On Crossing Ball Structure in Knot and Link Complements

Wei Lin

Abstract

We develop a word mechanism applied in knot and link diagrams for the illustration of a diagrammatic property. We also give a necessary condition for determining incompressible and pairwise incompressible surfaces, that are embedded in knot or link complements. Finally, we give a finiteness theorem and an upper bound on the Euler characteristic of such surfaces.

1 Introduction

1.1 Preliminary discussion

Let $L \subset S^3$ be a link and $\pi(L) \subset S^2(\subset S^3)$ be a regular link projection. Additionally, let $F \subset S^3 - L$ be an closed incompressible surface. In 1981 Menasco introduced his crossing ball technology for classical link projections\cite{7} that replaced $\pi(L)$ in $S^2$ with two 2-spheres, $S^2_{\pm}$, which had the salient features that $L$ was embedded in $S^2_{+} \cup S^2_{-}$ and $S^3 \setminus (S^2_{+} \cup S^2_{-})$ was a collection of open 3-balls—$B^3_{\pm}$ that correspond to the boundaries $S^2_{\pm}$ and a collect of crossing balls. (Please see \S \ref{meridian} for formal definition.) Using general position arguments, this technology allows for placing $F$ into normal position with respect to $S^2_{\pm}$ so that $F \cap S^2_{\pm}$ is a collection of simple closed curves (s.c.c.’s). When one imposes the assumption that $\pi(L)$ is an alternating projection, the normal position of an essential surface is exceedingly will behaved to the point where by direct observation one can definitively state whether the link is split, prime, cabled or a satellite. As such, any alternating knot can by direct observation be placed into one of William Thurston’s three categories—torus knot, satellite knot or hyperbolic knot\cite{10}.

One salient result from\cite{7} is that any essential surface in a non-split alternating link exterior will contain a meridional curve of a link component and, thus, studying such essential surfaces can be reduced to studying essential surfaces with meridional boundary curves that are meridionally incompressible or pairwise incompressible. The importance of studying pairwise incompressible surfaces has been reflected in the work of numerous scholars. To name a few, Bonahon and Seibenmann's work on arborescent knots\cite{3} Oertel’s work on star links\cite{9} Adams’ work on toroidally alternating links\cite{1} Adams’ et. al work almost alternating links\cite{2} Fa’s initial cataloging of incompressible pairwise incompressible patterns
Lozano-Przytycki work on 3-braid links [6] and Hass-Thompson-Tsvietkova results on growth of the number of essential surfaces in alternating link complements [5]. However, to-date there has not been a criterion for determining whether a closed incompressible surface in an arbitrary link complement, when presented in Menasco’s normal form, is pairwise incompressible. The main result of this note fills this gap in the literature by giving such a necessary condition for determining pairwise compressibility/incompressibility—Theorem 1.1.

For reader familiar with the arguments in [7] our approach to proving Theorem 1.1 should be of interest. The techniques [7] allow for multiple ways for arguing the existence of a meridional curve. The most direct way in given in the proof of Theorem 2 (The Meridian Lemma) of [7]. In brief, once a surface $F$ is in normal position, for an alternating projection the existence of such a meridional curve is manifested by an innermost s.c.c. on $S_2^2$ of $S \cap S_2^2$. An alternative indirect way of arguing comes from the proof of Lemma 2 of [7]. Again in brief, for $F$ in normal position one considers an arbitrary s.c.c. $c \subset F \cap S_2^2$. Then when one considers how $c'$s imposes a “nesting behavior” on the subset of curves in $F \cap S_2^2$ that intersect $c$, the assumption of an alternating projection again forces the existence of some s.c.c. from this subset manifesting a meridian.

This latter technique for detecting the existence of a meridional curve is the main tool employed in this note. Our concluding remark is that it seem to be an underutilized tool to-date in works that have exported the crossing ball technology to study pairwise incompressibility in other settings.

(a) The bubble.  
(b) A local view of the crossing ball with a saddle. The solid arcs on the crossing ball are part of $F \cap S_2^2$, while the dotted arcs on the crossing ball are part of $F \cap S_2^2$. 

Figure 1
1.2 Main results

To review, a surface $F$ properly embedded in the link $L$ complement in $S^3$ is called pairwise incompressible if for each disk $D$ in $S^3$ meeting the $L$ transversely in one point, with $D \cap F = \partial D$, there is a disk $D' \subset F \cup L$ meeting $L$ transversely in one point, with $\partial D' = \partial D$. We consider the following problem: Given the condition that a surface is incompressible in the link complements, how to characterize such a surface that is also pairwise incompressible?

As before, let $\pi(L)$ be a connected regular projection from a knot or a link $L \subset S^3$ to a sphere $S^2$. We place a ball at each crossing of $\pi(L)$, which we refer to as a bubble $B$ (See Figure 1a). At each crossing, both the overstrand and the understrand are in the $\partial B$. We define $S^2_+ \times \mathbb{Z}$ to be the sphere $S^2$ where the equatorial disk in each bubble is replaced by the upper hemisphere of the bubble, and $B^3_+ \times \mathbb{Z}$ to be the 3-ball bounded by $S^2_+ \times \mathbb{Z}$ that does not contain any bubble $B$. Similarly, we can define $S^2_- \times \mathbb{Z}$ and $B^3_- \times \mathbb{Z}$ when replacing equatorial disk in each bubble by lower hemisphere of the bubble. Since such operations can be performed on an arbitrary link diagram on $S^2$, we use this convention that all $\pi(L)$ of this note are assumed to be endowed with the crossing ball structure.

Let $F \subset S^3 - L$ be a separating sphere, a closed incompressible pairwise incompressible surface, or an incompressible pairwise incompressible surface with meridional boundaries (when $L$ is a knot, if $F$ is a sphere, it is reducible). $F$ can be isotoped to intersect each bubble in a set of saddles. We assume the surface $F$ is connected and is chosen to minimize the total number of saddles and curves in lexicographical order. We can replace $F$ (isotope when it is incompressible pairwise incompressible) with $F'$ such that it is in normal position (details are shown in §2.1). In other word, we can place $F$ in a position such that for each $C \subset F \cap S^2_\pm$, $C$ bounds a disk in either $B^3_\pm$, $C$ does not pass through a bubble twice, and all meridional boundaries of $F$, i.e. punctures are on the arcs of $F \cap S^2_+ \cap S^2_-$, away from the crossing balls.

With the above-mentioned concepts, we now introduce the word mechanism. We consider the nontrivial situation when $C$ has nonempty intersections with saddles and give the following definition:

**Definition 1.1** (cyclic word) For a loop $C \subset F \cap S^2_\pm$ we consider it the union of two types of arcs. Type I $A_1 \subset C \cap S^2_+ \cap S^2_-; \text{ and, Type II } A_2 \subset C \setminus C \cap S^2_+ \cap S^2_-$. (Notice a arc of type I is away from bubbles and an arc of type II is on the boundary of a bubble.). Arcs of type I whose ends encounter two overcrossings of $L$ on same side of $C$ are assigned a label $R$. Arcs of type I that have intersections with $L$ are assigned a label $P^i$, where $i$ is the number of times it intersects with $L$. Arcs of type II are assigned a label $S$. A cyclic word $\omega(C)$ is a word obtained by recording in order the labels of the arcs of $C$.

From a cyclic word $\omega(C)$ and the number of loops at each bubble $C$ crosses, we will be able to obtain a virtual word, $\omega^v(C)$ (see Definition 3.3). We will show that each
virtual words related to incompressible pairwise incompressible surfaces in normal position satisfies a condition defined in \( \S 3 \) as \( \omega \)-reducible. The diagrammatic property mentioned at the beginning of this note can be illustrated by the following two theorems:

**Theorem 1.1** Let \( L \subset S^3 \) be a link, \( \pi(L) \) be its link diagram on \( S^2 \), \( F \subset S^3 - L \) be a separating sphere or an incompressible pairwise incompressible surface, that is closed or with meridional boundaries in normal position, then \( \omega^v(C) \) is \( \omega \)-reducible for each simple closed curve \( C \subset F \cap S^2_\pm \).

If in the link complement, there exists a separating sphere or a closed incompressible pairwise incompressible surface \( F \subset S^3 - L \), we also give a theorem to show that the configurations of such surfaces are finite, and we can give an upper bound on the Euler characteristic of such surfaces through the pullback graph of \( F \), i.e. the surface \( F \) endowed with a 4-valent graph structure and labels on some of the edges (see refchapter final), where \( |R| \) stands for the total number of type I arcs of \( F \cap S^2_\pm \) (or \( F \cap S^2 \)) marked with \( R \), and \( n \)-gons correspond to disks of \( F \cap B^3_\pm \) which intersect with \( n \) saddles:

**Theorem 1.2** Let \( L \) be a link, \( F \) be a separating sphere, or a closed incompressible pairwise incompressible surface in \( S^3 - L \). Then:
(a) If $|R| = 4$ or $6$, $F$ is a sphere. If $|R| = 8$, $F$ is either a sphere or a torus.

(b) The maximum vertex number of a region in the pullback graph is bounded by $|R| - 2$.

(c) For fixed $|R|$, there are only finitely many such surfaces $F$ (up to isotopy when $F$ is not a sphere).

(d) The Euler characteristics $\chi(F)$ of $F$ and the number of $n$-gons $F_n$ subject to the following restriction:

\[
\chi(F) = \sum_{n=2}^{\frac{|R|-2}{2}} F_n - \sum_{n=2}^{\frac{|R|-2}{4}} \frac{n}{4} F_n = \sum_{n=2}^{\frac{|R|-2}{2}} F_n - |S| \leq |R| - |S|
\]

2 Normal Position and Motivating Examples

2.1 Normal position

Suppose $F \subset S^3 - L$ is a surface whose boundary curves are all meridians of $L$ which do not intersect the bubbles. To each component $C$ of $F \cap S^2_{\pm}$ can be associated a cyclic word. Note that Definition 1.1 modifies the original representation in [7]. When $L$ is alternating, $SP^{2i}S$ of [7] will be denoted as $SP^{2i+1}RS$ in this note, where $R$ represents successive saddles lie on the same side of $C$; If the puncture number is even, the notation of this note is $SP^{2i}S$, according to the alternating property of $\pi(L)$.

Proposition 2.1 Let $F \subset S^3 - L$ be a separating sphere or a closed incompressible pairwise incompressible surface, then $F$ can be replaced by another surface $F'$ of the same type (isotopic to $F$ when it is closed incompressible pairwise incompressible) that is in the following position:

1. No word $\omega(C)$ associated to $F$ is empty.

2. No loop of $S \cap S^2_{\pm}$ meets a bubble in more than one arc.

3. Each loop of $S \cap S^2_{\pm}$ bounds a disk in $B^3_{\pm}$.

See [7] Lemma 1 for proof. Notably it is independent of the alternatingness of $L$. We say $F$ is in normal position of closed surface if it satisfies all above conditions. Conditions (1) and (3) are the results of the incompressibility of $F$ and are independent of the alternatingness of $L$. Condition (2) is the result of both the pairwise incompressibility of $F$ and the choice of the isotopic class of $F$ to minimize saddles number, which are both independent on the alternating property of $\pi(L)$.

In addition to (1), (2) and (3) conditions for closed surface in normal position, we claim $F$ can be isotoped to satisfy additional three conditions if it has meridional boundaries:
Proposition 2.2 Let $F \subset S^3 - L$ be an incompressible and pairwise incompressible surface with meridional boundaries, then $F$ can be isotoped so that, in addition to (1), (2) and (3), $F$ satisfies the following conditions:

(4) No loop of $F \cap S^2_\pm$ meets both a bubble and an arc of $L \cap S^2_\pm \cap S^2_\pm$ having an endpoint on that bubble.

(5) No loop of $F \cap S^2_\pm$ meets a component of $L \cap S^2_\pm$ more than once.

(6) There does not exist two loops $\alpha \subset F \cap S^2_\pm$ and $\beta \subset F \cap S^2_\pm$, with arcs $a, b \subset \alpha \cap \beta$ such that the interiors of $a, b$ are contained in adjacent components of $S^2_\pm \cap S^2_\pm - L$, and $\partial a \cap \partial b = \emptyset$.

See [8] Lemma 3 for proof, as these properties are independent of the alternatingness of $L$. We say a surface with meridional boundaries is in normal position when it satisfies all (1)-(6) of the above conditions.

2.2 Motivating examples

Let $F \subset S^3 - L$ be a surface that is closed or with meridional boundaries. Suppose $F$ satisfies condition (1), (3), (4), (5), (6) of normal position, and $C \subset F \cap S^2_\pm$. We show a few examples which will explain the motive for some technical definitions in next section.

Example 2.3 Fig. 3a shows the situation $C$ intersects with saddles alternatingly. Assume $F$ is punctured, no matter how we “connect” the saddles on the same side of $C$ in $F \cap S^2_\pm$ through disk(s), on $F \cap S^2_\pm$ there is always a curve going through both sides of a bubble, contradicting the pairwise incompressibility of $F$ (see [1], the proof of Lemma 1).

In Fig. 3b, $F$ is closed and $C$ no longer intersects saddles alternatingly. However, as one
tries to connect the 1st arc end with the 2nd, 3rd, or 4th arc end, the resolving diagram represents the result of different “connection of saddles” through disk in \( F \cap S^2_+ \), and the resolving surface always contains a meridian curve.

We will describe such “connection of saddles” in the next section (see Definition 3.3). A natural question raised here is: How far away the link diagram is from alternating can we put a closed incompressible pairwise incompressible surface in the link complement?

**Example 2.4** Fig. 4 shows two examples when multiple saddles exist in the same bubble. In Fig. 4a, \( F \cap S^2_+ \), an arc goes through the same side of a bubble twice; While in Fig. 4b the surface \( F \) is placed in normal position.

![Figure 4](image)

In order to illustrate how certain arrangement of saddles can lead to the difference between Fig. 4a and Fig. 4b, we give Definition 3.5 to keep record of the appearance of multiple saddles.

### 3 Word Reduction and Virtual Diagram

**Definition 3.1** \((\omega\text{-reducible})\) We say a cyclic word is \(\omega\text{-reducible}\) if it can be reduced to \(\emptyset\) after we perform finitely many times of the following operations:

(I) \(R^{2i} \to \emptyset\),

(II) \(S^i RS^i \to R\),

(III) \((SR)^{2i} S \to R\),
where $i$ stands for a positive integer, $S_i$ means $i$ saddles in a sequence, and similarly for $R_i$, $P_i$. If a cyclic word is not $\omega$-reducible, we call it $\omega$-irreducible.

**Remark 3.1** We call the above operations $\omega$-reductions or reductions. An $\omega$-reduction is a modification of the cyclic word. A cyclic word without $P$ must be even length, a (I), (II), or (III) reduction does not change the parity of word length.

We focus on closed surface and cyclic word with no puncture for convenience, the results of the case with punctures are going to be similar due to reduction (IV). Assume $F \subset S^3 - L$ is a sphere, or a closed incompressible pairwise incompressible surface in normal position, $C \subset F \cap S^2_+$ (Similarly we can give definitions for $C \subset F \cap S^2_-$). We denote the set of all saddles contained in the bubbles that $C$ crosses as $\Lambda_C$. With these assumptions, we give the following technical definitions:

**Definition 3.2** (partial word) A partial word $\omega(C')$ is a record of an arc $C' \subset C$, in order, the intersections of $C'$ with the bubbles in the same sense as the cyclic word, where $C'$ is a union of type I or type II arc(s). We say a partial word of odd length is $R$-$\omega$-reducible if it can be $R$ after we perform finitely many times of (I), (II), (III) reductions. Otherwise, we call it $R$-$\omega$-irreducible.

![Figure 5: Two $S_1$'s in the figure are paired up through a connecting operation $f_1$, similarly for the three $S_2$'s. $S_3$ is not paired up with $S_2$, since the arcs connecting $S_2$ and $S_3$ in $F \cap S^2_-$ is either intersecting $C$ or intersecting $S_{1,1} \in \Lambda_C$.](image)

**Definition 3.3** (paired up saddles) Suppose two saddles that $C$ crosses both intersect with
an arc \( \mathcal{A} \subset F \cap S^2 \), and \( \mathcal{A} - \{C \cup \Lambda C\} \) is connected, then we these saddles are assigned with a connecting operation \( f_\lambda \). We can replace \( S \)'s in the cyclic word \( \omega(C) \) related to these saddles with \( S_\lambda \). If a saddle is assigned with two connecting operations, then we identify the associated connecting operations, and we say all the saddles are paired up through \( f_\lambda \) if all of them are assigned with \( f_\lambda \). See Fig. 5.

**Remark 3.2** This is well-defined since a saddle \( C \) intersects is associated to, if any, a unique connecting operation. And an \( S_\lambda \) still represents a saddle.

With the above definitions, we can now show a lemma to describe the situation when each bubble that a selected loop \( C \) crosses contains only one saddle:

**Lemma 3.1** Let \( L \subset S^3 \) be a link with a connected diagram \( \pi(L) \) on \( S^2 \), \( F \subset S^3 - L \) be a separating sphere or a closed incompressible pairwise incompressible surface in normal position. If each saddle intersecting \( C \subset F \cap S^2_\pm \) is paired up to some other saddle(s) intersecting \( C \), then \( \omega(C) \) is \( \omega \)-reducible.

**Proof.** Suppose \( F \subset S^3 - L \) is a closed pairwise incompressible surface in normal position. Without lost of generality, we assume \( C \subset F \cap S^2_\pm \) is a simple closed curve such that \( \omega(C) \)

---

**Figure 6:** Examples of two different situations where saddles of \( \omega(C^{(1)}) \) are paired up with each other.

(a) a R-\( \omega \)-reducible partial word \( \omega(C^{(1)}) \) reads as 
\[ RRSR\text{SSRSS} \]

(b) existence of a meridian curve

\[ F \cap S^2_\pm \]
Figure 7: A $\omega$-reducible partial word $\omega(C^{(k)})$ reads as $SSRSSSSSRSSR$ (omitting the subscripts). $\omega(C^{(k)})$ does not contain any $S_{\lambda_k}'s$. $S_{\lambda_1}, S_{\lambda_2}$ are two saddles contained in $\omega(C^{(k)})$ that are not paired up with any saddles contained in $\omega(C^{(k)})$, and they are not recorded in the partial word between any two paired up saddles contained in $\omega(C^{(k)})$.

is $\omega$-irreducible, and that each of the saddles in $\omega(C)$ is paired up to other saddle(s) in $\omega(C)$. Then each saddle can be assigned to a unique connecting operation $f_{\lambda_i}, 1 \leq \lambda_i \leq n$. $\omega(C)$ always consists of $S^i$ after we perform finitely many times of (I), (II), (III) reductions since it is by assumption $\omega$-irreducible.

Consider an adjacent pair of $S_{\lambda_1}$'s associated to $f_{\lambda_1}$. Denote the two partial words of the arcs separated by cutting the loop $C$ at two of the edge points of type-II arcs intersecting this pair of saddles as $\omega(C^{(1)})$ and $\omega(C^{(1)(1)})$, so that $\omega(C^{(1)})$ does not consist of any $S_{\lambda_1}'s$ and $\omega(C^{(1)(1)})$ contains all the $S_{\lambda_1}$'s of $\omega(C)$. Note that the length of both $\omega(C^{(1)})$ and $\omega(C^{(1)(1)})$ are odd. Therefore at least one of these partial words is $R$-$\omega$-irreducible, otherwise the word $\omega(C)$ would be $\omega$-reducible. We can assume $\omega(C^{(1)})$ is $R$-$\omega$-irreducible, because otherwise we can pick a different adjacent pair of $S_{\lambda_1}$'s associated to $f_{\lambda_1}$ so that $\omega(C^{(1)})$ is $R$-$\omega$-irreducible and it does not consist of any $S_{\lambda_1}$'s.

Not all of the saddles of $\omega(C^{1})$ are paired up with each other, otherwise either $\omega(C^1)$ is $R$-$\omega$-reducible, see Fig. 6a, or $F$ is not in normal position and it contains a meridian curve, see Fig. 6b. We claim that we can find a connecting operation $f_{\lambda_k}$ and two paired up saddles $S_{\lambda_k}$'s, together with the two partial words in between of this pair of saddles, $\omega(C^{(k)})$ and $\omega(C^{(k)(k)})$, $C = C^{(k)} \cup C^{(k)(k)}$, so that:

(i) $\omega(C^{(k)})$ consists of $\omega(C^{(k-1)})$ when $k > 1$, and $\omega(C^{(k)})$ is $R$-$\omega$-irreducible, and does not consist of any $S_{\lambda_k}$’s;

(ii) All partial words that’s contained in $\omega(C^{(k)})$, and in between each pair of adjacent paired up saddles are $R$-$\omega$-reducible.

Because otherwise we can take an adjacent paired up saddles $S_{\lambda_{k+1}}$’s contained in $\omega(C^{(k)})$,
associated to some connecting operation \( f_{k+1} \), such that the partial word \( \omega(C^{(k+1)}) \) that’s contained in \( \omega(C^{(k)}) \), and in between this pair of saddles is \( R-\omega \)-irreducible, and the finitiveness of word length will guarantee the existence of paired up saddles satisfying condition (i) and (ii). In other words, we can always find the “inner” partial word \( \omega(C^{(k+1)}) \) contained in \( \omega(C^{(k)}) \) until the “innermost” partial word satisfies condition (i) and (ii).

Let \( \{ S_{\lambda_j} \} \) be all the saddles contained in \( \omega(C^k) \) that are not paired up with any saddles contained in \( \omega(C^k) \), and each saddle of \( \{ S_{\lambda_j} \} \) is not recorded in the partial word between any two paired up saddles contained in \( \omega(C^k) \), see Fig. 7.

Suppose the cardinality of \( \{ S_{\lambda_j} \} \) is larger or equal to two. According to (ii), we can first perform (II), (III) reductions on \( S \)'s which correspond to saddles paired up within \( \omega(C^k) \) and the \( R-\omega \)-reducible partial words in between, then perform (II), (III) reduction(s) on the saddles of \( \{ S_{\lambda_j} \} \) and we are left with an odd number of \( R \)'s, see Fig. 7, then we perform the (I) reduction and we are left with only one \( R \), contradicting \( \omega(C^k) \) being \( R-\omega \)-irreducible;

Suppose the cardinality of \( \{ S_{\lambda_j} \} \) is one. We are in the situation similar to Fig. 6b, contradicting the pairwise incompressibility of \( F \).

The rest of definitions in this section are given in order to convert the word problem of multiple-saddle cases to the word problem that each bubble a selected loop \( C \) crosses contains only one saddle, so that Lemma 3.1 would apply. The purpose is to obtain a bookkeeping diagram, in which we obtain the virtual word \( \omega_v(C) \) by recording the virtual bubbles \( C \) crosses and the labels on type I arcs of \( C \). These definitions are unrelated to the concept of virtual crossing. First, to exclude the case that a diagram of \( F \cap S^2_\pm \) represents a disconnected surface in the link complement, we give the following definition. Furthermore, it has a strong connection with the virtual word in Definition 3.5.

**Definition 3.4 (C-onion)** A C-onion is a set of saddles satisfying either of the following conditions:

1. Suppose \( S_{\alpha_k} \) intersects \( C \), \( S_{\alpha_m} \) either intersects \( C \), or from \( S^2_+ \)'s side of view, lies beneath a saddle intersecting \( C \). If \( S_{\alpha_k} \) and \( S_{\alpha_m} \), both intersect an arc \( \mathcal{A}_{km} \subset F \cap S^2_\pm \), such that \( \mathcal{A}_{km} - \{ C \cup S_{\alpha_k} \cup S_{\alpha_m} \} \) is connected, then \( S_{\alpha_k} \) and \( S_{\alpha_m} \) belong to the same C-onion.

2. Suppose \( S_{\alpha_m} \) is in a C-onion, and from \( S^2_+ \)'s side of view, \( S_{\alpha_m'} \) lies beneath a saddle intersecting \( C \). If \( S_{\alpha_m} \) and \( S_{\alpha_m'} \), both intersect an arc \( \mathcal{A}_{mm'} \subset F \cap S^2_\pm \), such that \( \mathcal{A}_{mm'} - \{ C \cup S_{\alpha_m} \cup S_{\alpha_m'} \} \) is connected, then \( S_{\alpha_m} \) and \( S_{\alpha_m'} \) belong to the same C-onion. See Fig. 8

\[ \]

11
Figure 8: (a) and (b) are dual diagrams illustrating the same $C$-onion. One can check by definition that the five shadowed saddles in (a) are the only saddles not belong to this $C$-onion. $S_{\alpha_k}$ intersects $C$, $S_{\alpha_k}$ and $S_{\alpha_m}$ both intersect an arc $\mathcal{A}^{km} \subset F \cap S_2^+$, such that $\mathcal{A}^{km} - \{C \cup S_{\alpha_k} \cup S_{\alpha_m}\}$ is connected. Therefore $S_{\alpha_k}$ and $S_{\alpha_m}$ belong to the same $C$-onion; $S_{\alpha_m}$ is in a $C$-onion, $S_{\alpha_m}$ and $S_{\alpha_m'}$ both intersect an arc $\mathcal{A}^{mm'} \subset F \cap S_2^-$, such that $\mathcal{A}^{mm'} - \{C \cup S_{\alpha_m} \cup S_{\alpha_m'}\}$ is connected. Therefore $S_{\alpha_m}$ and $S_{\alpha_m'}$ belong to the same $C$-onion.

Remark 3.3 In particular, paired up saddles intersecting $C$ belong to the same $C$-onion. $C$-onions are disjoint sets if multiple $C$-onions exist.

The following proposition says it is easy to determine the saddles of a certain $C$-onion when $F$ is connected:

**Proposition 3.2** Suppose $F$ is a connected surface that is closed or with meridional boundaries, and $C \subset F \cap S_2^+$ (or $C \subset F \cap S_2^-$). Let $B$ be a bubble $C$ crosses, $S$ be a saddle in $B$ intersecting $C$, where $l$ means there are totally $l$ saddles beneath it from $S_2^+$’s side of view (or from $S_2^-$’s side of view, respectively), then all those $l$ saddles belong to the same $C$-onion with $S$. 

Proof. We assume \( C \subset F \cap S_2^+ \). By the connectness of \( F \), we can find arcs and saddles which satisfy the conditions given in the definition of \( C \)-onion, so that each saddle lies beneath \( \hat{S} \) from \( S_2^+ \)’s side of view belong to the same \( C \)-onion with \( \hat{S} \). For \( C \subset F \cap S_2^- \) the proof is similar.

\[
\text{Definition 3.5} \quad \text{(virtual cyclic word)} \quad \text{Virtual bubbles of a } \text{C}-\text{onion are a bunch of bubbles intersecting } C \text{ obtained by replacing a bubble } C \text{ crosses, that contains } 1 + l \text{ saddles in the } C\text{-onion } (l \geq 0), \text{ with a set of } 1 + 2l \text{ bubbles. These virtual bubbles intersect } C \text{ alternatingly in the following way: see Fig. 9. Each single saddle contained in a virtual bubble is called a virtual saddle. With all bubbles that } C \text{ crosses replaced by virtual bubbles, we obtain a virtual cyclic word } \omega^v(C), \text{ that is a record in order of virtual saddles intersecting } C, \text{ in the same sense as the record of cyclic word.}
\]

![Diagram](image)

(a) when \( l \) is odd  
(b) when \( l \) is even

Figure 9: Virtual bubbles. \( C \) is oriented in the same direction as \( \omega(C) \) or \( \omega^v(C) \) is recorded. The bubble in which \( \hat{S} \) lies contains a total of \( 1 + l \) saddles, all of which belong to the same \( C \)-onion.

In other words, we can obtain virtual words from cyclic words in the following way:

\[
\text{Definition 3.6} \quad \text{(l-reduction)} \quad \text{Assume } F \text{ is a connected surface in normal position, } C \text{ is a curve of } F \cap S_2^+ \text{ (or } F \cap S_2^-), \text{ } \hat{S} \text{ intersects } C, \text{ and from } S_2^+ \text{’s side of view (or from } S_2^- \text{’s side of view, respectively), there are totally } l \text{ saddles underneath it. We can replace the associated } S \text{ of } \hat{S} \text{ in the cyclic word } \omega(C) \text{ with } \hat{S}. \text{ We call the following operation performed on a cyclic word } \omega(C) \text{ an } l\text{-reduction:}
\]

13
(V) $\hat{S} \to S^{2l+1}$.

$l$ is a non-negative integer. The resulting word is a virtual cyclic word $\omega^v(C)$.

(a) We replace a $1 + l$-saddle-bubble with $2l + 1$ virtual bubbles arranged alternatingly. The dotted ellipse records the related arc-ends in order.

(b) Orient $C$, and direct the arc $\omega_{hg}^{km}$ to connect the related arc-ends.

Figure 10

The following definition of virtual diagram is the construction of a bookkeeping diagram, so that it and its dual diagram would satisfy Proposition 3.3.

**Definition 3.7** (virtual diagram) Suppose $F \subset S^3$ is a closed surface satisfying (1) and (3) of normal position, $C \subset F \cap S^2_+$. A positive virtual diagram of $C$, denoted as $F_C^v \cap S^2_+$ (or respectively, if $C_- \subset F \cap S^2_-$, a negative virtual diagram of $C_-$, $F_{C_-}^v \cap S^2_-$), is a diagram devoid of link and surface structure, obtained by modifying (part of) the diagram $F \cap S^2_+$ in the following way:

Suppose $S_{\alpha_{k,h}}$ and $S_{\alpha_{m,g}}$ both belong to a $C$-onion, and both intersect an arc $\omega_{hg}^{km} \subset F \cap S^2_+$, such that $\omega_{hg}^{km} - \{\text{all saddles in the C-onion}\}$ is connected, then we connect the virtual saddles manifested by the bubbles containing $S_{\alpha_{k,h}}$ and $S_{\alpha_{m,g}}$ as shown in Fig. 10. We perform such operations on each $C$-onion of $C$ to obtain $F_C^v \cap S^2_+$.  

14
Moreover, we can produce the dual diagram in the same way that is shown in Fig. 1b, i.e. \( F_v^+ \cap S_2^- \), of a positive virtual diagram \( F_C^+ \cap S_2^- \). Similar to the cyclic word, we can define \( \omega \)-reducible or \( \omega \)-irreducible virtual words.

**Definition 3.8** (\( \omega \)-reducible virtual cyclic word) We say a virtual cyclic word \( \omega^v(C) \) is \( \omega \)-reducible if it is \( \emptyset \) after we perform finitely many times of (I), (II), (III), (IV) reductions, otherwise we call it \( \omega \)-irreducible.

![Diagram](image)

Figure 11: (a) shows a torus \( F \) embedded in a link complement from \( S_2^+ \)’s side of view. (b) is the positive virtual diagram of \( C \), in which the 4 red slashes on \( C \) separate 2 arcs each meeting 3 virtual bubbles.

The following proposition transfers the word problem of multiple-saddle case to the case where each bubble a selected loop \( C \) crosses contains only one saddle, so that Lemma 3.1 would apply for the corresponding virtual word:

**Proposition 3.3** Suppose \( F \subset S^3 \) is a surface satisfying condition (1), (3), (4), (5), (6) of normal position, \( C \subset F \cap S_2^- \). Let \( \{C_i\} \) be all loops of \( F \cap S_2^- \) which intersect saddles that belong to the \( C \)-onion(s), and let \( \{C'_i\} \) be all loops of \( F_C^+ \cap S_2^- \). Then there is a natural correspondence between \( F_C^+ \cap S_2^- \) and \( F \cap S_2^- \):

1. If in \( F_C^+ \cap S_2^- \), there exists a loop going through a virtual bubble manifested from a bubble \( B \) twice, or two virtual saddles manifested from \( B \) are paired up, then there exists a loop in the diagram of \( F \cap S_2^- \) that goes through \( B \) twice.
(2) The diagrams \((\{C_i^+\}, S^2)\) and \((\{C_i^-\}, S^2)\) are ambient isotopic.

(3) Loops of \(\{C_i^-\}\) have an one-to-one correspondence with loops of \(\{C_i^+\}\).

**Proof.** This is by construction of the virtual diagrams, see Definition 3.5 Definition 3.7.

To illustrate (2), (3), see also Example 3.4.

**Example 3.4** A torus \(F\) in normal position embedded in a link complement, where \(C \subset F \cap S^2_+\), see Fig. 11a. The virtual diagram of \(C, F^v_C \cap S^2_+\), see Fig. 11b. See also Fig. 12 for the dual diagrams.

![Figure 12: The dual diagrams of Figure 11](image)

3.1 Proof of Theorem 1.1

**Proof.** According to Proposition 2.1 and Proposition 2.2, we can assume without loss of generality \(F\) is a surface in normal position. And suppose \(\omega^v(C)\) is \(\omega\)-irreducible for some \(C \subset F \cap S^2_+\), then there exists a loop in the dual virtual diagram \(F^v_C \cap S^2_+\) passing through a virtual bubble twice, according to Lemma 3.1. Then according to the Proposition 3.3 (1), either there is a loop in \(F \cap S^2_+\) passing through the same side of a bubble twice, contradicting the minimality of saddle number; Or there is a loop in \(F \cap S^2_-\) passing through different sides of a bubble twice, contradicting the pairwise incompressibility of \(F\).
4 An Upper Bound On The Euler Characteristic

![Diagram showing the intersection of a surface with $S^2_\pm$ and the pullback graph of $F$.]

Figure 13: The edges marked in $R$ are colored in red in the pullback graph of $F$. Disks above $S^2_+$ are marked with $+$, while disks below above $S^2_-$ are marked with $-$. Let $F$ be a separating sphere, or a closed incompressible pairwise incompressible surface in $S^3 - L$ that is in normal position. Consider $F$ with all intersection curves both in $F \cap S^2_+$ and $F \cap S^2_-$ projected on it. All saddles correspond to quadrilaterals, which we collapse to vertices to obtain a 4-regular graph on $F$. $F$ can be checkerboarded so that for any two adjacent faces in this graph, one corresponds to a region which is contained strictly above $S^2_+$, and the other to a region contained strictly below $S^2_-$, see Fig. 13. A region which has $n$ vertices ($n$ must be even) will be denoted as $D_n$. An edge can be marked with $R$ if it corresponds to a type-I arc labeled with $R$ in $F \cap S^2_+$, or equivalently, $F \cap S^2_-$. We call this graph a pullback graph of $F$. Denote $|R|$ as the total number of $R$'s in the graph, $|S|$ as the vertex number (also the number of saddles of $F$).

Lemma 4.1 Let $L \subset S^3$ be a link, $F \subset S^3 - L$ be a separating sphere or an incompressible pairwise incompressible surface, that is closed or with meridional boundaries and in normal position, then for each simple closed curve $C \subset F \cap S^2_\pm$ that intersects with saddles, $\omega(C)$ contains at least two $R$’s.

Proof. Assume $\omega(C)$ contains only $P$’s and $S$’s, then according to the construction of
virtual cyclic word (see Definition 3.5), $\omega(v)(C)$ contains only $P$'s and $S$'s, i.e. $C$ crosses virtual saddles in an alternating pattern, which makes any of its virtual cyclic word $\omega$-irreducible, contradicting Theorem 1.1.

This means in the link complement, if we put the surface of our interest $F$ in normal position, then in the diagrams of $F \cap S^2_+$ the arrangement of saddles intersecting each simple closed curve can never be alternating. The following lemma is also true if we replace $S^2_+$ with $S^2_−$.

**Lemma 4.2** Let $D_n$ be a disk component of $F \cap S^2_+\,$ intersecting $n$ saddles, $\partial D_n$ divides $S^2_+$ into two regions, then the region that is over $m_0$ saddles ($m_0 \leq n$) intersecting $\partial D_n$ also contains loop(s) of $F \cap S^2_+$ admitting $|R_{in}| \geq m_0$ type-I arcs which are labeled with $R$.

**Proof.** We assume by convention the region that contains $m_0$ saddles is the inner one. Suppose $m_0 = 1$, the region still contains at least one loop, admitting $2R$'s on the corresponding edges in pullback graph of $F$ by Lemma 4.1. Suppose $m_0 \geq 2$, and if the $m_0$ saddles are all paired up with each other, then the loop(s) contained in the region admits at least $m_0 R$'s. If $m_1$ of the saddles are not paired up, then the $m_0 - m_1$ saddles are paired up through loop(s) admitting at least $m_0 - m_1 R$'s, while the other saddles are going to induce loop(s) intersecting the not paired up saddles. These loop(s) induce $k_1 R$’s and contain at least $m_2$ saddles on the different side of these $m_1$ saddles, where $k_1$ is a positive integer and $k_1 + m_2 \geq m_1$. If $m_2 \geq 1$, we can repeat this counting process on the inner region(s) bounded by the loop(s) containing $m_2$ saddles. In the $(i+1)$th step we obtain $m_i - m_{i+1} R$’s through paired up saddles, $k_{i+1} R$’s through the not paired up saddles, and $m_{i+2}$ saddles in the inner region(s) obtained through the $(i + 1)$th step, where $k_{i+1} + m_{i+2} \geq m_{i+1}$. As we repeat this process, after finitely many steps we will obtain second innermost region(s) that are over $m_s$ saddle(s) in total, and innermost loop(s) admitting at least $m_s R$’s. In total, we can obtain more or equal to $m_0 R$’s.

4.1 Proof of Theorem 1.2

**Proof.** To prove (a), according to Lemma 4.1 when $|R| = 4$ or 6, we only need to check the situation in which $F \cap S^2_\pm$ contains two or three loops. When $|R| = 8$ the maximal loop number becomes 4. In each of the cases we can calculate the Euler characteristics of $F$ to show it is a sphere, see Fig.14 except for when $|R| = 8$ there are only two configurations of torus shown in Fig. 15.

To prove (b), we assume on the pullback graph, $D_N$ is an $N$-gon so that $N$ is the maximum vertex number of the $n$-gons. Note that on $F \cap S^2_\pm$ the corresponding loop of $\partial D_N$ divides.
Figure 14: The configurations of sphere when $|R| = 4, 6, \text{ or } 8$. 
$S^2_+$ into two regions, and this loop intersects with a total of $N$ saddles. Then according to Lemma 4.2, these two regions contain loops admitting at least $N$ type-I arcs which are labeled with $R$. Moreover, $\partial D_N$ admits at least 2 $R$'s. Therefore $N \leq |R| - 2$.

To prove (c), first we note that Lemma 4.1 implies the number of regions in the pullback graph is bounded by $|R|$. Then by (b) the maximum vertices number of a region in the pullback graph is bounded by $|R| - 2$. Therefore there are only finitely many types of the pullback graph, each type corresponds to finitely many (isotopic classes) of $F \subset S^3 - L$ due to the limited amount of crossing numbers in $\pi(L)$, and the finiteness of type-I arcs which are labeled with $R$.

To prove (d), we calculate the $\chi(F)$ through the pullback graph. $\chi(F)$ is equal to the vertex number $V$ subtract the edge number $E$, and plus the region number $F$ in the pullback graph. Thus, $\chi(F) = V - E + F = \sum_{n=2}^{\infty} \frac{n}{4} F_n - \sum_{n=2}^{\infty} \frac{n}{2} F_n + \sum_{n=2}^{\infty} F_n = \sum_{n=2}^{\infty} F_n - \sum_{n=2}^{\infty} \frac{n}{2} F_n$. Here $\sum_{n=2}^{\infty} F_n$ equals the region number $F$, and by Lemma 4.1 $F \leq |R|$. $\sum_{n=2}^{\infty} \frac{n}{4} F_n$ equals the vertex number $V$, which is equal to $|S|$. By (b) the maximum possible value of $n$ is $|R| - 2$. Therefore we have $\chi(F) = \sum_{n=2}^{\left| R \right| - 2} F_n - \sum_{n=2}^{\left| R \right| - 2} \frac{n}{4} F_n = \sum_{n=2}^{\left| R \right| - 2} F_n - |S| \leq |R| - |S|$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure15.png}
\caption{The configurations of torus when $|R| = 8$.}
\end{figure}

\section*{Acknowledgement}
I would like to thank William Menasco for his guidance and many helpful discussions.
References

[1] C. Adams, Toroidally alternating knots and links, *Topology* 33, no. 2,(1994): 353-369
[2] C.C. Adams, J. F.Brock, J. Bugbee, T.D. Comar, K.A. Faigin, A.M. Huston, A.M. Joseph and D. Pesikoff, Almost alternating links, *Topology and its Applications*, 46 (1992), no. 2, pp. 151–165.
[3] F. Bonahon and L. Siebenmann, New geometric splittings of classical knots, and the classification and symmetries of arborescent knots, (365 pages) available at http://www-rcf.usc.edu/~fbonahon/Research/Preprints/Preprints.html
[4] H. Fa, Incompressible pairwise incompressible surfaces in almost alternating knot complements, *Topology and its Applications*, 80 (1997), no. 3, pp. 239–249.
[5] J. Hass, A. Thompson and A. Tsvietkova, The Number of Surfaces of Fixed Genus in an Alternating Link Complement, *International Mathematics Research Notices*, 2017, August 2015.
[6] M.T. Lozano and J.H. Przytycki, Incompressible surfaces in the exterior of a closed 3-braid I, surfaces with horizontal boundary components, Math. Proc. Camb. Phil. Soc. 98 (1985), 275–299.
[7] W. Menasco, Closed Incompressible Surfaces In Alternating Knot And Link Complements. *Topology* 33, no.2(1984): 353-369
[8] W. Menasco, Determining Incompressibility Of Surfaces In Alternating Knot And Link Complements. *Pacific Journal Of Mathematics* Vol 117, No 2. 1985
[9] U. Oertel, Closed incompressible surfaces in complements of star links, *Pacific Journal of Mathematics*, 111 no. 1 (1984), pp. 209—230.
[10] W. Thurston, On the Geometry of Topology of 3-manifolds. *Princeton Notes.*