PROBE BRANES DYNAMICS: EXACT SOLUTIONS IN GENERAL BACKGROUNDS

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We consider probe $p$-branes and $Dp$-branes dynamics in $D$-dimensional string theory backgrounds of general type. Unified description for the tensile and tensionless branes is used. We obtain exact solutions of their equations of motion and constraints in static gauge as well as in more general gauges. Their dynamics in the whole space-time is also analyzed and exact solutions are found.

PACS number(s):11.25.-w; 11.27.+d; 11.30.-j

1 Introduction

The probe branes approach for studying issues in the string/M-theory uses an approximation, in which one neglects the back-reaction of the branes on the background. In this sense, the probe branes are multidimensional dynamical systems, evolving in given, variable in general, external fields.

The probe branes method is widely used in the string/M-theory to investigate many different problems at a classical, semiclassical and quantum levels. The literature in this field of research can be conditionally divided into several parts. One of them is devoted to the properties of the probe branes themselves, e.g., [1]-[29]. The subject of another part of the papers is to probe the geometries of the string/M-theory backgrounds, e.g., [30]-[42]. One another part can be described as connected with the investigation of the correspondence between the string/M-theory geometries and their field theory duals, e.g., [43]-[59]. Let us also mention the application of the probe branes technique in the 'Mirage cosmology'- approach to the brane world scenario, e.g., [60]-[70].

In view of the wide implementation of the probe branes as a tool for investigation of different problems in the string/M-theory, it will be useful to have a method describing their dynamics, which is general enough to include as many cases of interest as possible, and on the other hand, to give the possibility for obtaining explicit exact solutions.

In this article, we propose such an approach, which is appropriate for $p$-branes and $Dp$-branes, for arbitrary worldvolume and space-time dimensions, for tensile and tensionless branes, for different variable background fields with minimal restrictions on them, and finally, for different space-time and worldvolume gauges (embeddings).

The paper is organized as follows. In Section 2, we perform an analysis in order to choose the brane actions, which are most appropriate for our purposes - do not contain square roots, generate only independent constraints and give a unified description for tensile and tensionless branes. In Section 3, we formulate the conditions, under which the probe branes dynamics
reduces to a particle-like one. Then, we investigate the reduced dynamics for three different types of brane embeddings, starting with the usually used static gauge one. In the last part of this section, we find explicit exact solutions of the branes equations of motion. In Section 4, we discuss the obtained results.

2 Actions

Before considering the problem for obtaining exact brane solutions in general string theory backgrounds, it will be useful first to choose appropriate actions, which will facilitate our task. Generally speaking, there are two types of brane actions - with and without square roots. The former ones are not well suited to our purposes, because the square root introduces additional nonlinearities in the equations of motion. Nevertheless, they have been used when searching for exact brane solutions in fixed backgrounds, because there are no constraints in the Lagrangian description and one has to solve only the equations of motion. The other type of actions contain additional worldvolume fields (Lagrange multipliers). Varying with respect to them, one obtains constraints, which, in general, are not independent. Starting with an action without square root, one escapes the nonlinearities connected with the square root, but has to solve the equations of motion and the (dependent) constraints.

Independently of their type, all actions proportional to the brane tension cannot describe the tensionless branes. The latter appear in many important cases in the string theory, and it is preferable to have a unified description for tensile and tensionless branes.

Our aim in this section is to find brane actions, which do not contain square roots, generate only independent constraints and give a unified description for tensile and tensionless branes.

2.1 \(P\)-brane actions

The Polyakov type action for the bosonic \(p\)-brane in a \(D\)-dimensional curved space-time with metric tensor \(g_{MN}(x)\), interacting with a background \((p+1)\)-form gauge field \(b_{p+1}\) via Wess-Zumino term, can be written as

\[
S_p = - \int d^{p+1}\xi \left\{ \frac{T_p}{2} \sqrt{-\gamma} \left[ \gamma^{mn} \partial_m X^M \partial_n X^N g_{MN}(X) - (p - 1) \right] \right. \\
- \left. Q_p \varepsilon^{m_1 \ldots m_{p+1}} \partial_{m_1} X^{M_1} \ldots \partial_{m_{p+1}} X^{M_{p+1}} b_{M_1 \ldots M_{p+1}}(X) \right\},
\]

(2.1)

where \(\gamma\) is the determinant of the auxiliary worldvolume metric \(\gamma_{mn}\), and \(\gamma^{mn}\) is its inverse. The position of the brane in the background space-time is given by \(x^M = X^M(\xi^m)\), and \(T_p, Q_p\) are the \(p\)-brane tension and charge, respectively. If we consider the action (2.1) as a bosonic part of a supersymmetric one, we have to set \(Q_p = \pm T_p\). In what follows, \(Q_p = T_p\).

The requirement that the variation of the action (2.1) with respect to \(\gamma_{mn}\) vanishes, leads to

\[
\left( \gamma^{kl} \gamma_{mn} - 2 \gamma^{km} \gamma_{ln} \right) G_{mn} = (p - 1) \gamma^{kl},
\]

(2.2)

where \(G_{mn} = \partial_m X^M \partial_n X^N g_{MN}(X)\) is the metric induced on the \(p\)-brane worldvolume. Taking the trace of the above equality, one obtains

\[
\gamma^{mn} G_{mn} = p + 1,
\]

\footnote{Examples of these two type of actions are the Nambu-Goto and Polyakov actions for the string.}
i.e., \( \gamma^{mn} \) is the inverse of \( G_{mn} \): \( \gamma^{mn} = G^{mn} \). If one inserts this back into (2.1), the result will be the corresponding Nambu-Goto type action \( (G \equiv \text{det}(G_{mn})) \):

\[
S_p^{NG} = \int d^{p+1}\xi L^{NG} = -\frac{T_p}{\sqrt{-G}} \frac{\sqrt{G - \sum_{m} \partial_{m}X^{M_{m}} \cdot \partial_{m+1}X^{M_{m+1}}}}{(p+1)!} \int \frac{d^{p+1}\xi}{\sqrt{-G}} \left[ \sqrt{-G} - \frac{\sqrt{G - \sum_{m} \partial_{m}X^{M_{m}} \cdot \partial_{m+1}X^{M_{m+1}}}}{(p+1)!} \partial_{m}X^{M_{m}} \cdot \partial_{m+1}X^{M_{m+1}} \right].
\]

This means that the two actions, (2.1) and (2.3), are classically equivalent.

As already discussed, the action (2.3) contains a square root, the constraints (2.2), following from (2.1), are not independent and none of these actions is appropriate for description of the tensionless branes. To find an action of the type we are looking for, we first compute the explicit expressions for the generalized momenta, following from (2.3):

\[
P_{M}(\xi) = -T_p \left( \sqrt{-G} \sqrt{g^{MN}} \partial_{n}X^{N} - \partial_{i}X^{M_{i}} \cdot \partial_{p}X^{M_{p}}b_{M_{i}M_{p}} \right).
\]

It can be checked that \( P_{M}(\xi) \) satisfy the constraints

\[
C_{0} \equiv g^{MN}P_{M}P_{N} - 2T_p g^{MN}D_{M_{1}...M_{p}}P_{N} + T_p^2 \left[ G G^{00} + (-1)^{p} D_{1...pM}g^{MN}D_{N1...p} \right] = 0,
\]
\[
C_{i} \equiv P_{M} \partial_{i}X^{M} = 0, \quad (i = 1,\ldots,p),
\]

where we have introduced the notation

\[
D_{M_{1}...p} \equiv b_{M_{1}M_{2}...M_{p}} \partial_{1}X^{M_{1}} \cdot \partial_{p}X^{M_{p}}.
\]

Let us now find the canonical Hamiltonian for this dynamical system. The result is:

\[
H_{canon} = \int d^{p}\xi \left( P_{M} \partial_{0}X^{M} - L^{NG} \right) = 0.
\]

Therefore, according to Dirac [71], we have to take as a Hamiltonian the linear combination of the first class primary constraints \( C_{n} \): \(^2\)

\[
H = \int d^{p}\xi H = \int d^{p}\xi \left( \lambda^{0}C_{0} + \lambda^{i}C_{i} \right).
\]

The corresponding Hamiltonian equations of motion for \( X^{M} \) are

\[
(\partial_{0} - \lambda^{i}\partial_{i})X^{M} = 2\lambda^{0}g^{MN} \left( P_{N} - T_p D_{N1...p} \right),
\]

from where one obtains the explicit expressions for \( P_{M} \)

\[
P_{M} = \frac{1}{2\lambda^{0}}g_{MN} \left( \partial_{0} - \lambda^{i}\partial_{i} \right)X^{N} + T_p D_{M1...p}. \tag{2.4}
\]

With the help of (2.4), one arrives at the following configuration space action

\[
S_p = \int d^{p+1}\xi L_{p} = \int d^{p+1}\xi \left( P_{M} \partial_{0}X^{M} - H \right) \tag{2.5}
\]

\[
= \int d^{p+1}\xi \left\{ \frac{1}{4\lambda^{0}} \left[ g_{MN}(X) \left( \partial_{0} - \lambda^{i}\partial_{i} \right)X^{M} \left( \partial_{0} - \lambda^{j}\partial_{j} \right)X^{N} - \left( 2\lambda^{0}T_p \right)^{2} G G^{00} \right] + T_p b_{M_{0}...M_{p}} \left( X \right) \partial_{0}X^{M_{0}} \cdot \partial_{p}X^{M_{p}} \right\}
\]

\[
= \int d^{p+1}\xi \left\{ \frac{1}{4\lambda^{0}} \left[ G_{00} - 2\lambda^{i}G_{0i} + \lambda^{i}\lambda^{j}G_{ij} - \left( 2\lambda^{0}T_p \right)^{2} G G^{00} \right] + T_p b_{M_{0}...M_{p}} \left( X \right) \partial_{0}X^{M_{0}} \cdot \partial_{p}X^{M_{p}} \right\},
\]

\(^2\)In the case under consideration, secondary constraints do not appear. The first class property of \( C_{n} \) follows from their Poisson bracket algebra.
which does not contain square root, generates the independent \((p+1)\) constraints, as we will show below, and in which the limit \(T_p \to 0\) may be taken. For other actions, allowing for unified description of tensile and tensionless \(p\)-branes, see [72 - 74].

It can be proven that this action is classically equivalent to the previous two actions. It is enough to show that (2.3) and (2.5) are equivalent, because we already saw that this is true for (2.1) and (2.3).

Varying the action \(S_p\) with respect to Lagrange multipliers \(\lambda^m\) and requiring these variations to vanish, one obtains the constraints

\[ G_{00} - 2\lambda^i G_{0j} + \lambda^i \lambda^j G_{ij} + \left(2\lambda^0 T_p\right)^2 G G^{00} = 0, \quad (2.6) \]
\[ G_{0j} - \lambda^i G_{ij} = 0. \quad (2.7) \]

By using them, the Lagrangian density \(L_p\) from (2.5) can be rewritten in the form

\[ L_p = -T_p \sqrt{-GG^{00} \left[ G_{00} - G_{0i} \left(G^{-1}\right)^{ij} G_{j0}\right] + T_p b_{M_0...M_p}(X) \partial_0 X^{M_0} \ldots \partial_p X^{M_p}. \quad (2.8) \]

Now, applying the equalities

\[ GG^{00} = \det (G_{ij}) \equiv \mathbf{G}, \quad G = \left[ G_{00} - G_{0i} \left(G^{-1}\right)^{ij} G_{j0}\right] \mathbf{G}, \quad (2.9) \]

one finds that

\[ G^{00} \left[ G_{00} - G_{0i} \left(G^{-1}\right)^{ij} G_{j0}\right] = 1. \]

Inserting this in (2.8), one obtains the Nambu-Goto type Lagrangian density \(L^{NG}\) from (2.6). Thus, the classical equivalence of the actions (2.3) and (2.5) is established.

We will work further in the gauge \(\lambda^m = \text{constants}\), in which the equations of motion for \(X^M\), following from (2.5), are given by

\[ g_{LN} \left[ \left(\partial_0 - \lambda^i \partial_i\right) \left(\partial_0 - \lambda^j \partial_j\right) X^N - \left(2\lambda^0 T_p\right)^2 \partial_i \left(G G^{ij} \partial_j X^N\right)\right] \]
\[ + \Gamma_{L,MN} \left[ \left(\partial_0 - \lambda^i \partial_i\right) X^M \left(\partial_0 - \lambda^j \partial_j\right) X^N - \left(2\lambda^0 T_p\right)^2 G G^{ij} \partial_i X^M \partial_j X^N\right] \]
\[ = 2\lambda^0 T_p H^b_{LM_0...M_p} \partial_0 X^{M_0} \ldots \partial_p X^{M_p}, \]

where \(\mathbf{G}\) is defined in (2.9),

\[ \Gamma_{L,MN} = g_{LK} \Gamma^K_{MN} = \frac{1}{2} \left(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}\right) \]

are the components of the symmetric connection compatible with the metric \(g_{MN}\) and \(H^b_{p+2} = db_{p+1}\) is the field strength of the \((p+1)\)-form gauge potential \(b_{p+1}\).

### 2.2 Dp-brane actions

The Dirac-Born-Infeld type action for the bosonic part of the super-\(Dp\)-brane in a \(D\)-dimensional space-time with metric tensor \(g_{MN}(x)\), interacting with a background \((p+1)\)-form Ramond-Ramond gauge field \(c_{p+1}\) via Wess-Zumino term, can be written as

\[ S^{DBI} = -T_p \int d^{p+1}x \left\{ e^{-\alpha(p,D)\Phi} \sqrt{-\det (G_{mn} + B_{mn} + 2\pi \alpha' F_{mn})} \right\} \]
\[ - \frac{\varepsilon^{m_1...m_{p+1}}}{(p+1)!} \partial_{m_1} X^{M_1} \ldots \partial_{m_{p+1}} X^{M_{p+1}} \varepsilon_{M_1...M_{p+1}} \]
\( T_{Dp} = (2\pi)^{-(p-1)/2} g_s^{-1} T_p \) is the D-brane tension, \( g_s = \exp(\Phi) \) is the string coupling expressed by the dilaton vacuum expectation value \( \langle \Phi \rangle \) and \( 2\pi\alpha' \) is the inverse string tension. \( G_{mn} = \partial_m X^M \partial_n X^N g_{MN}(X) \), \( B_{mn} = \partial_m X^M \partial_n X^N b_{MN}(X) \) and \( \Phi(X) \) are the pullbacks of the background metric, antisymmetric tensor and dilaton to the Dp-brane worldvolume, while \( F_{mn}(\xi) \) is the field strength of the worldvolume \( U(1) \) gauge field \( A_m(\xi) \): \( F_{mn} = 2\partial_m A_n \). The parameter \( a(p, D) \) depends on the brane and space-time dimensions \( p \) and \( D \), respectively.

A Dp-brane action, which generalizes the Polyakov type p-brane action, has been introduced in \([73]\). Namely, the action, classically equivalent to (2.11), is given by

\[
S^{AZH} = -\frac{T_{Dp}}{2} \int d^{p+1} \xi \left\{ e^{-a(p,D)\Phi} \sqrt{\mathcal{K}} \left[ \mathcal{K}_{mn} (G_{mn} + B_{mn} + 2\pi\alpha' F_{mn}) - (p - 1) \right] - \frac{2 \epsilon^{m_1 \ldots m_{p+1}}}{(p + 1)!} \partial_{m_1} X^{M_1} \ldots \partial_{m_{p+1}} X^{M_{p+1}} \eta_{M_1 \ldots M_{p+1}} \right\},
\]

where \( \mathcal{K} \) is the determinant of the matrix \( \mathcal{K}_{mn} \), \( \mathcal{K}^{mn} \) is its inverse, and these matrices have symmetric as well as antisymmetric part

\[
\mathcal{K}^{mn} = \mathcal{K}^{(mn)} + \mathcal{K}^{[mn]},
\]

where the symmetric part \( \mathcal{K}^{(mn)} \) is the analogue of the auxiliary metric \( \gamma^{mn} \) in the p-brane action (2.11).

Again, none of these actions satisfy all our requirements. In the same way as in the p-brane case, just considered, one can prove that the action

\[
S_{Dp} = \int d^{p+1} \xi \mathcal{L}_{Dp} = \int d^{p+1} \xi \left\{ G_{00} - 2\lambda^i G_{0i} + \left( \lambda^i \lambda^j - \kappa^i \kappa^j \right) G_{ij} \left( 2\lambda^0 T_{Dp} \right)^2 G + 2\kappa^i \left( F_{0i} - \lambda^j F_{ji} \right) + 4\lambda^0 T_{Dp} e^{a\Phi} c_{M_0 \ldots M_p} \partial_0 X^{M_0} \ldots \partial_p X^{M_p} \right\},
\]

which possesses the necessary properties, is classically equivalent to the action (2.11). Here additional Lagrange multipliers \( \kappa^i \) are introduced, in order to linearize the quadratic term

\[
\left( F_{0i} - \lambda^k F_{ki} \right) \left( G^{-1} \right)_{ij} \left( F_{0j} - \lambda^l F_{lj} \right)
\]

arising in the action. For other actions of this type, see \([76] - [78]\).

Varying the action \( S_{Dp} \) with respect to Lagrange multipliers \( \lambda^m, \kappa^i \), and requiring these variations to vanish, one obtains the constraints

\[
G_{00} - 2\lambda^j G_{0j} + \left( \lambda^i \lambda^j - \kappa^i \kappa^j \right) G_{ij} + \left( 2\lambda^0 T_{Dp} \right)^2 G + 2\kappa^i \left( F_{0i} - \lambda^j F_{ji} \right) = 0,
\]

\[
G_{0j} - \lambda^i G_{ij} = \kappa^i F_{ij},
\]

\[
F_{0j} - \lambda^i F_{ij} = \kappa^i G_{ij}.
\]

Instead with the constraint (2.13), we will work with the simpler one

\[
G_{00} - 2\lambda^j G_{0j} + \left( \lambda^i \lambda^j + \kappa^i \kappa^j \right) G_{ij} + \left( 2\lambda^0 T_{Dp} \right)^2 G = 0,
\]

which is obtained by inserting (2.15) into (2.13).
We will use the gauge \((\lambda^m, \kappa^i) = \text{constants}\) and for simplicity, we will restrict our considerations to constant dilaton \(\Phi = \Phi_0\) and constant electro-magnetic field \(F_{mn} = F^0_{mn}\) on the Dp-brane worldvolume. In this case, the equations of motion for \(X^M\), following from (2.12), are

\[
g_{LN}
\left[
\left(\partial_0 - \lambda^i \partial_i\right) \left(\partial_0 - \lambda^j \partial_j\right) X^N - \left(2\lambda^0 T_{Dp}\right)^2 \partial_i \left( G^{ij} \partial_j X^N \right) - \kappa^i \kappa^j \partial_i \partial_j X^N \right]
+ \Gamma_{LMN}
\left[
\left(\partial_0 - \lambda^i \partial_i\right) X^M \left(\partial_0 - \lambda^j \partial_j\right) X^N \right]
- \left(2\lambda^0 T_{Dp}\right)^2 G^{ij} \partial_i X^M \partial_j X^N
- \kappa^i \kappa^j \partial_i X^M \partial_j X^N
\right] = \left(\partial_0 - \lambda^i \partial_i\right) X^M \partial_j X^N,
\]

(2.17)

where \(H^c_{p+2} = dc_{p+1}\) and \(H_3 = db_2\) are the corresponding field strengths.

3 Exact solutions in general backgrounds

The main idea in the mostly used approach for obtaining exact solutions of the probe branes equations of motion in variable external fields is to reduce the problem to a particle-like one, and even more - to solving one dimensional dynamical problem, if possible. To achieve this, one must get rid of the dependence on the spatial worldvolume coordinates \(\xi^i\). To this end, since the brane actions contain the first derivatives \(\partial_i X^M\), the brane coordinates \(X^M(\xi^a)\) have to depend on \(\xi^i\) at most linearly:

\[
X^M(\xi^0, \xi^i) = \Lambda_i^M \xi^i + Y^M(\xi^0), \quad \Lambda_i^M \text{ are arbitrary constants.}
\]

(3.1)

Besides, the background fields entering the action depend implicitly on \(\xi^i\) through their dependence on \(X^M\). If we choose \(\Lambda_i^M = 0\) in (3.1), the connection with the p-brane setting will be lost. If we suppose that the background fields do not depend on \(X^M\), the result will be constant background, which is not interesting in the case under consideration. The compromise is to accept that the external fields depend only on part of the coordinates, say \(X^a\), and to set namely for this coordinates \(\Lambda^a_i = 0\). In other words, we propose the ansatz \((X^M = (X^\mu, X^a))\):

\[
X^\mu(\xi^0, \xi^i) = \Lambda^\mu \xi^i + Y^\mu(\xi^0), \quad X^a(\xi^0, \xi^i) = Y^a(\xi^0),
\]

(3.2)

\[
\partial_\mu g_{MN} = 0, \quad \partial_\mu b_{MN} = 0, \quad \partial_\mu b_{M_0...M_p} = 0, \quad \partial_\mu c_{M_0...M_p} = 0.
\]

(3.3)

The resulting reduced Lagrangian density will depend only on \(\xi^0 = \tau\) if the Lagrange multipliers \(\lambda^m, \kappa^i\) do not depend on \(\xi^i\). Actually, this property follows from their equations of motion, from where they can be expressed through quantities depending only on the temporal worldvolume parameter \(\tau\).

Thus, we have obtained the general conditions, under which the probe branes dynamics reduces to the particle-like one. However, we will not start our considerations relaying on the generic ansatz (3.2). Instead, we will begin in the framework of the commonly used in ten spacetime dimensions static gauge: \(X^m(\xi^a) = \xi^a\). The latter is a particular case of (3.2), obtained under the following restrictions:

\[
\begin{align*}
(1): \mu = i = 1, \ldots, p; \quad (2): \Lambda^\mu_j = \Lambda_j^i = \delta_j^i; \\
(3): Y^\mu(\tau) = Y^i(\tau) = 0; \quad (4): Y^0(\tau) = \tau \in \{Y^a\}.
\end{align*}
\]

(3.4)

Therefore, the static gauge is appropriate for backgrounds which may depend on \(X^0 = Y^0(\tau)\), but must be independent on \(X^i, (i = 1, \ldots p)\). Such properties are not satisfactory in the lower
dimensions. For instance, in four dimensional black hole backgrounds, the metric depends on $X^1$, $X^2$ and the static gauge ansatz does not work. That is why, our next step is to consider the probe branes dynamics in the framework of the ansatz

$$X^\mu(\tau, \xi^i) = \Lambda_0^\mu \xi^m + \Lambda_i^\mu \xi^i, \quad X^a(\tau, \xi^i) = Y^a(\tau),$$

(3.5)

which is obtained from (3.2) under the restriction $Y^\mu(\tau) = \Lambda_0^\mu \tau$. Here, for the sake of symmetry between the worldvolume coordinates $\xi^0 = \tau$ and $\xi^i$, we have included in $X^\mu$ a term linear in $\tau$. At any time, one can put $\Lambda_0^\mu = 0$ and the corresponding terms in the formulas will disappear. Further, we will refer to the ansatz (3.5) as linear gauges, as far as $X^\mu$ are linear combinations of $\xi^m$ with arbitrary constant coefficients.

Finally, we will investigate the classical branes dynamics by using the general ansatz (3.2), rewritten in the form

$$X^\mu(\tau, \xi^i) = \Lambda_0^\mu \tau + \Lambda_i^\mu \xi^i + Y^\mu(\tau), \quad X^a(\tau, \xi^i) = Y^a(\tau).$$

(3.6)

Compared with (3.2), here we have separated the linear part of $Y^\mu$ as in the previous ansatz (3.5). This will allow us to compare the role of the term $\Lambda_0^\mu \tau$ in these two cases.

### 3.1 Static gauge dynamics

Here we begin our analysis of the probe branes dynamics in the framework of the static gauge ansatz. In order not to introduce too many type of indices, we will denote with $Y^a$, $Y^b$, etc., the coordinates, which are not fixed by the gauge. However, one have not to forget that by definition, $Y^a$ are the coordinates on which the background fields can depend. In static gauge, according to (3.3), one of this coordinates, the temporal one $Y^0(\tau)$, is fixed to coincide with $\tau$. Therefore, in this gauge, the remaining coordinates $Y^a$ are spatial ones in space-times with signature $(-, +, \ldots, +)$.

#### 3.1.1 Probe $p$-branes

In static gauge, and under the conditions (3.3), the action (2.5) reduces to (the over-dot is used for $d/d\tau$)

$$S_p^{SG} = \int d\tau L_p^{SG}(\tau), \quad V_p = \int d^p \xi,$$

$$L_p^{SG}(\tau) = \frac{V_p}{4\lambda^0} \left[ g_{ab}(Y^a) \dot{Y}^a \dot{Y}^b + 2 \left[ g_{0a}(Y^a) - \lambda^i g_{ia}(Y^a) + 2 \lambda^0 T_p b_{a1\ldots p}(Y^a) \right] \dot{Y}^a + g_{00}(Y^a) - 2 \lambda^i g_{i0}(Y^a) + \lambda^i \lambda^j g_{ij}(Y^a) - \left( 2 \lambda^0 T_p \right)^2 \det(g_{ij}(Y^a)) + 4 \lambda^0 T_p b_{01\ldots p}(Y^a) \right].$$

To have finite action, we require the fraction $V_p/\lambda^0$ to be finite one. For example, in the string case ($p = 1$) and in conformal gauge ($\lambda^1 = 0, (2\lambda^0 T_1)^2 = 1$), this means that the quantity $V_1/\alpha' = 2\pi V_1 T_1$ must be finite.

The constraints derived from the action (3.7) are:

$$g_{ab} \dot{Y}^a \dot{Y}^b + 2 \left( g_{0a} - \lambda^i g_{ia} \right) \dot{Y}^a + g_{00} - 2 \lambda^i g_{i0} + \lambda^i \lambda^j g_{ij} + \left( 2 \lambda^0 T_p \right)^2 \det(g_{ij}) = 0, \quad (3.8)$$

$$g_{0a} \dot{Y}^a + g_{00} - g_{ij} \lambda^j = 0. \quad (3.9)$$
The Lagrangian $L_{p}^{SG}$ does not depend on $\tau$ explicitly, so the energy $E_p = p_a^{SG} \dot{Y}^a - L_p^{SG}$ is conserved:

$$g_{ab} \dot{Y}^a \dot{Y}^b - g_{00} + 2 \lambda^i g_{i0} - \lambda^i \lambda^j g_{ij} + \left(2 \lambda^0 T_p \right)^2 \det(g_{ij}) = 4 \lambda^0 T_p = \text{constant.}$$

With the help of the constraints, we can replace this equality by the following one

$$g_{0a} \dot{Y}^a + g_{00} - \lambda^i g_{i0} + 2 \lambda^0 T_p b_{01...p} = -\frac{2 \lambda^0 E_p}{V_p}. \quad (3.10)$$

To clarify the physical meaning of the equalities (3.9) and (3.10), we compute the momenta (2.4) in static gauge

$$2 \lambda^0 P_{M}^{SG} = g_{Ma} \dot{Y}^a + g_{M0} - \lambda^i g_{Mi} + 2 \lambda^0 T_p b_{M1...p}. \quad (3.11)$$

The comparison of (3.11) with (3.10) and (3.9) shows that $P_0^{SG} = -E_p/V_p = \text{const}$ and $P_p^{SG} = \text{const} = 0$. Inserting these conserved momenta into (3.8), we obtain the effective constraint

$$g_{ab} \dot{Y}^a \dot{Y}^b = U^S, \quad (3.12)$$

where

$$U^S = -\left(2 \lambda^0 T_p \right)^2 \det(g_{ij}) + g_{00} - 2 \lambda^i g_{i0} + \lambda^i \lambda^j g_{ij} + 4 \lambda^0 (T_p b_{01...p} + E_p/V_p).$$

In the gauge $\lambda^m = \text{constants}$, the equations of motion following from $S_p^{SG}$ (or from (2.10) after imposing the static gauge) take the form:

$$g_{ab} \ddot{Y}^b + \Gamma_{a,0b} \dot{Y}^b \dot{Y}^c = \frac{1}{2} \partial_a U^S + 2 \partial_a A_0^S \dot{Y}^b, \quad (3.13)$$

where

$$A_0^S = g_{a0} - \lambda^i g_{ai} + 2 \lambda^0 T_p b_{a1...p}.$$ 

Thus, in general, the time evolution of the reduced dynamical system does not correspond to a geodesic motion. The deviation from the geodesic trajectory is due to the appearance of the effective scalar potential $U^S$ and of the field strength $2 \partial_a A_0^S$ of the effective $U(1)$-gauge potential $A_0^S$. In addition, our dynamical system is subject to the effective constraint (3.12).

### 3.1.2 Probe Dp-branes

In static gauge, and for background fields independent of the coordinates $X^i$ (conditions (3.3)), the reduced Lagrangian, obtained from (2.12), is given by

$$L_{p}^{SG}(\tau) = \frac{V_{Dp} e^{-a\Phi_0}}{4 \lambda^0} \left[ g_{ab} \dot{Y}^a \dot{Y}^b + g_{00} - 2 \lambda^i g_{i0} + \left(\lambda^i \lambda^j - \kappa^i \kappa^j \right) g_{ij} + 2 \left( g_{0a} - \lambda^i g_{ia} + 2 \lambda^0 T_{Dp} e^{a\Phi_0} c_{a1...p} + \kappa^i b_{ai} \right) \dot{Y}^a - \left(2 \lambda^0 T_{Dp}\right)^2 \det(g_{ij}) + 4 \lambda^0 T_{Dp} e^{a\Phi_0} c_{01...p} + 2 \kappa^i \left( b_{0i} - \lambda^j b_{ji} \right) + 4 \pi \alpha' \kappa^i \left( F_0^a - \lambda^j F_{ji}^a \right) \right].$$
As we already mentioned at the end of Section 2, we restrict our considerations to the case of constant dilaton $\Phi = \Phi_0$ and constant electro-magnetic field $F_{\mu \nu}^0$ on the D$p$-brane worldvolume.

Now, the constraints (2.16), (2.14) and (2.15) take the form

$$g_{ab} \dot{Y}^a \dot{Y}^b + 2 \left( g_{0a} - \lambda^i g_{ia} \right) \dot{Y}^a + g_{00} - 2 \dot{\lambda}^i g_{0i}$$

$$+ \left( \dot{\lambda}^i \lambda^j + \kappa^i \kappa^j \right) g_{ij} + \left( 2 \dot{\lambda}^0 T_{Dp} \right)^2 \det(g_{ij}) = 0,$$

$$g_{ij} \dot{Y}^a + g_{0j} - \dot{\lambda}^i g_{ij} - \kappa^j b_{ij} = 2 \pi \alpha' \kappa^i F_{ij}^0$$

$$b_{ij} \dot{Y}^a + b_{0j} - \lambda^i b_{ij} - \kappa^j g_{ij} = - 2 \pi \alpha' \left( F_{0j}^0 - \lambda^i F_{ij}^0 \right).$$

(3.14)

(3.15)

The reduced Lagrangian $L^{SG}$ does not depend on $\tau$ explicitly. As a consequence, the energy $E_{Dp}$ is conserved:

$$g_{ab} \dot{Y}^a \dot{Y}^b - g_{00} + 2 \dot{\lambda}^i g_{0i} - \left( \dot{\lambda}^i \lambda^j + \kappa^i \kappa^j \right) g_{ij} + \left( 2 \dot{\lambda}^0 T_{Dp} \right)^2 \det(g_{ij}) - 4 \dot{\lambda}^0 T_{Dp} e^{\alpha \Phi_0} c_{01...p}$$

$$- 2 \kappa^i \left( b_{0i} - \lambda^j b_{ji} \right) - 4 \pi \alpha' \kappa^i \left( F_{0i}^0 - \lambda^j F_{ji}^0 \right) = \frac{4 \dot{\lambda}^0 E_{Dp} e^{\alpha \Phi_0}}{V_{Dp}} = \text{constant}.$$

(3.16)

By using the constraints (3.14) and (3.15), the above equality can be replaced by the following one

$$g_{0a} \dot{Y}^a + g_{00} - \dot{\lambda}^i g_{0i} + 2 \dot{\lambda}^0 T_{Dp} e^{\alpha \Phi_0} c_{01...p} + \kappa^i \left( b_{0i} + 2 \pi \alpha' F_{0i}^0 \right) = \frac{-2 \dot{\lambda}^0 E_{Dp} e^{\alpha \Phi_0}}{V_{Dp}}.$$

(3.17)

Now, we compute the momenta, obtained from the initial action (2.12), in static gauge

$$2 \dot{\lambda}^0 e^{\alpha \Phi_0} P_{SG}^M = g_{Ma} \dot{Y}^a + g_{M0} - \dot{\lambda}^j g_{Mj} + 2 \dot{\lambda}^0 T_{Dp} e^{\alpha \Phi_0} c_{M1...p} + \kappa^j b_{Mj}.$$

(3.18)

Comparing (3.17) with (3.16) and (3.14), one finds that

$$P_{SG}^0 = - \frac{E_{Dp}}{V_{Dp}} + \frac{\pi \alpha'}{\lambda^0} e^{-\alpha \Phi_0} \kappa^j F_{0j}^0 = \text{constant},$$

$$P_{SG}^i = - \frac{\pi \alpha'}{\lambda^0} e^{-\alpha \Phi_0} \kappa^j F_{ij}^0 = \text{constants}.$$

As in the $p$-brane case, not only the energy, but also the spatial components of the momenta $P_{SG}^i$, along the $X^i$ coordinates, are conserved. In the D$p$-brane case however, $P_{SG}^i$ are not identically zero due the existence of a constant worldvolume magnetic field $F_{ij}^0$.

Inserting (3.15) and (3.16) into (3.14), one obtains the effective constraint

$$g_{ab} \dot{Y}^a \dot{Y}^b = \mathcal{U}^{DS},$$

(3.19)
where

$$A^D_{a} = g_{a0} - \lambda^i g_{ai} + 2\lambda^0 T_{Dp} e^{a\Phi_0} c_{a1\ldots p} + \kappa^i b_{ai}. $$

It is obvious that the equations of motion \[3.13\], \[3.19\] and the effective constraints \[3.12\], \[3.18\] have the same form for \(p\)-branes and for \(D_p\)-branes. The difference is in the explicit expressions for the effective scalar and 1-form gauge potentials.

### 3.2 Branes dynamics in linear gauges

Now we will repeat our analysis of the probe branes dynamics in the framework of the more general linear gauges, given by the ansatz \[3.10\]. The static gauge is a particular case of the linear gauges, corresponding to the following restrictions:

\[(1): \mu = i = 1, \ldots, p; \quad (2): \Lambda^\mu_0 = \Lambda^i_0 = 0; \quad (3): \Lambda^\mu_j = \Lambda^i_j = \delta^i_j; \quad (4): \gamma^0(\tau) = \tau \in \{\gamma^a\}.

#### 3.2.1 Probe \(p\)-branes

In linear gauges, and under the conditions \[3.3\], one obtains the following reduced Lagrangian, arising from the action \[3.3\],

$$L_p^{\text{LG}}(\tau) = \frac{V_p}{4\lambda^0} \left\{ g_{ab} \dot{Y}^a \dot{Y}^b + 2 \left[ (\Lambda^\mu_0 - \lambda^i \Lambda^\mu_i) g_{\mu a} + 2\lambda^0 T_p B_{a1\ldots p} \right] \dot{Y}^a \right\} \quad (3.20)$$

$$+ \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) (\Lambda^\nu_0 - \lambda^j \Lambda^\nu_j) g_{\nu a} - \left( 2\lambda^0 T_p \right)^2 \det(\Lambda^\mu_a \Lambda^\nu_b g_{\mu \nu})$$

$$+ 4\lambda^0 T_p \Lambda^\mu_0 B_{a1\ldots p} \}, \quad B_{M1\ldots p} \equiv b_{M1\ldots p} \Lambda^\mu_1 \ldots \Lambda^\mu_p.$$

The constraints derived from the Lagrangian \[3.20\] are:

$$g_{ab} \dot{Y}^a \dot{Y}^b + 2 \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) g_{\mu a} \dot{Y}^a + \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right)$$

$$\times \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\nu a} + \left( 2\lambda^0 T_p \right)^2 \det(\Lambda^\mu_a \Lambda^\nu_b g_{\mu \nu}) = 0,$$

$$\Lambda^\mu_i \left[ g_{\mu a} \dot{Y}^a + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\nu a} \right] = 0. \quad (3.22)$$

The Lagrangian \(L_p^{\text{LG}}\) does not depend on \(\tau\) explicitly, so the energy \(E_p = p_0^{\text{LG}} \dot{Y}^a - L_p^{\text{LG}}\) is conserved:

$$g_{ab} \dot{Y}^a \dot{Y}^b - \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\mu \nu} + \left( 2\lambda^0 T_p \right)^2 \det(\Lambda^\mu_a \Lambda^\nu_b g_{\mu \nu})$$

$$- 4\lambda^0 T_p \Lambda^\mu_0 B_{a1\ldots p} = \frac{4\lambda^0 E_p}{V_p} = \text{constant}.$$

With the help of the constraints \[3.21\] and \[3.22\], one can replace this equality by the following one

$$\Lambda^\mu_0 \left[ g_{\mu a} \dot{Y}^a + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\nu a} + 2\lambda^0 T_p B_{a1\ldots p} \right] = -\frac{2\lambda^0 E_p}{V_p}. \quad (3.23)$$

In linear gauges, the momenta \[2.4\] take the form

$$2\lambda^0 F^{\text{LG}}_M = g_{Ma} \dot{Y}^a + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{M a} + 2\lambda^0 T_p B_{M1\ldots p}. \quad (3.24)$$
The comparison of (3.21) with (3.23) and (3.22) gives
\[ \Lambda_0^\mu P_\mu^{LG} = \frac{E_p}{V_p} = \text{constant}, \quad \Lambda_i^\mu P_\mu^{LG} = \text{constants} = 0. \]

Therefore, in the linear gauges, the projections of the momenta \( P_\mu^{LG} \) onto \( \Lambda_0^\mu \) are conserved. Moreover, as far as the Lagrangian (3.20) does not depend on the coordinates \( X^\mu \), the corresponding conjugated momenta \( P^{LG}_\mu \) are also conserved.

Inserting (3.23) and (3.22) into (3.21), we obtain the effective constraint
\[ g_{ab} \dot{Y}_a \dot{Y}_b = U^L, \]
where the effective scalar potential is given by
\[ U^L = -\left(2\lambda^0 T_p\right)^2 \det(\Lambda_0^\mu - \lambda^i \Lambda_i^\mu) g_{\mu\nu} + \left(\Lambda_0^\mu - \lambda^i \Lambda_i^\mu\right) \left(\Lambda_0^\nu - \lambda^j \Lambda_j^\nu\right) g_{\mu\nu} + 4\lambda^0 \left(T_p \Lambda_0^\mu B_{\mu1...p} + E_p \right). \]

In the gauge \( \lambda^m = \text{constants} \), the equations of motion following from \( L^{LG}_P \) take the form:
\[ g_{ab} \ddot{Y}_a + \Gamma_{a,be} \dot{Y}_b \dot{Y}_c = \frac{1}{2} \partial_a U^L + 2 \partial_{\dot{a}} A^L_{\dot{a}} \dot{Y}_b, \]
where
\[ A^L_{\dot{a}} = \left(\Lambda_0^\mu - \lambda^i \Lambda_i^\mu\right) g_{\mu\dot{a}} + 2\lambda^0 T_p B_{a1...p}, \]
is the effective 1-form gauge potential, generated by the non-diagonal components \( g_{a\mu} \) of the background metric and by the components \( b_{a1...p} \) of the background \((p + 1)\)-form gauge field.

### 3.2.2 Probe Dp-branes

In linear gauges, and for background fields independent of the coordinates \( X^\mu \) (conditions (3.3)), the reduced Lagrangian, obtained from (2.12), is given by
\[ L^{LG}_{Dp}(\tau) = \frac{V_{Dp} e^{-\alpha \Phi_0}}{4\lambda^0} \left\{ g_{ab} \dot{Y}_a \dot{Y}_b + \left(\Lambda_0^\mu - \lambda^i \Lambda_i^\mu\right) \left(\Lambda_0^\nu - \lambda^j \Lambda_j^\nu\right) g_{\mu\nu} \right. \]
\[ + 2 \left(\Lambda_0^\mu - \lambda^i \Lambda_i^\mu\right) g_{\mu a} + 2\lambda^0 T_{Dp} e^{-\alpha \Phi_0} C_{a1...p} + \kappa^j \Lambda_i^\mu b_{a\mu} \right] \dot{Y}_a \]
\[ - \left(2\lambda^0 T_{Dp}\right)^2 \det(\Lambda_i^\mu \Lambda_i^\nu g_{\mu\nu} + 4\lambda^0 T_{Dp} e^{-\alpha \Phi_0} \Lambda_0^\mu C_{a1...p} \right) \]
\[ - 2\kappa^j \Lambda_i^\mu \left(\Lambda_0^\nu - \lambda^j \Lambda_j^\nu\right) b_{\mu\nu} + 4\alpha' \kappa^j \left(\tilde{F}_0^\mu - \lambda^j F_j^\mu\right) \right\}, \]

where the following shorthand notation has been introduced
\[ C_{M1...p} = c_{M1...p} \Lambda_i^{\mu_1} \ldots \Lambda_i^{\mu_p}. \]

Now, the constraints (2.16), (2.17), and (2.18) take the form
\[ g_{ab} \ddot{Y}_a \dot{Y}_b + 2 \left(\Lambda_0^\mu - \lambda^i \Lambda_i^\mu\right) g_{\mu\dot{a}} \dot{Y}_a + \left(2\lambda^0 T_{Dp}\right)^2 \det(\Lambda_i^\mu \Lambda_i^\nu g_{\mu\nu}) = 0, \]
\[ \Lambda_i^\mu \left[g_{b\mu} \dot{Y}_a + \left(\Lambda_0^\nu - \lambda^j \Lambda_j^\nu\right) g_{\mu\nu} + \kappa^j \Lambda_i^\mu b_{\mu\nu}\right] = 2\pi \alpha' \kappa^j F_{j\dot{a}}^\nu, \]
\[ \Lambda_i^\mu \left[b_{\mu a} \dot{Y}_a + \left(\Lambda_0^\nu - \lambda^j \Lambda_j^\nu\right) b_{a\nu} + \kappa^j \Lambda_i^\mu g_{\mu\nu}\right] = 2\pi \alpha' \left(\tilde{F}_0^\mu - \lambda^j F_j^\mu\right). \]
The reduced Lagrangian $L^\text{LG}_{Dp}$ does not depend on $\tau$ explicitly. As a consequence, the energy $E_{Dp}$ is conserved:

$$g_{ab} \hat{Y}^a \hat{Y}^b - \left[ \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) - \kappa^i \kappa^j \Lambda^\mu_i \Lambda^\nu_j \right] g_{\mu\nu} + (2\lambda^0 T_{Dp})^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}) - 4\lambda^0 T_{Dp} e^{a\Phi_0} \Lambda^\mu_0 C_{\mu1...p}$$

$$- 2\kappa^i \Lambda^\mu_i \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) b_{\mu\nu} - 4\pi\alpha' \kappa^i \left( F^\alpha_{0i} - \lambda^j F^\alpha_{j0} \right) = \frac{4\lambda^0 E_{Dp}}{V_{Dp}} e^{a\Phi_0} = \text{constant}.$$

By using the constraints (3.25) and (3.26), the above equality can be replaced by the following one:

$$\Lambda^\mu_0 g_{\mu\nu} + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\mu\nu} + 2\lambda^0 T_{Dp} e^{a\Phi_0} C_{\mu1...p} + \kappa^j \Lambda^\nu_j b_{\mu\nu}$$

$$+ 2\pi\alpha' \kappa^i F^\alpha_{0i} = -\frac{2\lambda^0 E_{Dp}}{V_{Dp}} e^{a\Phi_0}.$$  (3.27)

In linear gauges, the momenta obtained from the initial action (2.12), are

$$2\lambda^0 e^{a\Phi_0} P^\text{LG}_M = g_{M0} \hat{Y}^a + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{M\nu} + 2\lambda^0 T_{Dp} e^{a\Phi_0} C_{M1...p} + \kappa^j \Lambda^\nu_j b_{M\nu}.  \quad (3.28)$$

Comparing (3.25) with (3.27) and (3.26), one finds that the following equalities hold

$$\Lambda^\mu_0 P^\text{LG}_\mu = -\left( \frac{E_{Dp}}{V_{Dp}} + \frac{\pi\alpha'}{\lambda^0} e^{-a\Phi_0} \kappa^j F^\alpha_{0j} \right) = \text{constant},$$

$$\Lambda^\mu_0 P^\text{LG}_\mu = -\frac{\pi\alpha'}{\lambda^0} e^{-a\Phi_0} \kappa^j F^\alpha_{0j} = \text{constants}.  \quad (3.29)$$

They may be viewed as restrictions on the number of the arbitrary parameters, presented in the theory.

As in the $p$-brane case, the momenta $P^\text{LG}_\mu$ are conserved quantities, due to the independence of the Lagrangian on the coordinates $X^\mu$.

Inserting (3.26) and (3.27) into (3.25), one obtains the effective constraint

$$g_{ab} \hat{Y}^a \hat{Y}^b = U^{DL},$$

where

$$U^{DL} = \left[ \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) - \kappa^i \kappa^j \Lambda^\mu_i \Lambda^\nu_j \right] g_{\mu\nu}$$

$$- (2\lambda^0 T_{Dp})^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}) + 4\lambda^0 e^{a\Phi_0} \left( T_{Dp} \Lambda^\mu_0 C_{\mu1...p} + \frac{E_{Dp}}{V_{Dp}} \right)$$

$$+ 2\kappa^i \Lambda^\mu_i \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) b_{\mu\nu} + 4\pi\alpha' \kappa^i \left( F^\alpha_{0i} - \lambda^j F^\alpha_{j0} \right).$$

In the gauge $(\lambda^m, \kappa^i) = \text{constants}$, the equations of motion following from $L^\text{LG}_{Dp}$ take the form:

$$g_{ab} \ddot{Y}^b + \Gamma_{a,bc} \dot{Y}^b \dot{Y}^c = \frac{1}{2} \partial_a U^{DL} + 2\partial_b A^a_{Dp} \dot{Y}^b, \quad (3.30)$$

where

$$A^a_{DL} = \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) g_{a\mu} + 2\lambda^0 T_{Dp} e^{a\Phi_0} C_{a1...p} + \kappa^i \Lambda^\mu_i b_{a\mu}.$$  

It is clear that the equations of motion and the effective constraints have the same form for $p$-branes and for $Dp$-branes in linear gauges, as well as in static gauge. The only difference is in the explicit expressions for the effective scalar and 1-form gauge potentials.
3.3 Branes dynamics in the whole space-time

Working in static gauge \( X^\alpha(\xi^m) = \xi^m \), we actually imply that the probe branes have no dynamics along the background coordinates \( x^m \). The (proper) time evolution is possible only in the transverse directions, described by the coordinates \( x^a \).

Using the linear gauges, we have the possibility to place the probe branes in general position with respect to the coordinates \( x^b \), on which the background fields do not depend. However, the real dynamics is again in the transverse directions only.

Actually, in the framework of our approach, the probe branes can have 'full' dynamical freedom only when the ansatz (3.6) is used, because only then all of the brane coordinates \( X^M \) are allowed to vary nonlinearly with the proper time \( \tau \). Therefore, with the help of (3.6), we can probe the whole space-time.

We will use the superscript \( A \) to denote that the corresponding quantity is taken on the ansatz \( 3.6 \). It is understood that the conditions (3.3) are also fulfilled.

3.3.1 Probe \( p \)-branes

Now, the reduced Lagrangian obtained from the action (2.5) is given by

\[
L_p^A(\tau) = \frac{V_p}{4A^0} \left\{ g_{MN} Y^M Y^N + 2 \left( (\Lambda^\mu_0 - \lambda^j \Lambda^\mu_j) g_{\mu N} Y^N + (\Lambda^\mu_0 - \lambda^j \Lambda^\mu_j) \right) + \left( \Lambda^\mu_0 - \lambda^j \Lambda^\mu_j \right) \left( \Lambda^\mu_0 - \lambda^j \Lambda^\mu_j \right) g_{\mu \nu} - (2\lambda^0 T_p)^2 \det(\Lambda^\mu_0 \Lambda^\mu_j g_{\mu \nu}) + 4\lambda^0 T_p \Lambda^\mu_0 \Lambda^\mu_j B_{\mu 1...p} \right\}.
\]

The constraints, derived from the above Lagrangian, are:

\[
g_{MN} Y^M Y^N + 2 \left( (\Lambda^\mu_0 - \lambda^j \Lambda^\mu_j) g_{\mu N} Y^N + (\Lambda^\mu_0 - \lambda^j \Lambda^\mu_j) \right) \times \left( \Lambda^\mu_0 - \lambda^j \Lambda^\mu_j \right) g_{\mu \nu} + (2\lambda^0 T_p)^2 \det(\Lambda^\mu_0 \Lambda^\mu_j g_{\mu \nu}) = 0,
\]

\[
\Lambda^\mu_0 \left[ g_{\mu N} Y^N + (\Lambda^\mu_0 - \lambda^j \Lambda^\mu_j) g_{\mu \nu} \right] = 0.
\]

The corresponding momenta are \( P_M = P_M^A / V_p \)

\[
2\lambda^0 P_M = g_{MN} Y^N + \left( \Lambda^\mu_0 - \lambda^j \Lambda^\mu_j \right) g_{M \nu} + 2\lambda^0 T_p B_{M 1...p},
\]

and part of them, \( P_\mu \), are conserved

\[
g_{\mu N} Y^N + \left( \Lambda^\mu_0 - \lambda^j \Lambda^\mu_j \right) g_{\mu \nu} + 2\lambda^0 T_p B_{\mu 1...p} = 2\lambda^0 P_\mu = \text{constants},
\]

because \( L_p^A \) does not depend on \( X^\mu \). From (3.30) and (3.31), the compatibility conditions follow

\[
\Lambda^\mu_0 P_\mu = 0.
\]

We will regard on (3.32) as a solution of the constraints (3.30), which restricts the number of the arbitrary parameters \( \Lambda^\mu_0 \) and \( P_\mu \). That is why from now on, we will deal only with the constraint (3.29).

In the gauge \( \lambda^m = \text{constants} \), the equations of motion for \( Y^N \), following from \( L_p^A \), have the form

\[
g_{LN} Y^N + \Gamma_{L,MN} Y^M Y^N = \frac{1}{2} \partial_L U^m + 2\partial_L \mathcal{A}_N^{in} Y^N,
\]

(3.33)
where

\[ U^{\text{in}} = - \left( 2^0 T_p \right)^2 \det(\Lambda^\mu_0 \Lambda^\nu_i g_{\mu\nu}) + \left( \Lambda^\mu_0 - \lambda_i \Lambda^\mu_i \right) \left( \Lambda^\nu_j - \lambda^j \Lambda^\nu_j \right) g_{\mu\nu} + 4\lambda^0 T_p \Lambda^\mu_0 B_{\mu1...p}, \]

\[ A^{\text{in}}_N = \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) g_{N\mu} + 2\lambda^0 T_p B_{N1...p}. \]

Let us first consider this part of the equations of motion (3.33), which corresponds to \( L = \lambda \). It follows from (3.3) that the connection coefficients \( \Gamma_{\lambda, MN} \), involved in these equations, are

\[ \Gamma_{\lambda, ab} = \frac{1}{2} (\partial_a g_{b\lambda} + \partial_b g_{a\lambda}), \quad \Gamma_{\lambda, a\mu} = \frac{1}{2} \partial_a g_{\mu\lambda}, \quad \Gamma_{\lambda, \mu\nu} = 0. \]

Inserting these expressions in the part of the differential equations (3.33) corresponding to \( L = \lambda \) and using that \( \dot{g}_{MN} = \dot{Y}^a \partial_a g_{MN}, \dot{B}_{M1...p} = \dot{Y}^a \partial_a B_{M1...p} \), one receives

\[ \frac{d}{d\tau} \left[ g_{\mu N} \dot{Y}^N + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\mu\nu} + 2\lambda^0 T_p B_{\mu1...p} \right] = 0. \]

These equalities express the fact that the momenta \( P_\mu \) are conserved (compare with (3.31)). Therefore, we have to deal only with the other part of the equations of motion, corresponding to \( L = a \)

\[ g_{aN} \dot{Y}^N + \Gamma_{a, MN} \dot{Y}^M \dot{Y}^N = \frac{1}{2} \partial_a U^{\text{in}} + 2 \partial_a A^{\text{in}}_N \dot{Y}^N. \] (3.34)

Our next task is to separate the variables \( \dot{Y}^\mu \) and \( \dot{Y}^a \) in these equations and in the constraint (3.29). To this end, we will use the conservation laws (3.31) to express \( \dot{Y}^\mu \) through \( \dot{Y}^a \). The result is

\[ \dot{Y}^\mu = \left( g^{-1} \right)^{\mu\nu} \left[ 2\lambda^0 (P_\nu - T_p B_\nu1...p) - g_{\nu a} \dot{Y}^a \right] - (\Lambda^\mu_0 - \lambda^i \Lambda^\mu_i). \] (3.35)

We will need also the explicit expressions for the connection coefficients \( \Gamma_{a, \mu b} \) and \( \Gamma_{a, \mu\nu} \), which under the conditions (3.3) reduce to

\[ \Gamma_{a, \mu b} = - \frac{1}{2} (\partial_a g_{b\mu} - \partial_b g_{a\mu}) = - \partial_a g_{b\mu}, \quad \Gamma_{a, \mu\nu} = - \frac{1}{2} \partial_a g_{\mu\nu}. \] (3.36)

By using (3.35) and (3.36), after some calculations, one rewrites the equations of motion (3.34) and the constraint (3.29) in the form

\[ h_{ab} \dot{Y}^b + \Gamma_{a, bc}^{\text{h}} \dot{Y}^b \dot{Y}^c = \frac{1}{2} \partial_a U^A + 2 \partial_a A_{\mu}^{\text{h}} \dot{Y}^b, \] (3.37)

\[ h_{ab} \dot{Y}^a \dot{Y}^b = U^A, \] (3.38)

where a new, effective metric appeared

\[ h_{ab} = g_{ab} - g_{a\mu} (g^{-1})^{\mu\nu} g_{\nu b}. \]

\( \Gamma_{a, bc}^{\text{h}} \) is the connection compatible with this metric

\[ \Gamma_{a, bc}^{\text{h}} = \frac{1}{2} \left( \partial_b h_{ca} + \partial_c h_{ba} - \partial_a h_{bc} \right). \]
The new, effective scalar and gauge potentials are given by

$$U^A = - \left( 2 \lambda^0 T_p \right)^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}) - (2 \lambda^0)^2 \left( P_\mu - T_p B_{\mu11...p} \right) \left( g^{-1} \right)^{\mu\nu} \left( P_\nu - T_p B_{\nu11...p} \right),$$

$$A^A_a = 2 \lambda^0 \left[ g_{\mu\nu} \left( g^{-1} \right)^{\mu\nu} \left( P_\nu - T_p B_{\nu11...p} + T_p B_{a11...p} \right) \right].$$

We note that Eqs. (3.37), (3.38), and therefore their solutions, do not depend on the parameters $\Lambda^\mu_0$ and $\lambda^i$, in contrast to the previously considered cases. However, they have the same form as before.

### 3.3.2 Probe D$p$-branes

The reduced Lagrangian, obtained from (2.4), is given by

$$L^A_{Dp}(\tau) = \frac{V_{Dp} e^{-\alpha \phi_0}}{4 \lambda^0} \left\{ g_{MN} Y^M Y^N + \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) - \kappa^i \kappa^j \Lambda^\mu_i \Lambda^\nu_j \right\} g_{\mu\nu}$$

$$+ 2 \left[ \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) g_{\mu\nu} + 2 \lambda^0 T_{Dp} e^{\alpha \phi_0} C_{N11...p} + \kappa^i \Lambda^\mu_i b_{N\mu} \right] Y^N$$

$$- \left( 2 \lambda^0 T_{Dp} \right)^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}) + 4 \lambda^0 T_{Dp} e^{\beta \phi_0} A^0_{\mu11...p}$$

$$- 2 \kappa^i \Lambda^\mu_i \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) b_{\mu\nu} + 4 \pi \alpha' \kappa^i \left( F^\mu_0 - \lambda^j F^\mu_{ji} \right).$$

The constraints (2.16), (2.17), and (2.18) take the form

$$g_{MN} Y^M Y^N + 2 \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) g_{\mu\nu} Y^N + \left( 2 \lambda^0 T_{Dp} \right)^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu})$$

$$\left[ \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) + \kappa^i \kappa^j \Lambda^\mu_i \Lambda^\nu_j \right] g_{\mu\nu} = 0,$$

$$\Lambda^\mu_i \left[ g_{\mu\nu} Y^N + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\mu\nu} + \kappa^j \Lambda^\nu_j b_{\mu\nu} \right] = 2 \pi \alpha' \kappa^i F^\mu_0 \quad (3.40)$$

$$\Lambda^\mu_i \left[ b_{\mu\nu} Y^N + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) b_{\mu\nu} + \kappa^j \Lambda^\nu_j g_{\mu\nu} \right] = 2 \pi \alpha' \left( F^\mu_0 - \lambda^j F^\mu_{ji} \right). \quad (3.41)$$

Because of the independence of $L^A_{Dp}$ on $X^\mu$, the momenta $P^D_\mu = P^{DA}/V_{Dp}$ are conserved

$$2 \lambda^0 e^{\alpha \phi_0} P^D_\mu = g_{\mu\nu} Y^N + \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{\mu\nu} + 2 \lambda^0 T_{Dp} e^{\alpha \phi_0} C_{\mu11...p} + \kappa^j \Lambda^\nu_j b_{\mu\nu} = \text{constants}. \quad (3.42)$$

From (3.40) and (3.42), one obtains the following compatibility conditions

$$\Lambda^\nu_j F^\mu_\nu = \frac{\pi \alpha'}{\lambda^0} e^{-\alpha \phi_0} \kappa^i F^\mu_{ij},$$

which we interpret as a solution of the constraints (3.40).

In the gauge $(\lambda^m, \kappa^i) = \text{constants}$, the equations of motion for $Y^N$, following from $L^A_{Dp}$, take the form

$$g_{MN} Y^M Y^N + \Gamma_{L,MN} Y^M Y^N = \frac{1}{2} \partial_L U^{Din} + 2 \partial_L A^N_{Din} Y^N, \quad (3.43)$$

where

$$U^{Din} = - \left( 2 \lambda^0 T_p \right)^2 \det(\Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}) + \left[ \left( \Lambda^\mu_0 - \lambda^i \Lambda^\mu_i \right) \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) - \kappa^i \kappa^j \Lambda^\mu_i \Lambda^\nu_j \right] g_{\mu\nu}$$

$$+ 4 \lambda^0 T_{Dp} e^{\alpha \phi_0} \Lambda^\mu_0 C_{\mu11...p} - 2 \kappa^i \Lambda^\mu_i \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) b_{\mu\nu},$$

$$A^N_{Din} = \left( \Lambda^\nu_0 - \lambda^j \Lambda^\nu_j \right) g_{N\nu} + 2 \lambda^0 T_{Dp} e^{\alpha \phi_0} C_{N11...p} + \kappa^j \Lambda^\nu_j b_{N\nu}.$$
As in the $p$-brane case, this part of the equations of motion (3.43), which corresponds to $L = \lambda$, expresses the conservation of the momenta $P_\mu^D$, in accordance with (3.42). The remaining equations of motion, which we have to deal with, are

$$g_{aN} \dot{Y}^N + \Gamma_{aMN} Y^M \dot{Y}^N = \frac{1}{2} \partial_a U^{DA} + 2 \partial_{[a} A_{bN]}^{DA} \dot{Y}^N. \quad (3.44)$$

To exclude the dependence on $\dot{Y}^\mu$ in the Eqs. (3.43), we use the conservation laws (3.42) to express $\dot{Y}^\mu$ through $\dot{Y}^a$:

$$\dot{Y}^\mu = \left( g^{-1} \right)^{\mu\nu} \left[ 2 \lambda^0 e^{\alpha_0}(P_{\nu}^D - T_{Dp} C_{\nu1...p}) - \kappa^i \Lambda^\nu_\lambda b_{\nu\lambda} \right] - (\Lambda_0^\mu - \lambda^i \Lambda^\mu_i). \quad (3.45)$$

By using (3.45) and (3.36), one can rewrite the equations of motion (3.44) and the constraint (3.39) as

$$h_{ab} \dot{Y}^b + \Gamma^b_{a,be} \dot{Y}^b \dot{Y}^c = \frac{1}{2} \partial_a U^{DA} + 2 \partial_{[a} A_{bN]}^{DA} \dot{Y}^b, \quad (3.46)$$

$$h_{ab} \dot{Y}^a \dot{Y}^b = U^{DA}. \quad (3.47)$$

Now, the effective scalar and 1-form gauge potentials are given by

$$U^{DA} = - \left( 2 \lambda^0 T_p \right)^2 \det(\Lambda^\mu_\lambda \Lambda^\nu_i g_{\mu\nu}) - \kappa^i \Lambda^\mu_i \Lambda^\nu_j g_{\mu\nu}$$

$$- \left[ 2 \lambda^0 e^{\alpha_0}(P_{\nu}^D - T_{Dp} C_{\nu1...p}) - \kappa^i \Lambda^\nu_\lambda b_{\nu\lambda} \right] \left( g^{-1} \right)^{\mu\nu}$$

$$\times \left[ 2 \lambda^0 e^{\alpha_0}(P_{\nu}^D - T_{Dp} C_{\nu1...p}) - \kappa^j \Lambda^\nu_\mu b_{\nu\mu} \right],$$

$$A^{DA}_a = g_{a\mu} \left( g^{-1} \right)^{\mu\nu} \left[ 2 \lambda^0 e^{\alpha_0}(P_{\nu}^D - T_{Dp} C_{\nu1...p}) - \kappa^j \Lambda^\nu_\mu b_{\nu\mu} \right]$$

$$+ 2 \lambda^0 T_{Dp} e^{\alpha_0} C_{a1...p} + \kappa^i \Lambda^\mu_i b_{a\mu}.$$  

Eqs. (3.46), (3.47), have the same form as in static and linear gauges, but now they do not depend on the parameters $\Lambda^\nu_0$ and $\lambda^i$. Another difference is the appearance of a new, effective background metric $h_{ab}$ and the corresponding connection $\Gamma^b_{a,be}$. In the D-brane case, we have another set of constraints (3.41), generated by the Lagrange multipliers $\kappa^i$. With the help of (3.45), they acquire the form

$$\left\{ b_{a\nu} - g_{a\rho} \left( g^{-1} \right)^{\rho\mu} b_{\mu\nu} \right\} \dot{Y}^a + b_{\mu\nu} \left( g^{-1} \right)^{\mu\nu} \left[ 2 \lambda^0 e^{\alpha_0}(P_{\nu}^D - T_{Dp} C_{\nu1...p}) - \kappa^j \Lambda^\nu_\mu b_{\nu\mu} \right]$$

$$- \kappa^i \Lambda^\mu_i g_{\mu\nu} \Lambda^\nu_j = -2\pi \alpha' \left( F^0_{ij} - \lambda^i F^0_{ij} \right).$$

### 3.4 Explicit solutions of the equations of motion

All cases considered so far, have one common feature. The dynamics of the corresponding reduced particle-like system is described by effective equations of motion and one effective constraint, which have the same form, independently of the ansatz used to reduce the $p$-branes or Dp-branes dynamics. Our aim here is to find explicit exact solutions to them. To be able to describe all cases simultaneously, let us first introduce some general notations.

---

3The additional restrictions on the solutions, depending on the ansatz and on the type of the branes, will be discussed in the next section.
We will search for solutions of the following system of nonlinear differential equations

\[ G_{ab} \dddot{Y}^b + \Gamma^b_{a, bc} \dddot{Y}^b \dot{Y}^c = \frac{1}{2} \partial_a U + 2 \partial_{[a} A_{b]} \dddot{Y}^b, \]  
(3.48)

\[ G_{ab} \dddot{Y}^a \dddot{Y}^b = U, \]  
(3.49)

where \( G_{ab}, \Gamma^b_{a, bc}, U, \) and \( A_a \) can be as follows

\[ G_{ab} = (g_{ab}, h_{ab}), \quad \Gamma^b_{a, bc} = (\Gamma^b_{a, bc}, \Gamma^h_{a, bc}), \]
\[ U = (U^S, U^{DS}, U^L, U^{DL}, U^A, U^{DA}), \]
\[ A_a = (A^S_a, A^{DS}_a, A^L_a, A^{DL}_a, A^A_a, A^{DA}_a), \]

depending on the ansatz and on the type of the brane (p-brane or Dp-brane).

Let us start with the simplest case, when the background fields depend on only one coordinate \( X^a = Y^a(\tau) \). In this case the Eqs. (3.48), (3.49) simplify to

\[ \frac{d}{d\tau} (G_{aa} \dot{Y}^a) - \frac{1}{2} d_a G_{aa} (\dot{Y}^a)^2 = \frac{1}{2} d_a U, \]  
(3.50)

\[ G_{aa} (\dot{Y}^a)^2 = U, \]  
(3.51)

where we have used that

\[ G_{ab} \dddot{Y}^b + \Gamma^b_{a, bc} \dddot{Y}^b \dot{Y}^c = \frac{d}{d\tau} (G_{ab} \dddot{Y}^b) - \frac{1}{2} \partial_a G_{bc} \dddot{Y}^b \dot{Y}^c. \]

After multiplying with \( 2G_{aa} \dddot{Y}^a \) and after using the constraint (3.51), the Eq. (3.50) reduces to

\[ \frac{d}{d\tau} \left[ (G_{aa} \dddot{Y}^a)^2 - G_{aa} U \right] = 0. \]  
(3.52)

The solution of (3.52), compatible with (3.51), is just the constraint (3.51). In other words, (3.51) is first integral of the equation of motion for the coordinate \( Y^a \). By integrating (3.51), one obtains the following exact probe branes solution

\[ \tau (X^a) = \tau_0 \pm \int_{X^a_0}^{X^a} \left( \frac{U}{G_{aa}} \right)^{1/2} dx, \]  
(3.53)

where \( \tau_0 \) and \( X^a_0 \) are arbitrary constants.

When one works in the framework of the general ansatz (3.6), one has to also write down the solution for the remaining coordinates \( X^\mu \). It can be obtained as follows. One represents \( \dot{Y}^\mu \) as

\[ \dot{Y}^\mu = \frac{dY^\mu}{dY^a} \dot{Y}^a, \]

and use this and (3.51) in (3.35) for the p-brane, and in (3.45) for the Dp-brane. The result is a system of ordinary differential equations of first order with separated variables, which integration

\footnote{An example of such background is the generalized Kasner type metric, arising in the superstring cosmology [79] (see also [80], [81]).}
is straightforward. Replacing the obtained solution for $Y^\mu(X^a)$ in the ansatz (3.6), one finally arrives at

$$X^\mu(X^a, \xi^i) = X_0^\mu + \Lambda_\mu^i \left[ \lambda^i \tau(X^a) + \xi^i \right]$$

(3.54)

$$- \int_{X_0^a}^{X^a} \left( g^{-1} \right)^\mu\nu \left[ g_{\nu a} \mp 2\lambda^0 (P_\nu - T_\nu B_{\nu1...p}) \left( \frac{U^A}{h_{aa}} \right)^{-1/2} \right] dx$$

for the p-brane case, and at

$$X^\mu(X^a, \xi^i) = X_0^\mu + \Lambda_\mu^i \left[ \lambda^i \tau(X^a) + \xi^i \right]$$

(3.55)

$$- \int_{X_0^a}^{X^a} \left( g^{-1} \right)^\mu\nu \left\{ g_{\nu a} \mp 2\lambda^0 e^{\Phi_0} (P_\nu^D - T_{Dp} C_{\nu1...p}) - \kappa^i \Lambda_\rho^i b_{\nu \rho} \right\} \left( \frac{U^{DA}}{h_{aa}} \right)^{-1/2} dx$$

for the Dp-brane case correspondingly. In the above two exact branes solutions, $X_0^\mu$ are arbitrary constants, and $\tau(X^a)$ is given in (3.53). We note that the comparison of the solutions $X^\mu(X^a, \xi^i)$ with the initial ansatz (3.6) shows, that the dependence on $\Lambda_\mu^0$ has disappeared. We will comment on this later on.

Let us turn to the more complicated case, when the background fields depend on more than one coordinate $X^a = Y^a(\tau)$. We would like to apply the same procedure for solving the system of differential equations (3.48), (3.49), as in the simplest case just considered. To be able to do this, we need to suppose that the metric $G_{ab}$ is a diagonal one. Then one can rewrite the effective equations of motion (3.48) and the effective constraint (3.49) in the form

$$\frac{d}{d\tau} \left( G_{aa} \dot{Y}^a \right)^2 - \dot{Y}^a \partial_a (G_{aa} U) + \dot{Y}^a \sum_{b \neq a} \partial_a \left( \frac{G_{aa}}{G_{bb}} \right) \left( \dot{G}_{bb} \dot{Y}^b \right)^2 - 4\partial_{[a} A_{b]} G_{aa} Y^b \right] = 0,$$

(3.56)

$$G_{aa} \left( \dot{Y}^a \right)^2 + \sum_{b \neq a} G_{bb} \left( \dot{Y}^b \right)^2 = U.$$

(3.57)

To find solutions of the above equations without choosing particular background, we fix all coordinates $X^a$ except one. Then the exact probe brane solution of the equations of motion is given again by the same expression (3.53) for $\tau(X^a)$. In the case when one is using the general ansatz (3.6), the solutions (3.54) and (3.55) still also hold.

To find solutions depending on more than one coordinate, we have to impose further conditions on the background fields. Let us show, how a number of sufficient conditions, which allow us to reduce the order of the equations of motion by one, can be obtained.

First of all, we split the index $a$ in such a way that $Y^r$ is one of the coordinates $Y^a$, and $Y^\alpha$ are the others. Then we assume that the effective 1-form gauge field $A_a$ can be represented in the form

$$A_a = (A_r, A_\alpha) = (A_r, \partial_a f),$$

(3.58)

i.e., it is oriented along the coordinate $Y^r$, and the remaining components $A_\alpha$ are pure gauges. Now, the Eq. (3.56) read

$$\frac{d}{d\tau} \left( G_{aa} \dot{Y}^a \right)^2 - \dot{Y}^a \partial_a (G_{aa} U)$$

(3.59)
\[ +\dot{Y}^\alpha \left[ \partial_\alpha \left( \frac{G_{\alpha a}}{G_{rr}} \right) \left( G_{rr} \dot{Y}^r \right)^2 - 2G_{\alpha a} \partial_\alpha (A_r - \partial_r f) \dot{Y}^r \right] \\
+\dot{Y}^\alpha \sum_{\beta \neq \alpha} \partial_\alpha \left( \frac{G_{\alpha a}}{G_{\beta \beta}} \right) \left( G_{\beta \beta} \dot{Y}^\beta \right)^2 = 0, \]
\[ \frac{d}{d\tau} \left( G_{rr} \dot{Y}^r \right)^2 - \dot{Y}^r \partial_r (G_{rr} U) \]
\[ +\dot{Y}^r \sum_\alpha \left[ \partial_r \left( \frac{G_{rr}}{G_{\alpha \alpha}} \right) \left( G_{\alpha \alpha} \dot{Y}^\alpha \right)^2 + 2G_{rr} \partial_\alpha (A_r - \partial_r f) \dot{Y}^\alpha \right] = 0. \]

After imposing the conditions
\[ \partial_\alpha \left( \frac{G_{\alpha a}}{G_{aa}} \right) = 0, \quad \partial_\alpha \left( \frac{G_{rr}}{G_{\alpha \alpha}} \dot{Y}^\alpha \right)^2 = 0, \quad (3.61) \]
the Eq. (3.59) reduce to
\[ \frac{d}{d\tau} \left( G_{rr} \dot{Y}^r \right)^2 - \dot{Y}^r \partial_r \left\{ G_{rr} \left[ U + 2 (A_r - \partial_r f) \dot{Y}^r \right] \right\} = 0, \]
which are solved by
\[ \left( G_{\alpha \alpha} Y^\alpha \right)^2 = D_\alpha \left( Y^{a \neq \alpha} \right) + G_{\alpha \alpha} \left[ U + 2 (A_r - \partial_r f) \dot{Y}^r \right] = E_\alpha \left( Y^\beta \right) \geq 0, \quad (3.62) \]
where \( D_\alpha, E_\alpha \) are arbitrary functions of their arguments.  

To integrate the Eq. (3.60), we impose the condition
\[ \partial_r \left( \frac{G_{rr}}{G_{\alpha \alpha}} \dot{Y}^\alpha \right)^2 = 0. \quad (3.63) \]

After using the second of the conditions (3.61), the condition (3.63), and the already obtained solution (3.62), the Eq. (3.60) can be recast in the form
\[ \frac{d}{d\tau} \left[ \left( G_{rr} \dot{Y}^r \right)^2 + 2G_{rr} (A_r - \partial_r f) \dot{Y}^r \right] \\
= \dot{Y}^r \partial_r \left\{ G_{rr} \left[ (1 - n_\alpha) \left( U + 2 (A_r - \partial_r f) \dot{Y}^r \right) - \sum_\alpha \frac{D_\alpha \left( Y^{a \neq \alpha} \right)}{G_{\alpha \alpha}} \right] \right\}, \]
where \( n_\alpha \) is the number of the coordinates \( Y^\alpha \). The solution of this equation, compatible with (3.62), and with the effective constraint (3.57), is
\[ \left( G_{rr} \dot{Y}^r \right)^2 = G_{rr} \left[ (1 - n_\alpha) U - 2n_\alpha (A_r - \partial_r f) \dot{Y}^r - \sum_\alpha \frac{D_\alpha \left( Y^{a \neq \alpha} \right)}{G_{\alpha \alpha}} \right] = E_r \left( Y^r \right) \geq 0, \quad (3.64) \]
where \( E_r \) is again an arbitrary function.

Thus, we succeeded to separate the variables \( \dot{Y}^\alpha \) and to obtain the first integrals (3.62), (3.64) for the equations of motion (3.56), when the conditions (3.58), (3.61), (3.63) on the background are fulfilled. Further progress is possible, when working with particular background configurations, having additional symmetries (see, for instance, [60]).
4 Summary and discussion

In this paper we addressed the problem of obtaining explicit exact solutions for probe branes moving in general string theory backgrounds. We concentrated our attention to the common properties of the p-branes and Dp-branes dynamics and tried to formulate an approach, which is effective for different embeddings, for arbitrary worldvolume and space-time dimensions, for different variable background fields, for tensile and tensionless branes. To achieve this, we first performed an analysis in Section 2, with the aim to choose brane actions, which are most appropriate for our purposes.

In Section 3, we formulated the frameworks in which to search for exact probe branes solutions. The guiding idea is the reduction of the brane dynamics to a particle-like one. In view of the existing practice, we first consider the case of static gauge embedding, which is the mostly used one in higher dimensions. Then we turn to the more general case of linear embeddings, which are appropriate for lower dimensions too. After that, we consider the branes dynamics by using the most general ansatz, allowing for its reduction to particle-like one. The obtained results reveal one common property in all the cases considered. The effective equations of motion and one of the constraints, the effective constraint, have the same form independently of the ansatz used to reduce the p-branes or Dp-branes dynamics. In general, the effective equations of motion do not coincide with the geodesic ones. The deviation from the geodesic motion is due to the appearance of effective scalar and 1-form gauge potentials. The same scalar potential arises in the effective constraint.

In the last part of Section 3, we considered the problem of obtaining explicit exact solutions of the effective equations of motion and the effective constraint, without using the explicit structure of the effective potentials.

In the case when the background fields depend on only one coordinate $x^a = X^a(\tau)$, we show that these equations can always be integrated and give the probe brane solution in the form $\tau = \tau(X^a)$, where $\tau$ is the worldvolume temporal parameter. We also give the explicit solutions for the brane coordinates $X^\mu$ in the form $X^\mu = X^\mu(X^a, \xi^i)$. They are nontrivial when one uses the most general ansatz (3.6).\footnote{Let us remind that $x^\mu$ are the coordinates, on which the background fields do not depend.}

In the case when the background fields depend on more than one coordinate, and we fix all brane coordinates $X^a$ except one, the exact solutions are given by the same expressions as in the case just considered, if the metric $g_{ab}$ is a diagonal one. In this way, we have realized the possibility to obtain probe brane solutions as functions of every single one coordinate, on which the background depends. In the case when none of the brane coordinates is kept fixed, we were able to find sufficient conditions, which ensure the separation of the variables $\dot{X}^a = \dot{Y}^a(\tau)$. As a result, we have found the manifest expressions for $n_a$ first integrals of the equations of motion, where $n_a$ is the number of the brane coordinates $Y^a$.

In obtaining the solutions described above, it was not taken into account that some restrictions on them can arise, depending on the ansatz used and on the type of the branes considered. As far as we are interested here in the common properties of the probe branes dynamics, we will not make an exhaustive investigation of all possible peculiarities, which can arise in different particular cases. Nevertheless, we will consider some specific properties, characterizing the dynamics of the different type of branes for different embeddings.

We note that in static gauge, the brane coordinates $X^a$ figuring in our solutions, are spatial ones. This is so, because in this gauge the background temporal coordinate, on which the background fields can depend, is identified with the worldvolume time $\tau$.\footnote{Let us remind that $x^\mu$ are the coordinates, on which the background fields do not depend.}
The solutions $X^\mu(X^a, \xi^i)$, given by (3.54) for the $p$-brane and by (3.55) for the $Dp$-brane, depend on the worldvolume parameters $(\tau, \xi^i)$ through the specific combination $\Lambda_i^a (\lambda^i \tau + \xi^i)$. It is interesting to understand if its origin has some physical meaning. To this end, let us consider the $p$-branes equations of motion (2.10) and constraints (2.6), (2.7) in the tensionless limit $T_p \to 0$, when they take the form

$$
g_{LN} \left( \partial_0 - \lambda^i \partial_1 \right) \left( \partial_0 - \lambda^j \partial_1 \right) X^N + \Gamma_{L,MN} \left( \partial_0 - \lambda^i \partial_1 \right) X^M \left( \partial_0 - \lambda^j \partial_1 \right) X^N = 0,$$

$$
g_{MN} \left( \partial_0 - \lambda^i \partial_1 \right) X^M \left( \partial_0 - \lambda^j \partial_1 \right) X^N = 0,$$

$$
g_{MN} \left( \partial_0 - \lambda^i \partial_1 \right) X^M \partial_j X^N = 0.
$$

It is easy to check that in $D$-dimensional space-time, any $D$ arbitrary functions of the type $F^M = F^M (\lambda^i \tau + \xi^i)$ solve this system of partial differential equations. Hence, the linear part of the tensile $p$-brane and $Dp$-brane solutions (3.54) and (3.55), is a background independent solution of the tensionless $p$-brane equations of motion and constraints.

Let us point out here that by construction, the actions used in our considerations allow for taking the tensionless limit $T_p \to 0$ ($T_{Dp} \to 0$). Moreover, from the explicit form of the obtained exact probe branes solutions it is clear that the opposite limit $T_p \to \infty$ ($T_{Dp} \to \infty$) can be also taken.

We have obtained solutions of the probe branes equations of motion and one of the constraints, which have the same form for all of the considered cases. Now, let us see how we can satisfy the other constraints present in the theory. These are $p$ constraints, obtained by varying the corresponding actions with respect to the Lagrange multipliers $\lambda^i$. For the $Dp$-brane, we have $p$ additional constraints, obtained by varying the action with respect to the Lagrange multipliers $\kappa^i$. Actually, the constraints generated by the $\lambda^i$-multipliers are satisfied. Due to the conservation of the corresponding momenta, they just restrict the number of the independent parameters present in the solutions. The only exception is the $p$-brane in static gauge case, where the momenta $P_i^{SG}$ must be zero. Let us give an example how the problem can be resolved in a particular situation, which is nevertheless general enough. Let the background metric along the probe $p$-brane be a diagonal one. Then from (3.3) and (3.11), it follows that the momenta $P_i^{SG}$ will be identically zero, if we work in the gauge $\lambda^1 = 0$. In the general case, and this is also valid for the $\kappa^i$-generated constraints, we have to insert the obtained solution of the equations of motion into the unresolved constraints. The result will be a number of algebraic relations between the background fields. If they are not satisfied (on the solution) at least for some particular values of the free parameters in the solution, it would be fair to say that our approach does not work properly in this case, and some modification is needed.

Finally, let us say a few words about some possible generalizations of the obtained results.

As is known, the branes charges are restricted up to a sign to be equal to the branes tensions from the condition for space-time supersymmetry of the corresponding actions. In our computations, however, the coefficients in front of the background antisymmetric fields do not play any special role. That is why, to account for nonsupersymmetric probe branes, it is enough to make the replacements

$$
T_p b_{p+1} \to Q_p b_{p+1}, \quad T_{Dp} c_{p+1} \to Q_{Dp} c_{p+1}.
$$

In our $Dp$-brane action (2.12), we have included only the leading Wess-Zumino term of the possible $Dp$-brane couplings. It is easy to see that our results can be generalized to include other interaction terms just by the replacement

$$
c_{p+1} \to c_{p+1} + c_{p-1} \wedge b_2 + \ldots.
$$
This is a consequence of the fact that we do not used the explicit form of the background field \( c_{M_0...M_p} \). We have used only its antisymmetry and its independence on part of the background coordinates.

**Acknowledgments**

The author would like to acknowledge the hospitality of the ICTP-Trieste, where this investigation has been done. This work is supported by the Abdus Salam International Center for Theoretical Physics, Trieste, Italy, and by a Shoumen University grant under contract No.005/2002.

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