Quantum Stability of the Heisenberg Ferromagnet

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Work with Niklas Beisert
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[arXiv:0804.0324]
The Heisenberg Spin Chain

One of the oldest quantum mechanical models, set up by Heisenberg in 1928, describes a 1D-ferromagnet with nearest-neighbor interaction of $L$ spin-$1/2$ particles.

- The energy spectrum is bounded between
  - The ferromagnetic ground state $|\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$, energy $E = 0$.
  - The antiferromagnetic ground state "$|\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\rangle$", $E \approx L \log 4$.

- Fundamental excitations: Magnons $|p\rangle = \sum_k e^{i p k} |\ldots\downarrow\downarrow\uparrow\uparrow\ldots\rangle$.

- The spectrum of the closed Heisenberg spin chain is given by the **Bethe equations**:

$$
\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j=1, j \neq k}^M \frac{u_k - u_j + i}{u_k - u_j - i}, \quad k = 1, \ldots, L, \quad u_k = \frac{1}{2} \cot \left(\frac{p_k}{2}\right).
$$

[[Heisenberg 1928] Z. Phys. A49, 619]
[[Hulthén 1938]]
[[Bethe 1931] Z. Phys. 71, 205]
The Ferromagnetic Thermodynamic Limit

- Bethe equations hard to solve for more than a few excitations \( u_k \).
- Antiferromagnetic limit has been studied extensively (spinons). Focus on the **ferromagnetic limit**, ground state \( \ldots \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \ldots \rangle \).
- Analysis of the spectrum simplifies in **thermodynamic limit**:
  - Length of the chain (number of sites) \( L \to \infty \).
  - Number of excitations (flipped spins) \( M \to \infty \).
  - Filling fraction \( \alpha = M/L \) fixed.

- Keep only low-energy excitations above the ferromagnetic vacuum:
  - **Coherent** many-magnon excitations with [Sutherland, Phys. Rev. Lett. 74, 816 (1995)](https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.74.816) [Beisert, Minahan, Staudacher, Zarembo '03]
    - Magnon momenta peaked around collective momentum \( P \).
    - Energy \( E = \tilde{E}/L \sim 1/L \).

- In the ferromagnetic thermodynamic limit, the Heisenberg spin chain is equivalent to the **Landau-Lifshitz model**, a classical non-relativistic sigma model of closed strings on the sphere \( S^2 \) [Kruczenski, hep-th/0311203].
In the thermodynamic limit $L \to \infty$, rescaled roots $x_k = u_k / L$ of coherent states condense on contours in the complex plane (finite-gap solutions):

The Bethe equations turn into integral equations which describe the contours $C_i$ and the root densities $\rho_i$ along them. These equivalently can be described in terms of branch cuts of a spectral curve $p(x)$.

- $p(x)$ is a multivalued function on the complex plane.
- It is parameterized by a set of moduli.

Central Motivation: Validity of the spectral curve

Is the spectral curve a valid description for all values of its moduli?
Branch Cuts and Fluctuation Points

General picture around the ferromagnetic vacuum:

- Fluctuation points at $x_i \approx 1/2\pi n_i$ can be excited to branch cuts $C_i$.
- Integer mode numbers $n_i = -1, -2, -3, \ldots, +3, +2, +1$.
- Cuts $C_i$ have densities $\rho_i$ and fillings $\alpha_i = \int_{C_i} \rho_i$. 

\[
\begin{align*}
\alpha_{-1} & \quad \xleftarrow{\text{Mode numbers } n_i} \\
\alpha_{-2} & \quad \xleftarrow{\text{Fillings } \alpha_i} \quad \alpha_{-5} \\
\alpha_{+2} & \quad \xleftarrow{\text{fluctuation points}} \\
\alpha_{+3} & \quad \xleftarrow{\text{fluctuations: microscopic excitations}} \\
\end{align*}
\]

branch cuts: macroscopic excitations

Branch points

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The spectral curve for a single cut is algebraic:

\[ p(x) = \pi n + \frac{1 - 2\pi nx}{2x} \sqrt{1 + \frac{8\pi n\alpha x}{(1 - 2\pi nx)^2}}. \]

When the filling \( \alpha \) of a cut grows, its length and density increase.

As the cut grows, it attracts the neighboring fluctuation points:

What happens when

- Fluctuation point collides with cut: Density reaches \( |\rho| = 1/\Delta u = 1 \), Bethe equations singular. Spectral curve still valid?
- Two successive fluctuation points collide and diverge into the complex plane. Spectral curve still valid?
To investigate the validity of the spectral curve, study the **Landau-Lifshitz model**, which is an equivalent description of the Heisenberg spin chain in the thermodynamic limit.

- There is a state that corresponds to the one-cut spectral curve.
- Contact with the Heisenberg model: Semiclassically quantize around this state.
- Energy shift of fluctuation modes becomes complex at the point where two fluctuation points collide.
- Spectral curve invalid beyond this point?
- Point where fluctuation point collides with cut is not special.
The Two-Cut Spectral Curve

**Aim:** Understand what happens when fluctuation points collide with a cut or with each other.

For that purpose, replace the fluctuation point by a small but finite excitation:

Need to study the **two-cut spectral curve**.

- It is elliptic and can be constructed explicitly:

\[
p(x) = - \frac{\Delta n}{a_0 z(zs - u)} \sqrt{\frac{a_0^2 (b_0^2 - z^2)}{b_0^2 (a_0^2 - z^2)}} \left( u(a_0^2 - z^2) K(q) + a_0^2 (zs - u) \Pi \left( \frac{qz^2}{z^2 - a_0^2} \bigg| q \right) \right),
\]

\[
z = \frac{tx + u}{rx + s}, \quad q = 1 - a_0^2 / b_0^2.
\]

- It is given in terms of auxiliary parameters and cannot be solved in closed form for the mode numbers \(n_1, n_2\) and fillings \(\alpha_1, \alpha_2\).

\[\Rightarrow\] Needs to be solved numerically.
When the filling grows, neighboring cuts can collide and intersect:

Intersecting cuts form condensates with density $|\rho| = 1/\Delta u = 1$ (similar to “Bethe strings”). Cuts can even pass through each other:

The passing cut changes its contour such that the condensate persists.
Consider a very small second cut:

Compare this to a bare fluctuation point that passes through:

A closed loop with a condensate appears naturally. This prevents the density from exceeding unity, such that always $|\rho| \leq 1$.

Spectral curve stays valid beyond collision of fluctuation point with cut.
Collision of Fluctuation Points

Look at the point where two fluctuation points collide:

Consider again small cuts instead of bare fluctuation points:

Again, this naturally continues to the case of bare fluctuation points:

Spectral curve remains valid beyond this point as well.
Instability: Phase transition

- When two fluctuation points collide, the one-cut solution of the Landau-Lifshitz model appears to become unstable.
- Excitation of a mode means: Regular point $\rightarrow$ Two branch points.
  - Fluctuation point real:
    Excitation means addition of roots, energy increases.
  - Fluctuation points complex with loop cut:
    Excitation means taking roots away, energy decreases.

Energy is at a local minimum when third cut shrunk to zero.

Natural continuation of the ground state beyond the instability point: Two cuts, not (degenerate) three cuts. Phase transition.
### Summary and Conclusions

#### Thermodynamic limit
- In the ferromagnetic thermodynamic limit, macroscopic excitations (coherent states) are contours in the complex plane with mode numbers and fillings.
- Contours and root densities can be described by a spectral curve.

#### Validity of the spectral curve
- Apparent singularity of the Bethe equations in the thermodynamic limit is always hidden in a condensate.
- Apparently unstable classical solutions are degenerate three-cut solutions, a local minimum of the energy is given by a corresponding two-cut solution.
- The spectral curve appears to be valid for all values of its moduli.