Mathematical models for fake news

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Over the past decade it has become evident that intentional disinformation in the political context – so-called fake news – is a danger to democracy. However, until now there has been no clear understanding of how to define fake news, much less how to model it. This paper addresses both of these issues. A definition of fake news is given, and two approaches for the modelling of fake news and its impact in elections and referendums are introduced. The first approach, based on the idea of a representative voter, is shown to be suitable for obtaining a qualitative understanding of phenomena associated with fake news at a macroscopic level. The second approach, based on the idea of an election microstructure, describes the collective behaviour of the electorate by modelling the preferences of individual voters. It is shown through a simulation study that the mere knowledge that fake news may be in circulation goes a long way towards mitigating the impact of fake news.

I. INTRODUCTION

“Our democracy is at risk”, summarizes the interim report published in July 2018 by the UK House of Commons Digital, Culture, Media and Sport Committee (Collins et al. 2018) on the dissemination of disinformation on social media for the purpose of manipulating the public in election and referendum voting. The prevalence of false stories on the internet has made it difficult for many to distinguish what is true from what is false. The issue that lies at the heart of the current threat to the democratic process in the USA, the UK, and elsewhere is that, unlike in the physical sciences, in which the validity of a claim can be put to the test in a reproducible laboratory experiment, statements about past events are impossible to prove with the same degrees of scientific rigour. The existence of Holocaust denialists, for instance, illustrates how doubts about a major historical event can gain traction with certain individuals. To combat the forces of fake news it is important to view the issue through a scientific lens. Borrowing ideas from communication theory, the present paper aims at developing the mathematical theory that underlies the modelling of fake news.

In a broad sense the concept of “fake news” has been around for centuries. In ancient China, for instance, the military strategist Kongming famously made use of state-sponsored disinformation to his advantage (Shou c. 290). During the Medieval period in Europe, the spreading of fake news often left violence and death in its wake (Soll 2016). Once the technologies for mass printing had developed, fake news found a new kind of application in the form of sensationalist reporting to increase newspaper circulation. The demand for reliable information sources, however, was high in the twentieth century, especially in

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the post-war period, which made the running of a newspaper based on honest reporting a viable business model. This quasi-stable configuration of mainstream journalism based predominantly on honest reporting has been thrown out of its apparent equilibrium in the twenty-first century by the rapid growth of social media usage. As Soll (2016) puts it, “It wasn’t until the rise of web-generated news that our era’s journalistic norms were seriously challenged, and fake news became a powerful force again.”

Advances in communication technology have their advantages and disadvantages. While it is undoubtedly true that the internet, for instance, has made it possible to access information that previously would have been difficult to acquire, it is also a matter of fact that the internet serves as a platform for the propagation of irrelevant information, or “noise”. As noise becomes more pervasive, it becomes increasingly difficult to access reliable information. Ultimately, as Norbert Wiener explained in his insightful book *The Human Use of Human Beings* (Wiener 1954), one has to face the implications of the Second Law of Thermodynamics, which asserts that over time, noise will dominate (or, equivalently, in physicists’ terminology, entropy will increase). Just as in any physical system where entropy can be reduced by means of external inputs (such as energy or force), to combat the domination of noise, concerted efforts have to be made, because noise will not disappear spontaneously. This has important implications for policy makers.

Today, fake news, fuelled by its speed of dissemination, has become a serious concern to society – perhaps most importantly because it can endanger the democratic process. In response, academic research into various aspects of fake news has intensified recently, especially after the 2016 USA presidential election and the “Brexit” referendum in the UK on membership of the European Union. Broadly speaking, research carried out thus far has been primarily focused on the retrospective analysis of the impact of fake news (Allcott & Gentzkow 2017, Amador et al. 2017, Bovet & Makse 2018) and on the detection and prevention of fake news using deep learning and other related techniques (Conroy et al. 2015, Shu et al. 2017, Khajehnejad & Hajimirza 2018, Yang et al. 2018). However, to address the issues surrounding the impact of fake news, and to conduct a comprehensive scenario analysis, it is important that a consistent mathematical model should be developed that describes the phenomena resulting from the flow of fake news. For such a model to be useful, it should be intuitive (so that the model can be trusted as a plausible candidate) and tractable (so that model parameters can be calibrated against real data, and so that predictions can be made, either analytically or numerically). The purpose of this paper is to introduce a new framework for the mathematical modelling of fake news that fulfills these requirements. Our theory of fake news is straightforward to simulate and yet allows one to replicate qualitative features of empirical observations, as demonstrated below.

II. FAKE NEWS AND COMMUNICATION THEORY

Fake news is information that is inconsistent with factual reality. It is information that originates from the “sender” of fake news, is transmitted through a communication channel, and is then received, typically, by the general public. Hence any realistic model for fake news has to be built on the successful and well-established framework of communication theory. Indeed, this philosophy was already advocated by Wiener (1954), who wrote:

It is the thesis of this book that society can only be understood through a study
of the messages and the communication facilities which belong to it; and that in the future development of these messages and communication facilities, messages between man and machines, between machines and man, and between machine and machine, are destined to play an ever-increasing part.

The modelling framework we propose in this paper embraces Wiener’s philosophy. Specifically, we shall apply and extend techniques of filtering theory – a branch of communication theory that aims at filtering out noise in communication channels – in a novel way to generate models that are well-suited for the treatment of fake news.

The traditional applications of filtering theory are threefold: (i) extrapolation, or prediction, (ii) filtering, and (iii) interpolation, or smoothing (Wiener 1949, Kailath 1974). Over the past decade, however, a fourth application of filtering theory has been developed, namely, in phenomenology – the description and modelling of observed phenomena. Perhaps surprisingly, the domain of applicability of the phenomenological use of filtering techniques ranges from scales as small as elementary particles and atoms in physical systems (Brody & Hughston 2006) to scales as large as human activities in social systems (Brody, Hughston & Macrina 2007, 2008). In the latter context, to describe the phenomena associated with social systems, it is particularly natural to employ the mathematics of filtering theory because the actions of an individual are ultimately based on the result of filtering the noisy information available to that individual. In other words, it is not the predictive power of filtering theory that is relevant; rather, it is the fact that the behaviour of people is guided by their predictions via filters, and thus \textit{this behaviour is itself susceptible to a filtering description}.

These observations have opened up a promising avenue towards new discoveries and novel applications, including the one that we explore here for the modelling of fake news.

The paper is organized as follows. We begin by explaining how the techniques of filtering can be applied in the context of behavioural phenomenology of an individual. We then apply this idea to the modelling of fake news as a modification of noise, and introduce our key assumption that, to a good approximation, people are rational inasmuch as they follow Bayesian logic in their decision making. Hence, in our approach, those who are influenced by fake news are not viewed as being irrational as such, but rather they lack the ability to detect and mitigate the changes caused by the presence of fake news in the structure of the noise they are exposed to. In going from the behavioural model of an individual to that of the electorate we are in effect introducing the idea of a “representative voter” whose perception of the uncertain world represents the aggregation of the diverse views held by the public at large. We then examine the problem of estimating the release times of fake news, which in turn generates a new type of challenge in communication theory. This estimate is required for characterising a voter who is aware of the potential presence of fake news, but is unsure which precise items of information are fake. We show as an illustration the dynamics of the opinion-poll statistics in a referendum in the presence of a single piece of fake news. An application to an election in which multiple pieces of fake news are released at random times is then considered. Illustrative simulation results show that the qualitative behaviour of the dynamics of the opinion-poll statistics, seen for instance during the 2016 USA presidential election (Silver 2016), can be replicated by our model. To deepen the analysis further, we introduce what we call an “election microstructure” model in which we employ the same information-based scheme to describe the dynamical behaviour of individual voters and the resulting collective voting behaviour of the electorate under the influence of fake news. We conclude with a summary and a discussion of future directions.
Let us begin by explaining the phenomenological application of filtering techniques. In our decision making we are typically faced with uncertainties so that the most we can do is to arrive at a “best guess” for what the optimal decision might be. Such situations are commonly encountered in our every-day lives. Suppose, for instance, one wants to travel from one location in a city to another. Should one take the bus or travel by underground? The walk to the bus stop may be shorter, but there might be heavy traffic; on the other hand, signal failures on the underground might result in a delay. From experience one has an initial view concerning whether travelling by bus or by underground is better. To formalise this notion, we let \( p \) denote the \textit{a priori} probability that travelling by bus is the better choice. Correspondingly, \( 1 - p \) will be the \textit{a priori} probability that it would be better to travel by underground. In other words, we have a binary random variable \( X \) taking values, say, \((0, 1)\), with corresponding probabilities \((p, 1 - p)\), where \( X = 0 \) represents the bus being the better choice, and \( X = 1 \) represents the underground being the better choice.

The initial view, represented by the probability \( p \), however, changes over time. A colleague who has been travelling on the underground might complain about the signal failure he encountered. The traffic news on the radio might suggest a delay on the bus route – and so on. As time progresses, one’s knowledge increases, but uncertainties remain. We wish to model this type of dynamics. For this purpose, we assume for simplicity that reliable knowledge increases linearly in time, at a rate \( \sigma \). The uncertainty, or noise, is modelled by a Brownian motion, denoted by \( \{B_t\}_{t \geq 0} \), which is assumed to be independent of \( X \) because otherwise it cannot be viewed as representing pure noise. Hence, the flow of information, which we denote by the time series \( \{\xi_t\}_{t \geq 0} \), can be expressed in the form

\[
\xi_t = \sigma X_t + B_t. \tag{1}
\]

The quantity of interest is the actual value of \( X \). However, since there are two unknowns, \( X \) and \( \{B_t\} \), and only one known, \( \{\xi_t\} \), a rational individual will consider the probability that \( X = 0 \) (or \( X = 1 \)) conditional on the information contained in the time series \( \{\xi_s\}_{0 \leq s \leq t} \) gathered up to time \( t \). In other words, writing \((x_0, x_1) = (0, 1)\), one considers the conditional probability \( P(X = x_i | \xi_t) \). In this simple model the time series \( \{\xi_t\} \) is a Markov process, from which it follows that the conditional probability equals \( P(X = x_i | \xi_t) \).

The logical step of converting the prior probabilities \( P(X = x_i) \) into the posterior probabilities \( P(X = x_i | \xi_t) \) is captured by the Bayes formula:

\[
P(X = x_i | \xi_t) = \frac{P(X = x_i)P(\xi_t | X = x_i)}{\sum_j P(X = x_j)P(\xi_t | X = x_j)}. \tag{2}
\]

Here the conditional density function \( \rho(\xi_t | X = x_i) \) for the random variable \( \xi_t \) is defined by the relation

\[
P(\xi_t \leq y | X = x_i) = \int_{-\infty}^{y} \rho(\xi | X = x_i) \, d\xi, \tag{3}
\]

and is given by

\[
\rho(\xi | X = x_i) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{(\xi - \sigma x_it)^2}{2t} \right). \tag{4}
\]
This follows from the fact that, conditional on \( X = x_i \), the random variable \( \xi_t \) is normally distributed with mean \( \sigma x_i t \) and variance \( t \). Recalling that \( (x_0, x_1) = (0, 1) \) we thus obtain

\[
P(X = x_i | \xi_t) = \frac{p_i \exp \left( \sigma x_i \xi_t - \frac{1}{2} \sigma^2 x_i^2 t \right)}{p_0 + p_1 \exp \left( \sigma \xi_t - \frac{1}{2} \sigma^2 t \right)},
\]

where \( p_0 = p \) and \( p_1 = 1 - p \). Inferences based on the use of (5) are optimal in the sense that they minimize the uncertainty concerning the value of \( X \), as measured by the variance or entropic measures subject to the information available. Hence, a rational individual will at any given time act in accordance with the changing views expressed in (5). In the example mentioned above, for instance, the option to travel by bus would be chosen by a rational individual if at the time \( t \) of departure it holds that \( P(X = 0 | \xi_t) > \frac{1}{2} \). There are suggestions that people need not act rationally as anticipated by the Bayes rule (Kahneman & Tversky 1974, Grether & Plott 1979), but other studies suggest that the Bayes logic is nevertheless a dominant factor (El-Gamal & Grether 1995). It is our opinion that in the context of signal processing, given the prior, it is reasonable to assume that people intuitively follow a Bayesian line of thinking.

The mathematical framework outlined above is that of nonlinear filtering, familiar from communication theory. In communication theory, the random variable \( X \) in the first term of (1) represents the “signal” that one wishes to estimate in the presence of ambient noise represented by \( \{B_t\} \). The parameter \( \sigma \) then determines the signal-to-noise ratio. More generally, the signal typically changes in time, which can be represented by a time series \( \{X_t\}_{t \geq 0} \). The case of a fixed \( X \) considered here can thus be viewed as a special case, which was studied by Wonham (1965). It is important to emphasise, however, that in the context of communication theory, there is a “sender” actively transmitting the signal; whereas in our behavioural analysis, we often encounter circumstances where there are receivers but no senders of the signal, because the random variable \( X \) may represent the outcome of a future real-world event that is not known to anyone, and hence cannot be transmitted by anyone. Nevertheless, \( X \) does exist, and people make decisions in accordance with their best estimate about \( X \) based on partial information available to them. This is the sense in which mathematical techniques in communication theory can be applied to describe observed phenomena in science and in society. In what follows we shall extend the foregoing ideas to tackle the problem of modelling fake news and its impact.

Before we proceed, we remark, incidentally, that although we have considered here the situation involving two possible alternatives, represented by the binary random variable \( X \), the complexity of the analysis remains largely unchanged when \( X \) can take multiple values. Likewise, the assumption that knowledge concerning the value of \( X \) is revealed at a constant rate \( \sigma \) can be relaxed without affecting analytical tractability (Wonham 1965). In this case, the first term in (1) is replaced by \( X \int_0^t \sigma(s) ds \), where \( \sigma(s) \) represents the information flow rate at time \( s \). Circumstances where the value of \( X \) is revealed with certainty over a finite time horizon can be modelled by replacing the Brownian noise \( \{B_t\} \) in (1) with a Brownian bridge over that time horizon (Brody, Hughston & Macrina 2007). More generally, in situations where Brownian motion is not an appropriate model for uncertainties (if the noise process can have jumps, for instance), then one can model the noise term by a Lévy process, again without affecting analytical tractability (Brody & Hughston 2013).
IV. MODELLING FAKE NEWS

When fake news is released, for instance by a malicious individual aiming to mislead the public, the false information is superimposed on other information. However, fake news, by its nature, does not represent truth statements about the value of the quantity $X$ that people wish to determine, so it cannot be viewed as forming part of the signal that helps people discover the true value of $X$. On the other hand, from the point of view of signal processing, anything that is not part of the signal can be viewed as noise. Following this logic, we thus arrive at our model for the information process in the presence of fake news:

$$\eta_t = \sigma X_t + B_t + F_t,$$

where the time series $\{F_t\}_{t \geq 0}$ represents fake news. The noise term $\{B_t\}$, which has no bias, represents the aggregation of a large number of unsubstantiated rumours and speculations about the value of $X$. The central limit theorem then suggests the normality of the noise distribution, making Brownian motion a viable candidate for modelling noise. The time series $\{F_t\}$ thus introduces an additional bias. We can now offer a precise mathematical answer to the open issue raised in the UK House of Commons Committee Report on fake news (Collins, et al. 2018):

There is no agreed definition of the term ‘fake news’, which became widely used in 2016. Claire Wardle . . . told us . . . that “when we are talking about this huge spectrum, we cannot start thinking about regulation, and we cannot start talking about interventions, if we are not clear about what we mean”.

With this in mind, we propose the following:

**Definition (Fake News).** A time series $\{F_t\}$ appearing in the information process (6) represents “fake news” if it has a bias so that $E[F_t] \neq 0$, where $E$ denotes expectation.

The existence of bias here is important, for otherwise $\{F_t\}$ would merely represent further noise, rather than deliberate misinformation. It is certainly the case that additional unbiased noise is a nuisance, delaying the process of discovering the truth; but it cannot ultimately drive the public away from discovering the truth. That said, in some circumstances there are advantages in merely delaying the process of truths being uncovered, in which case a release of an unbiased noise with $E[F_t] = 0$ would suffice, and one could describe this situation as representing a mild form of disinformation. Such a scenario, however, is in effect equivalent to manipulating the information flow rate $\sigma(t)$, and corresponds to the disinformation model proposed in Brody & Law (2015). As regards the statistical dependency between $\{F_t\}$ and $X$, there are two situations that can arise: one in which no one knows the value of $X$, in which case $\{F_t\}$ should be independent of $X$, and one in which the value of $X$ is known to a small number of individuals who may wish to disseminate fake news, in which case $\{F_t\}$ may well be dependent on $X$.

The idea that information-based models of the kind represented in (1) can be extended to model deliberate misspecifications of the truth has previously been envisaged (Brody & Law 2015). The proposal there was that a malicious individual who wishes to manipulate the public can alter the value of the information flow rate $\sigma$. Hence the public would make their inferences based on a particular value of $\sigma$, whereas the actual value of $\sigma$ is in fact different, and as a consequence the public is misled. Such a scheme amounts to setting $F_t = \mu X_t$ for some $\mu$, which might arise in an election microstructure model described
below in which the value of $X$ may be known to the candidate but not to the public, thus allowing the candidate to transmit $X$-dependent fake news. More generally, taking into account the randomness in the release time, one might consider a fake-news structure of the form $F_t = \mu X(t - \tau)1\{t > \tau\}$. Here $\mathbb{1}\{A\} = 1$ if $A$ is true and $\mathbb{1}\{A\} = 0$ otherwise. This is equivalent to having the information process $\xi_t = \hat{\sigma}X(t - \tau)$ with a random $\hat{\sigma}$, for which analytic expressions for the conditional probabilities can be obtained (Brody & Law 2015).

In order to analyse the effects of fake news, it will be useful to classify members of the public into three categories. We define Category I to indicate those who are unaware of the potential existence of fake elements in the information they see. Nevertheless, they act rationally in the sense that they make their estimates in accordance with formula (5), except that $\eta_t$ is substituted in place of $\xi_t$. In other words, they “correctly” infer the posterior probability, but based on the mistaken belief that the information they are receiving is of the type (1), while in reality it is of the type (6). As we shall see, the people in this category are most vulnerable to exposure to fake news. We denote by Category II those members of the public who are aware of the potential existence of fake news, but without knowing precisely the times at which the items of fake news in the time series $\{F_t\}$ are released. These individuals face the most technically challenging task, because in their estimation they must deal with three unknowns, $X$, $\{B_t\}$, and $\{F_t\}$, but only one known, $\{\eta_t\}$. As we shall see, although analytic expressions for the conditional probability $P(X = x_i|\{\eta_s\}_{0 \leq s \leq t})$ can be obtained, the analysis is rather more involved than the one for Category I voters. Thus, the people in this category are considerably more cognisant of the uncertainties in their estimates than those in Category I. Finally, Category III consists of those people that are highly informed, to the extent that they know the values of the time series $\{F_t\}$. Because $\{F_t\}$ contains no information relevant to $X$, they can simply disregard $\{F_t\}$ from their information $\{\eta_t\}$ and use $\xi_t = \eta_t - F_t$ instead to work out their posterior belief according to (5). Like those in Category I, people in Category III would tend to be assertive in their judgements. We note however that a Category-III individual should be regarded to some extent as an idealization. After all, it is an almost insurmountable task for an individual to identify perfectly which items of news are fake and which ones are not.

V. ESTIMATING THE ARRIVAL TIMES OF FAKE NEWS

From the point of view of those belonging to Category II of our classification, there are two issues to address: first, one must estimate whether the information source has been contaminated with fake news. Based on this consideration, one must then determine the conditional probability $P(X = x_i|\{\eta_s\}_{0 \leq s \leq t})$, which gives the best estimate for the likelihood of the event $X = x_i$. Note that the former issue amounts to working out the conditional probability $f_t(u)du = P(\tau \in du|\{\eta_s\}_{0 \leq s \leq t})$ that $\tau$ takes a value within the small interval $[u, u + du]$, which we shall calculate.

Before we proceed, we remark that there is an extensively-studied research area within communication theory that is concerned with “change-point detection” or “disorder detection” (Page 1954, Shiryaev 1963a, 1963b). The nature of this problem is as follows. One observes a time series with the property that the structure of the series changes at some random time. In a situation where this transition is not immediately apparent from observed data the task is to detect whether a “regime change” has occurred, and if so, when it might have occurred. Stated mathematically, a prototype of such a problem is to detect the
random time \( \tau \) at which a Brownian motion acquires a drift (Karatzas 2003), or to detect the random time \( \tau \) at which the jump rate of a Poisson process changes from one to another (Galchuk & Rozovskii 1971, Davis 1976, Peskir & Shiryaev 2002). In the Brownian context, therefore, the observation is modelled by a process of the form \( \bar{B}_t + \mu(t - \tau) \mathbb{1}\{t \geq \tau\} \), which is the same as setting \( \sigma = 0 \) and \( \bar{F}_t = \mu(t - \tau) \mathbb{1}\{t \geq \tau\} \) in our information process \( \{\eta_t\} \). Thus, we see that the analysis of fake news introduces a new type of problem in signal detection, where one wishes to detect the unknown signal \( X \) in a noisy environment where the structure of the noise switches from one regime to another at a random time. In other words, one is trying to detect the moment of the switch – that is, the time \( \tau \) at which fake news emerges – while at the same time estimating \( X \). It should be intuitively clear that such a problem will have a wide range of applications beyond fake news analysis.

With this in mind, let us examine the conditional density for \( \tau \) in the context of at most one piece of fake news. For this purpose, let us consider the model for fake news given by

\[
F_t = m(t - \tau) \mathbb{1}\{t \geq \tau\},
\]

where \( m : \mathbb{R} \to \mathbb{R} \) is an arbitrary function which we assume to be at least once differentiable. This is the form of the fake news model that a Category II process assumes to be the true model. Then a calculation shows that the conditional density for \( \tau \) is given by

\[
f_t(u) = \frac{f_0(u) \sum_i p_i e^{f_0^i(\sigma x_i + m'(s-u)\mathbb{1}\{s \geq u\})} ds - \frac{1}{2} f_0^i(\sigma x_i + m'(s-u)\mathbb{1}\{s \geq u\})^2 ds}{\int_0^\infty f_0(w) \sum_i p_i e^{f_0^i(\sigma x_i + m'(s-w)\mathbb{1}\{s \geq w\})} ds - \frac{1}{2} f_0^i(\sigma x_i + m'(s-w)\mathbb{1}\{s \geq w\})^2 ds dw},
\]

(7)

where \( m'(u) = dm(u)/du \) and \( p_i = \mathbb{P}(X = x_i) \). Note that \( f_0(u) \) is the a priori density for \( \tau \), which reflects the initial view on how the release timing is distributed. Hence, the best estimate for the release time of fake news is given by \( \int_0^\infty u f_t(u) du \).

With the help of \( f_t(u) \) we are now in a position to address the estimation of \( X \) for the Category II public. From the tower property of conditional expectation we can write

\[
\mathbb{E}[X|\{\eta_s\}_{0 \leq s \leq t}] = \mathbb{E}[\mathbb{E}[X|\{\eta_s\}_{0 \leq s \leq t}, \tau]|\{\eta_s\}_{0 \leq s \leq t}].
\]

(8)

Our strategy is to work out the inner expectation first. Since conditional on \( \tau \) the information process \( \{\eta_t\} \) is Markov, this reduces to \( Y(\eta_t, \tau) = \mathbb{E}[X|\eta_t, \tau] \), which, on account of the Bayes formula, is given by

\[
Y(\eta_t, \tau) = \frac{\sum_k x_k p_k \exp \left( \sigma x_k \eta_t - \frac{1}{2} \sigma^2 x_k^2 t - \sigma x_k m(t - \tau) \mathbb{1}\{t \geq \tau\}\right)}{\sum_k p_k \exp \left( \sigma x_k \eta_t - \frac{1}{2} \sigma^2 x_k^2 t - \sigma x_k m(t - \tau) \mathbb{1}\{t \geq \tau\}\right)}.
\]

(9)

It follows that the best estimate for \( X \) is given by \( \int_0^\infty Y(\eta_t, u) f_t(u) du \). The foregoing analysis shows that although it is possible to deduce a closed-form expression for the best estimate of the quantity \( X \) that one wishes to determine, the procedure is somewhat intricate. Note that we have treated the case of a single piece of fake news being released at a random time. However, the mathematical formalism describing Category II voters can be extended in a straightforward manner to the case where multiple items of fake news are released. We shall see below simulations of this more general setup.
VI. REPRESENTATIVE VOTER FRAMEWORK

With the foregoing discussion in mind, the classification arising from our modelling setup is clear: there are those members of the public who are easily misled and manipulated since they ignore, by choice or otherwise, the possible existence of fake news; those who are wary of the potential existence of fake news but cannot be too certain about their judgements; and those who are able to detect and disregard fake news. The characteristics of these three categories can be captured most easily by means of simulation studies. For this purpose, let us consider a yes-or-no referendum scenario with a simple linear model for fake news: 

\[ F_t = \mu(t - \tau)1\{t \geq \tau \}. \]

Here, \( \mu \) is a constant whose sign corresponds to which way the public will be misled, and \( \tau \) is an exponentially distributed random variable, with density 

\[ f(u) = \lambda e^{-\lambda u}, \quad \lambda > 0 \text{ a constant}. \]

We let the realization \( X = 1 \) be associated with the “yes” vote and \( X = 0 \) with the “no” vote. In particular, we interpret the likelihoods for the events \( X = 1 \) and \( X = 0 \) to represent the aggregate opinion of the public. That is to say, \( p = \mathbb{P}(X = 1) \) represents the current percentage of the public who intend to vote “yes”, and conversely for \( 1 - p = \mathbb{P}(X = 0) \). Thus the a priori probability \( p \) can be calibrated from today’s opinion-poll statistics. The public opinion, however, changes over time in accordance with the flow of information, and hence the a priori probability will be updated accordingly.

It is worth remarking that whereas in the earlier discussion on the phenomenological application of filtering techniques we described the modelling of the behaviour of an individual, here we take an ensemble point of view in which the a priori probability \( p \) refers to the aggregation of the diverse opinions held by the public (see Brody & Hughston 2013 for a similar concept). The idea that we advocate here is that of a “representative voter”, in analogy with representative agent models in economics. This is useful for the purpose of obtaining a qualitative understanding of the observed phenomena. We shall later in the paper return to the level of individual behaviour by introducing an alternative “election microstructure” framework in analogy with market microstructure models in economics. This is useful for the purpose of developing countermeasures against fake news, and for policy making.

In the ensemble formalism, the event that the random variable \( X \) takes the value zero can be interpreted as the situation in which the totality of voters agree that they should be voting for the ‘no’ outcome. If the information flow-rate parameter \( \sigma \) were constant, then indeed the value of \( X \) would be revealed asymptotically, and the public opinion would thus eventually converge to one choice or the other, over an infinite time horizon. Of course, this rarely happens in real life, because \( \sigma \) will be time dependent and tends to vanish after the election has taken place. Nevertheless, it is not unreasonable in the present context to make the assumption that \( \sigma \) is a constant, since we are only interested in the dynamics up to the polling day.

With these preliminary remarks in mind, we have simulated typical sample paths corresponding to the three voter categories for various choices of the model parameters \((\mu, \sigma, \lambda, p)\). The results of the simulation are illustrated in Figure 1. The expected main feature of the simulation paths is that the voters unaware of the existence of fake news are indeed swayed in the intended direction. What is perhaps less obvious is that the Category II voters, who are aware of the possible existence of fake news, but do not know the timing of its release, tend to overcompensate for the possibility that the information they are receiving may be contaminated. As a consequence, their estimates deviate away from the “correct” (Category
FIG. 1: Referendum simulation. A typical sample path associated with the linear fake-news model is sketched. The left panel shows the information processes \( \{ \xi_t \} \) without fake news and \( \{ \eta_t \} \) with fake news, released here at time 0.4. The evolution of public opinion over time is shown on the right panel for the three types of voters: (I) those who are unaware of the existence of fake news; (II) those who are aware of the potential existence of fake news; and (III) those who are able to disregard fake news fully. Here, \( X = 0 \) corresponds to the “no” outcome, and \( X = 1 \) corresponds to the “yes” outcome. We see that in this particular sample path, a majority of Category III voters will vote for the “no” outcome. However, it is evident that a piece of fake news that attempts to sway the voters in favour of the “yes” outcome has had the effect of moving the percentage figure of Category I voters by a little over 2%, just enough to change the majority vote to “yes”. On the other hand, the relative proximity of Category II and Category III curves shows that Category II voters are able to largely correct for their exposure to fake news. The parameters chosen for the simulation are \( \mu = 0.5, \sigma = 0.3, \lambda = 3, p = 0.5 \), and for the random variable \( X \), the sample outcome is \( X = 0 \).

III) estimate before fake news is released. However, once the fake news is released, Category II voters do surprisingly well at removing the effects of fake news from their estimates. One can interpret this as an indication that mere knowledge of the possibility of fake news is already a powerful antidote to its effects. Further evidence of this will be seen below in our discussion of election microstructure models.

VII. APPLICATION TO OPINION-POLL STATISTICS IN AN ELECTION

We proceed to model the dynamics for opinion-poll statistics in an election where fake news is present. For simplicity, we shall assume that there are two dominant candidates, represented by a binary random variable \( X \), taking the values 0 and 1 that correspond to the two candidates, with a priori probabilities \( p \) and \( 1 - p \), respectively. As indicated earlier, in the representative voter framework the a priori probability \( p \) represents the diverse opinions and mixed views initially held by the public about which value \( X \) should take. The opinion of the public, however, evolves over time in accordance with the revelation of information, which we model by the process \( \{ \eta_t \} \). In the event in which the public at large favours candidate 0, a malicious individual supporting candidate 1 might in response decide to release a false statement about candidate 0. According to our previous model \( F_t = \mu(t - \tau) \mathbb{1}\{t \geq \tau\} \) the contribution of fake news continues to grow at a rate \( \mu > 0 \). However, in reality one might expect that over time the strength of any one fake news item
FIG. 2: Election dynamics without and with fake news. In this example, in the absence of fake news Candidate B, who initially struggles somewhat in the opinion poll, nevertheless manages to win the election comfortably (left panel). However, with persistent fake news (released in this example on nine occasions, indicated by the vertical lines), the public is misled sufficiently for Candidate A to secure a narrow victory (right panel). Of the 100K simulations we conducted with the same parameter choice, the losing candidate ended up winning the election owing to fake news in some 30% of the cases. The parameters are as follows: \( \sigma = 0.3, \mu = 2, \alpha = 5, p = 0.5, \) and \( \lambda = 10. \)

diminishes. Thus, we shall consider here a modification of \( F_t \) whereby the strength of fake news initially grows linearly in time, but is then damped exponentially. In other words, we let the contribution to \( F_t \) of a single item of fake news released at time \( \tau \) be given by 
\[
m(t - \tau) \mathbb{1} \{ t \geq \tau \},
\]
where \( m(u) = \mu u e^{-\alpha u} \) for some damping rate \( \alpha > 0 \), and \( \tau \) is distributed according to \( f(u) = \lambda e^{-\lambda u} \). Once the effects of fake news are sufficiently damped, the public support may revert back towards the direction of candidate 0. However, another item of fake news may be released, and so on. The process will be repeated until polling day.

The structure of our model should now be evident. At time \( \tau_1 \) comes the release of the initial piece of fake news, to be damped gradually, at time \( \tau_2 \) comes the next release, and so on. The waiting times between fake news releases are modelled by an exponential distribution. In other words, the release times are the jump times of a Poisson process. We are interested in a simulation study of the dynamical evolution of opinion-poll statistics in the presence of various pieces of fake news. For simplicity, we consider fake news to be one sided, i.e. it is released only by the supporters of one of the candidates. For the purpose of simulation we shall also assume that the damping rates for the various fake news releases are all equal, and similarly for the linear growth rates. Typical sample paths resulting from the simulation studies are shown in Figure 2. For clarity, we show only Category I and Category III voters. In this way, we can see the full, unmitigated effect of fake news. We see that for this sample path the candidate who would win the election in the absence of fake news ends up losing it. In fact, based on the parameter choice indicated in the figure caption, we found that the likelihood of the “losing” candidate (who would have lost the election in the absence of fake news) ending up winning is about 30%. This number, of course, depends on the choice of model parameters, and thus can be increased arbitrarily by increasing the fake news strength or frequency parameters. Thus, while each element of fake news is damped exponentially in time, a persistent attack on democracy can and will succeed if no action is taken against it.
VIII. ELECTION MICROSTRUCTURE MODELS

The representative voter framework presented above is highly effective in modelling the stylized aspects of fake news and its impact at a phenomenological level. It can also be applied in an efficient scenario analysis by means of simulation studies, for instance, the parameter dependence of the likelihood of fake news changing the outcome of an election. We shall now turn to an alternative formulation that focuses on the microscopic level of individual voters to deduce the macroscopic behaviour of the general public. While the mathematical ingredients used in this election microstructure model are essentially the same as those used for the representative voter framework, there is one important conceptual difference, namely, that in the election microstructure model the signal in the information process can be transmitted by a sender (e.g., the candidate). What follows is framed in the language of political elections, but the treatment is equally applicable to referendums.

Our model is based on the following idea, which represents a somewhat simplified characterization of the mechanism by which an individual arrives at their preferred choice of candidate: when deciding who to vote for in an election, an individual will attempt to evaluate candidates with respect to a number of issues, or factors. For instance, what are the views of the candidates on taxation, on social welfare, on freedom of the press, on immigration, on abortion, on transport, on gun control, on healthcare, on public spending, and so on? Nonpolitical elements may also be considered, such as the state of health of the candidate, or the level of personal integrity, etc. As we shall outline below, the positions of the candidates on these factors are then transformed into an overall score, and the voter ultimately picks the candidate with the highest score.

To proceed let us assume that there are $K$ independent such factors in an election with $L$ candidates and $N$ voters. We let $X^l_k$ denote the $l$-th candidate’s position on the $k$-th factor. However, the positions of the candidates on these factors, if they were in office, are not always transparent to voters. Nevertheless, during an election campaign, candidates will attempt to communicate their positions to the electorate. This communication flows through a number of channels: through advertising by the campaigns, through publications by news outlets, through word-of-mouth, through social media, and so on. The members of the electorate thus obtain partial information, which we shall model as before in the form

$$\eta^{k,l}_t = \sigma^{k,l}_t X^l_k t + B^{k,l}_t + F^{k,l}_t$$

for $k = 1, 2, \ldots, K$ and $l = 1, 2, \ldots, L$. In other words, we have a total of $KL$ information processes $\{\eta^{k,l}_t\}$. Some of these may be contaminated by fake news, which can take different forms. For instance, candidates may try to make themselves look more palatable to a wider section of the electorate than they perhaps are (such as by suppressing about their health, or by downplaying their views on taxation). They may run negative ads to spread untruths or half-truths about their opponents. Or there may be malicious agents spreading entirely fabricated stories supporting one candidate or the other. It is this latter type of fake news that can spread quickly on social media and has thus attracted much attention recently.

A member of the electorate has their own views on how attractive a candidate with a given set of factor values is. We model this by a score function, which can differ between individuals and assigns a measure of attractiveness to a candidate with a given factor vector $X^l_t$. Hence, a member of the electorate will vote for the candidate with the highest score. For the present paper, we consider linear score functions. Thus, the score $S^l_n$ of candidate $l$
determined by voter \( n \) is given by

\[ S'_n = \sum_{k=1}^{K} w^k_n X^l_k = w_n \cdot X^l, \]  

(11)

supposing the vector \( X^l \) were known. The weights \( \{w^k_n\} \) may be positive or negative, with magnitudes that may be large or small depending on how strongly the voter feels about a given issue. Keeping in mind that the factors need to be estimated from the available information, voter \( n \) will choose candidate \( l \) over candidate \( l' \) if and only if

\[ \sum_{k=1}^{K} w^k_n \left( \mathbb{E}_t[X^l_k] - \mathbb{E}_t[X^{l'}_k] \right) > 0, \]  

(12)

or more succinctly \( w_n \cdot (\mathbb{E}_t[X^l] - \mathbb{E}_t[X^{l'}]) > 0 \). Here we let \( \mathbb{E}_t[-] \) denote expectation conditional on having observed the information processes \( \{\eta^{k,l}_s\} \) up to time \( t \). Each component of the conditional expectations \( \mathbb{E}_t[X^l] \) can be worked out by following the methodology described earlier, on account of the independence of policy factors \( X^l_k \) and \( X^{l'}_k \) for \( k \neq k' \). We shall make the reasonable assumption that noise terms for independent factors are independent, and make a further simplifying assumption that the release times of fake news associated with independent factors are likewise independent.

A population of voters can be modelled by proposing a distribution over weight vectors \( \{w_n\} \). By randomly sampling from this distribution and computing, for each sample, which candidate is preferred, one can build up a population-level picture of voting patterns and investigate, in particular, the effect of fake news. In practice, such a distribution could be obtained by asking randomly chosen voters a series of questions in order to generate an approximation of their weight vectors. Moreover, large internet companies (such as search engines, social networks, and global retailers) possess data that is likely rich enough to reliably estimate these weights for individual users. This places such companies in a position of great responsibility. It should be evident that the knowledge of the weight vectors \( \{w_n\} \) is highly valuable in order to implement a targeted campaign. In particular, there is a concern that if such a targeted campaign were carried out by originators of fake news, democracy could be at risk. However, the simulation studies below support the notion that the mere knowledge that there might be fake news in circulation may be sufficient to eliminate the majority of the impact of fake news.

**IX. OPINION POLLS IN THE MICROSTRUCTURE MODEL**

We now present simulation results in the case where candidates are evaluated by three binary independent factors, in an election with two candidates. For concreteness, let us suppose that the three factors concern whether the candidate is liberal or conservative; whether the candidate is healthy or not; and whether the candidate is of good or bad character. For the purpose of this simulation, let us assume that the hidden characteristics of the two candidates, labelled here as \( A \) and \( B \), are such that \( X^A_1 = 1 \) (candidate \( A \) is liberal) and \( X^B_1 = -1 \) (candidate \( B \) is conservative); \( X^A_2 = 1 \) (candidate \( A \) is in good health) and \( X^B_2 = -1 \) (candidate \( B \) is in bad health); and \( X^A_3 = 1 \) (candidate \( A \) has good character) and \( X^B_3 = -1 \) (candidate \( B \) has bad character), the names and values being
FIG. 3: Proportion of population voting for candidate A as time passes. These curves are taken as averages over 1,000 runs. Category I voters are those that are unaware of the existence of fake news; category III voters are those who can fully eliminate fake news and thus represent the ‘correct’ estimates. We see that fake news, which is designed to support candidate B, has the intended effect on average, making an election which candidate A should win comfortably more competitive. These effects can be magnified or decreased by varying the parameters of the fake news terms in the information processes. We also recognize that Category II voters are successfully correcting for most of the effects of fake news. The parameters used in the simulation are: $\sigma = 0.2$, $|\mu| = 1.5$, $\alpha = 4$, $p = 0.5$, and $N = 1M$. Waiting times between fake news releases are drawn at random for each run from an exponential distribution with rate parameter 4. The code used for these simulations is available at github.com/dmmeier/Fake_News.

chosen for illustration. Voters try to estimate these factor values based on the information available to them.

We take the number of voters to be $N = 1$ million, and we sample the weight vectors according to the following prescription: we draw the weight vectors of 55% of voters from a three-dimensional normal distribution with standard deviation 0.4, Distribution 1, centred at $(1, 1, 1)$, but restricted to the positive values for the second dimension and restricted to the interval $[0, 1]$ for the third dimension. The truncation reflects the fact that voters are unlikely, for instance, to actively prefer candidates in ill health or with bad character. Members of this part of the population tend to prefer liberal, healthy candidates of good character. We then draw the weight vectors of the remaining 45% of the population from the same, truncated normal distribution (Distribution 2), but centred at $(-1, 1, 0)$, representing voters that prefer a conservative, healthy candidate, but that are indifferent in relation to the candidate’s character. The distributions are chosen purely for illustration.

It is evident that candidate A, whose factor vector $X^A = (1, 1, 1)$ coincides with the centre of Distribution 1 representing 55% of the population, tends to win the election in the absence
of fake news. However, we now assume that in the run-up to the election various pieces of fake news are released that purport to show that the health and character of candidate A are bad, that the health and character of candidate B are good, that candidate A is more conservative than they seem to be, and that candidate B is more liberal. The intended effect of these elements of fake news is clear: shifting the perceived vector of factors $X^A$ of candidate A out of the centre of Distribution 1 while at the same time moving candidate B towards it. Clearly, this increases the chances candidate B has of winning the election.

In order to isolate the effect of fake news in our simulations, we take the prior probabilities for all six random variables $\{X^A_k, X^B_k\}_{k=1,2,3}$ to be 0.5 for the value 1 and 0.5 for the value $-1$. This means that the population starts out in a fully agnostic state. To model fake news we employ our earlier model in which the effect of a piece of fake news initially grows linearly in time but is then damped exponentially. In Figure 3 we show the average over 1,000 simulations, calculating voting proportions over time for each of the voter categories. The code used for these simulations is available at github.com/dmmeier/Fake_News.

The results show that in the absence of fake news candidate A on average wins the election quite handsomely – this corresponds to the Category III curve. Fake news of the type constructed above, however, pushes the relative voting proportions of Category I voters, who are unaware of the existence of fake news, in favour of candidate B, as we had anticipated. What is perhaps most striking here is the curve showing estimates for Category II voters. These voters possess the knowledge that there may be fake news in circulation, but they do not know how many pieces of fake news have been released, or at what precise time. That is, they only know the statistical distribution of $\{F_t\}$, or equivalently, the prior distribution $f_0(u)$. In spite of this, Category II voters are able to reduce the effects of fake news significantly by developing a statistical understanding of the nature of fake news and correcting their estimates accordingly. Furthermore, preliminary simulation studies suggest that their performance is robust against modest misspecifications of the prior distribution.

X. DISCUSSION AND OUTLOOK

We have presented two approaches for the modelling of fake news in elections and referendums: one based on the idea of a representative voter, useful to obtain a qualitative understanding of the effects of fake news, and one based on the idea of an election microstructure, useful for practical implementation in concrete scenarios. In both cases the results illustrate rather explicitly the impact of fake news in elections and referendums. We have demonstrated that by merely possessing the knowledge of the possibility that pieces of fake news might be in circulation, a diligent individual (Category II) is able to largely mitigate the effects of fake news.

The models presented here invite further development in a number of directions. Our simulations, for instance, were based on random draws of waiting times for the release of fake news. However, in reality, a malicious individual trying to influence an election is likely to try to optimize release times to maximize impact (e.g., to maximize the chances of winning the election). It would be interesting to include such optimal release strategies in our models. Furthermore, as indicated above, in our simulations Category II voters are assumed to know the parameters of the fake news terms. This included, in particular, the value of the fake-news drift parameter $\mu$ and the damping rate $\alpha$. A natural extension of the model would allow for these parameters to be themselves random. Finally, the election microstructure approach could be developed further by allowing dependencies between the
various factors, or by introducing several different information processes reflecting the news consumption preferences of different sections of society. These generalizations will open up challenging but interesting new directions for research.

At any rate, the performance of Category II voters, which significantly exceeded the expectations of the present authors, leads to a hopeful conclusion indeed: namely, by ensuring that members of the electorate are made aware of the possibility and the nature of fake news in the information they consume, policy makers may find success in countering the dark forces of fake news.

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[1] Allcott, H. & Gentzkow, M. 2017 Social media and fake news in the 2016 election. Journal of Economic Perspectives, 31, 211–236.
[2] Amador, J., Oehmichen, A. & Molina-Solana, M. 2017 Characterizing political fake news in twitter by its meta-data. arXiv:1712.05999
[3] Beibel, M. 1996 A note on Ritov’s Bayes approach to the minimax property of the cusum procedure. Annals of Statistics 24, 1804-1812.
[4] Bovet, A. & Makse, H. A. 2019 Influence of fake news in Twitter during the 2016 US presidential election. Nature Communications 10, 7.
[5] Brody, D. C. & Hughston, L. P. 2006 Quantum noise and stochastic reduction. Journal of Physics A39, 833-876.
[6] Brody, D. C., Hughston, L. P. & Macrina, A. 2007 Beyond hazard rates: a new framework for credit-risk modelling. In Advances in Mathematical Finance (M. Fu, R. Jarrow, Ju-Yi Yen & R. Elliott, editors, Basel: Birkhäuser).
[7] Brody, D. C., Hughston, L. P. & Macrina, A. 2008 Dam rain and cumulative gain. Proceedings of the Royal Society London A464, 1801-1822.
[8] Brody, D. C., Hughston, L. P. & Yang, X. 2013 Signal processing with Lévy information. Proceedings of the Royal Society London A469, 20120433.
[9] Brody, D. C. & Hughston, L. P. 2013 Lévy information and the aggregation of risk aversion. Proceedings of the Royal Society London A469, 20130024.
[10] Brody, D. C. & Law, Y. T. 2015 Pricing of defaultable bonds with random information flow. Applied Mathematical Finance 22, 399-420.
[11] Collins, D., Efford, C., Elliott, J., Farrelly, P., Hart, S. Knight, J., Lucas, I. C., O’Hara, B., Pow, R., Stevens, J. & Watling, G. 2018 House of Commons Digital, Culture, Media and Sport Committee: Disinformation and ‘Fake News’: Interim Report, Fifth Report of Session 2017-19, www.parliament.uk/dcmscom
[12] Conroy, N. J., Rubin, V. L. & Chen, Y. 2015 Automatic deception detection: Methods for finding fake news. ASIS&T Annual Meeting Proceedings 52, 1-4.
[13] Davis, M. H. A. 1976 A note on the Poisson disorder problem. *Banach Centre Publications* **1**, 65-72.

[14] El-Gamal, M. A. & Grether, D. M. 1995 Are people Bayesian? Uncovering behavioral strategies. *Journal of the American Statistical Association* **90**, 1137-1145.

[15] Galchuk, L. I. & Rozovskii, B. L. 1971 The “disorder” problem for a Poisson process. *Theory of Probability and its Applications* **16**, 712-716.

[16] Grether, D. & Plott, C. 1979 Economic theory of choice and the preference reversal phenomenon. *American Economic Review* **69**, 623-638.

[17] Kahneman, D. & Tversky, A. 1974 Judgment under uncertainty: Heuristics and biases. *Science* **185**, 1124-1131.

[18] Kailath, T. 1974 A view of three decades of linear filtering theory. *IEEE Transactions of Information Theory* **20**, 146-181.

[19] Karatzas, I. 2003 A note on Bayesian detection of change-points with an expected miss criterion. *Statistics and Decisions* **21**, 3-14.

[20] Khajehnejad, A. & Hajimirza, S. 2018 A Bayesian model for false information belief impact, optimal design, and fake news containment. arXiv:1804.01576

[21] Silver, N. 2016 FiveThirtyEight: Who will win the presidency? https://projects.fivethirtyeight.com/2016-election-forecast/

[22] Page, E. S. 1954 Continuous inspection scheme. *Biometrika* **41**, 100-105.

[23] Peskir, G. & Shiryaev, A. N. 2002 Solving the Poisson disorder problem. In *Advances in Finance and Stochastics*. Essays in Honour of Dieter Sondermann (K. Sandmann & P. J. Schönbucher, eds., Springer).

[24] Shiryaev, A. N. 1963a On optimum methods in quickest detection problems. *Theory of Probability and its Applications* **8**, 22-46.

[25] Shiryaev, A. N. 1963b On the detection of disorder in a manufacturing process, I; II. *Theory of Probability and its Applications* **8**, 247-265; **8**, 402-413.

[26] Shou, C. c. 290 *Records of the Three Kingdoms*.

[27] Shu, K., Silva, A., Wang, S., Tang, J., & Liu, H. 2017 Fake news detection on social media: A data mining perspective. *ACM SIGKDD Explorations Newsletter* **19**, 22-36.

[28] Soll, J. 2016 The long and brutal history of fake news. *Politico Magazine*, 18 December 2016.

[29] Wiener, N. 1949 *Extrapolation, Interpolation, and Smoothing of Stationary Time Series*. (Boston: The Technology Press of the MIT).

[30] Wiener, N. 1954 *The Human Use of Human Beings*. A new and revised edition. (Boston: Houghton Mifflin Company).

[31] Wonham, W. M. 1965 Some applications of stochastic differential equations to optimal non-linear filtering. *Journal of the Society for Industrial and Applied Mathematics* **A2**, 347-369.

[32] Yang, Y., Zheng, L., Zhang, J., Cui, Q, Li, Z. & Yu, P. S. 2018 TI-CNN: Convolutional neural networks for fake news detection. arXiv:1806.00749