Refined geometric \( L^p \) Hardy inequalities

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Abstract

For a bounded convex domain \( \Omega \) in \( \mathbb{R}^N \) we prove refined Hardy inequalities that involve the Hardy potential corresponding to the distance to the boundary of \( \Omega \), the volume of \( \Omega \), as well as a finite number of sharp logarithmic corrections. We also discuss the best constant of these inequalities.

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1 Introduction

For a convex domain \( \Omega \subset \mathbb{R}^N \) the Hardy inequality

\[
\int_{\Omega} |\nabla u|^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{d^p} dx, \quad d(x) = \text{dist}(x, \partial \Omega) \quad u \in W^{1,p}_0(\Omega)
\]

is valid, where the constant \( \left( \frac{p-1}{p} \right)^p \) is optimal; cf [MMP], [MS]. Brezis and Marcus [BM] have established an improved version of (1.1) when \( p = 2 \): they showed that for bounded and convex \( \Omega \) there holds

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d^2} dx + \frac{1}{4 \text{diam}^2(\Omega)} \int_{\Omega} u^2 dx, \quad u \in H^1_0(\Omega).
\]

The question was asked in that paper as to whether it is possible to replace \( \text{diam}^{-2}(\Omega) \) by \( c|\Omega|^{-2/N} \), where \( |\Omega| \) denotes the volume of \( \Omega \). A positive answer was given by M. and T. Hoffmann-Ostenhof and Laptev [HHL], who showed that

\[
\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d^2} dx + k_2 \left( \frac{|\Omega|}{a_N} \right)^2 \int_{\Omega} u^2 dx, \quad u \in H^1_0(\Omega)
\]

where \( a_N \) is the volume of the unit ball and \( k_2 = N/4 \).

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In connection with this let us notice that when we take as \( d(x) \) the distance from a point of \( \Omega \), say the origin, the following improved Hardy inequality was established by Brezis and Vazquez \([BV]\)

\[
\int_{\Omega} |\nabla u|^2 \, dx \geq \left( \frac{N-2}{2} \right)^2 \int_{\Omega} \frac{u^2}{|x|^2} \, dx + \mu_2 \frac{a_N |\Omega|}{|\Omega|} \int_{\Omega} u^2 \, dx, \quad u \in H^1_0(\Omega); \quad (1.4)
\]

here \( \mu_2 \approx 5.783 \) is the first eigenvalue of the Dirichlet Laplacian for the unit disk in \( \mathbb{R}^2 \). This constant is optimal when \( \Omega \) is a ball centered at the origin, independently of the dimension \( N \geq 2 \), cf \([BV]\), whereas for general \( \Omega \) this constant is not optimal, cf \([FT, \text{Proposition 5.1}]\).

An \( L^p \)-version of (1.3) was recently obtained by Tidblom \([T]\) who showed that for convex \( \Omega \) there holds

\[
\int_{\Omega} |\nabla u|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{d^p} \, dx + k_p \left( \frac{a_N |\Omega|}{|\Omega|} \right)^{\frac{2}{N}} \int_{\Omega} u^2 \, dx, \quad u \in W^{1,p}_0(\Omega), \quad (1.5)
\]

with

\[
k_p = (p-1) \left( \frac{p-1}{p} \right)^p \frac{\sqrt{\pi} \Gamma(\frac{N+p}{2})}{\Gamma(\frac{p+1}{2}) \Gamma(\frac{N}{2})}.
\]

For \( p = 2 \) this reduces to (1.3); in particular \( k_2 = N/4 \).

In addition to (1.3) it was shown in \([HHL, \text{Theorem 3.4}]\) that if

\[
X_1(t) = (1 - \log t)^{-1}, \quad t \in (0, 1), \quad (1.7)
\]

the following more refined improvement of (1.3) is true: for any \( D \geq \text{diam}(\Omega)/2 \) there holds

\[
\int_{\Omega} |\nabla u|^2 \, dx \geq \frac{1}{4} \int_{\Omega} \frac{u^2}{d^2} \, dx + \frac{1}{4} \int_{\Omega} \frac{u^2}{d^2} X_1^2(d/D) \, dx
\]

\[
+ k_2 (1 - X_1(\text{diam}(\Omega)/(2D)))^2 \left( \frac{a_N |\Omega|}{|\Omega|} \right)^{2/N} \int_{\Omega} u^2 \, dx, \quad (1.8)
\]

for all \( u \in H^1_0(\Omega) \). Note that if we let \( D \to \infty \) in (1.8) we regain (1.3).

In our main result we extend both (1.5) and (1.8). More precisely, with \( X_1(t) \) as in (1.7) we define recursively

\[
X_k(t) = X_1(X_{k-1}(t)), \quad k = 2, 3, \ldots, t \in (0, 1). \quad (1.9)
\]

These are iterated logarithmic functions that vanish at an increasingly low rate at \( t = 0 \). Let us fix \( k \geq 1 \) and set

\[
a = \begin{cases} 
0, & \text{if } 1 < p \leq 2, \\
\frac{(p-2)k}{2(p-1)}, & \text{if } p > 2,
\end{cases} \quad (1.10)
\]

and

\[
\eta(t) = \sum_{i=1}^{k} X_1(t) \ldots X_i(t),
\]

whereas for \( k = 0 \) we set \( \eta = 0 \). For \( D \geq \text{diam}(\Omega)/2 \) we also set

\[
\eta_D = \eta\left( \frac{\text{diam}(\Omega)}{2D} \right).
\]
Then our main result reads:

**Theorem A** Assume that $\Omega$ is convex and bounded. Let $k \geq 0$ be a fixed integer. Then, there exists $D_0 = D_0(k, p, \text{diam}(\Omega)) \geq \text{diam}(\Omega)/2$ such that for $D \geq D_0$ there holds

$$
\int_{\Omega} |\nabla u|^p dx \geq \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{\partial p} dx + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \sum_{i=1}^k \int_{\Omega} \frac{|u|^p}{\partial ^i} X_i^2 \frac{\partial D}{\partial i} \ldots X_i^2 \frac{\partial D}{\partial i} dx 
+ k_p (1 - \eta_D - \eta^2) \left( \frac{a_N}{|\Omega|} \right)^{\frac{p}{p^2}} \int_{\Omega} |u|^p dx,
$$

(1.11)

for all $u \in W^{1,p}_0(\Omega)$. When $p = 2$ we can take as $D_0$ the unique solution of $\eta D_0 = 1$.

Note that if we let $D \to +\infty$ in (1.11) we recover (1.5). Also, for $p = 2$ and $k = 1$ we recover (1.8). Moreover, the terms in the series are sharp: it was shown in [BFT, Theorem A] that for each $k \geq 1$ the relation

$$
\int_{\Omega} |\nabla u|^p dx - \left( \frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{\partial p} dx + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \sum_{i=1}^k \int_{\Omega} \frac{|u|^p}{\partial ^i} X_i^2 \ldots X_i^2 dx
\geq c \int_{\Omega} \frac{|u|^p}{\partial ^k} X_1^2 \ldots X_k^2 dx
$$

(1.12)

is not valid for $\gamma < 2$; In addition, the best constant $c$ in (1.12) when $\gamma = 2$ is equal to $\frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1}$, for any $k = 1, 2, \ldots$.

A natural question is whether the constants appearing in (1.3) or (1.11) are optimal. Working towards this we consider the simplest case (1.3) (corresponding to $p = 2$, $k = 0$). Let $\Omega = B$, be the unit ball in $\mathbb{R}^N$, and denote by $C_N$ the best constant of (1.3), that is

$$
C_N = \inf_{u \in H^1_0(B)} \frac{\int_B |\nabla u|^2 dx - \frac{1}{4} \int_B u^2 dx}{\int_B u^2 dx}.
$$

(1.13)

We then show that in this case the constant $k_2 = \frac{N}{4}$ appearing in (1.3) is far from being optimal. In particular we have:

**Theorem B** For $N = 3$, $C_3 = \mu_2$, whereas for any $N \geq 2$ there holds:

$$
C_N \geq \mu_2 + \frac{(N-1)(N-3)}{4},
$$

(1.14)

where $\mu_2 \simeq 5.783$ is the best constant of inequality (1.4).

It is remarkable that when $\Omega$ is a ball and $N = 3$ inequalities (1.3) and (1.4) have the same best constant. For any $N \geq 2$ the lower bound (1.14) on $C_N$ improves the estimate $C_N \geq k_2 = \frac{N}{4}$.

To prove Theorem A we combine a vector field approach (cf [BFT]) along with ideas of [HHL] or [T]. It is worth noting that the “mean distance” method of Davies (cf [D1], [D2]) plays an essential role. For Theorem B after restricting to radial functions we use suitable change of variables.

### 2 Preliminary inequalities

In this section we will prove some auxiliary one-dimensional inequalities. Throughout this section $b \leq \frac{\text{diam}(\Omega)}{2}$ is a fixed positive constant. We have the following
Lemma 2.1 Let $\rho(t) = \min\{t, 2b - t\}$. For any function $g \in C^1((0,b))$ there holds

\begin{align*}
(i) & \quad \int_0^{2b} |u'(t)|^p dt \geq \int_0^{2b} \{g'(\rho(t)) - (p - 1)|g(\rho(t))|^{\frac{p}{p-1}} \} |u(t)|^p dt - 2g(b)|u(b)|^p, \\
(ii) & \quad \int_0^{2b} |u'(t)|^p dt \geq \int_0^{2b} \{g'(\rho(t)) - (p - 1)|g(\rho(t)) - g(b)|^{\frac{p}{p-1}} \} |u(t)|^p dt \\
\end{align*}

for all $u \in C^\infty_c(0,2b)$.

Proof. We first prove (i). For $u \in C^\infty_c(0,2b)$ we have

\[
\int_0^b g'(t)|u(t)|^p dt = g(b)|u(b)|^p - p \int_0^b g(t)|u|^p u' g dt \\
\leq g(b)|u(b)|^p + p \left( \int_0^D |u|^p dt \right) \left( \int_0^b |g|^{\frac{p}{p-1}} |u|^p dt \right)^{\frac{p-1}{p}} \\
\leq g(b)|u(b)|^p + \int_0^b|u'|^p dt + (p - 1) \int_0^b |g|^{\frac{p}{p-1}} |u|^p dt,
\]

hence

\[
\int_0^b |u'(t)|^p dt \geq \int_0^b \{g'(t) - (p - 1)|g(t)|^{\frac{p}{p-1}} \} |u(t)|^p dt - g(b)|u(b)|^p.
\]

A similar argument on $(b,2b)$ gives

\[
\int_b^{2b} |u'(t)|^p dt \geq \int_b^{2b} \{g'(2b-t) - (p - 1)|g(2b-t)|^{\frac{p}{p-1}} \} |u|^p dt - g(b)|u(b)|^p
\]

and (i) follows by adding up the last two inequalities.

Part (ii) follows immediately from (i) by using the function $g(x) - g(b)$ in the place of $g(x)$.

In order to apply the above lemma we fix a positive integer $k$ and define the functions

\[
\eta(t) = \sum_{i=1}^k X_1(t) \ldots X_i(t),
\]

\[
B(t) = \sum_{i=1}^k X_1^2(t) \ldots X_i^2(t), \quad t \in (0,1),
\]

where the $X_i$'s are given by (1.9). It is easy to check that both $\eta$ and $B$ are increasing functions of $t$ with $\eta(0^+) = B(0^+) = 0$ and $\eta(1^-) = B(1^-) = k$. We also note that

\[
\frac{1}{k} \eta^2(t) \leq B(t) \leq \eta^2(t), \quad t \in (0,1).
\]

For $0 < b \leq \frac{\text{diam}(\Omega)}{2} \leq D$ we define the following functions of $s \in (0,b)$:

\[
g(s) = -\left( \frac{p-1}{p} \right)^{p-1} s^{-(p-1)} \left( 1 - \eta(s/D) - a\eta^2(s/D) \right)
\]

\[
A(s) = g'(s) - (p - 1)|g(s) - g(b)|^{\frac{p}{p-1}} - \left( \frac{p-1}{p} \right)^p s^{-p} - \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} s^{-p} B(s/D).
\]

Recall that $a$ is defined in (1.11). We then have the following
Lemma 2.2 There exists $D_0 = D_0(k,p,diam(\Omega)) \geq \frac{diam(\Omega)}{2}$, such that for all $D \geq D_0$ there holds:

(i) \[ 1 - \eta\left(\frac{diam(\Omega)}{2D}\right) - an^2\left(\frac{diam(\Omega)}{2D}\right) \geq 0, \]

(ii) \[ g'(s) - \left(\frac{p-1}{p}\right)s^{-p} - \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}s^{-p}B(s/D) \geq (p-1)|g(s)|^{\frac{p}{p-1}}, \]

(iii) $A(s)$ is a decreasing function of $s \in (0,b)$.

For $p = 2$, (ii) becomes equality. Also, for $p = 2$, we can take as $D_0$ the unique solution of $1 = \eta\left(\frac{diam(\Omega)}{2D_0}\right)$.

Proof. A straightforward calculation shows that

\[ \frac{d}{ds} \eta(s/D) = \frac{1}{s} B(s/D) + \frac{\eta^2(s/D)}{2}, \] \hspace{1cm} (2.4)

Setting $\Gamma(t) = tB'(t)$ we also have

\[ \frac{d}{ds} B(s/D) = \frac{1}{s} \Gamma(s/D) > 0; \] \hspace{1cm} (2.5)

the positivity follows from the fact that $B(t)$ is an increasing function of $t$.

Since $\eta(t)$ is an increasing function of $t$ with $\eta(0) = 0$, (i) is immediate.

We shall henceforth omit the argument $s/D$ from $\eta, B, \Gamma$ in the subsequent formulas. We next prove (ii). For $p = 2$ an easy calculation shows that (ii) becomes equality. For $p \neq 2$ the left hand side of (ii) is equal to

\[ g'(s) - \left(\frac{p-1}{p}\right)s^{-p} - \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}s^{-p}B(s/D) = \left(\frac{p-1}{p}\right)^p(p-1)s^{-p} \times \]

\[ \times \left[ 1 - \frac{p\eta}{p-1} + \left(\frac{p}{2(p-1)^2} - \frac{ap}{p-1}\right)\eta^2 + \frac{ap}{(p-1)^2}\eta^3 + \frac{ap}{(p-1)^2}\eta B \right]. \] \hspace{1cm} (2.6)

On the other hand, taking the Taylor expansion of $(1 - t)^{p-1}$ about $t = 0$, we see that the right hand side of (ii) is written as (for $\eta$ small)

\[ \left(\frac{p-1}{p}\right)^p(p-1)s^{-p}(1 - \eta - an^2)^{\frac{1}{p-1}} = \left(\frac{p-1}{p}\right)^p(p-1)s^{-p} \times \]

\[ \times \left[ 1 - \frac{p\eta}{p-1} - \frac{ap}{p-1}\eta^2 + \frac{ap\eta^3}{2(p-1)^2} + \frac{ap\eta^3}{6(p-1)^3} + O(\eta^4) \right]. \] \hspace{1cm} (2.7)

Comparing (2.6) and (2.7) we see that the corresponding right-hand sides agree to order $O(\eta^2)$. Recalling (2.2) and the choice of $a$ (cf. 1.10) we see that the cubic term in (2.6) is larger than the cubic term of (2.7). Hence (ii) is true provided $\eta$ is small enough, which amounts to $D_0$ being large enough.

We now prove (iii). Note that (ii) implies that $g'$ is positive in $(0,b)$ if $D_0$ is large enough. Hence for $s \in (0,b)$ we have

\[ A'(s) = g''(s) + p[g(b) - g(s)]^{\frac{1}{p-1}}g'(s) + \left(\frac{p-1}{p}\right)^{p-1}(p-1)s^{-p-1} \]
\[ + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} p s^{-p-1} B - \frac{1}{2} \left( \frac{p-1}{p} \right) s^{-p-1} \Gamma \]
\[ \leq \ g''(s) + p |g(s)| \left( \frac{1}{p-1} \right)^{p-1} (p-1) s^{-p-1} \]
\[ + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} p s^{-p-1} B - \frac{1}{2} \left( \frac{p-1}{p} \right) s^{-p-1} \Gamma. \]  
(2.8)

Using Taylor’s expansion we have
\[ |g(s)|^{\frac{1}{p-1}} = \frac{p-1}{p} s^{-1} (1 - \eta - a \eta^2)^{\frac{1}{p-1}} \]
\[ = \frac{p-1}{p} s^{-1} \left\{ 1 - \frac{1}{p-1} \eta - \left[ \frac{a}{p-1} + \frac{p-2}{2(p-1)^2} \right] \eta^2 - \right. \]
\[ \left. - \left[ \frac{(p-2)a}{(p-1)^2} - \frac{(p-2)(3-2p)}{6(p-1)^3} \right] \eta^3 + O(\eta^4) \right\}. \]  
(2.9)

From (2.4), (2.5), (2.8) and (2.9) we obtain
\[ A'(s) \leq (p-1)^2 \left( \frac{p-1}{p} \right)^{p-1} s^{-p-1} \left\{ \frac{p(p-2)}{6(p-1)^3} \eta^3 - \frac{ap}{(p-1)^2} \eta B + O(\eta^4) \right\} \]  
(2.10)

From this and the fact that
\[ \frac{1}{k} \eta^2 \leq B \leq \eta^2, \quad s \in (0, b) \]
we end up with
\[ A'(s) \leq p \left( \frac{p-1}{p} \right)^{p-1} s^{-p-1} \left\{ \frac{p-2}{6(p-1)} - \frac{a}{k} + O(\eta) \right\}. \]  
(2.11)

To conclude the proof we distinguish various cases:
(a) \( 1 < p < 2 \). Then \( a = 0 \) and it follows from (2.11) that \( A'(s) < 0 \) in \((0, b)\), provided \( D_0 \) is chosen large enough.
(b) \( p = 2 \). Again \( a = 0 \). A straightforward calculation shows that the right hand side of (2.8) is identically equal to zero. The only restriction here comes from (i), whence the choice of \( D_0 \).
(c) \( p > 2 \). Now \( a = \frac{(p-2)k}{3(p-1)} \) and the result follows again from (2.11).

This completes the proof.

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3 The Hardy inequality

Throughout the rest of the paper we assume that \( \Omega \subset \mathbb{R}^N \) is convex and set \( d(x) = \text{dist}(x, \partial \Omega) \).

Following [HHL], for \( \omega \in S^{N-1} \) and \( x \in \Omega \) we define the following functions with values in \((0, +\infty)\):
\[ \tau_\omega(x) = \inf \{ s > 0 \mid x + s \omega \notin \Omega \} \]
\[ \rho_\omega(x) = \min \{ \tau_\omega(x), \tau_-\omega(x) \} \]
\[ b_\omega(x) = \frac{1}{2}(\tau_\omega(x) + \tau_-\omega(x)). \]
We denote by $dS(\omega)$ the standard measure on $S^{N-1}$ normalized so that the total measure is one. Let $K_p > 0$ be defined by

$$\int_{S^{N-1}} |v \cdot \omega|^p dS(\omega) = K_p |v|^p, \quad \forall v \in \mathbb{R}^N. \quad (3.1)$$

The constant $K_p$ is computable and with $k_p$ as in (1.6) we have

$$k_p = (p-1)\left(\frac{p-1}{p}\right)^p K_p^{-1} \quad (3.2)$$

We have the following

**Lemma 3.1** Assume that $\Omega$ is convex. Then for all $x \in \Omega$ there holds

$$\int_{S^{N-1}} \rho^{-p}(x) dS(\omega) \geq K_p d(x)^{-p}. \quad (3.3)$$

**Proof.** Let $y \in \partial \Omega$ be such that $|y - x| = d(x)$ and let $P_y$ be the supporting hyper-plane through $y$ which is orthogonal to $y - x$. We define the half-sphere

$$S^+ = \{\omega \in S^{N-1} \mid \omega \cdot (y - x) > 0\}$$

and for $\omega \in S^+$ define $\sigma_\omega(x) > 0$ by requiring that $x + \sigma_\omega(x) \omega \in P_y$, so that

$$\frac{\omega \cdot (y - x)}{|y - x|} = \frac{|y - x|}{\sigma_\omega(x)}.$$

The convexity of $\Omega$ implies that $\tau_\omega(x) \leq \sigma_\omega(x)$ and hence

$$\int_{S^{N-1}} \frac{1}{\rho_\omega(x)^p} dS(\omega) \geq \int_{S^+} \frac{1}{\tau_\omega(x)^p} dS(\omega) \geq \int_{S^+} \frac{1}{\sigma_\omega(x)^p} dS(\omega) = \frac{2}{d(x)^{2p}} \int_{S^+} |(y - x) \cdot \omega|^p dS(\omega) = \frac{K_p}{d(x)^p},$$

as required.

We now give the proof of Theorem A.

**Proof of Theorem A.** Following [HHL] let us fix a direction $\omega \in S^{N-1}$ and let $\Omega_\omega$ be the orthogonal projection of $\Omega$ on the hyper-plane perpendicular to $\omega$. For each $z \in \Omega_\omega$ we apply Lemma 2.4 on the segment defined by $z$ and $\omega$ and we then integrate over $z \in \Omega_\omega$. We conclude that for any $u \in C_c^\infty(\Omega)$ there holds

$$\int \nabla u \cdot \omega |u|^p dx \geq \int \left\{ g'(\rho_\omega(x)) - (p-1)\left| g(\rho_\omega(x)) - g(b_\omega(x)) \right|^{\frac{p}{p-1}} \right\} |u|^p dx,$$
Integrating over $\omega \in S^{N-1}$ and recalling definition (3.1) we obtain
\[
\int_\Omega |\nabla u|^p \, dx \geq K_p^{-1} \int_\Omega \int_{S^{N-1}} \left\{ g'(\rho_\omega(x)) - (p-1)g(\rho_\omega(x)) - g(b_\omega(x)) \right\} \frac{\rho_\omega}{p-1} dS(\omega) |u|^p \, dx.
\] (3.4)

Now, let us choose $g$ as in (2.3). Since $\Omega$ is bounded Lemma 2.2 implies the existence of a $D_0 > 0$ such that for $D \geq D_0$, each of the functions
\[
A_{\omega,x}(s) := g'(s) - (p-1)\frac{g(s) - g(b_\omega(x))}{p-1} - \left( \frac{p-1}{p} \right) s - \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} s^{-p} B(s/D)
\]
– defined for $s \in (0, b_\omega(x))$ – is a decreasing function of $s \in (0, b_\omega(x))$. In particular $A_{\omega,x}(\rho_\omega(x)) \geq A_{\omega,x}(b_\omega(x))$, i.e.
\[
g'(\rho_\omega(x)) - (p-1)\frac{g(\rho_\omega(x)) - g(b_\omega(x))}{p-1} \geq \left( \frac{p-1}{p} \right) \rho_\omega(x)^{-p} + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \rho_\omega(x)^{-p} B(\rho_\omega(x)/D) + A_{\omega,x}(b_\omega(x)).
\] Hence (3.3) yields
\[
\int_\Omega |\nabla u|^p \, dx \geq K_p^{-1} \int_\Omega \int_{S^{N-1}} \left\{ \left( \frac{p-1}{p} \right) \rho_\omega(x)^{-p} + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \rho_\omega(x)^{-p} B(\rho_\omega(x)/D) + g'(b_\omega(x)) - \left( \frac{p-1}{p} \right) b_\omega(x)^{-p} - \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} b_\omega(x)^{-p} B(b_\omega(x)/D) \right\} \frac{\rho_\omega}{p-1} dS(\omega) |u|^p \, dx.
\] (3.5)

We first estimate the first two terms of (3.5). For each $x \in \Omega$ and $\omega \in S^{N-1}$ there holds $B(\rho_\omega(x)/D) \geq B(d(x)/D)$, and Lemma 3.1 yields
\[
K_p^{-1} \int_{S^{N-1}} \left\{ \left( \frac{p-1}{p} \right) \rho_\omega(x)^{-p} + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \rho_\omega(x)^{-p} B(\rho_\omega(x)/D) \right\} dS(\omega) \geq \left( \frac{p-1}{p} \right)^p d(x)^{-p} + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} d(x)^{-p} B(d(x)/D),
\]
(3.6)
for all $x \in \Omega$. The remaining three terms in the right-hand side of (3.5) are estimated using Lemma 2.2(ii)
\[
g'(b_\omega(x)) - \left( \frac{p-1}{p} \right) b_\omega(x)^{-p} - \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} b_\omega(x)^{-p} B(b_\omega(x)/D) \geq (p-1)\frac{g(b_\omega(x))}{p-1} \geq \left( \frac{p-1}{p} \right)^p (p-1) (1 - \eta_D - a\eta_D^p) b_\omega(x)^{-p}.
\]
Combining this with (3.5a) and (8.11) and recalling (8.2) we obtain
\[
\int_\Omega |\nabla u|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \int_\Omega |u|^p \, dx + \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \int_\Omega |u|^p B(d/D) + k_p^{-1} (1 - \eta_D - a\eta_D^p) \int_\Omega \int_{S^{N-1}} \frac{1}{b_\omega(x)^p} b_\omega(x)^p dS(\omega) |u|^p \, dx.
\] (3.7)
We estimate the last integral using a variation of an argument of [HHL]. Elementary analysis shows that \( \min_{t > 0} \left( \frac{1 + t^N}{1 + t} \right)^{\frac{N}{N + p}} = 2 - \frac{(N - 1) p}{N + p} \) and therefore for \( x \in \Omega \)

\[
2^{- \frac{(N-1)p}{N+p}} \leq \int_{S^{N-1}} \frac{(\tau_\omega(x)^N + \tau_{-\omega}(x)^N)^{p/(N+p)}}{(\tau_\omega(x) + \tau_{-\omega}(x))^{Np/(N+p)}} dS(\omega) \\
\leq \left( \int_{S^{N-1}} (\tau_\omega^N + \tau_{-\omega}^N) dS(\omega) \right)^{\frac{p}{N+p}} \left( \int_{S^{N-1}} \frac{1}{(\tau_\omega + \tau_{-\omega})^p} dS(\omega) \right)^{\frac{N}{N+p}} = 2^{\frac{N-1}{N+p}} \left( \int_{S^{N-1}} \tau_\omega^N dS(\omega) \right)^{\frac{p}{N+p}} \left( \int_{S^{N-1}} \frac{1}{\tau_\omega^p} dS(\omega) \right)^{\frac{N}{N+p}},
\]

that is

\[
\int_{S^{N-1}} \frac{1}{b_\omega(x)^p} dS(\omega) \geq \left( \int_{S^{N-1}} \tau_\omega(x)^N dS(\omega) \right)^{-p/N}. \tag{3.8}
\]

The convexity of \( \Omega \) implies \( a_N \int_{S^{N-1}} \tau_\omega(x)^N dS(\omega) = |\Omega| \). Hence the proof is concluded by combining (3.7) and (3.8).

**Remark** We note that inequality (3.4) can be used to obtain Hardy type inequalities for non convex domains as in [HHL], [T].

### 4 On the best constant for \( p = 2 \)

In this section we will prove Theorem B. We recall that \( C_N \) is the best constant of inequality (1.3), in case \( \Omega \) is a ball, defined by:

\[
C_N = \inf_{u \in H^1_0(B)} \frac{\int_B |\nabla u|^2 dx - \frac{1}{4} \int_B u^2 dx}{\int_B u^2 dx}. \tag{4.1}
\]

We first establish

**Lemma 4.1** The infimum in (4.1) remains the same if it is taken over all radially symmetric functions \( u = u(r) \in H^1_0(B) \).

**Proof:** We may assume that \( \Omega \) is the unit ball. Let us denote by \( \hat{C}_N \) the infimum over radial functions. Clearly \( \hat{C}_N \geq C_N \). Suppose now that \( u \in H^1_0(B) \) and let

\[
u(x) = u_0(r) + \sum_{m=1}^{\infty} f_m(\sigma) u_m(r), \quad r = |x|,\]

be its decomposition into spherical harmonics; here \( u_m \) are radially symmetric functions in \( H^1_0(B) \) and \( f_m \) are orthonormal in \( L^2(S^{N-1}) \) eigenfunctions of the Laplace-Beltrami operator on \( \{|x| = 1\} \), with corresponding eigenvalues \( \lambda_m = m(N - 2 + m), m \geq 1 \). It is easily seen that

\[
\int_B |\nabla u|^2 dx = \int_B |\nabla u_0|^2 dx + \sum_{m=1}^{\infty} \int_B (|\nabla u_m|^2 + \frac{\lambda_m}{|x|^2} u_m^2) dx, \tag{4.2}
\]
and hence
\[
\int_B (|\nabla u|^2 - u^2) dx = \int_B \{|\nabla u_0|^2 - \frac{u_0^2}{4(1-|x|^2)} \} dx + \\
\sum_{m=1}^{\infty} \int_B \{|\nabla u_m|^2 + (\frac{c_m}{|x|^2} - \frac{1}{4(1-|x|^2)}) u_m^2 \} dx \\
\ge \tilde{C}_N \int_B u_0^2 + \tilde{C}_N \sum_{m=1}^{\infty} \int_B u_m^2 dx
\]

This implies \(C_N \ge \tilde{C}_N\) and the Lemma is proved.

**Proof of Theorem B:** By the previous Lemma we restrict attention to radially symmetric functions. Let \(u = u(r) \in C^\infty_0(B)\) be a radial function and define \(v\) by
\[
\quad u(r) = r^{\frac{N-1}{2}} (1-r)^{1/2} v(r), \quad r \in (0,1).
\]

Then \(v(0) = v(1) = 0\). We compute
\[
\frac{1}{Na_N} \int_B |\nabla u|^2 dx = \int_0^1 (u')^2 r^{N-1} dr \\
= \int_0^1 (1-r) \left( -\frac{(N-1)v}{2r} - \frac{v}{2(1-r)} + v' \right)^2 dr
\]

Using integration by parts for the terms involving \(vv' = (v^2)'/2\) we conclude after some simple calculations that
\[
\frac{1}{Na_N} \left( \int_B |\nabla u|^2 dx - \frac{1}{4} \int_B u^2 dx \right) = \int_0^1 (1-r)(v')^2 dr + \frac{(N-1)(N-3)}{4} \int_0^1 \frac{1-r}{r^2} v^2 dr \\
\ge \int_0^1 (1-r)(v')^2 dr + \frac{(N-1)(N-3)}{4} \int_0^1 (1-r) v^2 dr.
\]

But (cf [BV, Section 4])
\[
\inf_{v(0) = v(1) = 0} \frac{\int_0^1 (1-r)(v')^2 dr}{\int_0^1 v^2 dr} = \inf_{v(0) = v(1) = 0} \frac{\int_0^1 r(v')^2 dr}{\int_0^1 rv^2 dr} = \mu_2,
\]

and estimate (1.14) of Theorem B follows.

To prove that \(C_3 = \mu_2\), let us define
\[
\quad u_\epsilon(r) = r^{-1}(1-r)^{\frac{1}{2} + \epsilon} w(1-r), \quad r \in (0,1),
\]

where \(\epsilon > 0\) and \(w(|x|)\) is the first eigenfunction of the Dirichlet Laplacian for the unit disk in \(\mathbb{R}^2\). Then
\[
\quad u'_\epsilon(r) = -r^{-1}(1-r)^{\frac{1}{2} + \epsilon} \left\{ \frac{w}{r} + (\frac{1}{2} + \epsilon) \frac{w}{1-r} + w' \right\}
\]

and hence \(u_\epsilon \in H^1_0(B)\) and
\[
\int_0^1 (u'_\epsilon)^2 r^{N-1} dr = \int_0^1 (1-r)^{1+2\epsilon} \left\{ \frac{w^2}{r^2} + (\frac{1}{2} + \epsilon)^2 \frac{w^2}{(1-r)^2} + (w')^2 + (1+2\epsilon) \frac{w^2}{r(1-r)} + \frac{2ww'}{r} + \frac{(1+2\epsilon)ww'}{1-r} \right\} dr
\]

(where \(w = w(1-r)\))
To handle the terms containing $w w'$ we integrate by parts: the boundary terms are equal to zero and making the change of variables $s = 1 - r$ we eventually obtain
\[
\int_0^1 \left( (u'_r(r))^2 - \frac{1}{4} \frac{u^2_r(r)}{(1-r)^2} \right) r^2 dr = \int_0^1 s^{1+2\epsilon} \left( (w'(s))^2 - \epsilon^2 \frac{w^2(s)}{s^2} \right) ds.
\]
Now, there holds
\[
\epsilon^2 \int_0^1 s^{-1+2\epsilon} w^2 ds \to 0, \quad \text{as } \epsilon \to 0,
\]
hence
\[
\lim_{\epsilon \to 0} \frac{\int_B \left( |\nabla u_{\epsilon}|^2 - \frac{u^2_{\epsilon}}{4s^2} \right) dx}{\int_B u_{\epsilon}^2 dx} = \lim_{\epsilon \to 0} \frac{\int_0^1 (w')^2 s^{1+2\epsilon} ds}{\int_0^1 w^2 s^{1+2\epsilon} ds} = \frac{\int_0^1 (w')^2 s ds}{\int_0^1 w^2 s ds} = \mu_2.
\]
It follows that $\tilde{C}_3 \leq \mu_2$; in view of (1.14) and Lemma 4.1 we conclude that $C_3 = \mu_2$.

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