A Boundedness Result
for Toric Log Del Pezzo Surfaces

Dimitrios I. Dais and Benjamin Nill

Abstract. In this paper we give an upper bound for the Picard number of
the rational surfaces which resolve minimally the singularities of toric log Del
Pezzo surfaces of given index \( \ell \). This upper bound turns out to be a quadratic
polynomial in the variable \( \ell \).

Mathematics Subject Classification (2000). 14M25, 52B20; 14J26, 14Q10.
Keywords. Log del Pezzo surfaces, Picard number, toric varieties.

1. Introduction

A normal complex surface \( X \) with at worst log terminal singularities, i.e., quotient
singularities, is called log Del Pezzo surface if its anticanonical divisor \(-K_X\) is a
\( \mathbb{Q} \)-Cartier ample divisor. The index of such a surface is defined to be the smallest
positive integer \( \ell \) for which \( \ell K_X \) is a Cartier divisor. Every log Del Pezzo surface
is isomorphic to the anticanonical model (in the sense of Sakai [13]) of the rational
surface obtained by its minimal desingularization. The following Theorem is due
to Nikulin [8] (for related results cf. [1, 9]):

Theorem 1.1. Let \( X \) be a log Del Pezzo surface of index \( \ell \) and \( \tilde{X} \to X \) be its
minimal desingularization. Then the Picard number \( \rho(\tilde{X}) \) of \( \tilde{X} \) (i.e., the rank of
its Picard group) is bounded by

\[
\rho(\tilde{X}) < c \cdot \ell^2,
\]

where \( c \) is an absolute constant.

The toric log Del Pezzo surfaces, i.e., those which are equipped with an algebraic
action of a 2-dimensional algebraic torus \( \mathbb{T} \), and contain an open dense \( \mathbb{T} \)-orbit,
constitute a special subclass within the entire class of all log Del Pezzo surfaces.
(For instance, in the toric case, only cyclic quotient singularities can occur.) To
indicate how these two classes differ in practice, it would be enough to recall some known results for log Del Pezzo surfaces with Picard number = 1 and index $\ell \leq 2$:

(i) Excluding the “exceptional” 2$D_4$-case, there exist, up to isomorphism, exactly 30 surfaces of this kind having index $\ell = 1$ (see [2, Thm. 4.3] or [14, Thm. 1.2]). Among them there are 16 having at worst cyclic quotient singularities. By [4, Thm. 6.10] we see that only 5 out of these 16 surfaces are toric (associated to the 5 reflexive triangles).

(ii) Up to isomorphism, there exist exactly 18 surfaces of this kind having index $\ell = 2$ (see [2, Thm. 4.2] or [5, Thm. 1.1 (1)]). Among them there are 14 having only cyclic quotient singularities. By [4, Thm. 6.12] we see that only 7 out of these 14 surfaces are toric.

The purpose of this paper is to prove an analogue of (1.1) for toric log Del Pezzo surfaces of given index.

**Theorem 1.2.** Let $X_Q$ be a toric log Del Pezzo surface of index $\ell$ (associated to the lattice polygon $Q$) and $\tilde{X}_Q \to X_Q$ be its minimal desingularization. Then $\rho(\tilde{X}_Q)$ is bounded as follows:

\[ \rho(\tilde{X}_Q) \leq \begin{cases} 7, & \text{if } \ell = 1, \\ 8\ell^2 - 6\ell + 3, & \text{if } \ell \geq 2. \end{cases} \tag{1.2} \]

Our proof uses tools from toric and discrete geometry.

**Acknowledgment.** The second author is a member of the Research Group Lattice Polytopes, led by Christian Haase and supported by Emmy Noether fellowship HA 4383/1 of the German Research Foundation (DFG).

### 2. Toric log Del Pezzo surfaces

Let $Q \subset \mathbb{R}^2$ be a (convex) polygon. Denote by $\mathcal{V}(Q)$ and $\mathcal{F}(Q)$ the set of its vertices and the set of its facets (edges), respectively. $Q$ will be called an LDP-polygon if it contains the origin in its interior, and its vertices belong to $\mathbb{Z}^2$ and are primitive. If $Q$ is an LDP-polygon, we shall denote by $X_Q$ the compact toric surface constructed by means of the fan

\[ \Delta_Q := \{ \text{the cones } \sigma_F \text{ together with their faces } | \ F \in \mathcal{F}(Q) \}, \]

where $\sigma_F := \{ \lambda x | x \in F \text{ and } \lambda \in \mathbb{R}_{\geq 0} \}$ for all $F \in \mathcal{F}(Q)$. It is known (cf. [4, Remark 6.7]) that every toric log Del Pezzo surface is isomorphic to an LDP-polygon $Q$. Moreover, every cone $\sigma_F$ is lattice-equivalent to the cone $\mathbb{R}_{\geq 0}^{(1)} + \mathbb{R}_{\geq 0}^{(p_F, q_F)}$, for suitable relatively prime integers $p_F, q_F$, with $0 \leq p_F < q_F$. (These are uniquely determined, up to replacement of $p_F$ by its socius $\tilde{p}_F$, i.e., by the integer $0 \leq \tilde{p}_F < q_F$, satisfying $\gcd(\tilde{p}_F, q_F) = 1$ and $p_F\tilde{p}_F \equiv 1 \mod q_F$.) The affine toric variety $U_F := \text{Spec}(\mathbb{C}[\sigma_F^\vee \cap (\mathbb{Z}^2)^\vee])$ (where $\sigma_F^\vee$ denotes the dual cone of $\sigma_F$ and $(\mathbb{Z}^2)^\vee$ the dual lattice of $\mathbb{Z}^2$) is $\cong \mathbb{C}^2$ only if $q_F = 1$. Otherwise, the orbit $\text{orb}(\sigma_F) \in U_F$ of $\sigma_F$, i.e., the single point remaining fixed under the canonical
action of the algebraic torus $T := \text{Hom}_\mathbb{Z}(\mathbb{Z}^2, \mathbb{C}^*)$ on $U_F$, is a cyclic quotient singularity. In particular, $U_F \cong \mathbb{C}^2 / G_F = \text{Spec}(\mathbb{C}[z_1, z_2]^{|G_F|})$, with $G_F \subset \text{GL}(2, \mathbb{C})$ denoting the cyclic group of order $q_F$ which is generated by $\text{diag}(\zeta_{q_F}^{-p_F}, \zeta_{q_F})$ (for $\zeta_{q_F}$ a $q_F$-th root of unity). Hence, the singular locus of $X_Q$ equals

$$\text{Sing}(X_Q) = \{ \text{orb}(\sigma_F) | F \in I_Q \},$$

where $I_Q := \{ F \in \mathcal{F}(Q) | q_F > 1 \}$. Its subset $\{ \text{orb}(\sigma_F) | F \in \bar{I}_Q \}$, with $\bar{I}_Q$ defined to be $\bar{I}_Q := \{ F \in I_Q | p_F = 1 \}$, is the set of the Gorenstein singularities of $X_Q$.

The minimal desingularization of the surface $X_Q$ can be described as follows: Equip the minimal generators of $\Delta_Q$ with an order (e.g., anticlockwise), and assume that for every $F \in \mathcal{F}(Q)$ the cone $\sigma_F$ has $\nu(F), \nu'(F) \in \mathbb{Z}^2$ as minimal generators $(\sigma_F = \mathbb{R}_{\geq 0} \nu(F) + \mathbb{R}_{\geq 0} \nu'(F))$, with $\nu(F)$ coming first w.r.t. this order. Next, for all $F \in I_Q$, consider the negative-regular continued fraction expansion of $\frac{q_F}{q_F - p_F}$

$$\frac{q_F}{q_F - p_F} = [b_1^{(F)}, b_2^{(F)}, \ldots, b_{s_F}^{(F)}] := b_1^{(F)} - \frac{1}{b_2^{(F)} - \frac{1}{\ddots - \frac{1}{b_{s_F}^{(F)}}}}, \quad (2.1)$$

and define $u_0^{(F)} := \nu(F), u_1^{(F)} := \frac{1}{q_F - p_F}(q_F - p_F)\nu(F) + \nu'(F)$, and lattice points $\{u_j^{(F)} | 2 \leq j \leq s_F + 1 \}$ by the formulae

$$u_{j+1}^{(F)} := b_j^{(F)} u_j^{(F)} - u_{j-1}^{(F)}, \quad \forall j \in \{1, \ldots, s_F \}.$$

It is easy to see that $u_{s_F+1}^{(F)} = \nu'(F)$, and that the integers $b_j^{(F)}$ are $\geq 2$, for all $j \in \{1, \ldots, s_F \}$. The singularity $\text{orb}(\sigma_F) \subset U_F$ is resolved minimally by the proper birational map induced by the refinement $\{ \mathbb{R}_{\geq 0} u_j^{(F)} + \mathbb{R}_{\geq 0} u_{j+1}^{(F)} | 0 \leq j \leq s_F \}$ of the fan which is composed of the cone $\sigma_F$ and its faces. The exceptional divisor is $E^{(F)} := \sum_{j=1}^{s_F} E_j^{(F)}$, having

$$E_j^{(F)} := \overline{\text{orb}(\mathbb{R}_{\geq 0} u_j^{(F)})} (\cong \mathbb{P}^1_{\mathbb{C}}), \quad \forall j \in \{1, \ldots, s_F \},$$

(i.e., the closures of the $T$-orbits of the “new” rays) as its components, with self-intersection number $(E_j^{(F)})^2 = -b_j^{(F)}$ (see [12] Cor. 1.18 and Prop. 1.19, pp. 23-25).

**Note 2.1.** (i) If $F \in \mathcal{F}(Q)$, and $\eta_F \in (\mathbb{Z}^2)^\vee$ is its unique primitive outer normal vector, we define its local index to be the positive integer $l_F := \langle \eta_F, F \rangle$, where

$$\langle \cdot, \cdot \rangle : \text{Hom}(\mathbb{R}^2, \mathbb{R}) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

is the usual inner product. For $F \in \mathcal{F}(Q) \setminus I_Q$ we have obviously $l_F = 1$. For $F \in I_Q$, let $K(E^{(F)})$ be the local canonical divisor of the minimal resolution of
orb(σ_F) ∈ U_F (in the sense of [4, p. 75]). K(E^{(F)}) is a Q-Cartier divisor (a rational linear combination of E^{(F)}_j)'s), and

\[ l_F = \min \left\{ \xi \in \mathbb{N} \mid \xi K(E^{(F)}) \text{ is a Cartier divisor} \right\} = \frac{q_F}{\gcd(q_F, p_F - 1)}. \]  \hfill (2.2)

By virtue of (2.2) and (2.4) the index \( K \) of the (smallest) denominators of the (rational) coordinates of the vertices of \( \sigma_F \), and by

\[ m_F := \dim C(\mathfrak{m}_{X_\sigma, orb(\sigma_F)}/(\mathfrak{m}_{X_\sigma, orb(\sigma_F)})^2) - 1 \]

its multiplicity, it is known (cf. [3, Satz 2.11, p. 347]) that

\[ m_F = 2 + \sum_{j=1}^{s_F} (b_j^{(F)} - 2). \]  \hfill (2.3)

**Lemma 2.2.** For all \( F \in I_Q \) we have

\[ m_F \leq 2l_F. \]

*Proof.* See [7, Lemma 1.1 (iii), p. 235]. \( \Box \)

**Lemma 2.3.** For all \( F \in I_Q \) the self-intersection number of \( K(E^{(F)}) \) equals

\[ K(E^{(F)})^2 = -2 - \frac{(2 - p_F + p_F)}{q_F} + (m_F - 2). \]

*Proof.* Follows from [4, Corollary 4.6, p. 96] and formula (2.3). \( \Box \)

The minimal desingularization \( \varphi : \tilde{X}_Q \rightarrow X_Q \) of \( X_Q \) is constructed by means of the smooth compact toric surface \( \tilde{X}_Q \) which is defined by the fan

\[ \tilde{\Delta}_Q := \left\{ \text{the cones } \{ \sigma_F \mid F \in \mathcal{F}(Q) \setminus I_Q \} \text{ and } \{ \mathbb{R}_{\geq 0} u_j^{(F)} + \mathbb{R}_{\geq 0} u_{j+1}^{(F)} \mid F \in I_Q, j \in \{0, 1, \ldots, s_F\} \}, \right\} \]

together with their faces

(Refining each of the cones \( \{ \sigma_F \mid F \in I_Q \} \) of \( \Delta_Q \) as mentioned above). Furthermore, the corresponding discrepancy divisor equals

\[ K_{\tilde{X}_Q} - \varphi^* K_{X_Q} = \sum_{F \in I_Q} K(E^{(F)}). \]  \hfill (2.4)

(By \( K_{X_Q}, K_{\tilde{X}_Q} \) we denote the canonical divisors of \( X_Q \) and \( \tilde{X}_Q \), respectively.)

**Note 2.4.** By virtue of (2.2) and (2.3) the index \( \ell \) of \( X_Q \) (as defined in (11)) equals

\[ \ell = \text{lcm} \{ l_F \mid F \in \mathcal{F}(Q) \}. \]  \hfill (2.5)

(For simplicity, sometimes \( \ell \) is referred as index of \( Q \).) In fact, if we denote by

\[ Q^* := \{ y \in \text{Hom}_R(\mathbb{R}^2, \mathbb{R}) \mid \langle y, x \rangle \leq 1, \ \forall x \in Q \} \]

the polar of the polygon \( Q \), the index \( \ell \) is nothing but \( \min \{ k \in \mathbb{N} \mid \mathcal{V}(kQ^*) \subset \mathbb{Z}^2 \} \), where \( kQ^* := \{ ky \mid y \in Q^* \} \). In other words, \( \ell \) equals the least common multiple of the (smallest) denominators of the (rational) coordinates of the vertices of \( Q^* \).
3. Proof of main theorem

The proof follows from suitable combination of the two upper bounds given in Lemmas 3.1 and 3.2. (Henceforth we use freely the notation introduced in §2.)

Lemma 3.1. Let $X_Q$ be a toric log Del Pezzo surface of index $\ell \geq 1$. Then

$$\sharp(\mathcal{V}(Q)) \leq 4 \max \{ \ell_H \mid H \in \mathcal{F}(Q) \} + 2 \leq 4\ell + 2. \quad (3.1)$$

Moreover, $\sharp(\mathcal{V}(Q)) = 4 \max \{ \ell_H \mid H \in \mathcal{F}(Q) \} + 2$, if and only if $\ell = 1$, and $Q$ is the unique hexagon (up to lattice-equivalence) with one interior lattice point. This means, in particular, that for indices $\ell \geq 2$ we have

$$\sharp(\mathcal{V}(Q)) \leq 4\ell + 1. \quad (3.2)$$

Proof. Obviously, there exists a facet $F \in \mathcal{F}(Q)$ such that $\sum_{v \in \mathcal{V}(Q)} v \in \sigma_F$ (this is a special facet, in the sense of [11, Sect. 3]). In addition, since $Q$ is two-dimensional, we have for all integers $j$:

$$\sharp \{ v \in \mathcal{V}(Q) \mid \langle \eta_F, v \rangle = j \} \leq 2.$$

Writing $\mathcal{V}(Q)$ as disjoint union $\mathcal{V}(Q) = \mathcal{V}^{(F)}_{\geq 0}(Q) \cup \mathcal{V}^{(F)}_{< 0}(Q)$, where

$$\mathcal{V}^{(F)}_{\geq 0}(Q) := \{ v \in \mathcal{V}(Q) \mid \langle \eta_F, v \rangle \geq 0 \} \quad \text{and} \quad \mathcal{V}^{(F)}_{< 0}(Q) := \{ v \in \mathcal{V}(Q) \mid \langle \eta_F, v \rangle < 0 \},$$

we observe that

$$\sharp(\mathcal{V}^{(F)}_{\geq 0}(Q)) \leq 2(l_F + 1),$$

because $\langle \eta_F, v \rangle \in \{0, 1, \ldots, l_F\}$ for all $v \in \mathcal{V}^{(F)}_{\geq 0}(Q)$. On the other hand,

$$0 \leq \langle \eta_F, \sum_{v \in \mathcal{V}(Q)} v \rangle = \sum_{v \in \mathcal{V}^{(F)}_{\geq 0}(Q)} \langle \eta_F, v \rangle + \sum_{v \in \mathcal{V}^{(F)}_{< 0}(Q)} \langle \eta_F, v \rangle$$

$$= \sum_{j=0}^{l_F} \sum_{v \in \mathcal{V}^{(F)}_{\geq 0}(Q) \mid \langle \eta_F, v \rangle = j} \langle \eta_F, v \rangle + \sum_{v \in \mathcal{V}^{(F)}_{< 0}(Q)} \langle \eta_F, v \rangle$$

$$\leq \sum_{j=0}^{l_F} 2j + \sum_{v \in \mathcal{V}^{(F)}_{< 0}(Q)} \langle \eta_F, v \rangle.$$

This implies

$$a := -\sum_{v \in \mathcal{V}^{(F)}_{< 0}(Q)} \langle \eta_F, v \rangle \leq 2\binom{l_F + 1}{2}.$$

Setting $\mu := \sharp(\mathcal{V}^{(F)}_{< 0}(Q))$ we examine two cases: (i) If $\mu = 2\lambda$, for a $\lambda \in \mathbb{N}$, then

$$\sum_{j=0}^{\lambda} 2j \leq a \Rightarrow 2\binom{\lambda + 1}{2} \leq 2\binom{l_F + 1}{2} \Rightarrow \lambda \leq l_F \text{ and } \mu \leq 2l_F.$$
(ii) If \( \mu = 2\lambda + 1 \), for a \( \lambda \in \mathbb{Z}_{\geq 0} \), then \( \sum_{j=0}^{\lambda} 2j + (\lambda + 1) \leq a \), i.e.,
\[
2 \left( \frac{\lambda + 1}{2} \right) + (\lambda + 1) \leq 2 \left( \frac{lF + 1}{2} \right) \implies \lambda \leq lF - 1 \text{ and } \mu \leq 2lF - 1.
\]
Hence,
\[
z(V(Q)) = z(V_{\geq 0}(Q)) + z(V_{< 0}(Q)) \leq 2(lF + 1) + \mu \\
\leq 2(lF + 1) + 2lF = 4lF + 2 \leq 4 \max \{ lH \mid H \in \mathcal{F}(Q) \} + 2,
\]
with the latter upper bound \( \leq 4l' + 2 \) (by (2.5)), giving the inequality (3.1). Finally, we deal with the case of equality: Suppose that \( z(V(Q)) = 4l' + 2 \), where
\[
l' := \max \{ lH \mid H \in \mathcal{F}(Q) \}.
\]
From (3.3) we see that \( \mu = 2lF \), and \( \lambda = lF = l' \). Therefore, by the equalities in (i) we have for the integers \( j = -l', \ldots, 0, \ldots, l' \):
\[
z \{ v \in V(Q) \mid \langle \eta_F, v \rangle = j \} = 2.
\]
(3.4)
In particular, \( 0 = \langle \eta_F, \sum_{v \in V(Q)} v \rangle \), i.e., \( \sum_{v \in V(Q)} v = 0 \). Hence, the previous argument holds for any facet. Now let \( F' \) be another facet of \( Q \) having a common vertex, say \( v \), with \( F \). If \( V(F) = \{ u, v \} \) and \( V(F') = \{ v, w \} \), then applying (3.4) for both \( F \) and \( F' \) we get \( \langle \eta_F, w \rangle = l' - 1 \) and \( \langle \eta_F, u \rangle = l' - 1 \). This implies \( l' = 1 = \ell \), since otherwise the primitive vertex \( v \) equals \( (l'/(l' - 1))(w + u - v) \), a contradiction. Consequently, \( Q \) has to be the unique hexagon (up to lattice-equivalence) with just one interior lattice point (see [10, Proposition 2.1]).

\[\text{Lemma 3.2. If } X_Q \text{ is a toric log Del Pezzo surface of index } \ell \geq 2 \text{ and } \tilde{X}_Q \xrightarrow{\phi} X_Q \text{ its minimal desingularization, then}
\]
\[
\rho(\tilde{X}_Q) < 2z(I_Q \setminus \tilde{I}_Q)(\ell - 1) - \frac{1}{\ell} z(V(Q)) + 10.
\]
(3.5)

\[\text{Proof. By Noether's formula and (2.4) we deduce}
\]
\[
\rho(\tilde{X}_Q) = 10 - K_{\tilde{X}_Q}^2 = 10 - K_{X_Q}^2 - \sum_{F \in I_Q} K(E(F))^2.
\]
Since \(-\ell K_{X_Q} \) is an ample Cartier divisor on \( X_Q \), we can compute by [12 Proposition 2.10, p. 79] its self-intersection number:
\[
(-\ell K_{X_Q})^2 = 2 \operatorname{area}(\ell Q^*) \implies K_{X_Q}^2 = \frac{2}{\ell^2} \operatorname{area}(\ell Q^*) = 2 \operatorname{area}(Q^*).
\]
For any facet \( H \) of \( \ell Q^* \) the primitive outer normal vector is given by some vertex of \( Q \), i.e., the lattice distance of \( H \) from \( 0 \) equals \( \ell \). This implies
\[
\operatorname{area}(\ell Q^*) \geq \frac{1}{2\ell} z(\mathcal{F}(\ell Q^*)) = \frac{1}{2\ell} z(V(Q)).
\]
Hence,
\[
-K_{X_Q}^2 = -\frac{2}{\ell^2} \operatorname{area}(\ell Q^*) \leq -\frac{1}{\ell} z(V(Q)).
\]
On the other hand, by Lemma 2.3 we infer that
\[- \sum_{F \in I_Q} K(E^F)^2 = \sum_{F \in I_Q} \left( \frac{\beta_F \cdot q_F}{l_F} - (p_F + \tilde{p}_F) + (m_F - 2) \right).\]
Taking into account that \(m_F = 2\) for all \(F \in I_Q\), and that \(p_F + \tilde{p}_F \geq 2\) for all \(F \in I_Q\), which is valid as equality only for \(p_F = \tilde{p}_F = 1\), i.e., whenever \(F \in I_Q\), we obtain
\[- \sum_{F \in I_Q} K(E^F)^2 = - \sum_{F \in I_Q} K(E^F)^2 < \sum_{F \in I_Q} (m_F - 2)\]
\[\leq \sharp(I_Q \setminus \hat{I}_Q) \max \left\{ m_F - 2 \mid F \in I_Q \setminus \hat{I}_Q \right\}\]
\[\leq \sharp(I_Q \setminus \hat{I}_Q) \max \left\{ 2(l_F - 1) \mid F \in I_Q \setminus \hat{I}_Q \right\} \leq 2 \sharp(I_Q \setminus \hat{I}_Q)(\ell - 1),\]
where the last but one inequality follows from Lemma 2.2. Thus, \(\rho(\hat{X}_Q)\) is strictly smaller than the sum \(10 - \sharp(V(Q))/\ell + 2 \sharp(I_Q \setminus \hat{I}_Q)(\ell - 1)\). \(\square\)

**Proof of Theorem 1.2** If \(\ell = 1\), then \(\rho(\hat{X}_Q) \leq 7\) by the known classification of the reflexive polygons (see [6] or [10, Proposition 2.1]). If \(\ell \geq 2\), applying (3.2) and (3.4), and the inequality \(\sharp(I_Q \setminus \hat{I}_Q) \leq \sharp(V(Q))\), we get
\[\rho(\hat{X}_Q) < 2 \sharp(I_Q \setminus \hat{I}_Q)(\ell - 1) - \frac{1}{\ell} \sharp(V(Q)) + 10\]
\[\leq \sharp(V(Q)) \left(2(\ell - 1) - \frac{1}{\ell}\right) + 10 \leq (4\ell + 1) \left(2(\ell - 1) - \frac{1}{\ell}\right) + 10,\]
i.e., \(\rho(\hat{X}_Q) < 8\ell^2 - 6\ell + 4 - \frac{1}{\ell}\), which yields the bound for \(\ell \geq 2\). \(\square\)

**4. Discussion, improvements and examples**

First, let us note that from the proof of Theorem 1.2 we derive a linear upper bound on \(\rho(\hat{X}_Q)\), if the number of vertices of \(Q\) is fixed. It is therefore natural to ask for an example of an infinite family \(\{Q_i\}\) of LDP-polygons with increasing number of vertices, for which \(\rho(\hat{X}_Q)\) exhibits a non-linear growth with respect to the indices of its members. To the best knowledge of the authors, this seems to be an open question.

Now, in some specific cases we can further improve the bound (1.2). If \(Q\) is an LDP-polygon and \(F \in I_Q\), then, according to (2.2), there is a positive integer \(\beta_F\) such that
\[p_F - 1 = \beta_F \cdot \frac{q_F}{l_F} \implies l_F(p_F - 1) = \beta_F q_F.\]
Since \(l_F(p_F - 1) < l_F(q_F - 1) < l_F q_F\), we have \(\beta_F \in \{1, \ldots, l_F - 1\}\). In Proposition 1.1 we construct a better upper bound for \(\rho(\hat{X}_Q)\) provided that \(\beta_F\) takes one of the extreme values \(1, l_F - 1, \) and \(l_F^2 \cdot q_F\) for all \(F \in I_Q \setminus \hat{I}_Q\).
Proposition 4.1. Let $Q$ be an LDP-polygon such that $X_Q$ has index $\ell \geq 2$. Suppose that for all $F \in I_Q \setminus \hat{I}_Q$ the following conditions are satisfied:

(i) $\beta_F \in \{1, l_F - 1\}$, and

(ii) $l_F^2 \mid q_F$. Then

$$\rho(\tilde{X}_Q) \leq 4\ell^2 - 3\ell + 4. \quad (4.1)$$

Proof. For $F \in I_Q \setminus \hat{I}_Q$ define $\xi_F := \frac{\beta_F}{q_F}$. If $\beta_F = 1$, then $\frac{q_F}{q_F - p_F}$ equals

$$1 + \frac{1}{(\ell_F - 2) + \frac{1}{(\ell_F - 1) + \frac{1}{(\ell_F - 2) - \ell - 1}} \text{ times}} \in \left\{ \begin{array}{ll}
2, \ldots, 2, l_F + 2 \right\} 
\end{array}$$

if $\xi_F = 1$, and

$$1 + \frac{1}{(\ell_F - 2) + \frac{1}{(\ell_F - 1) + \frac{1}{(\ell_F - 2) - \ell - 1}} \text{ times}} \in \left\{ \begin{array}{ll}
2, \ldots, 2, 3, \ldots, 2, \ell + 1 \right\} 
\end{array}$$

if $\xi_F \geq 2$,

(see Proposition 3.1, pp. 83-84), $\tilde{p}_F = q_F - l_F\xi_F + 1$, and

$$m_F - 2 = \sum_{j=1}^{s_F} (b^{(F)} - 2) = l_F, \forall F \in I_Q \setminus \hat{I}_Q.$$

Correspondingly, if $\beta_F = l_F - 1$, then $\frac{q_F}{q_F - p_F}$ equals

$$\left\{ \begin{array}{ll}
(l_F + 1) + \frac{1}{(\ell_F - 2) + \frac{1}{(\ell_F - 1) + \frac{1}{(\ell_F - 2) - \ell - 1}} \text{ times}} \in \left\{ \begin{array}{ll}
l_F + 2, \ldots, 2 \right\} 
\end{array} \right. 
\end{array}$$

if $\xi_F = 1$, and

$$l_F + \frac{1}{(\ell_F - 2) + \frac{1}{(\ell_F - 1) + \frac{1}{(\ell_F - 2) - \ell - 1}} \text{ times}} \in \left\{ \begin{array}{ll}
\ell + 1, \ldots, 2, 3, \ldots, 2 \right\} 
\end{array} \right. \text{ times} \text{ times}$$

if $\xi_F \geq 2$,

$\tilde{p}_F = l_F\xi_F + 1$, and $m_F - 2 = \sum_{j=1}^{s_F} (b^{(F)} - 2) = l_F, \forall F \in I_Q \setminus \hat{I}_Q$. Thus,

$$- \sum_{F \in I_Q \setminus \hat{I}_Q} K(E(F))^2 = \sum_{F \in I_Q \setminus \hat{I}_Q} \left( \frac{2 - p_F + \tilde{p}_F}{q_F} + (m_F - 2) \right)$$

$$= \sum_{F \in I_Q \setminus \hat{I}_Q} (l_F - 1) \leq \sharp(I_Q \setminus \hat{I}_Q)(\ell - 1).$$

Since $\sharp(I_Q \setminus \hat{I}_Q) \leq \sharp(V(Q))$, applying Lemma 3.1 and the reasoning used in the proof of Lemma 3.2, we get

$$\rho(\tilde{X}_Q) < \sharp(I_Q \setminus \hat{I}_Q)(\ell - 1) - \frac{1}{\ell} \sharp(V(Q)) + 10$$

$$\leq \sharp(V(Q)) \left( \ell - 1 - \frac{1}{\ell} \right) + 10 \leq (4\ell + 1) \left( \ell - 1 - \frac{1}{\ell} \right) + 10.$$

The upper bound (4.1) follows from this inequality. \qed
By [1, Lemma 6.9, p. 107] we see that the conditions (i), (ii) in Proposition 4.1 are automatically satisfied for all toric log Del Pezzo surfaces of index $\ell = 2$. Hence, the upper bound 14 improves noticeably (1.2) (which equals 23 in this case). In fact, for $\ell = 2$, it can be shown (though, at the cost of passing through ad hoc classification results for the corresponding LDP-polygons) that the sharp upper bound equals 10.

Finally, in Proposition 4.3 we classify those LDP-triangles of arbitrary index, whose toric log Del Pezzo surfaces have at most one singularity. Somehow surprisingly, the Picard number of their minimal desingularizations is bounded; moreover, it takes always the smallest possible value, namely 2. Note that from Lemma 4.2 one only derives that the Picard numbers behave at most linearly with respect to the index, once the number of non-Gorenstein singularities $\sharp(I_Q \setminus \tilde{I}_Q)$ is fixed.

**Lemma 4.2.** If $X_Q$ is a toric log Del Pezzo surface with Picard number $\rho(X_Q) = 1$ (i.e., if $Q$ is an LDP-triangle) and $\sharp(I_Q) = 1$, then $Q$ is lattice-equivalent to the triangle $Q_p$ having $(\frac{1}{1}, (\frac{p}{p+1}), (\frac{1}{1})$ as its vertices, for some positive integer $p$.

**Proof.** If $I_Q = \{F\}$, setting $p := q_F$ and $q := q_F$, there is a unimodular transformation mapping $n(F)$ onto $n_1 := (\frac{1}{1})$, $n(F)$ onto $n_2 := (\frac{p}{q})$, and the third vertex of $Q$ onto an $n_3 := (\frac{2}{q})$ which belongs necessarily to the set \( \left\{ \left( \frac{x}{q} \right) \in \mathbb{Z}^2 \mid \frac{q}{p} x_1 < x_2 < 0 \right\} \). Since $|\det(n_2, n_3)| = |\det(n_3, n_1)| = 1$, we have $x_2 = -1$ and $x_1 = -\frac{p+1}{q}$. Hence, $q \mid p + 1$, which implies $q = p + 1$ (because $p < q$). □

**Proposition 4.3.** Let $X_Q$ be a toric log Del Pezzo surface which has Picard number $\rho(X_Q) = 1$, arbitrary index $\ell \geq 1$, and $\sharp(I_Q) = 1$. For $\ell \geq 3$ we have either $X_Q \cong X_{Q_{\ell-1}}$, or $X_Q \cong X_{Q_{\ell-3}}$, whereas for $\ell \in \{1\} \cup 2\mathbb{Z}$ we have $X_Q \cong X_{Q_{\ell-1}}$. Furthermore, for all $\ell \geq 1$, the Picard number of the rational surface $\tilde{X}_Q$ obtained by the minimal resolution of the singularity of $X_Q$ equals

$$\rho(\tilde{X}_Q) = 2.$$ 

**Proof.** By Lemma 4.2 the LDP-triangle $Q$ is lattice-equivalent to $Q_p$, for some positive integer $p$. Since $q = p + 1$ and gcd($p + 1, p - 1$) $\in \{1, 2\}$, the index $\ell$ of $X_Q \cong X_{Q_p}$ equals $\frac{p + 1}{2}$ whenever $p$ is odd and $p + 1$ whenever $p$ is even (see 2.4). This bears out our first assertion. On the other hand, since $\Delta_{Q_p}$ is obtained from $\Delta_{Q_p}$ by adding just one new ray (namely $\mathbb{R}_{\geq 0}(\frac{1}{1})$), we have

$$\rho(\tilde{X}_Q) = \rho(\tilde{X}_{Q_p}) = \sharp\{\text{rays of } \Delta_{Q_p}\} - 2 = 4 - 2 = 2,$$

(cf. [12, Corollary 2.5, p. 74]). Thus, the second assertion is also true. □

It would be interesting to generalize this result by regarding LDP-polygons of arbitrary index, whose toric log Del Pezzo surfaces have at most one singularity.
References

[1] Alekseev V.A.: Fractional indices of log Del Pezzo surfaces, Math. USSR-Izv. 33 (1989), 613–629; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 52 (1988), 1288–1304.

[2] Alekseev V.A. & Nikulin V.V: Del Pezzo and $K_3$ surfaces, M.S.J. Memoirs, Vol. 15, Mathematical Society of Japan, 2006.

[3] Brieskorn E.: Rationale Singularit"aten komplexer Fl"achen, Inventiones Math. 4 (1968), 336–358.

[4] Dais D.I.: Geometric combinatorics in the study of compact toric surfaces. In “Algebraic and Geometric Combinatorics” (edited by C. Athanasiadis et. al.), Contemporary Mathematics, Vol. 423, American Mathematical Society, 2007, pp. 71–123.

[5] Kojima H.: Rank one log Del Pezzo surfaces of index two, J. Math. Kyoto University 43 (2003), 101–123.

[6] Kreuzer, M. & Skarke, H.: On the classification of reflexive polyhedra, Commun. Math. Phys. 185 (1997), 495–508.

[7] Nikulin V.V.: Del Pezzo surfaces with log-terminal singularities I, Math. USSR-Sb. 66 (1990), 231–248; translation from Mat. Sb. 180 (1989), 226–243.

[8] ______: Del Pezzo surfaces with log-terminal singularities II, Math. USSR-Izv. 33 (1989), 355–372; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 52 (1988), 1032–1050.

[9] ______: Del Pezzo surfaces with log-terminal singularities III, Math. USSR-Izv. 35 (1990), 657–675; translation from Izv. Akad. Nauk SSSR, Ser. Mat. 53 (1989), 1316–1334.

[10] Nill B.: Gorenstein toric Fano varieties, Manuscr. Math. 116 (2005), 183–210.

[11] Øbro M.: Classification of terminal simplicial reflexive $d$-polytopes with $3d-1$ vertices, Manuscr. Math. 125 (2008), 69–79.

[12] Oda T.: Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties. Erg. der Math. und ihrer Grenzgebiete, dritte Folge, Bd. 15, Springer-Verlag, 1988.

[13] Sakai F.: Anticanonical models of rational surfaces, Math. Ann. 269 (1984), 389–410.

[14] Ye Q.: On Gorenstein log Del Pezzo surfaces, Japanese J. of Math. 28 (2002), 87–136.

Dimitrios I. Dais
University of Crete, Department of Mathematics, Division Algebra and Geometry, Knossos Avenue, P.O. Box 2208, GR-71409, Heraklion, Crete, Greece
e-mail: ddais@math.uoc.gr

Benjamin Nill
Freie Universität Berlin, Institut für Mathematik, Arbeitsgruppe Gitterpolytope, Arnimallee 3, 14195 Berlin, Germany
e-mail: nill@math.fu-berlin.de