Broken Pencils and Moving Rulers:  
After an unpublished book by Mitchell Feigenbaum

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Mitchell Feigenbaum discovered an intriguing property of viewing images through cylindrical mirrors or looking into water. Because the eye is a lens with an opening of about 5mm, many different rays of reflected images reach the eye, and need to be interpreted by the visual system. This has the surprising effect that what one perceives depends on the orientation of the head, whether it is tilted or not. I explain and illustrate this phenomenon on the example of a human eye looking at a ruler immersed in water.

I. INTRODUCTION: ANAMORPHIC IMAGES AND CAUSTICS

Mitchell Jay Feigenbaum (Dec 19, 1944-June 30, 2019) was a well-known mathematical physicist, whose work on period doubling\(^1,2\) is known to many as one of the founding papers of the theory of chaos. Towards the end of his life, he worked intensely on a book whose working title was “Reflections on a Tube.”

This book starts with a study of anamorphic images, \(i.e.,\) what you see in a tube placed on a table. The study of cylindrical mirrors was started in the 17\(^{th}\) century, as soon as people were able to make mirrors.\(^3\) I show a modern example of this in Fig. 1. Feigenbaum’s study starts with asking what one really sees when one views the image reflected on the tube. Is it inside the tube, on the tube, or even seen behind the tube?

As this paper cannot do justice to all the ramifications these questions generate, I will instead concentrate on a beautiful application of the underlying principles and consider an experiment (also suggested in Feigenbaum’s book) which can easily be done with very limited equipment.

But let me start at the beginning. To understand what is at stake, let me first have a look at the light reflected by a shiny cylinder. Each point of the image in the plane below the cylinder emits light rays in all directions, some of which hit the cylinder and get reflected toward the eye of the observer. How does one detect what is seen by the eye?

A natural, but simplistic, idea is to draw a line from the eye to the cylinder, and then, down to the paper, obeying the laws of reflection (incoming angle to the tangent plane = outgoing angle from that plane). This method is commonly called “ray-tracing”.\(^4\)

What this idea overlooks is that the eye is not a pinhole camera and that one should therefore consider all the rays emanating from the source which reach the eye.

Many different rays will enter the eye through its opening (which is about 5mm in the young adult). So what does the eye do with all these rays coming from just one point source? The eye measures \textit{intensity}, and this intensity is maximal at the \textit{caustic},\(^5\) which is the point where most rays accumulate. In each direction, there is one such point, and these points, together, form the viewable surface. (We will see later that there are actually \textit{two} such points, giving rise to two viewable surfaces.)

The effects of caustics are well-known in rainbows, where you see maximal intensity at the outer edge of each color\(^6,7\) and a lighter background inside the rainbow as shown in Fig. 2.

To define the notion of caustic more precisely, look at Fig. 3. Light is emitted from the source point \(S\) and is reflected at the circle \(C\). Each ray is reflected to an outside ray which, in the drawing, is also continued \textit{inside} the circle. The collection of rays forms a darker (cardioid) curve. This curve is called the \textit{caustic}. The rays involved are tangent to the caustic, and this defines it. The viewer perceives the rays as coming from a bright source, the caustic, inside the circle. A simpler but similar phenomenon can be observed in any coffee cup, when light strikes the inside of the cup.\(^8\)
FIG. 2. (Color online) A photograph of a rainbow under optimum conditions. Note that the sky is darker outside the rainbow than inside. Also note that the maximal intensity of each color is at the outside of the region of same color. Actually, looking from the center, the rainbow is a cross-section of the caustics.

FIG. 3. The caustics formed by light emanating from the source $S$ and reflected on the circle $C$. The circle $C$ is the cross-section of the tube shown in Fig. 1. Note the cardioid-shaped caustic in the interior of the circle.

II. LOOKING AT AN IMMERSED RULER

The aim of this paper is to explain how the eye perceives caustics when looking at a ruler which is immersed into water. This problem is mathematically simpler than the one of the cylindrical anamorphs. Furthermore, it is easy to make the experiment in a classroom with minimal material.

The setup (Fig. 10) will be described in detail below, but it is interesting to consider what was known in the XXth century about looking into water. An early reference is the 1907 book by Watson, of which pages are reproduced in Fig. 4. One can clearly see that people realized at the time that, depending on the height of the eye above the water, one sees the most intense point at different depths in the water (see Fig. 316 of Watson, reproduced in Fig. 4). But authors before Feigenbaum seem to have overlooked that there are two caustic points, which are shown in Fig. 5: The well-known ones, denoted $V_1$ (respectively $V_2$), and the new ones denoted $H_1$ (respectively $H_2$). This drawing shows the point emitting light at $B$.

III. WHAT DOES ONE SEE?

The fact that there are two candidates for an intense caustic point raises now the important, and quite novel, question: Which of the two does one see?

And here is the surprise: With the head upright, looking into the water, we see the $H$ points (which seem to move when we move the head up and down). This is what follows from the 2-dimensional theory as shown in Fig. 4. However, tilting one's head sideways (like the owl in Fig. 6) one prefers the $V$ points. The effect, if you do the experiment, is that when you tilt your head, the bottom of the ruler seems to move toward you. (I describe in more detail below what you should expect, when I show how to best carry out the experiment.)

Experimentation in a class will show that about 80% of people see the effect. Note that the effect does not depend on binocular vision as you can easily check by closing one eye.
FIG. 5. (Color online) The transparent square surface represents the surface of the water. The red dot is the position of the light source. The blue funneled surface is the well-known caustic (now shown in 3D). (It does not extend to the red dot.) The dark yellow vertical line is the locus of the \( H \) caustics. The views of 2 observers, \( O_1 \) and \( O_2 \) are shown. Each observer can see two points of high intensity corresponding to the point \( B \). The two points are called \( H \) respectively \( V \). Note that \( H \) and \( V \) slide on a vertical line, while the \( V \)'s (blue) lie on the curved surface. This is where the observer will see the \( V \) caustic, depending on height of the eye above the water. The \( V \) were known to physicists, as in Watson. The \( H \) appears also in some references such as Quick.

The apparent motion of the ruler will appear because, depending on the angle at which a point appears in the water, its distance will vary (from on \( V \) point to another).

FIG. 6. (Color online) The meaning of tilting the head by 90 degrees. Copyrighted photograph by Ron Dudley.

IV. CALCULATING THE CAUSTIC

While much of this material is already explained in and in many textbooks, we need to repeat it here, so that the reader understands how the second caustic, the \( H \) caustic, appears. And to understand its astigmatism, one really needs to do the 3d calculation. While it is very natural to do the 2d calculation as in Watson, it is just not enough, because the astigmatism at \( H \) extends in the direction orthogonal to the plane \((r, z)\) of Fig. 7.

FIG. 7. Because the index \( n = 1.33 \) of water is larger than that of air \((n = 1)\), an object at depth \( d \) sends rays which exit the water at \( z = 0 \) at the angle \( \sin \beta = n \sin \alpha \). Note that this figure appears also as Figure 315 in Fig. 4.

The calculation starts with the index of water, \( n \sim 1.33 \). By Snell’s law, Fig. 7, \( \sin \beta = n \sin \alpha \). We use complex coordinates in the plane \( z = 0 \) that defines the surface of the water, and write \( (x, y) = x + iy = re^{i\varphi} \). Any point \( x \in \mathbb{R}^3 \) can then be conveniently be written as \( x = (re^{i\varphi}, z) \). With this notation, a point on the outgoing ray can be represented as \( p = p_0 + \ell \hat{t} \in \mathbb{R}^3 \), with \( \ell \in \mathbb{R} \) and

\[
p_0 = \left( D \frac{t}{\mu} e^{i\varphi}, 0 \right), \quad \hat{t} = \frac{\left( te^{i\varphi}/\mu, \sqrt{1-t^2} \right)}{\sqrt{1+t^2}/\nu^2}.
\]

(The point \( p_0 \) is in the plane \( z = 0 \) and \( \hat{t} \) is the unit vector along the outgoing ray.) Here, \( \nu = \sqrt{n^2 - 1} \), \( \mu = \nu/n \), \( t = \nu \tan \alpha \), and \( D = d/n \). The angle \( \varphi \) is the angle in the \((x, y)\) plane (perpendicular to the \( z\)-axis). The caustic is that surface which is tangent to \( \hat{t} \) (at every point). Eliminating \( \ell \) and expressing the result \( x = (x, y) \) in terms of \( z \) one gets

\[
x = x(z) = \frac{t}{\mu} e^{i\varphi} \left( D + \frac{z}{\sqrt{1-t^2}} \right) \in \mathbb{R}^2.
\]

Note that \( x \) is parameterized by \( t \) and \( \varphi \), with \( D \) and \( \mu \) being fixed quantities related to the depth of the source point and the index of refraction of water.

The caustics are now found by requiring that the differential of Eq. (1) vanishes, because we want the caustics
Note that the two caustic points coincide when \( t = 0 \), i.e., when one looks vertically down into the water to the source point. This point has perfect focus, since the coefficient of \( dt \) must vanish. This produces the point

\[
    \begin{align*}
    r_V &= \frac{D}{\mu} t^3 \\
    z_V &= -D(1 - t^2)^{3/2}.
    
    \end{align*}
\]

(First solve for \( z_V \) and substitute into the coefficient of \( d\varphi \) to obtain \( r_V \).) These are the points indicated by \( V \) in Fig. 5. They are unsharp in the horizontal direction because \( d\varphi \) is a variation in the horizontal plane \((x,y)\). Similarly, setting \( dt = 0 \) will produce the \( H \) points

\[
    \begin{align*}
    r_H &= 0 \\
    z_H &= -D\sqrt{1 - t^2},
    \end{align*}
\]

which are vertically unsharp, as \( dt \) is a variation of the vertical angle \( \alpha \). The cusp (of the \( V \) caustic) in Fig. 4 obeys the equation

\[
    [(\mu r)^{2/3} + (z^2)^{2/3} = D^{2/3}.
\]

Note that the two caustic points \( V \) and \( H \) coincide when \( t = 0 \), i.e., when one looks vertically down into the water to the source point. This point has perfect focus, since the two caustics coincide.

Our description of \( d\varphi = 0 \) and \( dt = 0 \), shows that the rays coming out of \( H \) and \( V \) form two orthogonal fans, as illustrated in Fig. 8: The fan of rays coming out of point \( H \) is horizontal while the fan of rays coming out of the \( V \) points is vertical (relative to the surface of the water).

I insist again: The novelty of this approach is to have performed the calculation in 3 dimensions, not just in the \((r,z)\) plane. Without this extension, the combination of the \( H \) and the \( V \) caustics will not be discovered in just one differential, namely Eq. (2). This is Feigenbaum’s mathematical contribution to the question of imaging.

I want to end this section with some historical remarks. The existence of the \( H \) points (in addition to the \( V \) points) appears in Kinsler, Bartlett, Horvath. An interesting sequence of papers is Nassar’s view on “apparent depth,” on which Bartlett and Mosca commented. In particular, Bartlett cites Sears, which contains a calculation of the astigmatics (on page 42). A more recent, and very complete reference of importance in the subject is which discusses both caustics (unfortunately in German). These authors observed that both \( H \) and \( V \) are on the same line of sight as illustrated for the two observers in Fig. 5.

An important reference with more mathematical inclination is Berry, which shows clearly, in Fig. 28 on page 484 a sketch of what Feigenbaum showed in Fig. 5. But there is no mention of the role of astigmatism.

So, the novelty of Feigenbaum’s approach is to connect the existence of the two points with the difference in astigmatic direction. To discover this, the 3d calculation is essential. And the change of astigmatic direction from vertical to horizontal (at \( H \), resp. \( V \)) is the cause that the ruler seems to move when the head is tilted.

**V. THE ROLE OF THE EYE**

So far, I have mainly concentrated on the optics of looking into water. But to really understand what one sees, I need to explain certain properties of the human eye.

Our eyes are not perfect. Some people are near- or far-sighted or cannot accommodate without glasses. These imperfections appear when the focus of what we see lies in front of or behind the retina (which is the capturing device in our eyes). Since the lens and the retina are not a perfect camera, the brain will correct some of the errors, if they are not too large.

There are other imperfections of the eye, and the one of interest to us is astigmatism. This notion is used when the image of a point is mapped to a small line segment on the retina. Depending on how large this line segment is, glasses must be made to correct for this, so that the retina gets to perceive a perfect point.

Ophthalmologists know that the necessity for optical correction depends, astonishingly, on the direction of the small line segment. In fact, if the line segment is vertical (called “with the rule” (WTR)), a correction is much less needed than if it is horizontal, (called “against the rule” (ATR)). For the frequency of astigmatic prevalence, see Nemeth et al. The evolutionary origin of this asymmetry seems not known: Is it gravity, looking at far-away things at the horizon, or a remnant that mammals descend from aquatic animals? Still, for our experiment, this asymmetry of dealing with unsharp images is crucial:
After all, as I have shown above, both images, the $H$ and the $V$, are unsharp, since they move perpendicular to the fans of Fig. 8, when the eye is moved perpendicularly to the fans. But, since the eye has a non-vanishing opening, this moving effect happens even if the eye is in a fixed position. (It is like moving the eye by its opening, about 5mm.) The $H$ caustic produces an image which is vertically unsharp (relative to the surface of the water) while the $V$ image of any point is spread horizontally, when it reaches the eye.

Since, as I explained above, our eyes correct more easily vertical unsharpness (WTR), we will preferentially focus on the $H$ caustic when the head is upright. Tilting the head by 90 degrees has the effect of switching “vertical” and “horizontal.” And now the eye-brain system will prefer to focus on $V$. And, as I said, we do not understand the reason for the preference of WTR over ATR. The experiment of the ruler in the water can thus be understood in terms of this preferred focus. It is not an effect of binocular vision, as you should check by closing one eye.

VI. THE EXPERIMENT

One should note that this experiment is not the well-known phenomenon of the “broken pencil” of Fig. 9, which appears when one looks through the water (and air) and not into the water as I will describe.

A good setup is a plastic container, of dimension about $20 \times 15 \times 15$ cm, placing the ruler at the far end, parallel to the (blackened) wall, as in Fig. 10. The container should be filled as much to the rim as possible.

The viewer should look into the water through the surface at the flattest possible angle for which one still sees the bottom of the ruler. You can then see that the ruler is not straight, but slightly curved, as in Fig. 11. This is already described in Watson, as shown in Fig. 4. Since the points of vision $P_1, P_2, P_3$ “slide down” as a function of the angle at which you look into the water, inclined straight objects appear curved. This was certainly known to physicists at the end of the 19th century.

However, the new effect, discovered by Feigenbaum, appears if you tilt your head (keeping the eye in the same position relative to the box). What I mean by tilting is...
shown in Fig. 6.

Then the bottom corner of the ruler (the point near the '30' graduation of the ruler) seems to move towards you, quite a bit. The eye re-focuses as if on an object coming closer. And the top of the ruler, at '15' graduation across from the '30', seems to move less toward you (since it is less deep in the water). This is the V caustic you are seeing.

As Feigenbaum noted, this is what happens “naturally.” However, if you force yourself and change focus willfully, you can choose to see the other caustic.

A final remark: Note that no photograph can capture this effect, because the eye has a second property: It focuses on what the brain can sharpen best. And in this case, the upright eye will focus on the H position. This is because the image is vertically unsharp, and the eye-brain system can more easily make a sharper image in contrast to horizontal blurriness, see Fig. 5. The tilted eye will focus on the V caustic, which slides along the surface. An amusing consequence of the ambiguity on the choice of focus appears when you want to photograph the scene with a modern camera: Often, the auto-focus will have difficulty “deciding” what to focus on.

I will end by adding a suggestion of a variant setup by one of the referees.

“I drew a pattern of horizontal and vertical lines on a vertical plane and submerged that vertical plane under water. With my eye close to the waterline, I managed to get a vertical line in focus (H image) for the untilted head, and a horizontal line in focus (V image) for the tilted head. However, I also managed to make the opposite observation: I was able to get a horizontal line in focus (V image) for the untilted head, and a vertical line in focus (H image) for the tilted head. Finally, I also tried a motion from untilted to tilted head, and I was able to keep one type of line (horizontal or vertical) in focus, depending on my will. Therefore, I conclude that both types of image are successively visible with monocular vision, independent of head orientation. It solely depends on the intention of the viewer.”

The interested person can test this ambiguity. What this says is that a willful change of focus, allows one to see the picture which is unsharp in the “wrong” direction. As Feigenbaum points out, slight horizontal motion of the head makes the H image disappear. Like in the case of the cylinder of Fig. 1 the image of the square grid appears on a curved surface—the “viewable surface”–which is again difficult to compute.

VII. CONCLUSION

What should one carry home from this? To me, the study of Feigenbaum teaches us that it is worthwhile doing a careful calculation, well-adapted to the problem. But it also tells us that it is good to think beyond the problem at hand. While we still do not understand why Nature has given preference to automatically correcting vertical astigmatism, we certainly can see how suddenly the study of a simple physical effect can inspire astonishing connections between two seemingly unrelated disciplines, in this case optics and ophthalmology.

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