ENTROPY: MYSTERY AND CONTROVERSY
Plethora of Informational-Entropies and Unconventional Statistics

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Some general considerations on the notion of entropy in physics are presented. An attempt is made to clarify the question of the differentiation between physical entropy (the Clausius-Boltzmann one) and quantities called entropies associated to Information Theory, which are in fact generating functionals for the derivation of probability distributions and not thermodynamic functions of state. The role of them in the construction of the so-called Unconventional Statistical Mechanics, and the meaning and use of the latter, is discussed. Particular attention is given to the situation involving far-from-equilibrium systems.
1. INTRODUCTION: On Physical Entropy vs. Informational Entropies

Entropy – The Curse of Statistical Mechanics?, wrote Herman Haken at some point in his book on Synergetics [1]. G. N. Alekseev commented [2] that a popular-science author of the early twentieth century wrote that “Entropy is the name of the Queen’s shadow. Face to face with this phenomenon, Man cannot help feeling some vague fear: Entropy, like an evil spirit, tries to diminish or annihilate the best creations of that gracious spirit, Energy. We are all under the protection of Energy and all potential victims of the latent poison of Entropy ...”. On the other hand, we can recall the sentence of the great Ludwig Boltzmann that “Thus, the general struggle for life is neither a fight for basic material ... nor for energy ... but for entropy becoming available by the transition from the hot sun to the cold earth” [3].

Moreover, Louis de Broglie [4] expressed that “En Thermodynamique classique, on introduit pour énoncer le second principe de cette science la grandeur ‘Entropie’ dont la signification physique restait si obscure que Henry Poincaré la qualifiait de ‘prodigieusement abstraite’ ”. As expressed by de Broglie entropy is related to the second law of Thermodynamics, and on that the French philosopher Henri Bergson [5] called this second law (or Entropy law) the most “metaphysical” of all the laws of Nature. Georgescu-Roegen [6] commented that from the epistemological point of view, the Entropy Law may be regarded as the greatest transformation ever suffered in physics, and that it is the basis of the economy of life at all levels. Also, that nobody is able to live without the sensation of the flux of entropy, that is, of that sensation that under diverse forms regulate the activities related to the maintenance of the organism. In the case of mammals this includes not only the sensations of cold and warm, but, also, the pangs of hunger or the satisfaction after a meal, the sensations of being tired or rested, and many others of the same type.

Returning to de Broglie [4], he wrote that “C’est Boltzmann qui, en développant les idées de la Thermodynamique statistique, nos a donné le véritable sens de cette grandeur en montrant que l’entropie $S$ de l’état d’un corps est reliée à la probabilité $P$ de cet état par la célèbre formule: $S = k_B \ln P$”, and that “en Mécanique analytique [...] la véritable signification du principe de Hamilton est la suivante: Le mouvement d’un corps est celui que possède la plus grande probabilité thermodynamique dans les conditions auxquelles il est soumis. Je pense que cette conception de la nature profonde du principe de Hamilton jette une flot de lumière
sur sons véritable sens, analogue à celui que jette la formule de Boltzmann sur la signification de l’entropie”.

The law of increase of entropy addresses the question of irreversibility of processes in nature, that is, the famous “time-arrow” according to which all macroscopic events have a preferred time direction. The question has recently been addressed in two relevant papers in Physics Today. On the one hand, Joel Lebowitz [7] expressed that “Boltzmann’s thoughts on this question have withstood the test of time. Given the success of Ludwig Boltzmann’s statistical approach in explaining the observed irreversible behavior of macroscopic systems in a manner consistent with their reversible microscopic dynamics, it is quite surprising that there is still so much confusion about the problem of irreversibility”. In the other article, by Elliot Lieb and Jacob Yngvason [8], it is raised the point that “the existence of entropy, and its increase, can be understood without reference to either statistical mechanics or heat engines”.

Points of view associated to the latter one go back in time, and it can be mentioned the approach of Elias Gyftopoulos et al. [9, 10]: The basic idea seems to consist in that, instead of regarding mechanics and thermodynamics as two different theories, belonging to two different levels of fundamentality, can be considered a new hypothesis, namely, that mechanics and thermodynamics are two particular aspects of a more general fundamental theory [11]. On entropy it has been noticed [9] that “In his extensive and authoritative review, Wehr [12] writes ‘It is paradoxical that although entropy is one of the most important quantities in physics, its main properties are rarely listed in the usual textbooks on statistical mechanics’… The lack of specificity has resulted in a plethora of expressions purporting to represent the entropy of thermodynamics (emphasis is ours), and perhaps influenced von Neumann to respond to Shannon’s question ‘what should I call − \( \sum p_i \ln p_i \)’ by saying ‘You should call it ‘entropy’ for two reasons: first, the function is already in use in thermodynamics under that name; second, and more importantly, most people don’t know what entropy really is, and if you use that word entropy you will win every time’ ”. Moreover, [13] that quantum theory admits not only probability distributions described by wave functions or projectors, but also probability distributions that are described by density operators \( \rho \) which are not statistical averages of projectors. Said differently, \( \rho \) can be represented only by a homogeneous ensemble, namely, an ensemble every member of which is characterized by the same \( \rho \) as the whole ensemble. For projectors (wave functions) the concept of a homogeneous ensemble was conceived by von Neumann. This discovery plus all the requirements that the entropy of
thermodynamics must satisfy both for reversible and irreversible processes yield
that the only expression for entropy is \( S = -k_B Tr \{ \rho \ln \rho \} \), provided that \( \rho \) is
representable by a homogeneous ensemble. In addition, the discovery requires a
new complete equation of motion different from the Schrödinger equation. Such
an equation has been conceived by Beretta [14]. In both, an unified quantum
theory of mechanics and thermodynamics, and in a novel exposition of thermody-
namics without reference to quantum mechanics, it is proved that entropy is an
intrinsic (inherent) property of each constituent of any system (both macroscopic
and microscopic), in any state (both thermodynamic equilibrium and not ther-
modynamic equilibrium) in the sense that inertial mass is an intrinsic property
of each constituent of any system in any state [15]. We also noticed that, Edwin
Jaynes showed that the expression above can be identified with Clausius' entropy
[16], and with Boltzmann-Planck expression for isolated systems in equilibrium,
namely, the celebrated \( S = k_B \ln W \).

On this, in a recent journalistic article by A. Cho in Science [17] we can read
that “Near the middle of Vienna’s sprawling Central Cemetery stands the im-
posing tomb of Ludwig Boltzmann, the 19th century Austrian physicist who first
connected the motions of atoms and molecules to temperature, pressure, and other
properties of macroscopic objects. Carved in the gravestone, a single short equa-
tion serves as the great man’s epitaph: \( S = k_B \ln W \). No less important than Ein-
stein’s \( E = mc^2 \), the equation provides the mathematical definition of entropy, a
measure of disorder that every physical system strives to maximize. The equation
serves as the cornerstone of “statistical mechanics”, and it has helped scientists
decipher phenomena ranging from the various states of matter to the behavior of
black holes to the chemistry of life”. In continuation A. Cho comments on the ex-
istence of proposals of new and supposedly extended forms of Boltzmann entropy,
apparently necessary to deal with nonextensive systems. This does not seem to
be correct [18–19], and we are facing here the question posted before that the lack
of specificity has resulted in a plethora of expressions purporting to represent the
entropy of thermodynamics [9]. The propagation of this idea about a more gen-
eral and universal expression for entropy, is a result of the confusion arising from
the fact that the proposed ones are informational entropies (that is, generating
functionals of probability distributions): They provide a practical and useful tool
for handling problems where the researcher does not have access to the complete
necessary information on the relevant – for the problem in hands – characteristics
of the system, for properly applying the well established Boltzmann-Gibbs quite
general, and physically and logically sound, theory. That is, it provides a way
out to contour the failure we do have to satisfy Fisher’s *Criterion of Sufficiency* in statistics of 1922 [20], as shown as we proceed. There are infinitely-many possible generating functionals that can be introduced for that purpose, what was done around the middle of past century by a school of statisticians. This lead to the development of unconventional statistics, which are useful for making predictions in the cases when the general and extremely successful Boltzmann-Gibbs statistics has its application impaired once, as noticed, the researcher cannot satisfy Fisher’s *Criterion of Sufficiency* [20], that is, we are unable to introduce in the calculations a proper description of the characteristics of the system that are relevant for the experiment under consideration.

As said a large number of possibilities (in principle infinitely-many) can be explored, and Peter Landsberg quite properly titled an article *Entropies Galore!* [21]. An infinite family is the one that can be derived from Csiszer’s general measure of cross-entropies (see for example [22]); other family has been proposed by Landsberg [23]; and specific informational entropies are, among others, the one of Skilling [24] – which can be used in mathematical economy –, and of Kaniadakis [25] who used it in the context of special relativity [26]. They, being generating functionals of probability distributions, give rise to particular forms of statistics as the one of next section which we have dubbed *Unconventional Statistical Mechanics* [27, 28]; we do also have so-called *Superstatistics* proposed by C. Beck and E. G. D. Cohen for dealing with nonequilibrium systems with a stationary state and intensive parameter fluctuations [29, 30]; what can be called *Kappa Statistics* [25, 26], and so on.

What we do actually have behind this question is the possibility to introduce of such sophisticated method in physics, more precisely in statistical mechanics, what we do try to describe in continuation.

2. UNCONVENTIONAL STATISTICAL MECHANICS

We begin calling the attention to the fact that Statistical Mechanics of many-body systems has a long and successful history. The introduction of the concept of probability in physics originated mainly from the fundamental essay of Laplace of 1825 [31], who incorporated and extended some earlier seminal ideas (see for example [32]). As well known, Statistical Mechanics attained the status of a well established discipline at the hands of Maxwell, Boltzmann, Gibbs, and others, and went through some steps related to changes, not in its fundamental structure, but just on the substrate provided by microscopic mechanics. Beginning with classical
dynamics, statistical mechanics incorporated – as they went appearing in the realm of Physics – relativistic dynamics and quantum dynamics. Its application to the case of systems in equilibrium proceeded rapidly and with exceptional success: equilibrium statistical mechanics gave – starting from the microscopic level – foundations to Thermostatics, and the possibility to build a Response Function Theory. Applications to nonequilibrium systems began, mainly, with the case of local equilibrium in the linear regime following the pioneering work of Lars Onsager [33] (see also [34]).

For systems arbitrarily deviated from equilibrium and governed by nonlinear kinetic laws, the derivation of an ensemble-like formalism proceeded at a slower pace than in the case of equilibrium, and somewhat cautiously, with a long list of distinguished scientists contributing to such development. It can be noticed that Statistical Mechanics gained in the fifties an alternative approach sustained on the basis of Information Theory [32, 35, 36, 37, 38, 39, 40, 41, 42]: It invoked the ideas of Information Theory accompanied with ideas of scientific inference [43, 44], and a variational principle (the latter being Jaynes’ principle of maximization of informational uncertainty – also referred-to as informational-entropy – and called MaxEnt for short), compounding from such point of view a theory dubbed as Predictive Statistical Mechanics [32, 35, 36, 37, 38, 39, 40, 43]. It should be noticed that this is not a new paradigm in Statistical Physics, but a quite useful and practical variational method which codifies the derivation of probability distributions, which can be obtained by either heuristic approaches or projection operator techniques [46, 47, 48, 49]. It is particularly advantageous to build nonequilibrium statistical ensembles, as done here, when it systematizes the relevant work on the subject that renowned scientists provided along the past century. The informational-based approach is quite successful in equilibrium and near equilibrium conditions [35, 36, 41, 42], and in the last decades has been, and is being, also applied to the construction of a generalized ensemble theory for systems arbitrarily away from equilibrium [46, 47, 48, 49]. The nonequilibrium statistical ensemble formalism (NESEF for short) provides mechanical-statistical foundations to irreversible thermodynamics (in the form of Informational Statistical Thermodynamics – IST for short) [50, 51, 52, 53], a nonlinear quantum kinetic theory [47, 48, 54] and a response function theory [48, 55] of a large scope for dealing with many-body systems arbitrarily away from equilibrium. NESEF has been applied with success to the study of a number of nonequilibrium situations in the physics of semiconductors (see for example the review article of Ref. [56]) and polymers [57], as well as to studies of complex behavior of boson
systems in, for example, biopolymers (e.g. Ref. [58]). It can also be noticed that the NESEF-based nonlinear quantum kinetic theory provides, as particular limiting cases, far-reaching generalizations of Boltzmann [59], Mori (together with statistical foundations for Mesoscopic Irreversible Thermodynamics [60] [61], and Navier-Stokes [62] equations and a, say, Informational Higher-Order Hydrodynamics, linear [63] and nonlinear [64].

NESEF is built within the scope of the variational method on the basis of the maximization of the informational-entropy in Boltzmann-Gibbs-Shannon-Jaynes sense, that is, the average of minus the logarithm of the time-dependent – i.e. depending on the irreversible evolution of the macroscopic state of the system – nonequilibrium statistical operator. It ought to be further emphasized that informational-entropy – a concept introduced by Shannon – is in fact the quantity of uncertainty of information, and has the role of a generating functional for the derivation of probability distributions (for tackling problems in Communication Theory, Physics, Mathematical Economics, and so on). There is one and only one situation when Shannon-Jaynes informational-entropy coincides with the true physical entropy of Clausius in thermodynamics, namely, the case of strict equilibrium; e.g. [16, 21, 65, 66]. For short, we shall refer to informational-entropy as infoentropy. As already noticed the variational approach produces the well established equilibrium statistical mechanics, and is providing a satisfactory formalism for describing nonequilibrium systems in a most general form. This Boltzmann-Gibbs Statistical Mechanics properly describes the macroscopic state of condensed matter systems, being a well established one and logically and physically sound, but in some kind of situations, for example, involving nanometric-scale systems with some type or other of fractal-like structures or systems with long-range space correlations, or particular long-time correlations, it becomes difficult to apply because of a deficiency in the proper knowledge of the characterization of the states of the system in the problem one is considering (at either the microscopic or/macroscopic or mesoscopic level). This is, say, a practical difficulty (a limitation of the researcher) in an otherwise general and highly successful physical theory.

In fact, in a classical and fundamental paper of 1922 [20] by Sir Ronald Fisher, titled “On the Mathematical Foundations of Theoretical Statistics”, are presented the basic criteria that a statistics should satisfy in order to provide valuable results, that is, reliable predictions. In what regards present day Statistical Mechanics in Physics two of them are of major relevance, namely the Criterion of Efficiency and the Criterion of Sufficiency. This is so because of particular constraints that impose recent developments in physical situations involving small
systems (nanotechnology, nanobiophysics, quantum dots and heterostructures in semiconductor devices, one-molecule transistors, fractal electrodes in microbatteries, and so on), where on the one hand the number of degrees of freedom entering in the statistics may be small, and on the other hand boundary conditions of a fractal-like character are present which strongly influence the properties of the system, what makes difficult to introduce sufficient information for deriving a proper Boltzmann-Gibbs probability distribution. Other cases when sufficiency is difficult to satisfy is the case of large systems of fluids whose hydrodynamic motion is beyond the domain of validity of the classical standard approach. It is then required the use of a nonlinear higher-order hydrodynamics, eventually including correlations and other variances (a typical example is the case of turbulent motion). Also we can mention other cases where long-range correlations have a relevant role (e.g. velocity distribution in clusters of galaxies at a cosmological size, or at a microscopic size the already mentioned case of one-molecule transistors where Coulomb interaction between carriers is not screened and then of long range creating strong correlations in space).

Hence, we may say that the proper use of Boltzmann-Gibbs statistics is simply impaired because of either a great difficulty to handle the required information relevant to the problem in hands, or incapacity on the part of the researcher to have a correct access to such information. Consequently, out of practical convenience or the force of circumstances, respectively, a way to circumvent this inconvenience in such kind of “anomalous” situations, consists to resort to the introduction of modified forms of the informational-entropy, that is, other than the quite general one of Shannon-Jaynes which leads to the well established statistics of Boltzmann-Gibbs. These modified infoentropies which are built in terms of the deficient characterization one does have of the system, are dependent on parameters – called information-entropic indexes, or infoentropic indexes for short with the understanding that refer to the infoentropy. We reiterate the fundamental fact that these infoentropies are generating functionals for the derivation of probabilities distributions, and are not at all to be confused with the physical entropy of the system.

As already noticed, this alternative approach originated in the decades of the 1950’s and 1960’s at the hands of theoretical statisticians, being extensively used in different disciplines (economy, queueing theory, regional and urban planning, nonlinear spectral analysis, and so on). Some approaches were adapted for use in physics, and we present here an overall picture leading to what can be called Unconventional Statistical Mechanics (USM for short and fully described in Refs.
consisting, as said before, in a way to patch the lack of knowledge of characteristics of the physical system which are relevant for properly determining one or other property (see also P. T. Landsberg in Refs. [21] and [23]) thus impairing the correct use of the conventional one.

Use of the variational MaxEnt for building NESEF provides a powerful, practical, and soundly-based procedure of a quite broad scope, which is encompassed in what is sometimes referred-to as Informational-Entropy Optimization Principles (see for example Ref. [22]). To be more precise we should say constrained optimization, that is, restricted by the constraints consisting in the available information. Such optimization is performed through calculus of variation with Lagrange’s method for finding the constrained extremum being the preferred one.

Jaynes’ variational method of maximization of the informational-statistical entropy is connected – via information theory in Shannon-Brillouin style – to a principle of maximization of uncertainty of information. This is the consequence of resorting to a principle of scientific objectivity [44, 43], which can be stated as: Out of all probability distributions consistent with a given set of constraints, we must take the one that has maximum uncertainty. Its use leads to the construction of a Gibbs’ ensemble formalism, recovering the traditional one in equilibrium [35, 36, 41], and allowing for the extension to systems far from equilibrium [42, 46, 47, 48, 49].

Jaynes’ MaxEnt is a major informational-entropy optimization principle requiring, as noticed, that we should use only the information which is accessible but scrupulously avoiding to use information not proven to be available. This is achieved by maximizing the uncertainty that remains after all the given information has been taken care of.

Jaynes’ MaxEnt aims at maximizing uncertainty when subjected to a set of constraints which depend on each particular situation (given values of observables and theoretical knowledge). But uncertainty can be a too deep and complex concept for admitting a unique measure under all conditions: We may face situations where uncertainty can be associated to different degrees of fuzziness in data and information. As already noticed, this is a consequence, in Statistical Mechanics, of a lack of a proper description of the physical situation. This corresponds to being violated the Criterion of Sufficiency in the characterization of the system (“the statistics chosen should summarize the whole of the relevant information supplied by the sample”) [20]. This could occur at the level of the microscopic dynamics (e.g. lack of knowledge of the proper eigenstates, all important in the calculations), or at the level of macroscopic dynamics (e.g. when we are forced, because
of deficiency of knowledge, to introduce a low-order truncation in the higher-order hydrodynamics that the situation may require). Hence, in these circumstances it may arise the necessity of introducing alternative types of statistics, with the accompanying indexed (or structural) informational-entropies, (infoentropies for short) different of the quite general and well established one of Boltzmann-Gibbs, as it has been discussed in the first section.
TABLE I: Informational-Statistical Entropies

| Conventional (Universal) ISE                                                                 | Unconventional (entropic-index-dependent) ISEs                                           |
|-------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------|
| Boltzmann-Gibbs-Shannon-Jaynes ISE (from Külback-Leibler measure)                         | From Havrda-Charvat measure                                                               |
|                                                                                           | \( \{- \ln \{ \text{Tr} \{ g \} \} \} \)                                               |
|                                                                                           | \( \{- \frac{1}{\alpha-1} \text{Tr} \{ g^\alpha \} \} \) \((\alpha > 0 \text{ and } \alpha \neq 1)\) |
|                                                                                           | From Sharma-Mittal measure                                                               |
|                                                                                           | \( \{- \frac{W_{\alpha-1}^{\beta-1}}{\alpha-\beta} \text{Tr} \{ [W_{\alpha-\beta}^\alpha g^\alpha - g^\beta] g^\beta-1 \} \} \) |
|                                                                                           | \((\alpha > 1, \beta \leq 1 \text{ or } \alpha < 1, \beta \geq 1)\)                     |
|                                                                                           | From Renyi measure                                                                      |
|                                                                                           | \( \{- \frac{1}{\alpha-1} \ln \text{Tr} \{ g^\alpha \} \} \) \((\alpha > 0 \text{ and } \alpha \neq 1)\) |
|                                                                                           | From Kapur measure                                                                      |
|                                                                                           | \( \{- \frac{1}{\alpha-\beta} \left[ \ln \text{Tr} \{ g^\alpha \} - \ln \text{Tr} \{ g^\beta \} \right] \} \) |
|                                                                                           | \((\alpha > 0, \beta > 0 \text{ and } \alpha \neq \beta)\)                             |

Applying MaxEnt to any of these unconventional infoentropies we obtain a probability distribution deemed appropriate for the given problem in hands, namely, either the conventional probability distribution when Shannon-Jaynes infoentropy is used, or the so-called in Pearsons’ nomenclature heterotypical probability distributions when other infoentropies are used. There are infinitely-many of these infoentropies, which, as said, are dependent on one or more parameters (the infoentropic indexes), and in Table I we list four of these informational-statistical entropies (ISE), plus the Shannon-Jaynes one (of Boltzmann-Gibbs type) which is parameter free as it should for being the most general one [22, 27, 48]. In Table I \( W^{-1} \) is the constant value of probability in the uniform distribution, and \( \alpha \) and \( \beta \) infoentropic indexes (open parameters).

Renyi’s approach appears to be a particularly convenient one for dealing with fractal systems as discussed in Ref. [65], where it is pointed out that predictions obtained resorting to the approach of maximization in Shannon-Jaynes approach including fractality can be equivalently obtained using Renyi’s approach ignoring fractality. Renyi’s ISE has been studied by Takens and Verbitski [67], and a variation of it is Hentschel-Procaccia infoentropy [68]. For the Havrda-Charvat
structural $\alpha$-entropy, one akin to the case $\alpha = 2$ has been considered by I. Prigogine in connection with practical and theoretical difficulties with Boltzmann ideas when extending them from the dilute gas to dense gases and liquids \[69\]. Prigogine argues that to cope with such situations one would need a statistical expression of entropy that depends explicitly on correlations, as is the case of the Havrda-Charvat structural $\alpha$-entropy for $\alpha = 2$ (also in the case of Renyi infoentropy).

It can be noticed that taking $\beta = 1$ reduces Kapur ISE to the one of Renyi, and Sharma-Mittal ISE to the one of Havrda-Charvat. Moreover, taking also $\alpha = 1$, is obtained an ISE which is of the form of Boltzmann-Gibbs-Shannon-Jaynes ISE. What we do have in these ISE’s, or in any other one of the infinitely-many which are possible, is that when the adjustment of the parameters (the infoentropic indexes) on which they depend – let it be in a calculation or as a result of the comparison with the experimental data – produces Boltzmann-Gibbs result, this gives an indication that the principle of sufficiency is being satisfied, i.e., for such particular situation the description of the system we are doing includes all the relevant characterization that properly determines the physical property that is measured in the given experiment being analyzed.

Moreover, we again stress the fundamental fact that the structural informational-entropies (quantity of uncertainty of information) are not to be confused with the Clausius-Boltzmann physical entropy: There is one and only one case when there is an equivalence, consisting of Shannon infoentropy when the system is strictly in equilibrium \[16, 21, 65\]. Boltzmann-Gibbs-Shannon-Jaynes nonequilibrium informational entropy and its role in NESEF is extensively discussed in Refs. \[48, 53, 70\].

It is quite relevant to notice that for each kind of statistical entropy it is necessary in an ad hoc manner, to introduce definitions of average values of observables with particular forms, what is required to obtain a posteriori consistent results. For the case of Kullback-Leibler measure, or Shannon-Jaynes statistical informational-entropy, we must use the usual expression, i.e. the average of quantity $\hat{A}$ is given by
\[
\langle \hat{A} \rangle = Tr \left\{ \hat{A} \varrho \right\} ,
\]
while for the case of Renyi ISE, needs be introduced an average of the form
\[
\langle \hat{A} \rangle = Tr \left\{ \hat{A} \hat{D}, \left\{ \varrho \right\} \right\} ,
\]
that is, in terms of the so-called escort probability of order $\gamma$ \[71, 72\]

$$D_\gamma \{\rho\} = \frac{\rho_\gamma}{\text{Tr} \{\rho_\gamma\}}, \quad (3)$$

which is also the one to be used in the case of Havrda-Charvat statistics. A detailed discussion on the role of escort probability is given in Ref. \[27\].

We call the attention to the fact that USM is to be based on the use of both definitions, namely, the heterotypical probability distribution and the escort probability (notice that for probability distributions other than Renyi and Havrda-Charvat other definitions of escort probabilities should be introduced). The role of the escort probability accompanying the heterotypical-probability distribution is that both complement each other in order to redefine, in the sense of weighting, the values of the probabilities associated to the physical states of the system \[27\].

Let us now consider the use of Renyi informational entropy, i.e.

$$S_\alpha (t) = -\frac{1}{\alpha - 1} \ln \text{Tr} \{ [\bar{\rho}_\alpha (t, 0)]^\alpha \}, \quad (4)$$

in the case of a fluid of single molecules driven out of equilibrium and in contact with ideal reservoirs; a more general situation is discussed in \[27\]. We recall that a recent application of Renyi’s statistics for dealing with (multi)fractal systems is presented by Jizba and Arimitsu \[65\]: there it is addressed the question on how Renyi’s approach appears as a quite convenient one in such cases. Further considerations on Renyi’s approach can be consulted in the articles by Hentschel and Procaccia \[68\] and Takens and Verbitski \[67\].

We proceed to derive the nonequilibrium statistical operator in Renyi’s approach, beginning to look for the “instantaneously frozen” auxiliary statistical operator which, in the situation above described, consists of the product of the one of the system, say $\bar{\rho}_\alpha (t, 0)$, times the constant one of the thermal bath and ideal reservoirs. Let us now look for $\bar{\rho}_\alpha (t, 0)$ which follows by maximizing $S_\alpha$ of Eq. (4) subjected to the conditions of normalization

$$\text{Tr} \{\bar{\rho}_\alpha (t, 0)\} = 1 \quad , \quad (5)$$

and the constraints consisting of the average values, say $Q_j (r, t)$, as defined by Eq. (2), of the basic dynamical variables, say $\{\hat{P}_j (r)\}$, namely

$$Q_j (r, t) = \text{Tr} \left\{ \hat{P}_j (r) \bar{D}_\alpha \{\bar{\rho} (t, 0)\} \right\} \quad , \quad (6)$$
where
\[
D_\alpha \{ \bar{\varrho} (t, 0) \} = [\bar{\varrho}_\alpha (t, 0)]^\alpha / Tr \{ [\bar{\varrho}_\alpha (t, 0)]^\alpha \},
\]
(7)
is the corresponding escort probability \[71, 72\] (cf. discussion after Eq. (3) above); it can be observed that the order of the escort probability is the same infoentropic index in Renyi’s informational entropy \[71\].

It follows that
\[
\bar{\varrho}_\alpha (t, 0) = \frac{1}{\bar{\eta}_\alpha (t)} \left[ 1 + (\alpha - 1) \sum_j \int d^3r \, F_{j\alpha} (r, t) \Delta \hat{P}_j (r, t) \right]^{-\frac{1}{\alpha - 1}},
\]
(8)
where
\[
\Delta \hat{P}_j (r, t) = \hat{P}_j (r) - Q_j (r, t)
\]
(9)
with \(Q_j (r, t)\) given in Eq. (6),
\[
\bar{\eta}_\alpha (t) = Tr \left\{ \left[ 1 + (\alpha - 1) \sum_j \int d^3r \, F_{j\alpha} (r, t) \Delta \hat{P}_j (r, t) \right]^{-\frac{1}{\alpha - 1}} \right\},
\]
(10)
ensures the normalization condition, and \(F_{j\alpha}\) are the Lagrange multipliers that the variational method introduces, which are related to the basic variables through Eq. (6).

In terms of the auxiliary \(\bar{\varrho}_\alpha\), the statistical distribution is given by \[46, 47, 48, 49\]
\[
\varrho_{\alpha \epsilon} (t) = \epsilon \int_{-\infty}^{t} dt' e^{\epsilon (t - t')} \bar{\varrho}_\alpha (t', t' - t)
\]
(11)
where, we recall,
\[
\bar{\varrho}_\alpha (t', t' - t) = \exp \left\{ -\frac{1}{i\hbar} (t' - t) \hat{H} \right\} \bar{\varrho}_\alpha (t', 0) \exp \left\{ \frac{1}{i\hbar} (t' - t) \hat{H} \right\}
\]
(12)
and \(\epsilon\) is a positive infinitesimal that goes to zero after the trace operation in the calculation of averages has been performed. The statistical distribution of Eq. (11) satisfies the Liouville equation
\[
\frac{\partial}{\partial t} \varrho_{\alpha \epsilon} (t) + \frac{1}{i\hbar} \left[ \varrho_{\alpha \epsilon} (t), \hat{H} \right] = -\epsilon \left[ \varrho_{\alpha \epsilon} (t) - \bar{\varrho}_\alpha (t, 0) \right]
\]
(13)
with the presence of the infinitesimal source introducing Bogoliubov’s symmetry breaking procedure (quasiaverages) \[73\], in the present case the one of time reversal and in that way are discarded the advanced solutions of the full Liouville equation. Thus, the retarded solutions have been selected, and, \textit{a posteriori}, this is transmitted to the kinetic equations producing a \textit{fading memory} and irreversible behavior (cf. Refs. \[48\] \[54\]).

We also call the attention to the fact that for average values, as given by Eq. (2), we then have

\[
\langle \hat{A} \rangle = Tr \left\{ \hat{A} \mathcal{D}_{\alpha} \{ \varrho_{\alpha} (t) \} \right\}, \tag{14}
\]

where

\[
\mathcal{D}_{\alpha} \{ \varrho_{\alpha} (t) \} = \frac{\varrho_{\alpha} (t)}{Tr \{ \varrho_{\alpha} (t) \}}, \tag{15}
\]

and it is implicit the limit $\epsilon \to 0$ after the calculation of traces has been performed.

Because of the boundary condition \[ \varrho_{\alpha} (t_o) = \bar{\varrho}_{\alpha} (t_o, 0) \ (t_o \to -\infty) \] \[48\] \[49\], we have that \[ \mathcal{D}_{\alpha} \{ \varrho_{\alpha} (t_o) \} = \mathcal{D}_{\alpha} \{ \bar{\varrho}_{\alpha} (t_o, 0) \} \], where \[ \mathcal{D}_{\alpha} \] is given by Eq. (7). For $\epsilon \to 0$, $\varrho_{\alpha}$ satisfies a true Liouville equation [cf. Eq. (13)], and so does \[ \mathcal{D}_{\alpha} \], and we recall that the infinitesimal source on the right-hand side of Eq. (13) is selecting the retarded solutions of the true Liouville equation (via, then, Bogoliubov’s method of quasiaverages, as previously noticed). Hence, for the given initial condition and the imposition of discarding the advanced solutions, \[ \mathcal{D}_{\alpha} \{ \varrho_{\alpha} (t) \} \] also satisfies a modified Liouville equation, and we can write

\[
\mathcal{D}_{\alpha} \{ \varrho_{\alpha} (t) \} = \bar{\mathcal{D}}_{\alpha} \{ \bar{\varrho}_{\alpha} (t, 0) \} + \mathcal{D}_{\alpha}^{r} \{ \hat{\mathcal{D}}_{\alpha} \{ \bar{\varrho}_{\alpha} (t) \} \}, \tag{16}
\]

where \[ \bar{\mathcal{D}}_{\alpha} \{ \bar{\varrho}_{\alpha} (t, 0) \} \] is given by Eq. (7), and

\[
\mathcal{D}_{\alpha}^{r} \{ \hat{\mathcal{D}}_{\alpha} \{ \bar{\varrho}_{\alpha} (t) \} \} = - \int_{-\infty}^{t} dt' e^{\epsilon(t-t')} \frac{d}{dt'} \bar{\mathcal{D}}_{\alpha} \{ \bar{\varrho}_{\alpha} (t', t' - t) \}. \tag{17}
\]

Introducing Eq. (16) into Eq. (14), we can see that the averages are composed of an “instantaneously frozen” (at time $t$) contribution, plus a contribution associated to the irreversible processes and including historicity. For the basic dynamical quantities, and \textit{only} for them, it follows that

\[
Q_j (r, t) = Tr \left\{ \hat{P}_j \mathcal{D}_{\alpha} \{ \varrho_{\alpha} (t) \} \right\} = Tr \left\{ \hat{P}_j \bar{\mathcal{D}}_{\alpha} \{ \bar{\varrho}_{\alpha} (t) \} \right\}. \tag{18}
\]
with, as already noticed, being implicit the limit of $\epsilon$ going to $+0$ to be taken after the calculation of the trace operation has been performed [46, 47, 48, 49].

After the nonequilibrium distribution using a heterotypical index-dependent information-al-entropy has been derived, next step – like done in the conventional case [48, 53, 54, 55, 70, 74] – should consists in deriving for arbitrarily far-from-equilibrium systems, a nonlinear quantum kinetic theory, a response function theory, and, of course, a systematic study of experimental results, that is, a full collection of measurements of diverse properties of the system, amenable to be studied in terms of structural (infoentropic-index dependent) informational-entropies, what is fundamental for the validation of the theory (see for example Refs. [28, 75, 76, 77, 78]).

3. IRREVERSIBLE THERMODYNAMICS AND ENTROPY

As know, nonequilibrium thermodynamics is involved with finite and irreversible processes, and, for that reason, it is also denominated Irreversible Thermodynamics. As a general rule special interest is currently focused on the case of open systems, which are coupled to external sources which supply them energy and matter. This manifests itself in macroscopic changes of the thermodynamic variables which naturally become dependent on space position and on time.

The principal characteristic of the processes involved, whether they are stationary or time dependent, is that they evolve with a positive production of internal entropy. Here we make contact with the second fundamental principle of thermodynamics. According to Planck [79], “The second law of thermodynamics is essentially different from the first law, since it deals with a question in no way touched upon by the first law, viz., the direction in which a process takes place in nature”. The second law has several equivalent formulations: the one due to Clausius refers to heat conduction, and, again according to Planck [79], it can be expressed as “heat cannot by itself pass from a cold to a hot body. As Clausius repeatedly and expressly pointed out, this principle does not merely say that heat does not flow from a cold to a hot body – that is self-evident, and is a condition of the definition of temperature – but it expressly states that heat can in no way and by no process be transported from a colder to a warmer body without leaving further changes, i.e. without compensation”.

The concept of irreversibility clearly appears in Clausius’ work, who can be considered the founder of Thermodynamics as an autonomous and unified sci-
ence. His presentation of the second law appears precisely as a criterion for evolution, governing irreversible behavior. It implies the definition and use of the extremely difficult concept of the title, namely, *entropy*. Using the usual notation, for an infinitesimal process the state function entropy \( S \) suffers a modification \( dS = \vartheta Q/T \) if the process is reversible, while for irreversible processes there follows \( dS > \vartheta Q/T \), where \( \vartheta Q \) is the heat exchange in the process and \( T \) the absolute temperature, and \( \vartheta \) stands for a nonexact differential (thus, \( 1/T \) acts as an integrating factor). The last inequality can be expressed in the form of the balance equation

\[
dS = \frac{\vartheta Q}{T} + \frac{\vartheta Q'}{T}
\]

with \( \vartheta Q' > 0 \). In this form of the second law, the quantity \( \vartheta Q' \), called by Clausius *uncompensated heat*, can be considered as a first evaluation of the degree of irreversibility of a natural process.

Equilibrium thermodynamics describes states of matter that are greatly privileged: Planck [79] has emphasized that the second law distinguishes among the several types of states in nature, some of which act as attractors to others: Irreversibility is an expression of this attraction. For systems together with the reservoirs to which they are connected there is an attraction towards equilibrium (thermodynamic potentials at a minimum value compatible with the constraints imposed by the reservoirs). In nonequilibrium systems, while in a stationary state in a linear regime near equilibrium, the attractor is the state of minimum internal entropy production [53].

According to Eq. (19), the change of entropy is composed of two separate types of contributions: the term \( \frac{\vartheta Q}{T} \) related to the exchange of heat with the surroundings, expressed as \( d_e S \), and the term \( \frac{\vartheta Q'}{T} \), due exclusively to the irreversible processes that develop in the interior of the system, expressed as \( d_i S \) so we can write

\[
dS = d_e S + d_i S, \quad \text{with} \quad d_i S \geq 0.
\]

Equation (20) provides the framework for an entropy equation of balance, equivalent to Eq. (19). Let us emphasize that the term \( d_e S \) involves all the contributions resulting from exchanges (energy, matter, etc.) with the environment.

After what has been manifested in Section 1, it is certainly a truism to say that entropy has a very special status in physics, expressing, in a very general way, the tendency of physical systems to evolve in an irreversible way, characterizing the eventual attainment of equilibrium, and given a kind of measurement of the order that prevails in the system. It is important to further stress that it is a very well
established concept in equilibrium situations, however requiring an extension and clear comprehension in the case of open systems, mainly in far-from-equilibrium conditions, a question that remains controversial. On the basis of the use of entropy as a state function, the properties of systems in equilibrium are very well described when in conjunction with the two fundamental laws. Let us add some additional consideration to what have been said in Section 1.

Lawrence Sklar [80] has noted that the concept of entropy is the most purely thermodynamic concept of all, and J. Bricmont [81] has commented that there is some kind of mystique about entropy. Again according to Sklar, given the abstractness of entropy and its high place up in the theory as well as its unrelat-
edness to immediate sensory qualities or primitive measurements (as temperature is related to these), it is not surprising than in seeking the statistical mechanical correlate of nonequilibrium thermodynamic entropy we have the least guidance from the surrounding embedding theory. Exists an openness in what to choose as the surrogate for entropy. Thus, it should be expected to arise a wide variety (plethora) of “entropies”, each functioning well for the specific purposes for which it was introduced. J. Meixner [82] asks: Is the concept of nonequilibrium entropy superfluous?, for in continuation to comment that one is so much accustomed to the concept of entropy that one would like to retain it as a quantity of physical significance. He also points to the difficulty of a definition if one does not have a clearly defined physical state of the system. This is the main difficulty, as also pointed out by Bricmont in that we may define as many entropies as we can find sets of macroscopic variables. Also, with the coarse-graining procedure there is not a sharp distinction between microscopy and macroscopy, passing through mesoscopy, so that we can arrive at many values of the say “entropy”, including arriving at the zero value when a complete microscopic description is given, and we have a pure mechanical description and no thermodynamics exists ([82] [83], see also Jaynes in [16]). Jaynes rightly says that he does not know what is the entropy of a cat; the problem being that we are unable to precisely define a set of macrovariables that properly specify the thermodynamic state of the cat.

At this point it is worth to present, with some modifications, several quite appropriate remarks made by Bricmont in the cited reference, which we roughly summarize here:

(i) These entropies are not subjective but objective as are the corresponding macroscopic variables. Called “anthropomorphic” by Jaynes following Wigner, they may be referred to as “contextual”, i.e. they depend on the physical situation and on its level of description.
(ii) The “usual” or “traditional” entropy of Clausius corresponds to the particular choice of macroscopic variables for a free monatomic gas in equilibrium (energy, specific volume, number of particles). The derivative with respect to the energy of that entropy defines the reciprocal of the Kelvin’s absolute temperature.

(iii) The second law in the form “Entropy increases” becomes undetermined: which entropy? Between two states of equilibrium is the Clausius’ one. Otherwise is not clear.

(iv) Whichever the chosen functional form for an entropy, in most cases is hard to compute or estimate. One needs to begin with the equations of evolution for the chosen basic variables and solve them for appropriate initial and boundary conditions. Moreover, irreversibility as characterized by some $\mathcal{H}$-theorem – as the one of Boltzmann – does not directly relate the $\mathcal{H}$-function to whatever may be the entropy. It only ensures that the choice of the initial condition and some ad hoc nonmechanical hypotheses (a Stosszahlansatz) gives a time-arrow and relaxation towards final equilibrium.

(v) There is no difficulty with Liouville theorem of invariance of extension in phase space; this is a purely mechanical result. At the statistical level the extent of the volume of space points, compatible with the macroscopic (or mesoscopic) description, changes because these constraints change in time, and the characteristic set of microscopic points changes. The evolution of such set is a different thing that the set of trajectories of given points in a volume of phase space, whose volume is indeed conserved according to Liouville theorem. Moreover,

(vi) It ought to be noticed that Gibbs’ entropy is in fact constant in time, because NESEF conserves the initial information, being then the said fine-grained entropy. But, as Fig. 1 shows, as time elapses it gets outside the informational subspace and therefore is no longer describing the state of the system in terms of the chosen set of basic variables. It needs be projected at each time on the informational subspace, again as shown in Fig. 1, thus introducing loss of information. Of course equilibrium is a particular case when Gibbs’ entropy coincides with Clausius thermodynamic entropy, and in the NESEF formalism it remains always at the point of initial preparation (which is the one of equilibrium).

(vii) Bricmont makes a similar statement to that of Meixner: why should one worry so much about entropy for nonequilibrium states? To account for the irreversible behavior of the macroscopic variables is not necessary to introduce some kind of entropy function that evolves monotonically in time. It is not required to account for irreversibility, however it may be interesting or useful to do so. In the case we have presented of informational entropies applies this question of interest
and usefulness: it allows for a better clarification of the meaning and interpretation of the Lagrange multipliers; to introduce the production of informational entropy and the derivation of useful criteria for evolution and stability; to better characterize the dissipative processes that develops in the media (organizing them in increasing orders of covariances of the informational-entropy production operator); to better analyze fluctuations out of equilibrium and work out studies on complementarity of micro/macro descriptions; and so on (see Ref. [53] for details).

Therefore, in principle and to all appearances, a true thermodynamic entropy is only clearly defined, via Clausius approach, in strictly equilibrium conditions. Out of equilibrium, quasi-entropies (in our nomenclature) may be introduced and be of utility, but it needs be clearly stated which is the definition, and to devise a particular name characterizing each: say, entropy in Classical (Onsagerian) Irreversible Thermodynamics, entropy in Extended Irreversible Thermodynamics, NESEF-entropy of Informational Statistical Thermodynamics, and so on and so forth [53].

The informational-entropy, to be related to the one in phenomenological theories, depends on a chosen set of basic variables and arises from a projected part of the logarithm of the said nonequilibrium statistical operator ln $\rho_\varepsilon(t)$, which is $\ln \bar{\rho}(t,0)$ (as shown in Fig. 1). This process is a coarse-grained-type procedure which restricts us to have as only accessible microstates in phase space those in the subspace spanned by the chosen basic dynamic variables. Irreversible effects are contained in the complementary part of the nonequilibrium statistical operator, namely $\rho'_\varepsilon$, as it is demonstrated by the proven generalized $\mathcal{H}$-theorem presented in Ref. [53].

Furthermore, the local informational-entropy production function is predominantly positive definite, but in any case these results cannot be connected with the formulation of the second law, until a clear cut definition of the entropy function in nonlinear thermodynamics for systems arbitrarily away from equilibrium is obtained. This statement has to be further clarified since here the second law must necessarily be understood as some extension of Clausius formulation, with the latter, we recall, involving changes between two equilibrium states and $S$ is the calorimetric entropy which is uniquely defined. We emphasize once more that in nonequilibrium states it is very likely that many different definitions of surrogates of the entropy are feasible depending essentially on how to obtain – in some sense – a complete set of macrovariables that may unequivocally characterize the macrostate of the system under the given experimental conditions, and thus
agreeing with Meixner conjecture. In the Informational Statistical Thermodynamics – which can be considered as given microscopic, i.e. mechano-statistical foundations, to macroscopic Extended Thermodynamics –, the associated informational entropy (or quasi-entropy) has been discussed in detail in Refs. [53, 48], where are presented the derivation: of nonequilibrium equations of state; of a generalized $\mathcal{H}$-theorem; of evolution and (in)stability criteria; of a generalized Clausius relation; of fluctuations and Maxwell-like relations; of a Boltzmann-like relation: $\dot{S}(t) = k_B \ln W(t)$; and comparison of theory and experimental results is made.

As a results of this, it may be stated that NESEF seems to offer a sound formalism to give foundations to irreversible thermodynamics on a statistical-mechanical basis, an approach providing what has been referred-to as Informational Statistical Thermodynamics. Thus, for the case of systems under quite arbitrary dissipative conditions (no restriction to local equilibrium, linearity, etc.) a theoretical treatment of a very large scope follows for the thermodynamics, transport properties, and response functions of nonequilibrium systems. Paraphrasing Zwanzig [84], we remark that, seemingly, NESEF possesses a remarkable compactness and has by far a most appealing structure, being a very effective method for dealing with nonlinear and nonlocal in space and time transport processes in far-from-equilibrium many-body systems.

4. COMMENTS AND CONCLUDING REMARKS

Summarizing, we first notice the relevant point that in the construction of a statistical mechanics, the derivation of an appropriate (for the problem in hands) probability distribution – associated to a set of constraints imposed on the system – can be obtained in a compact and practical way by means of optimization (variational) principles in a context related to information theory. These are methods of maximization of the so-called informational-entropies (better called quantities of uncertainty of information) or minimization of distances in a space of probability distributions (MaxEnt and MinxEnt respectively) [22].

In the original formulation of Shannon and Jaynes use was made of Boltzmann-Gibbs statistical-entropy, which in MaxEnt provides the canonical-like (exponential) distributions of classical, relativistic, and quantum statistical mechanics. In Ref. [48] it is described its use for the case of many-body systems arbitrarily far removed from equilibrium, and the discussion of the dissipative phenomena that unfold in such conditions (mainly ultrafast relaxation processes; see Ref. [56]).
This approach has been exceedingly successful in conditions of equilibrium, and is a very promising one for nonequilibrium conditions. To have a reliable statistical theory in these situations is highly desirable since in very many situations – as for example are the case of electronic and optoelectronic devices, chemical reactors, fluid motion, and so on – the system is working in far-from-equilibrium conditions.

However, the enormous success and large application of Shannon-Jaynes method to Laplace-Maxwell-Boltzmann-Gibbs statistical foundations of physics, as it has been noticed, some cases look as difficult to be properly handled within the Boltzmann-Gibbs formulation, as a result of existing some kind of fuzziness in data or information, that is, the presence of a condition of insufficiency in the characterization of the (microscopic and/or macroscopic or mesoscopic) state of the system. Such, say, difficulty with the proper characterization of the system in the problem in hands, (which is a practical one and, we stress, not intrinsic to the most general and complete Boltzmann-Gibbs formalism) can be, as shown, patched with the introduction of peculiar parameter-dependent alternative structural informational-entropies (see Table I).

Particularly, to deal with systems with some kind of fractal-like structure the use of Boltzmann-Gibbs-Shannon-Jaynes infoentropy would require to introduce as information the highly correlated conditions that are in that case present. Two examples in condensed matter physics are “anomalous” diffusion [75] and “anomalous” optical spectroscopy [76], when fractality enters via the non-smooth topography of the boundary surfaces which have large influence on phenomena occurring in constrained geometries (nanometer scales in the active region of the sample). In the conventional approach, the spatial correlations that the granular boundary conditions introduce need be given as information (to satisfy the criterion of sufficiency, since they are quite relevant for determining the behavior of the system in the nanometric scales involved), but to handle them is generally a nonfeasible task. For example, in the second case above mentioned one has no easy access to the determination of the detailed topography of the surfaces which limit the active region of the sample (the nanometric quantum wells in semiconductor heterostructures), what can be done in the first case using atomic-force microscopy and the determination of the fractal dimension involved is possible. Hence, the most general and complete Boltzmann-Gibbs formalism in Shannon-Jaynes approach becomes hampered out and is difficult to handle, and then, as shown, use of other types of informational-entropies (better called generating functionals for deriving probability distributions) may help to circumvent such inconvenience
by introducing alternative algorithms (dependent on the so-called informational-entropic indexes), that is, the derivation of heterotypical probability distributions on the basis of the constrained maximization of unconventional informational-statistical entropies (quantity of uncertainty of information), to be accompanied, as noticed in the main text, with the use of the so-called escort probabilities.

In brief, we recall that, Unconventional Statistical Mechanics consists of two steps: 1. The choice of a deemed appropriate structural informational-entropy for generating the heterotypical statistical operator, and 2. The use of a escort probability in terms of the heterotypical distribution of item 1.

We reiterate that the escort probability introduces corrections to the insufficient description by including correlations and higher-order variances of the observables involved. On the other hand, the heterotypical distribution introduces corrections to the insufficient description (or incomplete probabilities in Renyi’s nomenclature) by modifying the statistical weight of the dynamical states of the conventional approach involved in the situation under consideration.

Moreover, we have considered a particular case, namely the statistics as derived from the use of Renyi informational entropy. We centered the attention on the derivation of an Unconventional Statistical Mechanics appropriate for dealing with far-removed-from-equilibrium systems.

In conclusion, we may say that USM appears as a valuable approach, in which the introduction of informational-entropic-indexes-dependent informational-entropies leads to a particularly convenient and sophisticated tool for fitting theory to experimental data for certain classes of physical systems, for which the criterion of sufficiency in its characterization cannot be properly satisfied. Among them we can pinpoint fractal-like structured nanometric-scale systems, which, otherwise, would be difficult to deal with within the framework of the conventional Statistical Mechanics. While in the latter case one would need to have a detailed description of the spatial characteristics of the structure of the system, the other needs to pay the price of having an open adjustable index to be fixed by best fitting with experimental results. It is relevant to notice the fact that the infoentropic index(es) is(are) dependent on the dynamics involved, the system’s geometry and dimensions, boundary conditions, its macroscopic-thermodynamic state (in equilibrium, or out of it when becomes a function of time), and the experimental protocol.

Finally, we call the attention to the fact that we have presented several alternatives of infoentropies (see Table I), for which, as stated in the main text, the uniform probability distribution is taken as the reference one, and such generating functionals provide a corresponding family of heterotypical probability
distributions. However, other choices of the reference probability can be made and then we have at our disposal very-many possibilities: It is tempting to look for the construction of a theory using for the probability of reference, instead of the uniform distribution, Shannon-Jaynes informational-entropy in its incomplete formalism, that is, when suffering from the deficiency that the researcher cannot satisfy Fisher’s criterion of sufficiency.

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Figure 1: An outline of the description of the non-equilibrium-dissipative macroscopic state of the system. The projection – depending on the instantaneous macrostate of the system – introduces the coarse-graining procedure consisting into the projection onto the subspace of the "relevant" variables associated to the informational constraints in NESEF.