Some Applications of Generalized Mountain Pass Lemma

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Abstract

The Ghoussoub-Preiss’s generalized Mountain Pass Lemma with Cerami-Palais-Smale type condition is a generalization of classical MPL of Ambrosetti-Rabinowitz, we apply it to study the existence of the periodic solutions with a given energy for some second order Hamiltonian systems with symmetrical and non-symmetrical potentials.

Key Words: Second order Hamiltonian systems, periodic solutions, Ghoussoub-Preiss’s Generalized Mountain Pass Lemma, Cerami-Palais-Smale condition at some levels for a closed subset.

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1. Introduction and Main Results

In 1948, Seifert([17]) studied the periodic solutions of the Hamiltonian systems using geometrical and topological methods; in 1978 and 1979, Rabinowitz([15,16]) studied the periodic solutions of the Hamiltonian systems using global variational methods; in 1980’s, Benci ([4]) and Gluck-Ziller([9]) and Hayashi([11]) used Jacobi metric and very complicated geodesic methods and algebraic topology to study the periodic solutions for second order Hamiltonian systems with a fixed energy:

\[
\begin{align*}
\ddot{q} + V'(q) &= 0 \\
\frac{1}{2} |\dot{q}|^2 + V(q) &= h
\end{align*}
\]

They proved the following very general theorem:

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Theorem 1.1 Suppose $V \in C^1(\mathbb{R}^n, \mathbb{R})$, if
\[ \{ x \in \mathbb{R}^n | V(x) \leq h \} \]
is bounded, and
\[ V'(x) \neq 0, \quad \forall x \in \{ x \in \mathbb{R}^n | V(x) = h \}, \]
then the (1.1)-(1.2) has a periodic solution with energy $h$.

For the existence of multiple periodic solutions for (1.1)-(1.2), we can refer Groessen([10]) and Long [12] and the references there.

Ambrosetti–Coti Zelati([1]) used Ljusternik-Schnirelmann theory with classical $(PS)^+$ compact condition to get the following Theorem:

Theorem 1.2 Suppose $V \in C^2(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies:
- $(A1). 3V'(x) \cdot x + V''(x)x \cdot x \neq 0, \forall x \in \Omega = \mathbb{R}^n \setminus \{0\}$;
- $(A2). V'(x) \cdot x > 0, \quad \forall x \in \Omega$;
- $(A3'). \exists \alpha \in (0, 2), such that
  \[ V'(x) \cdot x \geq -\alpha V(x), \quad \forall x \in \Omega; \]
- $(A4'). \exists \delta \in (0, 2) \text{ and } r > 0, such that
  \[ V'(x) \cdot x \leq -\delta V(x), \quad \forall 0 < |x| \leq r; \]
- $(A5'). \liminf_{|x| \to +\infty} \left[ V(x) + \frac{1}{2} V'(x) \cdot x \right] \geq 0.$

Then $\forall h < 0$ the system (1.1)-(1.2) has at least a non-constant weak periodic solution which satisfies (1.1)-(1.2) pointwise except on a zero-measurable set.

Ambrosetti-Coti Zelati ([2]) used a variant of the classical Mountain-Pass Lemma and a constraint minimizing method to get the following Theorems:

Theorem 1.3 Suppose $V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies:
- $(V1). V(-\xi) = V(\xi), \forall \xi \in \Omega = \mathbb{R}^n \setminus \{0\}$;
- $(V2). \exists \alpha \in [1, 2), such that
  \[ \nabla V(\xi) \cdot \xi \geq -\alpha V(\xi) > 0, \quad \forall \xi \in \Omega; \]
- $(V3). \exists \delta \in (0, 2) \text{ and } r > 0, such that
  \[ \nabla V(\xi) \cdot \xi \leq -\delta V(\xi), \quad \forall 0 < |\xi| \leq r; \]
- $(V4). V(\xi) \to 0, \quad \text{as } |\xi| \to +\infty.$

Then $\forall h < 0$, the problem (1.1) – (1.2) has a weak periodic solution.
\textbf{Theorem 1.4} Suppose $V$ satisfies (V1), (V3), (V4) and
\quad (V2'). \exists \alpha \in (0, 2), \text{ such that }
\quad \nabla V(\xi) \cdot \xi \geq -\alpha V(\xi) > 0, \ \forall \xi \in \Omega;
\quad (V5). V \in C^2(\Omega, \mathbb{R}) \text{ and }
\quad 3 \nabla V(\xi) \cdot \xi + V''(\xi) \cdot \xi > 0.

Then for any $h < 0$, (1.1) - (1.2) has a weak periodic solution.

\textbf{Theorem 1.5} Suppose $V \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies:
\quad (V_1). V(-q) = V(q);
\quad (V_2). There are constant $0 < \alpha < 2$ such that
\quad \langle V'(q), q \rangle \geq -\alpha V(q) > 0, \ \forall q \in \mathbb{R}^n \setminus \{0\};
\quad (V_3). \exists \delta \in (0, 2), r > 0, \text{ such that }
\quad \langle V'(q), q \rangle \leq -\delta V(q), \ \forall 0 < |q| \leq r;
\quad (V_4). V(q) \to 0, \text{ as } |q| \to +\infty.

Then for any given $h < 0$, the system (1.1) - (1.2) has at least a non-constant weak periodic solution which can be obtained by Mountain Pass Lemma.

Motivated by these papers, we use Ghoussoub-Preiss’s Generalized Mountain Pass Lemma with Cerami-Palais-Smale condition at some levels for a closed subset to study the new periodic solutions with symmetrical and non-symmetrical potentials, we obtain the following Theorems:

\textbf{Theorem 1.6} Suppose $V \in C^1(\mathbb{R}^n, R)$ and $h \in R$ satisfies
\quad (B_1). V(-q) = V(q).
\quad (B_2). \exists \mu_1 > 0, \mu_2 \geq 0, s.t. V'(q) \cdot q \geq \mu_1 V(q) - \mu_2.
\quad (B_3). V(q) \geq h, |q| \to +\infty.
\quad (B_4). \forall q \neq 0, 3V'(q) \cdot q + V''(q)q \cdot q \neq 0.

Then for any $h > \frac{\mu_2}{\mu_1}$, (1.1) - (1.2) has at least one non-constant periodic solution with the given energy $h$, which can be obtained by the generalized MPL method.

\textbf{Corollary 1.1} Suppose $a > 0, \mu_1 \geq 2, \mu_2 \geq 0, V(q) = a|q|^\mu_1 + \frac{\mu_2}{\mu_1},$ then the conditions of Theorem 1.1 hold and for any $h > \frac{\mu_2}{\mu_1}$, (1.1) - (1.2) has at least two non-constant periodic solution with the given energy $h$.

\textbf{Theorem 1.7} Suppose $V \in C^1(\mathbb{R}^n, R)$ and $h \in R$ satisfies (B_2), (B_3) and (B_5)
\quad \exists r > 0, s.t.
\quad \inf_{u \in F} \int_0^1 (h - V(u)) dt > 0,
\quad \text{where}
\quad F \triangleq \{ u \in H^1 | \| \dot{u} \|_{L^2} = r \}.

Then $\forall h > \frac{\mu_2}{\mu_1}$, (1.1)-(1.2) has at least one non-constant periodic solution with energy $h$. 

3
2 A Few Lemmas

Define Sobolev space:

\[ H^1 = W^{1,2}(R/TZ, R^n) = \{u : R \rightarrow R^n, u \in L^2, \dot{u} \in L^2, u(t+1) = u(t)\} \]

Then the standard \(H^1\) norm is equivalent to

\[ \|u\| = \|u\|_{H^1} = \left( \int_0^1 |\dot{u}|^2 dt \right)^{1/2} + |\int_0^1 u(t) dt|. \]

**Lemma 2.1** ([1,10]) Let \(f(u) = \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \int_0^1 (h-V(u)) dt\) and \(\bar{u} \in H^1\) be such that \(f'(\bar{u}) = 0\) and \(f(\bar{u}) > 0\). Set

\[ \frac{1}{T^2} = \frac{\int_0^1 (h-V(\bar{u})) dt}{\frac{1}{2} \int_0^1 |\dot{u}|^2 dt} \quad (2.1) \]

Then \(\bar{q}(t) = \bar{u}(t/T)\) is a non-constant \(T\)-periodic solution for (1.1)-(1.2).

By symmetry condition \((B_1)\), similar to Ambrosetti-Coti Zelati [2], let

\[ E_1 = \{u \in H^1 = W^{1,2}(R/Z, R^n), u(t+1/2) = -u(t)\} , \]

\[ E_2 = \{u \in H^1 = W^{1,2}(R/Z, R^n), u(-t) = -u(t)\} . \]

By the symmetrical condition \((B_1)\) and Palais’s symmetrical principle([14]) or similar proof of [1,2], we have

**Lemma 2.2** If \(\bar{u} \in E_i\) is a critical point of \(f(u)\) and \(f(\bar{u}) > 0\), then \(\bar{q}(t) = \bar{u}(t/T)\) is a non-constant \(T\)-periodic solution of (1.1)-(1.2).

Using the famous Ekeland’s variational principle, Ekeland proved

**Lemma 2.3** (Ekeland[7]) Let \(X\) be a Banach space, \(F \subset X\) be a closed (weakly closed) subset. Suppose that \(\Phi\) defined on \(F\) is Gateaux-differentiable and lower semi-continuous (or weakly lower semi-continuous) and bounded from below. Then there is a sequence \(x_n \subset F\) such that

\[ \Phi(x_n) \rightarrow \inf_{F} \Phi \]

\[ (1 + \|x_n\|)\|\Phi'(x_n)\| \rightarrow 0. \]

Motivated by the paper of Cerami[6], Ekeland [7], Ghoussoub-Preiss[8] presented a weaker compact condition than the classical \((CPS)_c\) condition:

**Definition 2.1** ([7,8]) Let \(X\) be a Banach space, \(F \subset X\) be a closed subset, let \(\delta(x, F)\) denotes the distance of \(x\) to the set \(F\). Suppose that \(\Phi\) defined on \(X\) is Gateaux-differentiable, if sequence \(\{x_n\} \subset X\) such that

\[ \delta(x_n, F) \rightarrow 0, \]

\[ \Phi(x_n) \rightarrow c, \]
\[(1 + \|x_n\|)\|\Phi'(x_n)\| \to 0,\]

then \(\{x_n\}\) has a strongly convergent subsequence.

Then we call \(f\) satisfies \((CPS)_{c,F}\) condition at the level \(c\) for the closed subset \(F \subset X\), we denote it as \((CPS)_{c,F}\).

We can give a weaker condition than \((CPS)_{c}\) condition:

**Definition 2.2** Let \(X\) be a Banach space. \(F \subset X\) be a weakly closed subset. Suppose that \(\Phi\) defined on \(X\) is Gateaux-differentiable, if sequence \(x_n\) such that

\[\delta(x_n, F) \to 0,\]

\[\Phi(x_n) \to \gamma,\]

\[(1 + \|x_n\|)\|\Phi'(x_n)\| \to 0,\]

then \(\{x_n\}\) has a weakly convergent subsequence.

Then we call \(f\) satisfies \((W CPS)_{c,F}\) condition.

Now by **Lemma 2.3**, it’s easy to prove

**Lemma 2.4** Let \(X\) be a Banach space,

(i). Let \(F \subset X\) be a closed subset. Suppose that \(\Phi\) defined on \(X\) is Gateaux-differentiable and lower semi-continuous and bounded from below, if \(\Phi\) satisfies \((CPS)_{\inf \Phi,F}\) condition, then \(\Phi\) attains its infimum on \(F\).

(ii). Let \(F \subset X\) be a weakly closed subset. Suppose that \(\Phi\) defined on \(F\) is Gateaux-differentiable and weakly lower semi-continuous and bounded from below, if \(\Phi\) satisfies \((W CPS)_{\inf \Phi,F}\) condition, then \(\Phi\) attains its infimum on \(F\).

**Definition 2.3** ([7,8]) Let \(X\) be a Banach space, \(F \subset X\) be a closed subset. If \(z_0, z_1\) belong different disjoint connected components in \(X \setminus F\), then we call \(F\) separates \(z_0\) and \(z_1\).

Motivated by the famous classical Mountain Pass Lemma of Ambrosetti-Rabinowitz [3], Ghoussoub-Preiss [8] gave a generalized MPL:

**Lemma 2.5** (Ghoussoub-Preiss’s generalized MPL [8],[7]) Let \(X\) be a Banach space. Suppose that \(\Phi(u) : X \to R\) is a continuous Gateaux-differentiable function with \(\Phi' : X \to X^*\) norm-to-weak* continuous. Take two points \(z_0, z_1\) in \(X\), and define

\[\Gamma = \{c \in C^0([0, 1]; X)|c(0) = z_0, c(1) = z_1\}\]

\[\gamma = \inf_{c \in \Gamma} \max_{0 \leq t \leq 1} \Phi(c(t))\]

Let \(F \subset X\) be a closed subset separating \(z_0\) and \(z_1\). Assume that

\[\Phi(x) > \max\{\Phi(z_0), \Phi(z_1)\}, \forall x \in F,\]

\(\Phi\) satisfies condition \((CPS)_{\gamma,F}\) on the level \(\gamma\) for the set \(F\). Then there is a critical point of \(\Phi\) on the level \(\gamma\).
3 The Proof of Theorem 1.6

We define weakly closed subsets of $H^1$:

$$F = \{ u \in H^1 \mid \int_0^1 (V(u) + \frac{1}{2} V'(u) u) dt = h \}.$$  

$$F_i = \{ u \in E_i \mid \int_0^1 (V(u) + \frac{1}{2} V'(u) u) dt = h \}, i = 1, 2.$$  

**Lemma 3.1** If $(B_2) - (B_4)$ hold, then $F, F_1, F_2 \neq \emptyset$.

**Proof** Similar to the proof of [1]. Let $u \in H^1, u \neq 0$ be fixed. For $a > 0$, let

$$g_a(u) = g(au) = \int_0^1 [V(au) + \frac{1}{2} V'(au) au] dt$$  

By $(B_4) \frac{d}{da} g_a(u) \neq 0$, so $g_a$ is strictly monotone. Notice that

$$g_a(0) = g(0) = V(0) \leq \frac{\mu_2}{\mu_1}$$

When $a$ is large, we use $(B_2) - (B_3)$ to have

$$g_a(0) = g(0) = V(0) \leq \frac{\mu_2}{\mu_1}$$

Therefore, we have

$$g_a(+\infty) = g(+\infty) > h$$

So for any given $u \in H^1, u \neq 0$, there is $a(u) > 0$ such that $a(u)u \in F$. Similarly we can prove that for any given $u \in E_i, u \neq 0$, there is $a(u) > 0$ such that $a(u)u \in F_i$.

**Lemma 3.2** If $(B_1), (B_2)$ and $(B_4)$ hold, then for any given $c > 0, f(u)$ satisfies $(CPS)_{c,F_i}$ condition, that is: If $\{ u_n \} \subset H^1$ satisfies

$$\delta(u_n, F_i) \to 0, f(u_n) \to c > 0, \quad (1 + \| u_n \|) f'(u_n) \to 0. \quad (3.1)$$

Then $\{ u_n \}$ has a strongly convergent subsequence.

**Proof** Notice that $\forall u \in E_i, \int_0^1 u(t) dt = 0$, so we know $\| u \|_{E_i} \triangleq (\int_0^1 |\dot{u}|^2 dt)^{1/2}$ is an equivalent norm on $E_i$. Now from $f(u_n) \to c$, we have

$$-\frac{1}{2} \| u_n \|_{E_i}^2 \cdot \int_0^1 V(u_n) dt \to c - \frac{h}{2} \| u_n \|_{E_i}^2 \quad (3.2)$$
By (B_2) we have

\[< f'(u_n), u_n > = \|u_n\|^2_{E_i} \cdot \int_0^1 (h - V(u_n) - \frac{1}{2} < V'(u_n), u_n >) dt \]

\[\leq \|u_n\|^2_{E_i} \int_0^1 [h + \frac{\mu_2}{2} - (1 + \frac{\mu_1}{2})V(u_n)] dt \] \hspace{1cm} (3.3)

By (3.2) and (3.3) we have

\[< f'(u_n), u_n > \leq (h + \frac{\mu_2}{2})\|u_n\|^2_{E_i} + (1 + \frac{\mu_1}{2}) (2c - h)\|u_n\|^2_{E_i} \]

\[= (-\frac{\mu_1}{2}h + \frac{\mu_2}{2})\|u_n\|^2_{E_i} + C_1 \] \hspace{1cm} (3.4)

Where \(C_1 = 2(1 + \frac{\mu_1}{2})c\)

Since \(h > \frac{\mu_2}{\mu_1}\), then (3.1)and (3.4) imply \(\|u_n\|_{E_i}\) is bounded.

The rest for proving \(\{u_n\}\) has a strongly convergent subsequence is standard.

**Remark 3.1** We notice that in our proof, we didn’t use the condition

\[\delta(u_n, F_i) \to 0. \] \hspace{1cm} (3.5)

It seems interesting to efficiently use this condition to weak our assumptions.

**Lemma 3.2** Let

\[G = \{u \in H^1 | \int_0^1 (V(u) + \frac{1}{2}V'(u)u) dt < h\}; \] \hspace{1cm} (3.6)

\[G_i = \{u \in E_i | \int_0^1 (V(u) + \frac{1}{2}V'(u)u) dt < h\}. \] \hspace{1cm} (3.7)

Then

(i). \(F, F_i, i = 1, 2\) are respectively the boundaries of \(G, G_i\).
(ii).If (B_1) holds, then \(F, F_i, G, G_i\) are symmetric with respect to the origin 0.
(iii).If \(V(0) < h\) holds, then \(0 \in G, G_i, i = 1, 2\).

It’s not difficult to prove the following two Lemmas:

**Lemma 3.3** \(f(u)\) is weakly lower semi-continuous on \(H^1\) and \(F, F_i\).

**Lemma 3.4** \(F, F_i, i = 1, 2\) are weakly closed subsets in \(H^1\).

**Lemma 3.5** The functional \(f(u)\) has positive lower bound on \(F_i\).

**Proof** By the definitions of \(f(u)\) and \(F_i\), we have

\[f(u) = \frac{1}{4} \int_0^1 |\dot{u}|^2 dt \int_0^1 (V'(u)u) dt, u \in F_i. \] \hspace{1cm} (3.8)

For \(u \in F_i\) and (B_2) ,we have

\[\frac{1}{2} V'(u)u = h - V(u) \geq h - \frac{1}{\mu_1} V'(u)u - \frac{\mu_2}{\mu_1}, \]

\[\Rightarrow \frac{1}{2} V'(u)u \geq h - \frac{1}{\mu_1} V'(u)u - \frac{\mu_2}{\mu_1} \]

\[\Rightarrow \frac{1}{2} V'(u)u \geq \frac{\mu_1}{\mu_1} V'(u)u - \frac{\mu_2}{\mu_1} \]

\[\Rightarrow \frac{1}{2} V'(u)u \geq \frac{\mu_1 - \mu_2}{\mu_1} V'(u)u \]

\[\Rightarrow \frac{1}{2} V'(u)u \geq a V'(u)u \]

\[\Rightarrow f(u) \geq \frac{a}{2} \int_0^1 |\dot{u}|^2 dt \int_0^1 (V'(u)u) dt, u \in F_i. \]
\[ V'(u)u \geq \frac{h - \frac{\mu_2}{\mu_1}}{2 + \frac{1}{\mu_1}} > 0. \]

So we have the functional \( f(u) \geq 0 \). Furthermore, we claims that
\[
\inf_{f_2} f(u) > 0,
\]
(3.9)
since otherwise, \( u(t) = \text{const} \) attains the infimum 0.

If \( u \in F_i \), then by the symmetry \( u(t + 1/2) = -u(t) \) or \( u(-t) = -u(t) \), we know \( u(t) = 0, \forall t \); by \( (B_2) \) we have \( V(0) \leq \frac{\mu_2}{\mu_1} \), by \( h > \frac{\mu_2}{\mu_1} \), we have \( V(0) < h \). By the definition of \( F_i \), \( 0 \notin F_i \). So
\[
\inf_{f_2} f(u) > 0.
\]
(3.10)

Now by Lemmas 3.1-3.5 and Lemma 2.4, we know \( f(u) \) attains the infimum on \( F_i \), and we know that the minimizer is nonconstant.

**Lemma 3.6** \( \exists z_1 \neq 0, z_1 \in H^1 \) s.t. \( f(z_1) \leq 0 \).

**Proof** For any given \( y_1 \neq \text{const}, \dot{y}_1 \neq 0 \), so \( \min |y_1(t)| > 0 \), we let \( z_1(t) = R y_1(t) \), then when \( R \) is large enough, by condition \( (B_3) \), we have
\[
\int_0^1 (h - V(z_1)) dt \leq 0,
\]
(3.11)
that is,
\[
f(z_1) \leq 0.
\]
(3.12)

**Lemma 3.7** \( f(0) = 0 \).

**Lemma 3.8** \( F_i \) separates \( z_1 \) and 0.

**Proof** By \( V(0) < h \), we have that \( 0 \in G_i \). By \( (B_2) \) and \( (B_3) \) and \( h > \frac{\mu_2}{\mu_1} \), we can choose \( R \) large enough such that
\[
z_1 = R y_1 \in \{ u \in H^1 | \int_0^1 (V(u) + \frac{1}{2} V'(u) u) dt \geq (1 + \frac{\mu_1}{2}) \int_0^1 V(u) dt - \frac{\mu_1}{2} \geq (1 + \frac{\mu_1}{2}) h - \frac{\mu_1}{2} > h \}.
\]

So \( F_i \) separates \( z_1 \) and 0.

Now by Lemmas 2.4-2.5, 3.1-3.8, we can prove Theorem 1.6.

4 The Proof of Theorem 1.7

Let
\[
F = \{ u \in H^1 | ||\dot{u}||_{L^2} = r \}.
\]
\[ G_1 = \{ u \in H^1 \| \dot{u} \|_{L^2} < r \}, \]
\[ G_2 = \{ u \in H^1 \| \dot{u} \|_{L^2} > r \}. \]

Then \( H^1 \setminus F = G_1 \cup G_2. \)

Notice that we can use \((B_5)\) to get that
\[ F \cap \{ u \in H^1 \| f(u) \geq c \} = \{ u \in H^1, \frac{1}{2} \int_0^1 (h - V(u))dt \geq c \}, \]
\[ H^1 \setminus (F \cap \{ u \in H^1 \| f(u) \geq c \}) = \{ u \in H^1 \| \dot{u} \|_{L^2} < r \} \cup \{ u \in H^1 \| \dot{u} \|_{L^2} > r \} \cup \{ u \in H^1 \| f(u) < c \}. \]

It’s easy to see \( u_1 = 0 \in G_1, \) we choose \( u_2 \) such that \( \| u_2 \|_{L^2} > r, \) so \( u_2 \in G_2. \) Now every path \( g(t) \) connecting \( u_1 \) and \( u_2 \) must pass \( F, \) so we have
\[ \max_{0 \leq t \leq 1} f(g(t)) \geq \inf_{u \in F} f(u) = \left( \frac{1}{2} r^2 \right) \inf_{u \in F} \int_0^1 (h - V(u)) \geq c > 0. \]

So from the above, in order to apply Ghoussoub-Preiss’s generalized MPL, now we only need to prove the closed set \( F \) separate \( u_1 \) and \( u_2 \) and \( f \) satisfies \((CPS)_{c,F}.\)

From the definitions of the set \( F \) and \( u_1 \) and \( u_2, \) we know \( F \) separate \( u_1 \) and \( u_2. \)

In order to prove \( f \) satisfies \((CPS)_{c,F}\) for any \( c > 0, \) firstly, from \((B_2),\) similar to the proof of Lemma 3.1, we can get \((\int_0^1 |\dot{u}_n|^2 dt)^{1/2}\) is bounded, then by \((B_3),\) we prove that \( |u_n(0)|\) is bounded. In fact, if otherwise, there exists a subsequence, we still denote it as \( \{u_n(0)\} \) satisfying
\[ |u_n(0)| \rightarrow +\infty. \]

By Newton-Leibniz formula and Cauchy-Schwarz inequality, we have
\[ \min_{0 \leq t \leq 1} |u_n(t)| \geq |u_n(0)| - \| \dot{u}_n \|_2 \rightarrow +\infty \]  
\[(4.13)\]

So by \((B_3)\) we have
\[ \int_0^1 V(u_n)dt \geq h, \quad \text{as} \ n \rightarrow +\infty, \quad (4.14) \]
\[ \lim_{n \rightarrow \infty} f(u_n) = \lim_{n \rightarrow \infty} \frac{1}{2} \int_0^1 |\dot{u}_n|^2 dt \int_0^1 (h - V(u_n))dt \leq 0, \quad (4.15) \]

which contradicts with \( f(u_n) \rightarrow c > 0. \)

We know that \( H^1 \) is a reflexive Banach space, so \( \{u_n\} \) has a weakly convergent subsequence. The rest that proving \( \{u_n\} \) has a strongly convergent subsequence is standard, we can refer to Ambrosetti-Coti Zelati [2].

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