Abstract. The aim of this paper is the study of Gorenstein global and weak dimensions of semi-primary rings.

1. Introduction

Throughout the paper, all rings are associative with identity, and all modules are unitary. Let \( R \) be a ring, and let \( M \) be an \( R \)-module. As usual we use \( \text{pd}_R(M), \text{id}_R(M) \) and \( \text{fd}_R(M) \) to denote, respectively, the classical projective dimension, injective dimension and flat dimension of \( M \).

For a two-sided Noetherian ring \( R \), Auslander and Bridger \cite{2} introduced the \( G \)-dimension, \( \text{Gdim}_R(M) \), for every finitely generated \( R \)-module \( M \). They showed that there is an inequality \( \text{Gdim}_R(M) \leq \text{pd}_R(M) \) for all finite \( R \)-modules \( M \), and equality holds if \( \text{pd}_R(M) \) is finite.

Several decades later, Enochs and Jenda \cite{7,8} defined the notion of Gorenstein projective dimension (\( G \)-projective dimension for short), as an extension of \( G \)-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (\( G \)-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas \cite{9} introduced the Gorenstein flat dimension. Some references are \cite{3,5,6,7,8,9,11}.

Recall that a left (resp., right) \( R \)-module \( M \) is called Gorenstein projective if, there exists an exact sequence of projective left (resp., right) \( R \)-modules:

\[
P : \ldots \to P_1 \to P_0 \to P_0 \to P_1 \to \ldots
\]
such that \( M \cong \text{Im}(P_0 \to P_0) \) and such that the operator \( \text{Hom}_R(-,Q) \) leaves \( P \) exact whenever \( Q \) is a left (resp., right) projective \( R \)-module. The resolution \( P \) is called a complete projective resolution.

The left and right Gorenstein injective \( R \)-modules are defined dually.

And an \( R \)-module \( M \) is called left (resp., right) Gorenstein flat if, there exists an exact sequence of flat left (resp., right) \( R \)-modules:

\[
F : \ldots \to F_1 \to F_0 \to F_0 \to F_1 \to \ldots
\]
such that $M \cong \text{Im}(P_0 \to P^0)$ and such that the operator $I \otimes_R - \text{ (resp., } - \otimes_R I \text{)}$ leaves $F$ exact whenever $I$ is a right (resp., left) injective $R$-module. The resolution $F$ is called complete flat resolution.

The Gorenstein projective, injective and flat dimensions are defined in term of resolution and denoted by $\text{Gpd}(-)$, $\text{Gid}(-)$ and $\text{Gfd}(-)$ respectively (please see [9, 10, 11]).

In the rest of this papers, the word $R$-module will mean left $R$- module unless explicitly stated otherwise.

In [8], the authors prove the equality

$$\sup \{ \text{Gpd}_R(M) | M \text{ is an } R - \text{ module} \} = \sup \{ \text{Gid}_R(M) | M \text{ is an } R - \text{ module} \}$$

They called the common value of the above quantities the left Gorenstein global dimension of $R$ and denoted it by $l.Gldim(R)$. Similarly, they set

$$l.wGldim(R) = \{ \text{Gfd}_R(M) | M \text{ is an } R - \text{ module} \}$$

which they called the left Gorenstein weak dimension of $R$.

Recall that a ring $R$ is called semi-primary if there is a two-sided nilpotent ideal $N$ of $R$, which we call the radical of $R$, such that $R/N$ is semi-simple. It is clear that if $R$ is semi-primary, its radical is unique (see [12]). Some familiars examples of semi-primary rings are:

\textbf{Example 1.1.} In the following cases the rings $R$ are semi-primary with radical $J$:

1. $R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$ where $F$ is a field, $J = \left( \begin{array}{cc} 0 & F \\ 0 & 0 \end{array} \right)$ with $J^2 = 0$.

2. $R = K[X]/(X^2)$ where $K$ is field, $J = (X)$ and $J^2 = 0$.

3. Camillo Example ([12, Example 2.6]).

For more different examples (non-Noetherian, non-commutative, non self-injective...) of semi-primary rings please see (12).

The purpose of this papers is to characterize the left Gorenstein global and weak dimensions of semi-primary rings.

2. \textbf{Main results}

We begin with the first main result which give a characterization of the left Gorenstein global dimension of semi-primary rings.

Recall that the word $R$-module will mean left $R$- module unless explicitly stated otherwise.

\textbf{Theorem 2.1.} Let $R$ be a semi-primary ring with radical $N$. Then,

\begin{align*}
\text{(a)} & \quad l.Gldim(R) = \text{Gpd}_R(R/N) \\
\text{(b)} & \quad = \text{Gid}_R(R/N) \\
\text{(c)} & \quad = \sup_C \text{Gpd}_R(C) \\
\text{(d)} & \quad = \sup_C \text{Gid}_R(C)
\end{align*}

where $C$ ranges ranges over all simple left $R$-modules.

If therefore $R$ is not quasi-Frobenius ring then,

\begin{align*}
\text{(e)} & \quad l.Gldim(R) = 1 + \text{Gpd}_R(N)
\end{align*}

To prove this theorem, we need the following Lemma.
Lemma 2.2. Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of $R$-modules. Then,

1. $\text{Gpd}_R(N') \leq \max\{\text{Gpd}_R(N), \text{Gpd}_R(N'')\}$ with equality if $\text{Gpd}_R(N'') \neq \text{Gpd}_R(N) + 1$.
2. $\text{Gpd}_R(N'') \leq \max\{\text{Gpd}_R(N'), \text{Gpd}_R(N) + 1\}$ with equality if $\text{Gpd}_R(N') \neq \text{Gpd}_R(N)$.

Proof. Using [11, Theorems 2.20 and 2.24] the argument is analogous to the one of [3, Corollary 2, p. 135].

Proof of Theorem 2.7. (a). Obviously, by definition, we have $lG.n.gldim(R) \geq \text{Gpd}_R(R/N)$. Then, only the other inequality need a proof and we may assume $\text{Gpd}_R(R/N) < \infty$. We claim $\text{Gpd}_R(M) \leq \text{Gpd}_R(R/N)$ for each $R$-module $M$. Firstly, let $M$ be a $R$-module such that $NM = 0$. Note that $M$ can be considered a left module over the semi-simple ring $R/N$ by setting $\bar{x}m = x.m$, for each $m \in M$ and $x \in R$. Clearly this modulation is well defined since $\bar{x} = \bar{y} \implies x - y \in N \implies (x - y).m = 0$

Then, $M$ is a left projective $R/N$-module (recall that $R/N$ is semi-simple). Therefore, it is a direct summand of a left free $R/N$-module $(R/N)^{(1)}$. Since $NM = 0$, we can consider $M$ as a direct summand of $(R/N)^{(1)}$ as an $R$-modules. Then, from [11, Proposition 2.19], $\text{Gpd}_R(M) \leq \text{Gpd}_R((R/N)^{(1)}) = \text{Gpd}_R(R/N)$. Now, let $M$ be an arbitrary $R$-module and let $k$ be the smaller positive integer such that $N^kM = 0$ (such integer exists since $N$ is nilpotent). Consider the family of short exact sequences of $R$-modules:

$$0 \to N^{k-i+1}M \to N^{k-i}M \to N^{k-i}M/N^{k-i+1}M \to 0$$

where $0 < i \leq k$. Then, by Lemma 2.2(1) we have

$$\text{Gpd}_R(N^{k-i}M) \leq \sup\{\text{Gpd}_R(N^{k-i+1}M), \text{Gpd}_R(N^{k-i}M/N^{k-i+1}M)\}$$

But, $\text{Gpd}_R(N^{k-i}M/N^{k-i+1}M) \leq \text{Gpd}_R(R/N)$ since $N(N^{k-i}M/N^{k-i+1}M) = 0$. Thus, $\text{Gpd}_R(N^{k-i}M) \leq \sup\{\text{Gpd}_R(N^{k-i+1}M), \text{Gpd}_R(R/N)\}$. So, we conclude that $\text{Gpd}_R(M) \leq \sup\{\text{Gpd}_R(N^{k-i}M, \text{Gpd}_R(R/N)\}$. Again we have $\text{Gpd}_R(N^{k-i}M) \leq \text{Gpd}_R(R/N)$ since $N(N^{k-i}M) = 0$. Hence, $\text{Gpd}_R(M) \leq \text{Gpd}_R(R/N)$, as desired.

(b). Similarly to (a) it suffices to prove that for every left $R$-module $M$ such that $NM = 0$ we have $\text{Gid}_R(M) \leq \text{Gid}_R(R/N)$. The rest of the proof is the same lines as (a) by using the dual of Lemma 2.2. Let $M$ be such module. Then, $M$ is a direct summand of a left free $R/N$-module $(R/N)^{(1)}$. If we identify $M$ to a submodule of $(R/N)^{(1)}$ we get

$$M \subseteq (R/N)^{(1)} \subseteq \Pi_I(R/N)$$

Then $M$ is a direct summand of $\Pi_I(R/N)$ (as an $R/N$-modules and also as an $R$-modules) since $M$ is an injective $R/N$-module (since $R/N$ is semi-simple). Then, Using the injective version of [11, Proposition 2.19], we have

$$\text{Gid}_R(M) \leq \text{Gid}_R(\Pi_I(R/N) = \text{Gid}_R(R/N)$$

as desired.

(c). Since $R/N$ is semi-simple, $R/N \cong \oplus C_i$, finite direct sum of simple left $R$-modules, where the $C_i$ have the property that if $C$ is a left simple $R$-module,
then $C \cong C_i$ for some $i$. Therefore, by [11 Proposition 2.19], $\sup_C \{ \text{Gpd}_R(C) \} = l\text{Gpd}_R(R/N) = l\text{Gldim}(R)$, where $C$ ranges over all left simple $R$-modules.

(d). Since the direct sum $R/N \cong \oplus C_i$ is finite, we can replace the direct sum by the direct product and use the injective version of [11 Proposition 2.19]. Thus, (d) is proved in an analogous fashion to (e).

(e). Suppose that $R$ is not quasi-Frobenius. Then $l\text{Gldim}(R) = \text{Gpd}_R(R/N) > 0$ ([3 Proposition 2.6]). Therefore, from Lemma 2.2(2) we deduce from the exact sequence

$$0 \rightarrow N \rightarrow R \rightarrow R/N \rightarrow 0$$

that $\text{Gpd}_R(R/N) = 1 + \text{Gpd}_R(N)$.

□

The next Proposition give a functorial description of the the left Gorenstein global dimension of semi-primary rings provided this value is finite.

**Proposition 2.3.** Let $R$ a semi-primary rings with radical $N$ and with finite left Gorenstein global dimension and let $n > 0$ be an integer. The following are equivalent:

1. $l\text{Gldim}(R) < n$,
2. $\text{Ext}^n_R(I, R/N) = 0$ for every injective $R$-module $I$,
3. $\text{Ext}^n_R(I, C) = 0$ for every injective $R$-module $I$ and every simple left $R$-module $C$,
4. $\text{Tor}^n_R(R/N, I) = 0$ for every injective $R$-module $I$ (and $R/N$ is consider as a right $R$-module),
5. $\text{Tor}^n_R(C, I) = 0$ for every injective $R$-module $I$ and every simple right $R$-module $C$,
6. $\text{Ext}^n_R(R/N, P) = 0$ for every projective $R$-module $P$, and
7. $\text{Ext}^n_R(C, P) = 0$ for every projective $R$-module $I$ and every simple left $R$-module $C$.

**Proof.** By [3 Theorem 1.1], we have

$l\text{Gldim}(R) = \sup \{ \text{Gdim}_R(M) | M \text{ is an } R-\text{module} \}$.

Then, by [11 Theorem 2.22], $l\text{Gldim}(R) < n$ if, and only if, $\text{Ext}^i(I, M) = 0$ for each $i \geq n$ and every $R$-module $M$ and every injective $R$-module $I$. Hence we conclude that

$l\text{Gldim}(R) < n \iff \text{pd}_R(I) < n$ for every injective $R$-module $I$.

So, using [11 Proposition 7], we have the equivalence of (1), (2), (3), (4) and (5). Using [3 Lemma 2.1] and [11 Proposition 10] we obtain the equivalence of (1), (6) and (7).

□

Now, we give our second main result which characterize the left Gorenstein weak dimension of coherent semi-primary rings:

**Theorem 2.4.** Let $R$ be a right coherent semi-primary ring with radical $N$. Then,

$l\text{wGldim}(R) = \text{Gfd}_R(R/N) = \sup_C \text{Gfd}_R(C)$

where $C$ ranges ranges over all simple left $R$-modules.

To prove this Theorem we need the following Lemma:
Lemma 2.5. Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of modules over a right coherent ring $R$. Then, $Gfd_R(N') \leq \max\{Gfd_R(N), Gfd_R(N')\}$ with equality if $Gfd_R(N'') \neq Gfd_R(N) + 1$.

Proof. Using [11, Theorem 3.15] and [11, Theorem 3.14] the proof is similar to [4, Corollary 2, p. 135]. □

Proof of Theorem 2.4. Using [11, Proposition 3.13] and Lemma 2.5, the proof is the same lines as of proof of the equality (a) and (c) of Theorem 2.1. □

The next Proposition is an application of Theorem 2.1:

Proposition 2.6. Let $R$ be a semi-primary ring such that each simple left $R$-module is isomorphic to a left ideal in $R$, then: \{$gldim(R), l.Ggldim(R)$\} $\in \{0, \infty\}$.

Proof. The classical result $gldim(R) \in \{0, \infty\}$ is exactly [1] Proposition 14 and the Gorenstein version is proved by the same way. For exactness we give the proof here. Suppose that $l.G.\text{gldim}(R) = n, 0 < n < \infty$. By Theorem 2.1 we have $l.G.\text{gldim}(R) = Gpd_R(C)$, where $C$ is a simple left $R$-module. By hypothesis, $C \cong I$, where $I$ is an ideal in $R$. Thus $Gpd_R(I) = n$. Consider the exact sequence of $R$-modules:

$$0 \to I \to R \to R/I \to 0$$

Since $n > 0$, $R/I$ is not Gorenstein projective ([11, Theorem 2.5]). Therefore by Lemma 2.2 $Gpd_R(R/I) = 1 + n$. Contradiction with the fact that $Gpd_R(R/I) \leq l.Ggldim(R) = n$. This contradiction finish the proof. □

Remark 2.7 (Proposition 15, [1]). The hypothesis of Proposition 2.6 is satisfied in each of the following cases:

1. $R$ is a direct sum of a finite number of primary rings (a semi-primary ring $R$ is primary if $R/N$ is a simple ring).
2. $R$ is a semi-primary commutative ring.
3. $R$ is a quasi-Frobenius ring (i.e; Noetherian and self-injective ring).

Corollary 2.8. Every commutative semi-primary rings with finite Gorenstein global dimension is quasi-Frobenius.

Proof. This Corollary is a direct consequence of Proposition 2.6 Remark 2.7 and [3, Proposition 2.6]. □

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