FUNDAMENTAL GROUP OF $\text{Symp}(M, \omega)$ WITH NO CIRCLE ACTION

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Abstract. We show that $\pi_1(\text{Symp}(M, \omega))$ can be nontrivial for $M$ that does not admit any symplectic circle action.

1. The question

Question 1.1. Suppose that $\pi_1(\text{Symp}(M, \omega))$ is nontrivial. Is it true that some nonzero element is represented by a loop $S^1 \to \text{Symp}(M, \omega)$ that is a homomorphism (a circle action on $M$)?

Remark 1.2. If $G$ is a compact Lie group then any element of $\pi_1(G)$ is represented by a loop that is a homomorphism. Elements of $\pi_1(\text{Symp}(M, \omega))$ which are not represented by a circle action were described in Anjos [1] as well as in McDuff [4].

Acknowledgements I was asked the above question by Yael Karshon during the conference in Stare Jabłonki. The argument relies on a work of Lalonde and Pinsonnault [3]. I thank Rafał Walczak for comments.

2. An answer

We shall show that the answer to the above question is negative in general. First we will give a little bit imprecise argument and then show a concrete example.

2.1. A heuristic argument. Philosophically speaking there are very few manifolds admitting a circle action. They are, so to speak, exceptional. On the other hand topology of groups of diffeomorphisms is rather complicated. Hence one can expect nontrivial fundamental groups.

The argument consists of several steps:
(1) Take a closed simply connected symplectic manifold \((K, \omega_K)\). Consider the evaluation fibration

\[
\text{Symp}(K, p) \to \text{Symp}(K) \xrightarrow{ev} K,
\]

where \(\text{Symp}(K, p) \subset \text{Symp}(K)_0\) denote the isotropy subgroup of a point \(p \in K\).

Observe that \(ev_* : \pi_2(\text{Symp}(K)) \to \pi_2(K)\) is trivial up to torsion. Indeed, If \(ev_*\) was nontorsion then the corresponding map on rational cohomology would be nonzero, say \(ev^*(\alpha) \neq 0\) for \(\alpha \in H^2(K, \mathbb{Q})\). Then we have that \(0 = ev^*(\alpha^{n+1}) = ev^*(\alpha)^{n+1}\), where \(\dim K = 2n\). But \(\text{Symp}(K)\) is a topological group so its rational cohomology is free graded algebra. Thus every element of even degree is of infinite order which contradicts the above calculation. Thus we get that the rank of \(\pi_1(\text{Symp}(K, p))\) is not smaller than the rank of \(\pi_2(K)\). In particular it is nonzero.

(2) The isotropy subgroup \(\text{Symp}(K, p)\) should be weakly homotopy equivalent to the group of symplectomorphisms of a one point blow-up of \((K, \omega_K)\) in a very small ball. This is proved for a range of 4-dimensional manifolds by Lalonde and Pinsonnault in [3]. It is interesting to what extent it is true.

(3) The final step is to find a simply connected symplectic closed manifold that its blow-up does not admit any symplectic circle action.



## 2.2. Examples.

**Theorem 2.1.** Suppose that \((K, \omega_K)\) is neither rational nor ruled surface up to blow-up. Let \((M, \omega)\) be a symplectic blow-up (in a small ball) of a closed simply connected Kähler surface \((K, \omega_K)\). Then \((M, \omega)\) admits no symplectic circle action and \(\pi_1(\text{Symp}(M, \omega))\) is nontrivial.

**Proof.** We follow the scenario described in the previous section.

(1) This step is obvious.

(2) It is proved in [2] (Proof of Theorem 1.2) that \(\pi_1(\text{Symp}(M, \omega))\) surjects onto \(\pi_1(\text{Symp}(K, p))\). Hence \(\pi_1(\text{Symp}(M, \omega))\) is nontrivial. The argument here relies on results of Lalonde and Pinsonnault [3].

(3) Suppose \((M, \omega)\) admits a symplectic circle action. Since \((M, \omega)\) is simply connected then it has positive Euler characteristic. Hence the action has a fixed point and is Hamiltonian. Thus, according to the classification of Hamiltonian \(S^1\)-actions on 4-manifolds (see Theorem 13.19 in [5]), we get that \(M\) has to be
rational or ruled surface up to blow-up. This is excluded by the hypothesis.

Example 2.2. To see a concrete example take \((K, \omega_K)\) to be K3 surface with any symplectic form. Then its small blow-up \((M, \omega)\) does not admit any symplectic circle action and \(\pi_1(\text{Symp}(M, \omega))\) is nontrivial.

References

[1] Silvia Anjos. Homotopy type of symplectomorphism groups of \(S^2 \times S^2\). Geom. Topol., 6:195–218 (electronic), 2002.
[2] Jaroslaw Kędra. Evaluation fibration and topology of symplectomorphisms. Proc. AMS, to appear, 2004.
[3] François Lalonde and Martin Pinsonnault. The topology of the space of symplectic balls in rational 4-manifolds. Duke Math. J., 122(2):347–397, 2004.
[4] Dusa McDuff. Symplectomorphism groups and almost complex structures. In Essays on geometry and related topics, Vol. 1, 2, volume 38 of Monogr. Enseign. Math., pages 527–556. Enseignement Math., Geneva, 2001.
[5] Dusa McDuff and Dietmar Salamon. Introduction to symplectic topology. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

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