Square root voting in the Council of the European Union: Rounding effects and the Jagiellonian Compromise

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Abstract

In recent years, enlargement of the European Union has brought with it renewed discussion of voting arrangements in the Council of the EU. During these negotiations, the Polish government proposed a voting scheme that gives each country a voting weight proportional to the square root of its population, and sets a quota according to an optimality condition ("Jagiellonian Compromise"). In this paper, the optimal quota is found exactly for the current population data from the 27 EU member states, and it is found that rounding of the voting weights can be used to improve the voting scheme.

Key words: Voting systems, Penrose square root voting, Rounding effects

1991 MSC: 91-04, 91F10

1 Introduction

Following the failure of the draft EU constitution in referenda in France and the Netherlands, and the subsequent negotiations on a Reform Treaty, the voting arrangements in the Council of the EU have received a significant amount of public attention, not least through the Polish proposal of a voting system that gives every member state a voting weight proportional to the square root of its population. EUobserver (2007). The idea of this voting scheme is to give every EU citizen the same influence on decisions in the Council, based on an analysis of their voting power. The concept of voting power used in this context was first introduced by Penrose (1946), and adapted to the EU framework by Życzkowski and Slomczynski (2004), Życzkowski and Slomczynski (2006).

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The purpose of this paper is twofold: To find an exact value for the optimal quota required to make the voting scheme as close as possible to the ideal of equal influence for all citizens, and to demonstrate the impact that rounding effects have on this quota.

The paper is organised as follows: In section 2 the main ideas behind square root voting are summarised, and some notation is introduced. Section 3 examines the effect of using rounded rather than exact voting weights by analysing two example scenarios. In the example in subsection 3.1 the rounding makes the voting system less fair, while the example in subsection 3.2 demonstrates that rounding can actually improve the voting scheme. Section 4 is devoted to voting in the Council of the EU. The current voting arrangements are briefly described in subsection 4.1 while square root voting is applied to the EU context in subsection 4.2 following Zyczkowski and Slomczynski (2004), and rounding effects are analysed. A similar analysis including member quotas is carried out in subsection 4.3. Section 5 summarises the results.

2 Voting power and square root voting

The concept of voting power was first introduced by Penrose (1946) for the UN General Assembly, but his analysis applies to all bodies where votes are cast by members elected in separate geographical areas with different numbers of voters, and the elected members from each area cast a joint vote. This is, for example, the case in the Electoral College in US Presidential elections, and in the Council of the EU, where the national governments of the (currently) 27 member states are represented.

As the numbers of citizens in the 27 member states differ hugely (between approximately 400,000 in Malta and 82.3 million in Germany), the task at hand is to create a “fair” voting system by assigning a voting weight to each country in the Council, where the most reasonable definition of “fair” is that every voter should have the same influence on the outcome of votes in the Council. Thus a mathematically rigorous definition of “influence” is needed.

The first step is to analyse the voting power of each voter within their own country. Penrose defines voting power as the probability for the vote of one individual voter to change the voting outcome, assuming that all other voters vote randomly.

In the following discussion, let us assume that there are only two voting options, “yes” and “no”, and that the outcome of the vote is positive if there are more “yes” votes than “no” votes, and negative otherwise.
In a country with \( N \) voters, the total number of possible results of the vote is \( 2^N \). “Random voting” means that all these results are equally likely.

If \( N \) is odd, the probability for a single voter to make a difference is the probability that the remaining \( N - 1 \) votes are evenly split between “yes” and “no” votes, which is

\[
P_{\text{odd}} = \frac{1}{2^{N-1}} \binom{N-1}{(N-1)/2}.
\]

If \( N \) is even, the probability for a single vote to be decisive is the probability that without this vote, there is one “yes” vote more than there are “no” votes, which means

\[
P_{\text{even}} = \frac{1}{2^{N-1}} \binom{N-1}{N/2}.
\]

Using the Stirling approximation for factorials, and the fact that

\[
\sqrt{N-1} \approx \sqrt{N} \quad (N \gg 1),
\]

we obtain for the probability of a single vote to be decisive (i.e., the voting power of an individual voter)

\[
P_N \propto \frac{1}{\sqrt{N}}.
\]

To give every voter the same influence on the voting outcome in the Council of the EU, voting weights there must be assigned in such a way that the voting power of each country is proportional to the square root of the number of its inhabitants. This is known as Penrose’s Square Root Law.

The analysis of voting power within the Council (or other body) is also due to Penrose, but was independently carried out by Banzhaf (1965). Let us assume the number of members in the decision taking body is \( n \) (in the case of the Council of the EU \( n = 27 \)). All countries will be assigned voting weights, and in the following the voting weight of country \( i \) will be denoted by \( v_i \), where the weights are normalised such that

\[
\sum_{i=1}^{n} v_i = 1.
\]
Any subset of countries is called a coalition $C$, and there will be a quota $R$ which has to be reached for a positive voting outcome, ie a coalition $C$ will be successful if

$$\sum_{i \in C} v_i \geq R.$$  \hfill (6)

The voting power of country $i$ is defined as the probability that its vote is critical (ie makes a difference to the voting outcome) under the assumption that the remaining $n-1$ countries vote randomly. If the total number of winning coalitions (ie coalitions that reach the quota) is $\omega$, and the number of winning coalitions that contain country $i$ is $\omega_i$, there are

$$\eta_i = 2\omega_i - \omega$$ \hfill (7)

coalitions where the vote of country $i$ is critical, ie which would not reach the quota without country $i$’s vote. As the vote of the remaining $n-1$ countries can yield $2^{n-1}$ different outcomes, which are all equally likely, the probability for country $i$’s vote to be critical is

$$B_i = \frac{\eta_i}{2^{n-1}}.$$ \hfill (8)

The number $B_i$ is called the absolute Banzhaf index. For our purposes, it is useful to introduce the relative Banzhaf index

$$\beta_i = \frac{B_i}{\sum_{j=1}^{n} B_j} = \frac{\eta_i}{\sum_{j=1}^{n} \eta_j}.$$ \hfill (9)

To make the voting system fair, we would ideally want

$$\beta_i = \frac{\sqrt{N_i}}{\sum_{j=1}^{n} \sqrt{N_j}}$$ \hfill (10)

for all countries $i$, where $N_i$ is the population of country $i$. Such weights may not exist, and even finding an approximation by minimising an appropriately defined deviation of the voting powers from their ideal values is a complicated multi-dimensional optimisation problem. To approximate the ideal voting power distribution, Penrose suggested to use voting weights proportional to the square root of the respective populations, ie

$$v_i = \frac{\sqrt{N_i}}{\sum_{j=1}^{n} \sqrt{N_j}}$$ \hfill (11)
for all countries \(i\). This is the scheme that Zyczkowski and Slomczynski (2004) and Zyczkowski and Slomczynski (2006) propose for the Council of the EU, where they choose the quota \(R\) such that

\[
\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (v_i - \beta_i)^2}
\]  

is minimal. The voting scheme constructed in this way is called the Jagiellonian Compromise.

A further useful notion is the efficiency \(\epsilon\) of a voting system (also known as the Coleman index), which is the probability for a randomly chosen coalition to be successful,

\[
\epsilon = \frac{\omega}{2^n}
\]  

The concept of square root voting power has been criticised by many authors (see Garret and Tsebelis (2001), Albert (2003), Bafumi et al. (2004) for recent criticism) on the grounds that actual voter behaviour is far from random. In Bafumi et al. (2004), it is shown that in US presidential elections, the random voting assumption seems to overestimate the probability of close voting outcomes in larger states, and the authors contrast actual voting power with the random voting concept. Slomczyński and Zyczkowski, as well as for example Leech (2003) (following Shapley and Shubik (1954), Banzhaf (1968), Coleman (1971)) point out that the voting power concept derived from the random voting assumption is an a priori voting power, based on the freedom of the voters to make their choice in any way they want, rather than on the way they actually vote. It is hard to see how a voting system could be based on actual voting power, as this would have to be determined from previous election results. This would mean that voters could influence the weight of their votes in future elections by the way they vote. A further problem with the idea of basing voting weights on results of earlier elections is that voter behaviour can change over time, and older election results may not reflect the actual voting power held by current voters. Actual voting power in Europe may be quite different from voting power in America. Since January 1997, the governments of all 27 EU member states have changed at least once as a result of a general election, which may indicate that close election outcomes are more common in European countries than in the larger US states. Determining actual voting power would be a very ambitious project, involving very large separate data sets from all EU member states, including a large number of election results and/or sufficiently accurate opinion polls to derive statistical models for voter behaviour in all countries currently in the EU. In the absence of such data, and for the reason of making voting weights independent from how voters
have voted in the past, the square root voting system seems to be the best one can do. At least it is based on clearly stated assumptions, that can in principle be tested empirically. This is not the case for the current EU voting arrangements.

The discussion on what definition of voting power to use is ongoing, and is not the subject of this paper. Here we will regard a voting system based on the Penrose definition of voting power as “fair”.

It can reasonably be argued that voting within the Council is not random either: Governments will, for example, form strategic alliances. However, such voting behaviour is hard to predict and to include in a model of actual voting power, and will not be considered here (see Hosli and Machover (2004) for a more detailed discussion).

3 Rounding errors

The voting weights defined in equation 11 will in general not be rational numbers, and some rounding rule will have to be applied; even if no precise rounding rule is given, the weights will be implicitly rounded by the accuracy of the floating point representation on the computer system used. In Zyczkowski and Slomczynski (2004), the authors round all voting weights to four decimal places behind the decimal point (although it is not entirely clear if their minimisation of $\sigma$ is carried out with these rounded weights, or if some more accurate values are used in the calculation). The purpose of this section is to show that the rounding can have a very large effect on the optimal quota $R$ and on the minimal value of $\sigma$ that can be achieved by the Penrose square root voting system. In the following, “$k$ digit rounding” means standard rounding of voting weights to $k$ decimal places, where voting weights are normalised according to equation $5$.

3.1 Example 1

To demonstrate the effect of rounding errors on the minimisation of $\sigma$, let us consider an example with 4 countries, and different numbers of voters, as shown in table 1.

In this example, 4 digit rounding is exact. Using the 4 digit voting weights, we can list the voting weights for all possible coalitions, see table 2.

From this table, we can read off the number $\omega$ of coalitions having a majority for any given quota $R$, as well as the numbers $\omega_i$ of these coalitions containing
Table 1  
Example 1: Populations and voting weights for 4 countries.

| Country | \( N_i \) | \( \sqrt{N_i} \) | \( v_i \) (3 digits) |
|---------|-----------|----------------|---------------------|
| 1       | 20,295,025| 4505           | 0.4505              |
| 2       | 6,265,009 | 2503           | 0.2503              |
| 3       | 4,012,009 | 2003           | 0.2003              |
| 4       | 978,121   | 989            | 0.099               |

Table 2  
Example 1: Coalitions and their voting weights without rounding.

| Countries | \( \sum v_i \) | Countries | \( \sum v_i \) |
|-----------|----------------|-----------|----------------|
| 1, 2, 3, 4| 1.0000         | 2, 3      | 0.4506         |
| 1, 2, 3   | 0.9011         | 1         | 0.4505         |
| 1, 2, 4   | 0.7997         | 2, 4      | 0.3492         |
| 1, 3, 4   | 0.7497         | 3, 4      | 0.2992         |
| 1, 2      | 0.7008         | 2         | 0.2503         |
| 1, 3      | 0.6508         | 3         | 0.2003         |
| 2, 3, 4   | 0.5495         | 4         | 0.0989         |
| 1, 4      | 0.5494         | None      | 0.0000         |

country \( i \). This means that for all quotas \( 0 \leq R \leq 1 \) and all countries \( i \), the numbers \( \eta_i \) can be calculated from equation 7 giving the Banzhaf index, and ultimately the deviation \( \sigma \) from a fair square root voting system. For example, for \( 0.9011 < R \leq 1 \), there is exactly one successful coalition, containing all four countries, which means that \( \eta_i = 1 \) for \( i = 1, \ldots, 4 \). The deviation \( \sigma \) then follows from equations 8, 9 and 12. The values of \( \sigma \) for all quotas \( R \) are shown in table 3.

A plot of \( \sigma \) as a function of the quota \( R \) is shown in figure 4. Only quotas \( R \geq 0.5 \) are shown here, as quotas less than 0.5 would mean that a coalition could be successful even if the opposing coalition has more votes.

The minimum of \( \sigma \) can be read off. We have

\[
\sigma_{\text{min}} = 0.031053, \quad (14)
\]

and the minimum occurs in the interval

\[
0.5494 < R \leq 0.5495. \quad (15)
\]
Table 3
Example 1: Deviation from fair voting without rounding.

| $R >$ | $R \leq$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\eta_4$ | $\sigma$ |
|-------|---------|---------|---------|---------|---------|---------|
| 0.9011 | 1.0000  | 1       | 1       | 1       | 1       | 0.127966 |
| 0.7997 | 0.9011  | 2       | 2       | 2       | 0       | 0.109660 |
| 0.7497 | 0.7997  | 3       | 3       | 1       | 1       | 0.083069 |
| 0.7008 | 0.7497  | 4       | 2       | 2       | 2       | 0.061850 |
| 0.6508 | 0.7008  | 5       | 3       | 1       | 1       | 0.061200 |
| 0.5495 | 0.6508  | 6       | 2       | 2       | 0       | 0.093088 |
| 0.5494 | 0.5495  | 5       | 3       | 3       | 1       | 0.031053 |
| 0.4506 | 0.5494  | 6       | 2       | 2       | 2       | 0.061580 |
| 0.4505 | 0.4506  | 5       | 3       | 3       | 1       | 0.031053 |
| 0.3492 | 0.4505  | 6       | 2       | 2       | 0       | 0.093088 |
| 0.2992 | 0.3492  | 5       | 3       | 1       | 1       | 0.061200 |
| 0.2503 | 0.2992  | 4       | 2       | 2       | 2       | 0.061850 |
| 0.2003 | 0.2503  | 3       | 3       | 1       | 1       | 0.083069 |
| 0.0989 | 0.2003  | 2       | 2       | 2       | 0       | 0.109660 |
| 0.0000 | 0.0989  | 1       | 1       | 1       | 1       | 0.127966 |

Fig. 1. Example 1: Deviation from fair voting without rounding.

The situation changes dramatically if $\sigma$ is calculated from the rounded voting weights. For this case, the possible coalitions and their voting weights are shown in Table 4.
| Countries | $\sum v_i$ | Countries | $\sum v_i$ |
|-----------|------------|-----------|------------|
| 1, 2, 3, 4 | 1.000      | 1         | 0.451      |
| 1, 2, 3   | 0.901      | 2, 3      | 0.450      |
| 1, 2, 4   | 0.800      | 2, 4      | 0.349      |
| 1, 3, 4   | 0.750      | 3, 4      | 0.299      |
| 1, 2      | 0.701      | 2         | 0.250      |
| 1, 3      | 0.651      | 3         | 0.200      |
| 1, 4      | 0.550      | 4         | 0.099      |
| 2, 3, 4   | 0.549      | None      | 0.000      |

Table 4
Example 1: Coalitions and their voting weights with 3 digit rounding.

Comparison with table 2 shows that the coalition of countries 2, 3 and 4, and the coalition of countries 1 and 4 have changed places. This leads to different results for $\eta$ and $\sigma$, as shown in table 5.

| $R >$ | $R \leq$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\eta_4$ | $\sigma$ |
|-------|----------|----------|----------|----------|----------|----------|
| 0.901 | 1.000    | 1        | 1        | 1        | 1        | 0.127966 |
| 0.800 | 0.901    | 2        | 2        | 2        | 0        | 0.109660 |
| 0.750 | 0.800    | 3        | 3        | 1        | 1        | 0.083069 |
| 0.701 | 0.750    | 4        | 2        | 2        | 2        | 0.061850 |
| 0.651 | 0.701    | 5        | 3        | 1        | 1        | 0.061200 |
| 0.550 | 0.651    | 6        | 2        | 2        | 0        | 0.093088 |
| 0.549 | 0.550    | 7        | 1        | 1        | 1        | 0.154031 |
| 0.451 | 0.549    | 6        | 2        | 2        | 2        | 0.061580 |
| 0.450 | 0.451    | 7        | 1        | 1        | 1        | 0.154031 |
| 0.349 | 0.450    | 6        | 2        | 2        | 0        | 0.093088 |
| 0.299 | 0.349    | 5        | 3        | 1        | 1        | 0.061200 |
| 0.250 | 0.299    | 4        | 2        | 2        | 2        | 0.061850 |
| 0.200 | 0.250    | 3        | 3        | 1        | 1        | 0.083069 |
| 0.099 | 0.200    | 2        | 2        | 2        | 0        | 0.109660 |
| 0.000 | 0.099    | 1        | 1        | 1        | 1        | 0.127966 |

Table 5
Example 1: Deviation from fair voting with 3 digit rounding.

Again, $\sigma$ can be plotted as a function of $R$, as shown in figure 2.
Reading off the minimum of $\sigma$, we find

$$\sigma_{\text{min}} = 0.061200,$$

occurring at

$$0.651 < R \leq 0.701,$$

which means that the minimum value of $\sigma$ has changed dramatically from the case without rounding (it has more than doubled), and also the interval where the minimum occurs has completely changed. This shows that rounding can have a significant impact on the minimum value of $\sigma$ and the associated quota. In this example, rounding increases the value of $\sigma_{\text{min}}$, but in general this is not necessarily the case. By coincidence, rounding can decrease the value of $\sigma_{\text{min}}$, which means that it can improve the Jagiellonian approximation for a voting system with square root voting power. This is illustrated in Example 2.

### 3.2 Example 2

This is again an example for four countries, with populations given by table 6. Again, the 4 digit voting weights are exact, and for these unrounded weights, we can list all possible coalitions with their voting weights as shown in table 7 and calculate $\sigma$ for all values of the quota $R$ (table 8 and figure 3).
| Country | $N_i$   | $\sqrt{N_i}$ | $v_i$  | $v_i$ (3 digits) |
|---------|---------|--------------|-------|-----------------|
| 1       | 20,376,196 | 4514         | 0.4514 | 0.451           |
| 2       | 6,280,036   | 2506         | 0.2506 | 0.251           |
| 3       | 4,024,036   | 2006         | 0.2006 | 0.201           |
| 4       | 948,676     | 974          | 0.0974 | 0.097           |

Table 6
Example 2: Populations and voting weights for 4 countries.

| Countries | $\sum v_i$ | Countries | $\sum v_i$ |
|-----------|------------|-----------|------------|
| 1, 2, 3, 4 | 1.0000     | 1         | 0.4514     |
| 1, 2, 3   | 0.9026     | 2, 3      | 0.4512     |
| 1, 2, 4   | 0.7994     | 2, 4      | 0.3480     |
| 1, 3, 4   | 0.7494     | 3, 4      | 0.2980     |
| 1, 2      | 0.7020     | 2         | 0.2506     |
| 1, 3      | 0.6520     | 3         | 0.2006     |
| 1, 4      | 0.5488     | 4         | 0.0974     |
| 2, 3, 4   | 0.5486     | None      | 0.0000     |

Table 7
Example 2: Coalitions and their voting weights without rounding.

![Example 2 without rounding](image)

Fig. 3. Example 2: Deviation from fair voting without rounding.

We can read off the minimum value of $\sigma$, which is

$$\sigma_{\text{min}} = 0.061093,$$

(18)
Table 8

Example 2: Deviation from fair voting without rounding.

and occurs at

\[ 0.6520 < R \leq 0.7020. \]  \hspace{1cm} (19)

As in the previous example, the situation changes significantly if the voting weights are rounded to three digits. Looking at the coalitions and their voting weights (table 9), it can be seen that the coalition consisting of countries 1 and 4, and the coalition consisting of countries 2, 3 and 4 have swapped places. This is reflected in the function \( \sigma \), as shown in table 10 and figure 4.

Reading off the minimum value of \( \sigma \), we now obtain

\[ \sigma_{\min} = 0.031004, \]  \hspace{1cm} (20)

occurring at

\[ 0.548 < R \leq 0.549. \]  \hspace{1cm} (21)

In this case, the rounding has actually improved the approximation, the opti-
Table 9
Example 2: Coalitions and their voting weights with 3 digit rounding.

| Countries | $\sum v_i$ | Countries | $\sum v_i$ |
|-----------|------------|-----------|------------|
| 1, 2, 3, 4 | 1.000      | 2, 3      | 0.452      |
| 1, 2, 3   | 0.903      | 1         | 0.451      |
| 1, 2, 4   | 0.799      | 2, 4      | 0.348      |
| 1, 3, 4   | 0.749      | 3, 4      | 0.298      |
| 1, 2      | 0.702      | 2         | 0.251      |
| 1, 3      | 0.652      | 3         | 0.201      |
| 2, 3, 4   | 0.549      | 4         | 0.097      |
| 1, 4      | 0.548      | None      | 0.000      |

Table 10
Example 2: Deviation from fair voting with 3 digit rounding.

| $R >$ | $R \leq$ | $\eta_1$ | $\eta_2$ | $\eta_3$ | $\eta_4$ | $\sigma$ |
|-------|----------|----------|----------|----------|----------|----------|
| 0.903 | 1.000    | 1        | 1        | 1        | 1        | 0.128734 |
| 0.799 | 0.903    | 2        | 2        | 2        | 0        | 0.109418 |
| 0.749 | 0.799    | 3        | 3        | 1        | 1        | 0.083351 |
| 0.702 | 0.749    | 4        | 2        | 2        | 2        | 0.062709 |
| 0.652 | 0.702    | 5        | 3        | 1        | 1        | 0.061093 |
| 0.549 | 0.652    | 6        | 2        | 2        | 0        | 0.092371 |
| 0.548 | 0.549    | 5        | 3        | 3        | 1        | 0.031004 |
| 0.452 | 0.548    | 6        | 2        | 2        | 2        | 0.061960 |
| 0.451 | 0.452    | 5        | 3        | 3        | 1        | 0.031004 |
| 0.348 | 0.451    | 6        | 2        | 2        | 0        | 0.092371 |
| 0.298 | 0.348    | 5        | 3        | 1        | 1        | 0.061093 |
| 0.251 | 0.298    | 4        | 2        | 2        | 2        | 0.062709 |
| 0.201 | 0.251    | 3        | 3        | 1        | 1        | 0.083351 |
| 0.097 | 0.201    | 2        | 2        | 2        | 0        | 0.109418 |
| 0.000 | 0.097    | 1        | 1        | 1        | 1        | 0.128734 |

The value of $\sigma$ has almost halved. This means that rounding can be used as a tool in finding the optimal quota, i.e., $\sigma$ can be calculated for several rounding schemes (1 digit, 2 digits, ...), and the minimum value of $\sigma_{\text{min}}$ found will then determine the optimal rounding of the voting weights, and the quota $R$ that should be chosen.
4 Voting in the Council of the EU

4.1 Current voting arrangements

The current voting procedures in the Council of the EU are laid down in the Treaty of Nice (European Union (2001)) and the accession treaties for the member states that joined the EU after the introduction of the Nice voting scheme (European Union (2003, 2005)). Under this voting scheme, each of the currently 27 member states is assigned a certain number of votes, and the votes for all member states add up to 345. For a proposal to be passed, the following criteria must be met:

(1) The votes of the member states voting “yes” must add up to at least 255.
(2) More than half of the member states (ie currently 14 countries) vote “yes”.
   If the Council is not acting on a proposal by the European Commission, at least two thirds of the member states (ie currently 18 countries) must vote “yes”.
(3) The countries voting “yes” must represent at least 62% of the population of the EU.

The last condition is only applied if a member state requests it, and there are only very few coalitions that satisfy the first two but not the third condition.

The current population numbers (European Statistics Office (2007)) and votes for each country are given in table 11. At the time of writing, the 2007 population data from Belgium, Ireland and Luxembourg were not available, and
for these countries, the 2006 data will be used in the calculations in this and the next two subsections.

| Country      | Population      | Votes (Nice) |
|--------------|-----------------|--------------|
| Germany      | 82,310,995      | 29           |
| France       | 63,392,140      | 29           |
| UK           | 60,798,438      | 29           |
| Italy        | 59,131,287      | 29           |
| Spain        | 44,474,631      | 27           |
| Poland       | 38,125,479      | 27           |
| Romania      | 21,565,119      | 14           |
| Netherlands  | 16,357,992      | 13           |
| Greece       | 11,170,957      | 12           |
| Portugal     | 10,599,095      | 12           |
| Belgium      | 10,511,382      | 12           |
| Czech Republic | 10,287,189    | 12           |
| Hungary      | 10,064,000      | 12           |
| Sweden       | 9,113,257       | 10           |
| Austria      | 8,298,923       | 10           |
| Bulgaria     | 7,679,290       | 10           |
| Denmark      | 5,447,084       | 7            |
| Slovakia     | 5,393,637       | 7            |
| Finland      | 5,276,955       | 7            |
| Ireland      | 4,209,019       | 7            |
| Lithuania    | 3,384,879       | 7            |
| Latvia       | 2,281,305       | 4            |
| Slovenia     | 2,010,377       | 4            |
| Estonia      | 1,342,409       | 4            |
| Cyprus       | 778,537         | 4            |
| Luxembourg   | 459,500         | 4            |

continued on next page
In 2004, a new voting scheme was suggested in the draft constitution for Europe (European Union (2004)), which subsequently failed to reach a majority in referenda in France and in the Netherlands. However, an Intergovernmental Conference in 2007 has drafted a new Reform Treaty (European Union (2007)), which, if accepted, will introduce the voting method suggested by the draft constitution in November 2014. According to this voting scheme, a proposal is passed, if it satisfies the “double majority” conditions:

1. At least 55% of the member states (currently 15) vote “yes”, or, if the Council is not acting on a proposal by the European Commission or the High Representative of the Union for Foreign Affairs and Security Policy, at least 72% of the member states (currently 20) vote “yes”.
2. The countries voting “yes” represent at least 65% of the population of the European Union.

During a transition period (2014–2017), the Nice voting system can be used if this is requested by a member state.

The introduction of a square root voting system was suggested by Sweden at the Intergovernmental Conference 2000 (Zyczkowski et al. (2006)), and by Poland in 2007 (Euobserver (2007)), but was turned down by a majority of EU member states on both occasions. The scheme has also been brought to the attention of governments in an open letter written by scientists, Scientists for a Democratic Europe (2007).

For both the Nice voting scheme and the draft constitution scheme, the relative Banzhaf indices $\beta_i$ can be calculated, and the ratio

$$r = \frac{\beta_i}{\sqrt{N_i} / \sum_{j=1}^{n} \sqrt{N_j}}$$

(22)
can be determined for each country, as a measure of “fairness”. In a voting system that is fair in the Penrose sense,

\[ r = 1 \]  

(23)

should hold for all countries. As shown in figure 5, this is not the case: Both schemes favour smaller countries (this is due to the allocation of a relatively large voting weight to small countries in the Nice scheme, and the 55% member quota in the draft constitution). Furthermore, the draft Constitution favours the largest countries due to the 65% population quota. This would be acceptable if these schemes were based on an alternative definition of fairness and voting power, which does not seem to be the case.

![Deviation from square root voting power for the Nice and draft Constitution voting schemes.](image)

Fig. 5. Deviation from square root voting power for the Nice and draft Constitution voting schemes.

For a detailed discussion of the shortcomings of the Nice and draft constitution schemes see Nurmi and Meskanen (1999), Leech (2002), Baldwin and Widgren (2003a), Baldwin and Widgren (2003b), Felsenthal and Machover (2004), Felderer et al. (2003), Plechanovová (2003), Barberà and Jackson (2004), Baldwin and Widgren (2004).

4.2 Square root voting in the EU

The square root voting scheme suggested by Poland follows the Jagiellonian Compromise paper Zyczkowski and Slomczynski (2004), which finds (for the
27 member EU) an optimal quota of approximately 0.615, in the sense that at this quota, the quantity \( \sigma \) defined in section 2 is minimised. The purpose of this section is to find the “exact” minimum of \( \sigma \) using double precision numbers (ie 8 byte real numbers), and to investigate the effect of rounding on the minimum found.

The method used to minimise \( \sigma \) is the same as in section 3. There are

\[
2^{27} = 134,217,728
\]  

(24)

different coalitions in the 27 member EU. For each coalition, the square root voting weight is calculated, and the coalitions that have a voting weight of 0.5 or more are sorted into three subintervals of the interval \([0.5, 1]\), and within each subinterval are sorted according to their voting weights using the quicksort algorithm \[\text{Hoare (1961), Hoare (1962), in the version presented in Press et al. (1993).}\] As in the four-country examples, the relative Banzhaf indices are then calculated for all values of the quota \( R \), and \( \sigma \) is then determined as a function of \( R \). On a Pentium 4 PC, this procedure takes a few minutes. Searching through the values of \( \sigma \), the minimum is found for

\[
0.614966335781001 < R \leq 0.614966337885155,
\]  

(25)

with a value of

\[
\sigma_{\min} = 4.334790 \times 10^{-5}.
\]  

(26)

The quantity \( \sigma \) as a function of the quota \( R \) around the minimum is shown in figures 6 to 9 with increasingly improved resolution, where resolution means the horizontal distance between individual points.

Only very fine resolution reveals the piecewise constant nature of the function \( \sigma \), the minimum is obtained in an interval of a length of just over \( 2 \times 10^{-9} \). This means that rounding to less than 10 digits may have a significant impact on the minimal value of \( \sigma \) as well as on the range of quotas for which it is obtained.

As pointed out in section 3, the rounding effect may increase or decrease the deviation from fair square root voting. Thus it is sensible to determine \( \sigma_{\min} \) as described above for weights rounded to \( k \) digits, \( k = 1, \ldots, k_{\text{max}} \approx 10 \), and choose the rounding scheme that returns the smallest value of \( \sigma_{\min} \). This calculation yields the results shown in figure 10.

Figure 10 shows that the rounding effect on \( \sigma_{\min} \) is very small provided the voting weights are rounded to more than four decimal places, but the effect
is clearly visible for the 4-digit rounding used in the Jagiellonian Compromise paper Zyczkowski and Slomczynski (2004). Careful analysis shows that the minimum of $\sigma_{\text{min}}$ is obtained with 7-digit rounding. In this case, we obtain

$$\sigma_{\text{min}} = 4.334644 \times 10^{-5},$$

with an optimal quota $R$ in the interval

$$0.6149670 < R \leq 0.6149671.$$
Fig. 8. Square root voting in the EU: Deviation from fair voting.

Both the optimal quota and optimal rounding can change even with small changes in the population numbers. The same method applied to the 2006 population data (European Statistics Office (2007)) reveals that with those data, 4-digit rounding is optimal. This means that the rounding scheme and the quota have to be adjusted annually, to make sure that the smallest possible deviation from fair square root voting is obtained.

Using 7-digit rounding on the 2007 data, the quotient \( r \) defined in equation 22 can be calculated for all 27 member states. This is shown in figure [111](#).

A final quantity that is of interest is the efficiency \( \epsilon \) of the voting scheme just
Fig. 10. Minimum value of $\sigma$ from rounded voting weights. The dotted line is the value of $\sigma_{\text{min}}$ calculated from double precision weights.

Fig. 11. Deviation from fair voting for all EU member states, for square root voting with 7-digit rounding.

described. It is

$$\epsilon = 0.1644.$$  \tag{29}$$

This makes the square root method considerably more efficient than the Nice scheme ($\epsilon = 0.02026$) and the draft constitution ($\epsilon = 0.1284$).
4.3 Square root voting with a member quota

Slomczyński and Zyczkowski point out (Zyczkowski and Slomczynski (2004)) that it is possible to combine the square root voting scheme with a requirement that a majority of member states has to vote “yes” for a proposal to be successful. In fact, it is possible to include any member quota (14 countries, 15 countries, ...) in the square root method, which could be used to make sure that the interests of smaller member states receive adequate attention. Member quotas can also be employed as a threshold when voting on proposals not emanating from the European Commission, as is currently the case with the 18 member quota under the Treaty of Nice.

As in the case without member quota, all possible coalitions are ordered according to their voting weights, and the relative Banzhaf indices can then be calculated for all quotas $R$, simply by demanding that a coalition is successful if its voting weight is at least equal to $R$, and the number of countries included in the coalition is at least the member quota. The minimum deviation $\sigma_{\text{min}}$ can then be found as above, and the calculation can be repeated for 1-digit rounding, 2-digit rounding, etc. The results of this calculation for a number of member quotas (14, 15, 18 and 20, to reflect the different member quotas in the Treaty of Nice and the draft constitution) are shown in figure 12.

![Fig. 12. Minimum value of $\sigma$ from rounded voting weights for different member quotas. The dotted lines are the values of $\sigma_{\text{min}}$ calculated from double precision weights.

The data show that the optimal rounding for a member quota of 14, 15 or 18 is six digits, while for the 20 member quota, two digit rounding gives the smallest value of $\sigma_{\text{min}}$. Table 12 shows the optimal values of $\sigma_{\text{min}}$, and the quota intervals where they occur.](image)
Table 12
Optimal quota intervals and deviation from fair voting for different member quotas.

| Member quota | $R >$   | $R \leq$ | $\sigma_{\text{min}}$ |
|--------------|---------|----------|----------------------|
| 14           | 0.646660| 0.646661 | $1.10654 \times 10^{-3}$ |
| 15           | 0.682884| 0.682885 | $2.17490 \times 10^{-3}$ |
| 18           | 0.784222| 0.784223 | $6.49661 \times 10^{-3}$ |
| 20           | 0.80    | 0.81     | $1.01626 \times 10^{-2}$ |

For each individual country, the ratio $r$ defined in equation 22 can be calculated, and compared with the corresponding ratio from the Nice Scheme and the draft constitution for the same member quota. The results are shown in figures 13 to 16. As expected, the member quota increases the voting power of the smaller countries, but it can be seen that even with a member quota, the square root voting system is fairer in the sense of Penrose.

Fig. 13. The ratio $r$ for square root voting and Nice with a member quota of 14.

A comparison between the square root voting system and the Nice/constitution schemes should include the efficiencies. Table 13 shows the efficiency of square root voting with member quotas, and compares them with the corresponding efficiencies in the other two schemes.

It can be seen that for a member quota of 14, the square root voting system improves the efficiency compared to Nice (the low efficiency of the Nice scheme was one of the main reasons for introducing a new voting scheme in the draft constitution). For the other member quotas, the efficiency of the square root
system is considerably lower than that of the two existing schemes, which may be a reason not to adopt square root voting with these quotas in the form described in this paper. The reason for the low efficiency is that the optimum range of quotas occurs at relatively large values of $R$. However, the function
Fig. 16. The ratio $r$ for square root voting and the draft constitution with a member quota of 20.

| Member quota | Square root | Nice | Draft constitution |
|--------------|-------------|------|-------------------|
| 14           | 0.104526    | 0.020256 | — |
| 15           | 0.057132    | —    | 0.128419 |
| 18           | 0.005479    | 0.015809 | — |
| 20           | 0.000904    | —    | 0.007965 |

Table 13
Efficiency of square root voting with member quota compared to Nice and the draft constitution.

$\sigma$ remains comparatively small even for values of $R$ that are somewhat less than in the optimal range, and the low efficiency problem may be cured by choosing such a smaller value of $R$. The price to pay is a somewhat increased value of $\sigma_{\min}$.

5 Summary and conclusions

The concept of voting power according to Penrose, and its implementation by the Jagiellonian Compromise for the Council of the European Union have been summarised.

It has been shown that the exact optimal quota for square root voting weights
can be found analytically, but may depend strongly on the rounding of the individual voting weights. It was demonstrated by two examples that rounding may increase or decrease the deviation from fair voting in the sense of Penrose, and that rounding can be used as a tool to find the optimal voting weights. The square root voting system has been analysed both on its own, and with the inclusion of member quotas. For the 2007 population data from the member states of the EU, the resulting voting weights, optimal quota ranges, and efficiencies are shown in tables 14 and 15.

| Member quota | None       | 14         | 15         |
|--------------|------------|------------|------------|
| Germany      | 0.0944338  | 0.094434   | 0.094434   |
| France       | 0.0828736  | 0.082874   | 0.082874   |
| UK           | 0.0811605  | 0.081160   | 0.081160   |
| Italy        | 0.0800400  | 0.080040   | 0.080040   |
| Spain        | 0.0694152  | 0.069415   | 0.069415   |
| Poland       | 0.0642697  | 0.064270   | 0.064270   |
| Romania      | 0.0483364  | 0.048336   | 0.048336   |
| Netherlands  | 0.0420982  | 0.042098   | 0.042098   |
| Greece       | 0.0347891  | 0.034789   | 0.034789   |
| Portugal     | 0.0338870  | 0.033887   | 0.033887   |
| Belgium      | 0.0337465  | 0.033746   | 0.033746   |
| Czech Republic | 0.0333846 | 0.033385   | 0.033385   |
| Hungary      | 0.0330205  | 0.033021   | 0.033021   |
| v_i | Sweden | 0.0314221 | 0.031422 | 0.031422 |
|          | Austria  | 0.0299854 | 0.029985 | 0.029985 |
|          | Bulgaria | 0.0288442 | 0.028844 | 0.028844 |
|          | Denmark  | 0.0242930 | 0.024293 | 0.024293 |
|          | Slovakia | 0.0241735 | 0.024173 | 0.024173 |
|          | Finland  | 0.0213545 | 0.021354 | 0.021354 |
|          | Ireland  | 0.0191501 | 0.019150 | 0.019150 |
|          | Lithuania| 0.0191501 | 0.019150 | 0.019150 |

continued on next page
Table 14 – continued from previous page

| Member quota | None      | 14        | 15        |
|--------------|-----------|-----------|-----------|
| Latvia       | 0.0157214 | 0.015721  | 0.015721  |
| Slovenia     | 0.0147583 | 0.014758  | 0.014758  |
| Estonia      | 0.0120598 | 0.012060  | 0.012060  |
| Cyprus       | 0.0091841 | 0.009184  | 0.009184  |
| Luxembourg   | 0.0070557 | 0.007056  | 0.007056  |
| Malta        | 0.0066324 | 0.006632  | 0.006632  |
| Quota >      | 0.6149670 | 0.614660  | 0.682884  |
| Quota ≤      | 0.6149671 | 0.614661  | 0.682885  |
| Efficiency   | 0.1644    | 0.1045    | 0.05713   |

Table 14: Summary of optimal voting weights, optimal quota ranges, and efficiencies in square root voting with different member quotas.

| Member quota | 18     | 20     |
|--------------|--------|--------|
| Germany      | 0.094434 | 0.09   |
| France       | 0.082874 | 0.08   |
| UK           | 0.081160 | 0.08   |
| Italy        | 0.080040 | 0.08   |
| Spain        | 0.069415 | 0.07   |
| Poland       | 0.064270 | 0.06   |
| Romania      | 0.048336 | 0.05   |
| Netherlands  | 0.042098 | 0.04   |
| Greece       | 0.034789 | 0.03   |
| Portugal     | 0.033887 | 0.03   |

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| Member quota | 18     | 20     |
|--------------|--------|--------|
| Belgium      | 0.033746 | 0.03   |
| Czech Republic | 0.033385 | 0.03   |
| Hungary      | 0.033021 | 0.03   |
| Sweden       | 0.031422 | 0.03   |
| Austria      | 0.029985 | 0.03   |
| Bulgaria     | 0.028844 | 0.03   |
| Denmark      | 0.024293 | 0.02   |
| Slovakia     | 0.024173 | 0.02   |
| Finland      | 0.023911 | 0.02   |
| Ireland      | 0.021354 | 0.02   |
| Lithuania    | 0.019150 | 0.02   |
| Latvia       | 0.015721 | 0.02   |
| Slovenia     | 0.014758 | 0.01   |
| Estonia      | 0.012060 | 0.01   |
| Cyprus       | 0.009184 | 0.01   |
| Luxembourg   | 0.007056 | 0.01   |
| Malta        | 0.006632 | 0.01   |
| Quota >      | 0.784222 | 0.80   |
| Quota ≤      | 0.784223 | 0.81   |
| Efficiency   | 0.005479 | 0.000904 |

Table 15: Summary of optimal voting weights, optimal quota ranges, and efficiencies in square root voting with different member quotas.

It has been shown that these voting schemes have a smaller deviation from an ideal square root voting power system than both the Nice voting scheme
and the draft constitution voting scheme. However, for larger member quotas, the square root voting schemes minimise this deviation for rather large vote quotas, resulting in very small efficiency. To use square root voting with large member quotas, a compromise between the desire for a “fair” voting scheme and an efficient system would have to be found.

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Abstract

Key words:

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