Large-N Summation of Chiral Logs for Generalized Parton Distributions

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Abstract

We demonstrate that in the region of Bjorken $x_{Bj} \sim m_\pi^2/(4\pi F_\pi)^2$ and/or $x_{Bj} \sim |t|/(4\pi F_\pi)^2$ the standard chPT for the pion GPDs fails and one must perform all order resummation of chPT. We perform such resummation in the large-$N$ limit of the $O(N+1)$ extension of the chiral theory. Explicit resummation allows us to reveal novel phenomena – the form of the leading chiral correction to pion PDFs and GPDs depends on the small $x$ asymptotic of the pion PDFs. In particular, if the pion PDF in the chiral limit has the Regge-like small $x$ behaviour $q(x) \sim 1/x^\omega$, the leading large impact parameter ($b_\perp \to \infty$) asymptotics of the quark distribution in the transverse plane has the form ($m_\pi = 0$) $q(x, b_\perp) \sim 1/x^\omega \ln \left( b_\perp^2 / b_\perp^{(1+\omega)} \right)$. This result is model independent and it is controlled completely by the all order resummed chPT developed in this paper. This asymptotic interweaves with small-$x$ behaviour of usual PDFs, hence it depends on the scale, at which the corresponding PDF is defined. This is a new and interesting result in which the chiral expansion meets the QCD evolution.

Introduction

The generalized parton distributions (GPDs) \cite{1} (see Refs. \cite{2}\cite{3}\cite{4}\cite{5} for recent reviews) are determined by the matrix elements between hadronic states of the well-defined QCD quark-gluon operators of the type:

$$O(\lambda) = \bar{q} \left( \gamma^\perp n \right) \gamma_+ q \left( \gamma^\perp n \right),$$

(1)

which is defined on the light-cone, i.e. $n^2 = 0$ (the gauge link between different points is assumed). The dependence of GPDs on soft momenta and/or pion mass can be controlled by Chiral Perturbation Theory (chPT), see Refs. \cite{6}\cite{7}\cite{8}\cite{9}\cite{10}\cite{11}. However, in the case of chPT for GPDs apart from external soft momenta scale, one has to take into account another momentum scale in the problem – the inverse light-cone distance $1/\lambda$ in the operator (1). When these two scales are of the same chiral order chPT requires resummation \cite{12}.

In Ref. \cite{12} it was shown that the chiral expansions of the pion PDFs $q(x)$ (isovector) and $Q(x)$ (isosinglet) can be written in leading logs as:

$$Q(x) = Q^{reg}(x) + \sum_{n \geq 2, \text{even}} D_n \left[ a_\chi \ln \left( \frac{1}{a_\chi} \right) \right]^n \delta^{(n-1)}(x),$$

(2)

$$q(x) = q^{reg}(x) + \sum_{n \geq 1, \text{odd}} D_n \left[ a_\chi \ln \left( \frac{1}{a_\chi} \right) \right]^n \delta^{(n-1)}(x),$$

(3)

where $a_\chi = (m_\pi/4\pi F_\pi)^2$ is the chiral expansion parameter, $F_\pi \approx 93$ MeV is the pion decay constant. The superscript \textit{reg} denotes the regular contributions which do not contain the $\delta-$functions, the summation index $n$ corresponds to the number of loops in chPT.\footnote{the presence of the derivatives of the $\delta-$functions requires the reorganization of the chiral expansion because for $x \sim a_\chi$ all terms in above sums are of the same chiral order.} The presence of the derivatives of the $\delta-$functions requires the reorganization of the chiral expansion because for $x \sim a_\chi$ all terms in above sums are of the same chiral order.

\footnote{Usual parton distributions are just forward limit of GPDs.}

\footnote{In Ref. \cite{12} the coefficients $D_{1,2,3}$ were computed performing three-loop calculations with the result:

$$D_1 = -1, \quad D_2 = \frac{5}{3} \langle x \rangle, \quad D_3 = -\frac{25}{108} \langle x^2 \rangle.$$}
For the case of GPDs the problem is even more obvious. It was demonstrated in Ref. 12 that GPDs contain singular contributions in the kinematical variable $\xi (\xi = x_{Bj}/(2 - x_{Bj}))$. For example, appearance of such singular terms in GPDs leads to the following contributions to the leading order amplitude for hard exclusive processes: \[ A(\xi, t) = \int_{-1}^{1} dx \frac{H(x, \xi, t)}{\xi - x} = F_{reg}(\xi, t) + \sum_{k=1}^{\infty} A_k \frac{1}{\xi^k} [b_{\chi} \ln(1/b_{\chi})]^k, \] where we introduced small expansion parameter $b_{\chi} = |t|/(4\pi F_{\pi})^2$. The presence of such strong singularities $\sim 1/\xi^k$ in region of small $\xi$ compensates the smallness of chiral expansion parameter and therefore standard chiral expansion fails. Note that in the case of the nucleon target such failure of the standard chiral expansion is imperative, because if $t \sim O(p^2)$ in the chiral counting than $\xi \sim O(p^2)$ by kinematical constraints. We see clearly that one has to perform the resummation of singular contributions in order to obtain correct chiral expansion of GPDs and PDFs.

In present paper we perform such resummation in the large-$N$ limit, where $N$ is the number of Goldstone bosons ($N = 3$ in real world). The chiral Lagrangian for pions is equivalent to $O(4)$ $\sigma$-model because of the homeomorphism $SU(2) \otimes SU(2) = O(4)$. The extension to arbitrary $N$ is done by the generalization $O(4) \rightarrow O(N + 1)$. The $O(N + 1)$ $\sigma$-model in the large $N$ limit can be solved by the semiclassical methods (see Appendix B for details on $1/N$ expansion). In such approach one performs partial summation of $\chi$PT diagrams to all orders in accordance with the large-$N$ counting rules. We stress that that this partial summation respects the chiral symmetry.

**Large–$N$ extension of chiral Lagrangian and resummation of $\chi$PT diagrams**

We start with the description of the large–$N$ extension of the standard $\chi$PT. The leading order chiral Lagrangian describing the low-energy dynamics of pions reads \[ L_2 = \frac{F_{\pi}^2}{4} \text{tr} \left[ (\partial_\mu U \partial_\mu U^\dagger) + m_\pi^2 (U + U^\dagger) \right] \] where $F_{\pi} \approx 93$ MeV. We write in the chiral Lagrangian the physical values of the pion mass and decay constant because their difference with the corresponding bare constants is irrelevant for the leading log approximation we use in the present paper. The $SU(2)$ matrix $U$ in Eq. (6) is parameterized in the form:

\[ U = \frac{1}{F_{\pi}} (\sigma + i \pi \cdot \tau), \quad \sigma = F_{\pi} \sqrt{1 - \pi^2/F_{\pi}^2} \]

which gives the following result for the chiral Lagrangian:

\[ L_2 = \frac{1}{2} [\partial_\mu \sigma \partial_\mu \sigma + \partial_\mu \pi^a \partial_\mu \pi^a] + \sigma F_{\pi} m_\pi^2, \quad \sigma^2 + \sum_{a=1}^{3} \pi^a \pi^a = F_{\pi}^2 \]

We have obtained the Lagrangian of $O(4)$ $\sigma$–model. Next we extend this model replacing the group $O(4)$ with $O(N + 1)$, where $N$ has a meaning of the number of Goldstone bosons:

\[ L_2 = \frac{1}{2} [\partial_\mu \sigma \partial_\mu \sigma + \partial_\mu \pi^a \partial_\mu \pi^a] + \sigma F_{\pi} m_\pi^2, \quad \sigma^2 + \sum_{a=1}^{N} \pi^a \pi^a = F_{\pi}^2 \]

Substituting the $\sigma$–field in terms of $\pi$-fields we obtain

\[ L_2 = -\frac{1}{2} \pi^a \partial^2 \pi^a + \frac{1}{2} (\pi^a \partial^a \pi^a)^2 + \sqrt{F_{\pi}^2 - \pi^2} F_{\pi} m_\pi^2 \]

\[ = -\frac{1}{2} \pi^a \partial^2 \pi^a + N \sqrt{\pi^2} \frac{V(\pi^2/N)}{N} \]

\( \text{For simplicity we consider the chiral limit } m_{\pi} = 0. \)
where we assume the sum over the repeated indexes of the pion fields and

\[ V(\pi^2/N) = \frac{1}{8} \left( \partial_\mu \pi^2/N \right)^2 + \sqrt{G_\pi^2 - \pi^2/N} \Gamma_\pi m_\pi^2, \]  
\[ G_\pi^2 = F_\pi^2/N. \]  

The Lagrangian (10) is written in the form which is convenient for the construction of the large\( -N \) perturbation theory. Note that large\( -N \) expansion implies that the introduced coupling \( G_\pi \) is of order \( N^0 \) with respect to \( N \), i.e. chiral coupling \( F_\pi \) scales as \( F_\pi \sim \sqrt{N} \) in the large \( N \) limit.

Now, we have to introduce the generalization for the operators \( O^{(L,R)}(\lambda) \) that define GPDs and PDFs, see Appendix A for all definitions and conventions. The isovector operator in \( O(4) \)---case has the form:

\[ O^c(\lambda) = -i\varepsilon[abc] \mathcal{F}(\beta, \alpha)[\beta, \alpha] \pi^n \left( \frac{1}{2}(\alpha + \beta)n \right) i \overrightarrow{\partial} + \pi^b \left( \frac{1}{2}(\alpha - \beta)n \right), \]  

We generalize this operator to an arbitrary \( N \) as

\[ O^{[ab]}(\lambda) = -\mathcal{F}(\beta, \alpha) AS \pi^n((\frac{1}{2}(\alpha + \beta)\lambda)n) \overrightarrow{i\partial} + \pi^b((\frac{1}{2}(\alpha - \beta)\lambda)n) \]  

where symbol "AS" denotes the antisymmetrization with respect to \( O(N+1) \) indices:

\[ AS \pi^n(x_1) \overrightarrow{\partial} + \pi^b(x_2) = \frac{1}{2} \left[ \pi^n(x_1) \overrightarrow{\partial} + \pi^b(x_2) - \pi^n(x_1) \overrightarrow{\partial} + \pi^a(x_2) \right]. \]  

It is clear that the correspondence between the two definitions is given by

\[ O^c(\lambda) = i\varepsilon[abc] \lim_{N\to 3} O^{[ab]}(\lambda). \]  

Let us consider now the following matrix element in the extended theory

\[ \int \frac{d\lambda}{2\pi} e^{-ip\cdot x\lambda} \langle \pi^b(p) | O^{[ab]}(\lambda) | \pi^a(p) \rangle = 4 \left[ \delta^{a'a'} \delta^{bb'} - \delta^{ab} \delta^{ba'} \right] q(x). \]  

It defines an analog of isovector PDF \( q(x) \) for the \( O(N+1) \) case, the isovector pion GPD is defined by obvious generalization of the above equation.

The isoscalar PDF (and its generalization to GPD) for an arbitrary \( N \) is defined as:

\[ \int \frac{d\lambda}{2\pi} e^{-ip\cdot x\lambda} \langle \pi^b(p) | O(\lambda) | \pi^a(p) \rangle = 2\delta^{ab} Q(x), \]  

where the operator is defined as:

\[ O(\lambda) = -\mathcal{F}(\beta, \alpha) \left[ \sigma(x_1\lambda) i \overrightarrow{\partial} + \sigma(x_2\lambda) + \pi^n(x_1\lambda) i \overrightarrow{\partial} + \pi^a(x_2\lambda) \right] \]  

with

\[ x_1 = \frac{1 + \beta}{2}, \quad x_2 = \frac{1 - \beta}{2}. \]

In order to understand the relation of the \( 1/N \) expansion to the standard \( \chi PT \) we have to inspect the diagrams of the large\( -N \) approach. To the \( 1/N \) order for GPDs one has to compute the diagrams shown in Fig[1]. The first diagram is the tree contribution to the pion PDF, the second and the third diagrams are the first non-trivial \( 1/N \) corrections\[4\]. The solid line denotes the usual pion propagator and the dashed line is the propagator of the auxiliary field \( \varphi \) which is suppressed by \( 1/N \) (see details of \( 1/N \)

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\[\text{To the 1/N order there are another tadpole-like diagrams, however they do not contribute to the leading logs.}\]
expansion technique in Appendix B). The explicit expression for the propagator of the auxiliary field is given by

\[ \langle \phi \phi \rangle = -\frac{1}{\Delta F^2} \left[ 1 - \frac{N}{2\Delta F^2} \right]^{-1}, \quad \Delta = \frac{i}{p^2 - m^2}, \]  

(21)

where we apply the dimensional regularization with \( D = 4 - 2\varepsilon \) and with renormalization scale \( \mu \). After expanding the above expression in powers of pion coupling \( N/F^2 \pi \):

\[ \langle \phi \phi \rangle = -\frac{1}{\Delta F^2} \left( 1 + \left( \frac{N}{2\Delta F^2} \right)^2 + \left( \frac{N}{2\Delta F^2} \right)^3 + \ldots \right), \]  

(23)

we observe that this expansion produces the infinite series of the bubble chains with the multiplicative factor proportional to \( N/F^2 \pi \). This expansion explains which diagrams of the usual \( \chi \)PT are taken into account at given order of \( 1/N \) expansion.

**Large–N resummation for the pion PDFs**

In this section we consider the large-\( N \) resummation of the singular contributions to the pion PDFs. We give detailed calculations for the case of the isovector PDF \( q(x) \). For the isoscalar case we outline main differences in the calculation comparing to the isovector case.

**Isovector PDF**

Let us compute the leading logarithms from the diagrams with the insertion of the bubbles and select the contributions with the derivatives of the \( \delta \)-functions. Only two first diagrams in Fig. 1 contribute to the isovector PDF, the third diagram obviously contributes obviously to the isoscalar operator only. The corresponding diagrams are UV–divergent and must be renormalized. The UV–subtractions is performed by the application of \( \mathcal{R} \)-operation.\(^5\) The series of the relevant diagrams originating from the loop diagram in Fig. 1 can be written as

\[ q(x) = \frac{1}{N} \sum_{n=0}^{\infty} \mathcal{R} F^{l=1}(\beta, \alpha) * \int \frac{d^Dk}{(2\pi)^D} \frac{\mu^{2\varepsilon} k_+ \delta(\beta k_+ - xp_+)}{[k^2 - m^2_\pi]^2} \left( \frac{N}{F^2 \pi} [(k + p)^2 - m^2_\pi] \right)^{n+1} \left[ \right]^{n} \]  

(24)

The \( \delta \)-function in the numerator arises from the non-local vertex of the operator. It is clear that each term in the sum (24) corresponds to \( n + 1 \)-loop diagram. The \( \mathcal{R} \)-operation removes the UV-divergencies in the \( n + 1 \)-loop diagram. The subtractions have following structure

\[ \mathcal{R} G^{(n+1)} = G^{(n+1)} + \sum_{\{I_k\}} \frac{1}{\varepsilon^k} G^{(n+1-k)} + \sum_{k=1}^{n} \frac{1}{\varepsilon^k} G^{(k)}_{\text{tree}}, \]  

(25)

where \( G^{(n+1)} \) denotes the bare \( (n + 1) \)-loop diagram, the second term describes the subtractions of the subdivergencies in the bare diagram, the sum in the second term runs over the set of the UV–divergent

\(^5\)An introduction to the renormalization in quantum field theory and detailed discussion of the \( \mathcal{R} \)-operation see in [14, 15].
k-loop subdiagrams \( \{ \Gamma_k \} \). The index \( k \) denotes the number of the loops in the subdiagram and the pole \( \frac{1}{\varepsilon} \) originates from its \( UV \)-divergence. It is clear that the index \( k \) takes values \( 1, \ldots, n \). The term \( G^{(n+1-k)} \) describes the \((n+1-k)\)-loops diagram with insertion of the counterterm vertex that results from the contraction of the \( k \)-loop subdiagrams. The last term denotes the subtraction of the total resulting divergence, \( G^{(k)}_{\text{tree}} \) is the notation for the tree level diagram with the appropriate vertex structure. The contributions to the coefficient in front of the leading logarithm arises from the terms in (25) except the last one. The \( \varepsilon \) pole structure of the \((n+1-k)\)-loop diagram \( G^{(n+1-k)} \) in Eq. (26) is the following:

\[
\frac{1}{\varepsilon^k} G^{(n+1-k)} = \frac{1}{\varepsilon^{n+1}} \left( \frac{\mu^2}{m_\pi^2} \right)^{\varepsilon(n+1-k)} G_k(\varepsilon) = \frac{1}{\varepsilon^{n+1}} \left( g_0^{(k)} + \varepsilon g_1^{(k)} \ln \left( \frac{\mu^2}{m_\pi^2} \right) + \cdots + \varepsilon^n g_n^{(k)} \ln^n \left( \frac{\mu^2}{m_\pi^2} \right) \right) + g_0^{(k)} \frac{(n+1-k)^{n+1}}{(n+1)!} \ln^{(n+1)} \left( \frac{\mu^2}{m_\pi^2} \right) + \cdots.
\]

In the above equation we took into account that \( G_k(\varepsilon) \) is the polynomial of order \((n+1-k)\) in \( \varepsilon \) and it is an analytical function of external momenta and the pion mass. In the sum of all contributions in Eq. (25) the poles in Eqs. (26) with non-analytic dependence on external momenta and \( m_\pi \) must cancel due to locality of the total counterterm. That imposes certain relations between the coefficients \( g_i^{(k)} \) with \( k = 0, \ldots, n \) and \( l = 0, \ldots, n \). One can easily show that these relations leave only one unknown coefficient that must be computed in order to find the total coefficient in front of \( \ln^{(n+1)}(\mu^2/m_\pi^2) \), and has the form:

\[
\mathbf{R} G^{(n+1)} = \ln^{(n+1)} \left( \frac{\mu^2}{m_\pi^2} \right) \sum_{k=0}^{n} g_0^{(k)} \frac{(n+1-k)^{n+1}}{(n+1)!} + \cdots.
\]

It is convenient to introduce coefficient \( g_0^n \) as an unknown quantity, i.e. one has to compute only the contributions \( \sum_{\{ \Gamma_n \}} \frac{1}{\varepsilon^n} G^{(1)} \). In other words, we have to compute only one loop diagram, but with all possible \( n \)-loop counterterms. An example of one of such diagrams is shown in Fig. 2. The structure of the \( n \)-bubble counterterm has the form:

\[
\frac{1}{\varepsilon^n} \left( \gamma_n(kp)^{n+1} + \left[ \gamma_n^{(1,0)} m_\pi^2 + \gamma_n^{(0,1)} k^2 \right] (kp)^n + \cdots \right)
\]

where the dots denote all other possible contributions constructed from the \( m_\pi^2, k^2 \) and \( (kp) \). The power \( n \) is dictated by dimensions. Inserting the expression into the original diagram instead of the bubbles chain one obtains

\[
G^{(1)} \sim F^{I=1}(\beta, \alpha) \propto \int \frac{d^Dk}{(2\pi)^D} \left[ \frac{\mu^2}{[k^2 - m_\pi^2]^2} \right] \left( \gamma_n(kp)^{n+1} + \left[ \gamma_n^{(1,0)} m_\pi^2 + \gamma_n^{(0,1)} k^2 \right] (kp)^n + \cdots \right)
\]

Now we observe that the most singular term (with the highest derivative of \( \delta \)-function) is produced from the contribution with the highest power of the scalar product \( (kp) \). Expanding the \( \delta \)-function in the numerator of Eq. (29)

\[
k_\perp \delta(\beta k_\perp - xp_\perp) = k_\perp^{n+1} \beta^n \delta^{(n)}(xp_\perp) + \cdots
\]
and neglecting the terms with lower power of \( k_+ \), one obtains

\[
G^{(1)} \sim \gamma_n \delta^{(n)}(x p_+) \left[ F^{I=1}(\beta, \alpha) \ast \beta^n \right] \int \frac{d^Dk}{(2\pi)^D} \frac{\mu^{2\varepsilon} k_+^{n+1}(kp)^{n+1}}{[k^2 - m_\pi^2]^2} \quad (31)
\]

Computing the one-loop integral over \( k \) we obtain the expression for \( g_0^{(n)} = (n+1)! \gamma_n \). Using Eq. (27) we obtain the desired singular contribution to PDF:

\[
q(x) \sim \gamma_n \delta^{(n)}(x) a_\chi^{n+1} \ln^{n+1}[1/a_\chi] + \ldots ,
\quad (32)
\]

where we fixed the renormalization scale \( \mu = (4\pi F_\pi)^2 \) for simplicity.

Going back to the general case, we compute the sum over all possible \( n \)-loop subdiagrams. For that, in addition to the considered case, one has to consider the subdiagrams with the operator vertex, for instance such as in Fig. 3. Obviously, the reduced diagrams have always at least one contracted line, this reduces possible power of derivatives of the \( \delta \)-functions in the resulting expression. Indeed, the contracted propagator is always proportional at least to one power of the pion mass \( m_\pi^2 \) that excludes one power in possible expansion of the \( \delta \)-function. In other words, there is no possibility to produce the singular contributions of type (32) from the contributions with vertex subdiagrams in the diagrams with the bubble chains.

From the above consideration one learns several important lessons. We expect that each term in the sum of diagrams (24) contains a singular contribution proportional to \( \delta^{(n)}(x) a_\chi^{n+1} \ln^{n+1}[1/a_\chi] \). Moreover, taking into account the relations due to the cancellation of the poles with non-local dependence on external momenta, we obtained that the calculations of these contributions can be reduced to a consideration of the diagram as in Fig. 2 with \( n \)-loop bubble chain counterterm. Important technical observation, is that the structure of the corresponding counterterm is restricted by the maximal power of the scalar product \( (kp) \) as in (29) and does not depend on the pion mass in to the subdiagram. Therefore in the practical calculations we can put the pion mass inside the bubble chain to zero. In addition, one can expand the \( \delta \)-function in expression (24) and pick up only the contribution with \( \delta^{(n)}(x) \). Using these simplification and denoting the most singular term as \( q^{\text{sing}}(x) \) one obtains

\[
q^{\text{sing}}(x) = \frac{1}{N} \sum_{n=0}^\infty \langle x^n \rangle \delta^{(n)}(xp_+) R \int \frac{d^Dk}{(2\pi)^D} \frac{k_+^{(n+1)}}{[k^2 - m_\pi^2]^2} \left[ \frac{N}{F_\pi^2}(kp)^2 \right]^{n+1} \left[ \frac{(\mu^2 + kp_+^2)^{\varepsilon}}{\mu^2} \right]^{n+1} , \quad (33)
\]

where the \( R \)-operation removes subdivergencies only from the bubble chain subdiagram and

\[
\langle x^n \rangle = F^{I=1}(\beta, \alpha) \ast \beta^n = \int_{-1}^1 dx x^n q^\circ(x) . \quad (34)
\]

It turns out that such simplified expressions can be computed directly, without the combinatorial analysis of the \( R \)-operation mentioned above. We provide the corresponding technical details in the Appendix B. The result is very simple

\[
q^{\text{sing}}(x) = -2 \sum_{n=0,2,4,\ldots}^\infty \frac{\langle x^n \rangle}{(n+1)!} \delta^{(n)}(x) \equiv \epsilon^{n+1} (n+1)! \quad (35)
\]

where \( \epsilon = \frac{2}{N} a_\chi \ln[1/a_\chi] \). This expression shows that one has the singular contributions to all orders and we can perform the summation of these terms. Let us represent the \( n \)-th derivative of the \( \delta \)-function as:

\[
\delta^{(n)}(x) = \int \frac{d\lambda}{2\pi} (i\lambda)^n e^{i\lambda x} . \quad (36)
\]
Taking the Regge-like small-$x$ sum of equation demonstrates explicitly that resummation solves the problem of the singular terms – the infinite and inserting it into (37) we obtain

This is our final result for large-$N$ we obtain non-analytic behavior of PDFs in the chiral coupling

Changing the order of the integration the summation we obtain

Computing the sum as

and inserting it into (37) we obtain

Taking into account that isovector PDF is symmetrical function $q(-x) = q(x)$ one can rewrite (39) as

This is our final result for large-$N$ resummation of chiral corrections to the isovector pion PDF. The above equation demonstrates explicitly that resummation solves the problem of the singular terms – the infinite sum of $\delta$-function derivatives results in the contribution of the type $f(x/a_\chi)/a_\chi$ as it was conjectured in Ref. [12]. Note that the large-$N$ resummation preserves all chiral properties of the considered quantities.

From (40), one can observe that resummation does not change the small-$x$ asymptotic of the PDF. Taking the Regge-like small-$x$ behaviour of PDF in the chiral limit:

we obtain non-analytic behavior of PDFs in the chiral coupling $\sim (a_\chi \ln(1/a_\chi))^\omega$, for the chiral correction to PDF in the small-$x$ region:

Interestingly, the leading power of chiral coupling is determined by the intercept of the Regge trajectory $\omega (\omega \approx 1/2$ for the isovectors case). This shows clearly importance of the resummation of singular chiral corrections for the derivation of the leading chiral counting of PDF. Naive chiral counting – without taking into account the second scale related to the light-cone distance – suggests that the leading chiral correction to the isovector PDF is $\sim a_\chi \ln(1/a_\chi)$. However, it is not correct, as the leading chiral corrections are dominated by the small-$x$ region and one must perform the resummation discussed in the present paper.

Isoscalar PDF

In the case of isoscalar ($C$ parity even) pion PDF one has to compute additionally the third diagram shown in Fig. 1. The dashed line in this diagram represents the propagator of the auxiliary field (see Eq. [21]) corresponding to the sum of bubble chains. The third diagram is more complicated as it contains two loop integral. The expression for this diagram has the form:

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$$
\frac{1}{N} \sum_{m,n=0}^{\infty} \left( -\frac{N}{F_\pi} \right)^{m+n+2} F(\beta, \alpha) \ast \frac{1}{(2\pi)^D} \frac{d^Dk}{|k^2 - m_{\pi}^2|^2} \frac{d^Dl}{|l^2 - m_{\pi}^2|^2} \frac{k_+ \delta(\beta \cdot \epsilon - x p_+)[l^2 - m_{\pi}^2]^{n+m+2}}{|(l - k)^2 - m_{\pi}^2|} = \frac{1}{\epsilon^{m+n}} \left( \frac{\Gamma(1 + \epsilon) \Gamma(2 - \epsilon)}{\Gamma(2 - 2\epsilon)} \right)^{m+n} \frac{1}{\epsilon^{m+n}} \left( \frac{\Gamma(1 + \epsilon) \Gamma(2 - \epsilon)}{\Gamma(2 - 2\epsilon)} \right)^{m+n} \frac{1}{\epsilon^{m+n}} \left( \frac{\Gamma(1 + \epsilon) \Gamma(2 - \epsilon)}{\Gamma(2 - 2\epsilon)} \right)^{m+n}
$$
Applying the renormalization procedure (\( R \) operation) and picking up the most singular in \( x \) terms we obtain the contribution of the third diagram to the singular part of the isoscalar PDF \( Q^{\text{sing}}(x) \):

\[
Q^{\text{sing}}_{\text{third}}(x) = -\frac{2}{N} \sum_{n,m=0}^{\infty} (x^{n+m+1}) \delta^{(m+n+1)}(x) \left( \frac{\chi(a_{N})}{a_{N}} \right)^{m+n+2} \frac{(m+n+3)!}{(n+m+1)! (m+n+1)} , \quad (m+n+1) = \text{odd} \quad (45)
\]

Each term in the above sum depends only on \( m+n \) combination of the summation indice; hence, this sum can be easily computed. The final result of the summation of both the second and the third diagrams in Fig. 1 is the following:

\[
Q^{\text{sing}}(x) = 4 \frac{\text{sign}(x)}{N} \theta(|x| < \epsilon) \int_{|x|/\epsilon}^{1} d\beta \frac{\partial}{\partial \beta} Q(\beta) \left( 3 - 2 \frac{|x|}{\epsilon \beta} \right) . \quad (46)
\]

Here, as before, \( \epsilon = \frac{N}{2} \chi \ln(1/a_{\chi}) \). As in the case of isovector PDF, the small \( x \) behaviour of the chiral correction has the same form as for the PDF in the chiral limit, e.g. the Regge-like that \( \sim 1/x^{\omega} \). One can easily calculate that the small-\( x \) asymptotic of the leading chiral correction has the form:

\[
Q^{\text{sing}}(x) \sim \left( \frac{\chi \ln(1/a_{\chi})}{x} \right)^{\omega} . \quad (47)
\]

Again we see that the correct dependence of the small-\( x \) PDF on the pion mass can be obtained only after the resummation of all orders of the standard \( \chi \)PT.

Eq. (42) and Eq. (47) show that the leading chiral dependence of the PDFs on \( m_{\pi} \) is determined by the small \( x \) behaviour of the PDFs. The dynamics of the chiral degrees of freedom (Goldstone bosons) and the dynamics of QCD degrees of freedom (quarks and gluons) intertwine. Noticeably, even the power of the chiral expansion parameter changes with QCD evolution of PDFs.

Large-\( N \) resummation for the pion GPD

In this section we consider the resummation as before but for the GPD matrix element. The diagrams describing the \( 1/N \) correction to the pion GPDs \( H(x, \xi, t) \) are the same as in Fig. 1 but with momentum flow depicted in Fig. 4. The expression for the \( 1/N \) correction to the isovector GPD has the following form:

\[
H^{1=1}(x, \xi, t) = \frac{1}{N} \sum_{n=0}^{\infty} R F^{l=1}(\beta, \alpha) * \int \frac{d^{D}k}{(2\pi)^{D}} \frac{k_{\perp} \delta(\beta k_{\perp} + \alpha \xi - xp_{\perp})}{[k - \Delta/2]^{2} - m_{\pi}^{2}} \left( \frac{N}{F_{R}^{2}} \right)^{n+1} \left[ \begin{array}{c}
\Delta \\
\end{array} \right]^{n} .
\]

Due to the presence of the momentum transfer instead of \( \delta \)-functions we obtain the contributions \( \sim \theta(|x| < \xi/\Delta) \). Such contributions in the forward limit (\( \xi \to 0 \)) are reduced to the derivatives of \( \delta \)-function. Therefore we shall separate them neglecting the others less singular terms. The explicit resummation can be done in the same way as for the PDF. But now the calculations are slightly more complicated.
due to additional parameters $\xi$ and $t$. At the end we obtain the following expression for the singular contributions to the isovector pion GPD:

$$H^{I=1}(x, \xi, t) = -\frac{1}{N} \sum_{n=0, 2, \ldots}^{\infty} \int_{-1}^{1} d\eta \left( \frac{N R \ln 1/R}{(n+1)!} \right)^{n+1} \frac{1}{(n+1)^{\xi+1}} \delta^\eta \eta F^{I=1}(\alpha, \beta) \ast \delta \left( \beta \eta + \alpha - \frac{x}{\xi} \right)$$

(49)

where

$$R = \frac{m^2_\pi - t(1 - \eta^2)/4}{(4\pi F_\pi)^2}$$

is the small chiral expansion parameter. One can easily perform summation of the singular contributions in Eq. (49) with the following result:

$$H^{I=1}(x, \xi, t) = -\frac{1}{2N} \int_{-1}^{1} d\eta \int_{-\xi}^{\xi} d\tau \left( 1 + \frac{\eta \xi}{\tau} \right) \frac{F^{I=1}(\beta, \alpha) \ast \delta \left( [\xi \eta + \tau] \beta + \xi \alpha - x \right)}{2}$$

(50)

where

$$\zeta = \frac{N}{2} R \ln(1/R).$$

The result of the large $-N$ resummation of the singular contributions to the isoscalar pion GPD has the form:

$$H^{I=0}(x, \xi, t) = -\frac{1}{N} \int_{-1}^{1} d\eta \int_{0}^{\zeta} d\tau \left( 3 - 2 \frac{\tau}{\zeta} \right) \frac{F^{I=0}(\beta, \alpha) \ast \delta \left( [\eta \xi + \tau] \beta + \xi \alpha - x \right)}{2}$$

(51)

Note that the small chiral expansion parameter $\epsilon = \frac{N}{2} R \ln(1/R)$ enters these expressions always divided by $\xi$, meaning that for $\xi \sim \epsilon$ the singular contributions are not suppressed by the small chiral parameter. In next sections we shall analyze obtained expression for resummed chiral corrections to GPDs and corresponding amplitude of hard exclusive processes on the pion target.

**Parton distribution in the transverse plane**

To analyze the distribution of partons in the transverse plane one has to take the limit $\xi \rightarrow 0$ keeping $t \neq 0$ in GPDs. One can easily perform this limit for the expressions with the results:

$$H^{I=1}(x, \xi = 0, t = -\Delta^2_\perp) \equiv q^{\text{sing}}(x, \Delta_\perp) = -\frac{2}{N} \theta(||x|| < \epsilon) \int_{-\epsilon}^{\epsilon} d\beta q^o(\beta) \sqrt{1 - \frac{x}{\epsilon \beta}},$$

(52)

and

$$H^{I=0}(x, \xi = 0, t = -\Delta^2_\perp) \equiv Q^{\text{sing}}(x, \Delta_\perp) = -\frac{4}{N} \theta(x < \epsilon) \int_{-\epsilon}^{\epsilon} d\beta Q^o(\beta)$$

(53)

$$\times \left[ 3 \sqrt{1 - \frac{x}{\epsilon \beta}} - \frac{x}{\epsilon \beta} \ln \left( 1 + \sqrt{1 - \frac{x}{\epsilon \beta}} \right) \right].$$

---

7In section we consider the case of $m_\pi = 0$, because we focus our discuss on the dependence of GPD on $t$ and $\xi$. For the real world this case corresponds to the parametrically wide domain of momentum transfer squared $m_\pi^2 \ll -t \ll (4\pi F_\pi)^2$. 

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where $\epsilon = \frac{N_c}{2} b_\chi \ln(1/b_\chi)$ with $b_\chi = |t|/(4\pi F_\pi)^2 = \Delta^2 \chi / (4\pi F_\pi)^2$.

The Fourier transform of the PDFs $q(x,t), Q(x,t)$ in $\Delta_\perp (\Delta^2_\perp = -t)$ gives the density of partons in transverse plane ($b_\perp$) with given momentum fraction $x$. The chiral expansion allows us to compute the asymptotics of the distribution in the transverse plane at $b_\perp \to \infty$. The one-loop $\chi$PT gives the following large $b_\perp$ behaviour of isovector quark density $q(x,b_\perp)\,^7$:

$$q(x,b_\perp) \sim \frac{1}{b_\perp} \delta(x). \quad (54)$$

The large $b_\perp$ behaviour of isoscalar distribution of quarks $Q(x,b_\perp)$ in two-loop $\chi$PT has the form:

$$Q(x,b_\perp) \sim \frac{\ln(b_\perp^2)}{b_\perp^6} \delta'(x). \quad (55)$$

We see that at large $b_\perp$ the density of (anti)quarks is formed by partons with small longitudinal momentum fraction $x$. The presence of the $\delta$-functions is the artifact of the finite order $\chi$PT calculations, we simply can not resolve the $\delta$-functions. This shows that we must perform the resummation of $\chi$PT in order to access the correct large $b_\perp$ asymptotic of quark density in the transverse plane. We can easily perform the Fourier transform of our results for $q(x,t)$ and $Q(x,t)$, see Eqs. (52,53). As we discussed above only partons with small $x$ are relevant for the large $b_\perp$ asymptotic of parton densities. Assuming the Regge-like behaviour for the forward parton density $q(x) \sim 1/x^\omega$ and $Q(x) \sim 1/x^\omega$ (the subscript $\pm$ stays for the C-parity [signature] of the Regge trajectory) we obtain the following result for $b_\perp \to \infty$:

$$q(x,b_\perp) \sim \frac{\ln^\omega-(b_\perp^2)}{(b_\perp^2)^{1+\omega-}} \frac{1}{x^{\omega-}}, \quad (56)$$

$$Q(x,b_\perp) \sim \frac{\ln^\omega+(b_\perp^2)}{(b_\perp^2)^{1+\omega+}} \frac{1}{x^{\omega+}}. \quad (57)$$

The intercepts of the Regge trajectories are $\omega_- \approx 1/2$ and $\omega_+ \approx 1.1$. We see that distribution of partons at large $b_\perp$ drops slower than the naive results (54,55) of finite order $\chi$PT. Also we observe very interesting phenomenon – the distributions of partons in the transverse plane at large impact parameter $b_\perp$ depend on the intercept of the corresponding Regge trajectory. This new phenomenon is revealed due to the all order resummation of $\chi$PT developed in this paper.

The average transverse square size of a hadron in the transverse plane is defined as:

$$\langle b_\perp^2 \rangle(x) = \int d^2b_\perp \ b_\perp^2 q(x,b_\perp). \quad (57)$$

For the isoscalar PDF $Q(x,b_\perp)$ the integral for $\langle b_\perp^2 \rangle(x)$ is IR-finite in the chiral limit (see Eq. 56), meaning that the corresponding radius is not determined by the chiral expansion. On contrary the integral for the isovector $\langle b_\perp^2 \rangle(x)$ is IR-divergent and, hence, its value is dominated by the large $b_\perp \sim 1/m_\pi$, so that we can obtain the $m_\pi$ behaviour of $\langle b_\perp^2 \rangle(x)$ in the isovector case:

$$\langle b_\perp^2 \rangle(x) \sim \frac{1}{x^{\omega-}} \frac{\ln^\omega-(1/m_\pi^2)}{m_\pi^{2(1-\omega-)}}. \quad (58)$$

We see again that at small $x$ the transverse size of the pion grows as the power\footnote{We remind that we consider the chiral limit $m_\pi = 0$.} of $1/x$, also its dependence on small pion mass is power like (compare this with $\sim \ln(1/m_\pi^2)$ behaviour of the isovector radius of the pion in the standard $\chi$PT). The growth of the nucleon transverse size at small $x$ has been discussed\footnote{Note that in the Regge-like Ansatz for the $t$-dependence of GPDs the transverse size of a hadron grows as $\sim \ln(1/x)$ at $x \to 0$.}.
in Ref. [17]. The power-like growth of the pion transverse size at small $x$ derived here implies that the similar phenomenon takes place for the nucleon case.[10]

Up to now we discussed the large $b_\perp$ asymptotic for the chiral limit $m_\pi = 0$. The case of $m_\pi \neq 0$ can be obtained from the general result given in Eqs. (50,51):

$$q(x,b_\perp) \sim \frac{1}{N} \frac{1}{b_\perp^2} \left( \frac{N \ln \left( b_\perp^2 \left( 4\pi F_\pi \right)^2 \right)}{|x| b_\perp^2 \left( 4\pi F_\pi \right)^2} \right)^\omega \left[ m_\pi b_\perp \right]^{\omega+1} \int_0^1 d\eta \left( 1 - \eta^2 \right)^{(\omega-1)/2} K_{\omega+1} \left[ \frac{2m_\pi b_\perp}{\sqrt{1 - \eta^2}} \right].$$

(59)

For $m_\pi b_\perp \to 0$:

$$q(x,b_\perp) \sim \frac{1}{N} \frac{1}{b_\perp^2} \left( \frac{N \ln \left( b_\perp^2 \left( 4\pi F_\pi \right)^2 \right)}{|x| b_\perp^2 \left( 4\pi F_\pi \right)^2} \right)^\omega + O(m_\pi b_\perp).$$

This result coincides with the asymptotics (56) for the massless pion. For $m_\pi b_\perp \to \infty$

$$q(x,b_\perp) \sim \frac{1}{N} \frac{1}{b_\perp^2} \left( \frac{N \ln \left( b_\perp^2 \left( 4\pi F_\pi \right)^2 \right)}{|x| b_\perp^2 \left( 4\pi F_\pi \right)^2} \right)^\omega \left( 1 + O \left( \frac{1}{m_\pi b_\perp} \right) \right).$$

### Chiral corrections to the amplitudes of hard exclusive processes on the pion

In this section we compute the contribution of the singular terms to the amplitude of hard exclusive processes. We consider the Mandelstam $t$ in the region $m_\pi^2 \ll -t \ll (4\pi F_\pi)^2$ in order to focus on the the $t$-dependence of the amplitude, i.e. we can put effectively $m_\pi = 0$.

The corresponding amplitude is computed in terms of GPDs as the following convolution integral:

$$A^{l=0,1} (\xi, t) = \int_{-1}^{1} \frac{H^{l=0,1} (x, \xi, t)}{\xi - x - i0}. \quad (60)$$

Let us start with the isovector amplitude. To calculate the contribution of the singular terms $|b_\chi \ln(1/b_\chi)|^k/|x|$ into the amplitude we substitute the expansion given by Eq. (49) into the integral (60). The result can be represented as the following series:

$$A^{l=1} (\xi, t) = -\frac{1}{N} \sum_{n=0,2,...}^{\infty} \left( Nb_\chi \ln \left( \frac{1}{b_\chi} \right) \right)^{n+1} \frac{1}{(n+1)} \frac{1}{\xi^{n+1}} \int_{-1}^{1} dz \frac{\Phi^{l=1}_{n+1} (z)}{1 - z}. \quad (61)$$

Here we introduced the distribution amplitude (DA) of two pions with angular momentum $l = n + 1$ defined as:

$$\Phi^{l=1}_{n+1} (z) = \int_{-1}^{1} d\eta \ P_{n+1} (\eta) \ \Phi^{l=1} (z, \eta). \quad (62)$$

where $P_{n+1} (\eta)$ are Legendre polynomials and the two pion DA [16] $\Phi^{l=1} (z, \eta)$ is expressed in terms of coefficient function $F(\beta, \alpha)$ as follows:

$$\Phi^{l=1} (z, \eta) = \eta F^{l=1}(\alpha, \beta) * \delta (\beta \eta + \alpha - z). \quad (63)$$

The sum (61) has the form of the partial wave expansion of the amplitude in the $t$-channel, see [11]. Each term of the sum (61) is real, however for $\xi \leq Nb_\chi \ln (1/b_\chi)$ the sum is divergent and the imaginary part of the amplitude is generated – the case typical for the duality. That naturally invites one to apply the methods of the resummation of the $t$-channel exchanges developed for the dual parametrization of [10]We note that the results obtained here for the isoscalar quark distribution are also valid for the gluon distributions.
GPD in Ref. [20]. The final result for the chiral expansion of the imaginary part of the amplitude is the following:

\[
\text{Im} \mathcal{A}^{I=0} (\xi, t) = \pi \theta (\xi \leq 2 \epsilon) \int_{\xi/2c}^{1} \frac{dx}{x} \; \bar{N} (x) \left( 1 - \sqrt{\frac{\epsilon}{2c \epsilon}} \right),
\]

\[
\text{Im} \mathcal{A}^{I=1} (\xi, t) = \pi \theta (\xi \leq 2 \epsilon) \int_{\xi/2c}^{1} \frac{dx}{x} \; \bar{N} (x) \sqrt{\frac{\epsilon}{2c \epsilon}} \left( 5 - \sqrt{\frac{\epsilon}{2c \epsilon}} \left( 2 + 6 \epsilon \right) \right).
\]

Here \( \epsilon = \frac{N}{2} b_\chi \ln (1/b_\chi) \mid b_\chi = |t|/(4\pi F_\pi)^2 \), \( \bar{N} (x) \) is the so-called GPD quintessence function [21, 22], which is expressed via the imaginary part of the amplitude at \( m_\pi^2 = t = 0 \) with help of Abel tomography methods [21, 23]:

\[
\bar{N} (x) = \frac{2}{\pi} \frac{x (1 - x^2)}{(1 + x^2)^{3/2}} \int_{\frac{2x}{1 + x^2}}^{\frac{1}{1 + x^2}} \frac{d\xi}{\xi^{3/2}} \frac{1}{\sqrt{\xi - \frac{2\epsilon}{1 + x^2}}} \left\{ \frac{1}{2} \text{Im} \; \bar{A} (\xi) - \epsilon \frac{d}{d\xi} \text{Im} \; \bar{A} (\xi) \right\},
\]

\[
\text{Im} \; \bar{A} (\xi) = \pi H (\xi, \epsilon).
\]

Given the Regge-like small \( \xi \) behaviour of the amplitude at \( t = 0 \) \( -\text{Im} \; \bar{A} (\xi) \sim 1/\xi^\omega \). The leading chiral correction to the amplitude has the form [12]:

\[
\text{Im} \; \mathcal{A} (\xi, t) \sim \frac{1}{\xi^\omega} \left[ b_\chi \ln \left( \frac{1}{b_\chi} \right) \right]^\omega,
\]

which depends on the intercept of the corresponding Regge trajectory (\( \omega \approx 1/2 \) for \( I = 1 \) and \( \omega \approx 1.1 \) for \( I = 0 \)). This novel phenomenon can be obtained only after all order resummation of \( \chi \)PT. The standard \( \chi \)PT gives the leading chiral correction of the form \( \text{Im} \; \mathcal{A}^{I=1} (\xi, t) \sim \frac{1}{\xi} \left[ b_\chi \ln \left( \frac{1}{b_\chi} \right) \right] \) and \( \text{Im} \; \mathcal{A}^{I=0} (\xi, t) \sim \frac{1}{\xi^2} \left[ b_\chi \ln \left( \frac{1}{b_\chi} \right) \right]^2 \).

**Conclusions and discussion**

We demonstrated that in the region of Bjorken \( x_{Bj} \sim m_\pi^2/(4\pi F_\pi)^2 \) and/or \( x_{Bj} \sim |t|/(4\pi F_\pi)^2 \) the ordinary chiral expansion for the pion PDFs and GPDs fails and one must perform all order resummation of the standard \( \chi \)PT. We perform such resummation in the large-\( N \) limit. Explicit resummation allowed us to reveal novel phenomena in quark mass expansion of PDFs, in low energy behaviour of GPDs and amplitudes of hard exclusive processes. The main qualitative (model independent) results are the following:

- The leading small \( m_\pi \) asymptotic in the region of small \( x \) of pion PDFs depends on the intercept \( (\omega) \) of the corresponding Regge trajectory:\(^{11}\)

\[
q(x) \sim \frac{1}{x^\omega} \left[ \frac{m_\pi^2}{(4\pi F_\pi)^2} \ln \left( \frac{1}{m_\pi^2} \right) \right]^\omega,
\]

- The leading large impact parameter \( (b_\perp) \) asymptotics of the quark distribution in the transverse plane has the form (we show result for \( m_\pi \), the result for \( m_\pi \neq 0 \) is given in Eq. [59]):

\[
q(x, b_\perp) \sim \frac{1}{x^\omega} \frac{\ln^\omega (b_\perp^2)}{(b_\perp^2)^{1+\omega}}.
\]

\(^{11}\)We consider the Regge-like behaviour of PDFs for simplicity, one can easily obtain corresponding leading chiral corrections for other types of small \( x \) behaviour of PDFs.
The distribution of quarks at large impact parameter is controlled completely by the all order resummed $\chi$PT developed in this paper. This asymptotic is determined by the small-$x$ behaviour of usual PDFs, hence this asymptotic depends on the scale, at which the corresponding PDF is defined. This is new and interesting result – the chiral expansion meets the QCD evolution.

- The leading small $t$ behaviour of the amplitude for hard exclusive processes on the pion target has the form:

$$\text{Im } A(\xi, t) \sim \frac{1}{\xi^\omega} \left[ \frac{|t|}{4(4\pi F_\pi)^2} \ln \left( \frac{1}{|t|} \right) \right]^\omega . \quad (71)$$

Measurements of such processes at small $x_{Bj}$ and small $t$ would allow us to probe the chiral dynamics in a completely new regime – the dynamics of chiral and quark–gluon degrees of freedom intertwines.

The complete results for the resummation of $\chi$PT for pion PDFs and GPDs are given in the main body of the paper.

In order to perform the resummation of $\chi$PT we used $1/N$ expansion. To assess the accuracy of the $1/N$ expansion we computed the first three coefficients in Eq. (2) for arbitrary $N$ with the result:

$$D_1 = -1, \quad D_2 = -\frac{5}{6} N \left( 1 - \frac{1}{N} \right) \langle x \rangle, \quad D_3 = -\frac{1}{24} N^2 \left( 1 - \frac{37}{18} \frac{1}{N} + \frac{49}{18} \frac{1}{N^2} \right) \langle x^3 \rangle . \quad (72)$$

For $N = 3$ the above result coincides with the three loop calculation (see Eq. (4)) of Ref. [12], also for $N = 1$ the $D_2$ coefficient is zero as it should be, because the $O(2)$ $\sigma$-model is a free field theory. All that provides strong check of the three loop calculations of Ref. [12]. From Eq. (72) we see that the $1/N$ corrections are sizable (30-40 %) and seem to increase with number of loops. To check this tendency we computed four and six loop $D_4$ and $D_6$ coefficients in the expansion (2) with the results:

$$D_4 = -\frac{7N^3}{480} \left( 1 - \frac{76}{27} \frac{1}{N} + \frac{113}{27} \frac{1}{N^2} - \frac{64}{27} \frac{1}{N^3} \right) \langle x^3 \rangle,$n  
$$D_6 = -\frac{N^5}{8960} \left( 1 - \frac{46208}{10125} \frac{1}{N} + \frac{9441001}{911250} \frac{1}{N^2} - \frac{36581227}{2733750} \frac{1}{N^3} + \frac{30246239}{2733750} \frac{1}{N^4} - \frac{2449121}{546750} \frac{1}{N^5} \right) \langle x^5 \rangle .$$

These coefficients are zero at $N = 1$, that provides a check of our calculations. We see that indeed the $1/N$ corrections increase with number of loops (they amount to 60 – 75%). However, we note that the total coefficients $D_n$ at $N = 3$ decrease rapidly with number of loops, that indicates that the resummation of singular contributions is possible beyond the large-$N$ expansion and that one expects that main qualitative conclusions of the present paper are valid also for the exact resummation.

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Appendix A. Twist-2 operators and their matrix elements

In this Appendix we briefly describe the definitions and some technical details used in the paper. We introduce two light-like vectors $n$ and $\bar{n}$:

$$n^2 = \bar{n}^2 = 0, n \cdot \bar{n} = 1, \ a_+ = a \cdot n . \quad (73)$$
There exist two QCD quark light-cone operators of twist-2:

\[ P_R = \frac{1}{2} (1 - \gamma_5), \quad P_L = \frac{1}{2} (1 + \gamma_5), \]

(74)

\[ [O_R]_{fg} = \bar{q}_g \left( \frac{1}{2} \lambda n \right) \gamma_\mu P_R \, q_f \left( -\frac{1}{2} \lambda n \right), \]

(75)

\[ [O_L]_{fg} = \bar{q}_g \left( \frac{1}{2} \lambda n \right) \gamma_\mu P_L \, q_f \left( -\frac{1}{2} \lambda n \right), \]

(76)

where indexes \( f, g \) stand for flavor. These operators transform under the global chiral rotations as

\[ O_L \to V_L O_L V_L^\dagger \, , \quad O_R \to V_R O_R V_R^\dagger. \]

(77)

In \( \chi \)PT these QCD operators are described by an effective chiral operator with unknown chiral constants. In the pure pion sector one finds [7–8]

\[ O_{fg}^L(\lambda) = -\frac{iF_\pi^2}{4} \mathcal{F}(\beta, \alpha) \star \left[ U \left( \frac{\alpha + \beta}{2} \lambda n \right) n \cdot \tilde{\partial} U^\dagger \left( \frac{\alpha - \beta}{2} \lambda n \right) \right]_{fg}, \]

(78)

\[ O_{fg}^R(\lambda) = -\frac{iF_\pi^2}{4} \mathcal{F}(\beta, \alpha) \star \left[ U^\dagger \left( \frac{\alpha + \beta}{2} \lambda n \right) n \cdot \tilde{\partial} U \left( \frac{\alpha - \beta}{2} \lambda n \right) \right]_{fg}, \]

(79)

where by asterisk we denote the integral convolution with respect to \( \beta \) and \( \alpha \):

\[ \mathcal{F}(\beta, \alpha) \star O(\beta, \alpha) \equiv \int_1^{1-|\beta|} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \mathcal{F}(\beta, \alpha) O(\beta, \alpha). \]

(80)

Here \( \mathcal{F}(\beta, \alpha) \) represents the real generating function for the tower of low-energy constants and \( \tilde{\partial}_\mu \) denotes a combination of derivatives \( \partial_\mu - \partial_\mu \). It is important to note that the expressions [78] and [79] describe correctly only the operator vertices with two attached pions (including the tadpoles loops). The low-energy constants \( \mathcal{F}(\beta, \alpha) \) characterizes the structure of the pion, they are not determined in the effective field theory. According to the isospin \( I = 0, 1 \) one can construct two independent functions:

\[ F^{I=0}(\beta, \alpha) = \frac{1}{2} (\mathcal{F} [-\beta, \alpha] - \mathcal{F} [\beta, \alpha]), \]

(81)

\[ F^{I=1}(\beta, \alpha) = \frac{1}{2} (\mathcal{F} [-\beta, \alpha] + \mathcal{F} [\beta, \alpha]), \]

(82)

which are convenient for description of the pion matrix elements. Pion PDFs are defined as

\[ \int \frac{d\lambda}{2\pi} e^{-i p_x + \lambda x} \left\langle \pi^b(p) \left| \text{tr} [r^c O_{L+R}(\lambda)] \right| \pi^a(p) \right\rangle = 4i\varepsilon[abc]g(x), \]

(83)

\[ \int \frac{d\lambda}{2\pi} e^{-i p_x + \lambda x} \left\langle \pi^b(p) \left| \text{tr} [O_{L+R}(\lambda)] \right| \pi^a(p) \right\rangle = 2\delta^{ab}Q(x), \]

(84)

which in the chiral limit can be written as

\[ 2 \int_{-1+|\beta|}^{1-|\beta|} d\alpha F^{I=0}(\beta, \alpha) = \left[ \theta(\beta) \quad \tilde{q} (\beta) - \theta(-\beta) \quad \tilde{q} (-\beta) \right] = \tilde{Q} (\beta), \]

(85)

\[ 2 \int_{-1+|\beta|}^{1-|\beta|} d\alpha F^{I=1}(\beta, \alpha) = \left[ \theta(\beta) \quad \tilde{q} (\beta) + \theta(-\beta) \quad \tilde{q} (-\beta) = \tilde{q} (\beta). \right. \]

(86)

The GPDs are defined as:

\[ \int \frac{d\lambda}{2\pi} e^{-i p_x + \lambda x} \left\langle \pi^b(p') \left| \text{tr} [r^c O_{L+R}(\lambda)] \right| \pi^a(p) \right\rangle = 4i\varepsilon[abc]H^{I=1}(x, \xi, t), \]

(87)

\[ \int \frac{d\lambda}{2\pi} e^{-i p_x + \lambda x} \left\langle \pi^b(p') \left| \text{tr} [O_{L+R}(\lambda)] \right| \pi^a(p) \right\rangle = 2\delta^{ab}H^{I=0}(x, \xi, t) \]

(88)

with \( P = \frac{1}{2} (p + p') \), \( \xi = -\frac{t_p (p'_+p)}{(p'_+p)^2} \), \( t = (p' - p)^2 \). In the forward limit \( \xi \to 0, t \to 0 \):

\[ H^{I=1}(x, 0, 0) = q(x), \quad H^{I=0}(x, 0, 0) = Q(x). \]

(89)
Appendix B. Large $N$ expansion in $O(N+1)$ model

In this Appendix we discuss the technique of $1/N$ expansion for the $O(N+1)$ sigma model. We follow closely the methods of $1/N$ expansion presented in the book by A.N. Vasiliev [14].

Diagram technique for the large−$N$ expansion

Our task is to construct the large−$N$ expansion for the field theory given by the Lagrangian

\begin{equation}
\mathcal{L} = -\frac{1}{2} \pi^a \partial^2 \pi^a + N V(\pi^2/N) + J^a \pi^a,
\end{equation}

\begin{equation}
V(\pi^2/N) = \frac{1}{8} (\partial_\mu \pi^2/N)^2 + m^2 G \sqrt{G^2 - \pi^2/N}, \quad G^2 = F^2/N \sim \mathcal{O}(N^0)
\end{equation}

where $J^a$ is a source for the pion field. The constants $m$ and $F$ represent the bare values of the pion mass and axial coupling.

Introducing two auxiliary fields $\psi$ and $\varphi$ and using the representation

\[ 1 = \int D\psi \, \delta(\pi^2 - N \psi) = \int D\psi \, D\varphi \exp\left[ i \varphi (N \psi - \pi^2) /2 \right] \]

one can rewrite the Lagrangian as

\[ \mathcal{L} = -\frac{1}{2} \pi^a \partial^2 \pi^a + \varphi \left( N \psi - \pi^2 \right) /2 + NV(\psi) + J^a \pi^a, \]

Performing the integration with respect to the pion fields we obtain the effective Lagrangian

\[ \mathcal{L}_{\text{eff}} = N \frac{i}{2} \text{tr} \ln \left[ (\partial^2 + \varphi) /\partial^2 \right] + N \varphi \psi/2 + NV(\psi) + J^a (\partial^2 + \varphi)^{-1} J^a /2. \]

Notice that the second operation $\partial^2$ arises from the normalization constant in the functional integral. Computing the resulting integral by the steepest descent method, one obtains the desired large−$N$ expansion.

The equations of motion define the constant solutions $\varphi = \varphi_0$ and $\psi = \psi_0$:

\[ \varphi_0 = \frac{m^2 G}{\sqrt{G^2 - \psi_0}}, \quad \psi_0 = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2}. \]  

(92)

Let us postpone the discussion of the solution and suppose simply that it exists. Then performing expansion around the classical values we obtain the effective action

\[ \mathcal{L}_{\text{eff}} = N \frac{i}{2} \text{tr} \ln \left[ (\partial^2 + \varphi_0 + \varphi) /\partial^2 \right] + N \left( \varphi + \varphi_0 \right) (\psi + \psi_0) /2 \]

\[ + NV(\psi + \psi_0) + J^a \left( \partial^2 + \varphi_0 + \varphi \right)^{-1} J^a /2 \]

\[ = -\frac{N}{4} \varphi(x) i \Delta^2(x,x) \varphi(x) + N \left( \varphi(x) \psi(x) /2 - \frac{N}{2} \psi K_\psi \psi \right) \]

\[ - J^a(x) \Delta(x,y) \varphi(y) \Delta(y,z) J^a(z)/2 + ... \]

(93) (94) (95) (96)

where we introduced convenient notations:

\[ K_\psi = \frac{1}{4} \left( G^2 - \psi_0 \right)^{-1} \partial^2, \quad (\partial^2 + \varphi_0)^{-1} = \frac{i}{\Delta(x,y)}, \]

(97)

In the second line (95) we show the relevant to our accuracy $\mathcal{O}(N^{-1})$ contributions only. The first term in (95) arises from the trace in the first line, and $\Delta^2(x,x)$ in graphical notation corresponds to the scalar loop

\[ \Delta^2(x,x) \equiv L = \int \frac{d^Dl}{(2\pi)^D} \frac{i}{l^2 - \varphi_0 (l+p)^2 - \varphi_0}. \]  

(97)

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The quadratic in $\psi$ term originates from the expansion of the interaction $NV(\psi)$ and the dots denote the terms that contribute to the higher order corrections $\sim \mathcal{O}(N^{-2})$. We also keep the contribution quadratic with respect to the sources because we consider only the two-pion matrix elements. Performing the diagonalization with the help of substitution $\psi \rightarrow \psi + \frac{1}{\pi} K_\psi^{-1} \varphi$ we obtain a simple expression for the relevant part of the effective action

$$\mathcal{L}_{\text{eff}} = -\frac{N}{2} \varphi \left[ \frac{i}{2} L - \frac{(G^2 - \psi_0)}{\partial^2} \right] \varphi - J^a \Delta \varphi \Delta J^a / 2 + \ldots.$$  \hspace{1cm} (98)

The quadratic term provides the propagator for the auxiliary field $\varphi$. Notice that insertion of this propagator produces the factor $1/N$. This provides the counting rules with respect to $N$. The diagrams in Fig.1 are produced when we contract all the auxiliary fields, differentiate with respect to sources and amputate obtained irreducible diagrams. One can easily check that the terms denoted as dots can provide only graphs which are suppressed as $1/N^2$.

Let us shortly discuss the solutions of the equations (92). It is clear that constant $\varphi_0$ provides the physical mass $m_\pi$ of the pion in the large-$N$ expansion and the second constant $\psi_0$ can be associated with the physical axial coupling $F_\pi$. On can easily see that all diagrams which contribute to the renormalization of the mass and coupling can not provide leading log’s to the singular terms because renormalization decreases the power of the logarithm in front of $\delta-$function. Therefore we can skip the discussion about the renormalization of the pion mass $m$ and axial coupling $F$. Moreover, from the consideration in the text, we know that in the four-pion blob we can neglect the mass of the pion because corresponding terms produce subleading logarithms only. Therefore, in the propagator of the auxiliary field $\varphi$ one can neglect the pion mass. That is equivalent to

$$\varphi_0 = m_\pi^2 \rightarrow 0.$$  \hspace{1cm} (99)

It is clear that we must keep the pion mass only in the propagators connecting the operator vertex with the four-pion subdiagram. Next, in the $\varphi-$propagator we substitute

$$(G^2 - \psi_0) = F_\pi^2 / N \equiv G_\pi^2$$  \hspace{1cm} (100)

Finally, we obtain the following simple Lagrangian

$$\mathcal{L}_{\text{eff}} \simeq -\frac{N}{2} \varphi \left[ \frac{i}{2} L_0 - \frac{G_\pi^2}{\partial^2} \right] \varphi - J^a \Delta \varphi \Delta J^a / 2 ,$$  \hspace{1cm} (101)

where $L_0$ denotes the massless scalar loop (97). Thus for the propagator of the auxiliary field one has

$$\langle \varphi \varphi \rangle = \frac{i p^2}{N G_\pi^2} \frac{1}{1 + \frac{p^2}{G_\pi^2} L_0}.$$  \hspace{1cm} (102)

### Calculation of Large-$N$ corrections to PDF

Our aim is to discuss the evaluation of $1/N$-corrections given by graphs in Fig.1. Consider one loop diagram which contributes to $q^{\text{sing}}$ in (33). Expanding the propagator (102) in series with respect to loop $L_0$ we obtain expansion of $q^{\text{sing}}(x)$ which can be written as

$$q^{\text{sing}}(x) = -\frac{2}{N} \sum_{n=0}^{\infty} \langle x^n \rangle \delta^{(n)}(x) J_n.$$  \hspace{1cm} (103)

Hence, we have to work with the expression which include insertion of the products of the $n$ massless loops $(\frac{p^2}{G_\pi^2} L_0)^n$ in each $J_n$. It is clear that such construction is associated with the $n+1-$loop graph. The renormalization of such graph includes the subtraction of various subdivergencies. To the leading logarithmic accuracy it is enough to consider the insertion of the $L_0-$counterterms only. The explicit expression for $L_0$ reads

$$\frac{N i p^2}{F_\pi^2} L_0 = \frac{p^2}{F_\pi^2} \frac{1}{(4\pi)^2} \left( \frac{\mu^2}{-p^2} \right)^\varepsilon \Pi(\varepsilon) = \frac{p^2}{m_\pi^2} \frac{1}{\varepsilon} \varepsilon + O(\varepsilon^0),$$  \hspace{1cm} (104)
Performing expansion of this expression with respect to $\varepsilon$

$$\epsilon = \frac{N}{2} \frac{m_2^2}{(4\pi F)^2}, \quad \Pi(\varepsilon) = (4\pi)^2 \frac{\Gamma(1 + \varepsilon)\Gamma(1 - \varepsilon)^2}{\Gamma(2 - 2\varepsilon)}$$

For the product of $n$ loops $L_0$ we obtain obviously:

$$\left[ \frac{p^2}{G^2 F^2} L_0 \right]^n = \left[ \frac{p^2}{m_2^2} \left( \frac{\mu^2}{-p^2} \right)^\varepsilon \frac{\Pi(\varepsilon)}{\varepsilon} \right]^n$$

The expression for the total unrenormalized graph $J^B_n$ with $n$ loops $L_0$ can be represented as

$$J^B_n = \varepsilon^{n+1} \frac{\Pi^n}{\varepsilon^n} \int \frac{d^D k}{\pi^{D/2}} \frac{i (-k_+ + p_+)^{(n+1)}}{\left[-k^2 + m_2^2\right]^{2(n+1)}} \frac{\left[2(pk)^2/m_2^2\right]^{(n+1)}}{(\Delta - \varepsilon)},$$

where $\Delta \equiv (n + 1) \epsilon$. Evaluation of this integral is standard. Rewriting the denominator with the help of Feynman parameters, shifting the integral momentum $k$ and evaluating the resulting integral we obtained

$$J^B_n(\varepsilon, \Delta) = \varepsilon^{n+1} \frac{\Pi^n}{\varepsilon^n} \frac{(\mu^2/m_2^2)^{\Delta}}{\Delta} \frac{\Gamma[1 - \Delta, 1 + \Delta, D/2 + n + 1, 3n + 5 - 2\Delta]}{\Gamma[n + 2 - \Delta, D/2, 3n + 5 - \Delta - \varepsilon]}$$

Subtraction of $L_0$-subdivergencies in the $n$-loop chain $[L_0]^n$ is described by the substitution (104):

$$\left( \frac{N ip^2}{F^2} L_0 - \frac{p^2}{m_2^2} \varepsilon \right)^n = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{N ip^2}{F^2} L_0 \right)^{n-k} \left( -\frac{p^2}{m_2^2} \varepsilon \right)^k$$

Each term in the last sum can be represented in the form (106) with $\Delta \rightarrow \Delta' \equiv (n + 1 - k)\varepsilon$. Hence after subtraction of the subdivergencies we obtain following expression

$$J_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{\varepsilon^k} J^B_n(\varepsilon, \Delta')$$

Substituting the $J$-integral as in (106) we obtain

$$J_n = \frac{\varepsilon^{n+1}}{\varepsilon^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{\mu^2}{m_2^2} \right)^{\Delta'} \frac{(-1)^k}{(n - k + 1)} E_n(\varepsilon, \Delta').$$

Performing expansion of this expression with respect to $\varepsilon$ we obtain the local poles and finite terms:

$$J_n = \frac{Z_{n+1}}{\varepsilon^{n+1}} + \ldots + \frac{Z_1}{\varepsilon} + J_n^{LL} \ln^{n+1} \left[ \mu^2/m^2 \right] + \mathcal{O}(\ln^{n} \left[ \mu^2/m^2 \right])$$

In order to compute coefficient $J_n^{LL}$ we take into account that the largest power $\ln^{n+1}$ arises only from the expansion

$$\frac{1}{\varepsilon^{n+1}} \left( \frac{\mu^2}{m_2^2} \right)^{\Delta'} = \frac{(n - k + 1)^{n+1}}{(n + 1)!} \ln^{n+1} \left[ \mu^2/m_2^2 \right] + \mathcal{O}(\ln^{n} \left[ \mu^2/m_2^2 \right])$$
Picking up coefficient in front of $\ln^{n+1}$ in (111) we obtain:

$$J_n^{LL} = (\epsilon \ln \left[ \frac{\mu^2}{m^2} \right])^{n+1} \frac{E_n[0,0]}{(n+1)!} \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k+1)^n$$

$$= \langle x^n \rangle \left( \epsilon \ln \left[ \frac{\mu^2}{m^2} \right] \right)^{n+1} \frac{n!}{(n+1)!},$$

where we used that

$$\sum_{k=0}^{n} (-1)^k (n+1-k)^n \binom{n}{k} = n!, \quad \lim_{\epsilon \to 0} E_n[\epsilon, \Delta] = \frac{1}{n!} \tag{113}$$

and redefining $\epsilon \ln \left[ \frac{\mu^2}{m^2} \right] \equiv \epsilon$ one has

$$J_n^{LL} = \frac{\epsilon^{n+1}}{(n+1)!}.$$

Then the expansion (103) reads:

$$q^{\text{sing}}(x) = -\frac{2}{N} \sum_{n=0}^{\infty} \delta^{(n)}(x) \langle x^n \rangle \frac{\epsilon^{n+1}}{(n+1)!}.$$

The evaluation of the second, two-loop graph with the two propagators (102) in Fig.1 is more complicated due to additional divergency of the box subgraph. But technically it can be done in the same way therefore we shall not repeat this discussion.

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