Quantum Information and Spacetime Structure

Igor V. Volovich

Steklov Mathematical Institute, Russian Academy of Sciences,
Gubkin St.8,117966 Moscow, Russia
E-mail: volovich@mi.ras.ru

Abstract

In modern quantum information theory one deals with an idealized situation when the spacetime dependence of quantum phenomena is neglected. However the transmission and processing of (quantum) information is a physical process in spacetime. Therefore such basic notions in quantum information theory as qubit, channel, composite systems and entangled states should be formulated in space and time. In this paper some basic notions of quantum information theory are considered from the point of view of quantum field theory and general relativity. It is pointed out an important fact that in quantum field theory there is a statistical dependence between two regions in spacetime even if they are spacelike separated. A classical probabilistic representation for a family of correlation functions in quantum field theory is obtained. A noncommutative generalization of von Neumann’s spectral theorem is discussed. We suggest a new physical principle describing a relation between the mathematical formalism of Hilbert space and quantum physical phenomena which goes beyond the superselection rules. Entangled states and the change of state associated with the measurement process in space and time are discussed including the black hole and the cosmological spacetime. It is shown that any reasonable state in relativistic quantum field theory becomes disentangled (factorizable) at large spacelike distances if one makes local observations. As a result a violation of Bell’s inequalities can be observed without inconsistency with principles of relativistic quantum theory only if the distance between detectors is rather small. We suggest a further experimental study of entangled states in spacetime by studying the dependence of the correlation functions on the distance between detectors.
1 Introduction

Recent important experimental and theoretical results obtained in quantum computing, teleportation and cryptography (these topics sometimes are considered as belonging to quantum information theory) are based on the investigation of properties of nonrelativistic quantum mechanics. Especially important are properties of nonfactorized entangled states discussed by Einstein, Podolsky and Rosen, Bohm, Bell and many others. Results and ideas of Shannon’s classical information theory play an important role in the modern quantum information theory as well as the notions of qubit, quantum relative entropy, quantum channel, and entangled states, see for example [1] - [3].

However the spacetime dependence is not explicitly indicated in this approach. As a result, many important achievements in modern quantum information theory have been obtained for an idealized situation when the spacetime dependence of quantum phenomena is neglected.

We emphasize an importance of the investigation of quantum information in space and time. Transmission and processing of (quantum) information is a physical process in spacetime. Therefore a formulation of such basic notions in quantum information theory as composite systems, entangled states and the channel should include the spacetime variables.

Ultimately, quantum information theory should become a part of quantum field theory (perhaps, in future, a part of superstring theory) since quantum field theory is our most fundamental physical theory.

Quantum field theory is not just an abstract mathematical theory of operators in a Hilbert space. Basic equations of quantum field theory such as the Maxwell, Dirac, Yang–Mills equations are differential equations for operator functions defined on the spacetime. The nonrelativistic Schrödinger equation is also a differential equation in spacetime. Therefore a realistic quantum information theory should be based on the study of the solutions of these equations propagated in spacetime including the curved spacetime.

Entangled states, i.e. the states of two particles with the wave function which is not a product of the wave functions of single particles, have been studied in many theoretical and experimental works starting from the paper of Einstein, Podolsky and Rosen, see e.g. [4].

In this paper entangled states in space and time are considered. We point out a simple but the fundamental fact that the vacuum state $\omega_0$ in a free quantum field theory is a nonfactorized (entangled) state for observables belonging to spacelike separated regions:

$$\omega_0(\varphi(x)\varphi(y)) - \omega_0(\varphi(x))\omega_0(\varphi(y)) \neq 0$$

Here $\varphi(x)$ is a free scalar field in the Minkowski spacetime and $(x - y)^2 < 0$. Hence there is a statistical dependence between causally disconnected regions.

However one has an asymptotic factorization of the vacuum state for large separations of the spacelike regions. Moreover one proves that in quantum field theory there is an asymptotic factorization for any reasonable state and any local observables. Therefore at large distances any reasonable state becomes disentangled. We have the relation

$$\lim_{|l| \to \infty} [\omega(A(l)B) - \omega(A(l))\omega(B)] = 0$$

The importance of the investigation of quantum information effects in space and time and especially the role of relativistic invariance in classical and quantum information theory was stressed in the talk by the author at the First International Conference on Quantum Information which was held at Meijo University, Japan, November 4-8, 1997.
Here \( \omega \) is a state from a rather wide class of the states which includes entangled states, \( A \) and \( B \) are two local observables, and \( A(l) \) is the translation of the observable \( A \) along the 3 dim vector \( l \). As a result a violation of Bell’s inequalities (see below) can be observed without inconsistency with principles of relativistic quantum theory only if the distance between detectors is rather small. We suggest a further experimental study of entangled states in spacetime by studying the dependence of the correlation functions on the distance between detectors.

There is no a factorization of the expectation value \( \omega_0(\varphi(x)\varphi(y)) \) even for the space-like separation of the variables \( x \) and \( y \) if the distance between \( x \) and \( y \) is not large enough. However we will prove that there exist a representation of the form

\[
\omega_0(\varphi(x)\varphi(y)) = E\xi(x)\xi^*(y)
\]

which is valid for all \( x \) and \( y \). Here \( \xi(x) \) is a classical (generalized) complex random field and \( E \) is the expectation value. Therefore the quantum correlation function is represented as a classical correlation function of separated random fields. This representation can be called a local realistic representation by analogy with the Bell approach to the spin correlation functions.

J. Bell proved \(^7\) that there are quantum spin correlation functions in entangled states that can not be represented as classical correlation functions of separated random variables. Bell’s theorem reads, see \(^8\):

\[
\cos(\alpha - \beta) \neq E\xi_\alpha\eta_\beta
\]

where \( \xi_\alpha \) and \( \eta_\beta \) are two random processes such that \( |\xi_\alpha| \leq 1, \ |\eta_\beta| \leq 1 \) and \( E \) is the expectation. Here the function \( \cos(\alpha - \beta) \) describes the quantum mechanical correlation of spins of two entangled particles. Bell’s theorem has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics \(^7\). For a recent discussion of Bell’s theorem and Bell’s inequalities see, for example \(^9\) - \(^14\) and references therein.

However if we want to speak about locality in quantum theory then we have to localize somehow our particles. For example we could measure the density of the energy or the position of the particles simultaneously with the spin. Only then we could come to some conclusions about a relevance of the spin correlation function to the problem of locality.

The function \( \cos(\alpha - \beta) \) describes quantum correlations of two spins in the two qubit Hilbert space when the spacetime dependence of the wave functions of the particles is neglected. Let us note however that the very formulation of the problem of locality in quantum mechanics prescribes a special role to the position in ordinary three-dimensional space. It is rather strange therefore that the problem of \textit{local in space observations} was neglected in discussions of the problem of locality in relation to Bell’s inequalities.

Let us stress that we discuss here not a problem of interpretation of quantum theory but a problem of how to make correct quantum mechanical computations describing an experiment with two detectors localized in space. Recently it was pointed out \(^8\) that if we make \textit{local} observations of spins then the spacetime part of the wave function leads to an extra factor in quantum correlations and as a result the ordinary conclusion from the Bell theorem about the nonlocality of quantum theory fails.

We present a modification of Bell’s equation which includes space and time variables. The function \( \cos(\alpha - \beta) \) describes the quantum mechanical correlation of spins of two entangled particles if we neglect the spacetime dependence of the wave function. It was shown in
that if one takes into account the space part of the wave function then the quantum correlation describing local observations of spins in the simplest case will take the form $g \cos(\alpha - \beta)$ instead of just $\cos(\alpha - \beta)$. Here the parameter $g = g(O_A, O_B)$ describes the location of the detectors in regions $O_A$ and $O_B$ in space. In this case we have to investigate a modified Bell’s equation. We will prove that there exists the following representation

$$g(O_A, O_B) \cos(\alpha - \beta) = E \xi(\alpha, O_A) \eta(\beta, O_B)$$

if the distance between $O_A$ and $O_B$ is large enough. We will show that in fact at large distances all reasonable quantum states become disentangled. This fact leads also to important consequences for quantum teleportation and quantum cryptography, [13, 14]. Bell’s theorem constitutes an important part in quantum cryptography. In [13] it is discussed how one can try to improve the security of quantum cryptography schemes in space by using a special preparation of the space part of the wave function.

It is important to study also a more general question: which class of functions $f(s,t)$ admits a representation of the form

$$f(s,t) = Ex_s y_t$$

where $x_s$ and $y_t$ are bounded stochastic processes and also analogous question for the functions of several variables $f(t_1, ..., t_n)$.

Such considerations could provide a noncommutative generalization of von Neumann’s spectral theorem. We suggest a new physical principle describing a relation between the mathematical formalism of Hilbert space and quantum physical phenomena.

In modern quantum information theory the basic notion is the two dimensional Hilbert space, i.e. qubit. We suggest that in a relativistic quantum information theory, when the existence of spacetime is taken into account, the basic notion should be a notion of an elementary quantum system, i.e. according to Wigner (see [16]) it is an infinite dimensional Hilbert space $H$ invariant under an irreducible representation of the Poincare group labelled by $[m, s]$ where $m \geq 0$ is mass and $s = 0, 1/2, 1, ...$ is spin (helicity).

In the next section the disentanglement at large distances in quantum field theory is considered. Local observations and modified Bell’s equations are considered in Sect.3. Non-commutative spectral theory and local realism are discussed in Sect.4. Some remarks on the properties of entangled states in curved spacetime are made in Sect.5.

### 2 Quantum Probability and Quantum Field Theory

In quantum probability (see [17]) we are given a *- algebra $A$ and a state (i.e. a linear positive normalized functional) $\omega$ on $A$. Elements from $A$ are called random variables. Two random variables $A$ and $B$ are called (statistically) independent if $\omega(AB) = \omega(A)\omega(B)$.

Quantum field in the Wightman formulation is an operator-valued distribution $\varphi(f)$ acting in a Hilbert space where $f$ is a Schwartz test function on $\mathbb{R}^4$. One uses the standard notations

$$\varphi(f) = \int_{\mathbb{R}^4} \varphi(x)f(x)dx$$

Quantum field satisfies simple transformation properties under a representation of the Poincare group. Moreover if $f$ and $g$ are test functions whose supports are space-like to each other then $\varphi(f)$ and $\varphi(g)$ shall commute:

$$[\varphi(f), \varphi(g)] = 0$$
This assumption rests on the principle that no physical effect can propagate in space-like directions. The assumption is called the microscopic or relativistic causality.

We point out an interesting fact that for the vacuum state in quantum field theory the relativistic causality does not lead to the statistical independence in the sense of quantum probability for quantum fields with weight functions whose supports are space-like to each other. We will prove the following

Proposition 1. There is a statistical dependence between two spacelike separated regions for the vacuum state in the theory of free scalar quantum field.

Proof. Let us consider a free scalar quantum field \( \varphi(x) \):

\[
\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{dk}{\sqrt{2k^0}} (e^{ikx}a^*(k) + e^{-ikx}a(k))
\]

Here \( kx = k^0x^0 - kx \), \( k^0 = \sqrt{k^2 + m^2} \), \( m \geq 0 \) and \( a(k) \) and \( a^*(k) \) are annihilation and creation operators,

\[
[a(k), a^*(k')] = \delta(k - k')
\]

The field \( \varphi(x) \) is an operator valued distribution acting in the Fock space \( \mathcal{F} \) with the vacuum \( |0> \),

\[
a(k)|0> = 0
\]

The vacuum expectation value of two fields is

\[
\omega_0(\varphi(x)\varphi(y)) = <0|\varphi(x)\varphi(y)|0> = W_0(x - y, m^2)
\]

where

\[
W_0(x - y, m^2) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{dk}{2k^0} e^{-ik(x-y)}
\]

The statistical independence of two spacelike separated regions in particular would lead to the relation

\[
\omega_0(\varphi(x)\varphi(y)) - \omega_0(\varphi(x))\omega_0(\varphi(y)) = 0
\]

if \((x - y)^2 < 0\). But since \( \omega_0(\varphi(x)) = 0 \) in fact we have

\[
\omega_0(\varphi(x)\varphi(y)) - \omega_0(\varphi(x))\omega_0(\varphi(y)) = W_0(x - y, m^2) \neq 0
\]

Therefore it is proved that there is a statistical dependence between the spacelike separated regions for the vacuum state in theory of free scalar quantum field. The proposition is proved.

Note however that the violation of the statistical independence vanish exponentially with the spacial separation of \( x \) and \( y \) since for large \( \lambda = m\sqrt{-x^2} \) the function \( W_0(x, m^2) \) behaves like

\[
\frac{m^2}{4\pi\lambda} \left( \frac{\pi}{2\lambda} \right)^{1/2} e^{-\lambda}
\]

Let us prove that any polynomial state is asymptotically disentangled (factorized) for large spacelike distances. Let \( \mathcal{A} \) be the algebra of polynomials in the Fock space \( \mathcal{F} \) at the field \( \varphi(f) \) with the test functions \( f \). Let \( C \in \mathcal{A} \) and \( |\psi> = C|0> \). Denote the state \( \omega(A) = <\psi|A|\psi>/||\psi||^2 \) for \( A \in \mathcal{A} \).

Theorem 2. One has the following asymptotic disentanglement property

\[
\lim_{|l| \to \infty} [\omega(A(l)B) - \omega(A(l))\omega(B)] = 0
\]
Here $A$ and $B$ belong to $\mathcal{A}$ and $A(l)$ is the translation of $A$ along the 3 dim vector $l$. One has also
\[
\lim_{|l| \to \infty} \left[ \omega(A(l)) - \langle 0 | A(l) | 0 \rangle \right] = 0
\]
The proof of the theorem is based on the Wick theorem and the Riemann-Lebesgue lemma.

Similar theorems take place also for the Dirac and the Maxwell fields. In particular for the Dirac field $\psi(x)$ one can prove the asymptotic factorization for the local spin operator
\[
S(\mathcal{O}) = \int_{\mathcal{O}} \psi^* \Sigma \psi dx
\]
Here $\Sigma$ is made from the Dirac matrices.

Finally let us show that some correlation functions in the relativistic quantum field theory can be represented as mathematical expectations of the classical (generalized) random fields.

**Theorem 3.** If $\varphi(x)$ is a scalar complex quantum field then one has a representation
\[
\langle 0 | \varphi(x_1) \ldots \varphi(x_n) \varphi^*(y_1) \ldots \varphi^*(y_n) | 0 \rangle = \mathbf{E} \xi(x_1) \ldots \xi(x_n) \xi^*(y_1) \ldots \xi^*(y_n).
\]
Here $\xi(x)$ is a complex random field.

The proof of the theorem follows from the positivity of the quantum correlation functions. It is interesting that we have obtained a functional integral representation for the quantum correlation functions in real time. Similar representation is valid also for the two-point correlation function of an interacting scalar field. It follows from the Kallen-Lehmann representation.

### 3 Local Observations and Modified Bell’s Equations

Bell’s theorem reads:
\[
\cos(\alpha - \beta) \neq E\xi_\alpha \eta_\beta
\]
where $\xi_\alpha$ and $\eta_\beta$ are two random processes such that $|\xi_\alpha| \leq 1$, $|\eta_\beta| \leq 1$ and $E$ is the expectation. In more details:

**Theorem 4.** There exists no probability space $(\Lambda, \mathcal{F}, d\rho(\lambda))$ and a pair of stochastic processes $\xi_\alpha = \xi_\alpha(\lambda)$, $\eta_\beta = \eta_\beta(\lambda)$, $0 \leq \alpha, \beta \leq 2\pi$ which obey $|\xi_\alpha(\lambda)| \leq 1$, $|\eta_\beta(\lambda)| \leq 1$ such that the following equation is valid
\[
\cos(\alpha - \beta) = E\xi_\alpha \eta_\beta
\]
for all $\alpha$ and $\beta$.

Here $\Lambda$ is a set, $\mathcal{F}$ is a sigma-algebra of subsets and $d\rho(\lambda)$ is a probability measure, i.e. $d\rho(\lambda) \geq 0$, $\int d\rho(\lambda) = 1$. The expectation is
\[
E\xi_\alpha \eta_\beta = \int_{\Lambda} \xi_\alpha(\lambda) \eta_\beta(\lambda) d\rho(\lambda)
\]

The theorem follows from the CHSH inequality presented below. Let us discuss a physical interpretation of this result.

Consider a pair of spin one-half particles formed in the singlet spin state and moving freely towards two detectors. If one neglects the space part of the wave function then one
has the Hilbert space \( C^2 \otimes C^2 \) and the quantum mechanical correlation of two spins in the singlet state \( \psi_{\text{spin}} \in C^2 \otimes C^2 \) is

\[
D_{\text{spin}}(a, b) = \langle \psi_{\text{spin}} | \sigma \cdot a \otimes \sigma \cdot b | \psi_{\text{spin}} \rangle = -a \cdot b
\]

Here \( a = (a_1, a_2, a_3) \) and \( b = (b_1, b_2, b_3) \) are two unit vectors in three-dimensional space \( R^3 \), \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices,

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma \cdot a = \sum_{i=1}^{3} \sigma_i a_i
\]

and

\[
\psi_{\text{spin}} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)
\]

If the vectors \( a \) and \( b \) belong to the same plane then one can write \(-a \cdot b = \cos(\alpha - \beta)\) and hence Bell’s theorem states that the function \( D_{\text{spin}}(a, b) \) Eq. (3) can not be represented in the form

\[
P(a, b) = \int \xi(a, \lambda) \eta(b, \lambda) d\rho(\lambda)
\]

i.e.

\[
D_{\text{spin}}(a, b) \neq P(a, b)
\]

Here \( \xi(a, \lambda) \) and \( \eta(b, \lambda) \) are random fields on the sphere, \( |\xi(a, \lambda)| \leq 1, \ |\eta(b, \lambda)| \leq 1 \) and \( d\rho(\lambda) \) is a positive probability measure, \( \int d\rho(\lambda) = 1 \). The parameters \( \lambda \) are interpreted as hidden variables in a realist theory. It is clear that Eq. (3) can be reduced to Eq. (4).

To prove Theorem 4 one uses the following theorem which is a slightly generalized Clauser-Horn-Shimony-Holt (CHSH) result.

**Theorem 5.** Let \( f_1, f_2, g_1 \) and \( g_2 \) be random variables (i.e. measured functions) on the probability space \( (\Lambda, \mathcal{F}, d\rho(\lambda)) \) such that

\[
|f_i(\lambda)g_j(\lambda)| \leq 1, \quad i, j = 1, 2.
\]

Denote

\[
P_{ij} = Ef_ig_j, \quad i, j = 1, 2.
\]

Then

\[
|P_{11} - P_{12}| + |P_{21} + P_{22}| \leq 2.
\]

The last inequality is called the CHSH inequality. By using notations of Eq. (4) one has

\[
|P(a, b) - P(a, b')| + |P(a', b) + P(a', b')| \leq 2
\]

for any four unit vectors \( a, b, a', b' \).

It will be shown below that if one takes into account the space part of the wave function then the quantum correlation in the simplest case will take the form \( g \cos(\alpha - \beta) \) instead of just \( \cos(\alpha - \beta) \) where the parameter \( g \) describes the location of the system in space and time. In this case one can get a representation

\[
g \cos(\alpha - \beta) = E\xi_\alpha \eta_\beta
\]

if \( g \) is small enough. The factor \( g \) gives a contribution to visibility or efficiency of detectors that are used in the phenomenological description of detectors.
3.1 Modified Bell’s equation

In the previous section the space part of the wave function of the particles was neglected. However exactly the space part is relevant to the discussion of locality. The Hilbert space assigned to one particle with spin 1/2 is \( C^2 \otimes L^2(R^3) \) and the Hilbert space of two particles is \( C^2 \otimes L^2(R^3) \otimes C^2 \otimes L^2(R^3) \). The complete wave function is \( \psi = (\psi_{\alpha\beta}(r_1, r_2, t)) \) where \( \alpha \) and \( \beta \) are spinor indices, \( t \) is time and \( r_1 \) and \( r_2 \) are vectors in three-dimensional space.

We suppose that there are two detectors (A and B) which are located in space \( R^3 \) within the two localized regions \( O_A \) and \( O_B \) respectively, well separated from one another. If one makes a local observation in the region \( O_A \) then this means that one measures not only the spin observable \( \sigma \) but also some another observable which describes the localization of the particle like the energy density or the projection operator \( P_O \) to the region \( O \). We will consider here correlation functions of the projection operators \( P_O \).

Quantum correlation describing the localized measurements of spins in the regions \( O_A \) and \( O_B \) is

\[
\omega(\sigma \cdot a P_{O_A} \otimes \sigma \cdot b P_{O_B}) = \langle \psi | \sigma \cdot a P_{O_A} \otimes \sigma \cdot b P_{O_B} | \psi \rangle \quad (8)
\]

Let us consider the simplest case when the wave function has the form of the product of the spin function and the space function \( \psi = \psi_{\text{spin}} \phi(r_1, r_2) \). Then one has

\[
\omega(\sigma \cdot a P_{O_A} \otimes \sigma \cdot b P_{O_B}) = g(O_A, O_B) D_{\text{spin}}(a, b) \quad (9)
\]

where the function

\[
g(O_A, O_B) = \int_{O_A \times O_B} |\phi(r_1, r_2)|^2 dr_1 dr_2 \quad (10)
\]

describes correlation of particles in space. It is the probability to find one particle in the region \( O_A \) and another particle in the region \( O_B \).

One has

\[
0 \leq g(O_A, O_B) \leq 1. \quad (11)
\]

If \( O_A \) is a bounded region and \( O_A(l) \) is a translation of \( O_A \) to the 3-vector \( l \) then one has

\[
\lim_{|l| \to \infty} g(O_A(l), O_B) = 0. \quad (12)
\]

Since

\[
\langle \psi_{\text{spin}} | \sigma \cdot a \otimes I | \psi_{\text{spin}} \rangle = 0
\]

we have

\[
\omega(\sigma \cdot a P_{O_A} \otimes I) = 0.
\]

Therefore we have proved the following proposition which says that the state \( \psi = \psi_{\text{spin}} \phi(r_1, r_2) \) becomes disentangled at large distances.

**Proposition 6.** One has the following property of the asymptotic factorization (disentanglement) at large distances:

\[
\lim_{|l| \to \infty} [\omega(\sigma \cdot a P_{O_A(l)} \otimes \sigma \cdot b P_{O_B}) - \omega(\sigma \cdot a P_{O_A(l)} \otimes I) \omega(I \otimes \sigma \cdot b P_{O_B})] = 0 \quad (13)
\]

or

\[
\lim_{|l| \to \infty} \omega(\sigma \cdot a P_{O_A(l)} \otimes \sigma \cdot b P_{O_B}) = 0.
\]
Now one inquires whether one can write a representation
\[ \omega(\sigma \cdot a P_{\mathcal{O}_A(l)} \otimes \sigma \cdot b P_{\mathcal{O}_B}) = \int \xi(a, \mathcal{O}_A, \lambda) \eta(b, \mathcal{O}_B, \lambda) d\rho(\lambda) \]  
(14)
where \(|\xi(a, \mathcal{O}_A(l), \lambda)| \leq 1, \ |\eta(b, \mathcal{O}_B, \lambda)| \leq 1.

**Remark.** A local modified equation reads
\[ |\phi(\mathbf{r}_1, \mathbf{r}_2, t)|^2 \cos(\alpha - \beta) = E\xi(\alpha, \mathbf{r}_1, t)\eta(\beta, \mathbf{r}_2, t). \]

If we are interested in the conditional probability of finding the projection of spin along vector \(a\) for the particle 1 in the region \(\mathcal{O}_A(l)\) and the projection of spin along the vector \(b\) for the particle 2 in the region \(\mathcal{O}_B\) then we have to divide both sides of Eq. (14) by \(g(\mathcal{O}_A(l), \mathcal{O}_B)\).

Note that here the classical random variable \(\xi = \xi(a, \mathcal{O}_A(l), \lambda)\) is not only separated in the sense of Bell (i.e. it depends only on \(a\)) but it is also local in the 3 dim space since it depends only on the region \(\mathcal{O}_A(l)\). The classical random variable \(\eta\) is also local in 3 dim space since it depends only on \(\mathcal{O}_B\). Note also that since the eigenvalues of the projector \(P_{\mathcal{O}_1}\) are 0 or 1 then one should have \(|\xi(a, \mathcal{O}_A)| \leq 1, \ |\eta(\mathcal{O}_B, \lambda)| \leq 1\).

Due to the property of the asymptotic factorization and the vanishing of the quantum correlation for large \(|l|\) there exists a trivial asymptotic classical representation of the form (14) with \(\xi = \eta = 0\).

We can do even better and find a classical representation which will be valid uniformly for large \(|l|\).

If \(g\) would not depend on \(\mathcal{O}_A\) and \(\mathcal{O}_B\) then instead of Eq (2) in Theorem 1 we could have a modified equation
\[ g \cos(\alpha - \beta) = E\xi(\alpha, \mathbf{r}_1, t)\eta(\beta, \mathbf{r}_2, t) \]  
(15)
The factor \(g\) is important. In particular one can write the following representation [12] for \(0 \leq g \leq 1/2:\)
\[ g \cos(\alpha - \beta) = \int_0^{2\pi} \sqrt{2g \cos(\alpha - \lambda)} \sqrt{2g \cos(\beta - \lambda)} \frac{d\lambda}{2\pi} \]  
(16)
Therefore if \(0 \leq g \leq 1/2\) then there exists a solution of Eq. (13) where
\[ \xi_\alpha(\lambda) = \sqrt{2g \cos(\alpha - \lambda)}, \ \eta_\beta(\lambda) = \sqrt{2g \cos(\beta - \lambda)} \]
and \(|\xi_\alpha| \leq 1, \ |\eta_\beta| \leq 1\). If \(g > 1/\sqrt{2}\) then it follows from Theorem 2 that there is no solution to Eq. (15). We have obtained

**Theorem 7.** If \(g > 1/\sqrt{2}\) then there is no solution \((\Lambda, \mathcal{F}, d\rho(\lambda), \xi_\alpha, \eta_\beta)\) to Eq. (15) with the bounds \(|\xi_\alpha| \leq 1, \ |\eta_\beta| \leq 1\). If \(0 \leq g \leq 1/2\) then there exists a solution to Eq. (13) with the bounds \(|\xi_\alpha| \leq 1, \ |\eta_\beta| \leq 1\).

**Remark.** Local variable models for inefficient detectors are presented in [10, 11].

Let us take now the wave function \(\phi\) of the form \(\phi = \psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2)\) where
\[ \int_{R^3} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 = 1, \ \int_{R^3} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2 = 1 \]
In this case
\[ g(\mathcal{O}_A(l), \mathcal{O}_B) = \int_{\mathcal{O}_A(l)} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 \cdot \int_{\mathcal{O}_B} |\psi_2(\mathbf{r}_2)|^2 d\mathbf{r}_2 \]
There exists such \(L > 0\) that
\[ \int_{B_L} |\psi_1(\mathbf{r}_1)|^2 d\mathbf{r}_1 = \epsilon < 1/2, \]
where $B_L = \{ r \in R^3 : |r| \geq L \}$. Let us make an additional assumption that the classical random variable has the form of a product of two independent classical random variables $\xi(a, O_A) = \xi_{space}(O_A)\xi_{spin}(a)$ and similarly for $\eta$. We will prove that there exists the following representation

$$g(O_A, O_B) \cos(\alpha - \beta) = E\xi(\alpha, O_A)\eta(\beta, O_B)$$

if the distance between $O_A$ and $O_B$ is large enough. We have the following

**Theorem 8.** Under the above assumptions and for large enough $|l|$ there exists the following representation of the quantum correlation function

$$g(O_A(l), O_B) \cos(\alpha - \beta) = (E\xi_{space}(O_A(l)))(E\eta_{space}(O_B))E\xi_{spin}(\alpha)\xi_{spin}(\beta)$$

where all classical random variables are bounded by 1.

**Proof.** To prove the theorem we write

$$g(O_A(l), O_B) \cos(\alpha - \beta) = \int_{O_A(l)} \frac{1}{\epsilon} |\psi_1(r_1)|^2 dr_1 \cdot \int_{O_B} |\psi_2(r_2)|^2 dr_2 \cdot \epsilon \cos(\alpha - \beta)$$

$$= (E\xi_{space}(O_A(l)))(E\eta_{space}(O_B))E\xi_{spin}(\alpha)\xi_{spin}(\beta)$$

Here $\xi_{space}(O_A(l))$ and $\eta_{space}(O_B)$ are random variables on the probability space $B_L \times R^3$ with the probability measure

$$dP(r_1, r_2) = \frac{1}{\epsilon} |\psi_1(r_1)|^2 \cdot |\psi_2(r_2)|^2 dr_1 dr_2$$

of the form

$$\xi_{space}(O_A(l), r_1, r_2) = \chi_{O_A(l)}(r_1), \quad \eta_{space}(O_B, r_1, r_2) = \chi_{O_B}(r_2)$$

where $\chi_{O}(r)$ is the characteristic function of the region $O$. We assume that $O_A(l)$ belongs to $B_L$. Further $\xi_{spin}(\alpha)$ is a random process on the circle $0 \leq \varphi \leq 2\pi$ with the probability measure $d\varphi/2\pi$ of the form

$$\xi_{spin}(\alpha, \varphi) = \sqrt{2\epsilon} \cos(\alpha - \varphi)$$

The theorem is proved.

### 4 Noncommutative Spectral Theory and Quantum Theory

As a generalization of the previous discussion we would like to suggest here a new general physical principle which describes a relation between the mathematical formalism of Hilbert space and physical quantum phenomena. It will use theory of classical stochastic processes which, as we suggest, expresses the condition of local realism. According to the standard view to quantum theory any hermitian operator in a Hilbert space describes a physical observable and any density operator describes a physical state. Here we would like to suggest that this view is too general and that in fact there should exist some additional restrictions to the family of Hermitian operators and to the density operators if they have to describe physical phenomena.
Let $\mathcal{H}$ be a Hilbert space, $\rho$ is the density operator, $\{A_\alpha\}$ is a family of self-adjoint operators in $\mathcal{H}$. One says that the family of observables $\{A_\alpha\}$ and the state $\rho$ satisfy to the condition of local realism if there exists a probability space $(\Lambda, \mathcal{F}, dP(\lambda))$ and a family of random variables $\{\xi_\alpha\}$ such that the range of $\xi_\alpha$ belongs to the spectrum of $A_\alpha$ and for any subset $\{A_{i_1}, ..., A_{i_n}\}$ of mutually commutative operators one has a representation

$$\text{Tr}(\rho A_{i_1}...A_{i_n}) = E\xi_{i_1}...\xi_{i_n}$$

The physical meaning of the representation is that it describes the quantum-classical correspondence. If the family $\{A_\alpha\}$ would be a maximal commutative family of self-adjoint operators then for pure states the previous representation can be reduced to the von Neumann spectral theorem [20]. In our case the family $\{A_\alpha\}$ consists from not necessary commuting operators. Hence we will call such a representation a noncommutative spectral representation. Of course one has a question for which families of operators and states a noncommutative spectral theorem is valid, i.e. when we can write the noncommutative spectral representation. We need a noncommutative generalization of von Neumann’s spectral theorem.

It would be helpful to study the following problem: describe the class of functions $f(t_1, ..., t_n)$ which admits the representation of the form

$$f(t_1, ..., t_n) = E x_{t_1}...z_{t_n}$$

where $x_t, ..., z_t$ are random processes which obey the bounds $|x_t| \leq 1, ..., |z_t| \leq 1$.

From the previous discussion (Bell’s theorem) we know that there are such families of operators and such states which do not admit the noncommutative spectral representation and therefore they do not satisfy the condition of local realism. Indeed let us take the Hilbert space $\mathcal{H} = C^2 \otimes C^2$ and operators $\sigma \cdot a \otimes \sigma \cdot b$. We know from Theorem 2 that the function $\langle \psi_{\text{spin}} | \sigma \cdot a \otimes \sigma \cdot b | \psi_{\text{spin}} \rangle$ can not be represented as the expected value $E\xi(a)\eta(b)$ of random variables.

However, as it was discussed above, the space part of the wave function was neglected in the previous consideration. It was proved that for the observables of the form $\sigma \cdot a P_{O_A} \otimes \sigma \cdot b P_{O_B}$ one can write a local spectral representation if the distance between the regions $O_A$ and $O_B$ is large enough. We suggest that in physics one could prepare only such states and observables which satisfy the condition of local realism. Perhaps we should restrict ourself in this proposal to the consideration of only such families of observables which satisfy the condition of relativistic local causality. If there are physical phenomena which do not satisfy this proposal then it would be important to describe quantum processes which satisfy the above formulated condition of local realism and also processes which do not satisfy to this condition.

5 Entangled States and General Relativity

Here we would like to make some comments on the study of entangled states and the reduction postulate in curved spacetime. It is especially interesting to consider properties of entangled states in curved spacetimes possessing a nontrivial causal structure in particular in a spacetime containing an event horizon. In particular entangled states and the reduction of the wave function in the context of black holes, Hawking radiation, inflationary models of the early universe, creation of particles and accelerated detectors [21]-[24] should be considered.
Inflation leads to the phenomena of the cosmic entanglement since the scalar field creates particles during the inflation \[23\]. Analysis of quantum teleportation of a state through the horizon can help to clarify the notion of the reduction of the wave function associated with the measurement process.

6 Conclusions

We have discussed some problems in quantum information theory which requires the inclusion of spacetime variables. In particular entangled states in space and time were considered. A modification of Bell’s equation which includes the spacetime variables is investigated. A general relation between quantum theory and theory of classical stochastic processes was proposed which expresses the condition of local realism in the form of a noncommutative spectral theorem. Entangled states in space and time are considered. It is shown that any reasonable state in relativistic quantum field theory becomes disentangled (factorizable) at large space-like distances if one makes local observations. As a result a violation of Bell’s inequalities can be observed without inconsistency with principles of relativistic quantum theory only if the distance between detectors is rather small.

There are many interesting open problems in the approach to quantum information in space and time discussed in this paper. Some of them related with the noncommutative spectral theory and theory of classical stochastic processes. We suggest a further experimental study of entangled states in spacetime by studying the dependence of the correlation functions on the distance between detectors. It is very interesting to investigate properties of entangled states in curved spacetime.

Acknowledgments

I am grateful to G.G. Emch, R. Gill, Y.S. Kim, B. Hiley, G.’t Hooft, A. Khrennikov, W. Philipp, H. Rauch, and A. Sadreev for useful discussions. This work is supported in part by RFFI-02-01-01084 and the grant for the leading scientific schools 00-15-96073.

References

[1] M. Ohya, I.V. Volovich, Quantum Computer, Information, Teleportation, Cryptography, Springer-Verlag, 2002.

[2] A. Peres, Quantum Theory: Concepts and Methods, (Kluver Academic Publishers, 1994).

[3] C.A. Fuchs, Quantum Foundations in the Light of Quantum Information, http://arxiv.org/abs/quant-ph/0106166.

[4] Igor V. Volovich, Quantum Information in Space and Time, http://arxiv.org/abs/quant-ph/0108073.

[5] N.N. Bogoliubov and D.V. Schirkov, Introduction to Theory of Quantized Fields, Nauka, Moscow, 1985.
[6] A. Afriat and F. Selleri, *The Einstein, Podolsky, and Rosen Paradox in Atomic, Nuclear, and Particle Physics*, Plenum Press, 1999.

[7] J.S. Bell, Physics, 1, 195 (1964).

[8] Igor V. Volovich, *Bell’s Theorem and Locality in Space*, http://arxiv.org/abs/quant-ph/0012010.

[9] Andrei Khrennikov, *Non-Kolmogorov probability and modified Bell’s inequality*, quant-ph/0003017.

[10] E. Santos, Phys. Lett. A212, 10 (1996).

[11] J-A. Larsson, Phys. Lett. A256, 245 (1999).

[12] I. Volovich, Ya. Volovich, *Bell’s Theorem and Random Variables*, http://arxiv.org/abs/quant-ph/0009053.

[13] Igor V. Volovich, *Quantum Cryptography in Space and Bell’s Theorem*, in : Foundations of Probability and Physics, Ed. A. Khrennikov, World Sci. 2001.

[14] I.V. Volovich and Ya.I. Volovich, *On Classical and Quantum Cryptography*, Lectures at the Volterra-CIRM International School ”Quantum Computer and Quantum Information”, Trento, Italy, July 25-31, 2001; http://arxiv.org/abs/quant-ph/0108133.

[15] Andrei Khrennikov, *Contextual viewpoint to quantum statistics*, http://arxiv.org/hep-th/0112076.

[16] N.N. Bogoliubov, A.A. Logunov, A.I. Oksak and I.T. Todorov, *General Principles of Quantum Field Theory*, Nauka, Moscow, 1987.

[17] L. Accardi, Yu.G. Lu, I.V. Volovich, *Quantum Theory and Its Stochastic Limit*, Springer-Verlag, 2002.

[18] A.I. Achiezer and V.B. Berestecky, *Quantum Electrodynamics*, Nauka, Moscow, 1969.

[19] T. Hida, *Brownian Motion*, Springer-Verlag, 1980.

[20] M.A. Naimark, *Normieren Rings*, Nauka, Moscow, 1968.

[21] V.P. Frolov and I.D. Novikov, *Physics of Black Holes*, Academic Press, 1998.

[22] I.V. Volovich, V.A. Zagrebnov and V.P. Frolov, *Quantum Field Theory in Asymptotically Flat Spacetime*, in: Physics of Elementary Particles and Atomic Nucleus, Atomizdat, Moscow, 1978, pp.147-211.

[23] A. Linde, *Particle Physics and Inflationary Cosmology*, Harwood, Switzerland, 1990.

[24] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, 1982.