MOTION OF A VECTOR PARTICLE IN A CURVED SPACETIME. II
FIRST ORDER CORRECTION TO A GEODESIC IN A SCHWARTZCHILD BACKGROUND.

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The influence of spin on a photon's motion in a Schwartzchild and FRW spacetimes is studied. The first order correction to the geodesic motion is found. It is shown that unlike the world-lines of spinless particles, the photons world-lines do not lie in a plane.

1. Introduction

The motion of spinning particles, test ones or tops, in a curved spacetime has been a subject of great interest for the last nearly 80 years. It is amusing to note that, even today, there is no general consensus on the behaviour of particles with spin in external gravitational fields. It seems that the first to discuss the motion of free extended material particles with spin in an external electromagnetic field was Frenkel\textsuperscript{1}. The classical test body was considered to be so small compared with the background curvature length-scale that all its multipoles beyond the dipole could be neglected. The equations of the free motion of such a 'pole-dipole' particle for the linearized theory of gravitation were obtained by Mathisson\textsuperscript{2}. Papapetrou\textsuperscript{3} worked with non-singular localized energy-momentum tensor $T_{ij}$ for a spinning particle (in the limit of a vanishing mass) and obtained equations for moments of $T_{ij}$. Dixon\textsuperscript{4} found a way to make Papapetrou's type of argument covariant at each step of the derivation and also found equations which referred to paths described by an arbitrary parameter rather than the world-line length. He came up with the equations

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of motion in the form
\[
\frac{Dp^i}{Dq} = \frac{1}{2} R^i_{\ jkm} v^j S^{km},
\]
\[
\frac{DS^{ij}}{Dq} = p^{[i} v^{j]},
\]
(1)

with \( q \) an arbitrary parameter along the curve, \( v^i = dx^i(q)/dq \), \( D/Dq \) a covariant derivative along the world-line \( x^i(q) \) and \( p^i \) and \( S^{ij} \) defined as in Papapetrou\(^3\). The explicit form of generalized momentum \( p \), containing the additional term depending only on spin and connection, was shown in \([5]\), in complete agreement with the definition of Papapetrou. In the derivation of these equations by Dixon nothing required the object to have a rest mass or required \( g_{ij} dx^i dx^j \) to be nonzero. They were, therefore, sufficiently general to relate to any localized object.

However there have not yet been satisfactory direct derivation of these equations for massless particles with spin. In \([15]\) Baylin and Ragusa treated massless particles as traceless, but the authors couldn’t come out with the final answer whether "traceless"-massless particles’ motion deviate from null geodesics. At the same time they mention that "we have been unable to prove that null geodesics necessarily follow for the general case of spin in curved spacetime. It even seems unlikely in view of complexity of [Papapetrou-Dixon] equations".

Explicit coupling to curvature actually turns out to be the general rule in the field theoretical case, so that strictly geodesic behaviour is not to ever be expected in general. Because fields are defined at all spacetime points and not just along the trajectories associated with point particles, the associated second order field equations are sensitive to the values of the connections at differing points. Both Dirac\(^7\) and Maxwell\(^8,9,7\) equations lead to an implicit expressly non-inertial coupling to the Riemann curvature tensor. And even though it is possible to remove the Christoffel symbols at any given point (or even along a single curve), it is nonetheless impossible to remove them from an entire region, with the curvature dependent terms we find in the second order wave equations being the field theoretic generalization of the (covariantly describable) geodesic deviation found for pairs of nearby falling particles.

In the previous paper (paper I\(^10\)) we have derived the equations of motion for a vector particle with spin from a Lagrangian. We have worked out the approach that gives a satisfactory approximation to the wave as some curves that can be called "rays" and, at the same time, takes into account the polarization. The obtained equation coincided with the Papapetrou equation and is independent of a mass of a particle. In the present work we apply equations derived in paper I for photons transversely propagating in a Schwartzchild metric. We develop the technique to calculate the corrections to the geodesic motion. We show the deviation from a geodesic in Schwartzchild metric and examine the consequences of the deviation in different gravitational backgrounds. We present a general discussion of the results.

2. Geodesic congruences for Schwartzchild metric
2.1. Hamilton-Jacobi Equation for photons

The Hamilton-Jacobi equation (HJE) for photons reduces to the eikonal equation,
\[ g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} = 0. \] (2)

In Schwartzchild geometry
\[ ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \]
with the coordinates: \( x^\mu = \{t, r, \theta, \phi\} = \{0, 1, 2, 3\} \), the metric coefficients are:
\[ g^{ij} = \text{diag} \left(\frac{1}{1 + 2\phi}, -(1 + 2\phi), \frac{1}{r^2}, -\frac{1}{r^2 \sin^2 \theta}\right), \]
where by \( \phi \) we denoted the “Newtonian” potential \( \phi = -\frac{m}{r} \). We employ here the Planck units \( c = \hbar = G = 1 \). Eq. 2 becomes
\[ \frac{1}{1 + 2\phi} \left(\frac{\partial \psi}{\partial t}\right)^2 - (1 + 2\phi) \left(\frac{\partial \psi}{\partial r}\right)^2 - \frac{1}{r^2} \left(\frac{\partial \psi}{\partial \theta}\right)^2 - \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial \psi}{\partial \phi}\right)^2 = 0. \] (4)

Following the usual approach to separating the variables in HJE, we represent the desired solution in the form
\[ \psi = Et + L_z \phi + R(r) + \Theta(\theta), \]
with the convention that \( \sqrt{a^2} = \pm a \). Substituting it in Eq. 4, we obtain the equations where variables have got separated indeed and produced an arbitrary constant (the constant of separation, total angular momentum) \( L \):
\[ \frac{E^2 r^2}{1 + 2\phi} - (1 + 2\phi) r^2 \left(R'\right)^2 = \left(\Theta'\right)^2 + \frac{L_z^2}{\sin^2 \theta} = L^2, \]
from where we find functions \( \Theta \) and \( R \), and get the solution in the form
\[ \psi_{\pm} = Et + L_z \phi \pm \int \sqrt{\frac{E^2 r^2 - L^2 (1 + 2\phi)}{r (1 + 2\phi)}} dr \pm \int \sqrt{\frac{L^2 - \frac{L_z^2}{\sin^2 \theta}}{r^2}} d\theta. \]

2.2. Associated Tetrad Basis

Any coordinate system \( \{x^i\} \) in the spacetime specifies the natural vector \( \{\partial_i\} \) and covector \( \{dx^i\} \) frames at each point. We defined the metric by the scalar products of natural covectors
\[ g^{ij} = \langle dx^i, dx^j \rangle, \]
because our main requirement to the metric is that it shall separate the HJE for geodesics, which looks much more simple in the contravariant components than in covariant ones.

We choose for simplicity the equatorial plane, \( \theta = \pi/2 \) (\( L = L_z = p_\varphi \)). Consequently, the dynamic phase \( \psi \) becomes
\[
\psi_{\pm} = Et \mp \int \frac{\sqrt{E^2 r^2 - L^2 (1 + 2\phi)}}{r(1 + 2\phi)} \, dr \pm L\varphi ,
\]
where we fixed the signs as
\[
\psi_+ = Et - R + L\varphi , \\
\psi_- = Et + R - L\varphi ,
\]
and write the gradients of the phase
\[
d\psi_{\pm} = E dt \pm L d\varphi \mp R' \, dr .
\]

Now we can construct the associated basis on which the connection form is identically zero. Taking the scalar product \( \langle d\psi_+, d\psi_- \rangle \), we obtain
\[
\langle d\psi_+, d\psi_- \rangle = E^2 \langle dt, dt \rangle - L^2 \langle d\varphi, d\varphi \rangle - \langle R', dr \rangle = \frac{2E^2}{1 + 2\phi} ,
\]
since from the metric coefficients (Eq. 3) we have \( \langle dt, dt \rangle = g_{tt} = 1/(1 + 2\phi) \), \( \langle d\theta, d\theta \rangle = g_{\theta\theta} = -r^{-2} \), \( \langle dr, dr \rangle = g_{rr} = -(1 + 2\phi) \) and \( \langle d\varphi, d\varphi \rangle = g_{\varphi\varphi} = -1/r^2 \sin^2 \theta \). The first pair of basis vectors is thus
\[
\nu^\pm = \frac{d\psi_\pm}{\sqrt{\langle d\psi_+, d\psi_- \rangle}} = \frac{d\psi_\pm \sqrt{1 + 2\phi}}{E \sqrt{2}} .
\]

Since \( \nu^\pm \) is a null vector, we need to add timelike vectors to our basis. Applying the Hamilton-Jacobi theorem\(^{11}\) to the solution (Eq. 5) we can construct the integral of motion in the form of a function whose values are constant on the trajectory
\[
V = \frac{\partial \psi_\pm}{\partial L} = \pm \left( \varphi - \frac{\partial R'}{\partial L} \right) .
\]
We choose
\[
dV = d\varphi - \frac{\partial R'}{\partial L} \, dr .
\]
The scalar product \( \langle dV, dV \rangle \) is
\[
\langle dV, dV \rangle = \langle d\varphi, d\varphi \rangle + \left( \frac{\partial R'}{\partial L} \right)^2 \langle dr, dr \rangle = -\frac{1}{r^2 - D^2(1 + 2\phi)} ,
\]
where we denoted \( L/E = D \), with \( D \) having the meaning of the impact parameter. Thus,
\[
\nu^2 = \frac{dV}{\sqrt{-\langle dV, dV \rangle}} = dV \sqrt{r^2 - D^2(1 + 2\phi)} .
\]
As a fourth basis covector we choose \( d\theta \):
\[
\nu^3 = \frac{d\theta}{\sqrt{-\langle d\theta, d\theta \rangle}} = r d\theta.
\]

Thus, we constructed the associated basis of 1-forms,
\[
\begin{align*}
\nu^\pm &= \frac{\sqrt{1+2\phi}}{E \sqrt{2}} [E dt \pm L d\varphi \mp R' dr] \\
\nu^2 &= \sqrt{r^2 - D^2(1+2\phi)} \left[ d\varphi - \frac{\partial R'}{\partial L} dr \right], \\
\nu^3 &= r d\theta.
\end{align*}
\]

This basis is orthonormal in the sense of \( \langle \nu^\pm, \nu^\pm \rangle = 0 \) and \( \langle \nu^\pm, \nu^\mp \rangle = 1 \). The basis dual to \( \{\nu^a\} \) is the vector basis \( \{\vec{n}_a\} \) \( (\nu^a(\vec{n}_b) = \delta^a_b) \)
\[
\begin{align*}
\vec{n}_0 &= \frac{1}{\sqrt{2(1+2\phi)}} \frac{\partial}{\partial t}, \\
\vec{n}_1 &= \frac{D(1+2\phi)}{r} \frac{\partial}{\partial r} + \frac{L}{r} \frac{\partial}{\partial \varphi}, \\
\vec{n}_2 &= \frac{1}{r} \frac{\partial}{\partial \theta}.
\end{align*}
\]

We also introduce orthonormal sub-frames, constructed from the associated basis,
\[
\nu^0 = \frac{\nu^+ + \nu^-}{2}, \quad \nu^1 = \frac{\nu^+ - \nu^-}{2};
\]

and
\[
\begin{align*}
\vec{n}_0 &= \frac{1}{\sqrt{2(1+2\phi)}} \frac{\partial}{\partial t}, \\
\vec{n}_1 &= \frac{\sqrt{1+2\phi}}{E \sqrt{2}} \left[ \frac{L}{r^2} \frac{\partial}{\partial \varphi} - (1+2\phi)R' \frac{\partial}{\partial \varphi} \right].
\end{align*}
\]

According to the definition \textbf{Def. 1} \( \{n_b\} : n_b = h^b_\alpha \frac{\partial}{\partial x^\alpha} \) the non-zero tetrad functions of the vector basis are
\[
\begin{align*}
h^t_0 &= \frac{1}{\sqrt{2(1+2\phi)}} \\
h^r_1 &= \frac{-\sqrt{1+2\phi}}{r \sqrt{2}} \sqrt{r^2 - D^2(1+2\phi)} \\
h^\varphi_2 &= \frac{D(1+2\phi)}{r} \\
h^\theta_3 &= \frac{1}{r}
\end{align*}
\]

\[2.3. \text{Polarization basis on the geodesic}\]
We assume a narrow beam of rays and neglect transversal derivatives—in the zeroth approximation polarization basis is propagating paralelly along the ray. We can write tensors as
\[ F = F^{ab} f_{ab} \],
where \( \{ f_{ab} \} \)—tensor basis
\[ f_{ab} = \vec{n}_a \otimes \vec{n}_b = \frac{1}{2} \left( h^i_a h^j_b - h^i_b h^j_a \right) \left[ \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \right] , \]
and square brackets mean antisymmetrization. We define vertical (v) and horizontal (h) linearly polarized waves
\[ \vec{A}_v = A_2 \vec{n}_2 = (a \sin \psi) \vec{n}_2 , \]
\[ \vec{A}_h = A_3 \vec{n}_3 = (a \cos \psi) \vec{n}_3 , \]
where \( a \) is an amplitude. The waves with left and right circular polarizations (LCP and RCP, correspondingly) are their linear combinations,
\[ \vec{A}_R = \vec{A}_v + \vec{A}_h = (a \sin \psi) \vec{n}_2 + (a \cos \psi) \vec{n}_3 , \]
\[ \vec{A}_L = \vec{A}_v - \vec{A}_h = (a \sin \psi) \vec{n}_2 - (a \cos \psi) \vec{n}_3 . \]
The derivatives of (9) are
\[ \dot{\vec{A}}_v = \dot{A}_2 \vec{n}_2 = (a \dot{\psi} \cos \psi) \vec{n}_2 , \]
\[ \dot{\vec{A}}_h = \dot{A}_3 \vec{n}_3 = -(a \dot{\psi} \sin \psi) \vec{n}_3 . \]
We take the derivative only along the beam, hence,
\[ df = (\vec{n} \circ f) \nu^a . \]
Finally, the 'dot' operator takes the form
\[ \dot{f} = \frac{df}{d\psi^+} = (\vec{n} \circ f) \frac{\sqrt{1 + 2\phi}}{E \sqrt{2}} . \]
The spin tensor is defined as (see Appendix A)

\[ S_{ab} = \frac{1}{2} \int \left[ \dot{A}_2 A_3 - \dot{A}_3 A_2 \right] d^3 x . \]  

(12)

The only non-zero component of the spin current

\[ J_{23}^0 = \frac{a^2 E}{\sqrt{2(1 + 2\phi)}} , \]

and the only non-zero component of spin for the LCP polarization we find from Eqs. 10, 11 and 12

\[ L S_{23} = \int \frac{a^2 E}{\sqrt{2(1 + 2\phi)}} (\cos^2 \psi + \sin^2 \psi) d^3 x = \int \frac{a^2 E}{\sqrt{2(1 + 2\phi)}} d^3 x = \text{const} = 1 . \]

For RCP, correspondingly, the sign is opposite,

\[ R S_{23} = -1 . \]

It is convenient to introduce the following notation for the spin-gravity coupling term, the tensor

\[ H^i_j = \frac{1}{2} R^i_{jkl} S^{kl} , \]

where in order to express the spin tensor in the coordinate frame, we use the tensor basis defined in Eq. 8,

\[ S_{23} = \pm 1 = S^{23} = S^{kl} h^i_k h^i_l \equiv \pm (f_{kl})^{23} , \]

and

\[ S^{kl} = S^{23} (f_{23})^{kl} . \]

From six components of \{f_{23}\} basis only two survive

\[ (f_{23})^{\theta r} = -\frac{1}{2} h^\theta_3 h^r_2 = -\frac{D(1 + 2\phi)}{2r^2} , \]

\[ (f_{23})^{\theta \varphi} = -\frac{1}{2} h^\theta_3 h^\varphi_2 = -\frac{\sqrt{r^2 - D^2(1 + 2\phi)}}{2r^3} , \]

where the tetrad functions are found from Def. 1).

Thus, for LCP the components of this tensor are

\[ L H^\theta_r = \frac{1}{2} R^\theta_{r\theta r} (f_{23})^{\theta r} = -\frac{mD}{4r^5} ; \]

\[ L H^\theta_\varphi = \frac{1}{2} R^\theta_{\varphi\theta \varphi} (f_{23})^{\theta \varphi} = -\frac{m\sqrt{r^2 - D^2(1 + 2\phi)}}{2r^4} , \]
where the corresponding components of the Riemann tensor are

\[ R^\theta_{r \theta r} = -\frac{m}{r^3(1 + 2\phi)}, \]
\[ R^\theta_{\varphi \theta \varphi} = \frac{2m}{r}. \]

3. The Papapetrou Equation

The first equation of (Eq. 1) can be written in the form

\[ \dot{p}^i = \frac{1}{2} R^i_{\ jkl} S^{kl} \dot{x}^j. \] (13)

With our definition for the circular polarization the equation transforms to

\[ \dot{p}^i = \pm H^i_{\ j} \dot{x}^j (S^{23}) \] (14)

with ‘+’ corresponding to LCP and ‘-’ to RCP. The geodesics we are considering lie in the equatorial orbit \( \theta = \pi/2 \) and, hence, \( \dot{\theta} = 0 \). Since the only surviving \( H \) tensors contain \( \theta \) components, in the Eq. 14 we have non-zero right hand side only for \( \theta \)-component of the momentum. We are considering the deviation \( \delta x^i(\psi) \) from the geodesic \( x^i(\psi) \) as our first approximation, \( \delta x^i(\psi) \equiv (0, 0, \delta \theta(\psi), 0) \). The Papapetrou equation for all coordinates but \( \theta \) coincides with the geodesic equation, and for the \( \theta \) coordinate we have:

\[ \dot{p}^\theta = 0, \]
\[ \delta \dot{p}^\theta = \pm H^\theta_{\ j} \dot{x}^j, \]

where in the zeroth approximation (geodesic motion) the spin-dependent term of the generalized momentum \( p^i \) vanishes. To find the components of the velocities we need to find the geodesics in the explicit parametrized form. If we express the solution as \( \psi(x^i, \alpha^a) \), where \( x^i \) represents variables and \( \alpha^a \) represents constants, then equations

\[ \frac{\partial \psi}{\partial \alpha^a} = \psi_a(x^i, \alpha^a) = \text{const}. \]

give the analytical representation of geodesics in parametrized form (Hamilton-Jacobi theorem\(^{11} \)). Applying this to the Eq. 5 gives geodesics in the form of the following system of equations with \( r \) as an independent parameter:

\[ \frac{\partial \psi_\pm}{\partial E} = t_0, \]
\[ t - t_0 = \pm \int \frac{E_{x^i} dr}{(1+2\phi)\sqrt{E^2 + L^2(1+2\phi)}} \equiv t(r); \]

\[ \frac{\partial \psi_\pm}{\partial L} = \varphi^0, \]
\[ \varphi - \varphi_0 = \pm \int \frac{L_{x^i} dr}{r\sqrt{E^2 + L^2(1+2\phi)}} \equiv \varphi(r). \] (15)
Now we need to find the relation between momentum and velocities. The components of the momentum vector are obtained from the solution of HJE,

\[ p_t = \frac{\partial \psi}{\partial t} = E; \quad p_r = R'; \quad p_\phi = L; \quad p_\theta = 0, \]

and, accordingly,

\[ p_t = \frac{E}{1 + 2 \phi}; \quad p_r = -(1 + 2 \phi)R'; \quad p_\phi = -\frac{L}{r^2}; \quad p_\theta = 0. \]

Now we find \((x^i)'\)-s from (15)

\[
\begin{align*}
(x^t)' &= \frac{Er}{(1 + 2\phi)\sqrt{E^2r^2 - L^2(1 + 2\phi)}}; \quad (x^r)' = 1; \quad (x^\theta)' = 0; \\
(x^\phi)' &= \frac{L}{r\sqrt{E^2r^2 - L^2(1 + 2\phi)}},
\end{align*}
\tag{16}
\]

where ‘prime’ denotes differentiation with respect to \((w.r.t)\) \(r\). By taking out the common multiplier, we express the momentum through velocities

\[ p^i = \sqrt{E^2r^2 - L^2(1 + 2\phi)}r (x^i)' = \frac{E\sqrt{r^2 - D^2(1 + 2\phi)}}{r} (x^i)'. \]

In our first approximation we can assume the same relation between the velocity and the momentum as on the geodesic,

\[ \delta p^\theta = \frac{E\sqrt{r^2 - D^2(1 + 2\phi)}}{r} \delta \theta'. \tag{17} \]

By ‘prime’ we mean the covariant differentiation with respect to \(r\). However, we can express \(\delta p^\theta\) through our associated basis,

\[ \delta p^\theta = \sqrt{-g^{\theta \theta}} \delta p^3 = \frac{1}{r} \delta p^3. \tag{18} \]

According to the zero-connection property of the associated basis, we can replace the covariant differentiation of \(\delta p^3\) w.r.t. \(r\) with ordinary differentiation w.r.t. \(r\). The same would hold for a product of \(\delta p^3\) with any scalar function, as in (18).

Hence, we write the Papapetrou equation as

\[ (\delta p^\theta)' = H^\theta_\vdash (x^3)'. \]

The contraction with the velocity gives the right-hand side of the equation

\[ H^\theta_\vdash (x^3)' = H^\theta_t (x^r)' + H^\theta_\phi (x^\phi)' = \frac{mD}{4r^5}, \]

with \((x^i)'\)-s from Eq. 16. Thus, the Papapetrou equation is

\[ (\delta p^\theta)' = \frac{mD}{4r^5}. \tag{19} \]
and, according to (17),
\[
\left( \frac{E \sqrt{r^2 - D^2(1 + 2\phi)}}{r} \delta \theta \right)' = \frac{mD}{4r^5}.
\]
Thus, the trajectory of a photon world-line is: \( t(r), \varphi(r), \theta(r) = \pi/2 + \delta \theta(r) \), where for LCP (+) and RCP (-), correspondingly,
\[
\pm \delta \theta(r) = \frac{mD}{16E} \int_0^r \frac{dr}{r^3 \sqrt{r^2 - D^2(1 + 2\phi)}}, \tag{20}
\]
with gravitational deflection term \( \varphi(r) \) defined in (15). With \( (1 + 2\phi) \equiv (1 - r_g/r) \) and \( m \equiv 2r_g \), Eq. 20 takes the form
\[
\pm \delta \theta(r) = \frac{r_gD}{8E} \int_0^r \frac{dr}{r^3 \sqrt{r^2 - D^2(1 - r_g/r)}}. \tag{21}
\]
This formula shows the effect of dispersion, the energy-dependent deviation from a geodesic motion due to the spin. To see whether this quantity will have any appreciable effect on the propagation of light, we have to make an estimate of the order of magnitude of this integral. It is more convenient to express the result in terms of the distance \( r_0 \) of closest approach, rather than the impact parameter \( D \).
At \( r_0 \)
\[
R' = \pm \sqrt{\frac{E^2 r^2 - L^2(1 + 2\phi)}{r(1 + 2\phi)}} = 0,
\]
\[
r^2 - D^2(1 - \frac{r_0}{r}) = 0,
\]
Since the difference between \( D \) and \( r_0 \) is small, we write \( D = r_0 + \delta r_0 \), and inserting it into the previous equation gives the first order \( \delta r_0 \approx r_g r_0/2(r_0 - r_g) \) and \( D \approx (2r_0 + r_g)/2 \). Using \( r_0 \) instead of \( D \) in the integral (21) gives
\[
\pm \delta \theta(r) = \frac{1}{8E} \int_{r_0}^{r_0 + \delta r_0} \frac{r_0 r_g dr}{r^3 \sqrt{r^2 - \frac{(2r_0 + r_g)^2}{4}(1 - \frac{r_0}{r})}} = \frac{1}{8E} \int_{r_0}^{r_0 + \delta r_0} \frac{r_0 r_g dr}{r^3 \sqrt{r^2 - r_0^2}} \left( 1 + \frac{r_0 r_g}{2r(r^2 - r_0^2)} \right).
\]
Up to the first order in \( r_g/r \) we obtain \( (E = 1/\lambda) \)
\[
\pm \delta \theta(r) = -\frac{r_g \lambda}{16r_0^2} \left[ \sin^{-1} \left( \frac{r_0}{r} \right) - \frac{r_0 \sqrt{r^2 - r_0^2}}{r^2} \right].
\]
And at the observer the deviation from the \( \theta = \pi/2 \) plane is
\[
\frac{L}{R} \delta \theta(\pm \infty) = \pm \frac{r_g \lambda}{16r_0^2}.
\]
We notice that at \( r_g \to 0 \), or at \( E \to \infty \), the spin-dependent deviation vanishes. Spinless particles always move on plane orbits, in the case of a circularly polarized photon with the spin (helicity) parallel to the velocity, the perturbed motion is not plane.

We feel that we shall comment here on the problem of the deflection of light. Contrary to the reports\(^{17,18}\) about the polarization dependence of light deflection in a Schwartzchild metric, and in concordance with the pioneering work of Corinaldesi and Papapetrou\(^ {16}\), we conclude that spin gives no contribution to the deflection (the deflection (Eqs. 15) is identical with that which is found from using the equations of geodesics).

Of course, the result with \( \lambda/r_0 \) has an extremely small value and it is evident that the effect certainly has no observational consequences in the gravitational fields appropriate to present-day astrophysics and cosmology. Nevertheless, we feel that the main significance of the result is theoretical and does not depend on its observability. Moreover, in view of recent findings of the appearance of a chaotic behaviour in the motion of a spinning particle in a Schwartzchild spacetime due to the spin-orbit coupling\(^ {13}\), and of a report of a measurement of a time delay between RCP and LCP signals from a pulsar PSR 1937+21\(^ {19}\), we believe that a study of spin effects on the orbital evolution of relativistic systems is very important from the viewpoint of observations as well as from an academic one.

3.1. **Robertson-Walker Background and Radial Motion**

From the general form of the Eq. 13 it is clear that in the flat spacetime \((R_{\mu\nu\rho\sigma} = 0)\) or for the spinless particle \((S^{\mu\nu} = 0)\), the equation reduces to the geodesic equation or, in case of a massless particle, to the null geodesic equation.

In general metric at every point of a spacetime there exists a Weyl principal tetrad in which the Riemann tensor takes its normal form\(^ {12}\) (it is strictly diagonal in pairs of indices). If the direction of the momentum of a spinning particle coincides with one of the principal tetrad directions, then due to the Tulczijew constraint\(^ {14}\) (one of the supplementary conditions to the equations of motion, introduced in order to reduce the number of spin independent degrees of freedom, \(p_\alpha S^{\alpha\beta} = 0)\) the Papapetrou force is identically zero. In the homogeneous gravitational background, or background with constant curvature, any direction is the principal direction, and hence the Papapetrou force is always zero in such metrics. Since the FRW spacetime is isotropic and homogeneous, the origin of polar spherical coordinate system, for example, can be any arbitrary point. If we place the origin on a geodesic, the geodesic becomes strictly radial in this coordinate system. Thus, the photon’s motion is geodesic in FRW spacetime. The same can be shown for any spacetime of constant curvature, for example, de-Sitter, the fact first discovered by Bailyn and Ragusa\(^ {6}\).

The radial direction in Schwartzchild metric is both principal (in the sense of Weyl principal tetrad) and geodesic and, thus, there is no deviation. The motion is
just pure radial geodesic with
\[ t(r) = t_0 \pm \int \frac{dr}{1 + 2\phi}, \]
\[ R(r) = \pm \int \frac{Edr}{1 + 2\phi}, \]
\[ \theta(r) = \text{Const}, \]
\[ \varphi(r) = \text{Const}. \]

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Appendix A Spin tensor derivation

In paper I we have defined the spin tensor as following
\[ mS^{ab} = \frac{1}{2} \left( \dot{A}^a A^b - \dot{A}^b A^a \right). \]
However, it was pointed to us recently\(^a\) that in that definition the dimensions are not correct. Removing the mass term restores the correct dimensionality. It should be noted, however, that neither the derivations of the equations of motion for spin and momentum nor the final results of the paper I suffer from this redefinition. The derivation of the equation of motion of spin does not depend on the mass term, and in the last equation for the momentum (Eq. 31, paper I) the mass term is absorbed in the momentum term
\[ g_{ij}m D\dot{x}^j \equiv g_{ij} \frac{Dp^j}{Ds} = \dot{z}^j R^k_{jil} S^l_k. \]

We will present here the proper derivation of the spin tensor which was just postulated in paper I. Using the Noether theorem we will derive the spin tensor as a current conserved under the rotation of a local orthonormal frame as an operation of internal symmetry (see, for example, [20]). The field Lagrangian is
\[ 2\mathcal{L}_F = -\dot{A}^2 + m^2 A^2, \]
where ‘dot’ means covariant derivative. Under the infinitesimal rotation of the frame \( \delta \alpha^a_b \), the new Lagrangian is the function of
\[ \tilde{\mathcal{L}}_F = \tilde{\mathcal{L}}_F(A, \dot{A}, \delta \alpha, \delta \dot{\alpha}), \]
where
\[ \delta A = \delta \alpha^{ab} \tilde{S}_{ab} A. \]

\(^a We express our thanks to Dr. M. M. Sheikh-Jabbari from IPM, Tehran, Iran.
Thus, the new Lagrangian is
\[ 2\tilde{\mathcal{L}}_F = \mathcal{L}_F + \delta \left(-\dot{\mathbf{A}}^2 + m^2 \mathbf{A}^2\right). \]

Let us compute it,
\[
\begin{align*}
\tilde{\mathcal{L}}_F &= -\frac{1}{2} (\mathbf{A} + \delta \mathbf{A}) \cdot (\mathbf{A} + \delta \mathbf{A}) + \frac{m^2}{2} (\mathbf{A} + \delta \mathbf{A})^2 \\
&= -\frac{1}{2} \left( \dot{\mathbf{A}} + (\delta \mathbf{A}) \right) \cdot \left( \dot{\mathbf{A}} + (\delta \mathbf{A}) \right) + \frac{m^2}{2} (\mathbf{A}^2 + 2 (\delta \mathbf{A}) \cdot \mathbf{A}) \\
&= \mathcal{L}_F - \dot{\mathbf{A}} \left( \delta \alpha^{ab} \hat{S}_{ab} \mathbf{A} \right). 
\end{align*}
\]

The variation is
\[ \delta \mathcal{L}_F = -\left( \dot{\mathbf{A}} \hat{S}_{ab} \mathbf{A} \right) \left( \delta \alpha^{ab} \right). \]

The variation of the action integral
\[ 0 = \delta I = \int d^4x \delta \mathcal{L}_F = \int d^4x \left( \dot{\mathbf{A}} \hat{S}_{ab} \mathbf{A} \right) \left( \delta \alpha^{ab} \right). \]

Evaluating this integral by parts and assuming that at infinity the fields go to zero, the variation is
\[ \delta I = \int d^4x \left( \delta \alpha^{ab} \right) \left( \dot{\mathbf{A}} \hat{S}_{ab} \mathbf{A} \right). \]

Using Noether theorem\(^{20}\), we define the current
\[ \dot{\mathbf{A}} \hat{S}_{ab} \mathbf{A} = \left( \dot{\mathbf{j}}_{ab} \right)^0. \]

In the world-tube, where we defined the field, the spin tensor will be
\[ S_{ab} = \int \left( \dot{\mathbf{j}}_{ab} \right)^0 d^3x. \quad (A.1) \]

Since the spin of the vector field is 1 by definition, we do not need to integrate (A.1) over the whole space and can express spin directly through the orthonormal basis, defined along the world-tube. Thus, since our spin matrix is
\[ (\hat{S}_{ab})_{cd} = \frac{1}{4} (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}), \]
we obtain the expression for spin current as
\[ \dot{A}^c A^d (\hat{S}_{ab})_{cd} = \frac{1}{2} \left( \dot{A}_a A_b - \dot{A}_b A_a \right). \]

In the world-tube we assume only plane waves and, thus, identify the spin current with the spin. Thus, our expression for the conserved quantity, spin of the field, is
\[ S_{ab} = \frac{1}{2} \int \left( \dot{A}_a A_b - \dot{A}_b A_a \right) d^3x. \]
We shall note that the spin tensor $S_{ab}$ is defined now only on a geodesic, in the same way as velocity and momentum vectors are defined, which completes our transition from field description to particles.

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