Lifting for manifold-valued maps of bounded variation

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January 18, 2022

Abstract

Let $\mathcal{N}$ be a smooth, compact, connected Riemannian manifold without boundary. Let $\mathcal{E} \to \mathcal{N}$ be the Riemannian universal covering of $\mathcal{N}$. For any bounded, smooth domain $\Omega \subseteq \mathbb{R}^d$ and any $u \in \text{BV}(\Omega, \mathcal{N})$, we show that $u$ has a lifting $v \in \text{BV}(\Omega, \mathcal{E})$. Our result proves a conjecture by Bethuel and Chiron.

1 Introduction

Let $\mathcal{N}$ be a smooth, compact, connected Riemannian manifold without boundary. Let

$$\pi: \mathcal{E} \to \mathcal{N}$$

be the (smooth) universal covering of $\mathcal{N}$. We endow $\mathcal{E}$ with the pull-back metric, so that $\pi$ is a local isometry. Given a bounded, smooth domain $\Omega \subseteq \mathbb{R}^d$ and measurable maps $u: \Omega \to \mathcal{N}$, $v: \Omega \to \mathcal{E}$, we say that $v$ is a lifting for $u$ if $\pi \circ v = u$ a.e. on $\Omega$. We are interested in the

Lifting problem. Given a regular map $u: \Omega \to \mathcal{N}$, is there a lifting $v: \Omega \to \mathcal{E}$ of $u$ that is as regular as $u$?

Of course, the answer depends on what we mean precisely by “regular”. If $u$ is of class $C^k$ (with $k = 0, 1, \ldots, \infty$) and $\Omega$ is simply connected, then the lifting problem has a positive answer. If other notions of regularity — for instance, Sobolev regularity — are considered, the problem may be more delicate. The lifting problem for non-continuous maps has been studied first when $\mathcal{N}$ is the unit circle, $\mathcal{N} = \mathbb{S}^1$, in connection with the Ginzburg-Landau theory of superconductivity. In this case, $\mathcal{E} = \mathbb{R}$ and the covering map $\pi: \mathbb{R} \to \mathbb{S}^1$ is given by $\pi(\theta) = \exp(i\theta)$. The study of this case was initiated in [8, 7] and culminated with the work by Bourgain, Brezis and Mironescu [9], who gave a complete answer to the lifting problem when $u \in W^{s,p}(\Omega, \mathbb{S}^1)$, $s > 0$, $1 < p < +\infty$. Their results have been extended to the Besov setting by Mironescu, Russ and Sire [28]. Another particular instance of the lifting problem is the case when $\mathcal{N}$ is the real projective plane, $\mathcal{N} = \mathbb{RP}^2$, which is obtained from the 2-dimensional sphere $\mathbb{S}^2$ by identifying pairs of antipodal points. The covering space $\mathcal{E}$ is then the sphere $\mathbb{S}^2$.
and \( \pi: \mathbb{S}^2 \to \mathbb{R}P^2 \) is the natural projection. \( \mathbb{R}P^2 \)-valued maps and their lifting have a physical interpretation e.g. in materials science, as they serve as models for a class of materials known as (uniaxial) nematic liquid crystals — see e.g. \(^5\) for more details. The lifting problem for \( \mathbb{R}P^2 \)-valued maps, in the context of Sobolev \( W^{1,p} \)-spaces, has been studied e.g. by Ball and Zarnescu \(^6\) and Mucci \(^30\).

For more general target manifolds \( \mathcal{N} \), the lifting problem in Sobolev spaces \( W^{s,p}(\Omega, \mathcal{N}) \) was studied Bethuel and Chiron \(^6\), and only very recently it has been completely settled by Mironescu and Van Schaftingen \(^29\). Among other results, Bethuel and Chiron proved that, if \( \Omega \) is simply connected and \( p \geq 2 \), then every map \( u \in W^{1,p}(\Omega, \mathcal{N}) \) has a lifting \( v \in W^{1,p}(\Omega, \mathcal{E}) \). However, there exist maps that belong to \( W^{1,p}(\Omega, \mathcal{N}) \) for any \( p < 2 \), and yet have no lifting in \( W^{1,p}(\Omega, \mathcal{E}) \) — for instance, we can take \( \mathcal{N} = \mathbb{S}^1 \), \( \Omega \) the unit disk in \( \mathbb{R}^2 \), and \( u(x) := x/|x| \). Bethuel and Chiron raised the conjecture \(^6\) Remark 1] that any map \( u \in W^{1,p}(\Omega, \mathcal{N}) \), with \( p \geq 1 \), has a lifting of bounded variation (BV).

In this paper, we consider the lifting problem when \( u \) is a BV-map. Previous works showed that the lifting problem for \( u \in BV(\Omega, \mathcal{N}) \) has a positive answer in case \( \mathcal{N} = \mathbb{S}^1 \) (Gi-aquinta, Modica and Souček \(^18\) Corollary 1 in Section 6.2.2), Davila and Ignat \(^16\), Ignat \(^24\)), \( \mathcal{N} = \mathbb{R}P^k \) (Bedford \(^4\), Ignat and Lamy \(^25\)) and more generally, if the fundamental group of \( \mathcal{N} \), \( \pi_1(\mathcal{N}) \), is abelian \(^13\). The aim of this paper is to prove a lifting result for maps \( u \in BV(\Omega, \mathcal{N}) \) without assuming that \( \pi_1(\mathcal{N}) \) is abelian. Examples of closed manifolds with non-abelian fundamental group are obtained by taking the quotient of \( SO(3) \), the set of rotations of \( \mathbb{R}^3 \), by the symmetry group of a regular, convex polyhedron. The elements of this quotient space describe the possible orientations of the given polyhedron in \( \mathbb{R}^3 \). Manifolds of this form appear in variational problems, arising from applications of different kinds. For instance, in material science, they appear in models for ordered materials, such as biaxial nematics (see e.g. \(^27\)). In numerical analysis, they are found in Ginzburg-Landau functionals with applications to mesh generation, via the so-called cross-field algorithms (see e.g. \(^14\)).

**Setting.** By Nash’s theorem \(^32\), we can embed isometrically both \( \mathcal{N} \) and \( \mathcal{E} \) into Euclidean spaces, \( \mathcal{N} \subseteq \mathbb{R}^m \), \( \mathcal{E} \subseteq \mathbb{R}^\ell \). Moreover, since \( \mathcal{N} \), \( \mathcal{E} \) are complete Riemannian manifolds, we can choose the embeddings so that the images of \( \mathcal{N} \), \( \mathcal{E} \) are closed subsets of \( \mathbb{R}^m \), \( \mathbb{R}^\ell \), respectively \(^31\). From now on, we will identify \( \mathcal{N} \), \( \mathcal{E} \) with their closed Euclidean embeddings. Given an open set \( \Omega \subset \mathbb{R}^d \), we define \( BV(\Omega, \mathcal{N}) \) as the set of maps \( u \in BV(\Omega, \mathbb{R}^m) \) that satisfy the pointwise constraint \( u(x) \in \mathcal{N} \) for a.e. \( x \in \Omega \). We also define \( SBV(\Omega, \mathcal{N}) \) as the set of maps \( u \in BV(\Omega, \mathcal{N}) \) such that the distributional derivative \( Du \) (taken in the sense of \( BV(\Omega, \mathbb{R}^m) \)) has no Cantor part. We define \( BV(\Omega, \mathcal{E}) \), \( SBV(\Omega, \mathcal{E}) \) in a similar fashion. We write \( |\mu|(\Omega) \) to denote the total variation of a vector-valued Radon measure \( \mu \) on \( \Omega \).

**Theorem 1.** Let \( \mathcal{N} \subseteq \mathbb{R}^m \) be a smooth, compact, connected manifold without boundary. Let \( \Omega \subseteq \mathbb{R}^d \) be a smooth, bounded domain with \( d \geq 1 \). Then, any \( u \in BV(\Omega, \mathcal{N}) \) has a lifting \( v \in BV(\Omega, \mathcal{E}) \) that satisfies

\[
||v||_{L^1(\Omega)} \leq C_{\Omega, \mathcal{N}} (||Du||(\Omega) + 1), \quad |Du|(\Omega) \leq C_{\Omega, \mathcal{N}} |Du|(\Omega),
\]

where the constant \( C_{\Omega, \mathcal{N}} \) depends only on \( \Omega \) and (the given Euclidean embedding of) \( \mathcal{N} \). Moreover, if \( u \in SBV(\Omega, \mathcal{N}) \) and \( v \in BV(\Omega, \mathcal{E}) \) is a lifting of \( u \), then \( v \in SBV(\Omega, \mathcal{E}) \).
Theorem 1 implies, in particular, that a map \( u \in W^{1,p}(\Omega, \mathcal{N}) \) with \( p \geq 1 \) has a lifting \( v \in \text{SBV}(\Omega, \mathcal{E}) \), thus proving Bethuel and Chiron’s conjecture in [6].

The proof of Theorem 1 relies on ideas from [13, Theorem 3]. However, the results of [13] were formulated in terms of flat chains with coefficients in the group \( \pi_1(\mathcal{N}) \). We do not take this point of view here, because the theory of flat chains requires the coefficient group \( \pi_1(\mathcal{N}) \) to be abelian. Nevertheless, the global structure of the proof is similar to that of [13]: we approximate a given map \( u: \Omega \to \mathcal{N} \) with piecewise-affine maps \( u_j: \Omega \to \mathbb{R}^m \); we project the \( u_j \)'s onto \( \mathcal{N} \), so to define maps \( \Omega \to \mathcal{N} \) with polyhedral singularities; we lift the re-projected maps to \( v_j: \Omega \to \mathcal{E} \), by means of topological arguments; and finally, we pass to the limit, thus obtaining a lifting \( v: \Omega \to \mathcal{E} \) of \( u \). Our approach is, by its nature, extrinsic — that is, it depends upon the choice of embeddings.

The paper is organised as follows. In Section 2.1, we revisit the construction of a locally Lipschitz retraction \( \mathbb{R}^m \setminus \mathcal{X} \to \mathcal{N} \), where \( \mathcal{X} \) is a lower-dimensional, compact subset of \( \mathbb{R}^m \) (Section 2.1), and recall some topological properties of the covering \( \pi \) (Section 2.2). Section 3 contains the core of the proof: we construct a lifting for a particular class of \( \mathcal{N} \)-valued maps, those that are obtained from piecewise-affine maps by projection onto \( \mathcal{N} \). We complete the proof of Theorem 1 in Section 4.

2 Preliminaries

2.1 Projecting onto \( \mathcal{N} \)

As in [13], our arguments rely on the following topological property (see e.g. [20, Lemma 6.1], [10, Proposition 2.1], [23, Lemma 4.5]). We recall that, given a topological space \( A \) and a subset \( B \subseteq A \), a retraction \( \varrho: A \to B \) is a continuous map such that \( \varrho(z) = z \) for any \( z \in B \).

Proposition 2.1. Let \( \mathcal{N} \) be a smooth, compact, connected submanifold of \( \mathbb{R}^m \), without boundary. Let \( M > 0 \) be such that \( \mathcal{N} \) is contained in the interior of cube \( Q_M^m := [-M, M]^m \). Then, there exist a closed set \( \mathcal{X} \subseteq Q_M^m \setminus \mathcal{N} \) and a locally Lipschitz retraction \( \varrho: Q_M^m \setminus \mathcal{X} \to \mathcal{N} \) with the following properties.

(i) \( \mathcal{X} \) is a finite union of polyhedra of dimension \( m - 2 \) at most.

(ii) \( \varrho \) is smooth in a neighbourhood of \( \mathcal{N} \).

(iii) There exists a constant \( C_0 > 0 \) such that

\[
|\nabla \varrho(z)| \leq C_0 \operatorname{dist}^{-1}(z, \mathcal{X})
\]

for a.e. \( z \in Q_M^m \setminus \mathcal{X} \).

(iv) There exists a constant \( C_1 > 0 \) such that, if \( \gamma: [0, 1] \to Q_M^m \setminus \mathcal{X} \) is an injective, Lipschitz map that parametrises a straight line segment, then

\[
\int_0^1 |(\varrho \circ \gamma)'(t)| \, dt \leq C_1.
\]
As mentioned above, there are several references that prove the existence of $\mathscr{K}$ and $q$ satisfying Properties (i)–(iii). However, we have not been able to find a reference for Property (iv) (although it is, to some extent, reminiscent of the Deformation Theorem for integral currents, see e.g. [17, Theorem 4.2.9]). Since Property (iv) is crucial for us, we provide a proof below. Given an integer $q \geq 1$, we define the grid $\mathscr{G}$ on $Q^n_M$ of size $M/q$ as the collection of cubes

$$
\mathscr{G} := \left\{ \left[ \frac{Mz}{q} + 0, \frac{M}{q} \right]^m : z \in \mathbb{Z}^m \cap [-q, q - 1]^m \right\}.
$$

For $j \in \{0, 1, \ldots, m\}$, we denote by $\mathscr{G}_j$ the collection of the (closed) $j$-faces of cubes in $\mathscr{G}$. We define the $j$-skeleton of $\mathscr{G}$ as $R_j := \bigcup_{K \in \mathscr{G}} K$. We define the dual grid to $\mathscr{G}$ as

$$
\mathscr{G}' := \left\{ \left[ \frac{Mz}{2q}, \frac{M}{2q}, \ldots, \frac{Mz}{2q} \right] + \left[ 0, \frac{M}{q} \right]^m : z \in \mathbb{Z}^m \cap [-q - 1, q - 1]^m \right\}.
$$

We denote by $R'_j$ the $j$-skeleton of $\mathscr{G}'$.

**Proof of Proposition 2.1.** Given a $j$-dimensional cube $K \in \mathcal{G}_j$ of centre $\bar{z}$, we denote by $\xi_K : K \setminus \{\bar{z}\} \to \partial K$ the radial retraction onto its boundary. If we rotate and dilate $K$ so that it has the following properties:

- Given an integer $q \geq 1$, we define $\mathscr{G}$ on $Q^n_M$ of size $M/q$ as the collection of cubes
- For $j \in \{0, 1, \ldots, m\}$, we denote by $\mathscr{G}_j$ the collection of the (closed) $j$-faces of cubes in $\mathscr{G}$. We define the $j$-skeleton of $\mathscr{G}$ as $R_j := \bigcup_{K \in \mathscr{G}} K$. We define the dual grid to $\mathscr{G}$ as
- We denote by $R'_j$ the $j$-skeleton of $\mathscr{G}'$.

Since $\mathcal{N}$ is compact and smooth, for $r > 0$ small enough the $r$-neighbourhood of $\mathcal{N}$ retracts smoothly onto $\mathcal{N}$ (by nearest-point projection onto $\mathcal{N}$, for instance). Let us fix an integer $q \geq 1$, and let us consider the grid $\mathscr{G}$ of size $M/q$, defined by (2.1). If $q$ is large enough, there exists a set $W \subseteq Q^n_M$ that is a finite union of cubes of $\mathscr{G}$, contains $\mathcal{N}$ in its interior, and retracts smoothly onto $\mathcal{N}$. Let $g_W : W \to \mathcal{N}$ be a smooth retraction. We extend $g_W$ to a Lipschitz map $R_1 \cup W \to \mathcal{N}$, still denoted $g_W$ for simplicity. To do so, we first take an arbitrary extension $R_0 \cup W \to \mathcal{N}$ of $g_W$. Then, for any (1-dimensional) edge $E$ of $\mathscr{G}$ with $E \not\subseteq W$, we define $g_W : E \to \mathcal{N}$ as a Lipschitz path that joins the values at the endpoints. Such a path exists, because $\mathcal{N}$ is connected.

We construct locally Lipschitz retraction $\sigma_2 : (R_2 \setminus R_{m-2}^{'}) \cup (R_2 \cap W) \to R_1 \cup (R_2 \cap W)$ in the following way. If $K \in \mathscr{G}_2$ is contained in $W$, then we must define $\sigma_2$ to be the identity on $K$. Take a 2-dimensional cube $K \in \mathscr{G}_2$ that is not contained in $W$. We observe that $K \cap R_{m-2}^{'}$ is
Figure 1: Estimate on $|z - z_0|$, in case $m = 3$ (see the proof of Proposition 2.1).

exactly the centre of $K$, and we define $\sigma_2(z) := \xi_K(z)$ for $z \in K \setminus R_{m-2}$. If $K_1, K_2 \in \mathcal{G}_2$ share a common edge $E \in \mathcal{G}_1$, then $\xi_{K_1}(z) = \xi_{K_2}(z) = z$ for any $z \in E$, so the definition of $\sigma_2$ is consistent. Moreover, $\sigma_2$ is locally Lipschitz and

$$|\nabla \sigma_2(z)| \leq C \operatorname{dist}^{-1}(z, R_{m-2}) \quad \text{for a.e. } z \in R_2 \setminus R_{m-2},$$

by (a) above.

We construct now a locally Lipschitz retraction $\sigma_3: (R_3 \setminus R_{m-2}) \cup (R_3 \setminus W) \to R_1 \cup (R_3 \setminus W)$, in a similar way. Given a cube $K \in \mathcal{G}_3$ that is not contained in $W$, we observe that $\xi_K^{-1}(\partial K \cap R_{m-2}) = K \setminus R_{m-2}$, and define $\sigma_3(z) := \sigma_2(\xi_K(z))$ for $z \in K \setminus R_{m-2}$. Again, this definition is consistent. Moreover, let $z_0$ be the centre of the cube $K$, let $z \in K \setminus \{z_0\}$, and let $F$ be a 2-dimensional face of $\partial K$, such that $\xi_K(z) \in F$. Using the chain rule, Property (a) and (2.2) above, we deduce that

$$|\nabla \sigma_3(z)| \leq |(\nabla \sigma_2)(\xi_K(z))| |\nabla \xi_K(z)| \leq C \operatorname{dist}^{-1}(\xi_K(z), R'_{m-2}) |z - z_0|^{-1}.$$

On the other hand, if $w$ denotes the projection of $z$ onto $R'_{m-2}$, we have

$$|z - z_0| \geq |w - z_0| = M \frac{\operatorname{dist}(z, R'_{m-2})}{2q \operatorname{dist}(\xi_K(z), R'_{m-2})}$$

(see Figure 1). As a result,

$$|\nabla \sigma_3(z)| \leq \frac{2Cq}{M} \operatorname{dist}^{-1}(z, R'_{m-2}).$$

By induction, we define a sequence of locally Lipschitz retractions $\sigma_j: (R_j \setminus R_{m-2}) \cup (R_j \setminus W) \to R_1 \cup (R_j \cap W)$, for $j = 4, \ldots, m$. We take $\mathcal{X}$ as the closure of $R_{m-2} \cap (Q_M \setminus W)$, and $\varrho := \varrho_W \circ \sigma_m$. Properties (i), (ii) is now immediate, while (iii) follows from (2.3) by an inductive
argument. Let \( \gamma : [0, 1] \to Q_M^m \setminus \mathcal{X} \) be an injective, Lipschitz map that parametrises a straight line segment \( L \subseteq Q_M^m \setminus \mathcal{X} \). By applying Property (b) iteratively, we see that

\[
\sigma_m(L) \subseteq (L \cap W) \cup R_1 \cup \bigcup_{i=1}^p L_i,
\]

where the \( L_i \)'s are straight line segments, each one contained in a \((m-1)\)-face of \( \partial W \). By the area formula, we have

\[
\int_0^1 |(\sigma_m \circ \gamma)'(t)| \, dt \leq \mathcal{H}^1(L \cap W) + \int_{R_1} \mathcal{H}^0((\sigma_m \circ \gamma)^{-1}(z)) \, d\mathcal{H}^1(z) + \sum_{i=1}^p \int_{L_i} \mathcal{H}^0((\sigma_m \circ \gamma)^{-1}(z)) \, d\mathcal{H}^1(z)
\]

By Properties (b), (c) and an inductive argument we deduce that, for \( \mathcal{H}^1 \)-a.e. \( z \in \bigcup_i L_i \cup R_1 \), \( \mathcal{H}^0(L \cap \sigma_m^{-1}(z)) \) is bounded in terms of \( m \) only. Since \( \gamma \) is injective, \( \mathcal{H}^0((\sigma_m \circ \gamma)^{-1}(z)) \) is also bounded in terms of \( m \). As a result,

\[
\int_0^1 |(\sigma_m \circ \gamma)'(t)| \, dt \leq C \left( \text{diam}(W) + \mathcal{H}^1(R_1) + \sum_{K \in \mathcal{G}_{m-1}, K \subseteq \partial W} \text{diam}(K) \right),
\]

where \( \text{diam} \) denotes the diameter. Now (iv) follows, because \( \varrho = \varrho_W \circ \sigma_m \) and \( \varrho_W \) is Lipschitz. \( \square \)

Let us choose \( M, \mathcal{X}^* \) and \( \varrho \) as in Proposition 2.1 once and for all. Let \( \sigma > 0 \) be a small parameter, such that \( \mathcal{N} \subseteq (-M + \sigma, M - \sigma)^m \). Let \( B^m_{\sigma} := \{ y \in \mathbb{R}^m : |y| < \sigma \} \), and

\[
\Lambda := M - \sigma.
\]

For any \( y \in B^m_{\sigma} \), the map \( \tilde{\varrho}_y : z \mapsto \varrho(z - y) \) is well defined and locally Lipschitz in \( Q^m_{\Lambda} \setminus (\mathcal{X} + y) \). Moreover, reducing the value of \( \sigma > 0 \) if necessary, the restriction \( \tilde{\varrho}_{y|_{\mathcal{X}}} \) is a small, smooth perturbation of the identity — in particular, it is a diffeomorphism. We define

\[
\varrho_y(z) := \left( (\tilde{\varrho}_{y|_{\mathcal{X}}})^{-1} \circ \varrho \right) (z - y) \quad \text{for } z \in Q^m_{\Lambda} \setminus (\mathcal{X} + y), \ y \in B^m_{\sigma}.
\]

**Lemma 2.2.** Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded domain. For any \( u \in W^{1,1}(\Omega, Q^m_{\Lambda}) \) and a.e. \( y \in B^m_{\sigma} \), the map \( \varrho_y \circ u \) belongs to \( W^{1,1}(\Omega, \mathcal{N}) \). Moreover, there exists a constant \( C_{\Lambda} \) (depending only on \( \mathcal{N} \), \( m \), \( \mathcal{X}^* \), \( \varrho \), \( \sigma \) and \( \Lambda \)) such that

\[
\int_{B^m_{\sigma}} \|\nabla(\varrho_y \circ u)\|_{L^1(\Omega)} \, dy \leq C_{\Lambda} \|\nabla u\|_{L^1(\Omega)}.
\]

This well-known result is based on an argument by Hardt, Kinderlehrer and Lin [19, Lemma 2.3], [20, Theorem 6.2] (a proof of this statement may also be found, e.g., in [13, Lemma 14]).
2.2 Automorphisms of $\pi$

Let $\text{Aut}(\pi)$ be the set of smooth isometries $\varphi: \mathcal{E} \to \mathcal{E}$ such that $\pi \circ \varphi = \pi$. The set $\text{Aut}(\pi)$ is a group with respect to the composition of maps. In fact, $\text{Aut}(\pi)$ is isomorphic to $\pi_1(\mathcal{N}, z_0)$ for any $z_0 \in \mathcal{N}$, although the isomorphism may not be canonical (it is canonical if and only if $\pi_1(\mathcal{N}, z_0)$ is abelian). For any $w_1, w_2 \in \mathcal{E}$ such that $\pi(w_1) = \pi(w_2)$, there exists a unique $\varphi \in \text{Aut}(\pi)$ such that

$$(2.6) \quad \varphi(w_1) = w_2$$

(see e.g. [21, Section 1.3, p. 70] or [26, Chapter 12]).

**Lemma 2.3.** There exists a compact subset $E_s \subseteq \mathcal{E}$ such that

$$\mathcal{E} \subseteq \bigcup_{\varphi \in \text{Aut}(\pi)} \varphi(E_s).$$

**Proof.** Fix base points $z_1 \in \mathcal{N}$, $w_1 \in \pi^{-1}(z_1) \subseteq \mathcal{E}$. For any $z \in \mathcal{N}$, choose a minimising geodesic $\gamma_z: [0, 1] \to \mathcal{N}$ with endpoints $\gamma_z(0) = z$, $\gamma_z(1) = z_1$. Since $\mathcal{N}$ is compact, we have

$$(2.7) \quad R := \sup_{z \in \mathcal{N}} \int_0^1 |\gamma_z'(t)| \, dt < +\infty.$$ 

Let $E_s \subseteq \mathcal{E}$ be the closed geodesic disk of centre $w_1$ and radius $R$. Given $w \in \mathcal{E}$, we consider the (unique) Lipschitz map $\hat{\gamma}: [0, 1] \to \mathcal{E}$ such that $\pi \circ \hat{\gamma} = \gamma_{\pi(w)}$ and $\hat{\gamma}(0) = w$. We have $\pi(\hat{\gamma}(1)) = z_1 = \pi(w_1)$ and hence, due to (2.6), there exists a (unique) $\varphi \in \text{Aut}(\pi)$ such that $\hat{\gamma}(1) = \varphi(w_1)$. Since $\pi$ is a local isometry, the geodesic distance between $\hat{\gamma}(0) = w$ and $\hat{\gamma}(1) = \varphi(w_1)$ must be less than or equal to $R$, by (2.7). Since $\varphi$ is an isometry, $w \in \varphi(E_s)$, and the lemma follows. \qed

3 The case of piecewise-affine maps

Towards the proof of Theorem 1, we first construct a lifting for maps of the form $g_y \circ u$, where $u$ is piecewise-affine (but not necessarily $\mathcal{N}$-valued). For any $d \geq 1$, let $Q^d := (-1, 1)^d$. We say that a map $u: Q^d \to \mathbb{R}^m$ is piecewise-affine if $u$ is continuous and there exists a triangulation $T$ of $Q^d$ such that, for any simplex $T$ of $T$, $u|_T$ is affine.

**Proposition 3.1.** Let $u: Q^d \to Q^m_{\sigma}$ be piecewise-affine. Then, there exist $y \in B^m_{\sigma}$ and a lifting $v \in \text{BV}(Q^d, \mathcal{E})$ of $g_y \circ u$ such that

$$\|v\|_{L^1(Q^d)} \leq C_\Lambda \left(\|\nabla u\|_{L^1(Q^d)} + 1\right), \quad |Dv|(Q^d) \leq C_\Lambda \|\nabla u\|_{L^1(Q^d)},$$

where $C_\Lambda$ is a constant that depends only on $d$, $\mathcal{N}$, $m$, $\mathcal{E}$, $\sigma$, $\varphi$, and $\Lambda$.

Before proving Proposition 3.1, we state some auxiliary results. Let $u: Q^d \to Q^m_{\sigma}$ be piecewise-affine. Let us take a constant $u_* \in \mathcal{N}$, and define $U: [0, 1] \times Q^d \to Q^m_{\sigma}$ by

$$(3.1) \quad U(t, x) := (1-t)u(x) + tu_* \quad \text{for } (t, x) \in [0, 1] \times Q^d.$$ 

Let $\tau: [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ denote the canonical projection, $\tau(t, x) := x$. For any $y \in B^m_{\sigma}$, we define

$$(3.2) \quad S_y := (u-y)^{-1}(\mathcal{E}), \quad T_y := \tau \left( (U-y)^{-1}(\mathcal{E}) \right).$$
Lemma 3.2. Suppose that $d \geq 2$. Then, for a.e. $y \in B^n_2$, the sets $S_y$, $T_y$ are finite unions of polyhedra, of dimension less than or equal to $d - 2$, $d - 1$ respectively. Moreover,

$$\int_{B^n_2} \mathcal{H}^{d-1} (T_y) \, dy \leq C \int_{Q^d} |u-u_*| |\nabla u| \, d\mathcal{L}^d$$

where $C$ is a positive constant that depends only on $\mathcal{N}$, $m$, $\mathcal{X}$, $\varrho$.

We will prove Lemma 3.2 with the help of the following

Lemma 3.3. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, with $d \geq 2$. Let $v: \mathbb{R}^d \to \mathbb{R}^2$ be an affine map, let $v_* \in \mathbb{R}^2$ be a constant, and let $V: [0, 1] \times \mathbb{R}^d \to \mathbb{R}^2$ be defined by $V(t, x) := (1-t)v(x)+tv_*$. Then,

$$\int_{\mathbb{R}^2} \mathcal{H}^{d-1} \left( \tau (V^{-1}(z)) \cap \Omega \right) \, dz \leq 2 \int_{\Omega} |v-v_*| |\nabla v| \, d\mathcal{L}^d.$$

Lemma 3.3 follows immediately from [13, Lemma 15]. However, for the convenience of the reader, we recall the proof here.

Proof of Lemma 3.3. We consider the case $d = 2$ first. We use $\nabla$ to denote the gradient in $\mathbb{R} \times \mathbb{R}^2$, with respect to the variables $(t, x)$, and we call $V$, $V^2$ the components of $V$, i.e. $V = (V^1, V^2)$. Let $z \in \mathbb{R}^2$ be a regular value for $V$. Then, $V^{-1}(z)$ is a smooth curve in $\mathbb{R} \times \mathbb{R}^2$ and the vector field $(\nabla V^1 \times \nabla V^2) / (|\nabla V^1 \times \nabla V^2|)$ is tangent to $V^{-1}(z)$. By the area formula,

$$(3.3) \quad \mathcal{H}^1 \left( \tau (V^{-1}(z)) \cap \Omega \right) \leq \int_{V^{-1}(z) \cap ([0,1] \times \Omega)} \left| \nabla V^1 \times \nabla V^2 \right| \, d\mathcal{H}^1$$

Let $e := (1, 0, 0) \in \mathbb{R} \times \mathbb{R}^2$. By the properties of the cross product, we deduce

$$\left| \tau (\nabla V^1 \times \nabla V^2) \right| \leq \left| e \times (\nabla V^1 \times \nabla V^2) \right| = \left| (\partial_t V^2) \nabla V^1 - (\partial_t V^1) \nabla V^2 \right|$$

The $t$-component of $(\partial_t V^2) \nabla V^1 - (\partial_t V^1) \nabla V^2$ vanishes, so

$$\left| \tau (\nabla V^1 \times \nabla V^2) \right| \leq \left| (\partial_t V^2) \nabla_x V^1 - (\partial_t V^1) \nabla_x V^2 \right| \leq 2 |\partial_t V| |\nabla_x V| \leq 2 |v-v_*| |\nabla_x v|$$

Injecting this inequality into (3.3), we obtain

$$\mathcal{H}^1 \left( \tau (V^{-1}(z)) \cap \Omega \right) \leq 2 \int_{V^{-1}(z) \cap ([0,1] \times \Omega)} \frac{|v-v_*| |\nabla_x v|}{|\nabla V^1 \times \nabla V^2|} \, d\mathcal{H}^1.$$

We have $|\nabla V^1 \times \nabla V^2|^2 = \det (\nabla V (\nabla V)^T)$, that is, $|\nabla V^1 \times \nabla V^2|$ is the Jacobian of $V$ (up to a sign). Then, the coarea formula implies

$$\int_{\mathbb{R}^2} \mathcal{H}^1 \left( \tau (V^{-1}(z)) \cap \Omega \right) \, dz \leq 2 \int_{[0,1] \times \mathbb{R}^d} |v-v_*| |\nabla_x v| \, d\mathcal{L}^{d+1},$$

which gives the desired estimate when $d = 2$. Now, suppose that $d \geq 3$. Up to a translation, we may assume that $v$ is a linear map. Then, the kernel of $v$ contains a $(d-2)$-linear subspace $\Pi$, and $V^{-1}(z) = (V|_{\Pi^1})^{-1}(z) \times \Pi$ where $\Pi^1$ is the orthogonal complement of $\Pi$. Therefore, the lemma follows by a slicing argument. \qed
Proof of Lemma 3.2. By assumption, there exists a triangulation $\mathcal{T}$ of $Q^d$ such that, for any simplex $H$ of $\mathcal{T}$ (of arbitrary dimension), the restriction $u_{iH}$ is affine. By Proposition 2.1, the set $\mathcal{A}$ is a finite union of polyhedra, say $K_1, \ldots, K_p$, of dimension $m - 2$ at most. For any $y \in B^m_\varepsilon$, any simplex $H$ of $\mathcal{T}$ and any $i \in \{1, \ldots, p\}$, $(u - y)^{-1}(K_i) \times H$ is a polyhedron, hence $S_y$ is a union of polyhedra. To show that $T_y$ is a union of polyhedra too, we assume that $u_s = 0$, up to a translation. For any $y \in \mathbb{R}^m$ such that $-y \notin \mathcal{A}$, there exists $\varepsilon > 0$ such that

$$T_y \cap H = \bigcup_{i=1}^{p} \left\{ x \in H : u(x) \in \tilde{K}_{i,y} \right\} \quad \text{where} \quad \tilde{K}_{i,y} := \bigcup_{t \in [0,1] - \varepsilon} \frac{K_i + y}{1 - t}$$

The set $\tilde{K}_{i,y}$ itself is a polyhedron (it is the convex hull of $(K_i + y) \cup (K_i + y)/\varepsilon$, so $T_i$ is a finite union of polyhedra.

Let $\Pi_i$ be the affine subspace of $\mathbb{R}^m$ spanned by $K_i$. For any simplex $H$ of $\mathcal{T}$, any $i$ and a.e. $y \in B^m_\varepsilon$, the maps $(u - y)|_H$, $(U - y)|_{[0,1] \times H}$ are transverse to $\Pi_i$. This follows by Thom’s parametric transversality theorem (see e.g. [22, Theorem 2.7 p. 79]). By transversality, for any simplex $H$ in $\mathcal{T}$, any $i$ and a.e. $y \in B^m_\varepsilon$, we have

$$\dim \left( (u - y)^{-1}(\Pi_i) \cap ([0,1] \times H) \right) = \dim(H) + 1 - m + \dim \Pi_i \leq d - 1$$

(3.4)

(unless the intersection is empty), with equality only if $\dim(H) = d$ and $\dim(\Pi_i) = m - 2$. Then, for a.e. $y \in B^m_\varepsilon$, $T_y$ has dimension $d - 1$ at most. In a similar way, we show that $\dim(S_y) \leq d - 2$ for a.e. $y \in B^m_\varepsilon$. Moreover, from (3.4) we deduce

$$\mathcal{H}^{d-1}(T_y) \leq \sum_{i : \dim \Pi_i = m - 2 \atop H \in \mathcal{T} : \dim(T) = d} \mathcal{H}^{d-1} \left( \tau((U - y)^{-1}(\Pi_i)) \cap \text{int}(H) \right),$$

where $\text{int}(H)$ denotes the interior of $H$. Now, take $i$ such that $\dim \Pi_i = m - 2$ and $H \in \mathcal{T}$ of dimension $d$. Let $\Pi_i^\perp \subseteq \mathbb{R}^m$ be the orthogonal $2$-plane to $\Pi_i$, passing through the origin. Let $\zeta$, $\zeta^\perp$ be the orthogonal projections of $\mathbb{R}^m$ onto $\Pi_i$, $\Pi_i^\perp$, respectively. We denote the variable $y \in \mathbb{R}^m$ as $(z, z^\perp) \in \Pi_i \times \Pi_i^\perp$. We have

$$\int_{B^m_\varepsilon} \mathcal{H}^{d-1} \left( \tau((U - y)^{-1}(\Pi_i)) \cap \text{int}(H) \right) \ dy$$

$$\leq \int_{\zeta(B^m_\varepsilon) \times \Pi_i^\perp} \mathcal{H}^{d-1} \left( \tau((\zeta \perp U)^{-1}(z^\perp)) \cap \text{int}(H) \right) \ d(z, z^\perp).$$

Then, by applying Lemma 3.3 to the map $\zeta \perp U$, we obtain

$$\int_{B^m_\varepsilon} \mathcal{H}^{d-1} \left( \tau((U - y)^{-1}(\Pi_i)) \cap \text{int}(H) \right) \ dy \leq 2 \int_{H} \left| \zeta \perp u - \zeta \perp (u_s) \right| \left| \nabla (\zeta \perp u) \right| \ d\mathcal{L}^d.$$

Using that $\zeta \perp$ is $1$-Lipschitz, taking the sum over $i$ and $H$, and applying (3.5), the lemma follows.
Proof of Proposition 3.1. Let us focus on the interesting case \( d \geq 2 \); we will deal with the case \( d = 1 \) later. We take a constant \( u_* \in \mathcal{N} \) and define \( U, S_y, T_y \) as in (3.1), (3.2). By Lemma 2.2, Lemma 3.2 and an average argument, we can choose \( y \in B^m_\sigma \) such that \( S_y, T_y \) are polyhedral, of dimension \( d - 2, d - 1 \) respectively, and

\[
(3.6) \quad \|\nabla (\varrho_y \circ u)\|_{L^1(Q^d)} + \mathcal{H}^{d-1}(T_y) \leq C_\Lambda \|\nabla u\|_{L^1(Q^d)}.
\]

(The constant \( C_\Lambda \) depends only on \( d, \mathcal{N}, m, \mathcal{X}, \varrho, \Lambda \).) We also define

\[
(3.7) \quad \Sigma_y := \{ (t, x) \in [0, 1] \times Q^d : \text{there exists } s \in [t, 1] \text{ such that } (s, x) \in (U - y)^{-1}(\mathcal{X}) \}
\]

(see Figure 2). The set \( \Sigma_y \) is closed. We have \( (U - y)^{-1}(\mathcal{X}) \subseteq \Sigma_y \) and hence, the map \( \varrho_y \circ U \) is well-defined and locally Lipschitz on \( ([0, 1] \times Q^d) \setminus \Sigma_y \).

![Figure 2](image_url)

Figure 2: Left: The set \( \Sigma_y \subseteq ([0, 1] \times Q^d) \) (in red), in case \( Q^d \) is a two-dimensional square (in gray). The complement \( ([0, 1] \times Q^d) \setminus \Sigma_y \) retracts by deformation onto \( \{1\} \times Q^d \). Right: The path we use in Step 2, for the proof of (3.10).

Step 1 (Construction of a lifting). We first construct a continuous lifting \( V : ([0, 1] \times Q^d) \setminus \Sigma_y \to \mathcal{E} \) of \( \varrho_y \circ U \) restricted to \( ([0, 1] \times Q^d) \setminus \Sigma_y \). A classical result in topology (see e.g. [21 Proposition 1.33] or [26 Theorem 11.18]) asserts that such a lifting exists if and only if, for any continuous loop \( \gamma : S^1 \to ([0, 1] \times Q^d) \setminus \Sigma_y \), the composition \( \varrho_y \circ U \circ \gamma : S^1 \to \mathcal{N} \) is homotopic to a constant. (It does not matter whether we consider free or based homotopies here, because a loop that is freely homotopic to a constant is also homotopic to a constant relative to its base point.) Let \( \gamma : S^1 \to ([0, 1] \times Q^d) \setminus \Sigma_y \) be a continuous loop. Let \( \xi : [0, 1] \times \mathbb{R}^d \to [0, 1], \tau : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d \) be the projections, \( \xi(t, x) := t \) and \( \tau(t, x) := x \). For any \( \omega \in S^1 \) and \( s \geq (\xi \circ \gamma)(\omega) \), we have \( (s, (\tau \circ \gamma)(\omega)) \notin (U - y)^{-1}(\mathcal{X}) \) because of (3.7). Then, the map \( H : [0, 1] \times S^1 \to \mathcal{N}, \)

\[
H(t, \omega) := (\varrho_y \circ U)((1 - t)(\xi \circ \gamma)(\omega) + t, (\tau \circ \gamma)(\omega)) \quad \text{for } (t, \omega) \in [0, 1] \times S^1
\]
is well-defined and continuous. This maps provides a (free) homotopy between \(H(0, \cdot) = \varrho_y \circ U \circ \gamma\) and \(H(1, \cdot) = u_\ast\), thus showing the existence of a continuous lifting \(V: ([0, 1] \times Q^d) \setminus \Sigma_y \to \mathcal{E}\) of \(\varrho_y \circ U\). Not only is \(V\) continuous, but also it is locally Lipschitz, because \(\varrho_y \circ U\) is locally Lipschitz on \(([0, 1] \times Q^d) \setminus \Sigma_y\) and the map \(\pi\) is a local isometry.

We define \(v(x) := V(0, x)\) for \(x \in Q^d \setminus T_y\). Then, \(v: Q^d \setminus T_y \to \mathcal{E}\) is a locally Lipschitz lifting of \(\varrho_y \circ u\), and because \(\pi\) is a local isometry, we deduce that

\[
|\nabla v| = |\nabla (\varrho_y \circ u)| \quad \text{a.e. on } Q^d \setminus T_y
\]

via the chain rule. As a consequence,

\[
\nabla v \in L^\infty_{\text{loc}}(Q^d \setminus S_y, \mathbb{R}^{\ell \times d})
\]

because \(\varrho_y \circ u\) is locally Lipschitz on \(Q^d \setminus S_y\). However, the distributional derivative \(Dv\) of \(v\) does not coincide with \(\nabla v\), in general; it will also contain a singular part, which is carried by \(T_y\).

**Step 2** (Bounds on the jump of \(v\)). Given two points \(w_1, w_2 \in \mathcal{E}\), we denote by \(\text{dist}_\mathcal{E}(w_1, w_2)\) the geodesic distance between them. Thanks to (3.9), the map \(v\) has well-defined traces \(v^+\), \(v^−\) on either side of \(T_y\), \(\mathcal{H}^{d−1}\text{-a.e.}\) on \(T_y\). We claim that there exists a constant \(C_\Lambda\), depending only on \(N\), \(m\), \(\mathcal{H}\), \(\varrho\) and \(\Lambda\), such that

\[
\text{dist}_\mathcal{E}(v^+(x), v^−(x)) \leq C_\Lambda \quad \text{for } \mathcal{H}^{d−1}\text{-a.e. } x \in T_y.
\]

For this purpose, take a point \(x \in T_y \setminus S_y\) that belongs to the interior of a \((d−1)\)-polyhedron of \(T_y\). Let \(L\) be a straight line segment that is orthogonal to \(T_y\) at \(x\), contains \(x\) in its interior, and intersects \(T_y\) only at \(x\). Let \(x^−, x^+\) be the endpoints of \(L\). Since \(\varrho_y \circ u\) is Lipschitz continuous in a neighbourhood of \(x \in Q^d \setminus S_y\), we may take \(L\) so small that

\[
\int_L |\nabla (\varrho_y \circ u)| \, d\mathcal{H}^1 \leq 1.
\]

We define \(\gamma: [0, 4] \to N\), \(\tilde{\gamma}: [0, 4] \to \mathcal{E}\) as

\[
\gamma(t) := \begin{cases} 
(\varrho_y \circ U)((1−t)x + tx^+) & \text{if } 0 \leq t \leq 1 \\
(\varrho_y \circ U)(t−1, x^+) & \text{if } 1 \leq t \leq 2 \\
(\varrho_y \circ U)(3−t, x^−) & \text{if } 2 \leq t \leq 3 \\
(\varrho_y \circ U)((4−t)x^− + (t−3)x) & \text{if } 3 \leq t \leq 4,
\end{cases}
\]

\[
\tilde{\gamma}(t) := \begin{cases} 
v((1−t)x + tx^+) & \text{if } 0 \leq t \leq 1 \\
V(t−1, x^+) & \text{if } 1 \leq t \leq 2 \\
V(3−t, x^−) & \text{if } 2 \leq t \leq 3 \\
v((4−t)x^− + (t−3)x) & \text{if } 3 \leq t \leq 4.
\end{cases}
\]

Using that \(U(0, \cdot) = u\) and \(U(1, \cdot)\) is constant, it can be checked that \(\gamma\) is indeed continuous — in fact, Lipschitz. The map \(\tilde{\gamma}\) is also well-defined and Lipschitz. Indeed, \(V\) is continuous.
on \( \{1\} \times Q^d \), because \( \{(1) \times Q^d \} \cap \Sigma_y = \emptyset \), and \( \pi \circ V \) is constant on \( \{1\} \times Q^d \), so \( V \) must be constant on \( \{1\} \times Q^d \), too. Now, \( \tilde{\gamma} \) is a lifting of \( \gamma \). Since \( \pi \) is a local isometry, we must have

\[
\text{dist}_E(v^+(x), v^-(x)) \leq \int_0^4 |\tilde{\gamma}'(t)| \, dt = \int_0^4 |\gamma'(t)| \, dt.
\]

The maps \( t \in [0, 1] \mapsto U(t, x_+) \), \( t \in [0, 1] \mapsto U(t, x_-) \) are well-defined and Lipschitz (because \( x_+ \notin T_y \), \( x_- \notin T_y \)), and parametrise injectively straight lines segments that are contained in \( Q^m \setminus \mathcal{X} \). Therefore, by applying Proposition 2.1(iv) and (3.11), we deduce that

\[
\text{(3.13)} \quad \int_0^4 |\gamma'(t)| \, dt \leq C_\Lambda.
\]

By combining (3.12) and (3.13), the claim (3.10) follows.

Step 3 (Conclusion, in case \( d \geq 2 \)). From (3.10), we immediately obtain

\[
\text{(3.14)} \quad |v^+(x) - v^-(x)| \leq C_\Lambda \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in T_y.
\]

From (3.6), (3.8), (3.9) and (3.14), we deduce that the distributional derivative of \( Dv \) is a bounded measure, with

\[
\text{(3.15)} \quad |Dv|(Q^d) \leq \|\nabla v\|_{L^1(Q^d \setminus T_y)} + C_\Lambda \mathcal{H}^{d-1}(T_y) \leq C_\Lambda \|\nabla u\|_{L^1(Q^d)}.
\]

By a Poincaré-type inequality in the space \( \text{BV} \) (see e.g. [15, Eq. (16)]), there exist \( w_\ast \in \mathcal{E} \) and a constant \( C \) (depending only on \( d \)) such that

\[
\text{(3.16)} \quad \int_{Q^d} \text{dist}_E(v(x), w_\ast) \, dx \leq C |Dv|(Q^d).
\]

By Lemma 2.3 and up to composition with an element of \( \text{Aut}(\pi) \), we may assume without loss of generality that

\[
\text{(3.17)} \quad w_\ast \in E_\ast,
\]

where \( E_\ast \) is the compact subset of \( \mathcal{E} \) given by Lemma 2.3. Now, the proposition follows from (3.15), (3.16) and (3.17).

Step 4 (The case \( d = 1 \)). In case \( d = 1 \), Lemma 2.2 implies that \( \varrho_y \circ u \) is continuous on \([-1, 1]\) for a.e. \( y \), via Sobolev embedding. As a consequence, for a.e. \( y \) the map \( \varrho_y \circ u \) has a continuous lifting \( v: [-1, 1] \to \mathcal{E} \) and actually, \( v \in W^{1,1}(-1, 1; \mathcal{E}) \) because \( \pi \) is a local isometry. Now Proposition 3.1 follows from the same arguments as above.

Remark 3.1. In case \( \pi_1(\mathcal{N}) \) is finite, the proof of Proposition 3.1 simplifies considerably. Indeed, if \( \mathcal{N} \) is compact and \( \pi_1(\mathcal{N}) \) is finite, then \( \mathcal{E} \) is compact and hence, the estimate (3.10) is immediate. The finiteness of \( \pi_1(\mathcal{N}) \) proves to be quite useful in other contexts too, for instance, in the asymptotic analysis of minimisers for variational problems [11, 12], or in the study of extension problems for manifold-valued maps [5, 29].
4 Proof of Theorem 1

Let \( \Omega \subseteq \mathbb{R}^d \) be a bounded, smooth domain, and let \( u \in \text{BV}(\Omega, \mathcal{N}) \). We first reduce to the case \( \Omega \) is a cube. Up to scaling, we may assume without loss of generality that \( \overline{\Omega} \subseteq Q^d := (-1, 1)^d \).

Let \( u_\Omega := \mathcal{L}^d(\Omega)^{-1} \int_{\Omega} u \in \mathbb{R}^m \) be the average of \( u \) over \( \Omega \). Thanks to [1, Proposition 3.21] and the BV-Poincaré inequality [1, Theorem 3.44], we can extend \( \lim \) \( u \) to a map \( \tilde{u} \in (L^\infty \cap \text{BV})(Q^d, \mathbb{R}^m) \) that satisfies \( |D\tilde{u}|(Q^d) \leq C |Du|(|\Omega|) \), for some constant \( C \) depending only on \( \Omega \).

By re-defining \( u := \tilde{u} + u_\Omega \), we obtain an extension of the map we had before. The new map belongs to \( (L^\infty \cap \text{BV})(Q^d, \mathbb{R}^m) \) and satisfies

\[
(4.1) \quad |Du|(Q^d) \leq C |Du|(|\Omega|).
\]

We can approximate \( u \) with a sequence of smooth maps \( \tilde{u}_j : Q^d \to \mathbb{R}^m \) that converge to \( u \) weakly in \( \text{BV}(Q^d) \), strongly in \( L^1(Q^d) \) and a.e., and moreover

\[
(4.2) \quad \lim_{j \to +\infty} \|\nabla \tilde{u}_j\|_{L^1(Q^d)} \leq \|\nabla u\|_{L^1(Q^d)}
\]

(see e.g. [1, Theorem 3.9]). By (2.4) and a truncation argument, we may also assume that

\[ \tilde{u}_j(x) \in Q^m_\Lambda \quad \text{for any } x \in Q^d \text{ and any } j. \]

Finally, for any \( j \in \mathbb{N} \) we may choose a piecewise-affine interpolant \( u_j : Q^d \to Q^m_\Lambda \) of \( \tilde{u}_j \) in such a way that

\[
(4.3) \quad \|u_j - \tilde{u}_j\|_{L^1(Q^d)} + \|\nabla u_j - \nabla \tilde{u}_j\|_{L^1(Q^d)} \leq 1/j.
\]

By applying Proposition 3.1, (4.2) and (4.3), for any \( j \) we find \( y_j \in B^m_\varrho \) and a lifting \( v_j \in \text{BV}(Q^d, \mathcal{E}) \) of \( \varrho_{y_j} \circ u_j \) such that

\[
\limsup_{j \to +\infty} \|v_j\|_{L^1(Q^d)} \leq C_\Lambda (|Du|(|\Omega|) + 1), \quad \limsup_{j \to +\infty} |Dv_j|(Q^d) \leq C_\Lambda |Du|(|\Omega|).
\]

Therefore, there exist \( y \in B^m_\varrho \) and \( v \in \text{BV}(Q^d, \mathcal{E}) \) such that, up to extraction of subsequences, \( y_j \to y, v_j \to v \) weakly in \( \text{BV}(Q^d) \), strongly in \( L^1(Q^d) \) and a.e. on \( Q^d \). (The set \( \text{BV}(\Omega, \mathcal{E}) \) is closed with respect to the strong \( L^1 \)-convergence, because we have embedded \( \mathcal{E} \) as a closed subset of \( L^1(\mathbb{R}^d) \); see [31].) Since \( \varrho \) is smooth in a neighbourhood of \( \mathcal{N} \) and \( \mathcal{N} \) is compact, \( \varrho_{y_j} \to \varrho_y \) uniformly in a neighbourhood of \( \mathcal{N} \). For a.e. \( x \in \Omega \), we have \( u_j(x) \to u(x) \in \mathcal{N} \) and hence, \( u_j(x) \) is arbitrarily close to \( \mathcal{N} \) for \( j \) large (depending on \( x \)). As a consequence,

\[ \pi \circ v_j = \varrho_{y_j} \circ u_j \to \varrho_y \circ u = u \quad \text{a.e. on } \Omega \]

and \( v_{|\Omega} \) is a lifting of \( u \). To complete the proof of Theorem 1, it only remains to check that \( v \in \text{SBV}(\Omega, \mathcal{E}) \) in case \( u \in \text{SBV}(\Omega, \mathcal{N}) \). This can be done, e.g., by repeating word by word the arguments of [13, Theorem 3, Step 4].
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