Topological orderings of weighted directed acyclic graphs

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Abstract

We call a topological ordering of a weighted directed acyclic graph non-negative if the sum of weights on the vertices in any prefix of the ordering is non-negative. We investigate two processes for constructing non-negative topological orderings of weighted directed acyclic graphs. The first process is called a mark sequence and the second is a generalization called a mark-unmark sequence. We answer a question of Erickson by showing that every non-negative topological ordering that can be realized by a mark-unmark sequence can also be realized by a mark sequence. We also investigate the question of whether a given weighted directed acyclic graph has a non-negative topological ordering. We show that even in the simple case when every vertex is a source or a sink the question is NP-complete.

Keywords: topological ordering, directed acyclic graph

1 Introduction

A directed acyclic graph (or DAG) is a directed graph with no directed cycles. A set $M$ of vertices of $G$ is inward closed if for every edge $uv \in E(G)$ we have that $v \in M$ implies $u \in M$, i.e., every edge between $M$ and $V(G) \setminus M$ is directed towards $V(G) \setminus M$. A prefix of length $k$ of a sequence $s$ is the subsequence of the first $k$ terms of $s$. A topological ordering of a DAG $G$ is an ordering of the vertices of $G$ such that every prefix of the ordering is inward closed. The following two processes yield topological orderings of a given DAG $G$ with $n$ vertices.

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A mark sequence of $G$ is a sequence $M_1, M_2, \ldots, M_n$ of subsets of $V(G)$ formed in the following way: first choose an arbitrary source $v$ and put $M_1 = \{ v \}$, i.e., mark $v$ in step 1. For $i = 2, 3, 4, \ldots, n$, choose a vertex $u \not\in M_{i-1}$ such that $\{ u \} \cup M_{i-1}$ is inward closed and put $M_i = \{ u \} \cup M_{i-1}$, i.e., mark $u$ in step $i$.

A mark-unmark sequence of $G$ is a sequence of subsets of $V(G)$ formed in the following way: first choose an arbitrary source $v$ and put $M_1 = \{ v \}$, i.e., mark $v$ in step 1. For $i = 2, 3, 4, \ldots, n$, either choose a vertex $u \not\in M_{i-1}$ such that $\{ u \} \cup M_{i-1}$ is inward closed and put $M_i = \{ u \} \cup M_{i-1}$, i.e., mark $u$ in step $i$ or choose a vertex $u \in M_{i-1}$ such that $M_{i-1} \setminus \{ u \}$ is inward closed and put $M_i = M_{i-1} \setminus \{ u \}$, i.e., unmark $u$ in step $i$. This process stops when $M_i = V(G)$.

Clearly mark-unmark sequences are a generalization of mark sequences. Furthermore, if we arrange the vertices of $G$ by the last step in which they were marked in the mark-unmark sequence, then we have a topological ordering.

A DAG $G$ is called weighted if there is an assignment of real numbers to each vertex of $G$. We call a topological ordering non-negative if the sum of the weights of the vertices in every prefix is non-negative. Similarly a mark-unmark (or mark) sequence is non-negative if at each step the sum of the weights in $M_i$ is non-negative.

Clearly a non-negative mark sequence is equivalent to a non-negative topological ordering. However, a non-negative mark-unmark sequence may yield a negative topological ordering. For example let $G$ be a weighted DAG on four vertices $\{ a, b, c, d \}$ with a single edge $bc$ and weights $w(a) = w(c) = w(d) = 1$, $w(b) = -1$. Consider the following non-negative mark-unmark sequence: mark $a$, $b$, $c$, then unmark $a$, then mark $d$ and $a$. This gives the topological ordering $b, c, d, a$ which is not non-negative. This suggests the following question of Erickson [4]: is there a weighted DAG $G$ that has a non-negative mark-unmark sequence but no non-negative mark sequence?

We answer this question in the negative with the following theorem.

**Theorem 1.** If a weighted DAG $G$ has a non-negative mark-unmark sequence, then $G$ also has a non-negative mark sequence.

A natural question is to determine the complexity to decide whether a weighted DAG $G$ has a non-negative topological ordering. However, it is easy to show (see [3]) that this is a subproblem of the NP-complete problem SEQUENCING TO MINIMIZE MAXIMUM CUMULATIVE COST\(^1\).

If we restrict the problem to weighted DAGs that consist of only sources and sinks we will prove that the problem is still NP-complete.

**Theorem 2.** Suppose $G$ is a weighted DAG such that every vertex is either a source or a sink. The decision problem to determine whether $G$ has a non-negative topological ordering is NP-complete.

We will prove Theorem 1 in Section 2 and prove Theorem 2 using a series of reductions in Section 3.

\(^1\)This is problem SS7 in the famous book of Garey and Johnson [5]
2 Marking and Unmarking

In this section we prove Theorem 1. In particular, given a weighted DAG $G$ and a non-negative mark-unmark sequence, we will construct a non-negative mark sequence for $G$. We begin with some definitions. By $M_X$ a set of vertices $X$ we denote the sum of the weights of the elements of $X$. We say that a set $Y \subset X$ is $X$-outward closed if for every edge $uv$ if $u \in Y$, $v \in X$, then $v \in Y$, i.e., every edge between $Y$ and $X \setminus Y$ is directed towards $Y$. Similarly, say that a set $Y \subset X$ is $X$-inward closed if for every $uv$ edge if $v \in Y$, $u \in X$, then $u \in Y$, i.e., every edge between $Y$ and $X \setminus Y$ is directed towards $X \setminus Y$. For simplicity we call a set of vertices $Y$ of a DAG $G$ inward closed (outward closed) if $Y$ is $V(G)$-inward closed ($V(G)$-outward closed).

Proof of Theorem 1. Let $G$ be a weighted DAG with a non-negative mark-unmark sequence. Let $M_1, M_2, \ldots, M_t$ be a mark-unmark sequence with at least one unmark step (otherwise we are done) of minimum length. For $i \in [t]$, put $U_i = \cup_{j=1}^{i-1} M_j \setminus M_i$, i.e., the set of elements that have been unmarked in any of the first $i$ steps. Note that $U_i$ is $(M_i \cup U_i)$-outward closed.

Claim 3. $w(U_i) > 0$ for all $i$.

Proof. We prove the stronger statement that the weight of any $U_i$-outward closed set is positive ($U_i$ is clearly $U_i$-outward closed). Suppose the statement is false and let $X$ be a minimal counterexample, i.e., $X$ is $U_i$-outward closed and $w(X) \leq 0$. If $Y$ is a non-empty $X$-inward closed set, then $X \setminus Y$ is $U_i$-outward closed and a proper subset of $X$, hence $w(X \setminus Y) > 0$ (by the minimality of $X$).

If $w(Y) > 0$, then $w(X) = w(Y) + w(X \setminus Y) > 0$ which is a contradiction. Thus we can suppose that for every $X$-inward closed set $Y$ we have $w(Y) < 0$. Now let $M'_1, M'_2, \ldots, M'_t$ be the subsequence of $M_1, M_2, \ldots, M_t$ that remains after removing each $M_1, M_2, \ldots, M_{i-1}$ that involves marking or unmarking an element of $X$.

We claim that this new sequence is also a mark-unmark sequence. First note that the elements of $X$ are in $U_i$ and are therefore marked at some step after $i$, i.e., each element of $X$ will eventually be marked in the new sequence. Now we show that every $M'_j$ is inward closed. Indeed, if it is not inward closed, then there is an edge $uv$ with $u \notin M'_j$ and $v \in M'_j$. The set $M_j$ is inward closed and contains $M'_j$ (thus contains $v$), therefore $u \in M_j$. Thus $u \in X$. Now, as $X$ is $U_i$-outward closed, either $v \in X$ (which contradicts $v \in M'_j$), or $v \notin U_i$. But $v \in M_j$ implies it is marked before the $i$th step in the original sequence, hence $v \notin U_i$ is possible only if it is never unmarked, i.e., $v \in M_i$. However, $u \in X \setminus U_i$ implies $u \notin M_i$, which contradicts the inward closed property of $M_i$.

Now we show that the new mark-unmark sequence $M'_1, M'_2, \ldots, M'_t$ is non-negative. Indeed, let $X'_j = X \cap M_j$, then $X'_j$ is an $X$-inward closed set, thus $w(X'_j) \leq 0$. Now $w(M'_j) = w(M_j) - w(X'_j) \geq w(M_j) > 0$. This new non-negative mark-unmark sequence is shorter than a minimal sequence, a contradiction. This completes the proof of Claim 3. \qed

We now construct a new sequence by starting with the original mark-unmark sequence $M_1, M_2, \ldots, M_t$ and skipping every step where a vertex is unmarked or marked beyond the first marking. Let $M''_1, M''_2, \ldots, M''_t$ be the new sequence. We claim that $M''_1, M''_2, \ldots, M''_t$ is
a mark sequence. Clearly every vertex will be marked at some point as every vertex will be marked at the end of the original sequence. Furthermore, every \( M''_i \) is inward closed. Indeed, if \( M''_i \) is not inward closed, then there is an edge \( uv \) with \( u \not\in M''_i \) and \( v \in M''_i \). But for \( v \) to be marked, \( u \) had to have been marked in a previous step. This is a contradiction, thus \( M''_i \) must be inward closed. Finally, to show the sequence is non-negative we must prove that \( w(M''_i) \) is non-negative for all \( i \). For each \( i \), there is some \( j \geq i \) such that \( M''_i = M_j \cup U_j \). The original sequence is non-negative and \( M_j \) and \( U_j \) are disjoint by definition thus by Claim 3 we have \( w(M''_i) = w(M_j) + w(U_j) > w(M_j) > 0 \) and therefore the mark sequence is non-negative. This completes the proof of Theorem 1.

We now construct a new sequence by starting with the original mark-unmark sequence \( M_1, \ldots, M_t \) and skipping every step where a vertex is unmarked or marked beyond the first marking. Let \( M'_1, M'_2, \ldots, M'_j \) be the new sequence. Then add \( |U| = k \) steps to the resulting sequence where we unmark the vertices of \( U \) in the order in which their first unmarking occurred in the original sequence. In other words, when an element of \( U \) is unmarked first we move that step to the end of the sequence, and skip all other steps where that element is chosen to be marked or unmarked. The resulting sequence is \( M'_1, M'_2, \ldots, M'_{j+k} \).

To finish the proof we need to show that \( M'_1, M'_2, \ldots, M'_{j+k} \) satisfies the definition of a mark-unmark sequence (we omit the details) and that the sequence is non-negative. To see that it is non-negative, observe that after any step of the new sequence the set of marked elements is the same as the set of marked elements after some step in the original sequence.

Proposition 4. Let \( G \) be a weighted DAG, and suppose \( M_1, M_2, \ldots, M_t \) is a non-negative partial mark-unmark sequence which stops after \( t \) steps. Then there is a non-negative partial mark-unmark sequence \( M'_1, M'_2, \ldots, M'_{j+k} \) such that the first \( j \) steps are markings, the last \( k \) steps are unmarkings, and \( M'_{j+k} = M_t \).

We omit some details from the proof of Proposition 4 as it is very similar to the proof of Theorem 1.

Proof sketch. For \( i \in [t] \), put \( U_i = \bigcup_{j=1}^{i-1} M_j \setminus M_i \), i.e., the set of elements that have been unmarked in any of the first \( i \) steps. As in the proof of Theorem 1 we have that the weight of any \( U_i \)-outward closed set is positive (from the proof of Claim 3). Let \( U \) be the set of vertices that are unmarked at some step in the mark-unmark sequence \( M_1, M_2, \ldots, M_t \).

We now construct a new sequence by starting with the original mark-unmark sequence \( M_1, M_2, \ldots, M_t \) and skipping every step where a vertex is unmarked or marked beyond the first marking. Let \( M'_1, M'_2, \ldots, M'_j \) be the new sequence. Then add \( |U| = k \) steps to the resulting sequence where we unmark the vertices of \( U \) in the order in which their first unmarking occurred in the original sequence. In other words, when an element of \( U \) is unmarked first we move that step to the end of the sequence, and skip all other steps where that element is chosen to be marked or unmarked. The resulting sequence is \( M'_1, M'_2, \ldots, M'_{j+k} \).

To finish the proof we need to show that \( M'_1, M'_2, \ldots, M'_{j+k} \) satisfies the definition of a mark-unmark sequence (we omit the details) and that the sequence is non-negative. To see that it is non-negative, observe that after any step of the new sequence the set of marked elements is the same as the set of marked elements after some step in the original sequence.
with the addition of an $U_i$-outward closed set (which is non-negative by Claim 3). This completes the proof of Proposition 4.

3 NP-completeness

In this section we will prove Theorem 2. The proof is by a series of reductions of the original problem to a known NP-complete problem. We restate the decision problem in Theorem 2 here.

Problem 5. Given a weighted DAG $G$ such that every vertex is either a source or a sink, determine whether $G$ has a non-negative topological ordering.

First we show that there is a reduction from the following problem posed by Rote (see [2]) to Problem 5.

Problem 6. Given a balanced bipartite graph $G$, determine whether edges can be added to $G$ to create a bipartite graph with a unique perfect matching.

We will need the following easy observation.

Observation 7. Let $G$ be a balanced bipartite graph with classes $A$ and $B$. The graph $G$ contains a unique perfect matching $M$ if and only if there is an ordering $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ such that for all $i \in [n]$ we have $a_ib_i \in M$ and $a_ib_j \notin E(G)$ if $1 \leq j < i \leq n$.

We now transform a given bipartite graph $G$ with classes $A$ and $B$ into a weighted DAG such that every vertex is a source or a sink. First orient all edges in $G$ such that they are directed to $B$. Then assign weight $-1$ to every vertex of $A$ and weight 1 to every vertex of $B$. Finally, add an isolated vertex, $v$, with weight 1 to $G$. If $G$ is extendable to a bipartite graph with a unique perfect matching, then $v$ together with the order given by Observation 7 gives a non-negative topological ordering of $G$. In particular, $v, a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ is a non-negative topological ordering of $G$. Furthermore, any non-negative topological ordering on $A$ and $B$ satisfies the requirements of Observation 7.

The following problem can be reduced to Problem 6 by adding $n - k$ isolated vertices to each class.

Problem 8. Let $G$ be a balanced bipartite graph with class sizes $n$ and let $k$ be a positive integer. Determine whether $G$ has an induced subgraph $H$ with $k$ vertices in each class of $G$ such that edges can be added to $H$ to create a bipartite graph with a unique perfect matching.

To complete the proof of Theorem 2 we must show Problem 8 is NP-complete. An equivalent problem was shown to be NP-complete by Dasgupta, Jiang, Kannan, Li and Sweedyk [1]. We include a new proof here that is shorter and less technical.
The reduction is from the NP-complete problem LARGEST BALANCED INDEPENDENT SET\footnote{In \cite{ref1} the equivalent problem of finding the largest BALANCED COMPLETE BIPARTITE SUBGRAPH is shown to be NP-complete.}. We call an independent set in a bipartite graph balanced if each class of the bipartite graph contains exactly half of the vertices of the independent set. We first state the LARGEST BALANCED INDEPENDENT SET problem.

**Problem 9.** Let $G$ be a bipartite graph and let $k$ be a positive integer. Determine whether $G$ contains a balanced independent set on $2k$ vertices.

Let $G$ be a bipartite graph with classes $A$ and $B$ and let $k$ be a positive integer. Construct a bipartite graph $G'$ as follows. The vertex set of $G'$ consists of $k + 1$ copies of each vertex $v$ in $G$, denoted by pairs $(v, 1), (v, 2), \ldots, (v, k + 1)$. We connect two vertices $(u, i)$ and $(v, j)$ in $G'$ by an edge if either of the following are satisfied:

1. $u \in A$ and $v \in B$ and $i < j$.
2. $uv \in E(G)$.

**Claim 10.** The graph $G'$ has a subgraph $H$ on $2k^2 + 2k$ vertices such that edges can be added to $H$ to create a bipartite graph with a unique perfect matching if and only if $G$ has a balanced independent set with $2k$ vertices.

**Proof.** If $G$ has a balanced independent set with $2k$ vertices, then call $H$ the induced subgraph of $G'$ spanned by the $k + 1$ copies of this independent set. Clearly $H$ has $2k^2 + 2k$ vertices. Furthermore, it is easy to see that adding the edges of a matching to each copy of the independent set results in a bipartite graph with a unique perfect matching.

Now suppose that $G'$ has a subgraph $H$ on $2k^2 + 2k$ vertices such that edges can be added to $H$ to create a bipartite graph with a unique perfect matching. Let $A_H$ and $B_H$ be the two classes of $H$ defined by the partition of $G$. Now order the vertices of $A_H$ and $B_H$ by Observation\footnote{In \cite{ref2} the equivalent problem of finding the largest BALANCED COMPLETE BIPARTITE SUBGRAPH is shown to be NP-complete.} such that if $a < b$, then there is no edge between the $a$th vertex in $A_H$ and the $b$th vertex in $B_H$.

Let $(w, i)$ be the first vertex in the ordering of $A_H$ such that among the vertices that appear earlier in the ordering there are $k - 1$ different values in the first coordinate. Let $a$ be the index of $(w, i)$ in the ordering of $A_H$. Let $m$ be the smallest value among the second coordinates of the vertices with index less than $a$ in $A_H$. Thus we have $a \leq (k - 1)(k + 1 - m + 1)$. Therefore $a \geq k^2 + k - (k - 1)(k + 1 - m + 1) = m(k - 1) + 1$.

Recall that if $b > a$, then there is no edge between the $a$th vertex in $A_H$ and the $b$th vertex in $B_H$. Furthermore, by (1), for $i < j$ there is an edge between each $(u, i) \in A_H$ and $(v, j) \in B_H$. Thus every vertex in $B_H$ with index $b > a$ has second coordinate at most $m$. There are at least $m(k - 1) + 1$ vertices in $B_H$ with index $b > a$. Therefore, by the pigeonhole principle, there is a set of $k$ of these vertices with the same second coordinate. In particular, these $k$ vertices represent distinct vertices in the original graph $G$. By the definition of $(w, i)$ there is a set of $k$ vertices in $A_H$ with index at most $a$ and distinct first coordinates, i.e., distinct vertices in $G$. These two sets of vertices form an independent set of size $2k$ in $G$. This completes the proof of Claim\footnote{In \cite{ref1} the equivalent problem of finding the largest BALANCED COMPLETE BIPARTITE SUBGRAPH is shown to be NP-complete.}.\qed
Thus we have established that Problem 8 is NP-complete and therefore we have proved Theorem 2.

We end this section with another problem posed by the fourth author [6] that is equivalent to Problem 6.

**Problem 11.** Suppose $M$ is an $n \times n$ matrix. Determine whether it is possible to reorder its rows and columns such that we get an upper-triangular matrix.

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