SPAN OF THE JONES POLYNOMIALS OF CERTAIN V-ADEQUATE VIRTUAL LINKS

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Abstract. It is known that the Kauffman-Murasugi-Thislethwaite type inequality becomes an equality for any (possibly virtual) adequate link diagram. We refine this condition. As an application we obtain a criterion for virtual link diagram with exactly one virtual crossing to represent a properly virtual link.

1. Introduction

The Kauffman-Murasugi-Thislethwaite inequality \cite{4, 9, 10}, KMT inequality for short, is known as an effective tool to estimate, and in some cases determine, the minimal crossing number of (classical) links in terms of the span of the Jones polynomial (or equivalently of the Kauffman bracket polynomial). The KMT inequality is a strict inequality for some links, and the defect is closely related to the Euler characteristics of the Turaev surface \cite{11} (also known as the atom \cite{8}). Thus the KMT inequality has a refined form involving the Euler characteristics of the Turaev surfaces (Theorem 4.1 see also \cite{1, 2}). Moreover it is known that the refined KMT inequality becomes an equality for adequate link diagrams. The proofs of these results seem easier than that of the original KMT inequality, and in fact the refined inequality provides a simple proof of the Tait conjecture \cite{11}.

In this paper we refine this sufficient condition for (possibly virtual) link diagram under which condition the refined KMT inequality becomes an equality. We moreover introduce the notion of v-adequate link diagrams to be diagrams obtained by virtualizing exactly one real crossing of some adequate diagrams, and as an application we prove that the refined KMT inequality becomes an equality for v-adequate link diagrams with certain condition. This means that we can determine the span of the Jones polynomials of an adequate diagram and a v-adequate one obtained from the former, and this allows us to show that one of the span of the Jones polynomials of these links cannot be divided by four. We therefore obtain a recipe for producing properly virtual links. The “certain condition” is valid for v-adequate diagrams derived from classical adequate diagrams, and hence we indeed have infinitely many examples to which our criterion is applied. These examples can be seen as generalizations of Kishino’s result \cite{6}.

This paper is organized as follows. In \S2 we review the Kauffman bracket and the Turaev surface of (possibly virtual) link diagrams. We introduce the notion of (pseudo-) adequacy in \S3 and we prove our main theorems in \S4.

2. Preliminaries

We follow the conventions in \cite{5} for virtual link diagrams. Two virtual link diagrams $D$ and $D'$ are said to be equivalent if $D$ can be transformed into $D'$ by a finite sequence of Reidemeister moves and virtual Reidemeister moves \cite[Figure 2]{5}.

2.1. The Kauffman bracket polynomial. The transformations of a virtual link diagram in a neighborhood of a real crossing as in Figure 2.1 are called respectively $A$-splice and $B$-splice. We draw a dotted line at each spliced real crossing as in Figure 2.1 which we call a connecting arc.
Let $D$ be a virtual link diagram. A state $s$ of $D$ is a map from the set of the real crossings of $D$ to the set \{A, B\}. We denote by $S$ the set of the states of $D$. For $s \in S$, let $D(s)$ be the virtual link diagram obtained from $D$ by performing $s(p)$-splice at each real crossing $p$ of $D$. Define the weight of $s \in S$, denoted by $\langle D/s \rangle$, by

$$
\langle D/s \rangle := A^{\alpha(s)} - A^{-2}\beta(s) \in \mathbb{Z}[A, A^{-1}],
$$

where $\alpha(s) := \#s^{-1}(A)$, $\beta(s) := \#s^{-1}(B)$ and $\#D(s)$ is the number of components of $D(s)$. The Kauffman bracket polynomial $\langle D \rangle$ of $D$ is defined by

$$
\langle D \rangle := \sum_{s \in S} \langle D/s \rangle.
$$

For a Laurent polynomial $f \in \mathbb{Z}[A, A^{-1}]$ we denote the maximal (resp. the minimal) degree of $f$ by $\deg f$ (resp. $\deg f$), and define $\text{span}(f)$ as

$$
\text{span}(f) := \deg f - \deg f.
$$

It is well known that $\text{span}(D)$ is invariant under the generalized Reidemeister moves. For a virtual link $L$ we define $\text{span}(L)$ as $\text{span}(D)$ for any diagram $D$ representing $L$. The following property is well known.

**Lemma 2.1.** If $L$ is a classical link, then $\text{span}(L)$ can be divided by 4.

### 2.2. The Turaev surface.

Recall the definition of the Turaev surface of a virtual link diagram $D$ with $c(D) > 0$. We replace all the real crossings of $D$ with disks and join these disks with bands, each of which corresponds to an arc of $D$, so that the resulting surface $F'(D)$ is orientable and is almost embedded into $\mathbb{R}^2$, except for the neighborhoods of virtual crossings (Figure 2.2). The surface $F'(D)$ with boundary has been introduced in [3] and is called the fat frame of $D$. Around the real crossings of $D$, we color $\partial F'(D)$ in checkerboard manner so that the components corresponding to curves obtained by A-splice (resp. B-splice) is colored by red (resp. blue), as in Figure 2.3. We twist each band of $F'(D)$ if necessary so that the checkerboard colorings of $\partial F'(D)$ near the real crossings extend consistently to whole boundary, as in Figure 2.4. The resulting surface with boundary is called the twisted fat frame of $D$, and denoted by $F(D)$. Now let $T_D$ be the closed surface obtained by attaching disks along $\partial F(D)$, and call $T_D$ the Turaev surface of $D$.

**Remark 2.2.** By definition the Turaev surface of a trivial link diagram is a union of $S^2$.

**Remark 2.3.** $T_D$ is orientable for any classical link diagram [11]. This is not necessarily the case for virtual diagrams; see Figure 2.4 for example. It is not hard to see that $F(D) = F'(D)$, and hence $T_D$ is orientable, if $D$ is alternating.
Definition 2.4. For a virtual link diagram $D$, let $s_A$ (resp. $s_B$) be the state of $D$ that maps every real crossing of $D$ to A (resp. B).

Remark 2.5. By construction, the red boundary of $\partial F(D)$ correspond to $D(s_A)$, and the blue boundary of $\partial F(D)$ correspond to $D(s_B)$. Thus we have a one-to-one correspondence between the components of $D(s_A) \cup D(s_B)$ and those of $\partial F(D)$.

Lemma 2.6. For a virtual link diagram $D$, let $\chi(D)$ be the Euler characteristic of $T_D$. Then we have $\# D(s_A) + \# D(s_B) = \chi(D) + c(D)$.

Proof. $T_D$ can be decomposed into a cell complex; 0-cells are in one-one correspondence to the real crossings of $D$, 1-cells correspond to the arcs of $D$, and 2-cells correspond to the disks attached to $F(D)$ along the boundary. The number of arcs is equal to $2c(D)$, since $D$ can be seen as a 4-valent graph whose vertices are the real crossings of $D$ and whose edges are the arcs of $D$. Since the number of 2-cells is $\# D(s_A) + \# D(s_B)$ as mentioned in Remark 2.5, we have

$$\chi(D) = c(D) - 2c(D) + \# D(s_A) + \# D(s_B)$$

$$= -c(D) + \# D(s_A) + \# D(s_B).$$

3. PSEUDO-ADEQUATE DIAGRAMS AND V-PSEUDO-ADEQUATE DIAGRAMS

Definition 3.1. A virtual link diagram $D$ is said to be pseudo-adequate if, for any $s \in S$,

(a) $\# D(s_A) \geq \# D(s_A(1))$

(b) $\# D(s_B) \geq \# D(s_B(1))$

where $s_A(1)$ (resp. $s_B(1)$) is a state that sends one real crossing to B (resp. A) and all the other crossings to A (resp. B); for notations see §4.

Remark 3.2. A virtual link diagram $D$ is said to be A-adequate (resp. B-adequate) if the condition (a) (resp. (b)) in Definition 3.1 holds after replacing “≥” with “>”, and adequate if $D$ is A-adequate and B-adequate. $D$ is adequate if and only if the four components of $\partial F(D)$ near each real crossing $p$ are all distinct.
Figure 3.1. $H_n$ is pseudo-adequate but not adequate, and $H'_n$ is v-pseudo-adequate obtained from $H'_n()$.  

Figure 3.2. Admissible connecting arcs

Clearly an adequate diagram is pseudo-adequate. Any pseudo-adequate classical diagram is adequate, but this is not the case for virtual diagrams; the diagram $H_n$ in Figure 3.1 is pseudo-adequate but not adequate, because $\#H_n(s_A) = \#H_n(s_A(1)) = 1$ and $\#H_n(s_B) = \#H_n(s_B(1)) = 1$.

A virtualization of a real crossing is the replacement of the real crossing with a virtual crossing.

**Definition 3.3.** A v-adequate (resp. v-pseudo-adequate) diagram is a diagram $D$ obtained from an adequate (resp. a pseudo-adequate) diagram $D'$ by virtualizing one real crossing of $D'$.

**Example 3.4.** The diagram $H_n$ in Figure 3.1 is v-pseudo-adequate; indeed $H_n$ can be obtained from $H'_n$ by virtualizing one of its real crossings, and $H'_n$ is pseudo-adequate because $\#H'_n(s_A) = \#H'_n(s_A(1)) = 1$ while $\#H'_n(s_B) = \#H'_n(s_B(1)) = 1$.

**Definition 3.5.** For $s \in S$, a connecting arc $\gamma$ in $D(s)$ is said to be admissible if

1. $\gamma$ connect two distinct components of $D(s)$, or  
2. both the endpoint of $\gamma$ are on a single component of $D(s)$ and any orientation of the component looks as in Figure 3.2.

4. The refined KMT inequality

**Theorem 4.1** (11, see also [2, 8]). Let $D$ be a virtual link diagram representing a virtual link $L$, and $\chi(D)$ be the Euler characteristic of $T_D$. Then we have

$$\text{span}(L) \leq 4c(D) + 2(\chi(D) - 2).$$

It is known that the equality holds in Theorem 4.1 for adequate diagrams. This condition can be refined as follows:
Theorem 4.2. The equality holds in Theorem 4.1 for pseudo-adequate diagrams.

The above Theorem 4.2 deduces our main theorem.

Theorem 4.3. Let $D$ be a $v$-pseudo-adequate virtual link diagram obtained from a pseudo-adequate diagram $D'$ by replacing a real crossing $p$ with a virtual crossing $vp$. Let $C_A$ and $C_B$ be the components of $D(s_A)$ and $D(s_B)$ including $vp$ respectively. If any connecting arcs of $D(s_A)$ (resp. $D(s_B)$) whose endpoints are both on $C_A$ (resp. $C_B$) are admissible, then $\text{span}(D) = 4c(D) + 2\chi(D) - 2$.

Corollary 4.4. Let $D$ be a $v$-adequate diagram obtained from an adequate classical diagram $D'$. Then $D$ does not represent any classical link.

Proof. Any connecting arcs of $D(s_A)$ and $D(s_B)$ whose endpoints are both on $C_A$ and $C_B$ (Theorem 4.3) are admissible, because both $C_A$ and $C_B$ contain exactly one virtual crossing; see Figure 3.2. Therefore implies $\text{span}(D) = 4c(D) + 2\chi(D) - 2$.

On the other hand, because $D'$ is adequate (and hence pseudo-adequate), Theorem 4.2 implies $\text{span}(D') = 4c(D') + 2\chi(D') - 2$. Since $c(D') = c(D) + 1$, $\sharp(D(s_A)) = \sharp(D(s_A)) + 1$ and $\sharp(D'(s_B)) = \sharp(D(s_B)) + 1$, we obtain $\chi(D') = \chi(D) + 1$ by Lemma 5.6. Thus

\[ \text{span}(D') = 4(c(D) - 1) + 2(\chi(D) - 1 - 2) = 4c(D) + 2\chi(D) - 6 \]

Since $\text{span}(D')$ is divisible by 4 by Lemma 2.1, $\text{span}(D)$ cannot be divided by 4. Thus $D$ cannot represent any classical link.

Remark 4.5. In his master’s thesis, Kishino [6] proved that the span $\langle D \rangle = 4c(D) - 2$ if a diagram $D$ is obtained from a proper alternating classical diagram, and hence $D$ does not represent any classical link. Corollary 4.4 generalizes his result; any proper alternating diagram is adequate. Theorem 4.3 has another application to proper alternating virtual diagrams; see [3].

Example 4.6. The diagram $H_n$ in Figure 3.3 is pseudo-adequate and $v$-pseudo-adequate. Moreover, the endpoints of any connecting arc of $H_n(s_A)$ (resp. $K_n(s_B)$) are admissible. Thus both Theorems 4.2 and 4.3 can be applied to deduce $\text{span}(H_n) = 4c(H_n) + 2\chi(H_n) - 2$. Since $c(H_n) = n$ and we can see that $\sharp(H_n(s_A)) = \sharp(H_n(s_B)) = 1$, Lemma 2.6 tells us that $\chi(K_n) = 2 - n$ and hence span($K_n$) = 2n. In particular, if $n$ is odd, then span($H_n$) cannot be divided by 4 and therefore $H_n$ cannot represent any classical links.

The rest of this paper is devoted to the proofs of Theorems 4.2 and 4.3.

For arbitrarily chosen real crossings $p_1, \ldots, p_j (1 \leq j \leq c(D))$, let $s_A(j)$ (resp. $s_B(j)$) be a state of $D$ that maps $p_1, \ldots, p_j$ to $B$ (resp. $A$) and the other real crossings to $A$ (resp. $B$). Note that any $s \in S$ other than $s_A$ and $s_B$ can be expressed as $s_A(j)$ or $s_B(j)$. By definition

\[ \deg \langle D/s_A(j) \rangle = c(D) - 2j + 2\sharp(D(s_A(j))) - 2, \]
\[ \deg \langle D/s_B(j) \rangle = -c(D) + 2j - 2\sharp(D(s_B(j))) + 2. \]

Lemma 4.7. If $j \geq 1$, then we have

\[ \deg \langle D/s_A(j) \rangle \geq \deg \langle D/s_A \rangle, \quad \deg \langle D/s_B(j) \rangle \geq \deg \langle D/s_B \rangle. \]

Thus $\sup(\langle D \rangle) = \sup(\langle D/s_A \rangle) - \sup(\langle D/s_B \rangle)$.

Proof. Figure 4.1 shows

\[ \sharp(D(s_A(j) - 1)) - 1 \leq \sharp(D(s_A(j))) \leq \sharp(D(s_A(j) - 1)) + 1. \]

By 4.2 we inductively deduce

\[ \sharp(D(s_A(j))) \leq \sharp(D(s_A(j) - 1)) + 1 \leq \sharp(D(s_A(j) - 2)) + 2 \leq \cdots \leq \sharp(D(s_A) + j). \]
Similarly, we have
\[(4.5)\] \[\sharp D(s_B(j)) \leq \sharp D(s_B) + j.\]

By definition
\[(4.6)\] \[\deg \langle D/s_A \rangle = c(D) + 2\sharp D(s_A) - 2,\]
\[(4.7)\] \[\deg \langle D/s_B \rangle = -c(D) - 2\sharp D(s_B) + 2.\]

By (4.4)-(4.7),
\[\deg \langle D/s_A(j) \rangle = c(D) - 2j + 2\sharp D(s_A(j)) - 2\]
\[\leq c(D) + 2\sharp D(s_A) - 2,\]
\[\deg \langle D/s_B(j) \rangle = -c(D) - 2j - 2\sharp D(s_B(j)) + 2\]
\[\geq -c(D) - 2\sharp D(s_B) + 2.\]

These inequalities and (4.6), (4.7) imply
\[\deg \langle D \rangle \leq \deg \langle D/s_A \rangle \text{ and } \deg \langle D \rangle \geq \deg \langle D/s_B \rangle.\]

Thus we have \(\text{span} \langle D \rangle \leq \deg \langle D/s_A \rangle - \deg \langle D/s_B \rangle.\)

**Proof of Theorem 4.7.** Lemma 2.8 and Lemma 4.7 imply
\[(4.8)\] \[\text{span} \langle D \rangle \leq \deg \langle D/s_A \rangle - \deg \langle D/s_B \rangle\]
\[= 2c(D) + 2(\sharp D(s_A) + \sharp D(s_B)) - 4\]
\[= 2c(D) + 2(\chi(D) + c(D)) - 4\]
\[= 4c(D) + 2(\chi(D) - 2).\]

**Proof of Theorem 4.2.** Pseudo-adequacy of \(D\) implies
\[(4.9)\] \[\sharp D(s_A(1)) \leq \sharp D(s_A), \quad \sharp D(s_B(1)) \leq \sharp D(s_B).\]
Thus (4.10) and (4.11) can be sharpened in this case as
\[
\#D(s_A(j)) \leq \#D(s_B) + j - 1, \quad \#D(s_B(j)) \leq \#D(s_B) + j - 1.
\]
(4.11) and (4.10) imply
\[
\deg (D/s_A(j)) = c(D) - 2j + 2\#D(s_A(j)) - 2 \leq c(D) + 2\#D(s_A) - 4,
\]
\[
\deg (D/s_B(j)) = -c(D) + 2j - 2\#D(s_B(j)) + 2 \geq -c(D) - 2\#D(s_B) + 4.
\]
These estimations deduce \(\deg (D/s_A(j)) < \deg (D)\) and \(\deg (D/s_B(j)) > \deg (D)\) for any choices of \(p_1, \ldots, p_j\) \((j \geq 1)\), and hence we have \(\deg (D) = \deg (D/s_A)\) and \(\deg (D) = \deg (D/s_B)\). Thus the inequality in (4.3) becomes an equality. \(\Box\)

**Proof of Theorem 4.3.** In general, it is not hard to see that a connecting arc in \(D(s)\) \((s \in S)\) corresponding to a real crossing \(p\) is admissible if and only if \(\#D(s(1)) \leq \#D(s)\) where \(s(1)\) is the state of \(D\) obtained from \(s\) by changing splice at \(p\). Thus if all the connecting arcs of \(D(s_A)\) and those of \(D(s_B)\) are admissible, then \(D\) is pseudo-adequate.

If \(D\) is such a diagram as in Theorem 4.3 all the connecting arcs of \(D(s_A)\) and \(D(s_B)\) are admissible; because \(D^*\) is (pseudo-)adequate, the connecting arc one of whose endpoints is not on \(C_A\) nor \(C_B\) are admissible. By assumption the other arcs are also admissible. \(\Box\)

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