Constant of motion for Nonsmooth Extremals of Time Delay
Isoperimetric Variational Problems

G. S. F. Frederico\textsuperscript{1,2}, J. Paiva\textsuperscript{3} and M. J. Lazo\textsuperscript{3}
gastao.frederico@ua.pt

\textsuperscript{1}Department of Mathematics,
Federal University of Santa Catarina, Florianopolis, SC, Brazil
\textsuperscript{2}Department of Science and Technology,
University of Cape Verde, Praia, Santiago, Cape Verde
\textsuperscript{3}Institute of Mathematics, Statistics and Physics, Federal University of Rio Grande, Rio Grande, RS, Brazil

Abstract

We prove the isoperimetric Euler–Lagrange and DuBois–Reymond type optimality condi-
tions and we obtain a nonsmooth extension of Noether’s symmetry theorem for isoperimetric
variational problems with delayed arguments. The result is proved to be valid in the class
of Lipschitz functions, as long as the delayed isoperimetric Euler–Lagrange extremals are
restricted to those that satisfy the isoperimetric DuBois–Reymond necessary optimality condi-
tion. The important case of delayed optimal control problems is considered as well.

Keywords: isoperimetric problems; time delays; symmetries; conservation laws; Euler–
Lagrange and DuBois–Reymond necessary optimality condition; Noether’s theorem; Pon-
tryagin Maximum Principle.

2010 Mathematics Subject Classification: 49K05; 49S05.

1 Introduction

The concept of symmetry plays an important role in science and engineering. Symmetries are
described by transformations, which result in the same object after the transformations are carried
out. They are described mathematically by parameter groups of transformations \cite{13,21}. Their
importance, as recognized by Noether in 1918 \cite{24}, is connected with the existence of conserva-
tion laws that can be used to reduce the order of the Euler–Lagrange differential equations \cite{9}.
Noether’s symmetry theorem is nowadays recognized as one of the most beautiful results of the
calculus of variations and optimal control \cite{5,20}, and becomes one of the most important theorems
for physics in the 20th century. Since the seminal work of Emmy Noether it is well know that all
conservations laws in mechanics, e.g., conservation of energy or conservation of momentum, are
directly related to the invariance of the action under a family of transformations.

Within the years, this theorem has been studied by many authors and generalized in different
directions: see \cite{1,6,8,15,22,23,27} and references therein. In particular, in a recent paper \cite{10},
Noether’s theorem was formulated for variational problems with delayed arguments. The result is
important because problems with delays play a crucial role in the modeling of real-life phenomena
in various fields of applications \cite{4,12,14}. In order to prove Noether’s theorem with delays, it was

\textsuperscript{*}This is a preprint of a paper whose final and definite form will be submitted in Applicable Analysis.
assumed that admissible functions are \( C^2 \)-smooth and that Noether’s conserved quantity holds along all \( C^2 \)-extremals of the Euler–Lagrange equations with time delay [10].

One of the oldest and interesting class of variational problems, with applications in several fields, are the isoperimetric problems [30]. Isoperimetric in mathematical physics has roots in the Queen Dido problem of the calculus of variations, and has recently been subject to several investigations. Here we extend the Noether’s theorem with delays for variational problems.

The text is organized as follows. In Section 2 the fundamental problem of variational calculus with delayed arguments is formulated and a short review of the results for \( C^2 \)-smooth admissible functions is given. The main contributions of the paper appear in Sections 3; we prove the isoperimetric Euler–Lagrange and DuBois–Reymond type optimality conditions (Theorem 10 and Theorem 14, respectively), an isoperimetric Noether symmetry theorem with time delay for Lipschitz functions (Theorem 15) and an isoperimetric weak Pontryagin maximum principle (Theorem 23).

2 Preliminaries

In this section we review some necessary results on the calculus of variations with time delay. For more on variational problems with delayed arguments we refer the reader to [2,3,11,10,18,19,26].

The fundamental problem consists of minimizing a functional

\[
J^*[\eta(t)] = \int_{t_1}^{t_2} L(t, \eta(t), \dot{\eta}(t), \eta(t - \tau), \dot{\eta}(t - \tau)) \, dt
\]

subject to boundary conditions

\[
\eta(0) = \delta(t) \text{ for } t \in [t_1 - \tau, t_1] \text{ and } \eta(t_2) = g_{t_2}.
\]

We assume that the Lagrangian \( L : [t_1, t_2] \times \mathbb{R} \to \mathbb{R} \), is a \( C^2 \)-function with respect to all its arguments, the admissible functions \( \eta(t) \) are \( C^2 \)-smooth, \( t_1 < t_2 \) are fixed in \( \mathbb{R} \), \( \tau \) is a given positive real number such that \( \tau < t_2 - t_1 \), and \( \delta \) is a given piecewise smooth function on \( [t_1 - \tau, t_1] \).

Throughout the text, \( \partial_t L \) denotes the partial derivative of \( L \) with respect to its \( i \)-th argument, \( i = 1, \ldots, 5 \). For convenience of notation, we introduce the operator \([\cdot]_t\), defined by

\[
[q]_t(t) = (t, \eta(t), \dot{\eta}(t), \eta(t - \tau), \dot{\eta}(t - \tau)).
\]

The next theorem gives a necessary optimality condition of Euler–Lagrange type for [11–22].

**Theorem 1** (Euler–Lagrange equations with time delay [10]). If \( \eta(\cdot) \in C^2 \) is a minimizer for problem [11–22], then \( \eta(\cdot) \) satisfies the following Euler–Lagrange equations with time delay:

\[
\begin{align*}
\frac{d}{ds} \{&\partial_t L[q]_t(t) + \partial_\eta L[q]_t(t + \tau)\} = \partial_2 L[q]_t(t) + \partial_4 L[q]_t(t + \tau), \quad t_1 \leq t \leq t_2 - \tau, \\
\frac{d}{ds} \partial_{\eta} L[q]_t(t) = &\partial_2 L[q]_t(t), \quad t_2 - \tau \leq t \leq t_2.
\end{align*}
\]

**Definition 2** (Extremals). The solutions \( \eta(t) \) of the Euler–Lagrange equations [3] with time delay are called extremals.

**Definition 3** (Invariance of [11]). Consider the following s-parameter group of infinitesimal transformations:

\[
\begin{align*}
\bar{t} &= t + s \eta(t, \eta) + o(s), \\
\bar{\eta}(t) &= \eta(t) + s \xi(t, \eta) + o(s),
\end{align*}
\]

where \( \eta \in C^1(\mathbb{R}^2) \) and \( \xi \in C^1(\mathbb{R}^2) \). Functional [11] is said to be invariant under [11] if

\[
0 = \frac{d}{ds} \left. \int_{\eta(\bar{t})}^{\eta(t)} L \left( t + s \eta(t, \eta(t)) + o(s), \eta(t) + s \xi(t, \eta(t)) + o(s), \frac{\dot{\eta}(t) + s \dot{\xi}(t, \eta(t))}{1 + s \eta(t, \eta(t))}, \right) \cdot (1 + s \eta(t, \eta(t))) dt \right|_{s=0}
\]

for any subinterval \( I \subseteq [t_1, t_2] \).
Definition 4 (Constant of motion/conservation law with time delay). We say that a quantity $C(t, t + \tau, q(t), q(t - \tau), \dot{q}(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau))$ is a constant of motion with time delay $\tau$ if
\[
\frac{d}{dt} C(t, t + \tau, q(t), q(t - \tau), \dot{q}(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau)) = 0
\]
along all the extremals $q(\cdot)$ (cf. Definition 2). The equality (5) is then a conservation law with time delay.

Next theorem extends the DuBois–Reymond necessary optimality condition to problems of the calculus of variations with time delay.

Theorem 5 (DuBois–Reymond necessary conditions with time delay [10]). If $q(\cdot) \in C^2$ is an extremal of functional (1) subject to (2) such that
\[
\partial_4 L[q]_\tau(t + \tau) \cdot \dot{q}(t) + \partial_5 L[q]_\tau(t + \tau) \cdot \ddot{q}(t) = 0
\]
for all $t \in [t_1 - \tau, t_2 - \tau]$, then it satisfies the following conditions:
\[
\begin{cases}
\left. \frac{d}{dt} \{ L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \} \right|_{t_1}^{t_2} = \partial_1 L[q]_\tau(t), & t_1 \leq t \leq t_2 - \tau, \\
\left. \frac{d}{dt} \{ L[q]_\tau(t) - \dot{q}(t) \cdot \partial_1 L[q]_\tau(t) \} \right|_{t_2 - \tau}^{t_2} = \partial_1 L[q]_\tau(t), & t_2 - \tau \leq t \leq t_2.
\end{cases}
\]

Remark 6. If we assume that admissible functions in problem (1)–(2) are Lipschitz continuous, then one can show that the DuBois–Reymond necessary conditions with time delay (21) are still valid (cf. [11]).

Theorem 7 establishes an extension of Noether’s theorem to problems of the calculus of variations with time delay.

Theorem 7 (Noether’s symmetry theorem with time delay [10]). If functional (1) is invariant in the sense of Definition 3 such that
\[
\partial_4 L[q]_\tau(t + \tau) \cdot \dot{q}(t) + \partial_5 L[q]_\tau(t + \tau) \cdot \ddot{q}(t) = 0
\]
and , then the quantity $C(t, t + \tau, q(t), q(t - \tau), \dot{q}(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau))$ defined by
\[
(\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \cdot \xi(t, q(t))
\]
\[
+ \left( L[q]_\tau(t) - \dot{q}(t) \cdot (\partial_3 L[q]_\tau(t) + \partial_5 L[q]_\tau(t + \tau)) \right) \eta(t, q(t))
\]
for $t_1 \leq t \leq t_2 - \tau$ and by
\[
\partial_3 L[q]_\tau(t) \cdot \xi(t, q(t)) + \left( L[q]_\tau(t) - \dot{q}(t) \cdot \partial_3 L[q]_\tau(t) \right) \eta(t, q(t))
\]
for $t_2 - \tau \leq t \leq t_2$, is a constant of motion with time delay (cf. Definition 2).

3 Main results

We prove some important results for isoperimetric variational problems with time delay: a generalized isoperimetric Euler–Lagrange necessary optimality condition (Theorem 10), a isoperimetric DuBois–Reymond necessary optimality condition (Theorem 11) and a Noether’s first theorem for isoperimetric variational problems with time delay (Theorem 18). In section 3.4 we adopt the Hamiltonian formalism to prove the weak Pontryagin maximum principle for more general isoperimetric action-like optimal control problems.
3.1 Isoperimetric Euler–Lagrange equations with time delay

We begin by defining the isoperimetric variational problem under consideration.

**Problem 1.** (The isoperimetric variational problem with time delay) The isoperimetric problem of the calculus of variations consists to find the stationary functions of the functional \( I[q(t)] = \int_{t_1}^{t_2} \mathcal{L}[q(t), \dot{q}(t)] dt \), subject to isoperimetric equality constraints

\[
I'[q(t)] = \int_{t_1}^{t_2} \mathcal{L}[q(t), \dot{q}(t)] dt = l, \quad l \in \mathbb{R}^k,
\]

and boundary conditions \( \mathcal{J} \).

We assume that \( l \) is a specified real constant and \( (t, s, y, u, v) \mapsto \mathcal{L}(t, s, y, u, v) \) are assumed to be functions of class \( C^2 \).

Theorem \( \ref{thm:isoperimetric} \) motivates the following definition.

**Definition 8.** An admissible function \( q(\cdot) \in C^2 \) is an extremal for problem \( \ref{prob:isoperimetric} \) if it satisfies the following Euler–Lagrange equations with time delay:

\[
\begin{align*}
\frac{d}{dt} \mathcal{D_1} g[q, \dot{q}] + \mathcal{D_2} g[q, \dot{q}] &= \mathcal{D_3} g[q, \dot{q}], & 0 \leq t \leq t_1 - \tau, \\
\frac{d}{dt} \mathcal{D_4} g[q, \dot{q}] &= \mathcal{D_5} g[q, \dot{q}], & t_1 - \tau \leq t \leq t_2.
\end{align*}
\]

The arguments of the calculus of variations assert that by using the Lagrange multiplier rule, Problem \( \ref{prob:isoperimetric} \) is equivalent to the following augmented problem \( \ref{prob:augmented} \): to minimize

\[
J'[q(\cdot), \lambda] = \int_{t_1}^{t_2} \mathcal{F}[q, \lambda, \dot{q}(t)] dt
\]

\[
:= \int_{t_1}^{t_2} \left[ \mathcal{L}[q, \dot{q}] - \lambda \cdot \mathcal{G}[q, \dot{q}] \right] dt
\]

subject to \( \mathcal{J} \), where \( \mathcal{J}[q, \lambda, \dot{q}] = (t, q(t), \dot{q}(t), q(t - \tau), \dot{q}(t - \tau), \lambda) \).

The augmented Lagrangian

\[
\mathcal{F} := \mathcal{L} - \lambda \cdot \mathcal{G},
\]

\( \lambda \in \mathbb{R}^k \), has an important role in our study.

The notion of extremizer (a local minimizer or a local maximizer) can be found in \( \ref{def:extremizer} \). Extremizers can be classified as normal or abnormal.

**Definition 9.** An extremizer of Problem \( \ref{prob:isoperimetric} \) that does not satisfy \( \ref{eq:euler-lagrange} \) is said to be a normal extremizer; otherwise (i.e., if it satisfies \( \ref{eq:euler-lagrange} \) for all \( t \in [t_1, t_2] \)), is said to be abnormal.

The following theorem gives a necessary condition for \( q(\cdot) \) to be a solution of Problem \( \ref{prob:isoperimetric} \) under the assumption that \( q(\cdot) \) is a normal extremizer.

**Theorem 10.** If \( q(\cdot) \in C^2 ([t_1 - \tau, t_2]) \) is a normal extremizer to Problem \( \ref{prob:isoperimetric} \) then it satisfies the following isoperimetric Euler–Lagrange equation with time delay:

\[
\begin{align*}
\frac{d}{dt} \{ \partial_3 \mathcal{F}[q, \lambda, \dot{q}] \} + \frac{d}{dt} \{ \partial_4 \mathcal{F}[q, \lambda, \dot{q}] \} &= \partial_5 \mathcal{F}[q, \lambda, \dot{q}], & 0 \leq t \leq t_1 - \tau, \\
\frac{d}{dt} \{ \partial_3 \mathcal{F}[q, \lambda, \dot{q}] \} &= \partial_2 \mathcal{F}[q, \lambda, \dot{q}], & t_1 - \tau \leq t \leq t_2.
\end{align*}
\]

\( t \in [t_1, t_2] \), where \( \mathcal{F} \) is the augmented Lagrangian \( \ref{def:augmented} \) associated with Problem \( \ref{prob:isoperimetric} \).

**Proof.** Consider neighboring functions of the form

\[
\dot{q}(t) = q(t) + \epsilon_1 h_1(t) + \epsilon_2 h_2(t),
\]

\( \epsilon_1, \epsilon_2 \in \mathbb{R} \), for all \( t \in [t_1, t_2] \).
where for each \( i \in \{1, 2\} \), \( \epsilon_i \) is a sufficiently small parameter, \( h_i \) are assumed to be functions of class \( C^2 ([t_1 - \tau, t_2]) \), \( h_i(t) = 0 \) for \( t \in [t_1 - \tau, t_1] \) and \( h_i(t_2) = h_i(t_2 - \tau) = 0 \).

First we will show that (14) has a subset of admissible functions for the variational isoperimetric problem with time delay. Consider the quantity

\[
I' [\hat{q}(\cdot)] = \int_{t_1}^{t_2} g(t, \hat{q}(t), \hat{q}(t - \tau), \hat{q}(t - \tau)) dt.
\]

Then we can regard \( I'[\hat{q}(\cdot)] \) as a function of \( \epsilon_1 \) and \( \epsilon_2 \). Define \( \hat{I}(\epsilon_1, \epsilon_2) = I'[\hat{q}(\cdot)] - l \). Thus,

\[
\hat{I}(0, 0) = 0.
\] (15)

On the other hand, we have

\[
\left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} = \int_{t_1}^{t_2} \left[ \partial_2 g[q]_\tau(t) \cdot h_2(t) + \partial_3 g[q]_\tau(t) \cdot \dot{h}_2(t) \right] dt \\
+ \int_{t_1}^{t_2} \left[ \partial_4 g[q]_\tau(t) \cdot h_2(t - \tau) + \partial_5 g[q]_\tau(t) \cdot \dot{h}_2(t - \tau) \right] dt.
\] (16)

Using the change of variable \( t = s + \tau \) in the second integral and recalling that \( h_2 \) is null in \([t_1 - \tau, t_1]\), we obtain that

\[
\left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} = \int_{t_1}^{t_2} \left[ \partial_2 g[q]_\tau(t) \cdot h_2(t) + \partial_3 g[q]_\tau(t) \cdot \dot{h}_2(t) \right] dt \\
+ \int_{t_1}^{t_2 - \tau} \left[ \partial_4 g[q]_\tau(t + \tau) \cdot h_2(t) + \partial_5 g[q]_\tau(t + \tau) \cdot \dot{h}_2(t) \right] dt.
\]

Applying integration by parts and since equation (16) holds for all admissible variations \( h_2 \) such that \( h_2 = 0 \) for all \( t \in [t_2 - \tau, t_2] \), we get

\[
\left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} = \int_{t_1}^{t_2 - \tau} \left[ \partial_2 g[q]_\tau(t) - \frac{d}{dt} \partial_3 g[q]_\tau(t) \right. \\
\left. + \partial_4 g[q]_\tau(t + \tau) - \frac{d}{dt} \partial_5 g[q]_\tau(t + \tau) \right] \cdot h_2(t) dt.
\]

Now, if we restrict ourselves to those admissible variations \( h_2 \) such that \( h_2 = 0 \) for all \( t \in [t_1, t_2 - \tau] \), we obtain

\[
\left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} = \int_{t_2 - \tau}^{t_2} \left[ \partial_2 g[q]_\tau(t) - \frac{d}{dt} \partial_3 g[q]_\tau(t) \right] \cdot h_2(t) dt.
\]

Since \( q(\cdot) \) is not a extremal for problem (9)–(2), by the fundamental lemma of the calculus of variations (see, e.g., [30]), there exists a function \( h_2 \) such that

\[
\left. \frac{\partial \hat{I}}{\partial \epsilon_2} \right|_{(0,0)} \neq 0.
\] (17)

Using (15) and (17), the implicit function theorem asserts that there exists a function \( \epsilon_2(\cdot) \), defined in a neighborhood of zero, such that \( I(\epsilon_1, \epsilon_2(\epsilon_1)) = 0 \). Consider the real function \( \hat{J}(\epsilon_1, \epsilon_2) = J^* [\hat{q}(\cdot)] \). By hypothesis, \( \hat{J} \) has minimum (or maximum) at \((0,0)\) subject to the constraint \( I(0,0) = \)
0, and we have proved that $\nabla \tilde{I}(0,0) \neq 0$. Then, we can appeal to the Lagrange multiplier rule (see, e.g., [30, p. 77]) to assert the existence of a number $\lambda$ such that $\nabla(\tilde{J}(0,0) - \lambda \cdot \tilde{I}(0,0)) = 0$. Repeating the calculations as before,

$$
\frac{\partial \tilde{J}}{\partial \epsilon_1} \bigg|_{(0,0)} = \int_{t_1}^{t_2-\tau} \left[ \partial_2 L[q] \tau(t) - \frac{d}{dt} \partial_3 L[q] \tau(t) + \partial_4 L[q] \tau(t + \tau) - \frac{d}{dt} \partial_5 L[q] \tau(t + \tau) \right] \cdot h_1(t) dt,
$$

and

$$
\frac{\partial \tilde{I}}{\partial \epsilon_1} \bigg|_{(0,0)} = \int_{t_1}^{t_2-\tau} \left[ \partial_2 g[q] \tau(t) - \frac{d}{dt} \partial_3 g[q] \tau(t) + \partial_4 g[q] \tau(t + \tau) - \frac{d}{dt} \partial_5 g[q] \tau(t + \tau) \right] \cdot h_1(t) dt,
$$

Therefore, one has

$$
\int_{t_1}^{t_2-\tau} \left[ \partial_2 L[q] \tau(t) + \partial_4 L[q] \tau(t + \tau) - \frac{d}{dt} (\partial_3 L[q] \tau(t) + \partial_5 L[q] \tau(t + \tau)) - \lambda \cdot \left( \partial_2 g[q] \tau(t) + \partial_4 g[q] \tau(t + \tau) - \frac{d}{dt} (\partial_3 g[q] \tau(t) + \partial_5 g[q] \tau(t + \tau)) \right) \right] \cdot h_1(t) dt = 0 \quad (18)
$$

and

$$
\int_{t_1}^{t_2} \left[ \partial_2 \dot{q}[\tau(t)] - \frac{d}{dt} \partial_3 \dot{q}[\tau(t)] - \lambda \cdot \left( \partial_2 g[\tau(t)] - \frac{d}{dt} \partial_3 g[\tau(t)] \right) \right] \cdot h_1(t) dt = 0. \quad (19)
$$

Since equations (18) and (19) hold for any function $h_1$, from the fundamental lemma of the calculus of variations (see, e.g., [12]), we obtain equations (13).

**Remark 11.** If one extends the set of admissible functions in Problem 1 to the class of Lipschitz continuous functions, then the Euler–Lagrange equations (13) remain valid (cf. [13]).

**Definition 12** (Isoperimetric extremals with time delay). The solutions $q(\cdot) \in C^2([t_1 - \tau, t_2])$ of the Euler–Lagrange equations (13) are called isoperimetric extremals with time delay.

**Remark 13.** Note that if there is no time delay, that is, if $\tau = 0$, then Problem 1 reduces to the classical isoperimetric variational problem:

$$
J[q(\cdot)] = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t)) dt \longrightarrow \min,
$$

$$
\int_{t_1}^{t_2} g(t, q(t), \dot{q}(t)) dt = l.
$$
3.2 The DuBois-Reymond necessary condition

The following theorem gives a generalization of the DuBois–Reymond necessary condition for classical variational problems [4] and generalizes the DuBois–Reymond necessary condition for isoperimetric variational problems with time delay of [11].

**Theorem 14** (Isoperimetric DuBois–Reymond necessary condition with time delay). If \( q(\cdot) \) is an isoperimetric extremals with time delay such that

\[
\frac{d}{dt} \{ F[q]_{\tau}(t) - \dot{q}(t) \cdot (\partial_3 F[q]_{\tau}(t) + \partial_5 F[q]_{\tau}(t + \tau)) \} = \partial_1 F[q]_{\tau}(t)
\]

for all \( t \geq t_1 - \tau, t_2 - \tau \), then it satisfies the following conditions:

\[
\frac{d}{dt} \{ F[q]_{\tau}(t) - \dot{q}(t) \cdot (\partial_3 F[q]_{\tau}(t) + \partial_5 F[q]_{\tau}(t + \tau)) \} = \partial_1 F[q]_{\tau}(t)
\]

for \( t_1 \leq t \leq t_2 - \tau \), and

\[
\frac{d}{dt} \{ F[q]_{\tau}(t) - \dot{q}(t) \cdot (\partial_3 F[q]_{\tau}(t) + \partial_5 F[q]_{\tau}(t + \tau)) \} = \partial_1 F[q]_{\tau}(t)
\]

for \( t_2 - \tau \leq t \leq t_2 \), where \( F \) is defined in [12].

**Proof.** We only prove the theorem in the interval \( t_1 \leq t \leq t_2 - \tau \) (the proof is similar in the interval \( t_2 - \tau \leq t \leq t_2 \)). We derive equation (21) as follows:

Let an arbitrary \( x \in [t_1, t_2 - \tau] \). Note that

\[
\int_{t_1}^{x} \frac{d}{dt} \{ F[q]_{\tau}(t) - \dot{q}(t) \cdot (\partial_3 F[q]_{\tau}(t) + \partial_5 F[q]_{\tau}(t + \tau)) \} dt
\]

\[
= \int_{t_1}^{x} \left[ \partial_1 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t)) + \partial_2 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t)) \cdot \dot{q}(t)
\]

\[
- \partial_3 (L[q]_{\tau}(t + \tau) + \lambda \cdot g[q]_{\tau}(t + \tau)) \cdot \dot{q}(t)
\]

\[
- \frac{d}{dt} \{ \partial_3 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t)) \cdot \dot{q}(t) + \partial_5 (L[q]_{\tau}(t + \tau) + \lambda \cdot g[q]_{\tau}(t + \tau)) \cdot \dot{q}(t) \} dt
\]

\[
+ \int_{t_1}^{x} [ \partial_4 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t)) \cdot \dot{q}(t - \tau)
\]

\[
+ \partial_5 (L[q]_{\tau}(t + \tau) + \lambda \cdot g[q]_{\tau}(t + \tau)) \cdot \dot{q}(t - \tau) \} dt.
\]

Observe that, by hypothesis (20), the last integral of (23) is null and by substituting the Euler–Lagrange equation with time delay (13), the equation (23) becomes

\[
\int_{t_1}^{x} \frac{d}{dt} \{ F[q]_{\tau}(t) - \dot{q}(t) \cdot (\partial_3 F[q]_{\tau}(t) + \partial_5 F[q]_{\tau}(t + \tau)) \} dt
\]

\[
= \int_{t_1}^{x} \left( \partial_1 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t))
\]

\[
- \left( \partial_4 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t)) \cdot \dot{q}(t) + \partial_5 (L[q]_{\tau}(t + \tau) + \lambda \cdot g[q]_{\tau}(t + \tau)) \cdot \dot{q}(t) \right) dt
\]

Using hypothesis (20) in the right hand side of the last equation, we conclude that

\[
\int_{t_1}^{x} \frac{d}{dt} \{ F[q]_{\tau}(t) - \dot{q}(t) \cdot (\partial_3 F[q]_{\tau}(t) + \partial_5 F[q]_{\tau}(t + \tau)) \} dt
\]

\[
= \int_{t_1}^{x} \left( \partial_1 (L[q]_{\tau}(t) + \lambda \cdot g[q]_{\tau}(t)) \right) dt. \quad (24)
\]

We finally obtain (21) by the arbitrariness \( x \in [t_1, t_2 - \tau] \).

**Remark 15.** If we assume that admissible functions in Problem [4] are Lipschitz continuous, then one can show that the DuBois–Reymond necessary conditions with time delay (21)–(22) are still valid (cf. [11]).
3.3 Variational isoperimetric Noether’s conservation laws with time delay

In [11] the authors remark that when one extends Noether’s theorem to the biggest class for which one can derive the Euler–Lagrange equations, i.e., for Lipschitz continuous functions, then one can find Lipschitz Euler–Lagrange extremals that fail to satisfy the Noether conserved quantity established in [10]. They show that to formulate Noether’s theorem with time delays for nonsmooth functions, it is enough to restrict the set of delayed Euler–Lagrange extremals to those that satisfy the delayed DuBois–Reymond condition.

The notion of invariance given in Definition 16 can be extended up to an exact differential.

**Definition 16** (Invariance up to a gauge-term). We say that functional (11) is invariant under the s-parameter group of infinitesimal transformations (4) up to the gauge-term if

\[
\int_I \Phi[q,t](t)dt = \frac{d}{ds} \int_{i(t)} F \left( t + s\eta(t,q(t)) + o(s), q(t) + s\xi(t,q(t)) + o(s), \frac{\dot{q}(t) + s\dot{\xi}(t,q(t))}{1 + s\eta(t,q(t))} \right) dt \bigg|_{s=0}
\]

for any subinterval \( I \subseteq [t_1,t_2] \) and for all \( q(\cdot) \in \text{Lip}([t_1-t_2]) \).

**Lemma 17** (Necessary condition of invariance). If functional (11) is invariant up to \( \Phi \) in the sense of Definition 16 then

\[
\int_{t_1}^{t_2-\tau} \left[ -\dot{\Phi}[q]_\tau(t) + \partial_t F[q]_\tau(t) \eta(t,q) + (\partial_2 F[q]_\tau(t) + \partial_4 F[q]_\tau(t + \tau)) \cdot \xi(t,q) \\
+ (\partial_3 F[q]_\tau(t) + \partial_5 F[q]_\tau(t + \tau)) \cdot (\dot{\xi}(t,q) - \dot{q}(t)\dot{\eta}(t,q)) + F[q]_\tau(t)\dot{\eta}(t,q) \right] dt = 0
\]

for \( t_1 \leq t \leq t_2 - \tau \) and

\[
\int_{t_2-\tau}^{t_2} \left[ -\dot{\Phi}[q]_\tau(t) + \partial_t F[q]_\tau(t) \eta(t,q) + \partial_2 F[q]_\tau(t) \cdot \xi(t,q) \\
+ \partial_3 F[q]_\tau(t) \cdot (\dot{\xi}(t,q) - \dot{q}(t)\dot{\eta}(t,q)) + F[q]_\tau(t)\dot{\eta}(t,q) \right] dt = 0
\]

for \( t_2 - \tau \leq t \leq t_2 \).

**Proof.** Without loss of generality, we take \( I = [t_1,t_2] \). Then, (25) is equivalent to

\[
\int_{t_1}^{t_2} \left[ -\dot{\Phi}[q]_\tau(t) + \partial_t (L[q]_\tau(t) + \lambda \cdot g[q]_\tau(t)) \eta(t,q) + \partial_2 (L[q]_\tau(t) + \lambda \cdot g[q]_\tau(t)) \cdot \xi(t,q) \\
+ \partial_3 (L[q]_\tau(t) + \lambda \cdot g[q]_\tau(t)) \cdot (\dot{\xi}(t,q) - \dot{q}(t)\dot{\eta}(t,q)) + (L[q]_\tau(t) + \lambda \cdot g[q]_\tau(t)) \dot{\eta}(t,q) \right] dt \\
+ \int_{t_1}^{t_2} \left[ \partial_4 (L[q]_\tau(t) + \lambda \cdot g[q]_\tau(t)) \cdot \xi(t-\tau,q(t-\tau)) \\
+ \partial_5 (L[q]_\tau(t) + \lambda \cdot g[q]_\tau(t)) \cdot (\dot{\xi}(t-\tau,q(t-\tau)) - \dot{q}(t-\tau)\dot{\eta}(t-\tau,q(t-\tau))) \right] dt = 0.
\]

Performing a linear change of variables \( t = \sigma + \tau \) in the last integral of (28), and keeping in mind...
that \( \xi = \eta = 0 \) on \([t_1 - \tau, t_1]\), equation (28) becomes

\[
\int_{t_1}^{t_2 - \tau} \left[ -\dot{\Phi}[t, q, \tau] (t) + \partial_t (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) \eta(t, q) \right. \\
+ (\partial_2 (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) + \partial_4 (L[q, \tau] (t + \tau) + \lambda \cdot g[q, \tau] (t + \tau))) \cdot \xi(t, q) \\
+ (\partial_3 (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) + \partial_5 (L[q, \tau] (t + \tau) + \lambda \cdot g[q, \tau] (t + \tau))) \cdot \left( \dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) \\
\left. + (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) \eta(t, q) \right] \, dt \\
+ \int_{t_2 - \tau}^{t_2} \left[ -\Phi[q, \tau] (t) + \partial_t (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) \eta(t, q) + \partial_2 (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) \cdot \xi(t, q) \\
+ \partial_3 (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) \cdot \left( \dot{\xi}(t, q) - \dot{q}(t) \dot{\eta}(t, q) \right) + (L[q, \tau] (t) + \lambda \cdot g[q, \tau] (t)) \dot{\eta}(t, q) \right] \, dt = 0. 
\] (29)

Taking into consideration that (29) holds for an arbitrary subinterval \( I \subseteq [t_1, t_2] \), equations (28) and (27) hold.

**Theorem 18** (Noether’s symmetry theorem with time delay for Lipschitz functions). If functional (11) is invariant up to \( \Phi \) in the sense of Definition 16 such that satisfy the condition (20), then the quantity \( C(t, t + \tau, q(t), q(t - \tau), q(t + \tau), \dot{q}(t), \dot{q}(t - \tau), \dot{q}(t + \tau)) \) defined by

\[
-\Phi[q, \tau] (t) + (\partial_3 F[q, \tau] + \partial_5 F[q, \tau] (t + \tau)) \cdot \xi(t, q(t)) \\
+ (F[q, \tau] - \dot{q}(t) \cdot (\partial_3 F[q, \tau] + \partial_5 F[q, \tau] (t + \tau))) \eta(t, q(t)) 
\] (30)

for \( t_1 \leq t \leq t_2 - \tau \) and by

\[
-\Phi[q, \tau] (t) + \partial_3 F[q, \tau] (t) \cdot \xi(t, q(t)) + (F[q, \tau] - \dot{q}(t) \cdot \partial_3 F[q, \tau] (t)) \eta(t, q(t)) 
\] (31)

for \( t_2 - \tau \leq t \leq t_2 \), is a constant of motion with time delay along any \( q(\cdot) \in \text{Lip}([t_1 - \tau, t_2]) \) satisfying both (13) and (24), i.e., along any Lipschitz Euler–Lagrange extremal that is also a Lipschitz DuBois–Reynold extremal.

**Proof.** We prove the theorem in the interval \( t_1 \leq t \leq t_2 - \tau \). The proof is similar for the interval \( t_2 - \tau \leq t \leq t_2 \). Noether’s constant of motion with time delay (30) follows by using in the interval \( t_1 \leq t \leq t_2 - \tau \) the DuBois–Reymond condition with time delay (21) and the Euler–Lagrange equation with time delay (18) into the necessary condition of invariance (28).
that is,
\[ \int_{t_1}^{t_2} \frac{d}{dt} \left[ -\Phi[q]_t(t) + (\partial_3 L[q]_t(t) + \partial_5 L[q]_t(t + \tau)) \cdot \xi(t, q(t)) \right. \\
\left. + \left( L[q]_t(t) - \dot{q}(t) \cdot (\partial_3 L[q]_t(t) + \partial_5 L[q]_t(t + \tau)) \right) \eta(t, q(t)) \right] dt = 0. \] (32)

Taking into consideration that (32) holds for any subinterval \( I \subseteq [t_1, t_2 - \tau] \), we conclude that
\[- \Phi[q]_t(t) + (\partial_3 L[q]_t(t) + \partial_5 L[q]_t(t + \tau)) \cdot \xi(t, q(t))
+ \left( L[q]_t(t) - \dot{q}(t) \cdot (\partial_3 L[q]_t(t) + \partial_5 L[q]_t(t + \tau)) \right) \eta(t, q(t)) \] is an isoperimetric Euler–Lagrange extremal, i.e., satisfies
\[ \text{Example 19. Consider the isoperimetric problem of the calculus of variations with time delay} \]
\[ J^1[q(\cdot)] = \int_0^3 (\dot{q}(t) + \dot{q}(t - \tau))^3 dt \rightarrow \min, \] (33)
subject to isoperimetric equality constraints
\[ P^1[q(\cdot)] = \int_0^3 (\dot{q}(t) - \dot{q}(t - 1))^2 dt = l \] (34)
in the class of functions \( q(\cdot) \in \text{Lip}([-1, 3]) \). For this example, the augmented Lagrangian \( F \) is given as
\[ F = (\dot{q}(t) + \dot{q}(t - \tau))^3 - \lambda (\dot{q} + \dot{q}(t - 1))^2. \] (35)

From Theorem 17 (see Remark 17), one obtains that any solution to problem (33)–(34) must satisfy
\[ 3 (\dot{q} + \dot{q}(t - 1))^2 + (\dot{q}(t + 1) + \dot{q}(t))^2 - 2\lambda (2\dot{q}(t) + \dot{q}(t - 1) + \dot{q}(t + 1)) = c_1, \quad 0 \leq t \leq 2, \] (36)
\[ 3 (\dot{q} + \dot{q}(t - 1))^2 - 2\lambda (\dot{q}(t) + \dot{q}(t - 1)) = c_2, \quad 2 \leq t \leq 3, \] (37)
where \( c_1 \) and \( c_2 \) are constants. Because problem (33)–(34) is autonomous, we have invariance, in the sense of Definition 17, with \( \eta \equiv 1 \) and \( \xi \equiv 0 \). Simple calculations show that isoperimetric Noether’s constant of motion with time delay (34)–(31) coincides with the DuBois–Reymond condition (21)–(22):
\[ (\dot{q}(t) + \dot{q}(t - \tau))^3 - \lambda (\dot{q} + \dot{q}(t - 1))^2 - \dot{q}(t) \left( 3 (\dot{q} + \dot{q}(t - 1))^2 + (\dot{q}(t + 1) + \dot{q}(t))^2 \right) \\
- 2\lambda (2\dot{q}(t) + \dot{q}(t - 1) + \dot{q}(t + 1)) = c_3, \quad 0 \leq t \leq 2, \] (38)
\[ (\dot{q}(t) + \dot{q}(t - \tau))^3 - \lambda (\dot{q} + \dot{q}(t - 1))^2 \\
- \dot{q}(t) \left( 3 (\dot{q} + \dot{q}(t - 1))^2 + 2(\dot{q} + \dot{q}(t - 1)) \right) = c_4, \quad 2 \leq t \leq 3, \] (39)
where \( c_3 \) and \( c_4 \) are constants.

One can easily check that function \( q(\cdot) \in \text{Lip}([-1, 3]) \) defined by
\[ q(t) = \begin{cases} 
-1 & \text{for } -1 < t \leq 0 \\
t & \text{for } 0 < t \leq 1 \\
-1 & \text{for } 1 < t \leq 2 \\
-2 & \text{for } 2 < t \leq 3 
\end{cases} \] (40)
is an isoperimetric Euler–Lagrange extremal, i.e., satisfies (36)–(37) and is also a isoperimetric DuBois–Reymond extremal, i.e., satisfies (33)–(31). Theorem 18 asserts the validity of Noether’s constant of motion, which is here verified: (30)–(31) holds along (40) with \( \Phi \equiv 0, \eta \equiv 1, \) and \( \xi \equiv 0. \)
3.4 Isoperimetric optimal control with time delay

Theorem [18] gives a Lagrangian formulation of isoperimetric Noether’s principle to the time delay setting. Now we give a Hamiltonian formulation to prove an isoperimetric weak Pontryagin maximum principle for the more general isoperimetric problems of optimal control with time delay (Theorem [23]).

The isoperimetric optimal control problem with time delay is defined in Lagrange form as follows: to minimize

$$I^* [q(\cdot), u(\cdot)] = \int_{t_1}^{t_2} L(t, q(t), u(t), q(t - \tau), u(t - \tau)) \, dt$$

subject to the delayed control system

$$\dot{q}(t) = \varphi(t, q(t), u(t), q(t - \tau), u(t - \tau))$$

isoperimetric equality constraints

$$\int_{t_1}^{t_2} g(t, q(t), u(t), q(t - \tau), u(t - \tau)) \, dt = l, \ l \in k$$

and initial condition

$$q(t) = \delta(t), \ \ t \in [t_1 - \tau, t_1],$$

where $q(\cdot) \in C^1([t_1 - \tau, t_2]), u(\cdot) \in C^0([t_1 - \tau, t_2])$, the functions $L, g : [t_1, t_2] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ and the velocity vector $\varphi : [t_1, t_2] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ are assumed to be $C^1$-functions with respect to all their arguments, $t_1 < t_2$ are fixed in $\mathbb{R}$, and $\tau$ is a given positive real number such that $\tau < t_2 - t_1$. As before, we assume that $\delta$ is a given piecewise smooth function.

**Remark 20.** In the particular case when $\varphi(t, q, q_\tau, u_\tau) = u$, problem (41)–(44) is reduced to the Problem (7).

**Notation.** We introduce the operators $[\cdot, \cdot, \cdot],_\tau$ and $[\cdot, \cdot, \cdot],_\tau$ defined by

$$[q, u],_\tau(t) = (t, q(t), u(t), q(t - \tau), u(t - \tau)),$$

where $q(\cdot) \in C^1([t_1 - \tau, t_2])$ and $u(\cdot) \in C^0([t_1 - \tau, t_2])$; and

$$[q, u, p, \lambda],_\tau(t) = (t, q(t), u(t), q(t - \tau), u(t - \tau), p(t), \lambda),$$

where $q(\cdot) \in C^1([t_1 - \tau, t_2]), p(\cdot) \in C^1([t_1, t_2]), u(\cdot) \in C^0([t_1 - \tau, t_2])$ and $\lambda \in \mathbb{R}$.

**Definition 21.** The delayed differential control system (42) is called an isoperimetric control system with time delay.

**Definition 22.** (Isoperimetric process with time delay) An admissible pair $(q(\cdot), u(\cdot))$ that satisfies the isoperimetric control system (42) and the isoperimetric constraints (43) is said to be a isoperimetric control system with time delay.

**Theorem 23.** (Isoperimetric Weak Pontryagin maximum principle) If $(q(\cdot), u(\cdot))$ is a minimizer of (41)–(44), then there exists a covector function $p(\cdot) \in C^1([t_1, t_2])$ such that for all $t \in [t_1 - \tau, t_2]$ the following conditions hold:

- the isoperimetric Hamiltonian systems with time delay

$$\begin{cases}
\dot{q}(t) = \partial_0 H[q, u, p, \lambda],_\tau(t) \\
\dot{p}(t) = -\partial_2 H[q, u, p, \lambda],_\tau(t) - \partial_4 H[q, u, p, \lambda],_\tau(t + \tau)
\end{cases}$$

for $t_1 \leq t \leq t_2 - \tau$, and

$$\begin{cases}
\dot{q}(t) = \partial_0 H[q, u, p, \lambda],_\tau(t) \\
\dot{p}(t) = -\partial_2 H[q, u, p, \lambda],_\tau(t)
\end{cases}$$

for $t_2 - \tau \leq t \leq t_2$. 

• the isoperimetric stationary conditions with time delay
  \[ \partial_t H[q, u, p, \lambda]_\tau(t) + \partial_{\tau} H[q, u, p, \lambda]_\tau(t + \tau) = 0 \quad (47) \]
  for \( t_1 \leq t \leq t_2 - \tau \), and
  \[ \partial_\tau H[q, u, p]_\tau(t) = 0 \quad (48) \]
  for \( t_2 - \tau \leq t \leq t_2 \).

where the isoperimetric Hamiltonian \( H \) is defined by

\[ H[q, u, p, \lambda]_\tau(t) = L[q, u]_\tau(t) - \lambda \cdot g[q, u]_\tau(t) + p(t) \cdot \varphi[q, u]_\tau(t). \quad (49) \]

**Proof.** We prove the theorem only in the interval \( t_1 \leq t \leq t_2 - \tau \) (the reasoning is similar in the interval \( t_2 - \tau \leq t \leq t_2 \)). Minimizing (41) subject to (42)–(44) is equivalent, by the Lagrange multiplier rule, to minimize

\[ \mathcal{T}^*[q(\cdot), u(\cdot), p(\cdot), \lambda] = \int_{t_1}^{t_2} \left[ H[q, u, p, \lambda]_\tau(t) - p(t) \cdot \dot{q}(t) \right] dt \quad (50) \]

with \( H \) given by (49). Theorem 23 follows by applying the isoperimetric Euler–Lagrange optimality condition (13) to the equivalent functional (50):

\[
\begin{cases}
\frac{d}{dt} \left( \mathbb{L}_q[q, u, p, \lambda]_\tau(t) + \mathbb{L}_q[q, u, p, \lambda]_\tau(t + \tau) \right) \\
\quad = \mathbb{L}_q[q, u, p, \lambda]_\tau(t) + \mathbb{L}_q[q, u, p, \lambda]_\tau(t + \tau) \\
\frac{d}{dt} \left( \mathbb{L}_u[q, u, p, \lambda]_\tau(t) + \mathbb{L}_u[q, u, p, \lambda]_\tau(t + \tau) \right) \\
\quad = \mathbb{L}_u[q, u, p, \lambda]_\tau(t) + \mathbb{L}_u[q, u, p, \lambda]_\tau(t + \tau) \\
\frac{d}{dt} \left( \mathbb{L}_p[q, u, p, \lambda]_\tau(t) + \mathbb{L}_p[q, u, p, \lambda]_\tau(t + \tau) \right) \\
\quad = \mathbb{L}_p[q, u, p, \lambda]_\tau(t) + \mathbb{L}_p[q, u, p, \lambda]_\tau(t + \tau) \\
\end{cases}
\]

\[ \Leftrightarrow \begin{cases} p(t) = -\partial_\tau H[q, u, p, \lambda]_\tau(t) - \partial_{\tau} H[q, u, p, \lambda]_\tau(t + \tau) \\
0 = \partial_\tau H[q, u, p, \lambda]_\tau(t) + \partial_\tau H[q, u, p, \lambda]_\tau(t + \tau) \\
0 = -\dot{q}(t) + \partial_\tau H[q, u, p, \lambda]_\tau(t) \end{cases} \]

where \( \mathbb{L}_q[q, u, p, \lambda]_\tau(t) = H[q, u, p, \lambda]_\tau(t) - p(t) \cdot \dot{q}(t) \), \( \mathbb{L}_p[q, u, p, \lambda]_\tau(t) \) denotes the partial derivative of \( \mathbb{L} \) with respect to \( \zeta \) argument and \( \zeta_\tau = \zeta(t - \tau) \).

**Definition 24.** A triplet \((q(\cdot), u(\cdot), p(\cdot))\) satisfying the conditions of Theorem 23 is called an isoperimetric Pontryagin extremal with time delay.

**Remark 25.** The first equation in the Hamiltonian system (45) and (46) is nothing but the control system with time delay \( \dot{q}(t) = \varphi[q, u]_\tau(t) \) given by (42).

**Remark 26.** In the language of optimal control, \( p \) is called the generalized momentum. In the language of optimal control, \( p \) is known as the adjoint variable.

**Remark 27.** In the particular case when \( \varphi[q, u, p, \lambda]_\tau = u \), Theorem 23 reduces to Theorem 14. We verify this here in the interval \( t_1 \leq t \leq t_2 - \tau \), the procedure being similar for the interval \( t_2 - \tau \leq t \leq t_2 \). The stationary condition with time delay (47) gives \( p(t) = -\partial_\tau H[q, u, p, \lambda]_\tau(t) \) and the second equation in the Hamiltonian system with time delay (48) gives \( \dot{q}(t) = -\partial_\tau H[q, u, p, \lambda]_\tau(t) \). Comparing both equalities, one obtains the first Euler–Lagrange equation with time delay in (13). In other words, isoperimetric Pontryagin extremals with time delay (Definition 24) are a generalization of the normal Euler–Lagrange extremals with time delay (Definition 2).

In classical optimal control the DuBois–Reymond necessary optimality condition is generalized to the equality \( \frac{dH}{dt} = \partial H / \partial t \) (14). Next, we extend Theorem 14 (the DuBois–Reymond necessary condition with time delay) to the more general isoperimetric optimal control setting with time delay.
Theorem 28. If a triplet \((q(\cdot), u(\cdot), p(\cdot))\) with \(q(\cdot) \in C^1([t_1, t_2])\), \(p(\cdot) \in C^1([t_1, t_2])\), and \(u(\cdot) \in C^1([t_1, t_2])\) is an isoperimetric Pontryagin extremal with time delay such that
\[
\partial_4 H[q, u, p, \lambda]_\tau(t + \tau) \cdot \dot{q}(t) + \partial_3 H[q, u, p, \lambda]_\tau(t + \tau) \cdot \dot{u}(t) = 0
\]
for all \(t \in [t_1, t_1]\), then it satisfies the following condition:
\[
\frac{d}{dt} H[q, u, p, \lambda]_\tau(t) = \partial_1 H[q, u, p, \lambda]_\tau(t), \quad t \in [t_1, t_2].
\]

Proof. We prove condition (52) by direct calculations:
\[
\int_{t_1}^{t_2} \frac{d}{dt} H[q, u, p, \lambda]_\tau(t) dt
\]
\[
= \int_{t_1}^{t_2} \left[ \partial_4 H[q, u, p, \lambda]_\tau(t) + \partial_2 H[q, u, p, \lambda]_\tau(t) \cdot \dot{q}(t) + \partial_3 H[q, u, p, \lambda]_\tau(t) \cdot \dot{u}(t) + \partial_6 H[q, u, p, \lambda]_\tau(t) \cdot \dot{p}(t) \right] dt
\]
and by performing a linear change of variables \(t = \nu + \tau\) in the last integral of (53) and using the conditions (51), equation (53) becomes
\[
\int_{t_1}^{t_2} \frac{d}{dt} H[q, u, p, \lambda]_\tau(t) dt
\]
\[
= \int_{t_1}^{t_2 - \tau} \left[ \partial_4 H[q, u, p, \lambda]_\tau(t) + \partial_2 H[q, u, p, \lambda]_\tau(t + \tau) \cdot \dot{q}(t) + \partial_3 H[q, u, p, \lambda]_\tau(t + \tau) \cdot \dot{u}(t) + \partial_6 H[q, u, p, \lambda]_\tau(t + \tau) \cdot \dot{p}(t) \right] dt
\]
\[
+ \int_{t_2 - \tau}^{t_2} \left[ \partial_4 H[q, u, p, \lambda]_\tau(t) + \partial_2 H[q, u, p, \lambda]_\tau(t) \cdot \dot{q}(t) + \partial_3 H[q, u, p, \lambda]_\tau(t) \cdot \dot{u}(t) + \partial_6 H[q, u, p, \lambda]_\tau(t) \cdot \dot{p}(t) \right] dt.
\]
We obtain condition (52) by substituting (45) and (47) into the first integral, and substituting (46) and (48) into the second integral of (54).

4 Conclusions and Open Questions

The isoperimetric problems are a mathematical area of a currently strong research, with numerous applications in physics and engineering. The theory of conservation law in variational calculus with delay systems was recently initiated in [10], with the proof of delayed Noether’s theorem. In this paper we go a step further: we prove an isoperimetric Noether’s theorem with time delay.

The isoperimetric variational theory with time delay is in its childhood so that much remains to be done. This is particularly true in the area of isoperimetric optimal control with time delay, where the results are rare. Here, an isoperimetric Hamiltonian formulation with time delay is obtained. To the best of the author’s knowledge, there is no general formulation of an isoperimetric version of Pontryagin’s Maximum Principle. Then, with an isoperimetric notion of Pontryagin extremal, one can try to extend the present results to the more general context of isoperimetric optimal control with time delay.
References

[1] Z. Bartosiewicz and D. F. M. Torres, Noether’s theorem on time scales, J. Math. Anal. Appl. 342 (2008), no. 2, 1220–1226. arXiv:0709.0400

[2] M. Basin, New trends in optimal filtering and control for polynomial and time-delay systems, Lecture Notes in Control and Information Sciences, 380, Springer, Berlin, 2008.

[3] G. V. Bokov, Pontryagin’s maximum principle in a problem with time delay, J. Math. Sci. (N. Y.) 172 (2011), no. 5, 623–634.

[4] J. Chiasson and J. J. Loiseau (Eds.) Applications of Time Delay Systems Springer-Verlag Berlin Heidelberg, 2007.

[5] D. S. Djukić Noether’s theorem for optimum control systems, Internat. J. Control (1) 18 (1973), 667–672.

[6] G. S. F. Frederico, Generalizations of Noether’s theorem in the calculus of variations and optimal control, Ph.D. thesis, University of Cape Verde, 2009.

[7] G. S. F. Frederico and M. J. Lazo, Fractional Noether’s Theorem with Classical and Caputo Derivatives: constants of motion for non-conservative systems, accepted for publication in Nonlinear Dynamics.

[8] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. 334 (2007), no. 2, 834–846. arXiv:math/0701187

[9] G. S. F. Frederico and D. F. M. Torres, Conservation laws for invariant functionals containing compositions, Appl. Anal. 86 (2007), no. 9, 1117–1126. arXiv:0704.0949

[10] G. S. F. Frederico and D. F. M. Torres, Noether’s symmetry theorem for variational and optimal control problems with time delay, Numer. Algebra Control Optim. 2 (2012), no. 3, 619–630. arXiv:1203.3656

[11] G. S. F. Frederico, T. Odzijewicz and D. F. M. Torres, Noether’s Theorem for Nonsmooth Extremals of Variational Problems with Time Delay, Applicable Analysis 93 (2014), no. 1, 153–170.

[12] E. Fridman, Introduction to Time-Delay Systems Springer International Publishing, Springer International Publishing Switzerland, 2014

[13] I. M. Gelfand and S. V. Fomin: Calculus of variations, Prentice-Hall, Englewood Cliffs, N.J. 1963.

[14] L. Göllmann, D. Kern and H. Maurer, Optimal control problems with delays in state and control variables subject to mixed control-state constraints, Optimal Control Appl. Methods 30 (2009), no. 4, 341–365.

[15] P. D. F. Gouveia, D. F. M. Torres and E. A. M. Rocha, Symbolic computation of variational symmetries in optimal control, Control Cybernet. 35 (2006), no. 4, 831–849. arXiv:math/0604072

[16] D. K. Hughes, Variational and optimal control problems with delayed argument, J. Optimization Theory Appl. 2 (1968), 1–14.

[17] J. Jost and X. Li-Jost, Calculus of variations, Cambridge Studies in Advanced Mathematics, 64, Cambridge Univ. Press, Cambridge, 1998.
[18] G. L. Kharatishvili, A maximum principle in extremal problems with delays, in Mathematical Theory of Control (Proc. Conf., Los Angeles, Calif., 1967), 26–34, Academic Press, New York, 1967.

[19] G. L. Kharatishvili and T. A. Tadumadze, Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments, J. Math. Sci. (N. Y.) 140 (2007), no. 1, 1–175.

[20] E. Noether, Y. Kosmann-Schwarzbach and L. Meersseman, Les thormes de Noether: invariance et lois de conservation au XXe sicle, Palaiseau (Essonne), Les ditions de l’cole Polytechnique, 2004.

[21] J. D. Logan, Applied mathematics, Wiley, New York, 1987.

[22] A. B. Malinowska and D. F. M. Torres, Introduction to the fractional calculus of variations, Imperial College Press, London & World Scientific Publishing, Singapore, 2012.

[23] N. Martins and D. F. M. Torres, Noether’s symmetry theorem for nabla problems of the calculus of variations, Appl. Math. Lett. 23 (2010), no. 12, 1432–1438. arXiv:1007.5178

[24] E. Noether, Invariant variation problems, Transport Theory Statist. Phys. 3 (1971), no. 1, 186–207.

[25] (MR0166037) L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, “The mathematical theory of optimal processes”, Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt Interscience Publishers John Wiley & Sons, Inc. New York, 1962.

[26] J. F. Rosenblueth, Systems with time delay in the calculus of variations: the method of steps,IMA J. Math. Control Inform. 5 (1988), no. 4, 285–299.

[27] D. F. M. Torres, On the Noether theorem for optimal control, Eur. J. Control 8 (2002), no. 1, 56–63.

[28] D. F. M. Torres, Proper extensions of Noether’s symmetry theorem for nonsmooth extremals of the calculus of variations, Commun. Pure Appl. Anal. 3 (2004), no. 3, 491–500.

[29] J. L. Troutman, Variational calculus and optimal control, second edition, Undergraduate Texts in Mathematics, Springer, New York, 1996.

[30] B. Van Brunt, The calculus of variations, Springer, New York, 2004.