THE CONGRUENCE SUBGROUP PROPERTY DOES NOT IMPLY INVARIABLE GENERATION

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Abstract. It was suggested in [KLS14] that for arithmetic groups Invariable Generation is equivalent to the Congruence Subgroup Property. In this paper we dismiss this conjecture by proving that certain arithmetic groups which possess the latter property do not possess the first one.

1. INTRODUCTION

In an attempt to give a purely algebraic characterisation for the Congruence Subgroup Property (CSP), it was asked in [KLS14] whether for arithmetic groups the CSP is equivalent to Invariable Generation (IG). This was partly motivated by the result of [KLS14] that the pro-finite completion of an arithmetic group with the CSP is topologically finitely invariably generated. Another result in the positive direction was obtained in [G15], namely that discrete subgroups, and in particular arithmetic subgroups, of rank one simple Lie groups are not IG. Recall the famous Serre conjecture that an irreducible arithmetic lattice \( \Gamma \leq G \) in a semisimple Lie group has the CSP iff \( \text{rank}_\mathbb{R}(G) \geq 2 \). In the current paper we give a negative answer to the KLS question by producing various counterexamples of arithmetic groups in higher rank which are not IG. Some of our examples are known to possess the CSP.

1.1. Invariable generation. Recall that a subset \( S \) of a group \( \Gamma \) invariably generates \( \Gamma \) if \( \Gamma = \langle s^{g(s)} | s \in S \rangle \) for every choice of \( g(s) \in \Gamma, s \in S \). One says that a group \( \Gamma \) is in invariably generated, or shortly IG if one of the following equivalent conditions holds:

1. There exists \( S \subseteq \Gamma \) which invariably generates \( \Gamma \).
2. The set \( \Gamma \) invariably generates \( \Gamma \).
3. Every transitive permutation representation on a non-singleton set admits a fixed-point-free element.
4. \( \Gamma \) does not have a proper subgroup which meets every conjugacy class.

The main results of this paper are that certain lattices are not IG.

A matrix \( g \in \text{GL}_n(\mathbb{C}) \) is net if the multiplicative group \( A(g) \) generated by the eigenvalues of \( g \) does not contain any nontrivial root of unity. A linear group \( \Gamma \leq \text{GL}_n(\mathbb{C}) \) is net if all its nontrivial elements are net. It is well known that every finitely generated linear group
admits a finite index net subgroup (see [R72, Theorem 6.11]). The assumption that $\Gamma$ is net is not crucial but very convenient for our purpose.

**Theorem 1.1.** Let $Q$ be a rational quadratic form of signature $(2, 2)$, let $G = SO(Q)$ and let $\Gamma$ be a finite index net subgroup in $G(\mathbb{Z})$. Then $\Gamma$ is not invariably generated.

The group $\Gamma$ in Theorem 1.1 is an irreducible lattice in the higher rank semisimple group $G = SO(2, 2) \cong PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$, and as such is super-rigid, possesses the normal subgroup property and the strong approximation theorem. By Kneser’s theorem [KM79] $\Gamma$ has the CSP (when the form is isotropic this also follows from Raghunathan’s theorem [R76, R86]). In particular the groups obtained in this way are counterexamples to the question of [KLS14].

We also prove a general theorem (see Theorem 2.2) that under a certain condition on $G$ every regular lattice in $G$ (see Section 2.1 for the definition) is not IG. This produces a large family of examples, for instance:

**Example 1.2.** (See [WM14, Proposition 6.61]) Let $L$ be a cubic Galois extension of $\mathbb{Q}$. Let $D$ be a central division algebra of degree 3 over $\mathbb{Q}$ which contains $L$ as a subfield. Let $\mathcal{O}_D$ be an order in $D$. Then $SL(1, \mathcal{O}_D)$ is isomorphic to an anisotropic arithmetic lattice in $SL(3, \mathbb{R})$.

Being a lattice in $SL(3, \mathbb{R})$, the group $SL(1, \mathcal{O}_D)$ possesses many rigidity properties such as those mentioned after Theorem 1.1 and even more; for instance it has Kazhdan’s property (T) and IRS-rigidity (the Stuck–Zimmer theorem). The corresponding lattice is cocompact. By Serre’s conjecture, $SL(1, \mathcal{O}_D)$ is expected to possess the CSP, but this is currently unknown.

**Theorem 1.3.** The arithmetic group $SL_1(\mathcal{O}_D)$ is not IG.

Theorem 2.2 is more general and applies to many other example.

1.2. **The motivation.** Apart from trying to better understand the notion of invariable generation in infinite groups, our motivation arose from an attempt to understand better ‘thin subgroups’ of arithmetic groups. For instance an important question in this theory is whether every element belongs to some thin subgroup.

**Definition 1.4.** Let $\Gamma$ be an arithmetic group. Let us say that an element $\gamma \in \Gamma$ is fat if every Zariski dense subgroup containing it is automatically of finite index, i.e. $\gamma$ does not belong to any thin subgroup of $\Gamma$.

The existence and classification problems of fat elements are extremely interesting and may shed light on many problems concerning thin subgroups. Being fat is clearly invariant under conjugacy, hence the existence of such elements is closely related to invariable generation. Indeed, if $\Gamma$ admits an infinite index Zariski dense subgroup which meets every conjugacy class then there are no fat elements in $\Gamma$. Therefore our results imply that in certain arithmetic groups all elements are ‘thin’:

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1The origin of this notion is from Galois theory, and it is well known that finite groups are always IG.
Corollary 1.5. The arithmetic groups from Theorem 1.1 and Theorem 2.2 do not admit fat elements.

This applies for instance to certain lattices in SL(3, R) (cf. Example 1.2). However, our method does not apply to SL(3, Z). New tools are required in order to study property IG or existence of fat elements for this group.

Question 1.6. Is SL(3, Z) invariably generated? What about SL(n, Z) for n ≥ 4?

1.3. About the proofs. As in [W76] and [G15] the idea is to construct an independent set consisting of one representative of each non-trivial conjugacy class. We use of course the dynamics on the associated projective space P, where we construct such an infinite family of ping-pong partners. In the proof of Theorem 2.2 we use the standard action on points in P. The dynamical picture in the proof of Theorem 1.1 is more involved and we are led to consider the action on lines in P rather than points. Yet the spirit of the proof is similar.

The main technical novelty of this paper is a method that allows us to conjugate certain elements, in particular R-regular elements, such that their dynamics will simulate unipotent dynamics — in the sense that all non-trivial powers contract the complement of some small open set into that set. We first allow ourselves to use elements from the enveloping group G and then apply Poincaré recurrence theorem in order to approximate the solution in Γ.

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2. R-regular lattices

Suppose that G = G(R) is a Zariski connected linear real algebraic group with a faithful irreducible representation in a d-dimensional real vector space, with associated projective space P ≅ P^{d-1}(R). We identify G with its image in GL_d(R). An element g ∈ G is regular if it is either a regular unipotent or all its eigenvalues are of distinct absolute value (in particular each has multiplicity one). A subgroup Γ ≤ G is regular if every nontrivial element of Γ is regular. For an element g ∈ G we denote by v_g and H_g the attracting point and repelling hyperplane of g in P, assuming they are unique. Let us say that a point or a hyperplane is G-defined if it is of the form v_g (resp. H_g) for some g ∈ G. In addition let us say that an incident pair (v, H) is G-defined if there is a regular element g ∈ G for which v = v_g and H = H_{g^{-1}}. Let us also say that an ordered pair (U, V) of open neighborhoods in P is G-applicable if there is a G-defined incident pair (v, H) with v ∈ U and H entirely contained in the interior of the complement of V. We shall further suppose the following:

Assumption 2.1. The set of G-defined incident pairs is not contained in a proper subvariety of the set of incident pairs.

Theorem 2.2. Let G be a Zariski connected linear real algebraic group with a faithful irreducible representation on a d-dimensional vector space which satisfies Assumption 2.1
and let $\Gamma \leq G$ be a regular lattice. Then $\Gamma$ admits an infinite rank free subgroup which intersects every conjugacy class. In particular, $\Gamma$ is not invariably generated.

**Remark 2.3.** For $G = \text{SL}_d(\mathbb{R})$ with its standard $d$-dimensional representation Assumption 2.1 is obviously satisfied. Moreover it is easy to check that the lattices in Example 1.2 are regular, by construction. Thus Theorem 1.3 is a special case of Theorem 2.2.

2.1. The proof of Theorem 2.2.

**Proposition 2.4 (Main Proposition).** Let $C$ be a nontrivial conjugacy class in $\Gamma$, and let $(U, V)$ be a $G$-applicable pair of open sets in $\mathbb{P}$. Then there is $\gamma \in C$ such that $\gamma^n \cdot V, n \in \mathbb{Z} \setminus \{0\}$ are all disjoint and contained in $U$.

**Lemma 2.5.** Let $\gamma \in \Gamma \setminus \{1\}$, and let $(U, V)$ be a $G$-applicable pair of open sets in $\mathbb{P}$. Then there is $\delta \in \Gamma$ such that

1. $H_{\delta^{-1}} \cap \overline{V} = \emptyset$,
2. $v_{\gamma}, v_{\gamma^{-1}} \not\in H_{\delta}$,
3. $v_{\delta^{-1}} \not\in H_{\gamma} \cup H_{\gamma^{-1}}$,
4. $v_{\delta} \in U$, and
5. $\gamma^n \cdot v_{\delta^{-1}} \not\in H_{\delta}$ for all $n \neq 0$.

**Proof.** As a first step, observe that there exists a regular $g \in G$ such that all the five conditions hold with $g$ instead of $\delta$. Indeed, as $(U, V)$ is $G$-applicable there is $g$ for which (1) and (4) are satisfied. Moreover, in view of Assumption 2.1, for every $n \in \mathbb{N} \setminus \{0\}$ the set of regular $g \in G$ such that $\gamma^n \cdot v_{g^{-1}} \in H_g$ is nowhere-dense. Hence by the Baire category theorem and the fact that $G$ admits an analytic structure, we may slightly deform $g$ in order to guarantee that condition (5) holds as well. Since $G$ is Zariski connected and irreducible, the set of $h \in G$ such that either (2) or (3) do not hold for $g^h$ is nowhere-dense. Thus, up to replacing $g$ by a conjugate $g^h$ with $h$ sufficiently close to 1, all the five conditions are satisfied.

Next note that $\gamma^n \cdot v_{\delta^{-1}}$ tends to $v_\gamma$ when $n \to \infty$ and to $v_{\gamma^{-1}}$ when $n \to -\infty$. Thus if we pick a sufficiently small symmetric identity neighborhood $\Omega \subset G$ and $\epsilon > 0$ sufficiently small, then we have:

1. $\Omega \cdot (H_g^{-1})_\epsilon \cap \overline{V} = \emptyset$,
2. $v_\gamma, v_{\gamma^{-1}} \not\in \Omega \cdot (H_g)_\epsilon$,
3. $\Omega \cdot (v_{g^{-1}})_\epsilon \cap (H_\gamma \cup H_{\gamma^{-1}}) = \emptyset$,
4. $\Omega \cdot (v_g)_\epsilon \subset U$,
5. $\gamma^n \Omega \cdot (v_{g^{-1}})_\epsilon \cap \Omega \cdot (H_g)_\epsilon = \emptyset$ for all $n \neq 0$, and
6. $\Omega \cdot (v_g)_\epsilon \cap \Omega \cdot (H_g)_\epsilon = \emptyset = \Omega \cdot (v_{g^{-1}})_\epsilon \cap \Omega \cdot (H_{g^{-1}})_\epsilon$.

Finally we wish to replace $g$ by an element $\delta \in \Gamma$ with similar dynamical properties. Since $\Gamma$ is a lattice, $G/\Gamma$ carries a $G$-invariant probability measure. Let $\pi : G \to G/\Gamma$ denote the quotient map. By the Poincaré recurrence theorem we have that for arbitrarily large $m$, $g^m \cdot \pi(\Omega) \cap \pi(\Omega) \neq \emptyset$, which translates to $\Omega g^m \Omega \cap \Gamma \neq \emptyset$, since $\Omega$ was chosen to be symmetric.
Note that replacing $g$ by a power $g^m$, $m > 0$ does not change the attracting points and repelling hyperplanes, while for larger and larger $m$ the element $g^m$ is becoming more and more proximal. In particular, if $m$ is sufficiently large then

$$g^m \cdot (\mathbb{P} \setminus (H_g)_\epsilon) \subset (v_g)_\epsilon, \text{ and } g^{-m} \cdot (\mathbb{P} \setminus (H_{g^{-1}})_\epsilon) \subset (v_{g^{-1}})_\epsilon.$$  

In view of Item (6), this implies that any element of the form $h = w_1 g^m w_2$ with $w_1, w_2 \in \Omega$ is very proximal and satisfies

$$v_h \in \Omega \cdot (v_g)_\epsilon, \quad v_{h^{-1}} \in \Omega \cdot (v_{g^{-1}})_\epsilon, \quad H_h \subset \Omega \cdot (H_g)_\epsilon, \quad H_{h^{-1}} \subset \Omega \cdot (H_{g^{-1}})_\epsilon.$$  

In particular, we obtain the desired $\delta$ by choosing any element from the nonempty set $\Gamma \cap \Omega g^m \Omega$.

**Proof of Proposition 2.4.** Pick $\gamma'$ in $C$ and let $\delta$ be the element provided by Lemma 2.5. Items (2,3,5) of the lemma together with the contraction property of high powers of $\gamma'$ guarantee that if $O$ is a sufficiently small neighborhood $v_{\delta^{-1}}$ then its $\langle \gamma' \rangle$ orbit stays away from $H_\delta$, i.e. there is a neighborhood $W$ of $H_\delta$ disjoint from all $\gamma^n \cdot O, \ n \in \mathbb{Z} \setminus \{0\}$. In particular the $\langle \gamma' \rangle$ translates of $O$ are all disjoint, i.e. $O$ is $\langle \gamma' \rangle$-wondering (in the terminology of \cite{G15}). If $k$ is sufficiently large then by Item (1) of the lemma $\delta^{-k} \cdot V \subset O$ while by Item (4), $\delta^k \cdot (\mathbb{P} \setminus W) \subset U$. It follows that $\gamma = \delta^k \gamma' \delta^{-k}$ is our desired element.

We shall also require the following:

**Lemma 2.6.** There is a countable family of disjoint open sets $U_k, \ k \in \mathbb{N}$ such that if we let $V_k = \bigcup_{i \neq k} U_i$ for any $k \in \mathbb{N}$ then the pairs $(U_k, V_k), \ k \in \mathbb{N}$ are all $G$-applicable.

**Proof.** Let $g \in G$ be a regular element and set $(v, H) = (v_g, H_g)$. Pick $h \in G$ near the identity such that $h \cdot v \notin H$ and $v \notin h \cdot H$. Let $c : [0, 1] \to G$ be an analytic curve with $c(0) = g$ and $c(1) = h$. Since the curve $c$ is analytic, the boundary conditions allow us to chose a sequence $t_k \to 1$ such that if we let $(v_k, H_k) = (c(t_k) \cdot v, c(t_k) \cdot H)$ and $(v_\infty, H_\infty) = (h \cdot v, h \cdot H)$, then for all $i \neq j \leq \infty$ we have $v_i \notin H_j$. In addition, the set of pairs $\{(v_k, H_k) : k < \infty\}$ is discrete and accumulates to $(v_\infty, H_\infty)$. Thus, we may choose neighborhoods $U_k \supset v_k$ sufficiently small so that $\overline{U_i} \cap H_j = \emptyset, \ \forall i \neq j$ and so that the only point in $\overline{U_k} \setminus \bigcup U_k$ is $h \cdot v$. It is evident that the sets $U_k, \ k \in \mathbb{N}$ satisfy the required property.

The strategy of the proof of Theorem 2.2 is as follows. Let $(U_k, V_k), \ k \in \mathbb{N}$ be as in Lemma 2.6. Let $C_k$ be an enumeration of the non-trivial conjugacy classes in $\Gamma$. In view of Proposition 2.4, we may pick $\gamma_k \in C_k$ so that

$$\gamma_k^n \cdot V_k \subset U_k, \ \forall n \in \mathbb{Z} \setminus \{0\}.$$  

In order to complete the proof of Theorem 2.2 we may address the following variant of the ping-pong lemma (cf. \cite[Proposition 3.4]{G15}):
Proposition 2.7. Let $\Gamma$ be a group and $X$ a compact $\Gamma$-space. Let $\gamma_k \in \Gamma$, $k \in \mathbb{N}$ be elements of $\Gamma$, and suppose there are disjoint subsets $U_k \subset X$ such that for all $k \neq j$ and all $n \in \mathbb{Z} \setminus \{0\}$ we have $\gamma_k^n \cdot U_j \subset U_k$. Then

$$\Delta = \langle \gamma_k, k = 1, 2, 3, \ldots \rangle = *_k \langle \gamma_k \rangle,$$

is the free product of the infinite cyclic groups $\langle \gamma_k \rangle, k = 1, 2, 3, \ldots$.

Furthermore, the limit set $L(\Delta)$ is contained in $\cup_k U_k$.

3. Arithmetic groups of type $\text{SO}(2, 2)$

Assumption 3.1. Let $q(x)$ be an integral quadratic form of signature $(2, 2)$ and let $(\cdot, \cdot)$ be the corresponding bilinear form. Let $\Gamma \leq \text{SO}(q, \mathbb{Z})$ be a finite index net subgroup.

Lemma 3.2. Let $f$ be the characteristic polynomial of some $g \in \Gamma$. Then $f$ is self reciprocal and one of the following two conditions holds:

- All the roots of $f$ are real.
- All the roots of $f$ are non-real complex numbers of absolute values different than 1.

Proof. We first show that $f$ is reciprocal. The free coefficient of $f$ is 1 and $f$ is a monic polynomial of degree 4. Thus, in order to show that $f$ is reciprocal it is enough to prove that if $\alpha$ is a root of $f$ then so is $1/\alpha$. Let $v_1 \in \mathbb{C}^4$ be an eigenvector of $g$ with eigenvalue $\alpha$. Denote $V := \{x \in \mathbb{C}^4 \mid (v_1, x) = 0\}$ and let $v_2 \in \mathbb{C}^4$ be such that $(v_1, v_2) = 1$. Then, $gv_2 = (1/\alpha)v_2 + v_3$ for some $v_3 \in V$. Moreover, $g$ preserves $V$ and acts on $\mathbb{C}^4/V$ as multiplication by $1/\alpha$. This implies that $1/\alpha$ is an eigenvalue of $g$.

The next step is to show that it is not possible for $f$ to have a real root $\alpha$ with absolute values different than 1 as well as a non-real root $re^{i\theta}$. Assume otherwise, since $f$ is real and reciprocal $r = 1$ and $f(x) = (x - \alpha)(x - \frac{1}{\alpha})(x - e^{i\theta})(x - e^{-i\theta})$ where $\alpha$ and $\theta$ are real numbers, $|\alpha| \neq 1$ and $\theta$ is not an integral multiple of $\pi$. Choose $v_1 \in \mathbb{R}^4$ such that $gv_1 = \alpha v_1$. Define $U := \{x \in \mathbb{R}^4 \mid (g - e^{i\theta})(g - e^{-i\theta})x = 0\}$, $V := \{x \in \mathbb{R}^4 \mid (v_1, x) = 0\}$ and $W := \{x \in \mathbb{R}^4 \mid (g - \alpha)(g - \frac{1}{\alpha})x = 0\}$. The non-trivial subspace $U \cap V$ of $U$ is invariant under $g$ so that the assumption on $\theta$ implies that it must be equal to $U$. In particular, every element of $U$ is orthogonal to $v_1$. Choose $v_2 \in W$ to be an eigenvector with eigenvalue $\frac{1}{\alpha}$. Let $x \in U$ be orthogonal to $v_2$, then

$$0 = (v_2, x) = (gv_2, gx) = \frac{1}{\alpha}(v_2, gx).$$

Thus, the assumption on $\theta$ implies that $(v_2, x) = 0$ for every $x \in U$. It follows that $\mathbb{R}^4$ is an orthogonal sum of $W$ and $U$ and either $W$ and $U$ both have signature $(1, 1)$ or one of them is positive definite and the other is negative definite. The second case is not possible since an orthogonal transformation of a definite space cannot have an eigenvalue with absolute value strictly greater than 1. The first case cannot happen since any orthogonal transformation of a quadratic space of signature $(1, 1)$ is diagonalisable over $\mathbb{R}$.

If $z \in \mathbb{C} \setminus \mathbb{R}$ is a root of $f \in \mathbb{Z}[x]$ then so is $\bar{z}$. Thus if $f(x)$ has a non-real root $z = re^{i\theta}$ with absolute value greater than 1 then $re^{-i\theta}, \frac{1}{r}e^{i\theta}$ and $\frac{1}{r}e^{-i\theta}$ are the other roots of $f$.
Finally, \( f(x) \) cannot have a non-real root with absolute value 1 since then the previous paragraph implies that all the roots of \( f(x) \) must have absolute value 1. Thus they are roots of unity which contradicts the assumption about the eigenvalues of elements in \( \Gamma \). □

**Lemma 3.3.** Let \( \gamma \in \Gamma \) be a non-trivial element and denote its eigenvalues by \( \alpha, \beta, \frac{1}{\beta}, \frac{1}{\alpha} \) where \( |\alpha| \geq |\beta| \geq 1 \). Then there exists a base \( B = (v_1, v_2, v_3, v_4) \) of \( \mathbb{R}^4 \) for which the representative matrices \([\gamma]_B\) and \([q]_B\) are as in Table 1.

**Proof.** We start by proving that the Jordan blocks of elements in \( \Gamma \) are of sizes \([1, 1, 1, 1], [2, 2]\) or \([3, 1]\). Assume that \( \gamma \) has Jordan blocks of sizes \([2, 1, 1]\). By replacing \( \gamma \) with \( \gamma^{-1} \) if necessary we can assume that there exists a base \( B = (v_1, v_2, v_3, v_4) \) of \( \mathbb{R}^4 \) such that \([\gamma]_B = \text{diag}(J_2(\alpha), \frac{1}{\alpha}, \frac{1}{\alpha})\). For every \( i \neq 2 \) and every \( m \)

\[
(2) \quad (v_2, v_1) = (\gamma^m v_2, \gamma^m v_1) = (\alpha^mv_2 + m\alpha^{m-1}v_1, \alpha^{\pm m}v_1)
\]

so \( v_1 \) is orthogonal to \( v_1, v_3 \) and \( v_4 \). Moreover, \( (v_2, v_1) = 0 \) since for every \( m \geq 1 \),

\[
(3) \quad (v_2, v_2) = (\gamma^m v_2, \gamma^m v_2) = \alpha^{2m}(v_2, v_2) + 2m\alpha^{m-1}(v_2, v_1) + m^2\alpha^{2m-2}(v_1, v_1).
\]

Thus, \( v_1 \) is orthogonal to \( \mathbb{R}^4 \), a contradiction.

Assume that \( \gamma \) has a unique Jordan block of size 4 and let \( B = (v_1, v_2, v_3, v_4) \) be a base of \( \mathbb{R}^4 \) such that \([\gamma]_B = J_4(1)\). Then for every \( m \geq 1 \):

\[
(4) \quad (v_1, v_4) = (\gamma^m v_1, \gamma^m v_4) = (v_1, v_4 + mv_3 + \frac{m(m-1)}{2}v_2 + \frac{m(m-1)(m-2)}{6}v_1)
\]

\[
(5) \quad (v_3, v_3) = (\gamma^m v_3, \gamma^m v_3) = (v_3 + mv_2 + \frac{m(m-1)}{2}v_1, v_3 + mv_2 + \frac{m(m-1)}{2}v_1)
\]

From Equation 4 it follows that \( v_1 \) is orthogonal to \( v_1, v_2 \) and \( v_3 \). Equation 5 then shows that \( v_2 \) is orthogonal to itself and to \( v_3 \). Thus, the 2-dimensional subspace spanned by \( v_1 \) and \( v_2 \) is orthogonal to the 3-dimensional linear subspace spanned by \( v_1, v_2 \) and \( v_3 \), a contradiction.

In the rest of the proof we exhibit the existence of the required base by a case by case verification (Lemma 3.2 and the first part of the proof imply that the cases below account for all possible cases).

Case 1: Let \( V := \{ x \mid (\gamma^2 - 2r \cos \theta \gamma + r^2)x = 0 \} \) and \( U := \{ x \mid (\gamma^2 - \frac{2}{r} \cos \theta \gamma + \frac{1}{r^2})x = 0 \} \). Then the restriction of \( q \) to \( V \) and to \( U \) is the zero form. There exists a base \( v_1, v_2 \) of \( V \) such that \([\gamma]_{v_1,v_2} = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\) where \( c \) and \( d \) are as in Table 1. Since \((\cdot, \cdot)\) is non-degenerate there exists a base \( v_3, v_4 \) of \( U \) such that \([q]_{v_1,v_2,v_3,v_4} \) has the required form. Since \( \gamma \) belongs to the orthogonal group of \( q \), \([\gamma]_{v_1,v_2,v_3,v_4} \) must have the required form.

Cases 2 and 3 can be readily verified.
Cases 4: Assume that the Jordan blocks of $\gamma$ are of sizes $[2,2]$ and $\alpha \neq 1$. Let $B := (v_1, v_2, v_3, v_4)$ be a base such that $[\gamma]_B$ has the required form. For every $m$:

\[
(v_2, v_2) = (\gamma^m v_2, \gamma^m v_2) = (\alpha^m v_2 + ma^{-m-1} v_1, \alpha^m v_2 + ma^{-m-1} v_1)
\]

so the restriction of $q$ to the span of $v_1$ and $v_2$ is zero and similarly the restriction of $q$ to the span of $v_3$ and $v_4$ is zero. Moreover, for every $m$

\[
(v_2, v_3) = (\gamma^m v_2, \gamma^m v_3) = (\alpha^m v_2 + ma^{-m-1} v_1, \alpha^{-m} v_3)
\]

so $(v_1, v_3) = 0$.

Moreover, $\alpha^2(v_2, v_3) = -(v_1, v_4)$ since for every $m$

\[
(v_2, v_4) = (\gamma^m v_2, \gamma^m v_4) = (v_2 + mv_1, v_2 + mv_1)
\]

so $v_1$ is orthogonal to $v_1$ and $v_2$ and similarly $v_3$ is orthogonal to $v_3$ and $v_4$. Moreover, for every $m$

\[
(v_2, v_3) = (\gamma^m v_2, \gamma^m v_3) = (v_2 + mv_1, v_3)
\]

so $(v_1, v_3) = 0$. Since $q$ is non-degenerate, $v_1$ is not orthogonal to $v_4$ and $v_3$ is not orthogonal to $v_2$. By replacing $v_2$ with $v_2 - \frac{(v_2, v_3)}{(v_2, v_3)} v_3$ and $v_4$ with $v_4 - \frac{(v_4, v_3)}{(v_4, v_3)} v_3$ we can assume that $(v_2, v_2) = (v_4, v_4) = 0$. By replacing $v_4$ with $v_4 - \frac{(v_2, v_4)}{(v_2, v_3)} v_3$ we can assume that $(v_2, v_4) = 0$

Now, $(v_2, v_3) = -(v_1, v_4)$ since for every $m$

\[
(v_2, v_4) = (\gamma^m v_2, \gamma^m v_4) = (v_2 + mv_1, v_4 + mv_3).
\]

Thus, $[\gamma]_B$ has the required form up to a scalar. In order to remedy the situation, we replace $v_3$ and $v_4$ by a scalar multiple.

Case 5: Assume that the Jordan blocks of $\gamma$ are of sizes $[2,2]$ and $\gamma$ is unipotent. Let $B := (v_1, v_2, v_3, v_4)$ be a base such that $[\gamma]_B$ has the required form. For every $m$:

\[
(v_2, v_2) = (\gamma^m v_2, \gamma^m v_2) = (v_2 + mv_1, v_2 + mv_1)
\]

so $v_1$ is orthogonal to $v_1$ and $v_2$ and similarly $v_3$ is orthogonal to $v_3$ and $v_4$. Moreover, for every $m$

\[
(v_2, v_3) = (\gamma^m v_2, \gamma^m v_3) = (v_2 + mv_1, v_3)
\]

so $(v_1, v_3) = 0$. Since $q$ is non-degenerate, $v_1$ is not orthogonal to $v_4$ and $v_3$ is not orthogonal to $v_2$. By replacing $v_2$ with $v_2 - \frac{(v_2, v_3)}{(v_2, v_3)} v_3$ and $v_4$ with $v_4 - \frac{(v_4, v_3)}{(v_4, v_3)} v_3$ we can assume that $(v_2, v_2) = (v_4, v_4) = 0$. By replacing $v_4$ with $v_4 - \frac{(v_2, v_4)}{(v_2, v_3)} v_3$ we can assume that $(v_2, v_4) = 0$

Now, $(v_2, v_3) = -(v_1, v_4)$ since for every $m$

\[
(v_2, v_4) = (\gamma^m v_2, \gamma^m v_4) = (v_2 + mv_1, v_4 + mv_3).
\]

Thus, $[\gamma]_B$ has the required form up to a scalar. In order to remedy the situation, we replace $v_3$ and $v_4$ by a scalar multiple.

Case 6: Assume that the Jordan blocks of $\gamma$ are of sizes $[3,1]$. Let $B := (v_1, v_2, v_3, v_4)$ be a base such that $[\gamma]_B$ has the required form. Then for every $m$:

\[
(v_2, v_2) = (\gamma^m v_2, \gamma^m v_2) = (v_2 + mv_1, v_2 + mv_1)
\]

\[
(v_3, v_2) = (\gamma^m v_3, \gamma^m v_3) = (v_3 + mv_2 + \frac{m(m-1)}{2} v_1, v_2 + mv_1)
\]

\[
(v_3, v_3) = (\gamma^m v_3, \gamma^m v_3) = (v_3 + mv_2 + \frac{m(m-1)}{2} v_1, v_3 + mv_2 + \frac{m(m-1)}{2} v_1)
\]

Equation 12 implies that $(v_1, v_1) = 0$ and $(v_1, v_2) = 0$. Equation 13 implies that $(v_2, v_2) = -(v_1, v_3)$. In turn, Equation 14 implies that $2(v_2, v_3) = (v_1, v_3)$. Since $q$ is non-degenerate,
\((v_1, v_3) \neq 0\). By replacing \(v_3\) with \(v_3 - \frac{(v_3, v_1)}{2(v_1, v_3)}v_1\) we can assume that \((v_3, v_3) = 0\). Moreover, for every \(m\)

\[(v_4, v_3) = (\gamma^m v_3, \gamma^m v_4) = (v_4, v_3 + m_2 + \frac{m(m - 1)}{2}v_1)\]

so \((v_1, v_4) = 0\) and \((v_2, v_4) = 0\). By replacing \(v_4\) with \(v_4 - \frac{(v_4, v_1)}{(v_1, v_3)}v_1\) we can assume that \((v_4, v_3) = 0\). \(\square\)

### 3.1. Completing the proof of Theorem 1.1

Let us now explain how to prove Theorem 1.1. The structure of the proof follows the exact same lines as the proof of Theorem 2.2, only that instead of using the action on the projective space, we use the action on the planes Grassmannian \(X := \text{Gr}(2, \mathbb{R}^4)\). The elements of \(\text{Gr}(2, \mathbb{R}^4)\) can be viewed as lines in projective space \(\mathbb{P}^3\). We define two functions \(\overline{d}, \underline{d} : X \times X \to \mathbb{R}\) by:

\[
\overline{d}(l_1, l_2) := \max\{\text{dist}_{\mathbb{P}^3}(x_1, x_2) \mid x_1 \in l_1 \text{ and } x_2 \in l_2\} \quad \text{and} \quad \underline{d}(l_1, l_2) := \min\{\text{dist}_{\mathbb{P}^3}(x_1, x_2) \mid x_1 \in l_1 \text{ and } x_2 \in l_2\}.
\]

Note that \(\overline{d}\) is a metric on \(X\). For every \(\varepsilon > 0\) and \(l \in X\) we define two open sets of \(X\):

\[
(l)_\varepsilon := \{t \in X \mid \overline{d}(l, t) < \varepsilon\} \quad \text{and} \quad [l]_\varepsilon := \{t \in X \mid \underline{d}(l, t) < \varepsilon\}.
\]

For linearly independent \(u, v \in \mathbb{R}^4\), \(l_{u,v} \in X\) denotes their projective linear span. An element \(l \in X\) is called \textit{isotropic} if the restriction to the corresponding plane is the zero form. The next lemma follows from Lemma 3.3 and Table 1:

**Lemma 3.4.** Let \(\gamma \in \Gamma\) be a non-trivial element. Let \(B = (v_1, v_2, v_3, v_4)\) be a base of \(\mathbb{R}^4\) for which \([\gamma]_B\) and \([q]_B\) are as in Table 1. Let \(l_0\), \(l_\infty\) and \(l_-\infty\) be the isotropic elements of \(X\) defined in Table 1. Then

\[
\lim_{n \to \infty} \gamma^n \cdot l_0 = l_\infty \quad \text{and} \quad \lim_{n \to -\infty} \gamma^n \cdot l_0 = l_-\infty
\]

where the convergence is with respect to the metric \(\overline{d}\). Moreover, if \(\varepsilon > 0\) is small enough then for every \(0 \neq m \in \mathbb{Z}\), \(\gamma^m \cdot (l_\varepsilon) \cap [l_\varepsilon] = \emptyset\).

**Lemma 3.5.** Let \(l_1, l_2 \in X\) be isotropic elements and let \(\varepsilon > 0\). There exists \(\delta \in \Gamma\) such that \(\delta \cdot (X \setminus [l_1]) \subseteq (l_2)_\varepsilon\) and \(\delta^{-1} \cdot (X \setminus [l_2]) \subseteq (l_1)_{\varepsilon}\).

**Proof.** Choose an isotropic element \(l_3 \in (l_2)_{\varepsilon}\) and \(\varepsilon^* > 0\) such that the span of \(l_1\) and \(l_3\) is \(\mathbb{R}^4\), \([l_3]_{\varepsilon} \subseteq (l_2)_{\varepsilon}\) and \([l_3]_{\varepsilon} \subseteq (l_2)_{\varepsilon}\). It is enough to prove the lemma for \(l_1\), \(l_3\) and \(\varepsilon^*\) instead of \(l_1\), \(l_2\) and \(\varepsilon\). For every \(r > 0\) there exists \(g_r \in \text{SO}(2, 2)(\mathbb{R})\) which preserves \(l_1\) and \(l_3\), acts on \(l_1\) by multiplication by \(r\) and on \(l_3\) by multiplication by \(1/r\). Denote \(\tilde{\varepsilon} := \varepsilon^*/2\) and fix a large enough \(r\) such that for every \(m > 0\), \(g_r^m \cdot (X \setminus [l_1]) \subseteq (l_3)_{\varepsilon}\) and \(g_r^{-m} \cdot (X \setminus [l_3]) \subseteq (l_1)_{\varepsilon}\). Let \(U\) be a symmetric neighborhood of the identity in \(\text{SO}(2, 2)(\mathbb{R})\) such that for every \(u \in U\) and every \(i \in \{1, 3\}\), \(u \cdot (l_i)_{\varepsilon} \subseteq (l_i)_{\varepsilon}\) and \(u \cdot [l_i]_{\varepsilon} \subseteq [l_i]_{\varepsilon}\). As in Section 2.1, by the Poincaré recurrence theorem there exist \(u_1, u_2 \in U\) and \(m > 0\) such that \(\delta := u_1 g_r^m u_2 \in \Gamma\). Evidently, \(\delta\) is the desired element. \(\square\)
Corollary 3.6. Let \( l \in X \) be an isotropic projective line. Then for every \( \epsilon > 0 \) and every conjugacy class \( C \neq \{Id\} \) of \( \Gamma \) there exists \( \gamma \in C \) such that \( \gamma \cdot (X \setminus [l]_{\epsilon}) \subseteq (l)_{\epsilon} \).

Proof. Choose \( \gamma_0 \in C \). Lemma 3.4 states that there exists an isotropic \( l_0 \in X \) and \( \epsilon_0 > 0 \) such that for every \( 0 \neq m \in \mathbb{Z}, \gamma_0^m \cdot (l_0)_{\epsilon_0} \cap [l_0]_{\epsilon_0} = \emptyset \). Denote \( \epsilon_1 := \min(\epsilon, \epsilon_0) \). Lemma 3.5 implies that there exists \( \delta \in \gamma \) such that \( \delta \cdot (X \setminus [l_0]_{\epsilon_1}) \subseteq (l)_{\epsilon_1} \) and \( \delta^{-1} \cdot (X \setminus [l]_{\epsilon_1}) \subseteq (l_0)_{\epsilon_1} \).

The element \( \gamma := \delta \gamma_0 \delta^{-1} \) satisfies the requirement. \( \square \)

Pick a sequence \( (l_i)_{i \geq 1} \) of isotropic elements of \( X \) and a sequence of positive numbers \( (\epsilon_i)_{i \geq 0} \) such that \( (l_i)_{\epsilon_i} \cap [l_j]_{\epsilon_j} = \emptyset \) whenever \( i \neq j \). By Corollary 3.6, there are \( (\gamma_i)_{i \geq 1} \in \Gamma \) representing all the non-trivial conjugacy classes such that for every \( i \), \( \gamma_i \cdot (X \setminus [l_i]_{\epsilon_i}) \subseteq (l_i)_{\epsilon_i} \).

It follows that Proposition 2.7 is satisfied with \( U_i = (l_i)_{\epsilon_i}, \ i \in \mathbb{N}. \) Hence \( \langle \gamma_i : i \in \mathbb{N} \rangle \) is a free group and in particular of infinite index in \( \Gamma \). This completes the proof of Theorem 1.1. \( \square \)
The congruence subgroup property does not imply invariable generation

| (1) | $\alpha = re^{i\theta}$ | $\beta = re^{-i\theta}$ | $r > 1$ | $\theta \not\in \pi \mathbb{Q}$ | $c := r \cos(\theta)$ | $d := r \sin(\theta)$ | $\gamma|_B$ | $q|_B$ | $l_0, l_{\pm \infty}$ |
|-----|------------------------|------------------------|--------|-----------------------------|------------------|------------------|---------|---------|---------------------|
|     | $\left( \begin{array}{cccc} c & d & 0 & 0 \\ -d & c & 0 & 0 \\ 0 & 0 & \frac{c}{r^2} & \frac{d}{r^2} \\ 0 & 0 & \frac{d}{r^2} & \frac{c}{r^2} \end{array} \right)$ | $\left( \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right)$ | $l_0 := l_{v_1 + v_3, v_2 - v_3}$ | $l_{\infty} := l_{v_1, v_2}$ | $l_{-\infty} := l_{v_3, v_4}$ |

| (2) | $\alpha > \beta \geq 1$ | $\left( \begin{array}{cccc} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{array} \right)$ | $\left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right)$ | $l_0 := l_{v_1 + v_3, v_2 - v_4}$ | $l_{\infty} := l_{v_1, v_2}$ | $l_{-\infty} := l_{v_3, v_4}$ |

| (3) | $\alpha = \beta > 1$ | $\left( \begin{array}{cccc} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \end{array} \right)$ | $\left( \begin{array}{cccc} 0 & 0 & 0 & \alpha \\ 0 & 0 & -\frac{1}{\alpha} & 0 \\ 0 & -\frac{1}{\alpha} & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{array} \right)$ | $l_0 := l_{v_1 + v_3, 3v_2 + v_4}$ | $l_{\infty} := l_{v_1, v_2}$ | $l_{-\infty} := l_{v_3, v_4}$ |

| (4) | $\alpha = \beta > 1$ | $\left( \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right)$ | $\left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right)$ | $l_0 := l_{v_1 + v_3, v_3 + v_4}$ | $l_{\infty} := l_{v_1, v_3}$ | $l_{-\infty} := l_{v_1, v_3}$ |

| (5) | $\alpha = \beta = 1$ | $\left( \begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right)$ | $\pm \left( \begin{array}{cccc} 0 & 0 & 2 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$ | $l_0 := l_{v_3 + \sqrt{2}v_4, 2v_3 + v_2 - \frac{1}{2}v_1}$ | $l_{\infty} := l_{v_1, v_2 - \sqrt{2}v_4}$ | $l_{-\infty} := l_{v_1, v_2 - \sqrt{2}v_4}$ |

Table 1
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