COMPOSITION OPERATORS ON $H(b)$ SPACES OF UNIT BALL IN $\mathbb{C}^n$

SİBEL ŞAHİN

Abstract. In this work the composition operators on deBranges-Rovnyak spaces $H(b)$ on the unit ball of $\mathbb{C}^n$ are studied. The boundedness and compactness of composition operators are characterized through Carleson conditions over the Clark measure associated to the defining function $b$.

CONTENTS

Introduction 1
1. Preliminaries 2
2. Boundedness of Composition Operators on $H(b)$ 6
3. Compactness of Composition Operators on $H(b)$ 7
References 9

INTRODUCTION

The behavior of composition operators over holomorphic function spaces is an area of great interest in the intersection of operator theory and complex analysis. Especially there is a vast literature over the boundedness and compactness of composition operators over Hardy spaces. In order to examine those properties various methods are used however some techniques became classical over the years. One of these methods is to use Nevanlinna counting functions, led by Shapiro’s eminent work [Sh93], boundedness and compactness of a composition operator $C_\varphi$ on $H^2(\mathbb{D})$ with the symbol $\varphi$ is determined by the logarithmic growth of the counting function

$$N_\varphi(\omega) = \sum_{\varphi(z) = \omega} - \log |z|.$$ 

Over the last two decades a new family of holomorphic functions takes considerable attention by being in the intersection of holomorphic function theory, operator theory and functional analysis, namely deBranges-Rovnyak spaces $H(b)$. The classical $H(b)$ space of the unit disc is a contractive subspace of $H^2(\mathbb{D})$ defined by $H(b) = (I - T_b T_b^*)^{1/2} H^2(\mathbb{D})$ where $T_b$ is the Toeplitz operator with symbol $b \in H^\infty(\mathbb{D})$ with $\|b\|_\infty \leq 1$.
In 2019, Fricain et al. [FKM19] gave the conditions for the compactness of composition operators on $H(b)$ of unit disc via Nevanlinna counting function. According to their result for some $\gamma \in (0, 1/3)$ if
\[
N_{\varphi}(z) \left( \frac{(1 - |b(z)|)^\gamma}{1 - |z|^2} \right)^2 \to 0 \text{ as } |z| \to 1
\]
then the operator $C_{\varphi} : H(b) \to H^2(\mathbb{D})$ is compact ([FKM19], Theorem 1).

One other classical method for understanding the behavior of composition operators is to use Carleson conditions. For both Hardy spaces of unit disc and unit ball Cowen-Maccluer gave the boundedness and compactness criteria through Carleson conditions on Lebesgue measure in their prominent work [CM95].

In [S20], Şahin took the definition of deBranges-Rovnyak spaces to the setting of unit ball of $\mathbb{C}^n$ and studied the structure of these spaces via Clark measures and angular derivatives. In that work a partial result about the compactness of composition operators was given through the angular derivative of the symbol function $\varphi$.

In this study we aim to give the full characterization of bounded and compact composition operators on $H(b)$ spaces of unit ball through Carleson conditions over Clark measure associated to the defining function $b$. In the first main result we will see that a composition operator on $H(b)$ of unit ball is bounded as long as the pullback of the Clark measure satisfies the Carleson condition. As a result we also see that for a bounded $C_{\varphi}$, the boundary value function of the symbol cannot take sets of positive Clark measure to zero measure sets. In the last part of the article we consider the compactness and see that a composition operator $C_{\varphi}$ is compact as long as the pullback Clark measure is of vanishing order throughout the boundary $S^n$.

1. Preliminaries

The classical deBranges-Rovnyak space of the unit disc $H(b)$ is defined as follows:

For $\psi \in L^\infty(T)$, let $T_\psi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be the Toeplitz operator defined as $T_\psi(f) = P_\psi(\psi f)$ with $P_\psi$ being the Riesz projection. For a non-constant $b \in H^\infty(\mathbb{D})$ with $\|b\|_\infty \leq 1$

\[
H(b) = (1 - T_0 T_b^*)^{1/2} H^2(\mathbb{D})
\]

where $H^2(\mathbb{D})$ is the classical Hardy-Hilbert space of the unit disc.

In $\mathbb{D}$, $H(b)$ is contractively contained in $H^2(\mathbb{D})$ therefore for any $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic function the composition operator $C_\varphi : H(b) \to H^2(\mathbb{D})$ is a bounded operator. When it comes to the compactness, in 2019 Fricain et al. gave the characterization of compact composition operators on $H(b)$ through Nevanlinna counting function

\[
N_{\varphi}(\omega) = \sum_{\varphi(z) = \omega} -\log |z|, \quad \omega \in \mathbb{D}
\]

analogous to Shapiro’s results ([Sh93], pg:180) on $H^2(\mathbb{D})$: 
Theorem 1.1 (FKM19, Theorem 1). Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic such that $\varphi(0) = 0$. Let $b$ be a function in the unit ball of $H^\infty(\mathbb{D})$. Consider the following statements:

(i) For some $\gamma \in (0, 1/3)$ we have
\[ N_\varphi(z) \left( \frac{(1 - |b(z)|^2)}{1 - |z|^2} \right)^2 \to 0 \text{ as } |z| \to 1 \]

(ii) The operator $C_\varphi : H(b) \to H^2(\mathbb{D})$ is compact

(iii) $N_\varphi(z) \frac{(1 - |b(z)|)}{1 - |z|^2} \to 0 \text{ as } |z| \to 1$

(iv) The operator $C_\varphi : X(b) \to H^2(\mathbb{D})$ is compact where $X(b)$ is a contractively embedded subspace of $H(b)$.

Then we have the following implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Remark 1.2. For a detailed description of the space $X(b)$ one may check [FKM19] section 2.2.

From now on we will focus on the multivariable setting, specifically the unit ball $B_n \subset \mathbb{C}^n$. First of all, following the notation of Rudin [R80], let $H^\infty(B_n)$ be the space of bounded holomorphic functions of the unit ball and $H^2(B_n)$ be defined as

\[ H^2(B_n) = \left\{ f \in \mathcal{O}(B_n) : \sup_{0<r<1} \int_{S^n} |f(r\xi)|^2 d\sigma(\xi) < \infty \right\} \]

where the inner product on $H^2(B_n)$ is defined as

\[ \langle f, g \rangle = \int_{S^n} f \overline{g} d\sigma, \quad f, g \in H^2(B_n). \]

For the classical Hardy-Hilbert space $H^2(B_n)$, the boundedness and the compactness of the composition operators can be characterized via Carleson conditions:

For $\xi \in S^n$ and $h > 0$ let
\[ S(\xi, h) = \{ z \in \mathbb{B}^n : |1 - \langle z, \xi \rangle| < h \} \]
\[ Q(\xi, h) = S(\xi, h) \cap S^n \]
then Cowen-Maccluer [CM95, Theorem 3.35] gave the full characterization of bounded and compact composition operators on $H^2(B_n)$ as follows:

Notation 1. For any function $\gamma$, the function $\gamma^*$ represents the boundary value of $\gamma$.

Theorem 1.3 (CM95, 3.35). Suppose $\varphi : B^n \to B^n$ is analytic. Define a Borel measure $\mu$ on $B^n$ by $\mu(A) = \sigma_n(\varphi^{-1}(A))$. Then,

1. $C_\varphi$ is bounded on $H^2(B^n)$ if and only if there exists $C < \infty$ so that $\mu(S(\xi, h)) \leq C h^n$ for $\xi \in S^n, h > 0$.

2. $C_\varphi$ is compact on $H^2(B^n)$ if and only if $\mu(S(\xi, h)) = o(h^n)$ as $h \to 0$ uniformly in $\xi$. 
In this work we will take this approach to the setting of the composition operators on deBranges-Rovnyak spaces $H(b)$ of $\mathbb{B}^n$. For a non-constant, holomorphic function $b \in H^\infty(\mathbb{B}^n)$ with $\|b\|_\infty \leq 1$, $H(b)$ of $\mathbb{B}^n$ is the subspace of $H^2(\mathbb{B}^n)$ defined by the inner product
\[
\|(I - T_b T_b^*)^{1/2} f\|_b = \|f\|_2, \quad f \in H^2(\mathbb{B}^n) \ominus Ker(I - T_b T_b^*)^{1/2}.
\]

In [S20], Şahin gave a partial result about the compactness of the composition operator via angular derivative of the symbol function $\varphi$:

**Theorem 1.6** ([S20], Theorem 2.1). The mapping $V_b$ is a partial isometry from $L^2(\mu)$ onto $H(b)$. Moreover, for an $f \in H(b)$ there exists a unique $g \in H^2(\mu)$ such that $f = V_b(g)$ i.e
\[
f(z) = (1 - b(z)) \int_{\mathbb{B}^n} \frac{g(\xi)}{(1 - \langle z, \xi \rangle)^n} d\mu(\xi), \quad z \in \mathbb{B}^n.
\]

Before ending this section, for the sake of completeness we will give the unit ball version of the Lebesgue decomposition result of Clark measure:

**Proposition 1.7.** Let $\mu$ be the Clark measure associated to $b$ and $\mu = h d\sigma + \mu_s$ be the Lebesgue decomposition of $\mu$. Then,
\[
\|\mu_s\| = \lim_{\epsilon \to 0} \lim_{r \to 1} \int_{|1 - b(\xi)| < \epsilon} \frac{1 - r^2 |b(\xi)|^2}{1 - r^2 |b(\xi)|^2} d\sigma(\xi).
\]

Furthermore, if $\frac{1}{1 - b} \in L^2(S^n)$ then $\mu$ is absolutely continuous with respect to $d\sigma$ on $S^n$. 

**Definition 1.5.** Given an $\alpha \in \mathbb{T}$ and a holomorphic function $\psi : \mathbb{B}^n \to \mathbb{D}$ the quotient
\[
\Re \left( \frac{\alpha + \psi(z)}{\alpha - \psi(z)} \right) = \frac{1 - |\psi(z)|^2}{|\alpha - \psi(z)|^2}
\]
is positive and pluriharmonic therefore there exists a unique positive measure $\mu$ satisfying
\[
\int_{S^n} P(z, \xi) d\mu(\xi) = \Re \left( \frac{\alpha + \psi(z)}{\alpha - \psi(z)} \right).
\]
where $P(z, \xi)$ is the invariant Poisson kernel of $\mathbb{B}^n$. This measure $\mu$ is defined to be the Clark measure associated to $\psi$. 

From this point on $b$ always refers to the defining function of the space $H(b)$. Let $\mu$ be the Clark measure associated to the defining function $b$. Then the operator $V_b$ on $L^2(\mu)$ is defined as follows
\[
V_b(f)(z) = (1 - b(z)) \int_{\mathbb{B}^n} \frac{f(\xi)}{(1 - \langle z, \xi \rangle)^n} d\mu(\xi).
\]

**Proposition 1.4** ([S20], Corollary 3.1). If the composition operator with symbol $\varphi$, $C_\varphi$, is compact on $H(b)$ and $\frac{1 - |b|}{1 - |b(\mathbf{w}_n)|^2}$ is uniformly bounded for any sequence $\mathbf{w}_n \to S^n$ then $\varphi$ cannot have finite angular derivative at any point of $S^n$. 

Our main results in this work will give the full characterization of boundedness and compactness via Carleson conditions but first we will give the necessary background information about the behavior of $H(b)$ spaces of $\mathbb{B}^n$. 

In [S20], Şahin gave a partial result about the compactness of the composition operator via angular derivative of the symbol function $\varphi$: 

**Proposition 1.7.** Let $\mu$ be the Clark measure associated to $b$ and $\mu = h d\sigma + \mu_s$ be the Lebesgue decomposition of $\mu$. Then,
\[
\|\mu_s\| = \lim_{\epsilon \to 0} \lim_{r \to 1} \int_{|1 - b(\xi)| < \epsilon} \frac{1 - r^2 |b(\xi)|^2}{1 - r^2 |b(\xi)|^2} d\sigma(\xi).
\]

Furthermore, if $\frac{1}{1 - b} \in L^2(S^n)$ then $\mu$ is absolutely continuous with respect to $d\sigma$ on $S^n$. 

**Theorem 1.6** ([S20], Theorem 2.1). The mapping $V_b$ is a partial isometry from $L^2(\mu)$ onto $H(b)$. Moreover, for an $f \in H(b)$ there exists a unique $g \in H^2(\mu)$ such that $f = V_b(g)$ i.e
\[
f(z) = (1 - b(z)) \int_{\mathbb{B}^n} \frac{g(\xi)}{(1 - \langle z, \xi \rangle)^n} d\mu(\xi), \quad z \in \mathbb{B}^n.
\]
Proof. Let $0 \leq r \leq 1$, for the holomorphic function $b$ the quotient

$$
\frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2}, \quad z \in \mathbb{B}^n
$$

is a pluriharmonic function and

$$
\int_{S^n} \frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2} = \frac{1 - r^2|b(0)|^2}{|1 - rb(0)|^2}
$$

As we know from the definition of Clark measure

$$
\int_{S^n} d\mu = \frac{1 - r^2|b(0)|^2}{|1 - rb(0)|^2}.
$$

Moreover, for $|1 - b(\xi)| > \varepsilon$

$$
\frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2} \rightarrow \frac{1 - |b(\xi)|^2}{|1 - b(\xi)|^2}
$$

uniformly as $r \rightarrow 1$ and we have,

$$
\|\mu\| - \int_{|1 - b(\xi)| \geq \varepsilon} \frac{1 - |b(\xi)|^2}{|1 - b(\xi)|^2} d\sigma(\xi) = \lim_{r \rightarrow 1} \int_{|1 - b(\xi)| < \varepsilon} \frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2} d\sigma(\xi).
$$

Now since the boundary value of the kernel function gives us the Radon-Nikodym derivative according to the Fatou Theorem, for unit ball we obtain

$$
\frac{1 - |b(\xi)|^2}{|1 - b(\xi)|^2} = h(\xi) \quad \text{and} \quad h \in L^1(S^n) \tag{\dagger}.
$$

Hence,

$$
\lim_{\varepsilon \rightarrow 0} \int_{|1 - b(\xi)| \geq \varepsilon} \frac{1 - |b(\xi)|^2}{|1 - b(\xi)|^2} d\sigma(\xi) = \int_{S^n} h(\xi) d\sigma(\xi).
$$

Therefore, we have

$$
\lim_{\varepsilon \rightarrow 0} \lim_{r \rightarrow 1} \int_{|1 - b(\xi)| \leq \varepsilon} \frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2} d\sigma(\xi) = \|\mu_s\|.
$$

Now for almost all $\xi \in S^n$, we have $|1 - b(\xi)| \leq 2^n|1 - rb(\xi)|$ and

$$
\frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2} \leq \frac{2^{n+1}}{|1 - b(\xi)|^2}.
$$

From the assumption we have $\frac{1}{1 - b} \in L^2(S^n)$ and by Lebesgue Dominated Convergence Theorem

$$
\lim_{r \rightarrow 1} \int_{|1 - b(\xi)| < \varepsilon} \frac{1 - r^2|b(\xi)|^2}{|1 - rb(\xi)|^2} d\sigma(\xi) = \int_{|1 - b(\xi)| < \varepsilon} \frac{1 - |b(\xi)|^2}{|1 - b(\xi)|^2} d\sigma(\xi)
$$

but the integrand on the right is in $L^1(S^n)$ by (\dagger) so the right hand side approaches to 0 as $\varepsilon \rightarrow 0$ and $\|\mu_s\| = 0$ hence the Clark measure is absolutely continuous with respect to Lebesgue measure.

$\square$
2. BOUNDEDNESS OF COMPOSITION OPERATORS ON $H(b)$

In this section we will give our first main result on the boundedness of composition operators on $H(b)$ spaces of unit ball $\mathbb{B}^n$. It is the complete description of bounded composition operators through Carleson condition on the Clark measure associated to the defining function $b$.

Before proceeding with the main result let us first give a very important and powerful result that is used repeatedly in the literature and will be a useful tool for us:

**Theorem 2.1** ([CM95], 2.37). For $\mu$ a finite, positive Borel measure on $\mathbb{B}^n$ the following are equivalent

1. There is constant $K < \infty$ so that $\mu(S(b,h)) < Kh^n$ for $|b| = 1$ and $0 < h < 1$.
2. There is a constant $C < \infty$ so that
   $$\int_{\mathbb{B}^n} |f|^2 d\mu \leq C\|f\|_2^2$$
   for all $f$ in $H^2(\mathbb{B}^n)$.

Now we can give the boundedness result on $H(b)$ spaces of $\mathbb{B}^n$:

**Theorem 2.2.** Let $\varphi : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic function. Set up a Borel measure on $\mathbb{B}^n$ by $\nu(A) = \mu(\varphi(A)^{-1})(A)$ where $A$ is any Borel set in $\mathbb{B}^n$ and $\mu$ is the Clark measure associated to $b$. Suppose that $\frac{1}{1-b} \in L^2(S^n)$.

Then $C_{\varphi}$ is bounded on $H(b)$ if and only if there exists $C > 0$ such that

$$\nu(S(\xi,h)) \leq Ch^n$$

for $\xi \in S^n$, $h > 0$.

**Proof.** First of all let us suppose that $C_{\varphi}$ is bounded on $H(b)$. For a fixed $\eta \in S^n$ and $0 < h < 1$ set $\omega = (1-h)\eta$ and consider the functions

$$f_{\omega}(z) = V_b K(z,\omega).$$

For these functions and for $\nu = \mu \varphi^{-1}$ since $C_{\varphi}$ is bounded on $H(b)$ there exists $C > 0$ such that

$$\|f_{\omega} \circ \varphi\|_b^2 = \|V_b K(\varphi(z),\omega)\|_b^2 = \|K(\varphi(z),\omega)\|_{L^2(\mu)}^2$$

is satisfied. Therefore,

$$\int_{S^n} |K(\varphi(z),\omega)|^2 d\mu = \int_{\mathbb{B}^n} |K(z,\omega)|^2 d\nu \leq C\|f_{\omega}\|_b^2 = C\|K(z,\omega)\|_{L^2(\mu)}^2$$

and $\frac{1}{1-b} \in L^2(S^n)$ we have

$$\nu(S(\eta,h))(2h)^{-2n} \leq C_1 h^{-n}$$

A simple computation shows that $|K(z,\omega)|^2 \geq (2h)^{-2n}$ on $S(\eta,h)$ and it was given in ([RS0], pg:18) that $\|K(z,\omega)\|_{\mathbb{B}^n}^2 \sim h^{-n}$ and since $\mu$ is absolutely continuous with respect to the Lebesgue measure because of the assumption that $\frac{1}{1-b} \in L^2(S^n)$ we have

$$\nu(S(\eta,h))(2h)^{-2n} \leq C_2 h^{-n}$$

and therefore $\nu(S(\eta,h)) \leq C_2 h^n$. 
Conversely, suppose that for $h > 0$,
\[ \nu(S(\eta, h)) \leq C h^n \text{ for all } \eta \in S^n. \]
Now take $f \in A(\mathbb{B}^n)$, where $A(\mathbb{B}^n)$ is the algebra of holomorphic functions in $\mathbb{B}^n$ and continuous on $\mathbb{B}^n$ then by (S20, Theorem 2.1) there exists a $g \in H^2(\mu)$ such that $f = V_b g$ and we have
\[ \|f \circ \varphi\|_b = \|V_b g \circ \varphi\|_{L^2(\mu)} = \|V_b g\|_{L^2(\nu)} \]
where the last equality is due to the change of variables formula. By (1) $\Rightarrow$ (2) part of Theorem 2.1 we know that there exists $C_1 > 0$ such that
\[ \|V_b g\|_{L^2(\nu)} \leq C_1 \|V_b g\|_2 \leq C_2 \|V_b g\|_b = C_2 \|f\|_b \]
where the second inequality is a result of the inclusion $H^1(b) \hookrightarrow H^2(\mathbb{B}^n)$. Hence,
\[ \|f \circ \varphi\|_b \leq C_2 \|f\|_b \text{ and since } A(\mathbb{B}^n) \text{ is dense in } H^1(b) \text{ we obtain that } C_\varphi \text{ is bounded on } H^1(b). \]

An immediate consequence of this result gives us that the boundedness of the composition operator also determines the behavior of the Clark measure over sets of positive measure:

**Corollary 2.3.** If $C_\varphi$ is bounded on $H^1(b)$ then $\varphi^* \text{ does not take a set of } \text{positive Clark measure in } S^n \text{ into a set of zero Lebesgue measure.}$

**Proof.** Suppose that $\varphi^*(A) \subset E \subset S^n$ and $\sigma(E) = 0$ where $\sigma$ is the Lebesgue measure. Let $\varepsilon > 0$ be given then there exist Carleson sets $Q(\xi_k, h_k)$ with
\[ E \subset \bigcup_{k=1}^{\infty} Q(\xi_k, h_k) \text{ and } \sum_{k=1}^{\infty} \sigma(Q(\xi_k, h_k)) \leq \varepsilon. \]
Since $\sigma(Q(\xi_k, h_k)) \sim h_k^n$, for some $c > 0$ we have $\sum_{k=1}^{\infty} h_k^n < c \varepsilon$. If $C_\varphi$ is bounded then by the previous theorem there is a $C > 0$ such that
\[ \nu(Q(\xi_k, h_k)) = \mu(\varphi^*(Q(\xi_k, h_k))) \leq C h_k^n \text{ for each } k. \]
Hence,
\[ A \subset \varphi^*^{-1}(E) \subset \varphi^*^{-1}(\bigcup_{k=1}^{\infty} Q(\xi_k, h_k)) \]
and
\[ \mu(\varphi^*^{-1}(E)) \leq \sum_{k=1}^{\infty} \mu(\varphi^*^{-1}(Q(\xi_k, h_k))) \leq C \sum_{k=1}^{\infty} h_k^n \leq C c \varepsilon \]
and the result follows.

\[ \square \]

3. Compactness of Composition Operators on $H^1(b)$

In this last part we will consider the compactness of composition operators, as we have mentioned before, in (S20) a sufficiency condition is given in terms of angular derivative of the symbol function and now we will give the complete characterization of compact composition operators from the perspective of Carleson conditions over Clark measure.
Theorem 3.1. Let $\varphi : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic function. Set up a Borel measure on $\overline{\mathbb{B}^n}$ by $\nu(A) = \mu(\varphi^{-1})(A)$ where $A$ is any Borel set in $\overline{\mathbb{B}^n}$ and $\mu$ is the Clark measure associated to $b$. Suppose that $\frac{1}{1 - b} \in L^2(S^n)$.

Then $C_\varphi$ is compact on $H(b)$ if and only if

$$\nu(S(\xi, h)) = o(h^n)$$

as $h \to 0$ uniformly in $\xi$.

Proof. First of all suppose that $\nu(S(\xi, h)) \neq o(h^n)$ then we can have sequences $\xi_\ell \in S^n$ and $h_\ell \searrow 0$ such that for some $\beta > 0$ we have $\nu(S(\xi_\ell, h_\ell)) \geq \beta h^n$. For $\omega_\ell = (1 - h_\ell)\xi_\ell$ take the functions $f_\ell(z) = V_bK(z, \omega_\ell)$. Let $k_\ell = \frac{f_\ell}{\|f_\ell\|_b}$ then $k_\ell$ tends to 0 uniformly on compacta.

Now,

$$\|k_\ell \circ \varphi\|_b = \frac{\|V_bK(\varphi(z), \omega_\ell)\|_b}{\|V_bK(z, \omega_\ell)\|_b} \geq \frac{\|K(\varphi(z), \omega_\ell)\|_{L^2(\mu)}}{\|K(z, \omega_\ell)\|_{L^2(\mu)}} \int_{S^n} |K(\varphi(z), \omega_\ell)|^2d\mu \geq \frac{(2h)^{-2n}\beta h^n}{h^{-n}} > 0$$

hence $\|k_\ell \circ \varphi\|_b$ is bounded away from $0$.

For the calculations of the converse direction we will divide the unit ball into two different regions. One near boundary and one that is compactly contained inside the unit ball. One may interchange the regions $S(\xi, h)$ by the "windows" defined as

$$W(\xi, h) = \left\{ z \in \overline{\mathbb{B}^n} : 1 - |z| < h, \frac{z}{|z|} \in Q(\xi, h) \right\}.$$ 

For $\varepsilon > 0$, choose a sufficiently small $h_0$ such that $\nu(W(\xi, h)) \leq \varepsilon h^n$ for $h \leq h_0$ and all $\xi \in S^n$. Let $\tilde{\nu}$ be the restriction of $\nu$ to $\overline{\mathbb{B}^n} \setminus (1 - h_0)\overline{\mathbb{B}^n}$ then $\tilde{\nu}$ is also a Carleson measure i.e $\tilde{\nu}(W(\xi, h)) \leq C\varepsilon h^n$ for some $C > 0$ (i††). [Details of this calculation can be found in [CM95], pg:163.] Now suppose that $\{f_n\}$ is a bounded sequence in $H(b)$ that converges uniformly on compacta to 0. By Theorem 2.2 we know that $C_\varphi$ is bounded on $H(b)$ therefore,

$$\|f_n \circ \varphi\|_b = \|V_bg_n \circ \varphi\|_{L^2(\mu)} = \|V_bg_n\|_{L^2(\nu)} = \int_{\overline{\mathbb{B}^n} \setminus (1 - h_0)\overline{\mathbb{B}^n}} |V_bg_n|^2d\nu + \int_{(1 - h_0)\overline{\mathbb{B}^n}} |V_bg_n|^2d\nu.$$

Since $\{f_n\}$ converges uniformly to 0 on compacta, the integral on the right can be made sufficiently small and the one on the left can also be made arbitrarily small by (i††) and (CM95, Theorem 2.37). Hence, the result follows.

$\square$
REFERENCES

[Sh93] J.H Shapiro: Composition operators and classical function theory. Springer-Verlag NewYork Inc, (1993).
[CM95] C.C. Cowen and B.D. Maccluer: Composition operators on spaces of analytic functions. CRC Press Inc, (1995).
[FKM19] E. Fricain, M. Karaki and J. Mashreghi: Composition operator on deBranges-Rovnyak spaces Results in Mathematics, 74 (1), 1-18, (2019).
[S20] S. Sahin: Angular derivatives and boundary values of $H(b)$ spaces of unit ball of $\mathbb{C}^n$. Complex Variables and Elliptic Equations, DOI:10.1080/17476933.2020.1715373, (2020).
[R80] W. Rudin: Function theory in the unit ball of $\mathbb{C}^n$. New York: Springer-Verlag, (1980).

DEPARTMENT OF MATHEMATICS, MIMAR SINAN FINE ARTS UNIVERSITY, ISTANBUL, TURKEY

Email address: sibel.sahin@msgsu.edu.tr