Concentration Inequalities from Likelihood Ratio Method *

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Abstract

We explore the applications of our previously established likelihood-ratio method for deriving concentration inequalities for a wide variety of univariate and multivariate distributions. New concentration inequalities for various distributions are developed without the idea of minimizing moment generating functions.

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1 Introduction

Bounds for probabilities of random events play important roles in many areas of engineering and sciences. Formally, let $E$ be an event defined in probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is the sample space, $\mathbb{P}$ denotes the probability measure, and $\mathcal{F}$ is the $\sigma$-algebra. A frequent problem is to obtain simple bounds as tight as possible for $\mathbb{P}\{E\}$. In general, the event $E$ can be expressed in terms of a matrix-valued random variable $\mathbf{X}$. In particular, $\mathbf{X}$ can be a random vector or scalar. Clearly, the event $E$ can be represented as $\{\mathbf{X} \in \mathcal{E}\}$, where $\mathcal{E}$ is a certain set of deterministic matrices. In probability theory, a conventional approach for deriving inequalities for $\mathbb{P}\{E\}$ is to bound the indicator function $\mathbb{I}_{\{\mathbf{X} \in \mathcal{E}\}}$ by a family of random variables having finite expectation and minimize the expectation. The central idea of this approach is to seek a family of bounding functions $w(\mathbf{X}, \vartheta)$ of $\mathbf{X}$, parameterized by $\vartheta \in \Theta$, such that

$$\mathbb{I}_{\{\mathbf{X} \in \mathcal{E}\}} \leq w(\mathbf{X}, \vartheta) \quad \text{for all } \vartheta \in \Theta.$$  

(1)

Here, the notion of inequality (1) is that the inequality $\mathbb{I}_{\{\mathbf{X}(\omega) \in \mathcal{E}\}} \leq w(\mathbf{X}(\omega), \vartheta)$ holds for every $\omega \in \Omega$. As a consequence of the monotonicity of the mathematical expectation $\mathbb{E}[\cdot]$, 

$$\mathbb{P}\{E\} = \mathbb{E}[\mathbb{I}_{\{\mathbf{X} \in \mathcal{E}\}}] \leq \mathbb{E}[w(\mathbf{X}, \vartheta)] \quad \text{for all } \vartheta \in \Theta.$$  

(2)

Minimizing the upper bound in (2) with respect to $\vartheta \in \Theta$ yields

$$\mathbb{P}\{E\} \leq \inf_{\vartheta \in \Theta} \mathbb{E}[w(\mathbf{X}, \vartheta)].$$  

(3)

Classical concentration inequalities such as Chebyshev inequality and Chernoff bounds (2) can be derived by this approach with various bounding functions $w(\mathbf{X}, \vartheta)$, where $\mathbf{X}$ is a scalar random variable. We call this technique of deriving probabilistic inequalities as the mathematical expectation (ME) method, in view of the crucial role played by the mathematical expectation of bounding functions. For the ME method to be successful, the mathematical expectation $\mathbb{E}[w(\mathbf{X}, \vartheta)]$ of the family of bounding functions $w(\mathbf{X}, \vartheta)$, $\vartheta \in \Theta$ must be convenient for evaluation and minimization. The ME method is a very general approach. However, it has two drawbacks. First, in some situations, the mathematical expectation $\mathbb{E}[w(\mathbf{X}, \vartheta)]$ may be intractable. Second, the ME method may not fully exploit the information of the underlying distribution, since the mathematical expectation is only a quantity of summary for the distribution.

Recently, we have proposed in [3, 4, 5] a more general approach for deriving probabilistic inequalities, aiming at overcoming the drawbacks of the ME method. Let $f(.)$ denote the probability density function (pdf) or probability mass function (pmf) of $\mathbf{X}$. The primary idea of the proposed approach is to seek a family of pdf or pmf $g(.)$, parameterized by $\vartheta \in \Theta$, and a deterministic function $\Lambda(\vartheta)$ of $\vartheta \in \Theta$ such that for all $\vartheta \in \Theta$, the indicator function $\mathbb{I}_{\{\mathbf{X} \in \mathcal{E}\}}$ is bounded from above by the product of $\Lambda(\vartheta)$ and the likelihood ratio $\frac{g(\mathbf{X}, \vartheta)}{f(\mathbf{X})}$. Then, the probability $\mathbb{P}\{\mathbf{X} \in \mathcal{E}\}$ is bounded from above by the infimum of $\Lambda(\vartheta)$ with respect to $\vartheta \in \Theta$. Due to the central role played by the likelihood ratio, this technique of deriving probabilistic inequalities is referred to as the likelihood ratio (LR) method. It has been demonstrated in [1] that the ME method is actually a special technique of the LR method.

B Proofs of Multivariate Inequalities

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In this paper, we shall apply the LR method to investigate the concentration phenomenon of random variables. Our goal is to derive simple and tight concentration inequalities for various distributions. The remainder of the paper is organized as follows. In Section 2, we introduce the fundamentals of the LR method. In Section 3, we apply the LR method to the development of concentration inequalities for univariate distributions. In Section 4, we apply the LR method to establish concentration inequalities for multivariate distributions. Section 5 is the conclusion. Most proofs are given in Appendices.

Throughout this paper, we shall use the following notations. Let $I_E$ denote the indicator function such that $I_E = 1$ if $E$ is true and $I_E = 0$ otherwise. We use the notation $\binom{t}{k}$ to denote a generalized combinatoric number in the sense that $\binom{t}{k} = \frac{\prod_{\ell=1}^k (t-\ell+1)}{k!} = \frac{\Gamma(t+1)}{\Gamma(k+1) \Gamma(t-k+1)}$, $\binom{t}{0} = 1$, where $t$ is a real number and $k$ is a non-negative integer. We use $\overline{X}_n$ to denote the average of random variables $X_1, \ldots, X_n$, that is, $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$. The notation $\top$ denotes the transpose of a matrix. The trace of a matrix is denoted by tr. We use pdf and pmf to represent probability density function and probability mass function, respectively. The other notations will be made clear as we proceed.

2 Likelihood Ratio Method

In this section, we shall introduce the LR method for deriving probabilistic inequalities.

2.1 General Principle

Let $E$ be an event which can be expressed in terms of matrix-valued random variable $\mathbf{X}$, where $\mathbf{X}$ is defined on the sample space $\Omega$ and $\sigma$-algebra $\mathcal{F}$ such that the true probability measure is one of two measures $\Pr$ and $\mathbb{P}_\vartheta$. Here, the measure $\Pr$ is determined by pdf or pmf $f(.)$. The measure $\mathbb{P}_\vartheta$ is determined by pdf or pmf $g(., \vartheta)$, which is parameterized by $\vartheta \in \Theta$. The subscript in $\mathbb{P}_\vartheta$ is used to indicate the dependence on the parameter $\vartheta$. Clearly, there exists a set, $\mathcal{E}^*$, of deterministic matrices of the same size as $\mathbf{X}$ such that $E = \{ \mathbf{X} \in \mathcal{E}^* \}$. The LR method for obtaining an upper bound for the probability $\Pr\{E\}$ is based on the following general result.

**Theorem 1** Assume that there exists a function $\Lambda(\vartheta)$ of $\vartheta \in \Theta$ such that

$$f(\mathbf{X}) I_{\{\mathbf{X} \in \mathcal{E}\}} \leq \Lambda(\vartheta) g(\mathbf{X}, \vartheta) \quad \text{for all } \vartheta \in \Theta.$$  \hfill (4)

Then,

$$\Pr\{E\} \leq \inf_{\vartheta \in \Theta} \Lambda(\vartheta) \mathbb{P}_\vartheta\{E\} \leq \inf_{\vartheta \in \Theta} \Lambda(\vartheta).$$  \hfill (5)

In particular, if the infimum of $\Lambda(\vartheta)$ is attained at $\vartheta^* \in \Theta$, then

$$\Pr\{E\} \leq \mathbb{P}_{\vartheta^*}\{E\} \Lambda(\vartheta^*).$$  \hfill (6)

The notion of the inequality in (4) is that $f(\mathbf{X}(\omega)) I_{\{\mathbf{X}(\omega) \in \mathcal{E}\}} \leq \Lambda(\vartheta) g(\mathbf{X}(\omega), \vartheta)$ for every $\omega \in \Omega$. The function $\Lambda(\vartheta)$ in (4) is referred to as likelihood-ratio bounding function. Theorem 1 asserts that the probability of event $E$ is no greater than the likelihood ratio bounding function.
2.2 Construction of Parameterized Distributions

In the sequel, we shall introduce two approaches for constructing parameterized distributions \( g(., \vartheta) \) which are essential for the application of the LR method.

2.2.1 Weight Function

A natural approach to construct parameterized distribution \( g(., \vartheta) \) is to modify the pdf or pmf \( f(.) \) by multiplying it with a parameterized function and performing a normalization. Specifically, let \( w(., \vartheta) \) be a non-negative function with parameter \( \vartheta \in \Theta \) such that \( E[w(X, \vartheta)] < \infty \) for all \( \vartheta \in \Theta \), where the expectation is taken under the probability measure \( Pr \) determined by \( f(.) \). Define a family of distributions as

\[
\begin{align*}
g(X, \vartheta) &= \frac{w(X, \vartheta) f(X)}{E[w(X, \vartheta)]} \end{align*}
\]

for \( \vartheta \in \Theta \) and \( X \) in the range of \( X \). In view of its role in the modification of \( f(.) \) as \( g(., \vartheta) \), the function \( w(., \vartheta) \) is called a weight function. Note that

\[
f(X) w(X, \vartheta) = E[w(X, \vartheta)] g(X, \vartheta) \quad \text{for all } \vartheta \in \Theta.
\]

For simplicity, we choose the weight function such that the condition (1) is satisfied. Combining (1) and (7) yields

\[
f(X) I_{\{X \in E\}} \leq f(X) w(X, \vartheta) = E[w(X, \vartheta)] g(X, \vartheta) \quad \text{for all } \vartheta \in \Theta.
\]

Thus, the likelihood ratio bounding function can be taken as

\[
\Lambda(\vartheta) = E[w(X, \vartheta)] \quad \text{for } \vartheta \in \Theta.
\]

It follows from Theorem 1 that

\[
Pr\{E\} \leq \inf_{\vartheta \in \Theta} \Lambda(\vartheta) \cdot Pr\{E\} \leq \inf_{\vartheta \in \Theta} \Lambda(\vartheta).
\]

Thus, we have demonstrated that the ME method is actually a special technique of the LR method.

By constructing a family of parameterized distributions and making use of the LR method, we have obtained the following result.

**Theorem 2** Let \( X \) be a random variable with moment generating function \( \phi(.) \). Let \( X_1, \ldots, X_n \) be i.i.d. samples of \( X \). Let \( C_{BE} \) be the absolute constant in the Berry-Essen inequality. Then,

\[
Pr\{X_n \geq z\} \leq \left( \frac{1}{2} + \Delta \right) \left[ e^{-z\tau \phi(\tau)} \right]^n,
\]

where

\[
\Delta = \min \left\{ \frac{1}{2}, C_{BE} \sqrt{n} \left( \frac{\phi(\tau)\phi''''(\tau) - 4z\phi''''(\tau) + 3\phi''''(\tau)^2}{\phi''(\tau)^2} - 3 \right)^{\frac{1}{2}} \right\}
\]

with \( \tau \) satisfying \( \frac{\phi'(\tau)}{\phi(\tau)} = z \).

See Appendix A.1 for a proof. Note that \( \Delta \to 0 \) as \( n \to \infty \). So, for large \( n \), the above bound is twice tighter than the classical Chernoff bound.
2.2.2 Parameter Restriction

In many situations, the pdf or pmf \( f(.) \) of \( X \) comes from a family of distributions parameterized by \( \theta \in \Theta \). If so, then the parameterized distribution \( g(., \vartheta) \) can be taken as the subset of pdf or pmf with parameter \( \vartheta \) contained in a subset \( \Theta \) of parameter space \( \Theta \). By appropriately choosing the subset \( \Theta \), the deterministic function \( \Lambda(\vartheta) \) may be readily obtained. As an illustrative example, consider the normal distribution.

A random variable \( X \) is said to have a normal distribution with mean \( \mu \) and variance \( \sigma^2 \) if it possesses a probability density function

\[
f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{|x - \mu|^2}{2\sigma^2}\right).
\]

Let \( X_1, \cdots, X_n \) be i.i.d. samples of the random variable \( X \). The following well-known inequalities hold true.

\[
\Pr\{X_n \leq z\} \leq \frac{1}{2} \exp\left(-\frac{n(z - \mu)^2}{2\sigma^2}\right) \quad \text{for} \quad z \leq \mu, \tag{8}
\]

\[
\Pr\{X_n \geq z\} \leq \frac{1}{2} \exp\left(-\frac{n(z - \mu)^2}{2\sigma^2}\right) \quad \text{for} \quad z \geq \mu. \tag{9}
\]

It should be noted that the factor \( \frac{1}{2} \) in these inequalities cannot be obtained by using conventional techniques of Chernoff bounds. By virtue of the LR method, we can provide an easy proof for inequalities (8) and (9). We proceed as follows.

Let \( X = [X_1, \cdots, X_n] \) and \( x = [x_1, \cdots, x_n] \). The joint probability density function of \( X \) is

\[
f_X(x) = \frac{1}{(\sqrt{2\pi\sigma})^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right).
\]

To apply the LR method to show (8), we construct a family of probability density functions

\[
g_X(x, \vartheta) = \frac{1}{(\sqrt{2\pi\sigma})^n} \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \vartheta)^2}{2\sigma^2}\right)
\]

for \( \vartheta \in (-\infty, z] \) with \( z \leq \mu \). It can be checked that

\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left[\exp\left(-\frac{2(\vartheta - \mu)z + \mu^2 - \vartheta^2}{2\sigma^2}\right)\right]^n.
\]

For any \( \vartheta \in (-\infty, z] \), we have \( \vartheta \leq z \leq \mu \) and thus

\[
\frac{f_X(x)}{g_X(x, \vartheta)} \leq \left[\exp\left(-\frac{2(\vartheta - \mu)z + \mu^2 - \vartheta^2}{2\sigma^2}\right)\right]^n \quad \forall \vartheta \in (-\infty, z] \text{ for } X_n \leq z.
\]

This implies that

\[
f_X(X) \mathbb{1}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) \cdot g_X(X, \vartheta) \quad \forall \vartheta \in (-\infty, z],
\]

where

\[
\Lambda(\vartheta) = \left[\exp\left(-\frac{2(\vartheta - \mu)z + \mu^2 - \vartheta^2}{2\sigma^2}\right)\right]^n.
\]

By differentiation, it can be readily shown that the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in (-\infty, z] \) is equal to

\[
\exp\left(-\frac{n(z - \mu)^2}{2\sigma^2}\right),
\]
which is attained at \( \vartheta = z \). By symmetry, it can be shown that

\[
P_z \{ X_n \leq z \} = \frac{1}{2}.
\]

Using these facts and invoking (8) of Theorem 1, we have

\[
\text{Pr} \{ X_n \leq z \} \leq P_z \{ X_n \leq z \} \Lambda(z) \quad \text{for } z \leq \mu.
\]

This implies that inequality (8) holds. In a similar manner, we can show inequality (9).

### 3 Concentration Inequalities for Univariate Distributions

In this section, we shall apply the LR method to derive bounds for tail probabilities for univariate distributions. Such bounds are referred to as concentration inequalities.

#### 3.1 Beta Distribution

A random variable \( X \) is said to have a beta distribution if it possesses a probability density function

\[
f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1, \quad \alpha > 0, \quad \beta > 0,
\]

where \( B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \). Let \( X_1, \ldots, X_n \) be i.i.d. samples of the random variable \( X \). Making use of the LR method, we have shown the following results.

**Theorem 3** Let \( z \in (0, 1) \) and \( \mu = \text{E}[X] = \frac{\alpha}{\alpha+\beta} \). Define \( \hat{\alpha} = \frac{\beta z}{1-z} \) and \( \hat{\beta} = \frac{\alpha(1-z)}{z} \). Then,

\[
\text{Pr} \{ X_n \leq z \} \leq \left[ \frac{B(\hat{\alpha}, \hat{\beta})}{B(\alpha, \beta)} z^n \right] \quad \text{for } 0 < z \leq \mu,
\]

\[
\text{Pr} \{ X_n \geq z \} \leq \left[ \frac{B(\alpha, \hat{\beta})}{B(\alpha, \beta)} (1-z)^n \right] \quad \text{for } \mu \leq z < 1.
\]

**Special case, if \( \beta = 1 \), then**

\[
\text{Pr} \{ X_n \leq z \} \leq \left( \frac{e^{\alpha z} \ln \frac{1}{z}}{\alpha} \right)^n \quad \text{for } 0 < z < \exp\left( \frac{1}{\alpha} \right).
\]

See Appendix A.2 for a proof.

#### 3.2 Beta Negative Binomial Distribution

A random variable \( X \) is said to have a beta distribution if it possesses a probability mass function

\[
f(x) = \text{Pr}(X = x) = \binom{n + x - 1}{x} \frac{\Gamma(\alpha + n)\Gamma(\beta + x)\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + n + x)\Gamma(\alpha)\Gamma(\beta)}, \quad x = 0, 1, 2, \ldots
\]

where \( \alpha > 1 \) and \( \beta > 0 \) and \( n > 1 \). By virtue of the LR method, we have obtained the following results.

**Theorem 4** Let \( z \) be a nonnegative integer no greater than \( \frac{n \beta}{\alpha+1} \). Then,

\[
\text{Pr} \{ X \leq z \} \leq \frac{\Gamma \left( \frac{z+\alpha}{\alpha} \right)}{\Gamma(\beta)} \frac{\Gamma(\beta + z)}{\Gamma \left( \frac{z+\alpha}{\alpha} + z \right)} \frac{\Gamma(\alpha + \frac{\alpha+\beta}{n} + n + z)}{\Gamma(\alpha + \beta + n + z)}.
\]

See Appendix A.3 for a proof.
3.3 Beta-Prime Distribution

A random variable $X$ is said to have a beta-prime distribution if it possesses a probability density function

$$f(x) = \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0.$$ 

Let $X_1, \cdots, X_n$ be i.i.d. samples of the random variable $X$. Making use of the LR method, we have obtained the following results.

Theorem 5  
Assume that $\beta > 1$ and $0 < z \leq \frac{\alpha}{\beta-1}$. Then,

$$\Pr\{X_n \leq z\} \leq \left[ \left( \frac{z}{1+z} \right)^{\alpha+\beta} \frac{B(\beta z - z, \beta)}{B(\alpha, \beta)} \right]^n$$  

(13)

$$\Pr\{X_n \leq z\} \leq \left[ \frac{B(\alpha, 1 + \frac{\alpha}{\beta})}{B(\alpha, \beta)} (1 + z)^{1+\frac{\alpha}{\beta}} \right]^n.$$  

(14)

See Appendix A.4 for a proof.

3.4 Borel Distribution

A random variable $X$ is said to possess a Borel distribution if it has a probability mass function

$$f(x) = \Pr\{X = x\} = \frac{(\theta x)^{x-1}e^{-\theta x}}{x!}, \quad x = 1, 2, \cdots,$$

where $0 < \theta < 1$. Let $X_1, \cdots, X_n$ be i.i.d. samples of the random variable $X$. Making use of the LR method, we have obtained the following result.

Theorem 6

$$\Pr\{X_n \leq z\} \leq \left[ \left( \frac{\theta e z}{1-z} \right)^{z-1} e^{-\theta z} \right]^n \text{ for } 1 < z < \frac{1}{1-\theta}.$$  

(15)

See Appendix A.5 for a proof.

3.5 Consul Distribution

A random variable $X$ is said to have a Consul distribution if it possesses a probability mass function

$$f(x) = \Pr\{X = x\} = \frac{1}{x} \left( \frac{m x}{x-1} \right)^{x-1} \left( \frac{\theta}{1-\theta} \right)^{m x}, \quad x = 1, 2, \cdots,$$

where $0 \leq \theta < 1$, $1 \leq m < \frac{1}{\theta}$. See, e.g., [4], for an introduction of this distribution. Let $X_1, \cdots, X_n$ be i.i.d. samples of the random variable $X$. Making use of the LR method, we have obtained the following result.

Theorem 7

$$\Pr\{X_n \leq z\} \leq \left[ \frac{\left( \frac{\theta}{1-\theta} \right)^{z-1} (1-\theta)^m}{\left( \frac{z-1}{z-mz} \right)^{z-1} (1-\frac{z-1}{mz})^m} \right]^n \text{ for } 1 \leq z < \frac{1}{1-m\theta}.$$  

(16)

See Appendix A.6 for a proof.
3.6 Geeta Distribution

A random variable $X$ is said to have a Geeta distribution if it possesses a probability mass function

$$f(x) = \Pr\{X = x\} = \frac{1}{\beta x - 1} \left( \frac{\beta x - 1}{x} \right) \theta^{x-1} (1 - \theta)^{\beta x - x}, \quad x = 1, 2, \ldots$$

where $0 < \theta < 1$ and $1 < \beta < \frac{1}{\theta}$. Making use of the LR method, we have obtained the following result.

**Theorem 8**

$$\Pr\{\bar{X}_n \leq z\} \leq \left[ \frac{\theta^{\bar{z}-1} (1 - \theta)^{\beta \bar{z} - z}}{\left( \frac{\beta - 1}{\beta z - 1} \right)^{\frac{z-1}{\beta z - 1}}} \right]^n \quad \text{for}\ 1 \leq z \leq \frac{1 - \theta}{1 - \beta \theta}. \quad (17)$$

See Appendix A.7 for a proof.

3.7 Gumbel Distribution

A random variable $X$ is said to have a Gumbel distribution if it possesses a probability density function

$$f(x) = \frac{1}{\beta} \exp \left[ \frac{\mu - x}{\beta} - \exp \left( \frac{\mu - x}{\beta} \right) \right], \quad -\infty < x < \infty,$$

where $\beta > 0$ and $-\infty < \mu < \infty$. Let $X_1, \cdots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 9**

$$\Pr\{\bar{X}_n \leq z\} \leq \left\{ \exp \left[ \frac{\mu - z}{\beta} + 1 - \exp \left( \frac{\mu - z}{\beta} \right) \right] \right\}^n \quad (18)$$

for $z \leq \mu$.

See Appendix A.8 for a proof.

3.8 Inverse Gamma Distribution

A random variable $X$ is said to have an inverse gamma distribution if it possesses a probability density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp \left( -\frac{\beta}{x} \right), \quad x > 0, \quad \alpha > 0, \quad \beta > 0.$$

Let $X_1, \cdots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following results.

**Theorem 10**

$$\Pr\{\bar{X}_n \leq z\} \leq \left[ \frac{\Gamma\left( \frac{\beta}{\alpha} + 1 \right)}{\Gamma(\alpha)} \left( \frac{z}{\beta} \right)^{\frac{\alpha}{\beta} - 1} \right]^n \quad \text{for}\ 0 < z \leq \frac{\beta}{\alpha - 1}. \quad (19)$$

$$\Pr\{\bar{X}_n \leq z\} \leq \left[ \frac{\beta^\alpha}{\alpha^z} \exp \left( \frac{\alpha z - \beta}{z} \right) \right]^n \quad \text{for}\ 0 < z \leq \frac{\beta}{\alpha}. \quad (20)$$

See Appendix A.9 for a proof.
3.9 Inverse Gaussian Distribution

A random variable $X$ is said to have an inverse Gaussian distribution if it possesses a probability density function

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\lambda(x - \theta)^2}{2\theta^2 x}\right), \quad x > 0$$

where $\lambda > 0$ and $\theta > 0$.

Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 11**

$$\Pr\{\bar{X}_n \leq z\} \leq \left[\exp\left(\frac{\lambda}{\theta} - \frac{\lambda}{2z} - \frac{\lambda z}{2\theta^2}\right)\right]^n \text{ for } 0 < z \leq \theta.$$  \hspace{1cm} (21)

See Appendix [A.10] for a proof.

3.10 Lagrangian Logarithmic Distribution

A random variable $X$ is said to have a Lagrangian logarithmic distribution if it possesses a probability mass function

$$f(x) = \Pr\{X = x\} = \frac{-\theta^x (1 - \theta)^{x(\beta - 1)} \Gamma(\beta x)}{\Gamma(x + 1) \Gamma(\beta x - x + 1) \ln(1 - \theta)} \ln(1 - \theta), \quad x = 1, 2, \ldots$$

where $0 < \theta \leq \theta \beta < 1$. Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 12**

$$\Pr\{\bar{X}_n \leq z\} \leq \left[\left(\frac{\theta}{\vartheta}\right)^{z} \left(\frac{1 - \theta}{1 - \vartheta}\right)^{z(\beta - 1)} \frac{\ln(1 - \theta)}{\ln(1 - \vartheta)}\right]^n \text{ for } 0 < z \leq \frac{\theta}{(\beta - 1) \ln(1 - \theta)},$$  \hspace{1cm} (22)

where $\vartheta$ satisfies the equation $z = \frac{\theta}{(\beta - 1) \ln(1 - \vartheta)}$.

See Appendix [A.11] for a proof.

3.11 Lagrangian Negative Binomial Distribution

A random variable $X$ is said to have a Lagrangian logarithmic distribution if it possesses a probability mass function

$$f(x, \theta) = \Pr\{X = x\} = \frac{\beta \alpha^x}{\alpha x + \beta} \left(\frac{\alpha x + \beta}{x}\right)^{\beta(1 - \theta)^{\beta + \alpha z - x}}, \quad x = 0, 1, 2, \ldots$$

where $0 < \theta < 1$, $\theta < \alpha \theta < 1$ and $\beta > 0$. Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 13**

$$\Pr\{\bar{X}_n \leq z\} \leq \left[\left(\frac{\theta}{\vartheta}\right)^{z} \left(\frac{1 - \theta}{1 - \vartheta}\right)^{\beta + \alpha z - z}\right]^n \text{ for } 0 \leq z \leq \frac{\beta \theta}{1 - \alpha \theta},$$  \hspace{1cm} (23)

where $\vartheta = \frac{z}{\beta + \alpha z}$.

See Appendix [A.12] for a proof.
3.12 Laplace Distribution

A random variable $X$ is said to have a Lagrangian-logarithmic distribution if it possesses a probability density function

$$f(x) = \frac{1}{2\beta} \exp\left(-\frac{|x - \alpha|}{\beta}\right), \quad -\infty < x < \infty,$$

where $-\infty < \alpha < \infty$ and $\beta > 0$. Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following results.

**Theorem 14**

$$\Pr\{X_n \geq z\} \leq \left[\frac{z - \alpha}{\beta} \exp\left(1 - \frac{z - \alpha}{\beta}\right)\right]^n \text{ for } z \geq \alpha + \beta, \quad (24)$$

$$\Pr\{X_n \leq z\} \leq \left[\frac{\alpha - z}{\beta} \exp\left(1 - \frac{\alpha - z}{\beta}\right)\right]^n \text{ for } z \leq \alpha - \beta. \quad (25)$$

See Appendix A.13 for a proof.

3.13 Logarithmic Distribution

A random variable $X$ is said to have a logarithmic distribution if it possesses a probability mass function

$$f(x) = \frac{q^x}{-x \ln p}, \quad x = 1, 2, \ldots$$

where $p \in (0, 1)$ and $q = 1 - p$. Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 15**

$$\Pr\{X_n \leq z\} \leq \left[\ln(1 - q) \ln(1 - \vartheta) \left(\frac{q}{\vartheta}\right)^z\right]^n \text{ for } 0 < z \leq \frac{q}{(1 - q) \ln \frac{1}{1 - q}}, \quad (26)$$

where $\vartheta \in (0, q]$ is the unique number such that $z = \frac{q}{(1 - \vartheta) \ln \frac{1}{1 - \vartheta}}$.

See Appendix A.14 for a proof.

3.14 Lognormal Distribution

A random variable $X$ is said to have a lognormal distribution if it possesses a probability density function

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left[-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right], \quad x > 0, \quad -\infty < \mu < \infty, \quad \sigma > 0.$$ 

Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 16**

$$\Pr\{X_n \leq z\} \leq \exp\left[-\frac{n}{2} \left(\frac{\mu - \ln z}{\sigma}\right)^2\right] \text{ for } 0 < z \leq e^\mu. \quad (27)$$

See Appendix A.15 for a proof.
3.15 Nakagami Distribution

A random variable $X$ is said to have a Nakagami distribution if it possesses a probability density function

$$ f(x) = \frac{2}{\Gamma(m) \sigma^{2m}} x^{2m-1} \exp\left(-\frac{x^2}{\sigma^2}\right), \quad x > 0 $$

where $m \geq \frac{1}{2}$ and $\sigma > 0$. Let $X_1, \cdots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following results.

**Theorem 17**

$$ \Pr\left\{ X_n \leq \frac{\Gamma(\vartheta + \frac{1}{2})}{\Gamma(\vartheta)} \sigma \right\} \leq \left\{ \frac{\Gamma(\vartheta)}{\Gamma(m) \left( \frac{\Gamma(\vartheta + 1/2)}{\Gamma(\vartheta)} \right)^{2(m-\vartheta)}} \right\}^n \quad \text{for } 0 < \vartheta \leq m, \quad (28) $$

$$ \Pr\{ X_n \geq z \} \leq \left( \frac{z^2}{m \sigma^2} \right)^m \exp\left( m - \frac{z^2}{\sigma^2} \right) \quad \text{for } z \geq \sqrt{m \sigma}. \quad (29) $$

See Appendix A.16 for a proof.

3.16 Pareto Distribution

A random variable $X$ is said to have a Pareto distribution if it possesses a probability density function

$$ f(x) = \frac{\theta}{a} \left( \frac{a}{x} \right)^{\theta+1}, \quad x > a > 0, \quad \theta > 1. $$

Let $X_1, \cdots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 18**

$$ \Pr\{ \overline{X}_n \leq \rho \mu \} \leq \left[ e \left( \theta - 1 \right) \ln \left( \frac{\rho \theta}{\theta - 1} \right) \right]^n \quad \text{for } 1 - \frac{1}{\theta} < \rho \leq \left( 1 - \frac{1}{\theta} \right) \exp\left( \frac{1}{\theta} \right), \quad (30) $$

where $\mu = \mathbb{E}[X] = \frac{\theta a}{(\theta-1)}$.

See Appendix A.17 for a proof.

3.17 Power-Law Distribution

A random variable $X$ is said to have a power-law distribution if it possesses a probability density function

$$ f(x) = \frac{x^{-\alpha}}{C(\alpha)}, \quad 1 \leq x \leq \beta, $$

where $\beta > 1$, $\alpha \in \mathbb{R}$ and

$$ C(\alpha) = \begin{cases} \frac{1 - \beta^{\alpha - 1}}{\alpha - 1} & \text{for } \alpha \neq 1, \\ \ln \beta & \text{for } \alpha = 1 \end{cases} $$

Let $X_1, \cdots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following result.

**Theorem 19** Let $\theta \geq \alpha > 1$ and $z = \frac{\theta - 1}{\beta - 1} \beta^{\alpha - 1} \beta^{-1}$. Then,

$$ \Pr\{ \overline{X}_n \leq z \} \leq \left( \frac{\alpha - 1}{\theta - 1} \beta^{\theta - 1} \beta^{-\alpha} z^{\alpha - \alpha} \right)^n. \quad (31) $$

See Appendix A.18 for a proof.
3.18 Stirling Distribution

A random variable is said to have a Stirling distribution if it possesses a probability mass function

\[
\Pr\{X = x\} = \frac{m!s(x, m)\theta^x}{x![-\ln(1 - \theta)]^m}, \quad 0 < \theta < 1, \quad x = m, m + 1, \ldots,
\]

where \(s(x, m)\) is the Stirling number of the first kind, with arguments \(x\) and \(m\). Let \(X_1, \ldots, X_n\) be i.i.d. samples of random variable \(X\). By virtue of the LR method, we have obtained the following result.

**Theorem 20**

\[
\Pr\{X_n \leq z\} \leq \left[\ln(1 - \vartheta)\right]^{nm} \left(\frac{\theta}{\vartheta}\right)^{nz} \quad \text{for } z \leq \frac{m\theta}{(\theta - 1)\ln(1 - \theta)}.
\]

where \(\vartheta \in (0, \theta]\) is the unique number such that \(z = \frac{m\vartheta}{(\vartheta - 1)\ln(1 - \vartheta)}\).

See Appendix [A.19] for a proof.

3.19 Snedecor’s F-Distribution

If random variable \(X\) has a probability density function of the form

\[
f(x) = \frac{\Gamma\left(n + m/2\right)\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(n + m/2\right)\Gamma\left(\frac{m}{2}\right)} x^{(m-2)/2} \frac{1}{(1 + \frac{m}{n}x)^{(n+m)/2}}, \quad \text{for } 0 < x < \infty,
\]

then the random variable \(X\) is said to possess an F-distribution with \(m\) and \(n\) degrees of freedom. Making use of the LR method, we have obtained the following results.

**Theorem 21**

\[
\Pr\{X \geq z\} \leq z^{m/2} \left(\frac{n + m}{n + mz}\right)^{(n+m)/2} \quad \text{for } z \geq 1
\]

\[
\Pr\{X \leq z\} \leq z^{m/2} \left(\frac{n + m}{n + mz}\right)^{(n+m)/2} \quad \text{for } 0 < z \leq 1.
\]

See Appendix [A.20] for a proof.

3.20 Student’s t-Distribution

If random variable \(X\) has a probability density function of the form

\[
f(x) = \frac{1}{\sqrt{n\pi}} \Gamma\left(\frac{n+1}{2}\right) (1 + \frac{x^2}{n})^{-(n+1)/2}, \quad \text{for } -\infty < x < \infty,
\]

then the random variable \(X\) is said to possess a Student’s t-distribution with \(n\) degrees of freedom. By virtue of the LR method, we have obtained the following results.

**Theorem 22**

\[
\Pr\{|X| \geq z\} \leq z \left(\frac{n + 1}{n + z^2}\right)^{(n+1)/2} \quad \text{for } z \geq 1,
\]

\[
\Pr\{|X| \leq z\} \leq z \left(\frac{n + 1}{n + z^2}\right)^{(n+1)/2} \quad \text{for } 0 < z \leq 1.
\]

See Appendix [A.21] for a proof.
3.21 Truncated Exponential Distribution

A random variable $X$ is said to have a truncated exponential distribution if it possesses a probability density function

$$f(x) = \frac{\theta e^{\theta x}}{e^{\theta} - 1}, \quad \theta \neq 0, \quad 0 < x < 1.$$

Let $X_1, \ldots, X_n$ be i.i.d. samples of random variable $X$. By virtue of the LR method, we have obtained the following results.

**Theorem 23**

$$\Pr\{X_n \leq z\} \leq \left[ \frac{\theta e^{\theta} - 1}{\theta e^{\theta} - 1 e^{(\theta - \vartheta)z}} \right]^n \text{ for } 0 < z \leq 1 + \frac{1}{e^{\theta} - 1} - \frac{1}{\theta} \text{ and } z \neq \frac{1}{2},$$

(37)

where $\vartheta \in (-\infty, \theta]$, $\vartheta \neq 0$ satisfies equation $z = 1 + \frac{1}{e^{\theta} - 1} - \frac{1}{\theta}$. Moreover,

$$\Pr\left\{X_n \leq \frac{1}{2}\right\} \leq \left( \frac{\theta e^{\theta/2}}{e^{\theta} - 1} \right)^n \text{ for } \theta > 0.$$

(38)

See Appendix A.22 for a proof.

3.22 Uniform Distribution

Let $X$ be a random variable uniformly distributed over interval $[0, 1]$. Let $X_1, \ldots, X_n$ be i.i.d. samples of the random variable $X$. By virtue of the LR method, we have obtained the following results.

**Theorem 24**

$$\Pr\{X_n \geq z\} \leq \left( \frac{e^{\vartheta} - 1}{\vartheta e^{\vartheta} z} \right)^n \text{ for } 1 > z > \frac{1}{2},$$

(39)

$$\Pr\{X_n \leq \frac{1}{2}\} \leq \left( \frac{\vartheta e^{\theta/2}}{e^{\theta} - 1} \right)^n \text{ for } \theta > 0.$$

(40)

where $\vartheta$ is a positive number such that $z = 1 + \frac{1}{e^{\theta} - 1} - \frac{1}{\theta}$. Similarly,

$$\Pr\{X_n \leq z\} \leq \left( \frac{e^{\vartheta} - 1}{\vartheta e^{\vartheta} z} \right)^n \text{ for } 0 < z < \frac{1}{2},$$

(40)

where $\vartheta$ is a negative number such that $z = 1 + \frac{1}{e^{\theta} - 1} - \frac{1}{\theta}$.

See Appendix A.23 for a proof.

3.23 Weibull Distribution

A random variable $X$ is said to have a Weibull distribution if it possesses a probability density function

$$f(x) = \alpha \beta x^{\beta - 1} \exp\left(-\alpha x^\beta\right), \quad x > 0, \quad \alpha > 0, \quad \beta > 0.$$

Let $X_1, \ldots, X_n$ be i.i.d. samples of the random variable $X$. By virtue of the LR method, we have obtained the following results.

**Theorem 25**

$$\Pr\{X_n \leq z\} \leq \left[ \alpha z^\beta \exp(1 - \alpha z^\beta) \right]^n \text{ for } \alpha z^\beta \leq 1 \text{ and } \beta < 1,$$

(41)

$$\Pr\{X_n \geq z\} \leq \left[ \alpha z^\beta \exp(1 - \alpha z^\beta) \right]^n \text{ for } \alpha z^\beta \geq 1 \text{ and } \beta > 1.$$

(42)

See Appendix A.24 for a proof.
4 Concentration Inequalities for Multivariate Distributions

In this section, we shall apply the LR method to derive concentration inequalities for the joint distributions of multiple random variables.

4.1 Dirichlet-Compound Multinomial Distribution

Random variables $X_1, \ldots, X_k$ are said to have a Dirichlet-compound multinomial distribution if they possess a probability mass function

$$f(x) = \binom{n}{x} \frac{\Gamma(\sum_{\ell=0}^{k} \alpha_{\ell})}{\Gamma(n + \sum_{\ell=0}^{k} \alpha_{\ell})} \prod_{\ell=0}^{k} \frac{\Gamma(x_{\ell} + \alpha_{\ell})}{\Gamma(\alpha_{\ell})},$$

where

$$x = [x_0, x_1, \ldots, x_k]^\top, \quad \binom{n}{x} = \frac{n!}{\prod_{\ell=0}^{k} x_{\ell}!}$$

and

$$\sum_{\ell=0}^{k} x_{\ell} = n$$

with $x_{\ell} \geq 0$ and $\alpha_{\ell} > 0$ for $\ell = 0, 1, \ldots, k$. Based on the LR method, we have obtained the following result.

**Theorem 26** Assume that $0 < z_{\ell} \leq \frac{n \alpha_{\ell}}{\sum_{i=0}^{k} \alpha_{i}}$ for $\ell = 1, \ldots, k$. Then,

$$\Pr\{X_{\ell} \leq z_{\ell}, \ell = 1, \ldots, k\} \leq \frac{\Gamma(\sum_{\ell=0}^{k} \alpha_{\ell})}{\Gamma(\sum_{\ell=0}^{k} \theta_{\ell})} \frac{\Gamma(n + \sum_{\ell=0}^{k} \theta_{\ell})}{\Gamma(n + \sum_{\ell=0}^{k} \alpha_{\ell})} \prod_{\ell=1}^{k} \frac{\Gamma(x_{\ell} + \alpha_{\ell}) \Gamma(\theta_{\ell})}{\Gamma(x_{\ell} + \theta_{\ell}) \Gamma(\alpha_{\ell})},$$

where $\theta_0 = \alpha_0$ and

$$\theta_{\ell} = \frac{\alpha_{0} z_{\ell}}{n - \sum_{i=1}^{k} z_{i}}, \quad \ell = 1, \ldots, k.$$

See Appendix B.1 for a proof.

4.2 Inverse Matrix Gamma Distribution

A positive-definite random matrix $X$ is said to have an inverse matrix gamma distribution \cite{9} if it possesses a probability density function

$$f(x) = \frac{|\Psi|^\alpha}{\beta^{p\alpha} \Gamma_p(\alpha)} |x|^{-(\alpha+(p+1)/2)} \exp\left(-\frac{1}{\beta} \text{tr}(\Psi x^{-1})\right),$$

where $\beta > 0$ is the scale parameter, $\Psi$ is a positive-definite real matrix of size $p \times p$. Here $x$ is a positive-definite matrix of size $p \times p$, and $\Gamma_p(\cdot)$ is the multivariate gamma function. The inverse matrix gamma distribution reduces to the Wishart distribution with $\beta = 2$, $\alpha = \frac{p}{2}$. Let $\preceq$ denote the relationship of two matrices $A$ and $B$ of the same size such that $A \preceq B$ implies that $B - A$ is positive definite. By virtue of the LR method, we have obtained the following result.

**Theorem 27**

$$\Pr\{X \preceq \rho Y\} \leq \frac{1}{\rho^{p\alpha}} \exp\left(-\frac{p}{2} \left(\frac{1}{\rho} - 1\right)(2\alpha - p - 1)\right) \quad \text{for } 0 < \rho < 1,$$

where $Y = \mathbb{E}[X] = \frac{2}{\beta} \frac{\Psi}{2\alpha - p - 1}$ is the expectation of $X$.

See Appendix B.2 for a proof.
4.3 Multivariate Normal Distribution

A random vector \( \mathbf{X} \) is said to have a multivariate normal distribution if it possesses a probability density function

\[
f(x) = (2\pi)^{-k/2} |\Sigma|^{-1/2} \exp \left( -\frac{1}{2} (x - \mu) \Sigma^{-1} (x - \mu) \right),
\]

where \( k \) is the dimension of \( \mathbf{X} \), \( x \) is a vector of \( k \) elements, \( \mu \) is the expectation of \( \mathbf{X} \), and \( \Sigma \) is the covariance matrix of \( \mathbf{X} \). Let \( \mathbf{X}_1, \ldots, \mathbf{X}_n \) be i.i.d. samples of \( \mathbf{X} \). Define

\[
\overline{\mathbf{X}}_n = \frac{\sum_{i=1}^n \mathbf{X}_i}{n}.
\]

Let \( \succcurlyeq \) denote the relationship of two vectors \( \mathbf{A} = [a_1, \ldots, a_k] \) and \( \mathbf{B} = [b_1, \ldots, b_k] \) such that \( \mathbf{A} \succcurlyeq \mathbf{B} \) implies \( a_\ell \geq b_\ell, \ell = 1, \ldots, k \). By virtue of the LR method, we have obtained the following result.

**Theorem 28**

\[
\Pr(\overline{\mathbf{X}}_n \succcurlyeq \mathbf{z}) \leq \left[ \exp \left( \mu \Sigma^{-1} \mathbf{z} - \frac{1}{2} \mathbf{z} \Sigma^{-1} \mathbf{z} + \mu \Sigma^{-1} \mu \right) \right]^n
\]

provided that \( \Sigma^{-1} \mathbf{z} \succcurlyeq \Sigma^{-1} \mu \).

See Appendix [B.3](#) for a proof.

4.4 Multivariate Pareto Distribution

Random variables \( X_1, \ldots, X_k \) are said to have a multivariate Pareto distribution if they possess a probability density function

\[
f(x_1, \ldots, x_k) = \left( \prod_{i=1}^k \frac{\alpha + i - 1}{\beta_i} \right) \left( 1 - k + \sum_{i=1}^k \frac{x_i}{\beta_i} \right)^{-(\alpha+k)}, \quad x_i > \beta_i > 0, \quad \alpha > 0.
\]

Let \( \mathbf{X} = [X_1, \ldots, X_k]^\top \). Let \( \mathbf{z} = [z_1, \ldots, z_k]^\top \). Let \( \mathbf{X}_1, \ldots, \mathbf{X}_n \) be i.i.d. samples of random vector \( \mathbf{X} \). Define

\[
\overline{\mathbf{X}}_n = \frac{\sum_{i=1}^n \mathbf{X}_i}{n}.
\]

Let the notation “\( \preceq \)” denote the relationship of two vectors \( \mathbf{A} = [a_1, \ldots, a_k]^\top \) and \( \mathbf{B} = [b_1, \ldots, b_k]^\top \) such that \( \mathbf{A} \preceq \mathbf{B} \) means \( a_\ell \leq b_\ell, \ell = 1, \ldots, k \).

By virtue of the LR method, we have the following results.

**Theorem 29** Let \( z_\ell > \beta_\ell, \ell = 1, \ldots, k \). The following statements hold true.

(I): The inequality

\[
\Pr(\overline{\mathbf{X}}_n \preceq \mathbf{z}) \leq \left[ \left( \prod_{i=1}^k \frac{\alpha + i - 1}{\beta_i} \right) \left( 1 - k + \sum_{i=1}^k \frac{z_i}{\beta_i} \right)^{\theta - \alpha} \right]^n
\]

holds for any \( \theta > \alpha \).

(II): The inequality [46] holds for \( \theta \) such that

\[
\sum_{\ell=0}^{k-1} \frac{1}{\theta + \ell} = \ln \left( 1 - k + \sum_{i=1}^k \frac{z_i}{\beta_i} \right)
\]
provided that
\[
\sum_{\ell=0}^{k-1} \frac{1}{\alpha + \ell} > \ln \left( 1 - k + \frac{k}{\alpha} \sum_{i=1}^{k} \frac{z_i}{\beta_i} \right). \tag{48}
\]

(III): The inequality (40) holds for
\[
\theta = 1 + \frac{1}{\left( \frac{1}{k} \sum_{i=1}^{k} \frac{z_i}{\beta_i} \right) - 1}
\]
provided that \( \alpha > 1 \) and \( \frac{1}{k} \sum_{i=1}^{k} \frac{z_i}{\beta_i} < \frac{\alpha}{\alpha - 1} \).

See Appendix B.4 for a proof.

5 Conclusion

We have investigated the concentration phenomenon of random variables based on the likelihood ratio method. A wide variety of concentration inequalities for various distributions are developed without using moment generating functions. The new inequalities are generally simple, insightful and fairly tight.

A Proofs of Univariate Inequalities

A.1 Proof of Theorem 2

Let \( f(\cdot) \) denote the pmf or pdf of random variable \( X \). Let \( \Theta \) be the set of non-negative real number such that the moment generating function \( \phi(\cdot) \) of \( X \) exists. Define
\[
g(x, \vartheta) = \frac{f(x)e^{\vartheta x}}{\phi(\vartheta)}, \quad \vartheta \in \Theta.
\]

Then, \( g(x, \vartheta) \) is a family of pmf or pdf, which contains \( f(x) = g(x, 0) \). Let \( \mathcal{X} = [X_1, \cdots, X_n] \) and \( \mathbf{x} = [x_1, \cdots, x_n] \). The joint pmf or pdf of \( \mathcal{X} \) is \( f_{\mathcal{X}}(x) = \prod_{i=1}^{n} f(x_i) \), which is contained in the family
\[
g_{\mathcal{X}}(x, \vartheta) = \left[ \frac{1}{\phi(\vartheta)} \right]^{n} \prod_{i=1}^{n} f(x_i) \exp(-\vartheta x_i), \quad \forall \vartheta \in \Theta.
\]

It can be checked that
\[
\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(x, \vartheta)} = [\phi(\vartheta)\exp(-\vartheta \overline{x}_n)]^{n}, \quad \forall \vartheta \in \Theta,
\]
where \( \overline{x}_n = \sum_{i=1}^{n} x_i/n \). Hence,
\[
\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(x, \vartheta)} \leq [\phi(\vartheta)e^{-z}]^{n}, \quad \forall \vartheta \in \Theta \text{ provided that } \overline{x}_n \geq z.
\]

This implies that
\[
f_{\mathcal{X}}(\mathcal{X}) \mathbb{I}_{\{\overline{x}_n \geq z\}} \leq \Lambda(\vartheta) g_{\mathcal{X}}(\mathcal{X}, \vartheta),
\]
where \( \Lambda(\vartheta) = [\phi(\vartheta)e^{-z}]^{n} \). By differentiation, it can be shown that the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in \Theta \) is attained at \( \tau \in \Theta \) such that \( \frac{\phi(\tau)e^{-z}}{\phi(\tau)} = z \). It follows from (6) of Theorem 1 that
\[
\Pr\{\overline{x}_n \geq z\} \leq \Lambda(\tau) \Pr\{\overline{x}_n \geq z\} \leq \Lambda(\tau) = [\phi(\tau)e^{-z\tau}]^{n}. \tag{50}
\]
Now we evaluate $P_\tau(\overline{X}_n \geq z)$. Let $E_\tau[.]$ denote the expectation of a function of random variable $X$ having pmf or pdf $g(x, \tau)$. Note that

$$E_\tau[X] = \int \frac{xf(x)e^{\tau x}}{\phi(\tau)} dx = \int \frac{1}{\phi(\tau)}xf(x)e^{\tau x}dx = \frac{\phi'(\tau)}{\phi(\tau)} = z.$$ 

Similarly,

$$E_\tau[X^2] = \frac{\phi''(\tau)}{\phi(\tau)}, \quad E_\tau[X^3] = \frac{\phi'''(\tau)}{\phi(\tau)}, \quad E_\tau[X^4] = \frac{\phi''''(\tau)}{\phi(\tau)}$$

So,

$$E_\tau[(X - z)^2] = E_\tau[X^2] - z^2 = \frac{\phi''(\tau)}{\phi(\tau)} - z^2.$$ 

Note that $(X - z)^4 = X^4 - 4zX^3 + 6z^2X^2 - 4z^3X + z^4$, Hence,

$$E_\tau[(X - z)^4] = \frac{1}{\phi(\tau)} \left[ \phi''''(\tau) - 4z\phi'''(\tau) + 6z^2\phi''(\tau) - 3z^4\phi(\tau) \right].$$

From Berry-Essen’s inequality [1, 8], we have

$$P_\tau(\overline{X}_n \geq z) \leq \left\{ \frac{1}{2} + \frac{C_{BE}}{\sqrt{n}} \left[ \frac{\phi''''(\tau) - 4z\phi'''(\tau) + 6z^2\phi''(\tau) - 3z^4\phi(\tau)}{(\phi''(\tau) - z^2\phi(\tau))^2/\phi(\tau)} \right]^{\frac{3}{2}} \right\}.$$ 

Making use of the above inequalities and [10] completes the proof of the theorem.

**A.2 Proof of Theorem 3**

Let $\mathcal{X} = [X_1, \ldots, X_n]$ and $\mathbf{x} = [x_1, \ldots, x_n]$. The joint probability density function of $\mathcal{X}$ is

$$f_{\mathcal{X}}(\mathbf{x}) = \frac{1}{[B(\alpha, \beta)]^n} \left( \prod_{i=1}^{n} x_i \right)^{\alpha-1} \left( \prod_{i=1}^{n} (1-x_i) \right)^{\beta-1}.$$ 

To apply the LR method, we construct a family of probability density functions

$$g_{\mathcal{X}}(\mathbf{x}, \vartheta) = \frac{1}{[B(\vartheta, \beta)]^n} \left( \prod_{i=1}^{n} x_i \right)^{\vartheta-1} \left( \prod_{i=1}^{n} (1-x_i) \right)^{\beta-1}$$

for $\vartheta \in (0, \alpha]$. It can be checked that

$$\frac{f_{\mathcal{X}}(\mathbf{x})}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} = \left[ \frac{B(\vartheta, \beta)}{B(\alpha, \beta)} \right]^n \left( \prod_{i=1}^{n} x_i \right)^{\alpha-\vartheta}.$$ 

Since the geometric mean is no greater than the arithmetic mean, we have

$$\prod_{i=1}^{n} x_i \leq \left( \overline{x}_n \right)^n,$$

where $\overline{x}_n = \frac{\sum_{i=1}^{n} x_i}{n}$. Hence,

$$\frac{f_{\mathcal{X}}(\mathbf{x})}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} \leq \left[ \frac{B(\vartheta, \beta)}{B(\alpha, \beta)} \left( \overline{x}_n \right)^{\alpha-\vartheta} \right]^n.$$
and it follows that
\[
\frac{f_X(x)}{g_X(x, \vartheta)} \leq \left[ \frac{B(\beta, \alpha)}{B(\alpha, \beta)} \right]^{\alpha-\beta} \quad \forall \vartheta \in (0, \alpha] \text{ provided that } x_n \leq z.
\]
Consequently,
\[
f_X(X) I_{\{x_n \leq z\}} \leq \Lambda(\vartheta) g_X(X, \vartheta) \quad \forall \vartheta \in (0, \alpha],
\]
where
\[
\Lambda(\vartheta) = \left[ \frac{B(\beta, \alpha)}{B(\alpha, \beta)} \right]^{\alpha-\beta}.
\]
It follows from Theorem 1 that
\[
\text{Pr} \{ x_n \leq z \} \leq \left[ \frac{1}{B(\alpha, \beta)} \inf_{\vartheta \in (0, \alpha]} B(\vartheta, \beta) z^{\alpha-\vartheta} \right]^n \quad \text{for } 0 < z < 1.
\] (51)
As a consequence of 0 < z ≤ μ and the definition of \( \hat{\alpha} \), we have that 0 < \( \hat{\alpha} \) ≤ α. Hence,
\[
\inf_{\vartheta \in (0, \alpha]} B(\vartheta, \beta) z^{\alpha-\vartheta} \leq B(\alpha, \beta) z^{\alpha-\hat{\alpha}},
\]
which leads to (11).

To show (11), we construct a family of probability density functions
\[
g_X(x, \vartheta) = \frac{1}{[B(\alpha, \beta)]^n} \left( \prod_{i=1}^n x_i \right)^{\alpha-1} \left( \prod_{i=1}^n (1-x_i) \right)^{\vartheta-1}
\]
for \( \vartheta \in (0, \beta] \). It can be checked that
\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{B(\alpha, \beta)}{B(\alpha, \beta)} \right]^{\alpha-\beta} \left[ \prod_{i=1}^n (1-x_i) \right]^{\beta-\vartheta} \leq \left[ \frac{B(\alpha, \beta)}{B(\alpha, \beta)} (1-z)^{\beta-\vartheta} \right]^n \quad \forall \vartheta \in (0, \beta] \text{ provided that } x_n \geq z.
\]
Hence,
\[
f_X(X) I_{\{x_n \geq z\}} \leq \Lambda(\vartheta) g_X(X, \vartheta) \quad \forall \vartheta \in (0, \beta],
\]
where
\[
\Lambda(\vartheta) = \left[ \frac{B(\alpha, \beta)}{B(\alpha, \beta)} (1-z)^{\beta-\vartheta} \right]^n.
\]
It follows from Theorem 1 that
\[
\text{Pr} \{ x_n \geq z \} \leq \left[ \frac{1}{B(\alpha, \beta)} \inf_{\vartheta \in (0, \beta]} B(\alpha, \beta)(1-z)^{\beta-\vartheta} \right]^n \quad \text{for } 0 < z < 1.
\]
As a consequence of \( \mu \leq z < 1 \) and the definition of \( \hat{\beta} \), we have that 0 < \( \hat{\beta} \) ≤ β. Hence,
\[
\inf_{\vartheta \in (0, \beta]} B(\alpha, \beta)(1-z)^{\beta-\vartheta} \leq B(\alpha, \hat{\beta})(1-z)^{\beta-\hat{\beta}},
\]
which leads to (11).

Finally, we need to show (12). Since \( \beta = 1 \), using \( \Gamma(z+1) = z\Gamma(z) \), we obtain from (51) the following inequality
\[
\text{Pr} \{ x_n \leq z \} \leq \left[ \inf_{\vartheta \in (0, \alpha]} \frac{\alpha z^{\alpha-\vartheta}}{\vartheta} \right]^n.
\] (52)
Consider function \( w(\vartheta) = \ln \alpha - \ln \vartheta + (\alpha - \vartheta) \ln z \). Note that the first and second derivatives are \( w'(\vartheta) = -\frac{1}{\vartheta} - \ln z \) and \( w''(\vartheta) = \frac{1}{\vartheta^2} \), respectively. By the assumption that \( 0 < z < \exp(-\frac{1}{\alpha}) \), the infimum is attained at \( \vartheta = \frac{1}{\ln z} \in (0, \alpha) \). Hence,

\[
\inf_{\vartheta \in [0, \alpha]} w(\vartheta) = \ln \alpha - \ln \left( \frac{1}{\ln z} \right) + \left( \alpha - \frac{1}{\ln z} \right) \ln z = 1 + \ln \left( \alpha \ln z \frac{1}{z} \right). \tag{53}
\]

Combining (52) and (53) yields

\[
\Pr \{ X_n \leq z \} \leq \left[ \exp \left( 1 + \ln \left( \alpha \ln z \frac{1}{z} \right) \right) \right]^n = \left( \alpha \ln z \frac{1}{z} \right)^n \quad \text{for} \ 0 < z < \exp \left( -\frac{1}{\alpha} \right).
\]

This proves (12). The proof of the theorem is thus completed.

### A.3 Proof of Theorem 4

To apply the LR method, we construct a family of probability mass functions

\[
g(x, \vartheta) = \binom{n + x - 1}{x} \frac{\Gamma(\alpha + n) \Gamma(\vartheta + x) \Gamma(\alpha + \vartheta) \Gamma(\alpha + \beta + n + x)}{\Gamma(\alpha + \vartheta + n + x) \Gamma(\alpha + \beta + n + x)} \quad x = 0, 1, 2, \ldots
\]

for \( \vartheta \in (0, \beta] \). Define

\[
L(x, \vartheta) = \frac{f(x)}{g(x, \vartheta)}, \quad x = 0, 1, 2, \ldots.
\]

Then,

\[
L(x, \vartheta) = \frac{\Gamma(\vartheta) \Gamma(\beta + x) \Gamma(\alpha + \vartheta + n + x)}{\Gamma(\beta) \Gamma(\vartheta + x) \Gamma(\alpha + \beta + n + x)} \quad x = 0, 1, 2, \ldots.
\]

It can be checked that

\[
\frac{L(x + 1, \vartheta)}{L(x, \vartheta)} = \frac{\beta + x \alpha + \vartheta + n + x}{\vartheta + x \alpha + \beta + n + x} \geq 1, \quad x = 0, 1, 2, \ldots
\]

for \( \vartheta \in (0, \beta] \). This implies that for any non-negative integer \( z \),

\[
L(x, \vartheta) \leq L(z, \vartheta), \quad \forall \vartheta \in (0, \beta]
\]

for any non-negative integer \( x \) no greater than \( z \). Hence,

\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta), \quad \forall \vartheta \in (0, \beta]
\]

for any non-negative integer \( x \) no greater than \( z \), where

\[
\Lambda(\vartheta) = \frac{\Gamma(\vartheta) \Gamma(\beta + z) \Gamma(\alpha + \vartheta + n + z)}{\Gamma(\beta) \Gamma(\vartheta + z) \Gamma(\alpha + \beta + n + z)}.
\]

Consequently,

\[
f(X) \mathbb{1}_{\{X \leq z\}} \leq \Lambda(\vartheta) g(X, \vartheta) \quad \forall \vartheta \in (0, \beta].
\]

By virtue of Theorem 4, we have

\[
\Pr \{ X \leq z \} \leq \inf_{\vartheta \in (0, \beta]} \Lambda(\vartheta).
\]

Since \( z \leq \mathbb{E}[X] = \frac{\alpha z}{\alpha - 1} \), we have

\[
0 \leq \frac{\alpha z - z}{n} \leq \beta
\]

and thus

\[
\Pr \{ X \leq z \} \leq \Lambda \left( \frac{\alpha z - z}{n} \right) = \frac{\Gamma(\alpha + \beta + n)}{\Gamma(\beta) \Gamma(\alpha + \beta + n + z)}.
\]

This completes the proof of the theorem.
A.4 Proof of Theorem 5

Let \( X = [X_1, \ldots, X_n] \) and \( x = [x_1, \ldots, x_n] \). The joint probability density function of \( X \) is

\[
f_X(x) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}(1 + x_i)^{-\alpha-\beta}}{[B(\alpha, \beta)]^n}.
\]

To apply the LR method to show (13), we construct a family of probability density functions

\[
g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{x_i^{\vartheta-1}(1 + x_i)^{-\vartheta-\beta}}{[B(\vartheta, \beta)]^n}, \quad \vartheta \in (0, \alpha].
\]

It can be checked that

\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{B(\vartheta, \beta)}{B(\alpha, \beta)} \right]^n \prod_{i=1}^{n} \left( \frac{x_i}{1 + x_i} \right)^{\alpha-\vartheta}, \quad \vartheta \in (0, \alpha].
\]

By differentiation, it can be shown that \( \ln \frac{x_i}{1 + x_i} \) is a concave function of \( x > 0 \). As a consequence of this fact, we have

\[
\prod_{i=1}^{n} \left( \frac{x_i}{1 + x_i} \right)^{\alpha-\vartheta} \leq \left( \frac{\overline{x}_n}{1 + \overline{x}_n} \right)^{n(\alpha-\vartheta)} \quad \forall \vartheta \in (0, \alpha],
\]

where \( \overline{x}_n = \frac{\sum_{i=1}^{n} x_i}{n} \). Since \( \frac{x_i}{1 + x_i} \) is an increasing function of \( x > 0 \), it follows that

\[
\prod_{i=1}^{n} \left( \frac{x_i}{1 + x_i} \right)^{\alpha-\vartheta} \leq \left( \frac{z}{1 + z} \right)^{n(\alpha-\vartheta)} \quad \forall \vartheta \in (0, \alpha] \quad \text{provided that} \quad 0 \leq \overline{x}_n \leq z.
\]

Therefore, we have established that

\[
f_X(X) \mathbb{1}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) \ g_X(X, \vartheta) \quad \forall \vartheta \in (0, \alpha],
\]

where

\[
\Lambda(\vartheta) = \left[ \frac{B(\vartheta, \beta)}{B(\alpha, \beta)} \left( \frac{z}{1 + z} \right)^{\alpha-\vartheta} \right]^n.
\]

Invoking Theorem 1, we have

\[
\Pr \{ X_n \leq z \} \leq \inf_{\vartheta \in (0, \alpha]} \Lambda(\vartheta).
\]

As a consequence of \( \beta > 1 \) and \( 0 < z \leq \frac{\alpha}{\beta-1} \), we have \( 0 < z(\beta-1) \leq \alpha \). Hence,

\[
\Pr \{ X_n \leq z \} \leq \Lambda(z - \beta) = \left[ \frac{B(\beta z - \beta - z, \beta)}{B(\alpha, \beta)} \left( \frac{z}{1 + z} \right)^{\alpha + z - \beta z} \right]^n.
\]

This proves (13).

To apply the LR method to show (14), we construct a family of probability density functions

\[
g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}(1 + x_i)^{-\alpha-\vartheta}}{[B(\alpha, \vartheta)]^n}, \quad \vartheta \in [\beta, \infty).
\]

It can be seen that

\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{B(\alpha, \vartheta)}{B(\alpha, \beta)} \right]^n \prod_{i=1}^{n} (1 + x_i)^{\vartheta-\beta}, \quad \vartheta \in [\beta, \infty).
\]
By differentiation, it can be shown that \( \ln(1 + x) \) is a concave function of \( x > 0 \). As a consequence of this fact, we have
\[
\prod_{i=1}^{n} (1 + x_i)^{\theta - \beta} \leq (1 + \mathbf{x})^{n(\theta - \beta)} \leq (1 + z)^{n(\theta - \beta)}, \quad \theta \in [\beta, \infty)
\]
provided that \( 0 \leq \mathbf{x} \leq z \). Hence, we have that \( f_X(\mathbf{X}) \mathbb{I}_{\{X_n \leq z\}} \leq \Lambda(\theta) \) \( g_X(\mathbf{X}, \theta) \) holds for any \( \theta \in [\beta, \infty) \), where
\[
\Lambda(\theta) = \left[ \frac{B(\alpha, \theta)}{B(\alpha, \beta)} (1 + z)^{\theta - \beta} \right]^n.
\]
Making use of Theorem 1 we have \( \Pr \{X_n \leq z\} \leq \inf_{\theta \geq \beta} \Lambda(\theta) \). As a consequence of \( 0 < z \leq \frac{\alpha}{\beta - 1} \), we have \( 1 + \frac{\alpha}{z} \geq \beta \). Hence,
\[
\Pr \{X_n \leq z\} \leq \Lambda(1 + \frac{\alpha}{z}) = \left[ \frac{B(\alpha, 1 + \frac{\alpha}{z})}{B(\alpha, \beta)} (1 + \frac{\alpha}{z})^{\theta - \beta} \right]^n \quad \text{for } 0 < z \leq \frac{\alpha}{\beta - 1}.
\]
This proves (14). The proof of the theorem is thus completed.

A.5 Proof of Theorem 6

Let \( \mathbf{X} = [X_1, \ldots, X_n] \) and \( \mathbf{x} = [x_1, \ldots, x_n] \). The joint probability mass function of \( \mathbf{X} \) is
\[
f_X(\mathbf{x}) = \prod_{i=1}^{n} \frac{(\theta x_i)_{x_i - 1} e^{-\theta x_i}}{x_i!}.
\]
To apply the LR method to show (15), we construct a family of probability mass functions
\[
g_X(\mathbf{x}, \theta) = \prod_{i=1}^{n} \frac{(\theta x_i)_{x_i - 1} e^{-\theta x_i}}{x_i!}, \quad \theta \in (0, \theta].
\]
It can be seen that
\[
\frac{f_X(\mathbf{x})}{g_X(\mathbf{x}, \theta)} = \left[ \frac{\theta}{\bar{x}} \right]^{\bar{x} - 1} \exp \left( (\theta - \bar{x}) \bar{x} \right) \left[ \frac{\theta}{\bar{x}} \exp \left( (\ln \theta - \bar{x} - \ln \theta + \bar{x}) \bar{x} \right) \right]^n,
\]
where \( \bar{x} = \sum_{i=1}^{n} x_i / n \). Noting that \( \ln x - x \) is increasing with respect to \( x \in (0, 1) \), we have that
\[
\ln \theta - \bar{x} - \ln \theta + \bar{x} \geq 0
\]
as a consequence of \( 0 < \theta \leq \theta \). It follows that
\[
\left[ \frac{\theta}{\bar{x}} \exp \left( (\ln \theta - \bar{x} - \ln \theta + \bar{x}) \bar{x} \right) \right]^n \leq \left[ \frac{\theta}{\bar{x}} \exp \left( (\ln \theta - \bar{x} - \ln \theta + \bar{x}) z \right) \right]^n \quad \forall \theta \in (0, \theta]
\]
provided that \( \bar{x} \leq z \). Hence,
\[
\frac{f_X(\mathbf{x})}{g_X(\mathbf{x}, \theta)} \leq \Lambda(\theta) \quad \forall \theta \in (0, \theta] \text{ provided that } \bar{x} \leq z,
\]
where
\[
\Lambda(\theta) = \left[ \frac{\theta}{\bar{x}} \exp \left( (\ln \theta - \bar{x} - \ln \theta + \bar{x}) z \right) \right]^n.
\]
Hence, we have that $f_{\mathbf{X}}(\mathbf{x}) \mathbb{I}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) = \mathbb{P}_{\mathbf{X}}(\mathbf{X}, \vartheta)$ holds for any $\vartheta \in (0, \theta]$. By virtue of Theorem 11, we have $\mathbb{P}\{X_n \leq z\} \leq \inf_{\vartheta \in (0, \theta]} \Lambda(\vartheta)$. By differentiation, it can be shown that the infimum of $\Lambda(\vartheta)$ with respective to $\vartheta \in (0, \theta]$ is attained at $\vartheta = 1 - \frac{1}{z}$. Therefore,

$$
\mathbb{P}\{X_n \leq z\} \leq \Lambda\left(1 - \frac{1}{z}\right) = \left[\left(\frac{e\vartheta z}{1 - \vartheta}\right)^{z-1} e^{-\vartheta z}\right]^n \quad \text{for} \quad 1 < z < \frac{1}{1 - \theta}.
$$

This completes the proof of the theorem.

### A.6 Proof of Theorem 7

Let $\mathbf{X} = [X_1, \cdots, X_n]$ and $\mathbf{x} = [x_1, \cdots, x_n]$. The joint probability mass function of $\mathbf{X}$ is

$$
f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{x_i} \left(\frac{m x_i}{x_i - 1}\right) \left(\frac{\theta}{1 - \theta}\right)^{x_i - 1} (1 - \theta)^{m x_i}.
$$

To apply the LR method to show (16), we construct a family of probability mass functions

$$
g_{\mathbf{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{1}{x_i} \left(\frac{m x_i}{x_i - 1}\right) \left(\frac{\vartheta}{1 - \vartheta}\right)^{x_i - 1} (1 - \vartheta)^{m x_i} \quad \vartheta \in (0, \theta].
$$

It can be verified that

$$
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left\{ \left[\frac{\theta(1 - \vartheta)}{\vartheta(1 - \theta)} \right]^m \vartheta_{-1} \left(\frac{1 - \theta}{1 - \vartheta}\right)^n \right\}.
$$

where $\vartheta = \frac{\sum_{i=1}^{n} x_i}{n}$. Define function

$$
h(x) = \ln \frac{x}{1 - x} + m \ln(1 - x)
$$

for $x \in (0, 1)$. Then,

$$
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left\{ \left[\frac{\theta(1 - \vartheta)}{\vartheta(1 - \theta)} \right]^m \vartheta_{-1} \left(\frac{1 - \theta}{1 - \vartheta}\right)^n \right\}.
$$

Note that the first derivative of $h(x)$ is $h'(x) = \frac{1 - x}{x} \left(\frac{1}{m} - m\right)$, which is positive for $x \in (0, 1)$. Hence, $h(\theta) - h(\vartheta) \geq 0$ for $\vartheta \in (0, \theta]$. It follows that

$$
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \theta] \quad \text{provided that} \quad 1 \leq \vartheta \leq \vartheta_n \leq z,
$$

where

$$
\Lambda(\vartheta) = \left\{ \left[\frac{\theta(1 - \vartheta)}{\vartheta(1 - \theta)} \right]^m \exp \left(\vartheta \left[h(\theta) - h(\vartheta)\right]\right) \right\}.
$$

This implies that $f_{\mathbf{X}}(\mathbf{x}) \mathbb{I}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) = g_{\mathbf{X}}(\mathbf{x}, \vartheta)$ holds for any $\vartheta \in (0, \theta]$. By virtue of Theorem 11, we have $\mathbb{P}\{X_n \leq z\} \leq \inf_{\vartheta \in (0, \theta]} \Lambda(\vartheta)$. By differentiation, it can be shown that the infimum of $\Lambda(\vartheta)$ with respective to $\vartheta \in (0, \theta]$ is attained at $\vartheta = \frac{1}{m z}$. So,

$$
\mathbb{P}\{X_n \leq z\} \leq \Lambda\left(\frac{z - 1}{m z}\right) = \left[\left(\frac{\theta}{1 - \theta}\right)^{z-1} (1 - \theta)^{m z}\right]^n \quad \text{for} \quad 1 \leq z < \frac{1}{1 - m \theta}.
$$

This completes the proof of the theorem.
A.7 Proof of Theorem 8

Let \( \mathbf{X} = [X_1, \cdots, X_n] \) and \( \mathbf{x} = [x_1, \cdots, x_n] \). The joint probability mass function of \( \mathbf{X} \) is

\[
f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\beta x_i - 1} \left( \frac{\beta x_i - 1}{x_i} \right)^{\vartheta - 1} (1 - \theta)^{(\beta - 1)x_i}.
\]

To apply the LR method to show (17), we construct a family of probability mass functions

\[
g_{\mathbf{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{1}{\beta x_i - 1} \left( \frac{\beta x_i - 1}{x_i} \right)^{\vartheta - 1} (1 - \theta)^{(\beta - 1)x_i}, \quad \vartheta \in (0, \theta].
\]

It can be verified that

\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left\{ \left( \frac{\vartheta}{\vartheta} \right)^{\frac{n}{\beta - 1}} \left( \frac{1 - \theta}{1 - \theta} \right)^{(\beta - 1)x_i} \right\}^{n},
\]

where \( \vartheta_n = \sum_{i=1}^{n} x_i \cdot \frac{\vartheta}{\beta n - 1} \). Define function

\[
h(x) = \ln x + (\beta - 1) \ln(1 - x)
\]

for \( x \in (0, 1) \). Then,

\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left\{ \left( \frac{\vartheta}{\vartheta} \right)^{\frac{n}{\beta - 1}} \left( \frac{1 - \theta}{1 - \theta} \right)^{(\beta - 1)x_i} \right\}^{n}.
\]

Note that the first derivative of \( h(x) \) is \( h'(x) = \frac{1 - \beta x}{x(1 - x)} \), which is positive for \( x \in (0, \frac{1}{2}) \). Hence, \( h(\theta) - h(\vartheta) \geq 0 \) for \( \vartheta \in (0, \theta] \). It follows that

\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \theta] \text{ provided that } \vartheta_n \leq z,
\]

where

\[
\Lambda(\vartheta) = \left\{ \left( \frac{\vartheta}{\vartheta} \right)^{\frac{n}{\beta - 1}} \left( \frac{1 - \theta}{1 - \theta} \right)^{(\beta - 1)x_i} \right\}^{n}.
\]

This implies that \( f_{\mathbf{X}}(\mathbf{x}) \mathbb{I}_{X_n \leq z} \leq \Lambda(\vartheta) g_{\mathbf{X}}(\mathbf{x}, \vartheta) \) holds for any \( \vartheta \in (0, \theta] \). By virtue of Theorem 4, we have \( \Pr \{ X_n \leq z \} \leq \inf_{\vartheta \in (0, \theta]} \Lambda(\vartheta) \). By differentiation, it can be shown that the infimum of \( \Lambda(\vartheta) \) with respective to \( \vartheta \in (0, \theta] \) is attained at \( \vartheta = \frac{z}{1 - \beta} \). Therefore,

\[
\Pr\{X_n \leq z\} \leq \left( \frac{z - 1}{\beta z - 1} \right)^n \left( \frac{\vartheta z - 1(1 - \theta)\beta z - z}{(\vartheta - 1)^{(\beta - 1)z} - 1} \right) \leq \left( \frac{z - 1}{\beta z - 1} \right)^n \text{ for } 1 \leq z \leq \frac{1 - \theta}{1 - \beta \theta}.
\]

This completes the proof of the theorem.

A.8 Proof of Theorem 9

Let \( \mathbf{X} = [X_1, \cdots, X_n] \) and \( \mathbf{x} = [x_1, \cdots, x_n] \). The joint probability density function of \( \mathbf{X} \) is

\[
f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\beta x_i - 1} \exp \left( \frac{\mu - x_i}{\beta} - \exp \left( \frac{\mu - x_i}{\beta} \right) \right).
\]

To apply the LR method to show (18), we construct a family of probability density functions

\[
g_{\mathbf{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{1}{\beta x_i - 1} \exp \left( \frac{\vartheta - x_i}{\beta} - \exp \left( \frac{\vartheta - x_i}{\beta} \right) \right), \quad \vartheta \in (-\infty, \mu].
\]
Note that
\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \prod_{i=1}^{n} \exp \left( \frac{\mu - \vartheta}{\beta} + \exp \left( \frac{\vartheta - x_i}{\beta} \right) - \exp \left( \frac{\mu - x_i}{\beta} \right) \right) \\
= \left[ \exp \left( \frac{\mu - \vartheta}{\beta} \right) \right]^{n} \exp \left\{ n \left[ \exp \left( \frac{\vartheta}{\beta} \right) - \exp \left( \frac{\mu}{\beta} \right) \right] \sum_{i=1}^{n} \exp \left( -\frac{x_i}{\beta} \right) \right\}.
\]

Observing that for \( \vartheta \in (-\infty, \mu) \),
\[
\left[ \exp \left( \frac{\vartheta}{\beta} \right) - \exp \left( \frac{\mu}{\beta} \right) \right] \exp \left( -\frac{x}{\beta} \right)
\]
is a concave function of \( x \), we have that
\[
\frac{f_X(x)}{g_X(x, \vartheta)} \leq \left[ \exp \left( \frac{\mu - \vartheta}{\beta} \right) \right]^{n} \exp \left\{ n \left[ \exp \left( \frac{\vartheta}{\beta} \right) - \exp \left( \frac{\mu}{\beta} \right) \right] \exp \left( -\frac{x_n}{\beta} \right) \right\},
\]
where \( x_n = \sum_{i=1}^{n} x_i \). In view of the fact that for \( \vartheta \in (-\infty, \mu) \),
\[
\left[ \exp \left( \frac{\vartheta}{\beta} \right) - \exp \left( \frac{\mu}{\beta} \right) \right] \exp \left( -\frac{x}{\beta} \right)
\]
is also an increasing function of \( x \), we have that
\[
\frac{f_X(x)}{g_X(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (-\infty, \mu) \text{ provided that } x_n \leq z,
\]
where
\[
\Lambda(\vartheta) = \left[ \exp \left( \frac{\mu - \vartheta}{\beta} \right) \right]^{n} \exp \left\{ n \left[ \exp \left( \frac{\vartheta}{\beta} \right) - \exp \left( \frac{\mu}{\beta} \right) \right] \exp \left( -\frac{x_n}{\beta} \right) \right\}.
\]
This implies that \( f_X(X) I_{(X_n \leq z)} \leq \Lambda(\vartheta) g_X(X, \vartheta) \) holds for any \( \vartheta \in (-\infty, \mu) \). By virtue of Theorem [1] we have \( \Pr\{X_n \leq z\} \leq \inf_{\vartheta \in (-\infty, \mu]} \Lambda(\vartheta) \). Note that
\[
\Lambda(\vartheta) = \left\{ \exp \left[ w(\vartheta) + \frac{\mu}{\beta} - \exp \left( \frac{\mu - z}{\beta} \right) \right] \right\}^{n},
\]
where
\[
w(\vartheta) = -\frac{\vartheta}{\beta} + \exp \left( \frac{\vartheta}{\beta} \right) \exp \left( -\frac{z}{\beta} \right), \quad \vartheta \in (-\infty, \mu].
\]
It can be checked that the first and second derivatives of \( w(\vartheta) \) are
\[
w'(\vartheta) = -\frac{1}{\beta} + \frac{1}{\beta} \exp \left( \frac{\vartheta}{\beta} \right) \exp \left( -\frac{z}{\beta} \right), \quad w''(\vartheta) = \frac{1}{\beta^2} \exp \left( \frac{\vartheta}{\beta} \right) \exp \left( -\frac{z}{\beta} \right).
\]
Obviously,
\[
w'(z) = 0, \quad w''(z) = \frac{1}{\beta^2} > 0.
\]
It follows that
\[
\inf_{\vartheta \in (-\infty, \mu]} \Lambda(\vartheta) = \Lambda(z).
\]
Therefore,
\[
\Pr\{X_n \leq z\} \leq \Lambda(z) = \left\{ \exp \left[ \frac{\mu - z}{\beta} + 1 - \exp \left( \frac{\mu - z}{\beta} \right) \right] \right\}^{n}
\]
for \( z \leq \mu \). This completes the proof of the theorem.
A.9 Proof of Theorem 10

Let $\mathcal{X} = [X_1, \cdots, X_n]$ and $\mathbf{x} = [x_1, \cdots, x_n]$. The joint probability density function of $\mathcal{X}$ is

$$f_{\mathcal{X}}(\mathbf{x}) = \prod_{i=1}^{n} \beta \alpha \Gamma(\alpha) x_i^{-\alpha-1} \exp \left( -\frac{\beta}{x_i} \right).$$

To apply the LR method to show (19), we construct a family of probability density functions

$$g_{\mathcal{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \beta \vartheta \alpha \Gamma(\alpha) x_i^{-\vartheta-1} \exp \left( -\frac{\beta}{x_i} \right), \quad \vartheta \in [\alpha, \infty).$$

Clearly,

$$\frac{f_{\mathcal{X}}(\mathbf{x})}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} = \prod_{i=1}^{n} \frac{\Gamma(\vartheta)}{\Gamma(\alpha)} \beta^{\vartheta-\alpha} x_i^{\vartheta-\alpha}$$

$$= \left[ \frac{\Gamma(\vartheta)}{\Gamma(\alpha)} \beta^{\vartheta-\alpha} \right]^{\frac{n}{\vartheta}} \left( \prod_{i=1}^{n} x_i \right)$$

$$\leq \left[ \frac{\Gamma(\vartheta)}{\Gamma(\alpha)} \beta^{\vartheta-\alpha} \right]^{\frac{n}{\vartheta}} (\overline{x}_n)^{n(\vartheta-\alpha)},$$

where $\overline{x}_n = \frac{\sum_{i=1}^{n} x_i}{n}$. It follows that

$$\frac{f_{\mathcal{X}}(\mathbf{x})}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [\alpha, \infty) \text{ provided that } \overline{x}_n \leq z,$$

where

$$\Lambda(\vartheta) = \left[ \frac{\Gamma(\vartheta)}{\Gamma(\alpha)} \left( \frac{\beta}{\vartheta} \right) \right]^{\frac{n}{\vartheta}}.$$

This implies that $f_{\mathcal{X}}(\mathbf{x}) I_{[\overline{x}_n \leq z]} \leq \Lambda(\vartheta) g_{\mathcal{X}}(\mathbf{x}, \vartheta)$ holds for any $\vartheta \in [\alpha, \infty)$. By virtue of Theorem 1, we have $\Pr \{ \overline{x}_n \leq z \} \leq \inf_{\vartheta \in [\alpha, \infty)} \Lambda(\vartheta)$. As a consequence of $0 < z \leq \frac{\beta}{\alpha}$, we have $\frac{\vartheta}{z} + 1 \geq \alpha$. It follows that

$$\Pr\{ \overline{x}_n \leq z \} \leq \Lambda \left( \frac{\beta}{z} + 1 \right) = \left[ \frac{\Gamma(\beta + 1)}{\Gamma(\alpha)} \left( \frac{\beta}{\alpha} \right) \right]^{\frac{n}{\vartheta}} \left( \frac{\beta}{\alpha} \right)^{\frac{n}{\vartheta}} \text{ for } 0 < z \leq \frac{\beta}{\alpha}.$$ 

This proves inequality (19).

To apply the LR method to show inequality (20), we construct a family of probability density functions

$$g_{\mathcal{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{\beta}{\Gamma(\alpha)} x_i^{-\vartheta} \exp \left( -\frac{\beta}{x_i} \right), \quad \vartheta \in (0, \beta].$$

It can be seen that

$$\frac{f_{\mathcal{X}}(\mathbf{x})}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} = \left( \frac{\beta}{\vartheta} \right)^{\alpha} \exp \left( \sum_{i=1}^{n} \frac{\vartheta - \beta}{x_i} \right).$$

Observing that for $\vartheta \in (0, \beta], \frac{\vartheta - \beta}{x}$ is a concave function of $x > 0$, we have that

$$\frac{f_{\mathcal{X}}(\mathbf{x})}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} \leq \left[ \left( \frac{\beta}{\vartheta} \right)^{\alpha} \exp \left( \frac{\vartheta - \beta}{\overline{x}_n} \right) \right]^n.$$
It follows that
\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \beta] \text{ provided that } \overline{X}_n \leq z,
\]
where
\[
\Lambda(\vartheta) = \left( \frac{\beta}{\vartheta} \right)^{\alpha} \exp \left( \frac{\vartheta - \beta}{z} \right)^n.
\]
This implies that \( f_{\mathbf{X}}(\mathbf{X}) \mathbb{I}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) g_{\mathbf{X}}(\mathbf{X}, \vartheta) \) holds for any \( \vartheta \in (0, \beta] \). By virtue of Theorem 1, we have
\[
\Pr \{X_n \leq z\} \leq \inf_{\vartheta \in (0, \beta]} \Lambda(\vartheta).
\]
By differentiation, it can be shown that the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in (0, \beta] \) is attained at \( \vartheta = \alpha z \) as long as \( 0 < z \leq \frac{\beta}{\alpha} \). Therefore,
\[
\Pr \{X_n \leq z\} \leq \Lambda(\alpha z) = \left[ \left( \frac{\beta}{\alpha z} \right)^{\alpha} \exp \left( \frac{\alpha z - \beta}{z} \right) \right]^n \text{ for } 0 < z \leq \frac{\beta}{\alpha}.
\]
This proves inequality (20). The proof of the theorem is thus completed.

A.10 Proof of Theorem 11

Let \( \mathbf{X} = [X_1, \ldots, X_n] \) and \( \mathbf{x} = [x_1, \ldots, x_n] \). The joint probability density function of \( \mathbf{X} \) is
\[
f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \left( \frac{\lambda}{2\pi x_i^3} \right)^{1/2} \exp \left( -\frac{\lambda(x_i - \theta)^2}{2\vartheta x_i} \right).
\]
To apply the LR method to show (21), we construct a family of probability density functions
\[
g_{\mathbf{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \left( \frac{\lambda}{2\pi x_i^3} \right)^{1/2} \exp \left( -\frac{\lambda(x_i - \vartheta)^2}{2\vartheta x_i} \right), \quad \vartheta \in (0, \theta].
\]
It can be verified that
\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left\{ \exp \left[ \frac{\lambda}{\vartheta} - \frac{\lambda}{\vartheta} + \left( \frac{\lambda}{2\vartheta^2} - \frac{\lambda}{2\vartheta^2} \right) \overline{X}_n \right] \right\}^n,
\]
where \( \overline{X}_n = \frac{\sum_{i=1}^{n} x_i}{n} \). It follows that
\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \theta] \text{ provided that } \overline{X}_n \leq z,
\]
where
\[
\Lambda(\vartheta) = \left\{ \exp \left[ \frac{\lambda}{\vartheta} - \frac{\lambda}{\vartheta} + \left( \frac{\lambda}{2\vartheta^2} - \frac{\lambda}{2\vartheta^2} \right) \overline{X}_n \right] \right\}^n.
\]
This implies that \( f_{\mathbf{X}}(\mathbf{X}) \mathbb{I}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) g_{\mathbf{X}}(\mathbf{X}, \vartheta) \) holds for any \( \vartheta \in (0, \theta] \). By virtue of Theorem 1, we have
\[
\Pr \{X_n \leq z\} \leq \inf_{\vartheta \in (0, \theta]} \Lambda(\vartheta).
\]
By differentiation, it can be shown that the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in (0, \theta] \) is attained at \( \vartheta = \alpha z \) as long as \( 0 < z \leq \frac{\theta}{\alpha} \). Therefore,
\[
\Pr \{X_n \leq z\} \leq \Lambda(\alpha z) = \left[ \left( \frac{\beta}{\alpha z} \right)^{\alpha} \exp \left( \frac{\alpha z - \beta}{z} \right) \right]^n \text{ for } 0 < z \leq \frac{\theta}{\alpha}.
\]
This completes the proof of the theorem.
A.11 Proof of Theorem 1

Let \( \mathbf{X} = [X_1, \cdots, X_n] \) and \( \mathbf{x} = [x_1, \cdots, x_n] \). The joint probability mass function of \( \mathbf{X} \) is

\[
f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{-\theta^{x_i}(1-\theta)^{x_i}(\beta-1)\Gamma(\beta x_i)}{\Gamma(x_i + 1)\Gamma(\beta x_i - x_i + 1)\ln(1-\theta)}.
\]

To apply the LR method to show (22), we construct a family of probability mass functions

\[
g_{\mathbf{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{-\theta^{x_i}(1-\vartheta)^{x_i}(\beta-1)\Gamma(\beta x_i)}{\Gamma(x_i + 1)\Gamma(\beta x_i - x_i + 1)\ln(1-\vartheta)}, \quad \vartheta \in (0, \theta].
\]

It can be seen that

\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left[ \left( \frac{\theta}{\vartheta} \right)^{x_n} \left( \frac{1-\theta}{1-\vartheta} \right)^{(\beta-1)x_n} \frac{\ln(1-\vartheta)}{\ln(1-\theta)} \right]^{n},
\]

where \( \bar{x}_n = \sum_{i=1}^{n} x_i \). Define function

\[
h(x) = \ln x + (\beta - 1) \ln(1-x)
\]

for \( x \in (0, 1) \). Then, we can write

\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left[ \exp \left( (h(\vartheta) - h(\vartheta))\frac{\ln(1-\vartheta)}{\ln(1-\theta)} \right) \right]^{n}.
\]

Note that the first derivative of \( h(x) \) is

\[
h'(x) = \frac{1 - \beta x}{x(1-x)},
\]

which is positive for \( x \in (0, \frac{1}{\beta}) \). Hence,

\[
\frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \theta] \text{ provided that } \bar{x}_n \leq z,
\]

where

\[
\Lambda(\vartheta) = \left[ \exp \left( (h(\vartheta) - h(\vartheta))z \right) \frac{\ln(1-\vartheta)}{\ln(1-\theta)} \right]^{n} = \left[ \left( \frac{\theta}{\vartheta} \right)^{z} \left( \frac{1-\theta}{1-\vartheta} \right)^{z(\beta-1)} \frac{\ln(1-\vartheta)}{\ln(1-\theta)} \right]^{n}.
\]

This implies that \( f_{\mathbf{X}}(\mathbf{X}) \leq \Lambda(\vartheta) \) \( g_{\mathbf{X}}(\mathbf{X}, \vartheta) \) holds for any \( \vartheta \in (0, \theta] \). By virtue of Theorem 4 we have \( \Pr \{ \bar{X}_n \leq z \} \leq \inf_{\vartheta \in (0, \theta]} \Lambda(\vartheta) \). By differentiation, it can be shown that, as long as \( 0 < z \leq \frac{\vartheta}{(\beta-1)\ln(1-\vartheta)} \), the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in (0, \theta] \) is attained at \( \vartheta \) such that \( z = \frac{\vartheta}{(\beta-1)\ln(1-\vartheta)} \).

Such number \( \vartheta \) is unique because the first derivative of \( \frac{\vartheta}{(\beta-1)\ln(1-\vartheta)} \) with respective to \( \vartheta \in (0, \frac{1}{\beta}) \) is equal to

\[
\frac{1}{[(1 - \beta \vartheta)\ln(1-\vartheta)]^2} \left[ -\ln(1-\vartheta) - \vartheta \frac{1 - \beta \vartheta}{1 - \vartheta} \right],
\]

which is no less than

\[
\frac{(\beta - 1)\vartheta^2}{(1 - \vartheta)((1 - \beta \vartheta)\ln(1-\vartheta))^2} > 0.
\]

This completes the proof of the theorem.
A.12 Proof of Theorem 13

Let $X = [X_1, \cdots, X_n]$ and $x = [x_1, \cdots, x_n]$. The joint probability mass function of $X$ is

$$f_X(x) = \prod_{i=1}^{n} \frac{\beta}{\alpha x_i + \beta} \left( \frac{\alpha x_i + \beta}{x_i} \right)^{\theta x_i} (1 - \theta)^{\beta + \alpha x_i - x_i}.$$  

To apply the LR method to show (23), we construct a family of probability mass functions

$$g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{\beta}{\alpha x_i + \beta} \left( \frac{\alpha x_i + \beta}{x_i} \right)^{\theta x_i} (1 - \vartheta)^{\beta + \alpha x_i - x_i}, \quad \vartheta \in (0, \theta].$$

It can be seen that

$$\frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{(\theta/\vartheta) x_n (1 - \theta)}{(1 - \vartheta) x_n} \right]^n,$$

where $x_n = \sum_{i=1}^{n} x_i$. Define function

$$h(x) = \ln x + (a - 1) \ln(1 - x)$$

for $x \in (0, 1)$. Then, we can write

$$\frac{f_X(x)}{g_X(x, \vartheta)} = \exp \left[ \frac{h(\theta) - h(\vartheta)}{x_n} \left( \frac{1 - \theta}{1 - \vartheta} \right)^{\beta + (a - 1) x_n} \right]^n.$$

Note that the first derivative of $h(x)$ is

$$h'(x) = \frac{1 - \alpha x}{x(1 - x)},$$

which is positive for $x \in (0, \frac{1}{a})$. Hence,

$$\frac{f_X(x)}{g_X(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \theta] \text{ provided that } x_n \leq z,$$

where

$$\Lambda(\vartheta) = \left[ \exp \left( \frac{h(\theta) - h(\theta)}{x_n} \left( \frac{1 - \theta}{1 - \vartheta} \right)^{\beta + (a - 1) x_n} \right) \right]^n.$$  

This implies that $f_X(X) \mathbb{1}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) g_X(X, \vartheta)$ holds for any $\vartheta \in (0, \theta]$. By virtue of Theorem 1, we have $\Pr \left\{ \hat{X}_n \leq z \right\} \leq \inf_{\vartheta \in (0, \theta]} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $0 < z \leq \frac{\theta}{\beta + \alpha z}$, the infimum of $\Lambda(\vartheta)$ with respect to $\vartheta \in (0, \theta]$ is attained at $\vartheta = \frac{z}{\beta + \alpha z}$. This completes the proof of the theorem.

A.13 Proof of Theorem 14

Let $X = [X_1, \cdots, X_n]$ and $x = [x_1, \cdots, x_n]$. The joint probability density function of $X$ is

$$f_X(x) = \prod_{i=1}^{n} \frac{1}{2\beta} \exp \left( - \frac{|x_i - \alpha|}{\beta} \right).$$

To apply the LR method to show (24), we construct a family of probability density functions

$$g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{1}{2\vartheta} \exp \left( - \frac{|x_i - \alpha|}{\vartheta} \right), \quad \vartheta \in [\beta, \infty).$$
It can be seen that for $\vartheta \in [\beta, \infty)$,

\[
\frac{f_{\mathbf{X}}(x)}{g_{\mathbf{X}}(x, \vartheta)} = \left(\frac{\vartheta}{\beta}\right)^n \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) \sum_{i=1}^{n} (x_i - \alpha)\right]
\]

\[
\leq \left(\frac{\vartheta}{\beta}\right)^n \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) \sum_{i=1}^{n} (\alpha - x_i)\right]
\]

\[
= \left(\frac{\vartheta}{\beta}\right)^n \left\{ \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) (\varpi_n - \alpha)\right]\right\}^n,
\]

where $\varpi_n = \sum_{i=1}^{n} \frac{x_i}{n}$. Since $\frac{1}{\vartheta} - \frac{1}{\beta} \leq 0$ for $\vartheta \in [\beta, \infty)$, it follows that

\[
\frac{f_{\mathbf{X}}(x)}{g_{\mathbf{X}}(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [\beta, \infty) \text{ provided that } \varpi_n \geq z,
\]

where

\[
\Lambda(\vartheta) = \left\{ \frac{\vartheta}{\beta} \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) (z - \alpha)\right]\right\}^n.
\]

This implies that $f_{\mathbf{X}}(\mathbf{X}) \mathbf{1}_{\{\varpi_n \geq z\}} \leq \Lambda(\vartheta) \ g_{\mathbf{X}}(\mathbf{X}, \vartheta)$ holds for any $\vartheta \in [\beta, \infty)$. By virtue of Theorem 1, we have $\Pr \left\{ X_n \geq z \right\} \leq \inf_{\vartheta \in [\beta, \infty]} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $z \geq \alpha + \beta$, the infimum of $\Lambda(\vartheta)$ with respect to $\vartheta \in [\beta, \infty)$ is attained at $\vartheta = z - \alpha$. This proves (24).

To show (25), note that for $\vartheta \in [\beta, \infty)$,

\[
\frac{f_{\mathbf{X}}(x)}{g_{\mathbf{X}}(x, \vartheta)} = \left(\frac{\vartheta}{\beta}\right)^n \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) \sum_{i=1}^{n} (x_i - \alpha)\right]
\]

\[
\leq \left(\frac{\vartheta}{\beta}\right)^n \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) \sum_{i=1}^{n} (\alpha - x_i)\right]
\]

\[
= \left(\frac{\vartheta}{\beta}\right)^n \left\{ \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) (\alpha - \varpi_n)\right]\right\}^n.
\]

Since $\frac{1}{\vartheta} - \frac{1}{\beta} \leq 0$ for $\vartheta \in [\beta, \infty)$, it follows that

\[
\frac{f_{\mathbf{X}}(x)}{g_{\mathbf{X}}(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [\beta, \infty) \text{ provided that } \varpi_n \leq z,
\]

where

\[
\Lambda(\vartheta) = \left\{ \frac{\vartheta}{\beta} \exp \left[\left(\frac{1}{\vartheta} - \frac{1}{\beta}\right) (z - \alpha)\right]\right\}^n.
\]

This implies that $f_{\mathbf{X}}(\mathbf{X}) \mathbf{1}_{\{\varpi_n \leq z\}} \leq \Lambda(\vartheta) \ g_{\mathbf{X}}(\mathbf{X}, \vartheta)$ holds for any $\vartheta \in [\beta, \infty)$. By virtue of Theorem 1, we have $\Pr \left\{ X_n \leq z \right\} \leq \inf_{\vartheta \in [\beta, \infty]} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $z \leq \alpha - \beta$, the infimum of $\Lambda(\vartheta)$ with respect to $\vartheta \in [\beta, \infty)$ is attained at $\vartheta = \alpha - z$. This proves (25). The proof of the theorem is thus completed.

### A.14 Proof of Theorem 15

Let $\mathbf{X} = [X_1, \cdots, X_n]$ and $\mathbf{x} = [x_1, \cdots, x_n]$. The joint probability mass function of $\mathbf{X}$ is

\[
f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{q^{x_i}}{x_i \ln \left(\frac{1}{1-q}\right)}.
\]
To apply the LR method to show (26), we construct a family of probability mass functions
\[ g_X(x, \vartheta) = \prod_{i=1}^{n} \vartheta^{x_i} \ln \frac{1}{1-\vartheta}, \quad \vartheta \in (0, q]. \]

Clearly,
\[ \frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{\ln(1-q)}{\ln(1-\vartheta)} \left( \frac{q}{\vartheta} \right)^{\sum_{i=1}^{n} x_i} \right]^n, \]
where \( \sum_{i=1}^{n} x_i = x_n \). Hence,
\[ \frac{f_X(x)}{g_X(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, q] \] provided that \( x_n \leq z \), where \( \Lambda(\vartheta) = \left[ \frac{\ln(1-q)}{\ln(1-\vartheta)} \left( \frac{q}{\vartheta} \right)^{z} \right]^n. \)

This implies that \( f_X(X) \mathbb{1}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) g_X(X, \vartheta) \) holds for any \( \vartheta \in (0, q] \). By virtue of Theorem 1, we have \( \Pr \{X_n \leq z\} \leq \inf_{\vartheta \in (0, q]} \Lambda(\vartheta) \). By differentiation, it can be shown that, as long as \( z \leq \frac{q}{(1-\vartheta) \ln \frac{1}{1-q}} \), the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in (0, q] \) is attained at \( \vartheta \in (0, q] \) such that \( z = \frac{\vartheta}{(1-\vartheta) \ln \frac{1}{1-q}} \). Such number \( \vartheta \) is unique because the function \( \frac{\vartheta}{(1-\vartheta) \ln \frac{1}{1-q}} \) is increasing with respect to \( \vartheta \in (0, 1) \). The proof of the theorem is thus completed.

A.15 Proof of Theorem 16

Let \( X = [X_1, \ldots, X_n] \) and \( x = [x_1, \ldots, x_n] \). The joint probability density function of \( X \) is
\[ f_X(x) = \prod_{i=1}^{n} \frac{1}{x_i \sqrt{2\pi\sigma}} \exp \left[ -\frac{1}{2} \left( \frac{\mu - \ln x_i}{\sigma} \right)^2 \right]. \]

To apply the LR method to show (27), we construct a family of probability density functions
\[ g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{1}{x_i \sqrt{2\pi\sigma}} \exp \left[ -\frac{1}{2} \left( \frac{\vartheta - \ln x_i}{\sigma} \right)^2 \right], \quad \vartheta \in (0, \mu]. \]

It can be seen that
\[ \frac{f_X(x)}{g_X(x, \vartheta)} = \exp \left[ \frac{\vartheta - \mu}{\sigma^2} \sum_{i=1}^{n} \left( \frac{\mu + \vartheta}{2} - \ln x_i \right) \right]. \]

It can be readily shown that for \( \vartheta \in (0, \mu], \)
\[ \frac{\vartheta - \mu}{\sigma^2} \left( \frac{\mu + \vartheta}{2} - \ln x \right) \]
is a concave function of \( x > 0 \). Hence,
\[ \frac{f_X(x)}{g_X(x, \vartheta)} \leq \left\{ \exp \left[ \frac{\vartheta - \mu}{\sigma^2} \left( \frac{\mu + \vartheta}{2} - \ln x \right) \right] \right\}^n \quad \text{for} \ \vartheta \in (0, \mu], \]
where $\overline{x}_n = \sum_{i=1}^{n} x_i$. It follows that
\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \mu] \text{ provided that } \overline{x}_n \leq z,
\]
where
\[
\Lambda(\vartheta) = \left\{ \exp \left[ \frac{\vartheta - \mu}{\sigma^2} \left( \frac{\mu + \vartheta}{2} - \ln \vartheta \right)^2 \right] \right\}^n.
\]
This implies that $f(x) \mathbb{I}_{[\overline{x}_n \leq z]} \leq \Lambda(\vartheta) \cdot g(x, \vartheta)$ holds for any $\vartheta \in (0, \mu]$. By virtue of Theorem 1, we have $\Pr \{ X_n \leq z \} \leq \inf_{\vartheta \in (0, \mu]} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $0 < z \leq e^\mu$, the infimum of $\Lambda(\vartheta)$ with respect to $\vartheta \in (0, \mu]$ is attained at $\vartheta = \ln z$. Therefore,
\[
\Pr \{ \overline{X}_n \leq z \} \leq \Lambda(\ln z) = \exp \left[ -\frac{n}{2} \left( \frac{\mu - \ln z}{\sigma} \right)^2 \right] \quad \text{for } 0 < z \leq e^\mu.
\]
The proof of the theorem is thus completed.

### A.16 Proof of Theorem 17

Let $X = [X_1, \cdots, X_n]$ and $x = [x_1, \cdots, x_n]$. The joint probability density function of $X$ is
\[
f(x) = \prod_{i=1}^{n} \frac{2}{\Gamma(m)} \frac{x_i^{2m-1}}{\sigma^{2m}} \exp \left( -\frac{x_i^2}{\sigma^2} \right).
\]
To apply the LR method to show (28), we construct a family of probability density functions
\[
g(x, \vartheta) = \prod_{i=1}^{n} \frac{2}{\Gamma(m)} \frac{x_i^{2\vartheta-1}}{\sigma^{2\vartheta}} \exp \left( -\frac{x_i^2}{\sigma^2} \right), \quad \vartheta \in (0, m].
\]
Clearly, for $\vartheta \in (0, m]$,
\[
\frac{f(x)}{g(x, \vartheta)} = \left[ \frac{\Gamma(\vartheta)}{\Gamma(m)} \right]^{n} \left( \prod_{i=1}^{n} x_i \right)^{2(m-\vartheta)} \leq \left[ \frac{\Gamma(\vartheta)}{\Gamma(m)} \right]^{n} \left( \overline{x}_n \right)^{2(n-m)}
\]
where $\overline{x}_n = \sum_{i=1}^{n} x_i$. It follows that
\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, m] \text{ provided that } \overline{x}_n \leq z,
\]
where
\[
\Lambda(\vartheta) = \left[ \frac{\Gamma(\vartheta)}{\Gamma(m)} \right]^{n} \left( \frac{x_i^2}{\sigma^2} \right)^{2(m-\vartheta)}
\]
This implies that $f(x) \mathbb{I}_{[\overline{x}_n \leq z]} \leq \Lambda(\vartheta) \cdot g(x, \vartheta)$ holds for any $\vartheta \in (0, m]$. By virtue of Theorem 1, we have $\Pr \{ \overline{X}_n \leq z \} \leq \inf_{\vartheta \in (0, m]} \Lambda(\vartheta)$. Letting $z = \frac{\Gamma(\vartheta+\frac{1}{2})}{\Gamma(\vartheta)^{1/2}} \sigma$ leads to (28).

To apply the LR method to show (29), we construct a family of probability density functions
\[
g(x, \vartheta) = \prod_{i=1}^{n} \frac{2}{\Gamma(m)} \frac{x_i^{2m-1}}{\vartheta^{2m}} \exp \left( -\frac{x_i^2}{\vartheta^2} \right), \quad \vartheta \in [\sigma, \infty).
\]
It can be seen that
\[
\frac{f(x)}{g(x, \vartheta)} = \left( \frac{\vartheta}{\sigma} \right)^{2mn} \exp \left[ \left( \frac{1}{\vartheta^2} - \frac{1}{\sigma^2} \right) \sum_{i=1}^{n} x_i^2 \right].
\]
Observing that for $\vartheta \in [\sigma, \infty)$, \(\frac{1}{\vartheta} - \frac{1}{\sigma^2} \) \(\vartheta^2\) is a concave function of $\vartheta > 0$, we have that
\[
\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(x, \vartheta)} \leq \left(\frac{\vartheta}{\sigma}\right)^{2m} \exp\left[\left(\frac{1}{\vartheta^2} - \frac{1}{\sigma^2}\right)(\vartheta_n)^2\right]^n, \quad \forall \vartheta \in [\sigma, \infty).
\]

It follows that
\[
\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [\sigma, \infty) \text{ provided that } \vartheta_n \geq z,
\]
where
\[
\Lambda(\vartheta) = \left[\frac{\vartheta}{\sigma}\right]^{2m} \exp\left(\frac{z^2}{\vartheta^2} - \frac{z^2}{\sigma^2}\right)^n.
\]

This implies that $f_{\mathcal{X}}(\mathcal{X}) \mathbb{I}_{\{X_n \geq z\}} \leq \Lambda(\vartheta) g_{\mathcal{X}}(\mathcal{X}, \vartheta)$ holds for any $\vartheta \in [\sigma, \infty)$. By virtue of Theorem 1, we have $\Pr\{X_n \geq z\} \leq \inf_{\vartheta \in [\sigma, \infty]} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $z \geq \sqrt{m\sigma}$, the infimum of $\Lambda(\vartheta)$ with respect to $\vartheta \in [\sigma, \infty)$ is attained at $\vartheta = \frac{z}{\sqrt{m}}$. Therefore,
\[
\Pr\{X_n \geq z\} \leq \Lambda\left(\frac{z}{\sqrt{m}}\right) = \left[\left(\frac{z^2}{m\sigma^2}\right)^n \exp\left(m - \frac{z^2}{\sigma^2}\right)\right]^n \text{ for } z \geq \sqrt{m\sigma}.
\]

This establishes (29) and completes the proof of the theorem.

### A.17 Proof of Theorem 18

Let $\mathcal{X} = [X_1, \ldots, X_n]$ and $x = [x_1, \ldots, x_n]$. The joint probability density function of $\mathcal{X}$ is
\[
f_{\mathcal{X}}(x) = \prod_{i=1}^{n} \frac{\vartheta}{a} \left(\frac{a}{x_i}\right)^{\vartheta+1}.
\]

To apply the LR method to show (30), we construct a family of probability density functions
\[
g_{\mathcal{X}}(x, \vartheta) = \prod_{i=1}^{n} \frac{\vartheta}{a} \left(\frac{a}{x_i}\right)^{\vartheta+1}, \quad \forall \vartheta \in [\theta, \infty).
\]

Clearly,
\[
\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(x, \vartheta)} = \left(\frac{\vartheta}{\theta}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\vartheta-\theta} \leq \left[\frac{\vartheta}{\theta}\left(\frac{\vartheta_n}{a}\right)^{\vartheta-\theta}\right]^n,
\]
where $\vartheta_n = \sum_{i=1}^{n} x_i$. It follows that
\[
\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [\theta, \infty) \text{ provided that } \vartheta_n \leq z,
\]
where
\[
\Lambda(\vartheta) = \left[\frac{\vartheta}{\theta}\left(\frac{z}{a}\right)^{\vartheta-\theta}\right]^n.
\]

This implies that $f_{\mathcal{X}}(\mathcal{X}) \mathbb{I}_{\{X_n \leq z\}} \leq \Lambda(\vartheta) g_{\mathcal{X}}(\mathcal{X}, \vartheta)$ holds for any $\vartheta \in [\theta, \infty)$. By virtue of Theorem 1, we have $\Pr\{X_n \leq z\} \leq \inf_{\vartheta \in [\theta, \infty]} \Lambda(\vartheta)$. Hence, for $\gamma > 1$,
\[
\Pr\{X_n \leq \gamma a\} \leq \inf_{\vartheta \geq \theta} \left(\frac{\vartheta}{\theta}\gamma^{\vartheta-\theta}\right)^n = \theta^n \inf_{\vartheta \geq \theta} \exp[n w(\vartheta)],
\]
where $w(\vartheta) = -\ln \vartheta + (\vartheta - \theta) \ln \gamma$. 

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Now consider the minimization of \( w(\vartheta) \) subject to \( \vartheta \geq \theta \). Note that the first and second derivatives of \( w(\vartheta) \) are \( w'(\vartheta) = -\frac{1}{\vartheta} + \ln \gamma \) and \( w''(\vartheta) = \frac{1}{\vartheta^2} \), respectively. Hence, the minimum is achieved at \( \vartheta^* = \frac{1}{\ln \gamma} \) provided that \( 1 < \gamma \leq e^{1/\theta} \). Accordingly, \( w(\vartheta^*) = 1 + \ln \ln \gamma - \theta \ln \gamma \) and

\[
\Pr\{X_n \leq \gamma a\} \leq \left( \frac{e^\theta \gamma}{\gamma \ln \gamma} \right)^n \text{ for } 1 < \gamma \leq e^{1/\theta}.
\]

Note that the mean of \( X \) is \( \mu = \frac{\theta a}{\theta - 1} \). Letting \( \gamma = \rho \mu \) yields

\[
\Pr\{X_n \leq \rho \mu\} \leq \left[ e^{\theta \left( \frac{\theta - 1}{\rho \theta} \right)} \ln \left( \frac{\rho \theta}{\theta - 1} \right) \right]^n \text{ for } 1 - \frac{1}{\theta} < \rho \leq \left( 1 - \frac{1}{\theta} \right) \exp(\frac{1}{\theta}).
\]

This establishes (30) and completes the proof of the theorem.

### A.18 Proof of Theorem 19

Let \( X = [X_1, \cdots, X_n] \) and \( x = [x_1, \cdots, x_n] \). The joint probability density function of \( X \) is

\[
f_X(x) = \prod_{i=1}^{n} \frac{x_i^{-\alpha}}{C(\alpha)}.
\]

To apply the LR method to show (31), we construct a probability density functions

\[
g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{x_i^{-\vartheta}}{C(\vartheta)}, \quad \vartheta \in [\alpha, \infty).
\]

Clearly,

\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{C(\vartheta)}{C(\alpha)} \right]^n \left( \prod_{i=1}^{n} x_i \right)^{\vartheta - \alpha} \leq \left[ \frac{C(\vartheta)}{C(\alpha)} \right]^n \left( \frac{x_n}{\vartheta} \right)^{\vartheta - \alpha} \leq \Lambda(\vartheta)
\]

provided that \( x_n \leq z \), where \( x_n = \frac{\sum_{i=1}^{n} x_i}{n} \) and

\[
\Lambda(\vartheta) = \left[ \frac{C(\vartheta)}{C(\alpha)} \right]^n \left( \frac{x_n}{\vartheta} \right)^{\vartheta - \alpha}.
\]

This implies that \( f_X(X) I\{X_n \leq z\} \leq \Lambda(\vartheta) g_X(X, \vartheta) \) holds for any \( \vartheta \in [\alpha, \infty) \). By virtue of Theorem 1 we have \( \Pr\{X_n \leq z\} \leq \Lambda(\vartheta) \). This completes the proof of the theorem.

### A.19 Proof of Theorem 20

Let \( X = [X_1, \cdots, X_n] \) and \( x = [x_1, \cdots, x_n] \). The joint probability mass function of \( X \) is

\[
f_X(x) = \prod_{i=1}^{n} \frac{m! [s(x_i, m)] \theta^x_i}{x_i! [-\ln(1 - \theta)]^m}.
\]

To apply the LR method to show (32), we construct a family of probability mass functions

\[
g_X(x, \vartheta) = \prod_{i=1}^{n} \frac{m! [s(x_i, m)] \vartheta^x_i}{x_i! [-\ln(1 - \vartheta)]^m}, \quad \vartheta \in (0, \theta].
\]

Clearly,

\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left[ \frac{\ln(1 - \vartheta)}{\ln(1 - \theta)} \right]^m \left( \frac{\vartheta}{\theta} \right)^x_n \leq \Lambda(\vartheta),
\]

where \( \Lambda(\vartheta) = \left[ \frac{\ln(1 - \vartheta)}{\ln(1 - \theta)} \right]^m \left( \frac{\vartheta}{\theta} \right)^x_n \) and \( \theta \in (0, 1) \). This completes the proof of the theorem.
where \( \overline{x}_n = \frac{\sum_{i=1}^{n} x_i}{n} \). It follows that

\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \vartheta] \text{ provided that } \overline{x}_n \leq z,
\]

where

\[
\Lambda(\vartheta) = \left[ \frac{\ln(1 - \vartheta)}{\ln(1 - m\vartheta)} \right]^{nm} \left[ \left( \frac{\theta}{\vartheta} \right)^{z} \right]^{n}.
\]

This implies that \( f(x) \mathbb{I}_{\{X \leq z\}} \leq \Lambda(\vartheta) g(x, \vartheta) \) holds for any \( \vartheta \in (0, \vartheta] \). By virtue of Theorem 1, we have \( \Pr \{ \overline{x}_n \leq z \} \leq \inf_{\vartheta \in (0, \vartheta]} \Lambda(\vartheta) \). By differentiation, it can be shown that, as long as \( z \leq \frac{m\vartheta}{(\vartheta - 1)\ln(1 - \vartheta)} \), the infimum of \( \Lambda(\vartheta) \) with respective to \( \vartheta \in (0, \vartheta] \) is attained at a number \( \vartheta \) such that \( z = \frac{m\vartheta}{(\vartheta - 1)\ln(1 - \vartheta)} \).

Such number \( \vartheta \) is unique because \( \frac{\vartheta}{(\vartheta - 1)\ln(1 - \vartheta)} \) is an increasing function of \( \vartheta \in (0, 1) \). This completes the proof of the theorem.

### A.20 Proof of Theorem 21

To apply the LR method, we introduce a family of probability density functions

\[
g(x, \vartheta) = \frac{1}{\vartheta} f \left( \frac{x}{\vartheta} \right), \quad \vartheta > 0.
\]

Clearly,

\[
\frac{f(x)}{g(x, \vartheta)} = \frac{f(x)}{\frac{1}{\vartheta} f \left( \frac{x}{\vartheta} \right)} = g^{m/2} \left( \frac{n + \frac{nx}{\vartheta}}{n + mz} \right)^{(n+m)/2} = g^{m/2} \left( 1 + \frac{1}{1 + \frac{m}{nz}} \right)^{(n+m)/2}.
\]

To show inequality (33), note that \( \frac{f(x)}{g(x, \vartheta)} \) is decreasing with respect to \( x > 0 \) for \( \vartheta \geq 1 \). Hence,

\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [1, \infty) \text{ provided that } x \geq z,
\]

where

\[
\Lambda(\vartheta) = g^{m/2} \left( 1 + \frac{1}{1 + \frac{m}{nz}} \right)^{(n+m)/2}.
\]

This implies that \( f(X) \mathbb{I}_{\{X \geq z\}} \leq \Lambda(\vartheta) g(X, \vartheta) \) holds for any \( \vartheta \in [1, \infty) \). By virtue of Theorem 1, we have

\[
\Pr \{ X \geq z \} \leq \inf_{\vartheta \in [1, \infty]} \Lambda(\vartheta) = \Lambda(z) = z^{m/2} \left( \frac{n + m}{n + mz} \right)^{(n+m)/2} \text{ for } z \geq 1.
\]

To show inequality (34), note that \( \frac{f(x)}{g(x, \vartheta)} \) is increasing with respect to \( x > 0 \) for \( 0 < \vartheta \leq 1 \). Hence,

\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, 1] \text{ provided that } x \leq z,
\]

where \( \Lambda(\vartheta) \) is defined by (54). This implies that \( f(X) \mathbb{I}_{\{X \leq z\}} \leq \Lambda(\vartheta) g(X, \vartheta) \) holds for any \( \vartheta \in (0, 1] \). By virtue of Theorem 1, we have

\[
\Pr \{ X \leq z \} \leq \inf_{\vartheta \in (0, 1]} \Lambda(\vartheta) = \Lambda(z) = z^{m/2} \left( \frac{n + m}{n + mz} \right)^{(n+m)/2} \text{ for } 0 < z \leq 1.
\]

This proves inequality (54) and completes the proof of the theorem.
A.21 Proof of Theorem 22

To apply the LR method, we introduce a family of probability density functions

\[ g(x, \vartheta) = \frac{1}{\vartheta} f \left( \frac{x}{\vartheta} \right), \quad \vartheta > 0. \]

Clearly,

\[ \frac{f(x)}{g(x, \vartheta)} = \frac{1}{\vartheta} \frac{f(x)}{f \left( \frac{x}{\vartheta} \right)} = \vartheta \left[ \frac{n + \left( \frac{x}{\vartheta} \right)^2}{n + x^2} \right]^{(n+1)/2} = \vartheta \left( 1 + \frac{1}{\frac{x}{\vartheta} + 1} \right)^{(n+1)/2}. \]

To show inequality (35), note that \( \frac{f(x)}{g(x, \vartheta)} \) is decreasing with respect to \( |x| \) for \( \vartheta \geq 1 \). Hence,

\[ \frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [1, \infty) \text{ provided that } |x| \geq z, \]

where

\[ \Lambda(\vartheta) = \vartheta \left( 1 + \frac{1}{\frac{x}{\vartheta} + 1} \right)^{(n+1)/2}. \tag{55} \]

This implies that \( f(X) I_{\{|X| \geq z\}} \leq \Lambda(\vartheta) g(X, \vartheta) \) holds for any \( \vartheta \in [1, \infty) \). By virtue of Theorem 1, we have

\[ \Pr \{ X \geq z \} \leq \inf_{\vartheta \in [1, \infty)} \Lambda(\vartheta) = \Lambda(z) = \frac{z^{n+1}}{(n+z^2)^{(n+1)/2}} \quad \text{for } z \geq 1. \]

This proves inequality (35).

To show inequality (36), note that \( \frac{f(x)}{g(x, \vartheta)} \) is increasing with respect to \( |x| \) for \( \vartheta \in (0, 1] \). Hence,

\[ \frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, 1] \text{ provided that } |x| \leq z, \]

where \( \Lambda(\vartheta) \) is defined by (55). This implies that \( f(X) I_{\{|X| \leq z\}} \leq \Lambda(\vartheta) g(X, \vartheta) \) holds for any \( \vartheta \in (0, 1] \). By virtue of Theorem 1, we have

\[ \Pr \{ X \leq z \} \leq \inf_{\vartheta \in (0, 1]} \Lambda(\vartheta) = \Lambda(z) = \frac{z^{n+1}}{(n+z^2)^{(n+1)/2}} \quad \text{for } 0 < z \leq 1. \]

This proves inequality (36) and completes the proof of the theorem.

A.22 Proof of Theorem 23

Let \( \mathbf{X} = [X_1, \ldots, X_n] \) and \( \mathbf{x} = [x_1, \ldots, x_n] \). The joint probability density function of \( \mathbf{X} \) is

\[ f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} \frac{\theta}{e^\theta - 1} e^{\theta x_i}. \]

To apply the LR method to show (37), we construct a family of probability density functions

\[ g_{\mathbf{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{\vartheta}{e^\vartheta - 1} e^{\vartheta x_i}, \quad \vartheta \in (-\infty, \theta], \quad \vartheta \neq 0. \]

Clearly,

\[ \frac{f_{\mathbf{X}}(\mathbf{x})}{g_{\mathbf{X}}(\mathbf{x}, \vartheta)} = \left( \frac{\vartheta}{e^\vartheta - 1} \right)^{n} \exp \left[ n(\theta - \vartheta) \mathbf{x}_n \right]. \]
where $\overline{x}_n = \frac{\sum_{i=1}^{n} x_i}{n}$. It follows that

$$\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (-\infty, \theta], \vartheta \neq 0 \text{ provided that } \overline{x}_n \leq z,$$

where

$$\Lambda(\vartheta) = \left( \frac{\vartheta e^\vartheta - 1}{\vartheta} e^\vartheta - 1 \right)^n \exp \left[ n(\vartheta - \vartheta)z \right].$$

This implies that $f_{\mathcal{X}}(\mathcal{X}) \uparrow_{(\overline{x}_n \leq z)} \Lambda(\vartheta) g_{\mathcal{X}}(\mathcal{X}, \vartheta)$ holds for any $\vartheta \in (-\infty, \theta], \vartheta \neq 0$. By virtue of Theorem 1, we have $\Pr \{ \overline{x}_n \leq z \} \leq \inf_{\vartheta \in (-\infty, \theta]} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $0 < z \leq 1 + \frac{1}{e^\vartheta - 1} - \frac{1}{\vartheta}$ and $z \neq \frac{1}{2}$, the infimum of $\Lambda(\vartheta)$ with respective to $\vartheta \in (-\infty, \theta], \vartheta \neq 0$ is attained at a number $\vartheta \in (-\infty, \theta], \vartheta \neq 0$ such that $z = 1 + \frac{1}{e^{\vartheta}} - \frac{1}{\vartheta}$. Such a number is unique because

$$\lim_{\vartheta \to -\infty} \left( 1 + \frac{1}{e^\vartheta - 1} - \frac{1}{\vartheta} \right) = 0$$

and $1 + \frac{1}{e^\vartheta - 1} - \frac{1}{\vartheta}$ is increasing with respect to $\vartheta \neq 0$. To show such monotonicity, note that the first derivative of $1 + \frac{1}{e^\vartheta - 1} - \frac{1}{\vartheta}$ with respective to $\vartheta$ is equal to

$$\left[ e^{\vartheta/2} - e^{-\vartheta/2} - \vartheta \right] \left( \frac{1}{\vartheta} + e^{\vartheta/2} - 1 \right),$$

where $e^{\vartheta/2} - e^{-\vartheta/2} - \vartheta$ is a function of $\vartheta$ with its first derivative assuming value 0 at $\vartheta = 0$, and its second derivative equal to $\frac{1}{4} (e^{\vartheta/2} - e^{-\vartheta/2})$. This establishes inequality 37. To show 38, it suffices to note that as $z \to \frac{1}{2}$, the root of equation $z = 1 + \frac{1}{e^\vartheta - 1} - \frac{1}{\vartheta}$ with respect to $\vartheta$ tends to 0. This completes the proof of the theorem.

A.23 Proof of Theorem 24

Let $\mathcal{X} = [X_1, \ldots, X_n]$ and $\mathbf{x} = [x_1, \ldots, x_n]$. The joint probability density function of $\mathcal{X}$ is $f_{\mathcal{X}}(\mathbf{x}) = 1$. To apply the LR method to show 39, we construct a family of probability density functions

$$g_{\mathcal{X}}(\mathbf{x}, \vartheta) = \prod_{i=1}^{n} \frac{\vartheta}{e^{\vartheta} - 1} e^{\vartheta x_i}, \vartheta > 0.$$

Clearly,

$$\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} = \left[ \frac{e^{\vartheta/2} - e^{-\vartheta/2} - \vartheta}{\vartheta (e^{\vartheta/2} - e^{-\vartheta/2})} \right] \left( \frac{1}{\vartheta} + e^{\vartheta/2} - 1 \right),$$

where $\overline{x}_n = \frac{\sum_{i=1}^{n} x_i}{n}$. It follows that

$$\frac{f_{\mathcal{X}}(x)}{g_{\mathcal{X}}(\mathbf{x}, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta > 0 \text{ provided that } \overline{x}_n \geq z,$$

where

$$\Lambda(\vartheta) = \left( \frac{e^{\vartheta/2} - e^{-\vartheta/2} - \vartheta}{\vartheta (e^{\vartheta/2} - e^{-\vartheta/2})} \right)^n.$$

This implies that $f_{\mathcal{X}}(\mathcal{X}) \downarrow_{(\overline{x}_n \geq z)} \Lambda(\vartheta) g_{\mathcal{X}}(\mathcal{X}, \vartheta)$ holds for any $\vartheta > 0$. By virtue of Theorem 1, we have $\Pr \{ \overline{x}_n \geq z \} \leq \inf_{\vartheta > 0} \Lambda(\vartheta)$. By differentiation, it can be shown that, as long as $1 > z \geq \frac{1}{2}$, the infimum of $\Lambda(\vartheta)$ with respective to $\vartheta > 0$ is attained at a positive number $\vartheta^*$ such that $z = 1 + \frac{1}{e^{\vartheta^*} - 1} - \frac{1}{\vartheta^*}$. Such a number is unique because

$$\lim_{\vartheta \to 0} \left( 1 + \frac{1}{e^{\vartheta} - 1} - \frac{1}{\vartheta} \right) = \frac{1}{2}.$$
and \(1 + \frac{1}{\vartheta - 1} - \frac{1}{\vartheta} \) is increasing with respect to \(\vartheta > 0\). Therefore, we have shown that

\[
\Pr\{X_n \geq z\} \leq \Lambda(\vartheta^*) \quad \text{for} \quad \frac{1}{2} < z < 1.
\]

On the other hand, it can be shown that

\[
\Lambda(\vartheta^*) = \left(\inf_{s > 0} e^{-zs} \mathbb{E}[e^{sX}]\right)^n.
\]

To establish an upper bound on \(\Lambda(\vartheta^*)\), we can use the following inequality due to Chen [6, Appendix H],

\[
\mathbb{E}[e^{sX}] < \exp\left(\frac{s^2}{24} + \frac{s}{2}\right), \quad \forall s \in (-\infty, \infty).
\]

By differentiation, it can be shown that

\[
\Pr\{X \geq z\} \leq \Lambda(\vartheta^*) \leq \left(\inf_{s > 0} \exp\left(\frac{s^2}{24} + \frac{s}{2} - zs\right)\right)^n \leq \exp\left(-6n\left(z - \frac{1}{2}\right)^2\right), \quad 1 > z > \frac{1}{2}.
\]

This establishes (39). By a similar argument, we can show (40). This completes the proof of the theorem.

**A.24 Proof of Theorem 25**

Let \(X = [X_1, \ldots, X_n]\) and \(x = [x_1, \ldots, x_n]\). The joint probability density function of \(X\) is

\[
f_X(x) = \prod_{i=1}^{n} \alpha \beta x_i^{\beta - 1} \exp\left(-\alpha x_i^\beta\right).
\]

To apply the LR method, we construct a family of probability density functions

\[
g_X(x, \vartheta) = \prod_{i=1}^{n} \vartheta \beta x_i^{\beta - 1} \exp\left(-\vartheta x_i^\beta\right), \quad \vartheta \in (0, \infty).
\]

Clearly,

\[
\frac{f_X(x)}{g_X(x, \vartheta)} = \left(\frac{\alpha}{\vartheta}\right)^n \exp\left((\vartheta - \alpha) \sum_{i=1}^{n} x_i^\beta\right).
\]

To show inequality (41) under the condition that \(\alpha \varepsilon^\beta \leq 1\) and \(0 < \beta \leq 1\), we restrict \(\vartheta\) to be no less than \(\alpha\). As a consequence of \(0 < \beta \leq 1\) and \(\vartheta \geq \alpha\), we have that \((\vartheta - \alpha) x_i^\beta\) is a concave function of \(x > 0\). By virtue of such concavity, we have

\[
\frac{f_X(x)}{g_X(x, \vartheta)} \leq \left\{\frac{\alpha}{\vartheta}\exp\left[(\vartheta - \alpha) (\vartheta_n)^\beta\right]\right\}^n, \quad \forall \vartheta \in [\alpha, \infty),
\]

where \(\vartheta_n = \sum_{i=1}^{n} x_i\). It follows that

\[
\frac{f_X(x)}{g_X(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in [\alpha, \infty) \text{ provided that } \vartheta_n \leq z,
\]

where

\[
\Lambda(\vartheta) = \left\{\frac{\alpha}{\vartheta}\exp\left[(\vartheta - \alpha) z^\beta\right]\right\}^n.
\]
This implies that \( f(x) \) holds for any \( \vartheta \in [\alpha, \infty) \). By virtue of Theorem 11 we have \( \Pr \{ x \leq z \} \leq \inf_{\vartheta \in [\alpha, \infty)} \Lambda(\vartheta) \). By differentiation, it can be shown that, as long as \( \alpha \beta \leq 1 \), the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in [\alpha, \infty) \) is attained at \( \vartheta = z^{-\beta} \). Therefore,

\[
\Pr \{ x \leq z \} \leq \Lambda(z^{-\beta}) = \left[ \alpha \beta \exp(1 - \alpha \beta) \right]^n \quad \text{for } \alpha \beta \leq 1 \quad \text{and} \quad \beta < 1.
\]

This proves inequality (11).

To show inequality (12) under the condition that \( \alpha \beta \geq 1 \) and \( \beta > 1 \), we restrict \( \vartheta \) to be a positive number less than \( \alpha \). As a consequence of \( \beta > 1 \) and \( 0 < \vartheta < \alpha \), we have that \( (\vartheta - \alpha) x^\beta \) is a concave function of \( x > 0 \). By virtue of such concavity, we have

\[
\frac{f(x)}{g(x, \vartheta)} \leq \left\{ \frac{\alpha \beta}{\vartheta} \exp \left[ (\vartheta - \alpha) x^\beta \right] \right\}^n, \quad \forall \vartheta \in (0, \alpha).
\]

It follows that

\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in (0, \alpha) \quad \text{provided that } x \geq z,
\]

where \( \Lambda(\vartheta) \) is defined by (13). This implies that \( f(x) \) holds for any \( \vartheta \in (0, \alpha) \). By virtue of Theorem 1 we have \( \Pr \{ X_n \geq z \} \leq \inf_{\vartheta \in (0, \alpha)} \Lambda(\vartheta) \). By differentiation, it can be shown that, as long as \( \alpha \beta \geq 1 \), the infimum of \( \Lambda(\vartheta) \) with respect to \( \vartheta \in (0, \alpha) \) is attained at \( \vartheta = z^{-\beta} \). Therefore,

\[
\Pr \{ X_n \geq z \} \leq \Lambda(z^{-\beta}) = \left[ \alpha \beta \exp(1 - \alpha \beta) \right]^n \quad \text{for } \alpha \beta \geq 1 \quad \text{and} \quad \beta > 1.
\]

This proves inequality (12). The proof of the theorem is thus completed.

#### B Proofs of Multivariate Inequalities

##### B.1 Proof of Theorem 26

To apply the LR method to show (13), we introduce a family of probability mass functions

\[
g(x, \vartheta) = \binom{n}{x} \frac{\Gamma(\sum_{\ell=0}^{k} \vartheta_{\ell})}{\Gamma(n + \sum_{\ell=0}^{k} \vartheta_{\ell})} \prod_{\ell=0}^{k} \frac{\Gamma(x_{\ell} + \vartheta_{\ell})}{\Gamma(\vartheta_{\ell})}, \quad \text{with } \vartheta_{0} = \alpha_{0} \text{ and } 0 < \vartheta_{\ell} \leq \alpha_{\ell}, \quad \ell = 1, \ldots, k
\]

where \( \vartheta = [\vartheta_{0}, \vartheta_{1}, \ldots, \vartheta_{k}]^\top \). Clearly,

\[
\frac{f(x)}{g(x, \vartheta)} = \frac{\Gamma(\sum_{\ell=0}^{k} x_{\ell})}{\Gamma(\sum_{\ell=0}^{k} \vartheta_{\ell})} \frac{\prod_{\ell=0}^{k} \Gamma(\vartheta_{\ell})}{\prod_{\ell=1}^{k} \Gamma(x_{\ell} + \vartheta_{\ell})},
\]

For simplicity of notations, define

\[
L(x, \vartheta) = \frac{f(x)}{g(x, \vartheta)}.
\]

Let \( y = [y_{0}, y_{1}, \ldots, y_{k}]^\top \) be a vector such that \( y_{i} = x_{i} + 1 \) for some \( i \in \{1, \ldots, k\} \) and that \( y_{\ell} = x_{\ell} \) for all \( \ell \in \{1, \ldots, k\} \) except \( \ell = i \). Then,

\[
L(y, \vartheta) = \frac{L(x, \vartheta)}{x_{i} \vartheta_{i}} \geq 1.
\]

Making use of this observation and by an inductive argument, we have that for \( z = [z_{0}, z_{1}, \ldots, z_{k}]^\top \) such that \( x_{\ell} \leq z_{\ell} \) for \( \ell = 1, \ldots, k \), it must be true that

\[
L(z, \vartheta) \geq L(x, \vartheta) \geq 1.
\]
It follows that
\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta) \quad \forall \vartheta \in \Theta \text{ provided that } x_\ell \leq z_\ell, \ \ell = 1, \ldots, k,
\]
where
\[
\Lambda(\vartheta) = \frac{\Gamma(n)}{\Gamma(n + \sum_{\ell=0}^{k} \vartheta_\ell)} \prod_{\ell=1}^{k} \frac{\Gamma(z_\ell + \alpha_\ell)}{\Gamma(z_\ell + \vartheta_\ell)}
\]
\[\Theta\] is the set of vectors \(\vartheta = [\vartheta_0, \vartheta_1, \ldots, \vartheta_k]^T\) such that \(\vartheta_0 = \alpha_0\) and \(0 < \vartheta_\ell \leq \alpha_\ell, \ \ell = 1, \ldots, k\). This implies that
\[
f(X) \mathbb{I}_{\{X \preceq z\}} \leq \Lambda(\vartheta) \ g(X, \vartheta) \quad \forall \vartheta \in \Theta,
\]
where \(X = [X_0, X_1, \ldots, X_k]^T\) and \(X \preceq z\) means \(X_\ell \leq z_\ell, \ \ell = 1, \ldots, k\). By virtue of Theorem 1, we have
\[
\Pr \{X_\ell \leq z_\ell, \ \ell = 1, \ldots, k\} = \Pr\{X \preceq z\} \leq \inf_{\vartheta \in \Theta} \Lambda(\vartheta).
\]
As a consequence of the assumption that \(0 < z_\ell \leq \frac{\alpha_0 \vartheta_0}{\sum_{i=0}^{n} \vartheta_i} \) for \(\ell = 1, \ldots, k\), we have that
\[
\theta_\ell = \frac{\alpha_0 \vartheta_0}{n - \sum_{i=1}^{\ell} \vartheta_i} \leq \frac{\alpha_0 \vartheta_0}{n - \sum_{\ell=1}^{k} \vartheta_i} = \alpha_\ell
\]
for \(\ell = 1, \ldots, k\). Define \(\theta = [\theta_0, \theta_1, \ldots, \theta_k]^T\). Then, \(\theta \in \Theta\) and
\[
\Pr \{X_\ell \leq z_\ell, \ \ell = 1, \ldots, k\} \leq \inf_{\vartheta \in \Theta} \Lambda(\vartheta) \leq \Lambda(\theta).
\]
This completes the proof of the theorem.

B.2 Proof of Theorem 27

To apply the LR method to show inequality (44), we introduce a family of probability density functions
\[
g(x, \vartheta) = \frac{|\vartheta|^\alpha}{\beta^p \Gamma_p(\alpha)} |x|^{-\alpha - (p+1)/2} \exp \left( -\frac{1}{\beta} \text{tr}(\vartheta x^{-1}) \right),
\]
where \(\vartheta\) is a positive-definite real matrix of size \(p \times p\) such that \(\vartheta \preceq \Psi\). Note that
\[
\frac{f(x)}{g(x, \vartheta)} = \frac{|\Psi|^\alpha}{|\vartheta|^\alpha} \exp \left( -\frac{1}{\beta} \text{tr}(|\Psi - \vartheta| x^{-1}) \right).
\]
For positive definite matrices \(x\) and \(z\) such that \(x \preceq z\), we have
\[
\text{tr}(|\Psi - \vartheta| x^{-1}) \geq \text{tr}(|\Psi - \vartheta| z^{-1})
\]
as a consequence of \(\vartheta \preceq \Psi\). If follows that
\[
\frac{f(x)}{g(x, \vartheta)} \leq \Lambda(\vartheta)
\]
for \(\vartheta \preceq \Psi\) and \(x \preceq z\), where
\[
\Lambda(\vartheta) = \frac{|\Psi|^\alpha}{|\vartheta|^\alpha} \exp \left( -\frac{1}{\beta} \text{tr}(|\Psi - \vartheta| z^{-1}) \right).
\]
Hence,
\[
f(X) \mathbb{I}_{\{X \preceq z\}} \leq \Lambda(\vartheta) \ g(X, \vartheta) \quad \text{provided that } \vartheta \preceq \Psi.
\]
By virtue of Theorem 1, we have \( \Pr\{X \preceq z\} \leq \inf_{\theta} \Lambda(\theta) \). In particular, taking \( z = \rho \mathbb{E}[X] = \frac{2\rho}{m^{2\alpha-p-1}} \) and \( \theta = \frac{\rho}{2}(2\alpha-p-1)z \), we have

\[
\Pr\{X \preceq \rho \Upsilon\} = \Pr\{X \preceq z\} \leq |\Psi|^{\alpha} |\beta|^{2(2\alpha-p-1)} |z|^{\alpha} \exp\left(-\frac{1}{\beta} \text{tr}(\Psi z^{-1})\right) = \frac{1}{\rho^{p\alpha}} \exp\left(-\frac{p}{2}(2\rho - 1)(2\alpha-p-1)\right).
\]

This completes the proof of the theorem.

### B.3 Proof of Theorem 28

For simplicity of notations, let \( \mathbf{X} = [X_1, \ldots, X_n] \).

Let \( \mathcal{X} = [x_1, \ldots, x_n] \), where \( x_1, \ldots, x_n \) are vectors of dimension \( k \). Since \( X_1, \ldots, X_n \) are identical and independent, the joint probability density of \( \mathbf{X} \) is

\[
f_{\mathbf{X}}(\mathcal{X}) = \prod_{i=1}^{n} (2\pi)^{-k/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(x_i - \mu)^\top \Sigma^{-1} (x_i - \mu)\right)^n.
\]

To apply the LR method to show (45), we introduce a family of probability density functions

\[
g_{\mathbf{X}}(\mathcal{X}, \vartheta) = \prod_{i=1}^{n} (2\pi)^{-k/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(x_i - \vartheta)^\top \Sigma^{-1} (x_i - \vartheta)\right)^n,
\]

where \( \vartheta \) is a vector of dimension \( k \) such that \( \Sigma^{-1} \vartheta \succeq \Sigma^{-1} \mu \). It can be checked that

\[
\frac{f_{\mathbf{X}}(\mathcal{X})}{g_{\mathbf{X}}(\mathcal{X}, \vartheta)} = \prod_{i=1}^{n} \exp\left((\mu^\top - \vartheta^\top) \Sigma^{-1} x_i + \frac{1}{2} x_i^\top \Sigma^{-1} \vartheta - \mu^\top \Sigma^{-1} \mu\right) = \left[\exp\left((\mu^\top - \vartheta^\top) \Sigma^{-1} x_n + \frac{1}{2} \vartheta^\top \Sigma^{-1} \vartheta - \mu^\top \Sigma^{-1} \mu\right)\right]^n,
\]

where

\[
x_n = \frac{\sum_{i=1}^{n} x_i}{n}.
\]

As a consequence of \( \Sigma^{-1} \vartheta \succeq \Sigma^{-1} \mu \), we have that

\[
(\mu^\top - \vartheta^\top) \Sigma^{-1} u \leq (\mu^\top - \vartheta^\top) \Sigma^{-1} v
\]

for arbitrary vectors \( u \) and \( v \) such that \( v \succeq u \). This implies that for \( \vartheta \) such that \( \Sigma^{-1} \vartheta \succeq \Sigma^{-1} \mu \),

\[
\frac{f_{\mathbf{X}}(\mathcal{X})}{g_{\mathbf{X}}(\mathcal{X}, \vartheta)} \leq \Lambda(\vartheta) \quad \text{provided that } x_n \succeq z,
\]

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where
\[ \Lambda(\vartheta) = \left[ \exp \left( (\mu^\top - \vartheta^\top) \Sigma^{-1} z + \frac{1}{2} [\vartheta^\top \Sigma^{-1} \vartheta - \mu^\top \Sigma^{-1} \mu] \right) \right]^n. \]

It follows that
\[ f_\mathbf{X}(\mathbf{x}) \mathbb{I}(\mathbf{x}_n \succeq z) \leq \Lambda(\vartheta) g_\mathbf{X}(\mathbf{x}, \vartheta) \]
for any \( \vartheta \) such that \( \Sigma^{-1} \vartheta \succeq \Sigma^{-1} \mu \). By virtue of Theorem 1, we have
\[ \Pr(\mathbf{X}_n \geq z) \leq \inf_{\Sigma^{-1} \vartheta \succeq \Sigma^{-1} \mu} \Lambda(\vartheta). \]

By differentiation, it can be shown that the infimum is attained at \( \vartheta = z \). Hence, \( \Pr(\mathbf{X}_n \geq z) \leq \Lambda(\vartheta) \).

This completes the proof of the theorem.

### B.4 Proof of Theorem 29

Let \( \mathbf{X} \) denote the random matrix of size \( k \times n \) such that the \( j \)-th column is \( \mathbf{x}_j \). Let \( \mathbf{x} \) denote a matrix of size \( k \times n \) such that the \( j \)-th column is \( [x_{1j}, x_{2j}, \ldots, x_{kj}]^\top \). Then, the joint probability density function of \( \mathbf{X} \) is
\[ f_\mathbf{X}(\mathbf{x}) = \prod_{j=1}^n \left[ \left( \prod_{i=1}^k \frac{\alpha + i - 1}{\beta_i} \right) \left( 1 - k + \sum_{i=1}^k \frac{x_{ij}}{\beta_i} \right)^{-(\alpha + k)} \right]. \]

To apply the LR method, we introduce a family of probability density functions
\[ g_\mathbf{X}(\mathbf{x}, \vartheta) = \prod_{j=1}^n \left[ \left( \prod_{i=1}^k \frac{\vartheta + i - 1}{\beta_i} \right) \left( 1 - k + \sum_{i=1}^k \frac{x_{ij}}{\beta_i} \right)^{-(\vartheta + k)} \right], \quad \vartheta > \alpha. \]

Clearly,
\[ \frac{f_\mathbf{X}(\mathbf{x})}{g_\mathbf{X}(\mathbf{x}, \vartheta)} = \left( \prod_{i=1}^k \frac{\alpha + i - 1}{\vartheta + i - 1} \right)^n \left[ \prod_{j=1}^n \left( 1 - k + \sum_{i=1}^k \frac{x_{ij}}{\beta_i} \right) \right]^{\vartheta - \alpha} \]

Using the fact that the geometric mean does not exceed the arithmetic mean, we have that
\[ \prod_{j=1}^n \left( 1 - k + \sum_{i=1}^k \frac{x_{ij}}{\beta_i} \right) \leq \left( 1 - k + \sum_{i=1}^k \frac{u_i}{\beta_i} \right)^n, \]
where
\[ u_i = \frac{1}{n} \sum_{j=1}^n x_{ij}. \]

Hence,
\[ \frac{f_\mathbf{X}(\mathbf{x})}{g_\mathbf{X}(\mathbf{x}, \vartheta)} \leq \left[ \left( \prod_{i=1}^k \frac{\alpha + i - 1}{\vartheta + i - 1} \right) \left( 1 - k + \sum_{i=1}^k \frac{u_i}{\beta_i} \right)^{\vartheta - \alpha} \right]^n. \]

This implies that
\[ f_\mathbf{X}(\mathbf{X}) \mathbb{I}(\mathbf{X}_n \succeq z) \leq \Lambda(\vartheta) g_\mathbf{X}(\mathbf{X}, \vartheta), \quad \forall \vartheta > \alpha, \]
where
\[ \Lambda(\vartheta) = \left[ \left( \prod_{i=1}^k \frac{\alpha + i - 1}{\vartheta + i - 1} \right) \left( 1 - k + \sum_{i=1}^k \frac{z_i}{\beta_i} \right)^{\vartheta - \alpha} \right]^n. \]
By virtue of Theorem 1, we have \( \Pr\{\overline{X}_n \leq z\} \leq \Lambda(\theta) \) for any \( \theta > \alpha \). This proves the first statement.

Note that if (48) holds, then \( \theta > \alpha \) for \( \theta \) satisfying (47). Moreover, by differentiation, it can be shown that \( \Pr\{\overline{X}_n \leq z\} \leq \inf_{\theta > \alpha} \Lambda(\theta) = \Lambda(\theta) \). This proves statement (II).

For \( \theta \) satisfying (49), it must be true that \( \theta > \alpha \) as a consequence of the assumption that \( \alpha > 1 \) and \( \frac{1}{k} \sum_{i=1}^{k} \frac{1}{m_i} < \frac{\alpha}{\alpha-1} \). This proves statement (III). The proof of the theorem is thus completed.

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