \) remains untwisted under the action of \( \breve{\mathcal{P}} \), then the moduli dependent threshold corrections associated with this \( T_2 \) are invariant under \( \Gamma = SL(2, \mathbb{Z})_{T,U} \). We will, in the following, stick to those \((0, 2)\mathbb{Z}_N\)-orbifolds for which \( T_6 = T_2 \oplus T_4 \) with the untwisted plane lying in \( T_2 \), and we will derive formulae for the gauge and gravitational threshold corrections associated with this untwisted plane \( T_2 \). As an example one can think of a \( \mathbb{Z}_4 \)-orbifold, for which \( \breve{\mathcal{P}} \) is
the \( \mathbb{Z}_2 \) generated by \( \tilde{\theta} = \theta^2 = (\Omega^2, \Omega, \Omega) \), where \( \Omega = e^{\pi i} \).

Let us consider the case where the gauge group \( G = \otimes G_a \) in the observable sector of the \((2, 2)\mathbb{Z}_N\)-orbifold gets broken to a subgroup \( \hat{G} \) by turning on Wilson moduli. Thus, the Wilson moduli act as Higgs fields. Let us assume that this breaking takes place when turning on Wilson moduli \( B \) and \( C \) associated with the \( T_2 \) (in the decomposition \( T_6 = T_2 \oplus T_4 \)). We will now determine the running of the gauge couplings in both the hidden and the observable sectors from equations (8.1), (8.3) and (8.4). We will throughout this section be working to lowest order in the Wilson moduli.

As a consequence, the results given below will only be invariant under the subgroup \( B \) when turning on Wilson moduli. Let us assume that this breaking takes place by turning on Wilson moduli. Thus, for this case the only relevant threshold scale is given

\[ C = \begin{pmatrix} T \otimes T \end{pmatrix} \quad SO(4, 2, \mathbb{Z}) \]

Also, for notational simplicity we will not include the Kac-Moody level \( k_a \) into the equations below. The tree-level gauge couplings in the observable and hidden sectors are given by

\[ g_a^2(M_{\text{string}}^2) = g_{E_8}^2(M_{\text{string}}^2) = \frac{S + \bar{S}}{2} \]

In the case of vanishing Wilson lines \( B \) and \( C \), the \( \sigma \)-model metric \( g_{\hat{C}} \) of a charged matter field/twisted modulus \( \hat{C} \) exhibits the following dependence on the moduli \( T, U \), namely

\[ g_{\hat{C}} = \left( (T + \bar{T})(U + \bar{U}) \right)^{n_{\hat{C}}} \]

where \( n_{\hat{C}} \) denotes the modular weight of \( \hat{C} \). In the presence of Wilson lines \( B \) and \( C \) one then expects the \( \sigma \)-model metric \( g_{\hat{C}} \) to be given by

\[ g_{\hat{C}} = \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{C})(C + \bar{B}) \right)^{n_{\hat{C}}} \] (8.5)

Let us first discuss the running, in the presence of Wilson lines \( B \) and \( C \), of the gauge couplings associated with the part of the gauge group which is not affected by turning on \( B \) and \( C \). For concreteness, take this to be the case for the \( E_8' \) in the hidden sector. Of the orbits discussed in section 5, there is one orbit which is the relevant one for the discussion of threshold corrections to \( g_{E_8'} \), namely the orbit \( 2nTm + q^T C q = 0 \). As also discussed in section 5, there are no finite points in the fundamental region of moduli space for which the massive states on this orbit might become massless and, hence, no additional threshold scales. Thus, for this case the only relevant threshold scale is given by \( M_X = M_{\text{string}} \). Inspection of (7.10) shows that the threshold corrections \( \Delta_{E_8'} \) should be given by

\[ \Delta_{E_8'} = -\alpha_{E_8'} \log \left| \eta(\bar{U}) \eta(T) \right|^4 \left( 1 - \frac{1}{2}BC \partial_U \log \eta^2(U) \partial_T \log \eta^2(T) \right)^{-2} \] (8.6)

where \( \alpha_{E_8'} = c(E_8') \) [8]. Inserting \( K = -\log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{C})(C + \bar{B}) \right) \) and (8.6) into (8.1) and (8.2) yields that

\[ \frac{1}{g_{E_8'}^2(p^2)} = \frac{S' + \bar{S}'}{2} + \frac{b_{E_8'}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} \]

\[ - \frac{\alpha_{E_8'}}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{C})(C + \bar{B}) \right) \]

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\[ - \frac{\alpha_{E_S}}{16\pi^2} \log \left( |\eta(U)\eta(T)|^4 \left| 1 - \frac{1}{2} BC \, \partial_U \log \eta^2(U) \, \partial_T \log \eta^2(T) \right|^2 \right) \]  

(8.7)

where \( b_{E_S} = -3c(E_S) \). Note that we have not yet taken into account the Green-Schwarz mechanism. The Green-Schwarz mechanism can remove an amount \( \delta_{GS} \) from the above \[6, 8\]. It might, in addition, also remove a modular invariant function \[56\], yielding

\[
\frac{1}{g_{E_S}^2(p^2)} = \frac{Y}{2} + \frac{b_{E_S}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} - \frac{(\alpha_{E_S} - \delta_{GS})}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2} (B + \bar{C})(C + \bar{B}) \right)
\]

\[
- \frac{(\alpha_{E_S} - \delta_{GS})}{16\pi^2} \log \left( |\eta(U)\eta(T)|^4 \left| 1 - \frac{1}{2} BC \, \partial_U \log \eta^2(U) \, \partial_T \log \eta^2(T) \right|^2 \right)
\]

(8.8)

where

\[ Y = S + S - \frac{\delta_{GS}}{8\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2} (B + \bar{C})(C + \bar{B}) \right) + (\text{modular inv. function}) \]

The effective gauge coupling \( [8, 8] \) has to be invariant under modular \( SL(2, \mathbb{Z})_{T,U} \) transformations. The quantity \( Y \) is invariant, when taking into account that the dilaton acquires a non-trivial transformation behaviour \( [8, 8] \) at the 1-loop level under \( SL(2, \mathbb{Z})_{T,U} \)-transformations, that is \( S \rightarrow S - \frac{1}{8\pi^2} \delta_{GS} \log(i\gamma U + \bar{\delta}) \) under \( (\text{8.14}) \), etc. Then it indeed follows that \( (8.8) \) is invariant under \( SL(2, \mathbb{Z})_{T,U} \)-transformations.

Next, let us look at the running of the gauge coupling constants in the observable sector. Consider a non-abelian factor \( G_a \) and assume that it gets broken down to \( \hat{G}_a = \otimes_i \hat{G}_{a,i} \) when turning on the Wilson moduli \( B \) and \( C \). Let us then discuss the running of the coupling \( \hat{g}_{a,i}(p^2) \) of one such subgroup \( \hat{G}_{a,i} \). In order to simplify the notation, we will, in the following, simply denote this subgroup \( \hat{G}_{a,i} \) by \( \hat{G}_a \) and its associated coupling constant \( \hat{g}_{a,i} \) by \( \hat{g}_a \). This time, the relevant orbit is the one for which \( n^T m = 0, q^2 = 1 \). Inspection of \( (\text{8.32}) \) shows that there are now 3 threshold scales in the presence of non-vanishing Wilson lines \( B \) and \( C \), namely \( M_X = M_{\text{string}}, M_I = |B + C|M_{\text{string}} \) and \( M_I' = |B - C|M_{\text{string}} \)

We will in the following set \( B = C \) for simplicity. Then the discussion simplifies and the remaining 2 threshold scales are given by \( M_X = M_{\text{string}}, M_I = |B + C|M_{\text{string}} \).

Furthermore, it follows from \( (\text{8.3}) \) that for \( B = C \)

\[ \hat{\alpha}_a K - 2 \sum_r T_a(\hat{r}) \log \det g_r = -\hat{\alpha}_a \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2} (B + \bar{B})^2 \right) \]

(8.9)

where \( \hat{\alpha}_a = -c(\hat{G}_a) + \sum_C \hat{T}_a(\hat{r}_C) (1 + 2n_C) \). Inspection of \( (\text{6.32}) \), on the other hand, shows

10 All moduli are taken to be dimensionless.
that the threshold corrections $\Delta_a$ should be given by

$$\Delta_a = -\alpha_a \log \left( |\eta(T)\eta(U)|^4 |1 - \frac{1}{2}B^2 \partial_U \log \eta^2(U)\partial_T \log \eta^2(T) + rB^2 \eta^4(T)\eta^4(U)|^{-2} \right)$$

(8.10)

where $r$ denotes an unknown coefficient which cannot be determined by symmetry considerations alone. $\alpha_a$ can either be written in terms of the gauge group $G_a$ as $\alpha_a = -c(G_a) + \sum_C T_a(r_C)(1 + 2n_C)$, or in terms of the unbroken gauge group $\hat{G}_a$ as $\alpha_a = -c(\hat{G}_a) - \sum_V \hat{T}_a(\hat{r}_V) + \sum_C \hat{T}_a(\hat{r}_C)(1 + 2n_C)$. Here the $n_C$ denote the modular weights of the light chiral superfields. Then, it follows from (8.3) and (8.4) that the effective gauge coupling associated with the unbroken subgroup $\hat{G}_a$ is given by

$$\frac{1}{g_a^2(p^2)} = \frac{S + \bar{S}}{2} + \frac{\hat{b}_a}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{\hat{b}_a - b_a}{16\pi^2} \log |B|^2$$

$$- \frac{\hat{\alpha}_a}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B}^2) \right)$$

$$- \frac{\alpha_a}{16\pi^2} \log \left( |\eta(T)\eta(U)|^4 |1 - \frac{1}{2}B^2 \partial_U \log \eta^2(U)\partial_T \log \eta^2(T) + rB^2 \eta^4(T)\eta^4(U)|^{-2} \right)$$

(8.11)

$r$ denotes a constant which cannot be determined by symmetry. Note that we have not yet taken into account the Green-Schwarz mechanism. The Green-Schwarz mechanism can, again, remove an amount $\delta_{\text{GS}}$ from the above $[7, 8]$ and possibly also a modular invariant function $[56]$, yielding

$$\frac{1}{g_a^2(p^2)} = \frac{Y}{2} + \frac{\hat{b}_a}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{\hat{b}_a - b_a}{16\pi^2} \log |B|^2$$

$$- \frac{(\hat{\alpha}_a - \delta_{\text{GS}})}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B}^2) \right)$$

$$- \frac{(\alpha_a - \delta_{\text{GS}})}{16\pi^2} \log \left( |\eta(T)\eta(U)|^4 |1 - \frac{1}{2}B^2 \partial_U \log \eta^2(U)\partial_T \log \eta^2(T) + rB^2 \eta^4(T)\eta^4(U)|^{-2} \right)$$

(8.12)

where $Y = S + \bar{S} - \frac{\delta_{\text{GS}}}{8\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B}^2) \right) + (\text{modular inv. function})$. The quantity $Y$ is invariant under $SL(2, \mathbb{Z})_{T,U}$-transformations, as discussed above. Requiring (8.12) also to be invariant under $SL(2, \mathbb{Z})_{T,U}$-transformations (8.13) leads to the following restriction on the spectrum

$$\hat{b}_a - b_a = \hat{\alpha}_a - \alpha_a$$

(8.13)
which can be rewritten as
\[ \sum_V \hat{T}_a(r_V) = -\sum_C n_C \hat{T}_a(r_C) + \sum_C n_C \hat{T}_a(r_C) \] (8.14)

Restrictions of a similar type on the spectrum where already considered in [55].

For completeness, let us consider the case of vanishing Wilson lines \( B = C = 0 \). Then, equation (8.12) turns into
\[ \frac{1}{g^2_a(p^2)} = \frac{Y}{2} + \frac{b_a}{16\pi^2} \log \frac{M^2_{\text{string}}}{p^2} \]
\[ -\frac{(\alpha_a - \delta_{GS})}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) \right) \]
\[ -\frac{(\alpha_a - \delta_{GS})}{16\pi^2} \log \left( |\eta(T)\eta(U)|^4 \right) \] (8.15)

where \( Y = S + \bar{S} - \frac{\delta_{GS}}{8\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) \right) + (\text{modular inv. function}) \). This describes the running of the gauge coupling associated with the unbroken group \( G_a \).

An example resembling the above situation is provided by a \( \mathbb{Z}_4 \)-orbifold of the type \( T_6 = T_2 \oplus T_4 \). In the (2, 2) case, the gauge group in the observable sector is given by \( G = E_6 \otimes SU(2) \otimes U(1) \). Now consider turning on complex Wilson lines \( B = C \) associated with the \( T_2 \). Then, the \( E_6 \otimes U(1) \) gets broken down to an \( SO(8) \otimes U(1)' \), whereas the \( SU(2) \) remains intact. At energies below \( M^2_I \), \( p^2 \ll M^2_I \), the running of the gauge coupling \( g_{SO(8)} \) is determined in terms of the truly massless representations of \( SO(8) \). At energies above \( M^2_I \), on the other hand, the running is determined in terms of the light representations of \( E_6 \).

Let us now consider a (2, 2)\( \mathbb{Z}_N \)-orbifold model with \( N = 2 \) spacetime supersymmetry. Let us again assume that the internal torus factorises as \( T_6 = T_2 \oplus T_4 \), and that the \( T_2 \) is not twisted under the action of the internal twist, thus leading to a model with \( N = 2 \) spacetime supersymmetry. Let us discuss the threshold corrections to the gauge couplings associated with the gauge group of the internal \( T_2 \). As discussed extensively in section [3], the orbit relevant to the discussion of the threshold corrections to these gauge couplings is given by \( n^T m = 1, q^2 = 0 \). Then, inspection of (6.13) shows that a different threshold scale \( M_I = |j(T) - j(U)|M_{\text{string}} \) arises in this context besides \( M_X = M_{\text{string}} \).

For points where \( T = U \) in the \( (T, U) \)-moduli space, the generic gauge group \( U(1) \otimes U(1) \) gets enlarged to \( SU(2) \otimes U(1) \), because 2 additional \( N = 2 \) vector multiplets become massless at these points. At generic values in the \( (T, U) \)-moduli space, the running of one such effective \( U(1) \)-gauge coupling should, in analogy to (8.11), be given by
\[ \frac{1}{\hat{g}^2_{U(1)}(p^2)} = \frac{1}{\hat{g}^2_{\text{tree}}} + \frac{\hat{b}_{U(1)}}{16\pi^2} \log \frac{M^2_{\text{string}}}{p^2} + \frac{(\hat{b}_{U(1)} - b_{U(1)})}{16\pi^2} \log |j(T) - j(U)|^2 \]
The gravitational coupling we will be considering in the following is the one associated with the light charged multiplets in the theory. Then, \( b_{U(1)} \) is given by \( b_{U(1)} = \hat{b}_{U(1)} + 2b_{\text{vec}}^{N=2} = 2b_{\text{vec}}^{N=2} \), where \( b_{\text{vec}}^{N=2} \) denotes the \( N = 2 \) \( \beta \)-function coefficient for one \( N = 2 \) vector multiplet. This is so, because at energies above the threshold \( M_I \) the 2 additional \( N = 2 \) vector multiplets are effectively massless. Finally, the coefficient \( \hat{\alpha}_{U(1)} \) is related to the \( N = 2 \) \( \beta \)-function coefficient \([1]\) as \( \hat{\alpha}_{U(1)} = \hat{b}_{U(1)} = 0 \). Note that (8.16) is manifestly invariant under modular \( SL(2,\mathbb{Z})_{T,U} \)-transformations\([7]\) and also that we have ignored a possible removal due to the Green-Schwarz mechanism.

Finally, at points where \( T = U \neq 1, \rho \) there is no threshold scale \( M_I \) anymore and the running of the effective \( SU(2) \) coupling is simply given by

\[
\frac{1}{g_{SU(2)}^2(p^2)} = \frac{1}{g_{\text{tree}}^2} + \frac{b_{SU(2)}^\text{SU(2)}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} - \frac{\alpha_{SU(2)}^\text{SU(2)}}{16\pi^2} \log(T + \bar{T})^2|\eta(T)|^8 \tag{8.17}
\]

Here, \( b_{SU(2)} = -2c(SU(2)) \) denotes the \( N = 2 \) \( \beta \)-function coefficient associated with the massless \( N = 2 \) \( SU(2) \)-vector multiplet in the theory. \( \alpha_{SU(2)} \) is again related to \( b_{SU(2)} \)\([4]\) by \( \alpha_{SU(2)} = b_{SU(2)} \).

Let us now turn to the discussion of moduli dependent threshold corrections to gravitational couplings in the context of \((0,2)\mathbb{Z}_N\)-orbifold theories. As in the gauge case, we will stick to those orbifolds for which the underlying \( T_6 \) torus factorises into \( T_6 = T_2 + T_4 \) and we will discuss the moduli dependent threshold corrections associated with this \( T_2 \).

The gravitational coupling we will be considering in the following is the one associated with a \( C^2 \) in the low energy effective action of \((0,2)\mathbb{Z}_N\)-compactifications of the heterotic string,\[ \mathcal{L} = -\frac{1}{2} \mathcal{R} + \frac{1}{4} g_{\text{grav}}^2 \mathcal{C}^2 + \frac{1}{4} \Theta_{\text{grav}} \mathcal{R}_{mnpq} \hat{\mathcal{R}}_{mnpq} + \frac{1}{\rho \sigma} \mathcal{R}_{mn}^2 + \frac{1}{\sigma^2} \mathcal{R}^2. \]

Here, \( \mathcal{C}^2 \) denotes the square of the Weyl tensor \( \mathcal{C}_{mnpq} \). The conventional choice\([52, 53, 54]\) for the tree-level couplings to quadratic gravitational curvature terms is taken to be the one where \( \frac{1}{g_{\text{grav}}^2} = -\frac{1}{2} \frac{1}{\rho^2} = \frac{3}{2} \frac{1}{\sigma^2} = \frac{s + s}{2} \). Then, in the gravitational sector, the dilaton only couples to the Gauss-Bonnet combination \( GB = C^2 - 2\mathcal{R}_{mn}^2 + \frac{2}{3} \mathcal{R}^2 \) at the tree-level,

\[ \mathcal{L} = -\frac{1}{2} \mathcal{R} + \frac{1}{4} \mathcal{R} \mathcal{S} \text{ } GB + \frac{1}{4} \Theta_{\text{grav}} \mathcal{R}_{mnpq} \hat{\mathcal{R}}_{mnpq}. \]

Let us first consider the running of the gravitational coupling \( g_{\text{grav}}^2 \) in the context of \((2,2)\mathbb{Z}_N\)-orbifold models with gauge group \( G \) \((B = C = 0)\). In analogy to the gauge case\[[8,16]\] as discussed in footnote\([4]\), we have ignored the issue of appearance of additional non covariant terms. This will be discussed in\([58]\). Note that such additional terms do not change the singular behaviour of \((8.16)\).
and ignoring a possible removal by the Green-Schwarz mechanism for the time being, the running is, in the presence of a threshold \( p^2 \ll M_f^2 \ll M_X^2 = M_{\text{string}}^2 \), given by

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{S + \bar{S}}{2} + \frac{\hat{b}_{\text{grav}}}{16\pi^2} \log \frac{M_f^2}{p^2} - \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}^2}{M_f^2} - \frac{\hat{\alpha}_{\text{grav}}}{16\pi^2} \log (T + \bar{T})(U + \bar{U}) - \frac{\alpha_{\text{grav}}}{16\pi^2} \log |\eta(T)\eta(U)|^4
\]

As already discussed, at the points in the \((T, U)\)-moduli space where \( T = U \), the gauge group occurring in the compactification sector of the model becomes enhanced to \( U(1) \) because an additional \( N = 2 \) vector multiplet becomes massless. Thus, the threshold scale \( M_f \) is to be identified with \( M_f = |j(T) - j(U)|M_{\text{string}} \), as discussed above. Equation (8.18) describes the running of \( g_{\text{grav}}^2 \) at generic points in the \((T, U)\)-moduli space. The 1-loop \( \beta \)-function coefficient \( \hat{b}_{\text{grav}} \) describes the running at low momenta \( p^2 \ll M_f^2 \) and is thus determined in terms of the truly massless modes in the theory. The 1-loop \( \beta \)-function coefficient \( b_{\text{grav}} \) describes the running between \( M_f \) and \( M_{\text{string}} \) and is thus determined in terms of all the light modes in the theory. Above the threshold \( M_f \) there is only one additional light multiplet around, namely the additional \( N = 2 \) vector multiplet. Decomposing this \( N = 2 \) vector multiplet into one \( N = 1 \) vector multiplet and one \( N = 1 \) chiral multiplet, it follows that \( b_{\text{grav}} = \hat{b}_{\text{grav}} = \delta_{\text{grav}}^V + \delta_{\text{grav}}^C \), where \( \delta_{\text{grav}}^V \) and \( \delta_{\text{grav}}^C \) denote the gravitational 1-loop \( \beta \)-function coefficient for an \( N = 1 \) vector and chiral multiplet, respectively. \( b_{\text{grav}} - \hat{b}_{\text{grav}} \) is also proportional to the trace anomaly coefficient of this additional \( N = 2 \) vector multiplet. The coefficient \( \hat{\alpha}_{\text{grav}} \) denotes the contribution from the truly massless modes due to non-vanishing Kähler and \( \sigma \)-model anomalies, and is given by \[ \hat{\alpha}_{\text{grav}} = \frac{1}{2\pi}(21 + 1 - \dim G + \gamma_M + \sum_C (1 + 2n_C)) \]

where the sum now runs over all the massless chiral matter/twisted moduli fields \( \hat{C} \) with modular weight \( n_C \), see equation (8.3). \( \gamma_M \) denotes the contribution from the untwisted modulinos. For the case that there is no Green-Schwarz mechanism for the \( T_2 \) under consideration, \( \hat{\alpha}_{\text{grav}} \) is also computed \[ 13 \] to be \( \hat{\alpha}_{\text{grav}} = \bar{b}_{\text{grav}}^N = b_{\text{grav}}^N \), where \( \bar{b}_{\text{grav}}^N = b_{\text{grav}}^N \) denotes the trace anomaly contribution to \( C^2 \) of all the truly massless \( N = 2 \) multiplets in the associated \( N = 2 \)-orbifold with gauge group \( \tilde{G} \) generated by \( \tilde{P}, T_m^m = -\frac{1}{16\pi^2} \bar{b}_{\text{grav}}^N C^2 \). This was checked explicitly for the example of a \( \mathbb{Z}_4 \)-orbifold in \[ 10 \]. Finally, the coefficient \( \alpha_{\text{grav}} \) denotes the contribution from all the massive states which decouple at \( M_{\text{string}} \) and is also given by \( \alpha_{\text{grav}} = \bar{b}_{\text{grav}}^N = b_{\text{grav}}^N \) \[ 13 \].

Inserting all of this into (8.18) yields

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{S + \bar{S}}{2} + \frac{\hat{b}_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} - \frac{(\delta_{\text{grav}}^V + \delta_{\text{grav}}^C)}{16\pi^2} \log |j(T) - j(U)|^2
\]
\[ - \frac{\hat{\alpha}_{\text{grav}}}{16\pi^2} \log(T + \bar{T})(U + \bar{U})|\eta(T)|^4 \] 

which is manifestly invariant under modular \( SL(2, \mathbb{Z})_{T,U} \)-transformations. If one also takes the Green-Schwarz mechanism into account, then (8.19) gets, in analogy to the gauge case, modified to

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{Y}{2} + \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} - \frac{(\delta^V_{\text{grav}} + \delta^C_{\text{grav}})}{16\pi^2} \log |j(T) - j(U)|^2 - \frac{(\hat{\alpha}_{\text{grav}} - \delta_{\text{GS}})}{16\pi^2} \log(T + \bar{T})(U + \bar{U})|\eta(T)|^4
\]

where \( Y = S + S - \frac{\delta_{\text{GS}}}{8\pi^2} \log((T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B})^2) + (\text{modular inv. function}) \), as pointed out by Kaplunovsky [56]. Note that (8.20) is again invariant under modular \( SL(2, \mathbb{Z}) \)-transformations.

The running of the effective gravitational coupling \( g_{\text{grav}} \) at the points where \( T = U \neq 1, \rho \) is, on the other hand, given as follows. Since \( T = U \), there is no additional threshold scale \( M_1 \), and thus it follows from (8.20) that

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{Y}{2} + \frac{b_{\text{grav}}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} - \frac{(\hat{\alpha}_{\text{grav}} + \hat{\alpha} - \delta_{\text{GS}})}{16\pi^2} \log(T + \bar{T})^2|\eta(T)|^8
\]

where now \( Y = S + S - \frac{\delta_{\text{GS}}}{8\pi^2} \log((T + \bar{T})^2 - \frac{1}{2}(B + \bar{B})^2) + (\text{modular inv. function}) \). Here, \( \hat{\alpha} \) denotes the contribution to the Kähler and \( \sigma \)-model anomalies due to the additional \( N = 2 \) vector multiplet which has become massless at \( T = U \neq 1, \rho \). Since a \( N = 2 \) vector multiplet can be decomposed into a \( N = 1 \) vector multiplet and into a \( N = 1 \) chiral multiplet \( C \), it follows that \( \hat{\alpha} = \frac{1}{24}(\hat{\alpha}_{\text{grav}} - \delta_{\text{GS}}) = -\frac{2}{24} \), which is precisely twice the trace anomaly of one \( N = 2 \) vector multiplet. Note that, in the associated \( N = 2 \) orbifold generated by \( \hat{P} \), the gauge group associated with the internal \( T_2 \) gets enhanced from \( U(1) \otimes U(1) \) to \( SU(2) \otimes U(1) \) at \( T = U \), resulting in 2 additional \( N = 2 \) vector multiplets. Thus, \( \hat{\alpha} \) is indeed equal to the additional trace anomaly contribution in the associated \( N = 2 \) orbifold generated by \( \hat{P} \). The coefficients \( b_{\text{grav}} \) and \( \hat{\alpha}_{\text{grav}} \) are the same as in the previous case of generic \( T \) and \( U \) values.

Next, consider turning on Wilson lines \( B \) and \( C \). Let us again set \( B = C \), for simplicity. This introduces a new threshold scale \( M'_1 = |B|M_{\text{string}} \) into the game, at which the gauge group \( G \) of the model gets broken down to some subgroup \( H \), thus changing the spectrum of the massless multiplets in the theory. Consider first the generic case where \( T \neq U \). The resulting changes to (8.19) read

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{S + S}{2} + \frac{\hat{b}_{\text{grav},H}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{(\hat{b}_{\text{grav},H} - b_{\text{grav},G})}{16\pi^2} \log |B|^2
\]

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If a possible Green-Schwarz removal is taken into account, then (8.22) turns into

\begin{align}
- \frac{(\delta \gamma_{\text{grav}} + \delta C_{\text{grav}})}{16\pi^2} \log |j(T) - j(U)| \\
+ \frac{1}{2} B^2 \left( \partial_U \log \eta^2(U) \ j'(T) - \partial_T \log \eta^2(T) \ j'(U) \right)^2 \\
- \frac{\hat{\alpha}_{\text{grav},H}}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B})^2 \right) \\
- \frac{\alpha_{\text{grav},G}}{16\pi^2} \log |\eta(U)\eta(T)|^4 |1 - \frac{1}{2} B^2 \partial_U \log \eta^2(U) \partial_T \log \eta^2(T) \\
+ \ r \ B^2 \eta^4(T) \eta^4(U) |^{-2} \\
\end{align}

(8.22)

The coefficient \( \hat{b}_{\text{grav},H} \) describes the running of the coupling constant at low energies and is thus determined entirely by the truly massless modes of the theory with gauge group \( H \). Above the threshold \( M' \) the running is described by \( b_{\text{grav},G} \), which is now determined by all the light modes in the theory. \( \hat{\alpha}_{\text{grav},H} \) denotes the contribution from the truly massless fields to the Kähler and \( \sigma \)-model anomalies, and should thus be given by \( \hat{\alpha}_{\text{grav},H} = \frac{1}{2\pi}(21 + 1 - \dim H + \gamma_M + \Sigma_C(1 + 2n_C)) \), where the sum goes over all the truly massless chiral matter/twisted moduli multiplets. Finally, \( \alpha_{\text{grav},G} \) describes the contribution of all the massive states which decouple at \( M_{\text{string}} \) and should thus still be given as before by \( \alpha_{\text{grav},G} = \hat{b}_{\text{grav}}^{N=2} \). As explained in section 6, \( r \) denotes a constant which cannot be determined by symmetry arguments. Note that all of the orbits discussed in section 5 are relevant for the gravitational case.

If a possible Green-Schwarz removal is taken into account, then (8.22) turns into

\begin{align}
\frac{1}{g_{\text{grav}}^2(p^2)} &= \frac{Y}{2} + \frac{\hat{b}_{\text{grav},H}}{16\pi^2} \log M_{\text{string}}^2 \frac{p^2}{p^2} + \frac{(\hat{b}_{\text{grav},H} - b_{\text{grav},G})}{16\pi^2} \log |B|^2 \\
- \frac{(\delta \gamma_{\text{grav}} + \delta C_{\text{grav}})}{16\pi^2} \log |j(T) - j(U)| \\
+ \frac{1}{2} B^2 \left( \partial_U \log \eta^2(U) \ j'(T) - \partial_T \log \eta^2(T) \ j'(U) \right)^2 \\
- \frac{(\hat{\alpha}_{\text{grav},H} - \delta_{GS})}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B})^2 \right) \\
- \frac{(\alpha_{\text{grav},G} - \delta_{GS})}{16\pi^2} \log |\eta(U)\eta(T)|^4 |1 - \frac{1}{2} B^2 \partial_U \log \eta^2(U) \partial_T \log \eta^2(T) \\
+ \ r \ B^2 \eta^4(T) \eta^4(U) |^{-2} \\
\end{align}

(8.23)

where \( Y = S + \hat{S} - \frac{\delta_{GS}}{8\pi^2} \log((T + \bar{T})(U + \bar{U}) - \frac{1}{2}(B + \bar{B})^2) + (\text{modular inv. function}) \) [56].

The effective coupling (8.23) should be invariant under modular \( SL(2, \mathbf{Z})_{T,U} \) transformations, which leads to the requirement that

\begin{align}
\hat{b}_{\text{grav},H} - b_{\text{grav},G} &= \hat{\alpha}_{\text{grav},H} - \alpha_{\text{grav},G} \\
\end{align}

(8.24)
Finally, let us discuss the running in the presence of a Wilson line $B$ for the case where $T = U \neq 1, \rho$. When taking into account the Green-Schwarz mechanism, the running (8.21) is modified to

\[
\frac{1}{g_{\text{grav}}^2(p^2)} = \frac{Y}{2} + \frac{(\hat{b}_{\text{grav},H} + \hat{b}_{\text{grav},G})}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{(\hat{b}_{\text{grav},H} - \hat{b}_{\text{grav},G})}{16\pi^2} \log |B|^2 \\
- \frac{(\tilde{\alpha}_{\text{grav},H} + \tilde{\alpha} - \delta_{\text{GS}})}{16\pi^2} \log \left( (T + \bar{T})^2 - \frac{1}{2} (B + \bar{B})^2 \right) \\
- \frac{(\tilde{\alpha}_{\text{grav},G} + \tilde{\alpha} - \delta_{\text{GS}})}{16\pi^2} \log |\eta(T)|^8 \left[ 1 - \frac{1}{2} B^2 (\partial_T \log \eta(T))^2 + \rho B^2 \eta^4(T) \right]^{-2}
\]

(8.25)

where $Y = S + \bar{S} - \frac{\delta_{\text{GS}}}{8\pi^2} \log((T + \bar{T})^2 - \frac{1}{2}(B + \bar{B})^2) + (\text{modular inv. function})$. The coefficients are all given in the above discussion.

9 Conclusions

In this paper we have computed string threshold corrections from the massive string spectrum for Abelian orbifold compactifications. Our discussion contains two main results which were not obtained before in the literature. First, we derive from the massive spectrum the dependence of the threshold functions on the continuous Wilson line moduli at least for small values of these fields. Second, we derive the threshold functions also for those gauge groups whose massless charged spectrum is enlarged at certain points in the orbifold moduli spaces. In terms of target space free energies these thresholds correspond to summation orbits which were neglected in previous considerations. This discussion includes the stringy Higgs effect, i.e. that a gauge group gets spontaneously broken to some subgroup when the moduli take values away from the critical points in the moduli space. Then the corresponding threshold functions possess logarithmic singularities at the critical points in $\mathcal{M}$. This also happens for the gravitational threshold corrections since the whole spectrum contributes to the loop corrections of the gravitational coupling constants. We have shown that if the appearance of the additional massless states is related to the six-dimensional orbifold, the correct singularity structure of the threshold functions is provided by the absolute modular invariant function $j$. (This result was already anticipated in [28] in quite general terms.) For the case that the enlargement of the massive spectrum is associated with the Wilson line fields, one encounters logarithmic singularities in these fields. (The leading Wilson line singularities in the threshold functions were discussed in [53, 59] by using renormalization group arguments.)
Our results can be applied in various ways. First, the complete moduli dependence of the threshold functions is relevant for several soft supersymmetry breaking parameters after breaking of local $N = 1$ supersymmetry. In particular, the Wilson line dependence is important for the $\mu$-problem [17, 30, 31]. Second one can use the threshold corrections for the non-perturbative gaugino condensation mechanism in the hidden gauge sector. Specifically, if the hidden gauge group is spontaneously broken by Wilson line moduli, the effective superpotential depends on these fields; thus one can determine the non-perturbatively fixed vacuum expectation values of these fields. Even more interesting effects may happen if the discontinuities in the spectrum related to the six-dimensional orbifold enter the non-perturbative superpotential. This might be the case when considering non-perturbative superpotential by gravitational instantons [62], since in this case the gravitational threshold corrections are most likely to be relevant for the non-perturbative dynamics. Then the non-perturbative superpotential will contain also the absolute modular invariant function $j$, and the vacuum structure of the moduli depends on the singularities of the threshold functions in an interesting way [28]. Finally, these types of superpotentials are also important for the formation of domain walls and other topological objects, as discussed in [63, 64].

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| Class | Twist | Order | Eigenvalues | Minimal GG | Maximal GG |
|-------|-------|-------|-------------|------------|------------|
| 1     | Ø     | 1     | [0]         | $E_8$      | $E_8$      |
| 2     | $A_1$ | 4*    | [2]         | $E_7$      | $E_7U_1$   |
| 3     | $A_1^2$ | 4*    | [2, 2]      | $D_6$      | $D_7U_1$   |
| 4     | $A_2$ | 3     | [1, 2]      | $E_6$      | $E_7U_1$   |
| 5     | $A_1^3$ | 4*    | [2, 2, 2]   | $D_4A_1$   | $E_6A_1U_1$ |
| 6     | $A_2A_1$ | 12*   | [4, 8, 6]   | $A_5$      | $D_6U_1^2$ |
| 7     | $A_3$ | 8*    | [2, 6, 4]   | $D_5$      | $D_6U_1^2$ |
| 8     | $A_1^{4,I}$ | 2    | [1, 1, 1]   | $D_4$      | $E_7A_1$   |
| 9     | $A_1^{4,II}$ | 4*   | [2, 2, 2]   | $A_1^4$    | $A_7U_1$   |
| 10    | $A_2A_1^2$ | 12*   | [4, 8, 6, 6] | $A_3U_1$   | $D_5A_1U_1^2$ |
| 11    | $A_2A_2$ | 3     | [1, 2, 1, 2]| $A_2A_2$   | $D_7U_1$   |
| 12    | $A_3A_1$ | 8*    | [2, 6, 4, 4]| $A_3A_1$   | $D_4A_2U_1^2$ |
| 14    | $D_4$ | 6     | [1, 5, 3, 3]| $D_4$      | $E_6U_1^2$ |
| 15    | $D_4(a_1)$ | 4    | [1, 3, 1, 3]| $D_4$      | $E_6A_1U_1$ |
| 16    | $A_1^5$ | 4*    | [2, 2, 2, 2]| $A_1^3$    | $D_6A_1U_1$ |
| 17    | $A_2A_1^3$ | 12*   | [4, 8, 6, 6, 6]| $A_1U_1^2$ | $A_5A_1U_1^2$ |
| 18    | $A_2^2A_1$ | 12*   | [4, 8, 4, 8, 6]| $A_2U_1$ | $D_5A_1U_1^2$ |
| 19    | $(A_3A_1^2)^I$ | 8*   | [2, 6, 4, 4, 4]| $A_3$ | $A_4A_1^2U_1^2$ |
| 20    | $(A_3A_1^2)^{II}$ | 8* | [2, 6, 4, 4, 4]| $A_1^2U_1$ | $D_5A_1^2U_1$ |
| 23    | $A_5$ | 12*   | [2, 10, 4, 8, 6]| $A_2A_1$ | $D_4A_1U_1^3$ |

Table 1: Table of unbroken gauge groups for vanishing and for generic Wilson lines. The notation is explained in the text. The data are taken from references [38,39,40].
| Class | Twist | Order | Eigenvalues | Minimal GG | Maximal GG |
|-------|-------|-------|-------------|------------|------------|
| 24    | $D_4A_1$ | 12*  | $[2, 10, 6, 6, 6]$ | $A_1^3$  | $A_5U_1^3$ |
| 25    | $D_4(a_1)A_1$ | 4 | $[1, 3, 1, 3, 2]$ | $A_1^3$  | $A_7U_1$ |
| 26    | $D_5$ | 8 | $[1, 7, 3, 5, 4]$ | $A_3$  | $D_5U_1^3$ |
| 28    | $A_1^6$ | 4*  | $[2, 2, 2, 2, 2, 2]$ | $A_1^2$  | $D_5A_3$ |
| 29    | $A_2A_1^4$ | 6 | $[2, 4, 3, 3, 3, 3]$ | $U_1^2$  | $D_6A_1U_1$ |
| 30    | $A_2^2A_1^2$ | 12*  | $[4, 8, 4, 8, 6, 6]$ | $U_1^2$  | $A_3^2U_2^2$ |
| 31    | $A_2^3$ | 3 | $[1, 2, 1, 2, 1, 2]$ | $A_2$  | $E_6A_2$ |
| 32    | $A_3A_1^3$ | 8*  | $[2, 6, 4, 4, 4, 4]$ | $A_1U_1$  | $A_3A_2A_1^2U_1$ |
| 34    | $(A_2^3)^I$ | 4 | $[1, 3, 1, 3, 2, 2]$ | $A_1^2$  | $D_6A_1U_1$ |
| 35    | $(A_3^2)^II$ | 8*  | $[2, 6, 2, 6, 4, 4]$ | $U_1^2$  | $A_3^2U_1^2$ |
| 38    | $(A_5A_1)^I$ | 6 | $[1, 5, 2, 4, 3, 3]$ | $A_2$  | $D_5A_2^2U_1$ |
| 39    | $(A_5A_1)^{II}$ | 12*  | $[2, 10, 4, 8, 6, 6]$ | $A_1U_1$  | $A_3A_2U_1^3$ |
| 40    | $A_6$ | 7 | $[1, 6, 2, 5, 3, 4]$ | $A_1U_1$  | $D_4A_2U_1^2$ |
| 41    | $D_4A_1^2$ | 12*  | $[2, 10, 6, 6, 6, 6]$ | $A_1^2$  | $D_4A_2^2U_1^2$ |
| 42    | $D_4A_2$ | 6 | $[1, 5, 3, 3, 2, 4]$ | $U_1^2$  | $A_6U_1^2$ |
| 43    | $D_4(a_1)A_2$ | 12 | $[3, 9, 3, 9, 4, 8]$ | $U_1^2$  | $A_5A_1U_1^2$ |
| 44    | $D_5A_1$ | 8 | $[1, 7, 3, 5, 4, 4]$ | $A_1U_1$  | $A_4A_2U_1^2$ |
| 47    | $D_6(a_1)$ | 8 | $[1, 7, 2, 6, 3, 5]$ | $A_1^2$  | $A_5U_1^3$ |
| 48    | $D_6(a_2)$ | 12*  | $[2, 10, 2, 10, 6, 6]$ | $A_1^2$  | $A_3A_2U_1^3$ |
| 49    | $E_6$ | 12 | $[1, 11, 4, 8, 5, 7]$ | $A_2$  | $D_4U_1^4$ |

Table 2: Table of unbroken gauge groups for vanishing and for generic Wilson lines (continued).
| Class | Twist       | Order | Eigenvalues     | Minimal GG | Maximal GG         |
|-------|-------------|-------|-----------------|------------|--------------------|
| 51    | $E_6(a_2)$  | 6     | $[1, 5, 1, 5, 2, 4]$ | $A_2$      | $D_5 A_1 U_1^2$   |
| 52    | $A_1^7$     | 4*    | $[2, 2, 2, 2, 2, 2]$ | $A_1$      | $A_7 A_1$         |
| 53    | $A_3^3 A_1^1$ | 12*  | $[4, 8, 4, 8, 4, 8, 6]$ | $U_1$      | $A_5 A_2 U_1$     |
| 54    | $A_3 A_1^4$ | 8*    | $[2, 6, 4, 4, 4, 4]$  | $U_1$      | $D_4 A_3 U_1$     |
| 56    | $A_3^2 A_1$ | 4     | $[1, 3, 1, 3, 2, 2, 2]$ | $A_1$      | $D_5 A_3$         |
| 59    | $A_5 A_1^2$ | 12*   | $[2, 10, 4, 8, 6, 6, 6]$ | $U_1$      | $A_3 A_1^2 U_1^2$ |
| 60    | $A_5 A_2$   | 12*   | $[2, 10, 4, 8, 6, 4, 8]$ | $A_1$      | $A_2^3 A_1 U_1$   |
| 62    | $(A_7)^f$   | 8     | $[1, 7, 2, 6, 3, 5, 4]$ | $A_1$      | $D_4 A_1^2 U_1^2$ |
| 64    | $D_4 A_1^3$ | 12*   | $[2, 10, 6, 6, 6, 6]$  | $A_1$      | $A_3^2 A_1 U_1$   |
| 66    | $D_4(a_1) A_3$ | 8*   | $[2, 6, 2, 6, 2, 6, 4]$ | $U_1$      | $A_3 A_2 A_1^2 U_1$ |
| 67    | $D_5 A_1^2$ | 8     | $[1, 7, 3, 5, 4, 4]$  | $U_1$      | $A_5 A_1 U_1^2$   |
| 71    | $D_6(a_2) A_1$ | 12* | $[2, 10, 2, 10, 6, 6, 6]$ | $A_1$      | $A_3 A_1^2 U_1^2$ |
| 72    | $E_6 A_1$   | 12    | $[1, 11, 4, 8, 5, 7, 6]$ | $U_1$      | $A_2^3 A_1 U_1^3$ |
| 74    | $E_6(a_2) A_1$ | 12* | $[2, 10, 2, 10, 4, 8, 6]$ | $U_1$      | $A_3 A_1^2 U_1^3$ |
| 80    | $E_7(a_2)$  | 12    | $[1, 11, 2, 10, 5, 7, 6]$ | $A_1$      | $A_3 A_1 U_1^4$   |
| 83    | $A_1^8$     | 2     | $[1, 1, 1, 1, 1, 1, 1, 1]$ | $-$        | $D_8$             |
| 84    | $A_2^4$     | 3     | $[1, 2, 1, 2, 1, 2, 1, 2]$ | $-$        | $A_8$             |
| 85    | $A_3^3 A_1^2$ | 4    | $[1, 3, 1, 3, 2, 2, 2, 2]$ | $-$        | $A_7 A_1$         |
| 87    | $A_5 A_2 A_1$ | 6    | $[1, 5, 2, 4, 3, 2, 4, 3]$ | $-$        | $A_5 A_2 A_1$     |
| 88    | $A_7 A_1$   | 8     | $[1, 7, 2, 6, 3, 5, 4, 4]$ | $-$        | $A_3^2 A_1 U_1$   |

Table 3: Table of unbroken gauge groups for vanishing and for generic Wilson lines (continued).
| Class | Twist | Order | Eigenvalues | Minimal GG | Maximal GG |
|-------|-------|-------|-------------|------------|------------|
| 90    | $D_4 A_1^4$ | 6     | $[1, 5, 3, 3, 3, 3, 3, 3]$ | –          | $A_7 U_1$  |
| 91    | $D_1^2$   | 6     | $[1, 5, 3, 3, 1, 5, 3, 3]$ | –          | $D_4 A_3 U_1$ |
| 92    | $D_4(a_1)^2$ | 4     | $[1, 3, 1, 3, 1, 3, 1, 3]$ | –          | $D_5 A_3$  |
| 93    | $D_5(a_1) A_3$ | 12    | $[2, 10, 3, 9, 6, 3, 9, 6]$ | –          | $A_2^3 A_1 U_1$ |
| 96    | $D_8(a_1)$ | 12    | $[1, 11, 3, 9, 3, 9, 5, 7]$ | –          | $A_2 A_1^4 U_1^2$ |
| 98    | $D_8(a_3)$ | 8     | $[1, 11, 3, 9, 3, 9, 5, 7]$ | –          | $A_3 A_2 A_1^2 U_1$ |
| 99    | $E_6 A_2$ | 12    | $[1, 11, 4, 8, 5, 7, 4, 8]$ | –          | $A_3^2 U_1^2$ |
| 100   | $E_6(a_2) A_2$ | 6     | $[1, 5, 1, 5, 2, 4, 2, 4]$ | –          | $A_5 A_2 U_1$ |
| 102   | $E_7(a_2) A_1$ | 12    | $[1, 11, 2, 10, 5, 7, 6, 6]$ | –          | $A_3 A_1^2 U_1^3$ |
| 103   | $E_7(a_4) A_1$ | 6     | $[1, 5, 1, 5, 1, 5, 3, 3]$ | –          | $A_5 A_1^2 U_1$ |
| 107   | $E_8(a_3)$ | 12    | $[1, 11, 1, 11, 5, 7, 5, 7]$ | –          | $A_2 A_1^3 U_1^3$ |
| 111   | $E_8(a_7)$ | 12    | $[1, 11, 2, 10, 2, 10, 5, 7]$ | –          | $A_2^2 A_1 U_1^3$ |
| 112   | $E_8(a_8)$ | 6     | $[1, 5, 1, 5, 1, 5, 1, 5]$ | –          | $A_4 A_3 U_1$ |

Table 4: Table of unbroken gauge groups for vanishing and for generic Wilson lines (continued).