On the linear processes of a stationary time series $AR(2)$

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Abstract. Our aim in this work is to give explicit formula of the linear processes solution of autoregressive time series $AR(2)$ with hint of generating functions theory by using the Horadam numbers and polynomials.

1. Introduction

The time series (see [1]) we observe are the realizations of Random variables $Y_1, \cdots, Y_t$ which are a part of a larger stochastic process $\{Y_t; t \in \mathbb{Z}\}$. Let $\{Y_t; t \in \mathbb{Z}\}$ a zero-mean time series. Consider $\mathcal{H}$ the Hilbert space spanned by the Random variables $Y_t$ with inner product

$$\langle X, Y \rangle = E(X, Y)$$

and the norm

$$\|X\| = \sqrt{E(X^2)}.$$

The mean function of a time series is defined by $\mu(t) = E(Y_t)$ and the autocovariance function is given by $\gamma(s, t) = cov(Y_s, Y_t)$. The mean and the autocovariance functions are fundamental parameters and it would be useful to obtain sample estimates of them. For general time series there are $2t + \frac{(t-1)}{2}$ parameters associated with $Y_1, \cdots, Y_t$ and it is not possible to estimate all these parameters from $t$ data values. To make any progress we consider the times series is stationary.

On general a times series $Y_t$ is strictly stationary if for $k > 0$ and any $t_1, \cdots, t_k$ of $\mathbb{Z}$, the distribution of $(Y_{t_1}, \cdots, Y_{t_k})$ is the same as that for $(Y_{t_1+u}, \cdots, Y_{t_k+u})$ for every $u$. If $Y_t$ is stationary then $\mu(t) = \mu$ and $\gamma(s, t) = \gamma(s-t, 0)$. We say $Y_t$ is weakly stationary if $E(Y_t^2) < \infty$, $\mu(t) = 0$ and $\gamma(t+u, t) = \gamma(u, 0)$. When time series are stationary it is possible to simplify the parameterizations of the mean and auto-covariance functions. In this case we can define the mean of the series to be $\mu = E(Y_t)$ and the autocovariance function to be $\gamma(u) = cov(Y_{t+u}, Y_t)$.

If the random variables which make up $Y_t$ are uncorrelated, have means 0 and

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2. MOULOUDE GOUBI

variance \( \sigma^2 \) (are so called white-noise series). Then \( Y_t \) is stationary with auto-

(1.1) \[ \gamma(u) = \begin{cases} \sigma^2 & \text{if } u = 0, \\ 0 & \text{otherwise.} \end{cases} \]

covariance function

2. Autoregressive series

If the time series \( Y_t \) satisfies the identity

\[
(2.1) \quad Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \epsilon_t
\]

where \( \epsilon_t \) is white noise and \( \phi_i \) are constants, then \( Y_t \) is called an autoregressive

series of order \( p \) and denoted by \( AR(p) \). These series are important, it explain how

the next value observed is a slight perturbation of a simple function of the most

recent observations. The solution; if exist in the form

\[
Y_t = \sum_{u=0}^{\infty} \phi_u \epsilon_{t-u}
\]

is called a linear processes with the condition \( \sum_{u=0}^{\infty} |\phi_u|^2 < \infty \) to assure the con-

vergence of the last series in \( \mathcal{H} \). The lag operator \( L \) for a time series \( Y_t \) is defined

by

\[
L(Y_t) = Y_{t-1}
\]

and is linear. The last relation (2.2) can be written in the form

\[
(2.2) \quad Y_t = \phi_1 L(Y_t) + \cdots + \phi_p L^p(Y_t) + \epsilon_t.
\]

To found \( Y_t \), we write

\[
(1 + \phi_1 L + \cdots + \phi_p L^p)Y_t = \epsilon_t.
\]

If the operator \( 1 + \phi_1 L + \cdots + \phi_p L^p \) is bijective we then write

\[
Y_t = (1 - \phi_1 L - \cdots - \phi_p L^p)^{-1} \epsilon_t.
\]

The linear processes of \( AR(1) \) is completely known. We have \( Y_t = (1 - L)^{-1} \epsilon_t \),

then \( Y_t = \sum_{u=0}^{\infty} \phi_1^u \epsilon_{t-u} \) with the condition \( \sum_{u=0}^{\infty} |\phi_1|^2 u < \infty \), which means that

\( |\phi_1| < 1 \). An equivalent condition is that the root of the equation \( 1 - \phi_1 z = 0 \) lies

outside the unit circle in the complex plane \( \mathbb{C} \).

The \( AR(2) \) model is defined by \( Y_t = \phi_1 L(Y_t) + \phi_2 L^2(Y_t) + \epsilon_t \). Then we write

\[
Y_t = (1 - \phi_1 L - \phi_2 L^2)^{-1} \epsilon_t.
\]

The decomposition gives

\[
1 - \phi_1 L - \phi_2 L^2 = (1 - c_1 L)(1 - c_2 L).
\]

We can invert the operator if we can invert each factor separately. This is possible

if and only if \( |c_1| < 1 \) and \( |c_2| < 1 \), or equivalently, if the roots of the polynomial

\( 1 - \phi_1 z - \phi_2 z^2 \) lie outside the unit circle. But from the symmetric relations of the

roots we have \( c_1 + c_2 = \phi_1 \) and \( c_1 c_2 = \phi_2 \). The strong conditions of convergence

are \( |\phi_1| < 2 \) and \( |\phi_2| < 1 \).
In this work we are interested by times series belong to AR(2) respecting the conditions \( |\phi_1| < 2 \) and \( |\phi_2| < 1 \). The linear processes \( Y_t \) can be computed by two different methods. The first expression; well-known by the statisticians which consist to compute each AR(1) separately and take the product. We reproduce it directly by the following theorem.

**Theorem 2.1.**

\[
Y_t = \sum_{u=0}^{\infty} \sum_{k=0}^{u} \phi_1^u (\phi_2 / \phi_1)^k \epsilon_{t+k-u} \epsilon_{t-k}.
\]

The identity (2.3) is interesting, but the second member of the equality contains the product of two white noise \( \epsilon_t \). Then it is not as the standard form of linear processes. To escape this problem the second way is given by the following theorem.

**Theorem 2.2.**

\[
Y_t = \sum_{u=0}^{\infty} \left( \sum_{k=0}^{\lfloor u \rfloor} \binom{u-k}{k} (\phi_2 / \phi_1^2)^k \phi_1^u \right) \phi_1^u \epsilon_{t-u}.
\]

The condition \( |\phi_1| < 2 \) and \( |\phi_2| < 1 \) for the convergence of the linear processes \( Y_t \) on \( \mathcal{H} \) can be replaced by the condition

\[
\sum_{u \geq 0} \left( \sum_{k=0}^{\lfloor u \rfloor} \binom{u-k}{k} (\phi_2 / \phi_1^2)^k \phi_1^u \right)^2 < \infty.
\]

Letting \( \phi_2 = \phi_1^2 \) in the expression (2.4) Theorem 2.2; the following corollary is immediate.

**Corollary 2.1.**

\[
Y_t = \sum_{u=0}^{\infty} \left( \sum_{k=0}^{\lfloor u \rfloor} \binom{u-k}{k} \phi_1^u \epsilon_{t-u} \right) \phi_1^u \epsilon_{t-u}.
\]

The symbol \( \binom{u-k}{k} \) (see [2]) occurs an important place in combinatorics. It is a particular case of \( \binom{u-k-l}{k-l} \) which is the number of \( k \)-blocks \( P \subset [u] = \{1, 2, \ldots, u\} \) with the following property; between two arbitrary points of \( P \) are at least \( l \) points of \( [n] \) which do not belong to \( P \).

**2.1. Proof of main results.** From the decomposition \( 1 - \phi_1 L - \phi_2 L^2 = (1 - c_1 L) (1 - c_2 L) \) we conclude that

\[
Y_t = \left( \sum_{u \geq 0} \phi_1^u \epsilon_{t-u} \right) \left( \sum_{u \geq 0} \phi_2^u \epsilon_{t-u} \right)
\]

Use Cauchy product of series to get the desired result (2.3) Theorem 2.1. For more details about this technique we refer to [4].
MOULOUD GOUBI

The proof of second theorem needs use techniques from generating functions theory (see [3]). We consider

$$\frac{1}{1 - \phi_1 xz - \phi_2 z^2} = \sum_{u \geq 0} A_u(x)t^u.$$ 

where $A_u(x)$ is a polynomial of degree $u$ to compute. Write $f(x, z) = \frac{1}{1 - \phi_1 xz - \phi_2 z^2}$ then

$$(1 - \phi_1 xz - \phi_2 z^2) f(x, z) = 1$$

and

$$\sum_{u \geq 0} A_u(x)z^n - \phi_1 x \sum_{u \geq 0} A_u(x)z^{n+1} - \phi_2 \sum_{u \geq 0} A_u(x)z^{n+2} = 1$$

Thus

$$\sum_{u \geq 0} A_u(x)z^n - \phi_1 x \sum_{u \geq 1} A_{u-1}(x)z^u - \phi_2 \sum_{u \geq 2} A_{u-2}(x)z^u = 1$$

and

$$A_0(x) + (A_1(x) - \phi_1 x A_0(x)) z + \sum_{u \geq 2} (A_u(x) - \phi_1 x A_{u-1}(x) - \phi_2 A_{u-2}(x)) z^u = 1$$

Since this entire series is constant we deduce that $A_0(x) = 1$, $A_1(x) = \phi_1 x$ and others are given by the recursion relation

$$A_u(x) = \phi_1 x A_{u-1}(x) + \phi_2 A_{u-2}(x).$$

These polynomials are well-known and so called Horadam polynomials. G. B. Djordjević and G. V. Milovanović (see [3]) provide the following explicit representation

$$A_u(x) = \sum_{k=0}^{\left\lfloor u/2 \right\rfloor} \frac{\phi_2^k (u-k)!}{k!(u-2k)!} (\phi_1 x)^{u-2k}.$$ 

Letting $x = 1$ we obtain Horadam numbers $A_n$ these are written by the form

$$A_u = \phi_1^u \sum_{k=0}^{\left\lfloor u/2 \right\rfloor} \frac{(u-k)!}{k!(u-2k)!} (\phi_2/\phi_1)^k.$$ 

and admit for generating function

$$\frac{1}{1 - \phi_1 t - \phi_2 t^2} = \sum_{u \geq 0} A_u t^u.$$ 

Numbers $A_n$ can be defined from the following recursion relation

$$A_u = \phi_1 A_{u-1} + \phi_2 A_{u-2}, \quad n \geq 2$$

with $A_0 = 1$ and $A_1 = \phi_1$. Instead of $z$ we take the operator $L$ and it is obvious to obtain the identity (2.4) Theorem 2.2.
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