ON THE FIXED LOCUS OF FRAMED INSTANTON SHEAVES ON \( \mathbb{P}^3 \)

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Abstract. Let \( \mathbb{T} \) be the three dimensional torus acting on \( \mathbb{P}^3 \) and \( \mathcal{M}_{\mathbb{P}^3}(c) \) be the fixed locus of the corresponding action on the moduli space of rank 2 framed instanton sheaves on \( \mathbb{P}^3 \). In this work, we prove that \( \mathcal{M}_{\mathbb{P}^3}(c) \) consist only of non locally-free instanton sheaves whose double dual is the trivial bundle \( \mathcal{O}_{\mathbb{P}^3} \oplus 2 \mathbb{P}^3 \). Moreover, we relate these instantons to Pandharipande–Thomas stable pairs and give a classification of their support. This allows to compute a lower bound on the number of components of \( \mathcal{M}_{\mathbb{P}^3}(c) \).

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1. Introduction

Framed Instanton sheaves have been the subject of study for more than four decades and by many authors of different backgrounds. One of the main reasons, is that they reflect a deep connection between algebraic geometry and mathematical physics; in the late 70’s, Atiyah, Drinfeld, Hitchin and Manin fully classified the Yang-Mills anti-self-dual solutions, known as Instantons \([4, 8, 2]\), on the four sphere \( S^4 \). The classification was given, first, by relating Instantons with certain holomorphic bundles on the projective space \( \mathbb{P}^3 \), over \( \mathbb{C} \), by means of Penrose-Ward correspondence. Then by using Horrocks Monads \([20]\), introduced in the late 60’s, the authors got linear algebraic data, called the ADHM data. Donaldson,
then, discovered that framed Instantons on the four sphere $S^4$ correspond to some framed holomorphic bundles on the projective plane $\mathbb{P}^2$ [7]. Moreover, during the 90’s Nakajima considered framed sheaves in order to provide a compactification [31, 32], of the moduli space of framed instanton bundles on surfaces. This led to the computation of many invariants [32, 33], such as Betti numbers and Euler characteristic of these moduli spaces, on one hand, and a connection to representation theory by means of Quiver varieties [34] and the infinite Heisenberg Algebra [32, 3], on the other hand. It is worth to mention that the rank 1 case gives an explicit description of the Hilbert scheme of points on $\mathbb{C}^2$ in terms of ADHM data, and is a basic model for the computations in the higher rank cases [3, 6].

On $\mathbb{P}^3$, the particular rank 2 instanton bundles corresponds to the $SU(2)$ gauge theoretic instantons on the four sphere $S^4$. Their moduli space have been studied for decades and some of its properties remained illusive for a long time. For instance, its irreducibility has been proved just few years ago, by Tikhomirov [39, 40]. Also, not long ago, its smoothness was showed by Jardim and Verbitsky [28]. Recently, there have been some interest in its compactification by using torsion-free sheaves [26, 27, 24].

In this work, we are interested in the moduli space of rank 2 framed instanton sheaves $M_{\mathbb{P}^3}(c)$, on the three dimensional projective space $\mathbb{P}^3$. More precisely we study its fixed locus $M_{\mathbb{P}^3}^T(c)$ with respect to the torus action inherited by the natural one on $\mathbb{P}^3$. We shall show that every fixed torsion-free instanton sheaf $E$ is an extension (non trivial in general) of ideal sheaves $I_C$ and $I_Z$, where $I_C$ is the ideal sheaf of a non-reduced Cohen Macaulay curve $C$, whose underlying reduced support is the line $l_0 = Z(z_2 = z_3 = 0)$, i.e., the unique line that does not intersect the framing line $l_\infty$ at infinity, and $I_Z$ is the ideal sheaf of points supported on $p_0 = [1; 0; 0; 0]$ or $/ and p_1 = [0; 1; 0; 0]$, in $l_0 \subset \mathbb{P}^3$. Moreover, using the fact that the double dual of such $E$ is the trivial bundle $O_{\mathbb{P}^3}$, we also show that every corresponding quotient $Q := O_{\mathbb{P}^3}/E$ is a pure sheaf of dimensional 1 on the curve $C$. This quotient sheaves $Q$ will be called rank 0—instanton sheaves. A similar phenomenon, that occurs on $\mathbb{P}^2$, is the fact that the fixed points in $M_{\mathbb{P}^2}(r, c)$, under the toric action inherited from the one on $\mathbb{P}^2$, split as the sum of ideal sheaves of points, all with the same topological support given by the origin $[0; 0; 1]$ [33, 6, §3]. The difference is that the set of fixed points, in the $\mathbb{P}^3$ case, might not be isolated in general, in other words, there might be continuous families of them.

This paper is organized as the following: in Sections 2 we recall the notion of ADHM data and their stabilities on $\mathbb{P}^3$ and how it relates to framed instanton sheaves through Horrocks monads. In section 3 we briefly describe the inherited action, of the three dimensional torus $\mathbb{T}$, on the ADHM data. In particular, we show that, for non vanishing second Chern class, the fixed framed instantsheaves are torsion-free but not locally-free sheaves, that their double dual is trivial and that their singularity locus is pure, of dimension 1.

In Section 4 we move on to give a relation of these fixed instanton sheaves with Pandharipande-Thomas stable pairs [36, 37]. More precisely, we show that to every fixed framed rank 2 instanton sheaf $F$, on $\mathbb{P}^3$, one may associate a PT-stable pair $(Q, s)$. Furthermore, we show that the Euler characteristic $\chi(M_{\mathbb{P}^3}(c))$ is zero, for any $c > 0$. 


Section 5 is devoted to completely classify the Cohen-Macaulay supports \( C \) associated to the fixed PT-stable pair \((Q, s)\), i.e., coming from a fixed instanton sheaf of rank 2, in \( \mathbb{P}^3 \). This is achieved by using results on monomial multiple structures provided by Vatne [42].

Finally, in Section 5.2, we compute a lower bound for the number of irreducible components of \( M_{\mathbb{P}^3}(c) \). Then, we use results provided by Drézet [9, 10], in order to give an explicit description of the first canonical filtration of the rank 0—instanton sheaf \( Q \), for primitive multiple structure support. For \( c = 1 \), we also compute the dimension of the tangent space and the obstruction at the specific fixed points.

We wonder if whether, or not, these fixed points can arise as degenerations of locally free framed instantons, i.e., if the fixed enumerated components intersect the closure of the framed locally free instanton moduli. We think that this problem is related to reachability of sheaves, on multiple structure [11] and hope to address this problem in future work.

2. ADHM Data and Instanton Sheaves

In this section we will gather useful results about ADHM data and instanton sheaves. Mostly, this material can be found in [18, 14, 23]. We consider in \( \mathbb{P}^3 \) the homogeneous coordinates \([z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3\) and the line \( \ell_\infty \) given by the equations \( z_0 = z_1 = 0 \). Set \( H_{\mathbb{P}^1} = \langle z_0, z_1 \rangle \subset H^0(\mathbb{P}^3) \).

Let \( V \) and \( W \) be complex vector spaces of dimension, respectively, \( c \) and \( r \). Set \( B := \text{End}(V)^{\otimes 2} \oplus \text{Hom}(W, V) \) and consider the affine spaces

\[
B_{\mathbb{P}^1} = B_{\mathbb{P}^1}(W, V) = B_{\mathbb{P}^1}(r, c) := B \otimes H_{\mathbb{P}^1}.
\]

A point of \( B_{\mathbb{P}^1} \) will be called in this paper an ADHM datum over \( \mathbb{P}^1 \).

One can write a point of \( X \in B_{\mathbb{P}^1} \) as

\[
X = (A, B, I)
\]

where the above components are

\[
A = A_0 \otimes z_0 + A_1 \otimes z_1
\]

\[
B = B_0 \otimes z_0 + B_1 \otimes z_1
\]

\[
I = I_0 \otimes z_0 + I_1 \otimes z_1 = J_0 \otimes z_0 + J_1 \otimes z_1
\]

with \( A_i, B_i, \in \text{End}(V), I_i \in \text{Hom}(W, V) \) and \( J_i \in \text{Hom}(V, W) \), \( i = 0, 1 \). Hence we naturally regard \( A, B \in \text{Hom}(V, V \otimes H_{\mathbb{P}^1}) \), and also \( I \in \text{Hom}(V, W \otimes H_{\mathbb{P}^1}) \) and \( J \in \text{Hom}(V, W \otimes H_{\mathbb{P}^1}) \).

For any \( P \in \mathbb{P}^1 \) we define the evaluation maps given on generators by

\[
ev^P_P : B_{\mathbb{P}^1} \rightarrow \mathbb{P}(B)
\]

\[
X_i \otimes z_i \mapsto [z_i(P)X_i]
\]

Note that \( z_i(P) \in \mathbb{C} \) depends on a choice of trivialization of \( O_{\mathbb{P}^1}(1) \) at \( P \) but the class on projective space does not. We set \( X_P := ev^P_P(X) \). In particular, \( A_P, B_P, \)
$I_p$ and $J_p$ are defined as well. For any subspace $S \subset V$, we are able to naturally well define the subspaces $A_p(S), B_p(S), I_p(W)$ and $\ker J_p$ of $V$.

We also consider the following stability and costability conditions:

**Definition 2.1.** [18] Let $X = (A, B, I) \in \text{B}_{p_1}$. Let also $P$ be a point in $\mathbb{P}^3$.

(i) $X_p$ is said stable if there is no proper subspace $S \subset V$ for which hold the inclusions $A_p(S), B_p(S), I_p(W) \subset S$;

(ii) $X_p$ is said costable if there is no nonzero subspace $S \subset V$ for which hold the inclusions $A_p(S), B_p(S), I_p(W) \subset S$;

(iii) $X_p$ is said weak stable if there is no subspace $S \subset V$ of codimension 1 for which hold the inclusions $A_p(S), B_p(S), I_p(W) \subset S$;

(iv) $X_p$ is said weak costable if there is no subspace $S \subset V$ of dimension 1 for which hold the inclusions $A_p(S), B_p(S) \subset S \subset \ker J_p$;

(v) $X$ is said stable if there is no proper subspace $S \subset V$ for which hold the inclusions $A(S), B(S), I(W) \subset S \otimes H_{p_1}$;

(vi) $X$ is said costable if there is no nonzero subspace $S \subset V$ for which hold the inclusions $A(S), B(S) \subset S \otimes H_{p_1}$ and $S \subset \ker J$;

(vii) $X$ is said locally (resp. globally) stable (corresp. weak stable) if $X_p$ is stable (corresp. weak stable) for some (resp. every) $P \in \mathbb{P}^1$;

(viii) $X$ is said locally (resp. globally) costable (corresp. weak costable) if $X_p$ is costable (corresp. weak costable) for some (resp. every) $P \in \mathbb{P}^1$.

(ix) regular (resp. locally regular, locally weak regular, globally regular, globally weak regular) if it is both stable and costable (resp. locally stable and locally costable, locally weak stable and locally weak costable, globally stable and globally costable, globally weak stable and globally weak costable).

We define $\text{B}_{p_1}^{st}, \text{B}_{p_1}^{lws}, \text{B}_{p_1}^{ls}, \text{B}_{p_1}^{gws}, \text{B}_{p_1}^{gs}, \text{B}_{p_1}^{gwr}$, and $\text{B}_{p_1}^{gr}$ as the subsets of $\text{B}_{p_1}$ consisting of stable, locally weak stable, locally stable, globally weak stable, globally stable, globally weak regular and globally regular ADHM data over $\mathbb{P}^3$, respectively.

Clearly, each of these sets are open subsets of $\text{B}_{p_1}$ (in the Zariski topology), and one has strict inclusions

$$
\text{B}_{p_1}^{gr} \subset \text{B}_{p_1}^{gs} \subset \text{B}_{p_1}^{lws} \subset \text{B}_{p_1}^{lsw} \subset \text{B}_{p_1}^{lsw} \subset \text{B}_{p_1}^{lwr} \subset \text{B}_{p_1}^{wr} \subset \text{B}_{p_1}^{w} \subset \text{B}_{p_1}^{st}
$$

An instanton sheaf on $\mathbb{P}^n$ is a torsion free coherent sheaf $E$ with $c_1(E) = 0$ satisfying the following cohomological conditions:

(i) for $n \geq 2$, $H^0(\mathbb{P}^n, E(-1)) = H^n(\mathbb{P}^n, E(-n)) = 0$;

(ii) for $n \geq 3$, $H^1(\mathbb{P}^n, E(-2)) = H^{n-1}(\mathbb{P}^n, E(1-n)) = 0$;

(iii) for $n \geq 4$, $H^p(\mathbb{P}^n, E(-k)) = 0$ for $2 \leq p \leq n-1, \forall k$.

The second Chern class $c := c_2$ is called the charge of $E$, and one can check that $c = -\chi(E) = h^1(\mathbb{P}^n, E(-1))$. An instanton sheaf is said to be of trivial splitting type if there exists a line $\ell$ in $\mathbb{P}^3$ such that the restriction $E|_\ell$ of $E$ on $\ell$ is trivial. A particular choice of trivialisation $\phi : E|_\ell \to \mathcal{O}_{\ell}^{\dim r}$ is called a framing, and the pair $(E, \phi)$ is called a framed instanton sheaf.

Now, we consider framed instantons on $\mathbb{P}^3$, and we fix the line $\ell_\infty$, given in the beginning of this section, as the framing line. Moreover, from [18 Proposition 3.6], [14, 23], we have that framed rank $r$ instantons $E$ of charge $c$ are cohomologies of
monads of the form:

$$\mathcal{M} : \quad V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} (V \oplus V \oplus W) \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}$$

$V$ is a $c$–dimensional vector space and can be identified with some homology group of $E$ twisted by a some differential sheaf, via the Beilinson spectral sequence construction of the monad [IS, §3]. $W$ is $r$–dimensional space (this can be identified with $\mathbb{C}^r$, given a fixed basis, due to the framing). The maps $\alpha$ and $\beta$ are given by:

$$\alpha = \begin{pmatrix} A_0 z_0 + A_1 z_1 + I z_2 \\ B_0 z_0 + B_1 z_1 + I z_3 \\ J_0 z_0 + J_1 z_1 \end{pmatrix},$$

$$\beta = (-B_0 z_0 - B_1 z_1 - I z_3, A_0 z_0 + A_1 z_1 + I z_2, I_0 z_0 + I_1 z_1),$$

where $A_0, A_1, B_0, B_1 \in \text{End}(V), I_0, I_1 \in \text{Hom}(W, V)$ and $J_0, J_1 \in \text{Hom}(V, W).$

These matrices satisfy the following equations

$$[A_0, B_0] + I_0 J_0 = 0$$
$$[A_1, B_1] + I_1 J_1 = 0$$
$$[A_0, B_1] + [B_0, A_1] + I_1 J_2 + I_2 J_1 = 0$$

which are equivalent to the complex condition $\beta \circ \alpha = 0$, in the monad $\mathcal{M}.$ Moreover there is a group action of $G = \text{Gl}(V),$ on the above data, given by

$$A_i \rightarrow g A_i g^{-1}$$
$$B_i \rightarrow g B_i g^{-1}$$
$$I_i \rightarrow g I_i$$
$$J_i \rightarrow g J_i$$

for $g \in G$ and $i = 0, 1.$

We denote by $V_{\mathbb{P}^3}(c, r)$ the space of the ADHM data satisfying the equations (2) and in which one can define the following subvarieties, according to stabilities in Definition 2.1

$$V^{\text{ss}}_{\mathbb{P}^3}(c, r) \subset V^{\text{sw}}_{\mathbb{P}^3}(c, r) \subset V^{\text{st}}_{\mathbb{P}^3}(c, r)$$

For an ADHM datum $X = (A_0, B_0, I_0, J_0, A_1, B_1, I_1, J_1),$ consider the following algebraic set

$$D_X = \{ z \in \mathbb{P}^3 | \alpha_X \text{ is not injective} \}.$$ 

Note that we always have $\text{codim}(D_X) \geq 2,$ by the framing condition. A simpler version of [IS, Proposition 3.3 & Proposition 3.4] can be written as the

**Theorem 2.2.** The complex (1) is a monad if and only if the corresponding ADHM datum is globally weak stable, and in this case $E,$ the middle cohomology of the monad, is torsion-free. Moreover, $E$ is a locally free framed instanton sheaf if and only if the ADHM datum $X$ is globally weak costable.
3. Torus action on the ADHM data

Now, we consider the following standard torus action of $T := \mathbb{T}^3$ on $\mathbb{P}^3$ given by

$$F_t : T \times \mathbb{P}^3 \rightarrow \mathbb{P}^3$$

$$((t_1, t_2, t_3), z) \mapsto [z_0: t_1 z_1: t_2 z_2: t_3 z_3]$$

This action can be lifted to the space of ADHM data as the following: Let $T := \mathbb{T} \times \hat{T}$, where $\hat{T}$ is the maximal torus of $GL(W)$, given by elements of the form $e = \text{diag}(e_1, \ldots, e_r)$. Let $\gamma_{e_1, \ldots, e_r}$ be the isomorphism $O|_{\ell}^r \ni (w_1, \ldots, w_r) \mapsto (e_1 w_1, \ldots, e_r w_r) \in O|_{\ell}^r$. For a framed instanton sheaf $(E, \phi : \xi_0 \rightarrow O|_{\ell}^r)$ one can define the following $(t, e_1, \ldots, e_r) \cdot (E, \phi) = ((F_t^{-1})^* E, \phi')$, where $\phi'$ is given by the composition

$$(F_t^{-1})^* E|_{\ell} \overset{(F_t^{-1})^* \phi}{\rightarrow} (F_t^{-1})^* O|_{\ell}^r \overset{\gamma_{e_1, \ldots, e_r}}{\rightarrow} O|_{\ell}^r.$$  

**Proposition 3.1.** The above action can be identified with the action on the ADHM data given by:

$$\begin{align*}
(4) & \quad A_0 \rightarrow t_2 A_0 & A_1 \rightarrow t_1^{-1} t_2 A_1 \\
& \quad B_0 \rightarrow t_3 B_0 & B_1 \rightarrow t_1^{-1} t_3 B_1 \\
& \quad J_0 \rightarrow t_3 e J_0 & J_1 \rightarrow t_1^{-1} t_3 e J_1 \\
& \quad I_0 \rightarrow t_2 I_0 e^{-1} & I_1 \rightarrow t_1^{-1} t_2 I_1 e^{-1}
\end{align*}$$

Moreover, the ADHM equations (2) and stability conditions are preserved.

**Proof.** Since any framed instanton sheaf $E$ is the middle cohomology of a monad as in (1), then the pull back $(F_t^{-1})^* E$ is the cohomology of a similar monad with maps $\alpha$ and $\beta$ given as below:

$$\alpha = \begin{pmatrix}
A_0 z_0 + A_1 t_1^{-1} z_1 + 1 t_2^{-1} z_2 \\
B_0 z_0 + B_1 t_1^{-1} z_1 + 1 t_3^{-1} z_3 \\
J_0 z_0 + J_1 t_1^{-1} z_1
\end{pmatrix},$$

$$\beta = \begin{pmatrix}
-B_0 z_0 - B_1 t_1^{-1} z_1 - 1 t_3^{-1} z_3, A_0 z_0 + A_1 t_1^{-1} z_1 + 1 t_2^{-1} z_2, I_0 z_0 + I_1 t_1^{-1} z_1
\end{pmatrix}.$$  

Under the isomorphism

$$V \otimes O_{\mathbb{P}^3} \overset{\otimes}{\rightarrow} \begin{pmatrix} v_1 \\ v_2 \\ w \end{pmatrix} \rightarrow \begin{pmatrix} t_3^{-1} v_1 \\ t_2^{-1} v_2 \\ t_2^{-1} w \end{pmatrix}$$

the kernel of $\beta$ is sent to the kernel of

$$-(t_3 B_0) z_0 - (t_1^{-1} t_3 B_1) z_1 - 1 t_3 z_3, (t_2 A_0) z_0 + (t_1^{-1} t_2 A_1) z_1 + 1 t_2 z_2, (t_2 I_0) z_0 + (t_1^{-1} t_2 I_1) z_1$$

and the image of $\alpha$ is sent to the image of

$$\begin{pmatrix}
(t_2 A_0) z_0 + (t_1^{-1} t_2 A_1) z_1 + 1 t_2 z_2 \\
(t_3 B_0) z_0 + (t_1^{-1} t_3 B_1) z_1 + 1 t_3 z_3 \\
(t_3 I_0) z_0 + (t_1^{-1} t_3 I_1) z_1
\end{pmatrix}.$$  

Composing with the action of $\gamma_{e_1, e_2}$ on the framing, the assertion follows. 
\qed
Hence one shows that \( J \) by induction, it follows that for any product \( \hat{z} \) in \( \mathcal{M}_{\mathbb{P}^3}(r,c) := \mathcal{V}_{\mathbb{P}^3}(r,c)/G \) (This quotient makes sense by means of [18 Section 2.3]), a datum \([X]\) is invariant under the toric action if and only if there exists an element \( g_t \in G \) such that \( t \cdot X = g_t \cdot X \). In other words, \([X] = [A_0,B_0,I_0, J_0, A_1,B_1, I_1, J_1] \) is \( T \)-invariant if and only if there exists a map
\[
\theta : T \rightarrow G
\]
\[
t \mapsto \theta(t) = g_t
\]
such that
\[
(5) \quad t_2 A_0 = g_t A_0 g_t^{-1} \quad t_1^{-1} t_2 A_1 = g_t A_1 g_t^{-1}
\]
\[
t_3 B_0 = g_t B_0 g_t^{-1} \quad t_1^{-1} t_3 B_1 = g_t B_1 g_t^{-1}
\]
\[
t_3 J_0 = J_0 g_t^{-1} \quad t_1^{-1} t_3 J_1 = J_1 g_t^{-1}
\]
\[
t_2 I_0 = g_t I_0 \quad t_1^{-1} t_2 I_1 = g_t I_1
\]

**Lemma 3.2.** If \([X]\) is fixed by the torus \( T \), then we have \( J_0 = J_1 = 0 \). Moreover, \( X \) is not globally weak costable.

**Proof.** Suppose \([X]\) is fixed by the torus \( T \), and let \( t = (t_1, t_2, t_3) \) Then one has \( J_0 I_0 = (J_0 g_t^{-1})(g_t I_0) = (t_3 J_0)(t_2 I_0) = t_2 t_3 J_0 I_0 \), hence \( J_0 I_0 = 0 \). In the same way, one shows that \( J_0 I_3 = 0 \), for all \( \alpha, \beta = 0, 1 \). Moreover, for \( A = z_0 A_0 + z_1 A_1 \) \( B = z_0 B_0 + z_1 B_1 \), \( I = z_0 I_0 + z_1 I_1 \) and \( J = z_0 J_0 + z_1 J_1 \) such that \([A, B] + IJ = 0, \forall z_0, z_1\), one has \( JBA = J[A, B] + JAB = J(-IJ) + JAB = -(JI)J + JAB = JAB \). Thus, by induction, it follows that for any product \( \hat{C} = C_{\alpha_1} \cdot C_{\alpha_2} \cdots BA \cdots C_{\alpha_m} \), where \( \alpha_i = 0, 1, \forall i = 1, \cdots, m \) and
\[
C_{\alpha_i} = \begin{cases} A & \alpha_i = 0 \\ B & \alpha_i = 1 \end{cases}
\]
one has \( J\hat{C} = JA^1 B^m \), where \( l \) and \( m \) are the numbers of \( A \)'s and \( B \)'s, respectively, appearing in \( \hat{C} \). On the other hand, we have
\[
J_0 A_0^l B_0^m I_0 = J_0 g_t^{-1} g_t A_0^l g_t^{-1} g_t B_0^m g_t^{-1} g_t I_0
\]
\[
= (t_3 J_0)(t_2 A_0^l)(t_2 B_0^m)(t_2 I_0)
\]
\[
= (t_2^{m+1} t_3^l) J_0 A_0^l B_0^m I_0,
\]
for all \( t \in T \). Hence \( J_0 A_0^l B_0^m I_0 = 0 \). In the same way, it follows that
\[
J_{\alpha_1} A_{\alpha_2}^l B_{\alpha_3}^m I_{\alpha_4} = 0,
\]
for all \( \alpha_i = 0, 1 \). By the stability condition, we have that \( V \otimes H_{\mathbb{P}^1} \) is generated by the action of \( C_{\alpha_1} C_{\alpha_2} \) on \( I(w_1) \) and \( I(w_2) \), where \( <w_1, w_2> = \mathbb{C}^2 \). Then every vector \( v \in V \otimes H_{\mathbb{P}^1} \) is of the form \( \Sigma_{\alpha_k} C_{\alpha_1} \cdots C_{\alpha_m} I(w_1) + \Sigma_{\alpha_k} C'_{\alpha_1} \cdots C'_{\alpha_m} I(w_2) \). Hence
\[
J_0 v = \Sigma_{\alpha_k} J_0 C_{\alpha_1} \cdots C_{\alpha_m} I(w_1) + \Sigma_{\alpha_k} J_0 C'_{\alpha_1} \cdots C'_{\alpha_m} I(w_2)
\]
\[
= 0 \text{ by } (5) \text{ and } (3).
Therefore, both $J_0$ and $J_1$ vanish identically. Moreover, it follows that the datum $X$ is not globally weak costable.

\[ \square \]

**Theorem 3.3.** A $T$-fixed framed instanton on $\mathbb{P}^3$ is torsion free.

**Proof.** By the correspondence in Theorem 2.2 we conclude that the instanton $E$ corresponding to $T$-fixed datum $X$ is not locally free. From the framing, we conclude that the singularity set of the sheaf is at least $2$-dimensional, hence the instanton sheaf is torsion-free, in this case.

\[ \square \]

In the rank 2 case we have the following result:

**Theorem 3.4.** Let $E$ be a rank 2 torsion-free instanton sheaf on $\mathbb{P}^3$. Then

(i) The singularity set $\text{Sing}(E)$ of $E$ is purely 1-dimensional.

(ii) The double dual $E^{**}$ is the trivial locally free instanton sheaf $\mathcal{O}_{\mathbb{P}^3}$.

**Proof.** Suppose $E$ is reflexive, then $E$ should be locally free since it is of rank two and has third chern class $c_3(E) = 0$ [17 Proposition 2.6]. This contradicts Theorem 3.3 for $c \neq 0$. Hence the singularity set $\text{Sing}(E)$ of $E$ is 1-dimensional, it remains to check purity. This is done by showing that the quotient sheaf $Q := \mathcal{O}_{\mathbb{P}^3}/E$ is pure. The sheaf $Q$ is supported in codimension 2, thus we have $\text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, for $q = 0, 1$. Moreover, by [22 Proposition 1.1.10], $Q$ is pure if, and only if, $\text{codim} (\text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}(-4))) \geq 3 + 1 = 4$. In other words, we need to show that $\text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}(-4))$ is the zero sheaf.

Note that $Q$ is a 1-dimensional sheaf, so by [22 Proposition 1.1.6] we have $\text{codim}(\text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}(-4))) \geq 3$. Hence $\text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3}(-4))$ is, supposedly, supported on a zero-dimensional subscheme in $\mathbb{P}^3$, lying inside $\text{Sing}(E)$.

From the local-to-global spectral sequence one has

$$\text{Ext}^3(Q, \mathcal{O}_{\mathbb{P}^3}) = H^0(\mathbb{P}^3, \text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3})) \oplus H^1(\mathbb{P}^3, \text{Ext}^2(E, \mathcal{O}_{\mathbb{P}^3})).$$

By Serre-Grothendieck duality can write

$$\text{Ext}^3(Q, \mathcal{O}_{\mathbb{P}^3}) = \text{Ext}^0(\mathcal{O}_{\mathbb{P}^3}, Q(-4))^* = H^0(\mathbb{P}^3, Q(-4))^*,$$

and from the long exact sequence in cohomology, associated to the short exact sequence $0 \to E(-4) \to \mathcal{O}_{\mathbb{P}^3}(-4) \to Q(-4) \to 0$ one has $H^0(\mathbb{P}^3, Q(-4)) = H^1(\mathbb{P}^3, E(-4))$. Twisting the monad, associated to $E$, by $\mathcal{O}_{\mathbb{P}^3}(-4)$ it is not difficult to check that $H^1(\mathbb{P}^3, E(-4)) = 0$.

Thus $\text{Ext}^3(Q, \mathcal{O}_{\mathbb{P}^3}) = 0$ and both contributing terms

$$H^0(\mathbb{P}^3, \text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3})), \quad H^1(\mathbb{P}^3, \text{Ext}^2(E, \mathcal{O}_{\mathbb{P}^3})),$$

to the local-global spectral sequence, must be trivial. Finally, observe that $\dim H^0(\mathbb{P}^3, \text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3})) = 0$ is the length of the sheaf $\text{Ext}^3(E, \mathcal{O}_{\mathbb{P}^3})$, which must be zero since any sheaf supported on a zero-dimensional subscheme of $\mathbb{P}^3$, with zero length is the zero sheaf. Hence $Q$ is pure.

The double dual of any sheaf is reflexive. But since $c_3(E^{**}) = 0$ and $E^{**}$ is of rank 2, it follows that it is locally free [17 Proposition 2.6]. Moreover, we just proved that there are no locally free fixed framed instantons sheaves. But $E^{**}$ is also fixed by the torus action. Hence it should be trivial.
Therefore we have the following

**Corollary 3.5.** The singularity locus of a $\mathbb{T}$–fixed framed instanton sheaf $E$ of rank 2 is topologically supported on the rational line given by $z_2 = z_3 = 0$. Moreover, the matrices $A_i, B_i$, for $i = 0, 1$, in the corresponding ADHM datum are nilpotent.

**Proof.** Since $E$ is $\mathbb{T}$–fixed then it is torsion free, by Theorem 3.3. Moreover, its singularity set is of purely 1–dimensional and by the framing condition it does not intersect the framing line. In particular, the singularity set is also $\mathbb{T}$–invariant. But the only invariant codimension 2 subscheme of $\mathbb{P}^3$, as a toric variety, which does not intersect the framing line is supported on the line $[z_0, z_1, 0, 0]$.

The singularity set is the locus on which the map $\alpha$, in the monad (1), is not injective. In particular, one can characterize it in terms of the eigenvalues equations

$$\det([A_0 z_0 + A_1 z_1]) + z_2 = 0, \quad \det([B_0 z_0 + B_1 z_1]) + z_3 = 0.$$

But we just showed that all the corresponding eigenvalues $z_2, z_3$ must be 0. Hence the matrices $(A_0 z_0 + A_1 z_1)$ and $(B_0 z_0 + B_1 z_1)$ must be nilpotent, for all $z_0, z_1$, and consequently, the result follows.

From the above, we see that if $[X] \in \mathcal{M}_{\mathbb{P}^3(r,c)}$ is a $\mathbb{T}$–fixed point, then is represented by a datum $X = (A_0, B_0, I_0, A_1, B_1, I_1)$ satisfying the equations:

$$[A_0, B_0] = 0$$
$$[A_1, B_1] = 0$$
$$[A_0, B_1] + [B_0, A_1] = 0$$

(9)

4. **Quotients and PT-stable pairs**

In this section we will adopt the following viewpoint: Let $E$ be a $\mathbb{T}$–invariant torsion-free instanton sheaf of rank 2, then $E$ fits in the short exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{Q} \rightarrow 0$. The Hilbert polynomials of sheaves involved in this sequence are $P_E(m) = 1/4 m^3 + 2 m^2 + (1/4 - c)m + (2 - 2c)$, $P_{\mathcal{Q}}(m) = cm + 2c$, and $P_{\mathcal{O}_{\mathbb{P}^3}}(m) = 2^m (3^m + m)$. Since every such sheaf $E$ is given by a datum $X \in \mathcal{V}_{\mathbb{P}^3}(r,c)$, one can think of $\mathcal{M}_{\mathbb{P}^3}(c)$ as an open subset of the scheme $\text{Quot}^{[cm+2c]}_{\mathcal{O}_{\mathbb{P}^3},[0]}$, which parametrizes quotients $\mathcal{Q} \rightarrow \mathbb{Q}$ with $\mathcal{Q}$ is $\mathbb{T}$–fixed 1–dimensional pure sheaf, topologically supported on the fixed line $l_0 : \mathbb{P}^1 \rightarrow \mathbb{P}^3$, not intersecting the framing line $l_\infty$, that is $l_0$ is given by $[z_0: z_1] \mapsto [z_0: z_1: 0: 0]$. The instanton cohomological conditions on $E$ imply the $\mathcal{Q}$ should satisfy $H^0(\mathbb{P}^3, \mathcal{Q}(-2)) = H^1(\mathbb{P}^3, \mathcal{Q}(-2)) = 0$. Obviously these are open conditions in flat families by semicontinuity.

Recall that a rank 0–instanton sheaf is a pure sheaf of codimension 2 satisfying $H^0(\mathbb{P}^3, \mathcal{Q}(-2)) = H^1(\mathbb{P}^3, \mathcal{Q}(-2)) = 0$. Thus $\mathcal{Q}$ is a rank 0–instanton sheaf.

**Lemma 4.1.** Every rank 0–instanton sheaf is $\mu$–semi-stable.

**Proof.** Let $T$ a subsheaf of $\mathcal{Q}$ with Hilbert polynomial $P_T(m) = am + b$. Note that $H^0(\mathbb{P}^3, \mathcal{Q}(-2)) = 0$ implies that $H^0(\mathbb{P}^3, T(-2)) = 0$. Thus $P_T(-2) = -2a + b = -H^1(\mathbb{P}^3, T(-2)) \leq 0$. Hence $\mu(T) = \frac{b}{a} \leq 2 = \frac{2a}{c} = \mu(\mathcal{Q})$. □
Following [36], let $q \in \mathbb{Q}[x]$ a degree 1 polynomial with positive leading coefficient. For $n \in \mathbb{Z}$ and $\beta \in H^2(\mathbb{P}^3, \mathbb{Z})$, we will denote by $P^n_\beta(\mathbb{P}^3, \beta)$ the moduli space of stable pairs $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{Q}$, on $\mathbb{P}^3$, where $\mathcal{Q}$ is a pure sheaf, of dimension 1 on $\mathbb{P}^3$, with Hilbert polynomial
\[
\chi(\mathcal{Q}(m)) = m \cdot \int c_1(\mathcal{Q}) + n.
\]
The polynomial $q$ is viewed as a stability parameter, and $s$ is a non-zero section. We also let $r_\tau$ denote, for any sheaf $\mathcal{T}$, the leading coefficient of the Hilbert polynomial. Since $\mathcal{Q}$ is pure, then any proper subsheaf $\mathcal{T}$ of $\mathcal{Q}$ is also pure of the same dimension. Therefore $r_\tau > 0$. We say that the pair $(\mathcal{Q}, s)$ is $q$-(semi-)stable if, for any proper subsheaf $\mathcal{T} \subset \mathcal{Q}$, the inequality
\[(10) \quad \frac{\chi(\mathcal{T}(m))}{r_\tau} < (\leq) \frac{\chi(\mathcal{Q}(m)) + q(m)}{r_\mathcal{Q}}, \quad m >> 0,
\]
holds, and for any proper subsheaf $\mathcal{T} \subset \mathcal{Q}$, through which the section $s$ factors, the inequality
\[(11) \quad \frac{\chi(\mathcal{T}(m)) + q(m)}{r_\tau} < (\leq) \frac{\chi(\mathcal{Q}(m)) + q(m)}{r_\mathcal{Q}}, \quad m >> 0,
\]
holds. Moreover the pair $(\mathcal{Q}, s)$ is said to be stable if is stable in the large $q$ limit, i.e., for sufficiently large coefficients of $q$. The moduli $P^n_\beta(\mathbb{P}^3, \beta)$ is constructed with respect to this stability in [29], and in [30] it is proved that it has a perfect obstruction and hence a well defined virtual class. In what follows, we will say that $(\mathcal{Q}, s)$ is a stable 0-instanton pair if it is stable and the sheaf $\mathcal{Q}$ is a rank 0-instanton sheaf.

Now we let $\mathcal{Q}$ be a rank 0-instanton sheaf, then we have

**Lemma 4.2.** For any proper subsheaf $\mathcal{T}$ of $\mathcal{Q}$, the inequality (10) is satisfied.

**Proof.** Let $q(m) = q_1 m + q_0$, where $q_i \in \mathbb{Z}$, $i = 0, 1$, $q_0 > 0$ and $\chi(\mathcal{T}(m)) = r_\tau m + n$. Then
\[
\frac{\chi(\mathcal{Q}(m)) + q(m)}{c} = m + 2 + \frac{q(m)}{c} = (1 + \frac{q_1}{c})m + (2 + q_0).
\]
It follows that
\[
\frac{\chi(\mathcal{T}(m))}{r_\tau} - \frac{\chi(\mathcal{Q}(m)) + q(m)}{r_\mathcal{Q}} = (\frac{-q_1}{c})m + (\frac{n}{r_\tau} - 2 - \frac{q_0}{c}).
\]
Since $q_1 > 0$, then for $m$ big enough one has $m > \frac{m}{q_1 r_\tau} - \frac{2}{q_1} - \frac{q_0}{q_1}$. Hence the result.

Moreover, one has the following commutative diagram:
\[(12) \begin{array}{c}
0 \\
\downarrow \\
\mathcal{O}_{\mathbb{P}^3} \\
\downarrow \\
n \\
0 \to E \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{Q} \to 0 \\
\downarrow \\
\mathcal{O}_{\mathbb{P}^3} \\
\downarrow \\
0
\end{array}\]
This define a section $s$ of $Q$. Let us put $G := \text{Im}(s)$ and $I := \ker(s)$. Then $I$ is an ideal sheaf in $O_{\mathbb{P}^3}$ of a subscheme $S$ of pure dimension 1 in $\mathbb{P}^3$, with structure sheaf $O_c = G$. Moreover, if $Q$ is a $\mathcal{T}$-fixed rank 0-instanton sheaf, then the theoretical support of $S$ is exactly the line $l_0$, in Corollary 3.5 defined by the locus $(z_2 = z_3 = 0)$, in $\mathbb{P}^3$.

**Proposition 4.3.** $(Q, s)$ is stable 0-instanton pair if, and only if, the instanton sheaf $E := \ker(O^2 \rightarrow Q)$ belongs to $\text{Ext}^1(I_Z, I_S)$, where $I_C$ is an ideal sheaf of a subscheme in $\mathbb{P}^3$, of pure dimension 1 and $I_Z$ is ideal sheaf of a zero-dimensional subscheme in $\mathbb{P}^3$, both not intersecting the line $l_{\infty}$.

**Proof.** First, notice that since $Q$ is a rank 0-instanton, one can complete the commutative diagram (12) above to get the following one:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
0 & I_C & O_{\mathbb{P}^3} & O_C & 0 \\
0 & E & O_{\mathbb{P}^3} & Q & 0 \\
0 & I_Z & O_{\mathbb{P}^3} & Z & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

where $O_C := \text{Im}(s)$, $I_C := \ker(s)$ and $Z := \text{coker}(s)$. $O_C$ a structure sheaf of subscheme in $\mathbb{P}^3$, which is purely dimensional and that (10) is satisfied. Moreover, by [36] Lemma 1.3 it follows that the pair $(Q, s)$ if and only if $I$ is a zero-dimensional sheaf. Hence, from the left column of the diagram (13), the rank two torsion free framed instanton sheaf $E$ belongs to $\text{Ext}^1(I_Z, I_S)$, where $I_C$ is an ideal sheaf of a subscheme in $\mathbb{P}^3$, of pure dimension 1 and $I_Z$ is ideal sheaf of a zero-dimensional subscheme in $\mathbb{P}^3$. Both $C$ and $\text{Supp}(Z)$ are supported on $\text{Sing}(E)$, which has vacuous intersection with the framing line $l_{\infty}$. \hfill \Box

**Corollary 4.4.** If $Q$ is $\mathcal{T}$-fixed rank 0-instanton, then $(Q, s)$ is stable 0-instanton pair.

This follows from proposition above and [25] Corollary 5. We remark that quasitrivial instantons in [25] are those whose double dual is trivial.

This realizes $\mathcal{M}_{0_{\mathbb{P}^3}}^\mathcal{T}(c)$ as a subset of $P^3_{\mathbb{P}^3}(\beta = [c \cdot H^2])$.

We end this section with the following

**Theorem 4.5.** The Euler characteristic of $\mathcal{M}_{0_{\mathbb{P}^3}}(c)$ is $\chi(\mathcal{M}_{0_{\mathbb{P}^3}}(c)) = 0$, $\forall c > 0$.

Moreover if $c = 1$, then the Poincaré Polynomial of $\mathcal{M}_{0_{\mathbb{P}^3}}(1)$ is

\[
P_{\mathcal{M}_{0_{\mathbb{P}^3}}(1)}(t) = \sum_{i=0}^{13} (1 - \delta_{1,i} - \delta_{12,i})t^i
\]

Before giving a proof, we recall now some useful definitions mostly from [38]:

**Definition 4.6.** Let $Y$ be an algebraic space endowed with a right (or left) action of a group $G$, and let $\pi: Y \rightarrow X$ be a morphism from $Y$ to the algebraic space $X$. We call the triple $(G, Y, X)$ (or just $Y$) a fibered system if $\pi$ satisfies $\pi(x \cdot g) = \pi(x)$ for all $x \in X$ and $g \in G$. 

A fibered system $Y$ is called **locally trivial** (resp. **locally isotrivial**) if for every Zariski open $U \subset X$, the restriction $Y|_U$, of $Y$ on $U$, is isomorphic to $U \times G$, with the endowed operations $(x, g)g' = (x, gg')$ and the canonical projection $U \times G \rightarrow U$ (resp. if for every open $U \subset X$ there is an unramified morphism $f : U' \rightarrow U$ over $U$ such that the inverse image $f^{-1}Y|_U$, of $Y|_U$, is trivial). $Y$ is called **trivial** if $Y$ is isomorphic to $X \times G$.

A group $G$ is called **special** if every locally isotrivial fibered system $(G, Y, X)$ is locally trivial. Finally, an isotrivial fibered system $(G, Y, X)$ is called a $G$–principal fibration. If, moreover, the morphism $\pi$ is flat and $(G, Y, X)$ is locally isotrivial, then $(G, \pi, X)$ (or just $Y$) is a called a $G$–principal bundle.

**Proof of Theorem 4.3.** Let $\mathcal{I}(c)$ is the moduli space of rank 2 locally free instantons on $\mathbb{P}^3$ (without framing). We first remark that, for all $c > 0$, $\mathcal{M}_{\mathbb{P}^3}(c)$ is an $\text{Sl}(2, \mathbb{C})$–bundle over $\mathcal{F}(c)$, where the projection is given by forgetting the framing. Since the group $\text{Sl}(2, \mathbb{C})$ is special $[15]$, in the sense above, we have that every $G$–principal bundle is locally trivial in the Zariski topology. In particular $\mathcal{M}_{\mathbb{P}^3}(c) \rightarrow \mathcal{F}(c)$ is a locally trivial $\text{Sl}(2, \mathbb{C})$–principal bundle. Hence, one can write the Poincaré polynomial of $\mathcal{M}_{\mathbb{P}^3}(c)$ as

$$P_{\mathcal{M}_{\mathbb{P}^3}(c)}(t) = P_{\mathcal{F}(c)}(t) \times P_{\text{Sl}(2, \mathbb{C})}(t),$$

and since $\text{Sl}(2, \mathbb{C}) \cong SU(2) \times \mathbb{R}^3$, one gets $P_{\text{Sl}(2, \mathbb{C})}(t) = 1 + t^3$. By putting $t = -1$, it follows that $\chi(\mathcal{M}_{\mathbb{P}^3}(c)) = 0$.

In [27] Section 6, the authors prove that $\mathcal{F}(1) \cong \mathbb{P}^5$. Hence, for $c = 1$ the Poincaré polynomial is computed from the product formula.

**5. Relation with multiple structures**

In this section we explore the relation of the rank 0–instanton sheaves and sheaves on multiple structures $[12]$ $[9]$ $[10]$ $[35]$. This allows us to give a concrete description in the lower charge cases $c = 1, 2$, as well as, the multiple primitive cases (see section 5.2). Moreover, we use such a description to compute the Euler Characteristic of $\mathcal{M}_{\mathbb{P}^3}(1)$. We also give a lower bound on the number of irreducible components.

**5.1. Monomial multiple structures.** Most of the material in this subsection is borrowed from [42], with the assumption that the ambient space is $\mathbb{P}^3$. Let $i : X = \mathbb{P}^1 \rightarrow \mathbb{P}^3$ be a linear subspace with saturated ideal $I_X$, $X^{(i)} \subset \mathbb{P}^3$ the $i$'th infinitesimal neighborhood of $X$, with ideal $(I_X)^{i+1}$, and $Y$ a Cohen-Macaulay multiple structure with $Y_{\text{red}} = X$, whose ideal is generated by monomials. Then the following filtration of $Y$ exists;

$$X = Y_0 \subset Y_1 \subset \cdots \subset Y_{k-1} \subset Y_k = Y; \quad Y_i = Y \cap X^{(i)},$$

for some $k$, and every term $Y_i$ is also Cohen-Macaulay since $X$ is a Cohen-Macaulay curve $[11]$ Corollary 2.6]. If $I_i$ is the ideal sheaf of $Y_i$, then there are two short exact sequences

$$0 \rightarrow I_{i+1}/I_X I_i \rightarrow I_i/I_X I_i \rightarrow L_i \rightarrow 0;$$

*For the multiplicative property of the Poincaré polynomial the reader might see [5] Introduction, for instance.
and

\[ 0 \to i^*L_i \to \mathcal{O}_{Y_{i+1}} \to \mathcal{O}_{Y_i} \to 0. \]

The first exact sequence define the \( \mathcal{O}_X \)-modules \( L_j \), see [35], and references therein, for more details. In general, the \( L_j' \)'s are torsion-free, but in the monomial case they are locally free.

One important result that will be used is the following\(^1\):

**Proposition 5.1.** [42, Proposition 1] There is a bijective (inclusion reversing) correspondence between Cohen-Macaulay monomial ideals in two variables and Young diagrams. Under this bijection, the number of boxes in the Young diagram is the multiplicity of the scheme defined by the corresponding ideal and whose reduced structure is a fixed line in \( \mathbb{P}^3 \).

For instance, if we choose \( I_X = \langle z_2, z_3 \rangle \subset S := \mathbb{C}[z_0, z_1, z_2, z_3] \), The Cohen-Macaulay monomial ideal \( J := \langle z_2^3, z_2^2z_3, z_3^3 \rangle \) will corresponds to the diagram

The number of boxes being 8, we have that \( J \) is an ideal of a Cohen-Macaulay multiplicity 8 structure on the line \( X \). Remark that the line \( X \) itself corresponds to the box \( \text{[ ]} \).

**Definition 5.2.**

- An inner box of a Young diagram will mean a box not in the diagram but such that the box bellow it and the box in its left are both in the diagram.
- An outer box of a Young diagram will mean a box not in the diagram and such that the box bellow it and the box in its left are both outside the diagram, but its lower left angle touches a box in the diagram.

**Example 5.3.** In the diagram associated to \( J := \langle z_2^3, z_2^2z_3, z_3^3 \rangle \), above, the red box is inner, while the green box is outer.

**Proposition 5.4.** [42, Proposition 4] Given a Cohen-Macaulay monomial ideal \( I \) with support on a line in \( \mathbb{P}^3 \), and its corresponding Young diagram \( T \). Then \( I \) fits in the exact sequence

\[ 0 \to \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(-n_{2j}) \to \bigoplus_i \mathcal{O}_{\mathbb{P}^3}(-n_{1i}) \to I \to 0, \]

where \( n_{1i} \) is the weight of the \( i \)'th inner box and \( n_{2j} \) is the weight of the \( j \)'th inner box, for some chosen indexing \( i \), (resp. \( j \)) of inner boxes (resp. outer boxes) in \( T \).

\(^1\)We only need, for our purpose, this restricted version of the more general result proved by Vatne.
This way, the syzygies correspond to the outer boxes.

**Example 5.5.** For the ideal $I$ corresponding to the Young diagram

```
 +-----+     +-----+
 | 1 1  |     | 2 2  |
 | 1 3  |     | 3 3  |
 | 1 4  |     | 4 4  |
```

one has four inner boxes with weights $n_{11} = 3, n_{12} = 3, n_{13} = 4, n_{14} = 4$, and three outer boxes with weights $n_{21} = 4, n_{22} = 5, n_{23} = 5$. Hence, one gets:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3}(-5)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 2} \to I \to 0,$$

**Theorem 5.6.** For a $\mathbb{T}$–fixed stable 0–instanton pair $(Q, s)$ of charge $c$, the associated scheme $C$ is a multiple structure that corresponds to a Young diagram $T$ of weight $c$. Moreover if the Young diagram is of the form $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k)$ of $c$, then $C$ has Hilbert polynomial

$$\chi_C(m) := \chi(\mathcal{O}_C(m)) = cm + 3c - \sum_{i=1}^{k} \frac{\nu_i(\nu_i + 2i + 1)}{2},$$

and $I_C$ is a smooth point in its Hilbert scheme of closed subschemes of $\mathbb{P}^3$. The dimension of the Hilbert scheme, of subschemes of $\mathbb{P}^3$, at $I_C$ is given by

$$D_{I_C} = \sum_{n_{2j} \geq n_{11}} \left( \frac{n_{2j} - n_{11} + 3}{3} \right) + \sum_{n_{11} \geq n_{2j}} \left( \frac{n_{11} - n_{2j} + 3}{3} \right)
- \sum_{n_{2j} \geq n_{2i}} \left( \frac{n_{2j} - n_{2i} + 3}{3} \right) - \sum_{n_{1j} \leq n_{1i}} \left( \frac{n_{1j} - n_{1i} + 3}{3} \right) + 1.$$

**Proof.** For a $\mathbb{T}$–fixed stable 0–instanton pair $(Q, s)$ the schematic support $C$ of $Q$ should also be invariant. By 3.5, it follows that it is a multiple structure on the unique line that does not intersect the framing line in $\mathbb{P}^3$. Hence its ideal $I_C$ should be generated by monomials. The Hilbert polynomial $\chi_C(m)$ can be computed according to [42, Corollary 2] and using the fact that the weight of a box $(i, j)$ is given by $w_{i, j} = i + j - 2$. The dimension $D_{I_C}$, of the Hilbert scheme of subschemes of $\mathbb{P}^3$ at $C$, follows from [42, Corollary 1].

The above result classifies all scheme theoretic supports of the $\mathbb{T}$–fixed stable 0–instanton pair $(Q, s)$.

**Corollary 5.7.** For a $\mathbb{T}$–fixed stable 0–instanton pair $(Q, s)$ of charge $c$, the associated sheaf $Z := \text{coker}(s)$ has length

$$l_Z = \frac{1}{2} \sum_{i=1}^{k} \nu_i^2 + \sum_{i=1}^{k} i \nu_i - \frac{c}{2},$$

where $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k)$ is the partition of $c$, represented by the Young diagram $T$ associated to the multiple structure $C$.

---

$^1$The Young diagram is associated to a partition $\nu$ of $c$, where the $i'$th column represents the $i'$th part $\nu_i$, $i = 1, \cdots, k$. 
Proof. This follows from Theorem 5.6 and the exact sequence

\[ 0 \to \mathcal{O}_C \to Q \to Z \to 0. \]

\[ \square \]

5.2. Sheaves on multiple structures. After classifying the possible schematic supports of the pair, we will now study the sheaf \( Q \), emanating from the \( T \)-fixed stable pair \((Q, s)\), as a sheaf on the monomial double structure \( C \) defined over the line \( l_0 = (z_2 = z_3 = 0) \). In order to achieve this goal we first recall some results from \( \text{[9, 10]} \).

For \( X = l_0 \subset Y \subset \mathbb{P}^3 \), as in \( \text{§5.1} \) with a filtration \( \text{[13]} \) we say that \( Y \) is primitive if for every \( x \in X \), there exists a surface \( S \) of \( \mathbb{P}^3 \) which is smooth at \( x \) and containing a neighborhood of \( x \) in \( Y \). In this case, \( L = \mathcal{I}_X / \mathcal{I}_Y \) is an invertible sheaf on \( X \) and we have \( \mathcal{I}_x / \mathcal{I}_y = L^i \) for \( 1 \leq i \leq c \). This means that for a point \( x \in X \), there are elements \( z_2, z_3, t \), of the maximal ideal \( m_{X,x} \) of \( x \) in \( \mathcal{O}_{X,x} \), such that their images in \( m_{X,x}/m^2_{X,x} \) form a basis and for all \( 1 \leq i \leq c \) one has \( \mathcal{I}_y = (z_2, z_3^i) \). Let \( F \) be a coherent sheaf over \( Y \).

**Definition 5.8.** The first canonical filtration of \( F \) is the filtration

\[ F_{i+1} = 0 \subset F_1 \subset \cdots \subset F_i \subset F_i \mid X \]

such that, for \( 1 \leq i \leq c \), \( F_{i+1} \) is inductively defined as the kernel of the restriction morphism \( F_i \mid X \to F_i \mid Y \).

In this way one has \( F_i / F_{i+1} = F_i \mid Y \) and \( F / F_{i+1} = F \mid Y \). The graded object \( \text{Gr}(F) = \bigoplus_{i=1}^c F_i / F_{i+1} \) is then an \( \mathcal{O}_X \)–module\( \text{[11]} \).

Some properties of these filtrations can be listed as it follows \( \text{[10, §3]} \):

- For the ideal \( I_X \), of \( X \), in \( \mathcal{O}_Y \) and a coherent sheaf \( F \), over \( Y \), one has \( F_i = I_X^i F \) so that \( \text{Gr}(F) = \bigoplus_{i=0}^{c-1} I_X^i F / I_X^{i+1} F \);
- \( F_i = 0 \) if and only if \( F \) is a sheaf over \( Y \);
- for each \( 0 \leq i \leq c \), \( F_i \) is a coherent sheaf over \( Y \) with first canonical filtration \( 0 \subset F_i \subset \cdots \subset F_i \mid X \subset F_i \);\footnote{A second canonical filtration, that we won’t use, is also defined in \( \text{[9, §4]} \). The interested reader might check the given reference.}
- Morphisms of coherent sheaves \( F \to G \), on \( Y \), induce morphisms of first canonical filtrations \( F_i \to G_i \), for all \( 0 \leq i \leq c \), and hence induce morphisms of the graded objects \( \text{Gr}(F) \to \text{Gr}(G) \).

**Definition 5.9.**

- The generalized rank is defined by the integer \( R(F) = \text{rk}(\text{Gr}(F)) \).
- The generalized degree is defined by the integer \( \text{deg}(F) = \text{deg}(\text{Gr}(F)) \).

The generalized rank and degree are defined so that they behave additively on exact sequence on \( Y \). In general the usual rank and degree fail to satisfy this condition. Moreover we have the following generalized Riemann-Roch Theorem:

**Theorem 5.10.** \( \text{[9, Theorem 4.2.1]} \) For a coherent sheaf \( F \), over \( Y \), we have

\[ \chi(F) = \text{Deg}(F) + R(F)(1 - g_Y). \]

Here, \( g_Y \) is the genus of the curve \( Y \).
5.2.1. **Stable rank 0—instanton pair of charge 1.** In this case the only possible support is the line \( l_0 \), the line that does not intersect the framing line \( l_\infty \). The sheaf \( Q \) sits in the short exact sequence

\[
0 \to O_{l_0} \to Q \to Z \to 0,
\]

where \( Z \) is the structure sheaf of one point. Hence the only possibilities are \( Q = O_{l_0}(p_i), i = 0, 1 \) where \( p_0 = [1; 0; 0; 0] \) and \( p_1 = [0; 1; 0; 0] \). We points out that the rank 2 fixed instanton bundles given by \( \ker(O^2_{P^3} \to O_{l_0}(p_i)) \) are nullcorrelation sheaves \([12]\). Moreover, we see that these \( T \)-fixed points are isolated.

**Corollary 5.11.** The moduli \( M_{P^3}(1) \) has at least two fixed components under the lifted toric action on \( P^3 \).

**Proof.** Since the \( T \)-fixed 0—rank instants can only be \( O_{l_0}(p_i), i = 0, 1 \), one gets two quotient maps

\[
O^2_{P^3} \to O_{l_0}(p_i) \quad i = 0, 1,
\]

with non isomorphic kernels \( E_i, i = 0, 1 \). These are fixed framed instanton sheaves and each one of them comes in a family, hence, we obtain at least two disconnected components.

\[\square\]

The **Conjecture 2** in \([37]\) states that the fixed locus of the stable PT-pairs is smooth. The next result gives the Tangent and Obstruction spaces at the fixed 0—instanton pairs

**Lemma 5.12.** For the stable pairs \( p_i = (Q_i = O_{l_0}(p_i), s) \in P_1(P^3, \beta = H^2)^T \), \( i = 0, 1 \), one has

\[
T_{p_i} P_1(P^3, \beta = H^2)^T = \mathbb{C}^5, \quad \text{Obs}_{p_i} P_1(P^3, \beta = H^2)^T = \mathbb{C}^3.
\]

**Proof.** Recall from \([36]\) that we have a triangle

\[
Q_i[-1] \to I^* \to O_{P^3} \to Q_i
\]

in \( D^b(P^3) \), where \( Q = O_{l_0}(p_i), i = 1, 2 \), and \( I^* := \{O_{P^3} \to Q\} \).

Applying \( \text{Hom}(\cdot, Q) \) on \((15)\), one gets the sequence

\[
\begin{align*}
\text{Ext}^{-1}(I^*, O_{l_0}(p_i)) & \to \text{End}(O_{l_0}(p_i)) \to \text{Ext}^0(O_{P^3}, O_{l_0}(p_i)) \\
\text{Ext}^0(I^*, O_{l_0}(p_i)) & \to \text{Ext}^1(O_{l_0}(p_i), O_{l_0}(p_i)) \to \text{Ext}^1(O_{P^3}, O_{l_0}(p_i)) \\
\text{Ext}^1(I^*, O_{l_0}(p_i)) & \to \text{Ext}^2(O_{l_0}(p_i), O_{l_0}(p_i))
\end{align*}
\]

where \( \text{Ext}^{-1}(I^*, O_{l_0}(p_i)) = 0 \), as in the proof of \([36]\ Lemma 1.5\). Observe that \( \chi(I^*, O_{l_0}(p_i)) = 2 \), and \( \chi(O_{l_0}(p_i), O_{l_0}(p_i)) = 4 \), by Hirzebruch-Riemann-Roch Theorem. One can easily compute that

\[
\text{End}(O_{l_0}(p_i)) = \mathbb{C}, \quad \text{Ext}^1(O_{l_0}(p_i), O_{l_0}(p_i)) = \mathbb{C}^4, \quad \text{Ext}^2(O_{l_0}(p_i), O_{l_0}(p_i)) = \mathbb{C}^3,
\]

and also

\[
\text{Ext}^1(O_{P^3}, O_{l_0}(p_i)) \cong H^1(O_{l_0}(p_i)) = 0, \quad \text{Hom}(O_{P^3}, O_{l_0}(p_i)) \cong H^0(O_{l_0}(p_i)) = \mathbb{C}.
\]
From (16), it follows that
\( \text{Ext}^1(I^{\bullet}, O_{l_0}(p_i)) \cong \text{Ext}^2(O_{l_0}(p_i), O_{l_0}(p_i)) \cong \mathbb{C}^3 \) and \( \text{Ext}^0(I^{\bullet}, O_{l_0}(p_i)) \cong \mathbb{C}^5 \)

\[ 5.2.2. \text{ Stable rank 0 – instanton pair of charge 2.} \]

For a T–fixed stable rank 0–instanton pair \((Q, s)\) of charge 2, the associated Cohen-Macaulay curve \(C\) is a primitive double curve with ideal generated by monomials, hence one can associate to it one of the following Young diagrams:

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\end{array}
\]

We will only treat the case \[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\end{array}
\]
the other case being very similar; The ideal sheaf of \(I\), of \(C\), in \(O_{\mathbb{P}^3}\) is \(I_C = \langle z_2, z_3 \rangle\), and \(C\) is clearly a complete intersection. Moreover, it is easy to see that we have
\[
0 \rightarrow O_{\mathbb{P}^3}(-3) \rightarrow O_{\mathbb{P}^3}(-2) \oplus O_{\mathbb{P}^3}(-1) \rightarrow I_C \rightarrow 0,
\]
with Hilbert polynomial \(\chi(m) = 2m - 1\), so that \(l_Z = 3\). Using [35, Lemma 1.3] one has a sequence
\[
0 \rightarrow I_C \rightarrow I_{l_0} \rightarrow L \cong O_{l_0}(-1) \rightarrow 0,
\]
hence the restriction sequence
\[
0 \rightarrow O_{l_0}(-1) \rightarrow O_C \rightarrow O_{l_0} \rightarrow 0.
\]
Thus the first canonical filtration of \(O_C\) is simply \(0 \subset O_{l_0}(-1) \subset O_C\), and the graded sheaf associated to it is \(\text{Gr}(O_C) = O_{l_0} \oplus O_{l_0}(-1)\). This gives the generalized rank and degree, respectively, \(R(O_C) = 2\), \(Deg(O_C) = -1\).

\(Q\) has first canonical filtration \(0 \subset Q_2 \subset Q\) with a graded object \(\text{Gr}(Q) = Q|_{l_0} \oplus Q_2\). Since \(Q\) is generically isomorphic to \(O_C\), and \(Z\) is also supported on \(l_0\), one obtains the diagram

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
0 \\
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
0 \\
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
0 \\
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
0 \\
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
0 \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
0 \rightarrow O_{l_0}(-1) \rightarrow O_C \rightarrow O_{l_0} \rightarrow 0
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
0 \rightarrow Q_2 \rightarrow Q \rightarrow Q|_{l_0} \rightarrow 0
\end{array}
\end{array} 
\begin{array}{c}
\begin{array}{c}
Z \rightarrow Z \rightarrow 0
\end{array}
\end{array}
\]

Thus \(Q_2 = O_{l_0}(-1)\), the generalized rank and degree of \(Q\) are, respectively, \(R(Q) = 2\) and \(Deg(Q) = 2\). This leaves us with the following possibility:

**Theorem 5.13.** \(Q|_{l_0} \cong O_{l_0} \oplus Z\).

**Proof.** Torsion free sheaves of generalized rank 2 on the double line are of three types [35, §8.2], namely line bundles, vector bundles on \(l_0\) and the ones which are not locally free.

If \(Q\) is a vector bundle on \(l_0\) then it is equal to its restriction, which contradicts the diagram (5.2.2).
If $Q$ is a line bundle on $C$, then its restriction is the line bundle $O_{l_0}(3)$, which is the only possibility compatible with the right column in (5.2.2). On the other hand $Q$ fits in the following exact sequence

$$0 \to D \otimes O_{l_0}(-1) \to Q \to D \to 0$$

where $D = O_{l_0}(3)$. But this means that $O_{l_0}(3) \otimes O_{l_0}(-1) \cong O_{l_0}(-1)$. Hence, $Q$ cannot be a line bundle on $C$.

In a more general situation $Q$ fits in a short exact sequence

$$0 \to D \otimes O_{l_0}(-1) \to Q \to D \oplus T \to 0,$$

where $D$ is a line bundle on $l_0$ and $T$ is a torsion sheaf, also on $l_0$. One can compare this sequence to the middle row in (5.2.2). Then, by calculating the generalized rank and degree, one finds that the only allowed case is $D = O_{l_0}$ and $T = \mathbb{Z}$.

Corollary 5.14. $\mathcal{M}_{P^3}^T(2)$ has at least 36 irreducible components.

Proof. From Theorem 5.6 and as we saw in the beginning of this section, there are two possible Young diagrams for the support. To each one of these curves there are 18 possibilities for the sheaf $\mathcal{Z}$; these are all possible ideal sheaves, fixed by the torus action of $T^3$, and supported on $p_0 = [1;0;0;0]$ or / and $p_1 = [0;1;0;0]$, in $l_0 \subset P^3$. These are given by plane partitions: 6 plane partitions for all three points supported on either $p_0$ or $p_1$, and 3 possibilities for each case in which, two points are supported on one of them and 1 point on the other.

Finally, by Theorem 5.13 and the fact that $Q$ might have non trivial deformation in each case, then the result follow.

We remark that the double curve $C$ can be deformed into two lines intersecting in a point, but we don’t know if one can deform the 0−rank instanton sheaves $Q$ into torsion free sheaves on the reducible curve formed by two intersecting lines. This is a hard problem and should be investigated in the future.

For a given integer $m$, we now denote by $PL(m)$ the number of its $(3−$dimensional) plane partitions and by $P(m)$ the number of its $(2−$dimensional) partitions. We recall that $l_Z(\nu(c))$ denotes the length of $Z$, for the multiple structure associated to a partition $\nu(c)$ of $c$.

If we consider the whole set of monomial multiple structures, not only the primitive ones, then we get the following generalization, of the above Corollary.

Lemma 5.15. The moduli space $\mathcal{M}_{P^3}^T(c)$ splits as a union

$$\mathcal{M}_{P^3}^T(c) = \bigcup_{\nu(c)} \bigcup_{a+b=l_Z(\nu(c))} \mathcal{M}(\nu(c), a, b).$$

Moreover the number of such components is $\mathcal{M}_{P^3}^T(c)$ is at least

$$P(c) \times \sum_{a+b=l_Z} PL(a)PL(b).$$
Proof. This is obtained by enumerating the possible Young diagrams, hence partitions \( \nu(c) \) of \( c \). Then for each one of them, the number of possible distributions of \( l(\mathcal{Z}) \) points on the supports \( p_0 = [1; 0; 0; 0] \) or / and \( p_1 = [0; 1; 0; 0] \), in \( \mathbb{P}^3 \). This is the least number of components because, in each such case, one can have non-trivial families, with possibly many irreducible components, of sheaves \( \mathcal{Q} \). Hence, components in \( \mathcal{M}_{\mathbb{P}^3}(c) \), of torsion-free sheaves \( E \) corresponding to the middle row of (13), will arise according to the combinatorial type of quotients \( \mathcal{Q} \). □

We remark that \( \mathcal{M}(\nu(c), a, b) \cap \mathcal{M}(\mu(c), a', b') = \emptyset \) for \( \nu(c) \neq \mu(c) \), \( a \neq a' \) or \( b \neq b' \). So they are completely disconnected components.

5.2.3. Stable rank 0—instanton pair of charge \( c \) with primitive support. We now describe the case in which the support is a primitive multiple line. For the pair \((\mathcal{Q}, s)\) of charge \( c \), the associated Cohen-Macaulay curve \( \mathcal{C} \) is a primitive multiple curve with ideal whom associated Young diagram is a column or a line. As in the last section we treat the case .

This time we have \( \mathcal{I}_c = \langle z_2, z_3^c \rangle \); also, for which \( \mathcal{C} \) is a complete intersection. Its resolution is

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-c-1) \to \mathcal{O}_{\mathbb{P}^3}(-c) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{I}_c \to 0,
\]

with Hilbert polynomial \( \chi(m) = cm - \frac{c(c-3)}{2} \), and length \( l_\mathcal{Z} = \frac{c(c+1)}{2} \).

The canonical filtration of supports is represented by:

\[
\begin{array}{cccc}
\mathcal{I}_0 & \supset & \mathcal{I}_2 & \cdots & \supset & \mathcal{I}_{c-1} & \supset & \mathcal{I}_c \\
\begin{array}{c}
\square \\
\mathcal{C}_0 \\
\mathcal{C}_2 \\
\vdots \\
\mathcal{C}_{c-1} \\
\mathcal{C}
\end{array}
\end{array}
\]

and we have sequences:

\[
0 \to \mathcal{I}_{c_2} \to \mathcal{I}_0 \to \mathcal{L} \cong \mathcal{O}_{\mathcal{I}_0}(-1) \to 0
\]

and

\[
0 \to \mathcal{I}_{c_{i+1}} \to \mathcal{I}_{c_i} \to \mathcal{L} \cong \mathcal{O}_{\mathcal{I}_0}(-1)^{\otimes i} \to 0,
\]

hence restrictions sequence

\[
0 \to \mathcal{O}_{\mathcal{I}_0}(-1) \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{I}_0} \to 0;
\]

\[
0 \to \mathcal{O}_{\mathcal{I}_0}(-i) \to \mathcal{O}_{\mathcal{C}_{i+1}} \to \mathcal{O}_{\mathcal{C}_{i}} \to 0,
\]

For \( 1 \leq i \leq c-1 \).

On the other hand, the first canonical filtration of \( \mathcal{O}_c \) reads as

\[
\mathcal{L}_{c+1} = 0 \subset \mathcal{L}_{c} \subset \cdots \subset \mathcal{L}_{2} \subset \mathcal{O}_c,
\]

where \( \mathcal{O}_c/\mathcal{L}_{i+1} \cong \mathcal{O}_{\mathcal{C}_i} \).
Lemma 5.16. The graded sheaf, the generalized degree and the generalized rank of $\mathcal{O}_C$ are given, respectively, by:

\[
\text{Gr}(\mathcal{O}_C) = \bigoplus_{i=0}^{c-1} \mathcal{O}_{l_0}(-i), \quad \text{Deg}(\mathcal{O}_C) = -\frac{c(c-1)}{2} \quad \text{and} \quad R(\mathcal{O}_C) = c.
\]

Proof. By using diagrams

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
L_{i+1} & L_{i+1} \\
\downarrow & \downarrow \\
0 & L_i \\
\downarrow & \downarrow \\
\mathcal{O}_C & \mathcal{O}_C \\
\downarrow & \downarrow \\
0 & \mathcal{O}_{l_0}(-i + 1) \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

one gets the graded sheaf. The generalized degree and rank follow easily by applying their definitions.

By Theorem 5.10 one gets $\chi(\mathcal{O}_C(m)) = cm - \frac{c(c-1)}{2}$. Thus the graded sheaf associated to $\mathcal{O}_C$ is $\text{Gr}(\mathcal{O}_C) = \bigoplus_{i=0}^{c-1} \mathcal{O}_{l_0}(-i)$, and the generalized rank and degree are, respectively, $R(\mathcal{O}_C) = c$, $\text{Deg}(\mathcal{O}_C) = -\frac{c(c-1)}{2}$.

$\mathcal{Q}$ has first canonical filtration $\mathcal{Q}_{c+1} = 0 \subset \mathcal{Q}_c \subset \cdots \subset \mathcal{Q}_2 \subset \mathcal{Q}$. The sheaf $\mathcal{Q}$ is generically isomorphic to $\mathcal{O}_C$, and, again, $\mathcal{Z}$ is also supported on $l_0$. Hence, one has a commutative diagram

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \mathcal{O}_{l_1+1} \\
\downarrow & \downarrow \\
\mathcal{Q}_1 & \mathcal{Q}_{l_1} \\
\downarrow & \downarrow \\
\mathcal{Q}_1 & 0 \\
\downarrow & \downarrow \\
\mathcal{Z} & 0 \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

By induction, it follows that $\mathcal{Q}_{i+1} = L_{i+1}$, for each $2 \leq i \leq c + 1$. Then the generalized rank and degree of $\mathcal{Q}$ are, respectively, $R(\mathcal{Q}) = c$ and $\text{Deg}(\mathcal{Q}) = c$.

Applying Lemma 5.15, it follows that the number of irreducible components of $\mathcal{M}_{\text{P}(c)}(c)$, with primitive support $\mathcal{C}$, is at least

$$2 \sum_{a+b=\frac{c(c+1)}{2}} \text{PL}(a)\text{PL}(b).$$

We remark that we do not know whether, or not, the above fixed components intersect the closure of the framed locally free instanton moduli. We think that
this problem is related to *reachability* of sheaves, on multiple structure \([11]\). Nevertheless, for charge \(c = 1\), the answer is positive; the sheaf \(\ker(O^{\oplus 2}_{\mathbb{P}^3} \to \mathcal{Q})\), in \([27, \S 6]\), is in the closure of the moduli of locally free framed instanton bundles \([27, \S 6]\). For higher values of the charge, this is a difficult problem to answer. For instance, if \(c = 2\) one can deform the (monomial) double curve into a union of two curves intersecting at a point. But we don’t know yet how to deform the 0-instanton sheaf, sitting on this double curve, to a sheaf on the reduced curve. We close this notes by writing

**Conjecture.** The fixed components, under the lifted toric action on \(\mathbb{P}^3\), intersect the closure of the locally free component in the moduli space of framed instantons.

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