FLOW BY GAUSS CURVATURE TO DUAL ORLICZ-MINKOWSKI PROBLEMS

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ABSTRACT. In this paper we study a normalised anisotropic Gauss curvature flow of strictly convex, closed hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$. We prove that the flow exists for all time and converges smoothly to the unique, strictly convex solution of a Monge-Ampère type equation. Our argument provides a parabolic proof in the smooth category for the existence of solutions to the Dual Orlicz-Minkowski problem introduced by Zhu, Xing and Ye.

Keywords: Gauss curvature flow, convex hypersurface, Monge-Ampère equation.
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1. Introduction

As we known, the Gauss curvature flow was introduced by Firey [14] to model the shape change of worn stones. The first celebrated result was proved by Andrews in [3] for Gauss curvature flow, where Firey’s conjecture that convex surfaces moving by their Gauss curvature become spherical as they contract to points was proved. Guan and Ni [15] proved that convex hypersurfaces in $\mathbb{R}^{n+1}$ contracting by the Gauss curvature flow converge (after rescaling to fixed volume) to a smooth uniformly convex self-similar solution of the flow. Soon, Andrews, Guan and Ni [7] extended the results in [15] to the flow by powers of the Gauss curvature $K^\alpha$ with $\alpha > \frac{n+2}{n+2}$. Recently, Brendle, Choi and Daskalopoulos [11] proved that round spheres are the only closed, strictly convex self-similar solutions to the $K^\alpha$-flow with $\alpha > \frac{n+2}{n+2}$. Therefore, the generalized Firey’s conjecture proposed by Andrews in [6] was completely solved, that is, the solutions of the flow by powers of the Gauss curvature converge to spheres for any $\alpha > \frac{n+2}{n+1}$. We also refer to [12, 14, 5] and the references therein.

As a natural extension of Gauss curvature flows, anisotropic Gauss curvature flows have attracted considerable attention and they provide alternative proofs for the existence of solutions to elliptic PDEs arising in geometry and physics, especially for the Minkowski-type problem. For example an alternative proof based on the logarithmic Gauss curvature flow was given by Chou-Wang in [13] for the classical Minkowski problem, in [21] for a prescribing Gauss curvature problem. Using a contracting Gauss curvature flow, Li-Sheng-Wang [17] have provided a parabolic proof in the smooth category for the classical Aleksandrov and dual Minkowski problems. Recently, two kinds of normalised anisotropic Gauss curvature flow are used to prove the $L_p$ dual Minkowski problems by Chen-Huang-Zhao [9] and Chen-Li [10], respectively. These results are major source of inspiration for us.

Let $M_0$ be a smooth, closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ enclosing the origin. In this paper, we study the long-time behavior of the following normalised anisotropic Gauss curvature flow which is a family of hypersurfaces $M_t$ given by smooth maps $X : M \times (0, T) \to \mathbb{R}^{n+1}$ satisfying the initial value problem

\[
\begin{align*}
\frac{\partial X}{\partial t} &= -\theta(t)f(\nu) \frac{\rho^{n+1}}{\varphi(r)} Ky + X, \\
X(\cdot, 0) &= X_0,
\end{align*}
\]

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where \( \nu \) is the unit outer vector of \( M_t \) at \( X \), \( K \) denotes the Gauss curvature of \( M_t \) at \( X \), \( r = |X| \) denotes the distance form \( X \) to the origin, \( f \in C^\infty(\mathbb{S}^n) \) with \( f > 0 \), and

\[
\theta(t) = \int_{\mathbb{S}^n} \varphi(\xi, t) d\xi \left[ \int_{\mathbb{S}^n} f(x) dx \right]^{-1}.
\]

Notice that \( u \) denotes the support function of \( M_t \) given by \( u = \langle X, \nu \rangle \) and \( \varphi \) is a positive smooth function.

The reason that we study the flow (1.1) is to explore the existence of the smooth solutions to the dual Orlicz-Minkowski problem introduced by Zhu-Xing-Ye [23], which is related to the following Monge-Ampère type equation

\[
(1.2) \quad \frac{u \varphi(r)}{r^{n+1}} \cdot \det(u_{ij} + \delta_{ij}) = f(x) \quad \text{on} \quad \mathbb{S}^n,
\]

where \( r = \sqrt{|Du|^2 + u^2} \). In deed, let \( \mathcal{K}_0 \) be the set of all convex bodies in \( \mathbb{R}^{n+1} \) which contain the origin in their interiors, \( \varphi : (0, +\infty) \to (0, +\infty) \) be a continuous function. Zhu-Xing-Ye [23] have introduced the definition of the dual Orlicz curvature measure \( \mathcal{C}_{\varphi}(K, \cdot) \), and posed the following dual Orlicz-Minkowski problem:

**Problem 1.1** (Dual Orlicz-Minkowski problem). Under what conditions on \( \varphi \) and a nonzero finite Borel measure \( \mu \) on \( \mathbb{S}^n \), there exists a constant \( c > 0 \) and a \( K \in \mathcal{K}_0 \) such that \( \mu = c \mathcal{C}_{\varphi}(K, \cdot) \)?

When \( \mu \) has a density \( f \), this Minkowski problem is equivalent to solve the Monge-Ampère type equation (1.2). When \( \varphi(r) = r^q \), this becomes the dual Minkowski problem for the \( q \)-th dual curvature considered by Huang-Lutwak-Yang-Zhang [16]. It is worth pointing out that they also proved the existence of symmetric solutions for the case \( q \in (0, n + 1) \) under some conditions. For \( q = n + 1 \), the dual Minkowski problem becomes the logarithmic Minkowski problem which studied in [8]. For \( q < 0 \), the existence and uniqueness of weak solution were obtained by Zhao [22].

It is to be expected that the flow (1.1) converges to the solution of the equation (1.2). The main idea is to find a suitable functional which is monotonic under the flow (1.1). The difficulty of our proof lies the inhomogeneous term \( \varphi(r) \). To statement our theorem, we need the following assumption.

**Assumption 1.1.** \( \Phi : (0, +\infty) \to (0, +\infty) \) is a continuous function such that

\[
\Phi(t) = \int_0^t \frac{\varphi(s)}{s} ds
\]

exists for every \( t > 0 \).

**Theorem 1.2.** Assume that \( f \in C^\infty(\mathbb{S}^n) \) is a positive smooth function and \( \varphi : (0, +\infty) \to (0, +\infty) \) is a smooth function. Let \( M_0 \subset \mathbb{R}^{n+1} \) be a strictly convex, closed hypersurface which contains the origin in its interior.

(i) If \( \max_{t \geq 0} \varphi(t) \varphi^{-1}(s) < 0 \) for any \( t \in (0, +\infty) \), then the normalised flow (1.1) has a unique smooth solution, which exists for any time \( t \in [0, \infty) \). For each \( t \in [0, \infty) \), \( M_t = X(\mathbb{S}^n, t) \) is a closed, smooth and strictly convex hypersurface and the support function \( u(x, t) \) of \( M_t = X(\mathbb{S}^n, t) \) converges smoothly, as \( t \to \infty \), to the unique positive, smooth and strictly convex solution of the equation (1.2) with \( f \) replaced by \( \lambda_0 f \) for some \( \lambda_0 > 0 \).

(ii) Under the assumption (1.1), if \( f \) is in addition even function and the initial hypersurface \( M_0 \) is origin-symmetric, then the normalised flow (1.1) has a unique smooth solution, which exists for any time \( t \in [0, \infty) \). For each \( t \in [0, \infty) \), \( M_t = X(\mathbb{S}^n, t) \) is a closed, smooth, strictly convex and origin-symmetric hypersurface and the support function \( u(x, t) \) of \( M_t = X(\mathbb{S}^n, t) \) converges smoothly, as \( t \to \infty \), to the unique positive, smooth, strictly convex and even solution of the equation (1.2) with \( f \) replaced by \( \lambda_0 f \) for some \( \lambda_0 > 0 \).
Similarly, we can also define the higher order covariant derivative of the coordinate expression of which is denoted by 

\[ D\alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r, X) = D_X\alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r) \]

We can continue to define the second covariant derivative of \( \alpha \) as follows:

\[ D^2\alpha(Y^1, \ldots, Y^s, X_1, \ldots, X_r, X, Y) = (D_Y(D\alpha))(Y^1, \ldots, Y^s, X_1, \ldots, X_r, X), \]

the coordinate expression of which is denoted by

\[ D^2\alpha = (\alpha^{l_1 \ldots l_s}_{k_1 \ldots k_r; k_{s+1} \ldots k_{s+r}}). \]

Similarly, we can also define the higher order covariant derivative of \( \alpha \):

\[ D^3\alpha = D(D^2\alpha), \]

and so on. For simplicity, the coordinate expression of the covariant differentiation will usually be denoted by indices without semicolons, e.g.,

\[ u_i, \ u_{ij} \text{ or } u_{ijk} \]

for a function \( u : M \rightarrow \mathbb{R} \).

Our convention for the Riemannian curvature (3,1)-tensor \( Rm \) is defined by

\[ Rm(X, Y)Z = -D_XD_YZ + D_YD_XZ + D_{[X,Y]}Z. \]

Pick a local coordinate chart \( \{x^i\}_{i=1}^n \) of \( M \). The component of the (3,1)-tensor \( Rm \) is defined by

\[ Rm\left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} = R_{ijkl}^l \frac{\partial}{\partial x^l}, \]

and \( R_{ijkl} = g_{lm}R_{ijkm} \). Then, we have the standard commutation formulas (Ricci identities):

\[ a_{l_1 \ldots l_s}_{k_1 \ldots k_r; j i} - a_{l_1 \ldots l_s}_{k_1 \ldots k_r; i j} = \sum_{a=1}^r R_{ijk}^{l_a} a_{l_1 \ldots l_s}_{k_1 \ldots k_{s+1} \ldots k_r} - \sum_{b=1}^s R_{ijm}^{l_b} a_{l_1 \ldots l_s ; l_{b+1} \ldots l_r}. \]

We list some facts which will be used frequently. For the standard sphere \( S^n \) with the sectional curvature 1,

\[ R_{ijkl} = \delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk}. \]

A special case of Ricci identity for a function \( u : M \rightarrow \mathbb{R} \) will be usually used frequently:

\[ u_{kji} - u_{kij} = R_{ijk}^m u_m. \]
In particular, for a function \( u : \mathbb{S}^n \to \mathbb{R} \),
\[
(2.2) \quad u_{kji} - u_{kij} = \delta_{ik}u_j - \delta_{jk}u_i.
\]

Let \((M, g)\) be an immersed hypersurface in \( \mathbb{R}^{n+1} \) and \( \nu \) be a given unit outward normal. The second fundamental form \( h_{ij} \) of the hypersurface \( M \) with respect to \( \nu \) is defined by
\[
(2.3) \quad h_{ij} = -\left( \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right)_{\mathbb{R}^{n+1}}.
\]

2.2. Basic properties of convex hypersurfaces.

We first recall some basic properties of convex hypersurfaces. Let \( M \) be a smooth, closed, uniformly convex hypersurface in \( \mathbb{R}^{n+1} \). Assume that \( M \) is parametrized by the inverse Gauss map \( X : \mathbb{S}^n \to M \).

The support function \( u : \mathbb{S}^n \to \mathbb{R} \) of \( M \) is defined by
\[
u(x) = \sup\{\langle x, y \rangle : y \in M\}.
\]
The supremum is attained at a point \( y \) such that \( x \) is the outer normal of \( M \) at \( X \). It is easy to check that
\[
X = u(x)x + Du(x),
\]
where \( D \) is the covariant derivative with respect to the standard metric \( \sigma_{ij} \) of the sphere \( \mathbb{S}^n \). Hence
\[
(2.3) \quad r = |X| = \sqrt{u^2 + |Du|^2}.
\]

Thus,
\[
(2.4) \quad u = \frac{r^2}{\sqrt{r^2 + |Dr|^2}}.
\]
The second fundamental form of \( M \) is given by, see e.g. [2, 20],
\[
(2.5) \quad h_{ij} = u_{ij} + \sigma_{ij},
\]
where \( u_{ij} = D_iD_ju \) denotes the second order covariant derivative of \( u \) with respect to the spherical metric \( \sigma_{ij} \). By Weingarten’s formula,
\[
(2.6) \quad \sigma_{ij} = \left( \frac{\partial \nu}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right) = h_{ik}g^{kl}h_{lj},
\]
where \( g_{ij} \) is the metric of \( M \) and \( g^{ij} \) is its inverse. It follows from \((2.5)\) and \((2.6)\) that the principal radii of curvature of \( M \), under a smooth local orthonormal frame on \( \mathbb{S}^n \), are the eigenvalues of the matrix
\[
b_{ij} = u_{ij} + u\delta_{ij}.
\]
In particular, the Gauss curvature is given by
\[
K = \frac{1}{\det(u_{ij} + u\delta_{ij})}.
\]
2.3. **Geometric flow and its associated functional.**

For reader's convenience, the associated Mong-Ampère equation (1.2) is restated here,

\[
\frac{u(r)}{r^{n+1}} \cdot \det(u_{ij} + u \delta_{ij}) = f(x) \quad \text{on} \quad \mathbb{S}^n.
\]

Recall the normalised anisotropic Gauss curvature flow (1.1)

\[
\begin{cases}
\frac{\partial X}{\partial t}(\xi, t) = -\theta(t) f(\xi) \frac{r^{n+1}}{\varphi(r)} K + X, \\
X(\cdot, 0) = X_0,
\end{cases}
\]

where

\[
\theta(t) = \int_{\mathbb{S}^n} \varphi(r(\xi, t)) d\xi \left[ \int_{\mathbb{S}^n} f(x) dx \right]^{-1}.
\]

By the definition of support function, we know

\[
u(x, t) = \langle x, X(x, t) \rangle.
\]
Hence,

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = -\theta(t) \frac{f(x)r^{n+1}}{\varphi(r)} K + u(x, t), \\
u(., 0) = u_0.
\end{cases}
\]

The normalised flow (1.1) can be also described by the following scalar equation for \( r(\cdot, t) \)

\[
\begin{cases}
\frac{\partial r}{\partial t}(\xi, t) = -\theta(t) \frac{f(x)r^{n+2}}{\varphi(r)u} K + r(\xi, t), \\
r(\cdot, 0) = r_0,
\end{cases}
\]

in view of

\[
\frac{1}{r(\xi, t)} \frac{\partial r(\xi, t)}{\partial t} = \frac{1}{u(x, t)} \frac{\partial u(x, t)}{\partial t},
\]

see Section 3 in [10] for the proof.

For a convex body \( \Omega \subset \mathbb{R}^{n+1} \), we define

\[
V_{\varphi}(\Omega) = \int_{\mathbb{S}^n} d\xi \int_0^{r(\xi, t)} \varphi(s) \frac{1}{s} ds.
\]

When \( \varphi(s) = s^q \), \( V_{\varphi}(\Omega) \) be the \( q \)-volume of the convex body \( \Omega \subset \mathbb{R}^{n+1} \), see [9, 10]. We show below that \( V_{\varphi}(\Omega_t) \) is unchanged under the flow (1.1), where \( \Omega_t \) is a compact convex body in \( \mathbb{R}^{n+1} \) with the boundary \( \mathcal{M}_t \).

**Lemma 2.1.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.1), then we obtain

\[
V_{\varphi}(\Omega_t) = V_{\varphi}(\Omega_0).
\]

**Proof.**

\[
\frac{d}{dt} V_{\varphi}(\Omega_t) = \int_{\mathbb{S}^n} \varphi(r) \frac{\partial r}{\partial t} d\xi = \int_{\mathbb{S}^n} \varphi(r) \left( -\theta(t) \frac{f(x)r^{n+2}}{\varphi(r)u} K + r(\xi, t) \right) d\xi = -\theta(t) \int_{\mathbb{S}^n} \frac{f(x)r^{n+1}}{u} K d\xi + \int_{\mathbb{S}^n} \varphi(r) d\xi = 0,
\]

where we use

\[
\frac{dx}{d\xi} = \frac{r^{n+1} K}{u},
\]
see e.g. [10, 16]. □

Next, we define the functional
\[ J[\varphi(X(\cdot, t))] = \int_{\mathbb{R}^n} \log u(x, t) f(x) dx. \]

The following lemma shows that the functional \( J[\varphi] \) is non-increasing along the flow (1.1).

**Lemma 2.2.** Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.1). For any \( \varphi \geq 0 \), the functional is non-increasing along the flow (1.1). In particular,
\[ \frac{d}{dt} J[\varphi(X(\cdot, t))] \leq 0, \]
and the equality holds if and only if \( X(\cdot, t) \) satisfies the elliptic equation (1.2) with \( f \) replaced by \( \theta(t)f \).

**Proof.**
\[
\begin{align*}
\frac{d}{dt} J[\varphi(X(\cdot, t))] &= \int_{\mathbb{R}^n} \frac{1}{u} \frac{\partial u(x, t)}{\partial t} f(x) dx \\
&= \int_{\mathbb{R}^n} \frac{1}{u} \left( -\theta(t) \frac{f(x)r^{n+1}}{\varphi(r)} K + u(x, t) \right) f(x) dx \\
&= \left[ \int_{\mathbb{R}^n} f(x) dx \right]^{-1} \left\{ -\int_{\mathbb{R}^n} \frac{u}{r^{n+1}K} K \int_{\mathbb{R}^n} \frac{r^{n+1}K}{u} f^2 dx + \int_{\mathbb{R}^n} f dx \int_{\mathbb{R}^n} f dx \right\} \\
&\leq 0
\end{align*}
\]

in view of
\[
\int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} f(x) dx \leq \int_{\mathbb{R}^n} \frac{u}{r^{n+1}K} K \int_{\mathbb{R}^n} \frac{r^{n+1}K}{u} f^2 dx + \int_{\mathbb{R}^n} f dx \int_{\mathbb{R}^n} f dx,
\]

which is implies by Hölder inequality, where \( d\sigma = f(x) dx \). Clearly, the equality holds if and only if
\[
\frac{f(x)r^{n+1}K}{u\varphi(r)} = \frac{1}{c(t)}.
\]

In this case, clearly, we have \( \theta(t) = c(t) \). Thus, \( X(\cdot, t) \) satisfies the elliptic equation (1.2) with \( f \) replaced by \( \theta(t)f \).

Before closing this section, we prove the following basic properties for any given \( \Omega \in \mathcal{K}_0 \), while smoothness of \( \partial \Omega \) is not required. First, we introduce the following Lemma for convex bodies, see Lemma 2.6 in [10] for the details.

**Lemma 2.3.** Let \( \Omega \in \mathcal{K}_0 \). Let \( u \) and \( r \) be the support function and radial function of \( \Omega \), and \( x_{\max} \) and \( \xi_{\min} \) be two points such that \( u(x_{\max}) = \max_{\mathbb{S}^n} u \) and \( r(\xi_{\min}) = \min_{\mathbb{S}^n} r \). Then
\[
\max_{\mathbb{S}^n} u = \max_{\mathbb{S}^n} r \quad \text{and} \quad \min_{\mathbb{S}^n} u = \min_{\mathbb{S}^n} r,
\]
\[
u(x) \geq x \cdot x_{\max} u(x_{\max}), \quad \forall x \in \mathbb{S}^n,
\]
\[r(\xi) \xi \cdot \xi_{\min} \geq r(\xi_{\min}), \quad \forall \xi \in \mathbb{S}^n.
\]

Let \( \mathcal{K}^{n+1} = [K|K] \) is convex body in \( \mathbb{R}^{n+1} \). Then, we have the following theorem (see also [18]).
Theorem 2.4. If \( K_i \in \mathcal{K}^{n+1} \) and there exists a constant \( R > 0 \) such that \( K_i \subset B_R \), then there exists a subsequence \( K_{i_j} \) and \( K_0 \in \mathcal{K}^{n+1} \) such that \( K_{i_j} \to K_0 \) in the Hausdorff metric.

To state the following theorem, we first recall the definition of the radial function of a convex body. (see also [18]).

Definition 2.1. Let \( K \in \mathcal{K}^{n+1}, 0 \in K \), a radial function \( r_K : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R} \) is defined as

\[
    r_K(x) = \max\{r \geq 0 | rx \in K\}.
\]

Now, the convergence of convex bodies imply the convergence of the corresponding radial functions.

Theorem 2.5. Let \( K_0, K_i \in \mathcal{K}^{n+1}, 0 \in \text{int}K_0 \) and \( K_i \to K_0 \), then \( r_{K_i} \Rightarrow r_{K_0} \).

For the proof of the theorem above, see [18].

3. \( C^0, C^1 \)-estimates

In this section, we will derive the \( C^0, C^1 \)-estimates of the flow (1.1). The key is the lower bound of \( u \). The difficulty of the proof lies in the inhomogeneous term \( \varphi(r) \).

3.1. The upper bound of \( u \) and gradient estimate. It is easy to obtain the upper bound of \( u \) and gradient estimate if we notice that the functional \( J_\varphi \) is non-increasing along the flow (1.1), see Lemma 2.2.

Lemma 3.1. Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.1), then we have

\[
    u(\cdot, t) \leq C, \quad \forall t \in [0, T).
\]

and

\[
    |Du(\cdot, t)| \leq C, \quad \forall t \in [0, T).
\]

Proof. Assume that \( x_i \) is a point at which \( u(\cdot, t) \) attains its spatial maximum, we know from Lemma 2.2

\[
    C \geq \int_{\mathbb{S}^n} \log u(x, t) f(x) dx \geq \int_{\{x \in \mathbb{S}^n : x_i > 0\}} \log[x \cdot x_i u(x_i, t)] f(x) dx,
\]

which implies

\[
    C \geq \max_{\mathbb{S}^n} u(\cdot, t).
\]

This yields the inequality (3.1). Since

\[
    \max_{\mathbb{S}^n} |Du(\cdot, t)| \leq \max_{\mathbb{S}^n} u(\cdot, t),
\]

we obtain (3.2). \( \square \)

3.2. The lower bound of \( u \). We get the lower bound of \( u \) by the following gradient estimate for Case (i) in Theorem 1.2 and the fact that \( f \) and \( u_0 \) are even functions for Case (ii) in Theorem 1.2.

Lemma 3.2. Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.1), if

\[
    \max_{s > 0} s \varphi'(s) \varphi^{-1}(s) < 0,
\]

then

\[
    \max_{\mathbb{S}^n} \frac{|Du|}{u}(\cdot, t) \leq C, \quad \forall t \in [0, T).
\]
Using the maximum principle, we get the gradient estimates of $\psi$.

Set $\psi = \frac{|Dz|^2}{2}$. By differentiating the $\psi$, we have

$$\frac{\partial \psi}{\partial t} = (\frac{\partial z}{\partial t}z^m)^m = (z_m)z^n = Q_m z^n.$$ 

Then,

$$\frac{\partial \psi}{\partial t} = Q^j z_{jm} z^m + Q^k z_{km} z^m + (e^z \sqrt{1 + |Dz|^2}) Q(D^2 z, Dz, z) + (D \log f, Dz) Q.$$ 

where

$$Q^{ij} = \frac{\partial Q}{\partial w_{ij}}, \quad Q_k = \frac{\partial Q}{\partial z_k}.$$ 

Interchanging the covariant derivatives, we have

$$\psi_{ij} = (z_m z^n)^j = z_{im} z^m + z_{im} z^n = z_{mj} z^m + z_{mj} z^n = z_{jm} z^m + z_{jm} z^n.$$ 

in view of (2.2). Thus, we have

$$\frac{\partial \psi}{\partial t} = Q^j \psi_{ij} + Q^k \psi_k - Q^{ij} (\delta_{ij} |Dz|^2 - z_{ij}) - Q^{ij} z_{mj} z^m + (e^z \sqrt{1 + |Dz|^2}) Q(D^2 z, Dz, z) + (D \log f, Dz) Q.$$ 

(3.5)

Since the matrix $Q^{ij}$ and $\delta_{ij} |D\varphi|^2 - \varphi_i \varphi_j$ are positive definite, the third and forth terms in the right of (3.5) are non-positive. And noticing that the fifth term in the right of (3.5) is nonpositive if (3.3) holds true and $|Dz| \geq \frac{C}{\max_{z > 0} \frac{\partial^2 \phi}{\partial \varphi \partial \varphi}}$. So we got the equation about $\psi$ as follows:

$$\begin{cases} 
\frac{\partial \psi}{\partial t} \leq Q^j \psi_{ij} + Q^k \psi_k \quad \text{on} \quad \mathbb{S}^n \times (0, \infty), \\
\psi(\cdot, 0) = \frac{|Dz(\cdot, 0)|^2}{2} \quad \text{on} \quad \mathbb{S}^n.
\end{cases}$$

Using the maximum principle, we get the gradient estimates of $z$. \hfill \Box

**Lemma 3.3.** Let $X(\cdot, t)$ be a strictly convex solution to the flow (1.1), then we have

$$\frac{1}{C} \leq u(x, t) \leq C, \quad \forall (x, t) \in \mathbb{S}^n \times [0, T).$$

if either (i) (3.3) holds true; or (ii) the assumption (1.1) holds true, $f$ and $u_0$ are even functions.
Proof. We only need prove the first inequality in (3.6) by noticing Lemma 3.1.

Case (i): If (3.3) holds true, we have by virtue of (3.4)
\[ \max_{\mathbb{S}^n} \log u(\cdot, t) - \min_{\mathbb{S}^n} \log u(\cdot, t) \leq C \max_{\mathbb{S}^n} |Du| \leq C, \]
which implies the positive lower bound of $u$ together with (3.1).

Case (ii): $f$ and $u_0$ are even. We have
\[ (3.7) \int_{\mathbb{S}^n} d\xi \int_0^{r(\xi, t)} \frac{\varphi(s)}{s} ds = \int_{\mathbb{S}^n} d\xi \int_0^{r(\xi, t)} \frac{\varphi(s)}{s} ds \]
by Lemma 2.1. Here we use the idea in [9] to complete our proof by contradiction. Assume $r(\xi, t)$ is not uniformly bounded away from 0 which means there exists $\inf_{\xi \in \mathbb{S}^n} r(\xi, t_i) \to 0$ as $i \to \infty$, where $t_i \in [0, T)$. Since $f$ and $u_0$ are even, $r(\xi, t)$ is even. Thus, $\Omega_i$ is a origin-symmetric body, where $\Omega_i$ is the convex body containing the origin and $\partial \Omega_i = M_i$. Thus, using Theorem 2.4, we have $\Omega_i$ (after choosing a subsequence) converges to a origin-symmetric convex body $\Omega_0$. Then, we have by Theorem 2.5
\[ \inf_{\xi \in \mathbb{S}^n} r_{\Omega_0}(\xi) = 0. \]
So, there exists $\xi_0 \in \mathbb{S}^n$ such that $r_{\Omega_0}(\xi_0) = 0$ and thus $r_{\Omega_0}(-\xi_0) = 0$, which implies $\Omega_0$ contained in a lower-dimensional subspace. This means that
\[ r(\xi, t_i) \to 0 \]
as $i \to \infty$ almost everywhere with respect to the spherical Lebesgue measure. Combined with bounded convergence theorem, we conclude
\[ \int_{\mathbb{S}^n} d\xi \int_0^{r(\xi, t)} \frac{\varphi(s)}{s} ds = \int_{\mathbb{S}^n} d\xi \int_0^{r(\xi, t_i)} \frac{\varphi(s)}{s} ds \to 0 \]
as $i \to \infty$, which is a contraction to (3.7). So, we complete our proof. $\square$

The $C^0$ and $C^1$ estimates of $u$ imply the corresponding $C^0$ and $C^1$ estimates of $r$ by using (2.4) and Lemma 2.4.

**Corollary 3.4.** Under the assumptions in Theorem 1.2 if $X(\cdot, t)$ is a strictly convex solution to the flow (1.1), then we have
\[ (3.8) \frac{1}{C} \leq r(\xi, t) \leq C, \quad \forall (\xi, t) \in \mathbb{S}^n \times [0, T), \]
\[ (3.9) |Dr|(\xi, t) \leq C, \quad \forall (\xi, t) \in \mathbb{S}^n \times [0, T), \]
and
\[ (3.10) \frac{1}{C} \leq \theta(t) \leq C, \quad \forall t \in [0, T). \]

4. $C^2$-estimates

In this section we establish uniformly positive and lower bounds for the principle curvatures for the normalised flow (1.1). We first use the technique that was first introduced by Tso [19] to derive the upper bound of the Gauss curvature along the flow (1.1), see also the proof of Lemma 4.1 in [17] and Lemma 5.1 in [9].
Lemma 4.1. Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.1) which encloses the origin for \( t \in [0, T) \). Then, there exists a positive constant \( C \) depending only \( \varphi, \max_{S^n \times [0, T)} u \) and \( \min_{S^n \times [0, T)} u \), such that

\[
\max_{S^n} K(\cdot, t) \leq C, \quad \forall t \in [0, T).
\]

Proof. We apply the maximum principle to the following auxiliary function defined on the unit sphere \( S^n \)

\[
W(x, t) = \frac{1}{\theta(t)} \frac{-u_t + u}{u - \varepsilon_0} = \frac{f(x)}{\varphi(r)} \cdot \frac{K}{u - \varepsilon_0},
\]

where

\[
\varepsilon_0 = \frac{1}{2} \min_{(x, t) \in S^n \times [0, T]} u(x, t) > 0.
\]

At the maximum \( x_0 \) of \( W \) for any fixed \( t \in [0, T) \), we have at \((x_0, t)\)

\[
0 = \theta(t)W_t = \frac{-u_{ti} + u_i}{u - \varepsilon_0} + \frac{u_t - u - u_{ij}u_{ij}}{(u - \varepsilon_0)^2}u_i,
\]

and

\[
0 \geq \theta(t)D^2_{ij}W = \frac{-u_{tij} + u_{ij}}{u - \varepsilon_0} + \frac{(u_t - u)u_{ij}}{(u - \varepsilon_0)^2},
\]

where (4.1) was used in deriving the second equality above. The inequality (4.2) should be understood in sense of positive-semidefinite matrix. Hence,

\[
u_{tij} + u_i \delta_{ij} \geq \theta(t)(-b_{ij} + \varepsilon_0 \delta_{ij})W + b_i,
\]

Thus,

\[
K_t = -Kb^{ij}(u_{tij} + u_i \delta_{ij}) \leq -nK - \theta(t)KW(-n + \varepsilon_0 H),
\]

where \( H \) denotes the mean curvature of \( X(\cdot, t) \). Noticing that \( H \geq nK^\frac{1}{n} \), we obtain

\[
K_t \leq CW(1 + W) - CW^{2 + \frac{1}{n}}.
\]

Using the equation (2.7) and the inequality above, we have

\[
W_t = \left[ \frac{f(x)}{\varphi(r)} \right] \left[ \frac{K}{u - \varepsilon_0} \right] K_t 
\leq CW^2 + CW - CW^{2 + \frac{1}{n}},
\]

in view of

\[
u_t \approx CW + C, \quad r_t = \frac{uu_t + u^i u_{ti}}{r} \approx CW + C.
\]

Without loss of generality we assume that \( K \approx W \gg 1 \), which implies that

\[
W_t \leq 0.
\]

Therefore, we arrive at \( W \leq C \) for some constant \( C > 0 \) depending on the \( C^1 \)-norm of \( r \) and \( \varepsilon_0 \). Thus, the priori bound follows consequently.

Now, we show the principle curvatures of \( X(\cdot, t) \) are bounded from below along the flow (1.1). The proof is similar to Lemma 4.2 in [17] and Lemma 5.1 in [9].

Lemma 4.2. Let \( X(\cdot, t) \) be a strictly convex solution to the flow (1.1) which encloses the origin for \( t \in [0, T) \). Then, there exists a positive constant \( C \) depending only \( \varphi, q, \max_{S^n \times [0, T)} u \) and \( \min_{S^n \times [0, T)} u \), such that the principle curvatures of \( X(\cdot, t) \) are bounded from below

\[
k_i(x, t) \geq C, \quad \forall (x, t) \in S^n \times [0, T), \text{ and } i = 1, 2, \ldots, n.
\]
Proof. We consider the auxiliary function
\[ \overline{\Lambda}(x, t) = \log \lambda_{\text{max}}((b_{ij})) - A \log u + B|Du|^2, \]
where \( A \) and \( B \) are positive constants which will be chosen later, and \( \lambda_{\text{max}}((b_{ij})) \) denotes the maximal eigenvalue of \((b_{ij})\). For convenience, we write \([b^{ij}]\) for \([b_{ij}]^{-1}\).

For any fixed \( t \in [0, T) \), we assume the maximum \( \Lambda \) is achieved at some point \( x_0 \in \mathbb{S}^n \). By rotation, we may assume \([b^{ij}(x_0, t)]\) is diagonal and \( \lambda_{\text{max}}((b_{ij}))(x_0, t) = b_{11}(x_0, t) \). Thus, it is sufficient to prove \( b_{11}(x_0, t) \leq C \).

Then, we define a new auxiliary function
\[ \Lambda(x, t) = \log b_{11} - A \log u + B|Du|^2, \]
which attains the local maximum at \( x_0 \) for fixed time \( t \). Thus, we have at \( x_0 \)
\[ 0 = D_i \Lambda = b_{11} \frac{u_i}{u} - A \sum_k u_k u_{ki}, \]
and
\[ 0 \geq D_i D_j \Lambda = b_{11} b_{11;j} - (b_{11})^2 b_{11;j} b_{11:j} - A \left( \frac{u_{ij}}{u} - \frac{u_i u_j}{u^2} \right) + 2B \sum_k \left( u_{kj} u_{ki} + u_k u_{kij} \right). \]

We can rewrite the equation (2.7) as
\[ \log(u - u_t) = -\log det(b) + \alpha(x, t), \]
where
\[ \alpha(x, t) = \log \left( \frac{(f(x))^n + 1}{u(x, t)} \right). \]

Differentiating (4.6) gives
\[ \frac{u_k - u_{kt}}{u - u_t} = -b^{ij} b_{ij;k} + D_k \alpha \]
and
\[ \frac{u_{11} - u_{11t}}{u - u_t} = \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} - b^{ij} b_{ij;11} + b^{ii} b_{i;1} (b_{i;1})^2 + D_1 D_1 \alpha. \]

Recalling the Ricci identity (2.1)
\[ b_{ii;1} = b_{11;ii} - b_{i;1} + b_{ii}, \]
which is taken into (4.8) implies
\[ \frac{u_{11} - u_{11t}}{u - u_t} = \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} - b^{ii} b_{11;ii} + \sum_i b^{ii} b_{11} - n + b^{ii} b_{i;1} (b_{i;1})^2 + D_1 D_1 \alpha. \]

So, we have
\[ \frac{\partial_t \Lambda}{u - u_t} = \frac{b^{ii} b_{i;11} - b^{ii} b_{i;11} (b_{i;1})^2}{u - u_t} - A \frac{u_i - u + u}{u - u_t} + 2B \frac{u_k u_{ki}}{u - u_t} \]
\[ = b^{ii} \left[ \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} + b_{i;11;ii} - \sum_i b^{ii} b_{11} - b^{ii} b_{i;1} (b_{i;1})^2 - D_1 D_1 \alpha \right] \]
\[ + \frac{1 - A}{u - u_t} + A \frac{\sum_k u_k u_{ki}}{u - u_t} + (n - 1)b^{ii}. \]

We know from (4.5) and (4.7)
\[ 0 \geq b^{ii} \left[ b^{ii} b_{11;ii} - b^{ii} b_{i;11} (b_{i;1})^2 \right] - A \frac{u_i + A}{u} \sum_i b^{ii} + Ab^{ii} \frac{u_i u_{ij}}{u^2}. \]
Thus, plugging the inequality above into (4.10) gives

\[ +2B \left[ b^{ii}(b_{ij} - u)^2 + \sum_k u_k(D_k \alpha - \frac{u_k - u_{ji}}{u - u_i}) - b^{ii} u_i u_j \right] \]

\[ \geq b^{11} [b^{ii} b_{11;ji} - b^{ij} b^{jj}(b_{ji;1})^2] - A \frac{n}{u} + A \sum_i b^{ii} + Ab^{ii} \frac{u_i u_j}{u^2} \]

\[ +2B \left[ \sum_k b^{ii} (b_{11}^2 - 2u b_{ij}) + \sum_k u_k(D_k \alpha - \frac{u_k - u_{ki}}{u - u_i}) - b^{ii} u_i u_j \right] \]

\[ \geq b^{11} [b^{ii} b_{11;ji} - b^{ij} b^{jj}(b_{ji;1})^2] - A \frac{n}{u} + A \sum_i b^{ii} + Ab^{ii} \frac{u_i u_j}{u^2} \]

\[ +2B \left[ \sum_i b_{ii} - 2nu + \sum_k u_k(D_k \alpha - \frac{u_k - u_{ki}}{u - u_i}) - b^{ii} u_i u_j \right]. \]

Thus, plugging the inequality above into (4.10) gives

\[
\frac{\partial_1 \Lambda}{u - u_i} \leq -b^{11} D_1 D_1 \alpha - 2B \sum_k u_k D_k \alpha + \frac{1 - A + 2B|Du|^2}{u - u_i}
\]

\[ + \frac{(n + 1)A}{u} + (n - 1)b^{11} + (2B|Du| - A - 1) \sum_i b^{ii} \]

\[ - Ab^{ii} \frac{u_i u_j}{u^2} - 2B \sum_i b_{ii} + 4Bu. \]

Now, we need estimate the first two terms in the inequality above. Clearly, a direct calculation results in

\[ r_i = \frac{uu_i}{r} + \sum_k u_k u_{ki} = \frac{uu_i}{r} \]

and

\[ r_{ij} = \frac{uu_{ij} + uu_{ij} + \sum_k u_k u_{kj} + \sum_k u_k u_{ki}}{r} - \frac{uu_i u_j u_{ij}}{r^3}. \]

Hence, we obtain by Lemma 3.1 Lemma 3.3 and Corollary 3.4

\[ -b^{11} D_1 D_1 \alpha - 2B \sum_k u_k D_k \alpha \]

\[ = -b^{11} \left[ f_{11} \frac{f_{11}}{f^2} - \frac{f_{11}^2}{f^2} - (n + 1) \frac{r_{11}^2}{r^2} + \frac{(\varphi')^2}{\varphi^2} \frac{r_{11}^2}{r} - \frac{\varphi''}{\varphi} \frac{r_{11}^2}{r} \right] - b^{11} \left[ (n + 1) \frac{1}{r} - \frac{\varphi'}{\varphi} \right] \]

\[ \leq C b^{11} (1 + b_{11}) + CB - \left[ (n + 1) \frac{1}{r} - \frac{\varphi'}{\varphi} \right] \left( b^{11} r_{11} + 2Bu_r \right) \]

\[ \leq C b^{11} (1 + b_{11} + b_{11}^2) + CB - \left[ (n + 1) \frac{1}{r} - \frac{\varphi'}{\varphi} \right] \left( b^{11} \frac{u_k u_{11}^2}{r} + 2B \frac{u_k u_{11} u_{11}^2}{r} \right). \]

Then, using (4.4), we have

\[ -b^{11} D_1 D_1 \alpha - 2B \sum_k u_k D_k \alpha \]

\[ \leq C b^{11} (1 + b_{11} + b_{11}^2) + CB - \left[ (n + 1) \frac{1}{r} - \frac{\varphi'}{\varphi} \right] \left( A \frac{u_k}{u} - b^{11} u_1 \delta_{kl} \right) \]

\[ \leq C b^{11} (1 + b_{11} + b_{11}^2) + CB + CA. \]
Thus, using the inequality above, we conclude from (4.11)
\[
\frac{\partial_t \Lambda}{u - u_t} \leq C(b^{11} + 1 + b_{11}) + CB + CA + \frac{(n + 1)A}{u} + (n - 1)b^{11} + (2B|Du| - A - 1) \sum_i b^{ii} \\
- Ab^{ii} \frac{u_i u_i}{u^2} - 2B \sum_i b^{ii} + 4nBu \\
< 0,
\]
provided \(b_{11} \gg 1\) and if we choose \(A \gg B\). So we complete the proof. \(\square\)

5. The convergence of the normalised flow

With the help of a prior estimates in the section above, we show the long-time existence and asymptotic behaviour of the normalised flow (1.1) which complete Theorem 1.2.

**Proof.** Since the equation (2.7) is parabolic, we have the short time existence. Let \(T\) be the maximal time such that \(u(\cdot, t)\) is a positive, smooth and strictly convex solution to (2.7) for all \(t \in [0, T)\). Lemmas 3.1, 3.2, 4.1 and Corollary 3.4 enable us to apply Lemma 4.2 to the equation (2.7) and thus we can deduce a uniformly lower estimate for the biggest eigenvalue of \((u_{ij} + u \delta_{ij})(x, t))\). This together with Lemma 4.2 implies
\[
C^{-1} I \leq (u_{ij} + u \delta_{ij})(x, t) \leq CI, \quad \forall (x, t) \in \mathbb{S}^n \times [0, T),
\]
where \(C > 0\) depends only on \(n, \alpha, f\) and \(u_0\). This shows that the equation (2.7) is uniformly parabolic. Using Evans-Krylov estimates and Schauder estimates, we obtain
\[
|\nabla u|_{C^{1,\alpha}(\mathbb{S}^n \times [0, T))} \leq C_{l,m}
\]
for some \(C_{l,m}\) independent of \(T\). Hence \(T = \infty\). The uniqueness of the smooth solution \(u(\cdot, t)\) follows by the parabolic comparison principle.

By the monotonicity of \(J_\phi\) (See Lemma 2.2), and noticing that
\[
|J_\phi(X(\cdot, t))| \leq C, \quad \forall t \in [0, \infty),
\]
we conclude that
\[
\int_0^\infty \left| \frac{d}{dt} J_\phi(X(\cdot, t)) \right| dt \leq C.
\]
Hence, there is a sequence \(t_i \to \infty\) such that
\[
\frac{d}{dt} J_\phi(X(\cdot, t_i)) \to 0.
\]
In view of Lemma 2.2 we see that \(u(\cdot, t_i)\) converges smoothly to a positive, smooth and strictly convex \(u_\infty\) solving (1.2) with \(f\) replaced by \(\lambda_0 f\) with \(\lambda_0 = \lim_{t_i \to \infty} \theta(t_i)\). \(\square\)

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