PRECOMPACT NONCOMPACT REFLEXIVE ABELIAN GROUPS

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Abstract. We present a series of examples of precompact, noncompact, reflexive topological Abelian groups. Some of them are pseudocompact or even countably compact, but we show that there exist precompact non-pseudocompact reflexive groups as well. It is also proved that every pseudocompact Abelian group is a quotient of a reflexive pseudocompact group with respect to a closed reflexive pseudocompact subgroup.

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1. Introduction

The Pontryagin–van Kampen duality theorem states that if $G$ is a LCA (locally compact Abelian) group then the canonical evaluation mapping $\alpha_G : G \to G^{\wedge\wedge}$ defined by $\alpha_G(x)(\chi) = \chi(x)$ for all $x \in G$ and $\chi \in G^{\wedge}$ is a topological isomorphism. Here $G^{\wedge}$ denotes the group of continuous characters $\chi : G \to T$, where $T = \{z \in \mathbb{C} : |z| = 1\}$ is the circle group in the complex plane $\mathbb{C}$, and the groups $G^{\wedge}$ and $G^{\wedge\wedge}$ carry, as usual, the compact-open topology $\tau_{co}$.

There have been many efforts to extend the Pontryagin–van Kampen duality theorem outside the realm of locally compact Abelian groups. The groups $G$ such that $\alpha_G$ is a topological isomorphism are usually called reflexive. It turns out that many non-locally compact topological Abelian groups are reflexive. The first examples were provided by Kaplan in [19], where he proved that any product of reflexive groups is reflexive. The main motivation of this paper is to answer the following question posed by M. J. Chasco and E. Martín Peinador in [6, p. 641]: Is a precompact reflexive Abelian group necessarily compact? We recall that a precompact group is just a subgroup of a compact group. Note that, since the dual of a metrizable precompact group is discrete (see [3, 4.10] or [5]), reflexive, precompact, noncompact groups can only be found within non-metrizable groups.

A topological space is said to be pseudocompact if every real-valued continuous function defined on it is bounded. According to [7, Theorem 1.1], pseudocompact groups form a proper subclass of the class of precompact groups; actually they can be characterized as the $G_\delta$-dense subgroups of compact groups. Being pseudocompact yields nice relations between a topological Abelian group and its dual. For example, Hernández and Macario established in [18, Proposition 3.4] that a precompact Abelian group $G$ is pseudocompact iff for every countable subgroup $H$ of $G^{\wedge}$, the largest precompact topological group topology of $H$ coincides with the topology $\omega(H, G)$ of pointwise convergence on elements of $G$.

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We show in Theorem 2.3 that every pseudocompact Abelian group with infinite compact subsets is reflexive. To establish the existence of infinite pseudocompact Abelian groups without infinite compact subsets, we use the notion of \( h \)-embedded subgroup of a topological group introduced in [23]. Let \( \mathcal{P}_h \) be the class of topological Abelian groups \( G \) which are pseudocompact and have the property that every countable subgroup \( C \) of \( G \) is \( h \)-embedded, i.e., every (not necessarily continuous) homomorphism from \( C \) to the circle group \( T \) admits an extension to a continuous homomorphism from \( G \) to \( T \). According to [23], the class \( \mathcal{P}_h \) contains many infinite topological groups. We show in Proposition 2.1 that all compact subsets of an arbitrary group \( G \in \mathcal{P}_h \) are finite.

Slightly refining our techniques, we present in Theorem 2.9 an example of a reflexive, precompact, non-pseudocompact group. Under Martin’s Axiom, a countably compact noncompact reflexive group is presented in Proposition 2.10. We show in Example 2, however, that \( \omega \)-bounded groups need not be reflexive.

In Section 3 we give some insight into the wideness and permanence properties of the class of reflexive pseudocompact groups. We prove in Theorem 3.3 that every pseudocompact Abelian group is a quotient of a reflexive group \( H \in \mathcal{P}_h \) with respect to a closed pseudocompact subgroup \( L \). In particular, the group \( L \) is reflexive. We also answer a question in [18] by showing that there exists an infinite reflexive pseudocompact group \( G \) such that the dual group \( G^\wedge \) is also pseudocompact and, hence, fails to be a \( \mu \)-space.

**Notation and preliminary facts.** We only consider Abelian groups. The **polar** of a subset \( A \) of a topological group \( G \) is the set

\[
A^\circ = \{ \chi \in G^\wedge : \chi(A) \subset T_+ \},
\]

where \( T_+ = \{ z \in T : Re z \geq 0 \} \). The **inverse** polar of a set \( B \subset G^\wedge \) is the set

\[
B^\triangleleft = \{ x \in G : \chi(x) \in T_+ \text{ for each } \chi \in B \}.
\]

Polars can be used to describe both the compact-open topology \( \tau_{co} \) and the topology of pointwise convergence, which we denote by \( \tau_p \). Polars of compact sets form a neighborhood basis at the identity of \( (G^\wedge, \tau_{co}) \), while polars of finite sets play the same role in \( (G^\wedge, \tau_p) \). For a topological Abelian group \( G \), a set \( E \subset G^\wedge \) is **equicontinuous** if there exists a neighborhood \( U \) of the neutral element in \( G \) such that \( E \subset U^\circ \).

The evaluation mapping \( \alpha_G \) is continuous if and only if the compact subsets of \( (G^\wedge, \tau_{co}) \) are equicontinuous [3, Proposition 5.10]. We will see below that this result has strong implications for precompact groups.

For an Abelian group \( A \), we denote by \( Hom(A, T) \) the group of all homomorphisms \( h : A \to T \) with the pointwise multiplication. Given a subgroup \( L \) of the group \( Hom(A, T) \), we will denote by \( \omega(A, L) \) the topology on \( A \) induced by the elements of \( L \). Comfort and Ross showed in their seminal paper [8] that the topology of a precompact Abelian group \( G \) always coincides with \( \omega(G, G^\wedge) \). In particular \( \omega(A, Hom(A, T)) \) is the maximal precompact topological group topology on a given abstract Abelian group \( A \). This topology is also known as the **Bohr topology** on \( A \).

A topological group \( G \) is **sequentially complete** if it is sequentially closed in any topological group \( H \) that contains \( G \) as a topological subgroup. In other words, no sequence of elements of \( G \) converges to an element of \( H \setminus G \). It is easy to see that \( G \) is sequentially complete iff \( G \) is sequentially closed in its completion \( \hat{G} \).
Precompact groups are subgroups of compact groups, which in turn satisfy the Pontryagin-van Kampen duality theorem. Hence it is easy to deduce that for each precompact group $G$, $\alpha_G$ is injective and open when considered as a mapping onto its image. However, for a precompact group $G$, $\alpha_G$ can fail to be continuous or surjective, as we see in the following examples.

**Example 1.** Let $H$ be any locally compact, noncompact Abelian group. Denote by $G$ the group $H$ endowed with the precompact topology $\omega(H, H^\wedge)$. It is clear that $\omega(H, H^\wedge)$ is strictly weaker than the original topology of $H$.

We claim that $\alpha_G$ is surjective but not continuous. Indeed, $G$ has the same continuous characters as $H$, and it is a consequence of the classical Glicksberg theorem in [16] that they have the same compact sets as well. Hence $G^\wedge = (G^\wedge, \tau_{co})$ is topologically isomorphic to $H^\wedge$ and, since $H$ is reflexive, $G^{\wedge\wedge} \cong H$. This implies our claim.

**Example 2.** Let $I$ be an arbitrary uncountable index set and $n \geq 2$ a natural number. For every point $x \in \mathbb{Z}(n)^I$, let $\text{supp}(x)$ be the set of those $i \in I$ for which $x(i) \neq 0$. Then

$$\Sigma = \{x \in \mathbb{Z}(n)^I : |\text{supp}(x)| \leq \omega\}$$

is a dense $\omega$-bounded subgroup of the compact topological group $\mathbb{Z}(n)^I$ (see [2 Corollary 1.6.34]). It is shown in [6] that the evaluation mapping $\alpha_\Sigma$ is continuous but not surjective. In fact, $\Sigma^\wedge$ is the direct sum $\mathbb{Z}(n)^I$ with the discrete topology, and hence the bidual group $\Sigma^{\wedge\wedge}$ is the full product $\mathbb{Z}(n)^I$ with the Tychonoff product topology.

Let us say that a subgroup $D$ of a topological Abelian group $G$ is *$h$-embedded* in $G$ if every (not necessarily continuous) homomorphism $f$ of $D$ to a compact Abelian topological group $K$ can be extended to a *continuous* homomorphism $\tilde{f}: G \to K$. This is equivalent to saying that every homomorphism $f: D \to \mathbb{T}$ can be extended to a *continuous* character $\tilde{f}: G \to \mathbb{T}$. Note that a subgroup $D$ of a precompact Abelian group $G$ is $h$-embedded in $G$ if and only if the topology induced on $D$ by $G$ coincides with the Bohr topology of $D$.

It is immediate from the definition that all homomorphisms of an $h$-embedded subgroup $D$ of a topological Abelian group $G$ to compact Abelian topological groups are continuous. In fact, since precompact topological groups are subgroups of compact groups, every homomorphism of such a subgroup $D \subseteq G$ to a precompact Abelian group is continuous. Discrete subgroups of topological groups are good candidates to be $h$-embedded in the enveloping groups, but an $h$-embedded subgroup need not be discrete, even if the enveloping group is compact.

### 2. Reflexivity of pseudocompact groups

In this section we present a sufficient condition for reflexivity of pseudocompact groups. It turns out that the absence of infinite compact subsets does the job (Theorem 2.3).

**Proposition 2.1.** Let $G$ be a topological Abelian group such that every countable subgroup of $G$ is $h$-embedded. Then the countable subgroups of $G$ are closed, the compact subsets of $G$ are finite, and $G$ is sequentially closed in its completion $\overline{\rho}G$. 


Proof. Let $K$ be a countable subgroup of $G$ and $x \in G \setminus K$. Denote by $K_x$ the subgroup of $G$ generated by the set $K \cup \{x\}$. The group $K_x$ is countable and $h$-embedded in $G$. Since $K_x$ is Abelian, there exists a homomorphism $h: K_x \to T$ such that $h(x) \neq 1$ and $h(K) = \{1\}$. Let $f$ be a continuous extension of $h$ to a homomorphism of $G$ to $T$. Then $U = f^{-1}(T \setminus \{1\})$ is an open neighborhood of $x$ in $G$ and $U \cap K = \emptyset$, i. e., $x \notin \overline{K}$. This proves that $K$ is closed in $G$.

Let us show that all compact subsets of $G$ are finite. Suppose to the contrary that $C$ is an infinite compact subset of $G$. Take a countable infinite subset $S$ of $C$ and consider the subgroup $K = \langle S \rangle$ of $G$ generated by $S$. Then $K$ is a countable subgroup of $G$ and, as we have just shown, $K$ is closed in $G$. Hence $P = C \cap K$ is a closed subset of $C$, whence it follows that $P$ is compact. Since $S \subset P \subset K$, the set $P$ is countable and infinite. Therefore, $P$ is metrizable and contains non-trivial convergent sequences. However, since $K$ is $h$-embedded in $G$, the topology that $K$ inherits from $G$ is finer than the largest precompact topology $\tau_h(K)$ on the abstract group $K$. It remains to recall a well-known fact that an Abelian group with the Bohr topology does not contain non-trivial convergent sequences (see [20] or [2] Theorem 9.9.30). Clearly, this contradicts the inclusion $P \subset K$. We have thus proved the second part of the proposition.

Finally, suppose that a sequence $\{x_n : n \in \omega\}$ of elements of $G$ converges to an element $y \in \varrho G \setminus G$. Then the nontrivial sequence $\{x_{n+1} - x_n : n \in \omega\}$ converges to the neutral element of $G$, which is impossible since $G$ does not contain infinite compact subsets. Hence $G$ is sequentially closed in $\varrho G$. \hfill \square

The following important fact was proved by Raczkowski and Trigos in [22, 3.1].

Lemma 2.2. If $G$ is a precompact Abelian group, then the evaluation mapping $\alpha_G$ is a topological isomorphism of $G$ onto $((G^\wedge, \tau_p)^\wedge, \tau_p)$.

Theorem 2.3. If every compact subset of a pseudocompact Abelian group $G$ is finite, then $G$ is reflexive.

Proof. It follows from [18] Proposition 4.4] that for any pseudocompact group $G$, the compact subsets of $(G^\wedge, \tau_p)$ are finite. In particular, $((G^\wedge, \tau_p)^\wedge, \tau_{co}) \cong ((G^\wedge, \tau_p)^\wedge, \tau_p)$.

Since all compact subsets of $G$ are finite, we have that $\tau_{co} = \tau_p$ on $G^\wedge$. Using Lemma 2.2 we conclude that $((G^\wedge, \tau_{co})^\wedge, \tau_{co}) \cong ((G^\wedge, \tau_p)^\wedge, \tau_p) \cong G$, i. e., $G$ is reflexive. \hfill \square

Remark 1. The condition in Theorem 2.3 that $G$ does not contain infinite compact subsets is sufficient but not necessary. Indeed, given a pseudocompact noncompact reflexive group $P$ and an infinite compact group $K$, the product group $P \times K$ is pseudocompact, noncompact, reflexive, and contains an infinite compact subset homeomorphic to $K$.

The following fact is an immediate consequence of Proposition 2.1 and Theorem 2.3.

Theorem 2.4. If $G$ is a pseudocompact Abelian group such that every countable subgroup of $G$ is $h$-embedded, then $G$ is reflexive.
The following result, proved in [23], will provide a first example of a noncompact group with the above properties and will be applied in our further arguments. Below, we denote by $c$ the power of the continuum, so that $c = 2^{\omega}$.

**Theorem 2.5.** There exists a dense pseudocompact subgroup $G$ of the product group $\mathbb{Z}(2)^c$ such that all countable subgroups of $G$ are $h$-embedded in $G$.

**Remark 2.** There are many other examples of pseudocompact groups all whose countable subgroups are $h$-embedded. Answering a question posed by D. Dikranjan and D. Shakhmatov in [10], J. Galindo and S. Macario ([14]) have recently found conditions under which a pseudocompact group admits another pseudocompact group topology which renders all countable subgroups $h$-embedded. This reference, which recently came to the authors’ attention, contains several generalizations of Theorem 2.4.

The following two results provide more information regarding the duals of precompact and pseudocompact groups. We start with considering pseudocompact groups:

**Lemma 2.6.** The following hold true for a pseudocompact Abelian group $G$:

a) All countable subgroups of $G^\wedge$ are $h$-embedded in $G^\wedge$.

b) Every compact subset of $G^\wedge$ is finite.

**Proof.** a) Consider a countable subgroup $H$ of $G^\wedge$ and an arbitrary homomorphism $f : H \to \mathbb{T}$. Since $G$ is pseudocompact, $\omega(H, G)$ is the maximal precompact topology on $H$ (see [18, Proposition 3.4]). This implies, by Comfort–Ross’ theorem in [8], that there exists an element $g \in G$ with $\alpha_G(g)|_H = f$. In particular, $\alpha_G(g)$ is a continuous character on $G^\wedge$ which extends $f$, and $H$ is $h$-embedded in $G^\wedge$.

b) By a), all countable subgroups of $G^\wedge$ are $h$-embedded. Hence Proposition 2.1 implies that every compact subset of $G^\wedge$ is finite. Alternatively, one can deduce the conclusion from [18, Proposition 4.4]. \[\square\]

In the case of a reflexive group $G$, the precompactness and pseudocompactness of $G$ can be completely characterized by corresponding properties of $G^\wedge$:

**Proposition 2.7.** Let $G$ be a reflexive group. Then:

a) $G$ is precompact iff the compact subsets of $G^\wedge$ are finite.

b) $G$ is pseudocompact iff the countable subgroups of $G^\wedge$ are $h$-embedded.

**Proof.** (a) If $(G, \tau)$ is precompact, then $\tau$ is the topology of pointwise convergence on elements of $G^\wedge$, that is, $\tau = \omega(G, G^\wedge)$. Since every compact set $K \subset G^\wedge$ is equicontinuous, there exists a finite set $F \subset G^\wedge$ with $K \subset (F^\circ)^\circ$. Since $F$ is finite, so is $(F^\circ)^\circ$ (see [3, 7.11]). The reflexivity of $G$ yields that $(F^\circ)^\circ = (F^\circ)^\circ$ and, hence, $K$ is finite.

Conversely, suppose that all compact subsets of $G^\wedge$ are finite. Then $(G^\wedge, \tau_p) \cong (G^\wedge, \tau_\omega) \cong G$.

(b) The necessity follows from a) of Lemma 2.6. Conversely, suppose that all countable subgroups of $G^\wedge$ are $h$-embedded. By Proposition 2.4 the compact subsets of $G^\wedge$ are finite, so $G^\wedge$ carries the topology of pointwise convergence on elements of $G^\wedge$. In other words, the compact-open topology of $G^\wedge$ coincides with $\omega(G^\wedge, G^\wedge)$. Since $\alpha_G : G \to G^\wedge$ is a topological isomorphism, we conclude that the original topology of $G$ is $\omega(G, G^\wedge) = \alpha_G^{-1}(\omega(G^\wedge, G^\wedge))$. Hence it suffices to show that the topology $\omega(G, G^\wedge)$ is pseudocompact.
Take any countable subgroup \( H \) of \( G^\land \). We claim that \( \omega(H, G) \) coincides with the maximal precompact topology on the abstract group \( H \). Indeed, let \( f : H \to \mathbb{T} \) be a homomorphism. The subgroup \( H \) is \( h \)-embedded in \( G^\land \), so there exists a continuous homomorphism \( \hat{f} : G^\land \to \mathbb{T} \) extending \( f \). Since the evaluation mapping \( \alpha_G \) is surjective, there is an element \( g \in G \) with \( \alpha_G(g) = \hat{f} \). This proves our claim. Hence [18 Proposition 3.4] implies the pseudocompactness of \( (G, \omega(G, G^\land)) \). \( \square \)

It seems strange at the first sight, but Proposition 2.1 and Theorem 2.3 enable us to find examples of reflexive precompact non-pseudocompact groups. Our construction makes use of the following result close to Proposition 2.1. As usual, we denote by \( (x) \) the cyclic subgroup of a group \( L \) generated by an element \( x \in L \).

**Lemma 2.8.** Let \( G \) be any infinite pseudocompact Boolean group all countable subgroups of which are \( h \)-embedded (Theorem 2.3). Suppose that \( C_0 \) is a countable infinite subgroup of \( G \). Consider \( K = (x_0) \) for some \( x_0 \in \overline{C_0} \setminus C_0 \) (the closure of \( C_0 \) is taken in the completion \( \varrho G \) of \( G \) and the quotient homomorphism \( \pi : \varrho G \to \varrho G/K \)). Then all compact subsets of the subgroup \( H = \pi(G) \subset \varrho G/K \) are finite and \( H \) is reflexive.

**Proof.** Since the group \( C_0 \) is infinite, its closure \( \overline{C_0} \) is an infinite compact group and, hence, \( |\overline{C_0}| \geq 2^\omega \). It follows from Proposition 2.1 that \( C_0 \) is closed in \( G \), so \( \overline{C_0} \cap G = C_0 \). Hence \( \overline{C_0} \setminus C_0 \) is an uncountable subset of \( \varrho G \setminus G \). This explains, in particular, why we can find an element \( x_0 \in \overline{C_0} \setminus C_0 \).

We claim that for every countable subgroup \( C \) of \( G \) with \( C_0 \subset C \), the image \( \pi(C) \) is closed in \( H \). Since the homomorphism \( \pi \) is open, it suffices to verify that \( \pi^{-1}(C) = C + K \) is closed in \( \pi^{-1}(G) = G + K \). If not, there exists \( y \in G + K \) such that \( y \in \overline{C + K} \setminus (C + K) \). There are two possibilities:

*Case I.* \( y \in G \). Since \( C \) is closed in \( G \), we deduce that \( y \notin \overline{C} \). It follows from \( \overline{C + K} = \overline{C} \cup \overline{C + x_0} \) that \( y \notin \overline{C + x_0} = \overline{C + x_0} \). In its turn, this implies that \( x_0 \notin \overline{C + y} \). Since \( C_0 \subset C \), we have that \( x_0 \in \overline{C} \cap (\overline{C + y}) \) and, hence, \( y \notin \overline{C} \). This is a contradiction.

*Case II.* \( y \in G + x_0 \). Then the element \( z = y + x_0 \in G + K \) satisfies \( z \in \overline{C + K} \setminus (C + K) \). We have just shown in Case I, however, that this is impossible. This proves that \( \pi(C) \) is closed in \( H \).

Let us see that all compact subsets of the subgroup \( H \) are finite. Suppose to the contrary that \( F \) is an infinite compact subset of \( H \). There exists a countable subgroup \( C \) of \( G \) such that the set \( \pi(C) \cap F \) is infinite and \( C_0 \subset C \). We have just proved that \( \pi(C) \) is a closed subgroup of \( H \). Therefore, \( P = \pi(C) \cap F \) is a countable, infinite, compact subset of \( H \). Since the projection \( \pi : \varrho G \to \varrho G/K \) is perfect, \( Q = \pi^{-1}(P) \) is a countable infinite compact subset of \( \varrho G \).

Let \( R = Q \cap G \). Clearly, \( R \subset C \) and \( P = \pi(R) \). It follows from the definition of \( Q \) and \( R \) that

\[
Q = R + K = R \cup (R + x_0).
\]

Since \( Q \) is countable, infinite, and compact, it contains a nontrivial convergent sequence. By Proposition 2.1, the group \( G \) and its subspace \( R \) do not contain infinite compact subsets. The same is also valid for the subspace \( R + x_0 \) of \( \varrho G \). Therefore, since the group \( \varrho G \) is Boolean and \( Q = R \cup (R + x_0) \), there exists a sequence \( D = \{x_n : n \in \omega \} \subseteq R \) converging to an element \( y \in Q \setminus R = Q \setminus (Q \cap G) \). This contradicts the sequential completeness of \( G \) (see Proposition 2.1) and implies that all compact subsets of \( H \) are finite.
Finally, the group $H$ is pseudocompact as a continuous image of the pseudocompact group $G$. Hence the reflexivity of $H$ follows from Theorem 2.3. 

In the following theorem we extend the frontiers of the class of reflexive precompact groups.

**Theorem 2.9.** There exist precompact reflexive groups which are not pseudocompact.

*Proof.* Let $G$ be any infinite pseudocompact Boolean group all countable subgroups of which are $h$-embedded (Theorem 2.4). Take a countable infinite subgroup $C$ of $G$ and pick a point $x_0 \in \overline{C} \setminus C$, where the closure is taken in the completion $\varrho G$ of $G$. Then the group $K = \langle x_0 \rangle$ has a trivial intersection with $G$. Let $\pi : \varrho G \to \varrho G/K$ be the canonical projection. By Lemma 2.8 the subgroup $H = \pi(G)$ of the compact group $\varrho G/K$ is reflexive. Hence its dual $D = H^\sim$ is reflexive as well.

We claim that the group $D$ is not pseudocompact. Indeed, otherwise item a) of Lemma 2.6 would imply that all countable subgroups of the dual group $D^\sim \cong H$ were $h$-embedded in $H$. However, our choice of $x_0 \in \overline{C} \setminus C$ together with the fact that the intersection $K \cap G$ is trivial imply that the topology of $\pi(C)$ is strictly coarser than the topology of $C$ or, in other words, the restriction of $\pi$ to $C$ is not open when considered as a mapping of $C$ onto $\pi(C)$. Indeed, by [17, 1.3], this restriction is open if and only if the intersection $K \cap C$ is dense in $K$, which is not our case. Thus the maximal precompact topology on the abstract group $\pi(C)$ is strictly finer than the topology of $\pi(C)$ inherited from $H = \pi(G)$ and, hence, $\pi(C)$ is not $h$-embedded in $H$. This finishes the proof.

According to Theorem 2.4 and Theorem 2.5, there are reflexive groups which are pseudocompact noncompact. By Theorem 2.9, there are reflexive groups which are precompact non-pseudocompact. It is natural to ask, therefore, if there exist countably compact noncompact reflexive groups. We show in Proposition 2.10 below that, consistently, the answer is “yes”.

**Proposition 2.10.** Under Martin’s Axiom, every Boolean group $G$ of cardinality $\mathfrak{c}$ admits a reflexive countably compact noncompact topological group topology.

*Proof.* Let $G$ be a Boolean group with $|G| = \mathfrak{c}$. It was established in the proof of [12, Theorem 3.9] that, under the assumption of Martin’s Axiom, there exists an (abstract) monomorphism $h : G \to \mathbb{Z}(2)^\mathfrak{c}$ satisfying the following conditions:

(a) every infinite subset $S$ of $h(G)$ is finally dense in $\mathbb{Z}(2)^\mathfrak{c}$, i.e., there exists $\alpha < \mathfrak{c}$ such that $\pi_{\mathfrak{c} / \alpha}(S)$ is dense in $\mathbb{Z}(2)^{\mathfrak{c} / \alpha}$, where $\pi_A : \mathbb{Z}(2)^\mathfrak{c} \to \mathbb{Z}(2)^A$ is the projection for any set $A \subset \mathfrak{c}$;

(b) $\pi_\alpha(h(G)) = \mathbb{Z}(2)^\alpha$, for every $\alpha < \mathfrak{c}$.

Let

$$\mathcal{T} = \{h^{-1}(U \cap h(G)) : U \text{ is open in } \mathbb{Z}(2)^\mathfrak{c}\}.$$ 

Then $\mathcal{T}$ is a Hausdorff topological group topology on $G$ and $h$ is a topological isomorphism of $H = (G, \mathcal{T})$ onto the subgroup $h(G)$ of $\mathbb{Z}(2)^\mathfrak{c}$. It was proved in [12, Theorem 3.9], using (a) and (b), that the group $H$ is countably compact.

We claim that all compact subsets of $H$ are finite. Suppose to the contrary that $F$ is an infinite compact subset of $H$. According to (a), there exists $\alpha < \mathfrak{c}$ such that $\pi_{\mathfrak{c} / \alpha}(F)$ is dense in $\mathbb{Z}(2)^{\mathfrak{c} / \alpha}$. Since $F$ is compact, we have that $\pi_{\mathfrak{c} / \alpha}(F) = \mathbb{Z}(2)^{\mathfrak{c} / \alpha}$. The latter is impossible since $|F| \leq \mathfrak{c}$ while $|\mathbb{Z}(2)^{\mathfrak{c} / \alpha}| = |\mathbb{Z}(2)^{\mathfrak{c}}| = 2^\mathfrak{c} > \mathfrak{c}$.
Thus, $H$ is a countably compact, noncompact Abelian group without infinite compact subsets. The reflexivity of $H$ follows from Theorem 2.3.

The countably compact group $H = (G, T)$ in Proposition 2.10 does not contain infinite compact subsets. It is an open problem still whether there exist in $\mathsf{ZFC}$ infinite countably compact groups without non-trivial convergent sequences. This makes it interesting to find out whether countably compact noncompact reflexive groups can exist in $\mathsf{ZFC}$ alone. Very recently this question was solved in the positive in [15].

3. Some properties of the class $\mathcal{P}_h$

Recall that $\mathcal{P}_h$ is the class of pseudocompact Abelian groups whose countable subgroups are $h$-embedded. We proved in Theorem 2.4 that the groups in this class are reflexive, and we will now establish some other properties of this class concerning the duality theory.

**Theorem 3.1.** If a topological group $G$ is in $\mathcal{P}_h$, so is $G^\wedge$.

**Proof.** Let $G \in \mathcal{P}_h$. Then $G$ is reflexive, by Theorem 2.4. Hence, according to item b) of Proposition 2.7, the topology $\omega(G^\wedge, G^{\wedge\wedge}) = \omega(G^\wedge; G)$ is pseudocompact, i. e., the group $G^\wedge$ is pseudocompact. The countable subgroups of $G^\wedge$ are $h$-embedded in $G^\wedge$, by Lemma 2.6. □

The following concept was introduced in [18]. A topological group $G$ is called countably pseudocompact if for every countable set $A \subset G$, there exists a countable set $B \subset G$ such that $A \subset B$ and $B$ is pseudocompact. It is easy to see that every countably pseudocompact group is pseudocompact, but not vice versa. Hernández and Macario proved in [18, Corollary 4.3] that for every countably pseudocompact Abelian group $G$, the dual group $G^\wedge$ endowed with the topology of pointwise convergence on elements of $G$ is a $\mu$-space, i. e., the closure of every functionally bounded subset of $(G^\wedge, \tau_p)$ is compact. In the same article they also raised the problem as to whether the conclusion remained valid for pseudocompact groups. We solve the problem in the negative:

**Corollary 3.2.** There exists a pseudocompact Abelian group $G$ such that the dual group $G^\wedge$ is pseudocompact and noncompact. Hence $G^\wedge$ fails to be a $\mu$-space.

**Proof.** Take any infinite group $G \in \mathcal{P}_h$ (see Theorem 2.5). By Theorem 3.1 the dual group $G^\wedge$ is in $\mathcal{P}_h$ as well, so $G^\wedge$ is pseudocompact. Clearly, $G^\wedge$ is not compact — otherwise the bidual group $G^{\wedge\wedge}$ would be discrete. The latter is impossible since, by Theorem 2.3, the groups $G^{\wedge\wedge}$ and $G$ are topologically isomorphic. □

Finally, we establish that the class $\mathcal{P}_h$ is wide and that the operation of taking quotient groups completely destroys reflexivity, even if the kernels of the respective quotient homomorphisms are pseudocompact.

**Theorem 3.3.** If $G$ is a pseudocompact Abelian group, there is a reflexive group $H \in \mathcal{P}_h$ such that $G \cong H/L$, with $L$ a closed reflexive pseudocompact subgroup of $H$.

**Proof.** By [11] Theorem 5.5, for every pseudocompact Abelian group $G$, one can find a pseudocompact sequentially complete Abelian group $H$ and a closed pseudocompact subgroup $L$ of $H$ such that the group $G$ is topologically isomorphic to the quotient group $H/L$. 

Sequential completeness of $H$ follows from a stronger property of $H$, namely, its countable subgroups are $h$-embedded. This was explicitly shown in the proof of [11, Theorem 5.5]. Hence $H \in \mathcal{P}_h$. Finally, the group $H$ and its closed pseudocompact subgroup $L$ are reflexive according to Theorem 2.4.

It is worth comparing the above result with Theorem 2.6 from [4] saying that taking quotients with respect to compact subgroups preserves reflexivity.

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