REGULARITY OF MINIMIZERS OF A TENSOR-VALUED VARIATIONAL
OBSTACLE PROBLEM IN THREE DIMENSIONS

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Abstract. Motivated by Ball and Majumdar’s modification of Landau-de Gennes model for nematic liquid crystals, we study energy-minimizer $Q$ of a tensor-valued variational obstacle problem in a bounded 3-D domain with prescribed boundary data. The energy functional is designed to blow up as $Q$ approaches the obstacle. Under certain assumptions, especially on blow-up profile of the singular bulk potential, we prove higher interior regularity of $Q$, and show that the contact set of $Q$ is either empty, or small with characterization of its Hausdorff dimension. We also prove boundary partial regularity of the energy-minimizer.

1. Introduction

1.1. Background and problem formulation. In this paper, we consider a tensor-valued variational obstacle problem originates from the Landau-de Gennes model for nematic liquid crystals. We shall study the regularity of the minimizer and estimate the size of the contact set.

In the Landau-de Gennes theory [4, 15], local state of nematic liquid crystals at spatial point $x \in \mathbb{R}^3$ is characterized by a $3 \times 3$-tensor-valued order parameter $Q(x)$ in

$$Q = \{Q \in \mathbb{R}^{3\times3} : Q_{ij} = Q_{ji}, Q_{ii} = 0\},$$

interpreted as traceless second moment of (formal) probability distribution function $f$ of local molecular orientation over $S^2$, i.e.,

$$Q(x) = \int_{S^2} \left( m \otimes m - \frac{1}{3} I \right) f(x, m) \, dm.$$

Here $I$ denotes the $3 \times 3$-identity matrix. It follows that all eigenvalues of $Q$ should belong to $[-\frac{1}{3}, \frac{2}{3}]$. If one of the eigenvalues is equal to $-1/3$ or $2/3$, formally the density $f(x, m) \, dm$ is concentrated on a measure-zero subset of $S^2$, in which case the bulk energy density will be infinity [1]. We introduce the set of admissible configuration of nematic liquid crystal

$$Q \in Q_{phy} := \left\{ Q \in \mathcal{Q} : \lambda_1(Q), \lambda_2(Q), \lambda_3(Q) \in \left[ -\frac{1}{3}, \frac{2}{3} \right] \right\},$$

where $\lambda_i(Q)$ ($i = 1, 2, 3$) are three eigenvalues of $Q$. $Q_{phy}$ is a bounded close subset of $\mathcal{Q}$ with

$$\partial Q_{phy} = \left\{ Q \in \mathcal{Q} : \lambda_i(Q) \in \left[ -\frac{1}{3}, \frac{2}{3} \right] \text{ for } i = 1, 2, 3; \text{ at least one of } \lambda_i(Q) = -\frac{1}{3} \right\}.$$

$$Q^0_{phy} = \left\{ Q \in \mathcal{Q} : \lambda_i(Q) \in \left( -\frac{1}{3}, \frac{2}{3} \right) \text{ for } i = 1, 2, 3 \right\}.$$

Here we used the fact that $\sum Q_{ii} = 0$. For more background on $Q$-tensor models, readers are referred to [4] and [15].

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Motivated by Ball-Majumdar model of nematic liquid crystals \[1,11\], we introduce

\[ E[Q] = \int_{\Omega} f_e(\nabla Q(x)) + f_b(Q(x)) \, dx. \]  

(1.4)

Here \( \Omega \) is a bounded smooth domain in 3-D, \( f_e(\nabla Q) \) represents elastic energy density and \( f_b(Q) \) is a singular bulk energy potential.

We assume

\[ f_e(\nabla Q) = \frac{1}{2} |\nabla Q|^2 + \frac{A}{2} |\text{div} \, Q|^2 = \frac{1}{2} Q_{ij,k} Q_{ij,k} + \frac{A}{2} Q_{ij,j} Q_{ik,k}. \]  

(1.5)

Here \( Q_{ij,k} \) is short for \( \partial_{x_k} Q_{ij} \), and summation is taken over repeated indices.

In fact, (1.5) is reduced from a more general form of elastic energy

\[ f_e(\nabla Q) = \frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} |\text{div} \, Q|^2 + \frac{L_3}{2} Q_{ij,k} Q_{ik,j}, \]  

(1.6)

where \( L_1, L_2, \) and \( L_3 \) are elastic constants, satisfying that \[14\]

\[ L_1 + L_3 > 0, \quad 2L_1 - L_3 > 0, \quad L_1 + \frac{5}{3} L_2 + \frac{1}{6} L_3 > 0. \]  

(1.7)

Under these conditions, \( f_e \) is coercive, i.e., there exist positive \( c < C \) such that

\[ c|\nabla Q|^2 \leq f_e(\nabla Q) \leq C|\nabla Q|^2. \]

It is well-known that with proper coefficients, the last two terms in (1.6) can be combined into a null Lagrangian (see e.g., \[12\] Lemma 1.1), which can be omitted under strong anchoring (Dirichlet) boundary conditions (see below). Hence, without loss of generality, we may simplify (1.6) into (1.5) by setting \( L_3 = 0 \) and \( L_1 = 1 \). Thanks to (1.7), when \( A > -3/5 \), \( f_e \) in (1.5) is coercive and strictly convex in \( \nabla Q \).

The following assumptions on \( f_b \) will be made throughout this work.

1. In \( Q_{\text{phy}}^o, \) \( 1 \leq f_b(Q) < +\infty \), and it is convex and smooth. The lower bound of \( f_b \) is not an essential constraint, since the constant 1 is irrelevant in minimizing (1.4) when \( \Omega \) is fixed.

2. \( f_b(Q) = \infty \) in \( Q \setminus Q_{\text{phy}}^o \), and \( f_b(Q) \to +\infty \) as \( Q \to \partial Q_{\text{phy}}^o \) \[1\].

More hypotheses on \( f_b \) will be proposed later.

To this end, we shall minimize (1.4) over the set of admissible configurations given by

\[ A = \{ Q(x) \in H^1(\Omega, Q_{\text{phy}}^o) : Q = Q_0 \text{ on } \partial \Omega \} \]

(1.8)

with some \( Q_0 \in C^\infty(\overline{\Omega}, Q_{\text{phy}}^o) \). Here the Dirichlet boundary condition is understood in the trace sense. We take \( Q_0 \) to be smooth for simplicity, which may be weaken. It is also noteworthy that \( Q_0 \) is separated from the obstacle \( \partial Q_{\text{phy}}^o \) throughout \( \overline{\Omega} \). Since \( Q_0 \in A \) with \( E(Q_0) < +\infty \) and \( A \) is closed, one can easily prove existence of energy minimizer of (1.4) in \( A \) by the direct method of calculus of variations. Moreover, the minimizer is unique thanks to the convexity of \( f_e \) and \( f_b \). We still denote it as \( Q \) with abuse of notation.

In spite of existence and uniqueness, it is not clear if \( Q \) will touch the obstacle \( \partial Q_{\text{phy}}^o \) in \( \Omega \). For this purpose, we introduce contact set \( \mathcal{C} \) of \( Q \),

\[ \mathcal{C} := \{ x \in \Omega : Q(x) \in \partial Q_{\text{phy}}^o \}. \]

Since \( E[Q] < +\infty \), by the assumption \[10\] on \( f_b \), \( \mathcal{C} \) has Lebesgue measure zero. However, one would naturally expect that \( \mathcal{C} = \emptyset \) — this will make more sense as we establish regularity of \( Q \) below since right now \( Q \) is only an \( H^1 \)-function but not well-defined pointwise. In this work, we shall propose a number of conditions to pursue emptiness of \( \mathcal{C} \). In the cases where \( \mathcal{C} = \emptyset \) cannot be guaranteed, we provide estimates for Hausdorff dimension of \( \mathcal{C} \).
Let us review some related works before stating our main results. As is mentioned earlier, the model we are studying originates from the work by Ball and Majumdar [1], where the authors proposed a thermotropic bulk potential that blows up as the eigenvalue of $Q$ approaches $-\frac{1}{3}$. They also proved in the one-elastic-constant case (i.e., $A = 0$) that, under weak assumptions on $f_b$ (see [2, Theorems 4 and 6, Corollaries 3 and 4]), the contact set is empty and $Q$ is smooth in the domain. Bauman and Phillips [2] studied the 2-D case of this problem with a Landau-de Gennes elastic energy density containing more elasticity terms. Using a hole-filling technique, they proved Hölder continuity of the energy minimizer $Q$ in general, and showed that if some of the elastic coefficients are zero, one should find $Q$ is smooth. Evans, Kneuss and Tran [7] established a partial regularity result for a general form of elastic energy, stating that $Q$ is smooth outside a zero-measure closed subset of $\Omega$. They also proved that under certain hypotheses on $f_b$, the singular set (contact set) has Hausdorff dimension at most $(n-2)$ in the $n$-dimensional case. Readers are referred to [8, 9, 16, 10] and references therein for results on dynamic problems with singular bulk potential modeling liquid crystal flows.

1.2. Main results. Our main strategy in this work is to derive the emptiness of $\mathcal{C}$ from its regularity. We start with a key result concerning interior regularity of $Q$.

**Proposition 1.1** (Interior regularity of $Q$). Assume $A > -3/5$ and $f_b$ is convex. Let $Q$ denote the unique minimizer of $E[Q]$ in $A$. Then

1. $Q \in H^2_{\text{loc}}(\Omega)$,
2. $\nabla Q \in L^q_{\text{loc}}(\Omega)$, for $q$ such that

\begin{align}
1 &\leq q < 6p(A),
\end{align}

where

\begin{align}
p(A) = \begin{cases}
1 + \frac{3}{4} + \frac{9}{5A^2}, & \text{if } A \in \left(-\frac{2}{3}, \frac{\sqrt{18}}{5}\right), \\
1 + \frac{3 + 5A}{2\sqrt{10A - 6}}, & \text{if } A \in \left(\frac{\sqrt{18}}{5}, \frac{3}{5} + \sqrt{\frac{18}{5}}\right), \\
1 + \frac{3 + \sqrt{9 + 6A^2}}{2A}, & \text{if } A > \frac{3}{5} + \sqrt{\frac{18}{5}}.
\end{cases}
\end{align}

In particular, $p(A)$ is decreasing in $A$ when $A > 0$, and $p(A) > 1$.

**Remark 1.1.** The first part of the Proposition is a special case of Theorem 4.1 in [7], while the second part is stronger than the $W^{1,6}$-regularity that directly follows from the first part by 3-D Sobolev embedding.

Proposition 1.1 shows that the minimizer $Q(x)$ can be realized as a Hölder continuous function in $\Omega$ and it is thus well-defined pointwise. Therefore, the contact set $\mathcal{C}$ is now well-defined.

As is mentioned above, the energy-minimizer in the special case $A = 0$ has been studied in [1, 2], where $f_b(\nabla Q)$ reduces to the Dirichlet energy and the uniform separation from the obstacle simply follows from the maximum principle. Then smoothness of $Q$ can be justified by elliptic regularity theory [3] applied to the Euler-Lagrange equation. When $A \neq 0$, this technique fails. Nevertheless, it is tempting to believe when $A$ is close to 0, the minimizer should behave like the one for $A = 0$. This leads to the following theorem with a perturbative nature.

**Theorem 1.2** (Small $A$ case). Assume $f_b$ satisfies the assumptions [1] and [11]. For any $V \subset \subset \Omega$, there exists $A_* > 0$ depending on $V$, such that for any $|A| \leq A_*$, the minimizer $Q_A \in A$ of (1.4) with parameter $A$ satisfies $\mathcal{C} \cap V = \emptyset$. Moreover, $Q_A$ is smooth in $V$. 
For general $A > -\frac{3}{5}$, it is natural to believe that whether $C = \emptyset$ should depend on the growth of $f_b$ near $\partial Q_{\text{phy}}$. Suppose $C \neq \emptyset$ and $x_0 \in C$. If $f_b$ grows fast near $\partial Q_{\text{phy}}$, in a neighborhood of $x_0$, $Q(x)$ must leave $\partial Q_{\text{phy}}$ sufficiently fast in order to not incur huge bulk energy. However, this may lead to large elastic energy, which prohibits $Q$ from being energy-minimizing. In this spirit, we propose the following hypothesis on the growth of $f_b$ near $\partial Q_{\text{phy}}$.

(iii) We assume there exists some $s > 0$ and $0 < m_s < M_s$, such that in $Q_{\text{phy}}$,

\begin{equation}
(1.11) \quad \frac{m_s}{d(Q)^s} \leq f_b(Q) \leq \frac{M_s}{d(Q)^s},
\end{equation}

or there exists $0 < k_0 \leq K_0$ and $m_0 \leq M_0$, such that for any $Q \in \partial Q_{\text{phy}}$, there exists $k(Q) \in [k_0, K_0]$ satisfying

\begin{equation}
(1.12) \quad k(Q) \ln(d(\lambda Q)) + m_0 \leq f_b(\lambda Q) \leq k(Q) \ln(d(\lambda Q)) + M_0, \quad \forall \lambda \in (0, 1).
\end{equation}

Here $d(Q) := \text{dist}(Q, Q_{\text{phy}}^c)$ is the distance measured in the Frobenius norm.

It can be shown that the Ball-Majumdar potential satisfies the assumption (1.12). See [1, (22)-(25)]. Then we have

**Theorem 1.3.** Assume $A > -3/5$ and $f_b$ satisfies the assumptions (i), (ii), and (iii) with (1.11). Let $Q$ be the unique minimizer of $E[Q]$ in $A$.

Then

(1) If $s > s(A)$, where

\begin{equation}
(1.13) \quad s(A) := \frac{2}{2p(A) - 1},
\end{equation}

we have $C = \emptyset$ and $Q$ is smooth in $\Omega$. Here $p(A)$ is defined in (1.10).

(2) Otherwise, if $0 < s \leq s(A)$,

\[
\dim_H C \leq 3 \left(1 - \frac{s}{s(A)}\right).
\]

Here $\dim_H C$ denotes the Hausdorff dimension of $C$.

**Remark 1.2.** We do not claim optimality of the borderline exponent $s(A)$, but we stress that $s(A) < 2$ for all $A > -\frac{3}{5}$ since $p(A) > 1$. As a result, Theorem 1.3, under slightly stronger assumptions on the energy functional, significantly improves the result of [7, Theorem 4.2] in 3-D case, where it was shown that $\dim_H C \leq 1$ in 3-D provided that $f_b$ grows like $d(Q)^{-2}$ near $\partial Q_{\text{phy}}$.

**Remark 1.3.** It is noteworthy that $s(A) \to 0$ as $A \to 0$. This echoes with the results that when $A = 0$, only very weak blow-up of $f_b$ near $\partial Q_{\text{phy}}$ is needed to achieve emptiness of $C$[1 2].

In [7], inspired by a model of the Ball-Majumdar singular bulk potential, where $f_b$ grows like $|\ln d(Q)|$ near $\partial Q_{\text{phy}}$, an additional assumption is proposed on $f_b$. It can be roughly stated as follows in our context: for some $c_0 > 0$,

\begin{equation}
(1.14) \quad \frac{\partial^2 f_b}{\partial Q_{ij} \partial Q_{mn}} y_{ij} y_{mn} \geq c_0 \left| \frac{\partial f_b}{\partial Q} \cdot y \right|^2 \quad \text{for all } y \in T Q_{\text{phy}}.
\end{equation}

Based on this, the authors proved that $\dim_H C \leq 1$ (see [7, Theorem 4.3]). This assumption has been verified rigorously in [9] for the Ball-Majumdar potential [1]. In what follows, we shall propose similar hypotheses on $f_b$ and improve our results in Theorem 1.3.
(iv) If \( f_b \) satisfies the assumption \( \text{(iii)} \) with \( (1.11) \), we assume that for some \( C_s, c_s > 0, \)
\[
|Df_b(Q)|^s \leq C_s f_b(Q)^{s+1} \quad \text{for all } Q \in Q_{phy},
\]
and
\[
\frac{\partial^2 f_b}{\partial Q_{ij}\partial Q_{mn}} y_{ij} y_{mn} \geq \frac{c_s}{f_b} \left| \frac{\partial f_b}{\partial Q} \right|^2 \quad \text{for all } y \in TQ_{phy}.
\]

If, otherwise, \( f_b \) satisfies the assumption \( \text{(iii)} \) with \( (1.12) \), we assume that for some \( C_0 > 0, \)
\[
|Df_b(Q)| \leq C_0 \exp(k_0^{-1} f_b(Q)) \quad \text{for all } Q \in Q_{phy},
\]
and also \( (1.14) \) holds.

Remark 1.4. Note that \( (1.15) \) and \( (1.16) \) are inspired by assuming that \( f_b(Q) \) behaves like \( d(Q)^{-s} \) near \( \partial Q_{phy} \); while \( (1.17) \) is derived by assuming \( f_b(Q) \) behaves like \( k_0 |\ln d(Q)| \) near \( \partial Q_{phy} \). We omit their derivations.

With the new assumption \( \text{(iv)} \), we may improve the results in part (2) of Theorem 1.3.

**Theorem 1.4.** Assume \( A > -3/5 \). Let \( Q \) be the unique minimizer of \( E[Q] \) in \( A \).

(1) Suppose \( f_b \) satisfies the assumptions \( \text{(i)}, \text{(ii)}, \text{(iii)} \) with \( (1.11) \) and \( \text{(iv)} \). When \( 0 < s \leq s(A), \)
\[
\dim_H C \leq \begin{cases} 
3(1 - \frac{3+2}{3s(A)+2}), & \text{if } p(A) \leq 2, \\
3 - \frac{s}{s(A)} - \frac{2(2+s)}{2+2s(A)}, & \text{if } p(A) > 2.
\end{cases}
\]

(2) Suppose \( f_b \) satisfies the assumptions \( \text{(i)}, \text{(ii)}, \text{(iii)} \) with \( (1.12) \) and \( \text{(iv)} \). Then
\[
\dim_H C \leq \begin{cases} 
3 - \frac{6p(A) - 3}{6p(A) - 2 + 2p(A) + 2}, & \text{if } p(A) \leq 2, \\
3 - \frac{6p(A) - 3}{6p(A) - 2 + 2p(A) + 2}, & \text{if } p(A) > 2.
\end{cases}
\]

Unfortunately, because of the limitation of our approach, Theorem 1.4 is not an improvement of Theorem 4.3 in [7].

Since no continuity of \( Q \) up to the boundary has been established, we are unable to extend the above discussion (even in the small \( A \) case) from the interior up to \( \partial \Omega \). Instead, we prove the following partial regularity result at \( \partial \Omega \).

**Theorem 1.5** (Boundary partial regularity). Assume \( A > -3/5 \) and \( f_b \) satisfies the assumptions \( \text{(i)} \) and \( \text{(ii)} \). The minimizer \( Q \in A \) of \( (1.3) \) is Hölder continuous in \( \Omega \) up to \( \partial \Omega \setminus S \) for some \( S \subset \partial \Omega \), with \( H^1(S) = 0 \). Here \( H^1(\cdot) \) denotes the 1-dimensional Hausdorff measure.

In particular, \( H^1(\overline{C} \cap \partial \Omega) = 0 \).

The rest of the paper is organized as follows. Section 2 is devoted to proving Proposition 1.1 on the interior regularity of the energy-minimizer and Theorem 1.2. Theorem 1.3 and Theorem 1.4 will be proved in Section 3. We show Theorem 1.5 on the boundary partial regularity in Section 4. In the Appendices, we derive the formula for \( p(A) \) in Proposition 1.1 in Appendix B. Useful properties of the distance function \( d(Q) \) are proved in Appendix C. Finally, in Appendix D, we shall present a construction of an approximating sequence of \( f_b \) that will be used in Section 3.
2. Interior Regularity of the Minimizer and Proof of Theorem 1.2

We first show Proposition 1.1. As is mentioned before, the $H^2_{\text{loc}}$-regularity has been established in [7, Theorem 4.1] under weaker assumptions. The proof there uses standard arguments in the calculus of variations [5], with special care of the singular bulk energy. For completeness, we still present it here in our context. Then we shall generalize its idea to prove the $W^{1,q}_{\text{loc}}$-regularity.

Proof of Proposition 1.1.

Step 1 (Basic setup). Fix an open subset $V$ of $\Omega$ such that $V \subset \subset \Omega$. We select another open subset $W$ such that $V \subset \subset W \subset \subset \Omega$. Take a nonnegative smooth cutoff function $\xi$ in $\mathbb{R}^3$ supported on $\overline{W}$, such that $\xi \equiv 1$ on $V$ and $\|\nabla \xi\|_{L^\infty(\Omega)} \leq C(V,W)$. Let $e_1, e_2, e_3$ be the standard coordinate vectors in $\mathbb{R}^3$. With $p \geq 1$ is to be determined, let

\begin{equation}
(2.20) \quad u_k = D^{-h}_k (\xi^2 |D^h_k Q|^{2p-2} D^h_k Q),
\end{equation}

where

\[ D^h_k Q(x) := \frac{Q(x + he_k) - Q(x)}{h}. \]

Here in order to make the above two quantities well-defined throughout $\Omega$, we make zero extension of $Q$ outside $\Omega$ (still denoted by $Q$).

Consider

\begin{equation}
(2.21) \quad (Q + \varepsilon u_k)(x) = \frac{\varepsilon(\xi^2 |D^h_k Q|^{2p-2})(x - he_k)}{h^2} Q(x - he_k) + \frac{\varepsilon(\xi^2 |D^h_k Q|^{2p-2})(x)}{h^2} Q(x + he_k) + \left(1 - \frac{\varepsilon(\xi^2 |D^h_k Q|^{2p-2})(x - he_k)}{h^2} + \frac{\varepsilon(\xi^2 |D^h_k Q|^{2p-2})(x)}{h^2}\right) Q(x).
\end{equation}

It is easy to show that $(Q + \varepsilon u_k) = Q = Q_0$ on $\partial \Omega$ if $h \leq C(W,\Omega)$.

Step 2 (Interior $H^2$-regularity). Let $p = 1$ in (2.21). Provided that $0 < \varepsilon \ll h^2$, $(Q + \varepsilon u_k)(x)$ is a convex combination of $Q(x)$, $Q(x + he_k)$, and $Q(x - he_k)$, which implies that $(Q + \varepsilon u_k)(x) \in Q_{\text{phy}}$ for all $x \in \Omega$. On the other hand, it is obvious that $(Q + \varepsilon u_k)$ has $H^1$-regularity and satisfies the boundary condition. Hence, $(Q + \varepsilon u_k) \in \mathcal{A}$. By the convexity of $f_b$,

\begin{align*}
f_b(Q + \varepsilon u_k) \leq & \frac{\varepsilon(\xi(x - he_k))^2}{h^2} f_b(Q(x - he_k)) + \frac{\varepsilon(\xi(x))^2}{h^2} f_b(Q(x + he_k)) + \left(1 - \frac{\varepsilon(\xi(x - he_k))^2}{h^2} - \frac{\varepsilon(\xi(x))^2}{h^2}\right) f_b(Q(x)).
\end{align*}

Hence,

\begin{align*}
&\int_{\Omega} f_b(Q + \varepsilon u_k) - f_b(Q) \, dx \\
\leq & \frac{\varepsilon}{h^2} \int_{\mathbb{R}^3} \xi(x - he_k)^2 [f_b(Q(x - he_k)) - f_b(Q(x))] + \xi(x)^2 [f_b(Q(x + he_k)) - f_b(Q(x))] \, dx \\
= & 0.
\end{align*}

In the last line, we used change of variables. Since $Q$ minimizes $E[Q]$ in $\mathcal{A}$, i.e.,

\begin{align*}
&\int_{\Omega} f_\varepsilon(\nabla(Q + \varepsilon u_k)) + f_b(Q + \varepsilon u_k) \, dx \geq \int_{\Omega} f_\varepsilon(\nabla Q) + f_b(Q) \, dx,
\end{align*}

we deduce that

\[(2.22) \quad \int_{\Omega} f_c(\nabla Q) \, dx \leq \int_{\Omega} f_c(\nabla (Q + \varepsilon u_k)) \, dx.\]

Sending \(\varepsilon \to 0^+\) yields a variational inequality,

\[\int_{\Omega} \frac{\partial f_c(\nabla Q)}{\partial Q_{ij,l}} \cdot \partial_h \left( D_k^{-h}(\varepsilon^2 D_k^h Q_{ij}) \right) \, dx \geq 0.\]

Here the summation convention applies to all repeated indices except for \(k\); it will always be this case in the rest of this proof. Using integration by parts for \(D_k^{-h}\),

\[\int_{\Omega} \xi^2 D_k^h \left( \frac{\partial f_c(\nabla Q)}{\partial Q_{ij,l}} \right) D_k^h Q_{ij,l} \, dx \leq - \int_{\Omega} D_k^h \left( \frac{\partial f_c(\nabla Q)}{\partial Q_{ij,l}} \right) \cdot 2\xi \partial_h \xi D_k^h Q_{ij} \, dx.\]

Recall that \(f_c(\nabla Q)\) is quadratic in \(\nabla Q\), and it is coercive. Hence, with \(a > 0\) to be determined,

\[c \int_{\Omega} \xi^2 |D_k^h \nabla Q|^2 \, dx \leq a \int_{\Omega} \xi^2 |D_k^h \nabla Q|^2 \, dx + \frac{C}{a} \int_{\Omega} |\nabla Q|^2 |D_k^h Q|^2 \, dx,\]

where \(c, C\) are universal constants only depending on \(A\). Taking \(a < c/2\), we end up having

\[\int_{\Omega} \xi^2 |D_k^h \nabla Q|^2 \, dx \leq C \int_{W} |D_k^h Q|^2 \, dx \leq C \int_{\Omega} |\nabla Q|^2 \, dx,\]

where \(h \ll 1\) and \(C > 0\) only depends on \(A\) and \(\xi\). This bound is independent of \(h\), which implies that \(5\)

\[\int_{V} |\nabla^2 Q|^2 \, dx \leq C(A, V) \int_{\Omega} |\nabla Q|^2 \, dx.\]

Hence, \(Q \in H^2_{\text{loc}}(\Omega)\).

**Step 3 (Interior \(W^{1,q}\)-regularity).** Take \(p > 1\) to be determined. By Sobolev embedding, in 3-D, \(Q \in H^2_{\text{loc}}(\Omega)\) implies \(Q\) is locally bounded. Hence, in the interior of \(\Omega\), if \(h \ll 1\), \(|D_k^h Q| \leq C/h\).

In fact, in our problem, \(Q\) should enjoy a trivial \(L^\infty\)-estimate since \(Q_{\text{phy}}\) is bounded. However, we avoid using this natural \(L^\infty\)-bound, so that the same proof can be applied to situations where no \(L^\infty\)-estimate is a priori available. This point will be useful later in Section 4. Hence, in the interior of \(\Omega\), \((Q + \varepsilon u_k)(x)\) is a convex combination of \(Q(x)\), \(Q(x + he_k)\), and \(Q(x - he_k)\), provided that \(0 < \varepsilon \ll h^{2p}\). This implies \((Q + \varepsilon u_k)(x) \in Q_{\text{phy}}\) for \(\forall x \in \Omega\). On the other hand, combining the interior \(H^2\)-regularity of \(Q\) with \(\ref{2.21}\), \((Q + \varepsilon u_k) \in H^1(\Omega, Q_{\text{phy}})\) for arbitrary \(p > 1\). Therefore, \((Q + \varepsilon u_k) \in A\). Then we argue as before by the convexity of \(f_h\) to find that \((2.22)\) still holds, which allows us to derive a similar variational inequality,

\[\int_{\Omega} D_k^h \left( \frac{\partial f_c(\nabla Q)}{\partial Q_{ij,l}} \right) \partial_h(\xi^2 |D_k^h Q|^{2p-2} D_k^h Q_{ij}) \, dx \leq 0.\]

Substituting the form of \(f_c\) in \(\ref{1.5}\) into this inequality, we calculate that

\[\int_{\Omega} D_k^h (Q_{ij,l} + \delta_{ij} A(\text{div } Q)_i) \left[ D_k^h Q_{ij,l} |D_k^h Q|^{2p-2} \xi^2 + \xi^2 D_k^h Q_{ij} \cdot (2p-2)|D_k^h Q|^{2p-4}(D_k^h \partial_h Q : D_k^h Q) \right] \, dx \leq - \int_{\Omega} D_k^h (Q_{ij,l} + \delta_{ij} A(\text{div } Q)_i) \cdot 2\xi \partial_h \xi |D_k^h Q|^{2p-2} D_k^h Q_{ij} \, dx.\]
Here $D^h_k \partial_t Q : D^h_k Q := D^h_k Q_{ij} : D^h_k Q_{ij}$. By Cauchy-Schwarz inequality, with $c > 0$ to be determined,

\[
\int_{\Omega} \xi^2 (|D^h_k \nabla Q|^2 + A|D^h_k \text{div } Q|^2) |D^h_k Q|^{2p-2} \, dx \\
+ \int_{\Omega} (2p - 2) \xi^2 |D^h_k \nabla Q : D^h_k Q|^2 |D^h_k Q|^{2p-4} \, dx \\
+ \int_{\Omega} A (2p - 2) \xi^2 |D^h_k Q|^{2p-4} (D^h_k \text{div } Q)_i \cdot (D^h_k Q)_j \cdot (D^h_k \nabla Q : D^h_k Q) \, dx \\
\leq c \int_{\Omega} \xi^2 |D^h_k \nabla Q|^2 |D^h_k Q|^{2p-2} \, dx + \frac{C}{c} \int_{\Omega} |\nabla \xi|^2 |D^h_k Q|^{2p} \, dx,
\]

where $C > 0$ is a universal constant only depending on $A$. Introducing another parameter $\omega \in [0,1]$ to be determined, which satisfies

\[
(2.24) \quad \frac{3}{5} \omega + A \geq 0,
\]

we derive that

\[
\begin{align*}
(\omega |D^h_k \nabla Q|^2 + A|D^h_k \text{div } Q|^2) |D^h_k Q|^{2p-2} \\
\geq \left( \frac{3}{5} \omega + A \right) |D^h_k \text{div } Q|^2 |D^h_k Q|^{2p-2} \\
\geq \frac{3}{2} \left( \frac{3}{5} \omega + A \right) |D^h_k \text{div } Q|^2 \|D^h_k Q\|^2 \|D^h_k Q\|^{2p-4},
\end{align*}
\]

where $\| \cdot \|_2$ denotes the matrix 2-norm. Here we used

\[
(2.25) \quad \|D^h_k Q\|^2 \leq \frac{2}{3} |D^h_k Q|^2, \quad |D^h_k \text{div } Q|^2 \leq \frac{2}{3} |D^h_k \nabla Q|^2.
\]

Indeed, they can be justified by using the fact that $D^h_k Q$ and $D^h_k \partial_t Q$ are symmetric traceless $3 \times 3$-matrices. We give a detailed proof of these two inequalities in Appendix [A].

On the other hand,

\[
(1 - \omega) |D^h_k \nabla Q|^2 |D^h_k Q|^{2p-2} \geq (1 - \omega) |D^h_k \nabla Q : D^h_k Q|^2 |D^h_k Q|^{2p-4}
\]

Hence, by Young’s inequality,

\[
\begin{align*}
(|D^h_k \nabla Q|^2 + A|D^h_k \text{div } Q|^2) |D^h_k Q|^{2p-2} + (2p - 2)|D^h_k \nabla Q : D^h_k Q|^2 |D^h_k Q|^{2p-4} \\
\geq 2 \sqrt{\frac{3}{2} \left( \frac{3}{5} \omega + A \right) ((1 - \omega) + (2p - 2)) |D^h_k Q|^{2p-4} |(D^h_k \text{div } Q)_i \cdot (D^h_k Q)_j \cdot (D^h_k \nabla Q : D^h_k Q)_j|.
\end{align*}
\]

If $p > 1$ satisfies that

\[
(2.26) \quad 1 < p < 1 + \frac{\frac{6}{5} (\omega + \frac{5}{3} A) + \sqrt{\frac{41}{86} (\omega + \frac{5}{3} A)^2 + \frac{18}{17} A^2 (\omega + \frac{5}{3} A) (1 - \omega)}}{2A^2} =: p(A, \omega),
\]

then we are able to take $c > 0$ suitably small in (2.23) to obtain that

\[
(2.27) \quad \int_{\Omega} \xi^2 |D^h_k \nabla Q|^2 |D^h_k Q|^{2p-2} \, dx \leq C \int_{\Omega} |\nabla \xi|^2 |D^h_k Q|^{2p} \, dx,
\]
where \( C > 0 \) depends only on \( A, p \) and \( \omega \).

By maximizing the right hand side of (2.28) over all admissible \( \omega \), we may take any \( p \in (1, p(A)) \), where
\[
p(A) := \sup_{\omega \in [0,1], \omega + \frac{1}{2} A \geq 0} p(A, \omega).
\]

We shall show in Lemma B.1 that \( p(A) \) has a more explicit form given by (1.10).

To this end, by (2.27), with \( h \ll 1 \),
\[
\int |\nabla (\xi |D^i_k Q(p)|^2) dx \leq C \int |\nabla \xi|^2 |D^i_k Q|^{2p} dx \leq C(\xi, A, p) \int |\nabla Q|^{2p} dx.
\]

Here \( W' \) is an open subset of \( \Omega \), such that \( W \subset W' \subset \Omega \). By Sobolev embedding,
\[
|D^i_k Q|_{L^{6p}(V)} \leq C(V, W, A, p) |\nabla Q|_{L^{2p}(W')},
\]

This estimate is independent of \( h \) as long as \( h \ll 1 \). Hence,
\[
|\nabla Q|_{L^{6p}(V)} \leq C(V, W, A, p) |\nabla Q|_{L^{2p}(W')}.
\]

Combining this with the interior \( H^2 \)-regularity in the previous step, we conclude, by making iterations if needed, that
\[
(2.28) \quad |\nabla Q|_{L^{6p}(V)} \leq C(V, A, p) |\nabla Q|_{L^2(\Omega)},
\]
as long as \( p \in (1, p(A)) \). This proves the interior \( W^{1,q} \)-regularity of \( Q \), where \( q = 6p \).

\( \square \)

Theorem 1.2 then follows from a compactness-type argument.

Proof of Theorem 1.2 Suppose the statement is false for some \( V \subset \subset \Omega \). Then there exists a sequence \( A_i \to 0 \), such that the minimizer \( Q_{A_i} \in A \) of (1.4) with parameter \( A_i \) admits \( d(Q_{A_i}(x_i)) = 0 \) at some \( x_i \in V \). We may assume \( x_i \to x_* \in \overline{V} \).

Take \( W \) such that \( V \subset \subset W \subset \subset \Omega \). Since the \( H^2(W) \)-estimate in Proposition 1.4 is uniform for all \( A \) sufficiently close to 0, \( Q_{A_i} \) has uniform bound in \( C^{1/2}(W) \). By Arzelà-Ascoli lemma, up to a subsequence, there exists \( Q_* \) such that \( Q_{A_i} \to Q_* \) uniformly in \( \overline{V} \). On the other hand, the uniform-in-\( A \) \( H^2_{loc}(\Omega) \)-estimate also implies that, up to a further sequence, the convergence is strong in \( H^1_{loc}(\Omega) \). Without loss of generality, we may assume that \( Q_{A_i} \to Q_* \) and \( \nabla Q_{A_i} \to \nabla Q_* \) almost everywhere in \( \Omega \). Hence,
\[
f_b(Q_{A_i}) \to f_b(Q_*) , \quad f_e(\nabla Q_{A_i}) \to f_e(\nabla Q_*), \quad \text{a.e. in } \Omega.
\]

Let \( Q^* \) denote the energy minimizer when \( A = 0 \). By Fatou’s Lemma,
\[
\int \nabla f_e(\nabla Q_*) \leq \liminf_{i \to +\infty} E_{A_i}[Q_{A_i}] \leq \lim_{i \to +\infty} E_{A_i}[Q^*] = E_0[Q^*].
\]

By the uniqueness of the energy minimizer, \( Q_* = Q^* \). Meanwhile, by the uniform convergence of \( Q_{A_i} \) to \( Q_* \) in \( \overline{V} \), the uniform \( C^{1/2}(W) \)-bound of \( Q_{A_i} \), and the fact that \( d(\cdot) \) is Lipschitz continuous (see Lemma C.2),
\[
d(Q_*(x_*)) = \lim_{i \to +\infty} d(Q_{A_i}(x_i)) \leq \lim_{i \to +\infty} d(Q_{A_i}(x_i)) + C|x_* - x_i|^{1/2} = 0,
\]
which implies \( x_* \in C \). This contradicts with the fact that \( C = \emptyset \) when \( A = 0 \).

Since \( C = \emptyset \) in \( V \), \( Q_A \) satisfies an Euler-Lagrange equation on \( V \),
\[
-\Delta Q_A - A \nabla \text{div } Q_A = -Df_b(Q_A).
\]
With $f_b$ smooth, the smoothness of $Q_A$ follows from the regularity theory of elliptic systems by a bootstrap argument.

3. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. With $a > 0$ sufficiently small, we define

$$\Omega_a = \{x \in \Omega : d(Q(x)) \leq a\}.$$  

The main idea of the proof is to derive a bound for the size of $\Omega_a$.

Proof of Theorem 1.3

Step 1. Let $\eta_a : [0, \infty) \rightarrow \mathbb{R}_+$ be defined by

$$\eta_a(x) = \min\left\{1, \frac{1 - \sqrt{6a}}{1 - \sqrt{6}}\right\}.$$  

By Lemma C.1 (in particular, (C.77)), it can be verified that the map $h_a : Q \mapsto \eta_a(d(Q))Q$ retracts $Q_{phy}$ to a smaller subset of $Q_{phy}$. To be more precise, $h_a(Q) \equiv Q$ if $d(Q) > a$, and $d(h_a(Q)) = a$ if $d(Q) \leq a$. Moreover, since $\eta_a \circ d$ is piecewise-smooth on $Q_{phy}$, by a limiting argument, $h_a(Q) \in H^1(\Omega)$.

Take arbitrary $U \subset \subset V \subset \subset \Omega$. Define a smooth cut-off function $\rho \in C^\infty_0(V)$ such that $\rho \in [0, 1]$ and $\rho \equiv 1$ on $U$. Let

$$Q_a(x) = \rho h_a(Q) + (1 - \rho)Q.$$  

Obviously when $a \ll 1$, $Q_a \in \mathcal{A}$. Also, by taking $a$ even smaller if needed, we may assume that $f_b(0) \leq f_b(Q)$ for all $Q \in Q_{phy}$ such that $d(Q) \leq a$. Then by the convexity of $f_b$, $f_b(Q) \geq f_b(Q_a)$ for all $x \in \Omega$, as $Q_a(x)$ can be viewed as a convex combination of $Q(x)$ and 0.

Step 2. We shall use $Q_a$ as a comparison configuration in (1.4). By the minimality of $Q$,

$$\int_{\Omega} f_b(Q(x)) - f_b(Q_a(x)) \, dx \leq \int_{\Omega} f_e(\nabla Q_a(x)) - f_e(\nabla Q(x)) \, dx.$$  

With $\lambda > 1$ to be determined, we derive that

$$\int_{\Omega} f_b(Q(x)) - f_b(Q_a(x)) \, dx \geq \int_{\Omega_{a/\lambda} \cap U} m_s \left(\frac{a}{\lambda}\right)^{-s} - M_s a^{-s} \, dx.$$  

Taking $\lambda = \Lambda_s$, with

$$\Lambda_s := \left(\frac{2M_s}{m_s}\right)^{\frac{1}{2}},$$

we find that

$$\int_{\Omega} f_b(Q(x)) - f_b(Q_a(x)) \, dx \geq \frac{m_s}{2} \cdot \left(\frac{a}{\Lambda_s}\right)^{-s} |\Omega_{a/\Lambda_s} \cap U|.$$  

On the right hand side of (3.30), since $f_e$ is quadratic,

$$\left|\int_{\Omega} f_e(\nabla Q_a(x)) - f_e(\nabla Q(x)) \, dx\right|$$

$$\leq C \int_{\Omega} |\nabla Q_a - Q|^2 + |\nabla Q||\nabla Q_a - Q| \, dx$$

$$= C \int_{\Omega_{a\cap V}} |\nabla (\rho Q(\eta_a(d(Q)) - 1))|^2 + |\nabla Q||\nabla (\rho Q(\eta_a(d(Q)) - 1))| \, dx.$$
Since $\eta \circ d$ is a piecewise-smooth map with Lipschitz constant 1 and $|\eta - 1| \leq Ca$,
\[
\left| \int f_c(\nabla Q_a(x)) - f_c(\nabla Q(x)) \, dx \right| \leq C \int_{\Omega_a \cap V} |\nabla Q|^2 + a^2|\nabla \rho|^2 \, dx.
\]
By Hölder’s inequality and Proposition 1.1,
\[
\|\nabla Q\|_{L^2(\Omega_a \cap V)} \leq \|\nabla Q\|_{L^2(\Omega_a \cap V)} |\Omega_a \cap V|^{\frac{1}{2} - \frac{q}{2}} \leq C \|\nabla Q\|_{L^2(\Omega)} |\Omega_a \cap V|^{\frac{1}{2} - \frac{q}{2}},
\]
where $q$ satisfies (1.9) and $C$ depends on $q$, $V$ and $A$. Therefore, for $a \ll 1$,
\[
\left(3.33\right) \quad \left| \int f_c(\nabla Q_a(x)) - f_c(\nabla Q(x)) \, dx \right| \leq C(|\Omega_a \cap V|^{1-2/q} + a^2|\Omega_a \cap V|) \leq C|\Omega_a \cap V|^{1-2/q},
\]
where $C$ depends on $A$, $q$, $U$, $V$ and $E[Q]$. Combining (3.30), (3.32), and (3.33), we obtain that
\[
\left(3.34\right) \quad \frac{|\Omega_a/\Lambda_s \cap U|}{(a/\Lambda_s)^s} \leq C|\Omega_a \cap V|^{1-2/q}.
\]

**Step 3.** We first show part (2) of Theorem 1.3.

Since $|\Omega_a \cap V| \leq C$, we find that for all $a \ll 1$, by virtue of (3.34), $|\Omega_a/\Lambda_s \cap U| \leq C(a/\Lambda_s)^s$.

With the notation changed, this is equivalent to that, for all $V \subset \subset \Omega$ and $a \ll 1$, $|\Omega_a \cap V| \leq Ca^s$. Applying this new bound in (3.34) yields that $|\Omega_a \cap V| \leq Ca^{s+\theta(1-2/q)}$, with a different constant $C$. By repeating this procedure, we can show that for all
\[
\left(3.35\right) \quad \beta < sq/2,
\]
all $V \subset \subset \Omega$,
\[
\left(3.36\right) \quad |\Omega_a \cap V| \leq Ca^\beta \quad \text{for all } a \ll 1,
\]
where $C$ depends on $\beta$, $A$, $M_s$, $m_s$, $q$, $V$, $E[Q]$ and $\Omega$, but not on $a$.

Now we assume $C \cap U \neq \emptyset$ for some $U \subset \subset V \subset \subset \Omega$, and $x_0 \in C \cap U$; otherwise, we have nothing to prove. By Proposition 1.1 and Sobolev embedding, $Q \in C_{loc}^{\alpha}(\Omega)$ where $\alpha = 1 - 3/q$. Hence, for all $x \in B_r(x_0)$ with $r \leq d(x_0, \partial V)/2$, $d(Q(x)) \leq Cr^\alpha$, where $C$ does not depend on $r$. Therefore, for $r \ll 1$,
\[
\left(3.37\right) \quad (C \cap U)_r \subset \{d(Q(x)) \leq Cr^\alpha\} \cap V,
\]
where $(C \cap U)_r := \cup_{x \in C \cap U} B_r(x)$ is the $r$-neighborhood of $C \cap U$. Combining this with (3.36), we find that for all $\gamma < s(q - 3)/2$, and $r \ll 1$,
\[
\left(3.38\right) \quad |(C \cap U)_r| \leq Cr^\gamma.
\]
Since $\cup_{x \in C \cap U} B_r(x)$ is a covering of $(C \cap U)_r$ with finite radius bound, by Vitali Covering Lemma, there exist a countable set $C_r' = \{x_i\} \subset C \cap U$, such that $B_r(x_i)$ are disjoint, and $\cup_{x \in C_r'} B_{5r}(x)$ is a covering of $C \cap U$. By the disjointness of $B_r(x_i)$, $\cup_{x \in C_r'} B_r(x) \subset (C \cap U)_r$, which together with (3.38) implies that $C_r'$ is a finite set with $|C_r'| \leq C r^{-\gamma}$. Since $C \cap U \subset \cup_{x \in C_r'} B_{5r}(x)$, we find $H^{3-\gamma}_{10r}(C \cap U) \leq C$ (see e.g. [13] for the notation), where $C$ is independent of $r$. Therefore, $H^{3-\gamma}(C \cap U) < +\infty$.

Since $\gamma$ is arbitrary as long as $\gamma < s(q - 3)/2$ with $q$ satisfying (1.9), we conclude that
\[
\dim_H(C \cap U) \leq 3 - \frac{3s}{s(A)}
\]
for any open subset $U \subset \Omega$. Applying this estimate to an exhaustion $\{U_i\}_{i=1}^\infty$ of $\Omega$, we can prove part (2) of Theorem 1.3.
Step 4. Finally, we prove part (1) of Theorem 1.3 by contradiction.

Suppose \( x_0 \in C \) for some \( s > s(A) \). Take \( x_0 \in V \subset W \subset \Omega \). By (3.37), for all \( a \ll 1 \), \( B_{Ca^{1/a}}(x_0) \subset \{ d(Q) \leq a \} \cap V \), where \( \alpha = 1 - 3/q \) and \( q \) satisfies (1.9). Hence,

\[
|\Omega_a \cap V| \geq |B_{Ca^{1/a}}(x_0)| = C a^{3/\alpha} = C a^{3q/(q-3)} =: C a^{q_0},
\]

where \( C \) is independent of \( a \). Applying this bound on the left hand side of (3.34) yields that, for all \( a \ll 1 \),

\[
|\Omega_a \cap W| \geq C a^{q_0 - 1 - 2/q} =: C a^{q_1}.
\]

Here \( C \) is a different universal constant. Repeating this procedure, we obtain a sequence \( \{q_n\} \) by

\[
q_n = \frac{q_{n-1} - s}{1 - 2/q} \quad \text{for } n \geq 1,
\]

such that for any \( U \subset \subset \Omega \), and \( a \ll 1 \), \( |\Omega_a \cap U| \geq C a^{q_0} \). Here the constant \( C \) should depend on \( n \) but not on \( a \). However, if \( s > 2q_0/q \), it is easy to verify that there exists \( N < \infty \), such that \( a \ll 1 \),

\[
|\Omega_a \cap U| \geq C a^{qN} \quad \text{for all } a \ll 1,
\]

which is obviously impossible.

By (1.9) and (1.13), whenever \( s > s(A) \), \( s > 2q_0/q = 6/(q-3) \) is achievable by suitably choosing \( q \). This proves \( C = \emptyset \).

\[ \square \]

4. Proof of Theorem 1.4

Recall that in the proof of Proposition 1.1, we got rid of the singular bulk energy term at the very beginning thanks to the convexity of \( f_b \). However, the new assumptions (1.14) and (1.16) on \( f_b \) allow us to make use of the bulk energy term in a better way, which leads to the improvement in Theorem 1.4.

In what follows, we shall first recast the proof of Proposition 1.1 to derive an estimate involving \( f_b(Q) \). However, as it is hard to directly work with \( f_b \) with singularity, we shall introduce an approximating sequence \( \{f_b^\varepsilon\}_{0<\varepsilon<1} \) of \( f_b \) with no singularity in the entire \( Q \). Indeed, we have

Lemma 4.1. Suppose \( f_b \) satisfies the assumptions (i) and (ii). Then there exists \( \{f_b^\varepsilon\}_{0<\varepsilon<1} \) satisfying the following conditions:

(i') For all \( 0 < \varepsilon \ll 1 \), \( f_b^\varepsilon(Q) \in [0,\infty) \) for all \( Q \in \Omega \);

(ii') \( f_b^\varepsilon \) are convex and smooth in \( Q \);

(iii') \( f_b^\varepsilon(Q) \leq f_b(Q) \) for all \( Q \in Q \).

(iv') Moreover,

\[
\lim_{\varepsilon \to 0^+} f_b^\varepsilon(Q) = f_b(Q), \quad \lim_{\varepsilon \to 0^+} Df_b^\varepsilon(Q) = Df_b(Q)
\]

locally uniformly in \( Q_{\text{phy}} \).

Here \( Df_b^\varepsilon(Q) \) denotes the gradient of \( f_b^\varepsilon \) with respect to \( Q \).

Its proof will be given in Appendix D.

Given \( \varepsilon > 0 \), let \( Q^\varepsilon \) denote the unique minimizer of

\[
E^\varepsilon[Q] = \int_{\Omega} f_\varepsilon(\nabla Q) + f_b^\varepsilon(Q) \, dx
\]
Lemma 4.2. For any $\delta > 0$, up to a subsequence, $Q^\varepsilon \to Q$ in $H^{2-\delta}_{loc}(\Omega)$.

Proof. The argument is similar to the proof of Theorem [12].

Since $f_b^\varepsilon$ are convex, Proposition [11] applies to $Q^\varepsilon$ with uniform-in-$\varepsilon$ interior $H^2$-bound. This implies that there exists $Q_*$ such that up to a subsequence, $Q^\varepsilon \to Q_*$ strongly in $H^{2-\delta}_{loc}(\Omega)$. We may assume that $Q^\varepsilon \to Q_*$, and $\nabla Q^\varepsilon \to \nabla Q_*$ almost everywhere in $\Omega$. Hence,

$$
\tilde{f}_b^\varepsilon(Q^\varepsilon) \to f_b(Q_*) , \ f_c(\nabla Q^\varepsilon) \to f_c(\nabla Q_*) , \ a.e. \ in \ \Omega.
$$

In justifying the first convergence, we need that

$$
E[Q^\varepsilon] = \int_{\Omega} f_c(\nabla Q^\varepsilon) + \tilde{f}_b^\varepsilon(Q^\varepsilon) \ dx \leq \int_{\Omega} f_c(\nabla Q) + f_b^\varepsilon(Q) \ dx \leq E[Q].
$$

By Fatou’s Lemma,

$$
\int_{\Omega} f_c(\nabla Q_*) + f_b(Q_*) \ dx \leq E[Q].
$$

Hence, $Q_* = Q$ by the uniqueness of minimizer, which completes the proof.

Now we are ready to derive an estimate for $f_b(Q)$.

Lemma 4.3. Assume $A > -\frac{3}{2}$ and $p \in [1, p(A))$. Suppose $f_b$ satisfies assumptions [11] - [13].

1. If $f_b$ additionally satisfies (1.16), then for any $V \subset \subset \Omega$,

$$
\int_{V} |\nabla Q|^{2p-2} |\nabla f_b(Q)|^2 \ dx \leq C(V, A, p, E[Q], c_\varepsilon).
$$

2. If $f_b$ additionally satisfies (1.14), then for any $V \subset \subset \Omega$,

$$
\int_{V} |\nabla Q|^{2p-2} |\nabla f_b|^2 \ dx \leq C(V, A, p, E[Q], c_\varepsilon).
$$

Proof. Up to minor adaptations, arguments in this proof are in the same spirit as those in the proof of Proposition [11]. See also [7, Theorem 4.3].

Step 1. Let $V \subset \subset W \subset \subset \Omega$ and $\xi$ be defined as in the proof of Proposition [11]. Similar to the proof of Proposition [11] define

$$
\tilde{u}_k^\varepsilon = D_h^{-h}(\xi^2 |D_h^\varepsilon Q^\varepsilon|^{2p-2} D_h^\varepsilon Q^\varepsilon),
$$

where

$$
D_h^\varepsilon Q^\varepsilon := (D_h^1 Q^\varepsilon, D_h^2 Q^\varepsilon, D_h^3 Q^\varepsilon)^T , \ |D_h^\varepsilon Q^\varepsilon|^2 = \sum_{m=1}^{3} |D_m^h Q^\varepsilon|^2.
$$

Since $Q^\varepsilon \in H^3_{loc}(\Omega)$, $\tilde{u}_k^\varepsilon \in H^1_h(\Omega, Q)$ for all $p \geq 1$. By the minimality of $Q^\varepsilon$, we argue as before to obtain that

$$
\int_{\Omega} \frac{\partial f_c(\nabla Q^\varepsilon)}{\partial Q_{ij,l}} \partial_i \tilde{u}_{k,ij}^\varepsilon + D_h f_b^\varepsilon(Q^\varepsilon) \cdot \tilde{u}_k^\varepsilon \ dx = 0,
$$

which gives

$$
\int_{\Omega} D_h^m \left( \frac{\partial f_c(\nabla Q^\varepsilon)}{\partial Q_{ij,l}} \right) \partial_l (\xi^2 |D_h^\varepsilon Q^\varepsilon|^{2p-2} D_h^\varepsilon Q^\varepsilon) + D_h^m (D_h f_b^\varepsilon(Q^\varepsilon))(\xi^2 |D_h^\varepsilon Q^\varepsilon|^{2p-2} D_h^\varepsilon Q^\varepsilon) \ dx = 0.
$$
By Lemma 4.2, it is easy to justify that up to a subsequence, as \( \varepsilon \to 0 \),
\[
\int_\Omega D_k^h \left( \frac{\partial f_\varepsilon(\nabla Q^\varepsilon)}{\partial Q_{ij,t}} \right) \partial_t (\xi^2 |D^h Q^\varepsilon|^{2p-2} D_k^h Q_{ij}^\varepsilon) \, dx \\
\rightarrow \int_\Omega D_k^h \left( \frac{\partial f_\varepsilon(\nabla Q)}{\partial Q_{ij,t}} \right) \partial_t (\xi^2 |D^h Q|^{2p-2} D_k^h Q_{ij}) \, dx.
\]
(4.42)

On the other hand, since \( f_b^\varepsilon \) are convex, for all \( w_1, w_2 \in Q_{phy}^0 \),
\[
(D f_b^\varepsilon(w_1) - D f_b^\varepsilon(w_2)) \cdot (w_1 - w_2) \geq 0,
\]
which implies that
\[
\xi^2 D_k^h(D f_b^\varepsilon(Q^\varepsilon)) D_k^h Q^\varepsilon \geq 0.
\]
It is assumed that \( D f_b^\varepsilon \to D f_b \) locally uniformly in \( Q_{phy}^0 \), and by Lemma 4.2, up to a subsequence \( Q^\varepsilon \to Q \) pointwise. Hence,
\[
D_k^h(D f_b^\varepsilon(Q^\varepsilon)) \to D_k^h(D f_b(Q)) \quad \text{a.e. in } \Omega.
\]

By Fatou’s Lemma, for that subsequence,
\[
\int_\Omega D_k^h(D f_b(Q))(\xi^2 |D^h Q|^{2p-2} D_k^h Q) \, dx \leq \lim_{\varepsilon \to 0^+} \int_\Omega D_k^h(D f_b^\varepsilon(Q^\varepsilon))(\xi^2 |D^h Q^\varepsilon|^{2p-2} D_k^h Q^\varepsilon) \, dx.
\]
This together with (4.41) and (4.42) implies that
\[
\int_\Omega D_k^h \left( \frac{\partial f_\varepsilon(\nabla Q)}{\partial Q_{ij,t}} \right) \partial_t (\xi^2 |D^h Q|^{2p-2} D_k^h Q_{ij}) + D_k^h(D f_b(Q))(\xi^2 |D^h Q|^{2p-2} D_k^h Q) \, dx \leq 0,
\]

Step 2. Using the form of \( f_e \) in (1.5), we rewrite the inequality above as
\[
\int_\Omega \xi^2 |D^h Q|^{2p-2} \cdot D_k^h(Q_{ij,t} + \delta_{jt} A(\text{div } Q)_i) \cdot D_k^h Q_{ij,t} \, dx \\
+ \int_\Omega \xi^2 |D^h Q|^{2p-2} \cdot D_k^h(D f_b(Q)) D_k^h Q \, dx \\
\leq - \int_\Omega D_k^h(Q_{ij,t} + \delta_{jt} A(\text{div } Q)_i) D_k^h Q_{ij} \cdot 2 \xi \partial_t \xi |D^h Q|^{2p-2} \, dx \\
- (2p - 2) \int_\Omega D_k^h(Q_{ij,t} + \delta_{jt} A(\text{div } Q)_i) D_k^h Q_{ij} \cdot \xi^2 |D^h Q|^{2p-4}(\partial_t D^h Q : D^h Q) \, dx.
\]

Here the summation convention applies to all repeated indices, including \( k \). With \( p \geq 1 \) and \( A \geq -\frac{3}{5} \), we derive that
\[
\int_\Omega \xi^2 |D^h Q|^{2p-2} |D^h \nabla Q|^2 \, dx + \int_\Omega \xi^2 |D^h Q|^{2p-2} \cdot D_k^h(D f_b(Q)) D_k^h Q \, dx \\
\leq C \int_\Omega |\xi||\nabla \xi||D^h Q|^{2p-1}|D^h \nabla Q| \, dx + C \int_\Omega \xi^2 |D^h \nabla Q||D^h Q||D^h Q|^{2p-3}|D^h \nabla Q| \, dx
\]
(4.43)
If \( p \geq 2 \),
\[
|D_k^h \nabla Q||D_k^h Q||D^h Q|^{2p-3}|D^h \nabla Q| \leq C(|D_k^h \nabla Q||D_k^h Q|^{p-1})^{\frac{1}{p-1}}(|D^h \nabla Q||D^h Q|^{p-1})^{\frac{2p-3}{p-1}}.
\]
Otherwise,
\[
|D_k^h \nabla Q||D_k^h Q||D^h Q|^{2p-3}|D^h \nabla Q| \leq C|D_k^h \nabla Q||D_k^h Q|^{p-1} : |D^h \nabla Q||D^h Q|^{p-1}.
\]

In either case, applying Young’s inequality to the right hand side of (4.43), we obtain that
\[
\int_\Omega \xi^2 |D^h Q|^{2p-2} |D^h \nabla Q|^2 \, dx + \int_\Omega \xi^2 |D^h Q|^{2p-2} \cdot D^h_k (Df_b(Q)) D^h_k Q \, dx \\
\leq C \int_\Omega |\nabla \xi|^2 |D^h Q|^{2p} \, dx + C \int_\Omega \xi^2 |D^h\nabla Q|^2 |D^h_k Q|^{2p-2} \, dx.
\]

By (2.27), for \( p \in [1, p(A)] \),
\[
(4.44) \quad \int_\Omega \xi^2 |D^h Q|^{2p-2} |D^h \nabla Q|^2 \, dx + \int_\Omega \xi^2 |D^h Q|^{2p-2} \cdot D^h_k (Df_b(Q)) D^h_k Q \, dx \leq C,
\]
where \( C = C(V, A, p, E[Q]) \).

**Step 3.** Take an arbitrary \( \delta \ll 1 \) and let \( \Omega_\delta \) be defined in (3.29). Since \( Q \) is Hölder continuous on \( \mathcal{W} \) by Lemma 4.2, \( Q \) is separated away from \( \partial Q_{\text{phy}} \) on an \( h \)-neighborhood of \( V \setminus \Omega_\delta \) provided that \( h \ll 1 \). Thus \( Q \) is smooth in this neighborhood, which means on \( V \setminus \Omega_\delta \),
\[
D^h Q \to \nabla Q, \quad D^h_k (Df_b(Q)) \to \partial_k (Df_b(Q)).
\]

By (4.44) and dominated convergence theorem, for any \( \delta \ll 1 \),
\[
(4.45) \quad \int_{V \setminus \Omega_\delta} |\nabla Q|^{2p-2} \partial_k (Df_b(Q)) \partial_k Q \, dx \leq C.
\]

To this end, if we assume (1.10), on \( V \setminus \Omega_\delta \),
\[
\partial_k (Df_b(Q)) \partial_k Q = \frac{\partial^2 f_b}{\partial Q_{ij} \partial Q_{mn}} \partial_k Q_{ij} \partial_k Q_{mn} \geq c_s \frac{|Df_b \cdot \nabla Q|^2}{f_b(Q)} = \frac{c_s |\nabla f_b(Q)|^2}{f_b(Q)}.
\]

Combining this with (4.45) and sending \( \delta \to 0 \), we obtain (4.39). (4.40) can be derived similarly if (1.14) is assumed.

\[\square\]

Recall that in the proof of Theorem 1.3 we used the distance function \( d(Q) \) in the construction of the comparison configuration \( Q_0 \). In the proof of Theorem 1.4 we would like the comparison configuration to depend more explicitly on \( f_0 \) so that the estimate in Lemma 4.3 may be used. For this purpose, we need the following technical construction.

Let us first consider the case where \( f_0 \) satisfies the assumptions (iii) with (1.11) and (iv) with (1.15) and (1.10). With \( m_s \) given in (1.11), define \( \tilde{\eta}_a \) \( : [0, \infty) \to \mathbb{R}_+ \) such that

1. \( \tilde{\eta}_a(y) \) is a decreasing \( C^1 \)-function on \( [0, +\infty) \).
2. \[
(4.46) \quad \tilde{\eta}_a(y) = \begin{cases} 
1, & \text{if } y \leq \frac{m_s}{2a^2}, \\
\min \left\{ 1, \frac{1 - \sqrt{6} \frac{m_s}{a^2}}{1 - \sqrt{6} \left( \frac{m_s}{a^2} \right)^{\frac{1}{2}}} \right\}, & \text{if } y \in \left[ \frac{m_s}{2a^2}, 2 \frac{m_s}{a^2} \right], \\
\frac{1 - \sqrt{6} \left( \frac{m_s}{a^2} \right)^{\frac{1}{2}}}{1 - \sqrt{6} \left( \frac{m_s}{a^2} \right)^{\frac{1}{2}}}, & \text{if } y \geq 2 \frac{m_s}{a^2}.
\end{cases}
\]
3. For all \( y \geq \frac{m_s}{2a^2} \),
\[
(4.47) \quad |\tilde{\eta}_a'(y)| \leq \frac{C m_s}{s} y^{\frac{1}{2} - 1}.
\]
Indeed, combining (4.46), (4.47) with (1.11) and (1.15), we deduce that for all
where \( \Lambda_s \)

\[ \min \left\{ 1, \frac{1 - \sqrt{6a}}{1 - \sqrt{6 \left( \frac{m_a}{y_s} \right)^2}} \right\}. \]

When \( a \ll 1 \), it is easy to verify that \(|1 - \tilde{\eta}_a| \leq C a \) and \( \tilde{\eta}_a \circ f_b \) is \( C^1 \) on \( Q^o_{phy} \) with bounded gradient. Indeed, combining (4.46), (4.47) with (1.11) and (1.15), we deduce that for all \( w \in Q^o_{phy} \),

\[ |\tilde{\eta}_a'(f_b(w)) \cdot Df_b(w)| \leq \frac{Cm_a}{s} \cdot C_s. \]

Define \( \tilde{h}_a(Q) := \tilde{\eta}_a(f_b(Q))Q \). It can be verified that when \( a \ll 1 \),

- \( \tilde{h}_a(Q) \equiv Q \) if \( d(Q) \geq \Lambda_s a \).
  - This directly follows from (1.11) and (4.46).
- \( \tilde{h}_a \) retracts \( Q_{phy} \) to a smaller subset of \( Q^o_{phy} \), such that \( d(\tilde{h}_a(Q)) \geq a \). Indeed, by Lemma C.1 (in particular, (C.77)),
  \[
  d(\tilde{h}_a(Q)) = \frac{\sqrt{6}}{2} \left( \tilde{\eta}_a(f_b(Q))\lambda_1(Q) + \frac{1}{3} \right) = \frac{\sqrt{6}}{6} - \tilde{\eta}_a(f_b(Q)) \left( \frac{\sqrt{6}}{6} - d(Q) \right).
  \]

In order to show \( d(\tilde{h}_a(Q)) \geq a \), it suffices to notice that

\[ \tilde{\eta}_a(f_b(Q)) \leq \frac{1 - \sqrt{6a}}{1 - \sqrt{6d(Q)}}, \]

which is true because of (1.11) and (4.46).
- \( \tilde{h}_a(Q) \in H^1(\Omega) \). This follows from \( C^1 \)-regularity of \( \tilde{\eta}_a \circ f_b \) and (4.48).

To this end, we are ready to show part (1) of Theorem 1.4.

**Proof of part (1) of Theorem 1.4.** We proceed as in the proof of Theorem 1.3 with minor modifications.

**Step 1.** Let \( U \subset V \subset \subset \Omega \) and a cut-off function \( \rho \) be defined as in the proof of Theorem 1.3. Let

\[ \tilde{Q}_a(x) = \rho \tilde{h}_a(Q) + (1 - \rho)Q. \]

By the argument above, \( \tilde{Q}_a \in A \) when \( a \ll 1 \).

We shall use \( Q_a \) as a comparison map in (1.13). By the minimality of \( Q \),

\[ \int_{\Omega} f_b(Q(x)) - f_b(\tilde{Q}_a(x)) \, dx \leq \int_{\Omega} f_e(\nabla \tilde{Q}_a(x)) - f_e(\nabla Q(x)) \, dx. \]

With \( a \ll 1 \), we may assume that \( f_b(0) \leq f_b(Q) \) for any \( Q \in Q^o_{phy} \) with \( d(Q) \leq \Lambda_s a \). For the same reason as in the proof of Theorem 1.3, \( f_b(Q) \geq f_b(\tilde{Q}_a) \) for all \( x \in \Omega \). Hence, we can still show that

\[ \int_{\Omega} f_b(Q(x)) - f_b(\tilde{Q}_a(x)) \, dx \geq \frac{m_s}{2} \left( \frac{a}{\Lambda_s} \right)^{-s} |\Omega_{a/\Lambda_s} \cap U|, \]

where \( \Lambda_s \) is defined in (3.31).
Step 2. On the right hand side of (4.39), since \( \tilde{Q}_a \equiv Q \) outside \( \Omega_{\Lambda_a} \cap V \),

\[
\left| \int_{\Omega} f_c(\nabla \tilde{Q}_a(x)) - f_c(\nabla Q(x)) \, dx \right|
\leq C \int_{\Omega_{\Lambda_a} \cap V} |\nabla (\tilde{Q}_a - Q)|^2 + |\nabla Q| |\nabla (\tilde{Q}_a - Q)| \, dx
\leq C \int_{\Omega_{\Lambda_a} \cap V} a |\nabla Q|^2 + a |\nabla \rho|^2 + |\nabla Q| |\nabla (\tilde{\eta}_a(f_b(Q)))| \, dx.
\]

(4.51)

Here we used the definition of \( \tilde{h}_a(Q) \) to derive that

\[
|\nabla (\tilde{Q}_a - Q)| \leq C(a |\nabla Q| + a |\nabla \rho| + |\nabla (\tilde{\eta}_a(f_b(Q)))|).
\]

Now we proceed in two different cases.

**Case 1.** Suppose \( p(A) \leq 2 \), i.e., we may only take \( p < 2 \) in Lemma 4.3. Thanks to (4.47), (4.48) and (4.51), with \( q = 6p \) and \( \delta \ll 1 \),

\[
\left| \int_{\Omega} f_c(\nabla \tilde{Q}_a(x)) - f_c(\nabla Q(x)) \, dx \right|
\leq C a |\Omega_{\Lambda_a} \cap V|^{\frac{1}{2} - \frac{2}{q}} + C \int_{\Omega_{\Lambda_a} \cap V} |\nabla Q|^2 \, dx + C \int_{(\Omega_{\Lambda_a} \setminus \Omega_b) \cap V} |\nabla Q| |\tilde{\eta}_a(f_b)| |\nabla f_b| \, dx
\leq C a |\Omega_{\Lambda_a} \cap V|^{\frac{1}{2} - \frac{2}{q}} + C |\Omega_{\tilde{\eta}} \cap V|^{1 - \frac{2}{q}} \cdot f_b^{-1+\frac{2}{q}} |\nabla f_b| \, dx.
\]

Combining this with (4.49), (4.50) and the assumption (iii), and sending \( \delta \to 0 \),

\[
|\Omega_{a/\Lambda_a} \cap U| \leq C a^{1+\varepsilon} |\Omega_{\Lambda_a} \cap V|^{1-\frac{1}{3p}} + C a^{1+\frac{2\varepsilon}{3}} \int_{\Omega_{\Lambda_a} \cap V} |\nabla Q| \cdot f_b^{-\frac{1}{2}} |\nabla f_b| \, dx.
\]

By Proposition 1.1, Lemma 4.3 and Hölder’s inequality,

\[
\int_{\Omega_{\Lambda_a} \cap V} |\nabla Q| \cdot f_b^{-\frac{1}{2}} |\nabla f_b| \, dx
\leq C |\nabla Q|^{\frac{2-p}{L^p(V)}} |\nabla Q|^{p-1} \cdot f_b^{-\frac{1}{2}} |\nabla f_b| \quad |\Omega_{\Lambda_a} \cap V|^{\frac{1}{2} - \frac{2-p}{2p}}
\leq C |\Omega_{\Lambda_a} \cap V|^{\frac{1}{2} - \frac{2-p}{2p}}.
\]

(4.52)

Hence,

\[
|\Omega_{a/\Lambda_a} \cap U| \leq C a^{1+\varepsilon} |\Omega_{\Lambda_a} \cap V|^{1-\frac{1}{3p}} + C a^{1+\frac{2\varepsilon}{3}} |\Omega_{\Lambda_a} \cap V|^{\frac{1}{2} - \frac{2-p}{6p}}.
\]

By a boot-strapping argument similar to the one in the proof of Theorem 1.3, we can show that for any

\[
\beta < \frac{3p(A)(3s+2)}{2(p(A)+1)},
\]

and any \( V \subset \subset \Omega \), we have \( |\Omega_a \cap V| \leq C a^\beta \) for all \( a \ll 1 \). Hence, (1.18) can be proved for the case \( p(A) \leq 2 \) by arguing as in the proof of Theorem 1.3.
Case 2. Now suppose $p(A) > 2$, i.e., we may take $p \geq 2$ in Lemma 4.3. Again by the assumption (iii) on $f_b$, (4.47), (4.48) and (4.51),
\[
\left| \int_{\Omega} f_b(\nabla Q_a(x)) - f_e(\nabla Q(x)) \, dx \right|
\leq C a|\Omega_{a,a} \cap V|^{\frac{2}{\sigma}} + C \int_{\Omega_b} |\nabla Q|^2 \, dx + C \int_{(\Omega_{a,a},\Omega_b) \cap V} |\nabla Q|^{\frac{2}{\sigma}} \left( |\hat{\eta}_a'(f_b)| |\nabla f_b| \right)^{\frac{2}{p}} \, dx
\leq C a|\Omega_{a,a} \cap V|^{\frac{2}{\sigma}} + C |\Omega_b \cap V|^{\frac{2}{\sigma}} + C a^{\frac{2+s}{p}} \int_{(\Omega_{a,a},\Omega_b) \cap V} |\nabla Q|^{\frac{2}{\sigma}} \left( |f_b|^{\frac{2}{p}} |\nabla f_b| \right)^{\frac{2}{p}} \, dx.
\]
Combining this with (4.49) and (4.50), sending $\delta \to 0$, and applying Hölder’s inequality as before, we find that
\[
|\Omega_{a/a} \cap U| \leq C a^{1+s} |\Omega_{a,a} \cap V|^{1-\frac{2}{3p}} + C a^{s+\frac{2+s}{p}} |\Omega_{a,a} \cap V|^{1-\frac{1}{p}}.
\]
Arguing as in Case 1, for any
\[
\beta < 2 + s + sp(A),
\]
and any $V \subset \subset \Omega$, we have $|\Omega_a \cap V| \leq C a^{\beta}$ for all $a \ll 1$. Then (1.18) can be proved as before for the case $p(A) > 2$.

Next, we consider the case when $f_b$ satisfies the assumptions (iii) with (1.12) and (iv) with (1.14) and (1.17). The argument is almost parallel.

Take $a \ll 1$. Let $\Lambda_0 := \exp(1 + k_0^{-1}(M_0 - m_0))$. Take $\hat{\eta}_a$ as a decreasing $C^1$-function on $[0, +\infty)$,
\[
\hat{\eta}_a(y) = \begin{cases} 
1, & \text{if } y \leq k_0|\ln a| + m_0 - \exp(m_0/k_0), \\
\frac{1 - \sqrt{6a} k_0}{1 - \sqrt{6a}}, & \text{if } y \geq k_0|\ln a| + m_0.
\end{cases}
\]
and for $y \in [k_0|\ln a| + m_0 - \exp(m_0/k_0), k_0|\ln a| + m_0]$, $\hat{\eta}_a$ needs to satisfy
\[
(4.53) \quad |\hat{\eta}_a'(y)| \leq 4\Lambda_0 \exp(-k_0^{-1}y).
\]
This is achievable since for $a \ll 1$,
\[
\exp(m_0/k_0) \cdot 4\Lambda_0 \exp(-k_0^{-1}(k_0|\ln a| + m_0)) > 1 - \frac{1 - \sqrt{6a}\Lambda_0 a}{1 - \sqrt{6a}}.
\]
We also note that (4.53) implies
\[
(4.54) \quad \hat{\eta}_a' \leq C a
\]
for some universal constant $C$.

We claim that
\begin{enumerate}
\item $\hat{\eta}_a \circ f_b$ is Lipschitz continuous in $Q_{p(h,y)}^a$. Indeed, by (1.17) and (4.53),
\[
|D(\hat{\eta}_a \circ f_b)| \leq |\hat{\eta}_a'(f_b)||Df_b| \leq 4\Lambda_0 C_0.
\]
\item $1 - \hat{\eta}_a \leq Ca$ for some universal $C > 0$, thanks to the monotonicity of $\hat{\eta}_a$ and $a \ll 1$.
\item Whenever $d(Q) \leq a$,
\[
(4.55) \quad f_b(Q) - f_b(\hat{\eta}_a \circ f_b(Q)) \geq k_0.
\]
\end{enumerate}
Indeed, by (1.12) and (C.77),
\[
\begin{align*}
f_b(Q) - f_b(\hat{\eta}_0 \circ f_b(Q)) \\
\geq k(Q)|\ln d(Q)| + m_0 - k(Q)|\ln d(\hat{\eta}_0 \circ f_b(Q))| - M_0 \\
\geq k_0 \ln \frac{\hat{\eta}_0 \circ f_b(Q)(d(Q) - \frac{\sqrt{a}}{a}) + \frac{\sqrt{a}}{a}}{d(Q)} - (M_0 - m_0).
\end{align*}
\]
Since \(d(Q) \leq a\) implies \(f_b(Q) \geq k_0|\ln a| + m_0\), we know that \(\hat{\eta}_0 = \frac{1-\sqrt{6}a}{1-\sqrt{6}a}\). Hence,
\[
\begin{align*}
f_b(Q) - f_b(\hat{\eta}_0 \circ f_b(Q)) \\
\geq k_0 \ln \frac{1-\sqrt{6}a}{1-\sqrt{6}a} (a - \frac{\sqrt{a}}{a}) + \frac{\sqrt{a}}{a} - (M_0 - m_0) \\
= k_0 \ln \Lambda_0 - (M_0 - m_0) \\
= k_0.
\end{align*}
\]
In particular, for arbitrary \(\nu > 1\), (4.55) holds when \(f_b(Q) \geq \nu K_0|\ln a| \geq K_0|\ln a| + M_0\) as long as \(a \ll 1\) (with the needed smallness depending on \(\nu, K_0\) and \(M_0\)).

Define \(\hat{\eta}_a(Q) := \hat{\eta}_a(f_b(Q))Q\). It is not difficult to justify that for \(a \ll 1\),
\[
\begin{itemize}
\item \(\hat{\eta}_a(Q) \equiv Q\) if \(f_b(Q) \leq k_0|\ln a| + m_0 - \exp(m_0/k_0)\).
\item \(f_b(\hat{\eta}_a(Q)) \leq f_b(Q)\). This follows from convexity of \(f_b\).
\item \(\hat{\eta}_a(Q) \in H^1(\Omega)\). This follows from \(C^1\)-regularity of \(\hat{\eta}_a \circ f_b\) and boundedness of \(\hat{\eta}_a(f_b) \cdot D f_b\).
\end{itemize}
\]

**Proof of part 2 of Theorem 7.4.** Define \(\hat{Q}_a\) as before, \(\hat{Q}_a(x) = \rho \hat{\eta}_a(Q) + (1 - \rho)Q\). We still have
\[
\int_\Omega f_b(Q(x)) - f_b(\hat{Q}_a(x)) \, dx \leq \int_\Omega f_e(\nabla \hat{Q}_a(x)) - f_e(\nabla Q(x)) \, dx.
\]
Denote
\[
S_y = \{x \in \Omega : f_b(Q(x)) \geq y\}
\]
For \(a \ll 1\), by (4.55) and the remark following that,
\[
\int_\Omega f_b(Q(x)) - f_b(\hat{Q}_a(x)) \, dx \geq k_0|\nabla \nu K_0|\ln a| \cap U|.
\]
On the other hand,
\[
\begin{align*}
\left| \int_\Omega f_e(\nabla \hat{Q}_a(x)) - f_e(\nabla Q(x)) \, dx \right|
\leq C \int_{S_0(1 + m_0 - \exp(m_0/k_0))} a|\nabla Q|^2 + a|\nabla p|^2 + |\nabla Q||\nabla (\hat{\eta}_a(f_b(Q)))| \, dx.
\end{align*}
\]
Take arbitrary \(\theta < k_0/(\nu K_0) < 1\). For \(a \ll 1\) with the smallness depending on \(\theta\), we have
\[
\theta(\nu K_0|\ln a|) \leq k_0|\ln a| + m_0 - \exp(m_0/k_0).
\]
Suppose \(p(A) \leq 2\). Arguing as before, we derive that for \(a \ll 1\),
\[
\begin{align*}
|S_\nu K_0|\ln a| \cap U| &\leq C a|S_{\theta \nu K_0}|\ln a| \cap V|^1 - \frac{2}{7} + C a \int_{S_{\theta \nu K_0}|\ln a| \cap V}|\nabla Q|^{2-p} : |\nabla Q|^{p-1}|\nabla f_b| \, dx \\
&\leq C a|S_{\theta \nu K_0}|\ln a| \cap V|^1 - \frac{1}{5p} + C a|S_{\theta \nu K_0}|\ln a| \cap V|^1 \frac{2}{p - \theta}. 
\end{align*}
\]
Here we used (1.54). Hence,

\[ |S_{\nu|a|} \cap U| \leq Ca|S_{\nu|a|} \cap V|^{\frac{2}{3}} \frac{1}{3p}. \]

By boot-strapping, for any \( \beta < \sum_{j=0}^{\infty} \left[ \theta \left( \frac{2}{3} - \frac{1}{3p(A)} \right) \right] j = \left[ 1 - \theta \left( \frac{2}{3} - \frac{1}{3p(A)} \right) \right]^{-1}, \)

and any \( V \subset \subset \Omega, \) we have \( |S_{\nu|a|} \cap V| \leq Ca^{\beta} \) for all \( a \ll 1. \) By (1.12), this implies

\[ |\Omega_{\exp(k_0^{-1}(\nu|a| + m_0))} \cap V| \leq Ca^{\beta}, \]

or equivalently, \( |\Omega_a \cap V| \leq Ca^{\beta/(\nu|a|)}, \) with \( C \) being a constant independent of \( a. \) This leads to (1.19) in the case \( p(A) \leq 2. \)

Finally, suppose \( p(A) > 2. \) Similarly,

\[
|S_{\nu|a|} \cap U| \leq Ca|S_{\nu|a|} \cap V|^{1 - \frac{1}{3p}} + Ca^{\frac{2}{p}} \int_{S_{\nu|a|} \cap V} |\nabla Q|^{\frac{2}{p}} |\nabla f_0| \frac{2}{p} dx \\
\leq Ca|S_{\nu|a|} \cap V|^{1 - \frac{1}{3p}} + Ca^{\frac{2}{p}} |S_{\nu|a|} \cap V|^{1 - \frac{1}{p}} \\
\leq Ca^{\frac{2}{p}} |S_{\nu|a|} \cap V|^{1 - \frac{1}{p}}
\]

Hence, for any \( \beta < 2/(p - \theta(p - 1)), \) and any \( V \subset \subset \Omega, \) we have \( |\Omega_a \cap V| \leq Ca^{\beta/(\nu|a|)} \) for all \( a \ll 1. \) Then (1.19) for the case \( p(A) > 2 \) follows. \( \square \)

5. BOUNDARY PARTIAL REGULARITY

This section is devoted to proving Theorem 1.5. First we introduce some notations. Let \( x_0 = (x_0^1, x_0^2, x_0^3) \in \mathbb{R}^3 \) and \( R > 0. \) Define

\[
B(x_0, R) = \{ x = (x^1, x^2, x^3) \in \mathbb{R}^3 : |x - x_0| < R \}, \\
B^+ (x_0, R) = \{ x \in B(x_0, R) : x^3 > x_0^3 \}, \\
\Gamma(x_0, R) = B(x_0, R) \cap \{ x^3 = x_0^3 \}, \\
Q_{x_0, R} = \int_{B^+(x_0, R)} Q(x) dx.
\]

In the special case \( x_0 = 0, \) we write them as \( B_R, B^+_R, \Gamma_R, \) and \( Q_R, \) respectively.

In order to prove \( Q \) is Hölder continuous near some point \( y_0 \in \partial \Omega, \) we first use a smooth local diffeomorphism \( \psi^{-1} \) to a ball \( U \) centered at \( y_0 \) to flatten the boundary, such that \( y_0, U \cap \Omega, \) and \( U \cap \partial \Omega \) are mapped to 0, \( B^+_R, \) and \( \Gamma_R, \) respectively. Moreover, up to a rotation, we may assume that

\[
\lim_{r \to 0} \| \nabla(\psi^{-1}) - Id \|_{L^\infty(B^+_R)} = 0.
\]

In other words, if we zoom in to smaller and smaller neighborhood of \( y_0, \) the deformation induced by \( \psi^{-1} \) is almost negligible, and \( \psi^{-1} \) behaves like an identity map.
To this end, under the change of variables $y = \psi(x)$, define
\begin{equation}
E[Q, U] := \int_U f_e(\nabla Q) + f_b(Q) \, dy
\end{equation}
subject to Dirichlet data $Q = Q_0$ on $\partial B^+_R$ for some $Q_0 \in C^\infty(\bar{B}_R, Q^0_{\text{phy}})$. Here $f_e$ is defined as in (1.4), and $J(x) : B^+_R \to \mathbb{R}^{3 \times 3}$ is a smooth function, satisfying
\begin{equation}
\lim_{r \to 0^+} \|J(x) - \text{Id}\|_{L^\infty(B^+_R)} = 0.
\end{equation}

We also used the notation that $(J(x) \nabla Q)_{ij,k} \triangleq J_{kl}(x) \nabla_l Q_{ij}$.

For $x_0 \in \Gamma_R$ and $0 < r < \text{dist}(x_0, \partial \Gamma_R)$, define an scaling-invariant quantity
\[ A_{x_0,r} = \frac{1}{r} \int_{B^+_r(x_0,r)} |\nabla Q(x)|^2 \, dx. \]

Denote $A_r := A_{0,r}$. Then we have

**Proposition 5.1.** Let $Q \in H^1(B^+_R, Q^0_{\text{phy}})$ be the unique minimizer of (5.58) subject to smooth Dirichlet boundary condition $Q = Q_0$ on $\partial B^+_R$. There exists $\varepsilon > 0$, such that if for $x_0 \in \Gamma_R$, $\liminf_{r \to 0^+} A_{x_0,r} < \varepsilon^2$, then $Q(x)$ is Hölder continuous in a neighborhood of $x_0$.

**Theorem 1.5** follows immediately from Proposition 5.1.

**Proof of Theorem 1.5** The diffeomorphism $\psi^{-1}$ for flattening the boundary is smooth, and it is sufficiently close to an identity map (up to a rotation in general) if we only consider sufficiently small boundary patches. Hence, a statement similar to Proposition 5.1 holds for the minimizer of (1.4) in $\Omega$ with curved boundary. This together with a classic covering argument implies that the minimizer of (1.4) is Hölder continuous up to $\partial \Omega \setminus S$, with $S \subset \partial \Omega$ satisfying $\mathcal{H}^1(S) = 0$.

Since $Q_0(x) \in Q^0_{\text{phy}}$ for all $x \in \partial \Omega$, by the continuity, $\mathcal{T} \cap \partial \Omega \subset S$. This completes the proof.

The rest of this section is devoted to proving Proposition 5.1. The proof closely follows the classical variational proof in [6] and the proof of interior partial regularity in [7], with necessary modifications to handle the boundary data. In what follows, without loss of generality, we shall assume $x_0 = 0$. We first prove a useful lemma.

**Lemma 5.2.** For all $r \in (0, R)$,
\[ |Q_r - Q_0(0)| \leq C \sqrt{A_r + r^2}, \]
where $C$ is a constant only depending on $\|\nabla Q_0\|_{L^\infty(B^+_R)}$. 

Proof. Since $Q - Q_0 = 0$ on $\Gamma_R$, by Poincaré inequality on domains with finite width,
\[ A_r \geq \frac{1}{2r} \int_{B_r^+} |\nabla (Q - Q_0)|^2 \, dx - \frac{1}{r} \int_{B_r^+} |\nabla Q_0|^2 \, dx \]
\[ \geq \frac{1}{2r^3} \int_{B_r^+} |Q - Q_0|^2 \, dx - \frac{1}{r} \int_{B_r^+} |\nabla Q_0|^2 \, dx \]
\[ \geq \frac{\pi}{3} (Q - Q_0)|^2 r - \frac{1}{r} \int_{B_r^+} |\nabla Q_0|^2 \, dx. \]
Hence, by the smoothness of $Q_0$,
\[ (5.61) \quad |(Q - Q_0)_r| \leq C \sqrt{A_r + r^2}. \]
In addition,
\[ |(Q_0)_r - Q_0(0)| \leq C r. \]
Then the desired estimate follows. □

To this end, we shall prove the so-called small energy regularity.

**Lemma 5.3.** Let $Q$ be the minimizer defined in Proposition 5.2. There exists $\theta \in (0, \frac{1}{4})$ and $\varepsilon > 0$, such that if
\[ r < \varepsilon, \quad A_r \leq \varepsilon^2, \]
then for any $x \in \Gamma_{r/4}$,
\[ A_{x, \theta r} \leq \frac{1}{2} A_r. \]

We shall prove Lemma 5.3 by contradiction. Suppose the statement is false. Then for a fixed $\theta \in (0, 1/4)$ which we will determine later, there exists $\{(\varepsilon_i, r_i, x_i)\}_{i=1}^{\infty}$ such that
\[ \varepsilon_i \to 0, \quad r_i \leq \varepsilon_i, \quad A_{r_i} \leq \varepsilon_i^2, \quad \text{and} \quad x_i \in \Gamma_{r_i/4}, \]
while
\[ A_{x_i, \theta r_i} > \frac{1}{2} A_{r_i}. \]

Define
\[ Q_i(x) = \varepsilon_i^{-1} (Q(r_i x) - Q_{r_i}). \]
It is straightforward to verify that
\[ \int_{B^+_1} |\nabla Q_i|^2 \, dx \leq 1, \quad (Q_i)_1 = 0, \]
and $Q_i$ minimizes
\[ (5.62) \quad W_i[Q] = \int_{B^+_1} \left[ f_c \left( \varepsilon_i \frac{r_i}{r} J(r_i x) \nabla Q \right) + f_b (\varepsilon_i Q + Q_{r_i}) \right] g(r_i x) \, dx. \]
Passing to a subsequence if necessary, there exists $x_* \in \Gamma_{1/4}$, and $\hat{Q} \in H^1(B^+_1, Q)$, such that
\begin{itemize}
  \item $x_i/r_i \to x_*$;
  \item $Q_i \to \hat{Q}$ weakly in $H^1(B^+_1, Q)$, and strongly in $L^2(B^+_1, Q)$;
  \item $(\hat{Q})_1 = 0$;
  \item $Q_i|_{r_1} \to \hat{Q}|_{r_1}$ in $C^2(\Gamma_1)$.
\end{itemize}
Indeed, it suffices to verify the last claim. Note that by (5.61),

\[ |(Q - Q_0)_{r_i}| \leq C \sqrt{A_{r_i} + r_i^2} \leq C \varepsilon_i. \]

By the definition of \( Q_i \), for \( \forall x \in \Gamma, \)

\[ |Q_i(x)| \leq \varepsilon_i^{-1}|Q_0(r_i x) - (Q_0)_{r_i}| + \varepsilon_i^{-1}|(Q_0 - Q)_{r_i}| \leq C(\varepsilon_i^{-1}r_i + 1) \leq C. \]

Moreover, for any \( k \in \mathbb{N} \),

\[ |\nabla^k_i Q_i(x)| \leq \varepsilon_i^{-1}r_i^k \|\nabla^k_i Q_0\|_{L^\infty(\Gamma)} \leq C_k. \]

The convergence in \( C^2(\Gamma_1) \) follows from Arzelà-Ascoli lemma.

The next lemma shows that the \( H^1 \)-convergence of \( Q_i \) to \( \tilde{Q} \) is in fact in the strong sense in smaller boundary patches.

**Lemma 5.4.** \( \nabla Q_i \) converges to \( \nabla \tilde{Q} \) strongly in \( L^2(B^+_r, Q) \) for any \( 0 < r < 1 \).

**Proof.** Define Radon measures \( \mu_i \) as follows,

\[ \mu_i(D) := \int_D |\nabla Q_i|^2 + |\nabla \tilde{Q}|^2 \, dx \text{ for any measurable set } D \subset B^+_1. \]

Up to a subsequence, there exists a Radon measure \( \mu \) such that \( \mu_i \to \mu \) in the sense of measures. For all but countably many \( r \in (0, 1) \), it holds that \( \mu(\partial B_r \cap B^+_1) = 0 \). It suffices to show

\[ \lim_{i \to \infty} \int_{B^+_r} |\nabla Q_i - \nabla \tilde{Q}|^2 \, dx = 0 \]

for any \( r \in (0, 1) \) such that \( \mu(\partial B_r \cap B^+_1) = 0 \).

Since \( f_e \) is strictly convex and quadratic in \( \nabla Q \), there exists \( \lambda > 0 \) such that

\[ \int_{B^+_1} \left[ f_e(\nabla Q_i(x)) - f_e(\nabla \tilde{Q}(x)) \right] \, dx \]

\[ \geq \int_{B^+_1} \left[ Df_e(\nabla \tilde{Q}(x)) \cdot (\nabla Q_i - \nabla \tilde{Q}) \right] \, dx + \lambda \int_{B^+_1} |\nabla Q_i - \nabla \tilde{Q}|^2 \, dx. \]

As \( i \to \infty \), the first term on the right hand side goes to 0. It suffices to prove

\[ (5.63) \quad \limsup_{i \to \infty} \int_{B^+_1} f_e(\nabla Q_i(x)) - f_e(\nabla \tilde{Q}(x)) \, dx \leq 0. \]

We shall use the minimality of \( Q_i \) to show this. Take \( R \in (r, 1) \) and let \( \xi(x) \) be a smooth cut-off function such that

\[ (5.64) \quad 0 \leq \xi \leq 1, \quad \xi \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_R, \quad \xi \equiv 1 \text{ in } B_r, \quad \text{and } |\nabla \xi| \leq \frac{2}{R - r}. \]

Then we define \( \{\tilde{Q}^i\} \) as truncations of \( \tilde{Q} \) at magnitude \( \frac{1}{\sqrt[\xi]{r}} \):

\[ \tilde{Q}^i(x) = \begin{cases} \tilde{Q}(x), & \text{if } |\tilde{Q}(x)| \leq \frac{1}{\sqrt[\xi]{r}}, \\ \tilde{Q}(x) \frac{1}{\sqrt[\xi]{|\tilde{Q}(x)|}}, & \text{if } |\tilde{Q}(x)| > \frac{1}{\sqrt[\xi]{r}}. \end{cases} \]

It is straightforward to verify that \( \tilde{Q}^i \to \tilde{Q} \) strongly in \( H^1(B^+_1) \). Define

\[ P_i = \xi \tilde{Q}^i + (1 - \xi)Q_i. \]

However, one can not use \( P_i \) as a comparison in (5.62), since \( P_i \) and \( Q_i \) do not agree on \( \Gamma_R \). We thus need the following technical lemma to make a correction. \( \square \)
Lemma 5.5. Given \( Q_i |_{\Gamma_1} \to Q |_{\Gamma_1} \) in \( C^2(\Gamma_1) \), for any \( R < 1 \), there exists a sequence of functions \( \{ F_i \} \subset H^1(B_R^+) \) such that

- \( F_i = \xi(Q_i - \tilde{Q}) \) on \( \Gamma_R \), \( \text{supp} \ F_i \subset B_R^+ \), and \( F_i \to 0 \) in \( H^1(B_R^+) \);
- \( \| F_i \|_{L^\infty} \leq C \), where \( C \) is independent of \( i \);
- In addition,

\[
\lim_{i \to \infty} \int_{B_R^+} |f_b(\varepsilon_i (Q_i + F_i) + Q_{r_i}) - f_b(\varepsilon_i Q_i + Q_{r_i})| \, dx = 0. \tag{5.65}
\]

Proof. Since \( \tilde{Q} \in C^2(\Gamma_1) \), we may assume \( i \) is sufficiently large such that \( \tilde{Q}^i |_{\Gamma_1} = \tilde{Q} |_{\Gamma_1} \). Since \( Q_i |_{\Gamma_1} \to \tilde{Q} |_{\Gamma_1} \) in \( C^2(\Gamma_1) \), we can easily find \( \tilde{F}_i \) (e.g., by making a constant extension in the \( x_3 \)-direction and making a suitable smooth cutoff) such that

\[
\tilde{F}_i = \xi(Q_i - \tilde{Q}) \text{ on } \Gamma_R, \quad \text{supp} \ \tilde{F}_i \subset B_R^+, \quad \text{and } \tilde{F}_i \to 0 \text{ in } C^2(B_R^+).
\]

By the assumption on the boundary data \( Q_0 \) on \( \Gamma_1 \), there exists a universal constant \( \eta > 0 \) such that \( d(Q(r_i x)) > \eta \) for all \( x \in \Gamma_1 \) and all \( i \).

Define a piecewise-linear function \( \phi(x) : [0, \infty] \to [0, 1] \) as

\[
\phi(x) = \begin{cases} 
1, & \text{if } x \geq \eta; \\
\frac{x - \eta/2}{\eta/2}, & \text{if } \eta/2 \leq x < \eta; \\
0, & \text{if } 0 \leq x < \eta/2.
\end{cases}
\]

Then we claim that \( F_i(x) := \tilde{F}_i \cdot \phi(d(Q(r_i x))) \) has the desired properties.

Firstly, thanks to the property of boundary data \( Q_0 \), \( F_i |_{\partial B_R^+} = \tilde{F}_i |_{\partial B_R^+} \). It then suffices to verify that \( F_i \to 0 \) in \( H^1(B_R^+) \), and (5.65).

For the \( H^1 \)-convergence, we simply calculate that

\[
\int_{B_R^+} |\nabla F_i|^2 \, dx \leq C \int_{B_R^+} \left[ |\nabla \tilde{F}_i|^2 \|\phi\|^2_{L^\infty} + |\nabla \phi(d(Q(r_i x)))|^2 \|\tilde{F}_i\|^2_{L^\infty} \right] \, dx \\
\leq C \int_{B_R^+} \left[ |\nabla \tilde{F}_i|^2 + \frac{1}{\eta^2} |\nabla Q(r_i x)|^2 \|\tilde{F}_i\|^2_{L^\infty} \right] \, dx \\
\leq C \int_{B_R^+} |\nabla \tilde{F}_i|^2 \, dx + C \eta^{-2} \|\tilde{F}_i\|^2_{L^\infty} \cdot A_{r_i} \\
\to 0 \quad \text{as } i \to \infty,
\]

where we used the property \( |\nabla d(Q)| \leq C |\nabla Q| \) in the second inequality (see Lemma C.2).

To show (5.65), we note that \( F_i = 0 \) if \( d(Q(r_i x)) = d(\varepsilon_i Q_i(x) + Q_{r_i}) < \eta/2 \). On the other hand, by assuming \( i \) to be sufficiently large and using the fact that \( F_i \) are uniformly bounded, we have

\[
d(Q(r_i x) + \varepsilon_i F_i) \geq \eta/4 \quad \text{if } d(Q(r_i x)) \geq \eta/2.
\]
Therefore, \( f_b(\varepsilon_i(Q_i + F_i) + Q_{ri}) < +\infty \) when \( f_b(\varepsilon_iQ_i + Q_{ri}) < +\infty \), and
\[
\int_{B_R^+} |f_b(\varepsilon_i(Q_i + F_i) + Q_{ri}) - f_b(\varepsilon_iQ_i + Q_{ri})| \, dx
\]
\[
= \int_{B_R^+ \cap \{d(Q_{ri}) \geq \eta/2\}} |f_b(Q_{ri}) + \varepsilon_iF_i| - f_b(Q_{ri})| \, dx
\]
\[
\leq \int_{B_R^+ \cap \{d(Q_{ri}) \geq \eta/2\}} |\varepsilon_iF_i| \sup_{\{Q : d(Q) \geq \eta/4\}} |Df_b| \, dx
\]
\[
\leq C\varepsilon_i.
\]
This completes the proof. \( \square \)

**Proof of Lemma 5.4 (continued).** To this end, we define
\[
G_i := P_i + F_i = \xi \tilde{Q}^i + (1 - \xi)Q_i + F_i
\]
as a comparison in \( \ref{5.62} \). Indeed, when \( i \) is sufficiently large, it can be shown that \( \varepsilon_iQ_i(x) + Q_{ri} + \varepsilon_iF_i(x) = Q(r_ix) + \varepsilon_iF_i(x) \in Q_{phy} \), and \( \varepsilon_i\tilde{Q}^i + Q_{ri} + \varepsilon_i\tilde{F}_i \in Q_{phy} \) since \( Q_{ri} \to Q_0(0) \) by Lemma \( \ref{5.2} \). This implies that \( \varepsilon_iG_i + Q_{ri} \in Q_{phy} \). Thanks to the assumptions on \( F_i \), it is also easy to verify that \( G_i|_{\partial B_R^+} = Q_i|_{\partial B_R^+} \), Hence, by the (local) minimality of \( Q_i \),
\[
\int_{B_R^+} \left[ f_e \left( \frac{\varepsilon_i}{r_i} J(r_ix)\nabla Q_i \right) + f_b(\varepsilon_iQ_i + Q_{ri}) \right] g(r_ix) \, dx
\]
\[
\leq \int_{B_R^+} \left[ f_e \left( \frac{\varepsilon_i}{r_i} J(r_ix)\nabla G_i \right) + f_b(\varepsilon_iG_i + Q_{ri}) \right] g(r_ix) \, dx.
\]
Since \( f_e \) is quadratic and \( P_i = \tilde{Q}^i \) on \( B_r^+ \), we may rewrite this inequality as
\[
\int_{B_R^+} \left[ f_e(J(r_ix)\nabla P_i) - f_e(J(r_ix)\nabla \tilde{Q}) \right] g(r_ix) \, dx
\]
\[
\leq \int_{B_R^+} \left[ f_e(J(r_ix)\nabla (P_i + F_i)) - f_e(J(r_ix)\nabla P_i) \right] g(r_ix) \, dx
\]
\[
+ \int_{B_R^+} \left[ f_e(J(r_ix)\nabla \tilde{Q}^i) - f_e(J(r_ix)\nabla \tilde{Q}) \right] g(r_ix) \, dx
\]
\[
+ \int_{B_R^+ \setminus B_r^+} \left[ f_e(J(r_ix)\nabla P_i) - f_e(J(r_ix)\nabla Q_i) \right] g(r_ix) \, dx
\]
\[
+ \frac{r_i^2}{\varepsilon_i^2} \int_{B_R^+ \setminus B_r^+} \left[ f_b(\varepsilon_iG_i + Q_{ri}) - f_b(\varepsilon_iQ_i + Q_{ri}) \right] g(r_ix) \, dx.
\]
\[
\equiv I_1 + I_2 + I_3 + \varepsilon_i^{-2}r_i^2I_4.
\]
Since \( F_i \to 0 \) in \( H^1(B_r^+) \) and \( \tilde{Q}^i \to \tilde{Q} \) in \( H^2(B_r^+) \), we use \( \ref{5.59} \) and \( \ref{5.60} \) to derive that \( I_1 + I_2 \to 0 \) as \( i \to \infty \). For \( I_3 \), we calculate by \( \ref{5.59} \) and \( \ref{5.60} \) again that
\[
I_3 \leq C \int_{B_R^+ \setminus B_r^+} (|\nabla P_i| + |\nabla Q_i|)|\nabla (P_i - Q_i)| \, dx
\]
\[
\leq C \int_{B_R^+ \setminus B_r^+} \left[ \frac{1}{R - r} |\tilde{Q}^i - Q_i||\nabla Q_i| + \frac{1}{(R - r)^2} |\tilde{Q}^i - Q_i|^2 \right] \, dx + C\mu_i(B_R^+ \setminus B_r^+).
\]
Hence,
\[
\limsup_{i \to \infty} I_3 \leq C \mu(B_R^+ \setminus B_r^+) \quad \text{as } i \to \infty.
\]
Here we used $L^2$-convergence of $Q_i$ and $H^1$-convergence of $\tilde{Q}^i$. For $I_4$, by the convexity of $f_b$,
\[
I_4 \leq \int_{B_R^+} (1 - \xi)[f_b(\varepsilon_i(Q_i + F_i) + Q_{r_i}) - f_b(\varepsilon_i Q_i + Q_{r_i})]g(r_i x) \, dx \\
+ \int_{B_R^+} \xi[f_b(\varepsilon_i(\tilde{Q}^i + F_i) + Q_{r_i}) - f_b(\varepsilon_i Q_i + Q_{r_i})]g(r_i x) \, dx \\
=: I_{4,1} + I_{4,2}.
\]
By the construction of $F_i$ and the boundedness of $g(r_i x)$, $I_{4,1} \to 0$ as $i \to \infty$. For $I_{4,2}$, noting that $|\varepsilon_i(\tilde{Q}^i + F_i)| \leq C \sqrt{\varepsilon_i}$ and $Q_{r_i} \to Q_0(0)$ by Lemma 5.2 when $i$ is sufficiently large, for all $x \in B_R^+$,\[
d(\varepsilon_i(\tilde{Q}^i + F_i) + Q_{r_i}) \geq \frac{1}{2} d(Q_0(0)) > 0.
\]
Hence, we may take $d_* \in (0, d(Q_0(0))/2)$, such that if $d(\varepsilon_i Q_i + Q_{r_i}) < d_*$,
\[
f_b(\varepsilon_i(\tilde{Q}^i + F_i) + Q_{r_i}) \leq f_b(\varepsilon_i Q_i + Q_{r_i}).
\]
Hence, for $i$ sufficiently large, with $g(r_i x) > 0$,
\[
I_{4,2} \leq \int_{B_R^+ \setminus \{d(\varepsilon_i Q_i + Q_{r_i}) \geq d_*\}} \xi[f_b(\varepsilon_i(\tilde{Q}^i + F_i) + Q_{r_i}) - f_b(\varepsilon_i Q_i + Q_{r_i})]g(r_i x) \, dx \\
\leq C \int_{B_R^+ \setminus \{d(\varepsilon_i Q_i + Q_{r_i}) \geq d_*\}} \sup_{d(Q) \geq d_*} |Df_b(Q)| \cdot |\varepsilon_i(\tilde{Q}^i + F_i - Q_i)| \, dx.
\]
Then $I_{4,2} \to 0$ as $i \to \infty$ by $L^2$-convergence of $\tilde{Q}_i$, $F_i$ and $Q_i$.

Therefore,
\[
\limsup_{i \to \infty} \int_{B_r^+} \left[f_e(J(r_i x) \nabla Q_i) - f_e(J(r_i x) \nabla \tilde{Q})\right] g(r_i x) \, dx \leq \mu(B_R^+ \setminus B_r^+),
\]
Sending $R \to r_+$, we prove that
\[
\limsup_{i \to \infty} \int_{B_r^+} \left[f_e(J(r_i x) \nabla Q_i) - f_e(J(r_i x) \nabla \tilde{Q})\right] g(r_i x) \, dx \leq 0.
\]
Then (5.63) immediately follows from (5.59) and (5.60).

**Lemma 5.6.** For $\forall r \in (0, 1)$, $\tilde{Q}$ is the minimizer of
\[
W[\tilde{Q}] = \int_{B_r^+} |\nabla \tilde{Q}|^2 + A |\text{div} \tilde{Q}|^2 \, dx.
\]

**Proof.** For any $\phi \in C^\infty_c(B_r^+, Q)$, we take
\[
H_i := \xi(\tilde{Q}^i + \phi) + (1 - \xi)Q_i + F_i
\]
as a comparison in (5.62). Here $\tilde{Q}^i$, $Q_i$ and $F_i$ are defined as before. Arguing as in the analysis of $I_4$ in the proof of Lemma 5.4, we find that as $i \to \infty$,
\[
\frac{r^2}{\varepsilon_i} \int_{B_R^+} [f_b(\varepsilon_i H_i + Q_{r_i}) - f_b(\varepsilon_i Q_i + Q_{r_i})] g(r_i x) \, dx \to 0.
\]
By the minimality of $Q_i$,
\[ \frac{r_i^2}{\varepsilon_i^2} \int_{B_R^+} f_\epsilon \left( \frac{\varepsilon_i}{r_i} J(r_i x) \nabla Q_i(x) \right) g(r_i x) \, dx \leq \frac{r_i^2}{\varepsilon_i^2} \int_{B_R^+} f_\epsilon \left( \frac{\varepsilon_i}{r_i} J(r_i x) \nabla H_i(x) \right) g(r_i x) \, dx + o(1). \]

Since $Q_i, \tilde{Q}^i \rightarrow \tilde{Q}$ strongly in $H^1(B_R^+)$, letting $i \to \infty$ yields that
\[ \int_{B_R^+} f_\epsilon (\nabla \tilde{Q}) \, dx \leq \int_{B_R^+} f_\epsilon (\nabla (\tilde{Q} + \phi)) \, dx. \]

Here we used the fact that $\phi$ is supported on $B_R^+$ while $\xi \equiv 1$ on $B_R^+$. By approximation we observe that (5.66) still holds for $\phi \in H^1_0(B_R^+, Q)$ and this completes the proof. \qed

**Proof of Lemma 5.3.** Thanks to Lemma 5.6, $\tilde{Q}$ satisfies the following Euler-Lagrange equation in $B_R^+$:
\[- \Delta \tilde{Q} - A \nabla \div \tilde{Q} = 0,\]
subject to smooth boundary data on $\Gamma_1$. By the elliptic regularity theory, $\tilde{Q}$ is smooth in $B_R^+$, and there exists $\theta_* \in (0, 1/4)$, such that for all $x \in \overline{\Gamma_1}$,
\[ A_{x, \theta_*}(\tilde{Q}) \leq \frac{1}{3} A_1(\tilde{Q}). \]

To this end, we go back to the argument that follows the statement of Lemma 5.3. We take $\theta$ there to be $\theta_*$. Since $\nabla Q_i \rightarrow \nabla \tilde{Q}$ in $L^2_{\text{loc}}(B_R^+)$, we find that $A_{x, \theta}(\tilde{Q}) \geq \frac{1}{3} A_1(\tilde{Q})$, which is a contradiction.

The proof of Proposition 5.1 is then straightforward.

**Proof of Proposition 5.1.** We take $x_0 = 0$ as before. First let $\varepsilon$ be defined as in Lemma 5.3. We may assume, by decreasing $R$ if needed, that
\[ \|\nabla (\psi^{-1}) - Id\|_{L^\infty(B_R^+)} \ll 1. \]

Then take $R \leq \varepsilon$ such that $A_R \leq \varepsilon^2$. Consider an arbitrary $x \in B_{R/4}^+ \cup \Gamma_{R/4}$. We shall show that for some $\alpha \in (0, 1)$ and $C > 0$,
\[ \frac{1}{r} \int_{B_r(x) \cap B_R^+} |\nabla Q|^2 \, dy \leq C \varepsilon^2 \left( \frac{r}{R} \right)^\alpha \quad \text{for all } r \in (0, R/4). \]

**Case 1.** When $x \in \Gamma_{R/4}$, it is readily proved by Lemma 5.3 that $A_{x, \theta R} \leq A_{R/2} \leq \varepsilon^2/2$. Then we may repeatedly apply Lemma 5.3 with fixed base $x$ to find that $A_{x, \theta^k R} \leq \varepsilon^2/2^k$ for all $k \in \mathbb{Z}_+$, which implies (5.69).

Next we consider $x \in B_{R/4}^+$. Let $x' \in \Gamma_{R/4}$ be the orthogonal projection of $x$ onto $\Gamma_{R/4}$. Denote $d_x = d(x, \Gamma_R) = |x - x'|$; here $d(\cdot, \cdot)$ denotes the usual Euclidean distance.

**Case 2.** If $r \geq d_x/2$, then $B_r(x) \cap B_R^+ \subset B_{3r}(x') \cap B_R^+$.

If $3r \leq R/4$, by the discussion in Case 1
\[ \frac{1}{r} \int_{B_r(x) \cap B_R^+} |\nabla Q|^2 \, dy \leq \frac{3}{3r} \int_{B_{3r}(x') \cap B_R^+} |\nabla Q|^2 \, dy \leq C \varepsilon^2 \left( \frac{r}{R} \right)^\alpha. \]

Otherwise, $r \in (R/12, R/4)$ and (5.69) is trivially true.
**Case 3.** Now let us assume $r < d_x/2$. Consider $\psi(B_r(x)) \subset \subset \psi(B_{d_x}(x)) \subset \subset \Omega$ in the original coordinate before flattening the boundary. By assumption (5.68), the deformation induced by $\psi$ is so small that we may assume that

$$d(\psi(B_r(x)), \partial(\psi(B_{d_x}(x)))) \geq C(d_x - r) \geq Cd_x.$$ 

Hence, the interior $H^2$-regularity established in Proposition (A.1) as well as its proof, implies that

$$\|\nabla Q\|_{L^2(B_r(x))} \leq C\|\nabla (Q \circ \psi^{-1})\|_{L^2(\psi(B_r(x)))}$$

$$\leq C\|\psi(B_r(x))\|^{1/3}\|\nabla (Q \circ \psi^{-1})\|_{L^p(\psi(B_r(x)))}$$

$$\leq Cr \cdot d_x^{-1}\|\nabla (Q \circ \psi^{-1})\|_{L^2(\psi(B_{d_x}(x)))}$$

$$\leq Cr \cdot d_x^{-1}\|\nabla Q\|_{L^2(B_{d_x}(x))}.$$ 

Note that with abuse of notations here, $Q$ denotes the minimizer of (5.58) with boundary flattened, while $Q \circ \psi^{-1}$ is the minimizer in the original coordinate. Therefore,

$$\frac{1}{r} \int_{B_r(x)} |\nabla Q|^2 \, dy \leq \frac{Cr}{d_x} \cdot \frac{1}{d_x} \int_{B_{d_x}(x)} |\nabla Q|^2 \, dy \leq \frac{Cr}{d_x} \cdot \varepsilon^2 \left(\frac{d_x}{R}\right)^\alpha \leq C\varepsilon^2 \left(\frac{r}{R}\right)^\alpha.$$ 

In the second inequality, we used the estimate from Case 2.

This completes the proof of (5.69), which implies that $Q \in C^\alpha(B_{R/4} \cup \Gamma_{R/4}).$ 

\[ \square \]

**Appendix A. Proof of (2.25)**

In order to prove (2.25), we first prove the following lemma.

**Lemma A.1.** Let $M, N, P$ be $3 \times 3$-symmetric traceless matrices. Then

(A.70) \quad $\|M\|^2 \leq \frac{2}{3} |M|^2,$

and

(A.71) \quad $\sum_{i=1}^3 (M_{i1} + N_{i2} + P_{i3})^2 \leq \frac{5}{3} (|M|^2 + |N|^2 + |P|^2).$

**Proof.** To prove (A.70), we can only consider the case when $M$ is diagonal, due to the fact that $M$ is symmetric and both Frobenius norm and matrix 2-norm are unitarily invariant. Without loss of generality we assume $M = \text{diag}\{\lambda_1, \lambda_2, -\lambda_1 - \lambda_2\}$ with $\lambda_1 \lambda_2 \geq 0$, we deduce that

$$2|M|^2 - 3\|M\|^2 = 2(\lambda_1^2 + \lambda_2^2 + (\lambda_1 + \lambda_2)^2) - 3(\lambda_1 + \lambda_2)^2 = 2(\lambda_1 - \lambda_2)^2 \geq 0.$$ 

Then (A.70) follows. The equality holds if and only if $M$ has two equal eigenvalues.

For (A.71), we compute

$$5(|M|^2 + |N|^2 + |P|^2) - 3 \sum_{i=1}^3 (M_{i1} + N_{i2} + P_{i3})^2$$

(A.72) \quad $\geq (5(M_{11}^2 + M_{22}^2 + M_{33}^2 + 2N_{12}^2 + 2P_{13}^2) - 3(M_{11} + N_{12} + P_{13})^2)$$

$+ (5(N_{12}^2 + N_{23}^2 + N_{31}^2 + 2M_{21}^2 + 2P_{23}^2) - 3(M_{21} + N_{23} + P_{23})^2)$$

$+ (5(P_{11}^2 + P_{22}^2 + P_{33}^2 + 2M_{31}^2 + 2N_{32}^2) - 3(M_{31} + N_{32} + P_{33})^2).$
The equality holds if and only if \( M_{23} = N_{13} = P_{12} = 0 \). It suffices to prove non-negativity of each term on the right hand side of (A.72). We only show this for the first term; the others can be handled similarly.

\[
5(M_{11}^2 + M_{22}^2 + M_{33}^2 + 2N_{12}^2 + 2P_{13}^2) - 3(M_{11} + N_{12} + P_{13})^2 \\
\geq \left( 5 + \frac{5}{2} - 3 \right) M_{11}^2 + 7N_{12}^2 + 7P_{13}^2 - 6M_{11}N_{12} - 6M_{11}P_{13} - 6N_{12}P_{13} \\
= \left( \frac{3}{2} M_{11} - 2N_{12} \right)^2 + \left( \frac{3}{2} M_{11} - 2P_{13} \right)^2 + 3(N_{12} - P_{13})^2 \geq 0.
\]

Here we used \( \sum_{i=1}^{3} M_{ii} = 0 \) in the second line. The equality holds if and only if

\[
N_{12} = P_{13} = \frac{3}{4} M_{11}, \quad M_{22} = M_{33} = -\frac{1}{2} M_{11}.
\]

This completes the proof of (A.71). \( \square \)

To this end, (2.25) follows immediately from the lemma if we take \( M = D_h^k Q \) in (A.70), and take \( M = D_h^k \partial_1 Q, N = D_h^k \partial_2 Q, P = D_h^k \partial_3 Q \) in (A.72).

**APPENDIX B. FORMULA OF \( p(A) \)**

**Lemma B.1.** For \( A > -\frac{3}{5} \), let

\[
p(A) := \sup_{\omega \in [0,1], \omega + \frac{5}{3} A \geq 0} p(A, \omega),
\]

where \( p(A, \omega) \) is defined in (2.26). Then \( p(A) \) is given by (1.10).

**Proof.** We rewrite (2.26) as

\[(B.73) \quad p(A, \omega) = 1 + \frac{2}{2A^2} \left( \omega + \frac{5}{3} A \right) + \sqrt{\frac{2}{2A^2}} \left( \omega + \frac{5}{3} A \right) \left( \left( \frac{9}{5} - 2A^2 \right) \left( \omega + \frac{5}{3} A \right) + 2A^2 \left( 1 + \frac{5}{3} A \right) \right) \frac{2A^2}{\omega + \frac{5}{3} A}.\]

It is easy to see that if \( \frac{9}{5} - 2A^2 \geq 0 \), \( p(A, \omega) \) achieves its supremum at \( \omega = 1 \), which gives

\[
p(A) = 1 + \frac{3}{A} + \frac{9}{5A^2}.
\]

It suffices to consider \( 2A^2 > \frac{9}{5} \), i.e., \( A > \frac{3\sqrt{10}}{10} \). Define

\[
g(y) := \frac{9}{5} y + \sqrt{\frac{9}{5}} y (B_1 y + B_2),
\]

where

\[
B_1 = \frac{9}{5} - 2A^2, \quad B_2 = 2A^2 \left( 1 + \frac{5}{3} A \right).
\]

Then

\[
p(A, \omega) = 1 + (2A^2)^{-1} g \left( \omega + \frac{5}{3} A \right).
\]

Since

\[
g'(y) = \frac{9}{5} + \sqrt{\frac{9}{5}} \frac{2B_1 y + B_2}{2\sqrt{y(B_1 y + B_2)}},
\]

\[
g(y) = \frac{9}{5} y + \sqrt{\frac{9}{5}} \frac{2B_1 y + B_2}{2\sqrt{y(B_1 y + B_2)}}.
\]
we find that $g'(y) < 0$ if and only if
\[ B_1y + \frac{B_2}{2} < -\sqrt{\frac{9}{5}y(B_1y + B_2)}, \]
which is equivalent to
\[ \left( B_1y + \frac{B_2}{2} \right)^2 > \frac{9}{5}y(B_1y + B_2) \quad \text{and} \quad B_1y + \frac{B_2}{2} < 0. \]
Solving these inequalities under the assumption $A > \frac{3\sqrt{10}}{10}$, we find that
\[ y > \frac{A \left( 1 + \frac{5}{3}A \right)}{2A - \sqrt{\frac{18}{5}}} =: y_+. \]
This implies that within the domain of $g(y)$, i.e.,
\[ y = \omega + \frac{5}{3}A \in \left[ \frac{5}{3}A, 1 + \frac{5}{3}A \right], \]
g($y$) is decreasing if and only if $y \geq y_+$.

1. When
\[ 1 + \frac{5}{3}A \leq y_+ \quad \iff \quad A \in \left( \frac{3\sqrt{10}}{10}, \sqrt{\frac{18}{5}} \right), \]
p($A$, $\cdot$) is increasing on $[0, 1]$. Hence,
\[ p(A) = p(A, 1) = 1 + \frac{3}{A} + \frac{9}{5A^2}. \]

2. When
\[ \frac{5}{3}A < y_+ < 1 + \frac{5}{3}A \quad \iff \quad A \in \left[ \sqrt{\frac{18}{5}}, \frac{3}{5} + \sqrt{\frac{18}{5}} \right], \]
then supremum of $p(A, \cdot)$ is achieved at $\omega_*$ such that $\omega_* + \frac{5}{3}A = y_+$. Combining this with (B.73) yields that
\[ p(A) = 1 + \frac{3 + 5A}{2\sqrt{10}A - 6}. \]

3. When
\[ \frac{5}{3}A \geq y_+ \quad \iff \quad A \geq \frac{3}{5} + \sqrt{\frac{18}{5}}, \]
p($A$, $\cdot$) is decreasing on $[0, 1]$. Hence,
\[ p(A) = p(A, 0) = 1 + \frac{3 + \sqrt{9 + 6A}}{2A}. \]
This completes the derivation. \qed
Appendix C. Study of \( d(Q) \)

We study the properties of \( d(Q) \) in this section. It is known that every \( Q \in Q_{\text{phy}} \) can be represented by

\[
Q = \lambda_1 n \otimes n + \lambda_2 m \otimes m + \lambda_3 p \otimes p, \tag{C.74}
\]

where

\[
\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_i \in \left[ -\frac{1}{3}, \frac{2}{3} \right], \quad \lambda_1 \leq \lambda_2 \leq \lambda_3,
\]

and where \((n, m, p)\) forms an orthonormal frame in \( \mathbb{R}^3 \). Then we have the following characterization of \( d(Q) \).

**Lemma C.1.** Let \( Q \in Q_{\text{phy}} \) be given by (C.74) and (C.75). Then \( d(Q) = |Q - Q'| \), where

\[
Q' = -\frac{1}{3} n \otimes n + \left( \lambda_2 + \frac{\lambda_1 + \frac{3}{4}}{2} \right) m \otimes m + \left( \lambda_3 + \frac{\lambda_1 + \frac{3}{4}}{2} \right) p \otimes p. \tag{C.76}
\]

As a result,

\[
d(Q) = \frac{\sqrt{6}}{2} \left( \lambda_1 + \frac{1}{3} \right). \tag{C.77}
\]

**Proof.** Since the distance between two matrices is invariant under orthogonal transforms, without loss of generality, we may assume \( n = (1, 0, 0) \), \( m = (0, 1, 0) \) and \( p = (0, 0, 1) \). Let \( s = 2\lambda_1 + \lambda_2 \) and \( r = 2\lambda_2 + \lambda_1 \). Then (C.74) becomes

\[
Q = s \left( n \otimes n - \frac{1}{3} I \right) + r \left( m \otimes m - \frac{1}{3} I \right), \quad r - \frac{1}{2} \leq s \leq r \leq 0. \tag{C.78}
\]

Now we are going to look for \( Q' \in \partial Q_{\text{phy}} \) such that \(|Q - Q'|\) is minimized. Assume

\[
Q' = s' \left( n' \otimes n' - \frac{1}{3} I \right) + r' \left( m' \otimes m' - \frac{1}{3} I \right),
\]

with

\[
n' = (a, b, c), \quad m' = (u, v, w), \quad r' - \frac{1}{2} \leq s' \leq r' \leq 0,
\]

and \((n', m')\) being an orthonormal pair.

First we are going to show that when \( s', r' \) is fixed, \(|Q - Q'|\) is minimized when \( n' = n, m' = m \). We calculate that

\[
|Q - Q'|^2 = |Q|^2 + |Q'|^2 - 2 \left[ s \left( n \otimes n - \frac{1}{3} I \right) + r \left( m \otimes m - \frac{1}{3} I \right) \right] \cdot \left[ s' \left( n' \otimes n' - \frac{1}{3} I \right) + r' \left( m' \otimes m' - \frac{1}{3} I \right) \right]
\]

\[
= C(s, r, s', r') - 2 \left[ ss' \left( (n, n')^2 - \frac{1}{3} \right) + sr' \left( (n, m')^2 - \frac{1}{3} \right) + rs' \left( (m, n')^2 - \frac{1}{3} \right) + rr' \left( (m, m')^2 - \frac{1}{3} \right) \right]
\]

\[
= C(s, r, s', r') - 2 (ss' a^2 + rs' b^2 + sr' u^2 + rr' v^2).
\]
Here $C(s, r, s', r')$ represents some constant depending only on $s$, $r$, $s'$ and $r'$, whose definition changes from line to line.

Then it suffices to show that
\[
ss' a^2 + rs' b^2 + sr' u^2 + rr' v^2 \leq ss' + rr'.
\]
(C.79)

Recall that
\[
a^2 + b^2 + c^2 = 1, \quad u^2 + v^2 + w^2 = 1, \quad au + bv + cw = 0.
\]
(C.80)

We claim that $u^2 \leq b^2 + c^2$. Indeed, by (C.80),
\[
a^2 u^2 = |bv + cw|^2 \leq (b^2 + c^2)(v^2 + w^2) = (1 - a^2)(1 - u^2),
\]
which implies that $u^2 \leq 1 - a^2 = b^2 + c^2$. Then we deduce that
\[
ss' + rr' - (ss' a^2 + rs' b^2 + sr' u^2 + rr' v^2)
\]
\[
= (s - r)s' (b^2 + c^2) - (s - r)r' u^2 + rr' w^2 + rs' c^2
\]
\[
\geq (s - r)(s' - r')u^2 \geq 0.
\]

Here we used (C.80) and the facts that $s \leq r \leq 0$ and $s' \leq r' \leq 0$.

To this end, we have showed that if $Q$ is given by (C.74) and if $Q' \in \partial Q_{\text{phy}}$ minimizes $|Q - Q'|$, $Q'$ should be represented by
\[
Q' = \mu_1 n \otimes n + \mu_2 m \otimes m + \mu_3 p \otimes p,
\]
for some $-1/3 = \mu_1 \leq \mu_2 \leq \mu_3 \leq 2/3$ such that $\mu_1 + \mu_2 + \mu_3 = 0$. The constraints on $\mu_i$ are due to the characterization of $\partial Q_{\text{phy}}$ in (1.2). Moreover,
\[
|Q - Q'| = \sqrt{(\lambda_1 - \mu_1)^2 + (\lambda_2 - \mu_2)^2 + (\lambda_3 - \mu_3)^2}.
\]
(C.81)

Therefore, $|Q - Q'|$ achieves its minimum if
\[
\mu_1 = -\frac{1}{3}, \quad \mu_2 = \lambda_2 + \frac{\lambda_1 + \frac{1}{3}}{2}, \quad \mu_3 = \lambda_3 + \frac{\lambda_1 + \frac{1}{3}}{2}.
\]
(C.77) follows immediately. This completes the proof.

An immediate consequence of Lemma C.1 is

**Lemma C.2.** $d(Q)$ is Lipschitz continuous in $Q_{\text{phy}}$.

**Proof.** The difference between the smallest eigenvalues of two matrices in $Q_{\text{phy}}$ can be bounded by their distance. Combining this fact with Lemma C.1, we complete the proof of the Lemma.

\[\square\]

**APPENDIX D. A CONSTRUCTION OF $\{f_b^\varepsilon\}$**

In this section, we provide a construction of $\{f_b^\varepsilon\}_{0<\varepsilon<1}$ used in Section 4 For convenience, we recall the conditions on $\{f_b^\varepsilon\}_{0<\varepsilon<1}$:

(i') For all $0 < \varepsilon < 1$, $f_b^\varepsilon(Q) \in [0, \infty)$ for all $Q \in Q$;

(ii') $f_b^\varepsilon$ are convex and smooth in $Q$;

(iii') $f_b^\varepsilon(Q) \leq f_b(Q)$ for all $Q \in Q$.

(iv') Moreover,
\[
\lim_{\varepsilon \to 0^+} f_b^\varepsilon(Q) = f_b(Q), \quad \lim_{\varepsilon \to 0^+} Df_b^\varepsilon(Q) = Df_b(Q)
\]
locally uniformly in $Q^o_{\text{phy}}$.

Here $Df_b(Q)$ denotes the gradient of $f_b^\varepsilon$ with respect to $Q$. 

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Proof of Lemma 4.4. Define
\[ \mathcal{Q}_{\mathrm{phy}}^\varepsilon = \{ Q \in \mathcal{Q}_{\mathrm{phy}}^0 : f_b(Q) < \varepsilon^{-1} \}. \]
Take \( \varepsilon \ll 1 \), such that \( \mathcal{Q}_{\mathrm{phy}}^\varepsilon \) is a non-empty open subset of \( \mathcal{Q}_{\mathrm{phy}} \). Then we define on the entire \( \mathcal{Q} \) that
\[ F_b^\varepsilon(Q) = \sup_{Q' \in \mathcal{Q}_{\mathrm{phy}}^\varepsilon} f_b(Q') + Df_b(Q')(Q - Q'). \]
It is not difficult to show that \( \{ F_b^\varepsilon \}_{0 < \varepsilon \ll 1} \) satisfies all the conditions above except for the smoothness issue. Indeed, \( F_b^\varepsilon \equiv f_b \) on \( \mathcal{Q}_{\mathrm{phy}}^0 \), while outside \( \mathcal{Q}_{\mathrm{phy}}^\varepsilon \), \( F_b^\varepsilon \) is only Lipschitz continuous and \( DF_b^\varepsilon \) exists in the \( L^\infty \)-sense but may not be well-defined pointwise. In particular, for all \( Q_1, Q_2 \in \mathcal{Q} \),
\[ |F_b^\varepsilon(Q_1) - F_b^\varepsilon(Q_2)| \leq |Q_1 - Q_2| \sup_{\mathcal{Q}_{\mathrm{phy}}^\varepsilon} |Df_b| =: |Q_1 - Q_2| \omega_\varepsilon. \]
Note that \( \omega_\varepsilon \to +\infty \) as \( \varepsilon \to 0^+ \).

We shall make a little modification of \( \{ F_b^\varepsilon \} \) to construct smooth \( \{ f_b^\varepsilon \} \). Let \( \phi \) be a non-negative \( C_0^\infty \)-mollifier in \( \mathcal{Q} \) supported on the unit ball, such that \( \int_\mathcal{Q} \phi(Q) \, dQ = 1 \).

Then we define
\[ f_b^\varepsilon(Q) = \int_\mathcal{Q} \phi(Q')F_b^\varepsilon(Q - \varepsilon\omega_\varepsilon^{-1}Q') \, dQ' - \varepsilon. \]

We derive that for arbitrary \( Q \in \mathcal{Q} \),
\[ |f_b^\varepsilon(Q) + \varepsilon - F_b^\varepsilon(Q)| \leq \int_\mathcal{Q} \phi(Q')|F_b^\varepsilon(Q - \varepsilon\omega_\varepsilon^{-1}Q') - F_b^\varepsilon(Q)| \, dQ' \]
\[ \leq \int_\mathcal{Q} \phi(Q') \cdot \varepsilon\omega_\varepsilon^{-1} \cdot \omega_\varepsilon \, dQ' = \varepsilon. \]

In the first inequality, we used the fact that \( \phi \) is non-negative and normalized; in the second inequality, we applied \( \text{D.82} \) as well as that \( \phi \) is supported on the unit ball in \( \mathcal{Q} \). \( \text{D.83} \) implies that \( F_b^\varepsilon(Q) - 2\varepsilon \leq f_b^\varepsilon(Q) \leq F_b^\varepsilon(Q) \leq f_b(Q) \).

It is then easy to verify that \( \{ f_b^\varepsilon \}_{0 < \varepsilon \ll 1} \) satisfies all the conditions we need. \( \square \)

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