Cut-Free ExpTime Tableaux
for Checking Satisfiability of a Knowledge Base
in the Description Logic SHI

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Abstract. We give the first cut-free ExpTime (optimal) tableau decision procedure for checking satisfiability of a knowledge base in the description logic $\text{SHI}$, which extends the description logic $\text{ALC}$ with transitive roles, inverse roles and role hierarchies.

Keywords: description logics, automated reasoning, tableaux, global caching.

1 Introduction

Ontologies provide a shared understanding of the domain for different applications that want to communicate to each other. They are useful for several important areas like knowledge representation, software integration and Web applications. Web Ontology Language (OWL) is a layer of the Semantic Web architecture, built on the top of XML and RDF. Together with rule languages it serves as a main knowledge representation formalism for the Semantic Web. The logical foundation of OWL is based on description logics (DLs). Some of the most well-known DLs, in the increasing order of expressiveness, are $\text{ALC}$, $\text{SH}$, $\text{SHI}$, $\text{SHIQ}$ and $\text{SROIQ}$ [17].

Description logics represent the domain of interest in terms of concepts, individuals, and roles. A concept is interpreted as a set of individuals, while a role is interpreted as a binary relation among individuals. A knowledge base in a DL consists of axioms about roles (grouped into an RBox), terminology axioms (grouped into a TBox), and assertions about individuals (grouped into an ABox). One of the basic inference problems in DLs, which we denote by $\text{Sat}$, is to check satisfiability of a knowledge base. Other inference problems in DLs are usually reducible to this problem. For example, the problem of checking consistency of a concept w.r.t. an RBox and a TBox (further denoted by $\text{Cons}$) is linearly reducible to $\text{Sat}$.

In this paper we study automated reasoning in the DL $\text{SHI}$, which extends the DL $\text{ALC}$ with transitive roles, inverse roles and role hierarchies. The aim is to develop an efficient tableau decision procedure for the $\text{Sat}$ problem in $\text{SHI}$. It should be complexity-optimal (ExpTime), cut-free, and extendable with useful
optimizations. Tableau methods have widely been used for automated reasoning in modal and description logics \cite{2} since they are natural and allow many optimizations. As \( \mathcal{SHI} \) is a sublogic of \( \mathcal{SROIQ} \) and \( \mathcal{REG}^c \) (regular grammar logic with converse), one can use the tableau decision procedures of \( \mathcal{SROIQ} \) \cite{7} and \( \mathcal{REG}^c \) \cite{10} for the Sat problem in \( \mathcal{SHI} \). However, the first procedure has suboptimal complexity (\( \text{NExpTime} \) when restricted to \( \mathcal{SHI} \)), and the second one uses analytic cuts.

The tableau decision procedure given in \cite{8} for the Cons problem in \( \mathcal{SHI} \) has \( \text{NExpTime} \) complexity. In \cite{4} together with Goré we gave the first ExpTime tableau decision procedure for the Cons problem in \( \mathcal{SHI} \), which uses analytic cuts to deal with inverse roles. In \cite{15} together with Szalas we gave the first direct ExpTime tableau decision procedure for the Sat problem in the DL \( \mathcal{SH} \).

In \cite{11} we gave the first cut-free ExpTime tableau decision procedure for the Sat problem in the DL \( \mathcal{ALCI} \).

In this paper, by extending the methods of \cite{4,15,11}, we give the first cut-free ExpTime (optimal) tableau decision procedure for the Sat problem in the DL \( \mathcal{SHI} \). We use global state caching \cite{5,6,11}, the technique of \cite{11} for dealing with inverse roles, the technique of \cite{4,15} for dealing with transitive roles and hierarchies of roles, and the techniques of \cite{15,14,16,11} for dealing with ABoxes.

The rest of this paper is structured as follows: In Section 2 we recall the notation and semantics of \( \mathcal{SHI} \). In Section 3 we present our tableau decision procedure for the Sat problem in \( \mathcal{SHI} \). In Section 4 we give proofs for the correctness of our procedure and analyze its complexity. Section 5 concludes this work.

## 2 Notation and Semantics of \( \mathcal{SHI} \)

Our language uses a finite set \( C \) of concept names, a finite set \( R \) of role names, and a finite set \( I \) of individual names. We use letters like \( A \) and \( B \) for concept names, \( r \) and \( s \) for role names, and \( a \) and \( b \) for individual names. We refer to \( A \) and \( B \) also as atomic concepts, and to \( a \) and \( b \) as individuals.

For \( r \in R \), let \( r^- \) be a new symbol, called the inverse of \( r \). Let \( R^- = \{ r^- | r \in R \} \) be the set of inverse roles. For \( r \in R \), define \( (r^-)^- = r \). A role is any member of \( R \cup R^- \). We use letters like \( R \) and \( S \) to denote roles.

An \( \mathcal{SHI} \) RBox \( \mathcal{R} \) is a finite set of role axioms of the form \( R \subseteq S \) or \( R \circ R \subseteq R \). By \( \text{ext}(\mathcal{R}) \) we denote the least extension of \( \mathcal{R} \) such that:

- \( R \subseteq R \in \text{ext}(\mathcal{R}) \) for any role \( R \)
- if \( R \subseteq S \in \text{ext}(\mathcal{R}) \) then \( R^- \subseteq S^- \in \text{ext}(\mathcal{R}) \)
- if \( R \circ R \subseteq R \in \text{ext}(\mathcal{R}) \) then \( R^- \circ R^- \subseteq R^- \in \text{ext}(\mathcal{R}) \)
- if \( R \subseteq S \in \text{ext}(\mathcal{R}) \) and \( S \subseteq T \in \text{ext}(\mathcal{R}) \) then \( R \subseteq T \in \text{ext}(\mathcal{R}) \).

By \( R \subseteq R \) \( S \) we mean \( R \subseteq S \in \text{ext}(\mathcal{R}) \). If \( R \subseteq R \) \( S \) then \( R \) is a subrole of \( S \) w.r.t. \( \mathcal{R} \). If \( R \circ R \subseteq R \in \text{ext}(\mathcal{R}) \) then \( R \) is a transitive role w.r.t. \( \mathcal{R} \).

Concepts in \( \mathcal{SHI} \) are formed using the following BNF grammar:

\[
C, D ::= \top | \bot | A | \neg C | C \cap D | C \cup D | \forall R.C | \exists R.C
\]
We use letters like $C$ and $D$ to denote arbitrary concepts.

A TBox is a finite set of axioms of the form $C \sqsubseteq D$ or $C \equiv D$. An ABox is a finite set of assertions of the form $a : C$ (concept assertion) or $R(a, b)$ (role assertion). A knowledge base in $SHI$ is a tuple $(\mathcal{R}, \mathcal{T}, \mathcal{A})$, where $\mathcal{R}$ is an RBox, $\mathcal{T}$ is a TBox and $\mathcal{A}$ is an ABox.

A formula is defined to be either a concept or an ABox assertion. We use letters like $\varphi$, $\psi$ and $\xi$ to denote formulas, and letters like $X$, $Y$ and $\Gamma$ to denote sets of formulas.

An interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \tau^\mathcal{I})$ consists of a non-empty set $\Delta^\mathcal{I}$, called the domain of $\mathcal{I}$, and a function $\tau^\mathcal{I}$, called the interpretation function of $\mathcal{I}$, that maps every concept name $A$ to a subset $\Delta^\mathcal{I}$ of $\Delta^\mathcal{I}$, maps every role name $r$ to a binary relation $r^\mathcal{I}$ on $\Delta^\mathcal{I}$, and maps every individual name $a$ to an element $a^\mathcal{I} \in \Delta^\mathcal{I}$.

The interpretation function is extended to inverse roles and complex concepts as follows:

$$(r^-)^\mathcal{I} = \{ (x, y) \mid (y, x) \in r^\mathcal{I}\} \quad \top^\mathcal{I} = \Delta^\mathcal{I} \quad \bot^\mathcal{I} = \emptyset$$

$$(\neg C)^\mathcal{I} = \Delta^\mathcal{I} \setminus C^\mathcal{I} \quad (C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I} \quad (C \equiv D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$$

$$(\forall R.C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} \mid \forall y [(x, y) \in R^\mathcal{I} \text{ implies } y \in C^\mathcal{I}] \}$$

$$(\exists R.C)^\mathcal{I} = \{ x \in \Delta^\mathcal{I} \mid \exists y [(x, y) \in R^\mathcal{I} \text{ and } y \in C^\mathcal{I}] \}$$

Note that $(r^-)^\mathcal{I} = (r^\mathcal{I})^{-1}$ and this is compatible with $(r^-) = r$.

For a set $\Gamma$ of concepts, define $\Gamma^\mathcal{I} = \{ x \in \Delta^\mathcal{I} \mid x \in C^\mathcal{I} \text{ for all } C \in \Gamma \}$.

The relational composition of binary relations $R_1$, $R_2$ is denoted by $R_1 \circ R_2$.

An interpretation $\mathcal{I}$ is a model of an RBox $\mathcal{R}$ if for every axiom $R \sqsubseteq S$ (resp. $R \circ R \sqsubseteq R$) of $\mathcal{R}$, we have that $R^\mathcal{I} \subseteq S^\mathcal{I}$ (resp. $R^\mathcal{I} \circ R^\mathcal{I} \subseteq R^\mathcal{I}$). Note that if $\mathcal{I}$ is a model of $\mathcal{R}$ then it is also a model of $\text{ext}(\mathcal{R})$.

An interpretation $\mathcal{I}$ is a model of a TBox $\mathcal{T}$ if for every axiom $C \sqsubseteq D$ (resp. $C \equiv D$) of $\mathcal{T}$, we have that $C^\mathcal{I} \subseteq D^\mathcal{I}$ (resp. $C^\mathcal{I} = D^\mathcal{I}$).

An interpretation $\mathcal{I}$ is a model of an ABox $\mathcal{A}$ if for every assertion $a : C$ (resp. $R(a, b)$) of $\mathcal{A}$, we have that $a^\mathcal{I} \in C^\mathcal{I}$ (resp. $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$).

An interpretation $\mathcal{I}$ is a model of a knowledge base $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ if $\mathcal{I}$ is a model of all $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{A}$. A knowledge base $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ is satisfiable if it has a model.

An interpretation $\mathcal{I}$ satisfies a concept $C$ (resp. a set $X$ of concepts) if $C^\mathcal{I} \neq \emptyset$ (resp. $X^\mathcal{I} \neq \emptyset$). A set $X$ of concepts is satisfiable w.r.t. an RBox $\mathcal{R}$ and a TBox $\mathcal{T}$ if there exists a model of $\mathcal{R}$ and $\mathcal{T}$ that satisfies $X$. For $X = Y \cup A$, where $Y$ is a set of concepts and $A$ is an ABox, we say that $X$ is satisfiable w.r.t. an RBox $\mathcal{R}$ and a TBox $\mathcal{T}$ if there exists a model of $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ that satisfies $X$.

3 A Tableau Decision Procedure for $SHI$

We assume that concepts and ABox assertions are represented in negation normal form (NNF), where $\neg$ occurs only directly before atomic concepts.\footnote{Every formula can be transformed to an equivalent formula in NNF.} We use $\overline{C}$ to denote the NNF of $\neg C$, and for $\varphi = a : C$, we use $\overline{\varphi}$ to denote $a : \overline{C}$. For
simplicity, we treat axioms of $T$ as concepts representing global assumptions: an axiom $C \sqsubseteq D$ is treated as $\overline{C} \cup D$, while an axiom $C \models D$ is treated as $(\overline{C} \cup D) \cap (\overline{D} \cup C)$. That is, we assume that $T$ consists of concepts in NNF. Thus, an interpretation $I$ is a model of $T$ iff $I$ validates every concept $C \in T$. As this way of handling the TBox is not efficient in practice, the absorption technique like the one discussed in [12,13] can be used to improve the performance of our algorithm.

From now on, let $(\mathcal{R}, T, \mathcal{A})$ be a knowledge base in NNF of the logic $\mathcal{SHI}$, with $\mathcal{A} \neq \emptyset$. In this section we present a tableau calculus for checking satisfiability of $(\mathcal{R}, T, \mathcal{A})$.

For a set $X$ of concepts and a set $Y$ of ABox assertions, we define:

\[
\begin{align*}
\text{SRTR}_R(R, S) &= (R \sqsubseteq R S \land S \circ S \sqsubseteq S \in \text{ext}(R)) \\
\text{Trans}_R(X, R) &= \{D \mid \forall R.D \in X\} \cup \{\forall S.D \in X \mid \text{SRTR}_R(R, S)\} \\
\text{Trans}_R(X, R, a) &= \{a : D \mid \forall R.D \in X\} \cup \{a : \forall S.D \in X \land \text{SRTR}_R(R, S)\} \\
\text{Trans}_R(Y, a, R) &= \{D \mid a : \forall R.D \in Y\} \cup \{\forall S.D \mid a : \forall S.D \in Y \land \text{SRTR}_R(R, S)\} \\
\text{Trans}_R(Y, a, R, b) &= \{b : D \mid a : \forall R.D \in Y\} \cup \\
&\quad \{b : \forall S.D \mid a : \forall S.D \in Y \land \text{SRTR}_R(R, S)\}
\end{align*}
\]

We call $\text{Trans}_R(X, R)$ the transfer of $X$ through $R$ w.r.t. $\mathcal{R}$, call $\text{Trans}_R(X, R, a)$ the transfer of $X$ through $R$ to $a$ w.r.t. $\mathcal{R}$, call $\text{Trans}_R(Y, a, R)$ the transfer of $Y$ starting from $a$ through $R$ w.r.t. $\mathcal{R}$, and call $\text{Trans}_R(Y, a, R, b)$ the transfer of $Y$ from $a$ to $b$ through $R$ w.r.t. $\mathcal{R}$.

In what follows we define tableaux as rooted “and-or” graphs. Such a graph is a tuple $G = (V, E, \nu)$, where $V$ is a set of nodes, $E \subseteq V \times V$ is a set of edges, $\nu \in V$ is the root, and each node $v \in V$ has a number of attributes. If there is an edge $(v, w) \in E$ then we call $v$ a predecessor of $w$, and call $w$ a successor of $v$.

The set of all attributes of $v$ is called the contents of $v$. Attributes of tableau nodes are:

- $\text{Type}(v) \in \{\text{state}, \text{non-state}\}$. If $\text{Type}(v) = \text{state}$ then we call $v$ a state, else we call $v$ a non-state (or an internal node). If $\text{Type}(v) = \text{state}$ and $(v, w) \in E$ then $\text{Type}(w) = \text{non-state}$.
- $\text{SType}(v) \in \{\text{complex, simple}\}$ is called the subtype of $v$. If $\text{SType}(v) = \text{complex}$ then we call $v$ a complex node, else we call $v$ a simple node. The graph never contains edges from a simple node to a complex node. If $(v, w)$ is an edge from a complex node $v$ to a simple node $w$ then $\text{Type}(v) = \text{state}$ and $\text{Type}(w) = \text{non-state}$. The root of the graph is a complex node.
- $\text{Status}(v) \in \{\text{unexpanded, expanded, incomplete, unsat, sat}\}$.
- $\text{Label}(v)$ is a finite set of formulas, called the label of $v$. The label of a complex node consists of ABox assertions, while the label of a simple node consists of concepts.
- $\text{RFms}(v)$ is a finite set of formulas, called the set of reduced formulas of $v$.

\footnote{If $\mathcal{A}$ is empty, we can add $a : \top$ to it, where $a$ is a special individual.}
\(DFmls(v)\) is a finite set of formulas, called the set of disallowed formulas of \(v\).

\(StatePred(v) \in V \cup \{null\}\) is called the state-predecessor of \(v\). It is available only when \(Type(v) = \text{non-state}\). If \(v\) is a non-state and \(G\) has no paths connecting a state to \(v\) then \(StatePred(v) = \text{null}\). Otherwise, \(G\) has exactly one state \(u\) that is connected to \(v\) via a path not containing any other states. In that case, \(StatePred(v) = u\).

\(ATPred(v) \in V\) is called the after-transition-predecessor of \(v\). It is available only when \(Type(v) = \text{non-state}\). If \(v\) is a non-state and \(v_0 = StatePred(v) \neq \text{null}\) then there is exactly one successor \(v_1\) of \(v_0\) such that every path connecting \(v_0\) to \(v\) must go through \(v_1\), and we have that \(ATPred(v) = v_1\). We define \(AfterTrans(v) = (ATPred(v) = v)\). If \(AfterTrans(v)\) holds then either \(v\) has no predecessors (i.e. it is the root of the graph) or it has exactly one predecessor \(u\) and \(u\) is a state.

\(CELabel(v)\) is a formula called the coming edge label of \(v\). It is available only when \(v\) is a successor of a state \(u\) (and \(Type(v) = \text{non-state}\)). In that case, we have \(u = StatePred(v)\), \(AfterTrans(v)\) holds, \(CELabel(v) \in Label(u)\), and

- if \(SType(u) = \text{simple}\) then
  \(CELabel(v)\) is of the form \(∃R.C\) and \(C \in Label(v)\)
- else \(CELabel(v)\) is of the form \(a:∃R.C\) and \(C \in Label(v)\).

Informally, \(v\) was created from \(u\) to realize the formula \(CELabel(v)\) at \(u\).

\(ConvMethod(v) \in \{0, 1\}\) is called the converse method of \(v\). It is available only when \(Type(v) = \text{state}\).

\(FmlsRC(v)\) is a set of formulas, called the set of formulas required by converse for \(v\). It is available only when \(Type(v) = \text{state}\) and will be used only when \(ConvMethod(v) = 0\).

\(AltFmlSets(v)\) is a set of sets of formulas, called the set of alternative sets of formulas suggested by converse for \(v\). It is available only when \(Type(v) = \text{state}\) and will be used only when \(ConvMethod(v) = 1\).

\(AltFmlSetsSCP(v)\) is a set of sets of formulas, called the set of alternative sets of formulas suggested by converse for the predecessor of \(v\). It is available only when \(v\) has a predecessor being a state and will be used only when \(ConvMethod(v) = 1\).

We define

\[
\text{AFmls}(v) = Label(v) \cup RFmls(v)\\
\text{NDFmls}(v) = \{\neg \varphi \mid \varphi \in DFmls(v)\}\\
\text{FullLabel}(v) = \text{AFmls}(v) \cup \text{NDFmls}(v)\\
\text{Kind}(v) = \begin{cases} 
\text{and-node} & \text{if } Type(v) = \text{state} \\
\text{or-node} & \text{if } Type(v) = \text{non-state} 
\end{cases}\\
\text{BeforeFormingState}(v) = v \text{ has a successor which is a state}
\]

The sets \(\text{AFmls}(v)\), \(\text{NDFmls}(v)\), and \(\text{FullLabel}(v)\) are respectively called the available formulas of \(v\), the negations of the formulas disallowed at \(v\), and the
Table 1. Some rules of the tableau calculus $C_{\text{SHI}}$
Function NewSucc(v, type, sType, ceLabel, label, rFmls, dFmls)

Global data: a rooted graph (V, E, ν).

Purpose: create a new successor for v.

1. create a new node w, \( V := V \cup \{w\} \), if \( v \neq \) null then \( E := E \cup \{(v, w)\} \);
2. Type(w) := type, SType(w) := sType, Status(w) := unexpanded;
3. Label(w) := label, RFmls(w) := rFmls, DFmls(w) := dFmls;
4. if type = non-state then
   5. if v = null or Type(v) = state then StatePred(w) := v, ATPred(w) := w
   6. else StatePred(w) := StatePred(v), ATPred(w) := ATPred(v);
7. if Type(v) = state then CELabel(w) := ceLabel, AltFmlSetsSCP(w) := ∅
   8. else ConvMethod(w) := 0, FmlsRC(w) := ∅, AltFmlSetsSC(w) := ∅
9. return w

Function FindProxy(type, sType, v₁, label, rFmls, dFmls)

Global data: a rooted graph (V, E, ν).

1. if type = state then \( W := V \) else \( W := \) the nodes of the local graph of \( v_1 \);
2. if there exists \( w \in W \) such that Type(w) = type and SType(w) = sType and Label(w) = label and RFmls(w) = rFmls and DFmls(w) = dFmls then
   3. return w
3. else return null

Function ConToSucc(v, type, sType, ceLabel, label, rFmls, dFmls)

Global data: a rooted graph (V, E, ν).

Purpose: connect v to a successor, which is created if necessary.

1. if type = state then \( v_1 := \) null else \( v_1 := ATPred(v) \)
2. w := FindProxy(type, sType, v₁, label, rFmls, dFmls);
3. if w \neq \) null then \( E := E \cup \{(v, w)\} \);
4. else w := NewSucc(v, type, sType, ceLabel, label, rFmls, dFmls);
5. return w

Function TUnsat(v)

1. return (⊥ \in Label(v) or there exists \{ϕ, ̸ϕ\} \subseteq Label(v))

Function TSat(v)

1. return (Status(v) = unexpanded and no rule except (conv) is applicable to v)

Function ToExpand

Global data: a rooted graph (V, E, ν).

1. if there exists a node v \in V with Status(v) = unexpanded then return v
2. else return null

Tableau rules are usually written downwards, with a set of formulas above the line as the premise, which represents the label of the node to which the rule is applied, and a number of sets of formulas below the line as the (possible) conclusions, which represent the labels of the successor nodes resulting from the
application of the rule. Possible conclusions of an “or”-rule are separated by |
, while conclusions of an “and”-rule are separated by &. If a rule is a unary

```plaintext
Procedure Apply(ρ, v)

Global data: a rooted graph (V, E, v).
Input: a rule ρ and a node v ∈ V s.t. if ρ ≠ (conv) then Status(v) = unexpanded
         else Status(v) = expanded and BeforeFormingState(v) holds.
Purpose: applying the tableau rule ρ to the node v.

1 if ρ = (forming-state) then
  2 ConToSucc(v, state, SType(v), null, Label(v), RFmls(v), DFmls(v))
else if ρ = (conv) then ApplyConvRule(v) // defined on page 9
else if ρ ∈ {(∃), (∃')} then
  5 ApplyTransRule(ρ, v) // defined on page 9
  6 if Status(v) = {incomplete, unsat, sat} then
    7 PropagateStatus(v), return
else
  8 let X₁, . . . , Xₖ be the possible conclusions of the rule;
  9 if ρ ∈ {(H), (H'), (v')} then Y := RFmls(v)
  else Y := RFmls(v) ∪ {the principal formula of ρ};
  10 foreach 1 ≤ i ≤ k do
    11     ConToSucc(v, non-state, SType(v), null, Xᵢ, Y, DFmls(v))
  12 Status(v) := expanded;
  13 foreach successor w of v with Status(w) ∉ {incomplete, unsat, sat} do
  14     if TUnsat(w) then Status(w) := unsat
  15     else if Type(w) = non-state then
  16         v₀ := StatePred(w), v₁ := ATPred(w);
  17         if SType(v₀) = simple then
  18             let ∃R.C be the form of CELabel(v₁);
  19             X := TransR(Label(w), R⁻) \ AFmls(v₀)
  20         else
  21             let a:∃R.C be the form of CELabel(v₁);
  22             X := TransR(Label(w), R⁻, a) \ AFmls(v₀)
  23     if X ≠ ∅ then
  24         if ConvMethod(v₀) = 0 then
  25             FmlsRC(v₀) := FmlsRC(v₀) ∪ X;
  26         else if X ∩ DFmls(v₀) ≠ ∅ then Status(v₀) := unsat, return
  27         else if X ∩ DFmls(v₀) ≠ ∅ then Status(v₀) := unsat
  28         else
  29             AltFmlSetsSCP(v₁) := AltFmlSetsSCP(v₁) ∪ {X};
  30         Status(w) := incomplete
  31     else if TSat(w) then Status(w) := sat
  32 UpdateStatus(v);
  33 if Status(v) ∈ {incomplete, unsat, sat} then PropagateStatus(v)
```
Procedure ApplyConvRule(v)

Global data: a rooted graph \((V, E, \nu)\).

Purpose: applying the rule \((\text{conv})\) to the node v.

1. let \(w\) be the only successor of v, \(E := E \setminus \{(v, w)\}\);
2. if \(\text{ConvMethod}(w) = 0\) then
   3. \(\text{newLabel} := \text{Label}(v) \cup \text{FmlsRC}(w)\);
   4. \(\text{ConToSucc}(v, \text{non-state}, \text{SType}(v), \text{null}, \text{newLabel}, \text{RFmls}(v), \text{DFmls}(v))\)
   5. else
      6. let \(\{\varphi_1\}, \ldots, \{\varphi_n\}\) be all the singleton sets belonging to \(\text{AltFmlSetsSC}(w)\),
         and let \(\text{remainingSetsSC}\) be the set of all the remaining sets;
      7. \(\text{foreach} \ 1 \leq i \leq n \ do\)
         8. \(\text{newLabel} := \text{Label}(v) \cup \{\varphi_i\},\)
         9. \(\text{newDFmls} := \text{DFmls}(v) \cup \{\varphi_j \ | \ 1 \leq j < i\};\)
         10. \(\text{ConToSucc}(v, \text{non-state}, \text{SType}(v), \text{null}, \text{newLabel}, \text{RFmls}(v), \text{newDFmls})\)
      11. \(Y := \{\varphi_i \ | \ 1 \leq i \leq n\};\)
      12. \(\text{foreach} \ X \in \text{remainingSetsSC} \ do\)
         13. \(\text{ConToSucc}(v, \text{non-state}, \text{SType}(v), \text{null}, \text{Label}(v) \cup X, \text{RFmls}(v), \text{DFmls}(v) \cup Y)\)

Procedure ApplyTransRule(\(\rho, u\))

Global data: a rooted graph \((V, E, \nu)\).

Purpose: applying the transitional rule \(\rho\), which is \((\exists)\) or \((\exists^*)\), to the state u.

1. let \(X_1, \ldots, X_k\) be all the conclusions of the rule \(\rho\) with \(\text{Label}(u)\) as the premise;
2. if \(\rho = (\exists)\) then
   3. let \(\exists R_1.C_1, \ldots, \exists R_k.C_k\) be the corresponding principal formulas;
      \(\text{foreach} \ 1 \leq i \leq k \ do\)
         4. \(v := \text{NewSucc}(u, \text{non-state, simple, } \exists R_i.C_i, X_i, \emptyset, \emptyset);\)
         5. \(\text{FmlsRC}(u) := \text{FmlsRC}(u) \cup (\text{Trans}(\text{Label}(v), R_i) \setminus \text{AFmls}(u))\)
   7. else
      8. let \(a_1 : \exists R_1.C_1, \ldots, a_k : \exists R_k.C_k\) be the corresponding principal formulas;
      9. \(\text{foreach} \ 1 \leq i \leq k \ do\)
         10. \(v := \text{NewSucc}(u, \text{non-state, simple, } a_i : \exists R_i.C_i, X_i, \emptyset, \emptyset);\)
         11. \(\text{FmlsRC}(u) := \text{FmlsRC}(u) \cup (\text{Trans}(\text{Label}(v), R_i, a_i) \setminus \text{AFmls}(u))\)
12. if \(\text{FmlsRC}(u) \cap \text{DFmls}(u) \neq \emptyset\) then \(\text{Status}(u) := \text{unsat}\);
13. while \(\text{Status}(u) \neq \text{unsat} \ and \ there \ exists \ a \ node \ w \ in \ the \ local \ graph \ of \ u \ such \ that \ \text{Status}(w) = \text{unexpanded} \ and \ a \ unary \ rule \ \rho \neq (\text{forming-state}) \ is \ applicable \ to \ w\) \ do \ Apply(\rho, w);
14. if \(\text{Status}(u) \neq \text{unsat}\) then
   15. if \(\text{FmlsRC}(u) \neq \emptyset\) then \(\text{Status}(u) := \text{incomplete}\)
   16. else \(\text{ConvMethod}(u) := 1\)
Function Tableau($R, T, A$)

Input: a knowledge base ($R, T, A$) in NNF in the logic $SHL$.
Global data: a rooted graph $(V, E, \nu)$.

1. $X := A \cup \{ (a:C) \mid C \in T \text{ and } a \text{ is an individual occurring in } A \}$
2. $\nu := \text{NewSucc}(\text{null, non-state, complex, null, } X, \emptyset, \emptyset)$
3. if $TUnsat(\nu)$ then $\text{Status}(\nu) := \text{unsat}$
4. else if $TSat(\nu)$ then $\text{Status}(\nu) := \text{sat}$
5. while $(v := \text{ToExpand}()) \neq \text{null}$ do
6. choose a tableau rule $\rho$ different from (conv) and applicable to $v$;
7. $\text{Apply}(\rho, v)$; // defined on page 5
8. return $(V, E, \nu)$

Procedure UpdateStatus($v$)

Global data: a rooted graph $(V, E, \nu)$.
Input: a node $v \in V$ with $\text{Status}(v) = \text{expanded}$.

1. if Kind($v) = \text{or-node}$ then
2. if some successors of $v$ have status sat then $\text{Status}(v) := \text{sat}$
3. else if all successors of $v$ have status unsat then $\text{Status}(v) := \text{unsat}$
4. else if every successor of $v$ has status incomplete or unsat then
5. if $v$ has a successor $w$ such that Type($w$) = state then
6. $\text{Apply}((\text{conv}), v)$
7. else $\text{Status}(v) := \text{incomplete}$
8. else // Kind($v) = \text{and-node}$
9. if all successors of $v$ have status sat then $\text{Status}(v) := \text{sat}$
10. else if some successors of $v$ have status unsat then $\text{Status}(v) := \text{unsat}$
11. else if $v$ has a successor $w$ with $\text{Status}(w) = \text{incomplete}$ then
12. $\text{AltFmlSetsSC}(v) := \text{AltFmlSetsSCP}(w), \text{Status}(v) := \text{incomplete}$

Procedure PropagateStatus($v$)

Global data: a rooted graph $(V, E, \nu)$.
Input: a node $v \in V$ with $\text{Status}(v) \in \{ \text{incomplete, unsat, sat} \}$.

1. foreach predecessor $u$ of $v$ with $\text{Status}(u) = \text{expanded}$ do
2. $\text{UpdateStatus}(u)$;
3. if $\text{Status}(u) \in \{ \text{incomplete, unsat, sat} \}$ then $\text{PropagateStatus}(u)$

rule (i.e. a rule with only one possible conclusion) or an “and”-rule then its conclusions are “firm” and we ignore the word “possible”. The meaning of an “or”-rule is that if the premise is satisfiable w.r.t. $R$ and $T$ then some of the possible conclusions are also satisfiable w.r.t. $R$ and $T$, while the meaning of an “and”-rule is that if the premise is satisfiable w.r.t. $R$ and $T$ then all of the conclusions are also satisfiable w.r.t. $R$ and $T$. 
Such a representation gives only a part of the specification of the rules.

We write $X, \varphi$ or $\varphi, X$ to denote $X \cup \{\varphi\}$, and write $X, Y$ to denote $X \cup Y$. Our tableau calculus $C_{SHI}$ for $SHI$ w.r.t. the RBox $\cal R$ and the TBox $\cal T$ consists of rules which are partially specified in Table 1 together with two special rules (forming-state) and (conv).

The rules (exists) and (exists′) are the only “and”-rules and the only transitional rules. The other rules of $C_{SHI}$ are “or”-rules, which are also called static rules. The transitional rules are used to expand states of tableaux, while the static rules are used to expand non-states of tableaux.

For any rule of $C_{SHI}$ except (forming-state) and (conv), the distinguished formulas of the premise are called the principal formulas of the rule. The rules (forming-state) and (conv) have no principal formulas. As usually, we assume that, for each rule of $C_{SHI}$ described in Table 1, the principal formulas are not members of the set $X$ which appears in the premise of the rule.

Expanding a non-state $v$ of a tableau by a static rule $\rho \in \{\top, \lor, \top', \lor'\}$ which uses $\varphi$ as the principal formula, we put $\varphi$ into the set $\text{RFmls}(w)$ of each successor $w$ of $v$. Recall that $\text{RFmls}(w)$ is called the set of the reduced formulas of $w$. If $w$ is a non-state, $v_1 = \text{ATPred}(w)$ and $v_1, v_2, \ldots, v_k = w$ is the path (of non-states) from $v_1$ to $w$, then an occurrence $\psi \in \text{RFmls}(w)$ means there exists $1 \leq i < k$ such that $\psi \in \text{Label}(v_i)$ and $\psi$ has been reduced at $v_i$. After that reduction, $\psi$ was put into $\text{RFmls}(v_{i+1})$ and propagated to $\text{RFmls}(v_k)$.

Expanding a simple (resp. complex) state $v$ of a tableau by the transitional rule (exists) (resp. (exists′)), each successor $w_i$ of $v$ is created due to a corresponding principal formula $\exists R_i.C_i$ (resp. $a_i : \exists R_i.C_i$) of the rule, and $\text{RFmls}(w)$ is set to the empty set.

For any state $w$, every predecessor $v$ of $w$ is always a non-state. Such a node $v$ was expanded and connected to $w$ by the static rule (forming-state). The nodes $v$ and $w$ correspond to the same element of the domain of the interpretation under construction. In other words, the rule (forming-state) “transforms” a non-state to a state. It guarantees that, if BeforeFormingState($v$) holds then $v$ has exactly one successor, which is a state.

The rule (conv) used for dealing with converses will be discussed shortly.

The priorities of the rules of $C_{SHI}$ are as follows (the bigger, the stronger): $(\top), (\top'), (\lor), (\lor'), (\top', \lor')$: 5; $(\top), (\lor)$: 4; (forming-state): 3; (exists), (exists′): 2; (conv): 1.

The conditions for applying a rule $\rho \neq \text{(conv)}$ to a node $v$ are as follows:

- the rule has $\text{Label}(v)$ as the premise (thus, the rules (top), (lor), (exists) are applicable only to simple nodes, and the rules (top′), (lor′), (top′, lor′), (exists′) are applicable only to complex nodes)
- all the conditions accompanying with $\rho$ in Table 1 are satisfied
- if $\rho$ is a transitional rule then Type($v$) = state
- if $\rho$ is a static rule then Type($v$) = non-state and
  - if $\rho \in \{\top, \lor, \top', \lor'\}$ then the principal formula of $\rho$ does not belong to $\text{RFmls}(v)$, else if $\rho \in \{(\top), (\top'), (\lor')\}$ then the formula occurring in the conclusion but not in the premise of $\rho$ does not belong to $\text{AFmls}(v)$
no static rule with a higher priority is applicable to \( v \).

We now explain the ways of dealing with converses, i.e., with inverse roles.

Consider the case when \( \text{Type}(v) = \text{state} \), \( \text{SType}(v) = \text{simple} \), \( \exists R.C \in \text{Label}(v) \) and \( v \) corresponds to an element \( x_v \in \Delta^2 \) of the interpretation \( \mathcal{I} \) under construction. We need to realize the formulas of \( \text{Label}(v) \) at \( v \) so that \( x_v \in (\text{Label}(v))^2 \). The formula \( \exists R.C \) is realized at \( v \) by making a transition from \( v \) to \( w \) with \( \text{Label}(w) = \{ C \} \cup \text{Trans}_R(\text{Label}(v), R) \cup T \). The node \( w \) corresponds to an element \( x_w \in \Delta^2 \) such that \( (x_v, x_w) \in R^2 \) and \( x_w \in C^2 \). If at some later stage we need to make \( x_w \in (\forall R^- \cdot D)^2 \) (for example, because \( (\forall R^- \cdot D) \in \text{Label}(w) \)) then we need to make \( x_v \in D^2 \), and hence we need to add \( D \) to \( \text{Label}(v) \) as a requirement to be realized at \( v \) if \( D \notin \text{AFmls}(v) \). Similarly, if at some later stage we need to make \( x_w \in (\forall S \cdot D)^2 \), where \( R^- \sqsubseteq R \) and \( S \circ S \sqsubseteq S \in \text{ext}(R) \), then we need to make \( x_v \in (\forall S \cdot D)^2 \), and hence we need to add \( \forall S \cdot D \) to \( \text{Label}(v) \) as a requirement to be realized at \( v \) if \( \forall S \cdot D \notin \text{AFmls}(v) \).

- If \( x_v \in D^2 \) (where \( D \) may be of the form \( \forall S \cdot D' \)) is a requirement but \( D \notin \text{AFmls}(v) \) then we record this by setting \( \text{ConvMethod}(v) := 0 \) and add \( D \) to the set \( \text{FmlsRC}(v) \). If \( \text{FmlsRC}(v) \cap \text{DFmls}(v) \neq \emptyset \) then the requirements at \( v \) are unrealizable and we set \( \text{Status}(v) := \text{unsat} \) (which means \( \text{FullLabel}(v) \) is unsatisfiable w.r.t. \( \mathcal{R} \) and \( \mathcal{T} \)). If \( \text{FmlsRC}(v) \neq \emptyset \) and \( \text{FmlsRC}(v) \cap \text{DFmls}(v) = \emptyset \) then we set \( \text{Status}(v) := \text{incomplete} \), which means the set \( \text{Label}(v) \) should be extended with \( \text{FmlsRC}(v) \) if \( v \) will be used.

- Consider the case when the computed set \( \text{FmlsRC}(v) \) is empty. In this case, we set \( \text{ConvMethod}(v) := 1 \). Each node \( w_i \) in the local graph of \( w \) is an “or”-descendant of \( w \) and corresponds to the same \( x_w \in \Delta^2 \) (for example, if \( C_1 \sqcup C_2 \in \text{Label}(w) \) then we may make \( w \) an “or”-node with two successors \( w_1 \) and \( w_2 \) such that \( C_1 \in \text{Label}(w_1) \) and \( C_2 \in \text{Label}(w_2) \)).

  • Consider the case \( (\forall R^- \cdot D) \in \text{Label}(w_i) \). Thus, \( x_w \in (\forall R^- \cdot D)^2 \) is one of possibly many alternative requirements (because \( w_i \) is one of possibly many “or”-descendants of \( w \)). If \( w_i \) should be selected for representing \( w \) and \( D \notin \text{AFmls}(v) \) then we should add \( D \) to \( \text{Label}(v) \). If \( D \in \text{DFmls}(v) \) then we set \( \text{Status}(w_i) := \text{unsat} \), which means the “combination” of \( v \) and \( w_i \) is unsatisfiable w.r.t. \( \mathcal{R} \) and \( \mathcal{T} \).

  • Consider the case when \( (\forall S \cdot D) \in \text{Label}(w_i) \), \( R^- \sqsubseteq R \) and \( S \circ S \sqsubseteq S \in \text{ext}(R) \). Thus, \( x_w \in (\forall S \cdot D)^2 \) is one of possibly many alternative requirements (because \( w_i \) is one of possibly many “or”-descendants of \( w \)). If \( w_i \) should be selected for representing \( w \) and \( \forall S \cdot D \notin \text{AFmls}(v) \) then we should add \( \forall S \cdot D \) to \( \text{Label}(v) \). If \( \forall S \cdot D \in \text{DFmls}(v) \) then we set \( \text{Status}(w_i) := \text{unsat} \), which means the “combination” of \( v \) and \( w_i \) is unsatisfiable w.r.t. \( \mathcal{R} \) and \( \mathcal{T} \).

If, for \( X = \text{Trans}_R(\text{Label}(w_i), R^-) \setminus \text{AFmls}(v) \), we have that \( X \neq \emptyset \) and \( X \cap \text{DFmls}(v) = \emptyset \), then we add \( X \) (as an element) to the set \( \text{AltFmlSetsSCP}(w) \) and set \( \text{Status}(w_i) := \text{incomplete} \), which means that, if the “or”-descendant \( w_i \) should be selected for representing \( w \) then \( X \) should be added (as a set) to \( \text{Label}(v) \).
Now consider the case when \( \text{Type}(v) = \text{state}, \text{SType}(v) = \text{complex} \) and \( a : \exists R.C \in \text{Label}(v) \). It is very similar to the previous one. We need to satisfy (the ABox) \( \text{Label}(v) \) in the interpretation \( \mathcal{I} \) under construction. To satisfy the formula \( a : \exists R.C \) in \( \mathcal{I} \) we make a transition from \( v \) to \( w \) with \( \text{Label}(w) = \{ C \} \cup \text{Trans}_R(\text{Label}(v), a, R) \cup \mathcal{T} \). The node \( w \) corresponds to an element \( x_w \in \Delta^2 \) such that \( (a^T, x_w) \in R^2 \) and \( x_w \in C^T \). If at some later stage we need to make \( x_w \in (\forall R^-.D)^T \) (for example, because \( (\forall R^- . D) \in \text{Label}(w) \)) then we need to make \( a^T \in D^2 \), and hence we need to add \( a : D \) to \( \text{Label}(v) \) as a requirement to be realized at \( v \) if \( a : D \not\in \text{AFmls}(v) \). Similarly, if at some later stage we need to make \( x_w \in (\forall S.D)^T \), where \( R^- \subseteq R \) and \( S \circ S \subseteq S \in \text{ext}(R) \), then we need to make \( a^T \in (\forall S.D)^T \), and hence we need to add \( a : \forall S.D \) to \( \text{Label}(v) \) as a requirement to be realized at \( v \) if \( a : \forall S.D \not\in \text{AFmls}(v) \).

− If \( a^T \in D^2 \) (where \( D \) may be of the form \( \forall S.D' \)) is a requirement but \( (a : D) \not\in \text{AFmls}(v) \) then we record this by setting \( \text{ConvMethod}(v) := 0 \) and add \( a : D \) to the set \( \text{FmlsRC}(v) \). If \( \text{FmlsRC}(v) \cap \text{DFmls}(v) \neq \emptyset \) then the requirements at \( v \) are unrealizable and we set \( \text{Status}(v) := \text{unsat} \) (which means \( \text{FullLabel}(v) \) is unsatisfiable w.r.t. \( R \) and \( T \)). If \( \text{FmlsRC}(v) \neq \emptyset \) and \( \text{FmlsRC}(v) \cap \text{DFmls}(v) = \emptyset \) then we set \( \text{Status}(v) := \text{incomplete} \), which means the set \( \text{Label}(v) \) should be extended with \( \text{FmlsRC}(v) \) if \( v \) will be used.

− Consider the case when \( \text{ConvMethod}(v) := 1 \). Each node \( w_i \) in the local graph of \( w \) is an “or”-descendant of \( w \) and corresponds to the same \( x_w \in \Delta^2 \).

  • Consider the case \( (\forall R^- . D) \in \text{Label}(w_i) \). Thus, \( x_w \in (\forall R^- . D)^T \) is one of possibly many alternative requirements (because \( w_i \) is one of possibly many “or”-descendants of \( w \)). If \( w_i \) should be selected for representing \( w \) and \( (a : D) \not\in \text{AFmls}(v) \) then we should add \( a : D \) to \( \text{Label}(v) \). If \( (a : D) \in \text{DFmls}(v) \) then we set \( \text{Status}(w_i) := \text{unsat} \), which means the “combination” of \( v \) and \( w_i \) is unsatisfiable w.r.t. \( R \) and \( T \).

  • Consider the case when \( (\forall S.D) \in \text{Label}(w_i) \), \( R^- \subseteq R \) and \( S \circ S \subseteq S \in \text{ext}(R) \). Thus, \( x_w \in (\forall S.D)^T \) is one of possibly many alternative requirements (because \( w_i \) is one of possibly many “or”-descendants of \( w \)). If \( w_i \) should be selected for representing \( w \) and \( (a : \forall S.D) \not\in \text{AFmls}(v) \) then we should add \( a : \forall S.D \) to \( \text{Label}(v) \). If \( (a : \forall S.D) \in \text{DFmls}(v) \) then we set \( \text{Status}(w_i) := \text{unsat} \), which means the “combination” of \( v \) and \( w_i \) is unsatisfiable w.r.t. \( R \) and \( T \).

If, for \( X = \text{Trans}_R(\text{Label}(w_i), R^-, a) \setminus \text{AFmls}(v) \), we have that \( X \neq \emptyset \) and \( X \cap \text{DFmls}(v) = \emptyset \), then we add \( X \) (as an element) to the set \( \text{AltFmlSetsSCP}(w) \) and set \( \text{Status}(w_i) := \text{incomplete} \), which means that, if the “or”-descendant \( w_i \) should be selected for representing \( w \) then \( X \) should be added (as a set) to \( \text{Label}(v) \).

When a node \( w \) gets status \text{incomplete}, \text{unsat} or \text{sat}, the status of every predecessor \( v \) of \( w \) will be updated as shown in procedure \text{UpdateStatus}(v) \) defined on page 113. In particular:

− If \( \text{Type}(w) = \text{state} \) and \( \text{Status}(w) = \text{incomplete} \) then \text{BeforeFormingState}(v) \) holds and \( w \) is the only successor of \( v \). In
this case, the edge \((v, w)\) will be deleted and the node \(v\) will be re-expanded by the converse rule \((\text{conv})\) as shown in procedure \(\text{ApplyConvRule}\) given on page \([1]\). For the subcase \(\text{ConvMethod}(w) = 0\), we connect \(v\) to a node with label \(\text{Label}(v) \cup \text{DFmls}(w)\). Consider the subcase when \(\text{ConvMethod}(w) = 1\). Let \(\text{AltFmlSets}(w) = \{\{\varphi_1\}, \ldots, \{\varphi_n\}, Z_1, \ldots, Z_m\}\), where \(Z_1, \ldots, Z_m\) are non-singleton sets. We connect \(v\) to successors \(w_1, \ldots, w_{n+m}\) such that:

- If \(\text{Type}(v) = \text{state}\) (i.e. \(\text{Kind}(v) = \text{and-node}\)) and \(v\) has a successor \(w\) such that \(\text{Status}(w) = \text{incomplete}\) then we set \(\text{AltFmlSets}(w) := \text{AltFmlSets}(w) \cup \{\text{DFmls}(w)\}\) and set \(\text{Status}(v) := \text{incomplete}\).

Example 3.1. This is an example about web pages, taken from [15] and adapted to our calculus. Let

\[
\mathcal{R} = \{\text{link} \subseteq \text{path}, \text{path} \circ \text{path} \subseteq \text{path}\}
\]

\[
\mathcal{T} = \{\text{perfect} \subseteq \text{interesting} \cap \forall \text{path}. \text{perfect}\}
\]

\[
\mathcal{A} = \{a: \text{perfect}, \text{link}(a, b)\}
\]

It can be shown that \(b\) is an instance of the concept \(\forall \text{link}. \text{interesting}\) w.r.t. the knowledge base \((\mathcal{R}, \mathcal{T}, \mathcal{A})\), i.e., for every model \(I\) of \((\mathcal{R}, \mathcal{T}, \mathcal{A})\), we have that \(b^I \in (\forall \text{link}. \text{interesting})^I\). To prove this one can show that the knowledge base \((\mathcal{R}, \mathcal{T}, \mathcal{A}')\), where \(\mathcal{A}' = \mathcal{A} \cup \{b : \exists \text{link}. \neg \text{interesting}\}\), is unsatisfiable. As abbreviations, let \(L = \text{link}, P = \text{path}, I = \text{interesting}, F = \text{perfect}, \) and \(\varphi = \neg F \cup (I \cap \forall P. F)\). We have

\[
\mathcal{R} = \{L \subseteq P, P \circ P \subseteq P\}
\]

\[
\mathcal{T} = \{\varphi\} \text{ (in NNF)}
\]

\[
\mathcal{A}' = \{a: F, L(a, b), b: \exists L. \neg I\}.
\]

An “and-or” graph for \((\mathcal{R}, \mathcal{T}, \mathcal{A}')\) is presented in Figure \([1]\)
∃ example, StatePred formulas of the node’s label. The node (11) is the only state. We have, for

In each node, we display the name of the rule expanding the node and the

Fig. 1. An “and-or” graph for the knowledge base \( \langle R, T, A' \rangle \), where \( R = \{ L \subseteq P, P \circ P \subseteq P \} \), \( T = \{ \varphi \} \), \( A' = \{ a:F, L(a,b), b: \exists L \sim I \} \), and \( \varphi = \neg F \cup (I \forall P.F) \).
In each node, we display the name of the rule expanding the node and the formulas of the node’s label. The node (11) is the only state. We have, for example, \( \text{StatePred}((15)) = (11) \), \( \text{ATPred}((15)) = (12) \) and \( \text{CELabel}((12)) = b: \exists L \sim I \).
Example 3.2. Let 
\[ R = \{ r \subseteq s, \ r^\bot \subseteq s, \ s \circ s \subseteq s \} \]
\[ T = \{ \exists r. (A \cap \forall s. \neg A) \} \]
\[ A = \{ a : \top \} . \]

In Figures 2 and 3 we give an “and-or” graph for the knowledge base \( \langle R, T, A \rangle \). The nodes are numbered when created and are expanded using DFS (depth-first search). At the end the root receives status \text{unsat}. Therefore, by Theorem 3.4, \( \langle R, T, A \rangle \) is unsatisfiable. As a consequence, \( \langle R, T, \emptyset \rangle \) is also unsatisfiable. \(<\)

Let \( \text{closure}(\mathcal{R}, \mathcal{T}, \mathcal{A}) \) be the union of

– the set of all formulas \( C \) and \( a : C \) such that \( C \) is a concept occurring in \( \mathcal{T} \) or \( \mathcal{A} \) as a formula or a subformula and \( a \) is an individual occurring in \( \mathcal{A} \)
– the set of all formulas \( \forall R. C \) and \( a : \forall R. C \) such that \( a \) is an individual occurring in \( \mathcal{A} \) and there exists a role \( S \) such that \( R \subseteq R S, S \circ S \subseteq S \in \text{ext}(R) \) and \( \forall S. C \) is a concept occurring in \( \mathcal{T} \) or \( \mathcal{A} \) as a formula or a subformula.

The size of \( \text{closure}(\mathcal{R}, \mathcal{T}, \mathcal{A}) \) is polynomial in the size of \( \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \), where the size of a set of formulas (resp. a knowledge base) is the sum of the lengths of its formulas (resp. formulas and axioms).

Lemma 3.3. Procedure Tableau\( \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \) runs in \( 2^{O(n)} \) steps and returns a rooted “and-or” graph \( G = (V, E, \nu) \) of size \( 2^{O(n)} \), where \( n \) is the size of \( \text{closure}(\mathcal{R}, \mathcal{T}, \mathcal{A}) \). Furthermore, for every \( v \in V : \)

1. the sets \( \text{Label}(v), RFmls(v) \) and \( DFmls(v) \) are subsets of \( \text{closure}(\mathcal{R}, \mathcal{T}, \mathcal{A}) \)
2. the local tree of \( v \) is a DAG (directed acyclic graph).

Proof. The assertion [1] should be clear. For the assertion [2] just observe that:

– if \( v \) is expanded by a static rule and \( w \) is a successor of \( v \) then \( RFmls(v) \subseteq RFmls(w) \) and \( AFmls(v) \subseteq AFmls(w) \) and \( DFmls(v) \subseteq DFmls(w) \)
– if \( v \) is expanded by a static rule \( \rho \not\in \{ (H), (H'), (\forall), (\text{conv}), (\text{forming-state}) \} \) and \( w \) is a successor of \( v \) then \( RFmls(v) \subset RFmls(w) \)
– if \( v \) is expanded by a rule \( \rho \in \{ (H), (H'), (\forall), (\text{conv}) \} \) and \( w \) is a successor of \( v \) then \( AFmls(v) \subset AFmls(w) \).

Note that, each tableau node is re-expanded at most once, by using the rule \((\text{conv})\). It is easy to see that \( G \) has size \( 2^{O(n)} \) and can be constructed in \( 2^{O(n)} \) steps. \(<\)

Theorem 3.4. Let \( \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \) be a knowledge base in NNF of the logic \( \text{SHI} \). Then procedure Tableau\( \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \) runs in exponential time (in the worst case) in the size of \( \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \) and returns a rooted “and-or” graph \( G = (V, E, \nu) \) such that \( \langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle \) is satisfiable iff \( \text{Status}(v) \neq \text{unsat} \). \(<\>

The complexity issue was addressed by Lemma 3.3. For the remaining assertion, see the proofs given in the next section.
Fig. 2. An illustration for Example 3.2 part I. The graph (a) is the “and-or” graph constructed until checking “compatibility” of the node (5) w.r.t. to the node (2). In each node, we display the name of the rule expanding the node and the formulas of the node’s label. The node (2) is the only state. We have, for example, StatePred((5)) = (2), ATPred((5)) = (3) and CELabel((3)) = a : ∃r.(A ∩ ∀s.¬A). Checking “compatibility” of the node (5) w.r.t. to the node (2), Status((2)) is set to incomplete and FmlsRC((2)) is set to {A, ∀s.¬A}. This results in the graph (b). The construction is then continued by applying the rule (conv) to (1). See Figure 3 for the continuation.
Fig. 3. An illustration for Example 3.2 part II. This is a fully expanded “and-or” graph for \((R, T, A)\). The node (1) is re-expanded by the rule \((conv)\). As in the part I, in each node we display the name of the rule expanding the node and the formulas of the node’s label. The nodes (2) and (8) are the only states. After the node (10) receives status \text{unsat}, the nodes (9)-(6) and (1) receive status \text{unsat} in subsequent steps.
4 Proofs of Soundness and Completeness of $C_{SHI}$

4.1 Soundness

If $X = \{C_1, \ldots, C_n\}$ then define $\text{Cnj}(X) = C_1 \sqcap \ldots \sqcap C_n$. If $X = \{a_1 C_1, \ldots, a_i C_n\}$ then define $\text{Cnj}(X) = a_1 (C_1 \sqcap \ldots \sqcap C_n)$. Furthermore, we define $\text{NegCnj}(X)$ to be the NNF of $\neg \text{Cnj}(X)$, and define $\text{NegAll} \{X_1, \ldots, X_k\} = \{\text{NegCnj}(X_1), \ldots, \text{NegCnj}(X_k)\}$.

Let $G$ be a $C_{SHI}$-tableau for $(\mathcal{R}, T, A)$. For each node $v$ of $G$ with Status$(v) \in \{\text{incomplete}, \text{unsat}, \text{sat}\}$, let $\text{DSTimeStamp}(v)$ be the moment at which Status$(v)$ was changed to its final value (i.e., determined to be incomplete, unsat or sat). $\text{DSTimeStamp}$ stands for “determined-status time-stamp”. For each non-state $v$ of $G$, let $\text{ETTimeStamp}(v)$ be the moment at which $v$ was expanded the last time.\footnote{Recall that, each non-state may be re-expanded at most once, using the rule (conv), and that, each state is expanded at most once.

Lemma 4.1. Let $G = (V, E, \nu)$ be a $C_{SHI}$-tableau for $(\mathcal{R}, T, A)$. For every $v \in V$:

1. if Status$(v) = \text{unsat}$ then
   a) case Type$(v) = \text{state}$: $\text{FullLabel}(v)$ is unsatisfiable w.r.t. $\mathcal{R}$ and $T$
   b) case Type$(v) = \text{non-state and StatePred}(v) = \text{null}$: $\text{FullLabel}(v)$ is unsatisfiable w.r.t. $\mathcal{R}$ and $T$
   c) case Type$(v) = \text{non-state}$, $v_0 = \text{StatePred}(v) \neq \text{null}$ and $\text{SType}(v_0) = \text{simple}$: if $v_1 = \text{ATPred}(v)$ and $\text{CELabel}(v_1)$ is of the form $\exists R.C$ then there do not exist any model $\mathcal{I}$ of both $\mathcal{R}$ and $T$ and any elements $x, y \in \Delta^2$ such that $(x, y) \in R^x$, $x \in (\text{FullLabel}(v_0))^T$, and $y \in (\text{Label}(v))^T$.
   d) case Type$(v) = \text{non-state}$, $v_0 = \text{StatePred}(v) \neq \text{null}$ and $\text{SType}(v_0) = \text{complex}$: if $v_1 = \text{ATPred}(v)$ and $\text{CELabel}(v_1)$ is of the form $a : \exists R.C$ then there do not exist any model $\mathcal{I}$ of $(\mathcal{R}, T, \text{FullLabel}(v_0))$ and any element $y \in \Delta^2$ such that $(a^x, y) \in R^x$ and $y \in (\text{Label}(v))^T$.

2. if Status$(v) = \text{incomplete and Type}(v) = \text{state}$ then
   a) case ConvMethod$(v) = 0$: $\text{FullLabel}(v) \cup \{\text{NegCnj}(\text{FmlsRC}(v))\}$ is unsatisfiable w.r.t. $\mathcal{R}$ and $T$
   b) case ConvMethod$(v) = 1$: $\text{FullLabel}(v) \cup \text{NegAll}(\text{AltFmlSetsSC}(v))$ is unsatisfiable w.r.t. $\mathcal{R}$ and $T$

3. if Type$(v) = \text{non-state and } w_1, \ldots, w_k$ are all the successors of $v$ then, for every model $\mathcal{I}$ of $\mathcal{R}$ and every $x \in \Delta^2$,
   a) case $\text{SType}(v) = \text{simple}$: $x \in (\text{FullLabel}(v))^T$ iff there exists $1 \leq i \leq k$ such that $x \in (\text{FullLabel}(w_i))^T$
   b) case $\text{SType}(v) = \text{complex}$: $\mathcal{I}$ is a model of $\text{FullLabel}(v)$ iff there exists $1 \leq i \leq k$ such that $\mathcal{I}$ is a model of $\text{FullLabel}(w_i)$.
Proof. We prove this lemma by induction on both DSTimeStamp(v) and ETimeStamp(v).

Consider the assertion 3. It should be clear for the cases when the rule expanding v is not (conv). So, assume that v was re-expanded by the rule (conv) and let w be the only successor of v before the re-expansion. We must have that Type(w) = state, Status(w) = incomplete, DSTimeStamp(w) < ETimeStamp(v), Label(w) = Label(v), RFmls(w) = RFmls(v) and DFmls(w) = DFmls(v). There are the following two cases:

- Case ConvMethod(w) = 0: By the inductive assumption 2a, FullLabel(w) ∪ {NegCnj(FmlsRC(w))} is unsatisfiable w.r.t. R and T. It follows that FullLabel(v) ∪ {NegCnj(FmlsRC(v))} is unsatisfiable w.r.t. R and T. After re-expansion v has only one successor w', with Label(w') = Label(v) ∪ FmlsRC(w), RFmls(w') = RFmls(v) and DFmls(w') = DFmls(v). Hence, the assertion 3 holds.

- Case ConvMethod(w) = 1: By the inductive assumption 2b, FullLabel(w) ∪ NegAll(AltFmlSetsSC(w)) is unsatisfiable w.r.t. R and T. It follows that FullLabel(v) ∪ NegAll(AltFmlSetsSC(v)) is unsatisfiable w.r.t. R and T. Using this, it can be observed that Steps 5-12 of procedure ApplyConvRule guarantees the assertion 3.

Consider the assertion 1. If Status(v) = unsat because ⊥ ∈ Label(v) or there exists {φ, ϕ} ⊆ Label(v) then FullLabel(v) is clearly unsatisfiable w.r.t. R and T. So, assume that Label(v) contains neither ⊥ nor a pair {φ, ϕ}.

Consider the assertion 1a and suppose that Status(v) = unsat and Type(v) = state. There are three cases: Status(v) was set to unsat either by Step 27 of procedure Apply (with v₀ = v) or by Step 12 of procedure ApplyTransRule (with u = v) or by Step 10 of procedure UpdateStatus. For the first two cases, we must have that ConvMethod(v) = 0 and FmlsRC(v) ∩ DFmls(v) ≠ ∅, which implies that FullLabel(v) is unsatisfiable w.r.t. R and T. The intuition behind the last inference is that FullLabel(v) = AFmls(v) ∪ NDFmls(v) and FmlsRC(v) is the set of formulas which must be added to AFmls(v). Consider the third case. Thus, v has a successor w with status Status(w) = unsat, Type(w) = non-state and DSTimeStamp(w) < DSTimeStamp(v). The inductive assumption 1c or 1d (depending on SType(w)) holds for w (in the place of v). If FullLabel(v) is satisfied in a model I of R and T, then I will violate this inductive assumption. Hence FullLabel(v) is unsatisfiable w.r.t. R and T.

Consider the assertion 1b and suppose that Status(v) = unsat and Type(v) = non-state and StatePred(v) = null. Let w₁, . . . , wₖ be all the successors of v. It must be that, for every 1 ≤ i ≤ k, Status(wᵢ) = unsat, Type(wᵢ) = non-state, StatePred(wᵢ) = null and DSTimeStamp(wᵢ) < DSTimeStamp(v). By the inductive assumption 1b, for every 1 ≤ i ≤ k, FullLabel(wᵢ) is unsatisfiable w.r.t. R and T. By the inductive assumption 3, it follows that FullLabel(v) is unsatisfiable w.r.t. R and T.

Consider the assertions 1c and 1d and suppose that Status(v) = unsat and Type(v) = non-state and StatePred(v) ≠ null. There are the following cases:
– Case $\text{Status}(v)$ was set to \texttt{unsat} by Step 28 of procedure \texttt{Apply}: The condition $Y \cap DFmls(v_0) \neq \emptyset$ of that step implies the assertions 1c and 1d.

– Case $\text{Status}(v)$ was set to \texttt{unsat} by Step 3 of procedure \texttt{UpdateStatus}: Let $w_1, \ldots, w_k$ be all the successors of $v$. It must be that, for every $1 \leq i \leq k$, $\text{Status}(w_i) = \text{unsat}$, $\text{Type}(w_i) = \text{non-state}$, $\text{StatePred}(w_i) = \text{StatePred}(v) \neq \emptyset$ and $\text{DSTimeStamp}(w_i) < \text{DSTimeStamp}(v)$. The inductive assumptions 1c and 1d for $w_1, \ldots, w_k$ imply the inductive hypotheses 1c and 1d for $v$.

The assertion 2a should be clear.

Consider the assertion 2b and suppose that $\text{Status}(v) = \text{incomplete}$, $\text{Type}(v) = \text{state}$ and $\text{ConvMethod}(v) = 1$. There must exist a successor $w$ of $v$ such that $\text{AfterTrans}_R(w)$ holds, $\text{Kind}(w) = \text{or-node}$, $\text{Status}(w) = \text{incomplete}$, and $\text{AltFmlSetsSCP}(w) = \text{AltFmlSetsSC}(v)$. Let $w_1, \ldots, w_k$ be all the nodes in the local graph of $w$ such that, for $1 \leq i \leq k$, $\text{Status}(w_i) = \text{incomplete}$ and when $\text{Status}(w_i)$ became \texttt{incomplete} a set $X_i$ of formulas was added (as an element) into $\text{AltFmlSetsSCP}(w)$ (i.e., $w_i$ got status \texttt{incomplete} not by propagation). The setting of $\text{Status}(w_i)$ and the addition of $X_i$ to $\text{AltFmlSetsSCP}(w)$ occur at Steps 30 and 31 of procedure \texttt{Apply}. We have that $\text{AltFmlSetsSCP}(w) = \{X_1, \ldots, X_k\}$. Note that, since $\text{Status}(w) = \text{incomplete}$, $k \geq 1$ and every node in the local graph of $w$ must have status \texttt{incomplete} or \texttt{unsat}. There are the following two cases:

– Case $\text{SType}(v) = \text{simple}$: Let $\text{CELLabel}(w) = \exists R.C$. For the sake of contradiction, suppose there exists a model $\mathcal{I}$ of $\mathcal{R}$ and $\mathcal{T}$ such that $(\text{FullLabel}(v) \cup \text{NegAll}(\text{AltFmlSetsSC}(v)))^\mathcal{I}$ is not empty and contains an element $x$. Since $\text{CELLabel}(w) \in \text{Label}(v)$, there exists $y \in \Delta^\mathcal{I}$ such that $(x, y) \in R^\mathcal{I}$ and $y \in C^\mathcal{I}$. Thus, $y \in (\text{Label}(w))^\mathcal{I}$, and hence $y \in (\text{FullLabel}(w))^\mathcal{I}$ (since $\text{RFmls}(w) = \text{DFmls}(w) = \emptyset$). For every node $w'$ in the local graph of $w$ with $\text{Status}(w') = \text{unsat}$, we have that $\text{DSTimeStamp}(w') < \text{DSTimeStamp}(v)$, and by the inductive assumption 1c, $y \notin (\text{Label}(w'))^\mathcal{I}$, and hence $y \notin (\text{FullLabel}(w'))^\mathcal{I}$. Since $y \in (\text{FullLabel}(w))^\mathcal{I}$, by the inductive assumption 3a, it follows that there exists $1 \leq i \leq k$ such that $y \in (\text{FullLabel}(w_i))^\mathcal{I}$. Since $X_i = \text{Trans}_R(\text{Label}(w_i), R^-)$ and $(x, y) \in R^\mathcal{I}$, it follows that $x \in X_i^\mathcal{I}$, which contradicts the fact that $x \in (\text{NegAll}(\text{AltFmlSetsSC}(v)))^\mathcal{I}$. Therefore $\text{FullLabel}(v) \cup \text{NegAll}(\text{AltFmlSetsSC}(v))$ must be unsatisfiable w.r.t. $\mathcal{R}$ and $\mathcal{T}$.

– Case $\text{SType}(v) = \text{complex}$: Let $\text{CELLabel}(w) = a : \exists R.C$. For the sake of contradiction, suppose there exists a model $\mathcal{I}$ of $\mathcal{R}$, $\mathcal{T}$ and $\text{FullLabel}(v) \cup \text{NegAll}(\text{AltFmlSetsSC}(v))$. Since $\text{CELLabel}(w) \in \text{Label}(v)$, there exists $y \in \Delta^\mathcal{I}$ such that $(a^\mathcal{I}, y) \in R^\mathcal{I}$ and $y \in C^\mathcal{I}$. Thus, $y \in (\text{Label}(w))^\mathcal{I}$, and hence $y \in (\text{FullLabel}(w))^\mathcal{I}$ (since $\text{RFmls}(w) = \text{DFmls}(w) = \emptyset$). For every node $w'$ in the local graph of $w$ with $\text{Status}(w') = \text{unsat}$, we have that $\text{DSTimeStamp}(w') < \text{DSTimeStamp}(v)$, and by the inductive assumption 1d, $y \notin (\text{Label}(w'))^\mathcal{I}$, and hence $y \notin (\text{FullLabel}(w'))^\mathcal{I}$. Since $y \in (\text{FullLabel}(w))^\mathcal{I}$, by the inductive assumption 3a, it follows that there exists $1 \leq i \leq k$ such that $y \in (\text{FullLabel}(w_i))^\mathcal{I}$. Since
If $X_i = \text{Trans}_R(\text{Label}(w_i), R^-, a)$ and $(a^x, y) \in R^x$, it follows that $I$ is a model of (the ABox) $X_i$, which contradicts the fact that $I$ is a model of (the ABox) $\text{NegAll}(\text{AltFmlSetsSC}(v))$. Therefore $\text{FullLabel}(v) \cup \text{NegAll}(\text{AltFmlSetsSC}(v))$ must be unsatisfiable w.r.t. $R$ and $T$.

Corollary 4.2 (Soundness of $C_{SHI}$). If $G = (V, E, \nu)$ is a $C_{SHI}$-tableau for $(R, T, A)$ and $\text{Status}(\nu) = \text{unsat}$ then $(R, T, A)$ is unsatisfiable. 

This corollary follows from the assertion 1b of Lemma 4.1.

4.2 Completeness

Lemma 4.3. Let $G = (V, E, \nu)$ be a $C_{SHI}$-tableau for $(R, T, A)$. Then no node with status incomplete is reachable from $\nu$.

Proof. This lemma follows from the observation that, after a state $w$ getting status incomplete, all edges coming to $w$ will be deleted (see Step 1 of procedure $\text{ApplyConvRule}$).

We prove completeness of $C_{SHI}$ via model graphs. The technique has been used in [17,3,9,4,14,13] for other logics. A model graph (also known as a Hintikka structure) is a tuple $\langle \Delta, C, E \rangle$, where:

- $\Delta$ is a finite set, which contains (amongst others) all individual names (occurring in the considered ABox)
- $C$ is a function that maps each element of $\Delta$ to a set of concepts
- $E$ is a function that maps each role to a binary relation on $\Delta$.

A model graph $\langle \Delta, C, E \rangle$ is $R$-saturated if every $x \in \Delta$ satisfies:

- if $C \sqcap D \in C(x)$ then $\{C, D\} \subseteq C(x)$ (1)
- if $C \sqcup D \in C(x)$ then $C \in C(x)$ or $D \in C(x)$ (2)
- if $\forall S.C \in C(x)$ and $R \subseteq R$ then $\forall R.C \in C(x)$ (3)
- if $(x, y) \in E(R)$ then $\text{Trans}_R(C(x), R^-) \subseteq C(y)$ (4)
- if $(x, y) \in E(R)$ then $\text{Trans}_R(C(y), R^-) \subseteq C(x)$ (5)
- if $\exists R.C \in C(x)$ then there exists $y \in \Delta$ s.t. $(x, y) \in E(R)$ and $C \in C(y)$ (6)

A model graph $\langle \Delta, C, E \rangle$ is consistent if no $x \in \Delta$ has $C(x)$ containing $\bot$ or containing both $A$ and $\neg A$ for some atomic concept $A$.

Given a model graph $M = \langle \Delta, C, E \rangle$, the $R$-model corresponding to $M$ is the interpretation $I = \langle \Delta, I \rangle$ where:

- $a^x = a$ for every individual name $a$
- $A^x = \{x \in \Delta \mid A \in C(x)\}$ for every concept name $A$
- $r^x = E'(r)$ for every role name $r \in R$, where $E'(R)$ for $R \in R \cup R^-$ are the smallest binary relations on $\Delta$ such that:
  - $E(R) \subseteq E'(R)$
Observe that each saturation path of the structure of $C$

**Lemma 4.4.** If $\mathcal{I}$ is the $\mathcal{R}$-model corresponding to a consistent $\mathcal{R}$-saturated model graph $\langle \Delta, C, \mathcal{E} \rangle$, then $\mathcal{I}$ is a model of $\mathcal{R}$ and, for every $x \in \Delta$ and $C \in C(x)$, we have that $x \in C^\mathcal{I}$.

**Proof.** Clearly, $\mathcal{I}$ is a model of $\mathcal{R}$. For the remaining assertion of the lemma, we first prove that if $(x, y) \in R^\mathcal{I}$ then $\text{Trans}_\mathcal{R}(\mathcal{C}(x), R) \subseteq \mathcal{C}(y)$ and $\text{Trans}_\mathcal{R}(\mathcal{C}(y), R^{-}) \subseteq \mathcal{C}(x)$. We prove this by induction on the timestamp of the addition of the pair $(x, y)$ to $\mathcal{E}'(R)$ when constructing $\mathcal{I}$ from the model graph. The base case is when $(x, y) \in \mathcal{E}(R)$ and follows from the assumption that $\langle \Delta, C, \mathcal{E} \rangle$ is an $\mathcal{R}$-saturated model graph. For induction step, there are the following cases:

- Case $(x, y)$ is added to $\mathcal{E}'(R)$ because $(y, x) \in \mathcal{E}'(R^{-})$: The induction hypothesis immediately follows from the inductive assumption.
- Case $(x, y)$ is added to $\mathcal{E}'(R)$ because $(x, y) \in \mathcal{E}'(S)$ and $S \subseteq R$: By 4, we have that $\text{Trans}_\mathcal{R}(\mathcal{C}(x), R) \subseteq \text{Trans}_\mathcal{R}(\mathcal{C}(x), S)$ and $\text{Trans}_\mathcal{R}(\mathcal{C}(y), R^{-}) \subseteq \text{Trans}_\mathcal{R}(\mathcal{C}(y), S^{-})$. The induction hypothesis follows from these properties and the inductive assumption with $S$ replacing $R$.
- Case $(x, y)$ is added to $\mathcal{E}'(R)$ because $R \circ R \subseteq R \in \text{ext}(R)$, $(x, z) \in \mathcal{E}'(R)$ and $(z, y) \in \mathcal{E}'(R)$: The induction hypothesis follows from the inductive assumption with $(x, z)$ replacing $(x, y)$ and from the inductive assumption with $(z, y)$ replacing $(x, y)$.

The remaining assertion of the lemma can then be proved by induction on the structure of $C$ in a straightforward way.

Let $G = (V, E)$ be a $C_{SHI}$ graph for $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ and $v \in V$ be a non-state with $\text{Status}(v) \notin \{\text{unsat}, \text{incomplete}\}$. A saturation path of $v$ is a sequence $v_0 = v, v_1, \ldots, v_k$ of nodes of $G$, with $k \geq 1$, such that $\text{Type}(v_k) = \text{state}$ and

- for every $1 \leq i \leq k$, $\text{Status}(v_i) \notin \{\text{unsat, incomplete}\}$
- for every $0 \leq i < k$, $\text{Type}(v_i) = \text{non-state}$ and $(v_i, v_{i+1}) \in E$.

Observe that each saturation path of $v$ is finite (by the assertion 2 of Lemma 4.3). Furthermore, if $v_i$ is a non-state with $\text{Status}(v_i) \notin \{\text{unsat, incomplete}\}$ then $v_i$ has a successor $v_{i+1}$ with $\text{Status}(v_{i+1}) \notin \{\text{unsat, incomplete}\}$. Therefore, $v$ has at least one saturation path.

**Lemma 4.5 (Completeness of $C_{SHI}$).** Let $G = (V, E, \nu)$ be a $C_{SHI}$-tableau for $(\mathcal{R}, \mathcal{T}, \mathcal{A})$. Suppose that $\text{Status}(\nu) \neq \text{unsat}$. Then $(\mathcal{R}, \mathcal{T}, \mathcal{A})$ is satisfiable.
Proof. By Lemma \textbf{lem}, \textit{Status}(\nu) \neq \text{incomplete}. Hence \nu has a saturation path \nu_0,\ldots,\nu_k with \nu_0 = \nu. We construct a model graph \(M = \langle \Delta, C, E \rangle\) as follows:

1. Let \(\Delta_0\) be the set of all individuals occurring in \(A\) and set \(\Delta := \Delta_0\). For each \(a \in \Delta_0\), set \(C(a)\) to the set of all concepts \(C\) such that \(a : C \in \text{AFmls}(\nu_k)\), and mark \(a\) as \text{unsolved}. (Each node of \(M\) will be marked either as unresolved or as resolved.) For each role \(R\), set \(E(R) := \{(a, b) \mid R(a, b) \in A\}\).

2. While \(\Delta\) contains unresolved nodes, take one unresolved node \(x\) and do:
   (a) For every concept \(\exists R.C \in C(x)\) do:
      i. If \(x \in \Delta_0\) then:
         - Let \(u = v_k\).
         - Let \(w_0\) be the node of \(G\) such that \(\text{CEL}(w_0) = x : \exists R.C\).
           (Note that \(C \in \text{Label}(w_0)\) and \(\text{Status}(w_0) \notin \{\text{unsat, incomplete}\}\) since \(\text{Status}(v_k) \notin \{\text{unsat, incomplete}\}\).
      ii. Else:
         - Let \(u = f(x)\). (\(f\) is a constructed mapping that associates each node of \(M\) not belonging to \(\Delta_0\) with a simple state of \(G\). As a maintained property of \(f\), \(\text{Status}(u) \notin \{\text{unsat, incomplete}\}\), \(\exists R.C \in \text{Label}(u)\) and \(C(x) = \text{AFmls}(u)\).)
         - Let \(w_0\) be the node of \(G\) such that \(\text{CEL}(w_0) = \exists R.C\). (Note that \(C \in \text{Label}(w_0)\) and \(\text{Status}(w_0) \notin \{\text{unsat, incomplete}\}\) since \(\text{Status}(u) \notin \{\text{unsat, incomplete}\}\).
   iii. Let \(w_0, \ldots, w_h\) be a saturation path of \(w_0\).
      (Note that \(\text{Status}(w_h) \notin \{\text{unsat, incomplete}\}\).
      iv. If there does not exist \(y \in \Delta \setminus \Delta_0\) such that \(C(y) = \text{AFmls}(w_h)\) then: add a new node \(y\) to \(\Delta\), set \(C(y) = \text{AFmls}(w_h)\), mark \(y\) as unresolved, and set \(f(y) = w_h\). (One can consider \(y\) as the result of sticking together the nodes \(w_0, \ldots, w_h\) of a saturation path of \(w_0\). The above mentioned properties of \(f\) still hold.)
      v. Add the pair \((x, y)\) to \(E(R)\).
   (b) Mark \(x\) as resolved.

The above construction terminates and results in a finite model graph because: for every \(x, x' \in \Delta \setminus \Delta_0\), \(x \neq x'\) implies \(C(x) \neq C(x')\), and for every \(x \in \Delta\), \(C(x)\) is a subset of closure\((R, T, A)\).

Note the following remarks for the remaining part of this proof:

- For any node \(v\) of \(G\), \(\text{RFmls}(v)\) may contain only formulas of the form \(C \cap D\), \(C \cup D\), \(a: (C \cap D)\), or \(a: (C \cup D)\). Hence, if \(\varphi\) is of the form \(\forall R.C\), \exists R.C, a: \forall R.C\) or \(a: \exists R.C\) and \(\varphi \in \text{AFmls}(v)\) then we must have that \(\varphi \in \text{Label}(v)\).
- After executing Step \textbf{2(a)iv}, \(\text{Label}(w_0) \subseteq \text{AFmls}(w_h) = C(y)\). Hence, if \(D \in \text{Label}(w_0)\) then \(D \in C(y)\).

\(M\) is a consistent model graph because \(\text{Status}(v_h) \neq \text{unsat}\) and if \(x \in \Delta \setminus \Delta_0\) and \(u = f(x)\) then \(C(x) = \text{AFmls}(u)\) and \(\text{Status}(u) \neq \text{unsat}\).

We show that \(M\) satisfies all Conditions \textbf{1}--\textbf{6} of being a saturated model graph. \(M\) satisfies Conditions \textbf{1}--\textbf{3} because the sequence \(v_0, \ldots, v_k\) is a saturation path of \(v_0\), and at Step \textbf{2(a)} the sequence \(w_0, \ldots, w_h\) is a saturation path
of \( w_0 \). \( M \) satisfies Condition 4 because at Step 2a, \( C \in \text{Label}(w_0) \), and hence \( C \in C(y) \).

Consider Condition 4:

- Assume \( x \in \Delta \) and \( (x, y) \in \mathcal{E}(R) \). We show that \( \text{Trans}(C(x), R) \subseteq C(y) \).
- Consider the case \( x \in \Delta_0 \) and \( \forall R.D \in C(x) \). We show that \( D \in C(y) \).
  Since \( \forall R.D \in C(x) \), we have that \( x : \forall R.D \in \text{Label}(v_k) \). If \( y \in \Delta_0 \), then \( R(x, y) \in A \) (since \( (x, y) \in \mathcal{E}(R) \)), and hence \( y : D \in \text{AFmls}(v_k) \) (due to the tableau rule \( \forall \)), and hence \( D \in C(y) \). Assume that \( y \notin \Delta_0 \) and \( y \) is created at Step 2(a)iv. Since \( (x : \forall R.D) \) belongs to the label of \( u = v_k \), by the tableau rule \( \exists \), \( D \in \text{Label}(w_0) \), and hence \( D \in C(y) \).
- Consider the case \( x \notin \Delta_0 \) and Step 2a at which the pair \( (x, y) \) is added to \( \mathcal{E}(R) \).
  - Suppose that \( \forall R.D \in C(x) \). We show that \( D \in C(y) \). Since \( \forall R.D \in C(x) \) and \( C(x) = \text{AFmls}(u) \), we have that \( \forall R.D \in \text{Label}(u) \). By the tableau rule \( \forall \), it follows that \( D \in \text{Label}(w_0) \), and hence \( D \in C(y) \).
  - Suppose that \( \forall S.D \in C(x) \). \( R \subseteq_R S \) and \( S \circ S \subseteq S \in \text{ext}(R) \). We show that \( \forall S.D \in C(y) \). Since \( \forall S.D \in C(x) \) and \( C(x) = \text{AFmls}(u) \), we have that \( \forall S.D \in \text{Label}(u) \). By the tableau rule \( \exists \), it follows that \( \forall S.D \in \text{Label}(w_0) \), and hence \( \forall S.D \in C(y) \).

Consider Condition 5:

- Assume \( y \in \Delta \) and \( (x, y) \in \mathcal{E}(R) \). We show that \( \text{Trans}(C(y), R^-) \in C(x) \).
- Consider the case \( y \in \Delta_0 \). We must have that \( x \in \Delta_0 \) and \( R(x, y) \in A \).
  - If \( \forall R^- . D \in C(y) \) then \( y : \forall R^- . D \in \text{Label}(v_k) \), which implies that \( x : D \in \text{AFmls}(v_k) \) (by the tableau rule \( \forall \)), and hence \( D \in C(x) \).
  - If \( \forall S.D \in C(y) \), \( R^- \subseteq_R S \) and \( S \circ S \subseteq S \in \text{ext}(R) \) then \( y : \forall S.D \in \text{Label}(v_k) \), which implies that \( x : \forall S.D \in \text{AFmls}(v_k) \) (by the tableau rule \( \forall \)), and hence \( \forall S.D \in C(x) \).
- Consider the case \( y \notin \Delta_0 \) and Step 2a at which the pair \( (x, y) \) is added to \( \mathcal{E}(R) \).
  - Suppose that \( \forall R^- . D \in C(y) \). We show that \( D \in C(x) \). Since \( \forall R^- . D \in C(y) \), we have that \( \forall R^- . D \in \text{Label}(w_h) \). Since \( \text{Status}(w_h) \notin \{\text{unsat, incomplete}\} \), we must have that:
    - if \( u \) is a simple node then \( x \notin \Delta_0 \) and \( D \in \text{AFmls}(u) \), and hence \( D \in C(x) \)
    - else \( v_k \), \( x \in \Delta_0 \) and \( x : D \in \text{AFmls}(u) \), and hence \( D \in C(x) \).
  - Suppose that \( \forall S.D \in C(y) \), \( R^- \subseteq_R S \) and \( S \circ S \subseteq S \in \text{ext}(R) \). We show that \( \forall S.D \in C(x) \). Since \( \forall S.D \in C(y) \), we have that \( \forall S.D \in \text{Label}(w_h) \). Since \( \text{Status}(w_h) \notin \{\text{unsat, incomplete}\} \), we must have that:
* if \( u \) is a simple node then \( x \notin \Delta_0 \) and \( \forall S.D \in \text{AFmls}(u) \), and hence \( \forall S.D \in C(x) \)
* else \( u = v_k \), \( x \in \Delta_0 \) and \( x : \forall S.D \in \text{AFmls}(u) \), and hence \( \forall S.D \in C(x) \).

Therefore \( M \) is a consistent saturated model graph.

By the definition of \( C_{SHI} \) graphs for \( (R, T, A) \) and the construction of \( M \): if \((a:C) \in A\) then \( C \in C(a)\); if \( R(u,b) \in A \) then \((a,b) \in E(R)\); and \( T \subseteq C(a) \) for all \( a \in \Delta_0 \). We also have that \( T \subseteq C(x) \) for all \( x \in \Delta \setminus \Delta_0 \). Hence, by Lemma 4.4, the interpretation corresponding to \( M \) is a model of \( (R, T, A) \).

5 Conclusions

We have given the first cut-free \( \text{ExpTime} \) (optimal) tableau decision procedure for checking satisfiability of a knowledge base in the description logic \( \text{SHI} \). Our decision procedure is novel: in contrast to \([4,5,6]\), it deals also with ABoxes; in contrast to \([4,16]\), it does not use cuts; in contrast to \([15,14]\), it deals also with inverse roles; and in contrast to \([11]\), it deals also with transitive roles and hierarchies of roles. The procedure can be implemented with various optimizations as in \([10]\) to give an efficient complexity-optimal program for checking satisfiability of a knowledge base in the popular DL \( \text{SHI} \). We intend to extend our methods for other DLs.

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