A TRIPLE BOUNDARY LEMMA 
FOR SURFACE HOMEOMORPHISMS

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Abstract. Given an orientation-preserving and area-preserving homeomorphism $f$ of the sphere, we prove that every point which is in the common boundary of three pairwise disjoint invariant open topological disks must be a fixed point. As an application, if $K$ is an invariant Wada type continuum, then $f^n|_K$ is the identity for some $n > 0$. Another application is an elementary proof of the fact that invariant disks for a nonwandering homeomorphism homotopic to the identity in an arbitrary surface are homotopically bounded if the fixed point set is inessential. The main results in this article are self-contained.

The aim of this short note is to prove the following result:

**Theorem 1.** If an orientation-preserving homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ has no wandering points and $B$ is a closed topological disk which intersects three pairwise disjoint $f$-invariant open topological disks, then $f(B) \cap B \neq \emptyset$.

As an immediate consequence one has the following:

**Corollary 2.** Under the same hypotheses, every point which is in the boundary of three pairwise disjoint open invariant topological disks must be a fixed point of $f$.

Since area-preserving homeomorphisms have no wandering points, these result apply to any area-preserving homeomorphism. As an example application, we state the following:

**Corollary 3.** If a continuum $\Lambda \subset \mathbb{S}^2$ is the boundary of three different connected components of its complement and $f$ is an area-preserving homeomorphism such that $f(\Lambda) = \Lambda$, then there is $n \in \mathbb{N}$ such that $f^n|_\Lambda = \text{Id}$.

**Proof.** The fact that $f$ is area-preserving implies that each connected component of $\mathbb{S}^2 \setminus \Lambda$ is periodic, so one may choose $n$ such that $f^n$ is orientation-preserving and leaves invariant three connected components of $\mathbb{S}^2 \setminus \Lambda$ which have $\Lambda$ as their boundaries and the result follows from the previous corollary. \hfill \Box

This applies in particular to any Wada-type continuum (i.e. a continuum $\Lambda$ whose complement has more than two connected components and the boundary of each such component is equal to $\Lambda$). Hence in the area-preserving setting, Wada-type continua can only appear as invariant sets if they have trivial dynamics. This is in contrast with the dissipative setting, where these types of continua appear frequently with a rich dynamics; for example the Plikyn attractor or more generally any transitive hyperbolic attractor on the sphere.

As a second application, we obtain an elementary proof of one of the main results of [KT14] and [KT17], which states that in an orientable surface $S$, if $f : S \to S$ is a homeomorphism homotopic to the identity with an invariant open topological disk
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$U$ and $f$ has no wandering points (for example, if $f$ is area-preserving), and if the fixed point set of $f$ is inessential (i.e. its inclusion in $S$ is homotopic to a point in $S$), then any open invariant topological disk must be homotopically bounded. The latter means that any lift of the disk to the universal covering space is relatively compact. The proofs given in [KT14] and [KT17] are involved and rely on the use of equivariant Brouwer theory [LC05] and maximal isotopies [Jaun]. Using the results from this paper, one obtains a direct and simple proof. See §3 for more details.

Theorem 1 is a special case of a more general result. In order to state it, let us introduce some definitions. Suppose $U$ is an open topological disk. A cross-cut of $U$ is an open-ended simple arc $\alpha$ in $U$ which extends to a compact arc joining two different points of $\partial U$. Each cross-cut $\alpha$ separates $U$ into exactly two connected components called cross-sections of $U$. Let $f: S \to S$ be an orientation-preserving homeomorphism such that $f(U) = U$. Let $W \subset U$ be an open set, and denote by $\text{cl}_U W$ the closure of $W$ in $U$. We say that $W$ is a positive boundary trapping region for $f$ in $U$ if $f(\text{cl}_U W) \subset W$ and $W$ is the union of a family of pairwise disjoint cross-sections of $U$ that is locally finite in $U$. A negative boundary trapping region for $f$ is a positive boundary trapping region for $f^{-1}$, and a boundary trapping region is a set which is either a positive or a negative boundary trapping region for $f$ in $U$. We say that a loop $\gamma$ in $S$ is trapping in $U$ if there is a boundary trapping region $W$ for $f$ in $U$ such that $\partial U W \subset \gamma$, and otherwise we say that $\gamma$ is non-trapping.

This is the key result of this note:

**Theorem 4.** Suppose $f: S^2 \to S^2$ is an orientation-preserving homeomorphism, and $B$ is a closed topological disk such that $f(B) \cap B = \emptyset$. If $B$ is a closed topological disk such that its interior intersects $\partial U$ and $f(B) \cap B = \emptyset$. If $\partial B$ is non-trapping, then there exists an arc $\sigma$ in $U$ joining a point $z \in B$ to $f(z) \in f(B)$ which intersects $B \cup f(B)$ only at its endpoints.

Note that if $U$ has a boundary trapping region $W$, then every point of $\partial U W$ is wandering, so in particular a homeomorphism without wandering points cannot have any boundary trapping regions. Thus Theorem 1 and its corollaries follow immediately from Theorem 4.

1. **Maximal cross-cuts.** A key element for the proof of our main result is the following lemma. Its proof is essentially contained in [KLN15, Lemma 4.6]. In order to keep this article self-contained we present a proof here as well.

**Lemma 5.** Let $S$ be an orientable surface, $f: S \to S$ an orientation-preserving homeomorphism, and $U$ be an invariant topological disk. Suppose that $B$ is a closed topological disk such that its interior intersects $\partial U$ and $f(B) \cap B = \emptyset$. If $\partial B$ is non-trapping, then there exists an arc $\sigma$ in $U$ joining a point $z \in B$ to $f(z) \in f(B)$ which intersects $B \cup f(B)$ only at its endpoints.

**Proof.** Consider the family $\mathcal{C}_0$ of all connected components of $\partial B \cap U$. Note that $\mathcal{C}_0$ is nonempty, since $B$ intersects $U$ and cannot contain $U$ entirely as $f(B) \cap B = \emptyset$. Each element of $\mathcal{C}_0$ is a free (i.e. disjoint from its image by $f$) cross-cut of $U$.

1 some definitions, as in [KLN15], allow these two points to coincide with an additional condition. Here it will suffice to use the more restrictive definition.
If \( \alpha \) is any free cross-cut of \( U \), we write \( U \setminus \alpha = D(\alpha) \cup D'(\alpha) \) where \( D'(\alpha) \) is the connected component of \( U \setminus \alpha \) containing \( f(\alpha) \) and \( D(\alpha) \) is the remaining component. One may easily verify that \( f(D(\alpha)) = D(f(\alpha)) \) and similarly for \( D'(\alpha) \).

We claim that for \( \alpha \in C_0 \) one has \( f(D(\alpha)) \subset D'(\alpha) \) (so \( D(\alpha) \) is free for \( f \)). Indeed, since \( f(\alpha) \) is disjoint from \( D(\alpha) \), the latter set is contained in a single connected component of \( U \setminus \alpha \), which is disjoint from \( f(\alpha) \). If \( D(\alpha) \) intersects \( f(D(\alpha)) \) then \( D(\alpha) \subset f(D(\alpha)) \), which implies \( \text{cl}_U D(\alpha) \subset f(D(\alpha)) \). But then \( f^{-1}(\text{cl}_U D(\alpha)) \subset D(\alpha) \), so \( D(\alpha) \) is a (negative) boundary trapping region contradicting our hypothesis.

Define a partial order among free cross-cuts of \( U \) by writing \( \alpha < \beta \) if \( D(\alpha) \subset D(\beta) \). Let \( C_1 = \{ f(\alpha) : \alpha \in C_0 \} \), and \( C = C_0 \cup C_1 \). Note that for \( \alpha \in C \) one still has \( f(D(\alpha)) \subset D'(\alpha) \), since we just showed this in the case that \( \alpha \in C_0 \), and if \( \alpha \in C_1 \) one has \( \alpha = f(\alpha') \) for some \( \alpha' \in C_0 \) so \( f(D(\alpha)) = f(D(f(\alpha'))) = f^2(D(\alpha')) \subset f(D'(\alpha')) = D'(f(\alpha')) \).

Let us note that for each \( c > 0 \) there are at most finitely many elements of \( C \) with diameter greater than \( c \), since \( C \) consists of pairwise disjoint arcs in \( \partial B \cup \partial f(B) \). As a consequence, the family \( C \) is locally finite in \( U \).

Denote by \( C^* \) the set of all elements of \( C \) which are maximal (in \( C \)) with respect to \( < \). We claim that for every \( \alpha \in C \) there exists \( \alpha^* \in C^* \) such that \( \alpha < \alpha^* \). In fact, one can show that there are finitely many elements \( \alpha' \in C \) such that \( \alpha < \alpha' \).

To see this, note that if \( \alpha < \alpha' \) we have \( f(D(\alpha)) \subset f(D(\alpha')) \subset D'(\alpha') \) whereas \( \alpha \subset D(\alpha') \), so \( \alpha' \) separates \( \alpha \) from \( f(\alpha) \) in \( U \). From this one may conclude that the diameter of \( \alpha' \) is bounded below by some positive number \( c \) which depends only on \( \alpha \), so \( \{ \alpha' : \alpha < \alpha' \} \) is finite as claimed.

We now claim that there is an element \( \alpha \in C^* \cap C_0 \) such that \( f(\alpha) \in C^* \). Indeed, assume for a contradiction that this is not the case, and let \( \alpha_0 \in C^* \) be any element. Since \( \alpha_0 \in C_0 \cup C_1 \), we consider two possibilities: suppose first that \( \alpha_0 \in C_0 \). Since \( f(\alpha_0) \notin C^* \) by our assumption, there must exist \( \alpha_1 \in C^* \) such that \( f(\alpha_0) < \alpha_1 \). Moreover, \( \alpha_1 \) cannot be in \( C_1 \), since that would mean that \( f^{-1}(\alpha_1) \subset C_0 \), contradicting the maximality of \( \alpha_0 \in C^* \). Thus \( \alpha_1 \in C_0 \), and we may repeat this argument inductively to obtain an infinite sequence \( (\alpha_i)_{i \in \mathbb{N}} \) of elements of \( C^* \cap C_0 \) such that \( f(\alpha_i) < \alpha_{i+1} \). Note that \( \{ D(\alpha_i) : i \in \mathbb{N} \} \) is a family of pairwise disjoint cross-sections (since each \( \alpha_i \) is maximal) and it is locally finite in \( U \) (which follows from the fact that \( \{ \alpha_i : i \in \mathbb{N} \} \subset C \) is locally finite in \( U \)). Thus \( W = \bigcup_{i \in \mathbb{N}} D(\alpha_i) \) is a (positive) boundary trapping region with \( \partial_U W = \bigcup_{i \in \mathbb{N}} \alpha_i \subset \partial B \), contradicting our hypothesis. Now suppose that \( \alpha_0 \in C_1 \). Then \( f^{-1}(\alpha_0) \in C_0 \), and there must exist \( \alpha_1 \in C^* \) such that \( f^{-1}(\alpha_0) < \alpha_1 \) (since otherwise \( \alpha = f^{-1}(\alpha_0) \) would be such that both \( \alpha \) and \( f(\alpha) \) belong to \( C^* \) contradicting our assumption). Moreover, \( \alpha_1 \in C_1 \), since in the case that \( \alpha_1 \in C_0 \) one has \( f(\alpha_1) \in C \) contradicting the maximality of \( \alpha_0 \). Thus by a similar argument we obtain a sequence \( (\alpha_i)_{i \in \mathbb{N}} \) in \( C^* \cap C_1 \) such that \( f^{-1}(\alpha_i) < \alpha_{i+1} \), and \( W = \bigcup_{i \in \mathbb{N}} D(\alpha_i) \) is a (negative) boundary trapping region, and moreover \( f^{-1}(W) \) is a negative boundary trapping region with \( \partial_U W \subset \partial B \), contradicting our hypotheses. This proves our claim.

Finally, given \( \alpha \in C^* \cap C_0 \) such that \( f(\alpha) \in C^* \), we claim that there exists an arc \( \sigma \) in \( U \) joining a point \( z \in \alpha \) to \( f(z) \in f(\alpha) \) which is disjoint from all elements of \( C \) except at its endpoints. To see this, let \( K \) be the union of all elements of \( C \) except \( \alpha \) and \( f(\alpha) \), which is a closed subset of \( U \). If the closed set \( K \) separates \( \alpha \) from \( f(\alpha) \) in \( U \approx \mathbb{R}^2 \) then some connected component \( \beta \) of \( K \) must separate \( \alpha \) from
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...f(α) (see for instance [New92 Theorem 14.3]). But the connected components of K are elements of C, so β ∈ C, and the fact that β separates α from f(α) implies that either α ≺ β (contradicting the maximality of α) or f(α) ≺ β (contradicting the maximality of f(α)). Thus α and f(α) are in the same connected component of U \ K, which means that there is an arc σ in U \ K joining any given z ∈ α to f(z) ∈ f(α). Of course σ may be chosen so that it only intersects α and f(α) at its endpoints. Thus σ only intersects ∂B or ∂f(B) at its endpoints, from which the required properties follow.

□

2. Proof of the main theorem. Only Theorem 4 requires a proof, since the other results follow as explained in the introduction. The proof follows from Lemma 5 and a simple observation using the fact that f preserves orientation.

Let f and B be as in the statement of Theorem 4 and let U₁, U₂, U₃ be three disjoint f-invariant open topological disks such that Uᵢ ∩ B ≠ ∅. Note that the interior of B must intersect each Uᵢ as well. Assume for a contradiction that ∂B is non-trapping in all three disks. Since the interior of B intersects both Uᵢ and its complement, it must intersect ∂Uᵢ. Applying Lemma 5 on each Uᵢ we know that there exists zᵢ ∈ ∂B ∩ Uᵢ and an arc σᵢ ⊂ Uᵢ joining zᵢ to zᵢ′ := f(zᵢ) which is disjoint from B ∪ f(B) except at its endpoints. Let γ = ∂B, oriented positively so that int B is locally on the left of γ, and let γ′ = f(γ). Since f preserves orientation, int f(B) is locally on the left of f(γ) as well. Note that since σᵢ is disjoint from int B, this implies that σᵢ is locally on the right of γ near zᵢ and on the right of γ′ near zᵢ′.

By permuting the points if necessary, we may assume that z₂ lies in the positive subarc γ₁₃ of γ joining z₁ to z₃. Since f preserves orientation, f(γ₁₃) is the positive subarc of γ′ joining z₁′ to z₃′. In particular, the positive subarc γ₁₃′ of γ′ joining z₁′ to z₃′ does not contain z₂′. Let η = γ₁₃ ∗ σ₃ ∗ γ₁₃′ ∗ σ₁⁻¹ (where σ₁⁻¹ denotes the arc σ₁ reversed). Then η is a simple loop. Let D be the component of the complement of η which lies locally to the right of η. Since int B is locally to the left of γ₁₃ and is disjoint from η, we have that int B is disjoint from D, so its closure B is also disjoint from D. For similar reasons, f(B) is disjoint from D, and in particular z₂′ ∉ D. Moreover, since z₂′ ∉ η we have z₂′ ∉ D.

On the other hand, if σ₂ denotes the arc σ₂ with its endpoints removed (which is disjoint from η = ∂D), we see that σ₂ is locally on the left of γ₁₃ near z₂, and therefore σ₂ ⊂ D. Hence σ₂ ⊂ D, which contradicts the fact that z₂′ ∉ D. This proves Theorem 4.

□

3. An application: homotopically bounded disks. Suppose that S is a closed orientable surface, and π: ̃S → S the universal covering map. We may endow S with a metric of constant curvature, and ̃S with the lifted metric. For a set X ⊂ ̃S we denote by diam(X) its diameter. The covering diameter of an open topological disk U ⊂ S is defined as D(U) = diam(U) ∈ ℝ₀ ∪ {∞}, where U is any connected component of π⁻¹(U) (and it is independent from the choice of U). When D(U) < ∞, we say that U is homotopically bounded. The next result was one of the key theorems in [KT14] and [KT17]. Using our main theorem, we are able to give a very direct proof of it.

Theorem 6. Let f: S → S be a homeomorphism homotopic to the identity of a closed orientable surface such that Fix(f) is inessential. Then there exists M > 0...
such that any open $f$-invariant topological disk $U$ without boundary trapping regions satisfies $D(U) \leq M$.

Proof. Denote by $G$ the group of deck transformations of $\pi$ (which are all isometries of $\tilde{S}$). Since $f$ is isotopic to the identity, there exists a lift $\tilde{f}_0$ of $f$ which commutes with every $T \in G$. Fix a connected component $\tilde{U}$ of $\pi^{-1}(U)$. Since $U$ is invariant there exists $T_0 \in G$ such that $\tilde{f}_0(\tilde{U}) = T_0 \tilde{U}$. Consider $\tilde{f} = T_0^{-1} \tilde{f}_0$, so that $\tilde{f}(\tilde{U}) = \tilde{U}$.

If $S = \mathbb{S}^2$ there is nothing to be done. Suppose now that $S = \mathbb{T}^2$. In that case, since $G$ is abelian, $\tilde{f}$ also commutes with every element of $G$, and in particular for each $T \in G$ one may find a compact connected subset $Q$ of $\tilde{S}$, such that $\pi(Q) = S$ and $\text{Fix}(\tilde{f}) \cap \partial Q = \emptyset$ (see [KT17, Remark 2.1]). Since $\partial Q$ is compact and has no fixed points of $\tilde{f}$, one may cover $\partial \tilde{U}$ by a finite family $B_1, \ldots, B_m$ of closed disks such that $\tilde{f}(B_i) \cap B_i = \emptyset$ for each $i$. If $\tilde{U}$ has diameter greater than $M := (2m + 1) \text{diam}(Q)$, then $\tilde{U}$ must intersect $T(\partial Q)$ for at least $2m + 1$ different values of $T \in G$. As a consequence, there exists $i \in \{1, \ldots, m\}$ such that $B_i$ intersects $T^{-1} \tilde{U}$ for at least three different values of $T \in G$. These are three pairwise disjoint $\tilde{f}$-invariant topological disks, so by Theorem 4 there is $T \in G$ such that $T^{-1} \tilde{U}$ has a boundary trapping region $\tilde{W}$ in $\tilde{f}$. Since $U$ is a topological disk, $\pi |_{T^{-1}U}$ is a homeomorphism onto $U$, and it follows that $\pi(W)$ is a boundary trapping region in $U$ for $f$ contradicting our hypothesis.

Finally, if $S$ is a hyperbolic surface and $T_0 = \text{Id}$ (which means that $\tilde{f}$ commutes with every element of $G$) the same proof used for $\mathbb{T}^2$ applies. Thus it remains to consider the case where $T_0 \neq \text{Id}$. In that case we know that $\tilde{f}$ commutes with $T_0$ but not necessarily with other elements of $G$. However, this case can be reduced to the case of $\mathbb{T}^2$ as in the proof of [KT17, Proposition 4.8]. Since the argument is identical, we omit these details and refer the reader to [KT17].

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References

[Jau14] O. Jaulent, Existence d’un feuilletage positivement transverse à un homéomorphisme de surface, Annales de l’institut Fourier 64 (2014), no. 4, 1441–1476.

[KLN15] A. Koropecki, P. Le Calvez, and M. Nassiri, Prime ends rotation numbers and periodic points, Duke Math. J. 164 (2015), no. 3, 403–472.

[KT14] A. Koropecki and F. A. Tal, Strictly toral dynamics, Invent. Math. 196 (2014), no. 2, 339–381.

[KT17] _____, Fully essential dynamics for area-preserving surface homeomorphisms, Ergodic Theory and Dynamical Systems, publ. online (2017) 1-46.

[LC05] P. Le Calvez, Une version feuilletée équivariante du théorème de translation de Brouwer, Publ. Math. Inst. Hautes Études Sci. (2005), no. 102, 1–98.

[New92] M. Newman, Elements of the topology of plane sets of points, Dover books on advanced mathematics, Dover Publications, 1992.