DISCRETE CALCULUS WITH CUBIC CELLS ON DISCRETE MANIFOLDS

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Abstract. This work is thought as an operative guide to "discrete exterior calculus" (DEC), but at the same time with a rigorous exposition. We present a version of (DEC) on "cubic" cell, defining it for discrete manifolds. An example of how it works, it is done on the discrete torus, where usual Gauss and Stokes theorems are recovered.

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1. Introduction

Discrete exterior calculus (DEC) is motivated by potential applications in computational methods for field theories (elasticity, fluids, electromagnetism) and in areas of computer vision/graphics, for these applications see for example [DDT15], [DMTS14] or [BSSZ08]. DEC is developing as an alternative approach for computational science to the usual discretizing process from the continuous theory. It considers the discrete mesh as the only thing given and develops an entire calculus using only discrete combinatorial and geometric operations. The derivations may require that the objects on the discrete mesh, but not the mesh itself, are interpolated as if they come from a continuous model. Therefore DEC could have interesting applications in fields where there isn’t any continuous underlying structure as Graph theory [GP10] or problems that are inherently discrete since they are defined on a lattice [AO05] and it can stand in its own right as theory that parallels the continuous one. The language of DEC is founded on the concept of discrete differential form, this characteristic allows to preserve in the discrete context some of the usual geometric and topological structures of continuous models, in particular the Stokes’ theorem

\[ \int_M d\omega = \int_{\partial M} \omega. \]  

Equation (1) can be considered the milestone to define the discrete exterior calculus since it contains the main objects to set a discrete exterior calculus, namely the concepts of discrete differentials form, boundary operator and discrete exterior derivative, moreover it is very natural in the discrete setting. A qualitative review for DEC is [DKT08], while to have a deeper view we suggest [Cra15] and [Hir03].

First we are establishing the necessary objects to introduce the discrete analogous of differential forms, i.e. the discrete exterior derivative \( d \) and its adjoint operator
\(\delta\) in the context of a cubic cellular complex for a lattice on a discrete manifold. In this way we can state the Hodge decomposition. This is done in section 2 here the ideas for the construction of DEC come from [DKT08] and [DHLV05], but respect to this works we present more precise definitions for a manifold setting and cubic cells instead of simplexes, the concepts of comparability, consistency and local orientation are introduced to define rigorously what is a discrete manifold. Here we are not exposing the continuous theory, good references for a treatment of Geometry with differential forms (and its application) to compare with DEC are [AMR88] [Fla89] and [Fra12]. Then, in section 3 we are illustrating how these operators work up to dimension three using as manifold the discrete torus.

2. Discrete exterior calculus on cubic mesh

Intuitively, \(k\)-differential forms are objects that can be integrated on a \(k\)-dimensional region of the space. For example 1-forms are like \(dF = f(x)dx\) or \(dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy + \frac{\partial G}{\partial z}dz\), which can be integrated respectively over an interval in \(\mathbb{R}\) or over a curve in \(\mathbb{R}^3\). With this idea in mind discrete differential forms are going to be defined. As we said we are working with an abstract cubic complex, instead of a simplicial one. This abstract complex can be thought as a collection of discrete sets of maximal dimension \(n\). This collection could be previously derived from a continuous structure on a manifold \(\mathcal{M}\) of dimension \(n\).

2.1. Primal cubic complex and dual cell complex. The next definitions fix in an abstract way the objects on which DEC operates, the language is the typical one in algebraic topology [Mun84].

**Definition 1.** A \(k\)-simplex is the convex span \(s_k = \{v_0, v_1, \ldots, v_k\}\) of \(k + 1\) geometrically independent points of \(\mathbb{R}^N\) with \(N \geq k\), they are called vertices of the \(k\)-simplex and \(k\) its dimension. A simplex \(s_k = (v_0, v_1, \ldots, v_{k+1})\) is oriented assigning one of the two possible equivalence classes of ordering of its vertices \(v_i\). Two orderings are in the same class if they differ for an even permutation, while they are not for an odd permutation. The \(k\)-simplex with same vertices but different ordering from \(s_k\) is said to have opposite orientation and denoted with \(-s_k\).

One orientation of a simplex can be called conventionally positive and the opposite one negative.

**Definition 2** (Orientation convention\(^1\) for simplexes). A way to define the sign of a \(k\)-simplex \(s_k = (v_0, \ldots, v_{k+1})\) is that of embedding it in \(\mathbb{R}^k\) equipped with a right handed orthonormal basis and saying it is positive oriented if \(\det(v_1 - v_0, v_2 - v_0, \ldots, v_{k+1} - v_0) > 0\) and negative in the opposite case.

"Inside" a \(k\)-simplex we can individuate some proper simplexes, we need to define this and how they relate with the "original" one.

\(^1\)This convention tell us the following. We consider \(\mathbb{R}^k\) with a right handed orthonormal basis \(e_1, \ldots, e_k\). A simplex \(s_1 = (v_0, v_1)\) embedded in \(\mathbb{R}^1\) can take orientation from \(v_0\) to \(v_1\), let’s call it positive assuming \(v_1 > v_0\), otherwise from \(v_1\) to \(v_0\), that is negative. A simplex \(s_2 = (v_0, v_1, v_2)\) embedded in \(\mathbb{R}^2\) can take anticlockwise orientation with the normal pointing outside the plane along the "right hand rule", let’s call it positive, otherwise clockwise with the normal pointing outside the plane along the "left-hand rule", that is negative. A simplex \(s_3 = (v_0, v_1, v_2, v_3)\) embedded in \(\mathbb{R}^3\) can take orientation along the "screw-sense" about the simplex embodied in the familiar "right-hand rule", let’s call it positive, otherwise orientation along the "left-hand rule", that is negative.
Definition 3. A \( j \)-face of a \( k \)-simplex is any \( j \)-simplex \((j < k)\) spanned by a proper subset of vertices of \( s_k \), this gives a strict partial order relation \( s_j \prec s_k \) and if \( s_j \) is a face of \( s_k \) we denote it \( s_j(s_k) \). A \( j \)-face is shared by two \( k \)-simplexes \((j < k)\) if it is a face of both.

We will need to say when it is possible and how to compare two \( k \)-simplexes (of the same dimension), i.e. their reciprocal orientations. The idea of next definition is that this is possible when there exists an hyperplane where both the \( k \)-simplexes lie.

**Definition 4.** We say that two \( k \)-simplexes in \( \mathbb{R}^N \) with \( N \geq k \) are comparable if they belong to the same \( k \)-dimensional hyperplane. Moreover, we say that two comparable oriented simplexes are consistent if they have same orientation sign.

The orientation sign refers to definition\(^2\) The condition for two \( k \)-simplexes to be in the same hyperplane is equivalent to ask that any convex span of \( k+2 \) points chosen from the union of their vertices is not a \((k+1)\)-simplex. Another relevant concept in DEC is that one of induced orientation, that is the orientation that a face of a simplex inherited when the last one is oriented.

**Definition 5.** We call induced orientation by an oriented \( k \)-simplex \( s_k \) on a \( j \)-face \( s_{ij}(s_k) \) the corresponding ordering of its vertices in the sequence ordering the vertices of \( s_k \). If two \( k \)-simplexes \( s_k \) and \( s'_k \) induce opposite orientations on a shared \( j \)-face we say that the \( j \)-face cancels.

Now we introduce the definition of \( k \)-cube.

**Definition 6.** A \( k \)-cube \( c_k = \{v_0, v_1, \ldots, v_{2^k-1}\} \) is the convex span of \( 2^k \) points of \( \mathbb{R}^N \) with \( N \geq k \) such that there exist \( k! \) different \( k \)-simplexes having their \( k+1 \) vertices chosen between the vertices \( v_i \) of \( c_k \) and sharing two by two only one \((k-1)\)-face. Moreover, vertices are extremal points of the convex combination. Each one of these \( k! \) simplexes \( s_k^i \) is said a proper \( k \)-simplex of \( c_k \) and we denote it \( s_k^i(c_k) \) where \( i \in \{0, 1, \ldots, k!\} \). The dimension of \( c_k \) is \( k \). Note that there is more than one way to choose these \( k! \) proper \( k \)-simplex and we call each of them a simplicial decomposition of \( c_k \) denoted \( \Delta c_k = \bigcup_{i=1}^{k!} s_k^i \).

When we don’t need to specify the index \( i \) of the these internal simplexes we omit it.

**Remark 7.** A shared \((k-1)\)-face \( s_{k-1}(s_k^i(c_k)) = s_{k-1}(s_k^{i'}(c_k)) \) with \( i \neq i' \) is inside \( c_k \) and not on its boundary. A precise definition of boundary for simplexes and cubes will be given later.

**Definition 8.** A \( j \)-face of a \( k \)-cube is any \( j \)-cube \((j < k)\) spanned by a proper subset of vertices of \( c_k \) and not intersecting its interior, this gives a strict partial order relation \( c_j \prec c_k \) and if \( c_j \) is a face of \( c_k \) we denote it \( c_j(c_k) \). A \( j \)-face is shared by two \( k \)-cubes \((j < k)\) if it is face of both.

The concept of orientation for \( k \)-cubes follows from that one for \( k \)-simplexes.

\(^2\)Extremal means that a vertex can not be written as convex combination of the other vertexes.
Definition 9. A $k$-cube $c_k = (v_0, v_1, \ldots, v_{2^k-1})$ is oriented assigning to each $k$-simplex $s^i_k$ in a simplicial decomposition of $c_k$ an orientation such that the $(k-1)$-faces $s_{k-1}(s^i_k(c_k))$ that they share cancel. Two oriented simplicial decompositions are in the same equivalence class of orientation if any two comparable not shared $(k-1)$-simplexes $s_{k-1}(s^i_k(c_k))$ (i.e. lying on the same $(k-1)$-face on the boundary of $c_k$) are consistent. We denote $\Delta c_k = \bigcup_{i=1}^{k!} s^i_k$ an oriented simplicial decomposition.

Definition 10. Analogously to definition 4 for simplexes, two $k$-cubes are comparable if they lie in the same $k$-dimensional hyperplane, while we say that they are consistent if the $k$-simplexes of the two simplicial decompositions are consistent.

It is enough to check the consistency between any $k$-simplex in the simplicial decomposition of one of the two $k$-cubes and any $k$-simplex in the decomposition of the other $k$-cube because of definition 9.

Example 11. Consider the 2-cube $c_2 = \{v_0, v_1, v_2, v_3\}$. A simplicial decomposition is given by $s^1_2 = (v_0, v_1, v_3)$ and $s^2_2 = (v_1, v_2, v_3)$. Indeed let $s_1 = \{v_1, v_3\}$ be the 1-simplex shared by $s^1_2$ and $s^2_2$, then they cancel on $s_1$ because $s_1(s^1_2) = (v_1, v_3)$ and $s_1(s^2_2) = (v_3, v_1) = -(v_3, v_1)$. Another decomposition is that one given by $s^C_2 = (v_0, v_2, v_3)$ and $s^D_2 = (v_0, v_1, v_2)$. These two decompositions are in the same equivalence class because they induce on the $1$-simplexes $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}$ and $\{v_3, v_0\}$ consistent orientations.

Proposition 12. There are only two possible equivalence classes of orientation\footnote{Namely, on them, it is induced an opposite orientation, see definition 4.} for a $k$-cube $c_k$, when an orientation $c_k$ is assigned the other one is denoted $-c_k$. One orientation can be conventionally defined to be positive and the other negative.

Proof. Once a single simplex $s_k(c_k)$ is oriented, the proposition follows from the facts that a simplex can be oriented in only two ways because the cancelling condition on the shared $(k-1)$-faces in a simplicial decomposition force all the others to assume an orientation propagating to the entire cubic cell. \hfill $\square$

Now the concept of induced orientation for simplexes can be transferred to cubes.

Definition 13. We call induced orientation by an oriented $k$-cube$^4$ $c_k$ on a $j$-face $c_j(c_k)$ the orientation assigned on it by the inductively oriented $j$-simplexes $s_j(s_k(c_k))$ of its simplicial decomposition. If two $k$-cubes induce opposite orientations on a shared $j$-face we say that the face cancels.

We do an example considering the oriented 2-cube $c_2 = (v_0, v_1, v_2, v_3)$. Let be $s^A_2 = (v_0, v_1, v_3)$ and $s^B_2 = (v_1, v_2, v_3)$ the 2-simplexes orienting a simplicial decomposition of $c_2$. Once $s_2^A$ is oriented, the orientation assigned on it by the inductively oriented $1$-simplexes $s_1(s_2^A(c_2))$ of its simplicial decomposition is $s_1(s_2^A) = (v_1, v_3)$. Another decomposition is that one given by $s^C_2 = (v_0, v_2, v_3)$. These two decompositions are in the same equivalence class because they induce on the $1$-simplexes $\{v_0, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}$ and $\{v_3, v_0\}$ consistent orientations.

\footnote{[Orientation convention for cubes] The result of this simplicial decomposition is that also a $k$-cube has only two possible orientations. In one dimensions a 1-cube is also a 1-simplex; in two dimensions a 2-cube $(v_0, v_1, v_2, v_3)$ can be anticlockwise oriented (positive) with the normal pointing outside the plane along the ”right hand rule” or clockwise oriented (negative) otherwise clockwise with the normal pointing outside the plane along the ”left-hand rule”; in three dimensions a 3-cube $(v_0, \ldots, v_7)$ can have, looking at it from outside, all the faces anticlockwise oriented (positive) or viceversa all clockwise (negative). These two possibilities corresponds respectively to have all the normals to its faces pointing outside or inside the volume.}

\footnote{We saw in definition 4 and proposition 12 that a $k$-cube $c_k$ is oriented trough the orientation of the $k$-simplexes of its decomposition, hence we talk equivalently of orientation induced by the oriented $k$-simplexes of the simplicial decomposition of $c_k$.}
decomposition of $c_2$. The inductively oriented 1-faces $c_1(c_2)$ are $(v_0, v_1), (v_1, v_2), (v_2, v_3)$ and $(v_3, v_0)$.

Our intention is to operate on object made by many cubes, like lattice. So we introduce collections of cubes suitable to define a discrete calculus. Later we will restrict ourselves to the case of discrete manifold.

**Definition 14.** A cubic complex $C$ of dimension $n$ is a finite collections of elementary cubes $c_k$, called also cells, such that $0 \leq k \leq n$, every face of an elementary cube is in $C$ and the intersection of any two cubes of $C$ is either empty or a face of both. To each cubes is assigned an orientation $c_k$. The local orientation of $C$ is the orientation of the cubes of dimension $n$. We denote with $|C|$ the topological set of $\mathbb{R}^n$ $(n \leq N)$ given by the union of all $n$-cubes $c_n$ in $C$.

Fixing an orientation of the $n$-cube is like orienting the $n$-volume of the $n$-hyperplane containing it or equivalently the $n$-volume of the space $\mathbb{R}^n$ where it can be embedded. For a discrete manifold is like to orient the tangent space for a continuous one. The meaning of having finite collections of cubes is that to deal with compact sets in the continuous case.

In particular we are interested in cubic complexes which are discrete version of an orientable compact boundaryless manifold $M$ of dimension $n$. The idea to define a discrete manifold of dimension $n$ is that to have a cubic complex (in local sense) topologically equivalent to a $n$-ball in $\mathbb{R}^n$. Moreover we want to orient a discrete manifold, this is possible using the "cancelling" notion of definition 13.

**Definition 15.** A cubic complex $C$ of dimension $n$ is a discrete manifold (boundaryless) if every $(n-1)$-cube is shared exactly by two $n$-cubes. A manifold is orientable if the orientations of all $n$-cubes can be chosen such that every shared $(n-1)$-face cancels.

The meaning of this definition is that if we consider a $(n-1)$-cube $c_{n-1} \in C$ the set $c_n : c_{n-1} \subseteq c_n$ is simply connected and homeomorphic to a unit $n$-dimensional ball.

In DEC important concepts are the ones of dual cell and dual complex. To define a centre of a cube we use barycentric coordinates, while in [DHL05] they use the concept of circumcentre for a simplicial complex. This last concept is very simple in the simplicial case but for our case we should introduce the concept of Voronoi diagram [JTI13] and we prefer to avoid this.

**Definition 16.** The centre of a $k$-cube $c_k$ is the barycentre of its vertices, denoted with $b(c_k)$.

**Remark 17.** The union $c_k \cup c'_k/s_k \cup s'_k$ of two comparable consistent $k$-cubes/$k$-simplexes sharing a $(k-1)$-face inherits an orientation that is consistent with that one of $c_k/s_k$ and $c'_k/s'_k$, in the sense that any $k$-simplex that can be generated by the vertices in the union and properly contained in it is defined to be consistent with $c_k/s_k$ and $c'_k/s'_k$.

**Definition 18.** We define the operation $+$ as $c_k + c'_k := c_k \cup c'_k$, when $c_k = c'_k$ we set $c_k + c_k = 2c_k := c_k \cup c_k$ as multiset\footnote{Multisets are sets that can differentiate for multiple instances of the same element, for example $a, b$ and $a, a, b$ are different multisets.} \{c_k, c_k\}. For oriented cubes $c_k$ we define an analogous $+$ operation and in addition we define the inverse element, that is...
The dual of a $k$-cube, called dual cell, is derived from the duality operator (see next definition 19) $\ast : c_k \to \ast(c_k)$ and the set of dual cells of a cubic complex will be the dual complex $\ast C$. Remember that when two $k$-cubes induce two opposite orientation on a shared $(k-1)$-face this last one cancels.

**Definition 19.** For a discrete manifold $C$ of dimension $n$ the duality map acts on a $k$-cube $c_k$ giving the (n-k)-dual cell $\ast(c_k)$ obtained with the following union of (n-k)-simplexes

$$\ast(c_k) = \bigcup_{c_n \prec c_{n-1} \prec \cdots \prec c_k} \bigcup \{b(c_k), b(c_{k+1}), \ldots, b(c_n)\},$$

where in both union $c_k$ is fixed, $c_n$ varies on the first union and fixed in the second one, while the (n-k)-tuple $c_{k+1}, \ldots, c_{n-1}$ vary on the second union according to the rule specified in subscript. For an oriented $k$-cube $c_k = (v_0, \ldots, v_{2k-1})$ the oriented dual cell $\ast(c_k)$ is obtained assigning to each (n-k)-simplex $\ast(c_k)$ an orientation $(b(c_k), \ldots, b(c_n))$ if the oriented $n$-cube $(v_0, \ldots, v_{k-1}, b(c_k), \ldots, b(c_n))$ is consistent with the local orientation of $c_n$ and $-(b(c_k), b(c_{k+1}), \ldots, b(c_n))$ otherwise. The dual complex $\ast C$, or dual discrete manifold, is the collection $\{\ast(c_k)\}_{c_k \in C}$.

**Remark 20.** The oriented (n-k)-simplexes in the union 2 share two by two exactly one (n-k)-face that cancels. In definition 19 with the oriented n-cubes $c_n$ and $-c_n$ we always associate a non-oriented object $c(c_n) = c_0$, while with a 0-cube $c_0$ we associate an oriented n-cube always consistent with the local orientation. This "asymmetry" is due to the fact that a vertex doesn’t have an intrinsic orientation. Formally we could fix this considering a vertex $c_0$ with two orientations $\pm c_0$ such that $\ast(\pm c_n) = \pm c_0$ and $\ast(\pm c_0) = \pm c_n$.

2.2. Discrete differential forms and exterior derivative. At this point we are ready to define discrete versions of differential forms and Stokes Theorem 1. Remind the idea of continuous $k$-form as object that can be integrated only on a $k$-submanifold and defined as a linear map from $k$-dimensional sets to $\mathbb{R}$. When $k$-dimensional sets are defined on a mesh of a discrete manifold we call them chains, a linear mapping from chains to real numbers is quite a natural discrete counterpart of a differential form.

**Definition 21.** Let $C$ be a cubic complex and $\{e_i^k\}_{i \in I_k}$ the collection of all elementary oriented $k$-cubes in $C$ indexed by $I_k$. The space of $k$-chains $C_k(C)$ is the space with basis $\{e_i^k\}_{i \in I_k}$ of the finite formal sums $\gamma_k = \sum_{i \in I_k} \gamma_i^k e_i^k$ where the coefficient $\gamma_i^k$ is an integer.

In defining the discrete $k$-forms we are not technical as usual in algebraic topology, we want just to stress that a discrete $k$-form is a map from the space of $k$-chains to $\mathbb{R}$.  

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These simplexes give a simplicial decomposition for the (n-k)-cube

$$\cup_{c_k \prec c_{k+1} \prec \cdots \prec c_n} \{b(c_k), \ldots, b(c_n)\}.$$
Definition 22. A discrete $k$-form $\omega^k$ is a linear mapping from $C_k(C)$ to $\mathbb{R}$, i.e.

$$\omega^k(\gamma_k) = \omega^k \left( \sum_{i \in I_k} \gamma^i \alpha^i_k \right) = \sum_{i \in I_k} \gamma^i \omega^k(c^i_k).$$  \hfill (3)

We add two forms $\omega_1$ and $\omega_2$ adding their values in $\mathbb{R}$, i.e. $(\omega_1 + \omega_2)(\gamma) = \omega_1(\gamma) + \omega_2(\gamma)$. The vector space of $k$-forms is denoted $\Omega^k(C)$.

Any discrete $k$-form can be written as finite linear combination respect to a basis $\{\alpha^i_k\}_i$ with same cardinality of $\{c^i_k\}_i$ and determined by the relation $\alpha^i_k(c^j_k) = \delta_{ij}$.

So we have a natural pairing between chains and discrete forms, that is the bilinear pairing

$$[\omega^k, \gamma_k] := \omega^k(\gamma_k).$$ \hfill (4)

Writing $\omega^k = \sum_i \omega^k_i \alpha^i_k$, here $\omega^k_i$ are real coefficient, the pairing (4) becomes $\sum_{i \in I_k} \omega^k_i c^i_k$.

So the natural pairing (4) leads to a natural notion of duality between chains and discrete forms. In DEC natural pairing plays the role that integration of forms plays in differential exterior calculus. The two can be related by a discretization procedure, for example in the manifold case, thinking to have a piecewise linear manifold that can be subdivided in cubes $\{\sigma^i_k\}_i$ and a differential $k$-form $\omega^k$, the integration of $\omega^k$ on each $k$-cube gives its discrete counterpart $\omega^k_d$, where the subscript $d$ is for discrete, defined as

$$\omega^k_d(\sigma_k) := \int_{\sigma_k} \omega^k.$$ \hfill (5)

In this way a discrete $k$-form is a natural representation of a continuous $k$-form.

Remark 23. A discrete $k$-form can be viewed as a $k$-field taking different values on different $k$-cubes of an oriented cubic complex, e.g. a $k$-form $\omega^k$ on a $k$-cube $c_k$ is such that

$$\omega^k(\sigma_k) := \int_{\sigma_k} \omega^k.$$  \hfill (6)

With this in mind, for a discrete 1-form we use also the name discrete vector field.

To define a discrete exterior derivative, that will give us a discrete version of (1), we have to introduce a discrete boundary operator. As we did so far the definition for cubes goes trough the one for simplexes.

Definition 24. The boundary operator $\partial_k : C_k(C) \to C_{k-1}(C)$ is the linear operator that acts on an oriented $k$-simplex $s_k = (v_0, \ldots, v_k)$ as

$$\partial_k s_k = \partial(v_0, \ldots, v_k) = \sum_{i=0}^k (-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_k),$$ \hfill (7)

where $(v_0, \ldots, \hat{v}_i, \ldots, v_k)$ is the oriented $(k-1)$-simplex obtained omitting the vertex $v_i$. Let $\Delta c_k$ be a simplicial decomposition (see definition 6) of $c_k$, the boundary operator on $k$-cubes acts as

$$\partial_k c_k = \sum_{s_k \in \Delta c_k} \partial_k s_k.$$ \hfill (8)

\footnote{In case of non-piecewise linear manifold the discretization process present some technicalities, but it is still possible to give meaning to \footnote{6}}
The boundary of non-oriented objects is obtained doing the boundary of the correspondent oriented objects and then considering the resulting sets without orientation.

**Example 25.** Consider \( \gamma_2 = (v_0, v_1, v_2, v_3) \) and \( \Delta c_k = (v_0, v_1, v_3) \cup (v_1, v_2, v_3) \), then \( \partial_2 \gamma_2 = \partial_2 (v_0, v_1, v_3) + \partial_2 (v_1, v_2, v_3) = (v_1, v_3) - (v_0, v_3) + (v_0, v_1) + (v_2, v_3) - (v_1, v_3) + (v_1, v_2) = (v_0, v_1) + (v_1, v_2) + (v_2, v_3) + (v_3, v_0) \).

In practice \( \partial_k \), applied to \( c_k \), gives back the faces of \( c_k \) with the orientation induced by \( c_k \). In other terms this \( \partial_k \) extracts the oriented border of an oriented \( k \)-cube. A remarkable property of this operator is that the border of a border is the void set, therefore\[
\partial_k \circ \partial_{k+1} = 0. \tag{9}\]

Now with the duality defined by the natural pairing \(^{14}\) the time is ripe to introduce the discrete exterior derivative (or coboundary operator) \( d^k : \Omega^k(\mathcal{C}) \to \Omega^{k+1}(\mathcal{C}) \) defined by duality \(^{14}\) and the boundary operator.

**Definition 26.** For a cubic complex \( \mathcal{C} \) the **discrete exterior derivative** (or **coboundary operator**) is the linear operator \( d^k : \Omega^k(\mathcal{C}) \to \Omega^{k+1}(\mathcal{C}) \) such that
\[
[d^k, \omega^k, c_{k+1}] = [\omega^k, \partial_{k+1} c_{k+1}] \tag{10}\]
where \( \omega^k \in \Omega^k(\mathcal{C}) \) and \( c_{k+1} \subset C_{k+1}(\mathcal{C}) \). Moreover we set \( d\Omega^n(\mathcal{C}) = 0 \). Definition \(^{10}\) is equivalent to \( d^k (\omega^k) := \omega^k \circ \partial_{k+1} \).

From definition \(^{26}\) and \(^{9}\) it is straightforward the property
\[
d^{k+1} \circ d^k = 0. \tag{11}\]

**Assumption 27.** Unless otherwise specified, we are omitting if operators are referred to the primal complex or its dual. We assume the right one at the right moment. Moreover for discrete manifolds in definition \(^{15}\) the operators of this section don’t change in the dual.

From definition \(^{10}\) and natural pairing \(^{5}\) we have a discrete Stokes theorem: consider a chain \( \gamma_k \) and a discrete form \( \omega^k \) then
\[
\int_{\gamma_k} d^k \omega^k \equiv [d^k, \omega^k, \gamma_k] = [\omega^k, \partial_{k+1} \gamma_k] \equiv \int_{\partial_k \gamma_k} \omega^k. \tag{12}\]

2.3. **Hodge star and codifferential.** The counterpart of \( d^k \), denoted with \( \delta^{k+1} \), mapping a \((k+1)\)-form into a \( k \)-form is the tool still missing to have all what we need from DEC. Given two \( k \)-forms \( \omega^k_1 \) and \( \omega^k_2 \), this operator is defined as the adjoint of \( d \) with respect to the scalar product
\[
\langle \omega^k_1, \omega^k_2 \rangle = \sum_{i \in I_k} \omega^k_{1,i} \omega^k_{2,i} \tag{13}\]
This scalar product is the discrete version of formula \(^{14}\) in footnote \(^{9}\) below.

**Definition 28.** The discrete codifferential operator \(^{14}\) \( \delta^k : \Omega^k(\mathcal{C}) \to \Omega^{k+1}(\mathcal{C}) \) is defined by \( \delta^0 \Omega^0(\mathcal{C}) = 0 \) and the equation
\[
\langle d^k \omega^k_1, \omega^k_{2,1} \rangle = \langle \omega^k_1, \delta^{k+1} \omega^k_{2,1} \rangle, \tag{15}\]

\(^{9}\)In the smooth case, for a manifold \( \mathcal{M} \) of dimension \( n \), the Hodge star is the map \( * : \Omega_k(\mathcal{M}) \to \Omega_{n-k}(\mathcal{M}) \), defined by its local metric and the local scalar product of \( k \)-forms \( \langle \omega^k_1, \omega^k_2 \rangle = \omega^k_1 \cdot \omega^k_2 \), such that
\[
\omega^k \wedge \omega^k := \langle (\omega^k_1, \omega^k_2) \rangle \text{vol}^n.\]
where \( \omega_k^1 \in \Omega^k(C) \) and \( \omega_2^{k+1} \in \Omega^{k+1}(C) \).

Also for this operator we have a property analogous to (11), that is

\[
\delta^k \circ \delta^{k+1} = 0.
\]

(16)

For a matrix description of the operators of this chapter see [DKT08]. The operators \(*, \partial, d, \star, \delta\) we introduced on \(C\) can be defined also on the dual complex \(*C\). In particular when a discrete manifold is considered and the inverse map of duality map \(*\) can be written as

\[
\star^{-1} = (-1)^{k(n-k)}\star
\]

(17)

where \(*\) acts on \(*C\) as on \(C\), i.e. when \(*C\) is still made of \(k\)-cubes, all definitions applies in the same way. In this case [9] the same notation for the discrete operators is used both on \(C\) and \(*C\).

2.4. Hodge decomposition. The original work of the Hodge decomposition theorem for finite dimensional complexes is [Eck44]. The Hodge decomposition in our context is as follows.

**Theorem 29.** Let \(C\) be a discrete manifold of dimension \(n\) and let \(\Omega^k(C)\) be the space of \(k\)-forms on \(C\). The following orthogonal decomposition holds for all \(k\):

\[
\Omega^k(C) = d^{k-1}\Omega^{k-1}(C) \oplus \delta^{k+1}\Omega^{k+1}(C) \oplus \Omega^k_H(C),
\]

(18)

where \(\oplus\) means direct sum and \(\Omega^k_H(C) = \{\omega_k|d^k\omega_k = \delta^k\omega_k = 0\}\) is the space of harmonic forms.

**Proof.** Consider the discrete forms \(\omega_1^{k-1} \in \Omega^{k-1}(C), \omega_2^{k+1} \in \Omega^{k+1}(C)\) and \(h^k \in \Omega^k_H(C)\) chosen arbitrary form their respective spaces. Since \(\delta^{k+1}\) is the adjoint of \(d^k\) using (11) and (15) we have \(\langle d^{k-1}\omega_1, \delta^{k+1}\omega_2 \rangle = \langle d^k d^{k-1}\omega_1, \omega_2 \rangle = 0\). By the definition of \(\Omega^k_H(C)\) and (15) we have \(\langle h^k, d^{k-1}\omega_1 \rangle = \langle \delta^k h^k, \omega_1 \rangle = 0\), likewise \(\langle h^k, \delta^{k+1}\omega_2 \rangle = \langle d^k h^k, \omega_2 \rangle = 0\). Therefore the spaces \(d^{k-1}\Omega^{k-1}(C), \delta^{k+1}\Omega^{k+1}(C)\) and \(\Omega^k_H(C)\) are each other orthogonal. A general \(k\)-form \(\omega^k \in \Omega^k(C)\)

where \(\text{vol}^n\) is the volume form on the manifold. Denoting with \(d\) the exterior derivative for differential forms, this operator can be computed through its action on the basis of \(k\)-forms \(dx^i = dx^{i_1} \wedge \cdots \wedge dx^{i_k} (i_1 < \cdots < i_k)\), that gives back the forms \(d^k dx^i = C dx^{i_1} \wedge \cdots \wedge dx^{i_k} (i_1 < \cdots < i_k)\) with \(C\) is such that \(dx^i \wedge C dx^{n-k} = \langle (dx^i, dx^{n-k})\rangle\text{vol}^n = \text{vol}^n\). On a smooth manifold \(\mathcal{M}\), for details see chapter 14 in [Tay12] or section 6.2 in [AMR88], the adjoint operator \(\delta\) of \(d\) is defined respect to the scalar product

\[
\langle \omega^k_1, \omega^k_2 \rangle := \int_{\mathcal{M}} \omega^k_1 \wedge \star \omega^k_2 = \int_{\mathcal{M}} \langle \omega^k_1, \omega^k_2 \rangle\text{vol}^n.
\]

(14)

Thinking about, (13) is the discrete version of the most right term of the last formula (13) and it can be introduced in a sophisticated way that "emulates" the continuous case of this footnote. Let’s try to sketch this parallelism. Since the duality \(\star\) maps a primal cell into an only one dual cell and vice versa, the most spontaneous thing to set a discrete Hodge star \(\star\) from \(k\)-forms into \((n-k)\)-forms is doing it from \(\Omega^k(C)\) into \(\Omega^{n-k}(\ast C)\), i.e. \(\ast : \Omega^k(C) \rightarrow \Omega^{n-k}(\ast C)\), this can be defined with the relation \((\omega^k, c_{k,n}) = (\ast \omega^k, \ast c_{k,n})\). With this definition (13) can be written as

\[
\langle \omega^k_1, \omega^k_2 \rangle := \sum_{i \in I_k} \langle \omega^k_1, c_{k,n} \rangle \langle \omega^k_2, \ast c_{k,n} \rangle = \sum_{i \in I_k} \omega^k_{1,i} \ast \omega^k_{2,i},
\]

that is the equivalent of the middle term in (13). For details how to define a discrete wedge product and the counterpart of the \(n\)-volume form see section 12 of [DHL08] and [DMS14].

In this case we have also the formula \(\delta^{k+1}\omega^{k+1} = (-1)^{nk+1} \ast \delta^{n-(k+1)} \ast \omega^{k+1}\), which is analogous to the one for \(\delta\) in smooth boundaryless manifolds. This is proved using \(\star^{-1} = (-1)^{k(n-k)}\star\), which follows from (17) and the definition of \(\star\) in footnote 9 see also subsection 5.5 in [DKT08].
Table 1. Left table: conceiving continuous and discrete forms. Right table: \(d\) and \(\delta\) applied to smooth forms.

| smooth case | discrete case |
|-------------|---------------|
| \(\omega^0\): scalar field | \(\omega^0\): vertex field |
| \(\omega^1\): vector field | \(\omega^1\): edge field |
| \(\omega^2\): vector field | \(\omega^2\): face field |
| \(\omega^3\): scalar field | \(\omega^3\): cell field |

\[
\begin{array}{ccc}
\text{form} & \text{smooth case} & \text{discrete case} \\
\omega^0 & \text{grad} \omega^0 & 0 \\
\omega^1 & \text{curl} \omega^1 & -\text{div} \omega^1 \\
\omega^2 & \text{div} \omega^2 & \text{curl} \omega^2 \\
\omega^3 & 0 & -\text{grad} \omega^3 \\
\end{array}
\]

belongs to \((d\Omega^{k-1}(C) \oplus \delta\Omega^{k+1}(C))^\perp\) if and only if \(\langle \omega^k, d^{k-1}\omega^{k-1} + \delta^{k+1}\omega^{k+1} \rangle = \langle \delta^k \omega^k, \omega^{k-1} \rangle + \langle d^k \omega^k, \omega^{k+1} \rangle = 0\) for all \(d^{k-1}\omega^{k-1} + \delta^{k+1}\omega^{k+1} \in d\Omega^{k-1}(C) \oplus \delta\Omega^{k+1}(C)\), namely \(d^k \omega^k = \delta^k \omega = 0\). Then we showed \((d\Omega^{k-1}(C) \oplus \delta\Omega^{k+1}(C))^\perp = \Omega^k_H\), so the decomposition is complete and generates all \(\Omega^k(C)\). \(\square\)

3. Discrete Operators on the Discrete Torus

In this section we show how the discrete exterior derivative and its adjoint work on a cubic complex of dimension three, namely we work with 0-forms, 1-forms, 2-forms, and 3-forms, setting discrete equivalent of gradient, curl and divergence operators. We consider a discrete mesh of edge 1 on the discrete torus \(\mathbb{T}^3_N = \mathbb{Z}^3/N\mathbb{Z}^3\) of side \(N\) and we refer explicitly to other dimensions whenever appropriate. Thinking about the bases \(dx_i, dx_j dx_k\) and \(dx_1 dx_2 dx_3\) respectively for 1-forms, 2-forms and 3-forms, the parallel of the left table 1 between the smooth case and the discrete case is really natural. For smooth from the action of \(d\) and \(\delta\) can be summarized as in the right table 1.

Now we are indicating also the discrete operators of last section 2 without the index \(k\), unless otherwise specified, it will be implicit to use the right one according to the \(k\)-form on which they act. We want to show how to compute \(d\) and \(\delta\). Calculations will be performed in some cases, the others follow mutatis mutandis. We recover also the usual divergence, Gauss and Stokes theorems in our discrete setting. A part proposition 33, all what we do in this section for the discrete torus can be done with some extra work and notation for general discrete manifold with non regular mesh (i.e. that can not be defined using the canonical basis \(\{e_1, e_2, e_3\}\)).

3.1. Notation. Let \(\{e_1, e_2, e_3\}\) be a canonical right handed orthonormal basis. We define the sets of all the vertices

\[
V_N := \{x = (x_1, x_2, x_3) : x_i = 0, e_i, \ldots, (N - 1)e_i\}.
\]

all the oriented edges \(E_N = \left( \bigcup_{i=1}^3 E_{N}^{i+} \right) \cup \left( \bigcup_{i=1}^3 E_{N}^{i-} \right)\) where

\[
E_{N}^{i\pm} = \{\text{positive/negative oriented } 1\text{-cubes } e = \pm(x, x + e_i), i \in 1, 2, 3\},
\]

all the oriented faces \(F_N = \left( \bigcup_{i=1}^3 F_{N}^{i+} \right) \cup \left( \bigcup_{i=1}^3 F_{N}^{i-} \right)\) where

\[
F_{N}^{1\pm} = \{\text{positive/negative oriented } 2\text{-cubes } f_1 = \pm(x, x + e_2, x + e_2 + e_3, x + e_3), \}
\]

\[
F_{N}^{2\pm} = \{\text{positive/negative oriented } 2\text{-cubes } f_2 = \pm(x, x + e_3, x + e_3 + e_1, x + e_1), \}
\]

\[
F_{N}^{3\pm} = \{\text{positive/negative oriented } 2\text{-cubes } f_3 = \pm(x, x + e_1, x + e_1 + e_2, x + e_2)\}
\]
and finally all the oriented cells \( C_N = (C^+_N) \cup (C^-_N) \) where
\[
C^+_N := \left\{ \text{positive/negative oriented 3-cubes } \pm c : \mathbf{b}(c) = x + \frac{e_1}{2} + \frac{e_2}{2} + \frac{e_3}{2} \right\}.
\]

**Remark 30.** Observe that for a face the orientation can be defined as for a 2-simplex, see definition \([\mathbf{1}]\) and \([\mathbf{2}]\) i.e. with the ordering given to the vertices in its sequence.

We indicate a general oriented edge, oriented face and oriented cell respectively with \( e = (x, y), f \) and \( c \), where \( y = x \pm e_i \) for some \( i \), while for the non-oriented ones we use a calligraphic writing, that is \( \epsilon = \{x, y\}, \xi \) and \( \zeta \). We indicate in calligraphic also the non-oriented collections just defined above, that is \( \mathcal{E}_N, \mathcal{F}_N \) and \( \mathcal{C}_N \). The collections of \( k \)-cube defining the discrete Manifolds \( \mathcal{C} \) on \( \mathbb{T}^k_N, \mathcal{T}^k_N \) and \( \mathcal{T}_N \) are respectively \( \{V_N, E^+_N, F^+_N, C^+_N\} \), \( \{V_N, E^{1, +}_N, E^{2, +}_N, F^+_N\} \) where \( F^+_N \) is defined as \( F^{2, +}_N \) and \( \{V_N, E^+_N\} \) where \( E^+_N \) is defined as \( E^{3, +}_N \).

A general vertex, edge, face and cell field is going to be denoted respectively with \( h(x), j(x, y), \psi(f) \) and \( \rho(c) \).

The dual complex \( \mathcal{G} \) is constructed on a mesh with the same geometrical structure of the original one and obtained translating each vertex of \( (e_1/2, e_2/2, e_3/2) \). We denote with \( \mathbb{T}^N_\mathcal{G} \) the dual torus obtained translating the vertexes. So we use an analogous notation and indicate with an index \( * \) the collections and the elements of the dual complex, that is respectively \( V^*_N, E^*_N, F^*_N \) and \( C^*_N \) and \( x^*, e^* = (x^*, y^*) \), \( f^* \) and \( c^* \).

**Remark 31.** With a vertex \( x \in V_N \) are associated the three edges \( \{x, x+e_1\}, \{x, x+e_2\} \) and \( \{x, x+e_3\} \), consequently we associate with \( x \) also the oriented edges \( \pm(x, x+e_1) \), \( \pm(x, x+e_2) \) and \( \pm(x, x+e_3) \). While in two dimension with \( x \in V_N \) are associated \( \{x, x+e_1\}, \{x, x+e_2\} \) and \( \{x, x+e_3\} \) and in one dimension only \( \{x, x+e_1\} \).

With a vertex \( x \in V_N \) are associated the three faces \( f_1, f_2 \) and \( f_3 \) such that \( \mathbf{b}(f_i) = x + \frac{a_i}{2} + \frac{a_j}{2} \) where \( i, j, k \in \{1, 2, 3\} \) and \( i \neq j \neq k \), consequently we associate with \( x \) also the oriented faces \( \pm f_1, \pm f_2 \) and \( \pm f_3 \). In two dimensions we don’t have a subscript on the faces and only one face \( \xi \), defined as \( f_3 \), is associated with \( x \).

Finally with a vertex \( x \in V_N \) is associated only one cell \( c \) such that \( \mathbf{b}(c) = x + \frac{a_1}{2} + \frac{a_2}{2} + \frac{a_3}{2} \), consequently the oriented cells \( \pm c \) are associated with \( x \).

### 3.2. \( \mathbf{d} : \Omega^0 \to \Omega^1 \)
Consider \( (x, y) \in E_N \) and compute \( \mathbf{d}h(x, y) = h \circ \partial(x, y) = h(y - x) = h(y) - h(x) \). Choosing \( (x, y) = (x, x + e_i) \) we define the discrete gradient
\[
\mathbf{d}h(x, x + e_i) = h(x + e_i) - h(x) = \nabla_i h(x).
\]
(19)

For a 1-chain \( \sum_{i=1}^m (x_k, y_k) \), where \( y_k = x_{k+1} \), setting \( \int_\gamma \nabla h \cdot dl := \sum_{k=1}^m \nabla_i h(x_k) \) we get
\[
\mathbf{d}h(\gamma) = \int_\gamma \nabla h \cdot dl = h \circ \partial(\gamma) = \int_{\partial\gamma} h = h(y_m) - h(x_1),
\]
where \( \partial\gamma \) is the boundary of \( \gamma \). This gives us the discrete versions of line integral. For completeness we translates in our language the well known result relating gradient fields to zero integrations on closed paths.
Proposition 32. A 1-form \( j(x, y) \in d\Omega^0 \) if and only if \( j(\gamma) = \oint_\gamma j = 0 \) for all closed path(chain) \( \gamma \) on \( V_N \). Moreover, the vertex function

\[
h^x(y) := \sum_{(w, z) \in \gamma_{x \rightarrow y}} j(w, z)
\]

(21)
is such that \( j(y, y') = h^x(y') - h^x(y) \) for every \( (y, y') \in E_N \) and it doesn’t depend on the particular path \( \gamma_{x \rightarrow y} \) from \( x \) to \( y \). Any two functions \( h^x(\cdot) \) and \( h^x(\cdot) \) differ for an additive constant.

Proof. If \( j(x, y) = dh(x, y) \) from (20) we get \( \oint_\gamma j = 0 \) since \( y_m = x_1 \). Now let’s prove the opposite implication. Let \( \gamma \) be a closed path and \( x, y \) any two points on it. Since \( \gamma = \gamma_{x \rightarrow y} + \gamma_{y \rightarrow x} \), from \( \sum_{(w, z) \in \gamma} j(w, z) = \sum_{(w, z) \in \gamma_{x \rightarrow y}} j(w, z) + \sum_{(w, z) \in \gamma_{y \rightarrow x}} j(w, z) = 0 \), we have

\[
\sum_{(w, z) \in \gamma_{x \rightarrow y}} j(w, z) = \sum_{(w, z) \in \gamma_{y \rightarrow x}} j(w, z), \text{ where } \gamma'_{x \rightarrow y} = -\gamma_{y \rightarrow x}.
\]

Hence the function (21) doesn’t depend on the particular path from \( x \) to \( y \) and for any \( (y, y') \in E_N \) its gradient is \( h^x(y') - h^x(y) = \sum_{(w, z) \in \gamma_{x \rightarrow y}} j(w, z) - \sum_{(w, z) \in \gamma_{y \rightarrow x}} j(w, z) = j(y, y') \). Taking a closed path \( \gamma = \gamma'_{x \rightarrow y} + \gamma_{y \rightarrow x} + \gamma_{x \rightarrow x'} \) we obtain \( h^x(y) = h^x(y) + \sum_{(w, z) \in \gamma_{x \rightarrow x'}} j(w, z) \) because of \( \oint_\gamma j = 0 \).

3.3. \( d : \Omega^1 \rightarrow \Omega^2 \). Let \( f_k \in F_N^{\pm} \) be the face with index \( k \) that is associated with \( x \) as in previous notation in subsection 3.1, see remark 31. We have \( df_k = j \circ \partial f_k = \sum_{(x, y) \in \partial f_k} j(x, y) \), that we write

\[
dj(f_k) = \sum_{(x, y) \in \partial f_k} j(x, y) =: \int_{\partial f_k} j \cdot dl.
\]

We compute \( dj(f_3) \) as \( dj(f_3) = j(x, x + e_1) + j(x + e_1, x + e_1 + e_2) - j(x + e_2, x + e_1 + e_2) - j(x, x + e_2) \), calling \( \nabla f_j(x) := j(x + e_1, x + e_1 + e_2) - j(x + e_2, x + e_1 + e_2) \), the last equation becomes \( dj(f_3) = \nabla f_{j_2}(x) - \nabla f_{j_1}(x) \), so \( dj(f_k) \) is also the curl defined as

\[
dj(f_k) = e^{klm} \nabla f_j(x) := (\text{curl} j)(f_k),
\]

where \( e^{klm} \) is the Levi-Civita symbol summed on repeated indexes with the Einstein convention. Let \( S \) be the 2-chain \( S = \sum_{i=1}^m f_i \), where \( \{f_i\}_{i=1}^m \) can be any collection of oriented faces in \( F_N \). Setting \( \int_S \text{curl} j = \int_S \sum_{i=1}^m \text{curl} j(f_i) \), we have

\[
\int_S \text{curl} j = \sum_{i=1}^m \int_{\partial f_i} j \cdot dl.
\]

where \( \Gamma = \sum_{i=1}^m \partial f_i \). If all the faces in the collection have same orientation, i.e. \( f_i \in F_N^\pm \) for all \( i = 1, \ldots, m \), the chain \( \Gamma = \sum_{i=1}^m \partial f_i \) is the closed path corresponding to the inductively oriented boundary of the oriented surface \( S \), indeed when a edge is shared by two faces \( f_i \) and \( f_j \) it cancels and gives two contributions with same modulo but opposite sign in \( \int_{\partial f_i} j \cdot dl \) and \( \int_{\partial f_j} j \cdot dl \). We obtained in this last case a discrete version of the usual Stokes theorem.
Two dimensions. In two dimensions \(dj(f) = \nabla_1 j_2(x) - \nabla_2 j_1(x)\), where \(f\) is the face associated with \(x\) as \(f_3\) in notation and \(\text{curl} j\) is a 2-form defined on the faces lying on the plane of \(e_1\) and \(e_2\). Proceeding like we did in three dimensions we have the discrete Green-Gauss formula.

3.4. \(d : \Omega^2 \rightarrow \Omega^3\). Consider the \(c \in C^+_N\), we compute and define the divergence \(d\) as follows

\[
d\psi(c) = \psi \circ \partial(c) = \sum_{f \in F_N : f \in \partial c} \psi(f) =: \text{div} \psi(c).
\]

As in notation the cell \(c\) is associated with the vertex \(x \in V_N\). Let \(f_i\) with \(i \in \{1, 2, 3\}\) the three positive faces associated with \(x\). We have then \(\sum_{f, j \in \partial c} \psi(f) = \sum^3_{i=1} \psi(f_i + e_i) - \psi(f_i)\). We define the flow across \(c\) as

\[
\int_{\partial c} \psi \cdot d\Sigma := \sum^3_{i=1} \Phi(f_i),
\]

where \(\Phi(f_i) := \psi(f_i + e_i) - \psi(f_i)\). Let \(V\) be the \(3\)-chain \(V = \sum^m_{i=1} c_i\), where \(\{c_i\}_{i=1}^m\) can be any collection of oriented cubes in \(C_N\). Setting \(\int_V \text{div} \psi \, dx := \sum^m_{i=1} (\text{div} \psi)(c_i)\), we have

\[
\int_V \text{div} \psi \, dx = \int_S \psi \cdot d\Sigma,
\]

where \(S = \sum^m_{i=1} \partial c_i\). If all the cubes in the collection have same sign orientation, i.e. \(c_i \in C^+_N\) for all \(i \in 1, \ldots, m\), the chain \(S = \sum^m_{i=1} \partial c_i\) is the closed surface corresponding to the inductively oriented boundary of the oriented volume \(V\), indeed when a face is shared by two cells \(c_i\) and \(c_j\) it cancels and gives two contributions with same modulo but opposite sign in \(\int_{\partial c_i} \psi \cdot d\Sigma\) and \(\int_{\partial c_j} \psi \cdot d\Sigma\). We obtained in this last case a discrete version of the usual divergence theorem.

Now we compute the codifferential operator \(\delta\), this can be done using the fact that it is the adjoint of \(d\) with respect to the scalar product \((13)\) or using the formula for \(\delta\) in footnote \((10)\) since \((17)\) is true. We don’t write the computation but we give just the explicit form for \(\delta\).

3.5. \(\delta : \Omega^3 \rightarrow \Omega^2\). Let \(f_i \in F^+_N\) be the face with index \(i\) that is associated with \(x\) as in notation of subsection 3.1 see remark \((11)\) we have

\[
\delta \rho(f_i) = \rho(c - e_i) - \rho(c) = \sum_{c' : f_i \in \partial c'} \rho(c')
\]

With the help of the dual complex we can recover line integrals analogously to subsection 3.2. To do it we should interpret \(\rho\) as 0-form on the dual vertex \(x^* = *c\) of a cell \(c\) (see the operator * defined in footnote \((10)\) and \(\delta \rho\) as discrete vector field of gradient type \((\nabla_i \rho(c) := \rho(c) - \rho(c - e_i))\) on the dual edges \((x^*, y^*) = *(f_i)\) of the faces \(f_i \in F^+_N\). The collection of cells \(c\) of \(C_N\) that would define the dual chain \(\gamma^*\) equivalent to \(\gamma\) in \((20)\) should be such that any two consecutive cells \(c_i, c_{i+1}\) in the collection share a common face that cancels.
3.6. \( \delta : \Omega^2 \to \Omega^1 \). On an edge \((x, y) \in E_N\) the codifferential operator is
\[
\delta \psi(x, y) = \sum_{f : (x, y) \in \partial f} \psi(f).
\]
(28)

Let \( \{f_j\}_{j=1}^3 \) be the faces associated with \( x \) and let’s set \( \nabla_i \psi(f_j) := \psi(f_j) - \psi(f_j - e_i) \), the codifferential can be rewritten \( \delta \psi(x, x + e_k) = \varepsilon_{klm} \nabla_i \psi(f_m) =: \text{curl} \psi(x, x + e_k) \) getting the relation \( \psi(x, x + e_k) = \sum_{f : (x, x + e_k) \in f} \psi(f) \). The definition of curl as integral on the border of a surface as \( (22) \) and a the discrete version of the Stokes theorem like \( (24) \) can be recovered with the dual complex. We should interpret (see the operator \( \ast \) defined in footnote \( 10 \)) \( \text{curl} \psi(x, x + e_k) \) as a 2-form on the dual face \( f^* = \ast(x, x + e_k) \) and \( \psi(f) \) as discrete vector fields on the dual edges \( (x^*, y^*) = \ast f \).

With a suitable collection of edges \( \{e_i\}_{i=1}^m \) we can define any desired surface \( S^* \).

**Two dimensions.** Formula \((28)\) is still true, calling \( f \) the face associated to \( x \), the codifferential is interpreted as an orthogonal gradient
\[
\begin{pmatrix}
\delta \psi(x, x + e_1) \\
\delta \psi(x, x + e_2)
\end{pmatrix} =
\begin{pmatrix}
\nabla_2 \psi(f) \\
\nabla_1 \psi(f)
\end{pmatrix} =: \nabla^\bot \psi(x).
\]
(29)

3.7. \( \delta : \Omega^1 \to \Omega^0 \). For a vertex \( x \in V_N \) the codifferential results to be
\[
\delta j(x) = \sum_{y : (y, x) \in E_N} j(y, x).
\]
(30)

To highlight the analogous nature of this operator to that of \( (24) \), we define \( \Phi_i(x) := j(x, x + e_i) - j(x - e_i, x) \). Here \( \Phi_i \) can be thought as the net flow passing through the ”source/sink” \( x \). We define a divergence operator as \( \text{div} j(x) := -\delta j(x) \), i.e.
\[
\text{div} j(x) = \sum_{y : (y, x) \in E_N} j(x, y) = \sum_i \Phi_i(x)
\]
(31)

To recover the usual definition of divergence like \( (24) \) with \( (25) \) and a discrete divergence theorem like \( (26) \) we should interpret the divergence in \( (31) \) with the help of the dual complex. We have to think about (see the operator \( \ast \) defined in footnote \( 10 \)) \( \text{div} j(x) \) as a 3-form on the cube \( c^* = \ast(x) \) and \( j(x, y) \) as 2-forms on the faces \( f^* = \ast(x, y) \). With a suitable collection of vertexes \( \{e_i\}_{i=1}^m \) we can define any desired volume \( V^* \).

**Other dimensions.** In other dimensions definition \( (31) \) doesn’t change.

We conclude determining the space of harmonic discrete vector fields on the discrete manifold of \( \mathbb{T}_N^d \).

**Proposition 33.** Consider on \( \mathbb{T}_N^d \) the collections \( V_N \) and \( E_N^+ \) in \( \mathcal{C} \) defined in notation \( [7] \), then in any dimension \( d \) the harmonic space \( \Omega^1_H(\mathbb{C}) \) has dimension \( d \) and it is generated by the discrete vector fields
\[
\varphi^{(i)}(x, x + e_j) := \delta_{ij}, \quad i, j = 1, \ldots, n.
\]
(32)

**Proof.** For each \( i \in \{1, \ldots, n\} \) we have both \( d^i \varphi^{(i)}(f) = 0 \) for all \( f \in F_N \) and \( \delta^i \varphi^{(i)}(x) = 0 \) for all \( x \in V_N \), then \( \varphi^{(i)} \in \Omega^1_H \) and the set of discrete vector fields \( \{\varphi^{(i)}(x)\}_i \) generate a \( d \)-dimensional subspace of \( \Omega^1_H \). Since \( d^1 = d^0 \mathbb{C}^0 \oplus \Omega^1_H \) we have \( \dim \Omega^1_H = \dim(\text{Im} d^0) \), this is also the dimension of the quotient space \( \text{Ker} d^1 / \im d^0 \) of closed discrete vector fields (1-forms) differing for an exact...
1-form. By duality (see natural pairing (4)) between $d$ and $\partial$, the quotient space $\text{Ker} \, d^1 / \text{Im} \, d^0$ is isomorphic to quotient space of closed 1-chains that differ for the boundary of a 2-chain $\text{Ker} \, \partial^1 / \text{Im} \, \partial^2$, which dimensionality is given by the number of independent circle $S^1$ in the cartesian product $S^1 \times \cdots \times S^1$ homeomorphic to the continuous torus $T^n$, that is $n$. Therefore $\{j_i(x)\}_{i=1}^n$ generates all $\Omega^1_H$. □

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