Convergence Analysis of Batch Normalization for Deep Neural Nets

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Abstract

Batch normalization (BN) is very effective in accelerating the convergence of a neural network training phase that it has become a common practice. We propose a generalization of BN, the diminishing batch normalization (DBN) algorithm. We provide an analysis of the convergence of the DBN algorithm that converges to a stationary point with respect to trainable parameters. We analyze a two layer model with linear activation. The main challenge of the analysis is the fact that some parameters are updated by gradient while others are not. In the numerical experiments, we use models with more layers and ReLU activation. We observe that DBN outperforms the original BN algorithm on MNIST, NI and CIFAR-10 datasets with reasonable complex FNN and CNN models.

1 Introduction

Deep neural networks (DNN) have shown unprecedented success in various applications such as object detection. However, it still takes a long time to train a DNN until it converges. Ioffe and Szegedy [12] identified an important problem involved in training deep networks, internal covariate shift, and then proposed batch normalization (BN) to decrease this phenomenon. BN addresses this problem by normalizing the distribution of every hidden layer’s input. In order to do so, it calculates the pre-activation mean and standard deviation using mini-batch statistics at each iteration of training and uses these estimates to normalize the input to the next layer. The output of a layer is normalized by using the batch statistics and two new trainable parameters per neuron are introduced that capture the inverse operation. It is now a standard practice [6] [11]. While this approach leads to a significant performance jump, to the best of our knowledge, there is no known theoretical guarantee for the convergence of an algorithm with BN. The difficulty of analyzing the convergence of the BN algorithm comes from the fact that not all of the BN parameters are updated by gradients. Thus it invalidates most of the classical studies of convergence for gradient methods.

In this paper, we propose a generalization of the BN algorithm, diminishing batch normalization (DBN), where we update the BN parameters in a diminishing moving average way. It essentially means that the BN layer adjusts its output according to all past mini-batches instead of only the current one. It helps to reduce the problem of the original BN that the output of a BN layer on a particular training pattern depends on the other patterns in the current mini-batch, which is pointed out in Bottou et al. [6]. By setting the layer parameter we introduce into DBN to a specific value, we recover the original BN algorithm.

We give a convergence analysis of the algorithm with a two-layer batch-normalized neural network and diminishing stepsizes. We assume two layers (the generalization to multiple layers can be done by using the same approach but substantially complicating the notation) and an arbitrary loss function. The convergence analysis does require linear activations. The main result shows that under diminishing stepsizes on gradient updates and updates on mini-batch statistics, and standard Lipschitz conditions on loss functions DBN converges to a stationary point. As already pointed out the main challenge is the fact that some trainable parameters are updated by gradient while others are updated.
by a mere recalculation. We also study the convex case where we assume that the overall loss function is convex (which holds if the regularization parameter is large). In this case, we show a sublinear convergence rate towards the global minimum.

**Contributions.** The main contribution of this paper is in providing a general convergence guarantee for DBN. Specifically, we make the following contributions.

- In section 4, we show the sufficient and necessary conditions for the stepsizes and diminishing weights to ensure the convergence of BN parameters.
- We show that the algorithm converges to a stationary point under a general nonconvex objective function.
- In section 5, we show that decreasing the stepsizes $\eta^{(m)}$ for trainable parameters sublinearly (i.e., at rate $O(1/m)$) ensures sublinear convergence to the optimal value with a convex objective function.

This paper is organized as follows. In Section 2, we review the related works and the development of the BN algorithm. We formally state our model and algorithm in Section 3. We present our main results in Sections 4 and 5. In Section 6, we numerically show that the DBN algorithm outperforms the original BN algorithm. Proofs are collected in Appendixes A and B in supplemental material.

## 2 Literature Review

Before the introduction of BN, it has long been known in the deep learning community that input whitening and decorrelation help to speed up the training process. In fact, Orr and Müller [16] show that preprocessing the data by subtracting the mean, normalizing the variance, and decorrelating the input has various beneficial effects for back-propagation. Krizhevsky et al. [14] propose a method called local response normalization which is inspired by computational neuroscience and acts as a form of lateral inhibition, i.e., the capacity of an excited neuron to reduce the activity of its neighbors. Gülçehre and Bengio [10] propose a standardization layer that bears significant resemblance to batch normalization, except that the two methods are motivated by very different goals and perform different tasks.

Inspired by BN, there are several new works taking BN as a basis for further improvements. Layer normalization [3] is much like the BN except that it uses all of the summed inputs to compute the mean and variance instead of the mini-batch statistics. Besides, unlike BN, layer normalization performs exactly the same computation at training and test times. Normalization propagation [2] uses data-independent estimations for the mean and standard deviation in every layer to reduce the internal covariate shift and make the estimation more accurate for the validation phase. Weight normalization also removes the dependencies between the examples in a minibatch so that it can be applied to recurrent models, reinforcement learning or generative models [19]. Cooijmans et al. [7] propose a new way to apply batch normalization to RNN and LSTM models.

In view of all these flavors, the original BN method is the most popular technique and for this reason our choice of the analysis. To the best of our knowledge, we are not aware of any prior analysis of BN.

BN has gradient and non-gradient updates. Thus nonconvex convergence results do not immediately transfer. Our analysis explicitly takes into account the workings of BN. However, nonconvex convergence proofs are relevant since some small portions of our analysis rely on known proofs and approaches.

Neural nets are not convex, even if the loss function is convex. For classical convergence results with a nonconvex objective function and diminishing learning rate, we refer to survey papers [4, 5, 6]. Bertsekas and Tsitsiklis [5] provide a convergence result with deterministic gradient with errors. Bottou et al. [5] provide a convergence result with stochastic gradient. The classic analyses showing the norm of gradients of the objective function going to zero date back to [9, 18, 17]. For strongly convex objective functions with a diminishing learning rate, we learn the classic convergence results from [6].
3 Model and Algorithm

The optimization problem for a network is a objective function consisting of a large number of component functions, that reads:

$$\min \bar{f}(\theta, \lambda) = \sum_{i=1}^{N} f_i(X_i : \theta, \lambda),$$

subject to $\theta \in P, \lambda \in Q,$

where $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}, i = 1, ..., N,$ are real-valued functions for any data record $X_i.$ Index $i$ associates with data record $X_i$ and target response $y_i$ (hidden behind the dependency of $f$ on $i$) in the training set. Parameters $\theta$ include the usual parameters updated by gradients directly associated with the loss function, i.e. behind the part that we have a parametric model, while BN parameters $\lambda$ are introduced by the BN algorithm and not updated by gradient methods but by the mini-batch statistics.

We define that the derivative of $f_i$ is always taken with respect to $\theta$:

$$\nabla f_i(X_i : \theta, \lambda) := \nabla_{\theta} f_i(X_i : \theta, \lambda).$$

The deep network we analyze here has 2 fully-connected layers with $D_j$ neurons each. Each hidden layer computes $y = kWu$ with slope $k$ for linear activation, i.e. we assume that the activation functions are linear, and $u$ is the input vector of the layer. We do not need to include an intercept term since the BN algorithm automatically adjusts for it. BN is applied to the output of the first hidden layer.

![Figure 1: The structure of our batch-normalized network model in the analysis.](image)

We next describe the computation in each layer to show how we obtain the output of the network. The notations introduced here is used in the analysis. Figure 1 shows the full structure of the network. The input data is vector $X$, which is one of $\{X_i\}_{i=1}^{N}.$ Vector $\lambda = ((\mu_j)_{j=1}^{D_1}, (\sigma_j)_{j=1}^{D_1})$ is the set of all BN parameters and vector $\theta = (W_1, W_2, (\beta_j^{(1)})_{j=1}^{D_1}, (\gamma_j^{(1)})_{j=1}^{D_1})$ is the set of all trainable parameters which are updated by gradients. Matrices $W_1, W_2$ are the actual model parameters and $\beta, \gamma$ are introduced by BN. The $j^{th}$ entry of output of the first hidden layer is

$$z_j^{(1)}(X : \theta) = k^{(1)}W_{1,j}X,$$

where $W_{1,j}$ denotes the weights of linear transformations for the $j^{th}$ neuron and $k^{(1)}$ is the slope of the linear activation function for the first hidden layer. The $j^{th}$ entry of batch-normalized output of the first layer is

$$y_j^{(1)}(X : \theta, \lambda) = \gamma_j^{(1)} \left( \frac{z_j^{(1)}(X, \theta) - \mu_j}{\sigma_j + \epsilon_B} \right) + \beta_j^{(1)},$$

where $\beta_j^{(1)}$ and $\gamma_j^{(1)}$ are trainable parameters updated by gradient and $\mu_j$ and $\sigma_j$ are batch normalization parameters for $z_j^{(1)}$. Trainable parameter $\mu_j$ is the mini-batch mean of $z_j^{(1)}$ and trainable parameter $\sigma_j$ is the mini-batch sample deviation of $z_j^{(1)}.$ Constant $\epsilon_B$ is a small offset term to keep the denominator from zero. The output of $j^{th}$ entry of the output layer reads:

$$z_j^{(2)}(X : \theta, \lambda) = k^{(2)}W_{2,j}y_j^{(1)}(X : \theta, \lambda).$$
We use two strategies to decide the values of $\alpha$ where both are based on SG and non-linear activation functions with many layers FNN and CNN. 

Algorithm 1 shows the algorithm studied herein. There are two deviations from the standard BN:

1. The main purpose of our work is to show that Algorithm 1 converges. In the general case, we focus on showing the convergence property as the full algorithm studied herein.

The second difference is that we update the BN parameters $\theta, \lambda$ by their moving averages with respect to diminishing $\alpha$. After introducing diminishing $\alpha(m)$, $\lambda(m)$ and hence the output of the BN layer is determined by the history of all past data records, instead of those solely in the last batch. Thus the output of the BN layer becomes more general that better reflects the distribution of the entire dataset.

The objective function $f_i(X_i : \theta, \lambda)$ is nonconvex with respect to $\theta$ and $\lambda$. However, when $c_2$ is large enough, the regularization term prevails and makes the objective function convex.

3.1 Algorithm

Algorithm 1 shows the algorithm studied herein. There are two deviations from the standard BN algorithm, one of them actually being a generalization. We use the full gradient instead of the more popular stochastic gradient (SG) method. It essentially means that each batch contains the entire training set instead of a randomly chosen subset of the training set. An analysis of SG is potential future research. Although the primary motivation for full gradient update is to reduce the burdensome go through the entire training set, while full gradient goes through it deterministically and the SG goes through it in expectation. Therefore, it is reasonable to speculate that the SG method has similar convergence property as the full algorithm studied herein.

**Algorithm 1 DBN: Diminishing Batch-Normalized Network Update Algorithm**

1: Initialize $\theta \in \mathbb{R}^{n_1}$ and $\lambda \in \mathbb{R}^{n_2}$
2: for iteration $m=1,2,...$  do
3: $\theta^{(m+1)} := \theta^{(m)} - \eta^{(m)} \sum_{i=1}^{N} \nabla f_i(X_i : \theta^{(m)}, \lambda^{(m)})$
4: for $j=1,...,D_1$ do
5: $\mu_j^{(m+1)} := \frac{1}{N} \sum_{i=1}^{N} z_j^{(1)}(X_i : \theta^{(m+1)})$
6: $\sigma_j^{(m+1)} := \sqrt{\frac{1}{N} \sum_{i=1}^{N} (z_j^{(1)}(X_i : \theta^{(m+1)}) - \mu_j^{(m+1)})^2}$
7: $\lambda^{(m+1)} := \lambda^{(m+1)} \left( (\mu_j^{(m+1)})_{j=1}^{D_1}, (\sigma_j^{(m+1)})_{j=1}^{D_1} \right) + (1 - \alpha^{(m+1)}) \lambda^{(m)}$

The second difference is that we update the BN parameters $(\theta, \lambda)$ by their moving averages with respect to diminishing $\alpha^{(m)}$. The original BN algorithm can be recovered by setting $\alpha^{(m)} = 1$ for every $m$. After introducing diminishing $\alpha^{(m)}$, $\lambda^{(m)}$ and hence the output of the BN layer is determined by the history of all past data records, instead of those solely in the last batch. Thus the output of the BN layer becomes more general that better reflects the distribution of the entire dataset. We use two strategies to decide the values of $\alpha^{(m)}$. One is to use a constant smaller than 1 for all $m$, the other one is to decay the $\alpha^{(m)}$ gradually, such as $\alpha^{(m)} = 1/m$.

In our numerical experiment, we show that Algorithm 1 outperforms the original BN algorithm, where both are based on SG and non-linear activation functions with many layers FNN and CNN models.

4 General Case

The main purpose of our work is to show that Algorithm 1 converges. In the general case, we focus on the nonconvex objective function.

4.1 Assumptions

Here are the assumptions we used for the convergence analysis.
Thus there exists constant $M$. We give a discussion of the above conditions for convergence. We first have the following lemma specifying sufficient conditions for convergence.

**Assumption 4.4** (Lipschitz continuity on $\theta$ and $\lambda$). For every $i$ we have

$$\|\nabla f_i(X : \hat{\theta}, \lambda) - \nabla f_i(X : \hat{\theta}, \hat{\lambda})\|_2 \leq \bar{L} \|\hat{\theta} - \hat{\lambda}\|_2, \forall \hat{\theta}, \hat{\lambda}, X.$$  

$$\|\nabla W_{1,j}, f_i(X : \hat{\theta}, \lambda) - \nabla W_{1,j}, f_i(X : \hat{\theta}, \lambda)\|_2 \leq \bar{L} \|W_{1,j} - W_{1,j}\|_2,$$  

$$\forall \lambda, \hat{\theta}, \hat{\lambda}, X, j \in \{1, ..., D_1\}.$$  

$$\|\nabla f_i(X : \hat{\theta}, \hat{\lambda}) - \nabla f_i(X : \hat{\theta}, \hat{\lambda})\|_2 \leq \bar{L} \|\hat{\lambda} - \hat{\lambda}\|_2, \forall \hat{\theta}, \hat{\lambda}, \lambda, X, j \in \{1, ..., D_1\}.$$  

Noted that the Lipschitz constants associates with each of the above inequalities are not necessarily the same. Here $\bar{L}$ is an upper bound for these Lipschitz constants for simplicity.

**Assumption 4.5** (bounded parameters). Sets $P$ and $Q$ are compact set, where $\theta \in P$ and $\lambda \in Q$. Thus there exists constant $M$ that weights $W$ and parameters $\lambda$ are bounded element-wise by this constant $M$.

$$\|W_1\| \leq M \text{ and } \|W_2\| \leq M \text{ and } \|\lambda\| \leq M.$$  

This also implies that the updated $\theta, \lambda$ in Algorithm remain in $P$ and $Q$, respectively.

**Assumption 4.3** (diminishing update on $\theta$). The stepsizes of $\theta$ update satisfy

$$\sum_{m=1}^{\infty} \eta^{(m)} = \infty \text{ and } \sum_{m=1}^{\infty} (\eta^{(m)})^2 < \infty.$$  

This is a common assumption for diminishing stepsizes in optimization problems.

**Assumption 4.4** (Lipschitz continuity of $l_i(\cdot)$). Assume the loss function $l_i(\cdot)$ for every $i$ is continuously differentiable. It implies that there exists $M$ such that

$$\|l_i(x) - l_i(y)\| \leq \bar{M} \|x - y\|, \forall x, y.$$  

**Assumption 4.5** (existence of a stationary point). There exists a stationary point $(\hat{\theta}^*, \hat{\lambda}^*)$ such that $\|\nabla f(\hat{\theta}^*, \hat{\lambda}^*)\| = 0$.

We note that all these are standard assumptions in convergence proofs. We also stress that Assumption 4.4 does not directly imply 4.1. Since we assume that $P$ and $Q$ are compact, then Assumptions 4.1, 4.4 and 4.3 hold for many standard loss function such as softmax and MSE.

### 4.2 Convergence Analysis

We first have the following lemma specifying sufficient conditions for $\lambda$ to converge. All the proofs are given in Appendix A.

**Theorem 4.6** Under Assumptions 4.1, 4.2 and 4.3 if $\{\alpha^{(m)}\}$ satisfies

$$\sum_{m=1}^{\infty} \alpha^{(m)} < \infty \text{ and } \sum_{m=1}^{\infty} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)} < \infty,$$

then sequence $\{\lambda^{(m)}\}$ converges to $\bar{\lambda}$.

We give a discussion of the above conditions for $\alpha^{(m)}$ and $\eta^{(m)}$ in the end of this section. With the help of Theorem 4.6, we can show the following convergence result.

**Lemma 4.7** Under Assumptions 4.4, 4.5 and the assumptions of Theorem 4.6 when

$$\sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha^{(i)} \eta^{(n)} < \infty \text{ and } \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha^{(n)} < \infty,$$

we have

$$\lim_{M \to \infty} \sup_{m=1}^{M} \sum_{m=1}^{M} \eta^{(m)} \|\nabla f(\hat{\theta}^{(m)}, \hat{\lambda})\|^2 < \infty.$$
This result is similar to the classical convergence rate analysis for the non-convex objective function with diminishing stepsizes, which can be found in [10].

Lemma 4.8 Under the assumptions of Lemma 4.7, we have
\[ \lim_{m \to \infty} \| \nabla f(\theta^{(m)}, \lambda) \|_2 = 0. \]

The statement of this theorem is that for the full gradient method with diminishing stepsizes the gradient norms cannot stay bounded away from zero. The following result characterizes more precisely the convergence property of Algorithm 3.3.

Lemma 4.9 Under the assumptions stated in Lemma 4.7, we have
\[ \lim_{m \to \infty} \| \nabla f(\bar{\theta}^{(m)}, \lambda^{(m)}) \|_2 = 0. \]

Our main result is listed next.

Theorem 4.10 Under the assumptions stated in Lemma 4.7, we have
\[ \lim_{m \to \infty} \| \nabla f(\bar{\theta}^{(m)}, \lambda^{(m)}) \|_2 = 0. \]

We cannot show that \{\theta^{(m)}\}’s converge (standard convergence proofs are also unable to show such a stronger statement). For this reason Theorem 4.10 does not immediately follow from Lemma 4.9 together with Theorem 4.6. The statement of Theorem 4.10 would easily follow from Lemma 4.9 if convergence of \{\theta^{(m)}\} is established and the gradient being continuous.

Considering the cases \eta^{(m)} = O\left(\frac{1}{m}\right) and \alpha^{(m)} = O\left(\frac{1}{m^k}\right). We show in Appendix B that the set of sufficient and necessary conditions to satisfy the assumptions of Theorem 4.6 are \( h > 1 \) and \( k \geq 1 \). The set of sufficient and necessary conditions to satisfy the assumptions of Lemma 4.7 are \( h > 2 \) and \( k \geq 1 \). For example, we can pick \eta^{(m)} = O\left(\frac{1}{m}\right) and \alpha^{(m)} = O\left(\frac{1}{m^{1.01}}\right) to achieve the above convergence result in Theorem 4.10.

5 Convex Case

In this section, we discuss the convergence of a convex objective function. We stress that even if \( l_i(\cdot) \)'s are convex, the overall \( f_i(\cdot) \)'s are not convex. However, if the \( l_2 \) penalty coefficient \( c_2 \) is large enough, then \( f_i(\cdot) \)'s become convex since the regularization term prevails. We have the following extra assumption for the convexity of our model.

Assumption 5.1 (strong convexity). The objective function \( f_i \) is strongly convex in \( \theta \) for every \( i \). It implies that there exists constant \( c > 0 \) such that
\[
f_i(X_i : \tilde{\theta}, \lambda) \geq f_i(X_i : \tilde{\theta}, \lambda) + \nabla f_i(X_i : \tilde{\theta}, \lambda)^T (\bar{\theta} - \tilde{\theta}) + \frac{1}{2} c \| \bar{\theta} - \tilde{\theta} \|_2^2, \forall \bar{\theta}, \tilde{\theta}, i.
\]

This inequality is a common result for strongly convex functions. Unfortunately, this convexity assumption is not met by most popular neural network models unless \( c_2 \) is large. In this section, we assume that \( c_2 \) is large enough to force such strong convexity.

5.1 Convex Objective Convergence Analysis

Here we perform the convergence analysis with the assumption that \( f(\theta, \lambda) \) is jointly convex in \( \theta \) for any fixed \( \lambda \) (which is implied by convexity of each \( f_i(X_i : \theta, \lambda) \)). The main result establishes a sublinear convergence rate if \( \eta^{(m)} \) are selected as \( O\left(\frac{1}{m}\right) \).

Theorem 5.2 Under Assumptions 4.1, 4.2, 4.3 and 5.1, suppose the stepsizes satisfy that, for all \( m \),
\[ \eta^{(m)} = \frac{\zeta}{\vartheta + m} \text{ for some } \zeta > \frac{1}{c\vartheta} \text{ and } \vartheta > 0. \]
If \( \alpha^{(m)} \) satisfies
\[ \sum_{m=1}^{\infty} \alpha^{(m)} \ln(m) < \infty, \]
then the optimality gap satisfies
\[ \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\vartheta^*, \bar{\lambda}) \leq \frac{\vartheta}{\vartheta + m}, \text{ where } \vartheta := \max \left\{ \frac{\zeta^2 (NM + M_1)}{2(\zeta c c_1 - 1)} \right\}, \right\} \]

\[ (\vartheta + 1)(\bar{f}(\theta^{(1)}, \lambda^{(1)}) - \bar{f}(\vartheta^*, \bar{\lambda})) \]
Note that \( \theta^* \) is the optimal \( \theta \) value with respect to the fixed \( \bar{\lambda} \). Point \((\theta^*, \bar{\lambda})\) is the point where \( \bar{f} \) attains its minimum. We learn from Theorem 5.2 that decreasing the stepsizes \( \eta^{(m)} \) sublinearly (i.e. at a rate \( O(1/m) \)) ensures the sublinear convergence to the optimal value. We also observe that the initial point determines the initial optimality gap, which appears prominently in the second term defining \( \nu \). With an appropriate initialization phase, we can easily diminish the role played by this term. Related discussions about the choice of constant parameters to ensure a sublinear convergence rate and the initialization can be found in Bottou et al. [6].

Considering the cases \( \eta^{(m)} = O(\frac{1}{m}) \) and \( \alpha^{(m)} = O(\frac{1}{m^2}) \). We show in Appendix B that the set of sufficient and necessary condition to satisfy the assumptions of Theorem 5.2 is \( h > 1 \). For example, we can pick \( \eta^{(m)} = O(\frac{1}{m}) \) and \( \alpha^{(m)} = O(\frac{1}{m^2}) \) to achieve the above convergence result.

6 Computational Experiments

We conduct the computational experiments with Theano and Lasagne on a Linux server with an Nvidia Titan-X GPU. We use MNIST [15], CIFAR-10 [13] and Network Intrusion (NI) [11] datasets to compare the performance between DBN and the original BN algorithm. For the MNIST dataset, we use a four-layer fully connected FNN \((784 \times 300 \times 300 \times 10)\) with the ReLU activation function and for the NI dataset, we use a four-layer fully connected FNN \((784 \times 50 \times 50 \times 10)\) with the ReLU activation function. For the CIFAR-10 dataset, we use a reasonably complex CNN network that has a structure of \((\text{Conv-Conv-MaxPool-Dropout-Conv-Conv-MaxPool-Dropout-FC-Dropout-FC})\), where all four convolution layers and the first fully connected layers are batch normalized. We use the softmax loss function and \( l_2 \) regularization with for all three models. All the trainable parameters are randomly initialized before training. For all 3 datasets, we use the standard epoch/minibatch setting with the minibatch size of 100, i.e., we do not compute the full gradient and the statistics are over the minibatch. We use AdaGrad [8] to update the learning rates \( \eta^{(m)} \) for trainable parameters, starting from 0.01.

We use two different strategies to decide the values of \( \alpha^{(m)} \) in DBN: constant values of \( \alpha^{(m)} \) and diminishing \( \alpha^{(m)} \) where \( \alpha^{(m)} = 1/m \) and \( \alpha^{(m)} = 1/m^2 \). We test the choices of constant \( \alpha^{(m)} \in \{1, 0.75, 0.5, 0.25, 0.1, 0.01, 0.001, 0\} \).

\[ \text{Figure 2: Comparison of predicted accuracy on test datasets for different choices of } \alpha^{(m)}. \text{ From left to right are FNN on MNIST, FNN on NI and CNN on CIFAR-10.} \]

\[ \text{Figure 3: Comparison of predicted accuracy on test datasets for the most efficient choices of } \alpha^{(m)}. \text{ From left to right are FNN on MNIST, FNN on NI and CNN on CIFAR-10.} \]
Figure 4: Comparison of the convergence of the loss function value on validation set for different choices of $\alpha^{(m)}$. From left to right are FNN on MNIST, FNN on NI and CNN on CIFAR-10.

We test all the choices of $\alpha^{(m)}$ with the performances presented in Figure 2. Figure 2 shows that all the non-zero choices of $\alpha^{(m)}$ converge properly. The algorithms converge without much difference even when $\alpha^{(m)}$ in DBN is very small, e.g., $1/m^2$. However, if we select $\alpha^{(m)} = 0$, the algorithm is erratic. Besides, we observe that all the non-zero choices of $\alpha^{(m)}$ converge at a similar rate. The fact that DBN keeps the batch normalization layer stable with a very small $\alpha^{(m)}$ suggests that the BN parameters do not have to be depended on the latest minibatch, i.e. the original BN.

We compare a selected sets of the most efficient choices of $\alpha^{(m)}$ in Figures 3 and 4. They show that DBN with $\alpha^{(m)} < 1$ is more stable than the original BN algorithm. The variances with respect to epochs of the DBN algorithm are obviously smaller than those of the original BN algorithm’s in each figure.

Table 1: Best results for different choices of $\alpha^{(m)}$ on each dataset, showing top three with heat map.

| Model | MNIST (200 epochs) | NI (100 epochs) | CIFAR-10 (50 epochs) |
|-------|------------------|----------------|---------------------|
|       | Epoch | Test Error | Epoch | Test Error | Epoch | Test Error |
| $\alpha^{(m)} = 1$ | 52 | 2.70% | 58 | 7.69% | 45 | 17.31% |
| $\alpha^{(m)} = 0.75$ | 69 | 1.91% | 67 | 7.37% | 49 | 17.03% |
| $\alpha^{(m)} = 0.5$ | 69 | 1.84% | 80 | 7.46% | 44 | 17.11% |
| $\alpha^{(m)} = 0.25$ | 46 | 1.91% | 38 | 7.32% | 43 | 17.00% |
| $\alpha^{(m)} = 0.1$ | 48 | 1.90% | 66 | 7.36% | 48 | 17.10% |
| $\alpha^{(m)} = 0.01$ | 51 | 1.94% | 74 | 7.47% | 43 | 16.82% |
| $\alpha^{(m)} = 0.001$ | 48 | 1.95% | 98 | 7.43% | 46 | 16.28% |
| $\alpha^{(m)} = 1/m$ | 39 | 2.10% | 78 | 7.45% | 37 | 17.26% |
| $\alpha^{(m)} = 1/m^2$ | 53 | 2.00% | 74 | 7.59% | 44 | 17.23% |
| $\alpha^{(m)} = 0$ | 199 | 24.27% | 53 | 26.09% | 2 | 79.34% |

Table 1 shows the best result obtained from each choice of $\alpha^{(m)}$. Most importantly, it suggests that the choices of $\alpha^{(m)} = 1/m$ and $1/m^2$ perform better than the original BN algorithm. Besides, all the constant less-than-one choices of $\alpha^{(m)}$ perform better than the original BN, showing the importance of taking the mini-batch history into consideration for the update of the BN parameters. The BN algorithm in each figure converges to similar error rates on test datasets with different choices of $\alpha^{(m)}$ except for the $\alpha^{(m)} = 0$ case. Among all the models we tested, $\alpha^{(m)} = 0.25$ is the only one that performs top 3 for all three datasets, thus the most robust choice.

To summarize, our numerical experiments show that the DBN algorithm outperforms the original BN algorithm on the MNIST, NI and CIFAR-10 datasets with typical deep FNN and CNN models.

Future Directions. On the analytical side, we believe an extension to more than 2 layers is doable with significant augmentations of the notation. A stochastic gradient version is likely to be much more challenging to analyze. A second open question concerns more general activation functions. Instead of the linear activation function assumed in this paper, it would be interesting to analyze more common choices such as Sigmoid, ReLU, and Maxout.
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Supplementary material for “Convergence Analysis of Batch Normalization for Deep Neural Nets”

7 Appendix A: Proofs

7.1 Nomenclature

| Symbol | Explanation |
|--------|-------------|
| $\bar{L}$ | The Lipschitz constant |
| $\eta$ | Learning rate for gradient algorithm |
| $\alpha$ | Weight parameter for batch-normalization update |
| $\vec{x}$ | The vector for one record of input data |
| $y_j$ | The target output vector for the input record $\vec{x}_j$ |
| $N$ | Size of training set, number of all $\vec{x}$ in one epoch |
| $D_1$ | Dimension of the hidden layer |
| $z_j(1)$, $z_j(2)$ | $j^{th}$ entry of output of the first and second hidden layer, respectively |
| $n_1$, $n_2$ | Numbers of parameters in $\theta$ and $\lambda$, respectively |
| $\theta$ | Set of all trainable parameters updated by its gradient |
| $W^{(1)}$, $W^{(2)}$ | $\in \theta$, weights of linear transformation between layers |
| $\gamma_j(1)$, $\beta_j(1)$ | $\in \theta$, trainable parameters for batch-normalized output $y_j^{(1)}$ |
| $\lambda$ | Set of all batch normalization parameters determined by previous updates |
| $\mu_j$ | $\in \lambda$, mean of previous values of $z_j(1)$ |
| $\sigma_j$ | $\in \lambda$, standard deviation of previous values of $z_j(1)$ |
| $\epsilon_B$ | The offset for batch normalization transformation |

7.2 Preliminary Results

**Proposition 7.1** There exists a constant $M$ such that, for any $\theta$ and fixed $\lambda$, we have

$$\|\nabla \bar{f}(\theta, \lambda)\|_2^2 \leq M.$$ 

**Proof.** By Assumption 4.5, we know there exists $(\theta^*, \lambda^*)$ such that $\|\nabla \bar{f}(\theta^*, \lambda^*)\|_2 = 0$. Then we have

$$\|\nabla \bar{f}(\theta, \lambda)\|_2 = \|\nabla \bar{f}(\theta, \lambda)\|_2 - \|\nabla \bar{f}(\theta^*, \lambda^*)\|_2$$

$$= \|\nabla \bar{f}(\theta, \lambda)\|_2 - \|\nabla \bar{f}(\theta, \lambda^*)\|_2 + \|\nabla \bar{f}(\theta^*, \lambda^*)\|_2 - \|\nabla \bar{f}(\theta^*, \lambda^*)\|_2$$

$$\leq \|\nabla \bar{f}(\theta, \lambda) - \nabla \bar{f}(\theta, \lambda^*)\|_2 + \|\nabla \bar{f}(\theta^*, \lambda^*) - \nabla \bar{f}(\theta^*, \lambda^*)\|_2$$

$$\leq \sum_{i=1}^{N} \|\nabla f_i(X_i : \theta, \lambda) - \nabla f_i(X_i : \theta, \lambda^*)\|_2 + \sum_{i=1}^{N} \|\nabla f_i(X_i : \theta, \lambda^*) - \nabla f_i(X_i : \theta^*, \lambda^*)\|_2$$

$$\leq N \bar{L}(\|\lambda - \lambda^*\|_2 + \|\theta - \theta^*\|_2),$$

where the last inequality is by Assumption 4.1. We then have

$$\|\nabla \bar{f}(\theta, \lambda)\|_2^2 \leq N^2 \bar{L}^2(\|\lambda - \lambda^*\|_2 + \|\theta - \theta^*\|_2)^2 \leq M,$$

because sets $P$ and $Q$ are compact by Assumption 4.2. □

**Proposition 7.2** We have

$$f_i(X : \hat{\theta}, \lambda) \leq f_i(X : \hat{\theta}, \lambda) + \nabla f_i(X : \hat{\theta}, \lambda)^T(\hat{\theta} - \hat{\theta}) + \frac{1}{2} \bar{L}\|\hat{\theta} - \hat{\theta}\|_2^2, \forall \hat{\theta}, \hat{\theta}, X.$$ 

**Proof.** This is a known result of the Lipschitz-continuous condition that can be found in [6]. We have this result together with Assumption 4.1.
7.3 Proof of Theorem 4.6

Lemma 7.3 When \( \sum_{m=1}^{\infty} \alpha(m) < \infty \) and \( \sum_{m=1}^{\infty} \sum_{n=1}^{m} \alpha(m) \eta(n) < \infty \),
\[
\tilde{\mu}_j^{(m)} := \frac{\mu_j^{(m)}}{(1 - \alpha(1))(1 - \alpha(2)) \ldots (1 - \alpha(m))}
\]
is a Cauchy series.

Proof: By Algorithm [1] we have
\[
\mu_j^{(m)} = \alpha(m) \frac{1}{N} \sum_{i=1}^{N} k(1) W_{1,j}^{(m)} X_i + (1 - \alpha(m)) \mu_j^{(m-1)}.
\] (3)

We define \( \tilde{\alpha}(m) := \frac{\alpha(m)}{(1 - \alpha(1))(1 - \alpha(2)) \ldots (1 - \alpha(m))} \) and \( \Delta W_{1,j}^{(m)} := W_{1,j}^{(m)} - W_{1,j}^{(m-1)} \). After dividing (3) by \( (1 - \alpha(1))(1 - \alpha(2)) \ldots (1 - \alpha(m)) \), we obtain
\[
\tilde{\mu}_j^{(m)} = \tilde{\alpha}(m) k(1) \frac{1}{N} \sum_{i=1}^{N} W_{1,j}^{(m)} X_i + \tilde{\mu}_j^{(m-1)}.
\]

Then we have
\[
|\tilde{\mu}_j^{(m+1)} - \tilde{\mu}_j^{(m)}| = \tilde{\alpha}(m) |k(1)| \frac{1}{N} \sum_{i=1}^{N} |W_{1,j}^{(m)} X_i|
\]
\[
= \tilde{\alpha}(m) |k(1)| \frac{1}{N} \sum_{i=1}^{N} \sum_{n=1}^{m} \Delta W_{1,j}^{(n)} X_i
\]
\[
= \tilde{\alpha}(m) |k(1)| \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \left[ \eta(n) \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right] \right)
\]
\[
\leq \tilde{\alpha}(m) |k(1)| \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \left[ \eta(n) \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right] \right)
\]
\[
\leq \tilde{\alpha}(m) |k(1)| \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \left[ \eta(n) \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right] \right)
\]
\[
\leq \tilde{\alpha}(m) |k(1)| \frac{1}{N} \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \left[ \eta(n) \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right] \right) \right)
\]
\[
\leq \tilde{\alpha}(m) |k(1)| \frac{1}{N} \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \left[ \eta(n) \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right] \right) \right)
\]
\[
\leq \tilde{\alpha}(m) |k(1)| \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \left[ \eta(n) \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right] \right) \right)
\]
\[
\leq \tilde{\alpha}(m) \sum_{n=1}^{m} \eta(n) \left| k(1) \sum_{i=1}^{N} \left( \sum_{l=1}^{N} \nabla W_{1,j}^{(n)} f_l(X_l : \theta(n), \lambda(n)) \right) \cdot X_i \right| \right]
\]
\[
\leq \tilde{\alpha}(m) \sum_{n=1}^{m} \eta(n) \tilde{M}_{L,M}.
\] (6)
Equation (4) is due to
\[ W_{1,i,j}^{(m)} = \sum_{n=1}^{m} \Delta W_{1,i,j}^{(n)}. \]

Therefore,
\[ |\tilde{\mu}_j^{(p)} - \tilde{\mu}_j^{(q)}| \leq \tilde{M}_{L,M} \cdot \sum_{m=p}^{q} \alpha^{(m)} \sum_{n=1}^{m} \eta^{(n)} = \tilde{M}_{L,M} \cdot \sum_{m=p}^{q} \sum_{n=1}^{m} \tilde{\alpha}^{(m)} \eta^{(n)}. \]

It remains to show that
\[ \sum_{m=1}^{\infty} \alpha^{(m)} < \infty, \]
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)} < \infty, \]
implies the convergence of \( \{ \tilde{\mu}^{(m)} \} \). By (8), we have
\[ \Pi_{m=1}^{\infty} (1 - \alpha^{(m)}) > 0, \]
since
\[ \ln(\Pi_{m=1}^{\infty} (1 - \alpha^{(m)})) = \sum_{m=1}^{\infty} \ln(1 - \alpha^{(m)}) > \sum_{m=1}^{\infty} -\alpha^{(m)} > -\infty. \]

It is also easy to show that there exists \( C \) and \( M_c \) such that for all \( m \geq M_c \), we have
\[ (1 - \alpha^{(1)})(1 - \alpha^{(2)}) \ldots (1 - \alpha^{(m)}) \geq C. \]

Therefore,
\[ \lim_{m \to \infty} (1 - \alpha^{(1)})(1 - \alpha^{(2)}) \ldots (1 - \alpha^{(m)}) \geq C. \]

Thus the following holds:
\[ \tilde{\alpha}^{(m)} \leq \frac{1}{C} \alpha^{(m)} \]

and
\[ \sum_{m=p}^{q} \sum_{n=1}^{m} \tilde{\alpha}^{(m)} \eta^{(n)} \leq \frac{1}{C} \sum_{m=p}^{q} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)}. \]

From (9) and (12) it follows that the sequence \( \{ \tilde{\mu}_j^{(m)} \} \) is a Cauchy series. \( \square \)

**Lemma 7.4** Since \( \{ \tilde{\mu}_j^{(m)} \} \) is a Cauchy series, \( \{ \mu_j^{(m)} \} \) is a Cauchy series.

**Proof.** We know that
\[ \mu_j^{(m)} = \tilde{\mu}_j^{(m)} (1 - \alpha^{(1)}) \ldots (1 - \alpha^{(m)}). \]

Since
\[ \lim_{m \to \infty} \tilde{\mu}_j^{(m)} \to \tilde{\mu}_j \]
and
\[ \lim_{m \to \infty} (1 - \alpha^{(1)}) \ldots (1 - \alpha^{(m)}) \to \tilde{C}, \]
we have
\[ \lim_{m \to \infty} \mu_j^{(m)} \to \tilde{\mu}_j \cdot \tilde{C}. \]

Thus \( \mu_j^{(m)} \) is a Cauchy series. \( \square \)
Lemma 7.5 If $\sum_{m=1}^{\infty} \alpha^{(m)} < \infty$ and $\sum_{m=1}^{\infty} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)} < \infty$, \{\sigma_j^{(m)}\} is a Cauchy series.

Proof. We define $\tilde{\sigma}_j^{(m)} := \tilde{\sigma}_j^{(m)} (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})$. Then we have

$$|\tilde{\sigma}_j^{(m+1)} - \tilde{\sigma}_j^{(m)}| = \tilde{\sigma}_j^{(m)} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( k^{(1)} W_{1,j,i}^{(m)} X_i - \mu_j^{(m)} \right)^2}$$

$$= \tilde{\sigma}_j^{(m)} \frac{1}{\sqrt{N}} \sqrt{\sum_{i=1}^{N} \left( k^{(1)} W_{1,j,i}^{(m)} X_i - \mu_j^{(m)} \right)^2}$$

$$= \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( W_{1,j,i}^{(m)} X_i - \mu_j^{(m)} \right)^2 \right)^{1/2}.$$

Since \{\mu_j^{(m)}\} is convergent, there exists $c_1$, $c_2$ and $N_1$ such that for any $m > N_1$, $-\infty < c_1 < \mu_j^{(m)} < c_2 < \infty$. Therefore,

$$|\tilde{\sigma}_j^{(m+1)} - \tilde{\sigma}_j^{(m)}| \leq \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \max \left\{ \left( \sum_{i=1}^{N} \left( W_{1,j,i}^{(m)} X_i - c_1 \right)^2 \right)^{1/2}, \left( \sum_{i=1}^{N} \left( W_{1,j,i}^{(m)} X_i - c_2 \right)^2 \right)^{1/2} \right\}.$$  \hfill (13)

For any $\tilde{C} \in \left\{ \frac{c_1}{k^{(1)}}, \frac{c_2}{k^{(1)}} \right\}$, we have

$$|\tilde{\sigma}_j^{(m+1)} - \tilde{\sigma}_j^{(m)}| \leq \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( W_{1,j,i}^{(m)} X_i - \tilde{C} \right)^2 \right)^{1/2}$$  \hfill (14)

$$\leq \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( |W_{1,j,i}^{(m)} X_i| + |\tilde{C}| \right)^2 \right)^{1/2}$$  \hfill (15)

$$= \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \Delta W_{1,j,i}^{(n)} \right) |X_i| + |\tilde{C}| \right)^{1/2}$$

$$\leq \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \eta^{(n)} \sum_{l=1}^{N} \nabla W_{1,j,i} \cdot f_l(X_l : \theta^{(n)}, \lambda^{(n)}) \cdot X_i \right) + |\tilde{C}| \right)^{1/2}$$  \hfill (16)

$$\leq \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \eta^{(n)} \sum_{l=1}^{N} \nabla W_{1,j,i} \cdot f_l(X_l : \theta^{(n)}, \lambda^{(n)}) \cdot \|X_i\| + |\tilde{C}| \right) \right)^{1/2}$$  \hfill (17)

$$\leq \tilde{\sigma}_j^{(m)} \frac{|k^{(1)}|}{\sqrt{N}} \left( \sum_{i=1}^{N} \left( \sum_{n=1}^{m} \eta^{(n)} (2NLM \|X_i\|_2) + |\tilde{C}| \right) \right)^{1/2}$$  \hfill (18)
\[ \leq \tilde{\alpha}^{(m)} |k(1)| \cdot \sqrt{N} \cdot \left( \tilde{M}_{L,M} \sum_{n=1}^{m} \eta^{(n)} + |\bar{C}| \right)^2 \] (19)

\[ = \tilde{\alpha}^{(m)} |k(1)| \cdot \left( \tilde{M}_{L,M} \sum_{n=1}^{m} \eta^{(n)} + |\bar{C}| \right) \] (20)

Inequality (14) is by plugging \( \bar{C} \in \{ c_1 k(1), c_2 k(1) \} \) into (13). Inequality (15) is by the following fact:

\[ \sqrt{\sum_{i=1}^{n} (a_i - c)^2} \leq \max \left\{ \sqrt{\sum_{i=1}^{n} (|a_i| - c)^2}, \sqrt{\sum_{i=1}^{n} (|a_i| + c)^2} \right\} = \sqrt{\sum_{i=1}^{n} (|a_i| + c)^2}, \] (21)

where \( b \) and \( a_i \) for every \( i \) are arbitrary real scalars. Besides, (21) is due to

\[-2a_i c \leq \max \{-2|a_i| c, 2|a_i| c\}.\]

Inequalities (16), (17) and (18) follow from the square function being increasing for nonnegative numbers. Besides these facts, (18) is also by the same techniques we used in (5)-(6) where we bound the derivatives with the Lipschitz continuity in the following inequality:

\[ \left\| \sum_{l=1}^{N} \nabla_{W_{l,j}} f_l(X_l : \theta^{(n)}, \lambda^{(n)}) \right\| \leq 2N \bar{L} M. \]

Inequality (19) is by collecting the bounded terms into a single bound \( \tilde{M}_{L,M} \). Therefore,

\[ |\bar{\sigma}_j^{(q)} - \bar{\sigma}_j^{(p)}| \leq \sum_{m=p}^{q-1} |\bar{\sigma}_j^{(m+1)} - \bar{\sigma}_j^{(m)}| \leq \sum_{m=p}^{q-1} \tilde{\alpha}^{(m)} |k(1)| \cdot \left( \tilde{M}_{L,M} \sum_{n=1}^{m} \eta^{(n)} + |\bar{C}| \right) \] (22)

Using the similar methods in deriving (8) and (9), it can be seen that a set of sufficient conditions ensuring the convergence for \( \{ \bar{\sigma}_j^{(m)} \} \) is:

\[ \sum_{m=1}^{\infty} \alpha^{(m)} < \infty, \]

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)} < \infty. \]

Therefore, the convergence conditions for \( \{ \sigma_j^{(m)} \} \) are the same as for \( \{ \mu_j^{(m)} \} \). \( \square \)

It is clear that these lemmas establish the proof of Theorem 4.6.

7.4 Consequences of Theorem 4.6

**Proposition 7.6** Under the assumptions of Theorem 4.6 we have

\[ |\lambda^{(m)} - \bar{\lambda}|_{\infty} \leq a_m, \]

where

\[ a_m = M_1 \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} + M_2 \sum_{i=m}^{\infty} \alpha^{(i)} \]

and \( M_1 \) and \( M_2 \) are constants.
Proof. For the upper bound of $\sigma_j^{(m)}$, by (20), we have

$$|\tilde{\sigma}_j^{(q)} - \tilde{\sigma}_j^{(p)}| \leq \sum_{m=p}^{q-1} \tilde{\alpha}^{(m)}|k^{(1)}| \left( M_{L,M} \sum_{n=1}^{m} \eta^{(n)} + |\bar{C}| \right).$$

We define $\tilde{\alpha}_j := \frac{\tilde{\sigma}_j}{(1 - \alpha^{(1)})... (1 - \alpha^{(u)})...}$. Therefore,

$$|\sigma_j - \tilde{\alpha}_j^{(m)}| \leq \sum_{i=m}^{\infty} \tilde{\alpha}_j^{(i)}|k^{(1)}| \left( M_{L,M} \sum_{j=1}^{i} \eta^{(j)} + |\bar{C}| \right) \leq \frac{|k^{(1)}|}{C} \sum_{i=m}^{\infty} \tilde{\alpha}_j^{(i)} \left( M_{L,M} \sum_{j=1}^{i} \eta^{(j)} + |\bar{C}| \right).$$

The first inequality comes by substituting $p$ by $m$ and by taking $\lim$ as $q \to \infty$ in (22). The second inequality comes from (10). We then obtain,

$$\left| \sigma_j^{(m)} - \tilde{\sigma}_j \right| = (1 - \alpha^{(1)})... (1 - \alpha^{(m)}) \left| \frac{\sigma_j^{(m)}}{(1 - \alpha^{(1)})... (1 - \alpha^{(m)})} - \tilde{\sigma}_j \right| \leq (1 - \alpha^{(1)})... (1 - \alpha^{(m)}) \left| \frac{\sigma_j^{(m)}}{(1 - \alpha^{(1)})... (1 - \alpha^{(m)})} - \tilde{\sigma}_j \right| + \left| \tilde{\sigma}_j \right| \left( \frac{1 - \alpha^{(1)}}{(1 - \alpha^{(1)})... (1 - \alpha^{(m)})} - \tilde{\sigma}_j \right) \left| (1 - \alpha^{(1)})... (1 - \alpha^{(u)})... \right|$$

$$\leq \left| \sigma_j^{(m)} - \tilde{\sigma}_j^{(\infty)} \right| + \left| \tilde{\sigma}_j \right| \left( \frac{1 - \alpha^{(1)}}{(1 - \alpha^{(1)})... (1 - \alpha^{(m)})} - \tilde{\sigma}_j \right) \left| (1 - \alpha^{(1)})... (1 - \alpha^{(u)})... \right|$$

$$= \left| \sigma_j^{(m)} - \tilde{\sigma}_j^{(\infty)} \right| + \tilde{\sigma}_j \left( \frac{1 - \alpha^{(m+1)}}{(1 - \alpha^{(1)})... (1 - \alpha^{(u)})...} - 1 \right)$$

$$\leq \left| \tilde{\sigma}_j^{(m)} - \tilde{\sigma}_j^{(\infty)} \right| + \frac{\tilde{\sigma}_j}{C} (1 - \alpha^{(m+1)})... (1 - \alpha^{(u)})...$$

$$\leq \left| \tilde{\sigma}_j^{(m)} - \tilde{\sigma}_j^{(\infty)} \right| + \frac{\tilde{\sigma}_j}{C} \sum_{n=m+1}^{\infty} \alpha^{(n)}.$$

The second inequality is by $(1 - \alpha^{(1)})... (1 - \alpha^{(m)}) < 1$, the third inequality is by (10) and the last inequality can be easily seen by induction. By (23), we obtain

$$|\sigma_j - \sigma_j^{(m)}| = \lim_{M \to \infty} |\sigma_j^{(M)} - \sigma_j^{(m)}| \leq |\tilde{\sigma}_j - \tilde{\sigma}_j^{(m)}| + \frac{\tilde{\sigma}_j}{C} \sum_{n=m+1}^{\infty} \alpha^{(n)}. \quad (26)$$
Therefore, we have
\[
|\tilde{\sigma}_j - \sigma_j^{(m)}| \\
\leq |\sigma_j - \sigma_j^{(m)}| + \frac{\tilde{\sigma}_j}{C} \sum_{n=m+1}^{\infty} \alpha(n) \\
\leq \sum_{i=m}^{\infty} \alpha(i) |k^{(1)}| \cdot \left( M_{E,M} \sum_{j=1}^{i} \eta(j) + |\tilde{C}| \right) + \frac{\tilde{\sigma}_j}{C} \sum_{i=m+1}^{\infty} \alpha(i) \\
\leq \sum_{i=m}^{\infty} \frac{1}{C} \alpha(i) |k^{(1)}| \cdot \left( M_{E,M} \sum_{j=1}^{i} \eta(j) + |\tilde{C}| \right) + \frac{\tilde{\sigma}_j}{C} \sum_{i=m}^{\infty} \alpha(i) \\
\leq \sum_{i=m}^{\infty} \frac{1}{C} \alpha(i) |k^{(1)}| \cdot \left( M_{E,M} \sum_{j=1}^{i} \eta(j) + |\tilde{C}| \right) + \frac{\tilde{\sigma}_j}{C} \sum_{i=m}^{\infty} \alpha(i) \\
= \frac{M_{E,M}}{C} |k^{(1)}| \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha(i) \eta(j) + \left( \frac{1}{C} \tilde{\sigma}_j |k^{(1)}| |\tilde{C}| \right) \sum_{i=m}^{\infty} \alpha(i).
\]

The first inequality is by (26), the second inequality is by (22), the third inequality is by (11) and the fourth inequality is by adding the nonnegative term \( \frac{\tilde{\sigma}_j}{C} \alpha^{(m)} \) to the right-hand side.

For the upper bound of \( \mu_j^{(m)} \), we have
\[
|\mu_j^{(m)} - \tilde{\mu}_j| \\
= (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)}) \left| \frac{\mu_j^{(m)}}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} - \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} \right| \\
\leq (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)}) \left| \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} \right| + (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(\infty)}) \left| \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} \right| \\
\leq |\tilde{\mu}_j^{(m)} - \tilde{\mu}_j^{(\infty)}| + \left| \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} - \tilde{\mu}_j^{(\infty)} \right|.
\]

Let us define \( A_m := |\tilde{\mu}_j^{(m)} - \tilde{\mu}_j^{(\infty)}| \) and \( B_m := \left| \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} - \tilde{\mu}_j^{(\infty)} \right| \). Recall from Theorem 4.6 that \( \{\tilde{\mu}_j^{(m)}\} \) is a Cauchy series, by (7),
\[
|\tilde{\mu}_j^{(p)} - \tilde{\mu}_j^{(q)}| \leq M_{E,M} \cdot \sum_{m=p}^{q} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)}.
\]

Therefore, the first term in (28) is bounded by
\[
|\tilde{\mu}_j^{(m)} - \tilde{\mu}_j^{(\infty)}| \leq M_{E,M} \cdot \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha^{(i)} \eta^{(n)} < \infty.
\]

For the second term in (28), recall that \( C := (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)}) \cdots \). Then we have
\[
C \cdot \left| \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} - \tilde{\mu}_j^{(\infty)} \right| \\
= |\tilde{\mu}_j| (1 - \alpha^{(m+1)}) \cdots (1 - \alpha^{(\infty)}) \\
\leq \tilde{\mu}_j \sum_{i=m+1}^{\infty} \alpha^{(i)}.
\]
where the last inequality can be easily seen by induction. Therefore, the second term in (28) is bounded by
\[ \left| \frac{\hat{\mu}_j}{(1 - \alpha^{(1)})(1 - \alpha^{(2)})} - \bar{\mu}_j \right| \leq \frac{\hat{\mu}_j}{C} \sum_{i=m+1}^{\infty} \alpha^{(i)}. \] (30)
From these we obtain
\[ \left| \mu_j^{(m)} - \bar{\mu}_j \left| \leq \left| \mu_j^{(m)} - \bar{\mu}_j^{(\infty)} \right| + \left| 1 \right| \left( 1 - \alpha^{(1)}(1 - \alpha^{(2)}) \right) \right| \leq \bar{\mu}_j \sum_{i=m+1}^{\infty} \alpha^{(i)}. \] (31)
The first inequality is by (28) and the second inequality is by (29) and (30). Combining (27) and (31), we have that
\[ |\lambda^{(m)} - \bar{\lambda}|_\infty = \max(|\mu_j^{(m)} - \bar{\mu}_j|, |\sigma_j^{(m)} - \bar{\sigma}|_\infty) \leq M_1 \sum_{i=m}^{\infty} \sum_{n=1}^{n} \alpha^{(i)} \eta^{(i)} + M_2 \sum_{i=m}^{\infty} \alpha^{(i)}, \]
where \( M_1 \) and \( M_2 \) are constants defined as
\[ M_1 = \max\left( \frac{\bar{\mu}_j}{C}, \bar{\mu}_j \right) \] and
\[ M_2 = \max\left( \frac{\bar{\sigma}_j + |k^{(1)}|}{C}, \bar{\mu}_j \right). \]

**Proposition 7.7** Under the assumptions of Theorem 4.6,
\[-\nabla \bar{f}(\theta^{(m)}, \bar{\lambda})^T \cdot \nabla \bar{f}(\theta^{(m)}, \lambda^{(m)}) \leq -\| \nabla \bar{f}(\theta^{(m)}, \bar{\lambda}) \|^2 - \tilde{L} M \sqrt{\bar{\n} a} m, \]
where \( a_m \) is defined in Proposition 7.6.

**Proof.** For simplicity of the proof, let us define
\[ x^{(m)} := \nabla \bar{f}(\theta^{(m)}, \bar{\lambda}), \quad y^{(m)} := \nabla \bar{f}(\theta^{(m)}, \lambda^{(m)}). \]
We have
\[ |x^{(m)} - y^{(m)}|_\infty \leq \| x^{(m)} - y^{(m)} \|_2 = \tilde{L} \| \lambda^{(m)} - \bar{\lambda} \|_2 \leq \tilde{L} \sqrt{\bar{\n} a} \| \lambda^{(m)} - \bar{\lambda} \|_\infty \leq \tilde{L} \sqrt{\bar{\n} a} m, \] (32)
where \( \sqrt{\bar{\n} a} \) is the dimension of \( \lambda \). The second inequality is by Assumption 4.1 and the fourth inequality is by Proposition 7.6. Inequality (32) implies that for all \( m \) and \( i \), we have
\[ |x_i^{(m)} - y_i^{(m)}| \leq \tilde{L} \sqrt{\bar{\n} a} m. \]
It remains to show
\[ -\sum_{i} y_i^{(m)} x_i^{(m)} \leq -\sum_{i} x_i^{(m)} + \tilde{L} M \sqrt{\bar{\n} a} m, \forall i, m. \] (33)
This is established by the following four cases.

1) If \( x_i^{(m)} \geq 0, x_i^{(m)} - y_i^{(m)} \geq 0, \) then \( x_i^{(m)} \leq \tilde{L} \sqrt{\bar{\n} a} m + y_i^{(m)}. \) Thus \( -x_i^{(m)} y_i^{(m)} \geq -x_i^{(m)} + \tilde{L} M \sqrt{\bar{\n} a} m \) by Proposition 7.1.
2) If \( x_i^{(m)} \geq 0, x_i^{(m)} - y_i^{(m)} \leq 0, \) then \( x_i^{(m)} \geq y_i^{(m)} - x_i^{(m)} \leq x_i^{(m)} - y_i^{(m)} \) and \( -x_i^{(m)} y_i^{(m)} \geq -x_i^{(m)} y_i^{(m)} \).
3) If \( x_i^{(m)} < 0, x_i^{(m)} - y_i^{(m)} \geq 0, \) then \( x_i^{(m)} \geq y_i^{(m)} - x_i^{(m)} \leq x_i^{(m)} - y_i^{(m)} \) and \( -x_i^{(m)} y_i^{(m)} \geq -x_i^{(m)} y_i^{(m)} \).
4) If \( x_i^{(m)} < 0, x_i^{(m)} - y_i^{(m)} \leq 0, \) then \( y_i^{(m)} - x_i^{(m)} \leq \tilde{L} \sqrt{\bar{\n} a} m, y_i^{(m)} x_i^{(m)} - x_i^{(m)} \geq \tilde{L} \sqrt{\bar{\n} a} m x_i^{(m)} \) and \( -y_i^{(m)} x_i^{(m)} \leq -x_i^{(m)} - \tilde{L} \sqrt{\bar{\n} a} m x_i^{(m)} \). The last inequality is by Proposition 7.1.
All these four cases yield (33). \( \square \)
Proposition 7.8 Under the assumptions of Theorem 4.6 we have

\[ f(\theta^{(m+1)}, \lambda) \leq f(\theta^{(m)}, \lambda) - \eta^{(m)} \|\nabla f(\theta^{(m)}, \lambda)\|_{\mathbb{H}}^2 + \eta^{(m)} \hat{L}M \sqrt{\eta^{(m)} \lambda} + \frac{1}{2} \eta^{(m)} \lambda \cdot \hat{L}M, \]

where \( \hat{L} \) is a constant and \( \lambda \) is defined in Proposition 7.6.

Proof. By Proposition 7.2,

\[ f_i(X_i : \theta, \lambda) \leq f_i(X_i : \hat{\theta}, \lambda) + \nabla f_i(X_i : \hat{\theta}, \lambda)^T (\hat{\theta} - \theta) + \frac{1}{2} L \|\hat{\theta} - \theta\|_{\mathbb{H}}^2. \]

Therefore, we can sum it over the entire training set from \( i = 1 \) to \( N \) to obtain

\[ f(\theta, \lambda) \leq f(\hat{\theta}, \lambda) + \nabla \hat{f}(\hat{\theta}, \lambda)^T (\hat{\theta} - \theta) + \frac{N}{2} \bar{L} \|\hat{\theta} - \theta\|_{\mathbb{H}}^2. \tag{34} \]

In Algorithm 1, we define the update of \( \theta \) in the following full gradient way:

\[ \theta^{(m+1)} := \theta^{(m)} - \eta^{(m)} \cdot \sum_{i=1}^{N} \nabla f_i(X_i : \theta^{(m)}, \lambda^{(m)}), \]

which implies

\[ \theta^{(m+1)} - \theta^{(m)} = -\eta^{(m)} \cdot \nabla \hat{f}(\theta^{(m)}, \lambda^{(m)}). \tag{35} \]

By (35) we have \( \hat{\theta} - \hat{\theta} = \theta^{(m+1)} - \theta^{(m)} = -\eta^{(m)} \nabla \hat{f}(\theta^{(m)}, \lambda^{(m)}). \) We now substitute \( \hat{\theta} := \theta^{(m+1)}, \)
\( \hat{\theta} := \theta^{(m)} \) and \( \lambda := \lambda \) into (34) to obtain

\[ f(\theta^{(m+1)}, \lambda) \]

\[ \leq \hat{f}(\theta^{(m)}, \lambda) - \eta^{(m)} \nabla \hat{f}(\theta^{(m)}, \lambda)^T \nabla \hat{f}(\theta^{(m)}, \lambda) + \eta^{(m)} \lambda \cdot \frac{N \hat{L} \|\nabla \hat{f}(\theta^{(m)}, \lambda)\|_{\mathbb{H}}^2}{2} \]

\[ \leq \hat{f}(\theta^{(m)}, \lambda) - \eta^{(m)} \nabla \hat{f}(\theta^{(m)}, \lambda)^T \nabla \hat{f}(\theta^{(m)}, \lambda) + \eta^{(m)} \lambda \cdot \frac{N \hat{L} \|\nabla \hat{f}(\theta^{(m)}, \lambda)\|_{\mathbb{H}}^2}{2} \]

\[ \leq \hat{f}(\theta^{(m)}, \lambda) - \eta^{(m)} \left( \|\nabla \hat{f}(\theta^{(m)}, \lambda)\|_{\mathbb{H}}^2 + \hat{L}M \sqrt{\eta^{(m)} \lambda} \right) + \frac{1}{2} \eta^{(m)} \lambda \cdot N \hat{L} \]

\[ = \hat{f}(\theta^{(m)}, \lambda) - \eta^{(m)} \|\nabla \hat{f}(\theta^{(m)}, \lambda)\|_{\mathbb{H}}^2 + \eta^{(m)} \lambda \cdot N \hat{L} \sqrt{\eta^{(m)} \lambda} + \frac{1}{2} \eta^{(m)} \lambda \cdot N \hat{L}. \]

The first inequality is by plugging (35) into (34), the second inequality comes from Proposition 7.1 and the third inequality comes from Proposition 7.7.

7.5 Proof of Theorem 4.10

Here we show Theorem 4.10 as the consequence of Theorem 4.6 and Lemmas 4.7, 4.8, 4.9 and 4.9.

7.5.1 Proof of Lemma 4.7

Here we show Lemma 4.7 as the consequence of Lemmas 7.9, 7.10 and 7.11.

Lemma 7.9

\[ \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha(i) \eta^{(n)} < \infty \]

and

\[ \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha(n) < \infty \]

is a set of sufficient condition to ensure

\[ \sum_{m=1}^{\infty} |\sigma_j - \sigma_j^{(m)}| < \infty, \forall j. \]
Proof. By plugging (26) and (24) into (37), we have the following for all \( j \):

\[
\sum_{m=1}^{\infty} \left| \bar{\sigma}_j - \sigma_j^{(m)} \right| \\
\leq \sum_{m=1}^{\infty} \left( \left| \bar{\sigma}_j - \sigma_j^{(m)} \right| + \frac{\bar{\sigma}_j}{C} \sum_{n=m+1}^{\infty} \alpha^{(n)} \right) \\
\leq \sum_{m=1}^{\infty} \left[ \frac{|k^{(1)}|}{C} \sum_{i=m}^{\infty} \alpha^{(i)} \left( \tilde{M}_{L,M} \sum_{j=1}^{i} \eta^{(j)} + |C| \right) \right] + \frac{\bar{\sigma}_j}{C} \sum_{n=m+1}^{\infty} \alpha^{(n)} \right) \\
\leq \frac{|k^{(1)}| \tilde{M}_{L,M}}{C} \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \alpha^{(i)} \sum_{j=1}^{i} \eta^{(j)} + \frac{\bar{\sigma}_j + |k^{(1)}||C|}{C} \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \alpha^{(n)}.
\]

(38)

It is easy to see that the following conditions are sufficient for the right-hand side of (38) to be finite:

\[
\sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{\infty} \alpha^{(i)} \eta^{(n)} < \infty
\]

and

\[
\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha^{(n)} < \infty.
\]

Therefore, we obtain

\[
\sum_{m=1}^{\infty} \left| \bar{\sigma}_j - \sigma_j^{(m)} \right| < \infty, \forall j.
\]

□

Lemma 7.10 Under Assumption 4.4,

\[
\sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{\infty} \alpha^{(i)} \eta^{(n)} < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha^{(n)} < \infty
\]

is a set of sufficient conditions to ensure

\[
\limsup_{M \to \infty} \sum_{m=1}^{M} \left| \tilde{f}(\theta^{(m)}, \lambda^{(m)}) - \tilde{f}(\bar{\theta}(m), \bar{\lambda}) \right| < \infty.
\]

Proof. By Assumption 4.4, we have

\[
||l_i(x) - l_i(y)|| \leq \tilde{M} ||x - y|| \leq \tilde{M} \sum_{i=1}^{D} |x_i - y_i|.
\]

(39)
By the definition of \( f_i(\cdot) \), we then have
\[
\sum_{m=1}^{\infty} \left| \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \check{f}(\theta^{(m)}, \check{\lambda}) \right| \\
= \sum_{m=1}^{\infty} \left| \sum_{i=1}^{N} \left( l_i(X_i : \theta^{(m)}, \lambda^{(m)}) + c_2 \| \theta^{(m)} \|_2^2 \right) - \sum_{i=1}^{N} \left( l_i(X_i : \theta^{(m)}, \check{\lambda}) + c_2 \| \theta^{(m)} \|_2^2 \right) \right| \\
= \sum_{m=1}^{\infty} \left| \sum_{i=1}^{N} \left( l_i(X_i : \theta^{(m)}, \lambda^{(m)}) - l_i(X_i : \theta^{(m)}, \check{\lambda}) \right) \right| \\
\leq \sum_{m=1}^{\infty} \sum_{i=1}^{N} \left( l_i(X_i : \theta^{(m)}, \lambda^{(m)}) - l_i(X_i : \theta^{(m)}, \check{\lambda}) \right) \\
\leq M_2 \sum_{m=1}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{D} \sum_{j=1}^{D} \left| \frac{k^{(1)}W_{1,j}^{(m)}X_i - \mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon_B} - \frac{k^{(1)}W_{1,j}^{(m)}X_i - \bar{\mu}_j}{\bar{\sigma}_j + \epsilon_B} \right| \\
= M_2 \sum_{m=1}^{\infty} \sum_{i=1}^{N} \left( \sum_{j=1}^{D} \left| \frac{k^{(1)}W_{1,j}^{(m)}X_i}{\sigma_j^{(m)} + \epsilon_B} - \frac{1}{\bar{\sigma}_j + \epsilon_B} \right| + \frac{\bar{\mu}_j}{\bar{\sigma}_j + \epsilon_B} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon_B} \right) \\
\leq M_2 \sum_{m=1}^{\infty} \sum_{i=1}^{N} \left( \sum_{j=1}^{D} \left| \frac{k^{(1)}W_{1,j}^{(m)}X_i}{\sigma_j^{(m)} + \epsilon_B} - \frac{1}{\bar{\sigma}_j + \epsilon_B} \right| + \frac{\bar{\mu}_j}{\bar{\sigma}_j + \epsilon_B} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon_B} \right) \\
\leq M_3 \sum_{m=1}^{\infty} \sum_{i=1}^{N} \left( \sum_{j=1}^{D} \left| \frac{\sigma_j - \sigma_j^{(m)}}{\epsilon_B (\bar{\sigma}_j + \epsilon_B)} \right| + \frac{\bar{\mu}_j}{\bar{\sigma}_j + \epsilon_B} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon_B} \right) \\
\leq M_3 \sum_{m=1}^{\infty} \sum_{i=1}^{N} \left( \sum_{j=1}^{D} \left| k^{(1)}W_{1,j}^{(m)}X_i \right| \frac{\sigma_j - \sigma_j^{(m)}}{\epsilon_B} \right) + N \frac{\bar{\mu}_j}{\bar{\sigma}_j + \epsilon_B} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon_B} \right). 
\]

The first inequality is by the Cauchy-Schwarz inequality, and the second one is by \((39)\). To show the finiteness of \((40)\), we only need to show the following two statements:

\[
\sum_{m=1}^{\infty} \sum_{i=1}^{N} \left| k^{(1)}W_{1,j}^{(m)}X_i \right| \left| \bar{\sigma}_j - \sigma_j^{(m)} \right| < \infty, \forall j 
\]

and

\[
\sum_{m=1}^{\infty} \left| \frac{\bar{\mu}_j}{\bar{\sigma}_j + \epsilon_B} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon_B} \right| < \infty, \forall j. 
\]

**Proof of** \((41)\): For all \( j \) we have
\[
\sum_{m=1}^{\infty} \sum_{i=1}^{N} \left| k^{(1)}W_{1,j}^{(m)}X_i \right| \left| \bar{\sigma}_j - \sigma_j^{(m)} \right| \\
\leq \sum_{m=1}^{\infty} \left| k^{(1)}|NDM\max_i X_i\| \frac{1}{\epsilon_B} \left| \bar{\sigma}_j - \sigma_j^{(m)} \right| \right| \\
= |k^{(1)}|NDM\max_i X_i\| \frac{1}{\epsilon_B} \sum_{m=1}^{\infty} \left| \bar{\sigma}_j - \sigma_j^{(m)} \right|. 
\]

The inequality comes from \(|W_{1,j}^{(m)}X_i| \leq D M \| X_i \|_2\), where \( D \) is the dimension of \( X_i \), and \( M \) is the element-wise upper bound for \( W_{1,j}^{(m)} \) in Assumption 4.2.
Finally, we invoke Lemma 7.3 to assert that \( \sum_{m=1}^{\infty} |\bar{\sigma}_j - \sigma_j^{(m)}| \) is finite.

**Proof of (42):** For all \( j \) we have

\[
\sum_{m=1}^{\infty} \left| \frac{\bar{\mu}_j}{\sigma_j + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right| \leq \sum_{m=1}^{\infty} \left| \frac{\bar{\mu}_j}{\sigma_j + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right| + \sum_{m=1}^{\infty} \left| \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right|. \tag{44}
\]

The first term in (44) is finite since \( \{\mu_j^{(m)}\} \) is a Cauchy series. For the second term, we know that there exists a constant \( M \) such that for all \( m \geq M \), \( \mu_j^{(m)} \leq \bar{\mu} + 1 \). This is also by the fact that \( \{\mu_j^{(m)}\} \) is a Cauchy series and it converges to \( \bar{\mu} \). Therefore, the second term in (44) becomes

\[
\sum_{m=1}^{M-1} \left| \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right| + \sum_{m=M}^{\infty} \left| \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right|. \tag{45}
\]

Noted that function \( f(\sigma) = \frac{1}{\sigma + \epsilon} \) is Lipschitz continuous since its gradient is bounded by \( \frac{1}{\epsilon^2} \).

Therefore we can choose \( \frac{1}{\epsilon^2} \) as the Lipschitz constant for \( f(\sigma) \). We then have the following inequality:

\[
\left| \frac{1}{\sigma_j + \epsilon} - \frac{1}{\sigma_j^{(m)} + \epsilon} \right| \leq \frac{1}{\epsilon^2} |\bar{\sigma}_j - \sigma_j^{(m)}|. \tag{46}
\]

Plugging (46) into (45), we obtain

\[
\sum_{m=1}^{M-1} \left| \frac{\mu_j^{(m)}}{\sigma_j + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right| + \sum_{m=M}^{\infty} \left| \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right|.
\]

where the first term is finite by the fact that \( M \) is a finite constant. We have shown the condition for the second term to be finite in Lemma 7.9. Therefore,

\[
\sum_{m=1}^{\infty} \left| \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} - \frac{\mu_j^{(m)}}{\sigma_j^{(m)} + \epsilon} \right| < \infty, \forall j.
\]

By (41) and (42), we have that the right-hand side of (40) is finite. It means that the left-hand side of (40) is finite. Thus,

\[
\sum_{m=1}^{\infty} \left| f(\theta^{(m)}, \lambda^{(m)}) - f(\bar{\theta}, \bar{\lambda}) \right| < \infty. \tag{40}
\]

**Lemma 7.11** If

\[
\sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{\infty} \alpha^{(i)}(n) < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha^{(n)} < \infty,
\]

then

\[
\lim_{M \to \infty} \sup_{m=1}^{M} \sum_{m=1}^{M} \eta^{(m)} \left\| \nabla f(\theta^{(m)}, \lambda^{(m)}) \right\|_2^2 < \infty.
\]
Proof. For simplicity of the proof, we define

\[ T^{(M)} := \sum_{m=1}^{M} \eta^{(m)} \| \nabla \bar{f}(\theta^{(m)}, \bar{\lambda}) \|_2^2, \]

\[ O^{(m)} := \bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^{(m)}, \lambda^{(m)}), \]

\[ \Delta_1^{(m+1)} := \bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^{(m+1)}, \bar{\lambda}), \]

\[ \Delta_2^{(m)} := \bar{f}(\theta^{(m+1)}, \bar{\lambda}) - \bar{f}(\theta^{(m)}, \bar{\lambda}), \]

where \( \bar{\lambda} \) is the converged value of \( \lambda \) in Theorem 4.6. Therefore,

\[ O^{(m)} = \Delta_1^{(m+1)} + \Delta_1^{(m)} + \Delta_2^{(m)} \leq |\Delta_1^{(m+1)}| + |\Delta_1^{(m)}| + \Delta_2^{(m)}. \] \hspace{1cm} (47)

By Proposition 7.8

\[ \Delta_2^{(m)} \leq -\eta^{(m)} \| \nabla \bar{f}(\theta^{(m)}, \bar{\lambda}) \|_2^2 + \eta^{(m)} \bar{L} M \sqrt{n_2} a_m + \frac{1}{2} (\eta^{(m)})^2 \cdot N \bar{L} M. \] \hspace{1cm} (48)

We sum the inequality (47) from 1 to \( K \) with respect to \( m \) and plug (48) into it to obtain

\[
\sum_{m=1}^{K} O^{(m)} \leq \sum_{m=1}^{K} |\Delta_1^{(m+1)}| + \sum_{m=1}^{K} |\Delta_1^{(m)}| - \sum_{m=1}^{K} \{ \eta^{(m)} \| \nabla \bar{f}(\theta^{(m)}, \bar{\lambda}) \|_2^2 \}
\]

\[ + \sum_{m=1}^{K} \eta^{(m)} \bar{L} M \sqrt{n_2} a_m + \sum_{m=1}^{K} \{ \frac{1}{2} (\eta^{(m)})^2 N \bar{L} M \}
\]

\[ = \sum_{m=1}^{K} |\Delta_1^{(m+1)}| + \sum_{m=1}^{K} |\Delta_1^{(m)}| - T^{(K)}
\]

\[ + \bar{L}^2 \sqrt{n_2} \cdot \sum_{m=1}^{K} \eta^{(m)} a_m + \sum_{m=1}^{K} \{ \frac{1}{2} (\eta^{(m)})^2 N \bar{L} M \}. \]

From this, we have:

\[
\limsup_{K \to \infty} T^{(K)} \leq \limsup_{K \to \infty} \frac{-1}{c_1} \left( \bar{f}(\theta^{(K)}, \lambda^{(K)}) - \bar{f}(\theta^{(1)}, \lambda^{(1)}) \right)
\]

\[ + \limsup_{K \to \infty} \frac{1}{c_1} \left( |\Delta_1^{(m+1)}| + |\Delta_1^{(m)}| \right)
\]

\[ + \limsup_{K \to \infty} \bar{L}^2 \sqrt{n_2} \sum_{m=1}^{K} \eta^{(m)} a_m
\]

\[ + \limsup_{K \to \infty} \frac{N \bar{L} K}{2c_1} \sum_{m=1}^{K} \eta^{(m)}^2. \] \hspace{1cm} (49)

Next we show that each of the four terms in the right-hand side of (49) is finite, respectively. For the first term,

\[
\limsup_{K \to \infty} \frac{-1}{c_1} \left( \bar{f}(\theta^{(K)}, \lambda^{(K)}) - \bar{f}(\theta^{(1)}, \lambda^{(1)}) \right) < \infty
\]

is by the fact that the parameters \( \{ \theta, \lambda \} \) are in compact sets, which implies that the image of \( f_i(\cdot) \) is in a bounded set.

For the second term, we showed its finiteness in Lemma 7.10.
For the third term, by (23), we have
\[
\limsup_{K \to \infty} \sum_{m=1}^{K} \eta^{(m)} a_m = \limsup_{K \to \infty} \sum_{m=1}^{K} \eta^{(m)} \left( K_1 \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} + K_2 \sum_{i=m}^{\infty} \alpha^{(i)} \right)
\]
\[
= K_1 \limsup_{K \to \infty} \sum_{m=1}^{K} \eta^{(m)} \left( \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} \right) + K_2 \limsup_{K \to \infty} \sum_{m=1}^{K} \eta^{(m)} \sum_{i=m}^{\infty} \alpha^{(i)}.
\]

The right-hand side of (50) is finite because
\[
\sum_{m=1}^{\infty} \eta^{(m)} \left( \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} \right) < \infty
\]
and
\[
\sum_{m=1}^{\infty} \eta^{(m)} \sum_{i=m}^{\infty} \alpha^{(i)} < \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \alpha^{(i)} < \infty.
\]

The second inequalities in (51) and (52) come from the stated assumptions of this lemma.

For the fourth term,
\[
\limsup_{K \to \infty} \frac{NLM}{2c} \sum_{m=1}^{K} \eta^{(m)}^2 < \infty
\]
holds, because we have \( \sum_{m=1}^{\infty} \eta^{(m)}^2 < \infty \) in Assumption 4.3. Therefore, \( T^{(\infty)} = \sum_{m=1}^{\infty} \eta^{(m)} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2^2 < \infty \) holds.

In Lemmas 7.9, 7.10 and 7.11 we show that \( \{ \sigma^{(m)} \} \) and \( \{ \mu^{(m)} \} \) are Cauchy series, hence Lemma 4.7 holds.

### 7.5.2 Proof of Lemma 4.8

This proof is similar to the the proof by Bertsekas and Tsitsiklis [5].

**Proof.** By Theorem 4.7 we have
\[
\limsup_{M \to \infty} \sum_{m=1}^{M} \eta^{(m)} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2^2 < \infty.
\]
If there exists a \( \epsilon > 0 \) and an integer \( \bar{m} \) such that
\[
\| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2 \geq \epsilon
\]
for all \( m \geq \bar{m} \), we would have
\[
\liminf_{M \to \infty} \sum_{m=\bar{m}}^{M} \eta^{(m)} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2^2 \geq \liminf_{M \to \infty} \epsilon^2 \sum_{m=\bar{m}}^{M} \eta^{(m)} = \infty
\]
which contradicts (53). Therefore, \( \liminf_{m \to \infty} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2 = 0 \). \( \square \)

### 7.5.3 Proof of Lemma 4.9

**Lemma 7.12** Let \( Y_t, W, t \) and \( Z_t \) be three sequences such that \( W_t \) is nonnegative for all \( t \). Assume that
\[
Y_{t+1} \leq Y_t - W_t + Z_t, \quad t = 0, 1, ..., \]
and that the series \( \sum_{t=0}^{T} Z_t \) converges as \( T \to \infty \). Then either \( Y_t \to \infty \) or else \( Y_t \) converges to a finite value and \( \sum_{t=0}^{\infty} W_t < \infty \).
This lemma has been proven by Bertsekas and Tsitsiklis [5].

**Lemma 7.13** When

\[ \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha^{(i)} \eta^{(n)} < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha^{(n)} < \infty, \]

it follows that \( f(\theta^{(m)}, \bar{\lambda}) \) converge to a finite value.

**Proof.** By Proposition 7.8 we have

\[ f(\theta^{(m+1)}, \bar{\lambda}) \leq f(\theta^{(m)}, \bar{\lambda}) - \eta^{(m)} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|^2 + \eta^{(m)} \bar{\lambda} \sqrt{\alpha_m} L \sqrt{\alpha_m} + \frac{1}{2} (\eta^{(m)})^2 \cdot N \bar{\lambda}. \]

Let \( Y^{(m)} := f(\theta^{(m)}, \bar{\lambda}), W^{(m)} := \eta^{(m)} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|^2 \) and \( Z^{(m)} := \eta^{(m)} \bar{\lambda} \sqrt{\alpha_m} L \sqrt{\alpha_m} + \frac{1}{2} (\eta^{(m)})^2 \cdot N \bar{\lambda} \). By (30) and (52), it is easy to see that \( \sum_{m=0}^{M} Z^{(m)} \) converges as \( M \to \infty \). Therefore, by Lemma 7.12 \( Y^{(m)} \) converges to a finite value. The infinite case can not occur in our setting due to Assumptions 4.1 and 4.2.

**Lemma 7.14** If

\[ \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha^{(i)} \eta^{(n)} < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha^{(n)} < \infty, \]

then \( \lim_{m \to \infty} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2 = 0. \)

**Proof.** To show that \( \lim_{m \to \infty} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2 = 0 \), assume the contrary; that is,

\[ \limsup_{m \to \infty} \| \nabla f(\theta^{(m)}, \bar{\lambda}) \|_2 > 0. \]

Then there exists an \( \epsilon > 0 \) such that \( \| \nabla f(\theta^{(m)}, \bar{\lambda}) \| < \epsilon/2 \) for infinitely many \( m \) and also \( \| \nabla f(\theta^{(m)}, \bar{\lambda}) \| > \epsilon \) for infinitely many \( m \). Therefore, there is an infinite subset of integers \( \mathbb{M} \), such that for each \( m \in \mathbb{M} \), there exists an integer \( q(m) > m \) such that

\[ \| \nabla f(\theta^{(m)}, \bar{\lambda}) \| < \epsilon/2, \]

\[ \| \nabla f(\theta^{(i(m))}, \bar{\lambda}) \| > \epsilon, \]

\( \epsilon/2 \leq \| \nabla f(\theta^{(i)} \bar{\lambda}) \| \leq \epsilon, \)

if \( m < i < q(m) \).

From

\[ \| \nabla f(\theta^{(m+1)}, \bar{\lambda}) \| - \| \nabla f(\theta^{(m)}, \bar{\lambda}) \| \leq \| \nabla f(\theta^{(m+1)}, \bar{\lambda}) - \nabla f(\theta^{(m)}, \bar{\lambda}) \| \]

\[ \leq \bar{\lambda} \| \theta^{(m+1)} - \theta^{(m)} \| \]

\[ = \bar{\lambda} \| \theta^{(m+1)} - \theta^{(m)} \|, \]

it follows that for all \( m \in \mathbb{M} \) that are sufficiently large so that \( \bar{\lambda} \eta^{(m)} < \epsilon/4 \), we have

\[ \epsilon/4 \leq \| \nabla f(\theta^{(m)}, \lambda^{(m)}) \|. \]

Otherwise the condition \( \epsilon/2 \leq \| \nabla f(\theta^{(m+1)}, \bar{\lambda}) \| \) would be violated. Without loss of generality, we assume that the above relations as well as (30) hold for all \( m \in \mathbb{M} \). With the above observations, we have for all \( m \in \mathbb{M} \),
\[ \epsilon \leq \| \nabla \tilde{f}(\theta^{(m)}, \bar{x}) \| - \| \nabla \tilde{f}(\theta^{(m)}, \bar{x}) \| \]
\[ \leq \| \nabla \tilde{f}(\theta^{(m)}, \bar{x}) - \nabla \tilde{f}(\theta^{(m)}, \bar{x}) \| \]
\[ \leq \tilde{L} \| \theta^{(m)} - \theta^{(m)} \| \]
\[ \leq \tilde{L} \sum_{i=m}^{q(m)-1} \| \theta^{(i+1)} - \theta^{(i)} \| \]
\[ \leq \tilde{L} \sum_{i=m}^{q(m)-1} \eta^{(i)} \| \nabla \tilde{f}(\theta^{(i)}, \lambda^{(i)}) \| \]
\[ \leq \tilde{L} \sum_{i=m}^{q(m)-1} \eta^{(i)} (\| \nabla \tilde{f}(\theta^{(i)}, \bar{x}) \| + \| \nabla \tilde{f}(\theta^{(i)}, \lambda^{(i)}) - \nabla \tilde{f}(\theta^{(i)}, \bar{x}) \|) \]
\[ \leq \tilde{L} \sum_{i=m}^{q(m)-1} \eta^{(i)} (\| \nabla \tilde{f}(\theta^{(i)}, \bar{x}) \| + \tilde{L} \sqrt{\eta_2 a_m}) \]
\[ \leq \tilde{L} \epsilon \sum_{i=m}^{q(m)-1} \eta^{(i)} + \tilde{L}^2 \sqrt{\eta_2} \sum_{i=m}^{q(m)-1} \eta^{(i)} a_m \]
\[ = \tilde{L} \epsilon \sum_{i=m}^{q(m)-1} \eta^{(i)} + \tilde{L}^2 \sqrt{\eta_2} \sum_{i=m}^{q(m)-1} \eta^{(i)} \left( M_1 \sum_{j=m}^{\infty} \sum_{k=1}^{j} \alpha^{(j)} \eta^{(k)} + M_2 \sum_{j=m}^{\infty} \alpha^{(j)} \right) \]
\[ = \tilde{L} \epsilon \sum_{i=m}^{q(m)-1} \eta^{(i)} + \tilde{L}^2 \sqrt{\eta_2} M_1 \sum_{i=m}^{q(m)-1} \eta^{(i)} \sum_{j=i}^{\infty} \sum_{k=1}^{j} \alpha^{(j)} \eta^{(k)} + \tilde{L}^2 \sqrt{\eta_2} M_2 \sum_{i=m}^{q(m)-1} \eta^{(i)} \sum_{j=m}^{\infty} \alpha^{(j)} \]

The first inequality is by (54) and the third one is by the Lipschitz condition assumption. The seventh one is by (52). By (2), we have for all \( m \in \mathbb{M} \),

\[ \sum_{i=m}^{q(m)-1} \eta^{(i)} \sum_{j=m}^{\infty} \sum_{k=1}^{j} \alpha^{(j)} \eta^{(k)} < \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \sum_{k=1}^{j} \alpha^{(j)} \eta^{(k)} < \infty \]

and

\[ \sum_{i=m}^{q(m)-1} \eta^{(i)} \sum_{j=m}^{\infty} \alpha^{(j)} < \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \alpha^{(j)} < \infty. \]

It is easy to see that for any sequence \( \{ \alpha_i \} \) with \( \sum_{i=1}^{\infty} \alpha_i < \infty \), if follows that \( \lim\inf_{M \to \infty} \sum_{i=M}^{\infty} \alpha_i = 0 \).

Therefore,

\[ \lim\inf_{m \to \infty} \sum_{i=m}^{q(m)-1} \eta^{(i)} \sum_{j=m}^{\infty} \sum_{k=1}^{j} \alpha^{(j)} \eta^{(k)} = 0 \]

and

\[ \lim\inf_{m \to \infty} \sum_{i=m}^{q(m)-1} \eta^{(i)} \sum_{j=m}^{\infty} \alpha^{(j)} = 0. \]

From this it follows that

\[ \lim\inf_{m \to \infty} \sum_{i=m}^{q(m)-1} \eta^{(i)} \geq \frac{1}{2\tilde{L}}. \]
By the triangle inequality, we have
\[ \| \nabla \tilde{f}(\theta^{(m)}, \lambda) \| \]
\[ = \| \nabla \tilde{f}(\theta^{(m)}, \lambda) - \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) + \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) \| \]
\[ \geq \left| \| \nabla \tilde{f}(\theta^{(m)}, \lambda) \| - \| \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) - \nabla \tilde{f}(\theta^{(m)}, \lambda) \| \right| . \]

By \((32)\) and \((55)\), if we pick \(m \in \mathbb{M}\) such that \(L \sqrt{n_2 a_m} \leq \frac{\epsilon}{8}\), we have \(\| \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) \| \geq \frac{\epsilon}{8}\). Using \((36)\), we observe that
\[ \tilde{f}(\theta^{q(m)}, \tilde{\lambda}) \leq \tilde{f}(\theta^{(m)}, \tilde{\lambda}) - \sum_{i=m}^{q(m)-1} \left( \eta_i c_1 \| \nabla \tilde{f}(\theta^{(i)}, \tilde{\lambda}) \|^2 \right) + \frac{1}{2} N \tilde{L} M \sum_{i=m}^{q(m)-1} (\eta_i)^2 \]
\[ \leq \tilde{f}(\theta^{(m)}, \tilde{\lambda}) - c_1 \left( \frac{\epsilon}{8} \right)^2 \sum_{i=m}^{q(m)-1} \eta_i + \frac{1}{2} N \tilde{L} M \sum_{i=m}^{q(m)-1} (\eta_i)^2, \forall m \in \mathbb{M}, \]
where the second inequality is by \((55)\). By Lemma \(7.13\), \(\tilde{f}(\theta^{q(m)}, \tilde{\lambda})\) and \(\tilde{f}(\theta^{(m)}, \tilde{\lambda})\) converge to the same finite value. Using this convergence result and the assumption \(\sum_{m=0}^{\infty} (\eta^{(m)})^2 < \infty\), this relation implies that
\[ \limsup_{m \to \infty, m \in \mathbb{M}} \sum_{i=m}^{q(m)-1} \eta_i = 0 \]
and contradicts \((56)\).

By Lemmas \(7.12\), \(7.13\) and \(7.14\) we show that Theorem \(4.10\) holds. To this end we write
\[ \lim_{m \to \infty} \| \nabla \tilde{f}(\theta^{(m)}, \lambda^{(m)}) \|_2 \]
\[ \leq \lim_{m \to \infty} \| \nabla \tilde{f}(\theta^{(m)}, \lambda^{(m)}) - \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) \|_2 + \lim_{m \to \infty} \| \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) \|_2 \]
By Theorem \(4.6\) we have
\[ \lim_{m \to \infty} \tilde{L} \| \lambda^{(m)} - \tilde{\lambda} \|_2 = 0 \]
and by Lemma \(4.9\) we have
\[ \lim_{m \to \infty} \| \nabla \tilde{f}(\theta^{(m)}, \tilde{\lambda}) \|_2 = 0. \]
Therefore, we have
\[ \lim_{m \to \infty} \| \nabla \tilde{f}(\theta^{(m)}, \lambda^{(m)}) \|_2^2 = 0, \]
which is the statement in Theorem \(4.10\).

### 7.6 Proof of Theorem \(5.2\)

In this section we assume that \(f_i(\cdot)\) is strongly convex.

**Lemma 7.15** If
\[ m^2 \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha_i \eta_i^{(n)} < \infty, \] (57)
there exists a constant \(M\) such that, for every \(m\) we have
\[ \frac{|\Delta_1^{(m+1)}| + (1 + \eta^{(m)} c) |\Delta_1^{(m)}| + \eta^{(m)} \tilde{L} M \sqrt{n_2 a_m}}{2 \eta^{(m)} a_m} \leq M. \] (58)
We define \( A \) and show that it is equivalent to showing constant upper bounds for \( \frac{\Delta_1^{(m)}}{\eta(m)^2} \) and \( \frac{a_m}{\eta(m)} \).

For an upper bound of \( \frac{\Delta_1^{(m)}}{\eta(m)^2} \), by (40) and (43), we have

\[
\frac{\Delta_1^{(m)}}{\eta(m)^2} = \frac{\|f(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^{(m)}, \bar{\lambda})\|}{\eta(m)^2} \leq M_3 \sum_{j=1}^{D} \left( |k(1)| N \frac{1}{6} \sum_{j=1}^{i} \eta^{(j)} \sum_{n=m}^{n} \alpha^{(n)} \big| \sigma_j - \sigma_j^{(m)} \big| + N \bigg| \frac{\bar{\mu}_j}{\bar{\sigma}_j} + \frac{\mu_j^{(m)}}{\bar{\sigma}_j^{(m)} + \epsilon_B} \bigg| \right).
\]

We can see that it is equivalent to showing constant upper bounds for all other terms in the right-hand side of (59) are finite constants.

By (38), we have

\[
|\sigma_j - \sigma_j^{(m)}| \leq \frac{|k(1)| \cdot \bar{M}_L M}{C} \sum_{i=m}^{\infty} \alpha^{(i)} \sum_{j=1}^{i} \eta^{(j)} + \bar{\sigma}_j \sum_{n=m+1}^{\infty} \alpha^{(n)}.
\]

Note that we have \( \eta^{(m)} = \frac{\zeta}{\bar{\theta} + m} \) and thus \( \eta^{(m)^2} = O\left(\frac{1}{m}\right) \). Therefore, (57) implies that

\[
\frac{1}{\eta^{(m)^2}} \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha^{(i)} \eta^{(n)} < \infty.
\]

Inequality (60) implies

\[
\frac{1}{\eta^{(m)^2}} \sum_{i=m+1}^{\infty} \alpha^{(i)} < \infty.
\]

This is by the fact that we assume \( \sum_{n=1}^{\infty} \eta^{(n)} = \infty \) in Assumption 4.3. We now apply the same kind of analysis to \( |\bar{\mu}_j - \bar{\mu}_j^{(m)}| \) to establish

\[
\left| \mu_j^{(m)} - \bar{\mu}_j \right| = (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)}) \left| \frac{\mu_j^{(m)}}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} - \frac{\bar{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} \right| \leq (1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)}) \left| \frac{\bar{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} - \frac{\bar{\mu}_j}{(1 - \alpha^{(1)}) \cdots (1 - \alpha^{(m)})} \right| + \left( \mu_j^{(m)} - \bar{\mu}_j \right). \]

We define \( A_m := |\bar{\mu}_j^{(m)} - \bar{\mu}_j^{(\infty)}| \) and \( B_m := |\bar{\mu}_j^{(m)} - \bar{\mu}_j - \bar{\mu}_j^{(\infty)}| \). Recall from Theorem 4.6 that \( \{\bar{\mu}_j^{(m)}\} \) is a Cauchy series. By (7), we have
Therefore, the first term in (62) is bounded by

\[ |\tilde{\mu}_j^{(m)} - \bar{\mu}_j^{\infty}| \leq \tilde{M}_{L,M} \cdot \sum_{m=p}^{q} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)}. \]

For the second term in (62), we first define

\[ C := \left(1 - \alpha^{(1)}\right)\ldots\left(1 - \alpha^{(u)}\right)\ldots. \]

Then we have

\[ |\mu_j^{(m)} - \bar{\mu}_j^{\infty}| \leq \tilde{\mu}_j \left[1 - \left(1 - \alpha^{(m+1)}\right)\ldots\left(1 - \alpha^{(\infty)}\right)\right] \leq \tilde{\mu}_j \sum_{i=m+1}^{\infty} \alpha^{(i)}, \]

where the last inequality can be easily checked by induction. Therefore, the second term in (62) is bounded by

\[ \left| \frac{\tilde{\mu}_j}{(1 - \alpha^{(1)})\ldots(1 - \alpha^{(m)})} - \bar{\mu}_j^{(\infty)} \right| \leq \frac{\tilde{\mu}_j}{C} \sum_{i=m+1}^{\infty} \alpha^{(i)}. \]

Hence (60) and (61) ensure \( \frac{|\tilde{\mu}_j - \mu_j^{(m)}|}{\eta^{(m)}^2} \) to be finite.

For an upper bound of \( \frac{a_m}{\eta^{(m)}} \), by (23), we have

\[ a_m = M_1 \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} + M_2 \sum_{i=m}^{\infty} \alpha^{(i)} \eta^{(m)}. \]

We know that

\[ M_1 \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} < M_1 \frac{1}{\eta^{(m)^2}} \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \eta^{(j)} < \infty \quad (63) \]

and

\[ M_2 \sum_{i=m}^{\infty} \alpha^{(i)} \eta^{(m)} < M_2 \frac{1}{\eta^{(m)^2}} \sum_{i=m}^{\infty} \alpha^{(i)} < \infty. \quad (64) \]

The second inequalities in (63) and (64) are by (60) and (61). Note that given that \( \eta^{(m)} = 1/m \), (60) is equivalent to

\[ \sum_{i=m}^{\infty} \sum_{j=1}^{i} \alpha^{(i)} \frac{1}{j} < \sum_{i=m}^{\infty} \alpha^{(i)} \ln(i) < \sum_{i=1}^{\infty} \alpha^{(i)} \ln(i) < \infty \]

This concludes the proof.

\[ \square \]

**Lemma 7.16** Under the assumptions of Lemma 7.15, Theorem 5.2 holds.

The proof for this Lemma of the high level follows the proof of Theorem 4.7 in Bottou et al. [6].

**Proof.** Assumption 5.1 implies that

\[ \bar{f}(\hat{\theta}, \lambda) \geq \bar{f}(\hat{\theta}, \lambda) + \nabla \bar{f}(\hat{\theta}, \lambda)^T (\hat{\theta} - \hat{\theta}) + \frac{1}{2} c \|\hat{\theta} - \hat{\theta}\|_2^2, \forall \hat{\theta}, \hat{\theta}. \]

Therefore, \( \bar{f} \) has a unique minimizer \( \bar{f}^* := \bar{f}(\theta^*, \lambda) \) for any \( \lambda \) fixed. Note that \( \theta^* = \theta^*(\lambda) \) but this dependency is irrelevant in the rest of the proof. Standard convex analysis argument establishes
We finally show by induction that
\[ 2c \left( \bar{f}(\theta^{(m)}, \lambda) - \bar{f}(\theta^*, \lambda) \right) \leq \| \nabla \bar{f}(\theta^{(m)}, \lambda) \|_2^2. \]  

Recall that \( \Delta_1^{(m+1)} := \bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^{(m+1)}, \lambda) \). We then have
\[
\begin{align*}
\bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^{(m)}, \lambda^{(m)}) &= \left[ \bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^{(m+1)}, \lambda) \right] - \left[ \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^{(m)}, \lambda) \right] \\
&\leq |\Delta_1^{(m+1)}| + |\Delta_1^{(m)}| + \bar{f}(\theta^{(m+1)}, \lambda) - \bar{f}(\theta^{(m)}, \lambda).
\end{align*}
\]

Therefore,
\[
\begin{align*}
\bar{f}(\theta^{(m+1)}, \lambda) - \bar{f}(\theta^{(m)}, \lambda) &\leq - \eta^{(m)} \| \nabla \bar{f}(\theta^{(m)}, \lambda) \|_2^2 + \eta^{(m)} \bar{L}M \sqrt{n_2} \alpha_m + \frac{1}{2} \eta^{(m)} N \bar{L} M \\
&\leq - \eta^{(m)} \eta \left( \bar{f}(\theta^{(m)}, \lambda) - \bar{f}(\theta^*, \lambda) \right) + \eta^{(m)} \bar{L}M \sqrt{n_2} \alpha_m + \frac{1}{2} \eta^{(m)} N \bar{L} M \\
&= - \eta^{(m)} \eta \left( \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda) \right) + \eta^{(m)} \bar{L}M \sqrt{n_2} \alpha_m + \frac{1}{2} \eta^{(m)} N \bar{L} M \\
&\leq - \eta^{(m)} \eta \left( \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda) \right) + \eta^{(m)} \left( \bar{f}(\theta^{(m)}, \lambda) - \bar{f}(\theta^{(m)}, \lambda^{(m)}) \right) \\
&\leq - \eta^{(m)} \eta \left( \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda) \right) + \eta^{(m)} \bar{L}M \sqrt{n_2} \alpha_m + \frac{1}{2} \eta^{(m)} N \bar{L} M.
\end{align*}
\]

The first inequality is by Proposition 7.8, while the second inequality is by the strong convexity property 65. Combining (66) and (67) yields
\[
\begin{align*}
\bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^{(m)}, \lambda^{(m)}) &\leq - \eta^{(m)} \eta \left( \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda) \right) + |\Delta_1^{(m+1)}| + (1 + \eta^{(m)} \eta) |\Delta_1^{(m)}| \\
&\leq - \eta^{(m)} \eta \left( \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda) \right) + \eta^{(m)} \bar{L}M \sqrt{n_2} \alpha_m + \frac{1}{2} \eta^{(m)} N \bar{L} M.
\end{align*}
\]

By Lemma 7.15, there exists an upper bound \( M_4 \) such that for all \( m \) sufficiently large,
\[
|\Delta_1^{(m+1)}| + (1 + \eta^{(m)} \eta) |\Delta_1^{(m)}| + \eta^{(m)} \bar{L}M \sqrt{n_2} \alpha_m \leq M_4.
\]

By subtracting \( \bar{f}(\theta^*, \lambda) \) from both side of (7.6), we obtain
\[
\begin{align*}
\bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^*, \lambda) &\leq (1 - \eta^{(m)} \eta) \left( \bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda) \right) + \frac{1}{2} \eta^{(m)} N \bar{L} M + M_4.
\end{align*}
\]

Inequality (68) has the exact same form used in classic convergence proofs for the strongly convex, diminishing step size case.

We finally show by induction that
\[
\bar{f}(\theta_m, \lambda_m) - \bar{f}(\theta^*, \lambda) \leq \frac{\nu}{\theta + m}
\]

Therefore, the conditions for \( v := \max\{\frac{\zeta^2(NLM + M_4)}{2(\zeta c - 1)}, (\vartheta + 1)[f(\theta^{(1)}, \lambda^{(1)}) - f(\theta^*, \lambda^*)]\} \).

First, the definition of \( \zeta \) ensures that it holds for \( m = 1 \). Assuming \( (68) \) holds for some \( m \geq 1 \), it follows from \( (68) \) that
\[
\bar{f}(\theta^{(m+1)}, \lambda^{(m+1)}) - \bar{f}(\theta^*, \lambda) \leq (1 - \eta^{(m)})(\bar{f}(\theta^{(m)}, \lambda^{(m)}) - \bar{f}(\theta^*, \lambda)) + \frac{1}{2} \eta^{(m)}^2 (LM + M_4)
\]
\[
\leq (1 - \eta^{(m)})(\frac{v}{\vartheta + m} + \frac{1}{2} \eta^{(m)}^2 (LM + M_4))
\]
\[
= (1 - \frac{\zeta c}{\vartheta + m}) \frac{v}{\vartheta + m} + \frac{\zeta^2 (LM + M_4)}{2(\vartheta + m)^2}
\]
\[
= \frac{\vartheta + m - \zeta c}{(\vartheta + m)^2} v + \frac{\zeta^2 (LM + M_4)}{2(\vartheta + m)^2}
\]
\[
\leq \frac{\vartheta + m - 1}{(\vartheta + m)^2} v
\]
\[
\leq \frac{v}{\vartheta + m + 1}.
\]

The first inequality is by \( (68) \), the second inequality is by the definition of \( \eta^{(m)} \), the third inequality is by the definition of \( v \), the sum of the latter two terms is non-positive, and the fourth inequality is because \( (\vartheta + m)^2 \geq (\vartheta + m + 1)(\vartheta + m - 1) \). This shows that the algorithm converges at a sublinear rate. \( \square \)

8 Appendix B: Discussions of conditions for stepsizes

Here we discuss the actual conditions for \( \eta^{(m)} \) and \( \alpha^{(m)} \) to satisfy the assumptions of Theorem 4.6, Lemma 4.7 and Theorem 5.2, respectively. We only consider the cases \( \eta^{(m)} = O\left(\frac{1}{m^r}\right) \) and \( \alpha^{(m)} = O\left(\frac{1}{m^r}\right) \), but the same analysis applies to the cases \( \eta^{(m)} = \bar{O}\left(\frac{1}{m^r}\right) \) and \( \alpha^{(m)} = \bar{O}\left(\frac{1}{m^r}\right) \).

8.1 Assumptions of Theorem 4.6

For the assumptions of Theorem 4.6, the first condition
\[
\sum_{m=1}^{\infty} \alpha^{(m)} < \infty
\]
requires \( h > 1 \).

Besides, the second condition
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{m} \alpha^{(m)} \eta^{(n)} = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \alpha^{(m)} \eta^{(n)} \approx \sum_{n=1}^{\infty} \eta^{(n)} \sum_{m=n}^{\infty} \alpha^{(m)} \approx \frac{1}{h - 1} \sum_{n=1}^{\infty} \eta^{(n)} \frac{1}{nh - 1} = \frac{1}{h - 1} \sum_{n=1}^{\infty} \frac{1}{n^{k+h-1}} < \infty
\]
requires \( k + h > 2 \). The approximation comes from the fact that for every \( p > 1 \), we have
\[
\sum_{k=n}^{\infty} k^{-p} \approx \int_{k=n}^{\infty} k^{-p} dx = \frac{1}{1 - p} x^{1-p}\bigg|_{n}^{\infty} = \frac{1}{p - 1} n^{1-p} - 1.
\]

Since \( k \geq 1 \) due to Assumption 4.3, we conclude that \( k + h > 2 \).

Therefore, the conditions for \( \eta^{(m)} \) and \( \alpha^{(m)} \) to satisfy the assumptions of Theorem 4.6 are \( h > 1 \) and \( k \geq 1 \).
8.2 Assumptions of Lemma 4.7

For the assumptions of Theorem 4.6, the first condition
\[\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \alpha^{(n)} \approx \sum_{m=1}^{\infty} \frac{1}{m^{h-1}} < \infty\]
requires \(h > 2\).

Besides, the second condition is
\[\sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \sum_{n=1}^{i} \alpha^{(i)} \eta^{(n)} = \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \alpha^{(i)} \sum_{n=1}^{i} \eta^{(n)} \leq C \sum_{m=1}^{\infty} \sum_{i=m}^{\infty} \alpha^{(i)} < \infty\.

The inequality holds because for any \(p > 1\), we have
\[\sum_{k=1}^{n} k^{-p} \approx \int_{1}^{n} k^{-p} dk = \frac{1}{1-p} k^{1-p}\left|_{1}^{n}\right. = \frac{1}{p-1} (1 - n^{1-p}) \leq C\]
Therefore, the conditions for \(\eta^{(m)}\) and \(\alpha^{(m)}\) to satisfy the assumptions of Lemma 4.7 are \(h > 2\) and \(k \geq 1\).

8.3 Assumptions of Theorem 5.2

Recall that we have let \(\eta^{(m)} = 1/m\). For the assumptions of Theorem 5.2, the condition
\[\sum_{m=1}^{\infty} \alpha^{(m)} \ln(m) < \infty\]
requires \(h > 1\). To see this, note that \(\ln(m) \leq C m^\epsilon\) for any \(\epsilon > 0\). Thus
\[\sum_{m=1}^{\infty} \alpha^{(m)} \ln(m) \leq C \sum_{m=1}^{\infty} m^{-h} m^\epsilon < \infty\]
if \(\epsilon - h < -1\). This yields \(h > 1\).

Therefore, the condition for \(\alpha^{(m)}\) to satisfy the assumptions of Theorem 5.2 is \(h > 1\).