SYMMETRIC JOINS AND WEIGHTED BARYCENTERS

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Abstract. Given a space \( X \), we study the homotopy type of \( B_n(X) \) the space obtained as the “union of all \((n-1)\)-simplexes spanned by points in \( X \)” or the space of “formal barycenters of weight \( n \) or less” of \( X \). This is a space encountered in non-linear analysis under the name of space of barycenters or in differential geometry in the case \( n = 2 \) as the space of chords. We first relate this space to more familiar symmetric join construction and then determine its stable homotopy type in terms of the symmetric products on the suspension of \( X \). This leads to a complete understanding of the homology of \( B_n(X) \) as a functor of \( X \), and to an expression for its Euler characteristic given in terms of that of \( X \). A sharp connectivity theorem is also established. Finally the case of spheres \( S \) is studied in details and the homotopy type of \( B_n(S) \) is described generalizing in this way an early and beautiful result of James, Thomas, Toda and Whitehead.

1. Introduction

An interesting construction of Bahri and Coron [3] associates to a given topological space \( X \) and integer \( n > 0 \), the space \( B_n(X) \) of “weighted barycenters” of \( X \) obtained by taking the union of all \((n-1)\)-simplexes \( \Delta(p_1, \ldots, p_n) \) on vertices \( p_i \in X \) with the topology that when two vertices of \( \Delta \) come together, the resulting \( n-2 \)-simplex is identified with a face of \( \Delta \) (precise definitions in [5]). In the notation of [3] these spaces were described according to

\[
B_n(X) = \left\{ \sum_{i=1}^{n} t_i \delta_{x_i} \mid (x_1, \ldots, x_n) \in X^n, (t_1, \ldots, t_n) \in \Delta_{n-1} \right\}
\]

where \( \delta_x \) is the “Dirac mass” at the point \( x \) and \( \Delta_{n-1} \) is the \((n-1)\)-simplex of all tuples \((t_1, \ldots, t_n)\), \( t_i \geq 0 \) and \( \sum t_i = 1 \). The formal sums are understood to be abelian; i.e. unordered, and if two \( x \)'s coincide their coefficients add up. An alternative geometric description can also be given as follows.

Let \( X \) be a finite CW complex. Then there is some choice of embedding \( i : X \hookrightarrow \mathbb{R}^N \), \( N \) large, whereby the space \( B_n(X) \) is the union of all \((n-1)\)-dimensional simplices with vertices in \( i(X) \) such that the intersection of any two such simplices is a common face if any (see [3]).

Spaces of barycenters play an important role in the study of limiting Sobolev exponent problems in non-linear analysis such as the Yamabe and the scalar-curvature equations. Symmetric products on the other hand (see below) appear in the study of singular solutions to these equations. The smooth solutions come as critical points of a functional, with nested level sets \( \cdots W_p \subset W_{p+1} \subset \cdots \), and when no critical point exists, the pair \((W_{p+1}, W_p)\) behaves topologically much like \((B_{p+1}(M), B_p(M))\), where \( M \) is the underlying domain of the equation. Induction formulae relating the \( \mathbb{Z}_2 \)-orientation class of the pair \((B_{p+1}(M), B_p(M))\) to that of \((B_p(M), B_{p-1}(M))\) one level lower, are then used to derive existence results for the solutions of such nonlinear elliptic equations (see [8], with an appendix by Jean Lannes, or [2]). More recently, [11] proved that barycenter spaces played a fundamental role in the two-dimensional scalar curvature problem and other conformal equations.

The spaces \( B_n(X) \) enter differently but fundamentally as well in work of Vassiliev as a tool to construct simplicial resolutions for complements of discriminant loci in algebraic geometry [13, 31]. This technique has been skillfully applied to provide stable splittings for so-called “Atiyah-Hitchin schemes” in [5].

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The space $B_n(X)$ is an even older concept. This is the so-called space of chords on $X$ and has been considered for instance in \cite{6} and much more recently in \cite{22}. The terminology comes from the fact that $B_2(X)$ can be viewed as the space of all “chords” $pq$, $p$ and $q$ in $X$, with the understanding that $pp$ is identified with the point $p$ (see \cite{8}).

The purpose of this paper is to investigate the homotopy type and homology of $B_n(X)$ for all $n \geq 1$ and for $X$ any reasonable topological space (i.e. based, connected, locally compact and hausdorff). The nature of these spaces is intimately related to that of more familiar constructions known as symmetric joins and symmetric products.

We will write $X \ast Y$ the join product of $X$ and $Y$. The $n$-fold symmetric join of $X$ is obtained as the quotient of the $n$-th iterated join $X^{*n} := X \ast \cdots \ast X$ by the permutation action of the symmetric group $\mathfrak{S}_n$ and is written $\text{Sym}^{*n}(X)$ (see \cite{2}). This space for $X$ a sphere and $n = 2$ for example is treated in \cite{21} where the following cute result was obtained (see corollary 1.6).

\begin{equation}
\text{Sym}^{*2}(S^k) \cong \Sigma^{k+1} \mathbb{R}P^k
\end{equation}

Here $\Sigma^{k+1} := S^k \ast X$ refers to the $k$-fold unreduced suspension of $X$ and $\mathbb{R}P^k$ is the $k$-th real projective space. Very little else seems to be known about the homotopy type and homology of symmetric joins and the purpose of this paper is to remedy to this.

The following is our first main observation.

\textbf{Theorem 1.1.} $B_n(X)$ and $\text{Sym}^{*n}(X)$ have the same homotopy type.

In particular when $X$ is a simplicial complex\footnote{in other terminology, a polyhedral space.}, we show that both spaces have the homotopy type of the quotient $S^{n-1} \wedge_{\mathfrak{S}_n} X^{(n)}$, where $X^{(n)}$ is the “smash product” of $X$ (see \cite{2}) and where $\mathfrak{S}_n$ acts on $S^{n-1} \subset \mathbb{R}^n$ by permutation of coordinates (details in \cite{35}). It is easy to see that $\text{Sym}^{*n}(X)$ has the structure of a CW complex of dimension $n(d+1) - 1$ if $X$ is a simplicial complex of dimension $d$. This shows for instance that the homology of $B_n(X)$ vanishes in degrees greater than $n(d+1)$. On the other hand and when $X = M$ is a closed orientable manifold of dimension $d$, $B_n(M)$ always has a top homology class mod-2 while with integer coefficients (see \cite{10})

\begin{equation}
H_{n(d+1)-1}(B_n(M)) = \begin{cases}
\mathbb{Z}, & \text{if } d \text{ is odd} \\
0, & \text{if } d \text{ even}
\end{cases}
\end{equation}

Using a handy description of a symmetric join as an iterated pushout, we show based on a Van-Kampen type argument that $\text{Sym}^{*n}(X)$, and hence $B_n(X)$, is simply connected as soon as $n > 1$ (Theorem 3.6). The connectivity of barycenter spaces is however more tricky to establish and we appeal to that end to some old results of Nakaoa \cite{24} and a relative Leray-Serre spectral sequence argument (see \S6). The optimal result is as follows.

\textbf{Theorem 1.2.} If $X$ is an $r$-connected CW complex, $r \geq 1$, then $B_n(X)$ is $(2n + r - 2)$-connected.

The above lower bound for the connectivity is sharp in light of \cite{11}.

It turns out that the best way to getting to the homology of $\text{Sym}^{*n}(X)$ or $B_n(X)$ is to analyze their stable homotopy type (i.e. after suspension). Write $\text{SP}^n X$ for the $n$-th symmetric product of $X$ obtained as the quotient of $X^n$ by the permutation action of $\mathfrak{S}_n$ on factors. Here $\text{SP}^0 X$ reduces to basepoint and $\text{SP}^1 X = X$. There is a topological embedding $\text{SP}^{n-1} X \hookrightarrow \text{SP}^n X$ which adjoins the basepoint to a configuration in $\text{SP}^{n-1} X$ and we write $\overline{\text{SP}}^n X := \text{SP}^n X/\text{SP}^{n-1} X$ for the cofiber of this embedding (also known as the reduced symmetric product). It is easily seen that $\overline{\text{SP}}^1 X$ is the symmetric smash product $X^{(n)}/\mathfrak{S}_n$, where $X^{(n)}$ is the $n$-fold smash product of $X$ with itself (see \cite{2}).

The next key result describes $\text{Sym}^{*n}(X)$ and thus $B_n(X)$ completely after one unreduced suspension.

\textbf{Theorem 1.3.} There is a homeomorphism $\Sigma \text{Sym}^{*n}(X) = \overline{\text{SP}}^n(\Sigma X)$. 
Symmetric products being well understood constructions, Theorem 1.3 gives a fairly complete understanding of the stable homotopy type of both $\text{Sym}^n(X)$ and $\mathcal{B}_n(X)$ and allows for extensive homology computations. In particular, classical considerations show that $H_*(\mathcal{B}_n(X))$ is a direct summand in
\[ H_*(\mathcal{B}_n(X)) \hookrightarrow \bigotimes H_{i+1}(K(\tilde{H}_i(X), i+1)) \]
where $K(G, i)$ is the Eilenberg-MacLane space whose only non-trivial homotopy group is $G$ in degree $i$ (see appendix). Based on this, we indicate in [10] how to recover some of the homological calculations of [3] alluded to earlier. Note that combining Theorem 1.3 with Theorem 1.2 produces sharp connectivity bounds for the reduced symmetric products of a simplicial complex (see [15]).

A few useful corollaries are stated next. We recall that $X^\vee n$ the one point union of $n$ copies of $X$.

**Corollary 1.4.**

(a) (Corollary 4.4) If $\chi(X)$ be the Euler characteristic of $X$, then
\[ \chi(\mathcal{B}_n(X)) = 1 - \frac{1}{n!}(1 - \chi(X)) \cdots (n - \chi(X)) \]
This has been obtained in the case of topological surfaces in [20].

(b) (Corollary 4.3) $\text{Sym}^n(S^1)$ is homeomorphic to $S^{2n-1}$. 

(c) (Lemma 8.2) If $C_\varnothing$ denotes a closed Riemann surface of genus $g \geq 0$, then
\[ \Sigma \mathcal{B}_2(C_\varnothing) \simeq (S^4)^{\vee (2g^2 + g)} \vee (S^5)^{\vee 2g} \vee \Sigma^4 \mathbb{R} P^2 \]

(d) (Lemma 8.4) There is as well a stable splitting of $\mathcal{B}_2(X \times Y)$ into a six term bouquet.

Our last and one of our most interesting results is that it is possible to give a streamlined generalization of the James-Thomas-Toda-Whitehead equivalence [1] to higher dimensional spheres. Key constructions from [28] happen to be tailor made for such a generalization.

**Theorem 1.5.** There is a homotopy equivalence $\mathcal{B}_n(S^k) \simeq \Sigma^{k+1} Q_{n,k}$ where $Q_{n,k}$ is the quotient of $S^{(k+1)(n-1)}$ the unit sphere in the linear subspace $\{(v_1, \ldots, v_n) \in (\mathbb{R}^{k+1})^n \mid \sum v_i = 0\}$ under the $S_n$-action given by permuting the $v_i$.

This theorem is established in [31] and in an appendix we completely determine the homology of the barycenter spaces of the 2-sphere.

We conclude this introduction by applying Theorem 1.5 to give a short novel proof of [1].

**Corollary 1.6.** [14] $\text{Sym}^2(S^k) \simeq \Sigma^{k+1} \mathbb{R} P^k$.

**Proof.** When $n = 2$ in Theorem 1.5, $Q_{2,k}$ is the unit sphere $S^k$ in $W = \{(v, -v) \in (\mathbb{R}^{k+1})^2\}$. The generator of $\mathbb{Z}_2$ acts on $W$ by permuting $v$ and $-v$ and hence is multiplication by $-1$ on that sphere. This is the antipodal action and the claim is immediate. \qed

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## 2. Notations and Definitions

This section sets up notation and basic constructions. Our spaces are assumed to be locally compact, hausdorff and unless otherwise stated connected (see [7]). For based spaces $X$ and $Y$ with basepoints $*$, we write $X \vee Y$ the “wedge” of $X$ and $Y$ consisting of $\{(x, y) \in X \times Y \mid x = *$ or $y = *\}$. We write $X \wedge Y$ the cofiber of the inclusion $X \vee Y \hookrightarrow X \times Y$. The “smash” product $X \wedge Y$ is naturally a based

\[ \text{Sym}^2(S^k) \simeq \Sigma^{k+1} \mathbb{R} P^k. \]

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2 According to [31], corollary 1.4 (a) is a result of Caratheodory. No further explicit reference was given.
construction. Throughout \((D^n, S^{n-1})\) refers to the pair (unit closed disk in \(\mathbb{R}^n\), its boundary sphere) and \(I = D^1 = [0, 1]\). We write the standard \((n-1)\)-simplex as

\[\Delta_{n-1} = \{(t_1, \ldots, t_n) | t_i \in [0, 1], \sum t_i = 1\}\]

The faces of this simplex correspond to when \(t_i = 0\). Of course \(\Delta_0 = \{1\}\).

**The Iterated Join.**

Given two connected topological spaces \(X\) and \(Y\), the join of \(X\) and \(Y\) is the space of all segments “joining points” in \(X\) to points in \(Y\). It is denoted by \(X \ast Y\) and is the identification space

\[X \ast Y := X \times I \times Y / (x, 0, y) \sim (x', 0, y), (x, 1, y) \sim (x, 1, y') \quad \forall x, x' \in X, \forall y, y' \in Y\]

The homotopy type of \(X \ast Y\) depends only on the homotopy type of \(X\) and \(Y\) (see [4]). Notice that the join of two based spaces \((X, x_0)\) and \((Y, y_0)\) is naturally based at \([x_0, \frac{1}{2}, y_0]\).

The join construction \(\ast\) can be iterated and since it is associative we can write \(X_1 \ast \cdots \ast X_n\) the result of performing \(\ast\) \(n\)-times. This is the same as the quotient construction

\[\left[ X_1 \times \cdots \times X_n \times \Delta_{n-1} \right]_{/\sim}\]

where

\[(x_1, \ldots, x_i, \ldots, x_n; t_1, \ldots, t_i = 0, t_{i+1}, \ldots, t_{n-1} \sim (x_1, \ldots, x'_i, \ldots, x_n; t_1, \ldots, t_i = 0, \ldots, t_n)\]

for all \(x_i, x'_i \in X_i\). When \(X_1 = \cdots = X_n = X\) we write \(X^n\) for the space so obtained. It is handy to write elements of \(X^n\) formally as \([t_1 \times_{x_1}, t_2 \times_{x_2}, \ldots, t_n \times_{x_n}]\) with the understanding that \(\sum t_i = 1\) and \(t_i \in [0, 1]\) and that \(0x = 0x'\) for all \(x, x' \in X\). Note that when collapsing the subspace of points of the form \([t_1 x_0, \ldots, t_n x_0]\) from \(X^n\), we obtain the reduced join, a space homotopy equivalent to \(X^n\) (see [32]).

The cone on \(X\) is

\[CX := I \times X / (0, x) \sim (0, x') = X \ast \{v\}\]

viewed as the join of \(X\) with a disjoint vertex \(v\). We will write elements of \(CX\) as \([t, x]\) with \([0, x] = [0, x']\). The base \(X \hookrightarrow CX\) corresponds to when \(t = 1\) and the vertex corresponds to when \(t = 0\).

The unreduced suspension is defined to be \(\Sigma X = S^0 \ast X\) or equivalently is the quotient \(CX/X\) obtained from collapsing out the base of the cone. Iterating this suspension \(n\)-times yields \(\Sigma^n X = S^n \ast X\). The reduced suspension on the other hand is the smash product \(S^1 \wedge X\). Since a few of our statements involve homotopy type, we must distinguish between both constructions.

**Example 2.1.** The following relevant facts and alternative descriptions will be useful:

1. The join is the strict pushout \(X \ast Y \cong CX \times Y /_{X \times Y} CX\).
2. There is a homeomorphism \(X \ast Y \cong CX \times Y /_{\sim} \), where \((x, y) \sim (x', y')\) for all \(x, x' \in X\) and \(y, y' \in Y\).
3. \(X^{n-1} \ast\) embeds in \(X^n\) via the map \([t_1 x_1, \ldots, t_{n-1} x_{n-1}] \mapsto [t_1 x_1, \ldots, t_{n-1} x_{n-1}, 0 x_0]\). The image of this embedding includes in the cone \(x_0 \ast X^{n-1}\) and hence is contractible in \(X^n\).
4. If \(C\) is a closed subset of \(\mathbb{R}^m\) then \(\mathbb{R}^{m+n} - C\) is homotopy equivalent to the join \(S^{n-1} \ast (\mathbb{R}^m - C)\).

**Further Notation.**

We write \(X^{(n)} := X \wedge \cdots \wedge X\) the \(n\)-fold smash product obtained from \(X^n\) after collapsing out the subspace of all tuples containing the basepoint. This is a space with a “canonical” basepoint. We write the fat diagonal in \(X^n\) (similarly in \(X^{(n)}\)) as

\[\Delta_{fat} := \{[x_1, \ldots, x_n] \in X^n | x_i = x_j \text{ for some } i \neq j\}\]

A based action of a group \(G\) on a pointed space \(Y\) is an action that fixes the basepoint. The action is based free if it is free away from that basepoint. If \(X\) admits a based right action of \(G\) and \(Y\) a based left action, then we can take the diagonal quotients \(X \times_G Y\), \(Y \vee_G Y = (X/G) \vee (Y/G)\) and \(X \wedge_G Y\) (the quotient of the previous two). The prototypical example of a based action we consider in this paper is that of the symmetric group \(G = S_n\) acting on the \(n\)-fold smash \(Y = X^{(n)}\) by permuting factors (the basepoint is the canonical basepoint).
The “homotopy colimit or pushout” of two maps \( f: Y \longrightarrow X, g: Y \longrightarrow Z \) is the double mapping cylinder
\[
X \sqcup (Y \times [0,1]) \sqcup Z / \sim, \quad (y, 0) \sim f(y), (y, 1) \sim g(y), \quad \forall y \in Y
\]
This is written \( \text{hocolim}(X \leftarrow f Y \longrightarrow g Z) \) so that for instance, Example 2.1 (1), becomes
\[
X * Y = \text{hocolim}(X \leftarrow X \times Y \rightarrow Y)
\]

3. **Symmetric Joins of Spaces**

The symmetric group on \( n \)-letters \( \mathfrak{S}_n \) acts on \( X^n \) by permuting factors
\[
\sigma[t_1x_1, \ldots, t_nx_n] = [t_{\sigma(1)}x_{\sigma(1)}, \ldots, t_{\sigma(n)}x_{\sigma(n)}], \quad \sigma \in \mathfrak{S}_n
\]
The quotient under this action is the \( n \)-th symmetric join
\[
\text{Sym}^n(X) := X^n / \mathfrak{S}_n. \quad \text{Alternatively and based on [14] we can write}
\]
\[
\text{Sym}^n(X) := \prod_{k=1}^{n} \Delta_{k-1} \times_{\mathfrak{S}_k} X^k / \sim
\]
where \( \mathfrak{S}_k \) acts on \( X^k \) and on \( \Delta_{k-1} \subset \mathbb{R}^k \) by permuting factors, and \( \times_{\mathfrak{S}_k} \) means taking the quotient with respect to the diagonal \( \mathfrak{S}_k \) action. The equivalence relation \( \sim \) indicates further identifications of the form
\[
\text{(i) } [x_1, \ldots, x_k; t_1, \ldots, t_k] \sim [\hat{x}_1, \ldots, \hat{x}_k; t_1, \ldots, t_k]
\]
whenever \( t_i = 0 \), here \( \hat{ } \) means deletion. Note that these identifications are made along the boundary of the simplex \( \Delta_{k-1} \).

**Notation:** Elements of \( \text{SP}^nX \) will be written as *formal abelian* sums \( \sum x_i = x_1 + \cdots + x_n \) and this represents the orbit of \( (x_1, \ldots, x_n) \in X^n \) under the \( \mathfrak{S}_n \) action. Similarly elements \( \zeta \) in \( \text{Sym}^n(X) \) are written in the form \( \sum t_i x_i, \sum t_i = 1 \), with the understanding that if one of the \( t_i \)'s is 0, then that entry is suppressed from the sum. This writing is unique if none of the \( t_i \)'s is zero.

**Remark 3.1.** By construction we have an inclusion \( \text{Sym}^{n-1}X \hookrightarrow \text{Sym}^nX \). This subspace is contractible in \( \text{Sym}^nX \). If \( x_0 \) is a basepoint in \( X \), it is fairly transparent how to write the contraction
\[
I \times \text{Sym}^{n-1}(X) \longrightarrow \text{Sym}^n(X)
\]
\[
(t, \sum t_i x_i) \longmapsto tx_0 + \sum (1-t) t_i x_i
\]
In particular the infinite symmetric join is weakly contractible (i.e. trivial homotopy groups).

**Example 3.2.** There is an embedding \( \text{Sym}^n(X) \hookrightarrow \text{SP}^n(CX) \) which sends \( \sum t_i x_i \) to \( \sum [t_i, x_i] \). This map is well-defined because the vertex of the cone is at \( t = 0 \). The space \( \text{SP}^n(CX) \) is of course contractible since \( CX \) is. In the case of the circle \( X = S^1 \) and \( C S^1 = D^2 \), we obtain an explicit embedding of \( \text{Sym}^n(S^1) \) into Euclidean space \( \text{SP}^n(D^2) \cong D^{2n} \).

It becomes now an interesting problem to determine the homotopy type of symmetric joins of some standard spaces. An early cute characterization was given in [14] for the “symmetric square” of spheres.

**Lemma 3.3.** [14] There is a homeomorphism \( \text{Sym}^2(S^n) = \Sigma^{n+1} \mathbb{R}P^n \).

**Proof.** This proof is so short and clever that we reproduce it. Let \( D = D^{n+1} \) be the unit \( (n+1) \)-disc and identify it with \( D^n = \text{CS}^{n-1} \) with vertex at the origin. By inserting \( D^n = \text{CS}^{n-1} = S^{n-1} \times I / \sim \) in the definition of the join [14], we can rewrite \( S^{n-1} \times X \) as \( D^n \times X \sim (x, y), \quad x \in S^{n-1} \) (that is \( \theta \times X \sim \theta \) for \( \theta \in \partial D^n \)).

Write \( (S^n)^2 \) as \( S^n \times D^{n+1} \cup D^{n+1} \times S^n \) as in (Example 2.1 (1)). To \( (x, y) \in D \times \mathbb{R}P^n \) draw a line through \( x \) parallel to \( y \). This meets \( S^n = \partial D \) in points \( z, z' \). Choose \( z \) to be closest to \( x \), and choose \( w \) (between \( x \) and \( z' \)) such that \( x \) is the midpoint of \( z \) and \( w \). We define
\[
f: D \times \mathbb{R}P^n \longrightarrow \text{Sym}^2(S^n), \quad (x, y) \mapsto [(z, w)]
\]
If \( z \) and \( z' \) are equidistant to \( x \), \( f(x, y) = [(z, z')] \) and there is no ambiguity so our map is well-defined. Notice that if \( x \in S^n \), then \( z = w = x \) (is independent of \( y \)). We can then factorize the map \( f \) through the quotient \( D \times \mathbb{R}P^n/(x, y) \sim (x, y'), \forall x \in \partial D \). This quotient is another description of \( \Sigma^n \ast \mathbb{R}P^n = \Sigma^{n+1} \mathbb{R}P^n \) (see Example 2.1 (2)). The map so obtained \( D \times \mathbb{R}P^n/\sim \rightarrow \Sigma^{n+2}(S^n) \) is continuous and bijective by construction, hence is bicontinuous (being closed). □

Note in particular that \( \Sigma^{n+2}(S^1) \cong S^3 \) which is a special case of corollary 4.3. For more general spaces \( X \) one has the following global description of \( \Sigma^{n+2}(X) \).

**Lemma 3.4.** There is a homotopy pushout \[ \Sigma^{n+2}X = \text{hocolim}(X \leftarrow X^2 \xrightarrow{\pi} SP^2X) \]
where \( p_2 \) is the projection onto the second factor, and \( \pi \) is the \( \mathbb{Z}_2 \)-quotient map.

*Proof.* An element of \( \Sigma^{n+2}(X) \) is written as \( t_1x + t_2y \), \( t_1 + t_2 = 1 \) with the identification \( 0x + 1y = y \). By using the order on the \( t_i \)'s in \( I \), we can write \[ \Sigma^{n+2}(X) = \{(t_1, t_2, x_1, x_2) \mid t_1 \leq t_2, t_1 + t_2 = 1\}/\sim \]
where \( I = \{0 \leq t_1 \leq t_2 \leq 1, t_1 + t_2 = 1\} \) is a copy of the one-simplex, and the identification \( \sim \) is such that \((0, 1, x, y) \sim y \) and \((\frac{1}{2}, \frac{1}{2}, x, y) \sim (\frac{1}{2}, x, \frac{1}{2}, y) \). But \((0, 1)\) and \((\frac{1}{2}, \frac{1}{2})\) are precisely the faces or endpoints of \( I \) and hence the claim. □

Pictorially this double mapping cylinder can be depicted as in the figure

\[
\begin{array}{ccc}
X & \xrightarrow{X \times X} & SP^2X \\
\downarrow & & \downarrow \\
X & \xrightarrow{X \times X \times X} & X \times X & \xrightarrow{X \times X} & SP^2X \\
\downarrow & & \downarrow & & \downarrow \\
SP^2X \times X & \xrightarrow{X \times X \times X} & X \times X & \xrightarrow{X \times X} & SP^2X
\end{array}
\]

As is clear it is possible to give a multiple pushout description of \( \Sigma^{n+2}(X) \) for \( n \geq 3 \) as in lemma 3.4. For the case \( n = 3 \) this proceeds as follows. Write elements as \((t_1, t_2, t_3; x_1, x_2, x_3)\) with \( t_1 + t_2 + t_3 = 1 \), \( t_1 \leq t_2 \leq t_3 \). The faces of this 2-simplex are: \( t_1 = 0 \) (and hence \( t_2 + t_3 = 1 \) forming the “right edge”), \( t_1 = t_2 \) and \( \frac{1}{3} \leq t_3 \leq 1 \) (the “left edge”), and thirdly \( t_2 = t_3 \) and \( 0 \leq t_1 \leq \frac{1}{3} \) (bottom edge). The top vertex corresponds to when \( t_1 = t_2 = 0, t_3 = 1 \), the bottom left as when \( t_1 = t_2 = t_3 = \frac{1}{3} \) and the bottom right vertex to when \( t_1 = 0, t_2 = t_3 = \frac{1}{2} \). The multiple homotopy pushout diagram representing \( \Sigma^{n+2}(X) \) is depicted below.

*Figure 1:* the colimit diagram for \( \Sigma^{n+2}(X) \)

The construction for \( n > 3 \) is predictable. Such a description is what we need to establish the simple connectivity of symmetric joins which is a first step towards Theorem 1.2.

**Lemma 3.5.** The following homotopy pushout is simply connected where the gluing on the left is via projection \( p_1 \) on the first coordinate, and on the right via the concatenation product \( \mu \)
\[ SP^iX \xrightarrow{SP^iX \times SP^jX} SP^{i+j}X \quad i, j \geq 1 \]
Proposition 4.1. A description is given next.

Then post composing and their opposite edges labeled \( n^* \). The pushout diagram and hence the pushout is also simply connected. In the general case one can use another useful property of symmetric products which is that \( \pi_1(\text{Sym}^n(X)) \) is simply connected for each \( \pi \) hypothesis. Since Sym \( n \) deformation retracts onto the basepoint inclusion induces a map \( \pi_1(X) \rightarrow \pi_1(\text{Sym}^n(X)) \) which is surjective for \( i \geq 2 \). It follows from there that any element of \( \pi_1(\text{Sym}^{i+j}(X)) \) is in the image of an element from \( \pi_1(\text{Sym}^n(X)) \) under the concatenation product \( \mu : \text{Sym}^n(X) \times \text{Sym}^n(X) \rightarrow \text{Sym}^{i+j}(X) \). But this element projects trivially in \( \pi_1(\text{Sym}^n(X)) \) under \( p_{1*} \) so that its image in the pushout \( G \) in trivial by the VK relations. On the other hand, these same relations show that any generator from \( \pi_1(\text{Sym}^n(X)) \) is identified to a generator coming from \( \pi_1(\text{Sym}^{i+j}(X)) \) so that its image in \( G \) is also trivial. This proves our claim.

Theorem 3.6. \( \text{Sym}^n(X) \) is simply connected for \( n \geq 2 \).

Proof. According to lemmas 3.4 and 3.5 \( \text{Sym}^2(X) \) is simply connected. We can then proceed by induction. Consider \( \text{Sym}^3(X) \) as in figure 1. Label the vertices \( V_i := X, V_2 = \text{Sym}^2(X), V_3 = \text{Sym}^3(X) \) and their opposite edges \( E_1, E_2, E_3 \), so for instance \( E_1 = X \times \text{Sym}^2(X) \). Let \( U_i = \text{Sym}^3(X) - V_i \). The \( U_i \)'s are open subspaces and each \( U_i \) deformation retracts onto \( E_i \) (this is proved further down). But each \( E_i \) is a space obtained by iterative pushout and is simply connected by lemma 3.5 and our inductive hypothesis. Since \( \text{Sym}^3(X) = U_1 \cup U_2 \cup U_3 \) is covered by three open simply connected subsets with non-empty intersection, Van-Kampen theorem implies that \( \text{Sym}^3(X) \) is simply connected.

The proof for \( n = 3 \) extends with little change to larger \( n \). The pushout diagram for \( \text{Sym}^n(X) \) is a labeled \( n - 1 \)-simplex with vertices \( V_i = \text{Sym}^i(X), 1 \leq i \leq n \). Opposite to each \( V_i \) is a face representing a pushout \( E_i \). The open set \( U_i = \text{Sym}^3(X) - V_i \) deformation retracts onto \( E_i \) and so it is simply connected by induction. Since the \( U_i \) cover \( \text{Sym}^n(X) \), the claim follows by Van-Kampen.

It remains to establish that indeed \( U_i \) deformation retracts onto \( E_i \). Figure 1 suggests an argument of proof. Pick a vertex \( v_i \) and write \( \Delta_n = \Delta_{n-1} \ast \{v_i\} \) a join, then there is an obvious geometric deformation of \( \Delta_n - \{v_i\} \) onto \( \Delta_{n-1} \) given by \( r_i : (\Delta_n - \{v_i\}) \times I \rightarrow \Delta_n - \{v_i\} \),

\[
 r_i(sx + (1-s)v_i, t) = (1-t)(sx + (1-s)v_i) + tx
\]

with \( x \in \Delta_{n-1}, sx + (1-s)v_i \in \Delta_{n-1} \ast v_i \) and \( t \in I \). This homotopy is well-defined since \( s \neq 0 \) (and hence \( x \) can be chosen) and it retracts the complement of the vertex \( v_i \) onto its opposite edge leaving the other faces invariant. Let \( U_i \) be the image of

\[
 \phi_i : (\Delta_n - \{v_i\}) \times X^{n+1} \rightarrow B_{n+1}(X)
\]

Then post composing \( \phi_i \) with the map

\[
 (\Delta_n - \{v_i\}) \times X^{n+1} \times I \rightarrow (\Delta_n - \{v_i\}) \times X^{n+1}
\]

which is \( r_i \) on \( \Delta_n - \{0\} \times I \) and the identity on \( X^{n+1} \), gives the desired retraction of \( U_i \) onto \( E_i \).

4. Symmetric Joins and Symmetric Products

The multiple pushout description of \( \text{Sym}^n(X) \), although useful for the fundamental group calculation, doesn't yield itself to an easy homological analysis. A much finer and surprisingly simple description is given next.

Proposition 4.1. For spaces \( X_1, \ldots, X_n \), there is a homeomorphism

\[
 C(X_1 \ast X_2 \ast \cdots \ast X_n) \cong CX_1 \times \cdots \times CX_n
\]

In particular \( C(X^n) \cong (CX)^n \). This homeomorphism is \( \Sigma \)-equivariant so that

\[
 C(\text{Sym}^n(X)) \cong \text{SP}^n(CX) \quad \text{and} \quad \Sigma \text{Sym}^n(X) \cong \text{SP}^n(\Sigma X)
\]
Proof. Start by writing elements of $CX$ as $[t, x]$ with $[0, x] = [0, x']$. The base of the cone corresponds to $X \rightarrow CX$, $x \mapsto [1, x]$. Let’s view $X_1 \ast \cdots \ast X_n$ as the quotient of $X_1 \times \cdots \times X_n \times \Delta_{n-1}$ as in (1), consider the map

$$
\Phi: C(X_1 \ast X_2 \ast \cdots \ast X_n) \longrightarrow \prod_{j=1}^n CX_j
$$

$$
[s, (x_1, \ldots, x_n; t_1, \ldots, t_n)] \quad \mapsto \quad ([\frac{t_1}{t}, x_1], \ldots, [\frac{t_n}{t}, x_n])
$$

where $t = \max_j \{t_j\}$. Notice that $t \neq 0$ since $\sum t_i = 1$ and $\Phi$ is well-defined. This map is a homeomorphism with inverse

$$
\Phi^{-1}(s_1, x_1, \ldots, s_n, x_n) = \begin{cases} 
\{\max_j \{s_j\}, (x_1, \ldots, x_n, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1})\}, & \text{if } \max_j \{s_j\} > 0 \\
\text{,} & \text{if } \max_j \{s_j\} = 0
\end{cases}
$$

where we have written $s = s_1 + \cdots + s_n$. This map is easily seen to be continuous.

Note that $\Phi$ carries the base of the cone on $X_1 \ast X_2 \ast \cdots \ast X_n$ homeomorphically onto the subspace

$$
D_n := \{([t_1, x_1], \ldots, [t_n, x_n] \mid t_j = 1 \text{ for at least one } j\} \subset \prod_{j=1}^n CX_i
$$

This is saying that there is a homomorphism of pairs

$$
\Phi: (C(X_1 \ast X_2 \ast \cdots \ast X_n), X_1 \ast X_2 \ast \cdots \ast X_n) \cong (\prod_{j} CX_j, D_n)
$$

and so collapsing out subspaces we obtain a homeomorphism

$$
\Phi: \Sigma(X_1 \ast X_2 \ast \cdots \ast X_n) \cong \Sigma X_1 \wedge \cdots \wedge \Sigma X_n
$$

where $\Sigma X = CX/X$. When all the $X_j$’s equal to $X$, the map $\Phi$ is evidently $\mathcal{G}_n$-equivariant and there is an induced homeomorphism $\Sigma(Sym^n X) \cong \Sigma^{P^n} (\Sigma X)$. \hfill $\square$

We give an alternate proof of the homotopy equivalence $\Sigma^{P^n} (\Sigma X) \simeq \Sigma Sym^n (X)$ in Corollary 6.2.

**Remark 4.2.** The earliest occurrence of the homeomorphism $\Sigma(X \ast X) \cong \Sigma X \wedge \Sigma X$ that we are aware of is in [17]. The above proof is similar to Cohen’s argument. The homeomorphism $C(X) \times C(Y) \cong C(X \ast Y)$ is quite standard and can be seen geometrically by thinking of $CX \times CY$ as a square with center the vertex of a cone on the boundary of this square $CX \times CY$.

As an immediate corollary of proposition 4.1 and of (1) we obtain the identification $\Sigma^{P^n} (S^n) = \Sigma^{n+1} RP^{n-1}$. For a generalization see Proposition 9.1. Another immediate corollary is the following.

**Corollary 4.3.** $Sym^n (S^1) \cong S^{2n-1}$.

**Proof.** The cone $CS^1$ is homeomorphic to the two dimensional closed disc $D^2$ so that $\Sigma^{P^n} (D^2) \cong C(Sym^n (S^1))$. It is well-known that $\Sigma^n D^2 \cong D^{2n}$ with boundary $\partial \Sigma^n D^2 = \Sigma^n D = \Sigma^n (int(D))$ the sphere $S^{2n-1} = \partial D^{2n}$ as is checked in [17]. This boundary sphere must correspond to unordered tuples with one entry in $\partial D$. Under the homeomorphism constructed in Proposition 4.1 this is precisely the base of the cone on $Sym^n(S^1)$ and the claim follows. \hfill $\square$

Another fairly useful corollary is the calculation of the Euler characteristic of symmetric joins (and hence of barycenter spaces in light of Theorem 4.1).

**Corollary 4.4.** If $\chi(X)$ is the Euler characteristic of a connected space $X$, then

$$
\chi(Sym^n(X)) = 1 - \frac{1}{n!}(1 - \chi(X)) \cdots (n - \chi(X))
$$
Proof. We recall that \( \chi(\Sigma X) = 2 - \chi(X) \), and that

(i) \( \chi(\text{SP}^n X) = \frac{1}{n!}(\chi(X) + n - 1) \cdots (\chi(X) + 1)\chi(X) \). This is extracted from the formula of MacDonald \cite{MacDonald} that the Euler characteristics for all \( n \) put together form the series

\[
\sum_{n \geq 0} \chi(\text{SP}^n X) t^n = \left( \frac{1}{1-t} \right)^{\chi(X)}
\]

so that \( \chi(\text{SP}^n X) \) is the number of non-negative solutions of \( j_1 + j_2 + \cdots + j_\chi(X) = n \); that is \( \chi(X) + n - 1 \). (ii) \( \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \). In particular if \( A \leftrightarrow X \) is a cofibration, then the quotient \( X/A \) is identified with the mapping cone \( X \cup_A CA \), and hence \( \chi(X/A) = \chi(X) + 1 - \chi(A) \). Applying this formula to \( \text{SP}^n X = \text{SP}^n X/\text{SP}^{n-1} X \), we see that

\[
\chi(\text{SP}^n X) = 1 + \left( \frac{\chi(X) + n - 1}{\chi(X) - 1} \right) - \left( \frac{\chi(X) + n - 2}{\chi(X) - 1} \right)
\]

From this we obtain that

\[
\chi(\text{Sym}^n(X)) = 2 - \chi(\text{SP}^n \Sigma X) = 1 - \left( \frac{n + 1 - \chi(X)}{1 - \chi(X)} \right) + \left( \frac{n - \chi(X)}{1 - \chi(X)} \right)
\]

and this gives precisely our formula. \(\square\)

Remark 4.5. It is not surprising that \( \text{SP}^n (\Sigma X) \cong \Sigma \text{Sym}^n(X) \) is again a suspension. Indeed let \( Y \) be a co-H space with comultiplication \( \nabla : Y \longrightarrow Y \vee Y \) (by definition of co-H space, this means that the composite \( Y \longrightarrow Y \vee Y \) is homotopic to the diagonal map). Then we claim that \( \text{SP}^n(Y) \) is also co-H. To see this set \( \text{SP}^0(Y) = S^0 \). It is not hard to check that

\[
\text{SP}^n(X \vee Y) = \bigvee_{r+s=n} \text{SP}^r(X) \wedge \text{SP}^s(Y)
\]

The co-H space structure on \( \text{SP}^n(Y) \) is obtained from the composite

\[
\text{SP}^n(Y) \longrightarrow \text{SP}^n(Y \vee Y) = \bigvee_{0 \leq i \leq n} \text{SP}^i(Y) \wedge \text{SP}^{n-i}(Y) \longrightarrow \text{SP}^n Y \vee \text{SP}^n Y
\]

with the second map being projection on the wedge summands corresponding to when \( i = n \) and \( i = 0 \).

5. Barycenter Spaces

We now turn to the spaces of most interest in this paper. The \( n \)-th barycenter space \( B_n(X) \) is defined as a quotient of the symmetric join \( \text{Sym}^n(X) \) under one further type of identifications of the form

\[
(\text{ii}) \quad t_1 x_1 + t_2 x_1 + t_3 x_3 + \cdots + t_n x_n \sim (t_1 + t_2) x_1 + t_3 x_3 + \cdots + t_n x_n
\]

More precisely we can write

\[
B_n(X) := \coprod_{k=1}^{n} \Delta_{k-1} \times \phi_k X^k / \approx
\]

where \( \approx \) consists of the identifications (i) as in (6) and (ii) as in (8).

Here too, it is useful to think of a point of \( B_n(X) \) as an abelian sum \( t_1 x_1 + \cdots + t_n x_n \), with the topology that when \( t_i = 0 \) the entry \( 0 \cdot x_i \) can be discarded from the sum, and when \( x_i \) moves in coincidence with \( x_j \), the coefficients add up. As before, there is an embedding \( B_{n-1}(X) \longrightarrow B_n(X) \) with contractible image.

Remark 5.1. If \( X \) is embedded in some Euclidean space \( \mathbb{R}^n \), then an element \( \sum t_i x_i, x_i \neq x_j \) can be viewed as the “barycenter” of the \( x_i \)’s with corresponding weights \( t_i \) and the resulting space of all weighted barycenters is like a convex hull in \( \mathbb{R}^n \). Ensuring that geometrically these weighted barycenters are distinct for different choices of finite tuples \( (x_1, \ldots, x_n) \), is requiring that \( X \) is embedded in “general
position” in $\mathbb{R}^N$ for large enough $N$. For example $B_2(\mathbb{R}^1)$ can be identified with the convex hull of the Veronese embedding $\mathbb{R}^1 \to \mathbb{R}^N, t \mapsto (t, t^2, t^3, \ldots, t^N)$ for $N \geq 3$.

**Example 5.2.** (Bunch of points) Let $n$ be a finite set which we can assume embedded in Euclidean space as the vertices of an $n - 1$-simplex $\Delta$. Then $B_n(n)$ is the union of all $(k - 1)$-faces of $\Delta$. This is a bouquet of spheres $\bigvee S^{k-1}$. For example $B_{n-1}(n) = S^{n-2}$.

In this section we give as simple as possible of a description of the homotopy type of $B_n(X)$ and observe that it is equivalent to $\text{Sym}^n(X)$. Recall again that $X^{(n)}$ is the quotient of $X^n$ by the fat wedge of all tuples with at least one entry at $x_0$.

**Theorem 5.3.** For a based connected simplicial complex $X$, there is a homotopy equivalence

$$B_n(X) \simeq S^{n-1} \wedge_{\mathfrak{S}_n} X^{(n)}$$

with $\mathfrak{S}_n$ acting on $S^{n-1} \subset \mathbb{R}^n$ by permuting coordinates, fixing the point $\frac{1}{n}(1, \ldots, 1)$.

**Proof.** With $B_n(X)$ as in (9), and $x_0 \in X$ the basepoint, consider the subspace

$$W_n := \{ \sum t_i x_i \in B_n(X), x_i = x_0 \text{ for some } i \}$$

Notice that $B_n-1X$ includes in $W_n$ as the subspace of all points of the form $\sum_{i=1}^{n-1} t_i x_i + 0x_0$. We claim that $W_n$ is contractible. We construct a contraction as follows. Write an element of $W_n$ in the form $\zeta = t_1y_1 + \cdots + t_ky_k + s_0x_0$, for some $k < n$, with $y_i \neq x_0$. We have a well-defined homotopy $F: W_n \times I \longrightarrow W_n$

$$(t_1y_1 + \cdots + t_ky_k + s_0x_0, t) \mapsto (1-t)t_1y_1 + \cdots + (1-t)t_ky_k + (s_0 + t(1 - s_0))x_0$$

This is well-defined because the sum of coefficients is always 1, and because it is continuous (we have to check continuity when one of the $y_i$’s approaches $x_0$ but this is an inspection). This homotopy is evidently a contraction with $F(0, -) = \text{id}$ and $F(1, -)$ the constant function at $x_0$. Since $W_n$ is contractible, we can collapse it out without changing the homotopy type.

Now $\approx$ identifies as in (i) the subspace of elements $\sum t_ix_i$ for which $(t_1, \ldots, t_n)$ lie in the boundary of $\Delta_{n-1}$ (i.e. for which $t_i = 0$ for some $i$) with a subspace of $W_n \subset B_n(X)$. Similarly configurations containing the basepoint $x_0$ are in $W_n$. What we obtain by identifying $W_n$ to point is then the quotient

$$B_n(X) \simeq \left( [\Delta \times \mathfrak{S}_n X^{(1)}]/[\partial \Delta \times \mathfrak{S}_n X^{(n)}] \right) \approx S^{n-1} \wedge_{\mathfrak{S}_n} X^{(n)}/\approx$$

and $\approx$ is the identification (8). Here $X \times Y := X \times Y \times \ast$ is the “half-smash” product. The smash product is taken with respect to the canonical basepoints in both $X^{(n)}$ and $S^{n-1} := \Delta/\partial \Delta$. The symmetric group $\mathfrak{S}_n$ acts on $S^{n-1} = \Delta/\partial \Delta$ by permuting the $t$’s of the simplex. Now note that (8) identifies points in the fat diagonal $\Delta_{fat}$; that is the subset of the form $\sum t_ix_i$ with $x_i = x_j$ for some $i \neq j$, to points in $W_n$ which we have again collapsed out. We can then write

$$B_n(X) \simeq (S^{n-1} \wedge_{\mathfrak{S}_n} X^{(n)})/(S^{n-1} \wedge_{\mathfrak{S}_n} \Delta_{fat})$$

$$= S^{n-1} \wedge_{\mathfrak{S}_n} (X^{(n)}/\Delta_{fat})$$

This last equivalence follows from the fact that $S^{n-1} \wedge X^{(n)}/S^{n-1} \wedge \Delta_{fat} = S^{n-1} \wedge (X^{(n)}/\Delta_{fat})$ and that this identification passes to the $\mathfrak{S}_n$-quotients.

It then remains to see that (12) is the same as the expression in the theorem. This would follow immediately if we knew that $S^{n-1} \wedge_{\mathfrak{S}_n} \Delta_{fat}$ were contractible. But this is precisely the content of lemma 5.3 next. To summarize then, we have shown the string of equivalences

$$B_n(X) \simeq B_n(X)/W_n \simeq S^{n-1} \wedge_{\mathfrak{S}_n} (X^{(n)}/\Delta_{fat}) \simeq S^{n-1} \wedge_{\mathfrak{S}_n} X^{(n)}$$

Finally to get the version of our theorem, we need replace this sphere and this action by $S^{n-1} \subset \mathbb{R}^n$ and the permutation action on coordinates, which is possible according to Lemma 6.3. $\square$
**Corollary 5.4.** There is a homotopy equivalence $\mathcal{B}_n(X) \simeq \text{Sym}^n X$.

We will derive this important corollary in two different ways.

**Proof.** (of corollary [5.4]) The first approach relies on a theorem of R. Vogt. Proposition 1.9 of [32] implies the existence of a $\mathfrak{S}_n$-equivariant equivalence (see remark 6.4)

\[ X^\ast \simeq_{\mathfrak{S}_n} S^{n-1} \wedge X^{(n)} \]  

(13)

Passing to the quotient under the symmetric group action gives that $\text{Sym}^n(X) \simeq S^{n-1} \wedge\mathfrak{S}_n X^{(n)}$ and hence the desired result. Here $X^\ast$ is the reduced join, $\frac{1}{\sqrt{n}}(1, \ldots, 1)$ and $(x_0, \ldots, x_0)$ are the basepoints in $S^{n-1} \subset \mathbb{R}^n$ and $X^n$ respectively.

The proof of the equivalence (13) in [32] takes some space and uses techniques different from ours. The problem is that it isn’t anymore continuous. The contraction we have written down for $\mathfrak{S}_n$, respectively.

To go around this, we go back to the quotient construction in (6) and make an extra identification as follows. We keep the usual identification (i) along the boundaries of the $\Delta_i$ but make the additional identification at the basepoint $x_0$ of $X$

\[ tx_0 + sx_0 + \ldots \sim (t + s)x_0 + \ldots \]  

(14)

This is of course the identification used in defining $\mathcal{B}_n$ but we only demand it holds true at $x_0$. Let’s refer to this new quotient by $T_n(X)$ with projection

\[ \pi : \text{Sym}^n X \rightarrow T_n(X) = \prod_{k=1}^{n} \Delta_{k-1} \times_{\mathfrak{S}_k} X^k / \sim \]

Suppose we show that $\pi$ is a homotopy equivalence. Then we can instead work with $T_n(X)$ and define the subspace $\pi(V_n)$ which again consists of all configurations in $T_n(X)$ where $x_0$ appears. We could then show that $\pi(V_n)$ is contractible since now the contraction we’ve written down in (10) works; i.e. it is continuous and this is the reason for introducing the relation (14).

What is left to do then is show that $\pi$ is an equivalence. To that end we use a version of the Begle-Vietoris theorem which states that a map between locally compact, locally contractible spaces such that the inverse image of any point is contractible, is necessarily a homotopy equivalence [27]. Now the preimage of a configuration $\sum t_iy_i = t_1y_1 + \cdots + t_ky_k$ is the same configuration again if no $x_0$ figures in this sum. In general, the inverse image of $\zeta = t_1y_1 + \cdots + t_ky_k + sx_0$, with $\sum t_i + s = 1$ and $y_i \neq x_0$, is

\[ \pi^{-1}(\zeta) = \left\{ \sum_{i=1}^{k} t_iy_i + s_1x_0 + \cdots + s_{n-k}x_0 \quad , \quad s_1 + \cdots + s_{n-k} = s = 1 - \sum t_i \right\} \]
This is a copy of a simplex hence contractible. Therefore
\[
\text{Sym}^n(X) \simeq T_n(X) \simeq T_n(X)/\pi(V_n) \simeq S^{n-1} \wedge G_n X^{(n)}
\]
and the proof is complete. \hfill \Box

Finally the following important lemma was needed in the proof of Theorem 5.3 above.

**Lemma 5.5.** Let \( \Delta_{fat} \) be the fat diagonal in \( X^{(n)} \). Then \( S^{n-1} \wedge G_n \Delta_{fat} \) is contractible.

**Proof.** The justification for this lemma is in Arone-Dwyer [1], proposition 7.11. We recall their statement. Given a simplicial \( G \)-space \( Y \), let \( Iso(Y) \) denote the collection of all subgroups of \( G \) which appear as isotropy of simplices of \( Y \). Let \( G = G_n \) and denote by \( P \) the subgroups of \( G_n \) of the form \( G_{n_1} \times \cdots \times G_{n_j} \) with \( j > 1 \) and \( \sum n_i = n \) with \( n_i > 1 \) for at least one \( i \). Let \( S^k \) be a sphere such that \( S^k/K \) is contractible for all \( K \in P \). This is the case for example when \( k = n \), \( S^n = (S^1)^{(n)} \) and \( G_n \) acts by permutation as in the case of \([1]\). Then by an elegant application of equivariant cell attachments, if \( Y \) is a pointed \( G_n \)-simplicial space with \( Iso(Y) \subset P \cup \{ G_n \} \), the quotient \( S^n \wedge G_n Y \) is contractible \([1]\). We apply this to \( Y = \Delta_{fat} \subset X^{(n)} \) and to the sphere \( \Delta/\partial \Delta \) with its permutation action of \( G_n \). First we can verify that \( \Delta/\partial \Delta \) is \( G_n \)-equivariantly homeomorphic to the unit sphere \( S^{n-1} \subset \mathbb{R}^n \) with its standard permutation of coordinates action (see Lemma 5.3). This permutation action has fundamental domain a simplex. In fact the quotient \( S^{n-1}/K \) is contractible for any \( \{1\} \neq K \subset G_n \). This is readily verified if we think of the permutation switching the \( i \)-th and \( j \)-th coordinate of the sphere as the reflection with respect to the hyperplane \( x_i = x_j \) in \( \mathbb{R}^n \). Next and given a simplicial structure on \( X \), there is a simplicial structure on \( X^n \) with vertices \( \varpi \) given as \( n \)-tuples of vertices of \( X \). The fat diagonal is sub-simplicial with vertices having at least two equal entries. Moreover the action of \( G_n \) is simplicial and permutes the entries of any \( \varpi \) (see \([10]\)). This simplicial action on \( \Delta_{fat} \) satisfies the hypothesis of proposition 7.11 in \([1]\) and lemma 5.5 follows. \hfill \Box

6. Some Equivariant Deformations

We clarify and complement some claims we made in the proofs above.

**Lemma 6.1.** Let \((S^1)^{(n)} \simeq S^n \) be the \( n \)-fold smash product, with \( G_n \) acting by permuting factors, and let \( \Sigma S^{n-1} \) be the unreduced suspension with \( G_n \) acting on the suspension coordinate trivially and on \( S^{n-1} \subset \mathbb{R}^n \) by permuting coordinates fixing the point \( p := (1/n, \ldots, 1/n) \). Then both spheres are \( G_n \)-equivariantly equivalent.

**Proof.** Represent \( S^1 \) as \( I = [0,1] \) with \( 0 \simeq 1 \), so that \( (S^1)^{(n)} \) is a solid cube with all sides identified to a point. We can think of this cube as inscribed in the unit sphere which via radial stretch is \( G_n \)-equivariantly identified with the unit disk \( D \) in \( \mathbb{R}^n \) and the boundary of the cube being identified to this unit sphere. The action of \( G_n \) on \( D \) is by reflections with respect to the hyperplanes \( x_i = x_j \). On the other hand, think of the unreduced suspension \( \Sigma S^{n-1} \) as \( I \times S^{n-1}/\sim \) with identifications at \( t = 0,1 \). Let \( G_n \) act on \( I \) trivially and on \( S^{n-1} \subset \mathbb{R}^n \) by permuting coordinates. Then the map \( \Phi : I \times S^{n-1} \rightarrow D/S, (t,x) \mapsto tx \) factorizes through \( \Sigma S^{n-1} \) and it is a \( G_n \)-equivariant equivalence. \hfill \Box

**Corollary 6.2.** Let \( Y \) be a pointed left \( G_n \)-space, and suppose \( G_n \) acts on \( S^n = (S^1)^n \) by permuting factors. Then \( S^n \wedge G_n Y \simeq \Sigma (S^{n-1} \wedge G_n Y) \) and \( G_n \) acts on \( S^{n-1} \subset \mathbb{R}^n \) by permuting coordinates. In particular we recover the (weaker) equivalence
\[
\overline{S} \Sigma X \simeq \Sigma \text{Sym}^n(X)
\]

**Proof.** According to lemma 6.1 \( S^n \wedge G_n Y \) can be replaced by \((S^1 \wedge S^{n-1}) \wedge G_n Y \) where the symmetric group action on \( S^1 \) is trivial and on \( S^{n-1} \) is via reflections as above. The first claim follows. The second claim is obtained through the series of identifications
\[
\overline{S} \Sigma X = (\Sigma X)^{(n)} \simeq S^n \wedge G_n X^{(n)} \simeq S^1 \wedge (S^{n-1} \wedge G_n X^{(n)}) \simeq \Sigma \text{Sym}^n X
\]
$\mathfrak{S}_n$ acting on the smash product by acting diagonally with action on $S^{n-1} \subset \mathbb{R}^n$ as described. \qed

Next we have been using interchangeably the unit sphere in $\mathbb{R}^n$ and the sphere $\Delta/\partial \Delta$ with $\Delta = \Delta_{n-1}$ as in \cite{in}. This is possible because of the following elementary observation.

**Lemma 6.3.** The sphere $\Delta/\partial \Delta$ is $\mathfrak{S}_n$-equivariantly homeomorphic to the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

**Proof.** Let $H^+ \subset S^{n-1}$ be the “positive hemisphere” \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1, x_i \geq 0 \forall i\}. This subspace is invariant under the permutation action of $\mathfrak{S}_n$. The point is to see that $H^+$ is a $\mathfrak{S}_n$ deformation retract of the unit sphere $S^{n-1}$, with $S^{n-1}\{-p\}$ being mapped onto the interior of $H^+$, here $p = (1/\sqrt{n}, \ldots, 1/\sqrt{n})$. An inverse deformation retraction can be given for example by bringing closer to the point $-p$ the “end points” of the hemisphere- those with a single non trivial entry- along great circles running through those points, and going from $p$ and $-p$. Such an inverse retraction extends to $H^+$ in the obvious way (we get at each time a portion of the sphere containing $p$ and bounded by geodesics between the images of these endpoints). On the other hand the simplex $\Delta$ is $\mathfrak{S}_n$-equivariantly homeomorphic to $H^+$ via the map $(t_1, \ldots, t_n) \mapsto (t_1, \ldots, t_n) / \sqrt{\sum t_i^2}$. Composing these equivalences we obtain an equivariant homeomorphism $\Delta \rightarrow S^{n-1}$, sending the interior of $\Delta$ to $S^{n-1}\{-p\}$ and sending the boundary to $-p$. This proves the claim. \qed

**Remark 6.4.** (About the splitting theorem of Vogt). In \cite{split}, it is shown that there is $\mathfrak{S}_n$-splitting, natural in $X$ up to homotopy and under $\Sigma^{n-1}D_nX$,

$$\Sigma^{n-1}X^n \simeq \Sigma^{n-1}D_nX \vee X^n$$

where $D_nX \subset X^n$ is the fat wedge consisting of all tuples with basepoint in at least one coordinate. More precisely, both sides of the equation are homotopy equivalent through $\mathfrak{S}_n$-equivariant maps. The case $n = 2$ of this theorem is a special case of the very classical stable splitting of $\Sigma(X \times Y)$ as the bouquet $\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. To get the $\mathbb{Z}_2$-equivariance of the splitting when $X = Y$, the involution must act on both $X^2$ and $X \times X$ by permuting factors, but also act on the suspension coordinate via $t \mapsto 1 - t$. To see why this is so, we recall that the equivalence $\Sigma(X \times Y) \rightarrow \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ is the wedge of two maps: (i) the projection $\Sigma X \times Y \rightarrow \Sigma X \wedge Y$, and (ii) the composite of the pinch map (on the suspension) with the projections $\pi_1, \pi_2$ of $X \times Y$ onto the first and second factors as in

$$\Sigma(X \times Y) \longrightarrow \Sigma(X \times Y) \vee \Sigma(X \times Y) \xrightarrow{\Sigma(\pi_1 \vee \pi_2)} \Sigma X \vee \Sigma Y$$

The point is that in order for this map to commute with the switch map interchanging $X$ and $Y$, one needs to switch the suspension coordinate from $t$ to $1 - t$ as well.

7. **Connectivity**

The aim of this section is to prove Theorem \cite{conn}. All spaces $X$ are assumed again to be connected and based simplicial complexes. We first establish a few basic facts about relative Leray-Serre spectral sequences for “singular” maps; that is maps $E \rightarrow X$ which might fail to be fibrations over some closed subspace $A \subset X$. This situation occurs when for example we collapse out some fibers from the total space of a fibration, or if we have the quotient map of a group action which is free away from some fixed set $A$. Such a situation is ubiquitous in topology.

**Definition 7.1.** We say $(F, F_0) \longrightarrow (E, W) \xrightarrow{f} X$ is a fibration relative to $A \subset X$ if:

(i) $A$ is closed and $f$ is a fibration away from $A$; that is

$$(F, F_0) \longrightarrow (E - f^{-1}(A), W - f^{-1}(A) \cap W) \longrightarrow X - A$$

is a fibration (typically in the examples below a bundle). This means that $(F, E - f^{-1}(A), X - A)$ is a fibration of which $(F_0, W - f^{-1}(A) \cap W, X - A)$ is a subfibration.

(ii) There is an open neighborhood $A \subset V$ such that $A$ is a deformation retract of $V$ and the deformation retraction lifts upstairs; that is the pair $(f^{-1}(V), f^{-1}(V) \cap W)$ deformation retracts onto $(f^{-1}(A), f^{-1}(A) \cap W)$.
Lemma 7.2. Let \((F, F_0) \longrightarrow (E, W) \longrightarrow X\) be a fibration relative to \(A \subset X\), with \(X\) simply connected. Then there is a spectral sequence with

\[ E^2 = H_*(X, A; H_*(F, F_0)) \implies H_*(E, W \cup f^{-1}(A)) \]

Proof. We have a fibration \((F, F_0) \longrightarrow (E', W') \longrightarrow X - A\) where \(E' = E - f^{-1}(A)\) and \(W' = W - f^{-1}(A) \cap W\). By the hypothesis we can choose a neighborhood \(V\) of \(A\) so that both \(V\) and its closure \(\overline{V}\) deformation retract onto \(A\) while \(f^{-1}(V)\) and \(f^{-1}(\overline{V})\) deformation retract onto \(f^{-1}(A)\). Now \(\overline{V} - A\) is closed in \(X - A\) and according to [23] there is a spectral sequence

\[ E^2 = H_*(X - A, \overline{V} - A; H(F, F_0)) \implies H_*(E - f^{-1}(A), f^{-1}(\overline{V} - A) \cup W') \]

We can replace \(H_*(X - A, \overline{V} - A) \cong H_*(X, \overline{V})\) by excision and then \(H_*(X, \overline{V}) \cong H_*(X, A)\) since \(\overline{V}\) deformation retracts onto \(A\). On the other hand

\[
H_*(E - f^{-1}(A), f^{-1}(\overline{V} - A) \cup W') \cong H_*(E - f^{-1}(A), f^{-1}(\overline{V} - A) \cup W - f^{-1}(A) \cap W) \\
\cong H_*(E - f^{-1}(A), f^{-1}(\overline{V}) \cup W - f^{-1}(A)) \\
\cong H_*(E, f^{-1}(\overline{V}) \cup W) \\
\cong H_*(E, f^{-1}(A) \cup W)
\]

The one before last isomorphism is obtained by excision again (\(f^{-1}(A)\) being closed), and the last isomorphism follows upon comparing the long exact sequences for \((f^{-1}(\overline{V}) \cup W, W)\) and \((f^{-1}(A) \cup W, W)\), and using the fact that \((f^{-1}(\overline{V}), f^{-1}(\overline{V}) \cap W)\) deformation retracts onto \((f^{-1}(A), f^{-1}(A) \cap W)\).

\[ \square \]

Corollary 7.3. Let \(F \longrightarrow E \longrightarrow X\) be a fibration relative to \(A\) which admits a section \(s : X \to E\). We view \(X\) as a subspace of \(E\) through the section and assume it is simply connected. Then there is a spectral sequence converging to \(H_*(E, f^{-1}(A) \cup X)\) with \(E^2\)-term \(H_*(X, A; H_*(F'))\). The proof is a direct application of lemma [72] applied to \((F, \ast) \longrightarrow (E, X) \longrightarrow X\) which is a fibration relative to \(A\). We are now in the position to prove Theorem [1.2] of the introduction.

Theorem 7.4. If \(X\) is \(r\)-connected, \(r \geq 1\), then \(\mathcal{B}_n(X)\) is \(2n + r - 2\)-connected.

Proof. Write \(\mathcal{B}_n(X) \cong S^{n-1} \wedge_{\mathfrak{S}_n} (X^{(n)}/\Delta_{fat})\) as in the proof of Theorem [5.3]. Notice that the action of \(\mathfrak{S}_n\) on \(X^{(n)}/\Delta_{fat}\) is free everywhere but at the basepoint \(*\) to which \(\Delta_{fat}\) is collapsed out. This is clear since fixed points of \(\mathfrak{S}_n\) acting on \(X^n\) consist precisely of tuples with two or more equal entries. Let \(E_n = S^{n-1} \times_{\mathfrak{S}_n} (X^{(n)}/\Delta_{fat})\) and set \(X_n := * \times_{\mathfrak{S}_n} (X^{(n)}/\Delta_{fat})\) := \((X^{(n)}/\Delta_{fat})\mathfrak{S}_n\) where \(* = (1/\sqrt{m}, \ldots, 1/\sqrt{m})\) is the fixed point under the \(\mathfrak{S}_n\)-action. Let \(x_0\) denote both the unique fixed point of \((X^{(n)}/\Delta_{fat})\mathfrak{S}_n\)-action and its image in the quotient \(X_n\). The projection \(f : E_n \longrightarrow X_n\) has as preimage \(S^{n-1}\) over every point of \(X_n - \{x_0\}\) and over \(x_0\) the fiber is \(S^{n-1} / \mathfrak{S}_n\), which is contractible as we pointed out earlier. We claim that \(f\) is a bundle relative to \(x_0\). This is straightforward. Let \(V\) be a neighborhood of \(x_0 \in X_n\) that deformation retracts onto \(V\) with deformation retraction \(G\). Write \(p_n : X^{(n)}/\Delta_{fat} \longrightarrow X_n\) the projection. Then \(p_n^{-1}(V - \{x_0\}) \longrightarrow V - \{x_0\}\) is a regular covering with a single ramification point at \(p_n^{-1}(x_0) = x_0\). The deformation \(G\) of \(V\) onto \(x_0\) lifts to a \(\mathfrak{S}_n\)-equivariant deformation of \(p_n^{-1}(V)\) onto \(x_0\) and hence lifts to a deformation of \(f^{-1}(V) = S^{n-1} \times_{\mathfrak{S}_n} p_n^{-1}(V)\) onto \(f^{-1}(x_0)\). Our hypothesis in definition [7.1] for \(f\) as above, \((F, F_0) = (S^{n-1}, \ast)\) and \(A = x_0\) are verified.

The space \(X_n\) has been studied by Nakaoa where he has shown [24, Proposition 4.3] that if \(X\) is \(r\)-connected, then \(X_n = (X^{(n)}/\Delta_{fat})\mathfrak{S}_n\) is \(r + n - 1\) connected and this only holds for \(r \geq 1\) hence our hypothesis.\(^3\) In particular \(X_n\) is always simply connected if \(X\) is.

Notice that \(E_n \longrightarrow X_n\) has a section and so according to corollary [7.3] we get a spectral sequence

\[ 15 \quad E^2 = \tilde{H}_*(X_n, \tilde{H}_*(S^{n-1})) \implies H_*(E, f^{-1}(x_0) \cup X_n) \cong H_*(B_n(X)) \]

\(^3\)It seems very likely that we can weaken the condition \(r \geq 1\) to \(r \geq 0\) if \(X\) is a manifold.
This $E^2$-term is necessarily trivial in total degree $r + n - 1 + n - 1 = 2n + r - 2$ using Nakaoka's aforementioned connectivity result so that $H_q(B_n(X)) = 0$ for $q \leq 2n + r - 2$. Since $B_n(X)$ is simply connected (Lemma 3.6), the claim follows.

Remark 7.5. Since $\Sigma B_n(X) \simeq \Sigma F^n(X)$ (Proposition 4.1) the above calculation shows that the connectivity of $\Sigma F^n(X)$ is $2n + r - 1$ if $X$ is $r$-connected. In fact we can deduce from here that $\Sigma F^n(X)$ is $2n + r - 2$ connected if $r \geq 1$ is the connectivity of $X$ \cite{15}.

8. The Space of Chords

Given a topological space $X$, then $B_2(X)$ is referred to as the space of chords on $X$ and can be viewed as the space of all segments $pq = qp$ between any two points $p, q \in X$. The length of $pq$ is assumed to be a continuous function of $p$ and $q$ so that the length approaches zero if $p$ and $q$ approach a common limit. Distinct segments do not intersect except at a common endpoint. This space has been looked at by Clark in 1944 \cite{6} who wrote down a homological description in terms of simplicial generators and relations. The method of Clark consisted in analyzing a Mayer-Vietoris exact sequence and is hard to work with. Below we use Proposition 4.1 and the identity

$$\Sigma F_2^n(X_1 \sqcup \cdots \sqcup X_k) \simeq \bigvee_{r_1 + \cdots + r_k = n} \Sigma F_1 X_1 \wedge \cdots \wedge \Sigma F^k X_k$$

(16)

to analyze some useful cases.

Lemma 8.1. For based spaces $X$ and $Y$, $B_2(X \times Y)$ is stably homotopy equivalent to a wedge of six terms

$$B_2(X) \vee B_2(Y) \vee B_2(X \wedge Y) \vee (X \ast Y) \vee (X \ast^2 \wedge Y) \vee (Y \ast^2 \wedge X)$$

the equivalence occurring after a single suspension.

Proof. The claim comes down to decomposing the symmetric smash

$$\Sigma B_2(X \times Y) = \Sigma F^2(\Sigma (X \times Y))$$

$$\simeq \Sigma F^2(\Sigma X \vee \Sigma Y \vee (\Sigma X \wedge Y))$$

$$\simeq \Sigma F^2(\Sigma X) \vee \Sigma F^2(\Sigma Y) \vee \Sigma F^2(\Sigma X \wedge Y) \vee \Sigma F^2(\Sigma X \wedge Y) \vee \Sigma^2(X \wedge X \wedge Y) \vee \Sigma^2(Y \wedge X \wedge Y)$$

and this is our claim after desuspending once. \hfill \Box

Let $C_g$ be a closed topological surface of genus $g$ and write as before $X^{\vee n}$ the one-point union of $n$-copies of $X$. Then

Lemma 8.2. $\Sigma B_2(C_g) \simeq (S^4)^{\vee (2g^2+g)} \vee (S^5)^{\vee 2g} \vee \Sigma^4 \mathbb{R}P^2$.

Proof. The surface $C_g$ is obtained by attaching a cell of dimension two to a bouquet of $2g$ circles $\sqcup S^1$. The suspension of the attaching map is trivial so that we have a standard splitting $\Sigma C_g \simeq S^3 \vee \Sigma^2 S^2$. We can then replace the symmetric smash $\Sigma F^2(\Sigma C_g)$ by $\Sigma F^2(S^3 \vee \Sigma^2 S^2)$ and apply (11). \hfill \Box

Example 8.3. As it should be, both lemmas 8.1 and 8.2 agree for the case of the torus $S^1 \times S^1$ and one has $B_2(S^1 \times S^1) \simeq S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4 \vee \Sigma^4 \mathbb{R}P^2$.

9. Barycenter Spaces of Spheres

The main objective of this section is to show that $B_n(S^k) \simeq \text{Sym}^n(S^k)$ is a $(k + 1)$-fold suspension on an explicit space. The special cases of $n = 2$ and of the circle were discussed in lemma 3.3 and in corollary 4.3 respectively. As a start we recall some useful constructions from \cite{28}.

There is a homeomorphism $S^m \ast S^m = \partial(D^{m+1} \times D^{m+1}) = S^{n+m+1}$ (see examples 2.1 (1)). In fact consider $S = S^{n+m+1}$ the unit sphere in $\mathbb{R}^{n+m+2}$ and choose an orthogonal decomposition $\mathbb{R}^{n+1} \oplus \mathbb{R}^{m+1}$
so that \( S^n = S \cap \mathbb{R}^{n+1}, S^m = S \cap \mathbb{R}^{m+1} \). Then by joining segments between these two spheres we obtain \( S^n \ast S^m \) and radial projection onto \( S \) yields the homeomorphism
\[
\Phi : S^n \ast S^m \longrightarrow \mathbb{R}^{n+m+1}, \quad [x, y, t] \mapsto \sqrt{1-t}x + \sqrt{t}y
\]  

Slightly more generally let \( D^l \) be the unit disk in \( \mathbb{R}^l \) and \( S^{l-1} \) its boundary. We write \( (D, S) \) for the corresponding pair and we usually omit to specify \( l \) when it is clear from the context. Suppose \( W_1 \oplus W_2 = \mathbb{R}^l \) is an orthogonal decomposition and set \( D_1 = W_1 \cap D, S_1 = W_1 \cap S \). Then if \( \ast \) is the join product as before, we have a homeomorphism of pairs
\[
(18) \quad \Phi : (D_1 \ast S_1, S_1 \ast S_2) \longrightarrow (D, S)
\]

We set \( l = nk \) and \( D = D^{nk} \subset (\mathbb{R}^k)^n \) the unit disk. There is an action of \( \mathfrak{S}_n \) on \((\mathbb{R}^k)^n\) by permuting vectors. This action is via orthogonal transformations and descends to an action on the pair \((D, S)\). Consider the \( \mathfrak{S}_n \)-invariant subspace
\[
(19) \quad W_1 = \{ v = (v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \mid v_1 = \ldots = v_n \}
\]
Its orthogonal complement \( W_2 = W_1^\perp \) is \( \mathfrak{S}_n \)-invariant, and \( \mathfrak{S}_n \) acts on the joins \( D_1 \ast S_2 \) and \( S_1 \ast S_2 \) by acting diagonally on factors. Since the action on \((D_1, S_1)\) is trivial, and since the homeomorphism \((18)\) is \( \mathfrak{S}_n \)-equivariant, one obtains
\[
(20) \quad (D^{nk}/\mathfrak{S}_n, S^{nk-1}/\mathfrak{S}_n) \simeq (D_1 \ast (S_2/\mathfrak{S}_n), S_1 \ast (S_2/\mathfrak{S}_n))
\]
This leads to the interesting consequence \([28]\).

**Proposition 9.1.** \( \Sigma^k S^k = S^k \ast (S_2/\mathfrak{S}_n) = \Sigma^{k+1}(S_2/\mathfrak{S}_n) \).

**Proof.** As in \([28]\), p. 370, the pair \((D, S)\) in \((\mathbb{R}^k)^n\) is \( \mathfrak{S}_n \)-equivariantly equivalent to the pair
\[
(21) \quad ((D^k)^n, \partial(D^k)^n) \quad \text{where} \quad \partial(D^k)^n := \bigcup_{i=1}^{k-1} (D^k)^i \times S^{k-1} \times (D^k)^{i-1}
\]
The quotient \( (D^k)^n/\partial(D^k)^n \) is \((S^k)^{(nk)}\) and the permutation action of \( \mathfrak{S}_n \) translates into a permutation of the smash factors so that the quotient is \( \Sigma^k S^k \). But according to \((20)\) this quotient must coincide with
\[
[D_1 \ast (S_2/\mathfrak{S}_n)]/[S_1 \ast (S_2/\mathfrak{S}_n)] \simeq (D_1/S_1) \ast (S_2/\mathfrak{S}_n) = S^k \ast (S_2/\mathfrak{S}_n)
\]
where again \( D_1 \) is a disc of dimension \( k = \dim W_1 \).

We have just shown that \( \Sigma \text{Sym}^{\ast n}(S^{k-1}) = \Sigma^k S^k \) is a \( k+1 \)-fold suspension. The main statement next is that this identification desuspends. With \( S_2 \) as above the unit \((k(n-1)-1)\)-sphere in \( W_2 = W_1^\perp \), we have

**Theorem 9.2.** There is a homotopy equivalence \( \text{Sym}^{\ast n}(S^{k-1}) \simeq \Sigma^k (S_2/\mathfrak{S}_n) \).

**Proof.** We first adopt some notation. Define the smash product quotient
\[
(21) \quad L_n(X) := S^{n-1} \wedge_{\mathfrak{S}_n} X^{(n)}
\]
where \( S^{n-1} \) is \( \Delta/\partial \Delta \) with the appropriate \( \mathfrak{S}_n \)-action. Of course \( L_n(X) \simeq \text{Sym}^{\ast n}(X) \) according to Theorem \([5,3]\). Write \( Y_{\mathfrak{S}_n} \) for the quotient of \( Y \) by \( \mathfrak{S}_n \) action. If \( A \) and \( B \) are compact manifolds with boundary \( \partial \), define as well
\[
A \wedge_{\partial} B := A \times B/\partial A \times B \cup A \times \partial B = (A/\partial A) \wedge (B/\partial B)
\]
For example \( I \wedge_{\partial} I = S^2 \). If moreover \( G \) is a group acting on the right of \( A \) and on the left of \( B \) preserving boundaries, then we write \( A \wedge_{\partial} G B \) for the quotient of \( A \wedge_{\partial} B \) under the diagonal action. For instance \( \Delta := \Delta_{n-1} \subset I^n \) and \( \Delta \times (D^{k-1})^n \) is a \( \mathfrak{S}_n \)-invariant subspace of \( I^n \times (D^{k-1})^n \) (under the diagonal action) so that we can write
\[
L_n(S^{k-1}) = S^{n-1} \wedge_{\mathfrak{S}_n} (S^{k-1})^{(n)} = \Delta \wedge_{\partial} (D^{k-1})^n
\]
We will transform this quotient in steps. We will use freely the identifications (equivariant with respect to the permutation action) $[0,1]^n \cong [-1,1]^n \cong D^n$. Also $(D^k)^n$ and $D^kn$ are $\Sigma_n$-equivariant subspaces of $(\mathbb{R}^k)^n$. Let now

\[(22)\quad V_n = \{(t_1, \ldots, t_n) \in [-1,1]^n \mid \sum t_i = 0\}\]

and consider the sequence of $\Sigma_n$-equivariant homeomorphisms

\[
\Delta \times (D^{k-1})^n \cong \Delta \times [-1,1]^{k-1})^n
\cong V_n \times [-1,1]^{k-1})^n
\cong \{(v_1, \ldots, v_n) \in [-1,1]^n \mid \sum v_i = 0\}
\cong \{(v_1, \ldots, v_n) \in D^{kn} \mid \sum v_i = 0\} = B_n
\]

where if $v \in [-1,1]^k$, then $v^i \in [-1,1]$ is its first coordinate. The identification \((22)\) is obtained by distributing coordinates in the obvious way. To see how to get \((24)\) we need check that the equivalences $([-1,1]^k)^n \cong (D^k)^n \cong D^{kn}$ preserve the subspace $\sum_{i=1}^n v^i = 0$. This is clear for the first equivalence. To see the second, we could observe that $(D^k)^n$ is the unit sphere in $(\mathbb{R}^k)^n$ but for a different norm. For $v = (v_1, \ldots, v_n) \in (\mathbb{R}^k)^n$ define (as in [28], p.370)

$$||v||' = \max_i ||v_i||$$

Then $(D^k)^n$ is the subspace of all such $v$ with $||v||' = 1$. The map $v \mapsto v' = \frac{||v||}{||v||'} v$ maps $D^{kn}$ homeomorphically and equivariantly onto $(D^k)^n$. Obviously if $\sum v_i = 0$, then $\sum v_i^i = 0$ as well. This establishes the last equivalence \((24)\).

Note that the space $B_n$ in \((22)\) is the unit sphere in the hyperplane $\{(v_1, \ldots, v_n) \mid \sum v_i = 0\}$. It is identified equivariantly with $\Delta \times (D^{k-1})^n$. Since this identification preserves boundaries we see that

$$L_n(S^{k-1}) = \Delta \langle \Sigma_n \rangle (D^{k-1})^n \cong (B_n/\partial B_n)\Sigma_n$$

The problem then boils down to identifying the $\Sigma_n$-equivariant homotopy type of the pair $(B_n, \partial B_n)$. This is where the methods of proposition \([9,1]\) come handy.

As in \((19)\), set $W_1 = \{v \in (\mathbb{R}^k)^n \mid v_1 = \ldots = v_n\}$ and the various disks

$$D = D^{nk}, \quad D_1 = D \cap W_1, \quad D'_1 = B_n \cap W_1$$

and $D_1^{+}$ the unit disk in $W_1^{+}$. The associated boundary spheres are denoted by $S, S_1, S'_1$ and $S_1^{+}$ respectively. Since $W_1^{+} = \{v = (v_1, \ldots, v_n) \in (\mathbb{R}^k)^n \mid \sum v_i = 0\}$, the condition $\sum v_i = 0$ says that $S_1^{+} \subset D_1^{+} \subset B_n$. Consider the homeomorphism $\Phi : (D_1 \ast S_1^{+}, S_1 \ast S_1^{+}) \to (D, S)$ constructed by means of the map \((17)\). Its restriction $\Phi_1$ makes the following diagram commute

$$\begin{array}{ccc}
(D_1^{+} \ast S_1^{+}, S_1^{+} \ast S_1^{+}) & \xrightarrow{\Phi_1} & (B_n, \partial B_n) \\
\downarrow & & \downarrow \\
(D_1 \ast S_1^{+}, S_1 \ast S_1^{+}) & \xrightarrow{\Phi} & (D, S)
\end{array}$$

and $\Phi_1$ is a homeomorphism of pairs as well (this is easy to see since $\Phi$ restricted to $D_1^{+} \ast S_1^{+}$ surjects onto $B_n$). We are at last ready to conclude. We have a series of equivalences

$$B_n/\partial B_n \cong D_1^{+} \ast S_1^{+}/S_1^{+} \ast S_1^{+} \simeq (D_1^{+}/S_1^{+}) \ast S_1^{+} = S^{k-1} \ast S_1^{+}, \quad \dim D_1^{+} = k - 1$$

The action of $\Sigma_n$ is trivial on $S^{k-1}$ since it is trivial on $D_1$ and thus on $D_1^{+}$, so by passing to quotients

$$L_n(S^{k-1}) = (B_n/\partial B_n)\Sigma_n \simeq S^{k-1} \ast (S_1^{+}/\Sigma_n) \simeq \Sigma^k(S_1^{+}/\Sigma_n)$$

and the proof is complete. \qed
10. THE CASE OF MANIFOLDS

We shed different light on calculations of Bahri-Coron and Lannes [3] on the homology of barycenters of manifolds. First we have the following description of the top homology group for symmetric products of general closed manifolds.

**Proposition 10.1.** Suppose $M$ is a closed manifold of dimension $d \geq 2$. If $M$ is orientable, then

$$H_{nd}(SP^n M; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & d \text{ even} \\ 0, & d \text{ odd} \end{cases}$$

For general closed manifolds $M$, $H_{nd}(SP^n M; \mathbb{Z}_2) = \mathbb{Z}_2$.

**Corollary 10.2.** If $M$ is closed orientable of dimension $d \geq 1$, then in top dimension $H_{n(d+1)−1}(B_n(M)) = \mathbb{Z}$ if $d$ is odd and is 0 if $d$ is even.

**Proof.** By Theorem 4.3 we need determine $H_{n(d+1)}(SP^n(\Sigma M))$. Since $\Sigma M$ is a CW complex with top integral homology group $H_{d+1} = \mathbb{Z}$; $\Sigma M$ has the homology of a wedge $S^{d+1} \vee Y$ where $Y$ is of dimension $d$. By a well-known result of Dold, the homology of symmetric products of simplicial complexes, and hence of their reduced symmetric products, only depends on the homology of the underlying space. This means

$$H_*(SP^n(\Sigma M)) \cong H_*(SP^n(S^{d+1} \vee Y)) \cong \bigoplus_{r+s=n} H_*(SP^n S^{d+1} \wedge SP^r Y)$$

according to the formula (7). Since the homological dimension of $Y$ is $d$, the term $H_*(SP^n S^{d+1} \wedge SP^r Y)$ is trivial in degrees larger than $r(d+1) + sd$. It follows that in top dimension

$$H_{n(d+1)}(SP^n(\Sigma M)) \cong H_{n(d+1)}(SP^n S^{d+1}) \cong H_{n(d+1)}(SP^n S^{d+1})$$

The claim now follows from Proposition 10.1 applied to $M = S^{d+1}$. \qed

**Example 10.3.** $B_2(S^k) \simeq \Sigma^{k+1} \mathbb{R}^k$ by (11) and hence $H_{2(k+1)−1}(B_2(S^k)) = H_2(\mathbb{R} P^k)$ and this is indeed $\mathbb{Z}$ or 0 according to whether $k$ is odd or even. Similarly and by lemma 8.2, the top class in dimension 5 of the space of chords of a closed Riemann surface is trivial as expected

$$H_5(B_2(C)) \cong H_6((S^4)^{\vee(2g+g)} \vee (S^5)^{\vee2g} \vee \Sigma^4 \mathbb{R} P^2) = H_6(\Sigma^4 \mathbb{R} P^2) = 0$$

In [3] a main point of consideration were tranfer morphisms

$$\Phi : H_{nd+n−1}(B_n M, B_{n−1} M) \longrightarrow H_{(n−1)d+n−2}(B_{n−1} M, B_{n−2} M)$$

sending orientation class to orientation class. In this case $M$ was allowed to have boundary and coefficients were in $\mathbb{Z}_2$. The map $\Phi$ was a mix of a transfer map, a cap product and a boundary morphism. We indicate below a streamlined construction of a transfer map which appeals as before to the identification of $\Sigma B_n(M)$ with $SP^n(\Sigma M)$. In our case and for closed oriented $M$ we seek to construct for each positive $n$ a map

$$\Theta : H_{n(d+1)}(SP^n(\Sigma M)) \longrightarrow H_{(n−1)(d+1)}(SP^{n−1}(\Sigma M))$$

with the right homological properties. The construction is due to L. Smith and has refinements in [9]. Consider the degree $n$ covering

$$\Sigma M \times SP^{n−1}(\Sigma M) \overset{p}{\longrightarrow} SP^n(\Sigma M)$$

$$(x, y_1 + \cdots + y_{n−1}) \mapsto x + y_1 + \cdots + y_{n−1}$$

The transfer map in homology for this covering can be achieved at the level of spaces as follows. Set $Y = \Sigma M$ to ease notation. To each element $y_1 + \cdots + y_n \in SP^n(Y)$, we associate the unordered tuple

$$\sum (y_i, y_1 + \cdots + y_{i−1} + y_{i+1} + \cdots + y_n) \in SP^n(Y \times SP^{n−1} Y)$$

The induced map in homology is a map $\tau : H_*(SP^n Y) \longrightarrow H_*(SP^n (Y \times SP^{n−1} Y))$. 

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18 SADOK KALLEL, RYM KAROUY
Next we invoke a homological splitting of Steenrod which asserts that for any based connected space $X$, $H_*(X,\mathbb{Z})$ is a direct summand of $H_*(\text{SP}^n X)$ (see [29]). On the other hand the projection $\text{SP}^n X \rightarrow \text{SP}^{n-1} X$ is an $n$-fold branched covering and hence on top dimension it is multiplication by $n$. It follows that

$$ p_*(\text{tr}(v_n)) = n v_n = p_*(v_1 \otimes v_{n-1}) $$

from which we deduce that $\text{tr}(v_n) = v_1 \otimes v_{n-1}$ since $v_1 \otimes v_{n-1}$ is the only generator in that dimension. The equality $\theta_*(v_n) = v_{n-1}$ is immediate.

It would be a good exercise to check whether this transfer agrees or not with the transfer map constructed in appendix C of [2][3], but we don’t pursue this further.

11. Appendix: Homology Computations

This final appendix summarizes for convenience some known results about the homology of the reduced symmetric products $\text{SP}^n X$ and derives information about the homology of barycenter spaces of spheres and manifolds.

As is standard $\text{SP}^\infty(X)$ will denote the direct limit under the inclusions $\text{SP}^n(X) \rightarrow \text{SP}^{n+1}(X)$, $\sum x_i \rightarrow \sum x_i + x_0$ where $x_0 \in X$ is a chosen basepoint ($X$ always assumed to be connected). Since $\text{SP}^\infty(X)$, is a connected abelian topological monoid (associative with unit), it has the homotopy type of a product of Eilenberg-MacLane spaces. In fact and as a consequence of a theorem of Dold and Thom

$$ \text{SP}^\infty(X) \simeq \prod_i K(\tilde{H}_i(X), i) $$

where $K(G, i)$ is the standard Eilenberg-MacLane space with $\pi_i(K(G, i)) = G$ (for a nice account see [29]). On the other hand and by a computation of Steenrod (see [10]) there is a splitting

$$ H_*(\text{SP}^n X) = H_*(\text{SP}^{n-1} X) \oplus H_*(\text{SP}^n X, \text{SP}^{n-1} X) $$

so that $H_*(\text{SP}^n X)$ embeds as a direct summand in $H_*(\text{SP}^n X)$. Replacing $X$ by a suspension $\Sigma X$ in the expressions above yield

**Corollary 11.1.** $H_*(B_n(X))$ embeds as a direct summand in $H_{*+1}(\text{SP}^n \Sigma X)$ which in turn embeds in

$$ \bigotimes_{i=0}^{\infty} H_{*+1}(K(\tilde{H}_i(X), i + 1)). $$

We propose to understand the image of this embedding for spheres. This follows principally from work of Nakaoka and Milgram [21]. Since $\text{SP}^\infty(S^n)$ is a $K(\mathbb{Z}, n)$, we can use the Steenrod splitting above and write

$$ \tilde{H}_*(K(\mathbb{Z}, n); \mathbb{F}) \cong \bigoplus_{j=1}^{\infty} \tilde{H}_*(\text{SP}^j S^n, \mathbb{F}) $$

where $\tilde{H}_*(K(\mathbb{Z}, n); \mathbb{F})$ is the reduced $\mathbb{F}$-homology of $K(\mathbb{Z}, n)$.
This splitting is valid in cohomology (additively) as well. The next step is to filter \( H_*(K(\mathbb{Z}, n); \mathbb{F}) \) over the positive integers so that \( H_*(\mathbb{Z}[S^n**]; \mathbb{F}) \) corresponds to the classes of filtration degree precisely \( j \).

This inductive procedure yields the following computations which can be deduced from [21] [25].

Recall by work of Serre (Theorem 3 of [29]) that for \( n \geq 2 \), \( H^*(K(\mathbb{Z}, n); \mathbb{F}_2) \) (resp. \( H^*(K(\mathbb{Z}_2, n); \mathbb{F}_2) \)) is a vector space having as basis elements all iterated Steenrod squares \( Sq^* \) \( u \) on an \( n \) dimensional class \( u \) and running over sequences of strictly positive integers \( I = \{i_1, \ldots, i_r\} \) which satisfy the *admissibility* conditions (i), (ii) and (iii) below (resp. (i) and (ii)):

(i) \( i_1 - i_2 \cdots - i_r < n \),
(ii) \( i_k \geq 2i_{k+1} \), \( k = 1, 2, \ldots, r - 1 \),
(iii) \( i_r > 1 \).

The graded group \( H^*(\mathbb{S}^p S^n**; \mathbb{F}_2) \) is a direct summand of \( H^*(K(\mathbb{Z}, n); \mathbb{F}_2) \) as we previously explained.

To describe it, let \( t_n \in H^*(\mathbb{S}^p S^n**; \mathbb{F}_2) \) be the bottom generator, and set the filtration degree of \( Sq^j(t_n) \) to be \( 2^j \) where \( j = \{i_1, \ldots, i_r\} \) as above. Then

**Theorem 11.2.** For \( n \geq 2 \), \( H^*(\mathbb{S}^p S^n**; \mathbb{F}_2) \) is the quotient of the polynomial algebra \( \mathbb{Z}_2[Sq^j(t_n)] \), \( I \) running over admissibles, by the ideal of elements having filtration degree greater than \( k \).

The subspace \( H^*(\mathbb{S}^p S^n**; \mathbb{F}_2) \) corresponds to elements of exact filtration \( k \).

**Proof.** This is a reformulation of the calculation of Nakaoka [22, 30].

A similar result holds modulo \( p \) for odd \( p \) with filtration degree set at \( p^\infty \). Note that the results in [21] were stated in homology and the above is the dual version. A more geometrical viewpoint on symmetric products of spheres can be found in work of Ucci [30].

For a filtered \( H \)-space \( X \), \( A := H_*(X; \mathbb{F}_p) \) becomes a bigraded algebra over \( \mathbb{F}_p \) and so we write \( x_{(i,k)} \in A \) for an element \( x \) of homological degree \( i \) and filtration degree \( k \). Both the homology and filtration degrees are additive under the product so that \( x_{(i,k)} \cdot y_{(j,s)} \) is a class of degree \( i + j \) and filtration \( k + s \).

**Example 11.3.** Since \( \text{SP}^\infty(S^1) \simeq S^1 \), \( H^*(\text{SP}^\infty(S^1); \mathbb{Z}) = E[\epsilon_{(1,1)}] \) where \( E \) stands for exterior algebra.

On the other hand, \( H^*(\text{SP}^\infty(S^2); \mathbb{Z}) \simeq \mathbb{Z}[b_{(2,1)}] \) and this is in turn consistent with the diffeomorphism \( \text{SP}^\infty(S^2) \simeq \mathbb{P}^\infty \). The case of \( \text{SP}^\infty(S^3) \) is much more delicate and is given with mod-2 coefficients by

\[
H^*(\text{SP}^\infty(S^3), \mathbb{F}_2) \equiv \mathbb{F}_2[\epsilon_{(1,1)}; \tilde{f}(3,1), \tilde{f}(5,2), \tilde{f}(9,4), \ldots; \tilde{f}(2^r+1,2^r), \ldots]
\]

where \( \epsilon_{(2^r+1,2^r)} := Sq^{2^r} \cdots Sq^4 Sq^2(i) \) with \( i \in H^3(\text{SP}^\infty S^3) \) is the bottom class. A similar formula mod-\( p \) is written up in [8] (see corollary 11.3 below).

For \( A \) a bigraded algebra as above, we will denote by \( [A]_k \) the submodule of elements with filtration degree precisely \( k \). Let \( \sigma \) be the formal suspension operator which raises homological degree by one. If \( E[\cdot] \) stands for an exterior algebra over the field with \( p \) elements \( \mathbb{F}_p \), then the following is a consequence of Theorem 11.3 and Example 11.3

**Corollary 11.4.** Let \( p \) be an odd prime.

(i) \( \sigma H_{**}(\mathbb{B}_n(S^2); \mathbb{F}_2) \equiv \mathbb{F}_2[\epsilon_{(1,1)}; \tilde{f}(3,1), \tilde{f}(5,2), \tilde{f}(9,4), \ldots; \tilde{f}(2^r+1,2^r), \ldots] \).

(ii) \( \sigma H_{**}(\mathbb{B}_n(S^2); \mathbb{F}_p) \) is given by

\[
\bigoplus_{r+s=n} E[\epsilon_{(1,1)}, h_{(2p+1,p)}, \ldots; h_{(2p'+1,p')} \cdots] \otimes \mathbb{F}_p[b_{(2p+2,p)}, \ldots; b_{(2p'+2,p')}, \ldots]
\]

(iii) For general spheres and \( p > n > 1 \), \( H_{**}(\mathbb{B}_n(S^k); \mathbb{F}_p) \) has a single non-trivial group \( \mathbb{F}_p \) for \( s = n(k+1) - 1 \) if \( k \) is odd, and is trivial if \( k \) is even. In particular and for \( p > n > 1 \), \( \mathbb{B}_n(S^{2k}) \) has no torsion of order \( p \) in its homology.

**Example 11.5.** The mod-\( p \) version of Theorem 11.2 shows that \( H^*(\text{SP}^n S^k, \mathbb{F}_p) \) for \( p > n \) is a truncated polynomial algebra \( \mathbb{F}_p[u]/u^{n+1} \) where \( m = 1 \) or \( n \) according to whether \( k \) is odd or even. This fact is recorded in [30], §1. The element \( u \) pulls back to a generator of \( H^k S^k \) under the inclusion \( S^k \hookrightarrow \text{SP}^n S^k \).
and so it is of bidegree \((k, 1)\). In particular and for \(p > n > 1\), \(H_*(\mathbb{F}_p)\) has a single non-trivial (reduced) homology group for \(\ast = nk\) and \(k\) even, and is entirely trivial if \(k\) is odd. This implies corollary \([\text{11.3}](iii).\) Note the consistency of this corollary with corollary \([\text{10.2}].\)

**Example 11.6.** As indicated in corollary \([\text{11.3}].\) \(\sigma H_*(B_*(S^2); \mathbb{F}_2)\) consist of elements of filtration degree \(n\) in \(\mathbb{F}_2[f_{(3,1)}, f_{(5,2)}, f_{(9,4)}, \ldots, f_{(2^i+1, 2^i)}, \ldots]\). When \(n = 2\), there are only two generators: \(f_{(5,2)}\) and \(f_{(3,1)}\) (of bidegree \((6, 2)\)) so that

\[
\tilde{H}_*(B_2(S^2); \mathbb{F}_2) = \begin{cases} 
\mathbb{F}_2, & \ast = 4 \\
\mathbb{F}_2, & \ast = 5 
\end{cases}
\]

and is zero otherwise. For \(p\) odd, \(\tilde{H}_*(B_2(S^2); \mathbb{F}_p)\) is trivial (i.e. no non-trivial class in filtration degree \(2\) in this case since \(f_{(3,1)} = 0\) with odd primes) and this is consistent with \(B_2(S^2) \simeq \Sigma^3 \mathbb{R}P^2\) as shown encore in \([\text{11}].\)

**References**

[1] G. Arone, B. Dwyer, **Partition complexes, Tits buildings and symmetric products**, Proc. London Math. Soc. (3) 82 (2001), no. 1, 229–256.

[2] A. Bahri, H. Brezis, **Non-linear elliptic equations on Riemannian manifolds with the Sobolev critical exponent**, in Topics in Geometry, in Memory of Joseph d’Atri, Progress in Nonlinear Differential Equations and their Applications (1996), Birkhauser.

[3] A. Bahri, J.M. Coron, **On a non-linear elliptic equation involving the critical sobolev exponent: the effect of the topology of the domain**, Comm. Pure and Applied Mathematics, Vol XLI, (1988), 253–294.

[4] R. Brown, **Elements of modern topology**, McGraw-Hill Book Co., New York-Toronto 1968.

[5] C. Cazanave, **Théorie homotopique des schémas d’Atiyah-Hitchin**, thèse de doctorat, École Polytechnique, Palaiseau, Octobre 2009.

[6] C.E. Clark, **The symmetric join of a complex**, Proc. Am. Acad. Sci. (1944), 81–88.

[7] D.E. Cohen, **Products and carrier theory**, Proc. London Math. Soc. 7 (1957), 219–248.

[8] F.R. Cohen, R.L. Cohen, B.M. Mann, R.J. Milgram, **The topology of rational functions and divisors of surfaces**, Acta. Math.. 166 (1991), 163-221.

[9] A. Dold, **Ramified coverings, orbit projections and symmetric powers**, Math. Proc. Cambridge Philos. Soc.. 99 (1986), no. 1, 65–72.

[10] A. Dold, **Decomposition theorems for \(S_n\)-complexes**, Annals of Math. 75, 1 (1962), 8–16.

[11] Z. Djadli, A. Malchiodi, **Existence of conformal metrics with \(Q\)-curvature**, Annals of Math. 168, no.3 (2008), 813–858.

[12] R. Fritsch and M. Golasinski, **Topological, Simplicial and Categorical Joins**, Contemp. Math. (2006), 147–161.

[13] A. Gorinov, **Real cohomology groups of the space of nonsingular curves of degree \(d\)**, Math. Proc. Cambridge Philos. Soc.. 138 (1989), 251–265.

[14] I.G. MacDonald, **The Poincaré polynomial of a symmetric product**, J. mathematics and mechanics, 12, 5 (1963), 771–776.

[15] S. Kallel, **Symmetric products, duality and homological dimension of configuration spaces**, Geometry and Topology Monographs 13 (2008), 499–527.

[16] S. Kallel, D. Sjerve, **Remarks on finite subset spaces**, Homology, Homotopy and Applications 11 (2009), No. 2, 229–250.

[17] S. Kallel, P. Salvatore, **Symmetric products of two dimensional complexes**, Contemp. Math. 407 (2006), 147–161.

[18] S.D. Liao, **On the topology of cyclic products of spheres**, Transactions AMS 77 (1954), 520–551.

[19] I.G. MacDonald, **The Poincaré polynomial of a symmetric product**, Proc. Cambridge Phil. Soc.. 58 (1962), 563–568.

[20] A. Malchiodi, **Morse theory and a scalar field equation on compact surfaces**, Adv. Diff. Eq. 13 (2008), 1109–1129.

[21] R.J. Milgram, **The homology of symmetric products**, Trans. Am. Math. Soc. 138 (1969), 251–265.

[22] I. Mineyev, **Flows and joins of metric spaces**, Geometry and Topology, 9 (2005), 403–482.

[23] B. Morin, **La classe fondamentale d’un espace fibré**, Séminal H. Cartan, tume 12, n.1 (1959-1960), exp. 8, 1–12.

[24] M. Nakaoka, **Cohomology of symmetric products**, J. Institute of Polytechnics, Osaka city University 8, no. 2, 121–140.

[25] M. Nakaoka, **Cohomology mod-p of symmetric products of spheres II**, J.Institute of Polytechnics, Osaka city University 10, (1956), 67–89.

[26] J.P. Serre, **Cohomologie modulo 2 des complexes d’Eilenberg-MacLane**, Comm. Math. Hel. 27 (1953), 198–232.

[27] S. Smale, **A Vietoris mapping theorem for homotopy**, Proceedings AMS 8 (1957), 604–610.

[28] V. Snaith, J. Ucci, **Three remarks on symmetric products and symmetric maps**, Pacific J. Math. 45 (1973), 369–377.
[29] E. Spanier, *Infinite symmetric products, function spaces and duality*, Ann. of Math 69 (1959), 142–198.
[30] J. Ucci, *On symmetric maps of spheres*, Inventiones 5 (1968), 8–18.
[31] V. Vassiliev, *Topological order complexes and resolutions of discriminant sets*, Publications Institut Mathématiques 66 80 (1999), 165–185.
[32] R. Vogt, *Splitting of spaces CX*, manuscripta math. 38 (1982), 21–39.

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