Derivation of the Ginzburg-Landau equations of a ferromagnetic $p$-wave superconductor

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We derive a Ginzburg-Landau free energy for a $p$-wave ferromagnetic superconductor. The starting point is a microscopic Hamiltonian including a spin generalised BCS term and a Heisenberg exchange term. We find that coexistence of magnetisation and superconductivity depends on the sign of the energy-gradient of the DOS at Fermi level. We also compute the tunneling contribution to the Ginzburg-Landau free energy, and find expressions for the spin-currents and Josephson currents across a tunneling junction separating two ferromagnetic $p$-wave superconductors.

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I. INTRODUCTION

In recent years, experimental evidence for materials that simultaneously feature superconductivity and ferromagnetism, has transpired from a number of independent studies. This has triggered much research into this class of materials, widely known as ferromagnetic superconductors (FMSC). There are still many unanswered questions concerning the properties of these systems featuring coexisting spontaneously broken symmetries (broken $U(1)$ and $O(3)$ symmetries). Recently, the superconductivity in ZrZn$_2$ has been found to be surface sensitive, and might in fact not be a bulk property of the material. A common feature for all the ferromagnetic superconductors is that they are superconducting only in the ferromagnetic phase. That is, when pressure is increased in such a way as to destroy the ferromagnetic phase transition, then superconductivity, if present, is also lost.

Ferromagnetic superconductivity is of great interest from a theoretical point of view. It offers a laboratory for studying condensed matter systems with multiple spontaneously broken symmetries, and their interplay. From a technological point of view, it is also hoped that heterostructures with FMSC will give rise to novel transport effects involving both the charge and the spin of the electron, which may have the potential for being exploited in novel types of devices. Recently, structures containing FMSC have been investigated theoretically. This work, however, considers tunneling effects between a spin-singlet superconductor in a Fulle-Ferrell-Larkin-Ovchinnikov state coexisting with helimagnetic order. On the other hand, little is known of the tunneling effects and interplay between ferromagnetism and superconductivity in spin-triplet superconductors coexisting with uniform magnetic order. It is the purpose of this paper to derive a Ginzburg-Landau model for such a system starting from a reasonable microscopic Hamiltonian, and then apply the results to computing tunneling effects both in the charge and spin sector, and thus to elucidate the interplay between the broken $U(1)$ and $O(3)$ symmetries form such systems. Previously, the Bogoliubov-de Gennes equations for such systems have been studied in some detail.

Phenomenological models designed to describe FMSC were proposed soon after their experimental discovery. These theories can be written down from symmetry considerations of the two order parameters in the problem, i.e. the theories make no reference to the underlying microscopic physics. In ferromagnetic superconductors it is believed that it is the same electrons that are responsible for both superconductivity and ferromagnetism. In such a scenario, the Cooper-pairs must have a magnetic moment, which means that a FMSC with uniform magnetic order must be a spin triplet superconductor. Microscopic theories attempting to explain the positive interaction between electrons with aligned spins have traditionally evolved around spin mediated interactions, i.e. magnon exchange.

In Ref. the phenomenological model of Refs. was employed in order to find stable solutions and the regions were these solutions of the model are stable. The phenomenological model used in Refs. is given by

$$f(\psi, M) = f_S(\psi) + f_F(M) + f_I(\psi, M) + \frac{1}{2} H^2, \quad (1)$$

were $\psi$ is a three dimensional complex vector describing superconducting order, $B = (H + 4\pi M) = \nabla \times A$ is the magnetic induction, $H$ the external magnetic field, and $A$ is the electromagnetic vector potential.

The term $f_S(\psi)$ describes superconductivity when $H = M = 0$, and is given by

$$f_S(\psi) = f_{\text{grad}}(\psi) + a_s |\psi|^2 + \frac{b_s}{2} |\psi|^4 + \frac{u_s}{2} |\psi|^2 + \frac{v_s}{2} \sum_{i=1}^3 |\psi_i|^4, \quad (2)$$

with

$$f_{\text{grad}}(\psi) = K_1 (D_i \psi_j)^* (D_i \psi_j) + K_2 (D_i \psi_{ji})^* (D_i \psi_{ji}) + (D_i \psi_{ji})^* (D_i \psi_{ji}) + K_3 (D_i \psi_{ji})^* (D_i \psi_{ji}), \quad (3)$$

where summation over the indices $i, j$ is assumed and the symbol

$$D_i = -i \frac{\partial}{\partial x_i} + 2eA_i, \quad (4)$$
of covariant differentiation is introduced. For a discussion of the superconductivity part of the free energy see Refs. 10. The term \( f_\ell(M) \) in (1) describes the ferromagnetic ordering of the material and is given by,

\[
f_\ell(M) = c_\ell \sum_{j=1}^{3} |\nabla_j M_j|^2 + a_\ell(T) M^2 + \frac{b_\ell}{2} M^4.
\]

(5)

This is the standard expression for an isotropic ferromagnet. Furthermore, \( f_\ell(\psi, M) \) describes the interaction between the order parameters for ferromagnetism and superconductivity, and is given by

\[
f_\ell(\psi, M) = i \gamma_\ell M \cdot (\psi \times \psi^*) + \delta M^2 |\psi|^2.
\]

(6)

In this paper, we will start out with a microscopic Hamiltonian that essentially reproduces the Ginzburg-Landau model proposed in Ref. 10. As a byproduct of such a derivation from a more microscopic model, we obtain analytic expressions for the coefficients in (1). The microscopic theory we start with is not trying to explain such a derivation from a more microscopic model, we observe.

As an application of the Ginzburg-Landau equations, we look at two FMSC in tunneling contact. We compute the free energy of the coupling and further find the Josephson current and the spin current in the direction perpendicular to the plane spanned by the two magnetisation directions.

II. MODEL

We start out with a Hamiltonian that is given by three terms, one describing free electrons, one BCS term to account for spin-triplet superconductivity, and finally a Heisenberg ferromagnetic exchange term to account for itinerant ferromagnetism. The Hamiltonian of the system is then given by

\[
\hat{H}[c, c^\dagger] = \sum_{k, \sigma} \varepsilon_{k} c_{k, \sigma}^\dagger c_{k, \sigma} + \frac{1}{2} \sum_{k, k', q} V_{k, k' q/2, \alpha} c_{k+q/2, \alpha}^\dagger c_{k+q/2, \beta} c_{-k+q/2, \beta}^\dagger c_{k' q/2, \alpha}^\dagger + \frac{1}{2} \sum_{q} J \gamma(q) \mathbf{S}_q \cdot \mathbf{S}_{-q},
\]

(7)

where \( c_{k, \sigma}^\dagger \) and \( c_{k, \sigma} \) annihilates and creates an electron in the state \( (k, \sigma) \) respectively. Here, \( \mathbf{S}_k \) is the usual spin operator given by, \( \mathbf{S}_k = \sum_{q, \alpha, \beta} f_{k+q/2, \alpha}^\dagger \alpha \beta c_{k+q/2, \alpha} c_{-k-q/2, \beta} \mathbf{\sigma}_{\alpha \beta} \), and \( \mathbf{\sigma} \) is the Pauli matrices. \( \gamma(q) = \sum_{\delta} \exp i q \cdot \delta \) is the structure factor of the underlying lattice, and \( \delta \) is a nearest neighbour vector. A summation over repeated Greek indices is implied.

We are interested in calculating the partition function of the system, formally given by

\[
Z = \text{Tr} \left( e^{-\beta \hat{H}[c, c^\dagger]} \right) = e^{-\beta F},
\]

(8)

were \( F \) is the exact free energy of the system. Introducing fermion coherent states, \( \xi_{k, \sigma} \), and performing a Hubbard-Stratonovich decoupling of the two last terms in (7), we arrive at an effective action (in Euclidean time) which reads,

\[
S_{\text{eff}} = -\frac{1}{2} \int_{0}^{3} d\tau \sum_{k, \sigma} \left[ \xi_{k, \sigma}^* \left( \partial_\tau + \varepsilon_{k} \right) \xi_{k, \sigma} + \xi_{k, \sigma}^* \left( \partial_\tau - \varepsilon_{k} \right) \xi_{k, \sigma} \right]
\]

\[
+ \sum_{k, q, \alpha, \beta} \left[ \xi_{k+q/2, \alpha}^* (\mathbf{\sigma} \cdot M_{q})_{\alpha \beta} \xi_{k-q/2, \beta} - \xi_{k-q/2, \beta}^* (\mathbf{\sigma} \cdot M_{q})_{\alpha \beta} \xi_{k+q/2, \alpha} \right]
\]

\[
+ \sum_{q} \left[ \Delta^\dagger_{\alpha \beta}(k, q) \xi_{k+q/2, \beta}^* \xi_{-k+q/2, \alpha} + \Delta_{\alpha \beta}(k, q) \xi_{k+q/2, \alpha}^* \xi_{-k+q/2, \beta} \right]
\]

\[
- \sum_{q} \frac{1}{J \gamma(q)} M_{q} \cdot M_{-q} - \sum_{k, k', q} \Delta^\dagger_{\alpha \beta}(k', q) V_{k', k}^{-1} \Delta_{\beta \alpha}(k, q)
\]

(9)

Here, \( \Delta_{\alpha \beta}(k, q) \) and \( M_{q} \) are auxiliary fields in the functional integral representation of the partition function, and physically represent the spin-triplet pairing fields and magnetisation, respectively.

The above action is Gaussian in the fermionic fields and hence the integral over the fermionic fields may be performed exactly. We next introduce a Majorana basis
\[\phi_k^\dagger = \left[\xi_{k,\uparrow} \xi_{k,\downarrow} \xi_{-k,\downarrow} - \xi_{-k,\uparrow}\right],\]

the effective action may now be written,

\[S_{\text{eff}} = -\frac{1}{2} \int_0^\beta d\tau \left\{ \sum_{k,q} \phi_k^\dagger \mathcal{G}^{-1} \phi_k - \frac{1}{\beta} \int_0^\beta \ln \mathcal{G}(q) M_q M_{-q} \right. \]
\[\left. - \sum_{k,k',q} \text{tr} \Delta^\dagger(k', q) V_{k,k'}^{-1} \Delta(k, q) \right\}. \tag{10}\]

Here, tr denotes trace over spin indices. Furthermore, \(\mathcal{G}^{-1}\) is a \(4 \times 4\) matrix given by, \(\mathcal{G}^{-1} = \mathcal{G}_0^{-1} - \Sigma\), where \(\mathcal{G}_0^{-1} = \text{diag}(-i\omega_n + \varepsilon_k, -i\omega_n + \varepsilon_k, -i\omega_n - \varepsilon_k, -i\omega_n - \varepsilon_k)\) describes a free electron gas. Here, \(\Sigma\) describes the interaction and pairing correlations, and is explicitly given by,

\[\Sigma = \begin{bmatrix} \mathcal{M} & \mathcal{D} \\ \mathcal{D}^\dagger & \mathcal{M} \end{bmatrix}, \tag{11}\]

where \(\mathcal{M} = M \cdot \sigma\) is a \(2 \times 2\) matrix order parameter describing magnetisation and \(\mathcal{D} = d \cdot \sigma = -i\Delta \sigma_y\) is a \(2 \times 2\) matrix order parameter describing triplet superconductivity.

The integral over \(\phi\) is Gaussian, and hence it may be performed analytically. The integral produces a fermion determinant which may be included in the exponent, i.e. in the Ginzburg-Landau free energy, which is given by

\[\beta F_{\text{GL}} = -\text{Tr} \ln \mathcal{G}^{-1} \]
\[= -\frac{1}{2} \int_0^\beta d\tau \left[ \sum_{k,q} \frac{1}{\beta} M_q M_{-q} \right. \]
\[+ \sum_{k,k',q} \text{tr} \Delta^\dagger(k', q) V_{k,k'}^{-1} \Delta(k, q) \right], \tag{12}\]

where Tr implies a trace over all variables. The GL free energy is defined by

\[\int \mathcal{D}[\xi, \xi^\dagger] e^{S_{\text{eff}}[\xi, \xi^\dagger]} = e^{-\beta F_{\text{GL}}}. \tag{13}\]

The trace over \(\ln \mathcal{G}^{-1}\) may formally be rewritten as \(\text{Tr} \ln \mathcal{G}^{-1} = \text{Tr} \ln \mathcal{G}_0^{-1} + \text{Tr} \ln (1 - \mathcal{G}_0 \Sigma),\) the first term, describing free theory, is neglected in the following. The second term is assumed small, and hence we may expand the logarithm to obtain

\[\text{Tr} \ln (1 - \mathcal{G}_0 \Sigma) \approx -\text{Tr} \left( \mathcal{G}_0 \Sigma + \frac{1}{2} \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma \right) \]
\[+ \frac{1}{3} \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma + \frac{1}{4} \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma + O(\mathcal{G}_0 \Sigma)^5 \]
\[= -E_1 - E_2 - E_3 - \ldots \tag{14}\]

The first term in the expansion, \(E_1 = \text{Tr} \mathcal{G}_0 \Sigma,\) is zero since the Pauli matrices are traceless. The terms \(E_2\) and \(E_3\) are second order in the ordering fields \(\Delta \sigma_y(k, q)\) and \(M_q,\) and we now proceed top discuss these terms in turn.

III. SECOND ORDER TERM

The second order term in the expansion of the trace is given as

\[E_2 = \frac{1}{2} \text{Tr} \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma = \frac{1}{2} \sum_n \langle n | \mathcal{G}_0 \Sigma \mathcal{G}_0 \Sigma | n \rangle \]
\[= \frac{1}{2} \sum_{k_1, k_2} \mathcal{G}_0 k_1 \Sigma k_1, k_2 \mathcal{G}_0 k_2 \Sigma k_2, k_1. \tag{15}\]

The last equality comes from inserting completeness relations and noting that \(\mathcal{G}_0\) is local in \(k\)-space. We have introduced the notation \(\mathcal{G}_0 k_1 = \langle k_1 | \mathcal{G}_0 | k_1 \rangle\) and \(\Sigma k_1, k_2 = \langle k_1 | \Sigma | k_2 \rangle.\) After changing variables, \(\{k_1, k_2\} \rightarrow \{k+q/2, k-q/2\},\) the second order term is

\[E_2 = \sum_{k,q} \left\{ \left( g_{0, k+q/2} g_{0, k-q/2} + c.c. \right) M_q M_{-q} \right. \]
\[\left. - \left( g_{0, k+q/2} g_{0, k-q/2} + c.c. \right) |d_{k,q}|^2 \right\}, \tag{16}\]

where \(g_{0, k} = (-i \omega_n + \varepsilon_k).\)

The trace of the superconducting order parameter is given by

\[\text{tr} \Delta^\dagger(k, q) \Delta(k, q) = \text{tr} \left[ i d_{k,q}^\dagger \sigma_y \sigma_y \right] \left[ i d_{k,q}^\dagger \sigma_y \sigma_y \right]^\dagger \]
\[= 2 |d_{k,q}|^2. \tag{17}\]

The complete second order term of the Ginzburg-Landau free energy is therefore given by

\[F_2 = \sum_{k,q} \left\{ \right. \]
\[\left. - \delta_{k,0} \frac{1}{E_1} \left( g_{0, k+q/2} g_{0, k-q/2} + c.c. \right) M_q M_{-q} \right. \]
\[\left. - \left[ \frac{1}{3} \delta_{k,0} \left( g_{0, k+q/2} g_{0, k-q/2} + c.c. \right) + 2/V \right] |d_{k,q}|^2 \right\}. \tag{18}\]

Now \(q\) is assumed to be small, and hence we may expand \(\varepsilon_{k+q/2} \approx \varepsilon_k + q/2 \cdot \frac{\partial \varepsilon_k}{\partial k} + O(q^2) = \varepsilon_k + q/2 \cdot v_F,\) the inverse electron propagator is to first order in \(q\) given by \(g_{0, k+q/2} \approx (-i \omega_n + \varepsilon_k).\) Hence, we find for the part containing magnetisation

\[g_{0, k+q/2} g_{0, k-q/2} + g_{k+q/2} g_{k-q/2} \]
\[= 2 \left( \frac{1}{(-i \omega_n + \varepsilon_k)^2} \right) \tag{19}\]

when keeping terms to second order in \(q.\) In addition, we expand the structure factor \(\gamma(q) \approx 6 - 2q^2,\) and assume for simplicity a cubic lattice in three dimensions. Now, we substitute \(\sum_k \rightarrow N_0 \int_{-\varepsilon_c}^{\varepsilon_c} d\varepsilon \int \frac{d^3 k}{V},\) where \(N_0\) is the
density of states at Fermi level and \( \epsilon_c \) is some cutoff. The homogeneous part of the magnetisation is given by

\[
F^c_{2,m} = \frac{\beta}{2} \sum_q \left( 4N_0 \tanh \left( \frac{\epsilon_m\beta}{2} \right) + \frac{1}{6J} \right) M_q M_{-q}. \tag{20}
\]

Similarly, we find for the superconducting order parameter, when introducing \( d_\mu(k,q) = A_\mu(q)\hat{k}\hat{k}_m \) and the indices \( \mu \) and \( i \) run from 1 to 3, that the coefficient in front of the term involving the superconducting order parameter is given by

\[
\sum_{k,\omega_n} 2\Re g_0\hat{k}_m = \frac{\beta}{2} \sum_q \left[ \frac{N_0(v_F/2)^2\epsilon_{n,m}^2}{72} \tanh \left( \frac{\epsilon_m\beta}{2} \right) \right] \left[ 1 - \tanh^2 \left( \frac{\epsilon_m\beta}{2} \right) \right] + \frac{1}{36J} \right] q^2 M_q M_{-q} \tag{21}
\]

The part containing derivatives is given by

\[
\alpha_d = -2N_0(v_F/2)^2q_iq_j \sum_{\omega_n} \int d\xi \int d\Omega \frac{2N_0\epsilon_m\hat{k}_i\hat{k}_j}{4\pi \omega_n^2 + \xi^2} \left[ 1 - \frac{(v_F/2)^2q_iq_m\hat{k}_i\hat{k}_m}{\omega_n^2 + \xi^2} \right] + \frac{q/2 \cdot v_F}{i\omega_n + \xi} - \frac{q/2 \cdot v_F}{-i\omega_n + \xi} \right]. \tag{22}
\]

Here, we have used \( v_F = v_F\hat{k} \). The terms linear in \( q \) will integrate to zero when integrated over the angles \( \Omega \).

We start with the constant term (independent of \( q \))

\[
\alpha_c = \sum_{\omega_n} \int d\xi \int d\Omega \frac{2N_0\epsilon_m\hat{k}_i\hat{k}_j}{4\pi \omega_n^2 + \xi^2} \left[ 1 - \frac{(v_F/2)^2q_iq_m\hat{k}_i\hat{k}_m}{\omega_n^2 + \xi^2} \right] + \frac{2\beta N_0j\delta_{ij}}{3} \sum_{n \geq 0} \frac{1}{n + 1/2} \tag{23}
\]

where \( \epsilon \) is an energy cut-off. In total the constant second order term is given by

\[
F^{c}_{2,S} = \frac{1}{2} \sum_q \left( \frac{2}{3V} - \frac{4N_0}{3} \sum_{n \geq 0} \frac{1}{n + 1/2} \right) \text{tr} \left( \mathcal{A} \mathcal{A}^t \right). \tag{24}
\]

After using the result for the critical temperature \( T_c \), the term may be written on the form

\[
F^{c}_{2,S} = \frac{1}{2} \sum_q \frac{4N_0 (T - T_c)}{3} \text{tr} \left( \mathcal{A} \mathcal{A}^t \right). \tag{25}
\]

The part containing derivatives is given by

\[
\alpha_d = -2N_0(v_F/2)^2q_iq_j \sum_{\omega_n} \int d\xi \int d\Omega \frac{2N_0\epsilon_m\hat{k}_i\hat{k}_j}{4\pi \omega_n^2 + \xi^2} \left[ 1 - \frac{(v_F/2)^2q_iq_m\hat{k}_i\hat{k}_m}{\omega_n^2 + \xi^2} \right] + \frac{q/2 \cdot v_F}{i\omega_n + \xi} - \frac{q/2 \cdot v_F}{-i\omega_n + \xi} \right]. \tag{26}
\]

so that the total second order term containing derivatives is given by

\[
F^{d}_{2,S} = -\frac{7\beta N_0\epsilon_m^2 \zeta(3)}{240\pi^2} \left[ q^2 \text{tr} \mathcal{A} \mathcal{A}^t + (q_i\mathcal{A}_{\mu i}) \left( \mathcal{A}_{\mu i} \right) + (q_j\mathcal{A}_{\mu j}) \left( \mathcal{A}_{\mu j} \right) \right]. \tag{27}
\]

Here we have assumed an isotropic quadratic dispersion relation, hence the coefficients in \( \mathcal{A} \) are all the same to this approximation. If one introduces an effective inverse mass tensor \((1/m^*)_{ij}\), one can use the result above by only replacing \( q_i \rightarrow q_i v_{Fi,j} \), where \( v_{Fi,j} \equiv 1/(1/m^*)_{ij} + (1/m^*)_{ji} \), and remove \( v_F \). In this way space is no longer isotropic.

### IV. Third Order Term

In this section we will find a coupling between the order parameters for magnetisation and superconductivity. The third order term will, however, only be nonzero when the system is in a non unitary phase, i.e. when \( d^* \times d \neq 0 \). In a ferromagnetic superconductor where the same electrons are responsible for superconductivity and ferromagnetism, the system, however, has to be in a
non-unitary state. If the system is in a unitary state the superconducting order parameter is invariant under the time inversion operator, and hence there can not be any magnetism associated with it.

The third order term in the expansion of the fermion determinant is given by

\[
E_3 = \frac{1}{3} \sum_{k_1, k_2, k_3} G_{0, k_1} \Sigma_{k_1, k_2} G_{0, k_2} \Sigma_{k_2, k_3} G_{0, k_3}. 
\]

(28)

Via a change of variables, \(k_1 \rightarrow \{k+q_1/2, k_2 \rightarrow \{k+q_2/2, k_3 \rightarrow \{k+q_3/2\}\}\) along with the constraint, \(q_1 + q_2 + q_3 = 0\), a multiplication of the matrices yields the following expression for the third order term

\[
E_3 = 4 \sum_{k, \{q_i\}} \left( g_{0, k} g_{0, k} \right) \cdot \left( \frac{\partial}{\partial q_1} \right) \cdot \left( \frac{\partial}{\partial q_2} \right) \cdot \left( \frac{\partial}{\partial q_3} \right) .
\]

(29)

were the combination \(m = i d \times d^*\) is interpreted as the average magnetisation due to the Cooper-pairs. The prime on the sum in Equation (29) denotes summation over configurations with the restriction \(\delta(q_1 + q_2 + q_3)\).

In the coefficient we have neglected the \(q\) dependence, since expanding in powers of \(q\) amounts to finding derivatives. The coefficient is given by, by introducing \(d_\mu = A_\mu \hat{k}_i\),

\[
\frac{\alpha_3}{3!} = 4 \sum_{\omega_n, k} \frac{1}{\omega_n + \varepsilon_k + \varepsilon_k + \omega_n + \varepsilon_k} \hat{k}_i \hat{k}_j \\
\approx 4 N_0 \sum_{\omega_n} \int_0^{\epsilon_c} d \xi \int \frac{d \Omega}{4 \pi} \frac{\xi}{\omega_n \omega_n + \xi^2} \hat{k}_i \hat{k}_j \\
= N_0 \beta \delta_{ij} / 3 \sum_{n \geq 0} \frac{1}{n + 1/2}.
\]

(30)

The final summation in the above expression is proportional to the logarithm of the cut-off frequency. The total third order term is thus given by

\[
F_3 = - \sum_{q_1, q_2, q_3} \frac{\alpha_3}{3!} i \xi_{\mu \nu \lambda} A_{\mu i} A_{\nu i} M_{\lambda}.
\]

(31)

were \(\alpha_3 = \tilde{\alpha}_3 / \beta\). As the order parameters \(M\) and \(d\) do not couple to second order in the GL free energy, the coupling in the third order term is expected to be of crucial importance as to whether the two order parameters will coexist in the system or not. Coexistence is favoured by the system if there is an energy gain by having \(M\) and \(d\) finite simultaneously. Hence, coexistence of magnetism and superconductivity depends on the sign of \(\alpha_3\), which again is given by the gradient of the DOS at Fermi level. In the system considered here, there will be a ferromagnetic coupling between magnetism and the spin magnetism of the Cooper-pairs if \(N_0^© > 0\), i.e the gradient of the DOS should be positive at the Fermi level for coexistence of FM and SC is preferred by the material.

Comparing Eq. (26) with the model Eq. (5), we find the value of the coefficient \(\gamma_0\) when we assume that \(\psi\) is in a \(p\)-wave state. The sign dependence of \(N_0^©\) is however more general, since a triplet state must necessarily be an odd function of \(k\). From (29), we observe that in an expansion of the DOS around Fermi level, \(N(\xi) \approx N_0 + \xi N_0^© + \xi^2/2 N_0^© + \ldots\), only the odd terms of the expansion give a contribution to the coefficient. Hence, to lowest order \(\gamma_0 \approx N_0^©\). We next determine the fourth order terms in the order-parameter expansion of the GL free energy.

V. FOURTH ORDER TERM

In this section, we will find three different types terms, namely a term involving only magnetisation, one only involving superconductivity, and finally one term involving a coupling between the magnetisation \(M\) and the superconducting order parameter \(d\). In total, we will find five different independent terms for a \(p\)-wave ferromagnetic superconductor. The fourth order term of the expansion of the fermion determinant is given by

\[
E_4 = \frac{1}{4} \sum_{k_1, k_2, k_3, k_4} G_{0, k_1} \Sigma_{k_1, k_2} G_{0, k_2} \Sigma_{k_2, k_3} G_{0, k_3} \Sigma_{k_3, k_4}.
\]

(32)

As before, we change variables and multiply out the matrices and take the trace. In addition, we make the approximation of neglecting the dependence on \(q_i\) in \(g_{0, k}\).

A. Terms containing only magnetic order parameter

The term involving the magnetisation only is given by

\[
\beta F_{4, m} = \sum_{\{q_i\}} \left( M_{q_1} \cdot M_{-q_2} \right) \left( M_{q_3} \cdot M_{-q_4} \right) \left( \Re \sum_{k, \omega_n} g_{0, k} g_{0, k} \right).
\]

(33)

Here, the coefficient of the \(M_q\)-factors is given by the trace over the electron propagators,

\[
\tilde{\alpha}_{4m} = \Re \sum_{k, \omega_n} (g_{0, k})^4 = N_0 \Re \sum_{\omega_n} \int_0^{\epsilon_m} d \xi \left( \frac{1}{-i \omega_n + \xi} \right)^4 \\
= \frac{\beta^2}{4!} \tanh \left( \frac{\epsilon_m \beta}{2} \right) \left( 1 - \tanh^2 \left( \frac{\epsilon_m \beta}{2} \right) \right).
\]

(34)
Thus, the fourth order term involving magnetisation only, is given by
\[
F_{4,m} = \sum_{\{q_k\}} \frac{\alpha_{4m}}{4!} (M_{q_1} \cdot M_{-q_2}) (M_{q_3} \cdot M_{-q_4})
\] (35)
where \(\alpha_{4m} = 4!\alpha_{2m}/\beta\).

**B. Terms containing only superconducting order parameter**

The term involving the superconducting order parameter only will contain five different terms. Again neglecting the \(q\) dependence in \(g_{0,k}\), we arrive at
\[
\beta F_{4,S} = \sum_{\{q\}} \Re \sum_{k,\omega_n} \left( \frac{k_i k_j \tilde{k}_i \tilde{k}_j}{(\omega_n^2 + \epsilon_k^2)^2} \right) \left[ 2 A_{\mu i}^* A_{\nu j} A_{\nu l} A_{\nu m}^* - A_{\mu i}^* A_{\mu j} A_{\nu l} A_{\nu m}^* \right].
\] (36)
Now let the sum over \(k\) go over to an integral over \(\xi\) and the angles \(\theta\) and \(\phi\). The integral over the angles produces Kronecker-\(\delta\)'s in Latin indices,
\[
\int \frac{d\Omega}{4\pi} k_i \tilde{k}_j \tilde{k}_i \tilde{k}_j = \frac{1}{15} (\delta_{i,j} \delta_{l,m} + \delta_{i,l} \delta_{j,m} + \delta_{i,m} \delta_{j,l}).
\] (37)
The coefficient is given by
\[
\frac{\alpha_{4S}}{4!} = \frac{N_0}{15\beta} \int d\xi \sum_{\omega_n} \left( \frac{1}{\omega_n^2 + \xi^2} \right)^2 = \frac{7N_0\zeta(3)}{120\pi^2} \beta^2
\] (38)
In total, the fourth order term involving superconductivity alone is given by,
\[
F_{4,S} = \frac{\alpha_{4S}}{4!} \sum_{\{q\}} \left\{ - |\text{tr} \mathcal{A}^T \mathcal{A}|^2 + 2 (\text{tr} \mathcal{A}^T \mathcal{A})^2 \\
+ 2 \text{tr} (\mathcal{A}^T \mathcal{A}^T) (\mathcal{A} \mathcal{A}^*) + 2 \text{tr} (\mathcal{A} \mathcal{A}^T)^2 \\
- 2 \text{tr} (\mathcal{A}^T \mathcal{A}^T) (\mathcal{A}^T \mathcal{A}^*) \right\},
\] (39)
where \(\alpha_{4S} = \frac{7\zeta(3)N_0}{5\pi^2} \beta^2\). This part of the free energy thus consists of five independent terms.

**C. Terms containing magnetic and superconducting order parameter**

The fourth order coupling term between the magnetisation and the superconducting order parameter is given by
\[
F_{4,Sm} = \sum_{\{q\}} \left\{ 8g_{0,k} g_{0,k} g_{0,k} g_{0,k} \left[ 2 (M_{q_1} d_{-q_2}) (M_{q_3} d_{-q_4}) - (M_{q_1} M_{q_3}) (d_{-q_2} d_{-q_4}) \right] \\
- 16g_{0,k} g_{0,k} g_{0,k} g_{0,k} \left( d_{q_1} d_{-q_4} \right) (M_{q_1} M_{-q_2}) \right\}
\] = \sum_{\{q\}} \frac{7N_0\zeta(3)\beta^3}{3\pi^2} (M_{\mu i} A_{\mu j}) (M_{\nu i} A_{\nu j}^*)
\] (40)
where the constant in the free energy is \(\alpha_{4Sm} = 72N_0\zeta(3)\beta^2/\pi^2\). This coupling term between magnetism and superconductivity differs from the fourth order coupling in Eq. (34). In Eq. the coupling term is proportional to \(|M \cdot \psi|^2\), whereas we find a term proportional to \((M \cdot M) |\psi|^2\). In addition, the coefficient is positive rather than having an indefinite sign as commented on in Ref. 13. We have, however, assumed that the system is in a \(p\)-wave state. In the case of a general spin-triplet state, both types of fourth order coupling terms between magnetism and superconductivity may be present, as also seen from Eq. (40).

**D. Complete fourth order term**

In total, the fourth order term is thus given by
\[ F_4 = \frac{1}{4!} \sum_{\{q_i\}} \left\{ \alpha_{4m} (M_{q_1} \cdot M_{-q_2}) (M_{q_3} \cdot M_{-q_4}) + \alpha_{4Sm} (M_{\mu} A_{\mu j}) (M_{\nu} A_{\nu j}^*) \right. \]
\[ + \alpha_{4S} \left[ - \text{tr} \mathbf{A} \mathbf{A}^T \right]^2 + 2 \left( \text{tr} \mathbf{A} \mathbf{A}^T \right)^2 + 2 \text{tr} \left( \mathbf{A} \mathbf{A}^T \right) \left( \mathbf{A} \mathbf{A}^T \right)^* + 2 \text{tr} \left( \mathbf{A} \mathbf{A}^T \right)^2 \]
\[ - 2 \text{tr} \left( \mathbf{A} \mathbf{A}^T \right) \left( \mathbf{A} \mathbf{A}^T \right)^* \right\} \] (41)

All in all the Ginzburg-Landau free energy for a ferromagnetic superconductor is given by
\[ F_{GL} = \int \frac{d^3 r}{2} \frac{\alpha S(T)}{2} \text{tr} \mathbf{A} \mathbf{A}^T + \frac{\beta S}{2} \left( D^2 \text{tr} \mathbf{A} \mathbf{A}^T + D_1 A_{\mu i} D_j A^\dagger_{\mu j} + D_1 A_{\mu i} D_1 A_{\mu j} \right) \]
\[ + \frac{\alpha_m(T)}{2} \mathbf{M} \cdot \mathbf{M} + \frac{\beta_m(A)}{2} \mathbf{D} \cdot \mathbf{D} + \frac{\alpha_4}{3!} \epsilon_{\mu\nu\lambda} A_{\mu i} A_{\nu i} M_{\lambda} \]
\[ + \frac{\alpha_{4S}}{4!} \frac{1}{2} \left( \text{tr} \mathbf{A} \mathbf{A}^T \right)^2 - \left| \text{tr} \mathbf{A} \mathbf{A}^T \right| + 2 \text{tr} \left( \mathbf{A} \mathbf{A}^T \right) \left( \mathbf{A} \mathbf{A}^T \right)^* + 2 \text{tr} \left( \mathbf{A} \mathbf{A}^T \right)^2 - 2 \text{tr} \left( \mathbf{A} \mathbf{A}^T \right) \left( \mathbf{A} \mathbf{A}^T \right)^* \]
\[ + \frac{\alpha_{4m}}{4!} \left( \mathbf{M} \cdot \mathbf{M} \right)^2 + \frac{\alpha_{4ms}}{4!} \left( M_{\mu} A_{\mu i} \right) \left( M_{\nu} A_{\nu i}^* \right) \right\} , \] (42)

where \( D = \nabla + 2ieA \), \( \mathbf{D} = \nabla + ieA \) and \( A \) is the electromagnetic vector potential.

**VI. TUNNELING**

As an application of the Ginzburg-Landau theory derived above, we will consider tunneling between two ferromagnetic p-wave superconductors. The Hamiltonian we use is the common choice when studying tunneling between systems in equilibrium
\[ H = H_L + H_R + H_T , \] (43)
where \( H_{L(R)} \) is given by (42) and
\[ H_T = \sum_{k,p} \left\{ T_{k,p}^{\alpha,\beta} \mathbf{c}_{k,\alpha} d_{p,\beta} + T_{k,p}^{\ast,\alpha,\beta} \mathbf{c}^\dagger_{k,\alpha} \mathbf{d}^\dagger_{k,\beta} \right\} . \] (44)

For a general Hamiltonian, it is straightforward to show that the part of the Ginzburg-Landau free energy that contains the tunneling elements, is given by
\[ F_J = -\frac{1}{\beta} \text{Tr} \ln \left( 1 - \mathbf{G}_L \mathbf{T} \mathbf{G}_R \mathbf{T} \right) , \] (45)
where \( \mathbf{G}_{L(R)} \) are the Green’s functions for the left(right) subsystem and \( \mathbf{T} \) is a tunneling matrix,
\[ \mathbf{T}_{k,p} = \left( \begin{array}{cc} T_{k,p} & 0 \\ 0 & -T_{k,-p}^\dagger \end{array} \right) \] (46)
with
\[ T_{k,p} = \left( \begin{array}{cc} T_{k,p}^{\uparrow \uparrow} & T_{k,p}^{\uparrow \downarrow} \\ T_{k,p}^{\downarrow \uparrow} & T_{k,p}^{\downarrow \downarrow} \end{array} \right) \] (47)

Furthermore, we will assume for simplicity that the magnetisation of our system is weak, hence we assume that the magnetisation is homogeneous. Introducing the basis \( \phi_k^\dagger = \left[ \xi_{k,\uparrow}, \xi_{k,\downarrow}, \xi_{-k,\uparrow}, \xi_{-k,\downarrow} \right] \) and choosing the quantisation axis along the magnetisation vector at each side of the junction, we have the following effective action,
\[ S_{\text{eff}} = S^L + S^R + S^T \] (48)
and the inverse Green’s functions are given by
\[ \mathbf{G}_{L(R)}^{-1} = \left( \begin{array}{cc} \mathbf{g}_{L(R)}^{-1} & -\Delta_{L(R)}^{-1} \\ -\Delta_{L(R)} & \mathbf{g}_{L(R)}^{-1} \end{array} \right) , \] (49)
with \( g_{L(R)\alpha\beta}^{-1} = [-i\omega_n + \epsilon_k - \alpha M_{L(R)}] \delta_{\alpha\beta} \). The direction of the magnetisation does not enter in the Greens functions for the isolated systems on the left and right, the angle between the magnetisations, and hence the quantisation axes, only enters through the tunneling elements since we have applied different rotations on the left and right. This difference shows in the part of the Hamiltonian where the left and right systems couple.

We now want to calculate the Ginzburg-Landau free energy for the junction, \( F_J \). To this end, we expand (45) to the lowest order in the tunneling elements,
\[ F_J \approx \frac{1}{\beta} \text{Tr} \left[ \mathbf{G}_L \mathbf{T} \mathbf{G}_R \mathbf{T} \right] . \] (50)

Next, we use the Dyson equation to find an expansion of \( \mathbf{G} \) i.e. \( \mathbf{G} \approx \mathbf{G}_0 + \mathbf{G}_0 \Sigma \mathbf{G}_0 + ... \) were
\[ \mathbf{G}_0 = \left( \begin{array}{cc} g & 0 \\ 0 & g^* \end{array} \right) \] (51)
and
\[
\Sigma = \begin{pmatrix} 0 & \Delta \\ \Delta^\dagger & 0 \end{pmatrix}. \tag{52}
\]

Since we are assuming that the magnetisation is homogeneous, \( \mathbf{g}_0 \) is local in \( \mathbf{k} \)-space, \( \Sigma \) however is nonlocal as we do not want to impose any restriction on the superconducting pairing state.

From the free energy of the junction we can find the current of various quantities associated with tunneling. To show this we introduce a Hamiltonian \( H(q,p) \) which is a function of the canonically conjugate variables \( q,p \), and recall that the velocity operator is given by \( \dot{q} = \frac{\partial H}{\partial p} \).

Consider now the quantity \( \frac{\partial F}{\partial p} = -\frac{1}{Z} \frac{\partial}{\partial p} \ln \left[ \text{Tr} e^{-\beta H} \right] = \frac{1}{Z} \text{Tr} \left[ \frac{\partial H}{\partial p} e^{-\beta H} \right] = \langle \dot{q} \rangle \). Hence we are easily able to find the current over the junction of for instance electrons. Since \( \langle \phi, N \rangle = i \), the current is then given by \( I_J = -e \langle \dot{N} \rangle = e \frac{\partial}{\partial \sigma} F_J \) and the contribution to the spin current coming from tunneling across the barrier, is given by \( \langle \dot{S}_n \rangle = -\mu_B \frac{\partial \beta}{\partial \theta} F_J \), where \( \theta, S_n^i = i \), and \( \theta \) is the angle in the plane perpendicular to the direction \( \hat{n} \).

### A. Ferromagnetic case

We start with the term corresponding to a junction between two ferromagnetic metals,

\[
F_J^{(0)} = \frac{1}{\beta} \sum_{k,p} \text{tr} \left( \mathbf{g}_{0k} \mathbf{T}_{k,p} \mathbf{g}_{0p} \mathbf{T}_{k,p}^\dagger \right) = \frac{1}{\beta} \sum_{k,p \pi} \text{tr} \left\{ \mathbf{g}_k \mathbf{T}_{k,p} \mathbf{g}_p \mathbf{T}_{k,p}^\dagger + \mathbf{g}_k^\dagger \mathbf{T}_{k,p} \mathbf{g}_p^\dagger \mathbf{T}_{k,p}^\dagger \right\}. \tag{53}
\]

We observe that upon letting \( k \to -k \) and \( p \to -p \) the second term above is the complex conjugate of the first one, hence the free energy is two times the real part of the first term,

\[
F_J^{(0)} = \frac{2}{\beta} \sum_{k,p} \Re \left[ g_k^+ g_k^+ \left| T_{k,p}^+ \right|^2 + g_k^- g_k^- \left| T_{k,p}^- \right|^2 \\ + g_k^+ g_k^- \left| T_{k,p}^- \right|^2 + g_k^- g_k^+ \left| T_{k,p}^+ \right|^2 \right]. \tag{54}
\]

Here + and - means parallel or anti parallel to the magnetisation respectively. The angle between the magnetisations on the left and right enters through the tunneling elements \( \left| T_{k,p}^\beta \right|^2 = 1/2 \left( 1 + \alpha \beta M^L \cdot M^R \right) \left| T_{k,p} \right|^2 \), when we assume that the spin in preserved across the tunnel barrier. The potential difference between the left and right subsystems are taken care of through the tunneling elements i.e. we assume that \( \mathbf{T}_{k,p} \) is nonlocal in \( \omega \)-space with a boson frequency \( \omega_\nu \), later we do an analytical continuation \( \omega_\nu \to eV + i\delta \). The structure of the Green’s functions is,

\[
g_{k}^\beta g_{\mu}^\beta = \frac{1}{-i(\omega_{\mu} - \omega_{\nu}) + \varepsilon_{k\alpha} - \varepsilon_{\mu\beta} + i\delta} \tag{55}
\]

were \( \varepsilon_{k\alpha} = \varepsilon_k - \alpha M^L \) and similarly for the right subsystem. The zero’th order term of the free energy may be written in two parts, one part independent of the direction of the magnetisations and one part proportional to the dot product of the directions of the magnetisations, that is, to the cosine of the angle between the magnetisations, \( F_J^{(0)} = \tilde{F}_J^{(0)} + \mathbf{M}^L \cdot \mathbf{M}^R \tilde{F}_J^{(0)} \).

With

\[
\tilde{F}_J^{(0)} = \frac{1}{\beta} \sum_{k,p} \Re \left[ g_k^+ g_k^+ + g_k^- g_k^- + g_k^+ g_k^- + g_k^- g_k^+ \right] \left| T_{k,p} \right|^2
\]

\[
= |T|^2 \int d\xi_k \int d\xi_p N(\xi_k)N(\xi_p) \begin{cases} f(\xi_k - M^L) - f(\xi_p - M^R) \\ eV + \xi_k - \xi_p + M^R - M^L \end{cases}
\]

\[
+ f(\xi_k - M^L) - f(\xi_p + M^R) \\
+ f(\xi_k - M^L) - f(\xi_p + M^R) \\
+ f(\xi_k + M^L) - f(\xi_p - M^R) \\
+ f(\xi_k + M^L) - f(\xi_p - M^R) \end{cases}
\]

\[
eV + \xi_k - \xi_p + M^L - M^R \]  

and

\[
\tilde{F}_{J,M}^{(0)} = \frac{1}{\beta} \sum_{k,p} \Re \left[ g_k^+ g_k^+ + g_k^- g_k^- - g_k^+ g_k^- - g_k^- g_k^+ \right] \]

\[
= |T|^2 \int d\xi_k \int d\xi_p N(\xi_k)N(\xi_p) \begin{cases} f(\xi_k - M^L) - f(\xi_p - M^R) \\ eV + \xi_k - \xi_p + M^R - M^L \end{cases}
\]

\[
+ f(\xi_k - M^L) - f(\xi_p + M^R) \\
+ f(\xi_k - M^L) - f(\xi_p + M^R) \\
+ f(\xi_k + M^L) - f(\xi_p - M^R) \\
+ f(\xi_k + M^L) - f(\xi_p - M^R) \end{cases}
\]

\[
eV + \xi_k - \xi_p + M^L - M^R \]  

Here \( N(\xi) \) is the density of states and \( f(x) = 1/(1 + e^{\beta x}) \) is the Fermi-Dirac distribution function. The spin current \( \left\langle \dot{S}_n \right\rangle \), were \( n \) denotes the direction perpendicular to the plane spanned by the magnetisation vectors, is now easily found by taking the derivative with respect to the angle between the magnetisations, i.e.

\[
\left\langle \dot{S}_n \right\rangle = -\mu_B \frac{\partial \beta}{\partial \theta} F_J^{(0)}
\]

\[
= -\mu_B F_{J,M} \sin \theta \tag{58}
\]

This is precisely the same result as found previously in Ref. 17 via a different route.
B. Ferromagnetic superconducting case

In this subsection, we calculate the free energy which is first order in the superconducting gap function on both the left and right side, i.e., we consider the ferromagnetic superconducting state. We want to find the Josephson current and also the two-particle contribution to the left and right side, i.e., we consider the ferromagnetic is first order in the superconducting gap function on both sides.

We now change variables, \( k \rightarrow k^+q/2 \) and \( p \rightarrow p^+q/2 \). We were interested in derivatives and neglect the \( \Delta \) dependence in all terms except \( \Sigma \), since we want to have the possibility \( \Delta_{++} \neq 0 \). With these assumptions the free energy reads,

\[
F_j^{(2)} = \frac{1}{\beta} \sum \text{tr} \left[ g_k \Delta (\hat{k}, \hat{q}) g^*_k T_{-k,-p} g^*_p \Delta \hat{q} T_{k,p}^\dagger \right].
\]

(60)

Here we observe that upon changing the sign of all impulses in the second term and using that \( \Delta (-\hat{k}) = -\Delta (\hat{k}) \) for a triplet superconductor, the second term is just the complex conjugate of the first one. Hence, the free energy is just two times the real part of the first term. After some straightforward algebra, we obtain

\[
F_j^{(2)} = -|T|^2 \sum_{q, \hat{q}} \left\{ (1 + M^L \cdot M^R) \left[ \left( A^L_{1+} F^L_{1+} + A^L_{1+} F^L_{1+} + A^L_{1+} (F^L_{1+} + F^L_{1+}) \right) \cos Δφ \right. \right.
\]

\[
+ \left( A^L_{1+} h^L_{1+} + A^L_{1+} h^L_{1+} + A^L_{1+} (h^L_{1+} + h^L_{1+}) \right) \sin Δφ \right]
\]

\[
+ \left( 1 - M^L \cdot M^R \right) \left[ \left( A^R_{1+} F^R_{1+} + A^R_{1+} F^R_{1+} - A^R_{1+} (F^R_{1+} + F^R_{1+}) \right) \cos Δφ \right.
\]

\[
+ \left( A^R_{1+} h^R_{1+} + A^R_{1+} h^R_{1+} - A^R_{1+} (h^R_{1+} + h^R_{1+}) \right) \sin Δφ \right]
\]

\[
+ \left| M^L \times M^R \right| \left[ \left( A^L_{1+} (F^L_{1+} + F^L_{1+}) - A^L_{1+} (F^L_{1+} + F^L_{1+}) + A^L_{1+} (F^L_{1+} + F^L_{1+}) - A^L_{1+} (F^L_{1+} + F^L_{1+}) \right) \cos Δφ \right.
\]

\[
+ \left( A^L_{1+} (h^L_{1+} + h^L_{1+}) - A^L_{1+} (h^L_{1+} + h^L_{1+}) + A^L_{1+} (h^L_{1+} + h^L_{1+}) - A^L_{1+} (h^L_{1+} + h^L_{1+}) \right) \sin Δφ \left\} \right\}
\]

(61)

were we have introduced the notation

\[
A_{αβ}^{L,R}(q, \hat{q}) = 2 \int_{\Omega_k>0} \frac{dΩ_k}{4π} \int_{\Omega_p>0} \frac{dΩ_p}{4π} \left| \Delta_{αβ}^{L,R} \right| \left| \Delta_{αβ}^{R,L} \right| \left| T_{Ω_k,Ω_p} \right|^2,
\]

(62)

and

\[
F_{αβ}^{L,R} \equiv \int dξ_k dξ_p \left\{ \left. \frac{N(\xi_k + \alpha M^L) + N(\xi_k + \beta M^L)}{4 \left( ξ_k - \frac{α+β}{2} M^L \right) \left( ξ_p + \frac{α+β}{2} M^R \right)} f(ξ_k) - f(ξ_p) \right\} e^V + ξ_k - ξ_p.
\]

(63)

In addition, we have assumed that \( φ_{αβ}^{L,R} = φ^{L,R}_{αβ} = Δφ \) is independent of the spin indices, this can be done since any coupling between the different components of the superconducting order parameter will give a phase locking. Further since \( T_{k,-p} = 0 \) the sum over \( k \) and \( p \) in Equation (59) may be restricted to positive values only, and hence there are no problems involved in splitting the superconducting order parameter into an amplitude and a phase in Equation
The Josephson current is now found by simply taking the derivative of the free energy with respect to the phase difference $\Delta \phi$. This leads to the flowing expression for the Josephson current

$$I_J = e |T|^2 \sum_{q,q'} \left\{ (1 + M^L \cdot M^R) \left[ \left( A^\alpha_+ F^\alpha_{+\alpha} + A^\alpha_- F^\alpha_{-\alpha} + A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) \right) \sin \Delta \phi \right. \\
- \left. \left( A^\alpha_+ h^\alpha_{+\alpha} + A^\alpha_- h^\alpha_{-\alpha} + A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) \right) \cos \Delta \phi \right] \\
+ \left( 1 - M^L \cdot M^R \right) \left[ \left( A^\alpha_+ F^\alpha_{+\alpha} + A^\alpha_- F^\alpha_{-\alpha} - A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) \right) \sin \Delta \phi \\
- \left. \left( A^\alpha_+ h^\alpha_{+\alpha} + A^\alpha_- h^\alpha_{-\alpha} - A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) \right) \cos \Delta \phi \right] \\
+ \leq M^L \times M^R \leq \left[ \left( A^\alpha_+ (F^\alpha_{++} + F^\alpha_{-\alpha}) - A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) + A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) - A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) \right) \sin \Delta \phi \\
- \left. \left( A^\alpha_+ (h^\alpha_{++} + h^\alpha_{-\alpha}) - A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) + A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) - A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) \right) \cos \Delta \phi \right] \right\}. \quad (65)$$

Similarly, we find the following expression for the two-particle contribution to the spin current

$$\langle \hat{S}_n \rangle_2 = - |T|^2 \sum_{q,q'} \left\{ \sin \theta \left[ \left( A^\alpha_+ F^\alpha_{+\alpha} + A^\alpha_- F^\alpha_{-\alpha} + A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) + F^\alpha_{+\alpha} + F^\alpha_{-\alpha} - A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) \right) \cos \Delta \phi \\
+ \left( A^\alpha_+ h^\alpha_{+\alpha} + A^\alpha_- h^\alpha_{-\alpha} + A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) - A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) \right) \sin \Delta \phi \right] \\
+ \cos \theta \left[ \left( A^\alpha_+ (F^\alpha_{++} + F^\alpha_{-\alpha}) - A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) + A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) - A^\alpha_0 (F^\alpha_{++} + F^\alpha_{-\alpha}) \right) \cos \Delta \phi \\
+ \left( A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) - A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) + A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) - A^\alpha_0 (h^\alpha_{++} + h^\alpha_{-\alpha}) \right) \sin \Delta \phi \right] \right\}. \quad (66)$$

The spin current is not well defined in the limit when $\theta \to 0$, since we are calculating the current along the direction perpendicular to the plane spanned by the two magnetisation directions. We observe that the total spin current is the sum of Eq. (58) and (66). Hence, Eq. (66) is a contribution originating with the interplay between superconductivity and magnetism. It is seen to disappear when superconductivity is lost. Moreover, we observe that there is a term in equation (66) that is proportional to $\cos \theta$. In Josephson currents there also exist a term that is proportional to $\cos \Delta \phi$, which however is dissipative and vanishes if the voltage across the junction is set to zero. In the case of the spin current, we find that the condition that must be fulfilled for the cosine term to go to zero, is given by $|\Delta^L_{\alpha \beta}| = |\Delta^R_{\alpha \beta}|$, $M^L = M^R$ and $eV = 0$. Furthermore, it is seen that the cosine term is associated with flipping one electron from (spin)state $\alpha$ on the left to $-\alpha$ on the right, hence the cosine term is proportional to $|\Delta^L_{\alpha \alpha}| / |\Delta^R_{-\alpha \alpha}|$ and similarly with the superscripts interchanged.

VII. CONCLUDING REMARKS

We have derived a Ginzburg-Landau functional from a microscopic Hamiltonian consisting of three terms, free fermias, an Heisenberg term and a spin generalised BCS term. We find two order parameters, local magnetisation and the superconducting gap. We expand to fourth order in the order parameters. The lowest order coupling between the two order parameters is a third order term, this term is only non zero when the material is in a non unitary superconducting state i.e. $d \times d \neq 0$. From an exact calculation of the coefficient of the third order term we find that coexistence of ferromagnetism and superconductivity is enhanced if the gradient of the density of states, $dN(\xi)/d\xi$, is positive at Fermi level.

We have also computed the tunneling contribution to the Ginzburg-Landau free energy. From this, we have found expressions for spin- and charge currentwts in the spin- and charge channels across a tunneling junction separating two ferromagnetic $p$-wave superconductors.

Superconductivity coexisting with ferromagnetism can be triggered by the magnetisation as a result of the presence of a third order term, i.e. superconductivity appears at a higher temperature than if magnetisation were not present. The sign of the third order term is given by the sign of the gradient of the DOS at Fermi level. Since the Fermi level may be tuned by applying pressure, it may be possible to change the sign of the third order term by applying pressure.

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