MAXIMAL TORI IN THE SYMPECTOMORPHISM GROUPS OF HIRZEBRUCH SURFACES

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Abstract. We count the conjugacy classes of maximal tori in the groups of symplectomorphisms of $S^2 \times S^2$ and of the blow-up of $\mathbb{CP}^2$ at a point.

Consider the group $\text{Ham}(M, \omega)$ of Hamiltonian symplectomorphisms of a symplectic manifold. A $k$-dimensional torus in $\text{Ham}(M, \omega)$ is a subgroup which is isomorphic to $(S^1)^k$. A maximal torus is one which is not contained in any strictly larger torus.

An action of $(S^1)^k$ on $(M, \omega)$ is called Hamiltonian if it admits a moment map, i.e., a map $\Phi: M \to \mathbb{R}^k$ such that $d\Phi = -\iota(\xi_i)\omega$ for all $i = 1, \ldots, k$, where $\xi_1, \ldots, \xi_k$ are the vector fields on $M$ that generate the action.

A Hamiltonian $(S^1)^k$-action defines a homomorphism from $(S^1)^k$ to $\text{Ham}(M, \omega)$. The action is effective if and only if this homomorphism is one to one. Its image is then a $k$-dimensional torus in $\text{Ham}(M, \omega)$. Every $k$-dimensional torus in $\text{Ham}(M, \omega)$ is obtained in this way, and two Hamiltonian actions give the same torus if and only if they differ by a reparametrization of $(S^1)^k$. Tori in $\text{Ham}(M, \omega)$ have dimension at most $\frac{1}{2}\dim M$. A Hamiltonian action of a $(\frac{1}{2}\dim M)$-dimensional torus is called toric.

Theorem 1. Let $(M, \omega)$ be a compact symplectic four-manifold. Suppose that $\dim H^2(M, \mathbb{R}) \leq 3$ and $\dim H^1(M, \mathbb{R}) = 0$. Then every Hamiltonian circle action on $(M, \omega)$ extends to a toric action.

Remark. Many symplectic four-manifolds do not admit Hamiltonian circle actions. The theorem does not say anything about such manifolds.

Two tori, $T_1$ and $T_2$, in $\text{Ham}(M, \omega)$ are conjugate if there exists an element $g \in \text{Ham}(M, \omega)$ such that $gT_1g^{-1} = T_2$. Two torus actions, viewed as homomorphisms $(S^1)^k \to \text{Ham}(M, \omega)$, give conjugate tori in $\text{Ham}(M, \omega)$ if and only if they differ by an equivariant symplectomorphism composed with a reparametrization of $(S^1)^k$.

On $S^2 \times S^2$, let $\omega_1$ and $\omega_2$ be the pullbacks of the standard area form on $S^2$ via the two projection maps, and let $\omega_{a,b} = a\omega_1 + b\omega_2$. The standard $(S^1)^2$-action has the moment map image shown in Figure 1 on the left.

Let $\mathbb{CP}^2$ be the blow-up of $\mathbb{CP}^2$ at a point. In it, let $E$ be the exceptional divisor and let $L$ be a $\mathbb{CP}^1$ which is disjoint from $E$. For $l > e > 0$, let $\tilde{\omega}_{l,e}$ be a symplectic form such that the symplectic area of $L$ is $l$ and the symplectic area of $E$ is $e$. (We can construct $(\mathbb{CP}^2, \tilde{\omega}_{l,e})$ explicitly in several ways. By [McD2], it is unique up to...
symplectomorphism.) The standard \((S^1)^2\)-action (induced from the action on \(\mathbb{CP}^2\)) has the moment map image shown in Figure 1 on the right.

**Theorem 2.** For \(a \geq b > 0\), the number of conjugacy classes of maximal tori in \(\text{Ham}(S^2 \times S^2, \omega_{a,b})\) is \(\lceil a/b \rceil\).

For \(l > e > 0\), the number of conjugacy classes of maximal tori in \(\text{Ham}(\mathbb{CP}^2, \tilde{\omega}_{l,e})\) is \(\lceil \frac{e}{l-e} \rceil\).

**Remark.** Here, \(\lceil r \rceil\) denotes the smallest integer greater than or equal to \(r\). We will prove Theorem 2 by enumerating the different conjugacy classes by the set of integers \(k\) that satisfy \(0 \leq k < r\) for \(r = a/b\) and \(r = \frac{e}{l-e}\), respectively.

**Remark.** The topology of \(\text{Ham}(S^2 \times S^2, \omega_{a,b})\) also changes when the ratio \(a/b\) crosses an integer. See \([\text{Gr}, \text{Ab}, \text{AM}, \text{McD3}, \text{An}]\). Also see the remark at the end of the paper.

**Remark.** There are infinitely many conjugacy classes of tori in the group of contactomorphisms of an overtwisted \(S^3\) or lens space. See \([\text{L1}]\).

For a count of the conjugacy classes of tori in the group of contactomorphisms of the pre-quantum line bundle over a Hirzebruch surface, see \([\text{L2}]\).

We recall standard facts about Hamiltonian circle actions on compact symplectic manifolds: By local arguments, each component of the fixed point set is a submanifold of even dimension and has even index. By Morse-Bott theory, each local extremum for the moment map is a global extremum, and these extrema are attained on connected sets. See \([\text{GS2}, \S 32]\). An interior fixed point is a fixed point which is not a minimum or maximum for the moment map. A fixed surface is a two dimensional connected component of the fixed point set.

**Lemma 1.** Let \(M\) be a closed symplectic four-manifold with a Hamiltonian circle action. The dimension of \(H^2(M)\) is equal to the number of interior fixed points plus the number of fixed surfaces. If \(\text{dim} \ H^1(M) = 0\), each fixed surface has genus zero.

**Proof.** We apply a standard Morse theory argument. The moment map is a perfect Morse-Bott function whose critical points are the fixed points for the circle action \([\text{GS2}, \S 32]\). Therefore, \(\text{dim} \ H^1(M) = \sum \text{dim} \ H^{3-i_F}(F)\), where we sum over the connected components of the fixed point set, and where \(i_F\) is the index of the component \(F\). The theorem follows by a simple computation of the summands, which is summarized in the table below.
Figure 2. A standard Hirzebruch trapezoid

Proof of Theorem 1. By [Ka, Prop. 5.21], a Hamiltonian circle action extends to a toric action if and only if each fixed surface has genus zero and each non-extremal level set for the moment map contains at most two non-free orbits.

By Lemma 1, since \( \dim H^1(M) = 0 \), each fixed surface has genus zero.

By [Ka, Theorem 5.1], if all the fixed points are isolated, the circle action extends to a toric action. Therefore, let us assume that there exists at least one fixed surface. Suppose that the moment map attains its maximum on this surface; the case of a minimum can be treated similarly. By Lemma 1, since \( \dim H^2(M, \mathbb{R}) \leq 3 \), there exist at most two interior fixed points.

A \( \mathbb{Z}_k \)-sphere is a 2-sphere inside \( M \) on which the circle acts by rotations with speed \( k \). A non-free orbit is either a fixed point or belongs to a \( \mathbb{Z}_k \)-sphere. See [Au] or [Ka, Lemma 2.2]. A \( \mathbb{Z}_k \)-sphere intersects each level set in at most one orbit. The north pole of a \( \mathbb{Z}_k \)-sphere is an isolated, hence interior, fixed point (because we assume that the maximal set of the moment map is not isolated). Different \( \mathbb{Z}_k \)-spheres have different north poles. These considerations show that the number of non-free orbits in a non-extremal level set for the moment map is at most the number of interior fixed points. The theorem follows.

For each non-negative integer \( m \), consider the family of trapezoids shown in Figure 2 parametrized by the height \( b \) and average width \( a > \frac{m}{2} b \). We call these standard Hirzebruch trapezoids. More generally, consider their images under the group \( AGL(2, \mathbb{Z}) \) of transformations of \( \mathbb{R}^2 \) of the form \( x \mapsto Rx + v \) with \( R \in GL(2, \mathbb{Z}) \) and \( v \in \mathbb{R}^2 \). These we call Hirzebruch trapezoids modulo \( AGL(2, \mathbb{Z}) \) are in natural one-to-one bijection with the set of parameters \( (a, b, m) \) with \( m \) a non-negative integer, \( a \) and \( b \) positive real numbers, and \( a > \frac{m}{2} b \).

Take a polygon in \( \mathbb{R}^2 \). Let \( u_1, \ldots, u_m \) be normal vectors to its edges, in counterclockwise order, pointing inward. The polygon is a Delzant polygon if we can choose these normals so that \( u_i \in \mathbb{Z}^2 \) and \( \det(u_i u_{i+1}) = 1 \) for all \( i \), the indices taken cyclically.

Lemma 2. Every Delzant polygon with four edges is a Hirzebruch trapezoid.

More generally, the Delzant condition for a polytope in \( \mathbb{R}^n \) is that exactly \( n \) facets meet at every vertex and the normals to these facets can be chosen to be generators of \( \mathbb{Z}^n \).
Proof. Fix a Delzant polygon with four edges. Let $u_1$, $u_2$, $u_3$, and $u_4$ be normals to its edges that satisfy the above Delzant condition. Because $\det(u_1 u_2) = 1$, we may assume, without loss of generality, that $u_1 = (1, 0)$ and $u_2 = (0, 1)$. The conditions $\det(u_2 u_3) = 1$ and $\det(u_3 u_1) = 1$ imply that $u_3 = (-1, k)$ and $u_4 = (l, -1)$ for some $k, l \in \mathbb{Z}$. The condition $\det(u_3 u_4) = 1$ then implies $kl = 0$. Each of the cases $k = 0$ and $l = 0$ gives a Hirzebruch trapezoid, as shown in Figure 3.

The convexity theorem of Atiyah, Guillemin and Sternberg, [At, Gu-St], states that, for a Hamiltonian torus action on a compact symplectic manifold, the image of the moment map is a convex polytope. By Delzant’s classification of Hamiltonian toric actions [De], the moment map images for toric actions are precisely the Delzant polytopes, and two toric actions are equivariantly symplectomorphic if and only if their moment map images are translates of each other.

The symplectic four-manifolds that correspond to Hirzebruch trapezoids are Hirzebruch surfaces. See [Au] or [Ka, section 6.3]. Specifically, when $m = 0$ or $m = 1$, these are $(S^2 \times S^2, \omega_{a,b})$ and $(\tilde{\mathbb{C}P}^2, \tilde{\omega}_{l,e})$, respectively.

Lemma 3. A Hirzebruch surface which corresponds to an integer $n \geq 2$ is symplectomorphic to the Hirzebruch surface which corresponds to the integer $n - 2$ with the same parameters $a$ and $b$.

Proof. Let $m = n - 1$. Take two Hirzebruch trapezoids with the same parameters $a$ and $b$, corresponding to the integers $m - 1$ and $m + 1$. After transforming by appropriate elements of $\text{AGL}(2, \mathbb{Z})$, they can be brought to the forms shown in Figure 4.

In [Ka, §2.1] we associate a labeled graph to every Hamiltonian $S^1$-space, such that two spaces are isomorphic if and only if their graphs are isomorphic [Ka, Theorem 4.1]. If the $S^1$-action is obtained by restriction of a toric action, the graph can be easily read from the Delzant polygon. See [Ka, §2.2]. The two polygons in Figure 4 give rise to the same graph (shown in Figure 4 on the right). Therefore, the spaces are $S^1$-equivariantly symplectomorphic.

Remark. Lemma 3 (which I learned from S. Tolman) seems to be well known. For instance, the new circle action on $S^2 \times S^2$ obtained by identifying this space with the Hirzebruch surface with parameters $(a, b, m = 2)$ plays a role in [McD1].

\[ \text{Figure 3. Hirzebruch trapezoids} \]

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Lemma 4. Among the symplectic manifolds \((S^2 \times S^2, \omega_{a,b})\), for \(a \geq b > 0\), and \((\mathbb{CP}^2, \tilde{\omega}_{l,e})\), for \(l > e > 0\), no two are symplectomorphic.

Proof. The manifolds \(S^2 \times S^2\) and \(\mathbb{CP}^2\) are not homeomorphic. For instance, the self intersection of the exceptional divisor in \(\mathbb{CP}^2\) is \(-1\) whereas in \(S^2 \times S^2\) every class in \(H_2\) has an even self intersection.

A diffeomorphism of \(S^2 \times S^2\) acts on \(H^2(S^2 \times S^2) = \mathbb{Z}^2\) by a \(2 \times 2\) matrix of integers, with determinant \(\pm 1\), which preserves the intersection form \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]
There are exactly sixteen matrices with this property; they are
\[
\begin{bmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & \pm 1 \\
\pm 1 & 0
\end{bmatrix}.
\]
These cannot take \(\omega_{a,b}\) to a different \(\omega_{a',b'}\) with \(a \geq b > 0\) and \(a' \geq b' > 0\).

A diffeomorphism of \(\mathbb{CP}^2\) acts on \(H_2(\mathbb{CP}^2) = \mathbb{Z}L \oplus \mathbb{Z}E \cong \mathbb{Z}^2\) by a \(2 \times 2\) matrix of integers, with determinant \(\pm 1\), which preserves the intersection form \[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]
There are exactly four matrices with this property; they are
\[
\begin{bmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{bmatrix}.
\]
These cannot take \(\tilde{\omega}_{l,e}\) to a different \(\tilde{\omega}_{l',e'}\) with \(l > e > 0\) and \(l' > e' > 0\).

Proof of Theorem 4. By Theorem 4, we only need to consider two dimensional tori.

The conjugacy classes of 2-tori in \(\text{Ham}(M, \omega)\) for all possible symplectic four-manifolds \((M, \omega)\) are given by all the Delzant polygons in \(\mathbb{R}^2\), modulo \(\text{AGL}(2, \mathbb{Z})\)-congruence. This follows immediately from Delzant’s classification of Hamiltonian toric actions [De] and the fact that a reparametrization of the 2-torus \(S^1 \times S^1\) transforms the moment map image by an element of \(\text{GL}(2, \mathbb{Z})\).

Hence, to find the conjugacy classes of 2-tori in the symplectomorphism group of a particular symplectic four-manifold \((M, \omega)\), we must identify which Delzant polygons correspond to a space which is symplectomorphic to \((M, \omega)\), and we must take these polygons modulo \(\text{AGL}(2, \mathbb{Z})\)-congruence.

The number of edges of a Delzant polygon is equal to 2 plus the second Betti number of the corresponding space. See Lemma 4 and [Ka, section 2.2]. Therefore, when \(M = S^2 \times S^2\) or \(M = \mathbb{CP}^2\), we only need to consider Delzant polygons with four edges. By Lemma 4, we only need to consider Hirzebruch trapezoids.

\[\text{Figure 4. Hirzebruch trapezoids with integers } m - 1 \text{ and } m + 1, \text{ width } b \text{ and average height } a.\]
Consider the Hirzebruch trapezoid with parameters \( m \geq 0 \) and \( a \geq b > 0 \). Iterate Lemma 3. If \( m \) is even, the corresponding Hirzebruch surface is symplectomorphic to \( (S^2 \times S^2, \omega_{a,b}) \). If \( m \) is odd, it is symplectomorphic to \( (\mathbb{CP}^2, \tilde{\omega}_{l,e}) \), where \( l = a + \frac{1}{b} \) and \( e = a - \frac{1}{b} \), so that the corresponding trapezoid (shown in Figure 3 on the right) still has height \( b \) and average width \( a \).

Numbers \( a \geq b \) can occur as the average width and the height of a Hirzebruch trapezoid with integer parameter \( m \geq 0 \) if and only if \( \frac{a}{b} > \frac{m}{2} \). When \( m = 2k \) is even, this becomes the condition \( k < \frac{1}{2} \). When \( m = 2k + 1 \) is odd, it becomes the condition \( k < \frac{1}{2} + \epsilon \), with \( l \) and \( e \) as above.

We have found the required number of distinct torus actions on each of the symplectic manifolds \( (S^2 \times S^2, \omega_{a,b}) \), \( a \geq b > 0 \), and \( (\mathbb{CP}^2, \tilde{\omega}_{l,e}) \), \( l > e > 0 \). By Lemma 4, this accounts for all possible toric actions on each of these symplectic manifolds. \( \square \)

**Remark.** To each non-negative integer \( m \) we have associated a torus action on \( S^2 \times S^2 \) if \( m \) is even and on \( \mathbb{CP}^2 \) if \( m \) is odd, for appropriate ranges of values of symplectic forms. Each of these torus actions is in fact obtained from an action of a larger, non-abelian, compact Lie group by restricting to its maximal torus. For instance, the standard actions of \( \text{SO}(3) \times \text{SO}(3) \) on \( S^2 \times S^2 \) and of \( U(2) \) on \( \mathbb{CP}^2 \) restrict to the torus actions that correspond to the integers \( m = 0 \) and \( m = 1 \), respectively. More generally, given a non-negative integer \( m \), consider the quotient of \( S^3 \times S^2 \) by the circle action \( \lambda: (z, p) \mapsto (z, \lambda^m \cdot p) \), where \( S^1 \) is the circle group of complex numbers of norm one, \( \lambda \in S^1 \) acts on \( z \in S^3 \subset \mathbb{C}^2 \) by scalar multiplication, and \( p \mapsto \lambda \cdot p \) is the circle action on \( S^2 \subset \mathbb{R}^3 \) by rotations. This space admits natural Kähler structures with which it is symplectomorphic (but not biholomorphic) to \( S^2 \times S^2 \) if \( m \) is even and to \( \mathbb{CP}^2 \) if \( m \) is odd, with appropriate symplectic forms. On this space there is a natural action of the quotient of \( U(2) \times S^1 \) by \( \{ (aI, a^m) \mid a \in S^1 \} \), where \( I \in U(2) \) is the identity matrix. Note that this quotient is a central extension of \( \text{SO}(3) \times S^1 \). The torus action that corresponds to the integer \( m \) comes from this action.

In this context we recall that the only four dimensional compact symplectic \( \text{SO}(3) \) manifolds are \( S^2 \times S^2 \) and \( \mathbb{CP}^2 \), by [11].

We also recall that the Hamiltonian symplectomorphism group of \( S^2 \times S^2 \) with equal areas of the two factors retracts to \( \text{SO}(3) \times \text{SO}(3) \) by [14] and that the aforementioned compact Lie subgroups of the Hamiltonian symplectomorphism groups of \( S^2 \times S^2 \) and of \( \mathbb{CP}^2 \) in some sense carry an essential part of the topology of these symplectomorphism groups [11, AM, McD3].

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**References**

[Ab] M. Abreu, *Topology of symplectomorphism groups of \( S^2 \times S^2 \)*, Inv. Math. 131 (1998), 1–23.

[AM] M. Abreu and D. McDuff, *Topology of symplectomorphism groups of rational ruled surfaces*, in J. Amer. Math. Soc. 13 (2000), no.4, 971–1009.

[An] S. Anjos, *Homotopy type of symplectomorphism groups of \( S^2 \times S^2 \)*, math.SG/0009220.
[At] M. Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1–15.

[Au] M. Audin, *The topology of torus actions on symplectic manifolds*, Birkhäuser, 1991.

[De] T. Delzant, *Hamiltoniens périodiques et image convexe de l’application moment*, Bulletin de la Société Mathématique de France **116** (1988), 315–339.

[Gr] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Inv. Math. **82** (1985), 307–347.

[GS1] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[GS2] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, 1984.

[I] P. Iglesias, *Les SO(3)-variétés symplectiques et leur classification en dimension 4*, Bull. Soc. Math. France **119** (1991), no.3, 371–396.

[Ka] Y. Karshon, *Periodic Hamiltonian flows on four dimensional manifolds*, Memoirs Amer. Math. Soc. **672**, 1999.

[L1] E. Lerman, *Contact cuts*, Israel J. Math. **124** (2001), 77–92.

[L2] E. Lerman, *Maximal tori in the contactomorphism groups of circle bundles over Hirzebruch surfaces*, math.SG/0204334.

[LT] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 4201–4230.

[McD1] D. McDuff, *Examples of symplectic structures*, Invent. Math. **89** (1987), no. 1, 13–36.

[McD2] D. McDuff, *From symplectic deformation to isotopy*, in: *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), 85–99, First int. Press Lect. Ser., I, internat. Press, Cambridge, MA, 1998.

[McD3] D. McDuff, *Almost complex structures on $S^2 \times S^2$*, Duke Math. J. **101** (2000), no. 1, 135–177.

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