Exponential Approximation of Band-limited Functions from Nonuniform Sampling by Regularization Methods

Yunfei Yang ∗ Haizhang Zhang †

Abstract

Reconstructing a band-limited function from its finite sample data is a fundamental task in signal analysis. A Gaussian regularized Shannon sampling series has been proved to be able to achieve exponential convergence for uniform sampling. Whether such an exponential convergence can also be achieved for nonuniform sampling by regularization methods was unresolved. In this paper, we give an affirmative answer to this question. Specifically, we show that one can recover a band-limited function by Gaussian or hyper-Gaussian regularized nonuniform sampling series with an exponential convergence rate. Our analysis is based on the residue theorem in complex analysis, which is used to represent the truncated error by a contour integral. Several concrete examples of nonuniform sampling with exponential convergence will be presented.

Keywords: Band-limited functions, Nonuniform sampling, Regularization, Approximation bounds

MSC codes: 41A25, 30E10, 94A20

1 Introduction

The classical Whittaker-Kotelnikov-Shannon sampling theorem [23] plays an important role in signal processing, because it establishes the fundamental relation between a band-limited signal and its samples. It states that, for any function $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ which is band-limited to $[-\pi, \pi]$ in the sense that the support of its Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

is contained in $[-\pi, \pi]$, we can reconstruct $f$ from its infinite sampling data \{f(n) : n \in \mathbb{Z}\} by the cardinal series:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc} (x - n) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x - n)}{\pi(x - n)}, \quad (1.1)$$

where the series converges absolutely and uniformly on $\mathbb{R}$. For practical reason, we can only sum up finite samples near the point $x$ to approximate $f(x)$. Thus, one has to consider the

∗Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, P.R. China. E-mail address: yyangdc@connect.ust.hk.

†School of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, P.R. China. Supported in part by Natural Science Foundation of China under grants 11971490 and 12126610. E-mail address: zhhaizh2.sysu.edu.cn.
truncation of the cardinal series. However, due to the slow decayness of the sinc function, truncating the cardinal series leads to a convergence rate of order $O(N^{-1/2})$. Furthermore, this truncated series is an optimal algorithm for recovering functions band-limited to $[-\pi, \pi]$ in the worst case scenario [17].

In order to achieve a fast convergence rate, one may consider functions band-limited to $[-\sigma, \sigma]$ with $\sigma < \pi$, so that the sampling rate of the uniform samples $\{f(n) : n \in \mathbb{Z}\}$ is strictly larger than the Nyquist sampling rate. In this case, it has been shown that regularized Whittaker-Kotelnikov-Shannon sampling series can achieve exponential convergence rates [5, 6, 15, 19, 20, 22]. Specifically, these papers considered the convergence rate of the regularized sampling series of the form

$$\sum_{n=-N}^{N} f(n) \text{sinc}(x-n) G_N(x-n).$$

When the regularizer $G_N$ is a Gaussian function

$$G_N(x) := \exp\left(-\frac{x^2}{2\delta_{\sigma,N}^2}\right), \quad x \in \mathbb{R},$$

with $\delta_{\sigma,N} > 0$ depending on both $\sigma$ and $N$, Qian [19, 20] first used the Fourier analysis method to establish the convergence rate

$$O\left(N^{1/2} \exp\left(-\frac{\pi-\sigma}{2} N\right)\right).$$

This rate was improved to

$$O\left(N^{-1/2} \exp\left(-\frac{\pi-\sigma}{2} N\right)\right)$$

in [15, 17] and [22] by Fourier methods and complex analysis methods, respectively. Recently, it was shown in [5] that exponential approximation can also be achieved by using a hyper-Gaussian regularizer

$$G_N(x) = \exp\left(-\frac{x^{2m}}{2\delta_{\sigma,N,m}^2}\right), \quad x \in \mathbb{R},$$

where $m \in \mathbb{N}$ and $\delta_{\sigma,N,m}$ depends on $\sigma, N$ and $m$.

Note that all these results are concerned about the regular and uniform sampling points $\Lambda = \mathbb{Z}$. Uniform sampling is impractical in real applications. Whether exponential convergence can also be achieved for the reconstruction of a band-limited function from its nonuniform sampling remained an open question. The goal of this paper is to study this question for nonuniform sampling sequences.

Sampling theorems on irregular or nonuniform sampling points have been extensively studied in the literature. For example, Higgins [8, 9] and Hinsen [11, 12] proved that a Shannon type sampling formula still hold true for irregular and nonuniform sampling points $\{\lambda_n\}$ satisfying

$$|\lambda_n - n| \leq L, \quad n \in \mathbb{Z}$$

for certain positive constants $L$. Margolis and Eldar [16] considered nonuniform sampling for periodic band-limited functions. The work of Annaby and Asharabi [1] gave upper bound estimates for truncated Shannon’s formula for a class of nonuniform sampling points and band-limited functions with fast decay. However, so far, it remains unknown whether exponential
convergence can also be achieved in approximating a general band-limited function from its
finite samplings on nonuniform points. In this paper, we give affirmative answer to this
problem. More precisely, we consider a general sampling sequence \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) and use the
complex analysis method to analyze the convergence rate of the regularized sampling series
\[
G_N f(z) := \sum_{n=-N}^{N} f(\lambda_n) \varphi_{\Lambda,n}(z) G_N(z - \lambda_n),
\]
where \( \varphi_{\Lambda,n} \) is a canonical product to be defined later and \( G_N \) is a Gaussian or hyper-Gaussian
function. Through careful analysis, we will show that several kinds of regularized nonuniform
sampling series \( G_N f \) can achieve exponential convergence.

The rest of the paper is organized as follows. We formulate the approximation error \( f(z) - G_N f(z) \) as a contour integral in Section 2. Error estimates for Gaussian and hyper-Gaussian
regularizers \( G_N \) are given in Sections 3 and 4, respectively. Finally, we present in Section 5
several examples of nonuniform sampling sequences such that exponential approximation is
achieved.

2 Regularized nonuniform sampling series

Let us begin with an introduction to the spaces of band-limited functions. For any \( \sigma > 0 \), we
denote by \( B_\sigma \) the set of all entire functions of exponential type at most \( \sigma \). In other words, \( B_\sigma \) consists of functions \( f \) that are analytic in the whole complex plane and satisfy
\[
\limsup_{r \to \infty} \frac{1}{r} \log \left( \max_{|z|=r} |f(z)| \right) \leq \sigma.
\]
For \( 1 \leq p \leq \infty \), the Bernstein space \( B^p_\sigma \) is the set of all \( f \in B_\sigma \) whose restrictions to the real
axis belong to \( L^p(\mathbb{R}) \). The norm of \( f \in B^p_\sigma \) is defined to be the \( L^p \)-norm of its restriction on \( \mathbb{R} \). Functions in the Bernstein spaces \( B^p_\sigma \) are band-limited in the sense that they have a Fourier transform with compact support in \([-\sigma, \sigma]\) by the Paley-Wiener theorem [21]. Furthermore, the Plancherel-Pólya theorem implies that
\[
B^p_\sigma \subseteq B^q_\sigma \subseteq B^\infty_\sigma \subseteq C(\mathbb{R})
\]
for all \( 1 \leq p \leq q \leq \infty \). It is well-known that a function \( f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \) is band-limited to
\([-\sigma, \sigma]\) if and only if \( f \) is the restriction to \( \mathbb{R} \) of a \( B^2_\sigma \) function.

We consider the problem of recovering a band-limited function from its samples on a sam-
pling sequence \( \Lambda \). For simplicity, throughout this paper, we will assume that \( \Lambda := \{\lambda_n\}_{n \in \mathbb{Z}} \subseteq \mathbb{R} \) satisfies
\[
\lambda_0 = 0, \quad \lambda_n < \lambda_{n+1}, \quad n \in \mathbb{Z},
\]
and \( \Lambda \) is separated in the sense that
\[
\delta_\Lambda := \inf_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_n| > 0. \quad (2.1)
\]
By the Weierstrass factorization theorem [14], there exist entire functions \( \varphi_\Lambda \) whose zeros are
exactly those points in \( \Lambda \). Since
\[
\sum_{n \neq 0} \frac{1}{|\lambda_n|^2} \leq \sum_{n \neq 0} \frac{1}{\delta_\Lambda n^2} < \infty,
\]
one of these entire functions is the canonical product
\[ \varphi_\Lambda(z) = z \prod_{n \neq 0} \left( 1 - \frac{z}{\lambda_n} \right) e^{z/\lambda_n}, \quad z \in \mathbb{C}. \]

If we further have \( \lim_{r \to \infty} \sum_{0 < |\lambda_n| < r} \frac{1}{\lambda_n} < \infty \), then
\[ \varphi_\Lambda(z) = z \lim_{r \to \infty} \prod_{0 < |\lambda_n| < r} \left( 1 - \frac{z}{\lambda_n} \right), \quad z \in \mathbb{C} \tag{2.2} \]
is another simpler choice. Such an entire function \( \varphi_\Lambda \) whose zeros are exactly those points in \( \Lambda \) is called a generating function of \( \Lambda \).

We now turn to the sampling series to be used for reconstruction of a function \( f \in B_\sigma \) from its sampling at \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \). First choose a generating function \( \varphi_\Lambda \) of \( \Lambda \). Since \( \varphi_\Lambda(z) \) is an entire function with zeros exactly at the points \( \lambda_n, n \in \mathbb{Z} \), we can define for every \( n \in \mathbb{Z} \)
\[ \varphi_{\Lambda,n}(z) = \frac{\varphi_\Lambda(z)}{\varphi_\Lambda(\lambda_n)(z - \lambda_n)}. \tag{2.3} \]

These are entire functions which solve the interpolation problem
\[ \varphi_{\Lambda,n}(\lambda_k) = \begin{cases} 1 & k = n, \\ 0 & k \neq n. \end{cases} \]

Following the idea of classical Lagrange interpolation, we define
\[ (A_N f)(z) := \sum_{n=-N}^{N} f(\lambda_n) \varphi_{\Lambda,n}(z). \tag{2.4} \]
Together with the idea of regularization, we shall study the regularized sampling series
\[ (G_N f)(z) := \sum_{n=-N}^{N} f(\lambda_n) \varphi_{\Lambda,n}(z) G_N(z - \lambda_n), \quad f \in B_\sigma^\infty, \tag{2.5} \]
where \( \sigma < \pi \) and \( G_N(z) \) is an entire function with \( G_N(0) = 1 \) that will serve as a regularizer. Note that if \( \Lambda = \mathbb{Z} \), then \( \varphi_\Lambda(z) = \sin \pi z \), \( \varphi_{\Lambda,n}(z) = \text{sinc} (z - n) \) and the series \( (2.4) \) is the cardinal series \( (1.1) \). This series with \( G_N \) being a Gaussian function and \( \Lambda = \mathbb{Z} \) was used in [22] to approximate a band-limited function \( f \in B_\sigma^\infty \) with \( \sigma < \pi \).

Next, we use the complex analysis method to estimate the error \( f(z) - (G_N f)(z) \). The key idea is that we can represent the error by a contour integral. More specifically, let \( \mathcal{L}_N \) be the positively oriented rectangle with vertices at \( T_N^+ + i S_N^+ \), where
\[ \lambda_N < T_N^+ < \lambda_{N+1}, \quad \lambda_{N-1} < T_N^- < \lambda_N, \quad S_N^+ > 0, \quad S_N^- < 0. \]
Then by the residue theorem, for \( z = x + iy \notin \Lambda \) with \( T_N^- < x < T_N^+ \) and \( S_N^- < y < S_N^+ \), we can write the error as
\[ f(z) - (G_N f)(z) = \frac{\varphi_\Lambda(z)}{2\pi i} \int_{\mathcal{L}_N} \frac{f(\zeta) G_N(z - \zeta)}{\varphi_\Lambda(\zeta)(\zeta - z)} d\zeta, \tag{2.6} \]
since \( G_N(z) \) is an entire function with \( G_N(0) = 1 \). Now, denote by \( I_{\text{hor}}^\pm \) the contributions to the last integral coming from the two horizontal parts of \( \mathcal{L}_N \), where + and − refer to the upper and the lower line segment, respectively. Similarly, denote by \( I_{\text{ver}}^\pm \) the contributions coming from the two vertical parts of \( \mathcal{L}_N \), where + and − refer to the right and the left line segment, respectively. Then

\[
|f(z) - G_N f(z)| \leq \frac{|\varphi_A(z)|}{2\pi} (|I_{\text{hor}}^+(z)| + |I_{\text{hor}}^-(z)| + |I_{\text{ver}}^+(z)| + |I_{\text{ver}}^-(z)|),
\]

(2.7)

We now have the following initial estimate of the error \( f(z) - G_N f(z) \).

**Lemma 2.1.** Let \( f \in B_a^\infty \), \( \varphi_A \) be a generating function of the sampling sequence \( \Lambda \), and \( G_N \) be an entire function satisfying \( G_N(0) = 1 \). Then for all \( z = x + iy \notin \Lambda \) with \( T_N^- < x < T_N^+ \) and \( S_N^- < y < S_N^+ \),

\[
|f(z) - G_N f(z)| \leq \frac{|\varphi_A(z)|}{2\pi} (|I_{\text{hor}}^+(z)| + |I_{\text{hor}}^-(z)| + |I_{\text{ver}}^+(z)| + |I_{\text{ver}}^-(z)|),
\]

(2.8)

where

\[
|I_{\text{hor}}^\pm(z)| \leq \frac{||f||_\infty}{|S_N^\pm - y|} \left| \min_{\zeta \in \mathcal{L}_N} \phi_A(\zeta) \right| \int_{T_N^-}^{T_N^+} |G_N(x - t + iy - iS_N^\pm)| dt,
\]

(2.9)

\[
|I_{\text{ver}}^\pm(z)| \leq \frac{||f||_\infty}{|T_N^\pm - x|} \left| \min_{\zeta \in \mathcal{L}_N} \phi_A(\zeta) \right| \int_{S_N^-}^{S_N^+} e^{-|\pi|y}|G_N(x - T_N^\pm + iy - is)| ds,
\]

(2.10)

with \( ||f||_\infty \) being the \( L^\infty \)-norm of \( f \) on \( \mathbb{R} \) and

\[
\phi_A(z) = |\varphi_A(z)| e^{-\pi |\text{Im} z|}, \quad z \in \mathbb{C}.
\]

(2.11)

**Proof.** By the Phragmén-Lindelöf principle [14],

\[
|f(z)| \leq ||f||_\infty e^{|\text{Im} z|}, \quad z \in \mathbb{C}, \quad f \in B_a^\infty.
\]

Note also that

\[
\phi_A(z) = |\varphi_A(z)| e^{-\pi |\text{Im} z|} > 0 \quad \text{for} \quad z \in \mathcal{L}_N.
\]

We now estimate \( I_{\text{hor}}^\pm \) and \( I_{\text{ver}}^\pm \) as follows:

\[
|I_{\text{hor}}^\pm(x + iy)| = \int_{T_N^-}^{T_N^+} \frac{f(t + iS_N^\pm)G_N(x + iy - t - iS_N^\pm)}{\varphi_A(t + iS_N^\pm)(t + iS_N^\pm - x - iy)} dt
\]

\[
\leq \int_{T_N^-}^{T_N^+} \frac{||f||_\infty e^{|\pi|y}|G_N(x - t + iy - iS_N^\pm)|}{\left| \min_{\zeta \in \mathcal{L}_N} \phi_A(\zeta) \right| e^{\pi|S_N^\pm|}|t - x + iS_N^\pm - iy|} dt
\]

\[
\leq \frac{||f||_\infty}{|S_N^\pm - y|} \left| \min_{\zeta \in \mathcal{L}_N} \phi_A(\zeta) \right| \int_{T_N^-}^{T_N^+} |G_N(x - t + iy - iS_N^\pm)| dt,
\]
\[ |I_{\text{ver}}(x + iy)| = \left| \int_{S_N^-}^{S_N^+} f(T_N^\pm + is)G_N(x + iy - T_N^\pm - is) \frac{\varphi_L(T_N^\pm + is)}{\varphi_L(T_N^\pm + is - x - iy)} ds \right| \leq \int_{S_N^-}^{S_N^+} \left\| f \right\|_\infty e^{\sigma|s|} |G_N(x - T_N^\pm + iy - is)| \left( \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta) \right) |T_N^\pm - x + is - iy| \, ds \leq \left\| f \right\|_\infty \left( \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta) \right) \int_{S_N^-}^{S_N^+} e^{-|\pi - \sigma| |s|} |G_N(x - T_N^\pm + iy - is)| ds. \]

The above two equations prove (2.9) and (2.10).

We shall give further upper bound estimates of the error by estimating the generating function \( \varphi_L \) and specifying the regularization function \( G_N \).

### 3 Gaussian regularization

When the regularizer \( G_N(z) \) is a Gaussian function, we can give explicit upper bounds for \( |I_{\text{hor}}^\pm(z)| \) and \( |I_{\text{ver}}^\pm(z)| \) so that we can estimate the approximation error \( f(z) - (G_N f)(z) \).

**Theorem 3.1.** Suppose \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) is separated, and for every \( \lambda_0 < T_N^+ < \lambda_{N+1}, \lambda_{N-1} < T_N^- < \lambda_{-N}, \) denote \( N_* = \min\{\lambda_0 - T_N^- , T_N^+ - \lambda_0\} \). Define the series \( (G_N f)(z) \) by (2.5), where

\[ G_N(z) = \exp(-r^2 z^2) \text{ with } r^2 = \frac{\pi - \sigma}{2N_*}. \]

Then, for \( N > 1 \) and \( z = x + iy \) satisfying \( \lambda_0 - T_N^- < x < \lambda_0 \) and \( |y| < N_* \), it holds for all \( f \in B_\sigma^\infty \) with \( 0 < \sigma < \pi \) that

\[ |f(z) - (G_N f)(z)| \leq C_N(y) \left( \frac{\left\| f \right\|_\infty}{\pi \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta)} \right) e^{-\frac{\pi - \sigma}{2} N_*}, \tag{3.1} \]

where \( \mathcal{L}_N \) is the rectangle with vertices at \( T_N^\pm + i(y \pm N_*) \) and

\[ C_N(y) = \sqrt{\frac{2\pi}{(\pi - \sigma)N_*}} \cosh((\pi - \sigma)y) + \frac{4}{(\pi - \sigma)N_*} e^{(\pi - \sigma)y^2/(2N_*)}. \]

**Proof.** By Lemma 2.1, we only need to bound \( |I_{\text{hor}}^\pm(z)| \) and \( |I_{\text{ver}}^\pm(z)| \). For every \( z = x + iy \) satisfying the condition of the theorem, we choose \( S_N^\pm = y \pm N_* \). Then \( |S_N^\pm - y| = N_* \). By inequality (2.9),

\[ |I_{\text{hor}}^\pm(z)| \leq \left( \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta) \right) |S_N^\pm - y| \int_{T_N^-}^{T_N^+} |G_N(x - t + iy - iS_N^\pm)| \, dt \leq \left( \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta) \right) |S_N^\pm - y| \int_{-\infty}^{\infty} e^{-r^2 (x-t)^2} \, dt \leq \left( \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta) \right) \frac{\sqrt{\pi}}{r} e^{-\frac{(\pi - \sigma)y}{2}} \int_{-\infty}^{\infty} e^{-\frac{(\pi - \sigma)y}{2}} \, dt = \left( \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta) \right) \frac{\sqrt{\pi}}{r} e^{-\frac{(\pi - \sigma)y}{2}} \int_{-\infty}^{\infty} e^{-\frac{(\pi - \sigma)y}{2}} \, dt \]

\[ = \sqrt{\frac{2\pi}{(\pi - \sigma)N_*}} \left( \frac{\left\| f \right\|_\infty}{\pi \min_{\zeta \in \mathcal{L}_N} \varphi_L(\zeta)} \right) e^{-\frac{\pi - \sigma}{2} N_*}. \]
By the definition of $N_\ast$, we have $|T_N^\pm - x| \geq N_\ast$. By inequality (2.10),

$$
|I^\pm_{ver}(z)| \leq \frac{\|f\|_\infty}{(\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta))|T_N^\pm - x|} \int_{S^-_N} e^{-|\pi - \sigma|s} |G_N(x - T_N^\pm + iy - is)| ds \\
\leq \frac{\|f\|_\infty}{(\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta))N_\ast} e^{-r^2N_\ast^2} \int_{y-N_\ast}^{y+N_\ast} e^{r^2(y-s)^2-(\pi - \sigma)s} ds \\
= \frac{\|f\|_\infty}{(\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta))N_\ast} e^{-\frac{\pi^2}{2}N_\ast} \int_{-N_\ast}^{N_\ast} e^{r^2s^2-(\pi - \sigma)s + y} ds.
$$

To estimate the last integral, we use the convexity of parabolas to get

$$r^2s^2 - (\pi - \sigma)s + y \leq \begin{cases} \\
\frac{-\pi^2}{2}[y + (1 - y/N_\ast)s], & -N_\ast \leq s \leq -y, \\
\frac{\pi^2}{2}[y + (1 + y/N_\ast)s], & -y \leq s \leq N_\ast.
\end{cases}$$

Consequently, we obtain

$$\int_{-N_\ast}^{-y} e^{r^2s^2-(\pi - \sigma)s + y} ds \leq \frac{2e^{(\pi - \sigma)y^2/2N_\ast}}{(\pi - \sigma)(1 - y/N_\ast)}$$

and

$$\int_{-y}^{N_\ast} e^{r^2s^2-(\pi - \sigma)s + y} ds \leq \frac{2e^{(\pi - \sigma)y^2/2N_\ast}}{(\pi - \sigma)(1 + y/N_\ast)}.$$ 

Therefore,

$$|I^\pm_{ver}(z)| \leq \frac{4\|f\|_\infty}{(\pi - \sigma)(\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta))N_\ast} e^{(\pi - \sigma)y^2/2N_\ast} e^{-\frac{\pi^2}{2}N_\ast}.$$ 

Combining these inequalities and using equality (2.7), we get the desired bound (3.1). □

In order to get a convergence rate of the regularized nonuniform sampling series, $\Lambda$ needs to be “dense enough” so that it is an oversampling for $B^\infty_\sigma$. In view of Theorem 3.1, it is natural to pose conditions on the function $\varphi_\Lambda$.

**Corollary 3.2.** Assume the conditions of Theorem 3.1 and let $T_N^\pm = \frac{\lambda_N + \lambda_{N+1}}{2}$, $T_N^- = \frac{\lambda_{-1} + \lambda_{-N-1}}{2}$. If there exists $0 < \delta < \delta_\Lambda/2$ such that

$$|\varphi_\Lambda(z)| \geq C|z|^{-p}e^{\pi |\text{Im } z|}, \quad \text{whenever } \text{dist}(z, \Lambda) := \inf_{n \in \mathcal{L}} |z - \lambda_n| > \delta, \quad (3.2)$$

for some constants $C > 0$ and $p \geq 0$. Then, for any $f \in B^\infty_\sigma$ with $\sigma < \pi$ and $\lambda_{-1} < x < \lambda_1$,

$$|f(x) - (G_N f)(x)| \leq \left(\sqrt{\frac{2\pi}{\pi - \sigma}} + \frac{4}{(\pi - \sigma)^{\sqrt{N_\ast}}}\right) \frac{\|f\|_\infty |\varphi_\Lambda(x)|^{\tilde{N}^p}}{C\pi \sqrt{N_\ast}} e^{-\frac{\pi^2}{2}N_\ast},$$

where $\tilde{N} = \sqrt{\max\{|T_N^\pm|^2, |T_N^-|^2\} + N_\ast^2}$.

**Proof.** By hypothesis, $\phi_\Lambda(\zeta) \geq C|\zeta|^{-p}$ for every $\zeta \in \mathcal{L}_N$. The result is a direct consequence of the bound (3.1). □
If we further know the growth of $|\lambda_N|$ and $|\lambda_{-N}|$, then we can estimate $N_*$ and $\tilde{N}$ to have a more explicit estimate of $|f(x) - G_N f(x)|$. We give some examples in Section 5.

The condition (3.2) can be seen as a requirement on the density of $\Lambda$. For instance, if $\Lambda = a\mathbb{Z}$ for some $a > 1$, then $\varphi_{\Lambda}(z) = \sin(\frac{\pi}{a} z)$ satisfying $|\varphi_{\Lambda}(z)| \leq e^{\frac{\pi}{a} |z|}$. Thus, it cannot satisfy condition (3.2) for any $p$. Actually, in this case, $\Lambda$ is an under-sampling for $B_2^\infty$ with $\pi > \sigma > \pi/a$, so there is no hope to reconstruct $f \in B_2^\infty$ from its samples on $\Lambda$.

4 Hyper-Gaussian regularization

In this section, we consider the nonuniform sampling series (2.5) with the hyper-Gaussian regularizer $G_N (z) = \exp(-r_m(N) z^{2m})$, where $m > 1$ is an integer and $r_m(N)$ will be chosen later. Recently, it has been proved in [5] by Fourier analysis methods that hyper-Gaussian regularized Whittaker-Kotel’nikov-Shannon sampling series are able to recover a band-limited function from its finite uniform oversampling data. We aim at achieving exponential convergence in reconstructing a band-limited function from its non-uniform sample data by complex analysis methods in this section.

We shall first use the Laplace’s method [18] to estimate $|I_{\text{hor}}^\pm|$.

Lemma 4.1 (Laplace’s method). Let $f$ be a twice differentiable real-valued function on a finite interval $[a, b]$. Assume $c \in (a, b)$ is the only maximum point of $f$ on $[a, b]$, $f'(c) = 0$, and $f''(c) < 0$. Then

$$
\int_a^b e^{Nf(t)} dt = e^{Nf(c)} \left( \sqrt{\frac{2\pi}{f''(c)}} + o \left( \frac{1}{\sqrt{N}} \right) \right), \quad N \to \infty.
$$

Lemma 4.2. Let $m > 1$ be an integer and $h_m(t) = -\text{Re}(t+i)^{2m}$. Then

$$
\int_0^\infty e^{Nh_m(t)} dt = e^{N(\sin \frac{\pi}{4m-2})^{1-2m}} \left( \sqrt{\frac{\pi}{m(2m-1)}} \left( \sin \frac{\pi}{4m-2} \right) \frac{m-\frac{3}{2}}{\sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right) \right)
$$

as $N \to \infty$. Consequently, there exists a constant $A_m$ such that for all $N > 0$,

$$
\int_0^\infty e^{Nh_m(t)} dt \leq \frac{A_m}{\sqrt{N}} e^{N(\sin \frac{\pi}{4m-2})^{1-2m}}.
$$

Proof. We first find the extrema of $h_m$ on $[0, \infty)$. Since

$$
h'_m(t) = -2m \text{Re}(t+i)^{2m-1},
$$

the critical points of $h_m$ are $t_k \in [0, \infty)$ such that

$$
\text{arg}(t_k + i) = \frac{\pi + 2k\pi}{4m-2}, \quad 0 \leq k \leq m - 1.
$$

We calculate the value of $h_m$ at $t_k$ as

$$
h_m(t_k) = -\left( \sin \frac{\pi + 2k\pi}{4m-2} \right)^{2m} \cos \left( 2m \frac{\pi + 2k\pi}{4m-2} \right) = (-1)^k \left( \sin \frac{\pi + 2k\pi}{4m-2} \right)^{1-2m}.
$$
Observe that \( t_0 = \cot \frac{\pi}{4m - 2} \) is the only maximum point of \( h_m \) on \([0, \infty)\) and

\[
h''_m(t_0) = -2m(2m - 1) \left( \sin \frac{\pi}{4m - 2} \right)^{3-2m} < 0.
\]

On the other hand, since

\[
\lim_{t \to +\infty} \frac{h_m(t)}{t} = -\infty,
\]

there exists some \( a > t_0 \) such that

\[
\int_a^{\infty} e^{Nh_m(t)} dt \leq \int_a^{\infty} e^{-Nt} dt \leq \frac{1}{N}.
\]

By lemma 4.1,

\[
\int_0^a e^{Nh_m(t)} dt = e^{N(\sin \frac{\pi}{4m - 2})^{1-2m}} \left( \sqrt{\frac{\pi}{m(2m - 1)}} \right) \left( \sin \frac{\pi}{4m - 2} \right)^{m-\frac{3}{2}} \frac{1}{\sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right)
\]

as \( N \to \infty \). Combining the last two estimates proves the lemma.

Similar to the Laplace’s method, we will use the following lemma to estimate \(|I_{\text{ver}}^\pm|\).

**Lemma 4.3.** Let \( f \) be a continuous real-valued function on a finite interval \([0, b]\). Assume that \( f(t) < f(0) \) for \( t \in (0, b] \) and \((f(t) - f(0))/t \to -k \) when \( t \to 0 \) with \( k > 0 \). Then

\[
\int_0^b e^{Nf(t)} dt = e^{Nf(0)} \left( \frac{1}{kN} + o \left( \frac{1}{N} \right) \right), \quad N \to +\infty.
\]

**Proof.** Without loss of generality, we suppose that \( f(0) = 0 \). Then for any \( \epsilon > 0 \), the maximum of \( f(t) \) when \( t \geq \epsilon \) is negative. Thus we can write

\[
\int_0^b e^{Nf(t)} dt = \int_0^\epsilon e^{Nf(t)} dt + O(e^{-D_\epsilon N})
\]

for some \( D_\epsilon > 0 \) depending on \( \epsilon \). Now, we expand \( f(t) \) around \( t = 0 \):

\[
f(t) = -kt + o(t).
\]

For any two real numbers \( t_1, t_2 \) we have the inequality

\[
|e^{t_1} - e^{t_2}| \leq |t_1 - t_2| e^{t_3}, \quad t_3 = \max(t_1, t_2).
\]

Applying this with \( t_1 = Nf(t), \ t_2 = -Nkt \) for \( t \in [0, \epsilon] \), we can take \( t_3 \leq -Nkt/2 \) by taking \( \epsilon \) small enough. With this choice of \( \epsilon \) we can write

\[
|e^{Nf(t)} - e^{-Nkt}| \leq N o(t) e^{-Nkt/2}, \quad 0 \leq t \leq \epsilon.
\]

Then,

\[
\int_0^\epsilon |e^{Nf(t)} - e^{-Nkt}| dt = o \left( \frac{1}{N} \right).
\]
Consequently,
\[
\left| \int_0^\varepsilon e^{Nf(t)} dt - \frac{1}{k\varepsilon} \right| = \left| \int_0^\varepsilon e^{Nf(t)} dt - \int_0^\infty e^{-Nkt} dt \right|
\leq \int_0^\varepsilon |e^{Nf(t)} - e^{-Nkt}| dt + \int_0^\infty e^{-Nkt} dt
= o\left( \frac{1}{N} \right) + \frac{1}{k\varepsilon} e^{-Nk\varepsilon},
\]
which proves the lemma. □

Now, we are ready to use similar arguments as those in Theorem 3.1 to deduce an estimate in the Hyper-Gaussian regularization case. Note that in the following theorem, we impose an extra assumption \( \sup_N |\lambda_N + \lambda_{-N}| < \infty \), which is not needed in Theorem 3.1.

**Theorem 4.4.** Suppose \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) is separated and \( \sup_N |\lambda_N + \lambda_{-N}| < \infty \), and for every \( \lambda_N < T_N^+ < \lambda_{N+1}, \lambda_{-N-1} < T_N^- < \lambda_{-N} \), denote \( N_*=\min\{\lambda_{-1} - T_N^- , T_N^+ - \lambda_1\} \). Define the series \( \left( G_N f \right)(z) \) by (2.5), where \( G_N(z) = e^{-r_m N^2 z^m} \) with the integer \( m > 1 \), \( r_m N = \mu_m N^{1-2m} \) and
\[
\mu_m = \frac{2m-1}{2m} (\pi - \sigma) b_m, \quad b_m = (2m-1) \left( \frac{\sin \frac{\pi}{4m-2} \frac{2m-1}{4m-2}}{\frac{2m-1}{2m}} \right)^{\frac{2m-1}{2m}},
\]
then for \( N > 1 \) and \( z = x + iy \) satisfying \( \lambda_{-1} < x < \lambda_1 \) and \( |y| < b_m N_* \), there exists a constant \( C_m \) depending on \( m, \sigma \) and \( \Lambda \) such that
\[
|f(z) - (G_N f)(z)| \leq C_m \|f\|_\infty \|\varphi_\Lambda(z)e^{(\pi - \sigma)|y|}\| \sqrt{N_*} e^{-\mu_m N_*},
\]
for every \( f \in \mathcal{B}_\sigma^\infty \), \( 0 < \sigma < \pi \), where \( \mathcal{L}_N \) is the rectangle with vertices at \( T_N^\pm + i(y \pm b_m N_*) \).

**Proof.** For every \( z = x + iy \) satisfying the condition of the theorem, we choose \( S_N^\pm = y \pm b_m N_* \) so that \( |S_N^\pm| = b_m N_* \pm y, |S_N^\pm - y| = b_m N_* \). Observing \( \text{Re} (at \pm ia)^{2m} = |a|^{2m} \text{Re} (t+i)^{2m} \) for every \( a, t \in \mathbb{R} \), we have
\[
|T_{hor}^\pm(z)| \leq \frac{\|f\|_\infty}{\min_{\zeta \in \mathcal{L}_N} |\varphi_\Lambda(\zeta)| |S_N^\pm - y|} e^{-\pi |\varphi_\Lambda(\zeta)| |S_N^\pm - y|} \int_{T_N^-}^{T_N^+} |G_N(x - t + iy - iS_N^\pm)| dt
\leq \frac{\|f\|_\infty}{\min_{\zeta \in \mathcal{L}_N} |\varphi_\Lambda(\zeta)| |S_N^\pm - y|} e^{-\pi |\varphi_\Lambda(\zeta)| |S_N^\pm - y|} \int_{T_N^-}^{T_N^+} e^{-r_m N_* \text{Re} (x-t+iy-iS_N^\pm)} 2^m dt
\leq \frac{\|f\|_\infty}{\min_{\zeta \in \mathcal{L}_N} |\varphi_\Lambda(\zeta)| |S_N^\pm - y|} e^{-\pi |\varphi_\Lambda(\zeta)| |S_N^\pm - y|} \int_0^{\infty} e^{-r_m N_* \text{Re} (t+iy-iS_N^\pm)} 2^m dt
\leq \frac{\|f\|_\infty e^{\pi |\varphi_\Lambda| y}}{\min_{\zeta \in \mathcal{L}_N} |\varphi_\Lambda(\zeta)|} e^{-\pi |\varphi_\Lambda| y} \int_0^{\infty} e^{-r_m N_* |y|^2} 2^m dt.
\]
By Lemma 4.2, there exists a constant $A_m$ such that
\[
|T_{hor}^+(z)| \leq \frac{2A_m \|f\|_\infty e^{\mp (\pi - \sigma)y}}{(\min_{\zeta \in L_N} \phi_\Lambda(\zeta)) \sqrt{\mu_m b_m^2 N_\ast}} e^{-\mu_m N_\ast}.
\]
On the other hand,
\[
|T_{ver}^+(z)| \leq \frac{\|f\|_\infty}{(\min_{\zeta \in L_N} \phi_\Lambda(\zeta)) |T_N^+ - x|} \int_{S_N^{-}}^{S_N^{+}} e^{-(\pi - \sigma)|s|} |G_N(x - T_N^+ + iy - is)| ds
\]
\[
= \frac{\|f\|_\infty}{(\min_{\zeta \in L_N} \phi_\Lambda(\zeta)) |T_N^+ - x|} \int_{S_N^{-}}^{S_N^{+}} e^{-(\pi - \sigma)|s|} - r_m N_\ast Re (x - T_N^+ + iy - is)^{2m} ds
\]
\[
\leq \frac{2\|f\|_\infty e^{(\pi - \sigma)|y|}}{(\min_{\zeta \in L_N} \phi_\Lambda(\zeta)) |T_N^+ - x|} \int_{0}^{b_m N_\ast} e^{-(\pi - \sigma)|s|} - r_m N_\ast Re (T_N^+ - x + is)^{2m} ds
\]
\[
= \frac{2\|f\|_\infty e^{(\pi - \sigma)|y|}}{(\min_{\zeta \in L_N} \phi_\Lambda(\zeta))} \int_{0}^{b_m N_\ast} e^{-(\pi - \sigma)s} - r_m N_\ast |T_N^+ - x|^{2m} Re (1 + is)^{2m} ds.
\]

By the assumption that $\sup |\lambda_N + \lambda_{-N}| < \infty$, we can decompose $|T_N^+ - x| = N_\ast + k_N^+ (x)$ for some bounded $k_N^+(x)$. Therefore,
\[-r_m N_\ast |T_N^+ - x|^{2m} = -\mu_m N_\ast - 2mk_N^+(x) + O(N_\ast^{-1}), \quad N_\ast \to \infty.
\]
There hence exists a constant $B_m$ such that
\[
|T_{ver}^+(z)| \leq \frac{2B_m \|f\|_\infty e^{(\pi - \sigma)|y|}}{(\min_{\zeta \in L_N} \phi_\Lambda(\zeta))} \int_{0}^{b_m} e^{N_\ast h_m(s)} ds,
\]
where $h_m(s) = -(\pi - \sigma)s - \mu_m Re (1 + is)^{2m}$. If $h_m(s) < h_m(0) = -\mu_m$ for all $s \in (0, b_m]$, then by Lemma 4.3, there exists a constant $B'_m$ such that
\[
|T_{ver}^+(z)| \leq \frac{2B'_m \|f\|_\infty e^{(\pi - \sigma)|y|}}{(\pi - \sigma)(\min_{\zeta \in L_N} \phi_\Lambda(\zeta))N_\ast} e^{-\mu_m N_\ast}.
\]
To prove the theorem, it remains to show that $h_m(s) < h_m(0)$ for $s \in (0, b_m)$.

Let $s = \tan \beta$, $\beta \in [0, \pi/2)$. We calculate that
\[h'_m(s) = \sigma - \pi - (-1)^m 2m \mu_m Re (s + i)^{2m-1} = \sigma - \pi + 2m \mu_m (\cos \beta)^{1-2m} \sin(2m - 1) \beta.
\]
Observe that $h'_m(s)$ is negative around $s = 0$ and increases on $[0, \tan \frac{\pi}{4m-2}]$. Since
\[(2m - 1) \sin \frac{\pi}{4m-2} \geq 1 > \left(1 - \sin^2 \frac{\pi}{4m-2}\right)^m = \left(\cos \frac{\pi}{4m-2}\right)^{2m},
\]
we have \( b_m < \tan \frac{\pi}{4m-2} \). So the maximum point of \( h_m(s) \) on \([0, b_m]\) is \( s = 0 \) or \( s = b_m \). Therefore, we need to show that \( h_m(b_m) < h_m(0) = -\mu_m \), which is equivalent to
\[
\Re (1 + ib_m)^{2m} + \frac{1}{2m-1} > 0.
\]
By the definition of \( b_m \), the above equation is equivalent to
\[
\Re \left( \left( (2m-1)\sin\frac{\pi}{4m-2}\right)^{2m} + i\sin\frac{\pi}{4m-2} \right)^{2m} > -\sin\frac{\pi}{4m-2}.
\]
We consider the function
\[
F_m(t) = \Re \left( t + i\sin\frac{\pi}{4m-2} \right)^{2m}, \quad t \geq 0
\]
and observe that \( F_m(\cos\frac{\pi}{4m-2}) = \Re e^{i\frac{2m}{4m-2}\pi} = -\sin\frac{\pi}{4m-2} \) and
\[
F_m'(t) = 2m \Re \left( t + i\sin\frac{\pi}{4m-2} \right)^{2m-1} = 2m \left( \sin\frac{\pi}{4m-2} \right)^{2m-1} \Re \left( \cot \theta + i \right)^{2m-1},
\]
where \( \cot \theta = t/\sin(\frac{\pi}{4m-2}) \). As a consequence, when \( t > \cos(\frac{\pi}{4m-2}) \), \( F_m'(t) > 0 \). Using inequality (4.1), we have \( F_m(((2m-1)\sin\frac{\pi}{4m-2})^{1/2m}) < F_m(\cos\frac{\pi}{4m-2}) \), which is the desired inequality.

We remark that \( \mu_m \) is monotonically decreasing as \( m \) increases with \( \mu_1 = \frac{\pi}{4m-2} \), which is the same exponent as that in Theorem 3.1. Thus, judging by the exponential term, the Gaussian regularizer is the best among hyper-Gaussian regularizers.

We also have the following corollary.

**Corollary 4.5.** Assume the conditions of Theorem 4.4 and let \( T_N^+ = \frac{\lambda N + \lambda_{N+1}}{2} \), \( T_N^- = \frac{\lambda_{N-1} + \lambda_{N-2}}{2} \). If there exists \( 0 < \delta < \delta_N/2 \) such that
\[
|\varphi_A(z)| \geq C |z|^{-p} e^{\pi |1mz|} \quad \text{whenever} \quad \text{dist}(z, \Lambda) > \delta
\]
for some constants \( C > 0 \) and \( p \geq 0 \). Then
\[
|f(x) - (G_N f)(x)| \leq C_m \frac{\|f\|_{\infty} |\varphi_A(x)| \tilde{N}^p}{C \sqrt{N}} e^{-\mu_m N_*}
\]
for every \( \lambda_{-1} < x < \lambda_1 \) and \( f \in B^\infty_\sigma \) with \( \sigma < \pi \), where \( \tilde{N} = \sqrt{\max\{|T_N^+|^2, |T_N^-|^2\} + N_*^2} \).

## 5 Examples

In this section, we provide several examples of nonuniform sampling and prove that the corresponding regularized sampling sequences achieve exponential convergence in reconstructing a band-limited function from its oversampling data.
5.1 Uniform sampling sequence

The fundamental example is $\Lambda = \mathbb{Z}$. In this case, $\varphi_\Lambda(z) = \sin(\pi z)$ and

$$(G_N f)(x) = \sum_{n=-N}^{N} f(n) \text{sinc}(x - n) G_N(x - n).$$

We can choose $T_N^+ = -T_N^- = N + 1/2$, $N_\ast = N - 1/2$, then $\phi_\Lambda \geq 1/2$ on the the rectangle $L_N$. Therefore, if $-1 < x < 1$ and $G_N(x) = \exp(-\frac{\pi - \sigma}{2N} x^2)$, then

$$|f(x) - (G_N f)(x)| \leq \left( \sqrt{\frac{2\pi}{\pi - \sigma}} + \frac{4}{(\pi - \sigma)\sqrt{N - \frac{1}{2}}} \right) \frac{2\|f\|_\infty |\sin(\pi x)|}{\pi \sqrt{N - \frac{1}{2}}} e^{-\frac{\pi - \sigma}{2}(N - \frac{1}{2})}$$

$$= O(N^{-\frac{1}{2}}e^{-\frac{\pi - \sigma}{2} N})$$

for every $f \in B_\sigma^\infty$ with $0 < \sigma < \pi$. If $G_N(x) = \exp(-\mu_m(N - 1/2)^{1-2m} x^{2m})$, we have

$$|f(x) - (G_N f)(x)| \leq C_m \frac{\|f\|_\infty |\sin(\pi x)|}{\sqrt{N - 1/2}} e^{-\mu_m(N - 1/2)} = O(N^{-1/2}e^{-\mu_m N}), \quad -1 < x < 1.$$

The focus of the paper is of course on nonuniform sampling. The above example is to show that our analysis also recovers those results established in [15, 19, 20, 22] and [5] for uniform sampling.

5.2 Zeros of a sine-type function

**Definition 5.1** (Sine-type functions). An entire function $\varphi$ of exponential type $\pi$ is said to be a sine-type function if it has simple and separated zeros and there exist positive constants $A, B, H$ such that

$$A e^{\pi |y|} \leq |\varphi(x + iy)| \leq B e^{\pi |y|} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| \geq H.$$

The zeros of a sine-type function lie in a horizontal strip and if we enumerate them in increasing order of their real parts then $\Lambda$ satisfies (2.1) and

$$\sup_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_n| < \infty.$$

Moreover, for each $\epsilon > 0$, there exist constants $M_1$ and $M_2$ such that

$$0 < M_1 < |\varphi(z)| e^{-\pi |\text{Im} z|} < M_2 < \infty, \quad \text{dist}(z, \Lambda) > \epsilon. \quad (5.1)$$

Any sine-type function $\varphi$ can be determined from its zero set $\Lambda$ by (2.2). If $\Lambda \subset \mathbb{R}$ is the zeros of a sine-type function, then $\Lambda$ is a complete interpolating sequence for $B_\sigma^2$. Consequently, it has a uniform density in the sense that for every $x \in \mathbb{R},$

$$D(\Lambda) = \lim_{r \to \infty} \frac{\#(\Lambda \cap [x, x + r])}{r} = 1.$$

Readers are referred to [4, 10, 14, 24] for more information on sine-type functions.
Here is a simple way to construct sine-type functions. For any function $g$ that can be represented as

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} h(\xi) e^{i\xi z} d\xi, \quad z \in \mathbb{C}$$

for some $h \in L^1[-\sigma, \sigma]$, we can define the sine wave crossings of $g$ as

$$\varphi_g(z) = A \sin(\pi z) - g(z), \quad z \in \mathbb{C}$$

where $A$ is a constant such that $A > \|h\|_{L^1}$. These functions are all sine-type functions. Moreover, the zeros of $\varphi_g$ are all real and simple if $g$ is real on the real axis $[0, 2\pi]$. Therefore, given any function $g \in B^2_{\sigma}$, we can construct corresponding sine-type functions in this way.

Note that the zeros of the sine-type function $\varphi(z) = \sin(\pi z)$ corresponding to $g = 0$ is the uniform sampling sequence $\Lambda = \mathbb{Z}$.

Now, suppose that $\Lambda \subset \mathbb{R}$ is the zeros of a sine-type function $\varphi_\Lambda$, then we know that $\delta_\Lambda := \inf_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_n| > 0$. If we choose a $\epsilon < \delta_\Lambda/2$ in (5.1), and

$$T^+_N = \frac{\lambda_N + \lambda_{N+1}}{2}, \quad T^-_N = \frac{\lambda_N + \lambda_{N-1}}{2}, \quad S^+_N > \epsilon, \quad S^-_N < -\epsilon,$$

then by (5.1), $\phi_\Lambda > M_1$ on the rectangle $L_N$. The density $D(\Lambda) = 1$ implies

$$\lim_{N \to \infty} \frac{N}{|T^-_N|} = 1.$$
Lemma 5.3. Suppose that $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ is a perturbed uniform sequence with $L < \frac{1}{2}$ and $\lambda_0 = 0$. Then the generating function $\varphi_\Lambda(z)$ of $\Lambda$ defined by (2.2) is an entire function, and there are constants $C_1, C_2$ such that for all $z = |z|e^{i\theta} \in \mathbb{C}$ with $|z|$ large enough,

$$C_1 H_1(z)H_2(L, z) \leq |\varphi_\Lambda(z)| \leq C_2 H_1(z)H_2(-L, z),$$

where

$$H_1(z) := e^{\pi|\text{Im}z|} \prod_{k=N(z)}^{N(z)+2} |\lambda_k - z|, \quad \text{Im} z \geq 1 \text{ and } \text{Re} z > 0,$$

$$H_2(d, z) := \begin{cases} 
|z|^{-4d}, & 0 \leq |\sin \theta| \leq \sin((\pi/(2|z|)), \\
|z|^{-2d} |\sin \theta|^{2d}, & \sin((\pi/(2|z|))) < |\sin \theta| \leq 1
\end{cases}$$

with $N(z)$ being a positive integer dependent on $z$.

If $\Lambda$ is a perturbed uniform sequence with $L < \frac{1}{2}$, then $\delta_\Lambda \geq 1 - 2L$. For every $\epsilon < \frac{1}{2} - L$, we have $|\lambda_k - z| > \epsilon$ whenever $\text{dist}(z, \Lambda) > \epsilon$. By Lemma 5.3, there exists a constant $C$ such that

$$\phi_\Lambda(z) = |\varphi_\Lambda(z)|e^{-\pi|\text{Im}z|} \geq C|z|^{-4L}, \quad \text{dist}(z, \Lambda) > \epsilon.$$ 

Thus, if we choose $T_N^+ = -T_N^- = N + \frac{1}{2}$, then $\phi_\Lambda(z) \geq C|z|^{-4L}$ on the rectangle $\mathcal{L}_N$ and $N - \frac{1}{2} - L \leq N \leq N - \frac{1}{2} + L$. Therefore, if $G_N(x) = \exp(-\frac{\pi - \sigma}{\sqrt{2}}x^2)$, then

$$|f(x) - (G_N f)(x)| = O(N^{4L - \frac{1}{2}}e^{-\frac{\pi - \sigma}{2}x^2}N), \quad \lambda_1 < x < \lambda_1.$$

If $G_N(x) = \exp(-\mu_m(N - 1)^{1-2m}x^{2m})$, we have

$$|f(x) - (G_N f)(x)| = O(N^{4L - \frac{1}{2}}e^{-\mu_mN}), \quad \lambda_1 < x < \lambda_1.$$

Note that when $L = 0$, the estimates reduce to the case $\Lambda = \mathbb{Z}$. When $L \neq 0$, the above two results are also new to our best knowledge.

References

[1] M. H. Annaby and R. M. Asharabi, Bounds for truncation and perturbation errors of nonuniform sampling series, BIT 56 (2016), no. 3, 807–832.

[2] I. Bar-David, An implicit sampling theorem for bounded bandlimited functions, Information and Control. 24 (1974), 36–44.

[3] H. Boche and U.J. Mönich, Convergence behavior of non-equidistant sampling series, Signal Processing 90 (1) (2010), 145–156.

[4] J. Bruna, Sampling in complex and harmonic analysis, Eur. Congr. Math. 1 (2000), 225–246.

[5] L. Chen, Y. Wang, and H. Zhang, Hyper-Gaussian regularized Whittaker-Kotel’nikov-Shannon sampling series, Analysis and Applications, online, 2022.
[6] L. Chen and H. Zhang, Sharp exponential bounds for the Gaussian regularized Whittaker-Kotelnikov-Shannon sampling series, *J. Approx. Theory* **245** (2019), 73–82.

[7] R. Duffin and A. C. Schaeffer, Some properties of functions of exponential type, *Bull. Amer. Math. Soc.* **44** (1938), no. 4, 236–240.

[8] J. R. Higgins, A sampling theorem for irregularly spaced sample points, *IEEE Trans. Inform. Theory.* **IT-22** (1976), no. 5, 621–622.

[9] J. R. Higgins, Sampling theorems and the contour integral method, *Appl. Anal.* **41** (1991), no. 1-4, 155–169.

[10] J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*, Clarendon Press, Oxford, UK, 1996.

[11] G. Hinsen, Explicit irregular sampling formulas, *J. Comput. Appl. Math.* **40** (1992), no. 2, 177–198.

[12] G. Hinsen, Irregular sampling of bandlimited $L^p$-functions, *J. Approx. Theory* **72** (1993), 346–364.

[13] D. Jagerman, Bounds for truncation error of the sampling expansion, *SIAM J. Appl. Math.* **14** (1966), 714–723.

[14] B. Y. Levin, *Lectures on Entire Functions*, American Mathematical Society, Providence, RI, 1997.

[15] R. Lin and H. Zhang, Convergence analysis of the Gaussian regularized Shannon sampling series, *Numer. Funct. Anal. Optim.* **38** (2017), no. 2, 224–247.

[16] E. Margolis and Y. C. Eldar, Nonuniform sampling of periodic bandlimited signals, *IEEE Trans. Signal Process.* **56** (2008), no. 7, part 1, 2728–2745.

[17] C. A. Micchelli, Y. Xu, and H. Zhang, Optimal learning of bandlimited functions from localized sampling, *J. Complexity.* **25** (2009), 85–114.

[18] M. A. Pinsky, *Introduction to Fourier Analysis and Wavelets*, American Mathematical Society, Providence, RI, 2009.

[19] L. Qian, On the regularized Whittaker-Kotel’nikov-Shannon sampling formula, *Proc. Amer. Math. Soc.* **131** (2003), 1169–1176.

[20] L. Qian, A modification of the sampling series with a Gaussian multiplier, *Sampl. Theory Signal Image Process.* **5** (2006), 1–19.

[21] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, Boston, 1991.

[22] G. Schmeisser and F. Stenger, Sinc approximation with a Gaussian multiplier, *Sampl. Theory Signal Image Process.* **6** (2007), 199–221.

[23] C. E. Shannon, Communication in the presence of noise, *Proc. I.R.E.* **37** (1949), 10–21.

[24] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, 1980.