SYMMETRIES IN LINEAR AND INTEGER PROGRAMS

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Abstract. The notion of symmetry is defined in the context of Linear and Integer Programming. Symmetric linear and integer programs are studied from a group theoretical viewpoint. We show that for any linear program there exists an optimal solution in the fixed point set of its symmetry group. Using this result, we develop an algorithm that allows for reducing the dimension of any linear program having a non-trivial group of symmetries.

1. Introduction

Order, beauty and perfection – these are the words we typically associate with symmetry. Generally, we expect the structure of objects with many symmetries to be uniform and regular, thus not too complicated. Therefore, symmetries usually are very welcomed in many scientific areas, especially in mathematics. However, in integer programming, the reverse seems to be true. In practice, highly symmetric integer programs often turn out to be particularly hard to solve. The problem is that branch-and-bound or branch-and-cut algorithms, which are commonly used to solve integer programs, work efficiently only if the bulk of the branches of the search tree can be pruned. Since symmetry in integer programs usually entails many equivalent solutions, the branches belonging to these solutions cannot be pruned, which leads to a very poor performance of the algorithm.

Only in the last few years first efforts were made to tackle this irritating problem. In 2002, Margot presented an algorithm that cuts feasible integer points without changing the optimal value of the problem, compare [6]. Improvements and generalizations of this basic idea can be found in [7, 8]. In [9, 10], Linderoth et al. concentrate on improving branching methods for packing and covering integer problems by using information about the symmetries of the integer programs. Another interesting approach to these kind of problems has been developed by Kaibel and Pfetsch. In [5], the authors introduce special polyhedra, called orbitopes, which they use in [4] to remove redundant branches of the search tree. Friedman’s fundamental domains in [2] are also aimed at avoiding the evaluation of redundant solutions. For selected integer programs like generalized bin-packing problems there exists a completely different idea how to deal with symmetries, see e.g. [1]. Instead of eliminating the effects of symmetry during the branch-and-bound process, the authors exclude symmetry already in the formulation of the problem by choosing an appropriate representation for feasible packings.

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All ideas in the aforementioned papers finally rely on the branch-and-bound algorithm, or they are only applicable to selected problems. In contrast to this optimizational or specialized point of view, we want to approach the topic from a more general and algebraic angle and detach ourselves from the classical optimization methods like branch-and-bound. In this paper we will examine symmetries of linear programs in their natural environment, the field of group theory. Our main objective aims at a better understanding of the role of symmetry in the context of linear and integer programming. In a subsequent paper we will discuss symmetries of integer programs.

2. Preliminaries

Optimization problems whose solutions must satisfy several constraints are called restricted optimization problems. If all constraints as well as the objective function are linear, we call them linear programs, LP for short. The linearity of such problems suggests the following canonical formulation for arbitrary LP problems.

\[
\begin{align*}
\max & \quad c^t x \\
\text{s.t.} & \quad Ax \leq b, \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m \) and \( c \in \mathbb{R}^n \setminus \{0\} \). We are especially interested in points that are candidates for solutions of an LP.

Definition. A point \( x \in \mathbb{R}^n \) is feasible for an LP if it satisfies all constraints of the LP. The LP itself and any set of points is feasible if it has at least one feasible point.

Hence, the set of feasible points \( X \) of (1) is given by

\[
X := \{ x \in \mathbb{R}^n \mid Ax \leq b \}.
\]

Convention. We call \( X \) the feasible region, \( c \) the utility vector and \( n \) the dimension of \( \Lambda \). The map \( x \mapsto c^t x \) is called the utility function, and the value of the utility function with respect to a specific \( x \in \mathbb{R}^n \) is called the utility value of \( x \).

We can interpret the feasible region of an LP in a geometric sense. The following definition is adopted from [11], p. 87.

Definition. A polyhedron \( P \subseteq \mathbb{R}^n \) is the intersection of finitely many affine half-spaces, i.e.,

\[
P := \{ x \in \mathbb{R}^n \mid Ax \leq b \},
\]

for a matrix \( A \in \mathbb{R}^{m \times n} \) and a vector \( b \in \mathbb{R}^m \).

Note that every row of the system \( Ax \leq b \) defines an affine half-space. Obviously, the set \( X \) is a polyhedron. Since every affine half-space is convex, the intersection of affine half-spaces – hence, any polyhedron – is convex as well. Therefore, we can now state the convexity of \( X \).

Remark 1. The feasible region of an LP is convex.

Whenever we consider linear programs, we are particularly interested in points with maximal utility values that satisfy all the constraints.
Definition. A solution of an LP is an element \( x^* \in \mathbb{R}^n \) that is feasible and maximizes the utility function.

If we additionally insist on integrality of the solution, we get a so-called integer program, IP for short. According to the LP formulation in (1), the appropriate formulation for the related IP is given by

\[
\max \quad c^t x \\
\text{s.t.} \quad Ax \leq b, \ x \in \mathbb{Z}^n,
\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \) and \( c \in \mathbb{R}^n \setminus \{0\} \).

Analogously, the set of feasible points \( X_I \) of (2) is given by

\[
X_I := \{ x \in \mathbb{R}^n \mid Ax \leq b, \ x \in \mathbb{Z}^n \} = X \cap \mathbb{Z}^n.
\]

We now want to clarify the meaning of the term symmetry in the context of linear and integer programming.

3. Symmetries

In general, symmetries are automorphisms, that is, operations that map an object to itself in a bijective way compatible with its structure. Concerning linear and integer programs, we therefore have to consider operations that preserve both the utility vector and the inequality system, thus, in particular, the polyhedron which is described by the inequality system. By the usage of matrix notation, this polyhedron is already embedded in Euclidean space \( \mathbb{R}^n \). This is the point where we have to decide whether we want to regard \( \mathbb{R}^n \) as an affine or as a linear space. In respect of the algorithms we are going to develop, we follow the general tendency in the literature and choose the linear perspective for the sake of a simpler group structure. Hence, the operations we consider are automorphisms of the linear space \( \mathbb{R}^n \), that is, elements of the general linear group \( \text{GL}_n(\mathbb{R}) \). Furthermore, it is reasonable to restrict the set of possible symmetries even to isometries taking into account that the angles and the lengths of the edges of the polyhedron need to be preserved. Since the set of all automorphisms of an object always is a group, we therefore suggest that the symmetries of a linear or an integer program form a subgroup of the orthogonal group \( \text{O}_n(\mathbb{R}) \).

In general, a linear program and the associated integer program need not have the same symmetries. The following two examples illustrate this fact.

Since we are forced to rely on the linear description of an integer program to gain information about its symmetries, we want to make sure that any symmetry of a linear program is a symmetry of the associated integer program as well. As integer programs are naturally confined to the standard lattice \( \mathbb{Z}^n \), we only consider orthogonal operations that leave the lattice invariant. In particular, such an operation represented by a matrix \( M \in \text{O}_n(\mathbb{R}) \) maps any standard basis vector \( e_j \) to an integer vector

\[
Me_j = (m_{1j}, \ldots, m_{nj})^t \in \mathbb{Z}^n,
\]

which is the \( j \)-th column of the matrix \( M \). Hence, all columns of \( M \) have to be integral, that is,

\[
M \in \text{O}_n(\mathbb{R}) \cap \text{GL}_n(\mathbb{Z})
\]

where \( \text{GL}_n(\mathbb{Z}) \) is the group of all integrally invertible matrices.
Figure 1: The symmetries of the linear and the associated integer program do not coincide.

Notation. The group of all orthogonal matrices with integral entries
\[ O_n(\mathbb{R}) \cap \text{GL}_n(\mathbb{Z}) \leq O_n(\mathbb{R}) \]
is denoted by \( O_n(\mathbb{Z}) \).

Note that orthogonal matrices with integral entries always are integrally invertible. We want to learn more about \( O_n(\mathbb{Z}) \). Since any map \( M \in O_n(\mathbb{Z}) \) preserves the distance, the column \((m_{ij}, \ldots, m_{nj})^t\) is an integer vector of length 1, thus
\[ (m_{ij}, \ldots, m_{nj})^t \in \{ \pm e_i \mid 1 \leq i \leq n \} \]
for \( 1 \leq j \leq n \). Therefore, the set \( O_n(\mathbb{Z}) \) only consists of signed permutation matrices. In fact, since every signed permutation matrix is orthogonal and integral, the set of signed permutation matrices is equal to \( O_n(\mathbb{Z}) \). Apparently, the group \( O_n(\mathbb{Z}) \) acts on the set
\[ \{ \mathbb{R}e_i \mid 1 \leq i \leq n \} . \]
The kernel of this action is the group of sign-changes denoted by \( D_{\text{sign}} \). Hence, the group \( D_{\text{sign}} \) is a normal subgroup of \( O_n(\mathbb{Z}) \). Furthermore, any element of \( O_n(\mathbb{Z}) \), that is, any signed permutation matrix \( M \) has a unique representation \( M = DP \), where \( D \) is a sign-changing matrix, thus a diagonal matrix with entries \( \pm 1 \) on its diagonal, and \( P \) is a permutation matrix. Therefore, we have
\[ O_n(\mathbb{Z}) = D_{\text{sign}} \mathcal{P}_n , \]
where \( \mathcal{P}_n \leq O_n(\mathbb{Z}) \) denotes the subgroup of all \( (n \times n) \)-permutation matrices. Since \( D_{\text{sign}} \) and \( \mathcal{P}_n \) intersect trivially, we finally conclude that \( O_n(\mathbb{Z}) \) splits over \( D_{\text{sign}} \).

Remark 2. The group \( O_n(\mathbb{Z}) \) is the semidirect product
\[ O_n(\mathbb{Z}) = D_{\text{sign}} \rtimes \mathcal{P}_n . \]

In the literature, the group \( O_n(\mathbb{Z}) \) appears in the context of finite reflection groups. More precisely, the group \( O_n(\mathbb{Z}) \) is the Coxeter group \( B_n \) of rank \( n \), compare [3], p. 5.

Due to the invariance of the standard lattice \( \mathbb{Z}^n \) under \( O_n(\mathbb{Z}) \), the elements of \( O_n(\mathbb{Z}) \) have the potential to satisfy the strict requirements we made on symmetries. That
is, elements of $O_n(\mathbb{Z})$ that preserve an LP, i.e., its inequality system and its utility vector, also leave the associated IP invariant.

**Remark 3.** The invariance of an LP under an element of $O_n(\mathbb{Z})$ implies the invariance of the related IP under the same element.

However, the reverse does not hold in general, since we can always add asymmetric cuts to an LP without affecting the set of feasible points of the corresponding IP, compare Figure 1b. We could now define symmetries of linear and integer programs as elements of $O_n(\mathbb{Z})$ that leave invariant the inequality system and the utility vector of the problem. But if we take into account the usual linear and integer programming constraint $x \in \mathbb{R}^n_{\geq 0}$, which forces non-negativity of the solutions, the set of possible symmetries shrinks from $O_n(\mathbb{Z})$ to the group of permutation matrices $P_n \leq O_n(\mathbb{Z})$.

Now, how should we imagine the action of a symmetry group $G \leq P_n$ on a linear or an integer program? Since $G$ is an automorphism group of $\mathbb{R}^n$, its elements permute the standard basis vectors $e_1, \ldots, e_n$. Considering the bijective $G$-equivariant mapping $e_i \mapsto i$, $1 \leq i \leq n$, the permutation of the subscripts of the basic vectors is an action of a group $G' \leq S_n$ on the set of indices $\{1, \ldots, n\}$ which is isomorphic to the action of $G$ on the standard basis. Hence, we can always think of symmetry groups of linear or integer programs as subgroups of $S_n$.

**Remark 4.** A group $G \leq S_n$ acts on the linear space $\mathbb{R}^n$ via the $G$-equivariant mapping

$$
\beta : \{1, \ldots, n\} \rightarrow B : \ i \mapsto e_i ,
$$

where $B$ is the set of the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

Due to Remark 3, we are able to formulate the definition of symmetries of linear programs and the corresponding integer programs simultaneously. Consider an LP of the form

$$
\begin{align*}
\text{max} \ c^t x \\
\text{s.t.} \ Ax &\leq b, \ x \in \mathbb{R}^n_{\geq 0} ,
\end{align*}
$$

and the corresponding IP given by

$$
\begin{align*}
\text{max} \ c^t x \\
\text{s.t.} \ Ax &\leq b, \ x \in \mathbb{R}^n_{\geq 0}, \ x \in \mathbb{Z}^n ,
\end{align*}
$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n \setminus \{0\}$. Note that the LP (3) and the IP (4) have the additional constraint $x \in \mathbb{R}^n_{\geq 0}$.

**Notation.** An LP of the form (3) is denoted by $\Lambda$.

 Apparently, applying a permutation to the matrix $A$ according to Remark 4 translates into permuting the columns of $A$. Since the ordering of the inequalities does not affect the object they describe, we need to allow for arbitrary row permutations of the matrix $A$. The following definition takes these thoughts into account.
Definition. A symmetry of a matrix \( A \in \mathbb{R}^{m \times n} \) is an element \( g \in S_n \) such that there exists a row permutation \( \sigma \in S_m \) with
\[
P_\sigma AP_g = A,
\]
where \( P_\sigma \) and \( P_g \) are the permutation matrices corresponding to \( \sigma \) and \( g \). The full symmetry group of a matrix \( A \in \mathbb{R}^{m \times n} \) is given by
\[
\{ g \in S_n \mid \exists \sigma \in S_m : P_\sigma AP_g = A \}.
\]
A symmetry of a linear inequality system \( Ax \leq b \), where \( A \in \mathbb{R}^{m \times n} \), and \( b \in \mathbb{R}^m \), is a symmetry \( g \in S_n \) of the matrix \( A \) via a row permutation \( \sigma \in S_m \) which satisfies \( b^\sigma = b \).

A symmetry of an LP \( \Lambda \) or its corresponding IP is a symmetry of the linear inequality system \( Ax \leq b \) that leaves the utility vector \( c \) invariant. The full symmetry group of \( \Lambda \) and the corresponding IP is given by
\[
\{ g \in S_n \mid c^g = c, \exists \sigma \in S_m : (b^\sigma = b \land P_\sigma AP_g = A) \}.
\]

This is a definition of symmetry as it can be found in literature as well, see e.g. [7].

Unfortunately, we cannot predict the effect on the symmetry group in general if we add constraints to the inequality system. This is impossible even in the special case where the corresponding polyhedron stays unaltered, as we will see in Example [10]. However, in some cases we can at least guarantee that the symmetry group of the inequality system does not get smaller.

Theorem 5. Given a symmetry group \( G \leq S_n \) of two inequality systems \( Ax \leq b \) and \( A'x \leq b' \), where \( A \in \mathbb{R}^{m \times n} \), \( A' \in \mathbb{R}^{m' \times n} \), \( b \in \mathbb{R}^m \), and \( b' \in \mathbb{R}^m \), the group \( G \) also is a symmetry group of the inequality system
\[
\begin{pmatrix} A \\ A' \end{pmatrix} x \leq \begin{pmatrix} b \\ b' \end{pmatrix}.
\]

Proof. Let \( g \in G \) be a symmetry of \( Ax \leq b \) via the row permutation \( \sigma \in S_m \), and a symmetry of \( A'x \leq b' \) via \( \sigma' \in S_{m'} \), that is,
\[
P_\sigma AP_g = A, \ P_{\sigma'}A'P_g = A', \ P_\sigma b = b, \ P_{\sigma'}b' = b' .
\]
Then we get
\[
\left( \begin{array}{cc} P_\sigma & 0 \\ 0 & P_{\sigma'} \end{array} \right) \left( \begin{array}{c} A \\ A' \end{array} \right) P_g = \left( \begin{array}{c} P_\sigma AP_g \\ P_{\sigma'}A'P_g \end{array} \right) = \left( \begin{array}{c} A \\ A' \end{array} \right)
\]
and
\[
\left( \begin{array}{cc} P_\sigma & 0 \\ 0 & P_{\sigma'} \end{array} \right) \left( \begin{array}{c} b \\ b' \end{array} \right) = \left( \begin{array}{c} P_\sigma b \\ P_{\sigma'}b' \end{array} \right) = \left( \begin{array}{c} b \\ b' \end{array} \right).
\]
Hence, the permutation \( g \) is a symmetry of the inequality system
\[
\begin{pmatrix} A \\ A' \end{pmatrix} x \leq \begin{pmatrix} b \\ b' \end{pmatrix}
\]
via the row permutation \( \left( \begin{array}{cc} P_\sigma & 0 \\ 0 & P_{\sigma'} \end{array} \right) \in \mathbb{R}^{(m+m') \times (m+m')} \). \qed
4. Orbits

The following basic terms, notations and first insights into actions of symmetry groups on linear programs will turn out to be useful.

**Definition.** Given a group $G \leq S_n$ and an element $x \in \mathbb{R}^n$, the *orbit* $x^G$ of $x$ with respect to $G$ is defined by

$$x^G := \{x^g | g \in G\}.$$

If $G$ is the symmetry group of an LP with the feasible region $X$, the group $G$ leaves $X$ invariant. Hence, a point $x$ is feasible if and only if all elements of $x^G$ are feasible as well.

**Remark 6.** Given a symmetry group $G \leq S_n$ of an LP $\Lambda$, a point $x$ is feasible for $\Lambda$ if and only if every element of the orbit $x^G$ is feasible for $\Lambda$.

The following theorem states that applying symmetries does not change the value of the utility function.

**Theorem 7.** Let $G \leq S_n$ be a symmetry group of an LP $\Lambda$. Given $x \in \mathbb{R}^n$ the utility function of $\Lambda$ is constant on the orbit $x^G$.

**Proof.** By definition, every symmetry $g \in G$ fixes the utility vector $c$. Therefore, we have

$$c^t x^g = (c^t)^t x^g = \sum_{i \in I_n} (c^t)_{i}(x^g)_i = \sum_{i \in I_n} c_{i}s x_{is} = \sum_{i \in I_n} c_{i}x_{is} = c^t x$$

for every element $x^g$ of $x^G$. □

The orbits of two elements $x, \tilde{x} \in \mathbb{R}^n$ are equal if and only if $x$ and $\tilde{x}$ are equivalent, i.e., there exists an element $g \in G$ with $x^g = \tilde{x}$. Acting on the standard basis $B := \{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, the group $G$ splits $B$ into $k$ disjoint orbits.

**Notation.** An orbit of a group action on $B$ is denoted by $O$, and the set of all orbits is denoted by $O$. The subspace spanned by an orbit is denoted by $V$.

Formulating an LP problem, the variables can be named in an arbitrary way. Therefore, we can always assume that the decomposition into orbits is aligned to the order of the basis $B$, in the following sense:

**Remark 8.** Without loss of generality, the orbits of $G$ on $B$ are given by

$$O_1 = \{e_1, \ldots, e_{n_1}\},$$

$$O_i = \{e_{s_{i-1}+1}, \ldots, e_{s_{i-1}+n_i}\},$$

for $i \in \{2, \ldots, k\}$, where $k$ is the number of orbits, $n_i$ the number of elements in orbit $O_i$, and $s_i$ is defined by $s_i := \sum_{j=1}^{i} n_j$.

**Convention.** The corresponding spans of the orbits $O_i$ are denoted by $V_i$.

Applying Theorem 7 to a unit vector $e_i$, we get some important information about the structure of the utility vector $c$.

**Corollary 9.** Let $e_i, e_j \in B$ be two elements of the same orbit $O$ under a group $G$. Then the entries $c_i$ and $c_j$ of the utility vector $c$ are equal.
Referring to Remark 8, the utility vector $c$ has the following structure:

$$c = \left( \gamma_1, \ldots, \gamma_{n_1}, \gamma_2, \ldots, \gamma_{n_2}, \ldots, \gamma_k, \ldots, \gamma_{n_k} \right)^t.$$  

We do not want to suppress the fact that there are other ways to define symmetries of linear programs. Apart from the question whether to consider affine or linear transformations, another important subject needs to be put up for discussion. As already mentioned in the introduction, the original motivation for the study of such symmetries was the unnecessarily large size of the branch-and-cut trees caused by symmetric solutions sharing the same utility value. Hence, we should focus on operations that leave invariant the utility vector and the feasible region, which is the polyhedron described by the inequality system of the linear program. Now obviously, many different inequality systems give rise to the same polyhedron. Therefore, the invariance of an inequality system implies the invariance of the polyhedron, but the reverse is not true, as the following example illustrates:

**Example 10.** Consider the LP given by the utility vector $c = (1, 1)^t$ and the inequality system

$$x_1 + x_2 \leq 2$$

$$x_1 \geq 0$$

$$x_2 \geq 0.$$  

Obviously, the permutation $g = (1\ 2) \in S_2$ is a symmetry of the inequality system, thus a symmetry of the feasible region. If we add the redundant constraint

$$x_1 \leq 2,$$

the new inequality system describes the same polyhedron. Hence $g$ still is a symmetry of the feasible region, but the inequality system itself does not show any symmetry anymore. Adding another redundant constraint

$$x_2 \leq 2,$$

we retrieve the original symmetry group of the inequality system, again without changing the symmetry group of the feasible region.

So why did we choose this restrictive definition of symmetries for linear and integer programs? The main problem is the lack of apposite descriptions of the feasible region. The inequality system is the only source of information in this context, and the conversion into a description that provides direct access to the symmetries of the feasible region might already be equivalent to solving the problem itself.

5. *The Set of Fixed Points*

Symmetries in linear programs do not attract much attention in the literature, maybe because they do not influence the performance of standard solving procedures like the simplex algorithm in a negative way. But even though linear programs are solvable in polynomial time, it is always worth looking for generic methods to save calculation time. In this section, we will focus on the question how symmetries can contribute to this goal.
As it will turn out later in this section, the points in $\mathbb{R}^n$ that are fixed by a permutation group $G \leq S_n$ play the key role in our approach. Hence, we will use the first part of this section to tame these points by means of linear algebra.

**Definition.** Given a subset $Y \subseteq \mathbb{R}^n$ and a group $G \leq S_n$ acting on $Y$, the set of fixed points of $Y$ with respect to an element $g \in G$ is defined by

$$\text{Fix}_g(Y) := \{y \in Y \mid g^y = y\}.$$  

Therefore, the set of fixed points of $Y$ with respect to $G$ is given by

$$\text{Fix}_G(Y) := \{y \in Y \mid g^y = y \text{ for all } g \in G\} = \bigcap_{g \in G} \text{Fix}_g(Y).$$

Recall that a group $G \leq S_n$ acts on $\mathbb{R}^n$ as described in Remark 4, thus we interpret $G$ as a linear group. In terms of linear algebra, the set of fixed points $\text{Fix}_g(\mathbb{R}^n)$ is the eigenspace $\text{Eig}_1(g)$ corresponding to the eigenvalue 1. Since $\text{Fix}_G(\mathbb{R}^n)$ is the intersection of all of those eigenspaces, the structure of $\text{Fix}_G(\mathbb{R}^n)$ is not arbitrary.

**Remark 11.** The set of fixed points $\text{Fix}_G(\mathbb{R}^n)$ with respect to a group $G \leq S_n$ is a subspace of $\mathbb{R}^n$.

Now let $G \leq S_n$ be a symmetry group of a linear program $\Lambda$, compare (3). Then the utility vector $c$ of the linear program is fixed by every $g$ in $G$. Since $G$ acts as a linear group, the line $l := \{rc \mid r \in \mathbb{R}\}$ is pointwise fixed by every $g$ in $G$, that is, the line $l$ is in $\text{Fix}_g(\mathbb{R}^n)$ for every $g$ in $G$, thus in the intersection of these sets.

**Remark 12.** The line $l$ through the origin spanned by $c$ is a subspace of the set of fixed points $\text{Fix}_G(\mathbb{R}^n)$.

We are particularly interested in the exact dimension of $\text{Fix}_G(\mathbb{R}^n)$. By Remark 12, we already know that $\text{Fix}_G(\mathbb{R}^n)$ is at least one-dimensional. To determine its dimension precisely, we first need to consider the dimension of a certain subspace of $\text{Fix}_G(\mathbb{R}^n)$.

**Lemma 13.** Let $O$ be a subset of the standard basis $B$ of $\mathbb{R}^n$ and $G \leq S_n$ a group acting transitively on $O$. Then the intersection $\text{Fix}_G(V)$ of the span $V := \langle O \rangle$ and $\text{Fix}_G(\mathbb{R}^n)$ is determined by

$$\text{Fix}_G(V) = \langle \sum_{e_i \in O} e_i \rangle.$$  

In particular, the subspace $\text{Fix}_G(V)$ is one-dimensional.

**Proof.** Without loss of generality, let $O = \{e_1, \ldots, e_m\}$. Since $O$ is invariant under $G$, the vector

$$v := \sum_{e_i \in O} e_i = (1, \ldots, 1, 0, \ldots, 0)^t \in V$$

is fixed by $G$, thus

$$v \in V \cap \text{Fix}_G(\mathbb{R}^n) = \text{Fix}_G(V),$$

and further $\langle v \rangle \subseteq \text{Fix}_G(V)$.

In order to prove the converse inclusion, it suffices to show that the dimension
of $\text{Fix}_G(V)$ is not greater than 1. To this end, we define the $(m - 1)$-dimensional subspaces $W_j \leq V$ by

$$W_j := \langle O \setminus \{e_j\} \rangle.$$  

Assume that the dimension of $\text{Fix}_G(V)$ is greater than 1. By the dimension formula, we then have

$$\dim(W_j \cap \text{Fix}_G(V)) = \dim W_j + \dim \text{Fix}_G(V) - \dim V = \dim \text{Fix}_G(V) - 1 \geq 1$$

for every $j = 1, \ldots, m$. In particular, there exists a vector

$$0 \neq w := \sum_{i=1}^{m-1} a_i e_i \in (W_m \cap \text{Fix}_G(V))$$

with at least one coefficient $a_l \neq 0$. Since $G$ acts transitively on $O$, we find an element $g \in G$ that maps $e_l$ to $e_m$. Being an element of $\text{Fix}_G(V)$, the vector $w$ is fixed by $g$. Thus, we can write

$$w = w^g = \sum_{i=1}^{m-1} a_i (g^{-1}) e_i + a_l e_m \notin W_m,$$

contradicting the fact that $w \in W_m \cap \text{Fix}_G(V)$. Consequently, we have $\dim \text{Fix}_G(V) \leq 1$, and therefore

$$\text{Fix}_G(V) = \langle v \rangle = \langle \sum_{e_i \in O} e_i \rangle.$$  

By Lemma 13, we are now able to establish a direct relation between the dimension of $\text{Fix}_G(\mathbb{R}^n)$ and the number of orbits generated by $G$. For orbits and the corresponding spans we use the notation we introduced in Section 4.

**Theorem 14.** Let $k$ be the number of orbits of $B$ under $G$. Then the following statements hold:

i) The set of fixed points with respect to $G$ can be written as

$$\text{Fix}_G(\mathbb{R}^n) = \bigoplus_{i=1}^{k} \text{Fix}_G(V_i).$$

ii) The set of fixed points $\text{Fix}_G(\mathbb{R}^n)$ is a subspace of $\mathbb{R}^n$ of dimension $k$.

**Proof.** In both parts of the proof we will use the fact that $\text{Fix}_G(\mathbb{R}^n)$ is a subspace of $\mathbb{R}^n$, which we already know by Remark 11. We start with the proof for the special representation of $\text{Fix}_G(\mathbb{R}^n)$.

i) Since the set of orbits $O = \{O_1, \ldots, O_k\}$ is a partition of the basis $B$ of $\mathbb{R}^n$, we have

$$V_i \cap V_j = \{0\}$$

for $i \neq j$, and further

$$\mathbb{R}^n = \bigoplus_{i=1}^{k} V_i.$$
Thus, we can write
\[ \text{Fix}_G(R^n) = R^n \cap \text{Fix}_G(R^n) = \left( \bigoplus_{i=1}^{k} V_i \right) \cap \text{Fix}_G(R^n) . \]

Hence, any point \( v \in \text{Fix}_G(R^n) \) has a unique representation \( v = \sum_{i=1}^{k} v_i \), where \( v_i \in V_i \). For this representation, we get for any \( g \in G \)
\[ \sum_{i=1}^{k} v_i = v = v^g = \sum_{i=1}^{k} v_i^g . \]
The uniqueness of the representation implies that \( g \) maps each \( v_i \) to a certain \( v_j \in V_j \) of the representation. But since every subspace \( V_i \) is invariant under \( G \), we get \( v_i^g \in V_i \), thus \( v_i^g = v_i \), due to (5). Hence, we have proved the inclusion
\[ \left( \bigoplus_{i=1}^{k} V_i \right) \cap \text{Fix}_G(R^n) \subseteq \bigoplus_{i=1}^{k} (V_i \cap \text{Fix}_G(R^n)) . \]
The converse inclusion is immediate, thus we finally get
\[ \text{Fix}_G(R^n) = R^n \cap \text{Fix}_G(R^n) = \left( \bigoplus_{i=1}^{k} V_i \right) \cap \text{Fix}_G(R^n) = \bigoplus_{i=1}^{k} (V_i \cap \text{Fix}_G(R^n)) = \bigoplus_{i=1}^{k} \text{Fix}_G(V_i) . \]

ii) In order to prove the statement on the dimension of \( \text{Fix}_G(R^n) \), we recall that \( G \) acts transitively on every orbit \( O_i \). Therefore, Lemma 13 yields
\[ \dim \text{Fix}_G(V_i) = 1 \]
for all \( i = 1, \ldots, k \). Using i), the dimension of \( \text{Fix}_G(R^n) \) can therefore be computed as
\[ \dim \text{Fix}_G(R^n) = \dim \left( \bigoplus_{i=1}^{k} \text{Fix}_G(V_i) \right) = \sum_{i=1}^{k} \dim \text{Fix}_G(V_i) = k . \]

The statement in Theorem 14 is particularly interesting if the group \( G \) generates only one single orbit.

**Corollary 15.** If \( G \) acts transitively on the standard basis \( B \), the set of fixed points \( \text{Fix}_G(R^n) \) is one-dimensional.

We complete our studies on the set of fixed points with a simple example.

**Example 16.** Consider the LP given by
\[ c^T x = x_1 + x_2 \]
subject to
\[ x_1 \leq 2.5 \]
\[ x_2 \leq 2.5 \]
\[ x_1 + x_2 \leq 3.7 , \]
where \( x_1, x_2 \in \mathbb{R}_{\geq 0} \).

Then the LP has the full symmetry group \( S_2 \). In this special case, the set of fixed points \( \text{Fix}_{S_2}(\mathbb{R}^2) \) coincides with the line \( l \) through the origin spanned by the utility vector \( c \), compare Remark 12. The following figure shows the graphical representation of the LP.

![Graphical representation of the LP](image)

**Figure 2.** Graphical representation of the LP

Now we focus on the solutions of the linear program given in Example 0.1. Obviously, the point \( x^* \) is the solution of the LP provided by the simplex algorithm. In fact, all points on the bold line parallel to the hyperplane \( c^T x = z \) are solutions of the LP. In particular, this is also true for the intersection point \( x^*_\text{fix} \in \text{Fix}_{S_2}(\mathbb{R}^2) \). Hence, by generalizing Example 0.1, we get to the assumption that for any \( n \)-dimensional linear program with full symmetry group \( G \leq S_n \), we can always find a solution in the associated set of fixed points \( \text{Fix}_G(\mathbb{R}^n) \). We will check this assumption in the following section.

### 6. Solutions in the Set of Fixed Points

Before we turn to the main issue, we need to introduce a special representation of the barycenter of an orbit, which plays an essential role in our approach.

**Lemma 17.** Given \( x \in \mathbb{R}^n \), the barycenter of the orbit \( x^G \) can be written as follows:

\[
\frac{1}{|x^G|} \sum_{y \in x^G} y = \frac{1}{|G|} \sum_{g \in G} x^g.
\]

**Proof.** Since the stabilizer \( G_x \) is a subgroup of \( G \), we have

\[ G = \bigcup_{g \in G} G_x g. \]

Let \( S = \{ s_1, \ldots, s_{|x^G|} \} \subseteq G \) be a set of representatives of the family of cosets \( G_x g \). Then

\[ \sum_{y \in x^G} y = \sum_{s \in S} x^s. \]
Furthermore, the orbit-stabilizer theorem yields the relation
\[ |x^G| = |G : G_x| = \frac{|G|}{|G_x|}, \]
so we get
\[ \frac{1}{|x^G|} \sum_{y \in x^G} y = \frac{|G_x|}{|G|} \sum_{s \in S} x^s = \frac{1}{|G|} \sum_{s \in S} G_x x^s. \]
Since \( x^g = x \) for all \( g \in G_x \), we have
\[ |G_x| x^s = \sum_{g \in G_x} (x^g)^s, \]
and therefore
\[ \frac{1}{|G|} \sum_{s \in S} |G_x| x^s = \frac{1}{|G|} \sum_{s \in S} \sum_{g \in G_x} (x^g)^s = \frac{1}{|G|} \sum_{s \in S} \sum_{g \in G_x} x^{(gs)}. \]
Considering the disjoint representation
\[ G = \bigcup_{s \in S} G_x s \]
of \( G \), we finally obtain
\[ \frac{1}{|G|} \sum_{s \in S} \sum_{g \in G_x} x^{(gs)} = \frac{1}{|G|} \sum_{g \in G} x^g. \]

The representation of the barycenter provided by Lemma 17 facilitates the proof of the following statement about feasible points in the set of fixed points.

**Theorem 18.** Let \( X \) be the feasible region of the LP \( \Lambda \). If \( x \in \mathbb{R}^n \) is feasible for \( \Lambda \), there exists a feasible point \( x_{fix} \) in \( \text{Fix}_G(\mathbb{R}^n) \) with the same utility value as \( x \).

**Proof.** We define
\[ x_{fix} := \frac{1}{|x^G|} \sum_{y \in x^G} y. \]
Since \( x_{fix} \) is the barycenter of \( x^G \), it belongs to the convex hull of \( x^G \). The feasibility of the elements of \( x^G \), compare Remark 6, and the convexity of \( X \) now imply that \( x_{fix} \) is feasible, too.
Applying Lemma 17 and the linearity of \( G \), we have
\[ x_{fix}^g = \left( \frac{1}{|x^G|} \sum_{y \in x^G} y \right)^g = \left( \frac{1}{|G|} \sum_{g \in G} x^g \right)^g = \frac{1}{|G|} \sum_{g \in G} x^g g = x_{fix} \]
for all \( g' \in G \). This proves that \( x_{fix} \) is a fixed point of \( G \), thus \( x_{fix} \in \text{Fix}_G(\mathbb{R}^n) \).
By Theorem 7 we already know that
\[ c^t x^g = c^t x \]
for all \( g \in G \), hence
\[ c^t x_{fix} = \frac{1}{|G|} \sum_{g \in G} c^t x^g = \frac{1}{|G|} \sum_{g \in G} c^t x = c^t x. \]
This shows that \( x_{fix} \) has the same utility value as \( x \). \( \square \)
The application of Theorem 18 to a solution $x^*$ of $\Lambda$ leads to a remarkable result. The following corollary, which records this result, is of vital importance, since it prepares the ground for the algorithm we are going to present subsequent to this theoretical part.

**Corollary 19.** If $\Lambda$ has a solution, there also exists a solution $x^*_{\text{fix}} \in \text{Fix}_G(\mathbb{R}^n)$.

In particular, this result shows that the existence of a solution of $\Lambda$ implies the existence of a solution of $\Lambda$ restricted to $\text{Fix}_G(\mathbb{R}^n)$. Furthermore, the point $x^*_{\text{fix}}$ and therefore every solution of the restricted LP – has the same objective value as a solution of $\Lambda$. Consequently, we only need to solve the restricted problem to get a solution for the original LP. As we will see in the next section, this kind of relationship between two LP problems can be very useful. Therefore, we introduce an appropriate partial order $\preceq$ on the family of LP problems of dimension $n$ reflecting this relationship.

**Definition.** Let $\Lambda_1$ and $\Lambda_2$ be two linear programs of dimension $n$ and $X_1, X_2 \subseteq \mathbb{R}^n$ the corresponding feasible regions. Then the linear program $\Lambda_2$ is less or equal than the linear program $\Lambda_1$ if the following three conditions are satisfied:

- The solvability of $\Lambda_1$ implies the solvability of $\Lambda_2$.
- $X_2$ is a subset of $X_1$.
- The maximal utility values on the feasible regions $X_1$ and $X_2$ coincide, that is, $\max c^t X_1 = \max c^t X_2$.

In this case, we write $\Lambda_2 \preceq \Lambda_1$.

Obviously, the relation $\preceq$ is reflexive, asymmetric and transitive, and thus a partial order.

Before we turn to practical aspects, we want to direct attention to a special property of the result in Corollary 19. The statement connotes that the symmetry of a linear program is always reflected in one of its solutions. This is what W. C. Waterhouse calls the *Purkiss Principle* in his studies on the question: Do symmetric problems have symmetric solutions? In one of his papers, see [12], he gives a list of concrete examples for this principle, but he also shows that this property can not be taken for granted.

### 7. Substitutions and Retractions

In this section our goal is to benefit from the results of the previous section by exploiting the ordering of an LP and its restriction to the set of fixed points with respect to $\preceq$. The following theorem yields a detailed insight into this relation.

**Theorem 20.** Let $G$ be a symmetry group of $\Lambda$. Then there exists a matrix $P$ only depending on the orbits of $G$ such that the LP

$$
\max c^t(Px) \\
\text{s.t. } A(Px) \leq b, \quad Px \in \mathbb{R}^n_{\geq 0}, \quad x \in \text{Fix}_G(\mathbb{R}^n)
$$

is less or equal than $\Lambda$ with respect to the order $\preceq$. 
Proof. Let \( O \) be the set of orbits \( O \) of \( G \) on \( B \). Referring to Remark 8, we consider the orbits \( O \) to be of the form

\[
O_i = \{ e_{s_{i-1}+1}, \ldots, e_{s_{i-1}+n_i} \}.
\]

Let \( \hat{V}_i \) be the subspace of \( \mathbb{R}^n \) defined by

\[
\hat{V}_i := \bigoplus_{j=1}^{i-1} V_j \oplus \text{Fix}_G(V_i) \oplus \bigoplus_{j=i+1}^{k} V_j.
\]

In order to project every \( V_i \) onto \( \text{Fix}_G(V_i) = \langle \sum_{j=1}^{n_i} e_{s_{i-1}+j} \rangle \), we define the linear maps \( f_{P_i} \) by

\[
f_{P_i} : \mathbb{R}^n \to \hat{V}_i,
\]

\[
f_{P_i}(e_l) := \begin{cases} 
\sum_{j=1}^{n_i} e_{s_{i-1}+j} & \text{if } l = s_{i-1} + 1 \\
0 & \text{if } l \in \{s_{i-1} + 2, \ldots, s_{i-1} + n_i\} \\
e_l & \text{otherwise}
\end{cases}.
\]

Hence, the first element of \( O_i \) is mapped to the sum of the elements of \( O_i \), while the other elements of \( O_i \) are mapped to 0.

The \( n \times n \)-matrix \( P_i \) corresponding to \( f_{P_i} \) is defined by

\[
P_i := \begin{pmatrix}
I_{s_{i-1}} & 0 & 0 \\
0 & P_i & 0 \\
0 & 0 & I_{n-s_i}
\end{pmatrix}, \quad \tilde{P}_i := \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
0 & 0 & \ddots & 0 \\
1 & 0 & \ldots & 0
\end{pmatrix},
\]

where \( \tilde{P}_i \in \mathbb{R}^{n_i \times n_i} \). According to Theorem 14, we have

\[
\text{Fix}_G(\mathbb{R}^n) = \bigoplus_{i=1}^{k} \text{Fix}_G(V_i).
\]

Therefore, we are now able to define the map \( f_P : \mathbb{R}^n \to \text{Fix}_G(\mathbb{R}^n) \) by

\[
f_P(x) = Px,
\]

where

\[
P := \prod_{i=1}^{k} P_i = \begin{pmatrix}
\tilde{P}_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \tilde{P}_k
\end{pmatrix} = (B_1, \ldots, B_k).
\]

By Corollary 19 we know that the restricted LP

\[
\max c^T x \\
\text{s.t. } Ax \leq b, \quad x \in \mathbb{R}_{\geq 0}, \quad x \in \text{Fix}_G(\mathbb{R}^n)
\]

is less or equal than \( \Lambda \). Since \( f_P \) is a projection onto \( \text{Fix}_G(\mathbb{R}^n) \), we have \( Px = x \) for all \( x \in \text{Fix}_G(\mathbb{R}^n) \). Hence, the LP (8) is equal to

\[
\max c^T (Px) \\
\text{s.t. } A(Px) \leq b, \quad Px \in \mathbb{R}_{\geq 0}, \quad x \in \text{Fix}_G(\mathbb{R}^n).
\]

The transitivity of \( \preceq \) now implies that the LP (8) is less or equal than \( \Lambda \). \qed

Variables of a linear program that are tied together in one orbit are closely related. Therefore, we introduce a notation for sets of such variables.
**Notation.** Let $O$ be an orbit on the standard basis $B$ of $\mathbb{R}^n$. The set of variables of an LP corresponding to the elements of $O$ is denoted by $X_O$.

In order to translate the result of Theorem 20 into an applicable algorithm, we perform the so-called substitution procedure computing $\hat{c}^t = c^t P$ and $\hat{A} = AP$ in the LP (6). According to the definition of $P$, see (7), the resulting LP is given by

$$\max \: \hat{c}^t x$$

subject to

$$\hat{A}x \leq b, \: Px \in \mathbb{R}^n_{\geq 0}, \: x \in \text{Fix}_G(\mathbb{R}^n)$$

(9)

where

$$\hat{c}^t = c^t P = \left( \sum_{j=1}^{n_1} c_{j}, 0, \ldots, 0, \ldots, \sum_{j=1}^{n_k} c_{s_{k-1} + j}, 0, \ldots, 0 \right)^t$$

(10)

and

$$\hat{A} = AP = (AB_1, \ldots, AB_i, \ldots, AB_k)$$

Straightforward computation yields

$$AB_i = A \begin{pmatrix} 0 \\ P_i \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n_i} a_{1,s_i-1+j} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n_i} a_{n_{i},s_i-1+j} & 0 & \ldots & 0 \end{pmatrix}.$$

(11)

Furthermore, we have

$$Px = (x_1, \ldots, x_1, \ldots, x_{s_{k-1}+1}, \ldots, x_{s_{k-1}+1})^t.$$

This representation reveals that – except for the constraint $x \in \text{Fix}_G(\mathbb{R}^n)$ – the inequality system of the new LP does not depend on the variables $x_{s_i-1+2}, \ldots, x_{s_i-1+n_i}$ for all $i \in \{1, \ldots, k\}$. Furthermore, the coefficient of the representative $x_{s_{i-1}+1}$ of each set $X_O$ accumulates the original coefficients of all variables in $X_O$. Therefore, we can interpret this procedure as a simultaneous substitution of the elements of each $X_O$, by the representatives $x_{s_{i-1}+1}$.

**Notation.** The LP that is derived from $\Lambda$ by simultaneously substituting the elements of each $X_O$, by the representatives $x_{s_{i-1}+1}$, and adding the constraint $x \in \text{Fix}_G(\mathbb{R}^n)$ is denoted by Sub($\Lambda$).

By Theorem 20, we already know that Sub($\Lambda$) is less or equal than $\Lambda$. Hence, we only need to solve Sub($\Lambda$) to obtain a solution of $\Lambda$. This fact can be expressed in the following way.

**Corollary 21.** Every solution of Sub($\Lambda$) is a solution of $\Lambda$ as well.

A first application of the substitution procedure to a basic example will shed light on the effectiveness and the potential of the algorithm.
Example 22. Consider the following LP $\Lambda_0$ given by the inequality system $Ax \leq b$ and the utility vector $c$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

and

$$c = (1, 1, 2, 2)^t.$$

We can expand this LP to

$$x_1 + x_2 \leq 1$$
$$x_3 + x_4 \leq 2$$
$$x_1 + x_3 \leq 3$$
$$x_2 + x_4 \leq 3$$

and

$$c^t x = x_1 + x_2 + 2x_3 + 2x_4.$$

Obviously, we can exchange $x_1$ and $x_2$ without affecting $c$ or the inequality system if we exchange $x_3$ and $x_4$ at the same time. Therefore, this LP has

$$G := \langle (1, 2)(3, 4) \rangle$$

as a symmetry group, and $G$ divides $B$ into the two orbits $O_1 = \{e_1, e_2\}$ and $O_2 = \{e_3, e_4\}$. Applying the substitution procedure to the set of orbits $O = \{O_1, O_2\}$, we obtain the new LP $\text{Sub}(\Lambda_0)$ defined by $\hat{A}x \leq \hat{b}$ and $\hat{c}$, where

$$\hat{A} = AP = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\hat{c}^t = c^t P = (1, 1, 2, 2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (2, 0, 4, 0).$$

The expanded version of the new LP is given by

$$2x_1 \leq 1$$
$$2x_3 \leq 2$$
$$x_1 + x_3 \leq 3$$
$$x_1 + x_3 \leq 3$$

and

$$\hat{c}^t x = 2x_1 + 4x_3,$$

where $x \in \text{Fix}_G(\mathbb{R}^n)$. According to Corollary 21, we only need to solve the LP $\text{Sub}(\Lambda_0)$ which is almost independent of the variables $x_2$ and $x_4$. 
Note that the LP $\text{Sub}(\Lambda_0)$ can actually be derived from the original LP by substituting $x_1$ for $x_2$ and $x_3$ for $x_4$. Furthermore, we observe that we do not use any detailed knowledge about the structure of the group $G$ except for the specific decomposition of $B$ into orbits. Therefore, we can apply the substitution procedure to any LP problem with known orbit decomposition even if we do not have any additional information about the group structure of the symmetry group $G$ of the linear program.

**Remark 23.** Regarding Theorem 20 and the substitution procedure, the assumption of having a certain group $G$ can be relaxed to the assumption of having a certain orbit decomposition.

Except for the constraint $x \in \text{Fix}_G(\mathbb{R}^n)$, the LP $\text{Sub}(\Lambda)$ is completely independent of certain variables. Therefore, we now focus on a reduction of the dimension of the LP. This reduction can be realized by a certain operator, which we are now going to introduce.

**Definition.** Given an LP $\Lambda$ with the set of orbits $\mathcal{O} = \{O_1, \ldots, O_k\}$, the **retraction** $r$ is defined by $r : \text{Sub}(\Lambda) \mapsto \Lambda'$, where

$$
\Lambda' = \left\{ \max \ c^t M_r y \right\}_{\text{s.t. } \hat{A} M_r y \leq b, \ y \in \mathbb{R}^k_{\geq 0}}
$$

and $M_r \in \mathbb{R}^{n \times k}$ is defined by

$$
M_r = (v_1, \ldots, v_k), \ v_i = \sum_{e_j \in O_i} e_j = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)^t
$$

The LP $\Lambda'$ is called the **retract** of $\text{Sub}(\Lambda)$, and we denote $\Lambda'$ by $\text{Ret}(\text{Sub}(\Lambda))$.

Note that in contrast to the $n$-dimensional LP $\text{Sub}(\Lambda)$, the dimension of $\text{Ret}(\text{Sub}(\Lambda))$ is equal to the number of orbits, which coincides with the dimension of the set of fixed points $\text{Fix}_G(\mathbb{R}^n)$, see Theorem 14.

**Remark 24.** Given a linear program $\Lambda$ with the set of orbits $\mathcal{O} = \{O_1, \ldots, O_k\}$, the retract $\text{Ret}(\text{Sub}(\Lambda))$ of $\text{Sub}(\Lambda)$ is a linear program of dimension $k$.

To justify the term retraction, we introduce an appropriate inclusion $\iota$ satisfying $r \circ \iota = \text{id}$.

**Definition.** The **inclusion** $\iota$ is defined by $\iota : \text{Ret}(\text{Sub}(\Lambda)) \mapsto \Lambda''$, where

$$
\Lambda'' = \left\{ \max \ c^t M_r M_i x \right\}_{\text{s.t. } \hat{A} M_r M_i x \leq b, \ x \in \mathbb{R}^n_{\geq 0}, \ x \in \text{Fix}_G(\mathbb{R}^n)}
$$

and $M_i \in \mathbb{R}^{k \times n}$ is defined by

$$
M_i = (e_1, e_{s_1+1}, \ldots, e_{s_k+1})^t
$$

The retraction $r$ applied to the LP $\text{Sub}(\Lambda)$ eliminates the zeros in the representations (10) and (11) of $\hat{c}$ and $\hat{A}$. Conversely, the inclusion $\iota$ reintroduces these zeros in the following sense: Obviously, $M_i$ can be written as

$$
M_i = (C_1, \ldots, C_k), \ C_i = (e_i, 0, \ldots, 0) \in \mathbb{R}^{k \times n_i},
$$
where $e_i$ is the $i$-th unit vector in $\mathbb{R}^k$. Referring to the block representation $P = (B_1, \ldots, B_k)$ given in (7), we have
\[ M_i C_i = B_i \]
for every $i \in \{1, \ldots, k\}$, and therefore
\begin{equation} 
M_i M_i = (M_i C_1, \ldots, M_i C_k) = (B_1, \ldots, B_k) = P.
\end{equation}
Using the property $PP = P$ of the projection matrix $P$, we finally get
\[ \hat{A} M_i C_i = \hat{A} P x = AP x = \hat{A} x. \]
This shows that $\Lambda'' = \text{Sub}(\Lambda)$, and thus
\[ (r \circ \iota)(\text{Ret}(\text{Sub}(\Lambda))) = r(\text{Sub}(\Lambda)) = \text{Ret}(\text{Sub}(\Lambda)). \]
With respect to this category theoretical property, we now want to show that we only need to solve the retract $\text{Ret}(\text{Sub}(\Lambda))$ of $\text{Sub}(\Lambda)$. For this, we analyze the linear maps
\[ r : \mathbb{R}^k \to \mathbb{R}^n, \ y \mapsto M_i y \]
and
\[ \iota : \mathbb{R}^n \to \mathbb{R}^k, \ x \mapsto M_i x \]
by considering the corresponding matrices $M_i$ and $M_i$. On the one hand, the retraction $r$ maps any element of $\mathbb{R}^k$ to an element of $\text{Fix}_G(\mathbb{R}^n)$. On the other hand, the map $\iota$ applied to a vector $x \in \mathbb{R}^n$ picks exactly the representative $x_{s_i-1+1}$ of each set $X_{O_i}$. Concerning the LP problems $\text{Sub}(\Lambda)$ and $\text{Ret}(\text{Sub}(\Lambda))$, this behavior has the following effect.

**Lemma 25.** Let $X$ be the feasible region of $\text{Sub}(\Lambda)$. Then the following statements hold:

i) If $y$ is feasible for $\text{Ret}(\text{Sub}(\Lambda))$, then $x := M_i y$ is feasible for $\text{Sub}(\Lambda)$.

ii) The feasible region of $\text{Ret}(\text{Sub}(\Lambda))$ is given by $Y := M_i X$.

iii) The LP problems $\text{Sub}(\Lambda)$ and $\text{Ret}(\text{Sub}(\Lambda))$ have the same maximal utility value.

**Proof.** We will use the statement in i) to prove ii), and the representation in iii) to show iii).

i) Let $y$ be a feasible point of $\text{Ret}(\text{Sub}(\Lambda))$. Since $y$ is in $\mathbb{R}^k$ and $r$ maps $\mathbb{R}^k$ to $\text{Fix}_G(\mathbb{R}^n)$, the point $x = M_i y$ is in $\mathbb{R}^n \cap \text{Fix}_G(\mathbb{R}^n)$. Moreover, we have
\[ \hat{A} x = \hat{A} M_i y \leq b, \]
that is, the point $x$ is feasible for $\text{Sub}(\Lambda)$.

ii) Let $x$ be in $X$. Then $x$ is feasible for $\text{Sub}(\Lambda)$, and thus
\[ x \in \mathbb{R}^n \cap \text{Fix}_G(\mathbb{R}^n). \]

Therefore, we have $Px = x$ and $M_i x \in \mathbb{R}^k$. By the equality $M_i M_i = P$, see (12), we obtain
\[ \hat{A} M_i (M_i x) = \hat{A} (M_i M_i) x = \hat{A} (Px) = \hat{A} x \leq b, \]
that is, $M_x$ is feasible for $\text{Ret}(\text{Sub}(\Lambda))$. Conversely, let $y$ be a feasible point of $\text{Ret}(\text{Sub}(\Lambda))$. According to ii), the point $x = M_y$ is feasible for $\text{Sub}(\Lambda)$. Straightforward computation yields

$$M_y M_r = I_k .$$

Hence, $y$ can be written as

$$y = M_r y = M_r M_y = M_x .$$

Therefore, any feasible point of $\text{Ret}(\text{Sub}(\Lambda))$ is in $M_X$. Conclusively, the set $Y = M_Y$ defines the feasible region of $\text{Ret}(\text{Sub}(\Lambda))$. 

iii) Since $X$ is a subset of $\text{Fix}_G(\mathbb{R}^n)$, the definition of $Y$ given in ii) yields

$$M_r Y = M_r M_X = PX = X .$$

Therefore, we can write

$$\max_{x \in X} \hat{c}^t x = \max_{y \in Y} \hat{c}^t M_r y ,$$

hence the optimal values of $\text{Sub}(\Lambda)$ and $\text{Ret}(\text{Sub}(\Lambda))$ are equal.

The relations $M_M = I_k$ and $M_r M_x = x$ for all $x \in \text{Fix}_G(\mathbb{R}^n)$ which we used in our proof reveal in particular that $\iota$ and $r$ are bijective and mutually inverse if we restrict $\iota$ to $\text{Fix}_G(\mathbb{R}^n)$ . The following corollary records this interesting relationship.

**Corollary 26.** The linear maps

$$r : \mathbb{R}^k \to \text{Fix}_G(\mathbb{R}^n), \ y \mapsto M_r y$$

and

$$\iota|_{\text{Fix}_G(\mathbb{R}^n)} : \text{Fix}_G(\mathbb{R}^n) \to \mathbb{R}^k, \ x \mapsto M_l x$$

are bijective and mutually inverse.

The following theorem proves that we only need to solve $\text{Ret}(\text{Sub}(\Lambda))$ instead of $\text{Sub}(\Lambda)$. Furthermore, it provides a method how to regain a solution of $\text{Sub}(\Lambda)$ from a solution of $\text{Ret}(\text{Sub}(\Lambda))$.

**Theorem 27.** Let $X$ be the feasible region of $\text{Sub}(\Lambda)$. Given that $\text{Sub}(\Lambda)$ has a solution, we can prove the following statements.

i) The LP $\text{Ret}(\text{Sub}(\Lambda))$ has a solution as well.

ii) Any solution $y^*$ of $\text{Ret}(\text{Sub}(\Lambda))$ induces a solution $x^*_{\text{fix}} := M_r y^*$ of the linear program $\text{Sub}(\Lambda)$.

**Proof.** The proof essentially relies on Lemma 25.

i) Let $x^*_{\text{fix}}$ be a solution of $\text{Sub}(\Lambda)$. We show that $y^*$ defined by

$$y^* := M_l x^*_{\text{fix}}$$

is a solution of $\text{Ret}(\text{Sub}(\Lambda))$. By part ii) of Lemma 25, the feasible region of $\text{Ret}(\text{Sub}(\Lambda))$ is given by $Y := M_Y$. Since $x^*_{\text{fix}}$ is in $X$, the point $y^*$ is feasible for $\text{Ret}(\text{Sub}(\Lambda))$. According to 25 iii), we have

$$\max_{y \in Y} \hat{c}^t M_r y = \max_{y \in Y} \hat{c}^t M_r y = \hat{c}^t M_r = \hat{c}^t P x^*_{\text{fix}} = \hat{c}^t (M_r M_r) x^*_{\text{fix}} = \hat{c}^t (M_r M_l) x^*_{\text{fix}} = \hat{c}^t M_r y^* ,$$

that is, the point $y^*$ is a solution of $\text{Ret}(\text{Sub}(\Lambda))$. 

ii) Let \( y^* \) be a solution of \( \text{Ret(Sub}(\Lambda)) \). Using 25i), the point \( x^*_{\text{fix}} = M_r y^* \) is feasible for \( \text{Sub}(\Lambda) \). By 25iii), we now get

\[
\max_{x \in X} c^t x = \max_{y \in Y} c^t M_r y = c^t M_r y^* = c^t x^*_{\text{fix}}.
\]

This shows that \( x^*_{\text{fix}} = M_r y^* \) is a solution of \( \text{Sub}(\Lambda) \).

\[\square\]

Combining Corollary 21 and Theorem 25, we conclude that it suffices to solve the \( k \)-dimensional retract \( \text{Ret(Sub}(\Lambda)) \), compare Remark 24, in order to obtain a solution of \( \text{Sub}(\Lambda) \), which then is a solution of the original \( n \)-dimensional linear program \( \Lambda \).

Finally, we resume Example 22 to study the effects of the final two steps of the algorithm.

**Example 22 (continued).** Consider the LP \( \text{Sub}(\Lambda_0) \) defined by

\[
\hat{A} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{c}^t = (2, 0, 4, 0).
\]

Then the LP \( \text{Ret(Sub}(\Lambda_0)) \) is given by

\[
\text{Ret(Sub}(\Lambda_0)) = \left\{ \max_{y \in \mathbb{R}^k_{\geq 0}} \hat{c}^t M_r y \right. \\
\text{s.t. } \hat{A} M_r y \leq b, \quad y \in \mathbb{R}^k_{\geq 0}
\]

where

\[
\hat{c}^t M_r = (2, 0, 4, 0) \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = (2, 4)
\]

and

\[
\hat{A} M_r = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Expanding \( \text{Ret(Sub}(\Lambda_0)) \), we get

\[
2y_1 \leq 1 \\
2y_2 \leq 2 \\
y_1 + y_2 \leq 3 \\
y_1 + y_2 \leq 3
\]

and

\[
\hat{c}^t M_r y = 2y_1 + 4y_2.
\]

Obviously, this LP can be solved at a glance. The solution is given by

\[
y^* = (0.5, 1)^t.
\]
In order to get a solution of \( \text{Sub}(\Lambda_0) \), we multiply \( y^* \) by \( M_r \). By Theorem 27, the point

\[
x_{\text{fix}}^* = M_r y^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \\ 1 \end{pmatrix}
\]

is a solution of \( \text{Sub}(\Lambda_0) \). Finally, Corollary 21 guarantees that \( x_{\text{fix}}^* \) is a solution of \( \Lambda_0 \) as well.

In the procedure we developed, we take advantage of symmetries by deriving a linear program of smaller dimension, which still contains enough information to extract a solution of the original LP. The elaboration of our method revealed that the complexity of the derived linear program solely depends on the number of orbits, not on the concrete structure of the symmetry group. Therefore, transitivity of the symmetry group suffices to obtain the best possible result.

But even the knowledge about one single symmetry of a linear program already effects a reduction of the dimension, since every symmetry generates a symmetry group of the linear program and reduces the number of orbits. Sometimes, the derived linear program \( \text{Ret}(\text{Sub}(\Lambda)) \) shows further symmetries, even if we already considered the full symmetry group of the original problem. In that case, we can apply the substitution algorithm iteratively.

Of course, it is not clear how to determine symmetries of arbitrary linear programs. But in practice, some of the symmetries already attract attention during the construction of the linear programs. For instance, think of the graph-coloring problem, where it is obvious that the variables representing the colors can be exchanged. Therefore, the substitution procedure or algorithm should be understood not so much as a part of the solving process, but as a pre-processing step in order to produce a lower-dimensional linear program. In this respect, it would be reasonable to formulate linear programs as symmetric as possible.

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