The physical observer II: Gauge and diff anomalies

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Abstract

In a companion paper we studied field theory in the presence of a physical observer with quantum dynamics. Here we describe the most striking consequence of this assumption: new gauge and diff anomalies arise. The relevant cocycles depend on the observer’s spacetime trajectory and can hence not appear in QFT, where this quantity is never introduced. Diff anomalies necessarily arise in every locally non-trivial, non-holographic theory of quantum gravity. Cancellation of the divergent parts of the anomalies only works if spacetime has four dimensions.

1 Introduction

In a companion paper [14], the notion of absolute and relative fields was introduced. QFT deals with absolute fields $\phi_A(t, x)$, where the location $x$ is measured relative to a fixed origin$^1$, using some measuring rods. In contrast, QJT (Quantum Jet Theory) deals with relative fields $\phi_R(t, x)$, labelled by a location measured relative to a physical observer’s position $q(t)$. A physical observer obeys some quantum dynamics, and hence its position at a given time is a complete observable, which can be predicted by

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$^1$A fixed origin may be regarded as the location of an infinitely massive observer. This is a hidden assumption about an infinite observer mass in QFT.
the theory and which becomes an operator after quantization. The difference between absolute and relative fields hence resides “at the other side of the measuring rod”; this is a fixed origin in QFT but a quantized observable in QJT.

Matrix elements in QJT depend on the observer’s physical properties, in particular on its mass $M$ and charge $e$. These properties are never mentioned in QFT, which means that some tacit assumption is made; QFT is recovered from QJT in the joint limit $M \to \infty$ and $e \to 0$. This limit is well defined for all interactions except gravity, where mass and charge are related; inert mass equals heavy mass. Hence the QFT limit of QJT does not exist specifically in the presence of gravity. This is the origin of the difficulties with applying QFT to gravity.

The most striking new feature in QJT is the appearance of new gauge and diff anomalies, which have no counterpart in QFT. In all known representations of the extended gauge and diffeomorphism algebras, the relevant cocycles\(^2\) are functionals of the observer’s trajectory in spacetime. They can not be formulated within a QFT framework, since the observer is never introduced in QFT, but they arise naturally in QJT. The presence of new anomalies proves that QJT is substantially different from QFT.

The new anomalies only appear if we consider gauge transformations or diffeomorphisms in spacetime; the spatial subalgebras are essentially anomaly free. Constraint algebras in canonical quantization on a fixed foliation do hence not see these anomalies. However, it is possible to recover the gauge anomalies in QJT by moving away infinitesimally from the equal-time surface. This point-splitting construction is the main result in this paper.

We end this paper with a discussion on gauge anomalies and consistency, which contrary to popular belief are not mutually exclusive.

2 Gauss’ law

2.1 Free electromagnetic field

In [14] we quantized the free electromagnetic field within QJT by fixing a gauge. However, it is often more convenient to quantize first and impose the constraints afterwards. As a warmup, we review how this is done for the free electromagnetic field, within QFT rather than QJT. All fields are hence absolute fields.

\(^2\)We use the terms “cocycle”, “anomaly” and “extension” interchangably.
The canonically conjugates are the gauge potential \( A_i(x) \) and the electric field \( E_i(x) \), with nonzero commutators

\[
[A_i(x), E_j(y)] = i\delta_{ij}\delta(x - y). \tag{2.1}
\]

The Hamiltonian reads

\[
H = \int d^3x \left( \frac{1}{2} E_i(x) E_i(x) + \frac{1}{4} F_{ij}(x) F_{ij}(x) \right), \tag{2.2}
\]

where \( F_{ij} = \partial_i A_j - \partial_j A_i \). The fields are not independent, but subject to the Gauss’ law constraint

\[
J(x) \equiv \partial_i E_i(x) \approx 0. \tag{2.3}
\]

We quantize the theory by replacing Poisson brackets by commutators, passing to Fourier space and demanding that negative-frequency modes annihilate the vacuum. If we introduce the magnetic field \( B_i = \epsilon_{ijk} F_{jk} \), the Hamiltonian becomes

\[
H = \frac{1}{2} \int d^3k \left( E_i(k) E_i(-k) + B_i(k) B_i(-k) \right), \tag{2.4}
\]

and

\[
[E_i(k), B_j(k')] = \epsilon_{ijm} k_m \delta(k + k'). \tag{2.5}
\]

These brackets are compatible with Gauss’ law in its Fourier form:

\[
J(k) = k_i E_i(k) \approx 0. \tag{2.6}
\]

We now introduce the oscillators

\[
a_i(k) = \frac{1}{\sqrt{2|k|}} (E_i(k) - i|k| A_i(k)), \tag{2.7}
\]

\[
a_i^\dagger(k) = \frac{1}{\sqrt{2|k|}} (E_i(k) + i|k| A_i(k)),
\]

with commutators

\[
[a_i(k), a_j^\dagger(k')] = \delta_{ij} \delta(k + k'). \tag{2.8}
\]

The normal-ordered Hamiltonian becomes a sum of noninteracting harmonic oscillators,

\[
H = \int d^3k |k| a_i^\dagger(k) a_i(-k). \tag{2.9}
\]
We posit that the vacuum $|0\rangle$ is annihilated by all negative frequency states, i.e. $a_i(k)|0\rangle = 0$. Unlike the oscillators constructed in our companion paper [14], the oscillators (2.7) are not immediately compatible with Gauss’ law (2.6), which takes the form

$$J(k) = \sqrt{\frac{|k|}{2}}(k_ia_i(k) + k_ia_i^\dagger(k)).$$

(2.10)

Instead of realizing the constraint $J(k) = 0$ as an operator equation, we impose it as a condition on physical states; by definition, a state $|\text{phys}\rangle$ is physical if it satisfies $J(k)|\text{phys}\rangle = 0$, and two physical states are equivalent if they differ by a state of the form $J(k)|\rangle$. For this definition to be self-consistent, the constraint must commute with the Hamiltonian and itself:

$$[J(k), J(k')] = [J(k), H] = 0.$$

(2.11)

It is readily verified that these relations continue to hold after quantization. It is clear that quantization can not destroy the validity of (2.11) because the Gauss law generators (2.10) are linear in the oscillators, so there is no need for normal ordering. In interacting theories the constraint generators are at least bilinear in oscillators, and normal ordering can potentially invalidate the analogue of (2.11).

### 2.2 Yang-Mills field

Let us generalize this well-known story to Yang-Mills theory based on a finite-dimensional Lie algebra $\mathfrak{g}$. We denote the generators by $J^a$ and structure constants by $f^{abc}$, and we assume that $\mathfrak{g}$ has a Killing metric $\delta^{ab}$. The Lie brackets are thus

$$[J^a, J^b] = if^{abc}J^c.$$

(2.12)

From our point of view, the important new feature is that the Gauss law constraint also contains a bilinear term:

$$J^a(x) \equiv \partial_i E^a_i(x) + f^{abc} A^b_i(x) E^c_i(x),$$

(2.13)

where the nonzero CCR read

$$[A^a_i(x), E^b_j(y)] = i\delta^{ab}\delta_{ij}\delta(x - y).$$

(2.14)

The constraints (2.13) satisfy the current algebra $\text{map}(d, \mathfrak{g})$ (algebra of maps from $d$-dimensional space to $\mathfrak{g}$):

$$[J^a(x), J^b(y)] = if^{abc}J^c(x)\delta(x - y).$$

(2.15)
We again pass to Fourier space, where CCR become
\[
[A^a_i(k), E^b_j(k')] = i \delta^{ab} \delta_{ij} \delta(k + k').
\] (2.16)

We now introduce the oscillators
\[
a^a_i(k) = \frac{1}{\sqrt{2|k|}} (E^a_i(k) - i |k| A^a_i(k)),
\]
\[
a^{\dagger a}_i(k) = \frac{1}{\sqrt{2|k|}} (E^a_i(k) + i |k| A^a_i(k)),
\] (2.17)

with commutators
\[
[a^a_i(k), a^{\dagger b}_j(k')] = \delta^{ab} \delta_{ij} \delta(k + k').
\] (2.18)

By definition, the vacuum \(|0\rangle\) is annihilated by all negative frequency states, i.e. \(a_i(k)|0\rangle = 0\). However, now we encounter a problem with the second term of (2.13), which after normal ordering reads in Fourier space
\[
J^a(k) = f^{abc} \int d^d k' : A^b_i(k') E^c_i(k - k'):
\]
\[
= if^{abc} \int d^d k' \left( a^b_i(k') a^c_i(k - k') - a^{\dagger b}_i(k') a^c_i(k - k') \right) +
\]
\[
a^{\dagger c}_i(k - k') a^b_i(k') - a^{\dagger b}_i(k') a^{\dagger c}_i(k - k').
\] (2.19)

These generators satisfy the gauge algebra with two normal-ordering contributions,
\[
[J^a(k), J^b(l)] = i f^{abc} J^c(k + l) + \text{ext}_1 + \text{ext}_2
\] (2.20)

where
\[
\text{ext}_1 = - \text{ext}_2 = Q \delta^{ab} \delta(k + l) \int d^d k',
\] (2.21)

and \(Q\) denotes the second Casimir operator in the adjoint representation: \(f^{acd} f^{bcd} = Q \delta^{ab}\). Since the two extensions in (2.20) cancel, there is no anomaly. However, care must be taken, because both terms are proportional to \(\int d^d k' 1 = \infty\), so \(\text{ext}_1 + \text{ext}_2\) is a constant of the form \(\infty - \infty\). We will explain in section 7 below how to turn this difference into a finite term within the framework of QJT.

That the total extension vanishes is of course not surprising, because the first and last term in (2.19) vanish. E.g., \(a^b_i(k') a^c_i(k - k')\) is symmetric under the replacement \(k' \rightarrow k - k', b \leftrightarrow c\), and yields zero when multiplied with the antisymmetric constant \(f^{abc}\). The strategy for constructing a nonzero extension therefore consists of avoiding this cancellation.
3 General bilinear gauge generators

The need for normal ordering is not unique to Yang-Mills theory. In fact, it is a generic feature of all interacting theories, whenever the constraint generators are at least bilinear in the oscillators. Only for the free electromagnetic field, where the constraint is linear in $E_i$, is normal ordering unnecessary, and the classical constraint runs no risk of breaking down upon quantization. To study gauge anomalies, we must hence turn to interacting theories.

Consider a set of fields $\phi_\alpha$ with canonical conjugate momenta $\pi_\beta$. The CCR read ($\hbar = 1$ throughout this paper)

$$[\phi_\alpha, \pi_\beta] = i\delta_{\alpha\beta}, \quad [\phi_\alpha, \phi_\beta] = [\pi_\alpha, \pi_\beta] = 0. \quad (3.1)$$

We use an abbreviated notation, where indices $\alpha, \beta$ are shorthand for both discrete and continuous indices; in particular, this includes the space coordinates. When we want to emphasize that some of the indices are continuous, we can always make the substitutions $\phi_\alpha \to \phi_\alpha(x), \pi_\beta \to \pi_\beta(y)$, etc., and remember that contraction also implies integration over continuous coordinates. The main difference between discrete and continuous is that the product of delta functions,

$$\delta_{\alpha\beta}\delta_{\beta\alpha} \to \delta(x - y)\delta(y - x) = \delta(0)\delta(x - y), \quad \text{(no sum on } \alpha, \beta) \quad (3.2)$$

is proportional to $\delta(0)$ and hence ill defined in the continuous case. Denote by $N$ the number of degrees of freedom that the index $\alpha$ runs over; if $\alpha$ also includes continuous degrees of freedom, $N = \infty$.

Assume that our constraint algebra takes the form

$$[J^a, J^b] = if^{abc}J^c. \quad (3.3)$$

This is formally of the form (2.12), but in view of our abbreviated notation it is a shorthand for the current algebra (2.15). For simplicity, we only consider the case that the constraint algebra is a proper Lie algebra. This evidently includes Yang-Mills theory, but also the constraint algebra of general relativity can be cast in Lie-algebraic form [5, 15]. Assume that the matrices $M^a = (M^a_{\alpha\beta})$ furnish a representation of our constraint algebra, i.e.

$$[M^a, M^b]_{\alpha\beta} = M^a_{\alpha\gamma}M^b_{\gamma\beta} - M^a_{\alpha\gamma}M^b_{\gamma\beta} = if^{abc}M^c_{\alpha\beta}. \quad (3.4)$$

Then the operators

$$J^a = M^a_{\alpha\beta}E_{\alpha\beta} \equiv iM^a_{\alpha\beta}\phi_\alpha\pi_\beta \quad (3.5)$$
satisfy the algebra (3.3). We have introduced the bilinear combinations
\[ E_{\alpha\beta} = i\phi_\alpha \pi_\beta, \] (3.6)
which satisfy the algebra \( \mathfrak{gl}(N) \):
\[ [E_{\alpha\beta}, E_{\gamma\delta}] = \delta_{\gamma\beta} E_{\alpha\delta} - \delta_{\alpha\delta} E_{\gamma\beta}. \] (3.7)
The field operators carry a representation of \( \mathfrak{gl}(N) \):
\[ [E_{\alpha\beta}, \phi_\gamma] = \delta_{\gamma\beta} \phi_\alpha, \]
\[ [E_{\alpha\beta}, \pi_\gamma] = -\delta_{\alpha\gamma} \pi_\beta, \] (3.8)
as well as a representation of \( \mathfrak{g} \):
\[ [J^a, \phi_\alpha] = M^a_{\alpha\beta} \phi_\beta = (M^a \phi)_\alpha, \]
\[ [J^a, \pi_\alpha] = -(M^a_{\beta\alpha} \pi_\beta) = -(\pi M^a)_\alpha. \] (3.9)
Introduce the oscillators
\[ a_\alpha = \frac{1}{\sqrt{2}} (\phi_\alpha + i\pi_\alpha), \quad a_\alpha^\dagger = \frac{1}{\sqrt{2}} (\phi_\alpha - i\pi_\alpha), \] (3.10)
so that
\[ \phi_\alpha = \frac{1}{\sqrt{2}} (a_\alpha + a_\alpha^\dagger), \quad \pi_\alpha = -\frac{i}{\sqrt{2}} (a_\alpha - a_\alpha^\dagger). \] (3.11)
Upon quantization we must normal order, and move the creation operators \( a_\alpha^\dagger \) to the left of the annihilation operators \( a_\alpha \). The \( \mathfrak{gl}(N) \) operators (3.6) are replaced by
\[ E_{\alpha\beta} = i :\phi_\alpha \pi_\beta: = \frac{1}{2} (a_\alpha a_\beta + a_\alpha^\dagger a_\beta^\dagger - a_\beta^\dagger a_\alpha - a_\alpha a_\beta^\dagger). \] (3.12)
This amounts to adding a constant to \( E_{\alpha\beta} \):
\[ E_{\alpha\beta} \rightarrow E_{\alpha\beta} + [a_\alpha, a_\beta^\dagger] = E_{\alpha\beta} + \delta_{\alpha\beta}. \] (3.13)
The bracket (3.7) receives two new contributions due to normal ordering:
\[ [a_\alpha a_\beta, a_\gamma a_\delta^\dagger] = (\delta_{\gamma\beta} \delta_{\alpha\delta} + \delta_{\beta\delta} \delta_{\alpha\gamma}) + ..., \]
\[ [a_\alpha^\dagger a_\beta^\dagger, a_\gamma a_\delta] = - (\delta_{\gamma\beta} \delta_{\alpha\delta} + \delta_{\beta\delta} \delta_{\alpha\gamma}) + ... \] (3.14)
where ellipses denote terms that are already present before normal ordering. The sum of these two contributions vanishes, and the normal-ordered operators (3.12) satisfy $gl(N)$ without any extra terms. This is also clear from (3.13).

However, we must be careful when we deal with infinitely many degrees of freedom, $N = \infty$. In particular, the bracket $[E_{\alpha\alpha}, E_{\beta\beta}]$ gets two contributions (3.14) that are proportional to the product of delta-functions (3.2), and this signals a problem in the continuous case. If we restore the space coordinates, the $gl(\infty)$ generators become

$$E_{\alpha\beta}(x, y) = i :\phi_\alpha(x)\pi_\beta(y):,$$

and the current algebra generators are

$$J^a(x) = M^a_{\alpha\beta}E_{\alpha\beta}(x, x) = \int d^d x' M^a_{\alpha\beta}E_{\alpha\beta}(x, x')\delta(x - x')$$

The normal ordering contributions (3.14) become

$$[a_\alpha(x)a_\beta(y), a^\dagger_\gamma(x)a^\dagger_\delta(y)] = (\delta_{\gamma\beta}\delta_{\alpha\delta} + \delta_{\beta\delta}\delta_{\alpha\gamma})\delta(x - y)\delta(x - y) +..., $$

$$[a^\dagger_\alpha(x)a^\dagger_\beta(x), a_\gamma(y)a_\delta(y)] = -(\delta_{\gamma\beta}\delta_{\alpha\delta} + \delta_{\beta\delta}\delta_{\alpha\gamma})\delta(x - y)\delta(x - y) +...$$

The important observation is that both terms are proportional to $\delta(x - y)\delta(x - y) = \delta(0)\delta(x - y)$. Although the sum of the two terms in (3.17) vanishes, each term is proportional to $\delta(0)$ and thus infinite. Again, this is a signal that care is needed.

4 Extensions of gauge algebras

It is in a sense surprising that the generators (3.16) satisfy the current algebra without anomalous terms, because mathematically such terms do exist, and we expect that anything that can happen will happen in quantum theory. It is well known and easy to verify that the spacetime version of (2.15) admits a central extension [2, 6, 7, 18]:

$$[J^a(t, x), J^b(t', x')] = i f^{abc}J^c(t, x)\delta(t - t')\delta(x - x') + K\delta^{ab}\delta(t - t')\delta(x - x').$$

Unlike the situation in one dimension, the extension is no longer central when we take Poincaré or diffeomorphism symmetry into account, because
the “central” term does not commute with spacetime transformations. However, (4.1) admits a covariant formulation, which can be written in Fourier space as

\[
\begin{align*}
[J^a (k), J^b (\ell)] &= i f^{abc} J^c (k + \ell) - K \delta^{ab} k_\mu S^\mu (k + \ell), \\
[J^a (k), S^\nu (\ell)] &= [S^\mu (k), S^\nu (\ell)] = 0, \\
k_\mu S^\mu (k) &\equiv 0. 
\end{align*}
\]

(4.2)

Here \( k = (k_\mu) \in \mathbb{Z}^{d+1} \) labels the Fourier modes on a \((d + 1)\)-dimensional torus; the constant is denoted by a capital \( K \) to avoid confusion with Fourier labels. Since (4.2) is a generalization of affine Kac-Moody algebras to \((d + 1)\) dimensions, we denote it by \( \mathcal{A}ff(d + 1, \mathfrak{g}) \); the usual affine algebra is \( \hat{\mathfrak{g}} = \mathcal{A}ff(1, \mathfrak{g}) \). To recover the delta-function form from this extension, assume that \( S^\mu (k) \) is of the form

\[
S^\mu (k) = \delta^\mu_0 \delta (k_0) S(k).
\]

(4.3)

This expression clearly satisfies \( k_\mu S^\mu (k) \equiv 0 \), and the extension in the \( JJ \) bracket takes the form \(-K \delta^{ab} k_0 \delta(k_0 + \ell_0) S(k + 1)\), which is the Fourier transform of (4.1) provided that we choose \( S(k) = \delta (k) \).

In the formulation (4.2), \( \mathcal{A}ff(d + 1, \mathfrak{g}) \) admits an intertwining action of diffeomorphisms. Denote the \( \text{vect}(d + 1) \) (algebra of vector fields on the \((d + 1)\)-dimensional torus) generators by \( L_\mu (m) = -i \exp (im \cdot x) \partial_\mu \). Its semi-direct product with the current algebra (4.2) is defined by the brackets

\[
\begin{align*}
[L_\mu (k), L_\nu (\ell)] &= \ell_\mu L_\nu (k + \ell) - k_\epsilon L_\mu (k + \ell) \\
&\quad + (c_1 k_\mu \ell_\nu + c_2 k_\nu \ell_\mu) k_\rho S^\rho (k + \ell), \\
[L_\mu (k), S^\nu (\ell)] &= \ell_\mu S^\nu (k + \ell) + \delta^\nu_\rho k_\mu S^\rho (k + \ell), \\
[L_\mu (k), J^a (\ell)] &= \ell_\mu J^a (k + \ell).
\end{align*}
\]

(4.4)

Note that we have included two abelian extensions, which makes this algebra a multi-dimensional generalization of the Virasoro algebra, denoted by \( \mathfrak{Vir}(d + 1) \) [8, 19]. It is straightforward to verify that (4.2) and (4.4) satisfy the axioms for a Lie algebra, and that in the one-dimensional case both extensions in (4.4) reduce to the usual Virasoro algebra:

\[
[L_k, L_\ell] = (\ell - k) L_{k+\ell} - \frac{c}{12} (k^3 - k) \delta_{k+\ell},
\]

(4.5)

apart from the trivial, linear cocycle which can be absorbed into a redefinition of \( L_0 \).
Since $S^\mu(m)$ does not commute with diffeomorphisms, the condition (4.3) is not compatible with the full algebra $\mathfrak{Vir}(d + 1)$. However, it is preserved by the spatial subalgebra $\text{vect}(d)$ with generators $L_i(k)$, where the time component $k_0 = 0; k = (0, k)$. The condition (4.3) is equivalent to demanding that $S^j(\ell) = 0$ for all $\ell$ including $\ell_0 \neq 0$. This is consistent because

$$[L_i(k), S^j(\ell)] = \ell_i S^j(k + \ell) + \delta^j_i (\ell_0 S^0(k + \ell) + \ell_n S^n(k + \ell)), \quad (4.6)$$

and the RHS vanishes since $S^0(k + \ell) \propto \delta(\ell_0)$ and $S^j(k + \ell) = 0$ by assumption. The time component $S^0(\ell) = 0$ unless $\ell_0 = 0$, and $S(1)$ in (4.3) transforms as a scalar density under spatial diffeomorphisms. The full anomalous algebra of spacetime gauge transformations and spatial diffeomorphisms becomes

$$[J^a(k), J^b(\ell)] = i f^{abc} J^c(k + \ell) - K \delta^{ab} k_0 \delta(k_0 + \ell_0) S(k + 1),$$

$$[L_i(k), J^a(\ell)] = \ell_i J^a(k + \ell),$$

$$[L_i(k), S(1)] = \ell_i S(k + 1),$$

$$[L_i(k), L_j(1)] = \ell_i L_j(k + 1) - k_j L_i(k + 1),$$

$$[J^a(k), S(1)] = [S(k), S(1)] = 0. \quad (4.7)$$

We can here consistently assume that $S(k) = \delta(k)$; this is the largest subalgebra of $\mathfrak{Vir}(d + 1) \times \mathfrak{Aff}(d + 1, g)$ for which the extension is central. In the subalgebra of spatial gauge transformations, generated by $J^a(k)$ with $k_0 = 0$, the extension disappears completely. This leads to the important conclusion that the new gauge anomalies only arise if we consider spacetime transformations; in a purely spatial constraint algebra, which arises if we only consider fields on a fixed foliation, the new gauge anomalies vanish. The conclusion is morally the same for the new diff anomalies, except that working on a fixed foliation does not make much sense in this case, since spacetime diffeomorphisms do not preserve the foliation. A better alternative would be to work with a covariant formalism. Indeed, this was the main motivation for MCCQ, cf. subsection 8.1.

It must be emphasized that these extensions are not equivalent to the usual types of gauge anomalies arising in QFT. In particular, the Virasoro extension (4.4) is defined in any number of dimensions, but in QFT there are no diff anomalies at all in four dimensions [1]. Moreover, gauge anomalies in Yang-Mills theory are proportional to the third Casimir $\delta^{abc} = \text{tr} \{ J^a, J^b \} J^c [20]$, whereas the $\mathfrak{Aff}(d + 1, g)$ extension is proportional to the second Casimir $\delta^{ab} \propto \text{tr} J^a J^b$. In the Hamiltonian formalism, conventional gauge anomalies in three dimensions give rise to a Mickelsson-Faddeev (MF) algebra [16],
which can be written in Fourier space as

\[
\begin{align*}
[J^a(k), J^b(l)] &= i f^{abc} J^c(k + l) + d^{abc} \epsilon^{ijmn} k_i j \ell_j A^c_n(k + l), \\
[J^a(k), A^b_i(l)] &= i f^{abc} A^c_i(k + l) + \delta^{ab} k_i \delta(k + l), \\
[A^a_i(k), A^b_j(l)] &= 0.
\end{align*}
\] (4.8)

Here \(A^a_i(k)\) denotes the Fourier modes of the gauge connection. It is clear that the extensions (4.2) and (4.8) are essentially different. Gauge anomalies of the MF form render the theory inconsistent and must be avoided. There is a simple mathematical reason for this: the MF algebra does not admit any nontrivial unitary representations on a separable Hilbert space [17], and hence it can not be a symmetry of a quantum theory. This no-go theorem does not apply to the substantially different algebras (4.2) and (4.4).

Why can the extensions in (4.2) and (4.4) only arise within QJT and not within QFT? In all known representations, the anomaly takes the form

\[
S^\mu(k) = \int dt \dot{q}^\mu(t) \exp(ik \cdot q(t)),
\] (4.9)

where \(q^\mu(t)\) denotes the observer’s trajectory in spacetime [9]. This can only be written down if the observer’s position has been introduced in the first place, i.e. if we pass from QFT to QJT. In this realization, the last condition in (4.2) corresponds to Stokes’ theorem.

5 Origin of gauge anomalies

We learned in the previous section that the algebra of gauge transformations admits extensions, but that these are not realized in QFT, because the two anomalous terms in (2.20) cancel. It is natural to ask if there is some way to avoid this conclusion and realize the anomalous terms. Indeed there is, and the crucial idea is chirality in a general sense. By starting from twice as many oscillators, but only acting with our symmetry algebra on half of them, we can avoid anomaly cancellation. Normal ordering in a chiral theory only gives rise one of the contributions in (3.14) or (3.17), and no cancellation occurs.

After making the general construction in this section, both for bosonic and fermionic degrees of freedom, we apply the idea to genuinely chiral theories in section 6. In section 7 we show how a very similar idea yields a central extension of the gauge algebra of the form (4.7). This construction, which only works in QJT, amounts to moving away an infinitesimal distance from the equal-time surface.
5.1 Bosons

We posit four types of oscillators \( a_\alpha, a_\alpha^\dagger, b_\alpha, b_\alpha^\dagger \), subject to the commutators

\[
\begin{align*}
[a_\alpha, b_\beta] &= c_{\alpha\beta}, \\
[a_\alpha^\dagger, b_\beta] &= -c_{\alpha\beta}^*, \\
[a_\alpha, a_\beta^\dagger] &= [b_\alpha, b_\beta^\dagger] = [a_\alpha, b_\beta] = [a_\alpha^\dagger, b_\beta^\dagger] = 0, \\
[a_\alpha, a_\beta] &= [b_\alpha, b_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = [b_\alpha^\dagger, b_\beta^\dagger] = 0.
\end{align*}
\] (5.1)

Here \( c_{\alpha\beta} \) are some complex constants and \( c_{\alpha\beta}^* \) their complex conjugates. As the notation suggests, we assume that hermitean conjugation acts as \( a_\alpha \to a_\alpha^\dagger, b_\alpha \to b_\alpha^\dagger \). Further assume that the structure constants have the form

\[
\begin{align*}
c_{\alpha\beta} &= \delta_{\alpha\beta} + i\gamma_{\alpha\beta}, \\
c_{\alpha\beta}^* &= \delta_{\alpha\beta} - i\gamma_{\alpha\beta},
\end{align*}
\] (5.2)

where \( \delta_{\alpha\beta} \) denotes the Kronecker delta, and both \( \delta_{\alpha\beta} \) and \( \gamma_{\alpha\beta} \) are real. The real combinations

\[
\begin{align*}
\phi_\alpha &= \phi_\alpha^\dagger = \frac{1}{\sqrt{2}} (a_\alpha + a_\alpha^\dagger), \\
\pi_\alpha &= \pi_\alpha^\dagger = -\frac{i}{\sqrt{2}} (b_\alpha - b_\alpha^\dagger),
\end{align*}
\] (5.3)

satisfy the usual CCR

\[
\begin{align*}
[\phi_\alpha, \pi_\beta] &= \frac{i}{2} (c_{\alpha\beta} + c_{\alpha\beta}^*) = i\delta_{\alpha\beta}, \\
[\phi_\alpha, \phi_\beta] &= [\pi_\alpha, \pi_\beta] = 0.
\end{align*}
\] (5.4)

The normal-ordered combinations

\[
E_{\alpha\beta} = i: \phi_\alpha \pi_\beta : = \frac{1}{2} (a_\alpha b_\beta + a_\alpha^\dagger b_\beta^\dagger - b_\beta^\dagger a_\alpha - a_\alpha b_\beta).
\] (5.5)

therefore satisfies some central extension of \( \mathfrak{gl}(N) \) (3.7). The extension is necessarily central since \( E_{\alpha\beta} \) is bilinear in the oscillators. There are two normal-ordering contributions to the extension:

\[
\begin{align*}
[a_\alpha b_\beta, a_\gamma^\dagger b_\delta^\dagger] &= c_{\gamma\beta}^* c_{\alpha\delta} + ..., \\
[a_\alpha^\dagger b_\beta^\dagger, a_\gamma b_\delta] &= -c_{\gamma\beta} c_{\alpha\delta}^* + ....
\end{align*}
\] (5.6)
The normal-ordered generators (5.5) hence satisfy the following central extension of \( \mathfrak{gl}(N) \)

\[
[E_{\alpha\beta}, E_{\gamma\delta}] = \delta_{\gamma\beta} E_{\alpha\delta} - \delta_{\alpha\delta} E_{\gamma\beta} + \frac{1}{4} (c^*_{\alpha\delta} c_{\gamma\beta} - c^*_{\gamma\beta} c_{\alpha\delta}).
\] (5.7)

In view of (5.2), the extension can alternatively be written as

\[
\frac{i}{2} (\delta_{\alpha\delta} \gamma_{\gamma\beta} - \delta_{\gamma\beta} \gamma_{\alpha\delta}).
\] (5.8)

In this form, it is clear that the extension is real and nonzero. Normal ordering corresponds to the replacement

\[
E_{\alpha\beta} \rightarrow E_{\alpha\beta} + \frac{1}{2} [a_{\alpha}, b^\dagger_{\beta}] = E_{\alpha\beta} + \frac{1}{2} c_{\alpha\beta}.
\] (5.9)

If the index \( \alpha \) only runs over a finite set of values, such a redefinition can of course not result in anything non-trivial. However, we will see that this results in a non-trivial current algebra extension in the limit of infinitely many degrees of freedom.

What gives us a central extension in this case is a kind of chirality. We have four oscillators \( a_{\alpha}, a_{\alpha}^\dagger, b_{\alpha}, b_{\alpha}^\dagger \) but only two fields \( \phi_{\alpha} \) and \( \pi_{\alpha} \). Hence it must be possible to define two more fields, which are inert under \( \mathfrak{gl}(N) \). Indeed, the full set of real fields that we can write down is

\[
\phi_{\alpha} = \frac{1}{\sqrt{2}} (a_{\alpha} + a_{\alpha}^\dagger), \quad \pi_{\alpha} = -\frac{i}{\sqrt{2}} (b_{\alpha} - b_{\alpha}^\dagger),
\]

\[
\psi_{\alpha} = \frac{i}{\sqrt{2}} (a_{\alpha} - a_{\alpha}^\dagger), \quad \chi_{\alpha} = \frac{1}{\sqrt{2}} (b_{\alpha} + b_{\alpha}^\dagger).
\] (5.10)

These fields satisfy the algebra

\[
[\phi_{\alpha}, \pi_{\beta}] = i \delta_{\alpha\beta}, \quad [\phi_{\alpha}, \chi_{\beta}] = i \gamma_{\alpha\beta},
\]

\[
[\psi_{\alpha}, \pi_{\beta}] = -i \gamma_{\alpha\beta}, \quad [\psi_{\alpha}, \chi_{\beta}] = i \delta_{\alpha\beta}.
\] (5.11)

It is not quite obvious that this is a Heisenberg algebra, due to the extra terms proportional to \( \gamma_{\alpha\beta} \) in the RHS. However, the equations (5.11) can be diagonalized, showing that they do indeed describe two independent canonically conjugate pairs. To this end, it is convenient to go over to index-free notation, and introduce the vectors \( \phi = (\phi_{\alpha}) \), etc. The algebra (5.11) then becomes

\[
[\phi, \pi] = i, \quad [\phi, \chi] = i \gamma,
\]

\[
[\psi, \pi] = -i \gamma, \quad [\psi, \chi] = i.
\] (5.12)
To diagonalize these relations, we introduce two canonically conjugate pairs \( \tilde{\phi}, \tilde{\pi} \) and \( \tilde{\psi}, \tilde{\chi} \), whose only nonzero brackets are \([\tilde{\phi}, \tilde{\pi}] = [\tilde{\psi}, \tilde{\chi}] = i\). Let

\[
\begin{align*}
\phi &= \frac{1}{\sqrt{2}}(\tilde{\phi} + \sigma_1 \tilde{\psi}), \\
\pi &= \frac{1}{\sqrt{2}}(\tilde{\pi} + \tilde{\chi}\sigma_1), \\
\psi &= \frac{1}{\sqrt{2}}(iA\sigma_2 \tilde{\psi} + A\sigma_3 \tilde{\phi}) \\
\chi &= \frac{1}{\sqrt{2}}(-i\tilde{\chi}\sigma_2 A + \tilde{\pi}\sigma_3 A),
\end{align*}
\]

where \( \sigma_i \) are the Pauli matrices satisfying \( \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk} \), and \( A \) is a unitary matrix anticommuting with \( \sigma_3 \), i.e. \( A\tilde{A} = \tilde{A}A = 1 \) and \( A\sigma_3 + \sigma_3 A = 0 \). It is straightforward to verify that the combinations (5.13) indeed satisfy the relations (5.12). An explicit representation in terms of \( 4 \times 4 \) Dirac matrices is given by

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

where each block is a \( 2 \times 2 \) matrix and \( \sigma_i \) denotes the Pauli matrices. \( A = \gamma^0\gamma^5 \) in the Dirac representation of the Dirac matrices.

### 5.2 Fermions

The analysis above readily carries over to fermionic oscillators with some sign changes. For brevity, we just list the results in index-free notation. We introduce oscillators \( a, a^\dagger, b, b^\dagger \) with nonzero canonical anticommutation relations (CAR)

\[
\{a, a^\dagger\} = c, \quad \{a^\dagger, b\} = \{b, a^\dagger\} = c^\dagger,
\]

where \( c = 1 + i\gamma \) and \( c^\dagger = 1 - i\gamma \). From these oscillators we can construct the four fields

\[
\begin{align*}
\phi &= \frac{1}{\sqrt{2}}(a + a^\dagger), \\
\pi &= \frac{1}{\sqrt{2}}(b + b^\dagger), \\
\psi &= \frac{i}{\sqrt{2}}(a - a^\dagger), \\
\chi &= \frac{i}{\sqrt{2}}(b - b^\dagger),
\end{align*}
\]

which have the nonzero brackets

\[
\begin{align*}
\{\phi, \pi\} = 1, & \quad \{\phi, \chi\} = \gamma, \\
\{\psi, \pi\} = -\gamma, & \quad \{\psi, \chi\} = 1.
\end{align*}
\]
To diagonalize these relations, we introduce two canonically conjugate pairs \( \tilde{\phi}, \tilde{\pi} \) and \( \tilde{\psi}, \tilde{\chi} \), whose only nonzero brackets are \( \{ \tilde{\phi}, \tilde{\pi} \} = \{ \tilde{\psi}, \tilde{\chi} \} = 1 \). The relation to the original fields is

\[
\phi = \frac{1}{\sqrt{2}}(\tilde{\phi} + \sigma_1 \tilde{\psi}), \quad \pi = \frac{1}{\sqrt{2}}(\tilde{\pi} + \tilde{\chi} \sigma_1),
\]

\[
\psi = \frac{1}{\sqrt{2}}(iA \sigma_2 \tilde{\psi} + A \sigma_3 \tilde{\phi}), \quad \chi = \frac{1}{\sqrt{2}}(-i\tilde{\chi} \sigma_2 \tilde{A} + \tilde{\pi} \sigma_3 \tilde{A}).
\] (5.18)

where \( \sigma_i \) and \( A \) are the same as in the bosonic case.

The \( \mathfrak{gl}(N) \) generators read

\[
E = :\phi \otimes \pi: = \frac{1}{2}(a \otimes b + a^\dagger \otimes b - (1 \otimes b^\dagger)(a \otimes 1) + a^\dagger \otimes b^\dagger).
\] (5.19)

These operators satisfy the algebra \( \mathfrak{gl}(N) \) (3.7), and the extension

\[
\frac{1}{4}(c^*_{\gamma\beta}c_{\alpha\delta} - c^*_{\alpha\delta}c_{\gamma\beta}) = \frac{i}{2}(\delta_{\gamma\beta}\gamma_{\alpha\delta} - \delta_{\alpha\delta}\gamma_{\gamma\beta})
\]

is the negative of the corresponding bosonic extension. This is a generic feature: if we interchange bosons and fermions, anomalies change sign but retain amplitudes.

6 Chiral theories

If the indices \( \alpha, \beta \) belong to some finite set, the central extensions in the previous section can be removed by a redefinition and are hence trivial. However, we are interested in the case that the index set is infinite, and includes the space coordinates. Therefore, we briefly repeat the analysis for bosons with the spatial coordinates explicitly exhibited. The fermionic case is completely analogous and will not be treated; suffice it to mention that a fermionic extension is always the negative of the corresponding bosonic extension.

We posit four types of oscillators \( a_\alpha(x), a^\dagger_\alpha(x), b_\alpha(x), b^\dagger_\alpha(x) \), subject to the nonzero commutators

\[
[a_\alpha(x), b^\dagger_\beta(y)] = c_{\alpha\beta}\delta(x - y),
\]

\[
[a^\dagger_\alpha(x), b_\beta(y)] = -c^*_{\alpha\beta}\delta(x - y).
\] (6.1)

Define the real-valued field operators

\[
\phi_\alpha(x) = \frac{1}{\sqrt{2}}(a_\alpha(x) + a^\dagger_\alpha(x)),
\]

\[
\pi_\alpha(x) = \frac{i}{\sqrt{2}}(b_\alpha(x) - b^\dagger_\alpha(x)).
\] (6.2)
which satisfy the CCR
\[ [\phi_\alpha(x), \pi_\beta(y)] = i\delta_{\alpha\beta}\delta(x - y). \] (6.3)

From the oscillators we can also construct two more linearly independent combinations
\[ \psi_\alpha(x) = \frac{i}{\sqrt{2}}(a_\alpha(x) - a_\alpha^\dagger(x)), \] \[ \chi_\alpha(x) = \frac{1}{\sqrt{2}}(b_\alpha(x) + b_\alpha^\dagger(x)). \] (6.4)

The normal-ordered combinations
\[ E_{\alpha\beta}(x, y) = i :\phi_\alpha(x)\pi_\beta(y): \] (6.5)
\[ = \frac{1}{2}(a_\alpha(x)b_\beta(y) + a_\alpha^\dagger(x)b_\beta^\dagger(y) - b_\beta^\dagger(y)a_\alpha(x) - a_\alpha^\dagger(x)b_\beta(y)), \]

satisfy the following central extension of \( \mathfrak{gl}(\infty) \):
\[ [E_{\alpha\beta}(x, x'), E_{\gamma\delta}(y, y')] = \delta_{\gamma\beta}E_{\alpha\delta}(x, y')\delta(y - x') - \delta_{\alpha\delta}E_{\gamma\beta}(y, x')\delta(x - y') + \frac{1}{4}(c_{\alpha\delta}c_{\gamma\beta} - c_{\gamma\beta}c_{\alpha\delta})\delta(x - y')\delta(y - x'). \] (6.6)

In particular, we know from (3.16) that what enters into the current algebra are the operators \( E_{\alpha\beta}(x, x) \), where both space indices are the same.
\[ [E_{\alpha\beta}(x, x), E_{\gamma\delta}(y, y)] = \delta_{\gamma\beta}E_{\alpha\delta}(x, y)\delta(y - x) - \delta_{\alpha\delta}E_{\gamma\beta}(y, x)\delta(x - y) + \frac{1}{4}(c_{\alpha\delta}c_{\gamma\beta} - c_{\gamma\beta}c_{\alpha\delta})\delta(x - y)\delta(0). \] (6.7)

If the matrices \( M_{\alpha\beta}^a \) define a representation of the finite-dimensional Lie algebra \( \mathfrak{g} \) as in (3.4), the smeared generators
\[ J_X = -\int d^dx X^a(x)M_{\alpha\beta}^aE_{\alpha\beta}(x, x) \] (6.8)
satisfy an extension of the current algebra map \( \mathfrak{map}(d, \mathfrak{g}) \) (2.15):
\[ [J_X, J_Y] = J_{[X,Y]} + \text{ext}(X,Y), \] (6.9)
where \([X,Y]^a = if^{abc}X^bY^c\) and the extension is
\[ \text{ext}(X,Y) = \frac{1}{4}\delta(0)\int \text{tr}(XcYc^\dagger - Xc^\dagger Yc) \] (6.10)
\[ = \frac{1}{4}\int\int d^dx d^dy X^a(x)Y^b(y)M_{\beta\alpha}^aM_{\delta\gamma}^b(c_{\alpha\delta}c_{\gamma\beta} - c_{\gamma\beta}c_{\alpha\delta})\delta(0)\delta(x - y). \]
We see that the extension vanishes provided that \( \text{tr} (XcYc^\dagger) = \text{tr} (Xc^\dagger Yc) \)
for all pairs \( X \) and \( Y \). If we write \( c_{\alpha\beta} = \delta_{\alpha\beta} + i\gamma_{\alpha\beta} \) as in (5.2), the condition for anomaly cancellation is equivalent to demanding that \( \text{tr} (X\gamma Y\delta) = \text{tr} (X\delta Y\gamma) \). If this condition does not hold, the current algebra (6.9) is anomalous with a cocycle proportional to \( \delta(0) \). Since there is no way to make sense of an infinite cocycle, the anomaly is inconsistent.

Another way to see that the extension (6.10) is inconsistent is to note that it is neither of the affine form (4.2) nor of the MF form (4.8). One can verify that if we replace \( \delta(0) \) by a finite number, the cocycle (6.10) is not compatible with the Jacobi identities.

An algebra that acts on the fields (6.2) but not on (6.4) is thus a “chiral”, in our sense of the word: a chiral symmetry only acts on half of the fields. The conclusion is that chiral theories have serious problems with inconsistent anomalies. Although the analysis has been carried out for bosonic fields, the same conclusion holds for fermions as well.

7 Point-splitting in time

We observed in section 4 that spatial gauge algebras are not anomalous. To see the gauge anomalies of QJT, we must therefore consider time-dependent gauge transformations. This is not so natural in the canonical formalism which deals with fields at fixed time. However, gauge anomalies do arise if we move away infinitesimally from the equal-time surface.

Recall from [14] that absolute and relative fields are related by

\[
\phi_R(t, x) = \phi_A(t, x + q(t)), \quad \phi_A(t, x) = \phi_R(t, x - q(t)).
\]

(7.1)

Fields with operator-valued arguments can be unambiguously defined by their Taylor series:

\[
\phi_A(t, x) = \sum_m \frac{1}{m!} \phi_m(t)(x - q(t))^m,
\]

\[
\phi_R(t, x) = \sum_m \frac{1}{m!} \phi_m(t)x^m.
\]

(7.2)

We employ multi-index notation which is explained e.g. in [14]. Since a \( p \)-jet is essentially the same thing as Taylor series truncated at order \( p \), this motivates the name QJT (Quantum Jet Theory).

Let us think about the physical meaning of normal ordering. An annihilation operator destroys a particle at time \( t \), and a creation operator...
recreates it an instant later. Denote the duration of this instant by $2\epsilon$, so that annihilation takes place at time $t - \epsilon$ and creation at time $t + \epsilon$, but both take place at the same absolute location $x$. Although the distance to the fixed origin remains the same, the distance from the observer changes; if the observer moves at constant velocity $u$, i.e. $q(t) = ut$, the relative location changes from $x + uc$ to $x - uc$. The relation between absolute and relative fields yields

$$\phi_A(t + \epsilon, x) = \phi_R(t + \epsilon, x - uc),$$

$$\phi_A(t - \epsilon, x) = \phi_R(t - \epsilon, x + uc).$$

(7.3)

In QJT we deal with relative rather than absolute fields, and hence we consider the relative creation and annihilation operators

$$a_\alpha(x) = \phi_\alpha(x + \epsilon u) \approx \phi_\alpha(x) + \epsilon u \phi_\alpha(x),$$

$$a_\alpha^\dagger(x) = \phi_\alpha(x - \epsilon u) \approx \phi_\alpha(x) - \epsilon u \phi_\alpha(x),$$

$$b_\alpha(x) = -i\pi_\alpha(x + \epsilon u) \approx -i\pi_\alpha(x) - i\epsilon u \pi_\alpha(x),$$

$$b_\alpha^\dagger(x) = -i\pi_\alpha(x - \epsilon u) \approx -i\pi_\alpha(x) + i\epsilon u \pi_\alpha(x).$$

(7.4)

where $\partial_u = u_i \partial_i$ is the directional derivative in the direction of the observer’s velocity. Here and henceforth we suppress the subscript “R” for relative, and equip the fields with a discrete index $\alpha$. From the canonical brackets

$$[\phi_\alpha(x), \pi_\beta(y)] = i\delta_{\alpha\beta}\delta(x - y),$$

(7.5)

we read off the following nonzero brackets, to lowest order in $\epsilon$:

$$[a_\alpha(x), b_\beta^\dagger(y)] = \delta_{\alpha\beta}(\delta(x - y) + 2\epsilon u \delta(x - y)), $$

$$[a_\alpha^\dagger(x), b_\beta(y)] = \delta_{\alpha\beta}(\delta(x - y) - 2\epsilon u \delta(x - y)),$$

$$[a_\alpha(x), b_\beta(y)] = \delta_{\alpha\beta}\delta(x - y),$$

$$[a_\alpha^\dagger(x), b_\beta^\dagger(y)] = \delta_{\alpha\beta}\delta(x - y).$$

(7.6)

This is reminiscent in (5.1), where we also have four types of oscillators. The situation is not completely identical, since the $[a, b]$ and $[a^\dagger, b^\dagger]$ brackets do not vanish, but what matters is that the combinations

$$\phi_\alpha(x) = \frac{1}{2}(a_\alpha(x) + a_\alpha^\dagger(x)),$$

$$\pi_\alpha(x) = \frac{i}{2}(b_\alpha(x) + b_\alpha^\dagger(x)).$$

(7.7)
satisfy (7.5) to lowest order in \( \epsilon \). We define the normal-ordered bilinears

\[ E_{\alpha\beta}(x, y) = i : \phi_\alpha(x) \pi_\beta(y) : \]

\[ = -\frac{1}{4} : (a_\alpha(x) + a_\alpha^\dagger(x))(b_\beta(y) + b_\beta^\dagger(y)) : \]

\[ = -\frac{1}{4} (a_\alpha(x)b_\beta(y) + a_\alpha^\dagger(x)b_\beta(y) + b_\beta(y)a_\alpha(x) + a_\alpha^\dagger(x)b_\beta^\dagger(y)). \tag{7.8} \]

Equivalently, normal ordering amounts to the redefinition

\[ E_{\alpha\beta}(x, y) \rightarrow E_{\alpha\beta}(x, y) - \frac{1}{4} (b_\beta^\dagger(y), a_\alpha(x)) \]

\[ = E_{\alpha\beta}(x, y) + \frac{1}{4} \delta_{\alpha\beta} (\delta(x - y) - 2\epsilon \partial_u \delta(x - y)). \tag{7.9} \]

The normal-ordered generators satisfy the following central extension of \( \mathfrak{gl}(\infty) \):

\[ [E_{\alpha\beta}(x, y'), E_{\gamma\delta}(y', y'')] = \]

\[ = \delta_{\gamma\beta} E_{\alpha\delta}(x, y') \delta(y' - y'') - \delta_{\alpha\delta} E_{\gamma\beta}(y, y'') \delta(x - y'') \]

\[ + \frac{1}{4} \epsilon \delta_{\alpha\beta} \delta_{\gamma\delta} (\delta(x - y') \partial_u \delta(x - y') - \delta(x - y') \partial_u \delta(y - y')). \tag{7.10} \]

The smeared map \( (d, g) \) generators (6.8) therefore satisfy an extension of the current algebra (6.9), with

\[ \text{ext}(X, Y) = \frac{1}{4} \epsilon \int d^d x \, d^d y \, \text{tr} (X(x)Y(y)) \times \]

\[ \times (\delta(y - x) \partial_u \delta(x - y) - \delta(x - y) \partial_u \delta(y - x)) \]

\[ = \frac{1}{2} \epsilon \delta(0) \int d^d x \, \text{tr} (X(x)\partial_u Y(x)). \tag{7.11} \]

If we define \( K = \epsilon \delta(0)/2 \), the algebra becomes

\[ [J_X, J_Y] = J_{[X,Y]} + K \int d^d x \, \text{tr} (X(x)\partial_u Y(x)). \tag{7.12} \]

This is equivalent the central extension \( \mathfrak{aff}(d, g) \) in the form (4.1), and the relation between \( J_X \) and \( J^a(t, x) \) is

\[ J_X = \int d^d x \, X^a(x - ut)J^a(t, x). \tag{7.13} \]

The extension (7.12) is the main result of this paper. It shows that although the gauge anomalies of QJT do not appear on a fixed foliation,
they do show up when we work with relative fields, provided that we move away an infinitesimal distance from the equal-time surface. Note that there is no assumption about chirality here; the central extension appears for all gauge groups and all non-trivial irreps. The extension can of course be made to cancel by matching bosonic and fermionic degrees of freedom.

The derivation above is certainly formal. The constant $K$ is a product of the form $0 \times \infty$, which can be anything. To give a definite value to this expression, we must consider a regularized theory, where the time-split $\epsilon$ is small but finite, and the delta-function $\delta(0)$ stands for large but finite number. Their product $K$ is then well defined, and by choosing $\epsilon$ suitably, $K$ can be given a finite but nonzero limit. This process is reminiscent of renormalization, and therefore one may expect that similar anomalies may arise if we consider point-splitting in a renormalizable field theory, formulated in terms of relative fields.

The method in this section does not generalize to general-covariant theories. This is not surprising, since the point-splitting prescription leads to the central extension of $\mathfrak{gl}(\infty)$ (7.10). Since the Virasoro extension of $\mathfrak{vect}(d+1)$ is not central, it can not possibly arise in this way. The underlying assumption that the fields are of the form (7.1) is not valid, because a spacetime diffeomorphism will in general modify the foliation of spacetime into space and time. The foliation is only preserved by spatial diffeomorphisms, but as we saw in section 4, the spatial algebra $\mathfrak{vect}(d)$ is anomaly free. The relevant $\mathfrak{Vir}(d+1)$ extensions do arise in the manifestly covariant formalism described in subsection 8.1 below, where a foliation is avoided altogether.

8 Quantum Jet Theory

8.1 Manifestly covariant canonical quantization

The decomposition of spacetime into space and time is a drawback of canonical quantization. This problem becomes particularly serious when one wants to study the constraint algebra of a background-independent theory like general relativity, since four-diffeomorphisms do not preserve the foliation. However, as was noted already by Lagrange, the notion of phase space is itself covariant; it is the space of histories which solve the equations of motion. Such a history can be coordinatized by the values of the position and velocity, or momentum, at time $t = 0$, but this is only one way to put coordinates on phase space.

The idea behind Manifestly Covariant Canonical Quantization (MCCQ), introduced in [10, 11], was to quantize in the history phase space first, and
then impose dynamics in a BRST-like manner afterwards. It is very similar to BV quantization as described in [4], except that there are twice as many degrees of freedom; in addition to the fields and antifields, we also introduce the corresponding momenta. This is necessary because in order to do canonical quantization we need an honest Poisson bracket, and in conventional BV quantization there is only an antibracket.

Let us return to the abbreviated notation where the index $\alpha$ stands for both discrete indices and spacetime coordinates, and contraction includes both contraction of discrete indices and integration over spacetime. Let us consider some theory with spacetime fields, or histories, $\phi^\alpha$ and action $S[\phi]$. The covariant phase space is the space of histories $\phi^\alpha$ which solve the Euler-Lagrange equations

$$E_\alpha = \frac{\delta S}{\delta \phi^\alpha} = 0.$$  \hspace{1cm} (8.1)

Unfortunately, this description of phase space is not very convenient, because it requires us to find the general solution to the Euler-Lagrange equations. Fortunately, we are not interested in phase space itself, but rather in the dual space of functions over it; this is what becomes our Hilbert space after quantization. This space has a nice cohomological description. For each Euler-Lagrange equation $E_\alpha$, introduce an antifield $\phi^*_\alpha$ of opposite Grassmann parity. For simplicity, we assume that $\phi^\alpha$ and thus $E_\alpha$ are bosonic, which means that $\phi^*_\alpha$ is a fermionic field. Furthermore, we introduce momenta both for the field and for the antifield, subject to the nonzero graded commutators

$$[\phi^\alpha, \pi^\beta] = i \delta^\alpha_\beta, \quad \{\phi^*_\alpha, \pi^\beta\} = \delta^\beta_\alpha \tag{8.2}$$

We can now impose dynamics by passing to the zeroth cohomology group of the fermionic BV operator

$$Q = E_\alpha \pi^\beta_\alpha.$$  \hspace{1cm} (8.3)

Indeed, we have

$$[Q, \phi^\alpha] = 0, \quad \{Q, \phi^*_\alpha\} = E_\alpha. \tag{8.4}$$

Hence all antifields disappear from the cohomology because they are not closed, whereas the fields which do not solve the Euler-Lagrange equation vanish because they are not exact.
Alas, this construction does not quite work even for the harmonic oscillator. The problem is that spurious cohomology arises due to an overcounting of the degrees of freedom. To eliminate this overcounting, one must first identify momenta and velocities, and also compensate for the existence of solutions; the second type of problem already arises in conventional BV quantization. These problems can be solved by introducing further, second-order antifields \[13\], at least for the harmonic oscillator, but the extra antifields are noncovariant and ugly, and the formalism becomes very complicated.

Despite these problems, the original formalism has great appeal when it comes to gauge theories, because the constraint algebras naturally act in spacetime. In particular, the spacetime form of the constraint algebra for general relativity is the four-diffeomorphism algebra \(\text{vect}(4)\). In noncovariant canonical quantization, we must respect the foliation of spacetime, which modifies \(\text{vect}(4)\) into the Dirac algebra (however, see \[5, 15\]). In contrast, in a manifestly covariant formalism, the constraint algebra is \(\text{vect}(4)\) itself, which means that the representation theory of \(\text{Vir}(4)\) applies.

Moreover, the notion of quantization is very natural in the history space; it simply amounts to introducing a vacuum that is annihilated by all negative frequency modes. When we introduce oscillators for the free electromagnetic field in (2.7), we associated \(a_i^\dagger(k)\) with positive energy and \(a_i(k)\) with negative energy, and demanded that the negative energy modes annihilate the vacuum: \(a_i(k)|0\rangle = 0\). The treatment in history space is analogous. Each spacetime history \(\phi^\alpha(t, x)\) can be Fourier transformed in the time coordinate, and the Fourier modes \(\phi^\alpha(k_0, x)\) with \(k_0 < 0\) are defined to annihilate the vacuum. Therefore we need to construct representations of the constraint algebra, be it \(\text{map}(4, g)\) or \(\text{vect}(4)\), that are of lowest-energy type.

### 8.2 The need for QJT

To build lowest-energy representations of \(\text{vect}(1)\) is easy. The recipe consists of the following steps:

- Start from a classical module, i.e. a scalar density a.k.a. a primary field.
- Introduce canonical momenta.
- Normal order.

As is well known, this recipe results in the central extension known as the Virasoro algebra (4.5).
There are two main obstructions which prevent a straightforward generalization of this technique from $d = 1$ to higher dimensions:

- Fields in $d > 1$ dimensions do not admit a unique polarization, analogous to the division into positive and negative frequency modes on the circle.

- Normal ordering does not suffice to eliminate all infinities; some kind of further renormalization is necessary.

The insight in the seminal paper [19], geometrically clarified in [9], is that one should not start from a classical representation on tensor fields. Instead, one must pass to the corresponding space of $p$-jets, and consider one-dimensional histories in this space. The crucial point is that a $p$-jet history consists of finitely many functions of a single variable, and therefore the two problems above do not arise. However, since the classical vect$(d)$ realization on $p$-jets is nonlinear, the resulting extension (4.4) is non-central; the $S^\mu(k)$ do not commute with diffeomorphisms except when $d = 1$.

The Vect$(d + 1)$ realization in QJT is naturally expressed in terms of covariant spacetime $p$-jet histories rather than non-covariant spatial jets (7.2); e.g.,

$$
\phi_A(x, t) = \sum_m \frac{1}{m!} \phi_{,m}(t)(x - q(t))^m.
$$

(8.5)

Here $m = (m_0, m_1, ..., m_d) \in \mathbb{Z}^{d+1}$ is a $(d + 1)$-dimensional multi-index and $t$ is a timelike parameter; spatial jets are recovered by demanding that $x^0 = q^0(t) = t$, which makes the Taylor series (8.5) independent of $m_0$. Covariant jets are spanned by the operator-valued functions $q^\mu(t), \phi_{,m}(t)$, which together with their canonical momenta $p_\mu(t), \pi_{,n}(t)$ satisfy the Heisenberg algebra

$$
[q^\mu(t), p^\nu(t')] = i\delta^\mu_\nu \delta(t - t'),
$$

$$
[\phi_{,m}(t), \pi_{,n}(t')] = i\delta^m_n \delta(t - t').
$$

(8.6)

In contrast to the spatial jets (7.2), these commutation relations live in the history phase space, so field operators at different values of $t$ commute. The vect$(d + 1)$ generators can be written as

$$
L_\mu(k) = \int dt \left( \exp(ik \cdot q(t))p_\mu(t) + \right.
$$

$$
\left. + \sum_{0 \leq |n| \leq |m| \leq p} \pi_{,m}(t)T^n_m(\exp(ik \cdot q(t))\partial_\mu)\phi_{,n}(t) \right),
$$

(8.7)
where we define $T^m_n(\xi)$ for every vector field $\xi = \xi^\mu \partial_\mu$ by

$$T^m_n(\xi) = \sum_{\mu, \nu=0}^{d} \left( \begin{array}{c} n \\ m \end{array} \right) \partial_{n-m+\mu} \xi^\mu T^\nu_\mu + \sum_{\mu=0}^{d} \left( \begin{array}{c} n \\ m-\mu \end{array} \right) \partial_{n-m+\mu} \xi^\mu - \sum_{\mu=0}^{d} \delta^m_\mu \xi^\mu,$$

and the matrices $T^\mu_\nu$ satisfy $\mathfrak{gl}(d)$:

$$[T^\mu_\nu, T^\rho_\sigma] = \delta^\mu_\sigma T^\rho_\nu - \delta^\rho_\nu T^\mu_\sigma. \quad (8.9)$$

The integrand in (8.7) only depends on functions of a single variable $t$, and it can therefore be Fourier transformed. Normal ordering with respect to the corresponding frequency then yields a realization of the multi-dimensional Virasoro algebra (4.4). Hence we obtain a new well-defined representation of $\mathfrak{vir}(d+1)$ for each $\mathfrak{gl}(d+1)$ representation $\rho$ and for each jet order $p$.

Analogously, a map $(d+1, \mathfrak{g})$ representation is given by

$$J^a(k) = \sum_{0 \leq |n| \leq |m| \leq p} \left( \begin{array}{c} m \\ n \end{array} \right) \prod_{i=0}^{d} \delta^{m-n}_i \times \int dt \exp(ik \cdot q(t)) \pi^m(t) M^a \phi_n(t),$$

where $k^m = k_0^m k_1^m \ldots k_d^m$ and the matrices $M^a$ satisfy $\mathfrak{g}$. After normal ordering w.r.t. frequency, this expression yields a representation of $\mathfrak{aff}(d+1)$ (4.2).

### 8.3 The need for gauge anomalies

According to conventional wisdom, gauge anomalies are a sign of inconsistency and must necessarily cancel. This viewpoint is natural if we think of gauge symmetries as redundancies of the description, but it is an oversimplified point of view. What is important is not whether a gauge symmetry is represented trivially or not, but whether the resulting quantum theory is unitary. It is quite possible that the Hilbert space of an anomalous gauge theory has a positive-definite inner product and is thus consistent; the subcritical free string is a well-known example [3]. It is of course not consistent to try to eliminate a gauge symmetry in the presence of anomalies; a gauge anomaly transforms a classical gauge symmetry into a quantum global symmetry, which acts on the Hilbert space rather than reducing it. The resulting
theory may (subcritical free string) or may not (anomalous chiral fermions, supercritical free string) be consistent.

Gauge anomalies may not only be consistent but even necessary. It was pointed out in [12] that local gauge symmetries must act nontrivially the presence of a nonzero charge, provided that we consider the natural completion of the gauge algebra which also contains generators that diverge at infinity. This completion inevitably arises if we expand the gauge symmetry in a Laurent series. E.g., consider Yang-Mills theory in three dimensions. The map(3, \mathfrak{g}) Laurent modes,

\[ J^a_{n,\ell,m} = r^n Y_{\ell,m}(\theta, \varphi) M^a, \]

where the matrices \( M^a \) define a finite-dimensional representation of \( \mathfrak{g} \), can be classified according to their behavior at \( r = \infty \): local \((n < 0)\), global \((n = 0)\), and divergent \((n > 0)\). Since

\[ [J^a_{n,0,0}, J^b_{-n,0,0}] = i f^{abc} J^c_{0,0,0}, \]

we cannot assume that the local generator \( J^b_{-n,0,0} \) is represented trivially if the global charge operator \( J^c_{0,0,0} \) is nonzero.

At this point, unitarity may seem to require a gauge anomaly, in the same way as nontrivial representations of affine algebras are only possible with a nonzero central extension. However, this only follows if we assume that the representations are of lowest-weight type. The spatial subalgebra is not expected to be of this type; after all, it is energy that is bounded from below, not the spatial components of energy-momentum. This is consistent with the observation in (4.7) that the spatial subalgebra \( \text{vect}(d) \rtimes \text{map}(d, \mathfrak{g}) \) is anomaly free. In contrast, the relevant representations of the spacetime algebras are of lowest-energy type, and hence we expect that gauge anomalies arise, of the form \( \text{Vir}(d + 1) \rtimes \text{Aff}(d + 1, \mathfrak{g}) \).

But even in the absence of gauge anomalies, the argument above shows that nonzero charge and divergent transformations inevitably lead to a nontrivial representation of the local gauge transformations. This argument relies on the passage to the natural completion of the gauge algebra. By doing so, we must enlarge the Hilbert space to encompass new states obtained by acting with divergent operators. These states are not gauge invariant, only gauge covariant. The local operators with \( n < 0 \) do of course still annihilate the original Hilbert space. Hence there is no contradiction between gauge invariance of the original Hilbert space of physical states, and anomalous gauge covariance of the completed Hilbert space which also carries an action of the divergent generators.
8.4 Finite anomalies and four dimensions

Although gauge and diff anomalies of the right kind are desirable, they must be finite; infinite anomalies are a disaster. QJT suggests a natural regularization, where we work with $p$-jets rather than infinite jets, i.e. the Taylor series (7.2) or (8.5) are truncated at order $p$. The abelian charges, i.e. the values of the parameters multiplying the cocycles, depend on $p$. They are always finite in the regularized theory, but generically diverge in the field theory limit $p \to \infty$. Taken at face value, this appears to be a serious problem for QJT.

However, it was noted in [10, 11, 13] that the divergent parts of the anomalies can be made to cancel between bosonic and fermionic degrees of freedom, leaving a finite but nonzero remainder, but only if spacetime has four dimensions. This result depends only on a few natural assumptions: the model has both bosonic and fermionic fields, the equations of motion are first order for fermions and second order for bosons, there are bosonic gauge symmetries leading to third order continuity equations, and there are no reducible gauge symmetries. The gauge symmetries can then only be implemented up to order $p - 3$, and require ghosts of order $p - 2$. Under these assumptions, the algebras of spacetime diffeomorphisms, gauge transformations, and reparametrizations of the observer’s trajectory can all be made to acquire finite but nonzero anomalies in exactly four spacetime dimensions.

Let us illustrate the counting for electrodynamics. There are some fermionic fields $\psi$ which satisfy the first-order Dirac equation $\gamma^\mu (i\partial_\mu + eA_\mu)\psi = 0$. The bosonic field is the gauge potential $A_\mu$, with second-order equation of motion $\partial_\nu F^{\mu\nu} = j^\mu$, and the continuity equation comes from the third-order (in $A$) identity $\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0$.

Unfortunately, on closer scrutiny the counting becomes less appealing, because it appears to predict a field content in disagreement with experimental data. However, the situation remains unclear, and I remain optimistic that the problems can be circumvented once QJT is better understood. The prediction of four spacetime dimensions is quite robust, because it makes it possible to cancel the infinite parts of no less than five different anomalies in QJT.

9 Conclusion

The existence of new diff and gauge anomalies proves that QJT is substantially different from QFT. Given that QFT is incompatible with gravity, this is very positive. As is well known from conformal field theory, regarded as
diffeomorphism-symmetric field theory on the circle, Virasoro-like anomalies are necessary to combine diffeomorphism symmetry with locality, in the sense of correlators depending on separation. A non-holographic quantum theory of gravity with local observables hence requires diff anomalies and QJT.

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