An homogenization approach for the inverse spectral problem of periodic Schrödinger operators

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Abstract We study the inverse spectral problem for periodic Schrödinger operators of kind
\(-\frac{1}{2}\hbar^2 \Delta_x + V(x)\) on the flat torus \(T^n := (\mathbb{R}/2\pi\mathbb{Z})^n\) with potentials \(V \in C^\infty(T^n)\). We show that if two operators are isospectral for any \(0 < \hbar \leq 1\) then they have the same effective Hamiltonian given by the periodic homogenization of Hamilton-Jacobi equation. This result provides a necessary condition for the isospectrality of these Schrödinger operators. We also provide a link between our result and the spectral limit of quantum integrable systems.

Keywords Inverse spectral problem · Homogenization · Hamilton-Jacobi

1 Introduction

Let \(T^n := (\mathbb{R}/2\pi\mathbb{Z})^n\), \(V \in C^\infty(T^n)\) and \(H(x,p) := \frac{1}{2}|p|^2 + V(x)\). The related class of Schrödinger operators is given by
\[
\text{Op}_\hbar^V(H) = -\frac{1}{2}\hbar^2 \Delta_x + V(x).
\]
This operator is selfadjoint on \(W^{2,2}(T^n)\) and exhibits discrete spectrum which is bounded from below (see for example [27]). The semiclassical inverse spectral problem is the study of the family of those \(H\) such that \(\text{Spec}(\text{Op}_\hbar(H))\) is the same for all \(0 < \hbar \leq 1\), namely with the same eigenvalues and the same multiplicity.

The target of our paper is to discuss the link between this semiclassical inverse spectral problem for \(\text{Op}_\hbar^V(H)\) and the periodic homogenization of the Hamilton-Jacobi equation of \(H\) (see for example [8], [9], [10], [19], [26]).

The first observation we provide is about the Egorov Theorem for the propagation of quantum observables (see [1], [4] and the references therein).

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For classical observables \( A : \mathbb{R}^{2n} \to \mathbb{R} \) given by symbols associated to Weyl operators \( \text{Op}_w^n(A) \) and for symbols \( B : \mathbb{R}^{2n} \to \mathbb{R} \), \(|\partial_z^\gamma B(z)| \leq C_\gamma \forall |\gamma| \geq 2\) with Hamiltonian flow \( \varphi^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \), the Egorov Theorem gives

\[
U_\hbar^* (t) \circ \text{Op}_w^n(A) \circ U_\hbar(t) = \text{Op}_w^n(A_{\varphi^t})
\]  

(2)

by the quantum dynamics \( U_\hbar(t) := \exp \left( -i \text{Op}_w^n(B) t / \hbar \right) \) and where

\[
A_{\varphi} \sim A \circ \varphi^t + \mathcal{O}(\hbar)
\]  

(3)

is the semiclassical asymptotics providing the leading term in the space of symbols on \( \mathbb{R}^{2n} \). In Section 2 we recover (2) and (3) in the setting of the Weyl quantization on the flat torus.

The second observation we underline is a picture of classical mechanics. It is well known that, without integrability or KAM assumptions on \( H \), the related Hamilton-Jacobi equation has not global and smooth solutions. However, we can consider (see Section 4) the viscosity solutions \( S(P, \cdot) : \mathbb{T}^n \to \mathbb{R} \) of

\[
H(x, P + \nabla_x S(P, x)) = \mathcal{H}(P), \quad P \in \mathbb{R}^n.
\]  

(4)

The function \( \mathcal{H} \), called effective Hamiltonian, is one of the main outcomes of weak KAM theory and homogenization of Hamilton-Jacobi equation. It is a convex function and can be represented or approximated in various ways (see for example [8], [9], [10]). Here we are interested in a symplectic invariance property of \( \mathcal{H} \). As shown in [3], for all the time one Hamiltonian flows \( \varphi \equiv \varphi^t : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n \) with \( C^1 \) regularity, i.e. Hamiltonian diffeomorphisms, we have the invariance

\[
\mathcal{H} \circ \varphi = \mathcal{H}.
\]  

(5)

Together with this property, we can to look at the inverse homogenization problem. This is given by looking at the family of those \( H \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n) \) such that the effective Hamiltonian \( \mathcal{H} \) is the same. Such a problem has been recently studied in [19], where a characterization in the case \( n = 1 \) is given and various results for \( n \geq 1 \) are provided.

In the Section 4 of [19] it is shown a result on the link between this inverse homogenization problem and the spectrum of the Hill operator \(-\frac{1}{2} \frac{d^2}{dx^2} + V(x)\), namely when \( n = 1 \) and \( \hbar = 1 \). In this setting, it is proved that the isospectrality implies the same effective Hamiltonian. Our paper aims to provide a result on this link in higher dimensions, but instead of setting \( h = 1 \) we will assume the stronger isospectrality for all \( 0 < \hbar \leq 1 \).

In view of the above two observations, we will make use of (2), (3) and (5) in order to prove the two main results of the paper. More precisely, we notice that the isospectrality of two Schrödinger operators of kind (1), which exhibit discrete spectrum, implies their conjugation by a unitary operator

\[
U_\hbar \circ \text{Op}_h^n(H_1) \circ U_\hbar = \text{Op}_h^n(H_2).
\]  

(6)
In the framework of our paper, we can prove (see Lemma AB) that this unitary operator takes the form $U_h = \exp (-i \text{Op}_h^{w}(b(h))/\hbar) + O_{L^2,\infty}(\hbar)$. The properties of the symbol $b(h, \cdot): \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{C}$ and its time-one Hamiltonian flow $\varphi_h: \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ do not allow a global (on $\mathbb{T}^n \times \mathbb{R}^n$) semiclassical asymptotics of $H_2 = H_1 \circ \varphi_h + O(\hbar)$ in the space of symbols as shown in [5]. Nevertheless, it is possible a local $C^0$ - semiclassical asymptotics and this is the content of the first result of the paper.

**Theorem 1** Let $V_\alpha \in C^\infty(\mathbb{T}^n)$ and $H_\alpha(x,p) := \frac{1}{2}|p|^2 + V_\alpha(x)$ with $\alpha = 1, 2$. We assume that

$$\text{Spec}(\text{Op}_{\hbar}^{w}(H_1)) = \text{Spec}(\text{Op}_{\hbar}^{w}(H_2)) \quad \forall 0 < \hbar \leq 1. \quad (7)$$

Let $E > \inf_{x \in \mathbb{T}^n} V_2(x)$ and $\Omega(E) := \{(x, p) \in \mathbb{T}^n \times \mathbb{R}^n | H_2(x, p) < E\}$. Then, for $0 < \varepsilon \leq 1$ there exist $\sigma_0(E, \varepsilon) \to 0^+$ as $\varepsilon \to 0^+$ and a family of Hamiltonian diffeomorphisms $\varphi_\sigma: \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ such that

$$\|H_1 \circ \varphi_\sigma - H_2\|_{C^0(\Omega(E))} \leq \varepsilon, \quad \forall 0 < \sigma \leq \sigma_0(E, \varepsilon). \quad (8)$$

This local asymptotics does not guarantee the link of $H_1$ and $H_2$ by a unique Hamiltonian diffeomorphism on $\Omega(E)$ without a remainder. Recalling Thm. 1 of [6], if the unitary conjugation in the lefthand side of (6) is an order preserving algebra isomorphism of semiclassical Pseudodifferential operators filtered by powers of $\hbar$ then there exists a symplectic diffeomorphism $\psi$ on $\mathbb{T}^n \times \mathbb{R}^n$ such that $H_2 = H_1 \circ \psi$. Unluckily, such a condition is not easy to check in our case since the symbol $b(h, z)$ with $z = (x, p)$ does not necessarily admits the semiclassical asymptotics $b(h, z) \sim \sum_{j \geq 0} \hbar^j b_j(z)$.

The inequality (5) together with (6) will be used to recover the equivalence of effective Hamiltonians, which is the second result of the paper.

**Theorem 2** The isospectrality

$$\text{Spec}(\text{Op}_{\hbar}^{w}(H_1)) = \text{Spec}(\text{Op}_{\hbar}^{w}(H_2)) \quad \forall 0 < \hbar \leq 1 \quad (9)$$

implies

$$\overline{\mathcal{H}}_1 = \overline{\mathcal{H}}_2. \quad (10)$$

The above result tell us that equality (10) is a necessary condition for the isospectrality (1). Conversely, the condition (1) is a sufficient condition to have the same homogenization given by (10). In the assumption of a quantum integrable system, we have $1 \leq i \leq n$ commuting semiclassical pseudodifferential operators $\text{Op}_h(f_i)$. The principal symbols are commuting with respect to the Liouville bracket and hence we have a (classical) completely integrable Hamiltonian system with momentum map $F(x, p) = (f_1, \ldots, f_n)(x, p)$. As shown in Thm. 2 of [23], if the symbols are bounded then the convex hull of the joint spectra linked to $\text{Op}_h(f_i)$ recovers, in the classical limit as $\hbar \to 0^+$, the convex hull of the image of the momentum map $F$. Such a result solves to the isospectrality problem for quantum toric systems with bounded symbols. In our paper, we focus the attention on the effective Hamiltonian $\overline{\mathcal{H}}(P)$ which
can be regarded as the generalization, beyond integrability assumptions, of the smooth Hamiltonian depending only from the Action variables $P$ provided by the Liouville-Arnold Theorem. In the case that $S(P,x)$ solving (48) is a smooth and global generating function of a canonical map, then the inversion of $p = P + \nabla_x S(P,x)$ provides the momentum map $F(x,p) := P(x,p)$. In this case the image of the momentum map, on the sublevel set for energy $E$, equals

$$F(\{(x,p) \in \mathbb{T}^n \times \mathbb{R}^n : H(x,p) \leq E\}) = \{P \in \mathbb{R}^n : \overline{H}(P) \leq E\}. \quad (11)$$

In view of our Theorem 2 and equality (56), we recover (see Section 5) the same kind of result shown [23], here in the setting of integrable Hamiltonian with mechanical form $\frac{1}{2}|p|^2 + V(x)$ on $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$.

The one dimensional version of Theorem 2 is shown in [25]. In particular, it is shown an exact version of the Egorov Theorem for the class of one dimensional periodic Schrödinger operators $-\frac{1}{2}\hbar^2 \frac{d^2}{dx^2} + V(x)$. We also stress that in this case the effective Hamiltonian is given by the inversion of the map

$$E \mapsto \mathcal{J}(E) := \frac{1}{2\pi} \int_{0}^{2\pi} \sqrt{2(E - V(x))} \, dx, \quad E \geq \max V, \quad (12)$$

namely $\overline{H}(P) = \mathcal{J}^{-1}(P)$ for $|P| \geq (2\pi)^{-1} \int_{0}^{2\pi} \sqrt{2(\max V - V(x))} \, dx$ and $\overline{P}(P) = \max V$ for $|P| < (2\pi)^{-1} \int_{0}^{2\pi} \sqrt{2(\max V - V(x))} \, dx$. Moreover it is easily seen that

$$\mathcal{J}(E) = \text{Vol}\{(x,p) \in \mathbb{T} \times \mathbb{R} : \frac{1}{2}|p|^2 + V(x) \leq E\} \quad (13)$$

which is the first spectral invariant of the Weyl Law (see [16]) on the number of the eigenvalues $(E_{\hbar,\ell})_{\ell \in \mathbb{N}}$ smaller than $E$. Namely $N(h,E) := \sharp\{E_{\hbar,\ell} : \min V \leq E_{\hbar,\ell} \leq E\}$ has the asymptotics

$$N(h,E) = (2\pi\hbar)^{-1} \left( \text{Vol}\{(x,p) \in \mathbb{T} \times \mathbb{R} : \min V \leq \frac{1}{2}|p|^2 + V(x) \leq E\} + \mathcal{O}(\hbar) \right). \quad (14)$$

A study of the link between the effective Hamiltonian $\overline{H}(P)$, viscosity solutions of Hamilton-Jacobi equation and eigenfunctions (or quasimodes) in the semiclassical framework should involve their phase space localization. Some preliminary results, beyond the one dimensional case, have been obtained in [5], [24], [31].

In order to avoid confusion with respect to the literature, we underline that the effective Hamiltonian approach provided in [13] is a completely different tool from the effective Hamiltonian function of the Hamilton-Jacobi homogenization used in the present paper.
2 Pseudodifferential operators on the flat torus

In this section we recall the Weyl quantization on the flat torus as discussed in section 2 of [24], which make use of the toroidal Pseudodifferential operators theory developed in [28]. As shown in Prop. 2.3 of [24], the Weyl quantization on the flat torus here applied coincides to the one used in [15].

Consider the class of \( (x \cdot \text{periodic}) \) Hörmander’s symbols \( b \in \mathcal{S}^m(\mathbb{T}^n \times \mathbb{R}^n) \), \( m \in \mathbb{R} \), consisting of those functions in \( C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) which are \( 2\pi\mathbb{Z}^n \)-periodic in \( x \) (\( 2\pi \)-periodic in each variable \( x_j \), \( 1 \leq j \leq n \)) and such that \( \forall \alpha, \beta \in \mathbb{Z}_+^n \) there exists \( C_{\alpha\beta} > 0 \) such that

\[
|\partial_x^\alpha \Delta_\eta^\beta b(x, \eta)| \leq C_{\alpha\beta m}(\eta)^{m-|\beta|}, \quad \forall (x, \eta) \in \mathbb{T}^n \times \mathbb{R}^n
\]  

(15)

where \( \langle \eta \rangle := (1 + |\eta|^2)^{1/2} \). Together with these symbols, one can introduce the class of toroidal symbols \( \tau \in \mathcal{S}^m(\mathbb{T}^n \times \mathbb{Z}^n) \). That is, by using the difference operator \( \Delta_\eta \tau(x, k) := \tau(x, \kappa + e_j) - \tau(x, \kappa) \) (\( e_j \) being the \( j \)th vector of the canonical basis of \( \mathbb{R}^n \)), we can require that \( \forall \alpha, \beta \in \mathbb{Z}_+^n \) there exists \( C_{\alpha\beta} \) such that

\[
|\partial_x^\alpha \Delta_\eta^\beta \tau(x, \kappa)| \leq C_{\alpha\beta m}(\kappa)^{m-|\beta|}, \quad \forall (x, \kappa) \in \mathbb{T}^n \times \mathbb{Z}^n.
\]  

(16)

**Remark 1** There is a full correspondence between symbols \( \mathcal{S}^m(\mathbb{T}^n \times \mathbb{R}^n) \) and \( \mathcal{S}^m(\mathbb{T}^n \times \mathbb{Z}^n) \). Indeed, we address the reader to Remark 4.1 and Thm. 5.2 of [28], and we stress that we have a toroidal symbol \( \tau \in \mathcal{S}^m(\mathbb{T}^n \times \mathbb{Z}^n) \) if and only if there exists \( b \in \mathcal{S}^m(\mathbb{T}^n \times \mathbb{R}^n) \) such that \( \tau = b|_{\mathbb{T}^n \times \mathbb{Z}^n} \) and such \( b \) is unique modulo a term in \( \mathcal{S}^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n) \). In particular, since we have the equality

\[
\partial_x^\alpha \Delta_\eta^\beta b(x, \eta) = \partial_x^\alpha \Delta_\eta^\beta b(x, \omega)
\]  

(17)

for some \( \omega \in \mathcal{Q} := [\eta_1 + \alpha_1, ..., \eta_n + \alpha_n] \) then

\[
|\partial_x^\alpha \Delta_\eta^\beta b(x, \eta)| = |\partial_x^\alpha \Delta_\eta^\beta b(x, \omega)| \leq C_{\alpha\beta m}(\omega)^{m-|\beta|} \leq C_{\alpha\beta m}(\eta)^{m-|\beta|}
\]  

(18)

with new constants \( C'_{\alpha\beta m} > 0 \).

For any \( \tau \in \mathcal{S}^m(\mathbb{T}^n \times \mathbb{Z}^n) \) we can thus associate a toroidal Pseudodifferential operator as

\[
\text{Op}(\tau)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i(x-y, \kappa)} \tau(x, \kappa)\psi(y)dy, \quad \psi \in C^\infty(\mathbb{T}^n).
\]  

(19)

The composition rule of these operators (see Thm. 4.3 in [28]) states that for any \( \tau \in \mathcal{S}^m(\mathbb{T}^n \times \mathbb{Z}^n) \) and \( w \in \mathcal{S}^k(\mathbb{T}^n \times \mathbb{Z}^n) \) we have

\[
\text{Op}(\tau) \circ \text{Op}(w) = \text{Op}(\tau \circ w)
\]  

(20)

with \( \tau \circ w \in \mathcal{S}^{m+k}(\mathbb{T}^n \times \mathbb{Z}^n) \) and

\[
\tau \circ w \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_\eta^\alpha \tau(x, \kappa) D_x^{(\alpha)} w(x, \kappa).
\]  

(21)
This asymptotics of symbols (21) means that the remainder
\[ R_N(x,\kappa) := \sum_{\alpha > N} \frac{1}{\alpha!} \Delta_\kappa^\alpha \tau(x,\kappa) D_\alpha^{(\alpha)} w(x,\kappa) \] (22)
belongs to \( S^{m+\ell}(T^n \times \mathbb{Z}^n) \) and \( \forall N \geq 1 \) fulfills
\[ |\partial_\mu \Delta^\kappa R_N(x,\kappa)| \leq C_{\gamma \mu m} m^{\pm|\mu|-N}, \quad \forall (x,\kappa) \in T^n \times \mathbb{Z}^n. \] (23)
Notice that for symbols of kind \( \tau = \tau(x,\hbar \kappa) \) we have that \( \Delta_\kappa^\alpha \tau(x,\hbar \kappa) = \hbar \Delta_\kappa^\alpha \tau(x,\hbar \kappa) \) and this provides series (21) with powers of \( \hbar \).

Following the computations (see Thm. 4.3 in [28]) of the estimate in (23), we recover in this case the constant \( \hbar^N C_{\gamma \mu m} \).

The semiclassical framework of toroidal operators deals with the following\[
\begin{definition}[Weyl quantization]
Given a symbol \( b \in S^m(T^n \times \mathbb{R}^n) \), we define the Weyl quantization
\[ \text{Op}_h^w(b)(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} e^{i(x-y,\kappa)} b(y,\frac{\hbar}{2}) \psi(2y-x) dy, \quad \psi \in C^\infty(T^n). \] (24)
\end{definition}

\begin{remark}
The link between Weyl quantization and standard quantization is given by
\[ \text{Op}_h^w(b) \psi = \text{Op}(\sigma(h)) \psi \] (25)
where \( \sigma(h,x,\kappa) = b(x,h\kappa) + \mathcal{O}(h) \) in \( S^m(T^n \times \mathbb{Z}^n) \). To prove (25), we observe that \( T_\omega \psi(y) := \psi(2y - \omega) \) can be written as
\[ T_\omega \psi(y) = (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{T^n} e^{i((2y-\omega)-z,\kappa)} \psi(z) dz, \quad \forall \psi \in C^\infty(T^n), \] (26)
and hence (thank to Thm. 4.2 in [28]) we have \( \text{Op}_h^w(b) \psi(x) = (\sigma(X,D) \circ T_{\omega=x} \psi)(x) \) with \( \sigma \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \Delta_\kappa^\alpha D_\alpha^{(\alpha)} b(y,h\eta/2)|_{y=z} \). By applying Theorem 8.4 of [28] it is finally recovered (25).
\end{remark}

Together with the above notion we can introduce the next\[
\begin{definition}[Wigner transform]
The Wigner transform \( W_h \psi(x,\eta) \) for \( x \in T^n \) and \( \eta \in \frac{\hbar}{2} \mathbb{Z}^n \) is defined as
\[ W_h \psi(x,\eta) = (2\pi)^{-n} \int_{T^n} e^{2i\hbar^{-1}(z,\eta)} \psi(x-z) \psi(x+z) dz, \] (27)
and the Wigner distribution is therefore given by
\[ \langle \text{Op}_h^w(b) \psi, \psi \rangle_{L^2} = \sum_{\eta \in \frac{\hbar}{2} \mathbb{Z}^n} \int_{T^n} b(x,\eta) W_h \psi(x,\eta) dx. \] (28)
\end{definition}
In view of (24), and by using the properties of the toroidal Fourier Transform
\[ \langle W \rangle_{\phi} \]
for any \( \phi \), where the so-called Moyal bracket reads
\[ \{ \alpha, \beta \} := \frac{\hbar}{2\pi} \langle \alpha, \beta \rangle \]
when \( H \). Moreover, we recall that the toroidal Fourier Transform which is
\[ \sum_{\gamma \leq \beta} \frac{(-1)^{\beta - \gamma} e^{i(x, \gamma)}}{\beta!} \frac{\hbar}{2\pi} \langle \alpha, \beta \rangle =: e^{i(x, \gamma)} s(x, \kappa; \beta). \]
The above terms reads
\[ \partial_{\alpha}^{\beta} \Delta_{\kappa}^{\beta}(W) = (2\pi)^{-n} \int_{\mathbb{T}^n} e^{i(x, \gamma)} s(x, \kappa; \beta) \partial_{x}^{\gamma} \langle \psi(x - z) \rangle dz. \]

Moreover, we recall that the toroidal Fourier Transform which is \( F(\phi) := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-i(x, \gamma)} \varphi(z) dz \) maps \( C^\infty(\mathbb{T}^n) \) into \( S(\mathbb{R}^n) \), i.e. the set of rapidly decaying functions from \( \mathbb{Z}^n \) to \( \mathbb{C} \). Thus, \( |F(\phi)(\kappa)| \leq C_N|\kappa|^{-N} \) and applying this inequality to (30)

\[ |\partial_{\alpha}^{\beta} \Delta_{\kappa}^{\beta}(W)\rangle = C_{N, \alpha, \beta}|\kappa|^{-N}, \quad \forall \kappa \in \mathbb{N}. \]

This means that \( (x, \kappa) \mapsto W_{h}(x, \frac{h}{2\kappa}) \) belongs to \( S^m(\mathbb{R}^n) \) for any \( m \in \mathbb{Z} \).

As in the Euclidean Weyl quantization (see [20], [34]) here we have that for any \( \phi_h \in C^\infty(\mathbb{T}^n) \) with \( \| \phi_h \|_{L^2} \leq 1 \) the projection operator \( \pi_h \phi := (\phi_h, \psi)_{L^2} \phi \) can be regarded as a Weyl operator whose symbol is the Wigner transform \( W_{h}\phi_h \), see Lemma [1]

\[ \pi_h \phi = Op^w_{\phi}(q(h)) \psi, \quad q(h, x, \eta) := W_{h}\phi_h(x, \eta). \]

In view of [24], and by using the properties of the toroidal Fourier Transform \( F(\psi)(\kappa) := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{-i(y, \kappa)} \psi(y) dy \) together with its inverse \( F^{-1}(g)(x) := \sum_{\kappa \in \mathbb{Z}^n} e^{i(x, \kappa)} g(\kappa) \) (see section 2 of [28]) it is easily seen that

\[ -\frac{1}{2} \hbar^2 \Delta + V = Op^w_{\phi}(H) \]

when \( H = \frac{1}{2} \hbar^2 \Delta + V(y) \). For a given pair \( a \in S^f(\mathbb{T}^n \times \mathbb{R}^n) \) and \( b \in S^m(\mathbb{T}^n \times \mathbb{R}^n) \) it is shown in Thm. 2.4 of [22] that, by using the composition formula of toroidal Pseudodifferential operators, see [21], we can write

\[ Op^w_{\phi}(a) \circ Op^w_{\phi}(b) = Op^w_{\phi}(a \circ b), \quad a \circ b \sim a \cdot b + O(\hbar) \in S^{d+m}(\mathbb{T}^n \times \mathbb{R}^n). \]

Moreover,

\[ [Op^w_{\phi}(a), Op^w_{\phi}(b)] = Op^w_{\phi}(a \circ b - b \circ a) \]

where the so-called Moyal bracket reads

\[ \{a, b\}_M := a \circ b - b \circ a \sim -i\hbar \{a, b\} + O(\hbar^2) \]
with terms in $S^{t+m-1}(T^n \times \mathbb{R}^n)$ and where $\{a, b\}$ is the usual Liouville bracket. The difference
\[ \{a, b\}_M - (-ih \{a, b\}) \]
belongs to $S^{t+m-2}(T^n \times \mathbb{R}^n)$ and exhibits a semiclassical asymptotics with leading order $O(h^2)$.

In the next we recall the toroidal version of Calderon-Vaillancourt Theorem.

**Theorem 3 (see [15])** Let $Op_N^w(b)$ as in [22] with $b \in S^0(T^n \times \mathbb{R}^n)$. Let $M = \frac{1}{2}n + 1$ when $n$ is even, $M = \frac{1}{2}(n + 1) + 1$ when $n$ is odd. Then, for any $\psi \in C^\infty(T^n)$
\[
\|Op_N^w(b)\psi\|_{L^2(T^n)} \leq \frac{2^{n+1}}{n + 2} \frac{\pi^{\frac{n+1}{2}}}{I(\frac{n+1}{2})} \sum_{|\alpha| \leq 2M} \|\partial_\alpha b\|_{L^\infty(T^n \times \mathbb{R}^n)} \|\psi\|_{L^2(T^n)}. \tag{38}
\]

3 **Egorov Theorem**

We provide the Egorov Theorem in the toroidal setting, namely in the framework of the toroidal Weyl quantization shown in Section 2. Here we are not interested in the full asymptotics of the symbols at all orders $O(h^N)$, since we need only to deal with the leading term involving the Hamiltonian flow. We mainly follow the same arguments showed in [4, 1] for the euclidean setting.

**Theorem 4** Let $b \in S^2(T^n \times \mathbb{R}^n)$, $a \in S^m(T^n \times \mathbb{R}^n)$ with $m \leq 0$. Let $U_h(t) := \exp(-iOp_h^w(b)t/h)$ and let $\varphi^t : T^n \times \mathbb{R}^n \rightarrow T^n \times \mathbb{R}^n$ be the Hamiltonian flow of $b$. Then,
\[
U^*_h(t) \circ Op_h^w(a) \circ U_h(t) = Op_h^w(a_\varphi) \tag{39}
\]
where $a_\varphi \in S^m(T^n \times \mathbb{R}^n) \subset S^0(T^n \times \mathbb{R}^n)$ depends on $h$ and
\[
\|Op_h^w(a_\varphi) - Op_h^w(a \circ \varphi^t)\|_{L^2 \rightarrow L^2} = O(h). \tag{40}
\]
Furthermore, $a_\varphi = a \circ \varphi^t + O(h)$ in $S^0(T^n \times \mathbb{R}^n)$ uniform in $0 \leq t \leq 1$, more precisely there exists a family of functionals $B_{a, b} > 0$ such that for any $0 \leq t \leq 1$ and $\forall(x, \eta) \in T^n \times \mathbb{R}^n$
\[
|\partial_\alpha^\beta \partial_\eta^\delta(a_\varphi - a \circ \varphi^t)(x, \eta)| \leq B_{a, b}[a, b]h. \tag{41}
\]

**Proof** Let us define $a_0(t, x, p) := a \circ \varphi^t(x, p)$, which be denoted by $a_0(t)$ in the next. The standard approach in the Euclidean setting (see for example Thm. 1.1 in [4] or Thm. 1.2 in [4]) is to use the functional equality
\[
U^*_h(t) \circ Op_h^w(a) \circ U_h(t) - Op_h^w(a_0(t)) \tag{42}
= \int_0^t U^*_h(t - s) \circ \left(\frac{i}{h}[Op_h^w(b), Op_h^w(a_0(s))] - Op_h^w([b, a_0(s)])\right) \circ U_h(t - s) ds
\]
An homogenization approach for the inverse spectral problem

which can be used also in the toroidal setting. In view of asymptotics of the Moyal bracket shown in (35) it follows

\[ \frac{i}{\hbar} [\text{Op}_h^w(b), \text{Op}_h^w(a_0(s))] - \text{Op}_h^w(\{b, a_0(s)\}) = \text{Op}_h^w(\phi(s)) \]  

(43)

where \( \phi(s) := \frac{i}{\hbar}\{b, a_0(s)\}_M - \{b, a_0(s)\} \) belongs to \( S^{m+2-2}(\mathbb{T}^n \times \mathbb{R}^n) = S^m(\mathbb{T}^n \times \mathbb{R}^n) \subset S^0(\mathbb{T}^n \times \mathbb{R}^n) \), see (37), and exhibits a semiclassical asymptotics with leading order \( O(\hbar) \). Any term of the semiclassical asymptotics of \( \phi(s, z) \sim \sum_{\gamma \geq 1} \phi_\gamma(s, z) \hbar^\gamma = \phi_1(s, z) \hbar + O(\hbar^2) \) in \( S^m(\mathbb{T}^n \times \mathbb{R}^n) \) depends polynomially from the derivatives of the \( C^\infty \) - flow \( \varphi^s \), from the derivatives of \( a \) and \( b \). We remind that here we have assumed \( m \leq 0 \), and hence we can apply the Calderon-Vaillancourt Theorem (see Thm. 3) to have

\[ \| \text{Op}_h^w(\phi(s)) \|_{L^2 \rightarrow L^2} \leq \frac{2n+1}{n+2} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+1}{2})} \sum_{|\gamma| \leq 2M} \| \partial_\gamma^s \phi(s, \cdot) \|_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)} \]  

(44)

where \( M = \frac{1}{2}n + 1 \) when \( n \) is even, \( M = \frac{1}{2}(n + 1) + 1 \) when \( n \) is odd.

Recalling (15), Remark 1 and (28), any \( \partial_\gamma^s \phi \) belongs to \( S^0(\mathbb{T}^n \times \mathbb{R}^n) \) and can be written as \( \partial_\gamma^s \phi_1(s, z) \hbar + O(\hbar^2) \) with remainder uniform in the variable \( s \). Thus, for some \( K_\gamma > 0 \) we have the estimate \( \| \partial_\gamma^s \phi(s, \cdot) \|_{L^\infty} \leq K_\gamma h \) for any \( s \in [0, t] \).

Since \( U_h \) are unitary operators, we have \( \| U_h \|_{L^2 \rightarrow L^2} = 1 \), and thus

\[ \| \text{Op}_h^w(a_\varphi) - \text{Op}_h^w(a \circ \varphi^t) \|_{L^2 \rightarrow L^2} \leq \sup_{s \in [0, t]} \| \text{Op}_h^w(\phi(s)) \|_{L^2 \rightarrow L^2} \]  

(45)

\[ \leq \sum_{|\gamma| \leq 2M} K_\gamma h = O(h). \]  

(46)

The estimate (41) easily follows from the computations given in Theorem 1.2 of [4] (done in the setting of euclidean Weyl quantization) which works also for toroidal operators and toroidal symbols \( S^m(\mathbb{T}^n \times \mathbb{Z}^n) \). Then, by recalling Remark 1 on the correspondence between toroidal symbols and \((x\text{-periodic})\) Hörmander symbols \( S^m(\mathbb{T}^n \times \mathbb{R}^n) \) one get the same estimate. □

**Corollary 1** Let \( m \leq 0 \) and \( a, b, a_\varphi \in S^m(\mathbb{T}^n \times \mathbb{R}^n) \) be as in Theorem 4 and \( t \in [0, 1] \). Then, there exists a functional \( K[a, b] > 0 \) such that

\[ \| a - a_\varphi \|_{C^0(\mathbb{T}^n \times \mathbb{R}^n)} \leq K[a, b] h. \]  

(47)

**Proof** Consider (11) for \( \alpha = 0 \) and \( \beta = 0 \) so that we can set \( K[a, b] := B_{00}[a, b] \).

**4 Hamilton-Jacobi equation**

Let \( H \) be a Tonelli Hamiltonian, namely \( H \in C^2(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R}) \) is such that the map \( p \mapsto H(x, p) \) is convex with positive definite Hessian and \( H(x, p)/\| p \| \rightarrow +\infty \) as \( \| p \| \rightarrow +\infty \).
For any $P \in \mathbb{R}^n$, there exists a unique real number $c = \hat{H}(P)$ such that the following cell problem on $\mathbb{T}^n$:

$$H(x, P + \nabla_x S) = c,$$

has a solution $S = S(P, x)$ in the viscosity sense (see [11] and the references therein). As shown in [11], any viscosity solution is also a weak KAM solution of negative type and belongs to $C^{0, 1}(\mathbb{T}^n)$. Moreover, as shown in [26], any viscosity solution $S$ exhibits $C^{1, 1}_{\text{loc}}$-regularity outside the closure of its singular set $\Sigma(S)$. In particular, $\mathbb{T}^n \setminus \Sigma(S)$ is an open and dense subset of $\mathbb{T}^n$. The function $H$ is called the effective Hamiltonian, it is a convex function and can be represented or approximated in various ways (see for example [3], [2], [8], [9], [10]). In particular (see [2] and references therein) we have the following inf-sup formula. Let $v \in C^{1, 1}(\mathbb{T}^n)$ and $\Gamma := \{(x, \nabla_x v(x)) \in \mathbb{T}^n \times \mathbb{R}^n \mid x \in \mathbb{T}^n\}$. Let $\mathcal{G}$ be the set of all such sets $\Gamma$.

$$\overline{H}(P) = \inf_{\Gamma \in \mathcal{G}} \sup_{(x, p) \in \Gamma} H(x, p + P).$$

(49)

The effective Hamiltonian equals the Mather’s $\alpha(H)$ function (see for example [11], [30]). The inf-sup formula can be equivalently computed over $v \in C^1(\mathbb{T}^n)$ (see for example [8]) but the Lipschitz regularity of $\nabla v$ has the advantage of being the highest one for which the infimum is a minimum (see [2]).

As shown in Proposition 1 of [3], for any time one Hamiltonian flows $\varphi \equiv \varphi^1 : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{T}^n \times \mathbb{R}^n$ with $C^1$-regularity we have the invariance property

$$H \circ \varphi = \hat{H}.$$

(50)

In the mechanical case $H = \frac{1}{2}|p|^2 + V(x)$ the effective Hamiltonian can be written as

$$\overline{H}(P) = \inf_{v \in C^{1, 1}(\mathbb{T}^n)} \sup_{x \in \mathbb{T}^n} \frac{1}{2}|P + \nabla_x v(x)|^2 + V(x)$$

(51)

It is easily seen that $\overline{H}(0) = \inf_{v \in C^{1, 1}(\mathbb{T}^n)} \sup_{x \in \mathbb{T}^n} \frac{1}{2} |\nabla_x v(x)|^2 + V(x) = \max V$. Now let us consider $\{P \in \mathbb{R}^n : \overline{H}(P) \leq E\} =: \mathcal{U}_E$. In view of the above definition is easy to see that if we denote by $\mathcal{G}(E)$ the set of those $\Gamma$ in $\{(x, p) \in \mathbb{T}^n \times \mathbb{R}^n \mid H(x, P + p) \leq E\}$ then

$$\overline{H}(P) = \inf_{\Gamma \in \mathcal{G}(E)} \sup_{(x, p) \in \Gamma} H(x, p + P), \quad \forall P \in \mathcal{U}_E.$$ 

(52)

5 The quantum Integrable case

For $M = T^*X$, where $X$ is a manifold of dimension $n$, a quantum integrable system is given by $n$ commuting operators $\text{Op}_h^w(f_i(h))$ with $f_i : M \rightarrow \mathbb{R}$ and $f_i(h) \simeq f_i + \mathcal{O}(h)$ are such that $\{f_i, f_j\} = 0$. The momentum map is given by:

$$F(x, p) := (f_1, ..., f_n)(x, p).$$

(53)
As shown in Thm. 2 of [23], the “classical spectrum” is recovered in the classical limit as $\hbar \to 0^+$.

Convex Hull(Joint Spec(Op$_\hbar$(f$_1$),...Op$_\hbar$(f$_n$)) $\mapsto$ Convex Hull $F(M)$. (54)

If we assume that, in our case, the map $(P,x) \mapsto S(P,x)$ solving

$$H(x,P + \nabla_x S(P,x)) = \overline{h}(P)$$

is smooth and generates a canonical transformation, then the inversion of $p = P + \nabla_x S(P,x)$ gives the momentum map $F(x,p) := P(x,p)$. Moreover, for any $\Omega_E := \{(x,p) \in \mathbb{T}^n \times \mathbb{R}^n : H(x,p) \leq E\}$ we have

$$F(\Omega_E) = \{P \in \mathbb{R}^n : H(P) \leq E\} =: \mathcal{U}_E.$$ (56)

**Remark 4**

1. The momentum map $F : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}^n$ does not exists if $H$ is not Liouville Integrable, anyway $\overline{h}$ exists for any Tonelli $H$.

2. $\mathcal{U}_E$ is a convex set (since $H$ is a convex function) and it is spectrally invariant (thanks to our Theorem 2).

3. In view of (54) and (56), the set $\mathcal{U}_E$ is a candidate for a notion of “classical spectrum” instead of $F(\Omega_E)$, for $H(x,p) = \frac{1}{2}|p|^2 + V(x)$ and beyond the quantum integrable case.

For 1D case, the effective Hamiltonian is given by the inversion of the map

$$E \mapsto J(E) := \frac{1}{2\pi} \int_0^{2\pi} \sqrt{2(E - V(x))} \, dx, \quad E \geq \text{max} \, V,$$

namely

$$\overline{h}(P) = \begin{cases} \text{max} \, V & \text{if } |P| \leq J(\text{max} \, V), \\ J^{-1}(P) & \text{otherwise}. \end{cases}$$

Moreover, we stress that $J(E)$ equals the first spectral invariant of the Weyl Law

$$J(E) = \text{Vol}\{(x,p) \in \mathbb{T} \times \mathbb{R} : \frac{1}{2}|p|^2 + V(x) \leq E\}$$

and recall that Bohr-Sommerfeld Rules (see [17], [29] and references therein) take the form

$$S_h(E_h,\ell) = 2\pi h \ell \quad \text{for } \ell = 1, 2, ...$$

(57)

where $E_h,\ell$ are the eigenvalues of $-\frac{1}{2}h^2 \Delta_x + V(x)$, and

$$S_h(E) \sim \sum_{j=0}^{\infty} h^j S_j(E) = 2\pi J(E) + \frac{1}{2}h\pi \mu(E) + O(h^2).$$

(58)

where $\mu(E)$ is the Maslov index of the curve at energy $E$. As shown in Prop. 5.2 of [29], such a semiclassical series is locally uniform in $E$. Thus, the above equalities (57) - (58) imply that two systems with the same Bohr-Sommerfeld Rules have necessarily the same effective Hamiltonian. Has already shown in
since \( \mathcal{T} \circ J(E) = E \) then the Bohr-Sommerfeld Rules, up to the order \( \mathcal{O}(\hbar^2) \), implies
\[
E_{h,t} = \mathcal{T}(\ell \hbar - \mu \hbar/4 + \mathcal{O}(\hbar^3)).
\]  
(59)

It is easy to see that, thanks to (59), the knowledge of the spectrum of Schrödinger operator allow to “reconstruct”, in the classical limit \( \hbar \to 0^+ \), the function \( \mathcal{T} \) (see [25]).

6 Main Results

Lemma 1 The operator \( \pi_h \psi := \langle \varphi_h, \psi \rangle_{L^2} \varphi_h \) is a Weyl operator whose symbol is the Wigner transform of \( \varphi_h \in C^\infty(\mathbb{T}^n) \) where \( \| \varphi_h \|_{L^2} \leq 1 \forall 0 < h \leq 1 \).

Proof We begin by the setting of the Weyl operator with the Wigner transform \( \mathcal{W}_h \varphi \) used as a symbol,
\[
\left( \text{Op}^w_h(\mathcal{W}_h \varphi) \psi \right)(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i(x-y,\kappa)} \mathcal{W}_h \varphi(y,\hbar/2\kappa) \psi(2y-x) dy
\]  
(60)
and we recall (see Remark 3) that the map \( (x,\kappa) \mapsto \mathcal{W}_h \psi(x,\hbar/2\kappa) \) belongs to \( S^m(\mathbb{T}^n \times \mathbb{Z}^n) \) for any \( m \in \mathbb{Z} \) (whence for any \( m < 0 \)) and fixed \( 0 < h \leq 1 \). In view of this observation, the expression (60) is well posed as a Weyl operator on the torus, and can be rewritten as
\[
(2\pi)^{-2n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i(x-y+z,\kappa)} \varphi_h(y-z) \mathcal{W}_h(y+z) \psi(2y-x) dz dy
\]  
(61)
where the two absolutely convergent integrals and the series can be exchanged,
\[
(2\pi)^{-2n} \int_{\mathbb{T}^n} \sum_{\kappa \in \mathbb{Z}^n} e^{i(x+z,\kappa)} \left( \int_{\mathbb{T}^n} e^{-i(y,\kappa)} \varphi_h(y-z) \mathcal{W}_h(y+z) \psi(2y-x) dy \right) dz.
\]  
(62)

The toroidal Fourier transform composed with its inverse is the identity map on \( C^\infty(\mathbb{T}^n) \), and thus (62) becomes
\[
(2\pi)^{-n} \int_{\mathbb{T}^n} \varphi_h(x+z-z) \mathcal{W}_h(x+z+z) \psi(2(x+z)-x) dz,
\]  
(63)
which reads as
\[
(2\pi)^{-n} \int_{\mathbb{T}^n} \mathcal{W}_h(x+2z) \psi(x+2z) dz \varphi_h(x).
\]  
(64)

Since \( T_x \psi(z) := \psi(2z+x) \) is linear and preserves scalar product in \( L^2(\mathbb{T}^n) \), (write \( \psi \) as a Fourier series and easily recover such property) then we get
\[
(2\pi)^{-n} \int_{\mathbb{T}^n} \mathcal{W}_h(y) \psi(y) dy \varphi_h(x) = \langle \varphi_h, \psi \rangle_{L^2} \varphi_h(x).
\]  
(65)
Lemma 2 Let \( V \in C^\infty(\mathbb{T}^n; \mathbb{R}) \) and let \( (a, b) \subset \mathbb{R}_+ \) so that \( a > \min V \). Let \( \varphi_h \in C^\infty(\mathbb{T}^n; \mathbb{C}) \) be an \( L^2 \)-normalized eigenfunction for the operator \(-\frac{1}{2}h^2 \Delta_x + \frac{1}{4}h^4 V(x) : W^{2,\infty}(\mathbb{T}^n; \mathbb{C}) \to L^2(\mathbb{T}^n; \mathbb{C})\) for an eigenvalue \( 0 < a < E_h < b \) for any \( 0 < h \leq 1 \). Let \( g(h) := 2b h^{-2} e^{1/h} \). Then,
\[
\varphi_h = \sum_{k \in \mathbb{Z}^n : |k|^2 \leq g(h)} \langle \varphi_h, e_k \rangle_{L^2} e_k + r_h
\]
where \( e_k(x) := \frac{1}{\sqrt{2\pi}} e^{ik \cdot x} \) and \( r_h \in C^\infty(\mathbb{T}^n; \mathbb{C}) \) fulfills
\[
\|r_h\|_{L^2} \leq C(b) e^{-1/(4h)}, \quad \forall 0 < h \leq 1.
\]
where \( C(b) \) is given by (77).

Proof Since \( \varphi_h \in L^2(\mathbb{T}^n) \) we can write \( \varphi_h = \sum_{k \in \mathbb{Z}^n} \langle \varphi_h, e_k \rangle_{L^2} e_k \). By introducing the cut-off \(|k|^2 \leq g(h)\) it follows
\[
\varphi_h = \sum_{k \in \mathbb{Z}^n : |k|^2 \leq g(h)} \langle \varphi_h, e_k \rangle_{L^2} e_k + r_h
\]
for some \( r_h \in C^\infty(\mathbb{T}^n) \). In view of the smoothness of the eigenfunction, \( \Delta_x \varphi_h \in C^\infty(\mathbb{T}^n) \subset L^2(\mathbb{T}^n) \) so that \( \Delta_x \varphi_h = -\sum_{k \in \mathbb{Z}^n} \langle \varphi_h, e_k \rangle_{L^2} |k|^2 e_k \).

The eigenvalue equation can be rewritten in terms of the Fourier components,
\[
\left( \frac{1}{2}h^2 |k|^2 - E_h \right) \langle \varphi_h, e_k \rangle_{L^2(\mathbb{T}^n)} = - \langle V \varphi_h, e_k \rangle_{L^2},
\]
and the equality \( \frac{1}{2}h^2 |k|^2 - E_h \) \( \| \langle \varphi_h, e_k \rangle_{L^2} \| = \| \langle V \varphi_h, e_k \rangle_{L^2} \| \) gives
\[
|\langle \varphi_h, e_k \rangle_{L^2}| \leq \frac{\| \langle V \varphi_h, e_k \rangle_{L^2} \|}{\frac{1}{2}h^2 |k|^2 - E_h} \leq \frac{\| V \|_{C^0(\mathbb{T}^n)}}{\frac{1}{2}h^2 |k|^2 - E_h}. \tag{70}
\]

Here we look for an estimate of the Fourier components of the remainder, and hence we now consider \(|k|^2 > 2b h^{-2} e^{1/h}\). Notice that we have \( \frac{1}{2}h^2 |k|^2 > b e^{1/h} \geq b \forall 0 < h \leq 1. \) It follows \( \frac{1}{2}h^2 |k|^2 - E_h \geq b - a > 0 \) and
\[
|\langle r_h, e_k \rangle_{L^2}| \leq \frac{\| V \|_{C^0(\mathbb{T}^n)}}{\frac{1}{2}h^2 |k|^2 - E_h} \leq \frac{\| V \|_{C^0(\mathbb{T}^n)}}{\frac{1}{2}h^2 |k|^2 - b}. \tag{71}
\]

We now look for \( \bar{C}(b) > 0 \) such that
\[
\frac{1}{2}h^2 |k|^2 - b \leq \frac{\bar{C}(b)}{|k|^2} e^{-1/(4h)}, \tag{72}
\]
for any \( k \in \mathbb{Z}^n \) such that \(|k|^2 > 2b h^{-2} e^{1/h}\). A simple computation shows that we can define
\[
\bar{C}(b) := 4(2b)^{-1/2} \sup_{0 < h \leq 1} h^{-2} e^{-1/(2b)}. \tag{73}
\]
To conclude, we have

\[ \| r_h \|_{L^2}^2 = \sum_{|k|^2 > g(h)} \| \langle \varphi_h, e_k \rangle_{L^2} \|^2 \leq \| V \|_{C^0(T^n)}^2 C(b)^2 h^{2N} \sum_{k \in \mathbb{Z}^n, |k| > 0} \frac{1}{|k|^4}, \]  

(74)

and thus we can define

\[ C(b) := \| V \|_{C^0(T^n)} \tilde{C}(b) \left( \sum_{k \in \mathbb{Z}^n, |k| > 0} \frac{1}{|k|^4} \right)^{1/2}. \]  

(75)

Remark 5 We stress that the hypothesis \( a > 0 \) used in the statement of the Lemma 2 is not restrictive. Indeed, here we look at intervals \((a,b)\) containing only positive values \( E_h \) of the spectrum. However, by taking a constant \( L > \min_{x \in T^n} V(x) \) we have that the translated operator \(-\frac{1}{2} h^2 \Delta_x + V(x) + L\) has the same eigenfunctions of \(-\frac{1}{2} h^2 \Delta_x + V(x)\) and moreover all positive eigenvalues. Thus, the estimate (67) for an arbitrary operator works with

\[ C(b) := \| V + L \|_{C^0(T^n)} \tilde{C}(b) \left( \sum_{k \in \mathbb{Z}^n} \frac{1}{|k|^4} \right)^{1/2}. \]  

(76)

Proposition 1 Let \((a,b) \subset \mathbb{R}^n\) and let \( E_{h,\alpha} \) with \( \alpha = 1, ..., N(h,a,b) \) be all the eigenvalues \( E_{h,\alpha} \) of \(-\frac{1}{2} h^2 \Delta_x + V(x)\) inside \((a,b) \subset \mathbb{R}^n\) repeated with their multiplicity. Let \( \varphi_{h,\alpha} \in C^\infty(T^n) \) be related eigenfunctions, \( L^2 \)-normalized and linearly independent. Then,

\[ I_h \varphi := \sum_{\alpha=1}^{N(h,a,b)} E_{h,\alpha} \langle \varphi, \varphi_{h,\alpha} \rangle_{L^2} \varphi_{h,\alpha} \]  

(77)

\[ = \sum_{|k|^2, |\mu|^2 \leq g(h)} \omega_{h,k,\mu} \langle \varphi, e_\mu \rangle_{L^2} e_k + R_h \varphi, \]  

(78)

where \( \omega_{h,k,\mu} := \sum_{\alpha} E_{h,\alpha} \langle \varphi_{h,\alpha}, e_\mu \rangle_{L^2} \langle \varphi_{h,\alpha}, e_k \rangle_{L^2} \) and

\[ \| R_h \|_{L^2 \to L^2} \leq K(b) h^{-n} e^{-1/(4b)}, \quad \forall 0 < h \leq 1, \]  

(79)

where \( K(b) := 2 b \tilde{K}(b) C(b) \), \( \tilde{K}(b) := \sup_{0 < h < 1} N(a,b,h) h^n < +\infty \), \( C(b) \) is given in (75).
Proof: Let \( G(h) \varphi := \sum_{|k| \leq g(h)} \langle \varphi, e_k \rangle_{L^2} e_k \). In view of Lemma 2

\[
\Pi_h \varphi = \sum_{\alpha=1}^{N(h,a,b)} E_{h,\alpha} \langle \varphi, \varphi_{h,\alpha} \rangle_{L^2} \varphi_{h,\alpha} = \sum_{\alpha=1}^{N(h,a,b)} E_{h,\alpha} \langle \varphi, G(h) \varphi_{h,\alpha} + r_{h,\alpha} \rangle_{L^2} (G(h) \varphi_{h,\alpha} + r_{h,\alpha})
\]

(80)

(81)

(82)

The leading terms equals

\[
\sum_{\alpha=1}^{N(h,a,b)} E_{h,\alpha} \langle \varphi, G(h) \varphi_{h,\alpha} \rangle_{L^2} G(h) \varphi_{h,\alpha} = \sum_{|k|^2, |\mu|^2 \leq g(h)} \omega_{h,k,\mu} \langle \varphi, e_\mu \rangle_{L^2} e_k
\]

(83)

where \( \omega_{h,k,\mu} := \sum_{\alpha} E_{h,\alpha} \langle \varphi_{h,\alpha}, e_\mu \rangle_{L^2} \langle \varphi_{h,\alpha}, e_k \rangle_{L^2} \). The remainder

\[
R_h \varphi = \sum_{\alpha=1}^{N(h,a,b)} E_{h,\alpha} \left( \langle \varphi, G(h) \varphi_{h,\alpha} \rangle_{L^2} r_{h,\alpha} + \langle \varphi, r_{h,\alpha} \rangle_{L^2} \varphi_{h,\alpha} \right)
\]

(84)

has the estimate

\[
\| R_h \varphi \|_{L^2} \leq N(h,a,b) \sup_{\alpha} |E_{h,\alpha}| \left( \| \varphi \|_{L^2} \| G(h) \varphi_{h,\alpha} \|_{L^2} \| r_{h,\alpha} \|_{L^2} + \| \varphi \|_{L^2} \| r_{h,\alpha} \|_{L^2} \| \varphi_{h,\alpha} \|_{L^2} \right).
\]

(85)

(86)

Notice that \( \| \varphi_{h,\alpha} \|_{L^2} = 1, \| G(h) \varphi_{h,\alpha} \|_{L^2} \leq \| \varphi_{h,\alpha} \|_{L^2} = 1, |E_{h,\alpha}| \leq b \) and by Lemma 2 we have \( \| r_{h,\alpha} \|_{L^2} \leq C(b) e^{-1/(4h)} \).

In view of the Weyl Law of eigenvalues (see for example [16]) applied for the operator \(-\frac{1}{2}h^2 \Delta_x + V(x)\) we have that

\[
N(h,a,b) = (2\pi h)^{-a} \text{Vol}(a < \frac{1}{2}|\mu|^2 + V < b) + O(h).
\]

(87)

This implies that \( \tilde{K}(h) := \sup_{0 < h \leq 1} N(a, b, h) h^a < +\infty \). We conclude that

\[
\| R_h \varphi \|_{L^2} \leq 2b \tilde{K}(b) C(b) h^{-a} e^{-1/(4h)} \| \varphi \|_{L^2}.
\]

(88)
Proof of Theorem 1. We begin by recalling that \(-\frac{1}{2}\hbar^2 \Delta_x + V_1(x)\) and \(-\frac{1}{2}\hbar^2 \Delta_x + V_2(x)\) defined on the flat torus, i.e. with domain \(W^{2,2}(\mathbb{T}^n)\), both exhibit discrete spectrum. Since we are assuming that these operators have the same spectrum, then this ensures that there exists a unitary operator \(U_h : W^{2,2}(\mathbb{T}^n) \to W^{2,2}(\mathbb{T}^n)\) such that

\[
U_h^* \circ (-\frac{1}{2}\hbar^2 \Delta_x + V_1(x)) \circ U_h = -\frac{1}{2}\hbar^2 \Delta_x + V_2(x) \tag{89}
\]

on the domain \(W^{2,2}(\mathbb{T}^n)\). In particular, \(U_h \varphi^{(2)}_{\hbar,\alpha} = \varphi^{(1)}_{\hbar,\alpha}\) for all the eigenfunctions of the two operators. In fact, we can localize such a unitary equivalence in a bounded subset of the spectrum. Namely, for \(\Pi_h\) as in Prop.4 we have

\[
U_h^* \circ \Pi_h^{(1)} \circ U_h = \Pi_h^{(2)}. \tag{90}
\]

We observe that (in view of its definition) \(\Pi_h\) are finite rank operators, and moreover thanks to (78) such a finite rank can be regarded with respect to the orthonormal set \(e_\kappa(x)\). Thus, for \(|k|^2, |\mu|^2 \leq g(h)\) we define the finite dimensional matrix \(U_h(k,\mu) := \langle e_k, U_\hbar e_\mu \rangle\) and \(P_h(k,\mu) := \langle e_k, \Pi_\hbar e_\mu \rangle\) and realize that

\[
U_h^* \circ P_h^{(1)} \circ U_h = P_h^{(2)} + \mathcal{O}(h^\infty). \tag{91}
\]

In fact,

\[
P_h(k,\mu) = \langle e_k, (-\frac{1}{2}\hbar^2 \Delta_x + V(x)) e_\mu \rangle + \mathcal{O}(h^\infty) \tag{92}
\]

\[
= \frac{1}{2} \hbar^2 |\mu|^2 \delta_{k\mu} + \langle e_k, V(x) e_\mu \rangle + \mathcal{O}(h^\infty). \tag{93}
\]

Notice the polynomial (hence \(C^\infty\) - type) behavior of the above leading term \(P_{h,0}(k,\mu) := \frac{1}{2} \hbar^2 |\mu|^2 \delta_{k\mu} + \langle e_k, V(x) e_\mu \rangle\). We stress again that \(|k|^2, |\mu|^2 \leq g(h)\) and that \(g(h) \to +\infty\) as \(h \to 0^+\); this is the reason why we cannot find the eigenvalues of \(-\frac{1}{2}\hbar^2 \Delta_x + V(x)\) simply by the eigenvalues of \(P_h\) with \(|k|^2, |\mu|^2 \leq L\) for some \(h\)-independent constant \(L > 0\). Anyway, any components of the eigenfunctions and all the eigenvalues of the matrix \(P_{h,0}\) have a \(C^0\) - dependence from \(h\).

We then rewrite, thanks to a unitary operator \(U_{h,0}\) the equality

\[
U_{h,0}^* \circ P_{h,0}^{(1)} \circ U_{h,0} = P_{h,0}^{(2)} + \mathcal{O}(h^\infty). \tag{94}
\]

which maps to eigenfunctions of \(P_{h,0}^{(2)}\) into the eigenfunctions of \(P_{h,0}^{(1)}\). We are now in the position to say that \(U_{h,0}\) is a finite dimensional unitary operator; and that its dependence from \(h\) is continuous. As a consequence, there exists a (finite dimensional) selfadjoint matrix \(A_h\) such that

\[
U_{h,0} = e^{-iA_h} \tag{95}
\]
where $A_h$ has a continuous dependence from $h$. Hence, there is a selfadjoint finite rank operator $A_h$ on the vector space of the functions in $L^2(\mathbb{T}^n)$ such that

$$\varphi(x) := \sum_{|k| \leq g(h)} c_k e_k(x), \quad |c_k| \leq 1,$$

such that

$$U_h \varphi = e^{-iA_h} \varphi + \mathcal{O}(h^\infty).$$

(97)

The entries of the matrix $\langle e_k, A_h e_\mu \rangle$ have continuous dependence from $h$. We stress that, in view of Lemma 2, the set of all those

$$\psi(x) := N(h,a,E) \sum_{\alpha=1} \alpha \varphi_{h,\alpha}(x), \quad |d_\alpha| \leq 1,$$

(98)

can always be written as $\psi = \varphi + \mathcal{O}(h^\infty)$.

By defining $B_h := h A_h$, and recalling Lemma 1 we can say that

$$U_h \psi = e^{-\frac{i}{h} B_h} \psi + \mathcal{O}(h\infty) = e^{-\frac{i}{h} \text{Op}_w^x(b(h))} \psi + \mathcal{O}(h^\infty)$$

(99)

where $b(h, x, \eta) := h \sum_{\alpha=1}^N (h,a,E) E_{h,\alpha} W_{h,\alpha} \tilde{\psi}_{h,\alpha}(x, \eta) \cdot \mathcal{X}_{E^{+}}(x, \eta)$ and $\mathcal{X}_{E^{+}}(x, \eta)$ is a $C^\infty$ compactly supported function which equals 1 on $\{(x, \eta) \in \mathbb{T}^n \times \mathbb{R}^n : H(x, \eta) \leq E\}$, and equals 0 on $\{(x, \eta) \in \mathbb{T}^n \times \mathbb{R}^n : H(x, \eta) > E + \epsilon\}$. The functions $\tilde{\psi}_{h,\alpha}$ provide a complete ortonormal set of the eigenfunctions related to $A_h$.

Notice that, for any fixed value of $0 < h < 1$, the map $z \mapsto b(h, z)$ is smooth, compactly supported into the same set

$$\lim_{h \to \sigma^-} \| b(h, \cdot) - b(\sigma, \cdot) \|_{L^\infty} = 0.$$

(101)

This ensures that

$$\lim_{h \to \sigma^-} \text{Op}_w^x(b(\sigma)) \psi - \text{Op}_w^x(b(h)) \psi = 0.$$

(102)

In view of (91) and (99), we recover

$$e^{\frac{i}{h} \text{Op}_w^x(b(h))} \circ \text{Op}_w^x(H_1 \mathcal{X}_{E^{+}}) \circ e^{-\frac{i}{h} \text{Op}_w^x(b(h))} \psi = \text{Op}_w^x(H_1 \mathcal{X}_{E^{+}}) \psi.$$

(103)
The limit (102) allows to recover, up to the remainder $\delta(h, \sigma)$,
\[
e^{\frac{i}{\hbar} \text{Op}_w^\hbar(b(\sigma))} \circ \text{Op}_w^\hbar(H_1 \sharp X_{\leq E + \varepsilon}) \circ e^{-\frac{i}{\hbar} \text{Op}_w^\hbar(b(\sigma))} \psi = \text{Op}_w^\hbar(H_2 \sharp X_{\leq E + \varepsilon}) \psi + \delta(h, \sigma).
\]
In particular,
\[
\lim_{h \to 0^-} \|\delta(h, \sigma)\|_{L^2} = 0. \tag{105}
\]
The application of Corollary 1 involves the constant $K[H_1, b(\sigma)]$ (possibly diverging as $\sigma \to 0^+$), and $\varphi_\sigma$ corresponding to the Hamiltonian flow of $b(\sigma, z)$. This gives
\[
\|H_1 \circ \varphi_\sigma - H_2\|_{C^0(\Omega(E))} \leq K[H_1, b(\sigma)] h \tag{106}
\]
The inequality we require is written for arbitrary $0 < u \leq 1$ and
\[
K[H_1, b(\sigma)] h \leq u \tag{107}
\]
which is fulfilled for $h \leq u \cdot K[H_1, b(\sigma)]^{-1}$. Notice that
\[
u \cdot K[H_1, b(\sigma)]^{-1} \leq \sigma \tag{108}
\]
is fulfilled for
\[
u \leq \sigma \cdot K[H_1, b(\sigma)]. \tag{109}
\]
In particular, if $\sigma \to 0$ and $\nu \to 0$ then $h \to 0$ and $\sigma - h \leq \sigma \to 0$. We conclude that for any $0 < \varepsilon \leq 1$ there exists an interval $0 < \sigma \leq \sigma_0(E, \varepsilon)$ such that
\[
\|H_1 \circ \varphi_\sigma - H_2\|_{C^0(\Omega(E))} \leq \varepsilon. \tag{110}
\]
□

**Proof of Theorem 2.** We apply the main result of Theorem 1 combined with (50) and (52). More precisely,
\[
\overline{H}_2(P) = \inf_{\Gamma} \sup_{(x, p) \in \Gamma} H_2(x, p + P), \quad \forall P \in U_{2, E}. \tag{111}
\]
The application of
\[
\|H_1 \circ \varphi_\sigma - H_2\|_{C^0(\Omega(E))} \leq \varepsilon, \quad 0 < \sigma \leq \sigma_0(E, \varepsilon) \tag{112}
\]
ensures that
\[
\overline{H}_2(P) = \inf_{\Gamma} \sup_{(x, p) \in \Gamma} H_1 \circ \varphi_\sigma(x, p + P), \quad \forall P \in U_{2, E}. \tag{113}
\]
On the other hand,
\[
\underline{H}_1(P) = \inf_{\Gamma} \sup_{(x, p) \in \Gamma} H_1 \circ \varphi_\sigma(x, p + P), \quad \forall P \in \mathbb{R}^n, \quad \forall \varphi_\sigma. \tag{114}
\]
As a consequence,
\[ \mathcal{H}_1(P) = \mathcal{H}_2(P), \quad \forall P \in \mathcal{U}_{2,E} \] (115)
and since \( E \) can be fixed arbitrary large then
\[ \mathcal{H}_1(P) = \mathcal{H}_2(P), \quad \forall P \in \mathbb{R}^n. \] (116)

\[ \blacksquare \]

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