Fair Allocation with Interval Scheduling Constraints

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Abstract

We study a fair resource scheduling problem, where a set of interval jobs are to be allocated to heterogeneous machines controlled by agents. Each job is associated with release time, deadline and processing time such that it can be processed if its complete processing period is between its release time and deadline. The machines gain possibly different utilities by processing different jobs, and all jobs assigned to the same machine should be processed without overlap. We consider two widely studied solution concepts, namely, maximin share fairness and envy-freeness. For both criteria, we discuss the extent to which fair allocations exist and present constant approximation algorithms for various settings.

1 Introduction

With the rapid progress of AI technologies, AI algorithms are widely deployed in many societal settings such as the distribution of job and education opportunities. To motivate our study, let us consider a problem faced by the Students Affairs Office (SAO). An SAO clerk is assigning multiple part-time jobs to the students who submitted job applications. Each part-time job occupies a consecutive time period within a possibly flexible interval. For example, a one-hour math tutorial needs to be given between 8:00am and 11:00am on June 26th. A feasible assignment requires that the jobs assigned to an applicant can be scheduled without mutual overlap. The students are heterogeneous, i.e., different students may hold different job preferences. It is important that the students are treated equally in terms of getting job opportunities, and thus the clerk’s task is to make the assignment fair.

The SAO problem falls under the umbrella of the research on job scheduling, which has been studied in numerous fields, including operations research Gentner et al. [2004], machine learning Paleja et al. [2020], parallel computing Drozdzowski [2009], cloud computing Al-Arasi and Saif [2020], etc. Following the convention of job scheduling research, each part-time job, or job for short, is associated with release time, deadline, and processing time. The students are modeled as machines, who have different utility gains for completing jobs. Traditionally, the objective of designing scheduling algorithms is solely focused on efficiency or profit. However, motivated by various real-world AI driven deployments where the data points of the algorithms are real human beings who should be treated unbiasedly, addressing the individual fairness becomes important. Accordingly, the past several years has seen considerable efforts in developing fair AI algorithms Chierichetti et al. [2017], where combinatorial structures are incorporated into the design, such as vertex cover Rahmattalabi et al. [2019], facility location Chen et al. [2019] and knapsack Amanatidis et al. [2020].

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It is noted that people have different criteria on evaluating fairness, and in this work, we consider two of the most widely accepted definitions. The first is motivated by the max-min objective, i.e., maximizing the worst-case utility, which has received observable attention for various learning scenarios Rahmattalabi et al. [2019]. However, for heterogeneous agents, optimizing the worst case is not enough, as different people have different perspectives and may not agree on the output. Accordingly, one popular research agenda is centered around computing an assignment such that everyone believes that it (approximately) maximizes the worst case utility. Budish [2010]. The second one is envy-freeness (EF), which has been very widely studied in social sciences and economics. Informally, an assignment is called EF if everyone believes she has obtained the best resource compared with any other agent’s assignment. We note that, due to the scheduling-feasible constraint, some jobs may not be allocated. Thus EF alone is not able to satisfy the agents as keeping all resources unallocated does not incur any envy among them, but the agents envy the charity where unallocated/disregarded items are assumed to be donated to a "charity". To resolve this issue, in this work, we want to understand how we can compute allocations that are simultaneously EF and Pareto efficient (PO), where an allocation is called PO if there does not exist another allocation that makes nobody worse off but somebody strictly better off.

Recently, Chiarelli et al. [2020] and Hummel and Hetland [2021] studied the fair allocation of conflicting items, where the items are connected via graphs. An edge between two items means they are in conflict and should be allocated to different agents. However, in our model, since we allow the time intervals to be flexible, the conflict among items cannot be described as the edges in a graph. For example, two one-hour tutorials between 9:00am and 11:00am can be feasibly scheduled, but three such tutorials are not feasible any more.

1.1 Main Results
We study the fair interval scheduling problem (FISP), where fairness is captured by MMS and EF. For each of them, we design approximation algorithms to compute MMS or EF1 schedules.

Maximin Share. Informally, a machine’s MMS is defined to be its optimal worst-case utility in an imaginary experiment: it partitions the items into \( m \) bundles but was the last to select one, where \( m \) is the number of agents. It is noted that as the machines are heterogeneous, they may not have the same MMS value. Our task is to investigate the extent to which everyone agrees on the final allocation. A job assignment is called \( \alpha \)-approximate MMS fair if every machine’s utility is no less than \( \alpha \) fraction of its MMS value. Our main result in this part is an algorithmic framework which ensures a 1/3-approximate MMS schedule, and thus improves the best known approximation of 1/5 which is proved for a broader class of valuation functions – XOS Ghodsi et al. [2018]. Interestingly, in the independent and parallel work Hummel and Hetland [2021], the authors also show the existence of 1/3-approximate MMS for graphically conflicting items. With XOS valuation oracles, Ghodsi et al. [2018] also designed a polynomial-time algorithm to compute a 0.125-approximate MMS allocation. As a comparison, by slightly modifying our algorithm, it returns a 0.24-approximate MMS allocation in polynomial time, without any oracle assumptions. When all jobs are rigid and utilities are identical, i.e., processing time = deadline - release time, our problem degenerates to finding a partition of an interval graph such that the minimum weight of the independent set for each subgraph is maximized. Recently, a pseudo-polynomial-time algorithm is given in Chiarelli et al. [2020] for a constant number of agents. In this sense, we generalize this problem to flexible jobs and design approximation algorithms for an arbitrary number of agents.

Main Result 1. For an arbitrary FISP instance, there exists a 1/3-approximate MMS schedule, and a (0.24 – \( \epsilon \))-approximate MMS schedule can be found in polynomial time, for any constant \( \epsilon > 0 \).

EF1+PO. EF is actually a demanding fairness notion, in the sense that any approximation of EF is not compatible with PO. Instead, initiated by Lipton et al. [2004], most research is focused on its relaxation, envy-freeness up to one item (EF1), which means the envy between two agents may exist but will disappear if some item is removed. Unfortunately, EF1 and PO are still not compatible even if all jobs are rigid and agents have unary valuations. However, the good news is, if all jobs have unit processing time, an EF1 and PO schedule is guaranteed to exist and can be found in polynomial time. This result continues to hold when agent valuations are weighted but identical, i.e., the jobs have different values. It is shown in Biswas and Barman [2018] that under laminar matroid constraint an EF1 and PO allocation exists when agents have identical utilities, but the efficient algorithm is not given. We improve this result in two perspectives. First, our feasibility constraints, even for unit jobs, do not necessary correspond to laminar matroid. Second, our algorithm runs in polynomial time.
We consider discrete time, and for $t$ and $\text{Earliest Deadline First}$ since computing feasible job sets to maximize the total weight is NP-hard, a milder requirement is individual optimality (IO). Intuitively, an allocation is called IO if every agent gets the best feasible subset of jobs from the union of her current jobs and unscheduled jobs. We show that EF1 is still not compatible with IO in the general case. But for unary valuations, we obtain positive results and design polynomial time algorithms for (1) computing an EF1 and IO schedule for rigid jobs, and (2) computing an EF1 and 1/2-approximate IO schedule for flexible jobs. To prove these results, we utilize two classic algorithms to prove this result, we consider Nash social welfare – the geometric mean of all machines’ utilities. We show that a Nash social welfare maximizing schedule satisfies the desired approximation ratio. This result is in contrast to the corresponding one in Caragiannis et al. [2016], which shows that without any feasibility constraints, such an allocation is EF1 and PO. We also show that both approximations are tight.

**Main Result 3.** For any FISP instance, the schedule maximizing Nash social welfare is PO and 1/4-approximate EF1. If all jobs have unit processing time, it is 1/2-approximate EF1.

**EF1+IO** By the above results, we observe that PO is too demanding to measure efficiency in our model. One milder requirement is individual optimality (IO). Intuitively, an allocation is called IO if every agent gets the best feasible subset of jobs from the union of her current jobs and unscheduled jobs. We show that EF1 is still not compatible with IO in the general case. But for unary valuations, we obtain positive results and design polynomial time algorithms for (1) computing an EF1 and IO schedule for rigid jobs, and (2) computing an EF1 and 1/2-approximate IO schedule for flexible jobs. To prove these results, we utilize two classic algorithms Earliest Deadline First and Round-Robin.

### 1.2 Related Works

Since computing feasible job sets to maximize the total weight is NP-hard, various approximation algorithms have been proposed, such as additive bar-Noy et al. [2001]; Berman and DasGupta [2000]; Chuzhoy et al. [2006], and the best known approximation ratio is 0.644 et al. [2020]. For rigid instances, the problem is polynomial-time solvable Schrijver [1999]. Recently, scheduling has been studied from the perspective of machine learning, including developing learning algorithms to empirically solve NP-hard scheduling problem Zhang et al. [2020]; Paleja et al. [2020], and predicting uncertain data in order to optimize the performance in the online setting. Fairness has been concerned in the scheduling community in the past decades Ajtai et al. [1998]; Baruah and Lin [1998]; Baruah [1995]. Most of these works aim at finding a fair schedule for the jobs, such as balancing the waiting and completion time. Unfortunately, it is shown in Kurokawa et al. [2018]; Ghodsi et al. [2018]; Feige et al. [2021] that an exact MMS fair allocation may not exist. Thereafter, a string of approximation algorithms for various valuation types are proposed, such as additive Garg and Taki [2020], submodular Barman and Krishnamurthy [2020]; Ghodsi et al. [2018], XOS and subadditive Ghodsi et al. [2018]. Regarding EF1, in the unconstrained setting, an allocation that is both EF1 and PO is guaranteed to exist Caragiannis et al. [2016]; Barman et al. [2018]. However, when there are constraints, such as cardinality and knapsack, the general compatibility is still open Biswas and Barman [2018, 2019]; Dror et al. [2020]; Wu et al. [2021].

### 2 Preliminaries

#### 2.1 Fair Interval Scheduling Problem

In a fair interval scheduling problem (FISP), we are given a job-machine system, which is denoted by tuple $(J, \mathcal{A}, \mathbf{u})$. $J$ is a set of $n$ jobs (also called resources or items) and $\mathcal{A} = \{a_1, \ldots, a_m\}$ is a set of $m \geq 2$ machines controlled by agents. In this work, machines and agents are used interchangeably.

We consider discrete time, and for $t \in \mathbb{N}_+$, let $[t, t+1)$ denote the $t$-th time slot. Each $j_i \in J$ is associated
with release time $r_i \in \mathbb{N}_+$, deadline $d_i \in \mathbb{N}_+$, and processing time $p_j \in \mathbb{N}_+$ such that $p_j \leq d_i - r_i + 1$. The $[r_i, d_i]$ is called a job interval, which can also be viewed as a set of consecutive time slots, $\{r_i, r_i + 1, \cdots, d_i\}$. Job $j_i$ can be processed successfully if it is offered $p_j$ consecutive time slots within $[r_i, d_i]$. Each machine can process at most one job at each time slot and a set of jobs $J' \subseteq J$ is called feasible if all jobs in $J'$ can be processed without overlap on a single machine. For a job $j_k \in J$, agent $a_i \in A$ gains utility $u_i(\{j_k\}) \geq 0$ if $j_k$ is successfully processed by $a_i$. We slightly abuse the notation and assume that $u_i(\{j_k\}) = u_i(\{j_k\})$. We use $u_i$ to denote $a_i$'s utility function, and define $u_A = (u_i)_{i \in A}$. For a feasible set of jobs $S$, the agent’s utility is additive, i.e., $u_i(S) = \sum_{j_k \in S} u_i(j_k)$. For an arbitrary set of jobs that may not be feasible, the agent’s utility is the maximum she can obtain by processing a feasible subset, i.e.,

$$u_i(S) = \max_{S' \subseteq S: S' \text{ is feasible}} \sum_{j_k \in S'} u_i(j_k).$$

It is noted that $u_i(\cdot)$'s are not additive for infeasible set of jobs and the computation of its value is NP-hard [1979]. In Appendix A, we show that they are actually XOS, which is a special type of subadditive functions. We call these $u_i(\cdot)$'s interval scheduling (IS) functions.

A schedule or allocation $X = (X_1, \cdots, X_m)$ is defined as an ordered partial partition of $J$, where $X_i$ is the jobs assigned to agent $a_i$, such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $X_1 \cup \cdots \cup X_m \subseteq J$. Let $X_0 = J \setminus \bigcup_{i \in [m]} X_i$ denote all unscheduled jobs, which is regarded as the donation to a charity. A schedule $X$ is called feasible if $X_i$ is feasible for all $a_i \in A$, i.e., all jobs in $X_i$ can be successfully processed by $a_i$. Note that since jobs in $X_0$ are not scheduled, $X_0$ is not necessarily feasible. Observe that any infeasible schedule $X$ is equivalent to a feasible schedule $X'$ by setting each $X_i'$ to be the feasible subset of $X_i$ that maximizes $a_i$'s utility and $X_0' = J \setminus \bigcup_{i \in [m]} X_i'$. We call an instance rigid if $p_i = d_i - r_i + 1$, for all $j_i \in J$, i.e., the jobs need to occupy the entire job intervals. For rigid instances, the feasibility constraints can be described via interval graphs and the computation of $u_i(S)$ for any $S \subseteq J$ can be done in polynomial time [Kleinberg and Tardos 2006].

As we will discuss the approximation algorithms and the existences of MMS/EF1/PO/IO schedules in different settings, we introduce the following notations to simplify the description of different settings.

Regarding agents’ utilities, FISP contains three cases, from the most special to the most general:

- Unweighted: $u_i(j_k) = 1$ for all $a_i \in A, j_k \in J$, i.e., agents have unary utility for jobs.
- Identical: $u_i(j_k) = u_r(j_k)$ for all $a_i, a_r \in A, j_k \in J$, i.e., all agents have the same utility for the same job.
- Non-identical: $u_i(j_k) \geq 0$ without any restrictions.

Regarding jobs, there are three cases:

- Unit: $p_i = 1$, for all $j_i \in J$, i.e., all jobs have unit processing time.
- Rigid: $r_i + p_i - 1 = d_i$, for all $j_i \in J$, i.e., the jobs need to occupy the entire time intervals between their release times and deadlines.
- Flexible: $r_i + p_i - 1 \leq d_i$, for all $j_i \in J$.

Note that unit jobs may not be rigid and rigid jobs may not be unit either. In the remainder of the paper, we use notation FISP with (utility type, job type) to denote a certain case of the general FISP, e.g., FISP with (unweighted, unit) represents the case where the processing time of each job is 1 and each agent has unweighted utility function.

### 2.2 Solution Concepts

We first define the maximin value for any utility function $u$, item set $S$ and the number of agents $k$. Let $\mathcal{F}(S, k)$ be the set of all $k$-partial-partitions of $S$ and

$$\text{MMS}^u(S, k) = \max_{(S_1, \cdots, S_k) \in \mathcal{F}(S, k)} \min_{i \in [k]} u(X_i).$$
When $\alpha^{1/8}$-MMS schedule can be computed in polynomial time, given X OS function oracle.

When the parameters are clear in the context, we write $\alpha$ which is weaker than PO and study the compatibility between E F1 and IO.

Observation 1 for MMS scheduling problems.

Before introducing our algorithmic framework, we first recall the best known existential and computation results

3 Approximately MMS Scheduling

For any FISP instance $(J, A, u_a)$ with $m = |A|$, agent $a_i \in A$’s maximin share (MMS) is given by

$$\text{MMS}_i(J, m) = \text{MMS}^{\alpha_i}(J, m).$$

When the parameters are clear in the context, we write MMS$_i = \text{MMS}_i(J, m)$ for simplicity. If a schedule $X$ achieves MMS$_i$, i.e., $\min_{k \in [m]} u_i(X_k) = \text{MMS}_i$, it is called an MMS schedule for $a_i$.

Definition 1 ($\alpha$-MMS Schedule). For $0 < \alpha \leq 1$, a schedule $X = (X_1, \ldots, X_m)$ is called $\alpha$-approximate MMS ($\alpha$-MMS) if $u_i(X_i) \geq \alpha \cdot \text{MMS}_i$. When $\alpha = 1$, $X$ is called an MMS schedule.

We next introduce envy freeness (EF). An EF schedule $X = (X_1, \ldots, X_m)$ requires everybody’s utility to be no less than her utility for any other agent’s bundle, i.e., $u_i(X_i) \geq u_i(X_k)$ for any $a_i, a_k \in A$. Since EF is over demanding for indivisible items, following the convention of fair division literature, in this work, we mainly consider EF1.

Definition 2 ($\alpha$-EF1 Schedule). For $0 < \alpha \leq 1$, a schedule $X = (X_1, \ldots, X_m)$ is called $\alpha$-approximate envy-free up to one item ($\alpha$-EF1) if for any two agents $a_i, a_k \in A$,

$$u_i(X_i) \geq \alpha \cdot u_i(X_k \setminus \{j\}) \text{ for some } j \in X_k.$$

When $\alpha = 1$, $X$ is called an EF1 schedule.

We observe that an empty schedule is trivially EF and EF1, i.e., $X_0 = J$ and $X_i = \emptyset$ for all $a_i \in A$. However, this is a highly inefficient schedule, and thus we also want the schedule to be Pareto optimal.

Definition 3 (PO schedule). A schedule $X = (X_1, \ldots, X_m)$ is called Pareto Optimal (PO) if there does not exist an alternative schedule $X' = (X'_1, \ldots, X'_m)$ such that $u_i(X'_i) \geq u_i(X_i)$ for all $a_i \in A$, and $u_k(X'_k) > u_k(X_k)$ for some $a_k \in A$.

We note that any approximation of EF is not compatible with PO, even in the very simple setting with two machines and a single job. In the following, we introduce another efficiency criterion, individual optimality (IO), which is weaker than PO and study the compatibility between EF1 and IO.

Definition 4 ($\alpha$-IO schedule). A feasible schedule $X = (X_1, \ldots, X_m)$ with $X_0 = J \setminus \bigcup_{i \in [m]} X_i$ is called $\alpha$-approximate individual optimal ($\alpha$-IO) if $u_i(X_i) \geq \alpha \cdot u_i(X_0 \cup X_i)$ for all $a_i \in A$, where $\alpha \in [0, 1]$ and when $\alpha = 1$, $X$ is called IO schedule.

It is not hard to see that a PO allocation is also IO, but not vice versa. To show the existences and approximation of EF1/PO/IO, we sometimes use the schedule which maximizes the Nash social welfare.

Definition 5 (MaxNSW Schedule). A feasible schedule $X = (X_1, \ldots, X_m)$ is called MaxNSW schedule if and only if

$$X \in \arg \max_{X' \in \mathcal{F}} \prod_{i=1}^m u_i(X'_i)$$

where $\mathcal{F}$ is the set of all feasible schedules and $X' = (X'_1, \ldots, X'_m)$.

Note that in the standard definition of Nash social welfare maximizing schedule, $X$ was supposed to be a member of $\arg \max_{X' \in \mathcal{F}} \left( \prod_{i=1}^m u_i(X'_i) \right)^\frac{1}{m}$. Here, we ignore the power of $\frac{1}{m}$ to simplify the formula.

3 Approximately MMS Scheduling

Before introducing our algorithmic framework, we first recall the best known existential and computation results for MMS scheduling problems.

Observation 1 (Ghodsi et al. [2018]). For an arbitrary FISP instance, there exists a 1/5-MMS schedule and a 1/8-MMS schedule can be computed in polynomial time, given XOS function oracle.
3.1 Algorithmic Framework

In this section, we present our algorithmic framework and prove that it ensures a 1/3-MMS schedule. The algorithm has two parameters, a threshold vector \((\gamma_1, \cdots, \gamma_m)\) with \(\gamma_i \geq 0\) and a \(\beta\)-approximation algorithm for IS functions, where \(0 \leq \beta \leq 1\). In this section, we set \(\gamma_i = \text{MMS}_i\) for each \(a_i \in A\). We can pretend that \(\beta = 1\) to understand the existential result easily. Note that the computations of each \(\text{MMS}_i\) and exact value for IS functions are NP-hard, and in Section 3.2, we show how to gradually adjust the parameters to make it run in polynomial time. The high-level idea of the algorithm is to repeatedly fill a bag with unscheduled jobs (which may not be feasible) until some agent values it for no less than a threshold and takes away the bag. Then this agent reserves her best feasible subset of the bag, and returns the remaining jobs to the algorithm. By carefully designing the thresholds, we show that everybody can obtain at least \(\frac{\beta}{3+\beta}\) of her MMS.

3.1.1 Pre-processing

As we will see, the above bag-filling algorithm works well only if the jobs are small, i.e., \(u_i(j_k) \leq \frac{\beta}{3+\beta} \cdot \gamma_i\) for all \(a_i \in A\) and \(j_k \in J\). We first introduce the following property, which is used to deal with large jobs. Intuitively, Lemma 1 implies that after allocating an arbitrary job to an arbitrary agent, the remaining agents’ MMS values to design Algorithm 1 imply that after allocating an arbitrary job to an arbitrary agent, the remaining agents’ MMS values in the reduced sub-instance do not decrease. A similar result for additive valuations is proved in Amanatidis et al. [2017].

Lemma 1. For any instance \(\mathcal{I} = (J, A, u_A)\) with \(|A| = m\), the following inequality holds for any \(a_i \in A\) and any \(j_k \in J\),
\[
\text{MMS}_i(J \setminus \{ j_k \}, m - 1) \geq \text{MMS}_i(J, m).
\]

Proof. Let \(\mathcal{I} = (J, A, u_A)\) be an arbitrary instance of FISP with \(J = \{ j_1, \cdots, j_m \}\) and \(|A| = m\). To show that \(\text{MMS}_i(J \setminus \{ j_k \}, m - 1) \geq \text{MMS}_i(J, m)\) holds for any \(j_k \in J\), we consider an arbitrary agent \(a_i\). Let \(X = (X_1, X_2, \cdots, X_m)\) be a feasible schedule for \(a_i\), i.e., \(\min_{X \in X} u_i(X) = \text{MMS}_i\). Consider an arbitrary job \(j_k\), assume that \(j_k \in X_i\). Then remove job set \(X_i\) from \(\text{MMS}_i\) schedule. This generates a new schedule, denoted by \(X' = \{ X_1', X_2', \cdots, X_{m-1}' \}\). It is easy to see that \(X'\) is a feasible schedule to the instance with \(m - 1\) agents and the job set \(J \setminus \{ j_k \}\). This implies that \(\text{MMS}_i(J \setminus \{ j_k \}, m - 1) \geq \min_{X \in X'} u_i(X')\). Note that \(\min_{X' \in X'} u_i(X') \geq \text{MMS}_i(J, m)\). Therefore, we have
\[
\text{MMS}_i(J \setminus \{ j_k \}, m - 1) \geq \min_{X' \in X'} u_i(X') \geq \text{MMS}_i(J, m).
\]

In the case where \(j_k \notin \bigcup_{r \in [m]} X_r\), we remove an arbitrary job set from \(X\) and the above analysis still works. \(\square\)

We use Lemma 1 to design Algorithm 1 which repeatedly allocates a large job to some agent and removes them from the instance until there is no large job.

Algorithm 1. Matching Procedure

**Input:** Arbitrary FISP instance \(\mathcal{I} = (J, A, u_A)\); Thresholds \((\gamma_1, \cdots, \gamma_m)\).

**Output:** (1) Sub-instance \(\mathcal{I}' = (J', A', u_{A'})\) such that \(u_i(j_k) \leq \frac{\beta}{3+\beta} \cdot \gamma_i\) for all \(a_i \in A'\) and \(j_k \in J'\); (2) Partial Schedule \((X_r)_{a_r \in A' \setminus A}\).

1: Initialize \(A' = A\) and \(J' = J\).
2: while there is an agent \(a_i \in A'\) and a job \(j_k \in J'\) with \(u_i(j_k) > \frac{\beta}{3+\beta} \cdot \gamma_i\) do
3: \hspace{1em} Set \(X_i = \{ j_k \}, A' = A' \setminus \{ a_i \}, \) and \(J' = J' \setminus \{ j_k \}\).
4: end while

By Lemma 1, it is straightforward to have the following lemma.

Lemma 2. For any instance \(\mathcal{I} = (J, A, u_A)\) with \((\gamma_1, \cdots, \gamma_m)\), the partial schedule \((X_r)_{a_r \in A' \setminus A}\) and the reduced instance \(\mathcal{I}' = (J', A', u_{A'})\) returned by Algorithm 1 satisfy \(u_r(X_r) \geq \frac{\beta}{3+\beta} \cdot \gamma_r\) for all \(a_r \in A' \setminus A\) and \(\text{MMS}_i(J', |A'|) \geq \text{MMS}_i(J, |A|)\) for all \(a_i \in A'\).
3.1.2 Bag-Filling Procedure

Let $\mathcal{I} = (J, A, u_A)$ be an instance such that $|A| = m$ and $u_i(j_k) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i$ for all $a_i \in A$ and $j_k \in J$. We show the Bag-Filling Procedure in Algorithm 2, with parameters $(\gamma_1, \cdots, \gamma_m)$ and $\beta$-approximation algorithm for IS functions. For each $a_i \in A$, we use $u'_i : 2^J \rightarrow \mathbb{R}_+$ to denote the approximate utility, and thus $u'_i(S) \geq \beta \cdot u_i(S)$ for any $S \subseteq J$. Intuitively, it keeps a bag $B$ and repeatedly adds an unscheduled job into it until some agent $a_i$ first values this bag (under the approximate utility function $u'_i$) for at least $\frac{\beta}{\beta + 2} \cdot \gamma_i$. If there are more than one such agents, arbitrarily select one of them. Then $a_i$ gets assigned a feasible subset $X_i \subseteq B$ with $\sum_{j \in X_i} u_i(j_i) = u'_i(B)$, and returns $B \setminus X_i$ to the algorithm. This step is crucial, otherwise the other remaining agents may not obtain enough jobs. It is obvious that if agent $a_i$ gets assigned a bag, then her true utility satisfies

$$u_i(X_i) = \sum_{j_i \in X_i} u_i(j_i) = u'_i(X_i) \geq \frac{\beta}{\beta + 2} \cdot \gamma_i.$$

The major technical difficulty of our algorithm is to prove that everyone can obtain a bag.

Algorithm 2. Bag-Filling Procedure

Input: An FISP instance $\mathcal{I} = (J, A, u_A)$ such that $u_i(j_k) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i$ for all $a_i \in A$ and $j_k \in J$; $\beta$-approximation algorithm for IS functions; Thresholds $(\gamma_1, \cdots, \gamma_m)$.

Output: $\frac{\beta}{\beta + 2}$-MMS schedule $X = (X_1, \cdots, X_m)$.

1. Initialize $A' = A, J' = J$, and obtain approximate utility functions $u'_i$ for all $a_i \in A$.
2. while $A' \neq \emptyset$ and $J' \neq \emptyset$ do
3. \hspace{0.5cm} Set $B = \emptyset$.
4. \hspace{0.5cm} while $u'_i(B) < \frac{\beta}{\beta + 2} \cdot \gamma_i$ for all $a_i \in A'$ and $J' \neq \emptyset$ do
5. \hspace{1cm} Let $j_k$ be an arbitrary job in $J'$. Set $B = B \cup \{ j_k \}$ and $J' = J' \setminus \{ j_k \}$.
6. \hspace{0.5cm} end while
7. \hspace{0.5cm} Let $a_i$ be an arbitrary agent such that $u'_i(B) \geq \frac{\beta}{\beta + 2} \cdot \gamma_i$.
8. \hspace{0.5cm} Let $X_i \subseteq B$ be a feasible subset such that $\sum_{j_i \in X_i} u_i(j_i) = u'_i(B)$.
9. \hspace{0.5cm} Set $J' = J' \cup (B \setminus X_i)$ and $A' = A' \setminus \{ a_i \}$.
10. end while

Lemma 3. Setting $\gamma_i = \text{MMS}_i$ for all $a_i \in A$, Algorithm 2 returns a $\frac{\beta}{\beta + 2}$-MMS schedule.

Proof. As we have discussed, it suffices to prove that at the beginning of any round of the outer while loop, there are sufficiently many remaining jobs in $J'$ for every remaining agent in $A'$, i.e.,

$$u'_i(J') \geq \frac{\beta}{\beta + 2} \gamma_i, \text{ for any } a_i \in A'.$$

To prove the above inequality, in the following, we actually prove a stronger argument.

Claim 1. For any $a_i \in A'$, let $X' = (X'_1, \cdots, X'_m)$ be a feasible MMS schedule for $a_i$. Then there exists $k \in [m]$, such that $u_i(X'_k \cap J') \geq \frac{\beta}{\beta + 2} \cdot \gamma_i$.

Given Claim 1 and the $\beta$-approximation of $u'_i$, $u'_i(X'_k \cap J') \geq \frac{\beta}{\beta + 2} \cdot \gamma_i$ and thus the lemma holds. We prove by contradiction and assume Claim 1 does not hold for agent $a_i$. Since $X' = (X'_1, \cdots, X'_m)$ is a feasible MMS schedule for $a_i$, $u_i(X'_k) \geq \text{MMS}_i = \gamma_i$ for all $k \in [m]$ and thus

$$\sum_{k \in [m]} u_i(X'_k) \geq m \cdot \gamma_i. \quad (1)$$

Denote by $(X_r)_{a_r \in A \setminus A'}$ the assignments that are allocated to $A \setminus A'$ in previous rounds by Algorithm 2, and for each $a_r$, let $j_{r_i}$ be the last item added to the bag $B$. Note that $j_{r_i} \in X_r$ otherwise $a_r$ will stop the inner while loop.
(Step 4) before \( j_i \) was added. Moreover, since \( a_i \) did not break the while loop either, 
\[ u'_i (X_r \setminus \{ j_i \}) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i. \]
Thus \( u_i (X_r \setminus \{ j_i \}) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i \) as \( u'_i \) is \( \beta \)-approximation of \( u_i \). By the assumption that all jobs are small, i.e.,
\[ u_i (j_i) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i, \]
we have the following
\[ u_i (X_r) = u_i (X_r \setminus \{ j_i \}) + u_i (j_i) < \frac{\beta + 1}{\beta + 2} \cdot \gamma_i. \tag{2} \]

If \( u_i (X'_k \cap J') < \frac{1}{\beta + 2} \cdot \gamma_i \) for all \( k \in [m] \), then
\[
\sum_{k \in [m]} u_i (X'_k) = \sum_{k \in [m]} \left( u_i (X'_k \cap J') + \sum_{a_r \in A \setminus A'} u_i (X'_k \cap X_r) \right)
= \sum_{k \in [m]} u_i (X'_k \cap J') + \sum_{a_r \in A \setminus A'} \sum_{k \in [m]} u_i (X'_k \cap X_r)
\leq \sum_{k \in [m]} u_i (X'_k \cap J') + \sum_{a_r \in A \setminus A'} u_i (X_r)
< m \cdot \frac{1}{\beta + 2} \cdot \gamma_i + (m - |A'|) \cdot \frac{\beta + 1}{\beta + 2} \cdot \gamma_i < m \cdot \gamma_i,
\]
where the first inequality is because the \( X'_k \)'s are disjoint and the second inequality is because of Equation (2). Thus we obtain a contradiction with Equation (1).

3.1.3 Main Existential Theorem

Combining Lemma 2 and Lemma 3, it is not hard to prove the main existential result.

**Algorithm 3.** Main Algorithm: Matching-BagFilling

*Input:* An arbitrary FISP instance \( I = (\mathcal{J}, A, u_A) \); \( \beta \)-approximation algorithm for IS functions; Thresholds \((\gamma_1, \ldots, \gamma_m)\).

*Output:* \( \beta + 2 \)-MMS schedule \( X = (X_1, \ldots, X_m) \).
1. Run Algorithm 1 on \( I \) with \((\gamma_1, \ldots, \gamma_m)\). Obtain \( I' = (\mathcal{J}', A', u_{A'}) \) and \( (X_r)_{a_r \in A \setminus A'} \).
2. Run Algorithm 2 on \( I' \) with \((\gamma_1, \ldots, \gamma_m)\) and the \( \beta \)-approximation algorithm. Obtain \((X_i)_{a_i \in A'}\).

**Theorem 1.** Algorithm 3 with the optimal algorithm for IS functions (i.e., \( \beta = 1 \)) and \( \gamma_i = \text{MMS}_i \) for all \( a_i \in A \) returns a \( 1/3 \)-MMS schedule for arbitrary FISP instance.

Interestingly, in the independent and parallel work Hummel and Hetland [2021], via a similar bag-filling algorithm, the authors prove the existence of \( 1/3 \)-approximate MMS allocations under the context of graphically conflicting items. However, the two models in our work and theirs are not compatible in general.

3.2 Polynomial-time Implementation

Note that, in general, Algorithm 3 is not efficient, because if \( P \neq \text{NP} \), the computation of exact values for IS functions and MMS values cannot be done in polynomial time. For the special case when jobs are rigid or unit, IS functions can be computed in polynomial time. If the number of machines is constant, MMS values for rigid jobs can be computed in pseudo-polynomial time Chiarelli et al. [2020]. Thus, in this section, we deal with the general case. Of course, for IS functions, we can directly use the \( \beta \)-approximation algorithms, and the best-known approximation ratio is 0.644 Im et al. [2020]. Regarding the MMS barrier, instead of using their approximate values, we utilize a combinatorial trick similar with one used in Barman and Krishnamurthy [2020] such that without knowing their values, we can still execute our algorithm.

First, an important corollary of Lemma 2 and Lemma 3 is that if \( \gamma_i \leq \text{MMS}_i \) for some \( a_i \), no matter what values are set for \( \gamma_j, j \neq i \), Algorithm 3 always assigns a bag to \( a_i \) such that \( u_i (X_i) \geq \frac{\beta}{\beta + 2} \gamma_i \).
Lemma 4. For any $a_i$, if $\gamma_i \leq \text{MMS}_i$, Algorithm 3 ensures that $u_i(X_i) \geq \frac{\beta}{\beta + 2}\gamma_i$, regardless of $\gamma_{-i}$.

We prove Lemma 4 in Appendix B. Now, we are ready to introduce the trick. First, we set each $\gamma_i$ to be sufficiently large such that $\gamma_i \geq \text{MMS}_i$ for all $a_i$. Then we run Algorithm 3. If we found some agent $a_i$ with $u_i(X_i) < \frac{\beta}{\beta + 2}\gamma_i$, it means $\gamma_i$ is higher than MMS$_i$ and we can decrease $\gamma_i$ by $0 < 1 - \epsilon < 1$ fraction and keep the other MMS values unchanged. We repeat the above procedure until everyone is satisfied $u_i(X_i) \geq \frac{\beta}{\beta + 2}\gamma_i$. By Lemma 4, it must be that $\gamma_i \geq (1 - \epsilon)\text{MMS}_i$ for all $a_i$. We summarize this in Algorithm 4, and it is straightforward to have the following theorem.

Algorithm 4. Efficient Implementation: Matching-BagFilling

**Input:** An arbitrary FISP instance $I = (J, A, u)$; $\beta$-approximation polynomial-time algorithm for IS functions; Thresholds $(\gamma_1 = \frac{u_1(J)}{\beta}, \ldots, \gamma_m = \frac{u_m(J)}{\beta})$; $0 < \epsilon < 1$.

**Output:** $\frac{\beta}{\beta + 2}(1 - \epsilon)$-MMS schedule $X = (X_1, \ldots, X_m)$.

1. Run Algorithm 3 on $I$ with $(\gamma_1, \ldots, \gamma_m)$. Obtain $X = (X_1, \ldots, X_m)$.
2. while there exist $a_i \in A$ such that $u_i(X_i) < \frac{\beta}{\beta + 2}\gamma_i$ do
3. set $\gamma_i = (1 - \epsilon)\gamma_i$.
4. Run Algorithm 3 on $I$ with $(\gamma_1, \ldots, \gamma_m)$ and update $X = (X_1, \ldots, X_m)$.
5. end while

Theorem 2. For any $0 < \epsilon < 1$, Algorithm 4 returns a $\frac{\beta}{\beta + 2}(1 - \epsilon)$-MMS schedule for arbitrary FISP instance with an $\beta$-approximation algorithm for IS functions. The running time is polynomial with $|J|, |A|$ and $1/\epsilon$. Particularly, using the $0.64$-approximation algorithm in Im et al. [2020], we have $0.24(1 - \epsilon)$-approximation polynomial-time algorithm.

4 Approximately EF1 and PO Scheduling

In this section, we investigate the extent to which there is a schedule that is both EF1 and PO. We first show that EF1 and PO are not compatible even if jobs are rigid and valuations are unary, i.e., $u_i(j_k) = 1$ for all $a_i \in A$ and $j_k \in J$. That is no algorithm can return an EF1 and PO schedule for all instances. Fortunately, if the jobs have unit processing time, an EF1 and PO schedule exists and can be computed in polynomial time. This result continues to hold if the agents have weighted but identical utilities, i.e., $u_i(j_k) = u_r(j_k)$ for any job $j_k$ and any two agents $a_i$ and $a_r$. We sometimes ignore the subscript and use $u(\cdot)$ to denote the identical valuation.

4.1 Incompatibility of EF1 and PO

Lemma 5. EF1 and PO are not compatible for FISP with unweighted, rigid, i.e., no algorithm can return a feasible schedule that is simultaneously EF1 and PO for all FISP with unweighted, rigid instances.

Proof. To prove Lemma 5, we show that any PO schedule must not be an EF1 schedule for the instance in Figure 1. We consider an arbitrary PO schedule, denoted by $X = (X_1, \ldots, X_m)$ and let $X_0 = J \setminus \bigcup_{i \in [m]} X_i$. We claim that $X$ must satisfy the following two properties:

1. $\exists i \in [m]$ such that $X_i = J_1$;
2. $X_0 = \emptyset$.

We first prove that there exists an $i \in [m]$ such that $X_i = J_1$. Suppose, towards to the contradiction, that there is no $i \in [m]$ such that $X_i = J_1$. Note that there must exist $i \in [m]$ such that $X_i \cap J_1 \neq \emptyset$ otherwise $X$ is not a PO schedule. Now consider the job set $X_i$ such that $X_i \cap J_1 \neq \emptyset$. In the case where no job set except $X_i$ in $X$ contains jobs in $J_1$, we can construct another feasible schedule $X' = X \cup \{J_1\} \setminus X_i$. It is easy to see that $u_1(X_i) \geq u_i(X_i)$ for all $a_i \in A$ and $u_1(J_1) > u_i(X_i)$ for agent $a_i$. This implies that $X$ is not a PO schedule. In the case where there exist another one or two subsets $X_r, X_p \in X$ such that $X_r, X_p \cap J_1 \neq \emptyset$. Since there
are only three jobs in \( J_1 \), there are at most three job sets in \( X \) that contains some job in \( J_1 \). Without loss of generality, we assume that both \( X_r \) and \( X_p \) exist. Since every job in \( J_2 \) overlaps with every job in \( J_1 \), we have \( X_t, X_r, X_p \cap J_2 = \emptyset \). Therefore, \( |X_t| = |X_r| = |X_p| = 1 \). Since there are \( m - 1 \) long jobs and every job set in \( X \setminus (X_t \cup X_r \cup X_p) \) contains only one job in \( J_2 \), \( X_0 \) contains two jobs from \( J_2 \). Now we can construct another feasible schedule \( X' = (X'_1, \ldots, X'_m) \) in following way: move all jobs in \( X_r \cup X_p \) to \( X_i \); assign one of two jobs in \( X_0 \) to \( X_r \) and another one to \( X_p \); keep the remaining job sets same as the corresponding one in \( X \). It is easy to see that \( u_i(X'_i) \geq u_i(X_i) \) for all \( a_i \in A \) and \( u_i(X'_i) = u_i(J'_1) > u_i(X_i) \). This implies that \( X \) is not a PO schedule.

\[
\begin{array}{c|ccc|c}
  \text{job} & j_1 & j_2 & j_3 & \ldots & j_{m+2} \\
  \text{time} & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

Figure 1: Instance for Lemma 5. There are \(|A| = m \) agents and \(|J| = m + 2 \) jobs with \( m \geq 2 \). Job set \( J \) can be partitioned as \( J_1 \cup J_2 \), where \( J_1 = \{ j_1, j_2, j_3 \} \) and \( J_2 = \{ j_4, j_5, \ldots, j_{m+2} \} \). Each job \( J_1 \) has unit processing time and each job \( J_2 \) has processing time 5. All jobs are rigid such that \( j_i \in J_1 \) needs to occupy the entire time slot \( 2i - 1 \), where \( i \in \{ 1, 2, 3 \} \). And \( j \in J_2 \) occupies the entire time period from 1 to 5.

Therefore, we can assume that there must exist an \( i \in [m] \) such that \( X_i = J_1 \). Without loss of generality, we assume that \( X_1 = J_1 \). Now we show that the second property holds. Since \( |J_2| = m - 1 \), \( X_0 \neq \emptyset \) implies that there must exist a job set \( X_i \in X \) such that \( X_i = \emptyset \). This would imply that \( X \) is not a PO schedule. Since \( X \) holds the above two properties, we assume that every remaining agent in \( X \setminus \{ a_1 \} \) will receive exactly one job in \( J_2 \). Without loss of generality, we assume that \( X_i = \{ j_{i+2} \} \). Therefore, we have \( X = (X_1, \ldots, X_m) \), where \( X_1 = J_1, X_2 = \{ j_4 \}, \ldots, X_m = \{ j_{m+2} \} \). Since \( u_i(X_1 \setminus \{ j \}) = 2 > u_i(X_i) = 1, \forall a_i \in A \setminus \{ a_1 \}, \forall j \in X_1, X \) is not an EF1 schedule.

### 4.2 Compatibility of EF1 and PO

**Theorem 3.** Given an arbitrary instance of FISP with (identical, unit), Algorithm 5 returns a schedule that is simultaneously EF1 and PO in polynomial time.

**Algorithm 5.** \( m \)-Matching + Inner-Greedy

**Input:** An FISP instance \( \mathcal{I} = (J, A, u_A) \), where all jobs have unit processing time and all agents have identical valuation.

**Output:** EF1 and PO schedule \( X = (X_1, \ldots, X_m) \).

1. Construct graph \( G(J \cup T, E) \), and compute a maximum weighted \( m \)-matching \( M^* \).
2. Define \( J_t = \{ j \in J \mid (j, t) \in M^* \} \) for each \( t \in T \).
3. Set \( X_1 = X_2 = \cdots = X_m = \emptyset \).
4. for \( p = 1 \) to \(|T|\) do
5. 
6. if \( J_p \neq \emptyset \) then
7. 
8. Sort \( A \) in non-decreasing order of \( u_i(X_i)'s \), and \( J_p \) in non-increasing order of \( u(j_k)'s \).
9. for \( i = 1 \) to \(|J_p|\) do
10. 
11. end for
12. end if
13. end for

10
Before giving the proof of Theorem 3, we first give the definition of the condensed instance which is used to improve the running time.

Given an arbitrary instance of FISP with (identical, unit), denoted by $I$, for each job $j_i \in J$, let $T_i$ be the set of time slots included in the job interval of $j_i$, i.e., $T_i = \{r_i, r_i + 1, \cdots, d_i\}$. Let $T$ be the set of condensed time slots (Definition 6). We construct another instance, denoted by $I'$, by condensing $T_i$, i.e., for every job in $J$, $T_i = T_i \cap T$. We show that these two instances are equivalent (Lemma 6). Let $J'$ be the set of jobs in the instance $I'$.

**Definition 6.** Let $T$ be the condensed time slots set.

$$T = \bigcup_{1 \leq i \leq n} \{d_i - n + 1, d_i - n + 2, \cdots, d_i\}$$

where $d_i$ is the deadline of job $j_i$.

To prove Lemma 7, it suffices to prove the following lemma.

**Lemma 6.** Let $\hat{J} \subseteq J$ be an arbitrary subset of jobs in the instance $I$. Let $\hat{J}' \subseteq J'$ be the corresponding jobs in the instance $I'$. Then, $\hat{J}$ is a feasible job set if and only if $\hat{J}'$ is a feasible job set.

**Proof.** ($\Rightarrow$) This direction is straightforward.

($\Leftarrow$) To prove this direction, we define a job block as a maximal set of consecutive jobs such that they are scheduled after each other. Since $\hat{J}$ is a feasible job set, there is a feasible schedule for all jobs in $\hat{J}$. We start from the first job $j_i \in \hat{J}$ which is scheduled in time slot $t_i$ such that $t_i \notin T$, we show that we can always shift this job block to the right. Time slot $t_i \notin T$ implies that $t_i$ is in a distance more that $n$ from any elements in deadline set $D = \bigcup_{j_i \in J} \{d_i\}$. We can shift the job block $j_i$ to the right. An example is shown in Figure 2. We show that we can always shift $j_i$ to the right until:

- Either job $j_i$ is scheduled in a time slot in $T$.
- Or the job block starting from $j_i$ reaches another scheduled job and form a bigger job block.

![Figure 2: Illustration of Lemma 6. Initially, job $j_i$ is scheduled in the time slot $t_i$ which is not in $T$. The job block starting from $j_i$ only includes two jobs: $j_i$ and $j_j$. We can always shift job $j_i$ to the right until $j_i$ is scheduled in a time slot in $T$. Or the leftmost time slot in $T$ is occupied by a certain job $j_r$. The job block starting from $j_r$ contains three jobs: $j_r$, $j_h$ and $j_f$. We can still shift the merged job block, which contains $j_l$, $j_o$, $j_r$, $j_h$, $j_f$, to the right, since the distance between time slot $t_r$ and any deadline in $D$ exceeds $n$.](image)

If job $j_i$ is scheduled in a time slot in $T$, then the lemma follows. If the job block starting from $j_i$ reaches another scheduled job and form a bigger job block, we keep shifting the bigger job block to the right unit $j_i$ is scheduled in a time slot $t_i' \in T$. Note that no job would miss its deadline, since the distance between $t_i'$ and any deadline in $D$ exceeds $n$ which implies that there is enough time slots to schedule all jobs in the current job block.

**Lemma 7.** For an arbitrary instance of FISP with (identical, unit), if there is a polynomial-time algorithm that returns an EF1 and PO schedule for all condensed instances, there also exists one for non-condensed instances.

 Arbitrarily fix a maximum weighted $m$-matching $M^*$. For any $t \in T$, let $J_t$ be the set of jobs which are matched with time slot $t$, i.e., $J_t = \{j \in J \mid (j, t) \in M^*\}$. Note that the $J_t$’s are mutually disjoint. Therefore we can refer $t$ as the type of jobs in $J_t$. An example can be found in Figure 3 (a). The key idea of Algorithm 5
Algorithm 5, maximizes social welfare holds, i.e., for Algorithm 5, suppose jobs or agents, which can also be done in polynomial time, we finished the proof.

Since schedule X returned by Algorithm 5 maximizes social welfare \( \sum_{i \in A} u_i(X_i) \), X must be PO. According to Lemma 8, X is EF1. For time complexity, we have already discussed that computing a maximum m-matching can be done in polynomial time. Further, as allocating jobs by types only needs to sort jobs or agents, which can also be done in polynomial time, we finished the proof.
We note that Algorithm 5 fails to return an EF1 and PO schedule if the agents’ utilities are not identical. Actually, the existence of EF1 and PO schedule for this case is left open in Biswas and Barman [2018]; Dror et al. [2020]; Wu et al. [2021] even when the scheduling constraints degenerate to cardinality constraints.

Remark 1. We noted that the proof of Lemma 8 only uses the ranking of jobs’ weight. Therefore, Algorithm 5 is able to return to a feasible schedule that is simultaneously EF1 and PO in the setting where agents value jobs in the same order but the concrete jobs’ weight are not known by the algorithm.

4.3 Approximate EF1 and PO

Although EF1 and PO are only compatible in special cases, in this section we show that approximate EF1 and PO can be always satisfied. In the following, we show that Nash social welfare maximizing schedule satisfies the desired properties.

Theorem 4. Given an arbitrary instance of general FISP, any schedule that maximizes the Nash social welfare is a 1/4-EF1 and PO schedule.

The proof of Theorem 4 is essentially in the same spirit with the corresponding one in Wu et al. [2021], and we include the proof for completeness in Appendix C. Although we show that the proof in Theorem 4 is tight in the appendix, when the jobs are unit we can improve the approximation ratio to 1/2.

Theorem 5. Given an arbitrary instance of FISP with (non-identical unit), a MaxNSW schedule is a 1/2-EF1 and PO schedule.

Proof. We show that a feasible schedule $X = (X_1, \cdots, X_m)$ that maximizes Nash social welfare is simultaneously 1/2-EF1 and PO. Since any MaxNSW schedule must be a PO schedule, we only prove that $X$ is a 1/2-EF1 schedule. Hence, we only show that $X$ is an 1/2-EF1 schedule, i.e., $\forall i, k \in [m]$, $u_i(X_i) \geq \frac{1}{2} \cdot u_i(X_k \setminus \{ j \})$, $\exists j \in X_k$.

We prove by contradiction and assume that there exists $i, k \in [m]$ such that $u_i(X_i) < \frac{1}{2} \cdot u_i(X_k \setminus \{ j \})$, $\forall j \in X_k$. Then, we have

$$u_i(X_i) + u_i(j) < u_i(X_k) - u_i(X_i), \forall j \in X_k. \quad (3)$$

Since $X_i, X_k$ are feasible job set, there is a maximum weighted matching in $G(X_i \cup T, E_i)$, $G(X_k \cup T, E_k)$ with size $|X_i|, |X_k|$, respectively. Let $M_i, M_k$ be the maximum weighted matching in $G(X_i \cup T, E_i)$, $G(X_k \cup T, E_k)$, respectively. An example can be found in Figure 4.

![Figure 4: Illustration for $M_i, M_k$.](image)

In the above example, we have $X_i = \{ (j_1, t_1), (j_2, t_2) \}$, $X_k = \{ (j_3, t_3), (j_4, t_4) \}$, and $T = \{ t_1, t_2, t_3, t_4 \}$. We have matching $M_i = \{ (j_1, t_1), (j_2, t_2), (j_3, t_4) \}$ and $M_k = \{ (j_4, t_3), (j_4, t_3), (j_5, t_5), (j_6, t_6) \}$. Moreover, we have $M_i(J) = X_i, M_i(T) = \{ t_1, t_2, t_4 \}$ and $M_k(J) = X_k, M_k(T) = \{ t_2, t_3, t_4 \}$.

For every time slot $t_1 \in M_i(T) \cup M_k(T)$, we find the pair $(j^i, t^i_1) \in M_i, (j^k, t^k_1) \in M_k$. Note that there may exist some time slot $t_1$ such that $t_1$ is only matched in $M_i$ or $M_k$, e.g., time slot $t_1, t_3$ in the example shown in Figure 4. In this case, we add a dummy pair to $M_k$ or $M_i$, e.g., in the example shown in Figure 4.
\( M_i = M_i \cup \{ (j_o, t^i) \}, M_k = M_k \cup \{ (j_o, t^k) \} \) and let \( u_i(j_o) = u_k(j_o) = 0, \forall i \in [m] \). For every time slot \( t_i \in M_i(T) \cup M_k(T) \), we find the pair \( (j^i, t^i) \in M_i \), \( (j^k, t^k) \in M_k \) and define the big pair \( [(j^i, t^i)], [(j^k, t^k)] \) as \( (j^i, j^k) \) for convenience. For each pair \( (j^i, j^k) \), we define \( |(j^i, j^k)| \) as its value, where

\[
|(j^i, j^k)| = \frac{u_i(j^k) - u_i(j^i)}{u_k(j^k) - u_k(j^i)}.
\]

Note that there may exist two pairs: \( (j^i, t^i) \in M_i \), \( (j^k, t^k) \in M_k \) such that \( u_i(j^k) - u_i(j^i) = 0 \) and \( u_k(j^k) - u_k(j^i) = 0 \). In this case, we have

\[
|(j^i, j^k)| = \begin{cases} 0, & \text{if } u_i(j^k) - u_i(j^i) = 0, u_k(j^k) - u_k(j^i) \neq 0; \\ \infty, & \text{if } u_i(j^k) - u_i(j^i) \neq 0, u_k(j^k) - u_k(j^i) = 0. \end{cases}
\]

Let \( \mathcal{P}_+ \), \( \mathcal{P}_- \) be the set of all \((j^i, j^k)\) such that \( u_i(j^k) - u_i(j^i) > 0 \) and \( u_k(j^k) - u_k(j^i) \leq 0 \), respectively. We consider an arbitrary pair \((j^i, j^k)\) in \( \mathcal{P}_+ \), i.e., \( u_i(j^k) - u_i(j^i) > 0 \). Note that \( u_k(j^k) - u_k(j^i) \) holds; otherwise, we can construct a new feasible schedule by swapping job \( j^i \) and \( j^k \) will have larger Nash Social Welfare. This would imply that \( X \) does not maximize the Nash Social Welfare. Let \( X^+_i, X^+_k \) be the set of jobs in \( X_i, X_k \) that are covered by some pair in \( \mathcal{P}_+ \), respectively, i.e., \( X^+_i = \{ j^i \in X_i \mid \exists (j^i, j^k) \in \mathcal{P}_+ \} \) and \( X^+_k = \{ j^k \in X_k \mid \exists (j^i, j^k) \in \mathcal{P}_+ \} \). Notations \( X^-_i, X^-_k \) are defined in similar ways. Note that \( u_i(X_k) - u_i(X_i) = (u_i(X^+_k) - u_i(X^+_i)) + (u_i(X^-_k) - u_i(X^-_i)) \).

Since \( u_i(X^-_k) - u_i(X^-_i) \leq 0 \), we have

\[
u_i(X_k) - u_i(X_i) \leq u_i(X^+_k) - u_i(X^+_i). \tag{4}\]

Then, we have:

\[
\frac{u_i(X^+_k) - u_i(X^+_i)}{u_k(X^+_k)} \geq \frac{u_i(X^+_k) - u_i(X^+_i)}{u_k(X_k)} \geq \frac{u_i(X_k) - u_i(X_i)}{u_k(X_k)}, \tag{5}\]

where the first inequality is due to \( u_i(X^+_k) - u_i(X^+_i) > 0 \) and \( u_k(X_k) \geq u_k(X^+_k) \), the last inequality is due to Equation (4). Now, we define \((g^i, g^k)\) as:

\[
(g^i, g^k) = \arg \max_{(j^i, j^k) \in \mathcal{P}_+} \{|(j^i, j^k)|\}.
\]

Note that \( \mathcal{P}_+ \neq \emptyset \), i.e., there must exist a pair \((j^i, j^k)\) such that \( u_i(j^k) - u_i(j^i) > 0 \) because of \( u_i(X_k) > u_i(X_i) \). Since every pair \((j^i, j^k)\) in \( \mathcal{P}_+ \) has property \( u_i(j^k) - u_i(j^i) > 0 \) and \( u_k(j^k) - u_k(j^i) > 0 \), we have:

\[
\frac{u_i(g^k) - u_i(g^i)}{u_k(g^k) - u_k(g^i)} \geq \frac{u_i(X^+_k) - u_i(X^+_i)}{u_k(X^+_k) - u_k(X^+_k)} \geq \frac{u_i(X_k) - u_i(X_i)}{u_k(X_k)}, \tag{6}\]

where the last inequality is due to \( u_i(X_k) - u_i(X_i) > 0 \) and \( u_k(X^+_k) \geq 0 \). By combining Equation (5) and Equation (6), we have

\[
\frac{u_i(g^k) - u_i(g^i)}{u_k(g^k) - u_k(g^i)} \geq \frac{u_i(X_k) - u_i(X_i)}{u_k(X_k)} > \frac{u_i(X_i) + u_i(g^k)}{u_k(X_k)}, \tag{7}\]

where the last inequality is due to Equation (3).

Since \( u_i(g^k) - u_i(g^i) > 0 \) and \( u_k(g^k) - u_k(g^i) > 0 \), we have:

\[
\left(\frac{u_i(g^k) - u_i(g^i)}{u_k(X_k)}\right) \cdot u_k(X_k) > \left(\frac{u_k(g^k) - u_k(g^i)}{u_k(X_k)}\right) \cdot \left(\frac{u_i(X_i) + u_i(g^k)}{u_k(X_k)}\right). \tag{8}\]
Holding Equation (8) on our hand, we are ready to prove that \( \mathbf{X} \) does not maximize the Nash social welfare. Now, we construct another feasible schedule, denoted by \( \mathbf{X}' = (X'_1, \ldots, X'_m) \). We construct \( \mathbf{X}' \) by swapping the job \( g' \) with \( g^k \), i.e., \( X'_o = X_o, \forall o \in [m] \) and \( o \neq j, k, X'_i = X_i \cup \{ g^k \} \setminus \{ g' \} \) and \( X'_k = X_k \cup \{ g' \} \setminus \{ g^k \} \). Note that all job sets in \( \mathbf{X}' \) except \( X'_j \) are the same as the corresponding job sets in \( \mathbf{X} \). Observe that if we can show that \( u_i(X'_i)u_k(X'_k) > u_i(X_k)u_k(X_k) \), then it implies that \( \mathbf{X} \) does not maximize the Nash social welfare. Note that

\[
\begin{align*}
u_i(X'_i) &= u_i(X_i) + u_i(g^k) - u_i(g') ;  \\
u_k(X'_k) &= u_k(X_k) + u_k(g') - u_k(g^k).
\end{align*}
\]

We define \( \Gamma \) as follows for convenience:

\[
\Gamma = \left( u_i(g^k) - u_i(g') \right) \cdot u_k(X_k) - \left( u_k(g^k) - u_k(g') \right) \cdot \left( u_i(X_i) + u_i(g^k) \right),
\]

where \( \Gamma > 0 \) because of Equation (8). Then, we have

\[
u_i(X'_i)u_k(X'_k) - u_i(X_i)u_k(X_k) = \Gamma + \left( u_k(g^k) - u_k(g') \right) \cdot u_i(g').
\]

Since \( u_k(g^k) - u_k(g') > 0 \) and \( \Gamma > 0 \), we have \( u_i(X'_i)u_k(X'_k) - u_i(X_i)u_k(X_k) > 0 \). Hence \( \mathbf{X} \) does not maximize the Nash social welfare which contradicts our assumption. Therefore, \( \forall i, k \in [m], u_i(X_i) \geq \frac{1}{2} \cdot (X_k \setminus \{ j \}), \exists j \in X_k. \]

In the following, we show that our proof of Theorem 5 is tight.

**Lemma 9.** The schedule which maximizes the Nash social welfare can only guarantee 1/2-EF1 and PO for FISP with (non-identical,unit).

**Proof.** To prove Lemma 9, we give an instance for which a schedule that maximizes the Nash social welfare is an 1/2-EF1 schedule.

We consider the job set \( J = \{ j_1, \ldots, j_n, j_{n+1}, \ldots, j_{2n} \} \) which contains 2n jobs. All jobs have the same release time 1 and deadline \( n \). Moreover, all jobs have unit processing time. The agent set \( A = \{ a_1, a_2 \} \) contains two agents. The utilities matrix is as follows:

\[
\begin{array}{cccccccc}
& j_1 & j_2 & \cdots & j_n & j_{n+1} & \cdots & j_{2n} \\
a_1 & 2 & 2 & \cdots & 2 & 1 & \cdots & 1 \\
a_2 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\end{array}
\]

To find the schedule that maximizes the Nash social welfare, we consider an arbitrary schedule \( \mathbf{X} = (X_1, X_2) \) and assume that \( X_0 = J \setminus (X_1 \cup X_2) \). We define \( J_1 = \{ j_1, \ldots, j_n \} \) and \( J_2 = \{ j_{n+1}, \ldots, j_{2n} \} \). Observe that \( X_0 = \emptyset \) otherwise \( \mathbf{X} \) does not maximize the value of \( u_1(X_1) \cdot u_2(X_2) \). We assume that \( x \) jobs in \( J_1 \) are assigned to \( a_1 \) and \( y \) jobs in \( J_2 \) are assigned to \( a_2 \), where \( 0 \leq x, y \leq n \). Then, we have

\[
f(x, y) = u_1(X_1) \cdot u_2(X_2) = (2x + y) \cdot (n - x).
\]

To find the maximum value of \( f(x, y) \) under the constraints \( 0 \leq x, y \leq n \), we compute partial derivative.

\[
\begin{align*}
\frac{\partial f(x, y)}{\partial x} &= 2n - 4x - y = 0 \\
\frac{\partial f(x, y)}{\partial y} &= n - x = 0
\end{align*}
\]

The solution to the above two equations is \((n, -2n)\). Since the point \((n, -2n) \notin \{ (x, y) \mid 0 \leq x, y \leq n \}\), the maximum value will be taken at a certain vertex. We can find that the maximum value will be taken at the point \((x, y) = (0, n)\) by computing the value of \( f(0, 0), f(0, n), f(n, 0), f(n, n) \).

Hence, we found the schedule \( \mathbf{X} = (X_1, X_2) \) maximizes the Nash social welfare, where \( X_1 = J_2, X_2 = J_1 \). Then, we have \( u_1(X_1) = 2n \) and \( u_1(X_1 \setminus \{ j \}) = 2(n - 1), \forall j \in X_1 \). Then, we have

\[
\lim_{n \to +\infty} \frac{u_1(X_1)}{u_1(X_1 \setminus \{ j \})} = \frac{n}{2(n - 1)} = \frac{1}{2}.
\]

\qed
5 EF1 and IO Scheduling

Lemma 5 shows that PO is very demanding since even if agents have unweighted utilities, EF1 and PO are not compatible. Accordingly, in this section, we will consider the weaker efficiency criterion – Individual Optimality. As we will see, although EF1 and IO are still not compatible for weighted utilities, they are when agents have unweighted utilities.

5.1 An Impossibility Result

We first show that EF1 and IO are not compatible even for FISP with (identical, rigid), i.e., given an arbitrary instance of FISP with (identical, rigid), there is no algorithm can always find a feasible schedule that is simultaneously EF1 and IO (Lemma 10).

Lemma 10. EF1 and IO are not compatible even for FISP with (identical, rigid).

Proof. To prove Lemma 10, it suffices to consider the instance in Figure 5, and prove the following two claims.

Claim 2. For any IO schedule $X = (X_1, \cdots, X_m)$, $X_0 = J \setminus \bigcup_{i \in [m]} X_i = \emptyset$.

We prove this claim by contradiction. If $X_0 \cap J_1 \neq \emptyset$, as $|J_2| = m - 1$, there will be at least one agent, without loss of generality say $a_1$, for whom $X_1 \cap J_2 = \emptyset$. Note that by the design of the instance, $X_1 \cup (X_0 \cap J_1)$ is feasible, and thus by allocating $X_0 \cap J_1$ to $a_1$, $a_1$’s utility strictly increases.

If $X_0 \cap J_2 \neq \emptyset$, as $|J_2| = m - 1$, there will be at least two agents, without loss of generality say $a_1$ and $a_2$, for whom $X_1 \cap J_2 = \emptyset$ and $X_2 \cap J_2 = \emptyset$. Furthermore, as $|J_1| = 4$ one of them gets at most two jobs in $J_1$. Again without loss of generality assume this is agent $a_1$. Accordingly, $u_1(X_1) \leq 4$ and by exchanging $X_1$ with one job in $X_0 \cap J_2$, $a_1$’s utility strictly increases.

Claim 3. For any EF1 schedule $X = (X_1, \cdots, X_m)$, $X_0 = J \setminus \bigcup_{i \in [m]} X_i = \emptyset$.

We note that the only possible and feasible schedule $X$ such that $X_0 = \emptyset$ is that some agent, say $a_1$, gets entire $J_1$ and every other agent gets one job in $J_2$. Then to prove this claim, it suffices to prove $X$ cannot be EF1.

It is not hard to check that under $X$, for any agent $a_i$ with $i \geq 2$ and any job $j \in X_1$,

$$u_i(X_i) = 6 - \epsilon < u_i(X_1 \setminus \{j\}) = 6.$$ 

That is all $a_1$ envies $a_1$ for more than one item.

Combing the above two claims, we complete the proof of Lemma 10.

![Figure 5: Instance for Lemma 10. There are $|A| = m$ agents and $|J| = m + 3$ jobs with $m \geq 2$. Job set $J$ can be partitioned as $J_1 \cup J_2$ with $J_1 = \{ j_1, j_2, j_3, j_4 \}$ and $J_2 = \{ j_5, \cdots, j_{m+3} \}$. Each job in $J_1$ has unit processing time with weight 2 and each job in $J_2$ has processing time 7 with weight $6 - \epsilon$. All jobs are rigid such that $j_i \in J_1$ needs to occupy the entire time slot $2i - 1$, and $j \in J_2$ occupies the entire time period from 1 to 7.]


5.2 A Polynomial-time Algorithm for FISP with \(\text{unweighted, rigid}\)

In the following, we design a polynomial-time algorithm to compute a schedule that is EF1 and IO for any instance of FISP with \(\text{unweighted, rigid}\).

**Theorem 6.** Given an arbitrary instance of FISP with \(\text{unweighted, rigid}\), Algorithm 6 returns a feasible schedule that is simultaneously EF1 and IO in polynomial time.

**Algorithm 6.** Earliest Deadline First + Round-Robin

**Input:** Agent set \(A\) and job set \(J\).

**Output:** EF1 schedule \(X = (X_1, \ldots, X_m)\).

1. Sort all jobs by their deadline in non-decreasing order.
2. \(X_1 = X_2 = \cdots = X_m = \emptyset\).
3. \(i = 1, k = 1\). // The index.
4. for all \(j_k \in J\) do
5. \(\text{if } X_i \cup \{ j_k \} \text{ is a feasible job set then}
6. \(X_i = X_i \cup \{ j_k \} \).
7. \(J = J \setminus \{ j_k \} \).
8. \(i = (i + 1) \mod m\).
9. else
10. \(i = i \mod m\).
11. end if
12. end for
13. \(X_0 = J \setminus \bigcup_{i \in [m]} X_i\).

Let \(X = (X_1, \ldots, X_m)\) be the schedule returned by Algorithm 6 and let \(X_0 = J \setminus \bigcup_{i \in [m]} X_i\). Suppose that all agents receive a job at every round of first \(L\) rounds of Algorithm 6, i.e., in the \((L + 1)\)-th round, \(\exists i \in [m]\) such that \(a_i\) receives nothing. Note that \(L \leq n\), where \(n\) is the number of jobs. Let \(j_l^i, 1 \leq i \leq m, 1 \leq l \leq L\), be the job assigned to agent \(a_i\) in the \(l\)-th round of Algorithm 6. Let \(r^l_i, d^l_i\) be the release time and deadline of job \(j_l^i\). Let \(X_i^1\) be the job set that is assigned to agent \(a_i\) after the \(l\)-th round of Algorithm 6.

**Lemma 11.** \(d^1_i \leq d^2_i \leq \cdots \leq d^m_i\), \(\forall l \in [L]\).

**Proof.** We consider two agents \(a_i, a_k\) such that \(1 \leq i < k \leq m\). Note that \(j^i_{l+1}, j^k_{l+1}\) must exist since, in the first \(L\) rounds, all agents receive a job. We prove by induction.

**Base case and induction hypothesis.** In the base case where \(l = 1\), it is straightforward to see that \(d^1_i \leq d^1_k\), otherwise \(j^k_{l+1}\) will be assigned to agent \(a_i\) in the first round of Algorithm 6. Now, we have induction hypothesis \(d^1_i \leq d^2_k\).

We need to prove \(d^2_i < d^2_k\). We prove by contradiction and assume that \(d^2_i = d^2_k\). Since agent \(a_i\) chooses \(j^i_{l+1}\) instead of \(j^k_{l+1}\) in the \((l+1)\)-th round, we know that \(X^i_l \cup \{ j^i_{l+1} \}\) is not a feasible job set which implies that \(r^i_{l+1} = d^i_l\). By induction hypothesis, we have \(r^i_{l+1} = d^i_l \leq d^k_k\). This implies that \(X^i_l \cup \{ j^i_{l+1} \}\) is not a feasible job set. This contradicts our assumption. Thus, \(d^2_i \leq d^2_k\).

**Lemma 12.** \(d^m_i \leq d^1_i, \forall l \in [L-1]\). Moreover, \(d^m_i \leq d^l_{i+1}\) if \(j^i_{l+1}\) exists.

**Proof.** Let \(a_i, a_k\) be two agents such that \(1 \leq i < k \leq m\). We assume that \(j^i_{l+1}\) exists and prove that the lemma holds for all \(l \in [L]\). We prove by induction.

In the base case where \(l = 1\), it is not hard to see that \(d^m_i \leq d^2_i\); otherwise \(a_m\) will choose \(j^i_2\) in the first round. Now, we have induction hypothesis \(d^m_i \leq d^2_i\).

Suppose, towards a contradiction, that there exist \(l \in [L]\) such that \(d^m_i > d^l_{i+1}\). In the \(l\)-th round, agent \(a_m\) selects \(j^m_i\) instead of \(j^i_{l+1}\) because \(X^m_{l-1} \cup \{ j^i_{l+1} \}\) is not a feasible job set; otherwise \(a_m\) will select \(j^i_{l+1}\). Since \(X^m_{l-1} \cup \{ j^i_{l+1} \}\) is not a feasible job set, we have \(d^l_{i+1} \leq d^m_i\). By induction hypothesis, we have \(d^m_i \leq d^1_i\).
Therefore, we have $r_{i+1}^l \leq d_i^l$ which implies that $X_i \cup \{j_i^{l+1}\}$ is not a feasible job set. This contradicts our assumption.

Lemma 13. $|X_i| - |X_k| \in \{-1, 0, 1\}, \forall i, k \in [m]$. 

Proof. We consider the $(L+1)$-th round of Algorithm 6 in which $\exists f \in [m]$ such that $a_f$ receives nothing in this round. Let $J_f^L$ be the set of remaining jobs in $J$ after $a_f$ chooses in the $(L+1)$-th round. We consider the agent $a_k$ such that $1 \leq f \leq k < m$. Since $a_f$ receives nothing, we have $r_j \leq d_j^L, \forall j \in J_f^L$. According to Lemma 11, we have $d_j^L \leq d_k^L$. Then, we have $r_j \leq d_j^L \leq d_k^L, \forall j \in J_f^L$ which implies that agent $a_k$ also receives nothing in this round. Therefore, $a_m$ must receive nothing in the $(L+1)$-th round because there exist an agent that does not receive job in the $(L+1)$-th round. Let $J_m^L$ be the remaining jobs in $J$ before $a_m$ chooses in the $(L+1)$-th round. Since $a_m$ receives nothing in the $(L+1)$-th round, we have $r_j \leq d_m^L, \forall j \in J_m^L$. According to Lemma 12, we have $d_m^L \leq d_i^{L+1}$ if $j_i^{L+1}$ exists. Therefore, we have $r_j \leq d_i^{L+1}, \forall j \in J_m^L$. Thus, $a_1$ will receive nothing in the $(L+2)$-th round. Now, we consider an arbitrary agent $a_h, 1 \leq h \leq m$, it is straightforward to see that if $a_h$ receives nothing in $(L+1)$-th round, then $a_h$ will receive nothing in any $L$'-th round, where $L+1 < L'$. Note that, in the $(L+1)$-th round, there may exist many agents that receive nothing. Without loss of generality, we assume that $a_f$ is the agent with the smallest index who receives nothing in the $(L+1)$-th round. Therefore, we have

$$|X_i| = \begin{cases} L, & \forall f \leq i \leq m; \\ L+1, & 1 \leq i < f. \end{cases}$$

Thus, we have $|X_i| - |X_k| \in \{-1, 0, 1\}, \forall i, k \in [m]$.

Lemma 14. $u_i(X_i) \geq u_i(X_0 \cup X_i), \forall i \in [m]$. 

We will use the optimal argument for classical interval scheduling to prove Lemma 14. We restate the problem and optimal argument for completeness.

In classical interval scheduling, we are given a set of intervals $\mathcal{I} = \{I_1, I_2, \ldots, I_n\}$. Each interval is associated with a release time and a deadline. A set of intervals $\mathcal{I}'$ is called a compatible set if and only if, for every two intervals $I_a, I_b \in \mathcal{I}'$, $I_a$ and $I_b$ do not intersect. The goal is to find the compatible set with the maximum size. This problem can be easily solved by Earlier Deadline First (EDF) [Kleiberg and Tardos 2006].

Proof. We consider an arbitrary agent $a_i$. We prove by constructing an instance of classical interval scheduling problem. Let $\mathcal{I} = X_0 \cup X_i$. Let ALGE be the interval set selected by EDF algorithm. Observe that if we can prove that ALGE $= X_i$, then it implies that $u_i(X_i) \geq u_i(X_0 \cup X_i)$ since ALGE is the optimal solution. Suppose that ALGE $= \{j_1, j_2, \ldots, j_h\}$ and assume that the interval is added to ALGE by EDF algorithm in this order. Suppose that $X_i = \{j_1, j_2, \ldots, j_k\}$ and assume that the job is added to $X_i$ by Algorithm 6 in this order. Note that $|X_i| \leq |\text{ALGE}|$ since ALGE is the compatible set with the maximum size.

We prove by comparison. Assume that ALGE and $X_i$ become different from the $R$-th element, i.e., $j_i = j_i', \forall i \in [R-1]$ and $j_R \neq j_R'$. This implies that $j_R$ is the job with the smallest deadline in $X_0 \cup X_i \setminus \{j_1, \ldots, j_{R-1}\}$ to make $\{j_1, \ldots, j_{R-1}\} \cup \{j_R\}$ be compatible. Note that both $\{j_1, \ldots, j_{R-1}\} \cup \{j_R'\}$ and $\{j_1, \ldots, j_{R-1}\} \cup \{j_R\}$ are feasible. Since $j_R'$ is left to charity, there is no agent takes it away. Therefore, Algorithm 6 will assign $j_R$ instead of $j_R$ to $a_i$. Hence, we proved $X_i \subseteq \text{ALGE}$. It is easy to see that there is no interval $j_i' \in \text{ALGE}$ such that $j_i' \notin X_i$ which implies that $X_i = \text{ALGE}$. 

Now, we are ready to prove Theorem 6.

Proof of Theorem 6. According to Lemma 13, we know that the feasible schedule $X$ returned by Algorithm 6 is an EF1 schedule. According to Lemma 14, $X$ is also an IO schedule. Hence, Algorithm 6 returns a feasible schedule that is simultaneously EF1 and IO.

Now, we prove the running time. Line 1 requires running time $O(n \log n)$, where $n$ is the number of jobs. Line 4-13 requires running time $O(n)$. Hence, the running time of Algorithm 6 can be bounded by $O(n \log n)$. 

\[\square\]
Now, we show an instance that Algorithm 6 returns a schedule that is not PO schedule. See Figure 6. By applying Algorithm 6 to the instance in Figure 6, let $X$ be the returned schedule. Then, we have $X = (X_1, X_2)$, where $X_1 = \{ j_1, j_4 \}$, $X_2 = \{ j_2, j_5 \}$ and $X_0 = \{ j_3, j_6 \}$. But a possible PO schedule is $X' = (X'_1, X'_2)$, where $X'_1 = \{ j_1, j_4, j_5 \}$, $X'_2 = \{ j_2, j_6 \}$ and $X'_0 = \{ j_3 \}$.

![Figure 6: Instance for which Algorithm 6 fails to return a PO schedule. In the above instance, we have $J = \{ j_1, j_2, j_3, j_4, j_5, j_6 \}$, $A = \{ a_1, a_2 \}$. The job windows are $T_1 = \{ 1, 2 \}$, $T_2 = \{ 3, 4 \}$, $T_3 = \{ 2, 3, 4, 5, 6 \}$, $T_4 = \{ 6, 7, 8 \}$, $T_5 = \{ 10, 11 \}$, $T_6 = \{ 8, 9, 10, 11, 12 \}$, respectively.](image)

5.3 A polynomial time algorithm for FISP with (unweighted, flexible)

Note that Algorithm 6 can be modified to run on instances of FISP with (unweighted, flexible). But this modified algorithm fails to return an IO schedule. This is not surprising as it has been proved in Garey and Johnson [1979] that even with a single machine, finding an IO schedule is NP-hard. Fortunately, the modified algorithm still runs in polynomial time and always returns a schedule that is EF1 and 1/2-IO.

Before giving the round-robin algorithm, we first re-state the following classical scheduling problem.

Scheduling to find the maximum compatible job set We are given a job set $J$ which contains $n$ jobs, i.e., $J = \{ j_1, j_2, \ldots, j_n \}$, with each job regraded as a tuple, i.e., $j_i = (r_i, p_i, d_i), i \in [n], 1 \leq p_i \leq d_i - r_i + 1$, where $r_i, p_i, d_i$ are the release time, processing time and deadline, respectively. There is one machine which is used to process jobs. A subset $J'$ of jobs is called compatible job set if and only if all jobs in $J'$ can be finished without preemption before their deadlines. The objective is to find a compatible job set with the maximum size.

The above scheduling problem is the optimization version of the scheduling problem SEQUENCING WITH RELEASE TIMES AND DEADLINES, which is strongly NP-complete Garey and Johnson [1979]. In Bar-Noy et al. [2001], they give an $\frac{(m+1)m}{m}$-approximation algorithm for $m$ identical machines case. In particular, the approximation ratio is 2 when $m = 1$. We restate the greedy algorithm for completeness (Algorithm 7).

**Theorem 7.** A schedule that is simultaneously EF1 and 1/2-IO exists and can be found in polynomial time for all instance of FISP (unweighted, flexible).

Now, we are ready to give the algorithm (Algorithm 8) for instance of FISP (unweighted, flexible).

Let $X = (X_1, \ldots, X_m)$ be the schedule returned by Algorithm 8 and $X_0 = J \setminus \bigcup_{i \in [m]} X_i$. We first show that $X$ is an 1/2-IO schedule and then prove that $X$ is an EF1 schedule.

**Lemma 15.** $u_i(X_i) \geq \frac{1}{2} \cdot u_i(X_i \cup X_0), \forall u_i \in A$.

**Proof.** We consider an arbitrary agent $a_i \in A$ and the job set $X_i \cup X_0$. Let ALGC be the set of jobs selected from $X_0 \cup X_i$ by Algorithm 7. Let OPTC be the set of jobs selected by the optimal algorithm. Since Algorithm 7 is a 2-approximation algorithm, we have $|ALGC| \geq \frac{1}{2} \cdot |OPTC|$. Observe that if we can prove that ALGC = $X_i$, then we have $u_i(X_i) \geq \frac{1}{2} \cdot u_i(X_i \cup X_0)$ since ALGC has the size at least half of the optimal solution. Let $ALGC = \{ j_1, j_2, \ldots, j_k \}$ and assume that the jobs are added to the solution by Algorithm 7 in this order. Let $X_i = \{ j'_1, j'_2, \ldots, j'_l \}$ and assume that the jobs are added to $X_i$ by Algorithm 8 in this order. We prove by comparison. Assume that ALGC and $X_i$ become different from the $R$-th element, i.e., $j_R = j'_1, \forall i \in [R - 1]$ and $j_R \neq j'_R$. Assume that the completion time of $j_{R-1}$ is $D_{R-1}$. Then, we have $j'_R \neq j_R = \arg \min_{j \in J_R} \{ \max(D_{R-1}, r_j) + p_j \}.$
Algorithm 7. 2-approximation for scheduling problem on single machine.

1: \text{ALGC} = \emptyset.
2: \text{J}^* = \text{J}.
3: \text{D} = 0.
4: \textbf{while} \text{J}^* \neq \emptyset \textbf{do}
5: \text{J}^* = \emptyset. // reset \text{J}^*.
6: \textbf{for} every job \( j \in \text{J} \) \textbf{do}
7: \quad \text{if} \( d_j \leq \max\{D, r_j\} + p_j \) \text{ then}
8: \quad \quad \text{J}^* = \text{J}^* \cup \{ j \}.
9: \textbf{end if}
10: \textbf{end for}
11: \text{J}^* = \arg\min_{j \in \text{J}} \{\max\{D, r_j\} + p_j\}.
12: \text{Schedule job } j^* \text{ at time slot } \max\{D, r_j\}.
13: \text{D} = \max\{D, r_j\} + p_j^*.
14: \text{ALGC} = \text{ALGC} \cup \{ j^* \}.
15: \textbf{end while}

where \( J_R^* \subseteq J_r = X_0 \cup X_i \setminus \{ j_1, \ldots, j_R \} \) is a set of jobs which can be feasibly scheduled after \( J_R \), i.e.,

\[ J_R^* = \{ j \in J_r \mid \max\{D_{R-1}, r_j\} + p_j \leq d_j \} \, . \]

Note that \( J_R^* \neq J_R^* \). Since \( J_R^* \) instead of \( J_R \) is assigned to agent \( a_i \) in a certain round, we know that \( J_R \) must be assigned to a certain agent before agent \( a_i \) chooses, i.e., \( J_R \in X_k, \exists k \in [m] \). This contradicts our assumption since \( J_R \in X_0 \).

\[ \square \]

Let \( J^*_l \) be the set job \( J^*_l \) for agent \( a_i \in A \) in the \( l \)-th round, where \( 1 \leq i \leq m \) and \( 1 \leq l \leq L \). Suppose that in first \( L \)-th rounds of Algorithm 8, \( J^*_l \neq \emptyset \), \( \forall i \in [m] \), i.e., in the \((L + 1)\)-th round, \( \exists i \in [m] \) such that \( J^*_l = \emptyset \). Let \( D^L_1 \) be the parameter \( D_i \) in Algorithm 8 for agent \( a_i \in A \) at the end of the \( l \)-th round, where \( 1 \leq i \leq m \) and \( 1 \leq l \leq L \).

**Lemma 16.** \( D^1_1 \leq D^1_2 \leq \cdots \leq D^1_m, \forall l \in [L] \). Moreover, we have \( D^L_m \leq D^{L+1}_1, \forall l \in [L-1] \), and \( D^L_m \leq D^{L+1}_1 \) if \( J^{L+1}_l \neq \emptyset \).

**Proof.** We first prove \( D^1_1 \leq D^1_2 \leq \cdots \leq D^1_m, \forall l \in [L] \). Let \( a_k, a_h \in A \) be two agents, where \( 1 \leq k < h \leq n \). We only need to prove \( D^1_k \leq D^1_h, \forall l \in [L] \). We prove by induction. In the base case where \( l = 1 \), \( D^1_k \leq D^1_h \) obviously holds since agent \( a_k \) chooses the job before \( a_h \). Now, we have induction hypothesis \( D^1_k \leq D^1_h \) and we need to prove \( D^{l+1}_k \leq D^{l+1}_h \). We prove by contradiction and assume that \( D^{l+1}_k > D^{l+1}_h \). Let \( j^{l+1}_k, j^{l+1}_h \) be the jobs that are selected by agent \( a_k, a_h \) in the \((l + 1)\)-round of Algorithm 8, respectively. Hence, we have

\[ D^{l+1}_k = \max\{D^l_k, r^{l+1}_{j^{l+1}_k}\} + p^{l+1}_{j^{l+1}_k}, \]
\[ D^{l+1}_h = \max\{D^l_h, r^{l+1}_{j^{l+1}_h}\} + p^{l+1}_{j^{l+1}_h}. \]

By induction hypothesis \( D^l_k \leq D^l_h \), we have

\[ \max\{D^l_k, r^{l+1}_{j^{l+1}_k}\} + p^{l+1}_{j^{l+1}_k} \leq \max\{D^l_h, r^{l+1}_{j^{l+1}_h}\} + p^{l+1}_{j^{l+1}_h} \leq d^{l+1}_{j^{l+1}_k}. \]

This implies that \( j^{l+1}_h \in J^{l+1}_k \). Since

\[ \max\{D^l_k, r^{l+1}_{j^{l+1}_k}\} + p^{l+1}_{j^{l+1}_k} \leq D^{l+1}_k < D^{l+1}_h, \]

we have

\[ \max\{D^l_k, r^{l+1}_{j^{l+1}_k}\} + p^{l+1}_{j^{l+1}_k} < \max\{D^l_h, r^{l+1}_{j^{l+1}_h}\} + p^{l+1}_{j^{l+1}_h}. \]
Algorithm 8. Round-Robin for FISP (unweighted, flexible)

**Input:** Agent set $A$ and job set $J$.

**Output:** EF1 schedule $X = (X_1, \cdots, X_m)$. 

1. $X_1 = \cdots = X_m = \emptyset$.
2. $J_1^i = \cdots = J_m^i = J$.
3. $D_1 = \cdots = D_m = 0$.
4. $i = 1$. // The index.
5. **while** there is a $J_i^*$ $\neq \emptyset$ **do**
6.   **for** all $a_i \in A$ **do**
7.     $J_i^* = \emptyset$. // reset $J_i^*$.
8.     **for** every job $j \in J$ **do**
9.       if $d_j \leq \max\{D_{i}, r_j\} + p_j$ **then**
10.          $J_i^* = J_i^* \cup \{j\}$.
11.       **end if**
12.   **end for**
13.   $J_i^* = \arg\min_{j \in I_i} \max\{D_{i}, r_j\} + p_j$.
14. $X_i = X_i \cup \{J_i^*\}$.
15. Schedule job $j_i^*$ at max $\{D_{i}, r_j\}$.
16. $D_i = \max\{D_{i}, r_{j_i^*}\} + p_{j_i^*}$.
17. $J = J \setminus \{J_i^*\}$.
18. **end for**
19. **end while**
20. $X_0 = J \setminus \bigcup_{i \in [m]} X_i$.

This implies that $j_{l+1}^{i+1}$ instead of $j_{l+1}^i$ will be chosen by agent $a_k$ in the $(l + 1)$-th round of Algorithm 8. This contradicts our assumption.

We assume that $J_{l+1}^L \neq \emptyset$ and prove that $D_{l+1}^l \leq D_{l+1}^{l+1}, \forall l \in [L]$. We prove $D_{l+1}^l \leq D_{l+1}^{l+1}$ holds for any $1 \leq l \leq L$. Note that $D_{l+1}^0 = 0, \forall r \in [m]$. We prove by induction. In the base case where $l = 1$, if $D_{l+1}^1 > D_{l+1}^2$, $a_m$ will choose $j_1^{i+1}$ instead of $j_1^i$ in the first round. Now, we have induction hypothesis $D_{l+1}^l \leq D_{l+1}^{l+1}$ and we need to prove that $D_{l+1}^{l+1} \leq D_{l+1}^{l+2}$ holds. We prove by contradiction and assume that $D_{l+1}^{l+1} > D_{l+1}^{l+2}$. Let $j_{m+1}^{l+1}$, $j_{m+2}^{l+2}$ be the jobs that are selected by agent $a_m, a_1$ in the $(l + 1)$, $(l + 2)$-th round of Algorithm 8, respectively. Note that in the case where $l = L - 1$, there always exists a job $j_{m+2}^{L+2}$ since $\{J_{L+1}^L\} \neq \emptyset$. Hence, we have

\[
\begin{align*}
D_{l+1}^l & = \max\{D_{l+1}^l, r_{j_{m+1}^{l+1}}\} + p_{j_{m+1}^{l+1}}, \\
D_{l+1}^{l+2} & = \max\{D_{l+1}^{l+1}, r_{j_{m+2}^{l+2}}\} + p_{j_{m+2}^{l+2}}.
\end{align*}
\]

By induction hypothesis $D_{l+1}^l \leq D_{l+1}^{l+1}$, we have

\[
\max\{D_{l+1}^l, r_{j_{m+2}^{l+2}}\} + p_{j_{m+2}^{l+2}} \leq \max\{D_{l+1}^{l+1}, r_{j_{m+2}^{l+2}}\} + p_{j_{m+2}^{l+2}} \leq d_{j_{m+2}^{l+2}}.
\]

This implies that $j_{m+2}^{l+2} \in J_{m+2}^{l+2}$. Since

\[
\max\{D_{l+1}^l, r_{j_{m+2}^{l+2}}\} + p_{j_{m+2}^{l+2}} \leq D_{l+1}^{l+2} < D_{l+1}^{l+1},
\]

we have

\[
\max\{D_{l+1}^l, r_{j_{m+2}^{l+2}}\} + p_{j_{m+2}^{l+2}} < \max\{D_{l+1}^l, r_{j_{m+1}^{l+1}}\} + p_{j_{m+1}^{l+1}}.
\]

This implies that $j_{m+2}^{l+2}$ instead of $j_{m+1}^{l+1}$ will be chosen by agent $a_m$ in the $(l + 1)$-th round of Algorithm 8. This contradicts our assumption.

**Lemma 17.** $J_1^l \supseteq J_2^l \supseteq \cdots \supseteq J_m^l, \forall l \in [L + 1]$. Moreover, we have $J_m^l \supseteq J_1^{l+1}, l \in [L]$.
Proof. We first prove that $J_1^k \supseteq J_2^k \supseteq \cdots \supseteq J_m^k, \forall l \in [L]$. Let $a_k, a_h \in A$ be two agents, where $1 \leq k < h \leq n$. We only need to prove $J_h^k \subseteq J_h^h$. To prove $J_h^k \subseteq J_h^k$, we consider an arbitrary job $j \in J_h^k$ and show that $j \in J_h^k$. Note that $J_k^k, J_h^h \neq \emptyset, \forall l \in [L]$. Let $J_k^h, J_h^h$ be the set of jobs that are already assigned to the agents before agent $a_k$ and $a_h$ select, respectively. Note that $J_h^k \subseteq J_h^h$. According to Algorithm 8, we have

$$J_h^k = \{ j \in (J \setminus J_h^k) \mid d_j \leq \max\{D_h^{l-1}, r_j\} \},$$

and

$$J_h^h = \{ j \in (J \setminus J_h^h) \mid d_j \leq \max\{D_h^{l-1}, r_j\} \}. $$

Since $J_h^k \subseteq J_h^h$, we have $J \setminus J_h^k \supseteq J \setminus J_h^h$. Now we consider an arbitrary job $j \in J_h^k$ and show that $j$ is also a member of $J_h^k$. Since $j \in (J \setminus J_h^k)$ and $J \setminus J_h^k \supseteq J \setminus J_h^h$, we have $j \in (J \setminus J_h^k)$. Since $j \in J_h^k$, we have

$$d_j \leq \max\{D_h^{l-1}, r_j\} + p_j.$$ 

According to Lemma 16, $D_h^{l-1} \supseteq D_h^{l-1}$, we have

$$d_j \leq \max\{D_h^{l-1}, r_j\} + p_j,$$

which implies that $j \in J_h^k$. Now we consider the case where $l = L + 1$. Note that in the $(L + 1)$-th round of Algorithm 8, $\exists i \in [m]$ such that $J_h^{l+1} = \emptyset$. Now we prove that $J_h^{l+1} \supseteq J_h^{L+1}$. If $J_h^{L+1} = \emptyset$, then we are done. Hence, we assume that $J_h^{L+1} \neq \emptyset$. By a similar argument, a job $j \in J_h^{L+1}$ has the property $d_j \leq \max\{D_h^0, r_j\} + p_j$. Then we have $d_j \leq \max\{D_h^0, r_j\} + p_j$ holds since $D_h^0 \subseteq D_h^1$. Then we have $j \in J_h^L + 1$.

Now, we prove that $J_m^k \supseteq J_k^k + 1, \forall l \in [L]$. Note that it is possible that $J_h^{L+1} = \emptyset$. In this case $J_h^k \supseteq J_h^{L+1}$ trivially holds. Hence, we assume that $J_h^{L+1} \neq \emptyset$. To prove $J_m^k \supseteq J_k^k + 1$, we consider an arbitrary job $j \in J_m^k$ and show that $j \in J_m^k$. Let $J_m^m, J_m^1$ be the set of jobs that are already assigned to the agents before agent $a_m$ and $a_l$ select in the $(l + 1)$-th round of Algorithm 8, respectively. Note that $J_m^m \subseteq J_1^m$. According to Algorithm 8, we have

$$J_m^k = \{ j \in (J \setminus J_m^k) \mid d_j \leq \max\{D_m^{l-1}, r_j\} \},$$

and

$$J_m^l = \{ j \in (J \setminus J_m^l) \mid d_j \leq \max\{D_m^{l-1}, r_j\} \}.$$ 

Since $J_m^m \subseteq J_m^1$, we have $J \setminus J_m^m \supseteq J \setminus J_m^1$. Now we consider an arbitrary job $j \in J_m^{l+1}$ and show that $j \in J_m^l$. Since $j \in J_m^{l+1}$, we have

$$d_j \leq \max\{D_m^1, r_j\} + p_j.$$ 

According to Lemma 16, we have $D_m^{l-1} \leq D_1^l$. Then, we have

$$\max\{D_m^1, r_j\} + p_j \geq \max\{D_m^{l-1}, r_j\} + p_j.$$ 

Hence, we have $d_j \leq \max\{D_m^{l-1}, r_j\} + p_j$ which implies that $j \in J_m^l$. 

Lemma 18. $|X_i| - |X_k| \in \{-1, 0, 1\}, \forall i, k \in [m]$. 

Proof. We consider the $(L + 1)$-th round of Algorithm 8 in which there exists an agent $a_i \in A$ such that $a_i$ does not choose any job for the first time. Note that there may exist many agents that do not choose any job for the first time in $(L + 1)$-th round. We assume that $a_f$ is the first agent that chooses nothing in the $(L + 1)$-th round. Since $a_f$ chooses nothing, we have $J_f^{L+1} = \emptyset$. According to Lemma 17, we have $J_f^{L+1} = \emptyset, \forall f \leq i \leq n$. Moreover, we have $J_f' = \emptyset, \forall i \in [m]$ and $L + 1 < L'$. Therefore, we have

$$|X_i| = \begin{cases} L, & \forall f \leq i \leq m; \\ L + 1, & \forall 1 \leq i < f. \end{cases}$$ 

This implies that Lemma 18 holds. 

Now we are ready to prove Theorem 7. 

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Proof of Theorem 7. According to Lemma 18 and Lemma 15, we know that Algorithm 8 will return a feasible schedule that is simultaneously EF1 and 1/2-IO.

Now we bound the running time. According to Lemma 18 and Lemma 15, we know that line 5-20 will be run at most $\lceil \frac{n}{m} \rceil$ times, where $n$ is the number of jobs and $m$ is the number of agents. In each while loop, line 6-18 will be run at most $m$ times. In each for loop, line 8-12 will be run at most $n$ times and the running time of line 13 can be bounded by $O(n)$. Hence, we have the running time of Algorithm 8 $O(\lceil \frac{n}{m} \rceil \cdot m \cdot (n^2 + n)) = O(mn^3)$. □

6 Experiment

We now empirically test the performance of Algorithm 4 when jobs are rigid, comparing it against a simple Round-Robin algorithm. In this simple Round-Robin algorithm, all jobs are sorted by their deadlines in non-decreasing order. Then every agent picks a job in round-robin manner. Finally, every agent computes the compatible intervals with the maximum weight and all the remaining jobs will be assigned to charity. The formal description can be found in Algorithm 9 with $J' = J$ and $A' = A$. For the experiments, we have implemented both Algorithm 4 and the above round-robin algorithm.

Algorithm 9. Round-Robin (RR)

**Input:** Agent set $A'$ and job set $J'$.
**Output:** EF1 schedule $X = (X_1, \ldots, X_{|A'|})$

1. Sort all jobs by their deadline in non-decreasing order.
2. $X_1 = X_2 = \cdots = X_{|A'|} = \emptyset$.
3. $i = 1, k = 1$. // The index.
4. for all $j_k \in J'$ do
5.  for all $a_i \in A'$ do
6.    if $k \mod |A'| = i$ then
7.      $X_i = X_i \cup \{j_k\}$.
8.    end if
9.  end for
10. end for
11. $i = 1$. // Reset the index
12. for all $X_i$ do
13.  Let $X_i' \subseteq X_i$ be the compatible job set with the maximum weight for agent $a_i$.
14.  $X_i = X_i'$.
15. end for
16. $X_0 = J \setminus \bigcup_{i \in [|A'|]} X_i$.

We run our experiments on three job sets with different sizes: 100 (Figure 1 (a)), 500 (Figure 1 (b)) and 1000 (Figure 1 (c)). The release time and deadline of each job is uniformly randomly sampled from the interval $[0,50]$. For each job set, we further set up three subgroups according to the agents’ utility of every job: (i) the utility gain is sampled uniformly randomly from $[1,20]$; (ii) the utility gain follows Poisson Distribution with means 50; (iii) the utility gain follows Normal Distribution with means 25 and variance 10. For each subgroup, we further set up three subsubgroups according to the size of agent set: 5, 10 and 15.

In total, our experiment contains $3 \times 3 \times 3$ groups. For each group, we run Algorithm 4 and Round-Robin algorithm on 1000 different instances. Noted that Algorithm 4 does not have a good performance when the number of jobs is much larger than the number of agents, e.g., the groups with 5 agents (U.1, P.1, N.1) in Figure 1. The reason Algorithm 4 performances unsatisfactorily is that Algorithm 4 stops at the threshold while there are a lot of remaining jobs. To fix this problem, we add the Round-Robin procedure at the end of Algorithm 4, i.e., if there exist some unallocated jobs at the end of Algorithm 4, we run Round-Robin algorithm on the remaining job set. Finally, every agent computes the maximum compatible job set from the union of the job set returned by Algorithm 4 and Round-Robin algorithm. The formal description can be found in Algorithm 10. Let BAG+ be the updated version of Algorithm 4 and BAG be the original one. With the help of the Round-Robin procedure,
the performance of Algorithm 4 is better than the Round-Robin algorithm in all groups. Note that BAG+ does not have better theoretical performance than BAG. We give a hard instance to prove above argument in the appendix.

Algorithm 10. Matching-BagFilling + Round-Robin (BAG+)

Input: Agent set $A$ and job set $J$.
Output: EF1 schedule $X = (X_1, \cdots, X_m)$

1: Run Algorithm 4.
2: Let $X = (X_1, \cdots, X_m)$ be the schedule returned by Algorithm 4.
3: Let $X_0 = J \setminus \bigcup_{i \in [m]} X_i$.
4: if $X_0 \neq \emptyset$ then
5: Run Algorithm 9 with job set $X_0$ and agent set $A$.
6: Let $X' = (X'_1, \cdots, X'_m)$ be the schedule returned by Algorithm 9.
7: end if
8: $i = 1$. // The index.
9: for all $X_i$ do
10: Let $X''_i \subseteq (X_i \cup X'_i)$ be the compatible job set with the maximum weight for agent $a_i$.
11: $X_i = X''_i$.
12: end for
13: $X_0 = J \setminus \bigcup_{i \in [m]} X_i$.

According to Figure 1, it is not hard to see that Algorithm 4 is not able to achieve a good performance when the number of jobs is much larger than the number of agents. When the size of the job set is 100, Algorithm 4 performs worse than Round-Robin only in the setting where the agent set is 5 (see Figure 1 (a), only U.1, P.1, N.1’s green interval is behind 1.0). When we increase the number of jobs to 500, the situation that Algorithm 4 is worse than Round-Robin begins to appear at $|A| = 10$ (see Figure 1 (b), part of green interval of U.2 begins to appear behind 1.0). When we further increase the number of jobs to 1000, Algorithm 4 performs better than Round-Robin only in the setting where there are 15 agents (see Figure 1 (c), only U.3, P.3, N.3’s green interval is above 1.0).

The reason is that Algorithm 4 stops at the case where every agent gets the threshold but there are a lot of remaining jobs. We can fix this issue by adding an extra round-robin procedure to allocate the remaining jobs, and thus yield BAG+ algorithm. According to Figure 1, we can find that the performance of BAG+ is better than Round-Robin in all settings as all red intervals are above 1.0. Thus, BAG+ algorithm can achieve a good performance in practices and guarantee the approximation in the worst case.

7 Conclusion and Future Directions

In this work, we studied the fair scheduling problem for time-dependent resources, and designed constant approximation algorithms for MMS, EF1&PO and EF1&IO schedules. There are many open problems and future directions. An immediate direction is to improve our approximation ratios and investigate the limit of approximation algorithms for different settings. It is also interesting to impose other efficiency criteria on EF1 schedules, such as computing an EF1 schedule that maximizes social welfare. In this work, we have assumed the jobs are resources that bring utility to agents, and leave the case when jobs are chores for future study. Finally, it is of both theoretical interest and practical importance to consider the online setting when jobs arrive dynamically and the strategic setting when agents’ valuations are private information.
Figure 1: The results of the evaluation of Algorithm 4, Algorithm 10 and Algorithm 9 on different settings. Every subfigure represents the groups with same job set size. Every notation in x-axis represents a setting. Notations "U.", "P.", "N.", represent the utility gain follows the Uniform, Poisson, Normal Distribution, respectively. Notations "1", "2", "3", represent the number of agents is 5, 10, 15, respectively. The top and bottom point of every solid red interval represent the maximum and minimum value of BAG+/RR among all the agents, where BAG+/RR is the ratio of total gain that the agents receive when we run BAG+ and RR algorithm. The top and bottom point of every dot-dashed green interval represent the maximum and minimum value of BAG/RR among all the agents, where BAG/RR is the ratio of total gain that the agents receive when we run BAG and RR algorithm.
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Appendix

A  Missing Materials in Section 2

A set function \( f : 2^V \to \mathbb{R} \) defined on \( V \) is called fractionally subadditive (XOS) if there is a finite set of additive functions \( \{ f_1, \ldots, f_m \} \) such that \( f(S) = \max_{1 \leq i \leq m} f_i(S) \) for any \( S \subseteq V \).

Lemma 19. IS functions are XOS.

Proof. Let \( u \) be an IS function defined on job set \( J = \{ j_1, \ldots, j_n \} \) with individual utility \( (v_1 = u(j_1), \ldots, v_n = u(j_n)) \). To show \( u \) is XOS, it suffices to define a finite set of additive functions on \( J \). For each feasible job set \( T \subseteq J \), define additive function \( f_T \) such that \( f_T(j) = v_i \) if \( j_i \in T \) and \( f_T(j) = 0 \) otherwise. Therefore, for any \( S \subseteq T \),

\[
  u(S) = \max_{T \subseteq S \text{ feasible}} \sum_{j_i \in T} v_i = \max_{T \subseteq S \text{ feasible}} f_T(T) = \max_{S \subseteq T \text{ feasible}} f_T(S) = \max_{S \subseteq T} f_T(S),
\]

where the last equality is because any subset of a feasible job set is also feasible. Thus \( u \) is XOS.

B  Missing Materials for MMS Scheduling in Section 3

Lemma 4 (restate). For any \( a_i \), if \( \gamma_i \leq \text{MMS}_i \), Algorithm 3 ensures that \( u_i(X_i) \geq \frac{\beta}{\beta + 2} \gamma_i \), regardless of \( \gamma_{-i} \).

Proof of Lemma 4. Note that the algorithm only ensures that agent \( a_i \) with \( \gamma_i \leq \text{MMS}_i \) can obtain a bag but not everyone. This is natural as if for some \( a_j \neq a_i \) and \( \gamma_j \) is super large compared with \( \text{MMS}_j \), \( a_j \) will never stop the algorithm and get a bag.

Recall that we can assume that there is no large job in the instance, i.e., \( u_i(j_k) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i \), where \( 0 \leq \beta \leq 1 \). Observe that if agent \( a_i \) gets assigned a bag, then her true utility satisfies:

\[
u_i(X_i) = \sum_{j_i \in X_i} u_i(j_i) = u'_i(X_i) \geq \frac{\beta}{\beta + 2} \gamma_i.
\]

The above inequality also holds no matter whether \( \gamma_i \leq \text{MMS}_i \) or not. Similar as the proof of Lemma 3, the core is to prove that \( a_i \) can be guaranteed to obtain a bag as long as \( \gamma_i \leq \text{MMS}_i \). We consider the \( R \)-th round of the outer while loop of Algorithm 4 (line 2-5) in which the value of \( \gamma_i \) is decreased below \( \text{MMS}_i \). In the \( R \)-th round of Algorithm 4 (line 2-5), we assume that the order of the agents that break the while loop of Algorithm 2 (line 4-10) is \( \{ a_1, \ldots, a_{i-1}, a_i, \ldots \} \). It suffices to prove that at the beginning of the \( \epsilon \)-th while loop of Algorithm 2 (line 4-10), there are sufficiently many remaining jobs in \( J' \) for the agent \( a_i \), i.e.,

\[
u'_i(J') \geq \frac{\beta}{\beta + 2} \cdot \gamma_i, \forall a_i \in A'.
\]

Similar as the proof of Lemma 3, we prove the following stronger claim. Given Claim 4 and the \( \beta \)-approximation of \( u'_i \), we have \( u'_i(X'_k \cap J') \geq \frac{\beta}{\beta + 2} \cdot \gamma_i \). Therefore Lemma 4 holds.

Claim 4. For any \( a_i \in A' \) with \( \gamma_i \leq \text{MMS}_i \), let \( X' = \{ X'_1, \ldots, X'_m \} \) be a feasible MMS schedule for \( a_i \). Then, there exists \( k \in [m] \) such that \( u_i(X'_k \cap J') \geq \frac{1}{\beta + 2} \cdot \gamma_i \) where \( \gamma_i \leq \text{MMS}_i \).

Proof. We consider an arbitrary agent \( a_i \). Since \( X' = (X'_1, X'_2, \ldots, X'_m) \) is a feasible MMS schedule for \( a_i \), we have \( u_i(X'_k) \geq \text{MMS}_i \geq \gamma_i, \forall k \in [m] \) and therefore

\[
\sum_{k=1}^{m} u_i(X'_k) \geq m \cdot \text{MMS}_i \geq m \cdot \gamma_i.
\]
Same as the proof of Lemma 3, the key idea of the proof is to show that agent $a_i$ values the bundles that are taken by the agents before $a_i$ less than $\frac{\beta + 1}{\beta + 2} \cdot \gamma_i$, i.e.,

$$u_i(X_r) < \frac{\beta + 1}{\beta + 2} \cdot \gamma_i, \forall r \in [i - 1].$$

(10)

We consider an arbitrary bundle that is taken by agent $a_r$, $r \in [i - 1]$ and assume that job $j_r$ is the last job added to the Bag. Since $a_i$ did not break the while loop, we have $u_i'(X_r \setminus \{j_r\}) < \frac{\beta}{\beta + 2} \cdot \gamma_i$. This implies that $u_i'(X_r \setminus \{j_r\}) < \frac{1}{\beta + 2} \cdot \gamma_i$. Since all jobs are small, i.e., $u_i(j_r) \leq \frac{\beta}{\beta + 2} \cdot \gamma_i$, we have

$$u_i(X_r) = u_i(X_r \setminus \{j_r\}) + u_i(j_r) < \frac{\beta + 1}{\beta + 2} \cdot \gamma_i.$$

Therefore, Equation (10) holds. To help understand the following proof, an example is shown in Figure 2. Every rectangle in Figure 2 represents a job in $J$. The area of every rectangle $j_l$ in Figure 2 represents the value of $u_i(j_l)$. The non-white rectangles represent the jobs that are assigned to some agents in $\{a_1, \ldots, a_{i-1}\}$. According to Equation (10), the total area of non-white rectangles in Figure 2 is at most $\frac{1}{\beta + 2} \cdot \gamma_i$, i.e., $\sum_{r=1}^{i-1} u_i(X_r) < \frac{1}{\beta + 2} \cdot \gamma_i$. According to Equation (9), the total area of rectangles in Figure 2 is at least $m \gamma_i$. Therefore, the total area of white rectangles in $\{X_1', \ldots, X_m'\}$ is at least $m \gamma_i = \frac{(\beta + 1)(i - 1)}{\beta + 2} \gamma_i, i.e.,$

$$\sum_{r=1}^{m} u_i(X_r') \setminus \bigcup_{l \in [i-1]} X_l > m \gamma_i - \frac{(\beta + 1)(i - 1)}{\beta + 2} \gamma_i \geq \frac{m + \beta + 1}{\beta + 2} \gamma_i,$$

(11)

where the last inequality is due to $i \leq m$.

Figure 2: Illustration of Claim 4. The schedule is the feasible schedule $X'$ which implies that job set $X'_r$ is a feasible for all $r \in [m]$. Every rectangle represents a job. The width of rectangle $j_l$ is the value of $u_i(j_l)$ while the height is 1. The area of rectangle $j_l$ is also the value of $u_i(j_l)$. The four agents $a_1, a_2, a_3, a_4 \in \{a_1, \ldots, a_{i-1}\}$. The non-white rectangles represent the jobs that are assigned in some agents in $\{a_1, \ldots, a_{i-1}\}$ in schedule $X$, e.g., the gray, blue, green, pink rectangles are the jobs that are assigned to $a_1, a_2, a_3, a_4$, respectively. Recall that $X$ is the schedule returned by Algorithm 2. The white rectangles are the jobs in $J'$. In Claim 4, we show that there exist a $r \in [m]$ such that total area of white rectangles in $X_r'$ is at least $\frac{m + \beta + 1}{m \gamma_i}$.

According to Equation (11), the total area of white rectangles is at least $\frac{m + \beta + 1}{m \gamma_i}$. There must exist an $r \in [m]$ such that $u_i(X'_r \cap J') \geq \frac{m + \beta + 1}{m(\beta + 2)} \gamma_i$. Therefore, Claim 4 holds. \qed
C Missing Proof for EF1 and PO Scheduling in Section 4

C.1 Proof of Theorem 4

Theorem 4 (restate) Given an arbitrary instance of general FISP, any schedule that maximizes the Nash social welfare is a 1/4-EF1 and PO schedule.

Proof of Theorem 4. Given an arbitrary instance of general FISP, let $X = (X_1, \cdots, X_m)$ be the MaxNSW schedule and let $X_0 = J \setminus \bigcup_{i \in [m]} X_i$. Since any MaxNSW schedule must be a PO schedule, we only prove that $X$ is a 1/4-EF1 schedule i.e., $\forall i, k \in [m]$, $u_i(X_i) \leq \frac{1}{4} u_i(X_k \setminus \{j\})$, $\forall j \in X_k$. Suppose, on the contrary, that there exists $i, k \in [m]$ such that $u_i(X_i) < \frac{1}{4} u_i(X_k \setminus \{j\})$, $\forall j \in X_k$.

Now, we sort all jobs in $X_k$ in non-increasing order according to the value of $u_k(j), j \in X_k$. Assume that $X_k = \{j_1, j_2, \cdots \}$ after sorting. Without loss of generality, we assume that $|X_k|$ is an odd number; otherwise, we add a dummy job $j$ to $X_k$ such that $u_i(X_i) < \frac{1}{4} u_i(X_k \setminus \{j\})$, $\forall j \in X_k$. Now we partition $X_k \setminus \{j_1\}$ into two subsets $X_k^1, X_k^2$, where $X_k^1 = \{j_2, j_3, j_4, \cdots \}$ and $X_k^2 = \{j_3, j_5, j_7, \cdots \}$. Note that $X_k = \{j_1\} \cup \{X_k^1\} \cup \{X_k^2\}$. Note that $u_k(X_k^1) \geq u_k(X_k^2)$ and $u_k(X_k^2 \cup \{j_1\}) \geq u_k(X_k^1)$ since all jobs in $X_k$ are sorted in non-increasing order. Since $u_k(X_k^1) \geq u_k(X_k^2)$, we have $u_k(j_1) + u_k(X_k^1) \geq u_k(X_k^2)$. Therefore, we have

$$u_k(X_k^d \cup \{j_1\}) \geq \frac{1}{2} u_k(X_k), \forall d \in \{1, 2\}. \quad (12)$$

Since $u_i(X_i) < \frac{1}{4} u_i(X_k \setminus \{j\}), \forall j \in X_k$, we have $u_i(X_i) < \frac{1}{4} u_i(X_k^1 \cup X_k^2)$. Since $X_k$ is a feasible job set, we have $u_i(X_k = 1 \cup X_k^2) = u_i(X_k^1) + u_i(X_k^2)$ which implies that either $u_i(X_k^1) \geq \frac{1}{4} u_i(X_k^1 \cup X_k^2)$ or $u_i(X_k^1) \geq \frac{1}{4} u_i(X_k^1 \cup X_k^2)$. Therefore, we have

$$u_i(X_i) < \frac{1}{4} u_i(X_k^1 \cup X_k^2) \leq \frac{1}{8} u_i(X_k^d), \forall d \in \{1, 2\}. \quad (13)$$

Now we construct a new schedule, denoted by $X' = (X'_1, \cdots, X'_m)$, where $X'_r = X_r, \forall r \in [m], r \neq i, k$. Let $X'_0 = J \setminus \bigcup_{i \in [m]} X'_i$. We discard all jobs in $X_i$, i.e., $X'_0 = X_0 \cup X_i$. If $u_k(X'_1) \geq \frac{1}{4} u_k(X_k^1 \cup X_k^2)$, let $X'_1 = X_k^1$ and $X'_k = X_k^2 \cup \{j_1\}$; otherwise, let $X'_1 = X_k^2$ and $X'_k = X_k^1 \cup \{j_1\}$. It is easy to see that $X'$ is a feasible schedule. Note that all job sets in $X'$ except $X'_0, X'_1, X'_k$ are the same as the corresponding job sets in $X$. Observe that if we can prove that $u_i(X'_1) u_k(X'_1) > u_i(X_i) u_k(X_k), X$ is not a MaxNSW schedule which will contradict our assumption. In the case where $u_k(X'_1) \geq \frac{1}{4} u_k(X_k^1 \cup X_k^2)$, we have $X'_1 = X_k^1$. By Equation (12), we have $u_k(X'_1) = u_k(X_k^1 \cup \{j_1\}) \geq \frac{1}{4} u_k(X_k)$. By Equation (13), we have $u_i(X'_1) = u_k(X_k^1) > 8 u_i(X_i)$. In the case where $u_k(X'_1) < \frac{1}{4} u_k(X_k^1 \cup X_k^2)$, we have $X'_1 = X_k^2$. By Equation (12), we have $u_k(X'_1) = u_k(X_k^1 \cup \{j_1\}) \geq \frac{1}{4} u_k(X_k)$. By Equation (13), we have $u_i(X'_1) = u_k(X_k^2) > 8 u_i(X_i)$. By combining above two cases, we have $4 u_i(X_i) u_k(X_k) < u_i(X'_1) u_k(X'_1)$. Hence, we have $u_i(X_i) u_k(X_k) < u_i(X'_1) u_k(X'_1)$. \qed

C.2 The tight instance for Theorem 4

Lemma 20. Given an arbitrary instance of general FISP, a MaxNSW schedule can only guarantee 1/4-EF1 and PO.

Proof. To prove Lemma 20, it is sufficient to give an instance such that MaxNSW schedule is exactly 1/4-EF1 schedule and PO. In this instance, all jobs in job set $J$ are rigid and $J$ can be partitioned into two sets $J_L$ and $J_S$. There is only one job in $J_L$, which is very long and has weight 1. There are $\frac{1}{\varepsilon}$ jobs in $J_S$ each of which has unit length and weight $\varepsilon$. Note that $\frac{1}{\varepsilon}$ is assumed to be an even integer number. All jobs in $J_S$ are disjoint and the job in $J_L$ intersects with all jobs in $J_S$. The agent set $A$ contains only two agents, i.e., $|A| = 2$. The instance can be found in Figure 3.

Note that the total weight of jobs in $J_S$ is 4. Let $X = (X_1, X_2)$ be the schedule, where $X_1 = J_L, X_2 = J_S$. Let $X_0 = (X_1, X_2)$, where $X_1 = \{j_2, \cdots, j_{2\varepsilon + 1}\}, X_2 = J_S \setminus X_1$, i.e., $J_S$ is partitioned into two subsets with equal size. Note that $X_0 = J_L$. It is not hard to see that $X'$ is a MaxNSW schedule. And we have
Section 6.

It is not hard to see that Matching-BagFilling+ does not have better theoretical performance than Matching-BagFilling as claimed in X. We consider the schedule MMS ϵ \{ j_p \} takes away all remaining jobs, everyone obtains exactly D Missing the Hard Instance in Section 6.

4 − 4m ǫ = 4 − ǫ, ∀j_p ∈ X 2. Therefore, we have

\lim_{\epsilon \to 0} \frac{1}{4} u_1(X_2 \setminus \{ j_p \}) = 1 = u_1(X_1), \forall j_p ∈ X_1.

This implies that X is a 1/4-EF1 schedule.

D Missing the Hard Instance in Section 6

In the following, we present an instance such that even without the preprocessing procedure and the last agent takes away all remaining jobs, everyone obtains exactly 1/4 MMS_1 + ǫ. Accordingly, the instance proves that “Matching-BagFilling+ does not have better theoretical performance than Matching-BagFilling” as claimed in Section 6.

Consider the following instance with |A| = m agents where m is a sufficiently large even number.

The set J can be classified into the following categories:

• J_1 = \{ j_1^1, j_1^2, \cdots, j_m^1 \}: There are m rigid jobs in J_1. Every job in J_1 has the same job interval [1, 2]. For every job in J_1, a_m has the same utility gain \frac{1}{m} + \frac{1}{m}. For every job in J_1, all agents in A \{ a_m \} have the same utility gain \frac{1}{m} + \frac{1}{m};

• J_2 = \{ j_1^2, j_2^2, \cdots, j_{m-1}^2 \}: There are m − 1 rigid jobs in J_2. Every job in J_2 has the same job interval [3, \frac{m}{2} + 2]. For every job in J_2, all agents in A have the same utility gain \frac{1}{2};

• J_3 = \{ j_1^3, j_2^3, \cdots, j_{m-1}^3 \}: There are m unit jobs in J_3. Every job in J_3 has the same job interval [3, m + 2]. For every job in J_3, all agents in A have the same utility gain \frac{1}{m};

• J_4 = \bigcup_{r \in [m]} J_r^4: There are m group rigid jobs in J_4. Each group J_r^4, r \in [m], contains m rigid jobs. Assume that J_r^4 = \{ j_r^1, j_r^2, \cdots, j_r^m \}, ∀r \in [m − 1]. A job j_r^i \in J_r^4, i \in [m] has the job interval [m + 3 + i, m + 4 + i]. Assume that J_r^m = \{ j_1^m, j_2^m, \cdots, j_m^m \}. A job j_m^i \in J_r^m has the job interval [m + 4 + i, m + 5 + i]. In total, there are m^2 jobs in J_4. For every job in J_4, a_m has the same utility gain \frac{1}{m}. For every job in J_4, all agents in A \{ a_m \} have the same utility gain 0.

Let us focus on a_m first. The upper bound of MMS_m is:

\frac{1}{m} \cdot \left( \frac{1}{3} + \frac{1}{m} \right) \cdot m + \frac{1}{m} \cdot m + \frac{1}{3m} \cdot m + \frac{1}{3m} \cdot m^2 = 1 + \frac{1}{m}.

We consider the schedule X = (X_1, \cdots, X_m), where X_r = \{ j_1^1, j_r^2 \} ∪ J_1^r, ∀r \in [m − 1] and X_m = \{ j_m^1 \} ∪ J_3 \cup J_m^m (See Figure 4). It is not hard to see that X is a feasible schedule and min_{i \in [m]} u_m(X_i) = u_m(X_m) = 1 + \frac{1}{m}.

Therefore, X is a feasible schedule that obtains the value 1 + \frac{1}{m} which is also the upper bound of MMS_m. Thus, MMS_m = 1 + \frac{1}{m}. Hence, once a_m values the bag greater than or equal to \frac{1}{m} + \frac{1}{3m}, a_m will take the bag away.

Now, we consider an arbitrary agent a_i ∈ A \{ a_m \}. Since all agents in A \{ a_m \} have utility gain 0 for all jobs in J_4, we can ignore the job set J_4. Therefore, the upper bound of MMS, ∀i \in [m − 1] is:

\frac{1}{m} \left( \frac{2}{3} + \frac{1}{m} \right) \cdot m + \left( \frac{m - 1}{3} \right) + \left( \frac{1}{3m} \right) \cdot \frac{1}{m} = 1 + \frac{1}{m}.
Algorithm 3 adds \( J^{4}_{1}\) to the bag, and then adds \( J^{3}_{1}, J^{4}_{m} \) to the bag, and then adds \( J^{3}_{1} \) to the bag, i.e., \( BAG = \{ J^{4}_{11}, J^{2}_{12}, \ldots, J^{4}_{1(m-2)} \} \cup \{ J^{3}_{1}, J^{4}_{m} \} \cup \{ J^{3}_{1} \} \). It is not hard to see that \( BAG \) is a feasible job set and all agents in \( A \) obtain the utility gain exactly \( \frac{1}{3} + \frac{1}{3m} \). Without loss of generality, we assume that \( a_{1} \) takes the bag away at the end of the first round. In the \( l \)-th round, \( 2 \leq l \leq m - 1 \), Algorithm 3 first adds \( J^{4}_{1(m-1)}, J^{4}_{1m} \), and then adds \( J^{3}_{1}, J^{4}_{m} \), and then adds \( J^{3}_{1} \) to the bag. Without loss of generality, we assume that \( a_{1} \) takes the bag away at the end of the \( l \)-th round, where \( 2 \leq l \leq m - 1 \). Note that, at the end of the \( (m - 1) \)-th round, all agents in \( A \) obtain the utility gain exactly \( \frac{1}{3} + \frac{1}{3m} \). It is not hard to see that, at the end of the \( (m - 1) \)-th round,

\[
J' = J_{1} \cup J^{3}_{m} \cup J^{4}_{m} \cup J^{4}_{(m-1)} \cup J^{4}_{(m-1)m}.
\]

See the red jobs in Figure 4. Thus, \( u_{m}(J') = \left( \frac{1}{3} + \frac{1}{3m} \right) + \frac{1}{3m} + \frac{3}{3m} = \frac{1}{3} + \frac{7}{3m} \).

Therefore, everyone obtains exactly \( \frac{1}{3} + \frac{1}{3m} + \epsilon \) at the end of Algorithm 3. Moreover, it is not hard to see that if we run round-robin procedure at the end of Algorithm 3, the utility gains of all agents in \( A \) will be increased but the utility gain of \( a_{m} \) is not able to be further improved. Thus, the above instance implies that “BAG+ does not have better theoretical performance than BAG+.”

Figure 4: Illustration for the tight instance for Algorithm 3. The above schedule is \( X \) which is also the MMS schedule for agent \( a_{m} \). The red jobs are the remaining jobs at the end of the \( (m-1) \)-th round of Algorithm 3 with the specified job sequence described in the "The specified job sequence" paragraph.

We consider the schedule \( X' = (X_{1}', \ldots, X_{m}') \), where \( X_{i}' = \{ j_{1}^{1}, j_{i}^{2} \}, \forall i \in [m - 1] \) and \( X_{m}' = \{ j_{m}^{1} \} \cup J_{3} \). It is not hard to see that \( X' \) is a feasible schedule and \( u_{i}(X_{i}') = u_{i}(X_{i}') = 1 + \frac{1}{m}, \forall k, r \in [m], \forall i \in [m - 1] \). Therefore, \( X' \) is a feasible schedule that obtains the value \( 1 + \frac{1}{m} \) which is also an upper bound of \( MMS_{i}, \forall i \in [m - 1] \). Thus, \( MMS_{i} = 1 + \frac{1}{m}, \forall i \in [m - 1] \). Hence, once agent \( a_{i}, \forall i \in [m - 1] \), values the bag greater than or equal to \( \frac{1}{3} + \frac{1}{3m} \), \( a_{i} \) will take the bag away.

The specified job sequence Now, we consider the following job sequence. In the first round, Algorithm 3 adds \( J^{4}_{1} \) to the bag, and then adds \( J^{3}_{1}, J^{4}_{m} \) to the bag, and then adds \( J^{3}_{1} \) to the bag, i.e., \( BAG = \{ J^{4}_{11}, J^{2}_{12}, \ldots, J^{4}_{1(m-2)} \} \cup \{ J^{3}_{1}, J^{4}_{m} \} \cup \{ J^{3}_{1} \} \). It is not hard to see that \( BAG \) is a feasible job set and all agents in \( A \) value the bag exactly \( \frac{1}{3} + \frac{1}{3m} \). Without loss of generality, we assume that \( a_{1} \) takes the bag away at the end of the first round. In the \( l \)-th round, \( 2 \leq l \leq m - 1 \), Algorithm 3 first adds \( J^{4}_{1(m-1)}, J^{4}_{1m} \), and then adds \( J^{3}_{1}, J^{4}_{m} \), and then adds \( J^{3}_{2} \) to the bag. Without loss of generality, we assume that \( a_{1} \) takes the bag away at the end of the \( l \)-th round, where \( 2 \leq l \leq m - 1 \). Note that, at the end of the \( (m - 1) \)-th round, all agents in \( A \) obtain the utility gain exactly \( \frac{1}{3} + \frac{1}{3m} \). It is not hard to see that, at the end of the \( (m - 1) \)-th round,