Spectral statistics for product matrix ensembles of Hermite type with external source

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Abstract

We continue investigating spectral properties of a Hermitised random matrix product, which, contrary to previous product ensembles, allows for eigenvalues on the full real line. When a GUE matrix with an external source is involved, we prove that the eigenvalues of the product form a determinantal point process and derive a double integral representation for correlation kernel. As the source changes, we observe a critical value and establish the existence of a phase transition for scaled eigenvalues at the origin. Particularly in the critical case, we obtain a new family of Pearcey-type kernels.

1 Introduction and main results

1.1 Motivations

In this paper we continue the investigation of spectral properties of Hermitised product matrix ensembles initiated in [23]. More specifically, suppose that (1) each $G_i$ ($i = 1, \ldots, M$) is a standard complex Ginibre matrix of size $(\nu_{m-1} + n) \times (\nu_m + n)$ with $\nu_0 = 0, \nu_1, \ldots, \nu_M \geq 0$, i.e. a matrix with i.i.d. standard complex Gaussian entries; (2) $H$ is an $n \times n$ matrix from the Gaussian unitary ensemble (GUE) with an external source which is specified by the probability measure on $\mathbb{R}^{n^2}$ with density

$$2^{\frac{1}{2}n(n-1)} \pi^{-\frac{1}{2}n^2} e^{-\text{tr}(H-B)^2},$$

where $B$ is a deterministic $n \times n$ Hermitian matrix with eigenvalues denoted by $b_1, \ldots, b_n$, we are devoted to studying the eigenvalues of the Hermitised product matrix

$$W_M = G_M^* \cdots G_1^* HG_1 \cdots G_M$$

under the assumption that all matrices, $H$ and $G_i$ ($i = 1, \ldots, M$), are independent. We will see that the eigenvalues of $W_M$ form a determinantal point process as in the situation without $H$ which was first studied by Akemann et al. [4, 5].

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When $B = 0$ in (1.1), the global and local spectral properties for the product (1.2) have recently been studied in [23]. In particular, a new family of Meijer G-function type kernels at the origin was found therein, which is defined for $x, y \in \mathbb{R} \setminus \{0\}$ by

$$K_{\nu_1, \ldots, \nu_M}^{\text{(sub)}}(x, y) = \frac{1}{2\pi i} \int_{C_R} dv G_{0, M+1}^{1, 0}(0, -\nu_1, \ldots, -\nu_M) \left| -\text{sgn}(xy)|x|v \right|$$

$$\times G_{0, M+1}^{M+1, 0}(0, \nu_1, \ldots, \nu_M) \left| |y|v \right|$$  \hspace{1cm} (1.3)

with $C_R$ denoting a path in the right half-plane from $-i$ to $i$; see e.g. [40] for definition of Meijer G-functions. This is slightly different from the Meijer $G$-kernel defined for $x, y > 0$ by

$$K_{\text{Meijer}}^M(x, y) = \int_0^1 du G_{0, M+1}^{1, 0}(0, -\nu_1, \ldots, -\nu_M) \left| xu \right| G_{0, M+1}^{M, 0}(0, \nu_1, \ldots, \nu_M) \left| |yu| \right|,$$  \hspace{1cm} (1.4)

which was first obtained in [34] for the product of independent Ginibre matrices, i.e., (1.2) but with $H = I_n$.

Actually, the past few years have witnessed a very rapid development in the topic of products of independent random matrices. A crucial advance was the derivation of exact eigenvalue density for the product (1.2) with $H = I_n$ by Akemann and his coworkers [5, 4], which shows that it forms a determinantal point process. Subsequently, it was shown by Kuijlaars and Zhang in [34] that the corresponding correlation kernel admits a double integral formula. These have opened up the possibility to investigate local statistical properties of eigenvalues. Actually, a new family of limiting kernels, so-called Meijer $G$-kernels (1.4), was found in [34] at the hard edge and the standard Sine and Airy kernels in the bulk and soft edge of the spectrum was proved in [37]. Even more interestingly, the Meijer $G$-kernel also appears in other product ensembles [22, 33, 30] and Cauchy matrix models [10, 11]. All these studies form part of a fast paced and very recent literature relating to the integrability and universality of random matrix products. We refer the reader to [3] for a recent survey.

In another special case when $M = 0$, (1.2) reduces to the well-known Gaussian Unitary Ensemble with external source (also called deformed GUE ensemble in the literature). The deformed GUE ensemble has been treated in a series of papers [1, 7, 13, 14, 16, 17, 12, 19, 20, 28, 41, 42, 43]. More generally, see [31, 35, 36] and references therein for deformed Wigner ensembles. As the eigenvalues of the source $B$ change at a certain critical rate, except that there exists a phase transition for largest eigenvalues due to Baik, Ben Arous and Péché [8, 41] (sometimes called BBP transition in the literature), another interesting phenomenon will appear at the origin and can be described by the so-called Pearcey kernel [17] (a very special case of (1.14) below where $M = 0$ and $p = 0$). See also [1, 14, 12, 39, 43] for the Pearcey kernel.

It is worth stressing that the product (1.2) with $H = (G_0 + A)^n(G_0 + A)$, where $G_0$ is a Ginibre matrix and $A$ is a deterministic matrix, has been studied in [24]. As the source matrix $A$ changes, a phase transition phenomenon for smallest singular values is observed at the origin. In particular, there exists a new family of kernels defined in terms of Meijer $G$-functions at the critical value. It’s our goal in the present paper to prove the existence of a phase transition at the origin for the product (1.2) with $H$ distributed according to the density (1.1).
1.2 Main results

Let $\Delta_n(x) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$ denote the Vandermonde determinant. We are ready to state our main results as follows.

The first is about the eigenvalue probability density function (PDF for short) of the product (1.2) and can be derived after a direct application of [23, Lemma 2].

**Proposition 1.** Let $\nu_0 = 0, \nu_1, \ldots, \nu_M$ be non-negative integers. Suppose that $H$ is a random $n \times n$ Hermitian matrix with density (1.1) and that $G_1, \ldots, G_M$ are independent standard complex Gaussian matrices where $G_m$ is of size $(\nu_m - 1 + n) \times (\nu_m + n)$, independent of $H$. Then the joint PDF for non-zero eigenvalues of the product $W_M$ defined in (1.2) is given by

$$P_{n,M}(x_1, \ldots, x_n) = \frac{1}{Z_{n,M}} \Delta_n(x) \det \left[ g_M(x_i, b_j) \right]_{i,j=1}^{n} x_1, \ldots, x_n \in \mathbb{R} \setminus \{0\},$$

where $g_M$ is a function of two variables defined for $(y, v) \in \mathbb{R} \setminus \{0\} \times \mathbb{C}$ by

$$g_M(y, v) = \int_0^\infty \frac{dt_1}{t_1} \cdots \int_0^\infty \frac{dt_M}{t_M} \prod_{l=1}^{M} t_l^{\nu_l} e^{-t_l} \exp \left\{ - \frac{y^2}{(t_1 \cdots t_M)^2} + \frac{2yv}{t_1 \cdots t_M} \right\},$$

and the normalisation constant

$$Z_{n,M} = n! e^{\sum_{j=1}^{M} \nu_j} \Delta_n(b) \prod_{m=1}^{M} \prod_{j=1}^{n} \Gamma(\nu_m + j).$$

When some of $b_j$’s coincide, L’Hospital’s rule provides an appropriate density.

Our second result is a double integral representation of correlation kernel for the bi-orthogonal ensemble (1.5) as a determinantal point process (see e.g. [15] for the bi-orthogonal ensemble with more details). For this, let us introduce one auxiliary function, which is defined for non-negative integers $\nu_1, \ldots, \nu_M$ and for $(x, u) \in \mathbb{R} \times \mathbb{C}$ by

$$f_M(x, u) = \frac{1}{(2\pi i)^M} \int_{C_0} \frac{ds_1}{s_1} \cdots \int_{C_0} \frac{ds_M}{s_M} \prod_{l=1}^{M} s_l^{-\nu_l} e^{s_l} \exp \left\{ \frac{x^2}{(s_1 \cdots s_M)^2} - \frac{2ux}{s_1 \cdots s_M} \right\}$$

where $C_0$ is an anticlockwise loop around the origin.

Note that when $M = 0 f_0(x, u) = e^{x^2 - 2ux}$ and $g_0(y, v) = e^{-y^2 + 2yv}$, by convention. Moreover, it is easy to verify two simple facts: (1) $|f_M(x, u)| \leq C(x)e^{|xu|}$ for some constant depending on $x$, just by letting each contour be a unit circle and noting the inequality $|e^z| \leq e^{|z|}$; (2) $g_M(y, v)$ is an analytic function of $v$ whenever $y \in \mathbb{R} \setminus \{0\}$.  

**Theorem 2.** With two functions defined in (1.6) and (1.8), the correlation kernel associated with the eigenvalue PDF (1.5) is given by

$$K_n(b; x, y) = \frac{1}{2(\pi i)^2} \int_{L} du \int_{C_b} dv f_M(x, u) g_M(y, v) e^{u^2 - v^2} \prod_{l=1}^{n} \frac{u - b_l}{v - b_l},$$

where $C_b$ encircles $b_1, \ldots, b_n$ in an anticlockwise direction, and $L$ is a path from $-i\infty$ to $i\infty$ not crossing $C_b$.  

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The third is the key result of the present paper. It describes a phase transition phenomenon of eigenvalues at the origin, as the source matrix $B$ changes. Specifically, except for finitely many eigenvalues of $B$, say $b_1, \ldots, b_r$, we assume that one half of the rest are equal to $\sqrt{n/2a}$ and the other half $-\sqrt{n/2a}$. As $n$ goes to infinity, we observe three different families of limiting kernels.

**Theorem 3** (Phase transition at the origin). With the kernel (1.9), let $r$ be a fixed nonnegative integer such that $n - r = 2n_0$ is even, and suppose that

$$b_{r+1} = \cdots = b_{r+n_0} = -b_{r+n_0+1} = \cdots = -b_n = \sqrt{n/2a}, \quad a \geq 0. \tag{1.10}$$

The following hold true uniformly for $x, y$ in a compact set of $\mathbb{R} \setminus \{0\}$.

(i) When $0 \leq a \leq \sqrt{2}/2$, let $b_l = \sqrt{n/2a_l}$ such that $|a_l| < a + 1$ for $l = 1, \ldots, r$, then

$$\lim_{n \to \infty} \frac{1}{\sqrt{2(1-a)^n}} K_n \left( b_l \frac{x}{\sqrt{2(1-a)^n}}, \frac{y}{\sqrt{2(1-a)^n}} \right) = K_{\nu_1, \nu_M}^{(\text{sub})} (x, y), \tag{1.11}$$

where $K_{\nu_1, \nu_M}^{(\text{sub})} (x, y)$ is defined by (1.3).

(ii) When $a = (1 - \frac{\tau}{2\sqrt{n}})^{-1}$ with real $\tau$, for $0 \leq p_0 \leq p \leq r$ let

$$b_1 = 2^{-\frac{1}{4}n^{\frac{1}{4}}a_1}, \ldots, b_p = 2^{-\frac{1}{4}n^{\frac{1}{4}}a_p}, \quad a_1 \leq \cdots \leq a_{p_0} < a < a_{p_0+1} \leq \cdots \leq a_p, \tag{1.12}$$

and let $b_l = \sqrt{n/2a_l}$ for $l = p + 1, \ldots, r$, then

$$\lim_{n \to \infty} \frac{1}{\sqrt{2n}} K_n \left( b_l \frac{x}{\sqrt{2n}}, \frac{y}{\sqrt{2n}} \right) = K_{\nu_1, \nu_M}^{(\text{crit})} (\tau; x, y) \tag{1.13}$$

where

$$K_{\nu_1, \nu_M}^{(\text{crit})} (\tau; x, y) = \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} du \int_{\Sigma_- \cup \Sigma_+} dv e^{\frac{1}{2}(u^2 - v^2) - \frac{1}{4}(u^4 - v^4)} \frac{1}{u - v} \prod_{j=1}^p \frac{u - a_j}{v - a_j} \times C_{0, M+1}^{1, 0} \left( 0, -\nu_1, \ldots, -\nu_M | xu \right) C_{0, M+1}^{1, 0} \left( 0, \nu_1, \ldots, \nu_M | -yv \right). \tag{1.14}$$

Here $\Sigma_-$ is a path in the left half-plane from $e^{-3i\pi/4}$ to $v_1^{n/4}$ with $a_1, \ldots, a_{p_0}$ to its left side, while $\Sigma_+$ is a path in the right half-plane from $e^{3i\pi/4}$ to $e^{-i\pi/4}$ with $a_{p_0+1}, \ldots, a_p$ to its right side.

(iii) When $a > 1$, for $1 < p \leq r$ let $b_l = a_l$ for $l = 1, \ldots, p$ and let $b_m = \sqrt{n/2a_m}$ with $a_m \neq 0$ for $m = p+1, \ldots, r$, then

$$\lim_{n \to \infty} K_n \left( b_l \frac{x}{\sqrt{n/2a_l}}, \frac{y}{\sqrt{n/2a_l}} \right) = K_{\nu_1, \nu_M}^{(\text{sup})} (x, y), \tag{1.15}$$

where

$$K_{\nu_1, \nu_M}^{(\text{sup})} (x, y) = \frac{1}{2(2\pi)^2} \int_{\mathcal{C}_n} du \int_{\mathcal{C}_M} dv g_M (y, v) e^{(1 - \frac{1}{4}n^{\frac{1}{4}})(u^2 - v^2)} \frac{1}{u - v} \prod_{l=1}^p \frac{u - a_l}{v - a_l}, \tag{1.16}$$

**Remark 1.** We believe part (i) of Theorem 3 holds true whenever $a \in [0, 1)$, as in the GUE ensemble with external source; see e.g. [7]. The reason that we impose restrictions on $a$ is mainly because of the choice of contours. If we could remove the restriction stated in Proposition 4 of Sect. 3, then part (i) of Theorem 3 holds true too.
The rest of the article is organised as follows. In Section 2, we derive the eigenvalue PDF for the product (1.2) as a bi-orthogonal ensemble and give an explicit double integral for correlation kernel. The scaling limits are at the origin are proved in Section 3. Finally, in Section 4 we give some discussion on the global density.

2 Eigenvalue PDF and double integral for correlation kernel

With Lemma 2 of [23] at hand, we are immediately ready to write down the eigenvalue PDF for the product (1.2) and thus give a proof of Proposition 1.

Proof of Proposition 1. It is sufficient to derive the eigenvalue PDF of the product \( H(G_1 \cdots G_M)(G_1 \cdots G_M)^* \). Since the product \((G_1 \cdots G_M)(G_1 \cdots G_M)^*\) is only involved, equivalently, we can suppose that each \( G_j \) is an \( n \times n \) random matrix with density proportional to \( \det(G_j^*G_j)^{\nu} \exp\{-\Tr(G_j^*G_j)\} \) according to the results from [4, 26]. It is well-known that the eigenvalue PDF of an \( n \times n \) GUE matrix with an external source is given by (1.5) with \( M = 0 \), i.e. \( g_0(y, v) = \exp\{-y^2 + 2yv\} \) (see e.g. [28] or [21]). Note that when \( G \) is a square matrix and is distributed as \( \det(G^*G)^{\nu} \exp\{-\Tr(G^*G)\} \) up to a normalisation constant, Theorem 1 of [23] holds true, so does Lemma 2 of [23]. We thus complete the proof after repeating the lemma \( M \) times. \( \square \)

Next, we settle down to the derivation of double contour integrals for correlation kernel of the bi-orthogonal ensemble (1.5).

Proof of Theorem 2. First, we need to compute the moment matrix \( B_n = (b_{i,j})_{i,j=1}^n \) via Hermite polynomials and their integral representations given by

\[
H_m(z) := (-1)^m e^z \frac{d^m e^{-z^2}}{dz^m} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (2ix)^m e^{-(x+i)^2} dx, \tag{2.1}
\]

and get

\[
b_{k,\ell} := \int_{-\infty}^{\infty} x^{k-1} g_M(x, b_{\ell}) dx = \sqrt{\pi} (2i)^{-k+1} e^{b_{\ell}^2} H_{k-1}(ib_{\ell}) \prod_{m=1}^{M} \Gamma(\nu_m + k). \tag{2.2}
\]

Let \( C_n = (c_{k,\ell}) \) be the inverse of \( B_n \), then the correlation kernel for the bi-orthogonal ensemble (1.5) can be rewritten as a summation

\[
K_n(b; x, y) = \sum_{k,\ell=1}^{n} c_{\ell,k} x^{k-1} g_M(y, b_{\ell}), \tag{2.3}
\]

see e.g. [15, Proposition 2.2].

The entries \( c_{i,j} \) of \( C_n \) satisfy the relation \( \sum_{k=1}^{n} c_{j,k} b_{k,\ell} = \delta_{j,\ell} \), which we specify for

\[
\sum_{k=1}^{n} \sqrt{\pi} (2i)^{-k+1} e^{b_{\ell}^2} H_{k-1}(ib_{\ell}) \prod_{m=1}^{M} \Gamma(\nu_m + k) c_{j,k} = \delta_{j,\ell}, \quad j, \ell = 1, \ldots, n. \tag{2.4}
\]

Without loss of generality, we assume that \( b_1, \ldots, b_n \) are pairwise distinct. The above equations immediately imply

\[
\sum_{k=1}^{n} \sqrt{\pi} (2i)^{-k+1} H_{k-1}(iu) \prod_{m=1}^{M} \Gamma(\nu_m + k) c_{j,k} = e^{-b_j^2} \prod_{l=1, l \neq j}^{n} \frac{u - b_l}{b_j - b_l}. \tag{2.5}
\]
These can be verified by noting that both sides are polynomials of degree \( n - 1 \) in \( u \) and take the same values at \( n \) different points since (2.4) holds true.

Using these implicit formulas for \( \{c_{j,k}\} \) we are ready to show that (2.3) implies the double contour integral formula (1.9). Use the integral representations

\[
\frac{1}{\Gamma(l)} = \frac{1}{2\pi i} \int_{C_0} s^{-l} e^s ds, \quad (2iz)^{k-1} = \frac{1}{\sqrt{\pi i}} \int_{\mathcal{L}} e^{(u-z)^2} H_{k-1}(iu) du,
\]

where \( \mathcal{L} \) is a path from \(-i\infty\) to \(i\infty\), combine the identity (2.5) and we rewrite

\[
\sum_{k=1}^{n} x^{k-1} c_{k,\ell} = \int_{C_0} \frac{ds_1}{2\pi i} \cdots \int_{C_0} \frac{ds_M}{2\pi i} \prod_{l=1}^{M} s_l^{-\nu_l-1} e^{s_l} \sum_{k=1}^{n} \left( \frac{2ix}{s_1 \cdots s_M} \right)^{k-1} \prod_{m=1}^{M} \Gamma(\nu_m + k) c_{j,k}
\]

\[
= \frac{1}{\pi i} \int_{\mathcal{L}} f_M(x,u)e^{u^2-b_i^2} \prod_{l=1}^{n} \frac{u-b_l}{b_\ell-b_l}
\]

where we have exchanged the order of integration and used the definition of \( f_M(x,u) \) (1.8).

Finally, recognising the summation in (2.3) over \( \ell \) as the summation of the residues at \( b_1, b_2, \ldots, b_n \) for the \( \nu \)-function

\[
g_M(y,v)e^{-v^2} \frac{1}{u-v} \prod_{l=1}^{n} \frac{u-b_l}{v-b_l}
\]

and using Cauchy residue theorem, we thus arrive at the formula (1.9) by choosing two disjoint contours \( \mathcal{L} \) and \( C_b \).

\[\square\]

3 Scaling limits at the origin

In this section we prove part (i), (ii) and (iii) of Theorem 3 in turn.

Proof of Theorem 3: part (i). By the assumptions on \( b_1, \ldots, b_n \), substitute \( u, v \) by \( \sqrt{n/2}u, \sqrt{n/2}v \) respectively in (1.9) and we obtain for \( \xi = x/\sqrt{1-a^2} \) and \( \eta = y/\sqrt{1-a^2} \)

\[
\frac{1}{\sqrt{2n}} K_n(b; \frac{\xi}{\sqrt{2n}}, \frac{\eta}{\sqrt{2n}}) = \frac{1}{(2\pi i)^2} \int_{\mathcal{L}} du \int_{\mathcal{C}} dv f_M(\xi/\sqrt{2n}, \sqrt{n/2}u) g_M(\eta/\sqrt{2n}, \sqrt{n/2}v)
\]

\[
\times e^{n(h(u)-h(v))} \frac{1}{u-v} \left( \frac{u^2-a^2}{v^2-a^2} \right)^{-r/2} \prod_{l=1}^{r} \frac{u-a_l}{v-a_l}
\]

(3.1)

where \( \mathcal{C} \) encircles \( \pm a, a_1, \ldots, a_r \) and \( \mathcal{L} \) is a path from \(-i\infty\) to \(i\infty\), and the phase function

\[
h(z) = \frac{1}{2} z^2 + \frac{1}{2} \log(a^2 - z^2).
\]

Since

\[
h'(z) = z - \frac{z}{a^2 - z^2},
\]

(3.2)
we easily know that the equation \( h'(z) = 0 \) has three solutions
\[
    z_0 = 0, \quad z_\pm = \pm i \sqrt{1 - a^2}, \quad (3.4)
\]
from which we distinguish three scenarios: (i) \( 0 \leq a < 1 \); (ii) \( a = 1 \); (iii) \( a > 1 \). When 
\( a = 1 \), the three simple saddle points coalesce into a third-order point at zero and thus this is a critical case.

Although both the functions \( f_M \) and \( g_M \) in the integrand of (3.1) depend on \( n \), we will see below that for the large \( n \) they do not enter the saddle point equation. So we may perform saddle-point approximations and this is what we will do next in details.

In order to investigate the case (i) with \( 0 \leq a < 1 \), we first proceed to consider the situation \( \eta < 0 \). For this, we need to deform the integral contours as follows. Given \( \delta \geq 0 \), let’s first define \( C_{R,\delta} \) as a great arc along the circle with radius \( \sqrt{1 + \delta^2 + 2a\delta} \) and centre at \( a + \delta \), which is entirely in the right-half plane and connects the two points \( -i\sqrt{1 - a^2} \) and \( i\sqrt{1 - a^2} \). Let \( C_{L,\delta} \) be the reflection of \( C_{R,\delta} \) about the \( y \)-axis. Choose \( \bar{C} = C_{L,0} \cup C_{R,0} \) and deform \( L \) as the union of the \( y \)-axis and \( \bar{C}_R := C_{R,0.1} \cup \{(0,\eta) : |\eta| \leq \sqrt{1 - a^2}\} \), with \( \bar{C}_R \) in a counterclockwise direction and the \( y \)-axis from \(-i\infty \) to \( i\infty \). Note the assumption on \( a_1, \ldots, a_r \), such a choice assures that \( \bar{C}_R \) encircles \( \pm a, a_1, \ldots, a_r \). Divide the integration over \( L \) into two parts, we further rewrite the double integral on the RHS of (3.1) as a sum of two integrals
\[
    \frac{1}{\sqrt{2\pi n}} K_\eta(b; \frac{\xi}{\sqrt{2\pi n}}, \frac{\eta}{\sqrt{2\pi n}}) = \text{P.V.} \int_{i\delta} du \int_{C} dv f_M(\xi/\sqrt{2\pi n}, \sqrt{n/2}v)g_M(\eta/\sqrt{2\pi n}, \sqrt{n/2}v). \quad (3.5)
\]
Here the notation P.V. denotes the Cauchy principal value integral.

As \( n \to \infty \), we claim that the integral over the range of \( v \in C_{R,0} \) and \( u \in \bar{C}_R \) gives rise to a leading contribution to the double integral on the RHS of (3.1). Actually, for \( I_2 \), when \( v \in C_{L,0} \) the \( u \)-integral vanishes by Cauchy’s theorem since the integrand does not have any singularity inside \( \bar{C}_R \), while for \( v \in C_{R,0} \) application of the residue theorem shows
\[
    I_2 = \frac{1}{2\pi i} \int_{C_{R,0}} dv f_M(\xi/\sqrt{2\pi n}, \sqrt{n/2}v)g_M(\eta/\sqrt{2\pi n}, \sqrt{n/2}v). \quad (3.6)
\]
Consideration of the definition (1.8) permits us to get as \( n \to \infty \)
\[
    f_M(\xi/\sqrt{2\pi n}, \sqrt{n/2}v) \sim \sum_{k=0}^{\infty} \frac{(-\xi v)^k}{k!} \prod_{l=1}^{M} \frac{1}{\Gamma(\nu_l + 1 + k)}, \quad (3.7)
\]
the RHS of which is recognized as a Meijer G-function via
\[
    \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \prod_{l=1}^{M} \frac{1}{\Gamma(\nu_l + 1 + k)} = G_{0,M+1}^{1,0}(0, -\nu_1, \ldots, -\nu_M | z). \quad (3.8)
\]
Here the notation \( f_{1,n} \sim f_{2,n} \) means that \( \lim_{n \to \infty} f_{1,n}/f_{2,n} = 1 \).

However, in order to obtain the leading asymptotic behaviour of \( g_M(\eta/\sqrt{2\pi n}, \sqrt{n/2}v) \), we need to derive a Mellin-type integral representation of \( g_M(y,v) \) for \( (y,v) \in \mathbb{R} \setminus \{0\} \times \mathbb{C} \)
\[
    g_M(y,v) = \frac{e^{v^2/2}}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \left( \sqrt{2} |y| \right)^{-s} U \left( s - \frac{1}{2}, -\sqrt{2} \text{sgn}(y)v \right) \prod_{l=0}^{M} \Gamma(\nu_l + s), \quad (3.9)
\]
where $c > 0$ and the parabolic cylinder function
\[
U(c, z) = e^{-\frac{1}{2}z^2} \frac{1}{\Gamma(c + \frac{1}{2})} \int_0^\infty e^{-\frac{1}{2}t} e^{-\frac{1}{2}t^2 - zt} dt, \quad \text{Re}(c) > -\frac{1}{2}.
\] (3.10)

This can be proved from (1.6) by applying Mellin and inverse Mellin transforms if $g_M(y, v)$ is treated as a function of the variable $y$ over $(0, \infty)$. When $y \in (-\infty, 0)$, we just turn to consider the variable $-y > 0$.

Using asymptotic expansion of the parabolic cylinder function (3.10) as $z \to \infty$ (see e.g. [40, Sect. 12.9])
\[
U(c, z) = \begin{cases} 
    z^{-c - \frac{1}{2}} e^{-\frac{1}{2}z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right), & |\text{ph}(z)| < \frac{3}{4} \pi, \\
    \frac{1}{\Gamma(c + \frac{1}{2})} \sqrt{2\pi} (-z)^{c + \frac{1}{2}} e^{\frac{1}{2}z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right), & \frac{3}{4} \pi < |\text{ph}(z)| < \frac{5}{4} \pi,
\end{cases}
\] (3.11)

we have
\[
g_M(\eta/\sqrt{2n}, \sqrt{n/2}v) \sim \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \left|\eta/v\right|^{-s} \prod_{l=0}^{M} \Gamma(\nu_l + s) = G_{0, M+1}^{0, 1}\left(0, \nu_1, \ldots, \nu_M \left|\eta/v\right.\right).
\] (3.12)

Combining (3.6), (3.7) and (3.12), after a change of variables we see that $I_2$ leads to the limiting kernel in part (ii).

Next, we deal with the integral $I_1$ and show that it is negligible compared to $I_2$. In this case because of different asymptotic forms of $g_M$, we divide $I_1$ into two parts again as
\[
I_1 = \text{P.V.} \int_{i\mathbb{R}} du \int_{C_{1,+}} dv \left|\cdot\right| + \int_{i\mathbb{R}} du \int_{C_{1,-}} dv \left|\cdot\right| := I_{11} + I_{12},
\] (3.13)

where $C_{1,+} = \{z \in C_{L,0} \cup C_{R,0} : |\text{ph}(z)| < \frac{3}{4} \pi\}$ and $C_{1,-} = \{z \in C_{L,0} : \frac{3}{4} \pi < |\text{ph}(z)| < \frac{5}{4} \pi\}$.

When $0 \leq a \leq \sqrt{2}/2$, Proposition 4 below shows that $\text{Re}(h(z))$ attains its global minimum at $\pm i\sqrt{1-a^2}$ over $C_{L,0} \cup C_{R,0}$, and attains its global maximum at $\pm i\sqrt{1-a^2}$ over $i\mathbb{R}$. Therefore, for $I_{11}$, combining (1.8), (3.9) and (3.11) we obtain
\[
I_{11} \sim \text{P.V.} \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} du \int_{C_{1,+}} dv \left|\cdot\right| e^{\eta h(u) - h(v)} \prod_{l=1}^{r} \frac{u-a_l}{v-a_l} \times G_{0, M+1}^{1, 0}\left(0, -\nu_1, \ldots, -\nu_M \left|\xi u\right.\right) G_{0, M+1}^{M+1, 0}\left(0, \nu_1, \ldots, \nu_M \left|\eta/v\right.\right).
\] (3.14)

For this, the standard steepest descent argument shows that the leading term for the integral $I_{11}$ comes from the neighbourhood of the saddle points $(z_+, z_+)$ and $(z_-, z_-)$ and can be estimated by
\[
I_{11} = O\left(\frac{1}{\sqrt{n}}\right).
\] (3.15)

Similarly, for $I_{12}$, combination of (1.8), (3.9) and (3.11) gives rise to
\[
I_{12} \sim \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} du \int_{C_{1,-}} dv \left|\cdot\right| e^{\eta h(u) - h(v)} \prod_{l=1}^{r} \frac{u-a_l}{v-a_l} G_{0, M+1}^{1, 0}\left(0, -\nu_1, \ldots, -\nu_M \left|\xi u\right.\right) \times \frac{e^{\eta h(u) - h(\sqrt{1-a^2})}}{u-v} \int_{c-i\infty}^{c+i\infty} ds \frac{\left|\cdot\right|}{\sqrt{2\pi t}} \left|\eta\right|^{-s} \left(-nv\right)^{s-\frac{1}{2}} \sqrt{n} e^{-\frac{1}{2}(\log(a^2-v^2)+1-a^2)} \prod_{l=1}^{M} \Gamma(\nu_l + s).
\] (3.16)
We claim that the integrals of \( u \) and \( v \) respectively afford us bounds \( O(n^{-\frac{3}{2}}) \) and \( O(n^{-\frac{3}{2}}e^{-\frac{1}{2}(1-a^2)n}) \). The former can be obtained via the steepest descent argument. For the latter, writing \( v = -a + e^{i\theta} \in \mathcal{C}_{L,0} \), it is seen from \( \cos \theta \leq a \) that

\[
\text{Re}\{\log(a^2 - v^2)\} + 1 - a^2 = \frac{1}{2} \log(1 + 4a^2 - 4a \cos \theta) + 1 - a^2 \geq 1 - a^2. \tag{3.17}
\]

Together, we arrive at an exponential decay estimation

\[
I_{12} = O(n^{-1}e^{-n(1-a^2)/2}). \tag{3.18}
\]

Combining (3.6), (3.15) and (3.18), note (3.5) and (3.13) and we complete the proof of part (i) for \( \eta < 0 \).

The proof in the case of \( \eta > 0 \) is very similar. But this time we need to deform \( \mathcal{L} \) as the union of the \( y \)-axis and \( \mathcal{C}_L := \mathcal{C}_{L,0.1} \cup \{(0, y) : |y| \leq \sqrt{1-a^2}\} \) with \( \mathcal{C}_L \) being counterclockwise.

Finally, it is easily seen that the previously derived estimates are valid uniformly for \( \xi, \eta \) in any given compact set of \( \mathbb{R} \setminus \{0\} \).

**Proof of Theorem 3: part (ii).** Substituting \( u, v \) by \( \sqrt{n/2}u, \sqrt{n/2}v \) in (1.9), by the assumptions we obtain

\[
\frac{1}{\sqrt{2\sqrt{n}}}K_n\left(b; \frac{x}{\sqrt{2\sqrt{n}}}, \frac{y}{\sqrt{2\sqrt{n}}}ight) = \frac{n^\frac{1}{2}}{(2\pi)^2} \int_{\mathcal{L}} dv e^{n(h(u) - h(v))} f_M(\frac{x}{\sqrt{2\sqrt{n}}}, \sqrt{n}u) 
\times \frac{1}{\sqrt{2\sqrt{n}}} g_M(\left(\frac{y}{\sqrt{2\sqrt{n}}}, \sqrt{n}v\right) \frac{1}{u-v} \left(\frac{u^2-a^2}{v^2-a^2}\right)^{\frac{\tau}{2n}} \prod_{l=1}^p n^\frac{1}{2}a_l \prod_{m=p+1}^r \frac{u-a_m}{v-a_m}, \tag{3.19}
\]

where the phase function

\[
h(z) = \frac{1}{2} z^2 + \frac{1}{2} \log(a^2 - z^2), \quad a = (1 - \frac{r}{2\sqrt{n}})^{-1}. \tag{3.20}
\]

Here if \( a \) is equal to the critical value 1, then the three simple saddle points coalesce into a third-order point \( z_0 = 0 \).

To use the steepest descent method to investigate asymptotic behaviour of large \( n \), we need to choose proper contours according to Propositions 4 and 5 below. For convenience, let’s introduce some notations: 1) \( \delta \) is a fixed small positive number; 2) \( \theta_0 \) is a bit larger than \( \pi/4 \), say \( \theta_0 = \frac{101}{400}\pi \), so that it satisfies the condition of part (ii) in Proposition 5; 3) \( q := 1 + \max\{2, |a_1|, |a_2|, \ldots, |a_r|\}; \) 4) \( \mathcal{L}(x_1, y_1) \to (x_2, y_2) \to \cdots \) denotes the union of line segments from points \( (x_1, y_1) \) to \( (x_2, y_2) \) to \( \cdots \). Define \( \Gamma_+ = \Gamma_{+1} \cup \Gamma_{+2} \) with an anticlockwise direction where

\[
\Gamma_{+1} = \mathcal{L}(\delta \cos \theta_0, \delta \sin \theta_0) \to (0,0) \to (\delta \cos \theta_0, -\delta \sin \theta_0), \tag{3.21}
\]

and

\[
\Gamma_{+2} = \mathcal{L}(\delta \cos \theta_0, -\delta \sin \theta_0) \to (1, -\tan \theta_0) \to (q, -\tan \theta_0) \to (q, \tan \theta_0) \to (1, \tan \theta_0) \to (\delta \cos \theta_0, -\delta \sin \theta_0). \tag{3.22}
\]
Let $\mathcal{C}_- = \Gamma_{-1} \cup \Gamma_{-2}$ be the reflection of $\mathcal{C}_+$ about the $y$-axis with an anticlockwise direction, and let $\mathcal{L}$ be the $y$-axis. We stress that it might be better to deform a small portion of $\Gamma_{+1}$ near the origin to the right a little such that it doesn’t intersect the $y$-axis, however, our choice above works well because they just touch each other at the “point of tangency”.

First, we divide the integral on the RHS of (3.19) into two parts

$$\text{LHS of (3.19)} = \int_{\mathcal{L}} du \int_{\Gamma_{+,1}\cup \Gamma_{-,1}} dv \left( \cdot \right) + \int_{\mathcal{L}} du \int_{\Gamma_{+,2}\cup \Gamma_{-,2}} dv \left( \cdot \right) := I_1 + I_2. \quad (3.23)$$

We claim that the dominant contribution comes from the neighbourhood of $(0,0)$, so we need to expand the function $h(z)$ at zero. With the double scaling in mind, we obtain the Taylor series

$$h(z) = \log a + \frac{1}{2} \left( \frac{\pi}{\sqrt{n}} - \frac{\pi^2}{4n} \right) z^2 - \frac{1}{4} \left( 1 - \frac{\tau}{2\sqrt{n}} \right) z^4 - \frac{1}{6} \left( 1 - \frac{\tau}{2\sqrt{n}} \right)^3 z^6 + \cdots, \quad (3.24)$$

from which combining (1.8), (3.9) and (3.11), together with the relation (3.8) and the definition of Meijer $G$-function, we see that

$$I_1 \sim \frac{n^{\frac{3}{2}}}{(2\pi i)^2} \int_{\mathcal{L}} du \int_{\Gamma_{+,1}\cup \Gamma_{-,1}} dv e^{n(h(u)-h(v))} G_{0,M+1}^{1,0} \left(0, -\nu_1, \ldots, -\nu_M \mid n^{\frac{3}{2}} xu \right)$$

$$\times \frac{1}{u-v} \left( \frac{u^2-a^2}{v^2-a^2} \right)^{-\frac{3}{2}} \prod_{l=1}^{p} \frac{n^{\frac{3}{2}}u-a_l}{n^{\frac{3}{2}}v-a_l} \prod_{l=p+1}^{r} \frac{u-a_l}{v-a_l}. \quad (3.25)$$

Rescaling $u, v$ by $n^{-\frac{1}{4}}$, use (3.24) and we conclude that the limit of $I_1$ leads to the kernel (1.14), uniformly for $x, y$ in a compact set of $\mathbb{R} \setminus \{0\}$.

Secondly, for the integral $I_2$, we divide it into two parts again

$$I_2 = \int_{\mathcal{L}} du \int_{\Gamma_{+,2}} dv \left( \cdot \right) + \int_{\mathcal{L}} du \int_{\Gamma_{+,2}} dv \left( \cdot \right) := I_{2,+} + I_{2,-}, \quad (3.26)$$

and show that they are ignorable compared to $I_1$. We just focus on the integral $I_{2,-}$ since both are similar. Because of different asymptotic behaviour of (3.11), write

$$I_{2,-} = \int_{\mathcal{L}} du \int_{\Gamma_{+,2}^{(1)}} dv \left( \cdot \right) + \int_{\mathcal{L}} du \int_{\Gamma_{+,2}^{(2)}} dv \left( \cdot \right) := I_{2,-}^{(1)} + I_{2,-}^{(2)}, \quad (3.27)$$

where $\Gamma_{+,2}^{(1)} = \{ z \in \Gamma_{+,2} : | \phi(h(z)) | < \frac{3}{4} \pi \}$ and $\Gamma_{+,2}^{(2)} = \{ z \in \Gamma_{+,2} : \frac{3}{4} \pi < | \phi(h(z)) | < \frac{5}{4} \pi \}$.

Application of (3.11) gives us the same asymptotic form as in the RHS of (3.25) but with the $v$-contour $\Gamma_{-,2}^{(1)}$, from which use of the steepest descent argument leads to an exponential decay. However, application of (3.11) to $I_{2,-}^{(2)}$ yields

$$I_{2,-}^{(2)} \sim \frac{1}{(2\pi i)^3} \int_{\mathcal{L}} du \int_{\Gamma_{+,2}^{(2)}} dv e^{n(h(u)-h(0))} G_{0,M+1}^{1,0} \left(0, -\nu_1, \ldots, -\nu_M \mid n^{\frac{3}{2}} xu \right)$$

$$\times \frac{1}{u-v} \left( \frac{u^2-a^2}{v^2-a^2} \right)^{-\frac{3}{2}} \prod_{l=1}^{p} \frac{n^{\frac{3}{2}}u-a_l}{n^{\frac{3}{2}}v-a_l} \prod_{l=p+1}^{r} \frac{u-a_l}{v-a_l}$$

$$\times e^{-\frac{3}{2} \log(1-\frac{2}{\tau})} \int_{c-i\infty}^{c+i\infty} ds |y|^{-s} (\text{sgn}(y)v)^{s-1} n^{(3s-1)/4} \prod_{l=1}^{M} \Gamma(\nu_l+s). \quad (3.28)$$
Note that the endpoints of $\Gamma_{2,-}^{(2)}$ are $(-\tan \theta_0, \pm \tan \theta_0)$, with $a = (1 - \frac{\tau}{2\sqrt{n}})^{-1}$ in mind, for sufficiently large $n$ we see that

$$\text{Re}\left\{ \log(1 - \frac{v^2}{a^2}) \right\} \geq -2 \log a + \frac{1}{2} \log ((a + \tan \theta_0)^2 + \tan^2 \theta_0)(a - \tan \theta_0)^2 + \tan^2 \theta_0)$$

$$\geq -2 \log a + 2 \log \tan \theta_0 > \log \tan \theta_0 > 0 \quad (3.29)$$

holds true uniformly for $\tau$ in a compact set of $\mathbb{R}$ and for for every $v \in \Gamma_{2,-}^{(2)}$, use of the steepest descent argument leads to an exponential decay

$$I_{2,-}^{(2)} = \mathcal{O}\left(n^{-\frac{3}{4}} e^{-\frac{3}{2} \log \tan \theta_0} \right). \quad (3.30)$$

Lastly, by combining the foregoing results for $I_1$ and $I_2$, we complete the proof of part (ii).

**Proof of Theorem 3: part (iii).** Under the assumptions we can rewrite (1.9) as

$$K_n(b; x, y) = \frac{1}{2(\pi i)^2} \int_L du \int_C dv e^{u^2 - v^2} f_M(x, u) g_M(y, v)$$

$$\times \frac{1}{u - v} \prod_{l=1}^{p} \frac{u - a_l}{v - a_l} \prod_{m=p+1}^{r} \frac{u - \sqrt{\tau} a_m}{v - \sqrt{\tau} a_m}. \quad (3.31)$$

Without loss of generality, we assume that $a_{p+1}, \ldots, a_r > 0$. Choose a fixed number $q$ such that $q > \max\{|a_1, \ldots, |a_p|\}$, and let

$$C_+ = \{z = q + te^{\pm i \frac{\pi}{4}} : t \geq 0\} \quad \text{and} \quad C_- = \{z = -q + te^{\mp i(1 \pm \frac{\pi}{8})} : t \geq 0\} \quad (3.32)$$

both with an anticlockwise direction. Then $C_-$ encircles $\sqrt{n/2a}$ and $C_+$ encircles $\sqrt{n/2a}$, $\sqrt{n/2a_{p+1}}, \ldots, \sqrt{n/2a_r}$, both not crossing $C_a$. For large $n$, we choose $C = C_- \cup C_a \cup C_+$ and divide the integral on the RHS of (3.31) into three parts according to the $v$-contour, denoted by $I_-, I_a, I_+$.\)

Note that $a_m \neq 0$ for $m = p + 1, \ldots, r$, we easily see that the limit of $I_a$ leads to the dominant contribution, while

$$I_{\pm} \rightarrow \frac{1}{2(\pi i)^2} \int_L du \int_{C_{\pm}} dv e^{(1 - \frac{1}{2}x^2)(u^2 - v^2)} f_M(x, u) g_M(y, v) \frac{1}{u - v} \prod_{l=1}^{p} \frac{u - a_l}{v - a_l} = 0 \quad (3.33)$$

since the integrand has no pole in $C_{\pm}$ for the $v$-integral.

This completes the proof of part (iii). \(

The following two propositions are of importance in choosing appropriate contours of integration for the method of steepest descent.

**Proposition 4.** Let $C_R = \{z = a + e^{i\theta} : -\frac{\pi}{2} - \arccos a \leq \theta \leq \frac{\pi}{2} + \arccos a\}$ and let $C_L$ be the reflection of $C_R$ about the $y$-axis. Then for $h(z) = \frac{1}{2}z^2 + \frac{1}{2} \log(a^2 - z^2)$, the following hold true.

(i) When $0 \leq a \leq \sqrt{2}/2$, $\text{Re}\{h(z)\}$ attains its global minimum at $\pm i\sqrt{1 - a^2}$ over $C_L \cup C_R$.

(ii) When $0 \leq a \leq 1$, $\text{Re}\{h(z)\}$ attains its global maximum at $\pm i\sqrt{1 - a^2}$ over $i\mathbb{R}$.
Proof. For (i), we first consider $z \in C_R$ and let $z = a + e^{i\theta}$. It is easy to obtain
\[
\text{Re}\{h\} = \frac{1}{2} (2t^2 + 2at + a^2 - 1) + \frac{1}{4} \log(4at + 1 + 4a^2), \quad t = \cos \theta.
\] (3.34)
For this, we know from $t \in [-a, 1]$ with $0 \leq a \leq \sqrt{2}/2$ that
\[
\frac{d}{dt} \text{Re}\{h\} = \frac{2}{4at + 1 + 4a^2} (t + a)(4at + 2a^2 + 1) \geq 0,
\] (3.35)
and thus prove (i). The proof in the case $z \in C_L$ is similar.

For (ii), let $z = iy$, we have
\[
\text{Re}\{h\} = -\frac{1}{2} y^2 + \frac{1}{2} \log(a^2 + y^2), \quad -\infty < y < \infty.
\] (3.36)
Since
\[
\frac{d}{dt} \text{Re}\{h\} = \frac{y}{a^2 + y^2} (1 - a^2 - y^2),
\] (3.37)
the maximum can be obtained at $y = \pm \sqrt{1 - a^2}$.

\[\square\]

**Proposition 5.** Let $h(z) = \frac{1}{2} z^2 + \frac{1}{2} \log(1 - z^2)$, for $\theta_0 \in \mathbb{R}$ the following hold true.

(i) When $\frac{x}{4} \leq |\theta_0| \leq \frac{x}{4}$, $\text{Re}\{h(te^{i\theta_0})\}$ is a strictly increasing function of $t$ over $[0, \infty)$.

(ii) When $\frac{x}{4} < |\theta_0| \leq \frac{x}{4} - \frac{1}{2} \arccos \frac{2 - \sqrt{2}}{2}$, $\text{Re}\{h(te^{i\theta_0})\}$ is a strictly increasing function of $t$ over $[0, t_{\text{max}}]$ with $t_{\text{max}} = \sqrt{2 \cos 2\theta_0 - 1}/\cos 2\theta_0$. Moreover, $t_{\text{max}} \cos \theta_0 \geq 1$.

(iii) For $0 \neq y \in \mathbb{R}$, $\text{Re}\{h(x + iy)\}$ is a strictly increasing function of $x$ over $[1, \infty)$.

(iv) When $x \geq 2$, $\text{Re}\{h(x + iy)\}$ is a strictly decreasing (resp. increasing) function of $y$ over $[0, \infty)$ (resp. $(-\infty, 0]$).

Proof. We see from
\[
\text{Re}\{h\} = \frac{1}{2} t^2 \cos 2\theta_0 + \frac{1}{4} \log(1 + t^4 - 2t^2 \cos 2\theta_0)
\] (3.38)
that
\[
\frac{d}{dt} \text{Re}\{h(te^{i\theta_0})\} = \frac{t^3}{1 + t^4 - 2t^2 \cos 2\theta_0} (t^2 \cos 2\theta_0 + 1 - 2 \cos^2 2\theta_0) > 0, \quad \forall t > 0,
\] (3.39)
for any given $|\theta_0| \in [\frac{x}{4}, \frac{x}{4}]$. Part (i) then follows.

For part (ii), the monotonicity follows from the simple fact $\frac{d}{dt} \text{Re}\{h(te^{i\theta_0})\} > 0, \forall t \in [0, t_{\text{max}}]$. Let $s = \cos 2\theta_0$, simple calculation shows
\[
t_{\text{max}} \cos \theta_0 \geq 1 \iff \left(1 - \frac{1}{s}\right) \left(s + \frac{2 + \sqrt{2}}{2}\right) \left(s + \frac{2 - \sqrt{2}}{2}\right) \geq 0,
\] (3.40)
from which and the assumption we complete part (ii).

Note that
\[
\frac{d}{dx} \text{Re}\{h(x + iy)\} = x + \frac{1}{2} \frac{x + 1}{(x + 1)^2 + y^2} + \frac{1}{2} \frac{x - 1}{(x - 1)^2 + y^2} > 0, \quad \forall x \geq 1,
\] (3.41)
when $y \neq 0$ and
\[
\frac{d}{dy} \text{Re}\{h(x + iy)\} = -y + \frac{1}{2} \frac{y}{(x + 1)^2 + y^2} + \frac{1}{2} \frac{y}{(x - 1)^2 + y^2} < 0, \quad \forall y > 0
\] (3.42)
whenever $x \geq 2$, we prove part (iii) and part (iv).

\[\square\]
4 Limiting eigenvalue density

The study of scaling limits at the origin investigated in the previous section introduces a scale in which the average spacing between eigenvalues is of order unity. A very different, but still well-defined, limiting process is the so-called limiting spectral measure in the global scaling regime. Usually, it has a density $\rho(x)$ with compact support $I \subset \mathbb{R}$ such that $\int_I \rho(x)dx = 1$. Here $\rho(x)$ is referred to as the global density or limiting eigenvalue density.

For squared singular values of the product of independent Ginibre matrices, i.e., eigenvalues of $W_M$ defined by (1.2) but with $H = I_n$, the global limit corresponds to a change of variables $x_j \mapsto n^M x_j$ and the global density is known to be the so-called Fuss–Catalan density with parameter $M$. Its $k$-th moment ($k = 0, 1, \ldots$) is specified by the Fuss–Catalan number

$$FC_M(k) = \frac{1}{Mk+1} \binom{(M+1)k}{k},$$  \hspace{1cm} (4.1)

see e.g. [9, 38]. The Catalan numbers are the case $M = 1$, corresponding to the moments of the Marchenko-Pastur law in a special case and also the even moments of the famous Wigner semicircle law (its odd moments vanishing).

Recently, Forrester, Ipsen and the author [23] turn to the product $W_M$ in (1.2) but with $H$ being a GUE matrix, i.e. $B = 0$ in (1.1). After the change of variables $x_j \mapsto \frac{1}{\sqrt{2}} n^{M+\frac{1}{2}} x_j$, they prove that the global density is an even function and its even moments are given by the Fuss–Catalan numbers with parameter $2M+1$. In this section we investigate the global density for the product matrix $W_M$ with source $B$.

Specifically, we assume that $n$ is even and $b_1 = \cdots = b_{n/2} = -b_{1+n/2} = \cdots = -b_n = \sqrt{n/2a}$, $a \geq 0$. To obtain the global density, we need to make the change of variables $x_j \mapsto \frac{1}{\sqrt{2}} n^{M+\frac{1}{2}} x_j$. To see this, we may use free probability techniques; see e.g. [38].

Suppose that two selfajoint non-commutative random variables $h$ and $w$ are free, and at least one, say, $w$ is positive. Recall that the Stieltjes transform of $h$ with distribution $\mu$ is defined by

$$G_h(z) = \int \frac{d\mu(x)}{z-x}, \quad \text{Im}(z) > 0.$$  \hspace{1cm} (4.2)

Let $S_w(z)$ denote the $S$-transform of $w$, see e.g. [38] for definition. If $G_h(z)$ satisfies a functional equation $P(z, G_h(z)) = 0$, then we know from [38] that the Stieltjes transform $G_{hw}(z)$ of the product $hw$ satisfies

$$P\left(zS_w(zG_{hw}(z) - 1), \frac{zG_{hw}(z)}{S_w(zG_{hw}(z) - 1)}\right) = 0.$$  \hspace{1cm} (4.3)

Moreover, we know that if $h$ is a free convolution of the standard semicircular law and $\frac{1}{2}(\delta_a + \delta_{-a})$ and if $w$ is given by the free Poisson distribution with parameter 1 (i.e. Marchenko–Pastur law), then $S_w(z) = 1/(1 + z)$ and $G_h$ satisfies the cubic equation

$$(G_h - z)^2G_h + (1 - a^2)G_h - z = 0.$$  \hspace{1cm} (4.4)

Thus, using (4.3) $M$ times, we see that the Stieltjes transform of limiting spectral measure for our product (1.2) indeed satisfies a functional equation

$$(z^{2M-1}g^{2M+1} - 1)^2 zg + (1 - a^2)z^{2M-1}g^{2M+1} - 1 = 0.$$  \hspace{1cm} (4.5)
Considering two special cases of (4.5), we can give explicit forms of the limiting eigenvalue densities denoted by $\rho(a; x)$ and further compare leading asymptotic behaviour near the origin.

**Case 1:** $a = 0$. Let

$$x^2 = \left( \frac{\sin((2M + 2)\varphi)}{\sin(2M + 1)\varphi} \right)^{2M+2} \frac{\sin\varphi}{\sin(2M + 1)\varphi}^{2M+1}, \quad -\frac{\pi}{2M + 2} \leq \varphi \leq \frac{\pi}{2M + 2},$$

(4.6)

(4.5) has two special solutions

$$xg_\pm = \frac{\sin((2M + 2)\varphi)}{\sin((2M + 1)\varphi)} e^{\pm i\varphi}$$

(4.7)

from which the density reads

$$\rho(0; x) = \frac{1}{\pi} \sqrt{\sin\varphi \sin(2M + 1)\varphi} \left( \frac{\sin(2M + 1)\varphi}{\sin(2M + 2)\varphi} \right)^M \sin\varphi, \quad -\frac{\pi}{2M + 2} \leq \varphi \leq \frac{\pi}{2M + 2}.$$  

(4.8)

Moreover, as $x \to 0$ we have the leading term

$$\rho(0; x) \sim \frac{1}{\pi} \sin\frac{\pi}{2M + 2} |x|^{-1 + \frac{1}{M+1}}.$$  

(4.9)

These results have been obtained in [23].

**Case 2:** $a = 1$. In this case (4.5) reduces to

$$\left( z^{2M-1} g^{2M+1} - 1 \right)^2 zg - 1 = 0.$$  

(4.10)

Let

$$x^2 = \left( \frac{\sin((4M + 3)\varphi)}{\sin((4M + 2)\varphi)} \right)^{4M+3} \frac{\sin\varphi}{\sin((4M + 2)\varphi)}^{4M+2}, \quad -\frac{\pi}{4M + 3} \leq \varphi \leq \frac{\pi}{4M + 3},$$

(4.11)

(4.10) has two special solutions

$$xg_\pm = \left( \frac{\sin(4M + 3)\varphi}{\sin(4M + 2)\varphi} \right)^2 e^{\pm 2i\varphi},$$

(4.12)

from which the density reads

$$\rho(1; x) = \frac{1}{\pi} \sqrt{\sin\varphi \sin(4M + 3)\varphi} \left( \frac{\sin(4M + 2)\varphi}{\sin(4M + 2)\varphi} \right)^{2M-1} \sin 2\varphi, \quad -\frac{\pi}{4M + 3} \leq \varphi \leq \frac{\pi}{4M + 3}.$$  

(4.13)

Moreover, as $x \to 0$ we have the leading term

$$\rho(1; x) \sim \frac{1}{\pi} \sin\frac{2\pi}{4M + 3} |x|^{-1 + \frac{4}{4M+3}}.$$  

(4.14)

Generally, we expect from the algebraic equation (4.5) that there exist exactly three families of blow-up exponents at the origin for $\rho(a; x)$, which reads as $x \to 0$

$$\rho(a; x) \sim \begin{cases} 
    c_a |x|^{-1 + \frac{a+1}{M+1}}, & 0 \leq a < 1; \\
    c_a |x|^{-1 + \frac{a+2}{2M+2}}, & a = 1; \\
    c_a |x|^{-1 + \frac{a+2}{2M+2}}, & a > 1.
\end{cases}$$  

(4.15)

If so, this will be consistent with the local scalings chosen in Theorem 3.

Finally, we stress that the above parametrization representations are of vital importance in proving the sine kernel in the bulk, see e.g. [37] for more details.
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