Asymptotic Isometry
and
Two Dimensional Anti-de Sitter Gravity

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Abstract

In low dimensional gravity on anti-de sitter space there exists a possibility that the asymptotic isometry is raised to a Virasoro symmetry living at spatial infinity. We discuss in detail asymptotic isometry of the most general anti-de sitter dilaton gravity in two dimensions. From analysis of the boundary dynamics it turns out that the Virasoro symmetry is not preserved in all the models. Moreover even anti-de sitter isometry cannot survive and asymptotic isometry consists of only time translation for all the models except Jackiw-Teitelboim (JT) model. Meanwhile there exists a hybride nonsinglet representation of the isometry in JT model.
1 Introduction

The conception of asymptotic isometry of anti-de sitter (AdS) space has recently attracted much attention from the AdS/CFT correspondence viewpoint\cite{2} and gives a lot of information about the black hole entropy problem, strong coupling behavior of gauge theory and string theory\cite{3,4}. Here AdS/CFT correspondence means that supergravity on AdS space is equivalent with a conformal field theory on the boundary.

Originally the asymptotic isometry of three dimensional anti-de sitter space was discussed in 1986 by Brown and Henneaux\cite{1}. They showed that the asymptotic symmetry of Einseiten gravity on AdS space is not just the $SO(2,2)$ AdS isometry but rather a two dimensional conformal (Virasoro) symmetry at spatial infinity which includes the isometry as a subgroup. They also identified the central charge of the Virasoro algebra with

$$c = \frac{3}{2G\sqrt{-\Lambda}}$$

where $G$ is the gravitational constant and $\Lambda$ negative cosmological constant.

In two dimensions there exists a possibility that the AdS asymptotic isometry is also raised to a conformal symmetry on the boundary just like in three dimensions.

In this paper we analyze asymptotic isometry of two dimensional gravity in detail. We focus on AdS dilaton gravity models in which all the spacetimes of solutions possess a negative constant curvature. The AdS gravity models include Liouville induced gravity model\cite{5} and Jackiw-Teitelboim (JT) model\cite{6} with negative cosmological constant.

We prove that nontrivial asymptotic isometry of general AdS dilaton gravity models except JT model is only the time translation. If the conformal symmetry on the boundary is respected, the canonical representation is forced to be singlet.

On the other hand, the situation slightly changes in JT model. We show that the AdS symmetry can survive as asymptotic isometry in a nonsinglet representation composed of different canonical representation sectors.

Our result suggests that even when the models are extented to supergravity and quantized the asymptotic states do not belong to any representaion
of the boundary conformal group and that AdS/CFT correspondence does not work well in two dimensions. This aspect is quite different from in three dimensions.

Here we want to comment on relations with other works already performed. A canonical representation of JT model has been already discussed by Henneaux[7]. However he concentrated on an AdS variant boundary condition and did not deal with any combined representations to recover the AdS symmetry.

Quantization of JT model has been also investigated[7, 8]. However our nonsinglet representation has not yet been treated explicitly. In this paper quantum realization of the AdS symmetry of JT model is beyond the scope.

2 AdS Space in Two Dimensions

Let us first review in brief two dimensional AdS space. This can be expressed as an embedded hypersurface in three dimensional flat spacetime with its metric signature \((-1, -1, 1)\).

\[-T_1^2 - T_2^2 + X^2 = -\frac{1}{\mu^2},\]  
\[ds^2 = -dT_1^2 - dT_2^2 + dX^2.\]

Here \(\mu\) is a constant related with the curvature \(R = -2\mu^2\).

Taking the following parametrization:

\[T_1 = \frac{1}{\mu} \cos(\mu t) \sin(\mu x),\]  
\[T_2 = \frac{1}{\mu} \sin(\mu t) \sin(\mu x),\]  
\[X = \frac{1}{\mu} \cot(\mu x),\]

a metric in the full covered chart is obtained as

\[ds^2 = \frac{1}{\sin^2(\mu x)}(-dt^2 + dx^2),\]
where the coordinate variables run between

\[-\infty < t < \infty, \]
\[-\frac{\pi}{\mu} < x < 0. \]

The spatial infinity stays at \(x = 0\) and \(x = -\frac{\pi}{\mu}\). In the following discussion, we will discuss asymptotic AdS space in which the space approaches to AdS space near \(x \sim -0\). So omit another infinity and spacetimes with no asymptotic region on the left side can be taken account of.

It is a well known fact that the metric eqn(6) has three killing vectors which forms \(SO(1, 2)\) group and its infinitesimal transformation can be explicitly written as follows.

\[x'\mu = x\mu - \epsilon^\mu(x),\]
\[\epsilon^t = \epsilon_o + \epsilon_1 \cos(\mu x) \sin(\mu t + \delta),\]
\[\epsilon^x = \epsilon_1 \sin(\mu x) \cos(\mu t + \delta),\]

where \(\epsilon_0, \epsilon_1\) and \(\delta\) are group element parameters. These equations are also rewritten in a compact form using the conformal coordinates as

\[x^\pm = t \pm x,\]
\[\epsilon^\pm = \epsilon_0 + \epsilon_1 \sin(\mu x^\pm + \delta).\]

Generators of these transformations are defined as

\[L_0 = \frac{i}{\mu} \partial_x,\]
\[L_1 = \frac{i}{\mu} e^{i\mu x} \partial_x,\]
\[L_{-1} = \frac{i}{\mu} e^{-i\mu x} \partial_x,\]

and form the \(SO(1,2)\) algebra:

\[[L_0, L_{\pm1}] = \mp L_{\pm1},\]
\[[L_1, L_{-1}] = 2L_0.\]
This isometry is called AdS symmetry and has a root in maximally symmetric nature of the AdS space. The algebra has a casimir operator which is defined to commute with all the generators, and is expressed as

\[ M^2 = \mu^2 \left[ L_0^2 - \frac{1}{2} (L_1 L_{-1} + L_{-1} L_1) \right]. \]  

(19)

There is another coordinate system frequently used because of simplicity of the metric form. Substitution of a parametrization form:

\[ T_1 = -\frac{1}{2\mu} \left( \frac{1}{\mu x} + \mu x \right) + \frac{t^2}{2x}, \]  

(20)

\[ T_2 = -\frac{t}{\mu x}, \]  

(21)

\[ X = -\frac{1}{2\mu} \left( \frac{1}{\mu x} - \mu x \right) - \frac{t^2}{2x}, \]  

(22)

into eqn(2) yields

\[ ds^2 = \frac{1}{\mu^2 x^2} (-dt^2 + dx^2). \]  

(23)

In this chart the coordinate variables take values between \(-\infty < x < 0, \infty < t < \infty\) and the spatial infinity stays at \(x = -0\).

Next we precisely define asymptotic anti-de sitter (AAdS) space in two dimensions in the same spirit of the three dimensional one proposed by Brown and Henneaux.[1]

To fix boundary conditions of AAdS spaces, we must analyze a little bit what should be a AAdS space. Let us divide a metric of a will-be AAdS space into the AdS background metric (23) and its deviation near the spatial infinity \((x \sim -0)\).

\[ g_{ab} = \bar{g}_{ab} + h_{ab}, \]  

(24)

\[ \bar{g}_{ab} dx^a dx^b = \frac{1}{\mu^2 x^2} (-dt^2 + dx^2). \]  

(25)

The asymptotic form of \(h_{ab}\) near \(x = -0\) should be specified by some boundary conditions for the space to be AAdS. Also it must be stressed that the boundary conditions are invariant under the \(SO(1, 2)\) AdS isometry. This is
a crucial point clarified by Brown and Henneaux. Thus let us AdS transform the metric at the infinity by using eqns(9)(10)(11).

\[
\delta_h = \nabla_a \epsilon_b + \nabla_b \epsilon_a = \bar{g}_{ac} \nabla_b \epsilon^c + \bar{g}_{bc} \nabla_a \epsilon^c + \epsilon^c \nabla_c h_{ab} + h_{ac} \nabla_b \epsilon^c + h_{bc} \nabla_a \epsilon^c + o(h^2). \tag{26}
\]

This relations take simple forms in the conformal coordinates:

\[
\delta_h^{\pm \pm} = 2h^{\pm \pm} \partial^{\pm \pm} + \epsilon^+ \partial_+ h^{\pm \pm} + \epsilon^- \partial_- h^{\pm \pm}, \tag{27}
\]

\[
\delta_h^{+-} = \left( -\frac{1}{2 \mu^2 x^2} + h_{+-} \right) \left[ \partial_+ \epsilon^+ + \partial_- \epsilon^- - \frac{\epsilon^+ - \epsilon^-}{x} \right] + \epsilon^+ \partial_+ h_{+-} + \epsilon^- \partial_- h_{+-} + \frac{\epsilon^+ - \epsilon^-}{x} h_{+-}. \tag{28}
\]

It is important to note in eqn (28) that

\[
\lim_{x \to 0} \left( -\frac{1}{2 \mu^2 x^2} + h_{+-} \right) \left[ \partial_+ \epsilon^+ + \partial_- \epsilon^- - \frac{\epsilon^+ - \epsilon^-}{x} \right] = -\frac{1}{3 \mu^2} \frac{d^3}{dt^3} [\epsilon_0 + \epsilon_1 \sin(\mu t + \delta_\epsilon)] = o(1). \tag{29}
\]

This quantity is of the zeroth order of \( x \) near \( x = -0 \) and thus nonvanishing in the asymptotic region. Consequently the weakest boundary condition for the \( g_{+-} \) component of AAdS metric should take a form as

\[
g_{+-} = -\frac{1}{2 \mu^2 x^2} + o(1). \tag{30}
\]

Inspired by the Brown-Henneaux argument, we next impose another constraint on the AAdS boundary condition search. We will discuss later the AdS gravity model in which spacetimes of the solutions is the AdS space with \( R = -2 \mu^2 \). Thus the model always possesses the two dimensional BTZ blackhole solution:

\[
ds^2 = -\left( \mu^2 r^2 - a \right) dt^2 + \frac{dr^2}{\mu^2 r^2 - a} \tag{31}
\]

5
where $a$ is a constant related with mass. To map $r = \infty$ to $x = -0$, we change the coordinate as

$$x = -\frac{1}{\mu^2 r}.$$  

Then the metric behaves near $x = -0$ like

$$ds^2 = \frac{1}{\mu^2 x^2}(-dt^2 + dx^2) + a(dt^2 + dx^2) + o(x^2)$$

$$= -\frac{dx^+ dx^-}{\mu^2 x^2} + \frac{a}{2}(dx^2 + dx^{-2}) + o(x^2).$$  \hspace{1cm} (32)

The second constraint is that this solution(32) must also be AAdS. This is a just similar requirement to the three dimensional one [1] that point particle solutions should be AAdS. Taking account of eqns(27)(32), the AAdS boundary condition for the $g_{\pm\pm}$ component is specified as

$$g_{\pm\pm} = o(1).$$  \hspace{1cm} (33)

Next we argue a possibility of asymptotic isometry extension. When some of general coordinate transformations make the AAdS metric boundary conditions(30)(33) invariant, the symmetry is called asymptotic isometry of the AdS space. From the two dimensional nature one can notice a chance for the asymptotic isometry to be raised to a Virasoro symmetry. Let us consider a conformal transformation:

$$x' = F(x),$$  \hspace{1cm} (34)

where $F(x)$ is an arbitrary function. Then it can be shown that metrics satisfying the boundary condition (30)(33) are transformed into those which also satisfy the same conditions (30)(33):

$$ds^2 = -\frac{dx' dx'^{-}}{\mu^2 x'^2} + o(1)dx'^{2} + o(1)dx'^{-2} + o(1)dx'^{+} dx'^{-}$$

$$= -dx^+ dx^- \left( \frac{1}{\mu^2 x^2} + \frac{1}{\mu^2} \left[ \frac{2}{3} \frac{dF(t)}{dt} \frac{d^2 F(t)}{dt^2} - \left( \frac{d^2 F(t)}{dt^2} \right)^2 \right] \right)$$

$$+ o(1)dx^2 + o(1)dx^{-2} + o(1)dx^+ dx^- + o(x^2).$$
Thus from the observation of asymptoticity of AAdS metrics, the asymptotic isometry can be conjectured the Virasoro symmetry to which the $SO(1,2)$ AdS isometry belongs as a subgroup.

This conjecture sounds quite fascinating. But rigorously speaking, in order to check whether the asymptotic conformal symmetry really survives or not, detailed analysis of the dynamics is needed. In fact, as we will argue later, the Virasoro symmetry does not realize in the following AdS dilaton gravity. Moreover even for the $SO(1,2)$ AdS isometry it is impossible to construct a nonsinglet representation except JT model.

### 3 AdS Dilaton Gravity

Because Einstein gravity in two dimensions has trivial dynamics, some matter fields are requested if one wants to construct meaningful gravitational models. Here we introduce a scalar dilaton field and write down the most general action of the models in which all spacetime solutions are the AdS space with $R = -2\mu^2$. The models are called AdS dilaton gravity. Imposing renormalizability, the action reads

$$S = \frac{1}{16\pi G} \int d^2 x \sqrt{-g} \left[ \phi R + 4\lambda (\nabla \phi)^2 + \frac{\mu^2}{4\lambda} \left( U_o e^{8\lambda \phi} - 1 \right) \right], \quad (35)$$

where $\phi$ is dilaton field, $G$ the gravitational constant, $\lambda$ and $U_o$ are real constants. When $U_o = 0$ and negative $\lambda$, the model is reduced into the Liouville induced gravity model, as is well known. While if we take $U_o = 1$ and $\lambda \to 0$, the model is equivalent with JT model.

Equations of motion are easily derived from the action.

$$R = 8\lambda \nabla^2 \phi - 2\mu^2 U_o e^{8\lambda \phi}, \quad (36)$$

$$\left( g_{ab} \nabla^2 - \nabla_a \nabla_b \right) \phi + 4\lambda \left[ \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla \phi)^2 \right] = \frac{\mu^2}{8\lambda} \left[ U_o e^{8\lambda \phi} - 1 \right] g_{ab}. \quad (37)$$
By taking the trace of eqn(37), we obtain
\[ \nabla^2 \phi = \frac{\mu^2}{4\lambda} \left[ U_o e^{8\lambda \phi} - 1 \right]. \]
Substitution of this into eqn (36) clarifies the nature of the AdS gravity:
\[ R = -2\mu^2. \] (38)

In order to find solutions, let us adopt the conformal gauge for convenience.
\[ x^\pm = t \pm x, \] (39)
\[ ds^2 = -e^{2\rho(x^+,x^-)} dx^+ dx^- . \] (40)

Then various useful relations are available as follows.
\[ \Gamma^+_{++} = 2\partial_+ \rho, \] (41)
\[ \Gamma^-_{--} = 2\partial_- \rho, \] (42)
\[ \nabla^2 = -4e^{-2\rho} \partial_+ \partial_- , \] (43)
\[ R = 8e^{-2\rho} \partial_+ \partial_- \rho. \] (44)

In the conformal gauge the equations of motion are reexpressed as
\[ \partial_+ \partial_- \rho + \frac{\mu^2}{4} e^{2\rho} = 0, \] (45)
\[ \partial_+ \partial_- \phi + \frac{\mu^2}{16\lambda} e^{2\rho} \left[ U_o e^{8\lambda \phi} - 1 \right] = 0, \] (46)
\[ \partial^2 \pm \phi - 2\partial_\pm \rho \partial_{\pm \phi} - 4\lambda (\partial_{\pm \phi})^2 = 0. \] (47)

Firstly we concentrate on the case with \( U_o \neq 0 \) and \( \lambda \neq 0 \). Then general static solutions in the conformal gauge are found with ease as follows.
\[ e^{2\rho} = \frac{M^2}{\mu^2 \sin^2(Mx)}. \] (48)
\[ \phi = -\frac{1}{4\lambda} \ln \left[ A \cot(Mx) + B \right], \] (49)
where \( B \) is a real constant, \( A \) and \( M \) are real or pure imaginary constants satisfying \( A^2 + B^2 = U_o \). Note that when \( U_o < 0 \) one cannot get any metric
form in a full covered chart with a real and static dilaton configuration. Because we have interest only in models with static background solutions, only positive $U_o$ case is later concentrated on. In this case a simple background solution is written explicitly as follows.

\begin{align}
e^{2\rho} &= \frac{1}{\sin^2(\mu x)}, \\
\phi &= -\frac{1}{4\lambda} \ln \left( -\sqrt{U_o \cot(\mu x)} \right) \\
&= -\frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_o}}{-\mu x} \right) + o(x^2). \tag{51}
\end{align}

We can also obtain general solutions in the conformal gauge:

\begin{align}
e^{2\rho} &= \frac{\dot{g}_+(x^+) \dot{g}_-(x^-)}{\left[ 1 + \frac{\mu^2}{4} g_+(x^+) g_-(x^-) \right]^2}, \\
\phi &= \frac{1}{8\lambda} \ln \left( \frac{4ab}{\mu^2 U_o} \right) - \frac{1}{4\lambda} \ln \left[ \frac{ag_+(x^+) - bg_-(x^-) + c}{1 + \frac{\mu^2}{4} g_+(x^+) g_-(x^-)} \right]. \tag{53}
\end{align}

The most general solutions are obtained by coordinate transforming solutions (52)(53) arbitrarily.

Here it should be commented that because the candidate of AAdS isometry (34) is a kind of conformal symmetry, the solutions in the conformal gauge form a partial representation sector of the candidate group (34) and can be investigated separately from nonconformal solutions from the group theoretical point of view. Moreover it can be shown straightforwardly that the results obtained in the sector remain true even for other nonconformal solutions. Therefore we focus later on what happens in the conformal gauge sector of the configuration space.

Among solutions of the sector, general AAdS solutions are picked up as follows.

\begin{align}
e^{2\rho} &= \frac{4}{\mu^2} \frac{\dot{F}(x^+) \dot{F}(x^-)}{\left[ e^{\frac{1}{2}(F(x^+)-F(x^-))} - e^{-\frac{1}{2}(F(x^+)-F(x^-))} \right]^2}, \\
\phi &= \frac{1}{8\lambda} \ln \left( \frac{AB}{U_o} \right) - \frac{1}{4\lambda} \ln \left[ \frac{Ae^{F(x^+)} + Be^{-F(x^-)} + C}{1 - e^{F(x^+)-F(x^-)}} \right]. \tag{55}
\end{align}
It is easy to see that the asymptotic forms:

\[ e^{2\rho} = \frac{1}{\mu^2 x^2} \left[ 1 + x^2 \left( 2 \frac{d^2 E}{dt^2} - \left( \frac{d^2 E}{dt^2} \right)^2 - \frac{1}{3} \left( \frac{dE}{dt} \right)^2 \right) + o(x^4) \right], \]

\[ \phi = -\frac{1}{4\lambda} \ln \left( \sqrt{\frac{U_o A e^F(t) + B e^{-F(t)} + C}{AB}} \right) + \frac{1}{4\lambda} \frac{C F}{A e^F(t) + B e^{-F(t)} + C} x + o(x^2). \]

really satisfy the AAdS conditions (30)(33).

Next we make up canonical formulation of the model. Firstly let us adopt ADM decomposition of the metric.

\[ ds^2 = -N^2 dt^2 + a^2 (dx + J dt)^2. \]

\[ g_{ab} = \begin{bmatrix} -N^2 + a^2 J^2 & a^2 J \\ a^2 J & a^2 \end{bmatrix}. \]  

(56)

Then inverse of the metric is given as

\[ g^{ab} = \begin{bmatrix} -\frac{1}{N^2} & \frac{J}{N^2} \\ \frac{J}{N^2} & \frac{1}{a^2} - \frac{J^2}{N^2} \end{bmatrix}. \]  

(57)

Also extrinsic curvature is defined as follows.

\[ K = \frac{1}{N} \left[ \partial_x J + \frac{J}{a} \partial_x a - \frac{\dot{a}}{a} \right]. \]  

(58)

Then the action is reduced into the following form.

\[ S = \frac{1}{8\pi G} \int d^2 x L, \]

\[ L = -\frac{1}{N} \left[ \dot{a} - \partial_x (aJ) \right] \left[ \dot{\phi} - J \partial_x \phi \right] + \frac{1}{a} \partial_x N \partial_x \phi \]

\[ -2\lambda \frac{a}{N} \left( \dot{\phi} - J \partial_x \phi \right)^2 + 2\lambda \frac{N}{a} (\partial_x \phi)^2 \]

\[ + \mu^2 N a \frac{8\lambda}{8\lambda} \left( U_o e^{8\lambda \phi} - 1 \right) + surface \ term. \]  

(59)
From eqn(59) conjugate momentums are defined in the usual way.

\[ \Pi_a = -\frac{1}{8\pi GN} \left[ \dot{\phi} - J \partial_x \phi \right], \]
\[ \Pi_\phi = -\frac{1}{8\pi GN} \left[ \dot{a} - \partial_x (aJ) + 4\lambda a \left( \dot{\phi} - J \partial_x \phi \right) \right]. \]

Hamiltonian and momentum constraints arise from differentiation of the action with respect to lapse and shift functions.

\[ \mathcal{H}_N = -\frac{\delta S}{\delta N} = -8\pi G \Pi_a (\Pi_\phi - 2\lambda a \Pi_a) \]
\[ + \frac{1}{8\pi G} \left[ \partial_x \left( \frac{1}{a} \partial_x \phi \right) - \frac{2\lambda}{a} (\partial_x \phi)^2 - \frac{\mu^2 a}{8\lambda} \left( U_0 e^{8\lambda \phi} - 1 \right) \right] \approx 0, \]
\[ \mathcal{H}_J = -\frac{\delta S}{\delta J} = \Pi_\phi \partial_x \phi - a \partial_x \Pi_a \approx 0. \]

It is also known that the canonical equations of motion come from the action:

\[ S_{can} = \int dt \left( \int_{-\infty}^{0} dx \left( \Pi_a \dot{a} + \Pi_\phi \dot{\phi} \right) - H[\epsilon^t = 1, \epsilon^x = 0] \right), \]
where

\[ H[\epsilon] = \int_{-\infty}^{0} dx \left( \epsilon^N \mathcal{H}_N + \epsilon^J \mathcal{H}_J \right) + Q[\epsilon], \]
\[ \epsilon^N = N \epsilon^0, \quad (61) \]
\[ \epsilon^J = \epsilon^1 + J \epsilon^0, \quad (62) \]
\[ Q[\epsilon] = \left[ \frac{1}{8\pi Ga^2} \epsilon^N \partial_x \phi (a - \bar{a}) \right]_{x=0} \]
\[ + \left[ \frac{1}{8\pi Ga} \left[ \epsilon^N \left( -\partial_x (\phi - \bar{\phi}) + 4\lambda \partial_x \phi (\phi - \bar{\phi}) \right) + \partial_x \epsilon^N (\phi - \bar{\phi}) \right] \right]_{x=0} \]
\[ + \left[ \epsilon^J \left[ \bar{a} (\Pi_a - \bar{\Pi}_a) - \bar{\Pi}_\phi (\phi - \bar{\phi}) \right] \right]_{x=0}, \]
\[ \epsilon^N = \bar{N} \epsilon^0, \quad (63) \]
\[ \epsilon^J = \epsilon^1 + \bar{J} \epsilon^0. \]
Here we need boundary conditions how the dynamical variables approach to the background values. For the general solutions of the dilaton field, we make dilaton configurations restricted as

$$\phi = -\frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_o}}{-\mu x} \right) + \sum_{n=0} \phi_n(t)x^n. \quad (64)$$

Unfortunately it is not known apriori how fast the dilaton field approaches to the background value. However, there are two typical cases as follows. Firstly we set in general $\phi_0 \neq 0$ in eqn (64). Taking account of AAdS conditions of the metric, the boundary condition is expressed as follows.

$$N = -\frac{1}{\mu x} + o(x), \quad (65)$$

$$a = -\frac{1}{\mu x} + o(x), \quad (66)$$

$$J = o(x^2), \quad (67)$$

$$\phi = \frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_o}}{-\mu x} \right) + \phi_0(t) + o(x). \quad (68)$$

This boundary condition is characterized by a fact that action of the asymptotic isometry candidate (53) on the specified configuration space is close, that is,

$$\phi = \frac{1}{4\lambda} \ln \left[ \frac{2\sqrt{U_o}}{\mu} \frac{1}{F(x^+) - F(x^-)} \right] + \phi_o \left( \frac{1}{2} \left( F(x^+) + F(x^-) \right) \right) + o(x)$$

$$= \frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_o}}{-\mu x} \right) + \phi_o(F(t)) - \frac{1}{4\lambda} \ln \dot{F}(t) + o(x). \quad (69)$$

From the boundary condition, variations of the dynamical variables in order to get the equations of motion are also constrained.

$$\delta a = o(x), \quad (70)$$

$$\delta \phi = \delta \phi_0(t) + o(x), \quad (71)$$

$$\delta \Pi_a = o(x), \quad (72)$$

$$\delta \Pi_\phi = o(1). \quad (73)$$
By making the action stationary, we get just one boundary equation of motion from the action:

$$\phi_1(t) = 0. \quad (74)$$

This condition means that

$$C = 0$$

in eqn(55).

Next let us discuss the second boundary condition in which $\phi_0$ is fixed to $0$, but $\phi_1$ does not vanish in general.

$$N = -\frac{1}{\mu x} + o(x), \quad (75)$$

$$a = -\frac{1}{\mu x} + o(x), \quad (76)$$

$$J = o(x^2), \quad (77)$$

$$\phi = -\frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_0}}{-\mu x} \right) + \phi_1(t)x + o(x^2). \quad (78)$$

It is noticed that because both AdS and Virasoro transformation (13) (34) generate a nonvanishing $\phi_0$ term, the condition breaks the symmetries explicitly. The variational conditions read

$$\delta a = o(x), \quad (79)$$

$$\delta \phi = \delta \phi_1(t)x + o(x^2), \quad (80)$$

$$\delta \Pi_a = o(x^2), \quad (81)$$

$$\delta \Pi_\phi = o(x). \quad (82)$$

In this case we do not have any nontrivial equation of motion on the boundary from the action.

For the above two cases let us consider charges of the asymptotic isometry candidate (34).

$$Q[\epsilon] = \frac{1}{16\pi G} \left[ e^{-2\tilde{\rho}(x)} \tilde{\phi} \left[ \epsilon^t (g_{xx} - \tilde{g}_{xx}) + 2\epsilon^x (g_{tx} - \tilde{g}_{tx}) \right] \right]_{x=0} - \frac{1}{8\pi G} \left[ \left( \epsilon^t \partial_x + \epsilon^x \partial_t - \epsilon^t (\tilde{\rho} + 4\lambda \tilde{\phi}) - (\partial_x \epsilon^t) \right) (\phi - \tilde{\phi}) \right]_{x=0}, \quad (83)$$
where
\[ e^{2\bar{\rho}} = \frac{1}{\sin^2(\mu x)}, \]  
\[ \bar{g}_{ab}dx^a dx^b = e^{2\bar{\rho}}(-dt^2 + dx^2), \]  
\[ \bar{\phi} = -\frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_o}}{-\mu x} \right), \]  
\[ \epsilon^t = \frac{1}{2} [\epsilon(x^+) + \epsilon(x^-)], \]  
\[ \epsilon^x = \frac{1}{2} [\epsilon(x^+) - \epsilon(x^-)]. \]  

Substituting the AAdS solutions into \( Q \) and manipulating a little bit, we acquire for the both cases
\[ Q[\epsilon] = -\frac{\epsilon(t)}{8\pi G} \partial_x \delta \phi(x = -0, t) \]
\[ = -\frac{\epsilon(t)}{32\pi G\lambda Ae^F + Be^{-F} + C}. \]

For the first boundary condition: \( \phi_0 \neq 0 \), all the charges of the Virasoro transformation (34) vanish exactly: \( Q[\epsilon] = 0 \) due to \( C = 0 \). Thus the Virasoro symmetry (34) realizes as AAdS isometry, however, the representation is just singlet and trivial.

In the second case with \( \phi_0 = 0 \), time independence of the charges demands that
\[ C \neq 0, \]  
\[ \epsilon(t) = \text{const}, \]  
\[ \frac{\dot{F}}{Ae^F + Be^{-F} + C} = \frac{\mu}{2\sqrt{AB}}. \]

Thus, as expected, it is concluded that the asymptotic isometry includes only time translation. Then the form of AAdS solution is constrained as
\[ e^{F(t)} = -\frac{1}{2A} \left[ C + \sqrt{C^2 - 4AB} \tanh \left( \frac{\mu}{4\sqrt{AB}} \sqrt{C^2 - 4AB} (t + t_o) \right) \right], \]
\[ \phi = -\frac{1}{4\lambda} \ln \left( \frac{\sqrt{U_o}}{-\mu x} \right) + \frac{\mu}{16\lambda \sqrt{AB}} x + o(x^2), \]
and the conserved energy is given as follows.

\[ E = -\frac{\mu}{64\pi G \lambda \sqrt{AB}} \]  

(93)

For negative \( \lambda \) and \( U_o = 0 \), the model is reduced into the Liouville theory with negative cosmological constant. It is a well known fact that by integrating out the dilaton field in the action:

\[ S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \left[ \phi R - 4|\lambda| (\nabla \phi)^2 + \frac{\mu^2}{4|\lambda|} \right] \]  

(94)

the original polyakov action\[5\] appears:

\[ S_p = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \left[ -\frac{1}{16|\lambda|} R \frac{1}{\nabla^2} R + \frac{\mu^2}{4|\lambda|} \right]. \]  

(95)

In the Liouville case, any static background solution with

\[ e^{2\rho} = \frac{1}{\sin^2(\mu x)} \]

cannot exist and the dilaton field has explicit time dependence. However this difference between the cases with \( U_o \neq 0 \) and the Liouville model is not so significant, and does not change the results for the case with \( U_o \neq 0 \) if we take another background solution:

\[ ds^2 = -\frac{1}{\mu^2 x^2} dx^+ dx^-, \]  

(96)

\[ \phi = -4 \ln(-\mu x). \]  

(97)

Repeating the same analysis of the \( U_o \neq 0 \) cases, it is easily shown that the Liouville theory with negative cosmological constant have a singlet representation of Virasoro algebra \[34\] as AAdS isometry and a representation where only time translation is preserved.

## 4 JT Model

When taking \( U_o = 1 \) and \( \lambda = 0 \) the model expresses the JT model:

\[ S = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \phi (R + 2\mu^2). \]  

(98)
In the case general AAdS solutions in the conformal gauge are also obtained.

\[ e^{2\rho} = \frac{4}{\mu^2} \left[ e^{\frac{\rho}{2}(F(x^+)-F(x^-))} - e^{-\frac{\rho}{2}(F(x^+)-F(x^-))} \right]^2, \]  
\[ \phi = A \frac{1 + e^{F(x^+)-F(x^-)}}{1 - e^{F(x^+)-F(x^-)}} + B \frac{1}{e^{-F(x^-)} - e^{-F(x^+)}} + C \frac{1}{e^{F(x^-)} - e^{F(x^+)}}. \]

They behave near \( x = -0 \) like

\[ e^{2\rho} = \frac{1}{\mu^2 x^2} + \frac{1}{\mu^2} \left( \frac{2}{3} \frac{d^3 F}{dt^3} - \frac{\ddot{F}}{F^2} - \frac{1}{3} \dot{F}^2 \right) + o(x^2), \]
\[ \phi = -\frac{1}{2F_x} \left( 2A + Be^F + Ce^{-F} \right) + \frac{x}{12F} \left[ \frac{1}{F} \frac{d^3 F}{dt^3} \left( 2A + Be^F + Ce^{-F} \right) - 3\dot{F} \left( Be^F - Ce^{-F} \right) - \dot{F}^2 \left( 4A - Be^F - Ce^{-F} \right) \right] + o(x^3). \]

In later analysis we take the following static solution among them as a background which AAdS solutions are approaching to.

\[ ds^2 = \bar{g}_{ab} dx^a dx^b = e^{2\bar{\rho}}(-dt^2 + dx^2) \]
\[ = \frac{1}{\sin^2(\mu x)} (-dt^2 + dx^2), \]  
\[ \bar{\phi} = -\cot(\mu x) = -\frac{1}{\mu x} + o(x). \]

The charge of the asymptotic isometry candidate is expressed in this case as

\[ Q[\epsilon] = \frac{1}{16\pi G} \left[ e^{-2\rho(x)} \bar{\rho} \left[ \epsilon^t (g_{xx} - \bar{g}_{xx}) + 2\epsilon^x (g_{tx} - \bar{g}_{tx}) \right] \right]_{x=-0} - \frac{1}{8\pi G} \left[ (\epsilon^t \partial_x + \epsilon^x \partial_t - (\partial_x \epsilon^t)) (\phi - \bar{\phi}) \right]_{x=-0}, \]
where
\[
\epsilon^t = \frac{1}{2} [\epsilon(x^+) + \epsilon(x^-)],
\]
\[
\epsilon^x = \frac{1}{2} [\epsilon(x^+) - \epsilon(x^-)].
\]

Just as \( U_o \neq 0 \) cases, we consider two independent boundary condition of the dilaton field. Near \( x = -0 \) the canonical variables are expanded with respect to \( x \) as follows.

\[
a = -\frac{1}{\mu x} + a_1(t)x + o(x^2)
\]
\[
N = -\frac{1}{\mu x} + N_1(t)x + o(x^2)
\]
\[
J = J_2(t)x^2 + o(x^3)
\]
\[
\phi = -\frac{\phi_{-1}(t)}{\mu x} + x\phi_1(t) + o(x^2)
\]
\[
\Pi_a = \frac{1}{8\pi G} \left[ -\dot{\phi}_{-1} + (\mu\dot{\phi}_0 - J_2\phi_{-1})x + o(x^2) \right]
\]
\[
\Pi_\phi = \frac{1}{8\pi G} \left[ J_2x + \mu\dot{a}_1x^2 + o(x^3) \right]
\]

Here a zeroth order term \( \phi_0(t) \) in the dilaton field is omitted in order to make the charge \( Q \) finite. For the first example, \( \phi_{-1} \) is not fixed to any specified value in the configuration space. This condition is naturally required if one respects the AAdS isometry candidate (34) in this level because the dilaton solution (102) is transformed as

\[
\delta \epsilon \phi = \frac{\dot{\epsilon}(t)}{\mu x} + o(x) = \frac{o(1)}{x} + o(x),
\]
\[
x^{\pm'} = x^\pm - \epsilon(x^\pm).
\]

Adopting the boundary conditions, the boundary equations of motion demand that

\[
a_1 = \tilde{a}_1 = -\frac{\mu}{6},
\]
\[
N_1 = \tilde{N}_1 = -\frac{\mu}{6},
\]
\[
\phi_{-1} = 1.
\]
These require that
\[ \dot{F} = \frac{\mu}{2} \left( 2A + Be^{F} + Ce^{-F} \right), \quad (115) \]
\[ BC - A^{2} = 1, \quad (116) \]
in eqns(99)(100). Consequently this yields the vanishing charges of the conformal transformation:
\[ Q[\epsilon] = 0. \]
Thus the Virasoro symmetry (34) including the AdS isometry (13) survives, but the representation remains trivially singlet.

The second proposal for the boundary condition is to fix the asymptotic dilaton field as
\[ \phi_{-1} = 1 \]
in the configuration space level. This breaks explicitly the Virasoro symmetry (34) except time translation. While no additional constraint arises from the boundary equation. Solutions belonging to the specified configuration space must satisfy
\[ \dot{F} = \frac{\mu}{2} \left( 2A + Be^{F} + Ce^{-F} \right), \quad (117) \]
\[ \phi = -\frac{1}{\mu x} + \frac{\mu x}{3}(BC - A^{2}) + o(x^{2}), \quad (118) \]
\[ e^{2\rho} = \frac{1}{\mu^{2}x^{2}} + \frac{1}{3}(BC - A^{2}) + o(x^{3}). \quad (119) \]
From these relations one can calculate the charge straightforwardly as follows.
\[ Q[\epsilon] = \frac{\mu \epsilon(t)}{16\pi G}(A^{2} - BC + 1). \]
Thus the representation of AAdS isometry is nonsinglet, but the symmetry consists of only time translation and only energy:
\[ E = \frac{\mu}{16\pi G}(A^{2} - BC + 1) \]
is conserved.

Though the standard canonical representations of JT model cannot keep nontrivially the $SO(1, 2)$ AdS and the Virasoro symmetry (34)(34) as shown
above, it is quite notable that the JT model has a noncanonical but nonsinglet representation of the $SO(1, 2)$ AdS isometry \([\mathbb{P}3]\) with finite conserved charges.

Let us consider AdS transformed backgrounds generated by acting $SO(1, 2)$ on the reference solution \([102]\). By selecting the canonical variables approaching to the transformed backgrounds different canonical representation sectors are generated. Moreover let us adopt the first boundary condition \([107]\) in each sector. Combining all the canonical sectors, a hybrid representation for the total system can be defined. The hamiltonian which expresses the time evolution of the total system is written as follows.

$$H_{\text{tot}} = \sum_L 0 \otimes \cdots 0 \otimes H[\epsilon^t = 1, \epsilon^x = 0, L] \otimes 0 \cdots \otimes 0$$  \hspace{1cm} (120)

where

$$H[\epsilon, L] = \int dx \left( \epsilon \mathcal{H}_N + \epsilon^J \mathcal{H}_J \right) + \tilde{Q}[\epsilon, L],$$

$$\tilde{Q}[\epsilon, L] = \frac{1}{8\pi G a_L^2} \left[ \epsilon^N \left( \partial_x \tilde{\phi}_L (a - a_v) - \bar{a}_L \partial_x (\phi - \phi_v) \right) + \partial_x \epsilon^N \bar{a}_L (\phi - \phi_v) \right]_{x=0}$$

$$+ \epsilon^J \left[ \bar{a}_L (\Pi_a - \Pi_{av}) - \bar{\Pi}_L (\phi - \phi_v) \right]_{x=0}.$$  \hspace{1cm} (123)

and the transformed background quantities by a $SO(1, 2)$ element $L$ are denoted $\bar{a}_L$, $\tilde{\phi}_L$ and so on. Note that due to the isometry all the metric quantities of backgrounds remain unchanged under the transformation, that is, $\bar{a}_L = a$, $J_L = J$ and $\bar{N}_L = N$. Here the vacuum in the representation is chosen as

$$ds_v^2 = \frac{1}{\sin^2(\mu x)} \left( -dt^2 + dx^2 \right),$$

$$\phi_v = 0.$$  \hspace{1cm} (121) \hspace{1cm} (122)

Due to the boundary equations of motion, the singlet representation is automatically selected and all the solutions take the same value of $\tilde{Q}$ in each sector. Thus the value of $\tilde{Q}$ for a certain solution depends only on which background sector the solution belongs to.

The charges of the AdS isometry can be written more explicitly as

$$\tilde{Q}[\epsilon, L] = -\frac{1}{8\pi G} \left[ \epsilon^t \left( \partial_x + \frac{1}{x} - \frac{1}{3} \mu^2 x \right) + \epsilon^x \partial_x - (\partial_x \epsilon^t) \right] \bar{\phi}_L.$$  \hspace{1cm} (123)
For the reference background solution (101)(102), the charge is calculated straightforwardly as

$$\tilde{Q}[\epsilon, L = 1] = -\frac{\mu}{8\pi G} \epsilon_0. \quad (124)$$

Thus the three charges of the AdS symmetry can be conserved. Among the charges of the solution, only energy is nonvanishing and takes a negative value of

$$E = \frac{\partial \tilde{Q}}{\partial \epsilon_0} = -\frac{\mu}{8\pi G}. \quad (125)$$

Meanwhile, modifying the reference solution (102) by the infinitesimal AdS transformation, the dilaton field responds as follows.

$$\delta \xi \phi = \frac{\dot{\xi}}{\mu x} + \frac{x}{6\mu} \left( \frac{d^3 \xi}{dt^3} + 2\mu^2 \dot{\xi} \right), \quad (126)$$

$$\xi(t) = \xi_0 + \xi_1 \sin(\mu t + \delta \xi). \quad (127)$$

This yields deviation of the charge from the value of the background (101)(102) like

$$\delta \xi \tilde{Q} = \frac{\mu^2}{8\pi G} \epsilon_1 \xi_1 \sin(\delta \xi - \delta \epsilon). \quad (128)$$

It means that the reference solution (101)(102) belongs to a nonsinglet representation of the AdS isometry. To see it in a finite transformation, let us take a AdS transformation $L_1$ :

$$T_1 = T'_1 \cosh \omega - X' \sinh \omega, \quad (129)$$

$$T_2 = T'_2, \quad (130)$$

$$X = X' \cosh \omega - T'_1 \sinh \omega, \quad (131)$$

in eqns (1)(2). Then the dilaton field (102) is transformed into

$$\tilde{\phi}_{L_1} = -\cot(\mu x) \cosh \omega + \frac{\cos(\mu t)}{\sin(\mu x)} \sinh \omega. \quad (132)$$

and its charge is evaluated as

$$\tilde{Q}[\epsilon, L_1] = -\epsilon_0 \frac{\mu}{8\pi G} \cosh \omega - \epsilon_1 \sin \delta \epsilon \frac{\mu}{8\pi G} \sinh \omega. \quad (133)$$
Thus it has been verified that the AdS charges are transformed exactly as a vector of the AdS transformation.

Therefore even though the AdS symmetry cannot be realized in each canonical representation sector, one can construct a nonsinglet representation by combining the sectors and defining the AdS transformation from one sector to another by use of

\[ \epsilon^t = \epsilon_0 + \frac{\epsilon_1}{2} (\sin(\mu x^+ + \delta) + \sin(\mu x^- + \delta)), \]
\[ \epsilon^x = \frac{\epsilon_1}{2} (\sin(\mu x^+ + \delta) - \sin(\mu x^- + \delta)), \]
\[ \delta \phi = \epsilon^a \nabla_a \phi, \]
\[ \delta g_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a. \]

Though this hybrid representation is not purely canonical, we want to stress that it is a well defined representation with finite and nonvanishing AdS casimir operator:

\[ M^2 = \left( \frac{\mu}{8\pi G} \right)^2. \]

Therfore it can be argued that JT model have a nontrivial representation of the AdS isometry.

Unfortunately the energy takes only negative values in the representation, positive energy excitations of matter fields cannot be incorporated. If one wants positive energy to take part in the dynamics, another representation discussed above in which the AdS isometry is explicitly broken must be used.

References

[1] J.D.Brown and M.Henneaux, Commun.Math.Phys.104,207-226(1986)
[2] J.Maldacena,hep-th/9711200
[3] A.Strominger, J.High Energy Phy.,02:009,(1998)
[4] E.Witten,hep-th/9802150,hep-th/9803131.
[5] A.M.Polyakov,Phys.Lett.,B103,207,(1981).
[6] R. Jackiw, in Quantum Theory of Gravity, ed. S. Christensen (Adam Hilger, Bristol, 1984) p. 403; C. Teitelboim, in Quantum Theory of Gravity, ed. S. Christensen (Adam Hilger, Bristol, 1984) p. 327.

[7] M. Henneaux, Phys. Rev. Lett., 54, 959, (1985).

[8] H. Terao, Nucl. Phys., B395, 623, (1993).