Fractional-flux vortices and spin superfluidity in triplet superconductors

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We discuss a novel type of fractional flux vortices along with integer flux vortices in Kosterlitz-Thouless transitions in a triplet superconductor. We show that under certain conditions a spin-triplet superconductor should exhibit a novel state of spin superfluidity without superconductivity.

Superconductors with triplet pairing allow for a rich variety of topological defects and phase transitions. For example in p-wave superconductors there exist different realizations of fractional vortices such as: the half-quantum vortex (the Alice string), which is a vortex where phase changes by $\pi$ and spin is reversed when we go around the vortex core, fractional vortices trapped on a grain boundary, fractional flux trapped by twisted wire and other possibilities (for an excellent review see [1,2]). Another subject which was intensively studied is the split phase transitions in triplet systems (e.g. in the presence of disorder or magnetic field). There have also been particularly interesting studies of split transitions in a neutral p-wave superfluid in connection with thin films of liquid $^3$He. Also the related questions of various partial symmetry breakdowns are relevant and were studied in spin-1 Bose condensates in optical traps and bilayer Quantum Hall systems. In this Letter we discuss the effect of flux fractionalization due to spin-orbit coupling, as well as the appearance of neutral vortices in triplet superconductors and discuss its influence on the Kosterlitz-Thouless (KT) transitions. Based on topological arguments we predict the existence of a novel type of ordering in spin-triplet system - the spin-superfluid nonsuperconducting state.

We consider a model spin triplet superconductor similar to one considered in [2,3] with order parameter $\Psi_a(x)$ where $(a = 1, 0, -1)$ and $\zeta$ is a normalized spinor $\zeta^\dagger \cdot \zeta = 1$. If one neglects mixed gradient terms, the free energy density can be written as [3,11,2] for $< F >^2$:

$$F = \frac{\hbar^2}{2M} (\nabla \sqrt{n})^2 + \frac{\hbar^2 n}{2M} \left( \nabla^2 + \frac{2e}{\hbar c} \Lambda \right) |\zeta_0|^2 - \mu n$$

$$+ \frac{n}{2} \left[ c_0 + c_2 < F >^2 \right] + \frac{\hbar^2}{8\pi},$$

(1)

where $< F > = \zeta_0^a F_{ab} \zeta_b$ is the spin. The presence of a crystal lattice makes the mass $M$ a symmetric tensor $M_{ij}$. Spinors are related to one another by gauge transformation $e^{i\theta}$ and spin rotations $U(\alpha, \beta, \tau)$, where $< F > = e^{-iF_{a} e^{-iF_{0} e^{-iF_{1} \tau}}} \cdot \zeta_0$. The matrices $\{F_x, F_y, F_z\}$ are:

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix},
\begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}$$

(2)

The ground state structure of $\Psi_a(r)$ can be found by minimizing the energy with fixed particle number $\| \zeta \|^2$. When $c_2 < 0$ the energy is minimized by $< F >^2 = 1$ and the ground state spinor and density are [11]

$$\zeta = e^{i(\theta - \tau)} \frac{e^{\frac{ia}{2} \cos \frac{\theta}{2}}}{\sqrt{2 \cos \frac{\beta}{2} \sin \frac{\beta}{2}}} e^{i\beta \sin \frac{x}{2}} \frac{\mu}{c_0 + c_2}$$

(3)

In this case there exists an equivalence between the gauge transformation associated with $\theta$ and spin rotations associated with $\tau$ and the vacuum manifold is $SO(3)$.

The equation for the supercurrent in this case depends on phase gradients and also on spin texture (the Mermin-Ho relation):

$$J = \frac{\hbar e}{2M} (\zeta_0^a \nabla \zeta_a - \zeta_a^\dagger \nabla \zeta_0^a) - \frac{4c_2^2 n}{Mc} A = \frac{2\hbar e}{M} (\nabla (\theta - \tau) - \cos \beta \nabla \alpha) - \frac{4c_2^2 n}{Mc} A$$

(4)

In [13] it was discussed that [11] can be rewritten in the form of an extended Faddeev model [14] related to that derived for two-gap superconductors [10]. That is, introducing notations $\nabla_i = \frac{\partial}{\partial x_i}$, $s = (\sin \beta \cos \alpha, \sin \beta \sin \alpha, \cos \beta)$, $\tilde{C} = \frac{\hbar}{2M} J$ and separating variables, the free energy density [11] can be rewritten in terms of only gauge-invariant variables as

$$F = \frac{\hbar^2 (\nabla \sqrt{n})^2}{2M} + \frac{\hbar^2 n (\nabla s)^2}{4M} - \mu n + \frac{n^2 [c_0 + c_2]}{2}$$

$$+ \frac{n\tilde{C}^2}{8M} + \frac{\hbar^2 c^2}{128\pi} \left( \frac{\nabla C_j - \nabla C_i}{\hbar} \cdot \nabla i s \times \nabla j s \right)^2$$

(5)

Let us now allow for a spin-orbit coupling breaking the $O(3)$ symmetry. In the simplest case one should add to $\bar{F}$ a term [3] $F_{so} = \gamma_{i} s_i s_j$ [6]

$$F_{so} = \gamma_{i} s_i s_j + \gamma_{i} s_i s_j + \gamma_{i} s_i s_j$$

(6)

The case when there is only one nonzero coefficient $\gamma_{i} < 0$ corresponds to the “easy-axis” situation, then $O(3)$ spin symmetry is broken down to a point. When $\gamma_{i} > 0$ we have the “easy-plane” case and the symmetry is broken down to a circle. Adding the terms $\gamma_{i} s_i s_i$ still leads to an easy-plane or an easy-axis case. However, in the general case there should be higher order corrections e.g. fourth order in $s_i$. Such corrections shift the...
ground state from easy-axis or easy plane and the energetically preferred state of spin direction can be a circle on the unit sphere as shown on Fig. 1. Such a situation is known in many magnetic systems. Let us now examine the consequences of the spin-orbit coupling for the vortices of $S^1 \to S^1$ map. Let us consider the case when spin-orbit coupling breaks spin rotation symmetry down to an arbitrary circle on $S^2$ characterized by $\beta = \beta_0$ (see Fig. 1) (we consider the range $0 \leq \beta_0 \leq \pi$). One can observe that when $\beta = \beta_0 = \text{const}$ we have

\[ \vec{s} \cdot \nabla \vec{s} \times \nabla_j \vec{s} \propto \sin \beta \nabla_i \vec{\nabla} \alpha - \nabla_j \vec{\nabla} \alpha = 0. \]

In this case, in the London limit, the eq. (6) becomes:

\[
F = \frac{\hbar^2}{4M} \sin^2 \beta_0 (\nabla \alpha)^2 + \frac{\hbar^2}{2M} \left( \nabla (\theta - \tau) - \cos \beta_0 \nabla \alpha - \frac{2e}{\hbar c} A \right)^2 + \frac{H^2}{8\pi} \tag{7}
\]

From (7) it follows that the system allows the following vortices of $S^1 \to S^1$ map: (i) A vortex where $\theta$ (or $\tau$) changes by $2\pi$ around the core. This is the analogue of Abrikosov vortex in ordinary superconductors and it carries one flux quantum. (ii) A vortex where $\alpha$ changes by $2\pi$ around the core. This vortex has a neutral vorticity with the stiffness $2\hbar^2 \sin^2 \beta_0$ described by the first term in (4) and also there is a supercurrent around this vortex due to Mermin-Ho relation, which is given by

\[
J = -\frac{2\hbar}{Mc} \cos \beta_0 \nabla \alpha - \frac{4e^2 n}{M c} A \tag{8}
\]

From here it follows that the magnetic flux of this vortex is determined solely by spin-orbit coupling and is an arbitrary fraction of magnetic flux quantum:

\[
\Phi = \oint \alpha \, dl = -\cos \beta_0 \Phi_0, \tag{9}
\]

where $\sigma$ is a path around the core at a distance much larger than $\lambda$ and $\Phi_0$ is the magnetic flux quantum. A special situation appears in the "easy-plane" case, when $\beta_0 = \pi/2$, and a vortex in spin degree of freedom does not carry magnetic flux.

Let us now discuss the consequences of the above topological excitations to phase diagram of a quasi-two-dimensional model spin-triplet superconductor where the $O(3)$ symmetry is broken to $O(2)$. We consider the case when this symmetry is broken strongly and restrict discussion to the range $0 \leq \beta \leq \pi/2$, with the main focus on the case $\beta = \pi/2$. Thus for simplicity we do not consider in this Letter the case of two degenerate minima with $\beta = \beta_0$ and $\beta = \pi - \beta_0$.

From now on we discuss a quasi-two dimensional system of thickness $d$ where the effective magnetic field penetration length $\lambda = \lambda_p$ is related to Ginzburg-Landau penetration length $\lambda_{GL}$ via the Pearl expression

\[
\lambda_p = \lambda_{GL}^2/d.
\]

We begin with the simplest limit when the magnetic field penetration length $\lambda$ is larger than the sample size and $\beta_0 = \pi/2$ (the easy plane case). In such a situation we can neglect coupling to the vector potential in (4). Then we can observe that vortices where $(\Delta \theta = \oint d\nabla \theta = 2\pi, \Delta \alpha = \oint d\nabla \alpha = 0)$ are mapped onto vortices in the ordinary one component $U(1)$ model with stiffness $J_\theta = \frac{\hbar^2}{M}$, while the vortices $(\Delta \alpha = \oint d\nabla \alpha = 2\pi, \Delta \theta = \oint d\nabla \theta = 0)$ are mapped onto vortices in a $U(1)$-model with stiffness $J_\alpha = \frac{\hbar^2}{2M}$. Thus in the limit $\lambda \to \infty$ there are no corrections to vortex interaction due to the Meissner effect, and the system undergoes two KT phase transitions at the temperatures:

\[
T_{KT(\theta)}^{\lambda \to \infty, \beta_0=\pi/2} = \frac{\pi \hbar^2}{2Mc} \left( T_{KT(\theta)}^{\lambda \to \infty, \beta_0=\pi/2} \right)^2 \frac{M}{\lambda},
\]

\[
T_{KT(\alpha)}^{\lambda \to \infty, \beta_0=\pi/2} = \frac{\pi \hbar^2}{4Mc} \left( T_{KT(\alpha)}^{\lambda \to \infty, \beta_0=\pi/2} \right)^2 \frac{M}{\lambda},
\]

(10)

(equations for $T_{KT}$ should be solved self-consistently with the equations for gap modulus, see e.g. the review (13)). This splitting of the phase transitions corresponds to separate onset of quasi-long-range order in phases $\theta$ and $\alpha$.

The regime of short penetration length is however principally different and more interesting. In this limit the interaction which is mediated by charged current between vortices and antivortices $(\Delta \theta = \pm \pi)$ is exponentially screened at the length scale $\lambda$ and such vortices do not undergo a KT transition. Thus the variable $\theta$ is disordered and the system is not superconducting. However in this limit the vortices with $(\Delta \alpha = \pm \pi)$ still have a long range interaction mediated by the neutral mode and characterized by stiffness:

\[
J_{c}^{\alpha} = \sin^2 \beta_0 \hbar^2 2M
\]

(as follows from the first term in (4)). Thus the "spin vortices" in the regime $\pi/3 < \beta_0 \leq \pi/2$ can undergo a true KT transition at the temperature

\[
T_{SSF}^{SSF} = \frac{\pi}{4} \sin^2 \beta_0 \hbar^2 M \left( T_{SSF}^{SSF} \right)^2 \frac{M}{\lambda},
\]

(12)

this phase transition is associated with establishment of quasi-long range order in the phase variable $\alpha$. Note that in this state the superconducting phase $\theta$ is still disordered so $T_{SSF}^{SSF}$ is not a superconducting phase transition.
but is a transition to a “spin superfluid state”. Let us emphasise that the vortices with \((\Delta \alpha = \pm 2\pi)\) have non-trivial contribution in the second term in (7) and also carry a fraction of flux quantum \(\frac{1}{2}\).

We should observe that the situation when \(0 < \beta_0 < \pi/3\) is very special. That is, in the short-\(\lambda\) limit the long range interaction of vortices is mediated by the neutral current (coming from the neutral mode described by the first term in (7)) and besides that there is a finite energy contribution from the charged current (described by the second term in (7)). Although the later contribution is irrelevant for vortex interaction (in Coulomb gas mapping it plays the same role as the core energy), however the charged mode is important in the following respect: In the regime \(0 < \beta_0 < \pi/3\) the composite vortices \((\Delta \alpha = \pm 2\pi, \Delta \theta = \pm 2\pi)\) have smaller energy than elementary vortices \((\Delta \alpha = \pm 2\pi, \Delta \theta = 0)\) since the former vortices carry a smaller fraction of magnetic flux quantum and thus have smaller kinetic energy of charged supercurrent. Thus when \(0 < \beta_0 < \pi/3\) there will be a KT transition of the composite vortices \((\Delta \alpha = 2\pi, \Delta \theta = \pm 2\pi)\) and antivortices \((\Delta \alpha = -2\pi, \Delta \theta = \pm 2\pi)\) at the temperature \(T_{KT(\theta)}\). This is a rather unique feature, rooted in the nontrivial coupling of phases by vector potential in the triplet superconductor, when a KT transition is governed by topological defects with high topological charge. Note that in this regime the system allows also “purely charged” integer flux vortices (that is, without neutral superflow) \((\Delta \alpha = 0, \Delta \theta = \pm 2\pi)\). These vortices have only screened interaction. In this regime they are always liberated and disorder superconducting phase. Thus, also in the regime \(0 < \beta_0 < \pi/3\) the phase transition \(T_{KT(\theta)}\) is a transition to a nonsuperconducting spin-superfluid state in spite it is mediated by composite vortices which have topological windings in both phases \((\Delta \alpha = \pm 2\pi, \Delta \theta = \pm 2\pi)\).

Let us now discuss qualitatively regimes of intermediate penetration length. In a general case of large but finite \(\lambda\), the interaction of vortices and antivortices, which is mediated by charged current, receives correction due to Meissner screening (for a detailed review and citations see \[19\]). Then, if \(\lambda\) is sufficiently large, a gauged purely \(U(1)\) system, although, strictly speaking, does not display a KT transition into a superconducting state, is however believed to undergo a “would be” KT crossover \[19\]. So, \(T_{KT}\) at finite \(\lambda\) is transformed into a characteristic temperature of a broader crossover and also is shifted down comparing to the situation when there is no Meissner screening \[19\]. In the system \(\lambda = \infty\), at finite-\(\lambda\), the vortices \((\Delta \alpha = \pm 2\pi)\) interact via charged mode with the effective stiffness:

\[
J_{\alpha c}^\lambda(T) = \frac{\hbar^2 n(T)}{M}
\]

and with charged mode, which effective stiffness in the case \(\pi/3 < \beta_0 \leq \pi/2\) is

\[
J_{\alpha c}^{\lambda, \pi/3 < \beta_0 \leq \pi/2}(T) = \cos^2 \beta_0 \frac{\hbar^2 n(T)}{M}
\]

When \(0 < \beta_0 < \pi/3\) the relevant topological excitation is the composite vortex \((\Delta \alpha = 2\pi, \Delta \theta = 2\pi)\) for which

\[
J_{\alpha c}^{\lambda, 0 < \beta_0 < \pi/3}(T) = [1 - \cos \beta_0]^2 \frac{\hbar^2 n(T)}{M}
\]

Let us now consider the easy-plane case \(\beta_0 = \pi/2\). In type-I limit, as discussed above, there is only spin-superfluid KT phase transition while superconductivity sets in only at \(T \rightarrow 0\). When we increase penetration length the system undergoes a washed out superconducting KT crossover \[19\] at some nonzero temperature. In the extreme limit \(\lambda \rightarrow \infty\) we arrive at the above limit of two true KT transitions and an intermediate state where there is no quasi-long range order in spin degrees of freedom while the system is superconducting. The case of arbitrary \(\beta_0\) and \(\lambda\) appears to be a very interesting subject for numerical simulations. A schematic phase diagram in the simplest case \(\beta_0 = \pi/2\) is given on Fig. 2

![FIG. 2: A schematic phase diagram of the system in the simplest regime \(\beta_0 = \pi/2\). The temperature of onset of superconductivity, at finite penetration length is a characteristic temperature of a washed out crossover.](image)

Let us discuss the physical interpretation of the “spin-superfluid” state. In this state a quasi-long-range order is retained in the spin angle \(\alpha\). We observe that all three complex components in Eq. (3) have a prefactor \(e^{i\theta}\). Thus in the spin-superfluid state, all three phases of the spin components are disordered. Consequently, there is no dissipationless supercurrent \[\lambda = \infty\]. However the system retains hidden order, that is, taking the difference of the total phases of the first and third components, the contributions from the phase \(\theta\) is cancelled, while the contribution of the ordered phase \(\alpha\) is not. That means that there is a quasi-long-range order in a composite order parameter. Such a composite order parameter can be identified though the principle that a squared modulus of its gradient should give the
kinetic term of the composite neutral mode in the GL functional after separation of variables (7). This term is the following combination of the fields $\zeta_a$ which is decoupled from the vector potential but is gauge invariant: 

$$\frac{1}{24\pi^2} \left[ \frac{\hbar^2}{2M} \left| \nabla \zeta_a \nabla^* \zeta_a + (1/4)(\zeta_a^* \nabla \zeta_a - \zeta_a \nabla^* \zeta_a)^2 \right| \right]_{\beta=\beta_0}.$$ 

This corresponds to the following order parameter for the composite neutral mode:

$$\Xi = |\Xi| e^{i\alpha} = \sqrt{n} \left| \zeta_1 \right| e^{i\alpha} = \sqrt{\frac{n}{2}} \sin \beta_0 e^{i\alpha}.$$  \hspace{1cm} (17)

In the hydrodynamic limit the spin superfluid velocity is given by $V_S = \frac{\hbar}{2\pi} \nabla^2 \sin^2 \beta_0 \nabla \alpha$. This expression has a physical interpretation: in the easy-axis limits (when $\beta_0 = 0$ or $\beta_0 = \pi$) there is no neutral $U(1)$ symmetry and we find that also $J_S = 0$. On the other hand in the easy-plane case when $\beta_0 = \frac{\pi}{2}$ we have $|\Psi_1| = |\Psi_{-1}| = |\Psi_0|/\sqrt{2}$. We note that $\Psi_0$ is neutral with respect to $\alpha$ and thus does not participate in spin superfluidity. This gives a natural explanation for the extra factor $1/2$ in front of the phase gradient in the expression for $V_S$ in the easy-plane situation, compared to the similar coefficient in front of $\nabla \theta$ in the expression for $(1/2e)J$ given by eq. 4.

The configuration of the phases of the three complex components of the order parameter in the spin-superfluid state is illustrated in Fig. 3.

FIG. 3: A schematic illustration of the configuration of phases of individual spin components in different points in space in the spin-superfluid state.

Possible experimental probes of the spin-superfluid state are: (i) When there are corrections shifting spin from the easy plane, the vortices in the $\alpha$-phase carry a fraction of flux quantum. The intrinsic flux noise of a dc SQUID is about $10^{-6} \Phi_0/\sqrt{2}$ which makes flux-noise (FN) measurements sensitive to spin-superfluid transition even if a tiniest “shifting correction” is present. An observation of a KT transition in the spin-superfluid state in an FN experiment should allow to distinguish it from a superconducting transition: Because of the existence of the neutral mode, the transition should be sharp and a true jump in superfluid density should be recoverable from flux-noise data, in contrast to similar measurements of wider KT crossovers in ordinary superconductors. (ii) Upon the transition to spin superfluid state a “true” superconductivity is absent. So after detection of the transition by a FN measurement, in subsequent measurements of the conductivity or application of an external field, the system will not behave as a superconductor (iii) If the corrections shifting the spin from the equator on $S^2$ are absent - there nonetheless is a possibility to observe the transition to the spin-superfluid state if an external field is applied. Then the Zeeman effect would result in a term linear in $s_z$ in $\zeta$ thus shifting spins from equator on $S^2$, consequently the spin vortices will acquire a fraction of flux quantum and the transition will be detectable via FN measurement.

In conclusion, a triplet superconductor is a multicomponent charged system, where the vector potential is coupled to phase variables in a nontrivial way which results in the existence of neutral modes. The nontrivial coupling to vector potential of the components of the order parameter becomes particularly important in (quasi-)two dimensions where the system undergoes topological phase transitions. We considered a simple model of a spin-1 superconductor and have shown that topological considerations lead to a novel physical state: the “spin-superfluidity” when only an opposite superflow of two of three spin components is allowed. The results hold true in the simplest case of the easy-plan situation, however we extended the discussion to a general case when the ground state is shifted from the equator on $S^2$ which, as we have shown, leads to flux fractionalization. We also pointed out that there is a range of parameters when the quasi-long range order sets in only in the superconducting phase and pointed out an unusual KT transition mediated by defects of high topological charge.

We thank A. Sudbø, N. W Ashcroft, E. Demler, O. Eriksson, M. Katsnelson, S. Ktitorov, M. Zhitomirsky and especially G. E. Volovik, D. Agerberg and S. Korshunov for many generous discussions. This work has been supported by STINT and Swedish Research Council, Research Council of Norway, Grant No. 157798/432 and National Science Foundation, Grant DMR-0302347.

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