Unipolar Induction of a Magnetized Accretion Disk around a Black Hole

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Abstract

The structure and magnitude of the electromagnetic field produced by a rotating accretion disk around a black hole were determined. The disk matter is assumed to be a magnetized plasma with a frozen-in poloidal magnetic field. The vacuum approximation is used outside the disk.

Key words: pulsars, neutron stars; black holes, quasars, jets, accretion disks.

1 INTRODUCTION

Recently, various models of particle acceleration near supermassive black holes (SMBHs) in galactic nuclei and near stellar-mass black holes (BHs) in the Galaxy have been widely discussed in connection with the studies of synchrotron radiation and inverse Compton scattering from narrow-beam jets observed over a wide spectral range, from radio to gamma rays. Nevertheless, the particularly high angular resolution provided by radio interferometers does not allow the central part of a quasar to be distinguished, suggesting that the jet width is extremely small (comparable to the gravitational radius). Previously (Shatskiy 2001), the mechanism of Blandford and Znajek (1977) and Blandford (2001) for electric-field generation through the interaction of the magnetic field from a ring current with the gravimagnetic field (GMF) of a Kerr BH located on the common axis with the ring current was used as the model of particle acceleration. Bisnovatyi-Kogan and Blinnikov (1972) and Shatskiy and Kardashev (2002) considered the mechanism of Deutsch (1955) or Goldreich and Julian (1969) for electric-field generation by the unipolar induction produced by the axial rotation of an accretion disk with a frozen-in magnetic field. Here, we suggest a mechanism that combines a unipolar inductor and strong gravitational SMBHs effects. Naturally, this mechanism is closer to the actual processes that take place in quasars. In contrast to the mechanism from Blandford and Znajek (1977), Beskin et al. (1992), and Beskin (1997), the mechanisms of Bisnovatyi-Kogan and Blinnikov (1972) used previously (Shatskiy 2001; Shatskiy and Kardashev 2002) operate in the vacuum approximation; the validity criterion for the latter is the condition of Goldreich and Julian (1969) for the number density of free charges:

\[ n_e < \frac{|\Omega H|}{(2\pi c e)}. \]  

Here, \( \Omega \) is the plasma angular velocity, \( H \) is the characteristic magnetic field, \( c \) is the speed of light, and \( e \) is the elementary charge. Shatskiy and Kardashev (2002) showed that condition (1) could be satisfied near a BH, because there are no stable orbits for the particles closer than three gravitational radii (in a Schwarzschild field). The matter for which the vacuum approximation breaks down must be in the accretion disk, which, because of the effect of Bardeen and Petterson (1975), must be located in the equatorial plane of a rotating BH. We do not consider models in which the vacuum approximation breaks down in the inner regions of the accretion disk.
approximation breaks down in the entire space outside a BH (models with the magnetohydrodynamic approximation). These were considered in detail, for example, in the review articles by Beskin et al. (1992) and Beskin (1997). Here, we determine the energy of the charged particles accelerated by the unipolar mechanism as well as the configurations of the electromagnetic field and the acceleration region.

2 CONSTRUCTING THE MODEL

Consider a Schwarzschild BH surrounded by an equatorial accretion disk. Let the disk have the characteristic size \( R \) and width \( 2a \) (see the figure). Because of the Bardeen-Petterson effect, the disk thickness can be disregarded. We use the following notation: \( M \) is the mass of the central body, \( R_g = 2M \) is the Schwarzschild radius, \( m \) is the mass of the test particle, \( u_j \) is its 4-velocity, \( F_{ij} = \partial_i A_j - \partial_j A_i \) is the electromagnetic-field (EMF) tensor, \( A_j \) is the EMF potential, and \( \Gamma^k_{ij} \) are the Christoffel symbols. Let us write the Schwarzschild metric and its determinant in spherical coordinate:

\[
ds^2 = (1 - \frac{r_g}{r})dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2,
\]

\[
g = -r^4\sin^2\theta.
\]

In general relativity, the following quantity for a charged particle that moves in stationary fields is conserved:

\[
\varepsilon = m(u_0 - 1) + eA_0,
\]

which matches the particle energy in the nonrelativistic case. To prove this, it will suffice to consider the equation of motion for a charged particle in general relativity (see Landau and Lifshitz 1988):

\[
m\frac{du_i}{ds} = mu^k u_l \Gamma^l_{ik} + eu^k F_{ik},
\]

where \( ds \) is the element of the particle proper time [see formulas (2)]. After transformation, this expression reduces to

\[
\frac{d}{ds} (mu_i + eA_i) = \frac{m}{2} u^l u^k \partial_l g_{kl} + eu^k \partial_l A_k.
\]

The conservation of energy throughout the particle evolution follows for \( i = 0 \) in stationary fields.

Since the magnetic field is frozen into the disk, its distribution inside the disk is determined only by the initial conditions of the problem. These conditions depend on the accretion-disk formation mechanism. If the disk is assumed to have been formed through the destruction of a star by BH tidal forces, then the magnetic field of this star in the disk will preserve its direction. This field can have the profile shown in the figure.

In the frame of reference comoving with the accretion disk, there is no electric field inside the disk because of its conductivity. Therefore, in a fixed frame of reference (with respect to distant stars), an electric field is induced by disk rotation inside the disk. Let the plasma in the disk rotate at an angular velocity \( \Omega \) relative to distant stars. The transformation of coordinates to a rotating frame is then:

\[
dx^i = dx'^k[\delta^i_k + \Omega \delta^i_\varphi \delta^\varphi_k].
\]

---

2Below, we use the system of units in which the speed of light and the gravitational constant are equal to unity: \( c = 1, G = 1 \).

3Unless otherwise specified, \( x_i = t, r, \theta, \varphi \); \( x_\alpha = R, \theta \) (Greek and Roman indices).
Because of axial symmetry, only the following EMF potential components are nonzero: \( A_0 \), the electric-field potential, and \( A_\phi \), the magnetic-field potential. According to (6) (see Landau and Lifshitz 1988), the EMF components transform as

\[
A'_0 = A_0 + \Omega A_\phi, \quad A'_\phi = A_\phi, \quad F'_{a0} = F_{a0} + \Omega F_{a\phi}, \quad F'_{a\phi} = F_{a\phi}.
\]  

(7)

Since \( F_{a0} = 0 \) in plasma, we have inside the disk

\[
F_{a0} = -\Omega F_{a\phi}, \quad A_0 = \text{const} - \Omega A_\phi.
\]  

(8)

On the disk surface, continuous boundary conditions exist for the tangential electric-field components and for the normal magnetic-field components. Outside the disk, there are no field sources by the definition of the model. Thus, determining the EMF reduces to solving the Laplace equation in the spacetime curved by gravity with the specified boundary conditions on the disk surface and on the BH horizon. The boundary conditions for the EMF tensor on the BH horizon were found previously (Shatskiy 2001):

\[
\lim_{r \to r_g} F_{0\theta} \propto g_{00} \to 0, \quad \lim_{r \to r_g} F_{r\phi} \propto g_{00} \to 0.
\]  

(9)

In turn, the boundary conditions on the disk for the magnetic and electric fields are determined solely by the magnetic-field distribution inside it. The specific form of this distribution is not that important for the solution of the problem. This is because at distances from the disk to the point of observation much larger than the disk thickness, the dipole field (the monopole field must be absent, because the total disk charge is zero) mainly contributes to the electric field of the disk element within position angles between \( \phi \) and \( \phi + d\phi \) when the field is expanded in multipoles. In this case, the total electric field obtained by integrating over the angle \( \phi \) has a quadrupole nature:

\[
\lim_{(r/r_g) \to \infty} A_0 = \text{const} \cdot (1 - 3 \cos^2 \theta)/r^3.
\]

3 THE MAXWELL EQUATIONS

The Maxwell equations for the EMF in general relativity are

\[
\frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} F^{ik}) = 4\pi j^k.
\]  

(10)

In the disk, the 4-vector of the current \( j^k \) can be determined from a given magnetic field. Let us introduce the physical components of the EMF vectors, their analogs in Euclidean space: ⁴

\[
\hat{E}^\alpha = -F_{\beta0} \sqrt{|g^{00} g^{\alpha\beta}|}, \quad \hat{H}_\alpha = -e_{\alpha\beta\gamma} F_{\gamma\phi} \sqrt{|g^{\gamma\beta} g^{\phi\phi}|}, \quad \hat{J}^\alpha = j^\beta \sqrt{|g_{\alpha\beta}|}.
\]  

(11)

Here, \( e^{\alpha\beta\gamma} = e_{\alpha\beta\gamma} \) is the Levi-Civita symbol. This form of the EMF physical components was chosen in order that Eq. (10) correspond to the classical Maxwell equations in Euclidean space:

\[
\text{div} \hat{E} = 4\pi j^0, \quad \text{rot} \hat{H} = 4\pi \hat{J}.
\]  

(12)

⁴We denote them by a hat.
4 THE ELECTROMAGNETIC FIELD NEAR A SMBH

The magnetic field of an accretion disk around a Schwarzschild BH was determined by Tomimatsu and Takahashi (2001). The electric field of the disk element within position angles between $\varphi$ and $\varphi + d\varphi$ can be represented as the field from two charges: $+q\frac{d\varphi}{2\pi}$ and $-q\frac{d\varphi}{2\pi}$, located inside the disk, at the system equator, and at distances $+a$ and $-a$ from its center ($r = R, \theta = \pi/2$), respectively. As a result, we obtain an electric dipole $2qad\varphi/\pi$ in the disk element between $\varphi$ and $\varphi + d\varphi$. In Euclidean space, the radial electric field at the disk center can be obtained by integrating over the angle $\phi$; for $a \ll R$, it is $E_0 = \hat{E}_r(r = R, \theta = \pi/2) = -\frac{2q}{\pi Ra}$. The corresponding magnetic field (which is responsible for the emergence of the electric field) can be found from the electric field. Note that nothing forbids the frozen-in magnetic field in the disk to have precisely such a profile (see the figure).

In the figure, the accretion disk is located at $r \approx 6M$. The contradictions related to the existence of stable orbits in this region can be removed by the following reasoning:

1 For the Kerr metrics, the nearest stable orbit is at radius $r = M$ (see, e.g., Landau and Lifshitz 1988).

2 Even if the orbit is not stable, it is spiral and goes under the horizon, while a new orbit can arrive in place of it. Thus, the pattern is quasi-stationary. According to (11) and (8), the quantity $q$ can be expressed in terms of the magnetic field at the disk center $^6$ as follows:

$$q = -\pi RaE_0/2 = \frac{\pi}{2} \Omega R^2 a H_0.$$  \hspace{1cm} (13)

In physical coordinates, the field of a point charge $e$ near a BH was presented by Thorne et al. (1998). It was obtained in a closed form by Linet (1976):

$$A_0 = \frac{e}{R^2} \left( \frac{M}{1 - \frac{R-M}{2M} t} \right),$$

$$\hat{E}_r = \frac{e}{R^2} \left( \frac{M}{1 - \frac{R-M}{2M} t} \right) + \frac{1}{D^3} \left[ \left( \frac{r-M}{2M} \right) \left( \frac{M}{1 - \frac{R-M}{2M} t} \right) \right],$$

$$\hat{E}_\theta = -\frac{e(R-2M)}{D^3} \sqrt{1 - \frac{2M}{R}} \partial_\theta t,$$

$$D^2 = (r-M)^2 + (R-M)^2 - M^2 - 2(r-M)(R-M)t + M^2 t^2.$$  \hspace{1cm} (14)

where $t$ is the cosine of the angle between the directions of the point charge and the point of observation of the field from the BH center. The electric field of a charged ring at the BH equator was found by Bicak and Dvorak (1996) in the form of a series. Here, this field is found in a quadrature form. To this end, we make the following substitutions: $t \rightarrow \sin \theta \cos \varphi, e \rightarrow Q \frac{4q}{2\pi}$ and integrate over the angle from $-\pi$ to $+\pi$. The model electric-field potential $A_0^{tot}$ is a superposition of the fields from two charged rings at radii $R + a$ and $R - a$ and with charges $+q$ and $-q$, respectively. The quadrature obtained can be expressed in terms of incomplete elliptic integrals. Since this quadrature is cumbersome, it makes no sense to write it here. Instead, we give an expression more useful for practical calculations, an expansion of this quadrature in terms of $a$. We retain only the first term

$^5$Naturally, there are no free charges in the disk; these were introduced for the convenience of representing the dipole field outside the disk.

$^6$H_0 = -\hat{H}_\theta(r=R,\theta=\pi/2)/\sqrt{1-r_g/R}.$
of the series (because \( a \ll R \) is small): \(^7\)

\[
A_{0}^{\text{tot}} = \int_{-\pi}^{+\pi} 2a\partial_{R}A_{0}d\varphi = \int_{0}^{+\pi} 4a(\partial_{r}A_{0})_{(R \rightarrow r)} d\varphi = \int_{0}^{+\pi} 4aE_{(R \rightarrow r)} r d\varphi.
\]

Substituting expression (14) here finally yields

\[
A_{0}^{\text{tot}} = 2aq\pi R^{2} \pi \int_{0}^{+\pi} d\varphi \left( \frac{M(D-r+M-Mt)}{D} + \frac{R[(r-M)(R-M)-(r+M)t]}{D^{2}}[R-M-(r+M)t] \right).
\] (15)

On the \( \Omega \) axis, the integration over \( \varphi \) is simple and it is easy to see that the only maximum of the potential \( A_{0}^{\text{tot}} \) (outside the BH) is on the horizon (\( r = 2M \)):

\[
|A_{0}^{\text{tot}}(r = 2M, \theta = 0)|_{M, \text{MAX}} = \frac{\pi \Omega a^{2} R^{2} H_{0}}{R^{2} - a^{2}}.
\] (16)

Consider the distribution of energies \( \varepsilon(r) \) in an ensemble of test particles at rest on the \( \Omega \) axis [see (3)]. This distribution has a weak maximum near the horizon and then slowly falls o. to zero at distances \( r \gg r_{g} \). Subsequently, the energy variation can be virtually disregarded.

5 DISCUSSION

The necessary condition for particle escape to infinity from rest is positiveness of the force \( m\frac{du}{ds} = m\frac{2\varepsilon}{ds} \) that acts on the test particle at the starting point. This requires that the charge of the test particle \( e \) have the sign opposite to that of \( q \):

\[
-\text{sign}(eq) = \text{sign}(e\Omega \hat{E}_{(\theta = 0)}) = -\text{sign}(e\Omega H_{0}) = 1.
\] (17)

The force that acts on an particle is proportional to the particle energy gradient (see Eq. (4)):

\[
m\frac{du}{ds} = -(\partial_{r}\varepsilon) \cdot \sqrt{1 - 2M/r}.
\] (18)

In the main order in \( \alpha = mM/(eq) \ll 1 \), the energy maximum is almost equal to the electric component of the particle energy on the horizon: \(^8\)

\[
\varepsilon_{\text{max}} = \left( \frac{1}{\alpha} \cdot \frac{2Ma}{R^{2} - a^{2}} - 1 \right) \cdot mc^{2} \sim 0.1 \cdot mc^{2}/\alpha
\] (19)

and the point at which this maximum is reached is at the following distance from the horizon:

\[
r_{\text{max}} - r_{g} = r_{g} \cdot \alpha^{2} \cdot \frac{(R^{2} - a^{2})^{2}[(R - M)^{2} - a^{2}]^{4}}{16M^{6}a^{2}(3R^{2} + M^{2} - 4MR + a^{2})^{2}} \sim r_{g} \cdot 10^{3} \cdot \alpha^{2}.
\] (20)

In conclusion, several more words can be said about the same model in a Kerr field. BH rotation gives rise to a GMF that interacts with the EMF of the disk and changes its components in magnitude and direction. In the linear approximation in BH angular velocity, the GMF gives an additive component

\(^7\)Here, we make use of the symmetry of the potential \( A_{0} \) in variables \( R \) and \( r \) and use their change (the subscript \( ^{(R \rightarrow r)} \)).

\(^8\)At \( a \sim r_{g} \sim R/3 \) [see (16)].
Figure 1: Accretion disk around black hole, which center at radius $R = 3$, and which width $a = 1$, with the frozen-in magnetic field — firm line. Electrical field (generated by magnetic) — dotted line. $Z$ and $\rho$ are expressed in fractions of $r_g = 2M$.

to expression (19) for the maximum energy of the charged particle accelerated by an electric field. This component was determined previously (Shatskiy 2001; it is convenient to represent it here as

$$\varepsilon_g \approx \frac{1}{2\pi^2} \cdot \left( \frac{\Omega_g r_g}{\Omega R} \right) \cdot \left( \frac{r_g^2}{a R} \right) \cdot mc^2/\alpha \sim \varepsilon_{\text{max}}. \quad (21)$$

Here, $\Omega_g$ is the angular velocity of the BH horizon (the falling test particles are drawn into rotation by the BH GMF). We see from (19) that at

$$\alpha^{-1} \approx \left( \frac{\Omega R}{c} \right) \cdot \left( \frac{a R}{r_g^2} \right) \cdot \left( \frac{H_0}{10^4 \text{Gauss}} \right) \cdot \left( \frac{M}{10^9 M_\odot} \right) \cdot \left( \frac{m_e}{m} \right) \cdot 10^{15} \quad (22)$$

the particle energy $^9$ accelerated by SMBHs can reach values larger than $10^{20}$eV.

$^9$Here, $m_e$ is the electron mass.
6 CONCLUSIONS

1. The model described above yields even a higher energy of the accelerated particles than does the Blandford-Znajek model.

2. The energy excess can be expended on the radiation reaction and on collisions with particles of the rarefied plasma near the axis.

3. The derived EMF configuration can be used to numerically calculate the dynamics of the charged particles far from the axis. It will make it possible to compare theoretical conclusions with observational data. (4) Far from the axis, a force-free field with the approximation of magnetohydrodynamic models can give a large contribution to the EMF amplitude. Therefore, in numerical calculations, the contributions to the EMF configuration from different models should be taken into account with different weights.

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