Role of temperature effects in the phenomenon of ultraslow electromagnetic pulses in Bose-Einstein condensates of alkali-metal atoms

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We study the temperature dependence of optical properties of dilute gases of alkali-metal atoms in the state with Bose-Einstein condensates. The description is constructed in the framework of the microscopic approach that is based on the Green-functions formalism. We find the expressions for the scalar Green functions describing a linear response of a condensed gas to a weak external electromagnetic field (laser). It is shown that these functions depend on the temperature, other physical properties of a system, and on the frequency detuning of a laser. We compare the relative contributions of the condensate and non-condensate particles in the system response. The influence of the temperature effects is studied by the example of two- and three-level systems. We show that in these cases, which are most commonly realized in the present experiments, the group velocity and the absorption rate of pulses practically do not depend on the gas temperature in the region from the absolute zero to the critical temperature. We discuss also the cases when the temperature effects can play a significant role in the phenomenon of slowing of electromagnetic pulses in a gas of alkali-metal atoms with Bose-Einstein condensates.

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I. INTRODUCTION

Bose-Einstein condensate (BEC) is one of the most impressive examples when the matter demonstrates its quantum nature on the macroscopic level. Now this system is interesting also due the possibility of observing the electromagnetic pulses propagating with extremely slow group velocities in it [1].

Up to now, in the theoretical investigations pretending to describe the mentioned phenomenon in a BEC (see Refs. [2, 3]) the authors assumed that the temperature of a gas is small in comparison to the critical temperature. In other words, this assumption corresponds to the consideration of the zero-temperature limit. But in real systems, the temperatures can be of the same order as the critical temperature. Therefore, one needs to study the account of temperature effects in the ultraslow light phenomenon. Naturally, it is also important to compare the theoretical results to the experimental data [1] describing the dependence of the group velocity of a signal on the temperature of a system.

In the present paper, we generalize the approach developed earlier in Ref. [3] for the uniform (nontrap) systems on the case of finite (nonzero) temperatures. This approach is based on the Green-functions formalism [1] and an approximate formulation of the second-quantization method [3]. An object of the mentioned generalization is a study of the influence of temperature effects on the dispersion characteristics of the system and, as a result, on the propagation properties of a signal in it.

II. LINEAR RESPONSE OF A GAS IN A BEC STATE AT FINITE TEMPERATURES: GREEN FUNCTIONS

To describe the optical properties of gases consisting of alkali-metal atoms that are used in the BEC-related experiments, it is most convenient to use the model of an ideal gas of hydrogenlike atoms in the stationary state (the limits of this approach are discussed in Ref. [3]). By the term “stationary state” we mean that the atoms can be found only in the states whose lifetimes are much greater than the relaxation time of the system (e.g., hyperfine levels of the ground state) and in the states whose occupation are stimulated by an external electromagnetic field (e.g., a laser radiation). In the case of a BEC presence in this gas, the density distribution of atoms in the quantum state \( \alpha \) by the momentum \( \mathbf{p} \) at nonzero temperatures (0 ≤ \( T \) ≤ \( T_c \), \( T \) is the temperature in energy units) can be set equilibrium, therefore, it can be written as follows (see also Ref. [1]):

\[
\nu_\alpha(\mathbf{p}) = \nu_\alpha \left[ 1 - \left( \frac{T}{T_{\alpha}} \right)^{3/2} \right] \delta(\mathbf{p}) + g_\alpha (2\pi \hbar)^{-3} \left\{ \exp \left[ (\varepsilon_\alpha - \mu_\alpha)/T \right] - 1 \right\}^{-1},
\]

(1)

where \( \nu_\alpha \) is the total density of atoms in the \( \alpha \) state

\[
\nu_\alpha = \int d\mathbf{p} \nu_\alpha(\mathbf{p}),
\]

where \( T_{\alpha} \) is the temperature of the transition of a gas of atoms in the \( \alpha \) state to the BEC phase, \( \delta(\mathbf{p}) \) is the Dirac delta function, and \( g_\alpha = (2F_\alpha + 1) \) is the degeneracy order of the state by the total momentum \( F_\alpha \) of an atom in this state. In the next calculations, we assume that an
external field is present in the system, i.e., we set $g_a = 1$. Here, also $\hbar$ is the Planck constant, $\varepsilon_\alpha(p) = \varepsilon_\alpha + p^2/2m$, where $\varepsilon_\alpha$ is the energy of an atom in the $\alpha$ state, $m$ is the atomic mass ($m = mp + me$; $mp$ and $me$ are masses of the atomic core and electron, respectively), and $\mu_\alpha$ is the chemical potential of an atom in this state.

Now we must note the following. It is known that a Bose-Einstein condensation is a collective effect. The atoms occupying the same quantum state $\alpha$ are identical. Because of this fact, one cannot definitely state what particles participate in the condensation process. Therefore, the terms “condensate” and “noncondensate” particles do not correspond directly to the selected atoms in the system. Strictly speaking, the term condensate particles correspond only to the fraction of all atoms which participate in the formation of the Bose-condensed component. Hence, the number of noncondensate particles can be found from the difference between the total number of atoms and number of condensate particles in the system. By the use of these terms, now we can say that the first summand in Eq. (1) corresponds to the contribution of condensate particles in the distribution function and the second summand corresponds to the normal component. Thus in the next calculations, we differentiate contributions from the different types of particles in the response of the system to the external perturbation by an electromagnetic field.

It is shown in Ref. [5] that the linear response of an ideal gas of hydrogenlike atoms with BECs to a weak electromagnetic field can be studied from the first principles in the framework of the Green-function formalism. There it was shown that the Fourier transform for the scalar Green function can be defined by the relation (see also Ref. [6] for details)

$$
G(k, \omega) = \frac{1}{V} \sum_p \sum_{\alpha, \beta} |\sigma_{\alpha\beta}(k)|^2 \times \frac{f_\alpha(p - k) - f_\beta(p)}{\varepsilon_\alpha(p) - \varepsilon_\beta(p - k) + \omega + i\gamma_{\alpha\beta}}.
$$

(2)

Here, by $k$ and $\omega$ we denote the wave vector and the frequency of the external perturbing field, respectively, and $\gamma_{\alpha\beta}$ denotes the linewidth related to the probability of a spontaneous transition between the $\beta$ and $\alpha$ states.

The quantity $\sigma_{\alpha\beta}(k)$ defines the matrix elements of the charge density of hydrogenlike atoms. This quantity can be expressed in terms of the wave functions $\varphi_\alpha$ and $\varphi_\beta$ of the atoms in $\alpha$ and $\beta$ states (see also Refs. [6] [5]), respectively,

$$
\sigma_{\alpha\beta}(k) = e \int dy \varphi_\alpha^*(y)\varphi_\beta(y) \times \left[ \exp \left( i \frac{m}{2m} ky \right) - \exp \left( -i \frac{m}{2m} ky \right) \right],
$$

where $e$ is the elementary charge. In particular, in the case when the dipole transition is allowed between the $\alpha$ and $\beta$ states, in the linear order over small term $ky \ll 1$, one gets

$$
\sigma_{\alpha\beta}(k) \approx ik d_{\alpha\beta}, \quad d_{\alpha\beta} = e \int dy \varphi_\alpha^*(y)\varphi_\beta(y),
$$

where $d_{\alpha\beta}$ is the matrix element of the atomic dipole moment. It should be mentioned that in next calculations, we use only the quantities that are proportional to $|\sigma_{\alpha\beta}(k)|^2$. Therefore, for the allowed dipole transitions, these quantities can be expressed in terms of the average dipole moment $d_{\alpha\beta}$,

$$
|\sigma_{\alpha\beta}(k)|^2 \approx k^2 d_{\alpha\beta}^2/3.
$$

(3)

The quantity $f_\alpha(p)$ in Eq. (2) corresponds to the Bose distribution function of atoms by the momentum $p$,

$$
f_\alpha(p) = \{\exp[(\varepsilon_\alpha(p) - \mu_\alpha)/T] - 1\}^{-1}.
$$

(4)

Note that one can use the Bose distribution in the form only in the case of the thermal equilibrium in the system. Thus, in the next calculations, we consider only the states whose lifetimes are much greater than the system relaxation time and the states whose existence in the system is stimulated by an external field. Hence, for a studied gas (a condensed phase can be formed by atoms with the energy $\varepsilon_\alpha$), one gets the condition for the chemical potential $\mu_\alpha$ (see Ref. [6] for details)

$$
\mu_\alpha(T \leq T_c) = \varepsilon_\alpha.
$$

At temperatures $0 \leq T \leq T_c$ in Eq. (2), one can substitute the distribution function by the density distribution. To this end, one needs also use the rule

$$
\frac{1}{V} \sum_p f_\alpha(p) \ldots = \int dp \nu_\alpha(p) \ldots.
$$

Next let us note that the relation (2) is linear over the distribution function $f_\alpha(p)$. Therefore, in accordance with Eq. (1), it can be divided into two summands that define the contributions of the condensate and noncondensate particles in the system response

$$
G(k, \omega) = G^{(c)}(k, \omega) + G^{(n)}(k, \omega),
$$

(5)

where the summands $G^{(c)}(k, \omega)$ and $G^{(n)}(k, \omega)$ are defined by

$$
G^{(c)}(k, \omega) = \sum_{\alpha, \beta} |\sigma_{\alpha\beta}(k)|^2 \frac{\nu_\alpha(1 - t_\alpha^{3/2})}{\delta\omega_{\alpha\beta} + i\gamma_{\alpha\beta}},
$$

$$
G^{(n)}(k, \omega) = (2\pi\hbar)^{-3} \sum_{\alpha, \beta} |\sigma_{\alpha\beta}(k)|^2 \times \int_0^\infty \frac{2\pi p^2 dp}{\exp(\varepsilon_p/T) - 1} \frac{1}{\delta\omega_{\alpha\beta} + pk y/m + i\gamma_{\alpha\beta}}.
$$

(6)

Here, $t_\alpha = T/T_{c\alpha}$ denotes the relative temperature of a gas (in this paper, we consider the case $t_\alpha \leq 1$).
\(\delta \omega_{\alpha \beta} = \omega + \Delta \varepsilon_{\alpha \beta}\) is the frequency detuning taken in the energy units, \(\Delta \varepsilon_{\alpha \beta} = \varepsilon_{\alpha} - \varepsilon_{\beta}, \varepsilon_{p} = p^2/2m\) denotes the kinetic energy of an atom, and the integration variable \(y\) denotes the cosine of the polar angle \(\theta, y \equiv \cos \theta.\) Here and below, we neglect of the recoil energy \(\varepsilon_r = h^2k^2/2m\) that is small enough to do a significant contribution into the effect \((\varepsilon_r \ll \gamma_{\alpha \beta};\) strictly speaking, it can be accounted by redefining the quantity \(\delta \omega_{\alpha \beta},\) i.e., by shifting the resonant frequency by \(\varepsilon_r).\) For the definiteness, in the next description, the summand \(G^{(c)}(k, \omega)\) in Eq. (5) is called as the condensate Green function and the summand \(G^{(n)}(k, \omega)\) is called as the noncondensate Green function.

A. Condensate Green function at finite temperatures

Firstly, we consider the scalar Green function corresponding to the contribution of condensate particles. For a convenience, let us study in detail the real and imaginary parts of it. It is shown below that in the region of transparency the real part makes a main contribution to the refractive index of a gas and imaginary part makes a main contribution to the absorption rate of light pulses. In accordance with Eq. (6), the real and imaginary parts of this function can be written as

\[
\begin{align*}
\text{Re} G^{(c)} &= \sum_{\alpha, \beta} |\sigma_{\alpha \beta}(k)|^2 \frac{\nu_{\alpha}(1 - t_{\alpha}^{3/2})\delta \omega_{\alpha \beta}}{(\delta \omega_{\alpha \beta})^2 + (\gamma_{\alpha \beta})^2}, \\
\text{Im} G^{(c)} &= -\sum_{\alpha, \beta} |\sigma_{\alpha \beta}(k)|^2 \frac{\nu_{\alpha}(1 - t_{\alpha}^{3/2})\gamma_{\alpha \beta}}{(\delta \omega_{\alpha \beta})^2 + (\gamma_{\alpha \beta})^2}.
\end{align*}
\]

To simplify the description, in the next calculations, we use a model of a two-level system (\(\alpha = 1, \beta = 2\)). Below, by the index “1” we denote the set of quantum numbers corresponding to the ground state and by the index “2” we denote the set of quantum numbers corresponding to the excited state. It is also assumed that the occupation of the exited state is stimulated by a low-intensity laser pulse. Due to a low intensity of the pumping field, we can consider that the density of the atoms in the excited state is small in comparison to the density of atoms in the ground state (see Ref. 3 for details)

\[\nu_2 \ll \nu_1 \equiv \nu.\]

According to this relation in Eq. (7) we neglect of the summands that are proportional to the density \(\nu_2.\) Therefore, one can get

\[
\begin{align*}
\text{Re} G^{(c)} &= |\sigma_{12}(k)|^2 \nu(1 - t^{3/2})\gamma^{-1} F'(\Delta), \\
\text{Im} G^{(c)} &= |\sigma_{12}(k)|^2 \nu(1 - t^{3/2})\gamma^{-1} F''(\Delta),
\end{align*}
\]

where we introduce the quantity \(\gamma \equiv \gamma_{12}\) and the dimensionless functions \(F'(x) = x/(x^2 + 1)\) and \(F''(x) = -1/(x^2 + 1).\) The dependencies of these functions on the relative frequency detuning \(\Delta (\Delta = \delta \omega/\gamma)\) are shown in Fig. 1.

Let us note that the functions \(F'\) and \(F''\) do not depend on the temperature. Hence, it is easy to see from Eq. (8) that the real and imaginary parts decrease with temperature by a power law as \(t_{\alpha}^{3/2}.\) Note also that the condensate Green function is proportional to the density of condensate particles in a gas, \(\nu^{(c)} = \nu(1 - t^{3/2}),\) and it equals to zero at the critical temperature, \(t = 1.\)

B. Noncondensate Green function at finite temperatures

Now we study the scalar Green function \(G^{(n)}\) corresponding to the non-condensate particles (see Eq. (5)). Taking the real and imaginary parts, for the two-level system (setting the indexes \(\alpha = 1\) and \(\beta = 2\), see above), we come to the relations

\[
\begin{align*}
\text{Re} G^{(n)} &= |\sigma_{12}(k)|^2 \frac{m^2T}{4\pi^2\hbar^4k} I'(\Delta, t), \\
\text{Im} G^{(n)} &= |\sigma_{12}(k)|^2 \frac{m^2T}{4\pi^2\hbar^4k} I''(\Delta, t),
\end{align*}
\]

where we introduce the dimensionless functions \(I'(\Delta, t)\) and \(I''(\Delta, t)\) that depend on the frequency of an external field and on the system parameters (including the
temperature of a gas)

\[ I'(\Delta, t) = \int_0^\infty \frac{xdx}{e^{x^2} - 1} \ln \left| \frac{1 + [\Delta + x(\kappa\sqrt{t})]^2}{1 + [\Delta - x(\kappa\sqrt{t})]^2} \right|, \]

\[ I''(\Delta, t) = 2 \int_0^\infty \frac{xdx}{e^{x^2} - 1} \{ \arctan \left[ \Delta - x(\kappa\sqrt{t}) \right] - \arctan \left[ \Delta + x(\kappa\sqrt{t}) \right] \}. \quad (10) \]

Here, analogously to the formulas for the condensate particles (see Eq. [5]), \( \Delta = \delta \omega / \gamma \) is the relative frequency detuning, \( t = T/T_c \) is the relative temperature,

\[ T_c = 2\pi[\zeta(3/2)]^{-2/3} \hbar^2 c^2 / 3 m \]

(11)
is the critical temperature of the transition to the BEC phase, and \( \zeta(x) \) is the Riemann zeta function. In Eq. (10), we also introduce the dimensionless parameter \( \kappa \),

\[ \kappa = \frac{\hbar k}{\gamma} \sqrt{\frac{2T_c}{m}} \sim \frac{\hbar k}{\gamma m} \hbar \nu^{1/3} \]

(12)
that depends on the physical properties of a gas and the wave number \( k \) of the external electromagnetic field. It is shown below that the value of this parameter defines the characteristics of the normal component response to an external perturbation.

Let us estimate the value of the parameter \( \kappa \) for the resonant radiation that corresponds to the sodium \( D_2 \) line. The laser pulses tuned to the components of this line were used in Ref. 1. Taking \( m = 3.82 \times 10^{-23} \) g, \( \gamma = 8.1 \times 10^{-21} \) erg, \( k = 1.07 \times 10^8 \) cm\(^{-1}\), and \( T_c = 435 \) nK, we get \( \kappa \approx 0.025 \). Using this parameter value it is easy to get the dependencies for the integrals \( I'(\Delta, t) \) and \( I''(\Delta, t) \) that are shown in Fig. 2. There, one can study also a behavior of the integral \( I' \) with a decrease of the temperature to the values 0.5\( T_c \) and 0.1\( T_c \), respectively (note that the integral \( I'' \) has an analogous temperature dependence).

From Fig. 2 one can see that the dependencies of the functions \( I' \) and \( I'' \) are similar to the dependencies of the condensate functions \( F' \) and \( F'' \) (cf. Fig. 1). The reasons for this similarity become evident below, but now it is easy to see that the values of the functions \( I' \) and \( I'' \) strongly depend on the gas temperature. In particular, from Eq. (9) and Fig. 2 we can conclude that the contribution of the non-condensate Green functions in the total Green function \( \chi \) increases with temperature. Let us study this effect in detail in the next subsection.

C. Dependence of the scalar Green function on the temperature

Let us note that the functions \( I' \) and \( I'' \) characterizing the contribution of the non-condensate particles to the total response of a gas in the case \( \kappa \ll 1 \) can be simplified at arbitrary values of the detuning \( \Delta \). Really, due to the strongly decreasing function \( \exp(-x^2) \) in the integrand, the main contribution to the integral give small values of the integration variable \( x, x \ll 1 \). This fact allows us to expand the integrands into series over \( (x \kappa \sqrt{t}) \ll 1 \). As a result, using Eqs. (8) and (10), accurate within quadratic summands, we get

\[ I'(\Delta, t) \approx 4\kappa\sqrt{t} F'(\Delta) \left\{ \int_0^\infty \frac{x^2dx}{e^{x^2} - 1} \right\}, \]

\[ + (\kappa\sqrt{t})^2 \left[ \frac{4}{3} F''(\Delta) + F''(\Delta) \left\{ \int_0^\infty \frac{x^4dx}{e^{x^2} - 1} \right\} \right], \quad (13) \]

\[ I''(\Delta, t) \approx 4\kappa\sqrt{t} F''(\Delta) \left\{ \int_0^\infty \frac{x^2dx}{e^{x^2} - 1} \right\}, \]

\[ + (\kappa\sqrt{t})^2 \left[ F''(\Delta) - \frac{1}{3} F''(\Delta) \left\{ \int_0^\infty \frac{x^4dx}{e^{x^2} - 1} \right\} \right]. \]

According to the relations

\[ \int_0^\infty \frac{x^2dx}{e^{x^2} - 1} = \frac{\sqrt{\pi}}{4} \zeta(3/2), \quad \int_0^\infty \frac{x^4dx}{e^{x^2} - 1} = \frac{3\sqrt{\pi}}{8} \zeta(5/2), \]

the Green functions \( \chi \) with account of Eqs. (11)–(13) can be written as

\[ \Re G^{(n)} \approx |\sigma_1| 2^{1/3} \frac{\kappa}{T} \sqrt{t} F'(\Delta) \left| 1 + \kappa^2 t S'(\Delta) \right|, \]

\[ \Im G^{(n)} \approx |\sigma_1| 2^{1/3} \frac{\kappa}{T} \sqrt{t} F''(\Delta) \left| 1 + \kappa^2 t S''(\Delta) \right|, \quad (14) \]
where we introduced the functions

\[ S'(\Delta) = 2 \frac{\zeta(5/2)}{\zeta(3/2)} \left[ F^2(\Delta) + \frac{3}{4} F''(\Delta) \right], \]

\[ S''(\Delta) = \frac{3}{2} \frac{\zeta(5/2)}{\zeta(3/2)} \left[ F^2(\Delta) - \frac{1}{3} F''(\Delta) \right]. \]

Note that Eq. (14) explains also the similarity in the behavior of the functions \( F', F'' \) and \( I', I'' \) at \( \kappa \sqrt{T} \ll 1 \) that is mentioned above (see Figs. 2 and 2).

It is easy to see that the “total” Green function in the region from the absolute zero to \( T_c \) weakly depends on the temperature. This dependence appears only in the quadratic terms over \( (\kappa \sqrt{T}) \ll 1 \). Really, by the use of Eqs. (3), (8), and (14), one can get

\[ \text{Re} G \approx |\sigma_1|^2 \nu \gamma^{-1} F'(\Delta)[1 + \kappa^2 t S'(\Delta)], \]

\[ \text{Im} G \approx |\sigma_1|^2 \nu \gamma^{-1} F''(\Delta)[1 + \kappa^2 t S''(\Delta)]. \]

But in the case \( \kappa \sqrt{T} \gtrsim 1 \), the expansions (13) become incorrect and analytical formulas for the functions \( I' \) and \( I'' \) cannot be found. Therefore, in this case, we use the numerical calculations at the definite values of the detuning \( \Delta \) and parameter \( \kappa \) (see Figs. 3 and 3).

In Fig. 3 one can see the dependencies of the condensate and noncondensate Green functions on the temperature. The real parts of these functions are normalized to a value of the real part of the condensate Green function at \( t = 0 \) and corresponding detuning \( \Delta \). Analogously, the imaginary parts of these functions are normalized to a value of the imaginary part of the condensate Green function at \( t = 0 \) and corresponding detuning \( \Delta \). In other words, in Fig. 3, the following dependencies are shown:

\[ \text{Re} G^{(c)}(t, \Delta)/\text{Re} G^{(c)}(t = 0, \Delta), \]

\[ \text{Im} G^{(c)}(t, \Delta)/\text{Im} G^{(c)}(t = 0, \Delta). \]

One can see that due to this normalization, the inequality \( \text{Im} G(t, \Delta)/\text{Im} G^{(c)}(0, \Delta) > 0 \) takes place. But note that the imaginary part of all Green functions is negative due to the function \( F''(\Delta) \) (see Eqs. (8) and (14)).

Analyzing the dependencies shown in Fig. 3 one can conclude that the real and imaginary parts of the condensate Green function demonstrate a similar behavior (drop-down bold curve). This dependence agrees with the obtained relations. But at the same time, the relative values of the real and imaginary parts of the non-condensate Green functions strongly depend on the parameter \( \kappa \) and detuning \( \Delta \). One can conclude that at \( \kappa \lesssim 1 \), by the use of the mentioned normalization, the real and imaginary parts are practically coincide. In the opposite case, \( \kappa > 1 \), as one can see in Fig. 5, the real and imaginary parts demonstrate different dependencies. This fact must have a strong impact on the dependence of the total Green function (the sum of the condensate and non-condensate Green functions) on the temperature.

In Fig. 4 the dependencies of the total Green function on the temperature are shown at different values of the parameters \( \Delta \) and \( \kappa \). Here, it is important to pay attention to the central curve that is practically horizontal (the case \( \kappa \ll 1 \)). This curve demonstrates a weak dependence of the total Green function (including its real and imaginary parts) on the temperature. Note that one can come to the same conclusion by analyzing Eq. (15). Thus at \( \kappa \ll 1 \), the influence of the temperature effects in the response of a BEC to an external field is insignificant. The problem on obtaining the response of this
system can be solved with a good accuracy by setting the temperature of a gas equal to zero (exactly this approximation was used in Refs. [3, 6, 8, 9]). In other cases (at \( \kappa > 1 \)), as one can see from Fig. 4, the influence of the temperature effects can be rather significant.

Evidently, the effects studied in this section in some way (significantly or not) can influence on the dispersion properties of a condensed gas. Let us recall that the resonant peculiarities of the dispersion characteristics play an essential role in the ultraslow-light phenomenon in a BEC of alkali-metal atoms.

### III. DISPERSION CHARACTERISTICS OF A GAS IN A BEC STATE AT FINITE TEMPERATURES

It is known that in the framework of the linear approach, the permittivity \( \epsilon(k, \omega) \) of a gas can be expressed in terms of the scalar Green function (see in this case Refs. [4, 6]) as

\[
\epsilon^{-1}(k, \omega) = 1 + \frac{4\pi}{k^2} G(k, \omega).
\]

Therefore, the real and imaginary parts of the permittivity \( \epsilon(k, \omega) \),

\[
\epsilon(k, \omega) = \epsilon'(k, \omega) + i\epsilon''(k, \omega)
\]

can be written as

\[
\begin{align*}
\epsilon'(k, \omega) &= \frac{1 + 4\pi k^{-2} \text{Re} G}{(1 + 4\pi k^{-2} \text{Re} G)^2 + (4\pi k^{-2} \text{Im} G)^2}, \\
\epsilon''(k, \omega) &= \frac{-4\pi k^{-2} \text{Im} G}{(1 + 4\pi k^{-2} \text{Re} G)^2 + (4\pi k^{-2} \text{Im} G)^2}.
\end{align*}
\]

(16)

At the same time, the refractive index and damping factor can be written in terms of the real and imaginary parts of the permittivity,

\[
\begin{align*}
n'(k, \omega) &= \frac{1}{\sqrt{2}} \sqrt{\epsilon'^2 + \epsilon''^2 + \epsilon'^2}, \\
n''(k, \omega) &= \frac{1}{\sqrt{2}} \sqrt{\epsilon'^2 + \epsilon''^2 - \epsilon'^2}.
\end{align*}
\]

(17)

Hence, basing on the derived equations for the scalar Green functions, one can study the propagation properties of weak electromagnetic pulses through a BEC at finite (nonzero) temperatures. As the derived analytical expressions for the refractive index and damping factor in the general case may have rather lengthy form (see Eq. (15)), let us consider some particular cases.

#### A. Temperature effects in two-level systems

Firstly, let us consider a case when the frequency of an external field is close to the energy spacing between two definite quantum states of atoms. Thus, using the most common method (see, e.g., Ref. [10]), in calculations we can consider only the resonant terms corresponding to this transition. In other words, we can consider atoms as a two-level system. Therefore, by the use of Eqs. (5), (8), (9), and (12), the real and imaginary parts of the permittivity (see Eq. (10)) take the form

\[
\epsilon'(k, \omega; t) = \frac{1}{1 + aG_1 + (aG_2)^2},
\]

\[
\epsilon''(k, \omega; t) = \frac{-aG_2}{1 + aG_1 + (aG_2)^2},
\]

(18)

where

\[
\begin{align*}
G_1 &= (1 - \epsilon^{3/2})F' + bt'F''(\Delta, t; b), \\
G_2 &= (1 - \epsilon^{3/2})F''(\Delta) + bt'F''(\Delta, t; b),
\end{align*}
\]

(19)

\[
a = 4\pi |\sigma_{12}(k)|^2 \nu/\kappa^2 \gamma, \quad b = [\kappa\sqrt{\pi}(3/2)]^{-1}.
\]

(20)

In particular, for the condensed sodium vapor with the density \( \nu = 1.44 \times 10^{12} \text{ cm}^{-3} \) that interacts with the resonant radiation corresponding to the \( D_2 \) line, in accordance with Eqs. (3) and (20), we get \( a \approx 4.19d^2\nu/\gamma \). Next, considering \( d^2 \approx S_{F'F}(3.52\epsilon_n)^2 \) (\( \epsilon_n \) is the Bohr radius, \( \epsilon \) is the elementary charge, and \( S_{F'F} \) is the relative strength of the dipole-allowed transition \( F \rightarrow F' \)), \( F = 2, F' = 2, \) and \( S_{22} = 1/4 \), we come to \( a \approx 0.015 \) and \( b \approx 20.38 \). For the two-level system with these parameters, the characteristic dependencies are shown in Fig. 5 (straight bold lines on the upper and lower graphs; there also the dependencies for the case \( \kappa = 10 \) are shown). Let us note that these dependencies (bold lines) also correspond to the case \( t = 0 \) independently of the parameter \( \kappa \) value (see also Eq. (15)).

![FIG. 5: (Color online) Dispersion dependencies of the two-level system in the BEC state. The used value for the parameter \( a = 0.015 \).](image)
temperatures lower than the critical temperature in the case of the propagation velocity of the light pulse at the temperature. As one can conclude from Fig. 6, the resonant frequency ($\Delta = 0$). The corresponding curves correspond to the signals tuned up exactly to the physical characteristics, one can obtain the dependence of the intensity of the transmitted light on the temperature for the two-level system with different values of the parameter $\kappa$.

Now we use the expressions that define the values of the group velocity and intensity of the transmitted light

$$v_g(t) = c\{\nu'(t) + \omega[\partial n'(t)/\partial \omega]\}^{-1},$$

$$I(t) = I_0 \exp[-n''(t)kL],$$

where $L$ is the characteristic size of the atomic cloud (in calculations we set $L = 0.004$ cm). For the mentioned physical characteristics, one can obtain the dependencies corresponding to the signals tuned up exactly to the resonant frequency ($\Delta = 0$). The corresponding curves are shown in Fig. 6. As one can conclude from Fig. 6, the propagation velocity of the light pulse at the temperatures lower than the critical temperature in the case $\kappa \ll 1$ (as it is expected) practically does not depend on the temperature.

Note that inequality $\kappa \ll 1$ usually takes place for dilute gases of alkali-metal atoms that interact with a laser field ($\kappa \approx 0.025$ for the parameters of the experiment [1]). Probably, this inequality takes place for the most BEC-related experiments at the present moment. Really, as it comes from the definition of $\kappa$, to increase its value in one order of magnitude, the density of atoms in a condensate must be increased by 3 orders of magnitude. This requirement results in the instability of the BEC phase due to an increase of the number of three-body collisions (see Ref. [12]). Also it is easy to see that the use of the high-frequency radiation (with the large wave number $k$) cannot result to the considerable increase of the parameter $\kappa$. It comes from the fact that the high-frequency transitions correspond to the high levels of the atomic spectrum. But the linewidth of these levels is much greater than the linewidth corresponding to the lower states. Hence, probably, the only way to a significant increase of the $\kappa$ value is to use the states with the smaller linewidth $\gamma$ (see Eq. [12]). It means that to increase the mentioned value, one needs to select the levels whose probability of a spontaneous transition is much less than was used in the above calculations.

Therefore, the cases $\kappa \gtrsim 1$ that are shown in Fig. 6 correspond to the “long-living” levels. One may relate these states to the levels with the forbidden dipole transitions (e.g., hyperfine levels of the ground state, see in this case Ref. [8]) or the sublevels, whose relative transition intensity is rather low. One can conclude that in this case, the temperature effects must have a strong impact on the ultraslow light phenomenon in a BEC.

B. Temperature effects in three-level systems

Note that two-level systems from the standpoint of the experiments dealing with the ultraslow-light phenomenon in a BEC can be inconvenient. As it is known, this phenomenon is realized mostly in three-level systems. These systems have some characteristic advantages: the absorption rate of the signal can be decreased by a special tuning of the laser frequency, one can get positive time delays of the pulses (positive sign of the group velocity), and also one can use the magnetic field to control the group velocity of the ultraslow pulses [9] or use an additional coupling laser to provide the electromagnetically induced transparency [13].

Therefore, in the framework of the developed approach let us study the temperature dependencies of the group velocity and intensity of the transmitted light in a three-level system. To this end, let us consider the system that is schematically illustrated in Fig. 7 (upper part).

Note that in the framework of the developed approach, we do not account quantum interferences in the system. Therefore, in this case, the derived above expressions change insignificantly. In particular, due to the additive contribution of all quantum states to the Green functions (see Eq. (6)), the functions $G_1$ and $G_2$ characterizing the response of all particles in the system can be written as follows (cf. Eq. (19)):

$$G_1 = (1 - i^{3/2})[F'/(\Delta + \Omega/2) + F'/(\Delta - \Omega/2)]$$
$$+ b[I'/(\Delta + \Omega/2; t; b) + I'/(\Delta - \Omega/2; t; b)],$$

$$G_2 = (1 - i^{3/2})[F''/(\Delta + \Omega/2) + F''/(\Delta - \Omega/2)]$$
$$+ b[I''/(\Delta + \Omega/2; t; b) + I''/(\Delta - \Omega/2; t; b)],$$

where $\Omega = \Delta_2\epsilon_{\text{mag}}/\gamma$ and $\Delta_2\epsilon_{\text{mag}}$ is the energy spacing between the excited states. Here and below, we consider that the splitted levels belong to the same multiplet, i.e., $|\sigma_{12}^2/\gamma_{12} = |\sigma_{13}^2/\gamma_{13}$. Hence, in accordance with Eq. (20), we can set $a_{12} = a_{13} = a$. Thus the formulas (19), (17), and (21) that define the dependence of the dispersion characteristics on the temperature have the same form.

To get the numerical estimates, let us consider the condensed gas of sodium atoms with the density $\nu = 5 \times 10^{12} \text{ cm}^{-3}$. For this gas, we get $a = 0.052$, $\kappa = 0.016$, and $b = 13.46$. Also we consider that there are two...
FIG. 7: (Color online) Scheme and dispersion dependencies of the three-level system in a BEC state. The used value for the parameter $a = 0.052$.

FIG. 8: (Color online) Dependencies of the group velocity and intensity of the transmitted light on the temperature for the three-level system with different values of the parameter $\kappa$.

nearby excited levels in the system (states $|2\rangle$ and $|3\rangle$ in Fig. 7). We also set the spacing between them several times larger than the linewidth $\gamma$, $\Delta \varepsilon_{\text{mag}} = 8\gamma$. For this system, one can get the graphs characterizing the dispersion characteristics of the system that are shown in Fig. 7 (bold curves). Note that in this case ($\kappa \ll 1$), the functions $n'(\omega)$ and $n''(\omega)$ practically do not depend on the temperature. Let us emphasize that we regard the system as a superposition of two two-level systems (Autler-Townes treatment). This is the reason that the absorption rate does not go to zero at $\Delta = 0$ (Fig. 4 bottom) as in the electromagnetically-induced transparency (EIT) regime [1, 13].

Within the developed approach, one can also study the dependencies in the case when the temperature effects have a significant influence on the system response (as it is mentioned above, this corresponds to the case $\kappa > 1$). In particular, taking $\kappa = 10$ and the same values of the other parameters, we get the dispersion curves at different temperatures that are shown in Fig. 7. It should be noted that in this case, the steepness of the slope of the refractive index decreases with the temperature. At the same time, the absorption in the central region increases with the temperature. This behavior can be explained by the fact that the density distribution of the non-condensate particles by the momentum $p$ in the case $\kappa > 1$ is not so “sharp” as for the condensate particles (see Eq. [11]). As a result, the absorption rate of the pulses tuned up to the resonance ($\Delta < 1$) decreases and the absorption rate of the pulses detuned from the resonance ($\Delta > 1$) increases (see Figs. 4, 5, and 7) with the temperature.

The mentioned behavior of the main macroscopic parameters with an account of Eq. (21) results in the dependencies that are shown in Fig. 8. As one can see from the graphs, the velocity of a light pulse at the temperature lower than the critical temperature in the case $\kappa < 1$ (analogously to the two-level case) is practically constant. It corresponds to the fact that the refractive index profile and absorption rate in this case changes insignificantly. Let us emphasize that in this case for the calculations we used the same parameters of a gas with a BEC as were realized in Ref. [1].

At $\kappa > 1$, the situation differs from the two-level case. Let us note that here one can see that both the group velocity and absorption rate of the pulse increase with the temperature. It can be explained by the fact that for the detuned pulses the steepness of the slope of the refractive index decreases while the absorption increases with the temperature (see Fig. 7). Thus the system becomes not so convenient for the realization of the ultraslow light phenomenon as in the zero-temperature limit.

IV. CONCLUSION

In this paper, we studied some optical properties of dilute gases in the BEC state at finite temperatures. The analysis is based on the Green-function formalism. We found analytical expressions for the scalar Green functions that characterize a linear response of a gas to an external electromagnetic field (laser). We studied the characteristic dependencies of these functions on the temperature and frequency detuning. We made a comparison
of the contribution of the condensate and non-condensate particles into the effect. The ultraslow-light phenomenon in a BEC is studied both on the examples of two-level and three-level systems.

In particular, it is shown that for a light pulse, which is tuned up close to the dipole-allowed transitions, the dispersion characteristics of a gas of alkali-metal atoms weakly depend on the temperature in the region from the absolute zero to the critical temperature. Therefore, the group velocity of the pulses in this system weakly depends on the temperature. This fact allowed us to conclude that the results of the previous works, where the authors studied the response of a gas in the limit of zero temperatures, $T \to 0$, are correct and can be used also in the region $0 \leq T \leq T_c$.

It is significant to note that in the experiment [1], data relating to the dependence of the group velocity of light pulses on the temperature of an ultracold gas of sodium atoms were obtained. From these results, one can conclude that the group velocity weakly depends on the temperature in the region $0 < T < T_c$. In the present paper, in the case of a three-level system, we use the same parameters of a gas as in Ref. [1] and we come to the same conclusion in our research. But the statement that the results of our theory are verified by the mentioned experiment is not quite correct. It should be noted that in Ref. [1], the effect of the electromagnetically induced transparency was used. But in the framework of the introduced approach that is based on the Green-functions formalism, we cannot account for quantum interference effects in the system (see in this case Ref. [3]). Probably, the similarity of the results may be a sign that the quantum interference effects in the mentioned experiment have a weak influence on the dependence of the group velocity on the temperature of a gas with a BEC. But, obviously, this statement requires an additional experimental verification.

We also studied the cases when the temperature effects can have a strong impact on the dispersion characteristics of gases with a BEC. In our opinion, this situation may be realized when the frequency of an external field is tuned close to the transitions between “long-living” states. These states correspond to the excited levels with the forbidden dipole transitions (upper hyperfine levels of the ground state of alkali-metal atoms) or the levels with a low relative intensity of the transition. In this case, as one can conclude from the results of the paper, the parameters of slowing impair with the temperature increase. Thus, the achievement of the lower temperatures of a gas is more necessary in this case.

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[1] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Nature (London) 397, 594 (1999).
[2] Z. Dutton and L. V. Hau, Phys. Rev. A 70, 053831 (2004).
[3] Y. Slyusarenko and A. Sotnikov, Phys. Rev. A 78, 053622 (2008); arXiv:0901.2425v1.
[4] A. I. Akhiezer and S. V. Peletminskii, Methods of Statistical Physics (Pergamon Press, Oxford, 1981).
[5] S. V. Peletminskii and Y. V. Slyusarenko, J. Math. Phys. 46, 022301 (2005); arXiv:quant-ph/0605159v1.
[6] Y. V. Slyusarenko and A. G. Sotnikov, Condens. Matter Phys. 9, 459 (2006); arXiv:cond-mat/0702637v3.
[7] Y. V. Slyusarenko and A. G. Sotnikov, Low Temp. Phys. 33, 30 (2007).
[8] Y. V. Slyusarenko and A. G. Sotnikov, J. Low Temp. Phys. 150, 618 (2008); arXiv:0706.3280v1.
[9] Y. Slyusarenko and A. Sotnikov, Phys. Lett. A 373, 1392 (2009).
[10] L. Allen and J. H. Eberly, Optical Resonance and Two-Level Atoms (Wiley, New York, 1975).
[11] D. A. Steck, Sodium D Line Data (2000), URL http://steck.us/alkalidata.
[12] E. A. Cornell, J. R. Ensher, and C. E. Wieman, in Proceedings of the International School of Physics "Enrico Fermi” Course CXL” (Italian Physical Society, Varenna, 1999), pp. 15–66; arXiv:cond-mat/9903109v1.
[13] S. E. Harris, Phys. Today 50, 36 (1997).