Integrability vs exact solvability in the quantum Rabi and Dicke models

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The Rabi model describes the simplest interaction between light and matter via a two-level quantum system interacting with a bosonic field. We find that the fully quantised version of the Rabi model is integrable in the Yang-Baxter sense at two parameter values. The model does not appear to be Yang-Baxter integrable in general. This is in contrast to the claim that the quantum Rabi model is integrable based on a phenomenological criterion of quantum integrability not presupposing the existence of a set of commuting operators. Similar Yang-Baxter integrable points are identified for the generalised Rabi model and the fully quantised Dicke model.

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Integrability is arguably the most powerful concept in the mathematical description of physical systems. A classical system is defined to be integrable when the number of degrees of freedom is smaller than the number of independent constants of the motion \cite{Braak}. However, for quantum systems, the definition of integrability is not so clear \cite{Kadomtsev, Perelomov}. Among the various definitions, the concept of Yang-Baxter integrability \cite{Johanna, Faddeev} seems most appropriate for (1+1)-dimensional quantum systems. Solutions of the Yang-Baxter relation along with associated monodromy matrices allow the construction of integrable models and their conserved charges \cite{Sklyanin}. Such Yang-Baxter integrable (YBI) models are synonymous with the term exactly solvable models \cite{Yang-Baxter}.

A general criterion of quantum integrability, inspired by the classically integrable hydrogen atom and not requiring the existence of a set of commuting operators, has been proposed in the context of the quantum Rabi model \cite{Braak}. The Rabi model \cite{Rabi} describes a two-level quantum system interacting with a bosonic field. It models the simplest interaction between light and matter and is thus a fundamental model in quantum physics. Applications include the interaction between light and trapped ions or quantum dots \cite{Heinzen}, and between microwaves and superconducting qubits \cite{Gambetta}. The Rabi model is applicable to both cavity QED and circuit QED.

Despite its simplicity, the fully quantised version of the Rabi model was solved only recently and claimed to be integrable \cite{Braak}. Briefly stated, Braak’s criterion of quantum integrability, involving \( f_1 \) discrete and \( f_2 \) continuous degrees of freedom, is that integrability is equivalent to the existence of \( f = f_1 + f_2 \) “quantum numbers” to classify eigenstates uniquely. For the Rabi model \( f_1 = f_2 = 1 \), giving \( f = 2 \) which is the same dimension as the global label (parity) used to uniquely label the eigenstates. The quantum Rabi model is thus deemed to be integrable.

A natural question arises: if the quantum Rabi model is integrable – is it YBI? Here we show that the quantum Rabi model is YBI at two distinct parameter values. Significantly, the Rabi model does not appear to be YBI in general. This raises a question with regard to the utility of Braak’s criterion of quantum integrability, which we further discuss here. We also identify corresponding YBI points in the Dicke model \cite{Dicke}, which is the extension of the Rabi model to \( N \) qubits, a model also of fundamental interest. For \( N = 2 \) the Dicke model constitutes the simplest model of the universal quantum gate \cite{Braak}. It may also be possible to realise the \( N = 3 \) Dicke model within circuit QED \cite{CircuitQED}. Most recently an analog-digital quantum simulation for all parameter regimes of the quantum Rabi and Dicke models has been proposed using circuit QED \cite{CircuitQED}. The Dicke model is also of interest for large \( N \) where it exhibits a phase transition to a super-radiant state for strong coupling \cite{Superradiance}.

The quantum Rabi model. The Hamiltonian of the fully quantised version of the Rabi model (with \( \hbar = 1 \)) is

\[
H = \Delta s^z + \omega a^\dagger a + g s^x (a + a^\dagger),
\]

where \( s^x \) and \( s^z \) are spin-\( \frac{1}{2} \) matrices for the two-level system with level splitting \( \Delta \). \( a^\dagger \) (\( a \)) denote creation (destruction) operators for a single bosonic mode with \( [a, a^\dagger] = 1 \) and frequency \( \omega \). \( g \) is the coupling between the two systems. The quantum Rabi model has \( Z_2 \) symmetry (parity).

Using the representation of the bosonic operators in the Bargmann space of analytic functions, the regular eigenvalues of the quantum Rabi model were shown to be given in terms of the zeros of a function \( G_\pm(x) \) \cite{Judd, Judd2}. Simple poles of \( G_\pm(x) \) at \( x = 0, \omega, 2\omega, \ldots \) correspond to the eigenvalues of the uncoupled bosonic modes. We will call models with solutions of this type Braak solvable. The conditions proposed by Braak are a type of sufficiency condition for determining the regular solutions. They also include the exceptional eigenvalues, which are the well known Juddian isolated exact solutions \cite{Judd}. Symmetric, anti-symmetric and asymmetric solutions for the eigenstates are given in terms of con-
fluential Heun functions \cite{23,24}, which involve an infinite number of terms. The isolated exact solutions appear naturally as truncations of the confluent Heun functions.

The rotating wave approximation (RWA) was used to treat the fully quantised version of the Rabi model \cite{11} in the form

$$H_{JC} = \Delta s^z + \omega a^\dagger a + g(s^+a + s^-a^\dagger),$$

with $s^\pm = s^\pm \pm i s^\theta$. This is the Jaynes-Cummings (JC) model \cite{12}. The conditions of near resonance $\Delta \approx \omega$ and weak coupling $g \ll \omega$ for the RWA apply in many experimental settings. The JC model is YBI \cite{26,27}. The excitation number operator $M = a^\dagger a + s^2$ and the Casimir operator $s^2 = s^+s^- + s^z(s^z - 1)$ commute with hamiltonian \cite{28}, i.e., $[H_{JC},M] = [H_{JC},s^2] = 0$.

Returning to the Rabi model, we begin by rewriting the hamiltonian (1) in the form

$$H_R = 2\Delta s^z + \omega a^\dagger a + g(s^+s^-)(a + a^\dagger),$$

entailing a harmless redefinition of the system parameters. We find that the model is YBI for the two cases (i) $\Delta = 0$ and (ii) $\omega = 0$. The case $\Delta = 0$ is known as the degenerate atomic limit. The case $\omega = 0$ is not included in Braak’s solution \cite{10,21}.

In both cases, there is an extra conserved quantity $C$, i.e., $[H_R,C] = 0$. For $\Delta = 0$, $C = s^+s^- = 1$ and for $\omega = 0$, $C = a^\dagger a + a$. To establish Yang-Baxter integrability, the key idea we introduce is an operator-valued twist, which in this setting yields a “trivial” twist solution to the Yang-Baxter relation.

First consider the case $\Delta = 0$. We construct the transfer matrix operator $\tau(u) = \text{tr} T(u)$, where the monodromy matrix $T(u) = W^a L^s(u)$ is a combination of the spin operator-valued “twist”

$$W^a = \begin{bmatrix} 1 & s^+ + s^- \\ s^+ + s^- & -1 \end{bmatrix},$$

and the bosonic $L$-operator \cite{27}

$$L^a(u) = \begin{bmatrix} 1 + \eta u + \eta^2 N & \eta a \\ \eta a^\dagger & 1 \end{bmatrix},$$

where $\eta$ is a free parameter and $N = a^\dagger a$. The elements of $T(u)$ can then be shown to satisfy the intertwining relation \cite{26,27}

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v),$$

with the (standard) $R$-matrix

$$R_{12}(u) = \begin{bmatrix} u + \eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u + \eta \end{bmatrix},$$

satisfying the Yang-Baxter relation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

Thus

$$\tau(u) = \eta[u + \eta N + (s^+ + s^-)(a^\dagger + a)] = \eta [u + g^{-1}H_R],$$

where we have identified $\eta = \omega/g$.

For the case $\omega = 0$ the monodromy matrix is of the form $T(u) = W^a L^s(u)$ with the bosonic operator-valued twist

$$W^a = \begin{bmatrix} 1 + \lambda & a + a^\dagger \\ a + a^\dagger & 1 - \lambda \end{bmatrix},$$

where $\lambda = \Delta/g$ and the spin $L$-operator is \cite{26,27}

$$L^s(u) = \begin{bmatrix} u + \eta g s^z & \eta s^- \\ \eta s^+ & u - \eta g s^z \end{bmatrix},$$

where $\eta$ is a free parameter. $T(u)$ can also be shown to satisfy the intertwining relation \cite{10} with the $R$-matrix \cite{27}. Here

$$\tau(u) = 2u + \eta g^{-1}H_R.$$
This is an obvious generalisation of the Rabi hamiltonian to N two-level qubits. The quantum Rabi model with two qubits and the N = 3 Dicke model have also been shown to be Braak solvable. According to Braak’s criterion, the Dicke model is non-integrable for all N ≥ 2.

For this model, applying the RWA leads to the Tavis-Cummings model, with Hamiltonian

\[ H_{TC} = \Delta S_z + \omega a_{\dagger} a + g (S^+ a + S^- a_{\dagger}). \]  

For this model the operators \( M = a_{\dagger} a + S_z \) and \( S^2 = S^+ S^- + S^z(S^z - 1) \) commute with \( H_{TC} \). The TC model reduces to the JC model for \( N = 1 \). It is YBI for general \( N \) and can be solved by the algebraic Bethe Ansatz.

Our results for the Rabi model can be generalised to the Dicke model. The Dicke model is also YBI for the two cases \( \Delta = 0 \) and \( \omega = 0 \). For \( \Delta = 0 \), the conserved quantity is \( C = S^+ S^- \), while for \( \omega = 0 \), \( C = a_{\dagger} a \).

In terms of the spin operators defined in equation (14) the monodromy matrix for \( \Delta = 0 \) is \( T(u) = W^S L^a(u) \), where

\[
W^S = \begin{bmatrix} 1 & S^+ + S^- \\ S^+ + S^- & -1 \end{bmatrix},
\]  

and \( L^a(u) \) is the same bosonic operator. The intertwining relation is satisfied with the R-matrix. Here again \( \eta = \omega/g \) and \( N = a_{\dagger} a \), with now

\[
\tau(u) = \eta u + g^{-1} H_D.
\]  

Thus \( \tau(0) = \eta g^{-1} H_D \).

The monodromy matrix for \( \omega = 0 \) is \( T(u) = W^a L^S(u) \), where the bosonic twist operator \( W^a \) is given in equation with \( \lambda = \Delta/g \). Now

\[
L^S(u) = \begin{bmatrix} u + \eta S^z & \eta S^- \\ \eta S^+ & u - \eta S^z \end{bmatrix},
\]  

with \( \eta \) a free parameter. The intertwining relation is satisfied with the R-matrix. In this case an alternative factorised form of the monodromy matrix is

\[
T(u) = W^a \prod_{j=1}^N L^a_j(u),
\]  

where the spin operator \( L^a_j(u) \) is defined in an obvious way from the single spin term (11) for each site \( j \). In this case the operator \( \tau(u) = \text{tr} T(u) \) is a polynomial of degree \( N \), with

\[
\tau(u) = 2u^N + \eta g^{-1} u^{N-1} H_D + \ldots.
\]  

For both cases \( \tau(u), \tau(v) = 0 \), by construction.

The parameter value \( \omega = 0 \) of the Dicke model has been identified as YBI using another approach. In particular, a Bethe Ansatz solution has been obtained from the elliptic Gaudin model through a limiting procedure.

**Generalised Rabi model.** Our approach using operator-valued twists may be applied to other models. Consider the generalised Rabi model

\[
H_\epsilon = 2\Delta s^2 + \omega a_{\dagger} a + g (s^+ s^-)(a + a_{\dagger}) + \epsilon s^z,
\]  

where the additional term \( \epsilon s^z \) allows tunnelling between the two atomic states. It breaks the parity symmetry. This model is relevant to the description of hybrid mechanical systems and is referred to as the driven Rabi model. It is also Braak solvable, with eigenstates given in terms of Heun functions.

The generalised Rabi model is considered to be non-integrable. The argument is that \( H_\epsilon \) has no discrete symmetry and there is only one quantum number (energy) corresponding to the sole conserved quantity. Since the number of degrees of freedom exceeds one the model does not satisfy Braak’s criterion of quantum integrability. It is thus claimed to be the first example of a non-integrable but exactly solvable system. However, one can also construct YBI points for this model at the parameter values \( \Delta = 0 \) and \( \omega = 0 \). To do this we need only extend the operator-valued twist matrices.

For the case \( \Delta = 0 \), we define the monodromy matrix \( T(u) = W^a L^a(u) \), where

\[
W^a = \begin{bmatrix} 1 & 2s^z \\ 2s^z & -1 + b s^z \end{bmatrix},
\]  

with \( L^a(u) \) as defined in (15). Here we have the commuting operator \( \tau(u) = \eta u + g^{-1} H_\epsilon \), with \( \eta = \omega/g \) and \( b = \epsilon g \). Similarly for \( \omega = 0 \), we take \( T(u) = W^a L^a(u) \), with

\[
W^a = \begin{bmatrix} 1 + \lambda & a + a_{\dagger} + c \\ a + a_{\dagger} + c & 1 - \lambda \end{bmatrix},
\]  

and \( L^a(u) \) as defined in (11). Here \( \tau(u) = 2u + \eta g^{-1} H_\epsilon \), with \( \lambda = \Delta/g \) and \( 2c = \epsilon g \).
The situation for the Dicke model integrable point appears analogous to the Heisenberg spin 

The existence of YBI points in these models has various implications. The result [20] following from the monodromy matrix [19] for the value $\omega = 0$ leads to the construction of a series of higher order conserved quantities with increasing $N$. The presence of these conserved quantities implies level crossing, therefore the energy level statistics at the Dicke model integrable point should follow Poissonian level statistics as anticipated by the Berry-Tabor criterion [34]. Away from an integrable point the energy level statistics should become more apparent at the Dicke model integrable point with increasing parameter values, $\Delta = 0$ and $\omega = 0$, the more general Dicke [13] and Rabi [21] models are YBI. The underlying $R$-matrix is seen to be a common feature of integrability for these and related models [26, 27].

Our results bring into question the usefulness and validity of Braak’s phenomenological criterion of quantum integrability [10]. The Rabi model only appears to be YBI at two points. If indeed the Rabi model is not YBI in general then Braak’s criterion does not fit with Yang-Baxter integrability. We have also shown that the Dicke model and the generalised Rabi model have YBI points, yet they are non-integrable models according to Braak’s criterion. Nevertheless the Rabi and generalised Rabi models have been solved analytically [10, 21, 23, 24, 36, 37]. For the Rabi model, the part of the eigenspectrum corresponding to the Juddian isolated exact solutions can also be derived algebraically using ideas from the notion of quasi-exact solvability [38, 39]. For this reason the Rabi model has been called a quasi-exactly solved model [39, 40]. It has also been called an exactly solved model. However, the term exactly solved model in our view should apply to models whose complete eigenspectrum can be described algebraically in terms of finite polynomials, which is a feature, e.g., of finite-sized systems solved in terms of the Bethe Ansatz and related $T − Q$ relations, or equivalently, models which are YBI.

That the solution of the Rabi model is not of this particular form lends weight to the Rabi model not being YBI in general. Yang-Baxter integrability is arguably a not necessary but sufficient condition to guarantee integrability. With this proviso, the terms exactly solved, integrable and Yang-Baxter integrable should all be synonymously interchangeable for quantum systems of this kind.

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