PERIODS OF MIRRORS AND MULTIPLE ZETA VALUES

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Abstract. In a recent paper, A. Libgober showed that the multiplicative sequence \{Q_i(c_1, \ldots, c_i)\} of Chern classes corresponding to the power series \(Q(z) = \Gamma(1+z)^{-1}\) appears in a relation between the Chern classes of certain Calabi-Yau manifolds and the periods of their mirrors. We show that the polynomials \(Q_i\) can be expressed in terms of multiple zeta values.

1. The multiplicative sequence

In [6], the (Hirzebruch) multiplicative sequence \(\{Q_i\}\) associated to the power series \(Q(z) = \Gamma(1+z)^{-1}\) is considered in connection with mirror symmetry. If \(e_i\) denotes the \(i\)th elementary symmetric function in the variables \(t_1, t_2, \ldots\), then

\[
\sum_{i=0}^{\infty} Q_i(e_1, \ldots, e_i) = \prod_{i=1}^{\infty} \frac{1}{\Gamma(1+t_i)}.
\]

As shown in [6], the polynomials \(Q_i(c_1, \ldots, c_i)\) in the Chern classes of certain Calabi-Yau manifolds \(X\) are related to the coefficients of the generalized hypergeometric series expansion of the period (holomorphic at a maximum degeneracy point) of a mirror of \(X\). In particular, if \(X\) is a Calabi-Yau hypersurface of dimension 4 in a nonsingular toric Fano manifold, then

\[
\int_X Q_4(c_1, c_2, c_3, c_4) = \frac{1}{24} K_{ijkl} \frac{\partial^4 c(0, \ldots, 0)}{\partial \rho_i \partial \rho_j \partial \rho_k \partial \rho_l},
\]

where the \(c(\rho_1, \ldots, \rho_r)\) are coefficients in the expansion of the period and \(K_{ijkl}\) is the (suitably normalized) 4-point function corresponding to a mirror of \(X\). In [6] it is shown that the polynomials \(Q_i\) have the form

\[
Q_1(c_1) = \gamma c_1 \quad \text{and} \quad Q_i(c_1, \ldots, c_i) = \zeta(i)c_i + \cdots, \quad i > 1.
\]

In this note we show that the polynomials \(Q_i\) have an explicit expression in terms of multiple zeta values (called multiple harmonic series in [4]), which have previously appeared in connection with Kontsevich’s invariant in knot theory [8,5], and in quantum field theory [1].

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2. The formula for the $Q_i$

Let $\text{Sym}$ be the algebra of symmetric functions in the variables $t_1, t_2, \ldots$ (with rational coefficients), and let $p_i$ be the $i$th power-sum symmetric function in these variables. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $m_\lambda$ be the corresponding monomial symmetric function and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$. It is well known that $\{m_\lambda\}$ and $\{e_\lambda\}$ are bases for $\text{Sym}$ as a vector space. In [4] it is shown (Theorem 5.1) that the homomorphism $\zeta : \text{Sym} \to \mathbb{R}$ such that $\zeta(p_1) = \gamma$ and $\zeta(p_i) = \zeta(i)$ for $i > 2$ satisfies

$$\zeta \left( \sum_{i \geq 0} e_i z^i \right) = \frac{1}{\Gamma(1+z)}.$$  

(2)

Our main result expresses the polynomials $Q_i$ in terms of $\zeta$.

**Theorem.** For any partition $\lambda$ of $i$, the coefficient of $e_\lambda$ in $Q_i(e_1, \ldots, e_i)$ is $\zeta(m_\lambda)$.

**Proof.** Using equations (1) and (2), we have

$$\sum_{i \geq 0} Q_i(e_1, e_2, \ldots) = \prod_{i=1}^\infty \frac{1}{1 + t_i} = \prod_{i=1}^\infty \sum_{j=0}^\infty \zeta(e_j)t_i^j = \sum_{\lambda} \zeta(e_\lambda)m_\lambda.$$  

Now the transition matrix $M$ from the basis $\{e_\lambda\}$ of $\text{Sym}$ to the basis $\{m_\lambda\}$, i.e.

$$e_\lambda = \sum_\mu M_{\lambda \mu} m_\mu$$

is known to be symmetric (see Ch. I, §6 of [7]), so we have

$$\sum_\lambda \zeta(e_\lambda)m_\lambda = \sum_\lambda \sum_\mu M_{\lambda \mu} \zeta(m_\mu)m_\lambda = \sum_\mu \zeta(m_\mu) \sum_\lambda M_{\lambda \mu} m_\lambda = \sum_\mu \zeta(m_\mu)e_\mu,$$

and the result follows.

3. Multiple zeta values

As shown in [4], $\zeta$ can be thought of as a homomorphism from the algebra of quasi-symmetric functions in the $t_i$ (as defined in [2]) to $\mathbb{R}$ that extends the multiple zeta values introduced in [3] and [8], i.e.

$$\zeta(i_1, i_2, \ldots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}},$$

where $i_1$ must be assumed greater than 1 for convergence. If we let $\mathcal{H}^1$ be the rational vector space of polynomials in the noncommuting variables $z_1, z_2, \ldots$, then $\mathcal{H}^1$ becomes isomorphic to the algebra of quasi-symmetric functions if we define the (commutative) multiplication $*$ by the inductive rule

$$z_i w_1 * z_j w_2 = z_i (w_1 * z_j w_2) + z_j (z_i w_1 * w_2) + z_{i+j} (w_1 * w_2).$$

$$z_i w_1 * z_j w_2 = z_i (w_1 * z_j w_2) + z_j (z_i w_1 * w_2) + z_{i+j} (w_1 * w_2).$$
for any words $w_1, w_2$ in the $z_i$; see [4] for details. The algebra $\text{Sym}$ of symmetric functions can be identified with the subspace of $\mathcal{H}^1$ generated by linear combinations of monomials invariant under permutation of subscripts, e.g. $z_2^2 = m_{22}$ and $z_1 z_2 + z_2 z_1 = m_{21}$; $z_i$ and $z_i^1$ correspond to $p_i$ and $e_i$ respectively. As an algebra $\mathcal{H}^1$ is generated by Lyndon words in the $z_i$, i.e. monomials $w$ such that for any nontrivial decomposition $w = uv$ one has $v > w$, where the $z_i$ are ordered as $z_1 > z_2 > \cdots$ and this order is extended to monomials lexicographically. Then the only Lyndon word that starts with $z_1$ is $z_1$ itself, and $\zeta$ is the homomorphism from $\mathcal{H}^1$ to $\mathbb{R}$ defined on Lyndon words $w = z_{i_1} z_{i_2} \cdots z_{i_k}$ by

$$\zeta(w) = \begin{cases} \gamma, & w = z_1, \\ \zeta(i_1, i_2, \ldots, i_k), & \text{otherwise.} \end{cases}$$

By the results of [4], $\zeta(z_{i_1} z_{i_2} \cdots z_{i_k})$ coincides with $\zeta(i_1, i_2, \ldots, i_k)$ as defined by equation (3) whenever $i_1 > 1$.

Since the power-sum symmetric functions $p_i$ generate the algebra $\text{Sym}$, we can compute $\zeta(m_{\lambda})$ by first expressing $m_{\lambda}$ in terms of power-sum functions (see [7, p. 109] for an explicit formula), and then applying the homomorphism $\zeta$. Hence the coefficient of each monomial $c_{\lambda}$ in $Q_i(c_1, \ldots, c_i)$ is a polynomial in the numbers $\gamma$ and $\zeta(i)$, $i \geq 2$. By this method we can obtain formulas (1.3)-(1.6) of [6] (with the following corrections: in (1.4) the coefficient of $c_2^1$ should be $\frac{1}{2}(\gamma^2 - \zeta(2))$, while in (1.5) the coefficient of $c_3^1$ should be $\frac{1}{6}(\zeta(3) - \frac{1}{2}\zeta(2) + \frac{1}{6}\gamma^3)$).

If $m_{\lambda}$ is a monomial symmetric function such that the partition $\lambda$ involves no 1’s, then $\zeta(m_{\lambda})$ is just a sum of ordinary multiple zeta values, e.g. $\zeta(m_{22}) = \zeta(2, 2) = \frac{3}{4}\zeta(4)$ and $\zeta(m_{62}) = \zeta(6, 2) + \zeta(2, 6) = \frac{5}{3}\zeta(8)$. In particular, such $\zeta(m_{\lambda})$ are the only coefficients needed to evaluate $Q_i(c_1, \ldots, c_i)$ on a Calabi-Yau manifold, since $c_1 = 0$ in that case.

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