REGULARITY OF THE FREE BOUNDARY IN A NONLOCAL ONE-DIMENSIONAL PARABOLIC FREE BOUNDARY VALUE PROBLEM

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Abstract. We consider one-dimensional parabolic free boundary value problem with a nonlocal (integro-differential) condition on the free boundary. Results on $C^m$-regularity of the free boundary are obtained. In particular, a necessary and sufficient condition for infinite differentiability of the free boundary is given.

1. Introduction

In this paper we study the regularity properties of the free boundary in the following one-dimensional parabolic free boundary value problem. Problem $P$. Find $s(t) > 0$ and $u(x,t)$ such that

\begin{align}
(1.1) & \quad u_t = u_{xx} - \lambda u, \quad \lambda = \text{const} > 0, \quad 0 < x < s(t), \quad t > 0, \\
(1.2) & \quad u(0,t) = f(t), \quad t \geq 0, \\
(1.3) & \quad u(x,0) = \varphi(x), \quad x \in [0,b], \quad s(0) = b > 0, \quad \varphi(0) = f(0), \\
(1.4) & \quad u_x(s(t),t) = 0, \quad t > 0, \\
(1.5) & \quad s'(t) = \int_0^{s(t)} (u(x,t) - \sigma)dx, \quad \sigma = \text{const} > 0, \quad t > 0.
\end{align}

Notice that (1.2)–(1.4) are mixed type boundary conditions for the parabolic equation (1.1), and (1.5) is an integro-differential condition on the free boundary $x = s(t)$. Similar free boundary value problems arise in tumor modeling and modeling of nanophasic thin films (see [6, 11, 12, 5]).

Our goal in this paper is to examine the relationship between the smoothness of the functions $f(t)$ and $s(t)$, and to show the essential impact of the nonlocal character of condition (1.5) on the regularity properties of the free boundary.

If one sets $\lambda = 0$ in (1.1), and replaces our conditions (1.4) and (1.5) by the conditions

\begin{align}
(1.6) & \quad u(s(t),t) = 0 \quad \text{for} \quad t > 0
\end{align}

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and

\[ s'(t) = -u_x(s(t), t) \quad \text{for} \quad t > 0 \]

respectively, the resulting problem (1.1) (with \( \lambda = 0 \), (1.2), (1.3), (1.6) and (1.7) is the classical one-dimensional Stefan problem (see [9, Ch. 8], [16], [1, Ch. 17]). In this context, the infinite differentiability of the free boundary has been established in [2, 3, 4, 17].

On the other hand, in [10] it is proved that if \( f(t) \) is an analytic function then \( s(t), t > 0 \) is also an analytic function. In addition, the analyticity of \( s(t) \) at \( t = 0 \) is studied in [8].

There is a vast literature on the regularity of free boundaries in multi-dimensional (multi-phase) Stefan problems and their generalizations (e.g., see [13, 7, 15] and the bibliography therein). But in general the one-dimensional free boundary problems cannot be treated as a partial case of multidimensional ones, and their handling requires specific methods.

Following the method in [17], our approach in studying the regularity of the free boundary in Problem \( P \) is based on the theory of anisotropic Hölder spaces. In Section 2 we estimate from below the Hölder and \( C^m \)-smoothness of the free boundary. In Theorem 2.1 we prove that if \( f(t) \) has continuous derivatives up to order \( m \) on \( (0, T] \), \( T > 0 \), then \( s(t) \) has continuous derivatives up to order \( m + 1 \) on \( (0, T] \). Therefore, if \( f(t) \) is infinitely differentiable on \( (0, \infty) \), it follows that \( s(t) \) is infinitely differentiable on \( (0, \infty) \) as well.

However, it turns out that \( s(t) \) may not have derivatives of order higher than two if we assume \( f(t) \in C^1([0, \infty)) \) only (see Section 3, where we estimate the smoothness of \( s(t) \) from above). More generally, in Theorem 3.3 we prove that if \( s(t) \) has on \( (0, T] \) continuous derivatives up to order \( m + 2 \) then \( f(t) \) has continuous derivatives up to order \( m \) on \( (0, T] \). Therefore, if \( f(t) \) is not infinitely differentiable on \( (0, \infty) \), then the free boundary is not infinitely differentiable curve as well.

This is in a striking contrast with the case of one-dimensional Stefan problem, where the infinite differentiability of the free boundary does not require infinite differentiability of the boundary data at \( x = 0 \) (see [2, 3, 4, 17]). In our Problem \( P \), due to the nonlocal character of condition (1.5), the smoothness of the free boundary is essentially related to the smoothness of \( f(t) \), namely the free boundary is an infinitely differentiable curve if and only if the function \( f(t) \) is infinitely differentiable.

2. Lower bounds for the smoothness of the free boundary

Results on global existence and uniqueness of classical solutions of Problem \( P \) are obtained in [20, Theorem 1.1] (see also [18, 19]). More precisely, the following holds.

Global solvability of Problem \( P \) : Suppose

\[ f(t) \in C^1([0, \infty)), \quad \varphi(x) \in C^2([0, b]), \quad f(0) = \varphi(0), \]
\[ f'(0) = \varphi''(0) - \lambda \varphi(0), \quad \varphi'(b) = 0. \]
Then there exists a unique pair of functions $u(x, t)$ and $s(t)$ such that

(i) $u(x, t)$ is defined, continuous and has continuous partial derivatives $u_x, u_t, u_{xx}$ in the domain $\{(x, t) : 0 \leq x \leq s(t), t \geq 0\}$;

(ii) $s(t) \in C^1([0, \infty))$, $s(t) > 0$ for $t \geq 0$;

(iii) the conditions $(1.7)$–$(1.9)$ hold.

Let the pair of functions $(u(x, t), s(t))$ be a classical solution of Problem $P$ satisfying (i)–(iii). Indeed, since $u_t(x, t)$ is defined and continuous for $0 \leq x \leq s(t), t > 0$, from $(1.5)$ it follows

$$s''(t) = (u(s(t), t) - \sigma)s'(t) + \int_0^{s(t)} u_t(x, t)dx.$$  

By $(1.1)$ and $(1.5)$,

$$\int_0^{s(t)} u_t(x, t)dx = \int_0^{s(t)} (u_{xx}(x, t) - \lambda u(x, t))dx$$

$$= u_x(s(t), t) - u_x(0, t) - \lambda s'(t) - \lambda \sigma s(t),$$

so using $(1.4)$ we obtain

$$(2.2) s''(t) = (u(s(t), t) - \lambda - \sigma) s'(t) - \lambda \sigma s(t) - u_x(0, t),$$

where the expression on the right is a continuous function for $t \geq 0$, i.e., $s(t) \in C^2([0, \infty))$.

In this section, our main result is the following statement.

**Theorem 2.1.** Suppose the pair of functions $(u(x, t), s(t))$ is a classical solution of Problem $P$ satisfying (i)–(iii). If $f(t) \in C^m([0, T])$, where $m \in \mathbb{N}, m \geq 2$ and $T = \text{const} > 0$, then $s(t) \in C^{m+1}([0, T])$.

In particular, if $f(t) \in C^\infty([0, \infty))$, then $s(t) \in C^\infty([0, \infty))$.

In the proof of Theorem 2.1 we need some preliminary results. First we use the change of variables $\xi = \frac{x}{s(t)}, t = t$ to transform $(1.1)$ to an equation in a cylindrical domain by setting

$$(2.3) v(\xi, t) = u(\xi s(t), t), \quad Q = \{(\xi, t) : 0 < \xi < 1, t > 0\}.$$  

Then, in view of (i)–(iii), it follows that $v, v_\xi, v_t, v_{\xi\xi} \in C(Q)$ and

$$(2.4)Lv := v_t - \frac{1}{s^2} v_{\xi\xi} - \frac{\xi s'}{s} v_\xi - \lambda v = 0 \quad \text{for} \quad (\xi, t) \in Q,$$

$$(2.5)v(0, t) = f(t), \quad v(\xi, 0) = \varphi(\xi s(0)), \quad v_\xi(1, t) = 0.$$  

From $(2.2)$ we obtain

$$(2.6)s''(t) = (v(1, t) - \lambda - \sigma) s'(t) - \lambda \sigma s(t) - \frac{1}{s(t)} v_t(0, t), \quad t \geq 0.$$  

For convenience, we set

$$(2.7)Q_{\delta_1, \delta_2}^{\varepsilon, T} = \{\xi, t) : \delta_1 < \xi < \delta_2, \varepsilon < t < T\}.$$
In order to prove Theorem 2.1 we are going to estimate the Hölder smoothness of \(v(\xi, t)\) and \(s(t)\) in terms of the Hölder smoothness of \(f(t)\). To this end we use anisotropic Hölder spaces \(H^{m+\ell,\frac{m+\ell}{m}}(Q_{\delta_1,\delta_2}^{\varepsilon,T})\), where \(m = 0, 1, 2, \ldots\) and \(\ell \in (0, 1)\).

Recall that \(H^{m+\ell,\frac{m+\ell}{m}}(G)\) is the Banach space of all functions \(v(\xi, t)\) that are continuous on \(G\) together with all derivatives of the form \(D^k_\xi D^r_t v\) for \(k + 2r \leq m\) and have a finite norm

\[
\|v\|_{G}^{(m+\ell)} = \sum_{j=0}^{m} \sum_{k+2r=j} \left|D^k_\xi D^r_t v\right|^{(0)} + \sum_{k+2r=m} \left(D^k_\xi D^r_t v \right)^{(\ell)}_{G} + \sum_{k+2r=m+1} \left(D^k_\xi D^r_t v \right)^{(\ell+1)}_{G},
\]

where \(G\) is a bounded rectangular domain, \(\langle v \rangle_{L}^{\ell} G\) and \(\langle v \rangle_{t}^{\ell} G\) are the Hölder constants of a function \(v(\xi, t)\) in \(\xi\) and \(t\) respectively in the domain \(G\) with the exponent \(\ell\), \(\ell \in (0, 1)\), and \(|D^k_\xi D^r_t v|^{(0)} = \max_{\varepsilon,T} |D^k_\xi D^r_t v|\). For more details about these definitions and notations we refer to the book [14, Intr., p. 7].

In the following the functions of one variable \(t\) are regarded as functions of two variables \(x\) and \(t\).

**Proposition 2.2.** (a) For every \(T > 0, \ell \in (0, 1)\) we have

\[
v(\xi, t) \in H^{1+\ell, \frac{1+\ell}{2}}(Q_{0,1}^{0,T}).
\]

(b) If \(f(t) \in H^{m+\ell, \frac{m+\ell}{m}}(Q_{0,1}^{\varepsilon,T})\), where \(m \in \mathbb{N}, m \geq 2, \ell \in (0, 1)\) and \(T > \varepsilon > 0\), then

\[
v(\xi, t) \in H^{m+\ell, \frac{m+\ell}{m}}(Q_{0,1}^{\varepsilon,T}) \quad \forall \varepsilon \in (\varepsilon, T),
\]

and

\[
s(t) \in H^{m+3+\ell, \frac{m+3+\ell}{m+3}}(Q_{0,1}^{\varepsilon,T}) \quad \forall \varepsilon \in (\varepsilon, T).
\]

The following lemma helps to make the inductive step in the proof of Proposition 2.2.

**Lemma 2.3.** Let \(\varepsilon > 0, \ell \in (0, 1), k \in \mathbb{N},\)

\[
w(\xi, t) \in H^{k+\ell, \frac{k+\ell}{k}}(Q_{0,1}^{\varepsilon,T})
\]

and \(w_{\xi} \) exists in \(Q_{0,1}^{\varepsilon,T}\) in the case \(k = 1\), and let \(w(\xi, t)\) satisfy the equation

\[
\dot{w} := w_t - a(\xi, t)w_{\xi} - b(\xi, t)w - c(\xi, t)w = F(\xi, t), \quad (\xi, t) \in Q_{0,1}^{\varepsilon,T},
\]

where

\[
a, b, c, F \in H^{k-1+\ell, \frac{k-1+\ell}{k-1}}(Q_{0,1}^{\varepsilon,T}), \quad a(\xi, t) \geq \text{const} > 0.
\]
If
\[(2.14)\quad w(0, t) = f(t), \quad w_\xi(1, t) = g(t), \quad \varepsilon \leq t \leq T,\]
with
\[(2.15)\quad f(t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_{\varepsilon, T}^{\varepsilon}), \quad g(t) \in H^{k+\ell, \frac{k+\ell}{2}}(Q_{0, T}^{\varepsilon}),\]
then
\[(2.16)\quad w(\xi, t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_{0, T}^{\varepsilon}) \quad \forall \varepsilon \in (\varepsilon, T).\]

Proof. Fix \(\varepsilon \in (\varepsilon, T)\) and choose arbitrary \(\varepsilon_1 \in (\varepsilon, \varepsilon)\) and \(\delta \in (1/2, 1)\). Consider a function \(\psi(\xi, t)\) of the form \(\psi(\xi, t) = \psi_1(\xi) \psi_2(t)\), where \(\psi_1, \psi_2 \in C^\infty(\mathbb{R}), \quad 0 \leq \psi_1(\xi), \psi_2(t) \leq 1\) and
\[(2.17)\quad \psi_1(\xi) = \begin{cases} 1 & \text{if } \xi \leq 1/2, \\ 0 & \text{if } \xi \geq \delta \end{cases}, \quad \psi_2(t) = \begin{cases} 1 & \text{if } t \geq \varepsilon, \\ 0 & \text{if } t \leq \varepsilon_1. \end{cases}\]

Then the function \(w_1(\xi, t) = w(\xi, t) \psi(\xi, t)\) is a solution of the boundary value problem
\[(2.18)\quad \tilde{L}w_1 = F_1(\xi, t), \quad (\xi, t) \in Q_{0, \delta}^{\varepsilon,T},\]
\[(2.19)\quad w_1(0, t) = f(t)\psi_2(t), \quad w_1(\delta, t) = 0, \quad \varepsilon \leq t \leq T,\]
\[(2.20)\quad w_1(\xi, \varepsilon) = 0, \quad 0 \leq \xi \leq \delta,\]
where
\[(2.21)\quad F_1(\xi, t) = F \psi + w(\psi_t - a\psi_\xi - bv_\xi) - 2aw_\xi \psi_\xi.\]

In view of (2.11) and (2.13), it follows that \(F_1(\xi, t) \in H^{k-1+\ell, \frac{k-1+\ell}{2}}(Q_{0, \delta}^{\varepsilon,T})\). Therefore, applying Theorem 5.2 of [14] Ch.4, we conclude that Problem (2.18)-(2.20) has a unique solution
\[(2.22)\quad w_1(\xi, t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_{0, \delta}^{\varepsilon,T}).\]

Thus, taking into account the construction of the function \(\psi(\xi, t)\), we obtain
\[(2.23)\quad w(\xi, t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_{0, T}^{\varepsilon}).\]

Next, we fix \(\varepsilon \in (0, 1/2)\) and choose, for arbitrary \(\delta_1 \in (\delta, 1/2)\), a function \(\tilde{\psi}_1(\xi) \in C^\infty(\mathbb{R})\) such that
\[0 \leq \tilde{\psi}_1(\xi) \leq 1, \quad \tilde{\psi}_1(\xi) = \begin{cases} 1 & \text{if } \xi \geq 1/2, \\ 0 & \text{if } \xi \leq \delta_1. \end{cases}\]

Now we set \(\tilde{\psi}(\xi, t) = \tilde{\psi}_1(\xi) \psi_2(t)\), where \(\psi_2(t)\) is given by (2.17). Then the function \(w_2(\xi, t) = w(\xi, t) \tilde{\psi}(\xi, t)\) is a solution of the boundary value problem
\[(2.23)\quad \tilde{L}w_2 = F_2(\xi, t), \quad (\xi, t) \in Q_{\delta, 1}^{\varepsilon,T},\]
(2.24) \( \partial_{\xi} w_{2}(\tilde{\delta}, t) = 0, \partial_{\xi} w_{2}(1, t) = g(t) \tilde{\psi}_{2}(t), \ \varepsilon \leq t \leq T, \)
\[
(2.25) \quad w_{2}(\xi, \varepsilon) = 0, \quad \tilde{\delta} \leq \xi \leq 1,
\]
where
\[
F_{2}(\xi, t) = F \tilde{\psi} + w \left( \tilde{\psi}_{t} - a \tilde{\psi}_{\xi\xi} - b \tilde{\psi}_{\xi} \right) - 2aw_{\xi} \tilde{\psi}_{\xi}.
\]
From (2.11) and (2.13) it follows that \( F_{2}(\xi, t) \in H^{k-1+\ell, \frac{k-1+4\ell}{2}}_{\delta, 1}(Q_{\delta, 1}^{\varepsilon, T}) \).

Therefore, by Theorem 5.3 of [14, Ch.4], Problem (2.23)–(2.25) has a unique solution
\[
w_{2}(\xi, t) \in H^{k+1+\ell, \frac{k+4\ell}{2}}_{\delta, 1}(Q_{\delta, 1}^{\varepsilon, T}).
\]
Now, taking into account the construction of the function \( \tilde{\psi}(\xi, t) \), we obtain
\[
(2.27) \quad w(\xi, t) \in H^{k+1+\ell, \frac{k+4\ell}{2}}_{\delta, 1}(Q_{\delta, 1}^{\varepsilon, T}).
\]
Finally, (2.22) and (2.27) imply that \( w(\xi, t) \in H^{k+1+\ell, \frac{k+4\ell}{2}}_{\delta, 1}(Q_{\delta, 1}^{\varepsilon, T}), \) which completes the proof of Lemma 2.3.

Proof of Proposition 2.2. Since \( v(\xi, t) \in C^{2,1}(Q_{0, 1}^{\varepsilon, T}) \), we have that \( v, v_{\xi}, v_{\xi\xi}, v_{t} \in L^{q}(Q_{0, 1}^{\varepsilon, T}) \) for every \( q > 1 \). Therefore, by Lemma 3.3 of [14, Ch. 2] it follows that \( v_{\xi}(\xi, t) \) is Hölder continuous in \( t \) with exponent \( 1 - 3/q \) for every \( q > 3 \). Hence (2.8) holds.

We prove the assertion (b) by induction in \( m \). Let \( m = 2 \); suppose that \( f(t) \in H^{2+\ell, \frac{2+4\ell}{2}}_{\varepsilon, 1}(Q_{0, 1}^{\varepsilon, T}), \ell \in (0, 1). \)

From (2.8) and (2.6) it follows that \( s(t) \in H^{3+\ell, \frac{3+4\ell}{2}}_{\varepsilon, 1}(Q_{0, 1}^{\varepsilon, T}), \ell \in (0, 1). \)

Now it is easy see that the coefficients of the operator \( L \) in (2.4) satisfy the assumption (2.13) in Lemma 2.3 for \( k = 1 \). Therefore, applying Lemma 2.3 in the case \( k = 1 \) to the Problem (2.4)–(2.5), we obtain that
\[
v(\xi, t) \in H^{2+\ell, \frac{2+4\ell}{2}}_{\varepsilon, 1}(Q_{0, 1}^{\varepsilon, T}), \quad \forall \varepsilon \in (\varepsilon, T).
\]
Then, in view of (2.6), we conclude that \( s(t) \in H^{5+\ell, \frac{5+4\ell}{2}}_{\varepsilon, 1}(Q_{0, 1}^{\varepsilon, T}), \forall \varepsilon \in (\varepsilon, T). \)

Hence, (2.9) and (2.10) hold for \( m = 2 \), i.e., the assertion (b) holds for \( m = 2 \).

Assume that (b) holds for some \( m \geq 2 \); we shall prove that (b) holds for \( m + 1 \). Let \( f(t) \in H^{m+1+\ell, \frac{m+4\ell}{2}}_{\varepsilon, 1}(Q_{0, 1}^{\varepsilon, T}). \) Then from the inductive hypothesis it follows that (2.9) and (2.10) hold. Therefore, the coefficients of the
operator $L$ in (2.4) satisfy (2.13) with $k = m$. Thus, by Lemma 2.3 we conclude that $v(\xi,t) \in H^{m+1+\ell, \frac{m+1+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}) \forall \epsilon \in (\epsilon, T)$, i.e., (2.19) holds for $m + 1$. Now, in view of (2.6), we obtain that $s(t) \in H^{m+4+\ell, \frac{m+4+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1})$, $\forall \epsilon \in (\epsilon, T)$. Hence, (2.10) holds for $m + 1$ as well. This completes the proof of Proposition 2.2.

**Proof of Theorem 2.1.** If $f(t) \in C^m((0, T])$ for some $m > 1$, then
\[
f(t) \in H^{2m-1+\ell, \frac{2m-1+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}) \forall \epsilon \in (0, T), \forall \ell \in (0, 1).
\]
Now, for every fixed $\epsilon > 0$, Proposition 2.2 implies that
\[
s(t) \in H^{2m+2+\ell, \frac{2m+2+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}) \forall \epsilon \in (\epsilon, T).
\]
Thus, it follows that $s(t) \in C^{m+1}((0, T])$. This completes the proof of Theorem 2.1.

3. **Upper bounds for the smoothness of $s(t)$**

Now we are going to explain that the smoothness of $s(t)$ (in terms of Hölder scale) is bounded above by the smoothness of $f(t)$.

**Proposition 3.1.** Let $v(\xi,t)$ be the function defined by (2.3) (and satisfying (2.4)–(2.6)). Then for every $m \in \mathbb{N}$, $\epsilon > 0$ and $T > \epsilon$ the following implication holds:
\[
s(t) \in H^{m+4+\ell, \frac{m+4+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}) \implies v(\xi,t) \in H^{m+1+\ell, \frac{m+1+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}) \forall \epsilon \in (\epsilon, T).
\]

In the proof of Proposition 3.1 we need the following statement.

**Lemma 3.2.** Let $\epsilon > 0$, $\ell \in (0, 1)$, $k \in \mathbb{N}$,
\[
w(\xi,t) \in H^{k+\ell, \frac{k+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1})
\]
and $w_{\xi \xi}$ exists in $Q^{\epsilon,T}_{0,1}$ in the case $k = 1$, and let $w(\xi,t)$ satisfy the equation
\[
\tilde{L}w := w_\xi - a(\xi,t)w_{\xi\xi} - b(\xi,t)w_\xi - c(\xi,t)w = F(\xi,t), \quad (\xi,t) \in Q^{\epsilon,T}_{0,1},
\]
where
\[
a, b, c, F \in H^{k-1+\ell, \frac{k-1+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}), \quad a(\xi,t) \geq \text{const} > 0.
\]
If
\[
w_\xi(0,t) = h(t), \quad w_\xi(1,t) = g(t), \quad \epsilon \leq t \leq T,
\]
with
\[
h(t) \in H^{k+\ell, \frac{k+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}), \quad g(t) \in H^{k+\ell, \frac{k+\ell}{2}}(\tilde{Q}^{\epsilon,T}_{0,1}),
\]
From (3.2) and (3.4) it follows that
\[ \tilde{h}(3.9) \]
where 
\[ m \]
i.e., (3.1) holds for
\[ \forall \varepsilon \in (\varepsilon, T). \]

Proof. The proof of this statement is similar to the proof of Lemma 2.3. Indeed, let \( \bar{\varepsilon} \in (\varepsilon, T) \); choose \( \varepsilon_1 \in (\varepsilon, \bar{\varepsilon}) \) and \( \hat{\psi}(t) \in C^\infty(\mathbb{R}) \) such that \( \hat{\psi}(t) = 1 \) for \( t \geq \bar{\varepsilon} \) and \( \hat{\psi}(t) = 0 \) for \( t \leq \varepsilon_1 \). Set \( \tilde{w}(\xi, t) = w(\xi, t) \cdot \hat{\psi}(t) \); then the function \( \tilde{w}(\xi, t) \) is a solution of the boundary value problem
\[ \tilde{L} \tilde{w} = \tilde{F}, \quad \tilde{F}(\xi, t) = F(\xi, t) \cdot \hat{\psi}(t) + w(\xi, t) \hat{\psi}'(t), \quad (\xi, t) \in Q_0^\varepsilon T. \]

From (3.2) and (3.4) it follows that \( \tilde{F}(\xi, t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_0^\varepsilon T) \). Now, by Theorem 5.3 of [14, Ch.4], we conclude that the above boundary value problem has a unique solution
\[ \tilde{w}(\xi, t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_0^\varepsilon T). \]

Thus, taking into account that \( \hat{\psi}(t) = 1 \) for \( t \geq \bar{\varepsilon} \), we obtain that \( w(\xi, t) \in H^{k+1+\ell, \frac{k+1+\ell}{2}}(Q_0^\varepsilon T) \). The proof of Lemma 3.2 is complete. \( \square \)

Proof of Proposition 3.1. We prove the claim by induction in \( m \).

Let \( m = 1 \); then we assume that \( s(\xi, t) \in H^{5+\ell, \frac{5+\ell}{2}}(Q_0^\varepsilon T) \) and prove that \( v(\xi, t) \in H^{2+\ell, \frac{2+\ell}{2}}(Q_0^\varepsilon T) \) \( \forall \varepsilon \in (\varepsilon, T) \). Indeed, in view of (2.5) and (2.6), the function \( v(\xi, t) \) satisfies the boundary conditions
\[ (3.8) \quad v_\xi(0, t) = h(t), \quad v_\xi(1, t) = 0, \quad \varepsilon \leq t \leq T, \]
where
\[ (3.9) \quad h(t) := s(\xi, t) \left[ (v(1, t) - \lambda - \sigma)s'(t) - \lambda s(t) - s''(t) \right]. \]

From the above assumptions on \( s(\xi, t) \), and from the assertion (a) in Proposition 2.2 it follows that
\[ h(t) \in H^{1+\ell, \frac{1+\ell}{2}}(Q_0^\varepsilon T). \]

Moreover, since \( s'(t) \in H^{3+\ell, \frac{3+\ell}{2}}(Q_0^\varepsilon T) \), we obtain in view of (2.4) that the coefficients of the operator \( L \) belong to the space \( H^{3+\ell, \frac{3+\ell}{2}}(Q_0^\varepsilon T) \).

Therefore, applying Lemma 3.2 in the case when \( k = 1, \tilde{L} = L, w = v \) and boundary conditions given by (3.8), we conclude that
\[ v(\xi, t) \in H^{2+\ell, \frac{2+\ell}{2}}(Q_0^\varepsilon T) \quad \forall \varepsilon \in (\varepsilon, T), \]
i.e., (3.1) holds for \( m = 1 \).
Assume that the assertion holds for some \( m \geq 1 \). We will prove that (3.1) holds for \( m + 1 \). Suppose

\[
s(t) \in H^{m+5+\ell, \frac{m+5+\ell}{2}} (Q_{0,1}^\varepsilon). \]

Then from the inductive hypothesis it follows that

\[
v(\xi, t) \in H^{m+1+\ell, \frac{m+1+\ell}{2}} (Q_{0,1}^\varepsilon) \quad \forall \varepsilon \in (\varepsilon, T).
\]

Therefore, in view of (3.9) one can easily see that

\[
h(t) \in H^{m+1+\ell, \frac{m+1+\ell}{2}} (Q_{0,1}^\varepsilon) \quad \forall \varepsilon \in (\varepsilon, T).
\]

Hence, applying Lemma 3.2 in the case \( k = m + 1 \) we conclude that

\[
v(\xi, t) \in H^{m+2+\ell, \frac{m+2+\ell}{2}} (Q_{0,1}^\varepsilon) \quad \forall \varepsilon \in (\varepsilon, T),
\]

i.e., (3.1) holds for \( m + 1 \). This completes the proof of Proposition 3.1. \( \square \)

**Theorem 3.3.** Suppose the pair of functions \((u(x, t), s(t))\) is a classical solution of Problem \( P \) satisfying (i)-(iii). If \( s(t) \in C^{m+2}([0, T]), m \in \mathbb{N}, T > 0 \), then \( f(t) \in C^m([0, T]) \).

Moreover, if \( s(t) \in C^\infty((0, \infty)) \), then \( f(t) \in C^\infty((0, \infty)) \).

**Proof.** Suppose \( s(t) \in C^{m+2}([0, T]) \); then

\[
s(t) \in H^{2m+3+\ell, \frac{2m+3+\ell}{2}} (Q_{0,1}^\varepsilon) \quad \forall \varepsilon \in (0, T).
\]

Therefore, by Proposition 3.1 we obtain that

\[
v(\xi, t) \in H^{2m+\ell, \frac{2m+\ell}{2}} (Q_{0,1}^\varepsilon) \quad \forall \varepsilon \in (\varepsilon, T).
\]

Since \( f(t) = v(0, t) \), it follows that

\[
f(t) \in H^{2m+\ell, \frac{2m+\ell}{2}} (Q_{0,1}^\varepsilon) \quad \forall \varepsilon \in (\varepsilon, T);
\]

thus \( f(t) \in C^m([-\varepsilon, T]) \quad \forall \varepsilon \in (\varepsilon, T), 0 < \varepsilon < T \), which implies that \( f(t) \in C^m([0, T]) \). The proof of Theorem 3.3 is complete. \( \square \)

Theorem 2.1 and Theorem 3.3 prove, respectively, that the condition \( f(t) \in C^\infty((0, \infty)) \) is necessary and sufficient for \( s(t) \in C^\infty((0, \infty)) \). In other words, the following holds.

**Corollary 3.4.** In Problem \( P \), the free boundary \( x = s(t), t \in (0, \infty) \) is an infinitely differentiable curve if and only if \( f(t) \in C^\infty((0, \infty)) \).
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