Complex sine-Gordon-2: a new algorithm for multivortex solutions on the plane

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Abstract

We present a new vorticity-raising transformation for the second integrable complexification of the sine-Gordon equation on the plane. The new transformation is a product of four Schlesinger maps of the Painlevé-V to itself, and allows a more efficient construction of the \( n \)-vortex solution than the previously reported transformation comprising a product of \( 2n \) maps.
The complex sine-Gordon equation, also known as the Lund-Regge model, was introduced in 1970s in several field-theoretic contexts [1, 2, 3, 4]. In $(2 + 0)$-dimensional space, the equation assumes the form

$$
\nabla^2 \psi + \frac{(\nabla \psi)^2 \overline{\psi}}{1 - |\psi|^2} + \psi(1 - |\psi|^2) = 0.
$$

(1)

(Here and below, $\nabla = i \partial_x + j \partial_y$.) We will be referring to eq.(1) as the complex sine-Gordon-1, in order to distinguish it from another integrable complexification of the sine-Gordon theory, the so-called complex sine-Gordon-2:

$$
\nabla^2 \psi + \frac{(\nabla \psi)^2 \overline{\psi}}{2 - |\psi|^2} + \frac{1}{2} \psi(1 - |\psi|^2)(2 - |\psi|^2) = 0.
$$

(2)

This model has also been known since the late 1970s, yet in $(1 + 1)$-dimensional space [5, 6]. The names stem from the fact that if we assume that $\psi$ is real and substitute $\psi = \sin(\alpha/2)$ in (1) and $\psi = \sqrt{2} \sin(\alpha/4)$ in (2), both systems reduce to the conventional, real, sine-Gordon equation $\nabla^2 \alpha + \sin \alpha = 0$. In the physics literature, it is common to define the two models by their action functionals:

$$
E_{SG1} = \int \left\{ \frac{|\nabla \psi|^2}{1 - |\psi|^2} + (1 - |\psi|^2) \right\} d^2 x,
$$

and

$$
E_{SG2} = \int \left\{ \frac{|\nabla \psi|^2}{1 - \frac{1}{2}|\psi|^2} + \frac{1}{2}(1 - |\psi|^2)^2 \right\} d^2 x,
$$

respectively.

Recently there has been an upsurge of interest in the complex sine-Gordon equations, motivated by the fact that they define integrable perturbations of conformal field theories [7, 8, 9, 10]. There is, however, yet another reason for considering these systems more closely; out of all vortex-bearing equations for one complex field, the complex sine-Gordon-1 and -2 are the only equations whose vortex (and multivortex) solutions are available in explicit analytic form [11, 12]. Consequently, they provide a unique source of insight into general properties of topological solitons on the plane. The latter can be of value for a whole range of models like the Gross-Pitaevski and easy-plane ferromagnet equations, where vortices are only available numerically.

The (coaxial) multivortices of the complex sine-Gordon-2 have been obtained via the Schlesinger transformations of the fifth Painlevé equation [11]. The procedure is cumbersome: even if the $(n - 1)$-vortex is already available, the construction of the $n$-vortex solution
requires applying the Schlesinger transformation $2n$ times anew. On the contrary, there is an
efficient recursive procedure for the complex sine-Gordon-1, allowing a one-step construction
of its solution with vorticity $n$ provided the $(n-1)$-vortex solution is known \[11, 12\]. The
purpose of the present note is to formulate a similar recursive procedure for the complex
sine-Gordon-2.

2. The coaxial $n$-vortex configuration has the form $\psi(r, \theta) = Q_n^{1/2}(r)e^{in\theta}$. Substituting
this Ansatz into (2) yields an equation for the radial “amplitude” $Q_n$ which we write as
\[
\frac{d^2 Q_n}{dr^2} + \frac{1}{r} \frac{dQ_n}{dr} + \frac{1 - Q_n}{Q_n(Q_n - 2)} \left( \frac{dQ_n}{dr} \right)^2 + Q_n(1 - Q_n)(2 - Q_n) \left[ \frac{(a^2 - b^2)Q_n}{r^2(2 - Q_n)} + \frac{4a^2(1 - Q_n)}{r^2Q_n(2 - Q_n)} + \frac{\gamma Q_n(2 - Q_n)}{2r} \right] = 0,
\]
with $a = \gamma = 0$ and $b = -2n$. The last two terms in (3) being equal to zero, this form may
appear to be somewhat artificial. However, there is an advantage in considering eq. (3) with
general $a$, $b$, and $\gamma$; namely, the availability of transformations connecting solutions with
different sets of parameters. Indeed, the change of variables \[11\]

\[Q_n = \frac{2}{1 - W},\]

brings eq. (3) to the fifth Painlevé equation,
\[
\frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} - \frac{3W - 1}{2W(W - 1)} \left( \frac{dW}{dr} \right)^2 = \frac{(W - 1)^2}{r^2} \left( \alpha W + \frac{\beta}{W} \right) + \gamma W + \delta \frac{W(W + 1)}{W - 1},
\]
with $\alpha = \frac{1}{2} a^2$, $\beta = -\frac{1}{2} b^2$ and $\delta = 2$. The Painlevé-V is covariant under the Schlesinger
transformation \[13, 14\] which takes a solution $W$ with the parameter values $a, b, \gamma$ and $\delta$ to
a solution $\hat{W}$ of the same equation with parameter values
\[
\hat{a} = \frac{1}{2}(a + b - 1 - \gamma/c), \quad \hat{b} = \frac{1}{2}(a + b - 1 + \gamma/c), \quad \hat{\gamma} = c(b - a),
\]
and $\hat{\delta} = \delta$. Here $c$ is one of the two values with $c^2 = -2\delta$; in our case we can set, without
loss of generality, $c = 2i$. Written in terms of $Q$ and $\hat{Q} = 2(1 - \hat{W})^{-1}$, the direct and inverse
Schlesinger transformations have the form
\[
\hat{Q} = 1 - \frac{i}{Q(Q - 2)} \left[ \frac{dQ}{dr} + \frac{Q(a + b) - 2a}{r} \right], \quad (6a)
\]
\[
Q = 1 + \frac{i}{\hat{Q}(\hat{Q} - 2)} \left[ \frac{d\hat{Q}}{dr} - \frac{(\hat{Q} - 1)(a + b - 1)}{r} + \frac{i\gamma}{2r} \right]. \quad (6b)
\]
If $Q^{(0)} = Q_n$ is a solution of eq. (3) with parameters $a^{(0)} = \gamma^{(0)} = 0$, $b^{(0)} = -2n$, then applying transformation (6a) we obtain a solution $\hat{Q} = Q^{(1)}$ with parameters

$$
a^{(1)} = \frac{1}{2}(a^{(0)} + b^{(0)} - 1 - \gamma^{(0)}/c) = -n - \frac{1}{2},
$$
$$
b^{(1)} = \frac{1}{2}(a^{(0)} + b^{(0)} - 1 + \gamma^{(0)}/c) = -n - \frac{1}{2},
$$
$$
\gamma^{(1)} = 2i(b^{(0)} - a^{(0)}) = -4in.
$$

Note that since $a^{(1)}$ and $\gamma^{(1)}$ are not zero and $b^{(1)}$ is not a negative even integer, $Q^{(1)}$ does not represent the amplitude of any multivortex. Using (6a) again, this time with $Q = Q^{(1)}$ and $\hat{Q} = Q^{(2)}$, yields a solution $Q^{(2)}$ with parameters

$$
a^{(2)} = \frac{1}{2}(a^{(1)} + b^{(1)} - 1 - \gamma^{(1)}/c) = -1,
$$
$$
b^{(2)} = \frac{1}{2}(a^{(1)} + b^{(1)} - 1 + \gamma^{(1)}/c) = -2n - 1,
$$
$$
\gamma^{(2)} = 2i(b^{(1)} - a^{(1)}) = 0.
$$

Thus, the effect of the product transformation $Q^{(0)} \rightarrow Q^{(2)}$ is to reduce both $a$ and $b$ by one. Although $\gamma$ is now zero, $Q^{(2)}$ is still not a multivortex (since $a^{(2)}$ is nonzero). The crucial observation now is that (3) depends only on the square of $a$; hence $Q^{(2)}$ is a solution of eq. (3) not only for $a = -1$, $b = -2n - 1$, but also for $\tilde{a} = +1$, $\tilde{b} = -2n - 1$. Repeating the above two transformations will decrease both $\tilde{a}$ and $\tilde{b}$ by one more, yielding $a^{(4)} = 0$, $b^{(4)} = -2n - 2$. The corresponding solution $Q^{(4)}$ will therefore be $Q_{n+1}$, the multivortex with vorticity $n + 1$. Thus, the transformation $Q^{(0)} \rightarrow Q^{(4)}$, comprising a product of four Schlesinger maps, is nothing but a vorticity-raising transformation $Q_n \rightarrow Q_{n+1}$.

Starting with the trivial “vortex” $Q_0 = 1$ and applying the transformation $Q_n \rightarrow Q_{n+1}$ recursively, one can construct multivortices of any desired vorticity. It follows from the form of the transformation that all $Q_n$’s will be rational functions of $r$. The one-, two- and three-vortex amplitudes are:

$$
Q_1 = \frac{r^2}{r^2 + 4},
$$
$$
Q_2 = \frac{r^4(r^2 + 24)}{r^8 + 64r^6 + 1152r^4 + 9216r^2 + 36864},
$$

and

$$
Q_3 = \frac{r^6(r^6 + 144r^4 + 5760r^2 + 92160)^2}{D_3},
$$

where $D_3$ is a constant.
where

\[
D_3 = r^{18} + 324r^{16} + 41472r^{14} + 2820096r^{12} + 114130944r^{10} + 2919628800r^8 \\
+ 50960793600r^6 + 61152952300r^4 + 4892236185600r^2 + 19568944742400.
\]

These solutions coincide with those constructed previously in [11].

3. As another application of the new vorticity-raising transformation, we construct one more class of vortex-like solutions. Unlike the solutions discussed above, these solutions of the complex sine-Gordon-2 decay to their asymptotic value in an oscillatory fashion. (The asymptotic value is now \(\sqrt{2}\) not 1.)

We start with the radially symmetric solution [11] of the complex sine-Gordon-1 with vorticity \(n = 2\):

\[
\Phi_2 = \frac{-I_0 I_2 - I_1^2}{I_0^2 - I_1^2}.
\]

Here, \(I_m = I_m(r)\) is the modified Bessel function of order \(m\). The change of variables

\[
\Phi_2 = 1 + \frac{W}{1 - W}
\]

transforms it to a solution

\[
W_2(r) = \frac{I_0 I_1 - r(I_0^2 - I_1^2)}{I_0 I_1}
\]

of the Painlevé-V, eq.(5) with parameter values

\[
\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}, \quad \gamma = 0, \quad \delta = -2.
\]

(In (4), we simplified, using the standard recurrence relation \(I_0 - I_2 = 2I_1/r\).) Next, making the replacement \(r \to ir\) amounts to replacing \(\gamma \to i\gamma, \delta \to -\delta\) in (4); hence

\[
\tilde{W}_2(r) = W_2(ir) = \frac{J_0 J_1 - r(J_0^2 + J_1^2)}{J_0 J_1}
\]

is a solution to the Painlevé-V with

\[
\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}, \quad \gamma = 0, \quad \delta = 2.
\]

Here, \(J_m = J_m(r)\) is the ordinary Bessel function of order \(m\).

For the parameter values in (8), the change of variables (4) brings the Painlevé-V to equation (3) with \(a = 1\) and \(b = -1\). Thus we can use (4) followed by two applications of
(3a) to reduce both $a$ and $b$ by one (as described in the previous section). The result is a solution to the complex sine-Gordon-2 with $n = 1$:

$$
\tilde{Q}_1 = \frac{2 [r (J_0^2 + J_1^2) - J_0 J_1]^2}{r^2 (J_0^2 + J_1^2)^2 + J_0^4}.
$$

(9)

Applying the vorticity-raising transformation to $\tilde{Q}_1$ gives the formula for $\tilde{Q}_2$, the 2-vortex amplitude:

$$
\tilde{Q}_2 = 2 \frac{P_2^2}{D_2},
$$

(10)

where

$$
P_2 = 4r^3 J_0^4 + 8r^3 J_0^2 J_1^2 + 4r^3 J_1^4 - 8r^2 J_0^3 J_1 - 8r^2 J_0 J_1^3 + 3r J_0^3
$$

$$
+ 2r J_0^2 J_1^2 - 5r J_1^4 - 6J_0^3 J_1 + 2J_0 J_1^3,
$$

and

$$
D_2 = 416 r^4 J_1^2 J_0^6 - 128 r^5 J_0 J_1^7 + 736 r^4 J_1^6 J_0^2 - 384 r^5 J_0^3 J_1^5 + 96 r^6 J_1^4 J_0^4 - 12 r^2 J_0^2 J_1^6
$$

$$
- 96 r J_0^3 J_1^5 - 320 r^3 J_0 J_1^7 - 32 r J_0^5 J_1^3 + 318 r^2 J_1^4 J_0^4 + 64 r^6 J_1^2 J_0^6 - 128 r^5 J_1^7 J_1
$$

$$
- 832 r^3 J_0^3 J_1^5 + 260 r^2 J_1^6 J_0^2 - 448 r^3 J_0^3 J_1^5 - 384 r^5 J_0^2 J_1^3 + 1008 r^4 J_1^4 J_0^4 + 64 r^3 J_1^7 J_1
$$

$$
+ r^2 J_1^8 + 152 r^4 J_1^8 + 9 r^2 J_0^8 + 16 r^6 J_1^8 + 16 r^6 J_0^8 + 16 J_1^4 J_0^4 - 8 r^4 J_0^8 + 64 r^6 J_1^2 J_0^6.
$$

(The actual moduli $\tilde{\Phi}_1 = \sqrt{\tilde{Q}_1(r)}$ and $\tilde{\Phi}_2 = \sqrt{\tilde{Q}_2(r)}$ are shown in figure II.) Proceeding recursively we can construct $\tilde{Q}_n$ with arbitrary $n$. For $n \geq 3$, the formulas become intractable, so we only produce the asymptotic behaviours, as $r \to \infty$:

$$
\tilde{Q}_n \to 2 + (-1)^{n+1} \frac{2n}{r} \cos(2r) + O \left( \frac{1}{r^2} \right).
$$

(11)

Eq. (11) can be easily proved by induction.

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FIG. 1: The moduli of the 1 and 2-vortex solutions $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$. (As $r \to \infty$, $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ tend to $\sqrt{2}$.)

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