GLOBAL EXISTENCE AND LIFESPAN FOR SEMILINEAR
WAVE EQUATIONS WITH MIXED NONLINEAR TERMS

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ABSTRACT. Firstly, we study the equation \( \Box u = |u|^q + |\partial_t u|^p \) with small data, where \( q_c \) is the critical power of Strauss conjecture and \( p \geq q_c \). We obtain the optimal lifespan \( \ln(T_\varepsilon) \approx \varepsilon - q_c(q_c - 1) \) in \( n = 3 \), and improve the lower-bound of \( T_\varepsilon \) from \( \exp(\varepsilon - (q_c - 1)^2/2) \) in \( n = 2 \). Then, we study the Cauchy problem with small initial data for a system of semilinear wave equations \( \Box u = |v|^q, \Box v = |\partial_t u|^p \) in 3-dimensional space with \( q < 2 \). We obtain that this system admits a global solution above a \( p - q \) curve for spherically symmetric data. On the contrary, we get a new region where the solution will blow up.

1. Introduction

In this paper, we want to study the global solvability and the blow up for some semilinear wave equations with nonlinear terms like \( |u|^q \) and \( |\partial_t u|^p \). Firstly we study the lifespan of the equation with mixed nonlinear terms

\[
\begin{aligned}
\Box u := \partial^2_t u - \Delta u &= |\partial_t u|^p + |u|^q, \\
(u, \partial_t u)|_{t=0} &= \varepsilon(f(x), g(x)),
\end{aligned}
\]

where \( p > 1, q > 1 \) and \( x \in \mathbb{R}^n \). This equation is in relation with the following equations which are well-investigated:

\[
\begin{aligned}
\Box v &= |v|^q, \quad t > 0, \quad x \in \mathbb{R}^n, \\
\Box w &= |\partial_t w|^p, \quad t > 0, \quad x \in \mathbb{R}^n.
\end{aligned}
\]

The first equation (1.2) is related to the Strauss conjecture, for which the critical power, denoted by \( q_c(n) \), is known to be the positive root of the quadratic equation

\[(n-1)q^2 - (n+1)q - 2 = 0\]

This conjecture was finally verified in [4, 18]. And the complete history can be found in [17]. As for the other equation (1.3), which is related to the Glassey conjecture, see [7] and the references therein for more information.

The equation (1.1) was almost understood for \( n = 2, 3 \) by [8, 5], but for a critical case \( q = q_c, p \geq q_c \). For now we know the lifespan \( T_\varepsilon \) satisfying

\[
\exp\left(c\varepsilon^{-(q_c - 1)}\right) \leq T_\varepsilon \leq \exp\left(C\varepsilon^{-(q_c - 1)}\right), \quad n = 2, 3,
\]

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Theorem 1.1. Let $\epsilon > 0$. Thus we get the improve result

$$Klainerman-Sobolev$$

energy inequality with $q = q_c$ from [8], we adapt these generalized Strichartz

estimates to the current setting, use energy inequality with $Klainerman-Sobolev$ inequality to deal with derivative term. Thus we get the improve result

Theorem 1.1. Let $n = 2, 3$, and suppose that $(f, g)$ satisfies

$$\Lambda := \| x^5 \partial_x^3 f \|_{L^p} + \| x^5 \partial_x^2 g \|_{L^p} < \infty.$$ 

Then there exists an $\epsilon_0(\Lambda, n, p) > 0$ and a constant $c$, such that for any $\epsilon \in (0, \epsilon_0)$, (1.1) has a unique solution $u \in C^0([0, T_\epsilon]; H^1(\mathbb{R}^n)) \cap C^1([0, T_\epsilon]; H^2(\mathbb{R}^n))$ where

$$T_\epsilon := \begin{cases} \exp(c \epsilon^{-(q_c - 1)/2}) & n = 2, \\ \exp(c \epsilon^{-(q_c - 1)/2}) & n = 3. \end{cases}$$

Next, we consider a coupled wave system with different nonlinear terms in each equation,

$$\begin{cases} \Box u = |u|^q, & \Box v = |\partial_t u|^p, \\ (u, \partial_t u)|_{t=0} = \epsilon (f(x), g(x)), & (v, \partial_t v)|_{t=0} = \epsilon (\tilde{f}(x), \tilde{g}(x)), \end{cases} \quad (1.4)$$

where $u, v$ depend on $(t, x) \in [0, T] \times \mathbb{R}^3$ for some $T \in (0, \infty)$, $\epsilon$ is positive and small enough.

This kind of system has been discussed in [9], which shows there exists a curve in $(1, \infty)^2$ of index-pairs $(p, q)$,

$$\begin{cases} (p - 1)(pq - 1) = p + 2, & 1 < q, \quad 1 < p < 3. \end{cases} \quad (1.5)$$

When $(p, q)$ lies below the curve, this system blow up in most cases whatever small $\epsilon$ is. On the contrary, when $(p, q)$ lies above the curve with $2 < p < 3, 2 < q$, this system (1.4) has a global solution at least for radially symmetric small data.

To analyze this equation, we compare it with some closely related systems. They are

$$(I) : \begin{cases} \Box u = |u|^q, & \Box v = |u|^p, \\ u, \partial_t u \mid_{t=0} = \epsilon (f(x), g(x)), \quad (v, \partial_t v \mid_{t=0} = \epsilon (\tilde{f}(x), \tilde{g}(x)), \end{cases} \quad (1.6)$$

For $(I)$, which relates to the Strauss conjecture, it is known that

$$\max\left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} = 1$$

is the critical curve of index-pairs $(p, q)$. These results can be found in [1] and [13].

As for $(II)$, which relates to the Glassey conjecture, such a curve is

$$\max\left\{ \frac{q + 1}{pq - 1}, \frac{p + 1}{pq - 1} \right\} = 1. \quad (1.7)$$
This is optimal at least for radially symmetric initial data, where the blow up part can be found in [2] and the existence part for symmetric situation was verified in [12]. We refer the interested readers to [9] for more information about these two problems.

It is naturally to infer that the critical curve for (1.4) should lies between the curves (1.6) and (1.7). However, curve (1.5) across with one of above curves (see the below figure, $CC''$ across $l_B$). This motivate us to improve the result when $q < 2$.

Here $l_A$ is the critical curve to problem (I), $l_B$ is the critical curve to problem (II) and $CC''$ is (1.5) with $q < 2$. We want to establish a global existence theorem to (1.4) with radially symmetric small data, for the region above $CC''$,

\begin{equation}
1 < q < 2, \quad 2 < p < 3.
\end{equation}

**Theorem 1.2.** Suppose (1.8) is satisfied. We also suppose $f, \tilde{f} \in C^2(\mathbb{R}^3)$, $g, \tilde{g} \in C^1(\mathbb{R}^3)$ are spherically symmetric and supported in $B_1(0)$. Then there exists a positive number $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$, there exists a global solution $(u, v)$ of (1.4) satisfying

$$u, v, \partial_t u \in C(\mathbb{R}^+ \times \mathbb{R}^3).$$

The key idea here is that a symmetric 3-D wave equation is equivalent to an 1-D equation essentially, in which case the solution of linear problem has a higher regularity. In order to match the situation $q < 2$, we use the weight functions different from [9].

As for the blow up part, we adopt the strategy of deriving a system of ordinary differential inequalities which causes blow up solutions. Since the technique is suitable for any dimensions, we give a general result rather than $n = 3$.

**Theorem 1.3.** Let

\begin{equation}
1 < q < \frac{n + 1}{n - 1}, \quad 1 < p.
\end{equation}
Suppose that all data are supported in $B_1(0)$ and $g - f, f, \tilde{g} - \tilde{f}, \tilde{f}$ are non-negative where $\tilde{f}$ does not vanish identically. Then for any $\varepsilon > 0$, there is no weak solutions

$$u, v, \partial_t u, \partial_t v \in C([0, \infty); L^1(\mathbb{R}^n)), \quad v \in C([0, \infty); L^q(\mathbb{R}^n)), \quad \partial_t u \in C([0, \infty); L^p(\mathbb{R}^n)),$$

such that $\text{supp}(u, v) \subset \{(t, x) : |x| \leq t + 1\}$.

This improves the result from the curve $CC''$ in [9] to $l_B$’s right part. Remark also that this result is better than that in [15].

For $n = 3$, we believe that the critical curve comes from Theorem 1.2, $p(q - 1) = 2 + \frac{1}{pq}, \quad 1 < q < 2, \quad 1 < p$

which is $CC'$ in the figure (since $(\sqrt{6}/2 + 1, 2)$ is critical but $(2, 2)$ is not). So we only show the blow up occurs with a kind of special data, without giving the explicit upper-bound of the lifespan in terms of epsilon.

This rest of paper is organized as follows. In Section 2 we introduce some notation we will used. Then we prove Theorem 1.1-1.3, in Section 3-5, respectively.

2. Notation

We list here some notations which will be used in our article. First, the Einstein summation convention is used, as well as the convention that Greek indices $\mu, \nu, \cdots$ range from 0 to $n$ while Latin indices $i, j, \cdots$ will run from 1 to $n$.

We also denote

$$\langle a \rangle := \sqrt{1 + |a|^2} \approx 1 + |a|,$$

$$\|f(x)\|_{L^p L^q} := \left(\int_0^\infty \left(\int_{S^{n-1}} |f(r\omega)|^q d\omega\right)^{p/b} r^{n-1} dr\right)^{1/p},$$

$$\partial u(t, x) := \{\partial_\mu u\} = (\partial_t u, \partial_x u),$$

$$\partial \leq k f := \{\partial^\alpha f\}_{0 \leq |\alpha| \leq k}.$$

Furthermore, we will use some kinds of special vector fields. There are spatial roation: $\Omega_{jk} = x^k \partial_j - x^j \partial_k$, Lorentz boost: $\Omega_{0k} = t \partial_k + x^k \partial_t$ and scaling: $L_0 = t \partial_t + x^i \partial_i$. Set $\Gamma = \{\partial_\mu, \Omega_{\mu\nu}, L_0\}$ be the well-known Klainerman vector fields. For such vector fields, we have

$$[\partial_\mu, \Gamma_\alpha] f = C^\beta_{\mu\alpha} \partial_\beta f, \quad [\Gamma_\alpha, \Gamma_\beta] f = C^\gamma_{\alpha\beta} \Gamma_\gamma f,$$

$$[\Gamma_\alpha, \Box] f = C_\alpha \Box f, \quad (\alpha, \beta = 0, 1, \cdots, (n^2 + 3n + 2)/2)$$

where $[X, Y]$ denotes the commutator $XY - YX$. And all coefficients are belong to $C^\infty_b$.

We denote a constant $C$ which may change from line to line, but not depend on $\varepsilon, t$ or $x$. And $A \lesssim B$ means $A \leq CB$ for some $C > 0$, $A \gtrsim B$ is similar, $A \approx B$ means $A \lesssim B \lesssim A$. 

3. Proof of Theorem 1.1

3.1. Preliminaries.
We firstly list some lemmas we will use later.

Lemma 3.1 (Local Existence). When \( n \leq 3 \). For the equation
\[
\begin{cases}
\partial_t^2 u - \Delta u = F(u, \partial u), \\
u(0, x) = f(x), u_t(0, x) = g(x).
\end{cases}
\]
If \((f, g) \in H^3 \times H^2, F \in C^2 \) and \( F(0, 0) = 0 \) then there is a \( T > 0 \), depending on the norm of the data, so that this Cauchy problem has a unique solution satisfying
\[
\|\partial_t \leq u(t, \cdot)\|_{L^2_t \dot{H}^s} < \infty, \quad 0 \leq t \leq T.
\]
Also, if \( T_* \) is the supremum over all such times \( T \), then either \( T_* = \infty \) or
\[
|\partial_t \leq u| \notin L^\infty_t L^2_x (0 \leq t < T_*).
\]

Proof. The result is classical, for a proof, see, e.g., Chapter 12 of [3]. \( \square \)

Lemma 3.2 (Theorem 6.4 of [14]). When \( n = 3 \), suppose \( u \) satisfies the equation
\[
\begin{cases}
\Box u(t, x) = F(t, x), \\
u(0, x) = f(x), u_t(0, x) = g(x)
\end{cases}
\]
\( u \) in \( [0, T_*) \times \mathbb{R}^3 \). Then there exists a constant \( C \), such that \( \forall T < T_* \)
\[
T^{1/2} \|u(T, \cdot)\|_{L^\infty_t \dot{L}^2_x} \leq C \left( \|r^{1-2/q} \partial_t \leq u\|_{L^\infty_t \dot{L}^2_x} + \|r^{1+1/q} \partial_t \leq u\|_{L^\infty_t \dot{L}^2_x} \right.
\]
\[
\left. + \|r^{2-2/q} g\|_{L^\infty_t \dot{L}^2_x} + \|r^{2+1/q} g\|_{L^\infty_t \dot{L}^2_x} + \|F\|_{L^\infty_t \dot{L}^2_x(t < T)} + T^{1/q} \|F\|_{L^\infty_t \dot{L}^2_x(T/4 < t < T)} \right).
\]

Lemma 3.3 (Theorem 1.4 of [10]). When \( n = 2 \), suppose \( u \) satisfies the equation
\[
\begin{cases}
\Box u(t, x) = F(t, x), \\
u(0, x) = f(x), u_t(0, x) = g(x)
\end{cases}
\]
in \( [0, T_*) \times \mathbb{R}^2 \). Let \( q_* = \frac{q-1}{2}, \ s_* = 1 - 1/q, b > \frac{1}{q_*}, \ X^b \equiv H_{s_*}^{s_*+\delta} \times H_{s_*}^{s_*+\delta} \), where
\[
H_{s_*}^{s_*+\delta} = \{ u \in H^s : \|(1-\Delta)\|^{s/2} u \|_{H^s} < \infty \}, \quad X^b = \sum_{1 \leq i < j \leq n} \Omega_{ij}.
\]

Then there exists a constant \( C \), such that \( \forall T < T_* \)
\[
\|u\|_{L^\infty_t X^b \dot{L}^2_x(t < T)} \leq C \left( \ln(2 + T) \right)^{(q-1)/q} \left( \|(f, g)\|_{X^s} + \|F\|_{L^\infty_t \dot{L}^2_x} \right).
\]

Lemma 3.4 (Energy Inequality). For any \( n \), suppose \( u \) satisfies the equation
\[
\begin{cases}
\Box u(t, x) = F(t, x), \\
u(0, x) = f(x), u_t(0, x) = g(x)
\end{cases}
\]
We have, \( \forall T > 0 \)
\[
\|\partial u(T, \cdot)\|_{L^2_x} \leq \|\partial u(0, \cdot)\|_{L^2_x} + \|F\|_{L^1_t \dot{L}^2_x(t < T)}
\]
Since the property of commutator (2.1), we also have for any $k \in \mathbb{Z}^+$
\[ \|\partial^\leq k u(T, \cdot)\|_{L^2_t} \leq C_k \left( \|\partial^\leq k u(0, \cdot)\|_{L^2} + \|\Gamma^\leq k F\|_{L^1_t L^2_x(t < T)} \right). \]

**Lemma 3.5** (Klainerman-Sobolev inequalities). Let $v \in C^{\infty}(\mathbb{R}^{1+n})$ vanish when $|x|$ is large, $1 \leq p < q \leq \infty$ and $k > n/p$, then if $t > 0$
\[ (t + r)^{(n-1)/p} \|t - r\|^{1/p} \|v(t, x)\| \leq C \|\Gamma^\leq k v(t, \cdot)\|_{L^q_t}. \]

Otherwise if $1 \leq p < q < \infty$ and $k \geq n/p - n/q$, then
\[ (t)^{(n-1)/(1-p/q)} \|v(t, \cdot)\|_{L^q_t} \leq C \|\Gamma^\leq k v(t, \cdot)\|_{L^q_t}. \]

**Proof.** The inequality (3.3) is well-known Klainerman-Sobolev inequality, which is proved in [11]. Heuristically, the inequality (3.4) can be viewed as a consequence of (3.3). We refer [6, (3.9)] for a complete proof. \qed

**Lemma 3.6** (Sobolev embedding). Let $x \in \mathbb{R}^n$, and $1 \leq p < q \leq \infty$, then
\[ \|f(x)\|_{L^q_t} \leq C \|\partial^\leq k f(x)\|_{L^p_t}, \]
where $k \geq n/p - n/q$ if $q < \infty$, or $k > n/p$ if $q = \infty$. When it comes to $S^{n-1}$, with $1 \leq p < q \leq \infty$, then
\[ \|f(\omega)\|_{L^q_t} \leq C \|\Omega^\leq k f(\omega)\|_{L^p_t}, \]
where $k \geq (n-1)/p + (n-1)/q$ if $q < \infty$, or $k > (n-1)/p - (n-1)/q$ if $q = \infty$. For mixed-norm, we have
\[ \|f(x)\|_{L^q_t L^r_x} \leq C \|\partial^\leq k f(x)\|_{L^p_t L^s_x}, \]
where $2 \leq p \leq q \leq 4$ and $k \geq n/4$.

**Proof.** Here inequality (3.5) is known as Sobolev embedding, and inequality (3.6) just the same inequality on sphere. Finally the inequality (3.7) is a simple Corollary of lemma 3.2 in [16] and normal Sobolev embedding in a small ball contains origin point. \qed

**Claim 3.7.** We claim here that there exists a constant $C_0$ depends on $\Lambda, n$ but not on $\varepsilon$ if $\varepsilon$ is small enough, then all of the initial norms we will use can be bounded by $C_0 \varepsilon$.

The proof of Claim 3.7 need some delicate calculation, so we will postpone it to the end.

### 3.2. Main Proof

For now, we are ready to prove the Theorem 1.1.

**Part 1: $n = 3$.**

By Lemma 3.1, we only need to prove that if $0 < T < T_\varepsilon \leq \bar{T}_\varepsilon$, then $\|\partial^\leq 3 u\|_{L^p_x L^q_t(t < T)} \leq M$. Here we use the continuity method.

Firstly, we want to prove that for some suitable $C_M, \varepsilon$ to be fixed later, if $0 \leq T < T_\varepsilon \leq \bar{T}_\varepsilon$ then
\[ A(T) = (T)^{1/q} \|\Gamma^\leq 2 u(T, \cdot)\|_{L^p_x L^q_t} \leq C_M \varepsilon, \]
\[ B(T) = \|\partial:\Gamma^\leq 2 u(T, \cdot)\|_{L^2} \leq C_M \varepsilon. \]

With the help of Claim 3.7, we choose $C_M$ satisfying that $A(0), B(0) \leq C_M \varepsilon/4$. Take $X = \{T \in [0, T_\varepsilon) : A(t), B(t) \leq C_M \varepsilon, \forall t \in [0, T]\}$. We want to prove $X = \{T \in [0, T_\varepsilon) : A(t), B(t) \leq C_M \varepsilon, \forall t \in [0, T]\}$.
Lemma 3.6, Part 1.2: Estimate of $A$ implies that Estimate of Part 1.1: Lemma 3.1, we assume $T > (3.9)\varepsilon$.

Then, we consider the spatial norm part of $\varepsilon$ Back to equation (3.9), assume By Lemma 3.2, we obtain

$$A(T) \leq C_M\varepsilon/4 + C\int_{\Gamma^\leq 2}\|u\|_{L^q_tL^r_x(t<T/4)} + C\int_{\Gamma^\leq 2}\|\partial u\|_{L^q_tL^r_x(t<T/4)} + C\int_{\Gamma^\leq 2}\|\partial u\|_{L^q_tL^r_x(t<T/4)} + C\int_{\Gamma^\leq 2}\|\partial u\|_{L^q_tL^r_x(t<T/4)}.$$

First we consider the spatial norm part of $|u|$ terms, By Hlder’s inequality and Lemma 3.6,

$$\|\Gamma^\leq 2u\|_{L^q_tL^r_x} \leq C\|\Gamma^\leq 2u\|_{L^q_tL^r_x} + C\|\Gamma^\leq 1u\|_{L^q_tL^r_x} \leq C\|\Gamma^\leq 2u\|_{L^q_tL^r_x} + C\|\Gamma^\leq 2u\|_{L^q_tL^r_x} \leq C\left(\langle t \rangle^{-1/q} A(t)\right)^{q_c} \leq CC_M\varepsilon^{q_c}(\langle t \rangle)^{-1/q_c}.$$

Then, we consider the spatial norm part of $|\partial u|$ terms, by Hlder’s inequality and Lemma 3.5, Lemma 3.6,

$$\|\Gamma^\leq 2\|\partial u\|_{L^q_tL^r_x} \leq C\|\Gamma^\leq 2\partial u\|_{L^q_tL^r_x} + C\|\Gamma^\leq 1\partial u\|_{L^q_tL^r_x} \leq C\|\Gamma^\leq 2\partial u\|_{L^q_tL^r_x} + C\|\Gamma^\leq 2\partial u\|_{L^q_tL^r_x} \leq C\left(\langle t \rangle^{-1/q} A(t)\right)^{q_c} \leq CC_M\varepsilon^{p-1}.\varepsilon^{p-1}.$$

Back to equation (3.9), assume $\varepsilon_0 < 1/C_M$. Since $p \geq q_c = 1 + \sqrt{2}$, $0 < \varepsilon < \varepsilon_0$, we have

$$A(T) \leq C_M\varepsilon/4 + CC_M^{q_c}\varepsilon^{q_c}\left(1 + \left(\int_0^{T/4} (t)^{-1} dt\right)^{1/q_c}\right)$$

$$+ CC_M^p\varepsilon^p\left(\langle T \rangle^{q_c-p} + \left(\int_0^{T/4} (t)^{-q_c(p-2)} dt\right)^{1/q_c}\right) \leq C_M\varepsilon\left(1/4 + CC_M^{q_c-1}\varepsilon^{q_c-1}(\ln(T+2))^{1/q_c} + CC_M^{p-1}\varepsilon^{p-1}\right) \leq C_M\varepsilon\left(1/3 + CC_M^{q_c-1}\varepsilon^{q_c-1} + CC_M^{q_c-1}\varepsilon^{q_c-1}\varepsilon_0^{-1}\right).$$

Part 1.2: Estimate of $B(t)$. 

[0, T\varepsilon). Consequently, it would suffice to show that, if $T$ is as above, then equation (3.8) implies that $A(T), B(T) \leq C_M\varepsilon/2$ must valid if $T\varepsilon$ as above. Because of Lemma 3.1, we assume $T > 1$ without loss of generality.
For this part, by the Lemma 3.4, we know that
\begin{equation}
B(T) \leq C_M \varepsilon / 4 + C \left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^1_t L^\infty_x (t < T)} + C \left\| \Gamma^{\leq 2} |\partial u|^p \right\|_{L^1_t L^2_x (t < T)}.
\end{equation}

Similarly, by Hölder’s inequality and Lemma 3.5. Let \( a = (q_c - 1)/(1/2 - 1/q_c) \), then for the \(|u|\) term,
\[
\left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^2_x} \leq C \left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^2_x} + C \left\| \Gamma^{\leq 1} u \right\|_{L^2_x}^2 \left\| u \right\|_{L^2_x}^{q_c - 2}
\leq C \left\| \Gamma^{\leq 2} u \right\|_{L^2_x}^{q_c} \left( t \right)^{-2(1/q_c - 1/a)(q_c - 1)} + \left\| \Gamma^{\leq 1} u \right\|_{L^2_x}^{q_c} \left( t \right)^{-2(1/q_c - 1/4) - 2(1/q_c)(q_c - 2)}
\leq C \left( t \right)^{-1/a} \left\| \partial u \right\|_{L^2_x}^{q_c} \left( t \right)^{-1}
\leq C \left( t \right)^{-1/a} A(t) \left( t \right)^{-1} + CC_{\varepsilon}^{q_c - 1} \varepsilon^{q_c - 1}.
\]

For the \(|\partial u|\) term, we use Hölder’s inequality and Lemma 3.5 again,
\[
\left\| \Gamma^{\leq 2} |\partial u|^p \right\|_{L^2_x} \leq C \left\| \Gamma^{\leq 2} \partial u \right\|_{L^p_x} \left\| \partial u \right\|_{L^2_x}^{p-1} + C \left\| \Gamma^{\leq 1} \partial u \right\|_{L^2_x}^2 \left\| \partial u \right\|_{L^2_x}^{p-2}
\leq C \left\| \Gamma^{\leq 2} u \right\|_{L^2_x} \left( t \right)^{-2(1/p - 1/a)(p - 1)} + \left\| \Gamma^{\leq 1} u \right\|_{L^2_x} \left( t \right)^{-2(1/2 - 1/a) - 2(1/2)(p - 2)}
\leq CC_{\varepsilon}^{p} \varepsilon^p \left( t \right)^{1-p}.
\]

Back to (3.10), we have
\[
B(T) \leq C_M \varepsilon \left( 1/4 + CC_{\varepsilon}^{q_c - 1} \varepsilon^{q_c - 1} + CC_{\varepsilon}^{p - 1} \varepsilon^p \right) \leq C_M \varepsilon \left( 1 + CC_{\varepsilon}^{q_c - 1} \varepsilon^{q_c - 1} \right).
\]

**Part 1.3:** The boundness of \( A(t), B(t) \).

Since \( p \geq 1 + \sqrt{2} \), we chose \( \varepsilon_0 \) and the constant \( c \) in \( \tilde{T}_\varepsilon \) small enough. Then \( A(T), B(T) \leq C_M \varepsilon/2 \), which completes the proof.

**Part 2:** \( n = 2 \).

In this part, we need to prove that for some suitable \( C_M, c, \varepsilon \) to be fixed hereafter, if \( 0 \leq T < T_\varepsilon \leq \tilde{T}_\varepsilon \) then
\begin{equation}
A(T) = \left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^1_t L^\infty_x (t < T)} \leq C_M \varepsilon^{1/q_c},
B(T) = \left\| \partial \Gamma^{\leq 2} u(T, \cdot) \right\|_{L^\infty_x} \leq C_M \varepsilon,
\end{equation}
where \( q = (q_c - 1)/2 \) as in Lemma 3.3. By Claim 3.7, we fix \( C_M \) such that \( B(0) \leq C_M \varepsilon/4 \). We need to show that equation (3.11) implies \( A(T) \leq C_M \varepsilon^{1/q_c} / 2 \) and \( B(T) \leq C_M \varepsilon/2 \) for above \( T \) and \( T_\varepsilon \).

**Part 1.2:** Estimate of \( A(t) \).

By Lemma 3.3, and \( T < \tilde{T}_\varepsilon \), we get
\begin{equation}
A(T) \leq cC_M \varepsilon^{1/q_c} + cC_{\varepsilon}^{1/q_c} \left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^1_t L^\infty_x (t < T)} + C \left\| \partial \Gamma^{\leq 2} |u|^p \right\|_{L^1_t L^2_x (t < T)}.
\end{equation}

For the second term, similar to \( n = 3 \), we have
\[
\left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^1_t L^\infty_x} \leq C \left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^1_t L^\infty_x} + C \left\| \Gamma^{\leq 1} u \right\|_{L^2_x} \left\| u \right\|_{L^2_x}^{q_c - 2}
\leq C \left\| \Gamma^{\leq 2} |u|^{q_c} \right\|_{L^1_t L^\infty_x} + CC_{\varepsilon}^{q_c} \varepsilon.
\]
and
\[
\| \Gamma^{\leq 2} |\partial u|^p \|_{L^1_t L^2_x} \leq C \left( \| \Gamma^{\leq 2} \partial u \|_{L^2_t L^p_x} \| \partial u \|_{L^p_t L^\infty_x} + \| \Gamma^{\leq 1} \partial u \|_{L^1_t L^2_x}^2 \| \partial u \|_{L^2_t L^\infty_x}^{p-2} \right) \| \partial u \|_{L^p_t L^\infty_x}^{p-2} \\
\leq C \| \Gamma^{\leq 2} \partial u \|_{L^2_t L^p_x}^2 \| \partial u \|_{L^p_t L^\infty_x}^{p-2} \\
\leq CC^p_M \varepsilon^p \langle t \rangle^{1-p/2}.
\]

Since \( p \geq q_c, (1 - q_c/2)q_* = -1, T < \bar{T}, \) we get
\[
A(T) \leq cC_M \varepsilon^{1/q_c} + cC C^q_M \varepsilon^{1/q_c-1} + cC C^p_M \varepsilon^{1/q_c-1+p} \left( \int_0^T \langle t \rangle^{(1-p/2)q_*} \, dt \right)
\leq cC_M \varepsilon^{1/q_c} + cC C^q_M \varepsilon^{1/q_c-1} + cC C^p_M \varepsilon^{1/q_c-1+p} (\ln(\bar{T}^2))^{1/q_*}
\leq C_M \varepsilon^{1/q_c} \left( c + cC^q_M \varepsilon^{-1} \right).
\]

**Part 2.2:** Estimate of \( B(t) \).

In this part, by Hölder’s inequality, Lemma 3.5 and Lemma 3.6,
\[
\| \Gamma^{\leq 2} |u|^{q_c} \|_{L^1_t L^2_x} = \| \Gamma^{\leq 2} |u|^{q_c} \|_{L^1_t L^2_x}
\leq C \langle u \|_{L^\infty_x}^{q_c-2} \left( \| \Gamma^{\leq 2} u \|_{L^p_t L^2_x} \| u \|_{L^{p+1}_t L^\infty_x} + \| \Gamma^{\leq 1} u \|_{L^1_t L^2_x} \| u \|_{L^2_t L^\infty_x} \right) \|_{L^1_t}^2
\leq C \langle \| \Gamma^{\leq 1} u \|_{L^2_t L^{q_c}_x}^{q_c-2} \| \Gamma^{\leq 2} u \|_{L^p_t L^4_x} \langle t \rangle^{(1/q_c)(q_c-2)} \|_{L^1_t} \| \Gamma^{\leq 2} u \|_{L^p_t L^4_x} \langle t \rangle^{q_c-4} \|_{L^1_t}^2
\leq C \langle \| \Gamma^{\leq 2} u \|_{L^p_t L^4_x} \| u \|_{L^\infty_x} \| \langle t \rangle^{q_c-4} \|_{L^1_t} \| \Gamma^{\leq 2} u \|_{L^p_t L^4_x} \langle t \rangle^{q_c-4} \|_{L^1_t} \| \Gamma^{\leq 2} u \|_{L^p_t L^4_x} \langle t \rangle^{q_c-4} \|_{L^1_t} \| \Gamma^{\leq 2} u \|_{L^p_t L^4_x} \langle t \rangle^{q_c-4} \|_{L^1_t}
\leq CC^q_M \varepsilon^{q_c}.
\]

and the same for the last term
\[
\| \Gamma^{\leq 2} |\partial u|^p \|_{L^2_x} \leq C \| \Gamma^{\leq 2} \partial u \|_{L^2_x} \| \partial u \|_{L^{p-1}_x}^{p-1} + C \| \Gamma^{\leq 1} \partial u \|_{L^1_x} \| \partial u \|_{L^p_x}^{p-2}
\leq C \| \Gamma^{\leq 2} \partial u \|_{L^2_x} \langle t \rangle^{-(1/2)(p-1)} \| \partial u \|_{L^p_x}^{p-2} \langle t \rangle^{-(1/2-1/4)(p-2)} (1/2)(p-2)
\leq CC^p_M \varepsilon^p \langle t \rangle^{(1-p)/2}.
\]

By the definition of \( \bar{T} \) and \( \varepsilon_0 \), we have
\[
B(T) \leq C_M \varepsilon \left( 1/4 + CC^{q_c}_M \varepsilon^{q_c-1} + CC^{p-1}_M \varepsilon^{p-1} \right) \leq C_M \varepsilon \left( 1/4 + CC^{q_c}_M \varepsilon^{q_c-1} \right).
\]

**Part 2.3:** The boundness of \( A(t), B(t) \).

Now, by choosing a suitable constant \( c \) (in \( T_c \)) small enough, we have \( A(T) \leq C_M \varepsilon^{1/q_c}/2 \) and \( B(T) \leq C_M \varepsilon/2 \), which completes the proof.
3.3. Proof of Claim 3.7.

Before the discussion, set \( h = \{ \partial_x^1 f, g \} \). By equation (1.1) we have
\[
\left\| \partial_x^{\leq 3} u, \partial_x^{\leq 2} \partial_t u \right\|_{t=0} \leq C \varepsilon \left| \partial_x^{\leq 2} h \right|,
\]
\[
\left| \partial_x^{\leq 1} \partial_t^2 u \right|_{t=0} = \left| \partial_x^{\leq 1} \left( |\partial_t u|^p + |u|^{q_c} + \Delta u \right) \right|_{t=0} \leq C \varepsilon \left\{ \partial_x^{\leq 1} h |h|^{p-1}, |h|^{q_c}, \partial_x^{\leq 2} h \right\},
\]
\[
\left| \partial_t^3 u \right|_{t=0} \leq C \left\{ \left( \partial_t u |\partial_t u|^{p-1}, \partial_t u |u|^{q_c-1}, \partial_t \Delta u \right) \right\}_{t=0} \leq C \left\{ |\partial_t u|^p + |u|^{q_c} + \Delta u |\partial_t u|^{p-1}, \partial_t u |u|^{q_c-1}, \partial_t \Delta u \right\} \leq C \varepsilon \left\{ |h|^{2p-1}, |h|^{q_c+p-1}, \partial_x^{\leq 1} h |h|^{p-1}, |h|^{q_c}, \partial_x^{\leq 2} h \right\}.
\]

Set \( M = A + \Lambda^p + \Lambda^{q_c} + \Lambda^{2p-1} + \Lambda^{q_c+p-1} \), we want to show that all the initial norms can be controlled by \( C \varepsilon M \), where \( C \) not depends on \( \varepsilon \) and \( M \).

We begin with \( A(0) = \| \Gamma^{\leq 2} u(0, \cdot) \|_{L^\infty_t L^2_x} \), 
\[
\| \Gamma^{\leq 2} u(0, \cdot) \|_{L^\infty_t L^2_x} \leq C \left\| \partial_t \Gamma^{\leq 3} u(0, \cdot) \right\|_{L^\infty_x} \leq C \left\| \langle x \rangle^5 \partial_t \Gamma^{\leq 3} u(0, \cdot) \right\|_{L^\infty_x} \| \langle x \rangle^{-2} \|_{L^\infty_x} \leq C \varepsilon M.
\]

For \( B(0) = \| \partial \Gamma^{\leq 2} u(0, \cdot) \|_{L^2_x} \), similarly
\[
\| \partial \Gamma^{\leq 2} u(0, \cdot) \|_{L^2_x} \leq C \left\| \langle x \rangle^5 \partial \Gamma^{\leq 3} u(0, \cdot) \right\|_{L^\infty_x} \| \langle x \rangle^{-3} \|_{L^2_x} \leq C \varepsilon M.
\]

For \( \| r^{1-2/q_c} \partial_x^{\leq 1} \Gamma^{\leq 2} u(0, \cdot) \|_{L^\infty_t L^{3/2}_x} \) comes from RHS of equation (3.1), we have
\[
\| r^{1-2/q_c} \partial_x^{\leq 1} \Gamma^{\leq 2} u(0, \cdot) \|_{L^\infty_t L^{3/2}_x} \leq C \left\| \langle x \rangle^{4-2/q_c} \partial \Gamma^{\leq 3} f \right\|_{L^\infty_x} \leq C \left\| \langle x \rangle^5 \partial \Gamma^{\leq 3} u(0, \cdot) \right\|_{L^\infty_x} \| \langle x \rangle^{-1+2/q_c} \|_{L^\infty_x} \leq C \varepsilon M.
\]

The other three terms form equation (3.1) can be controled by the same or even simpler way. At last, for the term comes from equation (3.2) that \( \| \Gamma^{\leq 2} (u, \partial_t u)(0, \cdot) \|_{X^4} \) \( (n = 2) \), we have
\[
\| \Gamma^{\leq 2} (u, \partial_t u)(0, \cdot) \|_{X^4} \leq \| \Gamma^{\leq 2} (u, \partial_t u)(0, \cdot) \|_{(H^1, L^2)}.
\]

So through the previous discussion, we have proved the Claim 3.7.

4. Proof of Theorem 1.2

Before the proof, we consider a simple coordinate transform such that \( u, u_t \rangle_{t=2} = (f, g), (v, v_t) \rangle_{t=2} = (\hat{f}, \hat{g}) \). Due to the property of symmetry, \( u, v \) solve the equivalent 1-D integral equations in \( t \geq 2 \)
\[
u(t, r) = \varepsilon u_{\ast}(t, r) + L|v|^q(t, r),
\]
\[
v(t, r) = \varepsilon v_{\ast}(t, r) + L|\partial_t u|^p(t, r),
\]
where we consider \( u_{t<2} = v_{t<2} = 0 \), then

\[
\begin{align*}
u_o(t+2,r) &= \frac{1}{2r} \left( \frac{r+t}{r-t} f(r+t) + (r-t) f(r-t) + \int_{r-t}^{r+t} \rho g(\rho) d\rho \right), \\
v_o(t+2,r) &= \frac{1}{2r} \left( \frac{r+t}{r-t} \tilde{f}(r+t) + (r-t) \tilde{f}(r-t) + \int_{r-t}^{r+t} \rho \tilde{g}(\rho) d\rho \right), \\
LF(t,r) &= \frac{1}{2r} \int_{t}^{t} \int_{r-t+s}^{r+t-s} \rho F(s,\rho) d\rho ds.
\end{align*}
\]

Here we denote \( f(|x|) = f(-|x|) = f(x) \) and the rest is similar. Then the lower limits of the integrals (to \( \rho \)) may be replaced by \( |r-t| \) or \( |r-(t-s)| \) because of the symmetric assumption. To control the iteration procedure, we need to estimate some derivatives. With the notation \( w = \partial_t u \), we find that

\[
\begin{align*}
w(t,r) &= \varepsilon \partial_t u_o(t,r) + r^{-1} K_+ |v|^{q}(t,r), \\
v(t,r) &= \varepsilon v_o(t,r) + L \|w\|^p(t,r), \\
\partial_r \{rv(t,r)\} &= \varepsilon \partial_r (rv_o(t,r)) + K_- \|w\|^p(t,r),
\end{align*}
\]

where

\[
K_\pm F(t,r) = \frac{1}{2} \int_{t}^{t} (r+s)F(s,r+t-s) \pm (r-t+s)F(s,r-t+s) ds.
\]

To control the norm of \( (w,v) \), we set

\[
\| (w,v) \| = \| \omega_1 w \|_{L^\infty_t} + \| \omega_2 v \|_{L^\infty_t} + \| \omega_3 r \partial_r v \|_{L^\infty_t}
\]

where weight functions \( \omega_1 , \omega_2 , \omega_3 \) are defined by

\[
\begin{align*}
\omega_1(t,r) &= \begin{cases} (r) (t-r)^{\frac{\mu}{p} + q - 2} & (r < t/2), \\
(t-r)^{\frac{\mu}{p} + q - 1} & (r \geq t/2); \\
\end{cases} \\
\omega_2(t,r) &= \begin{cases} (r)^{\frac{p-2}{p}} (t+r)^{3 - p + \mu/p q} & (r < t/2), \\
(t-r)^{\mu/p q} & (r \geq t/2); \\
\end{cases} \\
\omega_3(t,r) &= (t-r)^{\mu/p q - 2p - 2}
\end{align*}
\]

for \( t \geq 0 \) and \( r \geq 0 \), with a fixed \( \mu < 1 \) and satisfies

\[
\begin{align*}
-\mu - pq + 2p &\leq p - 3 - \mu/p q.
\end{align*}
\]

Now we consider the system of integral equations (4.1)-(4.3) in the close subset of complete metric space

\[
X_\varepsilon = \left\{ (w,v) : w,v,r \partial_r v \in C([2, \infty) \times \mathbb{R}), \| (w,v) \| \leq C_1 \varepsilon, \text{ supp}(w,v) \subset \{ t-r \geq 1, t \geq 2 \} \right\},
\]

where \( C_1 \) will be determined later.

**Lemma 4.1.** Suppose (1.8), \( (w,v) \in X_\varepsilon \). Then we have

\[
\begin{align*}
\| \omega_2 L |w|^p \|_{L^\infty_t} &\leq C \| \omega_1 w \|_{L^\infty_t}, \\
\| \omega_1 r^{-1} K_+ |v|^q \|_{L^\infty_t} &\leq C \| \omega_2 v \|_{L^\infty_t}^q + C \| \omega_2 v \|_{L^\infty_t} \| \omega_3 r \partial_r v \|_{L^\infty_t}, \\
\| \omega_3 K_- |w|^p \|_{L^\infty_t} &\leq C \| \omega_1 w \|_{L^\infty_t}^p.
\end{align*}
\]
Lemma 4.2. Suppose (1.8), \((w, v), (\tilde{w}, \tilde{v}) \in X_\varepsilon\). Set
\[
\tilde{\omega}_1 = \begin{cases} 
  r(t-r)^{\mu/p + q - 2} & (r < t/2), \\
  (t-r)^{\mu/p} (t+r)^{q-1} & (r \geq t/2);
\end{cases}
\]
\[
\tilde{\omega}_2 = \begin{cases} 
  r^p (t+r)^{2-p+\mu/pq} & (r < t/2), \\
  (t-r)^{\mu/pq} (t+r) & (r \geq t/2).
\end{cases}
\]
Then we have
\[
(4.8) \quad \|\tilde{\omega}_2 L|w|^p - |\tilde{w}|^p \|_{L_t^\infty_{r,v}} \leq C\|\tilde{\omega}_1 (w, \tilde{w})\|_{L_t^\infty_{r,v}}^{p-1} \|\tilde{\omega}_1 (w - \tilde{w})\|_{L_t^\infty_{r,v}},
\]
\[
(4.9) \quad \|\tilde{\omega}_1 r^{-1} K_+ (|v|^q - |\tilde{v}|^q) \|_{L_t^\infty_{r,v}} \leq C\|\tilde{\omega}_2 (v, \tilde{v})\|_{L_t^\infty_{r,v}}^{q-1} \|\tilde{\omega}_2 (v - \tilde{v})\|_{L_t^\infty_{r,v}}.
\]
Here, applying the fixed point theorem with mapping
\[
P : (w, v) \mapsto (Pw, Pv) := (\varepsilon \partial_t u_o + r^{-1} K_+ |v|^q, \varepsilon v_o + L|w|^p).
\]
Firstly we check that \(P\) is well defined in \(X_\varepsilon \to X_\varepsilon\). By expression (4.1)-(4.3), it is obvious that \(\text{supp}(Pw, Pv) \subset \{t-r \geq 1, t \geq 2\}\) and \(Pw, Pv, r\partial_t Pv \in C(\mathbb{R}^+ \times \mathbb{R})\).
To estimate \(\|(Pw, Pv)\|\), we have
\[
\|\omega_1 Pw\|_{L_t^\infty_{r,v}} \leq \varepsilon \|\omega_1 \partial_t u_o\|_{L_t^\infty_{r,v}} + \|\omega_1 r^{-1} K_+ |v|^q\|_{L_t^\infty_{r,v}}.
\]
Since \(f \in C^2, \ g \in C^1,\) and \(\text{supp} u_o \subset \{(t, r) : 3 \geq t - r \geq 1\}\) where \(\omega_1 \lesssim \langle t \rangle\). We have
\[
\|\omega_1 \partial_t u_o\|_{L_t^\infty_{r,v}} \leq C_{f,g}.
\]
Moreover, by (4.6), and \((w, v) \in X_\varepsilon,\) we have
\[
\|\omega_1 Pw\|_{L_t^\infty_{r,v}} \leq C_{f,g} \varepsilon + C\varepsilon^q.
\]
The estimates for the remaining terms are similar, notice \(r \partial_t Pv = \partial_t (r Pv) - Pv\) and \(\omega_3 \lesssim \omega_2,\) by (4.5) and (4.7) we finally have
\[
\|\omega_1 Pw\| \leq C_{f,g, g, g} \varepsilon + C\varepsilon^p + C\varepsilon^q \leq C_1 \varepsilon
\]
for \(C_1 \geq 2C_{f,g,f,g}\) and \(\varepsilon\) small enough.
Similiarly, by (4.8) and (4.9) we have \(P\) is contraction in a weaker sense. But it’s enough to obtain the fixed point \((u, v)\) solves (4.1)-(4.3) which completes the proof.

4.1. Proof of (4.5) and (4.8).

First we prove (4.5). Let \(r > 0,\) Consider \(D = \{(s, \rho) : t - r \leq s + \rho \leq t + r, 1 \leq s - \rho \leq t - r\}\) is the influence domain of \((t, r)\) intersect \(\{(s, \rho) : s - \rho \geq 1\}\). Set \(D_1 = D \cap \{(s, \rho) : \rho < s/2\}, \) \(D_2 = D \cap \{(s, \rho) : \rho \geq s/2\}\), then
\[
|\omega_2 L|w|^p| \leq \omega_2^\mu r \int_D \rho \omega_1^{-p}(s, \rho) \|\omega_1 w\|_{L_t^\infty_{r,v}}^p \, d\rho 
\]
\[
= C_{\omega_2} \|\omega_1 w\|_{L_t^\infty_{r,v}}^p \left( \int_{D_1} + \int_{D_2} \right) \rho \omega_1^{-p}(s, \rho) \, d\rho.
\]
Part 1: \((s, \rho) \in D_1,\)
Here \( \langle s \rangle \approx \langle s - \rho \rangle \approx \langle s + \rho \rangle \), take \( \tau = s + \rho, \sigma = s - \rho \), by (4.4) we have

\[
\frac{1}{r} \int_{D_1} \rho \omega_1^{-p}(s, \rho) \, d\rho \, ds = \frac{1}{r} \int_{D_1} \rho \langle \rho \rangle^{-p} \langle s - \rho \rangle^{-\mu - p q + 2p} \, d\rho \, ds
\]

(4.11)

\[
\leq \frac{C}{r} \int_{t-r}^{t-r} \int_{(t-r)/3}^{t-r} \langle \tau - \sigma \rangle^{1-p} \langle \tau \rangle^{\mu - pq + 2p} \, d\tau \, d\sigma
\]

\[
\leq \frac{C}{r} \int_{t-r}^{t-r} \langle \tau - t + r \rangle^{2-p} \langle \tau \rangle^{\mu - pq} \, d\tau.
\]

**Part 1.1:** \( r < t/2 \).

Here \( t - r \approx t \approx t + r \), we have

\[
RHS \ of \ (4.11) \leq C(t + r)^{p-3-\mu/pq} r^{-1} ((r + 1)^{3-p} - 1^{3-p})
\]

(4.12)

\[
\leq C(t + r)^{p-3-\mu/pq} (r)^{2-p}
\]

\[
= C\omega_2^{-1}.
\]

**Part 1.2:** \( r \geq t/2 \).

Here \( r \approx t \approx t + r \), we have

\[
RHS \ of \ (4.11) \leq C(t)^{-1} \left( (t - r)^{p-3-\mu/pq} \int_{t-r}^{t-r} \langle \tau - t + r \rangle^{2-p} \, d\tau \right.
\]

\[
+ \left. \int_{t-r}^{t-r} \langle \tau \rangle^{1-\mu/pq} \, d\tau \right)
\]

(4.13)

\[
\leq C(t)^{-1} (t - r)^{-\mu/pq}
\]

\[
\leq C\omega_2^{-1}.
\]

**Part 2:** \( (s, \rho) \in D_2 \).

Here \( \langle \rho \rangle \approx \langle s \rangle \approx \langle s + \rho \rangle \), take \( \tau = s + \rho, \sigma = s - \rho \), then

\[
\frac{1}{r} \int_{D_2} \rho \omega_1^{-p}(s, \rho) \, d\rho \, ds = \frac{1}{r} \int_{D_2} \rho \langle \rho \rangle^{-\mu} \langle s - \rho \rangle^{-p(q-1)} \, d\rho \, ds
\]

(4.14)

\[
\leq \frac{C}{r} \int_{t-r}^{t-r} \langle \tau \rangle^{1-p(q-1)} \, d\tau \int_{t-r}^{t-r} \langle \sigma \rangle^{-\mu} \, d\sigma.
\]

**Part 2.1:** \( r < t/2 \).

(4.15)

\[
RHS \ of \ (4.14) \leq C(t - r)^{1-p(q-1)} (t - r)^{1-\mu} \leq C\omega_2^{-1}.
\]

**Part 2.2:** \( r \geq t/2 \).

(4.16)

\[
RHS \ of \ (4.14) \leq C(t)^{-1} (t - r)^{2-p(q-1)} (t - r)^{1-\mu} \leq C\omega_2^{-1}.
\]

Thus (4.10)-(4.16) gives \( \|\omega_2 L[w]^p\| \leq C \|\omega_1 w\|^{p}_{L^\infty} \) and completes the proof. The proof of (4.8) is similar, since that \( \|a|^q - |b|^q\| \lesssim (|a|^{q-1} + |b|^{q-1}) |a - b| \) for \( q > 1 \).
4.2. Proof of (4.6) and (4.9).

For (4.6), we divide the proof into two main cases: \( r \geq 1/4 \) and \( r < 1/4 \).

**Part 1:** \( r \geq 1/4 \).

For this situation, we have \( r \approx \langle r \rangle \). Similar to the proof of (4.5) we have

\[
|\omega_1 r^{-1} K_+ [v]_q| \leq \|\omega_2 v\|_{L^q_{t,r}}^q \omega_1 r^{-1} \int_0^t \frac{(r + t - s)}{\omega_2(s, r + t - s)} + \frac{|r - t + s|}{\omega_2(s, |r - t + s|)} ds
\]

\[
\equiv \|\omega_2 v\|_{L^q_{t,r}}^q \omega_1 r^{-1} \int_0^t I + II ds
\]

**Part 1.1:** \( r < t/2 \). Estimates about \( I \).

In this part, we have \( \omega_1 r^{-1} \approx (t - r)^{\mu/p + q - 2} \).

**Part 1.1.1:** \( 0 \leq s < 2(r + t)/3 \).

This means \( r + t - s > s/2 \), then

\[
\int_0^{2(t+r)/3} I ds = \int_0^{2(t+r)/3} (r + t - s)(2s - r - t)^{-\mu/p} (r + t)^{-q} ds
\]

\[
\leq C(t + r)^{1-q} \int_0^{2(t+r)/3} (2s - r - t)^{-\mu/p} ds
\]

\[
\leq C(t - r)^{2-q-\mu/p}
\]

\[
\leq Cr^{\omega_1^{-1}}.
\]

**Part 1.1.2:** \( 2(r + t)/3 \leq s \leq t \).

This means \( r + t - s \leq s/2 \), then

\[
\int_0^t I ds = \int_0^t (r + t - s)(r + t - s)^{-q(p-2)} (r + t)^{-3q+pq-\mu/p} ds
\]

\[
\leq C(t + r)^{2-q(p-2)} (r + t)^{-3q+pq-\mu/p}
\]

\[
\leq Cr^{\omega_1^{-1}}.
\]

**Part 1.2:** \( r \geq t/2 \). Estimates about \( I \).

Here we always have \( r + t - s \geq s/2 \). Then it is similar to (4.18).

**Part 1.3:** \( r \geq t/2 \). Estimates about \( II \).

**Part 1.3.1:** \( 0 \leq s < t - r \).

Here \( |r - t + s| = t - s - r \). It’s similar to (4.18)-(4.19).

**Part 1.3.2:** \( t - r \leq s < 2(t - r) \).

Here \( r - t + s < s/2 \), then

\[
\int_{t-r}^{2(t-r)} II ds = \int_{t-r}^{2(t-r)} (r - t + s)(r - t + s)^{-q(p-2)} (r - t + 2s)^{-3q+pq-\mu/p} ds
\]

\[
\leq C(t - r)^{2-q(p-2)} (t - r)^{-3q+pq-\mu/p}
\]

\[
\leq Cr^{\omega_1^{-1}}.
\]

**Part 1.3.3:** \( 2(t - r) \leq s < t \).
Here $r-t+s \geq s/2$, then

$$\int_{2(t-r)}^{t} II \, ds = \int_{2(t-r)}^{t} (r-t+s)(t-r)^{-\mu/p}(r-t+2s)^{-q} \, ds$$

(4.21)

$\leq C(t-r)^{-\mu/p} \int_{2(t-r)}^{t} (r-t+2s)^{1-q} \, ds$

$\leq C(t-r)^{-\mu/p} (t+r)^{2-q}$

$\leq Cr\omega_{1}^{-1}$.

**Part 1.4:** $r < t/2$. Estimates about $II$.

**Part 1.4.1:** $0 \leq s < t - r$.

It’s similar to (4.18)-(4.19).

**Part 1.4.2:** $t - r \leq s \leq t$.

$$\int_{t-r}^{t} II \, ds = \int_{t-r}^{t} (r-t+s)(r-t+s)^{-q(p-2)}(r-t+2s)^{-3q+pq-\mu/p} \, ds$$

(4.22)

$\leq C(r)^{2-q(p-2)}(t+r)^{-3q+pq-\mu/p}$

$\leq Cr\omega_{1}^{-1}$.

**Part 2:** $r < 1/4$.

In this part $(r) \approx 1$. For the convenience of proof, we set $r^{+} = r + t - s$, $r^{-} = r - t + s$. Then

$$|\omega_{1}r^{-1}K_{+}\|v\|^{q}| \leq C\omega_{1}r^{-1} \int_{0}^{t} \left| r^{+}\|v\|^{q}(s,r^{+}) + r^{-}\|v\|^{q}(s,|r^{-}|) \right| \, ds$$

(4.23)

where

$$\left| r^{+}\|v\|^{q}(s,r^{+}) + r^{-}\|v\|^{q}(s,|r^{-}|) \right| \leq \left| r^{+}\|v\|^{q}(s,r^{+}) + r^{-}\|v\|^{q}(s,r^{+}) \right|$$

$$+ \left| r^{-}\|v\|^{q}(s,r^{+}) - r^{-}\|v\|^{q}(s,|r^{-}|) \right|$$

then

$$RHS \ of \ (4.23) \leq C\omega_{1} \left( \int_{0}^{t} \|v\|^{q}(s,r^{+}) \, ds + r^{-1} \int_{0}^{t} \left| r^{-}\|v\|^{q}(s,r^{+}) - \|v\|^{q}(s,|r^{-}|) \right| \, ds \right)$$

$\equiv C\omega_{1}I + C\omega_{1}II$.

**Part 2.1:** Estimates about $I$.

It’s similar to (4.18)-(4.19).

**Part 2.2:** Estimates about $II$.

For this part, we should consider $\partial_{r}v$. Since $r^{+} - |r^{-}| \leq 2r$,

$$r^{-1}\left| \|v\|^{q}(s,r^{+}) - \|v\|^{q}(s,|r^{-}|) \right| \leq C|\partial_{r}v(s,r^{+})|\|v\|^{q-1}(s,r^{+})$$
for $|r^-| < r^*(s) < r^+$. Then

$$II \leq C \int_0^t |r^-| |\partial_r v(s, r^*)| |v|^{q-1}(s, r^*)\, ds$$

$$\leq C \|\omega_2 v\|_{L^q_t}^{q-1} \|\omega_3 r \partial_r v\|_{L^\infty_{t,r}} \int_0^t \left(\omega_2^{-\frac{q-1}{2}} \omega_3^{-1}\right)(s, r^*)\, ds$$

$$\equiv C \|\omega_2 v\|_{L^q_t}^{q-1} \|\omega_3 r \partial_r v\|_{L^\infty_{t,r}} \int_0^t III\, ds$$

Here we notice $\langle t \rangle \lesssim \langle |r^-| + s + r \rangle \lesssim \langle r^* + s \rangle$.

**Part 2.2.1: $r^* < s/2$.**

Here $(s \pm r^*) \gtrsim (r^*) \gtrsim \langle t - s \rangle$, by (4.4), we have

$$\omega_1 \int_{2r^* < s < t} III\, ds \leq C \int_{2r^* < s < t} \langle r^* \rangle^{1-q}(p-2)$$

$$\times \langle s - r^* \rangle^{1/p-2} \langle s - r^* \rangle^{3-p+\mu/pq} \, ds$$

$$\leq C \int_{2r^* < s < t} \langle r^* \rangle^{1-p+2p+\mu/pq} \, ds$$

$$\leq C \int_{2r^* < s < t} \langle t - s \rangle^{p-4} \, ds$$

$$\leq C.$$  

**Part 2.2.2: $r^* \geq s/2$.**

Here $(s + r^*) \gtrsim (s - r^*) \gtrsim \langle 2s - t \rangle$, similarly we have

$$\omega_1 \int_{s < 2r^* \wedge t} III\, ds \leq C \int_{s < 2r^* \wedge t} \langle 2s - t \rangle^{p-4} \, ds \leq C.$$

In sum, we complete the proof. To verify (4.9), without distinguishing whether $r > 1/4$ or not we have a proof just like (4.10)-(4.22).

### 4.3. Proof of (4.7).

We follow the same process as before

$$|\omega_3 K_-| \leq ||\omega_1 v||_{L^p_t}^p \int_0^t \frac{r + t - s}{\omega_1(s, r + t - s)^p} + \frac{|r - t + s|}{\omega_1(s, |r - t + s|)^p} \, ds.$$

Both part in the integration is similar to the last proof. Here we only show the proof of $r \geq t/2$, $(t - r) \leq s$ for $|r - t + s|$ part.

**Part 1: $t - r \leq s < 2(t - r)$.**

Here $r - t + s < s/2$, then

$$\int_{t-r}^{2(t-r)} \frac{r - t + s}{\omega_1(s, r - t + s)^p} \, ds = \int_{t-r}^{2(t-r)} (r - t + s)(r - t + s)^{-p} \langle r - t + 2s \rangle^{-\mu - pq + 2p} \, ds$$

$$\leq C \langle t - r \rangle^{-\mu - pq + 2p}$$

$$\leq C \omega_3^{-1}.$$  

**Part 2: $2(t - r) \leq s \leq t$.**
Here \( r - t + s \geq s/2 \), then
\[
\int_{2(t-r)}^{t} \frac{r - t + s}{\omega_1(s, r - t + s)^p} \, ds = \int_{2(t-r)}^{t} (r - t + s)(t - r)^{-\mu} \, ds,
\]
\[
\leq C(t - r)^{-\mu} \int_{2(t-r)}^{t} (r - t + s)^{1-pq + p} \, ds,
\]
\[
\leq C(t - r)^{-\mu} (t - r)^{2-pq + p},
\]
\[
\leq C \omega_3^{-1},
\]
which complete the proof.

5. Proof of Theorem 1.3

Following [18], we introduce the two positive functions
\[
\phi(x) = \int_{S^{n-1}} e^{x \cdot \omega} \, d\omega, \quad \psi(t, x) = e^{-t} \phi(x).
\]
We already know some good properties of them.

**Lemma 5.1.** For \( \phi, \psi \) defined above, then
\[
\psi(t, x) \leq C(r)^{-(n-1)/2} e^{-t},
\]
\[
\|\psi(t, x)\|_{L^p_T(0, t+1)} \leq C(t)^{(n-1)(1/p - 1/2)}.
\]

**Proof.** The proof can be found in [18]. \( \square \)

As in [19], we define
\[
F(t) = \int_{\mathbb{R}^3} u(t, x) \psi(t, x) \, dx, \quad G(t) = \int_{\mathbb{R}^3} v(t, x) \psi(t, x) \, dx.
\]
Since \( u, v, \partial_t u, \partial_t v \in C(0, T; L^1(\mathbb{R}^n)) \), we find that \( F, G \in C^1(0, T) \). And by the definition of weak solution to (1.4), we have

\[
F'(t) = \int_{\mathbb{R}^3} u_t(t, x) \psi(t, x) + u(t, x) \psi_t(t, x) \, dx
\]
\[
= \int_{\mathbb{R}^3} u_t(t, x) \psi(t, x) \, dx - F(t),
\]
\[
\Rightarrow F''(t) = \int_{\mathbb{R}^3} u_{tt}(t, x) \psi(t, x) + u_t(t, x) \psi_t(t, x) \, dx - F'(t)
\]
\[
= \int_{\mathbb{R}^3} |\psi|^q(t, x) \psi(t, x) + u(t, x) \Delta \psi(t, x) \, dx - 2F'(t) - F(t)
\]
\[
= \int_{\mathbb{R}^3} |\psi|^q(t, x) \psi(t, x) \, dx - 2F'(t),
\]
similarly, we have
\[
G''(t) = \int_{\mathbb{R}^3} |\partial_t u(t, x)|^p \psi(t, x) \, dx - 2G'(t).
\]
Since $v \in C([0,T);L^q(\mathbb{R}^n))$, $\partial_t u \in C([0,T);L^p(\mathbb{R}^n))$, we conclude $F,G \in C^2(0,T)$. Then use Hlder’s inequality and Lemma 5.1, we can get

$$
\left\{ \begin{array}{c}
F''(t) + 2F'(t) = \int_{\mathbb{R}^n} |v(t,x)|^q \psi(t,x) \, dx \\
\quad \geq C(t)^{(n-1)(1-q)/2} |G(t)|^q,
\end{array} \right.
$$

(5.1)

$$
\left\{ \begin{array}{c}
G''(t) + 2G'(t) = \int_{\mathbb{R}^n} |\partial_t u(t,x)|^p \psi(t,x) \, dx \\
\quad \geq C(t)^{(n-1)(1-p)/2} |F'(t) + F(t)|^p.
\end{array} \right.
$$

Notice that by (5.1) we have $F''(t) + 2F'(t) \geq 0$, and by assumption we have $F'(0) = \int_{\mathbb{R}^n} (g - f) \phi \, dx \geq 0$, then it is easy to conclude that $F'(t) \geq 0$. Moreover, since $F(0) = \int_{\mathbb{R}^n} f \phi \, dx \geq 0$, we have $F(t) \geq 0$ for all $t \geq 0$. Then it is obvious that

$$
|F'(t) + F(t)| \geq \frac{1}{2} F'(t) + 2F(t).
$$

By a similar argument, we have $G'(t) \geq 0$, $G(t) \geq C \varepsilon$ since $\tilde{f}$ does not vanish identically. Set $H(t) = F'(t) + 2F(t)$, then we have $H, G \in C^1(0,T) \times C^2(0,T)$ and

$$
\left\{ \begin{array}{c}
H'(t) \geq C(t)^{(n-1)(1-q)/2} G(t)^q, \\
H(t) \geq 0, \\
G'(t) \geq 0; \\
G''(t) + 2G'(t) \geq C(t)^{(n-1)(1-p)/2} H(t)^p, \\
G(t) \geq C \varepsilon, \\
G'(t) \geq 0.
\end{array} \right.
$$

Lemma 5.2. For system (5.2), assume (1.9) is satisfied, then for any $M > 0$, there exists $\tilde{A}, \tilde{T}$ may depend on $\varepsilon$, such that for any $t \geq \tilde{T}$,

$$
G(t) \geq \tilde{A} t^M.
$$

Lemma 5.3. Under the same assumption of Lemma 5.2, the system (5.2) must blow up in finite time.

Since the blow up of $(H, G)$ in (5.2) actually means the blow up of $(u, v)$ in (1.4), we complete the proof.

5.1. Proof of Lemma 5.2.

We use the method of induction, first we have

$$
G(t) \geq A_0 t^\alpha_0 \quad \forall t > T_{0,0}, \quad A_0 = C \varepsilon, \quad \alpha_0 = 0, \quad T_{0,0} = 0.
$$

Now assuming

$$
G(t) \geq A_k t^{\alpha_k} \quad \forall t > T_{k,0},
$$

for some $\alpha_k, A_k, T_{k,0}$, then by system (5.2) we have

$$
H'(t) \geq CA_k t^{(n-1)(1-q)/2 + q\alpha_k} \quad \forall t > T_{k,0}
$$

$$
\Rightarrow H(t) \geq H(T_{k,0}) + CA_k \int_{T_{k,0}}^t s^{(n-1)(1-q)/2 + q\alpha_k} \, ds
$$

$$
\geq CA_k t^{(n-1)(1-q)/2 + 1 + q\alpha_k} \quad \forall t > T_{k,1},
$$
for some $T_{k,1}$ depends on $T_{k,0}$. By system (5.2) again, we then have
\[
G''(t) + 2G'(t) \geq CA_k^{pq}(t)^{(n-1)(1-p)/2+p((n-1)(1-q)/2+1+pq\alpha_k)} \quad \forall \ t > T_{k,1}
\]
\[
\Rightarrow (G'(t)e^{2t})' \geq CA_k^{pq}(t)^{(n-1)(1-p)/2+p(1+pq\alpha_k)} e^{2t} \quad \forall \ t > T_{k,1}
\]
\[
\Rightarrow G'(t)e^{2t} \geq CA_k^{pq}(t)^{(n-1)(1-p)/2+p+pq\alpha_k} e^{2t} \quad \forall \ t > T_{k,2}
\]
\[
\Rightarrow G(t) \geq CA_k^{pq}(t)^{(n-1)(1-p)/2+p+1+pq\alpha_k} \quad \forall \ t > T_{k,3},
\]
in a similarly way, where $T_{k,2}, T_{k,3}$ big enough. Set
\[
A_{k+1} = CA_k^{pq}, \quad \alpha_{k+1} = \frac{n-1}{2} (1 - pq) + p + 1 + pq\alpha_k, \quad T_{k+1,0} = T_{k,3}.
\]
Notice that $\alpha_0 = 0$ and (1.9) is satisfied, which means
\[
\frac{n-1}{2} (1 - pq) + p + 1 > 0.
\]
We get $\alpha_k \to \infty$ when $k \to \infty$ and complete the proof.

5.2. **Proof of Lemma 5.3.**

First of all we want to simplify the system (5.2). Take a $\tilde{T}$ which will be determined later, then for any $t > \tilde{T},$
\[
H(t)(G'(t) + 2G(t)) = \int_\tilde{T}^t H'(s)(G'(s) + 2G(s)) \, ds
\]
\[
+ \int_\tilde{T}^t H(s)(G''(s) + 2G'(s)) \, ds + H(\tilde{T})(G'(\tilde{T}) + 2G(\tilde{T}))
\]
\[
\geq C \int_\tilde{T}^t (s)^{(n-1)(1-q)/2} G(s)^q (G'(s) + 2G(s)) \, ds
\]
\[
\geq C\tilde{T}^{-1}(n-1)(1-q)/2 \int_\tilde{T}^t G(s)^q G'(s) \, ds
\]
\[
\geq C\tilde{T}^{-1}(n-1)(1-q)/2 \left( G^{q+1}(t) - G^{q+1}(\tilde{T}) \right).
\]

By (5.2) again, we have
\[
(G'(t) + 2G(t))^p \geq C\tilde{T}^{-1}(n-1)(1-p)/2 \left( H(t)(G'(t) + 2G(t)) \right)^p
\]
\[
\Rightarrow \partial_t (G'(t) + 2G(t))^{p+1} \geq C\tilde{T}^{-1}(n-1)(1-p)/2 \left( G^{q+1}(t) - G^{q+1}(\tilde{T}) \right)^p
\]
\[
\Rightarrow (G'(t) + 2G(t))^{p+1} \geq C \int_\tilde{T}^t (s)^{(n-1)(1-p)/2} \left( G^{q+1}(s) - G^{q+1}(\tilde{T}) \right)^p \, ds
\]
\[
\geq C\tilde{T}^{-1}(n-1)(1-p)/2 \int_\tilde{T}^t \left( G^{q+1}(s) - G^{q+1}(\tilde{T}) \right)^p \, ds
\]
for $t > \tilde{T}.$ Here
\[
\int_\tilde{T}^t \left( G^{q+1}(s) - G^{q+1}(\tilde{T}) \right)^p \, ds \geq C \int_\tilde{T}^t \left( G^{q+1}(s) - G^{q+1}(\tilde{T}) \right)^p \frac{(G(s)^q + 1)'}{(G'(s) + 2G(s))G(s)^q} \, ds
\]
\[
\geq C \frac{\int_\tilde{T}^t \left( G^{q+1}(s) - G^{q+1}(\tilde{T}) \right)^p (G(s)^q + 1)' \, ds}{(G'(t) + 2G(t))G(t)^q}.
\]
since $G, G' > 0$ for $t > 0$ and $G' + 2G, G$ are monotonically increasing to infinity. Then if we choose a $\tilde{T}_1 > \tilde{T}$ such that $G(\tilde{T}_1)^{q+1} \geq 2G(\tilde{T})^{q+1}$, we have for any $t > \tilde{T}_1$

$$(G'(t) + 2G(t))^{p+2} \geq C(t)^{(n-1)(1-pq)/2} \frac{G^{q+1}(t) - G^{q+1}(\tilde{T})}{G(t)^q}^{p+1}$$

$$\geq C(t)^{(n-1)(1-pq)/2} G(t)^{pq+p+1}$$

$$\Rightarrow G'(t) + 2G(t) \geq C(t)^{(n-1)(1-pq)/2(p+2)} G(t)^{(pq+p+1)/(p+2)}.$$ 

By Lemma 5.2, take $M$ large enough, we will have

$$G'(t) + 2G(t) \geq C_\epsilon G(t)^{1+\delta}$$

for some $\delta > 0$ and any $t > \tilde{T}_1$. Moreover we can always take a $\tilde{T}_2 > \tilde{T}_1$ such that $G(\tilde{T}_2)$ big enough, then the ODE system must blow up in finite time, and we complete the proof.

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