EXISTENCE, STABILITY AND LONG TIME BEHAVIOUR OF WEAK SOLUTIONS OF THE THREE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH POTENTIAL FORCE

ANTHONY SUEN

Abstract. We address the global-in-time existence, stability and long time behaviour of weak solutions of the three-dimensional compressible Navier-Stokes equations with potential force. We show the details of the $\alpha$-dependence of different smoothing rates for weak solutions near $t = 0$ under the assumption on the initial velocity $u_0$ that $u_0 \in H^\alpha$ for $\alpha \in (\frac{1}{2}, 1]$ and obtain long time convergence of weak solutions in various norms. We then make use of the Lagrangean framework in comparing the instantaneous states of corresponding fluid particles in two different solutions. The present work provides qualitative results on the long time behaviour of weak solutions and how the weak solutions depend continuously on initial data and steady states.

1. Introduction

1.1. Background and motivation. We are interested in the 3-D compressible Navier-Stokes equations with an external potential force in the whole space $\mathbb{R}^3$ ($j = 1, 2, 3$):

\begin{equation}
\begin{cases}
\rho_t + \text{div} (\rho u) = 0, \\
(\rho u_j)_t + \text{div}(\rho u^2) + (P)_{x_j} = \mu \Delta u_j + \lambda (\text{div} u)_{x_j} + \rho f_j.
\end{cases}
\end{equation}

(1.1)

Here $x \in \mathbb{R}^3$ is the spatial coordinate and $t \geq 0$ stands for the time. The unknown functions $\rho = \rho(x, t)$ and $u = (u^1, u^2, u^3)(x, t)$ represent the density and velocity vector in a compressible fluid. The function $P = P(\rho)$ denotes the pressure, $f = (f^1(x), f^2(x), f^3(x))$ is a prescribed time-independent external force and $\mu, \lambda$ are positive viscosity constants which satisfy

\begin{equation}
\mu, \lambda > 0, \quad \frac{\mu}{\lambda} > \frac{4}{5}.
\end{equation}

(1.2)

The system (1.1) is equipped with initial condition

\begin{equation}
(\rho(\cdot, 0) - \rho_s, u(\cdot, 0)) = (\rho_0 - \rho_s, u_0),
\end{equation}

(1.3)

where the non-constant time-independent function $\rho_s = \rho_s(x)$ (known as the steady state solution to (1.1)) can be obtained formally by taking $u \equiv 0$ in (1.1):

\begin{equation}
\nabla P(\rho_s) = \rho_s f(x).
\end{equation}

(1.4)

Date: March 30, 2021.
2000 Mathematics Subject Classification. 35Q30.
Key words and phrases. Navier-Stokes equations; compressible flow; potential force; stability; long time behaviour.
The well-posedness problem of the Navier-Stokes system (1.1) is an important but challenging research topic in fluid mechanics, and we now give a brief review on the related results. The local-in-time existence of classical solution to the full Navier-Stokes equations was proved by Nash [Nas62] and Tani [Tan77], and some Serrin type blow-up criteria for smooth solutions was recently obtained by Suen [Sue20d]. Later, Matsumura and Nishida [MN80] obtained the global-in-time existence of $H^3$ solutions when the initial data was taken to be small with respect to $H^3$ norm, the results were then generalised by Danchin [Dan00] who showed the global existence of solutions in critical spaces. In the case of large initial data, Lions [Lio98] obtained the existence of global-in-time finite energy weak solutions, yet the problem of uniqueness for those weak solutions remains completely open. In between the two types of solutions as mentioned above, a type of “intermediate weak” solutions were first suggested by Hoff in [Hof95, Hof02, Hof05, Hof06] and later generalised by Li and Matsumura in [LM11], Matsumura and Yamagata in [MY04], Suen in [Sue13, Sue14, CS16] and other systems which include compressible magnetohydrodynamics (MHD) [SH12, Sue12, Sue20b] and compressible Navier-Stokes-Poisson system [Sue20a]. Solutions as obtained in this intermediate class are less regular than those small-smooth type solutions obtained by Matsumura and Nishida [MN80] and Danchin [Dan00], which contain possible codimension-one discontinuities in density, pressure and velocity gradient. Nevertheless, those intermediate weak solutions would be more regular than the large-weak type solutions developed by Lions [Lio98], hence the uniqueness and continuous dependence of solutions may be possibly obtained; see [Hof06] and the compressible MHD [Sue20b].

In this present work, we address solutions of (1.1) from the class of intermediate weak solutions as mentioned above. Our results give a more qualitative description on the $\alpha$-dependence of different smoothing rates for weak solutions when the initial velocity $u_0$ is in $H^\alpha$ for $\alpha \in (\frac{1}{2}, 1]$, which include those previous results obtained in [LM11, Sue13, Sue14, CS16]. In particular, we remove the smallness assumption on $\|\rho_0 - \rho_s\|_{L^\infty}$ which was required in [LM11] for proving global-in-time weak solutions to (1.1). Moreover, our results extend those of Cheung and Suen [CS16], which proved uniqueness of weak solution to (1.1) with Hölder continuous density function $\rho$. Such condition on $\rho$ is too strong in the sense that this would exclude solutions with codimension-one singularities, which are physically interesting features for the weak solutions; see [Hof02] for a detailed discussion on the propagation of singularities. The main novelties of this current work can be summarised as follows:

1. We strengthen the previous results obtained in [LM11, Sue13, Sue14, CS16], in the way that we show the details of the $\alpha$-dependence of different smoothing rates for weak solutions near $t = 0$ under the assumption that $u_0 \in H^\alpha$ for $\alpha \in (\frac{1}{2}, 1]$; see Theorem 1.5. Such regularity requirement on $u_0$ is crucial in obtaining uniqueness of the weak solutions of (1.1). It also matches with the results given in Hoff [Hof02] for Navier-Stokes equations.

2. We provide detailed descriptions on the long time behaviour of weak solution to (1.1), including the long time averages of the “smooth part” of velocity $u$ and the effective viscous flux; see (1.39)-(1.41) in Theorem 1.5. Our results are consistent with the optimal decay rates of weak solutions to (1.1) which are recently obtained by Hu and Wu in [HW20].

3. We successfully extend the uniqueness and continuous dependence theory given in [Hof06] for compressible Navier-Stokes system to (1.1), and we generalise the
condition on pressure $P$ which can accommodate more general cases; see \[1.49\] in Remark 1.9.

4. We remove the unnecessary Hölder continuity restriction on density from \[CS10\] for proving uniqueness, which allows us to include a larger class of weak solutions.

1.2. Ideas and strategies. We give a brief description on the idea and strategies behind our work. First of all, we introduce two important functions, namely the effective viscous flux $F$ and vorticity $\omega$, which are defined by

\begin{equation}
\rho_sF = (\mu + \lambda)\text{div}(u) - (\rho P(\rho - P_s)), \quad \omega = \omega^{j,k} = u_x^j - u_x^k.
\end{equation}

By the definitions of $F$ and $\omega$, and together with \[1.2\], $F$ and $\omega$ satisfy the elliptic equations

\begin{align}
\Delta(\rho_sF) &= \text{div}(\rho \dot{u} - \rho f + \nabla P(\rho_s)), \label{eq:1.6} \\
\mu \Delta \omega &= \nabla \times (\rho \dot{u} - \rho f + \nabla P(\rho_s)). \label{eq:1.7}
\end{align}

The functions $F$ and $\omega$ play essential roles for studying intermediate weak solutions to compressible flows, see \[Ho95\], \[Ho02\], \[SH12\], \[Sue20b\] for some related discussions.

One of the key steps in the derivation of a priori estimates on approximate solutions is to control the space-time integral of $|\nabla u|^4$. Due to the lack of regularity when $u_0 \in L^2$, it is expected that $u(\cdot, t) \notin H^2$ and we need some other methods for controlling $\nabla u$ in $L^4$. Using the decomposition that

$$\Delta u^j = \text{div}(u_{x_j}) + \omega^{j,k}_{x_k},$$

we have from the definition of $F$ that

$$(\mu + \lambda)\Delta u^j = (\rho_sF)_{x_j} + (\mu + \lambda)\omega^{j,k}_{x_k} + (P - P_s)_{x_j}.$$

Hence in view of the above, we can obtain the desired estimate for $\nabla u$ in $L^4$ in terms of $F$, $\omega$ and $\rho - \rho_s$.

In the small-time regime, we automatically have uniform-in-time estimate on $\int_0^t \| (\rho \rho_s)(\cdot, \tau) \|_{L^4}^4 d\tau$ provided that $\| (\rho - \rho_s)(\cdot, \tau) \|_{L^2} < \infty$ and $\| (\rho - \rho_s)(\cdot, \tau) \|_{L^\infty} < \infty$ for all $\tau \geq 0$. In the long-time regime, however, it is not obvious to obtain uniform-in-time estimate on $\int_0^t \| (\rho - \rho_s)(\cdot, \tau) \|_{L^4}^4 d\tau$ when $t > 1$ without any smallness assumption on $\| (\rho - \rho_s)(\cdot, \tau) \|_{L^\infty}$. As inspired by the work \[LM11\], if we consider the auxiliary functional $A(t)$ given by

$$A(t) = \int_1^t \int_{\mathbb{R}^3} |\rho - \rho_s|^{\frac{4}{3}} dxd\tau,$$

then we can show that $A(t)$ is indeed bounded uniformly in time in terms of $F$ and $u$ under the assumption that $\rho$ is essentially bounded above and below. Such estimate on $A(t)$ will be used for giving uniform-in-time bound on $\int_1^t \| (\rho - \rho_s)(\cdot, \tau) \|_{L^4}^4 d\tau$, which is important for obtaining global-in-time a priori estimates on the approximate smooth solutions to \[1.1\].

The uniform-in-time estimate on $A(t)$ is also useful for studying the long time behaviour of $\| F(\cdot, t) \|_{L^p}$ for $p \in \left[\frac{10}{7}, 6\right]$. The details will be carried out in Section 2 and Section 3.
To address the uniqueness of weak solutions with respect to initial data, we need to have a better control on \( \int_0^t \| \nabla u(\cdot, \tau) \|_{L^\infty} d\tau \). Our attempt is to decompose \( u \) as \( u = u_F + u_P \), where \( u_F \) and \( u_P \) satisfy

\[
\begin{align*}
(\mu + \lambda)\Delta (u_F)_j &= (\rho_s F)_{x_j} + (\mu + \lambda)(\omega)_{x_k}^j, \\
(\mu + \lambda)\Delta (u_P)_j &= (P - \bar{P})_{x_j}.
\end{align*}
\]

Using the \textit{a priori} bounds on the effective viscous flux \( F \), we can bound the integral \( \int_0^t \| \nabla u_F(\cdot, \tau) \|_{L^\infty} d\tau \) in terms of \( F \). On the other hand, to bound the integral \( \int_0^t \| \nabla u_P(\cdot, \tau) \|_{L^\infty} d\tau \), we point out that \( (\rho - \rho_s) \in L^2 \cap L^\infty \) is not sufficient for bounding \( \| \nabla u_P(\cdot, \tau) \|_{L^\infty} \). One way to overcome such difficulty is to assume that \( P(\rho(\cdot, t)) \) is Hölder continuous, yet such assumption would be too strong and exclude solutions with codimension-one singularities. On the other hand, we notice that if \( P(\rho(\cdot, t)) \in L^\infty \), then \( u_F^j(\cdot, t) = (\mu + \lambda)^{-1}\bar{F}_{x_j} \ast (P(\rho(\cdot, t)) - \bar{P}(\cdot)) \) is log-Lipschitz. If we assume that the initial density is \textit{piecewise Hölder continuous} (see Definition 1.1), then using the mass equation (1.1), and invoking the definition of \( F \), it implies that the density is also \textit{piecewise} Hölder continuous for positive time. Hence with such improved regularity on the density, it allows us to obtain the desired bound on \( \int_0^t \| \nabla u_P(\cdot, \tau) \|_{L^\infty} d\tau \). As a by-product, we further address the regularity and long-time behaviour of the “smooth part” \( u_F \) of \( u \). The details will be carried out in Section 4.

Once we obtain the global-in-time existence of weak solutions, we proceed to address the continuous dependence on initial data of weak solutions of the Navier-Stokes equations (1.1). As suggested by Hoff in [Hof06], weak solutions with minimal regularity are best compared in a Lagrangian framework. Instead of writing the equations (1.1) in Lagrangian coordinates, we consider the \textit{particle trajectories} \( X(y, t, s) \) with respect to a weak solution \((\rho, u, f, \rho_s)\) to (1.1):

\[
\begin{align*}
\frac{\partial X}{\partial t}(y, t, t') &= u(X(y, t, t'), t) \\
X(y, t', t') &= y.
\end{align*}
\]

If \((\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)\) is another weak solution to (1.1) with corresponding particle trajectories \( \bar{X}(y, t, s) \), then we aim at estimating the difference \( u - \bar{u} \circ S \), where \( \bar{u} \circ S := \bar{u}(S(x, t), t) \) and the mapping \( S \) is given by

\[
S(x, t) := \bar{X}(X(x, 0, t), t, 0).
\]

The mapping \( S \) works well with \textit{convective differentiation}, in the sense that \( \bar{u} \circ S \) and \( \bar{u} \) enjoy the relation

\[
(\bar{u}_t + \nabla \bar{u}) \circ S = (\bar{u} \circ S)_t + \nabla (\bar{u} \circ S)u,
\]

hence one can obtain bounds for \( u - \bar{u} \circ S \) from estimates for solutions of the adjoint of the weak equation satisfied by \( u - \bar{u} \circ S \). Once we know the estimates on \( u - \bar{u} \circ S \), the difference \( u - \bar{u} \) can then be controlled by \( u - \bar{u} \circ S \) if we have

\[
\int_0^t \| \nabla \bar{u}(\cdot, \tau) \|_{L^\infty} d\tau < \infty,
\]

which can be achieved if the initial density \( \bar{\rho}_0 \) is piecewise
Hölder continuous, and together with the fact that
\[ \int_{\mathbb{R}^3} |x - S(x,t)|^2 dx \leq C t \int_0^t \int_{\mathbb{R}^3} |u - \bar{u} \circ S|^2 dx dt. \]

The remaining details will be carried out in Section 3.

It is also worth mentioning that the steady state solutions \( \rho_s \) has a crucial role in addressing the stability of weak solutions to (1.1) (see (1.44) in Theorem 1.8). This is not surprising: the steady state solutions \( \rho_s \) mainly depend on the external force \( f \) as appeared in (1.1), hence \( \rho_s \) should be controlled by \( f \) in some appropriate ways. In our present work, however, we choose to list out the effects of \( \rho_s \) and \( f \) in separate ways, which can give a clearer picture on how the stability of weak solutions depends on \( \rho_s \) and \( f \).

1.3. Notations and conventions. We introduce the following notations used in our work. For any \( r \in (1, \infty] \) and \( k \geq 0 \), we define the following function spaces:
\[
\begin{align*}
L^r \rho^r(\mathbb{R}^3), D^{k,r} &= \{ u \in L^1_{\text{loc}}(\mathbb{R}^3) : \| \nabla^k u \|_{L^r} < \infty \}, \| u \|_{D^{k,r}} := \| \nabla^k u \|_{L^r} \\
W^{k,r} &= L^r \cap D^{k,r}, H^k = W^{k,2}.
\end{align*}
\]

We adapt the following usual notations for Hölder seminorms: Given \( m \geq 1 \), for \( v : \mathbb{R}^3 \rightarrow \mathbb{R}^m \) and \( \alpha \in (0, 1] \),
\[ \langle v \rangle^\alpha = \sup_{x_1, x_2 \in \mathbb{R}^3} \frac{|v(x_2) - v(x_1)|}{|x_2 - x_1|^\alpha}; \]
and for \( v : Q \subseteq \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^m \) and \( \alpha_1, \alpha_2 \in (0, 1] \),
\[ \langle v \rangle_{Q, \alpha_1, \alpha_2} = \sup_{(x_1, t_1), (x_2, t_2) \in Q} \frac{|v(x_2, t_2) - v(x_1, t_1)|}{|x_2 - x_1|^{\alpha_1} + |t_2 - t_1|^{\alpha_2}}. \]

We introduce the usual convective derivative \( \frac{D}{Dt} \) with respect to a velocity field \( u \) as follows. For a given function \( w : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R} \), we define
\[ \frac{D}{Dt} (w) = \dot{w} := w_t + u \cdot \nabla w, \]
where \( w_t := \frac{\partial w}{\partial t} \) and \( \nabla w \) is the gradient of \( w \). For \( w : \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3 \), we define
\[ \frac{D}{Dt} (w) = \dot{w} := w_t + \nabla wu, \]
where \( \nabla w \) is the \( 3 \times 3 \) matrix of partial derivatives of \( w \). We also write \( (\cdot)_{x_j} := \frac{\partial}{\partial x_j} \) for simplicity.

We give the notion of piecewise Hölder continuous as follows (also refer to [Hof02] for more details):

**Definition 1.1.** We say that a function \( \phi(\cdot, \tau) \) is piecewise \( C^{\beta(t)} \) if it has simple discontinuities across a \( C^{1+\beta(t)} \) curve \( C(t) : C(t) = \{ y(s, t) : s \in I \subset \mathbb{R} \} \), where \( \beta(t) > 0 \) is a function in \( t \), \( I \) is an open interval and the curve \( C(t) \) is the \( u \)-transport of \( C(0) \) given by:
\[ y(s, t) = y(s, 0) + \int_0^t u(y(s, \tau), \tau) d\tau. \]
Here $C(0)$ is a $C^{\beta_0}$ curve with $\beta(0) = \beta_0 > 0$, which means that 
\[ C(0) = \{ y_0(s) : s \in \mathbb{R} \}, \]
where $y(s,0) = y_0(s)$ is parameterised in arc length $s$ and $y_0$ is $C^{\beta_0}$.

**Remark 1.2.** In view of Definition 1.1 if $\rho_0$ is piecewise continuous, then it may have simple jump discontinuities across $C(0)$. Hence $\rho_0$ may contain codimension-one singularities in $\mathbb{R}^3$.

1.4. **Steady state solutions.** We define $\rho_s$ as mentioned at the beginning of this section. For the pressure function $P = P(\rho)$ and the external force $f$, we assume that
\begin{equation}
(1.11) \quad P(\rho) \in C^2((0, \infty)) \text{ with } P(0) = 0; \quad P'(\rho) > 0, \quad P''(\rho) > 0 \text{ for } \rho > 0;
\end{equation}
\begin{equation}
(1.12) \quad \text{There exists } \Psi \in H^2 \text{ such that } f = \nabla \Psi \text{ and } \Psi(x) \to 0 \text{ as } |x| \to \infty.
\end{equation}
Given a constant density $\rho_\infty > 0$, we say that $(\rho_s, 0)$ is a steady state solution to (1.12) if $\rho_s \in C^2(\mathbb{R}^3)$ and the following holds
\begin{equation}
(1.13) \quad \begin{cases}
\nabla P(\rho_s(x)) = \rho_s(x) \nabla \Psi(x), \\
\lim_{|x| \to \infty} \rho_s(x) = \rho_\infty.
\end{cases}
\end{equation}
Hence $\rho_s$ satisfies the equation
\begin{equation}
\int_{\rho_\infty}^{\rho_s} \rho^{-1} P'(\rho) d\rho = \Psi(x).
\end{equation}
To avoid vacuum state for $\rho_s$, we further assume
\begin{equation}
(1.14) \quad - \int_0^{\rho_s} \frac{P'(\rho)}{\rho} d\rho < \inf_{x \in \mathbb{R}^3} \Psi(x) \leq \sup_{x \in \mathbb{R}^3} \Psi(x) < \int_{\rho_\infty}^{\infty} \frac{P'(\rho)}{\rho} d\rho.
\end{equation}
We recall the following result about the existence and uniqueness of $\rho_s$ (see for example [LM11]):

**Proposition 1.3.** Assume that $P$ and $\Psi \in H^3$ satisfy (1.11) and (1.14) respectively. Then there exists a unique solution $\rho_s$ of (1.13) satisfying $\rho_s - \rho_\infty \in H^2 \cap W^{2,6}$. Moreover, there exist positive constants $\bar{\rho}$ and $\tilde{\rho}$ depending on $\|\Psi\|_{H^3}$ such that
\begin{equation}
(1.15) \quad \bar{\rho} < \inf_{x \in \mathbb{R}^3} \rho_s(x) \leq \sup_{x \in \mathbb{R}^3} \rho_s(x) < \tilde{\rho}.
\end{equation}

From now on, for the sake of simplicity, we also write $P = P(\rho)$ and $P_s = P(\rho_s)$ unless otherwise specified.

**Remark 1.4.** It is clear that the pressure function $P$ satisfying (1.11) includes the typical poly-tropic model $P = a \rho^\gamma$ for $a > 0$ and $\gamma > 0$.

1.5. **Weak solutions.** Weak solutions to the system (1.1) can be defined as follows. We say that $(\rho, u, f, \rho_s)$ on $\mathbb{R}^3 \times [0, T]$ is a weak solution of (1.1) if the following conditions hold:
\begin{equation}
(1.16) \quad \rho_s \text{ is a steady state solution to } \begin{array}{ll}
\text{(1.13)} \end{array}, \quad \text{which satisfies } \begin{array}{ll}
\text{(1.14)} \end{array};
\end{equation}
\begin{equation}
(1.17) \quad \rho - \rho_s \text{ is a bounded map from } [0, T] \text{ into } L^1_{loc} \cap H^{-1} \text{ and } \rho \geq 0 \text{ a.e.};
\end{equation}
\begin{equation}
(1.18) \quad \rho u_0 \in L^2; \quad \rho u, P - P_s, \nabla u, \rho f \in L^2(\mathbb{R}^3 \times (0, T)); \quad \rho|u|^2 \in L^1(\mathbb{R}^3 \times (0, T));
\end{equation}
For all $t_2 \geq t_1 \geq 0$ and $C^1$ test functions $\varphi \in D(\mathbb{R}^3 \times (-\infty, \infty))$ which are Lipschitz on $\mathbb{R}^3 \times [t_1, t_2]$ with $\text{supp} \varphi(\cdot, \tau) \subset K$, $\tau \in [t_1, t_2]$, where $K$ is compact and

$$(1.19) \quad \int_{\mathbb{R}^3} \rho(x, \cdot)\varphi(x, \cdot)dx \bigg|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} (\rho \varphi_t + \rho u \cdot \nabla \varphi)dx d\tau;$$

The weak form of the momentum equation

$$(1.20) \quad \int_{\mathbb{R}^3} (\mu u \cdot \nabla \varphi + (\mu - \xi)(\text{div}(u))\varphi_x)dx d\tau$$

holds for test functions $\varphi$ which are locally Lipschitz on $\mathbb{R}^3 \times [0, T]$ and for which $\varphi, \varphi_t, \nabla \varphi \in L^2(\mathbb{R}^3 \times (0, T)), \nabla \varphi \in L^\infty(\mathbb{R}^3 \times (0, T))$, and $\varphi(\cdot, \tau) = 0$.

For the two solutions $(\rho, u, f, \rho_s)$ and $(\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)$ we compare, they will be assumed to satisfy

$$(1.21) \quad u, \bar{u} \in C(\mathbb{R}^3 \times (0, T)) \cap L^4((0, T); W^{1, \infty}) \cap L^{\infty}_{loc}((0, T); L^\infty);$$

$$(1.22) \quad \rho - \rho_s, \bar{\rho} - \bar{\rho}_s, u, \bar{u}, f, \bar{f} \in L^2(\mathbb{R}^3 \times (0, T)).$$

One of the solutions $(\rho, u, f, \rho_s)$ will have to satisfy

$$(1.23) \quad \|f\|_{L^\infty} < \infty,$$

$$(1.24) \quad \rho, \rho^{-1} \in L^\infty(\mathbb{R}^3 \times (0, T)),$$

and

$$(1.25) \quad \int_0^T \int_{\mathbb{R}^3} |u|^q dx d\tau < \infty$$

for some $q > 3$, and the other solution $(\bar{\rho}, \bar{u}, \bar{f}, \bar{\rho}_s)$ will have to satisfy

$$(1.26) \quad \|\bar{\rho}_s\|_{L^\infty} + \|\nabla \bar{\rho}_s\|_{L^\infty} < \infty,$$

$$
\int_0^T \left[ \tau^\frac{1}{2} \|\nabla \bar{F}(\cdot, \tau)\|_{L^2}^2 + \tau^\frac{3}{4} \|\nabla \bar{F}(\cdot, \tau)\|_{L^4}^4 \right] d\tau
+ \int_0^T \left[ \tau^\frac{1}{2} \|\nabla \bar{\omega}(\cdot, \tau)\|_{L^2}^2 + \tau^\frac{3}{4} \|\nabla \bar{\omega}(\cdot, \tau)\|_{L^4}^4 \right] d\tau
+ \int_0^T \left[ \|\bar{u}(\cdot, \tau)\|_{L^\infty}^2 + \tau^\frac{1}{2} \|\nabla \bar{u}(\cdot, \tau)\|_{L^\infty}^2 \right] d\tau < \infty,
$$

where $\bar{F}$ and $\bar{\omega}$ are as in $(1.6)$-$(1.7)$ and

$$(1.28) \quad \bar{f} \in L^{2\bar{q}},$$

for some $\bar{q} \in [1, \infty]$. Finally, we assume that

$$(1.29) \quad \rho_0 - \bar{\rho}_0 \in L^2 \cap L^{2\bar{q}^\prime},$$

where $\bar{q}^\prime$ is the Hölder conjugate of $\bar{q}$ in $(1.28)$. 

In (1.28)
1.6. Main results. We state the main results of this paper. First of all, the following theorem gives the global-in-time existence and long time behaviour of weak solution \((\rho, u)\) to (1.1):

**Theorem 1.5.** Given \(\rho_1, \rho_2, \rho_\infty > 0\), let \(P, f, \lambda, \mu\) be the system parameters in (1.1) satisfying (1.2), (1.11), (1.12) and (1.14). The system (1.1) has a global-in-time weak solution \((\rho, u)\) provided that

\[
\begin{align*}
\rho_2 &\leq \rho_0 \leq \rho_1 \text{ a.e.,} \quad \rho_0 \in L^\infty, \\
\int_{\mathbb{R}^3} |u_0|^q dx &< \infty, \\
C_0^\beta := \|\rho_0 - \rho_s\|^2_{L^2} + \|u_0\|^2_{L^\alpha} &< 1, \quad \alpha \in \left(\frac{1}{2}, 1\right],
\end{align*}
\]

where \(q \geq 6\) satisfies

\[
\frac{\mu}{\lambda} > \frac{(q - 2)^2}{4(q - 1)} > \frac{4}{5},
\]

and \(\rho_s\) is a steady state solution satisfying (1.13) and (1.15). The solution can be shown to satisfy conditions (1.16) - (1.22) and (1.24) - (1.25) and the energy estimates: there exist \(C > 0\) and \(\theta > 0\) such that

\[
\sup_{0 \leq \tau < \infty} \int_{\mathbb{R}^3} [\rho|u|^2 + |\rho - \rho_s|^2 + \sigma(\tau)^{1-\alpha} |\nabla u|^2 + \sigma(\tau)^{2-\alpha} |\dot{u}|^2] d\tau
\]

\[
\quad + \int_0^\infty \int_{\mathbb{R}^3} (|\nabla u|^2 + \sigma(\tau)^{1-\alpha} |u|^2 + \sigma(\tau)^{2-\alpha} |\nabla \dot{u}|^2) dx d\tau \leq CC_0^\theta,
\]

\[
\frac{1}{2} \leq \rho_2 \leq 2 \rho_1 \text{ a.e.}
\]

where \(\sigma(\tau) = \min\{1, \tau\}\), and \(C(\tau)\) may depend additionally on a positive lower bound for \(\tau\);

\[
\sup_{0 \leq \tau < \infty} \|u(\cdot, \tau)\|_{H^\alpha} \leq CC_0^\theta, \quad \sup_{0 \leq \tau < \infty} \|u(\cdot, \tau)\|_{L^r} \leq C(\tau)C_0^\theta,
\]

where \(r \in (3, \frac{5}{1+\alpha})\) with \(C(\tau) > 0\) being a positive constant which depends only on \(r\). Furthermore, if \(\rho_0\) and \(\rho_s\) are piecewise \(C^{3_0}\) for some \(\beta_0 > 0\) in the sense of Definition 1.1 that

\[
\|\rho_0(\cdot) - \rho_s\|_{C^{\beta_0}_p} \leq N,
\]

for some \(N > 0\), then for each positive time \(T\) and \(t \in [0, T]\), there exists function \(\beta(t) \in (0, \beta_0]\) and positive constant \(C(N, T, C_0)\) such that \(\rho\) is piecewise \(C^{3(\tau)}\) on \([0, T]\) with

\[
\sup_{0 \leq \tau \leq T} \|\rho(\cdot, \tau) - \rho_s\|_{C^{\beta(\tau)}_p} + \int_0^T \|\nabla u(\cdot, \tau)\|_{L^\infty} d\tau \leq C(N, T, C_0).
\]

Moreover, if we define

\[
u(\cdot, \tau) = (\mu + \lambda)^{-1} \Gamma_\mu \ast (P(\rho(\cdot, \tau)) - \rho_0(\cdot)),
\]
where $P_\rho = P(\rho_s)$ and $\Gamma$ is the fundamental solution of the Laplace operator on $\mathbb{R}^3$, and

$$u_F^k(\cdot, \tau) = (\mu + \lambda)^{-1}[\Gamma_{x_j} * F + \mu * \omega_j^{i,k}],$$

where $F$ and $\omega$ are as in (1.39), then $u = u_F + u_P$. In particular, for $\tilde{\alpha} \in (0, \alpha - \frac{1}{2})$, $\nu > \max\{\frac{1}{2}, \alpha - \frac{1}{2} - \tilde{\alpha}\}$, $q_1 \geq \frac{10}{3}$, $q_2 \in [\frac{10}{3}, 6]$ and $q_3 > 2$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|\rho - \rho_s(\cdot, \tau)\|_{L_\infty}^2 d\tau = 0,$$

and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|u_F(\cdot, \tau)\|_{C^{1+\tilde{\alpha}}} + \|F(\cdot, \tau)\|_{C^{\tilde{\alpha}}} d\tau = 0,$$

and

$$\lim_{T \to \infty} (\|u(\cdot, T)\|_{L^{\infty}} + \|\rho - \rho_s\|_{L^{\infty}(\mathbb{R}^3 \times [T, \infty))}) = 0.$$

Remark 1.6. Using the energy estimate (1.32), one can show that the long time average of $\|\rho - \rho_s(\cdot, \tau)\|^2_{L_2}$ on $[0, T]$ satisfies

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \|\rho - \rho_s(\cdot, \tau)\|^2_{L_2} d\tau \leq CC_0^\alpha.$$

However, it is unknown whether the limit in (1.42) vanishes or not.

Remark 1.7. By choosing $\nu = 1$ in (1.39), the result shows that the long time averages of $\|u_F(\cdot, \tau)\|_{C^{1+\tilde{\alpha}}}$ and $\|F(\cdot, \tau)\|_{C^{\tilde{\alpha}}}$ on $[0, T]$ vanish as $T$ goes to infinity. On the other hand, it is unknown whether there is any non-zero limit for the long time average in (1.39) if $\nu \leq \max\{\frac{1}{2}, \alpha - \frac{1}{2} - \tilde{\alpha}\}$.

Once Theorem 1.5 is proved, we are able to obtain the following stability result on weak solutions which is given in Theorem 1.8.

**Theorem 1.8.** Given $\alpha > 0$, let $f$, $\lambda$, $\mu$ be the system parameters in (1.1) satisfying (1.12), (1.12) and (1.14), and assume $P$ satisfies (1.23) $P(\rho) = \rho a$. Assume that $(\rho_0, u_0)$ and $(\tilde{\rho}_0, \tilde{u}_0)$ are functions satisfying (1.29), (1.30) and (1.36). Then for each $T > 0$ and $C > 0$, there is a positive constant $M = M(T, C)$ such that if $(\rho, u, f, \rho_s)$ and $(\tilde{\rho}, \tilde{u}, \tilde{f}, \tilde{\rho}_s)$ are weak solutions of (1.1) satisfying (1.24)-(1.28), and if all the norms occurring in the above conditions are bounded by $C$, then

$$\left(\int_0^T \int_{\mathbb{R}^3} |u - \tilde{u}|^2 d\tau dx\right)^{\frac{1}{2}} + \sup_{0 \leq \tau \leq T} \|\rho - \tilde{\rho}\|_{H^{-1}} \leq M \left[\|\rho_0 - \tilde{\rho}_0\|_{L^2} + \|\rho_0 u_0 - \tilde{\rho}_0 \tilde{u}_0\|_{L^2} \right]$$

+ $M \left[\left(\int_{\mathbb{R}^3} |\rho_s - \tilde{\rho}_s|^2 dx\right)^{\frac{1}{2}} + \left(\int_0^T \int_{\mathbb{R}^3} |f - \tilde{f} \ast S|^2 d\tau dx\right)^{\frac{1}{2}}\right]$, 

where $\tilde{q}'$ is given in (1.29) and $S = S(x, t)$ is defined in (1.8).
\textbf{Remark 1.9.} Similar to the case as in \cite{Hof06} and \cite{Sue20b}, under a more general condition on the pressure \( P \), namely

\begin{equation}
\sup_{0 \leq \tau \leq T} \left\| \nabla \left( \frac{P(\rho(\cdot, \tau)) - P(\bar{\rho}(\cdot, \tau))}{\rho(\cdot, \tau) - \bar{\rho}(\cdot, \tau)} \right) \right\|_{L^r} < \infty,
\end{equation}

for some \( r \in [3, \infty] \), one can still obtain the same conclusion \eqref{1.44} from Theorem \ref{1.3} Such condition on \( P \) includes the one given in \cite{Hof06} and \cite{Sue20b}.

\textbf{Remark 1.10.} If we further assume that \( \| \nabla \tilde{f} \|_{L^\infty} < \infty \), then the term \( f - \tilde{f} \circ S \) can be replaced by \( f - \tilde{f} \) in \eqref{1.44}; see Remark \ref{4.3} for the explanation.

The rest of the paper is organised as follows. In Section 2, we give some \textit{a priori} estimates on the smooth solutions to \eqref{1.1}. In Section 3 we make use of those estimates obtained in Section 2 to prove the global-in-time existence and long time behaviour of weak solution to \eqref{1.1} described in Theorem \ref{1.5}. Finally in Section 4, we address the stability of weak solutions given in Theorem \ref{1.3} by making use of the Lagrangean framework and bounds on the weak solutions.

\section{A priori estimates}

In this section we derive \textit{a priori} bounds for smooth, local-in-time solutions \((\rho, u)\) of \eqref{1.1}. We first derive \textit{a priori} bounds for local-in-time smooth solutions under the assumption that the density \( \rho \) is bounded above and below as given by \eqref{2.1}. We then proceed to close the estimates by obtaining the necessary bounds for density in a maximum principle argument along particle trajectories of the velocity.

Given a steady state solution \( \rho_s \) satisfying \eqref{1.13} and \eqref{1.15}, we define functionals \( \Phi_1(t), \Phi_2(t), \Phi_3(t) \) and \( \Phi(t) \) for a given solution \((\rho, u)\) by

\begin{align*}
\Phi_1(t) &= \sup_{0 \leq \tau \leq t} (\|u(\cdot, \tau)\|_{L^2}^2 + \|\rho(\cdot, \tau)\|_{L^2}^2) + \int_0^t \|\nabla u(\cdot, \tau)\|_{L^2}^2 d\tau, \\
\Phi_2(t) &= \sup_{0 \leq \tau \leq t} \sigma(\tau)^{1-\alpha} \|\nabla u(\cdot, \tau)\|_{L^2}^2 + \int_0^t \sigma(\tau)^{1-\alpha} \|\dot{u}(\cdot, \tau)\|_{L^2}^2 d\tau, \\
\Phi_3(t) &= \sup_{0 \leq \tau \leq t} \sigma(\tau)^{2-\alpha} \|\dot{u}(\cdot, \tau)\|_{L^2}^2 + \int_0^t \sigma(\tau)^{2-\alpha} \|\nabla \dot{u}(\cdot, \tau)\|_{L^2}^2 d\tau, \\
\Phi(t) &= \Phi_1(t) + \Phi_2(t) + \Phi_3(t),
\end{align*}

where we recall that \( \alpha \in (\frac{1}{2}, 1] \) and \( \sigma(\tau) = \min\{1, \tau\} \). We will obtain \textit{a priori} bounds on the above functionals and the results can be summarised as follows:

\textbf{Theorem 2.1.} Given \( \rho_1, \rho_2, \rho_\infty > 0 \) and \( \gamma \geq 1 \), let \( P, f, \lambda, \mu \) be the system parameters in \eqref{1.1} satisfying \eqref{1.2}, \eqref{1.11}, \eqref{1.12} and \eqref{1.14}. Given \( q \geq 6 \), there are positive constants \( M, \theta \) depending on the parameters and assumptions in \eqref{1.2}, \eqref{1.11}, \eqref{1.12} and \eqref{1.14}, such that: if \( \rho_s \) is a steady state solution satisfying \eqref{1.13} and \eqref{1.15}, and if \((\rho, u)\) is a solution of \eqref{1.1} on \( \mathbb{R}^3 \times [0, T] \) with initial data \((\rho_0, u_0) \in H^3(\mathbb{R}^3) \) satisfying \eqref{1.30} with the smallness assumption \( C_0 \ll 1 \), then we have

\begin{align}
&\frac{1}{2} \rho_2 \leq \rho(x, t) \leq 2 \rho_1 \text{ on } \mathbb{R}^3 \times [0, T], \tag{2.1} \\
&\Phi(t) \leq MC_0^\theta \text{ on } \mathbb{R}^3 \times [0, T], \tag{2.2}
\end{align}
where \( \alpha \in (\frac{1}{2}, 1] \) and \( C_0 := \| \rho_0 - \rho_s \|^2_{L^2} + \| u_0 \|^2_{H^s} \).

Theorem 2.1 will be proved in a sequence of lemmas. We first establish the bound (2.2) under the assumption that the pointwise bounds in (2.1) hold for the density \( \rho \). In order to control the space-time integral of \( \rho - \rho_s \), we introduce different methods in small-time and large-time regimes, which will be given in Subsection 2.1 and Subsection 2.2 respectively. In Subsection 2.3, we then close the estimates of Theorem 2.1 by deriving pointwise bounds (2.1) for \( \rho \) under the smallness assumption on \( C_0 \). This gives an noncontingent estimate for \((\rho, u)\) and thereby proving Theorem 2.1.

Unless otherwise specified, throughout this paper, \( C \) will denote a generic positive constant which depends on the same quantities as the constant \( M \) in the statement of Theorem 2.1 but independent of time \( t \) and the regularity of initial data. And for simplicity, we write \( P = P(\rho) \) and \( P_s = P(\rho_s) \), etc., without further referring.

We will make repeated use of the following Gagliardo-Nirenberg type inequalities and Sobolev imbeddings, the proof can be found in [Zie89]:

**Proposition 2.2.** For \( p \in [2, 6] \) and \( r \in (3, \infty) \), there exists some generic constants \( C > 0 \) and \( C(r) > 0 \) such that for any \( g \in H^1 \) and \( h \in W^{1,r}(R^3) \), we have

\[
\| g \|^p_{L^p} \leq C \| g \|^\frac{6-p}{2} \| \nabla g \|^\frac{3p-6}{2},
\]

\[
\langle h \rangle^\alpha \leq C(r) \| \nabla h \|_{L^r(R^3)},
\]

where \( \alpha = 1 - \frac{3}{r} \).

We begin with the following energy balance law which gives bounds on \( \Phi_1(t) \) for all \( t \in [0, T] \).

**Lemma 2.3.** Assume that the hypotheses and notations of Theorem 2.1 are in force. Suppose that \( \rho \) satisfies the pointwise bounds (2.1). Then for any \( t \in [0, T] \),

\[
\Phi_1(t) \leq CC_0.
\]

**Proof.** Using \( \nabla \Psi = -\rho_s^{-1} \nabla P_s \) on the momentum equation (1.1.2), we get

\[
\rho \dot{u} + \rho(\rho^{-1} \nabla P - \rho_s^{-1} \nabla P_s) - \mu \Delta u - \lambda \nabla (\text{div}(u)) = 0.
\]

Multiply (2.6) by \( u \) and integrate to obtain that for \( t \in [0, T] \),

\[
\int_{R^3} \frac{1}{2} \rho |u|^2 dx \bigg|_0^t + \int_0^t \int_{R^3} \rho u(\rho^{-1} \nabla P - \rho_s^{-1} \nabla P_s) dx d\tau + \int_0^t \int_{R^3} [\mu |\nabla u|^2 + \lambda (\text{div}(u))^2] dx d\tau = 0,
\]

(2.7)

where the divergence of a matrix is taken row-wise. Next we define

\[
G(\rho) = \int_{\rho_s}^\rho \int_{\rho_s}^\tau \tau^{-1} P'(\tau) d\tau d\rho,
\]

then since \( \rho \) satisfies the bounds in (2.1), we have

\[
C^{-1} |\rho - \rho_s|^2 \leq G(\rho) \leq C |\rho - \rho_s|^2.
\]
Using the mass equation, the second term on the left side of (2.7) can be written as follows
\[
\int_0^t \int_{\mathbb{R}^3} \rho u (\rho^{-1} \nabla P - \rho_s^{-1} \nabla P_s) \, dx \, dt
\]
\[
= \int_0^t \int_{\mathbb{R}^3} \rho u \cdot \nabla \left( \int_{\rho_s}^{\rho} r^{-1} P'(r) \, dr \right) \, dx \, dt
\]
\[
= \int_0^t \int_{\mathbb{R}^3} \rho \left( \int_{\rho_s}^{\rho} r^{-1} P'(r) \, dr \right) \, dx \, dt = \int_0^t \int_{\mathbb{R}^3} G(\rho) \, dx \, dt = \int_{\mathbb{R}^3} G(\rho) \, dx |^t_0.
\]
Putting the above into (2.7), the estimate (2.8) follows. \(\square\)

Next, we recall the following estimates on the effective viscous flux \(F\) and vorticity \(\omega\) for all \(t \in [0, T]\). A proof can be found in [LM11] using the Poisson equations (1.0) and (1.7) and the Marcinkiewicz multiplier theorem.

**Lemma 2.4.** Assume that \(\rho\) satisfies the bounds in (2.1). Then for \(t \in [0, T]\) and \(r_1, r_2 \in (1, \infty)\),
\[
\| \nabla F \|_{L^{r_1}} + \| \nabla \omega \|_{L^{r_1}} \leq C (\| \dot{u} \|_{L^{r_1}} + \| \nabla u \|_{L^{r_1}} + \| (\rho - \rho_s)^2 \|_{L^{r_1}}),
\]
(2.8)\[
\| \nabla u \|_{L^{r_2}} \leq C (\| F \|_{L^{r_2}} + \| \omega \|_{L^{r_2}} + \| (\rho - \rho_s) \|_{L^{r_2}}),
\]
(2.9)

To proceed further, we have to obtain higher order estimates on \(u\). In order to obtain better estimates on \(\rho\), we subdivide the estimates into two cases namely \(t \in [0, 1]\) and \(t \in (1, T]\). These will be illustrated in Subsection 2.1 and Subsection 2.2 as follows:

### 2.1. Estimates on \(u\) and \(\rho\) for \(t \in [0, 1]\)

In this subsection, we aim at estimating \(\Phi_1(t)\), \(\Phi_2(t)\) and \(\Phi_3(t)\) when \(t \in [0, 1]\). We first derive bounds on \(u\) in \(L^\infty([0, t]; H^1(\mathbb{R}^3))\) under some smallness conditions on \(C_0\), which gives a bound on \(\Phi_2(t)\).

**Lemma 2.5.** Assume that \(\rho\) satisfies (2.1) and \(C_0 \ll 1\). For \(t \in [0, 1]\) and \(\alpha \in [0, 1]\), we have
\[
\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla u\|^2 \, dx + \int_{\mathbb{R}^3} \int_0^t \tau^{1-\alpha} |\dot{u}|^2 \, dx \, dt \leq CC_0.
\]
(2.10)\[
\text{Proof.} \, \text{We apply the interpolation techniques given by Hoff [Hof02] or Suen [Sue20b].}
\]
We define differential operators \(\mathcal{L}\) acting on functions \(w : \mathbb{R}^3 \times [0, \infty) \rightarrow \mathbb{R}^3\) by
\[
(\mathcal{L}w)^j = (\rho u^j)_t + \text{div}(\rho u^j u) - (\mu \Delta u^j + \lambda \text{div}(u_{x_j})).
\]
Then we define \(w_1\) and \(w_2\) by
\[
\mathcal{L}w_1 = 0, \quad w_1(x, 0) = w_{10}(x),
\]
(2.11)
\[
\mathcal{L}w_2 = -\rho \left( \rho^{-1} \nabla P - \rho_s^{-1} \nabla P_s \right), \quad w_2(x, 0) = 0
\]
for a given \(w_{10}\), and if \(w_{10} = u_0\), then \(w_1 + w_2 = u\). Following the proof of the energy balance law for the bound (2.5), we readily have
\[
\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |w_1(x, \tau)|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} |\nabla w_1|^2 \, dx \, dt \leq C \int_{\mathbb{R}^3} |w_{10}|^2 \, dx,
\]
(2.12)
as well as

\[
\sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |w_2(x, \tau)|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} |\nabla w_2|^2 \, dx \, d\tau \\
\leq Ct \sup_{0 \leq \tau \leq t} \| (\rho - \rho_s)(\cdot, \tau) \|_{L^2_x}^2.
\]  

(2.13)

On the other hand, for \( k = 0, 1 \), we multiply equations (2.11) for \( w_1 \) and \( w_2 \) by \( \tau^k \dot{w}_1 \) and \( \tau^k \dot{w}_2 \) respectively and integrate to obtain

\[
\tau^k \int_{\mathbb{R}^3} |\nabla w_1(x, \tau)|^2 \, dx \bigg|_{\tau=0}^{\tau=t} + \int_0^t \int_{\mathbb{R}^3} \tau^k \dot{w}_1^2 \, dx \, d\tau \\
\leq C \int_0^t \int_{\mathbb{R}^3} k\tau^{k-1} |\nabla w_1|^2 \, dx \, d\tau + C \int_0^t \int_{\mathbb{R}^3} \tau^{3k} (|\nabla w_1|^3 + |\nabla u|^3) \, dx \, d\tau,
\]

(2.14)

and

\[
\tau^k \int_{\mathbb{R}^3} |\nabla w_2(x, \tau)|^2 \, dx \bigg|_{\tau=0}^{\tau=t} + \int_0^t \int_{\mathbb{R}^3} \tau^k \dot{w}_2^2 \, dx \, d\tau \\
\leq C \left| \int_{\mathbb{R}^3} (P - P_s) \text{div}(w_2)(x, \tau) \, dx \bigg|_{\tau=0}^{\tau=t} + C \int_0^t \int_{\mathbb{R}^3} \tau^k (|\rho - \rho_s| \dot{w}_2) \, dx \, d\tau \\
+ C \int_0^t \int_{\mathbb{R}^3} \tau^{3k} (|\nabla w_2|^3 + |\nabla u|^3) \, dx \, d\tau.
\]  

(2.15)

The term \( C \int_0^t \int_{\mathbb{R}^3} k\tau^{k-1} |\nabla w_1|^2 \, dx \, d\tau \) can be bounded by \( C \|w_{10}\|^2_{L^2} \) with the help of (2.12), and using (2.12)-(2.13), the term involving \( P \) can be bounded by

\[
C \left| \int_{\mathbb{R}^3} (P - P_s) \text{div}(w_2)(x, \tau) \, dx \bigg|_{\tau=0}^{\tau=t} \right| \\
\leq C \left( \int_{\mathbb{R}^3} |\rho - \rho_s|^2 (x, t) \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^3} |\nabla w_2|^2 (x, t) \, dx \right)^\frac{1}{2} \\
\leq C \|w_{10}\|^2_{H^\infty} \left( \int_{\mathbb{R}^3} |\nabla w_2|^2 (x, t) \, dx \right)^\frac{1}{2}.
\]

Moreover, using the bound (2.5) on \( \rho - \rho_s \) and the assumption that \( t \in [0, 1] \), we have

\[
C \int_0^t \int_{\mathbb{R}^3} \tau^k |\rho - \rho_s| \dot{w}_2 \, dx \, d\tau \\
\leq \left( \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\rho - \rho_s|^2 \, dx \right)^\frac{1}{2} \left( \int_0^t \int_{\mathbb{R}^3} \tau^k |\dot{w}_2|^2 \, dx \, d\tau \right)^\frac{1}{2} \\
\leq CC_0^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} \tau^k |\dot{w}_2|^2 \, dx \, d\tau \right)^\frac{1}{2}.
\]
Next we consider the term \( \int_0^t \int_{\mathbb{R}^3} \tau^\frac{3k}{2} |\nabla u|^3 \, dx \, d\tau \). By applying (2.8)–(2.9) and (2.5), for \( t \in [0, 1] \), we have

\[
\int_0^t \int_{\mathbb{R}^3} \tau^\frac{3k}{2} |\nabla u|^3 \, dx \, d\tau \leq C \int_0^t \int_{\mathbb{R}^3} \tau^\frac{3k}{2} (|F|^3 + |\omega|^3 + |\rho - \rho_0|^3) \, dx \, d\tau \\
\leq C \int_0^t \tau^\frac{3k}{2} \left( \int_{\mathbb{R}^3} |F|^2 \, dx \right)^\frac{3}{4} \left( \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \right)^\frac{3}{4} \, d\tau \\
+ C \int_0^t \tau^\frac{3k}{2} \left( \int_{\mathbb{R}^3} |\omega|^2 \, dx \right)^\frac{3}{4} \left( \int_{\mathbb{R}^3} |\nabla \omega|^2 \, dx \right)^\frac{3}{4} \, d\tau + CC_0 \\
\leq CC_0^3 \left( \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^\frac{3}{4} \left( \int_0^t \int_{\mathbb{R}^3} \tau^k |\dot{w}|^2 \, dx \, d\tau \right)^\frac{3}{4} \\
+ C_0^3 \left( \int_0^t \int_{\mathbb{R}^3} \tau^k |\dot{w}|^2 \, dx \, d\tau \right)^\frac{1}{4} + CC_0.
\]

Hence under suitable smallness conditions on \( C_0 \), the term \( \int_0^t \int_{\mathbb{R}^3} \tau^\frac{3k}{2} |\nabla u|^3 \, dx \, d\tau \) can be absorbed into the left sides of (2.14) and (2.15). Treating the terms \( \int_0^t \int_{\mathbb{R}^3} \tau^\frac{3k}{2} |\nabla w_1|^3 \, dx \, d\tau \) and \( \int_0^t \int_{\mathbb{R}^3} \tau^\frac{3k}{2} |\nabla w_2|^3 \, dx \, d\tau \) in a similar way, we can conclude that

\begin{align*}
(2.16) & \quad \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla w_1|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} |\dot{w}_1|^2 \, dx \, d\tau \leq C \|w_{10}\|_{L^1}^2, \\
(2.17) & \quad \sup_{0 \leq \tau \leq t} \tau \int_{\mathbb{R}^3} |\nabla w_1|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} \tau |\dot{w}_1|^2 \, dx \, d\tau \leq C \|w_{10}\|_{L^2}^2, \\
(2.18) & \quad \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla w_2|^2 \, dx + \int_0^T \int_{\mathbb{R}^3} |\dot{w}_2|^2 \, dx \, d\tau \leq CC_0.
\end{align*}

Since the operator \( \mathcal{L} \) is linear, we can apply Riesz-Thorin interpolation to deduce from (2.16)–(2.17) that for \( t \in [0, 1] \) and \( \alpha \in [0, 1] \)

\[
(2.19) \quad \sup_{0 \leq \tau \leq t} \tau^{1-\tau} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \tau^{1-\alpha} |\dot{u}|^2 \, dx \, d\tau \leq C \|w_{10}\|_{L^1}^2.
\]

By taking \( w_{10} = u_0 \) in (2.19), we conclude from (2.18) and (2.19) that the bound (2.10) holds for \( t \in [0, 1] \). \( \square \)

**Remark 2.6.** The result (2.10) from Lemma 2.8 holds for all \( \alpha \in [0, 1] \), in particular it holds for \( \alpha \in (\frac{1}{2}, 1] \).

We further derive preliminary bounds for \( \dot{u} \) in \( L^\infty([0, t]; L^2(\mathbb{R}^3)) \) when \( t \in [0, 1] \). Notice that we require \( \alpha \in (\frac{1}{2}, 1] \) on the time layer factor \( \tau^{2-\alpha} \) due to the lack of integrability in time near \( t = 0 \) for \( \dot{u} \).

**Lemma 2.7.** Assume that \( \rho \) satisfies (2.1) and \( C_0 \ll 1 \). For \( t \in [0, 1] \) and \( \alpha \in (\frac{1}{2}, 1] \), we have

\[
(2.20) \quad \sup_{0 \leq \tau \leq t} \tau^{2-\alpha} \int_{\mathbb{R}^3} |\dot{u}|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}|^2 \, dx \, d\tau \leq CC_0^\alpha,
\]

for some \( \theta > 0 \).
Proof. We use the method given in [LM11] with some modifications. We rewrite the momentum equation (1.12) as follows.

\begin{equation}
\rho \dot{u} - \mu \Delta u - \lambda \nabla (\text{div}(u)) = -\nabla (P - P_\ast) + (\rho - \rho_\ast) \nabla \Psi.
\end{equation}

We apply the material derivative \( \frac{D}{Dt}(\cdot) \) on (2.21), multiply by \( \tau^{-2-\alpha} \) and integrate to give

\begin{align*}
\int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \rho \frac{D}{Dt} \frac{1}{2} |\dot{u}|^2 dxd\tau - \mu \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \dot{u}^i (\Delta u^i_t + \text{div}(\Delta u^i_t) dxd\tau \\
- \lambda \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \dot{u}^i (\text{div}(u_{x,j})_t + \text{div}(u \text{div}(u_{x,j})) dxd\tau \\
(2.22) = - \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \dot{u}^i (P_{x,t} + \text{div}(u(P - P_\ast)) dxd\tau \\
+ \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \cdot \dot{u} (\Psi_{x,j} \rho_t + \text{div}(u(\rho - \rho_\ast) \Psi_{x,j})) dxd\tau.
\end{align*}

The left side of (2.22) can be bounded below by the terms

\begin{align*}
\frac{\tau^{2-\alpha}}{2} \int_{\mathbb{R}^3} \rho |\dot{u}|^2(x, \tau) dxdx + \mu \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}|^2 dxd\tau \\
+ \lambda \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} (\text{div}(\dot{u}))^2 dxd\tau - C \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}| |\nabla u|^2 dxd\tau \\
- C \int_0^t \int_{\mathbb{R}^3} \tau^{1-\alpha} |\dot{u}|^2 dxd\tau.
\end{align*}

For the first term on the right side of (2.22), it can be bounded by

\begin{align*}
\left| \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \dot{u}^i (P_{x,t} + \text{div}(u(P - P_\ast)) dxd\tau \right| \\
\leq C \left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}|^2 dxd\tau \right)^{\frac{1}{2}} \\
\times \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxd\tau + \int_0^t \left( \int_{\mathbb{R}^3} |u|^6 dxdx \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} |\nabla \rho_s|^3 dxd\tau \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \\
\leq C \left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}|^2 dxd\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxd\tau \right)^{\frac{1}{2}},
\end{align*}

and the second term on the right side of (2.22) is bounded by

\begin{align*}
\left| \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} \cdot \dot{u} (\Psi_{x,j} \rho_t + \text{div}(u(\rho - \rho_\ast) \Psi_{x,j})) dxd\tau \right| \\
\leq C \left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} (|\nabla \dot{u}|^2 + |\dot{u}|^2) dxd\tau \right)^{\frac{1}{2}} \\
\times \left( \int_0^t \left( \int_{\mathbb{R}^3} |u|^6 dxdx \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} (|\Psi|^3 + |\nabla \Psi|^3) dxd\tau \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \\
\leq C \left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} (|\nabla \dot{u}|^2 + |\dot{u}|^2) dxd\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dxd\tau \right)^{\frac{1}{2}}.
\end{align*}
Hence we obtain from (2.22) that
\begin{align*}
\tau^{2-\alpha} \int_{\mathbb{R}^3} \rho |\dot{u}(x, \tau)|^2 dx + \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla u|^2 dx d\tau \\
&\leq C \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}| |\nabla u|^2 dx d\tau + C \int_0^t \int_{\mathbb{R}^3} \tau^{1-\alpha} |\dot{u}(x, \tau)|^2 dx d\tau \\
&\quad + C \left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} (|\nabla \dot{u}|^2 + |\dot{u}|^2) dx d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx d\tau \right)^{\frac{1}{2}}.
\end{align*}
(2.23)

In view of the bounds given by (2.22) and (2.10), the terms \(\int_0^t \int_{\mathbb{R}^3} \tau^{1-\alpha} |\dot{u}(x, \tau)|^2 dx d\tau\) and \(\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx d\tau\) can be bounded by \(CC_0\), and by the Cauchy-Schwarz inequality, the term \(\int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}| |\nabla u|^2 dx d\tau\) is bounded by
\[
\left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}|^2 dx d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla u|^4 dx d\tau \right)^{\frac{1}{2}}.
\]

It remains to consider the space-time integral of \(\tau^{2-\alpha} |\nabla u|^4\). Using (2.9),
\begin{align*}
\int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla u|^4 dx d\tau &\leq C \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} (|F|^4 + |\omega|^4 + |\rho - \rho_s|^4) dx d\tau.
\end{align*}
(2.24)

Since \(t \in [0, 1]\), with the help of (2.3), we have
\[
\int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\rho - \rho_s|^4 dx d\tau \leq CC_0.
\]

For the integral of \(F\) in (2.24), using the Sobolev inequality (2.3) and the estimate (2.3) on \(F\), for \(\alpha \in (\frac{1}{2}, 1]\),
\begin{align*}
\int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |F|^4 dx d\tau &\leq C \int_0^t \tau^{2-\alpha} \left( \int_{\mathbb{R}^3} |F|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla F|^2 dx \right)^{\frac{1}{2}} d\tau \\
&\leq C \left( \sup_{0 \leq \tau \leq t} \tau^{1-\alpha} \int_{\mathbb{R}^3} |F|^2 dx \right)^{\frac{1}{2}} \left( \sup_{0 \leq \tau \leq t} \tau^{2-\alpha} \int_{\mathbb{R}^3} |\nabla F|^2 dx \right)^{\frac{1}{2}} \\
&\quad \times \left( \int_0^t \int_{\mathbb{R}^3} \tau^{1-\alpha} |\nabla F|^2 dx d\tau \right)^{\frac{1}{2}} \\
&\leq CC_0^\frac{1}{2} \left( \sup_{0 \leq \tau \leq t} \tau^{2-\alpha} \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} |\rho - \rho_s|^4 dx \right)^{\frac{1}{2}} \\
&\quad \times (\Phi_2(t) + C_0) \\
&\leq CC_0^\frac{1}{2} \left( \sup_{0 \leq \tau \leq t} \tau^{2-\alpha} \int_{\mathbb{R}^3} |\dot{u}|^2 dx + C_0 \right)^{\frac{1}{2}},
\end{align*}

where we have used the bounds (2.1), (2.3) and (2.10) and the fact that \(\frac{1}{2} + \frac{2}{3} \alpha + 1 - \alpha \leq 2 - \alpha\) for \(\alpha \in (\frac{1}{2}, 1]\). The integral of \(\omega\) in (2.24) can be treated in a similar way, and we obtain that
\begin{align*}
\int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla u|^4 dx d\tau &\leq CC_0^\frac{1}{2} \left( \sup_{0 \leq \tau \leq t} \tau^{2-\alpha} \int_{\mathbb{R}^3} |\dot{u}|^2 dx + C_0 \right)^{\frac{1}{2}} + CC_0.
\end{align*}
Therefore, we conclude from (2.23) that
\[ \sup_{0 \leq \tau \leq t} \tau^{2-s} \int_{\mathbb{R}^3} |\dot{u}|^2 dx + \int_0^t \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}|^2 dx d\tau \leq CC_0^3 + CC_0^2 + CC_0, \]
and the bound (2.20) follows. \qed

Combining the above results, we now have the following lemma which gives the bound for \( \Phi(t) \) when \( t \in [0,1] \).

**Lemma 2.8.** Assume that the hypotheses and notations of Theorem 2.7 are in force. Assume that \( \rho \) satisfies (2.1) and \( C_0 \ll 1 \). Then for any \( t \in [0,1] \),
\[ \Phi(t) \leq CC_0^0. \]

**Proof.** The bound (2.25) follows immediately from (2.5), (2.10) and (2.20) with the smallness assumption \( C_0 \ll 1 \).

We end this subsection by giving the following auxiliary \( L^q \) estimates on the velocity \( u \) when \( q > 6 \). It will be used for obtaining pointwise bound on \( \rho \) later.

**Lemma 2.9.** Assume that the hypotheses and notations of Theorem 2.7 are in force. Then for \( q > 6 \) satisfying (1.31) and \( t \in [0,1] \),
\[ \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} |u(x,\tau)|^q dx + \int_0^t \int_{\mathbb{R}^3} |u|^q \langle \nabla u \rangle^2 dx d\tau \]
\[ \leq C \left( C_0 + \int_{\mathbb{R}^3} |u_0|^q dx \right). \]

**Proof.** The proof can be found in [CS10]. We point out that the condition (1.31) on \( q, \mu \) and \( \lambda \) is used for controlling the integral \( \int_0^t \int_{\mathbb{R}^3} |u|^q \langle \nabla u \rangle^2 dx d\tau \) in (2.20). \qed

**Remark 2.10.** For the more general case when \( t \in [0,T] \), one can obtain
\[ \sup_{0 \leq \tau \leq T} \int_{\mathbb{R}^3} |u(x,\tau)|^q dx + \int_0^T \int_{\mathbb{R}^3} |u|^q \langle \nabla u \rangle^2(x,\tau)dx d\tau \]
\[ \leq C(T) \left( C_0 + \int_{\mathbb{R}^3} |u_0|^q dx \right), \]
where \( C(T) > 0 \) further depends on \( T \). Hence it gives \( \int_0^T \int_{\mathbb{R}^3} |u|^q dx d\tau < \infty \), which shows the bound (1.26) for \( u \) in \( L^q \) when \( q > 6 \).

2.2. **Estimates on \( u \) and \( \rho \) for \( t > 1 \).** In this subsection, we obtain estimates on \( u \) and \( \rho \) when \( t > 1 \). Specifically, we define
\[ \tilde{\Phi}_1(t) = \sup_{1 \leq \tau \leq t} (||u(\cdot,\tau)||_{L^2}^2 + ||(\rho - \rho_0)(\cdot,\tau)||_{L^2}^2) + \int_0^t \| \nabla u(\cdot, \tau) \|_{L^2}^2 d\tau, \]
\[ \tilde{\Phi}_2(t) = \sup_{1 \leq \tau \leq t} \| \nabla u(\cdot, \tau) \|_{L^2}^2 + \int_1^t \| \dot{u}(\cdot, \tau) \|_{L^2}^2 d\tau, \]
\[ \tilde{\Phi}_3(t) = \sup_{1 \leq \tau \leq t} \| \dot{u}(\cdot, \tau) \|_{L^2}^2 + \int_1^t \| \nabla \dot{u}(\cdot, \tau) \|_{L^2}^2 d\tau, \]
and we aim at bounding $\tilde{\Phi}(t)$ under the assumption that the pointwise bounds in (2.1) hold for the density $\rho$. As inspired by the work [LM11], we further introduce the following auxiliary functionals $\mathcal{H}(t)$ and $\mathcal{A}(t)$ given by

$$\mathcal{H}(t) = \int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 dx \, d\tau,$$

$$\mathcal{A}(t) = \int_1^t \int_{\mathbb{R}^3} |\rho - \rho_s|^4 dx \, d\tau.$$

To begin with, by the energy balance law given by Lemma 2.3, for all $t > 1$, we readily have

$$\tilde{\Phi}_1(t) \leq C C_0.$$  

In the next lemma, we will give the bounds on $\tilde{\Phi}_2$ and $\tilde{\Phi}_3$ in terms of $\mathcal{H}$ and $\mathcal{A}$.

**Lemma 2.11.** Assume that the hypotheses and notations of Theorem 2.1 are in force. Suppose that $\rho$ satisfies the pointwise bounds (2.1). Then for any $t > 1$,

$$\tilde{\Phi}_2(t) \leq C \left(C_0 + \mathcal{A}(t) + C_0^2 \mathcal{H}(t) \right),$$

$$\tilde{\Phi}_3(t) \leq C (C_0 \mathcal{H}(t) + \tilde{\Phi}_2(t) + \mathcal{A}(t)).$$

**Proof.** First of all, following steps given in [LM11, pp. 501], we can rewrite the momentum equation (1.1) as follows.

$$\rho \dot{u} - \mu \Delta u - \lambda \nabla (\nabla (u)) + \rho_s \nabla (\rho_s^{-1} (P - P_s))$$

$$+ (\rho - \rho_s)^2 \rho_s^{-1} \nabla \rho_s \int_0^1 \int_0^1 \xi P''(\rho_s + \xi \eta (\rho - \rho_s)) d\xi d\eta = 0.$$

Multiply (2.31) by $\dot{u}$, integrate the resulting equation and follow the analysis given in [Hof95] and [LM11], for $t > 1$, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2(x,t) \, dx + \int_1^t \int_{\mathbb{R}^3} |\dot{u}|^2 \, dx \, d\tau$$

$$\leq C \left( \int_1^t \int_{\mathbb{R}^3} |\nabla u|^3 \, dx \, d\tau + \tilde{\Phi}_1(t) + \Phi_1(1) \right)$$

$$+ \left| \int_{\mathbb{R}^3} \int_1^t \left( u (\rho - \rho_s)^2 \rho_s^{-1} \nabla \rho_s \int_0^1 \int_0^1 \xi P''(\rho_s + \xi \eta (\rho - \rho_s)) d\xi d\eta \right) \, dx \, d\tau \right|$$

$$+ \left| \int_{\mathbb{R}^2} \nabla u (P - P_s) \, dx \right|_{L^1} + \left| \int_{\mathbb{R}^2} \nabla \rho_s \rho_s^{-1} \rho \, dx \right|_{L^1}.$$

The first term on the right side of (2.32) can be readily bounded by

$$\left( \int_1^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, d\tau \right) \leq \tilde{\Phi}_1(t) \frac{1}{2} \mathcal{H}(t) \frac{1}{2},$$

and using the bounds (2.25) and (2.28), the last two terms on the far right side of (2.32) are bounded by

$$\left( \sup_{1 \leq t \leq 1} \|u(\cdot,t)\|_{L^2} + \|\nabla u(\cdot,t)\|_{L^2} \right) \tilde{\Phi}_1(t) \frac{1}{2} \leq (\tilde{\Phi}_1(t) \frac{1}{2} + \tilde{\Phi}_2(t) \frac{1}{2}) \tilde{\Phi}_1(t) \frac{1}{2}.$$
To estimate the fourth term on the right side of (2.32), using the pointwise bound (2.1) on $\rho$, 

$$
\left| \int_1^t \int_{\mathbb{R}^3} \dot{u}(\rho - \rho_s)^2 \rho_s^{-1} \nabla \rho_s \int_0^1 \int_0^1 \xi \eta^\prime (\rho_s + \xi \eta (\rho - \rho_s)) d\xi d\eta dx d\tau \right|
$$

$$
\leq C \left( \int_1^t \int_{\mathbb{R}^3} |\dot{u}|^2 dx d\tau \right)^{\frac{1}{2}} \left( \int_1^t \int_{\mathbb{R}^3} (\rho - \rho_s)^4 d\tau \right)^{\frac{1}{2}}
$$

$$
\leq C \tilde{\Phi}_2(t)^{1/2} A(t)^{1/2}.
$$

Therefore, together with the bound (2.28) on $\tilde{\Phi}_1(t)$ and the bound (2.5) on $\Phi_1(1)$, we conclude that (2.29) holds for $t > 1$.

Next, to prove the bound (2.30), we perform the similar analysis given in the proof of Lemma 2.7 and obtain, for $t > 1$,

$$
\int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 (\cdot, t) dx + \int_1^t |\nabla \dot{u}|^2 (\cdot, \tau) dx d\tau
$$

$$
\leq C \int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx d\tau + C \int_1^t \int_{\mathbb{R}^3} |\nabla u| |\nabla u|^2 dx d\tau
$$

$$
+ C \left( \int_1^t \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |\dot{u}|^2) dx d\tau \right)^{1/2} \left( \int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 dx d\tau \right)^{1/2} + \Phi(1).
$$

By the Cauchy-Schwarz inequality,

$$
\int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}| |\nabla u|^2 dx d\tau \leq \left( \int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx d\tau \right)^{1/2} \left( \int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 dx d\tau \right)^{1/2}
$$

$$
\leq \tilde{\Phi}_3(t)^{1/2} \mathcal{H}(t)^{1/2},
$$

hence the bound (2.30) follows with the help of the bound (2.28) on $\tilde{\Phi}_1(t)$ and the bound (2.29) on $\Phi(1)$. \qed

We close the estimates on $\tilde{\Phi}_2$ and $\tilde{\Phi}_3$ by bounding $A(t)$ and $\mathcal{H}(t)$ in terms of $\tilde{\Phi}$, which will be illustrated in the following lemma:

**Lemma 2.12.** Assume that the hypotheses and notations of Theorem 2.1 are in force. Suppose that $\rho$ satisfies the pointwise bounds (2.1). Then for any $t > 1$,

(2.33) \quad $\mathcal{H}(t) \leq C (\tilde{\Phi}(t)^2 + A(t)^2 + A(t) + C_0^2)$,

(2.34) \quad $A(t) \leq C \tilde{\Phi}(t)^{1/2} (\tilde{\Phi}(t) + A(t) + C_0) + CC_0^{3/2} + CC_0$.

**Proof.** To prove the bound (2.33) on $\mathcal{H}$, using (2.30),

$$
\int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 dx d\tau \leq C \int_1^t \int_{\mathbb{R}^3} (|F|^4 + |\omega|^4 + |\rho - \rho_s|^4) dx d\tau.
$$

Proof of (2.34) on $A(t)$. Using (2.33),

$$
\int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx d\tau \leq C \int_1^t \int_{\mathbb{R}^3} (|F|^4 + |\omega|^4 + |\rho - \rho_s|^4) dx d\tau.
$$

By the Cauchy-Schwarz inequality,

$$
\int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx d\tau \leq \left( \int_1^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 dx d\tau \right)^{1/2} \left( \int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 dx d\tau \right)^{1/2}
$$

$$
\leq \tilde{\Phi}_3(t)^{1/2} \mathcal{H}(t)^{1/2},
$$

hence the bound (2.34) follows with the help of the bound (2.28) on $\tilde{\Phi}_1(t)$ and the bound (2.29) on $\Phi(1)$. \qed
The integral of $|\rho - \rho_*|^4$ can be bounded by $C\mathcal{A}(t)$, and by the Sobolev inequality (2.3), the estimate (2.8) on $F$ and the bound (2.28) on $\Phi(t)$, we also have
\[
\int_1^t \int_{\mathbb{R}^3} |F|^4 \, dx \, dt \leq \left( \sup_{1 \leq \tau \leq t} \int_{\mathbb{R}^3} |F|^2 \, dx \right)^{\frac{3}{2}} \left( \sup_{1 \leq \tau \leq t} \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \right)^{\frac{1}{2}} \left( \int_1^t \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq C \left( \bar{\Phi}_2(t) + C_0 \right)^{\frac{3}{2}} \left( \bar{\Phi}_3(t) + C_0 \right)^{\frac{1}{2}} (\bar{\Phi}_2(t) + A(t) + C_0).
\]
We estimate the integral of $|\omega|^4$ in a similar way and we conclude from (2.35) that
\[
\int_1^t \int_{\mathbb{R}^3} |\nabla u|^4 \, dx \, dt \leq C \left( \bar{\Phi}(t) + C_0 \right) (\bar{\Phi}(t) + A(t) + C_0) + C\mathcal{A}(t),
\]
which gives (2.33). To prove (2.34), we make use of the definition of $F$ and rewrite the mass equation (2.11) as follows:
\[
(\mu + \lambda) \frac{D}{Dt}(\rho - \rho_*) + \rho(P - P_*) = -\rho \rho_* F - (\mu + \lambda) u \cdot \nabla \rho_*.
\]
We multiply (2.36) by $\text{sign}(\rho - \rho_*) |\rho - \rho_*|^2$; integrate and use the pointwise bound (2.1) on $\rho$ to obtain
\[
\int_1^t \int_{\mathbb{R}^3} |\rho - \rho_*|^4 \, dx \, dt \leq C \left\| \int_1^t \int_{\mathbb{R}^3} |\rho - \rho_*|^4 \, dx \, dt \right\|_1 + C \int_1^t \int_{\mathbb{R}^3} (|F|^4 + |u|^4) \, dx \, dt.
\]
Using the bounds (2.25) and (2.28), the terms $\left\| \int_1^t \int_{\mathbb{R}^3} |\rho - \rho_*|^4 \, dx \, dt \right\|_1$ are bounded by $C C_0$. To bound the integrals of $|F|^4$ and $|u|^4$, we use the Sobolev inequality (2.3), the estimate (2.8) and the bound (2.28) to get
\[
\int_1^t \int_{\mathbb{R}^3} (|F|^4 + |u|^4) \, dx \, dt \leq C \int_1^t \left( \int_{\mathbb{R}^3} |F|^2 \, dx \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |\nabla F|^2 \, dx \right) \, dt + C \int_1^t \left( \int_{\mathbb{R}^3} |u|^2 \, dx \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \, dt \leq C \bar{\Phi}(t)^{\frac{3}{2}} (\bar{\Phi}(t) + A(t) + C_0) + C C_0^{\frac{3}{2}}.
\]
Therefore, we conclude that, for $t > 1$,
\[
\int_1^t \int_{\mathbb{R}^3} |\rho - \rho_*|^4 \, dx \, dt \leq C \bar{\Phi}(t)^{\frac{3}{2}} (\bar{\Phi}(t) + A(t) + C_0) + C C_0^{\frac{3}{2}} + C C_0,
\]
which implies (2.34).

We summarise the above results and give the bound for $\bar{\Phi}(t)$ when $t > 1$.

\textbf{Lemma 2.13.} Assume that the hypotheses and notations of Theorem 2.7 are in force. Assume that $\rho$ satisfies (2.1) and $C_0 \ll 1$. Then for any $t > 1$,
\[
\mathcal{H}(t) + A(t) + \bar{\Phi}(t) \leq C C_0^0.
\]

\textbf{Proof.}
Proof: The bound (2.38) follows immediately from (2.28), (2.30), (2.33) and (2.34) with the smallness assumption \( C_0 \ll 1 \).

2.3. Pointwise bound on \( \rho \) and proof of Theorem 2.1. We now close the estimates on \((\rho,u)\) by proving the pointwise bounds (2.1) on \( \rho \). First of all, by the bounds (2.25) and (2.38), we have

\[
\Phi(t) \leq CC_0^\theta, \quad t \in [0,T].
\]

Next, we obtain some higher order estimates on \( \|u\|_{L^r}, \|u\|_{H^\alpha}, \langle u \rangle_{\tilde{L}^{3\frac{3}{2}}} \) for \( \alpha \in [0,1], r \in (3, \frac{3}{1-\alpha}) \) and \( \tau > 0 \).

Lemma 2.14. Assume that the hypotheses and notations of Theorem 2.1 are in force. Suppose that \( \rho \) satisfies the pointwise bounds (2.1). Then for any \( t \in (0,T] \) and \( \tau > 0 \),

\[
\sup_{0 \leq \tau < t} \|u(\cdot, \tau)\|_{H^\alpha} \leq CC_0^\theta, \quad \sup_{0 \leq \tau < t} \|u(\cdot, \tau)\|_{L^r} \leq C(r)C_0^\theta,
\]

and

\[
\langle u \rangle_{\tilde{L}^{3\frac{3}{2}}} \leq C(\tau)C_0^\theta, \quad t \geq \tau > 0,
\]

where \( \alpha \in [0,1] \) and \( r \in (3, \frac{3}{1-\alpha}) \). Here \( C(r) \) is a positive constant which depends only on \( r \) and \( C(\tau) > 0 \) may depend additionally on a positive lower bound for \( \tau \).

Proof: To show the estimate for \( \|u(\cdot, \tau)\|_{H^\alpha} \), we follow the proof of Lemma 2.9 to obtain, for \( k = 0, 1 \) and \( t \leq 1 \),

\[
\sup_{0 \leq \tau < t} \|w_1(\cdot, \tau)\|_{H^k} \leq C\|w_{10}\|_{H^k}.
\]

Hence by interpolation, we have

\[
\sup_{0 \leq \tau < t} \|w_1(\cdot, \tau)\|_{H^\alpha} \leq C\|w_{10}\|_{H^\alpha}
\]

for \( \alpha \in [0,1] \). Choosing \( w_{10} = u_0 \), then we have \( u = w_1 + w_2 \) and together with the estimates on \( w_2 \), we conclude that

\[
\sup_{0 \leq \tau < t} \|u(\cdot, \tau)\|_{H^\alpha} \leq C\|u_0\|_{H^\alpha}.
\]

For the case when \( t \in (1,T] \), by the bound (2.38), we readily have

\[
\sup_{1 < \tau \leq t} \int_{\mathbb{R}^3} (|u|^2 + |\nabla u|^2) dx d\tau \leq CC_0^\theta,
\]

hence it implies

\[
\sup_{1 < \tau \leq t} \|u(\cdot, \tau)\|_{H^\alpha} \leq CC_0^\theta,
\]

and we have the estimate on \( \sup_{0 \leq \tau < t} \|u(\cdot, \tau)\|_{H^\alpha} \) for all \( t \in [0,T] \). The estimate on \( \sup_{0 \leq \tau < t} \|u(\cdot, \tau)\|_{L^r} \) then follows from (2.32)-(2.43) and the imbedding \( H^\alpha \hookrightarrow L^r \) for \( r \in (3, \frac{3}{1-\alpha}) \). Finally, to show (2.41), we only consider the case for \( F \) since the
cases for \( u \) and \( \omega \) are somewhat simpler. Notice that using (2.38) for \( r_1 = 6 \) and the Sobolev imbedding (2.3),

\[
\| \nabla F \|_{L^6} \leq C(\| \ddot{u} \|_{L^6} + \| (\rho - \rho_s) \|_{L^6})
\]

\[
\leq C(\| \nabla \dot{u} \|_{L^2} + \| (\rho - \rho_s) \|_{L^2}).
\]

Fix \( \tau > 0 \), by the bound (2.39), the right side of (2.44) can be bounded in terms of \( C_0 \) and \( \tau \) whenever \( t \geq \tau \). Therefore, we have

\[
\| F(\cdot, t) \|_{L^\infty} + \langle F(\cdot, t) \rangle^{\frac{1}{2}} \leq CC_0^\theta, \quad t \geq \tau.
\]

For the Hölder continuity in time, we fix \( x \) and \( t_2 \geq t_1 \geq \tau \) and compute

\[
|F(x, t_2) - F(x, t_1)| \leq \frac{1}{|BR(x)|} \int_{BR(x)} |F(z, t_2) - F(z, t_1)| \, dz + C(\tau)C_0^{\theta}R^\frac{1}{2}
\]

\[
\leq R^{-\frac{3}{2}}|t_2 - t_1| \sup_{t \geq \tau} \left( \int |F_t|^2 \, dx \right)^{\frac{1}{2}} + C(\tau)C_0^{\theta}R^\frac{1}{2}
\]

\[
\leq C(\tau)C_0^{\theta} \left[ R^{-\frac{3}{2}}|t_2 - t_1| + R^\frac{1}{2} \right]
\]

by the bound (2.39). Taking \( R = |t_2 - t_1|^{\frac{1}{2}} \) we then obtain the estimate in (2.41) for \( F \).

Now we are ready to establish the pointwise bound (2.1) on \( \rho \) and complete the proof of Theorem 2.1. The proof of (2.1) consists of a maximum-principle argument applied along particle trajectories of \( u \), we only sketch it here and details can be found in [Sue13, Sue14, CS16]. Fix \( y \in \mathbb{R}^3 \) and define the corresponding particle trajectory \( x(t) \) by

\[
\begin{cases}
\dot{x}(t) = u(x(t), t) \\
x(0) = y.
\end{cases}
\]

We have from the definition (1.5) of \( F \) and the mass equation (1.1) that

\[
(\mu + \lambda) \frac{d}{dt} [\log \rho(x(t), t)] + P(\rho(x(t), t)) - P_s = -\rho_s F(x(t), t).
\]

For \( T \leq 1 \), we integrate from \( t_0 \) to \( t_1 \) for \( t_1, t_2 \in [0, T] \), and abbreviate \( \rho(x(t), t) \) by \( \rho(t) \), etc., we then obtain

\[
(\mu + \lambda) [\log \rho(s) - \log(\rho_s)] \bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} [P(\rho) - P_s] \, d\tau = -\int_{t_0}^{t_1} \rho_s F(\tau) \, d\tau.
\]

Since \( P \) is increasing, the integral of \( P \) on the left side of (2.45) is a dissipative term which is harmless. On the other hand, using the Poisson equation (1.6), we can rewrite the integral of \( F \) as

\[
-\int_{t_0}^{t_1} \rho_s F(\tau) \, d\tau = -\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(\tau) - y) \rho \dot{u}^j(y, \tau) \, dy \, d\tau
\]

\[
+ \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j}(x(\tau) - y) \left[ (P_s)_{x_j} \rho_s^{-1}(\rho_s - \rho) \right] \, dy \, d\tau,
\]

where \( \Gamma \) is the fundamental solution of the Laplace operator on \( \mathbb{R}^3 \). Invoking the method suggested in [HMS05], the first integral on the right side of (2.46) can be
bounded by
\[
\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j} (x(t) - y) \rho u_\tau^j (y, \tau) dy d\tau \right|
\]
(2.47)
\[
\leq ||\Gamma_{x_j} (\rho u_\tau^j) (t_1) ||_{L^\infty} + ||\Gamma_{x_j} (\rho u_\tau^j) (t_0) ||_{L^\infty} + \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j x_k} (x(\tau) - y) \left[ u_\tau^k ((x(\tau), \tau) - u_\tau^k (y, \tau) \right] (\rho u_\tau^j)(y, \tau) dy d\tau .
\]
To bound the term \( ||\Gamma_{x_j} (\rho u_\tau^j)(\cdot, \tau) ||_{L^\infty} \), for \( \tau = t_0, t_1 \) appeared in (2.47), we make use of the \( L^q \) estimate (2.26) given by Lemma 2.9 to obtain that
\[
||\Gamma_{x_j} (\rho u_\tau^j)(\cdot, \tau) ||_{L^\infty} \leq C(||u||_{L^2} + ||u||_{L^1}^{\xi} ||u||_{L^3}^{1-\xi})
\]
\[
\leq CC'_{0},
\]
for some positive constant \( \theta' \) and \( \xi \) depends on \( q \) with \( \xi \in (0, 1) \). For the space-time integral on the right side of (2.47), we can apply the estimate (2.11) on the semi-Hölder norm \( (u(\cdot, \tau))^{\frac{1}{2}} \) to bound it in terms of \( C_0 \). Hence we have
\[
\left| \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Gamma_{x_j} (x(\tau) - y) \rho u_\tau^j (y, \tau) dy d\tau \right| \leq CC'_{0}.
\]
For the second integral on the right side of (2.40), it can be readily estimated as follows: for \( \tau \in [t_0, t_1] \subset [0, T] \),
\[
\left| \int_{\mathbb{R}^3} \Gamma_{x_j} (x(\tau) - y) \left[ (P_s)_{\tau} \rho^{-1} (\rho_s - \rho) \right] dy \right|
\]
\[
\leq C \int_{\mathbb{R}^3} |\tau|^{-1} |(\rho - \rho_s)(y, \tau)|^2 dy
\]
\[
\leq C \left[ \int_{|\tau| \leq 1} |\tau|^{-\frac{1}{2}} dy \right] \left[ \int_{|\tau| \leq 1} |(\rho - \rho_s)(y, \tau)|^\frac{1}{2} dy \right] \Phi(\tau)^{\frac{1}{2}} + \Phi(\tau)^{\frac{1}{2}} \right]
\]
\[
\leq CC'_{0}.
\]
Hence we conclude that
\[
\int_{t_0}^{t_1} F(\tau) d\tau \leq CC'_{0},
\]
and by exploiting the smallness condition on \( C_0 \), we can see that the density \( \rho \) should remain inside the interval \( [\frac{7}{3} \rho_2, 2\rho_1] \) for all \( t \in [0, T] \), provided that the initial density satisfies \( \rho_0(x) \in [\rho_2, \rho_1] \) for \( x \in \mathbb{R}^3 \). For case of \( T > 1 \), the proof is just similar and simpler since we can ignore the initial time factor near \( t = 0 \). This completes the proof of Theorem (2.1).

3. Existence and long time behaviour of weak solutions: Proof of Theorem (1.5)

In this section, we complete the proof of Theorem (1.5) by proving the global-in-time existence and long time behaviour solution to (1.1). To achieve our goals, we make use of the a priori estimates derived in Section 2. Specifically, we let
initial data \((\rho_0, u_0)\) be given satisfying (1.30) with \(u_0 \in H^\alpha\) for \(\alpha \in (\frac{1}{2}, 1]\), and we fix those constants \(\theta, M\) defined in Theorems 2.1. Upon choosing \((\rho_0^\eta, u_0^\eta)\) as a smooth approximation of \((\rho_0, u_0)\) which can be obtained by convolving \((\rho, u_0)\) with the standard mollifying kernel of width \(\eta > 0\), we can apply the local existence results obtained by Nash [Nas62] or Tani [Tan77] to show that there is a smooth local solution \((\rho^\eta, u^\eta)\) of (1.1) with initial data \((\rho_0^\eta, u_0^\eta)\) defined up to a positive time \(T\). The \textit{a priori} estimates of Theorem 2.1 and Lemma 2.1.4 then apply to show that for \(t \in (0, T]\) and \(\tau > 0\),

\[
\Phi(t) \leq MC_0^\theta
\]

\[
\frac{1}{2} \rho_2 \leq \rho^\eta(x, t) \leq 2 \rho_1
\]

\[
\sup_{0 \leq \tau < t} \|u^\eta(\cdot, \tau)\|_{H^{\alpha}} \leq CC_0^\theta, \quad \sup_{0 \leq \tau < t} \|u^\eta(\cdot, \tau)\|_{L^r} \leq C(\tau)C_0^\theta,
\]

\[
\langle u^\eta \rangle_{\mathbb{R}^3 \times [t, \infty)}, \langle F^\eta \rangle_{\mathbb{R}^3 \times [t, \infty)}, \langle \omega^\eta \rangle_{\mathbb{R}^3 \times [t, \infty)} \leq C(\tau)C_0^\theta, \quad t \geq \tau,
\]

where \(\Phi(t)\) is defined by (2.2.7) but with \((\rho, u)\) replaced by \((\rho^\eta, u^\eta)\). By the compactness argument given in [Sue13, Sue14, CS16], the bounds (3.1)-(3.4) will then be sufficient for showing that, there is a sequence \(\eta_k \to 0\) and functions \(u, \rho\) such that as \(k \to \infty\),

\[
u \eta_k \to u \text{ uniformly on compact sets in } \mathbb{R}^3 \times (0, \infty);
\]

\[
\nabla u \eta_k(\cdot, \tau), \nabla \omega \eta_k(\cdot, \tau) \rightharpoonup \nabla u(\cdot, \tau), \nabla \omega(\cdot, \tau) \text{ weakly in } L^2(\mathbb{R}^3) \text{ for all } t > 0;
\]

\[
\sigma \frac{1+\omega}{2} \tilde{u} \eta_k, \sigma \frac{2+\omega}{2} \tilde{u} \eta_k \rightharpoonup \sigma \frac{1+\omega}{2} \tilde{u}, \sigma \frac{2+\omega}{2} \tilde{u} \text{ weakly in } L^2(\mathbb{R}^3 \times [0, \infty));
\]

and

\[
\rho \eta_k(\cdot, \tau) \rightharpoonup \rho(\cdot, \tau) \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^3) \text{ for every } t \geq 0,
\]

where \(\rho(t) = \min\{1, \rho(t)\}\) and \(\alpha \in (\frac{1}{2}, 1]\). The limiting functions \((\rho, u)\) then inherit the bounds in (1.32)-(1.33). It is also clear from the modes of convergence described in (3.5)-(3.8) that \((\rho, u)\) satisfies the weak forms (1.19)-(1.20) of (1.1).

To prove that \(\rho\) is piecewise \(C^{\beta(t)}\) on \([0, T]\) for the case when \(\rho_0\) is piecewise \(C^{\beta_0}\), it involves an argument which is based on the observation of “enhanced regularity” gained by the effective viscous flux \(F\), where \(F\) is given by (1.3) with \(\rho\) and \(u\) being the limiting functions in (3.5)-(3.8). Details of the proof can be found in [Hof02, Sue20b, Sue20c] and we only give a sketch here. We recall the decomposition of \(u\) which is given by \(u = u_F + u_P\), where \(u_F, u_P\) satisfy

\[
\left\{ \begin{array}{ll}
(\mu + \lambda) \Delta (u_F)^j = (\rho_F x_j + (\mu + \lambda))(\omega)^j_{x_k} \\
(\mu + \lambda) \Delta (u_P)^j = (P - P_x) x_j,
\end{array} \right.
\]

Using the estimates (2.8) on \(F\) and \(\omega\), and together with (1.32), we readily have

\[
\int_0^T ||\nabla u_F(\cdot, \tau)||_{L^r} d\tau \leq C(T)
\]

for some positive constant \(C(T)\). On the other hand, in order to control \(u_P\), by applying the results from Bahouri-Chemin [BC93] on Newtonian potential, we can...
make use of the pointwise bounds (1.24) on $\rho$ to show that $u_P$ is, in fact, log-Lipschitz with bounded log-Lipschitz seminorm. This is sufficient to guarantee that the integral curve $x(y, t)$ as defined by

$$
\begin{align*}
\dot{x}(t) &= u(x(t), t) \\
x(0) &= y,
\end{align*}
$$

is Hölder-continuous in $y$. Upon integrating the mass equation along integral curves $x(t, y)$ and $x(t, z)$, subtracting and recalling the definition (1.5) of $F$, we arrive at

$$
\log \rho(x(T, y), T) - \log \rho(x(T, z), T) = \log \rho_0(y) - \log \rho_0(z) + \int_0^T \left[ P(\rho(x(\tau, y), \tau) - P(\rho(x(\tau, z), \tau)) d\tau
\right.
\left. + \int_0^T |F(x(\tau, y), \tau) - F(x(\tau, z), \tau)| d\tau.
\right.$$

(3.11)

Since the pressure $P(\cdot)$ is an increasing function, the integral involving $P$ on the right side of the above can be dropped out. Moreover, with the help of the estimate (2.8) on $F$ and the Hölder-continuity of $x(y, t)$, the third term can be bounded by $C(T)$. Hence we can conclude from (3.11) that $\rho(\cdot, \tau)$ is $C^{\beta(t)}$ on $[0, T]$ for some $\beta(t) \in (0, \beta_0]$ with bounded modulus.

Finally, it remains to prove the long time behaviour of $\rho$, $u$ and $F$ as described in (1.38)-(1.41). To show (1.38), using the bounds (3.1)-(3.2) and recalling the estimate (2.34), for all $q_1 \geq 10^3$ and $t > 0$, we have

$$
\int_0^t \int_{\mathbb{R}^3} |\rho - \rho_s|^{q_1}(x, \tau) dx d\tau \leq C \int_0^t \int_{\mathbb{R}^3} |\rho - \rho_s|^{\frac{10}{3}}(x, \tau) dx d\tau \leq CC_0^0,
$$

hence we have

$$
\frac{1}{t} \int_0^t \int_{\mathbb{R}^3} |\rho - \rho_s|^{q_1}(x, \tau) dx d\tau \leq \frac{CC_0^0}{t},
$$

and (1.38) follows by taking $t \to \infty$.

To show (1.40), for $q_2 \in \left[\frac{10}{3}, 6\right]$ and $t \geq 1$, we define the functional $F^n(t)$ by

$$
F^n(t) = \int_{\mathbb{R}^3} |F^n(x(t), t)|^{q_2} dx,
$$

where $F^n = \rho_s^{-1}(\mu + \lambda) \text{div}(u)^n - \rho_s^{-1}(P(\rho^n) - P(\rho_s))$. From the bound (2.38), we readily have

$$
\int_1^\infty \int_{\mathbb{R}^3} |\nabla u^n|^4 dx d\tau + \int_1^\infty \int_{\mathbb{R}^3} |\rho^n - \rho_s|^{\frac{10}{3}} dx d\tau \leq CC_0^0.
$$

(3.12)
Using the bounds (3.1), (3.2) and (3.12) and the estimate (2.8) on $F$, the Sobolev inequality (2.3) gives

$$0 \leq F_\eta(t) \leq C \left( \int_{\mathbb{R}^3} |F_\eta(x, t)|^2 \, dx \right)^{\frac{6-q_2}{2}} \left( \int_{\mathbb{R}^3} |\nabla F_\eta(x, t)|^2 \, dx \right)^{\frac{3q_2-6}{2}}$$

$$\leq C \left( \int_{\mathbb{R}^3} (|\nabla u_\eta|^2 + |\rho_\eta - \rho_s|^2)(x, t) \, dx \right)^{\frac{6-q_2}{2}}$$

$$\times \left( \int_{\mathbb{R}^3} (|\dot{u}_\eta|^2 + |\nabla u_\eta|^2 + |\rho_\eta - \rho_s|^4)(x, t) \, dx \right)^{\frac{3q_2-6}{2}}$$

$$\leq C C_0^{\theta}, \quad t \geq 1,$$

and since $q_2 \geq \frac{10}{3}$,

$$\int_1^\infty F_\eta(\tau) \, d\tau \leq C \int_1^\infty \left( \int_{\mathbb{R}^3} |F_\eta(x, \tau)|^2 \, dx \right)^{\frac{6-q_2}{2}} \left( \int_{\mathbb{R}^3} |\nabla F_\eta(x, \tau)|^2 \, dx \right)^{\frac{3q_2-6}{2}} \, d\tau$$

$$\leq C \left( \sup_{\tau \geq 1} \int_{\mathbb{R}^3} (|\dot{u}_\eta|^2 + |\nabla u_\eta|^2 + |\rho_\eta - \rho_s|^4)(x, t) \, dx \right)^{\frac{6-q_2}{2}}$$

$$\times \left( \sup_{\tau \geq 1} \int_{\mathbb{R}^3} (|\nabla u_\eta|^2 + |\rho_\eta - \rho_s|^2)(x, t) \, dx \right)^{\frac{3q_2-6}{2}}$$

$$\times \left( \int_1^\infty \int_{\mathbb{R}^3} (|\dot{u}_\eta|^2 + |\nabla u_\eta|^2 + |\rho_\eta - \rho_s|^4) \, dx \, d\tau \right)^{\frac{3q_2-6}{2}}$$

$$\leq C C_0^{\theta}.$$

Recalling the fact that

$$|F_\eta| = |\dot{F}_\eta - u_\eta \cdot \nabla F_\eta| \leq C (|\nabla u_\eta|^2 + |\nabla u_\eta|^2 + |u_\eta|^2 |\nabla F_\eta|),$$

we further have

$$\int_1^\infty \left| \frac{dF_\eta}{dt}(\tau) \right| \, d\tau$$

$$\leq C \int_1^\infty \int_{\mathbb{R}^3} |F_\eta|^{q_2-1} |(\dot{F}_\eta - u_\eta \cdot \nabla F_\eta)| \, dx \, d\tau$$

$$\leq C \left( \int_1^\infty \int_{\mathbb{R}^3} |F_\eta|^{2q_2-2} \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_1^\infty \int_{\mathbb{R}^3} (|\nabla u_\eta|^2 + |\nabla \dot{u}_\eta|^4) \, dx \, d\tau \right)^{\frac{1}{2}}$$

$$\quad + C \left( \int_1^\infty \int_{\mathbb{R}^3} |F_\eta|^{2q_2-2} \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_1^\infty \int_{\mathbb{R}^3} (|\nabla u_\eta|^2 + |u_\eta|^2 |\nabla F_\eta|^2) \, dx \, d\tau \right)^{\frac{1}{2}}$$

$$\leq C C_0^{\theta}.$$

By the same method as shown above, the integral $\int_1^\infty \int_{\mathbb{R}^3} |F_\eta|^{2q_2-2} \, dx \, d\tau$ is bounded by $C C_0^\theta$ for $q_2 \geq \frac{10}{3}$. Moreover, using the bounds (3.3) and (3.12), the space-time integrals on $|\nabla u_\eta|^2$, $|\nabla \dot{u}_\eta|^4$ and $|\nabla u_\eta|^2$ can be bounded by $C C_0^{\theta}$. To bound the integral $\int_1^\infty \int_{\mathbb{R}^3} |u_\eta|^2 |\nabla F_\eta|^2 \, dx \, d\tau$, using (2.8) and (2.8), together with the bounds
we have
\[ \int_1^\infty \int_{\mathbb{R}^3} |u_\eta|^2 |\nabla F_\eta|^2 \, dx \, d\tau \leq C \int_1^\infty \left( \int_{\mathbb{R}^3} |u_\eta|^6 \, dx \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^3} |\nabla u_\eta|^3 \, dx \right)^{\frac{2}{3}} \, d\tau \]
\[ \leq C \int_1^\infty \left( \int_{\mathbb{R}^3} |\nabla u_\eta|^2 \, dx \right) \left( \int_{\mathbb{R}^3} (|\dot{u}_\eta|^3 + |\nabla u_\eta|^3 + |\rho_\eta - \rho_s|^6) \, dx \right)^{\frac{\alpha - 1}{2}} \, d\tau \leq C C_0^\alpha, \]

We thus obtain the following estimate on \( \frac{dF_\eta}{dt} \):
\[ \int_1^\infty \left| \frac{dF_\eta}{dt} (\tau) \right| \, d\tau \leq C C_0^\alpha. \]

Hence there is a subsequence \( \eta' \) such that \( F_\eta' \rightarrow F \) a.e. on \([1, \infty)\) and \( F \) has bounded variation. For each \( T > 0 \), by the bounded convergence theorem,
\[ \int_1^T F(\tau) \, d\tau = \lim_{\eta' \rightarrow 0} \int_1^T F_{\eta'}(\tau) \, d\tau \leq CC_0^\alpha, \]
which gives \( F \in L^1(1, \infty) \). Therefore it implies \( \lim_{T \rightarrow \infty} F(T) = 0 \). On the other hand, by the convergences (3.6) and (3.8), we also have
\[ F_{\eta'}(\cdot, t) \rightharpoonup F(\cdot, t) \text{ weakly in } L^2 \text{ for all } t > 0. \]

Since \( F_{\eta'}(\cdot, t) \) are uniformly bounded in \( L^2 \) for all \( t \geq 1 \), passing to a subsequence of \( \eta' \) if necessary, it further implies that
\[ F_{\eta'}(\cdot, t) \rightharpoonup F(\cdot, t) \text{ weakly in } L^2 \text{ for all } t \geq 1. \]

Therefore for almost all \( t \geq 1, \)
\[ \int_{\mathbb{R}^3} |F(x, t)|^q \, dx \leq \liminf_{\eta' \rightarrow 0} \int_{\mathbb{R}^3} |F_{\eta'}(x, t)|^q \, dx = F(t). \]

For each \( \varepsilon > 0 \), there is a time \( T > 1 \) such that \( F(t) \leq \varepsilon \) when \( t \geq T \), then by the Hölder continuity of \( F_\eta \) in \( t \) given by (3.4), for all \( t \geq T, \)
\[ \int_{\mathbb{R}^3} |F(x, t)|^q \, dx \leq C \varepsilon. \]

Since \( \varepsilon \) can be arbitrary, the above estimate shows that \( F(\cdot, t) \rightarrow 0 \) in \( L^q \) as \( t \rightarrow \infty \). The case for \( \omega \) follows by the same argument as for \( F \). This proves (3.40).

The long time behaviour of \( \rho \) and \( u \) given by (1.41) can be proved in the same way as given in [Hof95], so we now focus on the long time average convergence (1.39) for \( u_F \) and \( F \). For \( \alpha \in \left( \frac{1}{2}, 1 \right] \), we choose \( r \in (3, \infty) \) so that
\[ \alpha - \frac{1}{2} > 1 - \frac{3}{r}. \]
Define $\tilde{\alpha} = 1 - \frac{3}{4}$, then we have $\alpha - \frac{1}{2} > \tilde{\alpha}$. Using the Sobolev imbeddings (2.3) and (2.4), together with (2.8)-(2.9), we have for $\tau \in (0, T)$ that

$$
(3.13) \quad \left\| u_F(\cdot, \tau) \right\|_{C^{1+\tilde{\alpha}}} + \left\| F(\cdot, \tau) \right\|_{C^{\tilde{\alpha}}} \\
\leq C\left\| \dot{u}(\cdot, \tau) \right\|_{L^p} \\
\leq C\left( \int_{\mathbb{R}^3} |\dot{u}(x, \tau)|^2 dx \right)^{\frac{6-\alpha}{2}} \left( \int_{\mathbb{R}^3} |\nabla \dot{u}(x, \tau)|^2 dx \right)^{\frac{3r-6}{2r}}.
$$

Hence for $T \leq 1$, we integrate (3.13) and apply (3.1) to obtain

$$
\int_0^T (\left\| u_F(\cdot, \tau) \right\|_{C^{1+\tilde{\alpha}}} + \left\| F(\cdot, \tau) \right\|_{C^{\tilde{\alpha}}}) d\tau \\
\leq C \int_0^T \left( \int_{\mathbb{R}^3} |\dot{u}(x, \tau)|^2 dx \right)^{\frac{6-\alpha}{2}} \left( \int_{\mathbb{R}^3} |\nabla \dot{u}(x, \tau)|^2 dx \right)^{\frac{3r-6}{2r}} d\tau \\
\leq C \int_0^T \left( \tau^{1-\alpha} \int_{\mathbb{R}^3} |\dot{u}(x, \tau)|^2 dx \right)^{\frac{6-\alpha}{2}} \left( \int_{\mathbb{R}^3} \tau^{2-\alpha} |\nabla \dot{u}(x, \tau)|^2 dx \right)^{\frac{3r-6}{2r}} \\
\times \left( \int_{\mathbb{R}^3} \tau^{s+\frac{3}{2}-\frac{3}{4}} d\tau \right)^{\frac{1}{2}} \\
\leq CC_0^{\tilde{\alpha}} T^{\alpha - \tilde{\alpha} - \frac{1}{4}},
$$

where the last inequality follows since $s + \frac{3}{2} - \frac{3}{4} = \alpha - \tilde{\alpha} - \frac{1}{4} > -1$. For $T > 1$, we can repeat the above computation by integrating from 1 to $T$, which gives

$$
\int_1^T (\left\| u_F(\cdot, \tau) \right\|_{C^{1+\tilde{\alpha}}} + \left\| F(\cdot, \tau) \right\|_{C^{\tilde{\alpha}}}) d\tau \leq CC_0^{\tilde{\alpha}} T^{\frac{1}{2}}.
$$

If we pick $\nu > \max\left\{ \frac{5}{2}, \alpha - \frac{1}{2} - \tilde{\alpha} \right\}$, then we have

$$
\frac{1}{T^\nu} \int_0^T (\left\| u_F(\cdot, \tau) \right\|_{C^{1+\tilde{\alpha}}} + \left\| F(\cdot, \tau) \right\|_{C^{\tilde{\alpha}}}) d\tau \leq CC_0^{\tilde{\alpha}} (T^{\frac{1}{2} - \nu} + T^{\alpha - \tilde{\alpha} - \frac{1}{4}}) - \nu),
$$

and hence by taking $T \to \infty$, the long time average convergence (1.39) holds. This completes the proof of Theorem (1.5).

**Remark 3.1.** Under the assumption that the initial density $\rho_0$ is piecewise Hölder continuous, by Theorem (1.5) we can see that $\rho$ is piecewise Hölder continuous for positive time. As a consequence, it further implies that $\nabla u \in L^1((0, T); W^{1,\infty})$. To see how it works, we make use of the Poisson equation (3.9) again and apply properties of Newtonian potentials to conclude that the $C^{1+\beta(0)}(\mathbb{R}^3)$ norm of $u_P$ remains finite in finite time, hence there exists $C(T) > 0$ such that the following bound holds for $\nabla u_P$ as well:

$$
(3.14) \quad \int_0^T \|\nabla u_P(\cdot, \tau)\|_{L^\infty} d\tau \leq C(T).
$$

Together with the bound (3.10) on $u_F$, we conclude the estimate (1.37) for $\|\nabla u\|_{L^\infty}$, hence the regularity condition (1.18) holds for the weak solution to (1.1) with piecewise Hölder continuous initial density. The results of Theorem (1.8) therefore
do apply to this class of weak solutions, which includes solutions with Riemann-like initial data.

4. Stability and uniqueness of weak solutions: Proof of Theorem 1.8

In this section, we address the stability of weak solutions to (1.1) with respect to initial data and steady state solutions and give the proof of Theorem 1.8. As mentioned before, weak solutions with minimal regularity are best compared in a Lagrangian framework. In other words, we aim at comparing the instantaneous states of corresponding fluid particles in two different solutions. To this end, we employ some delicate estimates on particle trajectories. More precisely, for $T > 0$, the bound (3.10) and (3.14) guarantee the existence and uniqueness of the mapping $Y(y, t, t') \in C(R^3 \times [0, T]^2)$ satisfying

$$\begin{cases}
\frac{\partial X}{\partial t}(y, t, t') = u(X(y, t, t'), t) \\
X(y, t', t') = y
\end{cases}$$

where $(\rho, u, B)$ is a weak solution to (1.1). Moreover, the mapping $X(\cdot, t, t')$ is Lipschitz on $R^3$ for $(t, t') \in [0, T]^2$. The results are given in the following proposition and the proof can be found in [Hof06].

**Proposition 4.1.** Let $T > 0$ and $u$ satisfy (1.21). Then there is a unique function $X(\cdot, t, t') \in C(R^3 \times [0, T]^2)$ satisfying

$$\begin{cases}
\frac{\partial X}{\partial t}(y, t, t') = u(X(y, t, t'), t) \\
X(y, t', t') = y
\end{cases}$$

We then define $S(x, t)$, $S^{-1}(x, t)$ by

$$S(x, t) = \bar{X}(X(x, 0, t), t, 0),$$

and

$$S^{-1}(x, t) = X(\bar{X}(x, 0, t), t, 0).$$

The following proposition provides some properties of $S$ and $S^{-1}$, which will be crucial for later analysis.

**Proposition 4.2.** Let $S$ and $S^{-1}$ be as given in (1.21)-(1.34). Then we have:

(a) $S^{\pm 1}$ is continuous on $R^3 \times [0, T]$ and Lipschitz continuous on $R^3 \times [\tau, T]$ for all $\tau > 0$, and there is a constant $C$ such that

$$\|\nabla S^{\pm 1}(\cdot, \tau)\|_{L^\infty} \leq C, \quad t \in [0, T];$$

(b) $(S_t + \nabla S u)(x, t) = \bar{u}(S(x, t), t)$ a.e. in $R^3 \times (0, T)$.
(c) $\bar{\rho}(S(x, t), t) \rho_0(X(x, 0, t)) \det \nabla S(x, t) = \rho(x, t)\bar{\rho}_0(X(x, 0, t))$ a.e. in $\mathbb{R}^3 \times (0, T)$;
(d) If $u, \bar{u} \in L^2(\mathbb{R}^3 \times (0, T))$, then for all $t \in (0, T),$

\begin{align*}
(4.4) & \int_{\mathbb{R}^3} |x - S(x, t)|^2 dx \leq C t \int_0^t \int_{\mathbb{R}^3} |u(x, \tau) - \bar{u}(S(x, \tau), \tau)|^2 dx d\tau, \\
(4.5) & \int_{\mathbb{R}^3} |x - S^{-1}(x, t)|^2 dx \leq C t \int_0^t \int_{\mathbb{R}^3} |u(S^{-1}(x, \tau), \tau) - \bar{u}(x, \tau)|^2 dx d\tau.
\end{align*}

Proof. The proof can be found in [Ho06] pp. 1752].

We are now ready to give the proof of Theorem 1.8. Throughout this section, $C$ always denotes a generic positive constant which depends on the parameters $P$, $f$, $\lambda$, $\mu$, $T$, $a$, $r$ as described in Theorem 1.8. First, in view of the weak form (1.20) of the momentum equation, we let $\psi : \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R}^3$ be a test function satisfying

\begin{equation}
- \int_{\mathbb{R}^3} \rho_0(x) u_0(x) \psi(x, 0) dx
\end{equation}

\begin{equation}
= \int_0^T \int_{\mathbb{R}^3} \left[ p u \cdot (\psi_t + \nabla \psi u) + (P(\rho) - P_\lambda) \text{div}(\psi) - \mu \nabla u^j \cdot \nabla \psi^j - \lambda \text{div}(u) \text{div}(\psi) - \lambda (\text{div}(u) \text{div}(\psi) + (\rho - \rho_\lambda) f \cdot \psi) \right] dx d\tau,
\end{equation}

where we used the expressions (1.12) and (1.13) for $\nabla P$. Define $\tilde{\psi} = \psi \circ S^{-1}$. Then we have

\begin{equation}
- \int_{\mathbb{R}^3} \bar{\rho}_0(x) \bar{u}_0(x) \tilde{\psi}(x, 0) dx
\end{equation}

\begin{equation}
= \int_0^T \int_{\mathbb{R}^3} \left[ \bar{p} \bar{u} \cdot (\tilde{\psi}_t + \nabla \tilde{\psi} \bar{u}) + (P(\bar{\rho}) - P_\bar{\lambda}) \text{div}(\tilde{\psi}) - \mu \nabla \bar{u}^j \cdot \nabla \tilde{\psi}^j - \lambda \text{div}(\bar{u}) \text{div}(\tilde{\psi}) - \lambda (\text{div}(\bar{u}) \text{div}(\tilde{\psi}) + (\bar{\rho} - \bar{\rho}_\lambda) \bar{f} \cdot \tilde{\psi}) \right] dx d\tau,
\end{equation}

with $\bar{P}_\lambda = P(\bar{\rho}_\lambda)$. For the term involving $\tilde{\psi}_t$, we can rewrite it as

\begin{equation}
\int_0^T \int_{\mathbb{R}^3} \bar{\rho} \bar{u} \cdot (\tilde{\psi}_t + \nabla \tilde{\psi} \bar{u}) dx d\tau
\end{equation}

\begin{equation}
= \int_0^T \int_{\mathbb{R}^3} \bar{\rho}(S) \bar{u}(S) \cdot (\tilde{\psi}_t(S) + \nabla \tilde{\psi} \bar{u}(S)) |\det(\nabla S)| dx d\tau
\end{equation}

\begin{equation}
= \int_0^T \int_{\mathbb{R}^3} A_0 \bar{u}(S) \psi_t + \nabla \psi u dx d\tau,
\end{equation}

where $A_0$ is given by

\begin{equation}
A_0(x, t) = \frac{\bar{\rho}_0(X(x, 0, t))}{\rho_0(X(x, 0, t))}
\end{equation}

and we used the fact that $A_0 \rho = (\bar{\rho} \circ S) |\det(\nabla S)|$ from Proposition 1.2. And by using the definition of $\bar{F}$ and $\omega$ from (1.5) (replacing $F$ by $\bar{F}$, $u$ by $\bar{u}$, etc.), we can
further rewrite the terms on the right side of (4.7) as follows.

\[
\int_0^T \int_{\mathbb{R}^3} \left[ (\bar{P}_s - \bar{P}) \text{div}(\bar{\psi}) + \mu \nabla \bar{u}^j \cdot \nabla \bar{\psi}^j + \lambda \text{div}(\bar{u}) \text{div}(\bar{\psi}) \right] dx d\tau \\
= \int_0^T \int_{\mathbb{R}^3} \left[ (\mu + \lambda) \text{div}(\bar{u}) - \bar{P} + \bar{P}_s \right] \text{div}(\bar{\psi}) dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \mu (\bar{u}^j_{x_k} - \bar{u}^j_{x_k}) \bar{\psi}^j_{x_k} dx d\tau \\
= - \int_0^T \int_{\mathbb{R}^3} \left[ \nabla(\bar{\rho}_s \bar{F}) \cdot \psi + \mu \bar{\omega}^{j,k}_{x_k} \psi^j \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \left[ (P_s - P) \text{div}(\psi) + \mu \nabla \bar{u}^j \cdot \nabla \psi^j + \lambda \text{div}(\bar{u}) \text{div}(\psi) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \left[ (1 - A_0) \rho(\bar{u} \circ S) (\psi_t + \nabla \psi u) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \left[ \nabla(\bar{\rho}_s \bar{F}) \cdot (\psi - \psi \circ S^{-1}) + \mu \bar{\omega}^{j,k}_{x_k} (\psi^j - \psi^j \circ S^{-1}) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} (\bar{\psi}_t) (\mu \Delta \psi + \lambda \nabla \text{div}(\psi)) dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi) + \int_0^T \int_{\mathbb{R}^3} \bar{P}_s - P_s \text{div}(\psi) dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \bar{\rho}_s (f \circ S) \cdot (\psi - \rho_s f \cdot \psi) dx d\tau.
\]

Hence by taking the difference between (4.6) and (4.7), for all \( \psi \) and \( \bar{\psi} \), we have the following expression on \( \int_{\mathbb{R}^3} (\bar{\rho} \bar{u}_0 - \rho_0 u_0) \cdot \psi(x,0) dx \):

\[
\int_{\mathbb{R}^3} (\bar{\rho} \bar{u}_0 - \rho_0 u_0) \cdot \psi(x,0) dx \\
= \int_0^T \int_{\mathbb{R}^3} \left[ \rho(u - \bar{u} \circ S)(\psi_t + \nabla \psi u) + (1 - A_0) \rho(\bar{u} \circ S)(\psi_t + \nabla \psi u) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \left[ (P_s - P) \text{div}(\psi) + \mu \nabla \bar{u}^j \cdot \nabla \psi^j + \lambda \text{div}(\bar{u}) \text{div}(\psi) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \left[ (1 - A_0) \rho(\bar{u} \circ S)(\psi_t + \nabla \psi u) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \left[ \nabla(\bar{\rho}_s \bar{F}) \cdot (\psi - \psi \circ S^{-1}) + \mu \bar{\omega}^{j,k}_{x_k} (\psi^j - \psi^j \circ S^{-1}) \right] dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} (\bar{\psi}_t) (\mu \Delta \psi + \lambda \nabla \text{div}(\psi)) dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi) + \int_0^T \int_{\mathbb{R}^3} \bar{P}_s - P_s \text{div}(\psi) dx d\tau \\
\quad + \int_0^T \int_{\mathbb{R}^3} \bar{\rho}_s (f \circ S) \cdot (\psi - \rho_s f \cdot \psi) dx d\tau.
\]

Next we extend \( \rho, u \) to be constant in time \( t \) outside \( [0,T] \) and let \( \rho^\varepsilon \) and \( u^\varepsilon \) be the corresponding smooth approximation obtained by mollifying in both \( x \) and \( t \). Then we define \( \psi^\varepsilon : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3 \) to be the solutions satisfying

\[
\begin{cases}
\rho^\varepsilon (\psi^\varepsilon_t + u^\varepsilon \cdot \nabla \psi^\varepsilon) + \mu \Delta \psi^\varepsilon + \lambda \nabla \text{div}(\psi^\varepsilon) = G \\
\psi^\varepsilon (\cdot, \tau) = 0.
\end{cases}
\]
By simple elliptic estimates (or refer to [Hof06 Lemma 3.1]), it can be shown that
\( \psi^\varepsilon \) satisfies the following bounds in terms of \( G \):

\[
\begin{align*}
(4.9) \quad & \sup_{0 \leq \tau \leq T} \int_{\mathbb{R}^3} \left[ |\psi^\varepsilon(x, \tau)|^2 + |\nabla \psi^\varepsilon(x, \tau)|^2 \right] dx \\
& + \int_0^T \int_{\mathbb{R}^3} \left[ |\psi^\varepsilon_t + \nabla \psi^\varepsilon u|^2 + |D_2^2 \psi^\varepsilon|^2 \right] dx d\tau \\
& \leq C \int_0^T \int_{\mathbb{R}^3} |G|^2 dx d\tau,
\end{align*}
\]

\[
(4.10) \quad \sup_{0 \leq \tau \leq T} \|\psi^\varepsilon(\cdot, \tau)\|_{L^\infty} + \int_0^T \int_{\mathbb{R}^3} |\psi^\varepsilon|^q dx d\tau \leq C(G),
\]

where \( q > 6 \) is given by (4.31) and \( C(G) \) is a positive constant which depends on \( G \). We now take \( \psi = \psi^\varepsilon \) in (4.8) to obtain

\[
(4.11) \quad \int_{\mathbb{R}^3} (\tilde{\rho}u - \rho_0 u_0) \cdot \psi^\varepsilon(x, 0) dx = \int_0^T \int_{\mathbb{R}^3} z \cdot G dx d\tau + \sum_{i=1}^7 R_i,
\]

where \( z = u - \bar{u} \circ S \) and \( R_1, \ldots, R_7 \) are given by:

\[
\begin{align*}
R_1 &= \int_0^T \int_{\mathbb{R}^3} \left[ \nabla (\bar{s} \tilde{F}) \cdot (\psi^\varepsilon - \psi^\varepsilon \circ S^{-1}) + \mu \bar{\omega}_{i,k} (\psi^\varepsilon - \psi^\varepsilon \circ S^{-1}) \right] dx d\tau, \\
R_2 &= \int_0^T \int_{\mathbb{R}^3} \left( \rho(f - \bar{f} \circ D) \cdot \psi^\varepsilon + (1 - A_0) \rho(\bar{f} \circ S) \cdot \psi^\varepsilon \right) dx d\tau, \\
R_3 &= \int_0^T \int_{\mathbb{R}^3} (\bar{u} \circ S - \bar{u}) \cdot (\mu \Delta \psi^\varepsilon + \lambda \text{div}(\psi^\varepsilon)) dx d\tau, \\
R_4 &= \int_0^T \int_{\mathbb{R}^3} z \cdot [(\rho - \rho^f) \psi^\varepsilon_t + \nabla \psi^\varepsilon (\rho u - \rho^f u)] dx d\tau, \\
R_5 &= \int_0^T \int_{\mathbb{R}^3} (1 - A_0) \rho(\bar{u} \circ S) \cdot (\psi^\varepsilon_t + \nabla \psi^\varepsilon u) dx d\tau, \\
R_6 &= \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi^\varepsilon) + \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi^\varepsilon) dx d\tau, \\
R_7 &= \int_0^T \int_{\mathbb{R}^3} (\bar{s} (\bar{f} \circ S) \cdot \psi^\varepsilon - \rho_s f \cdot \psi^\varepsilon) dx d\tau.
\end{align*}
\]

The left side of (4.11) can be readily bounded by

\[
(4.12) \quad \left| \int_{\mathbb{R}^3} (\tilde{\rho} \bar{u}_0 - \rho_0 u_0) \cdot \psi^\varepsilon(x, 0) dx \right| \leq \|\rho_0 u_0 - \bar{\rho}_0 \bar{u}_0\|_{L^2} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 dx d\tau \right)^{\frac{1}{2}}.
\]

We now aim at controlling the terms on the right side of (4.11). Following the method given in [Hof06] and with the help of the bounds (4.9)-(4.10), the terms \( R_2, R_3, R_4 \) and \( R_5 \) satisfy the following estimates:

\[
(4.13) \quad |R_2| \leq C \left[ \left( \int_{\mathbb{R}^3} |f - \bar{f} \circ S|^2 dx \right)^{\frac{1}{2}} + \|\bar{f}\|_{L^2}\|\rho_0 - \bar{\rho}_0\|_{L^2 \cap L^{2q'}} \right]^2
\times \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 dx d\tau \right)^{\frac{1}{2}},
\]
(4.14) \[ |R_3| \leq C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, dt \right)^{\frac{1}{2}}, \]

(4.15) \[ \lim_{\varepsilon \to 0} R_4 = \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^3} z \cdot \left[ (\rho - \rho^\varepsilon) \psi^\varepsilon_t + \nabla \psi^\varepsilon (\rho u - \rho^\varepsilon u^\varepsilon) \right] \, dx \, dt = 0, \]

and

(4.16) \[ |R_5| \leq C \left[ \|\rho_0 - \tilde{\rho}_0\|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt \right)^\frac{1}{2} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, dt \right)^\frac{1}{2} \right], \]

where we recall that \( \tilde{q} \) and \( \tilde{q}' \) are given in (1.28)-(1.29). It remains to estimate the terms \( R_1, R_6 \) and \( R_7 \). For \( R_1 \), using Hölder inequality and the bound (1.30) on \( \int_0^T \int_{\mathbb{R}^3} |D^2_\varepsilon \psi^\varepsilon|^2 \, dx \, dt \), we have

\[ |R_1| \leq C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \tau^{\frac{1}{2}} \|\nabla (\tilde{\rho}_s \tilde{F})(\cdot, \tau)\|_{L^4} \|\nabla \psi^\varepsilon(\cdot, \tau)\|_{L^4} \, d\tau \right) + \]

\[ C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |D^2_\varepsilon \psi^\varepsilon|^2 \, dx \, dt \right)^{\frac{1}{2}} \]

\[ \times \left( \int_0^T \tau^{\frac{1}{2}} (\|\nabla (\tilde{\rho}_s \tilde{F})(\cdot, \tau)\|_{L^4} + \|\nabla \tilde{\omega}(\cdot, \tau)\|_{L^4}) \, d\tau \right) \]

\[ \leq C \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, dt \right)^{\frac{1}{2}} \]

\[ \times \left( \int_0^T \tau^{\frac{1}{2}} (\|\nabla (\tilde{\rho}_s \tilde{F})(\cdot, \tau)\|_{L^4} + \|\nabla \tilde{\omega}(\cdot, \tau)\|_{L^4}) \, d\tau \right) \frac{1}{2}. \]

Using (2.8) and the boundedness assumption (1.26) on \( \tilde{\rho}_s \), the term involving \( \tilde{F} \) and \( \tilde{\omega} \) can be bounded by \( \left( \int_0^T \tau^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{u}(x, \tau)|^4 \, dx \right)^{\frac{1}{2}} \, d\tau \right)^{\frac{1}{2}}, \) and with the help of (2.3) and the energy estimates (1.32), we further have

\[ \int_0^T \tau^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{u}|^4 \, dx \right)^{\frac{3}{8}} \, d\tau \leq C \int_0^T \tau^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\tilde{u}|^2 \, dx \right)^{\frac{3}{8}} \left( \int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \, dx \right)^{\frac{5}{8}} \, d\tau \]

\[ \leq C \left( \int_0^T \tau^{4\alpha-3} \, d\tau \right)^{\frac{1}{4}} \left( \int_0^T \tau^{1-\alpha} \int_{\mathbb{R}^3} |\tilde{u}|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_0^T \tau^{2-\alpha} \int_{\mathbb{R}^3} |\nabla \tilde{u}|^2 \, dx \, dt \right)^{\frac{1}{8}} \]

\[ \leq CT^{\frac{4\alpha-2}{5}}, \]
and we recall that \( \alpha \in (\frac{1}{2}, 1] \). Hence we conclude

\[
|\mathcal{R}_1| \leq C T^{\frac{1}{2} - 1} \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 dx d\tau \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 dx d\tau \right)^{\frac{1}{2}}.
\]

In particular, for \([t_1, t_2] \subseteq [0, T]\), if we define

\[
\mathcal{R}_1(t_1, t_2) = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} [\nabla (\bar{\rho} \Phi) \cdot (\psi^\varepsilon - \psi^\varepsilon \circ S^{-1}) + \mu \overline{\omega}_{x_k}^j (\psi^\varepsilon - \psi^\varepsilon \circ S^{-1})] \ dx d\tau,
\]

then we also have

\[
|\mathcal{R}_1(t_1, t_2)| \leq C |t_2 - t_1|^{\frac{1}{2} - 1} \left( \int_0^{t_2} \int_{\mathbb{R}^3} |z|^2 dx d\tau \right)^{\frac{1}{2}} \left( \int_0^{t_2} \int_{\mathbb{R}^3} |G|^2 dx d\tau \right)^{\frac{1}{2}}.
\]

For \( \mathcal{R}_6 \), we define the function \( \chi(x, t) \) by

\[
\chi(x, t) = \frac{P(\rho(x, t)) - P(\bar{\rho}(x, t))}{\rho(x, t) - \bar{\rho}(x, t)},
\]

then by the mean value theorem, we can further rewrite \( \chi(x, t) \) as

\[
\chi(x, t) = \int_0^1 P'(\rho(x, t) + \theta(\bar{\rho}(x, t) - \rho(x, t))) d\theta.
\]

For \( \psi \in \mathcal{D}(\mathbb{R}^3) \) and \( t \geq 0 \),

\[
\left| \int_{\mathbb{R}^3} [P(\rho) - P(\bar{\rho})] \psi(x, \tau) dx \right| = \left| \int_{\mathbb{R}^3} \chi(\rho - \bar{\rho}) \psi(x, \tau) dx \right| \leq C \|\chi\|_{H^{-1}} (\|\psi\|_{H^1} + \|\nabla \psi\|_{L^2}).
\]

If \( P(\rho) = a \rho \), then it is clear that \( \chi \) is a constant and \( \nabla \chi = 0 \). For more general pressure \( P \) satisfying \((1.45)\), notice that for \( r \in [3, \infty) \), if we take \( p = 2 + \frac{4}{r - 2} \), then \( p \in [2, 6) \) with \( r = \frac{2p}{p - 2} \). Hence by the Sobolev imbedding \((2.3)\) and the boundedness condition \((1.45)\) on \( \nabla \chi \),

\[
\|\psi \nabla \chi\|_{L^2} \leq \|\psi\|_{L^p} \left( \int_{\mathbb{R}^3} |\nabla \chi|^\frac{2p}{p - 2} dx \right)^\frac{p - 2}{p} \leq \|\psi\|_{H^1} \|\nabla \chi\|_{L^r} \leq C \|\psi\|_{H^1}.
\]

Therefore in either cases, we have

\[
\|(P - \bar{P})(\cdot, \tau)\|_{H^{-1}} \leq C \|(\rho - \bar{\rho})(\cdot, \tau)\|_{H^{-1}}.
\]

Hence together with the bound \((1.10)\) on \( \psi^\varepsilon \), it implies that

\[
\left| \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \ \text{div}(\psi^\varepsilon) dx d\tau \right| \leq \int_0^T \|(P - \bar{P})(\cdot, \tau)\|_{H^{-1}} \|\text{div}(\psi^\varepsilon)\|_{H^1} d\tau \leq C \sup_{0 \leq \tau \leq T} \|(\rho - \bar{\rho})(\cdot, \tau)\|_{H^{-1}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 dx d\tau \right)^{\frac{1}{2}}.
\]
Following the argument given in [Hof06], the term \( \sup_{0 \leq \tau \leq T} \| (\rho - \bar{\rho})(\cdot, \tau) \|_{H^{-1}} \) can be bounded by

\[
\sup_{0 \leq \tau \leq T} \| (\rho - \bar{\rho})(\cdot, \tau) \|_{H^{-1}} \leq C \left[ \| \rho_0 - \bar{\rho}_0 \|_{L^2} + T^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \right],
\]

and we conclude from (4.20) that

\[
\left| \int_0^T \int_{\mathbb{R}^3} (P - \bar{P}) \text{div}(\psi^\varepsilon) \, dx \, d\tau \right| \leq C \left[ \| \rho_0 - \bar{\rho}_0 \|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \right] \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{\frac{1}{2}}.
\]

On the other hand, the term \( \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi^\varepsilon) \, dx \, d\tau \) can be readily bounded by

\[
\left| \int_0^T \int_{\mathbb{R}^3} (\bar{P}_s - P_s) \text{div}(\psi^\varepsilon) \, dx \, d\tau \right| \leq C \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \, dx \, d\tau \right)^{\frac{1}{2}},
\]

and therefore the bounds (4.21) - (4.22) together imply

\[
|\mathcal{R}_6| \leq C \left[ \| \rho_0 - \bar{\rho}_0 \|_{L^2} + \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \right] \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \, dx \, d\tau \right)^{\frac{1}{2}}.
\]

Finally for the term \( \mathcal{R}_7 \), we can rewrite it as follows.

\[
\mathcal{R}_7 = \int_0^T \int_{\mathbb{R}^3} (\bar{\rho}_s - \rho_s) f \cdot \psi^\varepsilon \, dx \, d\tau + \int_0^T \int_{\mathbb{R}^3} \bar{\rho}_s (\hat{f} \circ S - f) \cdot \psi^\varepsilon \, dx \, d\tau.
\]

Using (4.9), the term \( \int_0^T \int_{\mathbb{R}^3} (\bar{\rho}_s - \rho_s) f \cdot \psi^\varepsilon \, dx \, d\tau \) can be bounded by

\[
\left| \int_0^T \int_{\mathbb{R}^3} (\bar{\rho}_s - \rho_s) f \cdot \psi^\varepsilon \, dx \, d\tau \right| \leq C \| f \|_{L^\infty} \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |\psi^\varepsilon|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \leq C \| f \|_{L^\infty} \left( \int_{\mathbb{R}^3} |\rho_s - \bar{\rho}_s|^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{\frac{1}{2}},
\]
and similarly, \( \int_0^T \int_{\mathbb{R}^3} \tilde{\rho}_s(\bar{f} \circ S - f) \cdot \psi^\varepsilon \, dx \, d\tau \) can be bounded by
\[
\int_0^T \int_{\mathbb{R}^3} \tilde{\rho}_s(\bar{f} \circ S - f) \cdot \psi^\varepsilon \, dx \, d\tau \\
\leq C \| \tilde{\rho}_s \|_{L^\infty} \left( \int_0^T \int_{\mathbb{R}^3} |f - \bar{f}|^2 \, dx \, d\tau \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{1/2}.
\]

Recalling the assumptions (1.23) and (1.26), we therefore obtain
\[
|\mathcal{R}_7| \leq C \left( \int_{\mathbb{R}^3} |\rho_{s} - \bar{\rho}_{s}|^2 \, dx \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{1/2} \\
+ C \left( \int_0^T \int_{\mathbb{R}^3} |f - \bar{f} \circ S|^2 \, dx \, d\tau \right)^{1/2} \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{1/2}.
\]

Combining the estimates (4.12), (4.13), (4.14), (4.15), (4.16), (4.18), (4.23) and (4.24), we arrive at
\[
\left| \int_0^T \int_{\mathbb{R}^3} z \cdot G \, dx \, d\tau \right| \leq C \left[ M_0 \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{1/2} + |\mathcal{R}_1(0, T)| \right],
\]

where \( M_0 \) is given by
\[
M_0 = \| \rho_0 - \bar{\rho}_0 \|_{L^2 \cap L^{2\varepsilon}} + \| \rho_0 u_0 - \bar{\rho}_0 \bar{u}_0 \|_{L^2} + T^\delta \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{1/2} \\
+ \left( \int_{\mathbb{R}^3} |\rho_{s} - \bar{\rho}_{s}|^2 \, dx \right)^{1/2} + \left( \int_0^T \int_{\mathbb{R}^3} |f - \bar{f} \circ S|^2 \, dx \, d\tau \right)^{1/2}
\]

for some \( \delta > 0 \), and \( C > 0 \) is now fixed. Following the analysis given in [Hof06, pp. 1758–1759], there exists a small time \( \hat{\tau} > 0 \) such that
\[
\left( \int_0^{\hat{\tau}} \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{1/2} \leq 2CM_0,
\]
and consequently
\[
|\mathcal{R}_1(0, \hat{\tau})| \leq M_0 \left( \int_0^{\hat{\tau}} \int_{\mathbb{R}^3} |G|^2 \, dx \, d\tau \right)^{1/2}.
\]

By applying (4.25) with \( T \) replaced by \( 2\hat{\tau} \), we get
\[
\left( \int_0^{2\hat{\tau}} \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{1/2} \leq 4CM_0.
\]

Since \( \hat{\tau} > 0 \) is fixed, we can exhaust the interval \( [0, T] \) in finitely many steps to obtain that
\[
\left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{1/2} \leq CM_0,
\]
for some new constant \( C > 0 \). Hence the term \( T^\delta \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, d\tau \right)^{1/2} \) can be eliminated from the definition of \( M_0 \) by a Gronwall-type argument. Therefore we
conclude that
\begin{equation}
\left| \int_0^T \int_{\mathbb{R}^3} z \cdot G \, dx \, dt \right| \leq CM_0 \left( \int_0^T \int_{\mathbb{R}^3} |G|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\end{equation}
Since the bound \(4.26\) holds for any \(G \in H^\infty(\mathbb{R}^3 \times [0,T])\), it shows that the term \(\|z\|_{L^2([0,T] \times \mathbb{R}^3)}\) can be bounded by \(M_0\). Finally, using the bound \(1.21\) on the time integral on \(\|\nabla \bar{u}\|_{L^\infty}\), we have
\begin{equation}
\int_0^T \int_{\mathbb{R}^3} |\bar{u} - \bar{u} \circ S|^2 \, dx \, dt \leq \int_0^T \|\nabla \bar{u}(\cdot,\tau)\|^2_L \int_{\mathbb{R}^3} |x - S(x,\tau)|^2 \, dx \, d\tau \leq C \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt,
\end{equation}
and hence \(1.44\) follows. This completes the proof of Theorem 1.8.

Remark 4.3. If we further assume that \(\|\nabla \tilde{f}\|_{L^\infty} < \infty\), then it gives
\begin{equation}
C \|\nabla \tilde{f}\|_{L^\infty} \left( \int_0^T \int_{\mathbb{R}^3} |f - \tilde{f} \circ S|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq C \|\nabla \tilde{f}\|_{L^\infty} \left( \int_0^T \int_{\mathbb{R}^3} |x - S|^2 \, dx \, dt \right)^{\frac{1}{2}} + CT \left( \int_{\mathbb{R}^3} |f - \tilde{f}|^2 \, dx \right)^{\frac{1}{2}}.
\end{equation}
Using \(1.44\) and \(4.27\), we have
\begin{equation}
C \|\nabla \tilde{f}\|_{L^\infty} \left( \int_0^T \int_{\mathbb{R}^3} |x - S|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq C \|\nabla \tilde{f}\|_{L^\infty} \left( \int_0^T \int_{\mathbb{R}^3} |z|^2 \, dx \, dt \right)^{\frac{1}{2}},
\end{equation}
and hence we can replace the difference \(f - \tilde{f} \circ S\) by \(f - \tilde{f}\) in \(1.44\) provided that \(\|\nabla \tilde{f}\|_{L^\infty} < \infty\).

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DEPARTMENT OF MATHEMATICS AND INFORMATION TECHNOLOGY, THE EDUCATION UNIVERSITY OF HONG KONG
Email address: acksuen@eduhk.hk