Constraint polynomial approach - an alternative to the functional Bethe Ansatz method?

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Abstract

Recently developed general constraint polynomial approach is shown to replace a set of algebraic equations of the functional Bethe Ansatz method by a single polynomial constraint. As the proof of principle, the usefulness of the method is demonstrated for a number of quasi-exactly solvable (QES) potentials of the Schrödinger equation, such as two different sets of modified Manning potentials with three parameters, an electron in Coulomb and magnetic fields and relative motion of two electrons in an external oscillator potential, the hyperbolic Razavy potential, and a (perturbed) double sinh-Gordon system. The approach enables one to straightforwardly determine eigenvalues and wave functions. Odd parity solutions for the modified Manning potentials are also determined. For the QES examples considered here, constraint polynomials terminate a finite chain of orthogonal polynomials in an independent variable that need not to be necessarily energy. In the majority of cases the finite chain of orthogonal polynomials is characterized by a positive-definite moment functional \( \mathcal{L} \), implying that a corresponding constraint polynomial has only real and simple zeros. Constraint polynomials are shown to be different from the weak orthogonal Bender-Dunne polynomials. At the same time the QES examples considered elucidate essential difference with various generalizations of the Rabi model. Whereas in the former case there are \( n + 1 \) polynomial solutions at each point of a \( n \)th baseline, in the latter case there are at most \( n + 1 \) polynomial solutions on entire \( n \)th baseline.
I. INTRODUCTION

The Schrödinger equation \((\hbar = 2m = 1)\)

\[
\left( -\frac{d^2}{dx^2} + V \right) \psi = E\psi
\]

(1)

for a number of quasi-exactly solvable potentials \(V\) can on using a suitable substitution be recast in the same basic form as [1–5, 8–10]

\[
(a_3 z^3 + a_2 z^2 + a_1 z) \frac{d^2 \phi(z)}{dz^2} + (b_2 z^2 + b_1 z + b_0) \frac{d\phi(z)}{dz} + (c_1 z + c_0) \phi(z) = 0,
\]

(2)

where \(a_3, a_2, a_1, b_2, b_1, b_0, c_1, c_0\) are constant parameters. This form corresponds to the general Heun equation [6–8], and its confluent [9] and bi-confluent [10] forms, provided that one of the regular singular points is at \(z = 0\). Eq. (2) is a particular type of more general ordinary differential equation (ODE) with polynomial coefficients for which a general concept of gradation slicing has been recently employed in order to analyze their polynomial solutions [11]. Gradation slicing is universal and easily applicable algorithmic recursive approach for obtaining polynomial solutions which does not require any a priori knowledge about hidden algebraic structure of ODE. Its usefulness has been so far demonstrated on the examples of various Rabi models [11].

In the present article we employ the gradation slicing approach of Ref. [11] to determine polynomial solutions of quasi-exactly solvable (QES) Schrödinger equation for Xie [12] and Chen et al. [13] three parameters modified Manning potentials [1, 3, 14], an electron in Coulomb and magnetic fields and relative motion of two electrons in an external oscillator potential [15, 16], the perturbed double sinh-Gordon system (DSHG) [1, 3, 17], and the hyperbolic Razavy potential [1, 3, 18]. All those QES potentials lend themselves to \(sl_2\) algebraization [6, 7]. At the same time the above QES examples are used to elucidate essential difference with various generalizations of the Rabi model [11]. Whereas in the former case there are \(n + 1\) polynomial solutions at each point of a \(n\)th baseline [defined by condition (4) below], in the latter case there are at most \(n + 1\) polynomial solutions on entire \(n\)th baseline. In both cases a given baseline characterizes the set of model parameter in which case an \(sl_2\) algebraization with a given spin is possible. The difference between QES examples and Rabi models arises due to a cardinally different qualitative behaviour under variations
of a spectral parameter. (The latter can be either energy, as in examples of Sec. IV or some other model parameter, as in the so-called coupling constant metamorphosis examples of Sec. III.) When corresponding spectral parameter is varied then, in the QES examples presented here, one remains at a fixed point of a baseline. In other words, a corresponding algebraic Heun operator remains unchanged. Contrary to that, for a number of generalizations of the Rabi model [11] variations of spectral parameter induce translation on the corresponding baseline. This has the effect that to different values of spectral parameter there correspond different algebraic Heun operators (cf. Sec. VA). It is deemed expedient to appreciate this difference as it has led to occasional confusion in published literature.

Another motivation behind the present article is to provide an alternative to the functional Bethe Ansatz method [1, 3, 12, 13, 15–17]. Indeed, the eigenvalues, eigenfunctions and the allowed potential parameters were previously given exclusively in terms of the roots of a set of algebraic Bethe Ansatz equations of the functional Bethe Ansatz method [1, 3, 12, 13, 15–17]. It is demonstrated here that the set of algebraic Bethe Ansatz equations can be efficiently replaced by a recurrence [cf. Eq. (6) below] together with a single polynomial constraint \( P = 0 \) [cf. Eq. (8) below]. In general solving for the roots of \( P(n) = 0 \) determines an isolated finite set of points in parameter space at which polynomial solutions are possible.

In what follows, we first recapitulate the gradation slicing approach of Ref. [11] in Sec. III Then the approach is illustrated on the coupling constant metamorphosis QES examples in Sec. III and QES examples with energy as spectral parameter in Sec. IV Some important issues are discussed in Sec. V In particular, a relation to \( sl_2 \) algebraization and an algebraic Heun operator is discussed in Sec. VA A discussion of when \( P(n) \) has necessarily only real and simple roots can be found in Sec. VB A comparison of \( P(n) \) and the so-called weak orthogonal polynomials of Lancosz-Haydock and Bender-Dunne is provided in Sec. VC We then conclude with Sec. VI For the sake or presentation, a number of intermediary calculations has been relegated to online supplementary material.

II. SUMMARY OF GRADATION SLICING APPROACH

General necessary and sufficient conditions for the existence of a polynomial solution have been recently formulated involving constraint relations [11]. In the terminology of Ref. [11], the grade of a term \( z^m d_x^l \) is integer \( m - l \). One can straightforwardly identify that
the respective terms of the differential operator $\mathcal{L}$ on the left-hand side of Eq. (2) have the highest grade $\gamma = 1$, the lowest grade $\gamma_* = -1$, and can be assembled into three slices with the grades $1, 0, -1$ with the respective multiplicators

$$
F_1(n) = n(n-1)a_3 + nb_2 + c_1, \quad F_0(n) = n(n-1)a_2 + nb_1 + c_0, \\
F_{-1}(n) = n(n-1)a_1 + nb_0.
$$

(3)

In general, the necessary conditions for the ODE (2) with the grade $\gamma = 1$ to have a polynomial solution is that for some $n \in \mathbb{N}$

$$
F_1(n) = 0.
$$

(4)

Solving the condition (4) usually imposes a constraint on model parameters, which may include energy [11, 21]. The condition $F_1(n) = 0$ is known as the baseline condition for the Rabi models [11, 22] and for Jahn-Teller systems [21], because it constraints allowable energies to a set of lines, or hyperplanes, in a parameter space. The necessary baseline condition reappears also in the functional Bethe Ansatz method (cf. Theorems 4 and Remark 9 of Ref. [11]; Eqs. (1.8-10) of Ref. [19]), or as one of the conditions of $sl_2$ algebraization [6, 11, 20] [cf. Sec. V A for more details].

The necessary conditions for the ODE (2) with the grade $\gamma = 1$ to have a unique polynomial solution of degree $n \geq 1$ is that (cf. Theorems 1 and 2 of [11]),

$$
F_1(n) = 0, \quad F_1(k) \neq 0, \quad 0 \leq k < n.
$$

(5)

The conditions enable one to determine unique set of coefficients $\{P_{nk}\}_{k=0}^n$, defined recursively by the three-term recurrence relations (TTRR) for $1 \leq k \leq n$, beginning with $P_{n0} = 1$
(cf. Eq. (11) of Ref. [11])

\[ P_{n_1} = -F_0(n)P_{n_0}/F_1(n-1), \]
\[ P_{n_2} = -[F_{-1}(n)P_{n_0} + F_0(n-1)P_{n_1}]/F_1(n-2), \]
\[ \vdots \]
\[ P_{n,k} = -[F_{-1}(n + 2 - k)P_{n,k-2} + F_0(n + 1 - k)P_{n,k-1}]/F_1(n - k), \]
\[ \vdots \]
\[ P_{nn} = -[F_{-1}(2)P_{n,n-2} + F_0(1)P_{n,n-1}]/F_1(0). \]  

(6)

If the unique (monic) polynomial solution exists, then it is necessarily given by (cf. Theorems 1 and 2 of [11])

\[ S_n(z) = \prod_{i=1}^{n} (z - z_i) = \sum_{k=0}^{n} P_{n,n-k} z^k \quad (P_{n0} \equiv 1). \]  

(7)

The parameters entering the recurrence coefficients \( F_g(k) \) in (6) are assumed to satisfy the \( F_1(n) = 0 \) constraint.

The conditions (5) become both necessary and sufficient conditions for the ODE (2) to have a unique polynomial solution, provided that some subset of model parameters satisfying (4) obeys additionally (cf. Eq. (16) of Ref. [11])

\[ \mathcal{P}(n) := F_{-1}(1)P_{n,n-1} + F_0(0)P_{nn} = b_0P_{n,n-1} + c_0P_{nn} = 0. \]  

(8)

This equation can be seen as continuation of the TTRR (6) one step further by formally defining \( P_{n,n+1} = -\mathcal{P}(n) \).

The coefficients \( F_g(k) \) are polynomials in model parameters [e.g. examples (14), (19), (26), (35), (30), (40), (44), (50) below]. Hence \( \mathcal{P}(n) \) multiplied by \( \prod_{k=n-1}^{0} F_g(k) \neq 0 \) is necessarily a polynomial in model parameters, too. For the examples considered here it will be shown that the coefficients \( F_g(k) \) of Eq. (3) confined to a given baseline generate by the TTRR (6) a finite orthogonal polynomial system \( \{P_{nk}, k = 0, 1, 2, \ldots, n, \mathcal{P}(n)\} \) in some spectral parameter. The spectral parameter is a model parameter that does not enter the constraint \( F_1(n) = 0 \), and in fact none of multiplicators \( F_1(k) \). Hence a multiplication of
\( \mathcal{P}(n) \) by \( \prod_{k=n-1}^{0} F_\gamma(k) \neq 0 \) is in fact not necessary, because \( \mathcal{P}(n) \) is already a polynomial in the spectral parameter.

For the models considered here we have the following \textit{dichotomy}.

\textbf{(A1)} \( F_1(n) \) does depend on energy. Hence by solving the constraint \( F_1(n) = 0 \) energy can be expressed as a function of model parameters, \( E = E(V_j) \), and thereby eliminated from recurrence (6) and from the constraint polynomial (8). In these examples \( E \) is not spectral parameter and we have the above mentioned coupling constant metamorphosis. It turns out that the corresponding spectral parameter is a model parameter that does \textit{not} enter the constraint \( F_1(n) = 0 \). (For example, in the Manning potential case of Sec. III A (i) one fixes \( V_1 \) and \( V_2 \) together with energy \( E(V_1, V_2) \) and (ii) searches for the roots of the constraint polynomial (8) as a function of \( V_3 \) - cf. Figs. 1 [3].)

\textbf{(A2)} If only the multiplicator \( F_0(k) \) depends on energy, and is a \textit{linear} function of it, then \( E \) \textit{is} the spectral parameter.

An important characteristics of all examples considered here is that as the spectral parameter varies one stays at a \textit{fixed} point of an \( n \)th baseline. Constraint polynomial \( \mathcal{P}(n) \) will be shown to terminate a finite orthogonal polynomial system in corresponding spectral parameter. In the case of alternative \( \textbf{(A1)} \), and is some examples of alternative \( \textbf{(A2)} \), \( \mathcal{P}(n) \) will be shown to have only \textit{real} and \textit{simple} roots. The constraint polynomial relation (8) then determines a \textit{discrete} set of \( n + 1 \)th spectral parameter values at which polynomial solutions exist at any given \textit{fixed} point of the \( n \)th baseline. Thereby a set of algebraic Bethe Ansatz equations can be replaced by a single polynomial constraint (8).

The constraint relation (8) in the case of alternative \( \textbf{(A2)} \) provides a kind of quantization rule for the energy levels. The latter sounds similar to the role played by a critical polynomial of the Lanczos-Haydock finite-chain of polynomials [23, 24] (more known as the Bender-Dunne polynomials [25, 26]). Yet, as discussed in Sec. V.C such a resemblance is only coincidental.
Figure 1. Constraint polynomial for the Xie generalized Manning potential in the even parity case as a function of $V_3$ with fixed $V_1 = 1$, $V_2 = -50$, and $n = 10$. There are 11 real roots for $V_3 = 50.6499, 62.9912, 85.016, 117.499, 158.65, 208.126, 265.78, 331.54, 405.368, 487.239, 577.141$. They all correspond to the real eigenvalue $\sqrt{-E_{10}} = 3$ [cf. Eq. (12)]. For the real roots we have $V_1 > 0$, $V_2 < 0$, $V_3 > 0$. The double-well condition is satisfied for the lowest three values of $V_3$. Corresponding wave functions are shown in Fig. 2.

III. EXAMPLES OF $F_1(n)$ DEPENDING ON ENERGY RESULTING IN A COUPLING CONSTANT METAMORPHOSIS

A. A modified Manning potential with three parameters

In this section we examine parity invariant potential

$$V(x) = -V_1 \text{sech}^6x - V_2 \text{sech}^4x - V_3 \text{sech}^2x$$  \hspace{1cm} (9)

studied by Xie \cite{12}, which for $V_1 = 0$ reduces to the Manning potential \cite{14}. Obviously $\lim_{|x| \to \infty} V(x) = 0$. This potential describes a double-well potential whenever $V_1 > 0$, $V_2 < 0$, $V_3 > 0$ and $-V_3/(2V_2) < 1$. The two minima of the potential are then located at $x_\pm = \pm \text{arcsech} \sqrt{-V_3/(2V_2)}$. 

\hspace{1cm}
Figure 2. Even parity polynomial eigenfunctions for the Xie generalized Manning potential with fixed $V_1 = 1$, $V_2 = -50$, and $n = 10$ for the values of $V_3 = 50.6499, 62.9912, 85.016, 117.499, 158.65, 208.126, 265.78, 331.54, 405.368, 487.239, 577.141$ as in Fig. 1. They all correspond to the real eigenvalue $\sqrt{-E_{10}} = 3$ [cf. Eq. (12)]. The double-well condition is satisfied for the lowest three values of $V_3$.

1. **Even parity solutions**

The substitution

$$
\psi(x) = \exp \left( \frac{\sqrt{V_1}}{2} \tanh^2 x \right) (1 - \tanh^2 x)^{\frac{1}{2}} \phi(x)
$$

followed by the change in variable through $z = \tanh^2 x$ transform the Schrödinger equation (1) into (2) with

$$
a_2 = 4, \quad a_1 = -4, \\
b_2 = 4\sqrt{V_1}, \quad b_1 = 6 + 4(\sqrt{-E} - \sqrt{V_1}), \quad b_0 = -2, \\
c_1 = V_1 + V_2 + 3\sqrt{V_1} + 2\sqrt{V_1}\sqrt{-E}, \quad c_0 = \sqrt{-E} - E - \sqrt{V_1} - V_1 - V_2 - V_3.
$$

In the Ansatz (10) and further below the principal branch of fractional powers is assumed.
Because $c_1$ is energy dependent, the necessary condition (4),

$$F_1(n) = 4n\sqrt{V_1 + V_1 + V_2 + 3\sqrt{V_1 + 2\sqrt{V_1\sqrt{-E}}}} = 0,$$

forces energy onto a $n$th baseline,

$$\sqrt{-E_n} = -2n - \frac{V_1 + V_2 + 3\sqrt{V_1}}{2\sqrt{V_1}} - \frac{3}{2} \rightarrow -2(n + 1) - \frac{V_2}{2} \quad (V_1 \rightarrow 1). \quad (12)$$

Because $\lim_{|x| \rightarrow \infty} \tanh^2 x = 1$ and $1 - \tanh^2 x = \cosh^{-2} x$, the solutions expressed by the Ansatz (10) are normalizable for any polynomial $\phi(x)$ as long as $\sqrt{-E} > 0$. With a fixed value of $V_1 > 0$, the normalizability condition requires

$$V_2 < - [(4n + 3)\sqrt{V_1 + V_1}]. \quad (13)$$

On the $n$th baseline one has in virtue of (3)

$$F_1(k) = -4(n - k)\sqrt{V_1}, \quad F_0(k) = 2k[2k + 1 + 2(\sqrt{-E_n} - \sqrt{V_1})] + c_0(n), \quad F_{-1}(k) = -2k(2k - 1), \quad (14)$$

where, given $\sqrt{-E - E} = \sqrt{-E}(\sqrt{-E} + 1),$

$$c_0(n) = \left(2n + \frac{V_1 + V_2 + 3}{2\sqrt{V_1}} + \frac{3}{2}\right) \left(2n + \frac{V_1 + V_2 + \frac{1}{2}}{2\sqrt{V_1}} + \frac{1}{2}\right) - \sqrt{V_1} - V_1 - V_2 - V_3. \quad (15)$$

Being a linear function, $F_1(k)$ has for each $n$ only single zero. Hence the conditions (3) are satisfied and there is always a unique polynomial solution for a given fixed set of parameters.

Given the above expression for $c_0$, an obvious choice of independent variable, or spectral parameter, is $V_3$. The choice of $V_3$ immediately implies that one remains at a fixed point of the baseline, because neither the baseline nor resulting energy does not depend on the value of $V_3$. The choice of any of $\sqrt{V_1}$ and $V_2$ as independent variable would be analogous to what happens in search of the exceptional spectrum of the Rabi model [11, 21, 22]. This option is discussed later in Sec. V A.

It turned out straightforward to reproduce the even parity roots $V_3$ of the constraint polynomial in Tab. 1 of [3] for $n = 0$, $V_1 = 1$, $V_2 = -6$, $n = 1$, $V_1 = 1$, $V_2 = -12$, and
n = 2, V_1 = 1, V_2 = -18. It took not much effort to produce results of Fig. 1 showing the constraint polynomial as a function of V_3 for fixed V_1 = 1, V_2 = -50, and n = 10. Fig. 2 shows wave functions corresponding to the roots of the constraint polynomial of Fig. 1.

2. Odd parity solutions

Given that the odd parity solution has to have only odd powers of tanh x, replacing φ(x) in the Ansatz (10) by tanh x φ(x) leads to a grade γ = 1 and width w = 3 differential operator for the odd parity solutions,

\[
4z(z - 1)d_z^2 + \left\{ z \left[ 4z \sqrt{V_1} + 4(\sqrt{-E} - \sqrt{V_1}) + 10 \right] - 6 \right\} d_z
+ z \left[ V_1 + V_2 + \sqrt{V_1} (5 + 2\sqrt{-E}) \right] - E + 3\sqrt{-E} + 2 - (V_1 + V_2 + V_3 + 3\sqrt{V_1}),
\]

where \( d_z = d/dz \). The Schrödinger equation (1) is again transformed into (2) with

\[
\begin{align*}
& a_2 = 4, \quad a_1 = -4, \\
& b_2 = 4\sqrt{V_1}, \quad b_1 = 10 + 4(\sqrt{-E} - \sqrt{V_1}), \quad b_0 = -6, \\
& c_1 = V_1 + V_2 + 5\sqrt{V_1} + 2\sqrt{V_1}\sqrt{-E}, \\
& c_0 = -E + 3\sqrt{-E} + 2 - 3\sqrt{V_1} - V_1 - V_2 - V_3.
\end{align*}
\]

Because \( c_1 \) is energy dependent, the necessary condition (4),

\[
F_1(n) = 4n\sqrt{V_1} + V_1 + V_2 + 5\sqrt{V_1} + 2\sqrt{V_1}\sqrt{-E} = 0,
\]

forces energy onto a \( n \)th baseline,

\[
\sqrt{-E_n} = -2n - \frac{V_1 + V_2 + 5\sqrt{V_1}}{2\sqrt{V_1}} \rightarrow -2n - 3 - \frac{V_2}{2} \quad (V_1 \rightarrow 1).
\]

With a fixed value of \( V_1 > 0 \), the normalizability condition requires [cf. (13)]

\[
V_2 < -\left[ (4n + 5)\sqrt{V_1} + V_1 \right].
\]
Figure 3. Constraint polynomial for the Xie generalized Manning potential in the odd parity case as a function of $V_3$ with fixed $V_1 = 1$, $V_2 = -50$, and $n = 10$. There are 11 real roots for $V_3 = 38.8277, 58.8256, 83.2712, 116.335, 157.819, 207.504, 265.299, 331.158, 405.056, 486.981, 576.924$. They all correspond to the real eigenvalue $\sqrt{-E_{10}} = 2$ [cf. Eq. (18)]. The double-well condition is satisfied for the lowest three values of $V_3$. Corresponding wave functions are shown in Fig. 4.

On the $n$th baseline one has in virtue of (3)

\[
F_{1}(k) = -4(n - k)\sqrt{V_1}, \quad F_{0}(k) = 2k[2k + 3 + 2(\sqrt{-E_n} - \sqrt{V_1})] + c_0(n), \\
F_{-1}(k) = -2k(2k + 1),
\]

where, given $3\sqrt{-E} - E + 2 = (\sqrt{-E} + 2)(\sqrt{-E} + 1),

\[
c_0(n) = \left(2n + \frac{V_1 + V_2}{2\sqrt{V_1}} + \frac{3}{2}\right) \left(2n + \frac{V_1 + V_2}{2\sqrt{V_1}} + \frac{1}{2}\right) - 3\sqrt{V_1} - V_1 - V_2 - V_3.
\]

Again, any solution expressed by such an amended Ansatz will be normalizable for any polynomial $\phi(x)$ whenever $\sqrt{-E} > 0$. Fig. 3 shows the constraint polynomial as a function of $V_3$ for fixed $V_1 = 1$, $V_2 = -50$, and $n = 10$. Fig. 4 shows wave functions corresponding to the roots of the constraint polynomial of Fig. 3.
Figure 4. Polynomial eigenfunctions for the Xie generalized Manning potential in the odd parity case with fixed $V_1 = 1$, $V_2 = -50$, and $n = 10$ for the eleven values of $V_3 = 38.8277$, 58.8256, 83.2712, 116.335, 157.819, 207.504, 265.299, 331.158, 405.056, 486.981, 576.924 as in Fig. 3. They all correspond to the real eigenvalue $\sqrt{-E_{10}} = 2$ [cf. Eq. (18)]. The double-well condition is satisfied for the lowest three values of $V_3$.

B. Chen et al. modified Manning potential with three parameters

In this section we examine parity invariant potential

$$V(x) = \frac{V_1}{\cosh^2 x} + \frac{V_2}{1 + g \cosh^2 x} + \frac{V_3}{(1 + g \cosh^2 x)^2}$$  \hspace{1cm} (21)

studied by Chen et al. [13], which approximates the Manning potential [14] in the limit $g \gg 1$. As in the previous case, $\lim_{|x|\to\infty} V(x) = 0$.

1. Even parity solutions

The change in variable through $z = -\sinh^2 x$ and the substitution [13]

$$\psi(x) = (\cosh^2 x)^{\lambda_1} (1 + g \cosh^2 x)^{\lambda_2} \phi(z),$$  \hspace{1cm} (22)

$$\lambda_1 = \frac{1}{4} \left(1 + \sqrt{1 - 4V_1}\right) \geq \frac{1}{4}, \quad \lambda_2 = \frac{1}{2} \left[1 - \sqrt{1 + V_3/(1 + g)}\right].$$  \hspace{1cm} (23)
transform the Schrödinger equation (1) into (2) with \[13\]

\[a_3 = 1, \quad a_2 = -2 - 1/g, \quad a_1 = 1 + 1/g,
\]

\[b_2 = 2\lambda_1 + 2\lambda_2 + 1,
\]

\[b_1 = -1 - \frac{1}{2g} - (2\lambda_1 + \frac{1}{2}) \left(1 + \frac{1}{g}\right) - 2\lambda_2 = -\left(2\lambda_1 + 2\lambda_2 + \frac{3}{2} + \frac{2\lambda_1 + 1}{g}\right),
\]

\[b_0 = \frac{1+g}{2g}, \quad c_1 = (\lambda_1 + \lambda_2)^2 + \frac{E}{4},
\]

\[c_0 = -\frac{1+g}{4g} \left[2\lambda_1 + \frac{2\lambda_2 g - V_2}{1+g} - V_1 - \frac{V_3}{(1+g)^2} - 2\right]. \tag{24}\]

The Ansatz (22) provides a normalizable solution on the interval \(x \in (-\infty, \infty)\) for a polynomial \(\phi(z)\) of \(n\)-th degree if and only if \(\lambda_1 + \lambda_2 + n < 0\).

Because \(c_1\) is energy dependent, the necessary condition (4),

\[F_1(n) = n(n-1) + n(2\lambda_1 + 2\lambda_2 + 1) + (\lambda_1 + \lambda_2)^2 + \frac{E}{4} = 0,
\]

forces energy onto a \(n\)th baseline,

\[E_n = -4[n^2 + 2n(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2)^2] = -4(n + \lambda_1 + \lambda_2)^2. \tag{25}\]

On the \(n\)th baseline one has in virtue of (3) for \(0 \leq k + k_g < n\)

\[F_1(k) = k^2 - n^2 + 2(k-n)(\lambda_1 + \lambda_2) > (n-k)^2 > 0,
\]

\[F_0(k) = -k(k-1) \left(2 + \frac{1}{g}\right) - k \left(2\lambda_2 + 2\lambda_1 + \frac{3}{2} + \frac{2\lambda_1 + 1}{g}\right) + c_0(n),
\]

\[F_{-1}(k) = k(k-1) \left(1 + \frac{1}{g}\right) + \frac{1}{2} k \left(1 + \frac{1}{g}\right) = \frac{1+g}{2g} k(2k-1), \tag{26}\]

where \(k_g = 1, 0, -1\) denotes the subscript of corresponding \(F_{k_g}\), and

\[c_0(n) = -\frac{1+g}{4g} \left[2\lambda_1 + \frac{2\lambda_2 g - V_2}{1+g} - V_1 - \frac{V_3}{(1+g)^2} - 4(n + \lambda_1 + \lambda_2)^2\right].
\]

\(F_1(k)\) is quadratic function of \(k\) which has only single nonnegative root \(k = n\) [the other is \(k = -n - 2(\lambda_1 + \lambda_2) < 0\)]. Because \(F_1(k)\) has for each \(n\) only single nonnegative zero, the conditions (5) are satisfied and there can always be only a unique polynomial solution.

Given the definition (23) of \(\lambda_1\) it is obvious that one has to have \(V_1 \leq 1/4\) in order that \(\lambda_1 \in \mathbb{R}\). The latter restriction has been satisfied by all the cases (I to III) considered by
Figure 5. Constraint polynomial for the Chen et al. generalized Manning potential in the even parity case as a function of $V_2$ with fixed $V_1 = 0.09$, $V_3 = 400$, $g = 0.25$, and $n = 7$. There is the maximum number of 8 real zeros of the constraint polynomial: $V_2 = -378.075, -346.334, -325.892, -306.113, -272.536, -228.953, -176.075, -114.078$. Cor-

Figure 6. Polynomial eigenfunctions for the Chen et al. generalized Manning potential in the even parity case with fixed $V_1 = 1$, $V_3 = 400$, $g = 0.25$, and $n = 7$ for the eight values of $V_2 = -378.075, -346.334, -325.892, -306.113, -272.536, -228.953, -176.075, -114.078$ as in Fig. 5.
Chen et al. [13].

It turned out straightforward to reproduce the even parity roots \(V_2\) of the constraint polynomial in Tab. 1 of [4] for \(V_1 = 0.09, V_3 = 10, g = 0.25\) and \(n = 0, 1, 2, 3\). Fig. 5 shows the constraint polynomial as a function of \(V_2\) for fixed \(V_1 = 1, V_3 = 400, g = 0.25\), and \(n = 7\). Fig. 6 displays wave functions corresponding to the roots of the constraint polynomial of Fig. 5.

2. Odd parity solutions

Obviously the Ansatz (22) can lead to only even parity solutions. In order to arrive at odd parity solutions it is, given \(z = -\sinh^2 x\), expedient to modify the Ansatz by adding an extra \(\sinh x\) factor,

\[
\psi(x) = (\cosh x)^{2\lambda_1}(1 + g \cosh^2 x)^{\lambda_2} \sinh x \phi(z),
\]

(27)

with \(\lambda_1\) and \(\lambda_2\) as in (22). The Ansatz (27) yields a normalizable solution on the interval \(x \in (-\infty, \infty)\) for a polynomial \(\phi(z)\) of \(n\)-th degree if and only if \(\lambda_1 + \lambda_2 + n < -1/2\).

According to (63) and (64)

\[
\Delta B(z) = z^2 - \left(\frac{1+2g}{g} + 1 + \frac{1}{g}\right)z,
\]

\[
\Delta C(z) = z \left(\lambda_1 + \lambda_2 + \frac{1}{4}\right) - \left(\lambda_1 + \lambda_2 + \frac{1}{4}\right) \frac{1+g}{g} + \frac{\lambda_2}{g}.
\]

Therefore in the expressions in (24) the coefficients \(a_j\) remain the same, whereas the \(b_j\) and \(c_j\) coefficients are amended to

\[
b_2 = 2(\lambda_1 + \lambda_2 + 1),
\]

\[
b_1 = -\left[2\lambda_1 + 2\lambda_2 + \frac{7}{2} + \frac{2(\lambda_1+1)}{g}\right],
\]

\[
b_0 = \frac{3(1+g)}{2g},
\]

\[
c_1 = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1) + \frac{E+1}{4},
\]

\[
c_0 = \frac{-1+g}{4g} \left[6\lambda_1 + 4\lambda_2 + 1 + \frac{2\lambda_2 g - V_2}{1+g} - V_1 - \frac{V_3}{(1+g)\sigma} + E\right] + \frac{\lambda_2}{g}.
\]

(28)

Because \(c_1\) is energy dependent, the necessary condition (4),

\[
F_1(n) = n(n - 1) + 2n(\lambda_1 + \lambda_2 + 1) + (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1) + \frac{E+1}{4} = 0,
\]
Figure 7. Constraint polynomial for the Chen et al. generalized Manning potential in the odd parity case as a function of $V_2$ with fixed $V_1 = 0.09$, $V_3 = 400$, $g = 0.25$, and $n = 7$. There is the maximum number of 8 real zeros of the constraint polynomial: $V_2 = -374.929, -342.812, -316.597, -287.269, -248.489, -200.236, -142.792, -76.2691$. Polynomial eigenfunctions are shown in Fig. 8.

Figure 8. Polynomial eigenfunctions for the Chen et al. generalized Manning potential in the odd parity case with fixed $V_1 = 1$, $V_3 = 400$, $g = 0.25$, and $n = 7$ for the values of $V_2 = -374.929, -342.812, -316.597, -287.269, -248.489, -200.236, -142.792, -76.2691$ as in Fig. 7.
forces energy onto a nth baseline,

\[ E_n = -1 - 4[n^2 + n(2\lambda_1 + 2\lambda_2) + (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + 1)] = -1 - 4(n + \lambda_1 + \lambda_2)(n + \lambda_1 + \lambda_2 + 1). \]  

(29)

On the nth baseline one has in virtue of (3)

\[ F_1(k) = k(k - 1) - n(n - 1) + 2(k - n)(\lambda_1 + \lambda_2 + 1) > (n - k)^2 > 0, \]

\[ F_0(k) = -k(k - 1) \left( 2 + \frac{1}{g} \right) - k \left[ 2\lambda_2 + 2\lambda_1 + \frac{7}{2} + \frac{2(\lambda_1 + 1)}{g} \right] + c_0(n), \]

\[ F_{-1}(k) = k(k - 1) \left( 1 + \frac{1}{g} \right) + \frac{3}{2} k \left( 1 + \frac{1}{g} \right) = \frac{1 + 2g}{2g} k(2k + 1), \]  

(30)

where

\[ c_0(n) = -\frac{1 + g}{4g} \left[ 6\lambda_1 + 4\lambda_2 + 1 + \frac{2\lambda_2 g - V_2}{1 + g} - V_1 - \frac{V_3}{(1 + g)^2} - 1 \right] + \frac{\lambda_2}{g}. \]

Fig. 7 shows the constraint polynomial as a function of \( V_2 \) for fixed \( V_1 = 1, V_3 = 400, g = 0.25, \) and \( n = 7 \). Fig. 8 displays wave functions corresponding to the roots of the constraint polynomial of Fig. 5.

C. Electron in Coulomb and magnetic fields and relative motion of two electrons in an external oscillator potential

After an appropriate change of parameters, (i) the Schrödinger equation for electron in Coulomb and magnetic fields, (ii) the Klein-Gordon equation for electron in Coulomb and magnetic fields, and (iii) the three-dimensional Schrödinger equation for two electrons (interacting with Coulomb potential) in an external harmonic-oscillator potential with frequency \( \omega_{\text{ext}} \) can all be shown to have the same basic form [16]

\[ \frac{1}{2} \frac{d^2}{dr^2} - \frac{g(g - 1)}{2r^2} - \frac{1}{2} \omega^2 r^2 + \frac{\beta}{r} + \alpha \]  

\[ u(r) = 0. \]  

(31)

Here \( \beta, g \) and \( \omega \) (\( g, \omega > 0 \)) are real parameters, and \( \alpha \) is the eigenvalue of Eq. (31) [16]. The potential in the Schrödinger equation (31) is the only one here without a parity symmetry. Obviously \( \lim_{r \to \infty} V(r) = \infty \).
Figure 9. Constraint polynomial as a function of $\beta$ with fixed $g = 0.5$ and $n = 10$ for the problem defined by Eq. (34). There is the maximum number of 11 real zeros of the constraint polynomial arranged symmetrically around $\beta = 0$, namely $\beta = \pm 24.8502, \pm 18.676, \pm 13.0012, \pm 7.89603, \pm 3.50671, 0$.

After the change of variables: $x = \sqrt{2}\omega r$ and rescaling $\beta \rightarrow (\sqrt{2}/\omega)\beta$, Eq. (31) becomes:

$$\left[ \frac{d^2}{dx^2} - \frac{g(g-1)}{x^2} - \frac{x^2}{4} + \frac{\beta}{x} + \frac{\alpha}{\omega} \right] u(x) = 0.$$  

(32)

On substituting Ansatz

$$u(x) = x^g \exp(-x^2/4)\phi(x)$$  

(33)

into (32) one obtains

$$\left[ x \frac{d^2}{dx^2} + (2g-x^2) \frac{d}{dx} + (\epsilon x + \beta) \right] \phi(x) = 0,$$

(34)

where $\epsilon = \alpha/\omega - (g + 1/2)$ [16], which has again the form of Eq. (2). The Ansatz (33) yields a normalizable solution on the interval $x \in (0, \infty)$ for any polynomial $\phi(x)$, provided that $g > -1/2$. 

18
Figure 10. Wave functions given by the Ansatz (33) for the roots of the constraint polynomial shown in Fig. 9 ordered from the lowest till the highest one.

The necessary condition $F_1(n) = -n + \epsilon = 0$ forces energy onto a $n$th baseline, $\epsilon = n$. On the $n$th baseline one has in virtue of (3)

$$F_1(k) = n - k, \quad F_0(k) = \beta, \quad F_{-1}(k) = k(k - 1) + 2kg.$$  \hspace{1cm} (35)

The choice of $\beta$ as the spectral parameter is in virtue of $F_0(k) = \beta$ unavoidable here. Being a linear function, $F_1(k)$ has for each $n$ only single zero. Hence the conditions (5) are satisfied and there can always be only a unique polynomial solution.

The resulting equation is symmetric under simultaneous transformation $\beta \to -\beta$ and $x \to -x$. The latter implies that if $\phi(x)$ solves (34) for some $\beta_0$, then also $\phi(-x)$ is a solution of Eq. (34), but with the eigenvalue $-\beta_0$. In particular, the eigenvalue $\beta = 0$ is possible only for $n$ even if all the roots of $\mathcal{P}(n)$ are simple [$\mathcal{P}(n)$ has $n + 1$ roots]. The latter is explicitly manifested in the distribution of eigenvalues in Fig. 9. Fig. 10 displays wave functions corresponding to the roots of the constraint polynomial of Fig. 9.
IV. EXAMPLES OF ONLY $F_0(n)$ DEPENDING ON ENERGY

A. The hyperbolic Razavy potential

In this section we examine parity invariant potential (cf. Eq. (2.6) of Ref. \[18\])

$$V(x) = \frac{1}{8} \xi^2 [\cosh(4x) - 1] - (M + 1) \xi \cosh(2x) = \frac{1}{4} \xi^2 \sinh^2(2x) - (M + 1) \xi \cosh(2x), \quad (36)$$

$$\lim_{|x| \to \infty} V(x) = \infty. \text{ The Ansatz } [1]$$

$$\psi(x) = \exp \left( -\frac{\xi}{4} \cosh 2x \right) (\cosh^\alpha x) (\sinh^\beta x) \phi(x) \quad \text{(37)}$$

transforms the Schrödinger equation in virtue of (67) into

$$\left[d_x^2 + (-\xi \sinh 2x + 2\alpha \tanh x + 2\beta \coth x) d_x + E + (\alpha + \beta)^2 + M \xi \cosh(2x) - 2\xi (\alpha \sinh^2 x + \beta \cosh^2 x) \right] \phi = 0,$$

where $\alpha(\alpha - 1) = \beta(\beta - 1) = 0$ (i.e. $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$). Assuming the substitution $z = \cosh^2 x$, the Ansatz (37) yields a normalizable solution on the interval $x \in (-\infty, \infty)$ for any polynomial $\phi(z)$. The substitution $z = \cosh^2 x$ transforms the differential operator in (38) in virtue of (65) into

$$4z(z - 1) d_z^2 + [-4\xi z^2 + 4(\alpha + \beta + \xi + 1)z - 2(2\alpha + 1)] d_z$$

$$+ [2\xi(M - \alpha - \beta)z + E + (\alpha + \beta)^2 - \xi(M - 2\alpha)],$$

which is (2) with

$$a_2 = 4, \quad a_1 = -4,$$

$$b_2 = -4\xi, \quad b_1 = 4(\alpha + \beta + \xi + 1), \quad b_0 = -2(2\alpha + 1),$$

$$c_1 = 2\xi(M - \alpha - \beta), \quad c_0 = E + (\alpha + \beta)^2 + \xi(2\alpha - M).$$

(39)

The necessary condition $F_1(n) = -4n\xi + 2\xi(M - \alpha - \beta) = 0$ is solved by

$$M = 2n + \alpha + \beta.$$
Figure 11. Constraint polynomial for the hyperbolic Razavy potential as a function of energy $E$ with fixed $\xi = 0.5$, $\alpha = 0$, odd parity $\beta = 1$, and $n = 10$. There is the maximum number of 11 simple real roots $E = -441.066, -361.073, -289.084, -225.099, -169.121, -121.157, -81.2206, -49.3476, -25.6452, -9.23983, 6.55323$.

On the $n$th baseline one has in virtue of (3)

$$F_1(k) = 4\xi(n-k), \quad F_0(k) = 4k(k+\alpha+\beta+\xi) + c_0(n),$$

$$F_{-1}(k) = -2k(2k-1+2\alpha),$$

where

$$c_0(n) = E + (\alpha + \beta)^2 - \xi(2n + \beta - \alpha).$$

The even (odd) parity solutions given by the Ansatz (37) correspond to $\beta = 0$ ($\beta = 1$).

It turned out straightforward to reproduce energy levels $E_{n,\alpha,\beta}$ for $n, \alpha, \beta = 0, 1$ of the hyperbolic Razavy potential given in Eqs. (45), (47), (49), (52), (56), (58), (60), (64), (65) of [1]. Note in passing that when comparing our energy levels $E_{n,\alpha,\beta}$ against those in Ref. [1] one has to interchange $\alpha$ and $\beta$. Fig. 11 shows constraint polynomial as a function of $E$ for fixed $\xi = 0.5$, $\alpha = 0$, odd parity $\beta = 1$, and $n = 10$. Fig. 12 displays wave functions
Figure 12. Odd parity polynomial eigenfunctions for the hyperbolic Razavy potential given by the Ansatz (37) with \( \phi \) there being a polynomial in \( z = \cosh^2 x \) for fixed \( \xi = 0.5 \) and \( n = 10 \) for the 11 **simple real** roots \( E = -441.066, -361.073, -289.084, -225.099, -169.121, -121.157, -81.2206, -49.3476, -25.6452, -9.23983, 6.55323 \) of the constraint polynomial of Fig. 11.

corresponding to the roots of the constraint polynomial of Fig. 11.

**B. A double sinh-Gordon system**

The double sinh-Gordon (DSHG) parity invariant system (also called the bistable Razavy potential [3]) is characterized by the potential

\[
V(x) = [\xi \cosh(2x) - M]^2,
\]

where \( \xi \) and \( M \) are positive real parameters and \( \lim_{|x|\to\infty} V(x) = \infty \). The potential is one of the few double well problems in quantum mechanics which is QES.

The change of independent variable \( z = e^{2x} \) and

\[
\psi(z) = z^{\frac{1-M}{2}} \exp \left[-\frac{\xi}{4} \left(z + \frac{1}{z}\right)\right] \phi(z)
\]
Figure 13. Constraint polynomial for the DSHG with $\xi = 2$ on the 11th baseline corresponding to $M = 12$ is shown to have the maximum number of 12 simple real roots $E = 22.59494691, 22.59496818, 61.34425227, 61.35804569, 89.87448537, 91.28081517, 106.4782162, 117.0076415, 131.6165721, 147.9807662, 166.0915272, 185.7777543$ reproducing the results of Tab. 3 of Ref. [3].

transform the Schrödinger equation (1) into (2) with (cf. Appendix VIII)

$$
\begin{align*}
    a_2 &= 4, & a_1 &= 0, \\
    b_2 &= -2\xi, & b_1 &= 8 - 4M, & b_0 &= 2\xi, \\
    c_1 &= 2\xi(M - 1), & c_0 &= E + 1 - 2M - \xi^2.
\end{align*}
$$

The Ansatz (43) yields normalizable solutions on the interval $x \in (-\infty, \infty)$ for any polynomial $\phi(z)$.

The baseline condition $F_1(n) = -2n\xi + 2\xi(M - 1) = 0$ is satisfied by $n = M - 1$. Hence the Ansatz (43) will comprise polynomial powers of $z$ between $z^{-n/2} = e^{-nx}$ up to $z^{n/2} = e^{nx}$.

On the $n$th baseline one has in virtue of (3)

$$
\begin{align*}
    F_1(k) &= 2\xi(n - k), & F_0(k) &= -4k(n - k) + c_0(n), & F_{-1}(k) &= 2k\xi,
\end{align*}
$$

(44)
where $c_0(n) = E - \xi^2 - 2n - 1$.

It turned out straightforward to reproduce energy levels for the double sinh-Gordon system in Tab. 2, 3 of [3], which contain numerous energy levels and the energy levels splitting with $\xi = 2$ and $M$ between 1 and 12. Fig. 13 shows constraint polynomial for a double sinh-Gordon system for $n = 11$, corresponding to $\xi = 2$ and $M = 12$ of Ref. [3]. Fig. 14 displays wave functions corresponding to the roots of the constraint polynomial of Fig. 13.

Because $V(x)$ in (42) has even parity, the solutions has to have definite parity. Yet it is difficult to identify the parity of solutions on using the Ansatz (43). The latter will be answered in Sec. IV C on using the Ansatz Eq. (46) for the special case when $\alpha(\alpha - 1) = \beta(\beta - 1) \equiv 0$ [cf. the condition (48)], i.e. when $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$.
Figure 15. Constraint polynomial for the perturbed DSHG on the \( n = 11 \)th baseline for \( \alpha = 2 \), \( \beta = 0 \) (i.e. even parity states), and \( \xi = 2 \), corresponding to \( g(g+1) = 2 \) and \( h(h+1) \equiv 0 \) in the respective numerators of the potential (45). There is the maximal number of twelve simple real zeros \( E = 48.5067, 140.039, 223.425, 298.596, 365.435, 423.725, 472.987, 511.035, 534.418, 566.233, 609.075, 658.526 \).

C. A perturbed double sinh-Gordon system

Khare and Mandal [17] showed that after adding a parity invariant perturbation

\[
V_p = -\frac{g(g+1)}{\cosh^2 x} + \frac{h(h+1)}{\sinh^2 x}
\]

(45)

term to the DSHG potential [42], the resulting potential is still QES potential (cf. Eq. (41) of Ref. [17]). Because \( \sinh^2 x \) is singular at the origin, the singularity is usually tamed by imposing the restriction \(-1 < h \leq 0 \) on \( h \in \mathbb{R} \) [17], which limits the product \( h(h+1) \in (-0.25, 0) \). (For \( h(h+1) \leq -0.25 \) one has the familiar textbook “fall to the center” - a particle falls in the origin and one cannot prevent the spectrum from collapse by any means [27, 28].) On the other hand, \( \cosh^2 x \) is regular at the origin and the potential parameter \( g \in \mathbb{R} \) is unrestricted.
Figure 16. Even parity polynomial eigenfunctions for the perturbed DSHG given by the Ansatz (46) with \( \phi \) there being a polynomial in \( z = \cosh^2 x \) and fixed \( \alpha = 2, \beta = 0, \xi = 2 \), corresponding to the twelve simple real roots of the constraint polynomial of Fig. 15.

The Ansatz
\[
\psi(x) = \exp \left( -\frac{\xi}{2} \cosh 2x \right) (\cosh^\alpha x) (\sinh^\beta x) \phi(x),
\]
which differs from that of Eq. (37) in \( \xi \to 2\xi \), transforms the Schrödinger equation (1) in virtue of (69) into
\[
\left[ d_x^2 + 2(-\xi \sinh 2x + \alpha \tanh x + \beta \coth x) d_x + E - M^2 - \xi^2 + (\alpha + \beta)^2 \\
+ 2\xi(2\alpha - M + 1) + 4\xi(M - \alpha - \beta - 1) \cosh^2 x \right] \phi = 0,
\]
provided that
\[
\alpha(\alpha - 1) = g(g + 1), \quad \beta(\beta - 1) = h(h + 1).
\]
The condition determines for a given \( g \) and \( h \) a quadruplet of energy values characterized by \( \alpha = g + 1, -g \) and \( \beta = h + 1, -h \). The solutions expressed by the Ansatz (46) are normalizable on the interval \( x \in (-\infty, \infty) \) for any polynomial \( \phi(x) \).

Similarly to the hyperbolic Razavy potential of Sec. IV A, either substitution \( z = \cosh^2 x \) or \( z = \sinh^2 x \) transforms the Schrödinger equation into (2). With \( z = \cosh^2 x \), Eq. (47) is
Figure 17. Constraint polynomial for the perturbed DSHG on the \( n = 11 \)th baseline for \( \alpha = 2 \), \( \beta = 1 \) (i.e. odd parity states), and \( \xi = 2 \), corresponding to \( g(g+1) = 2 \) and \( h(h+1) \equiv 0 \) in the respective numerators of the potential (45). There is the maximal number of twelve simple real zeros \( E = 50.5262, 146.083, 233.507, 312.742, 383.69, 446.18, 499.874, 544.126, 580.222, 617.352, 661.546, 712.152 \)

transformed in virtue of (65) into (2) with (17)

\[
\begin{align*}
    a_2 &= 4, \quad a_1 = -4, \\
    b_2 &= -8\xi, \quad b_1 = 4(\alpha + \beta + 2\xi + 1), \quad b_0 = -2(2\alpha + 1), \\
    c_1 &= 4\xi(M - \alpha - \beta - 1), \quad c_0 = E - M^2 - \xi^2 + (\alpha + \beta)^2 + 2\xi(2\alpha - M + 1).
\end{align*}
\]

Note for consistency that the \( a_j \) and \( b_j \) coefficients here differ from those in Eq. (39) by the substitution \( \xi \rightarrow 2\xi \).

The necessary condition \( F_1(n) = -8n\xi + 4\xi(M - \alpha - \beta - 1) = 0 \) is solved by

\[
    M = 2n + \alpha + \beta + 1. \quad (49)
\]
On the $n$th baseline one has in virtue of (3)

$$F_1(k) = 8\xi(n-k), \quad F_0(k) = 4k(k+\alpha+\beta+2\xi) + c_0(n),$$

$$F_{-1}(k) = -2k(2k-1+2\alpha),$$

where

$$c_0(n) = E - (2n+1)(2n+1+2\alpha+2\beta) - \xi^2 + 2\xi(\alpha-\beta-2n).$$

Being a linear function, $F_1(k)$ in Eq. (50) has for each $n$ only single zero. Hence the conditions (5) are satisfied and there can always be only a unique polynomial solution.

The parity of solutions is controlled by the value of $\beta$: for even (odd) parity solutions $\beta$ has to be an even (odd) integer. Yet $\beta$ need not be an integer here [cf. Eq. (48)], in which case one has solutions in a parity invariant system without any definite parity. This weird and paradoxical behaviour has its origin in the well-known fact that for $h(h+1) \in (-0.25, 0)$ the potential problem involving the perturbation $V_p$ can only be well-defined (i) on the semi-infinite interval $x \in (0, \infty)$ and (ii) after imposing boundary condition $\lim_{x \to 0} \psi(x)/\sqrt{x} = 0$ at $x = 0$ [27, 28]. In what follows we do not want to go into the technical details here and plot
wave functions merely for the case $h(h+1) = 0$. Fig. 15 shows constraint polynomial for the perturbed DSHG on the $n = 11$th baseline with fixed $\alpha = 2$, $\beta = 0$, and $\xi = 2$, corresponding to $g(g+1) = 2$ and $h(h+1) = 0$ in the respective numerators of the potential (45). Fig. 16 displays even parity polynomial eigenfunctions of the perturbed DSHG corresponding to the twelve simple real roots of the constraint polynomial of Fig. 15. Similarly, Fig. 17 shows constraint polynomial for the perturbed DSHG on the $n = 11$th baseline with fixed $\alpha = 2$, $\beta = 1$, and $\xi = 2$, again corresponding to $g(g+1) = 2$ and $h(h+1) = 0$ in the respective numerators of the potential (45). Fig. 18 displays the odd parity polynomial eigenfunctions of the perturbed DSHG corresponding to the twelve simple real roots of the constraint polynomial of Fig. 17.

At the end of this section we want to show that the Ansatz (46) can be used to disentangle parity of the algebraic spectrum of the unperturbed DSHG parity invariant system of Sec. IVB. The unperturbed DSHG is covered by the Ansatz (46) as a special case for $\alpha(\alpha - 1) = \beta(\beta - 1) \equiv 0$ [cf. the condition (48)], i.e. when $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$. With $z = \cosh^2 x$, the baseline condition (49) can be satisfied for $M = 12$ provided that $n = 5$ and either (i) $\alpha = 1$ and $\beta = 0$ yielding even parity solutions, or (ii) $\alpha = 0$ and $\beta = 1$ yielding odd parity solutions. One finds, without any need of plotting wave functions as in Fig. 14, that the eigenvalues on the $n = 11$ baseline in the caption of Fig. 13 correspond to interlaced even and odd parity solution, beginning with the lowest energy even parity state.

V. DISCUSSION

Earlier approaches in determining exact solutions of the QES solvable models discussed here employed without exception the functional Bethe Ansatz method [1, 3, 4]. However, the latter requires a whole set of $n$ coupled algebraic equations to be solved simultaneously. For instance, the use of Bethe Ansatz allows to write eigenvalues for the hyperbolic Razavy potential formally as

$$E_{n,\alpha,\beta} = 4\xi \sum_{i=1}^{n} z_i - (\alpha + \beta)^2 + \xi(\alpha - \beta) - 4n \left(n + \alpha + \beta + \frac{\xi}{2}\right),$$

yet the roots $z_i$ remain to be determined by a set of $n$ coupled equations of the Bethe Ansatz. (Note in passing that the range of applicability of the functional Bethe Ansatz
method \[19\] has been recently expanded - cf. Theorem 4 and Remark 9 of Ref. \[11\].) For general values of \(n\) solving the system of Bethe Ansatz equation is difficult, and one must resort to numerical methods of solving a coupled set of equations \[29\]. Contrary to that, the gradation slicing was shown to be universal and easily applicable algorithmic recursive approach for obtaining polynomial solutions.

The list of potential considered here is far from being exhaustive. For a complete list of the potentials that can be brought to the form \(2\) see recent work by Turbiner \[6, 7\] and Ishkhanyan \[8–10\]. For example, both Xie and Chen et al. modified Manning potentials with three parameters are nothing but particular representative of \((1/2, 1/2, 0)\) class considered in Ref. \[8\]. The list includes QES potentials associated with the Pöschl-Teller potential, the generalized Pöschl-Teller potential, the Scarf potential, sextic oscillator and an anharmonic oscillator potential \[3\], and many further potentials, such as a number of spherically symmetric potentials \[2\] including a non-polynomial oscillator defined as

\[
V(r) = r^2 + \frac{\alpha r^2}{1 + \beta r^2},
\]

the screened Coulomb potential defined by,

\[
V(r) = \frac{\lambda}{r} + \frac{\delta}{r + \kappa}, \quad \lambda < -\delta,
\]

a singular integer power potential,

\[
V(r) = \frac{\lambda}{r^2} + \frac{\mu}{r^3} + \frac{\chi}{r^4} + \frac{\tau}{r^4},
\]

and a singular anharmonic potential

\[
V(r) = \omega r^2 + \frac{\epsilon}{r^2} + \frac{\sigma}{r^4} + \frac{\chi}{r^6},
\]

where all quantities different from independent variable \(r\) are various potential parameters \[2\].

In the case of both Xie and Chen et al. modified Manning potentials with three parameters we have succeeded in determining odd parity eigenstates. Note that the original Ansatz \[10\] by Xie \[12\] and the Ansatz \[22\] of Chen et al. \[13\] can capture only even parity
solutions. The odd parity solutions can be obtained by replacing \( \phi(x) \) in the Ansatz (10) by \( \tanh x \phi(x) \), and by modifying the Ansatz (22) of Chen et al. to (27) by adding an extra \( \sinh x \) factor. (Computational details have been relegated to the online supplementary material Secs. VIII A and VIII B.) Parity resolved solution for the DSHG system can be obtained by going from the Ansatz (43) to the Ansatz (46).

For both the hyperbolic Razavy potential of Sec. IV A and the perturbed double sinh-Gordon system of Sec. IV C either substitution of independent variable \( z = \cosh^2 x \) or \( z = \sinh^2 x \) is possible to transform the Schrödinger equation into (2). That is illustrated in the online supplementary material Sec. IX.

A. The condition of \( sl_2 \) algebraization and an algebraic Heun operator

As alluded to earlier, the baseline condition (4) reappears in the functional Bethe Ansatz method (cf. Theorem 4 and Remark 9 of Ref. [11]; Eqs. (1.8-10) of Ref. [19]), or as one of the conditions of \( sl_2 \) algebraization [6, 11, 20] - see e.g. the condition \( 2\nu(2\nu-1)a_3+2\nu b_2+c_1 = 0 \) for the \( sl(2,\mathbb{R}) \) spin \( \nu \) representation of the Heun operator of Turbiner [6, Eq. (6)],

\[
H_e = (a_3 z^3 + a_2 z^2 + a_1 z) d_z^2 + (b_2 z^2 + b_1 z + b_0) d_z + c_1 z,
\]

when recast in our notation [cf. Eq. (2)]. The operator (52) is defined up to additive constant \( c_0 \) – it is the reference point for the spectral parameter and coincides with the accessory parameter in the Heun equation [6]. When the baseline condition is satisfied, \( H_e \) can be recast in terms of the generators \( J \)'s of the \( sl(2,\mathbb{R}) \)-Lie algebra [6, Eq. (2)]

\[
H = t^{+0}J_+J_0 + t^{-0}J_-J_0 + t^{00}J_0^2 + t^{-0}J_0J_- + B^+J_+ + B^0J_0 + B^-J_-,
\]

where \( t^{+0}, t^{-0}, t^{00} \) and \( B^+, B^0, B^- \) are constants, with the correspondence

\[
a_3 = t^{+0}, \quad b_2 = t^{+0}(1 - 3\nu) + B^+, \quad c_1 = 2\nu(\nu t^{+0} - B^+).
\]

To each two different points of the baseline there correspond two different algebraic Heun operators, simply because they are determined by different constants \( t^{00}, B^+ \) in the expansion in terms of the generators \( J \)'s of the \( sl(2,\mathbb{R}) \)-Lie algebra. On the \( n \)-th baseline
energy $E$, and hence also the parameter $c_1$, even if it were formally dependent on energy, remain constant for the coupling constant metamorphosis QES examples of Sec. III. In particular, we have

$$c_1 = -4n\sqrt{V_1}$$

(55)

for the modified Manning potential with three parameters for both even [cf. Eqs. (11), (12)] and odd parity cases [cf. Eqs. (17), (18)]. For the Chen et al. modified Manning potential we have

$$c_1 = -n^2 - 2n(\lambda_1 + \lambda_2), \quad c_1 = -n^2 - n(2\lambda_1 + 2\lambda_2) - \frac{1}{4}$$

in the respective even parity case [cf. Eqs. (24), (25)] and odd parity case [cf. Eqs. (28), (29)]. For an electron in Coulomb and magnetic fields and relative motion of two electrons in an external oscillator potential

$$c_1 = \epsilon = \alpha/\omega - (g + 1/2).$$

On the other hand, spectral parameter $c_0$ depends on one of the other model parameters.

An illustration of what happens in search of the exceptional spectrum of various Rabi models [11, 21, 22] can be provided by the modified Manning potential with three parameters of Sec. IIIA by selecting $\sqrt{V_1}$ as an independent spectral variable. Any change of $\sqrt{V_1}$ induces a translation on the corresponding baseline in both the even [cf. Eq. (12)] and odd [cf. Eq. (18)] parity cases. During those translations, the value of $c_1$ varies according to (55) and the value of $b_2$ changes according to Eqs. (11), (17). Because of (52), (54), each different value of $b_2$, or $c_1$, corresponds to a different $sl_2$ operator $H$ in (53).

B. $\mathcal{P}(n)$ has only real and simple roots

Let us first introduce $p_{nk}$, $1 \leq k \leq n + 1$, through

$$P_{nk} = \frac{p_{nk}}{\prod_{l=1}^{k} F_1(n - l)}, \quad -\mathcal{P}(n) = \frac{p_{n,n+1}}{\prod_{l=1}^{n} F_1(n - l)},$$

32
while reminding that $F_1(n-l) \neq 0$ has been assumed for $1 \leq l \leq n$. Now our original TTRR (56), together with the definition of the constraint polynomial (8), can be recast as a TTRR

$$p_{nk} = -F_0(n+1-k)p_{n,k-1} - F_{-1}(n+2-k)F_1(n+1-k)p_{n,k-2}, \quad 1 \leq k \leq n+1, \quad (56)$$

with the initial condition $p_{n,-1} = 0$, $p_{n0} = 1$.

In what follows we compare (56) against the canonical TTRR for monic polynomials,

$$P_k(x) = (x-d_k)P_{k-1}(x) - \lambda_k P_{k-2}(x), \quad k \geq 1, \quad (57)$$

with the initial condition $P_{-1} = 0$, $P_0 = 1$. For any given $\{\lambda_k\}_{k=2}^N$, $\{d_k\}_{k=1}^N \in \mathbb{C}$ the TTRR (57) generates an orthogonal polynomial system (OPS) if and only if $\lambda_k \neq 0$, $k \geq 2$ (cf. Favard’s theorem - e.g. Theorem 4.4 of Chihara’s book [30]). Moreover:

(a) If $\{P_k(x)\}$ satisfies the TTRR (57) with $\lambda_k \neq 0$ for $2 \leq k \leq N$, then $P_k(x)$ and $P_{k-1}(x)$ cannot have a common zero for $k \leq N$ [30, Exercise 4.3]. If they had a common zero $x_0$, then necessarily $P_{k-2}(x_0) = P_{k-3}(x_0) = \ldots = P_0(x_0) = 0$. But that contravenes the initial condition $P_0(x) \equiv 1$.

(b) A unique moment functional $\mathcal{L}$ is positive definite if and only if $d_k$ and $\lambda_k > 0$ are real, and additionally $\lambda_k > 0$ ($k \geq 1$) [30, p. 22]. In the latter case [30, p. 22]

$$\mathcal{L}[P_k^2(x)] = \prod_{j=1}^{k+1} \lambda_j > 0, \quad k \geq 0. \quad (58)$$

Under the above conditions the zeros of $P_k(x)$ are (i) all real and simple, and (ii) located in the interior of the supporting set for $\mathcal{L}$ [30, Theorem 5.2]. Obviously, if (i) holds for the zeros of $P_k(x)$, the same is true also for the zeros of $P_k(-x)$. But the latter are generated with $x$ replaced by $-x$ in (57).

We can associate TTRR (56) with TTRR (57) by identifying $\lambda_k = F_{-1}(n+2-k)F_1(n+1-k)$ for $2 \leq k \leq n+1$. The baseline condition $F_1(n) = 0$ implies $\lambda_1 = 0$. Because such a $\lambda_1$ multiplies $p_{n,-1} \equiv 0$ in (56) nothing changes there if one assumes formally $\lambda_1 \neq 0$. Indeed, once the initial condition $P_{-1} = 0$ is imposed (57) one has a freedom to select $\lambda_1$ according to one needs. One finds that the following applies for the TTRR (56) for $2 \leq k \leq n+1$:...
1. Xie [12] modified Manning potential: TTRR (56) is equivalent to TTRR (57) with $x = V_3$, $\lambda_k = F_{-1}(n + 2 - k)F_1(n + 1 - k) > 0$ in (14), (19);

2. Chen et al. [13] modified Manning potential: TTRR (56) is equivalent to TTRR (57) with $x = -V_2/(4g)$, $\lambda_k = F_{-1}(n + 2 - k)F_1(n + 1 - k) > 0$, provided that $g > 0$ in (26), (30);

3. an electron in Coulomb and magnetic fields: TTRR (56) is equivalent to TTRR (57) with $x = -\beta$, $\lambda_k = F_{-1}(n + 2 - k)F_1(n + 1 - k) > 0$, provided that $g > 0$ in (35);

4. the hyperbolic Razavy potential: TTRR (56) is equivalent to TTRR (57) with $x = -E$, $\lambda_k = F_{-1}(n + 2 - k)F_1(n + 1 - k) < 0$, provided that $\alpha > -1/2, \xi > 0$ in (40);

5. the double sinh-Gordon system (DSHG): TTRR (56) is equivalent to TTRR (57) with $x = -E$, $\lambda_k = F_{-1}(n + 2 - k)F_1(n + 1 - k) > 0$, provided that $\xi > 0$ in (44);

6. the perturbed DSHG: TTRR (56) is equivalent to TTRR (57) with $x = -E$, $\lambda_k = F_{-1}(n + 2 - k)F_1(n + 1 - k) < 0$, provided that $\beta > -1/2$ and $\xi > 0$ in (50).

Therefore, for all the cases considered here the TTRR (56) defines a finite OPS $\{p_{nk}, k = 0, 1, 2, \ldots, n + 1\}$ satisfying at least the condition (a). Furthermore, in the 1st to 3rd and 5th case the above property (b) is also satisfied. Thus in those cases each polynomial of the finite OPS $\{p_{nk}, k = 0, 1, 2, \ldots, n + 1\}$, and correspondingly $\{P_{nk}, k = 0, 1, 2, \ldots, n, \mathcal{P}(n)\}$, is guaranteed to have only real and simple roots in a corresponding independent variable $x$ for any $n$th baseline. Even if the above root property need not to hold in general in the 4th and 6th case of the hyperbolic Razavy potential of Sec. IV A and a perturbed double sinh-Gordon system of Sec. IV C respectively, we could still observe it for the parameters considered.

C. $\mathcal{P}(n)$ vs weak orthogonal polynomials of Lanczos-Haydock and Bender-Dunne

If some $\lambda_N = 0$ in the TTRR (57), then one speaks about the so-called weak orthogonal polynomials [30, p. 23]. Examples of weak orthogonal polynomials are provided by the Lanczos-Haydock finite-chains of polynomials [23, 24], later rediscovered by Bender and Dunne [25, 26]. In the above cases a corresponding TTRR (57) determines the coefficients of
a sought polynomial solution (7) beginning from that of its lowest degree upwards, reflected by the initial conditions on the two coefficients of the lowest degree (cf. Eq. (5) of Ref. 25).

\[ P_{nn} = P_0 = 1 \quad \text{and} \quad P_{n,n-1} = P_1(E) = E. \] (59)

Contrary to that, a corresponding TTRR (57) in our case determines the coefficients of a sought polynomial solution (7) beginning from that of its highest degree downwards, which is reflected by the initial condition \( P_{n0} = P_n = 1 \), i.e. involving the coefficient of the highest degree of a sought polynomial solution (7). This bring us to two important differences relative to the weak orthogonal Bender-Dunne polynomials:

(i) First, we cannot guarantee in our case that the conditions (59) will be satisfied, simply because our TTRR (6), (8), or (56), run in the opposite direction. Consequently, one may well end up with, and cannot exclude that, e.g. \( P_{nn} = P_0 = 0 \).

(ii) Second, with \( \lambda_N = 0 \) in the TTRR (57), the quasi-exact energy eigenvalues are the roots of a critical polynomial \( P_N \) of a corresponding weak orthogonal polynomial sequence that is determining \( N \) energy levels in the \( N \)-dimensional polynomial subspace \( \{1, z, z^2, \ldots, z^{N-1}\} \). Hence the polynomial degree of solutions need not to be \( N \). Yet in our case all the polynomial solution on the \( n \)-baseline are of \( n \)th degree by construction. Therefore, our constraint polynomials \( \mathcal{P}(n) \) are not necessarily the critical polynomials of the weak orthogonal Bender-Dunne polynomials.

A TTRR may possess a unique minimal (or dominated) solution [31, 32]. It is interesting to recall that in the case when only the minimal solutions are the required physical solutions [33], then the whole physical spectrum of the model (i.e. including non-algebraic part of the spectrum) coincides with the support \( \mathcal{S} \) of a positive-definite moment functional \( \mathcal{L} \) of corresponding discrete orthogonal polynomials [33]. Therefore not only the algebraic part of the spectrum may be closely related to orthogonal polynomials.

VI. CONCLUSIONS

Recently developed general constraint polynomial approach was shown to replace a set of algebraic equations of the functional Bethe Ansatz method by a single polynomial constraint. As the proof of principle, the usefulness of the method has been demonstrated for
a number of quasi-exactly solvable potentials of the Schrödinger equation, enabling one to straightforwardly determine eigenvalues and wave functions.

Our constraint polynomials, which were shown to be different from the weak orthogonal Bender-Dunne polynomials, appear to be yet another class of polynomials closely related to the spectrum of quasi-exactly solvable models. For the models considered here, constraint polynomials terminated a finite chain of orthogonal polynomials characterized by a positive-definite moment functional $\mathcal{L}$, implying that a corresponding constraint polynomial has only real and simple zeros.

VII. ACKNOWLEDGMENTS

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Online supplementary material
VIII. GENERIC COORDINATE TRANSFORMATION

\[ \psi(x) = Q(z)\phi(z), \quad d_z Q(z) = K(z)Q(z), \quad z = f(x), \]

implies \( d_z^2 Q(z) = [K'(z) + K^2(z)]Q(z) \) and

\[
\begin{align*}
  d_z [Q(z)\phi(z)] &= Q(z)[\phi'(z) + K(z)\phi(z)], \\
  d_z^2 [Q(z)\phi(z)] &= Q(z)\{\phi''(z) + 2K(z)\phi'(z) + [K'(z) + K^2(z)]\phi\}, \\
  d_x &= f'(x)d_z, \\
  d_x^2 &= [f'(x)]^2 d_z^2 + f''(x)dz. \\
\end{align*}
\]

(60)

The Schrödinger equation (1) then becomes

\[
\begin{align*}
  [f'(x)]^2\phi''(z) + \{2[f'(x)]^2K(z) + f''(x)\} \phi'(z) \\
  + \{E - V + [f'(x)]^2[K'(z) + K^2(z)] + f''(x)K(z)\} \phi(z) &= 0.
\end{align*}
\]

(61)

As a slight variation of (61) we have with \( z = f(x) \) for

\[ \psi(x) = Q(x)\phi(z), \quad d_x Q(x) = K(x)Q(x), \quad d_x^2 Q(x) = [K'(x) + K^2(x)]Q(x), \]

\[
\begin{align*}
  d_x &= f'(x)d_z, \\
  d_x^2 &= [f'(x)]^2 d_z^2 + f''(x)dz, \\
  d_x Q(x)d_x \phi(z) &= Q(x)[K(x)f'(x)\phi'(z)], \\
  d_x^2 [Q(x)\phi(z)] &= Q(x)\{[f'(x)]^2\phi''(z) + f''(x)\phi'(z) + 2K(x)f'(x)\phi'(z) \\
  &\quad + [K'(x) + K^2(x)]\phi(z)\}, \\
\end{align*}
\]

and

\[
\begin{align*}
  [f'(x)]^2\phi''(z) + [2f'(x)K(x) + f''(x)] \phi'(z) + \{E - V + K'(x) + K^2(x)\} \phi(z) &= 0.
\end{align*}
\]

(62)
A. Xie modified Manning potential with three parameters and \( z = \tanh^2 x \)

In the case of the Ansatz (10) for the Xie modified Manning potential (9) with three parameters of Sec. III A,

\[
K(z) = \frac{\sqrt{V_1}}{2} - \frac{\sqrt{-E}}{2(1-z)}, \quad K'(z) = -\frac{\sqrt{-E}}{2(1-z)^2},
\]

\[
K'(z) + K^2(z) = \frac{V_1}{4} - \frac{\sqrt{V_1}\sqrt{-E}}{2(1-z)} - \frac{E+2\sqrt{-E}}{4(1-z)^2},
\]

\[
f'(x) = 2 \tanh x \text{ sech}^2 x,
\]

\[
[f'(x)]^2 = 4z(1-z)^2, \quad f''(x) = 2(1-z)^2 - 4z(1-z) = (1-z)(2-6z),
\]

\[
2[f'(x)]^2K(z) + f''(x) = (1-z) \left[ 4z(1-z)\sqrt{V_1} - 4z\sqrt{-E} - 6z + 2 \right].
\]

Hence from (61)

\[
A(z) = \frac{1}{1-z}[f'(x)]^2 = 4z(1-z),
\]

\[
B(z) = \frac{1}{1-z} \left\{ 2[f'(x)]^2K(z) + f''(x) \right\} = 4z(1-z)\sqrt{V_1} - 4z\sqrt{-E} - 6z + 2.
\]

Given that

\[
[f'(x)]^2[K'(z) + K^2(z)] = (1-z) \left[ V_1z(1-z) - 2z\sqrt{V_1}\sqrt{-E} - \frac{E+2\sqrt{-E}}{1-z} \right],
\]

\[
f''(x)K(z) = \sqrt{V_1}(1-z)(1-3z) - \sqrt{-E}(1-3z),
\]

\[
\frac{E}{1-z} - \frac{E+2\sqrt{-E}}{1-z} z - \frac{\sqrt{-E}}{1-z} (1-3z) = E - \sqrt{-E},
\]

we have eventually from (61)

\[
C(z) = \frac{1}{1-z} \left\{ E - V + [f'(x)]^2[K'(z) + K^2(z)] + f''(x)K(z) \right\}
\]

\[
= E - \sqrt{-E} + V_1(1-z)^2 + V_2(1-z) + V_3 + \sqrt{V_1}(1-3z)
\]

\[
+ V_1z(1-z) - 2z\sqrt{V_1}\sqrt{-E}
\]

\[
= E - \sqrt{-E} + V_1 - V_1z + V_2(1-z) + V_3 + \sqrt{V_1}(1-3z) - 2z\sqrt{V_1}\sqrt{-E}
\]

\[
= z(-V_1 - V_2 - 3\sqrt{V_1} - 2\sqrt{V_1}\sqrt{-E}) + E - \sqrt{-E}
\]

\[
+ V_1 + V_2 + V_3 + \sqrt{V_1}.
\]

One recovers the polynomial coefficients (11) by multiplying the current \( A(z) \), \( B(z) \), \( C(z) \)
by minus one.

Provided that $\phi(x)$ in the Ansatz (10) is replaced by $\tanh x \phi(x)$, we have the following changes in the above formulas:

$$K(z) = \sqrt{V_1} - \frac{\sqrt{-E}}{2(1-z)} + \frac{1}{2z}, \quad K'(z) = -\frac{\sqrt{-E}}{2(1-z)^2} - \frac{1}{2z^2},$$

$$\Delta [K'(z) + K^2(z)] = -\frac{1}{2z^2} + \frac{1}{4z^2} + \frac{1}{z} \left[ \frac{\sqrt{V_1}}{2} - \frac{\sqrt{-E}}{2(1-z)} \right]$$

$$= -\frac{1}{4z^2} + \frac{\sqrt{V_1}}{2z} - \frac{\sqrt{-E}}{2z(1-z)},$$

$$\Delta \{2[f'(x)]^2 K(z) + f''(x)\} = 2[f'(x)]^2 \Delta K(z) = 8z(1-z)^2 \frac{1}{2z} = 4(1-z)^2.$$

In order to recover the polynomial coefficients (17) for the odd parity Ansatz of Sec. III A 2 it suffices to focus only on the above changes indicated by $\Delta$. One finds immediately

$$A(z) = \frac{1}{1-z} [f'(x)]^2 = 4z(1-z),$$

$$B(z) = 4z(1-z) \sqrt{V_1} - 4z \sqrt{-E} - 6z + 2 + 4(1-z)$$

$$= -4z^2 \sqrt{V_1} - z(4\sqrt{-E} - 4\sqrt{V_1} + 10) + 6.$$ 

Given that

$$\Delta [f''(x)]^2 [K'(z) + K^2(z)] = (1-z)^2 \left( -\frac{1}{z} + 2\sqrt{V_1} - \frac{2z}{1-z} \right),$$

$$\Delta [f''(x)K(z)] = (1-z) \frac{1-z}{z},$$

$$\Delta C(z) = \frac{1-z}{z} - \frac{1-z}{z} + 2(1-z) \sqrt{V_1} - 2\sqrt{-E}$$

$$= -2z \sqrt{V_1} + 2\sqrt{V_1} - 2\sqrt{-E} - 2.$$ 

One recovers the polynomial coefficients (17) after multiplication of the current $A(z)$, $B(z)$, $C(z)$ by minus one.

**B. Chen et al. modified Manning potential with three parameters and $z = -\sinh^2 x$**

For the Ansatz (22) in the case of the Chen et al. modified Manning potential (21) with three parameters of Sec. III A on arrives at (62). Now with $z = f(x) = -\sinh^2 x$ and the
Ansatz (27),

\[ f'(x) = -\sinh 2x, \quad [f'(x)]^2 = \sinh^2 2x = 4 \sinh^2 x \cosh^2 x = 4z(z - 1), \]

\[ K(x) = 2\lambda_1 \tanh x + \frac{\lambda_2 g \sinh 2x}{1 + g \cosh^2 x} + \coth x, \]
\[ K'(x) = \frac{2\lambda_1}{\cosh^2 x} + \frac{2\lambda_2 g \cosh 2x}{1 + g \cosh^2 x} - \frac{\lambda_2 g^2 \sinh^2 2x}{(1 + g \cosh^2 x)^2} - \frac{1}{\sinh^2 x}, \]
\[ K^2(x) = \left(2\lambda_1 \tanh x + \frac{\lambda_2 g \sinh 2x}{1 + g \cosh^2 x} + \coth x\right)^2, \]
\[ \Delta[K'(z) + K^2(z)] = -\frac{1}{\sinh^2 z} - \coth^2 z + 4\lambda_1 + \frac{4\lambda_2 g \cosh^2 x}{1 + g \cosh^2 x} \]
\[ = 4\lambda_1 + 1 + \frac{4\lambda_2 g \cosh^2 x}{1 + g \cosh^2 x} = 4\lambda_1 + 4\lambda_2 + 1 - \frac{4\lambda_2}{1 + g \cosh^2 x}, \]
\[ \Delta[2f'(x)K(x) + f''(x)] = -2 \sinh 2x \coth x = -4 \cosh^2 x = 4(z - 1). \]

Here and below \(\Delta\) indicates the change of the term preceded by \(\Delta\) obtained from the Ansatz (27) relative to that resulting from the Ansatz (22).

On multiplying (62) by \(1 + g \cosh^2 x = 1 + g(1 - z)\) one finds the polynomial coefficient of \(\phi''(z)\),

\[ 4z(z - 1)[1 + g(1 - z)] = -4z[gz^2 - z(1 + 2g) + 1 + g] \]
\[ = -4g \left[z^3 - z^2 \left(2 + \frac{1}{g}\right) + 1 + \frac{1}{g}\right]. \]

One can reproduce the polynomial coefficient \(A(z)\) of \(\phi''(z)\) in Eqs. (24) after factoring out the prefactor \(-4g\). Similarly one determines \(\Delta B(z)\) from

\[ \Delta B(z) = -\frac{1}{4g} \Delta[2f'(x)K(x) + f''(x)][1 + g(1 - z)] \]
\[ = \frac{1}{g} (1 - z)[1 + g(1 - z)] = z^2 - z \frac{1 + 2g}{g} + \frac{1 + g}{g}, \quad (63) \]
and \( \Delta C(z) \) from

\[
\Delta C(z) = -\frac{1}{4g} \Delta[K'(z) + K^2(z)][1 + g(1 - z)]
= -\frac{1}{4g} \left[ 4\lambda_1 + 4\lambda_2 + 1 - \frac{4\lambda_2}{1 + g(1 - z)} \right][1 + g(1 - z)]
= \frac{1}{4g} \{ 4\lambda_2 - (4\lambda_1 + 4\lambda_2 + 1)[1 + g(1 - z)] \}
= z \left( \lambda_1 + \lambda_2 + \frac{1}{4} \right) - \left( \lambda_1 + \lambda_2 + \frac{1}{4} \right) \frac{1 + g}{g} + \frac{\lambda_2}{g}.
\] (64)

\[d_{\xi} = 2 \cosh x \sinh x \, d_z = \sinh 2x \, d_z,
\]
\[d_x^2 = d_x(\sinh 2x \, d_z) = 2 \cosh 2x \, d_z + \sinh^2 2x \, d_z^2 = 2 \cosh 2x \, d_z + 4(\sinh^2 x \, \cosh^2 x) \, d_z^2;
\]
\[= 2(2z - 1) \, d_z + 4z(z - 1) \, d_z^2,
\]
\[\sinh 2x \, d_x = \sinh^2 2x \, d_z = 4 \sinh^2 x \, \cosh^2 x \, d_z = 4z(z - 1) \, d_z.
\] (65)

For the hyperbolic Razavy potential \cite{Razavy}, and with \( \xi \to \xi/2 \) in the expression for \( Q(x) \) above, one finds

\[
-\frac{\xi^2}{4} \sinh^2 2x + (N + 1)\xi \cosh(2x) + \frac{\xi^2}{4} \sinh^2 2x + (\alpha + \beta)^2
- 2\xi(\alpha \sinh^2 x + \beta \cosh^2 x) - \xi \cosh(2x)
= N\xi \cosh(2x) + (\alpha + \beta)^2 - 2\xi(\alpha \sinh^2 x + \beta \cosh^2 x),
\] (66)

and

\[
\left[ d_x^2 - \frac{\xi^2}{4} \sinh^2 2x + (N + 1)\xi \cosh(2x) \right] (Q\phi) =
Q \left[ d_x^2 + (\xi \sinh 2x + 2\alpha \tanh x + 2\beta \coth x) \, d_x
+ E + N\xi \cosh(2x) + (\alpha + \beta)^2 - 2\xi(\alpha \sinh^2 x + \beta \cosh^2 x) \right] \phi.
\] (67)
Eventually one makes use of (65) to deduce that

\[d^2_x + (-\xi \sinh 2x + 2\alpha \tanh x + 2\beta \coth x) \, dx = 4z(z - 1)d^2_z + [2(2z - 1) - 4\xi z(z - 1) + 4\alpha(z - 1) + 4\beta z] \, dz. \tag{68}\]

D. DSHG

For the Ansatz (43) we have with \(z = e^{2x}\)

\[K(z) = \frac{1-M}{2z} - \frac{\xi}{4z^2} \left( z - \frac{1}{z} \right), \quad K'(z) = -\frac{1-M}{2z^2} - \frac{\xi}{2z^3}, \]

\[\left[ f'(x) \right]^2 = 4z^2, \quad f''(x) = 4z, \quad V(z) = \left[ \frac{\xi}{2} \left( z + \frac{1}{z} \right) - M \right]^2. \]

Hence from (61)

\[A(z) = \left[ f'(x) \right]^2 = 4z^2, \]

\[B(z) = 2\left[ f'(x) \right]^2 K(z) + f''(x) = 4z \left[ 1 - M - \frac{\xi}{2} \left( z - \frac{1}{z} \right) \right] + 4z \]

\[= -2\xi z^2 + 4z(2 - M) + 2\xi. \]

Given that

\[\left[ f'(x) \right]^2 K^2(z) - V(z) = \left[ 1 - M - \frac{\xi}{2} \left( z - \frac{1}{z} \right) \right]^2 - \left[ \frac{\xi}{2} \left( z + \frac{1}{z} \right) - M \right]^2 \]

\[= 1 + 2M\xi z - 2M - \xi \left( z - \frac{1}{z} \right) - \xi^2, \]

\[f''(x)K(z) = 4zK(z) = 2(1 - M) - \xi \left( z - \frac{1}{z} \right), \]

\[\left[ f'(x) \right]^2 K'(z) = -2(1 - M) - \frac{2\xi}{z}, \]

we have eventually from (61)

\[C(z) = E - V + \left[ f'(x) \right]^2 [K'(z) + K^2(z)] + f''(x)K(z) \]

\[= 2\xi(M - 1)z + E + 1 - 2M - \xi^2. \]
E. Perturbed DSHG

\[ Q := e^{-\frac{\xi}{2}} \cosh 2x (\cosh x)^\alpha (\sinh x)^\beta, \]
\[ Q' := (-\xi \sinh 2x + \alpha \tanh x + \beta \coth x) Q, \]
\[ Q'' := Q \left[ (-\xi \sinh 2x + \alpha \tanh x + \beta \coth x)^2 - 2\xi \cosh 2x + \frac{\alpha}{\cosh^2 x} - \frac{\beta}{\sinh^2 x} \right], \]
\[ (-\xi \sinh 2x + \alpha \tanh x + \beta \coth x)^2 = \]
\[ \xi^2 \sinh^2 2x + \alpha^2 \tanh^2 x + \beta^2 \coth^2 x - 4\xi \alpha \sinh^2 x - 4\xi \beta \cosh^2 x + 2\alpha \beta, \]
\[ \alpha^2 \tanh^2 x + \frac{\alpha}{\cosh^2 x} = \alpha^2 - \frac{\alpha(\alpha - 1)}{\cosh^2 x}, \]
\[ \beta^2 \coth^2 x - \frac{\beta}{\sinh^2 x} = \beta^2 + \frac{\beta(\beta - 1)}{\sinh^2 x}, \]
\[ -(\xi \cosh 2x - M)^2 + \xi^2 \sinh^2 2x - 4\xi \alpha \sinh^2 x - 4\xi \beta \cosh^2 x + (\alpha + \beta)^2 - 2\xi \cosh 2x = -\xi^2 + 2\xi(2z - 1)M - M^2 + (\alpha + \beta)^2 - 4\xi \alpha(z - 1) - 4\xi \beta z - 2\xi(2z - 1) \]
\[ = -\xi^2 - M^2 + (\alpha + \beta)^2 + 2\xi(2\alpha - M + 1) + 4\xi z(M - \alpha - \beta - 1), \]

where \( z = \cosh^2 x \). Hence

\[ \left[ d_x^2 - (\xi \cosh 2x - M)^2 \right] (Q\phi) = \]
\[ Q \left[ d_x^2 + 2 (-\xi \sinh 2x + \alpha \tanh x + \beta \coth x) d_x - M^2 - \xi^2 + (\alpha + \beta)^2 + 2\xi(2\alpha - M + 1) + 4\xi z(M - \alpha - \beta - 1) \right. \]
\[ \left. - \frac{\alpha(\alpha - 1)}{\cosh^2 x} + \frac{\beta(\beta - 1)}{\sinh^2 x} \right] \phi. \]  

(69)

Eventually one makes use of (65) to deduce that

\[ d_x^2 + 2 (-\xi \sinh 2x + \alpha \tanh x + \beta \coth x) d_x = \]
\[ 4z(z - 1)d_x^2 + [2(2z - 1) - 8\xi z(z - 1) + 4\alpha(z - 1) + 4\beta z] d_z. \]

The latter differs from (68) by the substitution \( \xi \to 2\xi \).
IX. INDEPENDENT VARIABLE \( z = \sinh^2 x \)

For both the hyperbolic Razavy potential of Sec. [IV.A] and the perturbed double sinh-Gordon system of Sec. [IV.C] either substitution of independent variable \( z = \cosh^2 x \) or \( z = \sinh^2 x \) is possible to transform the Schrödinger equation into (2). The former substitution was used in the main text. Here we illustrate the possibility of the latter. The substitution of independent variable \( z = \sinh^2 x \) implies on recalling elementary formulas

\[
2 \cosh x \sinh x = \sinh 2x, \quad \cosh 2x = 2 \sinh^2 x + 1, \\
\sinh^2 2x = 4 \cosh^2 x \sinh^2 x, \\
\]

\[d_x = 2 \cosh x \sinh x \, d_x = \sinh 2x \, d_x, \]

\[d_x^2 = 2 \cosh 2x \, d_z + 4(\sinh^2 x \cosh^2 x) d_z^2 = 2(2z + 1) d_z + 4(z + 1) d_z^2, \]

\[\sinh 2x \, d_x = 4 \sinh^2 x \cosh^2 x \, d_z = 4z(z + 1) d_z. \tag{70}\]

For the hyperbolic Razavy potential of Sec. [IV.A] the neglected possibility of the substitution \( z = \sinh^2 x \) implies in virtue of (70) that the Schrödinger equation is transformed into

\[
4z(z + 1) d_z^2 + [-4\xi z^2 + 4(\alpha + \beta - \xi + 1) z + 2(2\beta + 1)] d_z \]

\[+ [2\xi(N - \alpha - \beta) z + E + (\alpha + \beta)^2 + \xi(N - 2\beta)], \]

which is (2) with

\[
a_2 = 4, \quad a_1 = 4, \\
b_2 = -4\xi, \quad b_1 = 4(\alpha + \beta - \xi + 1), \quad b_0 = 2(2\beta + 1), \\
c_1 = 2\xi(N - \alpha - \beta), \quad c_0 = E + (\alpha + \beta)^2 + \xi(N - 2\beta). \]

The necessary condition \( F_1(n) = -4n\xi + 2\xi(N - \alpha - \beta) = 0 \) remains the same and is solved as before by \( N = 2n + \alpha + \beta \). On the \( n \)th baseline one has a slightly modified versions of
and (41),

\[ F_1(k) = 4\xi(n - k), \quad F_0(k) = 4k(k + \alpha + \beta - \xi) + c_0(n), \]
\[ F_{-1}(k) = 2k(2k - 1 + 2\beta), \] \hspace{1cm} (71)

where \( c_0(n) = E + (\alpha + \beta)^2 + \xi(n + \alpha - \beta). \) Being a linear function, \( F_1(k) \) in Eqs. (40), (71) has for each \( n \) only single zero. Hence the conditions (5) are satisfied and there can always be only a unique polynomial solution for a given fixed set of parameters.

For the perturbed double sinh-Gordon system of Sec. [IV C], the substitution \( z = \sinh^2 x \) transforms Eq. (47) in virtue of (70) into (2) with

\[ a_2 = 4, \quad a_1 = 4, \]
\[ b_2 = -8\xi, \quad b_1 = 4(\alpha + \beta - 2\xi + 1), \quad b_0 = 2(2\beta + 1), \]
\[ c_1 = 4\xi(M - \alpha - \beta - 1), \quad c_0 = E - M^2 - \xi^2 + (\alpha + \beta)^2 + 2\xi(M - 2\beta - 1). \] \hspace{1cm} (72)

The necessary condition \( F_1(n) = -8n\xi + 4\xi(M - \alpha - \beta - 1) = 0 \) remains the same as before and is solved by \( M = 2n + \alpha + \beta + 1. \) On the \( n \)th baseline one has in virtue of (3)

\[ F_1(k) = 8\xi(n - k), \quad F_0(k) = 4k(k + \alpha + \beta - 2\xi + 1) + c_0(n), \]
\[ F_{-1}(k) = 2k(2k - 1 + 2\beta), \] \hspace{1cm} (73)

where

\[ c_0(n) = E - (2n + \alpha + \beta + 1)^2 - \xi^2 + (\alpha + \beta)^2 + 2\xi(\alpha - \beta + 2n) \]
\[ = E - (2n + 1)(2n + 1 + 2\alpha + 2\beta) - \xi^2 + 2\xi(\alpha - \beta + 2n) \]

is, up, to a different sign of \( 2n \) in the last parenthesis, the same as in Eq. (51).