A GENERALIZATION OF THE FIRST BEURLING–MALLIAVIN THEOREM

IOANN VASILYEV

Abstract. In this article we prove a generalization of the First Beurling–Malliavin Theorem. In more detail, we establish a new sufficient condition for a function to be a Beurling–Malliavin majorant. Our result is strictly more general than that of the First Beurling–Malliavin Theorem. We also show that our result is sharp in a number of senses.

1. Introduction

Let $\text{Lip}(\mathbb{R})$ denote the space of Lipschitz functions in $\mathbb{R}$ (i.e. functions $f$ satisfying for all $x, y \in \mathbb{R}$ the following inequality: $|f(x) - f(y)| \leq C|x - y|$ with $C > 0$ independent of $x, y$). By $\text{Lip}(\mathbb{R}, \xi)$ we shall denote all Lipschitz functions in $\mathbb{R}$ with the Lipschitz constant $\xi$.

The following theorem was first proved by A. Beurling and P. Malliavin, see [6].

**Theorem A.** (Beurling–Malliavin) Let $\omega : \mathbb{R} \to (0, 1]$ be a function such that $\log(1/\omega) \in L^1(\mathbb{R}, dx/(1 + x^2))$, and $\log(1/\omega)$ is a Lipschitz function. Then for each $\delta > 0$ there exists a function $f \in L^2(\mathbb{R})$, which is not identically zero and satisfying $\text{spec}(f) \subset [0, \delta]$ and $|f(x)| \leq \omega(x)$ for all $x \in \mathbb{R}$.

For a function $f \in L^2(\mathbb{R})$, by $\text{spec}(f)$ we mean the spectrum of $f$, i.e. the support of its Fourier transformation. Note that the spectrum is defined up to a set of the Lebesgue measure zero. Let us also remark that here, the term “not identically zero” means “not zero almost everywhere”. We shall further sometimes write just “nonzero” for brevity.

The Beurling–Malliavin theorems are considered by many experts to be among the most deep and important results of the 20th century Harmonic Analysis. Theorem A above is called the First Beurling–Malliavin Theorem. This result gives conditions for the majorant $\omega$ ensuring existence of nonzero function whose spectrum lies in an arbitrary small interval and whose modulus is majorized by $\omega$. This theorem is a crucial tool in the proof of the Second Beurling–Malliavin Theorem about the radius of completeness of an exponential system. Moreover, Theorem A was recently used by J. Bourgain and S. Dyatlov in the theory of resonances for hyperbolic surfaces, see [8]. Deep connections of the First Beurling–Malliavin Theorem with nowadays popular gap and type problems are discussed in papers [7], [21] and [17].

Note that Theorem A is in a certain sense a contradiction to the following postulate, called the uncertainty principle: “It is impossible for a nonzero function and its Fourier transform to be simultaneously very small, unless the function...
is zero”. Indeed, Theorem A shows that there exist nonzero functions that are “small” and whose Fourier transforms are also “small”. Of course these smallnesses are different from each other and from the smallness in $L^2(\mathbb{R})$. So in fact Theorem A does not contradict the most well known variant of uncertainty principle, the Heisenberg inequality. For recent violations of the uncertainty principle of a completely different nature, see papers [14] and [19].

In addition to the original proof of A. Beurling and P. Malliavin, there are many approaches to the proof of Beurling–Malliavin theorems due to H. Redheffer (see [22]), L. De Branges (see [9]), P. Kargaev (see [15]), N. Makarov and A. Poltoratskii (see [17]), to name just some of them. In 2005 V. Havin, J. Mashreghi and F. Nazarov [18] suggested a new proof of the First Beurling–Malliavin Theorem. An essential novelty of their proof was that it was done by (almost) purely real methods and did not use complex analysis except at one place, see [18] and the remark right after the formulation of Theorem B below.

Among the goals of the present paper is to give a proof of a new nontrivial generalization of Theorem A. Before stating our main results, we recall some classical definitions and fix some notations.

One of the principle objects of this paper is the class of BM majorants.

Definition 1. Let $\omega$ be a bounded nonnegative function on $\mathbb{R}$. This function is called a Beurling–Malliavin majorant (we shall further write “BM majorant” to save space), if for any $\sigma > 0$ there exists a nonzero function $f \in L^2(\mathbb{R})$ such that

\begin{align*}
(a) \ |f| \leq \omega; \quad (b) \ \text{spec}(f) \subset [0, \sigma].
\end{align*}

The set of all BM majorants will be further referred to as the BM class. If the conditions (a) and (b) just above are satisfied for a function $\omega$ with some fixed $\sigma > 0$, then we call such function $\omega$ a $\sigma$ admissible majorant. If we replace the condition (a) with a stronger two-sided condition $C \omega \leq |f| \leq \omega$ for some constant $C > 0$, then what we get is the definition of a strictly admissible majorant.

Recall that the Poisson measure $dP$ on $\mathbb{R}$ is defined by the following formula

\[ dP(x) := \frac{dx}{1 + x^2}. \]

The corresponding weighted Lebesgue space $L^1(dP)$ is the space of all functions $f$ satisfying $\int_{\mathbb{R}} |f|dP < \infty$. The expression $\int_{\mathbb{R}} \log(1/\omega)dP$ will be sometimes further referred to as the logarithmic integral of $\omega$.

Note that the condition $\log(1/\omega) \in L^1(dP)$ is necessary for $\omega$ to be a BM majorant, but not sufficient, see [18]. What Theorem A establishes is that some additional regularity suffices for admissibility.

We remind the reader of how one should modify the Cauchy kernel in order to extend the definition of the Hilbert transformation up to the space $L^1(dP)$.

Definition 2. The Hilbert transformation of a function $f \in L^1(dP)$ is defined as the following principal value integral

\[ \mathcal{H}f(x) := \int_{\mathbb{R}} \left( \frac{1}{x-t} + \frac{t}{t^2+1} \right) f(t) dt. \]

It is worth noting that the integral above converges for almost all $x \in \mathbb{R}$.
To avoid ambiguity, we stress that this definition coincides, up to an additive constant, with the classical one for functions in $L^1(\mathbb{R})$.

Let us now introduce function classes that will play an important role in what follows.

**Definition 3.** Let $r$ be a positive number. Consider the following system of intervals: $J_0 := [-2, 2)$, and for $j \in \mathbb{N}$, $J_j := [2^j, 2^{j+1})$ and $J_{-j} := [-2^{j+1}, -2^j)$. We say that an absolutely continuous function $\varphi$ belongs to the class $V_r$ if

$$\left( \sup_{j \in \mathbb{Z}} \frac{1}{|J_j|} \int_{J_j} |\varphi'(x)|^r \, dx \right)^{1/r} < \infty.$$  

The expression in the left hand part of the last line will be further denoted $\| \varphi \|_{V_r}$. In case when $r = \infty$, we use the following convention $V_\infty = \text{Lip}(\mathbb{R})$.

Note that these classes resemble homogeneous weighted Sobolev spaces. We are going to work with functions that belong to intersections $L^1(dP) \cap V_r$ with $r > 1$. From the functional-analytic point of view, these intersections are Banach spaces with respect to the norms $\| \cdot \|_{L^1(dP)} + \| \cdot \|_{V_r}$.

We are now in position to formulate the first main result of this paper to be proved in the next section.

**Theorem 1.** Let $\omega : \mathbb{R} \to (0, 1]$ be a function such that $\log(1/\omega) \in L^1(\mathbb{R}, dx/(1 + x^2))$, with $\log(1/\omega)$ absolutely continuous and satisfying $\log(1/\omega) \in V_r$ for some $r \in (1, \infty]$. Then for each $\delta > 0$ there exists a function $f \in L^2(\mathbb{R})$, not identically zero, such that $\text{spec}(f) \subset [0, \delta]$ and $|f(x)| \leq \omega(x)$ for all $x \in \mathbb{R}$.

In order to get some intuition of what an “average” function satisfying $\log(1/\omega) \in L^1(dP)$ and $\log(1/\omega) \in V_r$ looks like, the reader is welcomed to think of a function, whose graph consists of an infinite number of “pits” and “hills”, see the pictures of Section 1.5 in paper [18]. Of course, the same intuition applies to the functions with Lipschitz logarithm and finite logarithmic integral (i.e. those satisfying the conditions of the First Beurling–Malliavin Theorem). However, we shall shortly see that there are drastic differences between these classes of functions.

Let us compare our sufficient condition of Theorem 1 with these already known. First, it is obvious that our theorem is stronger than the First Beurling–Malliavin Theorem. Another available in literature sufficient condition for being a BM majorant belongs to A. Baranov and V. Havin, see article [1], Theorem 1.4. That result stemmed from an approach, earlier developed in papers [11] and [12]. In their paper, A. Baranov and V. Havin proved that a function is a BM majorant, if its Hilbert transform is mainly increasing. We remind to the reader that an absolutely continuous function $f$ is called mainly increasing if there exists an increasing sequence of real numbers $\{d_n\}_{n \in \mathbb{Z}}$ satisfying $\lim_{|n| \to \infty} |d_n| = \infty$, such that $C^{-1} \leq f(d_{n+1}) - f(d_n) \leq C$, $\text{Osc}(f, [d_n, d_{n+1}]) \leq C$ and for almost all $x \in [d_n, d_{n+1}]$ holds $(d_{n+1} - d_n)^{-1} \int_{d_n}^{d_{n+1}} |f'(x) - f'(t)| \, dt \leq C$, for some constant $C > 1$ independent of $n$. See [1] for a detailed discussion of mainly increasing functions. We shall see in Proposition 1 that follows, that our result applies to some majorants, that are not covered by the aforementioned Baranov–Havin Theorem.
Proposition 1. There exist functions in classes \( V_r \cap L^1(dP) \) whose Hilbert transform is not mainly increasing.

Among other things, Theorem 1 infers that the logarithm of a function in the class \( BM \) can possibly have almost linear “hills” at the points \( 2^n, n \in \mathbb{N} \) (see the propositions below). Note that there exists no function with Lipschitz logarithm and finite logarithmic integral having this property. For reader’s convenience, this is explained in Proposition 2. On the other hand, there are functions whose logarithm is Lipschitz, belongs to the space \( L^1(dP) \) and has a almost linear growth at infinity. However, in order to construct an example of such a function one has to consider “hills” at points \( 2^n \) with \( \gamma > 1 \) and natural \( n \). In other words, our Theorem 1 provides examples of functions in the \( BM \) class with more frequent “hills” than in the Lipschitz case.

Remark. One might think that Theorem 1 could be derived directly from Theorem A in the following way. Take a function \( \varpi \) as in Theorem 1. If there was another function \( \varphi \) such that:

\[
\log(1/\varphi) \quad \text{is Lipschitz,} \\
\log(1/\varphi) \in L^1(dP) \quad \text{and} \\
\log(1/\varpi) \geq \log(1/\varphi),
\]

then, obviously, we would have \( \varpi \geq \varphi \), and, thanks to Theorem A, \( \varpi \) would be \( \sigma \) admissible for any \( \sigma > 0 \). However, this argument does not apply (at least as is), in view of two propositions that follow.

Proposition 2. For each \( r > 1 \) there exists a function \( \Omega \) satisfying \( \Omega \in V_r, \Omega \in L^1(dP) \) and such that for all natural \( n \geq 2 \) holds \( \Omega(2^n) \geq 2^n/\sqrt{n} \).

Proposition 3. There exists no Lipschitz function \( \Omega \) satisfying \( \Omega \in L^1(dP) \) and such that for all natural \( n \geq 2 \) holds \( \Omega(2^n) \geq 2^n/\sqrt{n} \).

By reasons, similar to those discussed in the remark right before Proposition 2, Theorem 1 can not be proved via majorisation by \( \beta \) Hölder functions with \( \beta \in (0, 1) \). We mean the method described in Sections 1.16 and 1.17 of the paper by Havin, Mashreghi and Nazarov, see [18].

Let \( \omega \) be a function satisfying conditions of our Theorem 1. The author has been asked independently by Anton Baranov and Konstantin Dyakonov, whether then the function \( (\log \omega(\cdot))/(1 + (\cdot)^2)^{1/2} \) necessarily belongs to the homogeneous fractional Sobolev–Slobodecki space \( W^{1/2,2}(\mathbb{R}) \). If this condition is fulfilled for some function that has convergent logarithmic integral, then some regularization of this function is an admissible majorant, see [9]. This is hence a very natural question, since the regularity condition of our Theorem 1 also resembles one of the definitions of homogeneous Sobolev spaces. In fact, it turns out that there exist functions to which our Theorem 1 can be applied and that do not verify this condition.

Proposition 4. For each \( r > 1 \), there exists a function \( \Upsilon \in L^1(dP) \cap V_r \) such that \( \Upsilon(x) = 0 \) for \( x \in [-1, 1] \) and such that the quotient \( K \) defined for real \( x \) by \( K(x) := \Upsilon(x)/x \) satisfies

\[
\mathcal{E}(K) := \iint_{\mathbb{R} \times \mathbb{R}} \left( \frac{K(x) - K(y)}{x - y} \right)^2 \, dx \, dy = \infty.
\]

Note that the following approximation property of the spaces \( V_r \) with \( r > 1 \) is a direct consequence of our Theorem 1.
Corollary. By a theorem of A. Baranov and V. Havin (see paper [1], Section 6), we get that for any $r > 1, \sigma > 0$, and any $\omega \in V_r$ the space of all functions in $L^2(\mathbb{R})$ with the spectrum in $\mathbb{R}\setminus[0, \sigma]$ is not dense in the weighted Lebesgue space $L^1(\omega)$.

We hope that our main results will find other applications in harmonic and complex analysis, in particular for the uncertainty principle and for exponential systems.

The main step of the proof of Theorem 1 is the following lemma.

Lemma 1. (a new variant of the global Nazarov lemma) Let $\Omega \in L^1(dP) \cap V_r$ be positive. Then for each $\varepsilon > 0$ there exists a function $\Omega_1$, satisfying

(A) $\Omega(x) \leq \Omega_1(x)$ for all $x \in \mathbb{R}$;
(B) $\Omega_1 \in L^1(\mathbb{R}, dx/(1 + x^2))$;
(C) $H\Omega_1 \in \text{Lip}(\mathbb{R}, \varepsilon)$, where $H$ is the Hilbert transform on the real line.

Indeed, Theorem 1 follows from Lemma 1, thanks to the following sufficient condition for a function to be a BM majorant, which is a consequence of a more general result, proved by Mashreghi and Havin.

Theorem B. If $\omega : \mathbb{R} \to (0, 1]$, $\log(1/\omega) \in L^1(dP)$ and $\|H\log(1/\omega)\|_{\infty} < \pi\sigma$, then $\omega$ is a $\sigma$ admissible majorant.

Remark. The proof of Theorem B uses a one-dimensional construction coming from the classical (complex) theory of Hardy spaces on the unit circle. Namely, given a nonnegative function on the unit circle with convergent logarithmic integral there exists an analytic function whose modulus coincides with the former function. Such functions are called outer, see [20] for details.

For necessary conditions for $\sigma$ admissible majorants, see papers [3] and [11] by Y. Belov and paper [2] by A. Baranov and V. Havin.

We briefly discuss main ideas lying behind our proof of Lemma 1. Our proof is inspired by that of the Nazarov Lemma from paper [18]. Indeed, we use the beautiful idea of a so-called regularized system of intervals, which was first introduced by F. Nazarov and fruitfully used in [18]. Another important feature of the proof of the Nazarov Lemma in [18] is a version of the Hadamard–Landau inequality. We have had to modify this result drastically in order for it to fit the conditions of our Lemma 1. This culminated in Lemma 4 of this paper. On top of that, most estimates from the proof in [18] become considerably harder under the assumption $\log(1/\omega) \in V_r$ in comparison to the Lipschitz condition from [18].

Note that Nazarov’s Lemma is by itself a highly nontrivial and very interesting result in Harmonic Analysis. To illustrate this, we mention paper [23] by P. Zatitsky and D. Stolyarov, where the authors have utilized the main object of the Nazarov Lemma, the regularized system of intervals, in some special form. For a multidimensional version of the classical Nazarov Lemma, see paper [25].

Let us now discuss the second main result of this article. Our Theorem 2 gives an answer to the following question: “Can one put $r = 1$ in the formulation of Theorem 1?” The answer to this question is negative, which is shown in the following result.
Theorem 2. There are functions $\omega : \mathbb{R} \to (0, 1]$ satisfying $\log(1/\omega) \in V_1 \cap L^1(dP)$ that are not BM majorants.

In fact, in this case, both alternatives ($\omega$ in and not in BM class) may occur. Indeed the function $\omega \equiv 1$ gives an example of a majorant in $L^1(dP) \cap V_1$ which is a BM majorant. However, in the space $V_1$, there are BM majorants that do not belong to any class $V_r$ with $r > 1$.

Proposition 5. There are functions $\omega : \mathbb{R} \to (0, 1]$ satisfying

$$\log(1/\omega) \in (V_1 \setminus \bigcup_{r>1} V_r) \cap L^1(dP)$$

that are BM majorants.

The main reason why Theorem 2 holds is, vaguely speaking, that a function in the class $V_1$ can have linear growth at sufficiently long intervals, tending to infinity. Indeed, it is not hard to give an example of a system of intervals $\{J_j\}_{j \in \mathbb{Z}}$ and the corresponding space $V_r$ which is strictly wider than $V_r$. Moreover, one can give an example of a Hadamard lacunary sequence of positive reals, that can not play the role of the lengths of intervals $\{J_j\}$ from the definition of spaces $V_r$, see the two remarks right after the proof of Theorem 2. The same applies to the factor in front of the integrals in this definition. The reason why it is so is that under such general conditions the function $\Omega$ need not have linear growth at infinity, whereas this is indispensable within our framework, see estimate (3) in our Section 2 below. However, in many cases, the corresponding functions growing faster than linear at infinity will actually not be BM majorants, see the second remark in our fourth section.

The very same “stability of smallness” argument of Borichev yields the fact that “typical” functions in classes $V_r \setminus \text{Lip}(\mathbb{R})$ are not strongly admissible majorants, see [5] for a definition of the strong admissibility. This is discussed in our third remark after the proof of Theorem 2.

The paper is organized as follows. Theorem 1 is proved in Sections 2 and 3. The fourth Section is devoted to the proofs of Theorem 2 and Proposition 5. Elementary proofs of the first three propositions are left to the reader and are
not detailed here. The Appendix contains examples of functions verifying conditions of our Theorem 1 and whose weighted logarithms have unbounded fractional homogeneous Sobolev space \( \dot{W}^{1/2,2}(\mathbb{R}) \) seminorms.

We finally mention four open questions concerning Theorems 1 and 2. The first question consists in determining whether the condition \( \log(1/\omega) \in V_r \) in Theorem 1 can be weakened down to, roughly speaking, a condition of the kind “\( \omega \) belongs to some Orlicz type space, defined in the spirit of \( V_r \) spaces”. The second question concerns the system of intervals that are used in the definition of the spaces \( V_r \). Namely, we would like to find a necessary and sufficient condition on the system of intervals instead of the dyadic system in Definition 3 for which the first theorem still holds. Yet another question is to find a multidimensional version of Theorem 1 which seems unavailable at the present time, according to [24]. The fourth and the final question reads as follows. It would be also interesting to find counterparts of the main results of this paper in the context of the so-called model spaces, in spirit of Yu. S. Belov’s early papers. The author plans to attack the first three of the aforementioned questions in the nearest future.

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2. A NEW LOCAL NAZAROV LEMMA

We accumulate here the list of the frequently used technical abbreviations and notations. For an interval \( a \subset \mathbb{R} \) its length is denoted by \( l(a) \), \( c_a \) will stand for the center of \( a \) and \( \lambda a \) with \( \lambda \) positive will be the interval centered at \( c_a \) and whose edge length equals \( \lambda l(a) \). Let \( I \) be an interval on the real line. We will denote by \( T_I(x) \) the distance from \( x \in \mathbb{R} \) to \( \mathbb{R} \setminus I \). For a dyadic interval \( b \), we will denote by \( b^\sharp \) the dyadic parent of \( b \).

The main step of the proof of our new global Nazarov lemma is its following local variant.

**Lemma 2.** (a new local Nazarov lemma) Let \( I \subset \mathbb{R} \) be an interval, let \( r \in (1, \infty] \) and let \( 1 \geq \delta > 0 \) and \( \kappa \geq 1 \) be real numbers. Suppose that \( f \) is a nonnegative absolutely continuous function such that \( \|f'\|_{L^r(I)} \leq \kappa l(I)^{1/r} \) and \( \|f\|_{L^\infty(I)} \leq \delta l(I) \). Then there exists a nonnegative function \( F \in C^\infty(\mathbb{R}) \), such that

1. \( F = 0 \) outside \( 1.5I \),
2. \( f(x) \leq F(x) \) for all \( x \in I \),
3. \( \|(HF)'\|_{L^\infty(\mathbb{R})} \lesssim \delta \),
4. \( \left( \frac{1}{\kappa} \right)^{r/(r-1)} \int_\mathbb{R} F(x)dx \lesssim \int_I f(x)dx \).

In case when \( r = \infty \) in Lemma 2, the corresponding result coincides with Lemma 2.6 from paper [18]. Note that there is no difference between the powers \( r/(r-1) \) and 1 in this case.

In the formulation of Lemma 2 and until the end of the third section, the signs \( \lesssim \) and \( \gtrsim \) indicate that the left-hand (right-hand) part of an inequality is less than
the right-hand (left-hand) part multiplied by a constant independent of \( \delta, f, I \) and \( \kappa \).

The rest of this section is entirely devoted to the proof of Lemma 2.

**Proof:** (of the new local Nazarov lemma) First, assume that we have already proved Lemma 2 for \( \delta = 1 \) and \( I = [-1/2, 1/2] =: I^* \). Then, for any \( f \) satisfying conditions of Lemma 2 with arbitrary \( \delta \) and \( I \) consider the function \( \tilde{f}(\cdot) := f((\cdot)l(I) + c_I)/(\delta l(I)) \). Note that then \( \|\tilde{f}\|_{L^\infty(I^*)} \leq l(I^*) \equiv 1 \) and \( \|\tilde{f}\|_{L^\infty(I^*)} \leq \kappa/\delta \). If we manage to prove that there exists a majorant \( \tilde{F} \) of the function \( \tilde{f} \) which equals 0 outside \( 1.5I^* \) and satisfying properties 3 and 4 with \( \delta = 1 \) and \( \kappa \) replaced by \( \kappa/\delta \), then the function \( F(\cdot) := \delta l(I)\tilde{F}((\cdot) - c_I)/l(I) \) will be the needed majorant of the function \( f \). Thus, it is sufficient to prove Lemma 2 for \( \delta = 1 \) and \( I^* \).

The rest of this section is devoted to the proof of Lemma 2 in this case.

The following definition is very important.

**Definition 4.** We say that a dyadic interval \( I \subset I^* \) is essential, if \( \|f\|_{L^\infty(I)} \geq l(I)/2 \). Denote \( A \) the set of essential intervals.

It is straightforward to see that we have
\[
\{x \in I^* : f(x) > 0\} \subseteq \bigcup_{a \in A} a.
\]
However, we will not use this fact later on in our estimates.

Consider \( A^M \), the set of maximal by inclusion elements of \( A \). To each interval \( a \in A^M \) we associate its tail \( t(a) \). Informally, the tail \( t(a) \) is a family of dyadic intervals that is composed of a countable number of finite series \( t_p(a), p = 0, 1, 2, \ldots \) of dyadic intervals. For \( p = 0 \) we define \( t_0(a) := a \) and for a fixed \( p \geq 1 \), the intervals of the family \( t_p(a) \) all have length equal to \( l(a)/2^p \) and their unions form the set
\[
a \cup \bigcup_{1 \leq q \leq p} t_q(a) = \{x \in \mathbb{R} : l(a)/2 + l(a) \sum_{q=1}^{p-1} 3^q/2^q \leq |x - c_a| < l(a)/2 + l(a) \sum_{q=1}^{p} 3^q/2^q \}.
\]

For a detailed discussion of tails, see paper [18], Section 2.6.5. In fact, after that we will have added these tails, we will get a regularized system of intervals, see [18], Sections 2.6 and 2.7. Next, we define \( B := \bigcup_{a \in A^M} t(a) \), and then pose \( \tau := \{c \in B^M : c \subseteq I^* \} \). Here, \( B^M \) stands for the set of maximal by inclusion elements of \( B \). Note that the system \( \tau \) covers \( I^* \) and consists of dyadic intervals, see [18].

Define for an interval \( a \in \tau \) its neighborhood \( N(a) \) by
\[
N(a) := \{b \in \tau : d(a, b) \leq 2l(a), \frac{1}{2} \leq \frac{l(a)}{l(b)} \leq 2 \}.
\]
Note that \( \#N(a) \leq 9 \). We shall need the following property of the system \( \tau \).

**Lemma 3.** Suppose that \( a \in \tau \) and \( b \in \tau \setminus N(a) \). If \( l(b) \leq 2l(a) \) then \( d(2a, 2b) \geq l(a)/2 \), and if \( l(b) = 2^k l(a) \) for some natural \( k \geq 2 \), then \( d(2a, 2b) \geq 2 \cdot 3^{k-2} l(a) \).
**Proof:** The proof of this lemma is not detailed here, since it can be found in paper [18], Section 2.6.6. □

Denote $2\tau := \{2c : c \in \tau\}$. As a direct consequence of the lemma, we deduce that the multiplicity $\#\{b \in 2\tau : x \in b\}$ is uniformly bounded in $x \in \mathbb{R}$. Indeed, if $b \in \tau \setminus N(a)$, then $d(2a, 2b) > 0$ and

$$\sup_{x \in \mathbb{R}} \#\{b \in 2\tau : x \in b\} \leq \sup_{a \in \tau} \#N(a) \lesssim 1.$$

First, fix a bump function $\phi$, i.e. $\phi \in C^\infty(\mathbb{R})$ satisfying $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$, $\phi \equiv 0$ outside $1.5I^*$ and $\phi \equiv 1$ on $I^*$. Second, for an interval $a \in \tau$ pose

$$\phi_a(x) := \lambda(a)\phi\left(\frac{x - c_a}{\lambda(a)}\right).$$

Simple calculation shows that $\mathcal{H}\phi(x) = l(b)\mathcal{H}\phi((x - c_b)/l(b))$. Hence we infer the inequality $\|\mathcal{H}\phi_a\|_{L^\infty(\mathbb{R})} \lesssim 1$. We finally define $F$ by $F := \sum_{a \in \tau} \phi_a$.

Now we have to check the required properties of the majorant $F$. The first one follows readily from the definition of $F$. To prove the second one, note that for all $a \in \tau$ we have $\|f\|_{L^\infty(a)} \leq l(a)$. Indeed, suppose the contrary, i.e. that $\|f\|_{L^\infty(a_0)} > l(a_0)$ for some $a_0 \in \tau$. This means that

$$\|f\|_{L^\infty(a_0)} \geq \|f\|_{L^\infty(a_0)} \geq l(a_0) = \left(\frac{l(a_0)}{2}\right),$$

which in turn signifies that $a_0^*$ is an essential interval and hence $a_0^* \in \tau$. This contradicts to the definition of $\tau$. From here we deduce that if $x \in a \in \tau$, then

$$F(x) \geq l(a) \geq \|f\|_{L^\infty(a)} \geq f(x).$$

Next we estimate the integral of the function $F$. To this end, we prove a variant of the Hadamard–Landau inequality which is appropriate for our goals.

**Lemma 4.** Suppose that $a \in A^M$ and let $f$ satisfy $\|f\|_{L^r(a)} \leq \kappa l(a)^{1/r}$ with $\kappa$ and $r$ as above (i.e., $\kappa \geq 1$ and $r > 1$). Then we have

$$\|f\|^2_{L^\infty(a)} \leq C(r) \cdot \left(\int_a f\right)^r \cdot \kappa^{r/(r-1)},$$

where $C(r)$ is a positive constant, depending on $r$ only.

**Proof:** Let $x_0 \in a$ be a point such that $\|f\|_{L^\infty(a)} = f(x_0)$. Suppose with no loss of generality that $a_+ - x_0 \geq l(a)/2$, where $a_+$ is the right end of the interval $a$. Consider a point $x \in (x_0, a_+)$. Denote $\kappa_a := (\int_a |f|^r)^{1/r}$. We make use of the Newton–Leibniz formula and of the Hölder inequality to infer the following estimates

$$|f(x) - f(x_0)| = \left|\int_{x_0}^x f'(t)dt\right| \leq |x - x_0|^{1-1/r} \left(\int_{x_0}^x |f'(t)|^r dt\right)^{1/r} \leq \kappa_a |x - x_0|^{1-1/r}.$$

Denote $v := (f(x_0)/\kappa_a)^{r/(r-1)}$. We shall treat two cases separately, according to the value of $v$. 
First, we suppose that \( \nu < l(a)/2 \). Observe that in this case the point \( x_0 + \nu \) belongs to the interval \( a \). We integrate the inequality \( f(x) \geq f(x_0) - \kappa_a |x - x_0|^{1-1/r} \) using this observation and deduce that

\[
\int_a f \geq \int_{x_0}^{x_0 + \nu/2} f(x) \, dx \geq \int_{x_0}^{x_0 + \nu/2} f(x_0) - \kappa_a |x - x_0|^{1-1/r} \, dx
\]

(1) 

\[
= \frac{f(x_0)}{2} \left( \frac{f(x_0)}{\kappa_a} \right)^{r/(r-1)} - \frac{r\kappa_a}{2^{2-1/r}(2r - 1)} \left( \frac{f(x_0)}{\kappa_a} \right)^{(2r-1)/(r-1)} \geq \frac{\|f\|_{L^\infty(a)}^{2/(r-1)}}{\kappa_a^{r/(r-1)}},
\]

But since \( a \in A^M \), we have also \( \kappa_a \leq \kappa l(a)^{1/r} \leq \kappa (2\|f\|_{L^\infty(a)})^{1/r} \). This and line (1) give the claimed inequality in the first case.

Consider now the second case, where \( \nu \geq l(a)/2 \). Note that in this case the point \( x_0 + l(a)/2 \) belongs to the interval \( a \). Integrating the same inequality as in the first case yields

\[
\int_a f \geq \int_{x_0}^{x_0 + l(a)/2} f(x) \, dx
\]

(2) 

\[
\geq \frac{l(a)f(x_0)}{2} - \frac{r\kappa_a}{2^{2-1/r}(2r - 1)} \frac{l(a)}{2} \left( f(x_0) - \frac{r\kappa_a}{2r - 1} \cdot \left( \frac{l(a)}{2} \right)^{(r-1)/r} \right).
\]

Note that since \( \nu \geq l(a)/2 \), we have also that \( f(x_0)/\kappa_a \geq (l(a)/2)^{(r-1)/r} \). Let us use this in the following way

\[
\int_a f \geq \frac{l(a)}{2} \left( f(x_0) - \frac{r f(x_0)}{2r - 1} \right) \geq \frac{\|f\|^2_{L^\infty(a)}}{C(r)\kappa},
\]

thanks to the facts that \( \kappa > 1 \) and that \( a \in A^M \). Hence Lemma 3 is proved. \( \square \)

We use this result to estimate the integral of the function \( F \):

\[
\left( \int_R F \leq \sum_{b \in A^M} \int_R \phi_b + \sum_{c \in A^M} \sum_{b \in c \setminus b} \int_R \phi_b \leq \sum_{b \in A^M} l(b)^2 + \sum_{c \in A^M} \sum_{b \in c \setminus b} l(b)^2 \right)
\]

\[
\lesssim \sum_{b \in A^M} l(b)^2 + \sum_{c \in A^M} \sum_{p=1}^{\infty} \sum_{b \in c \setminus b} l(b)^2 \lesssim \sum_{b \in A^M} l(b)^2 + \sum_{c \in A^M} \sum_{p=1}^{\infty} 3^p \left( \frac{l(c)}{2^p} \right)^2 \lesssim \sum_{c \in A^M} l(c)^2 \lesssim \sum_{c \in A^M} \|f\|^2_{L^\infty(c)} \lesssim \sum_{c \in A^M} \left( \int_c f \right)^{r/(r-1)} \leq \kappa^{r/(r-1)} \int f.
\]

The last and the last but one inequalities just above are in need of explanation. The last estimate uses the fact that intervals of \( A^M \) are nonoverlapping, whereas the penultimate bound follows from Lemma 3.

It remains to derive the inequality on the derivative of the Hilbert transformation of the function \( F \). First, we shall obtain this estimate for \( x \in \bigcup_{b \in r} 2b \). Let
\(a(=a(x))\) denote the interval from \(\tau\) such that \(x \in 2a\). We isolate the neighborhood \(N(a)\) from its complement in \(\tau\) and infer the following inequality

\[
|\langle \mathcal{H} F \rangle'(x)| \leq \sum_{b \in N(a)} |\langle \mathcal{H} \phi_b \rangle'(x)| + \sum_{b \in N(a), l(b) \leq 2l(a)} |\langle \mathcal{H} \phi_b \rangle'(x)| + \sum_{k=2}^{\infty} \sum_{b \in N(a), l(b) = 2^k l(a)} |\langle \mathcal{H} \phi_b \rangle'(x)| =: S_1 + S_2 + S_3.
\]

We shall estimate the terms \(S_1, S_2\) and \(S_3\) separately. We start with the sum \(S_1\), whose estimate turns out to be easy

\[
S_1 \leq \#N(a) \sup_{b \in \tau} \|\langle \mathcal{H} \phi_b \rangle'\|_{L^\infty(\mathbb{R})} \lesssim 1.
\]

We further proceed to the second term. We use a simple estimate on the kernel of the Hilbert transformation, and the fact that the system of intervals \(\{2b\}_{b \in \tau}\) has finite multiplicity and Lemma 3 to get

\[
S_2 \lesssim \sum_{b \in \tau \setminus N(a), l(b) \leq 2l(a)} \int_{\mathbb{R}} \phi_b(t) \frac{\partial}{\partial x} \left( \frac{1}{t-x} \right) dt \lesssim \sum_{b \in \tau \setminus N(a), l(b) \leq 2l(a)} \int_{\mathbb{R}} \phi_b(t) \frac{1}{(t-x)^2} dt \lesssim \sum_{b \in \tau \setminus N(a), l(b) \leq 2l(a)} \int_{1.5b}^b \frac{l(a)}{t-x} \lesssim l(a) \int_{|u| \geq l(a)/2} \frac{du}{|u|^2} \lesssim 1.
\]

The third term can be estimated as well using Lemma 3

\[
S_3 \lesssim \sum_{k=2}^{\infty} \sum_{b \in \tau \setminus N(a), l(b) = 2^k l(a)} \int_{2b}^{\infty} \phi_b(t) dt \lesssim \sum_{k=2}^{\infty} 2^k l(a) \int_{2b}^{\infty} \frac{dt}{(t-x)^2} \lesssim \sum_{k=2}^{\infty} 2^k l(a) \int_{\{|u| \geq 2.3^k - 2l(a)\}} \frac{du}{u^2} \lesssim 1,
\]

and the lemma for \(x \in \bigcup_{b \in \tau} 2b\) follows.

Next, if a point \(z \in \mathbb{R}\) is situated at a positive distance from the set \(\bigcup_{b \in \tau} 2b\), then denote by \(x\) the point of this set closest to \(z\), and let \(a(=a(x))\) be an interval as above. We infer the following estimates

\[
|\langle \mathcal{H} F \rangle'(z)| \leq \sum_{b \in \tau \setminus N(a)} |\langle \mathcal{H} \phi_b \rangle'(z)| + \sum_{b \in N(a)} |\langle \mathcal{H} \phi_b \rangle'(z)| \lesssim \sum_{b \in \tau \setminus N(a)} \int_{\mathbb{R}} \phi_b(t) dt + \#N(a) \sup_{b \in \tau} \|\langle \mathcal{H} \phi_b \rangle'\|_{L^\infty(\mathbb{R})}.
\]

Thanks to the estimates of the terms \(S_1, S_2\) and \(S_3\), we conclude that the needed variant of the local Nazarov lemma is proved.
3. Proof of a New Global Nazarov Lemma

In this section, we shall derive the global Nazarov lemma from the local one.

**Proof:** Denote \( \mu := \| \Omega \|_{L^\infty} \). Note that we may assume in the global Nazarov lemma that \( \Omega(x) = 0 \) for \( |x| \leq R \) with \( R \) being an arbitrary large positive number. Indeed, if it is not the case, then consider the function \( \Omega(\cdot) := \max(0, \Omega - \mathcal{M})(\cdot) \), where \( \mathcal{M} := \max_{x \in B(0, R)} \Omega(x) \). If \( \Omega_I \) is a majorant of the function \( \Omega \), satisfying properties \( \{B\} \) and \( \{C\} \) then the function \( \Omega_I + \mathcal{M} \) will be the desired majorant of the function \( \Omega \).

Until the end of the third section, the signs \( \lesssim \) and \( \gtrsim \) indicate that the left-hand (right-hand) part of an inequality is less than the right-hand (left-hand) part multiplied by a constant that may depend only on \( r \) and \( \mu \).

Fix \( 1 \geq \varepsilon > 0 \) and choose \( R_1 > 0 \) so big that

\[
\int_{R(-R_1, R_1)} \Omega \, dP \leq \varepsilon^{(2r-1)/(r-1)}.
\]

Thanks to the previous paragraph, we may assume that \( \Omega \) equals to zero on the interval \( (-R_1, R_1) \). Next, we shall prove the following inequality

\[
(4) \quad \Omega(x) \lesssim \varepsilon|x|.
\]

First, note that

\[
(5) \quad \Omega(x) = |\Omega(x) - \Omega(0)| \leq \int_0^x |\Omega(y)| \, dy \leq |x|^{1-1/r} \| \Omega \|_{L^r(0, x)} \leq \mu |x|.
\]

Second, according to the previous paragraph the bound \( (4) \) is obvious once \( |x| \lesssim 1 \), and for \( 1 \lesssim |x| \) we will argue as in Lemma \( 4 \). We thus find a point \( x_0 \in (x/3, 3x) \) such that \( \Omega(x_0) = \| \Omega \|_{L^\infty(x/3, 3x)} \). Then, for any \( y \in (x/3, 3x) \) we have

\[
\Omega(y) \geq \Omega(x_0) - C(\mu)|x|^{1/r}|y - x_0|^{1-1/r},
\]

for some constant \( C(\mu) \) depending on \( \mu \) only.

Further, denote \( \varepsilon := (\Omega(x_0))/(|x|^{1/r}\mu)^{r/(r-1)} \). Due to the bound \( (5) \), one of the points \( x_0 \pm \varepsilon \) belongs to the interval \( (x/3, 3x) \). Without loss of generality, we suppose that \( x_0 + \varepsilon \in (x/3, 3x) \). We finally infer the following chain of inequalities

\[
\varepsilon^{(2r-1)/(r-1)} \geq \int \Omega \, dP \geq |x|^{-2} \int_{x/3}^{3x} \Omega(y) \, dy
\]

\[
\geq |x|^{-2} \int_{x_0}^{x_0 + \varepsilon} \left( \Omega(x_0) - C(\mu)|x|^{1/r}|y - x_0|^{1-1/r} \right) \, dy
\]

\[
\geq C(\mu, r)|x|^{-2} \varepsilon^{(2r-1)/(r-1)} \geq C(\mu, r) \left( \frac{\Omega(x)}{|x|} \right)^{(2r-1)/(r-1)},
\]

for some constant \( C(\mu, r) \) depending on \( \mu \) and \( r \) only. Hence, the bound \( (4) \) is proved.

Recall the above defined system of intervals: \( J_0 = [-2, 2] \), and for \( j \in \mathbb{N} \),

\[
J_j = [2^j, 2^{j+1}], \quad J_{-j} = [-2^{j+1}, -2^j]
\]
and apply the local lemma to each interval \( J_j \) and the corresponding restriction \( f_j = \Omega \mathbb{1}_{J_j} \). Indeed, Lemma \( \text{(2)} \) can be applied since these functions satisfy \( \|f_j\|_\infty \leq \varepsilon 2^j \leq \varepsilon l(J_j) \) by \( \text{(4)} \) and
\[
\|f_j^r\|_{L^r(J_j)} \leq \mu l(J_j)^{1/r} \leq \max(\mu, 1),
\]
thanks to the fact that \( \Omega \) is in \( V_r \). Thus we obtain functions \( F_j \) for \( j \in \mathbb{Z} \). The needed majorant \( \Omega_1 \) is defined by
\[
\Omega_1 = \sum_{j \in \mathbb{Z}} F_j.
\]

Now, we shall check the required properties of \( \Omega_1 \). The first property follows obviously from the local Lemma. We proceed to the second one:
\[
\int_{\mathbb{R}} \Omega_1(t)dP(t) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}} F_j(t)dP(t) \lesssim \sum_{j \in \mathbb{Z}} \int_{1.5J_j} F_j(t) \frac{dt}{2^{2j}} \\
\lesssim \frac{1}{\varepsilon^{r/(r-1)}} \sum_{j \in \mathbb{Z}} \left( \int_{1.5J_j} \Omega(t) \frac{dt}{2^{2j}} \right) \lesssim \frac{1}{\varepsilon^{r/(r-1)}} \int_{\mathbb{R}} \Omega(t)dP(t) \\
\lesssim \varepsilon < \infty,
\]
where in the third inequality above we have used the local Lemma.

So, it remains to check that the third conclusion holds. First, fix a point \( x \in \mathbb{R} \).
Second, denote by \( S(x) \) the interval from the system \( \mathcal{F} = \{J_j\}_{j \in \mathbb{Z}} \) such that \( x \in S(x) \). Next, denote by \( U(x) \) the subset of \( \mathcal{F} \) consisting of \( S(x) \) and its two neighbor intervals and by \( W(x) \) its complement: \( W(x) = \mathcal{F} \setminus U(x) \). Finally, write the function \( \Omega_1 \) as a sum of two functions as follows:
\[
\Omega_1 = \sum_{j \in W(x)} F_j + \sum_{j \in U(x)} F_j =: \omega_1 + \omega_2.
\]
Since there is only a finite number of intervals in the family \( U(x) \), we see that
\[
| (\mathcal{H}\omega_2)'(x) | \lesssim \# U(x) \sup_{j \in U(x)} \| (\mathcal{H}F_j)' \|_\infty \lesssim \varepsilon,
\]
where we have just used condition \( \text{(3)} \) of Lemma \( \text{(2)} \) in the last estimate. On the other hand, since \( \text{supp}(\omega_1) \subseteq \bigcup_{j \in W(x)} 1.5J_j \) we deduce that
\[
\text{supp}(\omega_1) \subseteq \{ t \in \mathbb{R} : |t - x| \geq \frac{l(S(x))}{4} \} \subseteq \{ t \in \mathbb{R} : |t - x| \geq \frac{|x|}{16} \}.
\]
Therefore, we arrive at the following chain of inequalities
\[
| (\mathcal{H}\omega_1)'(x) | = \left| \left( \int_{\mathbb{R}} \omega_1(t) \frac{1}{t - x} dt \right)' \right| = \int_{\mathbb{R}} \omega_1(t) \frac{\partial}{\partial x} \left( \frac{1}{t - x} \right) dt \\
= \int_{\mathbb{R}} \omega_1(t) \frac{1}{(t - x)^2} dt \lesssim \int_{\mathbb{R}} \Omega_1(t)dP(t) \\
\lesssim \frac{1}{\varepsilon^{r/(r-1)}} \int_{\mathbb{R}} \Omega(t)dP(t) \lesssim \varepsilon < \infty,
\]
thanks to the bound \( \text{(6)} \).
Hence, the needed variant of the Nazarov lemma is proved. \( \square \)

Thus, Theorem \( \text{(4)} \) is also proved, via Theorem \( \text{(3)} \).
4. The class $V_1$

We begin with the proof of Theorem 2 which is a direct consequence of the following proposition.

**Proposition 6.** Let $\gamma > 1$ and denote $I_n := [2^n - 2^n/n^\gamma, 2^n + 2^n/n^\gamma]$, for $n \geq 3$. Consider for $x \in \mathbb{R}$ the following function

$$\omega(x) := \begin{cases} \exp(-n^\gamma T_{I_n}(x)), & \text{if } x \in I_n \text{ with } n \geq 3, \\ 1, & \text{otherwise.} \end{cases}$$

(7) We claim that $\log(1/\omega) \in L^1(dP)$ and $\log(1/\omega) \in V_1$, though $\omega$ is not a BM majorant.

**Proof:** The first two claims are easy to verify, so we omit their proofs.

Let $\sigma$ be a positive constant and consider the Bernstein space $E_{\sigma,1}$, i.e. the space of all entire functions $f$ such that

$$|f(z)| \leq e^{\sigma|z|} \text{ for any } z \in \mathbb{C} \text{ and } |f| \leq 1 \text{ on } \mathbb{R}.$$ 

Recall Lemma 1 from paper [2].

**Lemma A.** For any $\sigma > 0$ there exist a (small) $\alpha(\sigma) \in (0,1/2)$ and a (big) $h(\sigma) > 2$ such that for any $h \geq h(\sigma)$, any $f \in E_{\sigma,1}$ and any compact interval $I \subset \mathbb{R}$

$$|f| \leq e^{-h \alpha} \text{ on } \mathbb{R} \implies |f| \leq e^{-Ch h} \text{ on } \tilde{I},$$

where $C > 0$ is an absolute constant and $\tilde{I}$ is the interval centered at $c(I)$ with $|\tilde{I}| = h^{\alpha(\sigma)}|I|$.

Let us prove that $\omega$ is not in the BM class. Suppose the contrary. Hence, for a fixed $\sigma > 0$ there exists a function, not identically zero, satisfying $f \in L^2(\mathbb{R})$, $\text{spec}(f) \subset [0,\sigma]$ and $|f(x)| \leq \omega(x)$ for all real $x$. We shall now use Lemma A. Notice that the function $f$ satisfies the conditions of this lemma with $h := n^\gamma$ and $I := I_n$ for $n \geq n(\gamma)$. We deduce from this lemma that there exists a universal constant $C$ and a power $\alpha \in (0,1/2)$, depending only on $\sigma$ such that $|f(x)| \leq \exp(-C2^n)$ on the interval $I_{n,1} := (n^{\alpha(\gamma)}/2) I_n$.

**Remark.** From now on until the end of the present article, the sign $X \asymp Y$ means that $C_1 Y \leq X \leq C_2 Y$ for some constants $C_1$ and $C_2$ depending only on $\gamma, \alpha, \sigma$ and $C$. In this case, we shall say that $X$ is of order $Y$.

Note that the length of this interval satisfies the bound $|I_{n,1}| \asymp n^{\gamma(\alpha-1)/2}$. As a consequence, we infer that the inequality

$$|f(x)| \leq \exp(-Cn^{\gamma(\alpha-1)} I_{n,1}(x))$$

is valid for $x \in I_{n,1}$. This means that we can apply Lemma A once again, now for $h := Cn^{\gamma(1-\alpha)}$ and $I := I_{n,1}$. This yields the bound

$$|f(x)| \leq e^{-Ch h_{I_{n,1}}} = e^{-C2^n},$$

which is true for $x \in I_{n,2} := ((Cn^{\gamma(1-\alpha)})/2) I_{n,1}$. It is not difficult to see that the corresponding interval $I_{n,2}$ has length of order

$$C^{\alpha} n^{\gamma(1-\alpha)+\gamma(\alpha-1)/2} 2^n = C^{\alpha} n^{-\gamma(1-\alpha)^2} 2^n.$$
Acting inductively, after \( m \) steps, we arrive at the estimate \(|f(x)| \leq \exp(-C^m 2^n)\), verified by \( f \) for \( x \in I_{n,m} \) with
\[
|I_{n,m}| \asymp n^{-(1-\alpha)m} 2^n.
\]
Maybe, it is worth noting that \( I_{k,m} \cap I_{n,m} = \emptyset \) for any natural \( m \), once \( n \neq k \). This results from the fact that we assume, as we can, that \( C < 1 \).

We are now in position to prove that \( f = 0 \) identically, which will lead to a contradiction. To this end, we estimate the logarithmic integral of \( f \). For each natural number \( m \) holds
\[
\int_{\mathbb{R}} \log |f(x)|dP(x) \leq - \sum_{n \geq 3} \int_{I_{n,m}} 2^{-2nC^m 2^n} dx \asymp - \sum_{n \geq 3} n^{-(1-\alpha)m}.
\]
Choosing \( m \) sufficiently large and recalling that \((1-\alpha) \in (0,1)\), we arrive at the formula \( \int_{\mathbb{R}} \log |f(x)|dP(x) = -\infty \). Since \( f \in L^2(\mathbb{R}) \) has the spectrum in the interval \([0,\sigma]\), it hence belongs to the Hardy class \( H^2(\mathbb{R}) \). From the Jensen inequality, see [13] we deduce that \( f = 0 \) identically, which contradicts our assumption. Hence, the second theorem is proved. \( \Box \)

**Remark.** Alas, our proof above does not work, if one replaces in (7) and in the definition of intervals \( I_n \) the powers \( n^\alpha \) by \( \theta^n \) with \( \theta \in (1,2) \).

**Remark.** Consider for \( n \in \mathbb{N} \) intervals \( I_{n,*} := [2^{2n}, 2^{2n} \cdot (1+n^{-2})] \). Define for \( x \in \mathbb{R} \) the following function
\[
\omega_*(x) := \begin{cases} 
  e^{-n^2 T_{I_{n,*}}(x)}, & \text{if } x \in I_{n,*} \text{ with } n \geq 3, \\
  1, & \text{otherwise}.
\end{cases}
\]

The proof just above can be used to show that this function is not a BM majorant. On the other hand, it is easy to see that \( \omega_* \in L^1(dP) \) and that it satisfies
\[
\sup_{n \in \mathbb{N}} \frac{1}{2^{2n+1} - 2^{2n}} \int_{2^n}^{2^{n+1}} |\omega'_*(x)|^* dx < \infty.
\]
This means that one can not replace \( 2^n \) in the definition of spaces \( V_r \) with \( 2^n \), and hence also with a general Hadamard lacunary sequence.

**Remark.** It can be seen exactly as above that the function \( \omega_* \) is not a strictly admissible majorant, recall Definition [1]. For a detailed discussion of strictly admissible majorants, see [5].

We next give a proof of Proposition [5]

**Proof:** Consider for natural \( n > 100 \) intervals \( \mathbb{I}_n := [2^n - 2^n / \exp(2^{2n}), 2^n + 2^n / \exp(2^{2n})] \) and a piecewise linear function \( \omega \) defined by the following formula
\[
\omega(x) := \begin{cases} 
  \exp \left(-\max(2^n - e^{2^n} \cdot |x - 2^n|, 0)\right), & \text{if } x \in \mathbb{I}_n, \\
  1, & \text{otherwise}.
\end{cases}
\]

Note that holds \( \log(1/\omega) \in L^1(dP) \), \( \log(1/\omega) \in V_1 \) and \( \log(1/\omega) \notin V_r \) for all \( r > 1 \).

Let us now show that \( \omega \in BM \). To this end, denote
\[
\psi_k(x) := \frac{e^{i\pi2^{-k}x} \sin(\pi 2^{-k}x)}{\pi 2^{-k}x}.
\]
It is straightforward to see that \( \text{spec}(\psi_k) \subset [0, 2^{-k}] \). Note that for any natural \( k \) and any nonnegative integer \( n \) one has

\[
|\sin(\pi 2^{-k} x)| \leq \pi |2^{-k} x - 2^n|.
\]

This means that for \( n \geq k \) we have the following estimate

\[
|\psi_k(x)| \leq \frac{|x - 2^n|}{x}.
\]

On the other hand, minorizing a logarithmic function by a piecewise linear function gives the bound

\[
\max\left( \log\frac{1}{e^{-2^n} + |x - 2^n|}, 0 \right) \geq \max\left( 2^n - \exp(2^n) : |x - 2^n|, 0 \right),
\]

and in turn the inequality \( e^{-2^n} + |x - 2^n| \leq \omega(x) \). Hence, holds \( |\psi_k(x)| \leq \omega(x) \), and we are done. \( \square \)

5. Appendix

We finally prove Proposition 4.

**Proof:** For a given \( r > 1 \), let us construct a function \( \Upsilon \) that is zero in a neighborhood of the origin, that is integrable against the Poisson measure, that belongs to the class \( V_r \) and such that \( \mathcal{E}(K) = \infty \). Recall that \( K \) stands for the following quotient \( K(x) = \Upsilon(x)/x \). Choose \( \alpha \in (0, 1/2] \) and \( \beta > 0 \) satisfying \( \alpha + \beta > 1 \) and \( r(\beta - \alpha) < \beta \). Define intervals \( I_n := [2^n, 2^n + 2^n/n^\beta] \). Let \( T_n(x) \) as before stand for the distance from \( x \) to \( \mathbb{R} \setminus I_n \). Consider for \( x \in \mathbb{R} \) the following function

\[
\Upsilon(x) := \begin{cases} n^{\beta - \alpha} T_n(x), & \text{if } x \in I_n \text{ with } n \geq 3, \\ 0, & \text{otherwise.} \end{cases}
\]

This function trivially satisfies conditions of our first theorem. Note that

\[
\mathcal{E}(K) = \iint_{\mathbb{R} \times \mathbb{R}} \left( \frac{K(x) - K(y)}{x - y} \right)^2 \, dx dy \geq \sum_{n \geq 3} \iint_{I_n \times I_n} \frac{(1/n^\alpha)^2}{(2^n/n^\beta)^2} \, dx dy \geq \sum_{n \geq 3} \frac{1}{n^{2\alpha}}.
\]

The series on the right-hand side of the last line diverges, so \( \mathcal{E}(K) = \infty \) and the result follows. \( \square \)

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Université Paris-Saclay, CNRS, Laboratoire de mathématiques d’Orsay, 91405, Orsay, France

Email address: ioann.vasilyev@u-psud.fr