On $L^1$ estimates of solutions of compressible viscoelastic system

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Abstract

We consider the large time behavior of solutions of compressible viscoelastic system around a motionless state in a three-dimensional whole space. We show that if the initial perturbation belongs to $W^{2,1}$, and is sufficiently small in $H^4 \cap L^1$, the solutions grow in time at the same rate as $t^{1/2}$ in $L^1$ due to diffusion wave phenomena of the system caused by interaction between sound wave, viscous diffusion and elastic wave.

Keywords: Compressible viscoelastic system; diffusion wave; large time behavior.

1 Introduction

This paper studies the initial value problem of compressible viscoelastic system

\begin{align}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu') \nabla \text{div} v + \nabla P(\rho) &= \beta^2 \text{div}(\rho F^\top F), \\
\partial_t F + v \cdot \nabla F &= (\nabla v) F, \\
(\rho, v, F)|_{t=0} &= (\rho_0, v_0, F_0).
\end{align}

(1.1) \quad (1.2) \quad (1.3) \quad (1.4)
in the whole space $\mathbb{R}^3$. Here $\rho = \rho(x,t)$, $v = (v^1(x,t), v^2(x,t), v^3(x,t))$, and $F = (F^{jk}(x,t))_{1 \leq j,k \leq 3}$ denote the unknown density, velocity field, and deformation tensor, respectively, at position $x \in \mathbb{R}^3$ and time $t \geq 0$; $P = P(\rho)$ is the given pressure; $\nu$ and $\nu'$ are the viscosity coefficients satisfying $\nu > 0$, $2\nu + 3\nu' \geq 0$; $\beta > 0$ is the propagation speed of elastic wave. We assume that $P'(1) > 0$, and set $\gamma = \sqrt{P'(1)}$.

We also impose the following conditions for $\rho_0$ and $F_0$

\begin{align}
\rho_0 \det F_0 &= 1, \quad (1.5) \\
\sum_{m=1}^{3} (F_0^{ml} \partial_{x_m} F_0^{jk} - F_0^{mk} \partial_{x_m} F_0^{jl}) &= 0, \quad j,k,l = 1,2,3, \quad (1.6) \\
\text{div}(\rho_0 \top F_0) &= 0. \quad (1.7)
\end{align}

It follows from [5, Appendix A] and [18, Proposition.1] that the quantities (1.5) and (1.6) are invariant for $t \geq 0$:

\begin{align}
\rho \det F &= 1, \quad (1.8) \\
\sum_{m=1}^{3} (F^{ml} \partial_{x_m} F^{jk} - F^{mk} \partial_{x_m} F^{jl}) &= 0, \quad j,k,l = 1,2,3. \quad (1.9)
\end{align}

Here the constraint (1.8) means compressibility of fluid and the constraint (1.9) called the Piola’s formula is derived from a certain symmetric property of the first order derivatives of $F$ in the Lagrangian coordinates. Furthermore we see from [7. Appendix A] and [8. Appendix A] that the constraints (1.8) and (1.9) lead to the time invariance of the quantity (1.7):

\begin{align}
\text{div}(\rho \top F) &= 0. \quad (1.10)
\end{align}

The purpose of this paper is to deduce the estimate of the $L^1$ norm of solutions of the problem (1.1)–(1.7) around a motionless state $(\rho, v, F) = (1, 0, I)$. Here $I$ is the $3 \times 3$ identity matrix.

The system (1.1)–(1.3) is derived from a motion of compressible viscoelastic fluid in the macroscopic scale under the Hookean linear elasticity by the
variational settings. We refer to [1, 13, 21] for more physical details. We can classify the system (1.1)–(1.3) in a quasilinear parabolic-hyperbolic system since the system (1.1)–(1.3) consists of the compressible Navier-Stokes equations and a first order hyperbolic system for $F$.

In the case $\beta = 0$, the large time behavior of the solutions around $(\rho, v) = (1, 0)$ has been investigated so far. In particular, if we set $\beta = 0$ formally, the system (1.1)–(1.3) becomes the usual compressible Navier-Stokes equation

$$\partial_t \rho + \text{div}(\rho v) = 0, \quad (1.11)$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - (\nu + \nu') \nabla \text{div} v + \nabla P(\rho) = 0, \quad (1.12)$$

$$(\rho, v)|_{t=0} = (\rho_0, v_0). \quad (1.13)$$

Matsumura and Nishida [15] showed the global existence of the solutions of the problem (1.11)–(1.13) provided that the initial perturbation is sufficiently small in $H^3 \cap L^1$, and derived the decay estimate:

$$\|\nabla^k(\phi(t), m(t))\|_{L^2} \leq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}}, \quad k = 0, 1,$$

where $(\phi, m) = (\rho - 1, \rho v)$. Hoff and Zumbrun [2] derived the following $L^p$ decay estimates and asymptotic properties in $\mathbb{R}^n$, $n \geq 2$:

$$\|\phi(t), m(t)\|_{L^p} \leq \begin{cases} C(1 + t)^{-\frac{3}{4} + \frac{n+1}{4} - \frac{1}{2}} L(t), & 1 \leq p < 2, \\ C(1 + t)^{-\frac{7}{4} + \frac{n+1}{4} - \frac{1}{2}}, & 2 \leq p \leq \infty, \end{cases}$$

$$\left\|\left(\phi(t), m(t) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t\hat{P}(\xi)}\hat{m}_0\right)\right)\right)\right\|_{L^p} \leq C(1 + t)^{-\frac{3}{4} + \frac{n+1}{4} - \frac{1}{2}}, \quad 2 \leq p \leq \infty,$$

provided that the initial perturbation is sufficiently small in $H^4 \cap L^1$, where $L(t) = \log(1 + t)$ when $n = 2$, and $L(t) = 1$ when $n \geq 3$. Here $\hat{P}(\xi) = I - \frac{\xi^T \xi}{|\xi|^2}$, $\xi \in \mathbb{R}^n$. According to [10], the solution of the linearized system is decomposed as the sum of two terms, one is the incompressible part given by $\mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t\hat{P}(\xi)}\hat{m}_0\right)$ which behaves pure diffusively, and the other is the compressible part $(\phi(t), m(t)) - \left(0, \mathcal{F}^{-1}\left(e^{-\nu|\xi|^2 t\hat{P}(\xi)}\hat{m}_0\right)\right)$ containing the diffusion wave which stands for the convolution of the heat kernel and the fundamental solution of the wave equation with sound speed $\gamma$. The authors
of [2] revealed the hyperbolic aspect of the system (1.11)–(1.12) by proving that the large time behavior of the compressible part is different from the heat kernel in $L^p$ as $t \to \infty$, except the case $p = 2$. See also [11] for the linearized problem.

In the case $\beta > 0$, the mathematical analysis of solutions of the initial value problem (1.1)–(1.7) around the motionless state have been developed so far. The local existence of the strong solution of the problem (1.1)–(1.7) was guaranteed by Hu and Wang [4]. The global existence of the strong solution of the initial value problem (1.1)–(1.7) was proved by Hu and Wang [5], Qian and Zhang [18], and Hu and Wu [6], provided that the initial perturbation $(\rho_0 - 1, v_0, F_0 - I)$ is sufficiently small. Hu and Wu [6] showed that if the initial perturbation $(\rho_0 - 1, v_0, F_0 - I)$ belongs to $L^1 \cap H^3$, the $L^p$ decay estimates hold for the case $2 \leq p \leq 6$:

$$\|u(t)\|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p})},$$

(1.14)

by using the Fourier splitting method and the Hodge decomposition. Here $u(t) = (\phi, w, G) = (\rho - 1, v, F - I)$. Moreover, the authors also derived the lower $L^2$ estimate

$$\|u(t)\|_{L^2} \geq c(1 + t)^{-\frac{3}{4}}, \quad t \gg 1,$$

(1.15)

provided that the following conditions satisfy in the low frequency part $|\xi| \ll 1$:

$$|\hat{\phi}_0(\xi)| \geq c_0, |\hat{m}_0(\xi)| + |\hat{G}_0(\xi) - \hat{\Gamma}_0(\xi)| \ll |\xi|^{\eta_0},$$

(1.16)

where $(\phi_0, m_0, G_0) = (\rho_0 - 1, \rho_0 v_0, \rho_0 F_0 - I)$; $c$, $c_0$ and $\eta_0$ are positive constants independent of $\xi$ and $t$. Li, Wei and Yao [12, 22] generalized the upper $L^p$ decay estimates (1.14) to the case $2 \leq p \leq \infty$, and obtained the $L^2$ decay estimates of higher order derivatives:

$$\|\nabla^k u(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}}, \quad k = 0, 1, \ldots, N - 1,$$

(1.17)

provided that $u_0 = (\rho_0 - 1, v_0, F_0 - I)$ belongs to $H^N$, $N \geq 3$, and is small in $L^1 \cap H^3$. We also refer to [3, 14, 23] in recent progresses.

One problem which interests us is that the decay rates in (1.14) reveal only the parabolic aspect of the system (1.1)–(1.3); it would be desirable to establish decay estimates which reflect the hyperbolic aspect of the system (1.1)–(1.3).
In view of the results in [2], it is expected that the system (1.1)–(1.3) has the diffusion wave phenomena affected by sound wave, viscous diffusion and elastic wave. For simplicity, let us consider the linearized system around (1, 0, I):

\[ \partial_t u + Lu = 0. \]  \hspace{1cm} (1.18)

Here \( L \) denotes the linearized operator given by

\[ L = \begin{pmatrix} 0 & \text{div} & 0 \\ \gamma^2 \nabla & -\nu \Delta - \tilde{\nu} \nabla \text{div} & -\beta^2 \text{div} \\ 0 & -\nabla & 0 \end{pmatrix}, \]

where \( \tilde{\nu} = \nu + \nu' \). We then see that the solenoidal part of the velocity \( w_s = \mathcal{F}^{-1}(\hat{P}(\xi)\hat{w}) \) satisfies the following linear symmetric parabolic-hyperbolic system:

\[
\begin{aligned}
\partial_t w_s - \nu \Delta w_s - \beta \text{div} \tilde{G}_s &= 0, \\
\partial_t \tilde{G}_s - \beta \nabla w_s &= 0,
\end{aligned}
\]

where \( \tilde{G}_s = \beta \mathcal{F}^{-1}(\hat{P}(\xi)\hat{G}) \), while the complimentary part \( w_c = w - w_s \) solves the following strongly damped wave equation:

\[ \partial_t^2 w_c - (\beta^2 + \gamma^2)\Delta w_c - (\nu + \tilde{\nu})\partial_t \Delta w_c = 0. \]

Owing to the results in [20], the large time behavior of the solution of (1.18) becomes different to the case \( \beta = 0 \) ([2,11]) since the additional hyperbolic aspect arises in the incompressible part due to elastic wave. As a result, the principal part of the linearized system (1.18) can be identified as a system of the strongly damped wave equation. In [9], the hyperbolic aspect of the system (1.1)–(1.3) is clarified by showing the following \( L^p \) decay estimates:

\[ \|u(t)\|_{L^p} \leq C(1 + t)^{-\frac{1}{2} \left(1 - \frac{1}{p}\right)} \left(1 - \frac{1}{p}\right). \]  \hspace{1cm} (1.19)

for the case \( 1 < p \leq \infty \), provided that the initial perturbation \((\rho_0 - 1, v_0, F_0 - I)\) is small in \( L^1 \cap H^3 \). This improves the results in [6, 12].

The main difficulty of the mathematical analysis is nonlinearity of the constraints (1.8)–(1.10). Therefore, straightforward application of the semigroup theory to the nonlinear problem does not seem valid. To bypass this
difficulty, Hu and Wu [6] found that the behavior of $G$ is controlled by its skew-symmetric part $G - \frac{\top}{\top}G$ due to the constraints (1.9) and (1.10). This property leads to the global in time existence theorem. The authors of [6] next used the Helmholtz decomposition of $w$ and the skew-symmetric part of $G$ to obtain (1.14) with $2 \leq p \leq 6$, (1.17) with $N = 2$ and (1.15). In [9], the author relied on a material coordinate transform which makes the constraint (1.10) a linear one to apply the analysis of the linearized problem to the nonlinear problem. Let us introduce a displacement vector $\tilde{\psi} = x - X$ as in [19, 21]:

$$\tilde{\psi}(x, t) = x - X(x, t).$$

Here $x = x(X, t)$ is the material coordinate defined under the flow map

$$\begin{cases}
\frac{dx}{dt} = v(x(X, t), t), \\
x(X, 0) = X,
\end{cases}$$

and $X = X(x, t)$ denotes the inverse of $x$. In the continuum mechanics theory, $F$ is given by the Jacobi matrix of $x$ in the material coordinate. Then we notice that $F$ is written as

$$F - I = \nabla \tilde{\psi} + h(\nabla \tilde{\psi}).$$

(1.20)

Here $h(\nabla \tilde{\psi})$ is a function satisfying $h(\nabla \tilde{\psi}) = O(|\nabla \tilde{\psi}|^2)$ for $|\nabla \tilde{\psi}| \ll 1$. The author next set the nonlinear transform

$$\psi = \tilde{\psi} - (-\Delta)^{-1} \text{div}^\top(\phi \nabla \tilde{\psi} + (1 + \phi)h(\nabla \tilde{\psi})).$$

(1.21)

Here $(-\Delta)^{-1} = \mathcal{F}^{-1} |\xi|^{-2} \mathcal{F}$. By straight computation and the constraint (1.10), we arrive at the linear condition $\phi + \text{tr}(\nabla \psi) = \phi + \text{div} \psi = 0$ which makes the semigroup $e^{-tL}$ generated by $-L$ tend to 0 as $t \to \infty$ in $L^p$, $p > \frac{5}{4}$. Furthermore, the decay estimate of the $L^p$ ($1 < p \leq \infty$) norm of $u = (\phi, w, G)$ is obtained from $\tilde{U} = (\phi, w, \nabla \psi)$. Consequently, the $L^p$ decay estimates of $u$ are obtained from the following integral equation

$$\tilde{U}(t) = e^{-tL}\tilde{U}(0) + \int_0^t e^{-(t-s)L}N(\tilde{U}(s))ds,$$

where $N(\tilde{U}) = (N_1(\tilde{U}), N_2(\tilde{U}), N_3(\tilde{U}))$ is a nonlinearity satisfying $N_1(\tilde{U}) + \text{tr}N_3(\tilde{U}) = 0$.}

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The another difficulty arises from the nonlinear transform (1.21) containing the nonlocal operator \((-\Delta)^{-1}\). We note that the operator \(\nabla(-\Delta)^{-1}\text{div} = F^{-1}\xi\xi^T F\) is not bounded from \(L^p\) into \(L^p\) if \(p = 1, \infty\) due to the Riesz operator. In case \(p = \infty\), the above difficulty is avoided by using the Sobolev inequality and the Plancherel theorem, while the case \(p = 1\), it is expected that the solution of \(L^1\) norm grows as \(t \to \infty\) due to the diffusion phenomena, however, it remains open.

In this paper, we find a different approach of the reformulation to show the following \(L^1\) estimate of \(u\)
\[
\|(\rho(t) - 1, v(t), F(t) - I)\|_{L^1} \leq C(1 + t)^{\frac{1}{2}}, \quad t > 0,
\]
provided that the initial perturbation \(u_0 = (\rho_0 - 1, v_0, F_0 - I)\) belongs to \(W^{2,1}\), and is sufficiently small in \(H^2 \cap L^1\). We also prove that if \((\rho_0, v_0, F_0)\) satisfies (1.16), then the following lower \(L^1\) estimate holds
\[
\|(\rho(t) - 1, v(t), F(t) - I)\|_{L^1} \geq C(1 + t)^{\frac{1}{2}}, \quad t \gg 1,
\]
This indicates that the obtained rate \((1 + t)^{\frac{1}{2}}\) in (1.22) is sharp.

We give an outline of the proof of the main result. We first notice that the constraint (1.8) is read as \(\rho = \det F^{-1}\). Then, by employing (1.20), we have
\[
\phi = -\text{div}\tilde{\psi} + O(|\nabla\tilde{\psi}|^2), \quad \|\nabla\tilde{\psi}\|_{C(0,\infty;L^\infty)} \ll 1.
\]
Therefore, the behavior of \(\phi\) can be handled by \(-\text{div}\tilde{\psi}\) under the small perturbation. For simplicity, we omit the tilde \(\tilde{\cdot}\) of \(\tilde{\psi}\) here.

We next consider the nonlinear problem for \(U = (\tilde{\phi}, w, \tilde{G}) = (-\text{div}\psi, w, \nabla\psi)\):
\[
\begin{cases}
\partial_t U + LU = N(U), \\
\tilde{\phi} + \text{tr}\tilde{G} = 0, \quad \tilde{G} = \nabla\psi, \\
U|_{t=0} = U_0.
\end{cases}
\]
where \(N(U) = (N_1(U), N_2(U), N_3(U))\) is a nonlinearity such that \(N_1(U) + \text{tr}N_3(U) = 0\). We see from (1.20) and (1.24) that the \(L^1\) norm of \(u = (\phi, w, G)\) is estimated by \(U = (-\text{div}\psi, w, \nabla\psi)\). We point out that since \(U\) and \(N(U)\) hold the same linear constraint as in [9], the linear semigroup \(e^{-tL}U_0\) and the
Duammel term $\int_0^t e^{-(t-s)L}N(U(s))ds$ do not include terms which are time-independent or unbounded in $L^1$. Consequently, the $L^1$ estimate (1.22) is obtained from the following integral equation of $U$

$$U(t) = e^{-tL}U(0) + \int_0^t e^{-(t-s)L}N(U(s))ds.$$ 

The lower $L^1$ estimate (1.23) is obtained by the lower $L^2$ estimate (1.15), the estimate (1.19) with $p = \infty$, and the interpolation inequality.

We mention that this proof does not need the non-local operator in this reformation. Therefore we remove the difficulty in the case $p = 1$, and simplify the analysis of the solutions around the motionless state.

This paper is organized as follows. In Section 2 we introduce some notations and function spaces. In Section 3 we state the main result of this paper on the $L^1$ estimates of the solution. In Section 4 we show the $L^1$ estimates. In Section 5, we derive the $L^1$ estimate of the high frequency part of the linear semigroup.

## 2 Notation

In this section, we prepare notations and function spaces which will be used throughout the paper. $L^p$ ($1 \leq p \leq \infty$) denotes the usual Lebesgue space on $\mathbb{R}^3$, and its norm is denoted by $\| \cdot \|_{L^p}$. Similarly $W^{m,p}(1 \leq p \leq \infty, m \in \{0\} \cup \mathbb{N})$ denotes the $m$-th order $L^p$ Sobolev space on $\mathbb{R}^3$, and its norm is denoted by $\| \cdot \|_{W^{m,p}}$. We define $H^m = W^{m,2}$ for an integer $m \geq 0$. For simplicity, we denote $L^p = L^p \times (L^p)^3 \times (L^p)^9$ (resp. $H^m = H^m \times (H^m)^3 \times (H^m)^9$).

The inner product of $L^2$ is denoted by

$$(f,g) := \int_{\mathbb{R}^3} f(x)\overline{g(x)}dx, \ f, g \in L^2.$$ 

Here the symbol $\tau$ stands for its complex conjugate. Partial derivatives of a function $u$ in $x_j$ ($j = 1, 2, 3$) and $t$ are denoted by $\partial_{x_j}u$ and $\partial_t u$, respectively. $\Delta$ denotes the usual Laplacian with respect to $x$. For a multiindex $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\{0\} \cup \mathbb{N})^3$ and $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we define $\partial_x^\alpha$ and $\xi^\alpha$ as $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\xi_3^{\alpha_3}$, respectively. For a function $u$ and a nonnegative integer $k$, $\nabla^ku$ stands for $\nabla^k u = \{\partial_x^\alpha u | |\alpha| = k\}$. 

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For a scalar valued function $\rho = \rho(x)$, we denote by $\nabla \rho$ its gradient with respect to $x$. For a vector valued function $w = w(x) = (w^1(x), w^2(x), w^3(x))$, we denote by $\text{div} w$ and $(\nabla w)^{jk} = (\partial_{x_k} w^j)$ its divergence and Jacobian matrix with respect to $x$, respectively. For a $3 \times 3$-matrix valued function $F = F(x) = (F^{jk}(x))$, we define its divergence $\text{div} F$, trace $\text{tr} F$ and determinant $\det F$ by $\text{div} F = \sum_{k=1}^{3} \partial_{x_k} F^{jk}$, $\text{tr} F = \sum_{k=1}^{3} F^{kk}$ and $\det F = \sum_{\sigma \in S_3} \text{sgn}(\sigma) F^{1\sigma(1)} F^{2\sigma(2)} F^{3\sigma(3)}$, respectively. Here $S_3$ denotes a third-order symmetric group; for a permutation $\sigma \in S_3$, we denote by $\text{sgn}(\sigma)$ its signature.

For functions $f = f(x)$ and $g = g(x)$, we denote by $f \ast g$ the convolution of $f$ and $g$:

$$(f \ast g)(x) = \int_{\mathbb{R}^3} f(x - y) g(y) dy.$$ 

We denote by $\hat{f}$ or $\mathcal{F}f$ the Fourier transform of a function $f = f(x)$:

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(x) e^{-i\xi \cdot x} dx \ (\xi \in \mathbb{R}^3).$$

The Fourier inverse transform is denoted by $\mathcal{F}^{-1}$:

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} f(\xi) e^{i\xi \cdot x} d\xi \ (x \in \mathbb{R}^3).$$

We recall the Sobolev inequalities.

**Lemma 2.1.** The following inequalities hold:

(i) $\|u\|_{L^p} \leq C\|u\|_{H^1}$ for $2 \leq p \leq 6$, $u \in H^1$.
(ii) $\|u\|_{L^p} \leq C\|u\|_{H^2}$ for $2 \leq p \leq \infty$, $u \in H^2$.

We next introduce the following elementary inequality to control the Duhamel term.

**Lemma 2.2.** The following estimates hold:

(i) $\int_0^t (1 + t - s)^{\frac{1}{2}} (1 + s)^{-2} ds \leq C(1 + t)^{\frac{1}{2}}, \ t \geq 0,$

Here $C$ is a positive constant independent of $t$. 

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3 Main Result

In this section, we state the main result of this paper.

We set \( u(t) = (\phi(t), w(t), G(t)) = (\rho(t) - 1, v(t), F(t) - I) \). Then \( u(t) \) satisfies the following initial value problem

\[
\begin{align*}
\partial_t \phi + \text{div} w &= g_1(u), \\
\partial_t w - \nu \Delta w - \tilde{\nu} \text{div} w + \gamma^2 \nabla \phi - \beta^2 \text{div} G &= g_2(u), \\
\partial_t G - \nabla w &= g_3(u), \\
\nabla \phi + \text{div}^\top G &= g_4(u), \\
\end{align*}
\]

\[u|_{t=0} = u_0 = (\phi_0, w_0, G_0).\]

Here \( g_j(u), j = 1, 2, 3, 4 \), denote the nonlinear terms;

\[
\begin{align*}
g_1(u) &= -\text{div}(\phi w), \\
g_2(u) &= -w \cdot \nabla w + \frac{\phi}{1 + \phi} (-\nu \Delta w - \tilde{\nu} \text{div} w + \gamma^2 \nabla \phi) - \frac{1}{1 + \phi} \nabla R(\phi) \\
&\quad - \frac{\beta^2 \phi}{1 + \phi} \text{div} G + \frac{\beta^2}{1 + \phi} \text{div}(\phi G + G^\top G + \phi G^\top G), \\
g_3(u) &= -w \cdot \nabla G + \nabla w G, \\
g_4(u) &= g_4(\phi, G) = -\text{div}(\phi^\top G),
\end{align*}
\]

where

\[
R(\phi) = \phi^2 \int_0^1 P''(1 + s\phi)ds, \quad \nabla R = O(\phi) \nabla \phi
\]

for \( |\phi| \ll 1 \).

The following proposition ensures the global in time existence and the \( L^2 \) decay estimates of solutions.

**Proposition 3.1.** ([6, 12]) Let \( u_0 \in H^N, N \geq 3 \). There is a positive number \( \epsilon_0 \) such that if \( u_0 \) satisfies \( \|u_0\|_{L^1} + \|u_0\|_{H^3} \leq \epsilon_0 \), then there exists a unique solution \( u(t) \in C([0, \infty); H^N) \) of the problem (3.1), and \( u(t) = (\phi(t), w(t), G(t)) \) satisfies

\[
\begin{align*}
\|u(t)\|_{H^N}^2 + \int_0^t (\|\nabla \phi(s)\|_{H^{N-1}}^2 + \|\nabla w(s)\|_{H^N}^2 + \|\nabla G(s)\|_{H^{N-1}}^2)ds &\leq C_N\|u_0\|_{H^N}^2, \\
\|\nabla^k u(t)\|_{L^2} &\leq C(1 + t)^{-3/2 - k/2}(\|u_0\|_{L^1} + \|u_0\|_{H^N})
\end{align*}
\]
for $k = 0, 1, 2, \ldots, N - 1$ and $t \geq 0$.

In addition, if there exists a positive number $r > 0$ such that the following condition satisfies

$$|\hat{\phi}_0(\xi)| > c_0, |\hat{m}_0(\xi)| + |\hat{G}_0(\xi) - ^t\hat{G}_0(\xi)| \leq c_1|\xi|^{\eta_0}$$

(3.2)

for $0 \leq |\xi| \leq r$, where $(m_0, G_0) := (\rho_0 v_0, \rho_0 F_0 - I)$; $c_0$, $c_1$ and $\eta_0$ are positive numbers independent of $t$, the following lower $L^2$ estimate holds

$$\|u(t)\|_{L^2} \geq c(1 + t)^{-\frac{3}{4}}$$

(3.3)

uniformly for $t \geq R_1$. Here $R_1$ is a large positive number, and $c$ is a positive number independent of $t$.

We next state the main result of this paper which gives the upper estimate of the $L^1$ norm of solutions.

**Theorem 3.2.** Assume that $\phi_0$ and $G_0$ satisfy $\nabla \phi_0 + \text{div}^T G_0 = g_4(\phi_0, G_0)$ and $(I + G_0)^{-1} = \nabla X_0$ for some vector field $X_0$. There is a positive number $\epsilon$ such that if $u_0 = (\phi_0, w_0, G_0)$ satisfies $\|u_0\|_{H^{\frac{1}{2}}} + \|u_0\|_{L^1} \leq \epsilon$ and $u_0 \in W^{2,1}$, then there exists a unique solution $u(t) \in C([0, \infty); H^1)$ of the problem (3.1) satisfying

$$\|u(t)\|_{L^1} \leq C(1 + t)^{\frac{1}{2}}(\|u_0\|_{W^{2,1}} + \|u_0\|_{H^1})$$

uniformly for $t > 0$. Here $C$ is a positive constant.

We also prove the lower $L^1$ estimate to show that the obtained rate $(1 + t)^{\frac{1}{2}}$ is sharp.

**Theorem 3.3.** Under the assumptions in Proposition 3.1 and Theorem 3.2, there exists a positive large number $R_1$ such that the lower $L^1$ estimate of the solution $u$ of the problem (3.1) holds

$$\|u(t)\|_{L^1} \geq c(1 + t)^{\frac{1}{2}}$$

uniformly for $t \geq R_1$. Here $c$ is a positive constant independent of time $t$.

**Proof of Theorem 3.3.** Since we assume the condition (3.2), the lower $L^2$ estimate (3.3) holds uniformly for $t \geq R_1$. Here $R_1$ is taken in Proposition
3.1. We next introduce the following $L^\infty$ decay estimate which can be proved in a similar manner to [9, Theorem 3.2 (i)] and Theorem 3.2

$$
\|u(t)\|_{L^\infty} \leq C(1 + t)^{-2}(\|u_0\|_{L^1} + \|u_0\|_{H^3}), \quad t \geq 0.
$$

By using (3.3), (3.4) and the interpolation inequality $\|u(t)\|_{L^2} \leq \|u(t)\|_{L^\infty}^{1/2} \|u(t)\|_{L^1}^{1/2}$, we have

$$
c(1 + t)^{-\frac{1}{4}} \leq \|u(t)\|_{L^2} \leq \|u(t)\|_{L^\infty}^{\frac{1}{2}} \|u(t)\|_{L^1}^{\frac{1}{2}} \leq C(1 + t)^{-1}\|u(t)\|_{L^1}^{\frac{1}{2}}.
$$

This yields

$$
\|u(t)\|_{L^1} \geq c(1 + t)^{\frac{1}{2}}.
$$

This complete the proof. ■

4 Proof of Theorem 3.2

This section is devoted to the proof of Theorem 3.2. The global existence of solutions is proved by Proposition 3.1. Hence we focus on the derivation of the $L^1$ estimate.

As we mentioned above, it is not valid to apply the semigroup theory to the problem (3.1) directly. To overcome this difficulty, we reformulate the problem (3.1).

Let $x = x(X,t)$ be a material coordinate which solves the following flow map

$$
\begin{cases}
\frac{dx}{dt}(X,t) = v(x(X,t),t), \\
x(X,0) = X.
\end{cases}
$$

We next define $X = X(x,t)$ as the inverse of $x = x(X,t)$ and $\psi = x - X$. According to [1,21], $F$ is defined as $F = \frac{\partial x}{\partial X}$. It is shown in [19] that its inverse $F^{-1}$ is written as $F^{-1}(x,t) = \nabla X(x,t)$ if $F_0^{-1}$ has the form $F_0^{-1} = \nabla X_0$ with some vector field $X_0$. Then $\psi$ solves

$$
\partial_t \psi - v = -v \cdot \nabla \psi, \quad (4.1)
$$
and satisfies

\[ G = \nabla \psi + h(\nabla \psi), \quad (4.2) \]

where \( h(\nabla \psi) = (I - \nabla \psi)^{-1} - I - \nabla \psi. \)

We note that (4.2) is equivalent to

\[ \nabla \psi = I - (I + G)^{-1}. \quad (4.3) \]

We next read the constraint (1.8) as

\[ 1 + \phi = \rho = \det F^{-1} = \det(I + G)^{-1}. \]

By using (4.3) and the following expansion for a \( 3 \times 3 \) matrix \( A \)

\[
\det(I + A) = 1 + \text{tr} A + \frac{1}{2}((\text{tr} A)^2 - (\text{tr}(A^2)) + \det A,
\]

we have

\[ \phi = -\text{div} \psi + \frac{1}{2}(\text{tr}(\nabla \psi)^2 - (\text{tr}(\nabla \psi))^2) - \det(\nabla \psi). \quad (4.4) \]

We set \( \psi_0 = \psi|_{t=0}. \) The following estimates hold for \( \phi, G \) and \( \nabla \psi. \)

**Lemma 4.1.** Assume that \( G \) and \( \psi \) satisfy (4.2). There is a positive number \( \epsilon_1 < 1 \) such that if \( \| G \|_{C([0, \infty); H^3)} \leq \epsilon_1, \) the following inequalities hold uniformly for \( t \geq 0: \)

\[ C^{-1} \| \nabla \psi(t) \|_{L^p} \leq \| G(t) \|_{L^p} \leq C \| \nabla \psi(t) \|_{L^p}, \quad p = 1, 2, \quad (4.5) \]

\[ \| \nabla^2 \psi(t) \|_{L^2} \leq C \| \nabla G(t) \|_{L^2}, \quad (4.6) \]

\[ \| \nabla^3 \psi(t) \|_{L^2} \leq C(\| \nabla G(t) \|_{H^1}^2 + \| \nabla^2 G(t) \|_{L^2}), \quad (4.7) \]

\[ \| \nabla^4 \psi(t) \|_{L^2} \leq C(\| \nabla G(t) \|_{H^1} \| \nabla^2 G(t) \|_{H^1} + \| \nabla^3 G(t) \|_{L^2}). \quad (4.8) \]

\[ \| \nabla^2 \psi_0 \|_{L^1} \leq C \| \nabla G_0 \|_{L^1}, \quad (4.9) \]

\[ \| \nabla^3 \psi_0 \|_{L^1} \leq C \| \nabla^2 G_0 \|_{L^1}, \quad (4.10) \]

\[ \| \phi(t) \|_{L^1} \leq \| \text{div} \psi(t) \|_{L^1} + C(1 + \| \nabla \psi(t) \|_{L^\infty}) \| \nabla \psi(t) \|_{L^2}^2. \quad (4.11) \]

**Proof.** The inequalities (4.5)–(4.8) are shown in [9, Lemma 4.1]. The inequalities (4.9) and (4.10) are established in a similar argument as in the proof of (4.6). We only derive (4.11) here.

In order to obtain (4.11), we use (4.4). Since

\[ \| (\text{tr}(\nabla \psi)^2 - (\text{tr}(\nabla \psi))^2) \|_{L^1} \leq C \| \nabla \psi \|_{L^2}^2, \]
\[ \| \det(\nabla \psi) \|_{L^1} \leq C \| \nabla \psi \|_{L^\infty} \| \nabla \psi \|_{L^2}^2, \]

we have
\[ \| \phi \|_{L^1} \leq \| \text{div} \psi \|_{L^1} + C(1 + \| \nabla \psi \|_{L^\infty}) \| \nabla \psi \|_{L^2}^2. \]

This completes the proof of Lemma 4.1. ■

Let \( U \) and \( U_0 \) be \( U = (\tilde{\phi}, w, \tilde{G}) = (-\text{div} \psi, w, \nabla \psi) \) and \( U_0 = (\tilde{\phi}_0, w_0, \tilde{G}_0) = (-\text{div} \psi_0, w_0, \nabla \psi_0) \), respectively. By taking \( \epsilon_0 \) in Proposition 3.1 such that \( C_3 \epsilon_0 \leq \epsilon_1 \), thanks to Lemma 4.1, the behavior of \( u \) is identified from \( U \):
\[ \| u(t) \|_{L^1} \leq C \| U(t) \|_{L^1}. \]

By coupling (3.1) and (4.1), we arrive at the following problem for \( U \)

\[
\begin{cases}
\partial_t \tilde{\phi} + \text{div} w = N_1(U), \\
\partial_t w - \nu \Delta w - \tilde{v} \text{div} w + \gamma^2 \nabla \tilde{\phi} - \beta^2 \text{div} \tilde{G} = N_2(U), \\
\partial_t \tilde{G} - \nabla w = N_3(U), \\
\tilde{\phi} + \text{tr} \tilde{G} = 0, \quad \tilde{G} = \nabla \psi, \\
U|_{t=0} = U_0.
\end{cases}
\tag{4.12}
\]

Here \( N_j(U), j = 1, 2, 3 \), denote nonlinear terms;
\[ N_1(U) = \text{div} (w \cdot \nabla \psi), \]
\[ N_2(U) = g_2(u) - \frac{\gamma^2}{2} \nabla (\text{tr}(\nabla \psi)^2 - (\text{tr}(\nabla \psi))^2) - \gamma^2 \text{div} (\nabla \psi) + \beta^2 \text{div} h(\nabla \psi), \]
\[ N_3(U) = -\nabla (w \cdot \nabla \psi). \]

We note that \( N_1(U) \) and \( N_3(U) \) also satisfy the same linear constraint as \( \tilde{\phi} \) and \( \tilde{G} \)
\[ N_1(U) + \text{tr} N_3(U) = 0. \tag{4.13} \]

In what follows, we omit tildes \( \tilde{\cdot} \) of \( \tilde{\phi} \) and \( \tilde{G} \) for simplicity. The problem (4.12) is reduced to
\[
\begin{cases}
\partial_t U + LU = N(U), \\
\phi + \text{tr} G = 0, \quad G = \nabla \psi, \\
U|_{t=0} = U_0,
\end{cases}
\tag{4.14}
\]

where
\[
L = \begin{pmatrix}
0 & \text{div} & 0 \\
\gamma^2 \nabla & -\nu \Delta - \tilde{v} \text{div} & -\beta^2 \text{div} \\
0 & -\nabla & 0
\end{pmatrix}, \quad N(U) = \begin{pmatrix}
N_1(U) \\
N_2(U) \\
N_3(U)
\end{pmatrix}. \]
By using the Duammel principle, $U$ satisfies the following integral equations

$$U(t) = e^{-tL}U_0 + \int_0^t e^{-(t-s)L}N(U(s))ds. \quad (4.15)$$

To analyze the linearized semigroup $e^{-tL}$, we introduce several notations. We set

$$Q = I - P = \mathcal{F}^{-1} \frac{\xi^T \xi}{|\xi|^2} \mathcal{F},$$

$$\mathcal{K}_t^\lambda(x) = \mathcal{F}^{-1} \left[ \frac{e^{\lambda+(\xi)t} - e^{\lambda-(\xi)t}}{\lambda+(\xi) - \lambda-(\xi)} \right] (x),$$

$$\mathcal{K}_t^\mu(x) = \mathcal{F}^{-1} \left[ \frac{e^{\mu+(\xi)t} - e^{\mu-(\xi)t}}{\mu+(\xi) - \mu-(\xi)} \right] (x).$$

Here $\lambda_{\pm}(\xi)$ and $\mu_{\pm}(\xi)$ are given by

$$\lambda_{\pm}(\xi) = \frac{-\nu|\xi|^2 \pm \sqrt{\nu^2|\xi|^4 - 4\beta^2|\xi|^2}}{2},$$

$$\mu_{\pm}(\xi) = \frac{-(\nu + \tilde{\nu})|\xi|^2 \pm \sqrt{(\nu + \tilde{\nu})^2|\xi|^4 - 4(\beta^2 + \gamma^2)|\xi|^2}}{2}.$$

We see that the following properties of $\lambda_{\pm}(\xi)$ and $\mu_{\pm}(\xi)$ hold:

$$\lambda_{\pm}(\xi) \sim -\frac{\nu}{2}|\xi|^2 \pm i\beta|\xi|, \text{ for } |\xi| \ll 1,$$

$$\lambda_{+}(\xi) \sim -\frac{\beta^2}{\nu}, \lambda_{-}(\xi) \sim -\nu|\xi|^2, \text{ for } |\xi| \gg 1,$$

$$\mu_{\pm}(\xi) \sim \frac{-\nu + \tilde{\nu}}{2}|\xi|^2 \pm i\sqrt{\beta^2 + \gamma^2}|\xi|, \text{ for } |\xi| \ll 1,$$

$$\mu_{+}(\xi) \sim -\frac{\beta^2 + \gamma^2}{\nu + \tilde{\nu}}, \mu_{-}(\xi) \sim -(\nu + \tilde{\nu})|\xi|^2, \text{ for } |\xi| \gg 1.$$

According to [9], the expression of the semigroup $U(t) = \mathcal{T}(\phi(t), w(t), G(t)) = e^{-tL}U_0$ is given by

$$\begin{pmatrix} \phi(x, t) \\ w(x, t) \\ G(x, t) \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{11}^1(t) & \mathcal{K}_{12}^1(t) & \mathcal{K}_{13}^1(t) \\ \mathcal{K}_{21}^1(t) & \mathcal{K}_{22}^1(t) & \mathcal{K}_{23}^1(t) \\ \mathcal{K}_{31}^1(t) & \mathcal{K}_{32}^1(t) & \mathcal{K}_{33}^1(t) \end{pmatrix} \begin{pmatrix} \phi_0(x) \\ w_0(x) \\ G_0(x) \end{pmatrix}, \quad (4.16)$$
provided that $\phi_0 + \text{tr}G_0 = 0$ and $G_0 = \nabla \psi_0$.

Here $K^j_k(t)$, $j, k = 1, 2, 3$, are the linear operators defined as

$K^{11}(t)\phi_0(x) = (\partial_t - (\nu + \tilde{\nu})\Delta)(K^\mu_\nu * \phi_0)(x)$

$$= \mathcal{F}^{-1} \left[ \frac{\mu_+(\xi)e^{\mu_-(\xi)t} - \mu_-(\xi)e^{\mu_+(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \tilde{\phi}_0(\xi) \right](x),$$

$K^{12}(t)w_0(x) = -\text{div}(K^\mu_\nu * w_0)(x)$

$$= i\mathcal{F}^{-1} \left[ \frac{e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \xi \cdot \tilde{w}_0(\xi) \right](x),$$

$K^{13}(t)G_0(x) = 0,$

$K^{21}(t)\phi_0(x) = -\gamma^2 \nabla(K^\mu_\nu * \phi_0)(x)$,

$$= i\gamma^2 \mathcal{F}^{-1} \left[ \frac{e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \tilde{\phi}_0(\xi) \xi \right](x),$$

$K^{22}(t)w_0(x) = \partial_t(K^\lambda_\nu * Pw_0)(x) + \partial_t(K^\mu_\nu * Qw_0)(x)$

$$= \mathcal{F}^{-1} \left[ \frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \left( I - \frac{\xi^\top \xi}{|\xi|^2} \right) \tilde{w}_0(\xi) \right](x) + \mathcal{F}^{-1} \left[ \frac{\mu_+(\xi)e^{\mu_+(\xi)t} - \mu_-(\xi)e^{\mu_-(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \tilde{w}_0(\xi) \right](x),$$

$K^{23}(t)G_0(x) = \beta^2 (PG_0 * \nabla K^\lambda_\nu)(x) + \beta^2 (QG_0 * \nabla K^\mu_\nu)(x)$

$$= -i\beta^2 \mathcal{F}^{-1} \left[ \frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \left( I - \frac{\xi^\top \xi}{|\xi|^2} \right) \tilde{G}_0(\xi) \xi \right](x) - i\beta^2 \mathcal{F}^{-1} \left[ \frac{e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \tilde{G}_0(\xi) \xi \right](x),$$

$K^{31}(t)\phi_0(x) = 0,$

$K^{32}(t)w_0(x) = (Pw_0 * \nabla K^\lambda_\nu)(x) + (Qw_0 * \nabla K^\mu_\nu)(x)$

$$= -i\mathcal{F}^{-1} \left[ \frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \left( I - \frac{\xi^\top \xi}{|\xi|^2} \right) \tilde{w}_0(\xi) \xi \right](x) - i\mathcal{F}^{-1} \left[ \frac{e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \frac{\xi^\top \xi}{|\xi|^2} \tilde{w}_0(\xi) \xi \right](x),$$

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\[ K^{33}(t)G_0(x) = (\partial_t - \nu \Delta)(K^\lambda_t * PG_0)(x) + (\partial_t - (\nu + \nu)\Delta)(K^\mu_t * QG_0)(x) \]
\[ = \mathcal{F}^{-1} \left[ \frac{\lambda_+(\xi)e^{\lambda_-(\xi)t} - \lambda_-(\xi)e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)} \left( I - \frac{\xi^T \xi}{|\xi|^2} \right) \hat{G}_0(\xi) \right](x) \]
\[ + \mathcal{F}^{-1} \left[ \frac{\mu_+(\xi)e^{\mu_-(\xi)t} - \mu_-(\xi)e^{\mu_+(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} \frac{\xi^T \xi}{|\xi|^2} \right] \hat{G}_0(\xi)(x). \]

In view of the asymptotic profiles of \( \lambda_\pm(\xi) \) and \( \mu_\pm(\xi) \) in the Fourier space, we decompose the solution \( U(t) \) of the problem (4.14) into its low and high frequency parts. Let \( \hat{\phi}_1, \hat{\phi}_M, \hat{\phi}_\infty \in C^\infty(\mathbb{R}^3; [0, 1]) \) be cut-off functions such that

\[
\hat{\phi}_1(\xi) = \begin{cases} 
1 & |\xi| \leq \frac{M_1}{2}, \\
0 & |\xi| \geq \frac{M_1}{\sqrt{2}},
\end{cases} \quad \hat{\phi}_1(-\xi) = \hat{\phi}_1(\xi),
\]
\[
\hat{\phi}_\infty(\xi) = \begin{cases} 
1 & |\xi| \geq 2M_2, \\
0 & |\xi| \leq \sqrt{2}M_2,
\end{cases} \quad \hat{\phi}_\infty(-\xi) = \hat{\phi}_\infty(\xi),
\]
\[
\hat{\phi}_M(\xi) = 1 - \hat{\phi}_1(\xi) - \hat{\phi}_\infty(\xi),
\]
where
\[
M_1 = \min \left\{ \frac{\beta}{\nu}, \frac{\sqrt{\beta^2 + \gamma^2}}{\nu + \nu} \right\}, \quad M_2 = \max \left\{ \frac{\beta}{\nu}, \frac{\sqrt{\beta^2 + \gamma^2}}{\nu + \nu} \right\}.
\]

We define operators \( P_j, j = 1, \infty, \) on \( L^2 \) as
\[
P_1u = \mathcal{F}^{-1}(\hat{\phi}_1\hat{u}), \quad P_\infty u = \mathcal{F}^{-1}((\hat{\phi}_M + \hat{\phi}_\infty)\hat{u}) \text{ for } u \in L^2.
\]

**Lemma 4.2.** \( P_j \) (\( j = 1, \infty \)) have the following properties.

(i) \( P_1 + P_\infty = I \).

(ii) \( \partial_x^\alpha P_1 = P_1 \partial_x^\alpha \), \( \| \partial_x^\alpha P_1 f \|_{L^2} \leq C_\alpha \| f \|_{L^2} \) for \( \alpha \in \{(0) \cup N\}^3 \) and \( f \in L^2 \).

(iii) \( \partial_x^\alpha P_\infty = P_\infty \partial_x^\alpha \), \( \| \partial_x^\alpha P_\infty f \|_{L^2} \leq C \| \nabla \partial_x^\alpha P_\infty f \|_{L^2} \) for \( \alpha \in \{(0) \cup N\}^3 \) with \( |\alpha| = k \geq 0 \) and \( f \in H^{k+1} \).

Lemma 4.2 immediately follows from the definitions of \( P_j, j = 1, \infty, \) and the Plancherel theorem. We omit the proof.

The solution \( U(t) \) of (4.14) is decomposed as
\[
U(t) = U_1(t) + U_\infty(t), \quad U_1(t) = P_1 U(t), \quad U_\infty(t) = P_\infty U(t).
\]
By applying $P_j$ to (4.15), $U_j(t) = (\phi_j(t), w_j(t), G_j(t))$, $j = 1, \infty$, satisfy

$$
\begin{cases}
U_j(t) = e^{-tL}U_j(0) + \int_0^t e^{-(t-s)L} P_j N(U(s)) ds, \\
\phi_j + \text{tr} G_j = 0, \ P_j N_1(U) + \text{tr} P_j N_3 = 0, \\
U_j|_{t=0} = P_j U_0.
\end{cases} \tag{4.17}
$$

Concerning to the $L^1$ estimate of $e^{-tL}U_0$, we have the following proposition.

**Proposition 4.3.** Let $\phi_0 + \text{tr} G_0 = 0$ and $G_0 = \nabla \psi_0$. Then the following estimates hold for $t > 0$:

(i) $\|e^{-tL}P_1 U_0\|_{L^1} \leq C(1 + t)^{\frac{1}{2}} \|U_0\|_{L^1}$,

(ii) $\|e^{-tL}P_\infty U_0\|_{L^1} \leq C e^{-ct} \|U_0\|_{W^{2,1}}$.

The estimate (i) is done in [9, Lemma 6.4]. For the estimate (ii), we will discuss in Section 5.

In order to estimate the Duammel terms $\int_0^t e^{-(t-s)L} P_j N(U(s)) ds$, $j = 1, \infty$, we give the following $L^2$ decay estimates for $\nabla^k U(t)$.

**Proposition 4.4.** There exists a positive number $\epsilon_1$ such that if $\|u_0\|_{L^1} + \|u_0\|_{H^4} \leq \epsilon_1$, then the following inequality holds for $k = 0, 1, 2, 3$ and $t \geq 0$:

$$
\|\nabla^k U(t)\|_{L^2} \leq C(1 + t)^{-\frac{3}{4} + \frac{k}{2}} (\|u_0\|_{L^1} + \|u_0\|_{H^4}),
$$

Proposition 4.4 follows from Proposition 3.1 and Lemma 4.1.

The estimates of $\int_0^t \|e^{-(t-s)L} P_j N(U(s))\|_{L^1} ds$, $j = 1, \infty$, are given as follows.

**Lemma 4.5.** There exists a positive number $\epsilon_1$ such that if $u_0 \in L^1 \cap H^4$ and $\|u_0\|_{H^4} \leq \epsilon_1$, then the following estimates hold:

$$
\int_0^t \|e^{-(t-s)L} P_j N(U(s))\|_{L^1} ds \leq C(1 + t)^{\frac{1}{2}} (\|u_0\|_{L^1} + \|u_0\|_{H^4}), \quad j = 1, \infty, \ t \geq 0.
$$

**Proof.** We first show the case $j = 1$. We obtain the following estimate in a similar argument as in the proof of Lemma 4.3:

$$
\|e^{-(t-s)L} P_1 N(U(s))\|_{L^1} \leq C(1 + t - s)^{\frac{1}{2}} \|N(U(s))\|_{L^1}. \tag{4.18}
$$
In view of Lemma 2.1 and Proposition 4.4, we have
\[ \|N(U(s))\|_{L^1} \leq C\|U(s)\|_{H^2}\|\nabla U(s)\|_{H^1} \leq C(1 + s)^{-2}(\|u_0\|_{L^1} + \|u_0\|_{H^4}). \]  
(4.19)

We see from Lemma 2.2, (4.18) and (4.19) that
\[ \int_0^t \|e^{-(t-s)L}P_1 N(U(s))\|_{L^1} \, ds \]
\[ \leq C \int_0^t (1 + t - s)^{\frac{1}{2}} \|N(U(s))\|_{L^1} \, ds \]
\[ \leq C \int_0^t (1 + t - s)^{\frac{1}{2}} (1 + s)^{-2} ds(\|u_0\|_{L^1} + \|u_0\|_{H^4}) \]
\[ \leq C(1 + t)^{\frac{1}{2}}(\|u_0\|_{L^1} + \|u_0\|_{H^4}). \]  
(4.20)

We next consider the case \( j = \infty \).
We obtain the following estimate in a similar argument as in the proof of Lemma 4.3:
\[ \|e^{-(t-s)L}P_{\infty} N(U(s))\|_{L^1} \leq C e^{-c(t-s)}\|N(U(s))\|_{W^{1,1}}. \]  
(4.21)

In view of Lemma 2.1 and Proposition 4.4, we have
\[ \|N(U(s))\|_{W^{2,1}} \leq C(\|U(s)\|_{H^3}^2 + \|u(s)\|_{L^2}\|\nabla^4 w(s)\|_{L^2}) \leq C\|u_0\|_{H^4}. \]  
(4.22)

It follows from (4.21) and (4.22) that
\[ \int_0^t \|e^{-(t-s)L}P_{\infty} N(U(s))\|_{L^1} \, ds \]
\[ \leq C \int_0^t e^{-c(t-s)}\|N(U(s))\|_{W^{2,1}} \, ds \]
\[ \leq C\|u_0\|_{H^4} \]
\[ \leq C(1 + t)^{\frac{1}{2}}\|u_0\|_{H^4}. \]

This completes the proof. \( \blacksquare \)

**Proof of Theorem 3.2.** By taking \( L^1 \) norm of the first equation of (4.17), we have
\[ \|U_j(t)\|_{L^1} \leq \|e^{-tL}U_j(0)\|_{L^1} + \int_0^t \|e^{-(t-s)L}P_j N(s)\|_{L^1} \, ds. \]  
(4.23)
Combining Lemma 4.3, Lemma 4.5 and (4.23), we arrive at

\[ \|U(t)\|_{L^1} \leq C(1 + t)^{1/2} (\|u_0\|_{W^{2,1}} + \|u_0\|_{H^4}), \quad t \geq 0. \]

By using Lemma 4.1, we obtain

\[ \|u(t)\|_{L^1} \leq C\|U(t)\|_{L^1} \leq C(1 + t)^{1/2} (\|u_0\|_{W^{2,1}} + \|u_0\|_{H^4}), \quad t \geq 0. \]

This completes the proof of Theorem 3.2. ■

5 Proof of Proposition 4.3 (ii).

In this section, we prove Proposition 4.3 (ii).

For \( j = M, \infty \), we set

\[
\mathcal{K}^\pm_j (t) f(x) = \mathcal{F}^{-1} \left[ \frac{e^{\lambda_\pm(x)t}}{\lambda_\pm(x) - \lambda_\mp(x)} \hat{\varphi}_j(x) \hat{f}(x) \right],
\]
\[
\mathcal{M}^\pm_j (t) f(x) = \mathcal{F}^{-1} \left[ \frac{\lambda_\pm(x)e^{\lambda_\pm(x)t}}{\lambda_\pm(x) - \lambda_\mp(x)} \hat{\varphi}_j(x) \hat{f}(x) \right],
\]
\[
\mathcal{K}^\pm_j (t) u(x) = \mathcal{F}^{-1} \left[ \frac{e^{\mu_\pm(x)t}}{\mu_\pm(x) - \mu_\mp(x)} \hat{\varphi}_j(x) \hat{f}(x) \right],
\]
\[
\mathcal{M}^\pm_j (t) u(x) = \mathcal{F}^{-1} \left[ \frac{\mu_\pm(x)e^{\mu_\pm(x)t}}{\mu_\pm(x) - \mu_\mp(x)} \hat{\varphi}_j(x) \hat{f}(x) \right].
\]

We first consider the high frequency part.

Lemma 5.1. The following estimates hold for \( \alpha \in (\mathbb{N} \cup \{0\})^3 \), \( j \geq k \geq 0 \) and \( t > 0 \):

(i) \[ \left\| \partial_t^j \partial_x^k [\mathcal{K}^\pm_\alpha (t) f] \right\|_{L^1} + \left\| \partial_t^j \partial_x^k [\mathcal{K}^\pm_\alpha (t) u] \right\|_{L^1} \leq C e^{-ct} \| f \|_{W^{(1/2)+, 1}}, \quad (5.1) \]

(ii) \[ \left\| \partial_t^j \partial_x^k [\mathcal{K}^\pm_\alpha (t) f] \right\|_{L^1} + \left\| \partial_t^j \partial_x^k [\mathcal{K}^\pm_\alpha (t) u] \right\|_{L^1} \leq C e^{-ct} t^{-j-k} \| f \|_{W^{2k + (1/2)+, 1}}, \quad (5.2) \]

(iii) \[ \left\| \partial_t^j \partial_x^k [\mathcal{M}^\pm_\alpha (t) f] \right\|_{L^1} + \left\| \partial_t^j \partial_x^k [\mathcal{M}^\pm_\alpha (t) u] \right\|_{L^1} \leq C e^{-ct} \| f \|_{W^{(1)+}}, \quad (5.3) \]
Since we obtain $m$

By here $a^+$ denotes $a^+ = \max\{0, a\}$ for $a \in \mathbb{R}$.

**Proof.** We see from [20, Theorem 4.2.] that (5.1)–(5.4) are true. Therefore it remains to show (5.5) and (5.6).

We first write $\partial_t^j \partial_x^{\alpha} \mathcal{K}_{\lambda}^\pm (t) Q f$

as

$$\partial_t^j \partial_x^{\alpha} \mathcal{K}_{\lambda}^\pm (t) Q f = \mathcal{F}^{-1} \left[ \frac{\lambda_+ (\xi)^j e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} \hat{\varphi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right] \cdot \partial_x^\alpha f.$$  

We use the formula

$$e^{ix \cdot x} = \sum_{|\eta| = m} \frac{(-ix)^\eta}{|x|^{2m}} \partial_x^\eta (e^{ix \cdot x}).$$  

By $m$-times integration by parts and (5.7), we have

$$\mathcal{F}^{-1} \left[ \frac{\lambda_+ (\xi)^j e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} \hat{\varphi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right]$$

$$= \frac{1}{(2\pi)^d} \sum_{|\eta| = m} \frac{(-ix)^\eta}{|x|^{2m}} \int_{|\xi| \geq \sqrt{2}m} e^{ix \cdot \xi} \partial_x^\eta \left[ \frac{\lambda_+ (\xi)^j e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} \hat{\varphi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right] d\xi.$$  

Since

$$\left| \partial_x^\eta \left( \frac{\lambda_+ (\xi)^j e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} \hat{\varphi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right) \right| \leq C_\eta (1 + t)^{|\eta|} e^{-c_1 t |\xi|^{-|\eta|-2}},$$

we obtain

$$\mathcal{F}^{-1} \left[ \frac{\lambda_+ (\xi)^j e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} \hat{\varphi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right]$$

$$\leq C_m |x|^{-m} e^{-c_1 t} \int_{|\xi| \geq \sqrt{2}m} |\xi|^{-m-2} d\xi$$

$$\leq C_m |x|^{-m} e^{-c_1 t}, \quad m \geq 2.$$  

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Therefore, by using (5.8), we have

\[
\|\partial_t^j \partial_x^\alpha [\mathcal{K}_\infty^{\lambda_+} (t) Q f]\|_{L^1} \\
\leq \left\| \mathcal{F}^{-1} \left[ \frac{\lambda_+ (\xi) e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} \hat{\phi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right] \right\|_{L^1} \| \partial_x^\alpha f \|_{L^1} \\
\leq C e^{-\alpha t \pi} \int_{|\xi| \leq 1} |\xi|^{-2} \left( \int_{|\xi| > \sqrt{\mathcal{M}_2}} \frac{1}{|\xi|^4} \, d\xi \right) \| f \|_{W^{1,1}} \\
+ C e^{-\alpha t} \int_{|\xi| > 1} |\xi|^{-4} \left( \int_{|\xi| > \sqrt{\mathcal{M}_2}} \frac{1}{|\xi|^6} \, d\xi \right) \| f \|_{W^{1,1}} \\
\leq C e^{-\alpha t} \| f \|_{W^{1,1}}.
\]

This completes the proof of (5.5).

We next show (5.6). We write \( \partial_t^j \partial_x^\alpha [\mathcal{K}_\infty^{\lambda_+} (t) Q f] \) as

\[
\partial_t^j \partial_x^\alpha [\mathcal{K}_\infty^{\lambda_+} (t) Q f] = \mathcal{F}^{-1} \left[ \frac{\lambda_+ (\xi) e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} (1 + |\xi|^2)^{-k} \hat{\phi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right] * \partial_x^\alpha (1 - \Delta)^k f,
\]

where \((1 - \Delta)^k = \mathcal{F}^{-1} (1 + |\xi|^2)^k \mathcal{F}\).

By \(m\)-times integration by parts and (5.7), we have

\[
\mathcal{F}^{-1} \left[ \frac{\lambda_+ (\xi) e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} (1 + |\xi|^2)^{-k} \hat{\phi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right] \\
= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{|\eta| = m} \int_{|\xi| \geq \sqrt{\mathcal{M}_2}} e^{ix \cdot \xi} \frac{\lambda_+ (\xi) e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} (1 + |\xi|^2)^{-k} \hat{\phi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \, d\xi.
\]

Since

\[
\left| \partial_\xi^\eta \left( \frac{\lambda_+ (\xi) e^{\lambda_+ (\xi) t}}{\lambda_+ (\xi) - \lambda_- (\xi)} (1 + |\xi|^2)^{-k} \hat{\phi}_\infty (\xi) \frac{\xi^T \xi}{|\xi|^2} \right) \right| \leq C_n e^{-c_1 t - c_2 |\xi|^2 |\xi| - |\eta|^{-2}}.
\]
we obtain
\[
|F^{-1}\left[ \frac{\lambda_+ - \lambda_-(\xi)}{\lambda_+ - \lambda_-(\xi)}(1 + |\xi|^2)^{-1}\right] | \leq C_m |x|^{-m}t^{-(j-k)}e^{-c_1 t} \int_{|\xi| \geq \sqrt{2}M_2} e^{-c_2 |\xi|^2 t} |\xi|^{-m-2} d\xi
\]
(5.9)

Therefore, by using (5.9), we have
\[
\|\partial_t^j \partial_x^\alpha [K^\lambda_{x,n}(t)Qf]\|_{L^1} \leq C t^{-(j-k)} e^{-c_1 t} \int_{|x| \leq 1} |x|^{-2} dx \|f\|_{W^{2k+|\alpha|,1}}
\]
\[
+ C t^{-(j-k)} e^{-c_1 t} \int_{|x| \geq 1} |x|^{-4} dx \|f\|_{W^{2k+|\alpha|,1}}
\]
\[
\leq C t^{-(j-k)} e^{-c_1 t} \|f\|_{W^{2k+|\alpha|,1}}.
\]

Similarly, we obtain
\[
\left\| \partial_t^j \partial_x^\alpha [K^\lambda_{x,n}(t)Qf] \right\|_{L^1} \leq C t^{-(j-k)} e^{-c_1 t} \|f\|_{W^{|\alpha|,1}}.
\]

This completes the proof of (5.6). □

We next investigate the middle frequency part.

**Lemma 5.2.** The following estimates hold for \( \alpha \in (\mathbb{N} \cup \{0\})^3 \), \( j \geq 0 \) and \( t \geq 0 \):

- (i) \( \|\partial_t^j \partial_x^\alpha [\mathcal{K}_{x,n}^\lambda (t) f]\|_{L^1} \leq C e^{-c_1 t} \|f\|_{L^1} \)

- (ii) \( \|\partial_t^j \partial_x^\alpha [\mathcal{M}_{x,n}^\lambda (t) f]\|_{L^1} \leq C e^{-c_1 t} \|f\|_{L^1} \)

- (iii) \( \|\partial_t^j \partial_x^\alpha [\mathcal{K}_{x,n}^\lambda (t) Qf]\|_{L^1} \leq C e^{-c_1 t} \|f\|_{L^1} \)

**Proof.** The following formulas hold for \( \frac{M_2}{\sqrt{2}} \leq |\xi| \leq \sqrt{2}M_2 \)
\[
eq \frac{1}{2\pi i} \int_{|z|^2 + \nu|\xi|^2 t + \beta^2 |\xi|^2 = 1} e^{zt} z^j dz,
\]

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\[
\frac{e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}}{\mu_+(\xi) - \mu_-(\xi)} = \frac{1}{2\pi i} \int_\Gamma e^{zt} z^2 + (\nu + \tilde{\nu})|\xi|^2 z + (\beta^2 + \gamma^2)|\xi|^2 \, dz.
\]

Here \( \Gamma \) is a closed path containing \( \lambda_\pm(\xi) \) and \( \mu_\pm(\xi) \), and included in \( \{z \in \mathbb{C} | \Re z \leq -c_3 \} \). Here \( c_3 \) is a positive number taken by

\[
\max_{\frac{M}{\sqrt{2}} \leq |\xi| \leq \sqrt{2}M} \Re \mu_j(\xi) \leq -2c_3, j = 1, 2, 3, 4.
\]

Hence, we can compute as in a similar manner to [11, 20] to obtain

\[
\left| \partial_t \partial_x^\alpha \mathcal{F}^{-1} \left[ \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt}}{z^2 + \nu|\xi|^2 z + \beta^2|\xi|^2} \, dz \eta(\xi) \hat{\varphi}_M(\xi) \right] \right| 
\leq C_{j,\alpha,N} e^{-ct}|x|^{-N}, \, j + |\alpha| \geq 1, \, N \geq 0,
\]

\[
\left| \partial_t \partial_x^\alpha \mathcal{F}^{-1} \left[ \frac{1}{2\pi i} \int_\Gamma \frac{e^{zt}}{z^2 + (\nu + \tilde{\nu})|\xi|^2 z + (\beta^2 + \gamma^2)|\xi|^2} \, dz \eta(\xi) \hat{\varphi}_M(\xi) \right] \right| 
\leq C_{j,\alpha,N} e^{-ct}|x|^{-N}, \, j + |\alpha| \geq 1, \, N \geq 0.
\]

Here \( \eta \) is a function such that \( \eta \in C^\infty(S^2), \, S^2 = \{\xi \in \mathbb{R}^3 \mid |\xi| = 1\} \) and \( \eta = \eta(|\xi|) \). Therefore we have (5.10)–(5.12). This completes the proof. 

**Proof of Proposition 4.3 (ii).** By using Lemma 5.1 and Lemma 5.2, we have the following estimates of \( \mathcal{K}^{j1}(t)P_\infty \phi_0, \, j = 1, 2, \) and \( \mathcal{K}^{j2}(t)P_\infty w_0, \, j = 1, 2, 3 \)

\[
\|\mathcal{K}^{11}(t)P_\infty \phi_0\|_{L^1} \leq Ce^{-ct}\|\phi_0\|_{L^1},
\]

\[
\|\mathcal{K}^{12}(t)P_\infty w_0\|_{L^1} \leq Ce^{-ct}\|w_0\|_{L^1},
\]

\[
\|\mathcal{K}^{21}(t)P_\infty \phi_0\|_{L^1} \leq Ce^{-ct}\|\phi_0\|_{L^1},
\]

\[
\|\mathcal{K}^{22}(t)P_\infty w_0\|_{L^1} \leq Ce^{-ct}\|w_0\|_{W^{2,1}},
\]

\[
\|\mathcal{K}^{32}(t)P_\infty w_0\|_{L^1} \leq Ce^{-ct}\|w_0\|_{W^{2,1}}.
\]

We next focus on \( \mathcal{K}^{j3}(t)P_\infty G_0, \, j = 2, 3. \) By using \( G_0 = \nabla \psi_0 \), we write...
\[ \mathcal{F}[\mathcal{K}^{j3}(t)G_0], \ j = 2, 3 \text{ as} \]
\[ \mathcal{F}[\mathcal{K}^{23}(t)G_0](\xi) \]
\[ = -i\beta^2 \frac{e^{\lambda_+}(\xi)t - e^{\lambda_-}(\xi)t}{\lambda_+ - \lambda_-}(\hat{G}_0(\xi) - \text{tr}(\hat{G}_0(\xi))I)\xi \]
\[ - i\beta^2 \frac{e^{\mu_+}(\xi)t - e^{\mu_-}(\xi)t}{\mu_+ - \mu_-}\text{tr}(\hat{G}_0(\xi))\xi, \]
\[ \mathcal{F}[\mathcal{K}^{33}(t)G_0](x) \]
\[ = \frac{\lambda_+(\xi)e^{\lambda_+}(\xi)t - \lambda_-(\xi)e^{\lambda_-}(\xi)t}{\lambda_+ - \lambda_-} \hat{G}_0(\xi) + \nu|\xi|^2 \frac{e^{\lambda_+}(\xi)t - e^{\lambda_-}(\xi)t}{\lambda_+ - \lambda_-} \hat{G}_0(\xi) \]
\[ + \left( \nu + \bar{\nu} \right) \frac{e^{\mu_+}(\xi)t - e^{\mu_-}(\xi)t}{\mu_+ - \mu_-} \hat{G}_0(\xi) \]
\[ + \left( \frac{\mu_+(\xi)e^{\mu_+}(\xi)t - \mu_-(\xi)e^{\mu_-}(\xi)t}{\mu_+ - \mu_-} - \frac{\lambda_+(\xi)e^{\lambda_+}(\xi)t - \lambda_-(\xi)e^{\lambda_-}(\xi)t}{\lambda_+ - \lambda_-} \right) \frac{\xi^T \xi}{|\xi|^2} \hat{G}_0(\xi). \]

It then follows from Lemma 5.1 and Lemma 5.2 that
\[ \|K^{23}(t)P_\infty G_0\|_{L^1} \leq Ce^{-ct}\|G_0\|_{W^{2,1}}, \]
\[ \|K^{33}(t)P_\infty G_0\|_{L^1} \leq Ce^{-ct}\|G_0\|_{W^{2,1}}. \]

Consequently we arrive at
\[ \|e^{-tL}P_\infty U_0\|_{L^1} \leq Ce^{-ct}\|U_0\|_{W^{2,1}}. \]

This completes the proof. □

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