Asymptotic and numerical analysis of resonance and lock-in by flow-acoustic interaction in an expansion chamber-pipe system

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Abstract
The paper is concerned with the generation of sound by the flow through a closed, cylindrical expansion chamber, followed by a long tailpipe. The sound generation is due to self-sustained flow oscillations in the expansion chamber which, in turn, may generate standing acoustic waves in the tailpipe. The main interest is in the interaction between these two sound sources. An analytical, approximate solution of the acoustic part of the problem is obtained via the method of matched asymptotic expansions. The sound-generating flow is represented by a discrete vortex method, which is modified to include the effects of acoustic feedback. It is demonstrated that lock-in of the self-sustained flow oscillations to the resonant acoustic waves in the tailpipe can take place.

Key words : Vortex sound, Matched asymptotic expansions, Discrete vortex method, Flow-sound interaction

1. Introduction

Expansion chambers (mufflers, cavities) are used in connection with silencers in engine exhaust systems, with the aim of attenuating the sound waves through destructive interference. But the gas flow through the chamber may generate self-excited oscillations, thus becoming a sound generator rather than a sound attenuator (Davies, 1981; Rubio et al., 2006; English and Holland, 2010). Similar geometries and thus similar problems may be found in, for example, solid propellant rocket motors (Howe, 1998), valves (Tamura et al., 2013), and in corrugated pipes (Goyder, 2013).

The aim of the present work is to contribute to the understanding of the interaction between oscillations of the flow field and the acoustic field. By oscillations of the flow field we mean the self-sustained oscillations of the jet shear layer.
The shear layer is unstable and rolls up into a large, coherent vortex structure (similar to a smoke ring) which is convected downstream with the flow. This large vortex cannot pass through the hole in the downstream end plate of the expansion chamber but hits this plate, where it creates a pressure disturbance. The disturbance is transferred back to the upstream end plate, where it disturbs the shear layer. This initiates the roll-up of a new coherent vortex - and the process is repeated. In this way a feedback loop is formed.

These so-called hole-tone feedback oscillations (Rayleigh, 1945; Chanaud and Powell, 1965; Howe, 1998; Langthjem and Nakano, 2005, 2010; Matsuura and Nakano, 2012) may interact with the acoustic axial and (to a much lesser extend) radial eigen-oscillations in the cavity and in the tailpipe (Davies, 1981; English and Holland, 2010). In the present paper we seek to understand the interaction with the axial waves in the tailpipe.

As indicated in Fig. 2 (in the following section), perfect axisymmetry is assumed, and a mathematical model is formulated in terms of cylindrical axisymmetric coordinates \((x, r)\). The sound-generating flow is represented by a discrete vortex approach (Cottet and Koumoutsakos, 2000; Majda and Bertozzi, 2002), based on (axisymmetric) vortex rings, as applied also in earlier papers (Langthjem and Nakano, 2005, 2010).

The acoustic part of the problem can be solved analytically, and completely, in terms of eigenfunction expansions. A travelling wave formulation for a single change in cross-sectional area was considered already more than seventy years ago by Miles (1944). The results of Miles (1944) were employed by El-Sharkawy and Nayfeh (1978) in an analysis of sound propagation through an expansion chamber. The problem of Miles was reconsidered by Dup`ere and Dowling (2001) in terms of Howe’s theory of vortex sound (Howe, 1998, 2003).

But an eigenfunction expansion approach, as mentioned above, will become quite complicated. A much more manageable - and also more informative - approach is possible by taking advantage of characteristic length-scales in the different regions of the problem: (i) tailpipe region, (ii) step (cross section change) regions, and (iii) cavity region. The simplified solutions for these three regions can then be coupled by employing the method of matched asymptotic expansions (Lesser and Crighton, 1975; Kevorkian and Cole, 1981; Nayfeh, 1993). Such an approach was used by Lesser and Lewis (1972a, 1972b) for a plane (two-dimensional) duct. In the present paper such an approach is employed for a real (axisymmetric) pipe geometry.

The paper is divided into nine sections. The acoustic problem is stated in section 2. Section 3 is concerned with scaling (nondimensionalization) in terms of parameters with appropriate length- and time-scales. A perturbation expansion of the governing equations, and the solution of these equations, is discussed in section 4. Asymptotic matching of these solutions is discussed (briefly) in section 5. The evaluation of the final time-domain expressions (in terms of inverse Fourier transform integrals) is discussed in section 6. The discrete vortex modeling of the flow field, and - in particular - the implementation of acoustic feedback, is discussed in section 7. Numerical results are presented and discussed in section 8. Conclusions are made in section 9. Finally, after section 9 there is a mathematical appendix to section 5 which gives a more detailed discussion of the asymptotic matchings.

2. The acoustic problem

2.1. Governing equations

The problem is, again, formulated in terms of axisymmetric cylindrical coordinates \((x, r)\), as shown in Fig. 2. Let \(\mathbf{i}\) and \(\mathbf{j}\) denote unit vectors in the directions of \(x\) and \(r\), respectively. The source field, i.e. the sound-generating field, is specified in terms of axisymmetric variables \((\hat{x}, \hat{r})\), where \(\hat{x}\) is the coordinate associated with the vortex motion in the plane \((x, \hat{x})\) and \(\hat{r}\) is the coordinate associated with steady flow in the plane \((\hat{r}, \hat{r})\). The coordinate \(\hat{x}\) is chosen as a multiple of the radius \(r\) of the pipe, and \(\hat{r}\) as a multiple of the radius \(L\) of the pipe. The variables \(\hat{x}\) and \(\hat{r}\) are related to the original variables \(x\) and \(r\) through the relations:

\[
\hat{x} = \frac{r}{L} \quad \text{and} \quad \hat{r} = \frac{r}{L}
\]

Fig. 2 (a) Sketch of the configuration of the acoustic problem, and indication of coordinates. (b) Blow-up of the step region around \(x = x_2\). The thickness of the inner (boundary layer) region around \(x = x_2\) is proportional to the small parameter \(\epsilon = r_0/L_0\) (see again part (a)).
incompressible jet flow (as sketched in Fig. 1), termed here the ‘background flow’, is governed by the Euler equation

\[ \frac{\partial v}{\partial t} + \frac{1}{\rho} \nabla p_h = -v \cdot \nabla v, \]  

(1)

Here \( \rho \) is the mean density of the fluid, \( t \) is the time, \( p_h \) is the ‘hydrodynamic’ (incompressible) pressure, and \( v = ui + vj \) is the flow velocity. The continuity equation is given by \( \nabla \cdot v = 0 \). The boundary condition \( n \cdot v = 0 \) is satisfied on the solid surfaces. As to the meaning of the right hand side of (1), it is useful to note the relation

\[ v \cdot \nabla v = \frac{1}{2} \nabla (v \cdot v) + \omega \times v, \]  

(2)

where \( \omega = \nabla \times v \). The discretization of (1), via a vortex method, is discussed in section 7.

The compressible acoustic field, on the other hand, which is assumed to be superimposed on the just described background flow, is governed by the linearized, inhomogeneous Euler equation (Hawe, 1998)

\[ \frac{\partial u}{\partial t} + \frac{1}{\rho} \nabla p = -\mathcal{G} H(x - x_1)H(x_2^2 - x), \]  

(3)

where \( u = (u, e) \) is the acoustic particle velocity, \( p \) is the acoustic pressure, and

\[ \mathcal{G} = \omega \times v \]  

(4)

is the vortex force, also known as the Lamb vector (Hawe, 1998, 2003). (The first term on the right hand side of (2) is assumed to be insignificant in relation to the generation of sound.) In (3), \( H(s) \) is the Heaviside step function, which equals 0 for \( s < 0 \) and 1 for \( s > 0 \). The significance of these step functions and the meaning of \( x_2^2 \) is discussed a bit later, below.

The continuity equation is

\[ \frac{\partial p}{\partial t} = -\nabla \cdot u = -\left[ \frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(ru) \right], \]  

where \( \kappa = \frac{1}{\rho c_0^2} \),

(5)

and \( c_0 \) is the speed of sound.

Equations (3) and (5) can be combined through elimination of \( u \) to give the non-homogeneous wave equation

\[ \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \rho \nabla \cdot \mathcal{G} H(x - x_1)H(x_2^2 - x). \]  

(6)

Equation (3) should be understood in the same way as the Powell-Howe equation (6) normally is understood (Hawe, 1998, 2003), that is, it is assumed that the observation point \( (x, r) \) is well away from the sound source domain, such that the fluid dynamical flow velocity \( v \approx 0 \) there (giving then also that \( \mathcal{G} \approx 0 \)). In the argument to the second \( H \) function, \( x_2^2 \) signifies a small downstream ‘extension’ of \( x_2 \) into the pipe. (Specifically the extension used is \( \frac{1}{2} \) of the tailpipe diameter.) The significance of the step functions \( H(x - x_1)H(x_2^2 - x) \) is that they include the free shear layer of the jet in the cavity domain and - in particular - its impingement onto the cavity wall at \( x = x_2 \), and makes this ‘mechanism’ the only source of sound. The small extension \( x_2^2 - x_2 \) is included to take the full effect of the turning of the vortex rings into the pipe into account.

The sound produced by the vortex structures that, after the impingement at the cavity wall, are convected downstream through the tailpipe is thus ignored. In this sense the present mathematical modeling approach is similar to the modeling of a recorder by Verge et al. (1997). In that paper the shear layer which impinges on the labium (the downstream sharp edge of the mouth) is represented in a very simple way by a single, semi-empirical function. This sound source is then coupled to a resonator pipe, which is analogous to the tailpipe in the present problem. In a full, large-scale numerical simulation of a recorder (e.g. Yokoyama et al., 2015) it can be seen that vortex structures generated at the mouth are convected also into the resonator pipe, but when comparing results from the two papers (Verge et al., 1997) and (Yokoyama et al., 2015) it is evident that these convected vortex structures have no significance on the main dynamics of the problem.

Technically, the influence (sound) of the vortex structures convected through the tailpipe could be included, yet it would complicate the asymptotic matching (to be described in Section 5). But a more serious point is that this will, in fact, make the whole mathematical model invalid. In order to use an acoustic analogy-type of approach (like the presently used theory of vortex sound) in connection with an acoustically non-compact body, as the tailpipe is, it is necessary that the sound-generating flow is in an isolated region and well away from the point of observation (Hawe, 2003, Ch. 2).

The sound-generating, impinging shear layer of the jet is representation by a ‘necklace’ of \( M \) discrete axisymmetric vortex rings, located at \( (x_m, r_m), m = 1, 2, \cdots, M \); see section 7. Let the vorticity \( \omega_m \) of the \( m \)th vortex ring be \( \omega_m = \frac{1}{2 \pi r_m} \frac{\partial}{\partial x_m} (\omega \times v) \approx \frac{1}{2 \pi r_m} \frac{\partial}{\partial x_m} (\omega \times v) \),
\[ k_\Gamma \delta(x - x_m(t))\delta(r - r_m(t))/\pi r, \] where \( k \) is a unit vector in azimuthal direction, \( \Gamma \) is the vortex strength, \( \delta(x - x_m) \) is the one-dimensional delta function (Jones, 1982, p. 55), and \( \delta(r - r_m)/\pi r \) the axisymmetric delta function (Jones, 1982, p. 306). Then, by use of (4), the Lamb vector \( \mathcal{G} = \psi_i + \psi_j \) is evaluated as

\[ \mathcal{G}(x, r, x_m(t), r_m(t), t) = \sum_{m=1}^{M} \Gamma_m(t) \delta(x - x_m(t)) \frac{\delta(r - r_m(t))}{\pi r} \left[ -iv_0(t) + jw(t) \right]. \]  

(7)

### 2.2. Fourier transform

Solution of the equations, and asymptotic matching of these solutions, is easier to carry out in the frequency domain than in the time domain. We thus employ Fourier transform, defined by

\[ P(\omega) = \int_{-\infty}^{\infty} p(t) e^{i\omega t} dt, \quad p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\omega) e^{-i\omega t} d\omega, \]  

(8)

to obtain the frequency domain Euler equation

\[ i\omega P = -\nabla \cdot \mathbf{U} \]  

where \( \mathbf{U}(\omega) \) is the Fourier transform of \( \mathbf{u}(t) \) and \( L(\omega) \) the Fourier transform of \( \mathbf{G}(t) \).

The continuity equation takes the form

\[ i\omega kP = \nabla \cdot \mathbf{U}, \]  

(10)

where \( i = \sqrt{-1} \). The frequency domain version of (6) takes the form

\[ \nabla^2 P + k^2 P = -\rho \nabla \cdot LH(x, x_m^2 - x), \]  

(11)

where \( k = \omega/c_0 \) is the acoustic wave number.

These expressions are very convenient for the analysis to follow. But care must be taken, particularly in the final inversion back to the time domain, since the sound sources (the vortex rings represented by (7)) are moving and, if a viscosity model is taken into consideration, have varying strengths as well.

### 3. Scaling

#### 3.1. Non-dimensional variables

The governing equations (9), (10), and (11) are made nondimensional by the use of appropriate length scales for each of the three types of domain: (i) pipe domain, (ii) step domain, and (iii) cavity domain. The pipe domain and the cavity domain both play the role of an ‘outer problem’, while the step domain is an ‘inner problem’. In the following, let \( \epsilon = r_0/L_0 \) be a small parameter and let \( \mathbf{U} = (U, V), \mathbf{V} = (U, \mathbf{G}) \).

In the pipe domain the pipe length \( L_0 \) is an appropriate length scale in the \( x \) direction, since the wave lengths for the lowest modes of acoustic resonance in the pipe is of this order of magnitude. The pipe radius \( r_0 \) is the appropriate length scale in the \( r \) direction. Suitable non-dimensional variables are thus

\[ \hat{x} = \frac{x}{L_0}, \quad \hat{r} = \frac{r}{r_0}, \quad \hat{u} = \frac{U}{\omega L_0}, \quad \hat{v} = \frac{V}{\omega L_0}, \quad \hat{p} = \frac{P}{\rho c_0^2 \omega L_0}, \quad \hat{k} = kL_0 = \frac{\omega}{c_0}. \]  

(12)

In the step domain the geometry is rapidly varying; thus both \( x \) and \( r \) are made nondimensional with \( r_0 \). On the other hand, the acoustic velocity components are only varying slowly across the step, and \( \omega L_0 \) is an appropriate velocity scale in both \( x \) and \( r \) directions. Suitable nondimensional variables here are thus

\[ x' = \frac{x}{r_0}, \quad \hat{r} = \frac{r}{r_0}, \quad u' = \frac{U}{\omega L_0}, \quad v' = \frac{V}{\omega L_0}, \quad \hat{\rho} = \frac{\rho}{\rho c_0^2 \omega L_0}, \quad k' = kL_0. \]  

(13)

For cavity domain the long length scale \( L_0 \) is applied for both the \( x \) and the \( r \) direction. The nondimensional variables here are thus

\[ \hat{x} = \frac{x}{L_0}, \quad \hat{r} = \frac{r}{L_0} = \hat{r}, \quad \hat{u} = \frac{U}{\omega L_0}, \quad \hat{v} = \frac{V}{\omega L_0}, \quad \hat{\rho} = \frac{\rho}{\rho c_0^2 \omega L_0}, \quad \hat{k} = kL_0, \]  

(14)

\[ \hat{L} = \frac{L}{c_0}, \quad \hat{\Gamma}_m = \frac{\Gamma_m}{c_0 L_0}, \quad \hat{\alpha} = \frac{U}{\omega L_0}, \quad \hat{\beta} = \frac{\mathbf{G}}{\omega L_0}, \]  

(15)
3.2. Scaled governing equations

Using the nondimensional parameters defined just above, we obtain the following scaled, nondimensional equations:

- for the pipe domain,
  \[
  i\tilde{k}\tilde{u} = \frac{\partial \tilde{p}}{\partial \tilde{x}}, \quad \epsilon^2 i\tilde{k}\tilde{v} = \frac{\partial \tilde{p}}{\partial \tilde{r}}, \quad i\tilde{k}\tilde{\tilde{p}} = \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\tilde{v}}{\tilde{r}} + \frac{\partial \tilde{\tilde{p}}}{\partial \tilde{r}}; \tag{15}
  \]

- for the step domain,
  \[
  \epsilon i k' u' = \frac{\partial p'}{\partial x'}, \quad \epsilon i k' v' = \frac{\partial p'}{\partial r'}, \quad \epsilon i k' p' = \frac{\partial u'}{\partial x'} + \frac{v'}{r'} + \frac{\partial v'}{\partial r'}; \tag{16}
  \]

and for the cavity domain,

- for the step domain,
  \[
  i\tilde{k}\tilde{u} = \frac{\partial \tilde{p}}{\partial \tilde{x}}, \quad \tilde{k}\tilde{v} = \frac{\partial \tilde{p}}{\partial \tilde{r}} + \tilde{L}_s H(\hat{x} - \hat{x}_1) H(\hat{x}_s^2 - \hat{x}) \tilde{k}, \quad i\tilde{k}\tilde{\tilde{p}} = \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\tilde{v}}{\tilde{r}} + \frac{\partial \tilde{\tilde{p}}}{\partial \tilde{r}}; \tag{17}
  \]

4. Perturbation expansion, simplified equations, and solutions

Next the dependent variables \(p, u, v\) (with a tilde, a hat, or an asterisk) are expanded in the form of asymptotic sequences as follows:

\[
\begin{align*}
  p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots, \\
  u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \\
  v &= v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots.
\end{align*} \tag{19}
\]

4.1. Pipe section

To the lowest (\(\epsilon^0\)) order, the governing equations in the pipe section are

\[
\begin{align*}
  i\tilde{k}\tilde{u}_0 &= \frac{\partial \tilde{p}_0}{\partial \tilde{x}}, \\
  \frac{\partial \tilde{p}_0}{\partial \tilde{r}} &= 0, \\
  i\tilde{k}\tilde{\tilde{p}}_0 &= \frac{\partial \tilde{u}_0}{\partial \tilde{x}} + \frac{\tilde{v}_0}{\tilde{r}} + \frac{\partial \tilde{\tilde{p}}_0}{\partial \tilde{r}}. \tag{20}
\end{align*}
\]

The second equation of (20) gives that \(\tilde{p}_0\) is a function of \(x\) only, i.e. \(\tilde{p}_0(x)\). Using this, one finds that the acoustic particle velocity components are governed by the equations

\[
\begin{align*}
  \tilde{u}_0 &= \frac{1}{i\tilde{k}} \frac{\partial \tilde{p}_0}{\partial \tilde{x}}, \\
  \tilde{v}_0 &= -\frac{\tilde{r}}{i2\tilde{k}} \left[ \frac{\partial^2 \tilde{p}_0}{\partial \tilde{x}^2} + \tilde{k}^2 \tilde{p}_0 \right]. \tag{21}
\end{align*}
\]

Using the second of these equations on the pipe wall \(\tilde{r} = \tilde{r}_0\) (where \(\tilde{v}_0 = 0\)), one finds that the pressure \(\tilde{p}_0\) is governed by the one-dimensional wave equation contained within the square brackets \([\ ]\) in (21). The solution to this equation that satisfies the boundary condition \(\tilde{p}_0 = 0\) at \(\tilde{x} = \tilde{x}_3\) is given by

\[
\tilde{p}_0 = \tilde{p}_0(\tilde{x}, \tilde{k}) = A_0 \left[ \cos \tilde{k} \tilde{x} - \cot \tilde{k} \tilde{x}_3 \sin \tilde{k} \tilde{x}_0 \right]. \tag{22}
\]

4.2. Step sections

The step sections are, to the lowest order, governed by

\[
\begin{align*}
  \frac{\partial p'_0}{\partial x'} &= 0, \\
  \frac{\partial p'_0}{\partial r'} &= 0, \\
  ik' u'_0 &= \frac{\partial p'_0}{\partial x'}, \\
  ik' v'_0 &= \frac{\partial p'_0}{\partial r'}.
\end{align*} \tag{23}
\]

The first two equations give that

\[
p'_0 = C'_0 = \text{constant}. \tag{24}
\]

As the next two equations show, the lowest order velocity components are governed by the next-order (\(\epsilon'\)) pressure term, \(p'_1\). This term itself is governed by the Laplace equation

\[
\frac{\partial^2 p'_1}{\partial x'^2} + \frac{1}{r'} \frac{\partial p'_1}{\partial r'} + \frac{\partial^2 p'_1}{\partial r'^2} = 0. \tag{25}
\]
A particular solution can be expressed as

\[ p_1^i(x^∗, r^∗) = \begin{cases} 
\sum a_n^i e^{-\xi_n x^∗} J_0 (\xi_n r^∗), & x^∗ < x^*_1, \\
\sum a_n^i e^{\xi_n x^∗} J_0 (\xi_n r^∗), & x^∗ > x^*_2, 
\end{cases} \]

(26)

where \( J_0 \) is the Bessel function of first kind and order zero, and \( \xi_n \) are the zeros of \( J_1 \), the Bessel function of order one. These zeros are ordered such that \( \xi_0 = 0 < \xi_1 < \xi_2 < \ldots \). Finally \( \xi_1 = r_0/r_1 \) (please refer again to Fig. 2).

On the step at \( x^* = x^*_1 \), the following boundary conditions must be satisfied:

\[ \begin{align*}
\frac{\partial p_1^i}{\partial x}(x^*_1, r^*) &= 0, \\
\frac{\partial p_1^i}{\partial x^*}(x^*_2, r^*) &= 0, \\
\frac{\partial p_1^i}{\partial x^*}(x^*_1, r^*) &= 0, \\
1 < r^* &< 1/\xi_1,
\end{align*} \]

where the superscripts – and + refer to just upstream (left) and just downstream (right) of the step. As it is not possible to impose these ‘strong conditions’ on a solution of the form (26), we will instead employ the following equivalent weak conditions,

\[ \int_0^{r^*_1} \frac{\partial p_1^i}{\partial x^*}(x^*_2, r^*) r^* dr^* = \int_0^{r^*_1} \frac{\partial p_1^i}{\partial x^*}(x^*_1, r^*) r^* dr^*, \quad \int_0^{1/\xi_1} \frac{\partial p_1^i}{\partial x^*}(x^*_2, r^*) r^* dr^* = \int_0^{1/\xi_1} \frac{\partial p_1^i}{\partial x^*}(x^*_1, r^*) r^* dr^*. \]

(27)

It is noted that the last two conditions can be combined into one, on the form \( \int_0^{1/\xi_1} \frac{\partial p_1^i}{\partial x^*}(x^*_2, r^*) r^* dr^* = \int_0^{1/\xi_1} \frac{\partial p_1^i}{\partial x^*}(x^*_1, r^*) r^* dr^*. \)

4.3. Expansion chamber section

For the expansion chamber, it is convenient to state the lowest \((e^0)\) order governing equation on the wave equation form

\[ \frac{\partial^2 \rho_0}{\partial r^2} + \frac{1}{r} \frac{\partial \rho_0}{\partial r} + \frac{\partial^2 \rho_0}{\partial \tau^2} + \hat{k}^2 \rho_0 = -\hat{\nabla} : \hat{LH}(\hat{x} - \hat{x}_1)H(\hat{x}_2^* - \hat{x}). \]

(29)

A particular solution can be expressed as

\[ \rho_0^{\text{part}}(\hat{x}, \hat{r}, \hat{\tau}) = 2\pi \int_{\hat{x}_1}^{\hat{x}_2} \int_{\hat{r}_1}^{\hat{r}_2} \hat{L}(\hat{\tau}) (H(\hat{x} - \hat{x}_1)H(\hat{x}_2^* - \hat{x}) \hat{r} \hat{r}_d \hat{r}_d \) \]

(30)

where the superscript part refers to particular, and

\[ G(\hat{x}, \hat{r}, \hat{\tau}, \hat{\xi}_n, \hat{\xi}_n^*) = \frac{i}{2\pi \delta^2} \sum_{n=0}^{\infty} \frac{1}{\hat{k}_n J_0^2(\xi_n \delta)} J_0 \left( \frac{\xi_n \hat{r}}{\delta} \right) J_0 \left( \frac{\xi_n \hat{r}_d}{\delta} \right) e^{i\hat{r}_d |\hat{k}_n|} \]

is the Green’s function which satisfies the boundary condition \( \partial G / \partial \hat{r} = 0 \) at \( \hat{r} = \hat{r}_1 \). Here \( \delta = r_1/L_0 \).

\[ \hat{k}_n = \begin{cases} 
\left[ \hat{k}^2 - (\xi_n / \delta)^2 \right]^{1/2} & \text{for } \hat{k} > \xi_n / \delta \\
i \left[ \xi_n / \delta^2 - \hat{k}^2 \right]^{1/2} & \text{for } |\hat{k}| < \xi_n / \delta \\
-\left[ \hat{k}^2 - (\xi_n / \delta)^2 \right]^{1/2} & \text{for } \hat{k} < -\xi_n / \delta
\end{cases} \]

(32)

and \( \xi_n \) are again the the zeros of \( J_1 \). The integrals with subscripts \( \hat{x}_1 \) and \( \hat{r}_1 \) indicate integration over all source (i.e. all vortex ring) positions. Likewise the integral with subscript \( \tau \) indicates integration over all source times. The nondimensional time \( \tau \) (and likewise \( r \), to be used in section 6) is defined such that \( \omega \tau = \hat{\tau} \), that is, \( \tau = t c_0 / L_0 \).

Evaluation of the spatial integrals in (30) gives

\[ \rho_0^{\text{part}} = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \hat{K}_0(\hat{r}, \hat{r}_m(\tau)) e^{i\hat{k}_n|\hat{k} - \hat{k}_n(\tau)|} e^{i\hat{r}_d \hat{\tau}}, \]

(33)
where \( (\tilde{x}_m(\tau), \tilde{r}_m(\tau)) \) are, again, the positions of the free vortex rings present within the cavity, and

\[
\hat{f}_m(\tilde{r}, \tilde{r}_m(\tau)) = \frac{\hat{r}_m(\tau)}{\delta} \left( \sum_{n=0}^{\infty} \phi_m(\xi) J_0 \left( \frac{\tau}{\delta} \right) + i \hat{u}_m(\tau) \frac{\xi}{k_0 \tau} J_1 \left( \frac{\tau}{\delta} \right) \right). \tag{34}
\]

The dash on the summation symbol \( (\Sigma_{\text{m'}}) \) is used to indicate that the summation runs over the vortices in the domain \( \tilde{x}_1 < \tilde{x} < \tilde{x}_2 \) only.

We will also include an ‘eigensolution’, i.e. a solution to the homogeneous version of (29), which likewise satisfies the boundary condition \( \partial G / \partial \tau = 0 \) at \( \tilde{r} = \tilde{r}_1 \). Such a solution can be written as

\[
\hat{p}_0^\text{hom} = \sum_{n=0}^{\infty} C_n J_0 \left( \frac{\xi}{\delta} \right) e^{-i \kappa \tilde{x}}, \tag{35}
\]

where the superscript hom refers to homogeneous. The complete solution is thus \( \hat{p}_0 = \hat{p}_0^\text{part} + \hat{p}_0^\text{hom} \). Evaluation of the acoustic particle velocity components (the acoustic feedback velocity) will be based on the ‘homogeneous solution’, or eigensolution, (35) only; that is, \( \tilde{i} \hat{u}_0 = \partial \hat{p}_0^\text{hom} / \partial \tilde{x} \) and \( \tilde{i} \hat{\kappa} \hat{u}_0 = \partial \hat{p}_0^\text{hom} / \partial \tilde{r} \). This is because the acoustic feedback comes from the tailpipe and it is natural to assume that this feedback field ‘drives’ the eigensolution. Mathematically this provides a convenient matching condition. (This is discussed in the Appendix to Section 5). In this way the non-physical ‘self-noise’ of the vortices is excluded as well. (By ‘non-physical’ is meant that, in the theory of vortex sound (and in Lighthill’s acoustic analogy) such ‘self-noise’ is meaningless.)

It is remarked, finally, that the solution form \( e^{-i \kappa \tilde{x}} \) is chosen in (35), and not \( e^{i \kappa \tilde{x}} \). This is because we consider the acoustic feedback velocity as an incoming wave.

5. Asymptotic matching of solutions

We first match the solutions (22) and (24), for the pipe and step regions, respectively. Here \( \hat{p}_0(\tilde{x}) \) (for the pipe) is considered as the outer expansion and \( p_0^\text{part}(x^*) \) (for the step) as the inner expansion. The outer variable is \( \tilde{x} = x/L_0 \), while the inner variable is \( x^* = (\tilde{x} - \tilde{x}_2)/\varepsilon \), which gives that \( \tilde{x} = \varepsilon x^* + \tilde{x}_2 \). The matching principle applied here is

\[
\text{Inner expansion of (outer expansion)} = \text{Outer expansion of (inner expansion)}, \tag{36}
\]

(Nayfeh, 1993, p. 266) which for the present pipe-step matching problem takes the form

\[
\lim_{\varepsilon \to 0} p_0^\text{part}(\varepsilon x^* + \tilde{x}_2) = \lim_{\varepsilon \to 0} \hat{p}_0(\varepsilon x^* + \tilde{x}_2). \tag{37}
\]

In this way we obtain the pressure within the pipe on the form

\[
\hat{p}_0(\tilde{x}, \tilde{k}) = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{1}{\delta} \int_0^\delta \hat{f}_m(\tilde{r}, \tilde{r}_m(\tau)) e^{i \kappa \tilde{x}_2 - \tau \kappa \tilde{r}} e^{i \kappa \tilde{x} \tilde{k} \tau} d\tau d\tilde{k} \sin \tilde{k} (\tilde{x}_3 - \tilde{x}) \sin \tilde{k} (\tilde{x}_3 - \tilde{x}_2). \tag{38}
\]

In a similar fashion we obtain the axial feedback velocity component within the cavity on the form

\[
\tilde{u}_0(\tilde{x}, \tilde{k}) = -ie \tilde{r} e^{-i \kappa |\tilde{x}_2 - \tilde{x}|} \cot \tilde{k} (\tilde{x}_3 - \tilde{x}_2) \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{1}{\delta} \int_0^\delta \hat{f}_m(\tilde{r}, \tilde{r}_m(\tau)) e^{i \kappa \tilde{x}_2 - \tau \kappa \tilde{r}} e^{i \kappa \tilde{x} \tilde{k} \tau} d\tau d\tilde{k} \sin \tilde{k} (\tilde{x}_3 - \tilde{x}) \sin \tilde{k} (\tilde{x}_3 - \tilde{x}_2). \tag{39}
\]

It is noticed that, just as by the pressure within the pipe (38), the amplitude of the feedback velocity field in the expansion chamber (39) will go to infinity at the pipe resonance frequencies \( k \tilde{r} = j \pi, \ j = 1, 2, \cdots \), \( \tilde{r} = \tilde{x}_3 - \tilde{x}_2 \).

Further mathematical details regarding these results and the matching are given in the Appendix.

6. Time domain expressions

6.1. Pressure

In the (non-dimensional) time domain, the pressure within the tailpipe, (38), is

\[
\hat{p}_0(\tilde{x}, \tilde{t}) = \Re \left\{ \hat{f}_m(\tilde{r}, \tilde{r}_m(\tau)) J_0 \left( \frac{\tau}{\delta} \right) + \frac{\xi}{k_0 \tau} J_1 \left( \frac{\tau}{\delta} \right) e^{-\tilde{r} e^{-i \kappa \tilde{x} \tilde{k}}} \right\}. \tag{40}
\]

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where $H_1$ is the Struve function of order unity (Abramowitz and Stegun, 1964). This function appears as a result of analytical evaluation of the integral over $\hat{t}$ in (38). Again, the nondimensional times $t'$ and $\tau$ are both scaled as $tc_0/L_0$. $\Re e$ indicates the real part of the complex expression. It is noted also that $\hat{k} = \hat{k}$; the different markers refer just to the different domains. Finally, it should be noticed that a damping term, on the form $\exp[-\mu(t' - \tau)]$, has been included in (40). It is seen that this term will "punish" the "old" contributions (where $0 < \tau < t'$) significantly, and the "young" contributions (where $\tau \ll t'$) only lightly. But the form of the solution (40) with this damping term included does, in fact, also correspond to the correct solution for the one-dimensional wave equation operator with a viscous friction term, on the form $L(p) = \partial^2 p/\partial t^2 + 2\mu \partial p/\partial t - c_0^2 \partial^2 p/\partial x^2.$ (Morse and Feshbach, 2004, p. 1335). However, in a physically correct model of viscosity, the term $2\mu \partial p/\partial t$ will be replaced by $\frac{1}{\nu} \partial^2 p/\partial x^2 \partial t$, where $\nu$ is the kinematic viscosity (Lamb, 2004, p. 188).

The integration over $\hat{k}$ in (40) is evaluated analytically by making use of contour integration. The integration contour runs from negative to positive values of $\hat{k}$ along the real axis, and clockwise along a semicircle in the lower half-space. The integrand in (40) has two types of singularities. One type is the simple poles at $\hat{k} = j\pi/\hat{t}$, $j = \pm 1, \pm 2, \cdots$, the zeros of sin $\hat{k}\hat{t}$, which, with the presence of small viscous damping, are all lying inside of the integration contour. Another type, occurring in the second bracket in the curly brackets $\{\}$, is the zeros of the square root function (32), $\hat{k} = \pm \zeta_n/\hat{\delta}$, $n = 1, 2, \cdots$, which are branch points. These points are connected to the point $\hat{k} = \infty$ via branch cuts, running upwards from the point $\hat{k} = -\zeta_n/\hat{\delta}$ on negative real axis (through second quadrant) and downwards from the points $\hat{k} = \zeta_n/\hat{\delta}$ on the positive real axis (through fourth quadrant), such that they are outside of the integration contour.

The method of residues gives the contributions, $\tilde{p}_{0n}$ say, from the simple poles as

$$\tilde{p}_{0n}(\hat{x}, t') = -\Re e \left[ \frac{\hat{t}}{\hat{\ell}} \sum_{j=0}^{\infty} \sum_{m=0}^{M} \sum_{n=0}^{1} - \frac{2\hat{\delta}}{J_0(\zeta_n)} \int_0^{\tau} i e^{j\kappa_{\hat{n}}(\hat{\ell}-\kappa_0(\zeta_n))} \right]$$

$$\times \left\{ \hat{f}_{nm}(\tau)\tilde{\bar{u}}_{m}(\tau)J_0(\zeta_n) \delta_{m} + \frac{\zeta_n}{\hat{\delta}} \hat{f}_{nm}(\tau)\tilde{\bar{u}}_{m}(\tau)J_1(\zeta_n) \right\} e^{-j\kappa(\hat{\ell}-\kappa_0(\zeta_n))} dt,$$

where $\hat{\ell} = \hat{k} - \hat{x}$ is the length of the tailpipe, and

$$\kappa_{\hat{n}} = \begin{cases} \left[ (j\pi/\hat{t})^2 - (\zeta_n/\hat{\delta})^2 \right]^{1/2} & \text{for } j\pi/\hat{t} > \zeta_n/\hat{\delta} > 0, \\ i \left[ (\zeta_n/\hat{\delta})^2 - (j\pi/\hat{t})^2 \right]^{1/2} & \text{for } j\pi/\hat{t} < \zeta_n/\hat{\delta}, \\ - i \left[ (j\pi/\hat{t})^2 - (\zeta_n/\hat{\delta})^2 \right]^{1/2} & \text{for } j\pi/\hat{t} < -\zeta_n/\hat{\delta} > 0. \end{cases}$$

It is noted that the minus on the right hand side of (41) appears because the integration round the contour is in clockwise direction. It should be noted also that the contributions for $j = 0$ (corresponding to the frequency zero) is to be excluded from (41).

The contribution, $\tilde{p}_{0n}$ say, of (40) due to the singularities (branch points) at $\hat{k} = \pm \zeta_n/\hat{\delta}$ are not easily evaluated exactly. Asymptotic approximations based on Watson’s lemma, valid for large values of $t'$, are derived in Murray (1984), Ch. 5. The form of Murray’s result is such that $\tilde{p}_{0n}(\hat{x}, t') \to 0$ for $t' \to \infty$; thus these branch points contribute significantly only to the transient (start-up) acoustic pressure signature, not to the acoustic pressure at large values of $t'$.

### 6.2. Feedback velocity

The final expression for the time domain version of (39), the axial feedback velocity component within the cavity, is

$$\tilde{u}_0(\hat{x}, t') = -\Re e \left[ \frac{1}{\hat{\ell}} \sum_{j=0}^{\infty} \sum_{m=0}^{M} \sum_{n=0}^{1} - \frac{2\hat{\delta}}{J_0(\zeta_n)} \cot \hat{k}\hat{\ell} e^{-j\kappa(\hat{x}-\zeta_n(\tau))} \int_0^{\tau} e^{j\kappa(\hat{\ell}-\kappa_0(\zeta_n))} dt \kappa d\hat{k} \right]$$

$$\times \left\{ \hat{f}_{nm}(\tau)\tilde{\bar{u}}_{m}(\tau)J_0(\zeta_n) \delta_{m} + \frac{\zeta_n}{\hat{\delta}} \hat{f}_{nm}(\tau)\tilde{\bar{u}}_{m}(\tau)J_1(\zeta_n) \right\} e^{-j\kappa(\hat{\ell}-\kappa_0(\zeta_n))} dt d\hat{k},$$

Just as by (40) the integrand in (43) has simple poles at $\hat{k} = j\pi/\hat{t}$, $j = \pm 1, \pm 2, \cdots$, and branch points at $\hat{k} = \pm \zeta_n/\hat{\delta}$, $n = 1, 2, \cdots$.

The contribution $\tilde{u}_{0n}$ to this integral due to the singularities (poles) at $\hat{k}_j = j\pi/\hat{t}$, $j = \pm 1, \pm 2, \cdots$, are given by

$$\tilde{u}_{0n}(\hat{x}, t') = -\Re e \left[ \frac{1}{\hat{\ell}} \sum_{j=0}^{\infty} \sum_{m=0}^{M} \sum_{n=0}^{1} - \frac{2\hat{\delta}}{J_0(\zeta_n)} e^{-j\kappa(\hat{x}-\zeta_n(\tau))} \int_0^{\tau} e^{j\kappa(\hat{\ell}-\kappa_0(\zeta_n))} dt \kappa d\hat{k} \right]$$

$$\times \left\{ \hat{f}_{nm}(\tau)\tilde{\bar{u}}_{m}(\tau)J_0(\zeta_n) \delta_{m} + \frac{\zeta_n}{\hat{\delta}} \hat{f}_{nm}(\tau)\tilde{\bar{u}}_{m}(\tau)J_1(\zeta_n) \right\} e^{-j\kappa(\hat{\ell}-\kappa_0(\zeta_n))} dt.$$
Also here, the contribution for \( j = 0 \) in (44) is to be excluded. Regarding the contribution \( \tilde{u}_{0b} \) to (43) due to the branch points at \( \tilde{k} = \pm \zeta_{a}/\tilde{h} \), just as by the pressure, this contribution \( \tilde{u}_{0b}(\tilde{k}, r') \rightarrow 0 \) for \( t \to \infty \). Thus \( \tilde{u}_{0b} \) given by (44) is the only significant contribution.

It may be useful, finally, to sum up the physical significance of (44). The impinging vortex structures in the expansion chamber generates an acoustic field which is represented in (44) by the summation over \( m \) and \( n \), where \( m \) refers to individual (discrete) vortex rings and \( n \) (in particular \( n > 0 \)) to the higher-order acoustic modes in the expansion chamber (which are likely not very significant). This acoustic field drives the acoustic oscillations in the tailpipe, which is represented by the summation over \( j \) and which refers to an infinite number of acoustic eigenmodes. The corresponding velocity field, coming back from the tailpipe into the expansion chamber, is now imposed on the (free) vortex rings present in the expansion chamber, to give a slight modification of (or perturbation to) the incompressible velocity field dictated by the vortex dynamics.

7. Implementation of acoustic feedback into a discrete vortex model of the flow field

An axisymmetric jet of radius \( r = r_0 \) is discharging from the nozzle at \( x = x_1 \) in the axisymmetric cylindrical coordinate system \((x, r)\), see Fig. 2 (section 2). The shear layer of the jet is impinging upon the edge of the circular hole, also of radius \( r_0 \), in the parallel end plate placed at \( x = x_2 \). In this section we discuss a discrete vortex discretization of this sound-generating jet flow and, in particular, how the discrete vortex formulation is modified by an imposed acoustic feedback velocity field. It is emphasized that, contrary to the previous sections, the following discussion is given in terms of dimensional variables.

7.1. Axisymmetric inviscid flow with acoustic feedback

Adding the acoustic velocity field \( \mathbf{u} \) to (1) gives, after some reductions,

\[
\frac{\partial}{\partial t}(\mathbf{v} + \mathbf{u}) + \frac{1}{r^2} \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) - \mathbf{v} \cdot \mathbf{\omega} + \nabla \cdot (\mathbf{v} \cdot \mathbf{u}) - \mathbf{u} \times \mathbf{\omega} = -\frac{1}{\rho} \nabla p_b. \tag{45}
\]

Let \( \mathbf{w} = \mathbf{v} + \mathbf{u} = (\tilde{u}, \tilde{v}, \tilde{w}) = \tilde{u} \hat{i} + \tilde{v} \hat{j} + \tilde{w} \hat{k} \) and let \( D/Dt = \partial/\partial t + \mathbf{w} \cdot \nabla \) denote the convective time derivative. Then, taking the curl on both sides of (45), this equation can be written as

\[
\frac{D\mathbf{\omega}}{Dt} = \mathbf{\omega} \times \nabla \mathbf{w} + \kappa \frac{\partial p}{\partial t}, \tag{46}
\]

where the last term is obtained by making use of the continuity equation (5). In the present case of a three-dimensional yet axisymmetric flow (where \( \mathbf{w} = \tilde{u} \hat{i} + \tilde{v} \hat{j} \), see also section 2.1), the only non-zero component of the vorticity vector \( \mathbf{\omega} = (\omega_r, \omega_\theta, \omega_z) \) is the azimuthal component \( \omega_\theta \), that is, \( \mathbf{\omega} = \omega \hat{k} \). In the following we will write just \( \omega \) for \( \omega_\theta \), for simplicity. Taking the scalar product between (46) and the gradient \( \nabla \theta = k \rho^{-1} \) gives, again after some reductions,

\[
\frac{D}{Dt} \left( \frac{\omega}{r} \right) = \omega r \frac{\partial p}{\partial t}, \quad \text{or written out,} \quad \frac{\partial}{\partial t} \left( \frac{\omega}{r} \right) + \tilde{u} \frac{\partial}{\partial x} \left( \frac{\omega}{r} \right) + \tilde{v} \frac{\partial}{\partial y} \left( \frac{\omega}{r} \right) = \kappa \omega \frac{\partial p}{\partial t}. \tag{47}
\]

These equations show that, when acoustic feedback is excluded (\( \kappa = 0 \)) the quantity \( \omega/r \) is constant and moves with the flow. On the other hand, when acoustic feedback is included, this quantity is modified in a way reminiscent of the action of viscosity. Thus, inspired by the viscous splitting algorithm used in viscous vortex methods (Cottet and Koumoutsakos, 2000, Ch. 5), where the Navier-Stokes equation is split up into a convection part and a diffusion part, (47) is now rewritten in the two-step splitting algorithm form

\[
(\text{I}): \quad \frac{D}{Dt} \left( \frac{\omega}{r} \right) = 0, \quad (\text{II}): \quad \frac{\partial}{\partial t} \left( \frac{\omega}{r} \right) = \omega \frac{\kappa}{r} \frac{\partial p}{\partial t}, \tag{48}
\]

where the equation (I) is a convection part, and the equation (II) an ‘infusion’ part (as opposed to diffusion). According to the first equation of (48) the quantity \( \omega/r \) still moves with the flow; but according to the second equation, this quantity is now no longer constant, but changes proportionally to the present value \( \omega/r \) and the gradient of the acoustic pressure, \( \partial p/\partial t \).

In the case where equation (I) in (48) is valid, there exists a stream function \( \Psi \) such that the induced velocity \( \mathbf{w} = \tilde{u} \hat{i} + \tilde{v} \hat{j} \) can be obtained from

\[
\tilde{u} = \frac{1}{r} \frac{\partial \Psi}{\partial r}, \quad \tilde{v} = -\frac{1}{r} \frac{\partial \Psi}{\partial x}. \tag{49}
\]
This stream function $\Psi$ satisfies the differential equation

$$L\Psi = \frac{\omega}{r} \quad \text{where} \quad L = -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial x^2}.$$  \hspace{1cm}(50)

7.2. Vortex ring representation of the shear layer

The shear layer of the jet issued from the nozzle has, up to this point, been thought of as a continuous, cylindrical vortex sheet. That is to say, it has been assumed that the vorticity $\omega$ (which, again, is a short-hand form for $\omega_0$) is concentrated in a thin axially symmetric discontinuity-surface. This cylindrical vortex sheet is now discretized into a ‘necklace’ of discrete vortex rings, with the vorticity distribution $\omega$ discretized as

$$\omega(x, r, t) = \sum_m \Gamma_m(t) \delta(x - x_m(t)) \frac{\delta(r - r_m(t))}{\pi r},$$  \hspace{1cm}(51)

where $\Gamma_m(t)$ is the strength of the single vortex ring located at $(x_m, r_m)$ and, as in (7), $\delta(x - x_m)$ is the one-dimensional delta function while $\delta(r - r_m)/\pi r$ is the axisymmetric delta function.

The temporal development of $\Gamma_m(t)$ is governed by the solution of the second equation, (II), of (48), which is given by $\omega_f = \exp(\kappa p(t) + K)$, where $K$ is a constant. Based on this solution, $\Gamma_m(t)$ will take the form $\Gamma_m(t) = r_m(t) \exp(\kappa p(t) + K)$, subject to the boundary conditions $\Gamma(r_{rel}^m) = \Gamma_{bm}$ and $r_m(r_{rel}^m) = r_0$. That is to say, vortex ring no. $m$ is released from the nozzle at time $r_{rel}^m$ with strength $\Gamma_{bm}$ and radius $r_0$. After release it will, at time $t$, have strength $\Gamma_m(t)$ and radius $r_m(t)$. These conditions give

$$\Gamma_m(t) = \Gamma_{bm} \frac{r_m(t)}{r_0} \exp[\kappa (p(t) - p(r_{rel}^m))],$$  \hspace{1cm}(52)

(The evaluation of $\Gamma_{bm}$ is discussed in section 7.3.) That is to say, the acoustic feedback (acoustic energy) implies a growth in the strengths of the vortex rings which, although exponential, is slow since $\kappa p$ is small; see (5) for the definition of $\kappa$. At the same time the vortex strengths are reduced (again very slightly) by viscosity. Here we apply Lamb’s model (Lamb 1993, p. 592), where the vortex strength is reduced by the factor

$$1 - \exp \left( \frac{\sigma^2 \text{Re}_m}{4 t_\eta \pi r_0} \right),$$  \hspace{1cm}(53)

where $\text{Re}_m = u_0 r_0 / v$ is a Reynolds number, $t_\eta = t - r_{rel}^m$ is the age of the vortex, and $a = r_m / r_0$. (The reader is referred to Langthjem and Nakano (2015) for a thorough discussion of this type of viscous dissipation model.)

Since (51) represents the vorticity distribution by singularities (a sum of delta functions), the solution of (50) (with (51) inserted) will have singularities at all $(x_m, r_m)$. Regularisation, or desingularization, is necessary in order to design a properly working numerical method. One possible desingularized solution to (50) is (Nitsche and Krasny, 1994)

$$\Psi(x, r, x_m, r_m) = \sum_m \Gamma_m(\eta_1 + \eta_2) \left[ K(\lambda) - E(\lambda) \right], \quad \text{where} \quad \lambda = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1},$$  \hspace{1cm}(54)

$$\eta_1 = \left( x - x_m \right)^2 + \left( r - r_m \right)^2 + \alpha \sigma_m^2 \right)^{1/2}, \quad \text{and} \quad \eta_2 = \left( x - x_0 \right)^2 + \left( r + r_m \right)^2 + \alpha \sigma_m^2 \right)^{1/2}.$$  \hspace{1cm}

Here $K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of first and second kind, respectively (Abramowitz and Stegun, 1964). The term $\alpha \sigma_m^2$ in the $\eta_1$ and $\eta_2$ functions is a smoothing, or regularization, function. Here $\sigma_m$ is again the vortex core radius and $\alpha$ is a parameter ($\alpha > 0$).

The volume of each vortex ring is kept constant, that is, $V_m = 2\pi r_m \cdot \pi \sigma_m^2 = V_0 = 2\pi r_0 \cdot \pi \sigma_0^2$ is constant. Thus the smoothing term $\alpha \sigma_m^2$ varies with time. This approach, which was introduced by Moore (1972), is different from the approach in Nietsche and Krasny (1994), where the smoothing term is taken to be constant. The approach with a constant smoothing term, introduced by Rosenhead (1930), was used also in earlier works by the present authors (Langthjem and Nakano, 2015). The approach with a time-varying smoothing term is ‘more physical’ since it is directly related to the vortex core size.

7.3. Modeling of vortex shedding

The rate of continuous shedding of circulation $\gamma$ (that is, the spatial integral over the vorticity $\omega$) from the nozzle is given by (Cottet and Kourtisakos, 2000, p. 109)

$$\frac{dy}{dt} = \frac{1}{2} \left( u_{i_{v}}^2 - u_{i_{v}}^2 \right).$$  \hspace{1cm}(55)
where $u_{tr}$ is the tangential velocity slightly downstream from the pipe exit, at station $x_e = x_1 + \mu$ ($\mu$ is a small positive number), and at radius $r_e = r_0 - \mu$, and $u_{ts}$ is the corresponding velocity at radius $r_e = r_0 + \mu$. This equation can be obtained by integrating the tangential component of the Euler equations over the tube surface, and using the Kutta condition (which demands that the pressure a little above the nozzle edge equals the pressure a little below, (Cottet and Koumoutsakos, 2000, p. 109)).

 Imagining again the shear layer as a continuous vortex sheet, the convection velocity a small distance downstream from the nozzle edge is given by $u_{tr} = \frac{1}{2}(u_{te} + u_{ts})$. During a small time increment $\delta t$ the vortex sheet is advanced a distance $u_{tr}\delta t$. After the discretization of the continuous vortex into a ‘necklace’ of discrete vortex rings, a ring is released at every time-step $\Delta t$ at the position $x_{release} = x_1 + u_{tr}\Delta t$. Its initial strength is obtained from (55) as

$$ \Gamma_{\text{free}} = \frac{1}{2}\Delta t \left( u_{te}^2 - u_{ts}^2 \right). $$

(56)

### 7.4. Modeling of the solid surface

The (single, continuous) solid surface which makes up the expansion chamber and the tailpipe is represented by bound vortex rings. The inviscid boundary condition of zero normal velocity is imposed at control points between these rings. The mean jet flow is also provided by a number of vortex rings placed, like a ‘lid’, on the upstream ‘back’ of a relatively short nozzle pipe, which proceeds the expansion chamber. The strengths of the bound vortex rings are dictated by the boundary condition and by specification of the mean jet velocity.

### 8. Numerical results

#### 8.1. Computational details

In the numerical examples to follow the length of the cavity (expansion chamber) $L_1 = 50$ mm. The diameter of the jet nozzle, as well as the end pipe, $d_0 = 50$ mm. The diameter of the cavity $d_1 = 100$ mm. The length $L_0$ of the tailpipe will vary between approximately $15d_0$ and $31d_0$.

The number of concentric vortex rings on the flow-providing ‘lid’ on the upstream end of the nozzle pipe (of length $2.5d_0$) is 20. The number of vortex rings representing the nozzle pipe surface is 40. The number of vortex rings representing the expansion chamber wall (at radius $r_1 = 2r_0 (= d_0)$) is 20. The upstream and downstream end plates that close the cavity section are represented by 30 and 60 vortex rings, respectively. Finally, the end pipe is represented by 400 vortex rings. It has been verified these discretizations are sufficiently fine, and that the system dynamics is insensitive to further increase of these parameters.

The time step is chosen as $\Delta t = 0.01d_0/u_0$, where the mean jet flow speed $u_0$ here will vary between 9 and 11 m/s. If the vortex roll-up is ignored for the moment, and it is assumed that the vortex convection velocity is $0.6u_0$, then the chosen $\Delta t$ will correspond to a discretization of the free shear layer in the cavity region into 167 equidistant vortex rings (by $u_0 = 10$ m/s). The initial vortex core radius (for any free vortex ring) is set as $r_m \equiv r_0 = r_0/5 = d_0/10$. Thus there is a significant overlap between the vortex rings that represent the shear layer. The smoothing parameter $\sigma$ in (54) is set to 0.35. (For a discussion of the significance of this parameter, see Langthjem and Nakano 2005.) In the viscous dissipation model (53), we set $\alpha^2Re_{inj} = \frac{1}{2}$, just as in Langthjem and Nakano (2015). With this value the dynamics of the vortices is not affected to any significance during the short time they are present in the cavity domain; they are only affected significantly during their long travel through the tailpipe. Thus the precise value of $\alpha^2Re_{inj}$ does not have any significant effect on the results.

In order to impose the acoustic feedback velocity field (44) onto the free vortex rings present within the cavity domain, this domain is divided lengthwise into ten sections. The acoustic velocity is computed in the center of each of these sections. (It is noted, from (44) that the acoustic velocity is not dependent on the radial variable $r$.) The vortex rings present within each of these ten sections is then subjected to the acoustic velocity computed for that particular section, in addition to the induced incompressible velocity field given by (49).

In a numerical evaluation of the acoustic pressure (41) and the acoustic feedback velocity (44), the summations over the axial modal functions ($j = 1, 2, \ldots$) and the radial modal functions ($n = 0, 1, 2, \ldots$) clearly have to be truncated. The numerical results to follow were obtained with 12 axial modal functions ($j_{\text{max}} = 24$) and 13 radial modal functions ($n_{\text{max}} = 12$). Results with higher values of these parameters have also been computed, but the resulting changes are insignificant. In particular, the higher radial modes have very little significance by the present shallow expansion chamber.

Finally the concept of cut-off frequency (below which no sound waves can propagate) should be mentioned. Only axisymmetric waves are considered here, and for these wave there are, in fact, no cut-off frequency (Morse and Ingard, 1986, p. 511).
8.2. Results

Figure 3 shows the appearance of the vortex rings present in the cavity and in the most upstream part of the tailpipe during one period of flow oscillation. Shown are side-views, represented by the vortex ring coordinates on the form \((x_m, \pm r_m), m = 1, 2, \ldots\), of all free and all fixed vortex rings which represent the shear layer and the boundaries. It is seen that the shear layer periodically rolls up into large-scale coherent vortex structures which impinge on the downstream end plate of the expansion chamber. These impacts drive the acoustic oscillations in the tailpipe, as specified by (41), and the acoustic feedback velocity field therefrom, as specified by (44). The length of the tailpipe is here chosen as \(L_0 = 15.11d_0\), which gives that the first acoustic resonance frequency (one half-wave) \(f_{\text{pipe}1} = f_0 = 225\) Hz. The acoustic feedback (44) is included in the simulation, which results in that the fundamental hole-tone frequency \(f_0\) takes the value 225 Hz as well. (This is discussed in more detail in the following.)

Figure 4 shows frequency spectra of the acoustic pressure fluctuations inside the tailpipe, at the position \(x = x_2 + 3L_0/8\) (refer to Fig. 2) for three different mean flow velocities. Shown is the sound pressure level (SPL) in decibels (dB),
defined as $20 \log_{10}(\mathfrak{B}[p/p_{\text{ref}}])$, where $\mathfrak{B}[]$ denotes (fast) Fourier transform and $p_{\text{ref}} = 20 \times 10^{-5} \text{N/m}^2$. These frequency spectra are based on simulations with 3500 time steps. Out of these, the last $2^{11} = 2048$ time steps are analyzed, while the first 1452 steps are considered as transients and discarded. The length of the tailpipe is here $L_0 = 15.5d_0$ (slightly longer than the one used in Fig. 3), giving the resonance frequencies $f_{\text{pipe } n} = 220n \text{ Hz, } n = 1, 2, \ldots$. Acoustic feedback is excluded in the graphs of the top rows, parts (a), (b), and (c), and included in those of the bottom rows, parts (d), (e), and (f).

Considering first part (a), where the mean jet flow speed $u_0 = 9 \text{ m/s}$, the dominating sound frequency is $f_0 \approx 195 \text{ Hz}$. In part (b), where $u_0$ has been increased to 10 m/s, this peak has moved to 215 Hz, and in part (c), where $u_0 = 11 \text{ m/s}$, to 235 Hz. Consider next the bottom row, parts (d), (e), and (f), where acoustic feedback ((44) and also (52)) is included, it can be seen that the sound frequency $f_0$ locks-in to the pipe frequency $f_{\text{pipe } 1} = 220 \text{ Hz}$ for $u_0 = 9 \text{ m/s}$. For $u_0 = 10 \text{ m/s}$ the peak value is already close to 220 Hz without feedback, so here the effect of the acoustic feedback is to bring about resonance between the self-sustained flow oscillations in the expansion chamber and the acoustic pressure oscillations in the tailpipe. This amplifies the pressure amplitude (in the tailpipe) significantly. (See, however, the discussion to Fig. 5 (below) regarding acoustic lock-in at $u_0 = 10 \text{ m/s}$.) For $u_0 = 11 \text{ m/s, } f_0$ stays at 235 Hz, but the pressure amplitude is again amplified significantly by the acoustic feedback.

![Fig. 4 Spectrum of the sound pressure fluctuations (in decibels) inside the tailpipe, at the position $x = x_2 + 3L_0/8$ (see Fig. 2(a)), for the pipe resonance frequency $f_{\text{pipe } 1} = 220 \text{ Hz.}$](image)

Figure 5 shows frequency spectra of the flow velocity fluctuations (VFL = velocity fluctuation level) in the shear layer just slightly upstream from position $x_2$ (see again Fig. 2), for the same three different mean flow velocities. Shown is $20 \log_{10}(\mathfrak{B}[(u-u_0)/u_0])$. The effect of the acoustic feedback is perhaps more clearly seen here. Again, acoustic feedback is excluded in the graphs of the top rows, parts (a), (b), and (c). The dominating velocity fluctuation frequencies, for the mean flow speeds $u_0 = 9, 10, \text{ and } 11 \text{ m/s, are } 195, 205, \text{ and } 160 \text{ Hz, respectively. For } u_0 = 11 \text{ m/s there is also a significant component at } 235 \text{ Hz. With inclusion of acoustic feedback (parts (d), (e), and (f)) the frequency peaks become locked-in to the tailpipe resonance frequency frequency } f_{\text{pipe } 1} = 220 \text{ Hz in the first two cases. For } u_0 = 11 \text{ m/s, the peak at } 235 \text{ Hz becomes the most significant.}
Some preliminary experiments have been carried out using a blow-down wind tunnel (for details, see Matsuura and Nakano (2012)) with a nozzle of diameter $d_0$, to which the cavity-tailpipe model shown in Fig. 2 was connected. The tailpipe is of length $L_0 = 31d_0$ (1550 mm), with corresponding resonance frequencies $f_{pipe n} = 110n$ Hz, $n = 1, 2, \ldots$

The sound pressure was measured with a microphone placed 20 mm downstream from the tailpipe exit at $x = x_3$ (refer to Fig. 2) and 100 mm away from the jet axis, that is, at $r = 100$ mm. Measured results are shown in Fig. 6, parts (a) - (c). At the mean jet speed $u_0 = 9$ m/s the measured dominating acoustic frequency $f_0 = 210$ Hz and at $u_0 = 10$ m/s it is $f_0 = 224$ Hz. Then at $u_0 = 11$ m/s, the dominating frequency jumps, however, up to $f_0 = 324$ Hz, which corresponds to a higher mode. (Modes and mode jumps are discussed in, e.g., Langthjem and Nakano (2005).) It is noted that at $u_0 = 10.5$ m/s, before the mode jump happens, the dominating frequency is $f_0 = 230$ Hz.

Computational results for the case of a tailpipe of length $L_0 = 31d_0$ and with acoustic feedback included are shown in Fig. 6, parts (d) - (f). It must be emphasized that the observation point is again taken inside the pipe, at $x = x_2 + 3L_0/8$. We cannot compare sound pressure levels since the present theory does not include the free field outside of the tailpipe where the measurements were done. (This will be taken up in a forthcoming paper.) Thus only the frequency peaks can be compared. The graphs show that, at $u_0 = 9$ m/s the dominating acoustic frequency $f_0 = 220$ Hz, at $u_0 = 10$ m/s it is $f_0 = 335$ Hz, and at $u_0 = 11$ m/s it is $f_0 = 235$ Hz. Thus here, a mode jump takes place between $u_0 = 9$ m/s and 10 m/s. Another one, which returns the dominating mode to its previous stage, takes place between $u_0 = 10$ m/s and 11 m/s. It is noted, however, that the level of low frequency ‘white noise’ is high in these cases, particularly for $u_0 = 10$ m/s.

Computational results with acoustic feedback excluded are shown in Fig. 6, parts (g) - (i). Here, the dominating frequency at $u_0 = 9$ m/s is $f_0 = 334$ Hz. At $u_0 = 10$ m/s it is $f_0 = 225$ Hz, but the component at 335 Hz is quite similar in magnitude. At $u_0 = 11$ m/s we obtain again $f_0 = 235$ Hz.
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Fig. 6 Experimentally and computationally obtained frequency spectra of sound pressure fluctuations (in decibels). The experimental results are recorded outside of and downstream from the tailpipe exit, while the computations are again for an observation point inside the pipe, at the position \( x = x_1 + 3L_0/8 \) (see Fig. 2(a)). The tailpipe resonance frequency \( f_{\text{pipe}1} = 110 \) Hz.

Results for velocity fluctuations in the shear layer are shown in Fig. 7. In the experiments, measurements were made with a hot-wire probe at a distance \( 0.4d_0 \) downstream from the upstream expansion chamber end plate at \( x = x_1 \), that is, at \( (x, r) = (x_1 + 0.4d_0, 0.5d_0) \). The computational results are for the position \( (x, r) = (x_1 + 0.5d_0, 0.5d_0) \); so there is a small difference. It is noted that what is shown is 20\( \log_{10}(\bar{u}/u_0) \), where \( u_0 = 5 \) m/s.

In the experiments, the dominating frequencies for \( u_0 = 9 \) m/s, 10 m/s, and 11 m/s are \( f_0 = 210 \) Hz, 224 Hz, and 324 Hz, respectively; that is, the same as for the sound pressure spectrum.

Agreement between peaks for pressure and velocity fluctuation spectra is also the case for the computed results in parts (d)-(f) (acoustic feedback included) and (g)-(i) (acoustic feedback excluded); yet there are some differences where several modes are in ‘close competition’.

Parts (d) and (g) of Fig. 7 do, together with parts (d) and (g) of Fig. 6, indicate that the acoustic feedback signal has a strong influence on the mode jump that takes place between \( u_0 = 9 \) m/s and 10 m/s. This is because there is a lock-in to the second pipe resonance frequency \( f_{\text{pipe}2} = 220 \) Hz at \( u_0 = 9 \) m/s but not at \( u_0 = 10 \) m/s. It is noted that, while the dominating shear layer frequency of 335 Hz is close to \( f_{\text{pipe}3} = 330 \) Hz, it is the dominating frequency component also for the case without acoustic feedback and is thus evidently not related to \( f_{\text{pipe}3} \). (It is noted also that the peak at 335 Hz in the sound pressure spectrum in Fig. 6(c) is relatively small.)

While the acoustic feedback thus can have influence on mode jumps, whether a mode jump takes place or not is thought to be determined mainly by the stability properties of the flow (see e.g. the discussion of this in Langthjem and Nakano (2015)). A stability analysis of the flow is thus necessary in order to fully understand the mechanism of mode jumps. This is not considered in the present work and is left for a future study.

It may be noted finally, that for the absolute magnitude of the shear layer velocity fluctuations, there is quiet good agreements between sound pressure and velocity fluctuation spectra.
agreement between theory and experiment, in particular for the cases where acoustic feedback is included.

![Graphs showing experimental and numerical results for flow velocity fluctuations with different speeds](image)

**Fig. 7** Experimentally and computationally obtained frequency spectra of the flow velocity fluctuations in the shear layer in the middle of the expansion chamber, with a tailpipe with resonance frequency $f_{\text{pipe}} = 110$ Hz.

### 9. Conclusion

The main contributions of the present work are (i) the formulation and analytical solution of an inherently coupled flow-acoustic problem within the framework of the classical acoustic analogy approach, (ii) implementation of acoustic feedback into the discrete vortex method, and (iii) successful demonstration of lock-in of the self-sustained flow oscillations to resonant acoustic oscillations.

From the numerical examples it can be concluded that the self-sustained flow oscillations in the expansion chamber and the acoustic oscillations in the tailpipe may not only reinforce each other, but may also become synchronized, due to the acoustic feedback velocity field.

Although not shown here, it has been verified that the amplitude of the feedback velocity is of the order of magnitude $1\%$ of the mean flow speed $u_0$. In one sense it is surprising that such a small velocity correction can be significant; yet it is known to be a typical feature of synchronized mechanical systems (Pikovsky et al., 2001).

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Appendix: Details of the asymptotic matching of solutions

Evaluation of (37) gives the relation

\[ C_0^* = \hat{A}_0 \left[ \cos \hat{k}\bar{x}_2 - \cot \hat{k}\hat{x}_3 \sin \hat{k}\bar{x}_2 \right]. \]  

(57)

Next, (24) for the (downstream) step is to be matched with (33) for the cavity. But in the cavity only the particular solution \( \hat{p}_0^{\text{part}} \) will be considered. (The homogeneous part of the cavity-solution, \( \hat{p}_0^{\text{hom}} \), will be determined in connection with matching of axial velocity components; see a little later.) The outer variable is now \( \hat{x} = x/L_0 \). The inner variable is \( x^* = (x - x_2)/r_0 = (\hat{x} - \hat{x}_2)/\epsilon, \) giving \( \hat{x} = \epsilon x^* + \hat{x}_2 \). A limit process similar to (37) gives

\[ C_0^* = \sum_n \sum_m \frac{1}{\delta} \int_0^\delta \int \hat{f}_n(\hat{r}, \hat{r}_n(\tau)) e^{ik\hat{x}_2 - i\epsilon x^* \hat{r}} e^{i\hat{k}\bar{x}_2} d\hat{r} d\tau. \]  

(58)

The pressure within the pipe then takes the form (38).

Next, matching of the axial velocity components will be considered. For the pipe section we have, from (21), that \( \partial\hat{p}_0/\partial \hat{x} = i\hat{k}\hat{u}_0 \), giving

\[ \hat{u}_0 = i\hat{A}_0 \left[ \sin \hat{k}\hat{x} + \cot \hat{k}\hat{x}_3 \cos \hat{k}\bar{x}_2 \right]. \]  

(59)

For the step we have, from (23), that \( \partial p_1^*/\partial x^* = ik\hat{x}_2 u_0^* \), which with (26) gives that

\[ u_0^* = b_0^{*+} + \sum_{n=1}^{\infty} b_n^{*+} e^{-\lambda_n x^*} J_0(\lambda_n r^*) \]  

(60)

just downstream of the step, with \( b_n^{*+} \) (and likewise \( b_n^{*-} \), used just upstream of the step) being constants. It is noted that the constant terms \( a_0^{*-} \) and \( a_0^{*+} \) in (26) disappear by differentiation. But the matching condition

\[ \lim_{\epsilon \to 0} u_0^* \left( \frac{\hat{x} - \hat{x}_2}{\epsilon} \right) \]  

(61)

\[ \hat{x} - \hat{x}_2 \text{ fixed} \quad x^* \text{ fixed} \]

demands the presence of a constant term in (60) since the remaining terms will be exponentially small. Addition a so-called switchback term is permissible in the present case, since a constant is a solution to the Laplace equation (25) with \( p_1^* \) replaced by \( u_0^* \). Thus (61) yields

\[ b_0^{*+} = i\hat{A}_0 \left[ \sin \hat{k}\hat{x}_2 + \cot \hat{k}\hat{x}_3 \cos \hat{k}\bar{x}_2 \right]. \]  

(62)

Next, for the cavity it used that \( \partial \hat{p}_0^{\text{hom}}/\partial \hat{x} = i\hat{k}\hat{u}_0 \). A matching at the upstream part of the step similar to (61) gives that

\[ i\hat{k}b_0^{*-} = -\sum_n \hat{k}_n J_0 \left[ \frac{\hat{r}_n}{\delta} \right] \hat{C}_n e^{-i\hat{k}_n \hat{x}_2}. \]  

(63)

The combined condition below (28) gives that \( b_0^{*-} \) is related to \( b_0^{*+} \) as

\[ b_0^{*-} = \varepsilon^2 b_0^{*+}. \]  

(64)

In order to determine the coefficients \( \hat{C}_n \) in (63), we multiply both sides of (63) by \( J_0 \left( \frac{\hat{r}_n}{\delta} \right) \hat{r}/\delta \), and integrate over the whole cross section, to obtain

\[ i\hat{k}_0 b_0^{*-} \int_0^\delta J_0 \left( \frac{\hat{r}_n}{\delta} \right) \frac{\hat{r}}{\delta} d\delta = -\sum_n \hat{k}_n \hat{C}_n e^{-i\hat{k}_n \hat{x}_2} \int_0^\delta J_0 \left( \frac{\hat{r}_n}{\delta} \right) J_0 \left( \frac{\hat{r}_n}{\delta} \right) \frac{\hat{r}}{\delta} d\delta. \]  

(65)

Since \( \hat{k}_0 = \hat{k} \) this gives the lowest order coefficient as

\[ \hat{C}_0 = -b_0^{*-}. \]  

(66)

Thus the axial feedback velocity component within the cavity takes the form (39).

It is noted that no boundary conditions have been imposed at the upstream end of the cavity, at \( \hat{x} = \hat{x}_1 \), in the acoustic model. Mathematically, these conditions come into play only at the order \( \epsilon^2 \), along with the terms \( n \geq 1 \) in (26). They do thus not appear in leading order terms the present paper restricts itself to. From a physical point of view, the reflection from the annular end plate present at \( \hat{x} = \hat{x}_1, \hat{r}_0 < \hat{r} < \hat{r}_1 \) may have an amplifying effect on the vortex sound sources, but the acoustic resonances in the cavity do not have a bearing on the present problem since the corresponding frequencies are much higher than those in the tailpipe. With the cavity length \( L_1 = d_0 = 50 \text{ mm} \), the lowest axial resonance frequency is 3400 Hz (considering the cavity as a pipe open at both ends). This is \( \sim 20 \) times larger than the resonance frequencies in the tailpipe, as the results in Section 8 show.
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