GLOBAL WELL-POSEDNESS OF SLIGHTLY SUPERCRITICAL SQG EQUATION AND EXPONENTIAL GRADIENT ESTIMATE

HYUNJUN CHOI

Abstract. We prove the global regularity of smooth solutions for a dissipative surface quasi-geostrophic equation with both velocity and dissipation logarithmically supercritical compared to the critical equation. By this, we mean that a symbol defined as a power of logarithm is added to both velocity and dissipation terms to penalize the equation’s criticality. Our primary tool is the nonlinear maximum principle which provides transparent proofs of global regularity for nonlinear dissipative equations. In addition, we prove an exponential gradient estimate for the critical surface quasi-geostrophic equation which improves the previous double exponential bound [13].

1. Introduction

The global well-posedness of the dissipative surface quasi-geostrophic (SQG) equation (1.1) has been widely studied in recent literature.

\begin{equation}
\begin{aligned}
\partial_t \theta + (u \cdot \nabla) \theta + \Lambda^\alpha \theta &= 0, \\
u &= \nabla^\perp \Lambda^{-1} \theta.
\end{aligned}
\end{equation}

(1.1)

The SQG equation first appeared in the mathematical literature in [2]. Due to the positivity of the operator \( \Lambda^\alpha \), it can be easily seen that the \( L^p \) norms of \( \theta \) are nonincreasing in time under smooth evolution. Indeed, the equation has \( L^\infty \)-maximum principle [7]. The situation differs by whether the order of the diffusion term is higher/identical/lower than the nonlinear transport term under \( L^\infty \)-rescaling of a solution. The case \( \alpha = 1 \) is termed “critical” since the order is identical. The weak formulation and the global well-posedness for the subcritical case (\( \alpha > 1 \)) were understood in the usual sense [3, 16]. In addition, a small initial data case was studied in [4]. The global well-posedness in the critical case was first proved in [1] (in \( \mathbb{R}^n \)) and [13, 14] (in \( \mathbb{T}^n \)). In particular, [13] analyzed a breakthrough scenario of a modulus of continuity, called the nonlocal maximum principle, which is studied further in [15]. Shortly after that, [16] presented the third proof. They introduced the nonlinear maximum principle, which can be applied to well-posedness proofs for various critical active scalar equations.

In the supercritical case (\( \alpha < 1 \)), the global well-posedness for generic initial remains an open problem. There have been several results regarding conditional regularity [5, 11] (a Hölder continuous solution is smooth) and eventual regularization of solutions [17, 18, 15]. The global regularity has been studied if velocity or dissipation is slightly supercritical. In particular, the slightly supercritical velocity was studied in [9], where the velocity is obtained from \( \theta \) by a Fourier multiplier with symbol \( i\zeta^\perp |\zeta|^{-1} m(\zeta) \). Here, \( m(\zeta) \) grows slower than \( \log \log |\zeta| \) as \( \zeta \to \infty \). Moreover, the slightly supercritical dissipation for several active scalar equations was studied in [10], where the dissipation term is given by multiplier with behavior \( P(\zeta) \sim |\zeta| (\log |\zeta|)^{-\alpha} \) (\( 0 \leq \alpha \leq 1 \)) for large \( \zeta \). These proofs were based on the nonlocal maximum principle introduced in [13, 15].

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In this paper, we consider the SQG equation, which is slightly supercritical in both velocity and dissipation:

\[
\begin{cases}
\partial_t \theta + (u \cdot \nabla) \theta + L \theta = 0, \\
u = \nabla^\perp \Lambda^{-1} m(\Lambda) \theta, \\
\theta(t = 0) = \theta_0.
\end{cases}
\] (1.2)

The nonlocal operator \(L\) is defined by a convolutional kernel as follows:

\[
L \theta(x) = \text{P.V.} \int_{\mathbb{R}^2} \frac{(\theta(x) - \theta(y)) k(x - y)}{|x - y|^2} dy.
\]

Note that \(L = \Lambda\) when \(k(x) = \frac{1}{2\pi |x|}\). The main example we have in mind is

\[
k(r) = \frac{1}{r(\log(10 + r^{-1}))^{\alpha_1}}, \quad m(\zeta) = (\log(10 + |\zeta|))^{\alpha_2},
\] (1.3)

for some \(\alpha_1, \alpha_2 \geq 0\).

We prove the global regularity with the help of the nonlinear maximum principle [9]. We show a conditional regularity result in the first step, then we prove that desired regularity is conserved by evolution in the second step. The main difference between our proof and [6] is that we provide a modulus of continuity while [6] argues only with uniform continuity (the only small shock condition). An explicit modulus of continuity allows us to construct an exponential gradient estimate, Main Theorem 2.

In section 2 we list some estimates regarding the relationship between the kernel and the Fourier multiplier. We provide suitable conditions for \(k\) and \(m\) to behave similarly as the main example (1.3). We also collect some elementary integral estimates for later use. In section 3 we provide conditional regularity result for (1.2). If \(\theta\) is a bounded weak solution to (1.2) in time \([0, T]\) and \(\theta(\cdot, t)\) admits a modulus of continuity \(\Omega\) for \(t \in [0, T]\) which satisfies

\[
\lim_{r \to 0^+} \frac{m(r^{-1}) \Omega(r)}{rk(r)} = 0,
\] (1.4)

then \(\theta\) is a smooth solution. In section 4 we prove that a modulus of continuity \(M\omega\) for \(\theta(\cdot, t)\) is conserved by (1.2) if a constant \(M\) depends on \(\theta_0\) is sufficiently large and

\[
\lim_{r \to 0^+} \frac{m(r^{-1}) \omega'(r)}{k(r) \omega(r)} = 0.
\] (1.5)

Combining two steps, we prove the following:

**Main Theorem 1.** Assume that \(\theta_0 \in \mathcal{S}(\mathbb{R}^2)\). Suppose there is a non-decreasing, continuous, concave function \(\omega : [0, \infty) \to [0, \infty)\) satisfying \(\omega(0) = 0\), (1.4), (1.5), and (2.1). Then there exists a global smooth solution \(\theta\) of the slightly supercritical SQG equation (1.2).

**Remark.** An initial \(\theta_0 \in W^{1,\infty} \cap H^s(\mathbb{R}^2)\) with \(s > 1\) is enough to guarantee a smooth solution. A typical example is a logarithmic multiplier (1.3).

**Corollary 1.** Assume that \(\theta_0 \in \mathcal{S}(\mathbb{R}^2)\). If

\[
\alpha_1 + \alpha_2 < 1,
\]

then there exists a global smooth solution \(\theta\) of

\[
\begin{cases}
\partial_t \theta + (u \cdot \nabla) \theta + \Lambda(\log^{-\alpha_1} \Lambda) \theta = 0, \\
u = \nabla^\perp \Lambda^{-1} (\log^{\alpha_2} \Lambda) \theta, \\
\theta(t = 0) = \theta_0.
\end{cases}
\] (1.6)
This result improves [9] which studied a slightly supercritical velocity, where \( m(\zeta) \) being growing slower than double logarithm. Moreover, we prove the global regularity when both velocity and dissipation are slightly supercritical. Hence this paper extends [10], which studied the slightly supercritical dissipation.

In section 5, we prove an exponential gradient estimate for the critical SQG equation:

\[
\begin{align*}
\partial_t \theta + (u \cdot \nabla) \theta + \Lambda \theta &= 0, \\
u &= \nabla^\perp \Lambda^{-1} \theta, \\
\theta(t = 0) &= \theta_0
\end{align*}
\]  
(1.7)

**Main Theorem 2.** The critical SQG equation (1.7) with initial \( \theta_0 \in S(\mathbb{R}^2) \) has a unique global smooth solution. In addition, for any \( \gamma > 0 \),

\[
\|\nabla \theta\|_{L^\infty_t L^\infty_x} \leq C\|\nabla \theta_0\|_{L^\infty} \exp(C\|\theta_0\|_{L^\infty}^{1+\gamma}),
\]  
(1.8)

where constants depend only on \( \gamma \).

Our result (1.8) improves the following double exponential estimate (1.9). In [13], they used a modulus of continuity \( \omega(r) \), which behaves like \( \log \log(r/\delta) \) for \( \delta \ll r \ll 1 \). While we use a modulus of continuity \( |\log r|^{-\beta} \) for \( r \ll 1 \) with some \( \beta > 0 \).

**Proposition 2** ([13]). The critical SQG equation (1.7) with initial \( \theta_0 \in C^\infty(T^2) \) has a unique global smooth solution. In addition,

\[
\|\nabla \theta\|_{L^\infty_t L^\infty_x} \leq C\|\nabla \theta_0\|_{L^\infty} \exp(\exp(C\|\theta_0\|_{L^\infty})).
\]  
(1.9)

Lastly, these results can be easily reproduced for other \( L^\infty \)-critical active scalar equations with nonlocal dissipation and a linear balance law \( u = T \theta \).

2. Preliminaries

The local regularity result for the SQG-type equation is standard. It is well-known that \( \|\nabla \theta\|_{L^1_t L^\infty_x} \) is a blow-up criterion.

**Proposition 3** (local existence of smooth solution, [12]). Suppose \( \theta_0 \in W^{1,\infty} \cap H^s(\mathbb{R}^2) \) and \( s > 1 \). Then there exists \( T > 0 \) and a solution \( \theta \in C^\infty((0,T] \times \mathbb{R}^2) \) of (1.2). Moreover, the smooth solution may be continued as long as

\[
\|\nabla \theta\|_{L^1_t L^\infty_x} = \int_0^T \|\nabla \theta(\cdot,t)\|_{L^\infty} < \infty.
\]

Next, we provide assumptions on \( k \) and \( m \) and arrange some estimates on the kernel of the operator \( \nabla \Lambda^{-1} m(\Lambda) \) and the kernel \( r^{-2}k(r) \). Throughout the paper, we assume on \( k \) and \( m \) as follows:

- The kernel \( k(x) = k(|x|) \) is radially symmetric, nonnegative, and nonincreasing. In addition, \( r^2k(r) \) is nondecreasing for \( r > 0 \).

- (Slightly supercritical dissipation) For any \( \epsilon > 0 \),

\[
r^{1-\epsilon}k(r)
\]

is nonincreasing for \( 0 < r < r_\epsilon \) and tends to \( \infty \) as \( r \to 0 \).
• There exists a constant $c \geq 1$ such that

$$k\left(\frac{r}{2}\right) \leq c k(r), \quad r > 0.$$  

• The Fourier multiplier $m(\zeta) = m(|\zeta|) \geq 1$ is radially symmetric, nondecreasing, and satisfies the Hörmander-Mikhlin condition:

$$|\zeta|^k |\nabla^k m(\zeta)| \leq C k m(\zeta), \quad \zeta \neq 0,$$

for some constant $C_k > 0$.

• (Slightly supercritical velocity) For any $\epsilon > 0$,

$$|\zeta|^\epsilon m(|\zeta|^{-1})$$

is nondecreasing in $0 < |\zeta| < r_\epsilon$ and tends to 0 as $|\zeta| \to 0$.

The following lemmas reveal relations between an operator’s kernel and the Fourier multiplier. Lemma 4 deals with velocity $u$ and the Lemma 5 deals with dissipation term $L\theta$.

**Lemma 4** (Lemma 4.1 of [9]). Suppose $K$ is the kernel corresponding to the operator $\partial_j \Lambda^{-1} m(\Lambda)$ and $m$ satisfies the prescribed conditions. Then

$$|K(x)| \leq C |x|^{-d} m(|x|^{-1}),$$

and

$$|\nabla K(x)| \leq C |x|^{-d-1} m(|x|^{-1}),$$

for all $x \neq 0$.

**Lemma 5** ([10] [18]). Suppose $P(\zeta) = P(|\zeta|)$ is a radially symmetric function that is smooth, nonnegative, nondecreasing from zero, $P(0) = 0$, and $P(\zeta) \to \infty$ as $|\zeta| \to \infty$. In addition, assume the following for $P$:

• There is a constant $c \geq 1$ so that $P(2\zeta) \leq c P(\zeta)$ for all $\zeta \in \mathbb{R}^d$.

• $P$ is of the Hörmander-Mikhlin type:

$$|\zeta|^k |\nabla^k P(\zeta)| \leq C_k P(\zeta), \quad \zeta \neq 0,$$

for some constant $C_k$.

• $P$ satisfies growth condition

$$\int_0^1 P(|\zeta|^{-1}) |\zeta| d\zeta < \infty.$$ 

Then, the corresponding radially symmetric kernel $K = \breve{P}$ satisfies

$$|K(y)| \leq C |y|^{-d} P(|y|^{-1}),$$

and

$$|\nabla K(y)| \leq C |y|^{-d-1} P(|y|^{-1}),$$

for all $y \neq 0 \in \mathbb{R}^d$. Moreover, if $P$ satisfies

• $(-\Delta)^{d+1} P(\zeta) \geq c |\zeta|^{-d-2} P(\zeta)$

for some constant $c > 0$. Then, $K$ is bounded below as

$$K(y) \geq c |y|^{-d} P(|y|^{-1}),$$

for all sufficiently small $y$. 
Remark. If we consider $L = \Lambda(\log^{-\alpha} \Lambda)$, i.e.,
\[
P(\zeta) = \zeta (\log(10 + |\zeta|))^{-\alpha}, \quad \alpha > 0,
\]
then Lemma 5 shows that it is as same as considering
\[
k(r) = \frac{1}{r (\log(10 + r^{-1}))^\alpha}.
\]

Lastly, we denote some technical integral estimates which we will use frequently.

**Lemma 6.** Suppose $k$ and $m$ satisfy the prescribed conditions and $\omega : [0, \infty) \to [0, \infty)$ is a non-decreasing, continuous, concave function with $\omega(0) = 0$. In addition, assume that there exists $0 < \gamma < 1$ such that
\[
\lim_{r \to 0^+} \frac{\omega(r)}{r^\gamma} = \infty, \quad \text{and} \quad \lim_{r \to \infty} \frac{\omega(r)}{r^\gamma} = 0. \tag{2.1}
\]

For sufficiently small $r$, namely $0 < r < r_0$, the following inequalities hold.

(i) $\int_r^\infty \frac{k(p)}{p} \frac{dp}{p} \geq \frac{k(r)}{2}$.

(ii) $\int_{r/2}^r \frac{\omega(p)k(p)}{p} \frac{dp}{p} \leq C \omega(r)k(r)$ for a constant $C > 0$.

(iii) $\int_r^\infty \omega(p) \left( -\frac{k'(p)}{p} + \frac{2k(p)}{p^2} \right) \frac{dp}{p} \leq C \frac{\omega(r)k(r)}{r}$ for a constant $C > 0$.

(iv) $\int_0^r \frac{\rho k(p)}{p^2} \frac{dp}{p} \leq \frac{4m(r^{-1})^2}{k(r)}$.

(v) $\int_r^\infty \frac{\omega(p)m(p^{-1})}{p^2} \frac{dp}{p} \leq \frac{C \omega(r)m(r^{-1})}{r}$ for a constant $C > 0$.

**Proof.**

(i) $\int_r^\infty \frac{k(p)}{p} \frac{dp}{p} \geq \frac{r^2k(r)}{2}$.

(ii) $\int_{r/2}^r \frac{\omega(p)k(p)}{p} \frac{dp}{p} \leq \omega(r)k \left( \frac{r}{2} \right) \int_{r/2}^r \frac{dp}{p} \leq c \log 2 \cdot \omega(r)k(r)$.

(iii) $\int_r^\infty \omega(p) \left( -\frac{k'(p)}{p} + \frac{2k(p)}{p^2} \right) \frac{dp}{p} \leq \int_r^\infty \omega(p) \frac{d}{dp} \left( \frac{2(1 - \gamma)^{-1}k(p)}{\rho^{1-\gamma}} \right) \frac{dp}{p^{1-\gamma}} \leq \frac{\omega(r)}{r^{1-\gamma}} 2(1 - \gamma)^{-1}k(r)$.

(iv) $\int_0^r \frac{\rho k(p)}{p^2} \frac{dp}{p} \leq \int_0^r (\rho^{1/4} m(r^{-1}))^2 \frac{dp}{r^{3/4}k(r)} \int_0^r \frac{dp}{r^{3/4}} \leq C \omega(r)k(r)$.

(v) $\int_r^\infty \omega(p)m(p^{-1}) \frac{dp}{p^2} \leq r^{-\gamma} \omega(r)m(r^{-1}) \int_r^\infty \frac{dp}{p^2} \leq \frac{(1-\gamma)^{-1} \omega(r)m(r^{-1})}{r}$. \hfill $\square$

3. Conditional regularity

In this section, we prove that some modulus of continuity for a solution of (1.2) implies global regularity.

**Theorem 7.** Suppose $\theta$ is a bounded weak solution of the slightly supercritical SQG equation
\[
\begin{align*}
\partial_t \theta + (u \cdot \nabla)\theta + \mathcal{L}\theta &= 0, \\
u &= \nabla^\perp \Lambda^{-1} m(\Lambda)\theta, \\
\theta(t = 0) &= \theta_0.
\end{align*}
\]
with \( k \) and \( m \) satisfying the prescribed conditions. If \( \theta \) admits a modulus of continuity \( \Omega \) for \([0, T]\) and \( k, m, \Omega \) satisfy
\[
\lim_{r \to 0^+} \frac{m(r^{-1})\Omega(r)}{rk(r)} = 0,
\]
then there is a uniform-in-time estimate for \( \|\nabla \theta(\cdot, t)\|_{L^\infty} \) and \( \theta \) is a smooth solution on \((0, T]\).

**Proof.** It suffices to prove the assertion for a smooth solution \( \theta \). One may consider regularized equation with the term \(-\epsilon \Delta \theta\), get a uniform estimate independent of \( \epsilon \), and then take the inviscid limit \( \epsilon \to 0^+ \) to get the desired result.

**Step 1.** Evolution of \( |\nabla \theta|^2 \)
We take gradient on both sides of the equation to get:
\[
\partial_t \nabla \theta + u \cdot \nabla \theta + \nabla u \cdot \nabla \theta + \mathcal{L} \nabla \theta = 0.
\]
Multiply \( \nabla \theta \) to both sides:
\[
\nabla \theta \cdot \partial_t \nabla \theta + \nabla \theta \cdot u \cdot \nabla \theta + \nabla \theta \cdot \nabla u \cdot \nabla \theta + \nabla \theta \cdot \mathcal{L} \nabla \theta = 0.
\]
Considering singular integral formulation for the nonlocal operator \( \mathcal{L} \) gives:
\[
\frac{1}{2} (\partial_t + u \cdot \nabla + \mathcal{L}) |\nabla \theta|^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla \theta(x, t) - \nabla \theta(y, t)|^2}{|x - y|^2} k(x - y) \, dy = -\nabla \theta \cdot \nabla u \cdot \nabla \theta. \tag{3.1}
\]
Due to the maximum principle for \( \mathcal{L} \), it suffices to show that
\[
(\partial_t + u \cdot \nabla + \mathcal{L}) |\nabla \theta|^2 (x, t) < 0
\]
whenever \( |\nabla \theta(x, t)| \) is sufficiently large in terms of \( \|\theta_0\|_{L^\infty}, k, m, \) and \( \Omega \).

**Step 2.** Pointwise lower bound of \( D(x, t) \)
A smooth function \( \varphi : [0, \infty) \to \mathbb{R} \) is a non-decreasing cutoff function such that
\[
\varphi(x) = 0 \text{ on } x \leq \frac{1}{2}, \quad \varphi(x) = 1 \text{ on } x \geq 1, \quad 0 \leq \varphi' \leq 4.
\]
For some sufficiently small \( R = R(x, t) > 0 \) which will be determined later, we have the following estimate on \( D(x, t) \):
\[
\frac{D(x, t)}{2\pi} \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\nabla \theta(x, t) - \nabla \theta(y, t)|^2}{|x - y|^2} k(x - y) \varphi\left(\frac{|x - y|}{R}\right) \, dy
\]
\[
\geq |\nabla \theta(x, t)|^2 \frac{1}{2\pi} \int_{|x - y| \geq R} \frac{k(x - y)}{|x - y|^2} \, dy - 2 \nabla \theta(x, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \theta(y, t) \frac{k(x - y)}{|x - y|^2} \varphi\left(\frac{|x - y|}{R}\right) \, dy
\]
\[
\geq |\nabla \theta(x, t)|^2 \int_{R} \frac{k(\rho)}{\rho} \, d\rho - 2 \nabla \theta(x, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} (\theta(y, t) - \theta(x, t)) \left\{ \nabla_v \left( \frac{k(|v|)}{|v|^2} \varphi\left(\frac{|v|}{R}\right) \right) \right\}_{v = x - y} \, dy
\]
\[
\geq \frac{1}{2} |\nabla \theta(x, t)|^2 k(R) - 2 |\nabla \theta(x, t)| \int_{R} \rho \Omega(\rho) \frac{d}{d\rho} \left( \frac{k(\rho)\varphi(\rho/R)}{\rho^2} \right) \, d\rho.
\]
By Lemma [ii] and (iii),
\[
\int_0^\infty \rho \Omega(\rho) \left| \frac{d}{d\rho} \left( \frac{k(\rho)\varphi(\rho/R)}{\rho^2} \right) \right| \, d\rho \\
\leq \frac{4}{R} \int_{R/2}^R \Omega(\rho) k(\rho) \, d\rho + \int_{R/2}^\infty \Omega(\rho) \left( -\frac{k'(\rho)}{\rho} + \frac{2k(\rho)}{\rho^2} \right) \, d\rho \\
\leq \frac{c_1}{8} \frac{\Omega(R)k(R)}{R}.
\]

Therefore,
\[
\frac{D(x, t)}{2\pi} \geq \frac{1}{2} |\nabla \theta(x, t)|^2 k(R) - |\nabla \theta(x, t)| \frac{c_1 \Omega(R)k(R)}{4R} = \frac{1}{2} |\nabla \theta(x, t)|^2 k(R) \left( 1 - \frac{c_1}{2 |\nabla \theta(x, t)|} \frac{\Omega(R)}{R} \right).
\]

Set \( R = R(x, t) > 0 \) to satisfy
\[
\frac{\Omega(R)}{R} = \frac{|\nabla \theta(x, t)|}{c_1}.
\]

Then
\[
D(x, t) \geq 4c_2 |\nabla \theta(x, t)|^2 k(R).
\]

Note that \( R(x, t) \) is arbitrarily small when \( |\nabla \theta(x, t)| \) is large enough.

\textit{Step 3. Estimate of } \nabla u
\[
\nabla u(x, t) = \nabla \Lambda^{-1} m(\Lambda) \nabla \theta(x, t) = P.V. \int_{\mathbb{R}^2} K(x-y)(\nabla \theta(x, t) - \nabla \theta(y, t)) \, dy.
\]

We estimate \( \nabla u \) by splitting \( \mathbb{R}^2 \) into two pieces; an inner piece \( |x-y| \leq r \) and an outer piece \( |x-y| > r \) for some \( r = r(x, t) > 0 \).

For the inner piece, we use the Cauchy-Schwartz inequality:
\[
|\nabla u_{in}(x, t)| \leq C \sqrt{\left( \int_{|x-y| \leq r} \frac{|\nabla \theta(x, t) - \nabla \theta(y, t)|^2}{|x-y|^2} k(x-y) \, dy \right) \left( \int_{|x-y| \leq r} K(x-y)^2 \frac{|x-y|^2}{k(x-y)} \, dy \right)} \\
\leq C \sqrt{D(x, t) \int_0^r \frac{m(\rho^{-1})^2}{\rho k(\rho)} \, d\rho \leq c_3 \sqrt{D(x, t) \frac{m(r^{-1})^2}{k(r)}}. (\text{Lemma } [iv])
\]

For the outer piece, we apply integration by parts and use the modulus of continuity of \( \theta \):
\[
|\nabla u_{out}(x, t)| \leq \left| \int_{|x-y| > r} \left( \theta(x, t) - \theta(y, t) \right) \nabla K(x-y) \, dy \right| \\
+ \left| \int_{|x-y| = r} \left( \theta(x, t) - \theta(y, t) \right) K(x-y) \nu(y) \, d\sigma(y) \right| \\
\leq C \int_r^\infty \frac{\Omega(\rho)m(\rho^{-1})}{\rho^2} \, d\rho + C \frac{\Omega(r)m(r^{-1})}{r} \\
\leq c_4 \frac{\Omega(r)m(r^{-1})}{r}. (\text{Lemma } [v])
\]

Therefore,
\[
|\nabla u| |\nabla \theta|^2 \leq |\nabla u_{in}| |\nabla \theta|^2 + |\nabla u_{out}| |\nabla \theta|^2 \\
\leq \frac{D}{4} + c_3 \frac{m(r^{-1})^2}{k(r)} |\nabla \theta|^4 + c_4 \frac{\Omega(r)m(r^{-1})}{r} |\nabla \theta|^2. (3.4)
\]
Step 4. Maximum principle

Combining (3.1), (3.3), and (3.4),

\[
\begin{align*}
(\partial_t + u \cdot \nabla + \mathcal{L}) \lvert \nabla \theta \rvert^2 &+ \frac{D}{2} + 2c_2 k(R) \lvert \nabla \theta \rvert^2 \\
\leq (\partial_t + u \cdot \nabla + \mathcal{L}) \lvert \nabla \theta \rvert^2 + D \\
\leq 2 \lvert \nabla u \rvert \lvert \nabla \theta \rvert^2 &\leq \frac{D}{2} + 2c_2^2 \frac{m(r^{-1})^2}{k(r)} \lvert \nabla \theta \rvert^4 + 2c_4 \frac{\Omega(r) m(r^{-1})}{r} \lvert \nabla \theta \rvert^2 .
\end{align*}
\]

Hence

\[
(\partial_t + u \cdot \nabla + \mathcal{L}) \lvert \nabla \theta \rvert^2 + 2c_2 k(R) \lvert \nabla \theta \rvert^2 \leq 2c_3^2 \frac{m(r^{-1})^2}{k(r)} \lvert \nabla \theta \rvert^4 + 2c_4 \frac{\Omega(r) m(r^{-1})}{r} \lvert \nabla \theta \rvert^2 .
\]

What we want to show are

\[
2c_3^2 \frac{m(r^{-1})^2}{k(r)} \lvert \nabla \theta \rvert^2 < c_2^2 k(R),
\] (3.5)

and

\[
2c_4 \frac{\Omega(r) m(r^{-1})}{r} < c_2^2 k(R),
\] (3.6)

whenever \( \lvert \nabla \theta \rvert \) is sufficiently large. Recall (3.2) that

\[
\lvert \nabla \theta \rvert = c_1 \frac{\Omega(R)}{R}.
\]

Hence (3.5) is equivalent to

\[
\frac{m(r^{-1})^2 \Omega(R)^2}{R^2 k(r) k(R)} < \frac{c_2}{2c_3^2}.
\]

And (3.6) is equivalent to

\[
\frac{m(r^{-1}) \Omega(r)}{r k(R)} < \frac{c_2}{2c_4}.
\]

Set \( r = R = R(x, t) > 0 \) and then the above two inequalities hold for sufficiently small \( R \), since we assumed

\[
\lim_{r \to 0^+} \frac{m(r^{-1}) \Omega(r)}{r k(r)} = 0.
\]

Consider the main example

\[
k(r) \sim \frac{1}{r(-\log r)^{\alpha_1}} , \quad m(\zeta) \sim (\log |\zeta|)^{\alpha_2}.
\]

Then, a modulus of continuity

\[
\Omega(r) \sim \frac{1}{(-\log r)^{\beta}}
\]

satisfies the condition

\[
\frac{m(r^{-1}) \Omega(r)}{r k(r)} \sim \frac{(-\log r)^{\alpha_1 + \alpha_2}}{(-\log r)^{\beta}} \to 0 , \quad \text{as } r \to 0^+ ,
\]

if \( \beta > \alpha_1 + \alpha_2 \).

Corollary 8. Suppose \( \theta \) is a bounded weak solution of the slightly supercritical SQG equation

\[
\begin{align*}
\partial_t \theta + (u \cdot \nabla) \theta + \Lambda(\log^{-\alpha_1} \Lambda) \theta &= 0 , \\
u &= \nabla^\perp \Lambda^{-1}(\log^{\alpha_2} \Lambda) \theta .
\end{align*}
\]

and \( \theta \) admits a modulus of continuity \( \Omega \) for \([0, T] \). If

\[
\lim_{r \to 0^+} \Omega(r)(-\log r)^{\alpha_1 + \alpha_2} = 0,
\]
then $\theta$ is a smooth solution on $(0,T]$.

4. CONSERVATION OF MODULUS OF CONTINUITY

In this section, we prove that some modulus of continuity of $\theta(\cdot, t)$ is conserved by $\Omega^2$.

**Theorem 9.** Suppose $\theta$ is a bounded weak solution of the slightly supercritical SQG equation

\[
\begin{cases}
\partial_t \theta + (u \cdot \nabla) \theta + \mathcal{L} \theta = 0, \\
u = \nabla^\perp \Lambda^{-1} m(\Lambda) \theta, \\
\theta(t = 0) = \theta_0.
\end{cases}
\]

with $k$ and $m$ satisfying the prescribed conditions. Suppose a $C^1$-function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is nondecreasing, concave, and satisfies (1.1) and

\[
\lim_{r \to 0^+} \frac{m(r^{-1}) \omega'(r)}{k(r) \omega(r)} = 0.
\]

Then there exists a constant $M > 0$ depending on $\|\theta_0\|_{L^\infty}$, $k$, $m$, and $\omega$ such that if $\theta_0 \in L^p \cap L^\infty$ satisfies the modulus of continuity $\frac{M}{2} \omega$, then $\theta$ satisfies the modulus of continuity $M \omega$ as long as the solution is defined.

**Proof.** For notational convenience, define $\delta_h f(x) = f(x + h) - f(x)$ for a function $f$. We consider the maximum principle for

\[
v(x, h; t) := \left( \frac{\delta_h \theta(x, t)}{\omega(|h|)} \right)^2 F(h).
\]

The function $F(h) = \exp(-G(|h|))$ is just for the decay of $v$ in $h$, so that $v \in L^p \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ for some $1 \leq p < \infty$. We set $G$ being smooth, nonnegative, nondecreasing, and $G(r) = 0$ for $0 \leq r \leq 1$.

**Step 1.** Evolution of $v$

Since

\[
\partial_t \theta(x) + u(x) \cdot \nabla \theta(x) + \mathcal{L} \theta(x) = 0,
\]

and

\[
\partial_t \theta(x + h) + u(x + h) \cdot \nabla \theta(x + h) + \mathcal{L} \theta(x + h) = 0.
\]

Subtracting two formulas gives:

\[
\partial_t (\theta(x + h) - \theta(x)) + u(x) \cdot \nabla (\theta(x + h) - \theta(x)) + (u(x + h) - u(x)) \cdot \nabla \theta(x) + (\mathcal{L} \theta(x + h) - \mathcal{L} \theta(x)) = 0,
\]

and thus the evolution of $\delta_h \theta$ is given by

\[
(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_x + \mathcal{L}_x) \delta_h \theta = 0.
\]

Multiply $\delta_h \theta$ to both sides:

\[
\frac{1}{2} (\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_x + \mathcal{L}_x) (\delta_h \theta)^2(x, h, t) + \int_{\mathbb{R}^2} \frac{|\delta_h \theta(x, t) - \delta_h \theta(y, t)|^2}{|x - y|^2} k(x - y) \, dy = 0.
\]

Therefore

\[
(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_x + \mathcal{L}_x) v(x, h, t) + D_h(x, t) \frac{F(h)}{\omega(|h|)^2} = -\left( \delta_h u \cdot \frac{h}{|h|} \left( G(|h|) + \frac{2\omega'(|h|)}{\omega(|h|)} \right) v. \right.
\]
Step 2. Breakthrough moment and maximum principle
Fix $M > 0$, which will be determined later. Suppose $\theta_0$ satisfies a modulus of continuity $\frac{M}{2} \omega$ and $\theta(t)$ admits a modulus of continuity $M \omega$ for $0 \leq t \leq t^*$. However, at the time $t = t^*$,
\[
|\delta_h \theta(x, t^*)| \geq M \omega(|h|) \tag{4.2}
\]
for some $x, h$. Suppose we prove that
\[
(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_x + \mathcal{L}_x) v(x, h, t^*) < 0,
\]
whenever the such event occurs, then we get contraction from the maximum principle of $\mathcal{L}_x$. Hence such an event is impossible, and the modulus of continuity $M \omega(\cdot)$ is conserved.

Therefore it suffices to show that
\[
(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_x + \mathcal{L}_x) v(x, h, t^*) < 0,
\]
whenever
\[
M \omega(|h|) \leq |\delta_h \theta(x, t^*)| \leq 2 M \omega(|h|).
\]
If (4.2) is satisfied,
\[
M \omega(|h|) \leq |\delta_h \theta(x, t^*)| \leq 2 \|\theta_0\|_{L^\infty}.
\]
Hence, we only need to consider $h$ with $M \omega(|h|) \leq 2 \|\theta_0\|_{L^\infty}$. Let $r_0 > 0$ be $M \omega(r_0) = 2 \|\theta_0\|_{L^\infty}$. We choose $M$ large so that $r_0 < 1$ and thus $F(h) = 1, G(\|h\|) = 0$ for $|h| \leq r_0$. Due to (4.1), it suffices to show that if $M$ is large enough, then
\[
D_h(x, t^*) \geq \frac{2 \omega(\|h\|)}{\omega(|h|)} |\delta_h u(x, t^*)| |\delta_h \theta(x, t^*)|^2,
\]
whenever $M \omega(|h|) \leq |\delta_h \theta(x, t^*)| \leq 2 M \omega(|h|)$. Let us denote $t^*$ by simply $t$ from here.

Step 3. Pointwise lower bound of $D_h$
A smooth function $\varphi : [0, \infty) \to \mathbb{R}$ is a non-decreasing cutoff function such that
\[
\varphi(r) = 0 \text{ on } r \leq \frac{1}{2}, \quad \varphi(r) = 1 \text{ on } r \geq 1, \quad 0 \leq \varphi' \leq 4.
\]
For some sufficiently small $R = R(x, h, t) \geq 6 |h|$ which will be determined later, we have the following estimate on $D_h(x, t)$:
\[
\frac{D_h(x, t)}{2\pi} \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\delta_h \theta(x, t) - \delta_h \theta(y, t)|^2}{|x - y|^2} k(x - y) \varphi \left( \frac{|x - y|}{R} \right) dy
\]
\[
\geq |\delta_h \theta(x, t)|^2 \frac{1}{2\pi} \int_{|x - y| \leq R} k(x - y) dy - 2 \delta_h \theta(x, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \delta_h \theta(y, t) \varphi \left( \frac{|x - y|}{R} \right) dy
\]
\[
\geq |\delta_h \theta(x, t)|^2 \int_{R}^{\infty} \frac{k(\rho)}{\rho} d\rho - 2 \delta_h \theta(x, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} (\theta(y, t) - \theta(x, t)) \left\{ \delta_h \left( \frac{k(|v|)}{|v|^2} \varphi \left( \frac{|v|}{R} \right) \right) \right\} _{v = x - y} dy.
\]
By the mean value theorem,
\[
|\delta_h \left( \frac{k(|v|)}{|v|^2} \varphi \left( \frac{|v|}{R} \right) \right) |_{v = x - y} \leq |h| \left| \nabla_v \left( \frac{k(|v|)}{|v|^2} \varphi \left( \frac{|v|}{R} \right) \right) \right| _{v = x - y + h^*}
\]
\[
\leq |h| \left\{ \frac{4 k(\rho^*)}{R (\rho^*)^2} R \rho^* \geq \frac{R}{2} + \frac{k(\rho^*)}{(\rho^*)^2} + \frac{2 k(\rho^*)}{(\rho^*)^3} \right\} \rho^* = |x - y + h^*|
\]
Since $\rho^* = |x - y + h^*| \geq \frac{R}{2}$ and $|h^*| \leq |h| \leq \frac{R}{6}$, it follows $\rho = |x - y| \geq \frac{R}{3}$ and $\rho^* \geq \rho - \frac{R}{6} \geq \frac{R}{2}$. Hence the above formula is less than equal to the same formula where $\rho^*$ replaced by $\rho/2$, $R \geq \rho^* \geq \frac{R}{2}$.
$R$ replaced by $\frac{7R}{6} \geq \rho \geq \frac{R}{3}$, and $\rho^* \geq \frac{R}{2}$ replaced by $\rho \geq \frac{R}{3}$. In addition, regarding the modulus of continuity of $\theta(\cdot, t)$, we have

$$|\theta(y,t) - \theta(x,t)| \leq M \omega(|x-y|).$$

Therefore, wrapping up the estimates with Lemma 3 we have the following estimate for $D_h$ with a constant $c_5(>2)$.

$$\frac{D_h(x,t)}{2\pi} \geq \frac{1}{2} \delta_h \theta(x,t)|^2 k(R) - 2 |\delta_h \theta(x,t)| |h| M \cdot c_5 \frac{\omega(R)k(R)}{R} \geq \frac{1}{2} |\delta_h \theta(x,t)|^2 k(R) \left(1 - \frac{4c_5 |h| M \omega(R)}{|\delta_h \theta(x,t)|} \right).$$

Set $R = R(x, h, t) > 0$ to satisfy

$$\frac{\omega(R)}{R} = \frac{1}{8c_5 M} |\delta_h \theta(x,t)| \leq \frac{1}{4c_5} \frac{\omega(|h|)}{|h|} < \frac{\omega(6|h|)}{6|h|},$$

we get $R \geq 6|h|$ as assumed. On the other hand,

$$\frac{\omega(R)}{R} = \frac{1}{8c_5 M} |\delta_h \theta(x,t)| \geq \frac{1}{8c_5} \frac{\omega(|h|)}{|h|},$$

and thus

$$8c_5 \frac{|h|^{1-\gamma}}{R^{1-\gamma}} \geq \frac{\omega(|h|)}{\omega(R)} \frac{|h|^{-\gamma}}{R^{-\gamma}} \geq 1.$$

Hence $R \leq (8c_5)^{1/(1-\gamma)} |h|$. Summing up,

$$D_h(x,t) \geq c_k(R) |\delta_h \theta(x,t)|^2 \geq 8c_6 k(|h|) |\delta_h \theta(x,t)|^2.$$  \hspace{1cm} (4.5)

**Step 4. Estimate of $\delta_h u$**

$$\delta_h u(x, t) = P.V. \int_{\mathbb{R}^2} K(x-y)(\delta_h \theta(x,t) - \delta_h \theta(y,t)) dy.$$  

We estimate $\nabla u$ by splitting $\mathbb{R}^2$ into two pieces; an inner piece $|x-y| \leq 3|h|$ and an outer piece $|x-y| > 3|h|$.

For the inner piece, we use the Cauchy-Schwartz inequality:

$$|\delta_h u_{in}(x,t)| \leq c_7 \sqrt{D_h(x,t) \frac{m(|h|^{-1})^2}{k(|h|)}}.$$  \hspace{1cm} (4.6)

For the outer piece:

$$|\delta_h u_{out}(x,t)| = \left| \int_{|x-y| > 3|h|} K(x-y)(\theta(y,t) - \theta(y+h,t)) dy \right| \leq \left| \int_{|x-y| > 3|h|} (K(x-y) - K(x-y+h)) \theta(y,t) dy \right|$$  

$$+ \left| \int_{\{y: |x-y| > 3|h|\} \Delta \{y: |x-y+h| > 3|h|\}} K(x-y+h) \theta(y,t) dy \right|,$$
The first term is estimated as follows:
\[
\leq \|\theta_0\|_{L^\infty} \int_{|x-y|>3|h|} |h| |\nabla K(x-y+h^*)| \, dy \quad (|h^*| \leq |h| < \frac{1}{3} |x-y|)
\]
\[
\leq c \|\theta_0\|_{L^\infty} |h| \int_{|x-y|>3|h|} \left(\frac{2}{3} |x-y| \right)^{-3} m(\frac{2}{3} (x-y)^{-1}) \, dy
\]
\[
\leq c \|\theta_0\|_{L^\infty} |h| \int_{|h|}^{\infty} \frac{m(\rho^{-1})}{\rho^2} \, d\rho \leq c_3 \|\theta_0\|_{L^\infty} m(|h|^{-1}).
\]
And the second term is estimated as follows:
\[
\leq \|\theta_0\|_{L^\infty} \int_{|2h|<|x-y+h|<4|h|} |K(x-y+h)| \, dy \leq C \|\theta_0\|_{L^\infty} \int_{|2h|}^{4|h|} \frac{m(\rho^{-1})}{\rho} \, d\rho \leq C \|\theta_0\|_{L^\infty} m(|h|^{-1}).
\]
Therefore,
\[
|\delta_h u_{\text{out}}(x,t)| \leq c_8 \|\theta_0\|_{L^\infty} m(|h|^{-1}). \quad (4.7)
\]
Step 5. Proof of (4.4)
By the estimates (4.5), (4.6), and (4.7), it is enough to show that
\[
\frac{D_h}{4} + 2c_k k(|h|) |\delta_h \theta|^2 \geq \left( c_7 \sqrt{D_h(x,t)} \frac{m(|h|^{-1})^2}{k(|h|)} + c_8 \|\theta_0\|_{L^\infty} m(|h|^{-1}) \right) \frac{\omega'(|h|)}{\omega(|h|)} |\delta_h \theta|^2. \quad (4.8)
\]
By Young’s inequality,
\[
\text{R.H.S. of (4.8)} \leq \frac{D_h}{4} + c_7 \frac{m(|h|^{-1})^2}{k(|h|)} \left( \frac{\omega'(|h|)}{\omega(|h|)} \right)^2 |\delta_h \theta|^4 + c_8 \|\theta_0\|_{L^\infty} m(|h|^{-1}) \left( \frac{\omega'(|h|)}{\omega(|h|)} \right)^2 |\delta_h \theta|^2.
\]
Our goal (4.3) is verified if
\[
c_7 \frac{m(|h|^{-1})^2}{k(|h|)} \left( \frac{\omega'(|h|)}{\omega(|h|)} \right)^2 |\delta_h \theta|^2 \leq c_6 k(|h|), \quad (4.9)
\]
and
\[
c_8 \|\theta_0\|_{L^\infty} m(|h|^{-1}) \left( \frac{\omega'(|h|)}{\omega(|h|)} \right) \leq c_6 k(|h|). \quad (4.10)
\]
Since $|\delta_h \theta| \leq 2 \|\theta_0\|_{L^\infty}$, it is enough to show
\[
\left( \frac{m(|h|^{-1}) \omega'(|h|)}{k(|h|) \omega(|h|)} \right)^2 \leq \frac{c_6}{4c_7^2 \|\theta_0\|_{L^\infty}^2},
\]
instead of (4.9). And (4.10) is equivalent to
\[
\frac{m(|h|^{-1}) \omega'(|h|)}{k(|h|) \omega(|h|)} \leq \frac{c_6}{c_8 \|\theta_0\|_{L^\infty}}. \quad (4.11)
\]
The above two inequalities hold for sufficiently small $|h|$ by our assumption on $k, m,$ and $\omega$. It completes the proof that a modulus of continuity $M \omega$ is conserved for sufficiently large $M$. \[\square\]

Consider the main example
\[
k(r) \sim \frac{1}{r(-\log r)^{\alpha_1}}, \quad m(\zeta) \sim (\log |\zeta|)^{\alpha_2}.
\]
Then, a modulus of continuity
\[
\omega(r) \sim \frac{1}{(-\log r)^3}
\]
satisfies the condition
\[ \frac{m(r^{-1}) \omega'(r)}{k(r)} \sim \frac{r(-\log r)^{\alpha_1+\alpha_2}}{r(-\log r)} \to 0 \text{ as } r \to 0^+, \]
if \( \alpha_1 + \alpha_2 < 1 \). Combining this with Corollary \ref{corollary:exponential_gradient} then we obtain Corollary \ref{corollary:global_smooth_solution} that \eqref{eq:1.6} admits a global smooth solution if \( \alpha_1 + \alpha_2 < 1 \).

5. Exponential Gradient Estimate

In this section, we prove the estimate \eqref{eq:1.8} for the critical SQG equation \eqref{eq:1.7}. Note that the critical SQG equation is a special case in the previous discussion that
\[ k(r) = \frac{1}{2\pi r}, \quad m(\zeta) = 1. \]
Refer to the Appendix for self-contained proof. Here, we present it as a consequence of previous discussions.

**Proof of Main Theorem 2.** A solution \( \theta \) to the critical SQG equation exhibits the following rescaling property:
\[ \theta_\lambda(x, t) = \theta(\lambda x, \lambda t). \]
Since this rescaling changes \( \|\nabla \theta\|_{L^\infty} \) by \( \lambda \) times and remains \( \|\theta\|_{L^\infty} \) the same, we may assume that \( \|\nabla \theta_0\|_{L^\infty} = 1 \). In addition, it suffices to show our estimate for \( \|\theta_0\|_{L^\infty} \geq 1 \) as we already have \eqref{eq:1.9}.

For a fixed \( \beta > 0 \), let \( \omega(r) = (-\log r)^{-\beta} \) for small \( r \) and set \( \Omega(r) = \min\{M\omega(r), 2\|\theta_0\|_{L^\infty}\} \) with a constant \( M > 0 \) which will be determined later. Define \( r_0 \in (0, 1) \) to be
\[ M\omega(r_0) = 2\|\theta_0\|_{L^\infty}. \]
Due to our assumptions on \( \theta_0 \), it automatically admits \( \Omega \) as a modulus of continuity. If \( M \) is large enough so that \eqref{eq:4.11} is satisfied for \( |h| \leq r_0 \), then the modulus of continuity \( \Omega \) is conserved. In the case of critical SQG,
\[ \frac{r\omega'(r)}{\omega(r)} = \beta(-\log r)^{-1} \leq \frac{1}{c\|\theta_0\|_{L^\infty}}, \quad \text{for } r \leq r_0. \]
Take \(-\log r_0 = c\beta\|\theta_0\|_{L^\infty}\) and \( M = 2(c\beta)^{\beta}\|\theta_0\|_{L^\infty}^{1+\beta} \), which satisfy the above conditions.

Let \( r_1 \in (0, 1) \) be
\[ \Omega(r_1) = \min\left\{ \frac{c_1^2}{2c_1 c_3}, \frac{c_2}{2c_4}, 1 \right\} =: 2(c')^{-1}, \]
where \( c_1, c_2, c_3, c_4 \) are numeric constants in the proof of Theorem \ref{theorem:global_smooth_solution}. According to the proof of Theorem \ref{theorem:global_smooth_solution} in particular the last constants in the proof of Theorem \ref{theorem:global_smooth_solution}, we get
\[ \|\nabla \theta\|_{L^\infty} \leq c_1 \frac{\Omega(r_1)}{r_1} \leq c_1 \exp \left( c(c')^{\frac{1}{\beta}} \|\theta_0\|_{L^\infty}^{1+\frac{1}{\beta}} \right). \]

**Remark.** Since there is no control for constants as \( \beta^{-1} \to 0 \), we do not know how to improve it to \( C\exp(C\|\theta_0\|_{L^\infty}) \).

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REFERENCES

[1] L. A. Caffarelli and A. Vasseur. “Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation.” *Annals of Mathematics* 171.3 (2010): 1903-1930. arXiv:math/0608447

[2] P. Constantin, A. Majda, and E. Tabak. “Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar.” *Nonlinearity* 7.6 (1994): 1495-1533.

[3] P. Constantin and J. Wu. “Behavior of solutions of 2D quasi-geostrophic equations.” *SIAM journal on mathematical analysis* 30.5 (1999): 937-948.

[4] P. Constantin, D. Córdoba, and J. Wu. “On the critical dissipative quasi-geostrophic equation.” *Indiana University Mathematics Journal* (2001): 97-107.

[5] P. Constantin and J. Wu. “Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation.” *Annales de l’Institut Henri Poincaré C, Analyse non linéaire.* 25.6. (2008): 1103-1110. arXiv:math/0701592

[6] P. Constantin and V. Vicol. “Nonlinear maximum principles for dissipative linear nonlocal operators and applications.” *Geometric And Functional Analysis* 22.5 (2012): 1289-1321. arXiv:1110.0179

[7] A. Córdoba and D. Córdoba. “A maximum principle applied to quasi-geostrophic equations.” *Communications in Mathematical Physics* 249.3 (2004): 511-528.

[8] M. Dabkowski. “Eventual regularity of the solutions to the supercritical dissipative quasi-geostrophic equation.” *Geometric and Functional Analysis* 1.21 (2011): 1-13. arXiv:1007.2970

[9] M. Dabkowski, A. Kiselev, and V. Vicol. “Global well-posedness for a slightly supercritical surface quasi-geostrophic equation.” *Nonlinearity* 25.5 (2012): 1525. arXiv:1106.2137

[10] M. Dabkowski, A. Kiselev, L. Silvestre, and V. Vicol. “Global well-posedness of slightly supercritical active scalar equations.” *Analysis & PDE* 7.1 (2014): 43-72. arXiv:1203.6302

[11] H. Dong and N. Pavlović. “A regularity criterion for the dissipative quasi-geostrophic equations.” *Annales de l’IHP Analyse non linéaire.* 26.5 (2009): 1607-1619. arXiv:0710.5201

[12] H. Dong. “Dissipative quasi-geostrophic equations in critical Sobolev spaces: smoothing effect and global well-posedness.” *Discrete Contin. Dyn. Syst.* 26.4 (2010): 1197-1211. arXiv:math/0701826

[13] A. Kiselev, F. Nazarov, and A. Volberg. “Global well-posedness for the critical 2D dissipative quasi-geostrophic equation.” *Inventiones mathematicae* 167.3 (2007): 445-453. arXiv:math/0604185

[14] A. Kiselev and F. Nazarov. “Variation on a theme of Caffarelli and Vasseur.” *Journal of Mathematical Sciences* 166.1 (2010): 31-39. arXiv:0908.0923

[15] A. Kiselev. “Nonlocal maximum principles for active scalars.” *Advances in Mathematics* 227.5 (2011): 1806-1826. arXiv:1009.0542

[16] S. G. Resnick. *Dynamical problems in non-linear advective partial differential equations.* 1996, Ph.D. Thesis, University of Chicago.

[17] L. Silvestre. “Eventual regularization for the slightly supercritical quasi-geostrophic equation.” *Annales de l’Institut Henri Poincaré C, Analyse non linéaire.* 27.2 (2010): 693-704. arXiv:0812.4901

[18] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions (PMS-30).* Vol. 30. Princeton University Press, 1970.

APPENDIX

This is supplementary to section 5. We repeat discussion on section 3 and 4, in the sprit of the critical SQG, which gives a self-contained proof for Main Theorem 2.

**Main Theorem 2.** The critical SQG equation

\[
\begin{align*}
\partial_t \theta + (u \cdot \nabla) \theta + \Lambda \theta &= 0, \\
\theta(t = 0) &= \theta_0
\end{align*}
\]

with initial \( \theta_0 \in \mathcal{S}(\mathbb{R}^2) \) has a unique global smooth solution. In addition, for any \( \gamma > 0 \),

\[
\|\nabla \theta\|_{L_{\infty}^{1/\gamma}} \leq C \|\nabla \theta_0\|_{L^\infty} \exp(C \|\theta_0\|_{L^1_{\infty}}^{1+\gamma}),
\]

(5.2)

where constants depend only on \( \gamma \).

A solution \( \theta \) to the critical SQG equation exhibits the following rescaling property:

\[
\theta_\lambda(x,t) = \theta(\lambda x, \lambda t).
\]
Since this rescaling changes \( \| \nabla \theta \|_{L^\infty} \) by \( \lambda \) times and remains \( \| \theta \|_{L^\infty} \) the same, we may assume that \( \| \nabla \theta_0 \|_{L^\infty} = 1 \). In addition, it suffices to show our estimate for \( \| \theta_0 \|_{L^\infty} \geq 1 \) as we already have a double exponential estimate.

Let \( \omega(r) = (-\log r)^{-\beta} \) for some \( \beta > 0 \) and let \( \Omega(r) = \min\{M \omega(r), 2\|\theta_0\|_{L^\infty}\} \) with a constant \( M \) which will be determined later. Note that \( \theta_0 \) admits \( \min\{M \omega(r), 2\|\theta_0\|_{L^\infty}\} \) as a modulus of continuity. We split the proof into two parts:

**Proposition 10.** Suppose \( \theta \in L^\infty([0,T]; L^p \cap L^\infty) \) is a weak solution of the critical SQG equation (5.1) with an initial \( \theta_0 \) satisfying prescribed assumptions. Then there exists constant \( M = C_\beta \| \theta_0 \|_{L^\infty}^{1+\beta} \) so that \( \theta(t,\cdot) \) admits modulus of continuity \( \Omega \) as long as the solution is defined.

**Proposition 11.** Suppose \( \theta \) is a bounded weak solution of the critical SQG equation (5.1). If \( \theta \) admits a modulus of continuity \( \Omega \) for \( [0,T] \), then

\[
\| \nabla \theta \|_{L^\infty_t L^\infty_x} \leq C \| \nabla \theta_0 \|_{L^\infty} \exp(C M^{\frac{1}{\beta}}),
\]

where constants depend on \( \beta \). Thus \( \theta \) is a smooth solution on \( (0,T] \).

**Proof of Proposition 10.** Let \( r_0 \in (0,1) \) be \( M \omega(r_0) = 2 \| \theta \|_{L^\infty} \). Since \( L^\infty \)-norm is conserved, it suffices to show the modulus of continuity for distance less than \( r_0 \).

For notational convenience, define \( \delta_h f(x) = f(x+h) - f(x) \) for a function \( f \). We consider the maximum principle for

\[
v(x,h;t) := \left( \frac{\delta_h \theta(x,t)}{\omega(|h|)} \right)^2 F(h).
\]

The function \( F(h) = \exp(-G(|h|)) \) is just for the decay of \( v \) in \( h \), so that \( v \in L^p \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \) for some \( 1 \leq p < \infty \). We set \( G \) being smooth, nonnegative, nondecreasing, and \( G(r) = 0 \) for \( 0 \leq r \leq 1 \).

**Step 1.** Evolution of \( v \)

Since

\[
\partial_t \theta(x) + u(x) \cdot \nabla \theta(x) + \Lambda \theta(x) = 0,
\]

and

\[
\partial_t \theta(x+h) + u(x+h) \cdot \nabla \theta(x+h) + \Lambda \theta(x+h) = 0.
\]

Subtracting two formulas gives:

\[
\partial_t (\theta(x+h) - \theta(x)) + u(x) \cdot \nabla (\theta(x+h) - \theta(x)) + (u(x+h) - u(x)) \cdot \nabla \theta(x+h) + (\Lambda \theta(x+h) - \Lambda \theta(x)) = 0,
\]

and thus the evolution of \( \delta_h \theta \) is given by

\[
(\partial_t + u \cdot \nabla + (\delta_h u) \cdot \nabla + \Lambda_x) \delta_h \theta = 0.
\]

Multiply \( \delta_h \theta \) to both sides:

\[
\frac{1}{2} (\partial_t + u \cdot \nabla + (\delta_h u) \cdot \nabla + \Lambda_x) (\delta_h \theta)^2(x,h,t) + \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\delta_h \theta(x,t) - \delta_h \theta(y,t)|^2}{|x-y|^3} dy = 0.
\]

Therefore

\[
(\partial_t + u \cdot \nabla + (\delta_h u) \cdot \nabla + \Lambda_x) v(x,h,t) + D_h(x,t) \frac{F(h)}{\omega(|h|)^2} = - (\delta_h u) \cdot \frac{h}{|h|} \left( G'(|h|) + \frac{2 \omega'(|h|)}{\omega(|h|)} \right) v, \tag{5.3}
\]
Step 2. Breakthrough moment and maximum principle

Fix $M > 0$, which will be determined later. Suppose $\theta_0$ satisfies a modulus of continuity $\frac{M}{2} \omega$ and $\theta(t)$ admits a modulus of continuity $M \omega$ for $0 \leq t \leq t^*$. However, at the time $t = t^*$,

$$|\delta_h \theta(x, t^*)| \geq M \omega(|h|)$$ \hspace{1cm} (5.4)

for some $x, h$. Suppose we prove that

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \Lambda_x) v(x, h, t^*) < 0,$$ \hspace{1cm} (5.5)

whenever the such event occurs, then we get contraction from the maximum principle of $\Lambda_x$. Hence such an event is impossible, and the modulus of continuity $M \omega(\cdot)$ is conserved.

Therefore it suffices to show that

$$(\partial_t + u \cdot \nabla_x + (\delta_h u) \cdot \nabla_h + \Lambda_x) v(x, h, t^*) < 0,$$

whenever

$$M \omega(|h|) \leq |\delta_h \theta(x, t^*)| \leq 2M \omega(|h|).$$

If (5.4) is satisfied,

$$M \omega(|h|) \leq |\delta_h \theta(x, t^*)| \leq 2\|\theta_0\|_{L^\infty}.$$ 

Hence, we only need to consider $|h| \leq r_0$. Due to (5.3), it suffices to show that

$$\frac{D_h(x, t^*)}{2\beta} \geq \frac{1}{|h|(-\log |h|)} |\delta_h u(x, t^*)| |\delta_h \theta(x, t^*)|^2,$$ \hspace{1cm} (5.6)

whenever $M \omega(|h|) \leq |\delta_h \theta(x, t^*)| \leq 2M \omega(|h|), |h| \leq r_0$. Let us denote $t^*$ by simply $t$ from here.

Step 3. Pointwise lower bound of $D_h$

A smooth function $\varphi : [0, \infty) \to \mathbb{R}$ is a non-decreasing cutoff function such that

$$\varphi(r) = 0 \text{ on } r \leq \frac{1}{2}, \quad \varphi(r) = 1 \text{ on } r \geq 1, \quad 0 \leq \varphi' \leq 4.$$

For some sufficiently small $R = R(x, h, t) \geq 6|h|$ which will be determined later, we have the following estimate on $D_h(x, t)$:

$$D_h(x, t) \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\delta_h \theta(x, t) - \delta_h \theta(y, t)|^2}{|x-y|^3} \varphi\left(\frac{|x-y|}{R}\right) dy$$

$$\geq |\delta_h \theta(x, t)|^2 \frac{1}{2\pi} \int_{|x-y| \geq R} \frac{1}{|x-y|^3} dy - 2|\delta_h \theta(x, t)| \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \delta_h \theta(y, t) \cdot \frac{1}{|x-y|^3} \varphi\left(\frac{|x-y|}{R}\right) dy$$

$$\geq |\delta_h \theta(x, t)|^2 \int_{R}^{\infty} \frac{1}{\rho^2} d\rho$$

$$- 2|\delta_h \theta(x, t)| \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} (\theta(y, t) - \theta(x, t)) \left\{ \delta_h \left( \frac{1}{|v|^3} \varphi\left(\frac{|v|}{R}\right) \right) \right\}_{v=x-y} dy.$$

By the mean value theorem,

$$\left| \delta_h \left( \frac{1}{|v|^3} \varphi\left(\frac{|v|}{R}\right) \right) \right|_{v=x-y} \leq |h| \left| \nabla v \left( \frac{1}{|v|^3} \varphi\left(\frac{|v|}{R}\right) \right) \right|_{v=x-y+h^*}$$

$$\leq |h| \left\{ \frac{4}{R} \frac{1}{(\rho^*)^3} \frac{1_{\rho^* \geq \frac{3}{2}}}{} + \frac{3}{(\rho^*)^4} \frac{1_{\rho^* \geq \frac{3}{2}}}{} \right\}_{\rho^* = |x-y+h^*|}$$

Since $\rho^* = |x-y+h^*| \geq \frac{R}{2}$ and $|h^*| \leq |h| \leq \frac{R}{6}$, it follows $\rho = |x-y| \geq \frac{R}{3}$ and $\rho^* \geq \rho - \frac{R}{6} \geq \frac{R}{2}$. Hence the above formula is less than equal to the same formula where $\rho^*$ replaced by $\rho/2$, $R \geq \rho^* \geq \frac{R}{2}$.
For the inner piece, we use the Cauchy-Schwartz inequality:

\[ |\theta(y,t) - \theta(x,t)| \leq M\omega(|x - y|). \]

Therefore, we have the following estimate for \( D_h \) with a constant \( c_5 > 3 \).

\[
D_h(x, t) \geq \frac{|\delta_h \theta(x, t)|^2}{R} - 2 |\delta_h \theta(x, t)| |h| M \cdot c_5 \frac{\omega(R)}{R^2} \\
\geq \frac{|\delta_h \theta(x, t)|^2}{R} \left( 1 - \frac{2c_5 |h| M \omega(R)}{|\delta_h \theta(x, t)| R} \right).
\]

Set \( R = R(x, h, t) > 0 \) to satisfy

\[
\frac{\omega(R)}{R} = \frac{1}{4c_5 M} \frac{|\delta_h \theta(x, t)|}{|h|} \leq \frac{1}{2c_5} \frac{\omega(|h|)}{|h|} < \frac{\omega(6 |h|)}{6 |h|},
\]

we get \( R \geq 6 |h| \) as assumed. On the other hand,

\[
\frac{\omega(R)}{R} = \frac{1}{4c_5 M} \frac{|\delta_h \theta(x, t)|}{|h|} \geq \frac{1}{4c_5} \frac{\omega(|h|)}{|h|},
\]

and thus

\[
4c_5 \frac{|h|^{1-\gamma}}{R^{1-\gamma}} \geq \frac{\omega(|h|) |h|^{-\gamma}}{\omega(R) R^{-\gamma}} \geq 1.
\]

Hence \( R \leq (4c_5)^{1/(1-\gamma)} |h| \). Summing up,

\[
D_h(x, t) \geq \frac{|\delta_h \theta(x, t)|^2}{2R} \geq 8c_6 \frac{|\delta_h \theta(x, t)|^2}{|h|}. \tag{5.7}
\]

**Step 4.** Estimate of \( \delta_h u \)

\[
\delta_h u(x, t) = P.V. \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^3} (\delta_h \theta(x, t) - \delta_h \theta(y, t)) \, dy.
\]

Name the kernel by \( K(z) = z^\perp / |z|^3 \). We estimate \( \nabla u \) by splitting \( \mathbb{R}^2 \) into two pieces; an inner piece \( |x - y| \leq 3 |h| \) and an outer piece \( |x - y| > 3 |h| \).

For the inner piece, we use the Cauchy-Schwartz inequality:

\[
|\delta_h u_{\text{in}}(x, t)| \leq C \sqrt{\left( \int_{|x-y| \leq 3|h|} \frac{|\delta_h \theta(x, t) - \delta_h \theta(y, t)|^2}{|x - y|^3} \, dy \right) \left( \int_{|x-y| \leq 3|h|} \frac{1}{|x - y|} \, dy \right)} \\
= c_7 \sqrt{D_h(x, t) |h|}. \tag{5.8}
\]
For the outer piece:

$$|\delta_h u_{\text{out}}(x, t)| = \left| \int_{|x-y|>3|h|} K(x-y)(\theta(y, t) - \theta(y+h, t)) \, dy \right|$$

$$\leq \left| \int_{|x-y|>3|h|} (K(x-y) - K(x-y+h))\theta(y, t) \, dy \right|$$

$$+ \left| \int_{\{y:|x-y|>3|h|\}} K(x-y+h)\theta(y, t) \, dy \right|$$

The first term is estimated as follows:

$$\leq \|\theta_0\|_{L^\infty} \int_{|x-y|>3|h|} \rho \|\nabla K(x-y+h^*)\| \, dy (|h^*| \leq |h| < \frac{1}{3} |x-y|)$$

$$\leq c\|\theta_0\|_{L^\infty} |h| \int_{|x-y|>3|h|} |x-y|^{-3} \, dy = c_3\|\theta_0\|_{L^\infty}.$$}

And the second term is estimated as follows:

$$\leq \|\theta_0\|_{L^\infty} \int_{2|h|<|x-y+h|\leq4|h|} |K(x-y+h)| \, dy \leq C\|\theta_0\|_{L^\infty} \int_{2|h|}^{4|h|} \frac{1}{\rho} \, d\rho \leq C\|\theta_0\|_{L^\infty}.$$}

Therefore,

$$|\delta_h u_{\text{out}}(x, t)| \leq c_8\|\theta_0\|_{L^\infty}. \quad (5.9)$$

**Step 5. Proof of (5.6)**

By the estimates (5.7), (5.8), and (5.9), it is enough to show that

$$\frac{D_h}{4\beta} + \frac{2c_6}{\beta} \frac{|\delta_h \theta|^2}{|h|} \geq \left( c_7 \sqrt{D_h(x, t)} |h| + c_8 \|\theta_0\|_{L^\infty} \right) \frac{1}{|h| (-\log |h|)} |\delta_h \theta|^2. \quad (5.10)$$

By Young’s inequality,

R.H.S. of (5.10) \leq \frac{D_h}{4\beta} + \frac{\beta c_7^2}{4} |h| \left( \frac{1}{|h| (-\log |h|)} \right)^2 |\delta_h \theta|^4 + c_8 \|\theta_0\|_{L^\infty} \left( \frac{1}{|h| (-\log |h|)} \right) |\delta_h \theta|^2. \quad (5.11)

If we verify that

$$\beta c_7 |\delta_h \theta| (-\log |h|)^{-1} \leq c_6, \quad (5.11)$$

and

$$\beta c_8 \|\theta_0\|_{L^\infty} (-\log |h|)^{-1} \leq c_6, \quad (5.12)$$

then (5.5) holds. Since \(|\delta_h \theta| \leq 2\|\theta_0\|_{L^\infty}, it suffices to show

$$-\log |h| \geq \beta c_9 \|\theta_0\|_{L^\infty}, \text{ for all } |h| \leq r_0,$$

instead of (5.11) and (5.12). We completes the proof by setting \(M = 2(\beta c_9)^2 \|\theta_0\|_{L^\infty}^{1+\beta}. \square$$

**Proof of Proposition 11** It suffices to prove the assertion for a smooth solution \(\theta\). One may consider regularized equation with the term \(-\epsilon \Delta \theta, get a uniform estimate independent of \(\epsilon, and then take the inviscid limit \(\epsilon \to 0+ to get the desired result.

**Step 1. Evolution of \(|\nabla \theta|^2**

We take gradient on both sides of the equation to get:

$$\partial_t \nabla \theta + u \cdot \nabla^2 \theta + \nabla u \cdot \nabla \theta + \Lambda \nabla \theta = 0.$$
Multiply $\nabla \theta$ to both sides:

$$\nabla \theta \cdot \partial_t \nabla \theta + \nabla \theta \cdot u \cdot \nabla^2 \theta + \nabla \theta \cdot \nabla u \cdot \nabla \theta + \nabla \theta \cdot \Lambda \nabla \theta = 0.$$  

Considering singular integral formulation for the nonlocal operator $\Lambda$ gives:

$$\frac{1}{2}(\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2 + \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\nabla \theta(x, t) - \nabla \theta(y, t)|^2}{|x - y|^3} \, dy = -\nabla \theta \cdot \nabla u \cdot \nabla \theta. \quad (5.13)$$

Our goal here is to prove

$$(\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2 (x, t) < 0$$

whenever $|\nabla \theta(x, t)|$ is sufficiently large in terms of $M$.

Step 2. Pointwise lower bound of $D(x, t)$

A smooth function $\varphi : [0, \infty) \to \mathbb{R}$ is a non-decreasing cutoff function such that

$$\varphi(x) = 0 \text{ on } x \leq \frac{1}{2}, \quad \varphi(x) = 1 \text{ on } x \geq 1, \quad 0 \leq \varphi' \leq 4.$$

For some sufficiently small $R = R(x, t) > 0$ which will be determined later, we have the following estimate on $D(x, t)$:

$$D(x, t) \geq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|\nabla \theta(x, t) - \nabla \theta(y, t)|^2}{|x - y|^3} \varphi \left( \frac{|x - y|}{R} \right) \, dy$$

$$\geq \frac{|\nabla \theta(x, t)|^2}{2\pi} \int_{|x - y| \geq R} \frac{1}{|x - y|^3} \, dy - 2\nabla \theta(x, t) \cdot \frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \theta(y, t) \varphi \left( \frac{|x - y|}{R} \right) \, dy$$

$$\geq \frac{|\nabla \theta(x, t)|^2}{\rho} \int_{0}^{\infty} \frac{d}{d\rho} \left( \frac{\varphi(\rho/R)}{\rho^3} \right) \, d\rho$$

The second term:

$$\int_{0}^{\infty} \frac{d}{d\rho} \left( \frac{\varphi(\rho/R)}{\rho^3} \right) \, d\rho \leq \frac{4}{R} \int_{R/2}^{R} \frac{\Omega(\rho)}{\rho^2} \, d\rho + 2 \int_{R/2}^{\infty} \frac{\Omega(\rho)}{\rho^3} \, d\rho \leq \frac{c_1 \Omega(R)}{4 R^2}.$$

Therefore,

$$D(x, t) \geq \frac{|\nabla \theta(x, t)|^2}{R} - |\nabla \theta(x, t)| \frac{c_1 \Omega(R)}{2 R^2} = \frac{|\nabla \theta(x, t)|^2}{R} \left( 1 - \frac{c_1}{2 |\nabla \theta(x, t)|} \frac{\Omega(R)}{R} \right).$$

Set $R = R(x, t) > 0$ to satisfy

$$\frac{\Omega(R)}{R} = \frac{|\nabla \theta(x, t)|}{c_1}. \quad (5.14)$$

Then

$$D(x, t) \geq \frac{|\nabla \theta(x, t)|^2}{2R}. \quad (5.15)$$

Note that $R(x, t)$ is arbitrarily small when $|\nabla \theta(x, t)|$ is large enough.

Step 3. Estimate of $\nabla u$

$$\nabla u(x, t) = \nabla \Lambda^{-1} \nabla \theta(x, t) = P.V. \int_{\mathbb{R}^2} K(x - y)(\nabla \theta(x, t) - \nabla \theta(y, t)) \, dy,$$

where $K(z) = z^3 / |z|^3$. We estimate $\nabla u$ by splitting $\mathbb{R}^2$ into two pieces; an inner piece $|x - y| \leq R$ and an outer piece $|x - y| > R$. 

For the inner piece, we use the Cauchy-Schwartz inequality:

$$|\nabla u_{in}(x, t)| \leq C \sqrt{\left( \int_{|x-y| \leq R} \frac{|\nabla \theta(x, t) - \nabla \theta(y, t)|^2}{|x-y|^2} dy \right) \left( \int_{|x-y| \leq R} \frac{1}{|x-y|} dy \right)}$$

$$= c_3 \sqrt{D(x, t) R}.$$  

For the outer piece, we apply integration by parts and use the modulus of continuity of $\theta$:

$$|\nabla u_{out}(x, t)| \leq \left| \int_{|x-y| > R} (\theta(x, t) - \theta(y, t)) \nabla K(x-y) dy \right|$$

$$+ \left| \int_{|x-y| = R} (\theta(x, t) - \theta(y, t)) K(x-y) \nu(y) d\sigma(y) \right|$$

$$\leq C \left( \int_{R}^{\infty} \frac{\Omega(\rho)}{\rho^2} d\rho + C \frac{\Omega(R)}{R} \right) \leq c_4 \frac{\Omega(R)}{R}.$$

Therefore,

$$|\nabla u| |\nabla \theta|^2 \leq |\nabla u_{in}| |\nabla \theta|^2 + |\nabla u_{out}| |\nabla \theta|^2$$

$$\leq D + c_2 R |\nabla \theta|^4 + c_4 \frac{\Omega(R)}{R} |\nabla \theta|^2.$$  

(5.16)

**Step 4. Maximum principle**

Combining (5.13), (5.15), and (5.16),

$$(\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2 + \frac{D}{2} + \frac{|\nabla \theta|^2}{2R}$$

$$\leq (\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2 + D$$

$$\leq 2 |\nabla u| |\nabla \theta|^2 + \frac{D}{2} + 2c_3 R |\nabla \theta|^4 + 2c_4 \frac{\Omega(R)}{R} |\nabla \theta|^2.$$

Hence

$$(\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2 + \frac{|\nabla \theta|^2}{2R} \leq 2c_3 R |\nabla \theta|^4 + 2c_4 \frac{\Omega(R)}{R} |\nabla \theta|^2.$$

Recall that (5.14) that

$$|\nabla \theta| = c_1 \frac{\Omega(R)}{R}.$$  

And let $r_1 \in (0, 1)$ be

$$M(- \log r_1)^{-\beta} = \Omega(r_1) = \min \left\{ \frac{1}{8^{1/2}c_1 c_3}, \frac{1}{8c_4}, 1 \right\} =: c'.$$

If $|\nabla \theta| > c_1 \frac{\Omega(r_1)}{r_1}$, then $R < r_1$,

$$2c_3^2 R \left( c_1 \frac{\Omega(R)}{R} \right)^2 < \frac{1}{4R},$$

and

$$2c_4 \frac{\Omega(R)}{R} < \frac{1}{4R}.$$

Hence,

$$(\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2(x, t) < 0.$$

Due to the maximum principle,

$$\|\nabla \theta\|_{L_{t,x}^\infty} \leq \max \left\{ c_1 \frac{\Omega(r_1)}{r_1}, 1 \right\} \leq \frac{C}{r_1} \leq C \exp \left( (c')^{-\frac{1}{\beta}} M^\frac{1}{\beta} \right).$$

□
Since $M = C\|\theta_0\|_{L^\infty}^{1+\beta}$ according to Proposition \[10\], we get \[5.2\]:

$$\|\nabla \theta\|_{L^\infty_{t,x}} \leq C\|\nabla \theta_0\|_{L^\infty} \exp(C\|\theta_0\|_{L^\infty}^{1+\frac{1}{\beta}}).$$

(Hyungjun Choi)

**Department of Mathematics**

**Princeton University, Princeton NJ 08544, USA**

*Email address: hyungjun.choi@princeton.edu*