SEIDEL'S LONG EXACT SEQUENCE ON CALABI-YAU MANIFOLDS

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Abstract. In this paper, we generalize construction of Seidel’s long exact sequence of Lagrangian Floer cohomology to that of compact Lagrangian submanifolds with vanishing Malsov class on general Calabi-Yau manifolds. We use the framework of anchored Lagrangian submanifolds developed in \cite{FOOO3} and some compactness theorem of smooth $J$-holomorphic sections of Lefschetz Hamiltonian fibration for a generic choice of $J$. The proof of the latter compactness theorem involves a study of proper pseudoholomorphic curves in the setting of noncompact symplectic manifolds with cylindrical ends.

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1. INTRODUCTION

To put the content of this paper in perspective, we first recall a long exact sequence for symplectic Floer cohomology of Lagrangian submanifolds, which was constructed by Seidel \cite{Se3} originally for the category of exact Lagrangian submanifolds on (non-compact) exact symplectic manifolds.

1.1. Dehn twists and Seidel’s long exact sequence. Let $(M, \omega, \alpha)$ be an exact symplectic manifold with contact type boundary: $\alpha$ is a contact one-form on $\partial M$ which satisfies $d\alpha = \omega|_{\partial M}$ and makes $\partial M$ convex. Assume $[\omega, \alpha] \in H^2(M, \partial M; \mathbb{R})$ is zero so that $\alpha$ can be extended to a one-form $\theta$ on $M$ satisfying $d\theta = \omega$.

**Theorem 1.1** (Seidel \cite{Se3}). Let $L$ be an exact Lagrangian sphere in $M$ together with a preferred diffeomorphism $f : S^2 \to L$. Denote by $\tau_L = \tau_{(L, [f])}$ be the Dehn twist associated to $(L, [f])$. For any two compact exact Lagrangian submanifolds $L_0, L_1 \subset M$, there is a long exact sequence of Floer cohomology groups

$$\cdots \to HF(\tau_L(L_0), L_1) \to HF(L_0, L_1) \to HF(L, L_1) \otimes HF(L_0, L) \to \cdots$$  \hspace{1cm} (1.1)

Due to the well-known difficulties in the construction of Lagrangian intersection Floer cohomology and a new non-trivial compactness issue arising from the singularities of the Lefschetz fibration used for the construction, Seidel put the exactness assumption to avoid these difficulties and work entirely with exact Lagrangian category and left the extension to the more general situation like for the closed Calabi-Yau manifolds as an open problem \cite{Se3}. While the limitation to the exact Lagrangian category simplifies the analysis of holomorphic disc bubbles, it also forces him to work entirely with the language of exact Lefschetz fibrations and to make sure that he does not go out of this domain largely for the consistency of
his exposition, as Seidel himself indicated. Because of this, \cite{Se3} develops a fair amount of geometry of exact Lefschetz fibrations some of which are not directly relevant to the construction of the long exact sequence. Partly due to this digression, it took some effort and time for the author of the present paper to get to the main point of Seidel’s construction in \cite{Se3}.

The cases of closed Calabi-Yau manifolds or Fano toric manifolds are the one that is physically most relevant to the mirror symmetry: According to Kontsevich \cite{K2} and Seidel \cite{Se3}, the symplectic Dehn-twists correspond to a particular class of autoequivalences, “twist functors along spherical objects”, of derived categories of coherent sheaves, and this long exact sequence corresponds to an exact sequence of the same form in the mirror Calabi-Yau. Therefore it is important to establish the long exact sequence for a class of Lagrangian branes that is \textit{closed} under the action of symplectic Dehn twist. The class of exact Lagrangian submanifolds in exact symplectic manifolds is one such class, which Seidel considered in \cite{Se3}.

One of the point Seidel tried to ensure by working with the exact Lagrangian category is to have not only single-valuedness of the action functional on the path space but \textit{coherence} of the definition of the action functional between different exact Lagrangian submanifolds: this then allows one to have the energy estimate for the Floer trajectories, and more importantly to allow one to have \textit{thick-thin decomposition} of the Floer moduli spaces entering in the construction. This decomposition then enables him to apply the spectral sequence argument and derive the desired conclusion based on the contribution coming from the \textit{thin} part of the Floer moduli spaces which can be explicitly analyzed.

1.2. \textbf{Calabi-Yau Lagrangian branes}. In regard to extending Seidel’s construction to closed Calabi-Yau manifolds, we highlight two points that we take in this paper.

The first point is our restriction to the class of Lagrangian submanifolds with zero Maslov class. This class is \textit{closed} under the action by symplectic Dehn twists and enables one to consider the involved cohomology as a \textit{Z-graded group} which is essential in the point of view of mirror symmetry. The second point is the usage of the notion of \textit{anchors} and \textit{anchored Lagrangian submanifolds} introduced in \cite{FOOO3}. This notion has its origin in the preprint \cite{FOOO1} when the authors take the \textit{based point of view} of Lagrangian submanifolds in relation to the coherence of the definitions of various Maslov-type indices and of action functionals when one considers several Lagrangian submanifolds altogether as one studies Fukaya category. In the technical point of view, consideration of anchored Lagrangian submanifolds enables one to keep consistency of the definitions both of action functionals and of the absolute gradings on the Calabi-Yau Lagrangian branes. Most importantly this also enables us to provide a coherent filtration in the relevant Floer complexes and to have thick-thin decomposition of the relevant Floer moduli spaces of the kind as Seidel considered in \cite{Se3}.

Now we introduce a class of decorated Lagrangian submanifolds on Calabi-Yau manifold \((M, \omega)\) which we call \textit{Calabi-Yau Lagrangian branes}. It is expected that this class of Lagrangian submanifolds ‘generates’ the Fukaya category of a Calabi-Yau manifold that is mirror to the derived category of coherent sheaves on the mirror Calabi-Yau. We refer readers to the main part of the paper for various undefined terms in the statement. We also omit the important datum of flat line
bundles on $L$ in this definition because it will not play much role in our proof but can be easily incorporated in the construction.

**Definition 1.2.** Let $y \in M$ be a base point and $\Lambda_y \subset T_y M$ a fixed Lagrangian subspace. Suppose $\Theta$ is a quadratic complex volume form on $(M, \omega, J)$ with $\langle \Theta(y), \Lambda_y \rangle = 1$. We consider the quadruple $((L, \gamma), s, [b])$, which we call an (anchored) Calabi-Yau Lagrangian brane, that satisfies the following data:

1. $L$ a Lagrangian submanifold of $M$ such that the Maslov index of $L$ is zero and $[\omega] \in H^2(M, L; \mathbb{Z})$. We also enhance $L$ with flat complex line bundle on it.
2. $\gamma$ is an anchor of $L$ relative to $y$.
3. $s$ is a spin structure of $L$.
4. $[b] \in M(L)$ is a bounding cochain described in subsection 8.2.

We denote by $\mathcal{E}^{CY}_{brane}$ the collection of Calabi-Yau Lagrangian branes and define $\text{Fuk}(\mathcal{E}^{CY}_{brane})$ to be the Fukaya category generated by $\mathcal{E}^{CY}_{brane}$.

We remark that the notion of anchor to $L$ is introduced to solve the problems of grading and filtration on the Floer complex in a uniform way in [FOOO3]. In particular it provides a canonical filtration on the associated Floer complex of anchored Lagrangian submanifolds which is needed to apply some spectral sequence argument in the proof. See the end of section 10 in particular.

1.3. **Statement of the main result and compactness issue.** The main purpose of the present paper is to construct an exact sequence for the Calabi-Yau Lagrangian branes on Calabi-Yau manifolds, which is the analog to Seidel’s [Se3].

We first note that each Dehn twist $\tau_L$ along a given Lagrangian sphere $L \subset M$ acts on $\mathcal{E}^{CY}_{brane}$. We denote this action by $(\tau_L)_*: \mathcal{E}^{CY}_{brane} \to \mathcal{E}^{CY}_{brane}$ and the image of $L$ under this action by $\tau_L(L) = (\tau_L)_*L$. This action defines an auto-equivalence on $\text{Fuk}(\mathcal{E}^{CY}_{brane})$, whose non-anchored versions should correspond to twist functors along spherical objects of derived categories of coherent sheaves alluded in the beginning of this introduction.

**Theorem 1.3.** Let $(M, \omega)$ be a compact (symplectic) Calabi-Yau and $y \in M$ be a base point $y$. Let $L \subset M$ be a Lagrangian sphere with $y \notin L$ together with a preferred diffeomorphism $f: S^2 \to L$ and $L = ((L, \gamma), s_{st}, 0)$ be the associated Calabi-Yau Lagrangian brane. Denote by $\tau_L = \tau_{(L,[f])}$ the Dehn twist associated to $(L,[f])$.

Consider any Calabi-Yau Lagrangian branes $\mathcal{L}_0, \mathcal{L}_1$. Then there is a long exact sequence of $\mathbb{Z}$-graded Floer cohomologies

$$\cdots \to HF((\tau_L)_*\mathcal{L}_0, \mathcal{L}_1) \to HF(\mathcal{L}_0, \mathcal{L}_1) \to HF(\mathcal{L}, \mathcal{L}_1) \otimes HF(\mathcal{L}_0, \mathcal{L}_1) \to \cdots$$

as a $\Lambda_{non}$-module where the Floer cohomologies involved are the deformed Floer cohomology constructed in [FOOO2].

We also have the long exact sequence for the non-anchored version of the Floer cohomology. See section 11.2.

Once we are to use these frameworks, construction of the long exact sequence largely follows Seidel’s strategy: We use the framework of Lefschetz fibration with Lagrangian boundary conditions for the construction of various operators appearing in the Floer theory, and use the spectral sequence for the $\mathbb{R}$-filtered groups
based on the thick-thin decomposition of the Floer moduli spaces. However unlike the exact Lagrangian case, the definition of Lagrangian Floer cohomology for Calabi-Yau Lagrangian branes meet obstruction as described in [FOOO2]. Because of this we have to use the Maurer-Cartan elements $b_i$ and use the associated deformed Floer cohomology appearing in the statement of the main theorem above. (Since Lagrangian submanifolds with zero Maslov class in Calabi-Yau manifolds are semi-positive, the related transversality issue is relatively standard which is one of the advantages considering this class of Lagrangian submanifolds.) For the readers’ convenience and the readability of the paper, we borrow a fair amount of materials from [FOOO2] in our exposition. For the same reason, we also borrow much exposition from [Se3] and refer to the two for further details. In a way, most of the materials used in this paper are not new but has already been present in the literature one way or the other. We organize them in a coherent way to be able to construct the required long exact sequence. Familiarity of the scheme in the paper [Se3] would be useful for the readers to follow the stream of the arguments used in this paper, especially those presented in sections 9-11.

However there is one nontrivial analytical issue that needs to be overcome. This concerns the issue of compactification of smooth pseudo-holomorphic section of Lefschetz (Hamiltonian) fibration when the fibration has non-empty critical fibers. By the definition of Lefschetz Hamiltonian fibration given in Definition 5.1, any smooth section will avoid critical points of the fibration. However a priori a sequence of smooth sections may approach critical points if the derivatives of the sections in the sequence blow up. When applied to a sequence of pseudo-holomorphic sections, the bubble could touch the critical points. Therefore to define the relative Gromov-Witten type invariants in the Lefschetz fibration setting, one should study the behavior of pseudo-holomorphic sections approaching the critical points. This compactness result may be mathematically the most novel part of the present paper which is carried out in section 7.

In this regard, we prove the following

**Theorem 1.4 (Theorem 7.7).** Let $\pi : E \to \Sigma$ be a Lefschetz Hamiltonian fibration with Lagrangian boundary $Q \subset E_{\partial \Sigma}$ such that $E$ is fiberwise Calabi-Yau and $Q$ has vanishing fiberwise Maslov class. Then there exists a dense subset of $j$-compatible $J$’s for which we have a constant $C > 0$ depending only on $(E, J, j)$ and the section class $B \in \pi^s_2 (E, Q)$, but independent of $s$, such that we have

$$\text{dist}(\text{Im} s, E^{crit}) \geq C$$

for any smooth section $s : \Sigma \to E$ with $[s, \partial s] \in B \in \pi^s_2 (E, Q)$.  

The proof of this theorem turns out to involve the compactification and the Fredholm theory in the setting of symplectic field theory [EGH] in which we regard a bubble touching a critical point $x_0$ as a proper pseudo-holomorphic curve on $C$ in a punctured fiber $E_{x_0} \setminus \{x_0\}$. See [FOOO4] and [OZ] for relevant studies of such compactification and Fredholm theory.

Once this theorem is established, study of compactification of smooth pseudo-holomorphic sections in the current case is essentially the same as the case of smooth Hamiltonian fibrations as studied in [En], [MS].

The result in the present paper was first announced in Eliashberg’s 60-th Birthday Conference: “New Challenges and Perspectives in Symplectic Field Theory” held at Stanford University, June 25 - 29, 2007 and then presented in various
seminars and in conferences afterwards. We apologize to readers for a long delay of appearance of the present paper.

As always, we would like to thank Fukaya, Ohta and Ono for our years-long collaboration on Lagrangian Floer theory, especially writing together the book [FOOO2] and the paper [FOOO3] whose frameworks we adopt throughout the present paper. We also thank Seidel for some useful e-mail communications in the early stage of the present work, concerning compactness of the moduli space of holomorphic sections of exact Lefschetz fibrations in his paper [Se3]. Our thanks also go to KIAS and NIMS in Korea for their financial supports and hospitality during our stay when a large chunk of writing of this paper was carried out.

2. Basic facts on symplectic Dehn twists

In this section, we summarize basic facts on the Dehn twists in the symplectic point of view which Seidel extensively studied in a series of papers [Se1, Se3]. We borrow the basic facts on the symplectic Dehn twists from them with a slight variation of the exposition that will be necessary for the purpose of the present paper.

Assume that \( L \subset (M, \omega) \) be an embedded Lagrangian sphere together with an equivalence class \([f]\) of diffeomorphisms \( f: S^2 \to L\): two \( f_1, f_2 \) are equivalent if and only if \( f_2^{-1} f_1 \) can be deformed inside \( \text{Diff}(S^n) \) to an element of \( O(n + 1) \). To any such \((L, [f])\) Seidel associates a Dehn twist \( \tau_L = \tau_L([f]) \in \text{Symp}(M) \) using a model Dehn twist on the cotangent bundle \( T^* S^n \).

Let \( f: S^n \to L \subset M \) be a representative of the equivalence class \([f]\). Denote by \( T^*(r) \subset T^* S^n \) the disc bundle of radius \( r \) in terms of the standard metric on the unit sphere \( S^n = S^n(1) \subset \mathbb{R}^{n+1} \).

Identifying \( T = TS^n \) with respect to the standard metric, one considers the map

\[
\sigma_t(u, v) = \left( \cos(t)u - \sin(t)\|u\|v, \cos(t)v + \sin(t)\frac{u}{\|u\|} \right)
\]

for \( 0 < t < \pi \). \( \sigma_\pi \) is the antipodal involution \( A(u, v) = (-u, -v) \).

Next we fix a function \( R \in C^\infty(\mathbb{R}, \mathbb{R}) \) such that

\[
\text{supp } R \subset T(1)
\]

\[
R(-t) = R(t) - t \quad \text{for } |t| \leq \frac{1}{2}.
\]

Then we consider the re-scaled function

\[
R_\lambda(t) = \lambda R \left( \frac{t}{\lambda} \right)
\]

for all \( 0 < \lambda \leq 1 \). Then \( R_\lambda \) is supported in \( T(\lambda) \) and satisfies

\[
R_\lambda(-t) = R_\lambda(t) - t \quad \text{for } |t| \leq \frac{\lambda}{2}.
\]

Insertion of one-parameter \( \lambda \) in our choice of \( R \) is deliberate which will be later explicitly related to the parameter that enters in the Lagrangian surgery.

The following lemma is a slight variation of Lemma 1.8 [Se3] whose proof is referred thereto.

**Lemma 2.1.** Let \( \mu: T \setminus T(0) \to \mathbb{R} \) be the length function \( \mu(u, v) = \|v\| \) and \( \nabla H_\lambda = R_\lambda \circ \mu \) on \( T \setminus T(0) \). Then \( \phi^{2\pi}_{H_\lambda} \) extends smoothly over \( T(0) \) to a symplectic
diffeomorphisms $\phi_\lambda$ of $T$. The function $K_\lambda = 2\pi(R'_\lambda \circ \mu - R \circ \mu)$ also extends smoothly over $T(0)$, and satisfies

$$\phi_\lambda^* \theta_T - \theta_T = dK_\lambda.$$  

These $\phi_\lambda$ are called model Dehn twists.

The model Dehn twists, denoted by $\tau_\lambda$, have the explicit formula

$$\tau_\lambda(y) = \begin{cases} 
\sigma_{2\pi R'(\mu(y))} & y \in T(\lambda) \setminus T(0) \\
A(y) & y \in T(0)
\end{cases}$$

(2.5)

where the angle of rotation goes from $2\pi R'(0) = \pi$ to $2\pi R'_\lambda(\lambda) = 0$: Note that as $\lambda \to 0$, we have $2\pi R'(0) = \pi$ to $2\pi R'(1) = 0$.

Now we take a Darboux-Weinstein chart, or a symplectic embedding $\iota : T(1) \to M$ such that $\iota|_{\partial D(\lambda)} = f$, $\iota^* \omega = \omega_T(-d\theta_T)$ for a representative of the framed Lagrangian sphere $(L, [f])$. Take a model Dehn twist $\tau$ supported in the interior of $T(1)$.

We denote $U = \text{im} \iota$ and fix the Darboux neighborhood once and for all, and consider the one-parameter family of Dehn twists $\tau_r$ any of which we denote by $\tau_L$

$$\tau_L = \tau(L,[f];r) = \begin{cases} 
\iota \circ \tau_r \circ \iota^{-1} & \text{on } \text{im}(\iota) = U \\
id & \text{elsewhere.}
\end{cases}$$

(2.6)

We quote the following basic fact on the Dehn twist $\tau_L([f])$ from [Se3] with a slight variation of the statements.

**Proposition 2.2** (Proposition 1.11 [Se3]). Let $(L, [f])$ be a framed Lagrangian sphere in $M$. There is a one-parameter family of Lefschetz fibrations $(E_L^\lambda, \pi_L^\lambda) \to D(\lambda)$ together with an isomorphism $\phi_L^\lambda : E_L^\lambda \to M$ of symplectic manifolds, such that

1. Consider the re-scaling map $R_\lambda : D(\lambda) \to D(1)$ defined by $z \mapsto \frac{z}{\lambda}$. Then

$$(R_\lambda)^* E_L^\lambda = E_L^\lambda.$$

2. If $\rho_L^\lambda$ is the symplectic monodromy around $\partial D(\lambda)$, then $\phi_L^\lambda \circ \rho_L^\lambda \circ (\phi_L^\lambda)^{-1}$ is a Dehn twist along $(L, [f])$.

3. There exists a decomposition

$$E_L = E \cup D(\lambda) \times (M \setminus \iota(T(\lambda)) \setminus V)$$

such that $E$ is the standard fibration $q : \mathbb{C}^{n+1} \to \mathbb{C}$ defined by

$$q(z_1, \cdots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2.$$ 

We denote any of these maps by $\tau_L$.

An important point on this Dehn twist $\tau_L([f];\lambda)$ is that its support can be put into a Darboux neighborhood of the given Lagrangian sphere $L$ which can be made
as close as to $L$ by choosing $\lambda > 0$ small, whose derivative can be controlled: One can choose $R$ so that for some $\delta > 0$ we have

$$R'(t) \geq 0 \quad \text{for all } t \geq 0$$

and then consider $R_{\lambda}$ for any sufficiently small $\lambda > 0$. According to Seidel’s terminology, the corresponding Dehn twist is $\delta$-wobbly.

### 3. Action, grading and anchored Lagrangian submanifolds

In this section, we consider the general Lagrangian submanifolds treated as in \cite{FOOO2}. For the fine chain level analysis of the Floer complex, it is essential to analyze the $\mathbb{R}$-filtration on $CF(L_0, L_1)$ that is provided by the action functional $A$ on $\Omega(L_0, L_1)$. This action functional is not single valued on $\Omega(L_0, L_1)$ itself even for the pair $(L_0, L_1)$ of Calabi-Yau Lagrangian branes, but single-valued only on some covering space. For the purpose of studying the Fukaya category and carrying out various constructions in the Floer homology in a coherent manner, we need to consider a whole collection of Lagrangian submanifolds and assign these auxiliary data to each pair of the given collection in a consistent way. For this purpose, \cite{FOOO2} uses the notion of anchored Lagrangian submanifolds. This auxiliary data is important later for consistency of definition of action functionals and in turn for the analysis of thick-thin decomposition of the various Floer moduli spaces entering in the construction of boundary map, chain map, chain-homotopy and pants products. The above mentioned covering space is defined in terms of a reference path $\ell_0$ chosen in $\Omega(L_0, L_1)$, which we denote by $\hat{\Omega}(L_0, L_1; \ell_0)$.

Furthermore to provide an absolute grading to each critical point $[p, w]$ of the action functional $A : \hat{\Omega}(L_0, L_1) \rightarrow \mathbb{R}$, we also need to fix a section $\lambda$ of $\ell_0^* \Lambda(M, \omega)$ where $\Lambda(M, \omega)$ is the bundle of Lagrangian Grassmanians on $(M, \omega)$.

From now on, we denote either $(L, \gamma)$ or $(L, \gamma, \lambda)$ just by $L$ depending on the circumstances.

#### 3.1. The $\Gamma$-equivalence

We consider a covering space of $\Omega(L_0, L_1; \ell_0)$ by modding out the space of paths in $\Omega(L_0, L_1; \ell_0)$ by an equivalence relation, which is weaker than the homotopy. The deck transformation group of this covering space is abelian by construction.

Note that when we are given two pairs $(\ell, w)$ and $(\ell, w')$ from $\Omega(L_0, L_1; \ell_0)$, the concatenation

$$\pi \# w' : [0, 1] \times [0, 1] \rightarrow M$$

defines a loop $c : S^1 \rightarrow \Omega(L_0, L_1; \ell_0)$. One may regard this loop as a map $C : S^1 \times [0, 1] \rightarrow M$ satisfying the boundary condition $C(s, 0) \in L_0$, $C(s, 1) \in L_1$. Obviously the symplectic area of $C$, denoted by

$$I_\omega(c) = \int_C \omega$$

depends only on the homotopy class of $C$ and so defines a homomorphism on $\pi_1(\Omega(L_0, L_1; \ell_0))$, which we also denote by

$$I_\omega : \pi_1(\Omega(L_0, L_1; \ell_0)) \rightarrow \mathbb{R}.$$
Next we note that for the map $C : S^1 \times [0, 1] \to M$ satisfying the given Lagrangian boundary condition, it associates a symplectic bundle pair $(\mathcal{V}, \lambda)$ defined by
\[ \mathcal{V}_C = C^*TM, \quad \lambda_C = c_0^*TL_0 \cup c_1^*TL_1 \]
where $c_i : S^1 \to L_i$ is the map given by $c_i(s) = C(s, i)$ for $i = 0, 1$. This allows us to define another homomorphism
\[ I_\mu : \pi_1(\Omega(L_0, L_1), \ell_0) \to \mathbb{Z}; \quad I_\mu(c) = \mu(\mathcal{V}_C, \lambda_C) \]
where $\mu(\mathcal{V}_C, \lambda_C)$ is the Maslov index of the bundle pair $(\mathcal{V}_C, \lambda_C)$. See section 2.2.1 [FOOO2] for details.

Using the homomorphisms $I_\mu$ and $I_\omega$, we define an equivalence relation $\sim$ on the set of all pairs $(\ell, w)$. For given such pair $w, w'$, we denote by $\overline{w} \# w'$ the concatenation of $\overline{w}$ and $w'$ along $\ell$, which defines a loop in $\Omega(L_0, L_1; \ell_0)$ based at $\ell_0$.

**Definition 3.1.** We say that $(\ell, w)$ is $\Gamma$-equivalent to $(\ell, w')$ and write $(\ell, w) \sim (\ell, w')$ if the following conditions are satisfied $I_\omega(\overline{w} \# w') = 0 = I_\mu(\overline{w} \# w')$. We denote the set of equivalence classes $[\ell, w]$ by $\widetilde{\Omega}(L_0, L_1; \ell_0)$ and call the Novikov covering space.

There is a canonical lifting of $\ell_0 \in \Omega(L_0, L_1; \ell_0)$ to $\widetilde{\Omega}(L_0, L_1; \ell_0)$: this is just $[\ell_0, \tilde{\ell}_0] \in \widetilde{\Omega}(L_0, L_1; \ell_0)$ where $\tilde{\ell}_0$ is the map $\tilde{\ell}_0 : [0, 1]^2 \to M$ with $\tilde{\ell}_0(s, t) = \ell_0(t)$. In this way, $\widetilde{\Omega}(L_0, L_1; \ell_0)$ also has a natural base point which we suppress from the notation.

We denote by $\Pi(L_0, L_1; \ell_0)$ the group of deck transformations of the covering space $\widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$. It is easy to see that the isomorphism class of $\Pi(L_0, L_1; \ell_0)$ depends only on the connected component containing $\ell_0$.

The two homomorphisms $I_\omega$ and $I_\mu$ push down to homomorphisms
\[ E : \Pi(L_0, L_1; \ell_0) \to \mathbb{R}, \quad \mu : \Pi(L_0, L_1; \ell_0) \to \mathbb{Z} \]
defined by
\[ E(g) = I_\omega[C], \quad \mu(g) = I_\mu[C] \]
for any map $C : S^1 \times [0, 1] \to M$ representing the class $g \in \Pi(L_0, L_1; \ell_0)$. The group $\Pi(L_0, L_1; \ell_0)$ is an abelian group.

We now define the Novikov ring $\Lambda(L_0, L_1; \ell_0)$ associated the abelian covering $\widetilde{\Omega}(L_0, L_1; \ell_0) \to \Omega(L_0, L_1; \ell_0)$ as a completion of the group ring $R[\Pi(L_0, L_1; \ell_0)]$. Here $R$ is a commutative ring with unit.

**Definition 3.2.** $\Lambda^R_k(L_0, L_1; \ell_0)$ denotes the set of all (infinite) sums
\[ \sum_{g \in \Pi(L_0, L_1; \ell_0), \mu(g) = k} a_g[g] \]
such that $a_g \in R$ and for each $C$, the set $\{g \in \Pi(L_0, L_1; \ell_0) \mid E(g) \leq C, \ a_g \neq 0\}$ is of finite order. We put
\[ \Lambda^R(L_0, L_1; \ell_0) = \bigoplus_k \Lambda^R_k(L_0, L_1; \ell_0). \]
The ring structure on $\Lambda^R(L_0, L_1; \ell_0)$ is defined by the convolution product

$$
\left( \sum_{g \in \Pi(L_0, L_1; \ell_0)} a_g[g] \right) \cdot \left( \sum_{g \in \Pi(L_0, L_1; \ell_0)} b_g[g] \right) = \sum_{g_1, g_2 \in \Pi(L_0, L_1; \ell_0)} a_{g_1} b_{g_2} [g_1g_2].
$$

It is easy to see that the term in the right hand side is indeed an element in $\Lambda^R(L_0, L_1; \ell_0)$. Thus $\Lambda^R(L_0, L_1; \ell_0) = \oplus_k \Lambda^R_k(L_0, L_1; \ell_0)$ becomes a graded ring under this multiplication. We call this graded ring the Novikov ring associated to the pair $(L_0, L_1)$ and the connected component containing $\ell_0$.

We also use the universal Novikov ring $\Lambda_{\text{nov}}$ in this paper. We recall its definition here. An element of $\Lambda_{\text{nov}}$ is a formal sum $\sum a_i T^{\lambda_i} e^{\mu_i}$ with $a_i \in \mathbb{C}$, $\lambda_i \in \mathbb{R}$, $\mu_i \in \mathbb{Z}$ such that $\lambda_i \leq \lambda_{i+1}$ and $\lim_{i \to \infty} \lambda_i = \infty$. $T$ and $e$ are formal parameters. We define a valuation $\nu : \Lambda_{\text{nov}} \to \mathbb{R}_{\geq 0}$ defined by

$$
\nu \left( \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \right) = \lambda_1.
$$

We denote the corresponding `valuation ring' by

$$
\Lambda_{0,\text{nov}} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} e^{\mu_i} \in \Lambda_{\text{nov}} \mid \lambda_i \geq 0 \right\}.
$$

It carries a unique maximal ideal consisting of $\sum a_i T^{\lambda_i} e^{\mu_i}$ with $\lambda_i > 0$ for all $i$ which we denote by $\Lambda_{0,\text{nov}}^\ast$. We have a natural embedding

$$
\Lambda^R(L_0, L_1; \ell_0) \to \Lambda_{\text{nov}}
$$

given by

$$
\sum_{g \in \Pi(L_0, L_1; \ell_0)} b_g[g] \mapsto \sum_{g \in \Pi(L_0, L_1; \ell_0)} b_g T^{\nu(g)} e^{\mu(g)/2}. \quad (3.1)
$$

Now for a given pair $(\ell, w)$, we define the action functional

$$
\mathcal{A} : \tilde{\Omega}(L_0, L_1; \ell_0) \to \mathbb{R}
$$

by the formula

$$
\mathcal{A}(\ell, w) = \int w^* \omega.
$$

It follows from the definition of $\Pi(L_0, L_1; \ell_0)$ that the integral depends only on the $\Gamma$-equivalence class $[\ell, w]$ and so pushes down to a well-defined functional on the covering space $\tilde{\Omega}(L_0, L_1; \ell_0)$.

**Lemma 3.3.** The set $\text{Cr}(L_0, L_1; \ell_0)$ of critical points of $\mathcal{A}$ consists of the pairs of the type $[\ell_p, w]$ where $\ell_p$ is the constant path with $p \in L_0 \cap L_1$ and $w$ as in (2.2.2). $\text{Cr}(L_0, L_1; \ell_0)$ is invariant under the action of $\Pi(L_0, L_1; \ell_0)$ and so forms a principal bundle over a subset of $L_0 \cap L_1$ with its fiber isomorphic to $\Pi(L_0, L_1; \ell_0)$.

We put

$$
\text{Cr}(L_0, L_1) = \bigcup_{\ell_{0,i}} \text{Cr}(L_0, L_1; \ell_{0,i})
$$

where $\ell_{0,i}$ runs over the set of base points of connected components of $\Omega(L_0, L_1)$.

Next, we assign an absolute Morse index to each critical point of $\mathcal{A}$. In general, assigning such an absolute index is not a trivial matter because the obvious Morse index of $\mathcal{A}$ at any critical point is infinite. For this purpose, we will use the Maslov
index of certain bundle pair naturally associated to the critical point \([\ell_p, w] \in Cr(L_0, L_1; \ell_0)\).

We call this Morse index of \([\ell_p, w]\) the \textit{Maslov-Morse index} (relative to the base path \([\ell_0]\)) of the critical point. The definition of the index will somewhat resemble that of \(A\). However to define this, we also need to fix a section \(\lambda^0\) of \(\ell_0^*\Lambda(M)\) such that
\[
\lambda^0(0) = T_{\ell_0(0)} L_0, \quad \lambda^0(1) = T_{\ell_0(1)} L_1.
\]

Here \(\Lambda(M)\) is the bundle of Lagrangian Grassmanians of \(TM\) 
\[
\Lambda(M) = \bigcup_{p \in M} \Lambda(T_p M)
\]
where \(\Lambda(T_p M)\) is the set of Lagrangian subspaces of the symplectic vector space \((T_p M, \omega_p)\).

Let \([\ell_p, w] \in Cr(L_0, L_1; \ell_0) \subset \Omega(L_0, L_1; \ell_0)\) be an element whose projection corresponds to the intersection point \(p \in L_0 \cap L_1\).

As before we associate a symplectic bundle pair \((V_w, \lambda_w)\) over the square \([0, 1]^2\), which will be defined uniquely up to the homotopy. We first choose 
\[
\alpha^p : [0, 1] \to \Lambda(T_p M, \omega_p)
\]
satisfying
\[
\begin{align*}
\alpha^p(0) &= T_p L_0, \quad \alpha^p(1) = T_p L_1 \subset T_p M, \\
(\alpha^p)(t) &\oplus T_p L_0 = T_p M, \\
\alpha^p(t) &\in U_0(T_p L_0) \quad \text{for small } t,
\end{align*}
\]
where \(U_0(T_p L_0)\) is as above.

Then we consider a continuous Lagrangian subbundle \(\lambda_w \to \partial[0, 1]^2\) of \(\mathcal{V}|_{\partial[0, 1]^2}\) by the following formula: the fiber at each point of \(\partial[0, 1]^2\) is given as
\[
\begin{align*}
\lambda_w(s, 0) &= T_{w(s, 0)} L_0, \quad \lambda_w(1, t) = \alpha^p(t), \\
\lambda_w(s, 1) &= T_{w(s, 1)} L_1, \quad \lambda_w(0, t) = \lambda^0(0, t).
\end{align*}
\]

It follows that the homotopy type of the bundle pair constructed as above does not depend on the choice of \(\alpha^p\) either.

**Definition 3.4.** We define the \textit{Maslov-Morse index} of \([\ell_p, w]\) (relative to \(\lambda^0\)) by
\[
\mu([\ell_p, w]; \lambda^0) = \mu(\mathcal{V}_w, \lambda_w).
\]

**3.2. Anchored Lagrangian submanifolds.** Now we would like to generalize this construction for a chain
\[
\mathcal{L} = (L_0, \cdots, L_k)
\]
of more than two Lagrangian submanifolds, i.e., with \(k \geq 2\).

To realize this purpose, we need the notion of \textit{anchors} of Lagrangian submanifolds. In this subsection, we briefly recall the definition of anchored Lagrangian submanifolds introduced in \[FOOO3\].

**Definition 3.5.** Fix a base point \(y\) of ambient symplectic manifold \((M, \omega)\). Let \(L\) be a Lagrangian submanifold of \((M, \omega)\). We define an \textit{anchor} of \(L\) to \(y\) is a path \(\gamma : [0, 1] \to M\) such that
\[
\gamma(0) = y, \quad \gamma(1) \in L.
\]

We call a pair \((L, \gamma)\) an \textit{anchored} Lagrangian submanifold.
It is easy to see that any homotopy class of path in \( \Omega(L, L') \) can be realized by a path that passes through the given point \( y \). We denote the set of homotopy classes of the anchors \( \gamma \) to \( y \in M \) by \( \pi_1(y, L) \).

The following lemma is easy to check.

**Lemma 3.6.** Suppose that \( L \) is connected. Then set \( \pi_1(y, L) \) of homotopy classes relative to the ends is a principal homogeneous space of \( \pi_1(M, L) \), i.e., it is a \( \pi_1(M, L) \)-torsor. We call an element of \( \pi_1(y, L) \) an anchor class of \( L \) relative to \( y \).

**Proof.** Since \( L \) is connected, the natural map \( \pi_1(M) \to \pi_1(M, L) \) is surjective and so \( \pi_1(M, L) \cong \pi_1(M) / \text{im}(\pi_1(L) \to \pi_1(M)) \) forms a group. It is obvious to see that \( \pi_1(M, L) \) acts on \( \pi_1(y, L) \) by concatenation of paths on the right. By definition, this action is free. Transitivity is obvious by definition. \( \square \)

For a given pair \((\mathcal{L}, \mathcal{L}')\) of anchored Lagrangians \( \mathcal{L} = (L, \gamma) \), \( \mathcal{L}' = (L', \gamma') \), we denote
\[
\Omega(\mathcal{L}, \mathcal{L}') := \Omega(L, L', \gamma \# \gamma')
\]
where the latter is the path component of \( \Omega(L, L') \) containing \( \gamma \# \gamma' \). We also denote
\[
\mathcal{L} \cap \mathcal{L}' = \{ p \in L \cap L' \mid \hat{p} \in \Omega(\mathcal{L}, \mathcal{L}') \}.
\]
(3.2)

Here \( \hat{p} \) is the constant path \( \hat{p}(t) \equiv p \).

When we are given a Lagrangian chain
\[
\mathcal{L} = (L_0, L_1, \ldots, L_k)
\]
we also consider a chain of anchors \( \gamma_i : [0, 1] \to M \) of \( L_i \) to \( y \) for \( i = 0, \ldots, k \). These anchors give a systematic choice of a base path \( \ell_{ij} \in \Omega(L_i, L_j) \) by concatenating \( \gamma_i \) and \( \gamma_j \) as
\[
\ell_{ij} = \gamma_i * \gamma_j
\]
where \( \gamma \) is the time-reversal of \( \gamma \) given by \( \gamma(t) = \gamma(1 - t) \). The upshot of this construction is the following overlapping property
\[
\ell_{ij}(t) = \ell_{i\ell}(t) \quad \text{for } 0 \leq t \leq \frac{1}{2}
\]
\[
\ell_{ij}(t) = \ell_{ij\ell}(t) \quad \text{for } \frac{1}{2} \leq t \leq 1
\]
(3.3)

for all \( j, \ell \).

**Definition 3.7.** Let \( \mathcal{E} = \{ (L_i, \gamma_i) \}_{0 \leq i \leq k} \) be a chain of anchored Lagrangian submanifolds. A homotopy class \( B \in \pi_2(\mathcal{L}, \hat{p}) \) is called **admissible** to \( \mathcal{E} \) if it can be obtained by a polygon that is a gluing of \( k \) bounding strips \( w_{i(i+1)} : [0, 1] \times [0, 1] \to M \) satisfying
\[
w_{i(i+1)}(0, t) =
\begin{cases} 
  \gamma_i(2t - 1) & 0 \leq t \leq \frac{1}{2} \\
  \gamma_{i+1}(2t - 1) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]
(3.4)
\[
w_{i(i+1)}(s, 0) \in L_i, \quad w_{i(i+1)}(s, 1) \in L_{i+1}
\]
(3.5)
\[
w_{i(i+1)}(1, t) = p_{i(i+1)}.
\]
(3.6)

When this is the case, we denote the homotopy class \( B \) as
\[
B = [w_{01}] \# [w_{12}] \# \cdots \# [w_{k0}]
\]
and the set of admissible homotopy classes by \( \pi_2^a(\mathcal{E}, \hat{p}) \).

We call such a tuple \( \mathcal{E} \) an anchored Lagrangian chain.
Remark 3.8. We remark that we denote by \( \mathfrak{L} \) a chain \((L_0, \cdots, L_k)\) of Lagrangian submanifolds and by \( E \) that of anchored Lagrangian submanifolds. When the collection \( \mathcal{E} = \{(L_i, \gamma_i)\}_{0 \leq i \leq k} \) is given, we note that not all homotopy classes in \( \pi_2(L; \vec{p}) \) is admissible. But we have the following basic lemma which will be enough for the construction of Fukaya category, whose proof is easy and so omitted.

Lemma 3.9. Let \( w_{i(i+1)} \) be given for \( i = 1, \cdots, k \) and \( B \in \pi_2(L; \vec{p}) \). Then there exists a unique \([w_{i0}]\) such that

\[
B = [w_{01}] \# \cdots \# [w_{(k-1)k}] \# [w_{k0}]
\]

The following basic identity immediately follows from definitions

Proposition 3.10. Suppose \( B \in \pi_2^\text{ad}(\mathcal{E}, \vec{p}) \) given as Lemma 3.7 and provide the analytic coordinates at the marked points \( z_j \) so that all \( z_j \) are outgoing. Then we have

\[
\omega(B) = \sum_{i=0}^{k} A([p_i, w_i]) \quad (3.7)
\]

\[
\mu(\mathcal{E}, \vec{v}; B) = \sum_{i=0}^{k} \mu([p_i, w_i]; \lambda_{i(i+1)}). \quad (3.8)
\]

In particular the sum in the right hand sides do not depend on the choice of \( \lambda_i \subset \gamma_i^* TM \).

Here the index \( \mu([p_i, w_i]; \lambda_{i(i+1)}) \) is the Maslov-Morse index relative to the path \( \lambda_{i(i+1)} \) of Lagrangian planes as defined in [FOOO2], which provides a coherent grading \( \mu : \text{Crit} \mathcal{A} \to \mathbb{Z} \).

The action functional provides a canonical \( \mathbb{R} \)-filtration on the set

\[
\mathcal{A} : \text{Crit} \mathcal{A} \to \mathbb{R}.
\]

In addition, inside the collection of anchored Lagrangian submanifolds \((L, \gamma)\) we are given a coherent system of single valued actions functionals

\[
\mathcal{A} : \widetilde{\Omega}_0(L_i, L_j; \tau_i \# \gamma_j) \to \mathbb{R}
\]

at one stroke for any pair from the given collection \( \mathcal{E} \) of anchored Lagrangian submanifolds.

3.3. Relation to the graded Lagrangian submanifolds. Now we go back to the collection of Lagrangian submanifolds with vanishing Maslov class on a Calabi-Yau manifolds. In this case, we will be able to obtain a canonical Lagrangian path \( \lambda \) along a given anchor \( \gamma \) of \( L \) to \( y \).

Let \( J \) be a compatible almost complex structure. The assumption \( 2c_1(M) = 0 \) implies that the bundle \( \Delta = \Lambda^n(TM, J)^\otimes 2 \) is trivial. Choose a section \( \Theta \) of \( \Delta^* \) that has length one everywhere in terms of the metric \( g = \omega(\cdot, J \cdot) \). This determines a map \( \det^2_{\Theta} : \Lambda(M, \omega) \to S^1 \), and then an \( \infty \)-fold Maslov covering by

\[
\Lambda_{\infty} = \{(\Lambda, t) \in \Lambda \times \mathbb{R} | \det^2_{\Theta}(\Lambda) = e^{2\pi it}\} \quad (3.9)
\]
where $s_L : L \rightarrow \Lambda(M, \omega)|_L$ is the natural section defined by the Gauss map $s_L(x) = T_xL$. An $\Lambda_\infty$-grading of a Lagrangian submanifold $L \subset (M, \omega)$ according to Seidel [Se2] is just a lift to $\mathbb{R}$ of the map 
$$\det^2_0 \circ s_L : L \rightarrow S^1.$$ 
First of all the condition on $\mu_L = 0$ implies that there is such a lifting to $\Lambda_\infty$.

We now explain how we give a coherent grading to Lagrangian submanifolds with vanishing Maslov class. We choose a Lagrangian path $\lambda$ over $\ell$ so that 
$$\det^2_0 \Theta(\gamma(t))(\lambda(t)) \equiv 1.$$ 
Then we choose a lifting $\widetilde{\det^2_0 \circ \gamma}$ of $\det^2_0 \Theta(\gamma) : [0, 1] \rightarrow S^1$ so that 
$$\widetilde{\det^2_0 \circ \gamma}(1) = 0.$$ 
The hypothesis $2c_1(M) = 0, \mu_L(0) = 0$ immediately implies that there exists a unique lifting $\widetilde{\det^2_0 \circ s_L} : L \rightarrow \mathbb{R}$ of $\det^2_0 \circ s_L : L \rightarrow S^1$ satisfying $\widetilde{\det^2_0 \circ s_L}(\gamma(1)) = 0$. We denote this lifting by $\alpha_{L, \gamma} : L \rightarrow \mathbb{R}$. Obviously this does not depend on the choice of $\lambda$ as long as $\lambda$ satisfies (3.10).

**Definition 3.11.** Let $(M, \omega)$ be such that $2c_1(M) = 0$ and fix a base point $(y, \Lambda_y)$. Let $(L, \gamma)$ be a Calabi-Yau anchored Lagrangian. We denote the above common lifting by $\alpha_{L, \gamma} : L \rightarrow \mathbb{R}$ and call the canonical grading of $(L, \gamma)$ relative to $(y, \Lambda_y)$.

We refer to section 9.2 [FOOO3] for more detailed explanation on the above discussion. Because of this presence of canonical grading associated to $(L, \gamma)$, we will drop $\lambda$ from our notation $L = (L, \gamma, \lambda)$ when we consider Calabi-Yau Lagrangian branes later in this paper.

Adapting to the convention from [K1] and [Se2], we denote by $\tilde{L}[0] = (L, \alpha(L, \gamma))$ and 
$$\tilde{L}[k] = (L, \alpha(L, \gamma) - k).$$

4. **CALABI-YAU LAGRANGIAN BRANES AND DEHN TWISTS**

We restrict ourselves to the case of $(M, \omega)$ with $2c_1(M, \omega) = 0$ and $L \subset M$ whose Maslov class vanishes from now on. We will give a precise definition of Calabi-Yau Lagrangian branes in this section. This is the case that is most relevant to mirror symmetry and to extension of Seidel’s long exact sequence of $\mathbb{Z}$-graded symplectic Floer cohomology.

Now we introduce a class of decorated Lagrangian submanifolds on Calabi-Yau manifold $(M, \omega)$ which we call Calabi-Yau Lagrangian branes.

**Definition 4.1.** Let $y \in M$ be a base point and $\Lambda_y \subset T_yM$ a fixed Lagrangian subspace. Suppose $\Theta$ be a quadratic complex volume form on $(M, \omega, J)$. Let $\mathcal{E}^{CY}$ be the Calabi-Yau Lagrangian collection of $(M, \omega)$. We consider the triple $(L, s, [b])$, $L = (L, \gamma)$ which we call an anchored Calabi-Yau Lagrangian brane, that satisfies the following data:

1. $L$ a Lagrangian submanifold of $M$ such that the Maslov index of $L$ is zero and $[\omega] \in H^2(M, L; \mathbb{Z})$. We also enhance $L$ with flat complex line bundle on it.
2. $\gamma$ is an anchor of $L$ to $y$. 

We refer to section 9.2 [FOOO3] for more detailed explanation on the above discussion. Because of this presence of canonical grading associated to $(L, \gamma)$, we will drop $\lambda$ from our notation $L = (L, \gamma, \lambda)$ when we consider Calabi-Yau Lagrangian branes later in this paper. 

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2. $\gamma$ is an anchor of $L$ to $y$. 

We refer to section 9.2 [FOOO3] for more detailed explanation on the above discussion. Because of this presence of canonical grading associated to $(L, \gamma)$, we will drop $\lambda$ from our notation $L = (L, \gamma, \lambda)$ when we consider Calabi-Yau Lagrangian branes later in this paper.
(3) $s$ is a spin structure of $L$.
(4) $[b] \in \mathcal{M}(L)$ is a bounding cochain described in subsection 8.2.

We denote the collection of Calabi-Yau Lagrangian branes by $\mathcal{E}_{\text{brane}}^{\text{CY}}$, and the Fukaya category generated by them by $\text{Fuk}(\mathcal{E}_{\text{brane}}^{\text{CY}})$.

The first simplification arising from considering the Calabi-Yau Lagrangian collection is that we have only to use the Novikov ring of the form

$$\Lambda_{\text{nov}}^{(0)} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \middle| a_i \in \mathbb{Q}, \lambda_i \in \mathbb{R}, \lambda_i \leq \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\}$$

(4.1)

which becomes a field. We also consider the subring

$$\Lambda_{\text{nov}, 0}^{(0)} = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \in \Lambda_{\text{nov}} \middle| \lambda_i \geq 0 \right\}$$

(4.2)

This is because the Maslov index satisfies $\mu(w) = \mu_L(\partial w) = 0$ for any disc map $w : (D^2, \partial D^2) \to (M, L)$ where $\mu_L \in H^1(L; \mathbb{Z})$ is the Maslov class of $L$.

**Remark 4.2.** Furthermore, as we mentioned in subsection 3.3, the anchor provides a canonical graded structure on a Calabi-Yau Lagrangian brane. Therefore it provides a canonical $\mathbb{R}$-filtration and a $\mathbb{Z}$-grading on $\text{CF}(L_0, L_1) = \text{CF}((L_0, \gamma_0), (L_1, \gamma_1)) := \text{CF}(L_0, L_1; \gamma_0 \# \gamma_1)$ for any pair $(L_0, L_1)$ of CY Lagrangian branes, and hence on its cohomology $HF(L_0, L_1)$. See section 8 for related discussion.

Now we examine the effect of Dehn twists on the CY Lagrangian collection.

**Proposition 4.3.** Let $\mathcal{E}_{\text{brane}}^{\text{CY}}$ be the associated collection of anchored CY Lagrangian branes. Then $\mathcal{E}_{\text{brane}}^{\text{CY}}$ is closed under the action of $\tau_L$‘s for all framed Lagrangian sphere $(L, [f])$, and so induces an auto-equivalence of $\text{Fuk}(\mathcal{E}_{\text{brane}}^{\text{CY}})$.

**Proof.** We note that the Dehn twist $\tau_L$ is a symplectic automorphism. Therefore it pushes the spin structure of $L_0$ to the image $\tau_L(L_0)$, and pull-backs the Maslov class. Therefore the Maslov class of $\tau_L(L_0)$ is also zero. Similarly we can push forward the anchor of $L$ to $\tau_L(L_0)$. Finally Theorem B (B.2) [FOOO2] states that the bounding cochain can also be canonically pushes under the symplectic automorphism and hence under the action of $\tau_L$. This finishes the proof. \qed

Therefore we can ask the question on how the Floer cohomology changes under the Dehn twist along a Lagrangian sphere. The answer is supposed to come from a long exact sequence that Seidel introduced in [Se3] for the context of exact Lagrangian submanifolds. The rest of the paper will be occupied with the construction of this long exact sequence for the Calabi-Yau Lagrangian branes on Calabi-Yau manifolds.

### 5. Lefschetz Hamiltonian fibration and coupling form

In this section, we first recall the basics on smooth Hamiltonian fibrations presented in [GLS, En] and extend our discussion to fibrations with Lefschetz-type singular fibers. Especially we generalize the notion of coupling form to the current
singular fibration and prove the uniqueness of the coupling form on a given Lefschetz Hamiltonian fibration, when the fibration \( \pi : E \to \Sigma \) is proper, e.g., when the fiber of \( E \) is compact.

The notion of Hamiltonian fibrations introduced by Guillemin-Lerman-Sternberg [GLS] is the family of symplectic manifolds of a fixed isomorphism type, which could be twisted on the parameter space \( \Sigma \). On the other hand, Seidel introduced the notion of exact Lefschetz fibrations which could have a finite number of singular fibers of type \( A_1 \)-singularity.

Combining [GLS] and [Se3], we give the following definition

**Definition 5.1** (Lefschetz Hamiltonian fibration). A Lefschetz Hamiltonian fibration over a compact surface \( \Sigma \) with boundary \( \partial \Sigma \) consists of the data \((E, \pi, \Omega, J_0, j_0)\) as follows:

1. \( \partial E = \pi^{-1}(\partial \Sigma) \) and \( \pi|_{\partial E} \to \partial \Sigma \) forms a smooth fiber bundle.
2. \( \pi : E \to \Sigma \) can have at most a finitely many critical points, and no two may lie on the same fiber. Denote \( E^{\text{crit}} \subseteq E \) and \( \Sigma^{\text{crit}} \subseteq \Sigma \).
3. \( J_0 \) is a complex structure on a neighborhood of \( E^{\text{crit}} \), \( j_0 \) is a positively oriented complex structure on a neighborhood of \( \Sigma^{\text{crit}} \), and \( \pi = (J_0, j_0) \)-holomorphic near \( E^{\text{crit}} \). And the Hessian \( D^2 \pi \) at any critical point is nondegenerate as a complex quadratic form.
4. \( \Omega \) is a closed two form on \( E \) which must be nondegenerate on \( \ker D\pi_x \) for each \( x \in E \), and a Kähler from for \( J_0 \) in some neighborhood of \( E^{\text{crit}} \).

We say that the fibration \( \pi : E \to \Sigma \) is (symplectically) Calabi-Yau if \( c_1(E^\vee) = 0 \) at all \( z \notin \Sigma^{\text{crit}} \).

We would like to remark that one may allow more than one critical points in the same fiber, which could be useful for the study of a family of Lefschetz Hamiltonian fibrations.

**Remark 5.2.** We would like to highlight that at a critical point \( x \in E \) of \( \pi \) Condition (4) implies that the form \( \Omega_x \) is required to be nondegenerate on the whole tangent space \( T_x E \) since \( \ker D\pi_x = T_x E \), while at a regular point \( \Omega_x \) it is required so only at on the vertical tangent space \( T^v E \) as \( \ker D\pi_x = T^v_x E \).

When a generic fiber \( E_z \) with \( z \in \Sigma \setminus \Sigma^{\text{crit}} \) is compact, it is proved in [GLS] for a smooth Hamiltonian fibration that one can choose \( \Omega \) is uniquely determined by the following additional requirement

\[
\pi_* \Omega^{n+1} = 0 \tag{5.1}
\]

where \( \pi_* \) is the integration along the fiber. Now we prove the following analog to this result for the case of Lefschetz Hamiltonian fibrations.

**Theorem 5.3.** Let \((E, \pi, \Omega, J_0, j_0)\) be a Lefschetz Hamiltonian fibration as in Definition 5.1. Then there exists a closed 2-form \( \Omega' \) smooth on \( E \setminus E^{\text{crit}} \) that satisfies the following:

1. \( \Omega'|_{T^v E} = \omega_v \) at all \( e \in E \setminus E^{\text{crit}} \),
2. it satisfies (5.1) on \( E \setminus \pi^{-1}(\Sigma^{\text{crit}}) \).

Furthermore such a form \( \Omega' \) is unique. We call such a form the coupling form of \((E, \pi, \Omega, J_0, j_0)\).
Proof. We first consider the subset $E \setminus E^{\text{crit}}$. Then the form $\Omega$ induces a splitting
\[
\Gamma : \quad T_x E = T_x E^v \oplus T_x E^h
\]
at any regular point $x \in E \setminus E^{\text{crit}}$ where the horizontal space is given by
\[
T_x E^h = \{ \eta \in T_x E \mid \Omega(\eta, \xi) = 0 \forall \xi \in T_x E^v \}
\]
and hence induces a natural (Ehresman) connection on $E \setminus E^{\text{crit}}$ whose monodromy is symplectic.

Because of the closedness of $\Omega$, the connection is Hamiltonian in that its curvature $\text{curv}(\Gamma)$ has its values contained in $\text{ham}(E^\pi_{\Sigma(z)})$, the set of Hamiltonian vector fields of the fiber $E^\pi_{\Sigma(z)}$ and hence induces a natural (Ehresman) connection on $E \setminus E^{\text{crit}}$ whose monodromy is symplectic.

Because of the closedness of $\Omega$, the connection is Hamiltonian in that its curvature $\text{curv}(\Gamma)$ has its values contained in $\text{ham}(E^\pi_{\Sigma(z)})$, the set of Hamiltonian vector fields of the fiber $E^\pi_{\Sigma(z)}$, and hence the restriction $\pi : E \setminus \pi^{-1}(\Sigma^{\text{crit}}) \to \Sigma \setminus \Sigma^{\text{crit}}$ is a smooth Hamiltonian fibration in the sense of [GLS]. In particular, if we restrict to $E \setminus \pi^{-1}(\Sigma^{\text{crit}}) \to \Sigma \setminus \Sigma^{\text{crit}}$ its fibers are all compact and so we can construct a closed 2-form $\Omega' = \Omega + \pi^* \alpha$ on $E \setminus \pi^{-1}(\Sigma^{\text{crit}})$ for some closed two form $\alpha$ on $\Sigma \setminus \Sigma^{\text{crit}}$ that satisfies (5.1) thereon. In fact $\Omega'$ (and so $d\beta$) can be explicitly constructed by requiring
\[
\Omega' (\eta_1^\sharp, \eta_2^\sharp) = H_{\eta_1, \eta_2}(5.2)
\]
where $H_{\eta_1, \eta_2}$ is the smooth function whose restriction to each fiber over a point in $\Sigma \setminus \Sigma^{\text{crit}}$ is uniquely determined by the two requirements

1. $H_{\eta_1, \eta_2}$ generates the Lie algebra element $\text{curv}_1(\eta_1, \eta_2)$ of $\text{Ham}(E^z, \omega_z)$ which is a Hamiltonian vector field
2. it satisfies the normalization condition
\[
\int_{E^z} H_{\eta_1, \eta_2} \omega_z^n = 0, \quad \omega_z = \Omega|_{E^z}
\]
for all $z \in \Sigma \setminus \Sigma^{\text{crit}}$.

This finishes the proof. \hfill \Box

Definition 5.4 (Coupling form). We call the above unique closed 2-form constructed in Theorem 5.3 the coupling form of the Lefschetz Hamiltonian fibration $E \to \Sigma$.

The following result was essentially proved by Seidel Lemma 1.6 [Se3]: Seidel proved this for the context of exact Lefschetz fibrations but the same proof applies if one ignores his consideration of generating functions of $Q$ therein.

Lemma 5.5 (Compare with Lemma 1.6 [Se3]). Let $(E, \pi, \Omega, J_0, j_0)$ be a Lefschetz Hamiltonian fibration, and $x_0$ be a critical point of $\pi$. Then there are smooth families $\Omega^\mu \in \Omega^2(E)$ $0 \leq \mu \leq 1$, such that

1. $\Omega^0 = \Omega$
2. for all $\mu$, $\Omega^\mu = \Omega^0$ outside a small neighborhood of $x_0$
3. each $(E, \pi, \Omega^\mu, J_0, j_0)$ is a Lefschetz Hamiltonian fibration
4. there is a holomorphic Morse chart $(\xi, \Xi)$ around $x_0$ with $\Xi : V \subset \mathbb{C}^{n+1} \to E$ such that $\Xi^* \Omega^1$ agree near the origin with the standard forms $\omega_{\mathbb{C}^{n+1}} = i \sum dx_k \wedge dx_k$.

In fact, if near $E^{\text{crit}}$ we are given a one form $\Theta$ with $\Omega = d\Theta$ as in the exact cases, we can also deform the one-form to $\Theta^\mu$ so that $\Xi^* \Theta^1$ becomes the standard one-form
\[
\theta_{\mathbb{C}^{n+1}} = \frac{i}{4} \sum x_k d\pi_k - \pi_k dx_k.
\]
6. Exact Lagrangian boundary condition and action estimates

Now we consider a subbundle \( i_Q : Q \to \partial \Sigma \) of the symplectic vector bundle \((E|_{\partial \Sigma}, \Omega|_{\partial \Sigma})\) whose fiber \( Q_z \) is a Lagrangian submanifold of \( \Omega_z \) for each \( z \in \partial \Sigma \). We call such \( Q \) a fiberwise-Lagrangian submanifold of \((E|_{\partial \Sigma}, \Omega|_{\partial \Sigma})\).

**Definition 6.1.** We call a fiberwise-Lagrangian submanifold \( Q \subset \partial E \) an exact Lagrangian boundary over \( \partial \Sigma \), if there exists a one-form \( \kappa_Q \) on \( Q \) such that

\[
\kappa_Q|_{T^v(\partial E)} \equiv 0, \quad \text{and} \quad i_Q^* \Omega = d\kappa_Q
\]

where \( i_Q : Q \to \partial E \) is the inclusion map and \( T^v(\partial E) \) is the vertical tangent space of \( \partial E \).

We note that when \( \Sigma \) is oriented and the boundary orientation on \( \partial \Sigma \) is provided by an orientation one-form, denoted by \( d\theta \) with \( \theta \in \partial \Sigma \), the connection induced by the form \( \Omega|_{\partial E} \) enables us to express any such one-form \( \kappa_Q \) as

\[
\kappa_Q(\theta, x) = h_i(\theta, x) \, d\theta
\]

for \((\theta, x) \in \partial E\) with \( \theta \in \partial \Sigma \) and \( h_i : \partial E \to \mathbb{R} \) where \( \partial \Sigma = \bigsqcup \partial_i \Sigma \). The function \( h_i \) is unique up to the addition of the function \( c_i : \partial \Sigma_i \to \mathbb{R} \).

**Definition 6.2.** We define \( \|\kappa_Q\|_{(1, \infty)} \) by

\[
\|\kappa_Q\|_{(1, \infty)} = \int_{\partial \Sigma} \text{osc}(h_\theta) \, d\theta
\]

with \( \text{osc}(h_\theta) := \max_{x \in E_i} h(\theta, x) - \min_{x \in E_i} h(\theta, x) \), and call it the \( L^{(1, \infty)} \)-norm of \( \kappa_Q \).

We remark that \( \|\kappa_Q\|_{(1, \infty)} \) does not depend on the choice of the function \( h \).

To give readers some insight on these definitions, we compare this with the classical notion of exact Lagrangian isotopy \([Gr]\).

**Example 6.3.** Let \( E = (\mathbb{R} \times [0, 1]) \times (M, \omega) \) with \( \pi : (\mathbb{R} \times [0, 1]) \times (M, \omega) \to \mathbb{R} \times [0, 1] \) the projection. Consider the two form \( \Omega = \pi^* \omega \). Let \( L_i \subset (M, \omega) \) be a Lagrangian submanifold and let \( \psi_i : [0, 1] \times L \to M \) be a Lagrangian isotopy for \( i = 1, 2 \). The isotopy \( \psi_i \) is an exact Lagrangian isotopy if there is a smooth function \( h_i : [0, 1] \times L \to \mathbb{R} \) such that \( \psi_i^* \omega = dh_i \wedge dt = d(h_i \, dt) \) \([Gr]\). This definition is a special case of Definition \(6.1\). Just consider the embedding

\[(h_i, \psi_i) : \mathbb{R} \times L_i \to \mathbb{R} \times [0, 1] \times M\]

and set \( Q_i = \text{im}(h_i, \psi_i) \) the Lagrangian suspension, and \( \kappa_{Q_i}(t, x) = h_i(t, x) \, dt \).

We now study the structure of \( \pi_2(E, Q) \). We start with the following relative version of section class. A class \( B \in \pi_2(E, Q) \) is a section class if \( \pi_2 \) is the positive generator with respect to the given orientation of \( \Sigma \). We say that \( B \) is a fiber class if it is in the image of \( \pi_2(M, L) \to \pi_2(E, Q) \) induced from the inclusion of the fiber. The following is proved in Lemma 2.2 \([HL]\) for the smooth Hamiltonian fibration but the same proof applies to the current case with singular fibers. We refer readers to \([HL]\) for its proof.
Lemma 6.4. The following sequence of homotopy groups is exact at the middle term:

\[ \pi_2(M, L) \to \pi_2(E, Q) \to \pi_2(\Sigma, \partial \Sigma). \]

The following proposition is the reason why the notion of exact Lagrangian boundary is relevant to the study of pseudo-holomorphic curves with boundary later. Similar estimates were previously obtained in [En], [Oh5] and [Se3] in somewhat different contexts.

Proposition 6.5. Suppose \( \Sigma \) is oriented and denote by \( d\theta \) a given orientation one-form on \( \partial \Sigma \). Let \( Q \subset \partial E \) over \( \partial \Sigma \) be an exact Lagrangian boundary of \( E \) and \( \kappa_Q \) be a corresponding Hamiltonian one-form. Consider a section \( s : \Sigma \to E \setminus E^{\text{crit}} \) with \( s(\partial \Sigma) \subset Q \subset \partial E \). Then for each given section class \( [s, \partial s] \in \pi_2(E, Q; \mathbb{Z}) \), there exists a constant \( C = C(\kappa_Q, [s, \partial s]) > 0 \) such that the integral bound

\[ \left| \int_{\Sigma} s^* \Omega \right| \leq C = C(B) \]

holds for any section \( s \) in a fixed class \( B = [s, \partial s] \in H_2(E, Q; \mathbb{Z}) \). In fact, we have

\[ \left| \int_{\Sigma} s_2^* \Omega - \int_{\Sigma} s_1^* \Omega \right| \leq \|\kappa_Q\|_{(1, \infty)} \]

(6.1)

for any two such sections with \( [s_1, \partial s_1] = [s_2, \partial s_2] \).

Proof. Recall \( s_i^* \Omega = d\kappa_i \) for a one-form \( \kappa_i \) which exists by definition of exact Lagrangian boundary \( Q \). Let \( s_i, i = 1, 2 \) be two sections of \( E \) with \( s_i(\partial \Sigma) \subset Q \) with \( [s_1, \partial s_1] = [s_2, \partial s_2] \). Then we have a geometric chain \((S, C)\) with

\[ \partial S = s_1 \coprod s_2 \coprod C. \]

and \( \partial C = \partial s_2 - \partial s_1 \) as a chain in \( Q \).

By Stokes’ and closedness of \( \Omega \), we have

\[ 0 = \int S^*(d\Omega) = \int_{\partial S} \Omega = \int_{\Sigma} s_2^* \Omega - \int_{\Sigma} s_1^* \Omega - \int C^* \Omega \]

and hence

\[ \int_{\Sigma} s_2^* \Omega - \int_{\Sigma} s_1^* \Omega = \int_{\Sigma} \Omega. \]

But by the exactness of the fiberwise-Lagrangian subbundle \( Q \) and since \( C \) has its image in \( Q \), we obtain

\[ \int_{\Sigma} \Omega = \int_{\Sigma} d\kappa_Q = \int_{\partial s_2} \kappa_Q = \int_{\partial s_1} \kappa_Q - \int_{\partial s_1} \kappa_Q. \]

Therefore we have obtained

\[ \int_{\Sigma} s_2^* \Omega - \int_{\Sigma} s_1^* \Omega = \int_{\partial s_2} \kappa_Q - \int_{\partial s_1} \kappa_Q = \int_{\partial \Sigma} (h \circ s_2 - h \circ s_1) \, d\theta \]

and so

\[ \left| \int_{\Sigma} s_2^* \Omega - \int_{\Sigma} s_1^* \Omega \right| \leq \int_{\partial \Sigma} (\max_{x \in E} h_{\theta}(x) - \min_{x \in E} h_{\theta}(x)) \, d\theta \leq \|\kappa_Q\|_{(1, \infty)}. \]

Since \( \|\kappa_Q\|_{(1, \infty)} \) does not depend on \( s \), this finishes the proof. \( \square \)
Next we consider the topological index associated to the section \((s,\partial s)\) for \(s\) which does not pass through critical points \(E^{\text{crit}}\). By considering the pull-back \(s^*(TE')\), it defines a symplectic bundle pair \((s^*TE',(\partial s)^*TQ')\) where \(TQ' = TQ = TQ|_{\partial \Sigma}\). Therefore we can associate the Maslov index, which we denote by \(\mu([s,\partial s])\). (See [KL, FOOO2].)

Now we examine topological dependence of \(\mu([s,\partial s])\). Note that each section \((s,\partial s)\) defines an element in \(\pi_2(E,Q)\). We denote the corresponding class by \(s_*([\Sigma,\partial \Sigma])\) where \([\Sigma,\partial \Sigma]\) is the fundamental class which is a generator of \(H_2(\Sigma,\partial \Sigma;\mathbb{Z}) \cong \mathbb{Z}\). The following lemma immediately follows from the definition of the Maslov index for the bundle pair.

**Lemma 6.6.** Suppose \((s_1)_*([\Sigma,\partial \Sigma]) = (s_2)_*([\Sigma,\partial \Sigma])\). Then we have \(\mu([s_1,\partial s_1]) = \mu([s_2,\partial s_2])\).

**Definition 6.7.** We denote by \(\pi^\text{sec}_2(E,Q) \subset \pi_2(E,Q)\) the subset of section classes \([s,\partial s]\) in \(\pi_2(E,Q)\). We say two section classes \(B_1, B_2\) are \(\Gamma\)-equivalent if they satisfy

\[
\Omega(B_1) = \Omega(B_2), \quad \mu(B_1) = \mu(B_2)
\]

and denote by \(\Pi(E,Q)\) the quotient group

\[
\Pi(E,Q) = \pi^\text{sec}_2(E,Q)/\sim.
\]

For the Calabi-Yau Lefschetz fibrations, one can proceed the study of Maslov indices following the exposition given in [Se4]. Consider the bundle of relative quadratic volume forms

\[
K^2_{E/\Sigma} = \pi^*\det^2\mathbb{C}(T\Sigma) \otimes \det^{-2}\mathbb{C}.
\]

By definition, if \(E \to \Sigma\) is Calabi-Yau, we have nowhere zero section \(\eta^2_{E/B}\) of this on \(E \setminus E^{\text{crit}}\). Furthermore, we can require \(\eta^2_{E/B}\) to satisfy

\[
\eta^2_{E/B} = \frac{(dz_1 \wedge \cdots \wedge dz_{n+1})^2}{(2z_1dz_1 + \cdots + 2z_{n+1}dz_{n+1})^{n/2}}
\]

in a neighborhood \(U \subset E\) of each critical point under the given identification \(\pi: U \setminus \{x\} \to \pi(U \setminus \{x\})\) with \(q: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C} \setminus 0\).

**Definition 6.8.** We say that a fiberwise Lagrangian submanifold \(Q \subset E\) is (relatively) graded if there exists a function \(\alpha: Q \to \mathbb{R}\) such that

\[
\exp(2\pi \sqrt{-1}\alpha(x)) = \eta^2_{E/B}(T^x_{E/B}Q), \quad x \in Q.
\]

We call \(\alpha\) an \(L^\infty\)-grading of \(Q\).

For a given pair of a fiberwise Lagrangian submanifold \(Q_1, Q_2 \subset E\) intersecting transversely, we can associate a natural \(\mathbb{Z}\)-grading on the intersection \(Q_1 \cap Q_2\) in the following way.

We consider the two form

\[
\omega_{E,\lambda} = \Omega + \lambda \pi^*\omega_{\Sigma}
\]

which is nondegenerate on \(E \setminus E^{\text{crit}}\). Existence of such \(\lambda > 0\) is easy to check. The following lemma also immediately follows whose proof we leave to the readers.

**Lemma 6.9.** Any fiberwise Lagrangian submanifold \(Q \subset E\) is Lagrangian for \(\omega_{E,\lambda}\).
7. Pseudo-holomorphic sections

In this section, we carry out various studies of geometry and analysis of pseudo-holomorphic sections with Lagrangian boundary condition in the setting of Lefschetz Hamiltonian fibrations.

Let $(E, \pi, \Omega, J_0, j_0)$ be a Lefschetz Hamiltonian fibration and let $x_0 \in E^{crit}$. Denote by $(\xi, \Xi)$ be a holomorphic Morse chart at $x_0$ i.e., a $j_0$-holomorphic coordinates $\xi : U \to S$ with $\xi(0) = z_0 = \pi(x_0)$ where $U \subset \mathbb{C}$ is a neighborhood of the origin and $\Xi : W \to E$ be a $J_0$-holomorphic chart with $W \subset \mathbb{C}^{n+1}$ a neighborhood of the origin in $\mathbb{C}^{n+1}$ with $\Xi(0) = x_0$ such that

$$(\xi^{-1} \circ \pi \circ \Xi)(x) = x_1^2 + \cdots + x_{n+1}^2. \quad (7.1)$$

With $(\xi, \Xi)$ fixed, we denote the model Lefschetz fibration by $q : \mathbb{C}^{n+1} \to \mathbb{C}$ defined by $q(z_1, \cdots, z_n) = z_1^2 + \cdots + z_{n+1}^2$.

Now for the given $(E, \pi, \Omega, J_0, j_0)$ and the holomorphic Morse charts $(\xi, \Xi)$ at each critical points of $E$, we consider an almost complex structure $J$ on $E$ that satisfies

1. $J = J_0$ in a neighborhood of $E^{crit}$,
2. $d\pi \circ J = j \circ d\pi$,
3. $\Omega(\cdot, J \cdot)|_{TE_x}$ is symmetric and positive definite for any $x \in E$.

Following Seidel [Se3], we call such $J$ compatible relative to $j$. An immediate consequence of the definition is the following lemma

**Lemma 7.1.** Let $J$ be compatible relative to $j$. Then for any given area form $\omega_\Sigma$ on $\Sigma$ with $\int_\Sigma \omega_\Sigma = 1$, the two form $\Omega + \lambda \pi^* \omega_\Sigma$ tames $J$ for all sufficiently large $\lambda$.

We refer to [Se3] for a more complete explanation of the structure of $J$’s compatible to $j$.

7.1. Energy estimates and Hamiltonian curvature. Next we study the energy estimates of pseudo-holomorphic sections in terms of the topological action $\int u^* \Omega$ and the contribution coming from the curvature integral of $\int_\Sigma \omega_\Sigma$ of a canonical symplectic connection of the Hamiltonian fibration $E \to \Sigma$ associated to the coupling form $\Omega$ defined in Definition 5.4. This kind of estimates has been studied in [Se3, Oh5, MS].

Using the connection associated to $\Omega$, we decompose $Du = (Du)^v + (Du)^h$ into the vertical and horizontal components. Now we consider the symplectic form $\omega_E = \Omega + \lambda \pi^* \omega_\Sigma$ with $\omega_\Sigma$, an area form on $\Sigma$ with $\int_\Sigma \omega_\Sigma = 1$. We like to remark that an almost complex structure $J$ compatible to $j$ is not compatible in the usual sense in that the bilinear form

$$\langle \cdot, J \cdot \rangle$$

may not be symmetric. However if $\lambda$ is sufficiently large, it is tame to $\omega_E$ (see Lemma 2.1 [Se3]). Therefore we can symmetrize this bilinear form and define the associate metric $g_J$ by

$$\langle V, W \rangle = g_J(V, W) := \frac{1}{2}(\Omega(V, JW) + \Omega(W, JV)).$$

(7.2)

We call (7.2) the metric associated to $J$ and denote

$$|V|^2 = |V|^2_J = g_J(V, V).$$
With respect to this metric, we still have the following basic identity whose proof we omit.

**Lemma 7.2.** Let \( s : \Sigma \to E \) be any \( J \)-holomorphic map \( v \). Then we have

\[
\frac{1}{2} \int |Ds|^2 = \int s^*\omega_E
\]

and

\[
\frac{1}{2} \int |Ds|^2 = \int s^*\omega_E + \int |\nabla J s|^2. \tag{7.3}
\]

We decompose \( Ds = (Ds)^v + (Ds)^h \) into vertical and horizontal parts and write

\[
|Ds|^2 = |(Ds)^v|^2 + |(Ds)^h|^2 + 2\langle (Ds)^v, (Ds)^h \rangle.
\]

Then it is straightforward to prove

\[
|(Ds)^h|^2\omega = 2(s^*\Omega + \lambda\omega) \tag{7.4}
\]

by the identity

\[
\sum_{i=1}^{2} |(Ds)^h(e_i)|^2 = \sum_{i=1}^{2} \omega_{E,\lambda}((Ds)^h(e_i), (Ds)^h(e_i))
\]

\[
= \sum_{i=1}^{2} (\Omega + \lambda\pi^*\omega)(e_i, (Ds)^h(e_i))
\]

\[
= 2(\omega_E((Ds)^h(e_1), (Ds)^h(e_2)) + \lambda\omega_{\Sigma}(e_1, e_2))
\]

for an orthonormal frame \( \{e_1, e_2\} \) and then applying the curvature identity

\[
d(\Omega(e_1^\# , e_2^\# )) = -[e_1^\# , e_2^\# ]|\Omega : \text{fiberwise}
\]

Here \( e_i^\# \) is the horizontal lift of \( e_i \) and we have

\[
\text{curv}_\Gamma(e_1, e_2) = \Omega(e_1^\# , e_2^\# )
\]

by the definition of coupling form (see (1.12) [GLS] but with caution on the sign convention). The following is an immediate corollary of (7.3).

**Proposition 7.3.** Let \( \Omega \) be the coupling form of \( E \). Suppose that \( \Omega + \lambda\pi^*\omega_{\Sigma} \) for \( \lambda > 0 \) is positive, i.e., symplectic for an area form \( \omega_{\Sigma} \) on \( \Sigma \). Then we have the inequality as a two-form

\[
s^*\Omega + \lambda\omega_{\Sigma} \geq 0 \tag{7.5}
\]

for any \( J \)-holomorphic section \( s \). In other words, if we write \( s^*\Omega = f(s)\omega_{\Sigma} \) for a function \( f : \Sigma \to \mathbb{R} \), then we have \( f(s) + \lambda \geq 0 \).

**Proof.** Just choose a complex structure \( j \) on \( \Sigma \) and a \( j \)-compatible \( J \) on \( E \). Positivity (7.5) immediately follows from (7.4). \( \square \)
7.2. **Gromov-Floer moduli space of $J$-holomorphic sections.** We first translate the anchor introduced in section 3 in the setting of pointed Lagrangian boundary conditions in Hamiltonian fibrations over the surface $\Sigma$ with boundary $\partial \Sigma \neq \emptyset$. This will be needed to study morphisms between two Floer chain modules constructed via the moduli space of pseudoholomorphic sections of Lefschetz fibrations over $\Sigma$.

Let $y \in M$ be a base point and let $(L, \gamma), (L', \gamma')$ be two anchored Lagrangian submanifolds of $(M, \omega)$, which intersect transversely. Let $\gamma$ and $\gamma'$ be the paths $\gamma(0) = \gamma'(0) = y$ and $\gamma(1) \in L, \gamma'(1) = L'$ given as in the anchor data.

Now to each intersection $p \in L \cap L'$, we associate a Hamiltonian fibration over $[0, 1]^2$. The paths $\gamma$ and $\gamma'$ provide a path in $M$ along $\{0\} \times [0, 1]$ via the obvious concatenation of $\gamma$ and $\gamma'$ with the midpoint given by $y$.

We take the trivial fibrations $E = [0, 1]^2 \times (M, \omega) \to [0, 1]^2$. Then for each given pair $[p, w]$ with $p \in L \cap L'$ and with a bounding strip $w : [0, 1]^2 \to M$ such that

$$
\begin{align*}
    w(0, t) &= \tilde{\gamma}(t), \quad w(\xi) \equiv p \\
    w(s, 0) &\in L_0, \quad w(s, 1) \in L_1,
\end{align*}
$$

we can associate a section of $s_{[p, w]} : [0, 1]^2 \to E$ by

$$(s, t) \mapsto ((s, t), w(s, t)) : [0, 1]^2 \to [0, 1]^2 \times (M, \omega).$$

We call this fibration over $[0, 1]^2$ with a section $s_{[p, w]}$ an anchor cap associated to $[p, w]$ relative to the given anchor. For the notational convenience, we denote the corresponding fibration with the fiberwise Lagrangian submanifolds by

$$(E_{[p, w]}; [0, 1] \times \{0\} \times L_0, [0, 1] \times \{1\} \times L_1; s_{[p, w]})$$

or simply as $(E_{[p, w]}; s_{[p, w]})$.

Note that the set of homotopy classes $[p, w]$ of $w$ relative to the boundary condition $[\tilde{s}]$ has one-one correspondence with the homotopy class of sections with the obvious corresponding boundary condition and asymptotic boundary condition at $\pm \infty$ respectively.

For a given compact surface $\Sigma$ with marked points $\tilde{\zeta}$, we consider the corresponding surface $\tilde{\Sigma} = \Sigma \setminus \tilde{\zeta}$ with punctures. We denote the given preferred holomorphic chart $\varphi_{\zeta} : D_{\zeta} \subset \Sigma \to D^+$ of the half disc $D^+ = D \cap \{\text{im}(z) \geq 0\}$ with $\varphi_{\zeta}(\zeta) = 0$.

We also have a local trivialization

$$
\Phi_{\zeta} : E|_{D_{\zeta}\setminus\{\zeta\}} \to D^+ \setminus \{0\} \times M
$$

lying over $\varphi_{\zeta}$. When $Q \subset E$ is a Lagrangian boundary condition, we have a unique pair $L_{\zeta, \pm}$ of Lagrangian submanifolds of $M$ such that

$$
\Phi_{\zeta}(Q) = [-1, 0) \times L_{\zeta, -} \cup (0, 1] \times L_{\zeta, +}.
$$

For the given ordered chain of Lagrangian boundaries $Q = (Q_0, Q_1, \ldots, Q_k)$, denote $Q = \cup_i Q_i$. Then we require the unique pair $L_{\zeta, \pm}$ at $\zeta_i$ to be

$$
\Phi_{\zeta_i}(Q) = [-1, 0) \times L_i \cup (0, 1] \times L_{i+1}.
$$

at each $\zeta_i$. In this way, for each given $(\pi : E \to \Sigma; Q)$, we consider the moduli space

$$
\mathcal{M}_J(E, Q; \pi^\# B)
$$

of smooth $J$-holomorphic sections for each section class $B \in \pi_2(E, Q; \pi^\# B)$.

The following lemma immediately follows from definition.
We note that when \( C \phi \), where \( \text{Aut} \) there is a natural

\[
\text{We consider the pairs} \quad M \quad \text{such that there are only finitely many section classes}
\]

\[
\text{Once we have this energy estimate, it follows by a standard compactness argument}
\]

\[
\text{We denote by} \quad \Omega(\Sigma) \quad \text{in general. We note that there is no non-trivial holomorphic automorphisms of} \quad \text{this energy estimate, it follows by a standard compactness argument}
\]

\[
\text{Finally we state the Gromov-Floer type compactness of} \quad \text{in the interior of} \quad \Sigma \quad \text{and on the boundary} \quad \partial \Sigma \quad \text{for the later purpose. For this we introduce additional marked points}
\]

\[
\text{Definition 7.5. A configuration on} \quad \Sigma \quad \text{is the set of of finite points consisting of}
\]

\[
\text{We denote by} \quad \tilde{C}_{m;i} \quad \text{the set of such configurations.}
\]

\[
\text{We denote such a configuration by}
\]

\[
\text{in general. We note that there is no non-trivial holomorphic automorphisms of} \quad C \quad \text{except the following cases:}
\]

\[
\text{We consider the pairs}
\]

\[
\text{For \( [n_1, w_1] \) with \( i = 0, \cdots, k \) realizing} \quad B = (-[w_0])\#([w_1] \# \cdots \# [w_k]), \quad \text{we have the identity}
\]

\[
\int w_0^* \omega = \sum_{i=1}^k \int w_i^* \omega - \Omega(B) \quad (7.7)
\]

\[
\text{or equivalently}
\]

\[
\Omega(B) = \sum_{i=1}^k \int w_i^* \omega - \int w_0^* \omega. \quad (7.8)
\]

\[
\text{Since any section} \quad s \quad \text{in class} \quad [s, \partial s] = B \quad \text{that satisfies the asymptotic condition}
\]

\[
\text{Finally we state the Gromov-Floer type compactness of} \quad \mathcal{M}_J(E, Q; \tilde{p}; B) \quad \text{in a}
\]

\[
\text{Definition 7.5. A configuration on} \quad \tilde{\Sigma} \quad \text{is the set of of finite points consisting of}
\]

\[
\text{We denote by} \quad \tilde{C}_{m;i} \quad \text{the set of such configurations.}
\]

\[
\text{We denote such a configuration by}
\]

\[
\text{in general. We note that there is no non-trivial holomorphic automorphisms of} \quad C \quad \text{except the following cases:}
\]

\[
\text{We consider the pairs}
\]

\[
\text{There is a natural} \quad \text{Aut}(\tilde{\Sigma})-\text{action on the product} \quad \tilde{\mathcal{M}}_J(E, Q; \tilde{p}; B) \times \tilde{C}_{(m;i)} \quad \text{defined by}
\]

\[
(s, C) \mapsto (s \circ \phi^{-1}, \phi(C))
\]

where \( \phi \in \text{Aut}(\tilde{\Sigma}) \). We define \( \mathcal{M}_{J,(m;i)}(E, Q; \tilde{p}; B) \) to be the quotient

\[
\mathcal{M}_{J,(m;i)}(E, Q; \tilde{p}; B) = \tilde{\mathcal{M}}_J(E, Q; \tilde{p}; B) \times \tilde{C}_{(m;i)}/\text{Aut}(\tilde{\Sigma}).
\]

We note that when \( C \neq \emptyset \) we have the natural evaluation maps

\[
ev: \mathcal{M}_{(m;i)}(E, Q; \tilde{p}; B) \to E^m \times \prod_{i=0}^k Q_i^{n_i}.
\]
which respect the above mentioned $Aut(\Sigma)$-action and so well-defined.

We denote by $\tilde{\mathcal{M}}_1(E_z, J_z; \alpha_z)$ the stable maps of genus 0 with one marked point, and by $\tilde{\mathcal{M}}_1(E_z, Q_z, J_z; \beta_z)$ the set of bordered stable maps with one marked point at a boundary of the disc. We consider the fiber product

$$\tilde{\mathcal{M}}_{(m; \beta)}(E, Q; \tilde{p}; B_0) = ev \times ev \left( \prod_i \mathcal{M}(E_z, J_z; \alpha_i) \times \prod_{j=0}^k \prod_{j \neq 1}^{n_j} \mathcal{M}(E_z, Q_z, J_z; \beta_j^{(i)}) \right) (7.10)$$

with respect to the obvious evaluation maps.

7.3. Bubble may hit critical points. In this subsection, we analyze the failure of convergence of a sequence of smooth pseudo-holomorphic sections $\mathcal{M}_f(E, Q; \tilde{p}; B)$ with

$$\text{vir. dim } \mathcal{M}_f(E, Q; \tilde{p}; B) = 0 \quad (7.11)$$
of symplectic Lefschetz fibrations $E \subset \Sigma$ with the asymptotic condition provided by $\tilde{p}$. We will be especially interested in bubble components passing through critical points.

By definition, the pull-back $\Xi \ast J$ is the standard complex structure on $\mathbb{C}^{n+1}$ and $\xi \ast J$ the standard one on $\mathbb{C}$ and $\xi^{-1} \circ \pi \circ \Xi = q$ on the neighborhoods $W \subset \mathbb{C}^{n+1}$, $U \subset \mathbb{C}$ of the origins respectively corresponding to the given holomorphic Morse chart $(\xi, \Xi)$ at each $x_0 \in E^{crit}$.

We denote by $W_{x_0} \subset E$, $U_{x_0} \subset \Sigma$ be the corresponding neighborhoods of $x_0$ and $\pi(x_0)$ respectively. We also denote $B^{2n+2}(r)$ the ball of radius $r > 0$ in $\mathbb{C}^{n+1}$ with its center at the origin, and $B_{x_0}(r)$ its image under $\Xi$ at $x_0 \in E^{crit}$, and similarly for $B^2(\varepsilon)$ and $B_{x_0}(\varepsilon)$.

Since we assume that there are finitely many critical points and different critical points lie in different fibers of $\pi : E \to \Sigma$, we have constants $\varepsilon$, $r > 0$ such that

$$B_{\pi(x)}(\varepsilon) \cap B_{\pi(x')}(<\varepsilon) = \emptyset \quad \text{for } x \neq x' \text{ with } x, x' \in E^{crit} \quad (7.12)$$

and

$$\pi(B_{x_0}(r)) \supset B_{x_0}(\varepsilon) \quad \text{for all } x_0 \in E^{crit}. \quad (7.13)$$

Lemma 7.6. The graph of any differentiable section does not intersect $E^{crit}$.

Proof. Since $s$ is a section, we have $\pi \circ s = Id$. By differentiating this, we obtain $d\pi \circ ds = Id$. In particular, $d\pi$ is surjective at any point $s(z)$, i.e., $s(z)$ must be a regular point of $\pi$ and hence $s(z) \in E \setminus E^{crit}$. $\square$

Obviously we have the inequality

$$\text{dist}(s(z), E^{crit}) \geq C > 0 \quad \text{for } z \in \Sigma \setminus \cup_{\varepsilon \in E^{crit}} B_{\pi(z)}(\varepsilon) \quad (7.14)$$

where $C = C(\xi, E^{crit})$ is a constant depending only on $\varepsilon$ and the holomorphic Morse chart $(\xi, \Xi)$ independent of $s$. Now we consider the restriction of $s$ on $\cup_{\varepsilon \in E^{crit}} B_{\pi(z)}(\varepsilon)$. On this neighborhoods, we can identify the section $s$ to the holomorphic map

$$f : B^2(\varepsilon) \subset \mathbb{C} \to B^{2n+2}(r) \subset \mathbb{C}^{n+1}$$
satisfying $q \circ f(z) = z$ for all $z \in B^2(\varepsilon)$, i.e.,

$$f_1^2(z) + \cdots + f_{n+1}^2(z) = z.$$

In particular, we have

$$|f(z)|^2 = \sum_{j=1}^{n+1} |f_j^2(z)| \geq |f_1^2(z) + \cdots + f_{n+1}^2(z)| = |z|$$
for all $z \in B^2(\varepsilon)$. Therefore we obtain
\[
|f(z)| \geq \sqrt{\varepsilon} \quad \text{for } z \in \partial B^2(\varepsilon).
\] (7.15)

The following theorem is the main theorem proved in this section.

**Theorem 7.7.** Suppose that the Lefschetz Hamiltonian fibration with Lagrangian boundary $Q \subset E|_{\partial \Sigma}$ such that $E$ is relative Calabi-Yau and $Q$ with vanishing fiberwise Maslov class. Then there exists a dense subset of $j$-compatible $J$’s such that for any such $J$, there exists a constant $C > 0$ depending only on $(E, Q, J, j)$, the section class $[s]$ and $\varepsilon > 0$ such that we have
\[
\text{dist}(\text{Im } s, E^{\text{crit}}) \geq C
\] (7.16)
for any smooth section $s : \Sigma \to E$.

We prove Theorem 7.7 by contradiction. Let $J$ be any $j$-compatible and suppose that there is a sequence $s_i$ of smooth $J$-holomorphic sections such that
\[
\min_{z \in \Sigma} \text{dist}(s_i(z), E^{\text{crit}}) \to 0.
\] (7.17)
Since $E^{\text{crit}}$ is a finite set and by (7.12), we may choose critical point $x_0 \in E^{\text{crit}}$ and a sequence $z_i \in \Sigma$ so that
\[
\text{dist}(s_i(z_i), x_0) \to 0.
\] (7.18)
By choosing a subsequence of $z_i$ if necessary, we may assume $z_i \to z_0$ and so
\[
z_i \in B_{z_0}(\varepsilon).
\] (7.19)
for all $i$. By the Gromov-Floer convergence applied to $J$-holomorphic curves $s_i : (\Sigma, j) \to (E, J)$, which are also $J$-holomorphic sections, there exists a subsequence which converges to
\[
s_\infty = s_0 + \text{“bubble components”}
\]
where $s_0$ is a smooth section of $E \to \Sigma$ and each bubble must be either a fiberwise pseudo-holomorphic sphere or a fiberwise pseudo-holomorphic disc. And each disc bubble has its boundary lying in the given Lagrangian boundary condition.

Due to the property (7.19), at least one bubble must pass through the critical point $x_0$ whose image is contained in $E_{z_0}$. By the connectedness of the image of the limit, this bubble is contained in a bubble tree rooted at a point $z_1 \in \Sigma$ in the principal component $(s_0, \Sigma)$. The image of this bubble tree itself must be contained in the same fiber $E_{z_0}$. Denote this bubble tree by $(v, (C, z))$, $z \in C$ which is a stable map in $E$ such that
\[
v(z) = s_0(z_1) \in E.
\]
However since $E_{z_0}$ contains a singularity $x_0$ and so is not a smooth manifold, we need further clarification on the bubble component passing through $x_0$. Since $\pi : E \to \Sigma$ is isomorphic to the standard Lefschetz fibration
\[
q(z_1, \cdots, z_{n+1}) = z_1^2 + \cdots + z_{n+1}^2
\]
near $x_0$, there is a well-defined “multiplicity” of the component at the critical point $x_0$. We now make this statement precise in the next subsection.
7.4. Proper holomorphic curves in $E_{z_0} \setminus \{x_0\}$. Using the holomorphic Morse chart $(\xi, \Xi)$ at the critical point $x_0 \in E_{z_0} \subset E$, we consider the decomposition

$$E = B_{x_0}(\delta) \coprod (E \setminus B_{x_0}(\delta))$$

where $B_{x_0}(\delta) = \Xi^{-1}(B^{2(n+1)}(\delta)) \cap E_{z_0}$ for $0 < \delta < \varepsilon$.

Now we consider the hypersurface

$$q^{-1}(0) = \{(x_1, \cdots, x_{n+1}) \mid x_1^2 + \cdots + x_{n+1}^2 = 0\}.$$ 

The only singularity of this hypersurface is $0 \in \mathbb{C}^{n+1}$ and so $q^{-1}(0) \setminus \{0\}$ is a smooth complex hypersurface of $\mathbb{C}^{n+1} \setminus 0$. We denote by $\theta_{\mathbb{C}^{n+1}}$ the one form

$$\theta_{\mathbb{C}^{n+1}} = \frac{i}{4} \sum x_k d\tau_k - \tau_k dx_k$$

and the standard Kähler form

$$\omega_{\mathbb{C}^{n+1}} = -d\theta_{\mathbb{C}^{n+1}} = \sum dq_k \wedge dp_k$$

where $x_k = q_k + ip_k$. We denote by $\theta$ and $\omega$ the restriction of these to $q^{-1}(0) \setminus \{0\}$.

Following [Se3], we denote $T = T^*S^n$ and $T(0) = \text{the zero section of } T$ and by $\theta_T$ and $\omega_T = -d\theta_T$ the standard Liouville one-form and the standard symplectic form on the cotangent bundle $T$. We identify $T$ with the subset

$$\{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \langle u, v \rangle = 0, \|v\| = 1\}$$

and then consider the map

$$\Phi : q^{-1}(0) \setminus \{0\} \to T \setminus T(0)$$

defined by

$$\Phi(x) = (\text{im}(\hat{x})\|\text{re}(\hat{x})\|, \text{re}(\hat{x})\|\text{re}(\hat{x})\|^{-1})$$

where $se^{i\alpha}$ are polar coordinates on the base of $q : \mathbb{C}^{n+1} \to \mathbb{C}$, and $\hat{x} = e^{-i\alpha/2}$. We note that this map is equivariant with respect to the canonical $O(n+1)$-actions on $q^{-1} \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$ and $T = T^*S^n$.

The following is a consequence of straightforward calculation, which is a restriction of the identity in p. 1014 [Se3].

**Lemma 7.8.** $\Phi$ is a diffeomorphism such that

$$\Phi^*\theta_T = \theta$$

and so $\Phi^*\omega_T = \omega$. In particular the symplectic manifold $(q^{-1}(0) \setminus \{0\}, \omega)$ is symplectomorphic to the cotangent bundle $T = T^*S^n$.

This lemma shows that $E_{z_0} \setminus \{x_0\}$ is a symplectic manifold with negative cylindrical end whose asymptotic boundary is symplectomorphic to the unit co-sphere bundle $S^1(T^*S^n)$. Furthermore complex structure on $\Xi^{-1}(W)$ is required to be induced from the standard complex structure from $q^{-1}(0) \setminus \{0\} \subset \mathbb{C}^{n+1} \setminus \{0\}$. Therefore any $J$-compatible $J$ provides a translational invariant almost complex structure with respect to the cylindrical structure and any $J$-holomorphic curve is genuinely holomorphic near the end of $q^{-1}(0) \setminus \{0\}$.

In particular, any such curve converges to a Reeb orbit of $S^1(T^*S^n)$ with a finite multiplicity. This motivates us to study the moduli problem of proper $J$-holomorphic curves from $\mathbb{C} \cong \mathbb{C}P^1 \setminus \{N\}$ with the asymptotic boundary condition given by $(\gamma, k)$ where $\gamma$ is a simple Reeb orbit of $S^1(T^*S^n)$ and a multiplicity $k \in \mathbb{Z}_+$. We denote by $\widehat{R}_1(S^1(T^*S^n))$ the set of parameterized Reeb orbits on
S^1(T^*S^n) with period 2\pi and by \mathcal{R}_1(S^1(T^*S^n)) the quotient by the natural S^1-action, i.e., the set of unparameterized Reeb orbits.

We denote by (s, \Theta) the cylindrical coordinates of T \setminus T(0) where s and \Theta are defined by

\begin{align*}
    s(q, p) = \|p\|, \quad \Theta(q, p) = \left(q, \frac{p}{\|p\|}\right).
\end{align*}

Note that all geodesics on S^n have the same period which implies that the contact manifold S^1(T^*S^n) is foliated by Reeb orbits all of which have the same period.

We denote the corresponding cylindrical coordinates on B_{2\eta}(\varepsilon) \setminus \{x_0\} \subset E \setminus \{x_0\} by the same letters (s, \Theta). By a suitable translation of s-coordinates, we may assume the identification

\begin{align*}
    (s, \Theta) : B_{2\eta}(\varepsilon) \setminus \{x_0\} \to (-\infty, 0) \times S^1(T^*S^n).
\end{align*}

Now we are ready to define the moduli space of our interest. For the simplicity of notation, we denote

\begin{align*}
    E_{z_0} \setminus \{x_0\} = E^*_z,
\end{align*}

and \tilde{S} = S \setminus \{z_0\} is either an open Riemann surface isomorphic to \mathbb{C} or an open Riemann surface with boundary isomorphic to \mathbb{C} \setminus D^1(1). We fix an analytic coordinates z = e^{r+i\tau} in a neighborhood of \varepsilon \in \eta. Let u : \tilde{S} \to E^*_z \setminus \{x_0\} be a pseudo-holomorphic curve with Lagrangian boundary condition

\begin{align*}
    u(\partial S) \subset Q_{z_0} \subset E^*_z \setminus \{x_0\}.
\end{align*}

Since the treatment of the latter is essentially similar to the former, we will focus on the former case in the following exposition. We will briefly mention the latter case in the end of our discussion.

By the properness and exponential convergence property of u as \tau \to -\infty, we have

\begin{align*}
    \lim_{\tau \to -\infty} u|_{(-\infty, -R] \times S^1} \subset B_{2\eta}(\varepsilon)
\end{align*}

for a sufficiently large R > 0. It is proved in [H] that

\begin{align*}
    T = \lim_{\tau \to -\infty} \int (\Theta \circ u_\tau)\star \lambda
\end{align*}

with T = 2\pi k for some integer k \geq 1 where u_\tau(t) := u(\tau, t). Then it is proved in [H, Bo] that there exist constants C, \delta > 0 depending only on (S^1(T^*S^n), \lambda) such that

\begin{align*}
    \lim_{\tau \to -\infty} \text{dist}(u(\tau/T, t/T), u_\gamma(\tau + t_0, t + t_0)) \leq C e^{-\delta|\tau|}
\end{align*}

for some simple Reeb orbit \gamma of S^1(T^*S^n) and \gamma_0 \in \mathbb{R} and \gamma_0 \in S^1. (See also [FOOO4] for a proof of similar exponential convergence result in the relative context.) Here u_\tau : [0, \infty) \times S^1 \to (-\infty, 0] \times S^1(T^*S^n) denotes the trivial cylinder map u_\tau(t) = (\tau, \gamma(t)).

By the above discussion, we can now define the following moduli spaces for each given integer k \geq 1 and a homotopy class A.

**Definition 7.9.** Let \gamma \in \mathcal{R}_1(S^1(T^*S^n)) and k \in \mathbb{Z}_+. For each given (\gamma, k), we define

\begin{align*}
    \mathcal{M}_{z_0}^{SFT}(E^*, J_0, \gamma; A, k) = \{ u : \hat{\Sigma} \to E^*_z \setminus \{0\} | J_0 u = 0, \quad \int u^*\Omega < \infty, \quad \lim_{\tau \to -\infty} u(\tau/2\pi k, t/2\pi k) = u_\gamma(t) | [u] = A \}
\end{align*}
We then define
\[ M_{SF T}^{z_0}(E^*, J_0; A, k) = \bigcup_{\gamma \in \mathcal{R}_1(S^1(T^*S^n))} M_{SF T}^{z_0}(E^*, J_0, \gamma; A, k). \]

The following general index formula can be derived from Corollary 5.4 [Bo]. In this regard, we note that the dimension of the space \( \mathcal{R}_1(S^1(T^*S^n)) \) of simple Reeb orbits of \( S^1(T^*S^n) \) is \( n \).

**Proposition 7.10.** Let \( u \in M_{SF T}^{z_0}(E^*, J_0; A, k) \). Then we have
\[ \text{Index } D_u \bar{\mathcal{F}}_J = \left( -\mu_{CZ}(\gamma) + \frac{n}{2} + (n - 3) + 2c_1(u; \phi_\gamma) \right) \]
where \( \mu_{CZ} \) is the generalized Conley-Zehnder index defined by Robbin and Salamon [RS].

For the reader’s convenience, we provide the precise definitions of \( \mu_{CZ}(\gamma) \), and \( c_1(u; \phi_\gamma) \) in the Appendix. Once the definitions are made precise, its proof follows from that of [Bo].

The following transversality result is an easy consequence of the standard argument whose proof is omitted: We first note that we have
\[ \mu_{CZ}(\gamma) = -\text{Morse}(\gamma) + \frac{\dim \mathcal{R}_{sim}}{2} \]
for the Reeb orbit at the negative end. (See e.g., [Mo] and Corollary 1.7.4 [EGH] for such a formula.)

**Proposition 7.11.** Let \( x_0 \in E^{crit} \). There exists a dense subset \( \mathcal{J}^{tr}(x_0) \) of the set \( \mathcal{J}(j) \) of \( \Omega \)-compatible almost complex structures such that \( M_{SF T}^{z_0}(E^*, J_0; A, k) \) is Fredholm-regular and so becomes a smooth manifold of dimension
\[ -\text{Morse}(\gamma) + (n - 3) + 2c_1(u; \phi_\gamma) \]
for any \( k \geq 1 \).

An immediate corollary of this proposition is the following vanishing result.

**Corollary 7.12.** Let \( x_0 \in E^{crit} \) and \( E^* = E \setminus \{x_0\} \). Suppose the relative Maslov class of \( E \to \Sigma \) is zero. Then for any \( J_0 \in \mathcal{J}^{tr}(x_0) \), \( M_{SF T}^{z_0}(E^*, J_0; A, k) = \emptyset \) for all \( A \) and \( k \).

**Proof.** By the assumption, we have \( c_1(A) = 0 \) for all \( A \in \pi_2(E^*_0; z_0) \). Furthermore the Morse index for the simple closed geodesic of \( S^n \) is given by \( n - 1 \) and greater than \( n - 1 \) for multiple geodesics. Therefore we derive
\[ \dim M_{SF T}^{z_0}(E^*_0, J_0; A, k) \leq -(n - 1) + n - 3 = -2 \]
for all \( k \geq 1 \) from (7.22) and hence the proof. \( \square \)

Finally we briefly mention the case in which \( \hat{S} \cong \mathbb{C} \setminus D^2(1) \) and \( u(\partial S) \subset Q_{z_0} \). Since we assume that \( Q \) has vanishing fiberwise Maslov class, the corresponding moduli space
\[ M_{SF T}^{z_0}(E^*_0, Q_{z_0}; J_0; A, k) \]
has its dimension exactly same as that of \( M_{SF T}^{z_0}(E^*_0, J_0; A, k) \). Therefore the same dimension counting argument applies in the exactly same way. (See e.g., [Mo] for a discussion on the dimension formula in the context with Lagrangian boundary.)
We would like to note that the above dimension counting argument strongly relies on the fact that $E$ is (fiberwise) Calabi-Yau so that $c_1(u; \phi_\ast) = 0$ and $Q$ has vanishing fiberwise Maslov class. It seems to be interesting to investigate how the long exact sequence will be transformed in other contexts like in the Fano case.

7.5. **Bubble does not hit critical points.** In this subsection, we restrict ourselves to the case of vanishing relative first Chern class $c_1(E) = 0$.

We prove Theorem 7.7 in this subsection. In fact, it will be enough to take

$$\mathcal{J}^{tr} := \bigcap_{x_0 \in E \setminus E^{\crit}} \mathcal{J}^{tr}(x_0)$$

for the dense subset of $j$-compatible $J$’s.

**Proof of Theorem 7.7.** Let $J \in \mathcal{J}^{tr}$ defined as above.

We have derived before by the Gromov-Floer compactness applied to $M_J(E, Q; \vec{p}; B)$ in $E$ that there exists a subsequence which converges to $s_\infty = s_0 + \text{“bubble components”}$ where $s_0$ is a smooth section of $E \to \Sigma$ and each bubble tree is contained in a fiber which consists of either fiberwise pseudo-holomorphic spheres or discs. By Corollary 7.12 there cannot be any bubble tree passing through a critical point of $E$ and so contained in $E \setminus E^{\crit}$. Since the principal component $s_0$, which is smooth, cannot pass through a critical point by Lemma 7.6, we have proved that the Gromov-Floer limit of $s_i : \Sigma \to E \setminus E^{\crit}$ does not pass any critical point of $E$. Therefore the compactification $\overline{M}_J(E, Q; \vec{p}; B)$ of $M_J(E, Q; \vec{p}; B)$ has image contained in $E \setminus E^{\crit}$.

Once we have achieved this, the rest of the proof follows, in the same way as in the case of smooth Hamiltonian fibration, by the standard dimension counting argument from the fact that $E \to \Sigma$ is a Hamiltonian Lefschetz fibration with vanishing relative Maslov class: We consider the evaluation maps

$$ev : \mathcal{M}_{(m; \vec{n})}(E, Q; \vec{p}; B) \to E^m \times \prod_{i=0}^k Q_{n_i} \times \prod_{j=0}^k \prod_{i=1}^{n_j} M(E_{z_i}, Q_{z_i}, J_{z_i}; \beta_{j}^{(i)})$$

and

$$ev_1 : \mathcal{M}_1(E_z, J_z; \alpha_z) \to E_z, \quad ev_1 : \mathcal{M}_1(E_z, Q_z, J_z; \beta_z) \to E_z$$

for $z \in \Sigma \setminus \Sigma^{\crit}$ and consider the fiber product

$$\overline{M}_{(m; \vec{n})}(E, Q; \vec{p}; B_0; \{\alpha_i\}, \{\beta_j\}) := \overline{M}_{(m; \vec{n})}(E, Q; \vec{p}; B_0)$$

$$ev \times ev_1 \left( \Pi_i \mathcal{M}(E_z, J_z; \alpha_i) \times \prod_{i=0}^k \prod_{j=1}^{n_j} \mathcal{M}(E_{z_i}, Q_{z_i}, J_{z_i}; \beta_{j}^{(i)}) \right)$$

with respect to the obvious evaluation maps.

Note that the dimension of the moduli space of holomorphic sphere in any class in a fiber $E_z \setminus E^{\crit}$ has virtual dimension given by $2n - 6$ and so

$$\text{vir. dim} \bigcup_{z \in \Sigma} \mathcal{M}_1(E_z, J_z; \alpha_z) = 2n - 4.$$
each fiber has at least codimension 4 which obviously avoids the images of pseudo- holomorphic sections in the classes whose associated moduli space has dimension zero. This proves that there cannot be any bubble passing through critical points of \( E \) in the limit and hence the above fiber product, for a generic choice of \( J \), becomes empty. The relevant Fredholm theory needed to carry out this kind of dimension counting argument is by now standard. We refer to section 2 \[Se3\] for an elegant exposition on this Fredholm theory in the context of exact Lefschetz fibrations which applies to the current context of Lefschetz Hamiltonian fibrations without change. This proves that there exist a dense subset \( \mathcal{J}^{reg,tr} \subset \mathcal{J}^{tr} \) for any \( \bar{p} \in \pi_2(E, Q) \).

In particular there exists a constant \( C = C(E, Q, \bar{p}) > 0 \)
\[
\min_{z \in \Sigma} \text{dist}(s_\infty(z), E^{crit}) > C.
\]
But this together with Hausdorff-convergence the image of \( s_i \) to that of \( s_\infty \) contradicts to the assumption \( s_i(z_i) \to x_0 \in E^{crit} \). This finishes the proof. \( \square \)

This proposition shows that as far as compactness property of the set of smooth pseudo-holomorphic sections is concerned, we can ignore presence of critical points of the fibration \( E \to \Sigma \).

7.6. **Gromov-Floer moduli space of \( J \)-holomorphic trajectories.** With Theorem 7.7 at our disposal, we can safely ignore the critical points in the study of smooth \( J \)-holomorphic sections and their degenerations for the Lefschetz Hamiltonian fibrations \( \pi : E \to \Sigma \). This, together with the energy estimates derived from the previous section, makes the study essentially the same as for the case of usual smooth Hamiltonian fibrations (without critical points) as studied in \[LM2\], \[MS\].

We formulate the definition of Fukaya-Oh-Ohta-Ono’s \( m_k \)-maps in the setting of fibrations. Obviously this discussion can be extended to the Lefschetz Hamiltonian fibration \( \pi : E \to \Sigma \) with punctures \( \bar{\zeta} = \{\zeta_0, \ldots, \zeta_k\} \subset \partial \Sigma \) and with a chain of Lagrangian boundary conditions
\[
Q = (Q_0, \ldots, Q_k)
\]
by considering the anchor caps attached to \([p_{ij}, w_{ij}]\) with \( p \in L_i \cap L_j \). Here \( Q_j \) is the parallel fiberwise Lagrangian submanifolds corresponding to \( L_j \). We decompose
\[
\partial \Sigma = \coprod_{j=0}^k \partial_j \Sigma
\]
where \( \partial_j \Sigma \) is the \( j \)-th connected component of \( \partial \Sigma \). We briefly add some necessary modification from that of \[FOOO2\] to accommodate the possible critical point in the fibration \( \pi : E \to \Sigma \) following the notations from \[Se3\].

For a given compact surface \( \Sigma \) with marked points \( \bar{\zeta} \), we consider the corresponding surface \( \bar{\Sigma} = \Sigma \setminus \bar{\zeta} \) with punctures. We fix a given preferred holomorphic chart \( \varphi_\zeta : D_\zeta \subset \Sigma \to D^+ \) of the half disc \( D^+ = D \cap \{ \text{im}(z) \geq 0 \} \) with \( \varphi_\zeta(\zeta) = 0 \). We considering the moduli space
\[
\mathcal{M}_J(E, Q; \bar{p}; B)
\]
as in subsection 7.2 for all the section class \( B \in \pi_2(E, Q; \bar{p}) \) with
\[
\text{vit. dim} \mathcal{M}_J(E, Q; \bar{p}; B) = 0.
\]
More specifically, the map is supposed to be given by
\[ \Phi^\text{rel}_0(E, \pi; Q)(\otimes_{j=1}^k [p_j, w_j]) = \sum_k \#(\mathcal{M}_f(E, Q; \bar{p}; B))[p_k, w_k] \]
with a suitable definition of the coefficient \( \#(\mathcal{M}_f(E, Q; \bar{p}; B)) \).

Finally we recall the notion of broken Floer trajectory moduli space i.e., the case corresponds to \( k + 1 = 2 \). See \( \text{[FOn]} \) for the corresponding definition for the closed case.

**Definition 7.13.** Let \( J = \{ J_i \}_{0 \leq i \leq 1} \) and \( x, y, \in L_+ \cap L_- \). A stable broken Floer trajectory from \( p \) to \( q \) is a triple

\[ u = ((u_1, \cdots, u_a); (\sigma_1, \cdots, \sigma_m), (\gamma^0, \cdots, \gamma^0_{m_0}), (\gamma^1, \cdots, \gamma^1_{n_1}); o) \]

that satisfies the following:

1. For \( i = 1, \cdots, a - 1, u_i \in \mathcal{M}(x_i, x_{i+1}) \) and satisfy

\[ u_1(-\infty) = p, u_a(\infty) = q \]

\[ u_i(\infty) = u_{i+1}(-\infty) \quad \text{for} \quad i = 1, \cdots, a - 1. \quad (7.24) \]

We call \( (7.24) \) the matching condition and say a pair \( (u, u') \) of Floer trajectories \( \text{gluable} \) if it satisfies the matching condition.

2. \( \sigma_i \in \overline{\mathcal{M}}_1(J_i; \alpha_i) \) for \( i = 1, \cdots, m \)

3. \( \gamma^0_j \in \overline{\mathcal{M}}_1(L_0, J_0; \beta^0_j) \) for \( j = 1, \cdots, n_0 \) and \( \gamma^1_k \in \overline{\mathcal{M}}_1(L_1, J_1; \beta^1_k) \) for \( k = 1, \cdots, n_1 \).

4. For each \( \ell = 1, \cdots, a \), either the map \( u_\ell \) is non-stationary or \( \Theta_\ell \cap \text{Im} \neq \emptyset \).

We denote the domain of \( u \) simply by \( \Theta_u \) which is the product of a broken configuration. This is the union of a finite copies of

\[ \mathbb{R} \times [0, 1], \]

the \textit{principal components}, and the pre-stable curves of closed or bordered Riemann surfaces of genus 0 the \textit{bubble components} with their roots attached to the principal components of \( \Theta_u \). Each broken Floer trajectory \( u : \Theta_u \to M \) can be regarded as a broken Floer trajectory into the fiber \( M_\zeta \) for \( \zeta \in \zeta \). We denote by \( \overline{\mathcal{M}}(p, q; B_0; \{ \alpha_i \}, \{ \beta_{0,j} \}, \{ \beta_{1,k} \}) \) the set of stable broken Floer trajectories in the prescribed topological types. The group \( \text{Aut}(\Sigma) \) acts on the moduli space \( \overline{\mathcal{M}}(p, q; B_0; \{ \alpha_i \}, \{ \beta_{0,j} \}, \{ \beta_{1,k} \}) \) by the simultaneous translation of the roots of the bubbles attached to the principal component. Then we denote by \( \mathcal{M}(p, q; B) \) the set of all stable broken trajectories in class \( B \in \pi_2(p, q) \).

We then define

\[ \overline{\mathcal{M}}(E, Q; \bar{p}; B) = \prod_i \mathcal{M}(m; n_i)(E, Q; \bar{p}'; B'_i; \{ \alpha_i \}, \{ \beta^i_j \}) \# \left( \prod_i \mathcal{M}(p'_i, q_i; B_i) \right) \]

for all choices of \( B_0, \{ \alpha_i \}, \{ \beta^i_j \} \) and \( B'_i \)'s satisfying

\[ B = B_0 + \sum_i \alpha_i + \sum_{i=0}^k \left( \sum_{j=1}^{n_i} \beta^{(i)}_j \right) + \sum_i B_i \]
and provide it with a topology of stable maps. We denote the corresponding decomposition of maps by
\[ s = s^0 \#(\Pi_i v_i) \# \left( \#_{i=0}^k (\Pi_j w_j^{(i)}) \right) \#(u_i) \]
and call \( s^0 \) the principal component and other fiberwise curves the bubble components.

At this stage, we would like to emphasize that this compactification is defined as a topological space for any choice of \((j, J)\) for a transversal chain \((L_0, \cdots, L_k)\) of Lagrangian submanifolds. The topological space \( M(E, Q; \vec{p}; B) \) will not be a smooth ‘manifold’ even for a generic choice of \( J \), but will be a space with Kuranish structure \([FO]_n\).

8. Anchored Floer cohomology; review

In this subsection, we recall the exposition from \([FOOO3]\) on the Lagrangian Floer theory of anchored Lagrangian submanifolds.

8.1. Floer chain complex. Let \((L_0, L_1)\) be a pair with \( L_0 \) intersecting \( L_1 \) transversely.

Let \((L_i, \gamma_i)\) \( i = 0, 1 \) be anchored Lagrangian submanifolds. Let \( p, q \in L_0 \cap L_1 \) be admissible intersection points. We defined the set \( \pi_2(p, q) = \pi_2((L_0, L_1), (p, q)) \) in section 3. We also defined \( \pi_2(\ell_{01}; p) \) there. We now define:

**Definition 8.1.** Let \( R \) be the underlying coefficient field. We define \( CF(L_1, L_0) = CF((L_1, \gamma_1), (L_0, \gamma_0)) \) to be a free \( R \)-module over the basis \([p, w]\) where \( p \in L_0 \cap L_1 \) is an admissible intersection points and \( w \) is a map from \([0, 1]^2 \to M\) connecting \( \ell_{01} \) and \( \vec{p} \).

Here \( R \) is a ground ring such as \( \mathbb{Q}, \mathbb{C} \) or \( \mathbb{R} \).

**Remark 8.2.** We remark that the set of \([p, w]\) where \( p \) is the admissible intersection point is identified with the set of the critical point of the action functional \( A \) defined on the Novikov covering space of \( \Omega(L_0, L_1; \ell_{01}) \). The group \( \Pi(L_0, L_1) \) defined in Section 3.1 acts freely on it so that the quotient space is the set of admissible intersection points.

We next take a grading \( \lambda_i \) to \((L_i, \gamma_i)\) as in Subsection 8.2. It induces a grading of \([p, w]\) given by \( \mu([p, w]; \lambda_{01}) \), which gives the graded structure on \( CF(L_1, L_0) \)

\[ CF(L_1, L_0) = \bigoplus_k CF^k(L_1, L_0; \lambda_{01}) \]

where \( CF^k(L_1, L_0; \lambda_{01}) = \text{span}_R \{[p, w] \in CF(L_1, L_0) | \mu([p, w]; \lambda_{01}) = k\} \).

For given \( B \in \pi_2(p, q) \), we denote by \( Map(p, q; B) \) the set of such \( w \)'s in class \( B \).

We summarize the extra structures added in the discussion of Floer homology for the anchored Lagrangian submanifolds in the following:

1. We assume that \((L_0, L_1)\) is a relatively spin pair. We consider a pair \((L_0, \gamma_0), (L_1, \gamma_1)\) of anchored Lagrangian submanifolds and the base path \( \ell_{01} = \pi_0 * \gamma_1 \).
2. We fix a grading \( \lambda_i \) of \( \gamma_i \) for \( i = 0, 1 \), which in turn induce a grading of \( \ell_{01} \), \( \lambda_{01} = \lambda_0 * \lambda_1 \).
3. We fix an orientation \( o_p \) of Index \( \delta_{\lambda_p} \) for each \( p \in L_0 \cap L_1 \) as in \([FOOO3]\).
Under these conditions, orientation of the Floer moduli space $M(p, q; B)$ is induced. Using virtual fundamental chain technique [FO0], Appendix A.1 [FOOO2] we can take a system of multisections and obtain a system of reduced. Using virtual fundamental chain technique [FO0], Appendix A.1 [FOOO2]

Finally we define the Floer ‘boundary’ map $\partial : CF(L_1, L_0) \to CF(L_1, L_0)$ by the sum

$$\partial([p, w]) = \sum_{q \in L_0 \cap L_1} \sum_{B \in \pi_2(p, q)} n(p, q; B)[q, w\#B]. \quad (8.1)$$

By Remark 8.2, $CF(L_1, L_0)$ carries a natural $\Lambda(L_0, L_1)$-module structure and $CF^k(L_1, L_0; \lambda_01)$ a $\Lambda^{(0)}(L_0, L_1)$-module structure where

$$\Lambda^{(0)}(L_0, L_1) = \left\{ \sum a_\gamma[g] \in \Lambda(L_0, L_1) \big| \mu([g]) = 0 \right\}.$$ 

We define

$$C(L_1, L_0) = CF(L_1, L_0) \otimes_{\Lambda(L_0, L_1)} \Lambda_{nov} \quad (8.2)$$

where we use the embedding $\Lambda(L_0, L_1) \to \Lambda_{nov}$ given in (3.1).

We write the $\Lambda_{nov}$ module (8.2) also as

$$C(L_1, L_0; \Lambda_{nov}).$$

**Definition 8.3.** We define the energy filtration $F^\lambda CF(L_1, L_0)$ of the Floer chain complex $CF(L_1, \gamma_1), (L_0, \gamma_0))$ (here $\lambda \in \mathbb{R}$) such that $[p, w]$ is in $F^\lambda CF(L_1, L_0)$ if and only if $\Lambda([p, w]) \geq \lambda$.

This filtration also induces a filtration on (8.2).

**Remark 8.4.** We remark that this filtration depends (not only of the homotopy class of) but also of $\gamma_1$ itself.

It is easy to see the following from the definition of $\partial$ above:

**Lemma 8.5.**

$$\partial \left( F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0)) \right) \subseteq F^\lambda CF((L_1, \gamma_1), (L_0, \gamma_0)) \right).$$

According to the definition (8.1) of the map $\partial$, we have the formula for its matrix coefficients

$$\langle \partial \partial [p, w], [r, w\#B] \rangle = \sum_{q \in L_0 \cap L_1} \sum_{B = B_1 \# B_2 \in \pi_2(p, r)} n(p, q; B_1)n(q, r; B_2)T^\omega(B) \quad (8.3)$$

where $B_1 \in \pi_2(p, q)$ and $B_2 \in \pi_2(q, r)$.

To prove, $\partial \partial = 0$, one needs to prove $\langle \partial \partial [p, w], [r, w\#B] \rangle = 0$ for all pairs $[p, w], [r, w\#B]$. On the other hand it follows from definition that each summand

$$n(p, q; B_1)n(q, r; B_2)T^\omega(B) = n(p, q; B_1)T^\omega(B_1)n(q, r; B_2)T^\omega(B_2)$$

and the coefficient $n(p, q; B_1)n(q, r; B_2)$ is nothing but the number of broken trajectories lying in $\mathcal{M}(p, q; B_1)\#\mathcal{M}(q, r; B_2)$. This number is nonzero in the general situation we work with.

To handle the problem of obstruction to $\partial \circ \partial = 0$ and of bubbling-off discs in general, a structure of filtered $A_\infty$ algebra $(C, m)$ with non-zero $m_0$-term is associated to each Lagrangian submanifold $L$ [FOOO1], [FOOO2].
8.2. $A_\infty$ algebra. In this subsection, we review the notion and construction of filtered $A_\infty$ algebra associated to a Lagrangian submanifold. In order to make the construction consistent to one in the last section, where $\Lambda(L_0, L_1)$ is used for the coefficient ring rather than the universal Novikov ring, we rewrite them using smaller Novikov ring $\Lambda(L)$ which we define below. Let $L$ be a relatively spin Lagrangian submanifold. We have a homomorphism

$$(E, \mu) : H_2(M, L; \mathbb{Z}) \to \mathbb{R} \times \mathbb{Z},$$

where $E(\beta) = \beta \cap [\omega]$ and $\mu$ is the Maslov index homomorphism. We put $g \sim g'$ for $g, g' \in H_2(M, L; : \mathbb{Z})$ if $E(g) = E(g')$ and $\mu(g) = \mu(g')$. We write $\Pi(L)$ the quotient with respect to this equivalence relation. It is a subgroup of $\mathbb{R} \times \mathbb{Z}$. We define

$$\Lambda(L) = \left\{ \sum c_g | g \in \Pi(L), c_g \in R, E(g) \geq 0, \right\} \forall E_0 \#\{g \mid c_g \neq 0, E(g) \leq E_0\} < \infty \right\}$$

We have the natural embedding $\Lambda(L) \to \Lambda_0,_{m} \text{nov}$ similarly as in (3.1).

Let $\overline{C}$ be a graded $R$-module and $\overline{C} = \overline{C} \otimes_R \Lambda(L)$. Here and hereafter we use symbol $\overline{C}$ for the modules over $\Lambda(L)$ or $\Lambda(L_0, L_1)$ and $C$ for the modules over the universal Novikov ring.

We denote by $CF[1]$ its suspension defined by $CF[1]^k = CF^{k+1}$. We denote by $\deg(x) = |x|$ the degree of $x \in C$ before the shift and $\deg'(x) = |x'|$ that after the degree shifting, i.e., $|x'| = |x| - 1$. Define the bar complex $\overline{B}(CF[1])$ by

$$\overline{B}_k(CF[1]) = (CF[1])^k, \quad B(CF[1]) = \bigoplus_{k=0}^\infty B_k(CF[1]).$$

Here $B_0(CF[1]) = R$ by definition. The tensor product is taken over $\Lambda(L)$. We provide the degree of elements of $B(CF[1])$ by the rule

$$|x_1 \otimes \cdots \otimes x_k|' = \sum_{i=1}^k |x_i'| = \sum_{i=1}^k |x_i| - k \quad (8.4)$$

where $| \cdot |'$ is the shifted degree. The ring $B(CF[1])$ has the structure of graded coalgebra.

**Definition 8.6.** A filtered $A_\infty$ algebra over $\Lambda(L)$ is a sequence of $\Lambda(L)$ module homomorphisms

$$m_k : \overline{B}_k(CF[1]) \to CF[1], \quad k = 0, 1, 2, \cdots,$$

of degree +1 such that the coderivation $d = \sum_{k=0}^\infty \hat{m}_k$ satisfies $dd = 0$, which is called the $A_\infty$-relation. Here we denote by $\hat{m}_k : B(CF[1]) \to B(CF[1])$ the unique extension of $m_k$ as a coderivation on $B(CF[1])$. A filtered $A_\infty$ algebra is an $A_\infty$ algebra with a filtration for which $m_k$ are continuous with respect to the induce non-Archimedean topology.

If we have $m_1m_1 = 0$, it defines a complex $(CF, m_1)$. We define the $m_1$-cohomology by

$$H(CF, m_1) = \text{Ker} m_1 / \text{Im} m_1. \quad (8.5)$$

The first two terms of the $A_\infty$ relation for a $A_\infty$ algebra are given as

$$m_1(m_0(1)) = 0 \quad (8.6)$$

$$m_1m_1(x) + (-1)^{|x|'} m_2(x, m_0(1)) + m_2(m_0(1), x) = 0. \quad (8.7)$$
In particular, for the case \( m_0(1) \) is nonzero, \( m_1 \) will not necessarily satisfy the boundary property, i.e., \( m_1 m_1 \neq 0 \) in general.

We now describe the \( A_\infty \) operators \( m_k \) in the context of \( A_\infty \) algebra of Lagrangian submanifolds. For a given compatible almost complex structure \( J \), consider the moduli space of stable maps of genus zero
\[
\mathcal{M}_{k+1}(\beta; L) = \{(w, (z_0, z_1, \cdots, z_k)) \mid \overline{\partial} w = 0, z_i \in \partial D^2, [w] = \beta \in \pi_2(M, L)\} / \sim
\]
where \( \sim \) is the conformal reparameterization of the disc \( D^2 \). We require that \( z_0, \cdots, z_k \) respects counter clockwise cyclic order of \( \overline{S}^1 \). (We wrote this moduli space \( \mathcal{M}_{k+1}^{\text{main}}(\beta; L) \) in Section 2.1 [FOOO2]. The symbol ‘main’ indicates the compatibility of \( z_0, \cdots, z_k \), with counter clockwise cyclic order. We omit this symbol in this paper since we always assume it.)

\( \mathcal{M}_{k+1}(\beta; L) \) has a Kuranishi structure and its dimension is given by
\[
n + \mu(\beta) - 3 + (k + 1) = n + \mu(\beta) + k - 2.
\]
(8.8)

Now let \([P_1, f_1], \cdots, [P_k, f_k] \in C_* (L; \mathbb{Q})\) be \( k \) smooth singular simplices of \( L \). (Here we denote by \( C_* (L; \mathbb{Q}) \) a suitably chosen countably generated cochain complex of smooth singular chains of \( L \).) We put the cohomological grading \( \deg P_i = n - \dim P_i \)
and consider the fiber product
\[
ev_0 : \mathcal{M}_{k+1}(\beta; L) \times_{(ev_1, \cdots, ev_k)} (P_1 \times \cdots \times P_k) \to L.
\]
A simple calculation shows that the expected dimension of this chain is given by \( n + \mu(\beta) - 2 + \sum_{j=1}^k (\dim P_j + 1 - n) \) or equivalently we have the degree
\[
\deg [\mathcal{M}_{k+1}(\beta; L) \times_{(ev_1, \cdots, ev_k)} (P_1 \times \cdots \times P_k), \ev_0] = \sum_{j=1}^n (\deg P_j - 1) + 2 - \mu(\beta).
\]

For each given \( \beta \in \pi_2(M, L) \) and \( k = 0, \cdots \), we define \( m_{1,0}(P) = \pm \partial P \) and
\[
m_{k,\beta}(P_1, \cdots, P_k) = [\mathcal{M}_{k+1}(\beta; L) \times_{(ev_1, \cdots, ev_k)} (P_1 \times \cdots \times P_k), \ev_0] \in C_* (L; \mathbb{Q})
\]
(8.9)

(More precisely we regard the right hand side of (8.9) as a smooth singular chain by taking appropriate multi-valued perturbation (multisection) and choosing a simplicial decomposition of its zero set.)

We put
\[
CF(L) = C_* (L; \mathbb{Q}) \widehat{\otimes} L(L).
\]

We define \( m_k : B_k CF(L)[1] \to B_k CF[1] \) by
\[
m_k = \sum_{\beta \in \pi_2(M, L)} m_{k,\beta} \otimes [\beta].
\]

Then it follows that the map \( m_k : B_k CF(L)[1] \to CF(L)[1] \) is well-defined, has degree 1 and continuous with respect to non-Archimedean topology. We extend \( m_k \) as a coderivation \( \hat{m}_k : BF[1] \to BF[1] \) where \( BF(L)[1] \) is the completion of the direct sum \( \oplus_{k=0}^\infty B_k CF(L)[1] \) where \( B_k CF(L)[1] \) itself is the completion of \( CF(L)[1] \otimes k \). \( BF(L)[1] \) has a natural filtration defined similarly as Definition 8.3.

Finally we take the sum
\[
\hat{d} = \sum_{k=0}^\infty \hat{m}_k : BF(L)[1] \to BF(L)[1].
\]

We then have the following coboundary property:
Theorem 8.7. Let $L$ be an arbitrary compact relatively spin Lagrangian submanifold of an arbitrary tame symplectic manifold $(M, \omega)$. The coderivation $\hat{d}$ is a continuous map that satisfies the $A_\infty$ relation $\hat{d}\hat{d} = 0$, and so $(CF(L), m)$ is a filtered $A_\infty$ algebra over $\Lambda(L)$.

We put

$$C(L; \Lambda_{0, \text{nov}}) = CF(L) \otimes_{\Lambda(L)} \Lambda_{0, \text{nov}}$$

on which a filtered $A_\infty$ structure on $C(L; \Lambda_{0, \text{nov}})$ (over the ring $\Lambda_{0, \text{nov}}$) is induced. This is the filtered $A_\infty$ structure given in Theorem A [FOOO2].

In the presence of $m_0$, $\hat{m}_1 \hat{m}_1 = 0$ no longer holds in general. This leads to consider deforming Floer’s original definition by a bounding cochain of the obstruction cycle arising from bubbling-off discs. One can always deform the given (filtered) $A_\infty$ algebra $(CF(L), m)$ by an element $b \in CF(L)[1]^0$ by re-defining the $A_\infty$ operators as

$$m^b_k(x_1, \cdots, x_k) = m(e^b, x_1, e^b, x_2, e^b, x_3, \cdots, x_k, e^b)$$

and taking the sum $\hat{d}^b = \sum_{k=0}^\infty \hat{m}^b_k$. This defines a new filtered $A_\infty$ algebra in general. Here we simplify notations by writing

$$e^b = 1 + b + b \otimes b + \cdots + b \otimes \cdots \otimes b + \cdots .$$

Note that each summand in this infinite sum has degree 0 in $CF(L)[1]$ and converges in the non-Archimedean topology if $b$ has positive valuation, i.e., $v(b) > 0$. (See Section 3.1 for the definition of $v$.)

Proposition 8.8. For the $A_\infty$ algebra $(CF(L), m^b_k)$, $m^b_0 = 0$ if and only if $b$ satisfies

$$\sum_{k=0}^\infty m_k(b, \cdots, b) = 0. \quad (8.10)$$

This equation is a version of Maurer-Cartan equation for the filtered $A_\infty$ algebra.

Definition 8.9. Let $(CF(L), m)$ be a filtered $A_\infty$ algebra in general and $BCF(L)[1]$ be its bar complex. An element $b \in CF(L)[1]^0 = CF(L)^1$ is called a bounding cochain if it satisfies the equation (8.10) and $v(b) > 0$. We denote by $\tilde{M}(L; \Lambda(L))$ the set of bounding cochains.

In general a given $A_\infty$ algebra may or may not have a solution to (8.10). In our case we define:

Definition 8.10. A filtered $A_\infty$ algebra $(CF(L), m)$ is called unobstructed over $\Lambda(L)$ if the equation (8.10) has a solution $b \in CF(L)[1]^0 = CF(L)^1$ with $v(b) > 0$.

One can define the notion of homotopy equivalence between two bounding cochains and et al as described in Chapter 4 [FOOO2]. We denote by $\tilde{M}(L; \Lambda(L))$ the set of equivalence classes of bounding cochains of $L$.

Remark 8.11. In Definition 8.9 above we consider bounding cochain contained in $CF(L) \subset C(L; \Lambda_0)$ only. This is the reason why we write $\tilde{M}(L; \Lambda(L))$ in place of $M(L)$. (The latter is used in [FOOO2].)
8.3. $A_{\infty}$ bimodule. Once the $A_{\infty}$ algebra is attached to each Lagrangian submanifold $L_i$ we then construct a structure of filtered $A_{\infty}$ bimodule on the module $\text{CF}(\Lambda(L_1), \Lambda(L_0)) = \text{CF}((L_1, \gamma_1), (L_0, \gamma_0))$, which was introduced in Section 8.1 as follows. This filtered $A_{\infty}$ bimodule structure is by definition a family of operators

$$n_{k_1, k_0} : B_{k_1}(\text{CF}(L_1)[1]) \otimes_{\Lambda(L_1)} \text{CF}((L_1, \gamma_1), (L_0, \gamma_0)) \otimes_{\Lambda(L_0)} B_{k_0}(\text{CF}(L_0)[1]) \rightarrow \text{CF}((L_1, \gamma_1), (L_0, \gamma_0))$$

for $k_0, k_1 \geq 0$. Here the left hand side is defined as follows: It is easy to see that there are embeddings $\Lambda(L_0) \rightarrow \Lambda(L_1, L_0)$, $\Lambda(L_1) \rightarrow \Lambda(L_0, L_1)$. Therefore a $\Lambda(L_0, L_1)$ module $\text{CF}((L_1, \gamma_1), (L_0, \gamma_0))$ can be regarded both as $\Lambda(L_0)$ module and $\Lambda(L_1)$ module. Hence we can take tensor product in the left hand side. ($\hat{\otimes}_{\Lambda(L_i)}$ is the completion of this algebraic tensor product.) The left hand side then becomes a $\Lambda(L_0, L_1)$ module, since the rings involved are all commutative.

We briefly describe the definition of $n_{k_1, k_0}$. A typical element of the tensor product

$$B_{k_1}(\text{CF}(L_1)[1]) \otimes_{\Lambda(L_1)} \text{CF}((L_1, \gamma_1), (L_0, \gamma_0)) \otimes_{\Lambda(L_0)} B_{k_0}(\text{CF}(L_0)[1])$$

has the form

$$P_{1,1} \otimes \cdots \otimes P_{1,k_1} \otimes [p, w] \otimes P_{0,1} \otimes \cdots \otimes P_{0,k_0}$$

with $p \in L_0 \cap L_1$ being an admissible intersection point. Then the image $n_{k_0, k_1}$ thereof is given by

$$\sum_{q, B} T^{\delta(B)} e^{B/2} \#(M(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0})) [q, B # w].$$

Here $B$ denotes homotopy class of Floer trajectories connecting $p$ and $q$, the summation is taken over all $[q, B]$ with

$$\dim M(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0}) = 0,$$

and $\#(M(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0}))$ is the ‘number’ of elements in the ‘zero’ dimensional moduli space $M(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0})$. Here the moduli space $M(p, q; B; P_{1,1}, \cdots, P_{1,k_1}; P_{0,1}, \cdots, P_{0,k_0})$ is the Floer moduli space $M(p, q; B)$ cut-down by intersecting with the given chains $P_{1,i} \subset L_1$ and $P_{0,j} \subset L_0$. (See Section 3.7 [FOOO2].) An orientation on this moduli space can be given in [FOOO2] [FOOO3].

**Theorem 8.12.** Let $(L_0, L_1)$ be a pair of anchored Lagrangian submanifolds. Then the family $\{n_{k_1, k_0}\}$ defines a left $(\text{CF}(L_1), \mathfrak{m})$ and right $(\text{CF}(L_0), \mathfrak{m})$ filtered $A_{\infty}$-bimodule structure on $\text{CF}(L_1, L_0)$.

See Section 3.7 [FOOO2] and [FOOO3] for the definition of filtered $A_{\infty}$ bimodules. (In [FOOO2] the case of universal Novikov ring as a coefficient is considered. It is easy to modify to our case of $\Lambda(L_0, L_1)$ coefficient.)

In the case where both $L_0, L_1$ are unobstructed, we can carry out this deformation of $n$ using bounding cochains $b_0$ and $b_1$ of $\text{CF}(L_0)$ and $\text{CF}(L_1)$ respectively, in a way similar to $\mathfrak{m}^b$. Namely we define $\delta_{b_1, b_0} : \text{CF}(L_1, L_0) \rightarrow \text{CF}(L_1, L_0)$ by

$$\delta_{b_1, b_0}(x) = \sum_{k_1, k_0} n_{k_1, k_0}(b_1^{\otimes k_1} \otimes x \otimes b_0^{\otimes k_0}) = \tilde{n}(e^{b_1}, x, e^{b_0}).$$

We can generalize the story to the case where $L_0$ has clean intersection with $L_1$, especially to the case $L_0 = L_1$. In the case $L_0 = L_1$ we have $n_{k_1, k_0} = n_{k_0 + k_1 + 1}$. So in this case, we have $\delta_{b_1, b_0}(x) = m(e^{b_1}, x, e^{b_0})$. 

We define Floer cohomology of the pair $\mathcal{L}_0 = (L_0, \gamma_0), \mathcal{L}_1 = (L_1, \gamma_1, \lambda_1)$ by

$$HF((\mathcal{L}_1, b_1), (\mathcal{L}_0, b_0)) = \text{Ker} \delta_{b_1, b_0}/\text{Im} \delta_{b_1, b_0}.$$ 

This is a module over $\Lambda(\mathcal{L}_0, \mathcal{L}_1)$.

**Theorem 8.13.** $HF((\mathcal{L}_1, b_1), (\mathcal{L}_0, b_0)) \otimes_{\Lambda(\mathcal{L}_0, \mathcal{L}_1)} \Lambda_{\text{nov}}$ is invariant under the Hamiltonian isotopies of $\mathcal{L}_0$ and $\mathcal{L}_1$ and under the gauge equivalence of bounding cochains $b_0, b_1$.

We refer to section 4.3 [FOOO2] for the definition of gauge equivalence and to Theorem 4.1.5 [FOOO2] for the proof of this theorem.

9. Definitions of Seidel’s maps $b, \epsilon$ and $\mathfrak{h}$

In this section, we recall the definition of Seidel’s cochain maps $b, \epsilon$ and the homotopy $\mathfrak{h}$ and give the definition of the analogs thereof in our general setting. They are the maps

$$b : CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(\tau_L(L_0), L_1)$$

$$\epsilon : CF(\tau_L(L_0), L_1) \to CF(L_0, L_1)$$

and the homotopy $\mathfrak{h} : CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(L_0, L_1)$ between the composition $\epsilon \circ b$ and the zero map.

We will generalize these maps to our non-exact case and describe all the necessary properties of the maps in the next section. We consider a quadruple of anchored Lagrangian submanifolds

$$\mathcal{L} = (L, \gamma), \mathcal{L}_0 = (L_0, \gamma_0), \mathcal{L}_1 = (L_1, \gamma_1), \tau_* \mathcal{L} = (\tau_L(L_0), \tau_L(L_0)),$$

For the simplicity of notations in this section, we will just denote the action functional associated to the pair $(L, \gamma)$ and $(L', \gamma')$ just by $\mathcal{A}_{\mathcal{L}, \gamma}$.

9.1. **The map $b$.** Let $\Sigma$ be a compact surface with boundary marked points $\bar{\zeta} = \{\zeta_0, \zeta_1, \zeta_2\}$. We denote $\tilde{\Sigma} = \Sigma \setminus \bar{\zeta}$ and $\partial \tilde{\Sigma} = \bigcup_{i=0}^2 \partial \Sigma$. We consider the three anchored Lagrangian submanifolds $(L_0, \gamma_0), (L, \gamma)$ and $(L_1, \gamma_1)$. Take the trivial Hamiltonian fibration $\pi : E = \tilde{\Sigma} \times M \to \tilde{\Sigma}$ with the two form $\Omega$ equal to the two form pulled back from $\omega$ in $M$. Equip this with the Lagrangian boundary condition

$$Q = (\partial_2 \tilde{\Sigma} \times \tau_L(L_0)) \cup (\partial_1 \tilde{\Sigma} \times L) \cup (\partial_3 \tilde{\Sigma} \times L_1).$$

$Q$ is an exact Lagrangian boundary with $\kappa_Q = 0$. We note that $E$ has the trivial connection given by $K \equiv 0$ and hence has zero curvature.

**Lemma 9.1.** Suppose that $\mathcal{L} = (L, \gamma), \mathcal{L}_0 = (L_0, \gamma_0), \mathcal{L}_1 = (L_1, \gamma_1)$ are given anchors of the type (9.1) and $\tau_* \mathcal{L} = (\tau_L(L_0), \tau_L(L_0))$. Let $s : \tilde{\Sigma} \to \tilde{\Sigma} \times M$ be a section with the given exact Lagrangian boundary condition $Q$ as above. Let $[p_0, w_0] \in \text{Crit} \mathcal{A}_{\tau_L(L_0), \mathcal{L}_1}, [p, w] \in \text{Crit} \mathcal{A}_{\mathcal{L}, \mathcal{L}_1}$, and $[p_1, w_1] \in \text{Crit} \mathcal{A}_{\tau_L(L_0), \mathcal{L}_1}$. Suppose the homotopy class $[s, \partial s]$ is admissible and satisfies

$$[w_0] = [s, \partial s] * [w] * [w_1] \quad \text{in} \quad \pi_2(\tau_L(L_0), L_1; \tau_L(L_0) \otimes \gamma_1).$$

Then we have

$$\int s_* \Omega = \mathcal{A}_{\tau_L(L_0), \mathcal{L}_1}(p_0, w_0) - \mathcal{A}_{\mathcal{L}, \mathcal{L}_1}(p, w) - \mathcal{A}_{\tau_L(L_0), \mathcal{L}_1}(p_1, w_1).$$

**Proof.** This immediately follows from Proposition 8.10. \hfill $\square$
Definition 9.2. Let $L, L' \subset M$ be a pair of Lagrangian submanifolds. We say that $J$ lies in $\mathcal{J}^{reg}(M; L, L')$ if all Floer trajectories associated to $J$ for the pair $(L, L')$ are Fredholm-regular.

At this point, we fix

$$J^{(1)} \in \mathcal{J}^{reg}(M; \tau_L(L_0), L)$$

$$J^{(2)} \in \mathcal{J}^{reg}(M; L, L_1)$$

$$J^{(3)} \in \mathcal{J}^{reg}(M; \tau_L(L_0), L_1).$$

By choosing a horizontal $J \in \mathcal{J}(E, \pi, j, J^{(1)}, J^{(2)}, J^{(3)})$, i.e., $J$ satisfying $J_*(TE^0_z) = TE^0_z$ for all $x \in E \setminus \text{Crit}$ with the above trivial fibration $E = \Sigma \times M$, this gives rise to the standard ‘pants product map’

$$CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(\tau_L(L_0), L_1)$$

in the cochain level. Seidel’s map $b$ in the cochain level is nothing but this pants product map.

By choosing bounding cochains $b_0, b_1$ of $L_0, L_1$ respectively and considering $(\tau_L), b_0$ on $\tau_L(L_0)$, we define the deformed $m_2$

$$m^\delta_2 : CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(\tau_L(L_0), L_1)$$

where $\delta = (0, (\tau_L), b_0, b_1)$ and $m^\delta_2$ is defined as in (8.15) [FOOO3].

According to [FOOO2] and [FOOO3], this induces a cochain map up to the higher homotopy map $m^\delta_2$ and induces a homomorphism in cohomology. For readers’ convenience, we summarize the construction of this product map $m^\delta_2$ in Appendix.

9.2. The map $c$. Let $(E^L, \pi^L)$ be the standard fibration over a disc $D(\frac{1}{2})$, whose monodromy around $\partial D(\frac{1}{2})$ is $\tau_L$, as defined in section 2 by choosing $r$ small. Let $0 < r < \frac{1}{3}$ be given, and choose a function $g$ with $g(t) = t$ for small $t$, $g(t) \equiv r$ for $t \geq r$ and $g'(t) \geq 0$ everywhere. We consider the map $p : D(\frac{1}{2}) \to D(\frac{1}{3})$ defined by $p(z) = g(|z|) \frac{1}{|z|}$ and consider the pull-back fibration

$$(E^p, \pi^p) = p^*(E^L, \pi^L).$$

This is flat on the annulus $D(\frac{1}{2}) \setminus \text{Int}(D(r))$.

Now take the surface

$$\Sigma^f = \mathbb{R} \times [-1, 1] \setminus \text{Int}(D(1/2)) \subset \mathbb{R}^2,$$

with coordinates $(s, t)$ and divide it into two parts $\Sigma^{f, \pm} = \Sigma^f \cap \{ t \in \mathbb{R}_\pm \}$ so that

$$\Sigma^{f, +} \cap \Sigma^{f, -} = ((-\infty, -1/2] \cup [1/2, \infty)) \times \{0\}.$$

Consider trivial fibrations

$$\pi^{f, \pm} : E^{f, \pm} = \Sigma^{f, \pm} \times M \to \Sigma^{f, \pm}$$

over the two parts, equip them with two forms $\Omega^{f, \pm}$ the pull-back of $\omega$. We define a fibration $(E^f, \pi^f)$ over $\Sigma^f$ by identifying the fibers $E_{(s,0)}^{f, +} \to E_{(s,0)}^{f, -}$ via $id_M$ for $s \geq \frac{1}{2}$ and via $\tau_L$ for $s \leq -\frac{1}{2}$. Because $\tau_L$ is symplectic, this defines a flat Hamiltonian fibration.

Using the fact that two fibrations $E^p$ and $E^f$ are flat close to the curve $|z| = \frac{1}{2}$, we now paste them along the curve. Denote the resulting fibration over

$$\Sigma = \Sigma^p \cup \Sigma^f = \mathbb{R} \times [-1, 1]$$
by \((E^c, \pi)\). Equip \((E^c, \pi)\) with the Lagrangian boundary condition
\[
Q^c = \begin{cases} \mathbb{R} \times \{1\} \times L_1 & \subset E^{f,+} \\ \mathbb{R} \times \{-1\} \times \tau_L(L_0) & \subset E^{f,-} \end{cases}
\]
This defines an exact Lagrangian boundary with the one form \(\kappa_{Q^c} = 0\). As explained in section 3.3 [Se3], \((E^c, \pi)\) is modeled on \((\tau_L(L_0), L_1)\) over the positive end of \(\Sigma\) and over the negative end, it is modeled on \((L_0, L_1)\) due to the monodromy effect around the critical value \((0, 0) \in \Sigma\).

Let \(j\) be some complex structure on \(\Sigma\), standard over the ends. Take some \(J^{(3)} \in \mathcal{J}^{reg}(M, \tau_L(L_0), L_1)\) as in the previous subsection and choose an additional \(J^{(5)} \in \mathcal{J}^{reg}(M; L_0, L_1)\). Using a regular \(J^{(6)} \in \mathcal{J}(E, \pi, Q, j, J^{(3)}, J^{(5)})\), we define a map
\[
\epsilon = C\Phi^{reg}_0(E^c, \pi, Q^c, J^{(6)}): CF(\tau_L(L_0), L_1) \to CF(L_1, L_0)
\]
as defined in section 3.3 [Se3].

Since we use the deformed Floer complex by bounding cochains, we need to construct the deformed version of the map \(\epsilon\). The construction will resemble that of \(A_{\infty}\)-bimodule structure on \(CF(L_1, L_0)\) carried out in subsection 8.3.

We first define a family of operators
\[
\epsilon_{k_1, k_0}: B_{k_1}(CF(L_1)[1]) \otimes_{\Lambda(L_1)} CF(L_1, \tau_L(L_0)) \otimes_{\Lambda(L_0)} B_{k_0}(CF(L_0)[1]) \to CF(L_1, L_0)
\]
for \(k_0, k_1 \geq 0\). A typical element of the tensor product
\[
B_{k_1}(CF(L_1)[1]) \otimes_{\Lambda(L_1)} CF(L_1, \tau_L(L_0)) \otimes_{\Lambda(L_0)} B_{k_0}(CF(L_0)[1])
\]
has the form
\[
P_{1,1} \otimes \cdots \otimes P_{1,k_1} \otimes [p, w] \otimes P_{0,1} \otimes \cdots \otimes P_{0,k_0}
\]
with \(p \in \tau_L(L_0) \cap L_1\) being an admissible intersection point. Then the image \(\epsilon_{k_0, k_1}\) thereof is given by
\[
\sum_{q, B} T^{\omega(B)} \mu^{\mu(B)/2} \#(\mathcal{M}(p^c, q; B; P_{1,1}, \ldots, P_{1,k_1}; P_{0,1}, \ldots, P_{0,k_0})) [q, B \# w].
\]
Here \(B\) denotes section class of \(J\)-holomorphic sections connecting \(p\) and \(q\), the summation is taken over all \([q, B]\) with
\[
\dim \mathcal{M}(p, q; B; P_{1,1}, \ldots, P_{1,k_1}; P_{0,1}, \ldots, P_{0,k_0}) = 0,
\]
and \(\#(\mathcal{M}(p, q; B; P_{1,1}, \ldots, P_{1,k_1}; P_{0,1}, \ldots, P_{0,k_0}))\) is the ‘number’ of elements in the ‘zero’ dimensional moduli space \(\mathcal{M}(p, q; B; P_{1,1}, \ldots, P_{1,k_1}; P_{0,1}, \ldots, P_{0,k_0})\). Here the moduli space \(\mathcal{M}(p, q; B; P_{1,1}, \ldots, P_{1,k_1}; P_{0,1}, \ldots, P_{0,k_0})\) is the moduli space \(\mathcal{M}(p, q; B)\) of \(J\)-holomorphic sections of \(\pi: E^c \to \Sigma\) cut-down by intersecting with the given chains \(P_{1,i} \subset L_1\) and \(P_{0,j} \subset L_0\).

**Theorem 9.3.** Let \((L_0, L_1) = ((L_0, \gamma_0), (L_1, \gamma_1))\) be a pair of anchored Lagrangian submanifolds. Then the family \(\{\epsilon_{k_1, k_0}\}\) defines a left \((CF(L_1), m)\) and right \((CF(L_0), m)\) filtered \(A_{\infty}\)-bimodule homomorphism from \(CF(L_1, \tau_L(L_0))\) to \(CF(L_1, L_0)\).

The proof of Theorem 9.3 is similar to that of Theorem 8.12 given in the proof of Theorem 3.7.21 [FOOO2].
When both $L_0$, $L_1$ are unobstructed, we can carry out this deformation of $c$ using bounding cochains $b_0$ and $b_1$ of $CF(L_0)$ and $CF(L_1)$ respectively, in a way similar to $\nabla^{b_0,b_1}$. Namely we define $\xi^{b_1,b_0} : CF(L_1, \tau_L(L_0)) \to CF(L_1, \tau_L)$ by

$$\xi^{b_1,b_0}(x) = \sum_{k_1,k_0} \xi_{k_1,k_0}(b_1^\otimes k_1 \otimes x \otimes b_0^{\otimes k_0}) = \tilde{\gamma}(e^{b_1}, x, e^{b_0}).$$

The following proposition is all we need to add to our context for the construction of Seidel's map $\xi^{b_1,b_0}$ in [Se3].

**Proposition 9.4.** Let $b_0$, $b_1$ be bounding cochains of $L_0$, $L_1$ respectively. Then $\xi$ defines a chain map

$$\xi^{b_1,b_0} = C\Phi_0^{rel}(E, \pi, Q, J^{(6)}): (CF(\tau_L(L_0), L_1), m_{(\tau_L)}^{(b_0)}) \to (CF(L_1, L_0), m_{(b_1)}^{(b_1)})$$

and hence induces a homomorphism

$$\xi^{b_1,b_0} : HF((\tau_L(L_0), (\tau_L), b_0), (L_1, b_1)) \to HF((L_1, b_1), (L_0, b_0)).$$

**Proof.** The proof proceeds in the same way as that of Theorem 8.12. The only difference from the latter is that the moduli space $M^c(p, q; B; P_1, \cdots, P_{k_1}; P_0, 1, \cdots, P_{0,k_0})$ does not have $\mathbb{R}$-action anymore and so we consider the moduli space of sections without considering the quotient.

9.3. A simple invariant and its vanishing theorem. Let $\Phi_1(E, \pi, Q)$ be the invariant

$$\Phi_1(E, \pi, Q) = (ev_{\zeta})_*[M_J] \in H_*(Q_\zeta; \mathbb{Z})$$

for the Calabi-Yau Lefschetz fibration. The following proposition replaces a similar proposition, Proposition 2.13 [Se3], for the present Calabi-Yau Lefschetz fibration setting.

For each given section class $A \in \pi_2^{sec}(E, Q)$, we define the moduli space $\mathcal{M}_J(A)$ of $J$-holomorphic section $s : D \to E$ with $[s, \partial s] = A$ and define an invariant

$$\Phi_1(E, \pi, Q) = \sum_{A \in \pi_2^{sec}(E, Q)} (ev_{\zeta})_*[M_J(A)] \tau^{\partial(A)} \in H_*(Q_\zeta; \Lambda_{0,00}).$$

We start with the following slight generalization of Proposition 2.2 in the general context of Lagrangian spheres in general symplectic manifolds.

**Proposition 9.5.** Let $(L, [f])$ be a framed Lagrangian sphere in $M$. There is a one-parameter family of Lefschetz Hamiltonian fibrations $(E_L^L, \pi_L^L) \to D(r)$ together with an isomorphism $\phi_L^r : E_L^L \to M$ of symplectic manifolds, such that

1. Consider the re-scaling map $\lambda_r : D(r) \to D(1)$ defined by $z \mapsto \frac{z}{r}$. Then $(\lambda_r)^* E_{1}^L = E_{r}^L$.

2. If $\rho_L^r$ is the symplectic monodromy around $\partial D(r)$, then $\phi_L^r \circ \rho_L^r \circ (\phi_L^r)^{-1}$ is a Dehn twist along $(L, [f])$.

We denote any of these maps by $\tau_L$ as before.

**Proof.** The proof follows from Seidel’s proof of Proposition 1.11 [Se3] stripping all the things related to the exactness requirement in the proof. In fact the proof is easier because we do not have to concern the exactness requirement in the construction.
We also recall that each fiber $E^L_z, z \neq 0$ of the fibration $E^L$ contains a distinguished Lagrangian sphere $\Sigma^L_z$. We call this fibration a standard Calabi-Yau Lefschetz fibration. The following is a crucial proposition needed in Seidel’s construction of long exact sequence in [Se3], whose proof goes through in our current context.

**Proposition 9.6.** Let $L \subset M$ be any Calabi-Yau Lagrangian brane and $\pi^L : E^L \to D(r)$ be the associated standard Calabi-Yau Lefschetz fibration. Then we have

$$\Phi_1(E^L, \pi^L, \pi^L) = 0.$$ 

**Proof.** The proof of this proposition will verbatim follow that of Proposition 2.13 [Se3] and so omitted. \(\Box\)

9.4. The gluing $b\#_\rho c$, composition $c \circ b$ and the homotopy $h$. We denote by $(E^b, \pi^b, \Sigma^b, Q^b)$ the fibration, surface and the boundary condition associated to the map $b$ and $(E^c, \pi^c, \Sigma^c, Q^c)$ those associated to the map $c$ constructed in the previous subsections. We glue the positive end of $E^b$ and the negative end of $E^b$ to obtain $(E^b_{\rho^c}, \pi^c_{\rho^b}, \Sigma^c_{\rho^b}, Q^c_{\rho^b})$ for a sufficiently large gluing parameter $\rho$ in the glued ends. This fibration provides a cochain map

$$b\#_\rho c : CF(L, L_1) \otimes CF(\tau_L(L_0), L) \to CF(L_1, L_0).$$

Here and henceforth we denote

$$b = m_2, \quad c = c_{b, b_0}$$

for notational simplicity.

**Lemma 9.7.** Let $(j, J)$ be such that $j$ is a complex structure on $\Sigma^c_{\rho^b}$ which is standard on its ends, and $J \in J(E^b_{\rho^c}, \pi^c_{\rho^b}, Q^c_{\rho^b}, j, J^{(0)}, J^{(2)}, J^{(5)})$. The cochain map $b\#_\rho c$ coincides with $c \circ b$ for any $\rho \geq \rho_0$ with $\rho_0$ sufficiently large.

**Proof.** We note that the composition $c \circ b$ is defined by counting the elements of the fiber product

$$M(E^b, Q^b, \pi^b, B^b)_{\rho^c} \#_{\rho^c} M(E^c, Q^c, \pi^c)$$

while the map $b\#_\rho c$ by counting those in the moduli space

$$M(E^b_{\rho^c}, Q^c_{\rho^b}, \pi^c_{\rho^b}, B^b_{\rho^c})$$

where $B^b_{\rho^c} = B^b_{\#_\rho c}B^c$ is the obvious glued homotopy class. By a gluing theorem in the Floer complex (see e.g., [FOOO2]), the two moduli spaces are diffeomorphic to each other and hence the proof. \(\Box\)

Finally we can construct a homotopy from $b\#_\rho c$ to the zero map verbatim following Seidel’s argument from [Se3]. (See Figure 9 and Figure 12 of [Se3] in particular.) We omit details of the construction.

10. Thick-thin decompositions

In this section, we will study the thick-thin decomposition of contributions in the cochain maps $b, c$ and the homotopy $h$.

For this purpose, we first note the Dehn twist $\tau_L^{-1} : M \to M$ acts by $(L, \gamma_i) \to \tau_L^{-1}(L_i, \gamma_i)$ and induces a one-one correspondence

$$\tau_L(L_0) \cap L \to L_0 \cap L; \quad x \mapsto \tau_L^{-1}(x)$$
Since \( \tau_L|_L = id|_L \). This lifts to a diffeomorphism
\[
\tilde{\Omega}(\tau_L(L_0), L) \to \tilde{\Omega}(L_0, L); \quad [p, w] \mapsto [\tau_L^{-1}(p), \tau_L^{-1}(w)].
\]
This latter diffeomorphism induces a filtration preserving isomorphism
\[
(\tau_L^{-1})_* : CF(\tau_L(L_0), L) \to CF(L_0, L)
\]
i.e., satisfies
\[
A_{\tau_L(L_0), L}([p, w]) = A_{L_0, L}([\tau_L^{-1}(p), \tau_L^{-1}(w)]).
\]
By perturbing \( L_0 \) and \( L_1 \) if necessary and choosing \( \varepsilon > 0 \) sufficiently small, we may assume:

1. \( L \cap L_0, L \cap L_1 \) and \( L_0 \cap L_1 = \emptyset \).
2. Each \( \iota^{-1}(L_k) \subset T(r) \) is a union of fibres; one can write this as
\[
\iota^{-1}(L_k) = \bigcup_{y \in \iota^{-1}(L_k \cap L)} T(r)_y.
\]
3. \( R \) satisfies \( 0 \geq 2\pi R(0) > -\varepsilon \), and is such that \( \tau_L \) is \( \delta \)-wobbly.

The following lemma is a part of Lemma 3.2 \[\text{[SE3]}\]. For readers’ convenience, we provide its proof.

Lemma 10.1. Suppose \( L_0 \cap L \cap L_1 = \emptyset \). Then we can choose \( \text{supp} \tau_L \) so close to \( L \) that \( L_0 \cap L_1 \subset M \setminus \text{im} \tau_L \) and \( \tau_L(L_0), \tau_L(L_1) \) intersect transversally, and there are injective maps
\[
p : (\tau_L(L_0) \cap L) \times (L \cap L_1) \to \tau_L(L_0) \cap L_1, \quad (10.1)
\]
\[
q : L_0 \cap L_1 \to \tau_L(L_0) \cap L_1 \quad (10.2)
\]
such that \( \tau_L(L_0) \cap L_1 \) is the disjoint union of their images.

Proof. The conditions (1), (2) right before Lemma 10.1 imply \( L_0 \cap L_1 \cap U = \emptyset \).
Since \( \tau_L \) is the identity outside \( U \), one has \( L_0 \cap L_1 = (\tau_L(L_0) \cap L_1) \setminus U \), so that \( q \) can indeed be defined to be the inclusion. There is a bijective correspondence between pairs \( (\tilde{x}_0, x_1) \in (\tau_L(L_0) \cap L) \times (L \cap L_1) \) and \( (y_0, y_1) \in \iota^{-1}(L_0 \cap L) \times \iota^{-1}(L \cap L_1) \), given by setting \( y_0 = \iota^{-1}(\tau_L^{-1}(\tilde{x}_0)), y_1 = \iota^{-1}(x_1) \). As a consequence of the condition (3) above,
\[
\iota^{-1}(\tau_L(L_0) \cap L_1) = \bigcup_{y_0, y_1} \tau(T(\lambda)_{y_0}) \cap T(\lambda)_{y_1}. \quad (10.3)
\]
It is clear from their definitions that \( p, q \) are injective. A point of \( \tau_L(L_0) \cap L_1 \) falls into \( \text{im}(q) \) or \( \text{im}(p) \) depending on whether it lies inside or outside \( \text{im}(\iota) \), hence the two images are disjoint and cover \( \tau_L(L_0) \cap L_1 \). The transversality follows from the definition of \( \tau_L \) for \( \text{im}(p) \) and from that of \( L_0 \cap L_1 \) for \( \text{im}(q) \).

We consider the triple
\[
L = (L, \gamma), \quad L_0 = (L_0, \tau_L^{-1} \circ \gamma), \quad L_1 = (L_1, \gamma_1).
\]
We note \( \tau_L(L_0) = (\tau_L(L_0), \gamma_0) = (\tau_L(L_0), \gamma) \). To make our discussion nontrivial, we may assume
\[
L \cap L_0 \neq \emptyset, \quad L \cap L_1 \neq \emptyset. \quad (10.4)
\]
We fix an element \( x_1 \in L \cap L_1 \) and \( \tilde{x}_0 \in \tau_L(L_0) \cap L \) and \( \tilde{z}_0 = p(\tilde{x}_0, x_1) \) where \( p \) is the injective map given in Lemma \[\text{[SE3]}\].

Lemma 10.2. Suppose \( x_1 \in L \cap L_1 \) and \( \tilde{x}_0 \in \tau_L(L_0) \cap L \) and \( \tilde{z}_0 = p(\tilde{x}_0, x_1) \). Then we have \( \tilde{z}_0 \in L_1 \cap \tau_L(L_0) \).
Proof. We first note that there is a canonical homotopy class $B_{can} = B(\tilde{x}_0, x_1, \tilde{z})$ spanned by a ‘thin’ triangle contained in $U = \text{im}\iota$. Choose a path $w_1$ from $\tau_0 * \gamma_1$ and $w_0$ from $\tau_0 * \gamma$. Then it follows that we can choose a path $w$ from $\tau_0 * \gamma_0$ defined by $u\# w_1 \# w_0$ where $u$ is the above thin triangle, i.e., any $w$ such that

$$[w] = B_{can} \# [w_1] \# [w_0].$$

This finishes the proof. 

Now we state the following lemma which is a variation of Lemma 3.2 [Se3] in our context.

**Proposition 10.3.** Let $L_0$, $L_1$ and $L$ be as in Lemma [10.1] and consider the maps $p, q$ defined therein. Then we can choose $\text{im}\tau_L$ so close to $L$ so that $\tau_L(L_0) \cap L_1$ satisfy the following properties in addition:

1. $q$ is the inclusion $q(x) = x$. Moreover, for any $z \in \tau_L(L_0) \cap L_1$ and $zf \in L_0 \cap L_1$ with $z \neq q(x)$, one has

$$A_{L_0 \cap L}(\{x, z\}) = A_{L_0 \cap L}(\{x, w\}) \neq 0; 3\varepsilon$$

whenever the corresponding Floer moduli space $M(x, z; \{\pi \# w\})$ is non-empty.

2. Set $\tilde{z} = p(\tilde{x}_0, x_1)$. Then there is a canonical homotopy class $B_{can} = B(\tilde{x}_0, x_1, \tilde{z})$ spanned by a ‘thin’ triangle contained in $U = \text{im}\iota$. And we have

$$|\Omega(B_{can})| < \varepsilon. \quad (10.5)$$

3. For any $z \in \tau_L(L_0) \cap L_1$, $z \neq \tilde{x}_0, x_1$, or for $z = \tilde{x} = p(\tilde{x}_0, x_1)$ with $B \neq B_{can}$, we have

$$\Omega(B) \geq C = C(\mathcal{E}; J) \quad (10.6)$$

independent of $\varepsilon > 0$ whenever $M_{J^*}(\tilde{x}_0, x_1; B) \neq \emptyset$ for some $J$.

4. Suppose that there are $x_k \in L \cap L_k$, $k = 0, 1$, whose preimages $y_k = \iota^{-1}(x_k)$ are antipodes on $S^n$. Since $\tau_l|S^n$ is the antipodal map, $\tilde{x}_0 = \tau_L(x_0)$ is equal to $x_1$ (hence $x_1 \in \tau_L(L_0) \cap L \cap L_1$ and these are all such triple intersection points). In that case $p(\tilde{x}_0, x_1) = \tilde{x}_0 = x_1$.

Proof. The proof is a slight modification of that of Lemma 3.2 [Se3]. Since we need to strip all the exact Lagrangian setting away from Seidel’s proof thereof and incorporate contributions coming from different choices of homotopy classes $B$, we give a complete proof of the proposition.

Now let $s(z) = (z, u(z))$ be the section of $E \to \tilde{\Sigma}$ in class $B_{can}$ satisfying the Lagrangian boundary condition. By definition of $B_{can}$ we can choose $u$ so that its image is contained in $\text{im}\iota$. Then (10.5) follows from the identity

$$\Omega(B_{can}) = \pi^* \omega(B_{can}) = \omega_0(\iota^{-1} \circ u)$$

which can be made as small as we want by choosing $r$ small in the definition of the Dehn twist $\tau_L$.

We now turn to (3). First we recall that since $E$ is trivial (and so of zero curvature), we have

$$\Omega(B) = \frac{1}{2} \int_{\tilde{\Sigma}} \|\langle Du \rangle \|^2$$
and hence whenever $\mathcal{M}_J(\bar{x}_0, x_1, z; B) \neq \emptyset$ for some $J$ we have $\Omega(B) \geq 0$. Define
\[
C(E; J) = \inf_u \{\omega(u) \mid u \neq \text{const.}, \ u \in \mathcal{M}_J(\bar{x}_0, x_1, z; B), \ B \neq B_{\text{can}} \text{ in } \Pi(E,Q)\}.
\]

**Proposition 10.4.** Let $B$ be an admissible class and $u \in \mathcal{M}_J(\bar{x}_0, x_1, z; B)$ with $\mu(B) = 0$ such that
\[
\text{im } u \subset U.
\]
Then we have $B = B_{\text{can}} \text{ in } \Pi(E,Q)$. In particular, we have
\[
C(E; J) > 0. \quad (10.7)
\]

**Proof.** Since we know $\mu(B_{\text{can}}) = 0$, it is enough to prove $\omega(B) = \omega(B_{\text{can}})$ by definition of $\Pi(E,Q)$. For this purpose, we compare the action for the paths whose image is contained in the Darboux neighborhood $U = \iota(V)$ with $\iota : (V, \omega_0) \rightarrow (M, \omega)$ and whose end points lie either on $L$ or on the fibers $F$ of the cotangent bundle $T^*L \cap V$ or in the Dehn twists $\tau_L(F)$. We recall that the model Dehn twist $\tau$ is a Hamiltonian diffeomorphism that satisfies
\[
\tau^* \theta_T - \theta_T = dK_{\tau}
\]
for $K_{\tau} = 2\pi(\mu R'(\mu) - R(\mu))$. On the cotangent bundle $T^*T$, the action functionals $A_{\text{or} F_1}$, $A_{\tau(F_0)\text{or}}$ and $A_{\tau(F_0)F_1}$ are defined by
\[
A_{\text{or} F_1}(z) = \int z^* \theta_T
\]
\[
A_{\tau(F_0)\text{or}}(z) = \int z^* \theta_T + K_{\tau} \circ \tau^{-1}(z(0))
\]
\[
A_{\tau(F_0)F_1}(z) = \int z^* \theta_T + K_{\tau} \circ \tau^{-1}(z(0))
\]
for a path $z : [0, 1] \rightarrow T^*L$. Here we use the fact that
\[
\theta_T|_{F} \equiv 0 \equiv \theta_T|_{\text{or}} \quad \theta_T|_{\tau(F)} = d(K_{\tau} \circ \tau^{-1}|_{\tau(F)}).
\]

(See (2.28) [Oh4] or (1.1) [Sc3].) Since it is easy to realize such $B = [u]$ as the gluing $-\frac{|w_0|}{\#|w|\#|w_1|}$ so that all $w_i, w$ have their images contained in $U$, we can write
\[
\omega(B) = A_{\text{or} F_1} (\hat{p}_0) - A_{\tau(F_0)\text{or}} (\tilde{p}) - A_{\tau(F_0)F_1} (\check{p}_1) = \omega(B_{\text{can}}).
\]

Here ‘hat’ denotes the constant path, e.g., $\hat{p}$ is the constant path $\hat{p}(t) \equiv p$. This proves $B = B_{\text{can}}$ in $\Pi(E,Q)$. For the proof of (10.7), it follows from the first part of the proof that for any $u \in \mathcal{M}_J(\bar{x}_0, x_1, z; B)$ with $B \neq B_{\text{can}}$ the image of $u$ must go out of $U$. Now a simple application of Gromov-type compactness theorem implies (10.7). This finishes the proof. \qed

This finishes the proof of (3). The statements in (4) are obvious from definition. \qed

Now we consider a Darboux neighborhood $U = \text{im } \iota$ of $L$ such that $\sup \tau_L \subset U$, and Lemma 2.1 and Proposition 2.2 hold. We denote by $\mathcal{M}_J(\tau_L(L_0), L, L_1)$ the moduli space of $J$-holomorphic sections of the fibration $(E^b, \pi^b, Q^b)$ where $J$ is
chosen as before. Then the arguments in subsection 3.2 [Se3] give rise to the following decomposition

$$b = \beta + (b - \beta) \quad (10.8)$$

that satisfies that $\beta$ is of order $[0; \varepsilon)$, while $(b - \beta)$ has order $[3\varepsilon; \infty)$. Furthermore $b$ is precisely the class induced by the canonical ‘small’ map continued from the constant map $p$.

Similar consideration as subsection 3.3 [Se3] also gives rise to the decomposition

$$c = \gamma + (c - \gamma) \quad (10.9)$$

such that $\gamma$ has order 0 while $(c - \gamma)$ has order $[3\varepsilon; \infty)$. We refer to [Se3] for the detailed proofs of the decomposition results.

Finally we can construct a homotopy from $b \#_p c$ to the zero map following Seidel’s argument from [Se3]. For this purpose we use Proposition 9.6 which is a slight generalization of Proposition 2.2 [Se3] in the general context of Lagrangian spheres in general symplectic manifolds. We omit the details of this construction referring readers to [Se3] for the details.

11. Construction of long exact sequence

In this section, we will combine all the results obtained in the previous section to construct the required long exact sequence. We first recall two basic lemmas that Seidel used in his construction of long exact sequence for the exact case.

**Lemma 11.1.** Let $D$ be an $\mathbb{R}$-graded vector space with a differential $d_D$ of order $[0; \infty)$. Suppose that $D$ has gap $[\varepsilon; 2\varepsilon)$ for some $\varepsilon > 0$. One can write $d_D = \delta + (d_D - \delta)$ with $\delta$ of order $[0; \varepsilon)$, satisfying $\delta^2 = 0$, and $(d_D - \delta)$ of order $[3\varepsilon; \infty)$. Suppose in addition, $H(D, \delta) = 0$; then $H(D, d_D) = 0$.

Seidel then applied this lemma to the direct sum

$$D = C' \oplus C \oplus C''$$

with the differentials given by

$$d_D = \begin{pmatrix} d_{C'} & 0 & 0 \\ b & d_C & 0 \\ h & c & d_{C''} \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix}$$

where

$$C' = CF(L_1) \otimes CF(\tau_L(L_0), L), \quad C = CF(\tau_L(L_0), L_1), \quad C'' = CF(L_1, L_0).$$

and the entries of the matrices are given as stated in the lemma below.

**Lemma 11.2** (Lemma 2.32 [Se3]). Take three $\mathbb{R}$-graded vector spaces $C'$, $C$, $C''$, each of them with a differential of order $(0; \infty)$. Suppose that we have the differential maps $b : C' \to C$, $c : C \to C''$ and a homotopy $h : C' \to C''$ between $c \circ b$ and the zero map, such that the following conditions are satisfied for some $\varepsilon > 0$:

1. $C'$, $C''$ have gap $(0, 3\varepsilon)$ and $C$ has gap $(0, 2\varepsilon)$.
2. For all $r \in \text{supp}(C')$ and $s \in \text{supp}(C'')$, $|r - s| \geq 4\varepsilon$. 


One can write
\[ b = \beta + (b - \beta), \quad c = \gamma + (c - \gamma) \]
with \( \beta \) of order \([0; \varepsilon]\) and \((b - \beta)\) of order \([2\varepsilon; \infty)\) and with the same properties for \( \gamma \) and \((c - \gamma)\). The lower order parts (which do not need be differential maps) fit into a short exact sequence of modules
\[ 0 \to C' \overset{\beta}{\to} C \overset{\gamma}{\to} C'' \to 0. \tag{11.1} \]

Then the maps on cohomology induced by \( b, c \) fit into a long exact sequence
\[ \cdots \to H(C'; d_{C'}) \overset{b^*}{\to} H(C; d_{C}) \overset{c^*}{\to} H(C''; d_{C''}) \overset{\delta}{\to} H(C'; d_{C'}) \to \cdots. \]

The proofs of both lemmata rely on an argument involving spectral sequences. For the exact case, all the complexes involved are finite dimensional vector spaces with \textit{bounded} filtration and \textit{gap} and so existence of spectral sequence such a complex is easy.

On the other hand, for the case of our current interest, the Floer complex as a \( Q \)-vector space is infinite dimensional with \textit{unbounded} filtration and \textit{without gap} on the vector space itself in general. Existence of spectral sequence in this case is much more non-trivial which has been studied by Fukaya-Oh-Ohta-Ono in \cite{FOOO1, FOOO2}. This is what we summarize in section 12.2.

In the mean time, we would like to remark that the proof of exactness of (11.1) is exactly the same as that of \cite{Se3} based on Lemmata 10.1, 10.2 and on some uniqueness result on small pseudoholomorphic triangle. (See section 3.2 \cite{Se3} for the details.)

11.1. \( CF(\mathcal{L}, \mathcal{L}') \) versus gapped \textit{d.g.c.f.z.} \( C(\mathcal{L}, \mathcal{L}'; \Lambda_{0,\text{nov}}) \). We first recall basic results on the structure of the Floer cochain group \( CF(\mathcal{L}, \mathcal{L}') \) as a module of the Novikov ring \( \Lambda(\mathcal{L}, \mathcal{L}') \). We note that in the current study of Calabi-Yau Lagrangian branes we can use the Novikov ring
\[ \Lambda_{0,\text{nov}}^{(0)} \]
as the coefficient ring where \( \Lambda_{0,\text{nov}}^{(0)} \) is the degree zero part of \( \Lambda_{0,\text{nov}} \) which is a field. Having this in mind, we first recall the basic construction on the spectral sequence of the \( \Lambda_{0,\text{nov}}^{(0)} \)-module \( C(L, L'; \Lambda_{0,\text{nov}}) \) from Chapter 6 \cite{FOOO2} (or Appendix \cite{FOOO1}) restricting to the \textit{finitely generated case}.

Under the given condition on \( L_0, L_1 \) and the given embedding \( f : S^n \to L \subset M, CF(\tau_L(L_0), \mathcal{L}), CF(\mathcal{L}, L_1) \) and \( CF(L_1, L_0) \) are all finitely generated over \( \Lambda_{\text{nov}} \). Here we note that there is a natural injective homomorphism
\[ I_{\mathcal{L}, \mathcal{L}'} : \Lambda(\mathcal{L}, \mathcal{L}') \to \Lambda_{0,\text{nov}} \]
and so we naturally extend their coefficient rings to \( \Lambda_{0,\text{nov}} \).

The thick-thin decompositions of the maps \( b \) and \( c \) given in section 10 implies that both maps are \textit{gapped} in the sense of Definitions 12.6 and 12.11. However these vector spaces, as they are, do not quite manifest the structure of \textit{d.g.c.f.z.} yet. Because of this, we follow the procedure given in section 12.4 \cite{FOOO2} turning these into a \textit{d.g.c.f.z.}. For readers' convenience, we collect the definition of \textit{d.g.c.g.z.} and construction of spectral sequence given in \cite{FOOO2} in Appendix.
Let \((\mathcal{L}_0, \mathcal{L}_1)\) be a general relatively spin pair of anchored Lagrangian submanifolds of \(M\). We first construct a \(\Lambda_{0,\text{nov}}\)-module \(C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}})\) which will have a filtered \(A_\infty\)-bimodule structure over \((C(\mathcal{L}_0; \Lambda_{0,\text{nov}}), \mathfrak{m}^{(0)})\) and \((C(\mathcal{L}_1; \Lambda_{0,\text{nov}}), \mathfrak{m}^{(1)})\) where the latter are the filtered \(A_\infty\) algebras defined in [FOOO2]. We consider the intersection \(\mathcal{L}_1 \cap \mathcal{L}_0\) and the \(\mathbb{R}\)-filtered set

\[
\tilde{I}(\mathcal{L}_0, \mathcal{L}_1) := \{ T^\lambda e^\mu[p, w] \mid p \in \mathcal{L}_0 \cap \mathcal{L}_1, \ [p, w] \in \tilde{\Omega}(\mathcal{L}_0, \mathcal{L}_1), \lambda \in \mathbb{R}, \mu \in \mathbb{Z} \}.
\]

This is a \(\mathbb{R} \times \mathbb{Z}\) principal bundle over \(\mathcal{L}_0 \cap \mathcal{L}_1\).

We define an equivalence relation \(\sim\) on \(\tilde{I}(\mathcal{L}_0, \mathcal{L}_1)\) as follows. We say

\[
T^\lambda e^\mu[p, w] \sim T'^\lambda e'^\mu[p', w']
\]

for \(T^\lambda e^\mu[p, w], T'^\lambda e'^\mu[p', w'] \in \tilde{I}(\mathcal{L}_0, \mathcal{L}_1)\), if and only if the following conditions are satisfied:

\[
\begin{align*}
p &= p' \\
\lambda + \int w^*\omega &= \lambda' + \int (w')^*\omega \\
2\mu + \mu([p, w]; \lambda_{01}) &= 2\mu' + \mu([p', w'; \lambda_{01}]).
\end{align*}
\]

Here \(\mu([p, w]; \lambda_{01})\) is the Maslov-Morse index. It is easy to see that this relation is compatible with the conditions of the \(\Gamma\)-equivalence given and so \(\sim\) defines an equivalence relation on \(\tilde{I}(\mathcal{L}_0, \mathcal{L}_1)\). Furthermore we define the action level on \(E : \tilde{I}(\mathcal{L}_0, \mathcal{L}_1) \to \mathbb{R}\) by

\[
E(T^\lambda e^\mu[p, w]) = \lambda + \int w^*\omega
\]

and the associated filtration on the set by setting

\[
T^\lambda e^\mu[p, w] \in F^\lambda(\tilde{I}(\mathcal{L}_0, \mathcal{L}_1))
\]

if

\[
\lambda + \int w^*\omega \geq \lambda'.
\]

We now define

\[
I(\mathcal{L}_0, \mathcal{L}_1) = \tilde{I}(\mathcal{L}_0, \mathcal{L}_1)/\sim
\]

and somewhat ambiguously denote an element thereof still by \(T^\lambda e^\mu/2[p, w]\) as long as no danger of confusion arises. The above mentioned filtration on \(\tilde{I}(\mathcal{L}_0, \mathcal{L}_1)\) obviously induces on on the quotient \(I(\mathcal{L}_0, \mathcal{L}_1)\). We now define

\[
I_{\geq 0}(\mathcal{L}_0, \mathcal{L}_1) := \left\{ T^\lambda e^\mu/2[p, w] \in I(\mathcal{L}_0, \mathcal{L}_1) \mid \lambda + \int w^*\omega \geq 0 \right\}.
\]

Consider the formal sum

\[
\alpha = \sum_{\lambda, \mu, [p, w]} a_{\lambda, \mu, [p, w]} T^\lambda e^\mu/2[p, w]
\]

for \(\lambda \in \mathbb{R}, \mu \in \mathbb{Z}\) and \([p, w] \in \text{Crit} \mathcal{A}\) and define \(\text{supp} \alpha\) to be

\[
\text{supp} \alpha = \{ T^\lambda e^\mu/2[p, w] \in I(\mathcal{L}_0, \mathcal{L}_1) \mid a_{\lambda, \mu, [p, w]} \neq 0 \}.
\]

**Definition 11.3.** We define by \(C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}})\) the \(\Lambda_{0,\text{nov}}\)-module

\[
C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}}) := \{ \alpha \mid E(\alpha) \geq 0, \#(\text{supp} \alpha \cap E^{-1}((-\infty, \lambda])) < \infty \text{ for all } \lambda \in \mathbb{R} \}
\]

Obviously \(C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}})\) has a structure of \(\Lambda_{0,\text{nov}}\)-module. In addition, we have
Proposition 11.4. \(C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}})\) is a d.g.c.f.z.

Given the grading of an element \(T^\lambda e^\mu[p, w] 2\mu + \mu([p, w]; \lambda_01)\), it becomes a filtered graded free \(\Lambda_{0,\text{nov}}\) module. Following section 5.1.3 [FOOO2], we write

\[
\langle p \rangle = \begin{cases} 
T^{-A_{\ell_0,\ell_1}}([p, w]) e^{-\mu([p, w]; \lambda_01)[p, w]} & \text{if } \mu([p, w]; \lambda_01) \text{ is even}, \\
T^{-A_{\ell_0,\ell_1}}([p, w]) e^{-(\mu([p, w]; \lambda_01)-1)/2[p, w]} & \text{if } \mu([p, w]; \lambda_01) \text{ is odd}.
\end{cases}
\]

Thus we have \(E(\langle p \rangle) = 0\) and \(\deg(\langle p \rangle)\) is either 0 or 1 depending on the parity of \(\mu([p, w]; \lambda_01)\).

It is easy to see that \(C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}})\) is isomorphic to the completion (with respect to the filtration on \(\Lambda_{0,\text{nov}}\)) of the free \(\Lambda_{0,\text{nov}}\) module generated by \(\langle p \rangle\) for the intersection points \(p \in \mathcal{L}_0 \cap \mathcal{L}_1\). Namely we have a canonical isomorphism

\[
C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}}) \cong \bigoplus_{p \in \mathcal{L}_0 \cap \mathcal{L}_1} \Lambda_{0,\text{nov}}(\langle p \rangle)
\]

as a \((\mathbb{Z}_2\text{-graded}) \Lambda_{0,\text{nov}}\) module.

Recall the definition of the Floer cochain module. By definition, we have an inclusion

\[
CF^*(\mathcal{L}_1, \mathcal{L}_0) \rightarrow C^*(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{\text{nov}})
\]

defined by

\[
[p, w] \mapsto e^{\mu([p, w])} e^{\mu([p, w])/2T^p(\Lambda_{\ell_0}(\langle p \rangle))}\langle p \rangle.
\]

It is compatible with the obvious inclusion \(I_{\mathcal{L}_0,\mathcal{L}_1} : \Lambda(\mathcal{L}_0, \mathcal{L}_1) \rightarrow \Lambda_{\text{nov}}\).

We take the coefficient \(R = \mathbb{Q}\) and recall that Floer cohomology

\[
HF(\mathcal{L}^{(1)}, \mathcal{L}^{(0)}) = \text{Ker} \delta^\beta/\text{Im} \delta^\beta, \quad \delta^\beta = (b_1, b_0)
\]

is defined as a \(\Lambda(\mathcal{L}^{(0)}, \mathcal{L}^{(1)})\)-module.

We remark that

\[
CF(\mathcal{L}_1, \mathcal{L}_0) \cong \Lambda(\mathcal{L}_1, \mathcal{L}_0)\#(\mathcal{L}_0 \cap \mathcal{L}_1),
\]

where \(#\mathcal{L}_0 \cap \mathcal{L}_1\) is finite by the transversality hypothesis. Therefore we have the isomorphism

\[
C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{\text{nov}}) \cong CF(\mathcal{L}_1, \mathcal{L}_0; \ell_01) \otimes \Lambda(\mathcal{L}_1, \mathcal{L}_0) \Lambda_{\text{nov}}.
\]

On the other hand, the Novikov ring \(\Lambda(\mathcal{L}_1, \mathcal{L}^{(0)})\) is a field if the ground ring is \(\mathbb{Q}\). Therefore, this leads to the isomorphism

\[
HF((\mathcal{L}_1, b_1), (\mathcal{L}_0, b_0); \Lambda_{\text{nov}}) \cong HF(\mathcal{L}_1, \mathcal{L}_0) \otimes \Lambda(\mathcal{L}_1, \mathcal{L}_0) \Lambda_{\text{nov}}.
\]

Finally, we explain how we combine the above discussed anchored versions into a single non-anchored version of Floer cohomology following section 5.1.3 [FOOO2].

We first note that the filtered \(\Lambda_{0,\text{nov}}\)-module structure of \(C(\mathcal{L}_1, \mathcal{L}_0; \Lambda_{0,\text{nov}})\) depends only on the homotopy class \(\ell_01\). So we form the completed direct sum

\[
C(L_1, L_0; \Lambda_{0,\text{nov}}) = \bigoplus_{[\ell_0] \in \pi_0(\text{Hom}(L_0, L_1))} C(L_1, L_0; \ell_0); \Lambda_{0,\text{nov}}).
\]

We note that we have the natural inclusion map

\[
C(\mathcal{L}_1, \mathcal{L}_0) \rightarrow C(L_1, L_0; \pi_0 \ast \gamma_1; \Lambda_{0,\text{nov}}) \subset C(L_1, L_0; \Lambda_{0,\text{nov}})
\]

defined as (11.2). We define the corresponding Floer cohomology by

\[
HF(L_1, L_0; \Lambda_{0,\text{nov}}) := \text{Ker} \delta^\beta/\text{Im} \delta^\beta.
\]
Then we have
\[ HF(L_1, L_0; \Lambda_{0, \text{nov}}) \cong \bigoplus_{[\alpha] \in \pi_0(\Omega(L_0, L_1))} HF(L_1, L_0) \otimes_{\Lambda(L_1, L_1; \alpha_0)} \Lambda_{\text{nov}}. \]

(See Proposition 5.1.17 [FOOO2].)

11.2. Wrapping it up. Now combining all the discussions in the previous subsections, we are ready to prove the main theorem which we now re-state here.

**Theorem 11.5.** Let \((M, \omega)\) be a compact (symplectically) Calabi-Yau. Let \(L\) be a Lagrangian sphere in \(M\) together with a preferred diffeomorphism \(f : S^2 \to L\). Denote by \(\tau_L = \tau(L, [f])\) be the Dehn twist associated to \((L, [f])\).

Consider any Calabi-Yau Lagrangian branes \(L_0, L_1\). Then for any pair \((b_0, b_1)\) of the MC-solutions \(b_0 \in \mathcal{M}(L_0; \Lambda(L_0)), b_1 \in \mathcal{M}(L_1; \Lambda(L_1))\), there is a long exact sequence of \(\mathbb{Z}\)-graded Floer cohomologies

\[ \cdots \to HF(\tau_L(L_0), (\tau_L)_*(b_0)), (L_1, b_1)) \to HF((L_0, b_0), (L_1, b_1)) \to HF((L_0, b_0), (L_1, b_1)) \to \cdots \]

as a \(\Lambda_{\text{nov}}\)-module where the Floer cohomologies involved are the deformed Floer cohomology constructed in [FOOO1, FOOO2]. We also have the non-anchored version of the exact sequence.

The same exact sequence still holds for any orientable relatively spin pair \((L_0, L_1)\) if they are just unobstructed whose Maslov classes do not necessarily vanish.

To highlight the main points of the construction, let us first assume that \(b_0 = b_1 = 0\) are MC-solutions. In this case, the Floer cohomology is the standard one which uses the Floer boundary map \(\delta\). We first state the following lemma which is a consequence of Corollary [12.13] and a variation of Lemma [11.2].

**Lemma 11.6.** Let \(D\) be an \([0; \infty)\)-graded vector space with a differential \(d_D\) of order \([0; \infty)\), which is not necessarily finite dimensional but forms a d.g.c.f.z. in the sense of Definition [12.3]. Suppose that one can then write \(d_D = \delta + (d_D - \delta)\) with \(\delta\) of order \([0; \varepsilon]\), satisfying \(\delta^2 = 0\), and \((d_D - \delta)\) of order \([2\varepsilon; \infty)\). Suppose in addition, \(H(D, \delta) = 0\); then \(H(D, d_D) = 0\).

Now we apply this lemma to the direct sum
\[ D = C' \oplus C \oplus C'' \]
with the differentials given by
\[ d_D = \begin{pmatrix} 0 & 0 & 0 \\ b & d_C & 0 \\ h & c & d_{C''} \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} \]
where this time we consider the \(\Lambda_{0, \text{nov}}\)-modules
\[ C' = C(L, L_1) \otimes C(\tau_L(L_0), L), \quad C = C(\tau_L(L_0), L_1), \quad C'' = C(L_0, L_1). \]
and the entries of the matrices are given as stated as before.

Now Lemma [11.2] and the thick-thin decomposition results in section [10] give rise to a long exact sequence
\[ \cdots \to HF(\tau_L(L_0), L_1; \Lambda_{0, \text{nov}}) \to HF(L_0, L_1; \Lambda_{0, \text{nov}}) \to \cdots \]

(11.5)
for $A_{0,\text{nov}}^{(0)}$-modules. Since we have

$$HF(\mathcal{L}, \mathcal{L}'; \Lambda^{\text{nov}}) \cong HF(\mathcal{L}, \mathcal{L}'; \Lambda_{0,\text{nov}}) \otimes_{\Lambda_{0,\text{nov}}} \Lambda^{\text{nov}}$$

from (11.3) and $\Lambda^{\text{nov}}$ is a field, tensoring (11.5) with $\Lambda^{\text{nov}}$ produces the exact sequence (11.4) for the case $b_0 = b_1 = 0$.

Now the same reasoning as for the case $b_i = 0$ induces the long exact sequence (11.4). This finishes the proof of Theorem 11.5.

12. Appendix

12.1. Index formula for $E_{x_0} \setminus \{x_0\}$. In this section, we prove the index formula (7.21). There is an index formula stated various literature in terms of the ‘capping surfaces’ stated as in [EGH], [Bo], which however does not fit our need. For this reason, we give a complete proof of (7.21).

In fact we will consider the following general set-up. Consider a symplectic manifold $W$ with a contact type boundary of the type $\partial W \cong S^1(T^\ast N)$ with negative end for an oriented compact manifold $N$. We attach the cylinder $\mathbb{R}_+ \times \partial W$ and also denote by $W$ the completed manifold. We denote by $(r(x), \Theta(x))$ for a point $x \in \mathbb{R}_+ \times S^1(T^\ast N)$. Composing this with the diffeomorphism $(s, \Theta) \mapsto (e^s, \Theta); \mathbb{R} \times \partial W \to \mathbb{R}_+ \times \partial W$

we put a translational invariant almost complex structure $J$ on the end.

Next let $\gamma$ be a Reeb orbit of $S^1(T^\ast N)$ with period $T$. We note that the symplectic vector bundle $\gamma^\ast T(T^\ast N)$ carries a splitting

$$\gamma^\ast T(T^\ast N) = \mathbb{C} \oplus \gamma^\ast \xi_N$$

where $\xi_N$ is the contact distribution of $S^1(T^\ast N)$. Furthermore we fix a Riemannian metric $g$ on $N$ and consider the canonical almost complex structure $J_g$ on $T^\ast N$. The projection of $\gamma$ to $N$ is nothing but a geodesic on $N$ with respect to $g$. Denote by $c = c_\gamma$ the associated geodesic on $N$. Since we assume $N$ is oriented, we can take a trivialization $\gamma^\ast T(S^1(T^\ast N))$ which is tangent to the vertical fibers of $T(T^\ast N)$. Using this we can define the Conley-Zehnder index of $\gamma$ when $\gamma$ is nondegenerate, which we denote by $\mu_{CZ}(\gamma)$. For the Bott-Morse case, one uses the generalized Conley-Zehnder index defined by Robbin and Salamon [RS].

Next this choice of trivialization of $\gamma^\ast (T(S^1(T^\ast N))) = \gamma^\ast (T(\partial W))$ also allows one to define a relative Chern number of a map $u : \Sigma \to W$ with the asymptotic condition

$$\lim_{\tau \to \infty} \Theta \circ u(\tau, t) = \gamma(t), \quad \lim_{\tau \to \infty} s \circ u(\tau, t) = -\infty. \quad (12.1)$$

Denote by $\overline{\pi} : (\hat{\Sigma}, \partial \hat{\Sigma}) \to (W, \gamma)$ the obvious compactified map.

Then $\overline{\pi}^\ast (TW)$ is a symplectic vector bundle with a trivialization $\phi_\gamma : \gamma^\ast (T(\partial W)) \to S^1 \times \mathbb{C}^{n-1}$ constructed above.

This gives rise to the main definition

**Definition 12.1.** We define the relative Chern number, denoted by $c_1(u; \gamma)$, by

$$c_1(u; \gamma) = c_1(u^\ast TW; \phi_\gamma).$$
Once we have made the notions of relative Chern number and Conley-Zehnder index precise, the following index formula can be derived from the formula in Corollary 5.4. [Bo]

**Theorem 12.2.** The expected dimension of $\mathcal{M}(W,J;\gamma;A)$ is given by

$$-\mu_{CZ}(\gamma) + (n-3) + 2c_1(u;\gamma), \quad [u] = A$$

for a non-degenerate geodesic.

For the Morse-Bott case in which $R_{\text{sim}}$ forms a smooth manifold, the expected dimension of the moduli space $\mathcal{M}(W,J;A;1)$ consisting of $J$-holomorphic $u$’s with asymptotics

$$\lim_{\tau \to \text{inf}} \Theta \circ u(e^{2\pi i(\tau+it)}) = \gamma(t)$$

where $\gamma$ is a simple Reeb orbit, is given by

$$-\mu_{CZ}(\gamma) + \frac{\dim R_{\text{sim}}}{2} + (n-3) + 2c_1(u;\gamma)$$

where $\mu_{CZ}$ is the generalized Conley-Zehnder index of $\gamma$.

### 12.2. d.g.c.f.z. and spectral sequence.

We first start from the following situation. Let $V = (\Lambda_{0,\text{nov}}^{(0)})^{\oplus I}$ be a free $\Lambda_{0,\text{nov}}^{(0)}$ module with $\#(I)$ finite. We define a filtration on $V$ in the obvious way which will induce a topology on $V$. Let $\hat{V}$ be the completion of $V$. We call such $\hat{V}$ a completed free filtered $\Lambda_{0,\text{nov}}^{(0)}$ module generated by energy zero elements, or in short c.f.z. If $V$ is finitely generated (as a $\Lambda_{0,\text{nov}}^{(0)}$ module) in addition, we say that it is a finite c.f.z. We define a function, which we call the (action) level,

$$E : \hat{V} \setminus \{0\} \to \mathbb{R}_{\geq 0}$$

such that

$$v \in F^{E(v)} V, \quad v \notin F^{\lambda} V \quad \text{if} \quad \lambda > E(v).$$

Let $\nabla = V/\Lambda_{0,\text{nov}}^{(0)} V \cong R^I$. We always take an embedding (splitting)

$$\nabla \subset V$$

as the energy 0 part of $V$ so that its composition with the projection $V \to \nabla$ is the identity map.

Let $v \in V$. We put

$$v = \sum T^{\lambda_i} v_i,$$

where $v_i \in \nabla$, $\lambda_i < \lambda_{i+1}$, $\lim_{i \to \infty} \lambda_i = \infty$ and $v_i \neq 0$. We call $T^{\lambda_i} v_i$ the components of $v$, $T^{\lambda_1} v_1$ the leading component and $v_1$ the leading coefficient of $v$. We denote the leading coefficient of $v$ by $\sigma(v)$. We also define the leading component and the leading coefficient of an element of $\Lambda_{0,\text{nov}}^{(0)}$ in the same way.

Now we consider the case of graded $\Lambda_{0,\text{nov}}^{(0)}$ modules.

**Definition 12.3** (Definition 6.3.8 [FOOO2]). Let $\hat{C}$ be a graded $\Lambda_{0,\text{nov}}^{(0)}$ module. We assume that $\hat{C}^k$ is a c.f.z. for each $k$. A differential graded c.f.z. (abbreviated as d.g.c.f.z) is a pair $(\hat{C}, \delta)$ with a degree 1 operator $\delta : \hat{C} \to \hat{C}$ such that

$$\delta \circ \delta = 0, \quad \delta(F^{\lambda} \hat{C}) \subseteq F^{\lambda} \hat{C}.$$  

We call the pair a finite d.g.c.f.z. if each $\hat{C}^k$ is a finite c.f.z.
The following proposition is essential for the proof of some convergence properties of the spectral sequence.

**Proposition 12.4** (Proposition 6.3.9 [FOOO2]). Let \( W \) be a finitely generated \( \Lambda_{0,\text{nov}}^{(0)} \) submodule of \( \hat{C}^k \). Then there exists a constant \( c \) depending only on \( W \) but independent of \( \lambda \) such that
\[
\delta(W) \cap F^\lambda \hat{C}^{k+1} \subset \delta(W \cap F^{\lambda-c} \hat{C}^k).
\]

Now let \((\hat{C}, \delta)\) be a d.g.c.f.z. and \( \hat{C}^k \) a completion of \( C^k \). We assume that \( C^k \) is free over \( \Lambda_{0,\text{nov}}^{(0)} \). We put
\[
\hat{C} = C / \Lambda_{0,\text{nov}}^{+(0)} C \cong \hat{C} / \Lambda_{0,\text{nov}}^{+(0)} \hat{C},
\]
and let \( \bar{\delta} \) be the induced derivation on \( \bar{C} \). We again embed \( \bar{C} \subseteq C \subseteq \hat{C} \) as the energy 0 part. In general \( C \) is not a differential graded subalgebra of \( \hat{C} \). Let \( \{ e_i \} \) be a basis of \( C \) (over \( \Lambda_{0,\text{nov}}^{(0)} \)) and \( e_i \) be the corresponding basis of \( C \) over \( \Lambda_{0,\text{nov}}^{(0)} / \Lambda_{0,\text{nov}}^{+(0)} \).

We put
\[
\delta(e_i) = \sum \delta_{0,ij} e_j,
\]
and define \( \delta_0 : \hat{C} \to \hat{C} \) by \( \delta_0 = \bar{\delta} \otimes 1 \) i.e., by
\[
\delta_0 e_i = \sum \delta_{0,ij} e_j.
\]

**Definition 12.5.** We say that \((\hat{C}, \delta)\) satisfies the gapped condition if \((\delta - \delta_0)\) has order \( [\lambda'', \infty) \), i.e., if there exists \( \lambda'' > 0 \) such that for any \( \lambda \) we have
\[
\delta v - \delta_0 v \in F^{\lambda + \lambda''} \hat{C}
\]
for all \( v \in F^\lambda \hat{C} \).

Under the gapped condition, we take a constant \( \lambda_0 \) with \( 0 < \lambda_0 < \lambda'' \) and define a filtration on \( \hat{C} \) by
\[
F^n \hat{C} = F^n \Lambda_{0,\text{nov}}^{(0)} \hat{C}.
\]
[FOOO2] then uses this filtration to define a spectral sequence.

**Lemma 12.6** (Lemma 6.3.20 [FOOO2]). Denote
\[
\Lambda^{(0)}(\lambda_0) = \Lambda_{0,\text{nov}}^{(0)} / F^{\lambda_0} \Lambda_{0,\text{nov}}^{(0)}.
\]
Then there exists a \( \Lambda^{(0)}(\lambda_0) \) module homomorphism
\[
\delta_p^{p-q} : E_r^{p-q}(\hat{C}) \to E_r^{p+1,q+r-1}(\hat{C})
\]
such that
\[
\begin{align*}
(1) \quad \delta_r^{p+1,q+r-1} \circ \delta_p^{p-q} &= 0. \\
(2) \quad \text{Ker}(\delta_r^{p,q}) / \text{Im}(\delta_r^{p+1,q+r+1}) &= E_r^{p+1,q+1}(\hat{C}). \\
(3) \quad e^{\pm 1} \circ \delta_r^{p,q} &= \delta_r^{p+2,q} \circ e^{\pm 1}.
\end{align*}
\]

Of course, the construction of \( E_r^{p-q}(\hat{C}) \) is quite standard. One difference from the standard case is that the filtration used here is not bounded. Namely we do not have \( F^n \hat{C} = 0 \) for large \( n \). Hence the convergence property of our spectral sequence is far from being trivial in general. However it is stable from below in that \( F^0 \hat{C} = \hat{C} \). As a consequence we have:
Lemma 12.7 (Lemma 6.3.22 [FOOO2]). There exists an injection
\[ E^{p,q}_{r+1}(\hat{C}) \to E^{p,q}_r(\hat{C}) \]
if \( q - r + 2 \leq 0 \).

An immediate consequence of Lemma 12.7 is the following convergence result.

**Proposition 12.8.** The projective limit
\[ E^{p,q}_\infty(\hat{C}) := \lim_{\leftarrow} E^{p,q}_r(\hat{C}) \]
exists.

Furthermore from the construction, we have the description of the \( E_2 \)-term of the associated spectral sequence

**Lemma 12.9** (Lemma 6.3.24 [FOOO2]). We have an isomorphism
\[ E_2^{p,q}(\hat{C}) \cong H\left(\overline{\mathcal{C}};\delta\right) \otimes_R \text{gr}_*(F\Lambda_{0,nov}) \]
as \( \text{gr}_*(F\Lambda_{0,nov}) \) modules.

**Proof.** By definition we have
\[ E_1^{p,q}(\hat{C}) \cong \overline{\mathcal{C}} \otimes_R \text{gr}_*(F\Lambda_{0,nov}). \]
It follows from the gapped condition that \( \delta_1 = \overline{\delta} \). Hence it finishes the proof. \( \square \)

**Definition 12.10.** We define \( F^qH(\hat{C},\delta) \) to be the image of \( H(F^q\hat{C},\delta) \) in \( H(\hat{C},\delta) \).

To relate the limit \( E^{p,q}_\infty \) of the spectral sequence and \( F^qH(\hat{C},\delta) \), we need some finiteness assumption which we now describe. Let \( (C,\delta) \) and \( (C',\delta') \) be d.g.c.f.z’s satisfying the gap condition. Let \( \varphi : C \to C' \) be a map such that \( \varphi_0 = \delta' \varphi \) and let \( \overline{\varphi} : \overline{\mathcal{C}} \to \overline{\mathcal{C}'} \) be the map induced on \( \overline{\mathcal{C}} = C/\Lambda_{0,nov}^{(0)}C \) and \( \overline{\mathcal{C}'} = C'/\Lambda_{0,nov}^{(0)}C' \) respectively. The induced map \( \overline{\varphi} \) lifts to \( \varphi_0 = \overline{\varphi} \otimes 1 : C \to C' \).

**Definition 12.11** (Definition 6.3.26 [FOOO2]). Under the situation above, we say that \( \varphi : C \to C' \) satisfies a gapped condition, or is a gapped cochain map, if there exists \( \lambda'' \) such that
\[ (\varphi - \varphi_0)(F^\lambda\hat{C}) \subset F^{\lambda + \lambda''}\hat{C}. \]

Using these definitions, [FOOO2] proves

**Theorem 12.12** (Theorem 6.3.28 [FOOO2]). If \( C \) is finite, then there exists \( r_0 \) such that:
\[ E^{p,q}_{r_0}(\hat{C}) \cong E^{p,q}_{r_0+1}(\hat{C}) \cong \cdots \cong E^{p,q}_\infty(\hat{C}) \cong F^qH(\hat{C},\delta)/F^{q+1}H(\hat{C},\delta) \]
as \( \Lambda^{(0)}(\lambda_0) = \Lambda_{0,nov}^{(0)}/F^\lambda\Lambda_{0,nov}^{(0)} \) modules.

We summarize the above discussion into the following vanishing result which will be crucial in our spectral sequence arguments.

**Corollary 12.13.** Let \( C \) be a finite d.g.c.f.z. Then if \( H(\overline{\mathcal{C}},\delta) = 0 \), then \( H(C,\delta) = 0 \).
12.3. Products. In this subsection, we recall the description of the deformed products $m_L^\lambda$ from \cite{FOOO3}. We refer to \cite{FOOO3} and \cite{FOOO3} for the relevant proofs of the statements we make without proofs here.

Let $\mathcal{L} = (L_0, L_1, \cdots, L_k)$ be a chain of compact Lagrangian submanifolds in $(M, \omega)$ that intersect pairwise transversely without triple intersections.

Let $\mathcal{Z} = (z_{0k}, z_{k(k-1)}, \cdots, z_{10})$ be a set of distinct points on $\partial D^2 = \{z \in \mathbb{C} \mid |z| = 1\}$. We assume that they respect the counter-clockwise cyclic order of $\partial D^2$. The group $\text{PSL}(2; \mathbb{R}) \cong \text{Aut}(D^2)$ acts on the set in an obvious way. We denote by $\mathcal{M}_{k+1}^{\text{main}, \circ}$ the set of $\text{PSL}(2; \mathbb{R})$-orbits of $(D^2, \mathcal{Z})$.

In this subsection, we consider only the case $k \geq 2$ since the case $k = 1$ is already discussed in the last subsection. In this case there is no automorphism on the domain $(D^2, \mathcal{Z})$, i.e., $\text{PSL}(2; \mathbb{R})$ acts freely on the set of such $(D^2, \mathcal{Z})$'s.

Let $\pi_j(j-1) \in L_j \cap L_{j-1}$ ($j = 0, \cdots, k$), be a set of intersection points.

We consider the pair $(w; \mathcal{Z})$ where $w : D^2 \to M$ is a pseudo-holomorphic map that satisfies the boundary condition

$$w(z_{j(j-1)} z_{(j+1)j}) \subset L_j,$$

$$w(z_{(j+1)j}) = \pi_{j+1} \in L_j \cap L_{j+1}.$$  \hspace{1cm} (12.2a)

$$w(z_{(j+1)j}) = \pi_{j+1} \in L_j \cap L_{j+1}.$$  \hspace{1cm} (12.2b)

We denote by $\tilde{\mathcal{M}}^\circ(\mathcal{L}, \mathcal{P})$ the set of such $(D^2, \mathcal{Z}, w)$.

We identify two elements $((D^2, \mathcal{Z}), w), ((D^2, \mathcal{Z}'), w')$ if there exists $\psi \in \text{PSL}(2; \mathbb{R})$ such that $w \circ \psi = w'$ and $\psi(z_{j(j-1)}) = z_{j(j-1)}$. Let $\mathcal{M}^\circ(\mathcal{L}, \mathcal{P})$ be the set of equivalence classes. We compactify it by including the configurations with disc or sphere bubbles attached, and denote it by $\mathcal{M}(\mathcal{L}, \mathcal{P})$. Its element is denoted by $((\Sigma, \mathcal{Z}), w)$ where $\Sigma$ is a genus zero bordered Riemann surface with one boundary components, $\mathcal{Z}$ are boundary marked points, and $w : (\Sigma, \partial \Sigma) \to (M, L)$ is a bordered stable map.

We can decompose $\mathcal{M}(\mathcal{L}, \mathcal{P})$ according to the homotopy class $B \in \pi_2(\mathcal{L}, \mathcal{P})$ of continuous maps satisfying (12.2a, 12.2b) into the union

$$\mathcal{M}(\mathcal{L}, \mathcal{P}) = \bigcup_{B \in \pi_2(\mathcal{L}, \mathcal{P})} \mathcal{M}(\mathcal{L}, \mathcal{P}; B).$$

In the case we fix an anchor $\gamma_i$ to each of $L_i$ and put $\mathcal{E} = ((L_0, \gamma_0), \cdots, (L_k, \gamma_k))$, we consider only admissible classes $B$ and put

$$\mathcal{M}(\mathcal{E}, \mathcal{P}) = \bigcup_{B \in \pi_2(\mathcal{E}, \mathcal{P})} \mathcal{M}(\mathcal{E}, \mathcal{P}; B).$$

Theorem 12.14. Let $\mathcal{L} = (L_0, \cdots, L_k)$ be a chain of Lagrangian submanifolds and $B \in \pi_2(\mathcal{L}, \mathcal{P})$. Then $\mathcal{M}(\mathcal{L}, \mathcal{P}; B)$ has an oriented Kuranishi structure (with boundary and corners). Its (virtual) dimension satisfies

$$\dim \mathcal{M}(\mathcal{L}, \mathcal{P}; B) = \mu(\mathcal{L}, \mathcal{P}; B) + n + k - 2,$$  \hspace{1cm} (12.3)

where $\mu(\mathcal{L}, \mathcal{P}; B)$ is the polygonal Maslov index of $B$.

We next take graded anchors $(\gamma_i, \lambda_i)$ to each of $L_i$. We assume that $B$ is admissible and write $B = \{w_{ij}\} = \sum_{i=0}^{k-1} |w_{ij}|$ as in Definition 5.4. We put $w_{i(i+1)}^-(s, t) = w_{i(i+1)}^-(1 - s, t)$. We also put $w_{i(i+1)}^+(s, t) = w_{i(i+1)}^+(s, 1 - t)$. (\cite{FOOO3}) We also put $\lambda_k(\ell) = \lambda_k(1 - \ell)$. 

Lemma 12.15 (Lemma 8.11 [FOOO3]). If \( \dim \mathcal{M}(\mathcal{L}, \mathcal{P}; B) = 0 \), we have

\[
(\mu([p_{k0}, w_{k0}^+]; \lambda_{0k}) - 1) = 1 + \sum_{i=1}^{k} (\mu([p_{i(i-1)}, w_{i(i-1)}^+]; \lambda_{i(i-1)i}) - 1).
\]  

(12.4)

Using the case \( \dim \mathcal{M}(\mathcal{L}, \mathcal{P}; B) = 0 \), we define the \( k \)-linear operator

\[ m_k : CF((L_k, \gamma_k), (L_{k-1}, \gamma_{k-1})) \otimes \cdots \otimes CF((L_1, \gamma_1), (L_0, \gamma_0)) \to CF((L_k, \gamma_k), (L_0, \gamma_0)) \]

as follows:

\[
m_k([p_{k(k-1)}, w_{k(k-1)}^+], [p_{(k-1)(k-2)}, w_{(k-1)(k-2)}^+], \cdots, [p_{10}, w_{10}^+]) = \sum \#(\mathcal{M}_{k+1}(\mathcal{L}, \mathcal{P}; B))[p_{k0}, w_{k0}^+].
\]

(12.5)

Here the sum is over the basis \([p_{k0}, w_{k0}^+]\) of \( CF((L_k, \gamma_k), (L_0, \gamma_0)) \), where \( \mathcal{P} = (p_{0k}, p_{k(k-1)}, \cdots, p_{10}), B \) is as in Definition 8.7, and \( w_{i(i-1)i}(s, t) = w_{i(i-1)}(1 - s, t) \).

The formula (12.4) implies that \( m_k \) above has degree one.

In general the operator \( m_k \) above does not satisfy the \( A_\infty \) relation by the same reason as that of the case of boundary operators (see Section 8.1). We need to use bounding chains \( b_i \) of \( L_i \) to deform \( m_k \) in the same way as the case of \( A_\infty \)-bimodules (Subsection 8.3.3), whose explanation is now in order.

Let \( m_0, \cdots, m_k \in \mathbb{Z}_{\geq 0} \) and \( \mathcal{M}_{m_0, \cdots, m_k}(\mathcal{L}, \mathcal{P}; B) \) be the moduli space obtained from the set of \( ((D^2, \mathcal{Z}), (z^{(0)}, \cdots, z^{(k)}), w)) \) by taking the quotient by \( PSL(2, \mathbb{R}) \)-action and then by taking the stable map compactification as before. Here \( z^{(i)} = (z_1^{(i)}, \cdots, z_k^{(i)}) \) and \( z^{(k)} \in \mathbb{R}^{\sum_{i=1}^k \gamma_i - \gamma_{i-1}} \) such that \( z_{(i+1)i}, z_{(i+1)i}^{(k)} \) respects the counter clockwise cyclic ordering.

\[
((D^2, \mathcal{Z}), (z^{(0)}, \cdots, z^{(k)}), w)) \mapsto (w(z_1^{(0)}, \cdots, w(z_k^{(k)}))
\]

induces an evaluation map:

\[ ev = (ev^{(0)}, \cdots, ev^{(k)}) : \mathcal{M}_{m_0, \cdots, m_k}(\mathcal{L}, \mathcal{P}; B) \to \prod_{i=0}^{k} L_i^{m_i}. \]

Let \( P^{(i)} \) be smooth singular chains of \( L_i \) and put

\[ P^{(i)} = (p_1^{(i)}, \cdots, p_{m_i}^{(i)}), \quad \mathcal{P} = (\mathcal{P}^{(0)}, \cdots, \mathcal{P}^{(k)}).
\]

We then take the fiber product to obtain:

\[ \mathcal{M}_{m_0, \cdots, m_k}(\mathcal{L}, \mathcal{P}; B) = \mathcal{M}_{m_0, \cdots, m_k}(\mathcal{L}, \mathcal{P}; B) \times_{ev} \mathcal{P}; \]

We use this to define

\[ m_{k;m_0, \cdots, m_k} : B_{m_k}(CF(L_k)) \otimes CF((L_k, \gamma_k), (L_{k-1}, \gamma_{k-1})) \otimes \cdots \otimes CF((L_1, \gamma_1), (L_0, \gamma_0)) \rightarrow B_{m_0}(CF(L_0)) \rightarrow CF((L_k, \gamma_k), (L_0, \gamma_0)) \]

by

\[
m_{k;m_0, \cdots, m_k}([P^{(k)}], [p_{k(k-1)}, w_{k(k-1)}^+], \cdots, [p_{10}, w_{10}^+], P^{(0)}) = \sum \#(\mathcal{M}_{k+1}(\mathcal{L}, \mathcal{P}; B))[p_{k0}, w_{k0}^+].
\]
Finally for each given \( b_i \in CF(L_i)[1] \) \((b_i \equiv 0 \mod \Lambda_i)\), \( \bar{b} = (b_0, \cdots, b_k) \), and \( x_i \in CF((L_i, \gamma_i), (L_{i-1}, \gamma_{i-1})) \), we put
\[
m_i^{\bar{b}}(x_k, \cdots, x_1) = \sum_{m_0, \cdots, m_k} m_{k;m_0, \cdots, m_k}(b_k^{m_k}, x_k, b_{k-1}^{m_{k-1}}, \cdots, x_1, b_0^{m_0}). \tag{12.6}
\]

**Theorem 12.16.** If \( b_i \) satisfies the Maurer-Cartan equation \((8.10)\) then \( m_i^{\bar{b}} \) in \((12.6)\) satisfies the \( A_\infty \) relation
\[
\sum_{k_1, k_2, i} (-1)^* m_{k_1}^{\bar{b}}(x_k, \cdots, m_{k_2}^{\bar{b}}(x_{k-1}, \cdots, x_{k-1-i}, k_i), \cdots, x_1) = 0 \tag{12.7}
\]
where we take sum over \( k_1 + k_2 = k + 1 \), \( i = -1, \cdots, k - k_2 \). (We write \( m_k \) in place of \( m_i^{\bar{b}} \) in \((12.7)\).) The sign \( * \) is \( * = i + \deg x_k + \cdots + \deg x_{k-1} \).

We summarize the above discussion as follows:

**Theorem 12.17.** We can associate an filtered \( A_\infty \) category to a symplectic manifold \((M, \omega)\) such that:

1. Its object is \((\mathcal{L}, b, sp)\) where \( \mathcal{L} = (L, \gamma, \lambda) \) is a graded anchored Lagrangian submanifold, \( [b] \in M(CF(L)) \) is a bounding cochain and \( sp \) is a spin structure of \( L \).
2. The set of morphisms is \( CF((L_1, \gamma_1), (L_0, \gamma_0)) \).
3. \( m_i^{\bar{b}} \) are the operations defined in \((12.6)\).

**Remark 12.18.** Here we spelled out the choice of orientations \( o_p \) of Index \( \overline{\partial} \lambda_p \) is included. This choice in fact does not affect the module structure \( CF((L_1, \gamma_1), (L_0, \gamma_0)) \) up to isomorphism: if we take an alternative choice \( o'_p \) at \( p \), then all the signs appearing in the operations \( m_k \) that involves \([p, w]\) for some \( w \) will be reversed. Therefore \([p, w] \mapsto -[p, w]\) gives the required isomorphism.

The operations \( m_k \) are compatible with the filtration. Namely we have

**Proposition 12.19.** If \( x_i \in F^{\lambda_k} CF((L_i, \gamma_i), (L_{i-1}, \gamma_{i-1})) \), then
\[
m_k^{\bar{b}}(x_k, \cdots, x_1) \in F^{\lambda_k} CF((L_k, \gamma_k), (L_0, \gamma_0))
\]
where \( \lambda = \sum_{i=1}^k \lambda_i \).

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