D-brane Configurations for Domain Walls and Their Webs

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Abstract.
Supersymmetric $U(N_C)$ gauge theory with $N_F$ massive hypermultiplets in the fundamental representation admits various BPS solitons like domain walls and their webs. In the first part we show as a review of the previous paper [3] that domain walls are realized as kingly fractional D3-branes interpolating between separated D7-branes. In the second part we discuss brane configurations for domain wall webs. This is a contribution to the conference based on the talk given by MN.

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INTRODUCTION

Domain walls (or kinks) are important solitons in various subjects of physics like high energy physics, cosmology and condensed matter physics. They are BPS states in supersymmetric gauge theories. Like other BPS solitons their solutions admit the moduli space. Recently multiple domain wall solutions have been obtained and their moduli space has been completely determined in non-Abelian $U(N_C)$ gauge theory coupled with the fundamental Higgs fields [2], generalizing the $U(1)$ cases [3]. Soliton dynamics can be often well understood when solitons are embedded in D-brane configuration in string theory [4]. Domain wall solutions in [2] has been realized in [1] as a kingly brane configuration, first suggested in [5] for the $U(1)$ case.

Since then there have been extensive developments about domain walls. 1/4 BPS configurations of vortices stretched between multiple walls has been found in [6], generalizing vortices ending on a single wall [7]. The negative monopole charge firstly found in [6], now called a boojum, has been further studied in [8]. In [9] domain walls in $U(1)^2$ gauge theory has been investigated, where attractive and repulsive force between walls has been found. Relations between the moduli space of domain walls and one of monopoles has been clarified [10]. Relations to flux tube has been discussed [11]. Domain walls serve a set up to prove a relation between Skyrmion and instantons [12]. Finally, it has been found that several domain walls with different angles in the real space can make a 1/4 BPS web [13]. In the present talk the kingly D-brane configuration for domain walls [1] is reviewed and then is generalized to their webs.

DOMAIN WALLS

In this section we discuss domain walls [3]. For notation see the original papers. The vector multiplet contains a gauge field $W_M$ and a complex scalar field $\Sigma = \Sigma_1 + i \Sigma_2$ in the adjoint representation of the $U(N_C)$ gauge group whereas the hypermultiplets contain Higgs fields of two $N_C \times N_F$ matrices $(H^A)^{ir} \equiv (H^{irA})$ with $SU(2)_r i = 1, 2$, color $r = 1, \ldots, N_C$ and flavor $A = 1, \ldots, N_F$ indices. The bosonic Lagrangian is (summation over repeated index $\alpha = 1, 2$ is implied in the following)

$$\mathcal{L} = -\frac{1}{2g^2} \text{Tr} \left( F_{MN} F^{MN} \right) + \frac{1}{g^2} \text{Tr} \left( \mathcal{D}_M \Sigma_\alpha \mathcal{D}^M \Sigma_\alpha \right)$$
$$+ \text{Tr} \left( \mathcal{D}^M H^i \left( \mathcal{D}_M H^j \right)^\dagger \right) - V,$$  \hspace{1cm} (1)

with $g$ the gauge coupling constant and $V$ the potential:

$$V = \frac{1}{4} \text{Tr} \left( \left( H^1 H^\dagger_1 - H^2 H^\dagger_2 \right)^2 + 4 H^2 H^\dagger_1 H^1 H^\dagger_2 \right)$$
$$+ \text{Tr} \left( \left( \Sigma_\alpha H^i - H_i M_\alpha \right) \left( \Sigma_\beta H^j - H_j M_\beta \right)^\dagger - \frac{1}{g^2} \left[ \Sigma_1, \Sigma_2 \right]_c^2 \right),$$

with a triplet of the Fayet-Iliopoulos (FI) parameters chosen to the third direction as $(0, 0, c)$, and with the mass matrix defined by $(M_\alpha)^A_B \equiv (m_A, n_A) \delta^A B$. We consider non-degenerate real masses $(n_A = 0)$ with ordering $m_{A+1} < m_A$ for domain walls, but complex masses $m_A + i n_A$ for their webs. In the massless case the moduli space of vacua becomes the Higgs branch, the cotangent bundle over the complex Grassmann manifold $\mathcal{M}_{\text{vacua}}^{M=0} \simeq T^* G_{N_f-N_C} \simeq T^* \left[ \frac{SU(N_f)}{SU(N_f) \times SU(N_C)} \times U(1) \right]$ with $N_C \equiv N_f - N_C$ [14]. Turning on masses for hypermultiplets, most points on $\mathcal{M}_{\text{vacua}}^{M=0}$ are lifted leaving the discrete SUSY vacua given by $H^{1rA} = \sqrt{\mathcal{C}} \delta^{1A}$, $H^{2rA} = 0$, $\Sigma = \text{diag}(m_A + i n_A, \ldots, m_{A_N} + i n_{A_N})$, denoted
by \( \langle A_1, \ldots, A_{N_C} \rangle \), and therefore the number of SUSY vacua is \( N_C C_{N_C} = \frac{N_C}{N_C - n_{NC}} \).

The BPS equations for 1/2 BPS domain walls are obtained as \[ (H^1 \equiv H, H^2 = 0) \]
\[ \mathcal{R}_y \Sigma = \frac{g_s^2}{2} \left( e \mathcal{N}_C - HH^f \right), \quad \mathcal{R}_y H = -\Sigma H + HM, \] 
with \( y \) the codimension of walls. The tension of multiple BPS walls, interpolating between a vacuum \( \langle A_1, \ldots, A_{N_C} \rangle \) at \( y \rightarrow +\infty \) and a vacuum \( \langle B_1, \ldots, B_{N_C} \rangle \) at \( y \rightarrow -\infty \), is obtained as
\[ Z = c \left[ \text{Tr} \Sigma \right]_{y=+\infty} = c \left( \sum_{r=1}^{N_C} m_{A_r} - \sum_{r=1}^{N_C} m_{B_r} \right). \] 
(3)
See the original papers \[ \text{[2]} \] for how to solve Eqs. \[ \text{[2]} \].

We now realize our theory on D-branes in string theory as follows. Let us consider parallel \( N_F \) D7-branes in type IIB string theory and divide four spatial directions of their world volume by \( Z_2 \) to form the orbifold \( C^2/Z_2 \). The orbifold singularity is blown up to the Eguchi-Hanson space by \( S^2 \) with the area
\[ A = c g_s l_s^4 = \frac{c}{\tau_3} \] 
(4)
with \( g_s \) the string coupling, \( l_s = \sqrt{\alpha'} \) the string length and \( \tau_3 = 1/g_s l_s^4 \) the D3-brane tension. We then realize our theory on \( N_C \) fractional D3-branes, that is, D5-branes wrapping around \( S^2 \). Then the configuration becomes
\[ \begin{align*}
N_C & \text{ frac. D3:} & 0123 \\
N_F & \text{ D7:} & 01234567 \\
C^2/Z_2 \text{ ALE:} & 4567.
\end{align*} \] 
(5)
A string connecting D3-branes provides the gauge multiplets whereas a string connecting D3-branes and D7-branes provides the hypermultiplets in the fundamental representation. The gauge coupling constant \( g \) of the gauge theory realized on the D3-brane is
\[ \frac{1}{g^2} = \frac{b}{g_s} \] 
(6)
with \( b \) the \( B \)-field flux integrated over the \( S^2 \), \( b \sim A B_{(i)} \).

The positions of the D7-branes in the \( x^8, x^9 \) plane gives the complex masses for the fundamental hypermultiplets whereas the positions of the D3-branes in the \( x^8, x^9 \) plane is determined by the VEV of \( \Sigma \):
\[ \begin{align*}
(x^8, x^9)_{D7} & = l_s^2 (m, n), \\
(x^8, x^9)_{D3} & = l_s^2 (\Sigma^1 (x^1, x^2), \Sigma^2 (x^1, x^2)).
\end{align*} \] 
(7)

Taking a T-duality along the \( x^4 \)-direction the ALE geometry is mapped to two NS5-branes separated in the \( x^4 \)-direction. The configuration becomes the Hanany-Witten type brane configuration \[ \text{[15]} \]
\[ \begin{align*}
N_C \text{ D4:} & \quad 01234 \\
N_F \text{ D6:} & \quad 0123 5677 \\
2 \text{ NS5:} & \quad 0123 89.
\end{align*} \] 
(8)
The relations between physical quantities and the positions of branes are summarized as follows:
\[ \begin{align*}
(x^8, x^9)_{D4} & = l_s^2 (\Sigma^1 (x^1, x^2), \Sigma^2 (x^1, x^2)), \\
(x^8, x^9)_{D6} & = l_s^2 (m, n), \\
(\Delta x^5 \Delta x^6 \Delta x^7)_{\text{NS5}} & = g_s l_s^5 (0, 0, c).
\end{align*} \] 
(9)

As vacuum states in the brane picture before taking T-duality, any D3-brane must lie in a D7-brane. At most one D3-brane can lie in each D7-brane (because of s-rule \[ \text{[15]} \]) so that the vacuum \( \langle A_1, \ldots, A_{N_C} \rangle \) is realized with \( A_r \) denoting positions of D3-branes, and therefore the number of vacua is \( N_C C_{N_C} \).

For domain wall states, we consider real mass \( n = 0 \) and \( \Sigma_2 = 0 \), and \( \Sigma_1 \) depends on one coordinate \( y \equiv x^1 \). All D3-branes lie in some \( N_C \) D7-branes in the limit \( y \rightarrow +\infty \), giving \( \langle A_1, \ldots, A_{N_C} \rangle \), but lie in another set of D7-branes in the opposite limit \( y \rightarrow -\infty \), giving another vacuum \( \langle B_1, \ldots, B_{N_C} \rangle \). The \( N_C \) D3-branes exhibit kinks somewhere in \( y \)-coordinate as illustrated in Fig. \[ \text{[1]} \].

Here we labeled \( B_r \) such that the \( A_r \)-th brane at \( y \rightarrow +\infty \)

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DOMIAN WALL WEBS

The BPS equations for domain wall webs can be obtained as $[13] (H^1 \equiv H, H^2 = 0)$

$$F_{12} = i \langle \Sigma_1, \Sigma_2 \rangle, \quad \partial_1 \Sigma_1 = \partial_2 \Sigma_1,$$

$$\partial_1 \Sigma_1 + \partial_2 \Sigma_2 = \frac{g^2}{2} (cA_Nc - HH^1),$$

$$\partial_1 H = HM_1 - \Sigma_1 H, \quad \partial_2 H = HM_2 - \Sigma_2 H,$$

with $x^1$ and $x^2$ the codimensions. The energy density is

$$\mathcal{E} = \mathcal{Y} + \mathcal{Z}_1 + \mathcal{Z}_2 + \partial_a J_a,$$

where each charge density is given by

$$\mathcal{Y} = \frac{2}{g^2} c_{\alpha} \text{Tr} (e^{\alpha \beta} \Sigma_2 \partial_1 \Sigma_1), \quad \mathcal{Z}_a = \phi c_{\alpha} \text{Tr} \Sigma_{\alpha} (\text{no sum}),$$

with $J_\alpha$ terms not contributing to the tension under the space integration. Wall tensions and the junction charge are defined by $Z_\alpha \equiv \int dx^a \alpha \theta$ and $Y \equiv \int dx^1 dx^2 \theta$, respectively. See the original papers $[13]$ for solutions to (10) and their properties. In the following we consider only $U(1)$ gauge group for simplicity.

Several walls form a 1/4 BPS wall junction when both $\Sigma^1$ and $\Sigma^2$ fully depend on both $x^1$ and $x^2$ as a solution of (10). So the configurations are given by maps from $(x^1, x^2)$ to $(x^3, x^9)$. An example of three wall junction in the $U(1)$ gauge theory with $N_F = 3$ is illustrated in Fig. 4. Along a circle with sufficiently large radius surrounding each charge density is given by $[13]$. An example of three wall junction in the $U(1)$ gauge theory with $N_F = 3$. Three flavors with complex masses correspond to three D7-branes successively interpolating between $d$ D7-branes. For smaller radius the D3-brane is detached from the D7-branes and goes inside the region surrounded by $n$ D7-branes. Varying $r$ the configuration sweeps inside that column.

Let us take a T-duality along $x^4$ as $[8]$ and $[9]$. The brane configuration for wall junction is illustrated in Fig. 5. In these figures, the dependence of $\Sigma$ to $x^1$ and $x^2$ is suppressed but is drawn in the same figure. Each D4-brane ends on a D6-brane at one spatial infinity of the $x^1 \times x^2$ plane with fixing $x^3 \times x^9$. It can move to another D6-brane at another spatial infinity. Then the position $\Sigma_1, \Sigma_2$ of a D4-brane at each point of the $x^1 \times x^2$ plane is drawn in the same figure. The D4-brane tension contributing to $\tau_4 \Delta x^4 (\Delta x^8, \Delta x^9) = \frac{1}{8} \tau_4 (\Delta m, \Delta n)$ diverges in the limit $t \to 0$ but this is the vacuum energy. The contribution to the shaded area in Fig. 5a)

$$\tau_4 \Delta x^4 (\Delta x^8, \Delta x^9) = c(\Delta m, \Delta n) = (Z_1, Z_2)$$

FIGURE 5. Brane configuration for domain wall junction.
recovers the wall tensions correctly. We can only see the wall in Fig. 3a) but not a junction. To see a wall junction and its junction charge $Y$, we ignore the dependence on $x^7, x^8, x^9$. Instead in Fig. 3b), we take into account the $x^3, x^9$ dependence. Just as in Fig. 3a), we draw D3-brane position at each $(x^3, x^9)$ in the same figure. Then it sweeps inside a triangular column. Interestingly the wall in Fig. 5-a) but not a junction. To see a wall junction and its junction charge $Y$, we can calculate from the product of the D3-brane tension and the volume of the triangular column:

$$Y \sim \tau_3 \text{Vol} (\text{triang. column}) = \tau_3 \Delta x^4 |dx^8 \wedge dx^9|. \quad (14)$$

$Y$ is actually negative for $U(1)$ gauge theory \cite{13} but it is not clear in the brane picture yet. Non-Abelian $U(N_c)$ case could be considered in the same way. As in domain wall case \cite{11} duality between junctions in $U(N_c)$ and $U(N_f - N_c)$ gauge theories would be explained by exchange of the two NS5-branes.

**DISCUSSION**

In this talk we have discussed only non-degenerate masses for hypermultiplets. When masses are degenerate more interesting physics appears \cite{16} \cite{12}. Other than domain walls, lots of composite BPS solitons in the Higgs phase have been found, like 1/4 BPS instantons with vortices \cite{17}, 1/4 BPS intersecting vortices \cite{18}, and various 1/8 BPS composite solitons \cite{19}. Reviews of some of these development can be found in \cite{20}. D-brane configurations for these newly found solitons are desired.

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