Some results of strongly primitive tensors

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Abstract In this paper, we show that an order $m$ dimension 2 tensor is primitive if and only if its majorization matrix is primitive, and then we obtain the characterization of order $m$ dimension 2 strongly primitive tensors and the bound of the strongly primitive degree. Furthermore, we study the properties of strongly primitive tensors with $n \geq 3$, and propose some problems for further research.

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1 Introduction

A nonnegative square matrix $A = (a_{ij})$ of order $n$ is nonnegative primitive (or simply, primitive) if $A^k > 0$ for some positive integer $k$. The least such $k$ is called the primitive exponent (or simply, exponent) of $A$ and is denoted by $\exp(A)$.

Since the work of Qi [7] and Lim [5], the study of tensors which regarded as the generalization of matrices, the spectra of tensors (and hypergraphs) and their various applications has attracted much attention and interest.

As is in [1], an order $m$ dimension $n$ tensor $\mathcal{A} = (a_{i_1i_2...i_m})_{1 \leq i_j \leq n}$ ($j=1,...,m$) over the complex field $\mathbb{C}$ is a multidimensional array with all entries

$$a_{i_1i_2...i_m} \in \mathbb{C} \{i_1, \ldots, i_m \in [n] = \{1, \ldots, n\}\}.$$

In [1] and [2], Chang et al investigated the properties of the spectra of nonnegative tensors, defined the irreducibility of tensors and the primitivity of nonnegative tensors (as Definition 1.1), and extended many important properties of primitive matrices to primitive tensors.

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Definition 1.1. (See [2]) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$, $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ a vector and $x^{[r]} = (x_1^r, x_2^r, \ldots, x_n^r)^T$. Define the map $T_{\mathbb{A}}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ as: $T_{\mathbb{A}}(x) = (Ax)^{[r]}$. If there exists some positive integer $r$ such that $T_{\mathbb{A}}^r(x) > 0$ for all nonnegative nonzero vectors $x \in \mathbb{R}^n$, then $\mathbb{A}$ is called primitive and the smallest such integer $r$ is called the primitive degree of $\mathbb{A}$, denoted by $\gamma(\mathbb{A})$.

Recently, Shao [8] defined the general product of two $n$-dimensional tensors as follows.

Definition 1.2. (See [8]) Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and $k \geq 1$), dimension $n$ tensor, respectively. Define the general product $\mathbb{A} \mathbb{B}$ to be the following tensor $\mathbb{D}$ of order $(m - 1)(k - 1) + 1$ and dimension $n$:

$$d_{i_0i_1\ldots i_{m-1}} = \sum_{i_2, \ldots, i_m=1}^n a_{i_0i_2\ldots i_m} b_{i_2i_1\ldots i_m} \ldots b_{i_mi_{m-1}i_0} \quad (i \in [n], \alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}).$$

The tensor product is a generalization of the usual matrix product, and satisfies a very useful property: the associative law ([8], Theorem 1.1). With the general product, when $k = 1$ and $\mathbb{B} = x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ is a vector of dimension $n$, then $\mathbb{A} \mathbb{B} = \mathbb{A}x$ is still a vector of dimension $n$, and for any $i \in [n]$, $(\mathbb{A} \mathbb{B})_i = (\mathbb{A}x)_i = \sum_{i_2, \ldots, i_m=1}^n a_{i_0i_2\ldots i_m} x_{i_2} \ldots x_{i_m}$.

As an application of the general tensor product defined by Shao [8], Shao presented a simple characterization of the primitive tensors. Now we give the definition of “essentially positive” which introduced by Pearson.

Definition 1.3. (See [6], Definition 3.1) A nonnegative tensor $\mathbb{A}$ is called essentially positive, if for any nonnegative nonzero vector $x \in \mathbb{R}^n, Ax > 0$ holds.

Proposition 1.4. (See [8], Proposition 4.1) Let $\mathbb{A}$ be an order $m$ and dimension $n$ nonnegative tensor. Then the following three conditions are equivalent:

1. For any $i, j \in [n], a_{ij \ldots} > 0$ holds.
2. For any $j \in [n], A e_j > 0$ holds (where $e_j$ is the $j$-th column of the identity matrix $I_n$).
3. For any nonnegative nonzero vector $x \in \mathbb{R}^n, Ax > 0$ holds.

By Proposition 1.4, the following Definition 1.5 is equivalent to Definition 1.3. What’s more, in Proposition 1.6 Shao showed a characterization of primitive tensors and defined the primitive degree by using the properties of tensor product and the zero patterns which defined by Shao in [8].

Definition 1.5. (See [8], Definition 4.1) A nonnegative tensor $\mathbb{A}$ is called essentially positive, if it satisfies one of the three conditions in Proposition 1.4.

Proposition 1.6. (See [8], Theorem 4.1) A nonnegative tensor $\mathbb{A}$ is primitive if and only if there exists some positive integer $r$ such that $\mathbb{A}^r$ is essentially positive. Furthermore, the smallest such $r$ is the primitive degree of $\mathbb{A}$, $\gamma(\mathbb{A})$. 
The concept of the majorization matrix of a tensor introduced by Pearson is very useful.

**Definition 1.7.** (See [6], Definition 2.1) The majorization matrix $M(\mathbb{A})$ of the tensor $\mathbb{A}$ is defined as $(M(\mathbb{A}))_{ij} = a_{ij...j}$, $i, j \in [n]$.

By Definition 1.5, Proposition 1.6 and Definition 1.7, the following characterization of the primitive tensors was easily obtained.

**Proposition 1.8.** (See [10], Remark 2.6) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. Then $\mathbb{A}$ is primitive if and only if there exists some positive integer $r$ such that $M(\mathbb{A}^r) > 0$. Furthermore, the smallest such $r$ is the primitive degree of $\mathbb{A}$, $\gamma(\mathbb{A})$.

On the primitive degree $\gamma(\mathbb{A})$, Shao proposed the following conjecture for further research.

**Conjecture 1.9.** (See [8], Conjecture 1) When $m$ is fixed, then there exists some polynomial $f(n)$ on $n$ such that $\gamma(\mathbb{A}) \leq f(n)$ for all nonnegative primitive tensors of order $m$ and dimension $n$.

In the case of $m = 2$ ($\mathbb{A}$ is a matrix), the well-known Wielandt’s upper bound tells us that we can take $f(n) = (n-1)^2 + 1$. Recently, the authors [10] confirmed Conjecture 1.9 by proving Theorem 1.10.

**Theorem 1.10.** (See [10], Theorem 1.2) Let $\mathbb{A}$ be a nonnegative primitive tensor with order $m$ and dimension $n$. Then its primitive degree $\gamma(\mathbb{A}) \leq (n-1)^2 + 1$, and the upper bound is tight.

They also showed that there are no gaps in the tensor case in [11], which implies that the result of the case $m \geq 3$ is totally different from the case $m = 2$. In [8], Shao also proposed the concept of strongly primitive for further research.

**Definition 1.11.** (See [8], Definition 4.3) Let $\mathbb{A}$ be a nonnegative tensor with order $m$ and dimension $n$. If there exists some positive integer $k$ such that $\mathbb{A}^k > 0$ is a positive tensor, then $\mathbb{A}$ is called strongly primitive, and the smallest such $k$ is called the strongly primitive degree of $\mathbb{A}$.

Let $\mathbb{A} = (a_{i_1i_2...i_m})$ be a nonnegative tensor with order $m$ and dimension $n$. It is clear that if $\mathbb{A}$ is strongly primitive, then $\mathbb{A}$ is primitive. For convenience, let $\eta(\mathbb{A})$ be the strongly primitive degree of $\mathbb{A}$. Clearly, $\gamma(\mathbb{A}) \leq \eta(\mathbb{A})$. In fact, it is obvious that in the matrix case ($m = 2$), a nonnegative matrix $A$ is primitive if and only if $A$ is strongly primitive, and $\gamma(A) = \eta(A) = \exp(A)$. But in the case $m \geq 3$, Shao gave an example to show that these two concepts are not equivalent in the case $m \geq 3$. In [11], the authors proposed the following question.

**Question 1.12.** ([11], Question 4.18) Can we define and study the strongly primitive degree, the strongly primitive degree set, the $j$-strongly primitive of strongly primitive tensors and so on?
Based on Question 1.2, we study primitive tensors and strongly primitive tensors in this paper, show that an order $m$ dimension 2 tensor is primitive if and only if its majorization matrix is primitive, and obtain the characterization of order $m$ dimension 2 strongly primitive tensors and the bound of the strongly primitive degree. Furthermore, we study the properties of strongly primitive tensors with $n \geq 3$, and propose some problems for further research.

2 Preliminaries

In [11], the authors obtained the following Proposition 2.1 and gave Example 2.3 by computing the strongly primitive degree.

**Proposition 2.1.** ([11], Proposition 4.16) Let $\mathbf{A} = (a_{i_1i_2...i_m})$ be a nonnegative strongly primitive tensor with order $m$ and dimension $n$. Then for any $\alpha \in [n]^{m-1}$, there exists some $i \in [n]$ such that $a_{i\alpha} > 0$.

Let $k(\geq 0)$, $n(\geq 2)$, $q(\geq 0)$, $r(\geq 1)$ be integers and $k = (n - 1)q + r$ with $1 \leq r \leq n - 1$ when $k \geq 1$. In [4, 10, 11], the authors defined some nonnegative tensors with order $m$ and dimension $n$ as follows:

$$\mathbf{A}_k = (a_{i_1i_2...i_m})_{1 \leq i_j \leq n} \ (j=1,...,m),$$

where

$$M(\mathbf{A}_k) = M_1 = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 & 1 \\
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix},$$

(1) $a_{i_1i_2...i_m}^{[0]} = 0$, if $i_2 \ldots i_m \neq i_2 \ldots i_2$ for any $i \in [n]$.

(2) $a_{i_1i_2...i_m}^{[k]} = 1$, if $i \in [n] \setminus \{r - q, r - q + 1, \ldots, r + 1 \} \ (\text{mod } n)$ and $\alpha = i_2 \ldots i_m \in [n]^{m-1}$ with $\{i_2, \ldots, i_m\} = \{r - q + 1, r \} \ (\text{mod } n)$;

(3) $a_{i_1i_2...i_m}^{[k]} = 0$, except for (1) and (3).

The authors showed the tensors $\mathbf{A}_k \ (k \geq 0)$ are primitive, the primitive degree $\gamma(\mathbf{A}_0) = (n - 1)^2 + 1$ ([10]) and $\gamma(\mathbf{A}_k) = k + n$ ([11], Theorem 3.3) for $1 \leq k \leq n^2 - 3n + 2$.

**Remark 2.2.** It is clear that for any $\mathbf{A}_k \ (0 \leq k \leq n^2 - 3n + 2)$, there exists some $\alpha \in [n]^{m-1}$, for any $i \in [n]$, $a_{i\alpha}^{[k]} = 0$. Thus for each $0 \leq k \leq n^2 - 3n + 2$, $\mathbf{A}_k$ is not a strongly primitive tensor by Proposition 2.1.

**Example 2.3.** ([11], Example 4.17) Let $m = n = 3$, $\mathbf{A} = (a_{i_1i_2i_3})$ be a nonnegative tensor with order $m$ and dimension $n$, where $a_{111} = a_{222} = a_{333} = a_{233} = a_{311} = 0$ and other $a_{i_1i_2i_3} = 1$. Then $\eta(\mathbf{A}) = 4$.

**Remark 2.4.** In fact, we can obtain $\gamma(\mathbf{A}) = \eta(\mathbf{A}) = 4$ because of $a_{111}^{(3)} = a_{222}^{(3)} = a_{333}^{(3)} = a_{233}^{(3)} = a_{311}^{(3)} = 0$, where $\mathbf{A}^3 = (a_{i_1i_2...i_m}^{(3)}).$
In the computation of Example 2.3, we note that the following equation is useful and will be used repeatedly. It can be easy to obtain by the general product of two n-dimensional tensors which defined in Definition 1.2 in [8].

Let $A$ be a nonnegative primitive tensor with order $m$ and dimension $n$, $\alpha_2, \ldots, \alpha_m \in [n]^{(m-1)k-1}$. Then we have

$$(A^k)_{i_1 \alpha_2 \ldots \alpha_m} = \sum_{i_2, i_3, \ldots, i_m=1}^n a_{i_2 i_3 \ldots i_m} (A^{k-1})_{i_2 \alpha_2} \cdots (A^{k-1})_{i_m \alpha_m}. \quad (2.1)$$

**Proposition 2.5.** (See [10], Proposition 2.7) Let $A$ be a nonnegative primitive tensor with order $m$ and dimension $n$, $M(A)$ be the majorization matrix of $A$. Then we have:

1. For each $j \in [n]$, there exists an integer $i \in [n] \setminus \{j\}$ such that $(M(A))_{ij} > 0$.
2. There exist some $j \in [n]$ and integers $u, v$ with $1 \leq u < v \leq n$ such that $(M(A))_{uj} > 0$ and $(M(A))_{vj} > 0$.

Let $\alpha = jj \ldots j \in [n]^{m-1}$, then $M(A)_{ij} = a_{i\alpha}$. We can see that Proposition 2.1 is the generalization of the result (1) of Proposition 2.5 from a primitive tensor to a strongly primitive tensor. We note that Proposition 2.5 played an important role in [10], and if $A$ is a nonnegative strongly primitive tensor, then $A$ must be a nonnegative primitive tensor, thus the result (2) of Proposition 2.5 also holds for nonnegative strongly primitive tensors.

**Proposition 2.6.** Let $A = (a_{i_1 i_2 \ldots i_m})$ be a nonnegative strongly primitive tensor with order $m$ and dimension $n$. Then there exists at least one $j \in [n]$ and integers $u, v$ with $1 \leq u < v \leq n$ such that $(M(A))_{uj} > 0$ and $(M(A))_{vj} > 0$.

**Proposition 2.7.** Let $A = (a_{i_1 i_2 \ldots i_m})_{1 \leq i_j \leq n, j=1, \ldots, m}$ be a nonnegative tensor with order $m$ and dimension $n$ and $A \neq I$. For given $i \in [n]$, if $a_{i\alpha} = a_{jji \ldots i} = 1$ for any $\alpha \in [n]^{m-1}$ and any $j \in [n] \setminus \{i\}$, then $A$ is strongly primitive with $\eta(A) = 2$.

**Proof.** By (2.1), for any $k \in [n]$ and $\alpha_2, \ldots, \alpha_m \in [n]^{m-1}$, we have

$$(A^2)_{k \alpha_2 \ldots \alpha_m} = \sum_{k_2, k_3, \ldots, k_m=1}^n a_{k_2 k_3 \ldots k_m \alpha_2} a_{k_2 \alpha_2} \cdots a_{k_m \alpha_m} \geq a_{k_i i \ldots i} a_{i \alpha_2} \cdots a_{i \alpha_m} = 1,$$

which implies $A$ is strongly primitive and $\eta(A) = 2$. $\square$

**Remark 2.8.** From Proposition 2.7, we can see that:

1. There exist at least $n(2^{(n-1)(m-1)} - 1)$ strongly primitive tensors such that its strongly primitive degree is equal to 2.
2. We cannot improve the result of Proposition 2.7 any more by the fact that there exists $i \in [n]$ such that $a_{i\alpha} = 1 > 0$ for any $\alpha \in [n]^{m-1}$ and there is exactly one $i$ such that $a_{i\alpha} > 0$ for any $\alpha \neq ii \ldots i$.
3. Similarly, we cannot improve the result of Proposition 2.6 any more by the fact that there is exactly one $i \in [n]$ such that $(M(A))_{ui} > 0$ for any $u \in [n]$ and for any other $j \in [n] \setminus \{i\}$, there exists only $i \in [n]$ such that $(M(A))_{ij} > 0$.
4. What’s more, combining the above arguments, we know whether a nonnegative tensor is a nonnegative strongly primitive tensor or not, and the value
of the strongly primitive degree of a nonnegative strongly primitive tensor do not depend on the number of nonzero entries, but the positions of the nonzero entries.

**Proposition 2.9.** Let \( \mathbb{A} = (a_{i_1i_2 \ldots i_m}) \) be a nonnegative strongly primitive tensor with order \( m \) and dimension \( n \). Then for any \( i \in [n] \), there exists some \( \alpha \in [n]^{m-1} \) such that \( a_{i\alpha} > 0 \).

**Proof.** Since \( \mathbb{A} \) is strongly primitive, there exists some \( k > 0 \) such that \( \mathbb{A}^k > 0 \) by Definition 1.1. Assume that there exists some \( i \in [n] \) such that \( a_{i\alpha} = 0 \) for any \( \alpha \in [n]^{m-1} \). Then by (2.1), we have

\[
(\mathbb{A}^k)_{i\alpha_2 \ldots \alpha_m} = \sum_{i_2, i_3, \ldots, i_m=1}^{n} a_{i_2i_3 \ldots i_m} (\mathbb{A}^{k-1})_{i_2\alpha_2} \cdots (\mathbb{A}^{k-1})_{i_m\alpha_m} = 0,
\]

which leads to a contraction. \( \square \)

**Remark 2.10.** Let \( \mathbb{A} = (a_{i_1i_2 \ldots i_m})_{1 \leq i_j \leq n; j=1, \ldots, m} \) be a nonnegative tensor with order \( m \) and dimension \( n \). For given \( i \in [n] \), we take \( a_{i\alpha} = a_{jii \ldots i} = 1 \) for any \( \alpha \in [n]^{m-1} \) and any \( j \in [n] \setminus \{i\} \), and any other entry \( a_{i_1i_2 \ldots i_m} = 0 \). Then \( \mathbb{A} \) is strongly primitive with \( \eta(\mathbb{A}) = 2 \) by Proposition 2.9. This implies that we cannot improve the result of Proposition 2.9 any more, and it indicates the importance of the positions of the nonzero entries again.

**Proposition 2.11.** Let \( \mathbb{A} \) be a nonnegative strongly primitive tensor and \( k = \eta(\mathbb{A}) \). Then for any integer \( t > k > 0 \), we have \( \mathbb{A}^t > 0 \).

**Proof.** It is clear that \( \mathbb{A}^k > 0 \) by \( k = \eta(\mathbb{A}) \). We only need to show \( \mathbb{A}^{k+1} > 0 \), say, for any \( i \in [n] \), and any \( \alpha_2, \ldots, \alpha_m \in [n]^{m-1} \), we show \( (\mathbb{A}^{k+1})_{i\alpha_2 \ldots \alpha_m} > 0 \).

By Proposition 2.9, there exists some \( \alpha = j_2j_3 \ldots j_m \in [n]^{m-1} \) such that \( a_{i\alpha} = a_{i_2j_2 \ldots j_m} > 0 \). By \( \mathbb{A}^k > 0 \) we have \( (\mathbb{A}^k)_{j_2\alpha_2} > 0 \), \ldots, \( (\mathbb{A}^k)_{j_m\alpha_m} > 0 \), then by (2.1), we have

\[
(\mathbb{A}^{k+1})_{i\alpha_2 \ldots \alpha_m} = \sum_{i_2, i_3, \ldots, i_m=1}^{n} a_{i_2i_3 \ldots i_m} (\mathbb{A}^k)_{i_2\alpha_2} (\mathbb{A}^k)_{i_3\alpha_3} \cdots (\mathbb{A}^k)_{i_m\alpha_m} \geq a_{i_2j_2 \ldots j_m} (\mathbb{A}^k)_{j_2\alpha_2} \cdots (\mathbb{A}^k)_{j_m\alpha_m} > 0. \]

\( \square \)

**Proposition 2.12.** Let \( \mathbb{A} \) be a nonnegative tensor with order \( m \) and dimension \( n \), and \( t \) be a positive integer. Then \( \mathbb{A} \) is strongly primitive if and only if \( \mathbb{A}^t \) is strongly primitive.

**Proof.** Firstly, the sufficiency is obvious. Now we show the necessity. Let \( k = \eta(\mathbb{A}) \). Then \( \mathbb{A}^k > 0 \) by \( \mathbb{A} \) is strongly primitive. Let \( s \) be a positive integer such that \( st \geq k \), then \( \mathbb{A}^{st} > 0 \) by Proposition 2.11. Thus \( (\mathbb{A}^t)^s = \mathbb{A}^{st} > 0 \), which implies \( \mathbb{A}^t \) is strongly primitive. \( \square \)
A characterization of the (strongly) primitive tensor with order $m$ and dimension 2

In this section, we study primitive tensors and strongly primitive tensors in this paper, show that an order $m$ dimension 2 tensor is primitive if and only if its majorization matrix is primitive, and obtain the characterization of order $m$ dimension 2 strongly primitive tensors and the bound of the strongly primitive degree.

Lemma 3.1. (See [8], Corollary 4.1) Let $A$ be a nonnegative tensor with order $m$ and dimension $n$. If $M(A)$ is primitive, then $A$ is also primitive and in this case, we have $\gamma(A) \leq \gamma(M(A)) \leq (n-1)^2 + 1$.

Theorem 3.2. Let $A$ be a nonnegative tensor with order $m$ and dimension $n = 2$. Then $A$ is primitive if and only if $M(A)$ is primitive.

Proof. Firstly, the sufficiency is obvious by Lemma 3.1. Now we only show the necessity. Clearly, all primitive matrices of order 2 are listed as follows:

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let $A$ be primitive. Then $\gamma(A) \leq 2$ by Theorem 1.10 and $M(A^2) > 0$ by Proposition 1.8. Now we assume that $M(A)$ is not primitive, we will show $A$ is also not primitive.

It is not difficult to find that

$(A^2)_{ij...j} = \sum_{i_2, i_3, ..., i_m=1}^2 a_{i_2i_3...im} a_{i_2jj...j} \ldots a_{imjj...j}$

$= a_{i_22}\ldots(2jj...j)^{m-1} + \sum_{i_2, i_3, ..., i_m=1, i_2i_3\ldots i_m\neq 22...2}^2 a_{i_2i_3...im} a_{i_2jj...j} \ldots a_{imjj...j}$ (3.2)

$= a_{i_11}\ldots(1jj...j)^{m-1} + \sum_{i_2, i_3, ..., i_m=1, i_2i_3\ldots i_m\neq 11...1}^2 a_{i_2i_3...im} a_{i_2jj...j} \ldots a_{imjj...j}$. (3.3)

In (3.2), we note that $i_2i_3\ldots i_m \neq 22\ldots2$, which implies that there exists at least one entry, say, $i_s = 1$ where $2 \leq s \leq m$, then $a_{i_1jj...j} \in \{a_{i_2jj...j}, \ldots, a_{imjj...j}\}$.

Similarly, in (3.3), we note that $i_2i_3\ldots i_m \neq 11\ldots1$, which implies that there exists at least one entry, say, $i_s = 2$ where $2 \leq s \leq m$, then $a_{i_2jj...j} \in \{a_{i_2jj...j}, \ldots, a_{imjj...j}\}$.

Thus, by (3.2), (3.3) and the above arguments, we have

$M(A^2)_{ij} = (A^2)_{ijj...j} = a_{i_22}\ldots(2jj...j)^{m-1} + a_{i_1jj...j}P$ (3.4)

$= a_{i_11}\ldots(1jj...j)^{m-1} + a_{2jj...j}Q$. (3.5)
Since $M(\mathbb{A})$ is not primitive, by (3.1), we can complete the proof by the following two cases.

**Case 1:** $M(\mathbb{A}) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$ or $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

**Subcase 1.1:** $M(\mathbb{A})_{12} = 0$.

Then $a_{12...2} = 0$. By (3.4), we have $M(\mathbb{A}^2)_{12} = (\mathbb{A}^2)_{12...2} = 0$, which implies $\mathbb{A}^2$ is not essential positive.

**Subcase 1.2:** $M(\mathbb{A})_{21} = 0$.

Then $a_{21...1} = 0$. By (3.5), we have $M(\mathbb{A}^2)_{21} = (\mathbb{A}^2)_{21...1} = 0$, which implies $\mathbb{A}^2$ is not essential positive.

**Case 2:** $M(\mathbb{A}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Then we have $M(\mathbb{A})_{11} = M(\mathbb{A})_{22} = 0$, that is $a_{11...1} = a_{22...2} = 0$, by (3.5) we have $M(\mathbb{A}^2)_{12} = (\mathbb{A}^2)_{12...2} = 0$, which implies $\mathbb{A}^2$ is not essential positive.

Based on the above two cases and Proposition 1.6, we complete the proof of the necessity.

A nature question is whether the result of Theorem 3.2 is true for $n \geq 3$ or not. The following Example 3.3 shows that the necessity of Theorem 3.2 is false with $n \geq 3$.

**Example 3.3.** Let $\mathbb{A} = (a_{i_1i_2...i_m})$ be a nonnegative tensor of order $m$ and dimension $n \geq 3$, where

$$a_{i_1i_2...i_m} = \begin{cases} 0, & \text{if } i_1 = 1, i_2 = i_3 = \ldots = i_m \neq 1; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\mathbb{A}$ is (strongly) primitive, but $M(\mathbb{A})$ is not primitive.

**Proof.** By direct calculation and Definition 1.2 we know that $\mathbb{A}^2$ is the tensor of order $(m - 1)^2 + 1$ and dimension $n$, and for any $1 \leq i \leq n$, we have

$$(\mathbb{A}^2)_{i_2i_3...i_m} = \sum_{i_2,i_3,...,i_m=1}^2 a_{i_2i_3...i_m}a_{i_2\alpha_2} \ldots a_{i_m\alpha_m}$$

$$\geq \begin{cases} a_{12...m}a_{2\alpha_2} \ldots a_{m\alpha_m} = 1, & \text{if } i = 1; \\ a_{i_1i_2...i_m}a_{i_1\alpha_2} \ldots a_{i_m\alpha_m} = 1, & \text{otherwise.} \end{cases}$$

Obviously, $\mathbb{A}^2$ is positive, then $\mathbb{A}$ is strongly primitive with $\eta(\mathbb{A}) = 2$ and thus $\mathbb{A}$ is primitive with $\gamma(\mathbb{A}) = 2$.

On the other hand, by the definition of $\mathbb{A}$, we have $M(\mathbb{A}) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \\ \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}$.

Since the associated digraph of $M(\mathbb{A})$ is not strongly connected, thus $M(\mathbb{A})$ is not primitive.

Next, we will study the strongly primitive degree of order $m$ and dimension 2 tensors. Firstly, we discuss an example with order $m = 5$ and dimension $n = 2$ tensor as follows.
Definition 3.4. (See [3]) Let $\mathbb{A}$ be a tensor with order $m$ and dimension $n$. The $i$-th slice of $\mathbb{A}$, denoted by $\mathbb{A}[i]$, is the subtensor of $\mathbb{A}$ with order $m - 1$ and dimension $n$ such that $(\mathbb{A}[i])_{i_2...i_m} = a_{i_1 i_2...i_m}$.

Example 3.5. Let $\mathbb{A} = (a_{i_1 i_2 i_3 i_4 i_5})_{1 \leq i_j \leq 2 (j = 1, \ldots, 5)}$ be a nonnegative tensor with order $m = 5$ and dimension $n = 2$, where $a_{12122} = a_{21121} = 0$ and other $a_{i_1 i_2 i_3 i_4 i_5} = 1$. Then there exists at least one zero element in each slice of $\mathbb{A}^2$.

Proof. Let $\alpha_1 = 2122$, $\alpha_2 = 1121$, and denote $\beta_2 = \beta_4 = \beta_5 = \alpha_1$, $\beta_3 = \alpha_2$. Then we have

$$(\mathbb{A}^2)_{1\beta_2\beta_3\beta_4\beta_5} = \sum_{i_2,i_3,i_4,i_5=1}^2 a_{i_1 i_2 i_3 i_4 i_5} a_{i_2\beta_2} a_{i_3\beta_3} a_{i_4\beta_4} a_{i_5\beta_5}$$
$$= \sum_{i_3,i_4,i_5=1}^2 a_{i_1 i_3 i_4 i_5} a_{i_1\beta_2} a_{i_3\beta_3} a_{i_4\beta_4} a_{i_5\beta_5} + \sum_{i_3,i_4,i_5=1}^2 a_{i_1 i_3 i_4 i_5} a_{i_2\beta_2} a_{i_3\beta_3} a_{i_4\beta_4} a_{i_5\beta_5}$$
$$= \sum_{i_4,i_5=1}^2 a_{i_2 i_4 i_5} a_{i_2\beta_2} a_{i_4\beta_4} a_{i_5\beta_5} + \sum_{i_4,i_5=1}^2 a_{i_2 i_4 i_5} a_{i_2\beta_2} a_{i_4\beta_4} a_{i_5\beta_5}$$
$$= \sum_{i_5=1}^2 a_{i_2 i_5} a_{i_2\beta_2} a_{i_5\beta_5} + \sum_{i_5=1}^2 a_{i_2 i_5} a_{i_2\beta_2} a_{i_5\beta_5}$$
$$= 0.$$

Similarly, we let $\gamma_2 = \gamma_3 = \gamma_5 = \alpha_2$ and $\gamma_4 = \alpha_1$, we can show $(\mathbb{A}^2)_{2\gamma_2\gamma_3\gamma_4\gamma_5} = 0$ and we omit it.

Combining the above arguments, we know there exists at least one zero element in each slice of $\mathbb{A}^2$ by $(\mathbb{A}^2)_{1\beta_2\beta_3\beta_4\beta_5} = (\mathbb{A}^2)_{2\gamma_2\gamma_3\gamma_4\gamma_5} = 0$. \[\Box\]

Similarly, the result of Example 3.5 can be generalized to any nonnegative tensor with order $m$ and dimension $n = 2$.

Lemma 3.6. Let $\mathbb{A} = (a_{i_1 i_2...i_m})_{1 \leq i_j \leq 2 (j = 1, \ldots, m)}$ be a nonnegative tensor with order $m$ and dimension $n = 2$. If there exist $\alpha_1 = j_2 j_3 \ldots j_m \in [2]^{m-1}$ and $\alpha_2 = k_2 k_3 \ldots k_m \in [2]^{m-1}$ such that $a_{i_1 \alpha_1} = a_{2 \alpha_2} = 0$. For any $2 \leq t \leq m$, let $\beta_t = \begin{cases} \alpha_2, & \text{if } j_t = 1; \\ \alpha_1, & \text{if } j_t = 2; \end{cases}$ and $\gamma_t = \begin{cases} \alpha_2, & \text{if } k_t = 1; \\ \alpha_1, & \text{if } k_t = 2. \end{cases}$ Then

$$(\mathbb{A}^2)_{1\beta_2...\beta_m} = 0, \quad (\mathbb{A}^2)_{2\gamma_2...\gamma_m} = 0. \quad (3.6)$$

Proof. We first show $(\mathbb{A}^2)_{1\beta_2...\beta_m} = 0$. For any $2 \leq t \leq m$, $j_t \in \{1, 2\}$, we denote $\bar{j}_t = \{1, 2\} \setminus \{j_t\}$. Then we have $a_{i_1 \beta_t} = 0$ by $a_{i_1 \alpha_1} = a_{2 \alpha_2} = 0$, and

$$(\mathbb{A}^2)_{1\beta_2...\beta_m}.$$
Proof. Let $\mathcal{A} = (a_{i_1i_2\ldots i_m})_{1 \leq i_j \leq 2 (j = 1, \ldots, m)}$ be a nonnegative tensor with order $m$ and dimension $n = 2$. If there exist $\alpha_1 = j_2j_3\ldots j_m \in [2]^{m-1}$ and $\alpha_2 = k_2k_3\ldots k_m \in [2]^{m-1}$ such that $a_{1\alpha_1} = a_{2\alpha_2} = 0$. Then $\mathcal{A}$ is not strongly primitive.

Proof. Now we show that there exists at least one zero element in each slice of $\mathcal{A}^r$ by induction on $r (\geq 2)$.

Firstly, by Lemma 3.6, we know there exists at least one zero element in each slice of $\mathcal{A}^2$. Now we assume that there exists at least one zero element in each slice of $\mathcal{A}^{r-1}$, say, there exist $\delta_1, \delta_2 \in [2]^{(m-1)r-1}$ such that $(\mathcal{A}^{r-1})_{1\delta_1} = (\mathcal{A}^{r-1})_{2\delta_2} = 0$. For any $2 \leq t \leq m$, let $\beta_t = \{ \begin{array}{ll} \delta_2, & \text{if } j_t = 1; \\ \delta_1, & \text{if } j_t = 2, \end{array}$ and $\gamma_t = \{ \begin{array}{ll} \delta_2, & \text{if } k_t = 1; \\ \delta_1, & \text{if } k_t = 2. \end{array}$ Then by (3.1) and the similar proof of Lemma 3.6 we have

$$(\mathcal{A}^r)_{1\beta_2\ldots \beta_m} = \sum_{i_2, i_3, \ldots, i_m = 1}^{n} a_{i_1i_2i_3\ldots i_m}(\mathcal{A}^{r-1})_{i_2\beta_2}\ldots(\mathcal{A}^{r-1})_{i_m\delta_2} = 0,$$  
and the similar process of the above arguments. Thus we complete the proof of (3.6). \hfill \Box

Theorem 3.7. Let $\mathcal{A} = (a_{i_1i_2\ldots i_m})_{1 \leq i_j \leq 2 (j = 1, \ldots, m)}$ be a nonnegative tensor with order $m$ and dimension $n = 2$. If there exist $\alpha_1 = j_2j_3\ldots j_m \in [2]^{m-1}$ and $\alpha_2 = k_2k_3\ldots k_m \in [2]^{m-1}$ such that $a_{1\alpha_1} = a_{2\alpha_2} = 0$. Then $\mathcal{A}$ is not strongly primitive.
and
\[(A^r)_{\gamma_2...\gamma_m} = \sum_{i_2,i_3,...,i_m=1}^n a_{2i_2...i_m}(A^{r-1})_{i_2\gamma_2} \cdots (A^{r-1})_{i_m\gamma_m} = 0. \tag{3.8}\]

By (3.7) and (3.8), we obtain there exists at least one zero element in each slice of \(A^r\), and thus we complete the proof. \(\square\)

Now we give the characterization of the strongly primitive tensor with order \(m\) and dimension 2.

**Theorem 3.8.** Let \(A = (a_{i_1i_2...i_m})_{1\leq i_j \leq 2(j=1,...,m)}\) be a nonnegative tensor with order \(m\) and dimension \(n = 2\). Then

1. If \(A\) is strongly primitive, then \(\eta(A) \leq 2\).

**Proof.** Firstly, we show the sufficient of (1). It is easy to see that \(A = J\) is strongly primitive with \(\eta(J) = 1\), and if \(A\) satisfies (b) or (c), \(A\) is strongly primitive with \(\eta(J) = 2\) by Proposition 2.7 immediately.

Now we show the necessity of (1), that is, if \(A\) is not satisfied the conditions of (a), (b) or (c), then we will show that \(A\) is not strongly primitive. We complete the proof by the following three cases.

**Case 1:** \(a_{1\alpha} = 1\) for any \(\alpha \in [2]^{m-1}\) and \(a_{211...1} = 0\).

It is not difficult to find that \(M(A) = \begin{pmatrix} 1 & 1 \\ 0 & * \end{pmatrix}\). Then \(A\) is not primitive by Theorem 3.2 and thus \(A\) is not strongly primitive.

**Case 2:** \(a_{2\alpha} = 1\) for any \(\alpha \in [2]^{m-1}\) and \(a_{122...2} = 0\).

Similarly, we can find that \(M(A) = \begin{pmatrix} * & 0 \\ 1 & 1 \end{pmatrix}\). Then \(A\) is not primitive by Theorem 3.2 and thus \(A\) is not strongly primitive.

**Case 3:** There is at least one zero element in each slice of \(A\).

Then there exist \(\alpha_1 = j_2j_3...j_m \in [2]^{m-1}\) and \(\alpha_2 = k_2k_3...k_m \in [2]^{m-1}\) such that \(a_{1\alpha_1} = a_{2\alpha_2} = 0\). Thus \(A\) is not strongly primitive by Theorem 3.7.

(2) If \(A\) is strongly primitive, by Definition 1.11 and the proof of (1), we obtain \(\eta(A) \leq 2\) immediately. \(\square\)

**Remark 3.9.** By Theorem 3.8, we can see that the strongly primitive degree \(\eta(A)\) of an nonnegative tensor with order \(m\) and dimension \(n = 2\) is irrelevant to its order \(m\).

## 4 Some properties and problems of order \(m\) dimension \(n(\geq 3)\) strongly primitive tensors

In this section, we will study some properties of the strongly primitive tensors with order \(m\) and dimension \(n \geq 3\) and propose some questions for further research.
Proposition 4.1. Let $A = (a_{i_1i_2...i_m})_{1 \leq i_j \leq n(j=1,...,m)}$ be a nonnegative tensor with order $m$ and dimension $n$. Let $s \in [n]$, $2 \leq t \leq m$, $j_t^{(s)} \in [n]$, and $\alpha_s = j_2^{(s)}j_3^{(s)}...j_m^{(s)}$. If $a_{i\alpha_s} = 0$ for any $i \in [n]$ and any $s \in [n] \setminus \{i\}$, then there exist $\gamma_1, \gamma_2, \ldots, \gamma_n \in [n]^{(m-1)^2}$ such that $(A^2)_{i\gamma_k} = 0$ for any $i \in [n]$ and any $k \in [n] \setminus \{i\}$.

Proof. For each $i \in [n]$, let $\beta_t^{(k)} = \alpha_{j_t^{(k)}} \in [n]^{m-1}$ for any $k \in [n]$ and $2 \leq t \leq m$, then $a_{i\beta_t^{(k)}} = 0$ for any $j_t^{(k)} \in [n] \setminus \{j_t^{(k)}\}$ by $a_{i\alpha_s} = 0$ for any $i \in [n]$ and any $s \in [n] \setminus \{i\}$. Let $\gamma_k = \beta_2^{(k)}\beta_3^{(k)}...\beta_m^{(k)} \in [n]^{(m-1)^2}$ for any $k \in [n]$. Now we show $(A^2)_{i\gamma_k} = 0$ for any $k \in [n] \setminus \{i\}$.

\[
(A^2)_{i\gamma_k} = \sum_{i_2,i_3,...,i_m=1}^{n} a_{i_2i_3...i_m} a_{i_2\beta_2^{(k)}} a_{i_3\beta_3^{(k)}} ... a_{i_m\beta_m^{(k)}}
\]

We note that $k \in [n] \setminus \{i\}$ which means there are $n - 1$ zero elements in $i$-th slice of $A^2$, thus we complete the proof by $i \in [n]$.

We note that Proposition 4.1 is the generalization of Lemma 3.6, now we will obtain the generalization of Theorem 3.7.

Theorem 4.2. Let $A = (a_{i_1i_2...i_m})_{1 \leq i_j \leq n(j=1,...,m)}$ be a nonnegative tensor with order $m$ and dimension $n$. Let $s \in [n]$, $2 \leq t \leq m$, $j_t^{(s)} \in [n]$, and $\alpha_s =$
If $a_{i_0s} = 0$ for any $i \in [n]$ and any $s \in [n] \setminus \{i\}$, then $\mathbb{A}$ is not strongly primitive.

**Proof.** Now we show that there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in [n]^{(m-1)r}$ such that $(\mathbb{A}^r)_{i\varepsilon_k} = 0$ for any $i \in [n]$ and any $k \in [n] \setminus \{i\}$ by induction on $r (\geq 2)$, say, there exist at least $n - 1$ zero elements in each slice of $\mathbb{A}^r$ and thus $\mathbb{A}$ is not strongly primitive.

Firstly, we show the following assertion:

Let $\eta_s(\varepsilon) = \varepsilon_j(\varepsilon)$ for any $\varepsilon \in [n]$ and $2 \leq t \leq m$, then $(\mathbb{A}^r)_{i\varepsilon_k} = 0$ for any $i \in [n]$ and any $k \in [n] \setminus \{i\}$.

Let $\varepsilon_s(\varepsilon) = \varepsilon_j(\varepsilon)$ for any $i \in [n]$ and any $k \in [n] \setminus \{i\}$. Let $\varepsilon_s(\varepsilon) = \varepsilon_j(\varepsilon) \cdots \varepsilon_m(\varepsilon) \in [n]^{(m-1)r}$ for any $k \in [n]$. Now we show $(\mathbb{A}^r)_{i\varepsilon_k} = 0$ for any $i \in [n]$ and any $k \in [n] \setminus \{i\}$.

By (2.1) and the similar proof of Proposition 4.1, we have

$$(\mathbb{A}^r)_{i\varepsilon_k} = (\mathbb{A}^r)_{i\varepsilon(k)_{i}} = \sum_{j_2, j_3, \ldots, j_m=1}^n a_{ij_2i_3\cdots i_m}(\mathbb{A}^{r-1})_{i_j2j_3\cdots j_m} = 0,$$

then we complete the proof. \qed

**Proposition 4.3.** Let $\mathbb{A} = (a_{i_1i_2\cdots i_m})_{1 \leq i_j \leq n (j=1, \ldots, m)}$ be a nonnegative tensor with order $m$ and dimension $n$, $M(\mathbb{A})$ be the majorization matrix of $\mathbb{A}$. If there exist $i, j \in [n]$, such that $(M(\mathbb{A}))_{ij} > 0, (M(\mathbb{A}))_{uj} = 0$ for any $u \in [n] \setminus \{i\}$, and $(M(\mathbb{A}))_{ji} > 0, (M(\mathbb{A}))_{vi} = 0$ for any $v \in [n] \setminus \{j\}$. Then $\mathbb{A}$ is not primitive, and thus $\mathbb{A}$ is not strongly primitive.

**Proof.** Firstly, we show the following assert:

If $k$ is odd, then $(M(\mathbb{A}^k))_{ij} > 0, (M(\mathbb{A}^k))_{ji} > 0, (M(\mathbb{A}^k))_{uj} = 0$ for any $u \in [n] \setminus \{i\}$, $(M(\mathbb{A}^k))_{vi} = 0$ for any $v \in [n] \setminus \{j\}$.

If $k$ is even, then $(M(\mathbb{A}^k))_{ii} > 0, (M(\mathbb{A}^k))_{jj} > 0, (M(\mathbb{A}^k))_{ui} = 0$ for any $u \in [n] \setminus \{i\}$, $(M(\mathbb{A}^k))_{vj} = 0$ for any $v \in [n] \setminus \{j\}$.

When $k = 1$, the above result holds is obvious. When $k = 2$, by Definition 1.7 and (2.1), we have

$$(M(\mathbb{A}^2))_{ii} = (\mathbb{A}^2)_{ii, \ldots, i} = \sum_{i_2, i_3, \ldots, i_m=1}^n a_{i_2i_3\cdots i_m}a_{i_2i_3\cdots i_m} = (M(\mathbb{A}))_{ii}((M(\mathbb{A}))_{ii})^{m-1} > 0,$$

and for any $u \in [n] \setminus \{i\}$, we have

$$(M(\mathbb{A}^2))_{ui} = (\mathbb{A}^2)_{ui, \ldots, i} = \sum_{i_2, i_3, \ldots, i_m=1}^n a_{ui_2i_3\cdots i_m}a_{i_2i_3\cdots i_m}.$$
$$= a_{u_j \cdots j}(a_{j_i \cdots i})^{m-1}$$
$$= (M(\mathbb{A}))_{u_j}((M(\mathbb{A}))_{j_i})^{m-1}$$
$$= 0.$$

Similarly, we can show $(M(\mathbb{A}^2))_{jj} > 0$ and $(M(\mathbb{A}^2))_{vj} = 0$ for any $v \in [n] \setminus \{j\}$.

Now we assume that for any $k$, the above assert holds. Then for $k + 1$, we consider the following two cases.

**Case 1:** $k$ is odd.

Then by (2.1), we have

$$n(M(\mathbb{A}^{k+1}))_{ui} = (\mathbb{A}^{k+1})_{ui} = a_{ii_{i_3} \cdots i_m}(\mathbb{A}^k)_{i_2i_{i_3} \cdots i_m} \cdots (\mathbb{A}^k)_{i_{m-1}i_i}$$
$$= a_{ij}((\mathbb{A}^k)_{ji})^{m-1}$$
$$= (M(\mathbb{A}))_{ij}((M(\mathbb{A}))_{ji})^{m-1}$$
$$> 0,$$

and for any $u \in [n] \setminus \{i\}$, we have

$$(M(\mathbb{A}^{k+1}))_{ui} = (\mathbb{A}^{k+1})_{ui} = a_{ui_{i_3} \cdots i_m}(\mathbb{A}^k)_{i_2i_{i_3} \cdots i_m} \cdots (\mathbb{A}^k)_{i_{m-1}i_i}$$
$$= a_{uj}((\mathbb{A}^k)_{ji})^{m-1}$$
$$= (M(\mathbb{A}))_{uj}((M(\mathbb{A}))_{ji})^{m-1}$$
$$= 0.$$

Similarly, we can show $(M(\mathbb{A}^{k+1}))_{jj} > 0$ and $(M(\mathbb{A}^{k+1}))_{vj} = 0$ for any $v \in [n] \setminus \{j\}$.

**Case 2:** $k$ is even.

By (2.1) and the similar proof of Case 1, we can show $(M(\mathbb{A}^{k+1}))_{ij} > 0$, $(M(\mathbb{A}^{k+1}))_{ji} > 0$, $(M(\mathbb{A}^{k+1}))_{uj} = 0$ for any $u \in [n] \setminus \{i\}$, and $(M(\mathbb{A}^{k+1}))_{vi} = 0$ for any $v \in [n] \setminus \{j\}$.

By Proposition 1.8 and the above assert, we know $\mathbb{A}$ is not primitive, and thus $\mathbb{A}$ is not strongly primitive.

Let $\mathbb{A} = (a_{i_1i_2 \cdots i_m})_{1 \leq i_j \leq n(j = 1, \ldots, m)}$ be a nonnegative strongly primitive tensor with order $m$ and dimension $n$. When $n = 2$, we know $\eta(\mathbb{A}) \leq 2$ by Theorem 3.3. When $n \geq 3$, we donot know the value or bound of $\eta(\mathbb{A})$. Even $n = 3$, we donot find out all strongly primitive tensors. Thus we think it is not easy to obtain the value or bound of $\eta(\mathbb{A})$. Based on the computation of the case $n = 3$, we propose the following problem for further research.

**Question 4.4.** Let $n \geq 3$, $\mathbb{A} = (a_{i_1i_2 \cdots i_m})_{1 \leq i_j \leq n(j = 1, \ldots, m)}$ be a nonnegative strongly primitive tensor with order $m$ and dimension $n$. Then $\eta(\mathbb{A}) < (n-1)^2 + 1$.

In [4, 10], the authors gave some algebraic characterizations of a nonnegative primitive tensor, and in [3], the authors showed that a nonnegative tensor is primitive if and only if the greatest common divisor of all the cycles in the associated directed hypergraph is equal to 1. It is natural for us to consider the following.
Question 4.5. Study the algebraic or graphic characterization of a nonnegative strongly primitive tensor.

We are sure the above two questions are interesting and not easy.

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