Optimized Amplify-and-Forward Relaying for Hierarchical Over-the-Air Computation

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Abstract—Over-the-air computation (AirComp) is an emerging wireless technique with wide applications (e.g., in distributed edge learning), which can swiftly compute functions of distributed data from different wireless devices (WDs) by exploiting the superposition property of wireless channels. Different from prior works focusing on the AirComp over one single cell in a small area, this paper considers a new hierarchical architecture to enable AirComp in a large area, in which a set of intermediate relays are exploited to help the fusion center to aggregate data from massive WDs for functional computation. In particular, we present a two-phase amplify-and-forward (AF) relaying design for hierarchical AirComp. In the first phase, the WDs simultaneously send their data to the relays, while in the second phase, the relays amplify the received signals and concurrently forward them to the fusion center for aggregation. Under this setup, we minimize the computation distortion measured by the mean squared error (MSE), by jointly optimizing the transmit coefficients at the WDs and relays and the de-noising factor at the fusion center, subject to their individual transmit power constraints. For the highly non-convex MSE minimization problem, we develop an alternating-optimization-based algorithm to obtain a high-quality solution. The optimized solution shows that for each WD, the phase of its transmit coefficient is opposite to that of the composite channel from the WD itself to the relays to the fusion center, such that they can be aligned at the fusion center, and its transmit power follows a regularized composite-channel-inversion structure to strike a balance between minimizing the signal misalignment error and the noise-induced error. Numerical results show that our proposed design achieves a significant MSE performance gain over benchmark schemes with full-power transmission at the WDs and/or relays.

Index Terms—Hierarchical over-the-air computation (AirComp), amplify-and-forward relaying, mean squared error (MSE), optimization.

I. INTRODUCTION

The proliferation of smart wireless devices (WDs) has recently enabled various new applications such as wireless sensor networks, internet-of-things (IoT), and edge machine learning [1]. For the success of these applications, dedicated fusion centers (or edge servers) are employed to swiftly aggregate massive data distributed in WDs to make inference about the physical environments for facilitating further actions. Towards this end, the fusion center normally needs to compute functional values based on the distributed data from different WDs. For instance, in distributed sensing [2], the fusion center is interested in retrieving common parameters that are sensed by different WDs subject to independent sensing noises. In distributed edge learning [3], the edge server needs to iteratively compute the (weighted) mean values of the local gradients or machine learning model parameters at the WDs (or edge devices) for global aggregation.

Conventionally, such distributed functional computation is implemented based on a separated communication and computation design principle, in which different WDs need to send their data to the fusion center individually before the computation. This design, however, generally has a very low spectrum utilization efficiency, and may induce serious communication delay, especially when the number of WDs becomes large. Recently, over-the-air computation (AirComp) [4] has emerged as a promising solution to compute such functions over the air, by exploiting the signal superposition properties of wireless channels. With AirComp, different WDs can simultaneously transmit their data to the fusion center over the same frequency band. Via proper pre-processing and phase/power control at WDs, the fusion center can then reconstruct/estimate the targeted function values from the superimposed signals directly. As such, AirComp integrates the wireless communications and functional computation into a joint design, thereby enhancing both the communication and computation resource utilization efficiency.

In general, there are two types of AirComp approaches in the literature, namely the uncoded (analog) [5] and coded (digital) AirComp [6], [7], respectively. In particular, under independent and identically distributed (i.i.d.) real-valued Gaussian sources and a standard Gaussian multiple access channel, the analog AirComp was shown in [5] to be optimal in terms of minimizing the attainable mean squared error (MSE) distortion. Under other setups with e.g., correlated Gaussian sources [6], [7], the coded AirComp with sophisticated joint source and channel coding is generally required for minimizing the average MSE distortion. Despite its sub-optimality in general, analog AirComp has been widely studied in the literature [8]–[13] due to its simplicity in implementation, and has also been applied in emerging distributed edge learning systems [14], [15]. Note that these prior works mainly focused on the AirComp in a single-cell multiple access channel over a small area.

In practice, to fully exploit the big data value, it is of great importance to aggregate massive data that are distributed in WDs over a large area. In this scenario, however, the conventional AirComp with direct transmission from the WDs to the fusion center may not work well, as the WDs far apart from the fusion center may suffer from severe signal propagation loss, thus significantly exacerbating the compu-
tation distortion. Motivated by the great success of relaying in wireless communications [16], [17], we propose a new hierarchical AirComp architecture to overcome this issue, in which a set of intermediate relays are exploited to help the distributed function computation in a large area, by combating against the path loss. Notice that besides dedicatedly deployed relays, as shown in Fig. 1(a), the hierarchical AirComp can also be implemented in a heterogeneous network, in which a macro base station (BS) serves as a fusion center and multiple small-cell BSs act as relays to enable more efficient data aggregation. As shown in Fig. 1(b), another application scenario of the hierarchical AirComp might be the integrated aerial-terrestrial network, in which a big balloon or an airship in the sky serves as a fusion center to collect data from WDs over a large area assisted by relays.

In this paper, we consider a particular hierarchical AirComp system, which consists of multiple WDs, multiple relays, and one fusion center. We present a two-phase amplify-and-forward (AF) relaying protocol for AirComp. In the first phase, the WDs simultaneously broadcast their data to the relays; in the second phase, the relays amplify the received signals and concurrently forward them to the fusion center for aggregation.

Under this setup, we aim to minimize the computation distortion measured by the MSE, by jointly optimizing the transmit coefficients at the WDs and relays and the de-noising factor at the fusion center, subject to the individual transmit power constraints at the WDs and relays, respectively. The formulated MSE minimization problem is highly non-convex. As such, we develop an alternating-optimization-based algorithm to obtain a high-quality solution. It is shown that at the optimized solution, the transmit phase at each WD is opposite to that of the composite channel from the WD itself to the relays to the fusion center, such that they can be aligned at the fusion center; meanwhile, the transmit power at each WD follows a regularized composite-channel-inversion structure to strike a balance between minimizing the signal misalignment error and the noise-induced error. Numerical results show that our proposed design achieves a significant MSE performance gain over benchmark schemes with full-power transmission at the WDs and/or relays.

**Notations:** The superscripts $T$ and $H$ denote the transpose and Hermitian operations, respectively; for a scalar $x$, $|x|$ and $\angle x$ denote its absolute value and polar angle, respectively; $x \sim \mathcal{CN}(\mu, \sigma^2)$ denotes the distribution of a circular symmetric complex Gaussian (CSCG) random variable $x$ with mean $\mu$ and variance $\sigma^2$; $j = \sqrt{-1}$ represents the imaginary unit; $\diag(x_1, \ldots, x_n)$ stands for a diagonal matrix whose diagonal entries starting in the upper left corner are $x_1, \ldots, x_n$.

II. SYSTEM MODEL AND PROBLEM FORMULATION

As shown in Fig. 1, we consider a hierarchical AirComp network, in which the fusion center is interested in aggregating the data distributed in $K$ WDs over a large area assisted by $M$ intermediate relays. For exposition, all the nodes in the system are assumed to be equipped with a single antenna.

1See, e.g., the Google’s Project Loon at https://loon.com/ and the Alibaba Cloud IoT in the Sky LoRa Station at https://www.alibabacloud.com/blog/alibaba-cloud-speeds-up-iot-strategy-at-the-computing-conference_594071.
the transmitted signal by each relay \( m \in M \) is
\[
x_{R,m} = \beta_m r_m, \quad \forall m \in M.
\]

Let \( P_{R,m} \) denote the maximum transmit power budget at relay \( m \). We thus have the individual power constraints at the \( M \) relays as
\[
E[|x_{R,m}|^2] = |\beta_m|^2 (\sum_{k=1}^{K} |\alpha_k|^2 |h_{m,k}|^2 \delta_k^2 + \sigma_m^2) \leq P_{R,m},
\]
\[
\forall m \in M.
\]

Furthermore, let \( g_m \in \mathbb{C} \) denote the channel coefficient from relay \( m \) to the fusion center. The received signal at the fusion center is given by
\[
y = \sum_{m=1}^{M} g_m x_{R,m} + z_0
\]
\[
= \sum_{m=1}^{M} \alpha_k \beta_m g_m h_{m,k} x_k + \sum_{m=1}^{M} \beta_m g_m z_m + z_0,
\]
where (6b) is obtained by substituting (4) into (6a), and \( z_0 \sim \mathcal{CN}(0, \sigma_0^2) \) denotes the AWGN at the fusion center receiver.

Upon receiving \( y \) in (6), the fusion center is interested in reconstructing the function value \( \bar{x} \) in (1), by applying a positive de-noising factor, denoted by \( \eta > 0 \). In this case, the estimation of \( \bar{x} \) is obtained by the fusion center as
\[
\hat{x} = \frac{y}{K \eta}.
\]

Correspondingly, the computation distortion can be measured by the MSE between the estimator \( \hat{x} \) and the ground truth \( x \), which is expressed as
\[
\text{MSE}(\{\alpha_k\}, \{\beta_m\}, \eta) = \mathbb{E}[|\hat{x} - \bar{x}|^2]
\]
\[
= \frac{1}{K^2} \left[ \sum_{k=1}^{K} \left( \frac{\alpha_k h_k^T A_{y|\beta} \beta}{\eta} - 1 \right)^2 \delta_k^2 + \frac{\beta^H A_{y|\beta} A_{y|x} A_{y|\beta} \beta + \sigma_0^2}{\eta^2} \right],
\]
where \( \beta \triangleq [\beta_1, \ldots, \beta_M]^T \), \( A_{y|x} \triangleq \text{diag}(g_1, \ldots, g_M) \), and \( A_{y|x} \triangleq \text{diag}(g_1, \ldots, g_M) \). The expectation is taken on the randomness of the \( K \) WDs’ data \( \{x_k\}_{k=1}^{K} \). Note that MSE(\( \{\alpha_k\}, \{\beta_m\}, \eta \) in (8) consists of two error-related components, i.e., the composite signal misalignment error (the first term) and the noise-induced error (the second term), respectively.

In this paper, we aim to minimize the MSE in (8), by jointly optimizing the transmit coefficients \( \{\alpha_k\} \) and \( \{\beta_m\} \) at the WDs and relays and the de-noising factor \( \eta \) at the fusion center, subject to the individual power constraints in (3) and (5) for the WDs and relays, respectively. Mathematically, by omitting the constant factor \( 1/K^2 \) in (8), the power-constrained MSE minimization problem is formulated as
\[
\begin{align*}
\min_{\{\alpha_k, \beta_m, \eta\} > 0} & \quad \sum_{k=1}^{K} \left( \frac{\alpha_k h_k^T A_{y|x} \beta}{\eta} - 1 \right)^2 \delta_k^2 + \frac{\beta^H A_{y|x} A_{y|x} \beta + \sigma_0^2}{\eta^2} \\
\text{s.t.} & \quad (3) \text{ and } (5).
\end{align*}
\]

Due to the coupling of \( \{\alpha_k\} \) and \( \{\beta_m\} \) in the objective function of (9a) and constraints (5), problem (P1) is a highly non-convex optimization problem, thus making the globally optimal solution difficult to obtain. To circumvent this issue, we employ an alternating-optimization-based approach to obtain a low-complexity high-quality (though suboptimal in general) solution to problem (P1), by alternately optimizing the transmit coefficients at the WDs and relays, and the de-noising factor at the fusion center, as will be shown in Section III.

**III. PROPOSED SOLUTION TO PROBLEM (P1)**

In this section, we present an alternating-optimization-based approach to solving problem (P1), by alternately optimizing \( \{\alpha_k\} \), \( \{\beta_m\} \), and \( \eta \).

A. Optimizing \( \{\alpha_k\} \) with Given \( \{\beta_m\} \) and \( \eta \)

In this subsection, under given transmit coefficients \( \beta_m \)'s at the relays and the de-noising factor \( \eta \) at the fusion center, we optimize the transmit coefficients \( \{\alpha_k\}_{k=1}^{K} \) at the WDs. Notice that the noise-induced error \( (\beta^H A_{y|x} A_{y|x} \beta + \sigma_0^2)/\eta^2 \) in (8) is independent of \( \{\alpha_k\} \). Therefore, the optimization of \( \{\alpha_k\} \) can be equivalently expressed as
\[
\min_{\{\alpha_k\}} \sum_{k=1}^{K} \left( \frac{\alpha_k h_k^T A_{y|x} \beta}{\eta} - 1 \right)^2 \delta_k^2
\]
\[
s.t. \quad \alpha_k^2 \delta_k^2 \leq P_{R,k}, \quad \forall k \in K.
\]

Let \( \{\alpha_k^{\text{opt}}\} \) denote the optimal solution to problem (10).

To start with, we establish the following proposition.

**Proposition 1:** At the optimal solution to problem (10), it must hold that
\[
\angle \alpha_k^{\text{opt}} = -\angle h_k^T A_{y|x} \beta
\]
\[
= -\angle \left( \sum_{m=1}^{M} h_{m,k} g_m \beta_m \right), \quad \forall k \in K.
\]

**Proof:** See Appendix A.

Proposition 1 indicates that, in order to compute the arithmetic average function of \( \{x_k\} \) from distributed WDs over the air, each WD \( k \in K \) should adjust the polar angle (a.k.a. phase) of transmit coefficient \( \alpha_k \) to be opposite to that of the composite channel from WD \( k \) to the relays to the fusion center (i.e., \( \sum_{m=1}^{M} h_{m,k} g_m \beta_m \)), such that the WDs’ signal phases can be aligned at the fusion center receiver, thus leading to a constructive addition to facilitate the computation.

Based on Proposition 1, we define \( \alpha_k = \tilde{\alpha}_k e^{j \angle \alpha_k^{\text{opt}}} \), \( \forall k \in K \), where \( \tilde{\alpha}_k \) denotes the amplitude of the transmit coefficient by WD \( k \) to be optimized. Accordingly, problem (10) is equivalently recast as
\[
\begin{align*}
\min_{\{\alpha_k\}} & \quad \sum_{k=1}^{K} \left( \frac{\alpha_k h_k^T A_{y|x} \beta}{\eta} - 1 \right)^2 \delta_k^2 \\
\text{s.t.} & \quad \tilde{\alpha}_k \leq \sqrt{P_{R,k}} / \delta_k, \quad \forall k \in K \quad (12b)
\end{align*}
\]
\[
\sum_{k=1}^{K} \tilde{\alpha}_k^2 |h_{m,k}|^2 \delta_k^2 + \sigma_m^2 \leq P_{R,m}, \quad \forall m \in M. \quad (12c)
\]
It can be observed that problem (12) is convex and satisfies the Slater’s conditions. Therefore, strong duality holds between (12) and its dual problem [18]. As a result, one can solve problem (12) by leveraging the Karush-Kuhn-Tucker (KKT) optimality conditions. Let \( \{ \alpha_k^{\text{opt}} \} \) denote the optimal primal solution to problem (12), and \( \{ \beta_m^{\text{opt}} \} \) the optimal dual solution associated with the constraints in (12c). Then, we have the following proposition.

**Proposition 2:** The optimal solution \( \{ \alpha_k^{\text{opt}} \} \) to problem (12) is given as

\[
\alpha_k^{\text{opt}} = \min \left\{ \frac{|h_k^T \Lambda_k \beta|}{\eta / \sqrt{P_m \sigma_k^2}} : \sum_{m=1}^{M} \mu_m^{\text{opt}} |h_{m,k}|^2 \right\}, \quad \forall k \in K, \tag{13}
\]

where \( \mu_m^{\text{opt}} \)'s are non-negative and satisfy the following complementary slackness conditions:

\[
\mu_m^{\text{opt}} \left( \sum_{k=1}^{K} |(\alpha_k^{\text{opt}})^2| |h_{m,k}|^2 \delta_k^2 + \sigma_m^2 \right) = 0, \quad \forall m \in M. \tag{14}
\]

**Proof:** See Appendix B.

Propositions 2 indicates a regularized channel-inversion structure for the optimal amplitude of transmit coefficient (or equivalently the transmit power) at each WD, based on the effective composite channel term \( \sum_{m=1}^{M} h_{m,k} g_m \beta_m \) and the regularization component \( \sum_{m=1}^{M} \mu_m^{\text{opt}} |h_{m,k}|^2 \) related to the individual power constraints at the relays (12c). In particular, if both the WDs and relays have sufficiently large power budgets, then it yields that \( \mu_m^{\text{opt}} = 0, \quad \forall m \in M \), and

\[
\alpha_k^{\text{opt}} = \frac{1}{|h_k^T \Lambda_k \beta| / \eta}, \quad \forall k \in K, \quad \text{i.e., the composite-channel-inversion power control scheme is employed based on the composite channels for minimizing the computation MSE.}
\]

By further combining Propositions 1 and 2, we finally obtain the optimal \( \{ \alpha_k^{\text{opt}} \} \) to problem (10), which is given as

\[
\alpha_k^{\text{opt}} = \alpha_k^{\text{opt}} e^{-j \angle h_k^T \Lambda_k \beta}, \quad \forall k \in K. \tag{15}
\]

**B. Optimizing \( \{ \beta_m \} \) with Given \( \{ \alpha_k \} \) and \( \eta \)**

In this subsection, under given transmit coefficients \( \{ \alpha_k \} \) at the WDs and the de-noising factor \( \eta \) at the fusion center, we optimize the transmit coefficients \( \{ \beta_m \}_{M=1}^{M} \) at relays. In this case, problem (P1) under given \( \{ \alpha_k \} \) and \( \eta \) is reduced into

\[
\min_{\{ \beta_m \}} \sum_{k=1}^{K} \left| \frac{\alpha_k h_k^T \Lambda_k \beta}{\eta} \right|^2 - \frac{2}{\eta^2} \left( \beta^H \Lambda_k^H \Lambda_k \beta + \sigma_0 \right), \tag{16a}
\]

s.t. \( |\beta_m|^2 \leq \bar{P}_{R,m}, \quad \forall m \in M, \tag{16b} \)

where \( \bar{P}_{R,m} \triangleq \frac{P_{R,m}}{\sum_{k=1}^{K} |(\alpha_k)^2| |h_{m,k}|^2 \delta_k^2 + \sigma_m^2}, \quad \forall m \in M \). Denote \( B \) as the (convex) feasible region of \( \{ \beta_m \} \) characterized by (16b). Given both \( \{ \alpha_k \} \) and \( \eta \) being fixed, problem (16) is thus a convex quadratically constrained quadratic program (QCQP) [18], where the constraints in (16b) are separable. Let \( \beta_m^{\text{opt}} \triangleq [\beta_1^{\text{opt}}, \ldots, \beta_M^{\text{opt}}]^T \) denote the optimal solution to problem (16). We establish the following proposition on the optimal solution to problem (16).

**Proposition 3:** The optimal \( \beta_m^{\text{opt}} \) to problem (16) is obtained as

\[
\beta_m^{\text{opt}} = \mathcal{P}_B \left( \hat{\beta} \right), \tag{17}
\]

where \( \hat{\beta} \triangleq [\hat{\beta}_1, \ldots, \hat{\beta}_M]^T \) is given as

\[
\hat{\beta} = \eta \Lambda^{-1}_g \sum_{k=1}^{K} \left| \alpha_k \right|^2 \delta_k^2 h_k^H h_k + \Lambda_{\sigma^2}^{-1} \left( \sum_{k=1}^{K} \alpha_k \delta_k^2 h_k^T \right)^H, \tag{18}
\]

and \( \mathcal{P}_B : \mathbb{C}^{N \times 1} \mapsto \mathbb{C}^{N \times 1} \) is the orthogonal projection operator such that

\[
\mathcal{P}_m \left( \hat{\beta}_m \right) = \begin{cases} \sqrt{P_{R,m} \beta_m / \| \beta_m \|} & \text{if } |\beta_m| > \sqrt{P_{R,m}} \\ \hat{\beta}_m & \text{if } |\beta_m| \leq \sqrt{P_{R,m}} \end{cases} \tag{19}
\]

for all \( m \in M \).

**Proof:** Leveraging the quadratic convexity of the objective function (16a) and the optimality of projection into a convex feasible region [18], the optimal solution \( \{ \beta_m^{\text{opt}} \} \) can be obtained by checking the gradient of (16a) with respect to \( \beta \), along with a feasible orthogonal projection operation.

**Proposition 4:** The optimal de-noising factor \( \eta^{\text{opt}} \) to problem (20) is obtained as

\[
\eta^{\text{opt}} = \frac{\beta^H \Lambda^H \Lambda_{\sigma^2} \beta + \sigma_0}{\sum_{k=1}^{K} \left| \alpha_k \right|^2 \delta_k^2 h_k^H h_k + \Lambda_{\sigma^2}}, \tag{20}
\]

Note that problem (20) is convex, since its objective function is a convex quadratic function with respect to \( \eta \). Let \( \eta^{\text{opt}} \) denote the optimal solution to problem (20). By checking the first-order derivative of (20) with respect to \( \eta \), we establish the following proposition on \( \eta^{\text{opt}} \), for which the proof is omitted for brevity; herein, we consider the term \( \sum_{k=1}^{K} \alpha_k h_k^H \Lambda_k \beta \) to be a real number without loss of optimality based on the optimization of \( \{ \alpha_k \} \) in Section III-A.

**Proposition 4:** At the optimality of problem (20), the optimal \( \eta^{\text{opt}} \) is obtained as

\[
\eta^{\text{opt}} = \frac{\sum_{k=1}^{K} \left| \alpha_k \right|^2 \delta_k^2 h_k^H h_k + \Lambda_{\sigma^2}}{\sum_{k=1}^{K} \left| \alpha_k \right|^2 \delta_k^2 h_k^H h_k + \Lambda_{\sigma^2}} \tag{21}
\]

Proposition 4 indicates that the optimized de-noising factor \( \eta^{\text{opt}} \) is the ratio of the aggregated-signals-plus-noise power divided by the sum-message power scaled by the transmit and channel coefficients. This is intuitively expected, since
the fusion center needs to counter against the channel fading and transmit coefficient coupling effects in order to minimize the computation MSE.

Finally, by combining Propositions 1–4, we propose an alternating-optimization-based approach to efficiently obtain a high-quality solution of problem (P1) in an iterative fashion, which is presented as Algorithm 1 in Table I. For notation convenience, we use $\text{MSE}(\alpha(n), \beta(n), \eta(n))$ to denote $\text{MSE}(\alpha(n), \beta(n), \eta(n))$.

| TABLE I |
|---|
| **Proposed Algorithm 1 for Solving Problem (P1)** |
| a) Initialization: $n = 1$, $\epsilon > 0$, $|\alpha_k(0)|^2 \leq P_k$, $\forall k \in K$, and $|\beta_m(0)|^2 + (\sum_{k=1}^{K} |\alpha_k(0)| h_{k,m}^2 + \sigma_m^2) \leq P_{R,m}$, $\forall m \in M$. |
| b) While $(\text{MSE}(n) - \text{MSE}(n-1))/\text{MSE}(n-1) > \epsilon$ do |
| - Obtain the optimal $\{\alpha_k(n)\}$ and $\{\beta_m(n)\}$ by (15), together with Propositions 1 and 2; |
| - Obtain the optimal $\{\beta_m^{(n)}\}$ of problem (P1) under given $\{\alpha_k(n)\}$ and $\{\beta_m^{(n)}\}$ by Proposition 3; |
| - Obtain the optimal $\eta^{(n)}$ of problem (P1) under given $\{\alpha_k(n)\}$ and $\{\beta_m(n)\}$ by Proposition 4; |
| - $n = n + 1$; |
| c) End while |
| e) Output: $\{\alpha_k(n)\}, \{\beta_m^{(n)}\}, \{\eta^{(n)}\}$ for problem (P1). |

### IV. Numerical Results

In this section, we provide numerical results to evaluate the performance of the proposed AF relaying design for hierarchical AirComp. In the simulations, we consider the distance-dependent Rayleigh fading channel models, in which we set $h_{k,m} = \sqrt{\Omega_m d_{k,m}^{-\kappa}}$ and $g_m = \sqrt{\Omega_n d_m^{-\kappa}} g_0$, where $d_{k,m}$ and $d_m$ denote the distances from WD $k$ to relay $m$ and from relay $m$ to the fusion center, respectively. $\Omega_m \sim \mathcal{CN}(0,1)$ and $g_0 \sim \mathcal{CN}(0,1)$ account for small-scale fading. Unless stated otherwise, we set the variances of WDs’ transmit messages to be $\sigma_k^2 = 1$, the maximum transmit power to be $P_k = P_{R,m} = 26$ dBm, the noise power to be $\sigma_n^2 = 10^{-9}$ Watt, and the distances to be $d_{k,m} = 350$ and $d_m = 150$ meters, $\forall k \in K$, $\forall m \in M$. The numerical results are obtained by averaging over $10^3$ randomized channel realizations. For performance comparison, we consider the following three benchmark schemes.

- **Full power transmission at WDs and relays**: In this scheme, each WD $k \in K$ and each relay $m \in M$ use their full transmit power in sending data for AirComp, respectively. This scheme corresponds to solving problem (P1) by setting $|\alpha_k|^2 \sqrt{\delta_k^2} = P_k$, $\forall k \in K$, and $|\beta_m|^2 (\sum_{k=1}^{K} |\alpha_k|^2) h_{k,m}^2 \delta_k^2 + \sigma_m^2 = P_{R,m}$, $\forall m \in M$. |

- **Full power transmission at WDs only**: In this scheme, each WD $k \in K$ uses the full transmit power to send data to the relays, which corresponds to solving problem (P1) by setting $|\alpha_k|^2 \delta_k^2 = P_k$, $\forall k \in K$. |

- **Full power transmission at relays only**: In this scheme, each relay $m$ employs the full transmit power to amplify and forward its received signal to the fusion center, which corresponds to solving problem (P1) by setting $|\beta_m|^2 (\sum_{k=1}^{K} |\alpha_k|^2) h_{k,m}^2 \delta_k^2 + \sigma_m^2 = P_{R,m}$, $\forall m \in M$. |

Fig. 3 shows the average MSE versus the number of WDs $K$, where $M = 10$. It is observed that the MSE performance of the four schemes decreases as $K$ increases, and the proposed AirComp design significantly outperforms the benchmark schemes. When $K$ is small (e.g., $K \leq 40$), the full-power-transmission-at-WDs-only scheme is observed to outperform the full-power-transmission-at-relays-only one, but it is not true as $K$ grows large. This implies the importance of power control at WDs to reduce the signal misalignment error as $K$ increases. In addition, the full-power-transmission-at-relays-only scheme achieves performance close to the proposed design at large $K$ values, which indicates the benefit of full-power operation at relays in reducing the MSE when the number of WDs is large.

Fig. 4 depicts the average MSE versus the number of relays $M$, where $K = 50$. It is observed that the MSE performance of the four schemes decreases as $M$ increases, and the proposed design outperforms the benchmark schemes. When $M$ is small (e.g., $M \leq 17$), the full-power-transmission-at-relays-only scheme outperforms the full-power-transmission-at-WDs-only one, but the reverse holds true as $M$ grows larger in this setup, which is because the full-power-transmission-at-WDs-only scheme can adapt the transmit coefficients at relays to reduce the composite signal misalignment error. The full-power-transmission-at-WDs-only scheme achieves performance close to the proposed design as $M$ increases. This implies the benefit for the WDs to employ full power transmission to reduce the computation MSE in the cases with a large relay number.
V. CONCLUSION

In this paper, we presented the two-phase AF relaying design for hierarchical AirComp, and investigated the joint transceiver optimization. Towards minimizing the computational MSE, we optimized the transmit coefficients at the WDs and relays and the de-noising factor at the fusion center in an alternating manner, subject to the individual transmit power constraints at the WDs and relays, respectively. It was shown that at the optimized design, the transmit power at each WD follows a new regularized composite-channel-inversion structure to strike a balance between minimizing the composite-signal-misalignment error and the noise-induced error. Numerical results evaluated the MSE performance gains of the proposed hierarchical AirComp design as compared with the benchmark schemes. The current work may motivate several interesting research directions on hierarchical AirComp scenarios with multi-antenna and/or multi-hop setups, and/or with multiple functions to be computed.

APPENDIX

A. Proof of Proposition 1

This proposition is proved by contradiction. We assume the optimal solution to problem (10) is expressed as \( \alpha_k' = \tilde{\alpha}_k e^{j\phi_k}, \forall k \in K \), where \( \tilde{\alpha}_k \) is a real number. There exists an index \( k' \in K \), such that \( \theta_{k'} \neq -\angle h_{k'}^H \Delta_n \beta \) and \( \theta_k = -\angle h_k^H \Delta_n \beta, \forall k \neq k' \). Therefore, we can always construct a feasible solution to problem (10) as \( \alpha_k = \tilde{\alpha}_k e^{j\phi_k}, \forall k \in K \), where \( \tilde{\alpha}_k = \theta_k, \forall k \neq k' \), and \( \theta_{k'} = -\angle h_{k'}^H \Delta_n \beta \). It can be verified that the solution \( \{\alpha_k\} \) achieves a smaller value than that by \( \{\alpha_k'\} \). Therefore, the presumption cannot be true. We thus complete the proof of Proposition 1.

B. Proof of Proposition 2

Let \( \lambda_k \geq 0 \) and \( \mu_m \geq 0 \) denote the dual variables associated with the \( k \)-th constraint in (12b) and the \( m \)-th constraint in (12c), respectively, \( \forall k \in K, m \in M \). The Lagrangian of problem (12) is given as

\[
\mathcal{L} = \sum_{k=1}^{K} \left( \tilde{\alpha}_k^2 |h_k^H \Delta_n \beta|^2 - \frac{1}{\eta} \right) \delta_k^2 + \sum_{m=1}^{M} \mu_m |h_m|_2^2 \delta_k^2 + \lambda_k \tilde{\alpha}_k - \sum_{k=1}^{K} \frac{\sqrt{P_k}}{\delta_k} + \sum_{m=1}^{M} \mu_m |h_m|_2^2 - \sum_{m=1}^{M} \mu_m P_{R,m}/|\beta_m|^2\right].
\]

(22)

The necessary and sufficient conditions for the optimal primal and dual variables are given by the KKT optimality conditions [18], which are expressed as follows.

\[
\left\{ \begin{array}{l}
\tilde{\alpha}_k^* \leq \sqrt{P_k}/\delta_k, \forall k \in K \\
\sum_{k=1}^{K} (\tilde{\alpha}_k^*)^2 |h_m|_2^2 \delta_k^2 + \sigma_m^2 \leq P_{R,m}/|\beta_m|^2, \forall m \in M
\end{array} \right.
\]

(23a)

(23b)

\[
\lambda_k^* \geq 0, \forall k \in K, \mu_m^* \geq 0, \forall m \in M
\]

(23c)

\[
\lambda_k^* (\tilde{\alpha}_k^* - \sqrt{P_k}/\delta_k) = 0, \forall k \in K
\]

(23d)

\[
\mu_m^* (\sum_{k=1}^{K} (\tilde{\alpha}_k^*)^2 |h_m|_2^2 \delta_k^2 + \sigma_m^2 - P_{R,m}/|\beta_m|^2) = 0, \forall m \in M
\]

(23e)

\[
\frac{\partial \mathcal{L}}{\partial \tilde{\alpha}_k} = 2\tilde{\alpha}_k^* \left( |h_k^H \Delta_n \beta|^2/\eta^2 + \sum_{m=1}^{M} \mu_m^* |h_m|_2^2 \right) \delta_k^2
\]

\[
- 2|h_k^H \Delta_n \beta| \delta_k^2/\eta + \lambda_k^* = 0, \forall k \in K.
\]

(23f)

Based on the KKT conditions in (23), we have the optimal \( \{\tilde{\alpha}_k^*\} \) for problem (12) as

\[
\tilde{\alpha}_k^* = \min \left\{ \frac{|h_k^H \Delta_n \beta|^2}{\eta^2} + \sum_{m=1}^{M} \mu_m^* |h_m|_2^2, \sqrt{P_k}/\delta_k \right\}, \forall k \in K,
\]

where the nonnegative \( \{\mu_m^*\} \) satisfy the complementary slackness conditions in (23e). Therefore, we complete the proof of Proposition 2.

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