Ising Model and $L$ – Function

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Abstract. The correlation functions are calculated for the two dimensional Ising model with free boundary conditions and the two dimensional Ising model with periodic boundary conditions.

1 Introduction

Onsager and Kaufman [1] – [3] invented the formula for the partition function of the two dimensional Ising model. Another method for the calculation of the partition function was proposed by Kac and Ward [4]. They considered simultaneously two formulae: the determinant of the special matrix $I + T$ ($I$ is the identity matrix) is proportional to the partition function of the Ising model and it is proportional to the square of the partition function. For the proof of the first formula they used a topological statement. Sherman [5], [6] constructed a counter – example for this statement. Hurst and Green [7] proposed to use for the calculation of the Ising model partition function not a determinant but a Pfaffian of some special matrix. This method was improved by Kasteleyn [8], Fisher [9], McCoy and Wu [10]. McCoy and Wu [10] obtained the wrong formula connecting the Pfaffians with the Ising model partition function. The proper formula of this type is obtained in the paper [11]. Sherman [8], [10] gave some arguments for the equality

$$Z^2 = C(\beta) \det(I + T)$$  \hspace{1cm} (1.1)

where $Z$ is the partition function of the two dimensional Ising model with the free boundary conditions and $C(\beta)$ is the positive function of the inverse temperature $\beta$. In the paper [12] the following formula

$$Z^2 = C(\beta) \det(I - T)$$  \hspace{1cm} (1.2)

is proved. For the rectangular lattice the expression (1.2) is independent of the sign of the matrix $T$. For an arbitrary lattice the formula (1.1) is wrong. If the matrices $T^k$ satisfy some estimate, then

$$\det(I - T) = \exp\{-\sum_{k=1}^{\infty} k^{-1} \text{tr} T^k\}.$$  \hspace{1cm} (1.3)

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By the definition the partition function $Z > 0$. If the numbers $\text{tr} \, T^k$ are real, then the equalities (1.2) and (1.3) imply

$$Z(C(\beta))^{-1/2} = \exp\{-1/2 \sum_{k=1}^{\infty} k^{-1} \text{tr} \, T^k\}. \quad (1.4)$$

The bulk of Sherman papers [5], [6] and the subsequent Burgoyne paper [13] were devoted to the proof of Feynman conjecture. Due to Feynman conjecture the Ising model partition function is proportional to some infinite formal product. The right hand side of the equality (1.3) is the product. By calculating the numbers $\text{tr} \, T^k$ it is possible to show that the equality (1.4) is a correct statement for Feynman conjecture.

Hashimoto studied some special infinite product. He called these products zeta functions of finite graphs in the paper [14] and $L$– functions of finite graphs in the paper [15]. It is possible to prove that the right hand side of the equality (1.3) is one over the Hashimoto $L$– function (zeta function). The definition (1.3) seems suitable since the series in (1.3) is convergent if the numbers $\text{tr} \, T^k$ satisfy some estimate.

The paper [4]–[6], [12], [13] used the well–known van der Waerden formula [16] for the Ising model partition function. Similar formula for the correlation functions was obtained in the paper [17]. By using these formulae and the formula (1.4) we calculate the thermodynamic limit of the free energy and the correlation functions of the two dimensional Ising model with free boundary conditions. Similar results are obtained for periodic boundary conditions.

In the second section we discuss Hashimoto results [14], [17]. The third section is devoted to formula [17] for the correlation functions. In the fourth section we study the two dimensional Ising model with free boundary conditions. The fifth section is devoted to the two dimensional Ising model with periodic boundary conditions.

## 2 $L$– Function

Let $s$ be a complex number. Then for $\Re s > 1$ Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (2.1)$$

is an analytic function of the variable $s$. Euler showed that for $\Re s > 1$ ([18], formula 17.7.2)

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (2.2)$$

where the product is extended over the set of all prime numbers $p = 2, 3, 5, 7,\ldots$.

Let $n > 1$ be a fixed natural number and let $m$ be any natural number. Let us consider the functions $\chi(m)$ such that

$\chi(m) = \chi(m')$, if $m = m' \mod n$,

$\chi(1) = 1$,

$\chi(m) = 0$, if the greatest common divisor $(m, n)$ of the natural numbers $m$ and $n$ is not one,

$$\chi(m)\chi(m') = \chi(mm'). \quad (2.3)$$

Such function $\chi(m)$ is called a modulo $n$ character.
Let \( n > 1 \) be a fixed natural number and let \( \chi \) be a modulo \( n \) character. Then the series

\[
L(s, \chi) = \sum_{m=1}^{\infty} \chi(m) m^{-s}
\]  

(2.4)

for \( \Re s > 1 \) is called the \( L \) – series. The \( L \) – series was introduced by Dirichlet. The \( L \) – series and Riemann zeta function have many common properties. The Euler product (2.2) analogue is ([13], formula 17.8.5)

\[
L(s, \chi) = \prod_{p} \left( 1 - \chi(p)p^{-s} \right)^{-1}
\]  

(2.5)

where \( \Re s > 1 \) and the product is extended over the set of all prime numbers.

A function \( \chi(m) \) of the natural number \( m \) satisfying the condition (2.3) only is called a character. The function \( \chi(m, s) = m^{-s} \) gives an example of a character. Let us consider the series (2.4) with the character \( \chi(m) \). When the character \( \chi(m) \equiv 1 \) the series (2.4) coincides with Riemann zeta function (2.1). When the character \( \chi(m) \) is a modulo \( n \) character the series (2.4) is Dirichlet \( L \) – series.

Let us introduce \( L \) – function of the finite graph. Let \( X \) be a finite graph and let \( e \) be an oriented edge of a graph \( X \). The oriented edge \( e \) is defined by the pair of the vertices of a graph \( X \): the beginning \( b(e) \) and the end \( f(e) \) of the oriented edge \( e \). A closed path is a sequence of the oriented edges \( C = (e_1, \ldots, e_k) \) such that

\[
b(e_{i+1}) = f(e_i), \quad i = 1, \ldots, k-1, \quad b(e_1) = f(e_k).
\]  

(2.6)

Let us denote \( b(C) = b(e_1) \) and \( f(C) = f(e_k) \). For a closed path \( b(C) = f(C) \). Let \( e^{-1} \) be such oriented edge that \( b(e^{-1}) = f(e) \), \( f(e^{-1}) = b(e) \). The following pairs of the closed paths are regarded homotopic equivalent

\[
(e, e^{-1}) \sim C(b(e))
\]  

(2.7)

where the path \( C(b(e)) \) consists of the only vertex \( b(e) \):

\[
(e_1, \ldots, e_{i-1}, e_i, e_i^{-1}, e_{i+2}, \ldots, e_k) \sim (e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_k)
\]  

(2.8)

where \( i = 2, \ldots, k-2, \ k > 3; \)

\[
(e_1, e_1^{-1}, e_3, \ldots, e_k) \sim (e_3, \ldots, e_k);
\]  

(2.9)

\[
(e_1, \ldots, e_{k-2}, e_{k-1}, e_{k-1}^{-1}) \sim (e_1, \ldots, e_{k-2});
\]  

(2.10)

\[
(e_1, e_2, \ldots, e_{k-1}, e_{k-1}^{-1}) \sim (e_2, \ldots, e_{k-1}).
\]  

(2.11)

Two closed paths are regarded homotopic if they are related by the equivalence relation generated by the relations (2.7) – (2.11).

**Lemma 2.1.** The closed paths \( (e_k, e_1, \ldots, e_{k-1}) \) and \( (e_1, \ldots, e_k) \) are homotopic.

**Proof.** The equivalence relation (2.11) implies

\[
(e_k, e_1, \ldots, e_k, e_k^{-1}) \sim (e_1, \ldots, e_k).
\]  

(2.12)

The equivalence relation (2.10) implies

\[
(e_k, e_1, \ldots, e_k, e_k^{-1}) \sim (e_k, e_1, \ldots, e_{k-1}).
\]  

(2.13)
It follows from the equivalence relations (2.12) and (2.13) that the closed paths \((e_1, ..., e_k)\) and \((e_k, e_1, ..., e_k)\) are homotopic. The lemma is proved.

The equivalence relations (2.7) - (2.11) imply that every closed path is homotopic to a path consisting of the only vertex or is homotopic to a reduced closed path \((e_1, ..., e_k)\):

\[
b(e_{i+1}) = f(e_i), \quad f(e_{i+1}) \neq b(e_i), \quad i = 1, ..., k - 1, \quad b(e_1) = f(e_k), \quad f(e_1) \neq b(e_k). \quad (2.14)
\]

In view of Lemma 2.1 \(k\) reduced closed paths \((e_1, ..., e_k), (e_k, e_1, ..., e_k), ..., (e_2, ..., e_k, e_1)\) are homotopic to each other. They define the equivalence class called the oriented reduced cycle.

We denote it by \([e_1, ..., e_k]\). An oriented reduced cycle, or a reduced closed path \((e_1, ..., e_k)\) representing it, is called non-primitive, if there exists a positive integer \(l\) (\(1 \leq l < k\)) such that \((e_1, ..., e_k) = (e_{l+1}, ..., e_k, e_1, ..., e_l)\); and otherwise it is called primitive.

If two closed paths \(C_1 = (e_1, ..., e_k), C_2 = (e_{k+1}, ..., e_{k+l})\) have the same beginning vertex: \(b(e_1) = b(e_{k+1})\), it is possible to define the product \(C_1 \cdot C_2 = (e_1, ..., e_k, e_{k+1}, ..., e_{k+l})\). If \(b(e_1) = b(e_{k+1})\) and \(e_{k+1} \neq e_1\), then the product \(C_1 \cdot C_3 = (e_1, ..., e_k, e_{k+1}, ..., e_1, e_{k+1})\) and \(C_3 = (e_{k+1}, ..., e_1)\) of two reduced closed paths \(C_1 = (e_1, ..., e_k)\) and \(C_3 = (e_{k+1}, ..., e_1)\) is not a reduced closed path. The product \(C_1 \cdot C_2 = (e_1, ..., e_{k+l})\) of two reduced closed paths \(C_1 = (e_1, ..., e_k)\) and \(C_2 = (e_{k+1}, ..., e_{k+l})\) with the same beginning edge: \(e_1 = e_{k+1}\) is the reduced closed path.

Let a unitary matrix \(\rho(C)\) in \(n\) dimensional space correspond with any reduced closed path \(C\) and satisfy the following conditions: if two reduced closed paths \(C_1 = (e_1, ..., e_k)\) and \(C_2 = (e_{k+1}, ..., e_{k+l})\) have the same beginning edge: \(e_1 = e_{k+1}\) then

\[
\rho(C_1 \cdot C_2) = \rho(C_1)\rho(C_2); \quad \rho(C_1 \cdot C_2) = \rho(C_1)\rho(C_2) \quad (2.15)
\]

there exists a unitary matrix \(\gamma\) for any reduced closed path \((e_1, ..., e_k)\) such that

\[
\rho((e_k, e_1, ..., e_{k-1})) = \gamma \rho((e_1, ..., e_k))\gamma^{-1}. \quad (2.16)
\]

**Lemma 2.2.** Let a unitary matrix \(\rho(e_1; e_2)\) correspond with any pair \(e_1, e_2\) of the oriented edges of the graph \(X\) such that \(f(e_1) = b(e_2), b(e_1) \neq f(e_2)\). For any reduced closed path \((e_1, ..., e_k)\) we define the unitary matrix

\[
\rho((e_1, ..., e_k)) = \rho(e_1; e_2)\rho(e_2; e_3)\cdots \rho(e_{k-1}; e_k)\rho(e_k; e_1). \quad (2.17)
\]

Then the matrix (2.17) satisfies the relations (2.13) and (2.16).

**Proof.** Let two reduced closed paths \(C_1 = (e_1, ..., e_k)\) and \(C_2 = (e_{k+1}, ..., e_{k+l})\) have the same beginning edge: \(e_1 = e_{k+1}\). Then the definition (2.17) implies the relation (2.13).

Let \((e_1, ..., e_k)\) be a reduced closed path. Then the definition (2.17) implies the relation (2.16) with the unitary matrix \(\gamma = \rho(e_k; e_1)\). The lemma is proved.

By a labelling on the set of non-oriented edges of the graph \(X\) we mean an assignment \(e \rightarrow u(e) = u(e) = u(e^{-1})\), where \(u(e)\) are independent variables for the different non-oriented edges. We denote them simply by \(u = \{u(e)\}\). We put

\[
u^C = \prod_{i=1}^{k} u(e_i) \quad (2.18)
\]

where \(C = (e_1, ..., e_k)\) is a reduced closed path. The \(L\) function of \(X\) attached to \(\rho\) is defined by the following formal infinite product similar to the product (2.5)

\[
L(u, \rho; X) = \prod_{[C] \text{: primitive}} \det(I_n - \rho(C)u^C)^{-1} \quad (2.19)
\]
where the product is extended over the set of primitive oriented reduced cycles \([C]\) and \(I_n\) is the identity matrix in the \(n\) dimensional space.

If the graph \(X\) is connected, then for any reduced closed path \(C\) there exists a homotopic closed path \(\langle C\rangle\) such that \(b(\langle C\rangle) = p\) where \(p\) is the fixed vertex. In view of the relation \((2.9)\) we may consider that the beginning edge of the closed path \(\langle C\rangle\) is the fixed oriented edge \(e\) and \(b(e) = p\). Hashimoto \([14, 15]\) considered the unitary representation \(\rho\) of the group of the classes of homotopically equivalent closed paths \(\langle C\rangle\) with the fixed initial vertex \(p\). If we substitute the matrix \(\rho(\langle C\rangle)\) instead of the matrix \(\rho(C)\) in the definition \((2.19)\), we obtain Hashimoto definition \([14]\). In the paper \([14]\) the function \(L(u, \rho; X)\) is denoted by \(Z_X(u; \rho)\) and it is called a zeta function.

The definition \((2.19)\) is formal. We transform it into another form. We denote by \(\alpha_i(C), i = 1, ..., n\), the eigenvalues of the unitary matrix \(\rho(C)\). Since the matrix \(\rho(C)\) is unitary,

\[
|\alpha_i(C)| = 1, i = 1, ..., n. \tag{2.20}
\]

Taking the logarithm of \((2.19)\) we have

\[
\ln L(u, \rho; X) = \sum_{[C] \text{ primitive}} \ln[\det(I_n - \rho(C)u^C)^{-1}] \tag{2.21}
\]

\[
\ln[\det(I_n - \rho(C)u^C)^{-1}] = -\sum_{i=1}^{n} \ln(1 - \alpha_i(C)u^C) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} k^{-1}(\alpha_i(C))^k(u^C)^k = \sum_{k=1}^{\infty} k^{-1}(\text{tr} \rho(C))^k(u^C)^k. \tag{2.22}
\]

The substitution of the equality \((2.22)\) into the equality \((2.21)\) gives

\[
\ln L(u, \rho; X) = \sum_{[C] \text{ primitive}} \sum_{k=1}^{\infty} k^{-1}(u^C)^k \text{tr} \rho(C)^k. \tag{2.23}
\]

Any primitive oriented reduced cycle \(([e_1, ..., e_l])\) consists of \(l\) different reduced closed paths: \((e_1, ... , e_l), (e_l, e_1, ..., e_{l-1}), ..., (e_2, ..., e_l, e_1)\). Due to the definition \((2.18)\)

\[
u^{(e_1, e_2, ..., e_{l-1})} = u^{(e_1, ..., e_l)} \tag{2.24}
\]

for the reduced closed paths \((e_i, e_1, ..., e_{l-1})\) and \((e_1, ..., e_l)\). It follows from the equalities \((2.16)\) and \((2.24)\) that the equality \((2.23)\) may be rewritten as

\[
\ln L(u, \rho; X) = \sum_{C \text{ primitive}} \sum_{k=1}^{\infty} (k|C|)^{-1}(u^C)^k \text{tr} \rho(C)^k \tag{2.25}
\]

where \(C\) runs over the set of primitive reduced closed paths \(C = (e_1, ..., e_l)\) and the length \(|C| = l\).

The definition \((2.18)\) implies

\[
(u^C)^k = u^{C^{\times k}}. \tag{2.26}
\]

The equalities \((2.13), (2.25), (2.26)\) and the equality \(|C^{\times k}| = k|C|\) imply

\[
\ln L(u, \rho; X) = \sum_{C \text{ primitive}} \sum_{k=1}^{\infty} |C^{\times k}|^{-1} u^{C^{\times k}} \text{tr} \rho(C^{\times k}). \tag{2.27}
\]
Any reduced closed path has the form $C^n k$ where $k$ is a natural number and $C$ is a primitive reduced closed path. Then the equality (2.27) may be rewritten as

$$\ln L(u, \rho; X) = \sum_{C \in RC(X)} |C|^{-1} u^C \tr \rho(C)$$

(2.28)

where $C$ runs over the set $RC(X)$ of all reduced closed paths on the graph $X$.

The number of the non-oriented edges which is incident to a vertex $p$ is called $v(p)$, the valency of $p$. Let

$$v = \max_p v(p).$$

(2.29)

Let us construct a reduced closed path with the initial vertex $p$. For the first edge we have $v(p)$ possibilities. For any other edge the number of possibilities is less than $v - 1$. Thus the total number of reduced closed paths of the length $l$ is less than $\#(V X) v(v - 1)^{l-1}$ where $\#(V X)$ is the total number of vertices of the graph $X$. The relations (2.20) imply that the series (2.28) is absolutely convergent if the following estimate is valid

$$\max_e |u(e)| < (v - 1)^{-1}.$$  

(2.30)

**Definition.** The $L$ – function of the finite graph $X$ attached to $\rho$ defined by

$$L(u, \rho; X) = \exp\{ \sum_{C \in RC(X)} |C|^{-1} u^C \tr \rho(C) \}$$

(2.31)

where $C$ runs over the set $RC(X)$ of all reduced closed paths on the graph $X$.

Let $\#(EX)$ be the total number of oriented edges of the graph $X$.

**Theorem 2.3.** Let unitary matrix $\rho(e_1; e_2) = \{(\rho(e_1; e_2))_{k_1, k_2}, k_1, k_2 = 1, ..., n\}$ correspond with any pair $e_1, e_2$ of oriented edges of the graph $X$, such that $f(e_1) = b(e_2), b(e_1) \neq f(e_2)$. Let us define $(n\#(EX)) \times (n\#(EX)) –$ matrix

$$T(u, \rho)(e_1, e_2) = \begin{cases} u(e_1)(\rho(e_1; e_2))_{k_1, k_2}, & f(e_1) = b(e_2), b(e_1) \neq f(e_2), \\ 0, & \text{otherwise}. \end{cases}$$

(2.32)

If the estimate (2.30) is fulfilled, then

$$L(u, \rho; X) = \det(I – T(u, \rho))^{-1}$$

(2.33)

where $L –$ function $L(u, \rho; X)$ is defined by the equality (2.31).

**Proof.** Analogously to the equality (2.22) we get

$$\ln[\det(I – T(u, \rho))^{-1}] = \sum_{k=1}^{\infty} k^{-1} \tr T(u, \rho)^k.$$  

(2.34)

Due to the definitions (2.17), (2.18) and (2.32) we have

$$\tr T(u, \rho)^k = \sum_{C \in RC(X), |C| = k} u^C \tr \rho(C)$$

(2.35)

where $C$ runs over the set of all reduced closed paths of the length $k$. The substitution of the equality (2.35) into the right hand side of the equality (2.34) gives

$$\ln[\det(I – T(u, \rho))^{-1}] = \sum_{C \in RC(X)} |C|^{-1} u^C \tr \rho(C)$$

(2.36)

where $C$ runs over the set $RC(X)$ of all reduced closed paths on the graph $X$. The equalities (2.31) and (2.36) imply the equality (2.33). The theorem is proved.
3 Ising Model

We consider a rectangular lattice on the plane formed by the points with integral Cartesian coordinates \( x = k_1, y = k_2, M'_1 \leq k_1 \leq M_1, M'_2 \leq k_2 \leq M_2 \), and the corresponding horizontal and vertical edges connecting these vertices. We denote this graph by \( G(M'_1, M'_2; M_1, M_2) \). Let a graph \( G \) be embedded in a rectangular lattice on the plane. Let all the vertices from the boundaries of all the edges of a graph \( G \) be included into the set of the vertices of a graph \( G \). The cell complex \( P(G) \) is called the set consisting of the cells (vertices, edges, faces). A vertex of \( P(G) \) is called a cell of dimension 0. It is denoted by \( s^0 \). An edge of \( P(G) \) is called a cell of dimension 1. It is denoted by \( s^1 \). A face of \( P(G) \) is called a cell of dimension 2. It is denoted by \( s^2 \). We suppose that \( P(G) \) contains all the faces whose all boundary edges are included into a graph \( G \). We denote by \( Z_2^{opp} \) the group of modulo 2 residuals. The modulo 2 residuals are multiplied by each other and the group \( Z_2^{opp} \) is a field.

The mapping \( \partial \) of the complex \( P(G) \) with the coefficients in the group \( Z_2^{opp} \) is a function on the \( p \) dimensional cells taking values in the group \( Z_2^{opp} \). Usually the cell orientation is considered and the cochains are the antisymmetric functions: \( c^p(-s^p) = -c^p(s^p) \). However, \( -1 = 1 \mod 2 \) and we can neglect the cell orientation for the coefficients in the group \( Z_2^{opp} \).

The cochains form an Abelian group

\[
(c^p + c^p)(s^p_i) = c^p(s^p_i) + c^p(s^p_i) \mod 2. \tag{3.2}
\]

It is denoted by \( C^p(P(G), Z_2^{opp}) \). The mapping

\[
\partial c^p(s^{p-1}_i) = \sum_j (s^j_i : s^{p-1}_i) c^p(s^p_j) \mod 2 \tag{3.3}
\]
defines the homomorphism of the group $C^p(P(G), \mathbb{Z}_2^{add})$ into the group $C^{p-1}(P(G), \mathbb{Z}_2^{add})$. It is called the boundary operator. The mapping

$$\partial^* c^p(s_i^{p+1}) = \sum_j (s_i^{p+1} : s_j^p)c^p(s_j^p) \mod 2 \quad (3.4)$$

defines the homomorphism of the group $C^p(P(G), \mathbb{Z}_2^{add})$ into the group $C^{p+1}(P(G), \mathbb{Z}_2^{add})$. It is called the coboundary operator. The condition (3.1) implies $\partial\partial = 0$, $\partial^*\partial^* = 0$. The kernel $Z_p(P(G), \mathbb{Z}_2^{add})$ of the homomorphism (3.3) on the group $C^p(P(G), \mathbb{Z}_2^{add})$ is called the group of cycles of the complex $P(G)$ with the coefficients in the group $\mathbb{Z}_2^{add}$. The image $B_p(P(G), \mathbb{Z}_2^{add})$ of the homomorphism (3.3) in the group $C^p(P(G), \mathbb{Z}_2^{add})$ is called the group of boundaries of the complex $P(G)$ with the coefficients in the group $\mathbb{Z}_2^{add}$. Since $\partial\partial = 0$, the group $B_p(P(G), \mathbb{Z}_2^{add})$ is the subgroup of the group $Z_p(P(G), \mathbb{Z}_2^{add})$. Analogously, for the coboundary operator $\partial^*$ the group of cocycles $Z^p(P(G), \mathbb{Z}_2^{add})$ and the group of coboundaries $B^p(P(G), \mathbb{Z}_2^{add})$ are defined.

It is possible to introduce the bilinear form on $C^p(P(G), \mathbb{Z}_2^{add})$:

$$\langle f^p, g^p \rangle = \sum_i f^p(s_i^p)g^p(s_i^p) \mod 2. \quad (3.5)$$

The definitions (3.3) and (3.4) imply

$$\begin{align*}
\langle f^p, \partial^* g^{p-1} \rangle &= \langle \partial f^p, g^{p-1} \rangle \\
\langle f^p, \partial g^{p+1} \rangle &= \langle \partial^* f^p, g^{p+1} \rangle.
\end{align*} \quad (3.6)$$

Let a cochain $\sigma \in C^0(P(G), \mathbb{Z}_2^{add})$. Let the energy be expressed in the form

$$H'(\partial^* \sigma) = \sum_{s_i^1 \in P(G)} h_i(\partial^* \sigma(s_i^1)) \quad (3.7)$$

where $h_i(\epsilon)$ is an arbitrary function on the group $\mathbb{Z}_2^{add}$.

$$h_i(\epsilon) = D_i - E_i(-1)^{i} \quad (3.8)$$

and the constants

$$\begin{align*}
D_i &= 1/2(h_i(1) + h_i(0)) \\
E_i &= 1/2(h_i(1) - h_i(0)).
\end{align*} \quad (3.9)$$

The substitution of the equality (3.8) into the equality (3.7) gives

$$H'(\partial^* \sigma) = \sum_{s_i^1 \in P(G)} D_i + H(\partial^* \sigma) \quad (3.10)$$

where the function

$$H(\partial^* \sigma) = -\sum_{s_i^1 \in P(G)} E_i(-1)^{i}\partial^* \sigma(s_i^1) \quad (3.11)$$

is called the energy for the Ising model with zero magnetic field. The number $E_i = E(s_i^1)$ is the interaction energy attached to the edge $s_i^1$. The edge $s_i^1$ is given by its initial vertex and by its direction. For example, the edges of a rectangular lattice on the plane may be horizontal or vertical. If the interaction energy $E_i = E(s_i^1)$ is independent of the initial vertex
of the edge \(s_1^1\), the Ising model is called homogeneous. If the interaction energy \(E_i = E(s_1^1)\) is independent of the direction of the edge \(s_1^1\), the Ising model is called isotropic.

The equality (3.10) implies

\[
Z'_G = \sum_{\sigma \in C^0(P(G),Z^2)} \exp\{-\beta H'(\partial^* \sigma)\} = Z_G \exp\{-\beta \sum_{s_1^1 \in P(G)} D_i\} \tag{3.12}
\]

where the function

\[
Z_G = \sum_{\sigma \in C^0(P(G),Z^2)} \exp\{-\beta H(\partial^* \sigma)\} \tag{3.13}
\]

is called the partition function of Ising model.

Let the cochain \(\chi \in C^0(P(G),Z^2)\) take the value 1 at the vertices \(x_1,\ldots,x_m\) and be equal to 0 at all other vertices of the graph \(G\). The correlation function at the vertices \(x_1,\ldots,x_m\) of the lattice \(G\) is the function

\[
W_G(\chi) = (Z'_G)^{-1} \sum_{\sigma \in C^0(P(G),Z^2)} (-1)^{\langle \chi,\sigma \rangle} \exp\{-\beta H'(\partial^* \sigma)\} = (Z_G)^{-1} \sum_{\sigma \in C^0(P(G),Z^2)} (-1)^{\langle \chi,\sigma \rangle} \exp\{-\beta H(\partial^* \sigma)\}. \tag{3.14}
\]

**Proposition 3.1.** The partition function of Ising model on the graph \(G\)

\[
Z_G = 2^{\#(V_G)} \left( \prod_{s_1^1 \in P(G)} \cosh \beta E(s_1^1) \right) \times \sum_{\xi \in Z_1(P(G),Z^2), s_1^1 \in P(G)} \prod_{s_1^1 \in P(G)} \left( \tanh \beta E(s_1^1) \right)^{1/2(1-(-1)^{d_1(s_1^1)})} \tag{3.15}
\]

where \(\#(V_G)\) is the total number of the vertices of the graph \(G\).

The correlation function of Ising model on the graph \(G\)

\[
W_G(\chi) = (Z_G)^{-1} 2^{\#(V_G)} \left( \prod_{s_1^1 \in P(G)} \cosh \beta E(s_1^1) \right) \times \sum_{\xi \in C^1(P(G),Z^2), s_1^1 \in P(G)} \prod_{s_1^1 \in P(G)} \left( \tanh \beta E(s_1^1) \right)^{1/2(1-(-1)^{d_1(s_1^1)})}. \tag{3.16}
\]

**Proof.** The definition (3.14) implies

\[
\exp\{-\beta H(\sigma^1)\} = \prod_{s_1^1 \in P(G)} \exp\{\beta E(s_1^1)(-1)^{\sigma^1(s_1^1)}\} \tag{3.17}
\]

where \(\sigma^1 \in C^1(P(G),Z^2)\). It is easy to verify that for \(\epsilon = 0, 1\)

\[
\exp\{\beta E(s_1^1)(-1)^{\epsilon}\} = (\cosh \beta E(s_1^1)) \sum_{\xi = 0, 1} (-1)^{\epsilon \xi} (\tanh \beta E(s_1^1))^{1/2(-1)^{\xi}}. \tag{3.18}
\]
The relations (3.17), (3.18) imply
\[
\exp\{-\beta H(\sigma^1)\} = (\prod_{s^1_i \in P(g)} \cosh \beta E(s^1_i)) \times \\
\sum_{\xi^1 \in C^1(P(G),Z_2^{add})} (-1)^{\xi^1,\sigma^1} \prod_{s^1_i \in P(g)} (\tanh \beta E(s^1_i))^{1/2(1-(-1)^{\xi^1(s^1_i)})}.
\]
(3.19)

The substitution of the equality (3.19) into the definition (3.13), the first relation (3.6) and the relation
\[
\sum_{\xi^1 \in Z_1(P(G),Z_2^{add})} u_{\xi^1,\sigma^1}(P_G,Z_{ad}) = \begin{cases} 2, & \epsilon = 0, \\ 0, & \epsilon = 1, \end{cases}
\]
(3.20)
give the equality (3.15). The substitution of the equality (3.19) into the definition (3.14), the first relation (3.6) and the relation (3.20) give the equality (3.16). The proposition is proved.

Here we used the definitions and the methods of the paper [17]. The equality (3.15) was proved for the first time in the paper [16]. The equality (3.16) for homogeneous isotropic Ising model was proved in the paper [19].

The equality (3.16) implies that the correlation function \( W_G(\chi) \) is not zero only for the cochains \( \chi \in B_0(P(G),Z_2^{add}) \). Therefore the cochain \( \chi \) takes the value 1 only at the ends of the broken lines. Any broken line has two ends. Hence the cochain \( \chi \) takes the value 1 at even number of vertices.

4 Free Boundary Conditions

We denote the one dimensional cell \( s^1_i \) as the non-oriented edge \( e_i \) corresponding with two oppositely oriented edges \( e_i \) and \( e_i^{-1} \). The interaction energy of Ising model is denoted by \( E(s^1_i) = E(e_i) = E(e_i^{-1}) \). The equality (3.15) may be rewritten in the form
\[
Z_G = 2^{#(V_G)} \left( \prod_{e \in P(G)} \cosh \beta E(e) \right) Z_{r,G}
\]
(4.1)

where
\[
Z_{r,G} = \sum_{\xi^1 \in Z_1(P(G),Z_2^{add})} u_{\xi^1},
\]
(4.2)
\[
u_{\xi^1} = \prod_{e \in P(G)} u(e)^{1/2(1-(-1)^{\xi^1(e)})},
\]
(4.3)
\[
u(e) = u(e) = u(e^{-1}) = \tanh \beta E(e).
\]
(4.4)

Let a graph \( G \) be embedded in a rectangular lattice \( \mathbf{Z} \times \mathbf{Z} \) on the plane. Let with any pair \( e_1, e_2 \) of the oriented edges of a graph \( G \) such that \( f(e_1) = b(e_2), b(e_1) \neq f(e_2) \) there correspond the number
\[
\rho(e_1; e_2) = \exp\{i/2(e_1, e_2)\}
\]
(4.5)

where \( (e_1, e_2) \) is the radian measure of the angle between the direction of the oriented edge \( e_1 \) and the direction of the oriented edge \( e_2 \). Due to the equalities (2.17) and (4.3) with any reduced closed path \( C \) on the graph \( G \) there corresponds the number \( \rho(C) = \exp\{i/2\phi(C)\} \).
where \( \phi(C) \) is the total angle through which the tangent vector of the path \( C \) turns along the path \( C \).

For a graph \( G \) embedded in the rectangular lattice \( \mathbb{Z}^2 \) on the plane the estimate \((2.30)\) has the following form
\[
|u(e)| = |\tanh \beta E(e)| < 1/3. \quad (4.6)
\]

**Theorem 4.1.** Let a finite graph \( G \) be embedded in the rectangular lattice \( \mathbb{Z}^2 \) on the plane and let the estimate \((4.6)\) be fulfilled. Then for the reduced partition function \((4.2)\)
\[
Z_{r,G} = \exp\{-1/2 \sum_{C \in RC(G)} |C|^{-1} u^C \rho(C)\} \quad (4.7)
\]
where \( C \) runs over the set \( RC(G) \) of all reduced closed paths on a graph \( G \) and the number \( u^C \) is defined by the relations \((2.36)\), \((4.8)\).

**Proof.** Due to the paper \([12]\) \((Z_{r,G})^2 = \det(I - T(u, \rho)) \quad (4.8)\)
where \((\#(EG)) \times (\#(EG))\) - matrix \( T(u, \rho) \) is defined by the equalities \((2.32)\), \((1.4)\) and \((4.5)\). In view of the definition the angle \( \phi(C) = 2\pi k \) where \( k \) is an integer. Hence the number \( \rho(C) = \exp\{i/2\phi(C)\} \) is real. Now the equality \((4.7)\) follows from the equalities \((2.34)\) and \((1.8)\). The theorem is proved.

For the homogeneous two dimensional Ising model the interaction energy \( E(e) \) does not depend on an initial vertex of an edge \( e \). Let us denote \( E_1 \) (\( E_2 \)) the interaction energy \( E(e) \) for horizontally (vertically) directed edges \( e \). The relations \((1.1)\) - \((1.4)\) imply for the homogeneous Ising model
\[
Z_{G(M'_1,M'_2;M_1,M_2)} = Z_{G(1,1;M_1-M'_1+1,M_2-M'_2+1)} \quad (4.9)
\]

**Theorem 4.2.** Let for the interaction energy of the homogeneous two dimensional Ising model the estimate \((1.6)\) be valid. Then for the partition function \((4.1)\) of the homogeneous Ising model on the rectangular lattice on the plane
\[
\lim_{M_i \to \infty \atop i=1,2} (M_1 M_2)^{-1} \ln Z_{G(1,1;M_1,M_2)} = \ln(2\cosh \beta E_1 \cosh \beta E_2) + 1/2(2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \ln[(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta_1 - 2z_2(1 - z_1^2) \cos \theta_2] \quad (4.10)
\]
where the variables \( z_i = \tanh \beta E_i, \ i = 1, 2, \)

**Proof.** It follows from the equalities \((4.1)\) and \((4.7)\) for the homogeneous Ising model on the rectangular lattice that
\[
\lim_{M_i \to \infty \atop i=1,2} (M_1 M_2)^{-1} \ln Z_{G(1,1;M_1,M_2)} = \ln(2\cosh \beta E_1 \cosh \beta E_2) - \lim_{M_i \to \infty \atop i=1,2} (2M_1 M_2)^{-1} \sum_{C \in RC(G)} |C|^{-1} u^C \rho(C) \quad (4.11)
\]
where \( C \) runs over the set \( RC(G) \) of all reduced closed paths on the graph \( G(1,1;M_1,M_2) \).

The total number of all reduced closed paths of the length \( l \) with the initial vertex \((0,0)\) on the lattice \( \mathbb{Z}^2 \) is less than \( 4 \cdot 3^{l-1} \). Due to definitions \((2.17)\), \((1.3)\) the number \( |\rho(C)| = 1 \).
Hence the estimate (1.6) implies the absolute convergence of the series in the right hand side of the equality (4.11) for finite $M_1, M_2$.

The interaction energy $E(e)$ does not depend on the initial vertex of the oriented edge $e$. Therefore due to definitions (2.17), (1.1), (2.1) and (1.4) the number $|C|^{-1}u^C\rho(C)$ does not depend on the initial vertex of the path $C$. Hence

$$\sum_{C \in RC(\tilde{G})} |C|^{-1}u^C\rho(C) = \sum_{C_0 \in RC_0(\mathbb{Z} \times 2)} N(C_0)|C_0|^{-1}u^{C_0}\rho(C_0) \quad (4.12)$$

where $C_0$ runs over the set $RC_0(\mathbb{Z} \times 2)$ of all reduced closed paths with the starting point at the vertex $(0,0)$ on the lattice $\mathbb{Z} \times 2$. The number $N(C_0)$ is the total number of shifted paths $C_0$ on the graph $G(1,1; M_1, M_2)$. If $|C_0| \leq M_1$, $|C_0| \leq M_2$, the following estimate is valid

$$(M_1 - |C_0|)(M_2 - |C_0|) \leq N(C_0) \leq M_1M_2. \quad (4.13)$$

The estimate (4.6) implies the absolute convergence of the series

$$\sum_{C_0 \in RC_0(\mathbb{Z} \times 2)} |C_0|^k u^{C_0}\rho(C_0) \quad (4.14)$$

for $k = 0, \pm 1$. Thus it follows from the equalities (4.11), (4.12) and estimates (4.13) that

$$\lim_{M_i \to \infty, i = 1, 2} (M_1M_2)^{-1}\ln Z_{G(1,1; M_1, M_2)} = \ln(2 \cosh \beta E_1 \cosh \beta E_2) - 1/2 \sum_{C_0 \in RC_0(\mathbb{Z} \times 2)} |C_0|^{-1}u^{C_0}\rho(C_0). \quad (4.15)$$

Let us rewrite the expression (1.17) in more traditional form. On the rectangular lattice $\tilde{G}(0,0; M_1, M_2)$ on the torus the number $u(e)$ is defined by the equality (4.4) and the number $\rho(e_1; e_2)$ is defined by the equality (4.5) taking into account the identification of the vertices and edges in the graph $\tilde{G}(0,0; M_1, M_2)$. The $(4M_1M_2) \times (4M_1M_2)$ - matrix $T(u, \rho)$ is defined by the relation (2.32). The equality (2.39) implies

$$(2M_1M_2)^{-1}\ln[\det(I - T(u, \rho))] = -(2M_1M_2)^{-1}\sum_{C \in RC(\tilde{G})} |C|^{-1}u^C\rho(C) \quad (4.16)$$

where $C$ runs over the set $RC(\tilde{G})$ of all reduced closed paths on the graph $\tilde{G}(0,0; M_1, M_2)$.

For the homogeneous Ising model the number $|C|^{-1}u^C\rho(C)$ does not depend on the starting point of the path $C$. The graph $\tilde{G}(0,0; M_1, M_2)$ is invariant under the shifts. Any reduced closed path $C$ is a shifted reduced closed path $C_0$ where a path $C_0$ has the starting point at the vertex $(0,0)$. Therefore the equality (4.16) implies

$$(2M_1M_2)^{-1}\ln[\det(I - T(u, \rho))] = -1/2 \sum_{C_0 \in RC_0(\tilde{G})} |C_0|^{-1}u^{C_0}\rho(C_0) \quad (4.17)$$

where $C_0$ runs over the set $RC_0(\tilde{G})$ of all reduced closed paths with the starting point at the vertex $(0,0)$ on the graph $\tilde{G}(0,0; M_1M_2)$. When $M_i \to \infty$, $i = 1, 2$, the terms in the series (4.17) corresponding with the long paths connecting the vertices $(0,k)$ and $(M_1, k)$ or the $(k,0)$ and $(k, M_2)$, etc tend to zero due to the estimate (1.6). Hence

$$\lim_{M_i \to \infty, i = 1, 2} (2M_1M_2)^{-1}\ln[\det(I - T(u, \rho))] = -1/2 \sum_{C_0 \in RC_0(\mathbb{Z} \times 2)} |C_0|^{-1}u^{C_0}\rho(C_0) \quad (4.18)$$
where \( C_0 \) runs over the set \( RC_0(\mathbb{Z}^2) \) of all reduced closed paths with the starting point at the vertex \((0, 0)\) on the lattice \( \mathbb{Z}^2 \).

Let us calculate the determinant of the matrix \( I - T(\mathbf{u}, \rho) \). The vertices of the graph \( \tilde{G}(0, 0; M_1, M_2) \) are defined by the vectors \( \mathbf{j} \in \mathbb{Z}^2 \), \( 1 \leq j_1 \leq M_1, 1 \leq j_2 \leq M_2 \). The oriented edge \( \mathbf{e} \) of the graph \( \tilde{G}(0, 0; M_1, M_2) \) is defined by its initial vertex \( b(\mathbf{e}) = \mathbf{j} \) and by its direction: the unit vector \( v \). The unit vector \( v \) is one of four vectors: \((\pm 1, 0)\), \((0, \pm 1)\). Thus an oriented edge \( \mathbf{e} \) is a pair \((\mathbf{j}, v)\).

Due to relations (4.22), (4.4) and (4.5)

\[
T(\mathbf{u}, \rho)_{(\mathbf{j}, v), (\mathbf{j}', v')} = u((\mathbf{j}, v))\delta_{j_1+v_1-j_1', M_1}z\delta_{j_2+v_2-j_2', M_2}z \times \\
(1 - \delta_{v_1+v_1', 0}\delta_{v_2+v_2', 0})\exp\left\{i/2\left(\mathbf{v}, \mathbf{v}'\right)\right\} 
\]  

(4.19)

where Kronecker symbol

\[
\delta_{j, M_k}z = M_k^{-1}\sum_{i=1}^{M_k}\exp\{i2\pi M_k^{-1}jl\}. 
\]  

(4.20)

The symbol (4.20) equals 1 if \( j = M_kl \) where \( l \) is an integer and it equals 0 if \( j \neq M_kl \).

For the homogeneous Ising model the number \( u((\mathbf{j}, v)) \) does not depend on the vector \( \mathbf{j} \). We denote it by \( u(v) \). The equalities (4.19) and (4.20) imply

\[
(I - T(\mathbf{u}, \rho))_{(\mathbf{j}, v), (\mathbf{j}', v')} = (CBC^{-1})_{(\mathbf{j}, v), (\mathbf{j}', v')} 
\]  

(4.21)

where the matrices

\[
C_{(\mathbf{j}, v), (\mathbf{j}', v')} = \delta_{v_1, v_1'}\delta_{v_2, v_2'}(M_1M_2)^{-1/2}\exp\{i2\pi(M_1^{-1}j_1 + M_2^{-1}j_2, j_2')\}, 
\]  

(4.22)

\[
B_{(\mathbf{j}, v), (\mathbf{j}', v')} = \delta_{j_1, j_1'}\delta_{j_2, j_2'}B(\mathbf{j} \mathbf{v}, v', v) \\
B(\mathbf{j} \mathbf{v}, v', v) = \delta_{v_1, v_1'}\delta_{v_2, v_2'} - \\
\exp\{i2\pi(M_1^{-1}v_1j_1 + M_2^{-1}v_2j_2)\}u(v)\exp\{i/2\left(\mathbf{v}, \mathbf{v}'\right)\}(1 - \delta_{v_1, -v_1'}\delta_{v_2, -v_2'}). 
\]  

(4.23)

The matrix \( B_{(\mathbf{j}, v), (\mathbf{j}', v')} \) is diagonal for the vectors \( \mathbf{j} \mathbf{j}' \). The second relation (1.23) defines \( 4 \times 4 \) - matrix \( B(\mathbf{j} \mathbf{v}, v', v) \) for any vector \( \mathbf{j} \in \mathbb{Z}^2 \), \( 1 \leq j_1 \leq M_1, 1 \leq j_2 \leq M_2 \). It follows from the relations (1.21) and (1.23) that

\[
\det(I - T(\mathbf{u}, \rho)) = \prod_{j_1=1}^{M_1} \prod_{j_2=1}^{M_2} \det B(\mathbf{j}). 
\]  

(4.24)

Due to relations (1.4) \( u((\pm 1, 0)) = \tanh \beta E_1 = z_1, u(0, \pm 1) = \tanh \beta E_2 = z_2 \). By using the definition (1.23) it is possible to calculate

\[
\det B(\mathbf{j}) = (1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2)\cos 2\pi M_1^{-1}j_1 - 2z_2(1 - z_1^2)\cos 2\pi M_2^{-1}j_2.  
\]  

(4.25)

The substitution of the equality (1.23) into the relation (1.24) yields

\[
\det(I - T(\mathbf{u}, \rho)) = \\
\prod_{j_1=1}^{M_1} \prod_{j_2=1}^{M_2} [(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2)\cos 2\pi M_1^{-1}j_1 - 2z_2(1 - z_1^2)\cos 2\pi M_2^{-1}j_2]. 
\]  

(4.26)
The equalities (4.13), (4.18) and (4.26) imply the equality (4.10). The theorem is proved.

Let a cochain $\xi^1 \in C^1(P(G), \mathbb{Z}_2^{add})$. The support $||\xi^1||$ is the set of all non-oriented edges of the graph $G$ on which a cochain $\xi^1$ takes the value 1. Let a cochain $\chi \in C_0(P(G), \mathbb{Z}_2^{add})$. The support $||\xi^1||$ is called $\chi$-connected if any connected component of the support $||\xi^1||$ contains the non-oriented edges incident to the vertices on which a cochain $\chi$ equals 1. Let $\partial( ||\xi^1||)$ be the set of all non-oriented edges incident to the vertices incident to the edges of the support $||\xi^1||$. By using the relations (4.1) – (4.4) we rewrite the correlation function (4.16) in the following form

$$W_G(\chi) = (Z_{r,G})^{-1} \sum_{\xi^1 \in C^1(P(G), \mathbb{Z}_2^{add}), \partial \xi^1 = \chi, \chi \text{-connected} ||\xi^1||} u^{\xi^1} Z_{r,G \setminus \partial( ||\xi^1||)}$$  \hspace{1cm} (4.27)

where the graph $G \setminus \partial( ||\xi^1||)$ is obtained by deleting all edges of the set $\partial( ||\xi^1||)$ from the graph $G$.

If $C$ is a closed path on the graph $G$, the support $||C||$ is the set of all non-oriented edges corresponding with the oriented edges from a closed path $C$.

Theorem 4.3. Let the estimate (4.6) be valid. Let the interaction energy $E(e)$ be non-negative. Let a cochain $\chi \in C^0(P(G), \mathbb{Z}_2^{add})$ be equal to 1 on the finite number of the vertices. Then for the correlation function (3.14) of the two dimensional Ising model with free boundary conditions

$$\lim_{M_k \to \infty, M'_k \to \infty, k=1,2} W_G(M'_1, M'_2, M_1, M_2)(\chi) = \sum_{\xi^1 \in C^1(P(G), \mathbb{Z}_2^{add}), \partial \xi^1 = \chi, \chi \text{-connected} ||\xi^1||} u^{\xi^1} \exp \{1/2 \sum_{C \in RC(\mathbb{Z}^2), ||\xi^1|| = ||C||} |C|^{-1} u^{C} \rho(C)\}$$ \hspace{1cm} (4.28)

where the number $u^{\xi^1}$ is defined by the relations (4.3), (4.4), the number $u^{C}$ is defined by the relations (2.14), (4.4) and the number $\rho(C)$ is defined by the relations (2.17), (4.3).

Proof. Let us consider the equality (4.27) for the graph $G = G(M'_1, M'_2, M_1, M_2)$. Theorem 4.1 implies

$$(Z_{r,G})^{-1} Z_{r,G \setminus \partial( ||\xi^1||)} = \exp \{1/2 \sum_{C \in RC(G), ||\xi^1|| = ||C|| \neq 0} |C|^{-1} u^{C} \rho(C)\}. \hspace{1cm} (4.29)$$

The total number of all reduced closed paths of the length $l$ with fixed initial vertex on the lattice $\mathbb{Z}^2$ is less than $4 \cdot 3^{l-1}$. Due to the estimate (4.6) the series (4.29) is absolutely convergent for $G \to \mathbb{Z}^2$. Thus every term of the sum (4.27) for the graph $G = G(M'_1, M'_2, M_1, M_2)$ converges to the term of the series (4.28) when $G \to \mathbb{Z}^2$. The equality (4.28) will be proved if the absolute convergence of the series (4.28) is proved when the estimate (4.6) is valid.

The correlation function (4.27) is not zero only for the cochains $\chi \in B_0(P(G), \mathbb{Z}_2^{add})$ taking the value 1 at the even number of vertices $m_1, ..., m_{2k}$ of the graph $G \subset \mathbb{Z}^2$. Let a cochain $\xi^1 \in C^1(P(G), \mathbb{Z}_2^{add})$ satisfy the condition $\partial \xi^1 = \chi$. Then the vertex $m_1$ is incident to one or three non-oriented edges on which a cochain $\xi^1$ takes the value 1. We take one such edge. It corresponds with the oriented edge $(m_1, v_1)$. If $m_1 + v_1 = m_{j_1}$, then we have constructed the path connecting the vertices $m_1$ and $m_{j_1}$. If $m_1 + v_1$ does not coincide with any vertices $m_2, ..., m_{2k}$, then due to the condition $\partial \xi^1 = \chi$ the vertex $m_1 + v_1$ is incident to two or four non-oriented edges on which a cochain $\xi^1$ takes the value 1. One of these non-oriented edges corresponds to the oriented edge $(m_1, v_1)$. We take another
edge. It corresponds with the oriented edge \((m_i + v_1, v_2)\), \(v_2 \neq -v_1\). By repeating this process we obtain the path \(P_i = ((m'_1, v_1), \ldots, (m'_{q_i}, v_{q_i}))\) where \(m'_1 = m_i, m'_{i+1} = m'_i + v_i, i = 1, \ldots, q_i - 1, m'_{q_i} = m_{q_i}\) and \(m_i\) is one of the vertices \(m_2, \ldots, m_{2k}\) on which the cochain \(\chi\) takes the value 1. Any non-oriented edge may correspond with only one oriented edge from the path \(P_i\). Let the cochain \(\xi^1[P] \in C^1(P(G), Z_2^{add})\) equal 1 on all non-oriented edges corresponding with the oriented edges from the path \(P_i\). It equals 0 on all other non-oriented edges from the graph \(G\). By construction \(\xi^1 = \xi^1[P] + \eta^1\) where the supports of the cochains \(\xi^1[P], \eta^1 \in C^1(P(G), Z_2^{add})\) do not intersect each other. By repeating this process we construct the paths \(P_1, \ldots, P_k\) connecting the vertices \(m_{i_1}, \ldots, m_{i_k}, 1 = i_1 < i_2 < \cdots < i_k\) with the vertices \(m_{j_1}, \ldots, m_{j_k}, i_l < j_l, l = 1, \ldots, k\). Any non-oriented edge may correspond with only one oriented edge from the paths \(P_l, l = 1, \ldots, k\). These paths correspond with the cochains \(\xi^1[P_l], l = 1, \ldots, k\), such that \(\xi^1 = \xi^1[P_1] + \cdots + \xi^1[P_k] + \eta^1\) where the supports of the cochains \(\xi^1[P_1], \ldots, \xi^1[P_k], \eta^1 \in C^1(P(G), Z_2^{add})\) do not intersect each other and \(\eta^1 \in Z_1(P(G), Z_2^{add})\). This decomposition is not unique in general. Therefore not an equality but the estimate is valid
\[
W_G(\chi) \leq (Z_{r,G})^{-1} \sum_{\{i_l,j_l\}} \sum_{l=1}^k (\prod_{l=1}^k u_{|P_l|})Z_{r,G\setminus\prod_{l=1}^k |P_l|} \tag{4.30}
\]
where \(\{i_l,j_l\}\) runs over the set of the subdivisions of the numbers 1, ..., 2k into k pairs: \(1 = i_1 < \cdots < i_k, i_l < j_l, l = 1, \ldots, k\), the paths \(P_l, l = 1, \ldots, k\), run over the set of the paths connecting the vertices \(m_{i_l}\) and \(m_{j_l}\), \(l = 1, \ldots, k\), any non-oriented edge may correspond with only one oriented edge from the paths \(P_l, l = 1, \ldots, k\) and the graph \(G \setminus (\prod_{l=1}^k |P_l|)\) is obtained from the graph \(G\) by deleting all edges from the supports \(|P_l|\), \(l = 1, \ldots, k\). Due to the definition \((4.4)\) the variable \(u(e)\) is non-negative when the interaction energy \(E(e)\) is non-negative. For the non-negative variables \(u(e)\) the definition \((4.2)\) implies the estimate
\[
(Z_{r,G})^{-1} Z_{r,G\setminus\prod_{l=1}^k |P_l|} \leq 1. \tag{4.31}
\]
It follows from the estimates \((4.30), (4.31)\) that
\[
W_G(\chi) \leq \sum_{\{i_l,j_l\}} \sum_{P_l} \prod_{l=1}^k u_{|P_l|} \tag{4.32}
\]
The total number of the reduced closed paths of the length \(l\) starting at the fixed vertex on the lattice \(Z^2\) is less than \(4 \cdot 3^{l-1}\). Hence the estimate \((4.4)\) implies that for \(G \rightarrow Z^2\) the sum \((4.32)\) converges to the absolutely convergent series. This series majorizes the series \((4.28)\). Therefore the series \((4.28)\) is absolutely convergent when the estimate \((4.4)\) is valid. The theorem is proved.

Let a cycle \(\xi^1 \in Z_1(P(G), Z_2^{add})\). By using the arguments of Theorem 4.3 we can construct the closed paths \(C_1, \ldots, C_m\) where the supports \(|C_1|, \ldots, |C_m|\) do not intersect each other and any non-oriented edge may correspond with only one oriented edge from the closed paths \(C_1, \ldots, C_m\). Any closed path on a rectangular lattice on the plane has an even number of the vertically directed edges and it has an even number of the horizontally directed edges. Let the interaction energy \(E(e)\) be non-negative for the vertically directed edges \(e\) and let it be non-positive for the horizontally directed edges \(e\). Due to the definitions \((4.3), (4.4)\) \(u^{e_i} \geq 0\). Hence the inequality \((4.31)\) is fulfilled in this case. Therefore it is possible
to prove the absolute convergence of the series (1.25). Thus Theorem 4.3 is valid when the interaction energy is non – negative for the vertically directed edges and it is non – positive for the horizontally directed edges. Theorem 4.3 is valid also for the case when the interaction energy is non – positive for all oriented edges, then Theorem 4.3 is also valid.

5 Periodic Boundary Conditions

Let us consider the rectangular lattice $\tilde{G}(M_1', M_2'; M_1, M_2)$ on the torus introduced in Section 3. The number $\rho(e_1; e_2)$ is given by the relation (1.7) taking into account the identification of the vertices and the edges in the graph $\tilde{G}(M_1', M_2'; M_1, M_2)$. With every reduced closed path $C$ on the graph $\tilde{G}(M_1', M_2'; M_1, M_2)$ there corresponds the number $\rho(C) = \exp\{i/2\phi(C)\}$ defined by the equality (2.17). Here $\phi(C)$ is the total angle through which the tangent vector of the path $C$ turns along the path $C$.

If the reduced closed path $C$ lies on the rectangular lattice $\mathbb{Z}^2$ and the number $\rho(C)$ is defined by the relations (2.17), (4.5), then due to (20) the number $\rho(C) = (-1)^n(C)$ where $n(C)$ is the total number of the transversal self – intersections of the path $C$. The papers [3], [5], [12], [13] used this Whitney formula. Let us consider the line $C$ connecting the vertices $(M_1', k)$ and $(M_1, k)$ on the graph $\tilde{G}(M_1', M_2'; M_1, M_2)$. The line has no self – intersections. It is easy to see that $\rho(C) = 1$. Thus Whitney formula is wrong for a torus in general. Therefore we can not use the results of the papers [3], [5], [12], [13] for a torus.

Let us study the properties of the $L – \text{function}$ (2.31) for a graph $G$ lying on the graph $\tilde{G}(M_1', M_2'; M_1, M_2)$. Let for the reduced closed path $C = (e_1, ..., e_p, e_{p+1}, ..., e_{p+q})$ the vertices $b(e_{p+1}) = b(e_1)$. Then $C = C_1 \cdot C_2$ where the closed paths $C_1 = (e_1, ..., e_p)$ and $C_2 = (e_{p+1}, ..., e_{p+q})$ may be not reduced. Indeed, if $e_1 = e_p^{-1}$, then the closed path $C_1$ is not reduced. If $e_{p+1} = e_{p+q}^{-1}$, then the closed path $C_2$ is not reduced. By the definition a reduced closed path does not contain the oppositely oriented edges $e, e^{-1}$ if they are subsequent or if they are the first and the last edges of the path. A closed path is called completely reduced if it does not contained the oppositely oriented edges $e, e^{-1}$ at any places. The multipliers $C_1$ and $C_2$ of any completely reduced closed path $C = C_1 \cdot C_2$ are also completely reduced closed paths. The set of all completely reduced closed paths on the graph $G$ is denoted by $CRC(G)$.

**Theorem 5.1.** Let a graph $G$ be embedded in the rectangular lattice $\tilde{G}(M_1', M_2'; M_1, M_2)$ on the torus. Let with any reduced closed path $C$ on the graph $G$ there correspond the number $\rho(C)$ given by the relations (2.17), (4.5) and the number $u^C$ given by the relations (2.18), (4.4). If the estimate (1.6) is valid, then

$$\sum_{C \in CRC(G)} |C|^{-1} u^C \rho(C) = \sum_{C \in CRC(G)} |C|^{-1} u^C \rho(C). \quad (5.1)$$

In the left hand side of the equality (5.1) the sum extends over the set $RC(G)$ of all the reduced closed paths on the graph $G$ and in the right hand side of the equality (5.1) the sum extends over the set $CRC(G)$ of all the completely reduced closed paths on the graph $G$.

**Proof.** Let the reduced closed path $C = (e_1, ..., e_p, e, e_{p+1}, ..., e_{p+q}, e^{-1}, e_{p+q+1}, ..., e_{p+q+r})$ contain the oppositely oriented edges $e$ and $e^{-1}$. Then the closed path $C' = (e_1, ..., e_p, e, e_{p+q}^{-1}, ..., e_{p+1}^{-1}, e_{p+q+1}^{-1}, ..., e_{p+q+r}^{-1})$ is also reduced.
The path length definition and the definitions (2.18), (4.4) imply

\[ |C| = |C'|, \quad u^C = u^{C'} . \]  

(5.2)

By using the definitions (2.17), (4.5) we get

\[ \phi(C) = \phi_1 + \phi_2 + \phi_3 , \]  

(5.3)

\[ \phi_1 = \sum_{i=1}^{p-1} (e_i, \hat{e}_{i+1}) + (e_p, e) , \]  

(5.4)

\[ \phi_2 = (e, \hat{e}_{p+1}) + \sum_{i=p+1}^{p+q-1} (e_i, \hat{e}_{i+1}) + (e_{p+q}, e^{-1}) , \]  

(5.5)

\[ \phi_3 = (e^{-1}, \hat{e}_{p+q+1}) + \sum_{i=p+q+1}^{p+q+r-1} (e_i, \hat{e}_{i+1}) + (e_{p+q+r}, e_1) . \]  

(5.6)

It is easy to verify the relation

\[ (e_1, e_2) = -(e_2^{-1}, e_1^{-1}) \]  

(5.7)

for the oriented edges \( e_1, e_2 \) such that \( f(e_1) = b(e_2) \), \( b(e_1) \neq f(e_2) \). The definitions (2.17), (4.5) and the relations (5.7) imply

\[ \phi(C') = \phi_1 - \phi_2 + \phi_3 . \]  

(5.8)

Since the directions of the oriented edges \( e \) and \( e^{-1} \) are opposite, due to the definition (5.3) \( \phi_2 = (2k + 1)\pi \) where \( k \) is an integer. Hence \( \exp\{ -i/2\phi_2 \} = -\exp\{ i/2\phi_2 \} \) and in view of the relations (5.3), (5.8)

\[ \exp\{ i/2\phi(C) \} = -\exp\{ i/2\phi(C') \} . \]  

(5.9)

Due to the relations (5.2), (5.9) all terms \( |C|^{-1}u^C\rho(C) \) in the left hand side sum (5.1) corresponding with the reduced closed paths \( C \) containing the oppositely oriented edges \( e \) and \( e^{-1} \) cancel each other. The theorem is proved.

Theorem 5.1 is valid also for any graph \( G \) embedded in a rectangular lattice on the plane. Hence it is possible to change the summing over the reduced closed paths \( C \) containing the oppositely oriented edges \( e \) and \( e^{-1} \) along the path \( (e_1, ..., e_k) \). The number \( \rho((e_1, ..., e_k)) = \exp\{ i/2\phi((e_1, ..., e_k)) \} \) where \( \phi((e_1, ..., e_k)) \) is the total angle through which the tangent vector of the path \( (e_1, ..., e_k) \) turns along the path \( (e_1, ..., e_k) \). Therefore \( \phi((e_1, ..., e_k)) = 2\pi m \) where \( m \) is an integer. Hence \( (\rho((e_1, ..., e_k)))^{-1} = \rho((e_1, ..., e_k)) \). The definitions (2.17), (4.5) and the relations (5.7) imply

\[ \rho((e_k^{-1}, e_{k-1}^{-1}, ..., e_1^{-1})) = \rho((e_1, ..., e_k)) . \]  

(5.11)
Due to the definitions (2.18), (4.4)

\[
\exp\{\frac{-1}{2} \sum_{C \in RC(G)} |C|^{-1} u^C \rho(C)\} = 1 + \sum_{k=1}^{\infty} \sum_{\{[C_i], i=1, \ldots, k\} \text{prime, disjoint}} (-1)^k \left( \prod_{i=1}^{k} u^{C_i} \right) \left( \prod_{i=1}^{k} \rho(C_i) \right)
\]  

(5.13)

for the reduced closed paths \((e_{1}^{-1}, e_{2}^{-1}, \ldots, e_{l}^{-1})\) and \((e_{1}, \ldots, e_{k})\). It follows from the equalities (2.24), (5.10) that the numbers \(|C|^{-1} u^C \rho(C)\) are equal to each other for \(k\) reduced closed paths: \((e_{1}, \ldots, e_{k})\), \((e_{k}, e_{1}, \ldots, e_{k-1}, \ldots)\), \((e_{2}, \ldots, e_{k}, e_{1})\) which form the oriented reduced cycle \([e_{1}, \ldots, e_{k}]\). If the closed path \((e_{1}, \ldots, e_{k})\) is completely reduced, the oriented cycle \([e_{1}, \ldots, e_{k}]\) is called completely reduced. The equalities (5.11), (5.12) imply that the numbers \(|C|^{-1} u^C \rho(C)\) are equal to each other for two oriented reduced cycles \([e_{1}, \ldots, e_{k}]\) and \([e_{k}, e_{1}, \ldots, e_{k-1}, \ldots]\). This pair of the oriented reduced cycles \([e_{1}, \ldots, e_{k}]\) and \([e_{k}, e_{1}, \ldots, e_{k-1}, \ldots]\) is called the non-oriented reduced cycle \([[e_{1}, \ldots, e_{k}]]\). If the closed path \((e_{1}, \ldots, e_{k})\) is completely reduced, the non-oriented cycle \([[e_{1}, \ldots, e_{k}]]\) is called completely reduced.

In other words by a non-oriented completely reduced cycle is meant a definite sequence of oriented edges. There are no the oppositely oriented edges \(e, e^{-1}\) in this sequence. Each succeeding edge starts at the vertex where the previous edge ended. The last edge must end at the vertex from which the first edge started. The direction in which the sequence of edges is traversed, and also the particular starting point are both immaterial. By a primitive non-oriented completely reduced cycle is meant one which can not be constructed by exactly repeating some subpath of \((e_{1}, \ldots, e_{k})\) two or more times.

A completely reduced closed path does not contain the oppositely oriented edges \(e, e^{-1}\). But it may contain the oriented edge \(e\) many times. Due to Lemma 2.1 such path is homotopic to the path \(C = (e, e_{1}, \ldots, e_{k}, e_{k+1}, \ldots, e_{k+l}) = C_{1} \cdot C_{2}\) where the closed paths \(C_{1} = (e, e_{1}, \ldots, e_{k})\) and \(C_{2} = (e_{k+1}, \ldots, e_{k+l})\) are completely reduced. In view of the equality (2.13) \(\rho(C) = \rho(C_{1})\rho(C_{2})\). A prime closed path is meant a completely reduced closed path which contains any of its oriented edge only once. A non-oriented completely reduced cycle is called a prime non-oriented cycle, if every its representative is a prime closed path. The prime non-oriented cycles \([[C_{1}]], \ldots, [[C_{k}]]\) are called disjoint if any its representatives \(C_{1}, \ldots, C_{k}\) have no common oriented edges.

**Theorem 5.2.** Let a graph \(G\) be embedded in the rectangular lattice \(\tilde{G}(M_{1}', M_{2}', M_{1}, M_{2})\) on the torus. Let with any reduced closed path \(C\) on a graph \(G\) there correspond the number \(\rho(C)\) given by the relations (2.17), (4.7) and the number \(u^{C}\) given by the relations (2.18), (4.4). If the estimate (4.0) is valid, then

\[
\exp\{\frac{-1}{2} \sum_{C \in RC(G)} |C|^{-1} u^{C} \rho(C)\} = 1 + \sum_{k=1}^{\infty} \sum_{\{[C_{i}], i=1, \ldots, k\} \text{prime, disjoint}} (-1)^k \left( \prod_{i=1}^{k} u^{C_{i}} \right) \left( \prod_{i=1}^{k} \rho(C_{i}) \right)
\]

(5.13)

where \(C\) runs over the set \(RC(G)\) of reduced closed paths on the graph \(G\), \([[C_{i}]], i = 1, \ldots, k\), run over the set of prime non-oriented cycles and the prime non-oriented cycles \([[C_{1}]], \ldots, [[C_{k}]]\) are disjoint.

**Proof.** Due to Theorem 5.1 it is possible to change the summing over the reduced closed paths on the graph \(G\) in the left hand side of the equality (5.13) for the summing over the completely reduced closed paths on the graph \(G\).

Every completely reduced closed path has the form \(C^{\times k}\) where \(k\) is an integer and \(C\) is a primitive completely reduced closed path. By using the equality (2.22) we get

\[
\exp\{\frac{-1}{2} \sum_{C \in RC(G)} |C|^{-1} u^{C} \rho(C)\} = \exp\{\frac{1}{2} \sum_{C \in RC(G), \text{primitive}} |C|^{-1} \ln(1 - u^{C} \rho(C))\}
\]

(5.14)
Let a closed path \((e_1, ..., e_k)\) be a primitive completely reduced one. Then \(k\) primitive completely reduced closed paths \((e_1, ..., e_k), (e_2, ..., e_k, e_1), ..., (e_k, e_1, ..., e_k-1)\) are different. They form a primitive oriented completely reduced cycle \([\langle e_1, ..., e_k \rangle]\). Due to the equalities (2.24), (5.10) the numbers are equal to each other for all representatives of this primitive oriented completely reduced cycle \([\langle e_1, ..., e_k \rangle]\). Hence

\[
\exp\left\{-\frac{1}{2} \sum_{C \in R \mathcal{C}(G)} |C|^{-1} u^C \rho(C) \right\} = \exp\left\{\frac{1}{2} \sum_{[C] \in \mathcal{R} \mathcal{C}(G), \text{primitive}} \ln(1 - u^C \rho(C)) \right\}. \tag{5.15}
\]

Due to the relations (5.11), (5.12) the numbers \(u^C \rho(C)\) are equal to each other for all representatives of the primitive non-oriented completely reduced cycle \([\langle e_1, ..., e_k \rangle]\). We change the summing over the set of primitive oriented completely reduced cycle in the right hand side of the equality \((5.15)\) for the summing over the set of primitive non-oriented completely reduced cycles. Thus the right hand side of the equality \((5.15)\) is the product of the multipliers \((1 - u^C \rho(C))\). The decomposition of this product into the series gives

\[
\exp\left\{-\frac{1}{2} \sum_{C \in R \mathcal{C}(G)} |C|^{-1} u^C \rho(C) \right\} =
\]

\[1 + \sum_{k=1}^{\infty} \sum_{i=1}^{k} (-1)^k \left( \prod_{i=1}^{k} u^{C_i} \right) \left( \prod_{i=1}^{k} \rho(C_i) \right) \tag{5.16}\]

where \([\langle C_i \rangle], i = 1, ..., k, \) run over the set of primitive non-oriented completely reduced cycles and the primitive non-oriented completely reduced cycles \([\langle C_1 \rangle], ..., [\langle C_k \rangle] \) differ from each other.

Let us choose an oriented edge \(e\). Any completely reduced closed path does not contain the oppositely oriented edges \(e\) and \(e^{-1}\) simultaneously. We choose the representatives \(C_i\) of the non-oriented completely reduced cycles \([\langle C_i \rangle]\) in the right hand side of the equality \((5.16)\) such that the paths \(C_i\) do not contain the oriented edge \(e^{-1}\). Thus the sum of all terms in the series \((5.16)\) which contains \(u(e)^n\) is proportional to the sum

\[
\sum_{k=1}^{n} \sum_{[C_i], C_i \in \mathcal{R} \mathcal{C}(G), \text{primitive, different}} (-1)^k \left( \prod_{i=1}^{k} u^{C_i} \right) \left( \prod_{i=1}^{k} \rho(C_i) \right) \tag{5.17}
\]

where the primitive completely reduced closed path \(C_i\) contains (it may be many times) the oriented edge \(e\). The different paths \(C_1, ..., C_k\) all together contain the oriented edge \(e\) exactly \(n\) times. Due to Lemma 2.1 we can choose such representatives \(C_i, ..., C_k\) of the primitive oriented completely reduced cycles \([\langle C_1 \rangle], ..., [\langle C_k \rangle]\) that all paths start with the oriented edge \(e\). Every such path \(C_j\) is the product \(C_j' \cdots C_{j_2}'\) where the completely reduced closed path \(C_{j_l}'\) starts with the oriented edge \(e\) and contains it exactly one time. All paths \(C_{j_l}'\) are primitive. Since the paths \(C_1, ..., C_k\) all together contain the oriented edge \(e\) exactly \(n\) times, they are decomposed into \(n\) paths \(C_{j_l}'\). We change the notation so that \(C_{j_l}' = C_{ij_l}'\) where the numbers \((i_{j_1}, ..., i_{j_{k}})\), \(j = 1, ..., k,\) give the subdivision of the numbers \(1, ..., n\) into \(k\) groups. Due to definitions (2.17), (4.5) the number \(\rho(C_j') = \rho(C_{ij_1}') \cdots \rho(C_{ij_{k}}')\). The definitions (2.18), (4.4) imply that \(u^{C_j} = u^{C_{ij_1}'}, ..., u^{C_{ij_{k}}}'\). Thus the sum \((5.17)\) has the following form

\[
\sum_{[C_j'], C_j' \in \mathcal{R} \mathcal{C}(G), \text{primitive}} \left( \prod_{i=1}^{n} u^{C_i'} \right) \left( \prod_{i=1}^{n} \rho(C_i') \right) \sum_{k=1}^{n} \sum_{(i_{j_1}, ..., i_{j_{k}}) \text{ primitive, different}} (-1)^k \tag{5.18}
\]
where the primitive completely reduced closed paths $C'_1, \ldots, C'_n$ start with the oriented edge $e$ and contain it exactly one time. The numbers $(i_{j1}, \ldots, i_{jq})$, $j = 1, \ldots, k$, give a subdivision of the numbers $1, \ldots, n$ into $k$ groups.

If the sum \((5.18)\) is equal to zero for $n > 1$, then the left hand sides of the equalities \((5.13)\) and \((5.16)\) coincide and the theorem is proved.

Let us consider first the case when all completely reduced closed paths $C'_1, \ldots, C'_n$ are different. With every oriented completely reduced cycle $[C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}]$ there corresponds the permutation $\pi[i_{j_1}, \ldots, i_{j_{q_j}}]: i_{j_l} \to i_{j_{l+1}}$, $l = 1, \ldots, q_j - 1$, $i_{j_{q_j}} \to i_{j_1}$. All the other numbers from $1, \ldots, n$ the permutation $\pi[i_{j_1}, \ldots, i_{j_{q_j}}]$ leaves invariant. Since all completely reduced closed paths $C'_1, \ldots, C'_n$ are different, any permutation of the multipliers in the product $C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}$ gives another completely reduced closed path. The permutation $\pi[i_{j_1}, \ldots, i_{j_{q_j}}]$ gives the completely reduced closed path $C'_{i_{j_2}} \cdots C'_{i_{j_{q_j}}} \cdot C'_{i_{j_1}}$. Due to Lemma 2.1 it is homotopic to the completely reduced closed path $C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}$.

But the permutations $\pi[i_{j_2}, \ldots, i_{j_{q_j}}, i_{j_1}]$ and $\pi[i_{j_1}, \ldots, i_{j_{q_j}}]$ coincide. Hence the correspondence between the oriented completely reduced cycles $[C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}]]$ and the permutations $\pi[i_{j_1}, \ldots, i_{j_{q_j}}]$ is one – to – one. If the groups of numbers $i_{j_1}, \ldots, i_{j_{q_j}}$ and $i_{j_1}, \ldots, i_{j_{q_j}}'$ do not intersect, then the permutations $\pi[i_{j_1}, \ldots, i_{j_{q_j}}]$ and $\pi[i_{j_1}, \ldots, i_{j_{q_j}}']$ commute with each other. Thus any subdivision of the numbers $1, \ldots, n$ into $k$ non-intersecting groups $i_{j_1}, \ldots, i_{j_{q_j}}$, $j = 1, \ldots, k$ corresponds with the set of $k$ oriented completely reduced cycles $[C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}]$, $j = 1, \ldots, k$, and with the permutation $\pi[i_{j_1}, \ldots, i_{j_{q_j}} \cdots i_{k_{q_k}}]$ of the numbers $1, \ldots, n$. Conversely, any permutation $\pi$ from the permutation group $S_n$ of the numbers $1, \ldots, n$ corresponds with a subdivision of the numbers $1, \ldots, n$ into the systems of transitivity of the permutation $\pi$: $(i_{j_1}, \pi(i_{j_1}), \ldots, \pi^{q_j-1}(i_{j_1}))$ where $\pi^{q_j}(i_{j_1}) = i_{j_1}$. The total number of these systems of transitivity is denoted by $t(\pi)$. Any system of transitivity of the permutation $\pi$ corresponds with an oriented completely reduced cycle $[C'_{i_{j_1}} \cdots C'_{\pi(i_{j_1})} \cdots C'_{\pi^{q_j-1}(i_{j_1})}]$. Therefore for the different completely reduced closed paths $C'_1, \ldots, C'_n$ the following relation is valid

$$\sum_{k=1}^{n} \sum_{[C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}]} (-1)^k = \sum_{\pi \in S_n} (-1)^t(\pi). \quad (5.19)$$

Since all completely reduced closed paths $C'_1, \ldots, C'_n$ are different, all oriented completely reduced cycles $[C'_{i_{j_1}} \cdots C'_{i_{j_{q_j}}}]$, $j = 1, \ldots, k$, are primitive and different.

Let us define $n \times n$ matrix $A$ whose matrix elements $A_{ij} = -1$. We calculate the determinant of this matrix

$$\det A = \sum_{\pi \in S_n} (-1)^{\sigma(\pi)} \prod_{i=1}^{t(\pi)} A_{p_i \pi(p_i)} A_{\pi(p_i) \pi^2(p_i)} \cdots A_{\pi^{q_i-1}(p_i) p_i} \quad (5.20)$$

where $\pi^{q_i}(p_i) = p_i$ and the numbers $(p_i, \pi(p_i), \ldots, \pi^{q_i-1}(p_i))$, $i = 1, \ldots, t(\pi)$, give the subdivision of the numbers $1, \ldots, n$ into $t(\pi)$ groups. The parity of the permutation $\pi$ is equal to

$$\sigma(\pi) = \sum_{i=1}^{t(\pi)} (q_i - 1) \mod 2. \quad (5.21)$$

The substitution of the relation \((5.21)\) and of the matrix elements $A_{ij} = -1$ into the relation \((5.20)\) gives

$$\det A = \sum_{\pi \in S_n} (-1)^t(\pi). \quad (5.22)$$
\[ \det A = 0 \] for \( n > 1 \). Hence the sum (5.19) equals zero for \( n > 1 \).

If all completely reduced closed paths \( C'_1, ..., C'_n \) coincide, then the completely reduced closed path \( C'_{i_1} \cdots C'_{i_j} \cdots \) is non-primitive for \( q_j > 1 \). If all \( q_j = 1 \), the completely reduced closed paths \( C'_1, ..., C'_n \) are not different. Thus for this case the sum (5.18) does not give the contribution into the sum (5.16).

Let us consider the case when there are \( m \) groups in which \( n_i > 1 \), \( i = 1, ..., m \), the completely reduced closed paths \( C'_j \) coincide with each other. We prove the following equality

\[
\sum_{k=1}^{n} \sum_{j=1}^{k} \left( -1 \right)^k = 
\left( \prod_{i=1}^{m} (n_i)! \right)^{-1} \left[ \sum_{\pi \in S_n} \left( -1 \right)^{t(\pi)} - \sum_{l>1, q>1} \sum_{\pi \in S_l} \left( -1 \right)^{t(\pi)} \sum_{\tau \in S_l} \left( -1 \right)^{t(\tau)} \right].
\] (5.23)

In the right hand side of the equality (5.23) the second sum extends over certain numbers \( l, q \) and over certain permutations \( \pi \in S_n \).

Now the right hand side sum (5.19) contains the permutations corresponding with the non-primitive and non-different oriented completely reduced cycles. Let the completely reduced closed paths \( C'_{i_1}, C'_{i_2}, ..., C'_{i_j} \), \( j = 1, ..., q \), coincide with each other. Let \( \tau \in S_l \) be a permutation of the numbers \( 1, ..., l \). The groups of the numbers \( (p_j, \tau(p_j), \tau(p_j-1), \cdots, \tau(p_j-q+1)) \), \( \tau(p_j) = p_j, j = 1, ..., t(\tau) \), give the subdivision of the numbers \( 1, ..., l \) into the systems of transitivity of the permutation \( \tau \). We consider the permutations \( \pi \{ \tau, j \} = \pi_p \{ i_{p_1}, i_{p_2}, ..., i_{p_j}, i_{\tau(p_j)}, i_{\tau(p_j)-1}, ..., i_{\tau(p_j-q+1)} \} \) of the numbers \( 1, ..., n \) where \( j = 1, ..., t(\tau) \). The permutations \( \pi \{ \tau, 1 \} \) and \( \pi \{ \tau, t(\tau) \} \) commute with each other. Let \( \pi \{ \tau \} = \pi \{ \tau, 1 \} \cdots \pi \{ \tau, t(\tau) \} \). The permutation leaves invariant \( n - q \) numbers from \( 1, ..., n \). By the construction \( t(\pi(\tau)) = t(\tau) + n - q \). Let a permutation \( \pi' \in S_n \) leave invariant the numbers \( i_1, ..., i_j, j = 1, ..., q \). Then

\[
t(\pi' \cdot \pi \{ \tau \}) = t(\pi') - lq + t(\tau).
\] (5.24)

The permutation \( \pi' \cdot \pi \{ \tau \} \in S_n \) corresponds with \( t(\pi') - lq + t(\tau) \) oriented completely reduced cycles. From these cycles \( t(\tau) \) oriented completely reduced cycles \( C'_i, C'_j, ..., C'_{i_j}, C'_{i_{\tau(p_j)}} \cdot \cdots \cdot C'_{i_{\tau(p_j)-1}}, C'_{i_{\tau(p_j)-2}}, ..., C'_{i_{\tau(p_j)-q+1}} \), \( j = 1, ..., t(\tau) \), are non-primitive for \( d_j > 1 \).

If all numbers \( d_j = 1 \) and \( t(\tau) = l \), then these \( t(\tau) \) oriented completely reduced cycles are non-different. Hence the permutation \( \pi' \cdot \pi \{ \tau \} \) must be subtracted from the right hand side sum (5.19). Now we explain the multiplier \( (\prod_{i=1}^{m} (n_i)!)^{-1} \) in the right hand side of the equality (5.23).

Let a permutation \( \lambda \in S_n \) rearrange only the numbers corresponding with the coinciding completely reduced closed paths \( C'_i, ..., C'_{i_j} \). For this permutation \( C'_i \lambda(i_{j_1}) \cdots C'_i \lambda(i_{j_q}) = C'_{i_{j_1}} \cdots C'_{i_{j_q}} \). But the permutations \( \pi \{ \lambda(i_{j_1}), ..., \lambda(i_{j_q}) \} \) and \( \pi \{ i_{j_1}, ..., i_{j_q} \} \) may coincide only in the case when the permutation \( \lambda \) acts on the numbers \( i_{j_1}, ..., i_{j_q} \) as a cyclic permutation. By the definition of the permutation \( \lambda \) it is possible only for the coinciding completely reduced closed paths \( C'_{j_1}, ..., C'_{i_{j_q}} \). But then the oriented completely reduced cycle \( C'_{i_{j_1}} \cdots C'_{i_{j_q}} \) is non-primitive. Thus every set of the different primitive oriented completely reduced cycles \( C'_{i_{j_1}} \cdots C'_{i_{j_k}} \), \( j = 1, ..., k \), corresponds with \( \prod_{i=1}^{m} (n_i)! \) different permutations
\[ \pi[\lambda(i_{11}), \ldots, \lambda(i_{1q_1})] \cdots \pi[\lambda(i_{k1}), \ldots, \lambda(i_{kq_k})] \] of the numbers 1, ..., \( n \). The number \( \prod_{i=1}^{m} (n_i)! \) is the total number of the permutations \( \lambda \) rearranging only the numbers corresponding with the coinciding completely reduced closed paths \( C'_1, \ldots, C'_n \). The equality (5.23) is proved.

The equalities (5.22), (5.23) imply that the left hand side sum (5.23) is equal to zero for \( n > 1 \). Hence the sum (5.18) equals zero for \( n > 1 \) and the right hand sides of the equalities (5.13) and (5.16) coincide. The theorem is proved.

For the proof of Theorem 5.2 we used Theorem 5.1 and the definitions (2.17), (4.5), and (2.18), (4.1). Therefore Theorem 5.2 is valid also for a graph \( G \) embedded in a rectangular lattice on the plane.

Let the left hand side of the inequality (2.30) be denoted by \(|u|\). The inequality (4.6) is a particular case of the inequality (2.30). It may be rewritten as \(|u| < 1/3\).

**Theorem 5.3.** Let a graph \( G \) be embedded in the rectangular lattice \( G(M'_1, M'_2; M_1, M_2) \) on the torus. Let the estimate (4.6) be fulfilled and let interaction energy \( E(e) \) be non-negative. Then for the reduced partition function (1.2) the following inequalities are valid

\[
1 - 8/3(\prod_{s=1}^{2}(M_s - M'_s))(1 - 3|u|)^{-1} \sum_{s=1}^{2} (3|u|)^{M_s - M'_s} \leq \exp\{-1/2 \sum_{C \in RC(G)} |C|^{-1} u^{C} \rho(C)\} \leq 1 \quad (5.25)
\]

where \( C \) run over the set \( RC(G) \) of reduced closed paths on the graph \( G \), the natural number \(|C|\) is the length of the path \( C \), the number \( u^C \) is given by the equalities (2.18), (4.4) and the number \( \rho(C) \) is given by the equalities (2.17), (4.1).

**Proof.** Due to Theorem 5.2 the equality (5.13) is valid. Let four different oriented edges \( e_i \), \( i = 1, \ldots, 4 \), of the graph \( G \) have the same initial vertex: \( b(e_1) = \cdots = b(e_4) \). Let the prime closed paths \( C_1 = (e_1, e_5, \ldots, e_m, e_2^{-1}) \) and \( C_2 = (e_3, e_{m+1}, \ldots, e_{m+n}, e_4^{-1}) \) on the graph \( G \) correspond with the disjoint prime non-oriented cycles \([C_1]\) and \([C_2]\). Then the products \( C_1 \cdot C_2 = (e_1, e_5, \ldots, e_m, e_2^{-1}, e_3, e_{m+1}, \ldots, e_{m+n}, e_4^{-1}) \) and \( C_1 \cdot C_2^{-1} = (e_1, e_5, \ldots, e_m, e_2^{-1}, e_4, e_{m+n}, \ldots, e_{m+1}, e_3^{-1}) \) are the prime closed paths. Let \( C_3, \ldots, C_k \) be the prime closed paths on the graph \( G \) such that the non-oriented cycles \([C_1],[C_2],[C_3],\ldots,[C_k]\) are disjoint. Then the prime non-oriented cycles \([C_1 \cdot C_2],[C_3],\ldots,[C_k]\) are disjoint and the prime non-oriented cycles \([C_1 \cdot C_2^{-1}],[C_3],\ldots,[C_k]\) are disjoint too.

Let us consider the sum of three terms in the right hand side sum (5.13)

\[
(-1)^k \left( \prod_{i=1}^{k} u^{C_i} \right) \left( \prod_{i=1}^{k} \rho(C_i) \right) + (-1)^{k-1} u^{C_1 \cdot C_2} \left( \prod_{i=3}^{k} u^{C_i} \right) \rho(C_1 \cdot C_2) \left( \prod_{i=3}^{k} \rho(C_i) \right) + (-1)^{k-1} u^{C_1 \cdot C_2^{-1}} \left( \prod_{i=3}^{k} u^{C_i} \right) \rho(C_1 \cdot C_2^{-1}) \left( \prod_{i=3}^{k} \rho(C_i) \right). \quad (5.26)
\]

The definitions (2.18), (4.4) imply that

\[
u^{C_1 \cdot C_2} = u^{C_1 \cdot C_2^{-1}} = u^{C_1} u^{C_2}. \quad (5.27)
\]

The definitions (2.17), (4.5) and the relations (5.7) imply

\[
\rho(C_1 \cdot C_2) = \gamma(e_1, e_2, e_3, e_4) \rho(C_1) \rho(C_2),
\]

\[
\rho(C_1 \cdot C_2^{-1}) = \gamma(e_1, e_2, e_4, e_3) \rho(C_1) \rho(C_2) \quad (5.28)
\]
where
\[ \gamma(e_1, e_2, e_3, e_4) = \rho(e_1^{-1}; e_2)\rho(e_2^{-1}; e_3)\rho(e_3^{-1}; e_4)\rho(e_4^{-1}; e_1). \] (5.29)

In the second equality \((5.28)\) we took into account the value of the angle through which the tangent vector of the path \(C_2\) turns along the path \(C_2: \phi(C_2) = 2\pi k\) where \(k\) is an integer. Hence \(\rho(C_2) = \exp\{i/2\phi(C_2)\} = (\rho(C_2))^{-1}\).

The substitution of the equalities \((5.27), (5.28)\) into the expression \((5.24)\) gives

\[ [1 - \gamma(e_1, e_2, e_3, e_4) - \gamma(e_1, e_2, e_4, e_3)](-1)^k(\prod_{i=1}^{k} u^{C_i})(\prod_{i=1}^{k} \rho(C_i)). \] (5.30)

The definition \((5.29)\) implies

\[ \gamma(e_2, e_3, e_4, e_1) = \gamma(e_1, e_2, e_3, e_4). \] (5.31)

Let for some index \(i = 1, \ldots, 3\) the oriented edges \(e_i\) and \(e_{i+1}\) have the opposite directions or let the oriented edges \(e_1\) and \(e_4\) have the opposite directions. We shall prove that for all these cases \(\gamma(e_1, e_2, e_3, e_4) = 1\). Due to the relation \((5.31)\) it is sufficient to prove this statement only for the case when the oriented edges \(e_1\) and \(e_2\) have the opposite directions. In this case the oriented edges \(e_3\) and \(e_4\) have also the opposite directions. Due to the definitions \((4.5), (5.29)\) we have in this case

\[ \gamma(e_1, e_2, e_3, e_4) = \rho(e_2^{-1}; e_3)\rho(e_4^{-1}; e_1). \] (5.32)

The direction of the oriented edge \(e_2^{-1}\) coincides with the direction of the oriented edge \(e_1\). The direction of the oriented edge \(e_4^{-1}\) coincides with the direction of the oriented edge \(e_3\). Hence the definition \((4.5)\) and the equality \((5.32)\) imply \(\gamma(e_1, e_2, e_3, e_4) = 1\).

It is easy to verify that when the directions of the oriented edges \(e_i\) and \(e_{i+1}\), \(i = 1, \ldots, 3\), \(e_4\) and \(e_1\) are orthogonal to each other the definitions \((4.5), (5.29)\) imply \(\gamma(e_1, e_2, e_3, e_4) = -1\).

Let the directions of the oriented edges \(e_i\) and \(e_{i+1}\), \(i = 1, \ldots, 3\), \(e_4\) and \(e_1\) be orthogonal to each other. Then the directions of the oriented edges \(e_2\) and \(e_4\) are opposite. Therefore \(\gamma(e_1, e_2, e_3, e_4) = -1\) and \(\gamma(e_1, e_2, e_4, e_3) = 1\). Hence due to the relations \((5.27), (5.28)\) in this case the expression \((5.30)\) is equal to

\[ (-1)^k(\prod_{i=1}^{k} u^{C_i})(\prod_{i=1}^{k} \rho(C_i)) = (-1)^{k-1}u^{C_1}C_2(\prod_{i=3}^{k} u^{C_i})\rho(C_1 \cdot C_2)(\prod_{i=3}^{k} \rho(C_i)). \] (5.33)

The paths \(C_1, C_2\) and \(C_1 \cdot C_2\) go subsequently through the oriented edges \(e_i^{\pm 1}\), \(i = 1, \ldots, 4\), having the directions orthogonal to each other.

Let the oriented edges \(e_1\) and \(e_2\), \(e_2\) and \(e_4\), \(e_4\) and \(e_3\), \(e_3\) and \(e_1\) have the directions orthogonal to each other. Thus the oriented edges \(e_2\) and \(e_3\) have the opposite directions. Therefore \(\gamma(e_1, e_2, e_3, e_4) = 1\) and \(\gamma(e_1, e_2, e_4, e_3) = -1\). Hence in this case the expression \((5.30)\) is equal to

\[ (-1)^k(\prod_{i=1}^{k} u^{C_i})(\prod_{i=1}^{k} \rho(C_i)) = (-1)^{k-1}u^{C_1}C_2^{-1}(\prod_{i=3}^{k} u^{C_i})\rho(C_1 \cdot C_2^{-1})(\prod_{i=3}^{k} \rho(C_i)). \] (5.34)

The paths \(C_1, C_2\) and \(C_1 \cdot C_2^{-1}\) go subsequently through the oriented edges \(e_i^{\pm 1}\), \(i = 1, \ldots, 4\), having the directions orthogonal to each other.
Let the oriented edges $e_1$ and $e_2$ have the opposite directions. Hence $\gamma(e_1, e_2, e_3, e_4) = \gamma(e_1, e_2, e_4, e_3) = 1$ and in this case the expression (5.30) is equal to
\[
(-1)^{k-1} u^{C_1-C_2} \left( \prod_{i=3}^{k} u^{C_i} \right) \rho(C_1 \cdot C_2) \left( \prod_{i=3}^{k} \rho(C_i) \right) = \n
\]
\[
(-1)^{k-1} u^{C_1-C_2^{-1}} \left( \prod_{i=3}^{k} u^{C_i} \right) \rho(C_1 \cdot C_2^{-1}) \left( \prod_{i=3}^{k} \rho(C_i) \right),
\]
(5.35)
Since the oriented edges $e_1$ and $e_2$ have the opposite directions, the oriented edges $e_3$ and $e_4$ have the opposite directions and the directions of the oriented edges $e_1$ and $e_4$, $e_1$ and $e_3$, $e_2$ and $e_3$, $e_2$ and $e_4$ are orthogonal to each other. Thus in this case the paths $C_1$, $C_2$ and $C_1 \cdot C_2^{-1}$ go subsequently through the oriented edges $e_{i \pm 1}$, $i = 1, \ldots, 4$, having the directions orthogonal to each other.

We have considered all possible directions of the oriented edges $e_1, e_2, e_3, e_4$. In any case the sum (5.24) is equal to one of the expressions (5.33) – (5.35) where the paths go subsequently through the oriented edges $e_{i \pm 1}$, $i = 1, \ldots, 4$, having the directions orthogonal to each other.

With the term
\[
(-1)^{k} \left( \prod_{i=1}^{k} u^{C_i} \right) \left( \prod_{i=1}^{k} \rho(C_i) \right)
\]
(5.36)
the right hand side sum (5.13) contains all the terms of type (5.36) where the disjoint prime non-oriented cycles $[[C_1]], \ldots, [[C_k]]$ contain the same non-oriented edges as the disjoint prime non-oriented cycles $[[C_1]], \ldots, [[C_k]]$. We have proved that the sum of these terms of type (5.36) is equal to the only term of type (5.36) where the prime closed paths $C_1, \ldots, C_k$ satisfy the condition: if four different oriented edges from the prime closed paths $C_1, \ldots, C_k$ are incident to one vertex, then the prime closed paths $C_1, \ldots, C_k$ go subsequently through the oriented edges from these four oriented edges having the directions orthogonal to each other.

Let us define the cochain $\xi^1[C] \in C^1(P(G), Z_2^{odd})$ equal 1 on all non-oriented edges from the prime $\{C\}$ on the graph $G$. The cochain $\xi^1[C]$ equals zero on all other non-oriented edges of the graph $G$. Since $[[C]]$ is a prime non-oriented cycle, $\xi^1[C] \in Z_1(P(G), Z_2^{odd})$. If the prime non-oriented cycles $[[C_1]], \ldots, [[C_k]]$ are disjoint, then the supports of the cycles $\xi^1[C_1], \ldots, \xi^1[C_k]$ do not intersect. By the definition $\rho(C_i) = \pm 1$, $i = 1, \ldots, k$. The interaction energy $E(e)$ is non-negative. Hence we obtain the estimate for the term (5.36)
\[
(-1)^{k} \left( \prod_{i=1}^{k} u^{C_i} \right) \left( \prod_{i=1}^{k} \rho(C_i) \right) \leq u \sum_{i=1}^{k} \xi^1[C_i].
\]
(5.37)
Since the sum of the terms of type (5.36) corresponding with the disjoint prime non-oriented cycles containing the same set of the non-oriented edges is again the term of type (5.36), it follows from the equality (4.2) and from the inequality (5.37) that
\[
\exp \{-1/2 \sum_{C \in RC(G)} |C|^{-1} u^C \rho(C)\} \leq Z_{r,G}.
\]
(5.38)
We shall prove that for any cycle $\xi^1 \in Z_1(P(G), Z_2^{odd})$ there exist the disjoint prime non-oriented cycles $[[C_1]], \ldots, [[C_k]]$ such that $\xi^1 = \xi^1[C_1] + \cdots + \xi^1[C_k]$. Let $(m, v)$ be an oriented
edge corresponding with a non–oriented edge on which the cycle $\xi^1$ takes the value 1. Then the vertex $m + v$ is incident to two or four non–oriented edges on which the cycle $\xi^1$ takes the value 1. Let us choose an oriented edge $(m + v, v_1)$, $v_1 \neq -v$, corresponding with a non–oriented edge on which the cycle $\xi^1$ takes the value 1. By repeating this process we obtain the prime closed path $C_1$ such that the support of the cycle $\xi^1|C_1|$ is contained in the support of the cycle $\xi^1$. Hence there exists a cycle $\eta^1 \in Z_1(P(G), Z_{2dd}^d)$ such that $\xi^1 = \xi^1|C_1| + \eta^1$ and the supports of the cycles $\xi^1|C_1|$ and $\eta^1$ do not intersect. By applying the above procedure for a cycle $\eta^1$ we construct a prime closed path $C_2$. By repeating this process we obtain the disjoint prime non–oriented cycles $[[C_1] ,...,[[C_k]]$ such that $\xi^1 = \xi^1|C_1| + \cdots + \xi^1|C_k|$.

For a cycle $\xi^1 \in B_1(P(G), Z_{2dd}^d)$ we can construct the disjoint prime non–oriented cycles $[[C_1] ,...,[[C_k]]$ such that $\xi^1 = \xi^1|C_1| + \cdots + \xi^1|C_k|$ and the prime closed paths satisfy the above condition. If $\xi^1 \in B_1(P(G), Z_{2dd}^d)$, then $\xi^1 = \partial \xi^2$ where a cochain $\xi^2 \in C^2(P(G), Z_{2dd}^d)$. The support of the cochain $\xi^2$ consists of the squares on which the cochain $\xi^2$ takes the value 1. Two squares $s_i^2$ and $s_j^2$ belong to one connected component of the support of the cochain $\xi^2$ if there exist the squares $s_i^2 ,...,s_k^2$ from the support of the cochain $\xi^2$ such that the boundaries of the squares $s_i^2$ and $s_j^2$, $s_i^2$, and $s_{i+1}^2$, $l = 1, ..., k - 1$, $s_k^2$ and $s_j^2$ contain the common non–oriented edges. The boundaries of the squares from the different connected components of the support of the cochain $\xi^2$ may contain the common vertices only. Thus $\xi^1 = \partial \xi^2 + \cdots + \partial \xi^2$ where the support of the cochain $\xi^2$ has the only connected component and for $i \neq j$ the supports of the cochains $\xi^2$ and $\xi^2$ do not intersect. The support of the cochain $\partial \xi^2$ corresponds with the prime non–oriented cycle $[[C_1]]$ on the graph $G$. By the construction the prime non–oriented cycles $[[C_1] ,...,[[C_k]]$ are disjoint. Moreover, if four different oriented edges from the prime closed paths $C_1 ,...,C_k$ are incident to one vertex, then the prime closed paths $C_1 ,...,C_k$ go subsequently through such oriented edges from these four oriented edges that have the directions orthogonal to each other.

Let a prime non–oriented cycle $[[C]]$ be the boundary of a connected set of the squares. We remove one square from this set, so that the new set is also connected. Let a prime non–oriented cycle $[[C']] be the boundary of this new connected set of the squares. Let $C$ and $C'$ be the representatives of the prime non–oriented cycles $[[C]]$ and $[[C']]$. By using the definitions (2.17), (4.5) it is possible to prove that $\rho(C) = \rho(C')$. By repeating this process we obtain $\rho(C'') = \rho(C)$ where $C''$ is a representative of the boundary of one square. It is easy to calculate that $\rho(C'') = -1$.

The previous arguments imply that the sum of the terms (5.36) corresponding with a cycle $\xi^1 \in B_1(P(G), Z_{2dd}^d)$ is equal to a unique term (5.36) where $\rho(C_i) = -1$, $i = 1, ..., k$. For a cycle $\xi^1 \notin B_1(P(G), Z_{2dd}^d)$ we obtain a unique term (5.36) where $\rho(C_i) = \pm 1$, $i = 1, ..., k$. The interaction energy $E(e)$ is non–negative. Therefore we get the estimate

$$Z_{r,G} - \exp\{-1/2 \sum_{C \in \mathcal{R}C(G)} |C|^{-1} u^C \rho(C)\} \leq 2 \sum_{\xi^1 \in \mathcal{Z}_1(P(G), Z_{2dd}^d) \setminus \xi^1 \notin B_1(P(G), Z_{2dd}^d)} u^{\xi^1}. \quad (5.39)$$

Let us evaluate the right hand side of the inequality (5.39). We have proved that any cycle $\xi^1 \in \mathcal{Z}_1(P(G), Z_{2dd}^d)$ has the form $\xi^1 = \xi^1|C_1| + \cdots + \xi^1|C_k|$ where the prime non–oriented cycles $[[C_1] ,...,[[C_k]]$ are disjoint. If $\xi^1 \notin B_1(P(G), Z_{2dd}^d)$, then at least for one prime non–oriented cycle $[[C_i]]$ the cochain $\xi^1|C_i| \notin B_1(P(G), Z_{2dd}^d)$. Hence any cycle $\xi^1 \notin B_1(P(G), Z_{2dd}^d)$ has the form $\xi^1 = \xi^1|C| + \eta^1$ where the prime non–oriented cycle $[[C]]$ corresponds with the cycle $\xi^1|C| \notin B_1(P(G), Z_{2dd}^d)$ and the supports of the path $C$ and of the cycle $\eta^1 \in \mathcal{Z}_1(P(G), Z_{2dd}^d)$ do not intersect. By using this decomposition we obtain the
estimate
\[ \sum_{\xi^1 \in Z_1(P(G), \mathbb{Z}_2^{add}), \xi^1 \notin B_1(P(G), \mathbb{Z}_2^{add})} \mathbf{u}^{\xi^1} \leq Z_{r,G} \sum_{\xi \in \text{prime}, \xi \notin B_1(P(G), \mathbb{Z}_2^{add})} \mathbf{u}^C. \]  \tag{5.40}

The total number of all reduced closed paths of the length \( l \) with the fixed initial vertex on the lattice \( \tilde{G}(M_1', M_2'; M_1, M_2) \) is less than \( 4 \cdot 3^{l-1} \). The total number of vertices of the lattice \( \tilde{G}(M_1', M_2'; M_1, M_2) \) is equal to \( (M_1 - M_1')(M_2 - M_2') \). Hence the total number of all reduced closed paths of the length \( l \) on the lattice \( \tilde{G}(M_1', M_2'; M_1, M_2) \) is less than \( (M_1 - M_1')(M_2 - M_2')4 \cdot 3^{l-1} \). Let for a prime closed path \( C \) on the lattice \( \tilde{G}(M_1', M_2'; M_1, M_2) \) the cochain \( \xi[C] \notin B_1(P(G), \mathbb{Z}_2^{add}) \). Hence the length \( |C| \) of a path \( C \) is more than \( M = \min(M_1 - M_1', M_2 - M_2') \). It implies the following estimate
\[ \sum_{\xi \in \text{prime}, \xi \notin B_1(P(G), \mathbb{Z}_2^{add})} \mathbf{u}^C \leq \sum_{C \in R_C(\tilde{G}), |C| \geq M} ||u||^{|C|} \leq 4/3(M_1 - M_1')(M_2 - M_2') \sum_{l=M}^{\infty} (3||u||)^l \leq 4/3 \left( \prod_{s=1}^{2} (M_s - M_s') \right) (1 - 3||u||)^{-1} \sum_{s=1}^{2} (3||u||)^{M_s - M_s'} \]  \tag{5.41}

where \( ||u|| \) denotes the left hand side of the inequality (2.30). The inequalities (5.38) - (5.41) imply the inequalities (5.25). The theorem is proved.

If a graph \( G \) is embedded in a rectangular lattice on the plane, then \( Z_1(P(G), \mathbb{Z}_2^{add}) = B_1(P(G), \mathbb{Z}_2^{add}) \). Therefore the inequality (5.39) becomes the equality (4.7). Thus we obtain a new proof of the equality (4.7) independent of the papers [5], [6], [12], [13].

Theorem 5.4. Let for the non-negative interaction energy of the homogeneous Ising model on a rectangular lattice on the torus the estimate (4.6) be valid. Then for the partition function (4.1) of the homogeneous Ising model on the rectangular lattice \( \tilde{G}(0, 0; M_1, M_2) \) on the torus

\[ Z_{\tilde{G}(M_1', M_2'; M_1, M_2)} = Z_{\tilde{G}(0, 0; M_1 - M_1', M_2 - M_2')} \]  \tag{5.42}

Let us denote \( E_1(E_2) \) the interaction energy \( E(e) \) for horizontally (vertically) directed edges of the lattice \( \tilde{G}(0, 0; M_1, M_2) \).

Theorem 5.4. Let for the non-negative interaction energy of the homogeneous Ising model on a rectangular lattice on the torus the estimate (4.6) be valid. Then for the partition function (4.1) of the homogeneous Ising model on the rectangular lattice \( \tilde{G}(0, 0; M_1, M_2) \) on the torus

\[ \lim_{M_1, M_2 \to \infty} (M_1 M_2)^{-1} \ln Z_{\tilde{G}(0, 0; M_1, M_2)} = \]

\[ \ln(2 \cosh \beta E_1 \cosh \beta E_2) + 1/2(2\pi)^{-2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \]

\[ \ln[(1 + z_1^2)(1 + z_2^2) - 2z_1(1 - z_2^2) \cos \theta_1 - 2z_2(1 - z_1^2) \cos \theta_2] \]  \tag{5.43}

where the variables \( z_i = \tanh \beta E_i \), \( i = 1, 2 \).

Proof. The equality (2.36) and the inequalities (5.23) imply

\[ \ln[1 - 8/3(M_1 M_2)(1 - 3||u||)^{-1} \sum_{s=1}^{2} (3||u||)^{M_s} \leq 1/2 \ln[\det(I - T(u, \rho))] - \ln Z_{\tilde{G}(0, 0; M_1, M_2)} \leq 0 \]  \tag{5.44}
where the matrix $T(u, \rho)$ is given by the equalities (2.32) and (1.3) for the lattice $\tilde{G}(0, 0; M_1, M_2)$. Now the equality (5.43) follows from the inequalities (5.44) and from the equalities (4.1) and (1.26). The theorem is proved.

The equality (5.43) is proved in the paper [11] for the arbitrary interaction energies $E_i, i = 1, 2$.

The relation (4.27) is valid also for the graph $\tilde{G} = \tilde{G}(M_1', M_2'; M_1, M_2)$.

**Theorem 5.5.** Let for Ising model on the rectangular lattice $\tilde{G}(M_1', M_2'; M_1, M_2)$ on the torus the estimate (4.3) be valid and let the interaction energy $E(e)$ be non-negative. Let a cochain $\chi \in C^0(P(\mathbb{Z}^2), Z^{add}_2)$ be equal to 1 on the finite number of the vertices. Then for the correlation function (3.14) of the two dimensional Ising model with periodic boundary conditions

$$
\lim_{M_2-M_2' \to \infty, s=1,2, (M_1-M_1')((M_2-M_2')^{-1}+(M_2-M_2')(M_1-M_1')^{-1}) \leq \text{const}} W_{\tilde{G}}(M_1', M_2'; M_1, M_2)(\chi) = \sum_{\xi^1 \in C^1(P(\tilde{G}), Z^{add}_2), \partial \xi^1 = \chi, \chi = \text{connected} ||\xi^1||} u^{\xi^1} \exp\{1/2 \sum_{C \in RC(\tilde{G}), ||C|| \cap \{||\xi^1||\} \neq \emptyset} |C|^{-1} u^C \rho(C)\} \leq \sum_{C \in RC(\tilde{G}), ||C|| \cap \{||\xi^1||\} \neq \emptyset} |C|^{-1} u^C \rho(C) \leq (1-8/3\sum_{s=1}^2 (M_s-M_s')((1-3||u||)-1) \sum_{s=1}^2 3||u||)^{M_s-M_s'} \leq Z_{r,\tilde{G}}(Z_{r,\tilde{G} \setminus \{||\xi^1||\}})^{-1} \exp\{1/2 \sum_{C \in RC(\tilde{G})} |C|^{-1} u^C \rho(C)\} \leq (1-8/3\sum_{s=1}^2 (M_s-M_s')((1-3||u||)-1) \sum_{s=1}^2 3||u||)^{M_s-M_s'}^{-1}. \quad (5.46)
$$

The interaction energy $E(e)$ is non-negative. Then the definition (4.2) implies the following estimate

$$
(Z_{r,\tilde{G}})^{-1} Z_{r,\tilde{G} \setminus \{||\xi^1||\}} \leq 1. \quad (5.47)
$$

It follows from the estimates (4.6) and (5.47) that for $M_s-M_s' \to \infty, s = 1, 2$, the non-zero contributions give only those terms of the sum (4.27) for the graph $\tilde{G}$ which correspond to the cochains $\xi^1 \in C^1(P(\mathbb{Z}^2), Z^{add}_2)$ with the finite supports $||\xi^1||$. Let a cochain $\xi^1 \in C^1(P(\mathbb{Z}^2), Z^{add}_2)$ satisfy the condition $\partial \xi^1 = \chi$ and let it have the finite $\chi$-connected support $||\xi^1||$. For sufficiently large $M_s-M_s', s = 1, 2$, the set $\{||\xi^1||\} \subset \tilde{G}(M_1', M_2'; M_1, M_2)$. Let us prove that

$$
\lim_{M_2-M_2' \to \infty, s=1,2, (M_1-M_1')((M_2-M_2')^{-1}+(M_2-M_2')(M_1-M_1')^{-1}) \leq \text{const}} \sum_{C \in RC(\tilde{G} \setminus \{||\xi^1||\})} |C|^{-1} u^C \rho(C) = \exp\{1/2 \sum_{C \in RC(\tilde{G} \setminus \{||\xi^1||\})} |C|^{-1} u^C \rho(C)\}. \quad (5.48)
$$
Indeed, for $M_s - M'_s \to \infty, s = 1, 2$, and $(M_1 - M'_1)(M_2 - M'_2)^{-1} + (M_2 - M'_2)(M_1 - M'_1)^{-1} \leq const$ the left and the right hand sides of the inequalities (5.46) tend to 1. Since the set $i(|||\xi|||)$ is finite, the last multiplier in the central part of the inequalities (5.48) tends to the right hand side of the equality (5.48). The series (5.49) coincides with the series (4.28). It was proved in Theorem 4.3 that the series (4.28) is absolutely convergent if the estimate (4.6) is fulfilled and the interaction energy is non–negative. Thus the sum (4.27) for the graph $\tilde{G}(M'_1, M'_2; M_1, M_2)$ converges to the series (5.45). The theorem is proved.

The correlation functions (5.43) and (4.28) coincide.

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