Heat Kernels in the Context of Kato Potentials on Arbitrary Manifolds

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Abstract By introducing the concept of Kato control pairs for a given Riemannian minimal heat kernel, we prove that on every Riemannian manifold \( (M, g) \) the Kato class \( \mathcal{K}(M, g) \) has a subspace of the form \( L^q(M, d\varrho) \), where \( \varrho \) has a continuous density with respect to the volume measure \( \mu_g \) (where \( q \) depends on \( \dim(M) \)). Using a local parabolic \( L^1 \)-mean value inequality, we prove the existence of such densities for every Riemannian manifold, which in particular implies \( L^q_{\text{loc}}(M) \subset \mathcal{K}_{\text{loc}}(M, g) \). Based on previously established results, the latter local fact can be applied to the question of essential self-adjointness of Schrödinger operators with singular magnetic and electric potentials. Finally, we also provide a Kato criterion in terms of minimal Riemannian submersions.

Keywords Heat kernel estimates · Kato potentials · Parabolic mean value inequality

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1 Introduction

Given a Riemannian manifold \( (M, g) \) with \( \mu_g \) the Riemannian volume measure, a Borel function \( w : M \rightarrow \mathbb{R} \) is said to be Kato class of \( (M, g) \), symbolically \( w \in \mathcal{K}(M, g) \), if

\[
\lim_{t \to 0^+} \sup_{x \in M} \int_0^t \int_M e^{(s/2)\Delta_g(x, y)}|w(y)|d\mu_g(y)ds = 0,
\]

where \((-1/2)\Delta_g \geq 0\) denotes the Friedrichs realization of \((1/2)\) times the Laplace-Beltrami operator in \( L^2(M, d\mu_g) \). In particular, \( e^{\xi^2\Delta_g(x, y)} \) is precisely the minimal nonnegative heat kernel \( p_g(s, x, y) \) on \( (M, g) \). Likewise, there is the local counterpart \( \mathcal{K}_{\text{loc}}(M, g) \).
which is given by all \( w \) such that \( Kw \in K(M, g) \) for all compact \( K \subset M \). Ever since its introduction, the Kato class has proved to be a convenient and large class of perturbations of \((-1/2)\Delta_g\), for which the following important results hold simultaneously: For every \( w = w_+ - w_- \) such that its positive part satisfies \( w_+ \in L^1_{\text{loc}}(M) \), its negative part satisfies \( w_- \in K(M, g) \).

I) \( w_- \) is an infinitesimally small perturbation of \((-1/2)\Delta_g\) (cf. [19]) in the sense of quadratic forms; in particular the form sum \( H^w_g = (-1/2)\Delta_g + w \) is a well-defined self-adjoint operator in \( L^2(M, d\mu_g) \) which is bounded from below.

II) One has \( L^q(M, d\mu_g) \rightarrow L^q(M, d\mu_g) \)-bounds of the form (cf. Proposition A.1 below)

\[
\left\| e^{-tH^w_g} \right\|_{L^q(M, d\mu_g) \rightarrow L^q(M, d\mu_g)} \leq \delta e^{C(\delta, w_-, g)}, \quad \text{for every } \delta > 1
\]

III) \( x \mapsto e^{-tH^w_g} f(x) \) is continuous [18] for all \( f \in L^\infty(M, d\mu_g) \) if \( w \in K(M, g) \) [18].

The remarkable fact about these results is that all of them do not require any additional assumptions on the Riemannian structure \( g \) on \( M \). The bound from II) with \( q = \infty \) has been used recently in the context of the Riemannian total variation by D. Pallara and the author in [8]. Let us also note that one can even establish a semigroup theory of perturbations given by Kato measures rather than Kato functions: Here there exist very subtle results by Sturm [20], Stollmann-Voigt [19], and Kuwae-Takahashi [14], and can even do more general than that [18].

There is another important result which is built on the local Kato class [7]:

IV) If \( (M, g) \) is geodesically complete, if \( \alpha \in \Gamma_{1,4}^1(M, T^*M) \) is a magnetic potential with \( \text{grad}(\alpha) \in L^2_{\text{loc}}(M) \), and if \( w \in K_{\text{loc}}(M, g) \cap L^2_{\text{loc}}(M) \) is an electric potential such that the corresponding magnetic Schrödinger operator \( H^{\alpha, w}_g \) is bounded from below on the smooth compactly supported functions, then \( H^{\alpha, w}_g \) is in fact essentially self-adjoint.

Apart from the above “success” of the Kato class from an abstract point of view, as one knows the explicit form of \( p_g(t, x, y) \) only in very few cases, the following question remains:

**When is a given Borel function \( w \) on \( M \) actually in \( K(M, g) \) or in \( K_{\text{loc}}(M, g) \)?**

In the Euclidean \( \mathbb{R}^m \) this question is usually easy to answer, as one has the characterization \( w \in K(\mathbb{R}^m) \), if and only if

\[
w \in L^1_{\text{unif,loc}}(\mathbb{R}), \quad \text{if } m = 1,
\]

\[
\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}^m} \int_{|x-y| \leq r} |w(y)| h_m(|x-y|) dy, \quad \text{if } m \geq 2,
\]

where \( h_m : [0, \infty) \rightarrow [0, \infty) \) is given by

\[
h_2(r) := \log^+(1/r), \quad h_m(r) := r^{2-m}, \quad \text{if } m > 2,
\]

In fact Kato has introduced \( K(\mathbb{R}^m) \) essentially in this “analytic” form in [13], and the equivalence of the latter definition to the above heat-kernel definition has been shown