A simplified version of the Sequent Calculus

\textbf{G3[\text{mic}]}

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Abstract

We show that the Replacement Rule in the sequent calculus \textbf{G3[\text{mic}]}, for first order languages with function symbols and equality, can be replaced by the simpler rule in which the transformed formula is not repeated in the premiss.

Keywords Sequent Calculus, Equality, Replacement Rule, Admissibility

Mathematical Subject Classification 03F05

1 Introduction

As it is well known, the modern \textbf{G3} sequent calculi free of structural rules have evolved from the work of Gentzen through Ketonen, Kleene, Dragalin and Troelstra, until, in the words of [4], “a gem emerged”. In particular such systems were obtained by showing that the repetition of the principal formula in the premis(ses) of the logical rules, that Kleene, in [1], had proposed in all reasonable cases, in most of them, could actually be dispensed with. An extension of such calculi to logic with equality was proposed by Negri and van Plato in [3]. As the authors write in [5], Troelstra appreciated so much their proposal, that it was adopted in the second edition [7] of [6], thus replacing the standard axiomatic treatment of equality of the first edition. The rules for equality proposed in [3] (see also [4]) were the following:

\[
\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \text{Ref} \quad \frac{a = b, P[v/a], P[v/b], \Gamma \Rightarrow \Delta}{a = b, P[v/a], \Gamma \Rightarrow \Delta} \quad \text{Repl}
\]
where \( \Gamma \) and \( \Delta \) are finite multisets of formulae, \( P \) is an atomic formula and \( P[v/a] \) and \( P[v/b] \) denote the result of the substitution of the variable \( v \) by the individual constants \( a \) and \( b \) respectively.

The rules adopted in [7] are the above Ref and Repl, with \( a \) and \( b \) replaced by arbitrary terms \( s \) and \( r \) and the proviso that \( v \) does not occur in \( s \) and \( r \).

Our purpose is to show that, in line with the evolution from Kleene’s rules to the present ones, also the repetition of the formula \( P[v/s] \) in the premiss of the rule Repl, meant to ensure the admissibility of the contraction rule, can actually be dispensed with.

More precisely, we will refer to the multisuccedent systems \( G_3[\text{mic}]^- \), for minimal, intuitionistic and classical logic, in [2], with the rules Ref and Repl extended to arbitrary terms, and will show that in \( G_3[\text{mic}]^- \) the rule Repl can be replaced by the, apparently weaker, rule Repl\(^-\):

\[
\frac{s = r, P[v/r], \Gamma \Rightarrow \Delta}{s = r, P[v/s], \Gamma \Rightarrow \Delta}
\]

by showing that the rule Repl is admissible in the systems \( G_3[\text{mic}]^- \) obtained by replacing in \( G_3[\text{mic}]^- \) the rule Repl by Repl\(^-\). As a consequence \( G_3[\text{mic}]^- \) and \( G_3[\text{mic}]^- \) are equivalent.

## 2 Admissibility of Repl in \( G_3[\text{mic}]^- \)

Let Repl\(^-\) and Repl\(_1\) be the rules Repl\(^-\) and Repl as represented in the Introduction, in which it is required that there is exactly one occurrence of \( v \) in \( A \), i.e. only one occurrence of \( r \) is replaced by \( s \).

**Lemma 1** Repl\(^-\) is derivable from Repl\(_1\).

**Proof** By induction on the number of occurrences of \( v \) in \( A \). If such a number is \( n + 1 \), with \( n \geq 1 \), then let \( A' \) be obtained by replacing one occurrence of \( v \) in \( A \) by a new variable \( v' \). \( A[v/r] \) coincides with \( (A'[v/r])[v'/r] \). Thus from

\[
s = r, A[v/r], \Gamma \Rightarrow \Delta
\]

by Repl\(_1\) we can obtain

\[
s = r, (A'[v/r])[v'/s], \Gamma \Rightarrow \Delta
\]

that coincides with

\[
s = r, (A'[v'/s])[v/r], \Gamma \Rightarrow \Delta
\]

Since there are \( n \) occurrences of \( v \) in \( A'[v'/s] \), by the induction hypothesis, from the latter sequent, using Repl\(_1\) we can derive

\[
s = r, (A'[v'/s])[v/s], \Gamma \Rightarrow \Delta
\]
Lemma 2 \textit{Repl is derivable from Repl$_1$ and the left weakening rule.}

Proof As in the proof of Lemma 1 let $A'$ be obtained by replacing one of the $n + 1$ occurrences of $v$ in $A$ by a new variable $v'$. The premiss

\[
s = r, A[v/s], A[v/r], \Gamma \Rightarrow \Delta
\]

of Repl coincides with

\[
s = r, (A'[v/s])[v'/s], (A'[v/r])[v'/r], \Gamma \Rightarrow \Delta
\]

from which by the left weakening rule we obtain

\[
s = r, (A'[v/s])[v'/s], (A'[v/r])[v'/r], \Gamma \Rightarrow \Delta
\]

Then an application of Repl$_1$ yields

\[
s = r, (A'[v/s])[v'/s], (A'[v/r])[v'/s], \Gamma \Rightarrow \Delta
\]

namely

\[
s = r, (A'[v'/s])[v'/s], (A'[v'/s])[v/r], \Gamma \Rightarrow \Delta
\]

from which, by the induction hypothesis, we can derive

\[
s = r, (A'[v'/s])[v/s], \Gamma \Rightarrow \Delta
\]

i.e.

\[
s = r, A[v/s], \Gamma \Rightarrow \Delta
\]

\[\square\]

Let $G3[\text{mic}]^{-}$ be obtained by replacing Repl$^{-}$ by Repl$_1^{-}$ in $G3[\text{mic}]^{-}$. For the sake of notational brevity in the following we will denote $G3[\text{mic}]^{-}$ and $G3[\text{mic}]_1^{-}$ also by $S$ and $S_1$ respectively.

Lemma 3 \textit{The weakening rules are height preserving admissible in $S$ and $S_1$, i.e. if $\Gamma \Rightarrow \Delta$ has a derivation in $S$ ($S_1$) of height $\leq h$, then also $F, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, F$ have a derivation in $S$ ($S_1$) of height $\leq h$.}

Lemma 4 \textit{a) Derivability in $S_1$ of Contr$^=$}

\[
s = r, \Gamma \Rightarrow \Delta \text{ is derivable in } S_1 \text{ from } s = r, s = r, \Gamma \Rightarrow \Delta
\]
b) Admissibility in $S_1$ of Symm:

In $S_1$ and the left weakening rule, $r = s, \Gamma \Rightarrow \Delta$ is derivable from $s = r, \Gamma \Rightarrow \Delta$. The same holds for $S$.

**Proof** a) Since $s = r$ coincides with $A[v/r]$, where $A$ is $s = v$, the following is a derivation in $S$ of $s = r, \Gamma \Rightarrow \Delta$ from $s = r, s = r, \Gamma \Rightarrow \Delta$:

\[
\begin{array}{c}
D \\
\frac{s = r, s = r, \Gamma \Rightarrow \Delta}{s = r, \Gamma \Rightarrow \Delta} \quad \text{Repl}_1 \\
\frac{s = r, s = s, \Gamma \Rightarrow \Delta}{s = r, \Gamma \Rightarrow \Delta} \quad \text{Ref}
\end{array}
\]

b) The following is a derivation in $S_1$ and the left weakening rule of $r = s, \Gamma \Rightarrow \Delta$ from $s = r, \Gamma \Rightarrow \Delta$:

\[
\begin{array}{c}
D \\
\frac{s = r, r = r, \Gamma \Rightarrow \Delta}{s = r, \Gamma \Rightarrow \Delta} \quad \text{Repl}_1 \\
\frac{s = r, r = s, \Gamma \Rightarrow \Delta}{s = r, \Gamma \Rightarrow \Delta} \quad \text{Repl}_1 \\
\frac{r = r, r = s, \Gamma \Rightarrow \Delta}{r = s, \Gamma \Rightarrow \Delta} \quad \text{Ref}
\end{array}
\]

\[\square\]

**Proposition 5** Repl is admissible in $G3[\text{mic}]$.

**Proof** We have to show that the applications of the rule Repl can be eliminated from the derivations in $S + \text{Repl}$. Since the left weakening rule is admissible in $S$, by Lemma 2, we can transform a derivation in $S + \text{Repl}$ into a derivation with the same endsequent in $S + \text{Repl}_1$. Thus by Lemma 1 it suffices to show that a derivation $D$ in $S_1$ of

\[
s = r, A[v/s], A[v/r], \Gamma \Rightarrow \Delta
\]

with $v$ that does not occur in $s, r$ and has a single occurrence in $A$, can be transformed into a derivation $D'$ in $S_1$ of $s = r, A[v/s], \Gamma \Rightarrow \Delta$. The proof is by induction on the height of derivations, but for the induction argument to go through we have to generalize the statement to be proved. In fact, assume that $A[v/s]$ has the form $A^o[u/q, v/s]$ and the given derivation of $s = r, A[v/s], A[v/r], \Gamma \Rightarrow \Delta$ has the form:

\[
\begin{array}{c}
D_0 \\
\frac{q = p, s = r, A^o[u/q, v/s], A^o[u/p, v/r], \Gamma \Rightarrow \Delta}{q = p, s = r, A^o[u/q, v/s], A^o[u/q, v/r], \Gamma \Rightarrow \Delta}
\end{array}
\]

Then since $A^o[u/q, v/s]$ and $A^o[u/p, v/r]$ do not have the form $B[v/s]$ and $B[v/r]$ we could not apply the induction hypothesis to $D_0$. 

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To overcome that problem, we generalize the statement to be proved as follows. Let \( \vec{q} \) and \( \vec{p} \) be the sequences of terms \( q_1, \ldots q_n \) and \( p_1, \ldots p_n \) and, similarly, let \( \vec{u} \) stand for the sequence of variables \( u_1, \ldots u_n \) assumed to be distinct from one another and from \( v \) and not occurring in \( \vec{q}, \vec{p}, s, r \). \( \vec{q} = \vec{p} \) stands for the sequence of equalities \( q_1 = p_1, \ldots q_n = p_n \) and \( [\vec{u}/\vec{q}] \) for the substitution \([u_1/q_1, \ldots, u_n/q_n]\) and similarly for \([\vec{u}/\vec{p}]\). We proceed by induction on the height \( h(D) \) of \( D \) to show that if \( D \) is a derivation in \( S_1 \) of

\[
\vec{q} = \vec{p}, s = r, A[\vec{u}/\vec{q}, v/s], A[\vec{u}/\vec{p}, v/r], \Gamma \Rightarrow \Delta
\]

where each one of the variables in \( \vec{u} \) and \( v \) has exactly one occurrence in \( A \), then \( D \) can be transformed into a derivation \( D' \) in \( S_1 \) of

\[
\vec{q} = \vec{p}, s = r, A[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta
\]

The statement we are actually interested in, that yields the admissibility of Repl\(_1\), therefore of Repl, in \( S \), follows by letting \( n = 0 \).

If \( h(D) = 0 \), then \( D \) reduces to a logical axiom. If \( \Gamma \cap \Delta \neq \emptyset \) or one of \( \vec{q} = \vec{p}, s = r, A[\vec{u}/\vec{q}, v/s] \) belongs to \( \Delta \), then also \( \vec{q} = \vec{p}, s = r, A[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta \) is an axiom and we are done. Otherwise \( A[\vec{u}/\vec{q}, v/r] \in \Delta \). But then also \( \vec{q} = \vec{p}, s = r, A[\vec{u}/\vec{q}, v/r], \Gamma \Rightarrow \Delta \) is an axiom and as \( D \) we can take:

\[
\begin{align*}
\vec{q} = \vec{p}, s = r, & A[\vec{u}/\vec{q}, v/r], \Gamma \Rightarrow \Delta \\
\vdots \\
\vec{q} = \vec{p}, s = r, & A[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta
\end{align*}
\]

If \( h(D) > 0 \) and \( D \) ends with a logical inference (see [2] for the complete list), since \( A \) is atomic, neither \( A[\vec{u}/\vec{q}, v/s] \) nor \( A[\vec{u}/\vec{q}, v/r] \) nor any of \( \vec{q} = \vec{p} \) and \( s = r \) can be the principal formula of such an inference and the conclusion follows immediately from the induction hypothesis. The same applies if \( D \) ends with a Ref-inference. If \( D \) ends with a Repl\(_1\)-inference we distinguish the following cases.

Case 1. The last inference of \( D \) does not introduce any of the shown occurrences of \( \vec{q}, \vec{p}, s \) or \( r \).

Case 1.1 \( A \) is of the form \( A^0[u/q] \) and \( D \) has the form:

\[
\begin{align*}
\vec{q} = \vec{p}, & s = r, A^0[u/q], A^0[u/p, \vec{u}/\vec{p}, v/r], \Gamma' \Rightarrow \Delta \\
q = p, & \vec{q} = \vec{p}, s = r, A^0[u/q, \vec{u}/\vec{q}, v/s], A^0[u/q, \vec{u}/\vec{p}, \vec{u}/\vec{q}, v/r], \Gamma' \Rightarrow \Delta
\end{align*}
\]

By the induction hypothesis applied to \( D_0 \) and \( \Gamma' \), \( q = p, \vec{q} = \vec{p} \) and \( A^0 \) in place of \( \Gamma \), \( \vec{q} = \vec{p} \) and \( A \) respectively, we have a derivation \( D'_0 \) in \( S_1 \) of

\[
q = p, \vec{q} = \vec{p}, s = r, A^0[u/q, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta
\]

that can be taken as \( D' \).
Case 1.2 \( A \) is of the form \( A^c[u/q] \) and \( D \) has the form:

\[
\frac{\text{Case 1.2}}{q = p, \bar{q} = \bar{p}, s = r, A^c[u/p, \bar{u}/\bar{q}, v/s], A^c[u/q, \bar{u}/\bar{p}, v/r], \Gamma' \Rightarrow \Delta}
\]

By height-preserving weakening we have a derivation \( D_0^w \) of the same height as \( D_0 \) of

\[
\frac{\text{Case 1.2}}{p = q, q = p, \bar{q} = \bar{p}, s = r, A^c[u/p, \bar{u}/\bar{q}, v/s], A^c[u/q, \bar{u}/\bar{p}, v/r], \Gamma' \Rightarrow \Delta}
\]

By induction hypothesis there is a derivation \( D_0^w \) in \( S_1 \) of

\[
\frac{\text{Case 1.2}}{p = q, q = p, \bar{q} = \bar{p}, s = r, A^c[u/p, \bar{u}/\bar{q}, v/s], \Gamma' \Rightarrow \Delta}
\]

Then \( D' \) can be obtained from the following derivation in \( S_1 + \text{Contr}^= + \text{Symm} \), thanks to the derivability in \( S_1 \) of \( \text{Contr}^= \) and the admissibility in \( S_1 \) of \( \text{Symm} \):

\[
\frac{\text{Case 1.2}}{\begin{array}{c}
p = q, q = p, \bar{q} = \bar{p}, s = r, A^c[u/p, \bar{u}/\bar{q}, v/s], \Gamma' \Rightarrow \Delta \\
q = p, \bar{q} = p, q = p, q = p, \bar{q} = \bar{p}, s = r, A^c[u/q, \bar{u}/\bar{q}, v/s], \Gamma' \Rightarrow \Delta \\
q = p, \bar{q} = \bar{p}, s = r, A^c[u/q, \bar{u}/\bar{q}, v/s], \Gamma' \Rightarrow \Delta \\
\end{array}}
\]

Case 1.3 The last \( \text{Repl}^-_1 \) inference of \( D \) acts by means of \( q = p \) inside \( \Gamma' \).

Then the conclusion follows by applying the induction hypothesis and then the same \( \text{Repl}^-_1 \)-inference.

Case 2. The last inference of \( D \) does introduce one of the shown occurrences of \( \bar{q}, \bar{p}, s \) or \( r \). Without loss of generality we may assume that is either \( r \) or \( s \).

Case 2.1 The last inference of \( D \) introduces \( r \).

Case 2.1.1 \( A \) has the form \( A^c[u/q], v \) occurs in \( q \) and \( D \) has the form:

\[
\frac{\text{Case 2.1.1}}{q[v/r] = p, q = p, \bar{q} = \bar{p}, s = r, A^c[u/p, \bar{u}/\bar{q}, v/s], A^c[u/q, \bar{u}/\bar{p}, v/r], \Gamma' \Rightarrow \Delta}
\]

By height-preserving weakening we have a derivation \( D_0^w \) of the same height as \( D_0 \) of

\[
\begin{array}{c}
q[v/r] = p, q[v/s] = p, \bar{q} = \bar{p}, s = r, A^c[u/q, \bar{u}/\bar{v}, v/s], A^c[u/p, \bar{u}/\bar{v}, v/r], \Gamma' \Rightarrow \Delta \\
\end{array}
\]

By induction hypothesis there is a derivation \( D_0^w \) in \( S_1 \) of

\[
q[v/r] = p, q[v/s] = p, \bar{q} = \bar{p}, s = r, A^c[u/q, \bar{u}/\bar{v}, v/s], \Gamma' \Rightarrow \Delta
\]
Then $D'$ can be obtained from:

$$D'_0$$

$$\frac{q[v/r] = p, \; q[v/s] = p, \; \bar{q} = \bar{p}, \; s = r, \; A^\circ[\bar{u}/\bar{q}, \; u/q[v/s]], \; \Gamma' \Rightarrow \Delta}{\text{Symm}}$$

$$\frac{q[v/r] = p, \; q[v/s] = p, \; \bar{q} = \bar{p}, \; r = s, \; \bar{A}^\circ[\bar{u}/\bar{q}, \; u/q[v/s]], \; \Gamma' \Rightarrow \Delta}{\text{RepI}_1^-}$$

$$\frac{q[v/r] = p, \; \bar{q} = \bar{p}, \; r = s, \; \bar{A}^\circ[\bar{u}/\bar{q}, \; u/q[v/s]], \; \Gamma' \Rightarrow \Delta}{\text{Contr}^-}$$

$$\frac{q[v/r] = p, \; \bar{q} = \bar{p}, \; s = r, \; \bar{A}[\bar{u}/\bar{q}, \; u/q[v/s]], \; \Gamma' \Rightarrow \Delta}{\text{Symm}}$$

Case 2.1.2 $r$ is of the form $r^\circ[u/q]$ and $D$ has the form:

$$D_0$$

$$\frac{q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/q], \; A[\bar{u}/\bar{q}, \; v/s], \; A[\bar{u}/\bar{p}, \; v/r^\circ[u/p]], \; \Gamma' \Rightarrow \Delta}{\text{Repl}_1^-}$$

By height-preserving weakening we have a derivation $D'_0$ of the same height as $D_0$ of

$$q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/p], \; A[\bar{u}/\bar{q}, \; v/s], \; A[\bar{u}/\bar{p}, \; v/r^\circ[u/p]], \; \Gamma' \Rightarrow \Delta$$

By induction hypothesis there is a derivation $D'_0$ in $S_1$ of

$$q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/q], \; A[\bar{u}/\bar{q}, \; v/s], \; \Gamma \Rightarrow \Delta$$

Then $D'$ can be obtained from:

$$D'_0$$

$$\frac{q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/q], \; A[\bar{u}/\bar{q}, \; v/s], \; \Gamma \Rightarrow \Delta}{\text{Repl}_1^-}$$

$$\frac{q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/q], \; A[\bar{u}/\bar{q}, \; v/s], \; \Gamma \Rightarrow \Delta}{\text{Contr}^-}$$

Case 2.1.3 $r$ is of the form $r^\circ[u/q]$ and $D$ has the form:

$$D_0$$

$$\frac{q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/p], \; A[\bar{u}/\bar{q}, \; v/s], \; A[\bar{u}/\bar{p}, \; v/r^\circ[u/q]], \; \Gamma' \Rightarrow \Delta}{\text{Repl}_1^-}$$

By height-preserving weakening we have a derivation $D'_0$ in $S_1$ of the same height as $D_0$ of

$$q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/p], \; A[\bar{u}/\bar{q}, \; v/s], \; A[\bar{u}/\bar{p}, \; v/r^\circ[u/q]], \; \Gamma' \Rightarrow \Delta$$

By induction hypothesis there is a derivation $D'_0$ in $S_1$ of

$$q = p, \; \bar{q} = \bar{p}, \; s = r^\circ[u/q], \; A[\bar{u}/\bar{q}, \; v/s], \; \Gamma' \Rightarrow \Delta$$

Then $D'$ can be obtained from:
By induction hypothesis there is a derivation $D$ of the same height as $D_0$ of

$$q = p, \overline{q} = \overline{p}, s = r^o[u/p], A[\overline{u}/\overline{q}, v/s], \Gamma' \Rightarrow \Delta$$

Then $D'$ can be obtained from:

$$D_0^{w'}$$

Case 2.2.2 $s$ is of the form $s^o[u/q]$ and $D$ has the form:

$$q = p, \overline{q} = \overline{p}, s^o[u/q] = r, A[\overline{u}/\overline{q}, v/s^o[u/p]], A[\overline{u}/\overline{p}, v/r], \Gamma' \Rightarrow \Delta$$

By height-preserving weakening we have a derivation $D_0^{w'}$ of the same height as $D_0$ of

$$q = p, \overline{q} = \overline{p}, s = r, A^o[\overline{u}/\overline{q}, u/p], A^o[\overline{u}/\overline{p}, u/q[v/r]], \Gamma' \Rightarrow \Delta$$

By induction hypothesis there is a derivation $D_0^{w'}$ in $S_1$ of

$$q = p, \overline{q} = \overline{p}, s = r, A^o[\overline{u}/\overline{q}, u/p], \Gamma' \Rightarrow \Delta$$

Then $D'$ can be obtained from:

$$D_0^{w'}$$

Case 2.2.1 $A$ has the form $A^o[u/q], v$ occurs in $q$ and $D$ has the form:

$$q[v/s] = p, q = q[v/r], \overline{q} = \overline{p}, s = r, A^o[\overline{u}/\overline{q}, u/p], \Gamma' \Rightarrow \Delta$$

By height-preserving weakening we have a derivation $D_0^{w'}$ of the same height as $D_0$ of

$$q[v/s] = p, q = q[v/r], \overline{q} = \overline{p}, s = r, A^o[\overline{u}/\overline{q}, u/p], A^o[\overline{u}/\overline{p}, u/q[v/r]], \Gamma' \Rightarrow \Delta$$

By induction hypothesis there is a derivation $D_0^{w'}$ in $S_1$ of

$$q[v/s] = p, q = q[v/r], \overline{q} = \overline{p}, s = r, A^o[\overline{u}/\overline{q}, u/p], \Gamma' \Rightarrow \Delta$$

Then $D'$ can be obtained from:

$$D_0^{w'}$$

By height-preserving weakening we have a derivation $D_0^{w'}$ of the same height as $D_0$ of

$$q = p, \overline{q} = \overline{p}, s^o[u/q] = r, s^o[u/p] = r, A[\overline{u}/\overline{q}, v/s^o[u/p]], A[\overline{u}/\overline{p}, v/r], \Gamma' \Rightarrow \Delta$$

By induction hypothesis there is a derivation $D_0^{w'}$ in $S_1$ of

$$q = p, \overline{q} = \overline{p}, s^o[u/q] = r, s^o[u/p] = r, A[\overline{u}/\overline{q}, v/s^o[u/p]], \Gamma' \Rightarrow \Delta$$

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Then $D'$ can be obtained from:

$$
\begin{align*}
D'_{0'} \\
q = p, \bar{q} = \bar{p}, s^0[u/q] = r, s^0[u/p] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], \Gamma' \Rightarrow \Delta \\
q = p, \bar{q} = \bar{p}, s^0[u/q] = r, s^0[u/p] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], \Gamma' \Rightarrow \Delta \\
q = p, \bar{q} = \bar{p}, s^0[u/q] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], \Gamma' \Rightarrow \Delta
\end{align*}
$$

Case 2.2.3 $s$ is of the form $s^0[u/q]$ and $D$ has the form:

$$
\begin{align*}
D_0 \\
q = p, \bar{q} = \bar{p}, s^0[u/p] = r, s^0[u/q] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], A[\bar{u}/\bar{p}, v/r], \Gamma \Rightarrow \Delta
\end{align*}
$$

By height-preserving weakening we have a derivation $D'_0$ of the same height as $D_0$ of

$$
\begin{align*}
q = p, \bar{q} = \bar{p}, s^0[u/p] = r, s^0[u/q] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], A[\bar{u}/\bar{p}, v/r], \Gamma \Rightarrow \Delta
\end{align*}
$$

By induction hypothesis there is a derivation $D'_0$ in $S_1$ of

$$
\begin{align*}
q = p, \bar{q} = \bar{p}, s^0[u/p] = r, s^0[u/q] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], \Gamma' \Rightarrow \Delta
\end{align*}
$$

Then $D'$ can be obtained from:

$$
\begin{align*}
D'_{0'} \\
q = p, \bar{q} = \bar{p}, s^0[u/p] = r, s^0[u/q] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], \Gamma' \Rightarrow \Delta \\
q = p, \bar{q} = \bar{p}, s^0[u/q] = r, s^0[u/p] = r, A[\bar{u}/\bar{q}, v/s^0[u/q]], \Gamma' \Rightarrow \Delta
\end{align*}
$$

$\square$

2.1 Equivalence between $G3[mic]^=$ and $G3[mic]^=\neg$

Theorem 6 A sequent is derivable in $G3[mic]^=$ if and only if it is derivable in $G3[mic]^=\neg$.

Proof Let $D$ be a derivation in $G3[mic]^=$ of $\Gamma \Rightarrow \Delta$. By induction on the height of $D$, thanks to Proposition 5, it is straightforward that $D$ can be transformed into a derivation in $G3[mic]^=\neg$. Conversely, given a derivation $D$ in $G3[mic]^=\neg$ of $\Gamma \Rightarrow \Delta$, it suffices to apply the admissibility of the left weakening rule, to show that $D$ can be transformed into a derivation in $G3[mic]^=$ of $\Gamma \Rightarrow \Delta$. $\square$
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