Classical Dynamics of Quantum Numbers with Arrow of Time

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Abstract

We study a quantum theory with complex time parameter and non-Hermitian Hamiltonian structure. In this theory, the real part of the complex time is equal to ‘usual’ physical time, whereas the imaginary one is proportional to inverse absolute temperature of the system. Then, the Hermitian part of the Hamiltonian coincides with conventional operator of energy; the anti-Hermitian part, which is taken as a symmetry operator, defines decay parameters of the theory. We integrate the equations of motion in a Hamiltonian proper basis, and detect a classical dynamics of the corresponding quantum numbers, using their continuous approximation and the zero limit of the Plank’s constant. It is proved, that this dynamics possesses a well defined arrow of time in the isothermal and adiabatic regimes of the thermodynamical evolution of the system.

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1 Introduction

Irreversibility (in time) is one of the most evident and mysterious property of the real world. One says about ‘arrow of time’ in evolution of the Universe and its closed subsystems to stress this fundamental non-equivalence between the real past and future [1]–[4], and operates by different kinetic and thermodynamical constructions to describe the corresponding unidirectional processes [5]. The main problem in this activity is related to realization of the formal thermodynamics on the base of some fundamental dynamical theory, like the standard quantum or classical mechanics. Note, that this theory must be taken without any ‘hand made’ arrow of time – i.e., it must be conservative in the conventional sense of the canonic formalism. Thus, one needs to derive, for example, the irreversible thermodynamics from the definitely reversible mechanics, which seems non-realistic program in the framework of any consistent mathematics. Of course, one can try to improve this hopeless situation by corresponding modification of the fundamental theory itself. Note, that it is exactly the way that we have chosen in our approach to solve the problem under consideration.

In [6], we have developed the modified quantum theory, which possesses arrow of time in various regimes of its dynamics. In fact, this theory provides a general theoretical framework for unification of the standard quantum and statistical mechanics. Its dynamics is described in terms of the complex evolutionary parameter \( \tau \neq \tau^* \), and the non-Hermitian Hamiltonian operator \( \mathcal{H} \neq \mathcal{H}^+ \). For the natural analogies of the standard conservative systems, we put \( \mathcal{H}, \tau = 0 \). The main dynamical equation of the theory is postulated in the its conventional Schrödinger’s form,

\[
i\hbar \Psi,\tau = \mathcal{H}\Psi.
\]

We consider the holomorphic variant of the theory, with \( \Psi,\tau^* = 0 \) and \( \mathcal{H},\tau^* = 0 \), which leads to the simplest generalization of the standard theoretical quantum scheme. Also, we restrict it by the theories with \([\mathcal{H}, \mathcal{H}^+] = 0 \). The last relation becomes an identity in the standard quantum theory case, when \( \mathcal{H} = \mathcal{H}^+ \), so our approach can be understood as the ‘minimal generalization’ of the standard one.

The complex evolutionary parameter \( \tau \) can be parameterized in terms of the real variables \( t \) and \( \beta \) in the following form:

\[
\tau = t - i\frac{\hbar}{2}\beta,
\]

whereas the non-Hermitian operator \( \mathcal{H} \) can be represented using the Hermitian operators \( E \) and \( \Gamma \) as

\[
\mathcal{H} = E - i\frac{\hbar}{2}\Gamma.
\]
Note, that

\[ [E, \Gamma] = 0, \]  

(1.4)

in accordance to the restriction imposed above. In [6] it was argued, that, if one identifies the quantities \( t \) and \( E \) with the ‘usual’ time and energy operator, respectively, then the remaining quantities \( \beta \) and \( \Gamma \) mean the inverse absolute temperature \( \beta = 1/kT \) (multiplied to the Boltzmann constant \( k \)), and the operator of inverse decay time parameters of the system. Then, this scheme of generalization of the quantum theory must be completed by introducing of a conception of thermodynamic regime, which has a form of fixation of the temperature function \( \beta = \beta(t) \). The simplest regimes are the isothermal \( \beta = \text{const} \) and the adiabatic \( E(t, \beta) = \text{const} \) ones (we use bar for all averaged quantities here). In [6] it was shown, that for these two regimes the function \( \bar{\Gamma} = \bar{\Gamma}(t, \beta(t)) \equiv \bar{\Gamma}[t] \) is non-increasing. This fact indicates the presence of arrow of time in the corresponding evolution of the system. In [7] it was studied the quantum dynamics with the operator \( \Gamma \) of the parity type. For this theory, the left-right asymmetry becomes the result of its unidirectional dynamics. The Lyapunov function \( \Gamma[t] \) detects it in the explicit form for the arbitrary thermodynamic regime.

In this work, we check an existence of the irreversible classical dynamics of the fundamental type, which can be derived from the quantum theory described above. We use for its formulation and study the most convenient set of the original quantum variables – the set of quantum numbers, related to the corresponding stationary Schrödinger’s problem. We consider this set in the natural continuous approximation, and perform the limit procedure \( h \to 0 \) in the appropriate form for the all theory structures. We define classical dynamics as the dynamics of maximum of the resulting probability density of the system. In doing so, we derive equations of motion (of the Hamilton’s type), which describe the dynamics of this maximum point, and prove the irreversible character of this dynamics. Namely, we show, that in the isothermal and adiabatic regimes this modified classical dynamics leads to the non-increasing classical function \( \Gamma[t] \), which must be calculated on the classical trajectory under consideration.

2 Hamilton Equations for Quantum Numbers

Let us express all significant quantities of the theory in terms of a common basis of the eigenvectors \( \psi(n) \) of the commuting operators \( E \) and \( \Gamma \). We take it in the orthonormal form (i.e., we mean, that the identity \( \psi(n)^+ \psi(k) = \delta(n, k) \) takes place). Here, of course, the indexes are understood in the appropriate ‘multi-sense’ (and all summations are of the
corresponding type). The eigenvalue problem under consideration reads:

\[ E\psi(n) = E(n)\psi(n), \quad \Gamma\psi(n) = \Gamma(n)\psi(n); \quad (2.1) \]

it can be reformulated in terms of the non-Hermitian operator \( H \). Actually, it is easy to see, that \( \psi(n) \) is the eigenvector for this operator, which corresponds to the complex eigenvalue \( H(n) = E(n) - i\hbar/2\Gamma(n) \). Then, the state vectors \( \Psi(\tau, n) = \exp[-iH(n)\tau/\hbar]\psi(n) \) satisfy the Schrödinger’s equation (1.1), and also form the complete (but \( \tau \)-dependent) basis. This basis can be used for representation of any solution \( \Psi(\tau) \) of the Schrödinger’s equation in the form of linear combination with some set of constant parameters \( C(n) \), i.e., as \( \Psi(\tau) = \sum_n C(n)\Psi(n) \) Using this decomposition formula and the orthonormal basis properties, one can calculate the probability \( P(t, \beta, n) \) to find the quantum system in its basis state \( \Psi(\tau, n) \), when it is described by the state vector \( \Psi(\tau) \). The result reads:

\[ P(t, \beta, n) = \frac{w(t, \beta, n)}{Z}, \quad (2.2) \]

where \( Z = Z(t, \beta) = \sum_n w(t, \beta, n) \), and

\[ w(t, \beta, n) = \exp[-S_2(t, \beta, n)], \]
\[ S_2(t, \beta, n) = \sigma(n) + E(n)\beta + \Gamma(n)t. \quad (2.3) \]

Here we have put \( \sigma(n) = -\log|C(n)|^2 \).

Now let us put, for convenience, \( \tilde{t}_1 = t, \tilde{t}_2 = \beta \), and also \( \tilde{E}_1 = \Gamma, \tilde{E}_2 = E \), and combine the corresponding quantities to the sets \( \tilde{t}_\alpha \) and \( \tilde{E}_\alpha (\alpha = 1, 2) \). We define the classical value \( n_c = n_c(\tilde{t}_\alpha) \) of the collective \( n \)-variable according to the relation

\[ P(\tilde{t}_\alpha, n_c) = \max_n P(\tilde{t}_\alpha, n). \quad (2.4) \]

Thus, we identify the classical theory with the theory which describes the dynamics of the ‘maximum point’ of probability density. In the ‘hard’ classical limit the probability density is described by the delta-functional distribution. It is clear, that the resonance character of the classical probability density must be guaranteed by the corresponding choice of the weight parameter \( \sigma(n) \). Actually, this parameter is some initial data of the theory, so it can be taken in the appropriate form at the beginning of evolution of the system.

We derive the defining \( n_c \)-relation using differentiation of the necessary extremum condition \( \dot{S}_2 [\tilde{t}, n(\tilde{t})] = 0 \) in respect to \( \tilde{t}_\alpha \):

\[ \frac{\partial}{\partial\tilde{t}_\alpha} \dot{S}_2 [\tilde{t}, n(\tilde{t})] = 0 \quad (2.5) \]
(here the dot means differentiation in respect to the $n$-components). Note, that this $\tilde{t}_\alpha$-differentiation is understood with taking into account the total $\tilde{t}_\alpha$-dependence of the corresponding quantity. It fact, Eq. (2.5) means a conservation of the extremal character of the classical trajectory of the system during its possible physical evolution.

It is not difficult to prove, that the dynamical equation for the quantity $n_c$, which follows from Eq. (2.5), reads:

$$ n_c, \tilde{t}_\alpha = -A_2^{-1} \dot{E}_\alpha, \quad (2.6) $$

where $A_2 = \tilde{S}_{2c} = \tilde{\sigma}_c + \sum_\beta \tilde{E}_\beta c \tilde{t}_\beta$. Here $\dot{E}_\alpha$ is the column of the derivatives of the first order $\partial E_/c / \partial n_k$, whereas the quantity (for example) $\tilde{S}_{2c}$ means the symmetric matrix which is constructed from the derivatives of the second order $\partial S_{2c} / \partial n_k \partial n_l$.

Then, the limit $\hbar \to 0$ is important for calculation of the set of canonical moments, which we define as

$$ p = \dot{S}_1 [\tilde{t}_\alpha, n(\tilde{t}_\alpha)]. \quad (2.7) $$

In this formula, $S_1$ means the real part of the phase $S$ of the wave-function $\Psi$ in the $n$-representation, which is given by the relation $\Psi (\tilde{t}_\alpha, n) = C(n) \Psi(\tau, n)$. It is not difficult to prove, that

$$ S_1 (\tilde{t}_\alpha, n) = \lambda(n) - E(n)t + \frac{\hbar^2}{4} \Gamma(n) \beta. \quad (2.8) $$

Here the function $\lambda(n)$ is defined according to the relation $C(n) = |C(n)| \exp(i \hbar \lambda)$. Then, in the explicit form, and after the taking of the limit mentioned above, one concludes, that

$$ p = \dot{\lambda} - \dot{E}t. \quad (2.9) $$

Then, for the time derivatives of the moment variable $p$ it is possible to establish the equation $p, \dot{t}_\alpha = -\dot{E}\delta_{1,\alpha} + A_1 n, \dot{t}_\alpha$, where $A_1 = \dot{S}_1 c = \lambda - \dot{E} t$. Finally, using the relation (2.6), one concludes, that

$$ p, \dot{t}_\alpha = -\dot{E}\delta_{1,\alpha} - A_1 A_2^{-1} \dot{E}_\alpha. \quad (2.10) $$

It is clear, that the relations (2.6) and (2.10) form the pair of Hamilton’s equations for the theory under consideration. We state, that this system is consistent, i.e., that the mixed time derivatives for the coordinate and moment variables are equal to the inverse ones. This means, that this system of classical equations is correct.
3 Arrow of Time in Dynamics of Quantum Numbers

Our main statement is related to irreversibility of the evolution of the system, defined by the Hamilton’s equations (2.6), (2.10), and by the temperature regime $\beta = \beta(t)$ taken in the appropriate form. Below we consider the isothermal and adiabatic cases, as it had been performed for the original quantum theory in [6]. To prove the presence of arrow of time in these physically important regimes, let us choose the representation, where $\Gamma = \Gamma(n_1)$ and $E = E(n_2)$. It is clear, that this special representation always exists for the theory taken (i.e., for the starting quantum theory with the commuting energy and decay operators).

From the system of Hamilton’s equations it follows, that

$$
n_{1,t} = -\Gamma'(A_2^{-1})_{11}, \quad n_{2,t} = -\Gamma'(A_2^{-1})_{21},$$

$$
n_{1,\beta} = -E'(A_2^{-1})_{12}, \quad n_{2,\beta} = -E'(A_2^{-1})_{22},$$

(3.1)

where the prime means derivative in respect to the corresponding (single) variable of the function under consideration. Then, for the total $t$-derivative of the function $\Gamma$, i.e., for the quantity $d\Gamma/dt = \Gamma' dn/dt = \Gamma'(n_{1,t} + \beta' n_{1,\beta})$, one can calculate its explicit form using Eq. (3.1). In the isothermal case (with $d\beta/dt = 0$), the result reads:

$$
\frac{d\Gamma}{dt} = -\left(\Gamma'ight)^2 \left(A_2^{-1}\right)_{11}.
$$

(3.2)

In the adiabatic regime (with $dE/dt = 0$), one obtains, that

$$
\beta' = -\frac{\Gamma'}{E'} \left(A_2^{-1}\right)_{21} \left(A_2^{-1}\right)_{22}
$$

(3.3)

for its ‘temperature curve’, and that

$$
\frac{d\Gamma}{dt} = -\frac{(\Gamma)^2}{\left(A_2^{-1}\right)_{22}} \left| \begin{array}{cc}
(A_2^{-1})_{11} & (A_2^{-1})_{12} \\
(A_2^{-1})_{21} & (A_2^{-1})_{22}
\end{array} \right|
$$

(3.4)

for the total time derivative of the decay function $\Gamma$. It is clear, that $d\Gamma/dt < 0$ in the both situations considered, because the coefficient $(A_2^{-1})_{11}$ and the determinant written above are positive quantities in view of the positive definiteness supposed for the quantity $d^2S_2$ on the whole classical trajectory. Thus, one deals with the strictly decreasing evolution of the decay function $\Gamma[t]$ for the arbitrary initial data taken. This proves the Lyapunov nature of this function, and the presence of arrow of time in the dynamics of the classical system under investigation.
4 Conclusion

In this work we have detected an existence of the classical dynamics of the Hamilton’s type, which possesses the well-defined arrow of time (in the isothermal and adiabatic regimes of the thermodynamical evolution, at least). In fact, it is established the unified theory of classical and statistical mechanics, which gives a natural realization for the standard classical thermodynamics. We have used for its derivation the original set of quantum numbers, which originate from the general solution of the stationary Schrödinger’s equation. Our starting quantum theory is taken as holomorphic in respect to the complex parameter of evolution; the dynamics is governed by the non-Hermitian Hamiltonian structure. We would like to note, that the results obtained in this work are the most general ones for the all theories under consideration. Actually, one can choose the ‘quantum numbers representation’ for the arbitrary theory of this type.

In the next publication we will develop the classical dynamics with arrow of time in terms of the arbitrary set of canonical variables of the theory. We think, that this modified canonical formalism can give the base for construction of fundamental and time-irreversible quantum field and string theories in the simple and natural way. Also, it seems actually promising for applications in particle physics, gravity and cosmology [8]-[12].

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