A CHARACTERIZATION OF DUAL QUERMASSINTEGRALS AND THE ROOTS OF DUAL STEINER POLYNOMIALS

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Abstract. For any \( I \subset \mathbb{R} \) finite with \( 0 \in I \), we provide a characterization of those tuples \( (\omega_i)_{i \in I} \) of positive numbers which are dual quermassintegrals of two star bodies. It turns out that this problem is related to the moment problem. Based on this relation we also get new inequalities for the dual quermassintegrals. Moreover, the above characterization will be the key tool in order to investigate structural properties of the set of roots of dual Steiner polynomials of star bodies.

1. Introduction and notation

A subset \( S \) of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) is called starshaped (with respect to the origin \( 0 \)) if \( S \neq \emptyset \) and the segment \([0,x] \subseteq S\) for all \( x \in S \). For a compact starshaped set \( K \) its radial function \( \rho_K \) is defined by

\[
\rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}.
\]

Moreover, a star body is a compact starshaped set with positive and continuous radial function. We observe that this property implies that any star body has non-empty interior. We will denote by \( S_0^n \) the set of all star bodies in \( \mathbb{R}^n \). In particular, convex bodies (compact and convex sets) containing the origin in its interior are star bodies, and we write \( B_2^n \) to denote the \( n \)-dimensional unit ball. The volume of a set \( M \subset \mathbb{R}^n \), i.e., its \( n \)-dimensional Lebesgue measure, is denoted by \( |M| \), or \( |M|^n \) if the distinction of the dimension is needed. Furthermore, we write \( \text{bd} M \) and \( \text{int} M \) to represent the boundary and the interior of \( M \), and we use \( \text{conv} M \) and \( \text{pos} M \) for its convex and positive hulls, respectively.

For two convex bodies \( K, L \) and a non-negative real number \( \lambda \), the volume of the Minkowski sum (vectorial addition) \( K + \lambda L \) is expressed as a
polynomial of degree at most \( n \) in \( \lambda \) (see \[14\]), and it is written as

\[
|K + \lambda L| = \sum_{i=0}^{n} \binom{n}{i} W_i(K, L) \lambda^i.
\]

This expression is called *relative Steiner formula* of \( K \), and the coefficients \( W_i(K, L) \) are the *relative quermassintegrals* of \( K \), special cases of the more generally defined *mixed volumes* (see e.g. \[12\] s. 5.1).

**Dual Brunn-Minkowski theory** goes back to Lutwak \[8, 9\], and it is a cornerstone of modern convex geometry. For the immense impact of this theory we refer e.g. to \[1, 6, 12\] and the references inside. In this context, and among others, convex bodies are replaced by star bodies, the Minkowski sum by the radial addition and the support function by the radial function.

For \( x, y \in \mathbb{R}^n \), the radial addition \( \tilde{x} + \tilde{y} \) is defined as

\[
\tilde{x} + \tilde{y} = \begin{cases} 
  x + y & \text{if } x, y \text{ are linearly dependent,} \\
  0 & \text{otherwise.}
\end{cases}
\]

Then, the *radial sum* \( K \tilde{+} L \) for \( K, L \in \mathcal{S}_n^0 \) is defined by

\[ K \tilde{+} L = \{ x \tilde{+} y : x \in K, y \in L \}, \]

and has the property that, for \( \lambda, \mu \geq 0 \),

\[
\rho_{\mu K \tilde{+} \lambda L} = \mu \rho_K + \lambda \rho_L.
\]

As in the classical case, the volume of the radial sum \( K \tilde{+} \lambda L \) is also expressed as a polynomial of degree \( n \) in \( \lambda \) (see e.g. \[12\] p. 508),

\[
|K \tilde{+} \lambda L| = \sum_{i=0}^{n} \binom{n}{i} \tilde{W}_i(K, L) \lambda^i.
\]

This expression is known as *dual Steiner formula* of \( K \). The coefficients \( \tilde{W}_i(K, L) \) are the *dual quermassintegrals* of \( K \) and \( L \), and can be expressed in terms of their radial functions as

\[
\tilde{W}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n \rho_L(u)^i \, d\sigma(u).
\]

Here \( \sigma \) is the usual spherical Lebesgue measure. In particular, the use of spherical coordinates immediately yields \( \tilde{W}_0(K, L) = |K|, \tilde{W}_n(K, L) = |L| \) and \( 2 \tilde{W}_{n-1}(K, B^n_2)/|B^n_2| \) is the average length of chords of \( K \) through the origin. Moreover, and in contrast to the classical quermassintegrals, the dual ones can be defined via \( 1.3 \) for any real index \( i \in \mathbb{R} \). Dual quermassintegrals are particular cases of the dual mixed volumes, which were introduced for the first time by Lutwak in \[8\] (see also \[12\] s. 9.3).

In \[3\] it was studied the problem whether Steiner polynomials can be characterized, i.e.: given a polynomial \( f(z) = \sum_{i=0}^{n} a_i z^i, a_i \geq 0 \), is it a Steiner polynomial for a pair of convex bodies? This question is equivalent to know when a set of \( n+1 \) non-negative real numbers \( W_0, \ldots, W_n \geq 0 \) arises as the set of relative quermassintegrals of two convex bodies. Shephard proved...
in [13] that it suffices that the numbers satisfy the well-known Aleksandrov-Fenchel inequalities (see e.g. [12 (9.40)]) in order to be quermaßintegrals of convex bodies (see also [5]).

This question is a particular case of a beautiful problem which is still open: Given \( r \geq 2 \) convex bodies \( K_1, \ldots, K_r \subset \mathbb{R}^n \), there are \( N = \binom{n+r-1}{n} \) mixed volumes \( V(K_{i_1}, \ldots, K_{i_n}) \), \( 1 \leq i_1 \leq \cdots \leq i_n \leq r \); for the definition and a deep study on mixed volumes we refer to [12, s. 5.1]. Then, a set of inequalities is said to be a full set if given \( N \) (non-negative) numbers satisfying the inequalities, they arise as the mixed volumes of \( r \) convex bodies. For \( n = 2 \) and \( r = 3 \), Heine [4] proved that the Aleksandrov-Fenchel inequalities together with the determinantal inequality \( \det(V(K_i, K_j)_{i,j=1}^3) \geq 0 \) are a full set. For \( n \geq 2 \), Shephard [13] investigated whether the known inequalities (Aleksandrov-Fenchel and some determinantal inequalities) are a full set, and solved it for \( r = 2 \). Moreover, he showed that for \( r = n + 2 \) they do not form a full set. For arbitrary \( r \), the problem is still open.

The main aim of this paper is to consider the corresponding question in the dual setting, i.e., to look for necessary and sufficient conditions for \( n + 1 \) positive real numbers to be the dual quermaßintegrals of two star bodies in \( \mathbb{R}^n \). The first substantial difference with the classical case is that dual quermaßintegrals can be defined for any real index. Hence we provide, for any finite subset \( I \subset \mathbb{R} \) with \( 0 \in I \), a characterization of those tuples of positive numbers \((\omega_i)_{i \in I}\) which are the dual quermaßintegrals of two \( n \)-dimensional star bodies.

In order to state our result we need the following notation: for \( 0 < a < b \) and for an index set \( I = \{0, i_1, \ldots, i_m\} \subset \mathbb{R} \) of cardinality \( \#I = m + 1 \), we write

\[
C_{a,b}^I = \text{pos}\{(1, t^{i_1}, t^{i_2}, \ldots, t^{i_m}) : t \in [a, b]\} \subset \mathbb{R}^{m+1}.
\]

From now on and for the sake of brevity, any index set \( I = \{0, i_1, \ldots, i_m\} \subset \mathbb{R} \) will be assumed to have cardinality \( \#I = m + 1 \).

**Theorem 1.1.** Let \( I = \{0, i_1, \ldots, i_m\} \subset \mathbb{R} \), let \((\omega_i)_{i \in I}\) be a sequence of \( m + 1 \) positive numbers and let \( n \geq 2 \). Then there exist star bodies \( K, L \in S_0^n \) such that

\[
\tilde{W}_i(K, L) = \omega_i, \quad \text{for all } i \in I,
\]

if and only if either there exist \( 0 < a < b \) such that \((\omega_0, \omega_1, \ldots, \omega_m) \in \text{int} C_{a,b}^I\), or \( \omega_i = \lambda^i \omega_0 \) for some \( \lambda > 0 \) and every \( i \in I \); in this case \( L = \lambda K \).

Our above characterization of dual quermaßintegrals is related to the moment problem (see e.g. [7]). In Section 2 we study this relation, which will allow us to get, in the particular case when \( I = \{0, 1, \ldots, m\} \subset \mathbb{N} \), new inequalities between dual quermaßintegrals as a direct consequence of the following more general result.
Theorem 1.2. Let $K, L \in S_0^n$ be two star bodies. Then, for every $m \in \mathbb{N}$ the Hankel matrices
\[ A_m = \left( \tilde{W}_{i+j}(K, L) \right)_{i,j=0}^m, \quad B_m = \left( \tilde{W}_{i+j+1}(K, L) \right)_{i,j=0}^{m-1} \]
are positive definite.

From the above theorem new determinantal inequalities for dual quermassintegrals are obtained.

Corollary 1.1. Let $K, L \in S_0^n$ be two star bodies. We write $\tilde{W}_i = \tilde{W}_i(K, L)$ and let, for every $m \in \mathbb{N}$,
\[ \Delta_m = \begin{pmatrix} \tilde{W}_0 & \tilde{W}_1 & \cdots & \tilde{W}_m \\ \tilde{W}_1 & \tilde{W}_2 & \cdots & \tilde{W}_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{W}_m & \tilde{W}_{m+1} & \cdots & \tilde{W}_{2m} \end{pmatrix}, \quad \Delta'_m = \begin{pmatrix} \tilde{W}_1 & \tilde{W}_2 & \cdots & \tilde{W}_m \\ \tilde{W}_2 & \tilde{W}_3 & \cdots & \tilde{W}_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{W}_m & \tilde{W}_{m+1} & \cdots & \tilde{W}_{2m-1} \end{pmatrix}. \]

Then we have the determinantal inequalities
\[ \det \Delta_m > 0 \quad \text{and} \quad \det \Delta'_m > 0. \]

In the classical setting, the validity of a family of determinantal inequalities for mixed volumes remains an open problem (see [2]).

In several recent articles (see e.g. [5] and the references therein), the characterization of the quermassintegrals of convex bodies became a key tool in order to study properties of the roots of the relative Steiner polynomial (regarded as a polynomial in a complex variable, cf. (1.1)): structural properties of the set of roots, convexity, closeness, monotonicity, stability, etc.

In this paper we also carry out the corresponding study for the roots of dual Steiner polynomials. In the following we regard the right hand side in (1.2) as a formal polynomial in a complex variable $z \in \mathbb{C}$, which we denote by
\[ \tilde{f}_{K,L}(z) = \sum_{i=0}^n \binom{n}{i} \tilde{W}_i(K, L) z^i. \]

We observe that 0 cannot be a root of any dual Steiner polynomial because $\text{int} K \neq \emptyset$ for all $K \in S_0^n$. Moreover, since $\tilde{W}_i(K, L) = \tilde{W}_{n-i}(L, K)$ (see (1.3)), we have $\tilde{f}_{K,L}(z) = z^n \tilde{f}_{K,L}(1/z)$, and thus, up to multiplication by real constants, $\tilde{f}_{K,L}(z)$ and $\tilde{f}_{L,K}(z)$ have the same roots.

Here we are interested in the location and the structure of the roots of $\tilde{f}_{K,L}(z)$. To this end, let $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) \geq 0\}$, and we denote by $\mathbb{R}_{<0}$ and $\mathbb{R}_{\geq 0}$ the negative and non-negative real axes, respectively. For any dimension $n \geq 2$, let
\[ (1.5) \quad \tilde{R}(n) = \{ z \in \mathbb{C}^+ : \tilde{f}_{K,L}(z) = 0 \text{ for some } K, L \in S_0^n \} \]
be the set of all roots of all dual Steiner polynomials in the upper half-plane. We prove the following result.
Theorem 1.3. The set of roots \( \tilde{\mathcal{R}}(n) \) satisfies the following properties:

(a) It is a convex cone, containing the negative real axis.
(b) It is half-open, i.e., it does not contain the ray of the boundary not consisting of \( \mathbb{R}_{<0} \).
(c) It is monotounous in the dimension, i.e., \( \tilde{\mathcal{R}}(n) \subseteq \tilde{\mathcal{R}}(n+1) \).

We observe that the dual Steiner polynomial shares properties (a) and (c) with the relative Steiner polynomial (see [5, Theorem 1.1 and Theorem 1.3]). However, property (b) provides a first structural difference between both polynomials, since the cone of roots of the classical Steiner polynomial is shown to be closed (see [5, Theorem 1.2]).

The above theorem will be proved in Section 3, along with several additional properties of the roots. In Section 2 we give the proof of Theorems 1.1 and 1.2, which are based on a relation with the moment problem.

2. Dual quermassintegrals and the moment problem

For any \( i \in \mathbb{R} \), the \( i \)-th dual quermassintegral is a monotonous and homogeneous functional of degree \( n-i \) in its first argument and of degree \( i \) in the second one (cf. (1.3)), i.e.: given \( K, K', L \in S^n_0 \) with \( K \subseteq K' \) and \( \lambda > 0 \), then
\[
\tilde{W}_i(K, L) \leq \tilde{W}_i(K', L) \quad \text{and} \quad \tilde{W}_i(\lambda K, L) = \lambda^{n-i}\tilde{W}_i(K, L), \quad \tilde{W}_i(K, \lambda L) = \lambda^i\tilde{W}_i(K, L),
\]
for any \( i \in \mathbb{R} \). It is also well-known that the dual quermassintegrals of two star bodies \( K, L \) satisfy the inequalities
\[
(2.1) \quad \tilde{W}_j(K, L)^{k-i} \leq \tilde{W}_i(K, L)^{k-j}\tilde{W}_k(K, L)^{j-i}, \quad i < j < k,
\]
the counterpart to the classical Aleksandrov-Fenchel inequalities (see e.g. [12, (9.40)]), but now \( i, j, k \in \mathbb{R} \) is allowed. In (2.1) equality holds if and only if \( K \) and \( L \) are dilates.

We start this section collecting some further easy properties that will be needed later on.

Lemma 2.1. Let \( K, L \in S^n_0 \) and let \( i, j, k \in \mathbb{R} \), with \( i < j < k \).

i) If \( L \subseteq K \) then
\[
(2.2) \quad \tilde{W}_i(K, L) \geq \tilde{W}_j(K, L).
\]
ii) \( \tilde{W}_i(K, L) = \mu^l\tilde{W}_0(K, L) \) for \( l = i, j, k \) and some \( \mu > 0 \) if and only if \( L = \mu K \).

Proof. i) is a direct consequence of the monotonicity. In order to prove ii) we observe that if \( \tilde{W}_i(K, L) = \mu^l\tilde{W}_0(K, L) \) for \( l = i, j, k \), then we get
\[
\tilde{W}_i(K, L)^{k-j}\tilde{W}_k(K, L)^{j-i} = \mu^{(k-i)}\tilde{W}_0(K, L)^{k-i} = \tilde{W}_j(K, L)^{k-i}.
\]
Hence, we have equality in the dual Aleksandrov-Fenchel inequality (2.1), which yields \( L = \mu K \). The converse is obvious. \( \square \)
2.1. Characterizing dual quermaßintegrals: proof of Theorem 1.1

The main purpose of this section is to prove Theorem 1.1. To this end we first show a couple of lemmas, for which we need the following notation: given a measure $\mu$ on an interval $[a, b] \subset (0, \infty)$ we write, for any $i \in \mathbb{R}$,

$$m_i(\mu) = \int_a^b t^i \, d\mu(t).$$

We observe that when $i \in \mathbb{N} \cup \{0\}$, the above numbers are the moments of $\mu$ on the interval $[a, b]$ (see Subsection 2.2 for a brief introduction to the moment problem).

**Lemma 2.2.** Let $[a, b] \subset (0, \infty)$. For $I = \{0, i_1, \ldots, i_m\} \subset \mathbb{R}$, let $(\omega_i)_{i \in I}$ be a sequence of $m + 1$ positive numbers with $\omega_0 = |B_{a,b}^n|$. Let $\mu$ be a positive measure on $[a, b]$ such that $\mu([c, d]) > 0$ for every $[c, d] \subset [a, b]$ and

$$m_i(\mu) = \omega_i \quad \text{for every } i \in I.$$

Then there exists $L \in S_{n-1}$ satisfying $\omega_i = \tilde{\omega}_i(B_{a,b}^n, L)$ for all $i \in I$.

**Proof.** Let $F : [a, b] \rightarrow [0, 1]$ be the function defined by

$$F(t) = \frac{\mu([t, b])}{\mu([a, b])}.$$

Our assumption ensures that $F$ is a strictly decreasing function and continuous from the left, and satisfies $F(a) = 1$ and $F(b) = \mu(\{b\})/\mu([a, b])$. Let $G : [0, 1] \rightarrow [a, b]$ be the function

$$G(s) = \sup\{ t \in [a, b] : F(t) \geq s \},$$

which coincides with $F^{-1}$ when $F$ is bijective. Since $F$ is strictly decreasing, it is easy to see that $G$ is decreasing and continuous, and so, the function $\rho_L : S^{n-1} \rightarrow [a, b]$ given by

$$\rho_L(u) = G\left(\frac{\sigma\left(\{v \in S^{n-1} : |v_1| \geq |u_1|\}\right)}{\sigma(S^{n-1})}\right)$$

is continuous on $S^{n-1}$ and hence defines a star body $L$. Moreover, it is clear that $\rho_L(v) \geq \rho_L(u)$ if and only if $|v_1| \geq |u_1|$. Therefore, if $t \in [a, b]$ then

$$\frac{\sigma\left(\{v \in S^{n-1} : \rho_L(v) \geq t\}\right)}{\sigma(S^{n-1})} = F(t) = \frac{\mu([t, b])}{\mu([a, b])},$$

whereas for $t \notin [a, b]$ we trivially have

$$\frac{\sigma\left(\{v \in S^{n-1} : \rho_L(v) \geq t\}\right)}{\sigma(S^{n-1})} = \frac{\mu\left(\{s \in [a, b] : s \geq t\}\right)}{\mu([a, b])}.$$
Finally, since \( \mu((a, b]) = m_0(\mu) = \omega_0 = |B_2^n| \) and \( \sigma(S^{n-1}) = n|B_2^n| \), we get, for every \( i \in I \),
\[
\bar{w}_i(B_2^n, L) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u) \, d\sigma(u)
= \frac{1}{n} \int_0^\infty \alpha i^{-1} \sigma \left( \{ v \in S^{n-1} : \rho_L(v) \geq t \} \right) \, dt
= \frac{|B_2^n|}{\mu([a, b])} \int_0^\infty \alpha i^{-1} \mu \left( \{ s \in [a, b] : s \geq t \} \right) \, dt
= \int_a^b s^i \, d\mu(s) = m_i(\mu) = \omega_i.
\]

A refinement of Riesz’s Theorem will be also a key tool in our proof (see e.g. [7, Theorem 3.5 and P. 3.9 in p. 17]). It provides the connection between our characterization of dual quermaßintegrals and the moment problem, for which we refer to Subsection 2.2.

**Theorem 2.1** (Riesz). Let \( \alpha : [a, b] \to \mathbb{R}^n \), \( \alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t)) \), be a curve in \( \mathbb{R}^n \) and let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). There exists a probability measure \( \mu \) on \([a, b]\) such that
\[
x_i = \int_a^b \alpha_i(t) \, d\mu(t), \quad \text{for every } i = 1, \ldots, n,
\]
if and only if \( x \in \text{conv} \{ \alpha(t) : t \in [a, b] \} \).

Moreover, \( x \in \text{int conv} \{ \alpha(t) : t \in [a, b] \} \) if and only if there exists a continuous function \( \phi : [a, b] \to (0, \infty) \) such that \( d\mu(t) = \phi(t) \, dt \).

The following lemma shows that the above property is also equivalent to the fact that the measure \( \mu \) can be assumed to be supported on the whole interval \([a, b]\).

**Lemma 2.3.** Let \( \alpha : [a, b] \to \mathbb{R}^n \) be a continuous curve in \( \mathbb{R}^n \) not contained in a hyperplane and let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). There exists a probability measure \( \mu \) on \([a, b]\) such that \( \mu([c, d]) > 0 \) for every \([c, d] \subset [a, b]\) and
\[
x_i = \int_a^b \alpha_i(t) \, d\mu(t), \quad \text{for every } i = 1, \ldots, n,
\]
if and only if \( x \in \text{int conv} \{ \alpha(t) : t \in [a, b] \} \).

**Proof.** First we suppose that \( x \in \text{int conv} \{ \alpha(t) : t \in [a, b] \} \). Then, by Theorem 2.1 there exists a probability measure \( \mu \) with a positive density \( \phi \) with respect to the Lebesgue measure satisfying (2.3), and thus \( \mu([c, d]) > 0 \) for all \([c, d] \subset [a, b]\).

Conversely, if we suppose the existence of a measure \( \mu \) satisfying our hypotheses, Theorem 2.1 ensures that \( x \in \text{conv} \{ \alpha(t) : t \in [a, b] \} \). So, let us assume that \( x \in \text{bd conv} \{ \alpha(t) : t \in [a, b] \} \). Then there exists a supporting hyperplane to \( \text{conv} \{ \alpha(t) : t \in [a, b] \} \) at \( x \) with outer normal vector \( u \in S^{n-1} \).
such that \( \langle y, u \rangle \leq \langle x, u \rangle \) for every \( y \in \text{conv}\{ \alpha(t) : t \in [a, b] \} \). Furthermore, since \( \alpha([a, b]) \) is not contained in a hyperplane, there exists \([c, d] \subset [a, b]\) such that \( \langle \alpha(t), u \rangle < \langle x, u \rangle \) for every \( t \in [c, d] \), and thus
\[
\langle x, u \rangle = \int_a^b \langle \alpha(t), u \rangle \, d\mu(t) < \langle x, u \rangle,
\]
a contradiction. Therefore, \( x \in \text{int conv}\{ \alpha(t) : t \in [a, b] \} \).

Now we are ready to prove Theorem \[\text{[2.1]}\] for which we will apply the above results to the curve \( \alpha(t) = (t^{i_1}, t^{i_2}, \ldots, t^{i_m}) \). This is called the moment curve when \( i_k = k \) for \( k = 1, \ldots, m \).

**Proof of Theorem \[\text{[2.1]}\].** We start assuming the existence of star bodies \( K, L \in S^n_0 \) such that \( \omega_i = \widetilde{W}_i(K, L) \) for all \( i \in I \).

First we observe that
\[
(2.4) \quad \widetilde{W}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho_L(u)}{\rho_K(u)} \right)^i \rho_K^n(u) \, d\sigma(u).
\]
We denote by \( f : S^{n-1} \rightarrow (0, \infty) \) the continuous function given by
\[
f(u) = \frac{\rho_L(u)}{\rho_K(u)},
\]
and let \( a = \min_{u \in S^{n-1}} f(u) \) and \( b = \max_{u \in S^{n-1}} f(u) \).

Let \( \nu \) be the measure on the sphere given by \( d\nu = (1/n) \rho_K^n(u) \, d\sigma(u) \) and let \( \mu \) be the push-forward measure of \( \nu \) by \( f \). Then \( \mu \) is supported on \([a, b]\) and is defined by
\[
\mu(A) = \nu(f^{-1}(A)) = \frac{1}{n} \int_{f^{-1}(A)} \rho_K^n(u) \, d\sigma(u)
\]
for any Borel subset \( A \subseteq [a, b] \). Consequently, if \( a < b \) we have
\[
\int_a^b t^i \, d\mu(t) = \int_{S^{n-1}} f(u)^i \, d\nu(u) = \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho_L(u)}{\rho_K(u)} \right)^i \rho_K^n(u) \, d\sigma(u)
\]
\[
= \widetilde{W}_i(K, L) = \omega_i
\]
for every \( i \in I \), and \( d\mu/\widetilde{W}_0(K, L) \) is a probability measure with support \([a, b]\). Then, the second part in Theorem \[\text{[2.1]}\] ensures that
\[
\left( \frac{\omega_i, \omega_{i_2}, \ldots, \omega_{i_m}}{\omega_0, \omega_0, \ldots, \omega_0} \right) \in \text{int conv}\{ (t^{i_1}, t^{i_2}, \ldots, t^{i_m}) : t \in [a, b] \}
\]
and hence
\[
(\omega_0, \omega_i, \ldots, \omega_{i_m}) \in \text{int } C^I_{a, b}.
\]
Now, if \( a = b \) then \( f(u) = a \) for every \( u \in S^{n-1} \), which implies that \( L = aK \), and hence \( \omega_i = \widetilde{W}_i(K, L) = a^{i} |K| = a^{i} \omega_0 \) for all \( i \in I \).

For the converse, we first assume that there exist \( 0 < a < b \) such that
\[
(\omega_0, \omega_i, \ldots, \omega_{i_m}) \in \text{int } C^I_{a, b}.
\]
Then
\[
(\frac{\omega_{i_1}}{\omega_0}, \frac{\omega_{i_2}}{\omega_0}, \ldots, \frac{\omega_{i_m}}{\omega_0}) \in \text{int conv}\{(t^{i_1}, t^{i_2}, \ldots, t^{i_m}) : t \in [a, b]\},
\]
and Lemma 2.3 ensures the existence of a probability measure \(\mu\) on \([a, b]\) such that \(\mu([c, d]) > 0\) for every \([c, d] \subset [a, b]\) and
\[
\frac{\omega_i}{\omega_0} = \int_a^b t^i d\mu(t), \quad i \in I.
\]
If we write
\[
\omega_i' = \frac{|B_{2}^n|\omega_i}{\omega_0} = \int_a^b t^i |B_{2}^n| d\mu(t) = m_i(|B_{2}^n|\mu),
\]
we can apply Lemma 2.2 to the set \(\{\omega_0', \omega_{i_1}', \ldots, \omega_{i_m}'\}\) and obtain the existence of a star body \(L' \in S_0^n\) such that
\[
\frac{|B_{2}^n|\omega_i}{\omega_0} = \omega_i' = \widetilde{W}_i(B_2^n, L')
\]
for \(i \in I\). Then, just taking
\[
K = \left(\frac{\omega_0}{|B_{2}^n|}\right)^{1/n} B_2^n \quad \text{and} \quad L = \left(\frac{\omega_0}{|B_{2}^n|}\right)^{1/n} L',
\]
we get
\[
\omega_i = \widetilde{W}_i(K, L) \quad \text{for all} \quad i \in I.
\]
Finally, if there exists \(\lambda > 0\) such that \(\omega_i = \lambda \omega_0\) for all \(i \in I\), any star body \(K \in S_0^n\) with \(|K| = \omega_0\) and \(L = \lambda K\) yield \(\omega_i = \widetilde{W}_i(K, L)\). \(\square\)

From now on, when \(I = \{0, \ldots, m\} \subset \mathbb{N}\), we will write \(C_{a, b}^m\) for the cone \(C_{a, b}^I\) in order to stress its dimension. At this point we would like to notice the following fact, which has been used in the above proof: a point
\[
(x_0, x_1, \ldots, x_m) \in \text{int} C_{a, b}^m
\]
if and only if
\[
\left(\frac{x_1}{x_0}, \ldots, \frac{x_m}{x_0}\right) \in \text{int conv}\{(t^1, t^2, \ldots, t^m) : t \in [a, b]\}.
\]
The set \(\text{conv}\{(t^1, t^2, \ldots, t^m) : t \in [a, b]\}\) is called the cyclic body associated to \([a, b]\), and it is just the union of all cyclic polytopes in \([a, b]\). So, in order to determine if a point lies or not in its interior, it is convenient to know its facial structure and supporting hyperplanes. This problem has been studied and solved in [11].

In the particular case when \(I = \{0, 1, \ldots, m\}, m \in \mathbb{N}\), another characterization of those sequences of positive numbers which are dual quermassintegrals of two star bodies can be obtained from Theorem 1.1 and the following
Let 0 < a < b and m ∈ N. Then \((x_0, \ldots, x_m) \in \text{int} \, C_{a,b}^m\) if and only if, for every polynomial \(\sum_{i=0}^m c_i t^i\) which is positive on \([a, b]\),

\[
\sum_{i=0}^m c_i x_i > 0.
\]

**Theorem 2.2.** Let \(m, n \in \mathbb{N}\), \(n \geq 2\), and let \((\omega_i)_{i=0}^m\) be a sequence of \(m+1\) positive numbers. Then there exist star bodies \(K, L \in S_0^n\) such that

\(\tilde{W}_i(K, L) = \omega_i\) for all \(i = 0, \ldots, m\)

if and only if, either there exist \(0 < a < b\) such that the Hankel matrices \((a_{j,k})_{j,k=0}^r\) and \((b_{j,k})_{j,k=0}^r\) given by

\[
a_{j,k} = \begin{cases} 
\omega_{j+k} & \text{if } m = 2r, \\
\omega_{j+k+1} - a\omega_{j+k} & \text{if } m = 2r + 1,
\end{cases}
\]

\[
b_{j,k} = \begin{cases} 
(a + b)\omega_{j+k+1} - ab\omega_{j+k} - \omega_{j+k+2} & \text{if } m = 2r, \\
b\omega_{j+k} - \omega_{j+k+1} & \text{if } m = 2r + 1,
\end{cases}
\]

are positive definite, or \(\omega_i = \lambda^i \omega_0\) for some \(\lambda > 0\) and every \(i = 1, \ldots, m\); in this case \(L = \lambda K\).

**Proof.** By Theorem 1.1, \(\omega_i, i = 0, \ldots, m\), are the dual quermassintegrals of two star bodies if and only if, either there exist \(0 < a < b\) such that

\(\omega_0, \omega_1, \ldots, \omega_m \in \text{int} \, C_{a,b}^m\)

or \(\omega_i = \lambda^i \omega_0\) for some \(\lambda > 0\), \(i = 1, \ldots, m\). So, we have to prove that (2.6) is equivalent to the fact that the Hankel matrices \((a_{j,k})_{j,k=0}^r\), \((b_{j,k})_{j,k=0}^r\) are positive definite.

First we assume (2.6) and consider the even case \(m = 2r\). On one hand, for any \(c_1, \ldots, c_r \in \mathbb{R}\), the polynomial

\[
\left(\sum_{k=0}^r c_k t^k\right)^2 = \sum_{j,k=0}^r c_j c_k t^{j+k}
\]

is always positive and so (2.5) yields

\[
\sum_{j,k=0}^r c_j c_k \omega_{j+k} > 0.
\]

It shows that the Hankel matrix \((\omega_{j+k})_{j,k=0}^r\) is positive definite. On the other hand, for any \(d_1, \ldots, d_r \in \mathbb{R}\), the polynomial

\[
(b - t)(t - a) \left(\sum_{k=1}^r d_k t^k\right)^2 = \sum_{j,k=0}^r d_j d_k \left[(a + b)t^{j+k+1} - abt^{j+k} - t^{j+k+2}\right]
\]
is positive on \([a, b]\), and so (2.5) yields
\[
\sum_{j,k=0}^{r} d_j d_k [(a + b)\omega_{j+k+1} - ab\omega_{j+k} - \omega_{j+k+2}] > 0.
\]
Therefore the matrix \([(a + b)\omega_{j+k+1} - ab\omega_{j+k} - \omega_{j+k+2}]_{j,k=0}^{r}\) is positive definite. One can argue the odd case \(m = 2r + 1\) in a similar way.

Conversely, we now assume that the Hankel matrices \((a_{j,k})_{j,k=0}^{r}\), \((b_{j,k})_{j,k=0}^{r}\) are positive definite. Markov-Lukács’ theorem (see [7, Theorem 2.2 in c. 3]) provides a representation of the non-negative polynomials on an interval \([a, b]\): any such polynomial \(P(t)\) of degree \(m\) can be expressed as
\[
(2.7) \quad P(t) = \begin{cases} 
\left( \sum_{t=0}^{r} c_i t^i \right)^2 + (b - t)(t - a) \left( \sum_{t=0}^{r-1} d_i t^i \right)^2 & \text{if } m = 2r, \\
(t - a) \left( \sum_{t=0}^{r} c_i t^i \right)^2 + (b - t) \left( \sum_{t=0}^{r-1} d_i t^i \right)^2 & \text{if } m = 2r+1
\end{cases}
\]
for \(c_i, d_i \in \mathbb{R}\). Then, in the even case \(m = 2r\), for any positive polynomial \(P(t)\) written as in (2.7) we have that
\[
\sum_{j,k=0}^{r} c_j c_k \omega_{j+k} + \sum_{j,k=0}^{r} d_j d_k [(a + b)\omega_{j+k+1} - ab\omega_{j+k} - \omega_{j+k+2}] > 0
\]
because the matrices \((\omega_{j+k})_{j,k=0}^{r}\) and \([(a + b)\omega_{j+k+1} - ab\omega_{j+k} - \omega_{j+k+2}]_{j,k=0}^{r}\) are positive definite and so both summands are positive. When \(m = 2r + 1\) we argue in the same way. Thus, in both cases (2.5) shows (2.6), which concludes the proof. \(\square\)

2.2. New inequalities for dual quermassintegrals and the moment problem. Next we prove Theorem 1.2, which allows us to obtain new inequalities for the dual quermassintegrals.

Proof of Theorem 1.2. Let \(f: S^{n-1} \rightarrow (0, \infty)\) be the function defined by
\[
f(u) = \frac{\rho_L(u)}{\rho_K(u)}
\]
and, for each sequence \((a_i)_{i \in \mathbb{N}} \subset \mathbb{R}\) and any \(m \in \mathbb{N}\), let \(P_{1,m}\) and \(P_{2,m}\) be the polynomials
\[
P_{1,m}(x) = \left( \sum_{i=0}^{m} a_i x^i \right)^2 = \sum_{i,j=0}^{m} a_i a_j x^{i+j},
P_{2,m}(x) = x \left( \sum_{i=0}^{m-1} a_i x^i \right)^2 = \sum_{i,j=0}^{m} a_i a_j x^{i+j+1},
\]
which are positive for all \( x \in (0, \infty) \). Consequently,

\[
\frac{1}{n} \int_{S^{n-1}} P_{1,m}(f(u)) \rho_{K}^{n}(u) \, d\sigma(u) > 0 \quad \text{and}
\]

\[
\frac{1}{n} \int_{S^{n-1}} P_{2,m}(f(u)) \rho_{K}^{n}(u) \, d\sigma(u) > 0,
\]

and thus, for every sequence \((a_{i,j})_{i,j \in \mathbb{N}} \) and any \( m \in \mathbb{N} \), we have (cf. (2.4))

\[
\sum_{i,j=0}^{m} a_{i,j} \left( \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho_{\ell}(u)}{\rho_{K}(u)} \right)^{i+j+1} \rho_{K}^{n}(u) \, d\sigma(u) \right) = \sum_{i,j=0}^{m-1} a_{i,j} \tilde{W}_{i+j}(K, L) > 0,
\]

\[
\sum_{i,j=0}^{m-1} a_{i,j} \left( \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho_{\ell}(u)}{\rho_{K}(u)} \right)^{i+j+1} \rho_{K}^{n}(u) \, d\sigma(u) \right) = \sum_{i,j=0}^{m-1} a_{i,j} \tilde{W}_{i+j+1}(K, L) > 0.
\]

Therefore, \( A_{m} \) and \( B_{m} \) are positive definite matrices. \( \square \)

We observe that Theorem 1.2 implies, in particular, that for every \( i < j, i, j \in \mathbb{N} \cup \{0\} \), the matrices

\[
\begin{pmatrix}
\tilde{W}_{2i}(K, L) & \tilde{W}_{i+j}(K, L) \\
\tilde{W}_{i+j}(K, L) & \tilde{W}_{2j}(K, L)
\end{pmatrix},
\begin{pmatrix}
\tilde{W}_{2i+1}(K, L) & \tilde{W}_{i+j+1}(K, L) \\
\tilde{W}_{i+j+1}(K, L) & \tilde{W}_{2j+1}(K, L)
\end{pmatrix}
\]

are positive definite, and hence have a positive determinant. Thus we obtain particular cases of the dual Aleksandrov-Fenchel inequalities (2.1):

\[
\tilde{W}_{2i}(K, L)\tilde{W}_{2j}(K, L) > \tilde{W}_{i+j}^{2}(K, L),
\]

\[
\tilde{W}_{2i+1}(K, L)\tilde{W}_{2j+1}(K, L) > \tilde{W}_{i+j+1}^{2}(K, L).
\]

Taking different submatrices we obtain a family of inequalities.

**Remark 2.1.** We would like to notice that the determinantal inequalities in Corollary 1.1 cannot be obtained from (2.1). For instance, \( \Delta_{2} > 0 \) does not hold if we consider a sequence of numbers \( \omega_{0} > \omega_{1} > \omega_{2} = \omega_{3} = \omega_{4} \) (cf. (2.3)) satisfying also (2.1).

We conclude this section by obtaining new inequalities for the dual quermassintegrals of two star bodies as a consequence of the moment problem.

The moment problem seeks necessary and sufficient conditions for a sequence \((m_{i})_{i \in \mathbb{N} \cup \{0\}} \) to be the *moments* of some measure \( \mu \) on the real line. There are different versions of this problem, our interest being focused to the case of a fixed interval \([a, b]\) (the Hausdorff moment problem).

The solution to the Hausdorff moment problem is given by the following result (see e.g. [7, c. 3, Theorem 2.5], cf. Theorem 2.2).

**Theorem 2.3.** Let \([a, b] \subset \mathbb{R}\) and let \((m_{i})_{i \in \mathbb{N} \cup \{0\}} \) be a sequence of positive numbers. Then, there exists a positive measure \( \mu \) on \([a, b]\) with \( m_{i}(\mu) = m_{i} \).
Proposition 2.1. Let $K, L \in S_0^n$ be two star bodies and let
\[ a = \min_{u \in S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)} \quad \text{and} \quad b = \max_{u \in S^{n-1}} \frac{\rho_L(u)}{\rho_K(u)}. \]
Then, for all $m \in \mathbb{N}$, the Hankel matrices $(a_{j,k})_{j,k=0}^m$ and $(b_{j,k})_{j,k=0}^m$ given by
\[ a_{j,k} = \tilde{W}_{j+k+1}(K, L) - a \tilde{W}_{j+k}(K, L), \]
\[ b_{j,k} = b \tilde{W}_{j+k+1}(K, L) - \tilde{W}_{j+k}(K, L), \]
are positive semi-definite.

Proof. Following the notation in the proof of Theorem 1.1, we denote by $f : S^{n-1} \rightarrow (0, \infty)$ the continuous function $f(u) = \rho_L(u)/\rho_K(u)$, by $\nu$ the measure on the sphere $d\nu = (1/n)\rho_K^n(u) d\sigma(u)$ and by $\mu$ the push-forward measure of $\nu$ by $f$. Then, for every $i \in \mathbb{N} \cup \{0\}$ we have
\[ m_i(\mu) = \int_{S^{n-1}} f(u)^i d\nu(u) = \frac{1}{n} \int_{S^{n-1}} \left( \frac{\rho_L(u)}{\rho_K(u)} \right)^i \rho_K^n(u) d\sigma(u) = \tilde{W}_i(K, L); \]
by Theorem 2.3 ii), the given Hankel matrices are positive semi-definite. \( \square \)

An analogous result can be obtained taking the Hankel matrices given by
\[ i) \] in Theorem 2.3.

3. The set of roots of dual Steiner polynomials

We start collecting some properties on the behavior of the roots of dual Steiner polynomials when the involved bodies slightly change. They will be used for the proof of Theorem 1.3. We notice that these properties are analogous to the ones of the relative Steiner polynomial; we include the proof for completeness.

Lemma 3.1. Let $\gamma$ be a root of the dual Steiner polynomial $\tilde{f}_{K,L}(z)$.
\[ i) \] Let $\lambda > 0$. Then $\lambda \gamma$ is a root of $\tilde{f}_{\lambda K,L}(z)$.
\[ ii) \] Let $\mu \geq 0$. Then $\gamma - \mu$ is a root of $\tilde{f}_{K,\mu L,L}(z)$.
\[ iii) \] Let $\gamma = a + bi$ with $a < 0$, and let $0 < \rho \leq 1$. Then $a + (\rho b)i$ is a root of $\tilde{f}_{\rho K, \gamma((\rho - 1)aL,L)}(z)$. 

Proof. Since $\tilde{W}_i(\lambda K, L) = \lambda^{n-i} \tilde{W}_i(K, L)$ for any $i = 0, \ldots, n$, we have $\tilde{f}_{K,L}(z) = \lambda^n \tilde{f}_{K,L}(z/\lambda)$, which shows i).

Now, for any non-negative numbers $\lambda, \mu \geq 0$, the radial addition of star bodies satisfies $\mu L + \lambda L = (\mu + \lambda)L$ (see e.g. [9, (2.2)]), and hence

$$\left| (K + \mu L) + \lambda L \right| = |K + (\mu + \lambda)L| = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) \tilde{W}_i(K, L)(\mu + \lambda)^i.$$ 

Therefore $\tilde{f}_{K+\mu L,L}(z) = \tilde{f}_{K,L}(\mu + z)$, which implies ii).

Finally, iii) is just a combination of ii) and i), because the radial addition also satisfies $\rho K + (\rho - 1) a L = \rho(K + [(\rho - 1) a/\rho] L)$ (see again [9, (2.2)]). □

The following proposition states that both the derivative and the anti-derivative of a dual Steiner polynomial are also dual Steiner polynomials. This result will be also crucial in the proof of Theorem 1.3.

**Proposition 3.1.** Let $K, L \in S_0^n$ be two star bodies.

i) There exist $K', L' \in S_0^{n-1}$ such that

$$\frac{d\tilde{f}_{K,L}}{dz}(z) = \tilde{f}_{K',L'}(z).$$

ii) There exist $K'', L'' \in S_0^{n+1}$ such that

$$\frac{d\tilde{f}_{K'',L''}}{dz}(z) = \tilde{f}_{K,L}(z).$$

**Proof.** We first notice that since $\tilde{W}_{n-i}(K, L) = \tilde{W}_i(L, K)$ for all $i = 0, \ldots, n$, Theorem 1.1 ensures that,

(a) either there exist $0 < a < b$ such that

$$\tilde{W}_n(K, L), \tilde{W}_{n-1}(K, L), \ldots, \tilde{W}_0(K, L) \in \text{int} C_{a,b}^n,$$

(b) or $\tilde{W}_{n-i}(K, L) = \lambda \tilde{W}_n(K, L)$ for some $\lambda > 0$ and all $i = 1, \ldots, n$.

First we prove i). We observe that the derivative of $\tilde{f}_{K,L}(z)$ can be expressed as

$$\frac{d\tilde{f}_{K,L}}{dz}(z) = \sum_{i=1}^{n} \left(\begin{array}{c} n \\ i \end{array}\right) i \tilde{W}_i(K, L)z^{i-1} = \sum_{i=0}^{n-1} \left(\begin{array}{c} n-1 \\ i \end{array}\right) n \tilde{W}_{i+1}(K, L)z^i.$$ 

If (a) holds then, in particular,

$$\tilde{W}_n(K, L), \tilde{W}_{n-1}(K, L), \ldots, \tilde{W}_1(K, L) \in \text{int} C_{a,b}^{n-1},$$

and thus also $(n \tilde{W}_n(K, L), n \tilde{W}_{n-1}(K, L), \ldots, n \tilde{W}_1(K, L)) \in \text{int} C_{a,b}^{n-1}$. So, there exist star bodies $L', K' \in S_0^{n-1}$ such that

$$n \tilde{W}_{n-i}(K, L) = \tilde{W}_i^{(n-1)}(L', K') = \tilde{W}_{n-i}(K', L').$$
for all \( i = 0, \ldots, n - 1 \), where \( \tilde{W}_i^{(j)} \) denotes the \( i \)-th dual quermassintegral in \( \mathbb{R}^j \). Therefore \((d\tilde{f}_{K,L}/dz)(z)\) is the dual Steiner polynomial in \( \mathbb{R}^{n-1} \) of the sets \( K', L' \).

If (b) holds, then \( K = \lambda L \) (see Lemma 2.1 ii), and hence
\[
\frac{d\tilde{f}_{K,L}}{dz}(z) = n|L|(\lambda + z)^{n-1}.
\]
Then, taking \( L' \in S_0^{n-1} \) such that \(|L'|_{n-1} = n|L|\) and \( K' = \lambda L' \), we obtain
\[
\tilde{f}_{K',L'}(z) = \sum_{i=0}^{n-1} \binom{n-1}{i} \tilde{W}_i^{(n-1)}(K', L')z^i = |L'|_{n-1}(\lambda + z)^{n-1} = \frac{d\tilde{f}_{K,L}}{dz}(z).
\]
It concludes the proof of i).

Now we prove ii). First we notice that the antiderivative of \( \tilde{f}_{K,L}(z) \) can be expressed, for any constant \( C \geq 0 \), as
\[
\int \tilde{f}_{K,L}(z) \, dz = C + \sum_{i=0}^{n} \binom{n}{i} \frac{1}{i+1} \tilde{W}_i(K, L)z^{i+1}
\]
\[
= C + \sum_{i=1}^{n+1} \binom{n+1}{i} \frac{1}{i+1} \tilde{W}_{i-1}(K, L)z^i.
\]
Again we start assuming that (a) holds. Let \( H = \text{span}\{e_1, \ldots, e_{n+1}\} \subset \mathbb{R}^{n+2} \) be the linear subspace spanned by the first \( n+1 \) canonical vectors \( e_i \). Since \( C_{a,b}^n = C_{a,b}^{n+1} |H \) is the orthogonal projection of the cone \( C_{a,b}^{n+1} \) onto \( H \), there exists a point \((\omega_0, \ldots, \omega_{n+1}) \in \text{int } C_{a,b}^{n+1}\) such that
\[
(\omega_0, \ldots, \omega_{n+1})|H = \frac{1}{n+1}(\tilde{W}_n(K, L), \tilde{W}_{n-1}(K, L), \ldots, \tilde{W}_0(K, L)).
\]
By Theorem 1.1 there exist star bodies \( L'', K'' \in S_0^{n+1} \) such that, for all \( i = 0, \ldots, n+1 \), \( \omega_i = \tilde{W}_i^{(n+1)}(L'', K'') \) and, consequently,
\[
\frac{1}{n+1} \tilde{W}_{n-i}(K, L) = \omega_i = \tilde{W}_i^{(n+1)}(L'', K'') = \tilde{W}_{n-i}^{(n+1)}(K'', L''),
\]
for all \( i = 0, \ldots, n \). Then, setting \( C = \omega_{n+1} \) we have
\[
\int \tilde{f}_{K,L}(z) \, dz = \sum_{i=0}^{n+1} \binom{n+1}{i} \tilde{W}_i^{(n+1)}(K'', L'')z^i = \tilde{f}_{K'',L''}(z).
\]
Thus, \((d\tilde{f}_{K'',L''}/dz)(z) = \tilde{f}_{K,L}(z)\), as required.

Finally, if (b) holds, then \( K = \lambda L \) (see Lemma 2.1 ii)). So it suffices to take \( L'' \in S_0^{n+1} \) such that \(|L''|_{n+1} = |L|/(n+1)\) and \( K'' = \lambda L'' \), which yields
\[
\frac{d\tilde{f}_{K'',L''}}{dz}(z) = \frac{d}{dz} \left[ \sum_{i=0}^{n+1} \binom{n+1}{i} \tilde{W}_i^{(n+1)}(K'', L'')z^i \right] = \frac{d}{dz} \left[ |L''|_{n+1}(\lambda + z)^{n+1} \right]
\]
\[
= (n+1)|L''|_{n+1}(\lambda + z)^n = |L|(\lambda + z)^n = \tilde{f}_{K,L}(z).
\]
This concludes the proof of the proposition. □

We are now ready for the proof of Theorem 1.3. We just introduce an additional notation: for complex numbers $z_1, \ldots, z_r \in \mathbb{C}$ let

$$\sigma_i(z_1, \ldots, z_r) = \sum_{J \subseteq \{1, \ldots, r\} \setminus \{j\}} \prod_{j \in J} z_j$$

denote the $i$-th elementary symmetric function of $z_1, \ldots, z_r$, $1 \leq i \leq r$, setting $\sigma_0(z_1, \ldots, z_r) = 1$.

**Proof of Theorem 1.3.** First we prove item (a).

Clearly $\tilde{f}_{L,L}(z) = |L|(z + 1)^n$ because $\tilde{W}_i(L, L) = |L|$ for all $i = 0, \ldots, n$, and hence $-1$ is a root of the polynomial $\tilde{f}_{L,L}(z)$. Thus, by Lemma 3.1 i), every $-c \in \mathbb{R}_{<0}$, $c > 0$, will be a root of $\tilde{f}_{cL,L}(z)$ for any $L \in S_0^n$, and so the negative real axis $\mathbb{R}_{<0}$ is contained in $\tilde{R}(n)$.

Lemma 3.1 i) also shows that $\tilde{R}(n)$ is a cone without apex, i.e., if $\gamma \in \tilde{R}(n)$ and $\lambda > 0$, then $\lambda \gamma \in \tilde{R}(n)$. It remains to show that $\tilde{R}(n)$ is convex. For the proof, let $\gamma_i = a_i + b_i i \in \tilde{R}(n)$, $i = 1, 2$, and let $\rho \in (0, 1)$.

If both roots $\gamma_1, \gamma_2 \in \mathbb{R}_{<0}$, i.e., $b_i = 0$ for $i = 1, 2$, then Lemma 3.1 i) ensures that $\rho \gamma_1 + (1 - \rho) \gamma_2$ is a root of $\tilde{f}_{cL,L}(z)$ for $c = \rho |a_1| + (1 - \rho) |a_2|$ and any $L \in S_0^n$. So we assume that at least one of them has strictly positive imaginary part.

First we show that there exist $K_i, L \in S_0^n$, $i = 1, 2$, such that $\gamma_i$ is a root of $\tilde{f}_{K_i,L}(z)$, $i = 1, 2$. We may assume, without loss of generality, that $b_1 \leq b_2$, which yields $b_2 > 0$. We suppose moreover that $a_1 \leq a_2 \leq 0$. Let $K_2, L \in S_0^n$ be such that $\tilde{f}_{K_2,L}(\gamma_2) = 0$. By Lemma 3.1 i), setting $\lambda = b_1/b_2$, we have that $\lambda \gamma_2 = \lambda a_2 + b_1 i$ is a root of $\tilde{f}_{K_2,L}(z)$ (see Figure 1).

![Figure 1](image)

**Figure 1.** Constructing $K_1 \in S_0^n$ such that $\tilde{f}_{K_1,L}(\gamma_1) = 0$.

Furthermore, since $\lambda \leq 1$, we clearly have that $a_1 \leq \lambda a_2 \leq 0$, and taking $\mu = \lambda a_2 - a_1 \geq 0$, Lemma 3.1 ii) ensures that $\gamma_1 = \lambda \gamma_2 - \mu$ is a root of $\tilde{f}_{\lambda K_2 + \mu L,L}(z)$. Thus, for $K_1 = \lambda K_2 + \mu L$ we get the desired property. The
other possibilities for \( a_1, a_2 \) can be argued in the same way by just choosing properly the path between the two roots.

Finally, in order to prove the convexity of the cone, we just have to construct a star body \( M \in S_0^n \) such that \( \rho \gamma_1 + (1 - \rho) \gamma_2 \) is a root of \( \tilde{f}_{M,L}(z) \). Since at least one of the roots has strictly positive imaginary part, we can take \( a_1/b_1 = \max\{a_i/b_i : b_i > 0, \; i = 1, 2 \} \) and let \( \mu = b_1/(\rho b_1 + (1 - \rho) b_2) \). Then

\[
\nu = \mu(\rho a_1 + (1 - \rho) a_2) = b_1 \frac{\rho a_1 + (1 - \rho) a_2}{\rho b_1 + (1 - \rho) b_2} \leq a_1,
\]

because the above function is increasing in \( \rho \in (0, 1) \) by the choice of \( a_1/b_1 \), and hence, Lemma \( 3.1 \) ii) ensures that \( \nu + b_1 i \) is a root of \( \tilde{f}_{K_{1,+(a_1-\nu)L,L}} \).

Finally, Lemma \( 3.1 \) i) shows that \( \rho \gamma_1 + (1 - \rho) \gamma_2 \) is a root of the dual Steiner polynomial \( \tilde{f}_{M,L}(z) \) for \( M = (1/\mu)(K_{1,+(a_1-\nu)L}) \).

Next we prove item (b).

In order to show that \( \bar{\mathcal{R}}(n) \) is half-opened, we are going to prove that \( C = \mathbb{C}^+ \setminus \bar{\mathcal{R}}(n) \) is closed. Let \( (z_i)_{i \in \mathbb{N}} \subset C \) be a convergent sequence and let \( z = \lim_{i \to \infty} z_i \). Clearly, \( z \notin \mathbb{R}_{<0} \), and since \( z_i \notin \bar{\mathcal{R}}(n) \), then \( z_i \) cannot be a root of any dual Steiner polynomial.

For the sake of brevity we denote by \( \Gamma \) any \((n-2)\)-tuple of complex numbers of the form

\[
\Gamma = \begin{cases} \left( \gamma_2, \gamma_2, \ldots, \gamma_n/2, \gamma_n/2 \right) \in \mathbb{C}^{n-2} & \text{if } n \text{ is even,} \\ \left( \gamma_2, \gamma_2, \ldots, \gamma(n-1)/2, \gamma(n-1)/2, c \right) \in \mathbb{C}^{n-3} \times \mathbb{R}_{<0} & \text{if } n \text{ is odd.} \end{cases}
\]

We also denote by \( \sigma : \mathbb{C}^n \to \mathbb{C}^n \times \{1\} \) the continuous map given by

\[
\sigma = \left( (-1)^n \sigma_n, (-1)^{n-1} \frac{\sigma_{n-1}}{n}, \ldots, (-1)^i \frac{\sigma_i}{i}, \ldots, -\frac{\sigma_1}{n}, 1 \right).
\]

Given an \((n-2)\)-tuple \( \Gamma \), let \( \sigma(z_i, \overline{z}_i, \Gamma) = (\omega^i_0, \ldots, \omega^i_{n-1}, 1) \) for all \( i \in \mathbb{N} \). On one hand, since \( z_i \) is not a root of any dual Steiner polynomial, \( \omega^i_j \), \( j = 0, \ldots, n \), (we set \( \omega^n_n = 1 \)) cannot be dual quermaßintegrals of any pair of star bodies, and hence, by Theorem \( 3.1 \) \((\omega^i_0, \ldots, \omega^i_{n-1}, 1) \notin \text{int} \; C_{a,b}^n \) for any \( 0 < a < b \). Moreover, there exists no \( \lambda > 0 \) such that \( \omega^i_j = \lambda^j \omega^i_0 \).

On the other hand, since \( \sigma \) is continuous, then \((\omega^i_0, \ldots, \omega^i_{n-1}, 1)_{i \in \mathbb{N}} \) is a convergent sequence and

\[
\lim_{i \to \infty} (\omega^i_0, \ldots, \omega^i_{n-1}, 1) =: (\omega_0, \ldots, \omega_{n-1}, 1) = \sigma(z, \overline{z}, \Gamma) \notin \text{int} \; C_{a,b}^n
\]

for any \( 0 < a < b \). Moreover, if there exists \( \lambda > 0 \) such that \( \omega_j = \lambda^j \omega_0 \) for all \( j = 0, \ldots, n \), then \( \omega_j = \widetilde{W}_j(\lambda K, K) \) for some \( K \in S_0^n \), and thus \( z \) would be a root of the dual Steiner polynomial \( \tilde{f}_{\lambda K,K}(z) \), i.e., \( z \in \mathbb{R}_{<0} \), which is not possible. Since this holds for any \((n-2)\)-tuple \( \Gamma \), we can conclude that \( z \notin \bar{\mathcal{R}}(n) \). It shows that \( C \) is closed and concludes the proof of (b).

Finally we show (c).
If \( \gamma \in \tilde{R}(n) \), there exists a dual Steiner polynomial \( \tilde{f}_{K,L}(z) \) for \( K, L \in S_0^n \), such that \( \tilde{f}_{K,L}(\gamma) = 0 \). Then, by Proposition 3.1 ii), we know there are star bodies \( K', L' \in S_{n+1}^0 \) satisfying
\[
\frac{d\tilde{f}_{K',L'}}{dz}(z) = \tilde{f}_{K,L}(z).
\]
Let \( \gamma_1, \ldots, \gamma_{n+1} \) be the roots of \( \tilde{f}_{K',L'}(z) \). Lucas’ theorem (see e.g. [10, Theorem (6.1)]) states that the roots of the derivative of a polynomial lie in the convex hull of the roots of the polynomial, and thus we get that
\[
\gamma \in \text{conv}\{\gamma_1, \ldots, \gamma_{n+1}\} \subset \tilde{R}(n+1),
\]
because \( \tilde{R}(n+1) \) is convex.

The different behavior of the roots of dual Steiner polynomials with respect to the Steiner polynomial shows up also in the stability. We recall that real polynomials whose zeros all have negative real part are called stable or Hurwitz. In the next proposition we show that, contrary to the classical case (cf. [5, Proposition 1.3]), \( \tilde{R}(n) \subset \{z \in \mathbb{C}^+ : \text{Re}(z) < 0\} \) if and only if \( n = 2 \).

**Proposition 3.2.** \( \tilde{R}(2) = \{z \in \mathbb{C}^+ : \text{Re}(z) < 0\} \). Moreover, a negative real number is root of \( \tilde{f}_{K,L}(z) \) for \( K, L \in S_0^2 \) if and only if \( K, L \) are dilates. For \( n \geq 3 \) there exist non-stable dual Steiner polynomials.

**Proof.** The roots of \( \tilde{f}_{K,L}(z) = \tilde{W}_0(K,L) + 2\tilde{W}_1(K,L)z + \tilde{W}_2(K,L)z^2 \), for \( K, L \in S_0^2 \), are
\[
\gamma_1, \gamma_2 = \frac{-\tilde{W}_1(K,L) \pm \sqrt{\tilde{W}_1(K,L)^2 - \tilde{W}_0(K,L)\tilde{W}_2(K,L)}}{\tilde{W}_2(K,L)},
\]
and the dual Aleksandrov-Fenchel inequality (2.1) for \( i = 0, j = 1, k = 2 \) shows that they cannot be real numbers unless \( K \) and \( L \) are dilates.

Next we observe that, given a triple \( (\omega_0, \omega_1, \omega_2) \) of positive numbers, the dual Aleksandrov-Fenchel inequality \( \omega_2^1 < \omega_0^2 \omega_2^2 \) is equivalent to the fact that the pair
\[
\left( \frac{\omega_1}{\omega_0}, \frac{\omega_2}{\omega_0} \right) \in \text{int conv}\{(t,t^2) : t > 0\},
\]
which implies that \( \omega_i, i = 0, 1, 2, \) are the dual quermaßintegrals of two planar star bodies \( K, L \in S_0^2 \).

Now, let \( a+bi \in \{z \in \mathbb{C}^+ : \text{Re}(z) < 0\}, b > 0 \). Then the positive numbers
\[
\omega_0 = a^2 + b^2, \quad \omega_1 = -a \quad \text{and} \quad \omega_2 = 1
\]
determine a polynomial \( \omega_0 + 2\omega_1 z + \omega_2 z^2 \) having \( a+bi \) as a root, and clearly satisfy the inequality \( \omega_1^2 < \omega_0^2 \omega_2 \), which implies that they are dual quermaßintegrals of two planar star bodies. Therefore \( a+bi \in \tilde{R}(2) \).
Finally, for \( n = 3 \), the Liénard-Chipart criterion for stability of polynomials (see e.g. [10] Theorem (40,3)) allows to check that there are dual Steiner polynomials having roots with positive real part.

Proposition 3.2 implies, in particular, that the inclusion \( \tilde{\mathcal{R}}(2) \subset \tilde{\mathcal{R}}(3) \) between the lowest dimensional cones is strict.

**Remark 3.1.** For \( n = 3 \), if \( a + bi \in \mathbb{C}^+ \), \( a, b > 0 \), is a root of a dual Steiner polynomial \( \tilde{f}_{K,L}(z) \) for some \( K, L \in \mathcal{S}_3 \), and \( -c, c \geq 0 \), is the real root, we immediately have the identities

\[
c - 2a = 3 \frac{\tilde{W}_2(K,L)}{\tilde{W}_3(K,L)}, \quad a^2 + b^2 - 2ac = 3 \frac{\tilde{W}_1(K,L)}{\tilde{W}_3(K,L)}, \quad c(a^2 + b^2) = \frac{\tilde{W}_0(K,L)}{\tilde{W}_3(K,L)}.
\]

Then, inequalities (2.1) allow to see that \( b > \sqrt{3}a \). Therefore, the cone \( \tilde{\mathcal{R}}(3) \subseteq \{ a + bi \in \mathbb{C}^+ : b > \sqrt{3}a \} \).

We have seen that two different negative real numbers cannot be the roots of a 2-dimensional dual Steiner polynomial. The same occurs in arbitrary dimension, which states another difference with the classical Steiner polynomial, where this situation is possible (see [5, Proposition 2.3]).

**Proposition 3.3.** For any \( K, L \in \mathcal{S}_n \), all roots of \( \tilde{f}_{K,L}(z) \) are real if and only if they are all equal.

**Proof.** We suppose there exist \( K, L \in \mathcal{S}_n \) and \( \gamma_1, \ldots, \gamma_n \in \mathbb{R}_{<0} \) such that \( \tilde{f}_{K,L}(\gamma_i) = 0 \) for all \( i = 1, \ldots, n \). Then, Newton inequalities (see e.g. [3]) ensure that the elementary symmetric functions of the \( \gamma_i \)'s satisfy

\[
\left( \frac{\sigma_j(\gamma_1, \ldots, \gamma_n)}{\binom{n}{j}} \right)^2 \geq \frac{\sigma_{j-1}(\gamma_1, \ldots, \gamma_n)}{\binom{n}{j-1}} \frac{\sigma_{j+1}(\gamma_1, \ldots, \gamma_n)}{\binom{n}{j+1}}.
\]

Since

\[
\sigma_j(\gamma_1, \ldots, \gamma_n) = (-1)^j \binom{n}{j} \frac{\tilde{W}_{n-j}(K,L)}{\tilde{W}_n(K,L)},
\]

the above inequality translates into

\[
\tilde{W}_{n-j}(K,L)^2 \geq \tilde{W}_{n-j+1}(K,L)\tilde{W}_{n-j-1}(K,L)
\]

which, together with the dual Aleksandrov-Fenchel inequalities (2.1) for \( i = j - 1 \) and \( k = j + 1 \), yields \( \tilde{W}_{n-j}(K,L)^2 = \tilde{W}_{n-j+1}(K,L)\tilde{W}_{n-j-1}(K,L) \).

Then \( K = \lambda L \) for some \( \lambda > 0 \), and hence \( \gamma_i = -\lambda \) for all \( i = 1, \ldots, n \), a contradiction.

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