A NOTE ON THE HIT PROBLEM FOR THE STEENROD ALGEBRA AND ITS APPLICATIONS

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Abstract. Let $P_k = H^*((\mathbb{R}P^\infty)^k)$ be the modulo-2 cohomology algebra of the direct product of $k$ copies of infinite dimensional real projective spaces $\mathbb{R}P^\infty$. Then, $P_k$ is isomorphic to the graded polynomial algebra $\mathbb{F}_2[x_1, \ldots, x_k]$ of $k$ variables, in which each $x_j$ is of degree 1, and let $GL_k$ be the general linear group over the prime field $\mathbb{F}_2$ which acts naturally on $P_k$. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements. We study the hit problem, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra $P_k$ as a module over the mod-2 Steenrod algebra, $A$.

In this Note, we explicitly compute the hit problem for $k = 5$ and the degree $5(2^s - 1) + 24.2^s$ with $s$ an arbitrary non-negative integer. These results are used to study the Singer algebraic transfer which is a homomorphism from the homology of the mod-2 Steenrod algebra, $\text{Tor}_{n+k}^{A}(\mathbb{F}_2, \mathbb{F}_2)$, to the subspace of $\mathbb{F}_2 \otimes_A P_k$ consisting of all the $GL_k$-invariant classes of degree $n$. We show that Singer’s conjecture for the algebraic transfer is true in the case $k = 5$ and the above degrees. This method is different from that of Singer in studying the image of the algebraic transfer. Moreover, as a consequence, we get the dimension results for polynomial algebra in some generic degrees in the case $k = 6$.

1. Introduction

Denote by $P_k = H^*((\mathbb{R}P^\infty)^k)$ the modulo-2 cohomology algebra of the direct product of $k$ copies of infinite dimensional real projective spaces $\mathbb{R}P^\infty$. Then, $P_k$ is isomorphic to the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$ of $k$ variables, in which each $x_j$ is of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

Being the cohomology of a group, $P_k$ is a module over the mod-2 Steenrod algebra, $A$. The action of $A$ on $P_k$ is determined by the elementary properties of the Steenrod squares $Sq^i$ and the Cartan formula (see Steenrod and Epstein [18]).

A polynomial $f$ in $P_k$ is called hit if it can be written as a finite sum $f = \sum_{u \geq 0} Sq^{2^u}(h_u)$ for suitable polynomials $h_u$. That means $f$ belongs to $A^+P_k$, where $A^+$ denotes the augmentation ideal in $A$.

Let $GL_k$ be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on $P_k$ by matrix substitution. Since the two actions of $A$ and $GL_k$ upon $P_k$ commute with each other, there is an action of $GL_k$ on $\mathbb{F}_2 \otimes_A P_k$.

Many authors study the hit problem of determination of a minimal set of generators for $P_k$ as a module over the Steenrod algebra, or equivalently, a basis of $\mathbb{F}_2 \otimes_A P_k$. This problem has first been studied by Peterson [11], Wood [22], Singer.
Priddy [15], who show its relationship to several classical problems in homotopy theory. Then, this problem was investigated by Nam [10], Silverman [17], Wood [22], Sum [19, 21]. Tin-Sum [22], Tin [26, 27, 29] and others.

For a positive integer $n$, by $\mu(n)$ one means the smallest number $r$ for which it is possible to write $n = \sum_{1 \leq i \leq r} (2^{u_i} - 1)$, where $u_i > 0$. This result implies a result of Wood, which originally is a conjecture of Peterson [11].

**Theorem 1.1** (Wood [22]). If $\mu(n) > k$, then $(\mathbb{F}_2 \otimes \mathbb{A} P_k)_n = 0$.

From the above result of Wood, the hit problem is reduced to the case of degree $n$ with $\mu(n) \leq k$.

Let $GL_k$ be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on $P_k$ by matrix substitution. Since the two actions of $\mathbb{A}$ and $GL_k$ upon $P_k$ commute with each other, there is an action of $GL_k$ on $\mathbb{F}_2 \otimes \mathbb{A} P_k$.

For a nonnegative integer $d$, denote by $(P_k)_d$ the subspace of $P_k$ consisting of all the homogeneous polynomials of degree $d$ in $P_k$ and by $(\mathbb{F}_2 \otimes \mathbb{A} P_k)_d$ the subspace of $QP_k$ consisting of all the classes represented by the elements in $(P_k)_d$. One of the major applications of hit problem is in surveying a homomorphism introduced by Singer. In [16], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : \text{Tor}^A_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (\mathbb{F}_2 \otimes \mathbb{A} P_k)_d^{GL_k}$$

from the homology of the Steenrod algebra to the subspace of $(\mathbb{F}_2 \otimes \mathbb{A} P_k)_d$ consisting of all the $GL_k$-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\text{Tor}^A_{k,k+d}(\mathbb{F}_2, \mathbb{F}_2)$. The hit problem and the algebraic transfer were studied by many authors (see Boardman [1], Hưng [5], Chơn and Hà [2], Nam [10], Phuc [14], Phuc and Sum [12], Sum [19, 20, 21], Sum and Tín [23] and others).

Singer showed in [16] that $\varphi_k$ is an isomorphism for $k = 1, 2$. Boardman showed in [1] that $\varphi_3$ is also an isomorphism. However, for any $k \geq 4$, $\varphi_k$ is not a monomorphism in infinitely many degrees (see Singer [16], Hưng [5]). Singer made the following conjecture.

**Conjecture 1.2** (Singer [16]). The algebraic transfer $\varphi_k$ is an epimorphism for any $k \geq 0$.

The conjecture is true for $k \leq 3$. However, for $k > 3$, it is still open. Recently, the hit problem and its applications to representations of general linear groups have been presented in the books of Walker and Wood [30, 31].

One of the extremely useful tools for computing the hit problem and studying Singer’s transfer is the Kameko squaring operation

$$\widetilde{Sq}^0_\ast := (Sq^0_\ast)_{(k,d)} : (\mathbb{F}_2 \otimes \mathbb{A} P_k)_{2k+d} \rightarrow (\mathbb{F}_2 \otimes \mathbb{A} P_k)_d,$$

which is induced by an $\mathbb{F}_2$-linear map $\varphi_k : P_k \rightarrow P_k$, given by

$$\varphi_k(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \ldots x_k y^2; \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in P_k$. The map $\varphi_k$ is not an $A$-homomorphism. However, $\varphi_k S q^{2i} = S q^i \varphi_k$ and $\varphi_k S q^{2i+1} = 0$ for any non-negative integer $i$. 
Theorem 1.3 (Kameko [6]). Let \( d \) be a non-negative integer. If \( \mu(2d + k) = k \), then
\[
(Sq^0_{d})_{(k, d)} : (\mathbb{F}_2 \otimes AP_k)_{2d + k} \rightarrow (\mathbb{F}_2 \otimes AP_k)_d
\]
is an isomorphism of \( GL_k \)-modules.

Thus, the hit problem is reduced to the case of degree \( n \) of the form
\[
n = a(2^s - 1) + 2^sb,
\]
where \( a, b, m \) are non-negative integers such that \( 0 \leq \mu(b) < a \leq k \).

Now, the \( \mathbb{F}_2 \)-vector space \( \mathbb{F}_2 \otimes AP_k \) was explicitly calculated by Peterson [11] for \( k = 1, 2 \), by Kameko [6] for \( k = 3 \) and by Sum [21] for \( k = 4 \). However, for \( k > 4 \), it is still unsolved, even in the case of \( k = 5 \) with the help of computers.

For \( a = k - 1 = 4 \) and \( b = 0 \), the vector space \( (\mathbb{F}_2 \otimes AP_k)_{n} \) is explicitly computed by Phuc and Sum [12], and in the case \( a = k - 2 = 3, b = 1 \) by Phuc [13] and by the present author [28] in the case \( a = k - 1 = 4, b \in \{8; 10; 11\} \) with \( s \) an arbitrary non-negative integer.

In this Note, we explicitly compute the hit problem for \( k = 5 \) and the degree \( 5(2^s - 1) + 24.2^s \) with \( s \) an arbitrary non-negative integer. Using this result, we show that Singer’s conjecture for the algebraic transfer is true in the case \( k = 5 \) and the above degrees. Moreover, as a consequence, we get the dimension results for polynomial algebra in some generic degrees in the case \( k = 6 \).

2. Preliminaries

In this section, we recall some needed information from Kameko [6], Singer [10] and Sum [19], which will be used in the next section.

Notation 2.1. We denote \( \mathbb{N}_k = \{1, 2, \ldots, k\} \) and
\[
X_J = X_{(j_1, j_2, \ldots, j_s)} = \prod_{j \in \mathbb{N}_k \setminus J} x_j, \quad J = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,
\]
Let \( \alpha_i(a) \) denote the \( i \)-th coefficient in dyadic expansion of a non-negative integer \( a \). That means \( a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots \), for \( \alpha_i(a) = 0 \) or \( 1 \) with \( i \geq 0 \).

Let \( x = x_1^{a_1}x_2^{a_2} \ldots x_k^{a_k} \in P_k \). Denote \( \nu_j(x) = a_j, 1 \leq j \leq k \). Set
\[
J_i(x) = \{j \in \mathbb{N}_k : \alpha_i(\nu_j(x)) = 0\},
\]
for \( i \geq 0 \). Then, we have \( x = \prod_{i \geq 0} X_{J_i(x)}^{2^i} \).

Definition 2.2. For a monomial \( x \) in \( P_k \), define two sequences associated with \( x \) by
\[
\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_k(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),
\]
where \( \omega_i(x) = \sum_{1 \leq j \leq k} \alpha_{i-1}(\nu_j(x)) = \deg X_{J_i(x)} \), \( i \geq 1 \). The sequence \( \omega(x) \) is called the weight vector of \( x \).

Let \( \omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots) \) be a sequence of non-negative integers. The sequence \( \omega \) is called the weight vector if \( \omega_i = 0 \) for \( i \gg 0 \).

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector \( \omega \), we define \( \deg \omega = \sum_{i \geq 0} 2^{i-1} \omega_i \). Denote by \( P_k(\omega) \) the subspace of \( P_k \) spanned by all monomials \( y \) such that \( \deg y = \deg \omega, \omega(y) \leq \omega, \)
and by $P_k^- (\omega)$ the subspace of $P_k$ spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

**Definition 2.3.** Let $\omega$ be a weight vector and $f, g$ two polynomials of the same degree in $P_k$.

i) $f \equiv g$ if and only if $f - g \in \mathcal{A}^+ P_k$. If $f \equiv 0$ then $f$ is called hit.

ii) $f \equiv_{\omega} g$ if and only if $f - g \in \mathcal{A}^+ P_k + P_k^- (\omega)$.

Obviously, the relations $\equiv$ and $\equiv_{\omega}$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_{\omega}$. Then, we have

$$QP_k(\omega) = P_k(\omega)/(\mathcal{A}^+ P_k \cap P_k(\omega)) + P_k^- (\omega).$$

For a polynomial $f \in P_k$, we denote by $[f]$ the class in $QP_k$ represented by $f$. If $\omega$ is a weight vector, then denote by $[f]_{\omega}$ the class represented by $f$. Denote by $|S|$ the cardinal of a set $S$.

**Definition 2.4.** Let $x, y$ be monomials of the same degree in $P_k$. We say that $x < y$ if and only if one of the following holds:

i) $\omega(x) < \omega(y)$;

ii) $\omega(x) = \omega(y)$ and $\sigma(x) < \sigma(y)$.

**Definition 2.5.** A monomial $x$ is said to be inadmissible if there exist monomials $y_1, y_2, \ldots, y_m$ such that $y_t < x$ for $t = 1, 2, \ldots, m$ and $x - \sum_{t=1}^m y_t \in \mathcal{A}^+ P_k$.

A monomial $x$ is said to be admissible if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree $n$ in $P_k$ is a minimal set of $\mathcal{A}$-generators for $P_k$ in degree $n$.

Now, we recall some notations and definitions in [21], which will be used in the next sections. We set

$$P_k^0 = \langle \{ x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} : a_1 a_2 \ldots a_k = 0 \} \rangle,$$

$$P_k^+ = \langle \{ x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} : a_1 a_2 \ldots a_k > 0 \} \rangle.$$

It is easy to see that $P_k^0$ and $P_k^+$ are the $\mathcal{A}$-submodules of $P_k$. Furthermore, we have the following.

**Proposition 2.6.** We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k = QP_k^0 \oplus QP_k^+$. Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.

**Definition 2.7.** For any $1 \leq i \leq k$, define the homomorphism $f_i : P_{k-1} \rightarrow P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leq j < i, \\ x_{j+1}, & \text{if } i \leq j < k. \end{cases}$$

Then, $f_i$ is a homomorphism of $\mathcal{A}$-modules.

For a subset $B \subset P_k$, we denote $[B] = \{ [f] : f \in B \}$. Obviously, we have

**Proposition 2.8.** It is easy to see that if $B$ is a minimal set of generators for $\mathcal{A}$-module $P_{k-1}$ in degree $n$, then $f(B) = \bigcup_{i=1}^k f_i(B)$ is a minimal set of generators for $\mathcal{A}$-module $P_k^0$ in degree $n$.

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree $n$ in $P_k$, $B_k^0(n) = B_k(n) \cap P_k^0$, $B_k^+(n) = B_k(n) \cap P_k^+$. For a weight vector $\omega$ of degree $n$, we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B_k^-(\omega) = B_k^+(n) \cap P_k(\omega)$. 
Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the bases of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$. 

3. The Main Results

We first recall a result in [24] the following: Let $d$ be an arbitrary non-negative integer. Set $t(k,d) = \max\{0,k - \alpha(d + k) - \zeta(d + k)\}$, where $\zeta(n)$ the greatest integer $u$ such that $n$ is divisible by $2^u$, that means $n = 2^u m$, with $m$ an odd integer.

**Theorem 3.1** (Tin-Sum [24]). Let $d$ be an arbitrary non-negative integer. Then

\[(S\eta_s)^{s-t}: (\mathbb{F}_2 \otimes_\mathcal{A} P_k)_{k(2s-1)+2^sd} \to (\mathbb{F}_2 \otimes_\mathcal{A} P_k)_{k(2s-1)+2^sd}\]

is an isomorphism of $GL_k$-modules for every $s \geq t$ if and only if $t \geq t(k,d)$.

It is easy to check that for $k = 5$ and $d = 53$ then

\[t(k,d) = \max\{0, k - \alpha(d + k) - \zeta(d + k)\} = 0.\]

Using the above theorem, we get $(\mathbb{F}_2 \otimes_\mathcal{A} P_5)_{5(2^s-1)+2^s53} \cong (\mathbb{F}_2 \otimes_\mathcal{A} P_5)_{5(2^s-1)+2^s53}$ for all $s \geq 0$.

Therefore, we need only to study $(\mathbb{F}_2 \otimes_\mathcal{A} P_5)_{5(2^s-1)+2^s53}$, for $s = 0$ and $s = 1$.

**Case s=0.**

Denote $\omega(1) = (4,4,3)$, $\omega(2) = (4,4,1,1)$, $\omega(3) = (4,2,2,1)$, $\omega(4) = (4,2,4)$. We give a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $(\mathbb{F}_2 \otimes_\mathcal{A} P_5)_{5(2^s-1)+2^s53}$ as follows:

**Theorem 3.2.** We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $(\mathbb{F}_2 \otimes_\mathcal{A} P_5)_{24} = (QP_5^0)_{24} \oplus QP_5^+ (\omega(1)) \oplus QP_5^+ (\omega(2)) \oplus QP_5^+ (\omega(3)) \oplus QP_5^+ (\omega(4))$.

Recall that $(\mathbb{F}_2 \otimes_\mathcal{A} P_4)_{10}$ is an $\mathbb{F}_2$-vector space of dimension 70 with a basis consisting of all the classes represented by the monomials $w_j$, $1 \leq j \leq 70$. Consequently, $|B_4(10)| = 70$, (see Sum [21]).

Since $\mu(24) = 4$, Theorem 1.3 implies that the squaring operation

\[S_{\eta_4}: (\mathbb{F}_2 \otimes_\mathcal{A} P_4)_{24} \to (\mathbb{F}_2 \otimes_\mathcal{A} P_4)_{10}\]

is an isomorphism of $GL_4$-module. Hence, $|B_4(24)| = |B_4(10)| = 70$, and therefore the set $[B_4(24)] = \{[\phi_4(w_j)] : w_j \in B_4(10)\}$ is a basis of the $\mathbb{F}_2$-vector space $(\mathbb{F}_2 \otimes_\mathcal{A} P_4)_{24}$, where $\phi_4(u) = x_1x_2x_3x_4u^2$.

Using Proposition 2.8 we obtain $[B_4^0(24)] = \{[a_t] : a_t \in \bigcup_{i=1}^{350} f_i,B_4(24)\}$, $1 \leq t \leq 350$ is a basis of the $\mathbb{F}_2$-vector space $(QP_5^0)_{24}$. Consequently, $\dim(QP_5^0)_{24} = |\bigcup_{i=1}^{350} f_i,B_4(24)| = 350$.

**Remark 3.3.** We recall a result in Mothebe-Kaelo-Ramatebele [3] as follows.

Set $M_{(k,r)} = \{J = (j_1,j_2,\ldots,j_r) : 1 \leq j_1 < \ldots < j_r \leq k\}$, $1 \leq r < k$. For $J \in M_{(k,r)}$, define the homomorphism $f_J : P_r \to P_k$ of algebras by substituting $f_J(x_t) = x_{j_t}$ with $1 \leq t \leq r$. Then, $f_J$ is a monomorphism of $\mathcal{A}$-modules. We have a direct summand decomposition of the $\mathbb{F}_2$-vector subspaces: 

\[QP_k^0 = \bigoplus_{1 \leq r \leq k-1} \bigoplus_{J \in M_{(k,r)}} (Qf_J(P_r^+))\]
where, $Qf_j(P_r^+) = F_2 \otimes_A f_j(P_r^+)$. 

In degree $n$, we have $\dim(Qf_j(P_r^+))_n = \dim(QP_r^+)_n$ and $|\mathcal{M}_{(k,r)}| = \binom{k}{r}$. Hence, combining with Theorem 1.1 we get

$$\dim(QP_k^0)_n = \sum_{\mu(n) \leq r \leq k-1} \binom{k}{r} \dim(QP_r^+)_n.$$ 

And therefore, using the results in Sum [21], one gets $\dim(QP_5^0)_{24} = 350$.

**Theorem 3.4.**

$$\dim(QP_5(\omega(d))) = \begin{cases} 
75, & \text{if } d = 1, \\
145, & \text{if } d = 2, \\
390, & \text{if } d = 3, \\
1, & \text{if } d = 4. 
\end{cases}$$

The proof of the above theorem is too long and very technical. It is proved by explicitly determining all admissible monomials of $P_5(\omega(d))$ for $d \in \{1, 2, 3, 4\}$.

From the above results, we get the corollary following.

**Corollary 3.5.** There exist exactly 961 admissible monomials of degree twenty-four in $P_5$. Consequently, $\dim(F_2 \otimes_A P_5)_{24} = 961$.

**Case s=1.**

Since Kameko’s homomorphism $(\widetilde{S}q_0)_{(5,24)}$ is an epimorphism, we have

$$(F_2 \otimes_A P_5)_{53} \cong \ker(\widetilde{S}q_*^0)_{(5,24)} \bigoplus \im(\widetilde{S}q_*^0)_{(5,24)}.$$ 

We have the following

**Theorem 3.6.** $\im(\widetilde{S}q_*^0)_{(5,24)}$ is isomorphic to a subspace of $(F_2 \otimes_A P_5)_{53}$ generated by all the classes represented by the admissible monomials of the form $x_1 x_2 \ldots x_5 u^2$, for every $u \in B_5(24)$. Consequently, $\dim(\im(\widetilde{S}q_*^0)_{(5,24)}) = 961$.

Next, we explicitly determine $\ker(\widetilde{S}q_*^0)_{(5,24)}$ by giving a direct summand decomposition of the $F_2$-vector spaces $\ker(\widetilde{S}q_*^0)_{(5,24)}$ as follows.

**Theorem 3.7.** We have a direct summand decomposition of the $F_2$-vector spaces

$$\ker(\widetilde{S}q_*^0)_{(5,24)} = (QP_5^0)_{53} \oplus QP_5^+(3, 3, 3, 2, 1).$$

**Remark 3.8.** From the result in [9] we have

$$\dim(QP_k^0)_{53} = \sum_{\mu(53) \leq r \leq 4} \binom{5}{r} \dim(QP_r^+)_{53}.$$ 

Since $\mu(53) = 3$, $\dim(QP_3^+)_{53} = 8$ (see Kameko [6]) and $\dim(QP_4^+)_{53} = 88$ (see Sum [21]), one gets

$$\dim(QP_5^0)_{53} = \binom{5}{3} \cdot \dim(QP_3^+)_{53} + \binom{5}{4} \cdot \dim(QP_4^+)_{53} = 520.$$
And therefore the set \( \{ [b_t] : b_t \in \bigcup_{i=1}^5 f_i(B_4(53)), 1 \leq t \leq 520 \} \) is a basis of the \( F_2 \)-vector space \((QP^+_5)_{53}\).

We denote by \( M^d(n) = \{[\bigcup_{i=1}^5 x_i^{d-1}f_i(x)] : x \in B_{k-1}(n - 2^d + 1) \} \) and set \( M = \text{Span}\{u : u \in \bigcup_{d=1}^5 M^d(53) \text{ and } \omega(u) = (3, 3, 3, 2, 1)\} \). Hence, one gets the theorem following.

**Theorem 3.9.** The following statements are true:

i) \( M \) is the \( F_2 \)-vector subspaces of \( QP^+_5(\omega) \), where \( \omega = (3, 3, 3, 2, 1) \).

ii) Assume \( QP^+_5(\omega) = M \oplus N \). Then, \( \dim(M) = 389 \quad \text{and} \quad \dim(N) \).

The above theorem is proved by explicitly determining all admissible monomials in \( P^+_5(\omega) \).

On the other hand, we see that \( |B_5(25)| = 1240 \) (see Sum [22]) and \( |B_5(19)| = 905 \) (see Tin [25]). Since \( \mu(25) = 3 = \alpha(25 + \mu(25)) \), using the result in Sum [21], one gets

\[
|B_6(5(2^v - 1) + 2^v25)| = (2^6 - 1)|B_5(25)|, \quad \text{for any integer } v \geq k - 1 = 5.
\]

So, we obtain the corollary following.

**Corollary 3.10.** There exist exactly 78120 admissible monomials of degree \( m_1 = 5(2^v - 1) + 252^v \) in \( P_6 \), for any \( v > 4 \). Consequently, \( \dim(F_2 \otimes_A P_6)_{m_1} = 78120 \).

Similarly, it is easy to check that \( \mu(19) = 3 = \alpha(\mu(19) + 19) \), using the result in Sum [21] we get the following.

**Corollary 3.11.** There exist exactly 57015 admissible monomials of degree \( m_2 = 5(2^v - 1) + 192^v \) in \( P_6 \), for any \( r \geq 5 \). Consequently, \( \dim(F_2 \otimes_A P_6)_{m_2} = 57015 \).

Next, the aim of our paper is to verify this conjecture for the case \( k = 5 \) and the internal degree \( 5(2^s - 1) + 24.2^s \), with \( s \) an arbitrary non-negative integer. We get the following theorem.

**Theorem 3.12.** Singer’s conjecture is true for \( k = 5 \) and the degree \( 5(2^s - 1) + 24.2^s \), with \( s \) an arbitrary non-negative integer.

**Remark 3.13.** We prove the above theorem by using the admissible monomial basis of degree \( 5(2^s - 1) + 24.2^s \) in \( P_5 \) to explicitly compute the vector space \((F_2 \otimes_A P_3)^{GL_5}(5(2^s - 1) + 24.2^s)\) and combining the computation of \( \text{Ext}^{5.5+*}_A(F_2, F_2) \) by Lin [7] and Chen [3]. Our method is different from Boardman’s method in [1] for \( k = 3 \). Recall that Boardman [1] computed the space \((F_2 \otimes_A P_3)^{GL_5}\) by using a basis consisting of all the classes represented by certain polynomials in \( P_5 \). It is difficult to use his method for \( k \geq 4 \). Moreover, our approach can be applied for \( k = 4 \) by using the admissible monomial basis for \( P_4 \) (see Sum [21]). We hope that the conjecture is also true in this case.

The proofs of the results of this Note will be published in detail elsewhere.

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