Qualitative Properties of Solutions for an Integral Equation

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Abstract Let $n$ be a positive integer and let $0 < \alpha < n$. In this paper, we continue our study of the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^{(n+\alpha)/(n-\alpha)} dy. \quad (0.1)$$

We mainly consider singular solutions in subcritical, critical, and super critical cases, and obtain qualitative properties, such as radial symmetry, monotonicity, and upper bounds for the solutions.

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1 Introduction

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space, and let $\alpha$ be a constant satisfying $0 < \alpha < n$. Consider the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} u(y)^p dy. \quad (1.1)$$

When $p = \alpha^* := \frac{n+\alpha}{n-\alpha}$, it is the so-called critical case. It arises as an Euler-Lagrange equation for a functional under a constraint in the context of the Hardy-Littlewood-Sobolev inequalities. In his elegant paper [L],

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Lieb classified the maximizers of the functional, and thus obtained the best constant in the Hardy-Littlewood-Sobolev inequalities. He then posed the classification of all the critical points of the functional – the solutions of the integral equation (1.1) as an open problem.

In our previous paper, we solved this open problem by using the method of moving planes. We proved that all the solutions of (1.1) are radially symmetric and assume the form

$$c \left( \frac{t}{t^2 + |x - x_o|^2} \right)^{(n-\alpha)/2}$$

with some constant $c = c(n, \alpha)$, and for some $t > 0$ and $x_o \in \mathbb{R}^n$. We also established the equivalence between the integral equation and the family of well-known semi-linear partial differential equations

$$(\triangle) u = u^{n+\alpha \over n-\alpha},$$

and therefore classified all the solutions of the PDE.

In this paper, we continue to study the integral equation. We consider subcritical cases $p < \alpha^*$, super critical cases $p > \alpha^*$, and singular solutions in all cases.

In section 2, we consider subcritical cases. We first prove the non-existence theorem.

**Theorem 1** For $p < \alpha^*$, there does not exist regular positive solutions of (1.1).

Then we consider solutions with one singularity, and use the method of moving planes to obtain the radial symmetry and monotonicity of the solutions. The result also applies to singular solutions in the critical case.

**Theorem 2** For $p \leq \alpha^*$, if a solution $u$ of (1.1) has only one singularity at a point $x^o$, then it must be radially symmetric about the same point.

In section 3, we study singular solutions in the critical case and obtain an upper bound for the solutions.
Theorem 3 Assume that \( u(x) \) is a positive solution of (1.1) with only one singularity at \( x_0 \), then there is a constant \( C \), such that

\[
u(x) \leq C \left| x - x_0 \right|^{-\frac{n}{2}}.
\] (1.3)

In section 4, we consider super critical cases and provide examples of non-radially symmetric solutions.

2 Subcritical Critical Cases

In this section, we prove Theorem 1 and 2. We first establish the non-existence theorem for regular solutions.

Theorem 2.1 For \( p < \alpha^* \), there does not exist regular positive solutions of (1.1).

Obviously, the integral equation possesses singular solutions. For instance, one can verify that, for \( p > \frac{n}{n - \alpha} \),

\[
u(x) = \frac{c}{\left| x \right|^{-\frac{n}{p-1}}}
\]

with some appropriate constant \( c \) is a singular solution. However, we can prove the following

Theorem 2.2 For \( p \leq \alpha^* \), if a solution \( u \) of (1.1) has only one singularity at a point \( x^* \), then it must be radially symmetric about the same point.

The main ingredient of the proofs are the Kelvin type transform and the method of moving planes.

Assume that \( u \) is a solution of integral equation (1.1). Let

\[
u(x) = \frac{1}{\left| x \right|^{n-\alpha}} u \left( \frac{x}{\left| x \right|^2} \right)
\] (2.4)
be the Kelvin type transform of $u(x)$. Then it is a straight forward calculation to verify that $v(x)$ satisfies the equation

$$v(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}|y|^{-(n-\alpha)(\alpha^*-p)}v^p(y)}dy.$$  

(2.5)

In the following, we will only present the method of moving planes in a sketchy way. For more details, please see our previous paper [CLO].

The Proof of Theorem 2.1.

Assume that $u(x)$ is a positive regular solution of the integral equation (1.1). Let $x_1$ and $x_2$ be any two points in $\mathbb{R}^n$. Since the integral equation is invariant under translations, we may assume that the midpoint $\frac{x_1 + x_2}{2}$ is at the origin. Let $v(x)$ be the Kelvin type transform as defined in (2.4). Then $v(x)$ has the desired asymptotic behavior at infinity

$$v(x) \leq \frac{C}{1 + |x|^{n-\alpha}}.$$  

Let

$$x_i^* = \frac{x_i}{|x_i|^2}, \quad i = 1, 2$$

be the inversions of $x_i$. Because of the presence of the singular term $|y|^{-(n-\alpha)(\alpha^*-p)}$ in the integral equation (2.5), similar to what we did in [CLO], we can use the method of moving planes to show that $v(x)$ must be radially symmetric about the origin. In particular, $v(x_1^*) = v(x_2^*)$; and therefore $u(x_1) = u(x_2)$. Since $x_1$ and $x_2$ are any two points in $\mathbb{R}^n$, we conclude that $u$ must be a constant. This is impossible. Therefore, (1.1) does not exist any positive regular solution.

The Proof of Theorem 2.2.

Without loss of generality, we may assume that $u(x)$ has only one singularity at point $e = (1, 0, \ldots, 0)$. We show that $u(x)$ is symmetric and monotone decreasing about any line passing through $e$. Since we do not know any asymptotic behavior of $u(x)$ at the infinity, we are not able to apply the method of moving planes on it. To overcome this difficulty, as usual, we make a Kelvin type transform (2.4) centered at the origin. Obviously,
$v(x)$ still has a singularity at point $e$ and a possible singularity at the origin. Let $\epsilon$ be any small positive number, make another Kelvin type transform centered at $\epsilon e$, i.e. let

$$w_\epsilon(x) = \frac{1}{|x|^{n-\alpha}} v\left(\frac{x}{|x|^2} + \epsilon e\right).$$

Now $w_\epsilon(x)$ has a singularity at $e_\epsilon := \frac{1}{1-\epsilon} e$ and a possible singularity at the inversion point of the origin $0^* := -\frac{1}{\epsilon} e$. Now, we are able to carry on the method of moving planes as we did in [CLO] to show that $w_\epsilon(x)$ is symmetric and monotone decreasing about the line $\overline{0^* e}$. Since

$$w_\epsilon(x) \to u(x) \text{ as } \epsilon \to 0,$$

We have shown that $u(x)$ is symmetric and monotone decreasing about the line $\overline{0 e}$. Because we can make the Kelvin type transform centered at any point around $e$, we prove that $u(x)$ is symmetric and monotone decreasing about any line passing through $e$. Therefore $u(x)$ must be radially symmetric and monotone decreasing about the point $e$. This completes the proof of the theorem.

### 3 Critical Case - Singular Solutions

In this section, we consider singular solutions of the integral equation (1.1) in the critical case when $p = \alpha^*$. As one has seen in the previous section,

$$u(x) = \frac{c}{|x|^{n-\alpha}}$$

with a suitable constant is a singular solution. We will show in fact that any singular solution can not grow faster than this power of $x$.

**Theorem 3.1** Assume that $u(x)$ is a positive solution of (1.1) with only one singularity at $x_0$, then there is a constant $C$, such that

$$u(x) \leq \frac{C}{|x - x_0|^{n-\alpha}}.$$  \hspace{1cm} (3.6)
Proof.

Without loss of generality, we may assume that the solution \( u(x) \) has only one singularity at the origin. Then as we have shown in section 2, \( u(x) \) is radially symmetric and monotone decreasing about the origin. Let \( e \) be any point such that \( |e| = 1 \). Then by the integral equation (1.1), we have, for any \( r > 0 \),
\[
\begin{align*}
    u(re) &\geq \int_{B_r(0)} \frac{1}{|re-s\omega|^{n-\alpha}} [u(s)]^{\frac{n+\alpha}{n-\alpha}} s^{n-1} ds d\omega \\
    &\geq [u(r)]^{\frac{n+\alpha}{n-\alpha}} \int_0^r \int_{\partial B_1(0)} \frac{1}{|re-r\omega|^{n-\alpha}} d\omega s^{n-1} ds \\
    &= [u(r)]^{\frac{n+\alpha}{n-\alpha}} r^\alpha \int_0^1 \int_{\partial B_1(0)} \frac{1}{|e-t\omega|^{n-\alpha}} d\omega t^{n-1} dt \\
    &= Cr^\alpha [u(r)]^{\frac{n+\alpha}{n-\alpha}},
\end{align*}
\]
with some constant \( C \). Here, by the radial symmetry of \( u \), \( u(re) = u(r) \) for any \( e \). It follows that
\[
    u(r) \geq \frac{C}{r^{\frac{\alpha}{2}}},
\]
This completes the proof of the theorem.

4 Super Critical Cases

In the super critical case when \( p > \alpha^* \), equation (1.1) possesses both symmetric solutions and non-symmetric solutions.

As we mentioned in the previous section,
\[
u(x) = \frac{c}{|x|^{\frac{\alpha}{p-\alpha}}},
\]
with some appropriate constant \( c \) is a singular symmetric solution.

Now we construct a non-radially symmetric solution. Let \( x' = (x_1, \cdots, x_{n-1}) \), let \( u(x) \) be a standard solution in \( \mathbb{R}^{n-1} \), i.e.
\[
u(x') = c(\frac{1}{1 + |x'|^2})^{\frac{\alpha-1}{2}}.
\]
Then it satisfies
\[
u(x') = \int_{\mathbb{R}^{n-1}} \frac{1}{|x' - y'|^{n-1-\alpha}} [u(y')]^{\frac{n-1+\alpha}{n-1-\alpha}} dy'. \quad (4.7)
\]
Let $x = (x', x_n)$, and define
\[
\tilde{u}(x) = u(x').
\]

Then one can verify that, for some constant $c$,
\[
\tilde{u}(x) = c \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-\alpha}} \tilde{u}(y)^{\frac{n+\alpha}{n-1+\alpha}} \, dy. \tag{4.8}
\]

It follows that a constant multiple of $\tilde{u}$ is an $n$-dimensional solution of the integral equation in super critical case, since \(\frac{n+\alpha}{n-\alpha} < \frac{n-1+\alpha}{n-1-\alpha}\). To see (4.8), one simply need to notice from elementary calculus that
\[
\int_{-\infty}^{\infty} \frac{1}{|x - y|^{n-\alpha}} \, dy_n = \frac{a}{|x' - y'|^{n-1-\alpha}},
\]
with some constant $a$.

**References**

[BN] H.Berestycki and L.Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brazil. Mat. (N.S.) 22 (1) (1991), 1-37.

[CL] W.Chen and C.Li Classification of solutions of some nonlinear elliptic equations, Duke Math. J., 63 (1991), 615-622.

[CLO] W.Chen, C.Li, and Biao Ou Classification of solutions for an integral equation, preprint, 2003.

[F] L.Fraenkel, An introduction to maximum principles and symmetry in elliptic problems, Cambridge University Press, New York, 2000.

[GNN] B.Gidas, W.Ni, and L.Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, (collected in the book Mathematical Analysis and Applications, which is vol. 7a of the book series Advances in Mathematics. Supplementary Studies, Academic Press, New York, 1981.)

[Li] C.Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, Invent. Math. 123(1996) 221-231.
[L] E.Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. 118(1983), 349-374.

[LL] E.Lieb and M.Loss, Analysis, 2nd edition, American Mathematical Society, Rhode Island, 2001.

[O] B.Ou, A Remark on a singular integral equation, Houston J. of Math. 25 (1) (1999), 181 - 184.

[S] E.Stein, Singular Integrals and Differentiability Properties of Functions Princeton University Press, Princeton, 1970,

[WX] J.Wei and X.Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 207-228(1999).

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