NUMERICAL SOLUTIONS OF VOLterra
INTEGRO-Differential EQUATIONS USING GENERAL
LINEar METHOD

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Abstract. In this paper, a third order General Linear Method for finding the
numerical solution of Volterra integro-differential equation is considered. The
order conditions of the proposed method are derived based on techniques of
B-series and rooted trees. The integral operator in Volterra integro-differential
equation approximated using Simpson’s rule and Lagrange interpolation is dis-
cussed. To illustrate the efficiency of third order General Linear Method, we
compare the method with a third order Runge-Kutta method.

1. Introduction. We consider solving the first order of Volterra integro-differential
equation (VIDE) given in the form

\[ y'(x) = F\left(x, y(x), \lambda \int_0^x K(x, t)y(t)dt\right), \quad y(0) = y_0, \]

where \(f(x)\) and \(K(x,t)\) are known functions while \(y(x)\) is the solution to be ob-
tained. Applications of VIDE arise in various fields such as biology, physics, en-
gineering and more (see [11]). Consequently abundant numerical methods have
appeared on finding the solutions to problem 1. Raftari in [10] solve the VIDE
using Homotopy perturbation method along with trapezoidal rule for the integral
operator and compared it with the finite difference method. Followed by Zarebnia
in [12], the author considered the numerical solutions of VIDE by means of the S-
inc collocation method where Sinc methods are direct solvers to integral equations.
Later, Filiz in [5] applied several Runge-Kutta methods with different order asso-
ciated with trapezoidal rule and Simpson’s rule to VIDE. Then, Filiz in [6] used a
higher order method which is the Runge-Kutta-Fehlberg method to solve the VIDE.
He developed a new numerical routine for the integral part by using Lagrange in-
terpolation and combination of various numerical quadrature rules to attain higher

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In this work, we focused on the General Linear Method (GLM) as a numerical method to obtain the solutions of VIDE. GLM was first introduced by Butcher in [1] as solver to initial value problems of ordinary differential equations. The GLM is known as natural generalizations of the classical Runge-Kutta and linear multistep methods. The combination of these methods is to achieve a high accuracy with high stage order method while incorporates good stability. Moreover, the order conditions of GLM are derived based on techniques of B-series and rooted trees (see [7], [3]). These techniques are alternatives to conventional way which is the Taylor method for finding order conditions to Runge-Kutta method and convenient for higher order method. Recently, Rabiei et al. in [9] proposed the General Linear Method for solving fuzzy differential equations and based on that, in this paper we construct the GLM adapted with suitable integration rules to solve VIDE problems.

The rest of the paper is organized as follows. In Section 2, we presented the general formula of GLM. In Section 3, implementation of GLM with suitable integration rules for Volterra integro-differential equation is given. Finally, numerical results and graphical illustrations are showed to test the performance of the proposed method.

2. General Linear Method. Consider the initial value problem of first order ordinary differential equation as follows:

\[ y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \]

The general formulae for General Linear Method presented by Butcher in [1],[7] is given by:

\[
\begin{align*}
Y_i &= \sum_{j=1}^{s} a_{ij} h F_j + \sum_{j=1}^{r} u_{ij} y_{i-1}^{[j]}, \\
y_i^{[n]} &= \sum_{j=1}^{s} b_{ij} h F_j + \sum_{j=1}^{r} v_{ij} y_{i-1}^{[j]}, \\
F_i &= \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}, \quad F &= \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_s \end{bmatrix}.
\end{align*}
\]

Butcher’s Tableau is used to represent the coefficients of GLM as given in Table 1.

The derivation of GLM can be further learned by understanding the concept of rooted trees. These rooted trees are first introduced by Butcher in [2] and used to obtain the order conditions which are needed to derive the GLM. Rabiei et al. in [9] use trees of order one to four. Aside from that, associated closely with the rooted trees is the composition law. This law is important in the construction of
Table 1. Matrix representation of coefficients of GLM.

\[
\begin{array}{c|c}
A_{s \times s} & U_{s \times r} \\
B_{r \times s} & V_{r \times r}
\end{array}
\]

different numerical methods such as general linear method, Runge-Kutta method, Rosenbrock-methods, multi-derivative methods, etc. (see [3]). The tedious calculations of Taylor series expansion can be avoided by conveniently apply this law in the derivation of algebraic order conditions.

Definition 2.1. (see [1],[7]) Let \( a : T^\# \rightarrow \mathbb{R} \) by a mapping, then the form of a formal series is given as follows:

\[
B(a, y(x)) = a(\emptyset)y(x) + \sum_{t \in T} \frac{1}{\gamma(t)\sigma(t)} h^{|t|} F(t)(y(x)), \tag{5}
\]

is the B-series, where

- \( T^\# \) = denoting the set of rooted trees together with an additional empty tree \( \emptyset \),
- \( a(\emptyset) = 1 \),
- \( |t| = \) order of tree,
- \( \gamma(t) = \) density of tree,
- \( \sigma(t) = \) symmetry of tree,
- \( F(t) = \) elementary differential.

In order to derive the order conditions for GLM, we need to satisfy theorem below:

Theorem 2.2. (see [1]) The general linear method \((A,U,B,V)\) has order \( p \) if there exists \( \xi \in X^r \) and \( \eta \in X^s_1 \), such that, for every tree \( t \) satisfying \( |t| \leq p \),

\[
\eta(t) = A(\eta D)(t) + U\xi(t), \tag{6}
\]

\[
(E\xi)(t) = B(\eta D)(t) + V\xi(t). \tag{7}
\]

From Theorem 2.2, \( \eta \) represents the stages of method while \( E\xi \) is the output approximations computed after a time-step. The general coefficients of GLM of order three with \( r = 2 \), \( s = 3 \) are given in Table 2.

Table 2. Matrix coefficients of GLM with \( s = 3 \), \( r = 2 \).

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
a_{21} & 0 & 0 \\
a_{31} & a_{32} & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
u_{11} & u_{12} \\
u_{21} & u_{22} \\
u_{31} & u_{32}
\end{bmatrix}
\]

\[
\begin{bmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23}
\end{bmatrix}
\begin{bmatrix}
v_{12} \\
1 & v_{12} \\
v_{21} & 0
\end{bmatrix}
\]

Based on Theorem 2.2, Rabiei et al. in [9] obtained the order conditions for a third order GLM as given in Table 3.
Based on the order conditions presented in Table 3, Rabiei et al. in [9] achieved the coefficients of third order with three stages of GLM method as shown in Table 4.

### Table 4. Coefficients Set 1 of third order GLM

| No | Order conditions                                                                 | $\mathbf{a}_{11}$ | $\mathbf{a}_{12}$ | $\mathbf{a}_{21}$ | $\mathbf{a}_{31}$ | $\mathbf{a}_{32}$ | $\mathbf{b}_{11}$ | $\mathbf{b}_{12}$ | $\mathbf{b}_{21}$ | $\mathbf{b}_{22}$ | $\mathbf{b}_{31}$ | $\mathbf{b}_{32}$ |
|----|-----------------------------------------------------------------------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1  | $b_{11}(u_{11} + u_{12}) + b_{12}(u_{21} + u_{22}) + b_{13}(u_{31} + u_{32}) - v_{12} = 1$ | $\frac{13}{17}$ | $\frac{7}{17}$ | $\frac{8}{17}$ | $\frac{8}{17}$ | $\frac{8}{17}$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ |
| 2  | $b_{21}(u_{11} + u_{12}) + b_{22}(u_{21} + u_{22}) + b_{23}(u_{31} + u_{32}) = 0$      |                   |                   |                   |                   |                   | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ |

In addition, we solve the order conditions in Table 3 to obtain another two new sets of coefficients of the third order GLM as presented in Tables 5 and 6.

Based on the existing coefficients set and two new sets of GLM, we compute the error norms for all sets. The principal error norm is expressed as follows (see [4]).
Table 5. Coefficients Set 2 of third order GLM

| a_{21} = \frac{3}{5}, a_{31} = \frac{3}{5}, a_{32} = 2 | u_{11} = 1, u_{12} = 0 | u_{21} = \frac{5}{6}, u_{22} = \frac{1}{3} |
| b_{11} = \frac{6}{7}, b_{12} = \frac{6}{7}, b_{13} = \frac{6}{7}, b_{21} = 0, b_{22} = 0, b_{23} = 0 | v_{11} = 1, v_{12} = 0 | v_{21} = 1, v_{22} = 0 |

Table 6. Coefficients Set 3 of third order GLM

| a_{21} = \frac{3}{5}, a_{31} = \frac{3}{5}, a_{32} = 2 | u_{11} = 1, u_{12} = 0 | u_{21} = \frac{5}{6}, u_{22} = \frac{1}{3} |
| b_{11} = \frac{6}{7}, b_{12} = \frac{6}{7}, b_{13} = \frac{6}{7}, b_{21} = 0, b_{22} = 0, b_{23} = 0 | v_{11} = 1, v_{12} = 0 | v_{21} = 1, v_{22} = 0 |

\[ ||\tau^{p+1}||_2 = \sqrt{\sum_{j=1}^{n_{p+1}} (\tau_j^{p+1})^2}. \quad (8)\]

The order conditions which are not satisfied by coefficients of GLM are as follows:

\[ \tau_5^4 : -b_{11}u_{12}^2\xi_{23} + b_{12}(a_{21}(u_{11} + u_{12}) - u_{22})(a_{21}u_{12}^2 + \xi_{23}u_{22}) + b_{12}(a_{21}(u_{11} + u_{12}) + a_{32}(u_{21} + u_{22}) - u_{42})(a_{31}u_{12}^2 + a_{32}(a_{21}(u_{11} + u_{12}) - u_{22})^2 + u_{32}\xi_{23}) + v_{12}\xi_{211} = \frac{1}{9} \]

\[ \tau_7^2 : -b_{11}u_{12}^2\xi_{24} + b_{12}(a_{21}(u_{11} + u_{12}) - u_{22})(\xi_{22}a_{21}u_{12} + E_{24}u_{22}) + b_{12}(a_{21}(u_{11} + u_{12}) + a_{32}(u_{21} + u_{22}) - u_{42})(a_{31}u_{12}\xi_{22} + a_{32}(\xi_{22}u_{22} - a_{21}u_{12}) + u_{32}\xi_{24}) + v_{12}\xi_{212} = \frac{1}{18} \]

\[ \tau_{15}^4 : b_{11}u_{12}\xi_{27} + b_{12}(\xi_{23}a_{21}u_{12} + \xi_{27}u_{22}) + b_{13}(a_{31}u_{12}\xi_{23} + a_{32}(a_{21}u_{12}^2 + \xi_{23}u_{22}) + u_{32}\xi_{27}) + v_{12}\xi_{216} = -\frac{1}{36} \]

\[ \tau_{17}^4 : b_{11}u_{12}\xi_{28} + b_{12}(\xi_{24}a_{21}u_{12} + \xi_{28}u_{22}) + b_{13}(a_{31}u_{12}\xi_{24} + a_{32}(\xi_{22}a_{21}u_{12} + \xi_{24}u_{22}) + u_{32}\xi_{28}) + v_{12}\xi_{217} = -\frac{1}{72} \]

Substituting coefficients set 1 of third order GLM into equations \( \tau_5^2, \tau_7^2, \tau_{15}^2, \) \( \tau_{17}^2 \) and assuming \( \xi_{23} = -\frac{1}{3}, \xi_{24} = -\frac{1}{5} \), the principal error norm is given as

\[ ||\tau^4||_2 = \sqrt{(\tau_5^4 + \tau_7^2 + \tau_{15}^2 + \tau_{17}^2)} = 0.004880058119. \quad (9) \]
Substituting coefficients set 2 of third order GLM into equations $\tau_5^2, \tau_7^2, \tau_{15}^2, \tau_{17}^2$ and assuming $\xi_{23} = -\frac{1}{3}, \xi_{24} = -\frac{1}{6}$, the principal error norm is given as

$$||\tau^4||_2 = \sqrt{(\tau_5^4 + \tau_7^4 + \tau_{15}^4 + \tau_{17}^4)} = 0.01464017435.$$  \hspace{1cm} (10)

Substituting coefficients set 3 of third order GLM into equations $\tau_5^2, \tau_7^2, \tau_{15}^2, \tau_{17}^2$ and assuming $\xi_{23} = -\frac{1}{3}, \xi_{24} = -\frac{1}{6}$, the principal error norm is given as

$$||\tau^4||_2 = \sqrt{(\tau_5^4 + \tau_7^4 + \tau_{15}^4 + \tau_{17}^4)} = 0.01464017435.$$  \hspace{1cm} (11)

Based on the error norms obtained in equations 9 - 11, it is clear that the Set 1 of coefficients of GLM have smaller error norm compared with two other sets. Therefore, we use Set 1 coefficients for the following sections.

3. Implementation. Let us denote the integral operator in equation 1 as follows

$$z(x) = \lambda \int_0^x K(x,t)y(t)dt,$$  \hspace{1cm} (12)

then the VIDE is given as

$$y'(x) = F(x, y(x), z(x)), y(0) = y_0.$$  \hspace{1cm} (13)

The third order GLM is combined with appropriate numerical quadrature rule in order to approximate the integral part given in equation 12. To evaluate the integral $z(x)$, we use the combination of composite Simpsons II rule (see [8]) for the interval $[x_0, x_n]$ and Lagrange interpolation for interval $[x_n, x(n+c_i)]$ at points $x = -1, x = 0, x = c_i$. The third order General Linear Method applied to approximate equation 13 on a set of interval $[0, X]$ of equally spaced grid points $x_0 < x_1 < x_2 < \cdots < x_N = X$ where $0 \leq n \leq N$ with step size $h = \frac{(X-x_0)}{N}$ may be written as:

For interval $[x_0, x_n]$, if $n = 1$, trapezoidal rule is used then

$$z_n = \frac{h}{2}(K(x_0+c_1, x_0)x_1y_1 + (x_0+c_1, x_n)y_1x_n)$$

else if $n$ is even, composite 1/3 Simpsons rule is used then

$$z_n = \frac{h}{3}\left(K(x_0+c_1, x_0)y_1 + 2\sum_{m=1}^{(n/2)-1} +K(x_0+c_1, x_2m)y_1(x_2m) + 4\sum_{m=1}^{n/2} +K(x_0+c_1, x_{2m-1})y_1(x_{2m-1}) + K(x_0+c_1, x_{2m}y_1(x_{2m}))\right),$$

else if $n$ is odd, composite Simpsons II rule is used then
\[ z_n = \frac{h}{3} \left( K(x_{n+c}, x_0) y_1(x_0) + 2 \sum_{m=1}^{n-1} K(x_{n+c}, x_{2m}) y_1(x_{2m}) \right) + 4 \sum_{m=1}^{n-3} K(x_{n+c}, x_{2m-1}) y_1(x_{2m-1}) + K(x_{n+c}, x_{n-3}) y_1(x_{n-3}) + \frac{3h}{8} K(x_{n+c}, x_{n-3}) y_1(x_{n-3}) + 3K(x_{n+c}, x_{n-2}) y_1(x_{n-2}) + 3K(x_{n+c}, x_{n-1}) y_1(x_{n-1}) + K(x_{n+c}, x_n) y_1(x_n) \right). \]

Therefore General Linear Method after employing composite Simpsons rule and Lagrange interpolation is given as follows

\[
Y_1 = u_{11} y_1(x_n) + u_{12} y_2(x_n), \\
Y_2 = a_{21} h F(x_n, Y_1, z_n) + u_{21} y_1(x_n) + u_{22} y_2(x_n), \\
z_{n+\frac{1}{2}} = z_n + h \left( -\frac{1}{72} K(x_{n+\frac{1}{2}}, x_{n-1}) y_1(x_{n-1}) + \frac{7}{24} K(x_{n+\frac{1}{2}}, x_n) y_1(x_n) + \frac{2}{9} K(x_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}) y_1(x_{n+\frac{1}{2}}) \right), \\
Y_3 = a_{31} h F(x_n, Y_1, z_n) + a_{32} h F(x_{n+\frac{1}{2}}, Y_2, z_{n+\frac{1}{2}}) + u_{31} y_1(x_n) + u_{32} y_2(x_n), \\
z_{n+1} = z_n + h \left( -\frac{1}{12} K(x_{n+1}, x_{n-1}) y_1(x_{n-1}) + \frac{2}{3} K(x_{n+1}, x_n) y_1(x_n) + \frac{5}{12} K(x_{n+1}, x_{n+1}) y_1(x_{n+1}) \right), \]

\[
y_1(x_{n+1}) = b_{11} h F(x_n, Y_1, z_n) + b_{12} h F(x_{n+\frac{1}{2}}, Y_2, z_{n+\frac{1}{2}}) + b_{13} h F(x_{n+1}, Y_3, z_{n+1}) + v_{11} y_1(x_n) + v_{12} y_2(x_n), \\
y_2(x_{n+1}) = b_{21} h F(x_n, Y_1, z_n) + b_{22} h F(x_{n+\frac{1}{2}}, Y_2, z_{n+\frac{1}{2}}) + b_{23} h F(x_{n+1}, Y_3, z_{n+1}) + v_{11} y_1(x_n) + v_{12} y_2(x_n). \]

4. **Numerical Results.** The performance of the third order General Linear Method in solving the Volterra integro-differential equation is determined through several test problems. We compared the third order GLM with a third order Runge-Kutta method given in [2]. The accuracy of proposed method and Runge-Kutta method are presented through estimates of maximum global errors. Notations below are used for the tabulated results.
Problem (1). Consider the Volterra integro-differential equation (see Filiz [5])
\[ y'(x) = 1 - \int_0^x y(t)\,dt, \quad y(0) = 0, \text{ with exact solution } y(x) = \sin(x). \]

Problem (2). Consider the Volterra integro-differential equation (see Filiz [5])
\[ y'(x) = 1 + \int_0^x y(t)\,dt, \quad y(0) = 0, \text{ with exact solution } y(x) = \sinh(x). \]

Problem (3). Consider the Volterra integro-differential equation (see Zarebnia [12])
\[ y'(x) = 1 + 2x - y(x) - \int_0^x x(1 + 2x)e^{(x-t)}y(t)\,dt, \quad y(0) = 1, \text{ with exact solution } y(x) = e^{x^2}. \]

Problem (4). Consider the Volterra integro-differential equation (see Zarebnia [12])
\[ y'(x) = -y(x) + \int_0^x e^{(t-x)}y(t)\,dt, \quad y(0) = 1, \text{ with exact solution } y(x) = e^{-x}\cosh(x). \]

Problem (5). Consider the Volterra integro-differential equation (see Zarebnia [12])
\[ y'(x) = -\ln(1 + x)\left(\frac{x}{2}\ln(1 + x) + 1\right) + \frac{1}{1 + x} + y(x) - \int_0^x \frac{y(t)}{t+1}\,dt, \quad y(0) = 0 \text{ with exact solution } y(x) = \ln(x + 1). \]

| Step size | GLM, s = 3 | RK, s = 3 |
|-----------|-------------|-----------|
| h = 0.1  | 1.2347 × 10^{-6} | 4.7137 × 10^{-6} |
| h = 0.025| 9.9859 × 10^{-9} | 7.1772 × 10^{-8} |
| h = 0.01 | 5.6041 × 10^{-10} | 4.6094 × 10^{-9} |
| h = 0.005| 6.7079 × 10^{-11} | 5.7715 × 10^{-10} |
| h = 0.001| 5.1845 × 10^{-13} | 4.6243 × 10^{-12} |

| Step size | GLM, s = 3 | RK, s = 3 |
|-----------|-------------|-----------|
| h = 0.1  | 2.4606 × 10^{-6} | 6.9906 × 10^{-6} |
| h = 0.025| 1.6319 × 10^{-8} | 1.0137 × 10^{-7} |
| h = 0.01 | 8.3870 × 10^{-10} | 6.4622 × 10^{-9} |
| h = 0.005| 9.7077 × 10^{-11} | 8.0749 × 10^{-10} |
| h = 0.001| 7.2935 × 10^{-13} | 6.4604 × 10^{-12} |

The result in Table 7 for problem 1 showed that the maximum error computed by the third order GLM with error exponent of 10^{-6} is similar with the RK method. However, as the step size decreases we observed that the maximum errors of GLM are lower than RK method indicating that the accuracy of GLM for approximating the Volterra integro-differential equation is better than that of RK. In this case, the error of GLM came out as ×10^{-13} which is better than the error of ×10^{-12} by RK.
Table 9. Maximum global errors for Problem 3

| Step size | GLM, $s = 3$ MAXE | RK, $s = 3$ MAXE |
|-----------|-------------------|------------------|
| $h = 0.1$ | $3.9332 \times 10^{-6}$ | $4.8141 \times 10^{-5}$ |
| $h = 0.025$ | $1.4323 \times 10^{-7}$ | $1.4400 \times 10^{-6}$ |
| $h = 0.01$ | $1.0939 \times 10^{-8}$ | $1.0134 \times 10^{-7}$ |
| $h = 0.005$ | $1.4325 \times 10^{-9}$ | $1.3052 \times 10^{-8}$ |
| $h = 0.001$ | $1.1851 \times 10^{-11}$ | $1.0688 \times 10^{-10}$ |

Table 10. Maximum global errors for Problem 4

| Step size | GLM, $s = 3$ MAXE | RK, $s = 3$ MAXE |
|-----------|-------------------|------------------|
| $h = 0.1$ | $4.5416 \times 10^{-6}$ | $1.4256 \times 10^{-5}$ |
| $h = 0.025$ | $1.7061 \times 10^{-8}$ | $2.5908 \times 10^{-7}$ |
| $h = 0.01$ | $1.4270 \times 10^{-9}$ | $1.7075 \times 10^{-8}$ |
| $h = 0.005$ | $2.1002 \times 10^{-10}$ | $2.1553 \times 10^{-9}$ |
| $h = 0.001$ | $1.8836 \times 10^{-12}$ | $1.7377 \times 10^{-11}$ |

Table 11. Maximum global errors for Problem 5

| Step size | GLM, $s = 3$ MAXE | RK, $s = 3$ MAXE |
|-----------|-------------------|------------------|
| $h = 0.1$ | $6.3429 \times 10^{-6}$ | $3.3432 \times 10^{-5}$ |
| $h = 0.025$ | $4.4142 \times 10^{-8}$ | $6.3237 \times 10^{-7}$ |
| $h = 0.01$ | $4.0164 \times 10^{-9}$ | $4.1259 \times 10^{-8}$ |
| $h = 0.005$ | $5.4245 \times 10^{-10}$ | $5.1811 \times 10^{-9}$ |
| $h = 0.001$ | $4.5689 \times 10^{-12}$ | $4.1572 \times 10^{-11}$ |

Table 12. Total number of function evaluations Problems 1 - 5

| Step size | GLM, $s = 3$ TFE | RK, $s = 3$ TFE |
|-----------|------------------|------------------|
| $h = 0.1$ | 34 | 34 |
| $h = 0.025$ | 124 | 124 |
| $h = 0.01$ | 304 | 304 |
| $h = 0.005$ | 604 | 604 |
| $h = 0.001$ | 3004 | 3004 |

method. Results from Table 8 and 9 for problems 2 and 3 further prove that our proposed method clearly outperforms the RK method by having smaller errors. On top of that, the solutions obtained by GLM in Table 10 and 11 for problems 4 and 5 apparently outshine the RK method as well.

Table 12 shows the total number of function evaluation used for GLM and RK GLM. GLM requires one step method to obtain the initial value for the second solution of GLM $y_2^{(0)}$. We use fourth order Runge-Kutta method (RK4) to obtain that initial value. Therefore GLM should has extra function evaluation than RK.
However in solving the VIDE, we use 3 point Lagrange interpolation for both GLM and RK. Therefore we use RK4 to obtain the initial value at \( y(x = -1) \) for both GLM and RK. Take note that for the GLM, we take \( y_2^{[0]} = y(x = -1) \). Since now both methods use RK4 as a starting method, GLM and RK have the same number of function evaluation. Graphical illustrations are presented from Figures 1 to 5.

5. **Conclusion.** In this paper, we constructed the third order General Linear Method adapted with Simpsons II rule. Handling the integral operator in Volterra integro-differential equations using Lagrange interpolation is demonstrated as well. Then we applied the GLM on some test problems and compared the results with RK method. Numerical results showed that the GLM is more accurate than RK method in solving VIDE. The GLM showed the better performance compared to the same order RK method for all tested problems. It is mainly due to the structure of GLM that naturally generalizes the linear multistep method. It can be caused by the existence of the second output component, \( y_2^{[n]} \) of GLM formulation.

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**Figure 3.** Log maximum error versus number of functions evaluations for Problem 3

**Figure 4.** Log maximum error versus number of functions evaluations for Problem 4

**Figure 5.** Log maximum error versus number of functions evaluations for Problem 5

**REFERENCES**

[1] J. C. Butcher, *General linear methods*, *Acta Numerica*, 15 (2006), 157–256.

[2] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*, John Wiley and Sons, Chichester, 2008.

[3] P. Chartier, E. Hairer and G. Vilmart, *Algebraic structures of B-series*, *Foundations of Computational Mathematics*, 10 (2010), 407–427.

[4] J. R. Dormand, *Numerical Methods for Differential Equations: A Computational Approach*, CRC Press, Florida, 1992.
[5] A. Filiz, A fourth-order robust numerical method for integro-differential equations, *Asian Journal of Fuzzy and Applied Mathematics*, 1 (2013), 28–33.

[6] A. Filiz, Numerical solution of linear volterra integro-differential equations using runge-kutta-felhberg method, *Applied and Computational Mathematics*, 1 (2014), 9–14.

[7] A. Filiz, General linear methods for ordinary differential equations, *Mathematics and Computers in Simulation*, 79 (2009), 1834–1845.

[8] P. Linz, *Analytical and Numerical Methods for Volterra Equations*, SIAM, Philadelphia, 1985.

[9] F. Rabiei, F. A. Hamid, M. M. Rashidi and F. Ismail, Numerical simulation of fuzzy differential equations using general linear method and B-series, *Advances in Mechanical Engineering*, 9 (2010), 1–16.

[10] B. Raftari, Numerical solutions of the linear volterra integro-differential equations: Homotopy perturbation method and finite difference method, *World Applied Sciences Journal*, 9 (2010), 7–12.

[11] A. M. Wazwaz, *Linear and Nonlinear Integral Equations*, Springer, Beijing, 2011.

[12] M. Zarebnia, Sinc numerical solution for the Volterra integro-differential equation, *Nonlinear Sci. Numer. Simulat.*, 15 (2010), 700–706.

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