A Critical Assessment of Turbulence Models for 1D Core-Collapse Supernova Simulations

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ABSTRACT
It has recently been proposed that global or local turbulence models can be used to simulate core-collapse supernova explosions in spherical symmetry (1D) more consistently than with traditional approaches for parameterised 1D models. However, a closer analysis of the proposed schemes reveals important consistency problems. Most notably, they systematically violate energy conservation as they do not balance buoyant energy generation with terms that reduce potential energy, thus failing to account for the physical source of energy that buoyant convection feeds on. We also point out other non-trivial consistency requirements for viable turbulence models. The model of Kuhfuss (1986) proves more consistent than the newly proposed approaches for supernovae, but complete consistency cannot be achieved unless the anelastic approximation is fully discarded. We perform numerical simulations for a 20 M⊙ progenitor to further illustrate problems of 1D turbulence models. If the buoyant driving term is formulated in a conservative manner, the explosion energy of \( \sim 2 \times 10^{51} \) erg for the corresponding non-conservative turbulence model is reduced to \(< 10^{48} \) erg even though the shock expands continuously. This demonstrates that the conservation problem cannot be ignored. Although plausible energies can be reached using an energy-conserving model when turbulent viscosity is included, it is doubtful whether the energy budget of the explosion is regulated by the same mechanism as in multi-dimensional models. We conclude that 1D turbulence models based on a spherical Reynolds decomposition cannot provide a more consistent approach to supernova explosion and remnant properties than other phenomenological approaches before some fundamental problems are addressed.

Key words: supernovae: general – convection – hydrodynamics – turbulence – stars: massive

1 INTRODUCTION
According to our present understanding, the explosions of massive stars as core-collapse supernovae depend critically on the breaking of spherical symmetry in the supernova core (Janka 2012; Foglizzo et al. 2015; Müller et al. 2016b) except in the case of the least massive progenitor stars (Kitaura et al. 2006). In the neutrino-driven paradigm, the breaking of spherical symmetry is mediated by two instabilities, namely buoyancy-driven convection (Herant et al. 1994; Burrows et al. 1995; Janka & Müller 1995) and the standing-accretion shock instability (Blondin et al. 2003; Foglizzo et al. 2007), which manifests itself in the form of global sloshing or spiral motions of the shock. The resultant multi-dimensional (multi-D) fluid flow aids neutrino heating through a variety of interrelated effects, e.g. by mixing hot neutrino-heated and colder material from the vicinity of the shock, by providing turbulent pressure (Burrows et al. 1995; Murphy et al. 2013), by providing heating close to the shock by secondary shocks (Müller et al. 2012b), and by turbulent dissipation (Mabanta & Murphy 2018).

There have been attempts to distil these effects back into an effective one-dimensional (1D) description using an appropriate turbulence model. On the one hand, such a 1D turbulence model for the supernova core may lead to a better conceptual understanding of the role of multi-D effects (Murphy & Meakin 2011; Mabanta & Murphy 2018). On the other hand, one might hope that effective 1D models of neutrino-driven supernovae could provide an efficient way to predict the “explodability” and even the explosion properties across a population of progenitor models at a cheaper cost than full-blown multi-D models, but with greater rigour and consistency than more parameterised approaches like those of O’Connor & Ott (2010); Ugliano et al. (2012); Perego et al. (2015); Sukhbold et al. (2016); Müller et al. (2016a).

The simplest approach of adopting the mixing-length theory (MLT) for stellar convection (Biermann 1932; Böhm-Vitense 1958) to the supernova problem already dates back to the 1980s (Mayle 1985; Wilson & Mayle 1988). However, the extra convective energy transport provided by convection within the framework of

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MLT alone does not improve the heating conditions to such a degree as to allow explosions in spherical symmetry (Hudepohl 2014; Mirizzi et al. 2016).

More general turbulence models have been proposed to capture multi-D effects in the heating region more adequately (Murphy et al. 2013; Mabanta & Murphy 2018). Only recently have there been efforts to use such turbulence models predictively. Mabanta & Murphy (2018) incorporated a 1D turbulence model into steady-state solutions for the accretion flow onto a proto-neutron star to derive the reduction of the critical neutrino luminosity for shock revival (Burrows & Goshy 1993) due to multi-D effects. Following up on their earlier work, Mabanta et al. (2019) went on to include a simple turbulence model in dynamical simulations with a view to studying the explodability of supernova progenitors. Using a light-bulb model for neutrino heating and cooling, they found that their turbulence model roughly reproduces the reduction of the critical luminosity in multi-D for a reasonable choice of model parameters. Shortly thereafter, Couch et al. (2019) presented a time-dependent 1D turbulence model, which they coupled to the neutrino transport code of O’Connor & Couch (2018) and then used to explore the systematics of explodability, and explosion and remnant properties across the stellar mass range. One might thus hope that such effective 1D turbulence models can furnish a more “consistent” approach to the progenitor-explosion connection than current phenomenological models.

In this paper, we critically examine this idea. We shall argue that consistency and crucial physical principles such as energy conservation are difficult to achieve, especially during the explosion phase. We show that even the time-dependent approach of Couch et al. (2019) still suffers from inconsistencies. To remedy these, one can draw on the extensive literature on generalisations of mixing-length theory, which have been studied since the 1960s (e.g. Spiegel 1963; Unno 1967; Eggleton 1983; Kuhfuss 1986; Gehmeyr & Winkler 1992; Canuto 1993; Canuto & Dubovikov 1998; Wachterl & Feuchtengl 1998). Classic time-dependent turbulence models as developed by Kuhfuss (1986) and Wachterl & Feuchtengl (1998) offer solutions to most of the inconsistencies in the models of Mabanta et al. (2019) and Couch et al. (2019), but even then the behaviour of 1D turbulence models during the explosion phase is not fully satisfactory. We illustrate the remaining problems for a 20 M⊙ progenitor by implementing the model of Kuhfuss (1986) in the neutrino hydrodynamics code CoCoNuT (Muller et al. 2010; Muller & Janka 2015) with some necessary adaptations for the core-collapse supernova problem. The goal of our analysis and our numerical experiments is primarily to illustrate the pitfalls that crop up when one seeks to model supernova explosions in 1D by including the effects of turbulence. We do not aim to present a consistent solution to all of these problems, which does not appear within reach at the moment.

This paper is organised as follows: In Section 2 we provide the necessary background on 1D turbulence models and the conditions of turbulent flow in supernovae cores, and identify the terms that need to be included in 1D supernova simulations with a turbulence model to ensure basic physical consistency. In Section 3 we briefly describe the hydrodynamics code CoCoNuT and the numerical implementation of the turbulence model. We then present results for the test case in Section 4 and conclude by discussing future perspectives for 1D explosion modelling in Section 5.

2 1D TURBULENCE MODELS FOR CORE-COLLAPSE SUPERNOVAE

Both Mabanta et al. (2019) and Couch et al. (2019) start from a spherical Reynolds decomposition of the fluid equations, from which they discard some higher-order terms before they apply closure relations. Specifically, they ignore the turbulent mass flux in the spirit of the anelastic approximation, so that no turbulent correlation terms appear in the continuity equation. The common starting point for both models can be written as

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \]  
\[ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P = \rho \mathbf{g} - \nabla \cdot (\rho \mathbf{R}), \]  
\[ \partial_t (\rho e) + \nabla \cdot [(\rho e + P) \mathbf{v}] = \rho \varepsilon \mathbf{g} - \nabla \cdot (\varepsilon (\rho \mathbf{R})). \]

Here \( \rho, \mathbf{v}, P, \) and \( \varepsilon \) denote the fluid density, velocity, pressure, and total (internal+kinetic) energy density, \( \mathbf{g} \) denotes the gravitational acceleration, and \( \varepsilon \) is the volumetric neutrino heating rate. \( \mathbf{R} \) and \( F_{\text{conv}} \) are the Reynolds stress tensor and the turbulent energy flux obtained from the Reynolds decomposition. Spherical Reynolds averages are denoted by angled brackets, or by carets for individual variables. Primes, as used in the term \( \rho' \mathbf{v}' \) for buoyant energy generation, denote fluctuating quantities.

2.1 Model of Mabanta et al. (2019)

Mabanta et al. (2019) impose a closure relation by essentially assuming that the magnitude of turbulent radial velocity fluctuations \( \mathbf{v}' \) is constant within the gain region, and that turbulent dissipation \( \varepsilon \) balances buoyant energy generation if integrated over the entire gain region. With buoyant driving parameterised as \( \beta \mathbf{Q}_0 \), in terms of the volume-integrated heating rate \( \mathbf{Q}_0 \) and a calibrated parameter \( \beta \), and turbulent dissipation scaling as \( \varepsilon \sim \mathbf{v}'^3/\Lambda \), an effective dissipation length \( \Lambda \) (taken to be the width of the gain region), one obtains

\[ \mathbf{v}' = \frac{1}{\sqrt{\frac{Q_0}{M_{\text{gain}}}}} \]  
\[ \mathbf{v}' = \frac{r - r_g}{h}, \]

as also derived by Müller & Janka (2015). The Reynolds stress tensor is assumed to be diagonal with equipartition between the radial and non-radial components so that \( \mathbf{R}_r = \rho' v'^2 \) and \( \mathbf{R}_{\theta\theta} = \rho' v'^2/2 \). The turbulent dissipation \( \varepsilon \sim \mathbf{v}'^3/\Lambda \) is also used to supply the source term \( \rho' \mathbf{v}' \mathbf{g} \) for buoyant energy generation in Equation (3).

To obtain the convective flux, Mabanta et al. (2019) use a fit for the convective luminosity \( L_{\text{conv}} \approx 4\pi r^2 F_{\text{conv}} \) that is inspired by an analysis of multi-D simulations,

\[ L_{\text{conv}} = \alpha Q_0 \tan \frac{r - r_g}{h}. \]

Here \( r \) is the radial coordinate, \( r_g \) is the gain radius, \( h \) is an appropriately chosen transition width, and \( \alpha \) is a calibrated dimensionless parameter. Mabanta & Murphy (2018) use Equation (6) only up to the shock, where \( L_{\text{conv}} \) plummets to zero. The term \( \nabla \cdot (\mathbf{v}' \cdot \mathbf{R}) \) for the work exerted by Reynolds stresses is neglected in the energy equation.

Although the prescriptions for the Reynolds stresses and the convective luminosity are in line with multi-D simulations, there is an obvious question about energy conservation in this model. Integrating Equation (3) under the assumption that turbulent dissipation...
locally balances buoyant energy generation results in

$$\frac{\partial}{\partial t} \int \rho_e dV = \int \rho \hat{\mathbf{v}} \cdot \mathbf{g} dV + \int \rho_e dV + \int q_r dV$$

The work done by gravitational forces on the spherically averaged flow does not violate energy conservation and cancels if gravitational potential energy is included in the budget (Shu 1992). Similarly, the contribution \( \dot{Q}_F \) of neutrino heating does not violate total energy conservation if it is obtained from a conservative neutrino transport scheme. However, the turbulence model introduces an extra term \( \dot{Q}_T \) that violates energy conservation. As we shall see, such a term also appears in the model of Couch et al. (2019).

It has been argued (Q. Mabanta, private communication; Couch et al. 2019) that energy is still effectively conserved if one accounts for the available free energy associated with convectively unstable gradients. Although this argument is not entirely incorrect, it does not convincingly justify the use of a 1D turbulence model that does not manifestly conserve energy, but rather points to a loophole in the model: In the full multi-D problem energy model that does not manifestly conserve energy, but rather points to unstable gradients. Although this argument is not entirely incorrect, it does not convincingly justify the use of a 1D turbulence model that does not manifestly conserve energy, but rather points to a loophole in the model: In the full multi-D problem energy model that does not manifestly conserve energy, but rather points to unstable gradients.

The equations come down to

$$\frac{\partial \rho v_r}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 (\rho v_r^2 + P + \rho v^2) \right) = -\rho g$$

where \( \rho = \rho_0 \) is chosen as a multiple of the pressure scale height \( \Lambda = a P/(\rho g) \), where \( a \) is a tunable parameter of order unity. Note that we have tacitly corrected a typo in their Equation (26), where the term \( \rho D V e \) should not be multiplied by the radial velocity \( v_r \).

The work done by gravitational forces on the spherically averaged energy equation in Equation (3) is formally correct, so how is it possible that the multi-D hydro equations conserve total energy whereas the spherically Reynolds-averaged equation does not? The answer is that the system (1-3) does not include all terms from the Reynolds decomposition consistently: Specifically, it ignores the turbulent mass flux \( \rho (\rho v_r) v_r \) in Equation (1) and thus neglects the (instantaneous) reduction of potential energy due to changes in the density profile, but includes it in the source term in Equation (3). In Section 2.3, we shall further discuss to what extent this problem can be fixed.

2.2 Model of Couch et al. (2019)

Couch et al. (2019) solve a separate evolution equation for the turbulent kinetic energy \( v^2 \) and compute the turbulent flux in the energy equation using an MLT closure. They assume the closure \( R_{\text{eff}} = 2R_{\text{typ}} = 2R_{\text{vis}} = \rho v^2 \) for the Reynolds stress tensor as Mabanta et al. (2019). In their model, the momentum and energy

where it is unclear whether such a feature can be handled properly by the hydrodynamics solver, i.e. whether it affects the propagation speed of the shock in an unphysical manner.
\[ \frac{d e}{d t} = -(P + P_f) \frac{d(1/\rho)}{d t}, \quad (11) \]

in the absence of heating and cooling terms from neutrinos and dissipation, instead of

\[ \frac{d e}{d t} = -P_f \frac{d(1/\rho)}{d t}. \quad (12) \]

This is tantamount to an artificial entropy source term

\[ T \frac{d s}{d t} = -P_f \frac{d(1/\rho)}{d t}. \quad (13) \]

It is noteworthy that this term can become negative, i.e. the turbulence model implicitly allows for a decrease of entropy even in the absence of physical source terms for cooling.

There is yet another problem with the model of Couch et al. (2019) that concerns the convective energy flux \( F_{\text{conv}} \). The model essentially assumes that \( F_{\text{conv}} \) can be computed by extrapolating the total energy density \( e \) to the original position of the convective bubbles using the local gradient to obtain the fluctuating part \( e' \):

\[ e' = \left( e + \frac{\nu^2}{2} \right) = \frac{\partial e}{\partial \rho} + \Lambda \frac{\partial e}{\partial \rho} + \frac{\partial^2 e}{\partial \rho^2}. \quad (14) \]

This, however, leads to unphysical results. Let us first consider the fluctuations of the internal energy density \( \epsilon \). To obtain the correct MLT flux, one needs to account for the \( P_{\text{DV}} \) done by the convective bubbles as they contract and expand while adjusting to the ambient pressure (Hudepohl 2014; Mirizzi et al. 2016). If the expansion/contraction is adiabatic, one obtains

\[ \epsilon' = \Lambda \frac{\partial e}{\partial \rho} + P_{\text{DV}} \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \right). \quad (15) \]

By expressing \( \partial e/\partial \rho \) in terms of the entropy and density gradients using the first law of thermodynamics, one obtains an important corollary: If the entropy gradient vanishes, then the convective energy flux also vanishes (assuming that there are no composition gradients). If one uses Equation (14) this is no longer guaranteed.

There is also a concern about the turbulent transport of bulk kinetic energy: This effect is included in Equation (9) via Equation (14), but there is no corresponding term for the turbulent transport of momentum (i.e. turbulent viscosity) in Equation (9).

### 2.3 The Energy-Conserving Model of Kuhfuss (1986)

Most of the problems discussed in Sections 2.1 and 2.2 are in fact nicely solved by time-dependent one-equation turbulence models that have been developed for stellar evolution (Stellingwerf 1982; Kuhfuss 1986; Wuchterl & Feuchtinger 1998), originally motivated by the problem of pulsations of RR Lyrae stars and Cepheids. Here we shall use the work of Kuhfuss (1986) (with some modifications by Wuchterl & Feuchtinger 1998) as a starting point. In their approach, the momentum equation in conservation form can be written as

\[ \frac{d \rho v_i}{d t} + \frac{\partial (\rho v_i v_j)}{\partial r} + \frac{\partial (P + P_f)}{\partial r} = -\rho \mathbf{g} \cdot \mathbf{v} + \frac{4}{3} \frac{\partial}{\partial r} \left[ \frac{\partial (\rho v_i / \rho)}{\partial r} \right]. \quad (16) \]

The underlying assumption about the form of the Reynolds stress tensor differs slightly from Mabanta et al. (2019) and Couch et al. (2019): it is decomposed into a trace component – the turbulent pressure \( P_f \) – and a trace-free component modelled after the the viscous Navier-Stokes equations. This trace-free term gives rise to the additional turbulent viscosity term on the RHS with a turbulent dynamic viscosity \( \mu \), which is expressed in the spirit of MLT as \( \alpha_g \rho \mathbf{v}^2 \), where \( \alpha_g \) is a dimensionless coefficient of order unity. Although the assumed form of the trace-free term can be criticised as ad hoc, it has the virtue of ensuring that it can be matched with a viscous term in the energy equation that can be expressed as a flux divergence (to ensure energy conservation) and always results in an increase of fluid entropy.

In the theory of Kuhfuss (1986), total energy conservation is ensured by consistently including the mass-specific turbulent kinetic energy \( \sigma \) in the energy equation. Kuhfuss (1986) originally formulated an extended internal energy equation including \( \sigma \) as the internal energy of an ‘eddy gas’, but this equation can be easily recast into a total energy equation analogous to Equation (3),

\[ \frac{\partial (e + \sigma)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \frac{r^2 (e + \sigma) v_r}{} + r^2 (P + P_f) v_r \right] + \frac{\partial^2 (F_{\text{conv}} + F_\alpha + F_{\text{visc}})}{\partial r} = -\rho \mathbf{v} \cdot \mathbf{g}. \quad (17) \]

Here, \( F_{\text{conv}} \) and \( F_\alpha \) denote the convective flux of internal energy and turbulent kinetic energy, and \( F_{\text{visc}} \) is the energy flux from viscous turbulent stresses. Equation (17) is manifestly conservative because all the turbulent effects are lumped into flux divergence terms.

Finally, Kuhfuss (1986) and Wuchterl & Feuchtinger (1998) also formulate an evolution equation for the turbulent kinetic energy \( \sigma \). With one important modification, the equation for \( \sigma \) can be written as

\[ \frac{\partial \rho v_i}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \frac{r^2 \rho \sigma v_r}{} + \frac{P_{\text{DV}}}{r^2} \frac{\partial \sigma}{\partial r} \right] + \frac{\partial^2 (F_{\text{conv}} + F_\alpha + F_{\text{visc}})}{\partial r} = -(\rho' v') g - C_D \frac{\rho v^2}{\Lambda}. \quad (18) \]

in Eulerian form. Here, \( \alpha_g \) and \( C_D \) are dimensionless coefficients for turbulent diffusion of turbulent kinetic energy and turbulent dissipation. Different from Kuhfuss (1986) and Wuchterl & Feuchtinger (1998), we have omitted the viscous dissipation term from the turbulent kinetic energy equation. Although it seems plausible that the turbulent viscosity should should initially feed kinetic energy from the bulk flow into disordered small-scale motion (i.e. into turbulent kinetic energy), this has undesirable consequences. The problem is that the viscous heating term is linear in \( v' \) just like the buoyant driving term. If included in the equation for \( \sigma \), it would act like a destabilising gradient wherever \( \partial v_i / \partial r - v_i / r \neq 0 \) (i.e. almost everywhere) with \( \partial v_i / \partial r - v_i / r \) taking the place of the Brunt-Väisälä frequency \( \omega_{BV} \) as the growth rate. From the viewpoint of numerical stability, such behaviour would be disastrous at the shock, but it is also clearly unphysical. Shifting the viscous heating term to the internal energy equation seems the only viable solution.

Like Equation (10), the turbulent energy equation includes terms for the advection and the turbulent diffusion of turbulent kinetic energy (the two flux divergence terms on the left-hand side). However, there is also a term that effectively accounts for \( P_{\text{DV}} \) term on the “eddy gas”. Including this term in Equation (18) ensures that the work exerted by the turbulent pressure correctly enters as \(-\nabla P_f\) in the corresponding equation for \( p e \) without turbulent kinetic energy, and hence does not change the internal energy density of the bulk flow. Apart from dimensionless coefficients of order unity, the source term for buoyant driving and the dissipation term are essentially the same as in Equation (10); the driving term can again be expressed in terms of the Brunt-Väisälä frequency as \(-\rho'(v')g \propto \rho v' \omega_{BV} \Lambda\).
For further analysis, it is useful to consider the internal energy equation for the spherical background flow as well, which reads

\[
\frac{\partial \rho \varepsilon}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 \rho v_r \varepsilon \right] + \frac{P}{r} \frac{\partial}{\partial r} \left[ r^2 \rho \sigma^{1/2} \left( \frac{\partial \varepsilon}{\partial r} + P \frac{\partial (1/\rho)}{\partial r} \right) \right] = \left( \rho' \varepsilon' \right) g + C_D \rho \sigma^{1/2} \frac{v}{\Lambda} + \frac{4}{3} \alpha_c P \Lambda \left[ \frac{\partial v_r}{\partial r} - \frac{v_r}{r} \right],
\]

where \( \sigma \) is another dimensionless coefficient of order unity and Equation (15) has been used to express the convective flux of internal energy. Note that the viscous dissipation term appears in the internal energy equation for reasons explained above.

Equations (17), (18), and (19) ensure that i) entropy is conserved during expansion and contraction in the absence of diffusive flux and source/sink term, that ii) the convective flux transports energy into the direction of negative entropy gradients, and that iii) total energy is conserved. Energy conservation is achieved by including a sink term due to buoyant driving in the internal energy equation, and not including a source term for buoyant driving in the total energy equation. This means that energy conservation still comes at a price: Within the model, turbulent kinetic energy is generated at the expense of a decrease in entropy in the bulk flow. However, in the context of supernova modelling this still appears preferable to sacrificing a crucial conservation law altogether. Since convection in the gain region is driven by neutrino heating (i.e. entropy production) in the first place, it is difficult to conceive of a situation where there is an unphysical runaway decrease of bulk entropy, whereas permanent artificial energy injection into the system is bound to critically affect the energy budget in the gain region, which is the very essence of what the turbulence models intend to reproduce in the first place.

One may justifiably wonder whether there is any possibility to construct a turbulence model that both conserves energy and can generate turbulent kinetic energy without affecting the entropy of the bulk flow. As already outlined in Section 2.1, one crucial obstacle for such an undertaking is that this would at least require additional terms in the continuity equation to effect a change in potential energy that exactly balances buoyant energy generation. However, the problem runs deeper: Once the anelastic approximation is relinquished to allow for a non-vanishing turbulent mass flux, one also needs to include more turbulent correlation terms in the turbulent fluxes of momentum and energy; otherwise one is likely bound to encounter consistency problems similar to those outlined before, e.g. spurious entropy-changing terms or spurious turbulent fluxes. Furthermore, constructing appropriate closures for the turbulent fluctuations that include compressibility effects is anything but trivial (see, e.g., Canuto 1993, and also Duffell 2016 for a thoughtfull attempt in the case of the compressible Rayleigh-Taylor instability). For this reason, we do not pursue our analysis of turbulence models any further beyond the model of Kuhfuss (1986) and Wuchterl & Feuchtinger (1998), and rather proceed to demonstrate some other problems by means of numerical simulations.

### 3 NUMERICAL METHODS

In order to illustrate the explosion dynamics of 1D models that incorporate effects of turbulence, we implement the model from Section 2.3 in the neutrino radiation hydrodynamics code CoCoNuT-FMT (Müller et al. 2010; Müller & Janka 2015) and run three different variations of the model for the 20M\(_{\odot}\) progenitor of Woosley & Heger (2007). CoCoNuT-FMT is a finite-volume code for general relativistic hydrodynamics in spherical polar coordinates with the xFCF approximation for the metric (Cordero-Carrion et al. 2009). It uses higher-order reconstruction and an approximate Riemann solver combined with a stationary, fast multi-group neutrino transport (FMT) scheme as described in Müller & Janka (2015).

Since CoCoNuT-FMT is a relativistic code, some care is required to implement the turbulence model from Section 2.3, which is formulated in the Newtonian approximation. Since general relativistic effects are unimportant and velocities are well below the speed of light in the gain layer as the main region of interest, one can dispense with a detailed re-derivation of the equations in full relativity and simply include a few heuristic modifications: All flux terms pick up an extra factor \(N\phi^4\) in terms of the lapse function \(N\) and the conformal factor \(\phi\) of the conformally flat metric, and all conserved quantities pick up an extra factor \(\phi^6\). Moreover, we use the relativistic expression for the MLT density contrast \(\rho'\) (cp. Müller et al. 2013),

\[
\rho' = \Lambda \left( \frac{\partial \rho (1 + e/c^2)}{\partial r} - \frac{1}{c^2} \frac{\partial P}{\partial r} \right),
\]

when computing the Brunt-Väisälä frequency and buoyant driving. Whether one computes the gravitational acceleration simply as the derivative of the lapse function or includes relativistic correction factors is immaterial in practice.

Reflecting the rough equipartition between radial and non-radial velocity fluctuations in multi-D simulations, we assume that the radial velocity fluctuations \(v'_r\) are related to the total turbulent kinetic energy as \(v'_r = \sigma^{1/2}\). Different from Kuhfuss (1986), we use \(P_1 = \rho v'_r^2 = \rho a_1 v'_r^2\) (as originally proposed by Stellingwerf 1982), which makes for a larger force from the turbulent pressure gradient in the momentum equation. Although this is somewhat inconsistent, it does not fundamentally alter the structure and properties of the turbulence model. All other dimensionless coefficients are set to unity for the sake of simplicity.

The terms in the turbulence model are implemented in an operator-split approach except for the convective energy flux and the corresponding turbulent fluxes in the equations for the mass fractions and the electron fraction \(Y_e\); these are added directly to the hydro fluxes obtained from the Riemann solver.

The operator-split update of the turbulent kinetic energy and the bulk fluid energy and momentum are staged as follows: We first integrate the terms for buoyant energy generation and dissipation,

\[
\frac{\partial \sigma}{\partial t} = \max(a_{\text{diss}} \Lambda \sigma^{1/2}, 0) - \rho \sigma^{3/2} \frac{\Lambda_{\text{diss}}}{A_{\text{diss}}},
\]

Here we deviate from Kuhfuss (1986) and Wuchterl & Feuchtinger (1998) by introducing a dissipation length \(\Lambda_{\text{diss}}\) that may be different from the mixing length \(\Lambda\). The rationale for this is that the identification of \(\Lambda_{\text{diss}}\) and \(\Lambda\) can lead to excessive overshoot into the convectively stable atmosphere of the proto-neutron star. We limit the dissipation length by balancing the turbulent kinetic energy and the work against buoyancy for overshooting by a distance \(\Lambda_{\text{diss}}\) (cp. Murphy et al. 2009),

\[
\Lambda_{\text{diss}} = \min \left[ \Lambda, \sqrt{\frac{-\sigma^{2}}{2a_{\text{diss}}}}, 10^{-10} \text{ cm} \right].
\]

Since the source term for buoyant driving vanishes for \(\sigma = 0\), care must be taken to ensure that convective motions can actually grow. One possibility is to impose a small seed value for \(\sigma\) as done by Couch et al. (2019). We instead circumvent this problem altogether by evolving \(v'_r\), whose evolution equation can be obtained using the
chain rule:
\[
\frac{\partial v'}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\omega_b^2 \Lambda}{2} \right) \frac{\partial}{\partial \rho} \left( \frac{\sigma^2}{\Lambda_{\text{diss}}} \right).
\]
(23)

No seed for \( v' \) is required since \( \partial v'/\partial t \) is non-zero even for \( v' = 0 \). Since this equation is stiff if \( \Lambda_{\text{diss}} \) is very small, we solve for the value \( v'_{\text{new}} \) at the next time step implicitly, and afterwards solve
\[
\frac{\partial \epsilon}{\partial t} = -\max(\frac{\omega_b^2 \Lambda \sigma^{1/2}}{2}, 0) + \rho \frac{\sigma^{3/2}}{\Lambda_{\text{diss}}}
\]
(24)
by updating the internal energy as
\[
\epsilon_{\text{new}} = \epsilon_{\text{old}} + (v'_{\text{old}}^2 - v'_{\text{new}}^2).
\]
(25)

We next update the fluid velocity according to
\[
\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial t},
\]
(26)
and then perform a step for diffusion, advection, and \( P \, dV \) work,
\[
\frac{\partial P \sigma}{\partial t} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2}{3} \frac{\partial v}{\partial r} \right) - \frac{P}{r^2} \frac{\partial v}{\partial r} \frac{\partial \sigma}{\partial r} - \frac{\alpha}{r} \frac{\partial}{\partial r} \left( \frac{r^2}{3} \frac{\partial \sigma}{\partial r} \right).
\]
(27)

Finally, we integrate the turbulent viscosity terms,
\[
\frac{\partial v}{\partial t} = \frac{4}{3} \Lambda^{1/2} \frac{\partial v}{\partial r} - \frac{v^2}{r},
\]
(28)
\[
\frac{\partial \epsilon}{\partial t} = \frac{4}{3} \Lambda^{1/2} \frac{\partial v}{\partial r} - \frac{v^2}{r},
\]
(29)
and convert back from the primitive variables to the conserved variables in general relativity.

4 NUMERICAL RESULTS

We evolve the 20M\(_{\odot}\) progenitor of Woosley & Heger (2007) until 13 ms after collapse and then switch on three variations of the turbulence model described in Sections 2.3 and 3:

- As a baseline model, we consider a case where turbulent viscosity is switched off.
- As a second case, we run another model without turbulent viscosity, where we switch off the sink term for buoyant driving in the internal energy equation at 42 ms after bounce. In other words, instead of Equation (24), we solve
\[
\frac{\partial \epsilon}{\partial t} = \rho \frac{\sigma^{3/2}}{\Lambda_{\text{diss}}}
\]
(30)
as in Couch et al. (2019). This case illustrates the effect of energy non-conservation.
- In a third simulation, we revert to the conservative formulation of buoyant driving and switch on turbulent viscosity.

Figure 1 compares the shock trajectories and diagnostics explosion energies \( E_{\text{diag}} \) for the three cases. Following the usual definition (Buras et al. 2006; Müller et al. 2012a), we compute \( E_{\text{diag}} \) as the total (internal+kinetic+potential) energy of the material that is nominally unbound at a given time. The turbulent kinetic energy \( \sigma \) is included in the total energy.

4.1 Impact of Energy Non-Conservation

The shock is revived in all three cases, but the explosion dynamics is significantly different. Comparing the two runs with and without the sink term for buoyant driving in the internal energy equation, we find that ignoring energy conservation in turbulence models has serious repercussions. Shock revival occurs more than 100 ms later for the energy-conserving turbulence model.

What is even more concerning, however, is that the rise of the explosion energy after shock revival is completely different. With the conservative formulation of the buoyant driving term, \( E_{\text{diag}} \) grows slowly to reach little more than of \( \sim 10^{51} \) erg. The diagnostic energy then decreases again so that the ejecta are only marginally unbound in the end. Even though the shock continuously propagates outward, the result is only a fizzle and not a full-blown explosion. Figure 2 provides more explanation for this behaviour by comparing profiles of the total specific energy \( \epsilon_{\text{tot}} \) and radial veloc-
Figure 2. Profiles of the radial velocity $v_r$ (left column) and the total specific energy $e_{\text{tot}}$ (excluding rest-mass contributions) in units of MeV per baryon mass $m_b$ (right column) for the two runs without turbulent viscosity at three different stages of shock propagation to 500 km, 100 km, and 3000 km (top to bottom). The post-bounce times at which these shock radii are reached are indicated in the legends. Black and red curves are used for the model with a conservative and non-conservative formulation of buoyant driving, respectively. In the model that respects total energy conservation, the post-shock velocity is consistently lower and the material in the outer part of the post-shock region is only marginally unbound.

ity from the conservative and non-conservative model for similar shock radii. For a given shock radius, the conservative model consistently exhibits lower $e_{\text{tot}}$ and radial velocity behind the shock. The turbulent flux terms and the turbulent pressure evidently alter the thermodynamic conditions on the downwind side of the shock sufficiently to maintain continuous shock expansion, but without pumping an appreciable amount of energy into the ejecta.

The late-time behaviour of the case with a non-conservative formulation of the buoyant driving term is particularly disturbing as it is clearly unrealistic. Initially, the rise of $E_{\text{diag}}$ slows down as physically expected from a declining neutrino heating rate with a visible break around a post-bounce time of 450 ms. A few hundreds of milliseconds later, however, $E_{\text{diag}}$ again starts to increase at an accelerated rate.

These findings lead to two conclusions: First, the question of total energy conservation clearly must not be ignored when formulating a 1D turbulence model for supernova explosions; it is crucial for the energetics of the explosion. Second, correctly reproducing the energetics within a 1D turbulence model is evidently a very different matter than merely reproducing shock revival and shock expansion in a seemingly realistic way. It is non-trivial to ensure that the shock propagation is coupled to the explosion energetics in the same manner as it is in multi-D simulations, where the shock velocity $v_{\text{sh}}$ roughly follows the analytic scaling law derived by Matzner & McKee (1999),

$$v_{\text{sh}} \sim \left( \frac{E_{\text{diag}} M_{\text{gain}}}{M_{\text{gain}} \rho_{\text{pre}} r_3^3} \right)^{0.19}$$

in terms of $E_{\text{diag}}$, the mass $M_{\text{gain}}$ between the shock and the gain radius, and the pre-shock density $\rho_{\text{pre}}$ as shown by Müller (2015). The conservative model without turbulent viscosity does not ensure this, and is therefore also unsatisfactory.

4.2 The Full Model with Turbulent Viscosity and its Limitations

But could 1D turbulence models fare better if they included more turbulent correlation terms, or if some of the dimensionless coefficients were adjusted? The conservative model with turbulent viscosity superficially points in this direction, as it respects energy conservation and reaches an explosion energy that is not far from the values observed for supernovae from progenitors with massive helium cores like SN 1987A with $\sim 1.5 \times 10^{51}$ erg (Arnett et al. 1989) and Cas A with $\sim 2.3 \times 10^{51}$ erg (Orlando et al. 2016).
Nonetheless, this turbulence model also fails to convince upon closer examination. The diagnostic energy rises significantly faster than in the exploding 3D model of Melson et al. (2015) for the same progenitor, and also faster than in other 3D explosion models of progenitors above 15M\(_{\odot}\) (Lentz et al. 2015; Müller et al. 2017; Vartanyan et al. 2019). One could assume that this is simply the consequence of early shock revival in this model, and might be fixed by a reasonable adjustment of the dimensionless parameters of the turbulence model, or by eliminating potential inconsistencies in the formulation of the turbulent viscosity term at the shock.

However, if the turbulence model could be tweaked to achieve more realistic explosion dynamics, this would still not imply that it captures the relevant physics of multi-D models consistently. There are several outstanding issues that would still need to be addressed. In multi-D, the explosion energy is supplied mostly by nucleon recombination of neutrino-driven outflows, which contributes an energy of \(\epsilon_{\text{nuc}}\) = 5-8 MeV per baryon\(^1\) in the outflows (Marek & Janka 2009; Müller 2015). The total enthalpy and mass flux in the outflows are remarkably constant over a wide range in radius; the total enthalpy flux in the downflows varies more strongly, but is a subdominant contribution to the angle-integrated total enthalpy flux outside a few hundred kilometres. What a turbulence model would need to reproduce to be viewed as consistent during the rise phase of the explosion energy is a convective total enthalpy flux of \(\epsilon_{\text{rec}}M_{\text{out}}\) from the recombination radius out to several thousands of kilometres, where most of the mass of the accumulated ejecta is located.

Given that turbulence models can produce convective velocities that match multi-D simulations reasonably well (see Figure 1 in Couch et al. 2019), one may grant that they can also be used to roughly predict the outflow rate as \(M_{\text{out}} \sim 2\pi r^2 \rho v'\). Since the energy carried by the outflows is split between internal energy and kinetic energy (whose contribution is more important at large radii), there is also some justification for interpreting the kinetic contribution to the energy flux into the ejecta as arising from “turbulent viscosity” in the framework of a spherical Reynolds decomposition, although this interpretation may not be very intuitive or useful.

But if the plausible energetics in the run with turbulent viscosity is more than a lucky coincidence resulting from a fortunate choice of non-dimensional model parameters, there would have to be a mechanism that ensures

\[
F_{\text{conv}} + F_{\text{visc}} + F_{\text{a}} \sim \rho v' \epsilon_{\text{nuc}}
\]

(32)

over a wide range of radii outside the recombination region. Since the \(F_{\text{conv}}\), \(F_{\text{visc}}\), and \(F_{\text{a}}\) are computed from local gradients of \(\epsilon\), \(P\), \(v\), and \(\sigma\), it is hard to conceive of such a mechanism: Why would the gradients adjust themselves in such a manner that the combined turbulent energy flux carries a specific energy per unit mass that corresponds to the energy liberated at the recombination radius?

This particular issue is, of course, only a part of a larger problem that needs to be investigated further: Although Murphy & Meakin (2011); Murphy et al. (2013); Mabanta & Murphy (2018) have studied the applicability of closure relations for the turbulent correlation terms in the Reynolds-averaged hydro equations during the pre-explosion phase, closures valid for subsonic convection on the background of the quasi-stationary accretion flow before shock revival may no longer be applicable during the explosion phase.

With a non-stationary background flow, large-scale overturn motions over many pressure scale heights, and the emergence of supersonic downflows (which clearly renders the anelastic approximation invalid), one should expect that the closure relations and non-dimensional parameters in the MLT fluxes need to change significantly during the explosion phase. For example, one likely needs to choose \(\alpha \gg 1\) in the definition of the mixing length \(L = \alpha P/\rho g\) during the explosion phase as the radial correlation length of the convective flow structures increases. To make matters worse, even the direction of unstable gradients changes already during the first seconds as a positive pressure gradient develops behind the shock, so that low-entropy material becomes susceptible to being mixed outward by the Rayleigh-Taylor instability (Chevalier 1976; Müller et al. 1991; Fryxell et al. 1991). Rather than providing a consistent recipe for 1D supernova simulations, the seeming “success” of the run with turbulent viscosity in fact underscores all of these concerns: It demonstrates that including additional turbulent correlation terms in the turbulence model can have a significant impact because the usual importance hierarchy of the correlation terms for low-Mach number convection no longer holds around shock revival and during the explosion.

Another lingering consistency problem concerns the explosive nucleosynthesis during the early explosion phase. In multi-D models, a sizeable fraction of the material synthesised by explosive burning in the shock is not entrained by the expanding neutrino-heated bubbles, but channelled into accretion downflows and not ejected (Müller et al. 2017; Harris et al. 2017). Capturing this process is crucial for consistently predicting the mass of \(^{56}\text{Ni}\) made in the explosion, but may be inherently beyond 1D turbulence models. Although 1D turbulence models may be able to shuffle \(^{56}\text{Ni}\) around via turbulent fluxes in the equations for the mass fractions, this would have no effect as long as the post-shock matter is still in nuclear statistical equilibrium (NSE). Whatever material is supplied in place of the \(^{56}\text{Ni}\) that is transported towards the protoneutron star by turbulent diffusion would simply be resupplied by \(\alpha\)-particles that are transported outward and recombine as required by the local NSE composition.

5 CONCLUSIONS

Prompted by the recent works of Mabanta et al. (2019) and Couch et al. (2019), we analysed the consistency of various 1D turbulence models for 1D supernova simulations and further bolstered this analysis by numerical experiments. Our analysis shows that the turbulence models of Mabanta & Murphy (2018) and Couch et al. (2019) implement buoyant driving of convection in a manner that violates energy conservation. This is because they do not treat the turbulent mass flux consistently, omitting it in the continuity equation while including it in the source term for buoyant energy generation. It is important to stress that this problem is not one of correct numerical discretisation; the non-conservation of energy is built into the analytic form of the model equations. We also point out that considerable care must be exercised when formulating the turbulent flux terms in order to ensure consistency between the energy and momentum equation (and the turbulent kinetic energy equation in time-dependent turbulence models), and the correct direction of the convective energy flux.

We point out that the energy-conserving turbulence model of Kuhfuss (1986) and Wuchterl & Feuchtinger (1998) already fixes the inconsistencies in the work of Mabanta et al. (2019) and Couch et al. (2019) except that it needs to tap internal energy instead of po-

\(^1\) This range of values also accounts for some losses from turbulent energy transfer between the outflows and downflows (Müller 2015).
tential energy to feed convective motions because it still relies on the anelastic approximation. This turbulence model has been implemented into the CoCoNuT-FMT neutrino hydrodynamics code with some minor modifications to avoid excessive convective overshoot and spurious generation of turbulent kinetic energy in non-homologously expanding or contracting flows.

We simulated the collapse and explosion of a 20\(M_\odot\) progenitor using three different variations of this 1D turbulence model in CoCoNuT-FMT to further illustrate the pitfalls and limitations of this approach. We find that including buoyant driving as an energy source term in a non-conservative manner without a corresponding sink term dramatically alters the explosion dynamics. Using the conservative model, the material behind the continuously expanding shock is barely unbound with a final explosion energy of less than 10\(^{48}\) erg as opposed to \(\sim 2 \times 10^{51}\) erg with the non-conservative turbulence model. This result suggests that non-conservative models as proposed by Mabanta et al. (2019) and Couch et al. (2019) should be used with considerable caution. Whether the non-conservative formulation of buoyant driving not only affects the dynamics of the explosion phase but also the systematics of “explodability”, which appears rather different in Couch et al. (2019) than in the parameterised models of Ugliano et al. (2012); Ertl et al. (2016); Sukhbold et al. (2016); Müller et al. (2016a); Ebing et al. (2019), also needs to be examined further in the future.

The low explosion energy obtained with our energy-conserving baseline model demonstrates another consistency problem of 1D turbulence models. Even if a turbulence model accurately captures the point of shock revival and predicts plausible shock trajectories, the energetics of the explosion may still be woefully off. A variation of the turbulence model including turbulent viscosity results in a plausible explosion energy for the 20\(M_\odot\) progenitor, but we view this as little more than a coincidence at this stage. It does not imply that 1D turbulence models can consistently capture the essential physics that governs the energetics of supernova explosions in multi-D.

Before 1D turbulence models can be considered significantly more consistent than other phenomenological approaches to the supernova progenitor-explosion connection (Ugliano et al. 2012; Pejcha & Thompson 2015; Perego et al. 2015; Sukhbold et al. 2016; Müller et al. 2016a), many critical issues still need to be addressed. Most importantly, one needs to account for:

(i) the coupling of the explosion energy to the recombination energy of the neutrino-heated ejecta,
(ii) the violation of the anelastic approximation due to the emergence of high-Mach number flow,
(iii) changes in the radial correlation length of turbulent fluctuations during the developing explosion,
(iv) the violation of the hydrostatic approximation and the emergence of an inverse pressure gradient behind the shock,
(v) the incomplete entrainment of \(^{56}\)Ni from explosive burning into the neutrino-heated ejecta.

None of the available turbulence models adequately deals with these daunting challenges yet. If there is a solution – which is not to be taken for granted – it will likely not materialise in the near future and will require a considerably more thorough analysis of multi-D explosion models than has been carried out so far. Because of the complexity of the problem, we cannot even hope to outline such a solution at this point.

On the other hand, the consistency problems and pitfalls that we pointed out should not lead to undue pessimism either. Even though 1D turbulence models have a long way to go before they can become a superior, more consistent method to predict supernova explosion and compact remnant properties than other approaches, they may complement other phenomenological supernova models and prove particularly useful for specific aspects of the progenitor-explosion connection as many of the other methods have. In order for 1D turbulence models to find their proper place, it is necessary, however, to investigate and incorporate the consistency requirements that follow from general physical principles and supernova explosion physics, to which the present work will hopefully contribute.

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