Recollements and stratifying ideals

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Abstract. Surjective homological epimorphisms with stratifying kernel can be used to construct recollements of derived module categories. These ‘stratifying’ recollements are derived from recollements of module categories. Can every recollement be put in this form, up to equivalence? A negative answer will be given after providing a characterisation of recollements equivalent to stratifying ones. Moreover, criteria for a ring epimorphism to be ‘stratifying’ will be presented as well as constructions of such epimorphisms.

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1. Introduction

Recollements of triangulated categories have been introduced by Beilinson, Bernstein and Deligne [7] in order to deconstruct a derived category of constructible sheaves into an open and a closed part. This concept is meaningful also for derived module categories of rings. A recollement

\[ \mathcal{D}(B) \rightarrow \mathcal{D}(A) \rightarrow \mathcal{D}(C) \]

can be viewed as a short exact sequence of derived module categories of rings \( A, B \) and \( C \), with the given derived category \( \mathcal{D}(A) \) as middle term.

Directly translating recollements from categories of perverse sheaves on flag manifolds to block algebras \( A \) of the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \) of a semisimple complex Lie algebra produces recollements of derived module categories of quasi-hereditary, or more generally stratified, algebras, as introduced and studied by Cline, Parshall and Scott [11, 12, 13]. By [13], a stratifying ideal \( AeA \) of a ring \( A \) and the associated ring epimorphism \( A \to A/AeA \) induce a recollement of derived module categories of the special form

\[ \mathcal{D}(A/AeA) \rightarrow \mathcal{D}(A) \rightarrow \mathcal{D}(eAe). \]

The ring epimorphism \( A \to A/AeA \) giving rise to this recollement has the additional property that it is a homological epimorphism and its kernel is a stratifying ideal. Most examples of recollements in the literature are known to be of this form, up to applying derived equivalences to the three rings \( A, B \) and \( C \).

One of the main applications of recollements is to relate homological data of the three rings, such as global or finitistic dimension [17, 3], K-theory [34, 10, 3] and Hochschild (co)homology [22, 16, 24]. From a practical point of view, for a recollement induced by a stratifying ideal, the resulting long exact sequences are much easier to handle, one reason being that the six functors in this case are derived from the obvious six functors associated with an idempotent on the level of module categories. Moreover, up to Morita equivalence of the algebras involved, these are

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exactly the recollements that can be produced by deriving recollements of module categories (see the classification of such recollements of abelian categories by Psaroudakis and Vitória [30]).

Motivated by a kind of folklore conjecture, we consider the question if all recollements of derived module categories are of this form. Taken naively, this question has an obvious negative answer, since one may hide the epimorphism $A \to A/AeA$ by replacing, for instance, $D(B)$ by an equivalent derived category $D(B')$, where there is no morphism at all from $A$ to $B'$. Moreover, there exist injective homological epimorphisms, which also induce recollements. A meaningful way to formulate the question is the following one, suggested by Changchang Xi:

**Question 1.1.** Given a recollement relating the derived module categories of three rings $A$, $B$ and $C$ as above, is there another - equivalent - recollement, obtained by replacing the derived module categories by equivalent ones, that is induced by a stratifying ideal?

In the following, we will call a recollement ‘stratifying’ when it is induced by a stratifying ideal, and thus derived from a recollement on module level. Then the question is whether ‘stratifying’ recollements do give a normal form of recollements, similar to the situation for module categories described in [30].

In the second Section, we will see that the answer to Question 1.1 is negative, even for finite dimensional algebras of finite global dimension, and even when allowing to change all three derived categories. More precisely, we are going to characterise - in Theorem A and Corollary 2.2 - the recollements, which up to derived equivalence are induced by a stratifying ideal. Using these characterisations, we will give a counterexample to Question 1.1.

In parallel work [31], Psaroudakis and Vitória have been able to provide a positive answer to this question for hereditary rings, using rather different methods.

The negative answer to Question 1.1 puts additional emphasis on the following questions, which we are going to address in Sections 3 and 4.

**Question 1.2.** Given a ring epimorphism $f : A \to B$, when is it a surjective homological epimorphism with a stratifying kernel?

The question of when an abstractly given ring epimorphism $f$ is surjective has been studied by several authors in the past, see e.g. [32, 37]. In Section 3 we provide some new criteria. The main result is Theorem B which gives a sufficient condition for surjectivity, and then also for the other desired properties, when $A$ is a perfect (e.g. an artinian) ring.

**Question 1.3.** Given a ring epimorphism that is not stratifying, when can it be replaced by a surjective homological epimorphism with a stratifying kernel?

In Section 4, we give two such constructions. The first one uses a connection with tilting theory from [15, 5] to show that certain recollements induced by injective ring epimorphisms are equivalent to stratifying recollements. The second construction, Theorem C, allows to form a ring epimorphism $A \to C$ with favourable properties from a given ring epimorphism $A \to B$.

Throughout the paper, rings are associative and unital. Modules by default are right modules and $D(A)$ denotes the unbounded derived category of the category of all modules over a ring $A$. 
2. Characterisations, and a counterexample

In this Section we are going to characterise recollements equivalent to stratifying ones, see Theorem [A] and Corollary [2.2]. As a consequence, we will answer Question [1.1] and variations of it, negatively, by giving an explicit counterexample.

2.1. Definitions and notations. We will use standard terminology for derived categories, derived equivalences and tilting complexes, see for instance [40].

Let \( C \) be a triangulated category with shift functor [1]. An object \( X \) of \( C \) is exceptional if \( \text{Hom}_C(X,X[n]) = 0 \) unless \( n = 0 \). Let \( S \) be a set of objects of \( C \). As usual, \( \text{thick } S \) denotes the smallest triangulated subcategory of \( C \) containing \( S \) and closed under taking direct summands. Assume further that \( C \) has all (set-indexed) infinite direct sums. An object \( X \) of \( C \) is compact if the functor \( \text{Hom}_C(X,-) \) commutes with taking direct sums.

A recollement [7] of triangulated categories is a diagram

\[
\begin{array}{ccc}
C' & \xleftarrow{i} & C \\
\downarrow{i^*} & & \downarrow{j^*} \\
C & \xrightarrow{j} & C''
\end{array}
\]

of triangulated categories and triangle functors such that

1. \((i^*,i_+),(j_+,j^*),(j^*,j_+)\) are adjoint pairs;
2. \(i_+,j_+\) are full embeddings;
3. \(i^* \circ j_+ = 0 \) (and thus also \( j^* \circ i_+ = 0 \) and \( i^* \circ j_+ = 0 \));
4. for each \( C \in C \) there are triangles

\[
\begin{array}{ccc}
i_+ i^*(C) & \longrightarrow & C \\
\downarrow & & \downarrow \quad \downarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
j_+ j^*(C) & \longrightarrow & C \\
\downarrow & & \downarrow \quad \downarrow \\
\end{array}
\]

where the maps are given by adjunctions.

Thanks to (1) and (3), the two triangles (often called the canonical triangles) in (4) are unique up to unique isomorphisms.

Two recollements involving categories \( C',C,C'' \) and \( D',D,D'' \), respectively, are called equivalent, if there exists a triangle equivalence \( C \simeq D \) inducing triangle equivalences \( C' \simeq D' \) and \( C'' \simeq D'' \) such that all squares commute.

An epimorphism \( \varphi : A \to B \) in the category of rings is called a ring epimorphism. Equivalently, the induced functor \( \varphi_* : \text{Mod-}B \to \text{Mod-}A \) is a full embedding. Another equivalent characterisation of \( \varphi \) being a ring epimorphism is that \( \text{Coker}(\varphi) \otimes_A B = 0 \), see e.g. [36, Chapter XI, Proposition 1.2].

Furthermore \( \varphi \) is a homological epimorphism if and only if the induced functor \( \varphi_* : D(B) \to D(A) \) is a full embedding, or equivalently, \( \varphi \) is a ring epimorphism with \( \text{Tor}_i^A(B,B) = 0 \) for all \( i \geq 1 \) (cf. [15, Theorem 4.4]). In this case, \( \varphi \) induces a recollement

\[
\begin{array}{ccc}
D(B) & \xleftarrow{i^*} & D(A) \\
\downarrow{i_*} & & \downarrow{i^*} \\
\end{array}
\]
for some triangulated category $X$, and the functors on the left hand side are induced by $\varphi$, that is, $i^* = - \otimes_A B$, $i^! = \mathsf{RHom}_A(B, -)$ and $i_* = \varphi_*$.  

Homological epimorphisms starting in a ring $A$ are closely related with recollements of $\mathcal{D}(A)$ where the left hand term is a derived category of a ring too. We say that a recollement of $\mathcal{D}(A)$ by triangulated categories $C'$ and $C''$ is induced by a homological epimorphism $\varphi : A \to B$ if there is an equivalence $F : C' \to \mathcal{D}(B)$ such that $i_* = \varphi_* \circ F$. The following result characterises such recollements.

**Proposition 2.1.** [1, 1.7] A recollement of $\mathcal{D}(A)$ is induced by some homological epimorphism $A \to B$ if and only if $i^*(A)$ is exceptional. In this case, $B$ is the endomorphism ring of $i^*(A)$.

A surjective homomorphism $\varphi : A \to B$ is an epimorphism. If it is homological and its kernel $I$ is of the form $AeA$ with $e = e^2 \in A$ an idempotent, then $I = AeA$ is called a stratifying ideal, see [13]. For the purpose of this article, we then call the induced recollement a stratifying recollement. It is of the following form

$$
\begin{array}{cccc}
\mathcal{D}(A/AeA) & \mathcal{D}(A) & \mathcal{D}(eAe) \\
i^* & = & \overset{\text{i}}{\leftarrow} & j^! = j^* \\
i^! & \overset{\text{i}}{\rightarrow} & \overset{\text{j}}{\rightarrow} & j^! = j^* \\
\end{array}
$$

where

$$
i^* = - \otimes_A A/AeA, \quad i^! = \mathsf{RHom}_A(A/AeA, -),
$$
$$i_* = \mathsf{RHom}_{A/AeA}(A/AeA, -) = - \otimes_{A/AeA} A/AeA = i^!,
$$
$$j_! = - \otimes_{eA} eA, \quad j_* = \mathsf{RHom}_{eA}(eA, -),
$$
$$j^! = \mathsf{RHom}_A(eA, -) = - \otimes_A eA = j^*.
$$

Using this language, Question 1.1 asks whether each recollement of derived module categories of rings is equivalent to a stratifying one.

At this point, it has to be noted that homological epimorphisms do not behave well under Morita or derived equivalences applied to the corresponding recollement. The following easy example shows that when formulating Question 1.1 it is necessary to allow for changes of the data by Morita or derived equivalences.

Let $A = M_2(k) \times M_3(k)$, $B = M_2(k)$, and $\lambda : A \to B$ the projection. Then $\lambda$ is a homological epimorphism with stratifying kernel, inducing a recollement with $B$ on the left hand side, $A$ in the middle and $M_3(k)$ on the right hand side. Let $B' = k$, which is Morita equivalent to $B$. Then there is no ring homomorphism from $A$ to $B'$ at all, and in particular no homological epimorphism with stratifying kernel.

2.2. **Characterisations of stratifying recollements.** In the first answer to Question 1.1 we keep the ring $C$ fixed and characterise when it can be realised as the ring $eA'e$ in a stratifying recollement equivalent to a given one:

**Theorem A.** Fix rings $A$, $B$, and $C$ and a recollement

$$
(R) \quad \begin{array}{ccc}
\mathcal{D}(B) & \mathcal{D}(A) & \mathcal{D}(C) \\
\end{array}
$$


Then the following statements are equivalent.

(a) There exist rings $A'$ and $B'$ that are derived equivalent to $A$ and $B$ respectively, and an idempotent $e \in A'$ such that $C = eA'e$, $B' = A'/A'eA'$, and the projection $\pi : A' \to B'$ is a homological epimorphism with stratifying kernel, which induces a recollement equivalent to (R).

(b) The complex $j_!(C)$ is a direct summand of a tilting complex $T$ over $A$ such that $i^*(T)$ is exceptional.

Proof. Suppose (a) is given. Then the homological epimorphism $\pi$ induces a stratifying recollement

$$D(B') \xrightarrow{\pi} D(A') \xrightarrow{j_!} D(C).$$

where $C = eA'e$ and $B' = A'/A'eA'$. The functor $j_!$ is the derived tensor functor $\otimes_{eA'e} eA'$, which sends $C$ to the projective module $j_!(C) = eA'$. Setting $T := A'$ shows that $j_!(C)$ is a direct summand of a tilting complex. The functor $i^*$ is the derived tensor functor $\otimes_{A'} B'$, which sends $A'$ to $i^*(A') = B'$, which is exceptional. Thus the recollement induced by $\pi$ satisfies the conditions in (b). Moving from $A'$ and $B'$ to $A$ and $B$, respectively, by derived equivalences, does not affect the conditions in (b), since a derived equivalence sends a tilting complex to a tilting complex and an exceptional object to an exceptional object. Hence the original recollement satisfies (b) as well.

Suppose (b) is satisfied. Set $A' := \text{End}_{D(A)}(T)$. Write $T = T_1 \oplus T_2$, where $T_1 = j_!(C)$. The tilting complex $T$ induces an equivalence $\alpha : D(A) \xrightarrow{\sim} D(A')$, which we use to change the recollement into one with middle term $D(A')$. The new functor $j_!$ sends $C$ to the image of $T_1$ under the derived equivalence $\alpha$. By construction of $\alpha$, this image $j_!(C) = \alpha(T_1)$ is a projective module $eA'$ for some idempotent $e$. Composing $j_!$ with a derived auto-equivalence of $C$, if necessary, we may assume that $C = eA'e$ and the new $j_!$ is the derived tensor functor $\otimes_{eA'e} eA'$. Therefore, the other two functors on the right hand side of the recollement, which are uniquely determined by being adjoints, are as required in a stratifying recollement, too.

The ring $A'$ may in general not be a flat algebra over $Z$. Therefore, we are now going to replace it by a flat dg ring $A''$, chosen as follows: Let $f : A'' \to A'$ be a cofibrant replacement of $A'$. This is provided as part of the model structure of the category of small dg categories constructed by Tabuada in [38]. Here the base ring is $Z$. As shown in the proof of [29] Lemma 5], $A''$ is flat over $Z$. By definition, $f$ is a quasi-equivalence in the sense of [21] Section 7]. In particular, the 0-cohomology of $A''$ is $A'$, and $f$ induces a derived equivalence $D(A'') \to D(A')$. Thus we obtain a recollement which has $D(A'')$ as the middle term and which is equivalent to the original one. By [29] Theorem 4] (which requires the dg ring $A''$ to be flat over $Z$), the new recollement is induced by a homological epimorphism of dg rings $\varphi : A'' \to B''$, where $B''$ is a dg endomorphism ring of $i^*(A'')$.

By construction of the new recollement, $i^*(A'')$ equals what was $i^*(T)$ in the old recollement. Therefore, $i^*(A'')$ is exceptional. As a consequence, the dg endomorphism ring $B''$ is quasi-isomorphic to its 0-cohomology $B' := H^0(B'')$, which is an ordinary ring. Returning from $A''$ to its derived equivalent 0-cohomology $A'$, we can replace the last recollement by an equivalent one with left hand term $D(B')$, middle term $D(A')$, and right hand term unchanged. Let $\pi : A' \to B'$
be the 0-cohomology of the homological epimorphism \( \varphi : A'' \to B'' \). Then \( \pi \) is a homological epimorphism, and the triangle

\[
A'e \otimes_{eA'e} eA' \to A' \xrightarrow{\pi} B' \to A'e \otimes_{eA'e} eA'[1]
\]
yields a short exact sequence

\[
0 \longrightarrow A'e \otimes_{eA'e} eA' \longrightarrow A' \xrightarrow{\pi} B' \longrightarrow 0,
\]
since both \( H^1(A'e \otimes_{eA'e} eA') \) and \( H^{-1}(B'') \) vanish. Here, the kernel of \( \pi \) is the multiplication map \( A'e \otimes_{eA'e} eA' \to A' \), whose image is \( A'\! eA' \). Therefore \( B' \cong A/A'\! eA' \) and up to this isomorphism \( \pi \) is identified with the quotient map \( A' \to A/A'\! eA' \), implying that the ideal \( A'\! eA' \) is stratifying.

The sequence of modifications to the given recollement is summarised in the following diagram

\[
\begin{array}{ccc}
\mathcal{D}(B) & \cong & \mathcal{D}(A) \\
\downarrow & & \downarrow \cong \\
\mathcal{D}(B) & \cong & \mathcal{D}(A') \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{D}(B'') & \cong & \mathcal{D}(A'') \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{D}(B') & \cong & \mathcal{D}(A') \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{D}(A'/\! eA') & \cong & \mathcal{D}(A') \\
\end{array}
\]

\[
\mathcal{D}(A') \cong \mathcal{D}(eA') \cong \mathcal{D}(eA'').
\]

When we relax the condition on \( C \) to be not necessarily isomorphic, but at least derived equivalent to \( eA' \) in the stratifying recollement, the answer to Question \[1\] is as follows:

**Corollary 2.2.** Fix rings \( A, B \) and \( C \) and a recollement

\[
(R) \quad \mathcal{D}(B) \cong \mathcal{D}(A) \cong \mathcal{D}(C).
\]

Then the following statements are equivalent.

(a) There exist rings \( A', B' \) and \( C' \) that are derived equivalent to \( A, B \) and \( C \) respectively, and an idempotent \( e \in A' \) such that \( C' = eA'e \), \( B' = A'/\! eA' \) and the projection \( \pi : A' \to B' \)
is a homological epimorphism with stratifying kernel which induces a recollement equivalent to \( (R) \).

(b) There exists a tilting complex \( T_0 \) over \( C \) such that the complex \( j_!(T_0) \) is a direct summand of a tilting complex \( T \) over \( A \) with \( i^*(T) \) being exceptional.

**Proof.** Denote \( \text{End}_{\mathcal{D}(C)}(T_0) \) by \( C' \). The tilting complex \( T_0 \) in \( (b) \) induces an equivalence of \( \mathcal{D}(C) \) with \( \mathcal{D}(C') \). Using this equivalence, the given recollement can be changed into one with \( \mathcal{D}(C') \) on the right hand side. Applying Theorem \[A\] to this recollement proves the equivalence of \( (a) \) and \( (b) \). \[\square\]
Remark. In special situations, part of condition (b) may be dropped. Here is an example: Suppose $A$ is a finite-dimensional algebra over a field with only two isomorphism classes of simple modules. Then (b) is equivalent to

(b') the complex $j_!(C)$ can be completed to a tilting complex $T$ over $A$.

In fact, in this case, both $B$ and $C$ are local algebras by [3, Proposition 6.5]. Therefore, any tilting complex $T_0$ over $C$ is a projective generator, so $j_!(T_0)$ can be completed to a tilting complex if and only if $j_!(C)$ can be completed to a tilting complex. Now assume that $j_!(C)$ can be completed to a tilting complex $T$. Then $i^*(T)$, being compact in $D(B)$, either is a shifted projective module or it has self-extensions in positive degrees, see for instance [33, 2.11-2.13]. But by [3] Proposition 6.6 (or the more general [18, Theorem 4.8]), $i^*(T)$ is a silting object of $K^b(projA)/thick j_!(C) \cong K^b(projB)$, so the latter case does not occur. That is, $i^*(T)$ is exceptional.

Remark. In [23], the existence of recollements of derived module categories has been characterised in terms of the existence of two exceptional complexes satisfying certain orthogonality conditions. The complex $j_!(C)$ always is exceptional. If $i_*(i^*(T))$ is exceptional, too, these two complexes together satisfy the conditions in the characterisation. Thus, exceptionality of $i_*(i^*(T))$, which corresponds to $i^*(T)$ being exceptional, can be understood as restating the existence of the recollement. When taking this point of view, the additional condition needed for this recollement to be stratifying (up to equivalence) is that $j_!(C)$ can be completed to a tilting complex.

Remark. Given $A$ and $C = eAe$, the exact functor $- \cdot e$ can be used to construct a ‘half recollement’, which is the right hand side (involving $A$ and $C$) of the stratifying recollement investigated here. The left hand side then can be completed by taking the derived category of some dg ring. The problem, however, is to construct the left hand side as the derived category of an ordinary ring. This is not always possible. There do exist examples of recollements, with given $A$ and $C = eAe$, where the left hand side cannot be a derived module category. This happens for instance, if $A$ has finite global dimension, but the endomorphism ring $C$ of some exceptional (or even projective) object has infinite global dimension, see e.g. [3, Proposition 2.14].

2.3. A counterexample. Here is an example of a recollement that cannot be turned into a stratifying one by replacing $A$, $B$ and $C$ by derived equivalent algebras. In other words, the following recollement does not satisfy condition (b) in Theorem A nor in Corollary [2.2].

Example 2.3. In [23] Example 4.4, the following algebra is studied:
Let \( k \) be a field and let \( A \) be the \( k \)-algebra given by quiver and relations

\[
\begin{array}{ccc}
  & 2 & \\
1 & \overset{\alpha}{\rightarrow} & \\
\beta & \overset{}{\rightarrow} & \\
\gamma & \overset{}{\rightarrow} & 3
\end{array}
\]

\[
\begin{array}{ccc}
  & \delta \nwarrow & \\
\uparrow & \alpha \searrow & \\
\downarrow & \beta \nearrow & \\
1 & \overset{}{\rightarrow} & 3
\end{array}
\]

\[
\beta \alpha = 0, \quad \alpha \delta = 0, \quad \delta \gamma = 0.
\]

The simple module \( S_1 \) supported at 1 is a compact exceptional module of projective dimension 2. It has a minimal projective resolution over \( A \) given by the exact sequence

\[
0 \rightarrow P_2 \overset{\beta}{\rightarrow} P_3 \overset{\delta}{\rightarrow} P_1 \overset{}{\rightarrow} S_1 \overset{}{\rightarrow} 0.
\]

As shown in [25], setting \( e = e_2 + e_3 \) the algebra \( A \) has a stratifying ideal \( AeA \) and thus a stratifying recollement

\[
D(A/AeA) \cong D(A) \cong D(eAe).
\]

where \( A/AeA \) is one-dimensional, i.e. isomorphic to the ground field \( k \), and as a right \( A \)-module isomorphic to the simple module \( S_1 \). The algebra \( eAe \) is isomorphic to the Kronecker algebra, hence hereditary. The algebra \( A \) has finite global dimension, and therefore it is possible to mutate (that is, extend) the above recollement downwards, by [3, Section 3]. Thus, there is a recollement

\[
D(B = eAe) \cong D(A) \cong D(C = k).
\]

where \( j_!(C) \) equals \( S_1 \). Extending earlier work of Rickard and Schofield, it has been checked in [25] that \( S_1 \) cannot be a direct summand of a tilting complex, that is, \( j_!(C) \) fails the condition in (b). Hence this recollement cannot be turned into a stratifying one by changing \( A \) and \( B \). Moreover, the auto-equivalences of \( D(k) \) are compositions of Morita equivalences and shifts. Therefore, replacing \( C \) by a derived equivalent algebra \( C' \) does not remove the obstruction to extending \( j_!(T_0) \).

It follows that condition (b) in Theorem [A] and also in Corollary [2.2] fails.

Note that \( i^*(A) \) as a module over the Kronecker algebra \( eAe \) is a direct sum of projective modules and a quasi-simple regular module \( M \). Since \( M \) has self-extensions, \( i^*(A) \) is not exceptional. Therefore, by Proposition [2.1] this recollement is not induced by a homological epimorphism.

This recollement also restricts to recollements on the level of bounded or left or right bounded derived categories by [3, Proposition 4.12]. Homotopy categories of projectives in this case are covered, too, since they coincide with the bounded derived categories. So, Question [1.1] has a negative answer for all these choices of derived module categories.

3. Surjective Homological Epimorphisms and Stratifying Ideals

When is a ring epimorphism \( \varphi : A \rightarrow B \) equivalent to a homological epimorphism \( A \rightarrow A/AeA \) with stratifying kernel? Of course, one first has to decide if \( \varphi \) is surjective. The main result of
this Section, Theorem 3 provides a criterion for that. Once surjectivity is known, well-known facts can be used to decide if the kernel is idempotent or even stratifying.

Recall that a ring $R$ is called semilocal if the quotient ring $R/\text{rad}(R)$ is semisimple artinian, and it is right perfect if in addition the Jacobson radical $\text{rad}(R)$ is a left-$t$-nilpotent ideal of $R$, i.e. for any sequence of elements $a_1,a_2,a_3,\ldots \in \text{rad}(R)$ there is an integer $n > 0$ such that $a_n a_{n-1} \ldots a_1 = 0$.

**Theorem B.** Let $\varphi : A \to B$ be a ring epimorphism with $A$ right (or left) perfect and $B$ semilocal. Suppose that $B$ is basic, that is, $B/\text{rad}(B)$ is a product of skew-fields. Then $\varphi$ is surjective. Moreover, $\varphi$ has a stratifying kernel if it is a homological epimorphism.

The crucial point here is to prove the surjectivity of $\varphi$. The proof will use the following characterisation of surjective ring epimorphisms as well as a consequence of this characterisation.

**Proposition 3.1.** Let $\varphi : A \to B$ be a ring epimorphism with $B$ semilocal. Then $\varphi$ is surjective if and only if each simple $B$-module is simple as an $A$-module.

**Proof.** The only-if-part is clear. To prove the converse, assume that all simple $B$-modules are simple as $A$-modules, too. Set $\tilde{B} = B/\text{rad}(B)$. Clearly, the composition $\pi : A \to \tilde{B}$ of $\varphi$ with the canonical projection $B \to \tilde{B}$ is a ring epimorphism such that all simple $\tilde{B}$-modules are simple as $A$-modules, and by Nakayama’s lemma it suffices to show that $\pi$ is surjective. So we can assume w.l.o.g. that $B$ is semisimple artinian.

Suppose now that there is an indecomposable direct summand $S$ of $B$, hence a simple $B$-module, which is not contained in the image of $\varphi$. Then the intersection $\text{Im}(\varphi) \cap S$ is a proper $A$-submodule of the simple $B$-module $S$, which by assumption also is a simple $A$-module. Thus, $\text{Im}(\varphi) \cap S = 0$ and $S$ is a direct summand of the cokernel of $\varphi$.

As mentioned above in Section 2.1, an equivalent condition of $\varphi$ being a ring epimorphism is that $\text{Coker}(\varphi) \otimes_A B = 0$, which implies $S \otimes_A B = 0$. But $S \otimes_A B = \varphi^* \varphi_*(S) \simeq S$, yielding a contradiction. So, $\varphi$ must be surjective. □

The following consequence of Proposition 3.1 is a special case of results by Storrer [37]. Storrer shows ([37, Corollary 5.4]) that self-injective rings, hence in particular semisimple rings, are saturated. Here, $R$ saturated means there is no non-trivial injective ring epimorphism starting in $R$.

**Lemma 3.2.** Injective ring epimorphisms between semisimple rings are isomorphisms.

**Proof.** Let $A,B$ be two semisimple rings and $\psi : A \to B$ an injective ring epimorphism. Let $S$ be a simple $B$-module. Its endomorphism ring $\text{End}_B(S)$ is local. Since $\psi$ is a ring epimorphism, the restriction functor $\psi_* : \text{Mod}-B \to \text{Mod}-A$ is fully faithful. Hence, $S$ must be indecomposable as an $A$-module, and thus simple over $A$. Now Proposition 3.1 can be applied. □

The following known statement will imply properties of $\text{Ker}(\varphi)$ in Theorem 3.
Lemma 3.3. Let $\varphi : A \to B$ be a surjective ring epimorphism. Then $\text{Tor}_1^A(B, B) = 0$ if and only if the kernel $\text{Ker}(\varphi)$ is an idempotent ideal of $A$.

In particular, if $A$ is right (or left) perfect, then every surjective homological epimorphism has a stratifying kernel.

Proof. The first statement is well known, see e.g. [8]. For the second assertion we use [27, Proposition 2.1], where it is shown that idempotent ideals of (one-sided) perfect rings are generated by idempotent elements. □

Proof of Theorem B.
First of all, $\varphi$ factors through its image $C$ as $\varphi = \tau \circ \psi$ where $\tau : C \to B$ is an injective ring epimorphism, and $\psi : A \to C$ is a surjective ring homomorphism, hence an epimorphism, too. Note that $C$ is again right perfect by [20, Corollary 11.7.3]. So, we can assume without loss of generality that $\varphi$ is injective and show that it is an isomorphism.

Since the quotient $B/\text{rad}(B)$ is a product of skew-fields, it does not contain non-zero nilpotent elements. Now the radical $\text{rad}(A)$ of the right perfect ring $A$ is left-t-nilpotent, thus its elements are nilpotent and so must be their images under $\varphi$. Hence they vanish in $B/\text{rad}(B)$.

Therefore we may pass to the quotients $\bar{A} = A/\text{rad}(A)$ and $\bar{B} = B/\text{rad}(B)$ and consider the ring homomorphism $\bar{\varphi} : \bar{A} \to \bar{B}$ between semisimple rings. It is also a ring epimorphism, for instance because $\text{Coker} \bar{\varphi} \otimes_{\bar{A}} \bar{B} = 0$. We claim that $\bar{\varphi}$ is injective. Since $\bar{A}$ is semisimple, the kernel $\text{Ker}(\bar{\varphi})$ is a direct summand of $\bar{A}$. If it is not zero, it must contain an idempotent $\bar{e}$. The radical $\text{rad}(A)$ is left-t-nilpotent, so by [20, Theorem 11.5.3] there is a lifting $e \in A$ such that $e^2 = e$ and $e + \text{rad}(A) = \bar{e}$. By the choice of $e$, the element $\varphi(e)$ is an idempotent element belonging to $\text{rad}(B)$, so it is zero, which implies $e = 0$ by the injectivity of $\varphi$. Hence also $\bar{e} = 0$. This proves the injectivity of $\bar{\varphi}$.

Now by Lemma 3.2 $\bar{\varphi}$ is an isomorphism. In particular, the set of simple $B$-modules coincides with the set of simple $A$-modules. Hence, by Proposition 3.1 $\varphi$ is surjective and thus an isomorphism.

To finish the proof, we just observe that the last statement follows from Lemma 3.3. □

The proof of Theorem B works as well when relaxing the assumption $B$ to be basic by requiring instead an inclusion $\tau(\text{rad}(C)) \subseteq \text{rad}(B)$.

As an application, a positive answer to Question 1.1 can be given in a particular situation:

Corollary 3.4. Let $A$ be right (or left) perfect and $B$ semilocal. Suppose there is a recollement of the derived module categories

\[
\begin{align*}
D(B) & \iff \text{Coker} \varphi \otimes_{\bar{A}} \bar{B} = 0 & \text{End}_{D(B)}(i^*(A)) \\
D(A) & \iff \text{rad}(B) \\
D(C) & \iff \text{rad}(C)
\end{align*}
\]

such that $i^*(A)$ is exceptional and basic. Then the recollement is equivalent to a stratifying one.

Artinian rings, for instance, are perfect and semilocal.

Proof. By [2.1] the recollement is induced by the homological epimorphism $\varphi : A \to \text{End}_{D(B)}(i^*(A))$. In order to be able to apply Theorem B we have to show:

Claim. The ring $\text{End}_{D(B)}(i^*(A))$ is semilocal.
Proof. The complex $X := i^*(A)$ is compact. Its entries are finitely generated projective $B$-modules $X_1, \ldots, X_l$ for some $l$. The endomorphism ring of $X$ as a complex is the subring $R$ of $R' := \text{End}_B(X_1) \times \cdots \times \text{End}_B(X_l)$ formed by $l$-tuples satisfying the commutativity condition in the definition of morphisms of complexes. Factoring out homotopies, a quotient ring $\bar{R}$ of $R$ is obtained that is isomorphic to $\text{End}_{D(B)}(i^*(A))$.

It is well known (see for instance [14, Section 1.2]) that if a ring $S$ is semilocal, so are all full matrix rings over $S$, all rings of the form $eSe$ for an idempotent element $e \in S$, and all quotient rings of $S$. Further, direct products of finitely many semilocal rings are semilocal, too.

Now, since $B$ is semilocal, we infer that $\text{End}_B(X_j)$ is semilocal for all $j$, and the direct product $R' = \text{End}_B(X_1) \times \cdots \times \text{End}_B(X_l)$ is so, too. The inclusion $R \subset R'$ is a local homomorphism in the sense that it carries non-units to non-units (or equivalently, $R$ is rationally closed in $R'$), because the inverse of an $l$-tuple of isomorphisms satisfying the commutativity conditions automatically satisfies the commutativity conditions as well. Therefore, a result by Camps and Dicks [9, Corollary 2] implies that $R$ is semilocal, too. Then so is its quotient $\bar{R}$, and the claim is proven.

Now, the statement follows from Theorem [13] \hfill \Box

Remark. In the proof of Corollary [5.4] we need to change the left and (in general also) the right hand terms of the recollement to get a stratifying one, while leaving the middle term $D(A)$ unchanged. The original and the modified recollement are in the same equivalence class of recollements of $D(A)$, according to the definition of equivalence of recollements in [11, 1.7].

4. Constructing homological epimorphisms with stratifying kernel

If $\lambda$ fails to be a surjective homological epimorphism with a stratifying kernel, one may try to replace $\lambda$ by a new homological epimorphism with better properties.

Can one change a homological epimorphism into a stratifying one in a way compatible with a given recollement?

A more precise formulation of this question is as follows: Suppose $\lambda : A \rightarrow B$ is a homological epimorphism. Are there rings $A'$ and $B'$ which are derived equivalent to $A$ and $B$, respectively, and a stratifying homological epimorphism $\lambda' : A' \rightarrow B'$ such that the following diagram commutes?

$$
\begin{array}{ccc}
D(\text{Mod}B) & \xrightarrow{\lambda_*} & D(\text{Mod}A) \\
\downarrow \cong & & \downarrow \cong \\
D(\text{Mod}B') & \xrightarrow{\lambda'_*} & D(\text{Mod}A')
\end{array}
$$

The results in Sections 2 and 3 suggest that some restrictions need to be imposed on the setup.

For instance, one can use the following connection with tilting theory from [15, 6]: if $\lambda : A \rightarrow B$ is an injective ring epimorphism such that $\text{Tor}_1^A(B, B) = 0$ and the right $A$-module $B_A$ has projective dimension at most one (which implies in particular that $\lambda$ is homological), then the $A$-module $T := B \oplus B/A$ is tilting. Tilting modules arising in this way are characterised by the
existence of a $T$-coresolution of $A$ of the form $0 \to A \to T_0 \to T_1 \to 0$ where $T_0, T_1 \in \text{Add}(T)$ satisfy $\text{Hom}_A(T_1, T_0) = 0$, see [5, Theorem 3.10].

Notice that such $T$ will not be finitely generated in general. Assuming $B_A$ to be finitely presented, however, gives a setup of interest in our context, since $T$ is then a classical tilting module and $A$ can be replaced by a derived equivalent ring $A'$.

This will be our first construction. The second construction will produce from $\lambda$ a new ring homomorphism $\mu : A \to C$, which will be a homological epimorphism under suitable assumptions.

**First construction.**

We present a case where the question above has a positive answer. In fact, it will be sufficient to change the ring $A$, while keeping $B$ unchanged.

**Proposition 4.1.** Suppose $\lambda : A \to B$ is a homological epimorphism. If $\lambda$ is injective and $B_A$ is finitely presented of projective dimension at most one, then there are a ring $A'$ which is derived equivalent to $A$ and a surjective homological epimorphism $\lambda' : A' \to B$, such that the two epimorphisms induce equivalent recollements.

**Proof.** Under the assumptions made, $T := B \oplus B/A$ is a tilting $A$-module and $\text{Hom}_A(B/A, B) = 0$, by [5, Theorem 3.5] and [15, Proposition 4.12]. The homological epimorphism $\lambda$ induces a recollement

$$
\begin{array}{ccc}
\mathcal{D}(B) & \subseteq & \mathcal{D}(A) \\
& \subseteq & \mathcal{D}(C)
\end{array}
$$

where $C := \text{End}_A(B/A)$ (see [1] Example 3.1 and [26, Theorem B]). Moreover,

$$
A' := \text{End}_A(T) = \begin{pmatrix} B = \text{End}_A(B) & \text{Hom}_A(B, B/A) \\ 0 & C = \text{End}_A(B/A) \end{pmatrix}
$$

is derived equivalent to $A$. This is a well studied situation, see for instance [23, Cor. 12 and 15]. The $T$-resolution of $A$ is $0 \to A \to B \to B/A \to 0$. Let $e \in A'$ be the idempotent corresponding to $B/A$. Since $\text{Hom}_A(B/A, B) = 0$, $eA'(1 - e) = 0$. Hence $eA' = eA'e = C$, $A'e = A'eA'$ and $A'e \otimes_{eA'e} eA' = A'eA'$, that is, the ideal $A'eA'$ generated by $e$ is stratifying. By construction, $A'/A'eA' = \text{End}_A(B) = B$. Hence the stratifying ideal $A'eA'$ induces a recollement

$$
\begin{array}{ccc}
\mathcal{D}(B) & \subseteq & \mathcal{D}(A') \\
& \subseteq & \mathcal{D}(C),
\end{array}
$$

which is equivalent to the original one. In particular, there is the desired commutative diagram of derived categories, involving the derived equivalence between $A$ and $A'$. \(\square\)

The following example illustrates how the injective homological epimorphism $\lambda$ gets enlarged to obtain a surjective homological epimorphism $\lambda'$ which is 'derived equivalent' to $\lambda$ in the above sense.

**Example 4.2.** Let $A$ be the path algebra of the quiver $A_2$ over a field $k$; in other words, $A$ is the algebra of $2 \times 2$ upper triangular matrices over $k$. Let $B$ be the algebra of all $2 \times 2$ matrices
over $k$. The inclusion

$$\lambda : A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \hookrightarrow \begin{pmatrix} k & k \\ k & k \end{pmatrix} = B$$

is a homological epimorphism such that the simple $B$-module gets identified with the projective-injective $A$-module $P$. So $B$ as a right $A$-module is isomorphic to $P \oplus P$. By [2, Theorem 5.1], $\lambda$ induces a recollement of $\mathcal{D}(A)$ in terms of $\mathcal{D}(B)$ and $\mathcal{D}(C)$ for some $k$-algebra $C$. Moreover, the image of $j_!$ (respectively, $j_*$) is generated by the simple injective (respectively, the simple projective) $A$-module, which is left (respectively, right) perpendicular to $P$. This is an easy example of a recollement not of ‘stratifying type’. In fact, $A$ has two non-trivial stratifying ideals, generated by the two primitive idempotents, and the resulting recollements are different from the current one, as can be checked directly on objects.

The tilting module $T := B \oplus B/A$ is the direct sum $P \oplus P \oplus S$. Hence

$$A' = \text{End}_A(T) = \begin{pmatrix} k & k & k \\ k & k & k \\ 0 & 0 & k \end{pmatrix}$$

which is an enlarged version of $A$; $A$ and $A'$ are Morita equivalent, but $T$ is not a progenerator. The new homological epimorphism $\lambda' : A' \to B$ is surjective with a stratifying kernel.

This example also shows that modifying $B$ while keeping $A$ does in general not allow for a solution of the modification problem.

Second construction.

The following general construction of a ring homomorphism, whose kernel and cokernel can be controlled, will be used to produce homological epimorphisms with stratifying kernel.

We start with a ring homomorphism $f : A \to B$ with cone $K_f$ in $\mathcal{D}(A)$, so that there is a triangle in $\mathcal{D}(A)$

$$(\dagger) \quad A \xrightarrow{f} B \to K_f \to A[1].$$

Denote by $C$ the endomorphism ring of $K_f$ in $\mathcal{D}(A)$. Then a ring homomorphism $\mu : A \to C$ can be defined as follows: any element $a \in A$ defines a module homomorphism $A \to A$ and its $f$-image $f(a)$ defines a module homomorphism $B \to B$ according to the diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow a & & \downarrow f(a) \\
A & \xrightarrow{f} & B
\end{array}$$

which is commutative, since $1_A$ gets sent to $f(a)$ in both ways. Therefore, the pair $(a, f(a))$ is an endomorphism of the complex $A \to B$ and induces an endomorphism of $K_f$ in $\mathcal{D}(A)$. In this way, we obtain a ring homomorphism $\mu : A \to C$.

**Theorem C.** Let $f : A \to B$ be a ring epimorphism whose cone $K_f$ satisfies $\text{Ext}_{\mathcal{D}(A)}^1(K_f, K_f) := \text{Hom}_{\mathcal{D}(A)}(K_f, K_f[-1]) = 0$. Assume that $\text{Tor}^A_1(B, B) = 0$. Then the ring homomorphism $\mu : A \to C$ defined above has kernel $\text{Hom}_A(B, A)$ and cokernel $\text{Ext}_A^1(B, A)$. 

Proof. Applying various Hom-functors to the triangle (†) yields long exact sequences, where we write \((X, Y) = \text{Hom}_{\mathcal{D}(A)}(X, Y)\) for short:

1. \(0 = (A, A[-1]) \rightarrow (K_f, A) \rightarrow (B, A) \rightarrow (A, A) \xrightarrow{\text{conn}} (K_f, A[1]) \rightarrow (B, A[1]) \rightarrow (A, A[1]) = 0\),
2. \(0 = (A, B[-1]) \rightarrow (K_f, B) \rightarrow (B, B) \xrightarrow{\text{conn}} (A, B) \rightarrow (K_f, B[1]) \rightarrow (B, B[1]) = \text{Ext}^1_A(B, B)\),
3. \(0 = (K_f, K_f[-1]) \rightarrow (K_f, A) \rightarrow (K_f, B) \rightarrow (K_f, K_f) = C \xrightarrow{\beta} (K_f, A[1]) \rightarrow (K_f, B[1])\).

In (1) and in (2), the starting terms vanish, since modules don’t have extensions in negative degrees. The starting term in (3) vanishes by assumption on \(K_f\).

In (2), the assumption \(\text{Tor}^1_A(B, B) = 0\) implies \(\text{Ext}^1_A(B, B) \simeq \text{Ext}^1_B(B, B) = 0\) by [35, Theorem 4.8]. Since \(f\) is a ring epimorphism, \(\text{Hom}_A(B, B) = \text{Hom}_B(B, B) \simeq B \simeq \text{Hom}_A(A, B)\), hence the map \(\alpha\) is an isomorphism and \((K_f, B) = 0 = (K_f, B[1])\).

Plugging this into (3) gives \((K_f, A) = 0\) and \(\beta : C \xrightarrow{\sim} (K_f, A[1])\).

Now the sequence (1) reduces to
\[
0 \rightarrow (B, A) \rightarrow (A, A) \xrightarrow{\text{conn}} (K_f, A[1]) \rightarrow (B, A[1]) \rightarrow 0.
\]
For any \(a \in A\), the following commutative diagram
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{\alpha} & \downarrow{f(a)} & \downarrow{\mu(a)} \\
A & \xrightarrow{f} & K_f \\
\downarrow{\alpha[1]} & \downarrow{\mu[1]} & \downarrow{\pi} \\
A & \xrightarrow{f} & K_f \\
\downarrow{\alpha[1]} & \downarrow{\mu[1]} & \downarrow{\pi} \\
A & \xrightarrow{f} & A[1]
\end{array}
\]
shows that \(\text{conn}(a) = a[1] \circ \pi = \pi \circ \mu(a) = (\beta \circ \mu)(a)\). Namely, under the identifications \((A, A) \simeq A\) and \(\beta : C \xrightarrow{\sim} (K_f, A[1])\), the connecting homomorphism \(\text{conn}\) gets identified with \(\mu\). This finishes the proof. \(\square\)

In the special case of injective ring epimorphisms, \(K_f = B/A\) is a module and thus it has no negative self-extensions. So Theorem 4.3 has the following consequence, generalising a construction from [15, p. 295].

**Corollary 4.4.** Let \(\lambda : A \rightarrow B\) be an injective ring epimorphism such that \(\text{Tor}^1_A(B, B) = 0\). Let \(C := \text{End}_A(B/A)\) be the endomorphism ring of \(B/A\) as right \(A\)-module. Then the left \(A\)-module structure on \(B/A\) induces a ring homomorphism \(\mu : A \rightarrow C\) such that \(\text{Ker}(\mu) \simeq \text{Hom}_A(B, A)\) and \(\text{Coker}(\mu) \simeq \text{Ext}^1_A(B, A)\).

Under additional assumptions, this leads to homological epimorphisms with stratifying kernel:

**Corollary 4.4.** Let \(A, B\) be artin algebras, and let \(\lambda : A \rightarrow B\) be an injective ring epimorphism such that \(B_A\) has projective dimension at most one and \(\text{Tor}^1_A(B, B) = 0\). Then \(\mu : A \rightarrow C\) is a homological epimorphism, and the projective dimension of \(AC\) as left \(A\)-module is at most one. Moreover, if also the projective dimension of \(A\text{Ext}^1_A(B, A)\) as left \(A\)-module is at most one, then \(\mu\) has a stratifying kernel.
Proof. The first statement follows from [13, Proposition 4.13]. For the second statement, we set \( I = \text{Ker}(\mu) \) and consider the surjective ring epimorphism \( \nu : A \to A/I \). The condition \( \text{Tor}_i^A(A/I, A/I) = 0 \) for \( i = 1 \) is verified as in the first part of the proof of [4, Lemma 4.5], using that the left \( A \)-modules \( C \) and \( \text{Coker} \mu \) have projective dimension at most one, and similarly one checks the cases \( i \geq 2 \). So \( \nu \) is a surjective homological epimorphism, and the claim follows from Lemma 3.3 since \( A \) is perfect. \( \square \)

Example 4.5. Let \( \lambda : A \to B \) be an injective homological epimorphism of hereditary artin algebras. From the Corollaries above we deduce that the homological epimorphism \( \mu : A \to C \) is

(i) surjective with a stratifying kernel if and only if \( B_A \) is projective,
(ii) injective if and only if \( B_A \) has no projective direct summand.

For case (i) see also [26, Theorem B].

More concretely, let \( A \) be the Kronecker algebra over a field \( k \), and let \( P_i \) be the indecomposable preprojective module of dimension vector \((i, i + 1)\) for \( i = 1, 2, 3 \). Consider the tilting module \( T = P_1 \oplus P_2 \). The minimal \( T \)-coresolution of \( A \) is given by \( 0 \to A \to P_1 \to P_2 \to 0 \), and \( T \) arises from the injective homological epimorphism \( \lambda : A \to B = \text{End}_A(P_1^3) \) as explained at the beginning of this Section. In this case \( B_A \) is projective, \( C = \text{End}_A(P_2) \simeq k \), and \( \mu : A \to C \) is the stratifying epimorphism induced by the idempotent element \( e \) of \( A \) corresponding to the projective module \( P_1 \).

Let us now consider the tilting module \( T' = P_2 \oplus P_3 \). The minimal \( T' \)-coresolution of \( A \) is given by \( 0 \to A \to P_2 \to P_3 \to 0 \), and \( T' \) arises from the injective homological epimorphism \( \lambda' : A \to B' = \text{End}_A(P_2^5) \). Here \( B'_A \) has no projective summand, \( C = \text{End}_A(P_3^3) \simeq M_3(k) \), and \( \mu : A \to C \) is injective.

Finally we remark that in the situation of Corollary 4.4 there is a ladder of height two as follows

\[
\mathcal{D}(B) \quad \equiv \quad \mathcal{D}(A) \quad \equiv \quad \mathcal{D}(C)
\]

where the ‘upper’ and ‘lower’ recollements are induced by the homological epimorphisms \( \lambda : A \to B \) and \( \mu : A \to C \) respectively (for the terminology of ladder see [3]). This shows that in some cases our second construction in Theorem C changes a homological epimorphism into a stratifying one, such that the induced recollements are not equivalent but lie on the same ladder. More precisely, the original recollement can be reflected one step downward to get the new one.

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