Option Pricing from Path Integral for Non-Gaussian Fluctuations. Natural Martingale and Application to Truncated Lévy Distributions

Hagen Kleinert
Institute for Theoretical Physics, FU-Berlin, Arnimallee 14, D-14195, Berlin, Germany

Within a path integral formalism for non-Gaussian price fluctuations we set up a simple stochastic calculus and derive a natural martingale for option pricing from the wealth balance of options, stocks, and bonds. The resulting formula is evaluated for truncated Lévy distributions.

I. INTRODUCTION

Financial markets exhibit non-Gaussian price fluctuations. For this reason they are incomplete markets and mathematicians are unable determine a unique riskless martingale distribution to calculate fair option prices.

As pointed out in Pareto’s 19th century work and reemphasized by Mandelbrot in the 1960s, the logarithms of assets prices in financial markets do not fluctuate with Gaussian distributions, but possess much larger tails which may be approximated by various other distributions such as truncated Lévy distributions, Meixner distributions, generalized hyperbolic distributions or simplifies versions thereof. This has the unpleasant consequence that the associated stochastic differential equations cannot be treated with the popular Ito calculus. The mathematicians call such markets incomplete, implying that there are different choices of riskless martingale distributions with which one can calculate option prices. Many of these have been discussed in the literature, and mathematicians have invented various sophisticated criteria under which one would be preferable over the others for calculating financial risks. Davis, for instance, has introduced a so-called utility function which is supposed to select optimal martingales for different purposes.

In this paper we want to point out that in a path integral formulation of the problem, a straightforward extension of the old chain of arguments which led Black-Scholes to their famous formula produces a specific simple martingale which is different from presently popular versions based on Esscher transforms. Before we come to this we briefly review the derivation of a stochastic calculus for non-Gaussian processes from path integrals which extends Ito’s calculus in the simplest way.

II. GAUSSIAN APPROXIMATION TO FLUCTUATION PROPERTIES OF STOCK PRICES

Let \( S(t) \) denote the price of some stock. On the average, stocks grow exponentially, and a first approximation considers a stock price as an exponential \( S(t) = e^{x(t)} \) of a fluctuating variable \( x(t) \) which obeys a stochastic differential equation

\[
\dot{x}_S(t) = r_S x + \sigma \eta(t),
\]

where \( \eta(t) \) is a white noise of unit strength with the correlation functions

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t) \eta(t') \rangle = \delta(t - t').
\]

The stock price \( S(t) \) itself satisfies the stochastic differential equation for exponential growth

\[
\frac{\dot{S}(t)}{S(t)} = r_S + \sigma \eta(t),
\]

where The standard deviation \( \sigma \) is a measure for the volatility of the stock price, which defined by the expectation value

*kleinert@physik.fu-berlin.de, http://www.physik.fu-berlin.de/~kleinert*
Hamiltonians are standard in elementary particle physics. A relativistic particle, for example, has a Gaussian distribution is only a rough approximation to the real stock prices. Given a certain time scale, in general, we may expand

\[
\hat{x}_S(t) = \frac{dx_S}{dS} \hat{S}(t) + \frac{1}{2} \frac{d^2 x_S}{dS^2} \hat{S}^2(t) dt + \ldots
\]

\[
= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2} \left( \frac{\dot{S}(t)}{S(t)} \right)^2 dt + \ldots,
\]

and replace the last term by its expectation value \(\langle \dot{S}^2 \rangle\):

\[
\hat{x}_S(t) = \frac{\dot{S}(t)}{S(t)} - \frac{1}{2} \left( \frac{\dot{S}(t)}{S(t)} \right)^2 dt
\]

\[
= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2} \sigma^2 + \ldots.
\]

Inserting Eq. (3), we obtain the well-known Ito relation

\[
r_{x_S} = \dot{r}_S - \frac{1}{2} \sigma^2.
\]

In practice, this relation implies that if we fit a straight line through a plot of the logarithms of stock prices, the forward extrapolation of the average stock price is given by

\[
S(t) = S(0) e^{r_st} = S(0) e^{(r_s \sigma^2/2)t}.
\]

III. NON-GAUSSIAN DISTRIBUTIONS

The description of the fluctuations of the logarithms of the stock prices around the linear trend by a Gaussian distribution is only a rough approximation to the real stock prices. Given a certain time scale, for instance days, they follow some distribution \(\tilde{D}(x)\) which is the Fourier transform of some function \(D(p)\), which we shall write as an exponential \(e^{-H(p)}\):

\[
\tilde{D}(x) = \int \frac{dp}{2\pi} e^{ipx} D(p) = \int \frac{dp}{2\pi} e^{ipx-H(p)}.
\]

In general, \(H(p)\) will have a power series expansion

\[
H(p) = ic_1 p + \frac{1}{2} c_2 p^2 - i \frac{1}{3!} c_3 p^3 - \frac{1}{4!} c_4 p^4 + i \frac{1}{5!} c_5 p^5 + \ldots,
\]

The coefficients \(c_n\) in the expansion \(\tilde{D}(x)\) are the cumulants of the distribution \(\tilde{D}(x)\), from which they can be obtained by the connected expectation values

\[
c_1 = \langle x \rangle_c \equiv \langle x \rangle \equiv \int dx x \tilde{D}(x), \quad c_2 = \langle x^2 \rangle_c \equiv \langle x^2 \rangle - \langle x \rangle^2 \equiv \int dx (x - \langle x \rangle)^2 \equiv \sigma^2 \tilde{D}(x), \ldots.
\]

By analogy with mechanics, we call \(H(p)\) the Hamiltonian of the fluctuations. A Gaussian Hamiltonian \(H(p) = \sigma^2 p^2 / 2\) coincides with the mechanical energy of a free particle of mass \(1/\sigma^2\). Non-Gaussian Hamiltonians are standard in elementary particle physics. A relativistic particle, for example, has a Hamiltonian \(H(p) = \sqrt{p^2 + m^2}\), and there is no problem in defining and solving the associated path integral [35].

We also introduce a modified Hamiltonian \(\tilde{H}(p)\), whose Fourier transform has zero average. Its expansion \(\tilde{H}(p)\) has the linear term \(ic_1 p\) subtracted. We also define a modified Hamiltonian \(H_r(p)\) by adding to \(\tilde{H}(p)\) a linear term \(irp\), i.e.,

\[
\tilde{H}(p) \equiv H(p) - H'(0)p, \quad H_r(p) \equiv \tilde{H}(p) + irp = H(p) - H'(0)p + irp.
\]
A. Path Integral for Non-Gaussian Fluctuations

It is easy to calculate the properties of a process \( \eta \) whose fluctuations are distributed according to a general non-Gaussian distribution. If we assume the rate \( x_S \) to coincide with the linear coefficient \( c_1 \) in the expansion (10), such that we can abbreviate \( H_{x_S}(p) \) by \( H(p) \), the stochastic differential equation reads

\[
\dot{x}_S(t) = \eta(t),
\]

and the probability distribution of the endpoints of paths starting at a certain initial point is given by the path integral

\[
P(x_b,t_b|a,t_a) = \int D\eta \int Dx \exp \left[ -\int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \delta[\dot{x} - \eta],
\]

with the initial condition \( x_S(t_a) = x_a \). The final point is, of course, \( x_b = x_S(t_b) \). The function \( \tilde{H}(\eta) \) is defined by the negative logarithm of the non-Gaussian distributions \( \tilde{D}(x) \), such that

\[ e^{-\tilde{H}(\eta)} \equiv \tilde{D}(x). \]

The path integral is defined \( \text{à la Feynman} \) by slicing the time axis at times \( t_n = cn \) with \( n = 0, \ldots, N \), and integrating over all \( x(t_n) \). At the end, the limit \( N \to \infty \), in particular it will be for the typical non-Gaussian distributions encountered in stock markets. This is completely analogous to Riemann’s procedure of defining ordinary integrals, which are approximated by sums over values of a function on a set of measure zero.

With the help of the path integral, the correlation functions of the noise in the path integral can be found by straightforward functional differentiation. For this purpose, we express the noise distribution \( P[\eta] \equiv \exp \left[ -\int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \) as a Fourier path integral

\[
P[\eta] = \int D\eta \int \frac{Dp}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt [ip(t)\eta(t) - H(p(t))] \right\},
\]

and note that the correlation functions can be obtained from the functional derivatives

\[
\langle \eta(t_1) \cdots \eta(t_n) \rangle = (-i)^n \int D\eta \int \frac{Dp}{2\pi} \left[ \frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{\int_{t_a}^{t_b} dt p(t)\eta(t)} \right] e^{-\int_{t_a}^{t_b} dt H(p(t))}.
\]

After \( n \) partial integrations, this becomes

\[
\langle \eta(t_1) \cdots \eta(t_n) \rangle = i^n \int D\eta \int \frac{Dp}{2\pi} e^{i\int_{t_a}^{t_b} dt p(t)\eta(t)} \left[ \frac{\delta}{\delta p(t_1)} \cdots \frac{\delta}{\delta p(t_n)} e^{-\int_{t_a}^{t_b} dt H(p(t))} \right]_{p(t) = 0}.
\]

By expanding the exponential \( e^{-\int_{t_a}^{t_b} dt H(p(t))} \) in a power series using (10), it is straightforward to calculate

\[
\langle \eta(t_1) \rangle \equiv Z^{-1} \int D\eta \eta(t_1) \exp \left[ -\int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] = c_1,
\]

\[
\langle \eta(t_1)\eta(t_2) \rangle \equiv Z^{-1} \int D\eta \eta(t_1)\eta(t_2) \exp \left[ -\int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] = c_2 \delta(t_1 - t_2) + c_1^2,
\]

\[
\langle \eta(t_1)\eta(t_2)\eta(t_3) \rangle \equiv Z^{-1} \int D\eta \eta(t_1)\eta(t_2)\eta(t_3) \exp \left[ -\int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] = c_2 \delta(t_1 - t_2) \delta(t_2 - t_3) + c_1^3,
\]

\[
\langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \rangle \equiv Z^{-1} \int D\eta \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \exp \left[ -\int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] = c_2 \delta(t_1 - t_2) \delta(t_2 - t_3) \delta(t_3 - t_4) + c_1^4.
\]
\[ c_3 \delta(t_1 - t_2) \delta(t_1 - t_3) + c_2 c_1 \left[ \delta(t_1 - t_2) + \delta(t_2 - t_3) + \delta(t_1 - t_3) \right], \]
\[ \langle \eta(t_1) \eta(t_2) \eta(t_3) \eta(t_4) \rangle \equiv Z^{-1} \int \mathcal{D} \eta \eta(t_1) \eta(t_2) \eta(t_3) \eta(t_4) \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right] \]
\[ = c_4 \delta(t_1 - t_2) \delta(t_1 - t_3) \delta(t_1 - t_4) + c_2 c_1 \left[ \delta(t_1 - t_2) + 5 \text{ pair terms} \right] + A c_2^2 \left[ \delta(t_1 - t_2) \delta(t_3 - t_4) \delta(t_1 - t_3) \delta(t_2 - t_4) \delta(t_1 - t_4) \delta(t_2 - t_3) \right], \]

where \( Z \) is the normalization integral
\[ Z \equiv \int \mathcal{D} \eta \exp \left[ - \int_{t_a}^{t_b} dt \tilde{H}(\eta(t)) \right]. \]

The higher correlation functions are obvious generalizations of these equations. The different contributions on the right-hand side of Eqs. (19)–(22) are distinguishable by their connectedness structure.

An important property of the probability (14) is that it satisfies the semigroup property of path integrals
\[ P(x_c t_c | x_a t_a) = \int dx_b P(x_c t_c | x_b t_b) P(x_b t_b | x_a t_a). \]

The experimental asset distributions do satisfy this property approximately. For truncated Lévy distributions this is shown in Fig. 1.

\[ \kappa \approx \text{const.} \times 1/(t_b - t_a)^{0.5} \]

FIG. 1. Left-hand side: Cumulative distributions obtained by repeated convolutions of the 15-min distribution (from Ref. [5]). Apart from the tails, the semigroup property (24) is reasonably well satisfied. Right-hand side: Falloff of kurtosis is slower than \( 1/(t_b - t_a) \) expected from convolution property (see Ref. [36]).

Apart from the far ends of the tails, the semigroup property (24) is reasonably well satisfied.

**B. Lévy-Khintchine Formula**

If is sometimes useful to represent the Hamiltonian in the form of a Fourier integral
\[ H(p) = - \int dx e^{ipx} F(x). \]
Due to the special significance of the linear term in $H(p)$ governing the drift, this is usually subtracted out of the integral by rewriting (25) as

$$H_r(p) = ip \int dx \left( e^{ipx} - 1 - ipx \right) F(x).$$

(26)

The first subtraction ensures the property $H_r(0) = 0$ which guarantees the unit normalization of the distribution. This subtracted representation is known as the Lévy-Khintchine formula, and the function $F(x)$ is the so-called Lévy distribution. This subtracted representation is known as the $\bar{H}$ function.

Integrating over all $x_S(t)$ with fixed end points enforces a constant momentum along the path, and we remain with a single integral

$$P(x_b,t_b|x_a,t_a) = \int \frac{dp}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt \left[ ip(t)\dot{x}_S(t) - ip(t)\eta(t) - \dot{H}(\eta(t)) \right] \right\}.$$  

(28)

Integrating over all $x_S(t)$ with fixed end points enforces a constant momentum along the path, and we remain with a single integral

$$P(x_b,t_b|x_a,t_a) = \int \frac{dp}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt \left[ ip(t)\dot{x}_S(t) - H(p(t)) \right] \right\}.$$  

(29)

From this representation it is easy to verify that this probability satisfies a Fokker-Planck-type equation

$$\partial_t P(x_b,t_b|x_a,t_a) = -i \partial_x P(x_b,t_b|x_a,t_a).$$  

(31)

The general solution $\psi(x,t)$ of this differential equation with the initial condition $\psi(x,0)$ is given by the path integral generalizing (14)

$$\psi(x,t) = \int D\eta \exp \left\{ - \int_{t_a}^{t_b} dt \dot{H}(\eta(t)) \right\} \psi \left( x - \int_{t_a}^{t} dt' \eta(t') \right).$$  

(32)

To verify that this satisfies indeed the Fokker-Planck-type equation (31) we consider $\psi(x,t)$ at a slightly later time $t + \epsilon$ and expand

$$\psi(x,t + \epsilon) = \int D\eta \exp \left\{ - \int_{t_a}^{t_b} dt \dot{H}(\eta(t)) \right\} \psi \left( x - \int_{t_a}^{t} dt' \eta(t') \right) - \int_{t_a}^{t_b} dt \psi \left( x - \int_{t_a}^{t} dt' \eta(t') \right) \dot{H}(\eta(t)) + \frac{1}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \eta(t') \dot{H}(\eta(t)) + \frac{1}{3!} \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \int_{t_a}^{t} dt'' \eta(t') \eta(t'') \eta(t_1) \eta(t_2) \dot{H}(\eta(t)) + \frac{1}{4!} \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \int_{t_a}^{t} dt'' \int_{t_a}^{t} dt''' \eta(t') \eta(t'') \eta(t_1) \eta(t_2) \eta(t_3) \dot{H}(\eta(t)) + \ldots$$  

(33)
Using the correlation functions (19)–(22) we obtain
\[
\psi(x, t + \epsilon) = \int D\eta \exp \left[ -\int_{t_0}^{t+\epsilon} dt \tilde{H}(\eta(t)) \right] \\
\times \left[ -\epsilon c_1 \partial_x + (\epsilon c_2 + \epsilon^2 c_1) \frac{1}{2} \partial_x^2 - \left( \epsilon c_3 + 3\epsilon^2 c_2 c_1 \right) \frac{1}{3!} \partial_x^3 \\
+ (\epsilon c_4 + \epsilon^2 3\epsilon c_3 c_1 + \epsilon^2 2\epsilon c_2 c_1^2 + \epsilon^4 c_1^3) \frac{1}{4!} \partial_x^4 + \ldots \right] \psi \left( x - \int_{t_0}^{t} dt' \eta(t') \right).
\]
(34)

In the limit \( \epsilon \to 0 \), only the linear terms in \( \epsilon \) contribute, which are all due to the connected parts of the correlation functions of \( \eta(t) \). The differential operators in the brackets can now be pulled out of the integral and we find the differential equation
\[
\partial_t \psi(x, t) = \left[ -\epsilon c_1 \partial_x + c_2 \frac{1}{2} \partial_x^2 - c_3 \frac{1}{3!} \partial_x^3 + c_4 \frac{1}{4!} \partial_x^4 + \ldots \right] \psi(x, t).
\]
(35)

We now replace \( c_1 \to r_x \) and express using (11) the differential operators in brackets as Hamiltonian operator \(-H_{r_x}(-i\partial_x)\). This leads to the Schrödinger-like equation
\[
\partial_t \psi(x, t) = -H_{r_x}(-i\partial_x) \psi(x, t).
\]
(36)

Due to the many derivatives in \( H(i\partial_x) \), this equation is in general non-local. This can be made explicit with the help of the Lévy-Khintchine weight function \( F(x) \) in the Fourier representation (22). In this case, the right-hand side
\[
-H(i\partial_x)\psi(x, t) = \int dx' e^{-x' \partial_x} F(x')\psi(x, t) = \int dx' F(x')\psi(x-x', t).
\]
(37)

and the Fokker-Planck-like equation (38) takes the form of an integral equation. Some people like to use the subtracted form (27) of the Lévy-Khintchine and arrive at the integro-differential equation
\[
\partial_t \psi(x, t) = \left[ -c_1 \partial_x - \frac{1}{2} \partial_x^2 \right] \psi(x, t) + \int dx' F(x')\psi(x-x', t).
\]
(38)

The integral term can then be treated as a perturbation to an ordinary Fokker-Planck equation.

By a similar procedure as in the derivation of Eq. (34), it is possible to derive a generalization of Ito’s rule (3) to functions of noise variable with non-Gaussian distributions. As in (3) we expand \( f(x(t+\epsilon)) \):
\[
\begin{align*}
f(x(t+\epsilon)) &= f(x(t)) + f'(x(t)) \int_{t_0}^{t+\epsilon} dt' \dot{x}(t') \\
&\quad + \frac{1}{2} f''(x(t)) \int_{t_0}^{t+\epsilon} dt_1 \int_{t_0}^{t+\epsilon} dt_2 \dot{x}(t_1) \dot{x}(t_2) \\
&\quad + \frac{1}{3!} f^{(3)}(x(t)) \int_{t_0}^{t+\epsilon} dt_1 \int_{t_0}^{t+\epsilon} dt_2 \int_{t_0}^{t+\epsilon} dt_3 \dot{x}(t_1) \dot{x}(t_2) \dot{x}(t_3) + \ldots,
\end{align*}
\]
(39)

where \( \dot{x}(t) = \eta(t) \) is the stochastic differential equation with a nonzero expectation value \( \langle \eta(t) \rangle = c_1 \). In contrast to (3) which had to be carried out only up to second order in \( \dot{x} \), we must now keep all orders in the noise variable. Evaluating the noise averages of the multiple integrals on the right-hand side using the correlation functions (19)–(22), we find the time derivative of the expectation value of an arbitrary function of the fluctuating variable \( x(t) \)
\[
\begin{align*}
\langle f(x(t+\epsilon)) \rangle &= \langle f(x(t)) \rangle + \langle f'(x(t)) \rangle \epsilon c_1 + \frac{1}{2} \langle f''(x(t)) \rangle (\epsilon c_2 + \epsilon^2 c_1^2) \\
&\quad + \frac{1}{3!} \langle f^{(3)}(x(t)) \rangle (\epsilon c_3 + \epsilon^2 c_2 c_1 + \epsilon^3 c_1^3) + \ldots \\
&= \epsilon \left[ -c_1 \partial_x + c_2 \frac{1}{2} \partial_x^2 - c_3 \frac{1}{3!} \partial_x^3 + \ldots \right] \langle f(x(t)) \rangle + O(\epsilon^2).
\end{align*}
\]
(40)
After the replacement $c_1 \to r_x$ the function $f(x(t))$ obeys therefore the following equation:

$$\langle \dot{f}(x(t)) \rangle = -H_{rs}(i\partial_x)(f(x(t))). \quad (41)$$

Taking out the lowest-derivative term this takes a form

$$\langle \dot{f}(x(t)) \rangle = \langle \partial_x f(x(t)) \rangle \langle \dot{x}(t) \rangle - \dot{H}_{rs}(i\partial_x)(f(x(t))). \quad (42)$$

In postpoint time slicing, this may be viewed as the expectation value of the stochastic differential equation

$$\dot{f}(x(t)) = \partial_x f(x(t)) \dot{x}(t) - \dot{H}_{rs}(i\partial_x)f(x(t)). \quad (43)$$

This is the direct generalization of Ito’s rule (3). For an exponential function $f(x) = e^{P_x}$, this becomes

$$\frac{d}{dt} e^{P_x(t)} = [\dot{x}(t) - H_{rs}(iP)] e^{P_x(t)}. \quad (44)$$

As a consequence of this equation for $P = 1$, the rate $r_S$ with which a stock price $S(t) = e^{x(t)}$ grows according to formula (3) is now related to $r_x$ by

$$r_S = r_x - \dot{H}(i) = r_x - [H(i) - iH'(0)] = -H_{rs}(i), \quad (45)$$

which replaces the simple Ito relation $r_S = r_x + \sigma^2/2$ in Eq. (1). Recall the definition $\dot{H}(p) = H(p) - H'(0)p$ in Eq. (12). The corresponding generalization of the left-hand part of Eq. (1) reads

$$\frac{\dot{X}}{S} = \dot{x}(t) - \dot{H}(i) = \dot{x}(t) - [H(i) - iH'(0)] = \dot{x}(t) - r_x - H_{rs}(i). \quad (46)$$

The forward price of a stock must therefore be calculated with the generalization of formula (3):

$$\langle S(t) \rangle = S(0)e^{r_S t} = S(0)\left(e^{r_x t + \int_0^t dt' \eta(t')}\right) = S(0)e^{-H_{rs}(i)t} = S(0)e^{(r_x - [H(i) - iH'(0)])t}. \quad (47)$$

Note that may derive the differential equation of an arbitrary function $f(x(t))$ in Eq. (47) from a simple mnemonic rule, expanding sloppily

$$f(x(t + dt)) = f(x(t) + \dot{x}dt) = f(x(t)) + f'(x(t))\dot{x}(t)dt + \frac{1}{2}f''(x(t))\dot{x}^2(t)dt^2 \right. \hfill (48)$$

and replacing (3),

$$\langle \dot{x}(t) \rangle dt \to c_1 dt, \quad \langle \dot{x}^2(t) \rangle dt^2 \to c_2 dt, \quad \langle \dot{x}^3(t) \rangle dt^3 \to c_3 dt, \ldots. \quad (49)$$

**IV. MARTINGALES**

In financial mathematics, an often-encountered concept is that of a martingale (3). The name stems from a casino strategy in which a gambler doubles his stake each time a bet is lost. A stochastic variable is called a martingale, if its expectation value is time-independent. The noise variable $\eta(t)$ with vanishing average is a trivial martingale.
A. Gaussian Martingale

For a harmonic noise variable, the exponential $e^{\int_0^t \sigma^2(t') \, dt'}$ is a nontrivial martingale, due to Eq. (8). For the same reason, a stock price $S(t) = e^{x(t)}$ with $x(t)$ obeying the stochastic differential equation (5), can be made a martingale by a time-dependent multiplicative factor the quantity $e^{-rS}e^{rt} = e^{-rS}e^{rt}$. An explicit distribution which makes $S(t) = e^{x(t)}$ a martingale is

$$P^M(x_b t_b | x_a t_a) \equiv \frac{e^{-rt}}{\sqrt{2\pi\sigma^2(t_b - t_a)^2}} \exp \left\{ -\frac{(x_b - x_a - r_x(t_b - t_a))^2}{2\sigma^2(t_b - t_a)} \right\}. \quad (50)$$

I can easily be verified by direct integration that the expectation value of $S(t) = e^{x(t)}$ is time-independent with this distribution:

$$\langle S(t_b) \rangle = \langle e^{x(t_b)} \rangle = \int dx_b e^{x^2} P^M(x_b t_b | x_a t_a) \quad (51)$$

is independent of the time $t_b$. At this point we can make an important observation: There exists an entire family of distributions for which $S(t) = e^{x(t)}$ is a martingale, namely

$$P^{Mr}(x_b t_b | x_a t_a) \equiv \frac{e^{-rt}}{\sqrt{2\pi\sigma^2(t_b - t_a)^2}} \exp \left\{ -\frac{(x_b - x_a - r_x(t_b - t_a))^2}{2\sigma^2(t_b - t_a)} \right\}. \quad (52)$$

for any $r$ and $r_x = r - \sigma^2/2$. Such distributions which differ by the drift $r$ are called equivalent. The prefactor $e^{-rt}$ is referred to as a discount factor with the rate $r$.

B. Non-Gaussian Martingales

For $S(t) = e^{x(t)}$ with an arbitrary non-Gaussian noise $\eta(t)$, there are many ways of constructing martingales. the relation (43) allows us to construct immediately the simplest martingale

1. Natural Martingale

An obvious generalization of the Gaussian expression $e^{\int_0^t \sigma^2(t') \, dt'}$ is the exponential $e^{\int_0^t \sigma^2(t') \, dt'}$, whose expectation value is time-independent due to Eq. (43). Expressed in terms of the associated stock price $S(t) = e^{x(t)}$, we obtain the simplest martingale

$$e^{-rS}S(t) = e^{-rS}e^{rt} \int_0^t \int D\eta \, D\xi \exp \left\{ -\int_{t_a}^{t_b} dt \tilde{H}_{rs} (\eta(t)) \right\} \delta[\tilde{\eta} - \eta]. \quad (53)$$

if $r_S$ and $r_{xs}$ are related by $r_S = r_{xs} - \tilde{H}(i) = -H_{xs}(i)$. It is easy to write down a distribution function which makes $S(t) = e^{x(t)}$ itself a martingale:

$$P^M(x_b t_b | x_a t_a) = e^{-rS} \int D\eta \int D\xi \exp \left\{ -\int_{t_a}^{t_b} dt \tilde{H}_{rs} (\eta(t)) \right\} \delta[\tilde{\eta} - \eta]. \quad (54)$$

There exists also here an entire family of equivalent distribution functions for which the integrals of the type (51) over $e^{x^2}$ are independent of $t_b$. This is done as follows. One obvious family of this type is a straightforward generalization of the Gaussian family (52) which we shall refer to as natural martingales:

$$P^{Mr}(x_b t_b | x_a t_a) = e^{-rt} \int D\eta \int D\xi \exp \left\{ -\int_{t_a}^{t_b} dt \tilde{H}_{rs} (\eta(t)) \right\} \delta[\tilde{\eta} - \eta], \quad (55)$$

with arbitrary $r = r_x - \tilde{H}(i) = -H_{rs}(i)$. Indeed, multiplying this with $e^{x_b}$ and integrating over $x_b$ gives rise to a $\delta$-function $\delta(p - i)$ and produces the same result $e^{x_a}$ for all times $t_b$. The path integral is solved by the Fourier integral [compare (51)]

$$P^{Mr}(x_b t_b | x_a t_a) = e^{-rt} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \exp \left\{ ip(x_b - x_a) - (t_b - t_a)H_{rs}(p) \right\}. \quad (56)$$
2. Esscher Martingale

In the literature on mathematical finance, much attention is given another family of equivalent martingale measures. It has been used a long time ago to estimate risks of actuaries \[38\] and introduced more recently into the theory of option prices \[39,40\] where it is now of wide use \[41\]– \[46\]. This family is constructed as follows. Let \( \tilde{D}(x) \) be an arbitrary distribution function with a Fourier transform

\[
\tilde{D}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-H(p)} e^{ipx},
\]

and \( H(0) = 0 \), to guarantee a unit normalization \( \int dx \tilde{D}(x) = 1 \). We now introduce an Esscher-transformed distribution function. It is obtained by slightly tilting the initial distribution \( \tilde{D}(x) \) by multiplication with an asymmetric exponential factor \( e^{\theta x} \):

\[
D^\theta(x) = e^{H(i\theta)} e^{\theta x} \tilde{D}(x).
\]

The constant prefactor \( e^{H(i\theta)} \) is necessary to conserve the total probability. This distribution can be written as a Fourier transform

\[
D^\theta(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-H^\theta(p)} e^{ipx}.
\]

with the Esscher-transformed Hamiltonian

\[
H^\theta(p) \equiv H(p + i\theta) - H(i\theta).
\]

Since \( H^\theta(0) = 0 \), the transformed distribution is properly normalized \( \int dx D^\theta(x) = 1 \). We now define the Esscher-transformed expectation value

\[
\langle F(x) \rangle^\theta \equiv \int dx D^\theta(x) F(x).
\]

It is related to the original expectation value by

\[
\langle F(x) \rangle^\theta \equiv e^{H(i\theta)} \langle e^{\theta x} F(x) \rangle \tag{62}
\]

For the specific function \( F(x) = e^x \), Eq. (62) becomes

\[
\langle e^x \rangle^\theta = e^{-H^\theta(i)} \equiv e^{H^\theta(i\theta)} \langle e^{(\theta+1)x} \rangle = e^{H^\theta(i\theta) - H^\theta(i\theta + i)} \tag{63}
\]

Applying the transformation (63) to each time slice in the general path integral (14), we obtain the Esscher-transformed path integral

\[
P^\theta(x_b,t_b|x_a,t_a) = e^{-\tau x(t) e^{H^\theta(r_{rs} t)}} \int D\eta \int Dx \exp \left\{ \int_{t_a}^{t_b} dt \left[ \theta \eta(t) - \tilde{H}_{rs}(\eta(t)) \right] \right\} \delta[\dot{x} - \eta]. \tag{64}
\]

The solution is given by the Fourier integral [compare (60)]

\[
P^\theta(x_b,t_b|x_a,t_a) = e^{-\tau x(t) e^{H^\theta(r_{rs} t)}} \int_{-\infty}^{\infty} dp \frac{dp}{2\pi} \exp \left[ ip(x_b - x_a) - (t_b - t_a) H_{rs}(p + i\theta) \right]. \tag{65}
\]

Let us denote the expectation values calculated with this probability by \( \langle \ldots \rangle^\theta \). Then we find for \( S(t) = e^{\tau(t)} \) the time dependence

\[
\langle S(t) \rangle^\theta = e^{-H^\theta_{rs}(t)} \tag{66}
\]

This equation shows, that the exponential of a stochastic variable \( x(t) \) can be made a martingale with respect to any Esscher-transformed distribution if we remove the exponentially growing factor \( \exp(r^\theta t) \) with
Thus, a family of equivalent martingale distributions for the stock price $S(t) = e^{x(t)}$ is

$$P^M_{\theta}(x_B t_B | x_A t_A) \equiv e^{-r_\theta t} P^\theta(x_B t_B | x_A t_A) \tag{68}$$

for any choice of the parameter $\theta$.

For a harmonic distribution function (50), the Esscher martingales and the previous ones are equivalent. Indeed, starting from (50) in which $r_S = r_{x_S} + \sigma^2/2$, the Esscher transform leads us after a quadratic completion to the family of natural martingales (50) with the rate parameter $r = r_{x_S} + \theta \sigma^2$.

3. Other Non-Gaussian Martingales

Many other non-gaussian martingales have been discussed in the literature. Mathematicians have invented various sophisticated criteria under which one would be preferable over the others for calculating financial risks. Davis has introduced a so-called utility function \[33] which is supposed to select optimal martingales for different purposes.

For the upcoming development of a theory of option pricing, only the initial natural martingale will turn out to be relevant.

V. OPTION PRICING

The most important use of path integrals in financial markets is made in the determination of a fair price of financial derivatives, in particular options. Options are an ancient financial tool. They are used for speculative purposes or for hedging major market transactions against unexpected changes in the market environment. These can sometimes produce dramatic price explosions or erosions, and options are supposed to prevent the destruction of huge amounts of capital. Ancient Romans, Grecians, and Phoenicians traded options against outgoing cargos from their local seaports. In financial markets, options are contracted between two parties in which one party has the right but not the obligation to do something, usually to buy or sell some underlying asset. Having rights without obligations has a financial value, so option holders must pay a price for acquiring them. The price depends on the value of the associated asset, which is why they are also called derivative assets or briefly derivatives. Call options are contracts giving the option holder the right to buy something, while put options entitle the holder to sell something. The price of an option is called premium. Usually, options are associated with stock, bonds, or commodities like oil, metals or other raw materials. In the sequel we shall consider call options on stocks, to be specific.

Modern option pricing techniques have their roots in early work by Charles Castelli who published in 1877 a book entitled The Theory of Options in Stocks and Shares. This book presented an introduction to the hedging and speculation aspects of options. However, it still lacked a sound theoretical base. Twenty three years later, Louis Bachelier offered the earliest known analytical valuation for options in his dissertation at the Sorbonne \[47]. It is curious that Bachelier discovered the treatment of stochastic phenomena five years before Einstein’s related but much more famous work on Brownian motion \[48], and 23 years before Wiener’s mathematical development \[49]. The stochastic differential equations considered by him still had an important defect of allowing for negative security prices, and for option prices exceeding the price of the underlying asset. Bachelier’s work was continued by Paul Samuelson, who wrote in 1955 an unpublished paper entitled Brownian Motion in the Stock Market. During that same year, Richard Kruizenga, one of Samuelson’s students, cited Bachelier’s work in his dissertation Put and Call Options: A Theoretical and Market Analysis. In 1962, another dissertation, this time by A. James Boness, focused on options. In his work, entitled A Theory and Measurement of Stock Option Value, Boness developed a pricing model that made a significant theoretical jump from that of his predecessors. More significantly, his work served as a precursor to that of Fischer Black and Myron Scholes, who in 1973 introduced their famous Black and Scholes Model \[50] which, together with the improvements introduced by R. Merton, earned the Nobel prize in 1997\[1].

\[1\]For F. Black the prize came too late—he had died two years earlier.
A. Black-Scholes Option Pricing Model

In the early seventies, Fisher Black was working on a valuation model for stock warrants and observed that his formulas resembled very much the well-known equations for heat transfer. Soon after this, Myron Scholes joined Black and together they discovered an approximate option pricing model which is still of wide use. It is an improved version of a previous model developed by A. James Boness in his Ph.D. thesis the University of Chicago.

The Black and Scholes Model is based on the following assumptions:

1. The logarithm of the returns is normally distributed. We remarked before that there are considerable deviations which call for improvement of the model to be developed below.

2. Markets are efficient. This assumption implies that the market operates continuously with share prices following a continuous stochastic process without memory. It also implies that different markets have the same asset prices.
   
   This is not quite true. Different markets do in general have slightly different prices. Their differences are kept small by the existence of arbitrage dealers. There also exist correlations over a short time scale which make it possible, in principle, to profit without risk from statistical arbitrage. This possibility is, however, strongly limited by transaction fees.

3. No commissions are charged.
   
   This is not true. Usually market participants have to pay a commission to buy or sell assets. Even floor traders pay some kind of fee, although this is usually very small. The fees paid by individual investors is more substantial and can distort the output of the model.

4. Interest rates remain constant and known.
   
   The Black and Scholes model assumes the existence of a risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate. As an approximation, one uses the discount rate on U.S. Government Treasury Bills with 30 days left until maturity. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

5. The stock pays no dividends during the option’s life.
   
   Most companies pay dividends to their share holders, so this is a limitation to the model since higher dividend lead to lower call premiums. There is, however, a simple possibility of adjusting the model to the real situation by subtracting the discounted value of a future dividend from the stock price.

6. European exercise terms are used. European exercise terms imply the exercise of an option only on the expiration date. This is in contrast to the American exercise terms which allow for this at any time during the life of the option. This greater flexibility makes an American option more valuable than the European one.

   The difference is, however, not dramatic in praxis because very few calls are ever exercised before the last few days of their life, since an early exercise means giving away the remaining time value on the call. Different exercise times towards the end of the life of a call are irrelevant since the remaining time value is very small and the intrinsic value has a small time dependence, barring a dramatic event right before expiration date.

Since 1973, the original Black and Scholes Option Pricing Model has been the subject of much attention. In the same year, Robert Merton included the effect of dividends. Three years later, Jonathan Ingerson relaxed the assumption of no taxes or transaction costs, and Merton removed the restriction of constant interest rates. At present we are in a position of being able to determine quite reliably the values of stock options.

The relevance of path integrals to this field was recognized first in 1988 by a theoretical physicist J.W. Dash, who wrote two unpublished papers on the subject entitled Path Integrals and Options I and II. Since then many theoretical physicists have entered the field, and papers on this subject have begun appearing on the Los Alamos server.
B. Evolution Equations of Portfolios with Options

The option price $O(t)$ has a larger fluctuations than the associated stock price. It usually varies with an efficiency factor $\partial O(S(t), t)/\partial S(t)$. For this reason it is possible, in the ideal case of Gaussian price fluctuations, to guarantee a steady growth of a portfolio by mixing $N_S(t)$ stocks with $N_O(t)$ options and a certain amount of cash which is usually kept in the form of bonds, whose number is denoted by $N_B(t)$. The composition $[N_S(t), N_O(t), N_B(t)]$ is referred to as the strategy of the portfolio manager. The total wealth has the value

$$W(t) = N_S(t)S(t) + N_O(t)O(S, t) + N_B(t)B(t).$$  \hfill (69)

The goal is to make it grow with a smooth exponential curve without fluctuations

$$\dot{W}(t) \approx r_W W(t).$$  \hfill (70)

As we shall see immediately, this is possible provided the short-term bonds (usually those with 30 days to maturity) grow without fluctuations.

$$\dot{B}(t) \approx r_B B(t).$$  \hfill (71)

Under this assumption, the rate $r_B$ is referred to as riskfree interest rate. This assumption is fulfilled in true markets only approximately since there can always be events which change the value of short-term bonds excessively.

The existence of arbitrage dealers will ensure that the growth rate $r_W$ is equal to that of the short-term bonds

$$r_W \approx r_B.$$  \hfill (72)

Otherwise the dealers would change from one investment to the other.

In the decomposition (63), the desired growth (70) reads

$$N_S(t)\dot{S}(t) + N_O(t)\dot{O}(S, t) + N_B(t)\dot{B}(t) + \dot{N}_S(t)S(t) + \dot{N}_O(t)O(S, t) + \dot{N}_B(t)B(t) = r_W \left[N_S(t)S(t) + N_O(t)O(S, t) + N_B(t)B(t)\right].$$  \hfill (73)

Due to (71) and (72), the terms containing $N_B(t)$ without a dot drop out. Moreover, if no extra money is inserted into or taken from the system, i.e., if stocks, options, and bonds are only traded against each other, this does not change the total wealth, assuming the absence of commissions. This so-called self-financing strategy is expressed in the equation

$$\dot{N}_S(t)S(t) + \dot{N}_O(t)O(S, t) + \dot{N}_B(t)B(t) = 0.$$  \hfill (74)

Thus the growth equation (70) translates into

$$\dot{W}(t) = N_S\dot{S} + N_O\dot{O} + N_B\dot{B} = r_W \left(N_S S + N_O O + N_B B\right).$$  \hfill (75)

Due to the equality of the rates $r_W = r_B$ and Eq. (71), the entire contribution of $B(t)$ cancels, and we obtain

$$N_S\dot{S} + N_O\dot{O} = r_W \left(N_S S + N_O O\right)$$  \hfill (76)

The important observation is now that there exists an optimal ratio between the number of stocks $N_S$ and the number of options $N_O$, which is inversely equal to the efficiency factor

$$\frac{N_S(t)}{N_O(t)} = \frac{\partial O(S(t), t)}{\partial S(t)}.$$  \hfill (77)

Then Eq. (76) becomes

$$N_S\dot{S} + N_O\dot{O} = r_W \left(-\frac{\partial O}{\partial x} + O\right) N_O.$$  \hfill (78)
The two terms on the left-hand side are treated as follows: First we use the relation (77) to rewrite

\[ N_S \dot S = - \frac{\partial O(S,t)}{\partial S} \dot S = - \frac{\partial O(S,t)}{\partial x} \frac{\dot S}{S}, \tag{79} \]

and further, with the help of Eq. (46), as

\[ N_S \dot S = - \frac{\partial O(S,t)}{\partial x} \left[ \dot x - \bar H(i) \right]. \tag{80} \]

In the second term on the left-hand side of (78), we expand the total time dependence of the option price in a Taylor series

\[ \frac{dO}{dt} = \frac{1}{dt} \left[ O(x(t) + \dot x(t) dt, t + dt) - O(x(t), t) \right] \]

\[ = \frac{\partial O}{\partial t} \dot x + \frac{1}{2} \frac{\partial^2 O}{\partial x^2} \dot x^2 dt + \frac{1}{3!} \frac{\partial^3 O}{\partial x^3} \dot x^3 dt^2 + \ldots. \tag{81} \]

We have gone over to the logarithmic stock price variable \( x(t) \) rather than \( S(t) \) itself. Some of the derivatives on the right-hand side are denoted by special symbols in financial mathematics: the quantities \( \vartheta \equiv \partial O/\partial t, \Delta \equiv \partial O/\partial S = \partial O/\partial xS, \) and \( \Gamma \equiv \partial^2 O/\partial S^2 = (\partial^2 O/\partial x^2 - \partial O/\partial x)/S^2 \) are called the “Theta”, “Delta”, and “Gamma” of the option. Another derivative with a standard name is the “Vega” \( V \equiv \partial O/\partial \sigma \).

The expansion (81) is carried to arbitrary powers of \( \dot x \) as in (48). It is, of course, only an abbreviated notation for the proper expansion in powers of a stochastic variable to be performed as in Eq. (39).

Inserting (81) and (80) on the left-hand side of Eq. (78), this becomes

\[ N_S \dot S + N_O \dot O = - N_O \frac{\partial O}{\partial x} [\dot x - \bar H(i)] \]

\[ + N_O \left( \frac{\partial O}{\partial t} \dot x + \frac{1}{2} \frac{\partial^2 O}{\partial x^2} \dot x^2 dt + \frac{1}{3!} \frac{\partial^3 O}{\partial x^3} \dot x^3 dt^2 + \ldots \right) \]

\[ = N_O \left[ \bar H(i) \frac{\partial O}{\partial x} + \frac{\partial O}{\partial t} \dot x + \frac{1}{2} \frac{\partial^2 O}{\partial x^2} \dot x^2 dt + \frac{1}{3!} \frac{\partial^3 O}{\partial x^3} \dot x^3 dt^2 + \ldots \right]. \tag{82} \]

Remarkably, the fluctuating variable \( \dot x \) drops out in this equation, which therefore loses its stochastic character. At the same time, it becomes independent of the growth rate \( r_S \) of the stock price. This is the reason why the total wealth \( W(t) \) increases without fluctuations.

We now treat the Taylor series

\[ \frac{1}{2} \frac{\partial^2 O}{\partial x^2} \dot x^2 dt + \frac{1}{6} \frac{\partial^3 O}{\partial x^3} \dot x^3 dt^2 + \ldots \tag{83} \]

in the same way as the expansion (48), using the rules (49), such that (83) becomes

\[ \frac{1}{2} \frac{\partial^2 O}{\partial x^2} \dot x^2 dt + \frac{1}{6} \frac{\partial^3 O}{\partial x^3} \dot x^3 dt^2 + \ldots = -\bar H(i \partial_x)O. \tag{84} \]

In this way we find for the option price \( O(x, t) \) the Fokker-Planck-like differential equation

\[ \frac{\partial O}{\partial t} = r_W O - r_{xw} \frac{\partial O}{\partial x} + \bar H(i \partial_x)O, \tag{85} \]

where we have defined, by analogy with (13), an auxiliary rate parameter

\[ r_{xw} \equiv r_W + \bar H(i). \tag{86} \]

Note, however, that in contrast to the relation between \( r_{xs} \) and \( r_S \) defined for a fluctuating stock price \( S(t) \) and its logarithm \( x(t) \), the parameter \( r_{xw} \) does not have the physical interpretation of governing the logarithm of \( W(t) \) since the absence of fluctuations in the wealth \( W(t) \) makes \( \log W(t) \) grow linearly with the same rate \( r_W \) that governs the exponential growth \( e^{r_W t} \) of \( W(t) \) itself.
If we rename \( t \) as \( t_a \). The general solution of the differential equation (85), which at some time \( t = t_b \) starts out like \( \delta(x - x_b) \), has the Fourier representation

\[
P(x_b|x_a t_a) = e^{-rw(t_b-t_a)} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_b-x_a)} \exp \left\{ - \left[ \bar{H}(p) + ir_{xW} p \right] (t_b-t_a) \right\},
\]

(87)

if the initial variables \( x_a \) and \( t_a \) are identified with \( x \) and \( t \), respectively. A convergent integral exists only for \( t_b > t_a \).

Comparing this expression with Eq. (53) we recognize it as a member of the family of equivalent martingale measures \( P^{M^r(x_b|t_a)} \), in which the discount factor \( r \) coincides with the risk-free interest rate \( r_W \).

C. Option Pricing for Gaussian Fluctuations

For Gaussian fluctuations where \( H(p) = \sigma^2 p^2 / 2 \), the integral in (87) can easily be performed and yields

\[
P(x_b|x_a t_a) = \Theta(t_b-t_a) \frac{e^{-rw(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b-t_a)}} \exp \left\{ - \frac{[x_b - x_a - r_{xW} (t_b-t_a)]^2}{2\sigma^2(t_b-t_a)} \right\}.
\]

(88)

This probability distribution is obviously the solution of the path integral

\[
P(x_b|x_a t_a) = \Theta(t_b-t_a) e^{-rw(t_b-t_a)} \int Dx \exp \left\{ - \frac{1}{2\sigma^2} \int_{t_a}^{t_b} [\dot{x} - r_{xW}]^2 \right\}.
\]

(89)

Recalling the discussion in Section IV, the distribution function (88) is recognized as a member of the equivalent family of martingale distributions (52) for the stock price distribution in which the discount factor contains the risk-free interest rate \( r_W \).

Comparing this expression with Eq. (56) we recognize it as a member of the family of equivalent martingale distribution (88). This distribution is referred to as the risk-neutral equivalent martingale distribution.

An option is written for a certain strike price \( E \) of the stock. The value of the option at its expiration date \( t_b \) is given by the difference between the stock price on expiration date and the strike price:

\[
O(x_b,t_b) = \Theta(S_b - E)(S_b - E) = \Theta(x_b - x_E)(e^{x_b} - e^{x_E}),
\]

(90)

where

\[
x_E = \log E.
\]

(91)

The Heaviside function accounts for the fact that only for \( S_b > E \) it is worthwhile to execute the option. From (90) we calculate the option price at an arbitrary earlier time using the time evolution amplitude (89)

\[
O(x_a,t_a) = \int_{-\infty}^{\infty} dx_b O(x_b,t_b) P^{M^r(x_b|x_a t_a)}.
\]

(92)

Inserting (90) we obtain the sum of two terms

\[
O(x_a,t_a) = O_S(x_a,t_a) - O_E(x_a,t_a),
\]

(93)

where

\[
O_S(x_a,t_a) = \frac{e^{-rw(t_b-t_a)}}{\sqrt{2\pi\sigma^2(t_b-t_a)}} \int_{\log E}^{\infty} dx_b \exp \left\{ x_b - \frac{[x_b - x_a - r_{xW} (t_b-t_a)]^2}{2\sigma^2(t_b-t_a)} \right\}.
\]

(94)

and

\[
O_E(x_a,t_a) = E e^{-rw(t_b-t_a)} \frac{1}{\sqrt{2\pi\sigma^2(t_b-t_a)}} \int_{\log E}^{\infty} dx_b \exp \left\{ - \frac{[x_b - x_a - r_{xW} (t_b-t_a)]^2}{2\sigma^2(t_b-t_a)} \right\}.
\]

(95)
In the second integral we set

\[ x_+ = x_a + r_{xW} (t_b - t_a) = x_a + \left(r_W - \frac{1}{2}\sigma^2\right) (t_b - t_a), \]  

(96)

and obtain

\[
O_E(x_a, t_a) = \int_{x_E-x_-}^\infty dx_b \exp\left\{-\frac{x_b^2}{2\sigma^2(t_b-t_a)}\right\}. \tag{97}
\]

After rescaling the integration variable \(x_b \to -\xi \sigma \sqrt{t_b-t_a}\), this can be rewritten as

\[
O_E(x_a, t_a) = e^{-r_W(t_b-t_a)} E N(y_-), \tag{98}
\]

where \(N(y)\) is the Gaussian distribution function

\[
N(y) = \int_{-\infty}^y \frac{d\xi}{\sqrt{2\pi}} e^{-\xi^2/2}. \tag{99}
\]

evaluated at

\[
y_- = \frac{x_- - x_E}{\sqrt{\sigma^2(t_a-t_b)}} = \frac{\log[S(t_a)/E] + r_{xW} (t_b - t_a)}{\sqrt{\sigma^2(t_a-t_b)}}
\]

\[
= \frac{\log[S(t_a)/E] + (r_W - \frac{1}{2}\sigma^2) (t_b - t_a)}{\sqrt{\sigma^2(t_a-t_b)}}. \tag{100}
\]

The integral in the first contribution \([34]\) to the option price is found after completing the exponent in the integrand quadratically as follows:

\[
x_b - \frac{[x_b - x_a - r_{xW} (t_b - t_a)]^2}{2\sigma^2(t_b-t_a)}
\]

\[
= -\frac{[x_b - x_a - (r_{xW} + \sigma^2) (t_b - t_a)]^2 - 2r_W \sigma^2 (t_b - t_a) - 2x_a \sigma^2 (t_b - t_a)}{2\sigma^2(t_b-t_a)}. \tag{101}
\]

Introducing now

\[
x_+ = x_a + (r_{xW} + \sigma^2) (t_b - t_a) = x_a + \left(r_W + \frac{1}{2}\sigma^2\right) (t_b - t_a), \tag{102}
\]

and rescaling \(x_b\) as before, we obtain

\[
O_S(x_a, t_a) = S(t_a) N(y_+), \tag{103}
\]

with

\[
y_+ = \frac{x_+ - x_E}{\sqrt{\sigma^2(t_a-t_b)}} = \frac{\log[S(t_a)/E] + (r_{xW} + \sigma^2) (t_b - t_a)}{\sqrt{\sigma^2(t_a-t_b)}}
\]

\[
= \frac{\log[S(t_a)/E] + (r_W + \frac{1}{2}\sigma^2) (t_b - t_a)}{\sqrt{\sigma^2(t_a-t_b)}}. \tag{104}
\]

The combined result

\[
O(x_a, t_a) = S(t_a) N(y_+) - e^{-r_W(t_b-t_a)} E N(y_-) \tag{105}
\]

is the celebrated Black-Scholes formula of option pricing.

In Fig. 3 we illustrate how the dependence of the call price on the stock price varies with different times to expiration \(t_b - t_a\) and with different volatilities \(\sigma\).
D. Option Pricing for Non-Gaussian Fluctuations

For non-Gaussian fluctuations, the option price must be calculated numerically from Eqs. (92) and (90). Inserting the Fourier representation (87) and using the Hamiltonian

$$H_{r_W}(p) \equiv \tilde{H}(p) + ir_{x_W}p$$

defined as in (12), this becomes

$$O(x_a, t_a) = \int_{x_E}^{\infty} dx_b (e^{x_b} - e^{x_E}) P(x_b t_b | x_a t_a)$$

$$= e^{-r_W(t_b-t_a)} \int_{x_E}^{\infty} dx_b (e^{x_b} - e^{x_E}) \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_b-x_a)} - H_{r_{x_W}}(p)(t_b-t_a).$$

(107)

The integrand can be rearranged as follows:

$$O(x_a, t_a) = e^{-r_W(t_b-t_a)} \int_{x_E}^{\infty} dx_b \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left[ e^{x_a e^{(p+1)(x_b-x_a)} - e^{x_E} e^{ip(x_b-x_a)}} - e^{-H_{r_{x_W}}(p)(t_b-t_a)} \right].$$

(108)

Two integrations are required. This would make a numerical calculation quite time consuming. Fortunately, one integration can be done analytically. For this purpose we write the integral in the form

$$O(x_a, t_a) = e^{-r_W(t_b-t_a)} \int_{x_E}^{\infty} dx_b \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_b-x_a)} f(x_a, x_E; p),$$

(109)

with

$$f(p) \equiv e^{x_a e^{-H_{r_{x_W}}(p+1)(t_b-t_a)} - e^{x_E} e^{-H_{r_{x_W}}(p)(t_b-t_a)}}.$$

(110)

We have suppressed the arguments $x_a, x_E, t_b - t_a$ in $f(p)$, for brevity. The integral over $x_b$ in (109) runs over the Fourier transform

$$\hat{f}(x_b - x_a) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_b-x_a)} f(p),$$

(111)
of the function \( f(p) \). It is then convenient to express the integral \( \int_{x_E}^{\infty} dx_b \) in terms of the Heaviside function \( \Theta(x_b - x_E) \) as \( \int_{-\infty}^{\infty} dx_b \Theta(x_b - x_E) \) and use the Fourier representation

\[
\Theta(x_b - x_E) = \int \frac{dq}{2\pi} \frac{i}{q + i\eta} e^{-iq(x_b - x_E)}. \tag{112}
\]

of the Heaviside function to write

\[
\int_{x_E}^{\infty} dx_b \tilde{f}(x_b - x_a) = \int_{-\infty}^{\infty} dx_b \int \frac{dq}{2\pi} \frac{i}{q + i\eta} e^{-iq(x_b - x_E)} \tilde{f}(x_b - x_a). \tag{113}
\]

Inserting here the Fourier representation \((111)\), we can perform the integral over \( x_b \) and obtain the momentum space representation of the option price

\[
O(x_a, t_a) = \int_{-\infty}^{\infty} dp \frac{e^{ip(x_E - x_a)}}{p + i\eta} f(p). \tag{114}
\]

For numerical integrations, the singularity at \( p = 0 \) is inconvenient. We therefore use the well-known decomposition

\[
\frac{i}{p + i\eta} = \frac{\mathcal{P}}{p} + \pi\delta(p), \tag{115}
\]

to write

\[
O(x_a, t_a) = \frac{1}{2} f(0) + i \int_{-\infty}^{\infty} dp \frac{e^{ip(x_E - x_a)} f(p) - f(0)}{p}. \tag{116}
\]

We have used the fact that the principal value of the integral over \( 1/p \) vanishes to subtract the constant \( f(0) \) from \( e^{ip(x_E - x_a)} f(p) \). After this the integrand is regular, does not need any more the principal-value specification, and allows for a numerical integration.

For \( x_a \) very much different from \( x_E \), we may approximate

\[
\int_{-\infty}^{\infty} dp \frac{e^{ip(x_E - x_a)} f(p) - f(0)}{p} \approx \frac{1}{2} \epsilon(x_a - x_E) f(0), \tag{117}
\]

where \( \epsilon(x) \equiv 1 + 2\Theta(x) \) is the step function, and obtain

\[
O(x_a, t_a) \approx \frac{1}{2} [1 + \Theta(x_a - x_E)] f(0). \tag{118}
\]

Using \((107)\) we have \( e^{-H_{rW}^{(i)}} = e^{rW(t_b - t_a)} \), and since \( e^{-H_{rW}^{(0)}} = 1 \), we see that \( O(x_a, t_a) \) goes to zero for \( x_a \to -\infty \) and has the large-\( x_a \) behavior

\[
O(x_a, t_a) \approx \left( e^{x_a} - e^{x_E} e^{-rW(t_b - t_a)} \right) = S(t_a) - e^{-rW(t_b - t_a)} E. \tag{119}
\]

This is the same behavior as in the Black-Scholes formula \((102)\).

In Fig. 3 we display the difference between the option prices emerging from our formula \((116)\), with a truncated Lévy distribution of kurtosis \( \kappa = 4 \), and the Black-Scholes formula \((103)\) for the same data as in the upper left of Fig. 3."
VI. CONCLUSION

The stochastic calculus and the option pricing formulas developed in this paper will be useful for estimating financial risks of a variety of investments. In particular, it will help developing a more realistic theory of fair option prices. More details can be found in the textbook Ref. [35].

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