Resurgent functions and nonlinear systems of differential and difference equations

By

Shingo KAMIMOTO *

Abstract

The principal aim of this article is to establish an iteration method on the space of resurgent functions. We discuss endless continuability of iterated convolution products of resurgent functions and derive their estimates developing the method in [KaS]. Using the estimates, we show the resurgence of formal series solutions of nonlinear differential and difference equations.

§ 1. Introduction

Resurgent analysis has its origin in the publications [E1] written by J. Écalle. It provides an effective method for the study of e.g. holomorphic dynamics, analytical differential equations, WKB analysis and it still fascinates many mathematicians and theoretical physicists. In this theory, the space of resurgent functions plays a central role: a formal series \( \varphi(x) := \sum_{j=0}^{\infty} \varphi_j x^{-j} \in \mathbb{C}[[x^{-1}]] \) is resurgent if its formal Borel transform

\[
\mathcal{B}(\varphi) := \varphi_0 \delta + \hat{\varphi}(\xi), \quad \hat{\varphi}(\xi) := \sum_{j=1}^{\infty} \varphi_j \frac{\xi^{j-1}}{(j-1)!}
\]

is convergent and \( \hat{\varphi}(\xi) \) is endlessly continuable (cf. [E1]). In this article, we adopt the definition of endless continuability in [CaNP]. As the Borel counterpart of Cauchy product in \( \mathbb{C}[[x^{-1}]] \), the convolution product \( \hat{\varphi} \ast \hat{\psi} \) of \( \hat{\varphi} \) and \( \hat{\psi} \) in \( \mathbb{C}\{\xi\} \) is defined as follows:

\[
\hat{\varphi} \ast \hat{\psi}(\xi) := \int_{0}^{\xi} \hat{\varphi}(\xi - \xi') \hat{\psi}(\xi') d\xi'.
\]
To discuss analytic continuation of such a convolution product, the notion of *symmetrically contractible path* was introduced in [E1]. Following the principle in [CaNP], systematic construction of such paths was given in [S3] and detailed estimates for the convolution product of an arbitrary number of endlessly continuable functions were obtained in [S4] when the set of singular points of the functions is a closed discrete subset in \( \mathbb{C} \) and closed under addition (see also [MS]). Further, it was generalized in [OD] and [KaS] to the case where the location of singular points of endlessly continuable functions is written by a discrete filtered set (see Definition 2.1 for its definition). Especially in [KaS], a rigorous foundation for the analysis on the space of such endlessly continuable functions was provided: a structure of Fréchet space on the space of the functions was precisely given by the aid of endless Riemann surfaces (see Section 2). It allows us to handle analytical problems related to the convergence of the functions, e.g. substitution of resurgent functions to convergent series, implicit function theorem for resurgent functions.

However, we have still a problem in applying resurgent analysis to the study of analytical differential equations: there is no universally applicable way of proving the resurgence of formal series solutions of differential equations. Especially, it is important to determine the location of singular points of the Borel transformed formal series solutions for the use of alien calculus, which is the main tool of resurgent analysis.

Having these backgrounds in mind, we discuss the following question in this article: Can we extend the principle in [KaS] so that we can show the resurgence of formal series solutions of differential equations? The main purpose of this article is to establish an iteration method on the space of resurgent functions by developing the method in [KaS] and to show the resurgence of formal series solutions of differential equations by applying it. More precisely, we consider a nonlinear differential equation

\[
\frac{d}{dx} \Phi = F(x^{-1}, \Phi)
\]

at \( x = \infty \) with \( F(x^{-1}, \Phi) \in \mathbb{C}^n \{ x^{-1}, \Phi \} \) satisfying the conditions \( F(0,0) = 0 \) and \( \det(\partial_\Phi F(0,0)) \neq 0 \). In this setting, (1.1) has a unique formal series solution \( \Phi \in \mathbb{C}^n[[x^{-1}]] \). In [E1], Écalle claims each entry of \( \Phi \) is resurgent. In [Co], Borel summability of transseries solutions of (1.1) was discussed and the singularity structure in the Borel plane of the solutions was precisely studied under non-resonance conditions. (See [Ku], [BKu1] and [BKu2] for the case of the difference equation (7.13).) However, our standpoint is close to Écalle’s *mould calculus* rather than [Co]. Mould calculus was developed in [E1] and applied to the classification of saddle-node singularities in [E2] (see also [S1] and [S2]). It uses expansions by resurgent monomials associated with words generated by e.g. \( \mathbb{Z} \) and resurgent properties of formal integrals was studied by the use of the mould expansions. In this article, we use an expansion of \( \hat{\Phi} \) by iterated
convolution products of meromorphic functions associated with *iteration diagrams* (see Definition 3.1) instead of words. Iterated convolution product is a combination of convolution product and Cauchy product determined by iteration diagrams (see Definition 3.6). Extending the estimates obtained in [KaS] to iterated convolution products of endlessly continuable functions, we obtain the following theorem as one of our main results:

**Theorem 1.1.** *The formal series solution* $\Phi \in \mathbb{C}[[x^{-1}]]$ *of (1.1) is resurgent.*

In Section 7, we describe detailed geometrical structure of singular points of $\hat{\Phi}$ by the use of discrete filtered set and reveal how the singular points are generated by the set of eigenvalues of $\partial_{\Phi}F(0,0)$.

The plan of this article is the following:

– Section 2 reviews the notions and the results related to Ω-resurgence.

– Section 3 introduces the notions of iteration diagram and iterated convolution. We give a key-estimate Theorem 3.9 for iterated convolution products of Ω-resurgent functions.

– Section 4 discusses the analytic continuation of iterated convolution products along a path $\gamma$ using a ($\gamma, T$)-adapted deformation.

– Section 5 and Section 6 are devoted to the proof of Theorem 4.6. We construct a ($\gamma, T$)-adapted deformation $(\Psi_t)_{t \in [a, 1]}$ in Section 5 and derive its estimates in Section 6.

– In Section 7, we show the resurgence of formal series solutions of nonlinear differential and difference equations using the estimate Theorem 3.9.

Some of the results in this article have been announced in [Ka].

§ 2. Preliminaries

In this section, we review the notions concerning Ω-resurgence of formal series discussed in [KaS].

**Definition 2.1.** We use the notation $\mathbb{R}_{\geq 0} = \{ \lambda \in \mathbb{R} \mid \lambda \geq 0 \}$.

1. A *discrete filtered set*, or *d.f.s.* for short, is a family $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ of subsets of $\mathbb{C}$ such that
   a) $\Omega_L$ is a finite set,
b) $\Omega_{L_1} \subseteq \Omega_{L_2}$ for $L_1 \leq L_2$,
c) there exists $\delta > 0$ such that $\Omega_\delta = \emptyset$.

2. Let $\Omega$ and $\Omega'$ be d.f.s. A d.f.s. $\Omega \ast \Omega'$ defined by the formula

$$(\Omega \ast \Omega')_L := \{ \omega_1 + \omega_2 \mid \omega_1 \in \Omega_{L_1}, \omega_2 \in \Omega'_{L_2}, L_1 + L_2 = L \} \cup \Omega_L \cup \Omega'_L$$

for $L \in \mathbb{R}_{\geq 0}$ is called the sum of d.f.s. $\Omega$ and $\Omega'$. We set $\Omega^* := \Omega \ast \cdots \ast \Omega$ for $n \geq 1$ and define a d.f.s. $\Omega^\ast\!\!\!\!\!\!\infty$ by

$$\Omega^\ast\!\!\!\!\!\!\infty := \lim_{n \to \infty} \Omega^*.$$  

3. A trivial d.f.s. $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ is a d.f.s. satisfying $\Omega_L = \emptyset$ for all $L \in \mathbb{R}_{\geq 0}$ and we denote it by $\emptyset$.

4. Given a d.f.s. $\Omega$, the distance to $\Omega$ is the number $\rho(\Omega) := \sup \{ \rho \in \mathbb{R}_{\geq 0} \mid \Omega_\rho = \emptyset \}$.

We define for a d.f.s. $\Omega$

$$S_\Omega := \{ (\lambda, \omega) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \omega \in \Omega_\lambda \},$$

$$M_\Omega := (\mathbb{R}_{\geq 0} \times \mathbb{C}) \setminus S_\Omega,$$

where $\overline{S}_\Omega$ denotes the closure of $S_\Omega$ in $\mathbb{R}_{\geq 0} \times \mathbb{C}$.

Let $\Pi$ be the set of all Lipschitz paths $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = 0$. We denote the restriction of $\gamma \in \Pi$ to the interval $[0, t]$ for $t \in [0, 1]$ by $\gamma|_t$ and the total length of $\gamma|_t$ by $L(\gamma|_t)$.

**Definition 2.2.** Given a d.f.s. $\Omega$, we call $\gamma \in \Pi \Omega$-allowed path if it satisfies

$$\check{\gamma}(t) := (L(\gamma|_t), \gamma(t)) \in M_\Omega \quad \text{for all } t \in [0, 1],$$

and denote the set of all $\Omega$-allowed paths by $\Pi_\Omega$.

**Remark 2.3.** When a piecewise $C^1$ path $t \in [0, 1] \mapsto \check{\gamma}(t) = (\lambda(t), \gamma(t)) \in M_\Omega$ with $\check{\gamma}(0) = (0, 0)$ is given, the $\Omega$-allowedness of $\gamma$ is characterized by the condition $\lambda'(t) = |\gamma'(t)|$ for a.e. $t \in [0, 1]$.

Recall that an $\Omega$-endless Riemann surface is a triple $(X, p, \emptyset)$ such that $X$ is a connected Riemann surface, $p : X \to \mathbb{C}$ is a local biholomorphism, $\emptyset \in p(0)$, and any path $\gamma : [0, 1] \to \mathbb{C}$ of $\Pi_\Omega$ has a lift $\overline{\gamma} : [0, 1] \to X$ such that $\overline{\gamma}(0) = \emptyset$. A
morphism \(q : (X, p, 0) \rightarrow (X', p', 0')\) of \(\Omega\)-endless Riemann surfaces is given by a local biholomorphism \(q : X \rightarrow X'\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
(X, 0) & \xrightarrow{q} & (X', 0') \\
\downarrow{p} & & \downarrow{p'} \\
(\mathbb{C}, 0)
\end{array}
\]

The existence of the initial object \((X_\Omega, p_\Omega, 0_\Omega)\) in the category of \(\Omega\)-endless Riemann surfaces was proved in \([\text{KaS}]\):

**Theorem 2.4 (\text{[KaS]})**. There exists an \(\Omega\)-endless Riemann surface \((X_\Omega, p_\Omega, 0_\Omega)\) such that \(X_\Omega\) is simply connected and, for any \(\Omega\)-endless Riemann surface \((X, p, 0)\), there is a unique morphism

\[q : (X_\Omega, p_\Omega, 0_\Omega) \rightarrow (X, p, 0),\]

Let \(\hat{\mathcal{R}}_\Omega\) denote the space of \(\Omega\)-continuable functions, i.e., holomorphic germs \(\hat{\varphi} \in \mathbb{C}\{\xi\}\) which can be analytically continued along any path \(\gamma \in \Pi_\Omega\). Then, there exists an isomorphism

\[p^*_\Omega : \hat{\mathcal{R}}_\Omega \xrightarrow{\sim} \Gamma(X_\Omega, \mathcal{O}_{X_\Omega}),\]

where \(\mathcal{O}_{X_\Omega}\) is the sheaf of holomorphic functions on \(X_\Omega\), and hence, a structure of Fréchet space is naturally introduced to \(\hat{\mathcal{R}}_\Omega\) as follows: We set for \(L, \delta > 0\)

\[
\mathcal{M}^{\delta,L}_\Omega := \{ (\lambda, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mid \text{dist}((\lambda, \xi), \mathcal{S}_\Omega) \geq \delta, \lambda \leq L \},
\]

\[
\Pi^{\delta,L}_\Omega := \{ \gamma \in \Pi_\Omega \mid (L(\gamma|t), \gamma(t)) \in \mathcal{M}^{\delta,L}_\Omega \text{ for all } t \in [0,1] \},
\]

where dist(\(\cdot, \cdot\)) is the Euclidean distance in \(\mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3\), and define compact subsets \(K^{\delta,L}_\Omega\) of \(X_\Omega\) by

\[K^{\delta,L}_\Omega := \{ \gamma(1) \in X_\Omega \mid \gamma \in \Pi^{\delta,L}_\Omega \}.
\]

Since \(X_\Omega\) is exhausted by \((K^{\delta,L}_\Omega)_{\delta,L>0}\), a family of seminorms \(\| \cdot \|^{\delta,L}_\Omega (\delta, L > 0)\) defined by

\[
\| \hat{\varphi} \|^{\delta,L}_\Omega := \sup_{\xi \in K^{\delta,L}_\Omega} |p^*_\Omega \hat{\varphi}(\xi)| \quad \text{for} \quad \hat{\varphi} \in \hat{\mathcal{R}}_\Omega,
\]

induces a structure of Fréchet space on \(\hat{\mathcal{R}}_\Omega\). Correspondingly, a family of seminorms \(\| \cdot \|^{\delta,L}_\Omega (\delta, L > 0)\) on the space of \(\Omega\)-resurgent series

\[
\mathcal{R}_\Omega := \mathcal{B}^{-1}(\mathbb{C} \delta \oplus \hat{\mathcal{R}}_\Omega)
\]

are defined by

\[
\| \varphi \|^{\delta,L}_\Omega := |\varphi_0| + \| \hat{\varphi} \|^{\delta,L}_\Omega \quad \text{for} \quad \varphi \in \mathcal{R}_\Omega,
\]

where \(\mathcal{B}(\varphi) = \varphi_0\delta + \hat{\varphi} \in \mathbb{C} \delta \oplus \hat{\mathcal{R}}_\Omega\).
Remark 2.5. Notice that \((X_\Omega, p_\Omega, 0_\Omega) \xrightarrow{\sim} (\mathbb{C}, \text{id}_\mathbb{C}, 0)\), and hence, \(\hat{\mathcal{R}}_\Omega \xrightarrow{\sim} \Gamma(\mathbb{C}; \mathcal{O}_\mathbb{C})\).

Since \(K^\delta_L = \{ \xi \in \mathbb{C} \mid |\xi| \leq L \}\) for \(\delta, L > 0\), we have \(\| \hat{\varphi} \|^{\delta,L}_\Omega = \sup_{|\xi| \leq L} |\hat{\varphi}(\xi)|\) for \(\hat{\varphi} \in \hat{\mathcal{R}}_\Omega\).

Now, let \(\Omega'\) be a d.f.s. satisfying \(\Omega \subset \Omega'\). From Theorem 2.4, we find that there exists a morphism \(q : (X_{\Omega'}, p_{\Omega'}, 0_{\Omega'}) \rightarrow (X_\Omega, p_\Omega, 0_\Omega)\). Since \(q(K^\delta_L_{\Omega'}) \subset K^\delta_L_\Omega\), we have \(\| \hat{\varphi} \|^{\delta,L}_{\Omega'} \leq \| \hat{\varphi} \|^{\delta,L}_\Omega\) for \(\hat{\varphi} \in \hat{\mathcal{R}}_\Omega\), and hence, \(\| \varphi \|^{\delta,L}_{\Omega'} \leq \| \varphi \|^{\delta,L}_\Omega\) for \(\varphi \in \mathcal{R}_\Omega\).

§ 3. Iterated convolution of resurgent functions

§ 3.1. Iteration diagram

Definition 3.1. Let \(T = (V, E)\) be a directed tree diagram, where \(V\) (resp. \(E\)) is the set of vertices (resp. edges) of \(T\). We call \(T\) iteration diagram if \(T\) satisfies the condition that any vertex \(v \in V\) has at most one outgoing edge. We denote the set of iteration diagrams by \(\mathcal{T}\).

Since \(T \in \mathcal{T}\) is connected and has no cycles, we immediately have the following

Lemma 3.2. Each \(T \in \mathcal{T}\) has a unique vertex \(\hat{v}\) such that there exists a path \(v \rightarrow \cdots \rightarrow \hat{v}\) from \(v\) to \(\hat{v}\) in \(T\) for any vertex \(v \in V\) and such a path is unique.

Definition 3.3. Let \(T = (V, E)\) be an iteration diagram.

1. We call \(\hat{v}\) in Lemma 3.2 root of \(T\).

2. We call a vertex \(v\) leaf of \(T\) if \(v\) has no edge \(e\) such that the terminal vertex of \(e\) is \(v\) and denote the set of leaves of \(T\) by \(L\).

3. The branch \(T_v = (V_v, E_v)\) of \(T\) at \(v \in V\) is the diagram that consists of the vertexes \(u \in V\) that have a path \(u \rightarrow \cdots \rightarrow v\) from \(u\) to \(v\) in \(T\) and the edges \(v_1 \xrightarrow{e} v_2 \in E\) such that \(v_1, v_2 \in V_v\).

From the definition of the branch, we obtain the following

Lemma 3.4. Given \(T \in \mathcal{T}\), the branch \(T_v\) of \(T\) at each vertex \(v \in V\) defines an iteration diagram with the root \(v\).

Notation 3.5. Let \(T = (V, E)\) be an iteration diagram.
1. We set $V^\circ := V \setminus \{\hat{v}\}$.

2. For each $v \in V^\circ$, there exists a unique vertex $u$ that has an edge $v \to u$. We denote such a vertex by $v_\uparrow$.

3. Given $v \in V$, we denote the set of vertices $u \in V$ that have an edge $u \to v$ by $V^1_v$.

We assign each vertex $v$ a weight $w_v$ defined as the cardinal of \{ $v' \in L \mid \exists$ a path $v' \to \cdots \to v$ \}. Notice that $w_v$ satisfies

$$
\begin{cases}
  w_v = 1 & (v \in L), \\
  w_v = \sum_{u \in V^1_v} w_u & (v \in V \setminus L).
\end{cases}
$$

Iteration diagrams are graded by the cardinal $|V|$ of vertexes:

$$
\mathcal{T} = \bigsqcup_{k=1}^\infty \mathcal{T}_k, \quad \mathcal{T}_k = \{ T = (V, E) \in \mathcal{T} \mid |V| = k \}.
$$

§ 3.2. Iterated convolution

Let $T = (V, E) \in \mathcal{T}_k$ ($k \geq 1$) be an iteration diagram and assume that analytic germs $\hat{f}_v, \hat{\varphi}_v \in \mathbb{C}\{\xi\}$ are assigned to each vertex $v \in V$. Starting from the leaves of $T$, we inductively construct $\{\hat{\psi}_v\}_{v \in V}$ from $\{\hat{f}_v\}_{v \in V}$ and $\{\hat{\varphi}_v\}_{v \in V}$ by the rule

$$
\hat{\psi}_v := \hat{\varphi}_v \cdot \left( \hat{f}_v \ast \prod_{u \in V^1_v}^* \hat{\psi}_u \right) \quad (v \in V),
$$

where $\prod_{u \in V^1_v}^* \hat{\psi}_u$ is the convolution product of $\hat{\psi}_u$ over all the vertices $u \in V^1_v$ and we regard it as the unit $\delta$ when $v \in L$.

**Definition 3.6.** Given $T \in \mathcal{T}$ and $\{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V} \subset \mathbb{C}\{\xi\}$, we call $\hat{\psi}_T := \hat{\psi}_\partial$ defined by the rule (3.1) *iterated convolution* of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$.

**Notation 3.7.** For an iteration diagram $T = (V, E) \in \mathcal{T}_k$ ($k \geq 1$), we set

$$
\Delta_T := \{(s_v)_{v \in V} \in \mathbb{R}^k_+ \mid \sum_{u \in V^1_v} s_u \leq s_v, s_\hat{v} = 1\}
$$

and $[\Delta_T] \in \mathcal{E}_{k-1}(\mathbb{R}^k)$ denotes the corresponding integration current, where the orientation of $\Delta_T$ is defined so that it satisfies

$$
\int_{\Delta_T} \wedge_{v \in V^\circ} ds_v = \frac{1}{(k-1)!}.
$$
We set for $\rho > 0$
\[
D_{\rho} := \{ \xi \in \mathbb{C} \mid |\xi| < \rho \}.
\]
Let $\Omega$ be a d.f.s. We define a map $\mathcal{L}_v : D_{\rho(\Omega)} \to X_{\Omega_v}$ by
\[
\mathcal{L}_v(\xi) := \gamma_{\xi}(1) \quad \text{for} \quad \xi \in D_{\rho(\Omega)},
\]
where $\Omega_v := \Omega^{w_v} (v \in V)$ and $\gamma_{\xi} : [0, 1] \to X_{\Omega_v}$ is the lift of the path $\gamma_{\xi} : t \in [0, 1] \mapsto t\xi$. Notice that $\mathcal{L}_v$ gives a local isomorphism from $D_{\rho(\Omega)}$ to an open neighborhood $\mathcal{L}_v(D_{\rho(\Omega)})$ of $0_{\Omega_v} \in X_{\Omega_v}$.

Now, assume that $\{\hat{f}_v\}_{v \in V} \subset \hat{\mathcal{R}}_0$ and $\hat{\varphi}_v \in \hat{\mathcal{A}}_{\Omega_v}$ for $v \in V$. We consider a map
\[
\mathcal{D}(\xi) : \bar{s} = (s_v)_{v \in V} \mapsto \mathcal{D}(\xi, \bar{s}) := (\mathcal{L}_v(s_v \xi))_{v \in V} \in X^T_{\Omega} \quad \text{for} \quad \xi \in D_{\rho(\Omega)}
\]
defined on a neighborhood of $\Delta_T$ in $\mathbb{R}^k$, where
\[
X^T_{\Omega} := \prod_{v \in V} X_{\Omega_v}.
\]
Let $\mathcal{D}(\xi)_{\#}[\Delta_T] \in \mathcal{E}_{k-1}(X^T_{\Omega})$ denote the push-forward of $[\Delta_T]$ by $\mathcal{D}(\xi)$. Then, we have the following representation of the iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$:

**Proposition 3.8.** Given $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$, define a holomorphic $(k-1)$-form $\beta_T$ on $X^T_{\Omega}$ by
\[
\beta_T := \left( \prod_{v \in V} (p_{\Omega_v}^* \hat{\varphi}_v)(\xi_v) \hat{f}_v(\xi_v - \sum_{u \in V^1_v} \xi_u) \right) \bigwedge_{v \in V^0} d\xi_v,
\]
where $\bigwedge_{v \in V^0} d\xi_v$ is the pullback of the $(k-1)$-form $\bigwedge_{v \in V^0} d\xi_v$ in $X^T_{\Omega}$ by $(p_{\Omega_v})_{v \in V} : X^T_{\Omega} \to \mathbb{C}^k$ and $\xi_v = p_{\Omega_v}(\xi_u) \quad (v \in V)$. Then, the following equality holds for $\xi \in D_{\rho(\Omega)}$:
\[
(3.3) \quad \hat{\psi}_T(\xi) = \mathcal{D}(\xi)_{\#}[\Delta_T](\beta_T).
\]

**Proof.** We prove (3.3) by induction. We first note that $\mathcal{D}(\xi)_{\#}[\Delta_T](\beta_T)$ is regarded as $(p_{\Omega}^* \hat{\varphi}_\xi)(\xi_v) \hat{f}_\xi(\xi_v) |_{\xi_v = \mathcal{L}(\xi)} = \hat{\varphi}_\xi(\xi) \hat{f}_\xi(\xi)$ when $T \in \mathcal{T}_1$, and hence, the equality (3.3) holds for the diagram $T_v \quad (v \in L)$. Next, take $v \in V$ and assume that (3.3) holds for all the branches $T_u \quad (u \in V^0_v)$. From the definition of the iterated convolution (3.1), we have the following representation of $\hat{\psi}_{T_v}$:
\[
\hat{\psi}_{T_v}(\xi) = \hat{\varphi}_v(\xi) \int_{\Delta_\ell} \hat{f}_\xi(\xi(1 - \sum_{u \in V^1_v} s_u)) \prod_{u \in V^1_v} \hat{\psi}_{T_u}(\xi s_u) \bigwedge_{u \in V^1_v} \xi ds_u,
\]
where $\ell = |V^1_v|$ and $\Delta_\ell$ is the $\ell$-dimensional simplex defined by
\[
\Delta_\ell := \{(s_u)_{u \in V^1_v} \in \mathbb{R}_{\geq 0}^\ell \mid \sum_{u \in V^1_v} s_u \leq 1\}.\]
Since $\Delta_{T_v}$ is rewritten as
\[
\left\{ (s_{\tilde{v}})_{\tilde{v} \in V_v} \mid s_{\tilde{v}} = s_u \tilde{s}_{\tilde{v}}, (\tilde{s}_{\tilde{v}}) \in \Delta_{T_v}, (s_u) \in \Delta_{l} \right\},
\]
we obtain (3.3) for the diagram $T_v$ from the induction hypothesis. It proves (3.3) for $T = T_{\tilde{v}}$.

We now state one of our main theorems:

**Theorem 3.9.** Let $\Omega$ be a d.f.s. and let $\delta, L > 0$ be reals such that $2\delta < \rho(\Omega)$. Then, there exist $c, \delta' > 0$ such that, for every $T = (V, E) \in T_k (k \geq 1), \{\hat{f}_v\}_{v \in V} \subset \hat{R}_\Omega$ and $\hat{\varphi}_v \in \hat{\Phi}_{\Omega_v} (v \in V)$, the iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ is $\Omega_{\tilde{v}}$-continuable and satisfies the following estimates:

\[
\|\hat{\psi}_T\|_{\Omega_v}^{\delta, L} \leq \frac{c^{k-1}}{(k-1)!} \sup_{s \in \Delta_{T_v}} \prod_{v \in V} \|\hat{\varphi}_v\|_{\Omega_v}^{\delta', s_v L} \|\hat{f}_v\|_{\hat{R}_\Omega}^{\delta', s_v L}.
\]

Since $\Omega \subset \Omega_v$ for all $v \in V$, we obtain the following

**Corollary 3.10.** Under the same assumptions with Theorem 3.9, there exist $c, \delta' > 0$ such that, for every $T = (V, E) \in T_k (k \geq 1), \{\hat{f}_v\}_{v \in V} \subset \hat{R}_\Omega$ and $\hat{\varphi}_v \in \hat{\Phi}_{\Omega_v} (v \in V)$, the iterated convolution $\hat{\psi}_T$ of $(T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})$ satisfies

\[
\|\hat{\psi}_T\|_{\Omega_v}^{\delta, L} \leq \frac{c^{k-1}}{(k-1)!} \prod_{v \in V} \|\hat{\varphi}_v\|_{\Omega_v}^{\delta', L} \|\hat{f}_v\|_{\hat{R}_\Omega}^{\delta', L}.
\]

The proof of Theorem 3.9 will be given in Section 4.

§ 4. $(\gamma, T)$-adapted deformation of $\mathcal{D}(\gamma(a))$

In this section, we introduce the notion of $(\gamma, T)$-adapted deformation of $\mathcal{D}(\gamma(a))$, which is a slight generalization of $\gamma$-adapted origin-fixing isotopies in [S4, Def. 5.1]. Let $T = (V, E)$ be an iteration diagram and let $\Omega$ be a d.f.s. We take $\rho > 0$ such that $2\rho < \rho(\Omega)$. We fix a path $\gamma : [0, 1] \to \mathbb{C}$ in $\Pi_{\Omega_{\tilde{v}}}^{\delta, L}$ with $L > 0$ and $\delta \in (0, \rho]$ satisfying the following condition:

\[
\text{there exists } a \in (0, 1) \text{ such that } \gamma|_{[a, 1]} \text{ is } C^1, |\gamma(a)| = \rho \text{ and } \gamma(t) = \gamma(a)a^{-1}t \\
\text{for } t \in [0, a].
\]
Notation 4.1. Given $T = (V, E) \in \mathcal{T}$, we set
\[
\Delta_{T,v}^0 := \{(s_u)_{u \in V} \in \Delta_T \mid s_v = 0\}, \\
\Delta_{T,v}^1 := \{(s_u)_{u \in V} \in \Delta_T \mid \sum_{u \in V_v^1} s_u = s_v\}, \\
\mathcal{N}_v^0 := \{(\xi_u)_{u \in V} \in X_T^v \mid \xi_v = 0\}, \\
\mathcal{N}_v^1 := \{(\xi_u)_{u \in V} \in X_T^v \mid \sum_{u \in V_v^1} p_{\Omega_u}(\xi_u) = p_{\Omega_v}(\xi_v)\},
\]
where $\Delta_{T,v}^0$ and $\mathcal{N}_v^0$ (resp. $\Delta_{T,v}^1$ and $\mathcal{N}_v^1$) are defined for $v \in V^\circ$ (resp. $v \in V \setminus L$).

Definition 4.2. We call a family of maps $\Psi_t : \Delta_T \rightarrow X_T^v$ ($t \in [a, 1]$) $(\gamma, T)$-adapted deformation of $\mathcal{D}(\gamma(a))$ in $X_T^v$ if it satisfies the following conditions:

1. $\Psi_a = \mathcal{D}(\gamma(a))$,
2. the map $(t, \tilde{s}) \in [a, 1] \times \Delta_T \mapsto \Psi_t(\tilde{s}) \in X_T^v$ is locally Lipschitz,
3. the $v$-th component $\xi_v^t$ of $\tilde{\xi}^t := \Psi_t(\tilde{s})$ depends only on the variables $s_u$ ($u \in W_v$),
   
   where $W_v := \{\hat{v}\} \cup \bigcup_{v'} V_v^1$

   and the union $\bigcup_{v'}$ is taken over all the vertexes $v'$ on the path from $v_1$ to $\hat{v}$,
4. $\xi_v^t(\tilde{s}) = \gamma(t)$ holds for any $t \in [a, 1]$ and $\tilde{s} \in \Delta_T$,
5. $\Psi_t(\Delta_{T,v}^0) \subset \mathcal{N}_v^0$ holds for any $t \in [a, 1]$ and $v \in V^\circ$,
6. $\Psi_t(\Delta_{T,v}^1) \subset \mathcal{N}_v^1$ holds for any $t \in [a, 1]$ and $v \in V \setminus L$.

We now show the following

Proposition 4.3. Consider an iterated convolution $\hat{\psi}_T$ of $(\gamma_{\tilde{v}} : \{\tilde{f}_{\tilde{v}}\}_{\tilde{v} \in V}, \{\tilde{\varphi}_{\tilde{v}}\}_{\tilde{v} \in V})$ with the data $T = (V, E) \in \mathcal{T}$, $\{\tilde{f}_{\tilde{v}}\}_{\tilde{v} \in V} \in \hat{\mathcal{R}}_\emptyset$ and $\tilde{\varphi}_{\tilde{v}} \in \hat{\mathcal{R}}_{\Omega_{\tilde{v}}}$ ($v \in V$) and assume that a $(\gamma, T)$-adapted deformation $(\Psi_t)_{t \in [a, 1]}$ of $\mathcal{D}(\gamma(a))$ is given. Then, the analytic continuation of $p_{\Omega_{\tilde{v}}}^* \hat{\psi}_T$ along $\gamma$ is written as follows:

\[
(p_{\Omega_{\tilde{v}}}^* \hat{\psi}_T)(\gamma(t)) = (\Psi_t) \# [\Delta_T](\beta_T) \quad \text{for} \quad t \in [a, 1].
\]

Notice that the situation discussed in [S4] and [KaS] is regarded as the case where the iteration diagram $T \in \mathcal{T}_k$ satisfies $V_{\hat{v}}^1 = V^\circ$ with $\tilde{f}_{\hat{v}} = \tilde{\varphi}_{\hat{v}} = 1$. We first note the following
Lemma 4.4 ([S4]). Let \( T = (V, E) \) be an iteration diagram such that \( V^1_\circ = V^\circ \) and assume that a \((\gamma, T)\)-adapted deformation \((\Psi_t)_{t \in [a, 1]}\) of \( D(\gamma(a)) \) is given. Then, the analytic continuation of \( p^*_\beta \tilde{\psi}_T \) along \( \gamma \) is written as follows:

\[
(p^*_\beta \tilde{\psi}_T)(\gamma(t)) = (\Psi_t)_{\#}[\Delta_T](\beta_T) \quad \text{for} \quad t \in [a, 1].
\]

Indeed, in this case, we can regard \( \beta_T \) as a holomorphic \((k - 1)\)-form on \( X^{k-1}_\Omega \) with a holomorphic parameter \( \xi_\delta \in X^{(k-1)}_\Omega \). Therefore, adapting the proof of [S4, Prop. 5.4] and restricting the parameter \( \xi_\delta \) to \( \gamma(t) \), we obtain Lemma 4.4.

Notation 4.5. Let a \((\gamma, T)\)-adapted deformation \((\Psi_t)_{t \in [a, 1]}\) of \( D(\gamma(a)) \) be given and assume that \((s_u)_{u \in V \setminus V_\circ} \) is taken so that it satisfies

\[
(4.2) \quad \{(s_u)_{u \in V \setminus V_\circ}, (s_v \tilde{s}_u)_{u \in V_v} \mid (\tilde{s}_u)_{u \in V_v} \in \Delta_{T_v}\} \subset \Delta_T.
\]

Let \( \text{pr}_{T_v} : X^T_\Omega \to X^{T_v}_\Omega \) be the natural projection. We define a map

\[
\Psi_t|_{T_v}((s_u)_{u \in V \setminus V_\circ}; \cdot) : \Delta_{T_v} \to X^{T_v}_\Omega
\]

by

\[
\Psi_t|_{T_v}((s_u)_{u \in V \setminus V_\circ}; s^\gamma) := \text{pr}_{T_v} \circ \Psi_t((s_u)_{u \in V \setminus V_\circ}, (s_v s_u)_{u \in V_v})
\]

for \( s^\gamma := (s_u)_{u \in V_v} \in \Delta_{T_v} \). We note that the map \( \Psi_t|_{T_v} \) depends only on the parameters \( s_u \ (u \in W_v) \), and hence, we write the map as \( \Psi_t|_{T_v}((s_u)_{u \in W_v}; \cdot) \). (See Definition 4.2.3.)

Since the path \( \gamma_v((s_u)_{u \in W_v}; \cdot) : [0, 1] \to X_{\Omega_v} \) obtained by concatenating the paths \( t \in [0, a] \mapsto \tilde{Z}_v(\gamma(a), s_t/a) \) and the \( v \)-th component of the path \( t \in [a, 1] \mapsto \Psi_t|_{T_v}((s_u)_{u \in W_v}; s^\gamma) \) is independent of the choice of \( (s'_u)_{u \in V_\circ} \), we find that, for fixed \( (s_u)_{u \in W_v} \),

\[
(4.3) \quad (\Psi_t|_{T_v})_{t \in [a, 1]} \text{ defines a } (\gamma_v, T_v)\text{-adapted deformation of } D(\gamma_v(a)).
\]

Proof of Proposition 4.3. We prove the following for every \( v \in V \) by the induction used in the proof of Proposition 3.8

\[
(4.4) \quad (p^*_\beta \tilde{\psi}_T_v)(\gamma_v(t)) = (\Psi_t|_{T_v})_{\#}[\Delta_{T_v}](\beta_{T_v})
\]

holds for every \( t \in [a, 1] \) and \((s_u)_{u \in W_v} \) satisfying the condition (4.2). For the case \( v \in L \), the equation (4.4) follows from the definition of \( \gamma_v \).

Now, assume that the equation (4.4) holds for all the vertexes \( u \in V^1_\circ \). Let \( \tilde{T}_v \) be an iteration diagram defined by the vertexes \( \tilde{V}_v := V^1_v \cup \{v\} \) and the edges of the form \( u \to v \ (u \in V^1_v) \). We set \( \Psi_t|_{\tilde{T}_v} := \text{pr}_{\tilde{T}_v} \circ \Psi_t|_{T_v} \), where \( \text{pr}_{\tilde{T}_v} : X^{T_v}_\Omega \to X^{T_{\tilde{T}_v}} := \prod_{u \in \tilde{V}_v} X_{\Omega_u}. \)
is the natural projection. Since \( \Psi_t|_{\mathcal{F}_v}((s_u)_{u \in W_v}; \tilde{s}) \) depends only on the variables \( s_u \) \((u \in \tilde{V}_v)\), it defines a map \( \Psi_t|_{\mathcal{F}_v} : \Delta_{\mathcal{F}_v} \to X_{\Omega}^{\mathcal{F}_v} \). We then obtain
\[
(\Psi_t|_{\mathcal{F}_v})_\# [\Delta_{\mathcal{F}_v}](\beta_{\mathcal{F}_v}) = (\Psi_t|_{\mathcal{F}_v})_\# [\Delta_{\mathcal{F}_v}](\tilde{\beta}_{\mathcal{F}_v})
\]
from the induction hypothesis, where \( \tilde{\beta}_{\mathcal{F}_v} \) is a holomorphic 1-form defined by
\[
\tilde{\beta}_{\mathcal{F}_v} := (p_{\Omega_v} \varphi_v)(\xi_v) f_v(\xi_v - \sum_{u \in V_v^1} \xi_u) \prod_{u \in V_v^1} (p_{\Omega_u} \psi_{T_u})(\xi_u) \bigwedge_{u \in V_v^1} d\xi_u.
\]
Applying Lemma 4.4, we have
\[
(\Psi_t|_{\mathcal{F}_v})_\# [\Delta_{\mathcal{F}_v}](\tilde{\beta}_{\mathcal{F}_v}) = p_{\Omega_v}^* \left( \varphi_v \cdot \left( f_v \prod_{u \in V_v^1} \psi_{T_u} \right) \right)(\gamma_v(t))
\]
\[
= (p_{\Omega_v}^* \psi_{T_v})(\gamma_v(t)).
\]
It proves (4.4) for all \( v \in V \). Since \( \gamma_{\bar{v}} = \gamma \) and \( T_{\bar{v}} = T \), we obtain Proposition 4.3 \( \square \)

The proof of Theorem 3.9 is reduced to the following

**Theorem 4.6.** Let \( T = (V, E) \) be an iteration diagram in \( \mathcal{F}_k \) \((k \geq 1)\) and assume that \( \gamma \in \Pi^{\delta, L}_{\Omega_v} \) for \( L > 0 \) and \( \delta \in (0, \rho] \) satisfying the condition (4.1) is given. Then, there exists a \((\gamma, T)\)-adapted deformation \( (\Psi_t)_{t \in [a, 1]} \) of \( \mathcal{D}(\gamma(a)) \) such that
\[
\Psi_t(\Delta_T) \subset \bigcup_{\tilde{\xi} \in \Delta_T} \prod_{v \in V} K_{\Omega_v}^{\delta(t), a_*, L(\gamma(t))}.
\]
Further, the partial derivatives \( \partial \xi_v^t / \partial s_u \) are defined almost everywhere on \( \Delta_T \) and satisfy
\[
\left| \det \left[ \frac{\partial \xi_v^t}{\partial s_u} \right]_{u, v \in V^2} \right| \leq \left( c(t) \right)^{k-1},
\]
where
\[
d'(t) := \rho e^{-2\sqrt{3}^{-1} L_\alpha(\gamma(t))}, \quad c(t) := \rho e^{3\delta^{-1} L_\alpha(\gamma(t))},
\]
\[
L_\alpha(\gamma(t)) = \int_a^t |\gamma'(t')| dt'.
\]
See [KaS] for the reduction of Theorem 3.9 to Theorem 4.6. The proof of Theorem 4.6 will be given in Section 5 and 6.

§ 5. Construction of a \((\gamma, T)\)-adapted deformation

In this section, we construct a \((\gamma, T)\)-adapted deformation of \( \mathcal{D}(\gamma(a)) \) satisfying the conditions in Theorem 4.6 by the method developed in [S3], [S4], [OD] and [KaS]. Let us assume that \( T = (V, E) \in \mathcal{F}_k \), a d.f.s. \( \Omega \) and \( \gamma \in \Pi^{\delta, L}_{\Omega_v} \) satisfying the assumptions in Theorem 4.6 are given.
Notation 5.1. We define functions \( \eta_v \) and \( D_v \ (v \in V) \) by
\[
\zeta = (\lambda, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \mapsto \eta_v (\zeta) := \text{dist} \left( (\lambda, \xi), \{(0,0)\} \cup \mathcal{S}_\Omega \right),
\]
\[(\zeta_v, (\zeta_u)_{u \in V_v^1}) \in (\mathbb{R}_{\geq 0} \times \mathbb{C})^{V_v^1 \times 1} \mapsto D_v (\zeta_v, (\zeta_u)_{u \in V_v^1}) := \sum_{u \in V_v^1} \eta_u (\zeta_u) + \left| \zeta_v - \sum_{u \in V_v^1} \zeta_u \right|,
\]
where \( \text{dist}(\cdot, \cdot) \) is the Euclidean distance in \( \mathbb{R} \times \mathbb{C} \simeq \mathbb{R}^3 \).

By the definition of \( \eta_v \), we find that \( \eta_v \) is Lipschitz continuous on \( \mathbb{R}_{\geq 0} \times \mathbb{C} \). More precisely, we have the following:
\[
|\eta_v (\zeta) - \eta_v (\zeta')| \leq |\zeta - \zeta'| \quad \text{holds for every} \quad \zeta, \zeta' \in \mathbb{R}_{\geq 0} \times \mathbb{C}.
\]
We then see that the following holds for every \( \zeta_v, \zeta'_v \) and \( \zeta_u, \zeta'_u \ (u \in V_v^1) \) in \( \mathbb{R}_{\geq 0} \times \mathbb{C} \):
\[
|D_v (\zeta_v, (\zeta_u)_{u \in V_v^1}) - D_v (\zeta'_v, (\zeta'_u)_{u \in V_v^1})| \leq |\zeta_v - \zeta'_v| + 2 \sum_{u \in V_v^1} |\zeta_u - \zeta'_u|.
\]

**Lemma 5.2.** The functions \( \eta_v \) and \( D_v \) satisfy the following inequality for every \( \zeta_v \) and \( \zeta_u \ (u \in V_v^1) \) in \( \mathbb{R}_{\geq 0} \times \mathbb{C} \):
\[
D_v (\zeta_v, (\zeta_u)_{u \in V_v^1}) \geq \eta_v (\zeta_v). \tag{5.1}
\]

**Proof.** For each \( \zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C} \ (u \in V) \), we can take \( \tilde{\zeta}_u \in \mathcal{S}_{\Omega_v} \cup \{(0,0)\} \) so that \( \eta_u (\zeta_u) = |\zeta_u - \tilde{\zeta}_u| \) holds. Therefore, we find the following inequality:
\[
D_v (\zeta_v, (\zeta_u)_{u \in V_v^1}) = \sum_{u \in V_v^1} |\zeta_u - \tilde{\zeta}_u| + |\zeta_v - \sum_{u \in V_v^1} \zeta_u| \geq |\zeta_v - \sum_{u \in V_v^1} \tilde{\zeta}_u|.
\]
Since \( \sum_{u \in V_v^1} \tilde{\zeta}_u \in \mathcal{S}_{\Omega_v} \cup \{(0,0)\} \), we obtain (5.1). \( \square \)

We now define a family of functions
\[
\zeta_v^t : \Delta_T \to \mathcal{M}_{\Omega_v} \quad (t \in [a, 1], v \in V)
\]
by the following process: We first define \( \zeta_v^t \) by \( \zeta_v^t (\bar{s}) := \bar{\gamma}(t) \) for all \( \bar{s} \in \Delta_T \). Next, assume that \( \zeta_v^t = (\lambda_v^t, \xi_v^t) \) has already determined and that
\[
\zeta_v^t (\bar{s}) \text{ is } C^1 \text{ on } [a, 1] \text{ and } \lambda_v^t (\bar{s}) \text{ is increasing for each } \bar{s} \in \Delta_T, \tag{5.2}
\]
\[
\lambda_v^t (\bar{s}) = 0 \text{ if and only if } s_v = 0. \tag{5.3}
\]
Then, we define \( \zeta_u^t \ (u \in V_v^1) \) as follows:
We then define a family of maps $\Psi_t$.

When $s_v = 0$, we set $\zeta_u^t(\vec{s}) := 0$ for $t \in [a, 1]$.

When $s_v > 0$, we define $\zeta_u^t(\vec{s})$ ($u \in V_v^1$) by the solution of the differential equation

\[
\frac{d\zeta_u^t}{dt} = \frac{\eta_u(\zeta_u)}{D_v(\zeta_u^t, (\zeta_u)_{u \in V_v^1})} \frac{\partial \zeta_u^t}{\partial t}
\]

with the initial condition $\zeta_u|_{t=a} = s_u\tilde{\gamma}(a)$.

Notice that, by the induction hypotheses and Lemma 5.2, we can take $\varepsilon > 0$ so that $D_v(\zeta_u^t, (\zeta_u)_{u \in V_v^1}) \geq \varepsilon$ holds for every $t \in [a, 1]$ and $\zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C}$ ($u \in V_v^1$) when $s_v > 0$.

We then see that the right hand side of (5.4.4) is locally Lipschitz continuous on the variables $\zeta_u$ ($u \in V_v^1$). Therefore, since $s_v\tilde{\gamma}(a) \notin \{\zeta_u \in \mathbb{R}_{\geq 0} \times \mathbb{C} | \eta_u(\zeta_u) = 0\} = \mathfrak{T}_{\Omega_u} \cup \{(0,0)\}$ when $s_u \neq 0$, the Cauchy-Lipschitz theorem yields the existence and uniqueness of the solutions $\zeta_u : [a, 1] \rightarrow \mathcal{M}_{\Omega_u} \setminus \{(0,0)\}$ ($u \in V_v^1$) of (5.4.4) satisfying the initial condition $\zeta_u|_{t=a} = s_u\tilde{\gamma}(a)$. We immediately derive from the construction of $\zeta_u^t (u \in V_v^1)$ that they also satisfy the induction hypotheses (5.2) and (5.3). In such a way, we can inductively determine a family of functions $\zeta_v^t (v \in V)$.

We now define a family of maps

$$
\Phi_t : \Delta_T \rightarrow \mathcal{M}_{\Omega}^T := \prod_{v \in V} \mathcal{M}_{\Omega_v} \quad (t \in [a, 1])
$$

by $\Phi_t := (\zeta_v^t)_{v \in V}$. Since each $\zeta_v^t(\vec{s})$ satisfies

$$
\lambda_v^t(\vec{s}) = s_v|\tilde{\gamma}(a)| + \int_a^t \left| \frac{\partial \zeta_v^t}{\partial t'}(\vec{s}) \right| dt',
$$

we find that the path $\tilde{\gamma}_v(\vec{s}) : [0, 1] \rightarrow \mathcal{M}_{\Omega_v}$ obtained by concatenating the paths $s_v\tilde{\gamma}|_{[0,a]}$ and $\zeta_v(\vec{s}) : [a, 1] \rightarrow \mathcal{M}_{\Omega_v}$ defines an $\Omega_v$-allowed path $\gamma_v(\vec{s})$ and its lift $\gamma_v(\vec{s})$ on $X_{\Omega_v}$.

We then define a family of maps $\Psi_t : \Delta_T \rightarrow X_{\Omega}^T$ ($t \in [a, 1]$) by $\Psi_t(\vec{s}) := (\zeta_v^t(\vec{s}))_{v \in V}$.

We now confirm that $(\Psi_t)_{t \in [a, 1]}$ gives a $(\gamma, T)$-adapted deformation of $\varnothing(\gamma(a))$. It suffices to show the following properties of $(\Phi_t)_{t \in [a, 1]}$:

**Lemma 5.3.** For each $v \in V$ and $t \in [a, 1]$, the function $\zeta_v^t$ on $\Delta_T$ only depends on the variables $s_u$ ($u \in W_v$).

**Lemma 5.4.** For every $t \in [a, 1]$, the followings hold:

\[
\zeta_v^t = 0 \quad \text{when} \quad s_v = 0 \quad (v \in V^\circ),
\]

\[
\sum_{u \in V_v^1} \zeta_u^t = \zeta_v^t \quad \text{when} \quad \sum_{u \in V_v^1} s_u = s_v \quad (v \in V \setminus L).
\]
Proposition 5.5. The map $\Phi : [a, 1] \times \Delta_T \to \mathcal{M}_\Omega^T$ is Lipschitz continuous.

Lemma 5.4 immediately follows from the construction of $\zeta^t_v$.

Proof of Lemma 5.4. Since $\eta_v(0) = 0$ for $v \in V^\circ$, we find that the hyperplane defined by $\zeta_v = 0$ is invariant by the flow of (5.4.v) when $s_{v_t} > 0$. Therefore, combining the case $s_{v_t} = 0$, we obtain (5.5). We then prove (5.6). We first consider the case $s_v > 0$. Since $\zeta^t_u \ (u \in V_v^1)$ satisfies (5.4.u), we find that $\Xi := \zeta^t_u - \sum_{u \in V_v^1} \zeta^t_u$ satisfies

$$
\frac{d\Xi}{dt} := \frac{\Xi}{D_v((\zeta^t_u)_{u \in V_v^1}, \Xi)} \frac{\partial \zeta^t_u}{\partial t}
$$

and an initial condition $\Xi|_{t=a} = (s_v - \sum_{u \in V_v^1} s_u)\gamma(a)$, where

$$
D_v((\zeta^t_u)_{u \in V_v^1}, \Xi) := \sum_{u \in V_v^1} \eta_u(\zeta^t_u) + |\Xi|.
$$

Therefore, since $\Xi = 0$ also satisfies (5.7) and the initial condition when $\sum_{u \in V_v^1} s_u = s_v$, we obtain (5.6) from the uniqueness of such solutions. On the other hand, the equation (5.6) is obvious when $s_v = 0$ because $\zeta^t_v = 0$ and $\zeta^t_u = 0 \ (u \in V_v^1)$.

Remark 5.6. Since the inequality

$$
\sum_{u \in V_v^1} \eta_u(\zeta^t_u) \leq D_v(\zeta^t_v, (\zeta^t_u)_{u \in V_v^1})
$$

holds for $\zeta_v, \zeta_u \in \mathbb{R}_{>0} \times \mathbb{C}$, we can derive from (5.4.u) the following inequality:

$$
\sum_{u \in V_v^1} \lambda^t_u \leq \Lambda_v^t \text{ holds for } t \in [a, 1].
$$

For the proof of Proposition 5.5, we use the following

Lemma 5.7. If $\lambda^t_v \ (v \in V \setminus L)$ satisfies $0 < \lambda^t_v(\vec{s}) \leq \rho$ at $\vec{s} \in \Delta_T$ and $t \in [a, 1]$, $\zeta^t_u(\vec{s}) \ (u \in V_v^1)$ is written as $\zeta^t_u(\vec{s}) = s_u s_v^{-1} \zeta^t_u(\vec{s})$.

Proof. Since $\lambda^t_u \ (u \in V_v^1)$ satisfies $\lambda^t_v(\vec{s}) \leq \lambda^t_v(\vec{s}) \leq \rho$ for $t' \in [a, t]$, we find that $\eta_u(\zeta^t_u(\vec{s})) = |\zeta^t_u(\vec{s})|$ holds by the definition of $\eta_u$. Therefore, since $\zeta_u = s_u s_v^{-1} \zeta^t_v$ also gives a solution of (5.4.u) satisfying the initial condition $\zeta_u|_{t=a} = s_u \gamma(a)$, we obtain the representation of $\zeta^t_v$ from the uniqueness of such solutions.

Proof of Proposition 5.5. We prove by induction that each of the functions $\zeta^t_v \ (v \in V)$ is Lipschitz continuous on $\Delta_T$. Since $\zeta^t_v$ is independent of $\vec{s} \in \Delta_T$, the statement holds for $v = \hat{v}$. We then assume that $\lambda^t_v$ is Lipschitz continuous on $\Delta_T$. Since the right
hand side of (5.4,u) \((u \in V^1_v)\) is Lipschitz continuous on the variables \(s_{v'} (v' \in W_v)\) by
the induction hypothesis, we find by the use of the Cauchy-Lipschitz theorem that \(\zeta^t_u\)
is locally Lipschitz continuous on the variables \(s_{v'} (v' \in W_u)\) when \(s_v > 0\). Further,
we see by the use of the representation of \(\zeta^t_u\) in Lemma 5.17 that the following holds for
\(\tilde{s}, \tilde{s}' \in \Delta_T\) with \(s_v, s'_v \) sufficiently small:

\[
|\zeta^t_u(\tilde{s}) - \zeta^t_u(\tilde{s}')| \leq \frac{\zeta^t_u(\tilde{s})}{s_v}|s_u - s'_v| + \frac{s_u'}{s_v} |\zeta^t_u(\tilde{s}) - \zeta^t_u(\tilde{s}')| + \frac{s_u'}{s_v} |\zeta^t_u(\tilde{s}')| |s'_v - s_v|.
\]

We may assume without loss of generality that \(s'_v \leq s_v\), and hence, \(s'_u \leq s_v\). Therefore,
we find by the induction hypothesis that we can take the Lipschitz constant of \(\zeta^t_u\)
uniformly in a neighborhood of \(s_v = 0\). Thus, we have the Lipschitz continuity of \(\zeta^t_u\) on
\(\Delta_T\). Since \(\zeta^t_u(\tilde{s})\) is \(C^1\) on \([a, 1]\) for each \(\tilde{s} \in \Delta_T\), we obtain the Lipschitz continuity of
the map \(\Phi\).

\[\square\]

§ 6. Estimates for the \((\gamma, T)\)-adapted deformation

In the previous section, we constructed the map \(\Phi: [a, 1] \times \Delta_T \to \mathcal{M}^T_{\Omega}\)
for \(\gamma \in \Pi^{\delta, L}_{\Omega_v}\) satisfying the assumptions in Theorem 4.6 and the properties of the
\((\gamma, T)\)-adapted deformation \((\Psi_t)_{t \in [a, 1]}\) was reduced to its properties. In this section, we derive estimates
for \(\Phi = (\zeta_v)_{v \in V}\) which are necessary to prove Theorem 4.6. We first show the following

Lemma 6.1. For every \(v \in V, t \in [a, 1]\) and \(\tilde{s} \in \Delta_T\), the function \(\zeta^t_v(\tilde{s})\) satisfies

\[\text{dist}(\zeta^t_v(\tilde{s}), \mathfrak{F}_{\Omega_v}) \geq \delta'(t), \]

where \(\delta'(t)\) is the function given in (4.7).

Proof. When \(s_v = 0\), (6.1) immediately follows from (5.5). We then assume that
\(s_v > 0\). Let \(\{v_j\}_{j=1}^\ell\) be vertexes such that \(v = v_1 \to v_2 \to \cdots \to v_\ell = \hat{v}\) gives
a path on \(T\) from \(v\) to \(\hat{v}\). Since \(s_{v_j} \geq s_v\) holds for \(j = 1, \cdots, \ell\), we find that \(\zeta^t_{v_j}\)
satisfies \(\eta_j(\zeta^t_{v_j}(\tilde{s})) > 0\) for every \(t \in [a, 1]\), where \(\eta_j := \eta_{v_j}\). We now consider estimates
of the function \(h^t_{v_j} := \eta_j(\zeta^t_{v_j}(\tilde{s}))\). Since \(h^t_{v_j}\) is Lipschitz continuous on \([a, 1]\), we find
by Rademacher’s theorem that the derivative of \(h^t_{v_j}\) exists a.e. on \([a, 1]\) and satisfies
\(|dh^t_{v_j}/dt| \leq |\partial \zeta^t_{v_j}/\partial t|\). We further obtain from Lemma 5.2 and (5.4.v_j) the following
sequence of inequalities:

\[\frac{1}{h^t_1} |\partial \zeta^t_1/\partial t| \leq \frac{1}{h^t_2} |\partial \zeta^t_2/\partial t| \leq \cdots \leq \frac{1}{h^t_\ell} |\partial \zeta^t_\ell/\partial t| \leq \frac{|\tilde{\gamma}'|}{\delta}.
\]

Thus, we have the following:

\[\frac{1}{h^t_1} \left| \frac{dh^t_1}{dt} \right| \leq \frac{|\tilde{\gamma}'|}{\delta} \quad \text{holds a.e. on } [a, 1].\]
Since \( h_1^a = \sqrt{2}s_v \rho \) and \( |\tilde{\gamma}'| = \sqrt{2}|\gamma'| \), we derive from (6.3) the following:
\[
\sqrt{2}s_v \rho e^{-\sqrt{2}\delta^{-1}L_a(\gamma_{t_1})} \leq h_1^t \leq \sqrt{2}s_v \rho e^{\sqrt{2}\delta^{-1}L_a(\gamma_{t_1})}.
\]

Therefore, we find that \( \rho e^{-\sqrt{2}\delta^{-1}L_a(\gamma_{t_1})} \leq h_1^t \) holds for any \( t \in [a, 1] \) when \( \sqrt{2}s_v \geq e^{-\sqrt{2}\delta^{-1}L_a(\gamma_{t_1})} \). Otherwise, we have \( h_1^t \leq \rho \), and hence, \( |\xi_1^t| \leq \rho \) holds for any \( t \in [a, 1] \). It proves (6.1). \( \square \)

We next show that \( \Phi_t = ((\lambda_v^t, \xi_v^t))_{v \in V} \) satisfies the estimate (4.6). We use the following

**Lemma 6.2.** Suppose that \( \bar{s}, \tilde{s}' \in \Delta_T \) satisfy \( s_u = s'_u \) for \( u \in V \setminus V_1^1 \) with \( v \in V \setminus L \). Then, we have for every \( t \in [a, 1] \)
\[
(6.4) \quad \sum_{u \in V_1^1} \left| \frac{\partial \xi_u^t}{\partial t}(\bar{s}) - \frac{\partial \xi_u^t}{\partial t}(\tilde{s}') \right| \leq \frac{3 |\gamma'|}{\delta} \sum_{u \in V_1^1} |\xi_u^t(\bar{s}) - \xi_u^t(\tilde{s}')|.
\]

**Proof.** Since the left hand side of (5.4.u) satisfies
\[
\sum_{u' \in V_1^1} \left| \left( \frac{\eta_v^t(\zeta_w')}{D_v(\zeta_v', (\zeta_w)_u \in V_1^1)} - \frac{\eta_v^t(\zeta'_w)}{D_v(\zeta_v', (\zeta'_w)_u \in V_1^1)} \right) \frac{\partial \xi_v^t}{\partial t} \right|
\]
\[
\leq \left| \frac{D_v(\zeta_v', (\zeta'_w)_u \in V_1^1) - D_v(\zeta_v', (\zeta_w)_u \in V_1^1)}{D_v(\zeta_v', (\zeta_w)_u \in V_1^1)} \right| \left| \frac{\partial \xi_v^t}{\partial t} \right| \sum_{u' \in V_1^1} \frac{\eta_v^t(\zeta'_w)}{D_v(\zeta_v', (\zeta'_w)_u \in V_1^1)}
\]
\[
+ \left| \frac{\partial \xi_v^t}{\partial t} \right| \sum_{u' \in V_1^1} \left| \eta_v^t(\zeta'_w) - \eta_v^t(\zeta'_w) \right| \frac{D_v(\zeta_v', (\zeta'_w)_u \in V_1^1)}{D_v(\zeta_v', (\zeta_w)_u \in V_1^1)}
\]
\[
\leq \frac{3}{D_v(\zeta_v', (\zeta_w)_u \in V_1^1)} \left| \frac{\partial \xi_v^t}{\partial t} \right| \sum_{u' \in V_1^1} \left| \zeta'_w - \zeta'_w \right|
\]

we obtain (6.4) by the use of (6.2) when \( s_v > 0 \). The inequality (6.4) is trivial when \( s_v = 0 \). \( \square \)

We now set for \( \bar{s}, \tilde{s}' \in \Delta_T \) and \( v \in V \setminus L \)
\[
\Xi_v(t) := \sum_{u \in V_1^1} \left| \xi_u^t(\bar{s}) - \xi_u^t(\tilde{s}') \right|
\]

Under the assumption in Lemma 6.2, we obtain from (6.4) the following:
\[
|\Xi_v(t) - \Xi_v(a)| \leq \frac{3}{\delta} \int_a^t \Xi_v(t')|\gamma'(t')|dt'.
\]

Therefore, Gronwall’s Lemma yields the following estimate:
\[
(6.5) \quad \Xi_v(t) \leq c(t) \Xi_v(a) \quad \text{holds for every} \quad t \in [a, 1],
\]
where \( c(t) \) is the constant given in (4.7). Since \( \Xi_v(a) = \rho \sum_{u \in V^1_v} |s_u - s'_u| \), we see by (6.5) that the following holds a.e. on \( \Delta_T \):

\[
(6.6) \quad \left| \det \left[ \frac{\partial \xi^t_{u'u}}{\partial s_u} \big|_{u,u' \in V^1_v} \right] \right| \leq \prod_{u \in V^1_v} \left( \sum_{u' \in V^1_v} \left| \frac{\partial \xi^t_{u'u}}{\partial s_u} \right| \right) \leq (c(t))^{\left| V^1_v \right|}.
\]

Notice that \( \frac{\partial \xi_v}{\partial s_u} = 0 \) for \( u / \notin W_v \), and hence, we find the following:

\[
(6.7) \quad \left| \det \left[ \frac{\partial \xi^t_{u'u}}{\partial s_u} \big|_{u,u' \in V^1_v} \right] \right| = \prod_{v \in V \setminus L} \left| \det \left[ \frac{\partial \xi^t_{u'u}}{\partial s_u} \big|_{u,u' \in V^1_v} \right] \right|.
\]

Since \( \sum_{v \in V \setminus L} \left| V^1_v \right| = k - 1 \), we obtain (4.6) from (6.6) and (6.7). Together with (5.8) and Lemma 6.1, Theorem 4.6 is validated.

§ 7. Applications to nonlinear differential and difference equations

In this section, we discuss the resurgence of formal series solutions

\[
\Phi = \begin{pmatrix} \Phi^{(1)} \\ \vdots \\ \Phi^{(n)} \end{pmatrix} \in \mathbb{C}^n[[x^{-1}]]
\]

of a nonlinear differential equation

\[
(7.1) \quad \frac{d}{dx} \Phi = F(x^{-1}, \Phi)
\]

at \( x = \infty \) with \( F(x^{-1}, \Phi) \in \mathbb{C}^n \{ x^{-1}, \Phi \} \) satisfying the conditions

\[
(7.2) \quad F(0, 0) = 0 \quad \text{and} \quad \det(\partial \Phi F(0, 0)) \neq 0.
\]

Under the assumption (7.2), there exists a unique formal series solution of the form

\[
\Phi(x) = \sum_{k=1}^{\infty} \Phi_k x^{-k}.
\]

We rewrite \( F(x^{-1}, \Phi) \) in the following form:

\[
F(x^{-1}, \Phi) = F_0(x^{-1}) + \partial \Phi F(0, 0) \Phi + \sum_{|\ell| \geq 1} F_{\ell}(x^{-1}) \Phi^\ell,
\]

where \( \Phi^\ell := (\Phi^{(1)})^{\ell_1} \cdots (\Phi^{(n)})^{\ell_n} \) with \( \ell = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}_{\geq 0}^n \) and \( |\ell| := \ell_1 + \cdots + \ell_n \).

Regarding (7.1) as an equation for \( \Phi(x) := x^{-1} (\Phi(x) - \Phi_1 x^{-1}) \), we may assume without loss of generality that

\[
(7.3) \quad F_{\ell}(0) = 0 \quad \text{for every} \quad \ell \in \mathbb{Z}_{\geq 0}^n.
\]
Applying the Borel transform, (7.1) is rewritten as follows:

\[(7.4) \quad P(\xi)\hat{\Phi} = \hat{F}_0 + \sum_{|\ell| \geq 1} \hat{F}_\ell \ast \hat{\Phi}^\ell,\]

where \(P(\xi) := -\xi - \partial_\nu F(0,0)\) and \(\hat{\Phi}^\ell := B(\Phi^\ell)\). We now inductively determine \(\hat{\Phi}_k\) \((k \geq 1)\) by the following procedure:

\[(7.5) \quad \hat{\Phi}_1 := P^{-1}\hat{F}_0,\]

\[(7.6) \quad \hat{\Phi}_{k+1} := P^{-1} \sum_{j=1}^{k} \sum_{|\ell|=j} \hat{F}_\ell \ast \sum_{k_1+\cdots+k_j=k} \hat{\Phi}^\ell_{k_1,\cdots,k_j},\]

where \(\hat{\Phi}^\ell_{k_1,\cdots,k_j}\) is the convolution product of functions in

\[\left\{ \hat{\Phi}^{(m)}_{k_i} \mid 1 + \sum_{p=1}^{m-1} \ell_p \leq i \leq \sum_{p=1}^{m} \ell_p, 1 \leq m \leq n \right\}.

We now introduce the following

**Definition 7.1.** Let \(T = (V, E) \in \mathcal{T}\) and consider a function \(\nu = (\nu_1, \nu_2) : V \rightarrow \{1, \cdots, n\}^2\). We call such a pair \(T = (T, \nu)\) \(n\)-decorated iteration diagram and denote the set of \(n\)-decorated iteration diagrams by \(\mathcal{T}^n\).

Let \(T = (T, \nu) \in \mathcal{T}^n\). We define an equivalence relation \(\sim_v\) on \(V_v^1\) for \(v \in V \setminus L\) as follows: \(u \sim_v u' (u, u' \in V_v^1)\) if \(T_u = T_{u'}\) and \(\nu|_{V_u} = \nu|_{V_{u'}}\). For each \([u] \in V_v^1/\sim_v\), we define an integer \(\#[u]\) as the cardinal of \(\{u' \in V_v^1 \mid u' \sim_v u\}\) and the multiplicity \(\mu_v\) of \(T\) at \(v\) by

\[\mu_v := \left( \prod_{[u] \in V_v^1/\sim_v} (\#[u])! \right)^{-1} \cdot (|V_v^1|)!.

We set \(\mu_T := \prod_{v \in V \setminus L} \mu_v\). We further define a map \(\lambda_{T,j} : V \rightarrow \mathbb{Z}_{\geq 0} (j = 1, \cdots, n)\) by \(\lambda_{T,j}(v) := |\{u \in V_v^1 \mid \nu_1(u) = j\}|\) when \(v \in V \setminus L\) and \(\lambda_{T,j}(v) := 0\) when \(v \in L\). We set \(\lambda_T = (\lambda_{T,1}, \cdots, \lambda_{T,n})\). By the use of the multinomial theorem, we obtain the following

**Lemma 7.2.** For every \(k \geq 1\) and \(j \in \{1, \cdots, n\}\), \(\hat{\Phi}_k^{(j)}\) is written as follows:

\[(7.7) \quad \hat{\Phi}_k^{(j)} = \sum_{T \in \mathcal{T}_k^{n,(j)}} \mu_T \cdot \psi_T,

where \(\mathcal{T}_k^{n,(j)} := \{(T, \nu) \in \mathcal{T} \mid T \in \mathcal{T}_k, \nu_1(\hat{v}) = j\}\) and \(\psi_T\) is the iterated convolution product of \((T; \{\hat{f}_v\}_{v \in V}, \{\hat{\varphi}_v\}_{v \in V})\) with \(\hat{f}_v := \hat{F}_{\nu_2(v)}(\nu_1(v))\) and \(\hat{\varphi}_v := (P^{-1})_{\nu(v)}((\nu_1(v), \nu_2(v))-th entry of P^{-1}).\)
We now show the following

**Lemma 7.3.** There exist positive constants $C$ and $\delta(<1/n)$ such that

\begin{equation}
\sum_{T \in \mathcal{T}_k^{n,(j)}} \mu_T \leq \delta B(k) C^k
\end{equation}

holds for ever $k \geq 1$ and $j \in \{1, \cdots, n\}$, where $B(k)$ is a constant defined by

\[ B(k) := \frac{3}{2\pi^2(k + 1)^2}. \]

**Proof.** Since the left hand side of (7.8) is independent of $j$, we denote it by $N_k$. We find by (7.6) that \{N_k\}_{k \geq 1} satisfy the following:

\begin{equation}
N_{k+1} = \sum_{j=1}^{k} \sum_{|\ell|=j} n \sum_{k_1+\cdots+k_j=k} N_{k_1} \cdots N_{k_j}.
\end{equation}

Taking $C > 0$ sufficiently large, we may assume that $N_1$ satisfies (7.8) for arbitrary small positive $\delta$. We then assume that (7.8) holds for $k \leq K$. Using the inequality

\[ \sum_{k_1+\cdots+k_j=k} B(k_1) \cdots B(k_j) \leq B(k), \]

we derive from (7.9) the following:

\[ N_{K+1} \leq nB(k)C^k \sum_{j=1}^{k} \sum_{|\ell|=j} \delta^j \leq nB(k)C^k \sum_{j=1}^{k} (n\delta)^j \leq nB(k)C^k \frac{n\delta}{1-n\delta}. \]

Therefore, taking $C$ sufficiently large so that it satisfies $n^2(1-n\delta)^{-1} \leq C$, we see that (7.8) holds for $k = K + 1$. We thus obtain (7.8) for $k \geq 1$. \[ \square \]

Let $\Omega = (\Omega_L)_{L \in \mathbb{R}_{\geq 0}}$ be a d.f.s. defined by the formula

\begin{equation}
\Omega_L = \{ \xi \in \mathbb{C} \mid \det(-\xi - \partial \Phi F(0,0)) = 0, |\xi| \leq L \}.
\end{equation}

We then find that each entry of $P^{-1}$ is $\Omega$-continuable, and hence, we obtain from Corollary 3.10 the following estimates: for any $\delta, L > 0$, there exist $c, \delta' > 0$ such that

\begin{equation}
\|P^\delta T\|_{\Omega_\delta} \leq \frac{c^{k-1}}{(k-1)!} \prod_{v \in V} \|\hat{\nu}(P^{-1})\|_{\Omega} \|\hat{F}(v)\|_{\Omega} \|\hat{F}(\nu_2(v))\|_{\Omega} \|\hat{F}(\nu_3(v))\|_{\Omega}
\end{equation}

holds for every $T \in \mathcal{T}_k^{n,(j)}$. Since $F(x^{-1}, \Phi) \in \mathbb{C}^n\{x^{-1}, \Phi\}$, we can take $A > 0$ so that $\|P^{-1}\|_{\Omega} \leq A$ and $\|\hat{F}(\nu)\|_{\Omega} \leq A^{1+|\ell|}$ hold for any $i, j \in \{1, \cdots, n\}$ and $\ell \in \mathbb{Z}_{\geq 0}$.\[ \square \]
Notice that $|\lambda_T(v)| = |V_v|$ when $v \in V \setminus L$, and hence, $\sum_{v \in V} |\lambda_T(v)| = k - 1$. Therefore, we derive from (7.7), (7.8) and (7.11) the following estimates: there exists a positive constant $C$ such that

$$
\frac{\|\hat{\Phi}^{(j)}_k\|_{\Omega^\infty}}{(k - 1)!} \leq \frac{C^k}{(k - 1)!}
$$

holds for every $k \geq 1$ and $j \in \{1, \cdots, n\}$. We then find that each entry of $\hat{\Phi} = \sum_{k \geq 1} \hat{\Phi}_k$ converges in $\hat{\mathcal{H}}_{\Omega^\infty}$ and $\hat{\Phi}$ gives a solution of (7.4). Thus, we obtain the following

**Theorem 7.4.** Let $\Omega = \{\Omega_L\}_{L \in \mathbb{R} \geq 0}$ be a d.f.s. defined by (7.10). Then, each entry of the formal series solution $\hat{\Phi} \in \mathbb{C}^n[[x^{-1}]]$ of (7.4) is $\Omega^\infty$-resurgent.

By the same discussion, we have the following

**Theorem 7.5.** Let us consider a nonlinear difference equation

$$
\Phi(x + 1) - \Phi(x) = F(x^{-1}, \Phi(x))
$$

at $x = \infty$ under the assumption (7.2). Then, there exists a unique formal series solution $\Phi(x) \in \mathbb{C}^n[[x^{-1}]]$ of (7.13) and each entry of $\Phi(x)$ is $\Omega^\infty$-resurgent, where $\Omega = \{\Omega_L\}_{L \in \mathbb{R} \geq 0}$ is a d.f.s. defined by

$$
\Omega_L = \{\xi \in \mathbb{C} \mid \det((e^{-\xi} - 1) - \partial_x F(0, 0)) = 0, |\xi| \leq L\}.
$$

**Acknowledgements.** The author expresses his gratitude to Prof. David Sauzin. His comments and suggestions have been helpful. The author also expresses his gratitude to Prof. Masafumi Yoshino and Prof. Yoshitsugu Takei for their encouragement. The author is grateful to all the staffs in Mathematical department of Hiroshima University for their kind hospitality.

**References**

[BKu1] B. Braaksma and R. Kuik: Resurgence relations for classes of differential and difference equations, Ann. Fac. Sci. Toulouse Math. (6) 13 (2004), no. 4, 479–492.

[BKu2] _______: Asymptotics and singularities for a class of difference equations, in Analyzable functions and applications, Contemp. Math., 373, Amer. Math. Soc., Providence, RI, 2005, pp. 113–135.

[CaNP] B. Candelpergher, J. C. Nosmas and F. Pham: Approche de la résurgence, Actualités Mathématiques., Hermann, Paris, 1993.

[Co] O. Costin: On Borel summation and Stokes phenomena for rank-1 nonlinear systems of ordinary differential equations, Duke Math. J. 93 (1998), no. 2, 289–344.
[E1] J. Écalle: Les fonctions résurgentes, Publ. Math. d’Orsay, Vol.1 (81-05), 2(81-06), 3(85-05), 1981 and 1985.

[E2] ______: Cinq applications des fonctions résurgentes. Publ. Math. d’Orsay 84-62, 1984.

[Ka] S. Kamimoto: Resurgence of formal series solutions of nonlinear differential and difference equations, Proc. Japan Acad. Ser. A Math. Sci. 92 (2016), no. 8, 92–95.

[KaS] S. Kamimoto and D. Sauzin: Iterated convolutions and endless Riemann surfaces, arXiv:1610.05453

[Ku] R. Kuik: Transseries in difference and differential equations, 2003, Doctoral Thesis (University of Groningen).

[MS] C. Mitschi, D. Sauzin: Divergent Series, Summability and Resurgence. Vol. 1: Monodromy and Resurgence. Lecture Notes in Mathematics 2153, Springer, Heidelberg, 2016.

[OD] Y. Ou and E. Deleabaere: Endless continuability and convolution product, arXiv:1410.1200

[S1] D. Sauzin: Initiation to mould calculus through the example of saddle-node singularities, Rev. Semin. Iberoam. Mat. 3 (2008), no. 5–6, 147–160.

[S2] ______: Mould expansions for the saddle-node and resurgence monomials, Renormalization and Galois theories, 83–163, IRMA Lect. Math. Theor. Phys., 15, Eur. Math. Soc., Zürich, 2009.

[S3] ______: On the stability under convolution of resurgent functions, Funkcial. Ekvac., 56 (2013), no. 3, 397–413.

[S4] ______: Nonlinear analysis with resurgent functions, Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 3, 667–702.