Generalized Umemura polynomials

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Abstract

We introduce and study generalized Umemura polynomials $U_{n,m}^{(k)}(z, w; a, b)$ which are a natural generalization of the Umemura polynomials $U_n(z, w; a, b)$ related to Painlevé VI equation. We will show that if $a = b$, or $a = 0$, or $b = 0$ then polynomials $U_{n,m}^{(0)}(z, w; a, b)$ generate solutions to Painlevé VI. We will describe a connection between polynomials $U_{n,m}^{(0)}(z, w; a, 0)$ and certain Umemura polynomials $U_k(z, w; \alpha, \beta)$.

§1. Introduction

There exists a vast body of literature about the Painlevé VI equation $P_{VI} := P_{VI}(\alpha, \beta, \gamma, \delta) :$

\[
\frac{d^2 q}{dt^2} = \frac{1}{2} \left( \frac{1}{q + 1} + \frac{1}{q - 1} + \frac{1}{q - t} \right) \left( \frac{dq}{dt} \right)^2 - \left( \frac{1}{t + 1} + \frac{1}{t - 1} + \frac{1}{q - t} \right) \left( \frac{dq}{dt} \right)
+ \frac{q(q - 1)(q - t)}{t^2(t - 1)^2} \left( \alpha - \beta \frac{t}{q^2} + \gamma \frac{(t - 1)}{(q - 1)^2} + \delta \frac{t(t - 1)}{(q - t)^2} \right)
\]  

(1.1)

where $t \in \mathbb{C}$, $q := q(t; \alpha, \beta, \gamma, \delta)$ is a function of $t$, and $\alpha, \beta, \gamma, \delta$ are arbitrary complex parameters. It is well-known and goes back to Painlevé that any solution $q(t)$ of the equation $P_{VI}$ satisfies the so-called Painlevé property:

- the critical points $0, 1$ and $\infty$ of the equation (1.1) are the only fixed singularities of $q(t)$.

- any movable singularity of $q(t)$ (the position of which depends on integration constants) is a pole.

In this paper we introduce and initiate the study of certain special polynomials related to the Painlevé VI equation, namely, the generalized Umemura polynomials $U_{n,m}^{(k)}(z, w; a, b)$. These polynomials have many interesting combinatorial and algebraic properties and in the particular case $n = 0 = k$ coincide with Umemura’s polynomials $U_{m}(z^2, w^2; a, b)$, see e.g. [U], [NOOU]. In the present paper we study recurrence relations between polynomials $U_{n,m}^{(k)}(z, w; a, b)$. Our main result is Theorem [4.1] which gives a generalization of the recurrence relation between Umemura’s polynomials [U]. In some particular cases the recurrence relation obtained in Theorem [4.1] coincides with that for Umemura’s polynomials. As a
corollary, we obtain that polynomials $U_{n,m}^{(0)}(z, w; a, 0)$ also generate solutions to the equation Painlevé $VI$. The main mean in our proofs is Lemma 4.2 from Section 4. For example, we prove a new recurrence relation between Umemura’s polynomials (Theorem 4.9), and describe explicitly connections between polynomials $U_{n,m}(0, b)$ and Umemura’s polynomials $U_m(b_1, b_2)$, see Lemma 4.7. Finally, in section 5 we state a conjectural description of the Plücker relations between certain Umemura’s polynomials.

§2. Painlevé $VI$

In this section we collect together some basic results about equation Painlevé $VI$. More detail and proofs may be found in familiar series of papers by K. Okamoto [OI-OIV]. We refer the reader to the Proceedings of Conference ”The Painlevé property. One century later” [P], where different aspects of the theory of Painlevé equations may be found.

2.1 Hamiltonian form

It is well-known and goes back to a paper by Okamoto [OI] that the sixth Painlevé equation (1.1) is equivalent to the following Hamiltonian system

$$\mathcal{H}_{VI}(b; t, q, p) : \begin{cases} \frac{dq}{dt} = \frac{\partial H}{\partial p}, \\ \frac{dp}{dt} = -\frac{\partial H}{\partial q} \end{cases} \tag{2.1}$$

with the Hamiltonian

$$H := H_{VI}(b; t, q, p) = \frac{1}{t(t-1)} \left[ q(q-1)(q-t)p^2 - \{ (b_1 + b_2)(q-1)(q-t) \\ + (b_1 - b_2)q(q-t) + (b_3 + b_4)q(q-1) \} p + (b_1 + b_3)(b_1 + b_4)(q-t) \right],$$

where $b = (b_1, b_2, b_3, b_4)$ belongs to the parameters space $\mathbb{C}^4$; the parameters $(\alpha, \beta, \gamma, \delta)$ and $(b_1, b_2, b_3, b_4)$ are connected by the following relations

$$\alpha = \frac{1}{2} (b_3 - b_4)^2, \quad \beta = -\frac{1}{2} (b_1 + b_2)^2, \quad \gamma = \frac{1}{2} (b_1 - b_2)^2, \quad \delta = -\frac{1}{2} (b_3 - b_4)(b_3 + b_4 - 2). \tag{2.2}$$

Proposition 2.1 (K. Okamoto, [OI].) If $(q(t), p(t))$ is a solution to the Hamiltonian system (2.1), the function

$$h(b, t) = t(t-1)H_{VI}(b; t, q(t), p(t)) + e_2(b_1, b_3, b_4) - \frac{1}{2} e_2(b_1, b_2, b_3, b_4)$$

satisfies the equation $E_{VI}(b)$:

$$\frac{dh}{dt} \left[ t(t-1)\frac{d^2 h}{dt^2} \right]^2 + \left[ \frac{dh}{dt} \left( 2h - (2t-1)\frac{dh}{dt} \right) + b_1 b_2 b_3 b_4 \right] = \prod_{k=1}^{4} \left( \frac{dh}{dt} + b_k^2 \right). \tag{2.3}$$
Conversely, for a solution \( h := h(b, t) \) to the equation \( E_{VI}(b) \) such that \( \frac{dh}{dt} \neq 0 \), there exists a solution \( (q(t), p(t)) \) to the Hamiltonian system (2.1). Furthermore, the function \( q := q(t) \) is a solution to the Painlevé equation (1.1), where parameters \((\alpha, \beta, \gamma, \delta)\) are determined by the relations (2.2).

We will call the equation \( E_{VI}(b) \) by the Painlevé–Okamoto equation.

2.2 Bäcklund transformation

Consider the following linear transformation of the parameters space \( \mathbb{C}^4 \):

\[
\begin{align*}
  s_1 &:= (b_1, b_2, b_3, b_4) \mapsto (b_2, b_1, b_3, b_4), \\
  s_2 &:= (b_1, b_2, b_3, b_4) \mapsto (b_1, b_3, b_2, b_4), \\
  s_3 &:= (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, b_4, b_3), \\
  s_0 &:= (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, -b_3, -b_4), \\
  l_3 &:= (b_1, b_2, b_3, b_4) \mapsto (b_1, b_2, b_3 + 1, b_4).
\end{align*}
\]

Denote by \( W = \langle s_0, s_1, s_2, s_3, l_3 \rangle \) the subgroup of \( \text{Aut}\mathbb{C}^4 \) generated by these transformations. It is not difficult to see, that \( W \cong W(D_4^{(1)}) \), i.e. \( W \) is isomorphic to the affine Weyl group of type \( D_4^{(1)} \).

**Proposition 2.2** (K. Okamoto, [OI].) For each \( w \in W \), there exists a birational transformation

\[
L_w : \{ \text{solutions to } \mathcal{H}_{VI}(b) \} \mapsto \{ \text{solutions to } \mathcal{H}_{VI}(w(b)) \}. 
\]

The birational transformations \( L_w, w \in W(D_4^{(1)}) \) are called by Bäcklund transformations associated to the equation Painlevé \( VI \).

2.3 \( \tau \)-function

Let \((q(t), p(t))\) be a solution to the Hamiltonian system (2.1), the \( \tau \)-function \( \tau(t) \) corresponding to the solution \((q(t), p(t))\) is defined by the following equation

\[
\frac{d}{dt} \log \tau(t) = H_{VI}(b; t, q(t), p(t));
\]

in other words,

\[
\tau(t) = (\text{constant}) \exp \left( \int H_{VI}(b; t, q(t), p(t))dt \right). 
\]
2.4 Umemura polynomials

Suppose that $b_3 = -\frac{1}{2}$, $b_4 = 0$, then it is well-known and goes back to Umemura’s paper [U], that the pair

$$(q_0, p_0) = \left( \frac{(b_1 + b_2)^2 - (b_1^2 - b_2^2)\sqrt{t(1-t)}}{(b_1 - b_2)^2 + 4b_1b_2t}, \frac{b_1q_0 - \frac{1}{2}(b_1 + b_2)}{q_0(q_0 - 1)} \right)$$

defines a solution to the Hamiltonian system (2.1) with parameters $b = (b_1, b_2, -\frac{1}{2}, 0)$. Note, see e.g. [U], that

$$H_0 = H_{VI}((b_1, b_2, -\frac{1}{2}, 0); t, q_0(t), p_0(t))$$

$$= \frac{1}{t(t-1)} \left\{ b_1(b_1 - 1)(1 - 2t) + 2b_1^2 \sqrt{t(t-1)} + 2b_2(b_1 - t) \
+ b_2(b_2 - 1)(1 - 2t) - 2b_2^2 \sqrt{t(t-1)} \right\},$$

and

$$\tau_0(t) = \exp \left\{ \int H_0(t) dt \right\}.$$
For example, \( T_0 = 1, T_1 = 1, T_2 = \frac{1}{2} (-4b_1^2 + 1)(2 - v)/4 + (-4b_2^2 + 1)(2 + v)/4 \).

It follows from the Toda equation (2.4) that polynomials \( T_n := T_n(v) \) satisfy the following recurrence relation [U]:

\[
T_{n-1}T_{n+1} = \left\{ \frac{1}{4} (-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2)v) + (n - \frac{1}{2})^2 \right\} T_n^2
+ \frac{1}{4} (v^2 - 4)^2 \left\{ T_n \frac{d^2 T_n}{dv^2} - \left( \frac{dT_n}{dv} \right)^2 \right\}
+ \frac{1}{4} (v^2 - 4) v T_n \frac{d T_n}{dv}
\]  

(2.5)

with initial conditions \( T_0 = T_1 = 1 \).

**Definition 2.4** Polynomials \( U_n := U_n(z, w, b_1, b_2) := 2^{n(n-1)} T_n(v) \), where \( z = \frac{2-v}{4}, w = \frac{2+w}{4} \), are called by Umemura polynomials.

The formula (2.6) below was stated as a conjecture by S. Okada, M. Noumi, K. Okamoto and H. Umemura [NOOU] and has been proved recently by M. Taneda, and A. N. Kirillov (independently):

\[
2^{n(n-1)} T_n(v) := U_n(z, w, b_1, b_2) = \sum_{I \subset [n-1]} d_n(I) c_I d_{[n-1] \setminus I} z^{|I|} w^{|I'|},
\]  

(2.6)

where

(i) \([n - 1] = \{1, 2, \ldots, n - 1\}\); for any subset \( I = \{i_1 > i_2 > \cdots > i_p\} \subset [n - 1] \),
\( d_n(I) = \dim_{GL(n)}^{\lambda(I)} \) stands for the dimension of irreducible representation of the general linear group \( GL(n) \) corresponding to the highest weight \( \lambda(I) \) with the Frobenius’ symbol \( \lambda(I) = (i_1, i_2, \ldots, i_p | i_1 - 1, i_2 - 1, \ldots, i_p - 1) \);

(ii) \( c = -4b_1^2, d = -4b_2^2, z = \frac{2-v}{4}, w = \frac{2+w}{4} \);

(iii) \( \bar{c} = c + (2k - 1)^2, \bar{d} = d + (2k - 1)^2, c_k = \bar{c}_1 \bar{c}_2 \cdots \bar{c}_k, d_k = \bar{d}_1 \bar{d}_2 \cdots \bar{d}_k \);

(iv) \(|I| = i_1 + i_2 + \cdots + i_p \).

Recall that Frobenius’ symbol \((a_1, a_2, \ldots, a_p | b_1, b_2, \ldots, b_p)\) denotes the partition which corresponds to the following diagram

![Diagram](image)
3 Generalized Umemura polynomials

Let $n, m, k$ be fixed nonnegative integers, $k \leq n$. Denote by $[n;m]$ the set of integers \{1, 2, \ldots, n, n+2, n+4, \ldots, n+2m\}. Let $I$ be a subset of the set $[n;m]$. Follow [DK], define the numbers

$$d_{n,m}(I) = \prod_{i \in I, j \in [n;m] \setminus I} \left| \frac{i + j}{i - j} \right|, \quad c(I) = \sum_{i \in I, j > n} \frac{i - n}{2}$$

(iii) for any subset $I \subset [n;m]$, we set $a_I = \prod_{i \in I} a_i$, $b_I = \prod_{i \in I} b_i$;

(iv) $e^{(n,m,k)}_I(z, w) = a_{I \setminus [k]} b_{[n,m] \setminus I} z^{|I|} w^{|[n;m] \setminus I|}$.

Note that the polynomial $U^{(0)}_{n,m}$ coincides with Umemura’s polynomial $T_m(z^2, w^2; a, b)$. The formula for generalized Umemura polynomials stated below follows from the Cauchy identity, and was used by J. P. van Diejen and A. N. Kirillov [DK] in their study of $q$-spherical functions.

Lemma 3.1 The generalized Umemura polynomials $U^{(k)}_{n,m}(a, b; z, w)$ admit the following determinantal expression

$$U^{(k)}_{n,m}(a, b; z, w) = \det \left| a_i w^i \prod_{s \in [k]} \left( \frac{i + s}{i - s} \right) \delta_{i,j} + \frac{2i}{i+j} c(i) \prod_{s \in [n,m], s \neq i} \left| \frac{i + s}{i - s} b_i z^i \right|_{i,j \in [n,m] \setminus [k]} \right|_{i,j \in [n,m] \setminus [k]}$$

where $c(i) = i$ if $i \leq n$, and $(i - n)/2$ if $i > n$.

In the particular case $k = 0$, $n = 0$ this formula gives a determinantal representation for Umemura’s polynomials and has many applications.
4 Main result

Let us introduce notation $U_{n,m} := U_{n,m}^{(0)}(z, w; a, b)$. The main result of our paper describes a recurrence relation between polynomials $U_{n,m}$.

Theorem 4.1

$$
U_{n,m} - U_{n,m+1} = \left( -\bar{a}_{n+2m+2}z^2 + \bar{b}_{n+2m+2}w^2 \right) U_{n,m}^2 + 8z^2w^2D_x^2U_{n,m} \circ U_{n,m}
$$

where for any two functions $f = f(x)$ and $g = g(x)$

$$
D_x^2f \circ g = f''g - 2f'g' + fg''
$$

denotes the second Hirota derivative, and $\bar{a}, \bar{b}$, here variables $z, w$ and $x$ are connected by the relations $z = \frac{1}{2}(e^x + e^{-x} - 2)^{1/2}$, $w = \frac{1}{2}(e^x + e^{-x} + 2)^{1/2}$.

Below we give a sketch of our proof of Theorem 4.1. Detailed exposition will appear elsewhere. The main step of the proof is to establish the following algebraic identity which appears to have an independent interest.

Lemma 4.2 For any two subsets $I, J$ of the set $[n; m]$, we have

$$
\prod_{\lambda \in I} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) \prod_{\lambda \in J} \left( \frac{x - \lambda}{x + \lambda} \right) + \prod_{\lambda \in I} \left( \frac{x - \lambda}{x + \lambda} \right) \prod_{\lambda \in J} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) = 2 + \sum_{\lambda \in I \cup J} \frac{b_{I,J}^\lambda}{(x + 2 - \lambda)(x + \lambda)},
$$

where $b_{I,J}^\lambda$ are some constants (depending on $\lambda, I$ and $J$) which may be computed explicitly.

One can prove this lemma by using the residue theorem.

Lemma 4.3 For any two subsets $I, J$ of the set $[n; m]$, we have

$$
\sum_{\lambda \in I \cup J} b_{I,J}^\lambda = 4(|I| - |J|)^2 - 4(|I| + |J|).
$$

This lemma follows from Lemma 4.2.

Lemma 4.4 For an element $\lambda \in I \cap J$, we have $b_{I,J}^\lambda = 0$ if and only if $\lambda - 2 \in I \cap J$. For an element $\lambda \in I \setminus (I \cap J)$, we have $b_{I,J}^\lambda = 0$ if and only if $\lambda - 2 \in J$.

This lemma follows from Lemma 4.2 by direct calculation.
Remarks 1. If $n = 0$, then $U_{0,m} = U_{m+1}(z^2, w^2; a, b)$ coincides with the Umemura polynomial, and $U^{(1)}_{0,m} = 0$. In this case the recurrence relation (4.1) has been used by M. Taneda in his proof of Okada–Noumi–Okamoto–Umemura’s Conjecture (2.6).

2. Note that $U_{0,m} = U_{2,m-1}/(2m + 1)$, and more generally
$$ U^{(k)}_{k,m} = U^{(k+1)}_{k+2,m-1}(2k+1)!!/(2k+2m+1)!! $$(4.1), where $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$.

3. "Unwanted term" in (4.1) which contains $(U^{(1)}_{m,m})^2$ vanishes if either $a = 0$, or $b = 0$, or $a = b$.

In the case $a = b$ and $k = 0$ the expression $e_{n,m,k}^{(n,m,k)}(z, w)$ doesn’t depend on a subset $I \subset [n; m]$ and is equal to $a_{[n;m]} z^{|I|} w^{|[n;m]\setminus I|}$. Hence, in this case
$$ U^{(0)}_{n,m}(z, w; a, a) = a_{[n;m]} \sum_{I \subset [n; m]} d_{n,m}(I) (-1)^{|I|} z^{|I|} w^{|[n;m]\setminus I|} $$
$$ = a_{[n;m]} (z + w) \left( \frac{n + m + 1}{2} \right) (z - w) \left( \frac{m}{2} + 1 \right). $$(4.2)

Recall that $\bar{a}_i = a + (i - 1)^2$, $a_{2i} = \bar{a}_2 \bar{a}_4 \cdots \bar{a}_{2i}$, $a_{2i+1} = \bar{a}_1 \bar{a}_3 \cdots \bar{a}_{2i+1}$ and $a_{[n;m]} = \prod_{i \in [n; m]} a_i$.

The last equality in (4.2) has been proved for the first time by J.P. van Diejen and A.N. Kirillov [DK]. On the other hand, we can show that polynomials
$$ X_{n,m}(z, w; a) = a_{[n;m]} z^{|I|} w^{|\bar{I}|} $$
also satisfy the recurrence relation (4.1) and coincide with polynomials $U^{(0)}_{n,m}(z, w; a, a)$ if $m = 0$. From this observation we can deduce the equality $X_{n,m}(z, w; a) = U^{(0)}_{n,m}(z, w; a, a)$, which is equivalent to the main identity from [DK]. Another case when “unwanted term” in (4.1) vanishes is the case when either $a = 0$, or $b = 0$. In this case we have

Corollary 4.5 Assume that $b = 0$, then polynomial $U_{n,m}(z, w; a, 0)$ defines a solution to the equation Painlevé VI.

Finally, we are going to compare polynomials $U_{n,m}(z, w; a, 0)$ and $U_m(z, w; a, 0)$. For this goal, let us consider functions
$$ h_0 := h_0(t) = \left\{ b_1^2 \left( \sqrt{t} - \sqrt{t - 1} \right)^2 + b_2^2 \left( \sqrt{t} + \sqrt{t + 1} \right)^2 \right\} / 4, $$
and
$$ h_{n,m} := h_{n,m}(b_1, b_2) = t(t - 1) \log (U_{n,m})' - h_0. $$

Proposition 4.6 (i) $h_{0,m}$ satisfies the Painlevé–Okamoto equation $E_{VI}(b_1, b_2, m+1/2, 0)$;

(ii) $h_{1,m} = -(2t - 1)(m + 1)^2/2$ satisfies the equation $E_{VI}(0, m + 1, b_3, b_4)$;

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(iii) \( h_{n,m}(0, b_2) \) satisfies the equation \( E_{VI}(0, b_2, \frac{n}{2}, \frac{n+2m+1}{2}) \).

Proposition 4.8 follows from Lemma 4.7 and Lemma 4.8 below. Let us define \( U_{n,m}(b_1, b_2) := U_{n,m}(z, w; -4b_1^2, -4b_2^2) \), then

**Lemma 4.7**

\[
U_{n,m}(0, b_2) = \begin{cases} 
  b_{[n;m]_o} w_{(\frac{1}{2})}^2 U_{0,m+\frac{1}{2}} \left( \frac{n}{2}, b_2 \right), & \text{if } n \text{ is even,} \\
  b_{[n;m]_o} w_{(\frac{1}{2}+m+1)}^2 U_{0,m+\frac{1}{2}} \left( m + \frac{n+1}{2}, b_2 \right), & \text{if } n \text{ is odd,}
\end{cases} \tag{4.3}
\]

where \( [n; m]_o = \{ i \in [n; m] \mid i \text{ is odd} \} \).

From Lemma 4.7 we can deduce the following

**Lemma 4.8**

\[
h_{n,m}(0, b_2) = \begin{cases} 
  h_{0,m+\frac{1}{2}} \left( \frac{n}{2}, b_2 \right), & \text{if } n \text{ is even}, \\
  h_{0,m+\frac{1}{2}} \left( m + \frac{n+1}{2}, b_2 \right), & \text{if } n \text{ is odd.}
\end{cases} \tag{4.4}
\]

It follows from Lemma 4.7, (4.4) and Theorem 4.1, that Umemura’s polynomials \( U_m(b_1, b_2) \) satisfy a new recurrence relation with respect to the first argument \( b_1 \).

**Theorem 4.9**

\[
U_m(b_1 - 1, b_2)U_m(b_1 + 1, b_2)(b_1^2 - b_2^2) = (b_1^2 - b_2^2)U_m^2(b_1, b_2) + 2z^2D_x^2U_m(b_1, b_2) \circ U_m(b_1, b_2).
\]

Recall that \( D_x^2 \) denotes the second Hirota derivative.

## 5 Conjecture

We define \( q_m := q_m(t) \) by

\[
q_m - t = 4U_m^2 \left\{ (m + \frac{1}{2}) t(t - 1) \frac{d}{dt} \log U_{m+1} - \left( m + \frac{3}{2} \right) t(t - 1) \frac{d}{dt} \log U_m \right. \\
- \frac{1}{2} b_1 b_2 + \frac{1}{4} \left( b_1^2 \frac{z}{w} + b_2^2 \frac{w}{z} \right) \left/ \left( U_{m+1}U_{m-1} - (2m + 1)U_m^2 \right) \right. \right\}.
\]

One can check that \( q_m \) is a solution to both equations \( P_{VI}(b_1, b_2, m + \frac{1}{2}, 0) \) and \( P_{VI}(b_1, b_2, 0, m + \frac{1}{2}) \). It follows from Okamoto’s theory [OI] that the function

\[
\bar{h}_{1,m} = t(t - 1) \frac{d}{dt} \log U_{m+1} - \frac{1}{4} \left( b_1^2 \frac{z}{w} + b_2^2 \frac{w}{z} \right) + \left( m + \frac{1}{2} \right) q_m - \frac{1}{2} \left( m + \frac{1}{2} \right) \quad (5.1)
\]

is also a solution to \( E_{VI}(b_1, b_2, n + \frac{1}{2}, 1) \). Based on the latter expression for the function \( \bar{h}_{1,m} \), and using Lemmas 5 and 6, we come to the following
Conjecture 5.1  If $b_1 = 0$, then we have

$$U_{m+1}U_{m-1} - (2m + 1)^2U_m^2 = \frac{1}{4b_2^2}U_{2,m-1}^2,$$

where $U_m := U_m(0, b_2)$ is a special case of Umemura’s polynomial, and $U_{2,m-1} = U_{2,m-1}(0, b_2)$.

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