On the spin-statistics theorem and the spin-half bosons with mass dimension three-half

A. R. Aguirre¹,*, B. A. Couto e Silva²,† and B. L. Sánchez-Vega²‡

¹Instituto de Física e Química, Universidade Federal de Itajubá, Av. BPS 1303, Itajubá – MG, CEP 37500-903, Brazil, and
²Departamento de Física, UFMG, Belo Horizonte, MG 31270-901, Brazil.

Abstract

This work is dedicated to a deep understanding of the relationship between the spin-statistics theorem and a very recent theoretical proposal of new particles residing in \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space, the so called spin-half bosons. We delve into the first principles of the quantum field theory, following Weinberg’s approach to causal fields, to bring to light the main characteristics carried by all spin-half particles in order to understand the reason why this proposed bosonic quantum field apparently escapes the very well established spin-statistics theorem. As we will see, the quantum fields associated with these half-spin bosons cannot be used to represent elementary particles with spin 1/2 in a quantum field theory that satisfies Lorentz invariance of the S-matrix.

*Electronic address: alexis.roaaguirre@unifei.edu.br
†Electronic address: brunoaces@ufmg.br
‡Electronic address: bruce@fisica.ufmg.br
I. INTRODUCTION

Wigner’s classic works established the very concept of particle in physics \([1, 2]\). It has been demonstrated, in a well-posed mathematical formulation, that a particle is nothing more than an irreducible representation of the Poincaré group. First, in Ref. \([1]\), Wigner developed all discussions within the orthochronous Lorentz subgroup itself, without taking into account the reflections carried out by discrete symmetries that lead to the complete Lorentz group. Later, in a more complete article \([2]\), a general treatment taking into account the states of a particle including reflections was developed.

As important results coming from the Wigner’s works, we evince four different cases, namely Wigner classes. That is, one case standing for a spin one-half one-particle state with well defined transformation rules under action of parity, time reversal, charge conjugation operations, that is, the Dirac spinors, and also three non-usual cases with respect to their behaviour under action of the aforementioned transformations, encoding, thus, certain doubling states, i.e. degeneracy, under reflections. The results obtained by Wigner were re-examined in \([3]\), bringing, thus, a concise interpretation of the results.

The foundations of Quantum Field Theory dictate that quantum fields are the result of engaging well defined one-particle states \([1]\) into interactions regulated through Lorentz invariance \([4]\). Furthermore, fundamental rules such as respect for the principle of cluster decomposition \([5]\) and causality, in the sense of Weinberg’s approach, do form the theoretical scope on which a quantum field emerges \([6]\). Weinberg argued, after a revision of the Wigner’s results that the non-usual fermionic classes found by Wigner cannot be used for representing elementary particles \([6]\). However, recent results might suggest some examples of particles populating such classes \([7, 8]\).

Concerning the usual spin one-half particles and its corresponding fields, the aforementioned formulation stands for the Weinberg’s well-known no-go theorem — the construction of Dirac’s quantum field from first principles of relativity and quantum mechanics is accomplished without invoking the Dirac equation and the work results in a no-go theorem: on the impossibility of constructing another spin one-half quantum field without violating Lorentz symmetries, and, consequently, locality. However, recently new possibilities concerning spin one-half fermionic field have been investigated in \([7]\), apparently bypassing the rigidity of the Weinberg theorem. Such formalization may lead to a local and Lorentz invariant field
endowed with a mass dimension one, thus supposedly evading the Weinberg’s no-go theorem through a freedom in the spinorial adjoint structure.

Recently, a series of results, such as the theoretical proposal of new spin one-half fermions with a mass dimension of one [9–11], widely explored in cosmology, mathematical physics and phenomenology [12]-[39], suggests that the fundamental structure of Quantum Field Theory (QFT) could be improved and extended. As far as we know, these mass dimension one fermions could belong to what is known as the Beyond the Standard Model (BSM) of particle physics. Certainly, understanding the fundamental aspects of the mentioned particles will help us to understand what the content of the BSM particles could be.

In the second decade of the 2000s, a new theoretical proposal involving new spin-half particles attracted attention due to its very peculiar characteristics [40]. It is claimed that such spinors belong to an entirely new class of spin-half particles that reside in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation space, endowed with a three-half mass dimension that escapes the spin-statistics theorem. In other words, the creation and annihilation operators of the associated local field satisfy commutation relations instead.

As it is well established, the spin-statistics theorem says that half-integral spin particles follow the Fermi-Dirac statistics, that is, they are fermions. On the other hand, the integral spin particles follow the Bose-Einstein statistics, that is, they are bosons. This spin-statistics connection is a fundamental principle of physics that has been demonstrated several times using different general assumptions. See Ref. [41] for a very complete review of the literature of the spin-statistics theorem. Among the most common assumptions we can mention microcausality, the positive semidefinite Hamiltonian, locality of fields and interactions and positive norms for states. In 1936 Pauli, Ref. [42], deduced the spin-statistics connection for scalar fields by imposing the condition that the charge density, \(\rho\), at two separate space-like points, \(x\) and \(x'\), must satisfy \(\rho(x) \cdot \rho(x') = 0\). In other words, the measurements of physical quantities, in this case the charge density, in space-like separation cannot influence each other. This condition is known as the classical locality condition (or classical microcausality condition) [43]. However, Pauli needed to impose one more condition, the positive semidefinite Hamiltonian, to extend this result to half-integral spin particles in 1940 [44].

S. Weinberg offered another proof in 1964 in a paper whose main objective was to discover Feynman’s rules for particles of any spin [4]. A more modern version of this proof can be found in Ref. [6]. Roughly speaking, Weinberg’s approach was motivated by the
construction of a general theory with a Lorentz invariant $S-$matrix. Although the algebraic constraint of this approach resembles the classical microcausality used by Pauli, in the Weinberg case space-like commutativity/anticommutativity of the fields is required by the Lorentz invariance of the $S-$matrix, and thus it is not really necessary the “causality” interpretation. The main advantage of this approach is that only this condition is sufficient to show the connection of spin-statistics for integer and half-integer spin particles. Another advantage is that it does not assume any particular Lagrangian or equation of motion to deduce the connection between spin and statistics. These are the main reasons why we will use this approach in this paper.

Therefore, if the spin-statistics theorem is so well established and has been reviewed by many authors for over 80 years, we consider, it is very important to study in detail the new theoretical proposal in Ref. [40] to find out which assumptions of spin-statistics theorem are violated for the proposed spin-half bosons.

In order to answer this question, we organize this article as follows: in section II, we follow Weinberg’s approach to causal fields [6] in order to find the statistics that spin one-half fields, in the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation space must satisfy. We consider only general principles, such as Lorentz invariance of the $S$-matrix and the cluster decomposition. We generalize the Weinberg’s result to fields with expansion coefficients that are not eigenstates of any discrete symmetry. In section III, we consider in detail the quantum field proposed in Ref. [40] in order to understand why this field apparently evades the spin-statistics theorem of the quantum field theory. Finally, in the last section, we present our conclusions.

II. SPIN ONE-HALF RELATIVISTIC QUANTUM FIELDS

In this section, we will consider relativistic spin one-half fields in order to show that this type of field satisfies the Fermi-Dirac statistic regardless of whether they are chosen as parity eigenstates, charge conjugation, or time inversion operators or not. To do this, we take into account two general principles: the Lorentz invariance of the $S$-matrix and the cluster decomposition as in Ref. [6]. The first one requires that the interaction term is the
spacetime integral of a Hamiltonian scalar density, \( \mathcal{H}(x) \), that satisfies:

\[
U(\Lambda, a)\mathcal{H}(x)U(\Lambda, a)^{-1} = \mathcal{H}(\Lambda x + a), \quad \text{i.e. } \mathcal{H}(x) \text{ is a Lorentz scalar,} \quad (1)
\]

and

\[
[\mathcal{H}(x), \mathcal{H}(x')] = 0, \quad \text{for } (x - x')^2 \geq 0, \quad (2)
\]

where \( \Lambda \) is a Lorentz transformation, \( a \) is a general space-time translation and, \( U \) is the corresponding operator representing the Lorentz transformation in the Hilbert space. Also, note that throughout this work we use the metric signature \((-++,++)\). Roughly speaking, the cluster decomposition principle imposes that \( \mathcal{H}(x) \) is constructed out the creation and annihilation operators. Because of the transformation of these operators under Lorentz transformations, \( \mathcal{H}(x) \) must be built out the fields with the following form:

\[
\psi^+_{\ell}(x) = \sum_{s,n} \int d^3p \ u_{\ell}(x, p, s, n) a(p, s, n), \quad (3)
\]

\[
\chi^-_{\ell}(x) = \sum_{s,n} \int d^3p \ v_{\ell}(x, p, s, n) b^*(p, s, n), \quad (4)
\]

where \( n \) denotes internal quantum numbers, \( s \) runs over the spin \( z \)-components and \( l \) runs over the representation index. From here on, we will omit the \( n \) index because this is not essential for our purposes.

Also, the coefficients \( u_{\ell}(x, p, s) \) and \( v_{\ell}(x, p, s) \) must be chosen such that the fields \( \psi^+_{\ell}(x) \) and \( \chi^-_{\ell}(x) \) transform under Lorentz transformations as

\[
U(\Lambda, a)\psi^+_{\ell}(x)U(\Lambda, a)^{-1} = \sum_{\ell} D_{\ell\tilde{\ell}}(\Lambda^{-1}) \psi^+_{\tilde{\ell}}(\Lambda x + a), \quad (5)
\]

\[
U(\Lambda, a)\chi^-_{\ell}(x)U(\Lambda, a)^{-1} = \sum_{\ell} D_{\ell\tilde{\ell}}(\Lambda^{-1}) \chi^-_{\tilde{\ell}}(\Lambda x + a), \quad (6)
\]

where the \( D \)-matrix furnishes a representation of the homogeneous Lorentz group \([45]\).

More specific properties of the coefficients \( u_{\ell}(x, p, s) \) and \( v_{\ell}(x, p, s) \) can be deduced applying Eqs. (5) and (6) for pure translations, boosts and rotations. First, applying pure translations it is found (Ref. [6]):

\[
u_{\ell}(x, p, s) = (2\pi)^{-3/2} e^{ip \cdot x} u_{\ell}(p, s), \quad (7)
\]

\[
v_{\ell}(x, p, s) = (2\pi)^{-3/2} e^{-ip \cdot x} v_{\ell}(p, s), \quad (8)
\]

where the factor \((2\pi)^{-3/2}\) is conventional. In other words, because of the translational invariance, the dependence on \( x \) of \( u_{\ell}(x, p, s) \) and \( v_{\ell}(x, p, s) \) is only an exponential factor.
We also have that due to the invariance under a general boost, \( L(p) \), these coefficients must satisfy

\[
u(0, s) = \sqrt{\frac{m}{p^0}} \sum_{\ell} D_{\ell}(L(p)) \nu(0, s),
\]

where \( D(L(p)) \) is the matrix representation of a general Lorentz boost and, \( u(0, s) \) and \( v(0, s) \) are the corresponding zero momentum coefficients. Finally, applying Eqs. (5) and (6) with \( \Lambda = R \) (where \( R \) is a general rotation) to the coefficients \( u(0, s) \) and \( v(0, s) \) it is straightforward see that these coefficients have to satisfy the following fundamental conditions

\[
\sum_{\ell} u_{\ell}(0, s) J_{ss}^{(j)} = \sum_{\ell} J_{\ell\ell} u_{\ell}(0, s),
\]

\[
-\sum_{\ell} v_{\ell}(0, s) J_{ss}^{(j)*} = \sum_{\ell} J_{\ell\ell} v_{\ell}(0, s),
\]

where \( J^{(j)} \) and \( J \) are the angular momentum matrices in the representations \( D^{(j)}(R) \) and \( D(R) \) [45, 46]. The conditions in Eqs. (10) and (11) mean that if \( \psi^+(x) \) and \( \chi^-(x) \) are supposed to describe particles with a spin \( j \) then the representation \( D(R) \) must contain among its irreducible components the spin-\( j \) representation \( D^{(j)}(R) \). At this point, it is important to remark that the conditions on the \( \psi^+_\ell(x) \) and \( \chi^-_{\ell}(x) \) fields given in Eqs. (5)-(6) and their respective consequences in Eqs. (7)-(11) are general in the sense that they do not depend neither on a particular adjoint structure, nor on Lagrangian (equations of motion) nor on the use of a discrete symmetry such a parity, time reversal or charge conjugation.

Now, it is time to use the general framework above discussed to our specific case of spin one-half fields. To do that, we will use the Weyl basis to write \( J^{(j)} \) and \( J \) matrices in Eqs. (10)-(11) in the specific representations \( j = 1/2 \) and \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \), respectively. In this basis we have [47, 48]

\[
J^{(1/2)} = \frac{1}{2} \sigma, \quad -J^{(1/2)*} = \frac{1}{2} \sigma_2 \sigma \sigma_2,
\]

and

\[
J_{i0} = -\frac{i}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \quad J_{ij} = \frac{1}{2} \epsilon_{ijk} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix},
\]
where $\sigma_k$, $k = 1, 2, 3$ are the Pauli matrices. Substituting the above matrix representations in conditions (10) and (11), we obtain

$$\sum_{\bar{s}} (u_{\pm}(0, \bar{s}))_i J^{(1/2)}_{\bar{s}s} \sigma_{ij} (u_{\pm}(0, s))_j,$$

$$- \sum_{\bar{s}} (v_{\pm}(0, \bar{s}))_i J^{(1/2)*}_{\bar{s}s} \sigma_{ij} (v_{\pm}(0, s))_j. \quad (14)$$

where we have defined $u(0, s) \equiv (u_+(0, s), u_-(0, s))^T$ and $v(0, s) \equiv (v_+(0, s), v_-(0, s))^T$.

By considering $(u_{\pm}(0, s))_i$ and $(v_{\pm}(0, s))_i$ as the $(i, s)$ elements of corresponding matrices $U_{\pm}$ and $V_{\pm}$, we can rewrite conditions (10) and (11) in matrix form

$$U_{\pm} J^{(1/2)} = \frac{1}{2} \sigma U_{\pm}, \quad (16)$$

$$-V_{\pm} J^{(1/2)*} = \frac{1}{2} \sigma V_{\pm}. \quad (17)$$

Then, Schur’s Lemma can be applied to straightforwardly show that there exist only two possible solutions for the $U_{\pm}$ and $V_{\pm}$ matrices [47, 49]. The first one is a trivial solution with vanishing matrices, and the second one tell us that $U_{\pm}$ and $V_{\pm} \sigma_2$ are proportional to the identity matrix. Thus, we find that the most general zero-momentum $u_\ell(0, s)$ and $v_\ell(0, s)$ spinors can take only the following forms

$$u \left( 0, \frac{1}{2} \right) = \begin{bmatrix} c_+ \\ 0 \\ c_- \end{bmatrix}, \quad u \left( 0, -\frac{1}{2} \right) = \begin{bmatrix} 0 \\ c_+ \\ 0 \end{bmatrix}, \quad (18)$$

$$v \left( 0, \frac{1}{2} \right) = \begin{bmatrix} 0 \\ d_+ \\ 0 \end{bmatrix}, \quad v \left( 0, -\frac{1}{2} \right) = -\begin{bmatrix} d_+ \\ 0 \\ d_- \end{bmatrix}, \quad (19)$$

where $c_\pm$ and $d_\pm$ are arbitrary constants, which in general can be complex numbers or even zero.

Now, we turn our attention to the condition in Eq. (2) which comes from the Lorentz invariance of the S-matrix. As shown in [6] this condition requires that $\mathcal{H}(x)$ is a function of the $\Psi(x)$ (and its Hermitian conjugate) which is written as

$$\Psi(x) = \kappa \psi^+(x) + \lambda \chi^-(x), \quad (20)$$

where $\kappa$ and $\lambda$ are constant which are chosen such that, for space-like intervals, the fields satisfy the following conditions

$$[\Psi_\ell(x), \Psi_\ell(y)]_\mp = 0, \quad [\Psi_\ell(x), \Psi_\ell^\dagger(y)]_\mp = 0, \quad (21)$$

where $(-, +)$ signs means commutator and anti-commutator, respectively. Note that the second condition in Eq. (21) makes use of the Hermitian conjugate field and then it is not necessary to define the adjoint structure at this point.

The first condition in Eq. (21) is satisfied for any $\kappa$ and $\lambda$ constants because

$$[a(p, s), a(p', s')]_\mp = [b(p, s), b(p', s')]_\mp = [a(p, s), b(p', s')]_\mp = 0, \quad (22)$$

and the corresponding relations for the Hermitian conjugate operators. However, the second condition in Eq. (21) is not satisfied in general, in this way imposing some constraints over the type of statistics that the field satisfies and, eventually, over the $\kappa$ and $\lambda$ constants. Using the canonical relations

$$[a(p, s), a^\dagger(p', s')]_\mp = \delta^{(3)}(p - p')\delta_{ss'}, \quad [b(p, s), b^\dagger(p', s')]_\mp = \delta^{(3)}(p - p')\delta_{ss'}, \quad (23)$$

we obtain

$$\left[\Psi_\ell(x), \Psi_\ell^\dagger(y)\right]_\mp = (2\pi)^{-3} \int d^3 p \left[|\kappa|^2 N_{\ell\ell}(p)e^{ip(x-y)} \mp |\lambda|^2 M_{\ell\ell}(p)e^{-ip(x-y)}\right], \quad (24)$$

where

$$N_{\ell\ell}(p) \equiv \sum_s u_\ell(p, s)u_\ell^\dagger(p, s), \quad (25)$$
$$M_{\ell\ell}(p) \equiv \sum_s v_\ell(p, s)v_\ell^\dagger(p, s). \quad (26)$$

Applying Eq. (9), we can write the above quantities as

$$N_{\ell\ell}(p) = \frac{m}{p^0} D(L(p))N_{\ell\ell}(0)D^\dagger(L(p)), \quad (27)$$
$$M_{\ell\ell}(p) = \frac{m}{p^0} D(L(p))M_{\ell\ell}(0)D^\dagger(L(p)), \quad (28)$$

where

$$N_{\ell\ell}(0) = \begin{bmatrix}
|c_+|^2 & 0 & c_+c_-^* \\
0 & |c_+|^2 & 0 & c_+c_-^* \\
c_-c_+^* & 0 & |c_-|^2 & 0 \\
0 & c_-c_+^* & 0 & |c_-|^2
\end{bmatrix}, \quad M_{\ell\ell}(0) = \begin{bmatrix}
|d_+|^2 & 0 & d_+d_-^* \\
0 & |d_+|^2 & 0 & d_+d_-^* \\
d_-d_+^* & 0 & |d_-|^2 & 0 \\
0 & d_-d_+^* & 0 & |d_-|^2
\end{bmatrix}. \quad (29)$$
From Eq. (29), we can see that $N_{\ell \ell}(0)$ and $M_{\ell \ell}(0)$ have a similar form and, for that reason, we will explicitly consider the calculations only for $N_{\ell \ell}(0)$, which can be spanned in terms of the gamma matrices as follows

$$N_{\ell \ell}(0) = \frac{1}{2} \left( |c_+|^2 + |c_-|^2 \right) \mathbb{1} + \frac{1}{2} \left( c_+ c_-^* + c_- c_+^* \right) i \gamma^0 + \frac{1}{2} \left( |c_+|^2 - |c_-|^2 \right) \gamma^5 + \frac{1}{2} \left( c_+ c_-^* - c_- c_+^* \right) i \gamma^0 \gamma^5. \quad (30)$$

It is worth mentioning that Eq. (30) is a general result, where no assumptions on parity symmetry or any other discrete symmetries have been used. Now, substituting Eq. (30) in Eq. (27), and using the following relations

$$D(L(p)) \mathbb{1} D^\dagger(L(p)) = \frac{1}{m} p_\mu \gamma^\mu \gamma^0, \quad (31)$$
$$D(L(p)) i \gamma^0 D^\dagger(L(p)) = i \gamma^0, \quad (32)$$
$$D(L(p)) \gamma^5 D^\dagger(L(p)) = \frac{1}{m} p_\mu \gamma^\mu \gamma^5 \gamma^0, \quad (33)$$
$$D(L(p)) i \gamma^0 \gamma^5 D^\dagger(L(p)) = i \gamma^5 \gamma^0, \quad (34)$$

we obtain

$$N_{\ell \ell}(p) = \frac{1}{2p^0} \left[ -i (|c_+|^2 + |c_-|^2) p_\mu \gamma^\mu + m \left( c_+ c_-^* + c_- c_+^* \right) \mathbb{1} + m \left( c_+ c_-^* - c_- c_+^* \right) \gamma^5 - i \left( |c_+|^2 - |c_-|^2 \right) p_\mu \gamma^\mu \gamma^5 \right] i \gamma^0. \quad (35)$$

By replacing $c_\pm \to d_\pm$ in Eq. (35) we get the expression for $M_{\ell \ell}(p)$, namely

$$M_{\ell \ell}(p) = \frac{1}{2p^0} \left[ -i (|d_+|^2 + |d_-|^2) p_\mu \gamma^\mu + m \left( d_+ d_-^* + d_- d_+^* \right) \mathbb{1} + m \left( d_+ d_-^* - d_- d_+^* \right) \gamma^5 - i \left( |d_+|^2 - |d_-|^2 \right) p_\mu \gamma^\mu \gamma^5 \right] i \gamma^0. \quad (36)$$

With these results we can finally write Eq. (24) as

$$\left[ \Psi_\ell(x), \Psi_\ell^\dagger(y) \right]_\mp = \left( |\kappa|^2 \left[ -\left( |c_+|^2 + |c_-|^2 \right) \gamma^\mu \partial_\mu + m \left( c_+ c_-^* + c_- c_+^* \right) \mathbb{1} + m \left( c_+ c_-^* - c_- c_+^* \right) \gamma^5 - \left( |c_+|^2 - |c_-|^2 \right) \gamma^\mu \gamma^5 \partial_\mu \right] i \gamma^0 \Delta(x - y) \right)_{\ell \ell}, \quad (37)$$

where

$$\Delta(x) \equiv \int \frac{d^3p}{2p^0(2\pi)^3} e^{ip.x}. \quad (38)$$
In order to satisfy the second condition in Eq. (21), it is necessary and sufficient that
\[
|\kappa|^2 |c_+|^2 = \mp |\lambda|^2 |d_+|^2, \quad |\kappa|^2 |c_-|^2 = \mp |\lambda|^2 |d_-|^2,
\]
and
\[
|\kappa|^2 c_+ c_-^* = \pm |\lambda|^2 d_+ d_-^*,
\]
where we have used that for \((x - y)\) space-like \(\Delta(x - y)\) and its first derivative are even and odd functions of \((x - y)\), respectively. We notice that the constraints in Eq. (39) have a non-trivial solution provided we choose the bottom sign, meaning that the spin-1/2 field \(\Psi(x)\) in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation must satisfy anti-commutation relations, i.e. the spin-1/2 field \(\Psi(x)\) is a fermion, in complete agreement with the spin-statistics theorem. This general result does not depend on the use of any discrete symmetry such as parity or the definition of any adjoint structure.

Once the statistics of the \(\Psi(x)\) field is setted, Eqs. (39) and (40) simplify a little more and we are taken to the following useful relation
\[
\frac{c_+}{c_-} = -\frac{d_+}{d_-}.
\]
Note that \(c_\pm\) and \(d_\pm\) are not completely determined from the condition in Eq. (41). Now, we can write \(\kappa = |\kappa|e^{i\theta_\kappa}\), \(\lambda = |\lambda|e^{i\theta_\lambda}\) and, redefining the creation and annihilation operators in Eqs. (3) and (4) as \(a(p, s) \rightarrow e^{i\theta_\kappa}a(p, s)\) and \(b(p, s) \rightarrow e^{-i\theta_\lambda}b(p, s)\) (note that these redefinitions do not modify the canonical anti-commutation relations of \(a\) and \(b\) operators in Eq. (23)), we obtain
\[
\Psi(x) = |\kappa| \left[ \psi^+(x) + \frac{|\lambda|}{|\kappa|} \chi^-(x) \right].
\]
Absorbing the overall factor \(|\kappa|\) in the normalization of the field \(\Psi(x)\) and using the constraints in Eqs. (39) and (40), we get
\[
\Psi(x) = \psi^+(x) + \frac{|c_\pm|}{|d_\pm|} \chi^-(x).
\]
Needless to say, we are assuming that \(|d_+| \neq 0\) or \(|d_-| \neq 0\) in Eq. (43).

To determine completely \(c_\pm\) and \(d_\pm\) additional physical or mathematical conditions must be imposed. For example, a default choice is to impose parity symmetry on the fields in Eqs. (3) and (4) which lead us to
\[
\left( \frac{c_+}{c_-} \right)^2 = \left( \frac{d_+}{d_-} \right)^2 = 1.
\]
Using this relation and the freedom of the overall factor, the form of the $\Psi(x)$ field is the standard Dirac field, i.e. $\Psi(x) = \psi^+(x) + \chi^-(x)$. It is also possible consider the charge-conjugation and time reversal properties of the field in Eq. (43) to obtain, for instance, the Majorana fields c.f. [6].

III. SPIN-HALF BOSONS WITH MASS DIMENSION THREE-HALF

We now turn our attention to the class of quantum fields presented in [40]. Roughly speaking, in this reference, it is claimed that this spin one-half field belongs to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation, however it satisfies the Bose-Einstein statistics which is in clear contradiction with the spin-statistics theorem. Therefore, in this section we will briefly review the main ideas behind this field and show why it does not satisfy the mentioned theorem.

The key idea behind the proposed field in Ref. [40] is to expand the coefficients of the quantum field as a combination of the eigenstates of one of the sixteen different roots of the $4 \times 4$ matrix. Note that $i\gamma_1, i\gamma_2, i\gamma_3, \gamma_0, i\gamma_2\gamma_3, i\gamma_3\gamma_1, i\gamma_1\gamma_2, \gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_0\gamma_3, i\gamma_0\gamma_2\gamma_3, i\gamma_0\gamma_1\gamma_2, \gamma_1\gamma_2\gamma_3$ and $i\gamma_0\gamma_1\gamma_2\gamma_3$ are all roots of the $4 \times 4$ matrix. From here on, these roots are denoted as $\Omega_i$ with $i = 1, ..., 16$. Then, the author chooses

$$\lambda_1(0) = \sqrt{\frac{m}{2}} (\mu_1 + i\mu_2),$$

(45)

$$\lambda_2(0) = \sqrt{\frac{m}{2}} (\mu_1 - i\mu_2),$$

(46)

and

$$\lambda_3(0) = \sqrt{\frac{m}{2}} (\mu_3 + i\mu_4),$$

(47)

$$\lambda_4(0) = \sqrt{\frac{m}{2}} (\mu_3 - i\mu_4),$$

(48)

as the expansion coefficients of the quantum field at momentum $p = 0$. Here $\mu_1, \mu_2, \mu_3$ and $\mu_4$ are the eigenstates of the $\Omega_4 = i\gamma_3$ that are written as

$$\mu_1 = \begin{bmatrix} 0, i, 0, 1 \end{bmatrix}^T, \quad \mu_2 = \begin{bmatrix} i, 0, 1, 0 \end{bmatrix}^T,$$

(49)

$$\mu_3 = \begin{bmatrix} 0, -i, 0, 1 \end{bmatrix}^T, \quad \mu_4 = -\begin{bmatrix} -i, 0, 1, 0 \end{bmatrix}^T,$$

(50)

Applying the Lorentz boost, $D(L(p))$, on the $\lambda_i(0)$ it is straightforward to obtain the
\( \lambda_i(p) \) for any \( p \neq 0 \). Once this is done, the quantum field is defined as

\[
\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left[ \sum_{i=1,2} \lambda_i(p)e^{-ip.x} a_i(p) + \sum_{i=3,4} \lambda_i(p)e^{+ip.x} b_i(p) \right],
\]

(51)

and its adjoint field

\[
\overline{\Phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left[ \sum_{i=1,2} \overline{\lambda}_i(p)e^{ip.x} a_i(p) + \sum_{i=3,4} \overline{\lambda}_i(p)e^{-ip.x} b_i(p) \right].
\]

(52)

The dual in Eq. (52) is not the standard one but \( \overline{\lambda}_i(p) = [\mathcal{P}\lambda_i(p)]^\dagger \) γ\( _0 \) where \( \mathcal{P} = m^{-1}p^\mu\gamma_\mu \)

is the parity operator in the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) representation. This redefinition of the dual is necessary in this construction because of the invariant orthogonality relations. It is also important to remark that in the momentum space the expansion coefficients of \( \Phi(x) \), \( \lambda_i(p) \), satisfy Dirac-like equations, i.e.

\[
(a_\mu p^\mu - m \mathbb{1}) \lambda_{1,2}(p) = 0,
\]

(53)

\[
(a_\mu p^\mu + m \mathbb{1}) \lambda_{3,4}(p) = 0,
\]

(54)

where \( a_\mu = i\gamma_\mu \) with \( \gamma = i\epsilon^{\mu\nu\lambda\sigma} \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \) and, \( a_\mu \) satisfies \( \{a_\mu, a_\nu\} = 2\eta_{\mu\nu} \mathbb{1} \). That means that the \( \lambda_i(p) \) are eigenvectors of the \( a_\mu p^\mu \) operator however they are not eigenstate of the parity operator \( \mathcal{P} \).

In order to set the statistics that \( \Phi(x) \) field satisfies, the author applies the condition in Eq. (21) finding that the \( \Phi(x) \) is a bosonic field. Then, the author claims that \( \Phi(x) \) is a new type of bosonic field endowed with spin-half, that evades the spin-statistics theorem of the quantum field theory. However, the field in Eq. (51), or its adjoint in Eq. (52), does not create or annihilate particles with spin 1/2. There are at least two ways to see that. The first one is to note that the expansion coefficients, \( \lambda_i(0) \), do not satisfy any of the fundamental conditions in Eqs. (14) and (15) as they should be if the \( \Phi(x) \) was for describing particles with spin 1/2 as it was explained in Sec. II.

The second way is to directly apply the spin operator \( \mathcal{J} \) in the specific representation \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \), Eq. (13), to (an)a (anti-)particle state created by the \( \Phi(x) \) field. For instance, if the \( \Phi(x) \) field in Eq. (51) is contracted in the vacuum state, \( |0\rangle \), it creates an anti-particle (or a type-b particle), which we denoted as \( |1_b\rangle \), with two possible spin projections given by \( \lambda_3(0) \) and \( \lambda_4(0) \). Note that, since the spin is a static physical property, it is more appropriate to go to center-of-momentum frame, i.e. the frame where \( p = 0 \), to measure the spin of this
particle state. Then, if the spin-z operator, i.e. $\hbar J_{z}$, is applied to this one particle state $|1_b\rangle$ it is not obtained $\pm \frac{\hbar}{2} |1_b\rangle$ as it must be if the $\Phi(x)$ field created particles/anti-particles with spin 1/2. Therefore, it is now absolutely clear that the field $\Phi(x)$ in Eq. (51) is neither suitable to describe particles with spin 1/2 in the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ nor there is an exception to the fundamental spin-statistics theorem.

It is our understanding that the main cause of the field proposed in reference [40] fails to describe particles with spin 1/2 is that although this construction mimics in some points Dirac’s historical construction, it uses as expansion coefficients of the field a linear combination of the eigenstates of the $\Omega_{4}$ operator, one of the sixteen roots of the $4 \times 4$ identity matrix. However, these are not eigenstates of the spin operator in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation.

IV. CONCLUSIONS

In this paper, we first show that any spinorial field in the representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the Lorentz group satisfies the Fermi-Dirac statistics, that is, it is a fermionic field, regardless of whether its expansion coefficients are eigenstates of the parity operator or any other discrete symmetry operator, or even the definitions of a particular adjoint structure. Next, we show that, despite the fact that the quantum field introduced in Ref. [40] satisfies some of the conditions arising from the Lorentz invariance of the $S$-matrix, such as translation, Eqs. (7) and (8), and boost invariance, Eq. (9), it fails to satisfy the invariance under rotations, Eqs. (10) and (11), which causes that this proposed field does not create particles with spin 1/2. Therefore, this field can not be used to represent elementary particles with spin 1/2 in a quantum field theory that satisfies Lorentz symmetry.

Acknowledgements

A.R. Aguirre and B. A. Couto e Silva thank CAPES for financial support. B. L. Sánchez-Vega thanks the National Council for Scientific and Technological Development of Brazil, CNPq, for the financial support through grant n° 311699/2020-0.

[1] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Annals Math. 40, 149 (1939).
[2] E. P. Wigner, *Unitary representations of the inhomogeneous Lorentz group including reflections*, in Group theoretical concepts and methods in elementary particle physics, Lectures of the Istanbul summer school of theoretical physics, 37-80 (1964).

[3] T. D. Lee and G. C. Wick, *Space inversion, time reversal, and other discrete symmetries in local field theories*, Phys. Rev. **148**, 1385 (1966).

[4] S. Weinberg, *Feynman rules for any spin*, Phys. Rev. **133**, B1318 (1964).

[5] E. H. Wichmann and J. H. Crichton, *Cluster decomposition properties of the S matrix*. Phys. Rev. **132**, 2788 (1963).

[6] S. Weinberg, *The Quantum Theory of Fields, Vol. I: Foundations*, Cambridge University Press, New York, (1996).

[7] D. V. Ahluwalia, *Evading Weinberg's no-go theorem to construct mass dimension one fermions: constructing darkness*, EPL **118**, 60001 (2017).

[8] J. M. Hoff da Silva, and R. J. Bueno Rogerio, *Massive spin-one-half one-particle states for the mass-dimension-one fermions*, EPL **128**, 11002 (2019).

[9] R. J. Rogerio, C. H. Coronado Villalobos, and A. R. Aguirre, *A hint towards mass dimension one Flag-dipole spinors*, Eur. Phys. J. C **79**, 12 (2019).

[10] D. V. Ahluwalia, *Mass dimension one fermions*, Cambridge University Press (2019).

[11] D. V. Ahluwalia, *A new class of mass dimension one fermions*, Proceedings of the Royal Society A **476**, 2240 (2020).

[12] D. V. Ahluwalia, *The theory of local mass dimension one fermions of spin one half*, Adv. Appl. Clifford Algebras **27**, 2247 (2017).

[13] R. da Rocha and J. M. Hoff da Silva, *From Dirac spinor fields to ELKO*, J. Math. Phys. **48**, 123517 (2007).

[14] R. da Rocha, A. E. Bernardini and J. M. Hoff da Silva, *Exotic Dark Spinor Fields*, JHEP **1104**, 110 (2011).

[15] L. Fabbri and S. Vignolo, *ELKO and Dirac Spinors seen from Torsion*, Int. J. Mod. Phys. D **23**, 1444001 (2014).

[16] R. da Rocha, J. M. Hoff da Silva and A. E. Bernardini, *Elko spinor fields as a tool for probing exotic topological spacetime features*, Int. J. Mod. Phys. Conf. Ser. **3**, 133 (2011).

[17] R. da Rocha, L. Fabbri, J. M. Hoff da Silva, R. T. Cavalcanti and J. A. Silva-Neto, *Flag-Dipole Spinor Fields in ESK Gravities*, J. Math. Phys. **54**, 102505 (2013).
[18] J. M. Hoff da Silva, C. H. Coronado Villalobos, R. J. Bueno Rogerio and E. Scatena, *On the bilinear covariants associated to mass dimension one spinors*, Eur. Phys. J. C **76**, 563 (2016).

[19] A. Alves, F. de Campos, M. Dias and J. M. Hoff da Silva, *Searching for Elko dark matter spinors at the CERN LHC*, Int. J. Mod. Phys. A **30**, 1550006 (2015).

[20] B. Agarwal, P. Jain, S. Mitra, A. C. Nayak and R. K. Verma, *ELKO fermions as dark matter candidates*, Phys. Rev. D **92**, 075027 (2015).

[21] A. Alves, M. Dias and F. de Campos, *Perspectives for an Elko Phenomenology using Monojets at the 14 TeV LHC*, Int. J. Mod. Phys. D **23**, 1444005 (2014).

[22] C.-Y. Lee and M. Dias, *Constraints on mass dimension one fermionic dark matter from the Yukawa interaction*, Phys. Rev. D **94**, 065020 (2016).

[23] C.-Y. Lee, *Symmetries and unitary interactions of mass dimension one fermionic dark matter*, Int. J. Mod. Phys. A **31**, 1650187 (2016).

[24] A. Alves, M. Dias, F. de Campos, L. Duarte and J. M. Hoff da Silva, *Constraining Elko Dark Matter at the LHC with Monophoton Events*, EPL **121**, 31001 (2018).

[25] J. M. Hoff da Silva and S. H. Pereira, *Exact solutions to Elko spinors in spatially flat Friedmann-Robertson-Walker spacetimes*, JCAP **1403**, 009 (2014).

[26] C. G. Boehmer, *The Einstein-Cartan-Elko system*, Annalen Phys. **16**, 38 (2007).

[27] C. G. Boehmer and J. Burnett, *Dark spinors with torsion in cosmology*, Phys. Rev. D **78**, 104001 (2008).

[28] L. Fabbri, *The most general cosmological dynamics for ELKO matter fields*, Phys. Lett. B **704**, 255 (2011).

[29] C. G. Boehmer, *The Einstein-Elko system - Can dark matter drive inflation?*, Annalen Phys. **16**, 325 (2007).

[30] C. G. Boehmer, *Dark spinor inflation - theory primer and dynamics*, Phys. Rev. D **77**, 123535 (2008).

[31] C. G. Boehmer, J. Burnett, D. F. Mota and D. J. Shaw, *Dark spinor models in gravitation and cosmology*, JHEP **07**, 053 (2010).

[32] H. M. Sadjadi, *On coincidence problem in ELKO dark energy model*, Gen. Relativ. Gravit. **44**, 2329 (2012).

[33] A. Basak, J. R. Bhatt, S. Shankaranarayanan and K. V. P. Varma, *Attractor behaviour in ELKO cosmology*, JCAP **04**, 025 (2013).
[34] S. H. Pereira, A. S. S. Pinho and J. M. Hoff da Silva, Some remarks on the attractor behaviour in ELKO cosmology, JCAP 1408, 020 (2014).
[35] S. H. Pereira and A. S. S. Pinho, ELKO applications in cosmology, Int. J. Mod. Phys. D 23, 1444008 (2014).
[36] S. H. Pereira and T. M. Guimarães, From inflation to recent cosmic acceleration: The fermionic Elko field driving the evolution of the universe, JCAP 1709, 038 (2017).
[37] S. H. Pereira and R. C. Lima, Creation of mass dimension one fermionic particles in asymptotically expanding universe, Int. J. Mod. Phys. D 26, 1730028 (2017).
[38] R. J. Bueno Rogerio, J. M. Hoff da Silva, M. Dias and S. H. Pereira, Effective lagrangian for a mass dimension one fermionic field in curved spacetime, JHEP 1802, 145 (2018).
[39] R. J. Bueno Rogerio, et.al, Mass dimension one fermions and their gravitational interaction, EPL 128, 20004 (2019).
[40] D. V. Ahluwalia, Spin-half bosons with mass dimension three-half: Towards a resolution of the cosmological constant problem, EPL 131, 41001 (2020).
[41] C. Curceanu, J. D. Gillaspy, and R. C. Hilborn, Resource Letter SS–1: The Spin-Statistics Connection Am. J. Phys. 80, 561 (2012).
[42] W. Pauli, Théorie quantique relativiste des particules obéissant à la statistique de Einstein–Bose. Annals de Institut Henry Poincaré, 6, 109–136 (1936).
[43] M. Massimi and M. Redhead, Weinberg’s proof of the spin-statistics theorem, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics, 34, 4 (2003).
[44] W. Pauli, The connection between spin and statistics. Physical Review, 58, 716–722 (1940).
[45] J. F. Cornwell, Group Theory in Physics Techniques of Physics, Vol. 2, Academic Press, (1986).
[46] S. Coleman, Aspects of Symmetry: selected Erice Lectures of Sydney Coleman, Cambridge Univ. Press, (1985).
[47] Howard Georgi, Lie Algebras in Particle Physics, Perseus Books, (1999).
[48] P. Ramond, Group Theory: A Physicists Surveys, Cambridge Univ. Press, (2010).
[49] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9, Springer, (1972).