Aspects of $N = 2$ Supersymmetric Gauge Theories in Three Dimensions

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We consider general aspects of $N = 2$ gauge theories in three dimensions, including their multiplet structure, anomalies and non-renormalization theorems. For $U(1)$ gauge theories, we discuss the quantum corrections to the moduli space, and their relation to “mirror symmetries” of 3d $N = 4$ theories. Mirror symmetry is given an interpretation in terms of vortices. For $SU(N_c)$ gauge groups with $N_f$ fundamental flavors, we show that, depending on the number of flavors, there are quantum moduli spaces of vacua with various phenomena near the origin.

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1. Introduction

Recent exact results in four dimensional supersymmetric theories have given insight into the quantum dynamics of strongly coupled gauge theories, revealing interesting phenomena and phases (for reviews see, e.g. [1,2]). Interesting phenomena have also been found recently for three dimensional $N = 4$ theories, in particular mirror symmetry which relates the IR behavior of two different field theories, interchanging their Higgs and Coulomb branches [3]. Additional examples of mirror symmetry and connections to string theory appeared in [4,5,6,7]. It is interesting to see how these phenomena depend on the dimension and the amount of supersymmetry, and how they behave upon compactification. Another motivation is that there are powerful connections between gauge theories in various dimensions and the dynamics of string theory. New information about field theory can lead to new insight into string theory and vice-versa. The analysis of three dimensional field theories may also have applications for statistical mechanics problems, which are beyond the scope of this paper.

We consider gauge theories in $d = 3$ dimensions with four supercharges, the same number as for $N = 1$ supersymmetry in $d = 4$, which corresponds to $N = 2$ supersymmetry in $d = 3$. In string theory, such theories arise as the low-energy theories of compactifications of M theory on Calabi-Yau fourfolds (as well as from dimensional reduction of $d = 4$ $N = 1$ theories). They also correspond to the low-energy field theories of membranes in M theory compactifications on fourfolds or on threefolds, and of D2-branes in type IIA compactifications on threefolds. We will not discuss these relations to string theory here.

Theories with $N = 1$ supersymmetry in three dimensions have no holomorphy properties, so we cannot control their non-perturbative dynamics. However, as we will discuss, with $N = 2$ supersymmetry there are holomorphic objects and non-renormalization theorems, which enable us to compute some properties of these theories exactly.

To summarize our results, we find exact superpotentials in $U(1)$, $SU(N_c)$, and $U(N_c)$ examples. Perturbative effects in Abelian theories can cause the Coulomb branch to split into several regions, described by different variables. At the intersections of different regions are RG fixed points for which we find dual descriptions. In non-Abelian cases, instantons (analogous to $d = 4$ monopoles) can lift most or all of the Coulomb branch, and sometimes also the Higgs branch. In 3d $N = 2$ $SU(N_c)$ SQCD with $N_f < N_c - 1$ flavors, for example, mirror symmetry in this context and its relation to mirror symmetry in string theory were recently discussed in [8].

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the moduli space is completely lifted and the theory has no stable vacuum. In some cases there is a quantum moduli space with the classically distinct Higgs and Coulomb branches smoothly merged together. This happens, for example, in $SU(N_c)$ SQCD theories with $N_f = N_c - 1$ flavors. In other cases, both for Abelian and Non-Abelian theories, there are distinct Higgs and Coulomb branches, with RG fixed points where they intersect. In some cases we find dual descriptions of these fixed points.

We begin in sect. 2 by discussing some general classical and quantum aspects of 3d $N = 2$ theories. For example, we discuss linear multiplets, central charges, Fayet-Iliopoulos terms, and non-renormalization theorems. We point out that the “parity anomaly” of [9,10] can be used to give an analog of the ’t Hooft anomaly matching condition for 3d theories.

In sect. 3 we analyze the dynamics of $U(1)$ gauge theories. For $N_f > 0$, the moduli space consists of different branches which meet at a point at the origin, where there is a RG fixed point. For $N_f = 1$, we find a dual description of the RG fixed point and verify that our discrete anomaly matching conditions are satisfied.

In sect. 4 we discuss dual descriptions of the RG fixed points at the origin of $N = 2$ SQED with $N_f > 0$ which are obtained from the “mirror symmetry” duality of the corresponding $N = 4$ theories [3]. We connect these duals with the results of sect. 3.

In sect. 5 we discuss vortices in $N = 4$ and $N = 2$ theories and their connection with mirror symmetry of $N = 4$ and $N = 2$ SQED.

In sect. 6 we analyze the dynamics of the $SU(2)$ gauge theory with various numbers of quark flavors and mass terms. For $N_f = 1$ there is a quantum moduli space of vacua, with the classically distinct Higgs and Coulomb branches smoothly merged together. For $N_f \geq 2$ there are distinct Higgs and Coulomb branches which intersect at a point at the origin, where there is a RG fixed point. For $N_f = 2$ we find a dual description of the RG fixed point at the origin. We discuss adding real mass terms, connecting with our results for $U(1)$ theories.

In sect. 7 we analyze $d = 4$ $N = 1$ $SU(2)$ gauge theories compactified on a circle of varying radius, showing how known non-perturbative effects in the 4d theories can be related to the quantum effects we found in 3d theories.

In sect. 8, we generalize the discussion of sect. 6 and sect. 7 to $SU(N_c)$ gauge theories.

Note added: As we completed this paper, we received [11], which considers the same theories.
2. General aspects of \( d = 3 \) \( N = 2 \) theories

\( N = 2 \) supersymmetry in three dimensions has four supercharges, with an algebra which follows simply from reducing \( d = 4 \) \( N = 1 \) supersymmetry down to three dimensions:

\[
\{ Q_\alpha, Q_\beta \} = \{ \overline{Q}_\alpha, \overline{Q}_\beta \} = 0, \quad \{ Q_\alpha, \overline{Q}_\beta \} = 2\sigma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z, \quad (2.1)
\]

where the \( \sigma^\mu \) are chosen to be real and symmetric. \( Z \) is a real central term, which in the dimensional reduction corresponds to the momentum \( P_3 \) in the reduced direction. The spinors \( Q \) and \( \overline{Q} \) are complex, and thus include twice the minimal amount of charges in three dimensions. As in four dimensions, the automorphism of this algebra is \( U(1)_R \), rotating the supercharges.

As in four dimensions, there are chiral superfields \( X \) which satisfy \( \overline{D}_\alpha X = 0 \), anti-chiral superfields \( \overline{X} \) which satisfy \( D_\alpha \overline{X} = 0 \), and vector superfields which satisfy \( V = V^\dagger \). In addition, in three dimensions there can be linear multiplets \( \Sigma \), which satisfy

\[
\epsilon^{\alpha\beta} D_\alpha D_\beta \Sigma = \epsilon^{\alpha\beta} \overline{D}_\alpha \overline{D}_\beta \Sigma = 0,
\]

whose lowest component is a real scalar field. Occasionally, we will find it convenient to dualize the vector and linear multiplets into chiral multiplets.

All states satisfy a BPS bound of the form \( M \geq |Z| \). General irreducible representations of the above algebra, such as the chiral and vector representations, contain two (real) bosonic and two (Majorana) fermionic degrees of freedom. Smaller representations with one real bosonic degree of freedom and one fermion can also exist, and must saturate the BPS bound \( M = |Z| \). However, as discussed in §2.3, non-zero \( Z \) can only occur for states which are charged under \( U(1) \) symmetries, and then CPT dictates the existence also of a conjugate representation.

2.1. Wess-Zumino theories in \( d = 3 \)

Consider a Wess-Zumino type Lagrangian for chiral and anti-chiral superfields \( X \) and \( \overline{X} \). The general Lagrangian we get by reduction from \( d = 4 \) is of the form

\[
\int d^4 \theta \ K(X, \overline{X}) + ( \int d^2 \theta \ W(X) + h.c.). \quad (2.2)
\]

A three dimensional chiral superfield \( X \) has engineering dimension \( 1/2 \), and thus the classically marginal superpotential is \( W = X^4 \), corresponding to an \( |X|^6 \) scalar potential. Unlike four dimensions, where WZ theories always flow to Gaussian fixed points, in three dimensions the WZ theory with superpotential \( W = X^3 \) flows to an interacting fixed point.
in the infrared. It follows from the three dimensional superconformal algebra that, as in \cite{13}, for any $N = 2$ theory the dimensions of all operators satisfy

$$D \geq |R|, \quad (2.3)$$

where $R$ is the charge under the $U(1)_R$ symmetry which is in the same multiplet as the stress tensor. The inequality (2.3) is saturated for chiral and anti-chiral operators. For $W = X^3$, $R(X) = 2/3$ and thus (2.3) gives

$$D(X) = \frac{2}{3}, \quad (2.4)$$

giving the exact anomalous dimension of $X$ at the infrared fixed point.

2.2. Gauge theories

The vector multiplet $V$ of $d = 3 \ N = 2$ supersymmetry contains, in addition to the $d = 3$ vector potential, a real scalar $\phi$ in the adjoint of the gauge group $G$; $\phi$ corresponds to the component of the $d = 4$ vector potential in the reduced direction. The massless vector multiplet also contains two real (which can be joined into one complex) fermion gauginos. As in four dimensions, there are chiral and anti-chiral field strengths, defined as $W_\alpha = -\frac{1}{4} DDe^{-V} D_\alpha e^V$ and $\overline{W}_\alpha = -\frac{1}{4} DDe^{-V} D_\alpha e^V$, and the classical gauge kinetic terms, including a kinetic term for the real scalar $\phi$, are

$$\frac{1}{g^2} \int d^2 \theta \ Tr \ W^2_\alpha + h.c. \quad (2.5)$$

The gauge field and real scalar are neutral under the $U(1)_R$ symmetry, while the complex fermion has a charge we will normalize to be +1.

When the gauge group $G$ is Abelian or has Abelian factors, it is possible also to add a Fayet-Iliopoulos term. As in 4d, the Fayet-Iliopoulos term is of the form $\zeta \int d^4 \theta V$ for some real Fayet-Iliopoulos parameter $\zeta$.

The moduli space of vacua has a “Coulomb branch” where the real scalar $\phi$ gets an expectation value in the Cartan subalgebra of the gauge group, breaking $G$ to the Cartan subgroup $U(1)^r$, with $r = \text{rank}(G)$. The Coulomb branch is thus a Weyl chamber, which is a wedge subspace of $\mathbb{R}^r$ parameterized by Cartan scalars $\phi^j$ in $\mathbb{R}^r / W$ ($W$ is the Weyl group of $G$), the expectation values of the scalars in the massless Cartan $U(1)^r$ vector multiplets $V^j$. At the boundaries of the Weyl chamber, at the classical level, there is enhanced gauge symmetry.
In the bulk of the Coulomb branch, the $U(1)^r$ gauge fields can be dualized to scalars via $F_{\mu\nu}^{(j)} = \epsilon_{\mu\nu\sigma} \partial^\sigma \gamma^j$, $j = 1 \ldots r$. Due to charge quantization, the scalars $\gamma^j$ live on an $r$-dimensional torus. It is the Cartan torus of the dual gauge group, whose size is of the order of the gauge coupling $g$ [14]. The currents $J_{\mu}^{(j)} = \epsilon_{\mu\rho\sigma} (F^{\nu\rho})^{(j)}$ generate “magnetic” $(U(1)_J)^r$ global symmetries, corresponding to shifts of $\gamma^j$; these symmetries are exact only for Abelian gauge groups. The $\gamma^j$ can be combined with the $\phi^j$ into chiral superfields $\Phi^j$, with scalar component $\Phi^j = \phi^j + i\gamma^j$. At the boundaries of the Coulomb wedge, where the gauge group is classically non-Abelian, there is no known way to dualize the gauge fields into scalars. In the quantum theory, these are the regions where we expect quantum effects to be especially important.

There can also be matter multiplets $Q_f$ in representations $R_f$ of the gauge group. The classical Lagrangian has terms

$$\sum_f \int d^4 \theta \, Q_f^\dagger e^V Q_f,$$

(2.6)

where $V$ includes a term $\phi \theta \bar{\theta}$ for the real adjoint scalar $\phi$. In particular, the Lagrangian includes a potential for the squarks of the form

$$\sum_f |\phi Q_f|^2,$$

(2.7)

and $\langle \phi \rangle$ looks like a “real mass” for the matter fields (we will discuss this further in §2.3).

In addition to the Coulomb branch, there can be a “Higgs” branch, where the squark components of the matter multiplets $Q_f$ get expectation values. Because of the coupling (2.7), $\langle Q_f \rangle \neq 0$ generally requires $\langle \phi \rangle = 0$ (at least for some of the components of $\phi$), and vice-versa. Therefore, the complete classical moduli space of vacua generally consists of distinct branches, Coulomb with $\langle \phi \rangle \neq 0$ and $\langle Q_f \rangle = 0$, and Higgs with $\langle Q_f \rangle \neq 0$ and $\langle \phi \rangle = 0$. This is similar to $N = 2$ supersymmetric theories in four dimensions. As is the case there, depending on the matter content there can also be mixed branches where some $\langle Q_f \rangle$ and $\langle \phi \rangle \neq 0$ such that (2.7) still vanishes.

There is a freedom in choosing the $U(1)_R$ charge assignments of the matter multiplets, as $U(1)_R$ can mix with other $U(1)$ global symmetries which act on the matter supermultiplets. Thus, generally it will not be simple to determine the dimensions of chiral fields at IR fixed points via (2.3), since we do not know which $U(1)_R$ current is in the same multiplet as the stress tensor there. In three dimensions there is no condition on the symmetries
that they be anomaly free. For convenience, we will choose the squarks to be neutral under \(U(1)_R\); the quarks then have charge \((-1)\).

As discussed in [13], non-perturbative effects can lead to a dynamically generated superpotential which lifts the classical moduli space degeneracy of the Coulomb branch. In addition, when there is also a Higgs branch, the intersection of the branches near the origin of the moduli space of vacua, where quantum effects are strong, can be interesting. We will discuss such effects in this paper.

2.3. Linear multiplets, central charges, Fayet-Iliopoulos terms and non-renormalization theorems

The vector multiplet in \(d = 3\) \(N = 2\) theories includes a vector, a scalar and two Majorana fermions. In §2.2, we defined the chiral superfield \(W_\alpha\) whose lowest component is a gaugino, and wrote the gauge kinetic term in terms of this superfield, which is gauge-invariant for Abelian theories (we will only discuss Abelian vector fields in this sub-section). However, we can also define a superfield whose lowest component is the scalar in the vector multiplet – this is a linear multiplet\(^2\), defined by \(\Sigma = \epsilon^{\alpha\beta} D_\alpha D_\beta V\). It is easy to check that this \(\Sigma\) satisfies \(D^2 \Sigma = \overline{D}^2 \Sigma = 0\) (where \(D^2 = \epsilon^{\alpha\beta} D_\alpha D_\beta\), and that it is gauge invariant (under \(V \rightarrow V + i(\Lambda - \Lambda^\dagger)\)). The lowest component of \(\Sigma\) is the scalar \(\phi\), and its expansion includes also a term \(\bar{\theta} \sigma_{\rho} \theta F_{\mu\nu} \epsilon^{\rho\mu\nu}\). The gauge kinetic term may be written simply in the form \(\int d^4 \theta \Sigma^2\). In three dimensions, we can dualize the vector into a scalar, and turn the linear (or vector) multiplets into chiral multiplets. Note, however, that chiral multiplets obtained in this way always have \(U(1)_J\) symmetries which act as shifts by an imaginary number on their scalar component.

Linear multiplets are useful also in describing global conserved currents. The conservation of the global currents, which follows from the equations of motion, implies that they can be viewed as components of linear multiplets, satisfying \(D^2 J = \overline{D}^2 J = 0\). In fact, the multiplet \(\Sigma\) we defined above is an example of this phenomenon, and it includes the conserved current \(J^\mu = \epsilon^{\mu\nu\rho} F_{\nu\rho}\), which generates the global \(U(1)_J\) symmetry. Generally, for a conserved current \(J_\mu\), there will be a linear multiplet \(J\) which includes the term \(\bar{\theta} \sigma^\mu \theta J_\mu\). However, since the current is only conserved on-shell, we cannot always find a vector superfield \(V\) such that \(J = \epsilon^{\alpha\beta} D_\alpha D_\beta V\).

\(^2\) These multiplets were called “scalar field strengths” in [10].
The SUSY algebra includes a real central charge $Z$, appearing in (2.1). In 4d $N = 2$ theories, $Z$ can get contributions from both local and global currents \cite{17}. However, in 3d the mass (and the central charge) of particles charged under local currents are divergent, even classically, due to the $1/r$ fall-off of electric fields in 3d. Thus, we cannot really discuss the masses of electrically charged fields, and we can only use the BPS formula for gauge-neutral states. Therefore, only global charges contribute to the central charge of physical states.

A simple example of a theory with a non-zero central charge is given by the Lagrangian

$$\int d^4\theta X^\dagger e^{\tilde{m}\theta\bar{\theta}} X, \quad (2.8)$$

where $X$ is a chiral multiplet. When written in components this Lagrangian includes terms of the form $(\tilde{m}^2 |X|^2 + i\tilde{m} \epsilon^{\alpha\beta} \overline{\psi}_\alpha \psi_\beta)$. The parameter $\tilde{m}$ appears here as a “real mass” term for $X$, which is distinct from the complex masses appearing in the superpotential. The central charge $Z$ (if we promote it to a background superfield) corresponds to the linear multiplet $J$ containing the global current under which $X$ is charged. Its scalar component is given by $Z = \tilde{m}$, and $X$ saturates the BPS bound. As discussed above, chiral multiplets do not have to saturate the BPS bound. For instance, if we add to (2.8) a superpotential of the form $W = mXY$ (where $Y$ is another chiral superfield needed to preserve the global $U(1)$), the (tree-level) mass of $X$ is $M = \sqrt{\tilde{m}^2 + |m|^2}$, and the BPS bound is no longer saturated.

More generally, $Z$ gets contributions from global Abelian currents, of the form

$$Z = \sum q_i m_i, \quad (2.9)$$

where $q_i$ is the charge of the field (or the state) under a global $U(1)_i$ symmetry, and $m_i$ is a parameter. As in \cite{18}, we promote every parameter to a background superfield. The parameters $m_i$ in (2.9) are then in background vector or linear multiplets. To see this, add to any system with a global $U(1)$ symmetry a charged chiral multiplet as in (2.8) with a large $U(1)$ charge. The large charge makes this field heavy and almost decoupled. Its mass is given by the BPS formula as its charge times $m$ where $m$ is the scalar of a linear multiplet. Hence, even without this heavy auxiliary state, the coefficient in $Z$ is in a linear multiplet. This sort of “background gauging” can only be performed if the global symmetry is exact, and only then will this type of “real mass” terms appear. The fact that we cannot write them down in general is consistent with the fact that we have no simple
superspace expression for these terms. (This is also related to the fact that $J$ is only in a linear multiplet when we use the equations of motion, and that generally we cannot view it as originating from a vector multiplet.)

Another example of a global symmetry which can contribute to the central charge is the $U(1)_J$ symmetry, which corresponds to shifting the dual photon $\gamma$. This is only a symmetry in Abelian theories. The current multiplet $J$ is exactly the linear multiplet $\Sigma$ defined above, and the added term is of the form $\int d^4\theta V_b \Sigma = \int d^4\theta \Sigma_b V$ by integrating by parts, where $\Sigma_b = \epsilon^{\alpha\beta} D_\alpha D_\beta V_b$ is a background linear multiplet. Thus, the scalar component of the background vector field is exactly the Fayet-Iliopoulos term $\zeta$, and $Z$ will get a contribution in (2.9) with $m_J = \zeta$. States with charge $q_J$ under $U(1)_J$ thus obey a BPS bound of the form $M \geq |q_J\zeta|$.

In the spirit of [18], the results of this section suggest two types of (non-perturbative) non-renormalization theorems for these theories. First, it is easy to see that (to preserve supersymmetry) the superpotential cannot include linear multiplets, but only chiral multiplets, and, therefore, it cannot depend on the “real mass” terms or on Fayet-Iliopoulos terms (which are all background linear multiplets). Alternatively, if we dualize the linear multiplets into chiral multiplets, holomorphy together with the shift symmetry mentioned above forbid these multiplets from appearing in the superpotential (as long as the shift symmetry is exact). Second, we found that the central charge $Z$ is a background linear multiplet. Any dependence on chiral multiplets, or any non-linear dependence on linear multiplets, would change this fact. Thus, the relation (2.9) between $Z$ and the “real mass” terms and Fayet-Iliopoulos parameter is, in fact, exact. This is analogous to the similar phenomenon in $d = 4$ $N = 2$ gauge theories [17], where the central charges are exactly equal to the scalar components of $N = 2$ vector multiplets.

2.4. Anomalies

Unlike the situation in four dimensions, in three dimensions there are no local gauge anomalies. However, as found in [24], gauge invariance can require the introduction of a classical Chern-Simons term, which breaks parity. This is referred to as a “parity anomaly.”

Consider first an Abelian $U(1)^r$ gauge theory. There can be classical Chern-Simons couplings,

$$\sum_{i,j=1}^r k_{ij} \int d^4\theta \Sigma_i V_j, \quad (2.10)$$
where $\Sigma_i = \epsilon^{\alpha\beta} \partial_{\alpha} D_{\beta} V_i$ are linear superfields; (2.10) is the supersymmetric completion of $\sum_{ij} k_{ij} A_i \wedge dA_j$. We work in a basis for $U(1)^r$ where all charges are integers. At the quantum level, if we integrate out the charged fermions, there is an additional induced contribution to the Chern-Simons term, coming from a one-loop diagram with charged fermions running in the loop:

$$(k_{ij})_{\text{eff}} = k_{ij} + \frac{1}{2} \sum_f (q_f)_i (q_f)_j \text{sign}(M_f);$$

the sum runs over all fermions, $(q_f)_i$ is the charge of fermion $f$ under $U(1)_i$ (in units of the quantized charge), and $M_f$ is the mass of fermion $f$. In the present context (without a superpotential), $M_f = \tilde{m}_f + \sum_{i=1}^r (q_f)_i \phi_i$.

Gauge invariance restricts the coefficients of the Chern-Simons term as $(k_{ij})_{\text{eff}} \in \mathbb{Z}$. From (2.11), we see that the bare Chern-Simons coefficients $k_{ij}$ must satisfy the quantization conditions

$$k_{ij} + \frac{1}{2} \sum_f (q_f)_i (q_f)_j \in \mathbb{Z},$$

where $f$ runs over all fermions. In particular, when $\sum_f (q_f)_i (q_f)_j$ is odd, $k_{ij} \neq 0$ and parity is necessarily broken. This is the “parity anomaly.” Without the classical Chern-Simons term, the fermion determinant in these cases would be multiplied by $(-1)$ under certain gauge transformations. The $k_{ij} \in \mathbb{Z} + \frac{1}{2}$ Chern-Simons term plays the role of a Wess-Zumino term whose lack of gauge invariance compensates for that of the fermion determinant.

If we introduce background gauge fields for the global $U(1)$ symmetries, as described in §2.3, and integrate out the fermions, similar Chern-Simons terms will appear (at one-loop) involving these background gauge fields as well. In particular, terms of the form $\int d^4 \theta \Sigma_b V$ where $\Sigma_b$ is a background linear multiplet and $V$ is a gauge field can be generated (by the same diagrams which generate (2.11)), which correspond to Fayet-Iliopoulos terms for the gauge field $V$. When such terms appear, the global symmetries corresponding to $\Sigma_b$ will be mixed with the $U(1)_j$ symmetry corresponding to $V$, and the chiral superfields corresponding to the dual photons will transform non-trivially under the other global symmetries as well. Below we will see this phenomenon in several examples.

There is a similar parity anomaly for non-Abelian theories [9,10]. Just as the 4d anomaly of [19] is associated with $\pi_4(G)$, the 3d parity anomaly is associated with $\pi_3(G)$. Under the large gauge transformations corresponding to non-trivial elements of $\pi_3(G)$, the
fermion determinant can pick up a minus sign, making the theory inconsistent. Unlike the situation in 4d, however, in 3d this anomaly can always be cancelled by adding a Chern-Simons term with a coefficient $k$ which is half-integral $[9,10]$. Again, the Chern-Simons term with half-integral $k$ plays the role of a Wess-Zumino term whose lack of gauge invariance compensates for that of the fermion determinant. The condition on the classical Chern-Simons term $k$ is

\[ k + \frac{1}{2} \sum_f d_3(R_f) \in \mathbb{Z}, \tag{2.13} \]

where the sum is over all fermions $f$ in representations $R_f$ of $G$ and $d_3(R_f)$ is the cubic index of $R_f$, normalized so that the $N$ of $SU(N)$ has $d_3(N) = 1$. In particular, when $\sum_f d_3(R_f)$ is odd, $k \neq 0$ and parity is necessarily broken; there is a “parity anomaly.”

It is interesting to note that the parity anomaly gives an analog of the ’t Hooft anomaly matching conditions in 3d. In 4d, the precise anomalies associated with gauging global symmetries must match between the microscopic and the low energy theories. In 3d, there is a weaker $\mathbb{Z}_2$ type condition: whether or not the gauged global symmetry would have a parity anomaly must match between the microscopic theory and any other theory which is equivalent to it in the IR.

The Chern-Simons term gives the photon a mass, lifting the Coulomb branch discussed earlier. In most of this paper we will not be interested in that situation, and will want to avoid turning on the Chern-Simons term. The matter content must then be chosen so that there is not a parity anomaly.

2.5. Instantons

In three dimensions, instantons are associated with $\pi_2$. Because $\pi_2(G) = 0$ for any gauge group, there can only be instantons on the Coulomb branch of non-Abelian theories, where the gauge group is broken to the Cartan torus $U(1)^r$. In that case, the relevant configurations are related to elements of $\pi_2(G/U(1)^r) = \mathbb{Z}^r$. (Also, because $\pi_2(G) = 0$, there is no analog of the 4d theta angle in three dimensions.) It follows from $\pi_2(G/U(1)^r) = \mathbb{Z}^r$ that there are $r = \text{rank}(G)$ independent fundamental instantons in $d = 3$, associated with the simple roots of the gauge group. This is the same as the $d = 4$ result that there are $r$ independent monopoles $[20]$ (which is not surprising since the $d = 3$ instantons are identical to $d = 4$ monopoles when ignoring the $d = 4$ time dimension). The $j$’th “fundamental” instanton is semi-classically weighted by $e^{-S_j} = e^{-\phi \beta_j / g^2}$, where $\phi$ is the adjoint scalar (in the Cartan subalgebra) defined above, and $\beta_j$, $j = 1, \ldots, r$, are the correct
basis of simple roots discussed in \cite{20}. In a given Weyl chamber, the \( \beta_j \) will be chosen so that \( \phi \cdot \beta_j \geq 0 \) for all \( j \); at the boundary of the Weyl chamber where a given \( \phi \cdot \beta_j = 0 \), there is an enhanced \( SU(2)_j \subset G \). Instantons corresponding to linear combinations of the \( \beta_j \) have additional zero modes corresponding, as with monopoles in \( d = 4 \) \cite{20}, to separating them into linear combinations of the \( r \) fundamental instantons.

It is natural to combine the above instanton factors with the dualized photons, which are similar to theta angles in \( d = 4 \), to make quantities holomorphic in \( \Phi^k = \phi^k + i \gamma^k \),

\[
Y_j \sim e^{\Phi \cdot \beta_j / g^2}.
\tag{2.14}
\]

The fields \( Y_j \) are normalized so that they have charge +1 under the (approximate) \( U(1)_J \) symmetries that these theories have on their Coulomb branch. They provide a natural set of coordinates for the Coulomb branch; the sign of the exponent in (2.14) is chosen so that large \( Y_j \) corresponds to being far out along the Coulomb branch. The \( j \)-th instanton contribution is weighted by \( Y_j^{-1} \), including correctly the dependence on the dual photon \cite{21}. The \( \sim \) in (2.14) is because this relation between \( Y_j \) and \( \Phi \) is valid only semi-classically, for large \( \phi \cdot \beta_j \); it can be modified near the boundaries of the Coulomb branch.

As discussed in §2.4, the fields \( Y_j \) can acquire charge under global symmetries due to one-loop effects. This is also consistent with the counting of fermionic zero modes in the instanton background. Then, \( \langle Y_j \rangle \) spontaneously breaks the global symmetry, with the dualized photon playing the role of the Goldstone boson \cite{15}. The fermion zero modes associated with the instantons are easily obtained: the quark zero modes are the same as the \( d = 4 \) fermionic zero modes in a monopole background \cite{22,23}, while the gluino zero modes are related by supersymmetry to the bosonic zero modes which were analyzed in \cite{20}. It thus follows that the \( j \)-th “fundamental” instanton always has two gaugino zero modes (this is half of the number of zero modes for the \( N = 4 \) theories analyzed in \cite{24}, since we have half the number of gluinos). In particular, in pure \( d = 3 \) \( N = 2 \) Super-Yang-Mills theory with no matter fields, each fundamental instanton has two gaugino zero modes (and no other fermionic zero modes), so the \( Y_j \) all carry charge \((-2)\) under the global \( U(1)_R \) (gaugino number) symmetry.
Consider $U(1)$ gauge theory with $N_f$ “flavors”, $Q^{i}, \tilde{Q}^{\tilde{i}} (i, \tilde{i} = 1, \cdots, N_f)$, with charge $\pm 1$. Unlike the situation in 4d, where this theory is IR free, in 3d it has interesting dynamics. Before taking into account the quantum corrections, there is a one dimensional Coulomb branch, parameterized by $\Phi = \phi + i\gamma$ which lives on a cylinder, $\phi \in \mathbb{R}$ and $\gamma \in S^1$ of period $g^2$ (the radius of $S^1$ is of order $g$). For $N_f > 0$ there is also a $(2N_f - 1)$-dimensional Higgs branch, which can be parameterized by the gauge invariant operators $M_{ij} = Q^i \tilde{Q}^j$ subject to the constraint that rank$(M) \leq 1$, i.e. $M_{ij}M_{kl} = M_{kj}M_{li}$. Classically the Higgs branch intersects the Coulomb branch at $\phi = 0$. Since there are no instanton corrections in this case, one might expect the classical picture to continue to hold also in the quantum theory. However, as we will see, perturbative effects change the topology of the moduli space, and new degrees of freedom will be needed to describe the quantum moduli space.

### 3.1. The quantum moduli space of $U(1)$ gauge theories

For large $\phi$, it appears that the Coulomb branch can be consistently parameterized by the vacuum expectation value of the chiral superfield $V = e^{\Phi/g^2}$. However, the metric for $\gamma$ receives quantum corrections. This has already been seen in [25,24] in $N = 4$ theories. Since the circumference of the circle which $\gamma$ lives on is $g$, the topology of the moduli space can be changed in perturbation theory. To determine the behavior of the quantum theory near the origin of moduli space, note that the Higgs branch (which classically intersects the Coulomb branch at $\phi = 0$) is invariant under the $U(1)_J$ symmetry which corresponds to rotations around the circle. Therefore, the radius of $\gamma$ must vanish where the two branches meet.

\[ \text{Fig. 1: } U(1) \text{ } N = 2 \text{ gauge theory in three dimensions with massless flavors;} \text{ the classical moduli space and the quantum corrected moduli space.} \]
Thus, the Coulomb branch splits into two distinct regions, and the moduli space looks (near the origin) like an intersection of three cones (as in fig. 1). The $\phi > 0$ half of the Coulomb branch is parameterized by a field $V_+$, which is semi-classically given by $V_+ \sim e^{\Phi/g^2}$. Because the region of $\phi = 0$ shrinks to a point, in the quantum theory $V_+ \to 0$ there, and can take values in the entire complex plane in this half of the Coulomb branch. Obviously, this means that a new variable $V_-$, which semi-classically is given by $V_- \sim e^{-\Phi/g^2}$, is needed to parameterize the second half of the Coulomb branch. So, our quantum picture of the Coulomb branch involves two unconstrained chiral superfields $V_{\pm}$ to describe the two distinct halves of the Coulomb branch. The topology of the Higgs branch is not changed by the quantum corrections, so we can still use the mesons $M_{ij}$ to parameterize this branch.

The metrics of the various branches in the semi-classical regions can easily be computed. Far along the Higgs branch (large $\langle M \rangle$), the Kähler potential of $M$ looks like $(M^\dagger M)^{1/2}$, while far along the Coulomb branch (large $V_{\pm}$) the field $\Phi \sim \pm \log(V_{\pm})$ has a canonical kinetic term, and the Coulomb branch looks like a cylinder. However, at a distance of order $g$ from the origin of moduli space, there will be strong corrections to these metrics, as described above.

The quantum dynamics is constrained by the global symmetries, which are

$$
\begin{array}{cccccc}
U(1)_R & U(1)_J & U(1)_A & SU(N_f) & SU(N_f) \\
Q & 0 & 0 & 1 & N_f & 1 \\
\bar{Q} & 0 & 0 & 1 & \bar{N}_f & \bar{N}_f \\
M & 0 & 0 & 2 & N_f & \bar{N}_f \\
V_{\pm} & N_f & \pm 1 & -N_f & 1 & 1 \\
\end{array}
$$

(3.1)

We chose the $U(1)_R$ charge of $Q$ and $\bar{Q}$ to be zero, so that their fermions have charge $(-1)$. The symmetry $U(1)_J$ corresponds to shifting the dual photon $\gamma$; its current is $J_\mu = \epsilon_{\mu\nu\lambda}F^{\nu\lambda}$. It follows from our semi-classical identifications that $V_{\pm}$ have charge $\pm 1$ under this symmetry. The charges of $V_{\pm}$ under the other global symmetries follow, as discussed in §2.4, from a one-loop diagram connecting the currents to the gauge field.

For $N_f = 1$ the three branches in fig. 1 are all one complex dimensional, and we propose that they are actually related near the origin by a triality exchange symmetry. At the origin, where the three branches meet, there is a RG fixed point. Another theory which flows to the same IR fixed point can be described by the fields $M, V_{\pm}$ with a superpotential

$$W = -MV_+V_-,$$

(3.2)
which correctly gives the moduli space consisting of three cones, parameterized by $V_\pm$ and $M$, which intersect at the point $V_\pm = M = 0$. (The sign and normalization in (3.2) are chosen for convenience.) The superpotential (3.2) respects the global symmetries (3.1). Since (3.2) is of degree 3, it is strongly coupled in the IR and flows to a fixed point which we claim is the same as that of $N_f = 1$ SQED.

A (rather weak) check that $N_f = 1$ SQED and the theory with $M$, $V_\pm$ in (3.2) flow to the same IR fixed point is that the parity anomaly matching conditions discussed in the previous section are satisfied. In both the original $U(1)$ theory with $N_f = 1$ and in the theory with the fields $M$ and $V_\pm$, eqn. (2.12) for the global $U(1)_R \times U(1)_J \times U(1)_A$ symmetries gives $k_{RR} \in \mathbb{Z} + \frac{1}{2}$, $k_{JJ} \in \mathbb{Z}$, $k_{AA} \in \mathbb{Z}$, $k_{RJ} \in \mathbb{Z}$, $k_{RA} \in \mathbb{Z}$, and $k_{AJ} \in \mathbb{Z}$.

We can further test (3.2) by giving the electron a complex mass $m$, leading to $W = -V_+ V_- M + mM$. Integrating out $M$, we find $V_+ V_- = m$; the two variables parameterizing the Coulomb branch are related as expected for the free $N_f = 0$ theory (where the moduli space is just a cylinder).

In addition to this “complex mass” deformation, there are several other deformations which cannot be described by a superpotential. Consider, for example, adding a Fayet-Iliopoulos term. In the original theory, this lifts the Coulomb branch, and we are left only with the Higgs branch. As discussed in §2.3, we can view this term as a background $U(1)_J$ vector field. Thus, we can turn it on also in the dual theory (3.2), where (using (3.1)) it corresponds to a real mass term for $V_+$ and $V_-$, which has the same effect on the moduli space.

Another possibility is to add “real masses” for $Q$ and $\tilde{Q}$. Opposite “real masses” for $Q$ and $\tilde{Q}$ may be absorbed in the definition of the origin of $\phi$. On the other hand, equal masses $\tilde{m}$ (which breaks CP), corresponding to a background $U(1)_A$ vector field, are physically significant. On the Coulomb branch, the “effective real mass” of $Q$ is now $(\phi + \tilde{m})$, and that of $\tilde{Q}$ is $(-\phi + \tilde{m})$. There is no Higgs branch since $Q$ and $\tilde{Q}$ are massless at different points on the Coulomb branch, but classically the Coulomb branch remains unlifted. However, if we integrate out the chiral multiplets for $-\tilde{m} < \phi < \tilde{m}$ (where the two “effective real masses” have the same sign), we will generate a Chern-Simons term for the gauge field (with coefficient one), that will give the gauge field a mass and lift this part of the Coulomb branch. On the other hand, if we integrate out the chiral multiplets for $\phi > \tilde{m}$ or for $\phi < -\tilde{m}$, the same diagrams generate a Fayet-Iliopoulos term for the gauge field (as discussed in §2.4), proportional to $\tilde{m}$. Thus, these regions of the Coulomb branch are also lifted, and at most we can remain with discrete vacua near $\phi = \pm \tilde{m}$. We
can perform the same analysis also in the dual theory (3.2). Here, a background $U(1)_A$ vector field corresponds to giving a real mass to $M, V_+$ and $V_-$, so only a single vacuum remains at the origin of moduli space, in agreement with the analysis of the original theory. A similar analysis may be performed for combinations of real mass and Fayet-Iliopoulos terms.

For $N_f > 1$, a similar analysis using the symmetries (3.4) gives (in a convenient normalization)

$$W = -N_f (V_+ V_- \det M)^{1/N_f}.$$  

(3.3)

This is reminiscent of the superpotentials found for $d = 4 \ N = 1$ SQCD theories with $N_f > N_c + 1$ [26]. This superpotential describes the moduli space correctly away from the origin and behaves in the expected way when we add (complex) masses for some of the quarks. However, (3.3) is singular at the origin of the moduli space, indicating the presence of new degrees of freedom there. This is also indicated by the fact that the fields $V_\pm$ and $M$ do not satisfy the parity anomaly matching conditions for even values of $N_f$.

We expect, again, that there is an interacting fixed point at the origin. A non-singular description of that fixed point would have to include additional fields. In the next section, we propose another theory which flows to the same fixed point.

Again, we can add “real mass” terms for the chiral multiplets. In principle, we can add independent real masses for all the fields $Q^i$ and $\tilde{Q}_i$. However, as discussed above, if we do not give real masses of equal magnitude and opposite signs to $Q^i$ and $\tilde{Q}_i$, CP is broken, and Fayet-Iliopoulos or Chern-Simons terms are generated at one-loop when integrating out the fermions. To avoid this, we discuss only the case where $Q^i$ ($\tilde{Q}_i$) has real mass $\tilde{m}_i$ ($-\tilde{m}_i$). Only the $N_f - 1$ relative mass terms are physically significant, as the average real mass can be absorbed by a shift in $\phi$. For simplicity, consider the theory with $N_f$ different real masses $\tilde{m}_i$. Classically, there are $N_f$ one dimensional Higgs branches, parameterized by the diagonal elements of $M^i_\i$, which intersect the Coulomb branch (parameterized by $\Phi$) at $\phi = \tilde{m}_i$.

At each such intersection point we have a $U(1)$ theory with $N_f = 1$, so, as discussed above, the Coulomb branch splits in the quantum theory, and the complete moduli space looks like figure 2. As before, near each intersection point there are fields $V_i, \pm$, which semi-classically behave as $e^{\pm (\Phi - \tilde{m}_i)/g^2}$, with a superpotential of the form $-\sum_{i=1}^{N_f} M^i_1 V_i, + V_i, -$. This description involves two fields in each of the middle ($N_f - 1$) regions of the Coulomb branch, while we expect only one – the semi-classical identifications suggest that we should
identify $V_{i,+}V_{i+1,-} = 1$ (up to a constant), which is consistent also with the global sym-
metries. The theory with $N_f$ quarks with different real masses is then described by

$$W = - \sum_{i=1}^{N_f} M_i^2 V_{i,+} V_{i,-} + \sum_{i=1}^{N_f-1} \lambda_i (V_{i,+} V_{i+1,-} - 1), \quad (3.4)$$

where the $\lambda_i$ are Lagrange multipliers. As discussed in §2.3, the superpotential cannot
depend on the real masses $\tilde{m}_i$, but we expect the Kähler potential to depend on the
$\tilde{m}_i$ in such a way that we recover the previous description when the $\tilde{m}_i$ are zero (or
equal). Indeed, starting with (3.4) and integrating out the fields $V_{i<N_f,+}$ and $V_{j>1,-}$
yields a superpotential of the form $W = -N_f (V_{1,-} V_{N_f,+} + M_1^2 M_2^2 \cdots M_{N_f}^2)^{1/N_f}$; this agrees
with (3.3) except for the absence of the off-diagonal mesons. This may be relevant for
understanding the additional degrees of freedom needed at the origin for $N_f > 1$.

4. Mirror symmetry for $U(1)$ gauge theories

$N = 4$ $U(1)$ gauge theory with $N_f > 1$ charged hypermultiplets has a RG fixed point
with a “mirror” dual description in terms of an $(U(1)^{N_f})/U(1)$ gauge theory with $N_f$
hypermultiplets $q_i$ of charge 1 under the $i$'th $U(1)$ and charge $-1$ under the $(i + 1)$'th
$U(1)$ (in cyclic order, with the sum of the $N_f$ $U(1)$’s ungauged) [3]. The $N = 4$ vector
multiplet includes an $N = 2$ vector multiplet and an $N = 2$ chiral multiplet $\Psi$. Giving a
mass to $\Psi$ breaks $N = 4$ to $N = 2$. The low energy theory is then the $N = 2$ SQED theory
discussed in the previous section. Duality is preserved under the RG flow, so mapping
the $N = 4$ breaking mass term to the dual theory gives a dual description of the low
energy $N = 2$ theory. Because these $N = 2$ duals came from the $N = 4$ mirror symmetry
of [3], we might still refer to them as “mirror symmetry.” (It is a misnomer, though:
Unlike the $N = 4$ case, where there are $SU(2)_{R_1} \times SU(2)_{R_2}$ global symmetries which provide an invariant way to distinguish between the Higgs and Coulomb branches, which are exchanged under mirror symmetry, in $N = 2$ theories there is no invariant distinction between the Higgs and Coulomb branches.) In principle, all $N = 4$ mirror symmetries can be turned into $N = 2$ mirror symmetries in this way\footnote{In the brane description of the mirror symmetry \cite{3}, the operation of adding a mass to the adjoint chiral superfield corresponds to rotating one of the NS 5-branes by ninety degrees in two planes (say, the 48 plane and the 59 plane in the conventions of \cite{3}), an operation which can be performed while preserving $d = 3 \ N = 2$ supersymmetry \cite{27}. Similar issues were recently discussed in \cite{28}. This leads to a brane configuration which is T-dual to the configurations discussed in \cite{29} and $SL(2, \mathbb{Z})$ duality in this configuration leads to mirror symmetry in $N = 2$ theories, as recently discussed also in \cite{30}.}.

Because the chiral multiplet $\Psi$ is neutral, there are two ways to give it a mass. We can either add a term $m\Psi^2$ to the superpotential, or pair it with another singlet field $S$ which also gets a mass by a superpotential term $mS\Psi$. At low energies, either choice will give the $N = 2$ theories discussed in the previous section, and the latter is simpler. In the superpotential $W = S\Psi$ the field $S$ acts as a dynamical Fayet-Iliopoulos term and is mapped by mirror symmetry \cite{3} to a dynamical mass term, $W = S \sum_{i=1}^{N_f} q_i \tilde{q}^i$. Including the $N_f - 1$ neutral chiral fields in the $U(1)^{N_f - 1}$ vector multiplets and their superpotential couplings to the matter fields, the $N = 2$ superpotential of the dual is now

$$W = \sum_{i=1}^{N_f} S_i q_i \tilde{q}^i,$$

where the $q_i$ are charged under the $U(1)^{N_f - 1}$ as discussed above. This theory gives a dual description of the $N = 2$ SQED with $N_f$ flavors RG fixed point discussed in the previous section. (A similar proposal appeared in \cite{30} via a brane construction.)

Let us compare the Coulomb branch of the original theory with the Higgs branch of the dual. In the previous section we saw that the Coulomb branch splits into two distinct regions, parameterized by $V_\pm$. The Higgs branch of the mirror theory, solving the classical D-term and F-term equations, is similarly given by two distinct regions, parameterized by the gauge invariant operators $N_- = q_1 q_2 \cdots q_{N_f}$ and $N_+ = \tilde{q}^1 \tilde{q}^2 \cdots \tilde{q}^{N_f}$, which intersect at $N_- = N_+ = 0$. Thus, we are led to identify $N_\pm$ with $V_\pm$. In the mirror theory, the quantum splitting of the Coulomb branch is visible already at the classical level.
The Higgs branch of the original SQED theory is obtained from the Coulomb branch of the mirror along with the $N_f$ chiral singlets $S_i$. In the original theory, the $2N_f - 1$ dimensional Higgs branch is parameterized by the mesons $M_{ij}$ subject to the classical constraints $M_{ij}M_{ik}^\dagger = M_{ik}^\dagger M_{ij}$, and intersects the Coulomb branch at $M_{ij}^\dagger = V_{\pm} = 0$. In the mirror theory, this moduli space arises from the $N_f$ singlets $S_i$ and the $N_f - 1$ dual photons, and classically it intersects the Higgs branch when all these variables, as well as $N_{\pm}$ vanish. Thus, the two theories reproduce the same moduli space.

In fact, we can now identify the operators $M_{ij}$ parameterizing the Higgs branch in the original theory with the operators parameterizing the mirror Coulomb branch. It follows from the map between mass terms of $N = 4$ SQED and Fayet-Iliopoulos terms in the mirror [3] that the diagonal elements of $M_{ij}$ of the $N = 2$ SQED theory are mapped to the $S_i$ of the dual, for $i = 1 \ldots N_f$. The map between the remaining components of $M_{ij}$, subject to the classical constraint, and the Coulomb branch moduli of the $U(1)^{N_f - 1}$ dual is a bit more difficult to obtain. As in the previous section, we identify the two fields $W_{i,\pm}$ corresponding to the photon of the $i$'th $U(1)$. These fields are mapped under the duality as $W_{i,+} = M_{i-1}^\dagger$ and $W_{i,-} = M_{i-1}^\dagger$. The $i$'th $U(1)$ has two charged fields with masses $S_{i-1}$ and $S_i$. Adding these mass terms to the analog of (3.3) for the $i$'th $U(1)$ and integrating out the massive matter yields the quantum relation $W_{i,+}W_{i,-} = S_iS_{i-1}$; this corresponds to a classical constraint in the original theory. The remaining mesons $M_{ij}$ can be identified using the permutation symmetry of the electrons, which acts in the dual theory via linear transformations on the $U(1)$ gauge fields. The remaining classical constraints on the mesons then follow from semi-classical relations involving products of $W_{i,\pm}$ and $W_{j,\pm}$.

This dual description can be connected with our results of the previous section. For example, consider the $N_f = 2$ case and flow to $N_f = 1$ by adding an operator $mM_2^2$ to the original theory. The mirror is then an $N = 2$ $U(1)$ with a superpotential $W = S_1q_1q_1 + S_2q_2q_2 + mS_2$. The equation $\partial W/\partial S_2 = 0$ causes the gauge symmetry to be broken, leaving three one-dimensional branches, parameterized by the gauge invariant operators $S_1, q_1q_2$ and $q_1^2q_2^2$. These were identified above with $M_{1,1}, V_+$ and $V_-$ in the original theory, and the superpotential in terms of these variables reproduces our result (3.2).
5. The Vortex Interpretation

\( N = 4 \) and \( N = 2 \) supersymmetric theories have semi-classical vortex solutions which can be BPS saturated. An understanding of these states provides additional insight into the physics of mirror symmetry in these theories.

Consider \( N = 2 \) theories of \( U(1) \) with \( N_f \) flavors, or their mirrors with products of \( U(1) \) gauge symmetries. In the presence of a real Fayet-Iliopoulos term, there are special vacua where only one of the expectation values for the \( Q^i, \tilde{Q}^i \) is non-vanishing; thus all holomorphic invariants of charged fields are zero. In such a vacuum, there are Nielsen-Olesen vortex solutions \cite{31} in which the non-vanishing expectation value winds at spatial infinity, as does the gauge field:

\[
Q \sim \sqrt{\zeta} e^{\pm i \theta}; \quad A_\theta \sim \pm \frac{1}{r}.
\] (5.1)

The covariant derivative \( D_\theta Q \) falls off faster than \( 1/r \) and so does not contribute a logarithmic divergence to the mass of the vortex solution. The BPS charge of this state is given by the winding number \( \pm 1 \) of the gauge field times \( \zeta \), and in this vacuum the BPS bound is satisfied. The existence of a BPS charge is directly tied \cite{12} to the existence of a Bogomolny bound \cite{32}.

However, other vacua on the Higgs branch do not have BPS vortices. In these more general vacua the vortex solutions still exist but do not saturate the BPS bound. This is possible because the vortices are not in small multiplets.

In \( N = 4 \) theories, there is an \( SU(2)_R \) triplet of Fayet-Iliopoulos terms which, in terms of \( N = 2 \), may be taken as a real parameter \( \zeta_r \) and a complex parameter \( \zeta_c \). Again vortices satisfy the BPS bound only at special points on the Higgs branch.

The BPS charge is easily seen to be proportional to the \( U(1)_J \) shift charge:

\[
Z = \zeta \int r d\theta A_\theta = \zeta \int d^2 x \epsilon^{0\mu\nu} F_{\mu\nu} = \zeta \int d^2 x \, j^0.
\] (5.2)

This means that on the Higgs branch there are massive vortex states of non-zero \( J \) charge. Since the only holomorphic operators carrying this charge are \( V_\pm \sim e^{\pm \Phi} \), we may expect

\[\text{As usual, there are solutions with any value of the winding number, but the multi-winding vortices are neutrally stable with respect to states with multiple vortices with winding number one.}\]

\[\text{5 These statements are related by dimensional reduction to similar statements about instantons in two dimensions} \cite{33} \text{and strings in four dimensions} \cite{34} \text{in Abelian gauge theories.}\]
that the vortices are related to them. In fact, if we take $\zeta$ to zero (so that the special vacuum where the vortices are BPS saturated becomes the origin of the Coulomb branch) we may then move smoothly onto the Coulomb branch, where $U(1)_J$ is spontaneously broken by expectation values for $V_\pm$. This leads to the natural interpretation that the Coulomb branch is associated with vortex condensation — the Higgs branch of the vortex theory.

The exchange of Higgs and Coulomb branches is characteristic of $N = 4$ mirror symmetry, which also implies that the masses of the mirror states should be given by the Fayet-Iliopoulos terms in the electron theory. This is true precisely for the BPS vortices, and it is therefore natural to interpret mirror symmetry (and its $N = 2$ analogue) as an exchange of electron and vortex descriptions.

Let us check this interpretation for $N = 4$ $U(1)$ with $N_f = 1$. For non-zero $\zeta$ this theory has a unique vacuum, in which the vortex $v_+$ and anti-vortex $v_-$ are BPS saturated and have mass $\zeta$. We may write an effective theory for these fields which, in terms of $N = 2$, includes kinetic terms, a real mass $\zeta_r$ for the vortices, and a superpotential $W = \zeta_c v_+ v_-$. (5.3)

Note that the $N = 4$ multiplet consisting of $v_\pm$, whose mass is $\sqrt{\zeta_r^2 + |\zeta_c|^2}$ is BPS saturated. It is not BPS saturated in $N = 2$.

To go from this to $N = 2$ with one flavor, we add a singlet $S$ and couple it to the field $\Psi$ through the superpotential $W = S \Psi$. This has the effect of replacing the complex Fayet-Iliopoulos term $\zeta_c$ with $S$. In addition the equation of motion for $\Psi$ sets $S = -Q \tilde{Q} = -M$. Identifying $V_\pm$ as $v_\pm$ and making these substitutions in (5.3), we recover our result (3.2). Here, the vortices are only BPS saturated at the origin of the Higgs branch, where $M = 0$. Indeed, the result (3.2) implies that for non-zero $M$ the vortex masses are given semiclassically (in the vortex effective theory) by $\sqrt{\zeta^2 + |M|^2}$ and therefore must be larger than $\zeta$ (even when quantum corrections are included.)

Next consider $N_f = 2$ for $N = 2$ supersymmetry; one can easily check that all of the following results can be lifted to the $N = 4$ case as well. The electron theory has mesons $M^i_j$ which form a $(2, 2)$ of the $SU(2) \times SU(2)$ flavor symmetry. The vortices are as described above, and correspond to the operators $V_\pm$. The new feature of this theory is that its mirror has a photon and a corresponding dual scalar $\tilde{\gamma}$. A shift in $\tilde{\gamma}$ is a global symmetry.

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*We emphasize that this photon is not the magnetic dual of the photon which couples to electrons – it is a physically distinct field.*
$U(1)_j$, which should correspond to a global symmetry in the electron theory. From \[3\] and the previous section, the generator of this global symmetry is $T_3$ in the diagonal $SU(2)$ subgroup of the $SU(2) \times SU(2)$ flavor symmetry. This generator rotates $M_1^2$ and $M_2^1$ in opposite directions (leaving $M_i^i$ unchanged) and so we identify these operators with $W_\pm \sim e^{\pm(i\tilde{\phi} + \gamma)}$. We can then see that the $SU(2) \times SU(2)$ flavor symmetry, which is not visible classically in the mirror theory, will appear in the far infrared in the following way: the fields $M_1^1$ and $M_2^2$ (of dimension $\frac{1}{2}$ in the classical mirror theory) will combine in the far infrared with the (classically dimensionless) fields $W_\pm$ to form a $(2, 2)$ representation. This type of realization of a hidden symmetry is analogous to many examples known in other dimensions.

We may also analyze the semi-classical vortices of the mirror theory to recover states containing the original electrons. The analogous arguments lead one to identify $V_\pm$ with bilinears of the mirror theory. Also, they show that the mesons $M_j^i$ represent BPS saturated states at any point on the Coulomb branch at which $Q^i$ is massive but $\tilde{Q}_j^i$ is not. One can think of $M_j^i$ as an electron $Q^i$ of bare mass $\tilde{m}_i$ which is dressed by a massless positron $\tilde{Q}_j^i$ to eliminate its logarithmic divergence.

As we mentioned earlier, these statements can easily be lifted back to the $N = 4$ theory, in which the hidden $SU(2)$ flavor symmetry of the mirror is realized in the far infrared by the triplet $V_+, \Psi, V_-$. For higher $N_f$ in either $N = 2$ or $N = 4$, the discussion is very similar. One can identify all of the scalars dual to the photons of the mirror theory as phases of off-diagonal mesons $M_i^{i+1}$. The identification of the BPS vortex $V_-$ with $q_1 q_2 \cdots q_{N_f}$ can again be made by considering the mirror gauge theory with a real mass $\zeta$; $q_1 q_2 \cdots q_{N_f}$ creates a state which is BPS saturated at the points on the Coulomb branch where there is only one massive field $q_i$, which when dressed with all the other $q_j$ forms a state of finite energy $\zeta$.

For any $N_f > 1$, one may add real masses to the original theory, along with a Fayet-Iliopoulos term. For unequal masses this leaves the theory with $N_f$ Higgs branches, each with a special vacuum where a vortex and corresponding anti-vortex are BPS saturated.

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7 A state with a single electron, which has divergent energy due to its electric field $F^{0r} = \partial_\theta \gamma = q/r$, causes $\gamma$ to wind at infinity: $V_+ \sim e^{iq\theta}$. The mirror of this statement is that a charged mirror field causes the phase of $W_+ = M_2^2$ to wind at infinity. This is precisely the winding one would expect for a global semi-classical vortex, which indicates a connection between global vortices and the charged fields in the mirror theory, a linkage already suggested by the brane picture of [34] and [3].
These vortices can be associated to the $2N_f$ operators $V_{i,\pm}$. In the mirror, all gauge symmetries are broken, and the fields $q_i, \tilde{q}^\dagger_j$ are gauge singlets which can be related directly to the $V_{i,\pm}$.

In summary, the mirror symmetries exchange Nielsen-Olesen vortices with logarithmically confined states of mirror electrons; similarly the confined states of the original theory appear as Nielsen-Olesen vortices of the mirror. The exchange of Higgs and Coulomb branches, masses and Fayet-Iliopoulos terms, etc., is natural in this language. Furthermore, in both the $N = 4$ and $N = 2$ $U(1)$ theories, the existence of hidden symmetries realized only in the far infrared predicts the presence of certain operators with certain charges, which we have identified as the chiral operators $V_{\pm}, W_{i,\pm}$. Here, we have understood these operators as associated with semi-classical Nielsen-Olesen vortices.\footnote{Further checks on this picture can be provided using the brane construction of\cite{footnote}, and one finds general consistency. However, interpretation of this construction is subtle, as no sign of the quantum corrections associated with logarithmic confinement is to be found at the level of classical branes.}

6. $SU(2)$ gauge theory

For $SU(2)$ gauge theories the Coulomb branch is parameterized by the expectation value of $\Phi$, whose scalar component is $\phi + i\gamma$ with $\phi \in \mathbb{R}^+$ and $\gamma \simeq \gamma + g^2$ (at tree level). As discussed in sect. 2.5, the natural coordinate on the Coulomb branch is $Y$, given semi-classically by $e^{\Phi/g^2}$.

To avoid having to include a Chern-Simons term, we will discuss theories with an even number, $2N_f$, of chiral multiplets $Q^i$ in the 2 representation. For $N_f > 0$ the classical moduli space also has a $4N_f - 3$ complex dimensional Higgs branch which attaches to the Coulomb branch at $\phi = 0$. As in 4d, the Higgs branch can be labeled by the mesons $M_{fg} = Q^e_f Q^d_g \epsilon_{cd}, f, g = 1, \ldots, 2N_f$. For $N_f \geq 2$ the $M_{fg}$ are classically constrained by $\text{rank}(M) \leq 2$, i.e. $\epsilon^{i_1 i_2 \cdots i_{2N_f}} M_{i_1 i_2} M_{i_3 i_4} = 0$ (which for $N_f = 2$ is just Pf $M = 0$).

The quantum corrections to the classical moduli space are constrained by the global flavor symmetries, which are:

$$
\begin{align*}
\begin{array}{ccc}
U(1)_R & U(1)_A & SU(2N_f)_F \\
Q & 0 & 1 \\
M & 0 & 2 \\
Y & 2N_f - 2 & -2N_f \\
\end{array}
\end{align*}
$$

(6.1)
The charges assigned to $Y$ follow from one-loop perturbative effects, similar to those described in §2.4. They are consistent with the fermion zero modes of an instanton on the Coulomb branch, which is weighed by $Y^{-1}$: there are two gluino zero modes and $2N_f$ quark zero modes, $\psi_{i=1...2N_f}$.

6.1. Pure Yang-Mills – no supersymmetric vacuum

This is the theory analyzed in [15]. The instanton has two zero modes, which is the correct number to generate a superpotential, and indeed the instantons generate a superpotential

$$W = \frac{1}{Y}. \quad (6.2)$$

(We do not explicitly write an overall scale needed on dimensional grounds.) This superpotential is exact, as it is the unique result which respects the global $U(1)_R$ symmetry (6.1). The classical degeneracy of the Coulomb branch is lifted by (6.2) and the theory has no vacuum. This is reminiscent of the $d = 4$ $SU(2)$ gauge theory with $N_f = 1$, where there is no Coulomb branch but a similar superpotential is generated on the Higgs branch [35].

6.2. $N_f = 1$: quantum merging of Higgs and Coulomb branches

When the quark flavor is massless, there is classically a one complex dimensional Higgs branch, labeled by the expectation value of the meson $M = Q_1Q_2$, which intersects the one complex dimensional Coulomb branch at the origin, $M = \phi = 0$. It is impossible to describe this in terms of $M$ and $Y$ in a holomorphic way.

Because all fields in (6.1) are neutral under $U(1)_R$, it is impossible to form a superpotential; therefore $W = 0$ and there is an exactly degenerate quantum moduli space of vacua.

The quantum moduli space of vacua, however, differs from the classical moduli space because of quantum effects associated with instantons. The quantum moduli space of vacua is given by the fields $M$ and $Y$ subject to the constraint

$$MY = 1. \quad (6.3)$$

With this constraint, (6.2) is reproduced at low energy upon adding $W = m_QM$ (and matching the scales in an appropriate way). The space (6.3) is a smooth quantum deformation of the singular classical moduli space. This is analogous to what happens in $d = 4$, $N = 1$ supersymmetric $SU(2)$ theories with $N_f = 2$ flavors [26].
To summarize, classically there are two distinct branches, Higgs and Coulomb, which intersect at the origin. Quantum-mechanically, they smoothly merge near the origin to form a single branch of moduli space subject to the constraint (6.3). The results derived here for this theory seem different from the solution proposed in [36].

6.3. $N_f \geq 2$ : moduli space with non-trivial RG fixed points at the origin

For $N_f = 2$, there is a unique superpotential consistent with holomorphy and the symmetries with the charges (6.1):

$$W = -YPf M.$$  \hfill (6.4)

This reproduces the quantum constraint (6.3) upon adding a mass term $W = mM_{34}$ and integrating out the massive fields. The theory without mass terms has a quantum moduli space of vacua given by expectation values of $M_{fg}$ and $Y$ subject to constraints coming from the equations of motion of (6.4):

$$Pf M = 0, \quad YM_{fg} = 0.$$  \hfill (6.5)

This is analogous to the situation for the $d = 4$ $SU(2)$ theory with $N_f = 3$ [26]. There are thus distinct Higgs and Coulomb branches, with $M \neq 0$ and $Y = 0$ on the Higgs branch and $Y \neq 0$, $M = 0$ on the Coulomb branch. This is similar to the classical moduli space, though now the branches touch at the origin, $M = Y = 0$, rather than at $M = \phi = 0$. Because (6.4) is of degree three in the fields $Y$ and $M_{fg}$, as discussed in sect. 2.1, this theory flows to an interacting RG fixed point. We argue that the original $SU(2)$ theory with $N_f = 2$ flavors flows to the same fixed point.

As a (rather weak) check that the original $SU(2)$ with $N_f = 2$ theory and the theory with fields $M_{fg}$ and $Y$ with superpotential (6.4) flow to the same fixed point, we note that their parity anomalies match. For the $U(1)_R \times U(1)_A$ part, in both theories we have $k_{RR} \in \mathbb{Z} + \frac{1}{2}$, $k_{AA} \in \mathbb{Z}$, and $k_{RA} \in \mathbb{Z}$. For the $SU(4)$, in both the original theory, which has two fields in the 4, and the dual, which has the field $M$ in the 6, there is no parity anomaly, $k_{SU(4)} \in \mathbb{Z}$.

For $N_f > 2$, the symmetries (6.4) determine the superpotential to be (with a convenient normalization)

$$W = -(N_f - 1) (YPf M)^{1/(N_f - 1)}.$$  \hfill (6.6)

The branch cut singularity is analogous to the effective superpotentials found in $d = 4$, $N = 1$ SUSY for $N_f \geq 4$. As in the $d = 4$ case [26], we interpret the singularity at the origin in (6.6) as a signature of important new degrees of freedom there. We expect that there is an interacting RG fixed point at the origin.
6.4. Adding real mass terms

Next, we consider adding real mass terms for the quarks. Unlike the U(1) case, we cannot absorb any such mass terms by shifts of the scalar field \( \phi \) in the vector multiplet. To preserve CP, whenever we give one doublet a real mass \( \tilde{m} \) we should give another doublet a real mass \( (-\tilde{m}) \). Otherwise, Chern-Simons and/or Fayet-Iliopoulos terms will be generated along the Coulomb branch, as we discussed in the U(1) case. For simplicity, we only discuss the case of equal masses \( \tilde{m} \) for \( N_f \) doublets, and masses \( (-\tilde{m}) \) for the other \( N_f \) doublets. Other cases may be obtained by a combination of the discussion here and the discussion of the previous section on U(1) theories with different real masses.

These real mass terms break the global flavor symmetry from \( SU(2N_f) \times U(1)_A \) to \( SU(N_f) \times SU(N_f) \times U(1)_A \times U(1)_B \) (they correspond to a background vector field in a \( U(1)_B \) subgroup of \( SU(2N_f) \)). The mesons which can obtain expectations values for \( \tilde{m} > 0 \) are now given by \( M_{ij} = Q_i \tilde{Q}_j \), where the \( Q_i \) are the doublets with positive real mass and the \( \tilde{Q}_j \) are the doublets with negative real mass term. The Higgs branch is now \( 2N_f - 1 \) dimensional, and classically it intersects the Coulomb branch at \( \phi = \tilde{m} \).

The region near \( \phi = \tilde{m} \) looks like a U(1) theory with \( N_f \) massless flavors. Our previous analysis thus shows that the Coulomb branch splits into two regions, parameterized by \( V_\pm \) with, at the perturbative level, a superpotential of the form \( W = -N_f (V_+ V_- \det M)^{1/N_f} \). In the semi-classical regimes, \( V_+ \sim e^{(\Phi - \tilde{m})/g^2} \) (for \( \phi \gg \tilde{m} \gg 0 \)) and \( V_- \sim e^{(\tilde{m} - \Phi)/g^2} \) (for \( \tilde{m} \gg \phi \gg 0 \)).

There can be additional instanton contributions to the superpotential. Semi-classically, for quarks of real mass \( \tilde{m} \), there are zero modes in the background of the instanton if and only if \( \phi > |\tilde{m}| \) [23]. Thus, we expect that in the region \( \phi < |\tilde{m}| \) an instanton will have only the two gluino zero modes and can contribute to the superpotential. For \( \phi > |\tilde{m}| \) because of the additional quark zero modes, there will be no instanton contribution to the superpotential. Without incorporating the splitting of the Coulomb branch described in fig. 1, these statements seem paradoxical (since the superpotential is holomorphic). Taking the splitting into account, the instanton contribution to the superpotential is simply given by an additional term \( W_{\text{inst}} = V_- \), which is indeed non-zero only in the region that classically corresponded to \( \phi < \tilde{m} \). Combining with the perturbative superpotential written above, yields the superpotential

\[
W = -N_f (V_+ V_- \det M)^{1/N_f} + V_-.
\] (6.7)
(Again, the normalization is chosen for convenience.) Upon taking into account the one-loop corrections to the global charges of the $V_\pm$ fields, discussed in §2.4, (6.7) is consistent with the global symmetries.

Consider, for example, the case of $N_f = 1$. Then (6.7) is $W = V_-(1 - V_+ M)$, which is similar to our previous description of this case (eqn. (6.3)) except that $V_-$ has become a dynamical field instead of a Lagrange multiplier. The moduli space is the same as in the massless case, and we expect the mass of $V_-$ to depend on $\tilde{m}$ such that it is massless as $\tilde{m} \to \infty$. Adding a complex mass $W = mM$ and integrating out the quarks, we find a constraint $V_+ V_- = m$ and a superpotential $W = V_- \sim 1/V_+$, both as expected for the low energy theory with $N_f = 0$. For higher $N_f$, we can no longer smoothly connect (6.7) to the Lagrangian with no real masses, since some degrees of freedom (corresponding to mesons of the form $Q_i Q_j$ and $\tilde{Q}_i \tilde{Q}_j$) become massless only when $\tilde{m} = 0$, and do not appear in our Lagrangian for the massive case. The qualitative behavior of the theory is, however, still the same as in the massless case. There is a one dimensional Coulomb branch parameterized by $Y = V_+$ which can take any value, which we identify with the semi-classical region of $\phi > \tilde{m}$, and a Higgs branch which intersects it at $Y = 0$.

7. $SU(2)$ theories from four to three dimensions

It is interesting to interpolate between the above results and those of $d = 4 \ N = 1$ theories by considering these theories on a circle of radius $R$. The scalars in the vector multiplet now live on a circle of radius $1/R$, and the $d = 4$ and $d = 3$ couplings are related classically by $1/g_3^2 = R/g_4^2$. In this section we discuss the superpotential for finite values of $R$. We can relate this to the $d = 4$ results by taking $R \to \infty$. In order to keep $g_4$ fixed, this means we need to take $g_3 \to 0$, and both circles corresponding to the Coulomb branch shrink to zero size in this limit. Thus, we should integrate out the fields corresponding to the Coulomb branch in order to get the $d = 4$ results. For any radius $R > 0$, the $d = 4$ instantons break the $U(1)_R$ symmetry and, for $N_f > 0$, also the $U(1)_A$ symmetry – but a linear combination of the two is preserved.
7.1. $SU(2)$ with $N_f = 0$

This case appeared in [24], where it was connected with the results for $N = 4$ theories by breaking to $N = 2$. The leading term associated with finite radius was found to modify (5.2) as

$$W = \frac{1}{Y} + \eta Y,$$

where $\eta \sim e^{-1/Rg_s^2}$ is the four dimensional instanton action, which in the four dimensional limit becomes $\eta \sim e^{-1/g_s^2} \sim \Lambda_4^{3N_c - N_f}$ (in our case this is simply $\Lambda_4^6$). However, we can also understand the second term directly, as arising from $d = 3$ instantons which are related by a large gauge transformation around the compact circle to the usual instantons. $d = 4$ instantons have too many zero modes in this case so they do not contribute to the superpotential. For finite $R$, the superpotential (7.1) has the two vacua expected in the $d = 4$ theory. In the $R \to \infty$ limit, we can integrate out $Y$ and get $W \sim \Lambda_4^3$, corresponding to $d = 4$ gaugino condensation. In the 3d limit $R = \eta = 0$, the two vacua run off to $Y \to \infty$, and the theory has no vacuum.

7.2. $SU(2)$ with $N_f > 0$

In all these theories, the $d = 3$ instanton with twisted boundary conditions around the circle still has no quark zero modes, so it will always contribute to the superpotential a term $W_\eta = \eta Y$, which will lift the Coulomb branch. Thus, for $N_f = 1$, the theory is described by the superpotential

$$W = \lambda(MY - 1) + \eta Y,$$

where $\lambda$ is a Lagrange multiplier to impose the constraint (6.3). For $N_f = 1$ only, the $d = 4$ instantons can also contribute to the superpotential (if two gluino zero modes and two quark zero modes are lifted together), and the global symmetries force their contribution to be of the form $W = \eta/M$, but this is identified with the second term in (7.2) using the constraint so we do not have to take it into account separately. Upon integrating out $Y$ by imposing the constraint, (7.2) leads to $W \sim \eta/M$, as in the four dimensional theory [33]. The theory has no supersymmetric vacua for any $R > 0$.

---

9 This is obvious in the brane construction of these theories.
As for $N_f = 0$, we can find this result also by starting with the curve describing the $N = 4$ theory and giving a mass to the adjoint chiral multiplet [24]. For $N_f = 1$, in order to repeat the computation of [24], we start with the superpotential

$$W = \lambda(y^2 - x^3 + x^2 u - (\Lambda_{N=2}^4)^6) + \epsilon u,$$  \hspace{1cm} (7.3)

which incorporates the curve as a constraint, and includes a mass $\epsilon$ for the adjoint field $u$. Integrating out $\lambda, u$ and $y$ as in [24], we find

$$W = \epsilon((\Lambda_{N=2}^4)^3 x + 1/x^2).$$  \hspace{1cm} (7.4)

Now, identifying $\epsilon/x$ with $e^{-\Phi/g^2}$ as in [24], and relating the $d = 4$ scales in the usual way by $(\Lambda_{N=1}^4)^5 = (\Lambda_{N=2}^4)^3\epsilon^2$, we find the three dimensional superpotential to be

$$W = e^{-1/Rg^2 + \Phi/g^2} + \frac{1}{\epsilon}e^{-2\Phi/g^2},$$  \hspace{1cm} (7.5)

and the last term vanishes in the limit $\epsilon \to \infty$ leaving us with the second term of (7.2). The same result may be obtained also for higher values of $N_f$. By using only the Coulomb branch description of the $N = 4$ theory we cannot obtain in a simple way the correct dependence on the meson fields.

For $N_f = 2$, the theory is described by the superpotential

$$W = -YPf M + \eta Y.$$  \hspace{1cm} (7.6)

Now, upon integrating out $Y$, (7.6) gives the quantum-modified moduli space constraint of [26]: $Pf M \sim \eta$. Note that again we find that the $d = 4$ instanton result is reproduced by $d = 3$ instantons. For finite $R$ this constraint is implemented as the equation of motion of $Y$, and $Y = 0$ in the vacuum.

For $N_f > 2$, adding $W_\eta = \eta Y$ to (6.6) and integrating out $Y$ leads to

$$W \sim (\eta^{-1}Pf M)^{1/(N_f-2)},$$  \hspace{1cm} (7.7)

which is the correct effective potential for the 4d theories [26]. Again, the moduli space for finite $R$ is the same as in the $d = 4$ theory, with $Y = 0$ at the vacuum.
7.3. $SU(2)$ with real masses on a circle

Real masses are also possible in the four dimensional theory compactified on a circle. In this case they should be thought of as Wilson loops of background vector fields, instead of VEVs of their scalar components, so the masses also naturally live on a circle of radius $1/R$.

Let us begin with the $N_f = 1$ case, with a real mass $\tilde{m}$ of opposite sign for the two doublets. Semi-classically, we can take $0 \leq \phi, \tilde{m} \leq 1/2R$. The standard instanton (with action $e^{-\Phi/g^2}$) has quark zero modes if and only if $\phi > \tilde{m}$, while the twisted boundary condition instanton (with action $e^{-(1/R-\Phi)/g^2}$) has quark zero modes if and only if $\phi < \tilde{m}$. Using our previous description for the $N_f = 1$ case with a real mass, we obtain that the superpotential on a circle is

$$W = -V_-V_+M + V_- + \eta V_+. \quad (7.8)$$

As before, there are no supersymmetric vacua for any value of $R$.

For $N_f > 1$ with equal real masses, we similarly obtain

$$W = -N_f(V_-V_+ \det M)^{1/N_f} + V_- + \eta V_+. \quad (7.9)$$

The Higgs branch now remains unlifted, with a constraint on $\det M$ for $N_f = 2$ and no constraints for $N_f > 2$. The vacuum obeys $V_- = (\det M \eta^{-1})^{1/(N_f-2)} = \eta V_+$, so the Coulomb branch is lifted. For $R \to \infty$ we can integrate out $V_-$ and $V_+$, and get the correct $d = 4$ limit except for the absence of some of the mesons, which became massive due to the Wilson loop in $d = 3$ but cannot be ignored in the $d = 4$ limit.

If we allow different real masses, some of the Coulomb branch may remain unlifted. For instance, for $N_f = 2$ with two different real masses, the superpotential describing the theory (derived by the same methods as above) is

$$W = -V_1_-V_1_+M_1^1 - V_2_-V_2_+M_2^2 + \lambda(V_1_+V_2_+ - 1) + V_1_- + \eta V_2_+, \quad (7.10)$$

where $\lambda$ is a Lagrange multiplier (as in the $U(1)$ theory with different real masses). In this case a stable vacuum exists, in which $V_1_- = V_2_+ = 0$ and $M_1^1M_2^2 = \eta$ (similar to the constraint we found before for $N_f = 2$). Note that classically the Higgs branches corresponding to $M_1^1$ and $M_2^2$ are disjoint (with the Coulomb branch connecting them), but they merge together in the quantum theory. Similar phenomena occur with more different real masses.
8. $SU(N_c)$ gauge theories with $N_c > 2$

Writing the adjoint scalar in the form $\phi = \text{diag}(\phi_1, \ldots, \phi_N)$, with $\sum_{j=1}^{N_c} \phi_j = 0$, the wedge of the Coulomb branch Weyl chamber can be taken to be $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_{N_c}$. The instanton factors (2.14) can be taken to be $Y_j^{-1}$ with, semi-classically,

$$Y_j \sim e^{(\Phi_j - \Phi_{j+1})/g^2}, \quad j = 1, \ldots, N_c - 1. \quad (8.1)$$

We consider the theories with $N_f$ flavors of $N_c + N_c$; with this choice we can avoid introducing a Chern-Simons term. For $N_f > 1$, in addition to the Coulomb branch, the theory has a Higgs branch with $SU(N_c)$ generically broken to $SU(N_c - N_f)$ for $N_f < N_c - 1$ and completely broken for $N_f \geq N_c - 1$. As in 4d, the Higgs branch can be parameterized by the $N_f^2$ mesons $M_j$ for $N_f < N_c$ or by mesons along with baryons, subject to classical constraints, for $N_f \geq N_c$.

Global symmetries, holomorphy and known limits, as usual [18], powerfully constrain the dynamics. The $j$-th instanton, which is weighted by $Y_j^{-1}$, has two gaugino zero modes. To begin with, we consider the theory with massless flavors, setting the complex and real masses to zero. At a generic point on the Coulomb branch, using the analysis of [23] (easily generalized to $SU(N_c)$ since the instantons are just embeddings of $SU(2)$ instantons), we find that the $j$-th instanton has $2N_f \delta_{j,K}$ quark zero modes, where $K$ is the value which satisfies $\phi_K > 0 > \phi_{K+1}$. The global symmetries and charges of the fields (including the one-loop corrections described in §2.4) are thus

$$
\begin{array}{ccccccc}
& U(1)_R & U(1)_B & U(1)_A & SU(N_f) & SU(N_c) \\
Q & 0 & 1 & 1 & N_f & 1 \\
\bar{Q} & 0 & -1 & 1 & 1 & \bar{N}_f \\
M & 0 & 0 & 2 & N_f & \bar{N}_f \\
Y_{j \neq K} & -2 & 0 & 0 & 1 & 1 \\
Y_K & 2(N_f - 1) & 0 & -2N_f & 1 & 1 \\
Y & 2(N_f - N_c + 1) & 0 & -2N_f & 1 & 1 \\
\end{array}
$$

(8.2)

---

10 It is easy to see the $K$ dependence of the quark zero modes in the brane construction of these theories, which is related by a rotation of one of the NS 5-branes to the construction of [3]. The $\phi_j$’s in this construction are positions of the D3-branes, while the instantons are Euclidean D-strings stretching between them, and they will have quark zero modes for a particular quark (semi-classically) if and only if the corresponding D-string intersects the D5-brane which gives rise to this quark.
Here $Y \equiv \prod_{j=1}^{N_c-1} Y_j \sim e^{(\Phi_1 - \Phi_{N_c})/g^2}$ is included for later convenience.

The dependence on $K$ in (8.2) reflects the fact that the theory with matter is defined in $N_c - 1$ sub-wedges of the Weyl chamber, corresponding to the choice of $K = 1 \ldots N_c - 1$. The fact that the description is not smooth at the $N_c - 2$ boundaries of the sub-wedges, $\phi_j = 0$ (for $j = 2, \ldots, N_c - 1$), is possible because there are massless fields on these boundaries: this is where components of the quarks classically become massless and where the Higgs branch connects to the Coulomb branch. Indeed, along these boundaries of the sub-wedges the low energy theory is governed by a particular $U(1) \subset SU(N_c)$ which has $N_f$ massless flavors. Thus, in accord with our analysis in §3, we expect the Coulomb branch to split along this sub-locus, with different variables describing the regions on either side of the boundary. Altogether, for $N_f > 0$ the bulk of the Coulomb branch splits into $N_c - 1$ regions, with (generally) different variables describing the various regions.

8.1. The pure SYM theory, $N_f = 0$.

For $N_f = 0$ we do not have the splitting described above of the Coulomb branch into sub-wedges. The instanton contributions to the superpotential along the coulomb branch are

$$W = \sum_{j=1}^{N_c-1} \frac{1}{Y_j}. \quad (8.3)$$

Every term in (8.3) is generated by the “fundamental” instantons in the same way as in the $SU(2)$ theory \[15\]. The superpotential (8.3) is not the most general one consistent with the symmetries (8.2). However, a simple analysis shows that in fact (8.3) is exact. First, since there are no interacting massless degrees of freedom on the Coulomb branch, $W$ must be single valued as each $Y_i \to e^{2\pi i Y_i}$. This leaves the possibility of terms like $Y_i/Y_j^2$. However, for every one of these terms, we can find a way to rescale all $\phi_i \to \infty$ consistent with $\phi_1 > \phi_2 > \cdots > \phi_{N_c}$ such that $Y_i/Y_j^2$ diverges. This leaves (8.3) as the only form which is compatible with the symmetries, holomorphy and the asymptotic behavior.

The superpotential (8.3) was obtained in \[38\] by considering M theory on Calabi-Yau fourfolds. For the non-supersymmetric version, the scalar potential corresponding to (8.3) was also obtained long ago by \[39\], generalizing the $SU(2)$ analysis of \[21\].

This theory has no stable supersymmetric vacuum.
8.2. Theories with massless quarks

It follows from (8.2) that there can be instanton contributions to the superpotential for any $N_f$ coming from the $N_c - 2$ instantons $Y_{j\neq K}, j = 1 \ldots N_c - 1$:

$$W_{\text{inst}} = \sum_{j \neq K} \frac{1}{Y_j}, \quad (8.4)$$

which will lift most of the Coulomb branch. Note that the variables $Y_i$ in the different regions (corresponding to different values of $K$) are generally not the same, due to the splitting effect described above. Similar contributions due to “fundamental” instantons will exist in any region of the moduli space where semi-classically there are two eigenvalues of the same sign. Thus, we expect that at most the one complex dimensional sub-locus of the Coulomb branch in which classically

$$\phi_1 > \phi_2 = \cdots = \phi_{N_c-1} = 0 > \phi_{N_c} = -\phi_1 \quad (8.5)$$

may remain unlifted.

In order to have a full description of the moduli space, we need to repeat our procedure in the previous sections. We should first find a superpotential which correctly describes the quantum moduli space before the non-perturbative corrections, and which will have (generally) different variables in each of the $N_c - 1$ regions of the Coulomb branch. Then, we should add to it the instanton corrections of the form (8.4). For the $SU(3)$ case, we will explicitly perform these computations below. For higher $N_c$, the superpotentials describing the quantum-corrected Coulomb branch are complicated, and we have not written them down explicitly.

There is always one instanton factor, corresponding to the sum of the simple roots, $Y = \prod_{i=1}^{N_c-1} Y_i \sim e^{(\Phi_1 - \Phi_{N_c})/g^2}$, which can be globally defined throughout the Coulomb branch. Whenever some part of the Coulomb branch (which will correspond to (8.5)) remains unlifted, it can be parameterized by $Y$. Thus, we can always integrate out all the other fields appearing in the superpotential, and remain with an effective description that includes only the field $Y$ and the fields parameterizing the Higgs branch.

For $N_f < N_c - 1$, (8.2) completely determines the superpotential for these fields to be (with a convenient normalization)

$$W = (N_c - N_f - 1)(Y \det M)^{1/(N_f - N_c + 1)}; \quad (8.6)$$
this is similar to the $d = 4$ superpotentials for $N_f < N_c$. With (8.6), there is no stable vacuum, so our procedure of integrating out only some of the massive fields is not really justified.

For $N_f = N_c - 1$, we obtain a quantum constraint on the moduli space, of the form $Y \det M = 1$, generalizing (6.3). This is similar to the quantum constraints found for $N_f = N_c$ in the $d = 4$ theories. The moduli space is again in a merged Higgs and Coulomb branch, as we found for $SU(2)$ with $N_f = 1$, with the small $\det M$ region of the Higgs branch corresponding to large values of $Y$, far out on the Coulomb branch. In particular, the points with $\det M = 0$, where part of the gauge symmetry would classically be unbroken, are at infinite distance on the Coulomb branch.

For $N_f \geq N_c$, the superpotentials can involve also baryonic operators $B$ and $\tilde{B}$ in addition to (8.6) continued to $N_f \geq N_c$. For $N_f = N_c$, we argue that, much as in (6.4), the theory is described by the fields $Y$, $M^f_i$, $B$, and $\tilde{B}$ with the superpotential

$$W = -Y(\det M - B\tilde{B}). \quad (8.7)$$

Because of the cubic $YB\tilde{B}$ term, this theory flows to a non-trivial fixed point in the IR; the original theory flows to the same fixed point, which seems to be the same fixed point as the one we found for the $U(1)$ theory with $N_f = 1$. One check of this is that the parity anomaly matching conditions discussed in §2.4 are satisfied: it follows from (8.2) and (2.12) and (2.13) that, in both the $N_f = N_c$ SQCD theory and the theory with fields $M^f_i$, $B$, and $\tilde{B}$ with (8.7), $k_{RR} \in \mathbb{Z} + \frac{1}{2}(1 + N_c^2)$, $k_{BB} \in \mathbb{Z}$, $k_{AA} \in \mathbb{Z}$, $k_{SU(N_f)L} \in \mathbb{Z} + \frac{1}{2}N_c$, $k_{SU(N_f)R} \in \mathbb{Z} + \frac{1}{2}N_c$. As another check, (8.7) is compatible with reproducing the correct Higgs branch. In particular, upon Higgsing to $SU(2)$ with $N_f = 2$ it properly reproduces (6.4). Also, (8.7) properly yields (8.6) upon adding mass terms and integrating out flavors. Finally, as will be discussed below, (8.7) properly connects to the 4d result of [26].

We can easily extend this discussion to $U(N_c)$ gauge groups with $N_f \leq N_c$. It is enough to study the case $N_f = N_c$; lower values of $N_f$ can be obtained by integrating out some quarks. Ignoring the $U(1)$ factor, we have the superpotential (8.7). We can now gauge the $U(1)$ factor. In this description the $U(1)$ dynamics is that of $U(1)$ with one flavor: $B$ and $\tilde{B}$. This theory is described by the gauge invariant combination $X = B\tilde{B}$, two more chiral fields $V_{\pm}$ and a superpotential $-XV_+V_-$. Adding this to (8.7) leads to

$$W = -Y(\det M - X) - XV_+V_. \quad (8.8)$$

Integrating out the massive fields $Y$ and $X$ we find

$$W = -V_+V_- \det M. \quad (8.8)$$
For \( N_f > N_c \), we may not be able to write a superpotential that will reproduce the classical Higgs branch, and a different description may be necessary. In any case, for \( N_f \geq N_c \) we expect to remain with a one dimensional Coulomb branch, parameterized by \( Y \), which intersects the Higgs branch (which is the same as the classical Higgs branch, of dimension \( 2N_f N_c - (N_c^2 - 1) \)) at \( Y = 0 \). There are also mixed branches, in which both \( Y \) and some of the mesons acquire VEVs. For instance, for \( N_f = N_c \), even when \( Y \) is non-zero \( M \) can obtain a VEV as long as its rank obeys \( \text{rank}(M) \leq N_c - 2 \), corresponding to the classical condition for having (at least) an unbroken \( U(1) \). We expect the origin of moduli space to correspond to some interacting SCFT for all \( N_f \geq N_c \).

Let us now analyze in detail how this description comes about for the case of \( N_c = 3 \). In this case the Coulomb branch is divided into two regions, according to the sign of \( \phi_2 \). Let us denote the instanton factors \((8.1)\) by \( Y_1 \) and \( Y_2 \) for \( \phi_2 > 0 \), and by \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) for \( \phi_2 < 0 \). As discussed above, \( Y = Y_1 Y_2 = \tilde{Y}_1 \tilde{Y}_2 \) may be continuously defined throughout the Coulomb branch. The global charges of these fields may be determined by looking at the instanton zero modes, and are given by (in the same conventions as above)

\[
\begin{array}{c|cc}
 & U(1)_R & U(1)_A \\
\hline
\text{det} \ M & 0 & 2N_f \\
Y & 2N_f - 4 & -2N_f \\
Y_1 & -2 & 0 \\
Y_2 & 2N_f - 2 & -2N_f \\
\tilde{Y}_1 & 2N_f - 2 & -2N_f \\
\tilde{Y}_2 & -2 & 0
\end{array}
\] (8.9)

Along the subspace \( \phi_2 = 0 \), we have a \( U(1) \) theory with \( N_f \) massless electrons, so we expect the perturbatively-corrected moduli space in this region to be described by a superpotential of the form \( W = -N_f (V_+ V_- \text{ det} \ M)^{1/N_f} \), for some variables \( V_\pm \) which semi-classically satisfy \( V_+ \sim e^{\alpha \phi_2/g^2} \) (for \( \phi_2 \gg 0 \)) and \( V_- \sim e^{-\alpha \phi_2/g^2} \) (for \( \phi_2 \ll 0 \)) for some constant \( \alpha \). For \( N_f \geq 3 \) there are also baryonic operators which can appear in the superpotential; we set these to zero here. In both regions of the moduli space, we can express \( e^{\Phi_2} \) (and, therefore, \( V_\pm \)) semi-classically in terms of the instanton factors of this region, and determine \( \alpha = 3/2 \) by using the global symmetries. Solving these relations for \( Y_2 \) and \( \tilde{Y}_1 \), and inserting the appropriate instanton contributions, we find

\[
W = -N_f (V_+ V_- \text{ det} \ M)^{1/N_f} + \lambda_1 (Y - Y_1 V_+^2) + \lambda_2 (Y - \tilde{Y}_2 V_-^2) + \frac{1}{Y_1} + \frac{1}{Y_2},
\] (8.10)

where \( \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers which enforce the relations between our variables, and reduce the number of independent variables on the Coulomb branch to three, as
expected (generally each splitting of the Coulomb branch adds one independent variable, so we expect to find \((N_c - 1) + (N_c - 2) = 2N_c - 3\) independent variables parameterizing the Coulomb branch). It is easy to see that this superpotential is compatible with adding mass terms \(W = \text{Tr} \, mM\) and integrating out massive quarks to reduce \(N_f\), including the flow to the \(N_f = 0\) theory described above. It is also compatible with giving one meson a VEV and flowing to the \(SU(2)\) theory described in §6.

8.3. Theories with real mass terms for the quarks

As discussed above, we can also add real mass terms for the quarks, and we will discuss here only mass terms corresponding to background vector-like fields, which do not break CP invariance. Again, it is easy to determine semi-classically the number of quark zero modes for each instanton, using the analysis of [23]. A quark with real mass \(\tilde{m}_f\) will have zero modes in the background of an instanton corresponding to a root \(\phi_j - \phi_k\) if and only if \(\phi_j > \tilde{m}_f > \phi_k\) (and, in this case, we will have two zero modes, one from the quark and one from the anti-quark chiral multiplet). Whenever some \(\phi_i = \tilde{m}_f\), we get a \(U(1)\) theory with a massless electron, and the Coulomb branch will split.

The analysis of these theories can be performed as described in the previous section. We should first find a superpotential describing the perturbatively-corrected moduli space, and then add to it the instanton corrections to determine the full dynamics of the theory. The construction of such theories is complicated in general, and will not be presented here. However, as with the massless case, the general form of the space of vacua of the theory can be determined by simple arguments. Any region of the Coulomb branch where semi-classically all masses are larger than \(\phi_i\) or smaller than \(\phi_{i+1}\) will be lifted by the instanton corresponding to the root \(\phi_i - \phi_{i+1}\), which has no quark zero modes there. Unlike the massless case, here higher dimensional subspaces of the Coulomb branch may also remain unlifted. For example, for \(N_f \geq N_c - 1\) with all masses different, we expect to have \((N_c - 1)\)-dimensional subspaces of the Coulomb branch which remain unlifted (though they may still mix with the Higgs branches), corresponding (for instance) to

\[
\phi_1 > \tilde{m}_1 > \phi_2 > \tilde{m}_2 > \cdots > \tilde{m}_{N_c - 1} > \phi_{N_c}.
\]

8.4. \(d = 4 \, N = 1\) SQCD theories compactified on a circle

The analysis of these theories is similar to the analysis of §7. For simplicity we will discuss here only the cases with no real masses. Then, the effect of the finite radius is (for

\[\text{This may also easily be seen in the brane construction of these theories.}\]
any $N_f$) just to add to the superpotential we wrote in the previous sections a term $W = \eta Y$
where $\eta = e^{-1/Rg^2} \sim e^{-1/g_4^2} \sim \Lambda_4^{3N_c-N_f}$. As in §7, this term arises from instantons twisted by a non-trivial gauge transformation on the circle [37].

For $N_f = 0$, as discussed in [38], this term leads to the existence of $N_c$ stable vacua for any value of $R$. Integrating out the massive fields we find $W \sim \Lambda_4^3$, as expected. For $0 < N_f < N_c$, adding this term leads to no stable vacua. We can integrate out all the fields parameterizing the Coulomb branch, and find the expected superpotential [35] $W \sim (\Lambda_4^{(3N_c-N_f)} / \det M)^{1/(N_c-N_f)}$.

For $N_f = N_c$, adding $W_\eta$ to (8.7) yields $W = Y(\eta - (\det M - B\tilde{B}))$. In the $d = 4$ limit, integrating out $Y$, this leads to the known quantum constraint on the moduli space [28]. For finite $R$, we find a similar constraint on the Higgs branch, which becomes the classical constraint as $R \to 0$, and at $R = 0$ the theory grows an additional one dimensional Coulomb branch. For $N_f = N_c + 1$, we can similarly find (if we put in the baryons in the correct way to reproduce the classical Higgs branch in the $d = 3$ theory) the $d = 4$ superpotential $W = (\det M - B_i M_i j \tilde{B} j) / \Lambda_4^{2N_c-1}$ [26].

For higher values of $N_f$, we have not been able to find a good description of the $d = 3$ theories, and also in $d = 4$ we do not have a description of the theory in terms of gauge invariant variables. In $d = 4$ there is a dual theory which flows to the same fixed point in the IR [10], but it is not clear if the same type of duality works in $d = 3$.

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