A remark on nearness spaces

Jan-David Hardtke

Abstract. We give a proof of the well-known fact that the category of nearness spaces is bireflective in the category of merotopic spaces which uses Zorn’s Lemma instead of the usual construction by transfinite induction.

First let us introduce some notation and definitions. If \( X \) is any set we denote by \( \mathcal{P}(X) \) the power-set of \( X \) and by \( c(X) \) the set of all non-empty coverings of \( X \). For \( A, B \in c(X) \) we say that \( A \) is a refinement of \( B \) (or \( A \) refines \( B \)), denoted by \( A \prec B \), if for every \( A \in A \) there is \( B \in B \) such that \( A \subseteq B \). We further put \( A \land B = \{ A \cap B : A \in A, B \in B \} \), which is obviously again a covering of \( X \). For any unexplained notions from category theory which we will use in the sequel the reader is referred to [6] and [7].

Now recall that a merotopic space is a pair \((X, \mu)\), where \( \mu \subseteq c(X) \) is non-empty and such that the following holds:

(i) \( A \in \mu, B \in c(X) \) and \( A \prec B \Rightarrow B \in \mu \)

(ii) \( A, B \in \mu \Rightarrow A \land B \in \mu \)

Then \( \mu \) is sometimes called a merotopic structure on \( X \) and the elements of \( \mu \) are called uniform coverings.

A map \( f : X \to Y \) between merotopic spaces \((X, \mu)\) and \((Y, \nu)\) is called uniformly continuous (with respect to \( \mu \) and \( \nu \)) if \( f^{-1}[A] \in \mu \) for every \( A \in \nu \), where \( f^{-1}[A] := \{ f^{-1}(A) : A \in A \} \).

Clearly, the merotopic spaces together with the uniformly continuous maps as morphisms form a concrete category over the category of sets, i.e. a construct, which will be denoted by \( \text{Mer} \).

For a merotopic space \((X, \mu)\) and a subset \( A \subseteq X \) we define the interior of \( A \) with respect to \( \mu \) as \( \text{int}_\mu(A) = \{ x \in X : \{ A, X \setminus \{x\} \} \in \mu \} \).

It is well-known and easily checked that the following assertions hold (cf. [6, Proposition 3.2.2.3], where merotopic spaces are called semi-nearness spaces):

(a) \( \text{int}_\mu(A) \subseteq A \ \forall A \subseteq X \)
(b) \( \text{int}_\mu(X) = X, \text{int}_\mu(\emptyset) = \emptyset \)

(c) \( A \subseteq B \subseteq X \Rightarrow \text{int}_\mu(A) \subseteq \text{int}_\mu(B) \)

(d) \( \text{int}_\mu(A \cap B) = \text{int}_\mu(A) \cap \text{int}_\mu(B) \ \forall A, B \subseteq X \)

A merotopic space \((X, \mu)\) is called a nearness space if \(\{\text{int}_\mu(A) : A \in \mathcal{A}\}\) belongs to \(\mu\) for every \(A \in \mu\). Then \(\mu\) is also called a nearness structure on \(X\). The nearness spaces (and uniformly continuous maps) induce a full subconstruct of \(\text{Mer} \) which will be denoted by \(\text{Near}\).

The merotopic spaces were originally introduced by Katětov in [5], though not in the formulation above but in an equivalent version using so called micromeric collections of subsets of \(X\) instead of uniform coverings. One can also use a concept of “near” collections of subsets of \(X\) and an associated closure operator for equivalent definitions of merotopic and nearness spaces. In such a way the nearness spaces were originally introduced by Herrlich in [2] and [4]. For details on the various equivalent formulations we refer the reader to [4].

It is also due to Herrlich that \(\text{Near}\) is bireflective in \(\text{Mer}\). This theorem is usually proved by constructing the bireflective modification of a given merotopic space with respect to \(\text{Near}\) via transfinite induction (cf. [4, Theorem 8.1] or [6, Theorem 3.2.2.5]).

We want to give a different proof here, which is based on Zorn’s Lemma. We begin with an easy lemma.

**Lemma.** If \(\mu_1\) and \(\mu_2\) are two nearness-structures on the set \(X\) then

\[ \mu = \{A \in \mathcal{C}(X) : \exists \mathcal{A}_1 \in \mu_1, \exists \mathcal{A}_2 \in \mu_2 \text{ such that } \mathcal{A}_1 \wedge \mathcal{A}_2 \prec A\} \]

is again a nearness-structure on \(X\) that contains \(\mu_1\) and \(\mu_2\). Moreover, every merotopic structure on \(X\) that contains \(\mu_1\) and \(\mu_2\) must also contain \(\mu\).

Note that if we already knew that \(\text{Near}\) is bireflective in \(\text{Mer}\) this lemma would be an immediate consequence of the general way of constructing initial objects in \(\text{Mer}\) (cf. [6, Remark 3.2.2.2 (2) and Theorem 3.2.2.1]), but since we want to use it to show the bireflectivity result we have to give a direct proof, which can be easily done as follows.

**Proof.** Obviously we have \(\mu_1, \mu_2 \subseteq \mu\) and \((X, \mu)\) is easily seen to be a merotopic space. Now pick any \(A \in \mu\). By definition there are \(\mathcal{A}_1 \in \mu_1\) and \(\mathcal{A}_2 \in \mu_2\) such that \(\mathcal{A}_1 \wedge \mathcal{A}_2 \prec A\). It follows that \(\mathcal{A}_i' = \{\text{int}_{\mu_i}(A) : A \in \mathcal{A}_i\} \in \mu_i\) for \(i = 1, 2\).

Take \(A_i \in \mathcal{A}_i\) arbitrarily for \(i = 1, 2\). If \(x \in \text{int}_{\mu_1}(A_1) \cap \text{int}_{\mu_2}(A_2)\) then \(\{A_i, X \setminus \{x\}\} \in \mu_i\) for \(i = 1, 2\) and since \(A_1, X \setminus \{x\}\) \(\wedge A_2, X \setminus \{x\}\) \(\prec \{A_1 \cap A_2, X \setminus \{x\}\}\) it follows that \(x \in \text{int}_\mu(A_1 \cap A_2)\).

Thus we have \(\text{int}_{\mu_1}(A_1) \cap \text{int}_{\mu_2}(A_2) \subseteq \text{int}_\mu(A_1 \cap A_2)\) and because of \(A_1 \wedge A_2 \prec \mathcal{A}\) we find \(A \in \mathcal{A}\) such that \(A_1 \cap A_2 \subseteq A\) and hence \(\text{int}_{\mu_1}(A_1) \cap \text{int}_{\mu_2}(A_2) \subseteq \text{int}_\mu(A_1 \cap A_2)\).
int\(_\mu(A)\). So we have that \(\mathcal{A}_1' \land \mathcal{A}_2'\) is a refinement of \(\{\text{int}_\mu(A) : A \in \mathcal{A}\}\) and hence the latter set belongs to \(\mu\) which shows that \((X, \mu)\) is indeed a nearness space. The “moreover” part is clear.

Now we are ready to prove the bireflectivity of \textbf{Near} in \textbf{Mer}.

**Theorem.** \textbf{Near} is bireflective in \textbf{Mer}.

**Proof.** Let \((X, \mu)\) be any merotopic space and put

\[\mathcal{M} = \{\nu \subseteq c(X) : (X, \nu) \in \textbf{Near} \text{ and } \nu \subseteq \mu\}\]

The set \(\mathcal{M}\) is partially ordered by inclusion and \(\mathcal{M}\) is non-empty, because \(\{A \subseteq \mathcal{P}(X) : X \in \mathcal{A}\}\) is an element of \(\mathcal{M}\).

If \(\mathcal{S}\) is any non-empty chain in \(\mathcal{M}\) then it is easy to see that \(\bigcup \mathcal{S}\) is again in \(\mathcal{M}\) and hence by Zorn’s Lemma there is a maximal element \(\tilde{\mu}\) of \(\mathcal{M}\).

Since \(\tilde{\mu} \subseteq \mu\) the identity map \(\text{id}_X : (X, \mu) \to (X, \tilde{\mu})\) is uniformly continuous.

Now if \((Y, \nu)\) is another merotopic space and \(f : (X, \mu) \to (Y, \nu)\) is uniformly continuous we can put

\[\mu_f = \{B \in c(X) : \exists A \in \nu \text{ with } f^{-1}[A] \prec B\}\]

and show exactly as in the proof from \[6, \text{Theorem 3.2.2.5}\] that \((X, \mu_f)\) is a nearness space. Next we define

\[\tilde{\mu} = \{A \in c(X) : \exists A_1 \in \tilde{\mu}, \exists A_2 \in \mu_f \text{ such that } A_1 \land A_2 \prec A\}\]

By the preceding lemma \((X, \tilde{\mu})\) is a nearness space and \(\tilde{\mu}, \mu_f \subseteq \tilde{\mu}\). Since \(f\) is uniformly continuous with respect to \(\mu\) and \(\nu\) it follows that \(\mu_f \subseteq \mu\) and because \(\tilde{\mu}\) is also contained in \(\mu\) it follows that \(\tilde{\mu} \subseteq \mu\), in other words \(\tilde{\mu} \in \mathcal{M}\) and by the maximality of \(\tilde{\mu}\) we must have \(\tilde{\mu} = \tilde{\mu}\). Hence \(\mu_f \subseteq \tilde{\mu}\) which implies that \(f : (X, \tilde{\mu}) \to (Y, \nu)\) is uniformly continuous. Thus we have shown that \((X, \tilde{\mu})\) is our desired bireflective modification of \((X, \mu)\) with respect to \textbf{Near}.

**References**

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DEPARTMENT OF MATHEMATICS
UNIVERSITÄT LEIPZIG
AUGUSTUSPLATZ 10, 04109 LEIPZIG
GERMANY

E-mail address: hardtke@math.uni-leipzig.de