LS-CATEGORY AND THE DEPTH OF RATIONALLY ELLIPTIC SPACES

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Abstract. Let $X$ be a finite type simply connected rationally elliptic CW-complex with Sullivan minimal model $(\Lambda V, d)$ and let $k \geq 2$ the biggest integer such that $d = \sum_{i \geq k} d_i$ with $d_i(V) \subseteq \Lambda^i V$. We show that:

$$\text{cat}(X \mathbb{Q}) = \text{depth}(\Lambda V, d_k)$$

if and only if $(\Lambda V, d_k)$ is elliptic.

This result is obtained by introducing two new spectral sequences that generalize the Milnor-Moore spectral sequence and its $E_{\infty}$-version [Mur94].

As a corollary, we recover a known result proved - with different methods - by L. Lechuga and A. Murillo in [LM02] and G. Lupton in [Lup02]: If $(\Lambda V, d_k)$ is elliptic, then

$$\text{cat}(X \mathbb{Q}) = \dim(\pi_{\text{odd}}(X) \otimes \mathbb{Q}) + (k - 2) \dim(\pi_{\text{even}}(X) \otimes \mathbb{Q}).$$

In the case of a field $K$ of $\text{char}(K) = p$ (an odd prim) we obtain an algebraic approach for $e_K(X)$ where $X$ is an $r$-connected ($r \geq 1$) finite CW-complex such that $p > \dim(X)/r$.

1. Introduction

LS-category was introduced since 1934 by L. Lusternik and L. Schnirelman in connection with variational problems [LS34]. They showed that for any closed manifold $M$, this category denoted $\text{cat}(M)$, is a lower bound of a number of critical points that any smooth function on $M$ must have. Later, it was shown that this is also right for a manifold with a boundary [Pal66].

If $X$ is a topological space, $\text{cat}(X)$ is the least integer $n$ such that $X$ is covered by $n + 1$ open subset $U_i$, each contractible in $X$. It is an invariant of homotopy type (c.f. [Jam78] or [FHT01] Prop 27.2 for example). Though its definition seems easy, the calculation of this invariant is hard to compute. In [FH82], Y. Félix and S. Halperin developed a deep approach for computing the LS-category for rational spaces. They defined the rational category of a 1-connected space $X$ in terms of its Sullivan minimal model and showed that it coincides with LS-category of its rationalization $X \mathbb{Q}$. Since then, much estimations of the rational LS-category were developed in connection with other numerical invariants. Later on, Y. Félix, S. Halperin and J.M. Lemaire [FHL] showed that for Poincaré duality spaces, the rational LS-category coincide with the rational Toomer invariant -denoted $e_\mathbb{Q}(X)$- which is difficult to define, but at first sight, it may look easier to compute it (c.f.

1991 Mathematics Subject Classification. Primary 55P62; Secondary 55M30 .

Key words and phrases. elliptic spaces, Lusternik-Schnirelman category, Toomer invariant,
The study of this topological invariants make use of tow other invariants of algebraic character which are defined for any augmented differential graded algebra \((A, d)\) over a field \(K\) as follow:

\[
\text{depth}(A, d) = \inf \{ n | \text{Ext}^n_{(A, d)}(K, (A, d)) \neq 0 \},
\]

\[
\text{gldim}(A, d) = \sup \{ n | \text{Ext}^n_{(A, d)}(K, K) \neq 0 \}.
\]

In this paper, we are focused on rational simply connected elliptic spaces. Recall that such spaces have both rational homotopy and rational cohomology of finite dimension. They satisfy Poincaré duality property and are also Gorenstein spaces. Such spaces have elliptic Sullivan minimal models denoted \((\Lambda V, d)\). (c.f. section 2 for details).

In what remain, we will denote by \(k\), the biggest integer \(k \geq 2\) such that \(d = \sum_{i \geq k} d_i\) with \(d_i(V) \subseteq \Lambda^i V\). We denote also by \((\Lambda V, d, \sigma)\) (resp. \((\Lambda V, d_k, \sigma)\)) the associated pure model of \((\Lambda V, d)\) (resp. \((\Lambda V, d_k)\)).

Using those properties and the machinery of spectral sequences, we show our first main result:

**Theorem 1.0.1.** Let \(X\) be a finite type simply connected CW-complex with an elliptic Sullivan minimal model \((\Lambda V, d)\). Then \(\text{cat}(X_Q) = \text{depth}(\Lambda V, d_k)\) if and only if \((\Lambda V, d_k)\) is also elliptic.

The proof make use of the Sullivan model \((\Lambda V, d)\) of \(X\), so replacing \((\Lambda V, d)\) by \((\Lambda V, d, \sigma)\), we deduce immediately from the equivalence: \((\Lambda V, d_k)\) is elliptic if and only if \((\Lambda V, d_k, \sigma)\) is, and the fact that \(d_{k, \sigma} = d_{\sigma, k}\), the:

**Corollary 1.0.2.** If \((\Lambda V, d)\) is elliptic, then

\[
\text{cat}(\Lambda V, d) = \text{depth}(\Lambda V, d_k) \iff \text{cat}(\Lambda V, d, \sigma) = \text{depth}(\Lambda V, d_{\sigma, k}) \iff (\Lambda V, d_{\sigma, k})\) is elliptic.
\]

Now using spectral sequences 1.0.3 and 1.0.4, we obtain our second main result:

**Theorem 1.0.3.** If \((\Lambda V, d)\) is elliptic, then \(\text{depth}(\Lambda V, d) = \text{depth}(\Lambda V, d, \sigma)\).

As an immediate consequence, we have the following corollary which is noting other than L. Lechuga and A. Murillo’result in [LM02] and G. Lupton’result in [Lup02].

**Corollary 1.0.4.** If \((\Lambda V, d_k)\) is elliptic, then:

\[
\text{cat}(\Lambda V, d) = \text{cat}(\Lambda V, d_k) = \text{dim}(V^{odd}) + (k-2)\text{dim}(V^{even}).
\]

**Remark 1.0.5.** The last corollary express also that, if \((\Lambda V, d_k)\) is elliptic, then:

\[
\text{depth}(\Lambda V, d_{k, \sigma}) = \text{dim}(V^{odd}) + (k-2)\text{dim}(V^{even}).
\]

On the other hand by ([LM02], Prop. 3) any elliptic pure model \((\Lambda V, d)\) in which, for each \(y_j \in V^{odd}\), \(d y_j\) is homogeneous of wordlength \(l_j\), one have \(\text{cat}(\Lambda V, d) \geq \text{dim}(V^{odd}) + (k-2)\text{dim}(V^{even})\). So by the equivalence of Theorem 1.0.1, it is natural to ask the following:

Does \(\text{depth}(\Lambda V, d_{k, \sigma})\) equal \(\text{dim}(V^{odd}) + (k-2)\text{dim}(V^{even})\) eventhough \((\Lambda V, d_k)\) is not elliptic?
Before giving some necessary ingredients to prove our main results, we point out the ulterior essential results in this area.

Let \( (\Lambda V, d) \) be a commutative differential graded algebra (cdga for short) over a field \( K \) of characteristic zero. We suppose that \( \dim(V) < \infty \), \( V^0 = K, V^1 = 0 \) and denote \( V^{\text{even}} = \bigoplus_k V^{2k} \) and \( V^{\text{odd}} = \bigoplus_k V^{2k+1} \). If \( d = \sum_{i \geq 2} d_i \) (where \( d_i \) designate the part of \( d \) such that \( d_i(V) \subseteq \Lambda^i V \)), \( (\Lambda V, d) \) is said to be minimal. Recall that to any nilpotent space (in particular any simply connected space), D. Sullivan in \[Sul78\] associated a minimal cdga \( (\Lambda V, d) \) over the rationales, which is unique up to quasi-isomorphism.

In their famous article \[FH82\] where rational LS-category is defined in terms of Sullivan minimal models, Y. Félix and S. Halperin had shown that a coformal space \( X \) (i.e. a space with minimal model \( (\Lambda V, d) \) such that \( d = d_2 \)) has \( \text{cat}(X_{\mathbb{Q}}) = \dim(V^{\text{odd}}) \). Longer after we showed in \[Ram99\], Prop B) by the use of the algebraic invariants \( \text{gldim} \) and \( \text{depth} \) (see, that for any rationally elliptic space \( X \), \( \text{gldim}(H_*(\Omega X, \mathbb{Q})) \) is finite, if and only if \( v_0(X) = \text{depth}(H_*(\Omega X, \mathbb{Q})) \) (notice that this remain true for any coefficient field \( K \) and any space \( X \) such that \( H^*(X, K) \) is a Poncaré duality algebra). However \( \text{gldim}(H_*(\Omega X, \mathbb{Q})) < \infty \iff \dim H^*(\Lambda V, d_2) < \infty \) and \( \text{depth}(H_*(\Omega X, \mathbb{Q})) = \dim(V^{\text{odd}}) \). Thus, according to \[FH82\], we have \( \text{cat}(X_{\mathbb{Q}}) = \dim(V^{\text{odd}}) \) if and only if \( (\Lambda V, d_2) \) is also elliptic. This last result is generalized in one direction by L. Lechuga and A. Murillo in \[LM02\] and with a different manner by G. Lupton in \[Lup02\] as follow: If \( X \) is an elliptic space with minimal model \( (\Lambda V, d) \), such that \( (\Lambda V, d_k) \) is also elliptic where \( k \geq 2 \) is the biggest integer for which \( d = \sum_{i \geq k} d_i \), then \( \text{cat}(X_{\mathbb{Q}}) = \dim(V^{\text{odd}}) + (k - 2) \dim(V^{\text{even}}) \).

The Milnor-Moore spectral sequence (cf. \[FH82\], Prop. 9.1) and its \( \mathcal{E}xt \)-version (cf. \[Mur94\]) are essential in major demonstrations. In this paper we introduce there generalizations:

\[
(1.0.1) \quad H^{p,q}(\Lambda V, d_k) \implies H^{p+q}(\Lambda V, d)
\]

\[
(1.0.2) \quad \mathcal{E}xt^{p,q}_{(\Lambda V, d_k)}(\mathbb{Q}, (\Lambda V, d_k)) \implies \mathcal{E}xt^{p+q}_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)).
\]

We will use also the odd spectral sequence \[Hal77\] and its \( \mathcal{E}xt \)-version \[Ram99\]:

\[
(1.0.3) \quad H^{p,q}(\Lambda V, d_\sigma) \implies H^{p+q}(\Lambda V, d)
\]

\[
(1.0.4) \quad \mathcal{E}xt^{p,q}_{(\Lambda V, d_\sigma)}(\mathbb{Q}, (\Lambda V, d_\sigma)) \implies \mathcal{E}xt^{p+q}_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)).
\]

Now let \( K \) be a field of \( \text{char}(K) \neq 2 \) and \( (L, \partial) \) a differential graded Lie algebra over \( K \). Consider the Cartan–Chevalley–Eilenberg cochain complex \( (C^*(L), d) = (\Lambda sL, d_1 + d_2) \) for \( (L, \partial) \) where:

\[
< d_1 v, sa > = (-1)^{|v|} < v, s \partial a >; \quad a \in L, \ v \in Hom(sL, K),
\]

\[
< d_2 v, sasb > = (-1)^{|b|+1} < v, s[a, b] >; \quad a, b \in L, \ v \in Hom(sL, K).
\]
In [Mur94] A. Murillo showed that if $L$ is 1-connected and finite-dimensional, then $(C^*(L),d)$ is Gorenstein and it has finite-dimensional cohomology if and only if $ev_{C^*(L),d} \neq 0$. In addition, if $\text{char} (\mathbb{K}) = p$ (an odd prime number) and $X$ is an $r$-connected finite CW-complex with $\dim(X) < rp$, then using the quasi-isomorphism $U(L,\partial) \cong C_\ast (\Omega X, \mathbb{K})$ which had with D. Anick [Ani89], S. Halperin [Hal92] associate to $X$ a minimal model $(\Lambda V, d)$. The homotopy Lie algebra $(C^*(\Lambda V), d_2)$ associated with it is then used to prove that $U(sV) \cong H_\ast (\Omega X, \mathbb{K})$ ($sV$ is the suspension of $V$).

Recall also ([Ram99], Prop. A) that for such a space, the condition of being $\mathbb{K}$-elliptic is equivalent to $\dim(sV) < \infty$. If so, $H^\ast (\Lambda V, d)$ is then a duality Poincaré algebra and a Gorenstein algebra. Adopting the same notations as before and applying the same approach as in the rational case, we have an algebraic approach for $e_\mathbb{K}(X)$:

**Theorem 1.0.6.** Let $\mathbb{K}$ be a field of $\text{char}(\mathbb{K}) = p > 2$, $X$ an $r$-connected finite CW-complex with $\dim(X) < rp$ and $(\Lambda V, d)$ its minimal model. If $\dim(V) < \infty$ then $e_\mathbb{K}(X) = \text{depth}(\Lambda V, d_k)$ if and only if $(\Lambda V, d_k, \sigma)$ is elliptic.

## 2. Preliminary

In this section we recall some notions we will use in other sections.

### 2.1. A Sullivan minimal model:

Let $\mathbb{K}$ a field of characteristic $\neq 2$.

A **Sullivan algebra** is a free commutative differential graded algebra (cdga for short) $(\Lambda V, d)$ (where $\Lambda V = \text{Exterior}(V^{\text{odd}}) \otimes \text{Symmetric}(V^{\text{even}}))$ generated by the graded $\mathbb{K}$-vector space $V = \bigoplus_{i=0}^{\infty} V^i$ which has a well ordered basis $\{x_\alpha\}$ such that $dx_\alpha \in \Lambda V^{\leq \alpha}$. Such algebra is said minimal if $\text{deg}(x_\alpha) < \text{deg}(x_\beta)$ implies $\alpha < \beta$. If $V^0 = V^1 = 0$ this is equivalent to saying that $d(V) \subseteq \bigoplus_{i=2}^{\infty} \Lambda^i V$.

A **Sullivan model** for a commutative differential graded algebra $(A, d)$ is a quasi-isomorphism (morphism inducing isomorphism in cohomology) $(\Lambda V, d) \cong (A, d)$ with source, a Sullivan algebra. If $H^0(A) = \mathbb{K}, H^1(A) = 0$ and $\dim(H^i(A, d)) < \infty$ for all $i \geq 0$, then ([Hal92], Th. 7.1), this minimal model exists. If $X$ is a topological space any (minimal) model of the algebra $C^\ast (X, \mathbb{K})$ is said a Sullivan (minimal) model of $X$.

The uniqueness (up quasi-isomorphism) in the rational case was assured by D. Sullivan in ([Sul78]). Indeed he associated to any simply connected space $X$ the cdga $A(X)$ of polynomial differential forms and showed that its minimal model $(\Lambda V_X, d)$ satisfied:

$$V^i_X \cong \text{Hom}_\mathbb{K}(\pi_i(X), \mathbb{Q}); \quad \forall i \geq 2,$$

that is, in the case where $X$ is a finite type CW-complex, the generators of $V_X$ corresponds to those of $\pi_\ast (X) \otimes \mathbb{Q}$.

If $\text{char}(\mathbb{K}) = p > 2$, for any $r$-connected CW-complex $X$ with $\dim(X) < rp$, there exists a sequence ([Hal92]):

$$(\Lambda V_X, d) \cong C^\ast (L) \cong B^\vee (C_\ast (\Omega X, \mathbb{K})) \cong C^\ast (X, \mathbb{K})$$
of quasi-isomorphisms, where $C^*(L)$ is the Cartan-Chevalley-Eilenberg cochain complex mentioned in the introduction and $B^\vee(C_*(\Omega X, K))$ is the dual of the bar construction of $C_*(\Omega X, K)$. The uniqueness of $(\Lambda V_X, d)$ is also assured under this restrictive conditions on $X$. As I know there is any relationship between $V_X$ and $\pi_*(X) \otimes K$.

In both cases, one can associate to $(\Lambda V_X, d)$ another cdga (called the pure model associated to $X$) denoted $(\Lambda V_X, d_\sigma)$ with $d_\sigma$ defined as follow:

$$d_\sigma(V_X^{even}) = 0 \quad \text{and} \quad (d - d_\sigma)(V_X^{odd}) \subseteq \Lambda V_X^{even} \otimes \Lambda^+ V_X^{odd}.$$  

2.2. The evaluation map: Let $(A, d)$ be an augmented $K$-dga and choose an $(A, d)$-semifree resolution (FHT88) $p : (P, d) \xrightarrow{\sim} (K, 0)$ of $K$. Providing $K$ with the $(A, d)$-module structure induced by the augmentation we define a chain map:

$$\text{ev}_{(A, d)} : \text{Ext}_{(A, d)}(K, (A, d)) \rightarrow H(A, d),$$

where Ext is the differential Ext of Eilenberg and Moore [Moo60]. Note that this definition is independent of the choice of $P$ and $z$ and it is natural with respect to $(A, d)$.

The authors of [FHT88] also defined the concept of a Gorenstein space at a field $K$. It is a space $X$ such that $\dim(\text{Ext}C^*(X, K)(K, C^*(X, K)) = 1$. If moreover $\dim H^*(X, K) < \infty$, then it satisfies Poincaré duality property over $K$ and its fundamental class is closely related to the evaluation map [Mur93].

2.3. The Toomer invariant: The Toomer invariant is defined by more than one way. Here we recall its definition in the context of minimal models. Let $(\Lambda V, d)$ any minimal cdga on a field $K$. Let $p_n : \Lambda V \rightarrow \Lambda V_{\leq n+1}$ denote the projection onto the quotient differential graded algebra obtained by factoring out the differential graded ideal generated by monomials of length at least $n+1$. The "commutative" Toomer invariant $e_{c, K}(\Lambda V, d)$ of $(\Lambda V, d)$ is the smallest integer $n$ such that $p_n$ induces an injection in cohomology or $+\infty$ if there is no such smallest $n$.

In [HL88], S. Halperin and J.M. Lemaire defined for any simply connected finite type CW-complex $X$ (and any field $K$) the invariant $e_K(X)$ in terms of its free model $(T(W), d)$ and showed that it coincides with the classical Toomer invariant. Consequently if $\text{char}(K) \neq 2$ and $X$ is an $r$-connected CW-complex with $\dim(X) < rp, \ (r \geq 1)$ then using both its Sullivan minimal model $(\Lambda V, d)$ and its free minimal model $(T(W), d)$, we have (cf [HL88], Th 3.3 ) that $e_K(X) = e_K(\Lambda V, d) = e_{c, K}(\Lambda V, d)$. 

Remark 2.3.1. In (FH82, Lemma 10.1), Y. Félix and S. Halperin showed that whenever $H(AV, d)$ has Poincaré duality, then $e_{c, Q}(AV, d) = \sup\{k / \omega \text{ can be represented by a cycle in } \Lambda^{2k}V\}$ where $\omega$ represents the fundamental class. This remains true when one replace $Q$ by any field $K$.

3. Spectral sequences and the proof of the results:

In this section we will work in a field $K$ of $\text{char}(K) \neq 2$. Let $(AV, d)$ a cdga with $\dim(V) < \infty$. Suppose that $d = \sum_{i \geq k} d_i$ and $k \geq 2$. The filtrations that induce the spectral sequences 1.0.1 and 1.0.2 given in the introduction are defined respectively as follow:

\[(3.0.1) \quad F^p = \Lambda^{2^p}V = \bigoplus_{i=p}^{\infty} \Lambda^i V\]

\[(3.0.2) \quad F^p = \{ f \in \text{Hom}_{AV}(AV \otimes \Gamma(sV), AV), / f(\Gamma(sV)) \subseteq \Lambda^{2^p}V \}\]

Recall that $\Gamma(sV)$ is the divided power algebra of $sV$ and the differential $D$ on $(\Gamma(sV) \otimes AV)$ is a $\Gamma$-derivation (i.e. $D(\gamma^p(sv)) = D(sv)\gamma^{p-1}(sv)$, $p \geq 1$, $sv \in (sV)^{even}$, and $D(sv) = v + s(dv)$) which restrict to $d$ in $AV$. With this differential, $(AV \otimes \Gamma(sV), D)$ is a dga called an acyclic closure of $(AV, d)$ hence it is $(AV, d)$-semifree. Therefore the projection $(AV \otimes \Gamma(sV), D) \overset{\sim}{\longrightarrow} K$ is a semifree resolution of $K$.

Recall also that de differential $\mathcal{D}$ of $\text{Hom}_{(AV,d)}((AV \otimes \Gamma(sV), D), (AV, d))$ is defined as follow: $\mathcal{D}(f) = d \circ f + (-1)^{|f|+1} f \circ D$.

Let us denote $A = AV$ (resp. $A = \text{Hom}_{AV}(AV \otimes \Gamma(sV), AV)$), $G^p = F^p$ (resp. $G^p = F^p$) and $\delta = d$ (the differential of $(AV, d)$) (resp. $\delta = D$).

The filtrations 3.0.1 and 3.0.2 verify the following lemma and then they define the tow spectral sequences below:

**Lemma 3.0.2.** (i) $(G^p)_{p \geq 0}$ is decreasing.

(ii) $G^0(A) = A$.

(iii) $\delta(G^p(A)) \subseteq G^p(A)$.

**Proof.** (i) and (ii) are immediate. The propriety (iii) follows from (a): the definition of $\mathcal{D}$ on $\text{Hom}_{(AV,d)}((AV \otimes \Gamma(sV), D), (AV, d))$, (b): $d$ is minimal and (c): $D(\gamma^p(sv)) = D(sv)\gamma^{p-1}(sv) = (v + s(dv))\gamma^{p-1}(sv)$. □

3.1. Determination of the first terms of the tow spectral sequences. The two filtrations are bounded, so they induce convergent spectral sequences. We calculate here there first terms.

Beginning with the filtration 3.0.1 one can check easily the following:

$$E^0_p = F^p / F^{p+1} \cong E^1_p \cong \ldots \cong E_{k-2}^p \cong \Lambda^p V.$$ and

$$d_0 = d_1 = \ldots = d_{k-2} = 0.$$
The first non-zero differential is $d_{k-1} : E^p_{k-1} \to E^{p+1}_{k-1}$, which coincides with the $k$-th term $d_k$ in the differential $d$ of $(\Lambda^v, d)$, and then: $(E_{k-1}, d_{k-1}) = (\Lambda^v, d_k)$.

So the first term in the induced spectral sequence is $E_k = H(\Lambda^v, d_k)$.

For the second spectral sequence, its general term is:

\[
\mathcal{E}_r^p = \frac{\{ f \in \mathcal{F}^p, \ D(f) \in \mathcal{F}^{p+r} \}}{\{ f \in \mathcal{F}^{p+1}, \ D(f) \in \mathcal{F}^{p+r} \} + \mathcal{F}^p \cap \mathcal{D}(\mathcal{F}^{p-r+1})}.\]

We first prove the following important lemma:

**Lemma 3.1.1.** Let $f \in \mathcal{F}^p$, then for any $p \geq 0$,
1. $D(f) \in \mathcal{F}^{p+2} \Leftrightarrow f \in Ker(D_2)$.
2. $f - D(g) \in \mathcal{F}^{p+1} \Leftrightarrow f - D_2(g) \in \mathcal{F}^{p+1}$

**Proof.** Using the relations $D(f) = df + (-1)^{df+1}f \circ D$, $D(\gamma^p(sv)) = D(sv)\gamma_{p-1}(sv) = (v + s(dv))\gamma_{p-1}(sv)$ and $D(sv) = v + s(dv)$, we have successively:
1. $D(f) \in \mathcal{F}^{p+2} \Leftrightarrow D(f)(\Gamma(sv)) \cap \Lambda^{p+1}V = \{0\} \Leftrightarrow D_2(f)(\Gamma(sv)) \cap \Lambda^{p+1}V = \{0\} \Leftrightarrow f \in Ker(D_2)$ and
2. $f - D(g) \in \mathcal{F}^{p+1} \Leftrightarrow (f - D(g))(|\Lambda^{p}V) \cap \Lambda^{p}V = \{0\} \Leftrightarrow (f - D_2(g)) \cap \Lambda^{p}V = \{0\}$. \(\square\)

Using this lemma, we have successively:

\[(\mathcal{E}_0^p, d_0) = (\mathcal{F}^p / \mathcal{F}^{p+1}, 0), \]
\[(\mathcal{E}_1^p, d_1) = (\mathcal{E}_0^p, d_1) = (\mathcal{F}^p / \mathcal{F}^{p+1}, d_1),\]

where $d_1 : \mathcal{F}^p / \mathcal{F}^{p+1} \to \mathcal{F}^{p+1} / \mathcal{F}^{p+2}$ such that $\forall f \in \mathcal{F}^p$, $\forall g \in \mathcal{F}^{p-1}$:

\[
\begin{cases}
    d_1(f) = 0 \Leftrightarrow D(f) \in \mathcal{F}^{p+2} \Leftrightarrow f \in Ker(D_2) \Leftrightarrow D_2(f) = 0 \\
    f - d_1(g) = 0 \Leftrightarrow f - D(g) \in \mathcal{F}^{p+1} \Leftrightarrow f - D_2(g) \in \mathcal{F}^{p+1} \Leftrightarrow f = D_2(g)
\end{cases}
\]

As a consequence:

If $k = 2$, $\mathcal{E}_2^p = Ext^*_\mathcal{F}(\mathcal{Q}, (\Lambda^v, d_2))$, which is exactly the first term of the Ext-Milnor-Moore spectral sequence introduced by A. Murillo in [Mur94].

If $k > 2$, the differential $D_2$ is reduced to $D_0$, where $D_0(f) = (-1)^{df+1}f \circ D_0$, $D_0(\gamma^p(sv)) = D_0(sv)\gamma_{p-1}(sv) = v\gamma_{p-1}(sv)$. Then $\mathcal{E}_2^p = Ext^*_\mathcal{F}(\mathcal{Q}, (\Lambda^v, 0))$.

Consider now the case were $k = 3$:

The general term:

\[
\mathcal{E}_3^p = \frac{\{ f \in \mathcal{F}^p, \ D(f) \in \mathcal{F}^{p+3} \}}{\{ f \in \mathcal{F}^{p+1}, \ D(f) \in \mathcal{F}^{p+3} \} + \mathcal{F}^p \cap \mathcal{D}(\mathcal{F}^{p-2})}.\]

Notice that:

\[f \in \mathcal{F}^p \text{ and } D(f) \in \mathcal{F}^{p+3} \Leftrightarrow f \in \mathcal{F}^p \cap Ker(D_3)\]

therefore the natural projection $\gamma : \mathcal{F}^p \cap Ker(D_3) \to \mathcal{E}_3^p$ is well defined and we have:

**Lemma 3.1.2.** The induced morphism of graded $\mathcal{Q}$-vector spaces:

\[\gamma : \mathcal{F}^p \cap Ker(D_3) \to \mathcal{E}_3^p.\]

is surjective with Kernel: $Ker(\gamma) = \mathcal{F}^p \cap Im(D_3) + \mathcal{F}^{p+1} \cap Ker(D_3)$.
Proof. $\gamma$ is clearly a surjective morphism of graded $\mathbb{Q}$ vector-spaces. Let $f \in Ker(\gamma) = \{ g \in \mathcal{F}_p \} \cap \mathcal{F}_p \cap D(\mathcal{F}_p^{-2})$. Write $f = g + D(h) = D_{t}(g) + (g + D_{\geq 4}(h))$, where $g \in \mathcal{F}_p \cap D(\mathcal{F}_p^{-2})$ and $h \in \mathcal{F}_p$. As $D_{t}(h) \in \mathcal{F}_p$ and because $D_{t}(f) = 0$ we have $g + D_{\geq 4}(h) \in \mathcal{F}_p \cap Ker(D_{t})$. \hfill $\square$

As a consequence of this Lemma, we have the isomorphisms of graded $\mathbb{Q}$-vector spaces:

$$\mathcal{E}_3^{p} \cong \frac{\mathcal{F}_p \cap Ker(D_{t})}{\mathcal{F}_p \cap Ker(D_{t}) + \mathcal{F}_p \cap Ker(D_{t})} \cong \frac{\mathcal{F}_p \cap Ker(D_{t}) + \mathcal{F}_p \cap Ker(D_{t})}{\mathcal{F}_p \cap Ker(D_{t})} \cong \frac{\mathcal{F}_p \cap Ker(D_{t})}{\mathcal{F}_p \cap Ker(D_{t})}.$$ 

That is:

$$\mathcal{E}_3^{p} \cong \frac{H(\mathcal{F}_p, D_{t})}{H(\mathcal{F}_p, D_{t})}.$$ 

Hence we deduce the isomorphism of graded algebras:

$$\mathcal{E}_3^{k} \cong \bigoplus_{p \geq 0} Ext_{(\Lambda V,d_{k})}^{p}(\mathbb{Q}, (\Lambda V,d_{k})).$$

The same arguments used for $k = 3$ can be applied term by term to conclude that for any $k \geq 3$, the first term of the second spectral sequence \[3.0.2\] is:

$$\mathcal{E}_k^{p} \cong Ext_{(\Lambda V,d_{k})}^{p}(\mathbb{Q}, (\Lambda V,d_{k})).$$

3.2. Proof of the main results. In this paragraph we give the proof of our main results such as the corresponding corollary.

For this purpose we use evaluation maps to relate each spectral sequence with its Ext-version.

Recall that the chain map

$$ev : Hom((\Lambda V,d_{k}))(\Lambda V \otimes \Gamma(V)), (\Lambda V,d_{k}) \rightarrow (\Lambda V,d_{k})$$

that induce $ev(\Lambda V,d_{k})$ is compatible with filtrations \[3.0.1\] and \[3.0.2\] so it is a morphism of filtered cochain complexes. We are then in the situation to apply the comparison theorem of spectral sequences.

Proof of theorem 1.0.1

Proof. We denote as in the introduction $(\Lambda V,d_{k})$ the Sullivan minimal model of $X$. Since $dim(V) < \infty$ $(\Lambda V,d_{k})$ are Gorenstein algebras, there exists a unique $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $Ext_{(\Lambda V,d_{k})}^{p}(\mathbb{Q}, (\Lambda V,d_{k})) = Ext_{(\Lambda V,d_{k})}^{p+q}(\mathbb{Q}, (\Lambda V,d_{k}))$, with a unique generator and depth $(\Lambda V,d_{k}) = p$. We have then the following diagram:

$$Ext_{(\Lambda V,d_{k})}^{p+q}(\mathbb{Q}, (\Lambda V,d_{k})) \quad \Rightarrow \quad Ext_{(\Lambda V,d_{k})}^{p+q}(\mathbb{Q}, (\Lambda V,d_{k}))$$

$$ev_{(\Lambda V,d_{k})} \downarrow \quad \Rightarrow \quad ev_{(\Lambda V,d_{k})} \downarrow$$

$$H^{p+q}(\Lambda V,d_{k}) \quad \Rightarrow \quad H^{p+q}(\Lambda V,d_{k})$$

Now the ellipticity of $(\Lambda V,d_{k})$ implies that it is a Poincaré duality space and so $dim H^{p+q}(\Lambda V,d_{k}) = 1$ with $N = p + q$ its formal dimension. Consequently the $E_{\infty}$ term of the convergent spectral sequence \[1.0.1\] is isomorphic to $H^{p+q}(\Lambda V,d_{k})$. On the other hand, because $(\Lambda V,d_{k})$ and $(\Lambda V,d_{k})$ are Gorenstein algebras, the $E_{\infty}$ term of \[1.0.2\] is exactly $Ext_{(\Lambda V,d_{k})}^{p+q}(\mathbb{Q}, (\Lambda V,d_{k}))$ and it is isomorphic to $Ext_{(\Lambda V,d_{k})}^{p+q}(\mathbb{Q}, (\Lambda V,d_{k}))$. Assume that $(\Lambda V,d_{k})$ is not elliptic or equivalently Mur94 $ev_{(\Lambda V,d_{k})} = 0$. The previous remark asserts that $0 = ev_{\infty} : E_{\infty} \rightarrow E_{\infty}$. Since $ev_{\infty}$ is identified
with the graded map \( G(H(ev)) \), we must have \( G(H(ev)) = 0 \) and consequently \( G(H(ev)) \neq ev_{(AV,d)} \) (which in non null because \((AV,d)\) is elliptic).

Suppose now that \( \mathcal{E}xt_{(AV,d)}^{p,q}(\mathbb{Q}, (AV,d)) = \mathcal{E}xt_{(AV,d)}^{p',q'}(\mathbb{Q}, (AV,d)) \) (where \((p', q') \in \mathbb{N} \times \mathbb{N}\)) and let \([h]\) its unique generator. Here \( p' = \text{depth}(AV,d) \), so we have \( p' \leq e_0(AV,d) \). On the other hand, it is immediate from the convergence of \( 1.0.2 \) that \( p \leq p' \), so if \( p = p' \), \( G(H(ev))([h]) = 0 \) implies that \( H(ev)([h]) \in F^{p+1}(H(AV,d)) \), consequently \( e_0(AV,d) \geq p + 1 \) and in the case \( p < p' \), \( ev_{(AV,d)}([h]) \in H^{p',q'}(AV,d) \) implies that \( e_0(AV,d) \geq p + 1 \). This show the first implication: \( e_0(AV,d) = p = \text{depth}(AV,d_k) \Rightarrow (AV,d_k) \) is elliptic.

The converse is clearly easier. It suffices to use the previous commutative diagram and the comparison theorem of first quadrant spectral sequences.

\( \square \)

**Proof of theorem 1.0.3**

**Proof.** It is an immediate use of the following diagram related to odd-spectral sequences 1.0.3 and 1.0.4 combined with the comparison theorem (recall here that \((AV,d)\) elliptic, so is \((AV,d_k)\)).

\[
\begin{align*}
\mathcal{E}xt_{(AV,d)}^{p,q}(\mathbb{Q}, (AV,d)) & \quad \Rightarrow \quad \mathcal{E}xt_{(AV,d)}^{p,q}(\mathbb{Q}, (AV,d)) \\
\text{ev}_{(AV,d)} \downarrow & \quad \Rightarrow \quad \text{ev}_{(AV,d)} \downarrow \\
H^{p,q}(AV,d) & \quad \Rightarrow \quad H^{p,q}(AV,d).
\end{align*}
\]

\( \square \)

**Proof of corollary 1.0.4**

**Proof.** The ellipticity of \((AV,d_k)\) gives the one of \((AV,d_k,\sigma)\) (resp. \((AV,d),(AV,d_\sigma)\)) and by applying theorem 1.0.1 we have successively: \( \text{cat}(AV,d) = \text{cat}(AV,d_k) = \text{depth}(AV,d_k) \) and \( \text{cat}(AV,d_\sigma) = \text{cat}(AV,d_{\sigma,k}) = \text{depth}(AV,d_{\sigma,k}) \). Now by using theorem 1.0.3 we have \( \text{depth}(AV,d_k) = \text{depth}(AV,d_{\sigma,k}) \) so that \( \text{cat}(AV,d) = \text{cat}(AV,d_{\sigma,k}) \), that is \( \text{cat}(AV,d) = e_0(AV,d_{\sigma,k}) = \text{dim}(V)^{\text{odd}} + (k - 2)\text{dim}(V)^{\text{even}} \) (by \( \text{LM02} \), Prop. 3).

\( \square \)

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