Higher Genus Correlators for the Complex Matrix Model

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Abstract

We describe an iterative scheme which allows us to calculate any multi-loop correlator for the complex matrix model to any genus using only the first in the chain of loop equations. The method works for a completely general potential and the results contain no explicit reference to the couplings. The genus $g$ contribution to the $m$–loop correlator depends on a finite number of parameters, namely at most $4g - 2 + m$. We find the generating functional explicitly up to genus three. We show as well that the model is equivalent to an external field problem for the complex matrix model with a logarithmic potential.
1 Introduction

In the matrix model approach to 2D quantum gravity one traditionally considers the hermitian matrix model. However, the complex matrix model is just as well suited for this purpose. The complex matrix model differs from the hermitian one in the discretized surface language by giving rise to only "chequered surfaces". Such short distance properties of the surfaces should however not matter in the continuum limit, and one do find that the two models lead to the same string equation.

The complex matrix model is in several aspects more appealing than the hermitian one. First it is possible to express in one formula all multi–loop correlators for genus zero and it might very well be possible to do so also for higher genera. Secondly it is possible to find a version of the complex matrix model for which the string equation has a pole–free solution with the same perturbative expansion as the the usual Painlevé solution. There is a possible drawback of the complex matrix model, though. Its Virasoro generators seem to be ill defined in the continuum limit. However one can identify in the correlators of the theory the term from which these divergencies originate. This term poses no problems for the double scaling limit of the correlators.

After having introduced the basic concepts in sections 2 and 3, we put forward in section 4 a conjecture about the form of the genus $g$ contribution to the generating functional of the complex matrix model. The conjecture is proven by induction. An outline of the proof including the first step is given in section 5. The details can be found in Appendix A. The proof provides us with an iterative scheme by means of which it is possible to calculate explicitly any multi–loop correlator to any genus. The method works for a completely general potential and the results contain no explicit reference to the couplings. The genus $g$ contribution to the $m$–loop correlator depends on a finite number of parameters, namely at most $4g - 2 + m$. Results for $g = 2$ and $g = 3$ are presented in section 6. Everything is based only on the first in the chain of loop equations.

Another iterative procedure based on work by Migdal has been advocated by David. However, this procedure involves the entire chain of loop equations and is applicable in practice only to potentials with a finite (small) number of couplings. It was applied to the hermitian matrix model with a quadratic and a cubic potential in. In reference an iterative procedure similar to the one described in this paper was developed for the hermitian matrix model. Since it was shown that the usual hermitian matrix model is equivalent to the so-called Kontsevich–Penner model, the iterative procedure can be formulated in the language of the latter. We show in Appendix B that an analogous property holds for the usual complex matrix model.
which is actually equivalent to an external field problem for the complex matrix model with a logarithmic potential (an analog of the Kontsevich–Penner model) but we formulate the iterative procedure independently of this.

2 Basic definitions

The complex matrix model is defined by the partition function

\[ Z = \int d\phi^d d\phi \exp(-NV(\phi^d)) \]  

(2.1)

where the integration is over complex \( N \times N \) matrices and

\[ V(\phi^d) = \sum_{j=1}^{\infty} g_j \text{Tr}(\phi^j)^2. \]  

(2.2)

We introduce the generating functional

\[ W(p) = \frac{1}{N} \sum_{k=0}^\infty \langle \text{Tr}(\phi^k)^2 \rangle / p^{2k+1} \]  

(2.3)

and the \( n \)-loop correlator \( (n \geq 2) \)

\[ W(p_1, \ldots, p_n) = N^{n-2} \sum_{k_1, \ldots, k_n=1}^\infty \langle \text{Tr}(\phi^{k_1}) \ldots \text{Tr}(\phi^{k_n}) \rangle_C / p_1^{2k_1+1} \ldots p_n^{2k_n+1}. \]  

(2.4)

The multi–loop correlators can be obtained from the generating functional by application of the loop insertion operator, \( \frac{d}{dV(p)} \):

\[ W(p_1, \ldots, p_n) = \frac{d}{dV(p_n)} dV(p_{n-1}) \ldots dV(p_2) W(p_1) \]  

(2.5)

where

\[ \frac{d}{dV(p)} \equiv -\sum_{j=1}^{\infty} \frac{j}{p^{2j+1}} dV(p). \]  

(2.6)

It is possible to show that the model defined by (2.1) and (2.2) is equivalent to an external field problem for the complex matrix model with a logarithmic potential (an analog of the hermitian Kontsevich–Penner model) in much the same way as was the case for the hermitian matrix model [3]. We defer the discussion of this to Appendix B since the iterative procedure can be formulated independently of this.

3 The loop equation

The first in the chain of loop equations can conveniently be written as

\[ \oint_C \frac{d\omega}{4\pi i} \frac{\omega V''(\omega)}{p^2 - \omega^2} W(\omega) = (W(p))^2 + \frac{1}{N^2} W(p, p) \]  

(3.1)
where $C$ is a curve which encloses singularities of $W(p)$ and $V(\omega) = \sum_j g_j \omega^{2j}/j$. With the normalization chosen above the genus expansion for the correlators reads

$$W(p_1, \ldots, p_n) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_g(p_1, \ldots, p_n) \quad (n \geq 1). \quad (3.2)$$

To leading order in $1/N^2$ one can thus ignore the last term in (3.1) and one finds

$$W_0(p) = \frac{1}{2} \oint_C \frac{d\omega}{4\pi i} \frac{\omega^2}{p^2 - \omega^2} \left( \frac{p^2 + c}{\omega^2 + c} \right)^{1/2} \quad (3.3)$$

where $c$ is given by

$$\oint_C \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{(\omega^2 + c)^{1/2}} = 2. \quad (3.4)$$

Inserting the genus expansion (3.2) in (3.1) it is seen that $W_g(p), \ g \geq 1$ obeys the following equation

$$\left\{ \hat{K} - 2W_0(p) \right\} W_g(p) = \sum_{g'=1}^{g-1} W_{g'}(p) W_{g-g'}(p) + \frac{d}{dV(p)} W_{g-1}(p) \quad (3.5)$$

where $\hat{K}$ is a linear operator, namely

$$\hat{K} f(p) \equiv \oint_C \frac{d\omega}{4\pi i} \frac{\omega V'(\omega)}{p^2 - \omega^2} f(\omega). \quad (3.6)$$

In equation (3.3) $W_g(p)$ is expressed entirely in terms of $W_{g_i}(p), \ g_i < g$. This indicates that one should be able to calculate $W_g(p)$ for any finite genus $g$ starting from $W_0(p)$. In the next section we describe an iterative procedure which makes this possible.

### 4 The conjecture

To characterize the matrix model potential we introduce, instead of the couplings $g_j$, the moments $I_n$ and $M_k$ defined by

$$M_k = \oint_C \frac{d\omega}{4\pi i \omega^{2k+1}(w^2 + c)^{1/2}}, \quad k \geq 0 \quad (4.1)$$

$$I_n = \oint_C \frac{d\omega}{4\pi i (w^2 + c)^{n+1/2}}, \quad n \geq 0 \quad (4.2)$$

The normalization condition (3.4) is then simply, $I_0 = 2$, and the $k^{th}$ multicritical point is reached when

$$I_1 = I_2 = \ldots = I_{k-1} = 0, \quad I_k \neq 0. \quad (4.3)$$
It is shown below that $W_g(p)$ can be expressed entirely in terms of the moments and that, for a given finite genus $g$, $W_g(p)$ depends only on a finite number of these. On the contrary $W_g(p)$ depends on all couplings $g_j$. Thus working with the moments instead of the couplings might facilitate calculations considerably.

Furthermore we introduce the basis vectors $\chi^{(n)}(p)$ and $\Psi^{(n)}(p)$ characterized by

\[
\begin{align*}
\{ \hat{K} - 2W_0(p) \} \chi^{(n)}(p) &= \frac{1}{(p^2 + c)^n}, \\
\{ \hat{K} - 2W_0(p) \} \Psi^{(n)}(p) &= \frac{1}{p^{2n}}.
\end{align*}
\]

It can be shown that $\chi^{(n)}(p)$ and $\Psi^{(n)}(p)$ can be expressed as

\[
\begin{align*}
\chi^{(n)}(p) &= \frac{1}{I_1} \left\{ \Phi^{(n)}(p) - \sum_{k=1}^{n-1} \chi^{(k)}(p) I_{n-k+1} \right\}, \\
\Psi^{(n)}(p) &= \frac{1}{M_0} \left\{ \Omega^{(n)}(p) - \sum_{k=1}^{n-1} \Psi^{(k)}(p) M_{n-k} \right\}.
\end{align*}
\]

where

\[
\begin{align*}
\Phi^{(n)}(p) &= \frac{1}{(p^2 + c)^{n+1/2}} \\
\Omega^{(n)}(p) &= \frac{1}{p^{2n}(p^2 + c)^{1/2}}.
\end{align*}
\]

We now put forward the following conjecture

\[
W_g(p) = \sum_{n=1}^{3g-1} A_g^{(n)} \chi^{(n)}(p) + \sum_{m=1}^{g} D_g^{(m)} \Psi^{(m)}(p).
\]

The coefficient $A_g^{(n)}$ is a sum of terms of the form

\[
a_g^{(n)} = I_{\alpha_1} I_{\alpha_2} \ldots I_{\alpha_k} M_{\beta_1} M_{\beta_2} \ldots M_{\beta_l} f(c, M_0, I_1)
\]

where

\[
\alpha_1, \alpha_2, \ldots, \alpha_k \in [2, 3g - n] \quad \beta_1, \beta_2, \ldots, \beta_l \in [1, g].
\]

Let us for an expression of the type (1.11) denote by $H_I$ and $H_M$ the degree of homogeneity in the $I$’s and $M$’s, respectively,

\[
H_I \equiv \sum_{j=1}^{k} (\alpha_j - 1), \quad H_M \equiv \sum_{j=1}^{l} \beta_j.
\]

Then the following inequalities hold for the $a_g^{(n)}$’s

\[
H_I (a_g^{(n)}) \leq 3g - n - 1, \quad H_M (a_g^{(n)}) \leq g.
\]
The structure of the coefficients $D_g^{(m)}$ is similar to that of the $A_g^{(n)}$‘s. However, the $f$’s will in general be different and for the $d_g^{(m)}$’s we have

$$\alpha_1, \alpha_2, \ldots, \alpha_k \in [2, 3g] \quad \beta_1, \beta_2, \ldots, \beta_l \in [1, g - m]$$

and

$$H_I(d_g^{(m)}) \leq 3g - 1; \quad H_M(d_g^{(m)}) \leq g - m.$$ (4.15)

The homogeneity requirements become more transparent if one considers the double scaling limit. For the $m^{th}$ multicritical model the double scaling limit is obtained by fixing the ratio between any given coupling and say $g_1$ to its critical value and setting

$$p^2 = z_c + a\pi \quad (4.17)$$
$$c = -z_c + a\Lambda^{1/m} \quad (4.18)$$

where $\Lambda$ is the cosmological constant. The $I$’s then scale as

$$I_n \sim a^{m-n}, \quad n \in [1, m-1]. \quad (4.19)$$

Furthermore it is well known that the genus $g$ contribution to the generating functional has the following scaling behaviour

$$W_g(\pi, \Lambda) \sim a^{(1-2g)(m+1/2)-1}$$

with the exception of $W_0(\pi, \Lambda)$ which also contains a non scaling part. From (4.6) – (4.9) it follows that the basis vectors behave as

$$\chi^{(n)} \sim a^{-m-n+1/2}, \quad \Psi^{(n)} \sim a^{-1/2}. \quad (4.20)$$

Bearing in mind that the coefficients $D_g^{(m)}$ and $A_g^{(n)}$ are of the form (4.11) and noting that the $I_1$ dependence of $f$ is also powerlike, including however negative powers, one finds for the $D$‘s and $A$‘s

$$A_g^{(n)}: \quad \sum_j (m - \alpha_j) \leq m(2 - 2g) - n - g - 1 \quad (4.22)$$
$$D_g^{(n)}: \quad \sum_j (m - \alpha_j) \leq (1 - 2g)(m + 1/2) - 1/2. \quad (4.23)$$

Here we have used the same notation as in (4.13). However $\alpha_j = 1$ is allowed and negative powers of $I_1$ give negative contributions to the sums. Since the generating functional away from the double scaling limit should look the same for all multicritical models the $D$’s and the $A$’s must satisfy the following conditions

$$A_g^{(n)}: \quad \sum_j 1 \leq 2 - 2g \quad \sum_j (\alpha_j - 1) \leq 3g - n - 1 \quad (4.24)$$
$$D_g^{(n)}: \quad \sum_j 1 \leq 1 - 2g \quad \sum_j (\alpha_j - 1) \leq 3g - 1. \quad (4.25)$$
Here we recognize the homogeneity requirements (4.14) and (4.16). The other two
requirements are useful for checking the outcome of the iteration process.

5 Proof of the conjecture

To prove that the conjecture is true for $g = 1$ we need only to calculate $W_0(p, p)$
according to (3.5). To do this we write the loop insertion operator as

$$\frac{d}{dV(p)} = \frac{\partial}{\partial V(p)} + \frac{dc}{dV(p)} \frac{\partial}{\partial c}$$

(5.1)

where

$$\frac{\partial}{\partial V(p)} = -\sum_{j=1}^{\infty} \frac{j}{p^{2j+1}} \frac{\partial}{\partial g_j}$$

and

$$\frac{dc}{dV(p)} = -\frac{c}{I_1 (p^2 + c)^{3/2}}$$

(5.2)

Noting that

$$\frac{\partial}{\partial V(p)} V'(\omega) = \frac{2 \omega p}{(p^2 - \omega^2)^2}$$

(5.3)

and

$$\frac{2p}{(p^2 - w^2)^3} = \frac{1}{4p} \left\{ \frac{\partial^2}{\partial p^2} \left( \frac{1}{p^2 - \omega^2} \right) - \frac{1}{p} \frac{\partial}{\partial p} \left( \frac{1}{p^2 - \omega^2} \right) \right\}$$

(5.4)

one gets

$$W_0(p, p) = \frac{c^2}{16p^2(p^2 + c)^2}$$

(5.5)

The genus one contribution to the generating functional is thus of the conjectured
form with

$$A_1^{(1)} = -\frac{1}{16}, \quad A_1^{(2)} = -\frac{c}{16}, \quad D_1^{(1)} = \frac{1}{16}.$$  

(5.6)

We see these coefficients do not depend on any moments. This is consistent
with the well known property of the 2-loop correlator that its scaling behaviour is
universal $\mathbb{I}$, i.e. the same for all multicritical models. Let us mention here that
the factor $\frac{1}{p^2}$ in $W_0(p, p)$ is actually the origin of all divergencies in the continuum
Virasoro generators of the theory. The divergencies appear because one expands
$W_0(p, p)$ in $\frac{1}{a\pi}$ before $a$ is sent to zero $\mathbb{I}$. The 2–loop correlator $W_0(p, p)$ itself is
however well defined in the double scaling limit.

To prove the conjecture in the general case we assume that it is true up to genus
$g = g_0$ and calculate the right hand side of the loop equation (3.5) for $W_{g_0+1}(p)$.
What we would like to show is of course that

$$\text{the r.h.s.} = \sum_{n=1}^{3(g_0+1)-1} A^{(n)} \frac{1}{(p^2 + c)^n} + \sum_{m=1}^{g_0+1} D^{(m)} \frac{1}{p^{2m}}$$

(5.7)

where the coefficients $A^{(n)}$ and $D^{(m)}$ fulfill the homogeneity requirements corresponding
to $g = g_0 + 1$. It is obvious how one treats the products $\sum_{g'=1}^{g_0} W_{g'}(p) W_{g_0+1-g'}(p)$. 

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As regards the term \( W_{g0}(p, p) = \frac{d}{dV(p)} W_{g0}(p) \) we make use of the fact that the loop insertion operator when acting on \( W_{g0}(p) \) can be written as

\[
\frac{d}{dV(p)} = \sum_i \frac{dI_i}{dV(p)} \frac{\partial}{\partial I_i} + \sum_j \frac{dM_j}{dV(p)} \frac{\partial}{\partial M_j} + \frac{dc}{dV(p)} \frac{\partial}{\partial c} \tag{5.8}
\]

and that

\[
\frac{dI_i}{dV(p)} = -i \left( 1 - \frac{1}{p^2 + c} \right)^{i+1/2} + (i + 1/2) \frac{c}{(p^2 + c)^{i+3/2}} - (i + 1/2) I_{i+1} \frac{dc}{dV(p)} \tag{5.9}
\]

\[
\frac{dM_j}{dV(p)} = \frac{1}{2} \left\{ - (2j + 1) \frac{1}{p^{2j+2}(p^2 + c)^{1/2}} - \frac{1}{p^{2j}(p^2 + c)^{3/2}} \right\} - \frac{1}{2} \left\{ \frac{1}{c} \sum_{l=0}^j M_{j-l} \frac{(-1)^l}{c^l} + \frac{(-1)^{j+1} I_1}{c^{j+1}} \right\} \frac{dc}{dV(p)} \tag{5.10}
\]

where \( \frac{dc}{dV(p)} \) is given by (5.2). The details of the general step can be found in Appendix A.

### 6 Results for \( g = 2 \) and \( g = 3 \)

It is straightforward to iterate the loop equation (3.5) starting from \( W_1(p) \) given by (5.6) (and (4.10)). To find \( W_g(p) \) one calculates the right hand side of (3.5) using the already known lower genus contributions \( W_1(p), \ldots, W_{g-1}(p) \) and the differentiation rules (5.9), (5.11) and (5.2). This gives an expression involving fractions of the type \( p^{-2m}(p^2 + c)^{-n} \). These fractions can be decomposed into fractions of the type \( p^{-2i}, (p^2+c)^{-j} \) allowing one to identify the coefficients \( A_g^{(n)} \) and \( D_g^{(m)} \) (cf. equations (4.4) and (4.5)). The results for \( W_2(p) \) and \( W_3(p) \) which are obtained using Mathematica read

\[
D_2^{(1)} = \frac{-9}{256 \ c^2 M_0^2} - \frac{1}{64 \ c^2 I_1 M_0} + \frac{I_2}{128 \ c I_1^2 M_0}
\]

\[
D_2^{(2)} = \frac{9}{256 \ c M_0^2}
\]

\[
A_2^{(1)} = -D_2^{(1)}
\]

\[
A_2^{(2)} = -(D_2^{(1)} c + D_2^{(2)})
\]

\[
A_2^{(3)} = -\frac{15}{256 \ I_1^2} + \frac{49 \ c^2 I_2}{256 \ I_1^4} + \frac{11 \ c \ I_2 - 20 \ c^2 I_3}{128 \ I_1^3} + \frac{5}{128 \ I_1 M_0}
\]

\[
A_2^{(4)} = -\frac{7 \ c}{128 \ I_1^2} - \frac{49 \ c^2 I_2}{128 \ I_1^4}
\]

\[
A_2^{(5)} = \frac{105 \ c^2}{256 \ I_1^2}
\]
\[
D_3^{(1)} = \frac{63 I_2^4}{512 I_1^4 M_0} - \frac{3 \left(9 I_2^3 + 25 c I_2^2 I_3\right)}{256 c J_1^6 M_0} + \frac{201 I_2^2 + 298 c I_2 I_3 + 145 c^2 I_3^2 + 308 c^2 I_2 I_4}{2048 c^2 I_1^5 M_0} + \frac{-85 c I_2 - 45 c^2 I_3 + 38 M_0}{1024 c^4 I_1^3 M_0^2} + \frac{110 c I_2^2 - 152 I_2 M_0 - 116 c I_3 M_0 - 91 c^2 I_4 M_0 - 105 c^3 I_5 M_0}{2048 c^3 I_1^4 M_0^2} + \frac{153 \left(2 M_0 + c M_1\right)}{2048 c^4 M_0^5} + \frac{9 \left(14 M_0 + c M_1\right)}{1024 c^4 I_1 M_0^4} + \frac{-45 c I_2 M_0 + 170 M_0^2 - 9 c^2 I_2 M_1}{2048 c^4 I_1^2 M_0^4}
\]

\[
D_3^{(2)} = \frac{-27}{1024 c^3 I_1 M_0^3} + \frac{27 I_2}{2048 c^2 I_1^2 M_0^3} - \frac{153 \left(2 M_0 + c M_1\right)}{2048 c^3 M_0^5}
\]

\[
D_3^{(3)} = \frac{-27}{1024 c^3 I_1 M_0^3} + \frac{27 I_2}{2048 c^2 I_1^2 M_0^3} - \frac{153 \left(2 M_0 + c M_1\right)}{2048 c^3 M_0^5}
\]

\[
A_3^{(1)} = -D_3^{(1)}
\]

\[
A_3^{(2)} = -(D_3^{(1)} c + D_3^{(2)})
\]

\[
A_3^{(3)} = \frac{5355 c^3 I_2^5}{512 I_1^9} + \frac{15 \left(171 c^2 I_2^4 - 533 c^3 I_2^3 I_3\right)}{512 I_1^9} + \frac{256 I_1^8}{1065 c I_2^3 - 43010 c^2 I_2 I_3^2 + 32845 c^3 I_2 I_3^2 + 35588 c^3 I_2^2 I_4}{2048 I_1^7} + \frac{-155}{2048 c^2 I_1^2 M_0^2} + \frac{235 c I_2 - 64 M_0}{2048 c^2 I_1^2 M_0^2} + \frac{138 I_2 - 19 c I_3 + 385 c^2 I_4}{2048 c I_1^4 M_0} + \frac{(840 c I_2^3 - 585 I_2^2 M_0 - 1015 c I_2 I_3 M_0 + 8975 c^2 I_3^2 M_0 - 19026 c^2 I_2 I_4 M_0 - 14280 c^3 I_3 I_4 M_0 - 16233 c^3 I_2 I_5 M_0)/2048 I_1^6 M_0}{2048 c^2 I_1^2 M_0^2} + \frac{(-18 I_2^2 - 1198 c I_2 I_3 + 465 I_3 M_0 + 126 c I_4 M_0 - 5439 c^2 I_5 M_0 + 4620 c^3 I_6 M_0)/2048 I_1^5 M_0}{2048 c^2 I_1^2 M_0^2} + \frac{(-18 I_2^2 - 1198 c I_2 I_3 + 465 I_3 M_0 + 126 c I_4 M_0 - 5439 c^2 I_5 M_0 + 4620 c^3 I_6 M_0)/2048 I_1^5 M_0}{2048 c^2 I_1^2 M_0^2} + \frac{45 \left(2 M_0 + c M_1\right)}{2048 c^2 I_1 M_0^4}
\]

\[
A_3^{(4)} = -\frac{5355 c^3 I_2^4}{256 I_1^8} + \frac{3 \left(-3105 c^2 I_2^3 + 7279 c^3 I_2^2 I_3\right)}{512 I_1^8} - \frac{-1590 c I_2^2 + 44116 c^2 I_2 I_3 - 17545 c^3 I_3^2 - 37373 c^3 I_2 I_4}{2048 I_1^6} - \frac{315}{2048 c^2 I_1^2 M_0^2} - \frac{21}{512 c I_1^3 M_0} + \frac{7 \left(-21 I_2 + 125 c I_3\right)}{2048 I_1^4 M_0} + \frac{(-1428 c I_2^2 + 945 I_2 M_0 + 131 c I_3 M_0 - 11109 c^2 I_4 M_0 + 10185 c^3 I_5 M_0)/2048 I_1^5 M_0}{2048 I_1^5 M_0}
\]

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We note that $A^{(1)}_g = -D^{(1)}_g$ and $A^{(2)}_g = -(D^{(1)}_g c + D^{(2)}_g)$ for both $g = 2$ and $g = 3$. The same is true for $g = 1$ according to (5.6). It is easy to show that the relations hold for any genus. Going through the induction proof in Appendix A one finds that $p^{-2}$, $p^{-4}$, $(p^2 + c)^{-1}$ and $(p^2 + c)^{-2}$ never appear in isolation on the right hand side of the loop equation (3.3) but only in the combinations $p^{-2}(p^2 + c)^{-1}$, $p^{-2}(p^2 + c)^{-2}$, $p^{-4}(p^2 + c)^{-1}$ and $p^{-4}(p^2 + c)^{-2}$. Decomposing these fractions one gets the relations between the coefficients stated above.

The expressions for the $A$'s and $D$'s look rather complicated. However, many terms in $W_g(p)$ can be ignored in the double scaling limit. In order for a given term to survive the double scaling limit the equality sign must hold in both homogeneity relations in (4.24) for $A$ terms and in (4.25) for $D$ terms. From the list of coefficients given above and from (5.6) one immediately finds that for $g = 1$, $g = 2$ and $g = 3$ all $D$ terms disappear. It actually holds that the second sum in (4.10) vanishes in the double scaling limit for all genera. This follows from the fact that (4.14) is always fulfilled and that $p^{-2m}$ never appears in isolation on the right side of (3.3) but always in combination with at least one power of $(p^2 + c)^{-1}$. In fact the induction proof can be carried through with the homogeneity requirement $H_1(d_g^{(m)}) \leq 3g - 2$ in stead of $H_1(d_g^{(m)}) \leq 3g - 1$.

### 7 Discussion

We have shown that it is possible to calculate for the complex matrix model any multi–loop correlator to any genus. This gives us the possibility of going beyond the planar approximation which has been used in most calculations to date. For
example it is interesting to note that in the double scaling limit $W_1(p)$ reduces to

$$W_1 = -\frac{c}{16} \chi^{(2)} \quad (d. s. l.) \tag{7.1}$$

which apart from a trivial factor is exactly the same as for the reduced hermitian matrix model \[2\]:

$$W_1^C(\pi, \Lambda) = \frac{1}{4} W_1^H(\pi, \Lambda). \tag{7.2}$$

It was conjectured long time ago that $W_1^C(\pi, \Lambda) = \frac{1}{4g} W_1^H(\pi, \Lambda) \ [2]$. The iterative procedure described above and the similar procedure in the hermitian matrix model case provide us with the possibility of testing this conjecture in detail. Presumably it is even possible within the iterative framework to give a rigorous proof of the conjecture. The iterative scheme also allows us to study higher genus contributions to the multi-loop correlators, for instance to investigate whether these can be expressed in a closed form as was the case for genus zero.

**Appendix A  Details of the proof**

We now go through the general step of the induction proof following the line of action described in section 5. Hence let us assume that the conjecture is true up to genus $g = g_0$ and let us calculate the right hand side of the loop equation \[3.5\] for $W_{g_0+1}(p)$. We consider first the products $\sum_{g'=1}^{g_0} W_{g'}(p)W_{g_0+1-g'}(p)$. For this purpose it is convenient to rewrite the basis vectors $\Psi^{(n)}(p)$ and $\chi^{(n)}(p)$ as

$$\Psi^{(n)}(p) = \sum_{i=1}^{n} \Omega^{(i)}(p) P_i^{(n)}(M_0, \ldots, M_{n-i}) \tag{A.1}$$

$$\chi^{(n)}(p) = \sum_{i=1}^{n} \Phi^{(i)}(p) Q_i^{(n)}(I_1, \ldots, I_{n-i+1}) \tag{A.2}$$

where

$$H_M(P_i^{(n)}) = n - i, \quad H_I(Q_i^{(n)}) = n - i \quad (A.3)$$

and

$$H_M(Q_i^{(n)}) = H_I(P_i^{(n)}) = 0 \quad (A.4)$$

The sum above consists of three types of terms: the one which products of $\chi$’s enter, the one which products of $\Psi$’s enter and the one which contains the mixed products $\chi \cdot \Psi$. The first of these gives rise to only terms of the type $K^{(n)}(p^2 + c)^{-n}$, $n \in [3, 3g_0 + 1]$ thus contributing only to the first sum in \[3.4\]. The constant $K^{(n)}$ is a sum (over $g', q, p$ and $i$) of products with the following structure

$$K^{(n)} \sim A_{q_0}^{(q)} A_{g_0+1-g'}^{(p)} Q_i^{(q)} Q_{n-i-1}^{(p)}$$
It is easily seen that

\[ H_I(K^{(n)}) \leq 3g_0 + 2 - n = 3(g_0 + 1) - n - 1, \quad H_M(K^{(n)}) \leq g_0 + 1. \]

The second type of terms, those which contain products of \( \Psi \)'s give rise to fractions of the type \( C^{(m)} p^{-2m}(p^2 + c)^{-1}, m \in [2, g_0 + 1] \). Decomposing a fraction like this one gets a sum of fractions \( p^{-2k}, k \in [1, m] \) plus the fraction \( (p^2 + c)^{-1}, \) all of them with some weight which depends only on \( c \). The constant \( C^{(m)} \) is a sum (over \( g', q, p \) and \( i \)) of products with the following structure.

\[ C^{(m)} \sim D_g^{(q)} D_{g_0+1-g'}^{(p)} P_i^{(q)} P_{m-i}^{(p)}. \]

One finds

\[ H_M(C^{(m)}) \leq g_0 + 1 - m \quad \text{and} \quad H_I(C^{(m)}) \leq 3(g_0 + 1) - 2 \]

which means that the homogeneity requirements (4.14) and (4.16) are fulfilled. Finally, let us consider the term with the mixed products \( \chi \cdot \Psi \). From this one we get fractions of the type \( B^{(n,m)} p^{-2m}(p^2 + c)^{-n}, m \in [1, g_0], n \in [2, 3g_0] \). After decomposition we are left with a sum of fractions of the type \( p^{-2i}, i \in [1, m] \) and \( (p^2 + c)^{-j}, j \in [1, n] \). Here the constant \( B^{(n,m)} \) is a sum (over \( g', q \) and \( p \)) of products like the following.

\[ B^{(n,m)} \sim D_g^{(q)} A_{g_0+1-g'}^{(p)} P_i^{(q)} P_{m-i}^{(p)}. \]

and one can show that

\[ H_I(B^{(n,m)}) \leq 3(g_0 + 1) - 1 - n, \quad H_M(B^{(n,m)}) \leq g_0 + 1 - m \]

which means that (4.14) and (4.16) are satisfied also in this case. This completes the treatment of the first term on the right hand side of the loop equation. Let us now turn to the term \( W_{g_0}(p, p) = \frac{d}{dv(p)} W_{g_0}(p) \). We start by recalling that the loop insertion operator when acting on \( W_{g_0}(p) \) can be written as in (5.8)–(5.11). Furthermore it is convenient to write \( W_{g_0}(p) \) in the following form.

\[ W_{g_0}(p) = \sum_{i=1}^{3g_0-1} \hat{A}^{(i)}_{g_0}(p) + \sum_{j=1}^{g_0} \hat{D}^{(j)}_{g_0}(p) \]

where

\[ \hat{A}^{(i)}_{g_0} = \hat{A}^{(i)}_{g_0}(M_0, \ldots M_{g_0}, I_1, \ldots, I_{3g_0-i}) = \sum_{n=1}^{3g_0-1} A^{(n)}_{g_0} Q_i^{(n)} \]  

\[ \hat{D}^{(j)}_{g_0} = \hat{D}^{(j)}_{g_0}(M_0, \ldots M_{g_0-j}, I_1, \ldots, I_{3g_0}) = \sum_{m=j}^{g_0} D^{(m)}_{g_0} P_j^{(m)} \]
and

\[ H_I(\hat{A}_{g_0}^{(i)}) \leq 3g_0 - i - 1, \quad H_M(\hat{A}_{g_0}^{(i)}) \leq g_0 \] (A.8)

\[ H_I(\hat{D}_{g_0}^{(j)}) \leq 3g_0 - 1, \quad H_M(\hat{D}_{g_0}^{(j)}) \leq g_0 - j \] (A.9)

Differentiation of \( \Phi^{(i)}(p) \) and \( \Omega^{(j)}(p) \) is simple since these expressions depend only on \( c \). One finds

\[ \frac{d\Phi^{(j)}(p)}{dV(p)} = \left( j + \frac{1}{2} \right) I_1 \frac{c}{(p^2 + c)^{j+3}} \] (A.10)

\[ \frac{d\Omega^{(j)}(p)}{dV(p)} = \frac{c}{2I_1} \frac{1}{p^{2j}(p^2 + c)^3}. \] (A.11)

Differentiation of the \( \Phi \)'s in (A.5) thus gives rise to terms of the type \( K^{(n)}(p^2 + c)^{-n} \), \( n \in [4, 3(g_0 + 1) - 1] \) where

\[ K^{(n)} \propto \hat{A}_{g_0}^{(n-3)} \]

while differentiation of the \( \Omega \)'s in (A.5) gives terms of the type \( C^{(m)}(p^2 + c)^{-3p^{-2m}} \), \( m \in [1, g_0] \) where

\[ C^{(m)} \propto \hat{D}_{g_0}^{(m)}. \]

It is easy to see that (4.14) and (4.16) are satisfied in both cases. The partial differentiation of \( \hat{A}_{g_0}^{(i)} \) and \( \hat{D}_{g_0}^{(j)} \) with respect to \( c \) results in terms which can be obtained from those just mentioned (except for trivial factors) by decreasing the power of \( (p^2 + c)^{-1} \) by one and replacing the coefficients by their derivatives with respect to \( c \). Since differentiation with respect to \( c \) does not change the degree of homogeneity, it follows that all resulting terms are in agreement with the conjecture.

We now turn to the partial differentiation of the \( \hat{D} \)'s and \( \hat{A} \)'s with respect to the moments \( I_j \). It is obvious that we do not have to worry about terms arising from the first term in (5.9) as long as we check that everything which comes from the second one is ok. However, checking the second one also takes care of the third. This is so because each term originating from the third term in (5.9) has a brother term originating from the second one from which it differs only by \( I_{i+1}(p^2 + c)^{-i} \) (apart from trivial factors).

Taking the partial derivative of the \( \hat{A} \)'s with respect to the \( I \)'s we get from the second term in (5.9) terms of the type \( K^{(n)}(p^2 + c)^{-n} \), \( n \in [4, 3(g_0 + 1) - 1] \) where

\[ K^{(n)} = \sum_{i=1}^{3g_0-1} \frac{\partial \hat{A}_{g_0}^{n-i-2}}{\partial I_i}(i + \frac{1}{2}) \]

for which it holds that

\[ H_I(K^{(n)}) \leq 3(g_0 + 1) - n - 1, \quad H_M(K^{(n)}) \leq g_0 \]
where we have used that differentiating $\hat{A}^{(i)}$ with respect to $I_i$ lowers $H_I$ by $i - 1$.

The terms which originate from the second term in (5.9) when we take the partial derivative of the $\hat{D}$'s with respect to the $I$'s are of the type $B^{(n,m)} p^{-2m} (p^2 + c)^{-n}, m \in [1, g_0], n \in [3, 3(g_0 + 1) - 1]$. The constants are given by

$$B^{(n,m)} \propto \frac{\partial \hat{D}^{(m)}_{g_0}}{\partial I_{n-2}}.$$  

It is easy to show that

$$H_I(B^{(n,m)}) \leq 3(g_0 + 1) - m - 1, \quad H_M(B^{(n,m)}) \leq g_0 - n$$

which means that (4.14) and (4.16) are satisfied for all terms. Now we only have left the partial differentiation of the $\hat{A}$ and the $\hat{D}$'s with respect to the $M$'s. In this case checking terms arising from the first line in (5.11) automatically takes care of terms arising from the second one. This is obvious for the last term in the second line. For the sum it follows from the fact that its most problematic term, the one with the $M_j$, has a brother term in the first line where $M_j$ is replaced by $p^{2j}$ (modulo some trivial factors).

Taking the partial derivative of the $\hat{A}$'s with respect to the $M$'s we get from the first term in (5.11) terms of the type $B^{(n,m)} p^{-2m} (p^2 + c)^{-n}, m \in [1, g_0 + 1], n \in [2, 3g_0]$ where

$$B^{(n,m)} \propto \frac{\partial \hat{A}^{(n-1)}_{g_0}}{\partial M_{m-1}}. \quad (A.12)$$

For $B^{(n,m)}$ it holds that

$$H_I(B^{(n,m)}) \leq 3(g_0 + 1) - n - 3, \quad H_M(B^{(n,m)}) \leq g_0 + 1 - m \quad (A.13)$$

where we have used that differentiating once with respect to $M_j$ lowers $H_M$ with $j$.

Hence (4.14) and (4.16) are satisfied. The second term in (5.11) gives rise to terms of the type $B^{(n,m)} p^{-2(m-1)} (p^2 + c)^{-n-1}, m \in [1, g_0 + 1], n \in [2, 3g_0]$ where $B^{(n,m)}$ is given by (A.12). It follows from (A.13) that the homogeneity conditions are satisfied also in this case.

Differentiating the $\hat{D}$'s partially with respect to the $M$'s we get from the first term in (5.11) terms of the type $K^{(m)} p^{-2m} (p^2 + c)^{-1}, m \in [3, g_0 + 1]$ where

$$K^{(m)} = - \sum_{i=0}^{g_0} \frac{\partial \hat{D}^{(m-i-1)}_{g_0}}{\partial M_i} (i + \frac{1}{2}). \quad (A.14)$$

One finds

$$H_I(K^{(m)}) \leq 3g_0 - 1, \quad H_M \leq g_0 + 1 - m \quad (A.15)$$
which is in accordance with (4.14) and (4.16). From the second term in (5.11) we get terms of the type $K(m) p^{-2(m-1)(p^2 + c)^{-2}}, m \in [3, g_0 + 1]$ where $K(m)$ is given by (A.14). It follows from (A.15) that the homogeneity conditions are fulfilled also in this case. This completes the proof of our conjecture.

Appendix B  Relation to external field problem

Let us now show that the complex matrix model considered so far is equivalent to an external field problem for the complex matrix model with a logarithmic potential. We start from the partition function

\[ Z[\eta^\dagger \eta] = e^{-N \text{Tr} (\eta^\dagger \eta)} \int d\phi^\dagger d\phi \exp \left\{ N \text{Tr} \left( -\phi^\dagger \phi + \phi^\dagger \eta + \eta^\dagger \phi + \alpha \log(\phi^\dagger \phi) \right) \right\} \quad (B.1) \]

where $\phi$ and $\eta$ are complex $N \times N$ matrices. Exploiting the invariance of the measure of integration, one finds in the standard way the following equation of motion

\[ \left\{ -\frac{\partial^2}{\partial \eta^\dagger_{ji} \partial \eta_{ik}} + 2\alpha N^2 \delta_{jk} - N \eta_{ij} \frac{\partial}{\partial \eta_{ik}} - N \eta^\dagger_{ki} \frac{\partial}{\partial \eta^\dagger_{ji}} \right\} Z[\eta^\dagger \eta] = 0 . \quad (B.2) \]

Since the partition function (B.1) depends only on (positive) eigenvalues of $\eta^\dagger \eta$ which we denote as $\lambda_i^2$, one can derive from this equation the so-called master equation of the theory

\[ \left\{ -\frac{1}{2} \frac{1}{\lambda_i} \frac{\partial}{\partial \lambda_i} \frac{\lambda_i}{\lambda_j} \frac{\partial}{\partial \lambda_j} - \sum_{j \neq i} \lambda_i \frac{\partial}{\partial \lambda_j} \lambda_i^2 - \lambda_j^2 - 2N \lambda_i \frac{\partial}{\partial \lambda_i} + 2\alpha N^2 \right\} Z[\lambda^2] = 0 . \quad (B.3) \]

Let us introduce the new variables $t_k$ which are related to the external field $\eta$ by a kind of the Miwa transformation

\[ t_k = \frac{1}{k} \text{Tr} (\eta^\dagger \eta)^{-k} + 2N \delta_{k,1} \quad k > 0 \quad (B.4) \]
\[ t_0 = \text{Tr} \log(\eta^\dagger \eta)^{-1} . \quad (B.5) \]

Using the chain rule one can write (B.2) (or (B.3)) as

\[ \sum_{n=1}^{\infty} (\eta^\dagger \eta)^{-n-1} L_n Z[\eta^\dagger \eta] = 0 \quad (B.6) \]

where

\[ L_{-1} = \alpha N^2 - 2N \frac{\partial}{\partial t_0} \quad (B.7) \]
\[ L_n = \sum_{k=0}^{n} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k}} + \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}} \quad n \geq 0 . \quad (B.8) \]
In the limit \( N \to \infty \) where the traces become independent variables we recover from these equations the Virasoro constraints \[2\] of the matrix model

\[
Z = \int d\phi^\dagger d\phi \exp\left(-\sum_{k=0}^{\infty} t_k \text{Tr} (\phi^\dagger \phi)^k\right)
\]

(B.9)

where the integration is over complex \( \frac{\alpha N}{2} \times \frac{\alpha N}{2} \) matrices. This matrix model is trivially related to the one defined by (2.1) and (2.2).

Similarly to \[10\] it can be shown that the model (B.1) is equivalent to a complex-matrix analog of the Kontsevich–Penner model defined by the partition function

\[
Z[\Lambda^\dagger \Lambda] = \int d\phi^\dagger d\phi \left\{ N \text{Tr} \left( -\Lambda \phi^\dagger \Lambda^\dagger \phi + \alpha \left[ \log(1 + \phi^\dagger)(1 + \phi) - \phi^\dagger - \phi \right] \right) \right\} .
\]

(B.10)

This partition function can be obtained (modulo a \( \Lambda \)-dependent factor) from (B.1) by substitution \( \phi \to (\Lambda^\dagger)^{\frac{\alpha}{2}} \phi(\Lambda)^{\frac{\alpha}{2}} + (\Lambda^\dagger \Lambda)^{\frac{\alpha}{2}} \) providing

\[
\eta = (\Lambda^\dagger \Lambda)^{\frac{\alpha}{2}} - \alpha(\Lambda^\dagger \Lambda)^{-\frac{\alpha}{2}}.
\]

(B.11)

References

[1] T. R. Morris, Nucl. Phys. B356 (1991) 703

[2] J. Ambjørn, J. Jurkiewicz and Yu. Makeenko, Phys. Lett. B251 (1990) 517

[3] S. Dalley, C. Johnson and T. Morris, Multicritical Complex Matrix Models and Nonperturbative 2D Quantum Gravity, SHEP 90/91-16, and Nonperturbative Two–Dimensional Quantum Gravity, SHEP 90/91-28

[4] Yu. Makeenko, A. Marshakov, A. Mironov and A. Morozov, Nucl. Phys. B356 (1991) 574

[5] A. A. Migdal, Phys. Rep. 102 (1983) 199

[6] F. David, Mod. Phys. Lett. A5 (1990) 1019

[7] J. Ambjørn and Yu. Makeenko, Mod. Phys. Lett. A5 (1990) 1753

[8] J. Ambjørn, L. Chekhov and Yu. Makeenko, Higher Genus Correlators and \( W \)-infinity from the Hermitian One-Matrix-Model, NBI-HE-92-22

[9] L. Chekhov and Yu. Makeenko, Phys. Lett. B278 (1992) 271

[10] L. Chekhov and Yu. Makeenko, Mod. Phys. Lett. A7 (1992) 1223

[11] Yu. Makeenko, Pis’ma v ZhETF 52 (1990) 885