Abstract

Resonant mode interactions in weakly nonlinear multi-dimensional lattices and related effects are described. We concentrate on formal description of the phenomenon and consider as examples mode interactions and evolution equations for quadratic solitons in two-dimensional lattices, and coherent structures in a double-chain as examples of the application of the theory.

1 Introduction

Elastic interactions among quasi-particles, like phonons, electrons, magnons, etc., in solids constitute fundamental phenomena allowing one to observe microscopic effects on the macroscopic level. Today they are very well studied and are described in numerous text-books (see e.g. Modeling, 1981). Although interactions mean nonlinearity and thus possibility of a complex dynamics such phenomena have been discussed, so far, on the language of quasi-particles. This means that usually only two-body interactions are taken into account. Mean-time, due to fundamental discoveries of sixties and early seventies, namely due to construction of the modern soliton theory, it became clear that the dynamic of nonlinear systems is much richer than one provided by the theory of two-body interactions. Namely, it became clear that together with linear quasi-particles one should consider solitons (or solitary waves) which are essentially nonlinear objects and cannot be constructed from the linear theory by any perturbation technique. In particular, speaking about nonlinear lattices one should consider not only phonons and interactions among them, but also solitary waves, called envelope solitons (Tsurui, 1972). Meantime solitons are approximate solutions of the nonlinear lattices and they are necessarily accompanied by generation of higher harmonics (Jimenez, 1999). The fact that corresponding dynamics
displays indeed a complicate behavior has been reported in (Economou, \textit{et al.} 1994; Kopidakis, \textit{et al.} 1994). The effect of high harmonics becomes especially important when the resonant conditions

\begin{equation}
\omega_3 = \omega_1 + \omega_2, \quad \mathbf{q}_3 = \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{Q}
\end{equation}

or

\begin{equation}
\omega_2 = 2\omega_1, \quad \mathbf{q}_2 = 2\mathbf{q}_1 + \mathbf{Q}
\end{equation}

are satisfied. On the language of solid state physics one can interpret the first and second equalities in (1) or in (2) as conservation laws of energy and momentum for phonons, respectively. Using the terminology of the nonlinear optics these formulas can be called matching conditions or conditions of resonant three mode generation and resonant generation of the second harmonic. The present work describes various phenomena in nonlinear multi-dimensional lattices when one of the above resonance conditions is satisfied.

The theory we develop is applied to lattices with a complex cell, to vector lattices (with a simple or complex cell), or to couple lattices of different physical nature (like for example to a system describing electron-phonon interactions (Konotop, 1997)). Also, subject to certain modification the theory can be adjusted to describe coupled nonlinear chains. In the context of the present work the last system is interesting since it does not belong either class of one-dimensional nor two-dimensional lattices. Also a double chain is an example of a system when one can provide resonant conditions between localized and extended degrees of freedom.

2 Properties of a linear lattice.

Let us recover some general properties of a linear $\mathcal{N}$D lattice [hereafter $\mathcal{N} (= 1, 2, 3)$ indicates the dimension of the space]. It will be assumed that the lattice is finite with $N_j$ ($j = 1, \ldots, \mathcal{N}$) cells in $j$th direction, so that the total number of cells is $N = N_1 \cdots N_{\mathcal{N}}$. It also will be assumed that each cell has $\alpha$ generalized degrees of freedom: they can be related either to different atoms, or to different components of the displacement vector. After all we will be interested in the limit $N_j \to \infty$. We start with a generic case, when the evolution equation is written as follows

\begin{equation}
\dot{\Psi}(\mathbf{n}) = -\sum_1 J(\mathbf{n}, 1)\Psi(1).
\end{equation}

Here $\Psi(\mathbf{n})$ is a $\mathcal{N} \times \mathcal{N}$ matrix with $\mathcal{N} = \alpha\mathcal{N}$ having elements $\psi_{ij}(\mathbf{n})$, $\mathbf{n}$ is an $\mathcal{N}$-dimensional vector indicating the position of a cell: it can be represented in the standard form $\mathbf{n} = \sum_j n_j\mathbf{e}_j$, where $\mathbf{e}_j$ are lattice vectors with $n_j = 0, 1 \cdots N_j - 1$, $J(\mathbf{n}, 1)$ is a $\mathcal{N} \times \mathcal{N}$ matrix of force constants which is considered to be hermitian, $J_{ij}(\mathbf{n}, 1) = \overline{J_{ji}(\mathbf{n}, 1)}$ (the overbar stands for the complex conjugation) and having only positive eigenvalues, and a dot standing for the derivative with respect to time. It will be assumed that the second index in $\psi_{ij}$ refers to a branch of the spectrum, while the first one is associated with a generalized degree
of freedom. Lattice \([\mathbb{Z}]\) is considered subject to periodic boundary conditions 
\[\Psi(n + N_j e_j) = \Psi(n)\] for all \(j\).

Evolution equation \([\mathbb{Z}]\) can be associated with the spectral problem
\[\Phi(n; q)\Lambda(q) = \sum_l J(n, l)\Phi(l; q)\] (4)
where \(\Phi(n; q) = (\Phi_1(n; q), \ldots, \Phi_N(n; q))\): \(\Phi_\eta(n; q)\) are column eigenfunctions, 
\(\Lambda(q) = \text{diag}[\lambda_1(q), \ldots, \lambda_N(q)]\), is the eigenvalue matrix: \(\omega^2_\eta(q) \equiv \lambda_\eta(q), (\eta = 1 \ldots N)\) and \(q\) is a vector in the reciprocal space. It will be the assumption that \(\lambda_\eta(q) \neq \lambda_{\eta'}(q)\) for any \(q\) if \(\eta \neq \eta'\).

We will be interested in translationally invariant lattices, this is \(J(n, 1) = J(|n - 1|)\). This property allows one to represent the eigenfunctions in the form
\[\Phi(n, q) = A(q)e^{i\mathbf{q}\cdot\mathbf{r}}\] (5)
where \(A(q)\) is a matrix depending on the wave vector only. It solves the equation
\[A(q)\Lambda(q) = J(q)A(q)\] (6)
with \(J(q) = \sum_n J(|n|)e^{-i\mathbf{q}\cdot\mathbf{r}}\). It will be assumed that the eigenfunctions \(\phi_\eta(n, q)\) make up an orthonormal complete set. Thus
\[\sum_n^\infty \Phi^\dagger(n; q')\Phi(n; q) = \Delta(q, q'), \quad \sum_q^\infty \Phi(n; q)\Phi^\dagger(l; q) = \Delta(n, l)\] (7)
where \(\Delta(q, q') = I\delta_{q, q'}\), \(\delta_{q, q'}\) is the Kronecker delta, \(I\) is a unit 4 \(\times\) 4 matrix and the dagger stands for the Hermitian conjugate. It follows from \([\mathbb{Z}]\) that \(A(q)\) is a unitary matrix \(A^\dagger(q)A(q) = I\) which means that \(\Phi(n, q)\) is a unitary matrix, as well. Now the dispersion relation of the linear lattice can be expressed as
\[\Lambda(q) = \sum_n \sum_l^\infty \Phi^\dagger(n; q)J(n, l)\Phi(l; q).\] (8)

Next we express the group velocity through the eigenfunctions of the linear spectral problem. To this end we consider a small variation \(q \rightarrow q + dq\) and the resulting changes of the eigenfunctions
\[\Phi(l; q) \rightarrow \Phi(l; q) + \left[i l \cdot dq - \frac{1}{2} (l \cdot dq)^2\right] \Phi(l; q)\]
This variation can be interpreted as one caused by the perturbation
\[\delta J(n, l) = \left[i (l - n) \cdot dq - \frac{1}{2} ((l - n) \cdot dq)^2\right] J(n, l)\] (9)
of the Kernel \(J(n, l)\). Thus on the one hand the Taylor expansion yields
\[\Lambda(q + dq) = \Lambda(q) + (dq\nabla_q)\Lambda(q) + \frac{1}{2} (dq\nabla_q)^2\Lambda(q) + O(|dq|^3)\] (10)
where $\nabla_q \equiv \partial/\partial q$. On the other hand we obtain from the standard perturbation technique

$$\Lambda(q + dq) = \Lambda(q) + \Lambda^{(1)}(q) + \Lambda^{(2)}(q) + O(|dq|^3)$$

where the diagonal matrices $\Lambda^{(j)}(q)$ have elements as follows

$$\Lambda^{(1)}_{\eta\eta}(q) = i \sum_{l,n}(dq \cdot (n - l))K_{\eta\eta}(n, l; q) \quad (11)$$

$$\Lambda^{(2)}_{\eta\eta}(q) = \sum_{l,n}(dq \cdot (n - l))^2K_{\eta\eta}(n, l; q)$$

$$- \sum_{l,n} \sum_{l',n'}(dq(n - l))(dq(n' - l')) \sum_{\alpha \neq \eta} \frac{K_{\eta\eta}(n, l; q)K_{\alpha\alpha}(n', l'; q)}{\lambda_{\eta} - \lambda_{\alpha}} \quad (12)$$

where we have introduced the matrix

$$K_{\eta\eta}(n, l; q) = \Phi^\dagger_{\eta\eta}(n; q)J(n, l)\Phi_{\eta\eta}(l; q) \quad (13)$$

Comparing (11) and (12) with (9) we compute the group velocity

$$v_{\eta}(q) \equiv \nabla_q \omega_{\eta}(q) = \frac{i}{2\omega_q(q)} \sum_{n_1, n_2} (n_2 - n_1)K_{\eta\eta}(n_1, n_2; q). \quad (14)$$

and the tensor of the group velocity dispersion

$$\omega_{\eta ij} = - \frac{1}{\omega_q} \frac{\partial \omega_{\eta}}{\partial q_i} \frac{\partial \omega_{\eta}}{\partial q_j} + \frac{1}{\omega_q} \sum_{l,n}(n_i - l_i)(n_j - l_j)K_{\eta\eta}(n, l; q)$$

$$- \frac{1}{\omega_q} \sum_{l,n} \sum_{l',n'}(n_i - l_i)(n_j' - l_j') \sum_{\alpha \neq \eta} \frac{K_{\eta\eta}(n, l; q)K_{\alpha\alpha}(n', l'; q)}{\lambda_{\eta} - \lambda_{\alpha}} \quad (15)$$

### 3 Nonlinear model

The subject of our main interest is the resonant mode interactions in a nonlinear lattice possessing quadratic nonlinearity. As in the previous section we start with the general model whose dynamics is governed by the equation

$$\ddot{\Psi}(n) = - \sum_{n_1} J(n, n_1)\Psi(n_1) - \sum_{n_1, n_2} \hat{J}_3(n, n_1, n_2) : \Psi(n_1)\Psi(n_2) \quad (16)$$

Here $\Psi(n) = \text{col}(\psi_1(n), \ldots, \psi_{N}(n))$, $J(n, n_1)$ is a $N \times N$ matrix possessing the properties described in the previous section and $\hat{J}_3(n, n_1, n_2)$ is understood to be a tensor with components $J_{3, jj_1, jj_2}$, such that the last term in the formal expression in (16) means that the jth component of the nonlinear term is given by

$$\left(\hat{J}_3(n, n_1, n_2) : \Psi(n_1)\Psi(n_2)\right)_j = \sum_{j_1, j_2} J_{3, jj_1, jj_2}(n, n_1, n_2)\psi_{j_1}(n_1)\psi_{j_2}(n_2) \quad (17)$$
Below we concentrate on the case of nearest neighbor interactions.

In order to describe mode interactions we employ the multiscale analysis. To this end we introduce a formal small parameter $\epsilon$, $\epsilon \ll 1$, which generates a set of temporal variables $t_\nu = \epsilon^\nu t$, ($\nu = 0, 1, \ldots$) and a set of spatial variables $n_0$ and $r_\nu = \epsilon^\nu n$, ($\nu = 1, 2, \ldots$) which are regarded as independent. $r_\nu$ will be considered as a continuous variable. Respectively, we look for the solution of (16) in the form of the expansion

$$
\Psi(n) = \epsilon \Psi^{(0)}(n_0, \{r_\nu\}; \{t_\nu\}) + \epsilon^2 \Psi^{(1)}(n_0, \{r_\nu\}; \{t_\nu\}) + \cdots
$$

(17)

where $\{r_\nu\}$ and $\{t_\nu\}$ stands for the whole set of variables. Being interested in the process of three wave interaction we represent

$$
\psi^{(0)}(n_0, \{r_\nu\}; \{t_\nu\}) = \sum_{\alpha=1}^3 A_\alpha(r_1; t_1) \phi_\alpha(n_0; q_\alpha)e^{-i\omega_\alpha(q_\alpha)t_0} + c.c.
$$

(18)

Here, $\phi_\alpha(n_0; q_\alpha)$ are the eigenfunctions of the eigenvalues $\omega_\alpha^2$, linked by (1), $A_\alpha(r_1; t_1)$ can be interpreted as the amplitude of the envelope of the $\alpha$-th mode and "c.c." stands for complex conjugation. In a generic situation the amplitude $A$ depends on all "slow variables" $(r_1, r_2, \ldots; t_1, t_2, \ldots)$. However being interested in the process of three wave interaction which corresponds to the scales defined by $r_1$ and $t_1$ we indicate only these variables in the argument of the amplitude. The scaling imposed implies neglecting the group velocity dispersion.

Substituting the new variables in evolution equation (14) and gathering terms of the same order of the parameter $\epsilon$ one recovers the dispersion relation (8) in the first order. In the second order with respect to $\epsilon$ subject to the matching condition (1) one obtains the following system of equations for slowly varying amplitudes

$$
\frac{\partial A_1}{\partial t_1} + (v_{\eta_1} \cdot \nabla_r)A_1 = i \frac{\chi_{n_1 n_2 n_3}(q_1, q_2)}{\omega_{\eta_1}} \bar{A}_2 A_3
$$

(19)

$$
\frac{\partial A_2}{\partial t_1} + (v_{\eta_2} \cdot \nabla_r)A_2 = i \frac{\chi_{n_1 n_2 n_3}(q_1, q_2)}{\omega_{\eta_2}} \bar{A}_1 A_3
$$

(20)

$$
\frac{\partial A_3}{\partial t_1} + (v_{\eta_3} \cdot \nabla_r)A_3 = i \frac{\chi_{n_1 n_2 n_3}(q_1, q_2)}{\omega_{\eta_3}} \bar{A}_1 A_2
$$

(21)

where $\nabla_r = \frac{\partial}{\partial r_1}$. The nonlinear coefficient $\chi_{n_1 n_2 n_3}(q_1, q_2)$ is given by

$$
\chi_{n_1 n_2 n_3}(q_1, q_2) = \sum_{n_1, n_2, n_3} \sum_{j_1, j_2, j_3} J_{3, j_1 j_2 j_3}(n_1, n_2, n_3) \\
\times \bar{\phi}_{j_1, n_1}(n_1, q_1) \bar{\phi}_{j_2, n_2}(n_2, q_2) \phi_{j_3, n_3}(n_3, q_1 + q_2)
$$

(22)

where $\phi_{j, n}(n, q)$ are entries of the column eigenvector $\Phi_j(n, q)$, $j = 1, 2, \ldots, \bar{N}$, (hereafter vectors of the reciprocal lattice $Q$ are dropped in the arguments of
the functions). In the particular case of the second-harmonic generation, when
conditions (2) are satisfied one has
\[ \frac{\partial A_1}{\partial t_1} + (v_\eta_1 \nabla_{r_1}) A_1 = \frac{i\chi_{\eta_1\eta_2}}{\omega_{\eta_1}} A_1, \]
\[ \frac{\partial A_2}{\partial t_1} + (v_\eta_2 \nabla_{r_1}) A_2 = \frac{i\chi_{\eta_1\eta_2}}{\omega_{\eta_2}} A_2^* \]
(23)
where for \( \chi_{\eta_1\eta_2} \equiv \chi_{\eta_1\eta_2}(q) \) one must use the formula
\[ \chi_{\eta_1\eta_2} = \sum_{n_1, n_2, n_3, j_1, j_2} J_{3,j_1j_1j_2}(n_1, n_2, n_3)(\phi_{j_1, \eta_1}(n_1, q))^2 \phi_{j_2, \eta_2}(n_2, 2q) \]  
(24)

4 Second harmonic generation in a two-dimensional lattice.

As the simplest example of the above approach we consider a 2D diatomic lattice
having atoms with masses \( M_1 \) and \( M_2 \) with linear interactions between nearest-
neighbors and subject to a nonlinear on-site potential. In order to introduce
atom numbering we can choose a cell as is shown in Fig. 1 Then the lattice
Hamiltonian reads
\[
H = \sum_n \left\{ \frac{p_1^2(n)}{2M_1} + \frac{p_2^2(n)}{2M_2} + \frac{K_{21}}{2} u_1^2(n) + \frac{K_{22}}{2} u_2^2(n) \right. \\
+ \frac{K_2}{2} [(u_1(n) - u_2(n))^2 + (u_1(n) - u_2(n + e_1))^2] \\
+ \left. \frac{K_{31}}{3} [u_{11}^3(n) + u_{12}^3(n)] + \frac{K_{32}}{3} [u_{21}^3(n) + u_{22}^3(n)] \right\} 
\]  
(25)
Here \( K_2 \) is a linear force constant, \( K_{2j} \) and \( K_{3j} \) are respectively the linear and
nonlinear constants of a force applied to the \( j \)-th atom in a cell and originated
by the on-site potential, \( u_{ij} \) is the \( i \)-th component of the displacement of the \( j \)-th atom. We assume that the lattice is square with a unitary lattice constant: \( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \). Then in order to represent the evolution equation in the form (23) we define \( \psi (\mathbf{n}) = \text{col} (\sqrt{M_1} u_{11}, \sqrt{M_2} u_{12}, \sqrt{M_3} u_{21}, \sqrt{M_4} u_{22}) \) and calculate the components of \( J_2 \) and \( J_3 \). Then the dispersion relation for the linear lattice can be written in the form

\[
\omega_{n}^{2}(\mathbf{q}) = \omega_{n,\sigma}^{2}\left[1 + (-1)^{n} \alpha_{\sigma} \sqrt{1 - \gamma_{\sigma} F(\mathbf{q})}\right]
\]

where \( \kappa = \frac{q + 1}{2} \), here \([\cdot]\) stands for the integer part, \( \sigma = 2^{\frac{1+(-1)^{n}}{2}} \),

\[
F(\mathbf{q}) = \sin^{2}\left(\frac{q_{1}}{2}\right) + \sin^{2}\left(\frac{q_{2}}{2}\right) + \sin^{2}\left(\frac{q_{1}}{2}\right) \sin^{2}\left(\frac{q_{2}}{2}\right),
\]

\[
\gamma_{\sigma} = \frac{4m}{(1 - m)^{2}(k_{\sigma}^{2} + 2k_{\sigma}) + (1 + m)^{2}}, \quad \alpha_{\sigma} = \sqrt{\frac{4m}{\gamma_{\sigma}} (1 + m)(1 + k_{\sigma})}
\]

\[
\omega_{0}^{2} = 2\omega_{0}^{2}(1 + k_{2})(1 + m), \quad k_{\sigma} = \frac{k_{2}}{\sqrt{M_{2}}}, \quad m = \frac{M_{2}}{M_{1}} \quad \text{and} \quad \omega_{0} = \sqrt{\frac{k_{2}}{M_{2}}}, \quad \text{and} \quad \eta = 1, 2
\]

correspond to the low frequency branches which are reduced to the acoustic modes at \( K_{2j} = 0 \), while \( \eta \) is 3, 4 denotes high frequency optical branches.

Peculiarities of the second harmonic generation in the 2D case follow from the dispersion relation. Firstly, in contrast to the 1D case the resonant conditions (23) are satisfied on a continuous line which can be seen from the fact that the equation for the line can be written as \( \sum_{ij} C_{ij} X^{i} Y^{j} = 0 \) with \( X = \sin^{2}\left(\frac{\mathbf{q}}{2}\right) \), \( Y = \sin^{2}\left(\frac{\mathbf{q}}{2}\right) \) and constants \( C_{ij} \) which are trivially found from (23). Secondly, energy transfer can occur between modes with the same direction of velocities or with different directions of velocities. In particular, the velocities can be orthogonal. The former case in many aspects is analogous to the second harmonic generation in the 1D chain (Konotop 1996, 1997). Finally, it is possible to have simultaneous processes when one low-frequency mode can resonantly interact with two other modes. In particular the processes \( 2\omega_{1}(\mathbf{q}_{1}) = \omega_{2}(2\mathbf{q}_{1}) \) and \( 2\omega_{1}(\mathbf{q}_{1}) = \omega_{3}(2\mathbf{q}_{1}) \) can occur simultaneously for \( \mathbf{q}_{1} = (\pi, 0) \) if the lattice parameters are chosen as following \( m = 12 \frac{k_{1}^{2} + 2k_{2} + 1}{3k_{1}^{2} + 6k_{1} + 4} \) and \( k_{2} = \frac{12k_{1}^{2} + 21k_{1} + 10}{3(1 + k_{1})} \).

Although in all other points the equations (23) are similar to the equations describing the second-harmonic generation in nonlinear optics, at such points equations (23) are not valid any more.

The effective nonlinearity controlling SHG is now given by

\[
\chi_{\eta, \eta_{2}}(\mathbf{q}) = \frac{1}{\sqrt{M_{1}}} [K_{31} A_{\eta, 1}^{\ast 2}(\mathbf{q}) A_{\eta_{2}, 1}(2\mathbf{q}) + K_{32} A_{\eta_{1}, 3}^{\ast 2}(\mathbf{q}) A_{\eta_{2}, 3}(2\mathbf{q})] + \frac{1}{\sqrt{M_{2}}} [K_{31} A_{\eta, 2}^{\ast 2}(\mathbf{q}) A_{\eta_{2}, 2}(2\mathbf{q}) + K_{32} A_{\eta_{1}, 4}^{\ast 2}(\mathbf{q}) A_{\eta_{2}, 4}(2\mathbf{q})]
\]

Here \( A_{\eta, j}(\mathbf{q}) \) is the amplitude which can be trivially found from (17).
If \( v_1 \) is parallel to \( v_2 \) then this reduces the system (23) describing the evolution of the envelope to an effectively one dimensional problem

\[
\frac{\partial A_1}{\partial t_1} + v_1 \frac{\partial A_1}{\partial x} = i \frac{\chi}{\omega_1} A_1 A_2, \quad \frac{\partial A_2}{\partial t_1} + v_2 \frac{\partial A_2}{\partial x} = i \frac{\bar{\chi}}{\omega_2} A_1^2
\] (29)

where \( x \) is the slow spatial coordinate in the direction of the group velocity vector and \( \chi = \chi_{n_1 n_2}(\mathbf{q}) \). This system is well studied in the context of nonlinear optics (see e.g. Suhorukov. 1988). It also reduces the 2D problem to quasi-one-dimensional one (Konotop, 1996).

4.1 Case of orthogonal velocities

If the velocities of the first and second harmonics are orthogonal then system (23) is reduced to

\[
\frac{\partial A_1}{\partial t_1} + v_1 \frac{\partial A_1}{\partial x} = i \frac{\chi}{\omega_1} A_1, \quad \frac{\partial A_2}{\partial t_1} + v_2 \frac{\partial A_2}{\partial y} = i \frac{\bar{\chi}}{\omega_2} A_1
\] (30)

where \( x \) and \( y \) are spatial coordinates in the direction of the group velocities of the first and second harmonic, respectively.

In a special case the system (30) can be linearized, using the ideas of (Suhorukov. 1988). Indeed, let us introduce normalized amplitudes

\[
a_1 = \frac{\bar{\chi}}{\sqrt{2\omega_1}} e^{-i\alpha_1} A_1, \quad a_2 = \frac{\chi}{\omega_1} e^{-i\alpha_2} A_2, \quad 2\alpha_1 - \alpha_2 = \pi/2
\]

and look for a solution of (30) such that the functions \( a_1 \) and \( a_2 \) are real. Let us also define operators

\[
\hat{L}_1 = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x}, \quad \hat{L}_2 = \frac{\partial}{\partial t} + v_2 \frac{\partial}{\partial y}
\]

Then system (30) can be rewritten in the form

\[
\hat{L}_1 a_1 = a_1 a_2, \quad \hat{L}_2 a_2 = -a_1^2
\] (31)

Next we apply \( \hat{L}_1 \) to the second of the above equations. As a result we arrive at \( \hat{L}_2(\hat{L}_1 a_2 - a_2^2) = 0 \). This means that \( \hat{L}_1 a_2 - a_2^2 = C(x, y - v_2 t) \) where \( C(\cdot, \cdot) \) is a function which is determined from the boundary (initial) conditions. Then the linear equation for \( 1/a_1 \) takes the form

\[
\hat{L}_1^2 \frac{1}{a_1} + C(x, y - v_2 t) \frac{1}{a_1} = 0
\] (32)

4.1.1 Solitary wave solution

The first simplification of (32) can be achieved by assuming that \( C(\cdot, \cdot) \) does not depend on \( x \): \( C(\cdot, \cdot) \equiv C(\xi_2) \). Here we introduce running variables: \( \xi_1 = x/v_1 - t \) and \( \xi_2 = y/v_2 - t \). Then one can look for the solutions depending on the running
variables only: \( a_j \equiv a_j(\xi_1, \xi_2) \). Here we represent one of them which is generated by the Lorentz pulse \( C(\xi_2) = \text{const}/(1 + \xi_2^2/v_2^2)^2 \). It reads

\[
A_1 = \sqrt{2}\omega_1 \chi \sqrt{1 + \xi_2^2} \left[ \sinh(\xi_1 \sinh(\arctan \xi_2)) - \cosh(\xi_1 \cosh(\arctan \xi_2)) \right]
\]

\[
A_2 = \frac{i\omega_1 \xi_2 \left[ \cosh(\xi_1 - \arctan \xi_2) - \sinh(\xi_1 - \arctan \xi_2) \right]}{\chi (1 + \xi_2^2) \cosh(\xi_1 - \arctan \xi_2)}
\]

The first harmonic represents a solitary wave pulse which is exponentially localized in the direction of the group velocity of the first mode and displays power-law-localization in the direction of propagation of the second harmonic. The second harmonic is delocalized in the direction of the group velocity of the first mode. In that direction it has a kink-like shape. At \( t > 0 \) the two modes being coupled with each other move without distortion in the direction defined by \( x/v_1 + y/v_2 = 0 \).

### 4.1.2 Spatial structure

In order to illustrate the possibility of existence of stationary spatial structures we consider solutions of \([\text{30}]\) or \([\text{31}]\) in a space defined by \( x \geq 0, y \geq 0 \) when \( \frac{\partial A_1}{\partial t} = 0 \) subject to proper boundary conditions and subject to the assumption that \( v_1 v_2 > 0 \). It reads \((x_0 \text{ and } y_0 \text{ are real constants})\) It reads

\[
A_1(x, y) = \frac{2\omega_1 \sqrt{v_1 v_2}}{\chi (e^{x+x_0} \sinh(y+y_0) + e^{-x-x_0} \cosh(y+y_0))}
\]

\[
A_2(x, y) = \frac{v_1 \omega_1 (e^{-x-x_0} \cosh(y+y_0) - e^{x+x_0} \sinh(y+y_0))}{\chi (e^{x+x_0} \sinh(y+y_0) + e^{-x-x_0} \cosh(y+y_0))}
\]

The respective solution represents energy transfer form the first harmonic, localized in the direction of the group velocity of the first mode, to the second harmonic, in the same direction having a kink-like shape, which occurs in the direction of the group velocity of the second mode.

### 5 Quadratic solitons in a lattice

Consideration provided in the previous section does not take into account the dispersion which is intrinsic property of the nonlinear lattices. This was justified by the choice of the scaling of the problem. Let us now turn to the case when the group velocity dispersion appears to be an important factor (Konotop and Malomed, 2000). Namely, assuming that resonant condition \([\text{4}]\) is satisfied let us look for the solution of nonlinear model \([\text{16}]\) in the form of the expansion \([\text{c.f. } [\text{17}]])

\[
\Psi(n) = \epsilon^2 \Psi^{(0)}(n_0, \{r_v\}; \{t_v\}) + \epsilon^3 \Psi^{(1)}(n_0, \{r_v\}; \{t_v\}) + \cdots
\]

\[
(33)
\]
Then the substitution
\[
\psi(0)({\bf n}_0; \{r_{\nu}\}; \{t_{\nu}\}) = \sum_{\alpha=1}^2 A_{\alpha}(r_1; t_1) \phi_{\eta_{\alpha}}(q_{\alpha}) e^{-i\omega_{\alpha} t_0} + \text{c.c.} \tag{34}
\]
results in the evolution equations for the amplitudes (they are obtained in the third order of the asymptotic expansion)
\[
i \partial A_{\eta_1}/\partial t_2 + \frac{1}{2} \sum_{i,j} \omega_{\eta_1}^{ij} \partial^2 A_{\eta_1}/\partial x^i \partial x^j + \chi_{\eta_1} A_{\eta_1} A_{\eta_2} = 0 \tag{35}
\]
\[
i \partial A_{\eta_2}/\partial t_2 + \frac{1}{2} \sum_{i,j} \omega_{\eta_2}^{ij} \partial^2 A_{\eta_2}/\partial x^i \partial x^j + \chi_{\eta_2} A_{\eta_2}^2 = 0 \tag{36}
\]
Some particular solutions of these equations (however in applications to the nonlinear optics) have been considered in (Malomed et al. 1997).

6 Dynamics of a double chains

In the present section we deal with phenomena originated by resonant interactions of longitudinal and transverse degrees of freedom in a system consisting of two coupled vector atomic chain depicted in Fig. 2.

![Figure 2: Schematic representation of a double chain. The axes x and y will be chosen along a and b, respectively.](image)

The potential energy of the interaction between the neighbor sites, say \( n \) and \( n+1 \), is considered to be \( U_L(|a+u_{n+1}-u_n|) \) in the \( u \)-chain and \( U_L(|a+v_{n+1}-v_n|) \) in the \( v \)-chain. The nearest neighbors in the both chains interact with the energy \( U_T(|b+u_n-v_n|) \).

Thus the hamiltonian of the system can be written down in the form
\[
H = \sum_n \left\{ \frac{p_{u,n}^2}{2m} + \frac{p_{v,n}^2}{2m} + U_L(|a+u_{n+1}-u_n|) + U_L(|a+v_{n+1}-v_n|) + U_T(|b+u_n-v_n|) \right\} \tag{37}
\]
The vectors $\mathbf{u}_n = (u_{x,n}, u_{y,n})$ and $\mathbf{v}_n = (v_{x,n}, v_{y,n})$ are displacements of the atoms from their equilibrium positions in the $u$- and $v$-lattices, respectively. Without restriction of generality, in what follows the atomic mass is considered to be one: $m = 1$.

If $\omega_u$ and $\omega_v$ are characteristic frequencies of excitations propagating along $u$- and $v$-chains, $\omega_l$ is a frequency of the transverse oscillations, it is natural to expect enhancement of the interaction when $n_t\omega_l + n_u\omega_u + n_v\omega_v = 0$, $n_t$, $n_u$, and $n_v$, being integers. The “lowest” resonant phenomena will be related to resonant three wave interactions either $\omega_l = 2\omega_l$ or $\omega_l = 2\omega_u$.

We will concentrate on excitation of a relatively small amplitude. Then the potential energy in (37) can be expanded in the Taylor series

$$U_L(|a + \mathbf{u}_{n+1} - \mathbf{u}_n|) = \frac{1}{2} K_2 (L_u(n) - a)^2 + \frac{1}{6} K_3 (L_u(n) - a)^3 + ...$$

$$U_T(|b + \mathbf{u}_n - \mathbf{v}_n|) = \frac{1}{2} Q_2 (L(n) - b)^2 + \frac{1}{6} Q_3 (L(n) - b)^3 + ...$$

where

$$L_u(n) = \left[ (a + u_{x,n} - u_{x,n-1})^2 + (u_{y,n} - u_{y,n-1})^2 \right]^{1/2}$$

is the distance between two neighbor sites in the $u$-chain ($L_v$ is defined by analogy)

$$L(n) = \left[ (u_{x,n} - v_{x,n})^2 + (b + u_{y,n} - v_{y,n})^2 \right]^{1/2}$$

is the distance between neighbor sites $u$- and $v$-lattices and the force coefficients are given by $K_n \equiv \frac{d^2 U_L(a)}{da^2}$, and $Q_n \equiv \frac{d^2 U_T(b)}{db^2}$ ($K_2$ and $Q_2$ being positive).

Since we are dealing with the phenomena originated by resonant three wave interactions the terms due to the nonlinearity of the central-force potentials $U_L$ and $U_T$ are not important in the leading order, since the lowest order nonlinearity results from the expansion of $L_u(n)$, $L_v(n)$ and $L(n)$ with respect to small displacements of atoms from the equilibrium positions. Then the equations of motion read

$$\ddot{u}_{x,n} = K_2 (u_{x,n+1} + u_{x,n-1} - 2u_{x,n}) - \frac{Q_2}{b} (u_{y,n} - v_{y,n})(u_{x,n} - v_{x,n})$$

$$+ \frac{K_2}{2a} [(u_{y,n+1} - u_{y,n})^2 - (u_{y,n-1} - u_{y,n})^2]$$

$$\ddot{v}_{x,n} = K_2 (v_{x,n+1} + v_{x,n-1} - 2v_{x,n}) - \frac{Q_2}{b} (v_{y,n} - u_{y,n})(v_{x,n} - u_{x,n})$$

$$+ \frac{K_2}{2a} [(v_{y,n+1} - v_{y,n})^2 - (v_{y,n-1} - v_{y,n})^2]$$

$$\ddot{u}_{y,n} = -Q_2 (u_{y,n} - v_{y,n}) - \frac{Q_2}{2b} (u_{x,n} - v_{x,n})^2 +$$

$$\frac{K_2}{a} [(u_{x,n+1} - u_{x,n})(u_{y,n+1} - u_{y,n}) - (u_{x,n} - u_{x,n-1})(u_{y,n} - u_{y,n-1})]$$

(40)
6.1 Resonance $\omega_l = 2\omega_l$.

As an example consider the situation when motion of $u$- and $v$- chains is synchronized, i.e. $\omega_u = \omega_v = \omega$ and respectively $\omega_l = 2\omega_l$. As far as lattices are considered identical the above assumption means that the wave vectors of the carrier wave (cw) of excitations propagating along both chains coincide. We will concentrate on the case when the wave vector of a single lattice borders the boundary of the Brillouin zone. Then solutions of the hamiltonian equations for the lattice displacements can be represented in the form

$$\ddot{v}_{y,n} = -Q_2(v_{y,n} - u_{y,n}) + \frac{Q_2}{2b}(u_{x,n} - v_{x,n})^2 - \frac{K_2}{a}[(v_{x,n} - v_{x,n-1})(v_{y,n} - v_{y,n-1}) + (v_{x,n+1} - v_{x,n})(v_{y,n+1} - v_{y,n})]$$

(41)

$$u_{x,n}(t) = \epsilon^2 \left[u_x^{(1)}(t_1; x_1)(-1)^n e^{-i\omega t_0} + O(\epsilon^2)\right] + \text{c.c.}$$

$$v_{x,n}(t) = \epsilon^2 \left[u_x^{(1)}(t_1; x_1)(-1)^n e^{-i\omega t_0} + O(\epsilon^2)\right] + \text{c.c.}$$

$$u_{y,n}(t) = \epsilon^2 \left[u_y^{(1)}(t_1; x_1)e^{-2i\omega t_0} + O(\epsilon^2)\right] + \text{c.c.}$$

$$v_{y,n}(t) = -\epsilon^2 \left[w_y^{(1)}(t_1; x_1)e^{-2i\omega t_0} + O(\epsilon^2)\right] + \text{c.c.}$$

Here we introduce a hierarchy of slow times $t_j = \epsilon^j t$ and spatial scales $n_j = \epsilon^j n$ ($j = 0, 1, \ldots$). The scales $n_j$ ($j \geq 1$) are slow ones and that is why we introduce $x_j = a n_j$ ($j \geq 1$) which are regarded as continuous variables. The symbol $O(\epsilon^2)$ stands for all the terms of order of $\epsilon^2$ which have frequencies either different from $\omega$ for longitudinal components or different from $2\omega$ for transverse component. Also in the expansion we use the convention that in the arguments of the functions only the most "rapid" scales are indicated [so, for example $u_1(t_1; x_1) \equiv u_1(t_1, t_2, \ldots; x_1, x_2, \ldots)$]. "c.c." stands for the complex conjugation, i.e. Quantity $2w(t; x_1, x_2)$ can be interpreted as an is an enlargement of the distance between neighbor atoms in different chains which oscillate with opposite phases.

In order to get solitonic effects this expansion should be provided up to the terms of order of $\epsilon^5$. After that the expansion for $u, v$ must be substituted in the Hamiltonian equations and terms of the same order of the small parameter must be collected. Dropping this straightforward standard calculus we pass to the description of the equations governing the dynamics of the double chain.

In the lowest order with respect to the small parameter (i.e. in the $\epsilon^2$ order) the chains appear to be independent. The respective terms of the expansion of the longitudinal and transverse components give the two values for the frequency $\omega = 2\sqrt{K_2}$ and $\omega = \sqrt{2K_2}$, which must coincide. In other words the phenomenon can be observed subject to the condition $\frac{d^2U_2(b)}{db^2} = \frac{d^2U_2(a)}{da^2}$. Subject to the above conditions, the equations of the third order with respect to
epsilon become zero identically. Thus, computing the terms of order of \( \epsilon^4 \) one arrives at the system of equations

\[
-i \frac{\partial U}{\partial \tau} + \frac{\partial^2 U}{\partial x_1^2} + 2(\bar{U} - \bar{V})W = 0, \quad -i \frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x_1^2} + 2(\bar{V} - \bar{U})W = 0 \quad (42)
\]

\[
-i \frac{\partial W}{\partial \tau} + \frac{1}{2}(U - V)^2 = 0 \quad (43)
\]

where \( U(\tau, x_1) = \frac{4}{\pi} u_x^{(1)}(t_2, x_1), \ V(\tau, x_1) = \frac{4}{\pi} v_x^{(1)}(t_2, x_1), \ W(\tau, x_1) = \frac{8}{\pi} w(t_2, x_1) \) and \( \tau = \sqrt{K_2 t_2/2} \).

Let us consider a particular solution of the above system, which corresponds to out-of-phase oscillations of the atoms in the \( u \)- and \( v \)-chains: \( U = -V \). Then the system (42)-(43) takes the form

\[
-i \frac{\partial U}{\partial \tau} + \frac{\partial^2 U}{\partial x_1^2} + 4\bar{U}W = 0, \quad -i \frac{\partial W}{\partial \tau} + 2U^2 = 0 \quad (44)
\]

We notice that the hamiltonian governing this system is as follows

\[
H = \int \left( |U_x|^2 + 2\bar{U}^2W + 2\bar{W}U^2 \right) dx \quad (45)
\]

System (44) admits solutions in a form of two coupled dark solitons

\[
U = \beta^2 e^{2i\beta^2 t} \tanh(\beta x), \quad W = -\frac{\beta^2}{2} e^{4i\beta^2 t} \tanh^2(\beta x) \quad (46)
\]

and two coupled bright solitons

\[
U = \frac{\beta^2}{\sqrt{2}} \frac{e^{-i\beta^2 t}}{\cosh(\beta x)}, \quad W = -\frac{\beta^2}{2} \frac{e^{-2i\beta^2 t}}{\cosh^2(\beta x)} \quad (47)
\]

These solutions describe localized in space narrowing and broadening of the double chain in transverse direction. The respective trajectories of the atoms are parabolas which are modulate by the envelopes (46) and (47) respectively.

### 7 Conclusion

In the present paper we have considered several effects in multidimensional lattices which are originated by resonant wave interactions. All the examples were given for nonlinear Klein-Gordon like lattices. The theory however allows for straightforward modification for the sake of description of lattices of the nonlinear Schrödinger type (see e.g. Konotop 1997; Konotop, et al. 1999). Also, resonant multi-wave interactions in lattice can be considered in the similar way.

Other relevant effects we did not discussed here are associated with the phase mismatch (i.e. when, for example, (11) is substituted by \( \omega_3 = \omega_1 + \omega_2 + \Delta \omega, \ q_3 = q_1 + q_2 + Q \), where \( \Delta \omega \ll \omega_j \ (j = 1, 2, 3) \).
Finally, taking into account that resonant mode interactions is a traditional subject of the nonlinear optics it is worth pointing out that the difference between lattices and optical media comes from the anisotropy, which is an intrinsic property of the lattice. Also, lattices provide physical systems where almost all relations among parameters are available (including change of the signs of nonlinear and dispersive terms).

Acknowledgments

Author is indebted to B. A. Malomed and to A. Gonçalves for collaboration. Support from FEDER and Program PRAXIS XXI, grant No Praxis/P/Fis/10279/1998 is acknowledged.

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