A new approach to tolerance analysis method based on the screw and the Lie Algebra of Lie Group

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Abstract. Tolerance analysis refers to the process of establishing mapping relations between tolerance features and the target feature along the dimension chain. Traditional models for tolerance analysis are all based on rigid body kinematics, and they adopt the Homogeneous Transformation Matrix to describe feature variation and accumulation. However, those models can hardly reveal the nature of feature variations. This paper proposes a new tolerance analysis method based on the screw and the Lie Algebra of Lie Group, which describes feature variation as the screw motion, and completely maps the twist, an element of the Lie Algebra, to the Lie Group that represents the feature configuration space. Thus, the analysis can be conducted in a more succinct and direct way. In the end, the method is applied in an example and proven to be robust and effective.

1. Introduction

In the product design phase, the geometric variations of a feature relative to its nominal size and state in the tolerance zone are known as errors. The introduction of GD&T specification made it possible to analyze feature variations in the three-dimensional space. Therefore, tolerance analysis should consider transmission and accumulation of errors in three-dimensional space.

Main models of tolerance analysis include the Matrix Method proposed by Desrochers et al.[1] in 1997; the State Transition Method put forward by Mantripragada et al.[2] in 1999; and the Jacobian Torsor Method presented by Desrochers et al.[3] in 2003. The above models all adopt Homogeneous Transformation Matrix (HTM) methods or methods derived from HTM[1][2, 3] to express the variation and transmission of features, which means that they all describe coordinate systems in a macroscopic way. However, such models lack microscopic analysis which is necessary for exploring the nature of feature variations. Moreover, the direction cosine matrix of HTM shows deficiency in describing special configuration of an object, for it can hardly express the position and orientation changes in a direct way. Therefore, this paper puts forward a tolerance analysis method based on the screw and the Lie Algebra of Lie Group, which describes Chasles' motion using the matrix exponential of the Lie Algebra, and further describes feature variations in the tolerance zone using Chasles' motion. What's more, instead of establishing sophisticated local coordinate systems, this method can express position and orientation variations of the functional feature simply by setting up the twist axis of each tolerance feature. Thus, the solution process is more succinct and direct.
2. Method based on the Screw and the Lie Algebra of Lie Group

2.1. Introduction to the method
The Lie Group is one of the cores of modern geometry, and the Lie Algebra is an algebra tool used for study of the Lie Group, differentiable manifolds and so on. There is a correspondence of the Lie Group and the Lie Algebra[4]. According to ASME Y14.5 Standards, the tolerance zone exerts constraints on the variations of tolerated features which form the Lie subgroup. Using the geometric algebraic theory, configuration variations of a rigid body in the space can be visually described, on which some scholars have already conducted studies. In 1999, Gou et al.[5] found that a tolerated feature demonstrates a symmetry subgroup under the action of the Lie Group, and the configuration space of a tolerance feature can be considered as the homogeneous space of the Lie Group. With the help of exponential mapping, mathematic models can be obtained for verification of form, profile and orientation tolerances. In 2000, Gou et al.[6] further pointed out that the establishment of datum reference frame and the tolerance verification in the position tolerance is similar to the sequential location problem in multi-featured workpieces, and a datum feature also exhibits a symmetry subgroup under the action of the Lie Group.

The above studies have proven the practicability of the Lie Algebra of Lie Group, but they mainly focused on the tolerance verification, instead of proposing a complete model for tolerance representation and analysis. Based on the outcomes of previous studies, this paper further introduces the screw theory[4, 7] in which the exponential matrix is surjective, and it possesses merits such as clear geometrical concepts, neat expression form and convenient algebraic operation.

2.2. Formula derivation
As is stated by the Chasles’ theorem[4], any rigid body movement can be achieved by screw motion which is a comprehensive one including rotating around and translating along an axis. The rigid body movement is equivalent to the screw motion. The infinitely small quantity of screw motion, twist $\hat{\xi}$, is also an element of the Lie Algebra. This concept combines the Lie Algebra of Lie Group, the screw and the screw motion together. As is shown in figure 1., in the screw motion, $\omega \in \mathbb{R}^3$ is the unit vector on the direction of the revolution axis, $\theta \in \mathbb{R}$ is the rotation angle around the revolution axis, and the $\hat{\xi}$ expression is manifested in figure 1.

$$\hat{\xi} = (v, \hat{\omega}) = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$$ (1)

Where: $\hat{\omega}$ is the skew-symmetric matrix of the direction vector $\omega$;
$v = -\omega \times q + h \omega$, $q \in \mathbb{R}^3$ is the coordinate of any point on the axis in the global coordinate system;
$h = d / \theta$, $d$ stands for the translation amount along the twist axis. The twist represents the pure rotational motion when $h=0$, the pure translational motion when $h=\infty$, and a mix of both rotational and translational motion when $h$ is a finite value.

So the screw motion representation determined by the twist $\hat{\xi}$ is
Therefore, the rigid body movement can be described by mapping the elements in $\hat{\xi} \in se(3)$ to $g \in SE(3)$. Since the exponential mapping is surjective[7], any rigid body movement can be represented as an exponent of a certain twist. When there is more than one rigid body, the rigid body movements of each kinematic pair can be combined, and such combination of movements is expressed by the mapping $g_o$:

$$g_o(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_n\theta_n} g_o(0)$$

(3)

$g_o(0)$ and $g_o(\theta)$ respectively stand for the initial configuration and the instantaneous configuration of coordinate system $T$ relative to the datum coordinate system $S$.

Therefore, the screw can express any rigid body movements as Chasles’ motion, while the exponential matrix can express them as screw matrix operator. Detailed derivation process is shown in the References[7].

3. Tolerance Representation

Tolerance representation is the first step of tolerancing. This paper takes the plane feature shown in figure 2.as an example to illustrate the tolerance representation of the screw. In figure 2., the red chain lines show the ideal position of the plane feature. In order to describe the screw motion, the reference coordinate system and revolution axis should be determined in the first place. Although the position of the reference coordinate system and the configuration of the revolution axis can be selected arbitrarily, the most convenient way is make the direction of the axis be in line with that of the reference coordinate system according to degree of freedom (DOF) of the features, as the revolution axes $\hat{\xi}_1, \hat{\xi}_2$ and $\hat{\xi}_3$ of the plane feature in figure 2. The DOF of a feature is basically determined as $(6-n)$ according to its degree of invariance $n[1]$. The number of DOF is influenced by the structure of the tolerance zone as well as feature, mating and datum types[8].

Features with multi degree of freedom can be seen as combinations of revolute pair and prismatic pair. Take the plane feature in figure 2. again as an example. Its three DOF in the tolerance zone include two revolute DOF and one translational DOF. Hence, the three twists, $\hat{\xi}_1, \hat{\xi}_2$ and $\hat{\xi}_3$, can be established, and the mapping of the plane feature can be expressed as:

$$g = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} = e^{\sum \hat{\xi}_i\theta_i}$$

(4)

Where: twists $\hat{\xi}_i$ and $\hat{\xi}_2$ are pure revolute variations whereas $\hat{\xi}_3$ is pure translational variation. $\theta_1$ and $\theta_2$ stand for rotation angles, and $\theta_3$ is the translation amount. Through changing the value of the three parameters, the plane feature can traverse the whole tolerance zone.

What's more, along the DOF, feature variations are also constrained by the boundaries of the tolerance zone. Therefore, feature variations in the tolerance zone can be expressed as products of exponentials (POE) of the twist, through which the complete geometric representation and parameterization of the feature variations can be realized. The tolerances of another six types of features in Reference[1]can be obtained through the similar way as well.

4. Tolerance Analysis

Tolerance analysis is in nature a repositioning of nominal features with errors being considered, and a description of various feature positions and orientations in the tolerance zone. Since the Lie Algebra of Lie Group involves the concept of set of variations, the method based on it is quite suitable for tolerance analysis.

The new method uses exponential mapping to represent the configuration of each feature, and expresses the variation of each feature in the tolerance zone as the exponent of its corresponding twist.
Hence, each feature can be mapped from its nominal position to the position with errors. When the feature $j$ has more than one DOF, the matrix exponential of the screw should be determined based on the DOF of the feature and the screw type.

$$ g_j(\theta) = e^{\sum_{i=1}^{n} \xi_i^j \theta^i} $$

(5)

Hence, mapping of the target feature configuration along the whole dimension chain can be expressed as

$$ g_{\alpha}(\theta) = e^{\sum_{i=1}^{n} \xi_i^\alpha \theta^i} \cdots e^{\sum_{i=1}^{n} \xi_i^\alpha \theta^i} \cdots e^{\sum_{i=1}^{n} \xi_i^\alpha \theta^i} : g_{\alpha}(0) = \prod_{j=1}^{n} e^{\sum_{i=1}^{n} \xi_i^j \theta^i} : g_{\alpha}(0) $$

(6)

Where: $g_{\alpha}(0)$ is the nominal configuration of the target feature when every feature is in the ideal position; $j$ stands for the sequence number of variable features along the dimension chain from the datum reference system; and $i$ is the number of DOF of the $j_{th}$ feature. Therefore, the entire tolerance analysis based on the screw and Lie Algebra of Lie Group can be realized.

5. Numerical Examples
The centering pin[9]is taken as an example to prove practicability of the new tolerance analysis method. As is shown in figure 3., the assembly is comprised of(a)Base,(b)Pin and(c)Block (see figure 4. for their detailed sizes and tolerance requirements). Based on the tolerance specification in Table 1., the functional requirement (FR) refers to the variable from the Pin point Q to the upper surface of the Base.

| Tolerance specifications | $t_a$ | $t_b$ | $t_c$ | $t_d$ | $t_e$ | $t_f$ | $t_g$ |
|--------------------------|------|------|------|------|------|------|------|
| Tolerance Value          | 0.2  | 0.1  | 0.2  | 0.1  | H11:0.00/0.13 | H8:-0.033/0.000 | 0.1  |
As is shown in figure 3. (b), the screw based on the reference coordinate system should be first established. Then the screws $\xi_1$, $\xi_2$ and $\xi_3$ of the tolerance feature N, $\xi_4$ and $\xi_5$ of H, as well as $\xi_6$ and $\xi_7$ of M should be set up, respectively. The screws $\xi_8$ and $\xi_9$ of the target feature Q should also established. Where: the two screws, $\xi_1$ and $\xi_2$, represent the pure rotational motion, and their corresponding exponential matrix is

$$e^{\dot{\xi}_i} = \begin{bmatrix} e^{\hat{\omega}_i} & (I - e^{\hat{\omega}_i})q \end{bmatrix}$$

(7)

Where: $i=1,2$.

The screw $\xi_3$ represents the pure translational motion, and its corresponding exponential matrix is

$$e^{\dot{\xi}_i} = \begin{bmatrix} I & \theta_i v_i \\ 0 & 1 \end{bmatrix}$$

(8)

Where: $i=3$.

The screws $\xi_4$, $\xi_5$, $\xi_6$, $\xi_7$, $\xi_8$, and $\xi_9$ contain both rotational and translational motions, and their corresponding exponential matrix is

$$e^{\dot{\xi}_i} = \begin{bmatrix} e^{\hat{\omega}_i} & (I - e^{\hat{\omega}_i})q_i + h_i \theta_i \omega_i \end{bmatrix}$$

(9)

Where: $h_i = \frac{d}{\theta_i}$ $i=4,5,6,7,8,9$.

Then the coordinate of the point $q_i$ on the direction vectors $\omega_i$ and revolute axis of each screw can be obtained, and the coordinate of the screw can further be acquired.

At last, through multiplying those exponential matrices, the ultimate mapping matrix $g_{st}$ can be obtained. Since the homogeneous coordinate of the Pin point in the reference coordinate system is
known as $p(\theta) = [0 \ 40 \ 45 \ 1]^T$, through multiplying it by $g_{st}$ using the formula (6), we can acquire the following expression:

$$p(\theta) = g_{st}(\theta)p(0) \quad (10)$$

Hence, the expression containing variation parameters $\theta_i$ and $d_i$ can be obtained. Taking into consideration the inequality constraint provided by the tolerance zone, and using the genetic algorithm toolbox of MATLAB software, we can get the variation range of the target feature as $(-0.579433, 0.647757)$

6. Conclusion
The new tolerance analysis method proposed in this paper demonstrates its merits in the following three aspects: first, this method has clear geometric concepts, defined physical significance, and simple expression form. It is also easy for describing the spatial movement of a rigid body from the perspective of geometry. Revolution axes rather than local coordinate systems can be constructed according to different types of features, with each screw having at most two parameters; second, this method has convenient and efficient algebraic operation. It describes the rigid body movement in global coordinate systems, avoiding the singularity that may be brought about by the local coordinate systems; third, this method has a wide scope of application. It can easily establish relationships with vector, matrix, and influence coefficient methods. To sum up, this new method can be well applied to tolerance analysis.

7. References
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