In the electromagnetic fields $\mathbf{B} = (\omega t)^{-2} \mathbf{b}(r/\omega t)$, $\mathbf{E} = -\mathbf{r} \times \mathbf{B}/(ct)$ the trajectories of non-relativistic charged particles conserve $(\mathbf{r} - \mathbf{v}t)^2$. The transformation $\tilde{\mathbf{r}} = \mathbf{r}/(\omega t)$, $\tau = \omega^{-2}t^{-1}$ maps such trajectories into orbits in the constant magnetic field $\tilde{\mathbf{B}} = -\mathbf{b}(\tilde{\mathbf{r}})$ all of which conserve $\tilde{\mathbf{v}}^2$. $\omega$ is a constant. The transformation may also be used to transform any fields obeying curl $\mathbf{E} = -c^{-1}\partial\mathbf{B}/\partial t$, div $\mathbf{B} = 0$ into others and relates the particle trajectories in them.

Introduction

Newton’s equal areas in equal times theorem for non-coplanar motions$^{1,2}$ stimulated me to ask under what circumstances could the magnitudes of other vector constants of the motion remain constant even when extra forces were introduced which change their direction$^3$. In particular $(\mathbf{r} - \mathbf{v}t)^2$ is conserved when the force is of the form

$$\mathbf{F} = (q/c)(\mathbf{v} \times \mathbf{B} - \mathbf{r} \times \mathbf{B}/t)$$

(1)

for any $\mathbf{B}(\mathbf{r}, t)$, which may or may not have zero divergence as in electricity.

The orbits in such a field were derived analytically for the special case of the electromagnetic field$^3$

$$\mathbf{B} = (\omega t)^{-2} \mathbf{b}, \mathbf{E} = -\mathbf{r} \times \mathbf{B}/(ct)$$

with $\mathbf{b}$ constant and $\omega$ a constant of dimension $t^{-1}$ put in to make $\mathbf{B}$ and $\mathbf{b}$ have the same dimensions.

Here we demonstrate an unexpected correspondence between orbits under any force law of the form (1) and those under the conservative force law

$$\mathbf{F} = (\mathbf{\bar{v}}/c) \times \tilde{\mathbf{B}}$$

(2)
where
\[ \vec{r} = r/\theta, \quad \theta = \omega t, \quad \tau = \omega^{-1}/\theta, \quad \vec{v} = d\vec{r}/d\tau , \] (3)
and
\[ \vec{B}(\vec{r}, \tau) = -(\omega \tau)^{-2}\vec{B}(\vec{r}/\omega \tau, \omega^{-2}\tau^{-1}) = -\theta^2\vec{B}(\vec{r}, t) . \] (4)
The conservation of \((r - vt)^2\) under the force (1) translates into the conservation of \(\vec{v}^2\) under force (2), because
\[ \vec{v} = d\vec{r}/d\tau = -\omega \theta^2 d(r/\theta)/d\theta = r - vt \ . \]
The equation of motion of a unit mass particle of charge \(q\) under the force law (1) is,
\[ d^2\vec{r}/dt^2 = \left(q/c\right) \left[d\vec{r}/dt \times \vec{B} - \vec{r} \times \vec{B}/t\right] . \] (5)
Now under the transformation (3) equation (5) can be written
\[ \omega^2 \theta^3 d^2(\vec{r}\theta)/d\theta^2 = \left(q/c\right)\omega^2 \theta^2 d\vec{r}/d\theta \times \vec{B} . \] (6)
But \(\theta^3 d^2(\vec{r}\theta)/d\theta^2 = \theta^2 d/d\theta(\theta^2 d\vec{r}/d\theta) = \omega^{-2} d^2\vec{r}/d\tau^2\) so we may rewrite (6) in the form,
\[ d^2\vec{r}/d\tau^2 = \left(q/c\right)d\vec{r}/d\tau \times \vec{B} , \] (7)
where \(\vec{B}\) is given by (4). Equation (7) is the equation of motion of a unit mass particle of charge \(q\) in a field \(\vec{B}\) of magnetic type with \(\tau\) as the time. So if \(r = R(t)\) is a solution of (1), \(\vec{r} = \omega \tau R(\omega^{-2} \tau^{-1})\) solves (7).

**Application to Electromagnetic orbits**

Under what circumstances can forces of the form (1) be delivered on a charge \(q\) by electromagnetic fields? Clearly the first term in the bracket of (1) is of magnetic type (provided \(\vec{B}\) has zero divergence) and for the other to be electric we need
\[ \vec{E} = -\vec{r} \times \vec{B}/(ct) \]
Thus \( \text{Curl } E = (2B + r \cdot \nabla B)/(ct) \).

Putting this equal to \(-\frac{1}{c} \partial B/\partial t\) leads to
\[
t \partial / \partial t (Bt^2) + r \cdot \nabla (Bt^2) = 0
\]
of which the general solution is
\[
B = \theta^{-2} b(r/\theta)
\]  
(8)

where \( b \) is any (vector) function of its argument subject to Maxwell’s condition \( \text{div } b = 0 \). Under the transformation (4) we find that (8) reduces to \( \tilde{B} = -b(\tilde{r}) \), so \( \tilde{B} \) can be any stationary magnetic field.

Thus, if \( \tilde{r} = \tilde{r}(\tau) \) is the trajectory of a charged test particle that moves non-relativistically in any stationary magnetic field \( \tilde{B} = -b(\tilde{r}) \), then the electromagnetic fields \( B = \theta^{-2} b(r/\theta), \ E = -r \times B/(ct) \) satisfy Maxwell’s curl \( \text{curl } E = -(1/c) \partial B/\partial t, \ \text{div } B = 0 \) and in them there is a corresponding trajectory \( r = \theta \tilde{r}(\omega^{-1}/\theta) \) which conserves \( (r - vt)^2 \).

Neither Maxwell’s equations, nor the equations of motion involve a particular zero point for time, so the same arguments hold if we write to \( t-t_0 \) wherever we have written \( t \) above and the trajectory then preserves \( [r - v(t-t_0)]^2 \).

A specific example is given by particles trapped by a dipolar field to form Van Allen belts; the field is \( \tilde{B} = (-M + 3M \cdot \hat{r}/\hat{r})/\hat{r}^3 \) where \( \hat{r} \) is the unit vector \( \tilde{r}/\tilde{r} = r/r \). Our theorem relates the orbits \( \tilde{r} = \tilde{r}(\tau) \) in this constant dipole to the orbits \( r = \omega(t-t_0)\tilde{r} (\omega^{-2}(t-t_0)^{-1}) \) in the dipolar field \( B \) with moment \( -\omega(t-t_0)M \)
\[
B = \omega(t-t_0)(M - 3M \cdot \hat{r}/\hat{r})/\hat{r}^3
\]  
(9)

with \( E = -r \times M/(cr^3) \). Now in a uniform field the orbits expand as the field weakens but in this dipolar field we see the orbits shrink as \( t \) approaches \( t_0 \) and the dipole weakens. This is because \( \tilde{r} = \tilde{r}(\tau) \) is confined to the ‘radiation belts’ for all \( \tau \) so the whole of that motion is now shrunk by the initial factor \( (t-t_0) \) in \( r \). The physics behind this shrinking of the orbit with the dipole strength lies in the \( cE \times B/B^2 \) drift of the orbit. The \( E \), which is an inevitable consequence of the changing \( B \), is directed toroidally. For a decreasing dipole the drift is directed inwards. It is then of interest to
ask whether the field strength at the guiding centre increases or decreases as the dipole strength diminishes. For an equatorial gyration the drift velocity is \( v_d = r/(t - t_0) \) and \( DB^2/Dt = 2B \cdot (\partial B/\partial t + v_d \cdot \nabla B) = -4B^2/(t - t_0) \) which is clearly positive for \( t \leq t_0 \) when the field is decreasing. Thus the drift pushes the orbits into regions of higher field strength even though the field strength at a given position decreases. This result emphasises the importance of considering the electric fields inevitably associated with changing magnetic fields. These electric fields have been included in our theorem on changing orbits. In this particular example the electric field is steady and toroidal but its lack of change depends crucially on the \( r^{-3} \) dependence of the field strength of a dipole and in general the \( E \) field is time dependent. Of course (9) is only the correct magnetic field of a changing dipole in the ‘near zone’ in which its radiation may be neglected. Notice that \( \text{div } E = 0 \) so no charge density is associated with this ‘pure induction’ \( E \) field.

*Transformations of more general electromagnetic fields*

Although we found the above results by looking at forces that preserve \((r - vt)^2)\), we have arrived at a strange transformation that preserves the two Maxwell equations that do not involve sources, so the transformation can be applied to any electromagnetic field. Under the transformation

\[
\ddot{r} = r/(\omega t) \\
\omega t = (\omega t)^{-1} \\
\ddot{B}(\ddot{r}, \tau) = -((\omega t)^2B(r, t) = -(\omega t)^{-2}B(\ddot{r}/\omega \tau, \omega^{-2}\tau^{-1}) \\
\ddot{E}(\ddot{r}, \tau) = (\omega t)^3 [E(r, t) + r \times B(r, t)/ct] \\
= (\omega)^{-3}[E(\dddot{r}\omega^{-1}\tau^{-1}, \omega^{-2}\tau^{-1}) + \dddot{r} \times B(\dddot{r}\omega^{-1}\tau^{-1}, \omega^{-2}\tau^{-1})/c] \\
\text{we write } \partial/\partial r \text{ in place of } \nabla \text{ and deduce}
\]

| \(r = \dddot{r}/(\omega \tau)\) | \(\dddot{r} = \ddot{r}/(\omega \tau)\) | \(\ddot{r} = \dddot{r}/(\omega \tau)\) | \(\dddot{r} = \ddot{r}/(\omega \tau)\) | \(\ddot{r} = \dddot{r}/(\omega \tau)\) |
|---|---|---|---|---|
| \(\dddot{r} = \ddot{r}/(\omega \tau)\) | \(\ddot{r} = \dddot{r}/(\omega \tau)\) | \(\dddot{r} = \ddot{r}/(\omega \tau)\) | \(\dddot{r} = \ddot{r}/(\omega \tau)\) | \(\dddot{r} = \ddot{r}/(\omega \tau)\) |
\[ \frac{\partial}{\partial \tilde{r}} \cdot \tilde{B} = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial r} \cdot B = 0 \]
\[ \frac{\partial}{\partial \tilde{r}} \times \tilde{E} + \frac{1}{c} \frac{\partial \tilde{B}}{\partial \tau} = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial r} \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0 \]

and
\[ d^2 \tilde{r} / d\tau^2 = q \left[ \frac{\tilde{v}}{c} \times \tilde{B} + \tilde{E} \right] \quad \Leftrightarrow \quad d^2 r / dt^2 = q (v / c \times B + E) \]

Thus if \( E \) and \( B \) obey Maxwell's equations with suitable sources \( \rho, j \) then \( \tilde{E} \) and \( \tilde{B} \) will also (with other sources) and the trajectories of classical non-relativistic particles \( r = R(t) \) map via the transformation into the trajectories \( \tilde{r} = \omega \tau R(\omega^{-2} \tau^{-1}) \) of particles of the same charge/mass ratio under the fields \( \tilde{E} \) and \( \tilde{B} \). We notice that the field of an electric dipole is invariant while that of a magnetic monopole reverses under the transformation. However if we added a time reversal to the transformation then both the electric dipole and the magnetic monopole would be invariant. For a somewhat related theorem on the gravitational \( N \) body problem see\(^4\), but note there should be a dot over \( f \) the fourth time it appears there in the mathematics.
References

(1) I. Newton, *Principia, Scholium to Proposition 2* (R.Soc., London), 1687.

(2) D. Lynden-Bell & M. Nouri-Zonoz, Revs of Mod Phys, 70, 427, 1999.

(3) D.Lynden-Bell, The Observatory, (accepted), 2000.

(4) D.Lynden-Bell, The Observatory, 102, No 1048, 86, 1982.