Oversampling and aliasing in de Branges spaces arising from Bessel operators

Julio H. Toloza and Alfredo Uribe

1 INMABB, Departamento de Matemática, Universidad Nacional del Sur (UNS) - CONICET, Bahía Blanca, Argentina
2 Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, México D.F.

Abstract

We show that a class of de Branges spaces, generated by means of generalized Fourier transforms associated with perturbed Bessel differential equations, has the properties of oversampling and aliasing.

Keywords: de Branges space, oversampling, aliasing, sampling theory, singular Schrödinger operator

2010 MSC: 46E22, 42C15, 34L40, 94A20

1 Introduction and main results

1.1 A hint at de Branges spaces

An entire function $E$ belongs to the Hermite-Biehler class if it has the property $|E(z)| > |E(\overline{z})|$ for all $z \in \mathbb{C}_+$. Given such a function, let us define

$$K(z, w) := \begin{cases} \frac{E^\#(z)E(\overline{w}) - E(z)E^\#(\overline{w})}{2\pi i(z - \overline{w})}, & z \neq \overline{w}; \\ \frac{E'(z)E(z) - E'(\overline{z})E(\overline{z})}{2\pi i}, & z = \overline{w}. \end{cases}$$

The de Branges spaces generated by $E$ is the linear set

$$\mathcal{B}(E) := \left\{ F \text{ entire : } \| F \|^2 := \int_{-\infty}^{\infty} \left| \frac{F(\lambda)}{E(\lambda)} \right|^2 d\lambda < \infty, \ |F(z)|^2 \leq \| F \|^2 K(z, z) \text{ for all } z \in \mathbb{C} \right\}$$

equipped with the inner product

$$\langle F, G \rangle_{\mathcal{B}} := \left( \int_{-\infty}^{\infty} \frac{\overline{F(\lambda)}G(\lambda)}{|E(\lambda)|^2} d\lambda \right)^{1/2}.$$

$\mathcal{B}(E)$ is a reproducing kernel Hilbert space whose reproducing kernel is precisely (1) [5, Thm. 20]. From now on, we will denote a de Branges space and its associated reproducing kernel as $\mathcal{B}$ and $K_\mathcal{B}(z, w)$, respectively. In passing, we note that there are alternative ways of
defining a de Branges space \([5, 23, 26]\). Also, there are many Hermite-Biehler functions that generate a given de Branges space \([4, \text{Thm. 1}]\).

Let \(S_B : D(S_B) \to \mathcal{B}\) denote the operator defined by

\[
D(S_B) = \{ F \in \mathcal{B} : zF(z) \in \mathcal{B} \}, \quad (S_B F)(z) := zF(z).
\]

It is well known that \(S_B\) is a regular, closed, symmetric operator with deficiency indices \((1, 1)\) \([15, \text{Prop. 4.2 and Lemma 4.7}]\).

Let \(S_{B, \gamma}, \gamma \in [0, \pi)\), denote the canonical selfadjoint extensions of \(S_B\) (viz., selfadjoint restrictions of \(S_B^*\)). Due to the regularity of \(S_B\), the spectra \(\sigma(S_{B, \gamma})\) consist of isolated eigenvalues of multiplicity equal to one, that moreover satisfy

\[
\bigcup_{\gamma \in [0, \pi)} \sigma(S_{B, \gamma}) = \mathbb{R}, \quad \sigma(S_{B, \gamma}) \cap \sigma(S_{B, \gamma'}) = \emptyset, \quad \gamma \neq \gamma'.
\]

It is straightforward to verify that \(K_B(z, w) \in \ker(S_B^* - \pi I)\) for all \(w \in \mathbb{C}\). It follows that \(\{K_B(z, \lambda)\}_{\lambda \in \sigma(S_{B, \gamma})}\) is an orthogonal basis. Hence, the sampling formula

\[
F(z) = \sum_{\lambda \in \sigma(S_{B, \gamma})} \frac{K_B(z, \lambda)}{K_B(\lambda, \lambda)} F(\lambda)
\]

holds true for all \(F \in \mathcal{B}\). The convergence of this series is in the norm, which in turn implies uniform convergence in compact subsets of \(\mathbb{C}\). The Parseval-Plancherel identity implies that any sequence \(\{\delta_\lambda\}_{\lambda \in \sigma(S_{B, \gamma})}\) obeying

\[
\sum_{\lambda \in \sigma(S_{B, \gamma})} \frac{|\delta_\lambda|^2}{K_B(\lambda, \lambda)} < \infty
\]

yields an approximation to \(F\) by means of the formula (2), when the samples \(\{F(\lambda)\}_{\lambda \in \sigma(S_{B, \gamma})}\) are replaced by \(\{F(\lambda) + \delta_\lambda\}_{\lambda \in \sigma(S_{B, \gamma})}\). In other words, (2) is stable under weighted \(\ell_2\)-perturbations.

There is a distinctive structural property of de Branges spaces related to the subject of this paper. For the class of spaces discussed here, this property can be stated as follows: Assume \(B_1, B_2\) and \(B_3\) are de Branges spaces such that \(B_1\) and \(B_2\) are both isometrically contained in \(B_3\). Then either \(B_1 \subset B_2\) or \(B_2 \subset B_1\). A more general form of this assertion is in the classical book by de Branges \([5, \text{Thm. 35}]\).

**1.2 Main results**

The class of de Branges spaces considered in this work are defined by means of generalized Fourier transforms associated with perturbed Bessel differential equations. Namely,

\[
B_s := \left\{ F(z) = \int_0^s \xi(z, x) \varphi(x) dx : \varphi \in L^2(0, s) \right\}, \quad \|F\|_{B_s}^2 = \int_0^s |\varphi(x)|^2 dx,
\]

where \(\xi(z, x)\) is the real entire solution (with respect to \(z\)) to the eigenvalue problem

\[-\varphi'' + \left(\frac{\nu^2 - 1/4}{x^2} + q\right) \varphi = z\varphi, \quad x \in (0, \infty), \quad \nu \in (0, \infty), \quad z \in \mathbb{C},\]

subject to the boundary condition

\[
\lim_{x \to 0^+} x^{1/2 - \nu/2} \left((\nu + 1/2)\varphi(x) - x\varphi'(x)\right) = 0
\]
when \( \nu \in (0, 1) \). A Hermite-Biehler function that generates \( \mathcal{B}_s \) is
\[
E_s(z) = \xi(z,s) + i\xi'(z,s),
\]
where \( \xi' \) denotes derivative with respect to \( x \). In this framework, our main results can be summarized as follows:

**Theorem** (oversampling). Fix \( \nu, b \in (0, \infty) \), \( a \in (0,b) \) and \( \gamma \in [0, \pi) \). Assume that \( q \) is a real-valued function belonging to \( A \mathcal{C}_{loc}(0,b) \) such that \( xq(x) \in L^r(0,b) \) for some \( r \in (2, \infty] \). Given \( \epsilon = \{ \epsilon_n \} \in \ell_\infty(\nu) \) and \( F \in \mathcal{B}_a \), define
\[
F_\epsilon(z) = \sum_{\lambda_n \in \sigma(S_{b,\gamma})} \frac{J_{ab}(z,\lambda_n)}{K_b(\lambda_n,\lambda_n)} (F(\lambda_n) + \epsilon_n),
\]
where \( \ell_\infty(\nu) \) and \( J_{ab}(z,w) \) are defined by (35) and (42) respectively. Then, for every compact set \( \mathbb{K} \subset \mathbb{C} \), there exists \( C(\mathbb{K}) > 0 \) such that
\[
|F(z) - F_\epsilon(z)| \leq C(\mathbb{K}) \| \epsilon \|_{\ell_\infty(\nu)}, \quad z \in \mathbb{K},
\]
uniformly for all \( F \in \mathcal{B}_a \).

**Theorem** (aliasing). Suppose the same hypotheses of the previous theorem, except that \( \gamma \in (0, \pi) \). For every \( F \in \mathcal{B}_b \), define
\[
\tilde{F}(z) = \sum_{\lambda_n \in \sigma(S_{a,\gamma})} \frac{K_a(z,\lambda_n)}{K_a(\lambda_n,\lambda_n)} F(\lambda_n).
\]
Then, for each compact set \( \mathbb{K} \subset \mathbb{C} \), there exists \( D(\mathbb{K}) > 0 \) such that
\[
|F(z) - \tilde{F}(z)| \leq D(\mathbb{K}) \| (I - P_{ab})F \|_{\mathcal{B}_b}, \quad z \in \mathbb{K},
\]
where \( P_{ab} : \mathcal{B}_b \to \mathcal{B}_a \) is the orthogonal projector onto \( \mathcal{B}_a \).

It is known that \( \mathcal{B}_s = \mathcal{B}_{\nu,s} \) setwise, where \( \mathcal{B}_{\nu,s} \) is the de Branges space associated with \( q \equiv 0 \) (and the same value of \( \nu \)), also under the hypothesis \( xq(x) \in L^r(0,s) \) with \( r \in (2, \infty] \) [25, Thm. 4.2]. Since it is natural to consider sampling formulas as regular for the case \( q \equiv 0 \) and irregular otherwise, we may state that our theorems above provide estimates for oversampling and aliasing error on a (relatively restricted) set of irregular sampling formulas for regularized Hankel transform of functions with compact support in \( \mathbb{R}_+ \). The restriction \( r > 2 \) is a technical limitation due to the perturbation methods involved in both [25] and the present paper; we believe that the results exposed here should hold under the weaker assumption \( xq(x) \in L^1(0,s) \).

### 1.3 A bit of history

The notions of oversampling and aliasing stem from the theory of Paley-Wiener spaces \([22,30]\), that is, the spaces of Fourier transform of functions with given compact support centered at zero,
\[
\mathcal{P}\mathcal{W}_a := \left\{ F(z) = \int_{-a}^{a} e^{-ixz} \phi(x) dx : \phi \in L^2(-a,a) \right\}.
\]
The linear set \( \mathcal{P}\mathcal{W}_a \), equipped with the norm \( \| F \| = \| \phi \| \), is a de Branges space generated by the Hermite-Biehler function \( E_a(z) = \exp(-iza) \). In this setting, the sampling formula (2) is known as the Whittaker-Shannon-Kotelnikov theorem, and takes the form
\[
F(z) = \sum_{n \in \mathbb{Z}} \mathcal{G}_a \left( z, \frac{n\pi}{a} \right) F \left( \frac{n\pi}{a} \right), \quad \mathcal{G}_a (z, w) := \frac{\sin \left[ a(z - \overline{w}) \right]}{a(z - \overline{w})},
\]
The function $G_a(z,w)$ is referred to as the sampling kernel, while the separation between sampling points $\pi/a$ is known as the Nyquist rate.

As shown in [22, Thm. 7.2.5], every $F \in \mathcal{PW}_a \subset \mathcal{PW}_b$ ($a < b$) admits the representation

$$F(z) = \sum_{n \in \mathbb{Z}} G_{ab}(z, n\pi/b) F \left(\frac{n\pi}{b}\right),$$

with the modified sampling kernel

$$G_{ab}(z,w) := \frac{2}{b-a} \frac{\cos((z-w)a) - \cos((z-w)b)}{(z-w)^2}.$$  \hspace{1cm} (5)

While the convergence of the sampling formula (3) is unaffected by $\ell_2$-perturbations of the samples $F\left(\frac{n\pi}{a}\right)$, the oversampling formula (4) is more stable in the sense that it converges under $\ell_\infty$-perturbations of the samples. That is, if the sequence $\{\delta_n\}_{n \in \mathbb{Z}}$ is bounded and one defines

$$F_\delta(z) := \sum_{n \in \mathbb{Z}} \left[F \left(\frac{n\pi}{b}\right) + \delta_n\right] G_{ab}(z, n\pi/b),$$

then $|F_\delta(z) - F(z)|$ is uniformly bounded in compact subsets of $C$ and, moreover, uniformly bounded on the real line. In other words, a more stable sampling formula is obtained at the expense of collecting samples at a higher Nyquist rate.

Aliasing, on the other hand, approximates a function $F \in \mathcal{PW}_b \setminus \mathcal{PW}_a$ by another one formally constructed using the sampling formula (3), namely,

$$\tilde{F}(z) = \sum_{n \in \mathbb{Z}} F \left(\frac{n\pi}{a}\right) G_a \left(z, \frac{n\pi}{a}\right).$$

As shown in [22, Thm. 7.2.9], the series in (6) is indeed convergent and, moreover, $|\tilde{F}(z) - F(z)|$ is uniformly bounded in compact subsets of $C$. Formula (6) yields in fact an approximation not only for functions in $\mathcal{PW}_b \setminus \mathcal{PW}_a$, but for the Fourier transform of elements in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Generalizations of sampling formula (3) have been a subject of research for quite some time within the theory of reproducing kernel Hilbert spaces [6, 11, 13, 14]; a classical result is the Kramer’s sampling theorem. The particular case of reproducing kernel Hilbert spaces related to the Bessel-Hankel transform ($q \equiv 0$ in the context of this paper) has been studied by Higgins in [12]. Sampling theorems associated with Sturm-Liouville problems (hence somewhat related to the methods involved in this paper) have been discussed in [29, 31, 32]. See also [33].

Analysis of error due to noisy samples and aliasing in Paley-Wiener spaces goes back at least to [21]. More recent literature on the subject is, for instance, [1–3, 7, 8, 13, 16, 28]. Oversampling in shift-invariant spaces is considered in [10].

To the best of our knowledge, oversampling and aliasing on de Branges spaces besides the Paley-Wiener class have not been discussed until recently, where this subject has been touched upon for de Branges spaces associated with regular Schrödinger operators using perturbative methods [27]. In this paper we extend the results of [27] to the larger class of de Branges spaces characterized by [25, Thm. 4.2] (see Theorem 2.7 below).

### 1.4 Organization of this paper

In Section 2 we summarize the necessary results concerning the perturbation theory on perturbed Bessel operators; the main source is (part of) the work by Kostenko, Sakhnovich and Teschl on a scalar singular Weyl-Titchmarsh theory [17, 18]. Section 3 is devoted to oversampling, while the results concerning aliasing are the subject of Section 4. Some necessary but tedious computations are presented in the Appendix.
2 de Branges spaces arising from Bessel operators

2.1 The unperturbed problem

Let us consider the differential expression
\[ \tau_\nu := -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2}, \quad x \in (0, s], \quad \nu \in [0, \infty), \]
with \( s \in (0, \infty) \). It is well-known that \( \tau_\nu \) is regular at \( x = s \), whereas at \( x = 0 \) it is in the limit point case if \( \nu \geq 1 \) or in the limit circle case if \( \nu \in [0,1) \). The eigenvalue problem \( \tau_\nu \varphi = z \varphi \) (\( z \in \mathbb{C} \)) has linearly independent solutions
\[ \begin{align*}
\xi_\nu(z,x) &= z^{-\nu/2} \sqrt{\frac{\pi x}{2}} J_\nu(\sqrt{zx}), \\
\theta_\nu(z,x) &= z^{\nu/2} \sqrt{\frac{\pi x}{2}} \left\{ \frac{1}{\sqrt{\pi}} \log(z) J_\nu(\sqrt{zx}) - Y_\nu(\sqrt{zx}) \right\}, \quad \nu \in \mathbb{R}_+ \setminus \mathbb{N}_0,
\end{align*} \]
where \( J_\nu \) and \( Y_\nu \) are the Bessel and Neumann functions; \( \sqrt{\cdot} \) denotes the main branch of the squared root. Both solutions are real entire functions with respect to \( z \), with Wronskian
\[ W_x(\theta_\nu(z), \xi_\nu(z)) := \theta_\nu(z,x) \xi_\nu'(z,x) - \theta_\nu'(z,x) \xi_\nu(z,x) \equiv 1. \]
Moreover, \( \xi_\nu(z, \cdot) \) is square-integrable and satisfies the boundary condition
\[ \lim_{x \to 0^+} x^{\nu-1/2} ((\nu + 1/2) \varphi(x) - x \varphi'(x)) = 0 \quad (9) \]
when \( \nu \in [0,1) \).

Remark 2.1. In order to simplify the discussion, in what follows we shall assume \( \nu > 0 \). This is because the case \( \nu = 0 \) entails the occurrence of logarithmic expressions that would require a somewhat clumsier, separated analysis. In our opinion, this extra workload would not add anything substantial to our results.

As shown in [17, Lemmas A.1 and A.2],
\[ |\xi_\nu(z,x)| \leq C \left( \frac{x}{1 + \sqrt{|z|x}} \right)^{\nu+1/2} e^{\text{Im}(\sqrt{z})|x|}, \]
\[ |\xi_\nu'(z,x)| \leq C \left( \frac{x}{1 + \sqrt{|z|x}} \right)^{\nu-1/2} e^{\text{Im}(\sqrt{z})|x|}. \]

Let \( H_{\nu,s} \) denote the closure of the minimal operator defined by \( \tau_\nu \), plus boundary condition (9) whenever \( \nu \in [0,1) \), on the interval \((0, s]\). This operator is regular, symmetric and has deficiency indices \((1,1)\). As a consequence, \( \xi_\nu(z, \cdot) \in \ker(H_{\nu,s}^*-zI) \) for all \( z \in \mathbb{C} \).

Let \( H_{\nu,s,\gamma} \) denote the selfadjoint extension of \( H_{\nu,s} \) associated with the boundary condition \( \varphi(s) \cos(\gamma) + \varphi'(s) \sin(\gamma) = 0, \gamma \in [0,\pi) \). For \( \gamma = 0 \) (Dirichlet boundary condition at \( x = s \)), the spectrum of \( H_{\nu,s,0} \) is given by the zeros \( \{j_{\nu,n}\}_{n=1}^\infty \) of \( J_\nu \), namely,
\[ \sigma(H_{\nu,s,0}) = \left\{ \left( \frac{j_{\nu,n}}{s} \right)^2 \right\}_{n \in \mathbb{N}}. \]
For $\gamma \in (0, \pi)$, the spectrum of $\sigma(H_{\nu,s,\gamma})$ is given by the zeros of the real entire function $\xi'_\nu(z, s) + \xi_\nu(z, s) \cot \gamma$. The lowest zero of this function, denoted $\lambda_{\nu,0}^\gamma$, has the same sign as $\nu + 1/2 + s \cot \gamma$. All the other zeros are positive, thus we can write

$$\sigma(H_{\nu,s,\gamma}) = \left\{ \left( \frac{j_{\nu,n}^\gamma}{s} \right)^2 \right\}_{n \in \mathbb{N}} \cup \{ \lambda_{\nu,0}^\gamma \},$$

(12)

where $\{ j_{\nu,n}^\gamma \}_{n=1}^\infty$ are the positive zeros of

$$w j_{\nu+1}(w) - \left( \nu + \frac{1}{2} + s \cot \gamma \right) j_\nu(w).$$

(13)

One has the asymptotic formulas (cf. [17, Eqs. 2.11 and 2.12])

$$j_{\nu,n} = \left( n + \frac{2\nu - 1}{4} \right) \pi + O\left( n^{-1} \right), \quad n \to \infty,$$

(14)

$$j_{\nu,n}^\gamma = j_{\nu+1,n} + O(n^{-1}) = \left( n + \frac{2\nu + 1}{4} \right) \pi + O(n^{-1}), \quad n \to \infty.$$  

(15)

It is convenient to recall asymptotic expansion

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left( \cos \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) e^{\left| \text{Im}(z) \right|} O \left( |z|^{-1} \right) \right), \quad |z| \to \infty,$$

(16)

where the error is uniform in sectors of the form $\{ z \in \mathbb{C} : |z| > r \wedge \text{arg}(z) \in [-\pi + \delta, \pi - \delta] \}$ [20, Eq. 10.7.8].

Associated with $H_{\nu,s}$ there is the de Branges space

$$B_{\nu,s} := \left\{ F(z) = \int_0^s \xi_\nu(z, x) \varphi(x) dx : \varphi \in L^2(0, s) \right\}, \quad \| F \|^2_{B_{\nu,s}} = \int_0^s |\varphi(x)|^2 dx;$$

(17)

$E_{\nu,s}(z) := \xi_\nu(z, s) + i \xi'_\nu(z, s)$ is a Hermite-Biehler function that generates $B_{\nu,s}$.

Increasing values of the parameter $\nu$ makes $H_{\nu,s}$ more singular; a precise meaning of this assertion can be found in [18]. This singular character is also reflected in the associated de Branges space, in the sense stated next.

**Theorem 2.2** (cf. [19, Thm. 5.1], [25, Thm. 3.1]). Fix $\nu, s \in (0, \infty)$. There exists a real, zero-free entire function lying in the set

$$\text{assoc}_{N(\nu)} B_{\nu,s} := B_{\nu,s} + z B_{\nu,s} + \cdots + z^{N(\nu)} B_{\nu,s},$$

where $N(\nu) := \min\{ n \in \mathbb{N} : n > \frac{\nu + 1}{2} \};$ no such a function exists within $\text{assoc}_k B_s$ for any $0 \leq k < N(\nu)$.

### 2.2 Adding a perturbation

Given $s \in (0, \infty)$, consider the differential expression

$$\tau := -\frac{d^2}{dx^2} + \frac{\nu^2 - 1/4}{x^2} + q(x), \quad x \in (0, s), \quad \nu \in [0, \infty).$$

(18)

We assume that $q \in L^1_{\text{loc}}(0, s)$ is a real-valued function such that $\tilde{q} \in L^1(0, s)$, where

$$\tilde{q}(x) := \begin{cases} x q(x) & \text{if } \nu > 0, \\ x (1 - \log(x)) q(x) & \text{if } \nu = 0. \end{cases}$$

(19)
As shown in [17, Thm. 2.4], \( \tau \) is regular at \( x = s \) whereas at \( x = 0 \) it is in the limit point case if \( \nu \geq 1 \) or in the limit circle case if \( \nu \in [0, 1) \).

The expression (18), along with the boundary condition (9) when \( \nu \in [0, 1) \), originates a closed, regular, symmetric operator whose deficiency indices are both equal to 1 [25, Sect. 4]. We denote this operator by \( H_s \) (the symbol \( H_{\nu,q,s} \) would be more accurate but also clumsier).

The corresponding one-parameter family of selfadjoint extensions \( H_{s,\gamma} \) \((0 \leq \gamma < \pi)\) is determined by the usual boundary condition at \( x = s \),

\[
D(H_{s,\gamma}) := \{ \varphi \in L^2(0, s) : \varphi, \varphi' \in AC(0, s), \tau \varphi \in L^2(0, s), \varphi(x) \cos \gamma = -\varphi'(x) \sin \gamma \}
\]

(20) plus the boundary condition (9) when \( \nu \in [0, 1) \). The spectrum of \( H_{s,\gamma} \) is purely discrete, of multiplicity one, with at most a finite number of negative eigenvalues [17, Thm. 2.4].

We henceforth assume \( \nu > 0 \) (see Remark 2.1). By [17, Lemma 2.2], the eigenvalue equation \( \tau \varphi = z \varphi \) \((z \in \mathbb{C})\) admits a solution \( \xi(z, x) \), real entire with respect to \( z \), with derivative \( \xi'(z, x) \) also real entire, that satisfy the estimates

\[
|\xi(z, x) - \xi(\nu, z)| \leq C \left( \frac{x}{1 + |\sqrt{z}|} \right)^{\nu + \frac{1}{2}} e^{\im \operatorname{Im}(\sqrt{z}) |x|} \int_0^x \frac{u |q(u)|}{1 + |\sqrt{z}| u} du,
\]

(21)

and

\[
|\xi'(z, x) - \xi'(\nu, z)| \leq C \left( \frac{x}{1 + |\sqrt{z}|} \right)^{-\frac{\nu - 1}{2}} e^{\im \operatorname{Im}(\sqrt{z}) |x|} \int_0^x \frac{u |q(u)|}{1 + |\sqrt{z}| u} du,
\]

(22)

for some constant \( C = C(\nu, q, s) \), with \( \nu, s \in (0, \infty) \), so the bounds above are uniform for \( x \in (0, s) \). Note that (10) and (21) (respectively, (11) and (22)) imply

\[
|\xi(z, x)| \leq C \left( \frac{x}{1 + |\sqrt{z}|} \right)^{\nu + \frac{1}{2}} e^{\im \operatorname{Im}(\sqrt{z}) |x|} \left( 1 + \| \tilde{q} \|_{L^1(0, s)} \right),
\]

(23)

\[
|\xi'(z, x)| \leq C \left( \frac{x}{1 + |\sqrt{z}|} \right)^{-\frac{\nu - 1}{2}} e^{\im \operatorname{Im}(\sqrt{z}) |x|} \left( 1 + \| \tilde{q} \|_{L^1(0, s)} \right),
\]

(24)

for all \( x \in (0, s) \).

**Lemma 2.3** (cf. [25, Lemma 4.1]). Assume \( \tilde{q} \in L^r(0, s) \) with \( r \in [1, \infty) \). Then,

\[
\|\xi(w^2, \cdot) - \xi(\nu, w^2, \cdot)\|_{L^2(0, s)} = e^{\im \operatorname{Im}(w)} \times \begin{cases} o(|w|^{-\nu - \frac{1}{2}}) & \text{if } r = 1, \\
O(|w|^{-\nu - 1}) & \text{if } 1 < r < \infty, \\
O(|w|^{-\nu - \frac{1}{2}} \log |w|) & \text{if } r = \infty,
\end{cases}
\]

as \( w \to \infty \), where \( p \) obeys \( 1/p + 1/r = 1 \).

**Remark 2.4.** The proof of [17, Lemma 2.2] leads to a more refined decomposition, namely,

\[
\xi(z, x) = \xi(\nu, z, x) + \xi_{\nu, 1}(z, x) + \Xi(z, x)
\]

(25)

for all \( z \in \mathbb{C} \) and \( x \in (0, s) \), where

\[
\xi_{\nu, 1}(z, x) = \xi(\nu, z, x) \int_0^x q(y) \theta_\nu(z, y) \xi(\nu, z, y) dy - \theta_\nu(z, x) \int_0^x q(y) (\xi(\nu, z, y))^2 dy
\]

obey\( s \)

\[
|\xi_{\nu, 1}(z, x)| \leq C \left( \frac{x}{1 + |\sqrt{z}|} \right)^{\nu + \frac{1}{2}} e^{\im \operatorname{Im}(\sqrt{z}) |x|} \int_0^x \frac{u |q(u)|}{1 + |\sqrt{z}| u} du,
\]

(27)

and

\[
|\Xi(z, x)| \leq C \left( \frac{x}{1 + |\sqrt{z}|} \right)^{\nu + \frac{1}{2}} e^{\im \operatorname{Im}(\sqrt{z}) |x|} \left( \int_0^x \frac{u |q(u)|}{1 + |\sqrt{z}| u} du \right)^2.
\]

(28)
From (28) one immediately obtains the following estimate:

**Lemma 2.5.** Fix \( \nu, s \in (0, \infty) \). Suppose \( \tilde{q} \in L^r(0, s) \) with \( r \in (2, \infty] \). Then, there exists positive constants \( C \) and \( \delta > 0 \) such that

\[
\left\| \Xi(t^2, \cdot) \right\|_{L^2(0, s)} \leq Ct^{-\nu - \frac{2}{r} - \delta},
\]

for all \( t > 0 \).

From (21) and (22), it follows that \( \xi(z, x) \) also obeys boundary condition (9) whenever \( \nu \in (0, 1) \). This in turn implies \( \xi(z, \cdot) \in \ker(H_* - zI) \). Hence, in view of (20), the spectrum of \( H_{s, \gamma} \) is given by

\[
\sigma(H_{s, \gamma}) = \{ \lambda \in \mathbb{R} : \xi(\lambda, s) \cos \gamma + \xi'(\lambda, s) \sin \gamma = 0 \}.
\]

(29)

In particular, if \( \lambda \in \sigma(H_{s, \gamma}) \), then \( \xi(\lambda, \cdot) \) is the corresponding eigenfunction (up to normalization).

Arrange the elements of \( \sigma(H_{s, \gamma}) \) according to increasing values. Let us denote (and enumerate) them as follows,

\[
\sigma(H_{s, \gamma}) = \begin{cases} \{ \lambda_n^2 \}_{n=1}^{\infty} & \text{if } \gamma = 0, \\ \{ \lambda_n^2 \}_{n=0}^{\infty} & \text{if } \gamma \neq 0 \end{cases}
\]

(the finitely many negative eigenvalues have imaginary values of \( t_n \)). According to [17, Thm. 2.5],

\[
t_n = \begin{cases} c_n^2 \left( n + \frac{2\nu - 1}{4} \right) \frac{\pi}{s} + \epsilon_n + O(n^{-1}) & \text{if } \gamma = 0, \\ c_n^2 \left( n + \frac{2\nu + 1}{4} \right) \frac{\pi}{s} + \epsilon_n' + O(n^{-1}) & \text{if } \gamma \neq 0, \end{cases}
\]

(30)

where

\[
\epsilon_n = O\left( \int_0^s \frac{x |q(x)|}{s + n\pi x} dx \right), \quad \epsilon_n' = O\left( \int_0^s \frac{x |q(x)|}{s + n\pi x} dx \right), \quad n \to \infty.
\]

Assuming \( \tilde{q} \in L^r(0, s) \) with \( r \in [1, \infty] \), it follows that

\[
\int_0^s \frac{x |q(x)|}{s + n\pi x} dx = \begin{cases} o(1) & \text{if } r = 1, \\ O\left(n^{-1+\frac{1}{r}}\right) & \text{if } 1 < r < \infty, \\ O\left(n^{-1} \log(n)\right) & \text{if } r = \infty, \end{cases}
\]

(31)

as \( n \to \infty \).

**Lemma 2.6.** Suppose \( \tilde{q} \in L^1(0, s) \). Then, there exist \( 0 < C < D < \infty \) and \( n_0 \in \mathbb{N} \) such that

\[
Cn^{-\nu - \frac{1}{2}} \leq \left\| \xi(t_n^2, \cdot) \right\|_{L^2(0, s)} \leq Dn^{-\nu - \frac{1}{2}}
\]

for all \( n \geq n_0 \).

**Proof.** We have the obvious inequalities

\[
\left\| \xi(t_n^2) - \xi(t_n^2) - \xi(t_n^2) \right\| \leq \left\| \xi(t_n^2) \right\| \leq \left\| \xi(t_n^2) \right\| + \left\| \xi(t_n^2) - \xi(t_n^2) \right\|
\]

In what follows we assume \( n \) so large that \( t_n^2 \) is positive. Then, on one hand,

\[
\left\| \xi(t_n^2) \right\|^2 = \frac{\pi}{2} t_n^{-2\nu} \int_0^s x (J_\nu(t_nx))^2 dx
\]

\[
= \frac{\pi}{4} s^2 t_n^{-2\nu} \left( (J_\nu(t_n s))^2 - J_{\nu-1}(t_n s) J_{\nu+1}(t_n s) \right)
\]
Combining (16), (30) and (31), one can see that
\[
(J_\nu(t_n s))^2 - J_{\nu - 1}(t_n s)J_{\nu + 1}(t_n s) = \frac{2}{\pi st_n} (1 + o(1)), \quad n \to \infty.
\]
That is,
\[
\|\xi_\nu(t_n^2)\| = \sqrt{\frac{s}{2}} n^{-\nu - \frac{1}{2}}(1 + o(1)), \quad n \to \infty.
\] (32)
On the other hand, Lemma 2.3 implies
\[
\|\xi(t_n^2) - \xi_\nu(t_n^2)\| = o \left( n^{-\nu - \frac{1}{2}} \right).
\] (33)
Finally, the assertion follows after combining (30), (32) and (33).

Associated with the symmetric operator \(H_s\), one has the de Branges space
\[
B_s := \left\{ F(z) = \int_0^s \xi(z, x) \varphi(x) dx : \varphi \in L^2(0, s) \right\}, \quad \|F\|_{B_s}^2 = \int_0^s |\varphi(x)|^2 dx.
\] (34)
The corresponding reproducing kernel is
\[
K_s(z, w) = \langle \xi(z, \cdot), \xi(w, \cdot) \rangle_{L^2(0, s)}.\]

**Theorem 2.7** (cf. [25, Thm. 4.2]). Fix \(\nu, s \in (0, \infty)\). Assume \(\tilde{q} \in L^r(0, s)\) with \(r \in (2, \infty]\).
Then \(B_s = B_{\nu,s}\) setwise. Consequently, \(\text{assoc}_{\nu}\) \(B_s\) contains a zero-free real entire function but no such a function lies in \(\text{assoc}_k B_s\) for any \(0 \leq k < N(\nu)\).

**Remark 2.8.** Let \(\Phi : L^2(0, s) \to B_s\) be the unitary operator defined by the rule \(F(z) = (\Phi \varphi)(z)\). Then \(S_s = \Phi H_s \Phi^{-1}\), where \(S_s\) is the operator of multiplication by the independent variable in \(B_s\). Moreover, the corresponding selfadjoint extensions are analogously related, viz., \(S_{s,\gamma} = \Phi H_{s,\gamma} \Phi^{-1}\). Thus, when referring to unitary invariants (such as the spectrum), we use interchangeably either \(S_{s,\gamma}\) or \(H_{s,\gamma}\) throughout this text.

3 Oversampling

As mentioned in the Introduction, a discussion of the oversampling property in de Branges spaces involves certain weighted \(\ell_p\) spaces. For the class of de Branges spaces treated in this paper, we need the following ones: Given \(1 \leq p \leq \infty\), define
\[
\ell_p(\nu) := \left\{ \{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{C} : \{\beta_n n^{-\nu - \frac{1}{2}}\}_{n \in \mathbb{N}} \in \ell_p \right\},
\] (35)
and
\[
\ell_p(\nu, s, q, \gamma) := \left\{ \{\beta_n\}_{n \in \mathbb{N}} \subset \mathbb{C} : \{\beta_n K_s(\lambda_n, \lambda_n)^{-\frac{1}{2}}\}_{n \in \mathbb{N}} \in \ell_p \right\},
\]
where \(\{\lambda_n\}\) is the (ordered) spectrum of \(H_{s,\gamma}\).

**Corollary 3.1.** Fix \(\nu, s \in (0, \infty)\) and \(\gamma \in [0, \pi)\). Suppose \(\tilde{q} \in L^1(0, s)\). Then, \(\ell_p(\nu, s, q, \gamma) = \ell_p(\nu)\) setwise.

**Proof.** Use Lemma 2.6.
For every $\varphi \in L^2(0, s)$, one clearly has
\[
\varphi(x) = \sum_{\lambda_n \in \sigma(H_{s, \gamma})} \frac{1}{K_s(\lambda_n)} \langle \xi(\lambda_n, \cdot), \varphi(\cdot) \rangle_{L^2(0, s)} \xi(\lambda_n, x), \quad \text{a.e. } x \in (0, s],
\] (36)
where the convergence takes place with respect to the $L^2$-norm. Hence the sampling formula
\[
F(z) = \sum_{\lambda_n \in \sigma(S_{s, \gamma})} K_s(\lambda_n) \frac{F(\lambda_n)}{K_s(\lambda_n)} F(\lambda_n), \quad F \in B_s,
\] (37)
holds true, where the convergence is with respect to the $B_s$-norm, which in turn implies uniform convergence in compact subsets of $C$ [24, Prop. 1]. Since
\[
\|F\|_{B_s}^2 = \sum_{\lambda_n \in \sigma(S_{s, \gamma})} \frac{|F(\lambda_n)|^2}{K_s(\lambda_n)}, \quad F \in B_s,
\] (38)
and taking into account Corollary 3.1, the sequence \{ $F(\lambda_n)$ : $\lambda_n \in \sigma(S_{s, \gamma})$ \} belongs to $\ell_2(\nu)$. Clearly, if one substitutes $F(\lambda_n)$ by $F(\lambda_n) + \delta_n$ with $\delta = \{\delta_n\} \in \ell_2(\nu)$, then (37) produces an approximation $F_\delta \in B_s$ that satisfies
\[
|F_\delta(z) - F(z)| \leq C(\mathbb{K}) \|\delta\|_{\ell_2(\nu)}
\]
uniformly for $z$ in any given compact subset $\mathbb{K} \subset \mathbb{C}$.

Fix $0 < a < b < \infty$. Any $\varphi \in L^2(0, a)$ can be regarded as an element of $L^2(0, b)$ since $\varphi = \varphi \chi(0, a] + 0\chi(a, b)$, where $\chi_E$ denotes the characteristic function of a set $E$. Define
\[
R_{ab}(x) := \chi(0, a](x) + \frac{b - x}{b - a} \chi(a, b](x).
\] (39)
In this way, $\varphi = \varphi R_{ab}$ for all $\varphi \in L^2(0, a)$. Hence, using (36) with $s = b$,
\[
\varphi(x) = \sum_{\lambda_n \in \sigma(H_{b, \gamma})} \frac{1}{K_b(\lambda_n)} \langle \xi(\lambda_n, \cdot), \varphi(\cdot) \rangle_{L^2(0, b)} R_{ab}(x) \xi(\lambda_n, x), \quad \text{a.e. } x \in (0, b],
\] (40)
where the series converges with respect to the norm of $L^2(0, b)$. Consider
\[
F(z) = \langle \xi(z, \cdot), \varphi(\cdot) \rangle_{L^2(0, b)}, \quad z \in \mathbb{C}.
\]
Plugging (40) in the previous equation, we arrive at
\[
F(z) = \sum_{\lambda_n \in \sigma(S_{b, \gamma})} \frac{1}{K_b(\lambda_n)} \langle \xi(z, \cdot), \varphi(\cdot) \rangle_{L^2(0, b)} F(\lambda_n),
\] (41)
which converges uniformly in compact subsets of $\mathbb{C}$.

Define
\[
J_{ab}(z, w) := \langle \xi(z, \cdot), R_{ab}(\cdot) \xi(w, \cdot) \rangle_{L^2(0, b)}.
\] (42)
Note that $J_{ab}(\cdot, w) \in B_s$ for every $w \in \mathbb{C}$, and $J_{ab}(w, z) = J_{ab}(z, w)$. We will prove that the spaces $B_s$ satisfy the following condition:

**sc1** Given $0 < a < b$ and any selfadjoint extension $S_{b, \gamma}$ of $S_b$, the series
\[
\sum_{\lambda_n \in \sigma(S_{b, \gamma})} \frac{|J_{ab}(z, \lambda_n)|}{\sqrt{K_b(\lambda_n)}}
\]
converges uniformly in compact subsets of $\mathbb{C}$. 


Proposition 3.2. Fix $\nu, b \in (0, \infty)$, $a \in (0, b)$, and $\gamma \in [0, \pi)$. Suppose $q \in AC_{loc}(0, b)$ such that $\tilde{q} \in L^r(0, b)$ with $r \in (2, \infty]$. Then, $\mathcal{B}_b$ satisfies (sc1).

Proof. Denote $\sigma(H_{b, \gamma}) = \{t_n^2\}$. In view of Lemma 2.6, it suffices to show that, given a compact subset $\mathbb{K} \subset \mathbb{C}$, there exist $n_0 \in \mathbb{N}$, positive constant $C = C(\mathbb{K}, \nu, \gamma, a, b)$ and $\delta > 0$ such that

$$\left| \langle \xi(\mathbb{K}), R_{ab} \xi(t_n^2) \rangle_{L^2(0, b)} \right| \leq Cn^{-\nu - \frac{1}{2} - \delta},$$

for all $z \in \mathbb{K}$ and $n \geq n_0$. For the purpose of this proof $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(0, b)}$. Resorting to (25), one can write

$$\langle \xi(\mathbb{K}), R \xi(t^2) \rangle = \langle \xi(\mathbb{K}), R \xi_{\nu}(t^2) \rangle + \langle \xi(\mathbb{K}), R \xi_{\nu, 1}(t^2) \rangle + \langle \xi(\mathbb{K}), R \Xi(t^2) \rangle$$

(43)

where we have abbreviated $R := R_{ab}$ and $t := t_n$.

An integration by parts (see A.1) reduces the first term in (43) to

$$\langle \xi(z), R \xi_{\nu}(t^2) \rangle = -\frac{1}{t^2} \frac{1}{b - a} \left( \xi(z, b) \xi_{\nu}(t^2, b) - \xi(z, a) \xi_{\nu}(t^2, a) \right)$$

$$- \frac{1}{t^2} \int_0^b R(x)(q(x) - z) \xi(z, x) \xi_{\nu}(t^2, x) dx$$

$$- \frac{2}{t^2} \int_a^b \frac{1}{b - a} \xi'(z, x) \xi_{\nu}(t^2, x) dx.$$  (44)

Using (10), (23), the fact that $\xi(z, b)$ and $\xi'(z, x)$ are entire with respect to $z$ for every $x \in (0, \infty)$, and noting that $\tilde{q} \in L^1(0, b)$ due to our hypotheses, one can see that (44) implies

$$\left| \langle \xi(\mathbb{K}), R \xi_{\nu}(t_n^2) \rangle \right| \leq C_1 n^{-\nu - \frac{1}{2}}, \quad z \in \mathbb{K}, \quad n \geq n_0,$$

for some $C_1 = C_1(\mathbb{K}, \nu, q, \gamma, a, b) > 0$.

The second term in (43) is computed in A.3 (assuming $q$ locally absolutely continuous), the result being

$$\langle \xi(\mathbb{K}), R \xi_{\nu, 1}(t^2) \rangle = -\frac{1}{t^2} \frac{1}{b - a} \left( \xi(z, b) \xi_{\nu, 1}(t^2, b) - \xi(z, a) \xi_{\nu, 1}(t^2, a) \right)$$

$$- \frac{1}{t^2} \int_0^b R(x)(q(x) - z) \xi(z, x) \xi_{\nu, 1}(t^2, x) dx$$

$$- \frac{1}{t^2} \int_a^b R(x)(q(x) - z) \xi(z, x) \xi_{\nu, 1}(t^2, x) dx$$

$$+ \frac{2}{t^2} \int_a^b \frac{1}{b - a} \xi'(z, x) \xi_{\nu, 1}(t^2, x) dx.$$  (45)

The estimates (23) and (27) imply

$$\left| \xi(z, s) \xi_{\nu, 1}(t^2, s) \right| \leq C_2 t^{-\nu - \frac{1}{2}}$$

for some $C_2 = C_2(\mathbb{K}, \nu, q, s) > 0$; this bound takes care of the first two terms in (45). Also,

$$\int_0^b \left| q(x) \xi(z, x) \xi_{\nu}(t^2, x) \right| dx \leq C_2 \int_0^b \left| xq(x) \right| x' e^{\text{Im}(\sqrt{x})} x^{\nu} (1 + |\sqrt{x}| x^{1 + \frac{1}{2}} (1 + tx)^{\nu + \frac{1}{2}}) dx \leq C_2 t^{-\nu},$$  (46)
where \( C_2 = C_2(\mathbb{K}, \nu, q, b) > 0 \); here we have used (10) and (23), along with fact our hypothesis on \( q \) implies \( \tilde{q} \in L^1(0, b) \). A similar argument shows

\[
\int_0^b \left| (q(x) - z)\xi(z, x)\xi_{\nu, 1}(t^2, x) \right| \, dx \leq C_2 t^{-\nu} \tag{47}
\]

and

\[
\int_a^b \left| \xi(z, x)\xi_{\nu, 1}(t^2, x) \right| \, dx \leq C_2 t^{-\nu - \frac{1}{2}},
\]

where the latter is due to (24) and (27). Therefore, taking into account (30), one has

\[
\left| \left< \xi(\cdot), R \right|_q \xi_{\nu, 1}(t^2) \right| \leq C_2 n^{-\nu - 2}, \quad z \in \mathbb{K}, \quad n \geq n_0,
\]

for some \( C_2 = C_2(\mathbb{K}, \nu, q, \gamma, a, b) > 0 \).

Finally, with the help of Lemma 2.5, the third term in (43) admits the bound

\[
\left| \left< \xi(\cdot), R \Xi(t_n^2) \right> \right| \leq \| \xi(\cdot) \|_{L^2(0, b)} \| \Xi(t_n^2) \|_{L^2(0, b)} \leq C_3 n^{-\nu - 3/2 - \delta},
\]

with \( C_3 = C_3(\mathbb{K}, \nu, q, \gamma, b) > 0 \) and some \( \delta > 0 \) (that depends on \( r > 2 \) from our hypothesis on \( q \)). \( \square \)

**Theorem 3.3.** Fix \( \nu, b \in (0, \infty), a \in (0, b) \) and \( \gamma \in [0, \pi) \). Assume that \( q \) is a real-valued function belonging to \( AC_{\text{loc}}(0, b) \) such that \( \tilde{q} \in L^r(0, b) \) for some \( r \in [2, \infty] \). Given \( \epsilon = \{ \epsilon_n \} \in \ell_\infty(\nu) \) and \( F \in \mathcal{B}_a \), define

\[
F_\epsilon(z) = \sum_{\lambda_n \in \sigma(S_{a, \gamma})} \frac{J_{ab}(z, \lambda_n)}{K_b(\lambda_n, \lambda_n)} (F(\lambda_n) + \epsilon_n).
\]

Then, for every compact set \( \mathbb{K} \subset \mathbb{C} \), there exists \( C(\mathbb{K}) = C(\mathbb{K}, \nu, q, \gamma, a, b) > 0 \) such that

\[
|F(z) - F_\epsilon(z)| \leq C(\mathbb{K}) \| \epsilon \|_{\ell_\infty(\nu)}, \quad z \in \mathbb{K},
\]

uniformly for all \( F \in \mathcal{B}_a \).

**Proof.** It is a straightforward consequence of Proposition 3.2 combined with Corollary 3.1. \( \square \)

**Remark 3.4.** A closer inspection to the estimates on (45) reveals that

\[
C(\mathbb{K}, \nu, q, \gamma, a, b) = \mathcal{O}((b - a)^{-1}), \quad a \to b.
\]

This fact, already known for Paley-Wiener space (cf. (5)), is somewhat expected since the stability of the oversampling formula depends on \( \mathcal{B}_a \) being a proper de Branges subspace of \( \mathcal{B}_b \).

4 Aliasing

A de Branges space \( \mathcal{B}_b \) has the aliasing (or undersampling) property if, given any de Branges subspace \( \mathcal{B}_a \subsetneq \mathcal{B}_b \), there exists a selfadjoint extension \( S_{a, *} \) of \( S_a \) such that the series

\[
\sum_{\lambda_n \in \sigma(S_{a, *})} \frac{K_a(z, \lambda_n)}{K_a(\lambda_n, \lambda_n)} F(\lambda_n),
\]

converges absolutely in compact subsets of \( \mathbb{C} \), for every function \( F \in \mathcal{B}_b \setminus \mathcal{B}_a \).
Suppose $B$ has the aliasing property. Then, for every $F \in B_b \setminus B_a$ one can define
\[
\tilde{F}(z) = \sum_{\lambda_n \in \sigma(B_{a,+})} \frac{K_a(z, \lambda_n)}{K_a(\lambda_n, \lambda_n)} F(\lambda_n).
\]

We expect $\tilde{F}$ be an approximation to $F$ obtained from samples that are more “sparse” than those required for the sampling formula (37). This vaguely worded claim can be made precise for the class of de Branges spaces under consideration. With this purpose in mind, we formulate a suitable condition for aliasing.

\textbf{(sc2)} Given $0 < a < b$, there exists $\gamma_s \in [0, \pi)$ such that the series
\[
\sum_{\lambda_n \in \sigma(H_{a,\gamma_s})} \frac{K_a(z, \lambda_n)}{K_a(\lambda_n, \lambda_n)} \xi(\lambda_n, x)
\]
converges absolutely and uniformly for $(z, x) \in \mathbb{K} \times [0, b]$, where $\mathbb{K}$ is any compact subset of $\mathbb{C}$.

The bulk of this section consists of showing that the de Branges spaces discussed in this work satisfy \textbf{(sc2)}. We start by defining
\[
Q_1(z, x) := \int_0^x q(y) \theta_\nu(z, y) \xi_\nu(z, y) dy, \quad Q_2(z, x) := \int_0^x q(y) (\xi_\nu(z, y))^2 dy.
\]
Clearly (26) becomes
\[
\xi_{\nu,1}(z, x) = \xi_\nu(z, x)Q_1(z, x) - \theta_\nu(z, x)Q_2(z, x).
\tag{48}
\]
Also,
\[
Q_1(t^2, a)\xi_{\nu+1}(t^2, a) - t^{-2}Q_2(t^2, a)\theta_\nu+1(t^2, a) = \int_0^a H_\nu(t^2, a, y)q(y)\xi_\nu(t^2, y) dy,
\]
where
\[
H_\nu(t^2, x, y) := \xi_{\nu+1}(t^2, x)\theta_\nu(t^2, y) - t^{-2}\xi_\nu(t^2, y)\theta_\nu+1(t^2, x).
\]
From (7) and (8), one can verify that
\[
H_\nu(t^2, x, y) = \frac{\pi}{2} t^{-1} \sqrt{xy} (J_{\nu+1}(tx)Y_\nu(ty) - J_\nu(ty)Y_{\nu+1}(tx))
\]

\textbf{Lemma 4.1.} Suppose $\tilde{q} \in L^r(0, a)$ with $r \in (2, \infty)$. There exist $C > 0$ and $\delta > 0$ such that
\[
|Q_1(t^2, a)\xi_{\nu+1}(t^2, a) - t^{-2}Q_2(t^2, a)\theta_\nu+1(t^2, a)| \leq C t^{-\nu^{\frac{3}{2}} - \frac{1}{2} - \delta}
\]
for all $t \geq 1$.

\textbf{Proof.} Resorting to an argument like in the proof of [17, Lemma A.1], one can prove (assuming $t \in \mathbb{R}$)
\[
|H_\nu(t^2, x, y)| \leq Ct^{-2} \left( \left( \frac{tx}{1+tx} \right)^{\nu+\frac{3}{2}} \left( \frac{1+ty}{ty} \right)^{\nu+\frac{1}{2}} + \left( \frac{ty}{1+ty} \right)^{\nu+\frac{3}{2}} \left( \frac{1+tx}{tx} \right)^{\nu+\frac{1}{2}} \right).
\]
Noting that the function $f(x) = x(1 + x)$ ($x \in \mathbb{R}_+$) is increasing and bounded, and recalling (10), it follows that
\[
\int_0^a |H(t^2, a, y)q(y)\xi(t^2, y)| dy \\
\leq C t^{-\nu - \frac{2}{a}} \left( \int_0^a \frac{ty}{1 + ty} |q(y)| dy + \frac{1}{ta} \int_0^a \left( \frac{ty}{1 + ty} \right)^{2^{\nu + 1}} |q(y)| dy \right).
\]
Assuming $t \geq 1$, it reduces to
\[
\int_0^a |H(t^2, a, y)q(y)\xi(t^2, y)| dy \leq C t^{-\nu - \frac{2}{a}} \int_0^a \frac{ty}{1 + ty} |q(y)| dy.
\]
Suppose $r \in (2, \infty)$. Then,
\[
\int_0^a \frac{ty}{1 + ty} |q(y)| dy \leq t \left( \int_0^a (1 + ty)^{-n} dy \right)^{\frac{1}{2}} \| \tilde{q} \|_{L^r(0, a)} \leq Ct^\gamma.
\]
Therefore,
\[
\int_0^a |H(t^2, a, y)q(y)\xi(t^2, y)| dy \leq C t^{-\nu - \frac{2}{a} - \frac{1}{n}}.
\]
The argument for $r = \infty$ is analogous hence omitted. □

**Proposition 4.2.** Given $\nu \in (0, \infty)$ and $0 < a < b < \infty$, (sc2) is satisfied for all $\gamma \in (0, \pi)$ whenever $q \in AC_{loc}(0, b]$ in addition to $\tilde{q} \in L^r(0, b)$ for some $r \in (2, \infty]$.

**Proof.** We show the statement assuming $r \in (2, \infty)$; the proof for the remaining case is similar save for some minor differences. Choose any $\gamma \in (0, \pi)$ and denote $\sigma(H_{a, \gamma}) = \left\{ t_n^2 \right\}_{n=0}^\infty$. Given a compact subset $K \subset \mathbb{C}$, it suffices to show that, for some $\delta > 0$,
\[
\frac{|K_a(z, t_n^2)|}{|K_a(t_n^2, t_n^2)|} \left| \xi(t_n^2, x) \right| \leq C n^{-1 - \delta}
\]
for all $z \in K$, $x \in [0, b]$, and $n$ sufficiently large. By Lemma 2.6, there exists $n_0 \in \mathbb{N}$ such that
\[
|K_a(t_n^2, t_n^2)| \geq C n^{-2\nu - 1}
\]
for $n \geq n_0$; we can assume $n_0$ so large that also $t_n^2 \notin K$ for $n \geq n_0$. Furthermore, in view of (23) and (30),
\[
\left| \xi(t_n^2, x) \right| \leq C n^{-\nu - \frac{1}{2}}
\]
for all $n \geq n_0$ and $x \in [0, b]$. Therefore, it suffices to show that
\[
|K_a(z, t_n^2)| \leq C n^{-\nu - \frac{1}{2} - \delta},
\]
for some $\delta > 0$.

Abbreviate $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(0, a)}$. We have
\[
|K_a(z, t_n^2)| \leq \left| \langle \xi(z), \xi(t_n^2) \rangle \right| + \left| \langle \xi(z), \xi_{\nu, 1}(t_n^2) \rangle \right| + \left| \langle \xi(z), \Xi(t_n^2) \rangle \right|.
\]
From (54),
\[
\langle \xi(z), \xi(t_n^2) \rangle = \xi(z, a)\xi_{\nu+1}(t_n^2, a) \\
+ \frac{1}{t^2} \left( \xi'(z, a) - (\nu + \frac{1}{2})a^{-1}\xi(z, a) \right) \xi_{\nu}(t_n^2, a) \\
- \frac{1}{t^2} \int_0^a (q(x) - z)\xi(z, x)\xi_{\nu}(t_n^2, x) \, dx.
\]

Note that (30) and (31) imply
\[
t_n a = \left( n + \frac{2\nu + 1}{4} \right) \pi + O(n^{-\frac{1}{r}}),
\]
since \( \gamma \neq 0 \) and \( p \in (1, 2) \), where \( 1/r + 1/p = 1 \). This in turn implies
\[
\left| \xi_{\nu}(t_n^2, a) \right| \leq Cn^{-\nu - \frac{1}{r}}, \quad \left| \xi_{\nu+1}(t_n^2, a) \right| \leq Cn^{-\nu - \frac{1}{2} - \frac{1}{p}},
\]
(50)

Also,
\[
\int_0^a \left| (q(x) - z)\xi(z, x)\xi_{\nu}(t_n^2, x) \right| \, dx \leq Cn^{-\nu}
\]
uniformly for \( z \in \mathbb{K} \). Therefore, there exists \( C_1 = C_1(\nu, \gamma, \mathbb{K}, a) > 0 \) such that
\[
\left| \langle \xi(\tau), \xi_{\nu+1}(t_n^2) \rangle \right| \leq C_1 n^{-\nu - \frac{1}{2} - \frac{1}{p}}, \quad z \in \mathbb{K}, \quad n \geq n_0.
\]

We now look at the second term in (49). As computed in A.2 (see (56)),
\[
\langle \xi(\tau), \xi_{\nu+1}(t^2) \rangle = \xi(z, a)Q_1(t^2, a)\xi_{\nu+1}(t^2, a) - \frac{1}{t^2} \xi(z, a)Q_2(t^2, a)\theta_{\nu+1}(t^2, a) \\
- \frac{1}{t^2} \left( (\nu + \frac{1}{2})a^{-1}\xi(z, a) - \xi'(z, a) \right) \xi_{\nu+1}(t^2, a) \\
- \frac{1}{t^2} \int_0^a (q(x) - z)\xi(z, x)\xi_{\nu+1}(t^2, x) \, dx \\
- \frac{1}{t^2} \int_0^a \xi(z, x)q(x)\xi_{\nu}(t^2, x) \, dx.
\]

As a consequence of Lemma 4.1,
\[
\left| \xi(z, a) \right| \left| Q_1(t^2, a)\xi_{\nu+1}(t^2, a) - t^{-2}Q_2(t^2, a)\theta_{\nu+1}(t^2, a) \right| \leq C_2 t^{-\nu - \frac{3}{2} - \frac{1}{p}}, \quad z \in \mathbb{K},
\]
where \( 1/r + 1/p = 1 \). Also,
\[
\left| (\nu + \frac{1}{2})a^{-1}\xi(z, a) - \xi'(z, a) \right| \left| \xi_{\nu+1}(t^2, a) \right| \leq C_2 t^{-\nu - \frac{1}{2}}, \quad z \in \mathbb{K}.
\]

By the same argument that leads to (46) and (47),
\[
\int_0^a \left| (q(x) - z)\xi(z, x)\xi_{\nu+1}(t^2, x) \right| \, dx \leq C_2 t^{-\nu}, \quad z \in \mathbb{K}
\]
and
\[
\int_0^a \left| \xi(z, x)q(x)\xi_{\nu}(t^2, x) \right| \, dx \leq C_2 t^{-\nu}, \quad z \in \mathbb{K}.
\]

Therefore, for all \( z \in \mathbb{C} \) and \( n \geq n_0 \),
\[
\left| \langle \xi(\tau), \xi_{\nu+1}(t_n^2) \rangle \right| \leq C_2 n^{-\nu - \frac{3}{2} - \frac{1}{p}}.
\]
for some $C_2 = C_2(\nu, q, \gamma, \mathcal{K}, a) > 0$.

Finally, the last term in (49) can be bounded as
\[ \left| \left\langle \xi(z), \Xi(t_n^2) \right\rangle \right| \leq \left\| \xi(z) \right\|_{L^2(0,a)} \left\| \Xi(t_n^2) \right\|_{L^2(0,a)} \leq C_3 n^{-\frac{3}{2} - \delta}, \]
for some $C_3 = C_3(\nu, q, \gamma, \mathcal{K}, a) > 0$ and all $n \geq n_0$. The proof is now complete. \qed

The proof of the following assertion is nearly identical to the proof of Lemma 4.2 in [27], hence omitted.

**Lemma 4.3.** Assume that (sc2) is met. Define
\[ \xi_{\text{ext}}^a(z,x) := \sum_{\lambda_n \in \sigma(H_{\text{ext}})} \frac{K_a(z,\lambda_n)}{K_a(\lambda_n,\lambda_n)} \xi(\lambda_n,x), \quad x \in [0,a], \quad z \in \mathbb{C}. \]
Then, for each $z \in \mathbb{C}$,

(i) $\xi_{\text{ext}}^a(z,\cdot)$ is continuous in $[0,b]$,

(ii) $\xi_{\text{ext}}^a(z,x) = \xi(z,x)$ for a.e. $x \in [0,a]$, and

(iii) the function $h_{ab}(z) := \sup_{x \in [a,b]} |\xi_{\text{ext}}^a(z,x) - \xi(z,x)|$ is continuous in $\mathbb{C}$.

Moreover,

(iv) if $F(z) = \langle \xi(z), \psi \rangle_{L^2(0,b)}$ with $\psi \in L^2(0,b)$, then
\[ \left\langle \xi_{\text{ext}}^a(z), \psi \right\rangle_{L^2(0,b)} = \sum_{\lambda_n \in \sigma(S_{\text{ext}})} \frac{K_a(z,\lambda_n)}{K_a(\lambda_n,\lambda_n)} F(\lambda_n), \quad z \in \mathbb{C}. \] (51)

Our main assertion concerning aliasing follows from Proposition 4.2 and Lemma 4.3:

**Theorem 4.4.** Suppose the hypotheses of Proposition 4.2. For every $F \in B_b$, define
\[ \tilde{F}(z) = \sum_{\lambda_n \in \sigma(S_{\text{ext}})} \frac{K_a(z,\lambda_n)}{K_a(\lambda_n,\lambda_n)} F(\lambda_n). \]
Then,
\[ |F(z) - \tilde{F}(z)| \leq h_{ab}(z) \int_a^b |\psi(x)| \, dx, \]
where $\psi \in L^2(0,b)$ satisfies $F(z) = \langle \xi(z), \psi \rangle_{L^2(0,b)}$.

**Remark 4.5.** We clearly have
\[ |F(z) - \tilde{F}(z)| \leq \sqrt{b-a} h_{ab}(z) \left( \int_a^b |\psi(x)|^2 \right)^{\frac{1}{2}} \leq C(\mathbb{K}, a, b) \left\| (I - P_{ab}) F \right\|_{B_b}, \]
for all $z \in \mathbb{K}$, where $\mathbb{K}$ is any compact subset of $\mathbb{C}$ and $P_{ab}$ denotes the orthogonal projector onto $B_a$. 

16
A Auxiliary results

A.1 Computation #1

Let us recall the identities

\[ C_\nu(tx) = t^{-1}x^{-\nu-1} \frac{d}{dx} \left( x^{\nu+1}C_\nu+1(tx) \right), \quad C_\nu+1(tx) = -t^{-1}x^{-\nu} \frac{d}{dx} \left( x^{-\nu}C_\nu(tx) \right), \]

where \( C_\nu \) denotes either \( J_\nu \) or \( Y_\nu \) (see [20, Eq. 10.22.1]). Applied to (7) and (8), they imply

\[ \xi_\nu(t^2, x) = x^{-\nu-\frac{1}{2}} \partial_x \left( x^{\nu+\frac{1}{2}} \xi_{\nu+1}(t^2, x) \right), \quad \xi_{\nu+1}(t^2, x) = -t^{-2}x^{-\nu+\frac{1}{2}} \partial_x \left( x^{-\nu-\frac{1}{2}} \xi_\nu(t^2, x) \right), \]

(52)

and

\[ \theta_\nu(t^2, x) = t^{-2}x^{-\nu-\frac{1}{2}} \partial_x \left( x^{\nu+\frac{1}{2}} \theta_{\nu+1}(t^2, x) \right), \quad \theta_{\nu+1}(t^2, x) = -x^{-\nu+\frac{1}{2}} \partial_x \left( x^{-\nu-\frac{1}{2}} \theta_\nu(t^2, x) \right). \]

Fix \( \epsilon > 0 \) small. A double integration by parts involving (52) yields

\[ \int_{\epsilon}^{a} \xi(z, x) \xi_\nu(t^2, x) dx = \xi(z, a) \xi_{\nu+1}(t^2, a) \left[ \frac{a}{\epsilon} + \frac{1}{\ell^2} \left( \xi'(z, a) - (\nu + \frac{1}{2}) a^{-1} \xi(z, a) \right) \xi_\nu(t^2, a) \right] - \frac{1}{\ell^2} \int_{\epsilon}^{a} x^{-\nu-\frac{1}{2}} \partial_x \left( x^{2\nu+1} \partial_x \left( x^{-\nu-\frac{1}{2}} \xi(z, a) \right) \right) \xi_\nu(t^2, x) dx. \]

Recalling (9), (10) and (27), taking the limit \( \epsilon \to 0 \), and using that \( (\tau - z) \xi(z, \cdot) = 0 \), one obtains

\[ \left< \xi(z), \xi_\nu(t^2) \right>_{L^2(0, a)} = \xi(z, a) \xi_{\nu+1}(t^2, a) \]

\[ + \frac{1}{\ell^2} \left( \xi'(z, a) - (\nu + \frac{1}{2}) a^{-1} \xi(z, a) \right) \xi_\nu(t^2, a) \]

\[ - \frac{1}{\ell^2} \int_{0}^{a} (q(x) - z) \xi(z, x) \xi_\nu(t^2, x) dx. \]

(54)

Similarly,

\[ \int_{a}^{b} \frac{b - x}{b - a} \xi(z, x) \xi_\nu(t^2, x) dx \]

\[ = - \xi(z, a) \xi_{\nu+1}(t^2, a) - \frac{1}{\ell^2} \left( \xi'(z, a) - (\nu + \frac{1}{2}) a^{-1} \xi(z, a) \right) \xi_\nu(t^2, a) \]

\[ + \frac{\xi(z, a)}{\ell^2(b - a)} \xi_\nu(t^2, a) - \frac{\xi(z, b)}{\ell^2(b - a)} \xi_\nu(t^2, b) \]

\[ - \frac{1}{\ell^2} \int_{a}^{b} \frac{b - x}{b - a} (q(x) - z) \xi(z, x) \xi_\nu(t^2, x) dx \]

\[ - \frac{2}{\ell^2} \int_{a}^{b} \frac{1}{b - a} \xi'(z, x) \xi_\nu(t^2, x) dx. \]

Thus,

\[ \left< \xi(z), R_{ab} \xi_\nu(t^2) \right>_{L^2(0, b)} = - \frac{1}{\ell^2} \frac{1}{b - a} \left( \xi(z, b) \xi_\nu(t^2, b) - \xi(z, a) \xi_\nu(t^2, a) \right) \]

\[ - \frac{1}{\ell^2} \int_{0}^{b} R(x)(q(x) - z) \xi(z, x) \xi_\nu(t^2, x) dx \]

\[ - \frac{2}{\ell^2} \int_{a}^{b} \frac{1}{b - a} \xi'(z, x) \xi_\nu(t^2, x) dx. \]

(55)
A.2 Computation #2

Fix some arbitrarily small $\epsilon > 0$. Using (52) and integration by parts, one obtains

$$\int_{\epsilon}^{a} \xi(z, x) Q_1(t^2, x) \xi_{\nu}(t^2, x) dx$$

$$= \xi(z, x) Q_1(t^2, x) \xi_{\nu+1}(t^2, x) \bigg|_{\epsilon}^{a}$$

$$- \frac{1}{t^2} \left( (\nu + \frac{1}{2}) x^{-1} \xi(z, x) - \xi'(z, x) \right) Q_1(t^2, x) \xi_{\nu}(t^2, x) \bigg|_{\epsilon}^{a}$$

$$+ \frac{1}{t^2} \xi(z, x) q(x) \theta_{\nu}(t^2, x) (\xi_{\nu}(t^2, x))^2 \bigg|_{\epsilon}^{a}$$

$$- \frac{1}{t^2} \int_{\epsilon}^{a} (q(x) - z) \xi(z, x) Q_1(t^2, x) \xi_{\nu}(t^2, x) dx$$

$$- 2 \int_{\epsilon}^{a} \xi'(z, x) q(x) \theta_{\nu}(t^2, x) (\xi_{\nu}(t^2, x))^2 dx$$

$$- \frac{1}{t^2} \int_{\epsilon}^{a} \xi(z, x) Q_1^\prime(t^2, x) \xi_{\nu}(t^2, x) dx.$$

Note that we require $q$ to be locally absolutely continuous in order to make sense of $Q_i^\prime(z, x)$ ($i = 1, 2$).

Similarly but now using (53),

$$\int_{\epsilon}^{a} \xi(z, x) Q_2(t^2, x) \theta_{\nu}(t^2, x) dx$$

$$= \frac{1}{t^2} \xi(z, x) Q_2(t^2, x) \theta_{\nu+1}(t^2, x) \bigg|_{\epsilon}^{a}$$

$$- \frac{1}{t^2} \left( (\nu + \frac{1}{2}) x^{-1} \xi(z, x) - \xi'(z, x) \right) Q_2(t^2, x) \theta_{\nu}(t^2, x) \bigg|_{\epsilon}^{a}$$

$$+ \frac{1}{t^2} \xi(z, x) q(x) \theta_{\nu}(t^2, x) (\xi_{\nu}(t^2, x))^2 \bigg|_{\epsilon}^{a}$$

$$- \frac{1}{t^2} \int_{\epsilon}^{a} (q(x) - z) \xi(z, x) Q_2(t^2, x) \theta_{\nu}(t^2, x) dx$$

$$- 2 \int_{\epsilon}^{a} \xi'(z, x) q(x) \theta_{\nu}(t^2, x) (\xi_{\nu}(t^2, x))^2 dx$$

$$- \frac{1}{t^2} \int_{\epsilon}^{a} \xi(z, x) Q_2^\prime(t^2, x) \theta_{\nu}(t^2, x) dx.$$

Since $W(\xi_{\nu}(z), \theta_{\nu}(z)) = 1$,

$$Q_i^\prime(t^2, x) \xi_{\nu}(t^2, x) - Q_i^\prime(t^2, x) \theta_{\nu}(t^2, x) = q(x) \xi_{\nu}(t^2, x).$$

Hence, recalling (48),

$$\int_{\epsilon}^{a} \xi(z, x) \xi_{\nu, 1}(t^2, x) dx = \xi(z, x) Q_1(t^2, x) \xi_{\nu+1}(t^2, x) \bigg|_{\epsilon}^{a}$$

$$- \frac{1}{t^2} \left( (\nu + \frac{1}{2}) x^{-1} \xi(z, x) - \xi'(z, x) \right) \xi_{\nu+1}(t^2, x) \bigg|_{\epsilon}^{a}$$

$$- \frac{1}{t^2} \int_{\epsilon}^{a} (q(x) - z) \xi(z, x) \xi_{\nu, 1}(t^2, x) dx$$

$$- \frac{1}{t^2} \int_{\epsilon}^{a} \xi(z, x) q(x) \xi_{\nu}(t^2, x) dx.$$
where

\[ f(0) = \nu, \quad \text{for every } z \in \mathbb{C} \text{ and } t > 0. \]

Thus, (23) in conjunction with

\[
\theta_{\nu}(t^2, x) = \frac{\Gamma(\nu + 1)2^{\nu - \frac{1}{2}}}{\nu! \pi} \left\{ \begin{aligned}
g_{\nu}((tx)^2), & \quad \nu \in \mathbb{R}_+ \setminus \mathbb{N}, \\
g_{\nu}((tx)^2) - \frac{\nu(tx)\log(tx)}{2(\nu + 1)^2}, & \quad \nu \in \mathbb{N},
\end{aligned} \right.
\]

where \( g_{\nu}(0) = g_{\nu}(0) = 0 \) [17, Eq. 2.7], imply

\[
\lim_{x \to 0^+} \xi(z, x)Q_2(t^2, x)\theta_{\nu+1}(t^2, x) = 0.
\]

Moreover, (9) and (27) yield

\[
\lim_{x \to 0^+} \left( (\nu + \frac{1}{2})x^{-1}\xi(z, x) - \xi'(z, x) \right) \xi_{\nu, 1}(t^2, x) = 0.
\]

Therefore,

\[
\left\langle \xi(z, x), \xi_{\nu, 1}(t^2) \right\rangle_{L^2(0,a)} = \xi(z, a)Q_1(t^2, a)\xi_{\nu, 1}(t^2, a) - \frac{1}{t^2} \xi(z, a)Q_2(t^2, a)\theta_{\nu+1}(t^2, a)
\]

\[
- \frac{1}{t^2} \left( (\nu + \frac{1}{2})a^{-1}\xi(z, a) - \xi'(z, a) \right) \xi_{\nu, 1}(t^2, a)
\]

\[
- \frac{1}{t^2} \int_0^a (q(x) - z)\xi(z, x)\xi_{\nu, 1}(t^2, x)dx
\]

\[
- \frac{1}{t^2} \int_0^a \xi(z, x)q(x)\xi_{\nu}(t^2, x)dx.
\]

\[ (56) \]

**A.3 Computation #3**

Much in the same vein as before, one obtains

\[
\int_a^b \frac{b-x}{b-a} \xi(z, x)Q_1(t^2, x)\xi_{\nu}(t^2, x)dx
\]

\[
= -\xi(z, a)Q_1(t^2, a)\xi_{\nu, 1}(t^2, a)
\]

\[
+ \frac{1}{t^2} \left( (\nu + \frac{1}{2})a^{-1}\xi(z, a) - \xi'(z, a) \right) Q_1(t^2, a)\xi_{\nu}(t^2, a)
\]

\[
- \frac{1}{t^2} \frac{1}{b-a} \xi(z, x)Q_1(t^2, x)\xi_{\nu}(t^2, x) \bigg|_a^b
\]

\[
- \frac{1}{t^2} \xi(z, a)q(a)\theta_{\nu}(t^2, a)(\xi_{\nu}(t^2, a))^2
\]

\[
- \frac{1}{t^2} \int_a^b \frac{b-x}{b-a} (q(x) - z)\xi(z, x)Q_1(t^2, x)\xi_{\nu}(t^2, x)dx
\]

\[
- \frac{2}{t^2} \int_a^b \frac{b-x}{b-a} \xi'(z, x)q(x)\theta_{\nu}(t^2, x)(\xi_{\nu}(t^2, x))^2dx
\]

\[
- \frac{1}{t^2} \int_a^b \frac{1}{b-a} \xi(z, x)Q_1^\nu(t^2, x)\xi_{\nu}(t^2, x)dx
\]

\[
+ \frac{2}{t^2} \int_a^b \frac{1}{b-a} \xi'(z, x)Q_1(t^2, x)\xi_{\nu}(t^2, x)dx.
\]

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Also,

\[
\int_a^b \frac{b - x}{b - a} \xi(z, x) Q_2(t^2, x) \theta_{\nu}(t^2, x) dx \\
= - \frac{1}{t^2} \xi(z, a) Q_2(t^2, a) \theta_{\nu+1}(t^2, a) \\
+ \frac{1}{t^2} \left((\nu + \frac{1}{2})a^{-1} \xi(z, a) - \xi'(z, a)\right) Q_2(t^2, a) \theta_{\nu}(t^2, a) \\
- \frac{1}{t^2} \frac{1}{b - a} \xi(z, x) Q_2(t^2, x) \theta_{\nu}(t^2, x) \bigg|_a^b \\
- \frac{1}{t^2} \xi(z, a) q(a) \theta_{\nu}(t^2, a) (\xi_{\nu}(t^2, a))^2 \\
- \frac{1}{t^2} \int_a^b \frac{b - x}{b - a} (q(x) - z) \xi(z, x) Q_2(t^2, x) \theta_{\nu}(t^2, x) dx \\
- \frac{2}{t^2} \int_a^b \frac{b - x}{b - a} \xi'(z, x) q(x) \theta_{\nu}(t^2, x) (\xi_{\nu}(t^2, x))^2 dx \\
- \frac{1}{t^2} \int_a^b \frac{b - x}{b - a} \xi(z, x) Q''_2(t^2, x) \theta_{\nu}(t^2, x) dx \\
+ \frac{2}{t^2} \int_a^b \frac{1}{b - a} \xi(z, x) q(x) \theta_{\nu}(t^2, x) (\xi_{\nu}(t^2, x))^2 dx \\
+ \frac{2}{t^2} \int_a^b \frac{1}{b - a} \xi'(z, x) Q_2(t^2, x) \theta_{\nu}(t^2, x) dx.
\]
Therefore, adding (56) to (57),

\[
\left\langle \xi(z, x), R_{ab} \xi_{\nu, 1}(t^2) \right\rangle_{L^2(0, b)} = -\frac{1}{t^2} \frac{1}{b - a} \left( \xi(z, b) \xi_{\nu, 1}(t^2, b) - \xi(z, a) \xi_{\nu, 1}(t^2, a) \right) \\
- \frac{1}{t^2} \int_0^b R(x)(q(x) - z)\xi(z, x)\xi_{\nu, 1}(t^2, x)dx \\
- \frac{1}{t^2} \int_0^b R(x)q(x)\xi(z, x)\xi_{\nu, 1}(t^2, x)dx \\
+ \frac{2}{t^2} \int_a^b \frac{1}{b - a} \xi'(z, x)\xi_{\nu, 1}(t^2, x)dx.
\]

\[\text{(58)}\]

Acknowledgments

Part of this work was done while J. H. T. visited IIMAS–UNAM (Mexico). He deeply thanks them for their kind hospitality.

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