Radiation Forces on Dust Envelopes

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ABSTRACT

We address in detail the radiation forces on spherical dust envelopes around luminous stars, and numerical solutions for these forces, as a first step toward more general dust geometries. Two physical quantities, a normalized force and a force-averaged radius, suffice to capture the overall effects of radiation forces. In addition to the optically thin and thick regimes, the wavelength dependence of dust opacity allows for an intermediate case in which starlight is easily trapped but infrared radiation readily escapes. Scattering adds a non-negligible force in this intermediate regime. We address all three regimes analytically and provide approximate formulae for the force parameters, for arbitrary optical depth and inner dust temperature. Turning to numerical codes, we examine the convergence properties of DUSTY and of the Monte Carlo code Hyperion. We find that Monte Carlo codes tend to underestimate the radiation force when the mean free path of starlight is not well resolved, as this causes the inner dust temperature, and therefore the inner Rosseland opacity, to be too low. We briefly discuss implications for more complicated radiation transfer problems.

Key words: keyword1 – keyword2 – keyword3

1 INTRODUCTION

Dusty gas is much more opaque to visible light than ionized gas lacking dust. As a result, radiation pressure forces become dominant in situations where luminous but sub-Eddington objects, like massive stars and AGN, are surrounded by dusty gas. Examples include individual massive star formation, where radiation forces present an obstacle to stellar accretion (Wolfire and Cassinelli 1987); massive star cluster formation (Fall et al. 2010; Matzner and Jumper 2015), in which matter may be expelled from the cluster-forming zone; the disruption of giant molecular clouds (Krumholz and Matzner 2009; Murray et al. 2010; Hopkins et al. 2012); the initial inflation of giant H\(\text{ii}\) regions (Draine 2011; Lopez et al. 2011; Yeh and Matzner 2012);1 and the ejection of gas from galaxies (Murray et al. 2011); as well as dust-driven winds and supernovae from AGB stars (Bowen and Willson 1991).

For each of these problems a detailed understanding of radiation forces on dust, and a calibration of numerical methods to estimate these forces, are clearly important. However, each scenario involves complicated dust distributions whose inhomogeneities make detailed comparisons difficult. For a first step toward the more general problem, we focus here on a dramatically simplified case: a spherically symmetric, power-law dust profile surrounding a central light source. What this scenario lacks in realism it makes up in precision and flexibility: it allows two numerical methods to be compared against one another, and against analytical estimates. This spherical calibration provides useful points to be carried into the more interesting, non-spherical setting.

On the numerical front, our simple scenario can be approached by the multi-group radiation transfer code DUSTY (Ivezic et al. 1999) whose adaptive spatial and frequency grids allow it to converge rapidly to a solution that we shall consider to be ground truth. But, DUSTY cannot treat complicated, three dimensional dust distributions. For these one needs more flexible codes like Monte Carlo and moment-closure methods. Monte Carlo methods should also converge to ground truth. However Monte Carlo convergence is rather different from DUSTY’s, as it relies on sufficiently high spatial and spectral resolution, as well as the propagation of sufficiently many photon packets. Furthermore, the weaknesses of moment-closure methods will be most apparent in non-spherical problems in which the radiation pressure tensor is anisotropic. For these reasons we shall compare DUSTY to the Monte Carlo code Hyperion (Robitaille 2011), focusing on Hyperion’s convergence criteria for radiation forces. We

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will address moment closure methods in a later paper on more complex dust distributions.

On the analytical front, radiative transfer through spherical dust envelopes, and the corresponding dust temperature profiles, have long been studied in relation to the emergent spectral energy distributions of protostars (Larson 1969; Adams and Shu 1985), star cluster-forming clumps in starburst galaxies (Chakrabarti and McKee 2005), and dusty winds from late-type stars (Ivezic and Elitzur 1995). However, analytical studies (including Krumholz and Matzner 2009, Murray et al. 2010, and our own work: Matzner and Jumper 2015) have so far considered only crude approximations to the radiation force term. We will develop more accurate analytical formulae, albeit restricted (for now) to the simple spherical geometry.

We delineate the problem to be solved below in §1.1. In §1.2 we introduce two integral quantities of interest, the net radiation force and the force-averaged radius, which can be combined to form a radiation force term in the virial equation. We address the theoretical problem in §2 and compare numerical solutions in §3.

1.1 Physical problem

We consider, for simplicity, a spherically symmetric dust envelope with inner radius \( r_\text{in} \), outer radius \( r_\text{out} \) (set to \( 4r_\text{in} \) in our fiducial case, purely for convenience) and radial profile \( \rho(r) \propto r^{-k} \) for \( r_\text{in} < r < r_\text{out} \), for some \( k > 1 \); in our fiducial case \( k = 1.5 \). This is representative of the density profile in low and high-mass star formation: van der Tak et al. (2000) finds \( k = 1.0 \) to \( k = 1.5 \), Jørgensen et al. (2002) finds \( k = 1.3 \) to \( k = 1.9 \), ±0.2, Shirley et al. (2002) finds \( k = 1.8 \pm 0.1 \), and Mueller et al. (2002) finds \( k = 0.75 \) to \( k = 2.5 \) with a mean value of \( k = 1.8 \pm 0.4 \).

We take the spectrum of the central source, \( L_\nu, \nu \), to correspond to a blackbody of color temperature \( T_\nu = 5772 \text{ K} \). We adopt the dust absorption opacity \( \kappa_\nu \) and albedo \( a_\nu \) for a dust mixture with \( R_\nu = 5.5 \) provided by Draine (2003a,b) from which we compute the total opacity \( \kappa = \kappa_\nu (1 + a_\nu) \). However, to treat scattered radiation identically in numerical and analytical calculations, we consider only isotropic scattering. This neglects a scattering angle \( \theta \) characterized by a mean angle of \( \langle \cos(\theta) \rangle \). While this is certain to alter our results to a small degree, it also permits a precise comparison between various approaches.

1.2 Quantities of Interest

We take the outward luminosity at radius \( r \) to be \( L(r) = \int_0^\infty L_\nu(r) \, d\nu \); photon momentum passes \( r \) at the rate \( L(r)/c \), which is independent of \( r \) in static equilibrium (whereas \( L_\nu \) can vary with radius). Given an extinction optical depth \( \tau_\nu \) (arising from a density and specific opacity, \( d\tau_\nu = \rho(r) \kappa_\nu \, dr \)), the radiation force satisfies

\[
dF_{\text{rad},\nu} = \frac{L_\nu}{c} \, d\tau_\nu; \tag{1}
\]

the frequency-integrated force within \( r \) is \( F_{\text{rad}}(r) = \int_0^{\infty} F_{\text{rad},\nu} \, d\nu \), and the total outward force is \( F_{\text{rad},\text{tot}} = F_{\text{rad}}(r = \infty) \).

Our first quantity of interest, then, compares the applied radiation force to the photon force:

\[
\Phi = \frac{F_{\text{rad},\text{tot}}}{L/c} = \int_0^{\infty} \frac{L_\nu}{L} \, d\tau_\nu \, d\nu. \tag{2}
\]

This can be less than unity, in the case of an optically thin dust envelope, or much greater than unity if the optical depth is very high. Indeed \( \Phi \) is a luminosity-weighted integral of \( \tau_\nu \), as the final expression in equation (2) shows.

However, one needs more information than just the net force in order to know what effect it will have. For instance, if the force acts on matter that is strongly bound, like a accretion disk, then its influence will be relatively minor, whereas the same force might blow away a loosely-bound envelope. Therefore we also need a measure of where the force is applied. For this we introduce a second quantity, the force-averaged radius:

\[
\langle r \rangle_F = \int_0^{F_{\text{rad},\text{tot}}} r \, dF_{\text{rad}} / F_{\text{rad},\text{tot}}. \tag{3}
\]

We are motivated here by the virial theorem, where the radiation force \( F_{\text{rad}} \) introduces the term \( \mathcal{R} = \int \mathcal{R} \, dF_{\text{rad}} \), equivalent to expression (4.24) of McKee and Zweibel (1992). For our spherically symmetric problem,

\[
\mathcal{R} = \Phi \frac{L}{c} \langle \nu \rangle_F. \tag{4}
\]

Our model density distributions are described by inner and outer radii \( r_\text{in} \) and \( r_\text{out} \). Dimensionless quantities like \( \Phi \) and \( \langle r \rangle_F / r_\text{in} \) are functions of the spectral shape of the input radiation (or its color temperature, if it is taken to be a blackbody) and the optical depth \( \tau_\nu \) at a chosen reference frequency \( \nu_0 \) (see Rowan-Robinson 1980 and Ivezic and Elitzur 1997).

For brevity we use \( \tau_\nu(r) \) to denote the dust extinction optical depth within \( r \), and \( \tau_r \) to denote the total optical depth through the dust distribution. Therefore \( \tau_r = \tau_r(\infty) \).

2 ANALYTICAL PREDICTIONS

Dusty radiative transfer has two quintessential features: first, the dust opacity \( \kappa_\nu \) increases with frequency over the relevant range of \( \nu \); and second, the stellar surface temperature is much hotter than dust grains can possibly be. Therefore, optical and ultraviolet starlight is always more readily absorbed than the infrared emission from heated grains. As a result, one can distinguish three regimes: I – optically thin to starlight (\( \tau_\nu < 1 \) where \( \tau_\nu = \int_0^{\nu} \tau_\nu(L_{\nu,\nu}/L) \, d\nu \) is the starlight-averaged opacity); II – thick to starlight but thin to dust emission (\( \tau_r > 1 > \tau_\nu(T_{\text{d,\nu}}) \) where \( \tau_\nu(T_{\text{d,\nu}}) \) is the Rosseland opacity at the temperature of the innermost dust); and III – optically thick to dust radiation (\( \tau_r > \tau_\nu(T_{\text{d,\nu}}) > 1 \)).

I. Optically thin case: In. Starlight. Of these, the optically thin case is of course the simplest. The direct starlight luminosity at \( r \) is \( L_{\text{dir},\nu}(r) = c \nu_0 \tau_\nu(r) L_{\nu,\nu} \), and therefore the contribution of direct irradiation to \( \Phi \) is

\[
\Phi_{\text{dir}} = \int_0^{\infty} (1 - e^{-\tau_\nu}) \frac{L_\nu}{L} \, d\nu = 1 - e^{-\tau_\nu} + O(\tau_\nu^2). \tag{5}
\]

Of course \( \Phi_{\text{dir}} = \tau_\nu + O(\tau_\nu^2) \) would be equally valid, but expression (5) is more accurate when \( \kappa_\nu \) varies slowly with \( \nu \).
In the limit \( \tau_s \ll 1 \) of very low optical depth, \( e^{-\tau_s} \to 1 \) and \( \langle r \rangle_F \) approaches a unique value \( \langle r \rangle_{F,\text{thin}} \); for our truncated power law profile this is

\[
\langle r \rangle_{F,\text{thin}} = \frac{k - 1 - 2\kappa}{k - 1} \left( 1 - \frac{r_{\text{out}}}{r_{\text{in}}} \right)^{2-k} \left( 1 - \frac{r_{\text{out}}}{r_{\text{in}}} \right)^{-1},
\]

which is unity for a thin shell \( (r_{\text{out}} \equiv r_{\text{in}}) \) and is intermediate between \( r_{\text{in}} \) and \( r_{\text{out}} \) even for very thick shells, so long as \( 1 < k < 2 \). In our fiducial case \( k = 1.5 \), \( \langle r \rangle_{F,\text{thin}} = (r_{\text{in}}/r_{\text{out}})^{1/2} \).

IIb. Optically thin dust emission. A minor but non-negligible contribution to the total force arises from the interaction of thermal dust radiation with other dust grains. The total infrared luminosity is approximately \( (1 - e^{-\tau_s}) L \) and the appropriate opacity is \( \kappa_{\text{dd}}(T_{\text{d,in}}) \), where

\[
\kappa_{\text{dd}}(T) = \frac{\int_0^\infty B_\nu(T)\kappa_{\nu,\text{d}} \, d\nu}{\int_0^\infty B_\nu(T)\kappa_{\nu,\text{d}} \, d\nu}
\]

is the opacity of dust grains to the emission of other grains at temperature \( T \). The associated optical depth is \( \tau_{\text{dd,in}} = \kappa_{\text{dd}}(T_{\text{d,in}})/\kappa_{\text{d}} \). Note that \( \kappa_{\text{dd}}(T) \) is relatively large compared to the Rosseland mean, because optically thin radiation is concentrated in frequencies where \( \kappa_{\nu} \) is maximized; otherwise optically thin infrared would be entirely negligible.

The contribution to \( \Phi \) due to recaptured infrared emission is therefore approximately

\[
\Phi_{\text{IR, thin}} = \Phi_{\text{dir}} + \Phi_{\text{IR, thin}} \]

and we can estimate the total force due to optically thin radiation as

\[
\Phi_{\text{thin}} = \Phi_{\text{dir}} + \Phi_{\text{IR, thin}}.
\]

II. Intermediate case: starlight scattering and absorption. As the starlight optical depth increases, \( \Phi_{\text{dir}} \to 1 \) but scattered starlight adds to the net force and hence a new component \( \Phi_{\text{sc}} \) to the force ratio \( \Phi \). We address this case in Appendix A by means of Eddington’s approximation. In the limit \( \tau_s \gg 1 \) (§ A1), \( \Phi_{\text{dir}} \to 1 \) and

\[
\Phi_{\text{sc}} \to \frac{a_v}{1 + \frac{3}{4}(1 - a_v)} \left( \frac{L}{L_s} \right) = \frac{a_s}{1 + \frac{3}{4}(1 - a_s)}
\]

where \( (\cdots)_L \) is a starlight-averaged value, i.e., a frequency average weighted by \( L_{\nu,\text{sc}} \) and \( a_s = (a_v)_L \).

In the asymptotic case \( \tau_s(T_{\text{d,in}}) \ll 1 \ll \tau_s \), all the force is applied at the inner boundary and therefore \( \langle r \rangle_F / r_{\text{in}} \to 1 \). Figure A1 demonstrates the dependence of \( \Phi_{\text{dir}} + \Phi_{\text{sc}} \) and the force-averaged radius due to (direct + scattered) starlight, as functions of optical depth and albedo.

III. Optically thick case. Here the dust is optically thick to starlight, and also to its own infrared thermal radiation. The direct and scattered radiation and their associated moments (like \( \Phi_{\text{dir}} \) and \( \Phi_{\text{sc}} \)) are as described above, but the self-force due to dust emission becomes appreciable and contributes a new term \( \Phi_{\text{IR}} \). In Appendix B we compute \( \Phi_{\text{IR}} \) using the diffusion approximation, deriving the thick limit

\[
\Phi_{\text{IR, thick}} \simeq \frac{4 - \beta}{k(4 - k)} \frac{\kappa_{\text{d}} r_{\text{in}}/r_{\text{out}}}{k(4 - k) + 2(4 - k) \kappa_{\text{d}} r_{\text{in}}/r_{\text{out}}}
\]

\[
= \frac{(4 - \beta)(k - 1)}{k(4 - k) + 2(4 - k) \kappa_{\text{d}} r_{\text{in}}/r_{\text{out}}}
\]

\[
\simeq \frac{4 - \beta}{k(4 - k)} \frac{\kappa_{\text{d}} r_{\text{in}}/r_{\text{out}}}{k(4 - k) + 2(4 - k) \kappa_{\text{d}} r_{\text{in}}/r_{\text{out}}}
\]

(10)

The latter expression (in terms of the inner conditions as well as a fiducial opacity and optical depth) is appropriate for comparison to DUSTY simulations. It demonstrates that the normalized radiation force depends on the Rosseland opacity at the inner boundary, and is therefore directly related to the inner dust temperature; this point will be relevant to our analysis of numerical methods.

In the thermal diffusion limit \( \langle r \rangle_F \) takes a limit \( \langle r \rangle_{F,\text{thick}} \) given in equation (B3) of Appendix B.

**Combined formulae.** To combine these asymptotic forms, we propose

\[
\Phi \simeq (1 - e^{-\tau_s}) \left( 1 + \frac{k_s}{\kappa_s} (1 - e^{-\tau_{\text{d, thin}}}) + \Phi_{\text{sc}} (1 - e^{-\tau_s}) + \Phi_{\text{IR, thick}} \right)
\]

(11)

and

\[
\langle r \rangle_F \simeq \frac{r_{\text{in}} + \langle (r \rangle_{F,\text{thick}} - r_{\text{in}}) e^{-\frac{\tau_{\text{d, thin}}}{\tau_s}} \right)}{1 - \frac{\Phi_{\text{IR}}'}{\Phi}} \left( \frac{r_{\text{in}} + \langle (r \rangle_{F,\text{thick}} - r_{\text{in}}) e^{-\frac{\tau_{\text{d, thin}}}{\tau_s}} \right)}{1 - \frac{\Phi_{\text{IR}}'}{\Phi}} \Phi_{\text{IR}}'.
\]

(12)

where \( \Phi_{\text{IR}}' \equiv (1 - e^{-\tau_s}) \Phi_{\text{IR, thick}} \). The radiation contribution \( \Phi \) in the virial theorem is then given by

\[
\frac{R}{L} \simeq \frac{r_{\text{in}} + \langle (r \rangle_{F,\text{thick}} - r_{\text{in}}) e^{-\frac{\tau_{\text{d, thin}}}{\tau_s}} \right)}{1 - \frac{\Phi_{\text{IR}}'}{\Phi}} \left( \Phi - \Phi_{\text{IR}}' \right) + \langle r \rangle_{F,\text{thick}} \Phi_{\text{IR}}'.
\]

(13)

Note that we include a prefactor \( 1 - e^{-\tau_s} \) on \( \Phi_{\text{IR, thick}} \) in equation (11), since IR luminosity is powered by absorbed starlight. The parameter \( q \) in equation (12) and (13) does not affect the asymptotes, so it is somewhat arbitrary; we find that \( q \approx 0.1 \) optimizes the fit.

### 3 COMPARING CODES AND SOLUTIONS

We now turn to the numerical codes DUSTY and Hyperion, to examine their convergence with respect to our quantities of interest, \( \Phi \) and \( \langle r \rangle_F \) and to check the accuracy of our analytical predictions.

#### 3.1 DUSTY: adaptive radiation transfer code

Ivezic and Elitzur (1997) found that the radiative transfer equations, if given a specified temperature for the inner edge of the dust envelope and the SED of the luminosity source, could be solved without further reference to dimensional values; everything else could be expressed in terms of dimensionless quantities varying with respect to a dimensionless radial profile. Using these methods, DUSTY (Ivezic et al. 1999) provides solutions to the radiative transfer problem parameterized by the inner dust temperature \( T_{\text{d,in}} \), the optical depth \( \tau_{\text{d}} \) at a fiducial frequency \( \nu_{\text{d}} \), and the spectral distributions of the central source \( (L_{\nu}\nu) \) and of the dust properties \( (\kappa_{\nu}, a_{\nu}) \). We inspect these solutions to construct \( \Phi \) and \( \langle r \rangle_F \). As DUSTY self-refines its computational grid to ensure flux conservation to a desired accuracy, our study of its convergence behavior is limited to the numerical parameters that control this process: the requested flux accuracy (default: 0.05), and the maximum relative change of optical depth (default: 0.3). For a quick survey we adopt values different from these defaults, starting at 0.01 and 0.025 respectively, and varying these values to explore the parameter space; for our truncation terms, we adopt values to match the fiducial case (default: 0.05), and the maximum relative change of optical depth (default: 0.3).
space. For these, we hold $T_{\text{d,lm}}$ fixed at 1500 K (roughly the dust sublimation temperature) and set $\tau_{\text{fid}} = 10$, which for $c/\nu_{\text{d}} = 1.95 \, \mu \text{m}$ corresponds to $\tau_{\text{r}} = 54.7$

The results are shown in Tables 1 and 2. DUSTY’s solutions appear to converge as the requested tolerances are made more strict and more grid points are required to determine the solution. Fitting the results with linear functions of the tolerances, we infer $\Phi \rightarrow 5.183$ and $(\langle \tau \rangle_{F}) \rightarrow 1.596_{\text{rm}}$ as these tend to zero.

### 3.2 Hyperion: Monte Carlo radiation transfer code

Hyperion (Robitaille 2011) represents the alternative Monte Carlo technique. Monte Carlo codes like Hyperion model radiation transfer by propagating a large number of photon packets throughout the medium, allowing them to scatter or be absorbed and re-emitted in a number of interactions. Hyperion has the advantage of handling arbitrary density configurations. However Hyperion is far slower for the spherical regime, which is also where starlight is absorbed very close to $r_{\text{m}}$. If the deposition of starlight energy is diluted by a lack of resolution, $T_{\text{d,lm}}$ will be systematically underestimated. The consequence is an artificial lowering of the Rosseland opacity $\kappa_R(T_{\text{d,lm}})$, as observed in Table 5. The radiation force $\Phi$ should be underestimated by the same factor, as in equation (10), even if the rest of the radiation transfer problem is handled perfectly.

We illustrate this point in Figure 3, where we compare $\Phi$ in a sequence of Hyperion runs against DUSTY – both for the problem they are meant to represent (in which $T_{\text{d,lm}} = 1500$ K) and also for problems with the same value of $T_{\text{d,lm}}$ as they actually achieved. In each case $\Phi$ is much closer to DUSTY’s value for the same value of $T_{\text{d,lm}}$ than for the ideal problem (although these converge as the resolution improves).

If correct, the effect of discretization should be much more noticeable at lower optical depths for which the starlight mean free path is not so short. We tested this out with Hyperion runs of varying resolution and with $\tau_{\text{fid}}$ as low as $10^{-3}$. As expected, the discrepancy in $\Phi$ depends strongly on optical depth: for $\log_{10} \tau_{\text{fid}} = (-3, -2, -1, 0, 1)$ we find that the 323 results underestimate Hyperion’s $\Phi$ by (0.57%, 1.6%, 0.7%, 2.8%, 8.7%) whereas the 256 results are (0.11%, 0.12%, 0.23%, 0.87%, 1.2%) low.

Resolution affects Hyperion’s determination of $(\langle \tau \rangle_{F})$ as well, but not as strongly: $(\langle \tau \rangle_{F})$ decreases by 4.2% going from 323 to 2563 in the model with $\tau_{\text{fid}} = 10$. This undoubtedly reflects the fact that $(\langle \tau \rangle_{F})$ approaches a unique asymptotic value in each of the three regimes discussed in §2, so it should be relatively insensitive to errors.

### 3.3 Parameter space survey

With the scaling behaviors examined, we now examine the overall behaviour of the radiation transfer solutions. Figures 1 and 2 present the variation of $\Phi$ and $(\langle \tau \rangle_{F})$, respectively, across the range $10^{-3} < \tau_{\text{fid}} < 10^{3}$ and 100 K $< T_{\text{d,lm}}$ $< 1500$ K. We plot DUSTY results (with the adopted parameters) against $\tau_{\text{r}}$ in panel (a) of Figure 1, and against $(\kappa_{R}(T_{\text{d,lm}}))_{\kappa_{\text{r}}}$ in panel (b); the two plots are meant to illustrate that $\Phi$ depends on $\tau_{\text{r}} = 5.47 \tau_{\text{r}}^{*}$ in the optically thin regime and on $(\kappa_{R}(T_{\text{d,lm}}))_{\kappa_{\text{r}}}^{\text{thick}}$ in the thick regime. Figure 2 makes a similar point. These dependences are hard-wired into the analytical predictions, equations (11) and (12), which we overplot in each figure. (To avoid clutter we plot these predictions only for $T_{\text{d,lm}} = 100$ K and 1500 K.)

These figures make very clear that the three radiation transfer regimes discussed in §2 apply to the real problem as
well as the idealized one. As for our analytical predictions: while for low temperatures the error in $\Phi$ is large, with a peak of $\approx 50\%$ for 200 K at $r_\tau = 274$ over the 100 K and 200 K range, there is also a peak error of $\approx 30\%$ at $r_\tau = 547$ at a temperature of 500 K on the range of 300 K to 800 K, and a peak error of 8.5\% also at $r_\tau = 547$ at a temperature of 900 K over the range of 900 K -1500 K.

### Table 1. Convergence of DUSTY radiative transfer for the problem with $T_{d,in} = 1500$ K and $r_{\text{FD}} = 10$ at $c/r_{\text{FD}} = 1.95 \mu m$. Here we vary the requested flux conservation accuracy and holding all other control parameters and physical parameters of the problem constant. For requested accuracy of 0.01 or worse, other control parameters provide more stringent refinement criteria and the solution is unchanged. At higher requested accuracy, more grid points are finally required, and $\Phi$ varies slightly (a 1\% change as the requested accuracy changes over an order of magnitude). Meanwhile, $(r)_F/r_{\text{FD}}$ remains insensitive to the required flux accuracy.

| Req. Accuracy | Grid Points | $\Phi$ | $(r)_F/r_{\text{FD}}$ | Flux Accuracy |
|---------------|-------------|--------|----------------------|---------------|
| 0.001         | 73          | 5.20   | 1.59                 | $3.0 \times 10^{-4}$ |
| 0.005         | 60          | 5.22   | 1.59                 | $1.7 \times 10^{-3}$ |
| 0.01          | 57          | 5.25   | 1.59                 | $3.6 \times 10^{-3}$ |
| 0.05          | 57          | 5.25   | 1.59                 | $3.6 \times 10^{-3}$ |
| 0.10          | 57          | 5.25   | 1.59                 | $3.6 \times 10^{-3}$ |

### Table 2. DUSTY convergence for the same problem as Table 1, here varying the maximum step in optical depth between consecutive grid points relative to the total optical depth, $\max \Delta r/r$, whilst holding all other control variables at default values. Allowing 8 times larger steps, number of steps decreases by a factor of 4.5; $\Phi$ changes by about 1.3\% from its starting value, while $(r)_F/r_{\text{FD}}$ chances by about 0.63\% from its starting value.

| Max $\Delta r/r$ | Grid Points | $\Phi$ | $(r)_F/r_{\text{FD}}$ | Flux Accuracy |
|------------------|-------------|--------|----------------------|---------------|
| 0.0125           | 108         | 5.23   | 1.59                 | $3.7 \times 10^{-3}$ |
| 0.025            | 57          | 5.25   | 1.59                 | $3.6 \times 10^{-3}$ |
| 0.05             | 33          | 5.29   | 1.59                 | $3.3 \times 10^{-3}$ |
| 0.1              | 24          | 5.32   | 1.58                 | $4.7 \times 10^{-3}$ |
| 0.2              | 23          | 5.32   | 1.58                 | $6.6 \times 10^{-3}$ |

### Table 3. Variation of Hyperion’s predictions for $\Phi$ as the numbers of photon packets per cell are varied, in the same problem as before. Cases labeled “Low”, “Medium”, and “High” correspond to averages of 10, $10^3/32^3 = 30.5$, and 100 photon packets per cell, respectively.

| # Cells | Low: $\Phi$ | Medium: $\Phi$ | High: $\Phi$ |
|---------|-------------|----------------|--------------|
| $32^3$  | 4.73462     | 4.73383        | 4.73498      |
| $64^3$  | 4.93232     | 4.93319        | 4.93246      |
| $128^3$ | 5.05638     | 5.057525       | —            |
| $256^3$ | 5.12479     | 5.12422        | —            |

### Table 4. Like Table 3, but for $(r)_F/r_{\text{FD}}$.

| # Cells | Low: $(r)_F/r_{\text{FD}}$ | Medium: $(r)_F/r_{\text{FD}}$ | High: $(r)_F/r_{\text{FD}}$ |
|---------|-----------------------------|--------------------------------|-----------------------------|
| $32^3$  | 1.67                        | 1.67                           | 1.67                        |
| $64^3$  | 1.63                        | 1.63                           | 1.63                        |
| $128^3$ | 1.61                        | 1.61                           | —                           |
| $256^3$ | 1.60                        | 1.60                           | —                           |

### Table 5. A convergence study of Hyperion with grid resolution, and a comparison with DUSTY (for a flux conservation accuracy parameter of 0.001 and a maximum $\Delta r/r_{\text{FD}}$ of 0.025), for a physical problem created to match that in Tables 1 and 2. Note that $T_{d,in}$ converges toward the desired value (1500 K) as $\Phi$ and $(r)_F/r_{\text{FD}}$ converge.

| Resolution | $\Phi$ | $(r)_F/r_{\text{FD}}$ | $T_{d,in}$ |
|------------|--------|-----------------------|------------|
| $32^3$     | 4.73   | 1.67                  | 1209       |
| $64^3$     | 4.93   | 1.63                  | 1296       |
| $128^3$    | 5.06   | 1.61                  | 1372       |
| $256^3$    | 5.12   | 1.60                  | 1420       |
| DUSTY      | 5.20   | 1.59                  | 1500       |

## 4 CONCLUSIONS

We start with our key findings. First, for a spherical dust envelope the overall importance of radiation force is captured by two integral quantities: a normalized radial force $\Phi$ (formally equivalent to the net flux-averaged optical depth) and a force-averaged radius $(r)_F$. These combine to give the reaction term in the virial theorem, $R \equiv \int r \cdot dF = \Phi (r)_F L/c$, and so they directly affect the dynamical evolution when radiation forces matter at all. We stress this point because radiation forces are frequently described only in terms of a net force. Assuming the force is applied on the largest scales gives an overestimate for $R$, because $(r)_F$ tends to be of order the innermost radius, or (in the optically thin case) intermediate between inner and outer radii.

Second, the difference in opacity between starlight and thermal infrared radiation opens an intermediate regime in which starlight is easily scattered and absorbed, but thermal emission escapes. For this reason, the radiation transfer problem breaks into thin, intermediate, and thick regimes. Each of these can be understood analytically in its asymptotic form, and we provide approximate formulae for $\Phi$, $(r)_F$, and $R$ that combine these forms into a single expression valid to a few tens of percent (and within 10\% at higher temperatures).

Third, we compare the DUSTY radiation transfer code, which is self-adaptive and optimized for our spherical problem, against the Hyperion Monte Carlo code. Although both codes converge toward a physical solution, this comparison reveals a weakness of the Monte Carlo technique when the starlight mean free path is not resolved. The stellar luminosity is then deposited in too thick a layer, leading to an
underestimate of the maximum dust temperature. Because of the wavelength dependence of dust opacity, the net result is an underestimate of the net radiation force.

This may generate a tension in the design of Monte Carlo simulations, between the desire to resolve the starlight mean free path and the need to allocate numerical resources. Although integral quantities like $\Phi$ and $(r)_F$ depend very weakly on the number of photon packets propagated, and although Monte Carlo codes parallelize efficiently with respect to the number of such packets, the total computational cost scales as the fourth power of grid resolution if the number of packets per cell is held constant. Furthermore, this parallelization requires a copy of the density grid to exist on each node, limiting the number of cells that can be defined. One solution would be to locally refine the grid using the starlight mean free path as a refinement criterion. Another would be to implement a sub-grid model for the dust temperature profile. A third would be to apply a correction factor to remove the systematic effect of poor resolution on the radiation forces.

Although we have only considered spherically symmetric dust profiles, we can comment on non-spherical effects. Clearly, segregating dust into clumps and opening paths of less optically thick regime. On the other hand, the same effects lower optical depth will reduce the trapping of radiation, effectively thick regime. In the intermediate scattering regime, the analytical approximation underestimates $\Phi$ and the diffusion limits. In the intermediate scattering regime, the analytical approximation underestimates DUSTY. (b): The same $\Phi_{\text{rad}}$ contours, but now plotted against the axis of $\tau_{\text{in}}$, the Rosseland mean optical depth of the dust to radiation reprocessed at the inner temperature. Once again, the 100 K (leftmost) and the 1500 K (rightmost) contours of the models are highlighted in red.

Figure 2. Plot of the $(r)_F$ parameter for radiation pressure force against the optical depth to starlight, $\tau_\star$, for 765 DUSTY models (15 different inner dust temperatures each at 51 different maximum optical depths), compared against analytical approximations across the direct, scattering, and diffusion regimes. The corresponding values of $\tau_{\text{in}}$ range from 0.001 to 100. The inner dust temperatures range over an interval from 100 K to 1500 K with 100 K increments. All models assume a geometry of $n = 4$ and $k = 1.5$. The 100 K (lower) and 1500 K (upper) contours are highlighted in red, with the intermediate contours shown in gray. Also shown in blue is an analytical approximation for $\Phi_{\text{rad}}$ for 100 K and 1500 K models.

Figure 3. Comparison of DUSTY calculations for $\Phi_{\text{rad}}$ and Hyperion Monte Carlo calculations for $\Phi_{\text{rad}}$ and an examination of the resulting errors. Four DUSTY model contour lines are plotted corresponding to four different dust shell interior temperatures: respectively, these are dark yellow (1200 K), orange (1300 K), red (1400 K), and dark red (1500 K). Here depicted are the results of four Hyperion models with the same geometry and dust properties, but at varying resolutions, to a DUSTY model at 1500 K and a fiducial optical depth of $\tau_{\text{in}} = 10$. The Monte Carlo results underestimate the force exerted by the radiation pressure with decreasing resolution, in addition to the central temperature. The force values are also compared against the corresponding DUSTY models at the same underestimated temperatures, denoted by starred data points and related to the corresponding Monte Carlo model with a dashed line.
APPENDIX A: SCATTERED LIGHT

Here we consider the scattered starlight. Because in our numerical investigations we implement isotropic scattering, the scattered light is close to isotropic at all radii. It can therefore be treated with Eddington’s approximation, in which the specific intensity in direction \( \hat{n} \) is a linear function of \( \mu = \hat{n} \cdot \hat{r} \) at every radius. The equations are exactly the same as those presented by Rybicki and Lightman, with the exception that the direct illumination by starlight adds a new source of scattered radiation. In this section, for additional clarity, we rename the total optical depth of the dust envelope \( \tau_{v,\text{max}} \) and the local optical depth from the centre as \( \tau_v \).

Defining \( \bar{\tau}_v = (3\epsilon_v)\tau_v \), where \((1-e_v) = a_v \) is the albedo, the mean scattered intensity \( J_v \) satisfies

\[
J''_v - J_v + (S_{\text{dir},v} + B_v) = 0 \tag{A1}
\]

where prime denotes \( d/d\bar{\tau}_v \), \( B_v \) is the thermal radiation at the local dust temperature, and

\[
S_{\text{dir},v} = \frac{1 - e_v}{e_v} \frac{L_v}{4\pi r^2} e^{-\bar{\tau}_v} \tag{A2}
\]

is the additional source term. With the inner boundary condition \( J'(0) = 0 \) corresponding to no net scattered flux at the origin (we take \( \bar{\tau}_v \) increasing outward), equation (A1) has the explicit solution

\[
J_v = C_v \cosh(\bar{\tau}_v) + \int_0^{\bar{\tau}_v} [S_{\text{dir},v}(\bar{\tau}') + B_v(\bar{\tau}')] \sinh(\bar{\tau}_v - \bar{\tau}') d\bar{\tau}'. \tag{A3}
\]

The integration constant \( C_v \) is determined by the condition of zero incoming flux at the outer boundary. In the two-stream approximation as described by Rybicki and Lightman this condition is \( 3^{1/2}J_v + dJ/d\bar{\tau}_v = 0 \), which corresponds to \( J_v + \epsilon_v^{1/2}J'_v = 0 \), at the maximum effective optical depth \( \bar{\tau}_{v,\text{max}} \). This implies

\[
C_v = \frac{\int_0^{\bar{\tau}_{v,\text{max}}} \sinh(\bar{\tau}_{v,\text{max}} - \bar{\tau}') + \epsilon_v^{1/2} \cosh(\bar{\tau}_{v,\text{max}} - \bar{\tau}') [S_{\text{dir},v}(\bar{\tau}') + B_v(\bar{\tau}')] \sinh(\bar{\tau}_v - \bar{\tau}') d\bar{\tau}'}{\cosh(\bar{\tau}_{v,\text{max}}) + \epsilon_v^{1/2} \sinh(\bar{\tau}_{v,\text{max}})}.
\tag{A4}
\]

Because the star is much hotter than the dust, and we are concerned here with the peak frequencies for scattered starlight, we neglect \( B_v \) in practice. This has the benefit that equations (A3) and (A4) can be evaluated directly, without solving self-consistently for the dust temperature distribution.

Our goals involve the radiation force due to scattered starlight. For this we require the scattered flux, which in Eddington’s approximation is given by \( F_{\text{sc},v} = -(4\pi/3)dJ_v/d\tau_v = -4\pi(\epsilon_v/3)^{1/2}J'_v \), where

\[
J'_v(\bar{\tau}_v) = C_v \sinh(\bar{\tau}_v) - \int_0^{\bar{\tau}_v} [S_{\text{dir},v}(\bar{\tau}') + B_v(\bar{\tau}')] \cosh(\bar{\tau}_v - \bar{\tau}') d\bar{\tau}'.
\]

The luminosity of this radiation is \( L_{\text{sc},v} = 4\pi r^2 F_{\text{sc},v} \), and its differential force is \( d^2F_{\text{sc},v} = (L_{\text{sc},v}(\epsilon_v) d\tau_v \) Its normalized total force at frequency \( \nu \) is then \( F_{\text{sc},\nu} = \mathcal{F}_{\text{sc},\nu}/L_{\text{sc},\nu} \), and its force-averaged radius is \( \langle r \rangle \mathcal{F}_{\text{sc},\nu} = \int r d^2F_{\text{sc},\nu} / \int d^2F_{\text{sc},\nu} \). Another important is the relation between \( r \) and \( \tau_v \), for which the simple cloud model \( p = \rho \Delta m/\Delta \tau \) implies \( r = r_{in} (1 - \tau_v/\tau_{\text{in},\nu})^{-1/\epsilon_v} \) where \( \tau_{\text{in},\nu} = \rho \Delta m \Delta \tau / (k/\lambda - 1) \). Finally, we have two equivalent expressions for the maximum optical depth:

\[
\tau_{v,\text{max}} \approx 1 - \left( \frac{r_{in}}{r_{\text{out}}} \right)^{k-1} \tau_{\text{in},\nu} \approx \frac{\nu^{1/2}}{\kappa_{\text{lfid}}}.
\]

The outcome of this analysis is plotted in Figure A1 for our fiducial case \( k = 1.5, r_{\text{out}} = 4r_{in} \). We see that the force is increased by about a factor of two over the direct force of starlight for reasonably high albedos, and that the force is applied at a location that is at most a couple times the inner boundary, decreasing toward \( r_{in} \) as the optical depth increases. Both of these trends are markedly different from the diffusive force due to dust emission.

A1 Limit of high optical depths

The case of a very optically thick cloud (to starlight) is of particular interest, as scattering is most important in this regime. So long as the cloud is also effectively optically thick, so that \( \bar{\tau}_{v,\text{max}} \gg 1 \), the force due to scattered light becomes analytical, because all the light is absorbed near the inner boundary and one can take \( r = r_{in} \) independent of \( \tau_v \). Then \( S_{\text{dir},v} = S_0 e^{-\tau_v} \) where \( S_0 = (\epsilon_v^{-1} - 1) L_{\text{sc},\nu}/(4\pi r_{\text{in}}^2) \). The requirement \( J' \rightarrow 0 \) as \( \bar{\tau}_v \rightarrow \infty \) implies \( C_v = \int_0^\infty S(\bar{\tau}_v) e^{-\tau_v} d\bar{\tau}_v = S_0/[1 + (3\epsilon_v)^{-1/2}] \), so \( J' = -S_0 (3\epsilon_v)^{1/2} (e^{-\tau_v} - e^{-\bar{\tau}_v})/(1 - 3\epsilon_v) \). Computing the force and comparing to the photon momentum, the ratio is

\[
\Phi_{\text{sc},\nu} = \frac{1 - \epsilon_v}{1 + (3\epsilon_v)^{1/2}}.
\tag{A5}
\]
and in this limit, all of the starlight momentum is transferred to the cloud \( \Phi_{\text{grav}} = 1 \). Equation (A5) shows that when scattered radiation is absorbed near the inner boundary, its force is comparable to that of the direct radiation. (For the net force of scattered radiation to become significantly larger, the scattered photons must penetrate beyond the inner radius.)

We note that, at the inner boundary, the mean intensity of scattered radiation is \( \left( 1 - e_\nu \right) / \left( e_\nu + \sqrt{e_\nu} / 3 \right) \) times that of the direct starlight in this limit. Backscatter should therefore push the sublimation radius outward by the factor \( \left( 1 + \sqrt{3} e_\nu / (1 + 1 / \sqrt{3} e_\nu) \right)^{1/2} \).

**APPENDIX B: DIFFUSION OF THERMAL INFRARED LIGHT**

At the risk of rehashing familiar material (e.g., Krumholz and Matzner 2009, eq. 33), we consider \( r_\tau(T_{\text{dust, in}}) \gg 1 \) and work in the diffusion approximation, \( dP_{\text{rad}} = -L dT_R / (4 \pi \nu c) \) where \( dT_R = \kappa_R dR \) for Rosseland opacity \( \kappa_R(T) \). Integration yields the profile of temperature and radiation pressure

\[
P_{\text{rad}} = \frac{dP_{\text{rad}}}{dR} = \frac{L}{4 \pi c} \int_{r_{\text{ph}}}^{1/r} \rho (r^{-1}) \, dr.
\]

The effective photosphere \( r_{\text{ph}} \) is the radius from which the Rosseland optical depth to infinity is roughly unity and the temperature is set by the Stefan-Boltzmann relation \( L \approx 4 \pi r_{\text{ph}}^2 \sigma_{\text{SB}} T_{\text{ph}}^4 \) so that \( P_{\text{rad}}(r_{\text{ph}}) \approx L / (3 \pi r_{\text{ph}}^3 \kappa_R(r_{\text{ph}})) \). Note that, for high enough optical depth, \( r_{\text{ph}} \) approaches the outer boundary \( r_{\text{out}} \) if one exists. If we adopt an opacity power law \( \kappa_R(T) \propto T^\beta \) (in addition to the density power law \( \rho(\tau) \propto \tau^\gamma \) then, integrating equation (B1),

\[
P_{\text{rad}} = \frac{1 - \beta/4}{1 + k} \frac{L \rho}{4 \pi (1 + k) c r} \left[ 1 - \left( \frac{r}{r_{\text{ph}}} \right)^{k+1} \right] + \frac{L}{3 \pi r_{\text{ph}}^3 \kappa_R(r_{\text{ph}})}.
\]

The last term is of order \( (r_{\text{ph}}/r)^2 (\kappa_R/T)^{-1} \) relative to everything else: it can safely be ignored in regions of high optical depth. Likewise the second term in brackets is negligible for \( (r_{\text{ph}}/r)^{k+1} \ll 1 \), so it can often neglected near the inner boundary. One is then left with the inner power law profile

\[
P_{\text{rad}} \approx \frac{1 - \beta/4}{1 + k} L \rho \frac{r_{\text{ph}}}{4 \pi (1 + k) c r}
\]

in which \( T \propto r^{-k(1+k)/(1+k)} \).

The force distribution is particularly simple as \( dF_{\text{IR}} = (L/c) dT_R \) in the diffusion approximation, so \( F_{\text{IR}} = r_\tau \) modulo a small offset arising from the photosphere and optically thin region. The inner power law solution suffices to estimate \( F_{\text{IR}} \), because the force is concentrated in the densest, hottest regions near the inner boundary; this gives equation (10).

The force-averaged radius \( \langle r \rangle_F \) is also determined by central conditions, although not at the same degree as \( \Phi_{\text{IR}} \): \( \langle r \rangle_F \) takes the optically thick limit

\[
\langle r \rangle_{\text{thick}} = \frac{\int_{r_{\text{ph}}}^{r_m} \left[ 1 - (r/r_{\text{ph}})^{k+1} \right] \frac{\rho}{\kappa_R} \left[ 1 - \left( \frac{r_{\text{ph}}}{r} \right)^{k+1} \right] \, dr}{\int_{r_{\text{ph}}}^{r_m} \left[ 1 - (r/r_{\text{ph}})^{k+1} \right] \frac{\rho}{\kappa_R} \left[ 1 - \left( \frac{r_{\text{ph}}}{r} \right)^{k+1} \right] \, dr}
\]

\[
\appropto \frac{2(k - 1) + \beta}{4 - \beta} r_{\text{in}}.
\]

The second expression is valid only insofar as the inner power law solution holds to radii well beyond \( \langle r \rangle_F \), and so should not be used in our fiducial problem.

**REFERENCES**

F. C. Adams and F. H. Shu. Infrared emission from protostars. ApJ, 296:655–669, September 1985. doi:10.1086/153448.

G. H. Bowen and L. A. Wilson. From wind to superson - The evolution of mass-loss rates for Mira models. Apj, 375:553–569, 1991. doi:10.1086/170021.

S. Chakrabarti and C. F. McKee. Far-Infrared SEDs of Embedded Protostars and Dusty Galaxies. I. Theory for Spherical Sources. ApJ, 631:792–808, October 2005. doi:10.1086/436259.

T. B. Draine. Interstellar Dust Grains. ARAA, 41:241–289, 2003a. doi:10.1146/annurev.astro.41.011802.094840.

T. B. Draine. Scattering by Interstellar Dust Grains. II. X-Rays. ApJ, 598:1026–1037, December 2003b. doi:10.1086/379123.

B. Draine. On Radiation Pressure in Static, Dusty H II Regions. ApJ, 732:100, May 2011. doi:10.1088/0004-637X/732/2/100.

S. M. Fall, M. R. Krumholz, and C. D. Matzner. Stellar Feedback in Molecular Clouds and its Influence on the Mass Function of Young Star Clusters. ApJ, 710:L142–L146, February 2010. doi:10.1088/2041-8205/710/2/L142.

T. J. Harries. Radiation-hydrodynamical simulations of massive star formation using Monte Carlo radiative transfer - I. Algorithms and numerical methods. MNRAS, 448:3156–3166, April 2015. doi:10.1093/mnras/stv158.

P. F. Hopkins, E. Quataert, and N. Murray. Stellar feedback in galaxies and the origin of galaxy-scale winds. MNRAS, 421:3522–3537, April 2012. doi:10.1111/j.1365-2966.2012.20593.x.

I. Ivezic and M. Elitzur. Infrared emission and dynamics of outflows in late-type stars. ApJ, 445:415–432, May 1995. doi:10.1086/175707.

I. Ivezic and M. Elitzur. Self-similarity and scaling behaviour of infrared emission from radiatively heated dust - I. Theory. MNRAS, 287:799–811, June 1997. doi:10.1093/mnras/287.4.799.

I. Ivezic, M. Nenkova, and M. Elitzur. User Manual for DUSTY. ArXiv Astrophysics e-prints, October 1999.

K. J. Jorgensen, F. L. Schöier, and E. F. van Dishoeck. Physical structure and CO abundance of low-mass protostellar envelopes. A&A, 389:908–910, July 2002. doi:10.1051/0004-6361:20020687.

M. R. Krumholz and C. D. Matzner. The Dynamics of Radiation-pressure-dominated H II Regions. ApJ, 703:1352–1362, October 2009. doi:10.1088/0004-637X/703/2/1352.

R. B. Larson. The emitted spectrum of a proto-star. MNRAS, 145:297, 1969. doi:10.1093/mnras/145.3.297.

L. A. Lopez, M. R. Krumholz, A. D. Bolatto, J. X. Prochaska, and E. Ramirez-Ruiz. What Drives the Expansion of Giant H II Regions?: A Study of Stellar Feedback in 30 Doradus. ApJ, 731:91, April 2011. doi:10.1088/0004-637X/731/2/91.

C. D. Matzner and P. H. Jumper. Star Cluster Formation with Stellar Feedback and Large-scale Inflow. ApJ, 815:68, December 2015. doi:10.1088/0004-637X/815/1/68.

C. F. McKee and E. G. Zweibel. On the virial theorem for turbulent molecular clouds. ApJ, 399:551–562, November 1992.
Radiation Forces on Dust Envelopes

K. E. Mueller, Y. L. Shirley, N. J. Evans, II, and H. R. Jacobson. The Physical Conditions for Massive Star Formation: Dust Continuum Maps and Modeling. ApJS, 143:469–497, December 2002. doi:10.1086/342881.

N. Murray, E. Quataert, and T. A. Thompson. The Disruption of Giant Molecular Clouds by Radiation Pressure and the Efficiency of Star Formation in Galaxies. ApJ, 709:191–209, January 2010. doi:10.1088/0004-637X/709/1/191.

N. Murray, B. Ménard, and T. A. Thompson. Radiation Pressure from Massive Star Clusters as a Launching Mechanism for Super-galactic Winds. ApJ, 735:66, July 2011. doi:10.1088/0004-637X/735/1/66.

T. P. Robitaille. HYPERION: an open-source parallelized three-dimensional dust continuum radiative transfer code. A&A, 536:A79, December 2011. doi:10.1051/0004-6361/201117150.

M. Rowan-Robinson. Radiative transfer in dust clouds. I - Hot-centered clouds associated with regions of massive star formation. ApJS, 44:403–426, November 1980. doi:10.1086/190698.

G. B. Rybicki and A. P. Lightman. Radiative Processes in Astrophysics, June 1986.

Y. L. Shirley, N. J. Evans, II, and J. M. C. Rawlings. Tracing the Mass during Low-Mass Star Formation. III. Models of the Submillimeter Dust Continuum Emission from Class 0 Protostars. ApJ, 575:337–353, August 2002. doi:10.1086/341286.

F. P. van der Tak, E. F. van Dishoeck, N. J. Evans, II, and G. A. Blake. Structure and Evolution of the Envelopes of Deeply Embedded Massive Young Stars. ApJ, 537:283–303, July 2000. doi:10.1086/309011.

M. G. Wolfire and J. P. Cassinelli. Conditions for the formation of massive stars. ApJ, 319:850–867, August 1987. doi:10.1086/165503.

S. C. Yeh and C. D. Matzner. Ionization Parameter as a Diagnostic of Radiation and Wind Pressures in H II Regions and Starburst Galaxies. ApJ, 757:108, October 2012. doi:10.1088/0004-637X/757/2/108.