Combinatorial Properties of Self-Overlapping Curves and Interior Boundaries

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Abstract

We study the interplay between the recently defined concept of minimum homotopy area and the classical topic of self-overlapping curves. The latter are plane curves which are the image of the boundary of an immersed disk. Our first contribution is to prove new sufficient combinatorial conditions for a curve to be self-overlapping. We show that a curve $\gamma$ with Whitney index 1 and without any self-overlapping subcurves is self-overlapping. As a corollary, we obtain sufficient conditions for self-overlappingness solely in terms of the Whitney index of the curve and its subcurves. These results follow from our second contribution, which shows that any plane curve $\gamma$, modulo a basepoint condition, is transformed into an interior boundary by wrapping around $\gamma$ with Jordan curves. Equivalently, the minimum homotopy area of $\gamma$ is reduced to the minimal possible threshold, namely the winding area, through wrapping. In fact, we show that $n+1$ wraps suffice, where $\gamma$ has $n$ vertices. Our third contribution is to prove the equivalence of various definitions of self-overlapping curves and interior boundaries, often implicit in the literature. We also introduce and characterize zero-obstinance curves, further generalizations of interior boundaries defined by optimality in minimum homotopy area.

1 Introduction

Classically, a curve $\gamma : S^1 \to \mathbb{R}^2$ is called self-overlapping if there is an orientation-preserving immersion $F : D^2 \to \mathbb{R}^2$ of the unit disk $D^2$, a map of full rank on the entire unit disk $D^2$, such that $F|_{\partial D^2} = \gamma$. One can think of such an immersion as distorting a unit disk that lies flat in the plane and stretching and pulling it continuously without leaving the plane and without twisting or pinching it [15]. If the disk is painted blue on top and pink on the bottom, then one only sees blue. If we also imagine the disk being semi-transparent,

Figure 1: A self-overlapping curve $\gamma$ with winding numbers for the faces circled. The Blank cuts, shown in red, slice $\gamma$ into a collection of simple positively oriented (counterclockwise) Jordan curves.

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then the blue will appear darker in the regions where it overlaps itself; see Fig. 1. That means, any self-overlapping curve $\gamma$ must have non-negative winding numbers, $wn(x, \gamma) \geq 0$ for every $x \in \mathbb{R}^2$. We call this condition **positive consistent**. Another simple and intuitive view originates from Blank [1]: The curve is self-overlapping when we can cut it along simple curves into simple positively oriented Jordan curves, i.e., a collection of blue topological disks. **Interior boundaries** are generalizations of self-overlapping curves that are defined similarly, except that $F$ is an interior map which allows finitely many branch points [12]. Interior boundaries are composed of multiple self-overlapping curves (of the same orientation); see Fig. 2 for an example. In this paper, all curves $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ are assumed to be closed, immersed, and generic, i.e., with only finitely many intersection points, each of which are transverse double points. We also assume $\gamma'(t)$ exists and is nonzero for all $t \in [0, 1]$. We show new combinatorial properties of self-overlapping curves and interior boundaries by revealing new connections to the minimum homotopy area of curves.

Figure 2: Example curves of different curve classes and inclusion relationships between the classes. $\gamma_{SO}$ is self-overlapping as indicated by the Blank cuts in red. $\gamma_{IB}$ is an interior boundary consisting of two self-overlapping curves (of the same orientation), one in blue the other in green. The bottom row shows curve classes that are introduced in this paper: $\gamma_{SI}$ is strongly irreducible as can be seen from the non-positive Whitney indices (shown in gray) of its direct split subcurves. Similarly, $\gamma_{I}$ is irreducible; note that $\gamma_{v}$ has Whitney index 1 but is not self-overlapping. Also note that $\gamma_{SO}$ is not irreducible since $\gamma_{u}$ is self-overlapping. $\gamma_{ZO}$ also consists of two self-overlapping curves but of different orientation and is therefore not an interior boundary, but it has zero obstinance.

1.1 Related work

**Self-Overlapping Curves and Interior Boundaries.** Self-overlapping curves and interior boundaries have a rather rich history, and have been studied under the lenses of analysis, topology, geometry, combinatorics, and graph theory [4, 10, 12, 13, 14, 15, 17, 19]. In the 1960s, Titus [19] provided the first algorithm to test whether a curve is self-overlapping (or an interior boundary), by defining a set of cuts that must cut the curve into smaller subcurves that are self-overlapping (or interior boundaries). In a 1967 PhD thesis [1], Blank proved that a curve is self-overlapping iff there is a sequence of cuts (different from Titus cuts) that
completely decompose the curve into simple pieces. He represents plane curves with words and showed that one can determine the existence of a cut decomposition by looking for algebraic decompositions of the word. In the 1970s, Marx [13] extended Blank’s work to give an algorithm to test if a curve is an interior boundary. In the 1990s, Shor and Van Wyk [17] expedited Blank’s algorithm to run in \( O(N^3) \) time for a polygonal curve with \( N \) line segments. Their dynamic programming algorithm is currently the fastest algorithm to test for self-overlappingness. It is not known whether this runtime bound is tight or whether a faster runtime might be achievable. In distantly related work, Eppstein and Mumford [4] showed that it is NP-complete to determine whether a fixed self-overlapping curve \( \gamma \) is the 2D projection of an immersed surface in \( \mathbb{R}^3 \) defined over a compact two-manifold with boundary. Graver and Cargo [10] approached the problem from a graph-theoretical perspective using so-called covering graphs. All of these algorithms also compute the number of inequivalent immersions. Another fact we glean from Blank is that any self-overlapping curve \( \gamma \) necessarily makes one full turn, i.e., it has Whitney index \( \text{WHIT}(\gamma) = 1 \). The necessity of Whitney index 1 and positive consistency to be self-overlapping are well-known and date back to [19].

Minimum Homotopy Area. The minimum homotopy area \( \sigma(\gamma) \) is the infimum of areas swept out by nullhomotopies of a closed plane curve \( \gamma \). The key link between minimum area homotopies and self-overlapping curves arose in [8] \([11]\), where the authors showed that any curve \( \gamma \) has a minimum area homotopy realized by a sequence of nullhomotopies of self-overlapping subcurves (direct split subcurves; see Section 3.1 for the definition). The minimum homotopy area was introduced by Chambers and Wang [3] as a more robust metric for curve comparison than homotopy width (i.e., Fréchet distance or one of its variants) or homotopy height [2]. The minimum homotopy area can be computed in \( O(N^2 \log N) \) time for consistent curves [3]. For general curves, Nie gave an algorithm to compute \( \sigma(\gamma) \) based on an algebraic interpretation of the problem that runs in \( O(N^6) \) time, while the self-overlapping decomposition result of [8] yields an exponential-time algorithm. The winding area \( W(\gamma) \) is the integral over all winding numbers in the plane. A simple argument shows that \( \sigma(\gamma) \geq W(\gamma) \); see [3]. Both self-overlapping curves and interior boundaries are characterized by positive consistency and optimality in minimum homotopy area, \( \sigma(\gamma) = W(\gamma) \). A curve possessing both of these properties is self-overlapping when \( \text{WHIT}(\gamma) = 1 \) and an interior boundary when \( \text{WHIT}(\gamma) \geq 1 \).

1.2 New Results

We ask the following question: what are the sufficient combinatorial conditions for a plane curve to be self-overlapping? Such conditions provide novel mathematical foundations that could pave the way for speeding up algorithms for related problems, such as deciding self-overlappingness or computing the minimum homotopy area of a curve. The first contribution of this paper is to answer this question in the affirmative (Theorems [17] and [19] in Section 4): We show that a curve \( \gamma \) with Whitney index 1 and without any self-overlapping subcurves is self-overlapping, and we obtain sufficient conditions for a curve to be self-overlapping solely in terms of the Whitney index of the curve and its subcurves. Here, we only consider direct split subcurves \( \gamma_v \) that traverse \( \gamma \) between the first and second appearance of vertex \( v \) in the plane graph induced by \( \gamma \). Our results apply to (strongly) irreducible curves; see Fig. 2. We call \( \gamma \) irreducible, if every (proper) direct split is not self-overlapping; if the Whitney index of each such direct split is non-positive, then we call \( \gamma \) strongly irreducible.

These results follow from our second contribution (Theorems [14] and [15] in Section 4), which shows that any plane curve \( \gamma \) is transformed into an interior boundary by wrapping around \( \gamma \) with Jordan curves. Equivalently, this means that the minimum homotopy area of \( \gamma \) is reduced to the minimal possible threshold, namely the winding area, through wrapping. See Fig. 3 for an example of wrapping. Of course it is easy to see that—after repeated wrapping—a curve becomes positive consistent, since a single wrap increases the winding numbers of each face by one. However, our result shows a new and non-trivial connection between wrapping and the minimum homotopy area.

The third contribution of this paper (in Section 3) is to unite the various definitions and perspectives on self-overlapping curves and interior boundaries. We prove the equivalence of five definitions of self-
overlapping curves and four of interior boundaries (Theorems 9 and 8). To this end, we define the new concept of obstinance of a curve \( \gamma \) as \( \text{obs}(\gamma) = \sigma(\gamma) - W(\gamma) \geq 0 \), and characterize zero-obstinance curves (Theorem 12), see Fig. 2. Rephrasing our earlier characterization, self-overlapping curves and interior boundaries are positive-consistent curves with zero-obstinance and positive Whitney index.

We conclude by defining a new operation called balanced loop insertion, a complementary notion to that of balanced loop deletion, the key trick to proving Theorem 17. As a parallel to our results on wraps, we show in Theorem 21 that careful iteration of balanced loop insertion turns any curve \( \gamma \) with \( \text{WHIT}(\gamma) = 1 \) (and positive outer basepoint) into a self-overlapping curve.

More supplementary details on the relationship between different curve classes studied in this paper are provided in Appendix 14.

2 Preliminaries

2.1 Regular and Generic Curves

We work with regular, generic, closed plane curves \( \gamma : [0, 1] \to \mathbb{R}^2 \) with basepoint \( \gamma(0) = \gamma(1) \). Let \( \mathbb{G} \) denote the set of such curves. A curve is regular if \( \gamma'(t) \) exists and is non-zero for all \( t \); a curve is generic (or normal) if the embedding has only a finite number of intersection points, each of which are transverse crossings. Being generic is a weak restriction, as normal curves are dense in the space of regular curves [20]. Viewing a generic curve \( \gamma \) by its image \( [\gamma] \subseteq \mathbb{R}^2 \), we can treat \( \gamma \) as a directed plane multigraph \( G(\gamma) = (V(\gamma), E(\gamma)) \). Here, \( V(\gamma) = \{p_0, p_1, \ldots, p_n\} \) is the set of ordered vertices (points) of \( \gamma \), with basepoint \( p_0 = \gamma(0) \) regarded as a vertex as well. An edge \( (p_i, p_j) \) corresponds to a simple path along \( \gamma \) between \( p_i \) and \( p_j \). The faces of \( G(\gamma) \) are the maximally connected components of \( \mathbb{R}^2 \setminus [\gamma] \). Each \( \gamma \in \mathbb{G} \) has exactly one unbounded face, the exterior face \( F_{ext} \). See Fig. 4. Two curves are combinatorially equivalent when their planar multigraphs are isomorphic. We may therefore define a curve just by its image, orientation, and basepoint. A curve is simple if it has no intersection points. We denote \( |\gamma| = |V(\gamma) \setminus \{p_0\}| \) as the complexity of \( \gamma \).

For any \( x \in \mathbb{R}^2 \setminus [\gamma] \), the winding number \( \text{wn}(x, \gamma) = \sum_i a_i \) is defined using a simple path \( P \) from \( x \) to \( F_{ext} \) that avoids the intersection points of \( \gamma \). Here, \( a_i = +1 \) if \( P \) crosses \( \gamma \) from left to right at the \( i \)-th intersection of \( P \) with \( \gamma \), and \( a_i = -1 \) otherwise. Since this number is independent of the path chosen and is constant over each face \( F \) of \( G(\gamma) \), we write \( \text{wn}(F, \gamma) \). If \( \text{wn}(F, \gamma) \geq 0 \) for every face \( F \) on \( G(\gamma) \), then we call \( \gamma \) positive consistent. If \( \text{wn}(F, \gamma) \leq 0 \) for every face, then \( \gamma \) is negative consistent. See Fig. 4 for an example curve illustrating these concepts. The winding area of a curve \( \gamma \) is given by \( \text{W}(\gamma) = \int_{\mathbb{R}^2} |\text{wn}(x, \gamma)| \, dx = \sum_F A(F) |\text{wn}(F, \gamma)| \), where \( A(F) \) is the area of the face \( F \) and \( \text{wn}(x, \gamma) = 0 \) for \( x \in [\gamma] \). The depth \( D(F, \gamma) \) of a face \( F \) is the minimum number of crossings of any simple path \( P \) from \( F \) to \( F_{ext} \) with \( \gamma \). The depth of \( \gamma \) is the sum \( D(\gamma) = \sum_F A(F) D(F, \gamma) \). The Whitney index \( \text{WHIT}(\gamma) \) of a curve \( \gamma \) is the winding number of the derivative \( \gamma' \) about the origin. A curve \( \gamma \) is positively oriented if \( \text{WHIT}(\gamma) > 0 \) and negatively oriented if \( \text{WHIT}(\gamma) < 0 \).

A basepoint \( p_0 = \gamma(0) \) is a positive outer basepoint if \( p_0 \) is incident to the two faces \( F_{ext} \) and \( F \), and \( \text{wn}(F, \gamma) = 1 \). If \( \text{wn}(F, \gamma) = -1 \), then \( p_0 \) is a negative outer basepoint. Several of our results require
To define the intersection sequence of \( \gamma \) consists of the sequence of all vertex labels along \( \gamma \). The signed intersection sequence of \( \gamma \) is 0, 1+, 2−, 2+, 1−, 3+, 4+, 4−, 5+, 6−, 4+, 5−, 6+, 3−, 0; vertex labels are shown, and the sign of each vertex is indicated with green (positive) or red (negative). The combinatorial relations are: \( p_2 \subset p_1; p_4, p_5, p_6 \subset p_3; p_1, p_2 \not\subset p_3, p_4, p_5, p_6; p_4, p_5 \not\subset p_6; p_4 \not\subset p_5 \).

\( \gamma \) to have a positive outer basepoint. A curve \( \gamma : S^1 \to \mathbb{R}^2 \) is (positive) self-overlapping when there is an orientation-preserving immersion \( F : \mathbb{D}^2 \to \mathbb{R}^2 \), a map of full rank, extending \( \gamma \) to a map on the entire two-dimensional unit disk \( \mathbb{D}^2 \). If the reversal \( \overline{\gamma} \) of a curve is self-overlapping, then we call \( \gamma \) negative self-overlapping. Unless stated otherwise, the term self-overlapping is used only to mean positive self-overlapping.

### 2.2 Combinatorial Relations and Intersection Sequences

Following Titus [18], we now describe how the intersection points of a curve \( \gamma \in \mathcal{C} \) relate to each other. See Fig. 4 for an illustration of these concepts. Let \( p_i, p_j \in V(\gamma) \) be two vertices such that \( p_i = \gamma(t_i) = \gamma(t_i^*) \) and \( p_j = \gamma(t_j) = \gamma(t_j^*) \) with \( t_i < t_i^* \) and \( t_j < t_j^* \). Then, one of the following must hold:

- **\( p_i \) links \( p_j \), or \( p_i \) \( L \) \( p_j \),** if \( t_i < t_j < t_j^* < t_i^* \) or \( t_j < t_i < t_i^* < t_j^* \)
- **\( p_i \) is separate from \( p_j \), or \( p_i \) \( S \) \( p_j \),** if \( t_i < t_i^* < t_j < t_j^* \) or \( t_j < t_j^* < t_i < t_i^* \)
- **\( p_i \) is contained in \( p_j \), or \( p_i \) \( \subset \) \( p_j \),** if \( t_j \leq t_i < t_i^* \leq t_j^* \)

To define the intersection sequence of \( \gamma \), the vertices are labeled in the order they appear on \( \gamma \), starting with 0 for the basepoint \( \gamma(0) \), and increasing by one each time an unlabeled vertex is encountered. The **signed intersection sequence** consists of the sequence of all vertex labels along \( \gamma \) starting at the basepoint; the first time vertex \( p_i \) is visited, the label is augmented with \( \text{sgn}(p_i) \), and the second time with \( -\text{sgn}(p_i) \). Here, \( \text{sgn}(p_i) := \text{sgn}(p_i, \gamma) \) is the **sign** of vertex \( p_i = \gamma(t_i) = \gamma(t_i^*) \), and is 1 if the vector \( \gamma' \) rotates clockwise from \( t_i \) to \( t_i^* \), and \(-1\) otherwise. Note that \( \text{sgn}(p_i) \) depends on the basepoint of the curve. As proved by Titus, interior boundariness is invariant with respect to signed intersection sequences [19].

### 2.3 Minimum Homotopies

A **homotopy** between two generic curves \( \gamma \) and \( \gamma' \) is a continuous function \( H : [0, 1]^2 \to \mathbb{R}^2 \) such that \( H(0, \cdot) = \gamma \) and \( H(1, \cdot) = \gamma' \). In \( \mathbb{R}^2 \), any curve is null-homotopic, i.e., homotopic to a constant map. Given a sequence of homotopies \( (H_i)_{i=1}^k \), we notate the concatenation of these homotopies in order as \( \sum_{i=1}^k H_i \). We use the notation \( \overline{H} \) for the reversal \( \overline{H}(i, t) = H(1 - i, t) \) of a homotopy. If \( H(0, \cdot) = \gamma \) and \( H(1, \cdot) = \gamma' \), we may write \( \gamma \overset{H}{\Rightarrow} \gamma' \).

**Homotopy moves** are basic local alterations to a curve defined by their action on \( G(\gamma) \). These moves come in three pairs [8], see Fig. 5. The I-moves destroy/create an empty loop, II-moves destroy/create a bigon, and III-moves flip a triangle. We denote the moves that remove vertices as \( I_a \) and \( II_a \), and moves that
create vertices as \( I_b \) and \( II_b \). See Fig. 5. It is well-known that any homotopy such that each intermediate curve is piecewise regular and generic, or almost generic, can be achieved by a sequence of homotopy moves. Thus, without loss of generality, we assume that each time the curve \( H(i, \cdot) \) combinatorially changes is through a single homotopy move.

![Figure 5: All three homotopy moves and their reversals. Figure from [8].](image)

Let \( \gamma \in \mathcal{C} \) and \( H \) be a nullhomotopy of \( \gamma \). Define \( E_H(x) \) as the number of connected components of \( H^{-1}(x) \). Intuitively, this counts the number of times that \( H \) sweeps over \( x \). The minimum homotopy area of \( \gamma \) is defined as \( \sigma(\gamma) = \inf \{ \int x^2 \, dx \mid H \) is a nullhomotopy of \( \gamma \} \). The following was shown in [3] [8]:

**Lemma 1** (Homotopy Area \( \geq \) Winding Area). Let \( \gamma \in \mathcal{C} \). Then \( \sigma(\gamma) \geq W(\gamma) \).

A straightforward proof by induction, similar to that of Lemma 1 shows the following.

**Lemma 2** (Homotopy Area \( \leq \) Depth). Let \( \gamma \in \mathcal{C} \). Then \( \sigma(\gamma) \leq D(\gamma) \).

On the directed multigraph \( G(\gamma) \), we can define the left and right face of any edge. We call a homotopy left (right) sense-preserving if \( H(i + \epsilon, t) \) lies on or to the left (right) of the oriented curve \( H(i, \cdot) \) for any \( i, t \in [0, 1] \) and any \( \epsilon > 0 \). The following two lemmas provide useful properties about sense-preserving homotopies; the first was proven in [3], the second in [8].

**Lemma 3** (Monotonicity of Winding Numbers). Let \( H \) be a homotopy. If \( H \) is left (right) sense-preserving, then for any \( x \in \mathbb{R}^2 \) the function \( a(i) = wn(x, H(i, \cdot)) \) is monotonically decreasing (increasing).

**Lemma 4** (Sense-Preserving Homotopies are Optimal). Let \( \gamma \in C \) be consistent. Then a nullhomotopy \( H \) of \( \gamma \) is optimal if and only if it is sense-preserving.

### 3 Equivalences

In this section, we show the equivalence of different characterizations of interior boundaries (Theorem 5) and of self-overlapping curves (Theorem 9). Our analysis of curve classes hinges around the concept of obstinance. In Theorem 12 we classify zero obstinance curves, which are generalizations of interior boundaries and of self-overlapping curves.

#### 3.1 Direct Splits

Let \( \gamma \in \mathcal{C} \) and \( p_i \in V(\gamma) \) with \( p_i = \gamma(t_i) = \gamma(t_i^*) \) and \( t_i < t_i^* \). Then, \( \gamma \) can be split into two subcurves at \( p_i \): The **direct split** is the curve with image \( [\gamma|_{t_i^*}] \) with basepoint \( p_i \), and the **indirect split** is the curve with image \( [\gamma|_{t_i}] \cup [\gamma|_{[0,t_i^*]}] \) with basepoint \( \gamma(0) \). We endow both of these curves with the same orientation as \( \gamma \). Given a direct (or indirect) split \( \gamma \) on a curve \( \gamma \), we write \( \gamma \setminus \gamma \) for the indirect (or direct) split complementary to \( \gamma \). We call a direct split \( \text{proper} \) if it is not the entire curve \( \gamma \). See Fig. 6. If \( v = p_i \in V(\gamma) \), we may notate the direct split as \( \gamma_v \). When removing multiple splits iteratively we write \( \gamma \setminus \left( \bigcup_{i=1}^{n} \gamma_i \right) \), where we require that \( \gamma_i \) is a direct split of \( \gamma \setminus \left( \bigcup_{j=1}^{i-1} \gamma_j \right) \). The direct splits carry a great deal of information about the curve. In fact, one can recover the combinatorial relations from the direct splits: \( p_i \) \( \Longleftrightarrow \) \( \gamma \) iff \( p_i \in [\gamma_j] \) and \( p_j \in [\gamma_i] \); \( p_i \) \( \not\Longleftrightarrow \) \( \gamma \) iff \( p_i \notin [\gamma_j] \) and \( p_j \notin [\gamma_i] \); and \( p_i \subseteq p_j \) iff \( p_i \) is a vertex of \( \gamma_j \). Being a direct split of a curve is a transitive property. I.e., if \( \gamma_i \in \mathcal{C} \) is a direct split on \( \gamma \), and \( \gamma_j \) is a direct split on \( \gamma_i \), then \( \gamma_j \) is a direct split on \( \gamma \). The parallel statement on indirect splits, however, is false.
Figure 6: A self-overlapping decomposition of a self-overlapping curve \( \gamma \). Here, \( \gamma_1 \) and \( \gamma_3 \) are (proper) direct splits of \( \gamma \), while \( \gamma_2, \gamma_4, \) and \( \gamma_5 \) are neither direct nor indirect splits of \( \gamma \).

### 3.2 Decompositions and Loops

A curve \( \gamma \in \mathcal{C} \) can be entirely decomposed by iteratively removing direct splits. For a sequence of subcurves \( \Omega = \gamma_k \), define \( C_0 = \gamma \) and inductively \( C_i = C_{i-1} \setminus \gamma_i \) for \( i \geq 1 \); the basepoint of \( \gamma_i \) is \( v_i = C_i \cap \gamma_i \). We call \( \Omega \) a **direct split decomposition** if \( \gamma_i \) is a direct split of \( C_{i-1} \), for all \( i \in \{1, 2, \ldots, k\} \), and \( \gamma_k = C_{k-1} \).

Note that \( \Omega \) nearly induces a partition of \( \gamma \), in the sense that \( \gamma = \bigcup_{i=1}^{k} [\gamma_i] \) and \( \gamma_i \cap \gamma_j \subset V(\gamma) \) for any \( i \neq j \). Given a direct split decomposition \( \Omega = \gamma_k \), we write \( V(\Omega) \) for the set of basepoints of all \( \gamma_i \in \Omega \). See Figure 6.

Observe that no two vertices \( v_i, v_j \in V(\Omega) \) may be linked. Hence, we obtain a partial order \( \prec \) on \( V(\Omega) \) by declaring \( v_i \prec v_j \) whenever \( v_i \subset v_j \). Define \( T_\Omega \) to be the rooted, directed tree with vertex set \( V(T_\Omega) = V(\Omega) \) and edges \( e = (v_i, v_j) \) whenever \( v_i \subset v_j \) and there is no other vertex \( v_k \neq v_i, v_j \) such that \( v_i \subset v_k \subset v_j \). The root of \( T_\Omega \) corresponds to the basepoint of \( \gamma \). We consider two subcurve decompositions \( \Omega, \Gamma \) equivalent, \( \Omega \sim \Gamma \), when \( T_\Omega = T_\Gamma \). One can easily verify that \( \sim \) is an equivalence relation on the set of direct split decompositions of \( \gamma \). This means that \( \Omega \) and \( \Gamma \) contain the same set of subcurves, just in a different order. If every \( \gamma_i \) is self-overlapping, we call \( \Omega \) a **self-overlapping decomposition**; it may contain self-overlapping subcurves of positive and negative orientations. We now observe that the vertex set of a decomposition already determines the subcurves in the decomposition:

**Observation 1.** Given a curve \( \gamma \in \mathcal{C} \) and a subset \( S \subset V(\gamma) \) such that \( p_0 \in S \) and no two vertices in \( S \) are linked, there is a unique equivalence class \( \mathcal{E} \) of direct split decompositions with \( V(\Omega) = S \) for all \( \Omega \in \mathcal{E} \).

The observation below follows directly from the definition of winding numbers.

**Observation 2.** Let \( \Omega \) be a direct split decomposition of a curve \( \gamma \in \mathcal{C} \). Then for any face \( F \) in the plane multigraph \( G(\gamma) \), \( \text{wn}(F, \gamma) = \sum_{v_i \in \Omega} \text{wn}(F, \gamma_i) \).

We define a **loop** as a simple direct split \( \gamma_v \) of a curve \( \gamma \in \mathcal{C} \). Intersection points of \( \gamma \) may lie on \( \gamma_v \), but none occur as intersections of \( \gamma_v \) with itself. Every non-simple plane curve has a loop; e.g., the direct split \( \gamma_w \), where \( w \) is the highest index vertex on \( \gamma \) in the signed intersection sequence. A loop \( \gamma_v \) is **empty** if \( v \) links no vertex \( w \in V(\gamma) \). Let \( \text{int}(\gamma_v) \) denote its interior. We call \( \gamma_v \) an **outwards loop** if the edges \( e_1, e_4 \), that are incident on \( v \) and lie on \( \gamma \setminus \gamma_v \), both lie outside \( \text{int}(\gamma_v) \). Otherwise \( \gamma_v \) is an **inwards loop**. See Fig. 7.

The lemma below follows from [9, 10]. Since it requires a digression from our main focus, its proof is given in Appendix [A].

**Lemma 5** (Whitney Index Through Decompositions). Let \( \gamma \in \mathcal{C} \) and \( \Omega \) be a direct split decomposition of \( \gamma \). Then \( \text{WHIT}(\gamma) = \sum_{C \in \Omega} \text{WHIT}(C) \).
A consequence of Lemma 5 is that iteratively removing loops and summing $\pm 1$ for their signs allows one to quickly compute Whitney indices. Assuming $\gamma$ is given as a directed plane multigraph, one can adapt a depth-first traversal to compute such a loop decomposition of $\gamma$ in $O(|\gamma|)$ time, which yields the following corollary:

**Corollary 6 (Compute Whitney Index).** Let $\gamma \in \mathcal{C}$ be of complexity $n = |\gamma| = |V(\gamma)|$. One can compute a loop decomposition of $\gamma$, and $WHIT(\gamma)$, in $O(n)$ time.

### 3.3 Well-Behaved Homotopies

Let $H$ be a nullhomotopy of a curve $\gamma$, and consider all the points $A = \{v_i\}_{i=1}^k$ of $\mathbb{R}^2$ such that $H$ performs a $L_i$ move to contract a loop to that point. All such points are called anchor points of the homotopy $H$. Following [8] we call a homotopy $H$ well-behaved when the anchor points $A$ of $H$ satisfy $A \subseteq V(\gamma)$, i.e., $H$ only contracts loops to vertices of the original curve, not to new vertices created along the way by $H$. The theorem below from [8] shows that computing minimum homotopy area is reduced to finding an optimal self-overlapping decomposition. The homotopy $H$ guaranteed in the following theorem is well-behaved.

**Theorem 7.** [Minimum Homotopy Decompositions] Let $\gamma \in \mathcal{C}$. Then there is a self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^k$ of $\gamma$ as well as an associated minimum homotopy $H_\Omega$ of $\gamma$ such that $H_\Omega = \sum_{i=1}^k H_i$ and each $H_i$ is a nullhomotopy of $\gamma_i$. In particular, $\sigma(\gamma) = \min_{\Omega \in \mathcal{D}(\gamma)} \sum_{C \in \Omega} W(C)$, where $\mathcal{D}(\gamma)$ is the set of all self-overlapping decompositions of $\gamma$.

### 3.4 Equivalence of Interior Boundaries

In this section, we unify different definitions and characterizations of interior boundaries by showing their equivalence. We call a curve $\gamma$ a $k$-interior boundary when (1) $\text{obs}(\gamma) = 0$, (2) $WHIT(\gamma) = k > 0$, and (3) $\gamma$ is positive consistent. We call $\gamma$ a $(-k)$-interior boundary when its reversal $\tau$ is a $k$-boundary. In accordance with Titus [19], we call a curve $\zeta : [0,1] \rightarrow \mathbb{R}^2$ a Titus interior boundary if there exists a map $F : \mathbb{D}^2 \rightarrow \mathbb{R}^2$ such that $F$ is continuous, light (defined as: pre-images are totally disconnected), open, orientation-preserving, and $F|_{\partial^2} = \zeta$. The map $F$ is called properly interior.

We prove the equivalence of these definitions and of two further characterizations below.

**Theorem 8 (Equivalence of Interior Boundaries).** Let $\gamma \in \mathcal{C}$ have a positive outer basepoint $\gamma(0)$, and suppose $WHIT(\gamma) = k > 0$. Then, the following are equivalent:

1. $\gamma$ is an interior boundary.
2. $\gamma$ is a Titus interior boundary.
3. $\gamma$ admits a self-overlapping decomposition $\Omega = (\gamma_i)_{i=1}^k$, where each $\gamma_i$ is positive self-overlapping.
4. $\gamma$ admits a well-behaved left sense-preserving nullhomotopy $H$ with exactly $k$ $L_i$-moves.

**Proof.** $1 \Rightarrow 2$. We proceed by induction on $k$. If $k = 1$, let $\gamma_1 = \gamma$. The proof of this follows from [19], as well as [1]. Let $k \geq 1$ and let our inductive assumption be that for all $c \in \{1,2,\ldots,k\}$, if $\gamma \in \mathcal{C}$, $WHIT(\gamma) = +c$, and $\gamma$ is an interior boundary, then $\gamma$ is a Titus interior boundary. Now, let $\gamma \in \mathcal{C}$ such that

![Figure 7: An outwards loop (left) and an inwards loop (right).](image)
\[ \text{WHIT}(\gamma) = + (k+1), \] and \( \gamma \) is an interior boundary. Since \( \text{WHIT}(\gamma) \geq 2 \), we can find a vertex \( v \) of \( \gamma \) such that the two direct splits at \( v, \gamma_1 \) and \( \gamma_2 \), have positive Whitney indices \( k_1 \) and \( k_2 \), respectively. Necessarily, \( k_1 \leq k \) and \( k_2 \leq k \), by Lemma [5]. By our inductive assumption, there exist properly interior maps \( F_1, F_2 : D^2 \to \mathbb{R}^2 \) such that \( F_1|_{\partial D^2} = \gamma_2 \) and \( F_1|_{\partial D^2} = \gamma_2 \). Finally, glue these two curves together at \( v \) by finding an arc interior to both. (We refer the reader to Titus’ paper [13] to see a comprehensive explanation of his trick of gluing two properly interior mappings together along an interior arc.) The resulting map \( F : D^2 \# D^2 \to \mathbb{R}^2 \), where \# denotes the connected sum, extends both \( F_1 \) and \( F_2 \) and represents the curve \( \gamma \). Moreover, \( F \) is a properly interior map. We conclude that \( 2 \Rightarrow 1 \).

Let \( \gamma \) be a Titus boundary interior. By [13] Lemma [1], we know that \( \gamma \) is consistent. Furthermore, by [19], we have \( |\gamma^{-1}(x)| = \text{wn}(x, \gamma) \) for all \( x \). Thus, the linear retraction of the disk \( D^2 \) induces a homotopy \( H \) with \( A(H) = \text{W}(\gamma) \). By Lemma [1] we have \( \sigma(\zeta) = \text{W}(\gamma) \). Thus, \( \gamma \) is an interior boundary, and so \( 2 \Rightarrow 1 \).

Let \( \gamma \) be an interior boundary. By Theorem [9], we have an optimal self-overlapping decomposition \( \Omega = (\gamma_i)_{i=1}^{k} \) of \( \gamma \). Suppose, by contradiction, that there exists an \( i \leq j \) such that \( \gamma_i \) is negative self-overlapping. Let \( F \) be any face contained in the interior \( \text{int}(\gamma_i) \). We know by Observation [2] that \( \text{wn}(F, \gamma) = \sum_{i=1}^{j} \text{wn}(F, \gamma_i) \), and since \( \gamma \) is positive consistent \( \text{wn}(F, \gamma) \geq 0 \). Thus there must exist a positive self-overlapping curve \( \gamma_i \in \Omega \) with \( F \subseteq \text{int}(\gamma_i) \). Consider the minimal homotopies \( H_i \) and \( H_1 \) that are part of the canonical optimal homotopy \( H_\Omega \). Then \( H_i \) contracts \( \gamma_i \) and is right sense-preserving, while \( H_1 \) contracts \( H_1 \) and is left sense-preserving. Thus by Lemma [3], \( H_i \) increases the winding number on \( F \) and \( H_1 \) decreases the winding number, which means \( F \) is swept more than \( W(F) \) times, a contradiction. Thus, no negative self-overlapping subcurve \( \gamma_i \) may exist in \( \Omega \). Since \( \text{WHIT}(C) = 1 \) for any positive self-overlapping subcurve and \( \text{WHIT}(\gamma) = \sum_{i=1}^{k} \text{WHIT}(\gamma_i) \) by Lemma [5], we must have \( k = j \).

Suppose \( \gamma \) has a decomposition \( \Omega \) into \( k \) positive self-overlapping subcurves. Then, the canonical homotopy \( H_\Omega \) associated to \( \Omega \) is left sense-preserving, since each minimum nullhomotopy of the subcurves is left sense-preserving. Since sense-preserving homotopies are optimal, see Lemma [4], we have \( \sigma(\gamma) = \text{W}(\gamma) \). For any self-overlapping decomposition \( \Omega = (\gamma_i)_{i=1}^{k} \), we may conclude that \( \text{wn}(x, \gamma) = \sum_{i=1}^{k} \text{wn}(x, \gamma_i) \) by Observation [2]. Thus, \( \text{wn}(x, \gamma) = \sum_{i=1}^{k} \text{wn}(x, \gamma_i) \geq 0 \) since each \( \gamma_i \) is positive consistent, as a positive self-overlapping curve.

If \( \gamma \) has a well-behaved left sense-preserving nullhomotopy \( H \) with exactly \( k \) \( L \)-moves, then \( H \) comes naturally with an associated self-overlapping decomposition \( \Omega \) of \( \gamma \) with \( |\Omega| = k \), and \( \text{WHIT}(\gamma) = k > 0 \) by Lemma [5]. We now show that \( \sigma(\gamma) = \text{W}(\gamma) \). Consider the reversal \( \overline{H} \) from the constant curve \( \gamma_{p_0}(t) = p_0 \) to \( \gamma \). Then, \( \overline{H} \) is right sense-preserving and by Lemma [3], the function \( a(t) = \text{wn}(x, \gamma_{p_0}(t)) \) is monotonically increasing for any \( x \in \mathbb{R}^2 \). Since \( \text{wn}(x, \gamma_{p_0}) = 0 \) for all \( x \in \mathbb{R}^2 \), we have that \( \text{wn}(x, \gamma) \geq 0 \) for all \( x \in \mathbb{R}^2 \). Thus, \( \gamma \) is an interior boundary. Conversely, if \( \gamma \) is a positive interior boundary, then \( \text{obs}(\gamma) = 0 \) and by Lemma [4] and since \( \gamma \) is positive, \( H \) is left sense-preserving. Again, by Lemma [5], \( \text{WHIT}(\gamma) = j \), where \( j \) is the number of \( L \)-moves in any well-behaved nullhomotopy \( H \) of \( \gamma \). Hence, we must have \( j = k \), as desired.

### 3.5 Equivalences of Self-Overlapping Curves

In this section, we study different characterizations of self-overlapping curves and show their equivalence in Theorem [9] which also shows that self-overlapping curves are 1-interior boundaries.

First, we describe a geometric formulation of self-overlappingness, inspired by the work of Blank and Marx [11, 13]. Let \( \gamma \in \mathcal{C} \) be self-overlapping. Let \( P : [0,1] \to \mathbb{R}^2 \) be a simple path so that \( P(0) = q = \gamma(t_q) \) and \( P(1) = p = \gamma(t_p) \) lie on \( [\gamma] \) but are not vertices of \( \gamma \). Without loss of generality, assume \( t_q < t_p \). Let \( \tilde{P} := \gamma|_{[t_q,t_p]} \), and suppose that (1) \( P \cap \tilde{P} = \{p,q\} \), (2) \( C = \tilde{P} \ast P \) is a simple closed curve, and (3) \( C \) is positively oriented; see Fig. [8] as well as Fig. [11]. Then we call \( P \) a \textbf{Blank cut} of \( \gamma \). By cutting along \( P \), \( \gamma \) is split into two curves of strictly smaller complexity, \( \gamma_1 \) and \( C \). We call a sequence \( (P_i)_{i=1}^{\infty} \) of Blank cuts a \textbf{Blank cut decomposition} if the final curve is a simple positively oriented curve.

**Theorem 9** (Equivalent Characterizations of Self-Overlapping Curves). Let \( \gamma \in \mathcal{C} \). Then the following are equivalent:
1. (Analysis) There is an immersion $F : D^2 \to \mathbb{R}^2$ so that $F|_{\partial D^2} = \gamma$.

2. (Geometry) $\gamma$ admits a Blank cut decomposition.

3. (Geometry/Topology) $\gamma$ is a 1-interior boundary, i.e., self-overlapping.

4. (Topology) $\gamma$ admits a left-sense preserving nullhomotopy $H$ with exactly one $I_a$-move.

5. (Analysis) $\gamma$ is a Titus interior boundary with $\text{WHIT}(\gamma) = 1$.

Proof. By property 3 in Theorem 8 self-overlapping curves are 1-interior boundaries, since any self-overlapping curve $\gamma$ has the trivial self-overlapping decomposition $\Omega = (\gamma)$. Thus, we have already established $\exists \Leftrightarrow 4 \Leftrightarrow 5$ in Theorem 8. We now prove $2 \Leftrightarrow 4$. Any Blank cut $P$ can be performed by a left sense-preserving homotopy that deforms $\bar{P}$ to $P$. Hence the Blank cut decomposition corresponds to a left sense-preserving homotopy to a simple positively oriented curve. Finally we perform a single $I_a$-move to complete a left sense-preserving nullhomotopy of $\gamma$. Conversely, let $\gamma$ have a left sense-preserving nullhomotopy $H$. From 3 to 4 we know that every intermediary curve $\gamma_i = H(i, \cdot)$ is self-overlapping since the subhomotopy $H_i = H|_{[i,1] \times [0,1]}$ is a left sense-preserving nullhomotopy of $\gamma_i$ with one $I_a$ move. As $H$ ends with a $I_a$ move, we may select a subhomotopy $H'$ such that $\gamma \overset{H'}{\approx} C$, where $C$ is a simple self-overlapping curve. Moreover, we see that $H = H' + H''$, where the unique $I_a$-move of $H'$ occurs during $H''$. Thus, $H'$ is regular, i.e., consists of a sequence of homotopy moves only of types $I_a$, $I_b$, or $I_3$, which deform $\gamma$ to $C$. Each of these homotopy moves can be performed by a Blank cut, as shown in Fig. 9. Since all of the intermediary curves are self-overlapping, this induces a Blank cut decomposition.

The following two lemmas provide useful properties of self-overlapping curves, the first of which was proved in [19, Theorem 5].

**Lemma 10 (Empty Positively Oriented Loop).** Let $\gamma \in \mathcal{C}$ have a positive outer basepoint and an empty positively oriented loop. Then, $\gamma$ is not self-overlapping.

We conclude this section with a simple yet powerful lemma.

**Lemma 11 (Sense-Preserving Homotopies).** Let $H$ be a regular homotopy with $\gamma \overset{H}{\approx} \gamma'$. Let $\gamma$ be right sense-preserving and $\gamma$ is self-overlapping, then $\gamma'$ is self-overlapping.

2. If $H$ is left sense-preserving and $\gamma$ is not self-overlapping, then $\gamma'$ is not self-overlapping.

Proof. To prove 1 assume $\gamma$ is self-overlapping. Then it has a left sense-preserving nullhomotopy $\gamma'$ by Theorem 9. Let us reverse our given homotopy $H$ to obtain $\overline{H}$ by $\overline{H}(s,t) = H(1-s,t)$. Then we note that the concatenation $H'' = \overline{H} + H'$ is a left sense-preserving nullhomotopy for $\gamma'$. Since sense-preserving
Figure 9: Homotopy moves II_a, II_b, and III each correspond to a Blank cut (shown in blue).

homotopies are optimal, \( \sigma(\gamma') = W(\gamma') \). Also, as \( H'' \) is regular, \( W(\gamma') = W(\gamma) = 1 \). Applying Theorem 9 again, we conclude that \( \gamma' \) is self-overlapping. Part 2 follows by contrapositive with a single application of 1 if \( \gamma' \) were self-overlapping then \( \gamma \) must be self-overlapping as well.

3.6 Zero Obstinance Curves

In this section, we classify curves \( \gamma \in \mathcal{C} \) with zero obstinance, \( \text{obs}(\gamma) := \sigma(\gamma) - W(\gamma) = 0 \). See Figures 2 and 10 for examples of zero-obstinance curves. We show that just as interior boundaries can be decomposed into self-overlapping curves, so too can zero-obstinance curves be decomposed into interior boundaries.

Figure 10: A zero obstinance curve, with its minimum homotopy decomposition, and winding numbers shown. Each curve in the decomposition is self-overlapping and shown in a different color. The vertices with labels 1, 2, 5, 8, 9 are sign-changing.

If a curve \( \gamma \) has zero obstinance, then there is a nullhomotopy \( H \) which sweeps each face \( F \) on \( G(\gamma) \) exactly \( wn(F, \gamma) \) times. Note that such a homotopy \( H \) is necessarily minimal by Lemma 1. Intuitively, this implies that the homotopy \( H \) should be locally sense-preserving. We expect it to sweep leftwards on positive consistent regions and rightwards on negative consistent regions. Hence, we might expect regions of the curve where the winding numbers change from positive to negative to be especially problematic.
Indeed, let \( v \in V(\gamma) \) be incident to the faces \( \{F_1, F_2, F_3, F_4\} \). We call \( v \) **sign-changing** when, as a multiset, \( \{wn(\gamma, F_1), wn(\gamma, F_2), wn(\gamma, F_3), wn(\gamma, F_4)\} = \{-1, 0, 0, 1\} \); see Figures 10 and 11.

![Image of a sign-changing vertex](image.png)

Figure 11: A sign-changing vertex \( v \). The winding numbers of the faces incident to \( v \), are up to cyclic reordering, -1, 0, 1, 0.

**Theorem 12** (Zero Obstinance Characterization). Let \( \gamma \in \mathcal{C} \) and let \( \mathcal{F} \) be the sign-changing vertices of \( \gamma \). Then \( \text{obs}(\gamma) = 0 \) if and only if no two vertices in \( \mathcal{F} \) are linked and any direct split subcurve decomposition \( \Omega \) with vertex set \( V(\Omega) = \mathcal{F} \cup \{p_0\} \) contains only interior boundaries.

**Proof.** Suppose \( \text{obs}(\gamma) = 0 \). By definition, any zero obstinance curve with consistent winding numbers must be an interior boundary and \( \mathcal{F} = \emptyset \). Hence, suppose \( \gamma \) is inconsistent so that \( \mathcal{F} \neq \emptyset \). We claim any sign-changing vertex \( v \) is anchor point of every well-behaved minimum homotopy \( H \) of \( \gamma \) of the form guaranteed by Theorem 7. Let us now proceed by contradiction. Suppose \( v \in V(\gamma) \) is a sign-changing vertex with incident faces labeled as in Fig. 11 such that \( v \) is not an anchor point of a minimum homotopy \( H \) for \( \gamma \). Write \( \Gamma(H) \) as the self-overlapping decomposition of \( H \). As \( \gamma \) has \( \text{obs}(\gamma) = 0 \), we know that \( W(\gamma) = \sigma(\gamma) = A(H) \).

In particular, the homotopy \( H \) sweeps each face \( F \in G(\gamma) \) precisely \( wn(\gamma, F) \) times. Of course, since our homotopy \( H \) consists of a sequence of nullhomotopies of self-overlapping subcurves, this means each face \( F \) must lie in the interior of \( wn(\gamma, F) \) distinct self-overlapping subcurves \( C \in \Gamma(H) \). In particular, if either face \( F_2, F_4 \) incident to \( v \) is contained in the interior of any curve \( C \in \Gamma(H) \), we have a contradiction. Let us now examine the edge \( e = (v, w_3) \). This edge must lay on precisely one subcurve \( C \in \Gamma(H) \) by our definition of a direct split subcurve decomposition. We now have two cases.

**Case 1:** \( C \) is positively self-overlapping. We now recall that for a positive self-overlapping curve, the interior of the curve always lies, locally at each edge, to the left. Since \( v \) is not an anchor point of \( H \), it must be the case that \( C \) also contains the edge \( e_1 = (w_1, v) \). As the face \( F_2 \) lies to the left of \( e_1 \), this implies \( F_2 \subset \text{int} \ C \).

**Case 2:** \( C \) is negative self-overlapping. Here, we use that the interior of a negative self-overlapping curve lies locally to the right. In this case, we see that \( F_4, \) lying to the right of edge \( e_1 \), satisfies \( F_4 \subset \text{int} \ C \).

We conclude that all sign-changing vertices are anchor points of \( H \). This is only possible if none of the sign-changing vertices link each other. Now, let \( \Theta = (\gamma_i)_{i=1}^k \) be any direct split subcurve decomposition with \( V(\Theta) = \mathcal{F} \cup \{p_0\} \). We claim that each curve \( \gamma_i \in \Theta \) is an interior boundary. It suffices to prove this claim for any decomposition \( \Omega \sim \Theta \). Hence, we may select the unique representative \( \Omega \) from the equivalence class of \( \Theta \) such that the ordering of \( \Omega \) is compatible with the ordering of \( \Gamma(\gamma) \). Since each sign-changing vertex is an anchor point of \( H \), it follows that \( \Gamma(H) \) is a refinement of the decomposition \( \Omega \). Thus, by Theorem 7 there is a subhomotopy \( H_i \) of \( H \) which is a nullhomotopy of \( \gamma_i \). As subhomotopies of a minimum homotopy \( H \), each \( H_i \) must be minimum as well, \( A(H_i) = \sigma(\gamma_i) \). Observe that

\[
\sum_{i=1}^k W(\gamma_i) = W(\gamma) = A(H) = \sum_{i=1}^k A(H_i) = \sum_{i=1}^k \sigma(\gamma_i).
\]

So, we must have equality, \( \sigma(\gamma_i) = W(\gamma_i) \), for every curve in the decomposition. By Lemma 4, this means each \( \gamma_i \in \Omega \) has \( \text{obs}(\gamma_i) = 0 \). Hence, if each \( \gamma_i \) is consistent, they are all interior boundaries, by definition. Of course, if some \( \gamma_i \) were inconsistent, then the winding numbers would change somewhere along the curve. Wherever the winding numbers of \( \gamma_i \) change, we will see a sign-changing vertex \( u \in V(\gamma_i) \). But since \( u \) is
not the basepoint of \( \gamma_i \), this is a contradiction. Indeed, by the fact that \( \mathcal{S} \cup \{p_0\} = V(\Omega) \), no sign-changing vertex can be a crossing point on a curve \( \gamma_i \in \Omega \).

Conversely, suppose no sign-changing vertices link each other and that each decomposition \( \Omega = (\gamma_i)_{i=1}^k \) of \( \gamma \) with vertex set \( V(\Omega) = \mathcal{S} \cup \{p_0\} \) contains only interior boundaries. Then let \( H_\Omega \) be the homotopy associated to \( \Omega \). Thus, we have \( W(\gamma) = \sum_{i=1}^k W(\gamma_i) = \sum_{i=1}^k \sigma(\gamma_i) = A(H) \). We conclude \( H \) is optimal and \( \sigma(\gamma) = W(\gamma) \). Thus, \( \text{obs}(\gamma) = 0 \).

4 Wraps and Irreducability

In this section, we show (Theorems 14 and 15) that wrapping around a curve \( \gamma \) until its obstinance is reduced to zero results in an interior boundary. This key result is used to prove sufficient combinatorial conditions for a curve to be self-overlapping based on the Whitney index of the curve and its direct splits (Theorems 17 and 19).

4.1 Wraps

Let us now define the construction of the wrap of a curve. Let \( \gamma \in \mathcal{C} \), and let \( I \) be its signed intersection sequence. Form \( I' \) by incrementing each label by one and removing the occurrences of 0 corresponding to the basepoint. If \( \gamma \) has a positive outer basepoint \( \gamma(0) \), then its (positive) wrap \( \text{Wr}_+^{(\gamma)} \) is the unique (class) of curves with signed intersection sequence \( 0, 1+, I', 1-, 0 \). This corresponds to gluing a simple positively oriented curve \( \alpha \) to \( \gamma \) at \( \gamma(0) \), where the interior \( \text{int}(\alpha) \supseteq [\gamma] \); the new basepoint \( p_0 = \text{Wr}_+^{(\gamma)}(0) \) is on \( \alpha \). See Fig. 12.

The negative wrap \( \text{Wr}_-^{(\gamma)} \) is defined analogously if \( \gamma \) has a negative outer basepoint. We write \( \text{Wr}^k_+^{(\gamma)} \) for the curve achieved from \( \gamma \) by wrapping \( k \) times.

![Figure 12](image-url)

To wrap a curve in the direction opposed to the sign of the basepoint, we must be more careful. Without loss of generality, we describe the construction of \( \text{Wr}_-^{(\gamma)} \) when \( \gamma \) has a positive outer basepoint. Perform a \( I_b \)-move to add a simple loop \( \hat{\gamma} \) of the opposite orientation tangent to the basepoint \( \gamma(0) \). Let \( \gamma' \) be the curve after the \( I_b \)-move, with a basepoint chosen to lie on \( \hat{\gamma} \). We then define \( \text{Wr}_-^{(\gamma)} = \text{Wr}_+^{(\gamma')} \). See Fig. 13.

Clearly one can always wrap any curve \( \gamma \in \mathcal{C} \) a sufficient number of times to make \( \text{Wr}^k_+^{(\gamma)} \) positive consistent. Indeed, setting \( k \) to be the maximum depth across all faces in \( G(\gamma) \) suffices. On the other hand, it is not at all obvious that wrapping always turns a curve into an interior boundary. We prove in Theorem 14 that, in fact, positively wrapping always transforms a curve \( \gamma \in \mathcal{C} \) with positive outer basepoint into a positive interior boundary. Thus one can think of wrapping as a rectifying operation with respect to minimum homotopy, as it always eventually removes all obstinance.
Figure 13: A curve $\gamma$ with positive outer basepoint and its transformation into its negative wrap $Wr_-(\gamma)$. First, we perform an $I_b$-move and then wrap normally on $\gamma'$.

### 4.2 Simple Path Decompositions

We now describe another type of decomposition for $\gamma \in \mathcal{C}$ that we will need for proving Theorem 15. First we prove a simple lemma which states that a curve $\gamma \in \mathcal{C}$ with an outer basepoint has an outwards loop.

**Lemma 13 (Existence of an Outwards Loop).** Let $\gamma \in \mathcal{C}$ have an outer basepoint. Then if $\gamma$ is non-simple, it has an outwards loop.

**Proof.** Let $v$ be the first self-intersection of $\gamma$. Then $\gamma_v$ is a loop. Write $\gamma^{-1}(v) = \{t, t^*\}$, where $t < t^*$. Since $\gamma(0)$ lies outside of int($\gamma_v$), as an outer basepoint, we note that if $\gamma_v$ were inwards, the path $P = \gamma_{[0, t]}$ would cross $[\gamma_v]$ to get from outside the simple curve to inside it. This is then a contradiction, for if the crossing occurred at a point $q$ on $[\gamma_v]$, then $q$ would be the first self-intersection of $\gamma$. Indeed, we would reach $q$ a second time before we reach $v$ a second time. Thus, $\gamma_v$ is outwards. □

Now, let $t_1 < t_1^* \in [0, 1]$ be the smallest value so that $\gamma_1 = \gamma_{[t_1, t_1^*]}$ is a loop. We call this the first loop of $\gamma$. Since $t_1^*$ is the first time that $\gamma$ self-intersects, we know that such a loop is outwards by the argument in Lemma 13. The sub-intersection sequence from $r = \gamma(0)$ until the second occurrence of $p_1 = \gamma(t_1) = \gamma(t_1^*)$ is of the following form: $r \cdots p_1 \cdots p_1$, where no intersection point occurs twice before the second occurrence of $p_1$. Let $P_1$ be the path from $r$ until the first occurrence of $p_1$, i.e., $\gamma_{[0, t_1]} = P_1 \ast \gamma_1$. We continue by recursively defining

$$t_i^* = \sup \{t \in [0, 1] \mid t > t_{i-1}^* \text{ and } \gamma_{[t_{i-1}^*, t]} \text{ is injective} \},$$

for $i > 1$. Then $\gamma(t_i^*) = p_i$ must be a basepoint of a loop and there exists a $t_i < t_i^*$ such that $p_i = \gamma(t_i) = \gamma(t_i^*)$. We define $P_i = \gamma_{[t_{i-1}^*, t_i]}$ and $\gamma_i = \gamma_{[t_i, t_i^*]}$. Let $k$ be the largest value such that $t_k^* < 1$, and set $P_{k+1} = \gamma_{[t_k^*, 1]}$. Then $\gamma = P_1 \ast \gamma_1 \ast \cdots \ast P_k \ast \gamma_k \ast P_{k+1}$, where $\ast$ denotes concatenation. The simple path decomposition of $\gamma$ is the sequence $(P_i, \gamma_i)_{i=1}^k$; see Fig. 14.

### 4.3 Wrapping Resolves Obstinance

We are now equipped to prove our second main result on wraps.

**Theorem 14 (Wrapping Resolves Obstinance).** Let $\gamma \in \mathcal{C}$ have positive outer basepoint. Then there is a positive integer $k$ so that $\text{obs}(Wr_k^+(\gamma)) = 0$. Moreover, $Wr_k^+(\gamma)$ is a positive interior boundary.
Proof. Let \( l \) be the number of negative vertices in \( V(\gamma) \). Set \( k = l + 1 \). We claim that \( \text{Wr}_k^+(\gamma) \) is an interior boundary. We will show this by iteratively constructing a left sense-preserving nullhomotopy \( H \) for \( \gamma \). By property 4 of Theorem 8 it then follows that \( \gamma \) is a positive interior boundary and \( \text{obs}(\gamma) = 0 \).

We first introduce a trick that we call balanced loop deletion. See Fig. 15 where all of the following objects are shown. Suppose that \( C \in \mathcal{C} \) is a curve that is positively wrapped, \( C = \text{Wr}_{T+}(C') \) for some curve \( C' \in \mathcal{C} \), and suppose that the first loop \( \gamma_- \) (shown in red) in the simple path decomposition of \( C \) is negatively oriented. Let \( b = C(t_b) = C(t^*_b) \), with \( t_b < t^*_b \), be the basepoint of \( \gamma_- \). Balanced loop deletion performs a left sense-preserving homotopy \( H \) so that \( C \overset{H}{\approx} C \setminus (\alpha \cup \gamma_-) \), where \( \alpha \) (shown in purple) is the positive outer wrap on \( C \).

Let \( P \) (shown in blue) be the simple subpath of \( C \) from \( a = C(t_a) = C(t^*_a) \) to \( b \), where \( a \) is the unique outer intersection point on \([C]\), i.e., the basepoint of the wrap \( \alpha \), and \( t_a < t^*_a \). For \( \varepsilon > 0 \) sufficiently small, let \( a' = C(t^*_a + \varepsilon) \) and \( b' = C(t^*_b - \varepsilon) \) and let \( P_\varepsilon \) (shown in dashed green) be a simple path between \( a' \) and \( b' \) that is \( \varepsilon \)-close to \( P \) in the Hausdorff distance. Let \( \tilde{P} = C\left[t^*_a + \varepsilon, t^*_b - \varepsilon\right] \) (shown in thick beige) be the simple subpath of \( C \) from \( a' \) to \( b' \). Then \( \tilde{P} \) is the concatenation of (i) the path from \( a' \) to \( a \) along \( \alpha \), (ii) the path \( P \) from \( a \) to \( b \), and (iii) the path from \( b \) to \( b' \) along \( \gamma_- \). The path \( \tilde{P} \) is simple because each of these subpaths are simple and none of them intersect each other since \( b \) is the first self-intersection point of the curve. We
now make a crucial observation: $\tilde{P} * P_\epsilon$ is a simple, positively oriented, closed curve. It follows that we can perform a Blank cut along $P_\epsilon$ that replaces $\tilde{P}$ on $C$ with the path $P_\epsilon$. The effect of this cut on $C$ is that both the outer wrap $\alpha$ and the negatively oriented loop $\gamma_-$ are deleted, and the path $P$ is replaced by $P_\epsilon$. This Blank cut can be performed by a left sense-preserving homotopy, so we have established the existence of left sense-preserving balanced loop deletion.

Now we construct a left sense-preserving nullhomotopy $H$ of $\text{Wr}^k_-(\gamma)$ by iteratively concatenating several left sense-preserving subhomotopies, so $H = \sum_i H_i$. We proceed inductively as follows. Suppose $H_1, \ldots, H_{i-1}$ have been defined and $\gamma_i$ is the current curve. Consider the first loop $C_i$ in the simple path decomposition of $\gamma_i$. If $C_i$ is positively oriented we let $H_i$ be the left sense-preserving nullhomotopy that contracts this loop. Otherwise $C_i$ is negatively oriented and we let $H_i$ be the homotopy performing balanced loop deletion.

We claim that we always have a wrap available to perform this balanced loop deletion. Each homotopy $H_j$ for $j = 1 \ldots i-1$ deletes one or two free subcurves of $\gamma_j$. Therefore the signs of the remaining intersection points are not affected. Observe that if a vertex $v$ is the basepoint of an outwards loop $\gamma_v$, then $\text{sgn}(v) = 1$ iff $\gamma_v$ is positively oriented and $\text{sgn}(v) = -1$ iff $\gamma_v$ is negatively oriented; see Fig. 16. By definition of $l$, we have $n_i \leq l$, where $n_i$ is the number of negative vertices on $\gamma_i$. Therefore there can be at most $l$ distinct integers $i_1, \ldots, i_l$ such that the first loop on $\gamma_{i_l}$ is negatively oriented, by Lemma 13 the first loop is always outwards. Since $k = l + 1$, we always have a wrap available on $\text{Wr}^k_+(\gamma)$.

The process of constructing homotopies $H_i$ never gets stuck, and $|\gamma_{i+1}| < |\gamma_i|$. Therefore we must eventually reach a point when the current curve $\gamma_m$ has $|\gamma_m| = 0$. We now show that this final curve $\gamma_m$ is positively oriented. Note that if $\gamma$ is simple then $\text{Wr}^k_+(\gamma)$ trivially is a positive $k$-interior boundary. So, assume $\gamma$ is not simple.

By the definition of wraps, the intersection sequence of $\text{Wr}^k_+(\gamma)$ has the form $0, 1+, 2+, \ldots, k+, I', k-, \ldots, 1-, 0$, where $I'$ is obtained from the signed intersection sequence of $\gamma$ by incrementing each label by $k$ and removing occurrences of $0$. The basepoint of the first loop on $\gamma_1 = \text{Wr}^k_+(\gamma)$ must be a vertex from $\gamma$. Then $H_1$ modifies the intersection sequence by removing two labels from $I'$ corresponding to this loop, and if the loop is negatively oriented then balanced loop deletion also removes a pair $a_+ \ldots a_-$ for the wrap. The same modification happens for each homotopy $H_i$ until $I'$ is empty, and the homotopies after that contract wraps $a_+ \ldots a_-$ which are all positively oriented loops. We know that $\gamma$ has at most $l = k - 1$ negative vertices, hence there can only be $k-1$ balanced loop deletions, but there are $k$ wraps. Thus, $\gamma_m$ must be a wrap that $\text{Wr}^k_+(\gamma)$ added to $\gamma$, so $\gamma_m$ is a positively oriented loop which can be contracted to its basepoint using a final left sense-preserving homotopy $H_m$. This shows that $H = \sum_{i=1}^m H_i$ is a left sense-preserving nullhomotopy of $\text{Wr}^k_+(\gamma)$ as desired.

The example in Fig. 17 shows that the number of wraps used in Theorem 14 is nearly tight. We now show that wrapping resolves obstinance in either direction of wrapping.

**Theorem 15** (Wrapping Resolves Obstinance (General)). Let $\gamma \in \mathcal{C}$ with outer basepoint and set $n = |\gamma|$. Then there are constants $k_-, k_+ \leq n + 1$ so that $\text{obs}(\text{Wr}^{k_-}_-(\gamma)) = \text{obs}(\text{Wr}^{k_+}_+(\gamma)) = 0$, $\text{Wr}^{k_-}_-(\gamma)$ is a negative interior boundary, and $\text{Wr}^{k_+}_+(\gamma)$ is a positive interior boundary.
Figure 17: A curve $\gamma$ that requires $k = l = 5$ wraps to resolve obstinance, where $l$ is the number of negative vertices in $V(\gamma)$.

Proof. If $\gamma(0)$ is a positive basepoint, then $k_+$ exists directly by Theorem 14. To prove the existence of $k_-$, consider the intermediary curve $\gamma'$ obtained from $\gamma$ by performing a $I_b$-move on the outer edge to create a negatively oriented loop that is entirely outer to the curve. Additionally, let us set the basepoint on $\gamma'$ on the new negatively oriented loop $\gamma_-$. Then $\gamma'$ has a negative outer basepoint. Here is the key observation: $WR^5(\gamma) \sim = WR_k^-(\gamma)$. Thus, by the existence of $\tilde{k}_+$ for $\gamma'$, we have $k_-$ for $\gamma$.

If $\gamma(0)$ is a negative outer basepoint, then since $k_-$ and $k_+$ exist for the reversal $\gamma$, we are done, as $WR^k(\gamma) \sim = WR^k_+(\gamma)$ and $WR^k_-(\gamma) \sim = WR^k_-(\gamma)$.

Let us make a simple observation: once $WR^k(\gamma)$ is an interior boundary, so too is $WR^j(\gamma)$ for any integer $j \geq k$. This holds because we can simply add the extra $j - k$ wraps to the self-overlapping decomposition $\Omega$ of $WR^k(\gamma)$. Consequently, Theorem 15 implies that interior boundaries are the equilibrium point for plane curves with respect to the action of wrapping. No matter where we begin, we will always eventually land and stick within the set of interior boundaries.

4.4 Irreducible and Strongly Irreducible Curves

We are now ready to apply Theorem 15 to prove sufficient combinatorial conditions for a curve $\gamma$ to be self-overlapping based on $WHIT(\gamma)$ and properties of its direct splits. If $\gamma \in C$ has no proper positive self-overlapping direct splits, we call $\gamma$ irreducible. A special case of irreducibility is of particular interest to us: If $WHIT(\gamma_v) \leq 0$ for all proper direct splits, we call $\gamma$ strongly irreducible. See Fig. 4, Fig. 6, and $\gamma_{SO}$ in Fig. 23 for examples of strongly irreducible curves. Note that a strongly irreducible curve is irreducible since any positive self-overlapping curve $\gamma$ has $WHIT(\gamma) = 1$.

We need one simple lemma before proving irreducible curves are self-overlapping.

Lemma 16 (Existence of a Direct Split). Let $\gamma \in C$ and $\Omega$ be a free subcurve decomposition of $\gamma$, with $|\Omega| \geq 2$. Then $\Omega$ contains a proper direct split.

Proof. A leaf $v_i$ in the tree $T_\Omega$ necessarily corresponds to the basepoint of a direct split $\gamma_i$ in the decomposition $\Omega$. Since $|\Omega| \geq 2$, this direct split $\gamma_i$ must be proper.

Theorem 17 (Irreducible Curves are Self-Overlapping). Assume $\gamma$ has $WHIT(\gamma) = 1$ and positive outer basepoint. If $\gamma$ is irreducible, then it is self-overlapping.

Proof. Apply Theorem 14 to find a $k \in \mathbb{Z}$ such that $WR^k_+(\gamma)$ is a positive interior boundary. We know from property 3 of Theorem 8 that there is a self-overlapping decomposition $\Omega$ of $WR^k_+(\gamma)$ into positive self-overlapping subcurves. By Lemma 16, we know that $\Omega$ must have a self-overlapping direct split of $WR^k_+(\gamma)$, and we will show that $\gamma$ is the only direct split of $WR^k_+(\gamma)$ that can be self-overlapping.
Let \( w_i \) be the vertex created by the \( i \)th wrap. The intersection sequence of \( Wr^k_+(\gamma) \) therefore has the prefix \( w_k, w_{k-1}, \ldots, w_1 \). Then the direct split \( Wr^k_+(\gamma)_{w_i} \) at \( w_i \) on \( Wr^k_+(\gamma) \) has \( \text{WHIT}(Wr^k_+(\gamma)_{w_i}) = 1 + (i - 1) = i \) by Lemma 5 and is therefore not self-overlapping for \( i \geq 2 \). And any direct split \( Wr^k_+(\gamma) \) at a vertex of \( \gamma \) which is also a proper direct split on \( \gamma \) cannot be self-overlapping since \( \gamma \) is irreducible. Note that by our notation \( w_1 \) is the vertex corresponding to the original basepoint \( \gamma(0) \). And this is the only vertex at which the direct split \( Wr^k_+(\gamma)_{w_1} = \gamma \) could potentially be self-overlapping. Thus, it follows with Lemma 16 that \( \gamma \) is self-overlapping.

**Corollary 18.** Let \( \gamma \) have \( \text{WHIT}(\gamma) = 1 \) and positive outer basepoint. Then if \( \gamma \) is not self-overlapping, it has a positive self-overlapping direct split \( \gamma_v \).

We now have a nice corollary: conditions on the Whitney indices of a curve and its subcurves alone can be sufficient for self-overlappingness.

**Theorem 19 (Strongly Irreducible Curves are Self-Overlapping).** Assume \( \gamma \) has \( \text{WHIT}(\gamma) = 1 \) and positive outer basepoint. If \( \gamma \) is strongly irreducible, then it is self-overlapping.

Note that strongly irreducible curves are a proper subset of irreducible curves, see \( \gamma_1 \) in Fig. 2. And Theorem 19 is false without the basepoint assumption, see Fig. 18.

Figure 18: This curve \( \gamma \) does not have an outer basepoint. It is not self-overlapping, yet \( \gamma \) is strongly irreducible due to the empty positively oriented loop on the indirect split \( \gamma_1 \).

One can decide whether a piecewise linear curve \( \gamma \) is (strongly) irreducible by checking the required condition for each direct split. Let \( N \) be the number of line segments of \( \gamma \) and \( n = |\gamma| = |V(\gamma)| \in O(N^2) \). Then irreducibility can be tested in \( O(nN^3) \) time, using Shor and Van Wyk’s algorithm to test for self-overlappingness in \( O(N^3) \) time [17]. Strong irreducibility can be decided in \( O(n^2) \) time by applying Corollary 6 to each direct split of \( \gamma \).

### 4.5 Global Balanced Loop Insertion

We now introduce an operation called balanced loop insertion, which is complementary to the balanced loop deletion applied in the proof of Theorem 14. We show in Theorem 21 that any curve \( \gamma \) with positive outer basepoint and \( \text{WHIT}(\gamma) = 1 \) can be transformed into a self-overlapping curve, more specifically, a strongly irreducible self-overlapping curve, through a sequence of balanced loop insertions. This result is a nice parallel to Theorem 14.

Let \( \gamma \in \mathcal{C} \) have a positive outer basepoint. Then given any edge \( e \) from \( G(\gamma) \), we define balanced loop insertion on \( \gamma \) with respect to \( e \) as follows: First, perform a I*-move to insert a negatively oriented loop, on the right side of the edge \( e \), and smooth the resulting curve \( \gamma' \) until it is normal and regular. Then, wrap around \( \gamma' \) to create \( \gamma'' = Wr_+(\gamma') \). See Fig. 19. The following is an interesting property of balanced loop insertion.
Figure 19: Balanced loop insertion on a self-overlapping curve $\gamma$ with respect to an edge $e$. Note that the curve produced, $\gamma''$, is also self-overlapping.

**Lemma 20.** Let $\gamma$ have $\text{WHIT}(\gamma) = 1$ and positive outer basepoint. If $\gamma$ is strongly irreducible, and $\gamma'$ is obtained from $\gamma$ by balanced loop insertion, then $\gamma$ is strongly irreducible as well.

**Proof.** Let $i : V(\gamma) \to V(\gamma')$ be the inclusion map, sending vertices on $\gamma$ to the corresponding vertices on $\gamma'$. Then the only possible change to the direct split was that we added a negatively oriented loop, so $\text{WHIT}(\gamma'_{i(v)}) \leq \text{WHIT}(\gamma_v) \leq 0$. The only other direct splits we need to check are those of the vertices $u,v$ created by balanced loop insertion. We note immediately that $\text{WHIT}(\gamma'_u) = -1$ where $u$ is the basepoint of the new negatively oriented loop. Meanwhile, if $v$ is the vertex created by the wrap, then $\text{WHIT}(\gamma'_v) = \text{WHIT}(\gamma) - 1 = 0$. Hence, $\gamma'$ is strongly irreducible.

While strong irreducibility is preserved by balanced loop insertion, and consequently, so too is self-overlappingness, if $\gamma$ has a positive outer basepoint and $\text{WHIT}(\gamma) = 1$, we need a stronger operation to transform a non-self-overlapping curve into a self-overlapping curve.

Let $\gamma \in \mathcal{C}$. **Global balanced loop insertion**, denoted by $M : \mathcal{C} \to \mathcal{C}$, applies balanced loop insertion simultaneously once on every edge of $\gamma$. Since there are $2|\gamma| + 1$ edges on $G(\gamma)$, the operator $M(\cdot)$ applies balanced loop insertion $2|\gamma| + 1$ times. Equivalently, $M(\gamma)$ can be obtained by performing a $I_b$-move to the right of every edge of $\gamma$, adding a new negatively oriented loop, and then wrapping the curve $2|\gamma| + 1$ times. See Fig. 20 for an example of global balanced loop insertion.

We now show that this operation transforms any curve $\gamma$ with positive outer basepoint and $\text{WHIT}(\gamma) = 1$ into a strongly irreducible curve and hence a self-overlapping curve by Theorem 19.

**Theorem 21.** Let $\gamma$ have positive outer basepoint and $\text{WHIT}(\gamma) = 1$. Then $M(\gamma)$ is strongly irreducible and self-overlapping.

**Proof.** Let $\gamma$ be a curve with positive outer basepoint. We will utilize the identity $\text{WHIT}(\gamma) = \sum_{v \in V(\gamma)} \text{sgn}(v)$ shown by Titus and Whitney [18, 20]; note the inclusion of the basepoint here. It follows immediately that $\text{WHIT}(\gamma) \leq |\gamma| + 1$ for $\gamma \in \mathcal{C}$ with outer basepoint. While a direct split $\gamma_v$ may not have an outer basepoint, we can instead consider a curve $\gamma'_v$ with the same image and orientation as $\gamma_v$, and hence the same number of intersection points, and an outer basepoint. We then have $\text{WHIT}(\gamma_v) = \text{WHIT}(\gamma'_v) \leq |\gamma'_v| + 1 = |\gamma_v| + 1$. On the other hand, since every edge of $\gamma_v$ received at least one negatively oriented loop, as edges of $\gamma_v$ may be further subdivided on $\gamma$, we note that we inserted at least $2|\gamma_v| + 1$ negatively oriented loops on the direct split $\gamma_v$. Thus, if we write $i : V(\gamma) \to V(M(\gamma))$ for the natural inclusion map and set $\tilde{v} = i(v)$, then we see $\text{WHIT}(M(\gamma)_{\tilde{v}}) \leq -|\gamma_v| \leq 0$. Hence, all the vertices on $M(\gamma)$ that came from $\gamma$ will not yield direct splits of Whitney index 1 or greater.
Figure 20: Global balanced loop insertion applied to a curve $\gamma$. Since $\gamma$ has empty positively oriented loops it is not self-overlapping (by Lemma 10). The curve $M(\gamma)$ is strongly irreducible and self-overlapping by Theorem 21.

Now, the only other vertices to consider are the basepoints of the new negatively oriented loops and the basepoints of the wraps. Clearly, for any vertex $u$ of the former kind, we have $\text{WHIT}(M(\gamma)_u) = -1$ for the direct split at $u$. We now address the basepoints of the wraps. Let $M(\gamma)_i$ be the direct split and $M(\gamma)_{i^*}$ be the indirect split at the $i$-th vertex of $M(\gamma)$. By definition $M(\gamma)$ contains $2|\gamma| + 1$ outer wraps, which implies $\text{WHIT}(M(\gamma)_{i^*}) = i$ for all $i \in \{1, \ldots, 2|\gamma| + 1\}$. And since direct splits and indirect splits are complementary, it follows from Lemma 5 that $\text{WHIT}(M(\gamma)) = \text{WHIT}(M(\gamma)_i) + \text{WHIT}(M(\gamma)_{i^*})$ and hence $\text{WHIT}(M(\gamma)_i) = 1 - i \leq 0$ for any $i \in \{1, \ldots, 2|\gamma| + 1\}$. Thus, $M(\gamma)$ is indeed strongly irreducible.

5 Discussion

We introduced new curve classes (zero-obstinance, irreducible, and strongly irreducible curves; see Fig. 2), which help us understand self-overlapping curves and interior boundaries. We proved combinatorial results and showed that wrapping a curve resolves obstinance. These new mathematical foundations for self-overlapping curves and interior boundaries could pave the way for related algorithmic questions. For example, is it possible to decide whether a curve is self-overlapping in $o(N^3)$ time? How fast can one decide self-overlappingness of a curve on the sphere? Can one decide irreducability in $o(n^2)$ time, even in the presence of a large number of linked subcurves?

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A Whitney Indices & Loop Decompositions

In this section are interested in a refinement of self-overlapping decompositions. Let $\Omega = (\gamma_i)_{i=1}^n$ be a self-overlapping decomposition. If each $\gamma_i$ is simple, i.e. a loop, then we call $\Omega$ a loop decomposition. See Fig. 21 for an example.

We now recall a construction of Seifert [9, 16]. He introduced a decomposition of plane curves by performing so-called “uncrossing moves”, which essentially split a curve $\gamma$ at a vertex $v$ into the two (nearly) disjoint pieces $\gamma_v$ and $\gamma_v^*$. See Fig. 22. If we cut the two curves at the vertex $v$, and smooth them, we obtain two completely disjoint plane curves. Iterating this process across many vertices of $\gamma$, one can achieve a decomposition of the original curve $\gamma$ into a set of Jordan curves.

Suppose $\gamma$ is decomposed into Jordan curves $\{C_i\}_{i=1}^k$ using these uncrossing moves. Then Seifert and Gauss [9, 16] showed that $\text{WHIT}(\gamma) = \sum_{i=1}^k \text{WHIT}(C_i)$. Using our terminology and replacing the Jordan curves with loops in a loop decomposition, we obtain the equivalent fact:

**Lemma 22** (Compute Whitney Index). Let $\gamma \in \mathcal{C}$, let $\Omega$ be a loop decomposition of $\gamma$, and let $n_+$ be the number of positively oriented loops and $n_-$ be the number of negatively oriented loops in $\Omega$. Then $\text{WHIT}(\gamma) = n_+ - n_-.$

As a consequence of this lemma, we can prove linearity of Whitney indices across a direct split decomposition.

**Lemma 5** (Whitney Index Through Decompositions). Let $\gamma \in \mathcal{C}$ and $\Omega$ be a direct split decomposition of $\gamma$. Then $\text{WHIT}(\gamma) = \sum_{C \in \Omega} \text{WHIT}(C)$.

Figure 21: A loop decomposition of a curve $\gamma$.

Figure 22: An uncrossing move applied to a vertex $v$, splitting the curve into two pieces.
Proof. The key is to piece together loop decompositions of each subcurve $C \in \Omega$. Write $\Omega = (C_i)_{i=1}^k$ and take $\Psi_i$ to be a loop decomposition of $C_i$. Then $\Psi = (\Psi_1, \ldots, \Psi_k)$ is a loop decomposition of $\gamma$. Since $\text{WHIT}(C_i) = p_i - n_i$, where $p_i$ is the number of positively oriented subcurves in $\Psi_i$ and $n_i$ the number of negatively oriented subcurves, we conclude that

$$\text{WHIT}(\gamma) = \sum_{i=1}^k p_i - \sum_{i=1}^k n_i = \sum_{i=1}^k (p_i - n_i) = \sum_{i=1}^k \text{WHIT}(C_i).$$

\[\square\]

B Lattice

We introduce two more classes of curves. We say a face $F$ is good when its depth is equal to its winding number. If a curve $\gamma \in \mathcal{C}$ is positive consistent and all faces on $G(\gamma)$ are good then we call the curve good. We call a curve basic if all of its self-overlapping decompositions are loop decompositions. That is, the only self-overlapping decompositions are decompositions into loops. By Theorem 12, basic curves with zero obstinance can be decomposed into good curves.

We define Simple, Basic-Zero-Obstinance, Zero-Obstinance as classes of those curves that have the property described by the class name. The classes SO$^+$, Interior-Boundary$^+$, Consistent$^+$, Good$^+$ consist of the curves with the positive property described by the class name (positive self-overlapping, positive interior boundary, positive consistent, and good curves that are positive consistent). Fig. 23 and Theorem 24 show the relationship between these curve classes. The curves in Fig. 23 show that the inclusions in parts 1, 2, 3 of Theorem 24 are proper. We first state a lemma that we need in the proof of the theorem.

Lemma 23 (Negatively Oriented Loop). Let $\gamma \in \mathcal{C}$ be a non-simple, positive self-overlapping curve with positive outer basepoint. Then $\gamma$ has a negatively oriented loop $\gamma_v$.

Proof. Let $\mathcal{V} = \{ v \in V(\gamma) \mid \gamma_v \text{ is a loop} \}$. By Lemma 13 we know $\mathcal{V} \neq \emptyset$. Define the relation $v \prec w$ for $v, w \in \mathcal{V}$ whenever $\text{int}(v) \subseteq \text{int}(w)$. It is straightforward to verify that $(\mathcal{V}, \prec)$ is a poset. Of course, since $|\mathcal{V}|$ is finite, we can choose $v_0 \in \mathcal{V}$ minimal with respect to $\prec$. Now, suppose that $\gamma_{v_0}$ were a positive loop. We show this is a contradiction to complete the proof. Indeed, by minimality of $v_0$, there are no loops of $\gamma$ completely contained inside $\text{int} \gamma_{v_0}$. This means every time a strand of $\gamma$ crosses from outside to inside $\gamma_{v_0}$, the strand does not cross itself inside of $\gamma$. Topologically, then, $\text{int}(\gamma_{v_0})$ looks like a disk with finitely many simple arcs traveling from boundary to boundary. By way of a (regular) right sense-preserving homotopy, we can sweep each such arc until it no longer intersects $\gamma_{v_0}$. As we need only sweep finitely many such arcs, let us denote $\gamma'$ as the result of this process. By Lemma 11 we know $\gamma'$ is self-overlapping. On the other hand, $\gamma'$ has an empty positive loop, namely the one we just emptied, which contradicts Lemma 10.

\[\square\]

Theorem 24 (Curve Classes). If $\gamma \in \mathcal{C}$ has a positive outer basepoint, then:

1. Simple $\subset$ SO$^+$ $\subset$ Interior-Boundary$^+$ $\subset$ Consistent$^+$
2. Simple $\subset$ Good$^+$ $\subset$ Basic-Zero-Obstinance $\subset$ Zero-Obstinance
3. Good$^+$ $\subset$ Interior-Boundary$^+$ $\subset$ Zero-Obstinance
4. Consistent$^+$ $\cap$ Zero-Obstinance = Interior-Boundary$^+$
5. Consistent$^+$ $\cap$ Basic-Zero-Obstinance = Good$^+$
6. SO$^+$ $\cap$ Good$^+$ = Simple

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Proof. By definition, a simple curve with a positive outer basepoint is positive self-overlapping, and \( +k \)-boundaries are positive consistent. By Theorem 9, a positive self-overlapping curve is a 1-boundary, which proves 1. A simple curve is trivially good. By Lemma 1 and Lemma 2, good curves have zero obstinance. We now show that good curves are basic by contradiction. Suppose \( \gamma \) admitted a self-overlapping decomposition \( \Omega = (\gamma_i)_{i=1}^k \) with a non-simple self-overlapping curve \( \gamma_j \). Then \( \gamma_j \) must contain a negatively oriented loop by Lemma 23. But we could then create a finer decomposition of \( \gamma \) by decomposing \( \gamma_j \) into loops. Precisely, let \( \Psi \) be a loop decomposition of \( \gamma_j \) and consider \( \Gamma = (\gamma_1, \ldots, \gamma_{j-1}, \Psi, \gamma_{j+1}, \ldots, \gamma_k) \). Then \( \Gamma \) is a free subcurve decomposition of \( \gamma \) refining \( \Omega \). Let \( C \) be a negatively oriented loop in \( \Psi \) and take any face \( F \) contained in the interior of \( C \). Now, take a path \( P \) from \( F \) to the exterior face on \( \Gamma(\gamma) \) such that the depth is monotonically decreasing along the path \( P \). Since \( F \) is contained inside \( C \), the path \( P \) must cross \( C \) to reach \( F_{ext} \). However, when \( P \) crosses past \( C \), we see the depth either decrease by 1 or remain unchanged, while the winding number increases by 1, since \( C \) is negatively oriented. We learn that \( wn(F, \gamma) < D(F, \gamma) \), which is a contradiction. This proves 2 with the last inclusion being trivial. Since a good curve is basic and positive consistent, it is a positive interior boundary by property 3 of Theorem 8. And interior boundaries have zero obstinance by definition, which proves 3.

By definition, interior boundaries are consistent and have zero obstinance, which proves 4. If a curve \( \gamma \) is basic and positive consistent, it follows that \( \gamma \) admits a loop decomposition \( \Omega \) with only positively oriented subcurves. Since by Observation 2, \( wn(F, \gamma) = \sum_{i \in \Omega} wn(F, \gamma_i) \) and each \( wn(F, \gamma_i) \in \{0, 1\} \), we must have \( wn(F, \gamma) \leq D(F, \gamma) \). The bound \( wn(F, \gamma) \leq D(F, \gamma) \) holds generally. Thus \( \gamma \) is good, which together with 1 and 2 proves 5. By Lemma 23, a good curve that is self-overlapping may not have a negatively oriented loop and hence must be simple, which together with 1 and 2 proves 6.

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Figure 23: Lattice of curve classes. Solid arrows are inclusions. Two dashed lines meet to form an inclusion. A member curve of each class is displayed, along with appropriate information to justify membership in its curve class: A set of blank cuts of \( \gamma_{SO} \) are shown in red, a self-overlapping decomposition of \( \gamma_{IB} \) in blue, the winding numbers of \( \gamma_C \), the loop decompositions of \( \gamma_G \) and \( \gamma_{BZO} \), and the self-overlapping decomposition of \( \gamma_{ZO} \). The inclusions are proper: \( \gamma_{BZO} \) is not consistent and consequently not good; \( \gamma_{IB} \) is not good, and since \( WHIT(\gamma_{IB}) = +2 \) it is not self-overlapping; \( \gamma_C \) is not an interior boundary because \( WHIT(\gamma_C) = 1 \) but since it has an empty positively oriented loop it is not self-overlapping; \( \gamma_{ZO} \) is not an interior boundary because it is not consistent. See Theorem 24.