Complex Hadamard matrices for prime numbers

Petre Dîta
Horia Hulubei National Institute of Physics and Nuclear Engineering, P.O. Box MG6, Magurele, Romania

In this paper we disprove the Haagerup statement that all complex Hadamard matrices of order five are equivalent with the Fourier matrix $F_5$ by constructing circulant matrices that lead to new Hadamard matrices. An important item is the construction of new mutually unbiased bases that are a basic concept of quantum theory and play an essential role in quantum tomography, quantum cryptography, teleportation, construction of dense coding schemes, classical signal processing, etc.

1. INTRODUCTION

Björck G. and Fröberg R., [1], seem to be the first authors who treated the problem of cyclic $n$-roots with applications to Hadamard matrices.

T. Durt and his coworkers, [2], studied the classification problem of Hadamard matrices of size $n \leq 5$. In particular they used the dephased form for all matrices. They says that the (rescaled) Fourier matrices are the unique example in order $n \leq 3$, and in order 5 one has uniqueness again, result which already is absolutely non-trivial! Their example for $n = 3$ is the following

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 \\ 1 & \gamma^2 & \gamma \end{bmatrix}$$ (1)

where $\gamma = e^{\frac{2\pi i}{3}}$, citing the paper [1].

Because this approach is spreading fast, see paper [4], we construct many three and five dimensional Hadamard matrices that disprove the Szöllősi assumption that only the Fourier matrices $F_3$ and $F_5$ have a real existence. Szöllősi in his thesis, [3], seems to agree that a complete classification of complex Hadamard matrices is only available up to order $n = 5$, and in this case it is equivalent with the Fourier matrix $F_5$.

In this paper we consider the cases $n = 2$, $n = 3$ and $n = 5$.

The orthogonality concept is essential for getting new complex Hadamard matrices and in the following we make use of the particular class of inverse orthogonal matrices, $O = (o_{ij})$, whose inverse is given by

$$O^{-1} = (1/o_{ij}) = (1/o_{ij})$$ (2)

where $t$ means transpose, and their entries, $0 \neq o_{ij} \in \mathbb{C}$, satisfy the relation

$$OO^{-1} = nI_n$$ (3)

When $o_{ij}$ entries take unimodular values, $O^{-1}$ coincides with the Hermitean conjugate $O^*$ of $O$, and in this case $O/\sqrt{n}$ is the definition of complex Hadamard matrices, see for example paper [5].

2. THE TWO DIMENSIONAL CASE

We start with the simplest case $n = 2$ and we use the next matrix in order to find new Hadamard matrices

$$C_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ (4)
This matrix is not Hadamard. Thus we make use of the relation (3) which provides the following constraint $bc + ad = 0$. By using it we find four solutions

$$H_1 = \begin{bmatrix} \frac{b}{a} & b \\ c & d \end{bmatrix}, \quad H_2 = \begin{bmatrix} a & -\frac{ad}{c} \\ c & d \end{bmatrix}, \quad H_3 = \begin{bmatrix} a & b \\ -\frac{bc}{a} & d \end{bmatrix}, \quad H_4 = \begin{bmatrix} a & b \\ c & -\frac{bc}{a} \end{bmatrix}$$ (5)

The above matrices generate many MUBs $(\mathbb{1}, H_1, H_2), (\mathbb{1}, H_1, H_3), (\mathbb{1}, H_2, H_4)$, etc.

3. THE THREE DIMENSIONAL CASE

As usual we make use of the Sylvester ortogonality, see paper [3], and we start with the circulant matrix $C_3$ whose form is

$$C_3 = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$ (6)

and $C_3$ matrix provides the following parameter constraints

$$a^2 b + b^2 c + ac^2 = 0, \quad ab^2 + a^2 c + bc^2 = 0$$ (7)

When they are satisfied $C_3$ transforms into new matrices which are not yet Hadamard. For example from the first equation (6) one gets

$$a = \frac{-c^2 \pm \sqrt{c^4 - 4b^2c}}{2b}$$ (8)

The choices $b = c$, followed by $c = 1$ give the following ten matrices

$$A_1 = \begin{bmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} \omega & \omega & 1 \\ 1 & \omega & \omega \\ \omega & 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ 1 & \omega & \omega \end{bmatrix}, \quad A_5 = \begin{bmatrix} \omega & 1 & \omega \\ 1 & \omega & 1 \\ \omega & 1 & \omega \end{bmatrix}$$ (9)

and respectively

$$B_1 = \begin{bmatrix} \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & \omega^2 & 1 \\ 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \omega^2 & \omega^2 & 1 \\ 1 & \omega^2 & \omega^2 \\ \omega^2 & 1 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 & \omega^2 \\ \omega^2 & 1 & 1 \\ 1 & \omega^2 & 1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} \omega^2 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \\ \omega^2 & 1 & \omega^2 \end{bmatrix}$$ (10)

All the above ten matrices are not Hadamard

4. NEW HADAMARD MATRICES

The $A_1$ matrix generates two complex Hadamard matrices

$$A_{11} = \begin{bmatrix} -(1)^{1/3} & 1 & 1 \\ 1 & -(1)^{1/3} & 1 \\ 1 & 1 & -(1)^{1/3} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} (1)^{2/3} & 1 & 1 \\ 1 & -(1)^{2/3} & 1 \\ 1 & 1 & (1)^{2/3} \end{bmatrix}$$ (11)

and $A_{11}$ and $A_{12}$ lead to a MUB set as $(\mathbb{1}, A_{11}, A_{12})$. The matrices $A_{11}$ and $A_{12}$, and respectively $A_{12}$ and $A_{11}$ generate the following two matrices

$$A_{1112} = \begin{bmatrix} -(1)^{1/6} & -i & -i \\ -i & -(1)^{1/6} & -i \\ -i & -i & -(1)^{1/6} \end{bmatrix}, \quad A_{1211} = \begin{bmatrix} (1)^{5/6} & -i & -i \\ -i & -(1)^{5/6} & -i \\ -i & -i & (1)^{5/6} \end{bmatrix}$$ (12)
In this case the MUB is \((\mathbb{I}, A_{1112}, A_{1211})\).

The \(A_2\) matrix generates also two complex Hadamard matrices

\[
A_{21} = \begin{bmatrix}
1 & -(-1)^{1/3} & 1 \\
1 & 1 & -(-1)^{1/3} \\
-(-1)^{1/3} & 1 & 1
\end{bmatrix}, \quad A_{22} = \begin{bmatrix}
1 & (-1)^{2/3} & 1 \\
1 & 1 & (-1)^{2/3} \\
(-1)^{2/3} & 1 & 1
\end{bmatrix} \quad (13)
\]

The matrices \(A_{21}\) and \(A_{22}\) and respectively \(A_{22}\) and \(A_{21}\) generate the same matrices \((12)\). The \(A_3\) matrix generates two complex Hadamard matrices

\[
A_{31} = \begin{bmatrix}
-(-1)^{1/3} & -(-1)^{1/3} & 1 \\
1 & -(-1)^{1/3} & -(-1)^{1/3} \\
-(-1)^{1/3} & 1 & -(-1)^{1/3}
\end{bmatrix}, \quad A_{32} = \begin{bmatrix}
(-1)^{2/3} & (-1)^{2/3} & 1 \\
1 & (-1)^{2/3} & (-1)^{2/3} \\
(-1)^{2/3} & 1 & (-1)^{2/3}
\end{bmatrix} \quad (14)
\]

Matrices \(A_{31}\) and \(A_{32}\) lead to the MUB set \((\mathbb{I}, A_{31}, A_{22})\). Similar to the preceding cases matrices \(A_{31}\) and \(A_{32}\) in this order, and respectively \(A_{32}\) and \(A_{31}\) generate the matrices

\[
A_{3132} = \begin{bmatrix}
i & -(-1)^{1/6} & -(-1)^{1/6} \\
-(-1)^{1/6} & i & -(-1)^{1/6} \\
-(-1)^{1/6} & -(-1)^{1/6} & i
\end{bmatrix}, \quad A_{3231} = \begin{bmatrix}
i & -(-1)^{5/6} & -(-1)^{5/6} \\
-(-1)^{5/6} & i & -(-1)^{5/6} \\
-(-1)^{5/6} & -(-1)^{5/6} & i
\end{bmatrix} \quad (15)
\]

The above matrices generate the MUB \((\mathbb{I}, A_{3132}, A_{3212})\).

The matrices \(A_{1112}\) and \(A_{3122}\), and respectively \(A_{3122}\) and \(A_{1112}\) generate the following matrices

\[
D_{11} = \begin{bmatrix}
(-1)^{1/6} & (-1)^{5/6} & (-1)^{5/6} \\
(-1)^{5/6} & (-1)^{1/6} & (-1)^{5/6} \\
(-1)^{5/6} & (-1)^{5/6} & (-1)^{1/6}
\end{bmatrix}, \quad D_{12} = \begin{bmatrix}
(-1)^{5/6} & (-1)^{1/6} & (-1)^{1/6} \\
(-1)^{1/6} & (-1)^{5/6} & (-1)^{1/6} \\
(-1)^{1/6} & (-1)^{1/6} & (-1)^{5/6}
\end{bmatrix} \quad (16)
\]

Thus the MUB is given by \((\mathbb{I}, D_{11}, D_{12})\).

The \(A_4\) matrix generates other two matrices whose form is

\[
A_{41} = \begin{bmatrix}
1 & 1 & -(-1)^{1/3} \\
-(-1)^{1/3} & 1 & 1 \\
1 & -(-1)^{1/3} & 1
\end{bmatrix}, \quad A_{42} = \begin{bmatrix}
1 & 1 & (-1)^{2/3} \\
(-1)^{2/3} & 1 & 1 \\
1 & (-1)^{2/3} & 1
\end{bmatrix} \quad (17)
\]

and the MU pair has the form \((\mathbb{I}, A_{41}, A_{42})\). The matrices \(A_{41}\) and \(A_{42}\), and respectively \(A_{42}\) and \(A_{41}\) generate again the matrices \((12)\).

In the next case \(A_5\) matrix leads also to two matrices

\[
A_{51} = \begin{bmatrix}
-(-1)^{1/3} & 1 & -(-1)^{1/3} \\
-(-1)^{1/3} & -(-1)^{1/3} & 1 \\
1 & -(-1)^{1/3} & -(-1)^{1/3}
\end{bmatrix}, \quad A_{52} = \begin{bmatrix}
(-1)^{2/3} & 1 & (-1)^{2/3} \\
(-1)^{2/3} & (-1)^{2/3} & 1 \\
1 & (-1)^{2/3} & (-1)^{2/3}
\end{bmatrix} \quad (18)
\]

with the MUB pair written as \((\mathbb{I}, A_{51}, A_{52})\). The matrices generated \(A_{51}\), and \(A_{52}\) coincide with the matrices \((11)\).

The matrices \(A_{51}, A_{52}\) generate the following matrices

\[
A_{5152} = \begin{bmatrix}
i & -(-1)^{1/6} & -(-1)^{1/6} \\
-(-1)^{1/6} & i & -(-1)^{1/6} \\
-(-1)^{1/6} & -(1)^{1/6} & i
\end{bmatrix}, \quad A_{5251} = \begin{bmatrix}
i & -(-1)^{5/6} & -(-1)^{5/6} \\
-(-1)^{5/6} & i & -(-1)^{5/6} \\
-(-1)^{5/6} & -(1)^{5/6} & i
\end{bmatrix} \quad (19)
\]

and the MUB is \((\mathbb{I}, A_{5152}, A_{5251})\).
With the $B_i$ matrices one get similar results. Thus the $B_1$ matrix leads to the following diagonal matrices

$$B_{11} = \begin{bmatrix} (-1)^{2/3} & 1 & 1 \\ 1 & (-1)^{2/3} & 1 \\ 1 & 1 & (-1)^{2/3} \end{bmatrix}, \quad B_{12} = \begin{bmatrix} -(-1)^{1/3} & 1 & 1 \\ 1 & -(-1)^{1/3} & 1 \\ 1 & 1 & -(-1)^{1/3} \end{bmatrix} \tag{20}$$

The MUB set is $(\mathbb{I}, B_{11}, B_{12})$. Similar to the preceding cases $B_{11}$ and $B_{12}$ generate the matrices $A_{1112}$ and $A_{211}$. The $B_2$ matrix leads to

$$B_{21} = \begin{bmatrix} (-1)^{2/3} & 1 & 1 \\ 1 & (-1)^{2/3} & 1 \\ 1 & 1 & (-1)^{2/3} \end{bmatrix}, \quad B_{22} = \begin{bmatrix} -(-1)^{1/3} & 1 & 1 \\ 1 & -(-1)^{1/3} & 1 \\ 1 & 1 & -(-1)^{1/3} \end{bmatrix} \tag{21}$$

and the MUB set is $(\mathbb{I}, B_{21}, B_{22})$. Matrices $B_{2122}$ and $B_{2221}$ coincide with the matrices $A_{1112}$ and $A_{211}$.

The $B_3$ matrix generates other two matrices

$$B_{31} = \begin{bmatrix} (-1)^{2/3} & (-1)^{2/3} & 1 \\ 1 & (-1)^{2/3} & (-1)^{2/3} \\ (-1)^{2/3} & 1 & (-1)^{2/3} \end{bmatrix}, \quad B_{32} = \begin{bmatrix} -(-1)^{1/3} & -(-1)^{1/3} & 1 \\ 1 & -(-1)^{1/3} & -(-1)^{1/3} \\ -(-1)^{1/3} & 1 & -(-1)^{1/3} \end{bmatrix} \tag{22}$$

The MU set is $(\mathbb{I}, B_{31}, B_{32})$. Matrices $B_{31}$ and $B_{32}$ generate the matrices $A_{5152}$ and $A_{5251}$.

The $B_4$ matrix generate the following two Hadamard matrices

$$B_{41} = \begin{bmatrix} 1 & 1 & (-1)^{2/3} \\ (-1)^{2/3} & 1 & 1 \\ 1 & (-1)^{2/3} & 1 \end{bmatrix}, \quad B_{42} = \begin{bmatrix} 1 & 1 & -(-1)^{1/3} \\ -(-1)^{1/3} & 1 & 1 \\ 1 & -(-1)^{1/3} & 1 \end{bmatrix} \tag{23}$$

and the MU form is $(\mathbb{I}, B_{41}, B_{42})$. The matrices $B_{41}$ and $B_{42}$ generate the matrices

$$B_{4142} = \begin{bmatrix} (-1)^{5/6} & -i & -i \\ -i & (-1)^{5/6} & -i \\ -i & -i & (-1)^{5/6} \end{bmatrix}, \quad B_{4241} = \begin{bmatrix} (-1)^{1/6} & -i & -i \\ -i & (-1)^{1/6} & -i \\ -i & -i & (-1)^{1/6} \end{bmatrix} \tag{24}$$

The MUB is given by $(\mathbb{I}, B_{4142}, B_{4241})$.

As usually the $B_5$ matrix generates other two matrices

$$B_{51} = \begin{bmatrix} (-1)^{2/3} & 1 & (-1)^{2/3} \\ (-1)^{2/3} & (-1)^{2/3} & 1 \\ 1 & (-1)^{2/3} & (-1)^{2/3} \end{bmatrix}, \quad B_{52} = \begin{bmatrix} -(-1)^{1/3} & 1 & -(-1)^{1/3} \\ -(-1)^{1/3} & -(-1)^{1/3} & 1 \\ 1 & -(-1)^{1/3} & -(-1)^{1/3} \end{bmatrix} \tag{25}$$

The MUB pair is $(\mathbb{I}, B_{51}, B_{52})$.

The matrices $A_{1112}$ and $B_{5152}$ generate the unitary diagonal matrix $I_{2/3}$

$$I_{2/3} = \begin{bmatrix} (-1)^{2/3} & 0 & 0 \\ 0 & (-1)^{2/3} & 0 \\ 0 & 0 & (-1)^{2/3} \end{bmatrix} \tag{26}$$

The corresponding MUB have the form $(I_{2/3}, A_{1112}, B_{5152})$. The same matrix is generated by $A_{4142}$ and $B_{5152}$, etc.

The matrices $B_{5132}$ and $A_{1112}$ generate another unitary diagonal matrix

$$I_{1/3} = \begin{bmatrix} -(-1)^{1/3} & 0 & 0 \\ 0 & -(-1)^{1/3} & 0 \\ 0 & 0 & -(-1)^{1/3} \end{bmatrix} \tag{27}$$
The MUB in this case has the form $(I_{m1/3}, B_{3152} A_{1112})$, where $m = -1$. The same matrix is generated by $B_{3152}$ and $A_{2122}$, and respectively by $B_{4122}$ and $A_{3132}$, etc.

Our approach has shown that the matrices $A_i$ and $B_i$, $i = 1, 2, 3, 4, 5$ are not complex Hadamard matrices. Thus the first step was to make use of the Sylvester trick, see equation (3), in order to transform all the matrices which depend on $\omega$ and $\omega^2$ into true Hadamard matrices. The final result is that with them we found many new MU bases.

5. THE FIVE DIMENSIONAL CASE

This case is very interesting because there is a 5-dimensional circulant matrix of the following form

$$C_5 = \begin{bmatrix}
1 & a & a^4 & a^4 & a \\
1 & a & a^4 & a & 1 \\
a^4 & a & 1 & a & 4 \\
a^4 & a^4 & a & 1 & a \\
a & a^4 & a^4 & a & 1
\end{bmatrix}$$

(28)

which leads to new complex Hadamard matrices. For this we make use of the relation (4) for getting complex Hadamard matrices from $C_5$ matrix. The diagonal entries are equal to 1. The off diagonal entries contain the polynomial $1 + a + a^2 + a^3 + a^4$.

For that we make use of the Sylvester relation (4), and the resulting matrix has 1 on the main diagonal and all the other entries have a common factor given by

$$1 + a + a^2 + a^3 + a^4$$

(29)

The solutions of (29) give 5-dimensional matrices, and they are

$$sol = \left[ a_1 = -1(-1)^{1/5}, a_2 = (-1)^{2/5}, a_3 = -1(-1)^{3/5}, a_4 = (-1)^{4/5} \right]$$

(30)

By using each solution in matrix (28) we get four different matrices denoted by $D_i$, $i = 1,2,3,4$. They are

$$D_1 = \begin{bmatrix}
1 & -(-1)^{1/5} & -(-1)^{4/5} & -(-1)^{3/5} & -(-1)^{3/5} \\
-(-1)^{1/5} & 1 & -(-1)^{4/5} & -(-1)^{3/5} & -(-1)^{3/5} \\
(-1)^{4/5} & -(-1)^{1/5} & 1 & -(-1)^{4/5} & -(-1)^{3/5} \\
-(-1)^{3/5} & -(-1)^{2/5} & -(-1)^{2/5} & -(-1)^{2/5} & -(-1)^{2/5}
\end{bmatrix}$$

$$D_2 = \begin{bmatrix}
1 & -(-1)^{2/5} & -(-1)^{3/5} & -(-1)^{3/5} & -(-1)^{3/5} \\
-(-1)^{2/5} & 1 & -(-1)^{3/5} & -(-1)^{3/5} & -(-1)^{3/5} \\
(-1)^{2/5} & -(-1)^{3/5} & -(-1)^{2/5} & -(-1)^{2/5} & 1
\end{bmatrix}$$

$$D_3 = \begin{bmatrix}
1 & -(-1)^{3/5} & -(-1)^{2/5} & -(-1)^{2/5} & -(-1)^{2/5} \\
-(-1)^{3/5} & 1 & -(-1)^{2/5} & -(-1)^{2/5} & -(-1)^{2/5} \\
(-1)^{2/5} & -(-1)^{3/5} & -(-1)^{2/5} & -(-1)^{2/5} & 1
\end{bmatrix}$$

$$D_4 = \begin{bmatrix}
1 & -(-1)^{4/5} & -(-1)^{1/5} & -(-1)^{1/5} & -(-1)^{1/5} \\
-(-1)^{4/5} & 1 & -(-1)^{1/5} & -(-1)^{1/5} & -(-1)^{1/5} \\
(-1)^{1/5} & -(-1)^{4/5} & 1 & -(-1)^{4/5} & -(-1)^{1/5} \\
(-1)^{1/5} & -(-1)^{4/5} & 1 & -(-1)^{4/5} & -(-1)^{1/5}
\end{bmatrix}$$

(31)

(32)

All the above four matrices are complex Hadamard and they generate a MUB of the following form $(I, D_1, D_2, D_3, D_4)$.

6. CONCLUSION

Our approach has shown that the matrices $A_i$ and $B_i$ are not complex Hadamard. So we used the necessary constraints to obtain new Hadamard matrices. In the same time we disproved the assertion that for 3- and 5-dimensional matrices they have the form of the corresponding Fourier matrices. An important result is that $C_5$ matrix generated four Hadamard matrices which disprove all assumptions risen in papers [1], [2] and [3].

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