Gaussian-Smoothed Optimal Transport: 
Metric Structure and Statistical Efficiency

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Abstract

Optimal transport (OT), and in particular the Wasserstein distance, has seen a surge of interest and applications in machine learning. However, empirical approximation under Wasserstein distances suffers from a severe curse of dimensionality, rendering them impractical in high dimensions. As a result, entropically regularized OT has become a popular workaround. However, while it enjoys fast algorithms and better statistical properties, it looses the metric structure that Wasserstein distances enjoy. This work proposes a novel Gaussian-smoothed OT (GOT) framework, that achieves the best of both worlds: preserving the 1-Wasserstein metric structure while alleviating the empirical approximation curse of dimensionality. Furthermore, as the Gaussian-smoothing parameter shrinks to zero, GOT Γ-converges to-wards classic OT (with convergence of optimizers), thus serving as a natural extension. An empirical study that supports the theoretical results is provided, promoting Gaussian-smoothed OT as a powerful alternative to entropic OT.

1 Introduction

In recent years optimal transport (OT) has been applied to a host of machine learning (ML) tasks as a powerful means of comparing probability measures. The Kantorovich OT problem between two probability measures μ and ν with cost c(x, y) is given by

$$\inf_{\pi \in \Pi(\mu, \nu)} \int c(x, y) \, d\pi(x, y),$$

(1)

where $$\Pi(\mu, \nu)$$ is the set of transport plans (or couplings) between μ and ν. Applications of the Kantorovich formulation include data clustering [2], density ratio estimation [3], domain adaptation [4, 5], generative models [6, 7], image recognition [8–10], word and document embedding [11–13], and many others.

This surge in popularity has been driven by some highly advantageous properties of OT. Beyond its robustness to mismatched supports of μ and ν (crucial for learning generative models), when $$c(x, y) = ||x - y||^2$$, (1) becomes the 1-Wasserstein distance, which (i) has the operational interpretation of minimizing work (or expected cost); (ii) metrizes weak (also known as, weak*) convergence of probability measures; and (iii) defines a constant speed geodesic in the space of probability measures (giving rise to a natural interpolation between measures). These advantages, however, come with a price as OT is generally hard to compute and suffers from the so-called curse of dimensionality.

Specifically, suppose we have n independent samples $$(X_i)_{i=1}^n$$ from a Borel probability measure μ on $$\mathbb{R}^d$$. Consider the fundamental question of how quickly the empirical measure $$\hat{\mu}_n \triangleq \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$ approaches μ in the 1-Wasserstein distance, i.e., the $$\mathbb{E} W_1(\hat{\mu}_n, \mu)$$ rate of decay. This quantity is at the heart of empirical approximation under $$W_1$$ since it controls the error in various additional approximation setups, such as $$\mathbb{E} |W_1(\hat{\mu}_n, \nu) - W_1(\mu, \nu)|$$ (one-sample goodness-of-fit test), $$\mathbb{E} |W_1(\hat{\mu}_n, \hat{\nu}_n) - W_1(\mu, \nu)|$$ (two-samples tests) and others; see [14] for a review on statistical applications of the Wasserstein distance. Since $$W_1$$ metriz es weak convergence [15] Cor. 6.18, the Glivenko-Cantelli theorem [16] implies $$W_1(\hat{\mu}_n, \mu) \rightarrow 0$$ as $$n \rightarrow \infty$$. Unfortunately, the convergence rate in n drastically deteriorates with dimension, scaling at best as $$n^{-\frac{1}{d}}$$ for any measure μ that is absolutely continu-

1Any p-Wasserstein distance has these properties.

2Note that while Wasserstein-type GANs in practice typically use the two-sample setup since the generator distribution is intractable to compute, fundamentally the GAN actually corresponds to a one-sample setup since infinite samples can be obtained from the generator network.
ous with respect to (w.r.t.) the Lebesgue measure. Note that the $n^{-\frac{d}{2}}$ rate is sharp for all $d > 2$ (see for sharper results). This renders empirical approximation under the Wasserstein distance infeasible in high dimensions—a disappointing shortcoming given the dimensionality of data in modern ML tasks.

In light of the above, entropic OT emerged as an appealing alternative to Kantorovich OT. Its popularity has been driven both by algorithmic advances and some better statistical properties it possesses. Entropic OT regularizes the expected cost by a Kullback-Leibler (KL) divergence, forming:

$$S_{\alpha}(\mu, \nu) \triangleq \inf_{\pi \in \Pi(\mu, \nu)} c(x, y) \, d\pi(x, y) + \alpha D(\pi \| \mu \times \nu),$$

where $c(x, y)$ is the cost and $D(\alpha \| \beta) \triangleq \int \log \left( \frac{d\alpha}{d\beta} \right) \, d\alpha$ if $\alpha \ll \beta$ and $+\infty$ otherwise. While the Wasserstein distance suffers from the curse of dimensionality, showed that if $c$ is Lipschitz and infinitely differentiable, then $E[S_{\alpha}(\mu, \nu)] = O(n^{-\frac{d}{2}})$, in all dimensions (see for sharper results specialized to quadratic cost). Despite this fast convergence in the two-sample test, sample complexity bounds in the strongest one-sample regime are not available. More importantly, the assumptions from exclude the triangle inequality. The expected value analysis is followed by a high probability claim derived through McDiarmid’s inequality. Numerical results that validate these theoretical findings are provided. We conclude that GOT is an appealing alternative to entropic optimal transport, both in terms of its analytic and its statistical properties.

2 Notation and Preliminaries

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of Borel probability measures on $\mathbb{R}^d$. Let $\mathcal{P}_1(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$ be those with finite first moments. Let $\mathcal{P}(\mathbb{R}^d)$ be the set of transport plans (or couplings) between measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$. Namely, any $\pi \in \mathcal{P}(\mu, \nu)$ is a probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ whose first and second marginals are $\mu$ and $\nu$, respectively.

The $n$-fold product extension of $\mu \in \mathcal{P}(\mathbb{R}^d)$ is $\mu^{\otimes n}$. The probability density function (PDF) of the isotropic Gaussian measure $\mathcal{N}_\sigma$ is $\varphi_\sigma$. Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, their convolution $\mu * \nu \in \mathcal{P}(\mathbb{R}^d)$ is $(\mu * \nu)(A) = \int \int 1_A(x + y) \, d\mu(x) \, d\nu(y)$, where $1_A$ is the indicator of $A$. For two independent random variables $X \sim \mu$ and $Y \sim \nu$, we have $X + Y \sim \mu * \nu$.

We use $E_\mu f$ for the expectation of a measurable $f$ w.r.t. $\mu$, sometimes writing $E_\mu f(X)$ to emphasize its dependence on $X \sim \mu$. When the underlying probability measure is clear from the context, the measures that metrizes the weak topology. Namely, a sequence of probability measures $(\mu_k)_{k \in \mathbb{N}}$ converges weakly to $\mu$ if and only if $W_1(\mu_k, \mu) \to 0$. We then turn to study properties of $W_1(\mu, \nu)$ as a function of $\sigma$ for fixed $\mu$ and $\nu$. We establish continuity and non-increasing monotonicity. These, in particular, imply convergence of the optimal transportation costs, i.e., $\lim_{\sigma \to 0} W_1(\mu, \nu) = W_1(\mu, \nu)$. Additionally, using the notion of $\Gamma$-convergence, we establish convergence of optimizing transport plans. Thus, if $(\pi_k)_{k \in \mathbb{N}}$ is sequence of optimal transport plans for $W_1(\pi_k) = W_1(\mu, \nu)$, where $\sigma_k \to 0$, then $(\pi_k)_{k \in \mathbb{N}}$ converges weakly to an optimal plan for $W_1(\mu, \nu)$.

Lastly, we explore the one-sample empirical approximation under GOT, i.e., the convergence rate of $\hat{W}_1(\hat{\mu}_n, \mu)$. It was shown in that Gaussian smoothing alleviates the curse of dimensionality, with $\hat{W}_1(\hat{\mu}_n, \mu)$ converging as $n^{-\frac{d}{2}}$ in all dimensions. Although GOT is specialized to Gaussian noise, we present a generalized empirical approximation result that accounts for any subgaussian noise density. This, in turn, implies fast convergence of $\hat{W}_1(\hat{\mu}_n, \mu) - W_1(\mu, \nu)$ and $\hat{W}_1(\hat{\mu}_n, \nu) - W_1(\mu, \nu)$ via the triangle inequality. The expected value analysis is followed by a high probability claim derived through McDiarmid’s inequality. Numerical results that validate these theoretical findings are provided. We conclude that GOT is an appealing alternative to entropic optimal transport, both in terms of its analytic and its statistical properties.
subscript is omitted. Accordingly, the characteristic function of \( \mu \in \mathcal{P}(\mathbb{R}^d) \) is \( \phi_{\mu}(t) \triangleq e^{it^\top X} \). For any \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \), we have \( \phi_{\nu \ast \mu}(t) = \phi_{\mu}(t)\phi_{\nu}(t) \); if \( \mu \times \nu \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \) is the product measure of \( \mu \) and \( \nu \), then \( \phi_{\nu \times \mu}(t, s) = \phi_{\mu}(t)\phi_{\nu}(s) \).

**Definition 1 (Weak Topology)** The weak topology on \( \mathcal{P}(\mathbb{R}^d) \) is induced by integration against the set \( C'_0(\mathbb{R}^d) \) of bounded and continuous functions. Accordingly, we say that \( (\mu_k)_{k \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d) \) converges weakly to \( \mu \in \mathcal{P}(\mathbb{R}^d) \), denoted by \( \mu_k \rightharpoonup \mu \), if
\[
\int_{\mathbb{R}^d} f(x) \, d\mu_k(x) \to \int_{\mathbb{R}^d} f(x) \, d\mu(x), \quad \text{for all } f \in C'_0(\mathbb{R}^d).
\]

It is a well-known fact that \( (\mathcal{P}(\mathbb{R}^d), \mathcal{W}_1) \) is a metric space, and that the 1-Wasserstein distance metrizes the weak topology (cf. [15] Thm. 6.9). As shown in the sequel, this statement remains true if the 1-Wasserstein distance is replaced with its Gaussian-smoothed version, as defined next.

**Definition 2 (Gaussian-Smoothed \( \mathcal{W}_1 \))** The Gaussian-smoothed 1-Wasserstein distance between \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \) is \( \mathcal{W}_1(\sigma)(\mu, \nu) \equiv \mathcal{W}_1(\mu \ast K_\sigma, \nu \ast K_\sigma) \).

Letting \( X \sim \mu, Y \sim \nu \) and \( Z, Z' \sim N_\sigma \) be independent random variables, \( \mathcal{W}_1(\sigma)(\mu, \nu) \) is the 1-Wasserstein distance between the probability laws of \( X + Z \sim \mu \ast N_\sigma \) and \( Y + Z' \sim \nu \ast N_\sigma \). Thus, \( \mathcal{W}_1(\sigma)(\mu, \nu) \) can be understood as a ‘smoothed’ version of \( \mathcal{W}_1 \), where ‘smoothing’ is applied to the probability measures via convolution with a Gaussian kernel (or, equivalently, via additive white Gaussian noise).

The theoretical results in this paper are organized as follows. Section 3 studies the metric properties of \( \mathcal{W}_1(\sigma) \). Section 4 establishes properties of \( \mathcal{W}_1(\sigma) \) as a function of \( \sigma \). One-sample empirical approximation rates under \( \mathcal{W}_1(\sigma) \) are explored in Section 5.

### 3 Metrizing the Weak Topology

Clearly, \( \mathcal{W}_1(\sigma)(\mu, \nu) < +\infty \), for any \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \). Furthermore, similar to the regular 1-Wasserstein distance, \( \mathcal{W}_1(\sigma) \) is a metric on \( \mathcal{P}(\mathbb{R}^d) \), whose convergence is equivalent to convergence in the weak topology.

**Theorem 1 (GOT Metric)** For any \( \sigma \geq 0 \), \( \mathcal{W}_1(\sigma) : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \to [0, +\infty) \) is a metric on \( \mathcal{P}(\mathbb{R}^d) \).

This result mostly follows from \( \mathcal{W}_1 \) being a metric. Some work is needed to establish the ‘identity of indiscernibles’ properties. See Section 7.1 for the proof.

**Theorem 2 (Weak Topology Metrization)** Let \( \sigma \geq 0 \) be in \( \mathcal{P}(\mathbb{R}^d) \) and \( \mu \in \mathcal{P}(\mathbb{R}^d) \). Then \( \mathcal{W}_1(\sigma)(\mu_k, \mu) \to 0 \) if and only if \( \mu_k \rightharpoonup \mu \). Consequently, \( \mathcal{W}_1(\sigma)(\mu_k, \mu) \to 0 \iff \mathcal{W}_1(\sigma)(\mu_k, \mu) \to 0 \).

**Theorem 3 (GOT Dependence on \( \sigma \))** Fix \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}^d) \). The following hold:

i) \( \mathcal{W}_1(\sigma)(\mu, \nu) \) is continuous and monotonically non-increasing in \( \sigma \in [0, +\infty) \);

ii) \( \lim_{\sigma \to 0} \mathcal{W}_1(\sigma)(\mu, \nu) = \mathcal{W}_1(\mu, \nu) \);

iii) \( \lim_{\sigma \to \infty} \mathcal{W}_1(\sigma)(\mu, \nu) = 0 \), for some \( \mu, \nu \in \mathcal{P}(\mathbb{R}^d) \).

While \( \mathcal{W}_1(\sigma)(\mu, \nu) \) is a monotonically non-increasing function of \( \sigma \), as \( \sigma \to \infty \) it is interestingly not true in general that \( \mathcal{W}_1(\mu \ast N_\sigma, \nu \ast N_\sigma) \) decays to zero. The proof of Theorem 3 (Section 7.3) shows this via a simple Dirac measure example.

A key technical tool (that may be of independent interest) for establishing item i) above is the following lemma, which ties GOT at different noise levels to one another. Its proof (Section 7.4) uses the Kantorovich-Rubinstein duality.

**Lemma 1 (Stability Across \( \sigma \))** Fix \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}^d) \), and \( 0 \leq \sigma_1 < \sigma_2 < +\infty \). We have
\[
\mathcal{W}_1(\sigma_2)(\mu, \nu) \leq \mathcal{W}_1(\sigma_1)(\mu, \nu) \leq \mathcal{W}_1(\sigma_1)(\mu, \nu) + 2\sqrt{d(\sigma_2^2 - \sigma_1^2)}.
\]

**Theorem 4 (Convergence of Optimal Plans)** Fix \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}^d) \) and let \( (\sigma_k)_{k \in \mathbb{N}} \) be a sequence with \( \sigma_k \searrow \sigma \geq 0 \). Let \( \pi_k \in \Pi(\mu \ast N_{\sigma_k}, \nu \ast N_{\sigma_k}) \), \( k \in \mathbb{N} \), be an optimal coupling for \( \mathcal{W}_1(\sigma_k)(\mu, \nu) \). Then there exists \( \pi \in \Pi(\mu \ast N_\sigma, \nu \ast N_\sigma) \) such that \( \pi_k \rightharpoonup \pi \) (weakly) as \( k \to \infty \) and \( \pi \) is optimal for \( \mathcal{W}_1(\sigma)(\mu, \nu) \).
The proof of Theorem 1 relies on the notion of $\Gamma$-convergence. Convergence of optimal transport plans then follows by standard tightness arguments. In particular, this theorem implies that a sequence of optimal transport plans for $W_{1}(\mu, \nu)$ converges to an optimal plan for the regular 1-Wasserstein distance $W_{1}(\mu, \nu)$ as $\sigma \to 0$.

5 Empirical Approximation

We now explore statistical properties of $W_{1}(\sigma)$. In fact, our derivation accounts for any isotropic noise distribution $G_{\sigma}$ that along each coordinate is $\sigma$-subgaussian with a bounded and monotone (in a proper sense) density $4$ Gaussian noise is captured as a special case.

Consider the fundamental one-sample empirical approximation, where $\mu \in P_{1}(\mathbb{R}^{d})$ is approximated by $\tilde{\mu}_{n} \triangleq \frac{1}{n} \sum_{i=1}^{n} \delta_{x_{i}}$, with $(X_{1}, \ldots, X_{n}) \sim \mu \otimes \delta_{x}$ as the Dirac measure centered at $x$. We study how fast $W_{1}(\sigma, \mu) \triangleq W_{1}(\tilde{\mu}_{n} \ast G_{\sigma}, \mu \ast G_{\sigma}) \to 0$ with $n$ in a remarkable contrast to the 1-Wasserstein curse of dimensionality, we show $\mathbb{E}_{\mu \otimes \sigma} W_{1}^{(\sigma)}(\tilde{\mu}_{n}, \mu) \leq O\left(n^{-\frac{1}{2}}\right)$ in all dimensions $4$, thus attaining the parametric rate.

To state the results, we first define subgaussianity.

**Definition 3 (Subgaussian Measure)** A probability measure $\mu \in P_{1}(\mathbb{R}^{d})$ is $K$-subgaussian, for $K > 0$, if for any $\alpha \in \mathbb{R}^{d}$, $X \sim \mu$ satisfies

\[
\mathbb{E}_{\mu} \left[ e^{\alpha^{T} (X - \mathbb{E}X)} \right] \leq e^{\frac{1}{2} K^{2} \|\alpha\|^{2}}. \tag{4}
\]

We begin with a bound on the expected value and then move to a high probability bound. The next theorem generalizes $4$ Prop. 1 to non-Gaussian noise models.

**Theorem 5 (GOT Empirical Approximation)** Fix $d \geq 1$, $\sigma > 0$ and $K > 0$. Let $G_{\sigma} \in P_{1}(\mathbb{R}^{d})$ have a density $g_{\sigma}$ that decomposes as $g_{\sigma}(x) = \prod_{j=1}^{d} \tilde{g}_{\sigma}(x_{j})$. Assume that $\tilde{g}_{\sigma}$ is $\sigma$-subgaussian, bounded and monotonically decreases as its argument goes away from zero in either direction. For any $K$-subgaussian $\mu \in P_{1}(\mathbb{R}^{d})$, we have

\[
\mathbb{E}_{\mu \otimes \sigma} W_{1}^{(\sigma)}(\tilde{\mu}_{n}, \mu) \leq c_{\sigma,d,K} n^{-\frac{1}{2}}, \tag{5}
\]

where $c_{\sigma,d,K} = e^{O(d)}$ is given in $4$. In particular $W_{1}^{(\sigma)}(\tilde{\mu}_{n}, \mu) \leq O\left(n^{-\frac{1}{2}}\right)$.

The proof of Theorem 5 is given in Section 7.6.

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$4$A further extension to nonisotropic noise is possible via similar techniques, but we do not delve into it here.

$5$Of course, $W_{1}(\sigma) = W_{1}(\sigma, \mu)$.

**Corollary 1 (Concentration Inequality)** Under the paradigm of Theorem 4 denote $X \triangleq \supp(\mu)$ and suppose $\text{diam}(X) < \infty$, where $\text{diam}(X) = \sup_{x \neq y \in X} \|x - y\|$. For any $t > 0$ we have

\[
\mathbb{P}_{\mu \otimes \sigma} \left( W_{1}(\sigma)(\tilde{\mu}_{n}, \mu) - \mathbb{E}W_{1}(\sigma)(\tilde{\mu}_{n}, \mu) \right) \leq t \leq 2e^{-\frac{2\sigma^{2}}{\text{diam}(X)^{2}}} \tag{6}
\]

and consequently,

\[
\mathbb{P}_{\mu \otimes \sigma} \left( W_{1}(\sigma)(\tilde{\mu}_{n}, \mu) \leq \omega \left( \frac{\log n}{\sqrt{n}} \right) \right) \leq \frac{1}{\text{poly}(n)}. \tag{7}
\]

The proof Theorem 4 is given in Section 7.7. It uses the $W_{1}$ duality and McDiarmid’s inequality.

6 Empirical Results

We turn to some numerical experiments demonstrating the difference in empirical approximation convergence rates between the regular 1-Wasserstein distance and GOT. Specifically, we compute $W_{1}(\tilde{\mu}_{n}, \mu)$ and $W_{1}(\sigma, \mu)$, for $\mu = \text{Unif}(0,1)^{d}$ the uniform measure on $[0,1]^{d}$, and $\mu_{n} \triangleq \frac{1}{n} \sum_{i=1}^{n} \delta_{x_{i}}$, the empirical measure based on i.i.d. samples $X_{1}, \ldots, X_{n} \sim \mu$. This simple setup also hints at how broad the class of distributions for which $W_{1}(\tilde{\mu}_{n}, \mu)$ attains the poor convergence rate.

The GOT framework corresponds to the 1-Wasserstein distance between two continuous (smooth) distributions. To evaluate this 1-Wasserstein distance we chose the neural network (NN) based dual optimization algorithm of Section 7. This approach seems to be better suited for continuous probability measures than, e.g., the Sinkhorn algorithm $19$. Starting from Kantorovich-Rubinstein duality

\[
W_{1}(\mu, \nu) = \sup_{\|f\|_{Lip} \leq 1} \mathbb{E}_{\mu} f - \mathbb{E}_{\nu} f, \tag{8}
\]

the function $f$ is first parametrized by a NN $f_{\theta}$, with parameter set $\theta \in \Theta^{4}$ and then the $\|f_{\theta}\|_{Lip}$ constraint is relaxed to a regularization penalty on the expected gradient of $f_{\theta}(x)$ (w.r.t. to $x$). In sum, as in $7$, we use the ADAM stochastic gradient ascent method to optimize

\[
\sup_{\theta \in \Theta} \mathbb{E}_{\mu} f_{\theta} - \mathbb{E}_{\nu} f_{\theta} + \lambda \mathbb{E}_{\eta} \left[ (\|\nabla_{x} f_{\theta}\| - 1)^{2} \right] \tag{9}
\]

where $\eta$ interpolates between $\mu$ and $\nu$ in a manner compatible with the gradient penalty (GP) theoretical justification $7$ Prop. 1. Specializing to $W_{1}(\sigma, \mu)$ and $W_{1}(\sigma, \nu)$ we used a fully connected network with 3 hidden ReLU layers, each comprising 1024 nodes. The network was trained until convergence of the estimated Wasserstein distance.
The measure $\mu$ is the uniform distribution over $[0, 1]^d$. Note that $\sigma = 0$ corresponds to the vanilla Wasserstein distance, which converges slower than GOT (note the difference in slopes), especially with larger $d$.

\( \nu \) above are replaced with their Gaussian-smoothed versions, i.e., $\mu \ast N_\sigma$ and $\nu \ast N_\sigma$, respectively. To approximate expectations with empirical sums, we sample from these Gaussian-smoothed measures by adding (sampled) Gaussian noise to the original samples. This makes use of the fact that convolution of probability measures corresponds to sums of independent random variables.

Figure 1 shows the results for $d = 5$ and $d = 10$, with each curve being the average of 10 random trials.\footnote{Error bars were omitted since they were too small to be visible.} Note the slower convergence to zero of the $W_1^{(\sigma)}$ for larger $d$. As NNs tend to overfit \( W_1 \), it remains to show that $W_1(\mu, \nu) = 0$ implies that $\mu = \nu$. Since $W_1$ is a metric, we know that if $W_1^{(\sigma)}(\mu, \nu) = 0$ then $\mu \ast N_\sigma = \nu \ast N_\sigma$. This implies pointwise equality between characteristic functions: $\phi_\mu \phi_{N_\sigma} = \phi_\nu \phi_{N_\sigma}$. Since $\phi_{N_\sigma} \neq 0$ everywhere, we get $\phi_\mu = \phi_\nu$ pointwise, implying $\mu = \nu$.
7.2 Proof of Theorem 2

The claim relies on the equivalence between weak convergence and pointwise convergence of characteristic functions. Since $W_1$ metrizes weak convergence:

$W_1^{(\sigma)}(\mu_k, \mu) \to 0$

$\iff \phi_{\mu_k}(t) \phi_{\mu}(t) \to \phi_{\mu}(t), \forall t \in \mathbb{R}^d$

$\iff \phi_{\mu_k}(t) = \phi_{\mu}(t), \forall t \in \mathbb{R}^d$.

7.3 Proof of Theorem 3

For Claim (ii), the fact that $\lim_{\sigma \to 0} W_1^{(\sigma)}(\mu, \nu) = W_1(\mu, \nu)$ follows from Lemma 1 by taking $\sigma = 0$ and $\sigma_2 = \sigma \to 0$.

For Claim (i), $W_1^{(\sigma)}(\mu, \nu)$ being monotonically non-increasing in $\sigma$ also follows directly from Lemma 1. To prove continuity at $\sigma = 0$, we consider left- and right-continuity separately. Let $\sigma_k \downarrow \sigma$ as $k \to \infty$. Lemma 1 gives

$W_1^{(\sigma)}(\mu, \nu) \leq W_1^{(\sigma)}(\mu, \nu) \leq W_1^{(\sigma)}(\mu, \nu) + 2d\sqrt{\sigma^2 - \sigma_k^2}$, \hspace{1cm} (11)

and left-continuity follows.

To see that $W_1^{(\sigma)}(\mu, \nu)$ is right-continuous in $\sigma$, let $\sigma_k \nearrow \sigma$ and denote $\epsilon_k = \sqrt{\sigma_k^2 - \sigma^2}$. We have

$W_1^{(\sigma_k)}(\mu, \nu) = W_1^{(\epsilon_k)}(\mu * \mathcal{N}_{\sigma}, \nu * \mathcal{N}_{\sigma}) \xrightarrow{k \to \infty} W_1^{(\sigma)}(\mu, \nu), \hspace{1cm} (12)$

where the last step uses $W_1^{(\sigma)}$ continuity at $\sigma = 0$.

Moving to Claim (iii), let $\mu = \delta_x$ and $\nu = \delta_y$ be two Dirac measures at $x \neq y \in \mathbb{R}^d$. For any $\sigma \in [0, +\infty)$, we have

$W_1^{(\sigma)}(\mu, \nu) = W_1(\mathcal{N}(x, \sigma^2 I_d), \mathcal{N}(y, \sigma^2 I_d))$

$\geq \left| E_{\mathcal{N}(x, \sigma^2 I_d)} X - E_{\mathcal{N}(y, \sigma^2 I_d)} Y \right| = \left| x - y \right|$, where the equality uses Jensen's inequality and convexity of norm.

7.4 Proof of Lemma 1

The first inequality immediately follows because $\mathcal{N}$ is non-increasing under convolutions and since $\mathcal{N}_{\sigma_2} = \mathcal{N}_{\sigma_1} * \mathcal{N}_{\sqrt{\sigma_2^2 - \sigma_1^2}}$.

For the second inequality, we use Kantorovich-Rubinstein duality to write

$W_1^{(\sigma_2)}(\mu, \nu) = \sup_{\|f_k\|_{\infty} \leq 1} \mathbb{E}_{\mu * \mathcal{N}_{\sigma_1}} f_k - \mathbb{E}_{\nu * \mathcal{N}_{\sigma_1}} f_k$.

Letting $f^*_1$ be optimal for $W_1^{(\sigma_2)}(\mu, \nu)$, we have

$W_1^{(\sigma_2)}(\mu, \nu) \geq \mathbb{E}_{\mu * \mathcal{N}_{\sigma_2}} f^*_1 - \mathbb{E}_{\nu * \mathcal{N}_{\sigma_2}} f^*_1$. \hspace{1cm} (13)

Set $X \sim \mu$, $Z_1 \sim \mathcal{N}_{\sigma_1}$ and $Z_{21} \sim \mathcal{N}_{\sqrt{\sigma_2^2 - \sigma_1^2}}$ as independent random variables; clearly, $Z_2 \triangleq Z_1 + Z_{21} \sim \mathcal{N}_{\sigma_2}$. Consider:

$\left| \mathbb{E}_{\nu * \mathcal{N}_{\sigma_2}} f^*_1 - \mathbb{E}_{\mu * \mathcal{N}_{\sigma_2}} f^*_1 \right| = \mathbb{E}_{f^*_1}(X + Z_1) - \mathbb{E}_{f^*_1}(X + Z_2) \leq \mathbb{E}\|Z_{21}\| = \sqrt{d(\sigma_2^2 - \sigma_1^2)}. \hspace{1cm} (14a)$

where the last in equality uses $\|f^*_1\|_{\text{lip}} \leq 1$. Similarly, one has

$\left| \mathbb{E}_{\nu * \mathcal{N}_{\sigma_2}} f^*_1 - \mathbb{E}_{\mu * \mathcal{N}_{\sigma_2}} f^*_1 \right| \leq \sqrt{d(\sigma_2^2 - \sigma_1^2)}. \hspace{1cm} (14b)$

Inserting (14) into (13) concludes the proof.

7.5 Proof of Theorem 4

We first include the definitions of tightness of measures and $\Gamma$-convergence of functionals.

Definition 4 (Tightness of Measures) A subset $S \subseteq \mathcal{P}(\mathbb{R}^d)$ is tight if for any $\epsilon > 0$ there is a compact set $K_\epsilon \subseteq \mathbb{R}^d$ such that $\mu(K_\epsilon) \geq 1 - \epsilon$, for all $\mu \in \mathcal{P}(\mathbb{R}^d)$.

Definition 5 (\(\Gamma\)-Convergence) Let $\mathcal{X}$ be a metric space and $F_k : \mathcal{X} \to \mathbb{R}$, $k \in \mathbb{N}$ be a sequence of functionals. We say $(F_k)_{k \in \mathbb{N}}$ $\Gamma$-converges to $F : \mathcal{X} \to \mathbb{R}$, and we write $F_k \rightharpoonup \Gamma F$, if:

i) For every $x_k, x \in \mathcal{X}$, $k \in \mathbb{N}$, with $x_k \to x$, we have $F(x) \leq \liminf_{k \to \infty} F_k(x_k)$;

ii) For any $x \in \mathcal{X}$, there exists $x_k \in \mathcal{X}$, $k \in \mathbb{N}$, with $x_k \to x$, and $F(x) \geq \limsup_{k \to \infty} F_k(x_k)$

By pointwise convergence of characteristic functions, $P_k \triangleq \mu * \mathcal{N}_{\sigma_k}$ and $Q_k \triangleq \nu * \mathcal{N}_{\sigma_k}$ are weakly convergent measures on $\mathbb{R}^d$. Prokhorov’s Theorem then implies they are tight. By [13, Lemma 4.4] we have that $\Pi(P_k)_{k \in \mathbb{N}}, (Q_k)_{k \in \mathbb{N}}$, the set of all couplings with marginals in $(P_k)_{k \in \mathbb{N}}$ and $(Q_k)_{k \in \mathbb{N}}$ is also tight. Hence, the sequence of optimal couplings $(\pi_k)_{k \in \mathbb{N}}$ is tight and weakly converges to some $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Taking the limit of the relation $\pi_k \in \Pi(P_k, Q_k)$ we obtain $\pi \in \Pi(P, Q)$, where $P \triangleq \mu * \mathcal{N}_\sigma$ and $Q \triangleq \nu * \mathcal{N}_\sigma$. 

With that in mind, recall that if \((F_k)_{k \in \mathbb{N}}\) \(\Gamma\)-converges to \(F\), then \(\lim_{k \to \infty} \inf F_k = \inf F\) [28 Thm. 7.8]. Furthermore, if \((x_k)_{k \in \mathbb{N}}\) is a sequence of minimizers of \(F_k\), for each \(k \in \mathbb{N}\), then any cluster (limit) point of \((x_k)_{k \in \mathbb{N}}\) is a minimizer of \(F\) [28 Cor. 7.20]. Thus, to conclude the proof of Theorem 4 it suffices to establish \(\Gamma\)-convergence of \(F_k : \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}\) to \(F : \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}\) defined as

\[
F_k(\pi) = \begin{cases} 
E_\pi \|X - Y\|, & \pi \in \Pi(\mu \ast \mathcal{N}_{\sigma_k}, \nu \ast \mathcal{N}_{\sigma_k}) \\
\infty, & \text{otherwise}
\end{cases}
\]

\[
F(\pi) = \begin{cases} 
E_\pi \|X - Y\|, & \pi \in \Pi(\mu \ast \mathcal{N}_\sigma, \nu \ast \mathcal{N}_\sigma) \\
\infty, & \text{otherwise}
\end{cases}
\]

(15)

We start with the \(\lim \inf\) \(\Gamma\)-convergence inequality. First observe that if \((\pi_k)_{k \in \mathbb{N}}\) does not contain a subsequence (without relabeling) such that \(\pi_k \in \Pi(\mu \ast \mathcal{N}_{\sigma_k}, \nu \ast \mathcal{N}_{\sigma_k})\), then the claim is trivial. Accordingly, assume (again, up to extraction of subsequences) that \(\pi_k \in \Pi(\mu \ast \mathcal{N}_{\sigma_k}, \nu \ast \mathcal{N}_{\sigma_k})\), for all \(k \in \mathbb{N}\). Since \(x \mapsto \|x\|\) is a non-negative and continuous, the \(\lim \inf\) condition directly follows from the Portmanteau Theorem:

\[
F(\pi) = \int \|x - y\| \, d\pi \\
\leq \lim \inf_{k \to \infty} \int \|x - y\| \, d\pi_k \\
= \lim \inf_{k \to \infty} F_k(\pi_k).
\]

(16)

For the \(\lim \sup\) let \(\pi \in \Pi(\mu \ast \mathcal{N}_\sigma, \nu \ast \mathcal{N}_\sigma)\). For convenience, we use random variable notation. There exists a tuple \((X, Y, Z', Z'')\) with marginal distributions \(X \sim \mu\), \(Y \sim \nu\) and \(Z', Z'' \sim \mathcal{N}_\sigma\), such that \((X, Z')\) are independent, \((Y, Z'')\) are independent, and \((X + Z', Y + Z'') \sim \pi\).

To construct the sequence \((\pi_k)_{k \in \mathbb{N}}\), let \(Z_k \sim \mathcal{N}_{\sqrt{\frac{\sigma^2}{\sigma_k} - \sigma^2}}\) be independent of \((X, Y, Z', Z'')\). Setting \(\pi_k\) as the joint probability law of \((X + Z', Z_k, Y + Z'' + Z_k)\), we have \(\pi_k \in \Pi(\mu \ast \mathcal{N}_{\sigma_k}, \nu \ast \mathcal{N}_{\sigma_k})\), \(k \in \mathbb{N}\). Evaluating \(F_k\) we obtain

\[
F_k(\pi_k) = \mathbb{E}\|X + Z' + Z_k - Y - Z'' - Z_k\| \\
= \mathbb{E}\|X + Z' - Y - Z''\| \\
= F(\pi),
\]

which in particular implies the \(\lim \sup\) condition.

7.6 Proof of Theorem 5

The 1-Wasserstein distance is upper bounded by weighted total variation (TV) as follows [19] Theorem 6.15:

\[
W_1(\mu_n \ast \mathcal{G}_\sigma, \mu \ast \mathcal{G}_\sigma) \leq \int_{\mathbb{R}^d} \|t\|\|r_n(t) - q(t)\| \, dt,
\]

(17)

where \(r_n\) and \(q\) are the densities of \(\hat{\mu}_n \ast \mathcal{G}_\sigma\) and \(\mu \ast \mathcal{G}_\sigma\), respectively. The inequality is proved using the maximal TV coupling of \(\hat{\mu}_n \ast \mathcal{G}_\sigma\) with \(\mu \ast \mathcal{G}_\sigma\).

Let \(a > 0\) (to be specified later) and set \(f_a : \mathbb{R}^d \to \mathbb{R}\) as the density of \(\mathcal{N}(0, \frac{1}{2a^2} I_d)\). By Cauchy-Schwarz, we have

\[
\mathbb{E}_{\mu \ast \mathcal{G}_\sigma} \int_{\mathbb{R}^d} \|t\|\|r_n(t) - q(t)\| \, dt \\
\leq \left( \int_{\mathbb{R}^d} \|t\|^2 f_a(t) \, dt \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} (\|q(t) - r_n(t)\|^2 - f_a(t))^2 \, dt \right)^{\frac{1}{2}}.
\]

(18)

The first term equals \(\frac{d}{2a^2}\). Turning to the second integral, note that \(r_n(t) = \frac{1}{n} \sum_{i=1}^n g_\sigma(t - X_i)\), where \(\{X_i\}_{i=1}^n\) are i.i.d. and \(\mathbb{E}_\mu g_\sigma(t - X) = q(t)\). Using the definition of subgaussianity (Definition 3), we have the following lemma (proven in Appendix A) that bounds \(g_\sigma\) everywhere in terms of the Gaussian density \(\varphi_\sigma\).

**Lemma 2** Let \(\delta = \min \{1, \frac{1}{4\sigma^2}\}\). There exists a constant \(c_1 > 0\) such that

\[
g_\sigma(t) \leq c_1^2 e^{\delta \|t\|^2} \varphi_\sigma(t), \quad \forall t \in \mathbb{R}^d.
\]

(19)

We now can bound the second integrand of (18):

\[
\mathbb{E}_{\mu \ast \mathcal{G}_\sigma} (q(z) - r_n(z))^2 = \mathbb{E}_{\mu \ast \mathcal{G}_\sigma} (r_n(z))^2 \\
= \mathbb{E}_{\mu \ast \mathcal{G}_\sigma} \left( \frac{1}{n} \sum_{i=1}^n g_\sigma(z - X_i) \right)^2 \\
= \frac{1}{n} \mathbb{E}_\mu g_\sigma^2(z - X) \\
\leq \mathbb{E}_\mu g_\sigma^2(z - X) \\
\leq c_2^2 \mathbb{E}_\mu e^{-\delta \|z - X\|^2} \varphi_\sigma^2(z - X) \\
\leq \frac{c_2^2}{n} \mathbb{E}_\mu e^{-\delta \|z - X\|^2}.
\]

(20)

c with \(c_2 = c_1^2 (2\pi \sigma^2)^{-d/2}\). This further implies

\[
\int_{\mathbb{R}^d} \mathbb{E}_{\mu \ast \mathcal{G}_\sigma} (q(t) - r_n(t))^2 \, dz \leq \frac{c_2^2}{n^{2d/2}} \mathbb{E}_{\mu \ast \mathcal{G}_\sigma} \frac{1}{f_a(X + Z)}.
\]

(21)

where \(X \sim \mu\) and \(Z \sim \mathcal{N}_\sigma\) are independent.

Starting from (21), we finish the proof via steps similar to [29]. Specifically, for \(c_3 = \left( \frac{\pi}{\sigma^2} \right)^{\frac{1}{2}}\), it holds that \(f_a(t)^{-\frac{1}{2}} = c_3 e^{a^2t^2}\). Since \(X\) is \(K\)-subgaussian and \(Z\) is \(\sigma\)-subgaussian, \(X + Z\) is \((K + \sigma)\)-subgaussian. Following [21], for any \(0 < a < \frac{1}{2(K + \sigma)}\), we have [30] Rmk. 2.3

\[
\frac{c_2^2}{n^{2d/2}} \mathbb{E}_{\mu \ast \mathcal{G}_\sigma} \frac{1}{f(X + Z)}.
\]
Let \( g(X) \) be an \( n \)-tuple of \( \mathcal{X} \)-valued independent random variables. Suppose \( g : \mathcal{X}^n \to \mathbb{R} \) is a map that for any \( i = 1, \ldots, n \) and \( x_1, \ldots, x_n, x'_i \in \mathcal{X} \) satisfies

\[
|g(x^n) - g(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)| \leq c_i, \quad (24)
\]

for some non-negative \( \{c_i\}_{i=1}^n \). Then for any \( t > 0 \):

\[
\Pr\left( g(X^n) - \mathbb{E}g(X^n) \geq t \right) \leq e^{-\frac{2t^2}{\sum_{i=1}^n c_i^2}}, \quad (25a)
\]

\[
\Pr\left( |g(X^n) - \mathbb{E}g(X^n)| \geq t \right) \leq 2e^{-\frac{2t^2}{\sum_{i=1}^n c_i^2}}, \quad (25b)
\]

Let \( g(X^n) \) and use Kantorovich-Rubinstein duality:

\[
g(X^n) = \sup_{\|f\|_{L^{\infty}} \leq 1} \mathbb{E}_{\mu^\ast \ast \sigma} f - \mathbb{E}_{\mu \ast \sigma} f = \sup_{\|f\|_{L^{\infty}} \leq 1} \frac{1}{n} \sum_{i=1}^n (f \ast g_\sigma(X_i) - \mathbb{E}_\mu[f \ast g_\sigma]).
\]

Fix \( i \in \{1, \ldots, n\} \) and \( x_1, \ldots, x_n, x'_i \in \mathcal{X} \). Property (24) follows by first observing that:

\[
n\left( g(x^n) - g(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n) \right)
\]

\[
= \sup_{\|f\|_{L^{\infty}} \leq 1} \left\{ \sum_{j \neq i} (f \ast g_\sigma(x_j) - \mathbb{E}_\mu[f \ast g_\sigma] + (f \ast g_\sigma)(x_i)) \right\}
\]

\[
- \sup_{\|h\|_{L^{\infty}} \leq 1} \left\{ \sum_{j \neq i} (h \ast g_\sigma(x_j) - \mathbb{E}_\mu[h \ast g_\sigma] + (h \ast g_\sigma)(x'_i)) \right\}
\]

\[
\leq \sup_{\|f\|_{L^{\infty}} \leq 1} \left( f \ast g_\sigma(x_i) - (f \ast g_\sigma)(x'_i) \right). \quad (26)
\]

Then we note that Lipschitzness of \( f \) implies that \( f \ast g_\sigma \) is also Lipschitz.
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Appendix

A Proof of Lemma 2

Recall that $g_\sigma(t) = \prod_{j=1}^{d} \tilde{g}_\sigma(t_j)$, where $\tilde{g}_\sigma$ is $\sigma$-subgaussian, zero mean, bounded, and monotonically decreasing as $t_j$ moves away from zero. We first analyze the one-dimensional densities $\tilde{g}_\sigma$, and show that there exists a constant $c > 0$, such that

$$\tilde{g}_\sigma(t) \leq ce^{2|t| - \delta^2 - \log \delta \tilde{\varphi}_\sigma(t)}, \quad \forall t \in \mathbb{R},$$

(27)

where $\tilde{\varphi}_\sigma$ is a scalar Gaussian density (zero mean and $\sigma^2$ variance). We prove (27) for $t > 0$; the $t < 0$ case is identical.

Note that the $\sigma$-subgaussianity of $\tilde{g}_\sigma$ (Def. 3) implies that

$$\mathbb{E}_{\tilde{g}_\sigma}[e^{\alpha X}] \leq e^{\frac{1}{2}\sigma^2\alpha^2}, \quad \forall \alpha \in \mathbb{R},$$

(28)

which by (31) yields

$$\mathbb{P}_{\tilde{g}_\sigma}((-\infty, t) \cup (t, \infty)) \leq \exp(1 - t^2/(2\sigma^2)) = c' \tilde{\varphi}_\sigma(t),$$

(29)

where $c' = \sqrt{2\pi\sigma^2}e^{t^2}$. Consequently, for any $t^*$,

$$\mathbb{P}_{\tilde{g}_\sigma}((t^* - \delta, t^*]) \leq \mathbb{P}_{\tilde{g}_\sigma}((t^* - \delta, \infty)) \leq c' \tilde{\varphi}_\sigma(t^* - \delta) = c'e^{(t^* - \delta)^2 \tilde{\varphi}_\sigma(t^*)} = c'e^{2\delta t^* - \delta^2 \tilde{\varphi}_\sigma(t^*)}.$$  

(30)

Now, since $\tilde{g}_\sigma(t)$ monotonically decreases as $t$ moves away from zero, for any $t^* \geq \delta$ we have $\mathbb{P}_{\tilde{g}_\sigma}((t^* - \delta, t^*]) \geq \delta \tilde{g}_\sigma(t^*)$. Substituting this into (30), we have for all $t^* \geq \delta$ that

$$\delta \tilde{g}_\sigma(t^*) \leq c'e^{2\delta t^* - \delta^2 \tilde{\varphi}_\sigma(t^*)},$$

$$\tilde{g}_\sigma(t^*) \leq c'e^{2\delta t^* - \delta^2 - \log \delta \tilde{\varphi}_\sigma(t^*)}.$$  

Repeating the argument for $t < 0$ then yields

$$\tilde{g}_\sigma(t) \leq c'e^{2\delta |t| - \delta^2 - \log \delta \tilde{\varphi}_\sigma(t)}$$

for all $|t| \geq \delta$. Since $\tilde{g}_\sigma$ is bounded, $\sup_{|t| \leq \delta} \tilde{g}_\sigma(t) \left(e^{2\delta t - \delta^2 - \log \delta \tilde{\varphi}_\sigma(t)}\right)^{-1}$ exists, and hence (27) holds for all $t \in \mathbb{R}$ with

$$c = \max \left[c', \sup_{|t| \leq \delta} \tilde{g}_\sigma(t) \left(e^{2\delta t - \delta^2 - \log \delta \tilde{\varphi}_\sigma(t)}\right)^{-1}\right].$$

Extending to the full $d$-dimensional distribution, note that since $t^2 + 1 > |t|$ for all $t$, we have that $\tilde{g}_\sigma(t) \leq ce^{2\delta t^2 + 2\delta^2 - \delta^2 - \log \delta \tilde{\varphi}_\sigma(t)}$ for all $t$. We can then write

$$g_\sigma(t) \leq (c')^d e^{2\delta \|t\|^2 + 2d\delta - d\delta^2 - d\log \delta \tilde{\varphi}_\sigma(t)},$$

(31)

which establishes the lemma after collecting terms.