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SOLUTIONS BLOWING UP ON ANY GIVEN COMPACT SET FOR THE ENERGY SUBCRITICAL WAVE EQUATION

THIERRY CAZENAVE\textsuperscript{1}, YVAN MARTEL\textsuperscript{2}, AND LIFENG ZHAO\textsuperscript{3}

Abstract. We consider the focusing energy subcritical nonlinear wave equation $\partial_t u - \Delta u = |u|^{p-1}u$, $u \in \mathbb{R}^N$, $N \geq 1$. For any given compact set $E \subseteq \mathbb{R}^N$, there exists a solution which blows up exactly on $E$. This is well-known for $N = 1, 2$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. It is also well-known that under similar conditions on $p$, the Cauchy problem for (1.1) is locally well-posed in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ (see [8, 9, 25]). For $H^1 \times L^2$ solutions, the energy

$$E(u(t), \partial_t u(t)) = \int \left\{ \frac{1}{2} |\partial_t u(t,x)|^2 + \frac{1}{2} |\nabla u(t,x)|^2 - \frac{1}{p+1} |u(t,x)|^{p+1} \right\} \, dx$$

is conserved through time. Moreover, it is known how to produce solutions blowing up in finite time (see e.g. [10, 18]).

Our main result states that for any given compact set $E \subseteq \mathbb{R}^N$, there exists a finite-energy solution of (1.1) which blows up in finite time exactly on $E$.

**Theorem 1.1.** Let $p$ satisfy (1.2) and let $E$ be any compact set of $\mathbb{R}^N$. There exists $\delta_0 > 0$ and a solution $(u, \partial_t u) \in C((0, \delta_0]; H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))$ of (1.1) which blows up at time $0$ exactly on $E$ in the following sense.

- If $x_0 \in E$ then for any $r > 0$,
  $$\lim_{t \downarrow 0} \|u(t)\|_{L^2(|x-x_0| < r)} = \infty \quad \text{and} \quad \lim_{t \downarrow 0} \|\partial_t u(t)\|_{L^2(|x-x_0| < r)} = \infty. \quad (1.3)$$

- If $x_0 \notin E$ then there exists $r > 0$ such that
  $$\sup_{t \in (0, \delta_0]} \left\{ \|u(t)\|_{L^2(|x-x_0| < r)} + \|\nabla u(t)\|_{L^2(|x-x_0| < r)} + \|\partial_t u(t)\|_{L^2(|x| < r)} \right\} < \infty. \quad (1.4)$$

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Remark 1.2. For \( t > 0 \), the function

\[
h(t) = \kappa t^{-\frac{2}{p-1}} \quad \text{where} \quad \kappa = \left[ \frac{2(p+1)}{(p-1)^2} \right]^{\frac{1}{p-1}}
\]  

is a solution of the ordinary differential equation \( h'' = h^p \) which blows up at time 0. It is also a solution of (1.1), but of course it fails to be in the energy space. The function \( h \) is the building block for our construction, it is thus relevant to compare it with the blow-up rate of the solutions constructed in Theorem 1.1. It follows from the proof that for any \( 0 < \mu < \frac{2}{p-1} \) there exist solutions \( u \) as in the statement of Theorem 1.1 satisfying in addition the following estimates: for any \( x_0 \in E \ r > 0 \), and all \( t \in (0, \delta_0) \),

\[
t^{-\mu} \lesssim \|u(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p+1}}, \quad (1.6)
\]

\[
t^{-\mu-1} \lesssim \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{2}{p+1}-1}. \quad (1.7)
\]

Moreover, if \( x_0 \in E \) and \( E \) contains a neighborhood of \( x_0 \) then it also holds, for any \( r > 0 \), and all \( t \in (0, \delta_0) \),

\[
\|u(t)\|_{L^2(|x-x_0|<r)} \gtrsim t^{-\frac{2}{p+1}} \quad \text{and} \quad \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \gtrsim t^{-\frac{2}{p+1}-1}. \quad (1.8)
\]

In contrast, if \( x_0 \) is an isolated point of the compact set \( E \), solutions \( u \) as in Theorem 1.1 can be chosen so that, for a small \( r > 0 \),

\[
\lim_{t \uparrow 0} \left\{ t^{\frac{2}{p+1}} \|u(t)\|_{L^2(|x-x_0|<r)} + t^{\frac{2}{p+1}+1} \|\partial_t u(t)\|_{L^2(|x-x_0|<r)} \right\} = 0.
\]

To prove Theorem 1.1, we follow the strategy developed in [4] to construct blow-up solutions of ODE type for a class of semilinear Schrödinger equations. First, we construct an approximate solution to the blow-up problem based on the explicit blow-up solution \( h \) defined by (1.5). The main order term of the approximate solution is \( U_0(t, x) = \kappa(t + A(x))^{-\frac{2}{p+1}} \), where \( A \) is a suitable nonnegative function which vanishes exactly on \( E \) and whose behavior at \( \infty \) ensures that \( U_0 \) belongs to the energy space. Typically, to obtain blowup at only one point \( x_0 \), it suffices to consider \( A(x) = |x - x_0|^k \) for \( k \) large enough. Compared to [4] where a simple ansatz such as \( U_0 \) is sufficient, at least for strong enough nonlinearities, the wave equation requires to introduce iterated refinements \( U_J \) of this ansatz (the number of iterations \( J \geq 1 \) depends on \( p \), see Remark 2.4). The basic idea is that for such blow-up profiles, the space derivatives are of lower order compared to time derivatives and to nonlinear terms. This allows to use only elementary arguments of ordinary differential equations for the construction of the refined ansatz \( U_J(t, x) \), at fixed \( x \). See Section 2.

Second, we consider the sequence \( (u_n) \) of solutions of the wave equation (1.1) with initial data \( u_n(\frac{1}{2^n}, x) = U_J(\frac{1}{2^n}, x) \). Using energy method in \( H^1 \times L^2 \), we prove uniform estimates on this sequence on intervals \( [\frac{1}{2^n}, \delta_0] \), where \( \delta_0 > 0 \) is uniform in \( n \) (see Section 3). Passing to the limit \( n \to \infty \) yields the solution \( u \) of Theorem 1.1.

We point out that this strategy by approximate solution and compactness is also reminiscent to [19, 20, 24] where global or blow-up solutions with special asymptotic behavior are constructed using the reversibility of the equation and suitable uniform estimates on backwards solutions.

For stability results concerning the solution \( h \) (1.2), we refer to [7]. For ODE-type blowup for quasilinear wave equations, see [26] and the references therein. We also refer to [5] where an ODE blow-up profile similar to \( U_0 \) is used to construct blow-up solutions of the nonlinear heat equation with applications to the Burgers equation.
In this article, we restrict ourselves to energy subcritical power nonlinearities for simplicity, since this framework allows us to use the energy method at the level of regularity $H^1 \times L^2$ only. However, the approximate solutions constructed in Section 2 are relevant for any power nonlinearity, and we expect that a higher order energy method (to estimate higher order Sobolev norms) should be sufficient to extend the construction to energy critical or supercritical nonlinearities (at least for integer powers to avoid regularity issues).

Remark 1.3. A more general question for nonlinear wave equations concerns the blow-up surface. For a solution of (1.1) with initial data at $t = 0$, which is assumed to blow up in finite time, there exists a 1-Lipschitz function $x \mapsto \phi(x) > 0$ such that the solution is well-defined in a suitable sense in the maximal domain of influence $D = \{(t, x) : 0 \leq t < \phi(x)\}$, see e.g. [1], Sections III.2 and III.3. The surface $\{\{(\phi(x), x) : x \in \mathbb{R}^N\} \}$ is called the blow-up surface. The question of the regularity of blow-up surface is addressed in [1, 2, 3, 21, 22]. The question of constructing solutions of the nonlinear wave equation with prescribed blow-up surface (with sufficient regularity and satisfying the space-like condition $\|\nabla \phi\|_{L^\infty} < 1$) is also a classical question, addressed in several articles and books, notably [15, 16], [11, 12, 13], [17] and [1]. The approach by Fuchsian reduction is especially well-described in the book [13]. First developed for analytic surfaces and exponential nonlinearity, this method was later extended to surfaces with Sobolev regularity and to some power nonlinearities. However, it is not clear to us whether the strategy described in [13] for constructing solutions with given blow-up surface can be extended to power nonlinearities $|u|^{p-1}u$ for any $p > 1$, or to more general nonlinearities.

Prescribing the blow-up set of a blow-up solution can be seen as a sub-product of prescribing its blow-up surface. This issue is discussed in [13, 14, 17]. However, the solutions constructed in [13, 14, 17] may only exist in a space-time region around the blow-up surface, which does not guarantee that the solution is globally defined in space at any one specific time.

We also would like to point out a difference between the above mentioned articles and our approach. Here, we resolutely work with finite energy solutions and the initial value problem for (1.1). It is often argued that finite speed of propagation and cut-off arguments allow to reduce to finite energy solutions. For example, the function (1.5) is used to claim that ODE-type blowup is easy to reach for finite initial data. Moreover, we hope that our somehow elementary approach can be of interest for its simplicity and its large range of applicability to other more complicated problems where ODE blowup is relevant.

Notation. We fix a smooth, even function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying:

$$\chi \equiv 1 \text{ on } [0, 1], \chi \equiv 0 \text{ on } [2, \infty) \text{ and } \chi' \leq 0 \leq \chi \leq 1 \text{ on } [0, \infty).$$

For $p > 1$ satisfying (2.2), recall the well-known inequality, for any $u \in H^1$,

$$\|u\|_{L^{p+1}}^{p+1} \lesssim \|u\|_{L^2}^{p+1-\frac{2}{N}(p-1)} \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^{2(p-1)}. \quad (1.10)$$

Let $f(u) = |u|^{p-1}u$ and $F(u) = \int_0^u f(v)dv$. For future reference, we recall Taylor’s formulas involving the functions $F$ and $f$. Let $\bar{p} = \min(2, p)$. First, we claim that for any $u > 0$ and $v \in \mathbb{R}$,

$$|F(u + v) - F(v) - F'(v)u - \frac{1}{2}F''(u)v^2| \lesssim |v|^{p+1} + u^{\bar{p}-\bar{p}'}|v|^\bar{p}+1. \quad (1.11)$$
Indeed, in the region $|v| \geq \frac{1}{2} u$, each term on the left-hand side is bounded by $|v|^{p+1}$. In the region $|v| \leq \frac{1}{2} u$, we use Taylor’s expansion to write
\[
|F(u+v) - F(u) - F'(u)v - \frac{1}{2} F''(u)v^2| \lesssim u^{-2}|v|^3.
\]
If $p \geq 2$, then $\bar{p} = 2$ and (1.11) is proved. If $1 < p < 2$, we finish by saying that in this case $u^{-2} |v|^3 \lesssim |v|^{p+1}$. The same argument shows that
\[
|(f(u+v) - f(u) - f'(u)v)| \lesssim |v|^{p+1} + u^{p-\bar{p}} |v|^{p+1}. \tag{1.12}
\]
Next, we claim that for any $u > 0$ and $v \in \mathbb{R}$,
\[
|f(u+v) - f(u) - f'(u)v - \frac{1}{2} f''(u)v^2| \lesssim u^{-1} |v|^{p+1} + u^{p-\bar{p}-1} |v|^{p+1}. \tag{1.13}
\]
Indeed, in the region $|v| \geq \frac{1}{2} u$, each term on the left-hand side is bounded by $u^{-1} |v|^{p+1}$, and (1.13) follows. In the region $|v| \leq \frac{1}{2} u$, we use Taylor’s expansion to write
\[
|f(u+v) - f(u) - f'(u)v - \frac{1}{2} f''(u)v^2| \lesssim u^{-3} |v|^3.
\]
If $p \geq 2$, then $\bar{p} = 2$ and (1.13) is proved. If $1 < p < 2$, we finish by saying that in this case $u^{-3} |v|^3 \lesssim u^{-1} |v|^{p+1}$.

In this article, we will use multi-variate notation and results from [6]. For any $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$, $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we set
\[
|\beta| = \sum_{j=1}^N \beta_j, \quad \beta! = \prod_{j=1}^N (\beta_j!), \quad x^\beta = \prod_{j=1}^N x_j^{\beta_j},
\]
\[
\partial_x^\beta = \text{Id} \quad \text{and} \quad \partial_x^\beta = -\frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}} \quad \text{for} \quad |\beta| > 0.
\]
For $\beta, \beta' \in \mathbb{N}^N$, we write $\beta' \lessdot \beta$ if $\beta'_j \leq \beta_j$ for all $j = 1, \ldots, N$. When $\beta' \leq \beta$, we set
\[
\binom{\beta}{\beta'} = \frac{\prod_{j=1}^N \beta_j!}{\beta'! (\beta - \beta')!}.
\]
With this notation, given two functions $a, b : \mathbb{R}^N \to \mathbb{R}$, Leibniz’s formula writes:
\[
\partial_x^\beta (ab) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left( \partial_x^\beta a \right) \left( \partial_x^{\beta - \beta'} b \right). \tag{1.14}
\]
We write $\beta' \prec \beta$ if one of the following holds
\begin{itemize}
  \item $|\beta'| < |\beta|$;
  \item $|\beta'| = |\beta|$ and $\beta'_1 < \beta_1$;
  \item $|\beta'| = |\beta|$, $\beta'_1 = \beta_1, \ldots, \beta'_k = \beta_k$ and $\beta'_{k+1} < \beta_{k+1}$ for some $1 \leq k < N$.
\end{itemize}
Finally, we recall the Faa di Bruno formula (see Corollary 2.10 in [6]). Let $n = |\beta| \geq 1$. Then, for functions $q : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R}^N \to \mathbb{R}$,
\[
\partial_x^\beta (q \circ a) = \sum_{\nu=1}^n \binom{\nu}{\beta} \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \frac{\partial^{\nu}(q \circ a)}{(\nu!)^\ell \nu!} \tag{1.15}
\]
where
\[
P(\beta, \nu) = \left\{ (\nu_1, \ldots, \nu_n; \beta_1, \ldots, \beta_n) : \text{there exists } 1 \leq m \leq n \text{ such that} \right. \]
\[
\nu_\ell = 0 \text{ and } \beta_\ell = 0 \text{ for } 1 \leq \ell \leq n - m; \nu_\ell > 0 \text{ for } n - m + 1 \leq \ell \leq n;
\]
\[
\text{and } 0 < \beta_{n-m+1} < \cdots < \beta_n \text{ are such that } \sum_{\ell=1}^n \nu_\ell = \nu, \sum_{\ell=1}^n \nu_\ell \beta_\ell = \beta \right\}.
\]
2. The blow-up ansatz

2.1. Preliminary. Recall that $h$ is the explicit solution (1.5) of the equation $h'' = h^p$ which blows up at 0. The linearization of this equation around the solution $h$ yields the linear equation

$$g'' = ph^{p-1}g = \frac{2p(p+1)}{(p-1)^2} t^{-2}g$$

which admits the following two independent solutions

$$g_1(t) = t^{-\frac{p+1}{p-1}}, \quad g_2(t) = t^{\frac{2p}{p-1}}, \quad \text{for } t > 0.$$ 

Since $\frac{p+1}{p-1} > \frac{2}{p-1}$, the function $g_1$, related to time invariance, is more singular at 0 than the function $h$. Note also that for a function $G$ satisfying $\int_0^1 s^{\frac{2p}{p-1}} |G(s)| ds < \infty$, a solution of the following linearized equation with source $G$

$$g'' = \frac{2p(p+1)}{(p-1)^2} t^{-2}g + G,$$

is given by

$$g(t) = -\frac{p-1}{3p} t^{-\frac{p+1}{p-1}} \left( t^{\frac{p+1}{p-1}} \int_0^t s^{\frac{2p}{p-1}} G(s) ds + t^{\frac{2p}{p-1}} \int_t^1 s^{\frac{2p}{p-1}} G(s) ds \right).$$

2.2. First blow-up ansatz. Set

$$J = \left\lfloor \frac{p+1}{p-1} \right\rfloor \quad \text{and} \quad k \geq 2J + 2 \quad (2.1)$$

where $x \mapsto \lfloor x \rfloor$ is the floor function which maps $x$ to the greatest integer less than or equal to $x$. (See Remark 2.4 below for the explanation of the numbers $J$ and $k$.)

We consider a function $A : \mathbb{R}^N \to \mathbb{R}$ of class $C^{k-1}$ on $\mathbb{R}^N$ and of class $C^k$ piecewise on $\mathbb{R}^N$ such that, for any $\beta \in \mathbb{N}^N$, with $|\beta| \leq k - 1$, the following hold

$$\begin{cases} A \geq 0 \text{ and } |\partial_x^\beta A| \lesssim A^{1-|\beta|} & \text{on } \mathbb{R}^N, \\ A(x) = |x|^k & \text{for } x \in \mathbb{R}^N, |x| \geq 2. \end{cases} \quad (2.2)$$

Remark 2.1. Typical examples of such functions are $A(x) := |x|^k$, which vanishes at 0 and

$$A(x) := \begin{cases} 0 & \text{if } |x| \leq 1 \\ (|x| - \chi(x))^k & \text{if } 1 < |x| \leq 2 \\ |x|^k & \text{if } |x| > 2 \end{cases}$$

(where $\chi$ is given by (1.9)) which vanishes on the closed ball of center 0 and radius 1. Another example, important for the proof of Theorem 1.1 is given in Section 4:

for any compact set $E$ of $\mathbb{R}^N$ included in the open ball of center 0 and radius 1, there exists a function $A$ satisfying (2.2) which vanishes exactly on $E$.

For $t > 0$ and $x \in \mathbb{R}^N$, set

$$U_0(t,x) = \kappa(t + A(x))^{-\frac{2}{p+1}} = h(W(t,x)) \quad \text{where } W(t,x) = t + A(x),$$

so that $U_0$ satisfies $\partial_t U_0 = f(U_0)$ on $(0, \infty) \times \mathbb{R}^N$. Let

$$\mathcal{E}_0 = -\partial_x U_0 + \Delta U_0 - f(U_0) = \Delta U_0.$$ 

We gather in the next lemma some estimates for $U_0$ and $\mathcal{E}_0$.

Lemma 2.2. The function $U_0$ satisfies

$$\partial_t U_0 = -\left( \frac{2}{p+1} U_0^{p+1} \right)^{\frac{2}{p+1}}, \quad (\partial_t U_0)^2 = \frac{2}{p+1} U_0^{p+1}, \quad \partial_{tt} U_0 = U_0^p. \quad (2.3)$$

Moreover, for any $\beta \in \mathbb{N}^N$, $\rho \in \mathbb{R}$, $0 < t \leq 1$, $x \in \mathbb{R}^N$, the following hold.
(i) If $0 \leq |\beta| \leq k - 1$ and $|x| \leq 2$,
\[
|\partial_x^2(U_0^0)| \lesssim U_0^{\rho + \frac{|\beta|}{p+1}} + \frac{|\beta|}{p+1}, \quad |\partial_x^2(U_0^0)| \lesssim U_0^{\rho + (1 + |\beta|)\frac{p-1}{p+1}},
\]
(ii) If $0 \leq |\beta| \leq k - 3$ and $|x| \leq 2$,
\[
|\partial_x^2 \mathcal{E}_0| \lesssim U_0^{1 + \frac{2 + |\beta|}{p-1}}.
\]
(iii) If $|x| > 2$,
\[
|\partial_x^2 U_0| \lesssim |x|^{-\frac{3p-2}{2p-1} + |\beta|}, \quad |\partial_x^2 \mathcal{E}_0| \lesssim |x|^{-\frac{3p-2}{2p-1} - |\beta| + 2}.
\]
Furthermore, for any $x_0 \in \mathbb{R}^N$ such that $A(x_0) = 0$, for any $r > 0$, $0 < t \leq 1$,
\[
t^{-\frac{3}{2p-1} + \frac{n}{p+1}} \lesssim \|U_0(t)\|_{L^2((|x-x_0|<r)} \lesssim t^{-\frac{3}{2p}}, \tag{2.7}
\]
\[
t^{-\frac{3p-2}{2p-1} - \frac{n}{p+1}} \lesssim \|\partial_t U_0(t)\|_{L^2(|x-x_0|<r)} \lesssim t^{-\frac{3p-2}{2p-1}}, \tag{2.8}
\]
where the implicit constants in (2.7) and (2.8) depend on $r$.

**Proof.** The identities in (2.3) follow from the definition of $U_0$ and direct calculations.

Proof of (2.4)-(2.5). For $0 < t \leq 1$ and $|x| \leq 2$, one has $0 < t + A(x) \lesssim 1$ and thus $U_0 \gtrsim 1$. From $U_0 = h \circ W$, setting $n = |\beta|$ and using (1.15), one has
\[
\partial_x^2 U_0 = \sum_{\nu=1}^n \left( h^{(\nu)} \circ W \right) \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \left( \frac{\partial^2 W}{\nu_\ell!} \right)^{\nu_\ell}.
\]
For $\nu \geq 1$, we have $|h^{(\nu)} \circ W| \lesssim W^{-\frac{3}{2p-1} - \nu}$. Moreover, using the assumption (2.2), we have, for $1 \leq |\beta| \leq k - 1$,
\[
|\partial_x^2 W| \lesssim |\partial_x^2 A| \lesssim A^{1 - |\beta|}.
\]
Since $\sum_{\ell=1}^n \nu_\ell = n, \sum_{\ell=1}^n \nu_\ell |\beta_\ell| = |\beta|$ and $|\beta| \leq k - 1$, we obtain
\[
|\partial_x^2 U_0| \lesssim \sum_{\nu=1}^n W^{-\frac{3}{2p-1} - \nu} \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \left( A^{1 - |\beta|} \right)^{\nu_\ell}
\]
\[
\lesssim \sum_{\nu=1}^n W^{-\frac{3}{2p-1} - \nu} A^{1 - |\beta|} \lesssim W^{-\frac{3}{2p-1} - \frac{|\beta|}{p+1}} \lesssim U_0^{1 + \frac{|\beta|}{p+1}},
\]
which proves the first estimate of (2.4) for $\rho = 1$. For $\rho \in \mathbb{R}$, using (1.15), we also have, for $1 \leq n = |\beta| \leq k - 1$,
\[
\partial_x^2(U_0^\rho) = \sum_{\nu=1}^n \left[ \rho \cdots (\rho - \nu + 1) \right] U_0^{\rho - \nu} \sum_{P(\beta, \nu)} (\beta!) \prod_{\ell=1}^n \left( \frac{\partial^2 U_0}{\nu_\ell!} \right)^{\nu_\ell}.
\]
Using the above estimate on $|\partial_x^2 U_0|$ and $\sum_{\ell=1}^n \nu_\ell = n, \sum_{\ell=1}^n \nu_\ell |\beta_\ell| = |\beta|$, we obtain
\[
|\partial_x^2(U_0^\rho)| \lesssim \sum_{\nu=1}^n U_0^{\rho - \nu} \sum_{P(\beta, \nu)} \prod_{\ell=1}^n U_0^{\nu_\ell \left[ 1 + \frac{|\beta_\ell|}{p+1} \right]} \lesssim \sum_{\nu=1}^n U_0^{\rho - \nu} U_0^{\nu + \frac{|\beta|}{p+1}} \lesssim U_0^{\rho + \frac{|\beta|}{p+1}}.
\]
Next, using the first identity in (2.3), we see that $\partial_t U_0^\rho = -\rho \left( \frac{n}{p+1} \right)^{\frac{3}{p-1}} U_0^{\rho + \frac{p-1}{p+1}}$; and so the second estimate in (2.4) follows from the first. Since $\mathcal{E}_0 = \Delta U_0$, (2.5) is an immediate consequence of the first estimate in (2.4).

Estimate (2.6) is a direct consequence of the definitions of $U_0$ and $\mathcal{E}_0$ and of the fact that $A(x) = |x|^k$ for $|x| > 2$.

Proof of (2.7)-(2.8). For any $x_0 \in \mathbb{R}^N$ and $r > 0$, the upper bounds in (2.7) and (2.8) are direct consequences of the estimates $0 \leq U_0 \lesssim t^{-\frac{3}{2p-1}}$ and $|\partial_t U_0| \lesssim t^{-\frac{3p-2}{2p-1}}$. Let $x_0 \in \mathbb{R}^N$ be such that $A(x_0) = 0$ and $r > 0$. By (2.2) and the fact that the function $A$ is of class $C^k$ piecewise, the Taylor formula implies that for any
Remark 2.4. Estimate (2.8) is proved similarly. □

2.3. Refined blow-up ansatz. Starting from $U_0$, we define by induction a refined ansatz to the nonlinear wave equation. Let $t_0 = 1$ and for any $j \in \{1, \ldots, J\}$, let $0 < a_j \leq 1$ and $0 < t_j \leq 1$ to be chosen later. Let

$$w_j = -\kappa \frac{p-1}{3p+1} \left( U_0 \frac{2}{p-1} \int_0^t U_0^{-p} \mathcal{E}_j^{-1} ds + U_0^{-p} \int_0^{t_j-1} U_0 \frac{2}{p-1} \mathcal{E}_j^{-1} ds \right),$$

$$U_j = U_0 + \sum_{t=1}^j \chi_t w_t, \quad \mathcal{E}_j = -\partial_t U_j + \Delta U_j + f(U_j),$$

where $\chi_j(x) = \chi(A(x)/a_j)$ and $\chi$ satisfies (1.9).

Lemma 2.3. There exist $0 < a_j \leq \cdots \leq a_1 \leq 1$ and $0 < t_j \leq \cdots \leq t_1 \leq 1$ such that for any $0 \leq j \leq J$, for any $\beta \in \mathbb{N}^N$, $0 < t \leq t_j$ and $x \in \mathbb{R}^N$, the following hold.

(i) If $1 \leq j \leq J$, $0 \leq |\beta| \leq k - 2j$, $|x| \leq 2$, then

$$|\partial_x^\beta w_j| \lesssim U_0^{1-j/p - \frac{2|j| + |\beta|}{p-1}},$$

$$|\partial_t \partial_x^\beta w_j| \lesssim U_0^{\frac{p+1}{2} - j/p - \frac{3|j| + |\beta|}{p-1}}.$$  

(ii) If $1 \leq j \leq J$, then

$$|U_j - U_0| \leq \frac{1}{4} (1 - 2^{-j}) U_0, \quad |U_j - U_0| \leq (1 - 2^{-j})(1 + U_0)^{-\frac{p+1}{2}} U_0,$$

$$|\partial_t U_j - \partial_t U_0| \lesssim U_0.$$  

(iii) If $0 \leq |\beta| \leq k - 3 - 2j$, $|x| \leq 2$, then

$$|\partial_x^\beta \mathcal{E}_j| \lesssim U_0^{1-j/p - \frac{2|j| + |\beta|}{p-1}}.$$  

(iv) If $|x| \geq 2$, then

$$|\partial_x^\beta U_j| \lesssim |x|^{-\frac{2\beta}{p-1}}, \quad |\partial_x^\beta \mathcal{E}_j| \lesssim |x|^{-\frac{2\beta}{p-1}}.$$  

Remark 2.4. We comment on the mechanism of the refined ansatz. For the energy control which we establish in the next section, we need an estimate on the error term $\|E_j\|_{L^2} \lesssim t(\frac{p-1}{p})^+. \quad \text{ (See formulas (3.20) and (3.21).)}$ By formula (2.14), this is achieved if $J > \frac{1}{p-1}$, which is the first condition in (2.1), and then $k$ sufficiently large (once $J$ is chosen), which is the second condition in (2.1). Note that for $p > 3$, $J = 1$ is enough, but one can never choose $J = 0$, so a refined ansatz is always needed. We see on formula (2.14) that at each step, the error estimate improves by a factor $U_0^{-p/(1-\frac{1}{p})} \sim t^2(1-\frac{1}{p})$. It is clear then that the number of steps goes to $\infty$ as $p \to 1$. 

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x such that $|x - x_0| < r$, $|A(x)| \leq C(r)|x - x_0|^k$. It follows that for such $x$, and for any $t \in (0, 1]$, $U_0^\alpha(t, x) = \kappa^2(t + A(x))^{-\frac{4}{p-1}} \geq (t + |x - x_0|^k)^{-\frac{4}{p-1}}$. The lower estimate in (2.7) then follows from

$$\int_{|y| < r} (t + |y|)^{-\frac{4}{p-1}} dy = t^{-\frac{4}{p-1} + \frac{N}{4}} \int_{|z| < t^{-\frac{1}{4} r}} (1 + |z|^k)^{-\frac{4}{p-1}} dz \geq t^{-\frac{4}{p-1} + \frac{N}{4} + \frac{3}{4}}.$$  

Estimate (2.8) is proved similarly. □
Proof of Lemma 2.3. Observe that (2.14) for \( j = 0 \) is exactly (2.5) in Lemma 2.2. Now, we proceed by induction on \( j \): for any \( 1 \leq j \leq J \), we prove that estimate (2.14) for \( E_{j-1} \) implies estimates (2.10)–(2.14) for \( w_j, U_j \) and \( E_j \), for an appropriate choice of \( a_j \) and \( t_j \).

Proof of (2.10)-(2.11). First, assuming (2.14) for \( E_{j-1} \), we show the following estimates related to the two components of \( w_j \): for \( |\beta| \leq k - 1 - 2j \), \( 0 < t \leq t_{j-1} \) and \( |x| \leq 2 \),

\[
\left| \partial_x^\beta \left( \int_0^t U_0^{-p} E_{j-1} ds \right) \right| \lesssim U_0^{-\frac{p-1}{2} - j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}}, \tag{2.16}
\]

\[
\left| \partial_x^\beta \left( \int_0^{t-j-1} U_0^{-p} E_{j-1} ds \right) \right| \lesssim U_0^{p+1 - j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}}. \tag{2.17}
\]

Indeed, we have by the Leibniz’s formula (1.14)

\[
\partial_x^\beta (U_0^{-p} E_{j-1}) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left( \partial_x^{\beta'} (U_0^{-p}) \right) \left( \partial_x^{\beta-\beta'} E_{j-1} \right),
\]

and thus, using (2.4) and (2.14),

\[
\left| \partial_x^\beta (U_0^{-p} E_{j-1}) \right| \lesssim \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left| U_0^{-p+|\beta'|} \frac{p-1}{2} \right| \left| U_0^{-1 - (j-1)(p-1) + \frac{2 + |\beta'|}{k} \frac{p-1}{2}} \right|
\]

\[
\lesssim U_0^{-j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}} \lesssim (t + A)^{\gamma},
\]

where for \( j \geq 1 \), \( |\beta| \leq k \),

\[
\gamma := 2j - \frac{2j + |\beta|}{k} = 2j \left( 1 - \frac{1}{k} \right) - \frac{|\beta|}{k} \geq 0.
\]

Integrating on \((0, t)\) for \( t \in (0, t_{j-1}] \), we obtain

\[
\left| \partial_x^\beta \left( \int_0^t U_0^{-p} E_{j-1} ds \right) \right| \leq (t + A)^{\gamma+1} \leq U_0^{-\frac{p-1}{2} - j(1 - \frac{1}{k})(p-1) + \frac{|\beta|}{k} \frac{p-1}{2}},
\]

which is (2.16). Similarly, using Leibniz’s formula, we check the following estimate

\[
\left| \partial_x^\beta (U_0^{p+1} E_{j-1}) \right| \lesssim U_0^{\frac{p+1}{p} - j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}} \lesssim (t + A)^{-\gamma'},
\]

where, using, \( 0 < j \leq J \leq \frac{p+1}{p-1} \),

\[
\gamma' := 3p + 1 - 2j + \frac{2j + |\beta|}{k} > 1.
\]

Thus, by time integration, for \( t \in (0, t_{j-1}] \),

\[
\left| \int_t^{t_{j-1}} \partial_x^\beta (U_0^{p+1} E_{j-1}) ds \right| \lesssim (t + A)^{-\gamma'+1} \lesssim U_0^{p+1 - j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}},
\]

which is (2.17).

Using Leibniz’s formula, (2.4), and (2.16)-(2.17), we deduce easily that, for any \( \beta \in \mathbb{N}^N \), \( |\beta| \leq k - 1 - 2j \),

\[
\left| \partial_x^\beta \left( \int_0^t U_0^{p+1} \int_0^t U_0^{-p} E_{j-1} ds \right) \right| \lesssim U_0^{1 - j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}},
\]

\[
\left| \partial_x^\beta \left( \int_0^t U_0^{-p} \int_t^{t-j-1} U_0^{p+1} E_{j-1} ds \right) \right| \lesssim U_0^{1 - j(p-1) + \frac{2 + |\beta|}{k} \frac{p-1}{2}}.
\]

Estimate (2.10) follows. Moreover, by the definition of \( w_j \) and setting \( b = \kappa \frac{p+1}{p} \frac{p-1}{4p+1} \),

\[
\partial_t w_j = -b \left( \partial_t (U_0^{p+1}) \right) \int_0^t U_0^{-p} E_{j-1} ds + \partial_t (U_0^{-p}) \int_t^{t-j-1} U_0^{p+1} E_{j-1} ds. \tag{2.18}
\]
Similarly as above, Leibniz’s formula, (2.4), and (2.16)-(2.17) yield (2.11). Note that we have proved estimates (2.10) and (2.11) for all $0 < t \leq t_j - 1$.

Proof of (2.12)-(2.13). For $0 < t \leq t_j - 1$ and $|x| \leq 2$, by the estimate (2.4) on $w_j$ for $\beta = 0$, the property $U_0 \gtrsim 1$ for $|x| \leq 2$, and the definition of $\chi_j$, we have

$$\chi_j w_j \lesssim \chi_j U_0^{1-j(1-\frac{1}{p})(p-1)} \lesssim \chi_j U_0^{1-(1-\frac{1}{p})(p-1)} \lesssim \chi_j (t + A)^{2-\frac{5}{2}} U_0 \lesssim (t + a_j) U_0.$$

Choosing $0 < a_j \leq 1$ and $0 < t_j \leq t_j - 1$ sufficiently small, for all $t \in (0, t_j]$, 

$$\chi_j w_j \lesssim 2^{-j-2} U_0 \quad \text{and} \quad \chi_j w_j \lesssim 2^{-j}(1 + U_0)^{-\frac{p-1}{2}} U_0.$$ 

From now on, $a_j$ and $t_j$ are fixed to such values. In the case $j = 1$, this proves (2.12) for $|x| \leq 2$. For $2 \leq j \leq J$, combining this estimate with (2.12) for $j - 1$, we find, for all $t \in (0, t_j]$ and $|x| \leq 2$,

$$\sum_{\ell=1}^j \chi_{\ell} |w_{\ell}| \leq \frac{1}{4} (1 - 2^{-j}) U_0 \quad \text{and} \quad \sum_{\ell=1}^j \chi_{\ell} |w_{\ell}| \leq (1 - 2^{-j})(1 + U_0)^{-\frac{p-1}{2}} U_0,$$

which implies (2.12) for $U_j$ and for $|x| \leq 2$.

To prove (2.13) for $|x| \leq 2$, we note that by (2.11) with $\beta = 0$ and $U_0 \gtrsim 1$,

$$\sum_{\ell=1}^j \chi_{\ell} \partial_{t} w_{\ell} \lesssim \sum_{\ell=1}^j \chi_{\ell} U_0^{1-j(1-\frac{1}{p})(p-1)} = U_0 \sum_{\ell=1}^j \chi_{\ell} U_0^{1-j(1-\frac{2}{p}(1-\frac{1}{p}))} \lesssim U_0.$$ 

For $|x| \geq 2$, (2.2) implies that $A(x) \gtrsim 2^k \gtrsim 2a_1 \geq \cdots \geq 2a_j$ and thus $\chi_j(x) = 0$ and $U_j(t, x) = U_0(t, x)$. The same applies to $\partial_t U_j$.

Proof of (2.14). Differentiating (2.18) with respect to $t$, using the relations (2.3),

$$\partial_t (U_0^{p-1}) = f'(U_0) U_0^{p-1} \quad \text{and} \quad \partial_t (U_0^p) = f'(U_0) U_0^p \quad \text{(these calculations are related to observations made in Section 2.1)},$$

we check that $w_j$ satisfies

$$\partial_t w_j = f'(U_0) w_j + E_{j-1}.$$ 

Using also $U_j = U_{j-1} + \chi_j w_j$ and the definition of $E_{j-1}$, we obtain

$$E_j = E_{j-1} - \chi_j \partial_{tt} w_j + \Delta (\chi_j w_j) + f(U_j) - f(U_{j-1}) = (1 - \chi_j) E_{j-1} + \Delta (\chi_j w_j) + f(U_j) - f(U_{j-1}) - f'(U_0) \chi_j w_j.$$ 

We estimate $\partial_t^k$ of each term on the right-hand side above for $|\beta| \leq k - 3 - 2j$ and $|x| \leq 2$. For the first term, recall that for $x$ such that $A(x) \leq a_j$, it holds $1 - \chi_j(x) = 0$ and for any $\beta$, $\partial^\beta \chi_j(x) = 0$. Moreover, for $0 < t \leq 1$, for $0 \leq x \leq 2$ such that $A(x) \geq a_j$, it holds $A(x) \approx 1$ and so $U_0(t, x) \approx 1$. Thus, using the Leibniz formula and (2.14) for $E_{j-1}$, we find

$$|\partial_t^k|(1 - \chi_j) E_{j-1}| \lesssim U_0^{1-j(1-\frac{1}{p})(p-1)} \frac{2^{j+k+1}|\beta|}{2^{j+k+2}} \lesssim U_0^{1-j(1-\frac{2}{p}(1-\frac{1}{p}))} \frac{2^{j+k+1}|\beta|}{2^{j+k+2}} \lesssim U_0^{1-j(1-\frac{2}{p}(1-\frac{1}{p}))} \frac{p-1}{2^{j+k+2}}.$$ 

Next, by the Leibniz’s formula, the properties of $\chi$ and $\chi_j$, the estimate (2.10) on $w_j$ and then $U_0 \gtrsim 1$, we have, for $0 < t < t_j$ and $|x| \leq 2$,

$$|\partial_t^k \Delta (\chi_j w_j)| \lesssim \sum_{|\beta| \leq |\beta|+2} |\partial_x^\beta w_j| \lesssim U_0^{1-j(1-\frac{2}{p}(1-\frac{1}{p}))} \frac{p-1}{2^{j+k+2}}.$$ 

Last, we estimate $\partial_t^k [f(U_j) - f(U_{j-1}) - f'(U_0) \chi_j w_j]$. We begin with the case $\beta = 0$. Recall that by (2.12), we have $0 < \frac{3}{4} U_0 \leq U_j \leq \frac{5}{4} U_0$, so that by elementary calculations

$$|f(U_j) - f(U_{j-1}) - f'(U_{j-1}) \chi_j w_j| \lesssim \chi_j U_0^{p-2} w_j^2$$ 

and

$$|f'(U_{j-1}) - f'(U_0)| \lesssim U_0^{p-2} \sum_{\ell=1}^{j-1} \chi_{\ell} |w_{\ell}|.$$
Moreover, by the Faà di Bruno’s formula (1.15), for \( \beta \leq 1 \), we have
\[
U_0^{-2} |w_j| |w_k| \lesssim U_0^{-2} \left[ U_0^{-j(1 - \frac{1}{2})(p - 1)} U_0^{-\ell(1 - \frac{1}{2})(p - 1)} \right] \lesssim U_0^{-\ell(1 - \frac{1}{2})(p - 1)} \lesssim U_0^{-j(1 - \frac{1}{2})(p - 1) + \frac{1}{2}(p - 1)}.
\]
Thus, \( |f(U_j) - f(U_j - 1) - f'(U_0)\chi_j w_j| \lesssim U_0^{0 - j(1 - \frac{1}{2})(p - 1) + \frac{1}{2}(p - 1)} \) is proved.

Now, we deal with the case \( 1 \leq |\beta| \leq k - 3 - 2j \). By the Taylor formula with integral remainder, we have, for any \( U \) and \( w \),
\[
f(U + w) - f(U) - f'(U)w = w^2 \int_0^1 (1 - \theta)f''(U + \theta w)d\theta.
\]
Thus, by the Leibniz’s formula (1.14)
\[
\partial_\beta^2 [f(U + w) - f(U) - f'(U)w] = \sum \left( \int \partial_\beta^2 (u^2) \right) \int_0^1 (1 - \theta)\partial_\beta^2 [f''(U + \theta w)]d\theta.
\]
Moreover, by the Faà di Bruno’s formula (1.15), for \( \beta' \neq 0 \), denoting \( n' = |\beta'| \),
\[
\partial_\beta^2 [f''(U + \theta w)] = \sum_{\nu = 1}^{n'} f^{(\nu + 2)}(U + \theta w) \sum_{P(\beta', \nu)} (\beta') \prod_{\nu = 1}^{n'} \frac{\partial_\beta^2 (U + \theta w)}{\nu! (\beta')^\nu}
\]
To estimate the term \( \partial_\beta^2 [f(U_j) - f(U_j - 1) - f'(U_0)\chi_j w_j] \), we apply these formulas to \( U = U_j \) and \( w = \chi_j w_j \). For \( \beta' \leq \beta \), using (2.10) and the properties of \( \chi \), we have
\[
|\partial_\beta^2 [(\chi_j w_j)^2]| \lesssim \sum_{\beta'' \leq \beta - \beta'} |\partial_\beta^2 (\chi_j w_j)| |\partial_\beta^{\beta''} (\chi_j w_j)| \lesssim U_0^{-2(\ell(1 - \frac{1}{2})(p - 1) + \frac{1}{2}(p - 1))}.\]
For \( \beta' = 0 \) and \( \theta \in [0, 1] \), using also (2.19), we obtain
\[
|\partial_\beta^2 [(\chi_j w_j)^2]| f''(U_0 + \theta \chi_j w_j)| \lesssim U_0^{-2j(1 - \frac{1}{2})(p - 1) + \frac{2j(1 + |\beta'|)}{p - 1} - 2} U_0^{-2} \lesssim U_0^{-j(1 - \frac{1}{2})(p - 1) + \frac{2j(1 + |\beta'|)}{p - 1} - 2}.
\]
For \( \beta' \neq 0 \), \( \beta' \leq \beta \) and \( \theta \in [0, 1] \), using (2.4), (2.10) and (2.19), we have (recall that the definition of \( P(\beta', \nu) \) implies that \( \sum_{\nu = 1}^{n'} \nu_\ell = \nu \) and \( \sum_{\nu = 1}^{n'} \nu_\ell |\beta_\ell| = |\beta'| \))
\[
|\partial_\beta^2 [(U_j - 1 + \theta \chi_j w_j)]| \lesssim \sum_{\nu = 1}^{n'} U_0^{- \nu - 2} \sum_{P(\beta', \nu)} \left( U_0^{1 + \frac{|\beta'|}{p - 1}} \right)^\nu \lesssim \sum_{\nu = 1}^{n'} U_0^{- \nu - 2 \nu^{\nu + \frac{|\beta'|}{p - 1}} \frac{1}{\nu_\ell}} \lesssim U_0^{-\nu - 2 \nu^{\nu + \frac{|\beta'|}{p - 1}} \frac{1}{\nu_\ell}}.
\]
Thus, similarly as before, it holds
\[
|\partial_\beta^2 [(\chi_j w_j)^2]| \partial_\beta^2 [(U_j - 1 + \theta \chi_j w_j)]| \lesssim U_0^{-j(1 - \frac{1}{2})(p - 1) + \frac{2j(1 + |\beta'|)}{p - 1} - 2}.
\]
Integrating these estimates in \( \theta \in [0, 1] \), we obtain
\[
|\partial_\beta^2 [(f(U_j) - f(U_j - 1) - f'(U_j - 1)\chi_j w_j)]| \lesssim U_0^{-j(1 - \frac{1}{2})(p - 1) + \frac{2j(1 + |\beta'|)}{p - 1} - 2}.
\]
By similar arguments, for any $U, W, w$, we have

$$ f'(U) - f'(W) = (U - W) \int_0^1 f''(W + \theta(U - W)) d\theta, $$

and thus

$$ \partial_x^\beta [w(f'(U) - f'(W))] = \sum_{\beta' \leq \beta} \left( \frac{\beta}{\beta'} \right) \left( \partial_x^{\beta - \beta'} [w(U - W)] \right) \int_0^1 \partial_x^{\beta'} [f''(W + \theta(U - W))] d\theta. $$

Moreover, for $\beta' \neq 0$,

$$ \partial_x^\beta [f''(W + \theta(U - W))] = \sum_{\nu=1}^{n'} f^{(\nu+2)}(W + \theta(U - W)) \sum_{P(\beta', \nu)} \left( \frac{\beta'}{\nu} \prod_{\ell=1}^{n'} \frac{(\partial_x^{\beta'} (W + \theta(U - W)))^{\nu_p}}{(\nu_p!)^{(\beta_p)\nu_p}}. $$

To estimate the term $\partial^\beta [\chi_j w_j (f'(U_{j-1}) - f'(U_0))]$, we apply these formulas to $U = U_{j-1}, W = U_0$ and $w = \chi_j w_j$.

For $\beta' \leq \beta$, using (2.10) and the properties of $\chi$, we have, for $1 \leq \ell \leq j - 1$,

$$ |\partial_x^{\beta - \beta'} [\chi_j w_j x_\ell w_\ell]| \lesssim U_0^{2-(j+\ell)(p-1)+\frac{2j+2\ell+|\beta'|}{p}-\frac{1}{p}}. $$

For $\beta' = 0$ and $\theta \in [0, 1]$, from (2.19), we obtain

$$ \left| \partial_x^\beta [\chi_j w_j (U_{j-1} - U_0)] f''(U_0 + \theta(U_{j-1} - U_0)) \right| \lesssim U_0^{1-j(p-1)+\frac{2j+2\ell+|\beta'|}{p}-\frac{1}{p}}. $$

For $\beta' \neq 0$, $\beta' \leq \beta$ and $\theta \in [0, 1]$, by the formula above, using (2.4), (2.10) and (2.19), we have as before

$$ |\partial_x^\beta [\chi_j w_j(U_{j-1} - U_0)] f''(U_0 + \theta(U_{j-1} - U_0))| \lesssim U_0^{1-j(p-1)+\frac{2j+2\ell+|\beta'|}{p}-\frac{1}{p}}. $$

Thus, we obtain

$$ \left| \partial_x^{\beta - \beta'} [\chi_j w_j(U_{j-1} - U_0)] \right| \lesssim U_0^{1-j(p-1)+\frac{2j+2\ell+|\beta'|}{p}-\frac{1}{p}}. $$

Integrating in $\theta \in [0, 1]$ and summing in $\beta' \leq \beta$, we obtain

$$ \left| \partial^\beta [\chi_j w_j (f'(U_{j-1}) - f'(U_0))] \right| \lesssim U_0^{1-j(p-1)+\frac{2j+2\ell+|\beta'|}{p}-\frac{1}{p}}. \quad (2.21) $$

Combining (2.20) and (2.21), we have proved for $t \in (0, t_j], |x| \leq 2$,

$$ \left| \partial^\beta [f(U_j) - f(U_{j-1}) - \chi_j w_j ] \right| \lesssim U_0^{1-j(p-1)+\frac{2j+2\ell+|\beta'|}{p}-\frac{1}{p}}. $$

In conclusion, we have estimated all terms in the expression of $\partial_x^\beta E_j$ and (2.14) is now proved.

Finally, for $|x| \geq 2$, (2.2) implies that $A(x) \geq 2^k \geq 2a_1 \geq \cdots \geq 2a_j$ and thus $\chi_j(x) = 0, U_j(t,x) = U_0(t,x)$ and $E_j(t,x) = E_0(t,x)$, so that (2.15) follows from (2.6).
3. Uniform bounds on approximate solutions

Let the function \( \chi \) be given by (1.9) and \( U_J \) be defined as in \S 2.3 with \( J \) and \( k \) as in (2.1). Set

\[
\lambda = \min \left( J - \frac{2}{p - 1}, \frac{1}{2} \right) \in \left( 0, \frac{1}{2} \right],
\]
and impose the following additional condition on \( k \)

\[
k \geq \frac{2(p+1)}{\lambda(p-1)} + 2.
\]

For any \( n \) large, let \( T_n = \frac{1}{n} < t_J \) and

\[
B_n = \sup_{t \in [T_n, t_J]} \|U_J(t)\|_{L^\infty} \quad \text{so that} \quad \lim_{n \to \infty} B_n = \infty.
\]

We let \( n \) be sufficiently large so that \( B_n > 1 \), and we define the function \( f_n : \mathbb{R} \to [0, \infty) \) by

\[
f_n(u) = f(u)\chi \left( \frac{u}{B_n} \right) \quad \text{so that} \quad f_n(u) = \begin{cases} f(u) & \text{for } |u| < B_n, \\ 0 & \text{for } u > 2B_n. \end{cases}
\]

Let \( F_n(v) = \int_v^u f_n(w)dw \). It follows from elementary calculations that for every \( \alpha \in \mathbb{N} \), there exists a constant \( C_\alpha > 0 \) independent of \( n \), such that for all \( u > 0 \),

\[
|f_n^{(\alpha)}(u)| \leq C_\alpha u^{p-\alpha}.
\]

In particular, we observe that Taylor’s estimates such as (1.11)–(1.13) still hold for \( F_n \) and \( f_n \) with constants independent of \( n \). We will refer to these inequalities for \( F_n \) and \( f_n \) with the same numbers (1.11), (1.12) and (1.13). In this proof, any implicit constant related the symbol \( \lesssim \) is independent of \( n \).

We define the sequence of solutions \( u_n \) of

\[
\begin{cases}
\partial_t u_n - \Delta u_n = f_n(u_n) \\
u_n(T_n) = U_J(T_n), \quad \partial_t u_n(T_n) = \partial_t U_J(T_n).
\end{cases}
\]

The nonlinearity \( f_n \) being globally Lipschitz, the existence of a global solution \((u_n, \partial_t u_n)\) in the energy space is a consequence of standard arguments from semigroup theory. Using energy estimates, we prove uniform bounds on \( u_n \) in the energy space. For this we set, for all \( t \in [T_n, t_J] \),

\[
u_n(t) = U_J(t) + \epsilon_n(t),
\]

so that \((\epsilon_n, \partial_t \epsilon_n) \in C([T_n, t_J], H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N))\).

**Proposition 3.1.** There exist \( C > 0 \), \( n_0 > 0 \) and \( 0 < \delta_0 < 1 \) such that

\[
\| (\epsilon_n(t), \partial_t \epsilon_n(t)) \|_{H^1 \times L^2} \leq C(t - T_n)^{\lambda}.
\]

for all \( n \geq n_0 \) and \( t \in [T_n, T_n + \delta_0] \), where \( \lambda \) is given by (3.1).

**Proof.** The equation of \( \epsilon_n \) on \([T_n, t_J] \times \mathbb{R}^N\) is

\[
\begin{cases}
\partial_t \epsilon_n - \Delta \epsilon_n = f_n(U_J + \epsilon_n) - f_n(U_J) + \mathcal{E}_J \\
\epsilon_n(T_n) = 0, \quad \partial_t \epsilon_n(T_n) = 0
\end{cases}
\]

where we have used from (3.3) and (3.4) that \( f(U_J) = f_n(U_J) \) on \([T_n, t_J] \times \mathbb{R}^N\).

Define the auxiliary function \( z \) as follows

\[
\epsilon_n = Q^{\frac{1}{4}} z \quad \text{where} \quad Q = (1 - \chi + U_0)^{p+1}.
\]
We note that $Q \gtrsim 1$, $Q \lesssim t^{-\frac{2(p+1)}{p-1}}$. Moreover, it follows from (2.4) that $|\nabla U_0| \lesssim Q^{\frac{p-1}{p+1}}$, from which we deduce easily that $|\nabla Q| \lesssim t^{-\frac{1}{2}} Q$. One proves similarly that $|\Delta Q| \lesssim t^{-3} Q$. To write the equation of $z$, we compute
\[\partial_t \varepsilon_n = \partial_t (Q^{\frac{1}{2}} z) = (\partial_t Q^{\frac{1}{2}}) z + (p+1)(1 - \chi + U_0)^{\frac{p-1}{p+1}} \partial_t U_0 \partial_z z + Q^{\frac{1}{2}} \partial_t z \]
Thus, setting $G = Q^{\frac{1}{2}} (f'(U_0) Q^{\frac{1}{2}} - \partial_t Q^{\frac{1}{2}})$, we obtain
\[\partial_t (Q \partial_t z) = Q^{\frac{1}{2}} \Delta (Q^{\frac{1}{2}} z) + Q^{\frac{1}{2}} \left( f_n(U_J + Q^{\frac{1}{2}} z) - f_n(U_J) - f'_n(U_J) Q^{\frac{1}{2}} z \right) + G z + Q^\frac{1}{2} E_J. \tag{3.10}\]
Let $\sigma = \frac{3}{4}$. We define the following weighted norm and energy functional for $z$,
\[
\mathcal{N} = \left( (Q \partial_t z)^2 + Q^2 |\nabla z|^2 + t^{-2\sigma} Q^2 z^2 \right)^{\frac{1}{2}},
\]
\[
\mathcal{H} = \int \left[ (Q \partial_t z)^2 + Q^2 |\nabla z|^2 + t^{-2\sigma} Q^2 z^2 \right. \\
- \left. Q \left( 2F_n(U_J + Q^{\frac{1}{2}} z) - 2F_n(U_J) - 2F'_n(U_J) Q^{\frac{1}{2}} z - F''_n(U_J) Q^2 z \right) \right].
\]
We remark that the first two terms in $\mathcal{H}$ are the energy for the linear part of equation (3.10). The third term yields the control of a weighted $L^2$ norm, and the last term is associated with the nonlinear terms in the equation.

**Step 1.** Coercivity of the energy. We claim that, for $0 < \delta \leq t_J$ and $0 < \omega \leq 1$ sufficiently small, for $n$ large, if $\mathcal{N} \leq \omega$ and $T_n \leq t \leq \delta$ then
\[
\mathcal{N}^2 \leq 2 \mathcal{H}, \tag{3.11}
\]
and
\[
\| (\varepsilon_n(t), \partial_t \varepsilon_n(t)) \|_{\dot{H}^1 \times L^2} \lesssim \mathcal{N}, \quad \| \varepsilon_n(t) \|_{L^2} \lesssim t^\sigma \mathcal{N}. \tag{3.12}
\]
Proof of (3.11). Let
\[
A_1 = |2F_n(U_J + Q^{\frac{1}{2}} z) - 2F_n(U_J) - 2F'_n(U_J) Q^{\frac{1}{2}} z - F''_n(U_J) Q^2 z|.
\]
The triangle inequality and the Taylor inequality (1.11) yield
\[
A_1 \lesssim \left[ 2F_n(U_J + Q^{\frac{1}{2}} z) - 2F_n(U_J) - 2F'_n(U_J) Q^{\frac{1}{2}} z - F''_n(U_J) Q^2 z \right] \\
+ |F''_n(U_0) - F''_n(U_J)| Q^2 z
\]
\[
\lesssim A_1
\]
where
\[
A_1 = Q^{\frac{p+1}{2}} |z|^{p+1} + U_J^{p-1} Q^{\frac{p+1}{2}} |z|^{p+1} + U_0^{p-2} |U_0 - U_J| Q^2 z. \tag{3.15}
\]
Using $U_J \lesssim U_0$ and $U_0 \lesssim Q^{\frac{p-1}{p+1}}$, we see that $U_J^{p-1} \lesssim Q^{\frac{p-1}{2}}$. Moreover, since $|U_0 - U_J| \lesssim (1 + U_0)^{-\frac{p-1}{p+1}} U_0 \lesssim Q^{-\frac{p-1}{p+1}} U_0$ (see (2.12)), we obtain $U_0^{p-2} |U_0 - U_J| \lesssim U_0^{p-1} Q^{-\frac{p-1}{p+1}} \lesssim Q^{\frac{p+1}{2}}$, and so
\[
A_1 \lesssim Q^{\frac{p+1}{2}} |z|^{p+1} + Q^{\frac{p+1}{2}} + Q^{\frac{p+1}{p+1}} |z|^{p+1} + Q^{\frac{p+1}{p+1}} z^2.
\]
It follows that
\[
\int Q A_1 \lesssim \int Q^{\frac{p+1}{2}} |z|^{p+1} + \int Q^{\frac{p+1}{2}} |z|^{p+1} + \int Q^{\frac{p+1}{p+1}} z^2.
\]
For the first term on the right-hand side above, we use $Q \gtrsim 1$, thus
\[
\int Q^{\frac{p+1}{2}} |z|^{p+1} = \int \lesssim \int |z|^{p+1} \lesssim \int |z|^{p+1}. \tag{3.17}
\]
Applying now (1.10), $|\nabla Q| \lesssim t^{-\frac{\delta}{2}} Q$, and the definition of $\mathcal{N}$,

$$
\int |Qz|^{p+1} \lesssim \left( \int |\nabla(Qz)|^2 \right)^{\frac{p}{2}} \left( \int Q^2 z^2 \right)^{\frac{p+1}{2}} \lesssim \left( \int Q^2 |\nabla z|^2 + t^{-\frac{\delta}{2}} \int Q^2 z^2 \right)^{\frac{p}{2}} \left( \int Q^2 z^2 \right)^{\frac{p+1}{2}} \tag{3.18}
$$

In the case $1 < p \leq 2$, one has $\tilde{p} = p$ and the second term is identical to the first one. In the case $p > \tilde{p} = 2$, the second term $Q^{\frac{\tilde{p}}{p-\tilde{p}}} |z|^3 = Q^{\frac{\tilde{p}}{p-\tilde{p}}} |z|^3$ is estimated as follows (using $|z|^3 \lesssim a^{p-2} |z|^{p+1} + \frac{1}{a} z^2$ with $a = Q^{\frac{\tilde{p}}{p-\tilde{p}}} Q^{-1} \lesssim 1$ and $Q^{\frac{\tilde{p}}{p-\tilde{p}}} \lesssim t^{-1}$)

$$
Q^{\frac{\tilde{p}}{p-\tilde{p}}} |z|^3 \lesssim Q^{\frac{\tilde{p}}{p-\tilde{p}}} Q^{p+1} |z|^{p+1} + Q^{\frac{\tilde{p}}{p-\tilde{p}}} z^2 \lesssim Q^{p+1} |z|^{p+1} + Q^{\frac{\tilde{p}}{p-\tilde{p}}} Q^2 z^2 \lesssim Q^{p+1} |z|^{p+1} + t^{-1} Q^2 z^2,
$$

and so

$$
\int Q^{\frac{p+1}{2}} Q^{\frac{p+1}{2}} |z|^{p+1} \lesssim \mathcal{N}^{p+1} + t^{2\sigma - 1} \mathcal{N}^2. \tag{3.19}
$$

Last, since $Q^{\frac{p+1}{2}} \lesssim t^{-1}$, we observe that

$$
\int Q^{\frac{p+1}{2}} Q^2 z^2 \lesssim t^{-1} \int Q^2 z^2 \lesssim t^{2\sigma - 1} \mathcal{N}^2.
$$

In conclusion, we have obtained $\int Q A_1 \lesssim \int Q A_1 \lesssim t^{2\sigma - 1} \mathcal{N}^2 + \mathcal{N}^{p+1}$, which implies that for $t$ and $\mathcal{N}$ small enough, $\mathcal{H} \geq \frac{1}{2} \mathcal{N}^2$.

**Proof of (3.12).** Since $\hat{\epsilon}_n = Q^2 z$, the inequality $\|\hat{\epsilon}_n\|_{L^2} \lesssim t^{-1} \mathcal{N}$ follows readily from the definition of $\mathcal{N}$ and $Q \gtrsim 1$. Next, using $|\nabla Q| \lesssim t^{-\frac{\delta}{2}} Q$, we see that

$$
\int |\nabla \hat{\epsilon}_n|^2 = \int \left| Q^{\frac{\delta}{2}} \nabla z + \frac{1}{2} Q^{-\frac{\delta}{2}} z \nabla Q \right|^2 \lesssim \int |\nabla z|^2 + t^{-\frac{\delta}{2}} \int Q^2 z^2 \lesssim \mathcal{N}^2.
$$

Last, using $|\partial_t Q| \lesssim Q^{\frac{p+1}{2}} Q \lesssim t^{-\frac{\delta}{2}} Q^{\frac{p+1}{2}}$, we have

$$
\int |\partial_t \hat{\epsilon}_n|^2 = \int \left| Q^{\frac{\delta}{2}} \partial_t z + \frac{1}{2} Q^{-\frac{\delta}{2}} z \partial_t Q \right|^2 \lesssim \int |\partial_t z|^2 + \int |\partial_t Q|^2 Q^{-1} z^2 \lesssim Q^2 |\partial_t z|^2 + t^{-1} \int Q^{\frac{p+1}{2}} z^2 \lesssim \mathcal{N}^2.
$$

This completes the proof of (3.12).

**Step 2.** Energy control. We claim that for $0 < \delta \leq t_j$ small enough and $C > 0$ large enough, for any $n$ large and for all $t \in [T_n, T_n + \delta]$

$$
\frac{d}{dt} \mathcal{H} \leq C \left[ t^{-1+\lambda} \mathcal{N} + t^{-\frac{\delta}{2}} \mathcal{N}^2 + \mathcal{N}^{p+1} \right]. \tag{3.20}
$$
Proof of (3.20). Taking the time-derivative of all the terms in $\mathcal{H}$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \mathcal{H} = \int (Q \partial_t z \partial_t (Q \partial_t z) + Q^2 \nabla z \cdot \nabla \partial_t z + t^{-2\sigma} Q^2 z \partial_t z) \\
- \int Q^2 \left( f_n(U_J + Q^{1/2} z) - f_n(U_J) - f_n'(U_0) Q^{1/2} z \right) \partial_t z \\
+ \int Q \partial_t Q |\nabla z|^2 + t^{-2\sigma} \int Q \partial_t Q z^2 - \sigma t^{-2\sigma-1} \int Q^2 z^2 \\
- \frac{1}{2} \int \partial_t Q \left( 2F_n(U_J + Q^{1/2} z) - 2F_n(U_J) - 2F_n''(U_0) Q^{1/2} z - F_n''(U_0) Q^2 z \right) \\
- \frac{1}{2} \int \partial_t Q \left( f_n(U_J + Q^{1/2} z) - f_n(U_J) - f_n'(U_0) Q^{1/2} z \right) Q^{1/2} z \\
- \frac{1}{2} \int Q \partial_t U_0 \left( 2f_n(U_J + Q^{1/2} z) - 2f_n(U_J) - 2f_n'(U_J) Q^{1/2} z - f_n''(U_0) Q^2 z \right) \\
- \frac{1}{2} \int Q \partial_t (U_J - U_0) \left( 2f_n(U_J + Q^{1/2} z) - 2f_n(U_J) - 2f_n'(U_J) Q^{1/2} z \right) \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\]

First, we note that $\partial_t Q = (p + 1)Q \frac{d}{dt} \partial_t U_0 \leq 0$, so that
\[
I_3 \leq -\sqrt{2(p + 1)} \int U_0 \frac{d}{dt} \left( \frac{Q^{p+1}}{2} |\nabla z|^2 - \sigma t^{-2\sigma-1} \int Q^2 z^2.\right.
\]

We now use equation (3.10) to replace the term $\partial_t (Q \partial_t z)$ in $I_1$, and we obtain
\[
I_1 + I_2 = \int \left( Q \frac{1}{2} \partial_t z \Delta (Q^{1/2} z) + Q^2 \nabla z \cdot \nabla \partial_t z \right) \\
+ \int \left( G_z + Q^{1/2} Q \partial_t z + t^{-2\sigma} \int Q^2 z \partial_t z = I_8 + I_9 + I_{10}.$
\]

The term $I_{10}$ is controlled using the Cauchy-Schwarz inequality,
\[
|I_{10}| \leq \frac{1}{10} |I_3| + Ct^{-2\sigma+1} \int (Q \partial_t z)^2 \leq \frac{1}{10} |I_3| + Ct^{-2\sigma+1} \|z\|^2 \leq \frac{1}{10} |I_3| + Ct^{-\frac{1}{2}} N^2.
\]

Next, integrating by parts,
\[
I_8 = -\int \nabla (Q^{1/2} z) \cdot \nabla (Q^{1/2} \partial_t z) + Q^2 \nabla z \cdot \nabla (\partial_t z) \\
= -\int \nabla (Q^{1/2}) \cdot \nabla (Q^{1/2} \partial_t z) - \int Q^{1/2} (\partial_t z) \nabla z \cdot \nabla (Q^{1/2}) \\
= -\int Q \partial_t z \nabla z \cdot \nabla Q + \int \Delta (Q^{1/2} Q^{1/2} z \partial_t z.
\]

By $|\nabla Q| \lesssim t^{-\frac{1}{2}} Q$ and the Cauchy-Schwarz inequality,
\[
\left| \int Q \partial_t z \nabla z \cdot \nabla Q \right| \lesssim t^{-\frac{1}{2}} N^2.
\]

Similarly, $|\Delta (Q^{1/2} Q^{1/2} z \partial_t z | \lesssim |\nabla Q|^2 + |\Delta Q|Q \lesssim t^{-\frac{1}{2}} Q^2$, and so
\[
\left| \int \Delta (Q^{1/2} Q^{1/2} z \partial_t z \right| \lesssim Q^2 |\partial_t z|^2 + t^{-\frac{1}{2}} \int Q^2 z^2 \lesssim N^2.
\]

We note that by Cauchy-Schwarz,
\[
|I_9| \lesssim \|G\|_{L^\infty} N^2 + \|Q^{1/2} Q \partial_t z\|_{L^2 N},
\]
and so, we only have to bound the $L^\infty$ norm of $G$ and the $L^2$ norm of $Q^{\frac{1}{2}}E_J$. We begin with $G = Q^{\frac{1}{2}} \left( pU_0^{p-1}Q^{\frac{1}{2}} - \partial_t Q^{\frac{1}{2}} \right)$. Using $Q = (1 - \chi + U_0)^{p-1}$ and the expressions of $\partial_t U_0$ and $(\partial_t U_0)^2$, we observe that

$$\partial_t Q^{\frac{1}{2}} = \frac{p-1}{2} U_0^{p-1}Q^{\frac{p-3}{2}} + \frac{p-1}{2} U_0^{p-1}Q^{\frac{p-3}{2}}.$$

Thus,

$$pU_0^{p-1}Q^{\frac{1}{2}} - \partial_t Q^{\frac{1}{2}} = \frac{p-1}{2} U_0^{p-1}Q^{\frac{p-3}{2}} + \frac{p-1}{2} U_0^{p-1}Q^{\frac{p-3}{2}} + \frac{p-1}{2} U_0^{p-1}Q^{\frac{p-3}{2}}(1 - \chi) + \frac{p-1}{2} U_0^{p-1}Q^{\frac{p-3}{2}}(1 - \chi)(1 - \chi + 2U_0).$$

Since for $|x| > 1$, we have $U_0 \lesssim 1$ and $Q \lesssim 1$, we obtain $\|G\|_{L^\infty} \lesssim 1$.

Now, we estimate $\|Q^{\frac{1}{2}}E_J\|_{L^2}$ from Lemma 2.3. For $|x| \geq 2$, it follows from (2.15) that

$$Q^{\frac{1}{2}}|E_J| \lesssim |E_J| \lesssim |x|^{-\frac{p+1}{2}}.$$ 

Note that for $N \geq 3$, $p - 1 \leq N_2$ and so $\frac{2k}{p-1} + 2 \geq N_2 k + 2 \geq N$. Thus, the following bound holds $\|Q^{\frac{1}{2}}E_J\|_{L^2(|x| \geq 2)} \lesssim 1$.

Next, using (2.14), we have for $|x| \lesssim 2$

$$Q^{\frac{1}{2}}|E_J| \lesssim Q^{\frac{1}{2}}U_0^{1-J(p-1)+\frac{(j+1)(p-1)}{k}} \lesssim U_0^{\frac{p+1}{2} - J(p-1)+\frac{(j+1)(p-1)}{k}}.$$

Note that by (3.1), $J \geq \gamma + \frac{2}{p-1}$, so that $-J(1 - \frac{1}{k}) + \frac{p-1}{k} \leq -2(1 - \frac{1}{k}) + \frac{p-1}{k} - \lambda(p - 1)(1 - \frac{1}{k})$; and so

$$Q^{\frac{1}{2}}|E_J| \lesssim U_0^{\frac{p+1}{2} - 2(1 - \frac{1}{k}) + \frac{p-1}{k} - \lambda(p - 1)(1 - \frac{1}{k}) \lesssim U_0^{\frac{p+1}{2} - \frac{p-1}{2} - \lambda(p - 1)(1 - \frac{1}{k}).$$

Moreover, the additional condition (3.2) is equivalent to $\frac{p+1}{2} - \lambda(p - 1)(1 - \frac{1}{k}) \leq -\frac{\lambda(p - 1)}{2}$. Thus, for $|x| \lesssim 2$,

$$Q^{\frac{1}{2}}|E_J| \lesssim U_0^{1-(\lambda + \frac{p-1}{2})} \lesssim (t + A(x))^{-1+\lambda} \lesssim t^{-1+\lambda}. \quad (3.21)$$

Therefore, one obtains $\|Q^{\frac{1}{2}}E_J\|_{L^2} \lesssim t^{-1+\lambda}$.

To complete the proof of (3.20), we estimate $I_4$, $I_5$, $I_6$, and $I_7$. First, using (3.13)–(3.16), and $|\partial_t Q| \lesssim |\partial_t U_0|Q^{\frac{1}{2}} \lesssim U_0^{\frac{p-1}{2}} Q^{\frac{1}{2}} \lesssim Q^{\frac{p+1}{2}}$, we obtain

$$|\partial_t Q| A_3 \lesssim Q^{\frac{p+1}{2} - \frac{(p-1)}{2}} \lesssim Q^{\frac{p+1}{2} - 2(1 - \frac{1}{k}) + \frac{p-1}{k} - \lambda(p - 1)(1 - \frac{1}{k}) \lesssim Q^{\frac{p+1}{2} - \frac{p-1}{2} - \lambda(p - 1)(1 - \frac{1}{k}).$$

Using $U_0 \gtrsim 1$ and the estimate (3.18), we treat the first term above as follows

$$\int Q^{p+1} \frac{p-1}{2} |z|^{p+1} \lesssim \int |Qz|^{p+1} \lesssim N^{p+1}.$$ 

In the case $1 < p \leq 2$, one has $p = p$ and the second term is identical to the first one. In the case $p > p = 2$, the second term $Q^{\frac{p}{2+1}}|z|^3$ is estimated as follows (using $|z|^3 \lesssim a^{-\frac{3}{2}}|z|^{p+1} + \frac{1}{2} z^2$ with $a = Q^{\frac{2}{p+1}}$, $Q^{-1} \lesssim 1$ and $Q^{\frac{p}{2+1}} \lesssim t^{-2}$)

$$Q^{\frac{p-1}{2+1}}|z|^3 \lesssim Q^{-\frac{p+1}{2}} Q^{p+1} |z|^{p+1} + Q^{\frac{p+1}{2}} Q^2 z^2 \lesssim Q^{p+1} |z|^{p+1} + t^{-2} Q^2 z^2.$$ 

Therefore

$$\int Q^{\frac{p+1}{2}} |z|^3 \lesssim N^{p+1} + t^{-2(1-\sigma)} N^2 \lesssim N^{p+1} + t^{-\frac{1}{2}} N^2.$$
Since $Q^{2+\frac{1}{2p-4}}z^2 \lesssim t^{-2}Q^2z^2$, we have proved

$$|I_4| \lesssim \int |\partial_t Q| \lesssim \int Q\frac{2}{2p-4} \Lambda_1 \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2.$$  \hspace{1cm} (3.22)

We proceed similarly for $I_5$. Indeed, setting

$$A_2 = |f_n(U_j + Q^2z) - f_n(U_j) - f'_n(U_0)Q^2z|Q^\frac{1}{2}z|$$

$$\leq |f_n(U_j + Q^2z) - f_n(U_j) - f'_n(U_j)Q^2z|Q^\frac{1}{2}z| + |f'_n(U_0) - f'_n(U_j)|Q^2z$$

we deduce from (1.12) and Taylor’s inequality that, with the notation (3.15),

$$A_2 \lesssim Q\frac{2}{2p-4}|z|^{p+1} + U_j^{2p-1}Q\frac{2}{2p-4}|z|^{p+1} + U_0^{p-2}|U_0 - U_j|Q|z|^2 \lesssim \Lambda_1.$$  

Using the last two inequalities in (3.22), we conclude that $|I_5| \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2$.

Now, we estimate $I_6$, and we set

$$A_3 = |2f_n(U_j + Q^2z) - 2f_n(U_j) - f'_n(U_j)Q^2z - f''_n(U_0)Q^2z|.$$  

By the triangle inequality, Taylor’s inequality (1.13), and $U_j^{-1} \lesssim U_0^{-1}$ (see (2.12)),

$$A_3 \lesssim |2f_n(U_j + Q^2z) - 2f_n(U_j) - f'_n(U_j)Q^2z - f''_n(U_0)Q^2z|$$

$$\lesssim |f'_n(U_0) - f'_n(U_j)|Q^2z$$

$$\lesssim U_j^{-1}Q\frac{2}{2p-4}|z|^{p+1} + U_j^{2p-1}Q\frac{2}{2p-4}|z|^{p+1} + U_0^{p-3}|U_0 - U_j|Q^2z$$

$$\lesssim U_0^{-1}Q\frac{2}{2p-4}|z|^{p+1} + U_j^{2p-1}Q\frac{2}{2p-4}|z|^{p+1} + U_0^{p-2}|U_0 - U_j|Q^2z$$

$$\lesssim U_0^{-1} \Lambda_1$$

with the notation (3.15). Using $|\partial_t U_0| \lesssim U_0^{2p-4}$ and $U_0 \lesssim Q\frac{1}{2p-4}$, we see that $Q|\partial_t U_0| \lesssim QU_0^{2p-4}U_0 \lesssim Q\frac{2}{2p-4}U_0$, hence $Q|\partial_t U_0|A_3 \lesssim Q\frac{2}{2p-4}\Lambda_1$. The last inequality in (3.22) yields $|I_6| \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2$.

Finally, we estimate $I_7$ and we set

$$A_4 = |f_n(U_j + Q^2z) - f_n(U_j) - f'_n(U_j)Q^2z|.$$  

By the triangle inequality Taylor’s expansion (1.13),

$$A_4 \lesssim |f_n(U_j + Q^2z) - f_n(U_j) - f'_n(U_j)Q^2z - \frac{1}{2}f''(U_j)Q^2z|$$

$$+ \frac{1}{2}|f''(U_j)|Q^2z$$

$$\lesssim U_j^{-1}Q\frac{2}{2p-4}|z|^{p+1} + U_j^{2p-1}Q\frac{2}{2p-4}|z|^{p+1} + U_0^{p-2}Q^2z^2$$

Using $Q|\partial_t(U_j - U_0)| \lesssim QU_0$ (see (2.13)), $U_j^{-1} \lesssim U_0^{-1}$, and $U_j \lesssim U_0$, we obtain

$$Q|\partial_t(U_j - U_0)|A_4 \lesssim Q\frac{2}{2p-4}|z|^{p+1} + U_0^{p-3}Q\frac{2}{2p-4}|z|^{p+1} + U_0^{p-1}Q^2z^2.$$  

Since $U_0 \lesssim Q\frac{2}{2p-4}$ and $U_0^{p-1} \lesssim t^{-2}$, we deduce that

$$Q|\partial_t(U_j - U_0)|A_4 \lesssim Q\frac{2}{2p-4}|z|^{p+1} + Q\frac{2}{2p-4} + U_0^{p-1}Q^2z^2.$$  

Applying (3.17)-(3.18) for the first term and (3.19) for the second term, we see that $|I_7| \lesssim \mathcal{N}^{p+1} + t^{-\frac{1}{2}}\mathcal{N}^2$. Collecting the above estimates, we have proved (3.20).

**Step 3.** Conclusion. The values of $\delta \in (0, t_3]$ and $0 < \omega \leq 1$ are now fixed so that (3.11), (3.12) and (3.20) hold. Since $\mathcal{N}(T_n) = 0$, the following is well-defined

$$T^*_n = \sup\{t \in [T_n, \delta] : \text{for all } s \in [T_n, t], \mathcal{N}(s) \leq \omega\}$$
and by continuity, \( T_n^* \in [T_n, \delta] \). For all \( t \in [T_n, T_n^*] \), using (3.20), we find (recall that \( \lambda \in (0, \frac{1}{2}) \))
\[
\frac{d}{dt} \mathcal{H} \leq C \left[ t^{-1+\lambda} + t^{-\frac{1}{2}} + 1 \right] \leq C t^{-1+\lambda}.
\]
Let \( t \in [T_n, T_n^*] \). Since \( \mathcal{H}(T_n) = 0 \), we obtain by integration on \([T_n, t] \)
\[
\mathcal{H}(t) \leq C(t^\lambda - T_n^\lambda) \leq C(t - T_n)^\lambda.
\]
Therefore, using the definition of \( T_n^* \) and (3.11), for all \( t \in [T_n, T_n^*] \),
\[
\mathcal{N}(t) \leq C(t - T_n)^{\frac{1}{2}}.
\]
In particular, there exists \( \delta_0 > 0 \) independent of \( n \) such that, for \( n \) large, it holds \( T_n^* \geq T_n + \delta_0 \). Moreover, using (3.12), for all \( t \in [T_n, T_n + \delta_0] \),
\[
\| (\varepsilon_n(t), \partial_t \varepsilon_n(t)) \|_{H^1 \times L^2} \lesssim \mathcal{N}(t) \lesssim (t - T_n)^{\frac{1}{2}},
\]
which completes the proof of Proposition 3.1. \( \square \)

4. End of the proof of Theorem 1.1

Let \( E \) be any compact set of \( \mathbb{R}^N \) included in the ball of center 0 and radius 1 (by the scaling invariance of equation (1.1), this assumption does not restrict the generality). It is well-known that there exists a smooth function \( Z : \mathbb{R}^N \to [0, \infty) \) which vanishes exactly on \( E \) (see e.g. Lemma 1.4, page 20 of [23]). For \( p \) as in (1.2), choose \( J \) and \( k \) satisfying (2.1) and (3.2). Define the function \( A : \mathbb{R}^N \to [0, \infty) \) by
\[
A(x) = (Z(x)\chi(x) + (1 - \chi(x))|x|)^k,
\]
where \( \chi \) is given by (1.9). It follows that the function \( A \) satisfies (2.2) and vanishes exactly on \( E \).

We consider the global solutions \( u_n \) of equation (3.6), \( \varepsilon_n \) defined by (3.7) and we set for \( 0 \leq t \leq t_J - T_n \),
\[
V_n(t) = U_J(T_n + t), \quad \eta_n(t) = \varepsilon_n(T_n + t), \quad F_n(t) = E_J(T_n + t).
\]
It follows from Proposition 3.1 that there exist \( 0 < \delta_0 < t_J, 0 < \lambda \leq \frac{1}{2} \), and \( C > 0 \) such that, for \( n \) large and for all \( t \in [0, \delta_0] \),
\[
\|(\eta_n(t), \partial_t \eta_n(t))\|_{H^1 \times L^2} \leq Ct^{\frac{1}{2}}.
\]
Moreover, it follows from (3.9) that
\[
\partial_t \eta_n - \Delta \eta_n = f_n(V_n + \eta_n) - f_n(V_n) + F_n.
\]
Using the estimate \( |f_n(u + v) - f_n(u)| \lesssim (|u|^{p-1} + |v|^{p-1})|v| \) and the embeddings \( H^1(\mathbb{R}^N) \hookrightarrow L^{p+1}(\mathbb{R}^N), L^{\frac{p+1}{2}}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N) \), we deduce that
\[
\| \partial_t \eta_n \|_{H^{-1}} \lesssim \| \eta_n \|_{H^1} + \| V_n \|_{L^{p+1}} \| \eta_n \|_{H^1} + \| \eta_n \|_{L^{p+1}} + \| F_n \|_{L^2},
\]
so that by the estimates of Lemmas 2.2 and 2.3, there exist \( C, c > 0 \) such that, for all \( t \in [0, \delta_0] \),
\[
\| \partial_t \eta_n \|_{H^{-1}} \leq Ct^{-c}.
\]
Given \( \tau \in (0, \delta_0) \), it follows from (4.1) and (4.3) that the sequence \( (\eta_n)_{n \geq 1} \) is bounded in \( L^\infty(\tau, \delta_0), H^1(\mathbb{R}^N)) \cap W^{1, \infty}((\tau, \delta_0), L^2(\mathbb{R}^N)) \cap W^{2, \infty}((\tau, \delta_0), H^{-1}(\mathbb{R}^N)) \).
Therefore, after possibly extracting a subsequence (still denoted by \( \eta_n \), there exists
\[ \eta \in L^\infty(\tau, \delta_0), H^1(\mathbb{R}^N) \cap W^{1, \infty}(\tau, \delta_0), L^2(\mathbb{R}^N) \cap W^{2, \infty}(\tau, \delta_0), H^{-1}(\mathbb{R}^N) \] such that
\[
\eta_n \rightarrow \eta \quad \text{in } L^\infty((\tau, \delta_0), H^1(\mathbb{R}^N)) \quad \text{weak}^* \tag{4.4}
\]
\[
\partial_t \eta_n \rightarrow \partial_t \eta \quad \text{in } L^\infty((\tau, \delta_0), L^2(\mathbb{R}^N)) \quad \text{weak}^* \tag{4.5}
\]
\[
\partial_t \eta_n \rightarrow \partial_t \eta \quad \text{in } L^\infty((\tau, \delta_0), H^{-1}(\mathbb{R}^N)) \quad \text{weak}^* \tag{4.6}
\]
\[
\eta_n \rightharpoonup \eta \quad \text{weakly in } H^1(\mathbb{R}^N), \quad \text{for all } t \in [\tau, \delta_0] \tag{4.7}
\]
\[
\partial_t \eta_n \rightharpoonup \partial_t \eta \quad \text{weakly in } L^2(\mathbb{R}^N), \quad \text{for all } t \in [\tau, \delta_0]. \tag{4.8}
\]

Since \( \tau \in (0, \delta_0) \) is arbitrary, a standard argument of diagonal extraction shows that there exists a function \( \eta \in L^\infty_{\text{loc}}((0, \delta_0), H^1(\mathbb{R}^N)) \cap W^{1, \infty}_{\text{loc}}((0, \delta_0), L^2(\mathbb{R}^N)) \cap W^{2, \infty}_{\text{loc}}((0, \delta_0), H^{-1}(\mathbb{R}^N)) \) such that (after extraction of a subsequence) (4.4)–(4.8) hold for all \( 0 < \tau < \delta_0 \). Moreover, (4.1) and (4.7)–(4.8) imply that
\[
\| (\eta(t), \partial_t \eta(t)) \|_{H^1 \times L^2} \leq Ct^\frac{1}{2}, \quad t \in (0, \delta_0), \tag{4.9}
\]
and (4.3) and (4.6) imply that
\[
\| \partial_t \eta \|_{L^\infty((\tau, \delta_0), H^{-1})} \leq C\tau^{-\gamma}, \quad \tau \in (0, \delta_0). \tag{4.10}
\]

In addition, it follows easily from (4.2), (3.3), (3.4) and the convergence properties (4.4)–(4.8) that
\[
\partial_t \eta - \Delta \eta = f(U_J + \eta) - f(U_J) + \mathcal{E}_J \tag{4.11}
\]
in \( L^\infty_{\text{loc}}((0, \delta_0), H^{-1}(\mathbb{R}^N)) \). Therefore, setting
\[
u(t) = U_J(t) + \eta(t), \quad t \in (0, \delta_0),
\]
we observe that the function \( u \in L^\infty_{\text{loc}}((0, \delta_0), H^1(\mathbb{R}^N)) \cap W^{1, \infty}_{\text{loc}}((0, \delta_0), L^2(\mathbb{R}^N)) \cap W^{2, \infty}_{\text{loc}}((0, \delta_0), H^{-1}(\mathbb{R}^N)) \) and satisfies \( \partial_t u - \Delta u = f(u) \) in \( L^\infty_{\text{loc}}((0, \delta_0), H^{-1}(\mathbb{R}^N)) \). It is a well-known property of the energy subcritical wave equation (corresponding to assumption (1.2)) that then it holds the stronger property
\[
u \in C((0, \delta_0), H^1(\mathbb{R}^N)) \cap C^1((0, \delta_0), L^2(\mathbb{R}^N)) \cap C^2((0, \delta_0), H^{-1}(\mathbb{R}^N)). \tag{4.12}
\]

We refer for example to Proposition 3.1 and Lemma 2.1 in [8].

Finally, we prove estimates (1.3) and (1.4). For \( x_0 \notin E \), there exist \( r > 0 \) and \( C > 0 \) such that \( A(x) \geq C \) for all \( x \in \mathbb{R}^N \) such that \( |x - x_0| < r \). In particular, for such \( x \), by (2.12) and (2.13), \( |U_J(x)| + |\partial_t U_J(x)| \leq C' \) for some constant \( C' > 0 \). Estimate (1.4) then follows from (4.9). For \( x_0 \in E \), (2.7), (2.8), (2.12) and (2.13) imply, for \( t \in (0, \delta_0) \),
\[
\begin{align*}
& t^{-\mu} \lesssim \| U_J(t) \|_{L^2(|x - x_0| < r)} \lesssim t^{-\frac{\mu}{2\tau}}, \\
& t^{-\mu - 1} \lesssim \| \partial_t U_J(t) \|_{L^2(|x - x_0| < r)} \lesssim t^{-\frac{\mu}{2\tau} - 1},
\end{align*}
\]
where \( \mu = \frac{2}{p-1} - \frac{N}{2p} \). Estimate (1.3), and more precisely estimates (1.6) and (1.7) then follow from (4.9).

Now, we justify the last part of Remark 1.2. If \( x_0 \in E \) and \( E \) contains a neighbourhood of \( x_0 \) then \( A(x) = 0 \) on this neighborhood and the lower estimate easily follows. In the case where \( x_0 \in E \) is isolated, the function \( A \) can be chosen so that \( A(x) = |x|^k \) in a neighbourhood of \( x_0 \) (see Remark 2.1). In particular, by (2.9) and a similar estimate for \( \partial_t U_0 \), we obtain for small \( r > 0 \),
\[
\begin{align*}
& \| u(t) \|_{L^2(|x - x_0| < r)} \lesssim t^{-\frac{k}{p} + \frac{N}{2p}} \quad \text{and} \\
& \| \partial_t u(t) \|_{L^2(|x - x_0| < r)} \lesssim t^{-1 - \frac{k}{p} + \frac{N}{2p}}.
\end{align*}
\]
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