ANISOTROPIC VERSIONS OF THE BREZIS-VAN SCHAFTINGEN-YUNG APPROACH AT $s = 1$ AND $s = 0$

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Abstract. In 2014, Ludwig showed the limiting behavior of the anisotropic Gagliardo $s$-seminorm of a function $f$ as $s \to 1^-$ and $s \to 0^+$, which extend the results due to Bourgain-Brezis-Mironescu (BBM) and Maz’ya-Shaposhnikova (MS) respectively. Recently, Brezis, Van Schaftingen and Yung provided a different approach by replacing the strong $L^p$ norm in the Gagliardo $s$-seminorm by the weak $L^p$ quasinorm. They characterized the case for $s = 1$ that complements the BBM formula. The corresponding MS formula for $s = 0$ was later established by Yung and the first author. In this paper, we follow the approach of Brezis-Van Schaftingen-Yung and show the anisotropic versions of $s = 1$ and $s = 0$. Our result generalizes the work by Brezis, Van Schaftingen, Yung and the first author and complements the work by Ludwig.

1. Introduction

For $1 \leq p < \infty$ and $0 < s < 1$, the Gagliardo $s$-seminorm of a function $f \in L^p(\mathbb{R}^n)$ is defined as

$$||f||_{W^{s,p}(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}dxdy,$$

(1.1)

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. This seminorm arises in connection with many problems in the theory of partial differential equations (e.g. [6, 12, 31]).

The limiting behavior of $||f||_{W^{s,p}(\mathbb{R}^n)}$ for $s \to 1^-$ was firstly studied by Bourgain, Brezis, and Mironescu [5], in which they obtained the following BBM formula: for $1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{R}^n)$,

$$\lim_{s \to 1^-} (1 - s)||f||_{W^{s,p}(\mathbb{R}^n)}^p = \frac{1}{p} k(p, n)||\nabla f||_{L^p(\mathbb{R}^n)}^p,$$

(1.2)

where

$$k(p, n) = \int_{S^{n-1}} |e \cdot \omega|^p d\omega = \frac{2\Gamma((p + 1)/2)n^{(n-1)/2}}{\Gamma((n + p)/2)}.$$  

(1.3)

Here $e \in S^{n-1}$ is any fixed unit vector and $e \cdot \omega$ is the inner product of $e, \omega \in S^{n-1}$.

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The limiting behavior of $\|f\|_{W^{s,p}(\mathbb{R}^n)}^p$ for $s \to 0^+$ was considered by Maz’ya and Shaposhnikova [32], in which they obtained the following MS formula: if $f \in W^{s,p}(\mathbb{R}^n)$ for all $s \in (0,1)$, then

$$\lim_{s \to 0^+} s\|f\|_{W^{s,p}(\mathbb{R}^n)}^p = \frac{2n}{p} |B^n| \|f\|_{L^p(\mathbb{R}^n)}^p,$$  

(1.4)

where $|B^n|$ is the volume of the Euclidean unit ball $B^n$ in $\mathbb{R}^n$.

We say that a set $K \subset \mathbb{R}^n$ is a convex body if it is compact and convex and has non-empty interior. For an origin-symmetric convex body $K \subset \mathbb{R}^n$, the Minkowski functional $\|\cdot\|_K$ is defined by:

$$\|x\|_K = \inf\{\lambda \geq 0 : x \in \lambda K\}$$  

(1.5)

defines a norm on $\mathbb{R}^n$ for all $x \in \mathbb{R}^n$. Moreover, we use $Z^*_p K$ to denote the polar $L^p$ moment body of $K$, whose norm $\|\cdot\|_p^{Z^*_p K}$ is given by

$$\|z\|^{Z^*_p K}_p = \frac{n+p}{2} \int_K |z \cdot y|^p dy, \quad z \in \mathbb{R}^n.$$  

(1.6)

In recent years, the research on anisotropic Sobolev spaces have received considerable attention (see e.g. [1, 2, 3, 4, 10, 11, 13, 19, 30, 33, 39]). In particular, the anisotropic Gagliardo $s$-seminorm is obtained by replacing the Euclidean norm $|x - y|$ by $\|x - y\|_K$ in (1.1). The limiting behavior of the anisotropic Gagliardo $s$-seminorm for $s \to 1^-$ and $s \to 0^+$ were established by Ludwig [22, 23]. More specifically, Ludwig proved the following two formulas:

The anisotropic version of BBM formula. If $1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{R}^n)$ has compact support, then

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_K^{n+sp}} dx dy = \frac{2}{p} \int_{\mathbb{R}^n} \|\nabla f(x)\|^{p}_{Z^*_p K} dx.$$  

(1.7)

The anisotropic version of MS formula. If $1 \leq p < \infty$, $f \in W^{s,p}(\mathbb{R}^n)$ for all $s \in (0,1)$ and $f$ has compact support, then

$$\lim_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\|x - y\|_K^{n+sp}} dx dy = \frac{2n}{p} |K| \|f\|_{L^p(\mathbb{R}^n)}^p.$$  

(1.8)

From the fact that $\|z\|^{p}_{Z^*_p B^n} = k(p,n)z^p/2$, it is easy to see that (1.7) recovers (1.2) and (1.8) recovers (1.4) if $K = B^n$. We shall mention the $L_p$ moment body $Z_p K$, the polar body of $Z_p K$, and the $L_p$ centroid body $\frac{2}{(n+p)(p)} Z_p K$ introduced by Lutwak and Zhang [29] have become important tools used in convex geometry, probability theory, and the local theory of Banach spaces (see e.g. [14, 17, 18, 20, 21, 24, 25, 26, 27, 28, 34, 35, 36, 40]).
For $1 \leq p < \infty$, the Marcinkiewicz (i.e. weak $L^p$) quasinorm $\| \cdot \|_{M^p(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}^{2n})}$ is defined by
\[
[g]_{M^p(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}^{2n})}^p := \sup_{\lambda > 0} \lambda^p \mathcal{L}^{2n}((x \in \mathbb{R}^n \times \mathbb{R}^n: |g(x)| \geq \lambda)),
\] (1.9)

$L^{2n}$ denotes the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$. The Marcinkiewicz space $M^p(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}^{2n})$ modeled on $L^p$ contains all the functions $g$ with $[g]_{M^p(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}^{2n})} < \infty$ (e.g. [9, 16]).

Recently, Brezis, Van Schaftingen and Yung [7] provided a different approach by replacing the strong $L^p$ norm in the Gagliardo $s$-seminorm by the weak $L^p$ quasinorm, which complements the BBM formula. Precisely, they proved that there exist a positive constant $c = c(n)$ such that for all $f \in C^s_c(\mathbb{R}^n)$ and $1 \leq p < \infty$,
\[
\frac{1}{n} k(p, n) \| \nabla f \|_{L^p(\mathbb{R}^n)}^p \leq \frac{\left[ \frac{f(x) - f(y)}{|x-y|^{\frac{n}{p}+1}} \right]^p}{M^p(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}^{2n})} \leq c \| \nabla f \|_{L^p(\mathbb{R}^n)}^p,
\] (1.10)

where $k(p, n)$ is defined in (1.3). Moreover, if we denote
\[
E_\lambda = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{|x-y|^{\frac{n}{p}+1}} \geq \lambda \right\},
\]
then
\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^{2n}(E_\lambda) = \frac{1}{n} k(p, n) \| \nabla f \|_{L^p(\mathbb{R}^n)}^p.
\] (1.11)

Clearly, the first inequality in (1.10) is a direct consequence of (1.11).

Inspired by this idea, Yung and the first author [15] established the corresponding MS formula for $s = 0$, i.e. they showed that for all $1 \leq p < \infty$ and all $f \in L^p(\mathbb{R}^n)$,
\[
2 |B^n| \| f \|_{L^p(\mathbb{R}^n)}^p \leq \frac{\left[ \frac{f(x) - f(y)}{|x-y|^{\frac{n}{p}}} \right]^p}{M^p(\mathbb{R}^n \times \mathbb{R}^n, \mathcal{L}^{2n})} \leq 2^{p+1} |B^n| \| f \|_{L^p(\mathbb{R}^n)}^p.
\] (1.12)

Moreover, if we denote
\[
\overline{E}_\lambda = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{|x-y|^\frac{n}{p}} \geq \lambda \right\},
\]
then
\[
\lim_{\lambda \to 0^+} \lambda^p \mathcal{L}^{2n}(\overline{E}_\lambda) = 2 |B^n| \| f \|_{L^p(\mathbb{R}^n)}^p.
\] (1.13)

Similarly, the first inequality in (1.12) can be deduced from (1.13). The approach of Brezis-Van Schaftingen-Yung also inspires the work [8, 27].

Motivated by the two formulas of Ludwig ((1.7) and (1.8)), it is natural to ask what happens if we replace the $|x-y|$ in (1.10) and (1.12) by $\| x - y \|_K$?

In this paper, we will give a positive answer to this question by the following theorem, which establishes anisotropic versions of formulas (1.10) and (1.12) and their limiting behaviors (1.11) and (1.13).
Theorem 1.1. Let $1 \leq p < \infty$ and let $K$ be an origin-symmetric convex body in $\mathbb{R}^n$. Then

(a) there exist a positive constant $C = C(n)$ such that for all $f \in C^\infty_c(\mathbb{R}^n)$,

$$
\frac{2}{n} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{L^p_{Z^*K}}^p \, dx \leq \left[ \frac{f(x) - f(y)}{\|x - y\|^2_K} \right]^{p+1} \leq C \int_{\mathbb{R}^n} \|\nabla f(x)\|_{L^p_{Z^*K}}^p \, dx. \quad (1.14)
$$

Moreover, if we denote

$$
E_{\lambda,K} = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{\|x - y\|^{2+1}_K} \geq \lambda \right\},
$$

then

$$
\lim_{\lambda \to \infty} \lambda^p L^{2n}(E_{\lambda,K}) = \frac{2}{n} \int_{\mathbb{R}^n} \|\nabla f\|_{L^p_{Z^*K}}^p \, dx. \quad (1.15)
$$

(b) for all $f \in L^p(\mathbb{R}^n)$,

$$
2\|K\| f \|_{L^p(\mathbb{R}^n)}^p \leq \left[ \frac{f(x) - f(y)}{\|x - y\|^2_K} \right]^{p+1} \leq 2^{p+1} \|K\| f \|_{L^p(\mathbb{R}^n)}^p. \quad (1.16)
$$

Moreover, if we denote

$$
\widetilde{E}_{\lambda,K} = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{\|x - y\|^{2}_K} \geq \lambda \right\},
$$

then

$$
\lim_{\lambda \to 0^+} \lambda^p L^{2n}(\widetilde{E}_{\lambda,K}) = 2\|K\| f \|_{L^p(\mathbb{R}^n)}^p. \quad (1.17)
$$

Also, the first inequalities in (1.14) and (1.16) can be deduced from (1.15) and (1.17), respectively. Furthermore, we shall mention that by letting $K = B^n$, the fact that $\|z\|_{Z^*B^n}^p = k(p,n)|z|^p/2$ yields that formula (1.14) recovers formula (1.10) (the second inequality in (1.14) differs from (1.10) up to the constant $k(p,n)$). Similarly formula (1.16) recovers formula (1.12) if $K = B^n$.

2. Proof of Theorem 1.1

Throughout the paper, we always assume $p \in [1, \infty)$ and $K$ denotes an origin-symmetric convex body in $\mathbb{R}^n$. 

4
Proof of Theorem 1.1(a). We first give some notations. For \( \omega \in S^{n-1} \), let \( \omega^\perp = \{ x \in \mathbb{R}^n : x \cdot \omega = 0 \} \) and \( L_\omega = \{ s\omega : s \in \mathbb{R} \} \). Let \( \text{Aff}(n, 1) \) be the Grassmannian of lines in \( \mathbb{R}^n \). Then for any \( L \in \text{Aff}(n, 1) \), we can write \( L = \hat{x} + L_\omega \) for some \( \omega \in S^{n-1} \) and \( \hat{x} \in \omega^\perp \). Furthermore, we may write \( x = \hat{x} + s_\omega \) for any point \( x \in L \). The following Blaschke-Petkantschin formula is from integral geometry (see e.g. [38, Theorem 7.2.7]).

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g(x, y) d\mathcal{H}^n(x) d\mathcal{H}^n(y) = \int_{\text{Aff}(n, 1)} \int_{L} \int_{L} g(x, y)|x - y|^{n-1} d\mathcal{H}^1(x) d\mathcal{H}^1(y) dL, \tag{2.1}
\]

where \( \mathcal{H}^k \) denotes the \( k \)-dimensional Hausdorff measure on \( \mathbb{R}^n \) and \( dL \) denotes a suitably normalized rigid motion invariant Haar measure on \( \text{Aff}(n, 1) \). Moreover, it follows from [38, Theorem 13.2.12] that for every measurable function \( h : \text{Aff}(n, 1) \to [0, \infty) \),

\[
\int_{\text{Aff}(n, 1)} h(L) dL = \frac{1}{2} \int_{S^{n-1}} \int_{\omega^\perp} h(\hat{x} + L_\omega) d\mathcal{H}^{n-1}(\hat{x}) d\mathcal{H}^{n-1}(\omega). \tag{2.2}
\]

The following remarkable Proposition for the one-dimensional case established by Brezis, Van Schaftingen, and Yung [7, Proposition 2.1] plays a central role in the proof of formula 1.10.

**Proposition 2.1.** There exists a universal constant \( C \) such that for all \( \gamma > 0 \) and \( F \in C_c(\mathbb{R}) \) that

\[
\iint_{E(F, \gamma)} |x - y|^{\gamma-1} d\mathcal{H}^1(x) d\mathcal{H}^1(y) \leq C \frac{5^\gamma}{\gamma} \|F\|_{L^1(\mathbb{R})},
\]

where

\[
E(F, \gamma) := \left\{ (x, y) \in \mathbb{R} \times \mathbb{R} : x \neq y, \left| \int_y^x F \right| \geq |x - y|^{\gamma+1} \right\}. \tag{2.3}
\]

Now we apply the Blaschke-Petkantschin formula to the above Proposition and obtain an inequality generalizing [7, Proposition 2.2] (see the following Proposition 2.2). It will be used in the proof of Theorem 1.1(a).

For this purpose, we let \( F \) be an operator

\[
F : \text{Aff}(n, 1) \to C_c(\mathbb{R}),
\]

i.e., \( F \) maps a line \( L \) to a compactly supported continuous function \( F_L \) defined on the line \( L \). Note that for any \( L \in \text{Aff}(n, 1) \), we can write \( L = \hat{x} + L_\omega \) for some \( \omega \in S^{n-1} \) and \( \hat{x} \in \omega^\perp \). Thus, we may use \( F_L(\hat{x} + s\omega) \) to denote the function of \( s \in \mathbb{R} \).

**Proposition 2.2.** For any positive integer \( n \), there exists a constant \( C = C(n) \) such that for any operator \( F : \text{Aff}(n, 1) \to C_c(\mathbb{R}) \),

\[
\mathcal{L}^n(E(F)) \leq C \int_{S^{n-1}} \int_{\omega^\perp} \int_{\mathbb{R}} |F_{\hat{x} + L_\omega}(\hat{x} + s\omega)| d\mathcal{H}^1(s) d\mathcal{H}^{n-1}(\hat{x}) d\mathcal{H}^{n-1}(\omega),
\]

where \( E(F, \gamma) \) is defined in Proposition 2.1.
where

\[ E(F) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \left| \int_{s_t}^{s_f} F_L(\hat{x} + s\omega)ds \right| \geq |x - y|^{n+1}, L \text{ passing through } x \text{ and } y \right\}. \]

**Proof.** Denote by \(1_{E(F)}\), the indicator function of the set \(E(F)\). The Blaschke-Petkantschin formula (2.1) with \(g(x, y) = 1_{E(F)}\) yields that

\[ \mathcal{L}^n(E(F)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{E(F)}(x)1_{E(F)}(y)dx\ dy = \int_{\text{Aff}(n, 1)} \int_L \int_L 1_{E(F)}(x)1_{E(F)}(y)dx\ dy. \]

Denote by \(\tilde{F}_L(s) = F_L(\hat{x} + s\omega)\). Then \(E(F)\) restricted on \(L\) is just \(E(\tilde{F}_L, n)\) defined in (2.3). Applying Proposition 2.1 to \(\tilde{F}_L\), we get

\[
\int_L \int_L 1_{E(F)} \cdot |x - y|^{n-1} d\mathcal{H}^1(x)d\mathcal{H}^1(y) = \int_{E(\tilde{F}_L, n)} \int_L |x - y|^{n-1} d\mathcal{H}^1(x)d\mathcal{H}^1(y)
\]

\[
\leq C \frac{S^n}{n} \int \tilde{F}_L(s) d\mathcal{H}^1(s) = \frac{C S^n}{n} \int_\mathbb{R} \tilde{F}_L(\hat{x} + s\omega) d\mathcal{H}^1(s). \tag{2.5}
\]

Substituting (2.5) into (2.4), together with (2.2), we obtain

\[
\mathcal{L}^n(E(F)) \leq C \frac{S^n}{n} \int_{\text{Aff}(n, 1)} \int_\mathbb{R} |F_L(\hat{x} + s\omega)| d\mathcal{H}^1(s)dL
\]

\[
= C \frac{S^n}{n} \int_\mathbb{R} \int_\mathbb{R} |F_L(\hat{x} + s\omega)| d\mathcal{H}^1(s)d\mathcal{H}^1(\hat{x})d\mathcal{H}^n-1(\omega),
\]

as desired. \(\square\)

Inspired by the technique developed in [7], the proof of Theorem 1.1(a) can be divided by the following two lemmas. The second inequality of (1.14) follows from Lemma 2.3. The limiting behavior (1.15) will be established in Lemma 2.4.

**Lemma 2.3.** There exists \(C = C(n) > 0\) such that for all \(f \in C^\infty_c(\mathbb{R}^n)\),

\[
\left[ \frac{|f(x) - f(y)|^p}{\|x - y\|^{\frac{n+1}{k}}_K} \right] \leq C \int_{\mathbb{R}^n} \|\nabla f\|^p_{L^p(\mathbb{R}^n)} dx. \tag{2.6}
\]

**Proof.** On one hand, for any \(L \in \text{Aff}(n, 1)\), we can write \(L = \hat{x} + L_\omega\) for some \(\omega \in S^{n-1}\) and \(\hat{x} \in \omega^+\). Define

\[
F_L(\hat{x} + s\omega) := \frac{\|\nabla f(\hat{x} + s\omega)\| \omega^{p+1}}{\lambda^{p+1}}, \quad s \in \mathbb{R}.
\]

Applying Proposition 2.2 to the above \(F_L\), we see that

\[
E(F) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \left| \int_{s_1}^{s_f} \frac{|\nabla f(\hat{x} + s\omega)\cdot \omega|^p}{\lambda^{p+1}} ds \right| \geq |x - y|^{n+1} \right\}.
\]
Together with Fubini’s theorem, the polar coordinate and (1.6), we obtain
\[
L^m(E(F)) \leq C \int_{S^{n-1}} \int_{\mathbb{R}^n} \frac{\nabla f(\hat{x} + s\omega) \cdot \omega}{\lambda^p ||\omega||_{K}^{n+p}} d\mathcal{H}^{1}(s)d\mathcal{H}^{n-1}(\omega) \\
= \frac{C}{\lambda^p} \int_{S^{n-1}} \int_{\mathbb{R}^n} |\nabla f(x) \cdot \omega|^p ||\omega||_{K}^{n-p} d\mathcal{H}^{n-1}(\omega) \\
= \frac{C}{\lambda^p} \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} |\nabla f(x) \cdot \omega|^p ||\omega||_{K}^{n-p} d\mathcal{H}^{n-1}(\omega) \right) dx \\
= \frac{2C}{\lambda^p} \int_{\mathbb{R}^n} ||\nabla f(x)||_{L_{x}^{p}}^p dx. \tag{2.7}
\]

On the other hand, let \( L = \hat{x} + L_{\omega} \) be the line passing through \( x, y \in \mathbb{R}^n \). Let \( \tilde{f}(s) = f(\hat{x} + s\omega) \), then \( \tilde{f}'(s) = \nabla f(\hat{x} + s\omega) \cdot \omega \). From Hölder’s inequality and the fact that \(|x - y| = |s_x - s_y|\), we have
\[
|f(x) - f(y)| = |\tilde{f}(s_x) - \tilde{f}(s_y)| \\
\leq \int_{s_y}^{s_x} |\nabla f(\hat{x} + s\omega) \cdot \omega| ds \\
\leq |x - y|^{\frac{n}{p+1}} \int_{s_y}^{s_x} |\nabla f(\hat{x} + s\omega) \cdot \omega|^p ds \tag{2.8}.
\]
Note that \(||x - y||_K = |x - y| \cdot ||\omega||_K\). Thus, (2.8) implies that
\[
\begin{align*}
\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{||x - y||_K^{\frac{n}{p+1}}} \geq \lambda \} & \subset \\
\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \int_{s_y}^{s_x} |\nabla f(\hat{x} + s\omega) \cdot \omega|^p ds \geq |x - y|^{n+1} \}. \tag{2.9}
\end{align*}
\]
Combining (2.7) and (2.9), we get
\[
\lambda^p L_{x}^{2n} \left( \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{||x - y||_K^{\frac{n}{p+1}}} \geq \lambda \} \right) \leq 2C \int_{\mathbb{R}^n} ||\nabla f(x)||_{L_{x}^{p}}^p dx. \tag{2.10}
\]
Hence by (1.9), the desired inequality (2.6) follows by taking the supremum of (2.10) for all \( \lambda > 0 \).

**Lemma 2.4.** For \( f \in C_{c}^{\infty}(\mathbb{R}^n) \), if we denote
\[
E_{\lambda,K} = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{||x - y||_K^{\frac{n}{p+1}}} \geq \lambda \right\}, \tag{2.11}
\]
then
\[
\lim_{\lambda \to \infty} \lambda^p \mathcal{L}^2(E_{\lambda,K}) = \frac{2}{n} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{L^p_{\mathbb{K}}}^p dx. \tag{2.12}
\]

**Proof.** For \( f \in C_c^\infty(\mathbb{R}^n) \), denote by \( a := \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \) and \( b := \|\nabla^2 f\|_{L^\infty(\mathbb{R}^n)} \). Obviously,
\[
|f(x) - f(y)| \leq a|x - y|, \quad \text{for any } x, y \in \mathbb{R}^n,
\]
and
\[
|f(x) - f(y) - \nabla f(x) \cdot (x - y)| \leq b|x - y|^2, \quad \text{for any } x, y \in \mathbb{R}^n. \tag{2.14}
\]

For any \( x \in \mathbb{R}^n \), \( \omega \in S^{n-1} \) such that \( \nabla f(x) \cdot \omega \neq 0 \) and small \( \delta \in (0, 1) \), define the interval
\[
I_{\lambda,\delta}(x, \omega) = \left\{ y = x + s\omega: 0 < s \leq \rho, \quad \rho^n = \min \left\{ \frac{\delta}{b} |\nabla f(x) \cdot \omega|^n, \frac{(1 - \delta)^p |\nabla f(x) \cdot \omega|^p}{\lambda^p \|\omega\|_{L^p_{\mathbb{K}}}^{p+n+p}} \right\} \right\}.
\]

**Claim 1.** For any \( x \in \mathbb{R}^n \) and \( y \in I_{\lambda,\delta}(x, \omega) \) such that \( \nabla f(x) \cdot \omega \neq 0 \), it holds that \((x, y) \in E_{\lambda,K}\).

Indeed, from the definition of \( I_{\lambda,\delta}(x, \omega) \), we have
\[
br \leq \delta |\nabla f(x) \cdot \omega| \quad \text{and} \quad \lambda^p \|\omega\|_{L^p_{\mathbb{K}}}^{p+1} \rho^p \leq (1 - \delta) |\nabla f(x) \cdot \omega|,
\]
and hence
\[
br + \lambda^p \|\omega\|_{L^p_{\mathbb{K}}}^{p+1} \rho^p \leq |\nabla f(x) \cdot \omega|.
\]
This together with (2.14) and the fact that \( s = |x - y| \leq \rho \), we obtain
\[
|f(x) - f(y)| \geq |\nabla f(x) \cdot (x - y)| - b|x - y|^2 \geq \lambda|x - y|^\frac{n}{p+1} \|\omega\|_{L^p_{\mathbb{K}}}^{\frac{n}{p+1}} = \lambda|x - y|^\frac{n}{p+1}.
\]
Hence it follows from (2.11) that \((x, y) \in E_{\lambda,K}\), which proves Claim 1.

By using Claim 1 and the polar coordinate, we obtain
\[
\lambda^p \mathcal{L}^2(E_{\lambda,K}) \geq \frac{\lambda^p}{n} \int_{\mathbb{R}^n} \int_{S^{n-1}} \mathbf{1}_{\nabla f(x) \cdot \omega \neq 0} \cdot \rho^n d\omega dx
\]
\[
= \frac{1}{n} \int_{\mathbb{R}^n} \int_{S^{n-1}} \mathbf{1}_{\nabla f(x) \cdot \omega \neq 0} \cdot \min \left\{ \frac{\lambda^p \delta^p}{b^n} |\nabla f(x) \cdot \omega|^n, \frac{(1 - \delta)^p |\nabla f(x) \cdot \omega|^p}{\|\omega\|_{L^p_{\mathbb{K}}}^{p+n+p}} \right\} d\omega dx.
\]
By the monotone convergence theorem, we further get
\[
\liminf_{\lambda \to \infty} \lambda^p \mathcal{L}^2(E_{\lambda,K}) \geq \frac{(1 - \delta)^p}{n} \int_{\mathbb{R}^n} \int_{S^{n-1}} |\nabla f(x) \cdot \omega|^p \|\omega\|_{L^p_{\mathbb{K}}}^{n-p} d\omega dx.
\]
Since \( \delta > 0 \) is arbitrary small, it follows from the polar coordinate and (1.6) that
\[
\liminf_{\lambda \to \infty} \lambda^p \mathcal{L}^2(E_{\lambda,K}) \geq \frac{1}{n} \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} |\nabla f(x) \cdot \omega|^p \|\omega\|_{L^p_{\mathbb{K}}}^{n-p} d\omega \right) dx
\]
\[
= \frac{2}{n} \int_{\mathbb{R}^n} \|\nabla f(x)\|_{L^p_{\mathbb{K}}}^p dx. \quad \tag{2.15}
\]
Indeed, if \((x)\), Substituting (2.19) into (2.18), we obtain
\[
\limsup_{\lambda \to \infty} \lambda^p \mathcal{L}^{2n} (E_{1,K}) \leq \frac{2}{n} \int_{\mathbb{R}^n} ||\nabla f(x)||^p_{\|\rho\|_K} dx.
\]
For any \(x \in \mathbb{R}^n\) and \(\omega \in S^{n-1}\), define the interval
\[
J_\lambda(x, \omega) = \left\{ y = x + s\omega : 0 < s \leq R, R^n = \frac{1}{\lambda^p\|\omega\|^{n+p}_K} \left( \|\nabla f(x) \cdot \omega \| + b \left( \frac{a}{\lambda\|\omega\|^{n+1}_K} \right)^\frac{n}{2} \right)^p \right\}.
\]

**Claim 2.** If \((x, y) \in E_{1,K}\) with \(\lambda > a\|\omega\|^{n-1}_K\) and \(\omega = (y - x)/|y - x|\), then \(y \in J_\lambda(x, \omega)\) and \(\text{dist}(x, \text{supp } f) \leq 1\). Here \(\text{dist}(x, \text{supp } f)\) is the Euclidean distance between \(x\) and the support of \(f\).

Indeed, it follows from (2.11) that \((x, y) \in E_{1,K}\) implies
\[
|f(x) - f(y)| \geq \lambda\|x - y\|^{n+1}_K. \tag{2.16}
\]

From (2.14), we get
\[
|f(x) - f(y)| \leq |\nabla f(x) \cdot (x - y)| + b|x - y|^2. \tag{2.17}
\]

Hence by (2.16) and (2.17),
\[
\lambda\|\omega\|^{n+1}_K s^{\frac{n}{2}} \leq |\nabla f(x) \cdot \omega| + bs, \tag{2.18}
\]
where \(s = |x - y|\) and \(\omega = (y - x)/|y - x|\). Using (2.13) and (2.16), we further have
\[
\lambda\|\omega\|^{n+1}_K s^{\frac{n}{2}} \leq a. \tag{2.19}
\]

Substituting (2.19) into (2.18), we obtain
\[
\lambda\|\omega\|^{\frac{n}{2}+1}_K s^{\frac{n}{2}} \leq |\nabla f(x) \cdot \omega| + b \left( \frac{a}{\lambda\|\omega\|^{n+1}_K} \right)^{\frac{n}{2}},
\]
which implies \(y \in J_\lambda(x, \omega)\).

It remains to show that if \((x, y) \in E_{1,K}\) with \(\lambda > a\|\omega\|^{n-1}_K\), then
\[
\text{dist}(x, \text{supp } f) \leq 1. \tag{2.20}
\]

Indeed, if \((x, y) \in E_{1,K}\), then it follows from (2.19) and the assumption \(\lambda > a\|\omega\|^{n-1}_K\) that \(s = |x - y| < 1\). On the contrary, suppose \(\text{dist}(x, \text{supp } f) > 1\). Then, from the above observation, we must have \(f(x) = f(y) = 0\). Together with (2.16), we get \(\lambda\|\omega\|^{n+1}_K |x - y|^{\frac{n}{2}+1} \leq |f(x) - f(y)| = 0\), which further implies that \(x = y\). However, it contradicts the fact that \((x, x) \notin E_{1,K}\) for any \(x \in \mathbb{R}^n\), and hence (2.20) follows. This completes the proof of Claim 2.
By Claim 2 and the polar coordinate, we obtain
\[ \lambda^p L^{2n}(E_{1,K}) \leq \frac{\lambda^p}{n} \int_{\mathbb{R}^n} \int_{S^{n-1}} 1_{\text{dist}(x, \text{supp } f) \leq 1} \cdot R^n d\omega dx \]
\[ = \frac{1}{n} \int_{\mathbb{R}^n} \int_{S^{n-1}} 1_{\text{dist}(x, \text{supp } f) \leq 1} \cdot \frac{1}{||\omega||^{n+p}_K} \left( |\nabla f(x) \cdot \omega| + b \left( \frac{a}{\lambda ||\omega||^{n+p}_K} \right)^\beta \right)^p d\omega dx. \]

By the dominated convergence theorem and (1.6), we further get
\[ \lim_{\lambda \to \infty} \lambda^p L^{2n}(E_{1,K}) \leq \frac{1}{n} \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} |\nabla f(x) \cdot \omega| ||\omega||^{-n-p}_K d\omega \right) dx = \frac{2}{n} \int_{\mathbb{R}^n} ||\nabla f(x)||^p_{L^p} dx, \]

together with (2.15), the desired result (2.12) follows. □

**Proof of Theorem 1.1(b).** Inspired by the technique developed in [15], the proof of Theorem 1.1(b) can be divided by the following two lemmas. The second inequality of (1.16) follows from Lemma 2.5. The limiting behavior (1.17) will be established in Lemma 2.6.

**Lemma 2.5.** For \( f \in L^p(\mathbb{R}^n) \),
\[ \left[ \frac{f(x) - f(y)}{||x - y||^p_K} \right]^p \leq 2^{p+1}|K||f|^p_{L^p(\mathbb{R}^n)}. \quad (2.21) \]

**Proof.** Let \( f \in L^p(\mathbb{R}^n) \) and \( \lambda > 0 \), denote
\[ \widetilde{E}_{1,K} = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{||x - y||^p_K} \geq \lambda \right\}. \quad (2.22) \]

Clearly,
\[ \widetilde{E}_{1,K} \subseteq \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f(x)|}{||x - y||^p_K} \geq \lambda \right\} \bigcup \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f(y)|}{||x - y||^p_K} \geq \lambda \right\}, \]

and by symmetry, the two sets on the RHS have the same \( 2n \)-Lebesgue measure. On the other hand, the change of variable \( z = y - x \) and the definition of the Minkowski functional (1.5), we have, for any given \( x \in \mathbb{R}^n \),
\[ \int_{\mathbb{R}^n} 1_{\left\{ y : ||y - x||_K \leq \left( \frac{2|f(x)|}{\lambda} \right)^{\frac{1}{p}} \right\}} dy = \int_{\mathbb{R}^n} 1_{\left\{ z + x : \frac{||z||}{\left( \frac{2|f(x)|}{\lambda} \right)^{\frac{1}{p}}} \leq 1 \right\}} dz = \left( \frac{2|f(x)|}{\lambda} \right)^p |K|. \quad (2.23) \]

Therefore,
\[ L^{2n}(\widetilde{E}_{1,K}) \leq 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{\left\{ y : ||y - x||_K \leq \left( \frac{2|f(x)|}{\lambda} \right)^{\frac{1}{p}} \right\}} dy dx \]
\[ = 2 \int_{\mathbb{R}^n} \left( \frac{2|f(x)|}{\lambda} \right)^p |K| dx. \]
\[
\frac{2^{p+1}|K|}{\lambda^p} \|f\|_{L^p(\mathbb{R}^n)}^p.
\] (2.24)

Multiplying by \(\lambda^p\) on both sides of (2.24) and taking supremum on \(\lambda\), the desired inequality (2.21) follows from the definition of the weak \(L^p\) quasinorm (1.9). \(\square\)

**Lemma 2.6.** For \(f \in L^p(\mathbb{R}^n)\), if we denote

\[
\tilde{E}_{\lambda,K} = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y, \frac{|f(x) - f(y)|}{\|x - y\|_K^{\frac{p}{2}}} \geq \lambda \right\},
\]

then

\[
\lim_{\lambda \to 0^+} \lambda^p \mathcal{L}^{2n}(\tilde{E}_{\lambda,K}) = 2|K|\|f\|_{L^p}^p.
\] (2.25)

**Proof.** We first deal with the case that \(f\) is compactly supported. Then we prove the lemma for general case that \(f \in L^p(\mathbb{R}^n)\) by using suitable truncations of \(f\).

**Case 1.** \(f\) is compactly supported. For \(\lambda > 0\), let

\[
H_{\lambda}^+, \quad H_{\lambda}, \quad H_{\lambda}^-,
\]

and

\[
H_{\lambda,K}^+, \quad H_{\lambda,K}^-.
\]

Note that it follows from (1.5) that \(\mathcal{L}^{2n}(H_{\lambda,K}) = 0\). Due to the fact that \(K\) is origin-symmetric, we further have \(\mathcal{L}^{2n}(H_{\lambda,K}^+) = \mathcal{L}^{2n}(H_{\lambda,K}^-)\). Recall the definition of \(\tilde{E}_{\lambda,K}\) in (2.22), clearly we have

\[
\mathcal{L}^{2n}(\tilde{E}_{\lambda,K}) = 2\mathcal{L}^{2n}(H_{\lambda,K}^+).
\] (2.26)

Since \(f\) is compactly supported, we may assume

\[
\text{supp } f \subseteq rK
\]

for some \(r > 0\). Observe that if \((x,y) \in H_{\lambda,K}^+\), then we must have \(x \in rK\). Otherwise, it means that \(\|y\|_K > \|x\|_K > r\). Thus, both \(x, y\) are outside \(rK\). Now, our assumption on \(\text{supp } f\) yields that \(f(x) = f(y) = 0\), and hence \((x,y) \notin \tilde{E}_{\lambda,K}\), contradicting the fact that \((x,y) \in H_{\lambda,K}^+ \subseteq \tilde{E}_{\lambda,K}\). For any given \(x \in rK\), let

\[
H_{\lambda,K,x}^+ := \left\{y \in \mathbb{R}^n : \|y\|_K > \|x\|_K, \frac{|f(x) - f(y)|}{\|x - y\|_K^{\frac{p}{2}}} \geq \lambda \right\}
\]
and

\[ H_{\lambda,K,x,r}^+ := \left\{ y \in \mathbb{R}^n : \|y\|_K > r, \frac{|f(x) - f(y)|}{\|x - y\|_{K^2}^{\frac{2}{n}}} \geq \lambda \right\} \]

\[ = \left\{ y \in \mathbb{R}^n : \|y\|_K > r, \frac{|f(x)|}{\|x - y\|_{K}^{\frac{2}{n}}} \geq \lambda \right\}. \]

Here we use the fact that for \( \|y\|_K > r, f(y) = 0 \). Therefore,

\[ H_{\lambda,K,x,r}^+ = H_{\lambda,K,x}^+ \setminus rK \subseteq H_{\lambda,K,x}^+ \subseteq H_{\lambda,K,x,r}^+ \cup rK. \quad (2.27) \]

Like the computation (2.23), it follows from the first inclusion in (2.27) that

\[ \mathcal{L}^n(H_{\lambda,K,x}^+) \geq \mathcal{L}^n(H_{\lambda,K,x,r}^+) \geq |K| \left( \frac{|f(x)|^p}{\lambda^p} - r^n \right). \quad (2.28) \]

Also, it follows from the second inclusion in (2.27) that

\[ \mathcal{L}^n(H_{\lambda,K,x}^+) \leq |K| \left( \frac{|f(x)|^p}{\lambda^p} + r^n \right). \quad (2.29) \]

Note that Fubini’s theorem implies

\[ \mathcal{L}^{2n}(H_{\lambda,K}^+) = \int_{rK} \mathcal{L}^n(H_{\lambda,K,x}^+) dx. \quad (2.30) \]

Now integrating (2.28) and (2.29) over \( x \in rK \), formula (2.30) yields

\[ \frac{|K|}{\lambda^p} \|f\|_{L^p(\mathbb{R}^n)}^p - |K|^2 r^{2n} \leq \mathcal{L}^{2n}(H_{\lambda,K}^+) \leq \frac{|K|}{\lambda^p} \|f\|_{L^p(\mathbb{R}^n)}^p + |K|^2 r^{2n}. \quad (2.31) \]

Multiplying both sides by \( \lambda^p \) and letting \( \lambda \to 0^+ \), we obtain

\[ \lim_{\lambda \to 0^+} \lambda^p \mathcal{L}^{2n}(H_{\lambda,K}^+) = |K| \|f\|_{L^p(\mathbb{R}^n)}^p. \quad (2.32) \]

Finally, it follows from (2.26) and (2.32) that

\[ \lim_{\lambda \to 0^+} \lambda^p \mathcal{L}^{2n}(\mathcal{E}_{\lambda,K}) = 2|K| \|f\|_{L^p(\mathbb{R}^n)}^p, \]

which completes the proof for case 1.

**Case 2.** \( f \in L^p(\mathbb{R}^n) \) is not necessarily compactly supported. Let \( f_r = f \cdot 1_{rK} \) be the truncation of \( f \) in \( rK \) for some \( r > 0 \) and let \( g_r = f - f_r \). Since \( f \in L^p(\mathbb{R}^n) \) for some \( 1 \leq p < \infty \), we must have \( \|g_r\|_{L^p(\mathbb{R}^n)} \to 0 \) as \( r \to \infty \).

Since \( f = f_r + g_r \), the triangle inequality yields that

\[ \frac{|f(x) - f(y)|}{\|x - y\|_{K^2}^{\frac{2}{n}}} \leq \frac{|f_r(x) - f_r(y)|}{\|x - y\|_{K^2}^{\frac{2}{n}}} + \frac{|g_r(x) - g_r(y)|}{\|x - y\|_{K}^{\frac{2}{n}}}. \]
Hence, for any $\sigma \in (0, 1)$, we have

$$
\bar{E}_{\lambda,K} = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f(x) - f(y)|}{\|x - y\|_K^p} \geq \lambda \right\} \subseteq A_f \cup A_g,
$$

where

$$
A_f := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f_r(x) - f_r(y)|}{\|x - y\|_K^p} \geq \lambda(1 - \sigma) \right\}
$$

and

$$
A_g := \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|g_r(x) - g_r(y)|}{\|x - y\|_K^p} \geq \lambda\sigma \right\}.
$$

Moreover,

$$
\mathcal{L}^{2n}(\bar{E}_{\lambda,K}) \leq \mathcal{L}^{2n}(A_f) + \mathcal{L}^{2n}(A_g). \quad (2.34)
$$

Since $f_r$ is compactly supported in $rK$, replacing $\lambda$ by $\lambda(1 - \sigma)$ in the second inequality of $2.31$ and $2.26$, we get

$$
\mathcal{L}^{2n}(A_f) \leq \frac{2|K|}{\lambda^p(1 - \sigma)^p} \|f\|_{L^p(\mathbb{R}^n)}^p + 2|K|^2 r^{2n}. \quad (2.35)
$$

For $A_g$, replacing $\lambda$ by $\lambda\sigma$ in $2.24$ for $g_r$, we have

$$
\mathcal{L}^{2n}(A_g) \leq \frac{2^{p+1}|K|}{\lambda^p\sigma^p} \|g_r\|_{L^p(\mathbb{R}^n)}^p. \quad (2.36)
$$

Together with $2.34$, $2.35$ and $2.36$, multiplying by $\lambda^p$, we obtain

$$
\lambda^p \mathcal{L}^{2n}(\bar{E}_{\lambda,K}) \leq \frac{2|K|}{(1 - \sigma)^p} \|f_r\|_{L^p(\mathbb{R}^n)}^p + 2\lambda^p|K|^2 r^{2n} + \frac{2^{p+1}|K|}{\sigma^p} \|g_r\|_{L^p(\mathbb{R}^n)}^p.
$$

Now first let $\lambda \to 0^+$, then let $r \to \infty$ and finally let $\sigma \to 0^+$, then the facts

$$
\lim_{r \to \infty} \|f_r\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \lim_{r \to \infty} \|g_r\|_{L^p(\mathbb{R}^n)} = 0
$$

yield that

$$
\limsup_{\lambda \to 0^+} \lambda^p \mathcal{L}^{2n}(\bar{E}_{\lambda,K}) \leq 2|K|\|f\|_{L^p(\mathbb{R}^n)}^p. \quad (2.37)
$$

Similarly, the triangle inequality also yields that

$$
\frac{|f(x) - f(y)|}{\|x - y\|_K^p} \geq \frac{|f_r(x) - f_r(y)|}{\|x - y\|_K^p} - \frac{|g_r(x) - g_r(y)|}{\|x - y\|_K^p}.
$$

Hence, for any $\sigma > 0$, we have

$$
\bar{E}_{\lambda,K} = \left\{ (x,y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f(x) - f(y)|}{\|x - y\|_K^p} \geq \lambda \right\} \supseteq A_f \setminus A_g
$$
where
\[
\overline{A}_f := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|f_r(x) - f_r(y)|}{\|x - y\|^n_K} \geq \lambda (1 + \sigma) \right\}
\]
and \(A_g\) is defined in (2.33). Hence
\[
\mathcal{L}^{2n}(\overline{E}_{\lambda,K}) \geq \mathcal{L}^{2n}(\overline{A}_f) - \mathcal{L}^{2n}(A_g).
\] (2.39)
Since \(f_r\) is compactly supported in \(rK\), replacing \(\lambda\) by \(\lambda(1 + \sigma)\) in the first inequality of (2.31) and (2.26), we get
\[
\mathcal{L}^{2n}(\overline{A}_f) \geq \frac{2|K|}{\lambda^p(1 + \sigma)^p} \|f_r\|_{L^p(\mathbb{R}^n)}^p - 2|K|^2 r^{2n}.
\] (2.40)
Together with (2.36), (2.39) and (2.40), multiplying by \(\lambda^p\), we obtain
\[
\lambda^p \mathcal{L}^{2n}(\overline{E}_{\lambda,K}) \geq \frac{2|K|}{(1 + \sigma)^p} \|f_r\|_{L^p(\mathbb{R}^n)}^p - 2\lambda^p|K|^2 r^{2n} - \frac{2^{p+1}|K|}{\sigma^p} \|g_r\|_{L^p(\mathbb{R}^n)}^p.
\]
Now first let \(\lambda \to 0^+\), then let \(r \to \infty\) and finally let \(\sigma \to 0^+\), together with (2.37), we obtain
\[
\liminf_{\lambda \to 0^+} \lambda^p \mathcal{L}^{2n}(\overline{E}_{\lambda,K}) \geq 2|K| \|f\|_{L^p(\mathbb{R}^n)}^p.
\] (2.41)
Consequently, (2.25) follows from (2.38) and (2.41).

\[\square\]

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