Anomalous finite-size scaling in the Fortuin-Kasteleyn clusters of the five-dimensional Ising model with periodic boundary conditions

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(Dated: September 11, 2019)

We present a Monte Carlo study of the Fortuin-Kasteleyn Ising model on a five-dimensional ($d = 5$) hypercubic lattice with linear size $L$ and periodic boundary conditions. Our numerical results show that, at the critical point, the size of the largest cluster $C_1 \sim L^{d_1}$ with $d_1 = 3d/4$, while the size of other clusters $s(R) \sim R^d$ with $R$ the radius of gyration and $d_2 = 1 + d/2$. This anomalous two-scaling behaviour motivates us to study the following two quantities calculated from all clusters except $C_1$: a reduced susceptibility $\chi' = L^{-d} \sum_{k\neq 1} C_k^d$ with $C_k$ the size of $k$-th cluster, and the cluster size distribution $n'(s, L)$. Our results demonstrate that $\chi' \sim L^{2d_1-d} = L^2$ and $n'(s, L) \sim s^{-1} n(s/L^{d_2})$ where the standard scaling relation $\tau = 1 + d/d_1$ holds. To identify the thermal exponent, we study $C_1$ and $\chi'$ within the critical window and show that $C_1(t, L) \sim L^{d_1} \tilde{C}_1(tL^{1/d_1})$ and $\chi' \sim L^2 \tilde{\chi}'(tL^{1/d_1})$ with $t = 1 - K/K_c$ and $1/\nu' = d/2$, $1/\nu'' = 2$. Our numerical results suggest that in order to have a complete description of the scaling of the 5D Ising model in the FK and other representations, two sets of renormalization exponents, $(2/d, 3d/4)$ and $(1/2, 1 + d/2)$, are needed, where the first set of exponents corresponds to those for the critical Ising model on a finite complete graph while the second set of exponents comes from the Gaussian fixed point in the framework of the renormalization group.

I. INTRODUCTION

In the field of phase transitions and critical phenomena, it is generally believed that many models exhibiting a continuous phase transition possess a model-dependent upper critical dimension $d_c$, such that above $d_c$ the critical exponents, characterizing the critical behaviours in the thermodynamic limit, take standard mean-field values. For the Ising model, field theory predicts $d_c = 4$ and above $d_c$, the mean-field theory applies and predicts the thermal and magnetic critical exponents $y_{\text{MF}}^\text{MF} = 1/\nu' = 2$ and $y_{\text{MF}}^\text{MF} = 1 + d/2$, which is consistent with the prediction of the Gaussian fixed point. However, the finite-size scaling (FSS) is surprisingly subtle above $d_c$ and remains the subject of an active debate [11].

The zero-field ferromagnetic Ising model on a finite lattice is defined via the Hamiltonian

$$H/(k_B T) = -K \sum_{\langle x,y \rangle} \sigma_x \sigma_y,$$  \hspace{1cm} (1)

where the non-negative parameters $k_B$, $T$, and $K$ are respectively the Boltzmann constant, temperature and the coupling constant. Here $\sigma_x \in \{-1,1\}$ denotes the spin variable at vertex $x$ and the sum is over all pairs of adjacent vertices. It has been numerically observed [7]–[11] that above $d_c$ the FSS of the Ising model depends sensitively on the imposed boundary conditions. For example, on five-dimensional boxes with periodic boundary conditions (PBC) and at the critical point $K_c$, the susceptibility was observed to have an anomalous scaling $\chi \sim L^{d/2}$ [3]–[8], contradicting the mean-field prediction $\chi \sim L^{2y_{\text{MF}}^\text{MF} - d} = L^2$, as observed on boxes with free boundary conditions (FBC) [3]–[7] [11]. In terms of the two-point function $g(x) := \langle \sigma_0 \sigma_x \rangle$, the anomalous scaling of $\chi$ originates from the distance-independent part of $g(x)$ when $\|x\|$ is larger than a certain crossover length. An explicit form of $g(x)$ was firstly conjectured by Papathanakos [14] to be

$$g(x) \sim \begin{cases} \|x\|^{2-d}, & \|x\| \leq O(L^{d/2 (d-2)}) \\ L^{-d/2}, & \|x\| \geq O(L^{d/2 (d-2)}). \end{cases}$$  \hspace{1cm} (2)

which has been confirmed numerically [9] via the Monte Carlo simulation of the Ising model in the high-temperature expansion representation. In words, $g(x)$ still follows standard mean-field prediction for moderate distances, but then enters a plateau when $\|x\|$ is large.

The anomalous FSS behavior of $\chi \sim L^{d/2}$ in PBC, which is caused by the anomalous FSS of $g(x)$ in Eq. (2), was firstly attempted to be explained using renormalization group arguments, by introducing an extra irrelevant dangerous field [2]. Later on, the authors in Ref. [9] show that the anomalous scaling is due to the proliferation of windings when $d \geq d_c$, and consequently can be repaired by simply considering an appropriate length scale which correctly accounts for the windings.

On the infinite lattice $\mathbb{Z}^d$ with sufficiently high $d$, the
two-point function of the Ising model has been proved to exhibit the same behaviour as the Green’s function of the simple random walk \([15]\), that is, \(g(x) \propto |x|^{-d-2}\). In order to understand \((2)\), it is natural to study a random-length random walk model \([11]\).

In \([11]\), it was proved that on boxes with \(d \geq 3\), the Green’s function of RLRW depends on the given mean walk length. This leads to a conjecture that the two-point function of the Ising model depends on the boundary conditions via its boundary-dependent mean walk length. One sensible choice of the Ising walk length is via the Aizenman random walk representation \([16, 17]\), and it was numerically observed that it scales as \(L^{d/2}\) on the five-dimensional torus \([18]\). Indeed, the Green’s function of RLRW with a mean walk length \(L^{d/2}\) was proved \([18]\) to recover the Papathanakos’ claim in Eq. \((2)\). Another way to predict the walk length of the Ising model is via its counterpart, the self-avoiding walk model in the variable-length (grand-canonical) ensemble, which is believed to exhibit the same FSS as the Ising model when \(d \geq d_c\). The mean walk length of self-avoiding walk on the high-dimensional torus is expected to have the same asymptotics as it on the complete graph; the latter was studied explicitly not only at the critical point but also at a broad family of pseudo-critical points \([19, 20]\).

The high-temperature expansion of the Ising model provides a nice geometric representation for the two-point function. However, another representation which possesses richer clusters information is via the \(q\)-state Fortuin-Kasteleyn (FK) random cluster model \([21]\), and the Ising model corresponds to \(q = 2\). The general FK random cluster model on a finite and connected graph \(G = (V, E)\) is defined by randomly choosing a spanning subgraph \((V, A)\) via the probability

\[
\pi(A) = \frac{1}{Z(p, q)} p^{|A|} (1 - p)^{|E\setminus A|} q^{-(A)},
\]

where \(p\) relates to the coupling \(K\) via \(p = 1 - e^{-2K}\) and \(c(A)\) is the number of connected components in \(A\). The partition function \(Z(p, q)\) is a normalisation factor. Compared to the high-temperature expansion, much less is known for the FSS of the FK clusters on high-dimensional boxes. Note that, in the FK representation, the two-point function \(g(x)\) can be written in terms of the probability that the origin \(0\) and the vertex \(x\) are in the same cluster. Thus, conceivably the anomalous FSS of \(g(x)\) will result in anomalous FSS of quantities related to FK clusters, such as cluster sizes and their distributions. The central motivation of this paper is to investigate the effect of the anomalous FSS on the FK clusters, by simulating the Ising model on five-dimensional tori.

Before discussing the high-dimensional torus, we first recall some known asymptotics on the complete graph (CG) with \(n\)-vertices, \(K_n\). The \(K_n\) case was firstly studied by Bollobás, Grimmett and Janson \([22]\) and then re-examined by Luczak and Luczak \([23]\). The latter proved that the size of the largest cluster \(C_1 \sim n^{3/4}\) within a critical window of width \(n^{-1/2}\), i.e., for all couplings \(K\) satisfying \(|K - K_c| = O(n^{-1/2})\). Mapping back to the lattice by setting \(n = L^d\), it suggests that \(C_1 \sim L^{3d/4}\) within a critical window of width \(L^{-d/2}\). In other words, the complete graph asymptotics suggest the magnetic and thermal exponents in high-dimensional torus are \(y_{h}^{CG} = 3d/4\) and \(y_{t}^{CG} = 1/\nu = d/2\).

In this paper, we first precisely estimate that the fractal dimension of \(C_1\) is \(d_f = 3.74(1)\) on five-dimensional tori at \(K = K_c\), consistent with \(y_{h}^{CG}\) as predicted from the CG asymptotics and also the numerical observation in \([24]\). In terms of FK clusters, the susceptibility can be written as

\[
\chi := \sum_i C_i^2 / L^d
\]

where \(C_i\) denotes the size of the \(i\)-th largest cluster. Recall that \(\chi \sim L^{2d_{\chi} - d} = L^{d/2}\). To quantify the contribution of \(C_1\) to \(\chi\), we define a reduced susceptibility \(\chi' = \sum_{i \neq 1} C_i^2 / L^d\) and numerically observe that \(\chi' \sim L^{2d_{\chi} - d} = L^2\), where \(d_{\chi}' = 1 + d/2\), consistent with the standard mean-field prediction \(y_{h}^{MF}\). We then go further, to study the critical windows of \(C_1\) and \(\chi'\), for the purpose of identifying the thermal exponent \(y_t\). Near \(K_c\), we numerically show that \(C_1 \sim L^{d_{\chi}'} \tilde{C}_1(tL^{d_{\chi}'} )\) while \(\chi' \sim L^2 \tilde{\chi}'(tL^{d_{\chi}'} )\), where \(t = (K - K_c)/K_c\) and \(\tilde{\chi}'(\cdot)\) are the corresponding scaling functions. Our results strongly suggest that, in terms of both thermal and magnetic exponents, the largest cluster follows the complete graph prediction, but the remaining clusters follow the standard mean-field prediction.

The above results for \(C_1\) and \(\chi'\) suggest that \(C_1\) dominates the other clusters. To investigate the fractal dimensions of all clusters in a collective way, we measure the size of a cluster in terms of its radius of gyration \(R\) (defined in Eq. \((3)\)), rather than the system size \(L\), and the corresponding fractal dimension is defined via \(C_i(R) \sim R^{d_{\chi}'}\). Our numerical results indicate that \(d_{\chi}' = y_{h}^{MF}\) for \(C_i\) with \(i \neq 1\), while for \(C_1\) we have \(d_{\chi}' = y_{h}^{CG}\).

Another interesting quantity to study is the cluster-size distribution \(n(s, L)\), defined by interpreting \(n(s, L)L^{d_{s}}\) as the number of clusters with size between \([s, s+ds]\). One expects the finite-size scaling \(n(s, L) \sim s^{-\tau} f(s/L^{d_{s}})\), with \(d_s\) the fractal dimension of the largest cluster, and the Fisher exponent \(\tau = 1 + d/2d_{\chi}'\). This has been observed in percolation clusters in various dimensions \([25, 26]\) and also Ising FK clusters in dimensions 2 and 3 \([27, 30]\). On the five-dimensional torus, however, the fractal dimension of other clusters is different from \(C_1\), which motivates us to study \(n'(s, L)\) defined in the same way as \(n(s, L)\) but excluding \(C_1\) in statistics. We expect \(n'(s, L)\) follows mean-field prediction, i.e., \(n'(s, L) \sim s^{-\tau'} \tilde{n}'(s/L^{d_{\chi}'})\) with \(\tau = 1 + d/d_{\chi}'\) and \(d_{\chi}' = y_{h}^{MF}\). We numerically confirm this in Section \([11, 12]\).

The remainder of this paper is organized as follows. In Section \([11]\) we summarize the observables and simulation details. Main numerical results are presented in Section \([11, 12]\). Finally we conclude with a discussion in Section \([14]\).
II. OBSERVABLES AND SIMULATIONS

The well-known cluster algorithms for simulating the Ising model are Wolff and Swendsen-Wang (SW) algorithms [31, 32]. The update of SW algorithm involves two steps. For a given bond configuration, identify all connected components and for each of them, randomly and uniformly assign either + or − spins to all vertices on it. This step thus maps configurations from bonds to spins. Given a spin configuration, for each vertex, adding a bond between this vertex and its neighbours with probability \( p = 1 - e^{-2K} \) if they have the same spin; otherwise do not add a bond. Given a spin configuration, the Wolff algorithm updates the spins as follows. Grow only one cluster from a uniformly and randomly chosen vertex, using the same connectivity rule as SW. Then generate a new spin configuration by flipping all spins on the cluster.

In two and three dimensions, it was numerically observed [33] that the Wolff algorithm has smaller dynamic exponent than SW. Thus, in simulations, we used Wolff algorithm to update and decrease the correlations of spin configurations and only used the SW update to create clusters for sampling. The number of Wolff updates between two consecutive SW steps is chosen to be approximately the volume divided by the averaged size of clusters, such that every spin has a decent chance to be updated during these steps. In both Wolff and SW updates, to speed up the process of adding bonds between neighbouring vertices with the same spin, we adopt the procedure described in detail in [20] (Eq.(3)-(5) and the context around).

In simulations, each time the SW step maps configurations from spins to bonds, the following observables are sampled:

(a) The size \( C_1 \) of the largest cluster. Moreover, we sample \( \sum_{i \neq 1} C_i^2 \) where \( C_i \) is the size of the cluster \( i \). The sum is over all cluster other than the largest cluster.

(b) For a cluster \( C \), its radius of gyration \( R(C) \) is defined as

\[
R(C) = \sqrt{\frac{\sum_{u \in C} (x_u - \bar{x})^2}{|C|}},
\]

where \( \bar{x} = \sum_{u \in C} x_u / |C| \). Here \( x_u \in \mathbb{Z}^d \) is defined algorithmically as follows. First choose the vertex, say \( o \), in \( C \) with the smallest vertex label, according to some fixed but arbitrary vertex labelling. Set \( x_0 = 0 \). Start from the vertex \( o \) and search through the cluster \( C \) using breadth-first growth. Iteratively \( x_v = x_u + e_i (e_i) \) if the vertex \( v \) is traversed from \( u \) along(against) the \( i \)-th direction, with \( e_i \) the unit vector in the \( i \)-th direction. We note this is a natural definition of radius of gyration when putting clusters in boxes with periodic boundary conditions.

(c) \( N(s) \), the number of clusters with size in \( [s, s+\Delta s) \).

(d) The averaged radius of gyration of clusters with size in \( [s, s+\Delta s) \), i.e.,

\[
R(s) = \frac{\sum_{C:|C| \in [s,s+\Delta s)} R(C)}{N(s)}.
\]

In our simulations, we appropriately choose the bin size \( \Delta s \) such that there are sufficient data in each bin for statistics. We then calculate the following quantities, by taking ensemble average of the observable defined above.

(i) The mean size of the largest cluster \( C_1 = \langle C_1 \rangle \).

(ii) A reduced susceptibility \( \chi' = L^{-d} \langle \sum_{i \neq 1} C_i^2 \rangle \).

(iii) \( R(s) = \langle R(s) \rangle \).

(iv) The cluster size density distribution \( n(s,L) = L^{-d} \langle N_s \rangle / \Delta s \).

The critical point we simulated in this paper is \( K_c = 0.1139150 \), using the known estimates reported in [13, 34]. The largest system size we achieved is \( L = 51 \).

III. NUMERICAL RESULTS

We report in this section the analysis of various quantities, by performing the least-square fits of the data to the corresponding finite-size scaling formula. As a precaution against correction-to-scaling terms that we fail to include in the fitting ansatz, we impose a lower cutoff \( L \geq L_{\text{min}} \) on the data points admitted in the fit, and we systematically study the effect on the \( \chi^2 \) value of increasing \( L_{\text{min}} \). Generally, the preferred fit for any given ansatz corresponds to the smallest \( L_{\text{min}} \) for which the goodness of fit is reasonable and for which subsequent increases in \( L_{\text{min}} \) do not cause \( \chi^2 \) value to drop by vastly more than one unit per degree of freedom. In practice, by “reasonable” we mean that \( \chi^2 / \text{DF} \lesssim 1 \), where DF is the number of degree of freedom. The systematic error is estimated by comparing estimates from various sensible fitting ansatz.

A. Fractal dimension of the largest and other clusters

We first analyze the data of \( C_1 \), the mean size of the largest cluster. At the critical point, we expect the scaling \( C_1 \sim L^{d_f} \). In Fig. 1, we plot \( C_1 \) versus \( L \) in log-log scale, and find that the slope is quite close to \( 3d/4 \). To better estimate \( d_f \), we perform the least-square fits of \( C_1 \) data to the following fitting ansatz,

\[
C_1 = L^{d_f} (a_0 + a_1 L^{d_1} + a_2 L^{d_2}).
\]
The exponents $y_1, y_2 < 0$, and thus the two terms associated with $a_1, a_2$ are for finite-size corrections. We find that, the fitting without including any correction terms (setting $a_1 = a_2 = 0$) cannot converge even for very large $L_{\min}$, i.e., the value of $\chi^2$ is always much larger than the degree of freedom. Including only one correction-to-scaling term $a_1$ with $y_1$ free gives reasonable fits when $L_{\min} = 8$; the result is listed in Tab. I. Including two correction-to-scaling terms with $y_1, y_2$ free cannot produce stable fits. We also perform the fitting by setting $y_1, y_2$ to be various values between 0 and $-2$, to test the effect to the estimate of $d_1$. We list the case $y_1 = -1$ and $y_2 = -2$ in Tab. I for comparison. For $C_1$, our final estimate is $d_1 = 3.74(1)$, which is consistent with $y_{\text{CG}}^C = 3d/4$ as predicted from the complete graph asymptotics.

For completeness, we also measure the radius of gyration for $C_1$. By performing the fits in a similar procedure as above, we obtain $R(C_1) \sim L^{0.99(1)}$. Thus $C_1 \sim L^{d_1} \sim R^{d_1}$.

We then study the size of all other clusters. To show the fractal dimension of all other clusters in a collective way, it turns out to be convenient to study the scaling of the cluster size $s$ versus the radius of gyration $R$, at the critical point. The scaling function $\tilde{\chi}(s/R)$ is first a power-law with exponent $d_{\chi}$, we list the case $d_{\chi} = 3.74(1)$, which are significantly smaller than $5/2$. So, in words, the standard mean-field prediction from CG asymptotics but all other clusters follow the prediction from standard mean-field theory. Thus, it is natural to study a reduced susceptibility $\chi'$ defined from the second moments of all other clusters, see Sec. II, and we expect $\chi' \sim L^{d_{\chi}}$. In Fig. 1 we plot $\chi'$ versus $L$ in a log-log scale, and data points show a slope close to 2. We then perform the least-square fits to the data of $\chi'$ to an ansatz $\chi' = L^{d_{\chi}}(a_0 + a_1 L^{y_1} + a_2 L^{y_2})$, to carefully estimate $y_{\chi'}$. However, it turns out that the fits are quite sensible to the correction-to-scaling terms. According to the different choices of $y_1, y_2$, the estimates of the $y_{\chi'}$ varies from 1.81(4) to 2.03(4), which are significantly smaller than $5/2$.

![FIG. 2: Log-log plot of the cluster size $s$, for all clusters except $C_1$, versus its radius of gyration $R$, at the critical point for various system sizes. Our data suggest a power-law relation between $s$ and $R$, with an exponent consistent with 7/2, rather than 15/4.](image)

Our estimate of $d_{\chi}, d_{\chi}'$ seemingly suggest that $C_1$ follows the prediction from CG asymptotics but all other clusters follow the prediction from standard mean-field theory. Thus, it is natural to study a reduced susceptibility $\chi'$ defined from the second moments of all other clusters, see Sec. II, and we expect $\chi' \sim L^{2}$, the standard mean-field scaling for susceptibility. In Fig. 1 we plot $\chi'$ versus $L$ in a log-log scale, and data points show a slope close to 2. We then perform the least-square fits to the data of $\chi'$ to an ansatz $\chi' = L^{d_{\chi}'}(a_0 + a_1 L^{y_1} + a_2 L^{y_2})$, to carefully estimate $y_{\chi'}$. However, it turns out that the fits are quite sensible to the correction-to-scaling terms. According to the different choices of $y_1, y_2$, the estimates of the $y_{\chi'}$ varies from 1.81(4) to 2.03(4), which are significantly smaller than $5/2$.

### B. Cluster-size distribution

We next study the cluster-size distribution $n(s, L)$, defined in Sec. II at the critical point. The standard scaling formula for $n(s, L)$ is

$$n(s, L) \sim s^{-\tau} \tilde{n} \left( s/ L^{d_{\chi}} \right),$$

where $\tilde{n}(\cdot)$ is the universal scaling function. The exponent $\tau$ is called Fisher exponent and $d_{\chi}$ is the fractal dimension of the largest cluster. The scaling function $\tilde{n}(x)$ is roughly a constant when $x \ll O(1)$ and then decays quickly to zero when $x \gg 1$. So, in words, the standard behaviour of $n(s, L)$ is first a power-law with exponent $\tau$,

![FIG. 1: Log-log plot of the size of the largest cluster $C_1$ and the reduced susceptibility $\chi'$ versus $L$, at the critical point.](image)

| $L_{\min}$ | $a_0$  | $a_1$  | $a_2$  | $d_1$ | $y_1$ | $y_2$ | $d_{\chi}/DF$ |
|-----------|--------|--------|--------|-------|-------|-------|----------------|
| 8         | 1.18(3) | -0.35(16) | 0      | 3.73(6) | -1.2(5) | 0      | 3.1/5          |
| 10        | 1.19(7) | -0.31(29) | 0      | 3.74(1) | -1(1)  | 0      | 3.1/4          |
| 8         | 1.19(4) | -0.26(32) | -0.14(129) | 3.73(8) | -1     | -2     | 3.1/4          |
| 10        | 1.19(4) | -0.26(32) | -0.14(129) | 3.73(8) | -1     | -2     | 3.1/4          |
then starts to decay around $s \sim L^{d_f}$. The scaling relation $\tau = 1 + d/d_f$ is believed to hold, as numerically observed for percolation models in various dimensions and for the FK Ising model in dimensions 2 and 3.

In Fig. 3, a direct log-log plot of the $n(s, L)$ data versus $s$ shows a clear power-law behaviour, following by a plateau before it decays significantly. The slope of the straight line is consistent with $17/7$. Setting $\tau = 17/7$ and applying the scaling relation, we obtain $d_f = 7/2$, which equals to $d_f' = 1 + d/2$ (or $y_{\text{MF}}$), the fractal dimension of clusters other than $C_1$. Recall our results in the above section, that the fractal dimension of all clusters except $C_1$ follows mean-field prediction. It then motivates us to study a reduced cluster-size distribution $n'(s, L)$, which is defined in the same way as $n(s, L)$ but only use clusters without $C_1$. We expect $n'(s, L) \sim s^{-\tau} n' \left( s/L^{d_f'} \right)$ with $\tau = 1 + d/d_f'$. Fig. 3 shows the log-log plot of $n'(s, L)$ versus $s$. A clear power-law behaviour is observed with exponent $\tau = 17/7$. To show the scaling function, we plot in log-log scale the $n'(s, L)s^\tau$ versus $s/L^{d_f'}$, shown as an inset in Fig. 4. The clear data collapse for systems with different sizes confirms the scaling formula for $n'(s, L)$.

By comparing Fig. 3 and Fig. 4, it is clear that the plateau part is contributed by $C_1$. Drawing a line tangential to the ending points of the plateau for systems with various sizes gives another power-law but with exponent $7/3$, shown in Fig. 3. Setting $\tau = 7/3$ and using the scaling relation, we obtain $d_f = 15/4$ which is equal to $d_f'$ (or $y_{\text{CG}}$), the fractal dimension of $C_1$.

### C. Critical window - thermal exponent

In this section we study the finite-size scaling near the critical point, to estimate the thermal exponent $y_t$. The exponent $y_t$ defines the size of the critical window, within which the finite-size scaling is expected to be the same as at the critical point.

We first study $C_1$. Results shown in the above two sections support the consequence in a consistent way, that is, at the critical point the property of $C_1$ follows complete graph prediction. Conceivably, the critical window of $C_1$ on five-dimensional torus exhibits the same size as on the complete graph. The rigorous result shown in Section 7 in Ref. 23 implies that, for $C_1$, the size of the critical window is of order $n^{-1/2}$, with $n$ the number of vertices on the complete graph. Mapping back to the lattice by matching the volume, we conjecture $C_1$ has a critical window with size of order $L^{-d_f/2}$, i.e., $C_1(t, L) \sim L^{d_f} \tilde{C}_1(tL^{h_{\text{CG}}})$ with $t = (K - K_c)/K_c$. For each system size $L$, we simulated at several values of $K$ such that $tL^{h_{\text{CG}}} \in [-2, 2]$. The scaling formula of $C_1$ conjectured above is confirmed from the data collapse shown in Fig. 5.

We finally consider $\chi'$. The above sections also provide a consistent conjecture for clusters excluding $C_1$: their behaviour follows standard mean-field prediction. We thus conjecture that $\chi'(t, L) \sim L^{2d_f - d} \tilde{\chi}'(tL^{h_{\text{MF}}})$ with $y_{\text{MF}} = 2$ and $2d_f' - d = 2$. For a fixed system size $L$, we perform simulations at several values of $K$ such that $tL^{h_{\text{MF}}} \in [-2, 2]$. Fig. 6 plots $\chi'(t, L)L^{-2}$ versus $tL^{d_f/2}$, and the conjectured scaling formula of $\chi'$ is observed from collapsing data for different system size $L$. However, compared with $C_1$, a much stronger finite-size correction was observed for $\chi'$, which also appears in the fits of $\chi'$ at the critical point, discussed in Sec. IIIA.
In this paper we study the Fortuin-Kasteleyn clusters of the Ising model on five-dimensional hypercubic lattice with periodic boundary conditions. At the critical point, we numerically show that the size of the largest cluster scales as $C_1 \sim L^{y_{CG}^\prime}_n \tilde{C}_1(tL^{y_{CG}^\prime})$, where $y_{CG}^\prime = 3d/4$ and $y_{CG}^\prime = d/2$ can be predicted from the complete graph asymptotics.

However, compared with $C_1$, more interesting results are found for the behaviour of other clusters. Our numerics show that the fractal dimension of all other clusters is $d_i^\prime = y_{MF}^\prime = 1 + d/2$. We also study a reduced susceptibility $\chi'$, defined in the same way as the standard susceptibility but excluding the contribution from $C_1$, and our numerical results suggest that $\chi' \sim L^{2y_{MF}^\prime - d} \chi'(tL^{y_{MF}^\prime})$. Furthermore, we numerically show the scaling of the reduced cluster-size distribution $n'(s, L) \sim s^{-\tau} n'(s/L^{d_i^\prime})$ with $\tau = 1 + d/d_i^\prime$. All of the above leads to a consistent consequence, that is, for the FK Ising model on five-dimensional torus, the critical behaviour of the largest cluster follows the complete graph asymptotics, while the rest follows the standard mean-field prediction.

V. ACKNOWLEDGMENTS

Y. D. acknowledges the support by the National Key R&D Program of China under Grant No. 2016YFA0301604 and by the National Natural Science Foundation of China under Grant No. 11625522. Z.Z. and S.F. thank the Research Support Scheme from ACEMS for providing financial support to hospitalize S. F in Monash University, during which this work was written.

IV. DISCUSSION

In this paper we study the Fortuin-Kasteleyn clusters of the Ising model on five-dimensional hypercubic lattice

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