The associated hyperringoid to a Krasner hyperring

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ABSTRACT

“Ends Lemma” is used to construct a hypergroupoid from a (quasi) partially ordered groupoid. But this lemma does not work well for creating a hyperringoid from a (partially) ordered ringoid. In this paper, we plan to gain that by modifying this lemma, called modified “Ends Lemma”. Then we construct a EL\(^2\)-hyperring, as a generalization of a “EL-hyperring”, by applying on a (partially) ordered Krasner hyperring.

1. Introduction

Hyperstructure theory as a natural generalization of algebraic structure theory was born by F. Marty at the 8th Congress of Scandinavian Mathematicians in 1934 [1]. He defined the concept of hypergroups based on the notion of hyperoperation. Since then, many mathematicians have widely studied a number of different hyperstructures. For instance, P. Corsini wrote one of the first books about hypergroups in 1993 [2], and a recent book on hyperstructures was written by B. Davvaz in 2012 [3].

The applications of hyperstructures to other areas have been extensively studied such as optimization theory, graph theory, physics, chemistry, theory of discrete event dynamical systems, generalized fuzzy computation, automata theory, formal language theory, coding theory and analysis of computer programs, for example, see [4–6].

In this paper, a relationship between (partially) ordered sets and algebraic hyperstructures would be studied. This topic was first studied by Vougiouklis in 1987 [7]. Since then, many researchers, such as Vougiouklis [8–10], Corsini [2, 11, 12], Hoskova [13] and Heidari and Davvaz [14], have analysed the connection between hyperstructures and (partially) ordered sets. One special aspect of this issue, known as EL-hyperstructures, was touched upon by Chvalina [15]. Also, Rosenberg in [16], Hoskova in [13], Rackova in [17] and Novak in [18–23] extended some results on the ordered semigroups and ordered groups connected with EL-hyperstructures. M. Novak mainly studied EL-hyperstructures that constructed from a (partially) quasi-ordered (semi) groups. He considered subhyperstructures of EL-hyperstructures in [21]. Also, he discussed some interesting results of important elements in this family of hyperstructures [19]. Then, in [20] Novak studied some basic properties of EL-hyperstructures such as invertibility, normality, being closed (ultra closed) and so on.

In 2015, El-(semi)hypergroups constructed based on a (partially) quasi-ordered (semi)hypergroups were studied [24]. This paper helps us construct EL-(semi)hyperrings based on a given partially ordered Krasner hyperrings. These hyperstructures are called EL\(^2\)-hyperstructures.

1.1. Definitions and preliminaries

In the following, we present some basic definitions and ideas from the hyperstructure theory. The hyperstructures are algebraic structures equipped with at least one multi-valued operation, called a hyperoperation. A non-empty set \(H\), endowed with a hyperoperation, \(\circ : H \times H \rightarrow \phi^*(H)\) is called a hypergroupoid. \(\phi^*(H)\) denotes the set of all non-empty subsets of \(H\). In this definition, if \(A\) and \(B\) are two non-empty subsets of \(H\) and \(x \in H\), then we define

\[ A \circ B = \bigcup \{a \circ b : a \in A, b \in B\}, \quad x \circ A = \{x\} \circ A \quad \text{and} \quad A \circ x = A \circ \{x\}. \]

An element \(e \in H\) is called an identity of \((H, \circ)\) if \(x \in x \circ e \cap e \circ x\), for all \(x \in H\) and it is called a scalar identity of \((H, \circ)\) if \(x \circ e = e \circ x = x\), for all \(x \in H\). If \(e\) is a scalar identity of \((H, \circ)\), then \(e\) is the unique identity of \((H, \circ)\). A hypergroupoid which verifies the condition \((x \circ y) \circ z = x \circ (y \circ z)\), for all \(x, y, z \in H\), is called a semihypergroup. If the semihypergroup \(H\) satisfies \(x \circ H = H \circ x\), for all \(x \in H\), it is called a hypergroup [3]. This condition is known as reproduction axiom.
Because of dealing with the theory of ordered structures, we recall that an ordered semihypergroup $(H, \circ, \leq)$ is a semihypergroup $(H, \circ)$ together with a partial order $\leq$ such that satisfies the monotone condition as follows:

$$x \leq y \Rightarrow z \circ x \leq z \circ y \quad \text{and} \quad x \circ z \leq y \circ z,$$

for all $x, y, z \in H$,

which is here $z \circ x \leq z \circ y$ means that for all $a \in z \circ x$, there exists $b \in z \circ y$ and for all $b \in z \circ y$ there exists $a \in z \circ x$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly. Indeed, the concept of ordered semihypergroups is a generalization of the concept of ordered semigroups. To read the concept and properties of ordered semigroups, we refer the reader to [25].

There are several definitions for a hyperring, if we replace at least one of the two operations by a hyperoperation. In general case, $(R, +, \cdot)$ is a good hyperring (hyperring), if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup and the hyperoperation $\cdot$ is distributive (weak distributive) over the hyperoperation $+$, which means that for all $x, y, z$ of $R$ we have $x.(y+z)=x.y+x.z$ and $(x+y).z=x.z+y.z$ $(x.(y+z) \subseteq x.y+x.z$ and $(x+y).z \subseteq x.z+y.z$). We call $(R, +, \cdot)$ a (good) hyperfield if $(R, +, \cdot)$ is a (good) hyperring and $(R, \cdot)$ is a hypergroup. We say that $(R, +, \cdot)$ is an additive hyperring, if the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a usual operation. A special case of this type is the Krasner hyperring. To read more details about hyperrings, see [26].

### 1.2. Modified Ends Lemma

In the following of this section, we present the concept of modified Ends Lemma and some theorems related to it.

**Definition 1.1:** ([27]) Let $R$ be a ring. We say that $R$ is partially ordered when there exists a partial order $\leq$ on the underlying set $R$ that it satisfies:

i) $a \leq b$ implies $a + c \leq b + c$,

ii) $0 \leq a$ and $0 \leq b$ imply that $0 \leq a.b$, for any $a, b, c$ in $R$.

If any two arbitrary elements $a, b$ of $R$ are comparable, then $R$ is ordered.

**Note:** In any (partially) ordered ring with unit element $1 \neq 0$, we have $0 < 1$.

**Example 1.2:**

i) The ring $(\mathbb{Z}, +, \cdot, \leq)$ with the ordinary addition and multiplication operation and the natural order relation is an ordered ring.

ii) The ring $\mathbb{Z}^I$ of integral-valued functions on set $I$, with pointwise order, is partially ordered (when $I$ has at least two elements).

**Remark 1.1:**

(1) There is no (partially) ordered and finite non-trivial group $(H, +, \leq)$, unless at least two distinct elements $a, b \in H$ are in relation $\leq$ (i.e. $\leq$ is not trivial). Suppose to the contrary there exists $n \in \mathbb{N}$ such that $|H| = n$. Now, if $0 \neq x \in H^+$, then there exists $m \in \mathbb{N}$ such that $|x| = m$ and $m \cdot n$. So, according to the monotone condition, we have

$$0 < x \Leftrightarrow x < x + x \Leftrightarrow \cdots \Leftrightarrow x + x + \cdots + x = 0,$$

$m$ times

Now, due to the transitivity of $\leq$, we have $x < 0$. It is a contradiction.

(2) There is no (partially) ordered ring with unit element $1 \neq 0$, that is finite. Consider (partially) ordered ring $(R, +, \cdot, \leq)$. Due to Definition 1.1, for every $n > 0$, we have

$$n \cdot 1 = 1 + 1 + \cdots + 1 > 0,$$

$n$ times

Thus, the identity 1, as an element of the group $(R, +)$, has infinite order. As a result, $R$ with an identity 1 is an infinite ring.

In any (partially) ordered ring $R$, the absolute value $|x|$ of an element $x$ can be defined as follows:

$$|x| = \begin{cases} x, & \text{OR} < x, \\ -x, & x \leq 0. \end{cases}$$

By applying original “Ends Lemma”, hypergroupoids are created from (quasi, partially) ordered groupoids. But about ringoids, we are faced with structures that are equipped with two addition and multiplication operations. After applying original “Ends Lemma” to (partially) ordered ringoids, the distribution multiplication hyperoperation by the ratio of addition hyperoperation on the right and left in creating hyperringoids will not be established. To accomplish this important, we will have the following well-defined hyperoperations:

**Definition 1.3:** Let $R$ be an ordered ring. For all $a, b \in R$, we define

$$a \oplus b = [|a| + |b|]_\leq,$$

(1)

$$a \odot b = [|a||b|]_\leq.$$  

(2)
Note: If \((R, \leq)\) is a (partially) ordered set and \(a \in R\), then the subset \(\{x \in R : |a| \leq x\}\) of \(R\) is called principal end generated by \(|a| \in R\) and denoted by \(|a|_\leq\).

In the following, we present another statements of the original "Ends Lemma".

**Lemma 1.4:** Let \(R\) be an ordered ring. By the definitions which are presented in (1) and (2), \((R, \oplus)\) is a commutative semihypergroup and \((R, \odot)\) is a semihypergroup.

**Proof:** There are eight cases to show the associativity of the defined hyperoperation. But since proofs of all cases are similar, we only consider the case in which \(a, b, c < 0\). In this case, we show that \(\bigcup_{s \in [−a−b], [s−c]_\leq} s \oplus c = \bigcup_{s \in [−|a|+|b|]_\leq} ([s]+|c|)_\leq\). Consider \(x \in \bigcup_{s \in [−a−b], [s−c]_\leq} s \oplus c\). Then there are \(s_0 \in R\) such that \(-a−b \leq s_0\) and \(s_0−c \leq x\). Due to associativity of operation +, we have \(-a+(−b−c)= (−a−b)−c \leq s_0−c \leq x\). Now by attention to \(-b−c \leq x\) we have \([−a−b], [s−c]_\leq\).\(\bigcup_{s \in [−a−b], [s−c]_\leq} [−a+t]_\leq\). The proof of the other side is similar. At the end we have

\[
(a \oplus b) \oplus c = \bigcup_{s \in [−a−b], [s−c]_\leq} s \oplus c = \bigcup_{s \in [−|a|+|b|]_\leq} [s]+|c| = \bigcup_{s \in [−|a|+|b|]_\leq} [s] \oplus c = a \oplus (b \oplus c).
\]

It is easy to see the commutativity of the hyperoperation \(\oplus\). For the multiplication hyperoperation \(\odot\), we will have the same proof.

**Theorem 1.5:** Let \(R\) be an ordered ring. Then \((R, \oplus, \odot)\) is a good semihypergroup.

**Proof:** It is sufficient to prove the the distribution multiplicative hyperoperation by the ratio of addition hyperoperation on the right (or left). So, let \(a, b \in R_\leq\) and \(c \in R_\geq\). Then

\[
a_1 \odot (a_2 \odot a_3) = \bigcup [a_1 \odot a_2 | a' \geq |a_2| + |a_3|] = \bigcup [a_1 \odot a' | a' \geq a_2 + a_3] = [a|a \geq |a_1| |a'|, a' \geq a_2 + a_3] = [a|a \geq -a_1 a' + a_3] = [a|a \geq -a_1 a_2 - a_3].
\]

On the other hand,

\[
(a_1 \odot a_2) \oplus (a_3 | a_1 a_3) = \bigcup [a_2 | a \geq |a_2| + |a_3|, b \geq |a_1| |a_3|] = \bigcup [a_2 \oplus b | a \geq a_2, b \geq -a_1 a_3] = [a|a \geq |a_1| + |b|, a \geq a_1 a_2, b \geq -a_1 a_3] = [a|a \geq a + b, a \geq a_1 a_2, b \geq -a_1 a_3] = [a|a \geq a_1 a_2 - a_1 a_3].
\]

The proofs of other cases are similar.

Notice that if \(R\) is not ordered, then there exists an element \(a_0 \in R\) such that \(a_0 \not\in R_\leq \cup R_\geq\), so \(|a_0|\) is meaningless. Therefore, the definitions of hyperoperations \(\oplus\) and \(\odot\) in (1) and (2), respectively, are not efficient for \(a \oplus b\) or \(a \odot b\) when at least one of \(a\) or \(b\) is not belonging to \(R_\leq \cup R_\geq\). So, we modify definitions of \(\oplus\) and \(\odot\) in (1) and (2), respectively, in the following way.

**Definition 1.6:** Let \((R, +, \leq)\) be a partially ordered ring. For \(a, b \in R\), we define

\[
a \oplus_1 b = [a|b|]_\leq \cup [a, b], \text{ eq} 3
\]

\[
a \odot_1 b = [a|b|]_\leq \cup [a, b]. \text{ eq} 4
\]

With the hyperoperations \(\oplus_1\) and \(\odot_1\) presented in (3) and (4), for any two elements \(a, b\) of partially ordered ring \(R\), we have 

\[
a \oplus_1 b \in \psi^* (R) \text{ and } a \odot_1 b \in \psi^* (R).
\]

**Example 1.7:** The hyperstructure \(R = ([a, b], \oplus, \odot)\) is defined as follows:

\[
\oplus \quad a \quad b \quad \odot \quad a \quad b
\begin{align*}
a &\quad a &\quad (a, b) &\quad a &\quad (a, b) &\quad b &\quad (a, b) &\quad b &\quad (a, b)
\end{align*}
\]

is a good hyperring. For instance, since the unions of all rows and columns of the table \(\oplus\) are equal to \(R\), so the reproduction principle is hold.

**Theorem 1.8:** The following propositions are hold:

1. Let \((R, +, \leq)\) be a partially ordered group. Then \((R, \oplus_1)\) is a hypergroup.
2. Let \((R, \leq)\) be a partially ordered group. Then \((R, \odot_1)\) is a hypergroup.

**Proof:**

1. It is sufficient to show the associative property of hyperoperation \(\oplus_1\) and the reproduction principle for the case in which at least one element is not belonging to \(R_\leq \cup R_\geq\). Let \(a \not\in R_\leq \cup R_\geq\), and
We only show that the distribution multiplication hyperoperation of Proposition (1).

\[ (a \odot b) \oplus_1 c = (|a| + |b|) \cup (a, b) \odot_1 c \]
\[ = (0 \cup (a, b)) \oplus_1 c \]
\[ = a \oplus_1 c \cup b \oplus_1 c \]
\[ = \{a, c\} \cup \{b, c\} \]
\[ = \{a, b, c\} \]
\[ = (r | r \geq b + c) \cup (a, b, c) \]

On the other hand,

\[ a \oplus_1 (b \oplus_1 c) \]
\[ = a \oplus_1 (r | r \geq b + c) \cup (b, c) \]
\[ = \{a \oplus_1 r | r \geq b + c\} \cup a \oplus_1 b \cup a \oplus_1 c \]
\[ = \{a, b, c\} \]
\[ = (r | r \geq b + c) \cup (a, b, c) \]

It is easy to see the reproduction principle is hold. (2) The proof of this proposition is similar to the proof of Proposition (1).

Also, we can see that

**Theorem 1.9:** The following propositions are hold.

(i) Let \((R, +, \leq)\) be an ordered ring. Then \((R, \oplus_1, \odot)\) is a good hyperring.

(ii) Let \((R, +, \leq)\) be a partially ordered ring. Then \((R, \oplus_1, \odot_1)\) is a hyperring.

**Proof:**

- We only show that the distribution \(\odot\) by the ratio of \(\oplus_1\) of the left. Then the case in which \(a \in R_+\) and \(b, c \in R_-\) is considered. So we have

\[ a \odot (b \oplus_1 c) \]
\[ = a \odot (r | r \geq b + c) \cup (b, c) \]
\[ = \{a \odot r | r \geq b + c\} \cup a \odot b \cup a \odot c \]
\[ = \{r | r \geq ab, r \geq -b + c\} \]
\[ = \{r | r \geq ab - ac\} \cup \{r | r \geq -ab\} \cup \{r | r \geq -ac\} \]

On the other hand,

\[ (a \odot b) \oplus_1 (a \odot c) \]
\[ = \{r | r \geq ab, r \geq ac\} \]
\[ = \{r | r \geq ab\} \cup \{r | r \geq ac\} \]
\[ = \{r | r \geq ab - ac\} \cup \{r | r \geq -ab\} \cup \{r | r \geq -ac\} \]

The distribution \(\odot\) by the ratio of \(\oplus_1\) of the right is proved similarly. Therefore, \((R, \oplus_1, \odot)\) is a good hyperring.

- We show the distribution multiplication hyperoperation \(\odot_1\) by the ratio of addition hyperoperation \(\oplus_1\) on the right (or left), only in the case that \(a, c \in R_+\) and \(b \in R_+ \cup R_-\).

\[ a \odot_1 (b \oplus_1 c) = a \odot_1 (r | r \geq b + c) \cup (b, c) \]
\[ = a \odot_1 (r | r \geq b + c) \cup a \odot_1 b \cup a \odot_1 c \]
\[ = \{a, b, c\} \]
\[ = (r | r \geq b + c) \cup (a, b, c) \]

On the other hand,

\[ (a \odot_1 b) \oplus_1 (a \odot_1 c) \]
\[ = \{r | r \geq ab\} \cup \{r | r \geq ac\} \]
\[ = \{r | r \geq ab - ac\} \cup \{r | r \geq -ab\} \cup \{r | r \geq -ac\} \]

The proofs of the other cases are done similarly. Therefore, \((R, \oplus_1, \odot_1)\) is a hyperring.

**Example 1.10:** Let \(G = (\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}, +)\) with ordinary addition of complex numbers. Put \(\mathbb{Z}[i]_+ = \{a \mid a \in \mathbb{Z}, 0 \leq a\} \cup \{bi \mid b \in \mathbb{Z}, b \leq 0\}\). Then \((\mathbb{Z}[i], \mathbb{Z}[i]_+)\) is a partially ordered group.

**Example 1.11:** Consider the partially ordered ring \(\mathbb{Z}^l\) of integral-valued functions on a set \(l = \{a, b\}\), with pointwise order. Then \((\mathbb{Z}^l, \oplus_1, \odot)\) is a hyperring.

**Definition 1.12:** [26] A canonical hypergroup is a non-empty set \(H\) endowed with an additive
hyperoperation $+: H \times H \rightarrow \wp^*(H)$, satisfying the following properties:

1. for all $x, y, z \in H$, $x + (y + z) = (x + y) + z$.
2. for all $x, y \in H$, $x+y=y+x$.
3. there exists $0 \in H$ such that $0+x=x+0=x$, for all $x \in H$.
4. for every $x \in H$, there exists a unique element $x_0 \in H$, such that $0 \in x + x_0$ (we shall write $-x$ for $x_0$ and we call it the opposite of $x$).
5. $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(H, +)$ is reversible.

**Definition 1.13:** [26] A Krasner hyperring is an algebraic hyperstructure $(R, +, .)$ which satisfies the following axioms:

1. $(R, +)$ is a canonical hypergroup,
2. $(R, .)$ is a semigroup having zero as a bilaterally absorbing element, i.e. $x.0=0,x=0$.
3. The multiplication $.$ is distributive with respect to the hyperoperation $+$.

We say that a Krasner hyperring is commutative (with unit element) if $(R, .)$ is a commutative semigroup (with unit element). Also, we say that a Krasner hyperring $R$ is a Krasner hyperfield, if $(R \setminus \{0\}, .)$ is a group. A Krasner hyperring $R$ is called a hyperdomain, if $R$ is a commutative hyperring with unit element and $a.b=0$ implies that $a=0$ or $b=0$ for all $a, b \in R$.

**Example 1.14:** Let $(A, .)$ be a semigroup with zero 0 such that $(A \setminus \{0\}, .)$ is a group. Define the hyperoperation $+$ on $A$ by

$$x + y = y + x = [x], y = 0, A \setminus \{x\}, x = y \neq 0, [x, y], x, y \in A \setminus \{0\}, x \neq y.$$ 

Then $(A, +, .)$ is a Krasner hyperring ([22]).

**Example 1.15** Consider the unit interval $[0, 1]$ and define the hyperoperation $+$ on it by

$$x + y = [\max(x, y)], x \neq y; [0, x], x = y.$$ 

Then, $([0, 1], +, .)$ is a Krasner hyperring where $.$ is the usual multiplication ([28]).

We now give an example of a finite hyperfield with two elements 0 and 1, as follows.

**Example 1.16:** Let $F_2 = \{0, 1\}$ be the finite set with two elements. Then $F_2$ becomes a Krasner hyperfield with the following hyperoperation $+$ and binary operation $\cdot$ such that

$$
\begin{array}{cccc}
+ & 0 & 1 & . \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
$$

In the following, we are trying to create new hyperstructures of (partially) ordered hyperstructures by aid of modified Ends Lemma.

**Definition 1.17:** [30] Let $(H, \circ)$ be a (partially) ordered hypergroupoid. For $a, b \in H$, we define the new hyperoperation $\ast : H \times H \rightarrow \wp^*(H)$ as follows:

$$a \ast b = [a \circ b] = \bigcup ([m], m \in a \circ b).$$

**Remark 1.2:** We name $(H, \ast)$ as the EL2-hypergroupoid associated to (partially) ordered hypergroupoid $(H, \circ, \leq)$.

**2. (Partially) ordered Krasner hyperrings**

**2.1. Ordered Krasner hyperring**

In this section, we introduce the notion of ordered Krasner hyperring and present several examples that illustrate the significance of this hyperstructure. Then we create the new hyperrigoids from that by the modified "Ends Lemma".

**Definition 2.1:** An algebraic hyperstructure $(R, +, ., \leq)$ is called a (partially) ordered Krasner hyperring if $(R, +, .)$ is a Krasner hyperring with a (partial) order relation $\leq$, such that for all $a, b$ and $c$ in $R$:

1. If $a \leq b$, then $a + c \leq b + c$, meaning that for any $x \in a + c$, there exists $y \in b + c$ and for any $y \in b + c$, there exists $x \in a + c$ such that $x \leq y$.
2. If $a \leq b$ and $0 \leq c$, then $a.c \leq b.c$ and $c.a \leq c.b$.

Indeed, the concept of ordered Krasner hyperrings is a generalization of the concept of ordered rings.

**Remark 2.1:** If a Krasner hyperring is arranged in a (partial) order, then the existence of a positive ordered Krasner hyperring is excluded. Indeed, for an element $x$ in an (partially) ordered Krasner hyperring, if $0 \leq x$, then $x + (-x) \geq 0 + (-x) = -x$. On the other hand, by attention to $0 \in x + (-x)$, we have inevitably $-x \leq 0$. 
Example 2.2: Let \( H = \{a, b, c, d\} \) be a set with the hyperoperation + defined as follows:

\[
\begin{array}{cccccccc}
+ & a & b & c & d \\
\hline
a & a & b & c & d \\
b & b & b & a, b, c, d \\
c & c & \{a, b, c\} & c & d \\
d & d & d & d & H \\
\end{array}
\]

Then, \((H, +)\) is a canonical hypergroup. Triple \((H, +, \leq)\) is a partially ordered canonical hypergroup where the order relation \(\leq\) is defined by

\[
\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (c, a), (c, b), (b, c), (b, a), (a, c)\}.
\]

Example 2.3: Let \( R = \{a, b, c\} \) be a set with the hyperaddition + and the multiplication \(\cdot\) defined as follows:

\[
\begin{array}{cccccccc}
+ & a & b & c & a b c \\
\hline
a & a & b & c & a a a a \\
b & b & b & a b c \\
c & c & R & c & c a b c \\
\end{array}
\]

Then, \((R, +, \cdot)\) is a Krasner hyperring [31]. \((R, +, \leq)\) is a quasi-ordered Krasner hyperring where the quasi-order relation \(\leq\) is defined by

\[
\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (g, g), (h, h), (a, e), (c, g), (f, b), (b, f), (e, a), (g, c), (d, h), (h, d)\}.
\]

Example 2.4: In Example 2.2, if we define the multipication \(\cdot\) on the canonical hypergroup \((R, +)\) as for any \((x, y) \in \mathbb{R}^2, x, y = a\), the resulting algebraic hyperstructure is a Krasner hyperring. Also, if the order relation \(\leq'\) is defined by

\[
\leq' := \{(a, a), (b, b), (c, c), (d, d), (a, b), (c, a), (c, b), (b, c), (b, a), (a, c)\}.
\]

then \((R, +, \cdot, \leq')\) is an ordered Krasner hyperring.

Example 2.5: Let \( R = \{a, b, c, d, e, f, g, h\} \) be a set with the hyperaddition + and the multiplication \(\cdot\) defined as follows:

\[
\begin{array}{cccccccc}
+ & a & b & c & d & e & f & g & h \\
\hline
a & a & b & c & d & e & f & g & h \\
b & b & b & a, b, c, d & b & f & f & [e, f, g, h] & f \\
c & c & \{a, b, c, d\} & c & c & g & [e, f, g, h] & g & g \\
d & d & d & c & a & h & f & g & e \\
e & e & f & g & h & [a, e] & [b, f] & [c, g] & [d, h] \\
f & f & f & [e, f, g, h] & f & [b, f] & [b, f] & R & [b, f] \\
g & g & [e, f, g, h] & g & g & [c, g] & R & [c, g] & [c, g] \\
h & h & f & g & e & [d, h] & [b, f] & [c, g] & [a, e] \\
\end{array}
\]

and

\[
\begin{array}{cccccccc}
+ & a & b & c & d & e & f & g & h \\
\hline
a & a & b & c & d & e & f & g & h \\
b & b & b & a, b, c, d & b & c & d & c a c b d & a c b d \\
c & c & \{a, b, c\} & c & d & d a a a a a a a a e a a a e e e e \\
d & d & d & d & d a a a a a a a a e a a a e e e e \\
\end{array}
\]

Then, \((R, +, \cdot)\) is a Krasner hyperring. \((R, +, \leq)\) is a quasi-ordered Krasner hyperring where the quasi-order relation \(\leq\) is defined by

\[
\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (g, g), (h, h), (a, e), (c, g), (f, b), (b, f), (e, a), (g, c), (d, h), (h, d))\}.
\]

In the following, we try to modify Definition 1.17. for (partially) ordered hyperringoids. First of all, we present some necessary lemmas.

Lemma 2.6: Let \((R, \leq)\) be a (partially) ordered set and \(A \subseteq R\). Then we have

1. \(A \leq \{A\}_{\leq}\),
2. the set \(\{A\}_{\leq}\) is the maximum element of the family of subsets \(A'\) of \(R\) such that \(A \leq A'\).

Proof:

1. We have \(A \subseteq \{A\}_{\leq} = \bigcup_{a \in A} \{a\}_{\leq}\). So, according to the reflexivity of the (partial) order relation \(\leq\), for every \(a \in A\), there exists \(y = a \in \{a\}_{\leq}\) such that \(a \leq y\). On the other hand, if \(y \in \{a\}_{\leq}\) is an arbitrary element, then there exists \(a \in A\) such that \(y \in \{a\}_{\leq}\), or equivalently, there exists \(a \in A\) such that \(a \leq y\).
2. Suppose to the contrary, there exists \(B \subseteq R\) such that \(\{A\}_{\leq} \subset \bigcup_{a \in A} \{a\}_{\leq}\). As a result for every \(a \in A\), \(x_0 < a\). On the other hand, from \(A \leq B\) and \(x_0 \in B\), it follows that there exists \(a \in A\) such that \(a \leq x_0\). It is a contradiction.
**Lemma 2.7:** Let \((R, \leq)\) be a (partially) ordered set and \(A \subseteq R\). Then we have \([A]_\leq = ([A])_\leq\).

**Proof:** According to the maximality of \([A]_\leq\) among all the subsets \(X\) of \(R\) satisfying the property \(A \leq X\), it is sufficient that prove \(A \leq ([A])_\leq\). Since \(A \subseteq [A]_\leq \subseteq ([A])_\leq\), from \(a \in A\), it follows that \(a \in ([A])_\leq\), and so by the reflexivity, for every \(a \in A\), there exists \(a \in ([A])_\leq\) such that \(a \leq a\). On the other hand, if \(x \in ([A])_\leq = \bigcup_{y \in ([A])_\leq} [y]_\leq\), then there exists \(y \in [A]_\leq\) such that \(y \leq x\). Also, from \(y \in [A]_\leq\), it follows that there exists \(a \in A\) such that \(a \leq y\). As a result, according to the transitivity of the (partial) order relation, there exists \(a \in A\) such that \(a \leq x\).

Due to the maximality of \([A]_\leq\) among all the subsets \(X\) of \((R, \leq)\) satisfying the property \(A \leq X\), the following corollary is easy to see.

**Corollary 2.8** For non-empty subsets \(A\) and \(B\) of a (partially) ordered canonical hypergroup \((H, +, \leq)\), we have

1. \([A]_\leq + [B]_\leq = [A + B]_\leq\) and \([A + B]_\leq = [A]_\leq + [B]_\leq\);
2. \([A + B]_\leq = [A]_\leq + [B]_\leq\);
3. \([A]_\leq + [B]_\leq \leq [A + B]_\leq\).

**Lemma 2.9:** Let \((R, +, \leq)\) be a (partially) ordered Krasner hyperring, \(a \in R_1\) and \(B \subseteq R\). Then we have \(a \cdot [B]_\leq \subseteq ([a]_\leq \cdot B)_\leq\). If \(R\) is a (partially) ordered Krasner hyperfield, then \(a \cdot [B]_\leq = ([a]_\leq \cdot B)_\leq\).

**Proof:** If \(y \in a \cdot [B]_\leq\), then there exists \(x \in [B]_\leq\) such that \(y = ax\). Since \(x \in [B]_\leq\), so there exists \(b \in B\) such that \(b \leq x\). Therefore \(ab \leq ax\) and \(ab \in a \cdot B\). Thus \(y = ax \in [a \cdot B]_\leq\). Now, if \(R\) is a (partially) ordered Krasner hyperfield and \(y \in [a \cdot B]_\leq\), then there exists \(b \in B\) such that \(ab \leq y\). Since \(a\) and therefore \(a^{-1}\) are positive, we have \(b = a^{-1}ab \leq a^{-1}y\). Also, we have \(ab \leq a(a^{-1}y) = y\). Now, by putting \(x = a^{-1}y\), we have \(b \leq x\), and as a result \(y \in a \cdot [B]_\leq\).

**2.2. \(EL^2\)-hyperringoids**

In the following, we are going to construct new hyperringoids from (partially) ordered Krasner hyperrings, through which we have achieved so far.

**Definition 2.10:** Let \((R, +, \leq)\) is an ordered Krasner hyperring. For \(a, b \in R\), we define the new hyperoperation \(\oplus : R \times R \rightarrow \mathcal{P}^\ast(R)\) as follows:

\[
a \oplus b = [\langle a \rangle + \langle b \rangle]_\leq = \bigcup_{m \in \langle a \rangle + \langle b \rangle} \{m\}_\leq.
\]

**Remark 2.2**

1. It is easy to see that Definition 2.10 is not useful for satisfying the reproduction principle. Indeed, there exists no positive-ordered Krasner hyperring. On the other hand, this definition is meaningless for partially ordered Krasner hyperrings which are not ordered. So we improve Definition 2.10 as follows:

\[
a \oplus_1 b = [\langle a \rangle + \langle b \rangle]_\leq = \bigcup_{m \in \langle a \rangle + \langle b \rangle} \{m\}_\leq.
\]

2. If \(\cdot\) is the multiplicative operation in a (partially) ordered Krasner hyperring, then due to Definition 1.1, we define the new multiplicative operation as follows:

\[a \cdot_1 b = [a]_\leq \cdot_1 [b]_\leq = [\langle a \rangle \cdot_1 \langle b \rangle]_\leq.
\]

**Theorem 2.11:** Let \((H, +, \leq)\) is an ordered canonical hypergroup. Then the hyperoperation \(\oplus\) is associative.

**Proof:** Suppose that \(0 \leq a, c\) and \(b \leq 0\). Then

\[
(a \oplus b) \oplus c = [\langle a \rangle + \langle b \rangle]_\leq \oplus c = [\langle [a]_\leq + [b]_\leq + [c]_\leq\)
\]

\[
= [\langle a \cdot_1 b \rangle]_\leq + c]_\leq = [\langle a \cdot_1 (c - b) \rangle]_\leq = a \oplus (b \cdot_1 c).
\]

Other states are proved in the same way.

**Theorem 2.12:** Let \((H, +, \leq)\) is a partially ordered canonical hypergroup. Then \(EL^2\)-hyperringoid \((H, \cdot_1)\) is a hypergroup.

**Proof:** Suppose that \(0 \leq a, c\) and \(b \leq 0\). Then

\[
(a \oplus_1 b) \oplus_1 c = ([\langle a \rangle + \langle b \rangle]_\leq \cdot_1 \oplus_1 c)
\]

\[
= ([\langle a \cdot_1 b \rangle]_\leq \cdot_1 \oplus_1 c)
\]

\[
= ([\langle a - b \rangle]_\leq \oplus_1 c) \cup \{a \cdot_1 c \cup (b \cdot_1 c)
\]

\[
= ([\langle a - b \rangle]_\leq \oplus_1 c) \cup \{a \cdot_1 c \cup (\langle a \cdot_1 (c - b) \rangle)]_\leq
\]

\[
= [\langle a \cdot_1 (b \cdot_1 c) \rangle]_\leq
\]

Now, if \(0 \leq a, c \leq 0\) and \(\langle (0, b), (b, 0) \rangle \leq\), then we
have
\[(a \oplus_1 b) \oplus_1 c = (|a| + |b|) \cup |a, b| \oplus_1 c = (\emptyset \cup |a, b|) \oplus_1 c = (a \oplus_1 c) \cup (b \oplus_1 c).\]

on the other hand,
\[a \oplus_1 (b \oplus_1 c) = a \oplus_1 (\emptyset \cup (b, c)) = (a \oplus_1 c) \cup (b \oplus_1 c).\]

Other states are proved in the same way. Due to the definition \(\oplus_1\), it is easy to see that reproduction principles are established.

**Theorem 2.13:** Let \((R, +, \leq, \cdot)\) is an ordered Krasner hyperring. Then the algebraic structure \((R, \cdot)\) is a semigroup.

**Proof:** Suppose that \(0 \leq a, b, c \leq 0\). Then
\[
\begin{align*}
(a \cdot b) \cdot c &= (|a|, |b|) \cdot c = (-a) \cdot c = (-ab)(-c) \\
&= abc = ab(c) = |a|(|b|\cdot c) = a \cdot (b \cdot c).
\end{align*}
\]

**Theorem 2.14:** Let \((R, +, \leq, \cdot)\) is an ordered Krasner hyperring. Then the hyperstructure \((R, \cdot)\) is an additive hyperring.

**Proof:** Due to the Theorems 2.7, 2.8 and 2.9, it is sufficient that we show the distribution \(\cdot\) by the ratio of \(\oplus_1\) on the left. So, suppose that \(b \leq 0\) and \(0 \leq a, c\):
\[
\begin{align*}
(a \cdot (b \oplus_1 c)) &= a \cdot ((-b + c) \cup (b, c)) \\
&= a \cdot (-b + c) \cup (a \cdot b, a \cdot c) \\
&\subseteq [a(-b + c)] \subseteq [-ab, ac] \\
&= [ac - ab] \cup [-ab, ac],
\end{align*}
\]

on the other hand,
\[
\begin{align*}
(a \cdot b) \oplus_1 (a \cdot c) &= (-ab) \oplus_1 (ac) \\
&= [-ab + ac] \cup [-ab, ac].
\end{align*}
\]

The distribution \(\cdot\) by the ratio of \(\oplus_1\) on the right is proved, similarly.

**Theorem 2.15:** Let \((R, +, \leq, \cdot)\) is an ordered Krasner hyperfield. Then the hyperstructure \((R, \oplus_1, \cdot)\) is a good additive hyperring.

**Proof:** Due to Lemma 2.5, it is easy to prove.

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