Noisy One-Way Quantum Computations: The Role of Correlations

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A scheme to evaluate computation fidelities within the one-way model is developed and explored to understand the role of correlations in the quality of noisy quantum computations. The formalism is promptly applied to many computation instances, and unveils that a higher amount of entanglement in the noisy resource state does not necessarily imply a better computation.

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I. INTRODUCTION

Since the advent of quantum computation [1], entanglement, a clear-cut quantum mechanical feature, is commonly believed the key resource behind it. Not surprisingly, the importance of correlations for quantum computations has been a much debated subject. For pure state quantum computations certainly some entanglement is necessary if the quantum protocol is not to be efficiently simulated by classical means [2 3]. However, entanglement is only necessary, but not a sufficient condition for an exponential gain of quantum computations over classical ones. There are quantum protocols that despite of producing highly entangled states can still be efficiently simulated classically [2 3].

The scenario for mixed state quantum computations is far more subtle [2]. A model for mixed state quantum computation introduced in [5], in which the input state consists of a single qubit in a pure state and all the others in a uniform incoherent sum of classical alternatives – and therefore not entangled–, offers an exponential speed-up to problems that are believed intractable by classical computers [6]. Also, room temperature NMR implementations of quantum information tasks [7] which employ a rather noisy state where entanglement is known not to be present [8], seem to still present gain over classical computations [9]. Nevertheless, in these cases, generation of entanglement during the computation itself cannot be ruled out [6]. A definitive statement about the influence of entanglement is thus challenging.

A clean investigation of the entanglement role in noisy quantum computations is however possible within the One-Way model [10]. In this model, local (projective) measurements on a highly entangled resource state are responsible for input preparation, the required computation and final read-out. No entanglement is created during the computation. We have thus a clear distinction between the entanglement creation and its use as a resource.

Employing this model of computation we address here still another facet of the entanglement role in noisy quantum computations: How does the noise affecting the entangled resource state impact on the “quality” (fidelity) of the computation? Does a more entangled resource state always empower better computations? Or in more practical terms: should one always try to minimize the influence of the environment over the entanglement such as to maximize the fidelity of a computation? To answer in the negative to these questions, we derive an expression for the fidelity of any one-way computation when the resource state undergoes various types of decoherence. Our results extend to noisy computations the assertion [11] that a high entangled state is not always advantageous for a measurement based quantum computation.

This article is organized as follows: In Sec. II we briefly review the one-way model for quantum computations. In Sec. III we discuss the effects of various models of decoherence in one-way computations, and derive the expression for the fidelity in such cases. In Sec. IV we apply the developed formalism to various computation instances, showing that a higher amount of entanglement in the noisy resource state does not necessarily imply better computations. In particular we obtain that some instances of the Deutsch-Jozsa [12] algorithm, which in the circuit model of quantum computation generically create entangled states [13], within the one-way framework require no entanglement for their execution. In Sec. V we analyze the effects of decoherence in the ancilla-driven quantum computation proposed in Ref. [14]. In Sec. VI we summarize our results, and draw some conclusions.

II. ONE-WAY MODEL OF QUANTUM COMPUTATION

In the one-way model [10] all the interactions between qubits, and local unitary transformations needed by the protocol are exchanged by a prior entanglement in a graph-state, and the possibility to make adaptive local measurements. A graph-state is defined by a set of vertices $V$, and a collection of edges $E$. In each vertex $i$ sits...
a qubit \( (\mathcal{H}_2) \) initialized in a state \( |+i\rangle = (|0_i\rangle + |1_i\rangle)/\sqrt{2} \), and an edge represents an interaction between vertices \( \{i,j\} \) given by \( C Z_{ij} = |0_i\rangle\langle 0_j| \otimes |1_i\rangle\langle 1_j| \). Hereafter \( \{1, X, Y, Z\} \) will represent the usual \( \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \) Pauli matrices. The N-qubit graph state is then

\[
|G_{(\mathcal{V}, \mathcal{E})}\rangle = \prod_{\{i,j\} \in \mathcal{E}} C Z_{ij} |+k\rangle^{\otimes |\mathcal{V}|}. \tag{1}
\]

Starting with a N-qubit graph state, an one-way computation is carried out by measuring M qubits, and the remaining \( N-M \) qubits encode the protocol answer. The algorithm is then defined by a triple \( \{(\theta_i, \alpha_i, s_i)\} \), with instructions on the measurement basis \( |M_{\theta_i}^{\alpha_i}(\theta_i, \alpha_i)\rangle \) for the \( i \)-th qubit. Here

\[
|M_{\theta_i}^{\alpha_i}(\theta_i, \alpha_i)\rangle = \cos \frac{\alpha_i}{2} |0\rangle + \sin \frac{\alpha_i}{2} e^{-i(1)^r \phi_i} |1\rangle \tag{2}
\]

\[
|M_{\theta_i}^{\alpha_i}(\theta_i, \alpha_i)\rangle = \sin \frac{\alpha_i}{2} |0\rangle - \cos \frac{\alpha_i}{2} e^{-i(1)^r \phi_i} |1\rangle \tag{1}
\]

with \( s_i \equiv s_i(\vec{k}) = s_i(k_1, \ldots, k_M) \in \{0, 1\} \) the adaptation parameter that depends on the outcome \( k_j \in \{0, 1\} \) of previous measurements and implicitly on the algorithm being considered (hereafter, the notation \( \vec{x} \) represents the M-tuple \( x_1, \ldots, x_M \)). The need for adaptations stems from the requirement of turning every computation deterministic, despite of the intrinsic randomness associated with each quantum measurement. Adaptations introduce a temporal order for the measurements, and thus classical correlations others than the already present in the initial state. The only two instances of \( \alpha_i \) necessary for all computations are: \( \alpha_1 = 0 \) for measurements along the z direction, and \( \alpha_1 = \pi/2 \) for measurements in x-y plane (equator) of the Bloch sphere. The measurements are non-adaptive when the basis are given by the Pauli operators \( \{X, Y, Z\} \) eigenvectors, and possibly adaptive otherwise. Lastly, the desired answer is given aside some local unitary transformations, final by-products (BP) of the computation, determined by the classical outcomes \( \vec{k} \) of all measured qubits. These by-products can be dealt with by classical post-processing.

Defined a computation to be performed on a graph state \( |G_{(\mathcal{V}, \mathcal{E})}\rangle \), the M qubits to be measured can be immediately written in their measurement basis (adaptations included) what explicit the answer:

\[
|G_{(\mathcal{V}, \mathcal{E})}\rangle = \frac{1}{2^M} \sum_{i=1}^{M} |M_{\theta_i}^{\alpha_i}(\theta_i, \alpha_i)\rangle |A_{\vec{\theta}}(\vec{\alpha})\rangle, \tag{3}
\]

with \( |A_{\vec{\theta}}(\vec{\alpha})\rangle = B P_{\vec{k}} |A_{\vec{\theta}}(\vec{\alpha})\rangle \), and \( |A_{\vec{\theta}}(\vec{\alpha})\rangle \) the desired answer without the by-products. The latter, in turn, can be expressed as local unitaries

\[
B P_{\vec{k}} = \bigotimes_{i=N-M+1}^{N} (-1)^{f_i(s_i(\vec{k}))} X_i f_i(X(\vec{k})) Z_i f_i(Z(\vec{k})), \tag{4}
\]

with \( f_{Sig}, f_X \) and \( f_Z \) boolean functions of the outcomes \( \vec{k} \) defined by the protocol at hands. In a noise free computation, the sign boolean function \( f_{Sig} \) only introduces a global phase on each answer, and it is then of no importance. However, under the action of the environment, some of the answers will be mixed, and this phase turns a relative one. As such, it cannot be neglected. It is important to notice that out of \( 2^M \) possible measurement outcomes in a given protocol, only \( 4^{N-M} \) of them possibly lead to different answers (modulo a global phase), meaning that in general many of answers are the same.

### III. FIDELITY OF NOISY ONE-WAY QUANTUM COMPUTATIONS

Once hardware (graph state) and algorithm (measurement basis + possible adaptations) are defined, we want to gauge how does the quality of a computation decay due to different noisy environments, and how this is related to the decay of the initial entanglement resource.

The standard measure for the computation quality is the output fidelity \( F \) \[15-20\]. This measure compares the desired state, \(|\Psi_{out}\rangle\), with the actually obtained one, \(|\psi_{out}\rangle\), returning \( F(|\Psi_{out}\rangle, |\psi_{out}\rangle) \equiv \langle \Psi_{out}|\psi_{out}|\Psi_{out}\rangle \). The fidelity measure is also used to define the error threshold for quantum computations \[21\].

In what follows, we consider that each qubit of the initial graph state is individually coupled to its own environment. A great variety of single qubit open dynamics are encompassed by the map

\[
A_k(\bullet) = \sum_{j=0}^{3} \lambda_j^k(t) \sigma_j \cdot \sigma_j + \mu_j^k(t) \left[ -\sigma_3 \sigma_3 + \sigma_1 \sigma_2 + i\sigma_2 \sigma_1 \right] \tag{5}
\]

with \( \lambda_j^k(t) = (1 + 2e^{-C_k t} + e^{-B_k t})/4 \), \( \lambda_j^0(t) = (1 - e^{-C_k t})/4 \), \( \lambda_j^1(t) = (1 - e^{-B_k t})/4 \), \( \lambda_j^2(t) = (1 - e^{-B_k t})/4 \), and \( \mu_j^k(t) = (2S_k - 1) (1 - e^{-B_k t})/4 \). For the k-th qubit, the parameters \( B_k \) and \( C_k \) are, respectively, the decay rate of inversion and polarization, and \( S_k \in [0, 1] \) depends on the temperature of the bath \[22\].

The decohered state after time \( t \) is obtained by the composition of the individual evolutions \( A_k \), i.e., \( \rho(t) = \Lambda(G_{(\mathcal{V}, \mathcal{E})})\langle G_{(\mathcal{V}, \mathcal{E})}\rangle = A_1 \otimes \ldots \otimes A_N(G_{(\mathcal{V}, \mathcal{E})})\langle G_{(\mathcal{V}, \mathcal{E})}\rangle \). The assumption of mutually independent environments is well justified whenever the separation between the vertices is large enough so that collective effects need not be taken into account.

To evaluate the computation fidelity after the noisy evolution, we note that, once defined a measurement basis, only the diagonal terms in such basis are relevant to the measurement outcome. The action of the channel \( \Lambda \) on a general measurement basis term \( |M_k\rangle \langle M_k| \) is such that non-diagonal \( |k_i \neq k_i'| \) terms are mapped onto non-diagonal terms, while diagonal \( |k_i = k_i'| \) terms evolve as

\[
|M_k\rangle \langle M_k| \rightarrow (1 - p_i) |M_k\rangle \langle M_k| + p_i |M_{k_i \oplus 1}\rangle \langle M_{k_i \oplus 1}| + \text{off diagonal terms}; \tag{6}
\]
where henceforth we use a simplified notation whenever ambiguities are not possible.

In the equation above, \( p_i \) is a function of time, and of the parameters describing the channel \([5]\). In the case of a measurement in \( x-y \) plane

\[
p_{x-y}^i = \lambda_i^1(t) + \lambda_i^2(t),
\]

(7)

and for a measurement on the \( z \) direction it reads

\[
p_{z}^i = 2\lambda_i^1(t) + (-1)^{k_i+1}2\mu_i^x(t).
\]

(8)

As an example, consider the following two particular noise instances (which will be used later):

i) phase-flip error (pf) – with a probability \( p_{pf}/2 \) the state \( |0\rangle + |1\rangle \) is mapped onto \( |0\rangle - |1\rangle \) and vice versa. This is obtained by setting \( B_i = 0 \) and \( C_i = 2\Gamma_{pf} \) in \([5]\). Accordingly, the state evolution maps \( \rho_i \rightarrow (1 - p_{pf}/2)\rho_i + p_{pf}(Z\rho_i Z)/2 \), with \( p_{pf} = [1 - \exp(-2\Gamma_{pf}t)] \).

The impact of the decoherence on the measurement basis is then given by \([7]\) and \([8]\)

\[
p_{x-y} \rho_{pf} = p_{pf}/2,
\]

\[
p_{z} \rho_{pf} = 0.
\]

(9)

ii) white noise (w) – add to the previous case the possibility of errors into the other independent directions \( x \) and \( y \). This amounts to exchange the initial state with a maximally mixed one with probability \( p_{w} \). This is described by setting \( S_i = 1/2 \) and \( B_i = C_i = 4\Gamma_{w} \) in \([5]\). Under this dynamics, the state evolves to \( (1 - p_{w})\rho_i + p_{w}\mathbb{I}/2 \), with \( p_{w} = 1 - \exp(-4\Gamma_{w}t) \). In this case, we have

\[
p_{x-y} \rho_{w} = p_{w}/2.
\]

(10)

It is important to notice that depending on the required measurement by an algorithm, some noisy maps might have no effect, and the corresponding measurement outcomes are thus not disturbed. This feature will be further exploited in section III A.

Note also that these decoherence processes can be interpreted as if the performed measurement was not perfect, being unable to distinguish between the two possible outcomes with probability \( p_i \).

Now we are set to evaluate the fidelity of any one-way noisy quantum computation. Given a result \( \vec{r} \) for the measurements, we want to determine \( F(|A_{\vec{r}}(\vec{\theta},\vec{\alpha})\rangle,|a_{\vec{r}}\rangle) \), with \( \vec{a}_{\vec{r}} \) the \((N-M)\) qubit state encoding the noisy protocol answer. We are thus interested on the projection of \( \Lambda(|G_{(\vec{r},\vec{z})}\rangle|G_{(\vec{r},\vec{z})}\rangle) \) onto \( \otimes_{i=1}^{M} |M_{s_i}^{(r)}\rangle\langle M_{s_i}^{(r)}| \).

However, since \( M_{s_i}^{(r)}|M_{s_i}^{(k_i)}\rangle \neq 0 \) in general, we first note that

\[
|M_{s_i}^{(k_i)}\rangle = \frac{1}{2} \left\{ |1 + (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|M_{s_i}^{(k_i)}\rangle + |1 - (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|M_{s_i}^{(k_i)}\rangle \right\},
\]

(11)

and therefore, the state in \([3]\) can be rewritten as:

\[
|G\rangle = \frac{1}{2\sqrt{M}} \sum_{\vec{k},\vec{l}=1}^{M} \left\{ |1 + (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|M_{s_i}^{(k_i)}\rangle + |1 - (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|M_{s_i}^{(k_i)}\rangle \right\} |A_{\vec{k}}\rangle.
\]

(12)

In the last two expressions above we used that when the measure is in the \( z \) direction there is no adaptation, i.e., \( s_i = 0 \).

Due to the noise, the components \( |M_{s_i}^{(r)}\rangle\langle M_{s_i}^{(r)}| \) and \( |M_{s_i}^{(r)}\rangle\langle M_{s_i}^{(r)}| \) mix according to the prescription in Eq.(6), reducing the fidelity of the protocol. The state after the action of the noise reads:

\[
\frac{1}{2\sqrt{M}} \sum_{\vec{k},\vec{l}=1}^{M} \left\{ |1 + (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|1 + (-1)^{l_i}e^{2i\theta_i}(s_i(\vec{l})\oplus s_i(\vec{r}))\rangle + |1 - (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|1 - (-1)^{l_i}e^{2i\theta_i}(s_i(\vec{l})\oplus s_i(\vec{r}))\rangle \right\} \Lambda(|A_{\vec{k}}\rangle\langle A_{\vec{l}}|).
\]

(13)

The projection onto the subspace corresponding to \( \vec{r} \) leads the remaining \( N-M \) qubits in the state:

\[
\theta_{\vec{r}} = \frac{1}{2\sqrt{M}} \sum_{\vec{k},\vec{l}=1}^{M} \left\{ (1 - p_i)|1 + (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|1 + (-1)^{l_i}e^{2i\theta_i}(s_i(\vec{l})\oplus s_i(\vec{r}))\rangle + p_i|1 - (-1)^{k_i}e^{-2i\theta_i}(s_i(\vec{k})\oplus s_i(\vec{r}))\rangle|1 - (-1)^{l_i}e^{2i\theta_i}(s_i(\vec{l})\oplus s_i(\vec{r}))\rangle \right\} \Lambda(|A_{\vec{k}}\rangle\langle A_{\vec{l}}|);
\]

(14)

with \( Z_{\vec{r}} \) the probability of obtaining the outcome \( \vec{r} \). Note that, in the above expression, the channels acting in each qubit can be different, and might influence the state for different time intervals. This is an important feature, for the qubits might be of different nature, and could be measured at different times.
We are thus in position to evaluate the computation fidelity \( F_{\text{OneWay}}(\vec{r}) = \langle A_\vec{r} | \tilde{G} | A_\vec{r} \rangle \) for the decohered graph state, to get:

\[
F_{\text{OneWay}}(\vec{r}) = \frac{1}{2^M Z_{\vec{r}}} \sum_{\vec{k}, \vec{l}} \prod_{i} \left( (1 - p_i) [1 + (-1)^{s_i + s_i} e^{-2i\theta_i} (s_k(\vec{k}) \otimes s_l(\vec{l})) (1 + (-1)^{s_k + s_l} e^{2i\theta_i} (s_k(\vec{k}) \otimes s_l(\vec{l})) ] ] \right) A_{\vec{k}, \vec{l}}
\]

with \( A_{\vec{k}, \vec{l}} = \langle A_\vec{r} | \Lambda(\{A_{\vec{k}}\}) | A_\vec{r} \rangle \).

Instead of looking for the fidelity of a particular outcome \( \vec{r} \) of the computation, one is often more interested on the average fidelity over all the outcomes, simply given by

\[
F_{\text{OneWay}} = \sum_{\vec{r}} Z_{\vec{r}} F_{\text{OneWay}}(\vec{r}).
\]

With this expression in hands and the results in \([23]\), which allow for the evaluation of noisy graph-state entanglement, one can compare the dynamics of entanglement in the state used as resource with the fidelity dynamics of any noisy one-way computation. Furthermore, from the expression (15) one can immediately infer that the computation fidelity shows a continuous decay in time – as each \( p_i \) is continuous function of time. That is in clear contrast with a generic entanglement evolution, where the amount of entanglement may vanish in finite time \([24]\). As noted in \([17]\) (considering the particular effects of individual decoherence in a four-qubit cluster state), the finite time disentanglement does not cause changes in the behaviour of the computation fidelity. Entanglement decay is shown here, in general generality, to not display a one-to-one correlation with computation quality.

### A. Computations without adaptations

Computations that need no adaptations (NA) turn out to be a very interesting subset of all possible computations. If a computation requires no adaptations, \( s_i(\vec{k}) = 0 \) for all \( \vec{k} \), then all the outcomes happen with the same probability, \( Z_{\vec{r}} = 1/2^M \) for all \( \vec{r} \). In this case, the expression (15) simplifies to:

\[
F_{\text{OneWay}}^{\text{NA}}(\vec{r}) = \prod_{\vec{k}} \left( (1 - p_i) [1 + \exp(-i\theta_i) \cdot \vec{r}_i \cdot \hat{1}]_\vec{k} \right),
\]

Expression (17) shows that the overall effect of quite general decoherence processes, as parametrized in (5), over the non-adaptive measured qubits is to incoherently combine all the possible noisy answers associated with such measurements. The sign boolean function \( f_{\text{Sig}} \) in the definition of the by-products \( \vec{r} \) can, also in such cases, be safely ignored.

Further simplification is possible for non-adaptive computations when the channel acting on the answer qubits is given by a Pauli map \( \Gamma \), obtained setting \( \mu(t) = 0 \) in the general expression (6). In this case

\[
\Gamma \left( BP_{\vec{k}} | A_\vec{r} \rangle \langle A_\vec{r} | BP_{\vec{k}} \right) = BP_{\vec{k}} \Gamma \left( | A_\vec{r} \rangle \langle A_\vec{r} | BP_{\vec{k}} \right)
\]

and

\[
A_{\vec{k}, \vec{l}} = A_{\vec{k}, \vec{l}} \Gamma(\vec{r}),
\]

which implies that \( F_{\text{OneWay}}^{\text{NA}}(\vec{r}) = F_{\text{OneWay}}^{\text{NA}}(\vec{r}) \). The average fidelity (16) is then the same as the fidelity of any outcome, tremendously simplifying its evaluation.

A greater insight on the nature of correlations necessary for an one-way computation without adaptations is possible when considering noise of the form

\[
\Lambda_j(\bullet) = (1 - p_j) \bullet + p_j R_{\vec{r}_j}(\phi_j) \bullet R_{\vec{r}_j}^\dagger(\phi_j)
\]

with \( R_{\vec{r}_j}(\phi_j) = \exp(-i\phi_j \cdot \hat{\vec{n}}_j / 2) \) a rotation around the axis \( \hat{\vec{n}}_j \) in the Bloch sphere. This map has two invariant states, the eigenvectors \( |\vec{n}_j \cdot 1(\phi_j)\rangle \) of the rotation.

Now remember that a measurement in a certain basis is to be done directly on it, or by first applying a unitary transformation to a convenient basis and then the measurement. It stands for a simple relabelling. Therefore, for all non-adaptive measurements (NAMs), if the noise is of the type in (15), it is possible to find \( U(\phi_j, \theta_j, \alpha_j) \) that transforms \( |\vec{n}_j \cdot 1(\phi_j)\rangle \) into \( |\vec{n}_j \cdot k(\phi_j)\rangle \). This perfectly protects the outcome probability distribution of the NAMs, even for highly mixed resource states, as the measurement outcomes won’t be affected. For computations in which all the \( N \) qubits are measured in a non-adaptive fashion, this procedure protects the whole computation, and the graph state can be exchanged by a mixed state with only classical correlations. The noisy computation without adaptations can thus be classically simulated.

To exemplify this, consider the state in which the \( i \)-th qubit is to be measured in the \( X \) basis, namely:

\[
|G_N\rangle = \frac{1}{\sqrt{2^N}} \left( |+i\rangle |A_+\rangle + |-i\rangle |A_-\rangle \right),
\]

with \( |A_\pm\rangle = (1/\sqrt{2}) \left( |G_{N-1}\rangle \pm \bigotimes_{j \in N_i} Z_j |G_{N-1}\rangle \right) \), and \( N_i \) the neighbours of the \( i \)-th qubit. Applying a bit-flip channel \( \Lambda_{\vec{r}}(\bullet) = (1 - p) \bullet + p \vec{X} \cdot \vec{X} \rangle \langle \vec{X} \bullet \rangle \) on the \( i \)-th qubit decreases the entanglement between this qubit and the rest of the graph, while it does not affect the measurement outcome. Therefore, since applying \( X \) on a qubit \( i \) of a graph is equivalent to apply \( Z \)'s in all its neighbors \( N_i \) \([22]\), preparing the state

\[
\frac{1}{2} \left( |G_N\rangle \langle G_N| + \bigotimes_{j \in N_i} Z_j |G_N\rangle \langle G_N| \bigotimes_{j \in N_i} Z_j \right) = \frac{1}{2} (|+i\rangle \langle +i|) |A_+\rangle \langle A_+| + |-i\rangle \langle -i| |-i\rangle \langle -i| |A_-\rangle \langle A_-|)
\]

(20)
which only bears classical correlations between the \(i\)-th qubit and the rest of the graph state, and is invariant under the application of bit flip to the \(i\)-th qubit, leads to the same probability distribution of outcomes for a measurement of the \(i\)-th qubit on the \(X\) basis. Moreover it generates the same answers \(\{|A_\pm\}\), and this procedure can be iterated to the remaining qubits to be measured non-adaptively.

The NAMs are related to the so-called Clifford-group transformations part of a quantum protocol, and as such can be simulated efficiently in a classical computer \([1] [10] [25]\). From the framework here presented, it is clear that the entanglement between the qubits to be measured non-adaptively and the rest of the graph state can be interchanged by simple classical correlations encoded in a mixed state without compromising the computation. When adaptations are necessary this scheme cannot prevail. Adaptive measurements are thus related to the quantum part of the computation, where some resilient entanglement may be of use.

### IV. APPLICATIONS

In the following we apply the formulae developed above to specific examples, and compare the fidelity dynamics of noisy one-way quantum computations to the entanglement decay of the resource state. In general, a mismatch between the two dynamics is found: higher entanglement is not connected with higher quality computations.

#### A. Remote State Preparation (RSP)

Take the simplest possible one-way protocol, i.e. to remotely prepare the single qubit state \(\cos(\phi/2)|0\rangle - i\sin(\phi/2)|1\rangle\) \([15]\). Within the one-way model we start with a graph state of two qubits, \(|G_2\rangle = (|00\rangle + |01\rangle + |10\rangle - |11\rangle)/\sqrt{2}\), and apply a measurement on the first qubit as to produce the desired state on the second one. For this task, one chooses to measure the observable with eigenvectors \(|M_k(\theta, \pi/2)\rangle = |0\rangle + (-1)^k \exp(-i\theta)|1\rangle\) \(/\sqrt{2}\), where \(k \in \{0, 1\}\) represents the two possible measurement outcomes. After the measurement, the second qubit is left in the state \(|\Psi_{out}\rangle = X^k \cos(\phi/2)|0\rangle - i\sin(\phi/2)|1\rangle\) \(/\sqrt{2}\). Apart from the by-product \(BP_k = X^k\), which is known, the desired state is obtained with maximal fidelity. However, quantum computations are prone to errors. Let’s assume, for simplicity, a favourable scenario where only the first qubit (the one to be measured) is subjected to noise. The initial two-qubit state evolves then to \((A_1 \otimes I)(|G_2\rangle|G_2\rangle)\). Since no adaptations are necessary, and the assumption that the second qubit is not under the action of an environment, the mean fidelity for this protocol, as evaluated by \([17]\) reads:

\[
\mathcal{F}_{\text{RSP}} = F_{\text{RSP}}(0) = (1 - p_1)A_{0,0,0} + p_1A_{0,1,1}.
\] (21)

![FIG. 1. (Color online) RSP: Entanglement decay vs. computation fidelity \(\Gamma_w = 0.375 \Gamma_{pf} = \Gamma\). A less entangled state may lead to a higher computation fidelity (the same order is obtained if the negativity \([27]\) is used instead of concurrence). In the white noise case, entanglement vanishes when the fidelity reaches \(2/3\) (horizontal dashed line) \([28]\). Entanglement is thus superfluous for achieving fidelities below this threshold.](image1)

![FIG. 2. (Color online) RSP: Quantum discord decay vs. computation fidelity \(\Gamma_w = 0.57 \Gamma_{pf} = \Gamma\). As for entanglement, a state with less quantum discord may lead to a better quality computation. Quantum discord vanishes only asymptotically.](image2)

For the measurement is on the \(x-y\) plane then \(p_1 = p_{xy}^1\). Furthermore, \(A_{0,0,0} = |\langle A_0|A_0\rangle|^2 = 1\), and \(A_{0,1,1} = |\langle A_0|X|A_0\rangle|^2 = 0\). This leads to \(\mathcal{F}_{\text{RSP}} = (1 - p_{xy}^1)\), which now depends only on the specific nature of the noise acting on the first qubit.

To address the connection between computation fidelity and entanglement, consider the two instances of open system dynamics introduced in Sec. \([10]\)

\(i\) phase-flip error (pf) – the state evolution maps \(|G_2\rangle|G_2\rangle \mapsto (1 - p_{pf}/2)|G_2\rangle|G_2\rangle + p_{pf}/2(Z \otimes 1)|G_2\rangle|G_2\rangle(Z \otimes 1)\), with \(p_{pf} = [1 - \exp(-2\Gamma_{pf}t)]\). The entanglement dynamics of the noisy resource state can be inferred by its concurrence \([29]\). \(C_{\text{RSP}}(t) = \exp(-2\Gamma_{pf}t)\).

Now, applying the state preparation protocol described above to the decohered state, the output fidelity, given
that \( p_{1}^{yw} = p_{pf}/2 \), is \( F_{RSP}^{pf}(t) = [1 + \exp(-2\Gamma_{pf}t)]/2 \). The correlation between decreasing entanglement with decreasing fidelity is as supposed.

ii) white noise (w) – by adding noise to the other independent directions, the resource state evolves to 
\((1 - p_{w})|G_{2}\rangle|G_{2}\rangle + p_{w}1/4\), with \( p_{w} = 1 - \exp(-4\Gamma_{w}t) \). As before, we evaluate the entanglement dynamics via concurrence, \( C_{w}^{RSP}(t) = \max \{0, [3\exp(-4\Gamma_{w}t) - 1]/2 \} \). Finally, the protocol fidelity with white noise, \( p_{1}^{yw} = p_{w}/2 \), reads: \( F_{RSP}^{w}(t) = [1 + \exp(-4\Gamma_{w}t)]/2 \). Once again, a smaller value of entanglement leads to a worst computation.

Nevertheless, a comparison between both situations shows unexpected behaviour (see Fig. 7): the computation fidelity is higher when the entanglement is more fragile against disturbances. Note that even after the entanglement is fully exhausted, the white noise case still outperforms the always entangled phase-flip case. Even in the simplest one-way protocol the entanglement is neither sufficient nor necessary signature of higher quality for the noisy quantum computation.

In fact, this reasoning can be extended to quantum discord, a recently proposed measure of quantum correlations which does not include only entanglement [29]. This measure attracted lots of attention lately, since it seems to pin-point efficient quantum computations even in the apparent absence of entanglement [6]. Quantum discord is defined as \( Q(\rho) = I(\rho) - C(\rho) \), where the mutual quantum information:

\[
I(\rho) = S(\rho_{A}) + S(\rho_{B}) - S(\rho)
\]

is a measure of total correlations, and

\[
C(\rho) = \sup_{\{\Pi_{i}\}} S(\rho_{A}) - S(\rho_{A}^{k}_{A}|\Pi_{B}^{k}_{B})
\]

is a measure of classical correlations, with the supremum taken over all sets of orthogonal projectors \( \{\Pi_{B}^{k}_{B}\} \). As usual, \( S(\rho) = -\text{Tr}(\rho \log \rho) \) denotes the von Neumann entropy, and \( \rho = \text{Tr}_{j \neq i}(\rho) \) with \( i, j = A, B \) is the partial density matrix. For classical systems \( I = C \), thus equivalent definitions for the mutual information, resulting zero quantum discord.

Evaluating the quantum discord, as shown in Ref. [30], for the toy-protocol described above shows that a lower quantum discord may lead to a higher computation fidelity (see Fig. 3). Also quantum discord cannot be employed to signal higher quality in a noisy quantum computation scenario.

As a last possible signature of a higher quality noisy quantum computation we may employ the measure of non-classicality very recently proposed in [31], namely the minimum entanglement potential (MEP) of a given state \( \rho_{A} \). As a result of the following activation protocol (see [31] for details):

i) act with local unitaries \( U_{A} \) in each sub-system of \( \rho_{A} \), any non-classical state becomes entangled (for any choice of \( U_{A} \)) with the ancillary system \( A' \). The minimum entanglement generated across the \( A : A' \) split quantifies the non-classicality of state, that is,

\[
MEP(\rho_{A}) = \min_{U_{A}} E_{A:A'}(\rho'_{A:A'})
\]

where \( \rho'_{A:A'} \) is the system-ancilla state generated in the end of the activation protocol.

Choosing as a measure of entanglement the negativity [27], we can readily compute the MEP for the resource state in the two noise scenarios above mentioned. In these cases we get \( MEP(A^{pf|w}|G_{2}(G_{2})) = (1 - p_{pf|w}) = MEP^{pf|w}_{RSP} \). For the RSP protocol then the relation \( MEP^{pf|w}_{RSP} = 2\Gamma_{RSP}^{pf|w} - 1 \) holds, and therefore a higher MEP implies higher computation quality. At least in this simple example, the non-classicality of the resource state, quantified by the minimum entanglement potential, seems to be correlated to the quality of the computation.

B. Primitives for Universal Quantum Computation

The basic gates for universal quantum computation are a generic single qubit rotation (\( R \)), and a two-qubit controlled operation, say, a controlled-not (\( CNOT \)) gate. How is the fidelity of these basic building blocks related to the entanglement of their resource states?

Any \( U(2) \) rotation can be decomposed into successive rotations over three different angles, known as the Euler angles, as \( R(\phi_{1}, \phi_{2}, \phi_{3}) = R_{y}(\phi_{3})R_{z}(\phi_{2})R_{z}(\phi_{1}) \). Any qubit state can be created out of any other qubit state via this operation. Within the one-way framework this task is implemented via adaptive measurements, in a five qubit cluster state [10]. The measurement pattern and required adaptations are depicted in Fig. 5 a). Given that the input qubit was initially in the state \( |\psi_{in}\rangle \), after a execution of this protocol with outcomes \( \vec{r} \), the output qubit is left on \( BP_{\vec{r}} R(\phi_{1}, \phi_{2}, \phi_{3}) |\psi_{in}\rangle \). The by-products \( BP_{\vec{r}} \) of this computation are defined as in 4 with:

\[
\begin{align*}
f_{5,S_{i}(\vec{k})} &= k_{3}k_{2}; \\
f_{5,X(\vec{k})} &= k_{4} + k_{2}; \\
f_{5,Z(\vec{k})} &= k_{3} + k_{1}.
\end{align*}
\]

For the \( CNOT \) gate a cluster of 15 qubits is necessary, but no adaptations are required [19]. The measurement pattern defining the algorithm is shown in Fig. 5 b). The \( CNOT \) acts on two qubits, called target and control, such that if the control is in the state \( |1\rangle|1\rangle \) the target qubit is flipped and nothing happens otherwise, that is, \( CNOT_{ij} = |0\rangle_{i}|0\rangle_{j} \otimes 1_{j} + |1\rangle_{i}|1\rangle_{j} \otimes X_{j} \). If the initial
The control-target state in the qubits 1 and 9 is $|\chi_{in}\rangle$, the outcome state in the qubits 7 and 15, after measurements with outcome $\vec{r}'$, is $|\chi_{out}\rangle = BP_{\vec{r}'} CNOT |\chi_{in}\rangle$. The by-products of this operation are given by setting:

$$f_{7,X}(\vec{k}) = k_2 + k_4 + k_5 + k_6;$$
$$f_{15,X}(\vec{k}) = k_2 + k_3 + k_8 + k_{10} + k_{12} + k_{14};$$
$$f_{7,Z}(\vec{k}) = k_1 + k_3 + k_4 + k_5 + k_8 + k_9 + k_{11} + 1;$$
$$f_{15,Z}(\vec{k}) = k_9 + k_{11} + k_{13}.$$  

As no adaptations are necessary for this gate, we can ignore the boolean function $f_{Sig}$.  

Now consider that all measured qubits, for both protocols, are under the influence of identical local environments ($p_i = p$). As before we consider the cases of phase-flip and white noise. The noisy evolution of the fidelity for the rotation protocol, $F_{R}^{wf}$, can be assessed by expressions (15) and (16). While for the controlled-not gate, the dynamics of $F_{CNOT}^{w}$, is obtained via Eq. (17), with no need of averaging over the possible measurement outcomes. For the protocols do not need measurements along the $z$ direction, if we have $p_{xy}^{w} = p_{xy}^{w}$, then the fidelity decay under the two kinds of noise is exactly the same. That is, for both noisy scenarios $F_{R}^{pf} = F_{R}^{w}$ and $F_{CNOT}^{pf} = F_{CNOT}^{w}$. This is in stark contrast to the entanglement decay of the resource states which will be generally different under the two noise instances. See Fig. 4 for a quantitative account of this fact. This once more shows that entanglement dynamics is detached from the fidelity dynamics, and as such a less entangled state can lead to higher (or equal) quality computations.

C. Deutsch-Jozsa (DJ) algorithm

Let $f : \{0, 1\}^N \rightarrow \{0, 1\}$ be an unknown boolean function which can be either constant – with all entries giving the same answer, or balanced – with half of the entries yielding 0 and the other half 1. What is the minimum number of times one has to query an Oracle that implements $f$ to discover the function’s type? Classically, in general, one needs at least $2^{N-1} + 1$ queries. Quantum mechanically, a single query via the DJ algorithm is sufficient [12]. In the quantum version, the Oracle applies a unitary $U_f$ on the $N$ qubits of the input state plus an auxiliary qubit $(A)$, as follows: $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$, where $x$ and $y$ are the decimal representations of binary strings. The DJ algorithm takes advantage of the superposition principle to evaluate all the entries at once, and can be cast as the unitary $DJ = H^\otimes (N+1)U_f H^\otimes (N+1)$, with $H = 1/\sqrt{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ the Hadamard gate. It is a simple calculation to deduce that $DJ(|0\rangle^\otimes N|1\rangle)$ leads to the outcome $\vec{0}$, after measurement of the $N$ qubits in the computational basis, if and only
if \( f \) is constant. This is clearly spelled out by the state below,

\[
DJ(|0\rangle^{\otimes N}|1\rangle) = H^{\otimes N} + \frac{1}{2^{N/2}} \sum_{i=0}^{2^{N-1}} (-1)^{f(i)} |i\rangle |\overline{\text{–}}\rangle.
\]

If \( f \) is constant the final state gets only a global phase \((-1)^{f(0)}\) in relation to the initial state (remember that \( H^2 = 1 \)). In its noiseless implementation, the DJ is known to generically create entanglement [13]. Note that along all the protocol the auxiliary qubit is never entangled with the \( N \)-qubit principal system. However, depending on the function \( f \) that the Oracle implements, the \( N \)-qubit system can become entangled. For a constant function \( f \) the state is never entangled as can be readily seen from (25). Despite of that, for example, the balanced function \( f : \{0,1\}^3 \rightarrow \{0,1\} \) with truth table:

| \( x \)  | \( f(x) \) |
|--------|---------|
| 0 = 000 | 0       |
| 1 = 001 | 0       |
| 2 = 010 | 0       |
| 3 = 011 | 1       |
| 4 = 100 | 1       |
| 5 = 101 | 1       |
| 6 = 110 | 1       |
| 7 = 111 | 0       |

generates an entangled state during the execution of the DJ protocol [32]. For three qubits this balanced function can be performed by the unitary operation \( U_f = CNOT_{1,A}CZ_{2,3} \), and can be easily generalized for \( N \) qubits as \( \prod_{i=0}^{N-1} CNOT_{i,A}CZ_{i+1} \). Note that the operation \( U_f \) can be done within the one-way framework by measurements without any adaptations (see Fig. 5).

On the other hand, a noisy implementation of the DJ, still within the circuit model, was analyzed in [33] and shown to present a small but finite advantage over its classical counterpart even for fully separable states. This indicates that to unveil the role of the entanglement in the mixed DJ protocol one has to split the generation and use of entanglement. In the one-way setting the DJ problem can be posed as follows [34]: the Oracle prepares a graph state that allows her to implement a function \( f \) by local measurements. She hands in to Neo (the user) a set of input qubits, and a set of output qubits. By encoding (through measurements) a certain value \( x \) into the input qubits, Neo can read out \( f(x) \) in the output qubits. As before, an appropriate choice of measurements allows Neo to discover whether the function is constant or balanced in a single run of the protocol.

In the example shown in Fig. 5, neither the implementation of \( f \) nor the measurements by Neo require adaptations. It is thus possible for the Oracle to interchange the NAMs in a entangled graph state with a mixed state without any entanglement, and even though Neo can decide whether \( f \) is constant or balanced in a single query. To design such classically correlated state the Oracle proceeds as follows: i) think of the original cluster state needed to encode the desired function for any input; ii) apply it to an hypothetical noise of the type in [18], but protecting the measurement outcomes by rotating the qubits to a convenient basis. In this step the Oracle can only protect a single instance of input states, and she does that for the \( H^{\otimes N}|0\rangle \). iii) finally, she evaluates the stationary state of the decoherence process, and rotates the qubits back to their original basis. The resulting (theoretical) state can be effectively prepared by the Oracle only by classical means, and it is ready to be hand in to Neo. If now Neo performs the correct set of measurements on the input qubits, the output qubits will encode whether the function is constant or balanced. It is interesting to notice that despite the fact the state bears only classical correlations, the quantum possibility of measuring on different basis entails advantages over a fully classical implementation. In fact, many boolean functions, that in a circuit model generate entangled states, can be decomposed into combinations of \( CZ \)'s, \( CNOT \)'s, \( H \)'s and possible relabelling of the qubits, transformations that can be attained within the one-way model without adaptations. As such, they can be simulated by a classically correlated resource state. These ideas, therefore, extend to many algorithms, for instance to the Simon’s [35] algorithm, as long as the function evaluated by the Oracle does not require adaptations. An interesting question, that we leave open, is to determine the fraction of boolean functions that can be evaluated without adaptations.
produced density matrix \( \rho \) is given by
\[
\rho = (1 - p_2) \rho_0 + p_2 \rho_1,
\]
where \( \rho_0 \) and \( \rho_1 \) are the initial states of the system and the ancilla, respectively. The fidelity of the measurement is then given by
\[
F = \frac{1}{2} \text{Tr} \left( \sqrt{\rho_0} \sqrt{\rho_1} \right).
\]

D. Ancilla-Driven Quantum Computation

The idea used in Sec. III can be readily applied to the ancilla-driven model of quantum computation [14]. In this model all the necessary unitary transformations to be realized in a quantum register are done through measurements in an ancillary qubit coupled to the register. A coupling interaction given simply by \( E = H_a H_r CZ_{ar} \) is sufficient to allow for universal quantum computation. The index \( a \) represents the ancilla, and \( r \) the \( i \)-th qubit of the quantum register. After the interaction the ancilla is measured in the same basis \( \{|0\rangle, |1\rangle\} \) as a result the two qubits on the register undergo the transformation \( (X_i^a H_r \otimes H_r) CZ_{ar} \).

FIG. 6. (Color online) (a) Single qubit rotation. After the interaction \( E = H_a H_r CZ_{ar} \), between the \( i \)-th qubit on the register and the ancilla, the latter is measured in the \( 1/\sqrt{2} \{ |0\rangle \pm e^{i\phi} |1\rangle \} \) basis. As a result the transformation \( X^a_i H_r |\phi\rangle \) is applied to the qubit in the register (with \( k \) the classical outcome of the measurement). (b) Two qubit entangling gate. After the interaction the ancilla is measured in the basis \( \{ |0\rangle, |1\rangle \} \) and as a result the two qubits on the register undergo the transformation \( (X^a_i H_r \otimes H_r) CZ_{ar} \).

V. CONCLUSION

In this paper we have shown that the effect of very general local decoherence maps on the measurement bases employed by the one-way model is to mix the two measurement directions with a certain weight \( p \), which characterizes the map. With this observation the fidelity of any computation within the one-way model can be readily obtained. This allowed us to conclude that the impact of the noise on the entanglement in the resource state is not generically related to loss of computation quality. Even for the simplest one-way protocol, the remote state preparation (Sec. IV A), a state with more entanglement (or discord) does not necessarily yield a higher quality computation. In other words, robust entanglement does not mean robust one-way quantum computations. Rather on the contrary, for the ancilla-driven measurement based computation model (Sec. IV D), the fidelity sensitivity to decoherence is bigger the higher is the entanglement in the resource state.
Even more surprisingly, the framework developed here made clear that the parts of algorithms that do not require adaptations can be replaced by a classically correlated resource state. This implies, for example, that some instances of the Deutsch-Josza algorithm can be realized in the one-way paradigm without any entanglement (Sec. IV C).

Thus, if entanglement (discord) cannot be assigned as the signature of efficient noisy quantum computations, which other quantities may assume this role? For the RSP protocol, the amount of non-classicality in the resource state (quantified by the minimum entanglement potential) seems to also point out the computation quality (Sec. IV A). We believe that would be interesting to extend or falsify this connection between non-classicality and computation fidelity to more general cases.

All these results show that entanglement may not be the most important resource for the quality of noisy quantum computations. The mere use of a quantum logic, in a mixed state scenario, seems to entail considerable gain over classical computations. This has obvious implications to experimental implementation of quantum information processes, for it relaxes the required isolation of the quantum system from its environment. In fact, this can shed some light on the functioning of biological systems, which, despite of being strongly influenced by the surroundings, are mesoscopic systems that may profit from their quantum nature [30].

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