Study of Chaos and Scaling in Classical SU(2) Gauge Theory

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Abstract

Following a recent suggestion by Nielsen, Rugh, and Rugh, we study the energy scaling of the maximal Lyapunov exponent of classical Hamiltonian SU(2) lattice gauge theory. It is shown that the conjectured scaling behavior \( \lambda_0 \sim E^{1/4} \) at small energies on the lattice is a finite-time artifact. New numerical results for the maximal Lyapunov exponent are presented for lattices up to size \( 20^3 \) and over two orders of magnitude in the energy per plaquette.
I. INTRODUCTION

Numerical solutions of the classical Yang-Mills equation in Minkowski space have recently attracted some interest as a nonperturbative tool for studying the dynamics of gauge fields. Much of this interest was motivated by the desire to calculate the rate of baryon number fluctuations in the high-temperature phase of the electro-weak gauge theory \[1–3\]. Other studies have concentrated on the spectrum of Lyapunov exponents of the Hamiltonian gauge field dynamics defined on a spatial lattice \[4–6\].

In a recent manuscript, Nielsen, Rugh, and Rugh \[7\] have analyzed the problems associated with the ultraviolet divergence of classical gauge fields and their implications for the Lyapunov exponents of these nonlinear dynamical systems. They showed that, when the divergences are regulated by a short-distance lattice cut-off, the high energy behavior of the Lyapunov exponents is dominated by lattice artifacts. For low energies they conjectured that the energy scaling of the maximal Lyapunov exponent \(\lambda_0\) is the same as that found in the limit of homogeneous gauge potentials \[8\]: \(\lambda_0 \sim E^{1/4}\). They argued further that the approximately linear energy dependence observed at intermediate energies does not have significance in terms of “continuum physics”.

By contrast, Biró et al. \[9\] had argued that the linear energy dependence of the maximal Lyapunov exponent could be related to the linear temperature dependence of the plasmon damping rate \[10\] in the thermal quantum gauge theory. The apparent deviations from this scaling behavior were interpreted as numerical artifacts caused by the loss of reliability of the procedure used in \[4\] for the determination of the maximal Lyapunov exponent.

The present study was motivated by the desire to resolve this controversy by a more careful investigation of lattice artifacts, combined with a more reliable algorithm for the calculation of Lyapunov exponents. As will be seen, this study clearly exhibits the presence of finite-size and finite-time effects at small energies. It shows the need for very long runs on large lattices in order to obtain accurate results in the low energy limit.
II. GENERAL CONSIDERATIONS

The numerical studies reported here were performed by integrating the classical SU(2) gauge fields on a spatial lattice in the Hamiltonian formulation (in the temporal gauge) with a continuous time variable. These equations are covariant against arbitrary space-dependent gauge transformations, but not against gauge transformations that depend on time. The equations of motion for the link variables $U_\ell$ and the numerical algorithm employed in their solution are discussed in detail in [6]. We denote the lattice spacing by $a$, the linear extent of the cubic lattice by $(Na)$, and the gauge coupling constant by $g$. As noted in [6], $g$ and $a$ can be scaled out of the Hamiltonian of the classical Yang-Mills theory. The combination $g^2 Ha$ is dimensionless, and the equations for the lattice gauge field can be written in parameter-free form. However, in order to facilitate physical arguments we will retain $g$ and $a$ as parameters throughout our discussion here.

In the original work on this topic [4] the maximal Lyapunov exponent was obtained by following two initially close gauge field configurations for a period of time and observing the exponential growth of an appropriately determined distance between the gauge fields. The accuracy of this “slope method” is principally limited because the distance between two gauge fields is bounded from above due to the compactness of phase space for a fixed total energy. The method loses its reliability completely at small energies because the exponential divergence is then superseded by distance fluctuations. A more reliable method for the calculation of the maximal Lyapunov exponent uses the rescaling algorithm (see [6], section 3.6). Here one follows two neighboring field configurations for an arbitrarily long time, periodically rescaling the distance to a small value. The accuracy of the obtained value for $\lambda_0$ here is only limited by the time one is willing to spend on the calculation. In the calculation reported here we have used a gauge non-invariant distance measure

$$D [U_\ell, E_\ell; U'_\ell, E'_\ell] = \sum_\ell \left( |U_\ell - U'_\ell|^2 + |E_\ell - E'_\ell|^2 \right).$$

This leads to the same results for the Lyapunov exponent as a gauge invariant distance.
measure, because the equations of motion conserve Gauss’ law, hence gauge degrees of freedom do not contribute to the exponential growth of the distance $D$.

Before I begin to discuss the new results, it is useful to recall some peculiarities of the classical gauge field theory defined on a lattice. As numerical calculations have shown, the distribution of the energy density in the gauge field rapidly becomes thermal. In the weak coupling limit, the “temperature” $T$—i.e. the slope parameter of the energy density distribution—is simply related to the total energy per elementary plaquette, $E_p = H/3N^3$,

$$E_p = \frac{2}{3}(N_c^2 - 1)T = 2T \quad (g^2E_p a \ll 1).$$ (2)

where $N_c$ is the number of colors (here $N_c = 2$). The energy density of the gauge field is

$$\varepsilon = 3E_p/a^3 = 6T/a^3.$$ (3)

This expression, which diverges in the limit $a \to 0$, is indicative of the fact that most of the energy contained in the thermalized classical gauge field resides in short wavelength modes.

The next quantity of interest is the Debye screening length $\mu^{-1}$ of the thermal gauge field. On the lattice one finds in the $N \to \infty$ limit:

$$\mu^2 = 2N_c g^2 \int \frac{d^3k}{(2\pi)^3} \left( \frac{\partial}{\partial \omega} \right) n(\omega) \approx 1.22 N_c \frac{g^2T}{\pi a},$$ (4)

where $\omega = |\vec{k}|$ and $n(\omega) = T/\omega$ is the classical limit of the Bose distribution function. The combination $g^2T$ has the dimension of an inverse length. The Debye screening length $\mu^{-1}$ vanishes in the limit $a \to 0$ as $\mu^{-1} \propto \sqrt{a} \to 0$.

The length scale $(g^2T)^{-1}$, which is associated with the correlation length of static magnetic fields in the thermal quantum gauge theory, is a classical length scale that remains finite in the limit $a \to 0$. It also appears in the damping rate of plasmons at rest:

$$\gamma_0 = \frac{6.635}{24\pi} N_c g^2 T,$$ (5)

and in the winding number fluctuation rate at high temperature, which is proportional to $(g^2T)^4$. 

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For quantities that depend only on $g^2T$, the physical limit requires that the length scale $(g^2T)^{-1}$ is much larger than the lattice spacing $a$ and much smaller than the total length of the lattice, $Na$. Hence, the physical limit is characterized by the two conditions

$$g^2Ta \ll 1, \quad Ng^2Ta \gg 1.$$  

(6)

This evidently requires $N \gg 1$, which is not a surprise. The first condition can also be considered as a weak coupling limit; the second one ensures that finite size effects are small.

As an explicit example, consider the plasmon damping rate $\gamma_0$, which involves an integral over the dimensionless variable $\xi = k/\mu$, which for $N_c = 2$ has an upper limit

$$\xi_{\text{max}} = \frac{\pi}{a\mu} \approx \frac{5.04}{N_c^{1/2}}(g^2Ta)^{-1/2}.$$  

(7)

On the other hand, the integration over $\xi$ becomes a discrete sum on a finite lattice with a spacing

$$\Delta\xi = \frac{\pi}{Na\mu} \approx \frac{5.04}{NN_c^{1/2}}(g^2\sqrt{a})^{-1/2}.$$  

(8)

In the case of the analytical evaluation of $\gamma_0$, the continuum limit requires that

$$g^2Ta \ll 1, \quad N^2g^2Ta \gg 1,$$

(9)

which is somewhat less restrictive than (6).

### III. RESULTS

We begin the discussion of the results with the extrapolation ($t \to \infty$) of the maximal Lyapunov exponent. Figure 1 shows the time evolution of

$$\lambda_0(t) = \frac{1}{t} \sum_{k=1}^{t/\tau} \ln s_k$$

(10)

where $s_k$ is the rescaling factor obtained in scaling step $k$, and $\tau$ is the rescaling interval. The choice of $t^{-1/2}$ as abscissa is motivated by the desire to be able to easily extrapolate to
\( t \to \infty \), combined with the empirical finding that \( \lambda_0(t) \) varies approximately linearly in this variable. The results are for \( N = 10 \).

As can be seen, little extrapolation is needed for \( g^2E_pa \geq 1 \) (curves a–c), because \( \lambda_0(t) \) rapidly approaches a constant value. For smaller values of \( g^2E_pa \) a considerable amount of extrapolation is required, because \( \lambda_0(t) \) still systematically decreases even at \( t/a = 5000 \) (curves d–g). For the lowest energy studied here, \( g^2E_pa = 0.032 \) (curve h), the fluctuations in \( \lambda_0(t) \) are so large that a reliable extrapolation is quite impossible. It is also clear from this figure that the slope method used in [4] significantly overestimates the maximal Lyapunov exponent for \( g^2E_pa < 1 \), since it determines \( \lambda_0 \) through the exponential growth of the distance between trajectories on the time scale \( t/a \leq 100 \).

Although the linear extrapolation in the variable \( t^{-1/2} \) gives statistically quite precise values for \( \lambda_0 \), except at the smallest energies, the extrapolated values have a rather large systematic error depending on the extrapolation scheme. In order to significantly reduce this error much longer runs, up to \( t/a \approx 10^5 \) would be required. Unfortunately, such runs were impossible with our currently available computing resources.

We next turn to the finite size scaling of the Lyapunov exponents. Figure 2a shows the time dependence of \( \lambda_0(t) \) for lattices of size \( N = 4, 6, 10, \) and 20 for the energy \( g^2E_pa \approx 0.125 \). The extrapolated value of \( \lambda_0 \) drops by a factor 3 from \( N = 4 \) to \( N = 20 \). Figure 2b, which shows the extrapolated \( \lambda_0 \) versus \( N^{-3/2} \), demonstrates that the size dependence is much weaker for larger energies, such as \( g^2E_pa \approx 1.8 \). Our results show systematically that finite size effects grow as \( g^2E_pa \) decreases. This result is in accord with conditions (6) and (9).

All our results for the extrapolated Lyapunov exponents \( \lambda_0 \) are shown in Figure 3 on

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1 A \( t/a = 10^5 \) run for a \( 20^3 \) lattice takes about 300 hours at 300 Mflops on a Cray T-90.

2 We have chosen this scaling because it interpolates between the \( N^{-1} \) and \( N^{-2} \) dependences suggested by (6) and (9), respectively, and because it represents the scaling of finite size fluctuations on a lattice with \( N^3 \) sites.
a double logarithmic scale. It is obvious from this figure that the deviation of $\lambda_0$ from the straight line $\lambda_0 \approx \frac{1}{6} g^2 E_a$ at small energies is a finite size effect. The value for $\lambda_0$ at $g^2 E_a = 0.125$ obtained on a $20^3$ lattice (open square) is still slightly below the conjectured scaling line. Note, however, that this point has a considerable systematic uncertainty due to extrapolation (see Figure 2a).

The published data from [4] are also shown in Figure 3 as the open circles. Not unexpectedly, those values, which were obtained on a $20^3$ lattice, clearly overestimate the correct results for $g^2 E_a < 0.5$. A crude estimate of this effect for $g^2 E_a = 0.125$ on the basis of Figure 2a indicates that the value for $\lambda_0$ obtained by the slope method should be about a factor three higher. This agrees quite nicely with the ratio between the old and the new results, both for $20^3$ lattices at this energy. In order to better resolve the results for the region $g^2 E_a \geq 1$, we show the same data on a linear scale in Figure 4. The new results clearly confirm the published values in this energy region, on which the conjectured linear scaling behavior was based [4].

**IV. CONCLUSION**

Our new results do not lend support to the conjecture by Nielsen, Rugh, and Rugh [7] that the maximal Lyapunov exponent of the classical Yang-Mills field scales as $E^{1/4}$ at small energies. Rather, we find that the two values of $\lambda_0$ for $g^2 E_a < 0.5$ published in [4] were distorted by short-time effects, i.e. the exponential divergence of the field configurations was not measured over a sufficiently long time.

We also have found a significant finite size effect for small energies, as already anticipated in [3] (section 3.5). Our value of $\lambda_0$ for the smallest energy ($g^2 E_a = 0.125$) on a $20^3$ lattice is still consistent with the scaling law [4]

$$\lambda_0 \approx \frac{1}{6} g^2 E_a$$

within the systematic uncertainties introduced by the extrapolation $t \to \infty$. 
In summary, our results show that a precise calculation of the energy dependence of the maximal Lyapunov of the classical SU(2) gauge theory requires lattices with $N \geq 20$ and very long evolution times ($t/a \gg 10^4$). Hopefully, such calculations will be feasible in the near future.

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FIG. 1. Time dependence of the measured Lyapunov exponent $\lambda_0$ on a $10^3$ lattice for eight energies. The curves are labeled in alphabetical order; the associated values of $g^2 E_p a$ and the extrapolated Lyapunov exponents are given in Table 1. The dashed lines represent linear least-squares fits to the curves. (Some lines look curved due to the logarithmic representation.)
FIG. 2. Finite size dependence of the Lyapunov exponents. The upper part (a) shows the evolution of the Lyapunov exponents obtained by the rescaling method together with the linear extrapolation \( t \to \infty \) for lattices of size \( N = 4, 6, 10, 20 \) at the energy \( g^2 E_p \approx 0.125 \). The lower part (b) shows the dependence of the extrapolated Lyapunov exponents for two different energies.
FIG. 3. Energy dependence of the extrapolated Lyapunov exponents for lattices of size $N = 2, 4, 6, 10, 20$. The results obtained by Müller and Trayanov [4] are labelled MT92. The dotted line is the fit $\lambda_0 = \frac{1}{g} g^2 E_p$. 
FIG. 4. Same as Figure 3, but on a linear scale.
| \( N \) | Label | \( g^2E_p a \) | \( \lambda_0 a \) | \( t_f/a \) |
|-------|-------|----------------|----------------|-----------|
| 4     | a     | 2.958          | 0.4981         | 1000      |
| 4     | b     | 1.836          | 0.2501         | 1000      |
| 4     | c     | 0.9288         | 0.1525         | 1000      |
| 4     | d     | 0.5265         | 0.1121         | 2000      |
| 4     | e     | 0.2542         | 0.07065        | 2000      |
| 4     | f     | 0.116          | 0.05719        | 4000      |
| 6     | a     | 3.376          | 0.5803         | 1000      |
| 6     | b     | 1.844          | 0.276          | 1000      |
| 6     | c     | 0.894          | 0.1291         | 1000      |
| 6     | d     | 0.5691         | 0.09648        | 2000      |
| 6     | e     | 0.2571         | 0.06043        | 5000      |
| 6     | f     | 0.1249         | 0.04199        | 5000      |
| 6     | g     | 0.06372        | 0.03948        | 5000      |
| 10    | a     | 3.244          | 0.5691         | 1000      |
| 10    | b     | 1.794          | 0.2843         | 1000      |
| 10    | c     | 0.9172         | 0.1246         | 1000      |
| 10    | d     | 0.5796         | 0.08039        | 1000      |
| 10    | e     | 0.2535         | 0.04366        | 5000      |
| 10    | f     | 0.126          | 0.02716        | 5000      |
| 10    | g     | 0.06717        | 0.0235         | 5000      |
| 10    | h     | 0.03291        | 0.02096        | 5000      |
| 20    | f     | 0.126          | 0.0176         | 2000      |

**TABLE I.** Lattice size \( N \), energy per plaquette \( g^2E_p a \), maximal Lyapunov exponent \( \lambda_0 a \), and total evolution times \( t_f/a \) for the results presented in Figures 1–4. The label refers to the labeling used in Figure 1.