Supervised Linear Regression for Graph Learning from Graph Signals

Arun Venkitaraman*, Hermina Petric Maretic†, Saikat Chatterjee*, and Pascal Frossard†

* Department of Information Science and Engineering, School of Electrical Engineering and Computer Science, KTH Royal Institute of Technology, Sweden
† Signal Processing Laboratory (LTS4), Ecole Polytechnique Fédérale de Lausanne (EPFL)

arunv@kth.se, hermina.petric.maretic@epfl.ch, sach@kth.se, pascal.frossard@epfl.ch

Abstract

We propose a supervised learning approach for predicting an underlying graph from a set of graph signals. Our approach is based on linear regression. In the linear regression model, we predict edge-weights of a graph as the output, given a set of signal values on nodes of the graph as the input. We solve for the optimal regression coefficients using a relevant optimization problem that is convex and uses a graph-Laplacian based regularization. The regularization helps to promote a specific graph spectral profile of the graph signals. Simulation experiments demonstrate that our approach predicts well even in presence of outliers in input data.

I. INTRODUCTION

Graph learning in the context of graph signal processing refers to the problem of learning associations between different nodes/agents of a graph or network. A network structure is inferred from the given signal values at the different nodes. Graph learning is part of many analysis and processing tasks, such as clustering, community detection, prediction of signal values, or for predicting entire graph signals. Various models have been proposed to infer a graph from a set of signals [1]. Most notable works include graph inference from smooth signals [2] [3] [4], based on the assumption that signals vary slowly over the graph structure. Pasdeloup et al. [5] and Segarra et al. [6] assume signals are given as a result of an arbitrary graph filtering process while learning the graph. Similarly, Mei et al. [7] propose a polynomial autoregressive model for graph signals and a method to infer both the graph and coefficients of the polynomial. We note that the aforementioned works take a one-shot approach by learning the graph
that best describes a given set of graph signals under suitable constraints. They do not explicitly use a
training dataset with labeled graphs and graph signals, and hence, may be seen as unsupervised learning
approaches for graph inference.

In this paper, we propose a supervised learning approach for predicting graphs from graph signals.
A motivating example of supervised graph learning approach can arise in a social network scenario. In
social networks, nodes represent different individuals / persons. Let us assume that we have a training
dataset comprising graph signals and underlying graphs. The graph signals may comprised of different
features, such as age, height, salary, food tastes, consumer habits, etc of the individuals. An underlying
graph could be the one formed by a rule based on who follows whom, or a friendship list of individuals.
Now, in the case of test data, we may have privacy, security or legal reasons for not revealing the true
underlying graph. The task is then to estimate the underlying graph from observed graph signals for the
test case.

To the best of authors’ knowledge, there exist no prior work on exploring supervised learning approach
for graph learning. A supervised learning approach incorporates prior knowledge through training. In our
approach, we model the edge-weights of the graph adjacency matrix as the predicted output of a linear
regression model with an input consisting of a set of graph signal observations. We compute the optimal
regression coefficients from training data by solving an optimization problem with a regularization based
on the graph spectral profile of the graph signals. In order to make that the optimization problem convex,
we use graph spectral profiles in the form of second order polynomial of the graph-Laplacian matrix. We
then discuss how for a suitably constructed input feature, the regression coefficients represent a weighting
of the different graph signals in the input for the prediction task. Simulation experiments reveal that our
approach gives good performance for graph learning under difficult conditions, for examples, if training
dataset is limited and noisy, and test input is also noisy. A block scheme summarizing our approach is
shown in Figure [I]

II. LINEAR REGRESSION FOR GRAPH LEARNING

We first review the relevant basics from graph signal processing and thereafter propose linear regression
for graph prediction.

A. Graph signal processing preliminaries

A graph signal refers to a vector whose components denote the values of a signal over different nodes
of an associated graph. The relation between the different nodes are quantified in using a weighted edge,
and the graph is described using the adjacency matrix $A = [a_{ij}]_{i,j}$ whose $(ij)$th entry $a_{ij}$ denotes the
edge-weight between $i$th and $j$th nodes. In this work, we consider only undirected graphs, which means $a_{ij} = a_{ji}$. The smoothness of a graph signal $x \in \mathbb{R}^N = [x(1), \cdots , x(N)]^\top$ over a graph with $N$ nodes is typically defined using the quantity $x^\top L x = \sum_{i,j} a_{ij} (x(i) - x(j))^2$, where $L \triangleq D - A$ is the graph-Laplacian matrix\[^{[8]}\], \[^{[9]}\], and $D = \text{diag}(d_1, d_2, \cdots , d_N) = \text{diag}(A 1_N)$ is the diagonal degree matrix with $d_i = \sum_j a_{ij}$, $1_N$ being the $N$-dimensional vector of all ones. A small value of $x^\top L x$ implies that the values across connected nodes are similar, leading to the notion of a smooth graph signal. A graph signal $x$ is also equivalently described in terms of its graph-Fourier transform which is defined as

$$\hat{x} \triangleq V^\top x,$$

where $V$ denotes the eigenvector matrix of $L = \Lambda \Lambda \Lambda V^\top$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots , \lambda_N)$ is the eigenvalue matrix arranged according to ascending values. By construction, $\lambda_1 = 0$, and the eigenvectors belonging to the smaller $\lambda_i$ vary smoothly over the graph and represent low-frequencies, and those of larger $\lambda_i$ vary more rapidly, denoting the high-frequencies.

In order to impose that $x$ follows a particular graph-spectral profile (in terms of the distribution of its graph Fourier spectral coefficients), the regularization $x^\top h(L) x$ is often employed, where $h(x) = \sum_{l=0}^{L-1} h_l x^l$ is a polynomial of order $L < N$. This is because the regularization penalizes the different components of $\hat{x}$ as

$$x^\top h(L) x = x^\top V h(\Lambda) V^\top x = \hat{x}^\top h(\Lambda) \hat{x} = \sum_{i=1}^{N} h(\lambda_i) \hat{x}(i)^2.$$

In the case of smooth graph signals, $H = L$ is usually employed since $x^\top h(L) x = \hat{x}^\top \Lambda \hat{x} = \sum_{i=2}^{N} \lambda_i \hat{x}(i)^2$, which penalizes the high-frequency components of $x$ more than the low-frequency ones. Similarly, setting
\( H = L^\dagger \) where \( L^\dagger \) is the pseudo-inverse of \( L \) leads to \( x^\top h(L)x = \hat{x}^\top A^\dagger \hat{x} = \sum_{i=2}^{N} \frac{1}{x_i} \hat{x}(i)^2 \), which promotes \( x \) to have high-frequency behaviour. We refer the reader to [9], [10] and the references therein for a more comprehensive view of graph signal processing framework.

**B. Linear regression model for graph prediction**

Linear and kernel regression form the workhorse of a gamut of applications which involves learning from from support vector machines [11] to deep learning [12] to prediction and reconstruction of graph signals [13], [14], [15], [16], [17], [18]. In this Section, we propose a graph prediction approach using linear regression. We note that in this paper we use the terms graph prediction and graph learning interchangeably.

Let us assume that we have a training set of one or more graphs indexed by \( 1 \leq g \leq G \), \( G \geq 1 \). Let the \( g \)th graph have \( N_g \) nodes. We further assume that we have \( M \) graph signals for each of the \( G \) graphs as input. Let \( A^{(g)} \) denote the weighted adjacency matrix of the \( g \)th graph. Then the input-output pairs are given by \( \{X^{(g)}, A^{(g)}\}_{g=1}^{G} \) where \( X^{(g)} \in \mathbb{R}^{N_g \times M} \) denotes the matrix with the \( M \) graph signals as columns. We consider the following model for predicting weight of the edge between the \( i \)th and \( j \)th nodes:

\[
a_{i,j}^{(g)} = w^\top \phi \left( x^{(g)}(i), x^{(g)}(j) \right) + \text{model noise.} \tag{1}
\]

Here \( w \in \mathbb{R}^K \) is the regression coefficient vector, \( \phi \) is a \( K \)-dimensional feature vector where \( x^{(g)}(i) \) is the \( i \)th row vector of \( X^{(g)} \) as follows

\[
x^{(g)}(i) = [x_1^{(g)}(i), \ldots x_m^{(g)}(i), \ldots x_M^{(g)}(i)]^\top \in \mathbb{R}^M.
\]

Thus, the estimate of \( a_{i,j}^{(g)} \) is given by

\[
\hat{a}_{i,j}^{(g)} = w^\top \phi \left( x^{(g)}(i), x^{(g)}(j) \right). \tag{2}
\]

The input feature vector \( \phi(\cdot) \) is assumed to be known. In the general case, it could be an arbitrary function of the input signal. Intuitively, for our problem it is desirable that the values of \( \phi(x(i), x(j)) \) should reflect on the similarity of the signal values between the nodes \( i \) and \( j \). The smaller the values of \([ (x_1(i) - x_1(j))^2 \cdots (x_m(i) - x_m(j))^2 \cdots (x_M(i) - x_M(j))^2 ] \), the larger \( \phi \) must be in order to ensure a strong edge between nodes \( i \) and \( j \). Similarly, dissimilar values across the nodes with large values of \([ (x_1(i) - x_1(j))^2 \cdots (x_M(i) - x_M(j))^2 ] \) should result in a \( \phi \) with small values. Though multiple such \( \phi \) could be constructed, we use a simple choice with the \( m \)th component of \( \phi \) defined by

\[
\phi(x(i), x(j))(m) = \frac{\sigma}{\max((x_m(i) - x_m(j))^2, \sigma)}, \quad 1 \leq m \leq M
\]
σ is a parameter introduced to avoid φ being unbounded when the signal values at nodes i and j are very similar. Thus, we observe that the mth component of φ reflects the similarity of the values of the mth graph signal at the i’th and j’th nodes. Correspondingly, the components of w represent the relative importance of the M graph signals in predicting the graph. In order to ensure that the graphs have no self loops, that is, \( a_{ii}^{(g)} = 0 \), \( \forall i, g \), we make the additional definition that \( \phi(x^{(g)}(i), x^{(g)}(j)) = 0 \) when \( i = j \).

Then, by collecting all the edge-weights predicted by the regression model (2) for the gth graph as a matrix, we have the adjacency matrix estimate for the gth graph given by

\[
\hat{A}^{(g)} = \Phi^{(g)}W_g, \quad \forall g, \tag{3}
\]

\[
\Phi^{(g)} = \begin{bmatrix}
\phi(x^{(g)}(1), x^{(g)}(1))^\top & \cdots & \phi(x^{(g)}(1), x^{(g)}(N_g))^\top \\
\vdots & \ddots & \vdots \\
\phi(x^{(g)}(N_g), x^{(g)}(1))^\top & \cdots & \phi(x^{(g)}(N_g), x^{(g)}(N_g))^\top
\end{bmatrix},
\]

\[
W_g = \begin{bmatrix}
w & 0 & \cdots & 0 \\
0 & w & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & w
\end{bmatrix} = I_{N_g} \otimes w;
\]

\( \otimes \) denotes the Kronecker product operation and \( I_{N_g} \) is the identity matrix of size \( N_g \). Then, the corresponding graph-Laplacian estimate is given by

\[
\hat{L}^{(g)} = \hat{D}^{(g)} - \hat{A}^{(g)} = \text{diag}(\Phi^{(g)}W_g 1_{N_g}) - \Phi^{(g)}W_g, \quad \forall g
\]

\[
= \text{diag}(A^{(g)}1_{N_g}) = \tilde{D}^{(g)}
\]

\[
\overset{(a)}{=} \sum_{n=1}^{N_g} e_n^\top (\Phi^{(g)}W_g 1_{N_g}) e_n - \Phi^{(g)}W_g, \quad \forall g \tag{4}
\]

where \( 1_{N_g} \) is the all ones column vector of length \( N_g \) and \( e_n \) is the column vector with all zeros except one at the \( n \)th component. The equality \( (a) \) follows from the matrix identity \( \text{diag}(a) = \sum_{n=1}^N (e_n^\top a) e_n e_n^\top \) where \( a \in \mathbb{R}^N \).
\begin{align}
g &= 2 \text{vec} \left( \sum_g \Phi^{(g)\top} A^{(g)} \right) - \alpha \text{vec} \left( \sum_{g=1}^G h_1 \Phi^{(g)\top} \sum_{n=1}^N e_n^1 \text{tr} \left( e_n^e X^{(g)\top} X^{(g)} - X^{(g)} X^{(g)\top} \right) \right) \\
F &= 2 \left[ I_N \otimes \sum_g \Phi^{(g)\top} \Phi^{(g)} \right] + 2 \alpha \sum_{g=1}^G h_2 \left[ X^{(g)\top} X^{(g)} \right] \otimes \left[ \Phi^{(g)\top} \Phi^{(g)} \right] + \beta I_{MN} \\
&+ \alpha \sum_{g=1}^G h_2 \sum_{n_1=1}^N \sum_{n_2=1}^N \left[ (1_N e_{n_1}^e) \otimes \Phi^{(g)\top} 1_N e_{n_2}^e \Phi^{(g)} \right] + \left( e_{n_1}^e 1_N \right) \otimes \left( \Phi^{(g)\top} 1_N \right) \text{tr} \left( e_{n_1}^e e_{n_2}^e e_{n_2}^e X^{(g)\top} X^{(g)} \right) \\
&+ \alpha \sum_{g=1}^G h_2 N \sum_{n=1}^N \text{vec} \left( \Phi^{(g)\top} 1_N e_n^e \right) \text{vec} \left( \left[ X^{(g)\top} X^{(g)} e_n^e \Phi^{(g)} \right] \right) ^\top + \text{vec} \left( \Phi^{(g)\top} e_n^e e_n^e X^{(g)\top} X^{(g)} \right) \text{vec} \left( \Phi^{(g)\top} 1_N \right) \\
\tilde{\rho} &= [\rho_1 \rho_2 \cdots \rho_K] \in \mathbb{R}^{N^2 K \times K} \text{ where } \rho_k = I_N \otimes e_k \\
\Omega &= [\Omega_1 \Omega_2 \cdots \Omega_N], \text{ where } \Omega_j \triangleq [(j-1)(N+1)K+1 : (j-1)(N+1)K+K]
\end{align}

C. Linear Regression for Graph Prediction

Given Eq. \((\ref{eq:linear_regression})\), \((\ref{eq:regularized_regression})\), our goal is to compute the optimal regression coefficients \(w\) such that the following cost is minimized

\[
J(w) = \sum_{g=1}^G \| A^{(g)} - \tilde{A}^{(g)} \|^2_F + \alpha \sum_{g=1}^G \text{tr} \left( X^{(g)\top} h(\tilde{L}^{(g)}) X^{(g)} \right) \\
+ \frac{\beta}{G} \sum_{g=1}^G \text{tr}(W_g^\top W_g),
\]

where the first regularization term imposes the learnt graphs to have the desired graph-spectral profile (as discussed in Section II-A). The second regularization ensures that \(w\) remains bounded. In imposing the regularization, we have implicitly assumed that the graph signals follow a particular graph spectral profile over the associated graph. This assumption is reasonable in cases such as social networks where the different communities in the graph might still have similar dynamics or distribution of features across the nodes. Now, if we make the further simplifying assumption that all training graphs have the same size \(N_g = N\), \(J(w)\) is expressible as follows:

\[
J(w) = \sum_{g=1}^G \| A^{(g)} - \Phi^{(g)} W \|^2_F \\
+ \alpha \sum_{g=1}^G \text{tr} \left( X^{(g)\top} h(\tilde{L}^{(g)}) X^{(g)} \right) + \beta \text{tr}(W^\top W),
\]

where \(W = I_N \otimes w\). We note that \(\Phi\) is not convex in \(w\) in general. Convex optimization problems have a global minimum and often resulting in tractable closed form solutions. This makes it desirable
that $J(w)$ in Eq. (6) be convex. This directly translates to the requirement that $h(\cdot)$ be a second order polynomial of the form $h(x) = h_0 + h_1x + h_2x^2$. A second order $h(\cdot)$ is nevertheless fairly generic and can represent various kinds of graph signal behaviour such as low-pass, high-pass, etc. As $J(w)$ is now convex, the unique global optimal value of $w$ is obtained by setting the derivative of $J(w)$ with respect to $w$ equal to zero. This leads to the following proposition:

**Proposition 1.** The optimal regression coefficients $w_{opt}$ that minimizes the cost in Eq. (6) satisfies

$$C_\Omega(\bar{\rho}^T F)(1_N \otimes w_{opt}) = \bar{\rho}^T g$$

where $C_\Omega(X)$ denotes the matrix operation that returns the submatrix of $X$ with only columns indexed by set $\Omega$, and quantities $\Omega$, $\bar{\rho}$, $F$, and $g$ are as defined in Eq. (5).

**Proof.** The proof follows from the use of matrix calculus to take the gradient of $J(w)$ with respect to $w$, and uses chain rule and other standard properties of Kronecker product and vectorization.

$$J(w) = \sum_{g=1}^G \|A^{(g)} - \Phi^{(g)}W\|_F^2 + \alpha \sum_{g=1}^G \text{tr} \left( X^{(g)^T} h_g(L^{(g)}) X^{(g)} \right) + \beta N \text{tr}(w^T w)$$

We shall hereafter use $\sum_g$ and $\sum_n$ to denote $\sum_{g=1}^G$ and $\sum_{n=1}^N$, to keep the notation simple. Then, from (3) and (4) we have that

$$J(w) = \sum_{g=1}^G \|A^{(g)} - \Phi^{(g)}W\|_F^2 + \underbrace{\alpha \sum_{g=1}^G \text{tr} \left( X^{(g)^T} h_g \left( \sum_{n=1}^N e_n^T (\Phi^{(g)}W)(1_N e_n) e_n^T - \Phi^{(g)}W \right) X^{(g)} \right) + \beta \text{tr}(W^T W)}_{J_2(W)}$$

$$= J_1(W) + \alpha J_2(W) + \beta J_3(W)$$

(7)
D. Simplifying cost function

We now analyze these terms separately:

\[
J_1(W) = \sum_g \|A^{(g)} - \Phi^{(g)} W\|_F^2 = \sum_g \text{tr}(\left[ A^{(g)} - \Phi^{(g)} W \right]^T \left[ A^{(g)} - \Phi^{(g)} W \right])
\]

\[
= \sum_g \text{tr}(\left[ A^{(g)} \right]^T A^{(g)}) - 2\sum_g \text{tr}(\left[ A^{(g)} \right]^T \Phi^{(g)} W) + \sum_g \text{tr}(\left[ \Phi^{(g)} W \right]^T \Phi^{(g)} W)
\]

\[
= \sum_g \text{tr}(\left[ A^{(g)} \right]^T A^{(g)}) - 2\sum_g \text{tr}(\left[ A^{(g)} \right]^T \Phi^{(g)} W) + \sum_g \text{tr}(W^T \Phi^{(g)} \Phi^{(g)} W)
\]

\[
= \sum_g \text{tr}(\left[ A^{(g)} \right]^T A^{(g)}) - 2\text{tr}(\sum_g \left[ A^{(g)} \right]^T \Phi^{(g)} W) + \text{tr}(W^T \sum_g \Phi^{(g)} \Phi^{(g)} W)
\]

(8)
\[ J_2(W) = \sum_{g=1}^{G} \text{tr} \left( X^{(g)^T} h_g \left( L^{(g)} \right) X^{(g)} \right) = \sum_{g=1}^{G} \text{tr} \left( h_g \left( L^{(g)} \right) X^{(g)} X^{(g)^T} \right) \]

\[ = \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)^T} \right) + \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} L^{(g)} X^{(g)} X^{(g)^T} \right) + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} L^{(g)^2} X^{(g)} X^{(g)^T} \right) \]

\[ = \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ \sum_{n=1}^{N} e_n^{T} \left( \Phi^{(g)} W_{1N} \right) e_n - \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ \sum_{n=1}^{N} e_n^{T} \left( \Phi^{(g)} W_{1N} \right) e_n - \Phi^{(g)} W \right]^2 X^{(g)} X^{(g)^T} \right) \]

\[ = \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ \sum_{n=1}^{N} e_n^{T} \left( \Phi^{(g)} W_{1N} \right) e_n - \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} e_{n_1}^{T} \left( \Phi^{(g)} W_{1N} \right) e_{n_1} e_{n_2}^{T} \left( \Phi^{(g)} W_{1N} \right) e_{n_2} - \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( -2h_2^{(g)} \left[ \sum_{n=1}^{N} e_n^{T} \left( \Phi^{(g)} W_{1N} \right) e_n - \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ \Phi^{(g)} W \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]
\[ J_2(W) \stackrel{(a)}{=} \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ \sum_{n=1}^{N} e_n^T (\Phi^{(g)} W_{1_N}) e_n e_n^T \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ -\Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} e_{n_1}^T (W^T \Phi^{(g)^T} 1_N) e_{n_1} e_{n_2}^T (\Phi^{(g)} W_{1_N}) e_{n_2} \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( -2h_2^{(g)} \left[ \sum_{n=1}^{N} e_n^T (W^T \Phi^{(g)^T} 1_N) e_n \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ W^T \Phi^{(g)^T} \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ J_2(W) \stackrel{(a)}{=} \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} h_1^{(g)} \sum_{n=1}^{N} e_n^T (\Phi^{(g)} W_{1_N}) \text{tr} \left( e_n e_n^T X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ -\Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} e_{n_1}^T (W^T \Phi^{(g)^T} 1_N) e_{n_1} e_{n_2}^T (\Phi^{(g)} W_{1_N}) \text{tr} \left( e_{n_1} e_{n_2} e_{n_1} e_{n_2}^T X^{(g)} X^{(g)^T} \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( -2h_2^{(g)} \sum_{n=1}^{N} e_n^T (W^T \Phi^{(g)^T} 1_N) \text{tr} \left( e_n e_n^T \Phi^{(g)} W X^{(g)} X^{(g)^T} \right) \right) \]

\[ + \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ W^T \Phi^{(g)^T} \Phi^{(g)} W \right] X^{(g)} X^{(g)^T} \right) \]
\[
J_2(W) = (a) \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} h_1^{(g)} \sum_{n=1}^{N} \text{tr} \left( e_n^\top (\Phi^{(g)} W 1_N) \right) \text{tr} \left( e_n^\top e_n^\top X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ -\Phi^{(g)} W \right] X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \text{tr} \left( e_{n_1}^\top (W^\top \Phi^{(g)\top} 1_N) e_{n_2}^\top (\Phi^{(g)} W 1_N) \right) \text{tr} \left( e_{n_1}^\top e_{n_2}^\top e_n^\top e_n^\top X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} -2h_2^{(g)} \sum_{n=1}^{N} \text{tr} \left( e_n^\top (W^\top \Phi^{(g)\top} 1_N) \right) \text{tr} \left( X^{(g)} X^{(g)\top} e_n^\top e_n^\top \Phi^{(g)} W \right) \\
+ \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ W^\top \Phi^{(g)\top} \Phi^{(g)} W \right] X^{(g)} X^{(g)\top} \right)
\]

\[
J_2(W) = (a) \sum_{g=1}^{G} \text{tr} \left( h_0^{(g)} X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} h_1^{(g)} \sum_{n=1}^{N} \text{tr} \left( W 1_N e_n^\top \Phi^{(g)} \right) \text{tr} \left( e_n^\top e_n^\top X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} \text{tr} \left( h_1^{(g)} \left[ -\Phi^{(g)} W \right] X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \text{tr} \left( W^\top \Phi^{(g)\top} 1_N e_{n_2}^\top \Phi^{(g)} W 1_N e_{n_1}^\top \right) \text{tr} \left( e_{n_1}^\top e_{n_2}^\top e_n^\top e_n^\top X^{(g)} X^{(g)\top} \right) \\
+ \sum_{g=1}^{G} -2h_2^{(g)} \sum_{n=1}^{N} \text{tr} \left( W^\top \Phi^{(g)\top} 1_N e_n^\top \right) \text{tr} \left( X^{(g)} X^{(g)\top} e_n^\top e_n^\top \Phi^{(g)} W \right) \\
+ \sum_{g=1}^{G} \text{tr} \left( h_2^{(g)} \left[ W^\top \Phi^{(g)\top} \Phi^{(g)} W \right] X^{(g)} X^{(g)\top} \right)
\] (10)
E. Taking derivatives of the cost function parts with respect to $W$

In order to keep the mathematics self-contained, we list here some properties of matrix calculus which we shall be using later:

\[
\text{vec}(AXB) = (B^\top \otimes A)\text{vec}(X)
\]

\[
\frac{\partial \text{tr}(AX)}{\partial X} = A^\top
\]

\[
\frac{\partial \text{tr}(X^\top AXB)}{\partial X} = AXB + A^\top XB^\top
\]

\[
\text{tr}(A^\top B) = (\text{vec}A)^\top \text{vec}B
\]

Then, from (8) we have that

\[
\frac{\partial J_1(W)}{\partial W} = -2 \sum_g \Phi(g)^\top A(g) + 2 \sum_g [\Phi(g)^\top \Phi(g)] W
\]

Similarly, from (10) we have that

\[
\frac{\partial J_2(W)}{\partial W} = (a) \sum_{g=1}^G h_1^{(g)} \sum_{n=1}^N \Phi(g)^\top e_n 1_N^\top \text{tr} \left( e_n e_n^\top X(g)^\top X(g)^\top \right)
\]

\[
- \sum_{g=1}^G h_1^{(g)} \Phi(g)^\top X(g)^\top X(g)^\top
\]

\[
+ \sum_{g=1}^G h_2^{(g)} \sum_{n=1}^N e_{n_2} 1_N^\top \Phi(g)^\top W e_{n_1} 1_N^\top \text{tr} \left( e_n e_n^\top X(g)^\top X(g)^\top \right)
\]

\[
+ \sum_{g=1}^G h_2^{(g)} \sum_{n=1}^N \sum_{n_1=1}^N \Phi(g)^\top e_{n_1} 1_N^\top \Phi(g)^\top W e_{n_2} 1_N^\top \text{tr} \left( e_n e_n^\top X(g)^\top X(g)^\top \right)
\]

\[
+ \sum_{g=1}^G -2 h_2^{(g)} \sum_{n=1}^N \Phi(g)^\top 1_N e_n^\top \text{tr} \left( X(g)^\top X(g)^\top e_n e_n^\top \Phi(g)^\top W \right)
\]

\[
+ \sum_{g=1}^G -2 h_2^{(g)} \sum_{n=1}^N \text{tr} \left( W^\top \Phi(g)^\top 1_N e_n^\top \right) \Phi(g)^\top e_n e_n^\top X(g)^\top X(g)^\top
\]

\[
+ 2 \sum_{g=1}^G h_2^{(g)} \left[ \Phi(g)^\top \Phi(g)^\top \right] W \left[ X(g)^\top X(g)^\top \right]
\]

And finally,

\[
\frac{\partial J_3(W)}{\partial W} = 2W
\]
Then, from (11), (12), and (13), we have that

\[
\frac{\partial J(W)}{\partial W} = -2 \sum_g \Phi(g)^T A(g) + 2 \sum_g [\Phi(g)^T \Phi(g)] W
\]

\[
+ \alpha \sum_g h_1(g) \sum_{n=1}^N \Phi(g)^T e_n 1_N^T tr \left( e_n e_n^T X(g) X(g)^T \right)
\]

\[
- \alpha \sum_g h_1(g) \Phi(g)^T X(g) X(g)^T
\]

\[
+ \alpha \sum_g h_2(g) \sum_{n_1=1}^N \sum_{n_2=1}^N \Phi(g)^T 1_N^T e_{n_2} e_{n_2}^T W 1_N^T e_{n_1}^T tr \left( e_{n_1} e_{n_1}^T e_{n_2} e_{n_2}^T X(g) X(g)^T \right)
\]

\[
+ \alpha \sum_g h_2(g) \sum_{n_1=1}^N \sum_{n_2=1}^N \Phi(g)^T e_{n_1} 1_N^T \Phi(g)^T W e_{n_1} e_{n_1}^T tr \left( e_{n_1} e_{n_1}^T e_{n_2} e_{n_2}^T X(g) X(g)^T \right)
\]

\[
+ \alpha \sum_g -2h_2(g) \sum_{n=1}^N \Phi(g)^T 1_N^T \Phi(g)^T W e_n^T tr \left( X(g) X(g)^T e_n e_n^T \Phi(g) W \right)
\]

\[
+ \alpha \sum_g -2h_2(g) \sum_{n=1}^N \Phi(g)^T 1_N^T e_n^T tr \left( W^T \Phi(g)^T 1_N^T \Phi(g)^T e_n e_n^T X(g) X(g)^T \right)
\]

\[
+ 2 \alpha \sum_g h_2(g) \left[ \Phi(g)^T \Phi(g)^T \right] W \left[ X(g) X(g)^T \right]
\]

\[
+ 2 \beta W
\]

(14)

\[\text{vec} \left( \frac{\partial J_1(W)}{\partial W} \right) = -2 \text{vec} \left( \sum_g \Phi(g)^T A(g) \right) + 2 \left[ I_N \otimes \sum_g \Phi(g)^T \Phi(g) \right] \text{vec} (W)\]
Similarly, we have

\[
\operatorname{vec} \left( \frac{\partial J_2(W)}{\partial W} \right) = \operatorname{vec} \left( \sum_{g=1}^{G} h_1^{(g)} \Phi^{(g)T} \left[ \sum_{n=1}^{N} e_n e_n^T \operatorname{tr} \left( e_n e_n^T X^{(g)} X^{(g)T} \right) - X^{(g)} X^{(g)T} \right] \right) \\
+ \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left( (1_N e_{n_1}^T \otimes \Phi^{(g)T} 1_N e_{n_2}^T \Phi^{(g)}) \right) \operatorname{tr} \left( e_n e_n^T e_{n_1}^T e_{n_2}^T X^{(g)} X^{(g)T} \right) \operatorname{vec} W \\
+ \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left( (e_n e_{n_1}^T \otimes \Phi^{(g)T} e_{n_2}^T \Phi^{(g)}) \right) \operatorname{tr} \left( e_n e_n^T e_{n_1}^T e_{n_2}^T X^{(g)} X^{(g)T} \right) \operatorname{vec} W \\
+ \sum_{g=1}^{G} \left( 2 h_2^{(g)} \sum_{n=1}^{N} \operatorname{vec} \left( \Phi^{(g)T} e_n^T X^{(g)} X^{(g)T} \right) \operatorname{vec} \left( \Phi^{(g)T} e_n^T \Phi^{(g)} \right) \right) \operatorname{vec} W \\
+ \sum_{g=1}^{G} \left( 2 h_2^{(g)} \sum_{n=1}^{N} \operatorname{vec} \left( \Phi^{(g)T} e_n^T X^{(g)} X^{(g)T} \right) \operatorname{vec} \left( \Phi^{(g)T} e_n^T \Phi^{(g)} \right) \right) \operatorname{vec} W
\]

Putting together vectorized partial derivatives of $J_1$, $J_2$, and $J_3$, we have that the vectorized partial derivative of $J$ with respect to $W$ is given by:

\[
\operatorname{vec} \left( \frac{\partial J(W)}{\partial W} \right) = -2 \operatorname{vec} \left( \sum_{g} \Phi^{(g)T} A^{(g)} \right) + 2 \left( I_N \otimes \sum_{g} \Phi^{(g)T} \Phi^{(g)} \right) \operatorname{vec} (W) \\
+ \alpha \operatorname{vec} \left( \sum_{g=1}^{G} h_1^{(g)} \Phi^{(g)T} \left[ \sum_{n=1}^{N} e_n e_n^T \operatorname{tr} \left( e_n e_n^T X^{(g)} X^{(g)T} \right) - X^{(g)} X^{(g)T} \right] \right) \\
+ \alpha \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left( (1_N e_{n_1}^T \otimes \Phi^{(g)T} 1_N e_{n_2}^T \Phi^{(g)}) \right) \operatorname{tr} \left( e_n e_n^T e_{n_1}^T e_{n_2}^T X^{(g)} X^{(g)T} \right) \operatorname{vec} W \\
+ \alpha \sum_{g=1}^{G} h_2^{(g)} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \left( (e_n e_{n_1}^T \otimes \Phi^{(g)T} e_{n_2}^T \Phi^{(g)}) \right) \operatorname{tr} \left( e_n e_n^T e_{n_1}^T e_{n_2}^T X^{(g)} X^{(g)T} \right) \operatorname{vec} W \\
+ \alpha \sum_{g=1}^{G} \left( 2 h_2^{(g)} \sum_{n=1}^{N} \operatorname{vec} \left( \Phi^{(g)T} e_n^T X^{(g)} X^{(g)T} \right) \operatorname{vec} \left( \Phi^{(g)T} e_n^T \Phi^{(g)} \right) \right) \operatorname{vec} W \\
+ \beta \operatorname{vec} W \\
= F \operatorname{vec} W - g
\]
The derivative of $J(W)$ with respect to $k$th component of $w$ denoted by $w(k)$ is then given by

$$\frac{\partial J(W)}{\partial w(k)} = \text{tr} \left( \left[ \frac{\partial J(W)}{\partial W} \right]^\top \frac{\partial W}{\partial w(k)} \right) = \left( \text{vec} \frac{\partial J(W)}{\partial W} \right)^\top \text{vec} \frac{\partial W}{\partial w(k)}$$

Now, we have that

$$\frac{\partial W}{\partial w(k)} = \frac{\partial (I_N \otimes w)}{\partial w(k)} = I_N \otimes \frac{\partial w}{\partial w(k)} = I_N \otimes e_k$$

Let $\rho_k \in \mathbb{R}^{N^2 K}$ denote $\text{vec} \frac{\partial W}{\partial w(k)}$. Then, we have that

$$\rho_k(i) = \begin{cases} 1, & \text{if } i = m((N + 1)K) + k, \ m = 0, 1, \ldots, N - 1, \\ 0, & \text{otherwise} \end{cases}$$

Then, the gradient of $J(W)$ with respect to $w$ may be written as

$$\rho^\top \text{vec} \frac{\partial J(W)}{\partial W} = 0$$

where $\bar{\rho} = [\rho_1 \rho_2 \cdots \rho_K] \in \mathbb{R}^{N^2 K \times K}$. Then, from above we have that

$$\frac{\partial J(W)}{\partial w} = \rho^\top \text{vec} \frac{\partial J(W)}{\partial W} = \bar{\rho}^\top (F \text{vec}W + g) = 0$$

Since $\text{vec}W = [w^\top 0 \cdots 0 \ w^\top 0 \cdots 0 \cdots w^\top]^\top$, the values of $w$ depend only on those columns of $\bar{F}$ and $g$ which correspond to the $N \times K$ nonzero components of $\text{vec}W$. The component sets of $\text{vec}W$ where $w$ is present are the following:

$$\Omega_1 \triangleq [1 : K] \text{(first } K \text{ corresponding to the first occurrence of } w)$$
$$\Omega_2 \triangleq [(N + 1)K + 1 : (N + 1)K + K]$$
$$\Omega_3 \triangleq [2(N + 1)K + 1 : 2(N + 1)K + K]$$
$$\Omega_4 \triangleq [3(N + 1)K + 1 : 3(N + 1)K + K]$$
$$\vdots$$
$$\Omega_N \triangleq [(N - 1)(N + 1)K + 1 : (N - 1)(N + 1)K + K]$$
$$\equiv [N^2 K - K + 1 : N^2 K] \text{(the last } K \text{ corresponding to the last occurrence of } w)$$

Then, from Proposition 1, the optimal coefficients $w$ are obtained by solving:

$$\bar{F}w_{\text{opt}} = \bar{\rho}^\top g$$

where $\bar{F} \triangleq [C_{\Omega_1} (\bar{\rho}^\top F) + C_{\Omega_2} (\bar{\rho}^\top F) + \cdots + C_{\Omega_N} (\bar{\rho}^\top F)]$. Finally, we have that

$$w_{\text{opt}} = \bar{F}^\top \bar{\rho}^\top g.$$
where $\dagger$ denotes the pseudo-inverse operation.

Once the regression coefficients are obtained, the prediction of the edge weight between two nodes $i$ and $j$ for a possibly new graph $g_{new}$, given their corresponding $M$ graph signal values as input, is given by

$$a_{i,j}^{(g_{new})} = w_{opt}^T \phi(x^{(g_{new})(i)}, x^{(g_{new})(j)}).$$

(16)

We also note here that since we model the edge-weights separately as Eq. (1), we do not make any assumptions regarding the size of the graphs in the test data, they could be of different number of nodes than that of the training data graphs. That is, the size of the graphs may be different in training and test datasets. Our approach remains applicable as long as the spectral profile of the graph signals of all graphs is assumed to be the same, as discussed earlier.

III. EXPERIMENTAL RESULTS

We first consider the application of our approach to synthesized graph datasets. Our goal is to learn the optimal regression coefficients from training graphs to make predictions for the adjacency matrices of test graphs. We consider the case when the input observations are smooth graph signals but a fraction of them are outlier signals which are high-frequency graph signals. We assume the outliers to occur at

Fig. 2. Results from synthesized graph signal data with outliers (a) Plot showing $w$ for sparse graph of size $N = 6$ for $M = 10N = 100$ graph signals, (b) NMSE of both approaches with 10% outliers, and (c) with 25% outliers.
the same observation indices among the $M$ graph signals in the input, in all the training and test graphs. Such an example simulates the case where the graph signals represent features which may be not equally relevant to graph learning process. We then use $w_{\text{opt}}$ to make predictions for the graphs in the test set using Eq. (2). Our expectation is that values of $w_{\text{opt}}$ will exhibit different trends for the smooth graph signals and the outliers signals.

We consider $32 (G = 16)$ graphs of size $N = 10$, out of which 16 are used for training and the remaining for testing. We first construct a sparse connected graph $A_0$ with only 40% of the total number of non-diagonal entries being non-zero with values drawn from uniform random distribution over $[0, 1]$. The training and test graphs $A^{(g)}$ are then generated by randomly inserting values at 10% of non-diagonal entries of $A_0$, with values again drawn from the uniform random distribution over $[0, 1]$. The adjacency matrices are scaled so that they all have unit Frobenius norm. For each of these graphs, we generate smooth graph signals by drawing realizations from zero-mean Gaussian with covariance matrix $[L^{(g)}]^{\dagger}$. The outlier signals are generated from zero-mean multivariate Gaussian distribution with covariance matrix $[L^{(g)}]^2$, which correspond to high-frequency graph signals (following discussion in Section II-A).

We consider two cases: outliers being 10% and 25% of the $M$ input signals for different values of $M$. We use $\alpha = 0.1/M$ and $\beta = 10/M$ which are values set by crossvalidation. We measure the performance of our approach in terms of the normalized mean square error (NMSE) of the prediction for test data defined as

$$\text{NMSE} = \frac{\mathbb{E}(\|A - \hat{A}\|_F^2)}{\mathbb{E}(\|A\|_F^2)}$$

where $A$ and $\hat{A}$ denote the true and estimated adjacency matrices, respectively; and the expectation is taken over all testing matrices for 100 Monte Carlo runs. In Figure 2(a), we show an instance of $w$ for $M = 10$, averaged over 100 Monte Carlo realizations for both 10% and 25% outlier cases. We observe that the computed regression coefficients clearly shows a difference in trend between smooth graphs signals and outliers. This is because we impose the resulting model to minimize the graph smoothness regularization. In Figure 2(c), we plot the NMSE for test prediction at different values of input observation size $M$. Since no prior supervised learning approaches exist for the problem, we restrict ourselves to the comparison of our method with that of Kalofolias et al [3]. The approach is a popular approach used in learning graphs from smooth graph signals. The hyperparameters of the method are set by choosing the values which minimize the training NMSE. In the case of both approaches, we threshold the estimated matrices in order to obtain sparse matrices.

We notice that, with 10% outliers in the input, our approach outperforms graph-learning when $M$ is small, but nearly coincides with the latter at larger $M$ values. When the fraction of outliers in the input is
increased to 25%, we observe that the other approach performs poorly in comparison with our approach. Further, our approach results in an NMSE which is comparable with that obtained in the 10% outlier case. Such a consistency in prediction NMSE of our approach may be attributed to the regression model which allows for differential treatment of the various input graph signals. The classical graph-learning on the other hand explicitly assumes all the signals to be smooth over the graph that is being inferred. We note that in both 10% and 25% outlier cases, the NMSE of both approaches decreases as \( M \) increases.

In Table I, we list the F values of the obtained matrices for test data from both methods. We observe that the F scores of both approaches increase with \( M \) and are close to each other.

We further note here that we also performed the experiment with Erdos-Renyi graphs of size \( N = 10 \), and observed similar trends in NMSE both with \( M \) and in comparison with \( M \). However, we omit the corresponding NMSE plots here for brevity.

### IV. Conclusions

We proposed a supervised learning approach based on linear regression for predicting graphs from graph signals based. We formulated a convex optimization problem to solve for the regression coefficients. Our approach was shown to result in a good prediction performance when not all the graph signals may be equally relevant or may have noise/ corruptions. This is because the linear regression model allows for the different graph signals be weighed differently. The learnt regression coefficients were seen to reflect the presence of outliers in the graph signals. In future work, we will pursue the application of our approach to real-world datasets such as weather measurements, and Yelp, and functional magnetic resonance imaging data.

### REFERENCES

[1] X. Dong, D. Thanou, M. Rabbat, and P. Frossard, “Learning graphs from data: A signal representation perspective,” *arXiv preprint arXiv:1806.00848*, 2018.
[2] X. Dong, D. Thanou, P. Frossard, and P. Vandergheynst, “Learning laplacian matrix in smooth graph signal representations,” *IEEE Transactions on Signal Processing*, vol. 64, no. 23, pp. 6160–6173, 2016. [Online]. Available: http://arxiv.org/abs/1406.7842

[3] V. Kalofolias, “How to learn a graph from smooth signals,” in *Artificial Intelligence and Statistics*, 2016, pp. 920–929.

[4] H. E. Egilmez, E. Pavez, and A. Ortega, “Graph learning from data under laplacian and structural constraints,” *IEEE Journal of Selected Topics in Signal Processing*, vol. 11, no. 6, pp. 825–841, 2017.

[5] B. Pasdeloup, V. Gripon, G. Mercier, D. Pastor, and M. G. Rabbat, “Characterization and inference of graph diffusion processes from observations of stationary signals,” *IEEE Transactions on Signal and Information Processing over Networks*, 2017.

[6] S. Segarra, A. G. Marques, G. Mateos, and A. Ribeiro, “Network topology inference from spectral templates,” *IEEE Transactions on Signal and Information Processing over Networks*, vol. 3, no. 3, pp. 467–483, 2017.

[7] J. Mei and J. Moura, “Signal processing on graphs: Causal modeling of unstructured data,” *IEEE Transactions on Signal Processing*, vol. 65, no. 8, pp. 2077–2092, 2017.

[8] F. K. Chung, *Spectral Graph Theory*, ser. CBMS Regional Conference Series in Mathematics, vol. 92. AMS, 1997.

[9] D. I. Shuman, S. Narang, P. Frossard, A. Ortega, and P. Vandergheynst, “The emerging field of signal processing on graphs: Extending high-dimensional data analysis to networks and other irregular domains,” *IEEE Signal Process. Mag.*, vol. 30, no. 3, pp. 83–98, 2013.

[10] A. Ortega, P. Frossard, J. Kovačević, J. M. F. Moura, and P. Vandergheynst, “Graph signal processing: Overview, challenges, and applications.” *Proc. IEEE*, vol. 106, no. 5, pp. 808–828, May 2018.

[11] C. M. Bishop, *Pattern Recognition and Machine Learning (Information Science and Statistics).* Secaucus, NJ, USA: Springer-Verlag New York, Inc., 2006.

[12] Y. Cho and L. K. Saul, “Kernel methods for deep learning,” in *Adv. Neural Inform. Process. Syst.* Curran Associates, Inc., 2009, pp. 342–350.

[13] D. Romero, M. Ma, and G. B. Giannakis, “Kernel-based reconstruction of graph signals,” *IEEE Trans. Signal Process.*, vol. 65, no. 3, pp. 764–778, Feb 2017.

[14] D. Romero, V. N. Ioannidis, and G. B. Giannakis, “Kernel-based reconstruction of space-time functions on dynamic graphs,” *CoRR*, vol. abs/1612.03615, 2016. [Online]. Available: http://arxiv.org/abs/1612.03615

[15] A. Venkitaraman, S. Chatterjee, and P. Händel, “Kernel Regression for Signals over Graphs,” *ArXiv e-prints*, Jun. 2017.

[16] A. Venkitaraman, S. Chatterjee, and P. Handel, “Multi-kernel regression for graph signal processing,” in *IEEE Int. Conf. Acoust. Speech Signal Process. Calgary, Canada*, 2018, pp. 4644–4648. [Online]. Available: https://doi.org/10.1109/ICASSP.2018.8461643

[17] A. Venkitaraman, S. Chatterjee, and P. Händel, “Gaussian processes over graphs,” *https://arxiv.org/abs/1803.05776*, 2018.

[18] V. N. Ioannidis, M. Ma, A. N. Nikolakopoulos, G. B. Giannakis, and D. Romero, “Kernel-based inference of functions over graphs,” in *Adaptive Learning Methods for Nonlinear System Modeling*, D. Comminiello and J. C. Príncipe, Eds. Butterworth-Heinemann, 2018, pp. 173 – 198.

[19] A. Sandryhaila and J. M. F. Moura, “Big data analysis with signal processing on graphs: Representation and processing of massive data sets with irregular structure,” *IEEE Signal Processing Magazine*, vol. 31, no. 5, pp. 80–90, Sept 2014.