SOME REMARKS ON THE SCHWEITZER COMPLEX

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Dedicated to the memory of J.-P. Demailly

ABSTRACT. We prove that the Schweitzer complex is elliptic and its cohomologies define cohomological functors. As applications, we obtain finite dimensionality, a version of Serre duality, restrictions of the behaviour of cohomology in small deformations, and an index formula which turns out to be equivalent to the Hirzebruch-Riemann-Roch relations.

INTRODUCTION

The Bott-Chern and Aeppli cohomology are classical and well-established invariants of complex manifolds [1, 5]. Given a complex manifold \(X\), with double complex of forms \((\mathcal{A}_X(X), \partial, \bar{\partial})\) they are defined as

\[
H^{p,q}_{BC}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im} \partial + \text{im} \bar{\partial}} \quad \text{and} \quad H^{p,q}_A(X) := \frac{\ker \partial \bar{\partial}}{\text{im} \partial + \text{im} \bar{\partial}}.
\]

M. Schweitzer and J. P. Demailly [10, 6] have shown that they arise as the cohomology groups in certain degrees of a differential complex \(L^{p,q}_\cdot(X) := (L^{p,q}_\cdot(X), d_L)\), the definition of which we recall below. The cohomologies of this complex appear naturally when considering higher-length Aeppli-Bott-Chern Massey products [9], but have received relatively little attention otherwise. In this short note, we will establish some basic properties of the Schweitzer complex. More precisely, we show:

**Theorem A.** Let \(X\) be a complex manifold and \(L^{p,q}_\cdot(X)\) its Schweitzer complex.

1. The complex \(L^{\cdot,q}_\cdot(X)\) is elliptic.
2. For every \(p, q, k \in \mathbb{Z}\), the association \(X \mapsto H^k(L^{p,q}_\cdot(X))\) defines a cohomological functor in the sense of [11, 12].

From now on, let us assume \(X\) to be compact and of dimension \(n\). Then we obtain:

**Corollary A.1.** The dimensions \(s_{p,q}^k(X) := \dim H^k(L^{p,q}_\cdot(X))\) are finite.

This has also been shown by Demailly [6, Thm. 12.4], using a different argument. Further, we have

**Corollary A.2.** For \(X\) connected, wedge product and integration induce a perfect pairing

\[
H^k(L^{p,q}_\cdot(X)) \times H^{2n-k-1}(L^{n-p+1,n-q+1}_\cdot(X)) \to \mathbb{C}.
\]

**Corollary A.3.** The numbers \(s_{p,q}^k(X_t)\) vary upper semi-continuously in holomorphic families.

We will illustrate that this last result gives new restrictions on the \(E_1\)-isomorphism-type under small deformations. In particular, one obtains semi-continuity results even for classical objects such as certain differentials in the Frölicher spectral sequence.
By ellipticity, we may apply the Atiyah-Singer index theorem and obtain equalities between the Euler-characteristics $\chi_{p,q}(X)$ of $L_{p,q}(X)$ and certain expressions $td_{p,q}(X)$ in characteristic numbers.

**Theorem B.** The relations $\chi_{p,q}(X) = td_{p,q}(X)$ are equivalent to the Hirzebruch-Riemann-Roch relations.

Theorem B is in accord with a conjecture made in [12], stating that any universal linear relation between cohomological invariants and Chern numbers of compact complex manifolds of a given dimension is a consequence of the Hirzebruch-Riemann-Roch relations.

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**Proofs of the Main Results**

**The Schweitzer complex.** Throughout, we fix an $n$-dimensional complex manifold $X$ and we denote by $A^{p,q}_X$ the sheaf of smooth complex valued $(p,q)$-forms. Given any fixed pair of integers $p, q \in \mathbb{Z}$, Schweitzer and Demailly [10, 6], define a simple complex $L^\cdot_{p,q}$ of locally free sheaves as follows:

$$ L^k_{p,q} := \bigoplus_{r+s=k \atop r<p, s<q} A^r,s_X $$

if $k \leq p + q - 2$

$$ L^k_{p,q} := \bigoplus_{r+s=k+1 \atop r\geq p, s\geq q} A^r,s_X $$

if $k \geq p + q - 1$

with differential $d_L$ given by:

$$ \ldots \xrightarrow{\text{pr}_\od} L^{p+q-3}_{p,q} \xrightarrow{\text{pr}_\od} L^{p+q-2}_{p,q} \xrightarrow{\partial\bar{\partial}} L^{p+q-1}_{p,q} \xrightarrow{d} L^{p+q}_{p,q} \xrightarrow{d} \ldots $$

We illustrate which components of $A_X$ contribute to $L^\cdot_{p,q}$ for $n = 3$, $(p, q) = (2, 1)$:

$$ A^{2,3}_X, A^{3,3}_X $$

$$ A^{2,2}_X, A^{3,2}_X $$

$$ A^{2,1}_X, A^{3,1}_X $$

$$ A^{0,0}_X, A^{1,0}_X $$

By construction, one has

$$ H^p_{BC}(X) = H^{p+q-1}(L^\cdot_{p,q}(X)) := H^{p+q-1}(L^\cdot_{p,q}) $$

and

$$ H^p_A(X) = H^{p+q-2}(L^\cdot_{p,q}(X)) := H^{p+q-2}(L^\cdot_{p,q}) $$

We will mainly be interested in the differential complex of global sections $L^\cdot_{p,q}(X)$.\footnote{Up to a shift in total degree, this complex is denoted $S_{p,q}(X)$ in [9]. Here we follow the indexing convention used in [10, 6].}
**Cohomological functors.** We recall [11, 12] that a cohomological functor on the category of compact complex manifolds is a functor to the category of finite dimensional vector spaces which factors as $H \circ A$, where $A : X \mapsto A_X := (\mathcal{A}_X(X), \partial, \bar{\partial})$ is the Dolbeault double complex functor and $H$ is a linear functor which vanishes on ‘squares’, i.e. double complexes of the form

$$
\begin{array}{ccc}
\mathbb{C} & \rightarrow & \mathbb{C} \\
\uparrow & & \uparrow \\
\mathbb{C} & \rightarrow & \mathbb{C}
\end{array}
$$

It is clear from the construction that $\mathcal{L}_{p,q}(X)$ depends only on $A_X$. For any double complex $A$, denote by $\mathcal{L}_{p,q}(A)$ the Schweitzer complex formed using $A$ instead of $\mathcal{A}_X(X)$ and denote $H^k_{S_{p,q}}(A) := H^k(\mathcal{L}_{p,q}(A))$. It is then clear that $H^k_{S_{p,q}}(A)$ is a linear functor (in particular it commutes with direct sums of double complexes) and what remains to show is the following:

**Lemma 1.** For any square $S_q$, we have $H^k_{S_{p,q}}(S_q) = 0$ for all $p, q, k \in \mathbb{Z}$.

**Proof.** Any square $S_q$ is concentrated in four bidegrees:

$$
S := \text{supp } S_q := \{(r, s), (r + 1, s), (r, s + 1), (r + 1, s + 1)\}.
$$

The structure of $\mathcal{L}_{p,q}(S_q)$ (and hence $H^k_{S_{p,q}}(S_q)$) depends on the relative position of $S$ and $T = T_A \cup T_B$ where

$$
T_A := \{(a, b) \in \mathbb{Z}^2 | a < p, b < q\}
$$

$$
T_B := \{(a, b) \in \mathbb{Z}^2 | a \geq p, b \geq q\}.
$$

There are three possibilities: $\#(S \cap T) \in \{0, 2, 4\}$, surveyed in the following table:

| Case 1: $S \cap T = \emptyset$ | Case 2: $\#(S \cap T) = 2$ | Case 3: $\#(S \cap T) = 4$ |
|---------------------------------|-----------------------------|-----------------------------|
| ![Diagram 1](null)             | ![Diagram 2](null)          | ![Diagram 3](null)          |
| $\mathcal{L}_{p,q}(S_q) = 0$   | $\mathcal{L}_{p,q}(S_q) = \mathbb{C} \xrightarrow{z} \mathbb{C}$ | $\mathcal{L}_{p,q}(S_q) = \mathbb{C} \hookrightarrow \mathbb{C}^2 \twoheadrightarrow \mathbb{C}$ |

In each case, clearly $H^k_{S_{p,q}}(S_q) = H^k(\mathcal{L}_{p,q}(S_q)) = 0$ for all $p, q, k$. □

Since for any compact $X$, there exists a decomposition $A_X = A^{eq}_X \oplus A^{zig}_X$ where $A^{eq}_X$ is a direct sum of squares and $A^{zig}_X$ is finite dimensional [11], this implies Corollary A.1.

**Proof of Corollary A.2.** We recall from [11, Cor. 20] that for any cohomological functor $H$, the integration pairing induces an isomorphism $H(A_X) \cong H(DA_X)$, where $DA_X$ denotes the dual double complex as in [11]. The bigraded components of the dual complex are given by $(DA_X)^{p,q} = (A_X^{p,q})^\vee$ and the differentials (up to sign) by pullback with the differentials of $A_X$. Therefore,

---

2If we consider all complex manifolds, one should modify this definition by lifting the dimension restriction on the target of $H$, but imposing the condition that $\dim H(Z)$ is finite dimensional for any (bounded) indecomposable double complex.
we have $\mathcal{L}_{p,q}^k(DA_X) = (\mathcal{L}_{n-p+1,n-q+1}^{2n-k-1}(A_X))^\vee$. But cohomology of a complex of vector spaces commutes with dualization, so

$$H^k(\mathcal{L}_{p,q}(A_X)) \cong H^k(\mathcal{L}_{p,q}(DA_X)) = H^{2n-k-1}(\mathcal{L}_{n-p+1,n-q+1}(X))^\vee.$$

\[\square\]

**Ellipticity.** Let $\pi : TX^\vee \longrightarrow X$ denote the projection map of the cotangent bundle and for any $\xi = \xi_x \in TX_x^\vee$ denote by $(L^\ast, \sigma) := (\pi^\ast\mathcal{L}_{p,q}(X), \sigma(d\mathcal{L}_{p,q}))(\xi)$ the symbol complex (c.f. [2]). To show ellipticity, we have to prove:

**Lemma 2.** For any $\xi \neq 0$, the symbol complex $(L^\ast, \sigma)$ is exact.

**Proof.** As is well-known, we have $\sigma(\partial)(\xi) = \xi^{1,0} \wedge \omega$ and $\sigma(\bar{\partial})(\xi) = \xi^{0,1} \wedge \omega$, where the superscripts mean projection to the corresponding bidegree. Without loss of generality, we may pick a local coordinate system $z_1, \ldots, z_n$ around $x$ such that $\xi = dz_1 + \xi_1$.

Let us check exactness of the following part of the complex. Exactness at stages of lower or higher degree is only notationally more cumbersome.

The first map is given by

$$\sigma^{p+q-4} : \omega^{p-3,q-1} + \omega^{p-2,q-2} + \omega^{p-1,q-3} \longrightarrow \xi^{1,0} \omega^{p-3,q-1} + \xi^{0,1} \omega^{p-2,q-2} + \xi^{0,1} \omega^{p-1,q-3}.$$

The second map is given by

$$\sigma^{p+q-3} : \omega^{p-2,q-1} + \omega^{p-1,q-2} \longrightarrow \xi^{1,0} \omega^{p-2,q-1} + \xi^{0,1} \omega^{p-1,q-2}.$$

The third map is given by

$$\sigma^{p+q-2} : \omega \longrightarrow \xi^{1,0} \xi^{0,1} \omega.$$

The fourth and fifth maps are given by $\omega \longmapsto \xi \omega$.

Now assume $\omega = \omega^{p-2,q-1} + \omega^{p-1,q-2} \in \ker \sigma^{p+q-3}$. Write

$$\omega^{p-2,q-1} = \xi^{1,0} \xi^{0,1} \omega_A + \xi^{1,0} \omega_B + \xi^{0,1} \omega_C + \omega_D$$

$$\omega^{p-1,q-2} = \xi^{1,0} \xi^{0,1} \omega_A' + \xi^{1,0} \omega_B' + \xi^{0,1} \omega_C' + \omega_D',$$

with $\omega_A, \omega_A', \ldots$ having no summand which contains a factor of $\xi^{1,0}$ or $\xi^{0,1}$. Then $\sigma(\omega) = 0$ translates into the three equations

$$\xi^{1,0} \xi^{0,1} \omega_C + \xi^{0,1} \xi^{1,0} \omega_B' = 0$$

$$\xi^{1,0} \omega_D = \xi^{0,1} \omega_D' = 0.$$
The second and third equation imply $\omega_D = \omega_D' = 0$ and the first implies $\omega_C = \omega_C'$. Hence, defining $\eta \in L^{p+q-4}_x$ as follows
\begin{align*}
\eta^{p-3,q-1} &= \xi^{0,1} \omega_A + \omega_B \\
\eta^{p-2,q-2} &= \omega_C \\
\eta^{p-1,q-3} &= -\xi^{1,0} \omega_A' + \omega_C',
\end{align*}

one obtains $\sigma(\eta) = \omega$.

Now, let $\omega \in L^{p+q-2}_x = A_{X,x}^{p-1,q-1}$. Write $\omega = \xi^{1,0} \xi^{0,1} \omega_A + \xi^{1,0} \omega_B + \xi^{0,1} \omega_C + \omega_D$ as before. Then $0 = \sigma(\omega) = \xi^{1,0} \xi^{0,1} \omega$ implies $\omega_D = 0$. Hence, defining $\eta \in L^{p+q-3}_x$ by
\begin{align*}
\eta^{p-2,q-1} &= \omega_B + \frac{1}{2} \xi^{0,1} \omega_A \\
\eta^{p-1,q-2} &= \omega_C - \frac{1}{2} \xi^{1,0} \omega_A,
\end{align*}

we get $\sigma(\eta) = \omega$.

Next, let $\omega \in L^{p+q-1}_x = A_{X,x}^{p,q}$. Write $\omega = \xi^{1,0} \xi^{0,1} \omega_A + \xi^{1,0} \omega_B + \xi^{0,1} \omega_C + \omega_D$ as above. Then sorting $0 = \sigma(\omega) = \xi^{1,0} \omega + \xi^{0,1} \omega$ by bidegree yields
\begin{align*}
\xi^{1,0} \xi^{0,1} \omega_C + \xi^{1,0} \omega_D &= 0 \\
\xi^{0,1} \xi^{1,0} \omega_B + \xi^{0,1} \omega_D &= 0.
\end{align*}

In particular, $\omega_D = -\xi^{0,1} \omega_C$ and therefore $0 = \omega_D = \omega_C = \omega_B$. Thus, $\omega = \sigma(\omega_A)$. Finally, let $\omega = \omega^{p,q+1} + \omega^{p+1,q} \in A_{X,x}^{p,q+1} \oplus A_{X,x}^{p+1,q} = L^{p+q}_x$ s.t. $0 = \sigma(\omega) = \xi \land \omega$. Again, write
\begin{align*}
\omega^{p,q+1} &= \xi^{1,0} \xi^{0,1} \omega_A + \xi^{1,0} \omega_B + \xi^{0,1} \omega_C + \omega_D \\
\omega^{p+1,q} &= \xi^{1,0} \xi^{0,1} \omega_A' + \xi^{1,0} \omega_B' + \xi^{0,1} \omega_C' + \omega_D'.
\end{align*}

Then $\sigma(\omega) = 0$ translates into
\begin{align*}
0 &= \xi^{0,1} \xi^{1,0} \omega_B + \xi^{0,1} \omega_D \\
0 &= \xi^{1,0} \xi^{0,1} \omega_C + \xi^{1,0} \omega_D + \xi^{0,1} \xi^{1,0} \omega_B' + \xi^{0,1} \omega_D' \\
0 &= \xi^{1,0} \xi^{0,1} \omega_C' + \xi^{1,0} \omega_D'.
\end{align*}

This implies $\omega_D = \omega_B = \omega_C = \omega_D' = 0$ and $\omega_C = \omega_C'$. Defining
$$\eta := -\xi^{1,0} \omega_A + \omega_C + \xi^{0,1} \omega_A' \in L^{p+q-1}_x = A_{X,x}^{p,q},$$

we have $\sigma(\eta) = \omega$. \hfill \Box

**Remark 1.** Ellipticity immediately gives a second proof of Corollary A.1, see [2, Prop. 6.5.]

**Small deformations.** Let us prove Corollary A.3:

**Proof.** Pick a hermitian metric. For any $k \neq p + q$, the order of $d^k_{L_{p,q}}$ is 1 and thus for $k \neq p + q, p + q - 1$, the operators $\Delta^k_{p,q} := (d^k_{L_{p,q}})^* d^k_{L_{p,q}} + d^{k-1}_{L_{p,q}} (d^{k-1}_{L_{p,q}})^*$ are elliptic and their kernel is isomorphic to $H^k(L^*_x (X))$. They vary smoothly in families and hence the result follows. For $k = p + q, p + q - 1$, i.e. for Bott-Chern and Aeppli cohomology, the Corollary is known. (In that case, $\Delta^k_{p,q}$ as defined above is not elliptic, but an appropriate modification of it is, see [2, sect. 6] for a general statement or [10, 6] for an explicit construction in this case.) \hfill \Box
As an example we use this result to obtain semicontinuity properties for a classical object: the Frölicher spectral sequence [7]. We denote by \( e^{p,q}_r := \dim E^{p,q}_r \) the dimensions of the bigraded pieces on the \( r \)-th page of this spectral sequence. The dimensions on the first page \( e^{p,q}_1 = h^{p,q}_0 \) are known to behave upper semi-continuously, but for later pages this is false in general. However, denoting by \( FD^{p,q} := e^{p,q}_1 - e^{p,q}_\infty \), we may show:

**Corollary 1.** Let \( X_t \) be a small deformation with \( X_0 = X \) compact of dimension \( n \). Then for \( t \) sufficiently close to zero, there are inequalities

\[
FD^{0,1}(X_0) \geq FD^{0,1}(X_t) \quad \text{and} \quad FD^{0,n-1}(X_0) \geq FD^{0,n-1}(X_t).
\]

**Proof.** In order to avoid heavy notation which obscures the idea of the proof, we do the proof in the case of 3-folds, where it is easier to draw all necessary diagrams. We use [8, 11] that \( A_X \) may be decomposed into a direct sum of indecomposable double complexes, namely infinitely many squares and finitely many zigzags. The multiplicities \( \text{mult}_Z(A_X) =: \#Z(A_X) \) of these summands are an isomorphism invariant of \( A_X \). Furthermore, for each zigzag, its images under the involutions \( \tau : (p,q) \mapsto (q,p) \) ‘flipping along the diagonal’ and \( \sigma : (p,q) \mapsto (n-p, n-q) \) ‘flipping along the main antidiagonal’ appear with the same multiplicity, so, denoting by \( Z' \) the sum of the elements in the \( \langle \sigma, \tau \rangle \)-orbit of \( Z \) we may write \( \#Z' = \#Z \). Finally, certain zigzags always have zero multiplicity (‘only dots in the corners’), see [11, Ch. 4]. With this understood we compute:

\[
b_1(X) := 2 \cdot \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) \quad \text{and} \quad h^{0,1}_{BC}(X) = \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X),
\]

whereas

\[
e^{0,1}_\infty (X) = \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X).
\]

Therefore \( e^{0,1}_\infty = b_1 - h^{0,1}_{BC} \). Since \( b_1 \) stays constant in families and \( h^{0,1}_{BC} \) behaves upper semicontinuously, the first inequality, \( FD^{0,1}(X) \geq FD^{0,1}(X_t) \), follows. For the second one, we compute

\[
b_3(X) = 2 \cdot \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + 2 \cdot \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + 2 \cdot \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X),
\]

and, writing as before \( s_{p,q}^k(X) := \dim H^k(\mathcal{L}_{p,q}(X)) \),

\[
s_{1,0}^2(X) = \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + 2 \cdot \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + 2 \cdot \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + \#
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + \#
\begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X).
\]

On the other hand, \( FD^{0,2}(X) = h^{0,2}(X) - e^{0,2}_\infty(X) \) counts the dimensions of all differentials in the Frölicher spectral sequence starting at \((0,2)\), i.e.

\[
FD^{0,2}(X) = \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X) + \# \begin{array}{c}
\vdots \\
\vdots \\
\vdots
\end{array} (A_X).
\]
Therefore,

$$FD^{0,2} = h_{BC}^{0,3} + s_{1,0}^2 - b_3,$$

which is upper semi-continuous since the first two terms are and $b_3$ is constant. \hfill \Box

Remark 2. The proof also shows that $e_{\infty}^{0,1}$ behaves lower semi-continuously. The result should be compared to the fact that $e_{\infty}^k = \bigoplus_{p+q=k} e_{\infty}^{p,q}$ is constant along deformations and the total Frölicher defect $FD^k = \bigoplus_{p+q=k} FD^{p,q} = h^k - b_k$ is upper semi-continuous.

**Index formulae.** Denote by $\chi_p(X) := \sum_q (-1)^q h^{p,q}(X)$ and by $td_p(X) := \int_X Td(X)ch(\Omega^p)$, where $Td(X)$ denotes the Todd class and $ch$ the Chern character. The Hirzebruch-Riemann-Roch relations can be expressed as $\chi_p(X) = td_p(X)$ for all $p \in \mathbb{Z}$, see [3].

Similarly, let us denote by $\chi_{p,q}(X) := \sum_k (-1)^k \dim H^k(L_{p,q}^*)$ the analytical index of $L_{p,q}^*$ and by $\text{td}_{p,q}(X)$ its topological index, which is the pairing of the fundamental class $[X]$ with a universal characteristic class which depends only on the $K$-theory class $[L_{p,q}^*] := \sum_k (-1)^k [L_{p,q}^k] \in K(X)$. The Atiyah-Singer index theorem [3, 4] then yields:

**Theorem 1 (ABC index formulae).** For any compact complex manifold $X$ and $p, q \in \mathbb{Z}$,

$$\chi_{p,q}(X) = \text{td}_{p,q}(X).$$

Remark 3. Strictly speaking, [3, 4] treat only elliptic complexes where all operators have order one, which is not the case for $L_{p,q}(X)$. However, the validity of Theorem 1 also follows a posteriori from Theorem B.

**Proof of Theorem B.** Recall from [6, 10] that there are subcomplexes $(S_p^*, \partial)$ and $(\bar{S}_q^*, \bar{\partial})$ of $L_{p,q}$ defined as follows (if $p, q \geq 1$):

$$S_p^k := \begin{cases} \Omega_X^k & \text{if } 0 \leq k \leq p - 1 \\ 0 & \text{else.} \end{cases} \quad \bar{S}_q^k := \begin{cases} \bar{\Omega}_X^k & \text{if } 0 \leq k \leq q - 1 \\ 0 & \text{else.} \end{cases}$$

If $p = 0$ or $q = 0$, one sets instead $S_0^{0} = \mathbb{C}$, resp. $\bar{S}_0^{0} = \mathbb{C}$ and all other components equal to 0. Set $S_{p,q}^k := S_p^k + \bar{S}_q^k$, the sum being direct except for $k = 0$. It is then known [6, 10] that $S_{p,q} \hookrightarrow L_{p,q}$ is a quasi-isomorphism. Further, there is a short exact sequence of complexes

$$0 \longrightarrow \mathbb{C} \longrightarrow S_p^* \oplus \bar{S}_q^* \longrightarrow S_{p,q}^* \longrightarrow 0.$$

For a complex $C$ of abelian sheaves on $X$, denote by $\chi(C) := \sum_k (-1)^k \dim \Omega^k(C)$ its hypercohomology Euler characteristic. Then, using $\chi_p = (-1)^n \chi_{n-p}$, there is an equality of Euler characteristics:

$$\chi_{p,q}(X) = \chi(S_{p,q}^*) = \chi(S_p^*) + \chi(\bar{S}_q^*) - \chi(\mathbb{C})$$

$$= \sum_{k=0}^{p-1} (-1)^k \chi_k + \sum_{k=0}^{q-1} (-1)^k \chi_k - \sum_{k=0}^{n} (-1)^k \chi_k$$

$$= \sum_{k=0}^{p-1} (-1)^k \chi_k + \sum_{k=n-q}^{n} (-1)^k \chi_k - \sum_{k=0}^{n} (-1)^k \chi_k = \sum_{k=p}^{n-q} (-1)^{k+1} \chi_k$$

To identify the characteristic number expressions, we instead identify the $K$-theory classes of the relevant complexes, using only the relation $\langle A_{X,q}^{n,q} \rangle \cong A_{X,n-q}^{n,q,n-p}$. Assume for simplicity $p + q \leq n$
and $p > q$, the other cases are similar. The following chain of equalities in $K(X)$ might be easier to follow with the following picture in mind, which illustrates the case $n = 3$, $(p, q) = (2, 1)$:

\begin{equation}
\mathcal{L}^\bullet_{p,q} = \sum_{k \in \mathbb{Z}} (-1)^k \mathcal{L}^k_{p,q} = \sum_{k \leq p+q-2} (-1)^k \sum_{r+s=k \atop r<s<p<q} \mathcal{A}^r_s + \sum_{k \geq p+q-1} (-1)^k \sum_{r+s=k+1 \atop r \geq p, s \geq q} \mathcal{A}^r_s
\end{equation}

\[= \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} (-1)^{r+s} \mathcal{A}^r_s + \sum_{r=p}^{n} \sum_{s=q}^{n} (-1)^{r+s} \mathcal{A}^r_s + \sum_{r=p}^{n} \sum_{s=q}^{n} (-1)^{r+s-1} \mathcal{A}^r_s + \sum_{r=n-q+1}^{n-p+1} \sum_{s=n-p+1}^{n-q} (-1)^{r+s-1} \mathcal{A}^r_s\]

\[= \sum_{r=p}^{n-q} \sum_{s=q}^{n} (-1)^{r+s} \mathcal{A}^r_s + \sum_{r=n-q+1}^{n-p} \sum_{s=n-p+1}^{n-q} (-1)^{r+s-1} \mathcal{A}^r_s + \sum_{r=p}^{n-q} \sum_{s=0}^{n-q} (-1)^{r+s-1} \mathcal{A}^r_s = (n-q)^2 \mathcal{A}^r_s + \sum_{r=p}^{n-q} \sum_{s=0}^{n-q} (-1)^{r+s-1} \mathcal{A}^r_s\]

And therefore:

\[td_{p,q} = \sum_{r=p}^{n-q} (-1)^{r+1} td_p,\]

which implies the theorem since $\chi_{p,q} = td_{p,q}$ for all $p, q$ if and only if $\chi_p = td_p$ for all $p$. \qed

**Remark 4.** In particular, we have shown the following relation between Dolbeault and Schweitzer cohomologies:

\[\chi_{p,q}(X) = \sum_{k=p}^{n-q} (-1)^{k+1} \chi_p(X).\]

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