Unitarity Cuts: NLO Six-Gluon Amplitudes in QCD

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We report on a technique for evaluating finite unitarity cut for one-loop amplitudes in gauge theories, and discuss its application to the cut-constructible part of six-gluon amplitude in QCD.

1. INTRODUCTION

The availability of theoretical results required to control multi-leg scattering beyond the leading order (LO) certainly does not cover the demand for describing the landscape of multi-particle final-state processes and backgrounds that will be delineated at the turn-on of the next generation colliders. Great efforts over the past 10 years have been made to push the status of theoretical results beyond processes with five external particles at next-to-leading order (NLO) \(^2\); and we are witnessing nowadays the overpass of the threshold represented by six-leg scattering \(^2\) \(^3\) \(^4\).

Evaluation of scattering amplitudes in perturbation theory means computation of Feynman diagrams. Currently, the evaluation of high-tensor-rank one-loop multi-leg Feynman integrals represents the bottleneck for NLO calculations.

One of the most efficient approach to tackle the problem of analytic computation of one-loop multileg-amplitudes is the unitarity cut method proposed by Bern, Dixon, Dunbar and Kosower \(^2\) \(^5\) \(^6\) \(^7\) with the spinor-helicity formalism \(^9\) \(^10\) \(^11\). The technique hereby described deals with a new way of performing the cut-integration, which was developed by Britto, Buchbinder, Cachazo and Feng \(^12\) in the context of supersymmetry, and we have further extended \(^1\) to deal with the complications due to the tensor structure of propagators of non-supersymmetric theory, like QCD. In carrying through the integration over the phase-space, we make use of ‘twistor motivated’ methods and ideas, initiated in the seminal work of Witten \(^13\) and further developed in \(^14\) \(^15\) \(^16\) \(^17\): exploiting the properties of analytic continued amplitudes with complex spinors, to reduce the cut-integration to the extraction of residues.

1.1. Setup

1.1.1. Supersymmetry Decomposition

One-loop amplitudes with all external gluons and a gluon circulating around the loop, \(\mathcal{A}^g\), can be rewritten \(^2\) \(^5\) \(^6\) by decomposing the inner loop as Super-Yang-Mills (SYM) contributions of a \(\mathcal{N} = 4\) and a chiral \(\mathcal{N} = 1\) multiplets, plus a complex scalar loop, often referred as to \(\mathcal{N} = 0\) contribution,

\[
\mathcal{A}^g = \mathcal{A}^{\mathcal{N}=4} - 4\mathcal{A}^{\mathcal{N}=1} + \mathcal{A}^{\mathcal{N}=0}.
\]

The main advantage of this decomposition is that supersymmetric amplitudes, \(\mathcal{A}^{\mathcal{N}=4}\) and \(\mathcal{A}^{\mathcal{N}=1}\), are four-dimensional cut-constructible \(^5\) \(^6\). Whereas, the term \(\mathcal{A}^{\mathcal{N}=0}\) contains both a polylogarithmic structure which can be reconstructed from its absorptive part, and a rational remainder not detected by 4-dimension cuts.

In \(^1\), we have completed the program of computing the cut-constructible piece of the NLO six-gluon amplitudes in QCD, by showing a systematic way to evaluate the cut-constructible piece of \(\mathcal{A}^{\mathcal{N}=0}\).
1.1.2. Integral Basis

By standard reduction techniques, it is known that any one-loop gluon amplitude can be expressed in a basis of scalar integral functions known as boxes ($I_4$), triangles ($I_3$), and bubbles ($I_2$), which are analytically available \[12\]. Indeed, we may exploit the knowledge about its singular behaviour \[12\], to express the amplitude in terms of a basis of scalar integral functions (with no one-mass or two-mass triangle functions):

\[
A_n^{N=0} = \sum_i c_{i} A_{i}^{\text{tree}} + \sum_j c_{j} A_{j}^{\text{tree}} + \sum_k c_{k} A_{k}^{\text{tree}},
\]

where the subfix $F$ stands for the finite part of the corresponding function.

To compute the amplitude, it is sufficient to compute each of those coefficients separately and the principle of the unitarity-based method \[5,6,10,11\], to express the amplitude in terms of a basis of finite scalar integrals (with no one-mass or two-mass triangle functions):

\[
A_n^{N=0} = \sum_i \left( c_{i2} I_{4}^{(m)} + c_{i3} I_{3}^{(m)} + c_{i4} I_{4}^{(r)} \right),
\]

where $I_4^{(m)}$ is to exploit the unitarity cuts of the scalar integrals to extract their coefficients, see Fig. 2.

2. THE METHOD (in a nutshell)

The discontinuity of the amplitude in the $P = k_i + \ldots + k_j$ momentum channel is computed through the integral

\[
C_P = \int d\mu A_L^{\text{tree}} A_R^{\text{tree}}
\]

with the tree-level amplitudes

\[
A_L^{\text{tree}} = A^{\text{tree}}(\ell_1, \ell_2), \quad A_R^{\text{tree}} = A^{\text{tree}}((-\ell_2), j + 1, \ldots, j - 1, (-\ell_1))
\]

very efficiently obtained via BCFW recurrence relation \[12\] and

\[
d\mu = d^4 \ell_1 d^4 \ell_2 \delta^{(+)P}(\ell_1^2) \delta^{(+)P}(\ell_2^2) \delta^{(4)}(\ell_1 + \ell_2 - P)
\]

being the Lorentz invariant phase-space measure of two light-like vectors $(\ell_1, \ell_2)$ constrained by momentum conservation, see Fig. 2.

\[
\int d^4 t \delta^{(+)P}(\ell_2^2) \delta^{(+)P}(\ell_1 - P)^2 = \int \frac{\langle \lambda | \ell \lambda \rangle}{\langle \lambda | \ell | \lambda \rangle} \int dt \delta \left( t - \frac{P^2}{\langle \lambda | \ell | \lambda \rangle} \right)
\]

with $\ell = \ell_1$, and where the integration contour for the spinors is the diagonal defined by $\lambda = \lambda$. Meanwhile, the integrand, i.e. the product of the two tree amplitudes assumes the generic shape\[3\],

\[
A_L^{\text{tree}} A_R^{\text{tree}} = \sum_i \mathcal{I}_i,
\]

\[
\mathcal{I}_i(\ell, \ell) = \frac{1}{\langle \ell | P | \ell \rangle^n} \left[ a_1 \ell \ldots a_n | Q_1 | \ell \ldots a_2 \ell \ldots a_1 \ell \right],
\]

where $Z$ is the integration measure, $\langle \lambda | d\lambda | \lambda | d\lambda \rangle$, the degrees of $Z$ in both $\lambda$ and $\bar{\lambda}$ amount to $-2$.\[3\]

\[3\] In Eq. (6), the scaling with the intensity of the integration measure, $\langle \lambda | d\lambda | \lambda | d\lambda \rangle$, the degrees of $Z$ in both $\lambda$ and $\bar{\lambda}$ amount to $-2$.\[3\]
with \(a, b, c, d\)'s on-shell spinor, and the off-shell vectors \(Q\)'s \(\neq P\). After rescaling \(|\ell| = \sqrt{t} \, |\lambda|, |\ell| = \sqrt{t} \, |\lambda|\), one can trivially perform the \(t\)-integration, with the help of the \(\delta\) function in Eq. (4). At this stage, the cut in Eq. (3) appears as an integral over the spinor variables,

\[
C_P = \sum_i \int \langle \lambda \, d\lambda \rangle \lambda \, d\lambda \, \mathcal{I}_i(\lambda, \bar{\lambda})
\]

### 2.1. Canonical Decomposition

In order to perform the spinor integration, it is useful to write the integrand as a derivative w.r.t. either \(\lambda\) or \(\bar{\lambda}\). Without loss of generality, one can decide for \(\bar{\lambda}\), and reduce each term of the integrand to a product of two factors: one carrying the dependence solely on \(\lambda\); and a second one, which depends on both, which will be soon rewritten as a derivative w.r.t. \(\bar{\lambda}\),

\[
\mathcal{I}_i(\lambda, \bar{\lambda}) = \mathcal{H}_i(\lambda, \bar{\lambda})
\]

The decomposition in Eq. (8) can be achieved algebraically by partial fraction of spinor products, using a generalization of the following Schouten identity

\[
\frac{[\lambda a]}{[\lambda b] [\lambda c]} = \frac{[b a]}{[b c] [\lambda b]} + \frac{[c b]}{[c b] [\lambda c]},
\]

for instance,

\[
\frac{[\lambda a]}{[V \lambda] [\lambda c]} = \frac{[V a]}{[V c] [V \lambda]} + \frac{[c V \lambda]}{[c V c] [\lambda c]},
\]

which holds for any off-shell vector \(V\).

The specific shape of \(\mathcal{H}_i(\lambda, \bar{\lambda})\) in Eq. (8) carries the signature of the cut of the polylogarithm which is associated to, and therefore of its corresponding topology. In particular we know that: 1) the cut of bubble-functions is rational; 2) the cut of triangle- and box-functions is logarithmic (with arguments which distinguish unequivocally among them).

### 2.2. Rational terms of the cut

One possibility is:

\[
\mathcal{H}_i(\lambda, \bar{\lambda}) = \frac{[\eta \lambda]^n}{[\lambda |P| \lambda]^{n+2}}.
\]

In this case one can perform the integration over the \(\lambda\)-variable by parts,

\[
\frac{[\lambda a \, \partial_{\lambda} |\lambda \rangle ] [\eta \lambda]^{n+1}}{[\lambda |P| \lambda]^{n+2}} = \frac{d\lambda}{(n+1)} \frac{[\eta \lambda]^{n+1}}{[\lambda |P| \lambda]^{n+1} [\lambda |P| \eta]}.
\]

### 2.2.1. Cauchy residue theorem

Finally, one performs the last integration over the \(\lambda\)-variable using Cauchy residue theorem, in the fashion of the holomorphic anomaly [16,17], according to which the following relation does hold,

\[
[d\lambda \, \partial_{\lambda} |\lambda \rangle ] [\eta \lambda]^{n+1} = d\lambda \frac{\partial}{\partial \lambda} \frac{[\eta \lambda]}{[\lambda |P| \lambda]^{n+1}} = 2\pi \delta([\lambda a])
\]

Therefore the contribution to Eq. (7) reads,

\[
\int \langle \lambda \, d\lambda \rangle [\lambda \, d\lambda] \, \mathcal{I}_i(\lambda, \bar{\lambda}) = \int \frac{\langle \lambda \, d\lambda \rangle [d\lambda \, \partial_{\lambda} |\lambda \rangle ] [\eta \lambda]^{n+1}}{(n+1)} \frac{[\lambda |P| \lambda]^{n+1} [\lambda |P| \eta]}{[\lambda |P| \lambda]^{n+2}} = \frac{1}{(n+1)} \left\{ \mathcal{G}_i(\lambda) \, \left[ \frac{[\lambda |P| \lambda]^{n+1} [\lambda |P| \eta]}{([\lambda |P| \lambda]^{n+2})} \right] \right\}
\]

where \(\lambda_{ij}\) are the simple poles of \(\mathcal{G}_i(\lambda)\). Since the final result is a rational number, it can be taken as the coefficient of the finite cut of a bubble-function with external momentum \(P\).

### 2.3. Logarithmic terms of the cut

The other possibility for the expression of \(\mathcal{H}_i\) can be represented by,

\[
\mathcal{H}_i(\lambda, \bar{\lambda}) = \frac{1}{[\lambda |Q_1| \lambda] [\lambda |Q_2| \lambda]}. \tag{15}
\]

#### 2.3.1. Feynman parameterization

In this case, to walk along the same path as in the previous section, we have to introduce a Feynman parameter and write,

\[
\mathcal{H}_i(\lambda, \bar{\lambda}) = \int_0^1 dx \frac{1}{[\lambda |R| \lambda]^2}, \tag{16}
\]

\[
R = R(x) = xQ_1 + (1-x)Q_2, \tag{17}
\]

so that the whole integral becomes

\[
\int_0^1 dx \int \langle \lambda \, d\lambda \rangle [\lambda \, d\lambda] \frac{\mathcal{G}_i(\lambda)}{[\lambda |R| \lambda]^2} \tag{18}
\]

#### 2.3.2. Cauchy residue theorem

Having the integrand in this form, one can proceed integrating-by-parts over the \(\lambda\)-variable with
the help of Eq. (12), and over the $\lambda$-variable using Cauchy residue theorem as in Eq. (14) (setting $n = 0$), obtaining

$$
\frac{G_i(\lambda)}{(R^2)} \bigg|_{|\lambda| = R[\eta]} + \sum_j \lim_{\lambda \to \lambda_j} \langle \lambda_{ij} \rangle \frac{G_i(\lambda)}{(\lambda|R|\lambda)} \frac{[\eta \lambda]}{(\lambda|R|\eta)}
$$

(19)

2.3.3. Feynman integration

The left over integration over the Feynman parameter is the source of the logarithmic part. Since $R^2$ is quadratic in $x$, it can be written as,

$$
R^2 = (x - x_1)(x - x_2),
$$

(20)

with $x_{1,2}$ being the two real roots of the quadratic equation $R^2 = 0$. The term $G_i(\lambda)$ at $|\lambda| = R[\eta]$ is instead a sum of ratios of polynomial$^4$ in $x$, therefore the first term in the above integrand will behave like,

$$
\frac{G_i(\lambda)}{(R^2)} \bigg|_{|\lambda| = R[\eta]} \sim \frac{1}{(1 + \xi x)^{\alpha}} \frac{1}{(1 + \xi_2 x)^{\alpha}} \frac{1}{(x - x_1)(x - x_2)}
$$

(21)

with $\alpha \geq 0$, and $\xi, \xi_2$ being ratios of spinor products. Carrying out the integration over the Feynman parameter does generate indeed logarithms of the form $\ln(f(x_1, x_2))$. The explicit expression of the argument is the character of the corresponding cut-function: if $f$ is rational (in the Mandelstam invariants), then it is a signature of 1m-, 2m- and 3m-box function; else if $f$ is irrational, then it is a signature of either 3m-triangle or 4m-box function. As a matter of fact one encounters triangle-functions when $Q_1 = P$ and $Q_2 \neq P$; while box-functions arise when $Q_1 \neq Q_2 \neq P$.

$^4$Notice that in Eq. (13), $G_i(\lambda)$ is of degree zero in $\lambda$, therefore it can be written as a combination of terms like $(\eta_2 \lambda)^{\alpha}$, with $\alpha \geq 0$, and generic spinors $\eta$'s.

2.4. Treatment of higher poles

In applying Cauchy residue theorem in Eq. (14), we have implicitly assumed the integrand as having only simple poles. That is indeed the case for scattering amplitudes in $\mathcal{N} = 4, 1$ SYM. But certainly not, within the framework of less supersymmetry, like in QCD. The problem of lifting up the simple poles hidden beneath the higher poles can be tackled algebraically, as well. In fact, one can iteratively use the parity-conjugated $([\rangle \leftrightarrow \langle])$ of Schouten identities Eqs. [11], for partial fractioning the spinor products in the denominator. The effect is an algebraic disentangling of the poles, and allows to give the integrand, which carries a degree -2 in $\lambda$, the following shape

$$
g_{i-1}(\lambda, \tilde{\lambda}) (\lambda \omega) + \sum_{j=2}^{\beta} \frac{\lambda(\phi)^{j-2}}{(\lambda \omega)^j} g_{i,j}(\tilde{\lambda})
$$

with $g_{i-1}(\lambda, \tilde{\lambda})$ having degree -1 in $\lambda$, and at most simple poles in $|\lambda| \neq \omega$; and $g_{i,j}(\tilde{\lambda})$ independent of $\lambda$. Therefore, when the index of the sum over $j$ in Eq. (14) hits the higher pole $\lambda_{ij} = \omega$, only the first term in the above expression will have the non-vanishing residue, $g_{i-1}(\lambda = \omega, \tilde{\lambda} = \tilde{\omega})$.

2.5. Out of the Cut

To summarize, every double-cut of a given amplitude does contain box-, triangle- and bubble-cut, each multiplied by its own coefficient, as depicted in Fig[14]. The canonical decomposition disentangles in a purely algebraic way the rational terms from the logarithmic terms: the former are associated to the cut of two-point functions; the latter, to the cut of three- and four-point functions. The argument of each logarithm specifies unequivocally the topology of the functions it is associated to. Alternatively, the coefficient of the box-function can be algebraically determined by

Figure 3. Quadruple-cuts of a one-loop six-point amplitude are associated to box-function coefficients.
freezing the loop momenta with the four conditions imposed by quadruple-cutting the amplitude, as extensively discussed in \[20\].

### 3. NLO Six-Gluon Amplitude

The helicity amplitudes for the six-gluon scattering have been recently computed via semi-numerical technique\[5\] \[1\]. The status of the efforts in computing them analytically is reported in Tab. 1. The completion of the cut-constructible piece required the computation of the three-minus $\mathcal{N} = 0$ contributions: the coefficients of box-functions had already been computed via quadruple-cuts \[23\], see in Fig.3 the answer for the cut-constructible $A(\ldots ++)$ has been obtained by recursive technique\[6\] \[25\]; while we have computed \[1\] via double cuts, see Fig.4 the coefficients of bubbles- and 3m-triangles of $A(\ldots + + +)$ and $A(\ldots - - +)$ according to the method discussed in the previous section.

The completion of the analytical result demands the computation of the rational term, not detected by the cuts in 4-dimension. On one side, the exploitation of properties like soft-collinear singularities and factorization, and the knowledge of the polylogarithmic terms (which are cut constructible), on the other side, have lead to uncover the recursive behavior of the rational term of one-loop gluon amplitude \[26,27,28,29\]. Thanks to this progress, the rational terms of $A(\ldots ++)$ and $A(\ldots - - +++)$ have been recently obtained by the bootstrap method \[28,29\], and the rational contribution coming from the other helicity configurations is at the horizon.

### 4. Squeezing Out of the Bottleneck

The importance of the method here outlined \[12,1\] is that it is a general method for computing finite cuts of one-loop multi-leg amplitudes. It has the non-trivial advantage of not encountering at all the main difficulties which arise from the standard tensor reduction; and the computational problem is algebraically reduced by trivial spinor algebra to the the extraction of residues. Such a method is therefore suitable for cut-constructible amplitudes, for instance in Super-Yang-Mills and Gravity\[30\], and to be used in ping-pong with techniques like the bootstrap method \[26,27,28,29\], or any other one which could provide the reconstruction of the rational remainder.

The six-point amplitudes obtained in \[1\] contain the complete polylogarithm structure of the all-$n$ gluon amplitude at NLO (except for $I_4^{3m}$ which can be anyhow computed via 4ple-cut), therefore it could be used as bootstrap point to tackle problems with the higher number of legs - once the recursive behavior of loop amplitudes will be completely sorted out.

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\[7\]see Z. Bern and D. Kosower, in this Proceedings
\[8\]see N.E.J. Bjerrum-Bohr, in this Proceedings
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