Vanishing threshold uncertainties and prediction for proton decay in non-supersymmetric $E_6$ GUT with intermediate trinification symmetry

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We consider a class of non-supersymmetric $E_6$ Grand Unified Theories (GUTs) with trinification symmetry $SU(3)_C \times SU(3)_L \times SU(3)_R \times D$ ($D$ denoted as D-parity for discrete left-right symmetry) at the highest intermediate scale and establish a useful theorem that the electroweak mixing angle $\sin^2 \theta_W$ has vanishing contributions due to one-loop GUT threshold corrections arising from every class of superheavy particles. This theorem can be extended to show that the one loop GUT threshold correction vanishes at the intermediate mass scale $M_I$ due to the presence of the discrete symmetry between the left and right handed Higgs fields. We quantify the effect of one loop GUT threshold corrections at the intermediate mass scale $M_I$ and the unification mass scale $M_U$ for the model. We observe that the modified $M_U$, provides crucial predictions on proton lifetime which can be verifiable within the foreseeable experiments.

I. INTRODUCTION

The discovery of Higgs boson at the Large Hadron Collider (LHC) has revealed an intriguing consistency of experimental results with almost all the predictions of Standard Model (SM), which is one of the greatest treasures that the particle physics community has been rich with. It holds the potential to explain almost all that is known to the subatomic world: from identifying the elementary particles to explaining all the ways they can interact. Thus it enables us to understand the complex structures of the nature including us, from predicting new elementary particles to accurately calculating their masses. Besides its phenomenological success as a fundamental theory of particle physics, the prediction of new interactions and particles such as Dark Matter and sub-eV scale neutrinos along with its inability to explain some theoretical questions like parity violations of weak interactions, baryon asymmetry of the universe etc compels the particle physicists to accept the SM as an effective low-energy approximation of a larger Grand-Unified-Theory (GUT) or part of any other theory operative at high scale.

Grand Unified Theories are believed to be the most elegant theoretical frameworks aiming to unify the fundamental forces like strong, weak and the electromagnetic interactions and are associated with a single gauge coupling $g_G$ corresponding to the gauge group $G$ at some high energy scale $M_G$. However it is observed that all GUTs, without supersymmetry (SUSY) and without an intermediate symmetry, fail to unify the three gauge couplings of the SM. With one or more intermediate symmetries, although the gauge unification is possible, it may not always comply with the present proton decay constraint which is believed to be a key prediction of most GUTs. Thus, in order to ensure favourable unification one has to rely on the possibilities, i.e either by introducing supersymmetry in the GUT model or by modifying the coupling constants through non-renormalisable operators (which induces gravitational correction) or through the threshold effect at the symmetry breaking scale. However most GUTs, without supersymmetry and with threshold effects seem to be more plausible to deal with these issues due to the following reasons. Since so far no hints of supersymmetric particles are experimentally observed, there is an urge for non-supersymmetric GUTs like $SU(5)$ $E_6$ $SO(10)$ etc with intermediate trinification symmetry $SU(3)_C \times SU(3)_L \times SU(3)_R \times D$ $D$-parity $\times \times \times$ invoked with unlike the conventional GUT models, where for simplicity, the superheavy fields are assumed to be exactly degenerate with the symmetry breaking scales. It has been shown in a recent work that the electroweak mixing angle $\sin^2 \theta_W$ and the intermediate mass scale $M_I$ have vanishing contributions due to one loop, two loop as well as gravitational corrections arising from mass scales greater than the intermediate symmetry breaking scale $M_I$ valid for all class of GUTs including $E_6$ GUT with trinification symmetry. In the present work, we intend to establish another remarkable property that GUT threshold corrections(arising out of superheavy scalars, fermions and gauge bosons) for $\sin^2 \theta_W$ and $M_I$, identically vanish. In this context, a general theorem has been proposed for vanishing of the threshold contributions for the electroweak mixing angle $\sin^2 \theta_W$ and the intermediate mass scale. It is observed that the additional particles like exotic color fermions, vector-like lepton doublets and two neutral fermions contained in the fundamental representation of $E_6$ GUT along with the threshold corrections, ensure successful gauge coupling unification, in tune with the present experimental limit of proton decay lifetime.

The paper is organized as follows. The next section is...
devoted to the model building. In section-III, we discuss the gauge coupling evolution of the associated gauge couplings with appropriate threshold corrections. The subsequent section is aimed to establish a generalised theorem (along with its proof), for vanishing of the threshold contributions to the electroweak mixing angle and the intermediate mass scale. In Sections-V and VI, numerical estimation of the mass scales $M_I$, $M_U$ and the GUT coupling constant $\alpha_G$ and the proton decay lifetime are done with specific choice of threshold masses in tune with the experimental constraint. The last section is devoted to concluding remarks on the phenomenological viability of the model.

II. THE MODEL FRAMEWORK

We briefly discuss here the non-supersymmetric $E_6$ Grand Unified Theory (GUT) with one intermediate trinification symmetry $SU(3)_L \otimes SU(3)_L \otimes SU(3)_R$. Due to the presence of $SU(N)_L \otimes SU(N)_R$ (here $N = 3$ for trinification symmetry) structure, there are possibility of two different scenarios of symmetry breaking chain—one with D-parity conserved and other with D-parity broken. Here we focus only on the trinification symmetry with D-parity, given as,

$$E_6 \xrightarrow{M_5} SU(3)_C \otimes SU(3)_L \otimes SU(3)_R \otimes D(G_{33\Delta_D})$$
$$\xrightarrow{M_I} SU(3)_C \otimes SU(2)_L \otimes U(1)_Y (G_{32\Delta})$$
$$\xrightarrow{M_U} SU(3)_C \otimes U(1)_Q (G_{31})$$

(1)

Here D-parity stands for discrete left-right symmetry [14, 21, 22] which acts mostly on Higgs fields ensuring equal coupling between $g_{3L}$ and $g_{3R}$ corresponding to $SU(3)_L$ and $SU(3)_R$ symmetry. Below this D-parity breaking scale, the asymmetry in the Higgs sector gives different contributions to the beta function of Renormalization Group Equations (RGEs) and thereby, yields asymmetry in the gauge couplings.

The first step of spontaneous symmetry breaking in eqn. (1) from $E_6$ GUT to $G_{33\Delta_D}$ is achieved by giving a GUT scale VEV to D-parity even singlet scalar $(1,1,1)$ contained in $650_H \subset E_6$ leading to $g_{3L} = g_{3R}$. The next stage of spontaneous symmetry breaking i.e from $G_{33\Delta_D} \rightarrow G_{3SM}$ is done by assigning a non-zero VEV to the neutral component of trinification multiplet $(1,3,3)$ of $27_H$ and $(1,8,8)$ of $650_H$ of $E_6$. It has been shown that, the minimal Higgs $(1,3,3) \subset 27_H$ is not enough for proper gauge coupling unification (which comes out to be the scale beyond Planck energy) [22]. So we need to introduce $(1,8,8) \subset 650_H$ in the present model to ensure viable unification. The last stage of symmetry breaking i.e SM to low energy theory ($G_{31}$) is done by assigning a non-zero VEV to SM Higgs doublet contained in $27_H \subset E_6$. We follow the “Extended Survival Hypothesis” for Higgs scalars responsible for spontaneous symmetry breaking and their contributions to RGEs by deriving one-loop beta functions along with one loop GUT threshold corrections in the following discussion.

III. GAUGE COUPLING EVOLUTION

Here we use the standard Renormalization Group Equations (RGEs) [24] to obtain the evolution of the gauge coupling $g_i$ for different range of mass scale corresponding to the channel in eqn. (1). The one-loop solution to the standard RGE for the inverse coupling constant valid for $\mu - M$ (M can be of any scale $> \mu$)

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(M)} + \frac{b_i}{2\pi} \ln \left( \frac{M}{\mu} \right)$$

(2)

Here, $\alpha_i = g_i^2/(4\pi)$ and $b_i$ is the one loop beta coefficients in the mass range $\mu - M$. In the present model, we include the GUT threshold effect in the above solution which is explicitly shown below. For different range of mass scale, we have the solutions:

(i) Between the mass scale $M_Z$ to $M_I$:--

$$\alpha^{-1}_i(M_Z) = \alpha^{-1}_i(M_U) + \frac{b_i}{2\pi} \ln \left( \frac{M_I}{M_U} \right)$$

(3)

with $i = 3C, 2L, 1Y$ and $b_i (i=3C,2L,1Y)$ are one-loop beta coefficients within mass range $M_Z - M_I$. We ignore here the threshold correction at intermediate mass scale $M_I$.

(ii) Between the mass scale $M_I$ to $M_U$:--

The analytic formula for one-loop RGEs including GUT threshold corrections between mass scales $M_I$ and $M_U$ is given by

$$\alpha^{-1}_i(M_I) = \alpha^{-1}_G(M_U) + \frac{b'_i}{2\pi} \ln \left( \frac{M_I}{M_U} \right) - \frac{\lambda^U_i}{12\pi}$$

(4)

where $i = 3C, 3L, 3R$. Here $b'_i (i=3C,3L,3R)$ are one-loop beta coefficients within the mass range $M_I - M_U$. Here we use the modified matching condition due to GUT threshold effects [24], given by,

$$\alpha^{-1}_i(M_U) = \alpha^{-1}_G(M_U) - \frac{\lambda^U_i}{12\pi}$$

where $\alpha^{-1}_G$ is the inverse GUT coupling constant and $\lambda^U_i$ is the GUT threshold contributions.

Here it is to be noted that physically threshold effects arise from the modification of light gauge boson propagator in the effective theory due to superheavy gauge bosons, scalars and fermions in the loop. In the present model, threshold effect $\lambda^U_i$ arises due to the superheavy fields around $M_U$. The general expression for $\lambda^U_i$ is explicitly written in the appendix [15]. For simplicity, we have assumed that the superheavy gauge bosons have degenerate mass with that of GUT scale symmetry breaking and there is no superheavy fermions in the model. Thus referring to eqn. (3), the modified one-loop GUT threshold corrections at $M_U$ for the present model is given by

$$\lambda^U_i = \text{Tr} (n_i^2) + 2\text{Tr} (n_{S}^2 \ln \frac{M^U_i}{M_U})$$

(5)

In eqn. (5), the first term is due to superheavy gauge bosons and the second term is due to superheavy scalars.
Here $t_{IV}$ and $t_{IS}$ denote the generators of the superheavy vector gauge bosons as well as scalars, respectively, under the gauge group $G_i (i = 3C, 3L, 3R)$. Further $M_U^{M}$ is the degenerate mass of superheavy scalars (the detail will be discussed in section IX).

Now using the standard procedure, we obtain the two simultaneous equations for the parameters $M_I$ (the intermediate mass scale) and $M_U$ (the unification mass scale) using GUT threshold effects, given as

\[
A_I \ln \left( \frac{M_I}{M_Z} \right) + A_U \ln \left( \frac{M_U}{M_Z} \right) = D_S - J_\lambda \tag{6}
\]

\[
B_I \ln \left( \frac{M_I}{M_Z} \right) + B_U \ln \left( \frac{M_U}{M_Z} \right) = D_W - K_\lambda \tag{7}
\]

Here the parameters $A_I$, $A_U$, $B_I$ and $B_U$ correspond to the one-loop effects. Similarly, the parameters $J_\lambda$ and $K_\lambda$ correspond to the threshold effects.

\[
A_I = \left[ (8b_{3C} - 3b_{2L} - 5b_Y) - 8b_{3C}' - 4b_{3L}' - 4b_{3R}' \right] \tag{8}
\]

\[
A_U = 4 \left( 2b_{3C} - b_{3L} - b_{3R} \right) \tag{9}
\]

\[
D_S = 16\pi \left[ \alpha_s^{-1}(M_Z) - \frac{3}{8} \alpha_{em}^{-1}(M_Z) \right] \tag{10}
\]

\[
J_\lambda = \frac{2}{3} \left( \lambda_{U3L}^U + \lambda_{3R}^U - 2\lambda_{3C}^U \right) \tag{11}
\]

\[
B_I = \left[ 5 \left( b_{2L} - b_Y \right) - 4 \left( b_{3L} - b_{3R} \right) \right] \tag{12}
\]

\[
B_U = 4 \left( b_{3L} - b_{3R} \right) \tag{13}
\]

\[
D_W = 16\pi \alpha_{em}^{-1}(M_Z) \left[ \sin^2 \theta_W - \frac{3}{8} \right] \tag{14}
\]

\[
K_\lambda = \frac{2}{3} \left( \lambda_{3R}^U - \lambda_{3L}^U \right) \tag{15}
\]

Now solving the equations (6) and (7), we obtain the analytic formula for the intermediate mass scale and unification mass scale.

\[
\ln \left( \frac{M_I}{M_Z} \right) = \frac{B_I}{A_U} D_S - A_U D_W + A_U K_\lambda - B_U J_\lambda
\]

\[
= \ln \left( \frac{M_I}{M_Z} \right) + \Delta \ln \left( \frac{M_I}{M_Z} \right)_{\text{1-loop}} \tag{16}
\]

\[
\ln \left( \frac{M_U}{M_Z} \right) = \frac{A_I}{B_U} D_W - B_I D_S + B_U A_I - B_I A_U
\]

\[
= \ln \left( \frac{M_U}{M_Z} \right) + \Delta \ln \left( \frac{M_U}{M_Z} \right)_{\text{GUT-Th.}} \tag{17}
\]

Further we note that, the inverse GUT coupling constant $\alpha_G^{-1}$ can also be obtained using the eqn. (10).

\[
\alpha_G^{-1} = \frac{3}{8} \left[ \alpha_{em}^{-1}(M_Z) - \frac{C_I}{2\pi} \ln \left( \frac{M_I}{M_Z} \right) - \frac{C_U}{2\pi} \ln \left( \frac{M_U}{M_Z} \right) \right]_{\text{1-loop}}
\]

\[
+ \frac{3}{8} \left[ - \frac{C_I}{2\pi} \Delta \ln \left( \frac{M_I}{M_Z} \right) - \frac{C_U}{2\pi} \Delta \ln \left( \frac{M_U}{M_Z} \right) + F_\lambda \right]_{\text{GUT-Th.}}
\]

\[
= (\alpha_G^{-1})_{\text{1-loop}} + \Delta (\alpha_G^{-1})_{\text{GUT-Th.}} \tag{18}
\]

The parameters $C_I$, $C_U$ contains one-loop effect and $F_\lambda$ contains the threshold contribution, given as

\[
F_\lambda = \frac{1}{12\pi} \left[ \frac{4}{3} \lambda_{3L}^U + \frac{4}{3} \lambda_{3R}^U \right] \tag{19}
\]

\[
C_I = \left[ \frac{5}{3} b_Y + b_{2L} - \frac{4}{3} b_{3L}' + \frac{4}{3} b_{3R}' \right] \tag{20}
\]

\[
C_U = \frac{4}{3} \left( b_{3L} + b_{3R} \right) \tag{21}
\]

In the next section, we propose a general theorem corresponding to the electroweak mixing angle $\sin^2 \theta_W$ and the intermediate mass scale $M_I$.

IV. USEFUL THEOREM FOR $\sin^2 \theta_W$ AND $M_I$

Theorem 1

Within a class of non-supersymmetric grand unified theories (GUTs) with trinification symmetry SU(3)$_C$ $\otimes$ SU(3)$_L$ $\otimes$ SU(3)$_R$ as the highest intermediate step, the electroweak mixing angle $\sin^2 \theta_W$ and the intermediate trinification breaking mass $M_I$ have vanishing contributions due to one-loop GUT threshold effects (due to presence of superheavy scalars, fermions and gauge bosons) arising from mass scales greater than the intermediate mass $M_I$.

Now in order to prove the above theorem we use the standard procedure (referring eqn. (A3)) to obtain the analytical formula for $\sin^2 \theta_W$ due to one-loop GUT threshold effect.

Analytical proof for $\sin^2 \theta_W$:

The formula for the electroweak mixing angle $\sin^2 \theta_W$ in terms of one-loop beta functions and one loop GUT threshold effects is given by

\[
\sin^2 \theta_W = \frac{1}{A_U} \left[ \frac{3}{8} A_I + \frac{\alpha_{em}}{\alpha_s} \left( \frac{3}{8} \right) B_U \right] + \frac{\alpha_{em} (A_I B_I - A_I B_U)}{16\pi} \ln \left( \frac{M_I}{M_Z} \right)_{\text{GUT-Th.}} \tag{22}
\]

Here the one-loop parameters $A_I, B_I, A_U, B_U, K_\lambda, J_\lambda$ are already defined in the previous section. It has been shown that $\sin^2 \theta_W$ has vanishing contribution due to one-loop effects as expressed in the first square bracketed term of eqn. (22). In the present work, we focus to the second term containing $K_\lambda$ and $J_\lambda$ corresponding to the threshold correction.

The one-loop GUT threshold contributions to the electroweak mixing angle $\sin^2 \theta_W$ is obviously expressed as,

\[
\Delta (\sin^2 \theta_W)_{\text{GUT-Th.}} = \frac{1}{A_U} \left[ \frac{\alpha_{em}}{\alpha_s} (A_I B_I - B_U J_\lambda) \right] \tag{23}
\]
where $J_3$ and $K^i_{A}$ are defined in the eqns. (11) and eqn(15), respectively. In a class of GUT with trinification symmetry invoked with D-parity as the highest intermediate symmetry, the parameters $B_U$ and $K^i_{A}$ as appeared in eqns. (13) and (15) identically vanish due to the following reason. Since $B_U = 4(b_{UL} - b_{UR})$ for $b_{UL} = b_{UR}$. This is due to the left-right symmetric Higgs used for conserved D-parity. Similarly, $K^i_{A} = \frac{1}{2} (\lambda^U_{AR} - \lambda^U_{AL}) = 0$ for $\lambda^U_{AR} = \lambda^U_{AL}$ as a result of D-parity. Even though the factor $J_3$ and $A_U$ are non-vanishing, the expression for $\Delta \sin^2 \theta_W^{GUT-Th.}$ (given in eqn. (23)) gives zero threshold contributions. Hence, we prove that the electroweak mixing angle $\sin^2 \theta_W$ has vanishing contributions from one-loop GUT threshold effects emerging from mass scale $\mu > M_1$.

**Analytical proof for intermediate mass scale $M_1$:**

The analytic formula for intermediate mass scale $M_1$ due to one-loop effects as well as GUT threshold effects is given by

$$
\ln \left( \frac{M_1}{M_Z} \right) = \ln \left( \frac{M_1}{M_Z} \right)_{1-loop} + \Delta \ln \left( \frac{M_1}{M_Z} \right)_{GUT-Th.}
$$

where

$$
\ln \left( \frac{M_1}{M_Z} \right)_{1-loop} = \frac{B_U D_S - A_U D_W}{B_V A_T - B_T A_U}
$$

$$
\Delta \ln \left( \frac{M_1}{M_Z} \right)_{GUT-Th.} = \frac{A_U K^i_{A} - B_U J_3}{B_V A_T - B_T A_U}
$$

as in eqn. (15). Here we focus on the threshold contributions in order to prove the theorem. Here the numerator of eqn. (24) vanishes identically as $B_U = 0$ and $K^i_{A} = 0$ as shown in the proof for $\sin^2 \theta_W$. Thus, the proof of the theorem for the intermediate mass scale $M_1$ is automatically established. Here, it is noteworthy to mention that the vanishing of the threshold uncertainty of $\sin^2 \theta_W$ and the intermediate mass scale $M_1$ are solely due to (i) the intermediate trinification symmetry invoked with D-parity GSMAD, (ii) the key matching condition between the gauge couplings, i.e., $\alpha^{-1}_G(M_1) = \frac{1}{4} \alpha^{-1}_{EM}(M_1) + \frac{1}{4} \alpha^{-1}_{EM}(M_1)$.

In the subsequent section, we shall estimate the values of $M_1$ and $\sin^2 \theta_W$ in the non-susy $E_6$ model to show the numerical proof.

### V. MODEL PREDICTIONS FOR THE MASS SCALES AND THE GUT COUPLING CONSTANT

V. MODEL PREDICTIONS FOR THE MASS SCALES AND THE GUT COUPLING CONSTANT

In our earlier discussion we have established the theorem with analytical proof for vanishing threshold contributions to $\sin^2 \theta_W$ and $M_1$ within a class of non-SUSY $E_6$ GUT with intermediate trinification symmetry invoked with conserved D-parity. Now including the threshold effects we estimate numerically the intermediate mass scale $M_1$ and the electroweak mixing angle $\sin^2 \theta_W$ in the present model for a numerical proof. Further we calculate $M_1$ (the unification mass scale) and the inverse GUT scale coupling constant $\alpha^{-1}_G$ for prediction of the proton decay.

Now we give the one-loop beta coefficients corresponding to the Higgs fields used in the model in Table I using the one-loop beta coefficients and the parameters $J_3$ and $K^i_{A}$ (from eqns. (11) and (15)) in eqns. (23) and (25), we have the threshold uncertainties of electroweak mixing angle $\sin^2 \theta_W$ and the intermediate mass scale $M_1$, given as

$$
\Delta \sin^2 \theta_W = \frac{\alpha_{EM}}{4 \pi} \left[ \lambda^U_{AR} - \lambda^U_{AL} \right]
$$

TABLE I: Higgs fields and one-loop beta coefficients for different range of masses

| Group | Range of masses | Higgs Content | $b_i$ values |
|-------|----------------|---------------|--------------|
| $G_{321}$ | $M_Z - M_1$ | $\phi(1, 2, -\frac{1}{3})_{27}$ | $b_{32C} = -7$ |
| $G_{333}$ | $M_1 - M_U$ | $\phi(1, 3, 3)_{27}$ | $b_{32L} = +\frac{10}{3}$ |

where $\Delta \sin \theta_W$ can be obtained

$$
\Delta \ln \left( \frac{M_1}{M_Z} \right) = \frac{2}{109} \left[ \lambda^U_{AR} - \lambda^U_{AL} \right]
$$

which obviously vanishes for $\lambda^U_{AR} = \lambda^U_{AL}$. Thus, $\sin^2 \theta_W$ and $M_1$ remain unaffected by the threshold contributions at the $M_1$. Similarly, the uncertainty of the unification mass scale $M_U$ and the inverse GUT coupling constant $\alpha^{-1}_G$ can be obtained as

$$
\Delta \ln \left( \frac{M_U}{M_Z} \right) = \frac{1}{51} \left[ \lambda^U_{AR} - \lambda^U_{AL} \right]
$$

$$
\Delta \alpha^{-1}_G = \frac{1}{204 \pi} \left[ 10 \lambda^U_{SR} + 7 \lambda^U_{SC} \right]
$$

For numerical proof of the theorem, we include the GUT threshold contributions $\lambda^U_i$ for $i = 3C, 3L, 3R$. The superheavy components (fermions, scalars and vector bosons) used in our calculation transforms under trinification symmetry as,

$27_F \supset$ No superheavy fermions
$27_H \supset \{ \Phi_2(3, 3, 1), \Phi_3(3, 1, 3) \}$
$650_H \supset \{ \Sigma_2(1, 1, 1), \Sigma_3(1, 3, 1), \Sigma_4(1, 1, 8), \Sigma_5(8, 1, 1) \}
\Sigma_6(3, 3, 3), \Sigma_7(3, 3, 3), \Sigma_8(3, 3, 3), \Sigma_9(3, 3, 3)
\Sigma_{10}(3, 6, 3), \Sigma_{11}(3, 6, 3), \Sigma_{12}(3, 3, 6), \Sigma_{13}(3, 3, 6)
\Sigma_{14}(3, 6, 3), \Sigma_{15}(3, 3, 6), \Sigma_{16}(3, 3, 6)
\Sigma_{17}(3, 3, 6), \Sigma_{18}(8, 1, 8), \Sigma_{19}(8, 8, 1)
\Sigma_{20}(3, 3, 3), \Sigma_{21}(3, 3, 3), \Sigma_{22}(3, 3, 3)
\Sigma_{23}(3, 3, 3), \Sigma_{24}(3, 3, 3), \Sigma_{25}(3, 3, 3)
\Sigma_{26}(3, 3, 3), \Sigma_{27}(3, 3, 3), \Sigma_{28}(3, 3, 3)
\Sigma_{29}(3, 3, 3), \Sigma_{30}(3, 3, 3), \Sigma_{31}(3, 3, 3)
\Sigma_{32}(3, 3, 3), \Sigma_{33}(3, 3, 3), \Sigma_{34}(3, 3, 3)
\Sigma_{35}(3, 3, 3), \Sigma_{36}(3, 3, 3), \Sigma_{37}(3, 3, 3)
\Sigma_{38}(3, 3, 3), \Sigma_{39}(3, 3, 3), \Sigma_{40}(3, 3, 3)
\Sigma_{41}(3, 3, 3), \Sigma_{42}(3, 3, 3), \Sigma_{43}(3, 3, 3)
\Sigma_{44}(3, 3, 3), \Sigma_{45}(3, 3, 3), \Sigma_{46}(3, 3, 3)
\Sigma_{47}(3, 3, 3), \Sigma_{48}(3, 3, 3), \Sigma_{49}(3, 3, 3)
\Sigma_{50}(3, 3, 3), \Sigma_{51}(3, 3, 3), \Sigma_{52}(3, 3, 3)
\Sigma_{53}(3, 3, 3), \Sigma_{54}(3, 3, 3), \Sigma_{55}(3, 3, 3)
\Sigma_{56}(3, 3, 3), \Sigma_{57}(3, 3, 3), \Sigma_{58}(3, 3, 3)$

The explicit expression for the GUT threshold contributions has been written in the appendix by using Table I and eqns. (13), (15), (19). We obtain $\lambda^U_i$ ($i = 3C, 3L, 3R$), given by

$$
\lambda^U_{3C} = 9 + 306 \eta^U_S
$$

$$
\lambda^U_{3L} = 9 + 255 \eta^U_S
$$

$$
\lambda^U_{3R} = 9 + 255 \eta^U_S
$$

where $\eta^U_S = \ln X$, with $X = \frac{M_2}{M_U}$. Using eqn. (31) in eqns. (20), we have

$$
\Delta \sin^2 \theta_W = 0
$$

$$
\Delta \ln \left( \frac{M_1}{M_Z} \right) = 0
$$

$$
\Delta \ln \left( \frac{M_U}{M_Z} \right) = 0
$$

$$
\Delta \alpha^{-1}_G = \frac{1}{4 \pi} \left[ 3 + 92 \eta^U_S \right]
$$

Here we assume that all superheavy scalars have degenerate mass $M_2$. Now using the specific choice of the $X$ ($\frac{M_2}{M_U} \leq \frac{1}{2}$) the values of $M_1$, $M_U$, $\alpha^{-1}_G$ can be calculated by using
the experimental measured value of parameters $\alpha_{S}^{-1}$, $\sin^2 \theta_W$, $\alpha_{I}^{-1}$ and $M_Z$ [17]. For numerical calculation we use eqns. (10), (17), (18) and (22).

### TABLE II: Numerical results for $M_I$, $M_U$, $\alpha_{G}^{-1}$ and $\sin^2 \theta_W$ with different ranges of $X$.

| $X$   | $\eta_{S}'$ | $M_I$ (GeV) | $M_U$ (GeV) | $\alpha_{G}^{-1}$ | $\sin^2 \theta_W$ |
|-------|-------------|-------------|-------------|-------------------|-------------------|
| 1     | 0           | 10^{13.007} | 10^{14.81}  | 40.1059           | 0.23129           |
| 1.5   | 0.6093      | 10^{13.007} | 10^{15.11}  | 25.7211           | 0.23129           |
| 2     | -2.3035     | 10^{13.007} | 10^{15.80}  | 23.484            | 0.23129           |
| 3     | -2.996      | 10^{13.007} | 10^{16.11}  | 18.4105           | 0.23129           |

Here in the above table the choice $X = 1$ corresponds to no threshold effects as mass of these superheavy fields are degenerate with GUT scale i.e., $M_S' = M_U$. We can see from the above table that the value of the intermediate mass scale $M_I$ and electroweak mixing angle $\sin^2 \theta_W$ remain unaffected by threshold corrections which automatically proves the theorem. However the unification mass scale $M_U$ increases with it and the inverse GUT coupling constant $\alpha_{G}^{-1}$ decreases. We note here that the threshold contributions at the intermediate mass scale $M_I$ may affect the mass scales $M_I$ and $M_U$ as well as $\sin^2 \theta_W$, $\alpha_{G}^{-1}$. We discuss explicitly the corresponding threshold corrections $\lambda_i' (i = 3C, 2L, 1Y')$ in eqns. (15) and (55). Now considering the corrections both at $M_U$ and $M_I$, the threshold uncertainties are given as

\[
\Delta \sin^2 \theta_W = \frac{\alpha_{em}}{48 \pi} \left[ 5 + \eta_S' \right]
\]

\[
\Delta \ln \left( \frac{M_I}{M_S} \right) = \frac{1}{109} \left[ 5 + \eta_S' \right]
\]

\[
\Delta \ln \left( \frac{M_U}{M_S} \right) = \frac{1}{1853} \left[ 83 + 82 \eta_S' - 1853 \eta_S' \right]
\]

\[
\Delta \alpha_{G}^{-1} = \frac{1}{22236\pi} \left[ 20187 + 2664 \eta_F' + 14824 \eta_F' + 511428 \eta_S' \right]
\]

where $\eta_S' = \ln \frac{M_I}{M_S}$ and $\eta_F' = \ln \frac{M_U}{M_S}$. Also $M_S'$ and $M_F'$ denote the mass of superheavy scalars and fermions, respectively, around the intermediate scale $M_I$. For the specific choice of $M_S'$, $M_F'$ and $M_S''$ presented in Table. III, the threshold uncertainties for the electroweak mixing angle $\sin^2 \theta_W$ and $M_I$ are very much suppressed. However, with the above choice, the predicted $M_U$ ($\alpha_{G}^{-1}$) is clearly in agreement with the viable phenomenology.

### VI. PREDICTIONS ON PROTON DECAY WITH GUT THRESHOLD EFFECTS

We aim to calculate the proton decay lifetime with and without one loop GUT threshold effects and wish to examine how the model predictions are closer or farther from the current experimental limit set by the present experiments. It is mediated mostly by the exchange of lepto-quark gauge bosons, which gives baryon and lepton number violation simultaneously. These lepto-quark gauge bosons are getting their masses through spontaneous symmetry breaking with scale VEV around mass scale $M_U$. That is the reason why one loop GUT threshold effects are particularly important which modifies the mass scale $M_U$ and the GUT coupling constant $\alpha_G$ leading to important prediction for proton decay lifetime. We wish to estimate the RG E effects of this effective dimension-6 operators using Standard Model fermions till the unification scale using the relevant anomalous dimensions.

The dimensional-6 effective operators which can induce proton decay within trinification symmetry with fermions transforming under $SU(3)_C \times SU(3)_L \times SU(3)_R \rightarrow L \equiv (1, 3, 3), Q \equiv (3, 1, 1), \bar{Q}^D \equiv (3, 1, 3)$ is given below

\[
\mathcal{O}_L^{\alpha = 6}(\epsilon, d) \subset \left( \overline{Q}^c \gamma^\mu Q \right) \left( \mathcal{T}_{\alpha, Q} \right)
\]

\[
\mathcal{O}_R^{\alpha = 6}(\epsilon, d^c) \subset \left( \overline{Q}^c \gamma^\mu Q \right) \left( \mathcal{T}_{\alpha, Q} \right)
\]

While the dimension-6 effective operator generating proton decay in terms of Standard Model fermions is as follows,

\[
\mathcal{O}_L^{\alpha = 6}(\epsilon, d) \subset C_l \epsilon^{ijk} \gamma^5 u_i g_{\epsilon}^2 \gamma_{\epsilon} \gamma_{\epsilon} d_k
\]

\[
\mathcal{O}_R^{\alpha = 6}(\epsilon, d^c) \subset C_2 \epsilon^{ijk} \gamma^5 u_i g_{\epsilon}^2 \gamma_{\epsilon} \gamma_{\epsilon} d_k
\]

with their respectively Wilson coefficients $C_{1,2}$.

The master formula for the inverse of proton decay width for the gauge-induced dimension-6 proton decay in the chain $p \rightarrow e^+ \pi^0$ (as discussed in refs. [11, 27, 32]) is given by

\[
\tau_p = \frac{1}{12 \pi} \left( p \rightarrow e^+ \pi^0 \right) = \frac{64 \pi f_\pi^2}{m_p} \left( \frac{M_I^4}{g_C^2} \right) \times \frac{1}{|A_L|^2 |R|^2 (1 + F + D)^2} \ \mathcal{R}
\]

where the representative set of parameters are defined as follows,

- $A_L$ is the long distance enhancement factor which is estimated from the RG evolution from the proton mass scale ($m_p \simeq 1$ GeV) to the electroweak scale ($M_Z$). This enhancement factor below SM for the effective dimension-6 operator is expressed as,

\[
A_L = \left[ \frac{\alpha_s (1 \ GeV)}{\alpha_s (m_t)} \right]^{2/9} \left[ \frac{\alpha_s (m_t)}{\alpha_s (m_b)} \right]^{1/5} \left[ \frac{\alpha_s (m_b)}{\alpha_s (m_c)} \right]^{1/3} \left[ \frac{\alpha_s (m_c)}{\alpha_s (m_t)} \right]^{1/3} \approx 1.25
\]

- $\mathcal{R}$ is the renormalization factor which can be expressed as,

\[
\mathcal{R} = \left[ \frac{\alpha_s (1 \ GeV)}{\alpha_s (m_t)} \right]^{2/9} \left[ \frac{\alpha_s (m_t)}{\alpha_s (m_b)} \right]^{1/5} \left[ \frac{\alpha_s (m_b)}{\alpha_s (m_c)} \right]^{1/3} \left[ \frac{\alpha_s (m_c)}{\alpha_s (m_t)} \right]^{1/3} \approx 1.25
\]

### TABLE III: Numerically estimated values for $M_I$, $M_U$, $\alpha_{G}^{-1}$ and $\sin^2 \theta_W$ by considering one loop threshold effects both at $M_I$ and $M_U$ with different choices of $\eta_S'$, $\eta_F'$ and $\eta_S''$.

| $\eta_S'$ | $\eta_F'$ | $\eta_S''$ | $M_I$ (GeV) | $M_U$ (GeV) | $\alpha_{G}^{-1}$ | $\sin^2 \theta_W$ |
|---------|---------|----------|-------------|-------------|-------------------|-------------------|
| 0       | 0       | 0        | 10^{13.007} | 10^{14.81}  | 40.1059           | 0.23129           |
| 0       | 0       | 0        | 10^{13.007} | 10^{15.11}  | 25.7211           | 0.23129           |
| -0.6093 | -0.6093 | -0.6093  | 10^{15.007} | 10^{15.80}  | 23.484            | 0.23129           |
| 1.5     | 1.5     | 1.5      | 10^{15.007} | 10^{16.11}  | 18.4105           | 0.23129           |
\[
\mathcal{R} = \left( (\alpha_{\text{SL}}^2 + \alpha_{\text{SR}}^2) (1 + |V_{ud}|^2)^2 \right),
\]

where, \(V_{ud} = 0.974\) is the (1, 1) element of \(V_{CKM}\) mixing matrix and \(\alpha_{\text{SL}}(\alpha_{\text{SR}})\) is the short-distance renormalization factor in the left (right) sectors derived by calculating the RGE effects from unification scale to electroweak scale.

The short distance renormalization factor \(\alpha_{\text{SL(R)}}\) – both for left as well as right-handed effective dimension-6 operator- derived in the presence of all possible intermediate scales and is a model dependent factor as,

\[
\alpha_{\text{SL(R)}} = \alpha_{\text{SL(R)}}^{33d} \cdot \alpha_{\text{SL(R)}}^{213}
\]

where,

\[
\alpha_{\text{SL(R)}}^{33d} = \left( \frac{\alpha_{\text{L}}^{-1}(M_{\text{L}})}{\alpha_{\text{U}}^{-1}(M_{\text{U}})} \right)^{\gamma_{\text{L}}(M_Z)} \]
\[i = 3C, 3L, 3R;\]

\[
\alpha_{\text{SL(R)}}^{213} = \left( \frac{\alpha_{\text{L}}^{-1}(M_{\text{L}})}{\alpha_{\text{U}}^{-1}(M_{\text{U}})} \right)^{\gamma_{\text{R}}(M_Z)} \]
\[i = 2L, 1Y, 3C.\]

Here \(\alpha = g^2/4\pi\) is the fine structure constant for gauge group \(G\). Further \(\gamma_{\text{L(R)}}(M_Z)\) are the anomalous dimensions from [10, 11, 33, 34] given by

For \(G_{3c2L1Y}\),
\[
\left\{ \begin{array}{c}
\gamma_{\text{L}}(M_Z) = (2, \frac{Z}{2}) \\
\gamma_{\text{R}}(M_Z) = (2, \frac{7}{2})
\end{array} \right.
\]

For \(G_{3c3L3R}\),
\[
\left\{ \begin{array}{c}
\gamma_{\text{L}}^\prime(M_{R}) = (2, 2, 4) \\
\gamma_{\text{R}}^\prime(M_{L}) = (2, 4, 2)
\end{array} \right.
\]

and \(b_0 = (-7, -19/6, 41/10) (b'_0 = (-5.7/2, 7/2))\) are the one loop beta coefficients at different stages of RGEs from \(M_Z - M_{1} (M_{1} - M_{\nu})\), respectively, in the model given in Table [1].

Other parameters are taken from refs [3, 33] as \(D = 0.81, F = 0.47, f_L = 139 \text{ MeV}\) and \(m_p = 938.3 \text{ MeV}\).

Redefining \(\alpha_H = \alpha_{\text{H}} (1 + F + D) = 0.012 \text{ GeV}^{-1}\) and \(A_H^2 \simeq \alpha_{\text{S}}^2 (\alpha_{\text{SL}}^2 + \alpha_{\text{SR}}^2)\), the modified expression for proton life time can be expressed as,

\[
\tau_{p \to \pi^0\nu_{\mu}} = \frac{4}{\pi} \left( \frac{M_D}{m_p} \right) \left( \frac{M_D}{\alpha_{G}^{-1}} \right) \frac{1}{\alpha_{H}^2 A_{H}^2 (1 + |V_{ud}|^2)^2}. \tag{43}
\]

The precision gauge coupling unification by solving RGEs for gauge coupling constants and without taking into account threshold effects gives unification mass scale and inverse GUT coupling constant as,

\[M_U = 10^{14.81} \text{ GeV} \quad \text{and} \quad \alpha_{G}^{-1} = 40.1059.\]

Using the numerical values of short distance renormalization factors for both the effective dimension-6 operators as \(\alpha_{\text{SL}} = 2.46\) and \(\alpha_{\text{SR}} = 2.34\), the estimated proton life time for the present scenario (without threshold effects) is \(\tau_{p} = 1.55 \times 10^{31} \text{ yrs}\). This prediction is well below the current Super-Kamiokande experiment which sets bound on the proton lifetime for \(p \to e^+\pi^0\) channel is \(\tau_{p}(p \to e^+\pi^0) > 1.6 \times 10^{34} \text{ yrs}\) [55] while it can be accessible to future planned experiments that can reach a bound [57, 58].

\[
\tau_{p}(p \to e^+\pi^0)|_{H_{K,2025}} > 9.0 \times 10^{34} \text{ yrs} \\
\tau_{p}(p \to e^+\pi^0)|_{H_{K,2040}} > 2.0 \times 10^{35} \text{ yrs}. \tag{44}
\]

**Table IV**: Estimated values for unification mass \(M_U\), inverse GUT coupling constant, \(\alpha_{G}^{-1}\) and the predicted value of proton lifetime \(\tau_{p}\) by taking with and without one loop GUT threshold effects. In the last column, the bold face values for proton lifetime are in agreement with the limit set by the present Super-Kamiokande experiment.

| \(X\) | \(M_U\) (GeV) | \(\alpha_{G}^{-1}\) | \(\tau_{p}\) |
|-------|---------------|-----------------|---------|
| \(1\) | \(10^{14.81}\) | 40.1059 | 1.55 \times 10^{31} \text{ yrs} |
| \(2\) | \(10^{15.11}\) | 35.2711 | 1.90 \times 10^{32} \text{ yrs} |
| \(3\) | \(10^{15.80}\) | 23.484 | 4.86 \times 10^{34} \text{ yrs} |
| \(4\) | \(10^{16.11}\) | 18.4105 | 5.19 \times 10^{35} \text{ yrs} |

VII. CONCLUSION

We have computed the one loop GUT threshold uncertainties for the electroweak mixing angle \(\sin^2 \theta_{W}\), intermediate mass scale \(M_I\), unification mass scale \(M_U\) and inverse GUT coupling constant \(\alpha_{G}^{-1}\) within a class of nonsupersymmetric \(E_6\) Grand Unified Theory with D-parity conserving trinification symmetry \(SU(3)_C \otimes SU(3)_I \otimes SU(3)_R \otimes D\). The purpose of the paper is two fold: i.e. proposing a general theorem for threshold uncertainty and investigating the effect on proton decay prediction. First we establish a useful theorem with robust proof that all one loop GUT threshold uncertainties vanish for the electroweak mixing angle \(\sin^2 \theta_{W}\) and intermediate mass scale \(M_I\) arising from superheavy fields (scalars or fermions or gauge bosons sitting around GUT scale) at scale \(\mu > M_I\). The generalised theorem is very crucial in order to ensure the stability of \(\sin^2 \theta_{W}\) and \(M_I\) for consistent theoretical prediction of phenomenological observables. The origin behind these vanishing contributions, is primarily because of (a) Grand Unified Theories accommodating D-parity conserving trinification symmetry, (b) the key matching condition between the gauge couplings, \(\alpha_{G}^{-1}(M_I) = 4 \alpha_{L}^{-1}(M_I) + 4 \alpha_{R}^{-1}(M_I)\). Even with the conservative estimation—the with the unification mass scale and inverse GUT coupling constant, \(M_U = 10^{14.81} \text{ GeV}\) and \(\alpha_{G}^{-1} = 40.1059\)—the predicted proton lifetime without considering GUT threshold effects are well below the current Super-Kamiokande experiment which sets bound on the proton lifetime for \(p \to e^+\pi^0\) channel is \(\tau_{p}(p \to e^+\pi^0) > 1.6 \times 10^{34} \text{ yrs}\) [55]. In order to circumvent the problem, one loop GUT threshold effect has been included in the model, which resulted modification in the unification mass scales, \(M_U = 10^{15.80} \text{ GeV}\) \((M_I = 10^{16.11} \text{ GeV})\). The above estimation is with the specific choice of masses for sup-
perheavy scalars which are few times lighter (10, 20 respectively) than the GUT scale $M_{G}$. The estimated proton lifetime $\tau_p$ as, $4.86 \times 10^{34}$ yrs ($5.19 \times 10^{34}$ yrs ), respectively, is consistent with the Super-Kamiokande experiments. For completeness, we have also examined the threshold correction around the intermediate mass scale $M_I$, which shows that the threshold uncertainty for $\sin^2 \theta_W$ and $M_I$ are very much suppressed (nearly vanishing) when superheavy mass of scalars are 150 times lighter than $M_I$. It is nice to observe that, with this choice, unification mass $M_U$ still complies with proton decay constraint. Thus the present model provides an important window of opportunity for the Exceptional group $E_6$ as an attractive unification model.

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Appendix A: Analytical expressions for mass scales

Here we aim to derive all the necessary analytical formulas which have been used in the text for proof of the theorem as well as for prediction of the proton decay lifetime with threshold corrections. The symmetry breaking channel considered here is given by,

$$E_6 \to G_{333D} \to G_{SM} \to G_3,$$  \hspace{1cm} (A1)

Now using the evolution equations for the gauge couplings between the mass scale $M_2 - M_I$ and $M_I - M_U$ (as noted in eqn.(A4) and (A5) of the text), we obtain the following relations,

$$\alpha_M^{-1}(M_2) = \alpha_G^{-1} + \frac{b_G}{2\pi} \ln \left( \frac{M_I}{M_2} \right) + \frac{b_M}{2\pi} \ln \left( \frac{M_U}{M_I} \right) - \frac{\lambda_G}{12\pi}$$

$$\alpha_M^{-1}(M_1) = \alpha_G^{-1} + \frac{b_G}{2\pi} \ln \left( \frac{M_I}{M_1} \right) + \frac{b_M}{2\pi} \ln \left( \frac{M_U}{M_I} \right) - \frac{\lambda_G}{12\pi} + \frac{\lambda_M}{12\pi}$$

$$\alpha_M^{-1}(M_Z) = \alpha_G^{-1} + \frac{b_G}{2\pi} \ln \left( \frac{M_U}{M_Z} \right) - \frac{\lambda_G}{12\pi} + \frac{\lambda_M}{12\pi}$$ \hspace{1cm} (A2)

We use the following two key relations in order to find the analytical expression for $M_I$, $M_U$, $\sin^2 \theta_W$ along with inverse GUT coupling constant $\alpha_G^{-1}$ by taking the threshold contributions,

$$8 \left( \alpha_S^{-1} - \frac{3}{8} \alpha_M^{-1} \right) = 8\alpha_S^{-1}(M_Z) - 3\alpha_M^{-1}(M_2) - 5\alpha_M^{-1}(M_1)$$

$$\sin^2 \theta_W - \frac{3}{8} = \frac{5}{8} \left( \alpha_S^{-1}(M_Z) - \alpha_M^{-1}(M_2) \right)$$  \hspace{1cm} (A3)

Using relations given in (A2) in eqn.(A3) one can deduce the expression for $M_I$ and $M_U$ in terms of one loop beta coefficients as well as one loop threshold effects as follows:

$$A \ln \left( \frac{M_I}{M_Z} \right) + A_U \ln \left( \frac{M_U}{M_Z} \right) = D_S - J_S$$ \hspace{1cm} (A4)

$$B \ln \left( \frac{M_I}{M_Z} \right) + B_U \ln \left( \frac{M_U}{M_Z} \right) = D_W - K_S$$ \hspace{1cm} (A5)

where the relevant parameters have already been defined in the text. Now solving equations (A4) and (A5), we obtain the following analytic formulas for intermediate mass scale $M_I$, unification scale $M_U$, electroweak mixing angle $\sin^2 \theta_W$ as,

$$\ln \left( \frac{M_I}{M_Z} \right) = \frac{B_I D_S - A_I D_W}{B_I A_R - A_I B_R} + \frac{A_I K_S - B_I J_S}{B_I A_R - A_I B_R} \ln \left( \frac{M_I}{M_Z} \right)$$

$$\ln \left( \frac{M_U}{M_Z} \right) = \frac{A_S D_I - B_S D_R}{B_S A_R - B_S B_R} + \frac{B_I J_S - A_I K_S}{B_I A_R - A_I B_R} \ln \left( \frac{M_U}{M_Z} \right)$$

In order to derive the inverse GUT coupling constant $\alpha_G^{-1}$ due to one loop threshold effects, we use the following relation,

$$\alpha_G^{-1}(M_Z) = \frac{5}{3} \alpha_G^{-1}(M_2) + \alpha_G^{-1}(M_1)$$ \hspace{1cm} (A9)

which results,

$$\alpha_G^{-1} = \frac{3}{8} \left[ \alpha_G^{-1}(M_Z) - \frac{C_I}{2\pi} \ln \left( \frac{M_I}{M_Z} \right) + \frac{C_U}{2\pi} \ln \left( \frac{M_U}{M_Z} \right) \right]$$

$$+ \frac{3}{8} \left[ - \frac{C_I}{2\pi} \Delta \ln \left( \frac{M_I}{M_Z} \right) + \frac{C_U}{2\pi} \Delta \ln \left( \frac{M_U}{M_Z} \right) + F_S \right]$$

$$= \left( \alpha_G^{-1} \right)_{1-loop} + \left( \Delta \left( \alpha_G^{-1} \right) \right)_{GUT-Th.}$$ \hspace{1cm} (A10)

Appendix B: Analytic formula for Threshold Effects

The general expression for one-loop threshold corrections for GUT model is given by

$$\lambda_{i}(\mu) = \text{Tr} \left( i_{\tilde{t}} \bar{t} \right) - 21 \text{Tr} \left[ \bar{t} \left( \frac{M_{V}}{\mu} \right) \right]$$

$$+ 2k \text{Tr} \left[ i_{\tilde{t}} \bar{t} \ln \left( \frac{M_{S}}{\mu} \right) \right] + 8k \text{Tr} \left[ i_{\tilde{t}} \bar{t} \ln \left( \frac{M_{F}}{\mu} \right) \right]$$ \hspace{1cm} (B1)

In eqn. (B1), the first two terms represent threshold effects due to superheavy gauge bosons, the third term is the threshold effects due to superheavy scalars while the fourth term accounts for threshold effects due to superheavy fermions. And $i_{\tilde{t}}$, $i_{\tilde{s}}$, and $i_{\tilde{f}}$ are denoting generators of the superheavy vector gauge bosons, scalars and fermions, respectively, under the gauge group $G_I$. Here $k = \frac{4}{3} (=1)$ for real scalar fields (for complex scalar fields) while $k = \frac{2}{3} (=1)$ for Weyl fermions (for Dirac fermions) in the last term of eqn. (B1). The notations $M_{V}$, $M_{S}$, and $M_{F}$ in eqn. (B1) are the masses of the superheavy vector gauge bosons, scalars and fermions respectively. It may be noted that the superheavy fields with masses around the symmetry breaking scale contribute to the threshold corrections. Here the mass scale $\mu = M_I, M_U$. In the Table below we show the specific fields contributing to one-loop beta function (with bold letter) and the remaining
fields contribute to the threshold corrections as per the “Extended Survival Hypothesis”. We assume that, all the standard model fermions along with the exotic fermions belonging from $27_{R}$ remains light and contribute to the one-loop beta function for $\mu = M_{T} - M_{U}$.

Before deriving one loop threshold corrections for the present work, the following assumptions are made.

- All the vector gauge bosons have degenerate mass at the unification mass scale and their masses are same as the GUT scale symmetry breaking. Thus, the second term in eqn. (B1) vanishes completely.
- Since all the SM fermions that contained in $Q_{L}$, $Q_{R}$ and $L$ of trinification symmetry belonging to 27 fundamental representation of $E_{6}$ remain light and thus, there are no superheavy fermions present in our model. As a result of this, the last term of eqn. (B1) vanishes completely if we take the threshold effects at the unification mass scale $M_{U}$ only.

The one loop threshold corrections at GUT symmetry breaking scale (or at $M_{U}$) are given by

$$\lambda_{SL}^{U} = 9 + 3\eta \rho_{\eta_{1}} + 6\eta \rho_{\eta_{2}} + 9\eta \rho_{\eta_{3}} + 9\eta \rho_{\eta_{4}} + 9\eta \rho_{\eta_{5}} + 9\eta \rho_{\eta_{6}} + 9\eta \rho_{\eta_{7}} + 9\eta \rho_{\eta_{8}} + 9\eta \rho_{\eta_{9}} + 9\eta \rho_{\eta_{10}} + 9\eta \rho_{\eta_{11}} + 9\eta \rho_{\eta_{12}} + 9\eta \rho_{\eta_{13}} + 45\rho_{\eta_{14}} + 45\rho_{\eta_{15}} + 45\rho_{\eta_{16}} + 48\rho_{\eta_{17}} = 9 + 306\rho_{U}^{U} \quad (B2)$$

### TABLE V: The superheavy scalars, fermions and gauge bosons at different symmetry breaking scales arising from $E_{6}$ representations $27_{R}, 27_{H}, 650_{H}, 78_{V}$. The superheavy fields denoted in normal text transforming under trinification symmetry are presented in third column while for SM symmetry in fourth column. Here the light fields (scalars, fermions and gauge bosons) denoted in bold face in third and fourth column are not contributing to one loop threshold effects but take part in the RG evolution of gauge couplings.

| Fields | $E_{6}$ | $G_{3C,3L,3R,D}$ (Fields at $M_{U}$) | $G_{3C,2L,1V}$ (Fields at $M_{I}$) |
|--------|--------|----------------------------------|----------------------------------|
| Fermion | 27_{R} | $L(1,3,3)$, $Q_{L}(3,3,1)$, $Q_{R}(3,1,3)$ | $Q_{L}(3,2,1), u_{R}(3,1,3), d_{R}(3,1,3)$ |
| Scalar | 27_{H} | $\phi_{1}(1,3,3)$, $\phi_{3}(3,1,3)$, $\phi_{5}(3,1,3)$ | $\phi_{1}(1,2,1), \phi_{3}(1,2,1)$ |
| $\Sigma_{0}(1,1,1), \Sigma_{1}(1,8,8), \Sigma_{8}(1,8,1)$ | | $\Sigma_{0}(3,3,3), \Sigma_{7}(3,3,3), \Sigma_{8}(3,3,3)$ | $(8,1,0), (1,3,0), (1,1,0)$ |
| $\Sigma_{0}(3,3,3), \Sigma_{7}(3,3,3), \Sigma_{8}(3,3,3), \Sigma_{8}(3,6,3), \Sigma_{13}(3,3,6)$ | | | $(1,1,0), (1,2,1), (1,2,2), (1,2,3)$ |
| $\Sigma_{12}(3,6,3), \Sigma_{13}(3,6,3), \Sigma_{14}(3,6,3), \Sigma_{15}(6,3,3), \Sigma_{16}(6,3,3), \Sigma_{17}(8,8,1)$ | | | $(1,1,0), (1,1,1), (1,1,0), (1,1,0), (1,1,1), (1,1,1), (1,1,1), (1,1,0), (1,1,1), (1,1,1), (1,1,1), (1,1,0)$ |
| Gauge Boson | 78_{V} | $(1,1,8), (1,8,1), (8,1,1), V_{1}(3,3,3), V_{2}(3,3,3)$ | $(1,1,1), (1,1,0), (1,1,0), (1,1,0), (1,1,0), (1,1,0), (1,1,0), (1,1,0), (1,1,0), (1,1,0), (1,1,0)$ |

$\lambda_{SL}^{U} = 9 + 3\eta \rho_{\eta_{1}} + 6\eta \rho_{\eta_{2}} + 9\eta \rho_{\eta_{3}} + 9\eta \rho_{\eta_{4}} + 9\eta \rho_{\eta_{5}} + 9\eta \rho_{\eta_{6}} + 9\eta \rho_{\eta_{7}} + 9\eta \rho_{\eta_{8}} + 9\eta \rho_{\eta_{9}} + 9\eta \rho_{\eta_{10}} + 9\eta \rho_{\eta_{11}} + 9\eta \rho_{\eta_{12}} + 9\eta \rho_{\eta_{13}} + 45\rho_{\eta_{14}} + 45\rho_{\eta_{15}} + 45\rho_{\eta_{16}} + 48\rho_{\eta_{17}} = 9 + 306\rho_{U}^{U} \quad (B3)$

$\lambda_{SR}^{U} = 9 + 3\eta \rho_{\eta_{1}} + 6\eta \rho_{\eta_{2}} + 9\eta \rho_{\eta_{3}} + 9\eta \rho_{\eta_{4}} + 9\eta \rho_{\eta_{5}} + 9\eta \rho_{\eta_{6}} + 9\eta \rho_{\eta_{7}} + 9\eta \rho_{\eta_{8}} + 9\eta \rho_{\eta_{9}} + 9\eta \rho_{\eta_{10}} + 9\eta \rho_{\eta_{11}} + 9\eta \rho_{\eta_{12}} + 9\eta \rho_{\eta_{13}} + 45\rho_{\eta_{14}} + 45\rho_{\eta_{15}} + 45\rho_{\eta_{16}} + 48\rho_{\eta_{17}} = 9 + 305\rho_{U}^{U} \quad (B4)$

where $\rho_{U}^{U} = \ln \frac{M_{U}}{M_{C}}$.

If we include the one-loop threshold corrections $\lambda_{i}^{U}$’s (i = 3C, 2L, 1Y) at the intermediate mass scale $M_{I}$, the RGE relations, given in eqn. (A2), are modified to,

$$\alpha_{3C}^{-1}(M_{X}) = \alpha_{G}^{-1} + \frac{b_{AC}}{2\pi} \ln \left( \frac{M_{I}}{M_{C}} \right) + \frac{b_{AC}}{2\pi} \ln \left( \frac{M_{I}}{M_{U}} \right)$$

$$- \frac{\lambda_{SL}^{U}}{12\pi} - \frac{\lambda_{SR}^{U}}{12\pi} \quad (B5)$$

$$\alpha_{2L}^{-1}(M_{X}) = \alpha_{G}^{-1} + \frac{b_{2L}}{2\pi} \ln \left( \frac{M_{I}}{M_{C}} \right) + \frac{b_{2L}}{2\pi} \ln \left( \frac{M_{I}}{M_{U}} \right)$$

$$- \frac{\lambda_{SL}^{U}}{12\pi} - \frac{\lambda_{SR}^{U}}{12\pi} \quad (B6)$$
\[
\alpha^{-1}_V(M_Z) = \alpha^{-1}_G - \frac{b_V}{2\pi\ln} \left( \frac{M_T}{M_Z} \right) + \frac{\frac{3}{2}b^0_{\nu L} + \frac{3}{2}b_{\nu R}}{12\pi} \left( \frac{\frac{3}{2}b^0_{\nu L} + \frac{3}{2}b_{\nu R}}{12\pi} \right)
\]

(B7)

Here, the individual one loop threshold contributions \( \lambda'_I \) are given by,

\[
\lambda'_{\lambda_C} = 4\eta_{\lambda_C} + 4\eta_{\lambda R} \\
\lambda'_{\lambda_L} = 1 + \eta_{\lambda_L} + 4\eta_{\lambda_L} + 4\eta_{\nu L} \\
\lambda'_{\nu} = 3 + \frac{3}{5}\eta_{\nu_L} + \frac{3}{5}\eta_{\nu_R} + \frac{8}{5}\eta_{\nu_L} + \frac{8}{5}\eta_{\nu_R} \\
\lambda'_{\psi} = 1 + \frac{12}{5}\eta_{\psi_L} + \frac{12}{5}\eta_{\psi_R}
\]

where we use Table(V) for various term stated above. If we assume that the superheavy fermions (fermions and scalars) take degenerate mass \( M_f \) and \( M_{\psi} \) respectively, the \( \lambda'_{\psi} \) reduces to

\[
\lambda'_{\psi} = 8\eta_{\psi} \\
\lambda'_{\nu} = 1 + 2\eta_{\psi} + 8\eta_{\nu}
\]

where \( \eta_{\psi} = \ln \frac{M_{\psi}}{M_f} \) and \( \eta_{\nu} = \ln \frac{M_{\psi}}{M_f} \) are defined around the intermediate mass scale \( M_I \). Now we obtain the analytical expression for the modified threshold parameters \( J_\lambda \) and \( K_\lambda \) as in eqns\[14,15\],

\[
J_\lambda = \frac{1}{6} \left[ (5\lambda'_{\nu} + 3\lambda'_{\lambda_L} - 8\lambda'_{\lambda_C}) + (4\lambda'_{\nu} + 4\lambda'_{\lambda_L} - 8\lambda'_{\lambda_C}) \right]
\]

\[
K_\lambda = \frac{1}{6} \left[ (5\lambda'_{\nu} - 5\lambda'_{\lambda_L}) + (4\lambda'_{\nu} - 4\lambda'_{\lambda_L}) \right]
\]

(B10)

Considering the threshold contributions at \( M_T \) \[10,16\] and \( M_f \) \[15\] and following the standard procedure, we have the threshold uncertainties,

\[
\Delta \sin^2 \theta_W = \frac{5\alpha_{em}}{96\pi} \left[ \lambda'_{\lambda_L} - \lambda'_{\lambda_L} \right] - \frac{\alpha_{em}}{48\pi} \left[ 5 + \eta_{\psi} \right]
\]

(B11)

\[
\Delta \ln \left( \frac{M_T}{M_Z} \right) = \frac{5}{218} \left( \lambda'_{\lambda_L} - \lambda'_{\lambda_L} \right) = \frac{1}{109} \left[ 5 + \eta_{\psi} \right]
\]

(B12)

\[
\Delta \ln \left( \frac{M_T}{M_Z} \right) = \frac{1}{8559} \left[ 70\lambda'_{\nu} + 39\lambda'_{\lambda_L} - 109\lambda'_{\lambda_C} + 109\lambda'_{\lambda_R} - 109\lambda'_{\lambda_C} \right]
\]

\[
= \frac{1}{1853} \left[ 83 + 82\eta_{\nu} - 1853\eta_{\nu} \right]
\]

(B13)

\[
\Delta \alpha_{em}^{-1} = \frac{1}{223651} \left[ 1210\lambda'_{\nu} - 120\lambda'_{\lambda_L} + 763\lambda'_{\lambda_C} + 1090\lambda'_{\lambda_R} + 763\lambda'_{\lambda_C} \right]
\]

\[
= \frac{1}{223651} \left[ 20187 + 2664\eta_{\nu} + 14824\eta_{\psi} + 511428\eta_{\nu} \right]
\]

(B14)
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