REFINEMENTS OF TOPOLOGICAL INVARIANTS OF FLOWS

Tomoo Yokoyama*

Applied Mathematics and Physics Division, Gifu University
Yanagido 1-1, Gifu, 501-1193, Japan

(Communicated by Shaobo Gan)

Abstract. We construct topological invariants, called abstract weak orbit spaces, of flows and homeomorphisms on topological spaces. In particular, the abstract weak orbit spaces of flows on topological spaces are refinements of Morse graphs of flows on compact metric spaces, Reeb graphs of Hamiltonian flows with finitely many singular points on surfaces, and the CW decompositions which consist of the unstable manifolds of singular points for Morse flows on closed manifolds. Though the CW decomposition of a Morse flow is finite, the intersection of the unstable manifold and the stable manifold of closed orbits need not consist of finitely many connected components. Therefore we study the finiteness. Moreover, we consider when the time-one map reconstructs the topology of the original flow. We show that the orbit space of a Hamiltonian flow with finitely many singular points on a compact surface is homeomorphic to the abstract weak orbit space of the time-one map by taking an arbitrarily small reparametrization and that the abstract weak orbit spaces of a Morse flow on a compact manifold and the time-one map are homeomorphic. In addition, we state examples whose Morse graphs are singletons but whose abstract weak orbit spaces are not singletons.

1. Introduction. Birkhoff introduced the concepts of non-wandering points and recurrent points [5]. Using these concepts, we can describe and capture sustained or stationary dynamical behaviors and conservative dynamics. Moreover, Conley [11] defined a weak form of recurrence, called chain recurrence, for a flow on a compact metric space. The Conley theory states that dynamical systems on compact metric spaces can be decomposed into blocks, each of which is a chain recurrent one or a gradient one. Then this decomposition implies a directed graph, called a Morse graph, which can capture the gradient behaviors. The vertices, called Morse sets, correspond to recurrent parts, and the edges correspond to gradient parts. Moreover, the Conley indices of the Morse sets are used to capture information about the local dynamics near the Morse sets. Since the Conley-Morse graphs (i.e. Morse graph with Conley indices) represent topological behaviors of dynamical systems, applying graph algorithms, one can analyze the dynamics automatically if the Morse graph is finite. The finite representability of Conley-Morse graphs is crucial for graph algorithms because only finite data can be computed by algorithms.

2020 Mathematics Subject Classification. Primary: 37B35, 37C55; Secondary: 37J46, 37E35, 54B15.

Key words and phrases. Morse graph, topological invariant, quotient space.

This work was partially supported by the JST PRESTO Grant Number JPMJPR16E and JSPS Kakenhi Grant Number 20K03583, 18H01136.

*Corresponding author: Tomoo Yokoyama.
In fact, the Conley-Morse graphs are implemented as a computer software CHomP (Computational Homology Project software) [2], and there are several relative works for analyzing dynamical systems using algorithms [15, 25, 27]. The finite realizability of the orbit class spaces is studied [7, 8]. Therefore the question of how to reduce dynamical systems into finite topological invariants is essential from a theoretical and application point of view. On the other hand, Hamiltonian flows, which are typical examples of recurrent dynamics, with finitely many singular points on compact surfaces can be classified by Reeb graphs of Hamiltonians with information about critical levels, because Reeb graphs are the dual graphs of the multi-saddle connection diagrams (i.e. the union of multi-saddles and orbits from or to multi-saddles) and the complement of the multi-saddle connection diagrams consist of periodic annuli. Here a periodic annulus is an annulus consisting of periodic orbits. In fact, Hamiltonian flows with finitely many singular points on closed surfaces are classified by a molecular, which is a finite labelled graph [6]. The set of structurally stable Hamiltonian flows on compact surfaces is open dense in the set of Hamiltonian flows, and such Hamiltonian flows are characterized as Hamiltonian flows with nondegenerate singular points and with self-connected saddle connections and are classified by the multi-saddle connection diagrams, which are finite invariants [20]. Similar results hold for the non-compact surfaces [45].

Thom proved that the unstable manifolds of the gradient flow of a Morse function are open cells [40]. Smale proved that the gradient flow of a generic Morse function is a Morse flow [38]. From Milnor’s work [24], Franks extracted that the set of the unstable manifolds of singular points of a Morse flow on a closed manifold is a CW decomposition up to homotopy equivalence [13]. As Abbondandolo and Majer pointed out (see “Some history” in [1] for details), from the works in [4, 16, 19, 32, 36, 37], it is possible to deduce that the set of the unstable manifolds of singular points of a Morse flow on a closed manifold is a finite CW decomposition (cf. [1, Theorem 1]).

It is known that finite topological spaces are in a one-to-one correspondence with finite pre-ordered sets via the specialization orders. Here the specialization order $\leq$ is defined by $x \leq y$ if $x \in \{y\}$, where the closure of a subset $A$ is denoted by $\overline{A}$. In particular, finite $T_0$ spaces are in a one-to-one correspondence with finite posets via the specialization orders. McCord showed that there is a correspondence between finite topological spaces and finite simplicial complex up to weak homotopy equivalence [22]. Thus one can analyze topologies of dynamical systems using finite quotient spaces. For instance, the relation between Hasse diagrams and orbit class spaces was studied [8]. Note that a Morse graph is a quotient space of the orbit space with the directed structure. In other words, the orbit space is a covering space of the underlying space of a Morse graph. To analyze both gradient parts and recurrent parts of dynamical systems and describe global dynamics in detail, we define a directed structure on the orbit space, and construct topological invariants of flows on topological spaces which are refinements of Morse graphs of flows on metric spaces, CW decompositions of Morse flows on closed manifolds, and Reeb graphs of Hamiltonian flows on surfaces. We state triviality of the Morse graphs of chaotic flows in the sense of Devaney, Hamiltonian flows on compact surfaces, pseudo-Anosov homeomorphisms, and non-identical non-pointwise periodic non-minimal volume-preserving On the other hand, the topological invariants of them are not singletons as shown in § 7.
Whitney Embedding Theorem says that a generic map from an $n$-dimensional manifold to $2n + 1$ dimensional Euclidean space is an embedding [42]. Moreover, Takens embedding theorem states that a generic diffeomorphism can be reconstructed from time series of observables [39]. In addition, applications of embedding time-series data have been extensive [17, 31]. These results imply that generic discrete dynamical systems can be reconstructed by observables. On the other hand, a flow need not be reconstructed by observables of the time-one map. In other words, the time-one map of a flow need not have the information of the original flow. Since practically physical observations can capture only discrete data from continuous physical phenomena in various cases, we would like to ask when the time-one map of a flow know the topology of the original flow. More precisely, when does the orbit space $X/v$ of a flow $v$ on a topological space $X$ become the orbit space $X/v_1$ of the time-one map $v_1$ of $v$? In other words, we consider the following problem.

**Problem.** Find an equivalence relation $\sim_{v_1}$ generated by the time-one map $w_1$ of a flow $w$ and a large class $C$ of flows such that, for any flow $v \in C$ on a topological space $X$, the orbit space $X/v$ is homeomorphic to the quotient space $X/ \sim_{v_1}$ of the time-one map $v_1$ of the flow $v$, where the orbit space $X/v$ is the quotient space collapsing orbits into singletons.

Since the orbit space of a flow is not any quotient space of a the orbit space of the time-one map in general, we also consider the following weak form of the previous problem: When does the orbit space $X/v$ of a flow $v$ on a topological space $X$ become the orbit space $X/v_1$ of the time-one map $v_1$ of $v$ after collapsing operation $\approx$ for orbit spaces? More precisely, when is $(X/v)/ \approx$ homeomorphic to $(X/v_1)/ \approx$? Here $(X/v)/ \approx$ (resp. $(X/v_1)/ \approx$) is the quotient space of the orbit space $X/v$ (resp. $X/v_1$) collapsed by the operation $\approx$. In other words, we consider the following problem.

**Problem.** Find a small equivalence relation $\approx_w$ generated by dynamical systems $w$ and a large class $C$ of flows such that, for any flow $v \in C$ on a topological space $X$, the quotient space $X/ \approx_w$ is homeomorphic to the quotient space $X/ \approx_{v_1}$, where $v_1: X \to X$ is the time-one map of $v$.

In general, the time-one map of a flow loses topological information of the original flow. However, in some cases, the time-one map has the topological information of the original flow and so we can reconstruct original flows in some classes. In fact, the equivalence relation canonically induced by the abstract weak orbit space is one of the desired relations. For instance, on Problem 1, we show that the orbit space of a Hamiltonian flow with finitely many singular points on a compact surface is homeomorphic to the abstract weak orbit space of the time-one map up to an arbitrarily small reparametrization (see Theorem 6.3). On Problem 1, we show that the abstract weak orbit spaces of a Morse flow on a compact manifold and the time-one map are homeomorphic (see Theorem 6.5).

The present paper consists of eight sections. In the next section, as preliminaries, we introduce the notions of combinatorics, topology, dynamical systems, and decomposition theory. In particular, we introduce topological invariants of flows, called abstract weak orbit space and abstract orbit space. In §3, we show that CW decompositions of unstable manifolds of singular points of Morse flows on closed manifolds are quotient spaces of the abstract weak orbit spaces. In §4, we state that the Morse graph is a reduction of the abstract (weak) orbit space. In §5, we describe that the abstract weak orbit spaces of Hamiltonian surfaces flows with
finitely many singular points can be reduced into the Reeb graphs. In §6, we re-
construct orbit spaces and abstract weak orbit spaces of flows from the time-one
maps. In the final section, we state triviality, non-triviality, and incompleteness of
abstract weak orbit spaces using several examples. In particular, the Morse graphs
in Examples 2–9 are singletons but the abstract weak orbit spaces are not singletons.

2. Preliminaries.

2.1. Combinatorial and topological notions.

2.1.1. Decomposition. By a decomposition, we mean a family $\mathcal{F}$ of pairwise disjoint
nonempty subsets of a set $X$ such that $X = \bigsqcup \mathcal{F}$, where $\bigsqcup$ denotes a disjoint union. Note
that the sets of orbits of flows are decompositions of connected elements.
Since connectivity is not required, the sets of orbits of homeomorphisms are also
decompositions.

2.1.2. Elements of topological notation. For a subset $A$ of a topological space, denote
by $\overline{A}$ the closure of $A$. Then $\partial A := \overline{A} \setminus \text{int}A$ is the boundary of $A$. Here $B \setminus C$ is
used instead of $B \setminus C$ when $C \subseteq B$.

2.1.3. Separation axiom. A point $x$ of a topological space $X$ is $T_0$ (or Kolmogorov)
if for any point $y \neq x \in X$ there is an open subset $U$ of $X$ such that $|\{x, y\} \cap U| = 1$,
where $|A|$ is the cardinality of a subset $A$. A point is $T_1$ if its singleton is closed.
Here a singleton is a set consisting of a point. A topological space is $T_0$ (resp. $T_1$)
if each point is $T_0$ (resp. $T_1$).

2.1.4. Graphs. An ordered pair $G := (V, D)$ is an abstract directed graph (or a
directed graph) if $V$ is a set and $D \subseteq V \times V$. An ordered triple $G := (V, E, r)$ is an
abstract multi-graph (or a multi-graph) if $V$ and $E$ are sets and $r : E \rightarrow \{\{x, y\} \mid x, y \in V\}$.

2.1.5. Orders. A binary relation $\leq$ on a set $P$ is a pre-order if it is reflexive (i.e.
$a \leq a$ for any $a \in P$) and transitive (i.e. $a \leq c$ for any $a, b, c \in P$ with $a \leq b$ and
$b \leq c$). For a pre-order $\leq$, the inequality $a < b$ means both $a \leq b$ and $a \neq b$.
A pre-order $\leq$ on $X$ is a partial order if it is antisymmetric (i.e. $a = b$ for any $a, b \in P$
with $a \leq b$ and $b \leq a$). A poset is a set with a partial order. A pre-order $\leq$
is a total order (or linear order) if either $a < b$ or $b < a$ for any points $a \neq b$. Note
that a poset is totally ordered if and only if there is no pair of two incomparable
points. A chain is a totally ordered subset of a pre-ordered set with respect to the
induced order.

2.1.6. Heights of points. Let $(X, \leq)$ be a pre-ordered set. For a point $x \in X$,
define the upset $\uparrow_{\leq} x = \uparrow x := \{y \in X \mid x \leq y\}$, the downset $\downarrow_{\leq} x = \downarrow x :=
\{y \in X \mid y \leq x\}$, and the class $\hat{x} := \downarrow x \cap \uparrow x$. For a subset $A \subseteq X$, define the
upset $\uparrow_{\leq} A = \uparrow A := \bigcup_{x \in A} \uparrow x$, the downset $\downarrow_{\leq} A = \downarrow A := \bigcup_{x \in A} \downarrow x$, and the class
$\hat{A} := \bigcup_{x \in A} \hat{x}$. Define the height $\text{ht} x$ of $x$ by $\text{ht}_{\leq} x = \text{ht} x := \sup\{\lvert C \rvert - 1 \mid C :\text{chain containing } x \text{ as the maximal point}\}$. Define the height of the empty set is
$-1$. The height $\text{ht} A$ of a nonempty subset $A \subseteq X$ is defined by $\text{ht}_{\leq} A = \text{ht} A :=
\sup_{x \in A} \text{ht} x$. 

2.1.7. Multi-graphs as posets. A poset $P$ is said to be multi-graph-like if the height of $P$ is at most one and $|\downarrow x| \leq 3$ for any element $x \in P$. For a multi-graph-like poset $P$, each element of $P_0$ is called a vertex and each element of $P_1$ is called an edge. Then an abstract multi-graph $G$ can be considered as a multi-graph-like poset $(P,\leq_G)$ with $V = P_0$ and $E = P_1$ as follows: $P = V \sqcup E$ and $e \leq_G e$ if $x \in r(e)$, where $\sqcup$ denotes a disjoint union. Conversely, a multi-graph-like poset $P$ can be considered as an abstract multi-graph with $V = P_0$, $E = P_1$, and $r : P_1 \rightarrow \{\{x, y\} | x, y \in V\}$ defined by $r(e) := \downarrow e - \{e\}$. Therefore we identify a multi-graph-like poset with an abstract multi-graph.

2.1.8. Specialization order. For a subset $A$ and a point $x$ of a topological space $(X,\tau)$, an abbreviated form of the singleton $\{x\}$ (resp. the difference $A - \{x\}$, the point closure $\overline{\{x\}}$) will be $x$ (resp. $A - x$, $\overline{x}$). The specialization order $\leq_\tau$ on a topological space $(X,\tau)$ is defined as follows: $x \leq_\tau y$ if $x \in \overline{y}$. Note that $\downarrow x = \overline{x}$ and that the set $\min X$ of minimal points in $X$ is the set of points whose classes are closed. Then $\hat{x} = \{y \in X | \overline{x} = \overline{y}\}$ for any point $x \in X$.

2.1.9. $T_0$-tication of a topological space. Let $X$ a topological space with the specialization order. Then the set $\hat{X} := \{\hat{x} | x \in X\} = \{y \in X | \overline{x} = \overline{y}\}$ of classes is a decomposition of $X$ and is a $T_0$ space as a quotient space, which is called the $T_0$-tication (or Kolmogorov quotient) of $X$.

2.1.10. Reeb graph of a function on a topological space. For a function $f : X \rightarrow \mathbb{R}$ on a topological space $X$, the Reeb graph of a function $f : X \rightarrow \mathbb{R}$ on a topological space $X$ is a quotient space $X/\sim_{\text{Reeb}}$ defined by $x \sim_{\text{Reeb}} y$ if there are a number $c \in \mathbb{R}$ and a connected component of $f^{-1}(c)$ which contains $x$ and $y$.

2.2. Fundamental notion of dynamical systems. A mapping $v : \mathbb{R} \times X \rightarrow X$ on a topological space $X$ is an $\mathbb{R}$-action on $X$ if the restriction $v(t, \cdot) : X \rightarrow X$ by $v(t, \cdot)(x) := v(t, x)$ for any $t \in \mathbb{R}$ is homeomorphic such that the restriction $v(0, \cdot)$ is an identity mapping on $X$ and $v(t, v(s, x)) = v(t + s, x)$ for any $t, s \in \mathbb{R}$. A flow is a continuous $\mathbb{R}$-action on a topological space. A flow $w$ is a reparametrization of a flow $v$ if $O_v(x) = O_w(x)$ for any $x \in X$. Let $v : \mathbb{R} \times X \rightarrow X$ be a flow on a topological space $X$. For $t \in \mathbb{R}$, define $v_t : X \rightarrow X$ by $v_t := v(t, \cdot)$. The homeomorphism $v_1 : X \rightarrow X$ by $v_1(x) = v(1, x)$ is called the time-one map of $v$. A subset of $X$ is invariant (or saturated) if it is a union of orbits. For a subset $A \subseteq X$, the saturation $v(A)$ of $A$ is the union $\bigcup_{x \in A} O(x)$. A nonempty closed invariant subset $A$ is minimal if there are no proper nonempty closed invariant subsets. $A$ is minimal with respect to the inclusion order. Recall that a point $x$ of $X$ is singular if $x = v_t(x)$ for any $t \in \mathbb{R}$ and is periodic if there is a positive number $T > 0$ such that $x = v_T(x)$ and $x \neq v_t(x)$ for any $t \in (0, T)$. An orbit is singular (resp. periodic) if it contains a singular (resp. periodic) point. An orbit is closed if it is singular or periodic. Denote by $\text{Sing}(v)$ the set of singular points and by $\text{Per}(v)$ (resp. $\text{Cl}(v)$) the union of periodic (resp. closed) orbits. A point is wandering if there are its neighborhood $U$ and a positive number $N$ such that $v_t(U) \cap U = \emptyset$ for any $t > N$. Then such a neighborhood $U$ is called a wandering domain. A point is non-wandering if it is not wandering (i.e. for any its neighborhood $U$ and for any positive number $N$, there is a number $t \in \mathbb{R}$ with $|t| > N$ such that $v_t(U) \cap U \neq \emptyset$). Denote by $\Omega(v)$ the set of non-wandering points, called the non-wandering set.
2.2.1. Chain recurrence. Let \( w: \mathbb{R} \times M \to M \) be a flow on a metric space \((M,d)\). For any \( \varepsilon > 0 \) and \( T > 0 \), a pair \( \{(x_i)_{i=0}^{k+1}, (t_i)_{i=0}^{k}\} \) is an \((\varepsilon,T)\)-chain from a point \( x \in M \) to a point \( y \in M \) if \( x_0 = x \), \( x_{p+1} = y \), \( t_i > T \) and \( d(w_{t_i}(x_i), x_{i+1}) < \varepsilon \) for any \( i = 0, \ldots, k \). Define a binary relation \( \sim_{CR} \) on \( M \) by \( x \sim_{CR} y \) if for any \( \varepsilon > 0 \) and \( T > 0 \) there is an \((\varepsilon,T)\)-chain from \( x \) to \( y \). A point \( x \in M \) is chain recurrent \([12]\) if \( x \sim_{CR} x \). Denote by \( CR(w) \) the set of chain recurrent points, called the chain recurrent set. It is known that the chain recurrent set \( CR(w) \) of a flow on a compact metric space is closed and invariant and contains the non-wandering set \( \Omega(w) \) \([11, \text{Theorem 3.3B}]\), and that connected components of \( CR(w) \) are equivalence classes of the relation \( \approx_{CR} \) on \( CR(w) \) \([11, \text{Theorem 3.3C}]\), where \( x \approx_{CR} y \) if \( x \sim_{CR} y \) and \( y \sim_{CR} x \).

From now on, we assume that \( v \) is a flow on a topological space \( X \) unless otherwise stated.

2.2.2. \( \alpha \)-limit sets, \( \omega \)-limit sets, and recurrent orbits. Recall that the \( \omega \)-limit set of a point \( x \in X \) is \( \omega(x) := \bigcap_{n \in \mathbb{N}} v_n(x) \mid t > n \big] \), and that the \( \alpha \)-limit set of \( x \) is \( \alpha(x) := \bigcap_{n \in \mathbb{N}} \{v_n(x) \mid t < n \} \). By definitions, the \( \alpha \)-limit set and the \( \omega \)-limit set of \( x \) are closed and invariant. We have the following observation.

**Lemma 2.1.** Let \( v \) be a flow on a Hausdorff space \( X \). The following statements hold for any point \( x \in X \):
1. If \( x \in Cl(v) \), then \( \overline{O(x)} = O(x) \).
2. If \( x \in Cl(v) \), then \( \overline{O(x)} = O(x) \).

**Proof.** By definition of \( \alpha \)-limit set and \( \omega \)-limit set, the orbit closure \( \overline{O(x)} \) contains \( \alpha(x) \cup O(x) \cup \omega(x) \). Define a continuous mapping \( v_x: \mathbb{R} \to X \) by \( v_x := v(\cdot, x) \). Since a closed interval \( I \subset \mathbb{R} \) is compact, the image \( v_x(I) \) is closed. The Hausdorff separation axiom of \( X \) implies that the image of a closed interval by \( v_x \) is closed. This implies the assertion (2). Assume that there is a point \( y \in \overline{O(x)} - (\alpha(x) \cup O(x) \cup \omega(x)) \). By \( y \notin \alpha(x) \cup O(x) \cup \omega(x) \), there is a number \( T > 0 \), such that \( y \notin v_x([R_{<T}]) \) and \( y \notin v_x([R_{>T}]) \). Then \( y \notin v_x([R_{<T} - [-T,T]]) \) and so there is an open neighborhood \( U \) of \( y \) such that \( U \cap v_x([R_{<T} - [-T,T]]) = \emptyset \). Since the image \( v_x([-T,T]) \subseteq O(x) \) of the interval \([-T,T] \) is closed and \( y \notin O(x) \), the difference \( V := U \setminus v_x([-T,T]) \) is an open neighborhood of \( y \). Then \( V \cap O(x) = (U \setminus v_x([-T,T])) \cap O(x) = U \cap v_x([-T,T]) = U \cap v_x(R_{<T}) = \emptyset \) and so \( y \notin \overline{O(x)} \), which contradicts \( y \in \overline{O(x)} \). Thus \( \overline{O(x)} = \alpha(x) \cup O(x) \cup \omega(x) \). \( \square \)

Define \( \alpha(A) := \bigcup_{x \in A} \alpha(x) \) and \( \omega(A) := \bigcup_{x \in A} \omega(x) \) for a subset \( A \) of \( X \). By definition, we obtain that \( \alpha(O) = \alpha(x) \) and \( \omega(O) = \omega(x) \) for any point \( x \in O \). A separatrix is a non-singular orbit whose \( \alpha \)-limit or \( \omega \)-limit set is a singular point. A point \( x \) is recurrent if \( x \in \alpha(x) \cup \omega(x) \). An orbit is recurrent if it contains a recurrent point. Denote by \( R(v) \) the set of recurrent points. By definition, we have that \( Cl(v) \subseteq R(v) \subseteq \Omega(v) \). Moreover, denote by \( R(v) \) (resp. \( P(v) \)) the union of non-closed recurrent orbits (resp. non-recurrent orbits). By definition, for a flow \( v \) on a topological space, we have a decomposition \( X = R(v) \cup P(v) = Sing(v) \cup Per(v) \cup P(v) \cup R(v) \).

2.2.3. \( \alpha' \)-limit sets and \( \omega' \)-limit sets. Define \( \alpha'(x) \) (resp. \( \omega'(x) \)) for a point \( x \in X \) as follows \([9, 21]\):

\[
\alpha'(x) := \alpha(x) \setminus O(x)
\]

\[
\omega'(x) := \omega(x) \setminus O(x)
\]
2.3. Quotient spaces and binary relations for flows.

2.3.1. Orbit classes. For any point \( x \in X \), define the orbit class \( \hat{O}(x) \) as follows:

\[
\hat{v}(x) = \hat{O}(x) := \{ y \in X \mid \overline{O(x)} = \overline{O(y)} \} \subseteq \overline{O(x)}
\]

We call \( \hat{O}(x) \) the orbit class of \( x \). Note that \( \hat{O}(x) = \hat{O}(y) \) for any point \( y \in \hat{O}(x) \). It’s known that the following conditions are equivalent for an orbit \( O \) on a paracompact manifold: (1) The orbit \( O \) is embedded; (2) \( O = \hat{O} \) [44, Corollary 3.4].

2.3.2. Orbit spaces and orbit class spaces. For a flow \( v \) on a topological space \( X \), the orbit space \( X/v \) of \( X \) is a quotient space \( X/\sim_v \) defined by \( x \sim_v y \) if \( O(x) = O(y) \). Similarly, the orbit class space \( X/\hat{v} \) of \( X \) is a quotient space \( X/\sim_{\hat{v}} \) defined by \( x \sim_{\hat{v}} y \) if \( \overline{O(x)} = \overline{O(y)} \). Note that the orbit class space \( X/\hat{v} \) is the refinement of the orbit space \( X/v \). The orbit class space is also called the quasi-orbit space in [7]. For a saturated subset \( T \) of \( X \), the subset \( \pi_v(T) \) is denoted by \( T/v \), where \( \pi_v : X \to X/v \) is the quotient map. Notice that an orbit space \( T/v \) is the set of orbits contained in \( T \) as a set.

2.3.3. Morse graph. Recall that a directed graph is a pair of a set \( V \) and a subset \( D \subseteq V \times V \). For a flow \( w \) on a compact metric space \( M \) with a set \( \mathcal{M} = \{ M_i \}_{i \in \Lambda} \) of pairwise disjoint compact invariant subsets, a directed graph \((V,D)\) with the vertex set \( V := \{ M_i \mid i \in \Lambda \} \) and with the directed edge set \( D := \{ (M_j, M_k) \mid D_{j,k} \neq \emptyset \} \) is a Morse graph of \( \mathcal{M} \) if \( M - \bigcup_j M_i = \bigcup D_{j,k} \), where \( D_{j,k} := \{ x \in M \mid \alpha(x) \subseteq M_j, \omega(x) \subseteq M_k \} = W^u(M_j) \cap W^s(M_k) \) for any distinct indices \( j \neq k \). Then such a graph is denoted by \( G_{\mathcal{M}} \), and \( D_{j,k} \) is called a connecting orbit set from \( M_j \) to \( M_k \). We also call that the vertex \( M_k \) is a connecting orbit set from \( M_j \) to \( M_k \).

If \( \mathcal{M} \) is the set of connected components of the chain recurrent set \( CR(w) \), then the graph \( G_{\mathcal{M}} \) is called the Morse graph of the flow \( w \) and denoted by \( G_w \), and vertices are called Morse sets. We will show that a Morse graph \( \mathcal{M}_w \) of a flow \( w \) is a quotient space of the orbit space \( M/w \) with a directed structure (see Theorem 4.4 and Corollary 5 for details). To describe dynamics in detail, we will introduce an intermediate quotient spaces, called abstract weak orbit space and weak orbit class space, of the orbit space which are refinements of the Morse graph. In other words, the Morse graph is a quotient space of such quotient spaces.

2.3.4. The specialization orders of flows. The specialization order \( \leq_v \) on \( X/v \) of the flow \( v \) is defined as follows:

\[
O_1 \leq_v O_2 \text{ if } O_1 \subseteq O_2 \text{ as subsets of } X
\]

By abuse of terminology, define the specialization order \( \leq_v \) on \( X \) of the flow \( v \) defined as follows:

\[
x \leq_v y \text{ if } O(x) \subseteq \overline{O(y)}
\]

Moreover, the specialization (partial) order \( \leq_v \) on \( X/\hat{v} \) is induced as follows:

\[
\hat{O}_1 \leq_v \hat{O}_2 \text{ if } \hat{O}_1 \subseteq \overline{\hat{O}_2} \text{ as subsets of } X
\]
For any orbit $O$, since $O \subseteq \hat{O} \subseteq \bar{O}$, we have that $\hat{O} = \bar{O}$, and so that $\hat{O}_1 \leq_v \hat{O}_2$ if and only if $O_1 \subseteq \bar{O}_2$. Then we have the following observation.

**Lemma 2.2.** The following conditions are equivalent:

1. $O(x) \subseteq \hat{O}(y)$
2. $x \leq_v y$
3. $O(x) \leq_v O(y)$
4. $\hat{O}(x) \leq_v \hat{O}(y)$

Let $\tau_v$ be the quotient topology on the orbit space $X/v$ and $\tau_{\bar{v}}$ the quotient topology on the orbit space $X/\bar{v}$. Then the binary relation $\leq_v$ on the orbit space $X/v$ (resp. orbit class space $X/\bar{v}$) corresponds to the specialization order $\leq_{\tau_v}$ (resp. partial order $\leq_{\tau_{\bar{v}}}$).

2.3.5. **Height of an orbit space.** For a point $x \in X$, define the height of $x$ by the height of the element $O(x)$ in the orbit space $X/v$ and write $\mathrm{ht}(x) := \mathrm{ht}_{\tau_v}(O(x))$, where $\mathrm{ht}_{\tau_v}$ is the height with respect to the specialization order $\leq_{\tau_v}$ on the orbit space $X/v$ induced by $v$. Since the orbit class space $X/\bar{v}$ is the $T_0$-ification of the orbit space $X/v$, we have $\mathrm{ht}(x) = \mathrm{ht}_{\tau_v}(O(x)) = \mathrm{ht}_{\tau_{\bar{v}}}(\hat{O}(x))$. Moreover, we have the following observation.

**Lemma 2.3.** The following conditions are equivalent for a point $x \in X$:

1. $\mathrm{ht}(x) \geq k$.
2. There is a sequence $\bar{O}_0 \subseteq \bar{O}_1 \subseteq \cdots \subseteq \bar{O}_{k-1} \subseteq \bar{O}_k = \hat{O}(x)$ as subsets of $X$.
3. There is a sequence $\bar{O}_0 \subseteq \bar{O}_1 \subseteq \cdots \subseteq \bar{O}_{k-1} \subseteq \bar{O}_k = \hat{O}(x)$ as subsets of $X$.

The height $\mathrm{ht} A$ of a subset $A \subseteq X$ is defined by $\mathrm{ht} A := \sup_{x \in A} \mathrm{ht}(x)$. Define the height $\mathrm{ht} v := \mathrm{ht} X$ of the flow $v$ on a topological space $X$. The height of the flow $v$ is defined by $\mathrm{ht}(v) := \mathrm{ht}(X)$.

2.3.6. **Abstract weak orbits.** Define a saturated subset $[x]$ for a flow on a topological space $X$ as follows:

$$[x] = \begin{cases} 
\text{the connected component of } \text{Sing}(v) \text{ containing } x & \text{if } x \in \text{Sing}(v) \\
\text{the connected component of } \text{Per}(v) \text{ containing } x & \text{if } x \in \text{Per}(v) \\
\text{the connected component of } \{y \in P(v) \mid \omega(x) = \omega(y)\} & \text{if } x \in P(v) \\
\{y \in R(v) \mid \omega(x) = \omega(y)\} & \text{if } x \in R(v) 
\end{cases}$$

We call that $[x]$ is the abstract weak orbit of $x$. Note that $[x] = [y]$ for any point $y \in [x]$. In general, the closure of an abstract weak orbit is saturated but is not a union of abstract weak orbits. The abstract weak orbit of a subset $A$ is defined by $[A] := \bigcup_{a \in A} [a]$.

2.3.7. **Abstract orbits.** Define a saturated subset $\langle x \rangle$ of $X$ as follows:

$$\langle x \rangle = \begin{cases} 
\text{the connected component of } \text{Sing}(v) \text{ containing } x & \text{if } x \in \text{Sing}(v) \\
\text{the connected component of } \text{Per}(v) \text{ containing } x & \text{if } x \in \text{Per}(v) \\
\text{the connected component of } \{y \in P(v) \mid \omega(x) = \omega(y)\} & \text{if } x \in P(v) \\
\{y \in R(v) \mid O(x) = O(y)\} & \text{if } x \in R(v) 
\end{cases}$$
We call that \( \langle x \rangle \) is the abstract orbit of \( x \). Note that \( \langle x \rangle = \langle y \rangle \) for any point \( y \in \langle x \rangle \).

In general, the closure of an abstract orbit is saturated but is not a union of abstract orbits. The abstract orbit of a subset \( A \) is defined by \( \langle A \rangle := \bigcup_{a \in A} \langle a \rangle \). We have the following observations.

**Lemma 2.4.** Let \( v \) be a flow on a topological space \( X \). The following statements hold for any point \( x \in R(v) \):

1. If \( x \in \alpha(x) \), then \( \overline{O(x)} = \alpha(x) \).
2. \( \overline{O(x)} = \alpha(x) \cup \omega(x) = \alpha(y) \cup \omega(y) = \overline{O(y)} \) for any point \( y \in [x] \cup \langle x \rangle \).
3. \( [x] \subseteq \langle x \rangle \).

**Proof.** Fix any point \( x \in X \). Suppose that \( x \in \alpha(x) \). If \( x \in \text{Cl}(v) \), then \( \overline{O(x)} = O(x) = \alpha(x) \). Thus we may assume that \( x \notin \text{Cl}(v) \). This means that \( x \in R(v) \).

The closedness and invariance of \( \alpha(x) \) imply that \( \alpha(x) = \overline{O(x)} \).

Fix any point \( y \in [x] \cup \langle x \rangle \). Suppose that \( x \in R(v) \). By definition of abstract (weak) orbit, we have \( y \in R(v) \). The assertion (1) implies that \( \overline{O(x)} = \alpha(x) \cup \omega(x) \) and \( \overline{O(y)} = \alpha(y) \cup \omega(y) \). Therefore \( \overline{O(x)} = \alpha(x) \cup \omega(x) = \alpha(y) \cup \omega(y) = \overline{O(y)} \). This means that \( [x] \subseteq \langle x \rangle \).

**Corollary 1.** The abstract weak orbit of any point is contained in its abstract orbit.

Since any subset \( B \) with \( A \subseteq B \subseteq \overline{A} \) for some connected subset \( A \) is connected, Lemma 2.4 implies the following observation.

**Corollary 2.** Any abstract weak orbit and any abstract orbit of a point are connected.

The Hausdorff separation axiom implies the following observation.

**Lemma 2.5.** Let \( v \) be a flow on a Hausdorff space \( X \). For a point \( x \in R(v) \), we have \( [x] = \overline{O(x)} \).

**Proof.** By definition, we obtain \( O(x) \subseteq \langle x \rangle = \{ y \in R(v) \mid \overline{O(x)} = \overline{O(y)} \} \subseteq \{ y \in X \mid \overline{O(x)} = \overline{O(y)} \} = \overline{O(x)} \). If \( O(x) = \overline{O(x)} \), then \( \langle x \rangle = \overline{O(x)} \). Thus we may assume that \( \overline{O(x)} - O(x) \neq \emptyset \). Fix a point \( y \in \overline{O(x)} - O(x) \). We claim that \( y \in R(v) \). Indeed, Lemma 2.1 implies \( x \in \overline{O(x)} = \overline{O(y)} = \alpha(y) \cup O(y) \cup \omega(y) \). Since \( O(y) \cap O(x) = \emptyset \), we obtain \( x \in \alpha(y) \cup \omega(y) \). The closedness and invariance of \( \alpha \)-limit set and \( \omega \)-limit set of a point imply \( y \in \overline{O(y)} = \overline{O(x)} \subseteq \alpha(y) \cup \omega(y) \). This means that \( y \in R(v) \).

Therefore \( [x] = \{ y \in R(v) \mid \overline{O(x)} = \overline{O(y)} \} = \{ y \in X \mid \overline{O(x)} = \overline{O(y)} \} = \overline{O(x)} \).

2.3.8. Abstract weak orbit space and abstract orbit space of a flow. Define the abstract weak orbit space \( X/\sim[v] \) as a quotient space \( X/\sim[v] \) defined by \( x \sim[v] y \) if \( [x] = [y] \). Similarly, define the abstract orbit space \( X/\sim[v] \) as a quotient space \( X/\sim[v] \) defined by \( x \sim[v] y \) if \( \langle x \rangle = \langle y \rangle \). Since \( O(x) \subseteq [x] \subseteq \langle x \rangle \) for any \( x \in X \), the abstract weak orbit space is a quotient space of the orbit space, and the abstract orbit space is a quotient space of the abstract weak orbit space. We have the following observation.

**Lemma 2.6.** Let \( v \) be a flow on a topological space \( X \) with \( R(v) = \emptyset \). Then the abstract orbit and the abstract weak orbit of a point coincide with each other. Moreover, we obtain \( X/\sim[v] = X/\langle v \rangle \).

By definitions of abstract weak orbit space and abstract orbit space, Lemma 2.4 implies the following reductions as in Figure 1.
Corollary 3. Let $v$ be a flow on a Hausdorff space $X$. Then the abstract weak orbit space $X/\lbrack v \rbrack$ is a quotient space of the weak orbit class space $X/v$, and the abstract orbit space $X/\langle v \rangle$ is a quotient space of the orbit class space $X/\hat{v}$ and of the abstract weak orbit space $X/\lbrack v \rbrack$.

3. Refinement of the CW decomposition of unstable manifolds. In this section, we show that the abstract weak orbit space of a Morse flow on a closed manifold is a refinement of the CW decomposition of unstable manifolds of singular points.

3.1. Fundamental concepts and typical classes of dynamical systems. Recall several concepts to state refinements.

3.1.1. Topological equivalence. A flow $v$ on a topological space $X$ is topologically equivalent to a flow $w$ on a topological space $Y$ if there is a homeomorphism $h: X \to Y$ whose image of any orbit of $v$ is an orbit of $w$ and which preserves the direction of the orbits.

3.1.2. Limit cycles and limit circuits. A cycle is a periodic orbit. A limit cycle $O$ is a periodic orbit which is an $\omega$-limit set or $\alpha$-limit set (i.e. $O = \omega(x)$ or $O = \alpha(x)$) of a point $x \notin O$. A trivial circuit is a singleton. A non-trivial circuit is an immersed image of a circle. A circuit is a trivial or non-trivial circuit. A non-trivial circuit $\gamma$ of a flow $v$ is limit if it is either a limit cycle or a finite union of orbits in $\mathrm{Sing}(v) \sqcup \mathrm{P}(v)$ which is an $\omega$-limit set or ant $\alpha$-limit set (i.e. $\gamma = \omega(x)$ or $\gamma = \alpha(x)$) of a point $x \notin \gamma$. A limit circuit $\gamma$ is repelling (resp. attracting) if there is a neighborhood $U$ of $\gamma$ such that $\alpha(x) = \gamma$ (resp. $\omega(x) = \gamma$) for any $x \in U - \gamma$.

3.1.3. Flow of finite type. A singular point is quasi-nondegenerate if it has a neighborhood which intersects at most finitely many abstract weak orbits. A flow $v$ on a topological space $X$ is a flow of finite type on the topological space if it satisfies the following conditions:
(1) Any singular point is quasi-nondegenerate.
(2) There are at most finitely many limit cycles.
(3) Any recurrent orbit is closed (i.e. $X = \mathrm{Cl}(v) \sqcup \mathrm{P}(v)$).

When the whole space $X$ is a surface, we require quasi-regularity of singular points for definition of “of finite type on a surface” (see the details in 5.1.1). Lemma 2.6 implies the following statement.

Corollary 4. Let $v$ be a flow of finite type on a topological space $X$. Then the abstract orbit and the abstract weak orbit of a point of coincide with each other. Moreover, we obtain $X/\lbrack v \rbrack = X/\langle v \rangle$. 

3.1.4. Unstable subsets and stable subsets. For a subset $A$ of $X$, the unstable set $W^u(A)$ and the stable set $W^s(A)$ of $A$ are defined by $W^u(A) := \{ x \in X \mid \alpha(x) \subseteq A \}$ and $W^s(A) := \{ x \in X \mid \omega(x) \subseteq A \}$. The unstable set $W^u(A)$ is the unstable manifold if it is an immersed manifold (i.e. the inclusion mapping $W^u(A) \to X$ is an injective immersion). Here an immersion is a mapping between manifolds whose derivative at any point is injective. Similarly, the stable set $W^s(A)$ is the stable manifold if there is an immersion $W^s(A) \to X$ between manifolds.

3.1.5. Structural stability. For a subset $\chi$ of the set $\chi'(M)$ of $C^r$ vector fields for any $r \in \mathbb{Z}_{\geq 0}$ on a manifold $M$, a vector field $X \in \chi$ is structurally stable with respect to $\chi$ if there is a $C^1$ neighborhood $\mathcal{U} \subseteq \chi$ of $X$ such that any vector field $Y$ in $\mathcal{U}$ is topologically equivalent to $X$ (i.e. there is a homeomorphism $h: M \to M$ whose image of any orbit of $Y$ is an orbit of $X$ and which preserves the direction of the orbits).

3.1.6. Gradient flows. A $C^r$ vector field $X$ for any $r \in \mathbb{Z}_{\geq 0}$ on a Riemannian manifold $(M, g)$ is gradient if there is a $C^{r+1}$ function $h: M \to \mathbb{R}$ such that $X = -\text{grad}(h)$, where the gradient of $h$ is defined by $\text{grad}(h) = g(\partial h, \cdot)$. In other words, locally the gradient vector field $X$ is defined by $X := \sum_{i,j} g^{ij}(\partial_i h) \partial_j$ for any local coordinate system $(x_1, \ldots, x_n)$ of a point $p \in M$, where $\partial_i := \partial/\partial x_i$, $g_{ij} := g(\partial_i, \partial_j)$, and $(g^{ij}) := (g_{ij})^{-1}$. A flow is gradient if it is topologically equivalent to a flow generated by a gradient vector field.

3.1.7. Morse-Smale flows on close manifolds. A $C^r$ vector field $X$ for any $r \in \mathbb{Z}_{\geq 0}$ on a closed manifold is Morse-Smale if (1) the non-wandering set $\Omega(X)$ consists of finitely many hyperbolic closed orbits; (2) any point in the intersection of the stable and unstable manifolds of closed orbits are transversal (i.e $W^s(O) \cap W^u(O')$ for any orbits $O, O' \subset \Omega(X)$, where $W^s(O)$ is the stable manifold of $O$ and $W^u(O')$ is the unstable manifold of $O'$). Here the transversality of submanifolds $A$ and $B$ on a manifold $M$ means that the submanifolds $A$ and $B$ span the tangent spaces for $M$ (i.e. $T_{A \cap B}M = T_{A \cap B}A + T_{A \cap B}B$). Therefore we say that a flow on a closed manifold is Morse-Smale if it is topologically equivalent to a flow generated by a Morse-Smale vector. A flow on a closed manifold is Morse if it is a Morse-Smale flow without limit cycles. Palis and Smale showed that a Morse-Smale $C^r$ vector field on a closed manifold is structurally stable with respect to the set of $C^r$ vector fields [33, 34].

3.2. Refinements of CW decompositions of unstable manifolds of singular points of Morse flows. As mentioned above, it is known that the set of the unstable manifolds of singular points of a Morse flow on a closed manifold is a finite CW decomposition (cf. [1, Theorem 1]). We show that the abstract weak orbit spaces of Morse flows on closed manifold are refinements of CW decompositions of unstable manifolds of singular points.

Theorem 3.1. The CW decomposition of unstable manifolds of singular points of a Morse flow on a closed manifold is a quotient space of the abstract weak orbit space. Moreover, the set of connecting orbit sets of singular points is also a quotient space of the abstract weak orbit space.

Proof. Let $v$ be a Morse flow on a closed manifold $M$. [1, Theorem 1] implies that both the set $\{W^u(x) \mid x \in \text{Sing}(v)\}$ of unstable manifolds of singular points and the set $\{W^s(x) \mid x \in \text{Sing}(v)\}$ of stable manifolds of singular points are finite.
Consider a vector field $X$ on a closed three dimensional manifold $M$. Then $M = \bigcup_{x \in \text{Sing}(v)} W^u(x) = \bigcup_{x \in \text{Sing}(v)} W^s(x) = \bigcup_{x,y \in \text{Sing}(v)} W^u(x) \cap W^s(y)$. In particular, we obtain $W^u(x) = \bigcup_{y \in \text{Sing}(v)} W^u(x) \cap W^s(y)$ for any $x \in \text{Sing}(v)$. This implies that the $\omega$-limit sets and the $\alpha$-limit set of any point are singular points. Then $W^u(\alpha(x)) \cap W^s(\omega(x)) = \{x\}$ for any $x \in \text{Sing}(v)$. This implies that $M = \text{Sing}(v) \sqcup \bigcup_{x \in \text{Sing}(v)} W^u(x) \cap W^s(y)$. For any $x \in \text{Sing}(v)$, we obtain $W^u(x) = \{x\} \sqcup \bigcup_{y \in \text{Sing}(v)} W^u(x) \cap W^s(y)$. By definition of Morse flow, we have $\text{Cl}(v) \subseteq M - P(v) \subseteq \mathcal{R}(v) \subseteq \Omega(v) = \text{Cl}(v)$ and so $M = \text{Sing}(v) \cup P(v)$. Therefore $P(v) = \bigcup_{x \in \text{Sing}(v)} W^u(x) \cap W^s(y)$. Fix any pair $x \neq y \in \text{Sing}(v)$. Then $W^u(x) \cap W^s(y) = \{z \in P(v) \mid \alpha(z) = W^u(x), \omega(x) = W^s(y)\}$. For any $z \in W^u(x) \cap W^s(y)$, the abstract weak orbit $[z]$ is the connected component of $\{z \in P(v) \mid \alpha(z) = W^u(x), \omega(x) = W^s(y)\}$ containing $z$. This means that the connecting orbit set $W^u(x) \cap W^s(y)$ is a union of abstract weak orbits. Since the unstable manifold of a singular point is a union of connecting orbit sets, the unstable manifold of a singular point is a union of abstract weak orbits.

3.3. **Infiniteness of Morse-Smale flows.** Notice that the connected components of connecting orbit sets are not finite in general. In other words, we have the following infinite property.

**Theorem 3.2.** There is a smooth Morse-Smale flow $v$ on a closed three dimensional manifold whose abstract orbit space with the partial order $\leq_\alpha$ is an abstract multigraph with directed edges which have infinitely many connected components.

**Proof.** Consider a vector field $X_0 = (-x, -y, -z)$ on a unit closed ball $M_0 := \mathbb{D}^3$ in $\mathbb{R}^3$ as on the left of Figure 2 and a vector field $Y_1 = (-x, -y, z)$ on a solid cylinder $C := [0, 1] \times \mathbb{D}^2$ as on the middle of Figure 2. Attaching the handle $C$ to the ball $\mathbb{D}^3$, the resulting space is a solid torus $M_1 := M_0 \cup C$. Identify $M_1$ with $\mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$. Smoothing the resulting vector field on $M_1$, we can obtain the resulting vector field $X_1 = (\sin(x/2\pi), -y, -z)$ on $M_1 = \mathbb{R}/\mathbb{Z} \times \mathbb{D}^2$ as on the right of Figure 2. Replacing the saddle with the index one of $Y_1$ by a pair of a source and a hyperbolic periodic orbit $\gamma$ with the index one, denote by $Y_1'$ the resulting vector field as on the left of Figure 3. Attaching a copy $C'$ of the handle $C$ with the vector field $Y_1$ to the solid torus $M_1$ as on the left of Figure 4, the resulting space is denote by $M_2 := M_0 \cup C \cup C'$ on the right of Figure 4. Smoothing the resulting vector field on $M_2$, we can obtain the resulting vector field $X_2$ on $M_2$ as on the right of Figure 3. Let $s_1$ be the saddle with index one of $X_2$ and $O$ the hyperbolic periodic
Figure 3. Left, the vector field $Y'_1$; middle, the projection of the vector field $Y'_1$ into the $x$-$y$ plane; right, the vector field $X_2$ on $M_2$.

Figure 4. Left, attaching a copy $C'$ of the handle $C$ with the vector field $Y_1$; right, attaching a copy $C'$ of the solid torus $M_0 \cup C$ with the vector field $X_1$.

Figure 5. Left, attaching two copies of $C$ to a copy of the ball $\mathbb{D}^3$ with the vector field $-X_0$; right, attaching a copy of $C$ to the union of a copy of the ball $\mathbb{D}^3$ with the vector field $-X_0$ and a copy of $C$ with the vector field $-Y_1$.

Orbit of $X_2$. Then the stable manifold $W^s(s_1)$ and the unstable manifold $W^u(O)$ intersect transversally. Here the transversality of a pair of differential submanifolds means that the sum of the tangent spaces of $W^s(s_1)$ and $W^u(O)$ at a point $x$ in the intersection $W^s(s_1) \cap W^u(O)$ is the tangent space of one of the whole space $M_2$ (i.e. $T_xW^s(s_1) + T_xW^u(O) = T_xM_2$).

On the other hand, attaching two copies of the handle $C$ with the vector field $-Y_1$ to a copy of the ball $\mathbb{D}^3$ with the vector field $-X_0$ as on the left of Figure 5, the resulting space is denoted by $M'_1$. Smoothing the resulting vector field on $M'_1$, we can obtain the resulting vector field $X'_1$ on $M'_1$ as on the right of Figure 6. Denote by $s_2$ a saddle of index two of $X'_1$ on $M'_1$. Let $\Sigma_2$ be an orientable closed surface of genus
two. Considering a torus attached with a handle as on the middles of Figure 7 and by rotating a handle \( \pi/2 \) on a torus, we can define a \( C^\infty \) diffeomorphism \( f : \Sigma_2 \to \Sigma_2 \) which maps two simple closed curves as in Figure 7. Pasting \( M_2 \) and \( M'_1 \) by the diffeomorphism \( f \) such that \( W^u(s_2) \) and \( W^s(s_1) \) intersect transversally and infinitely many times as in Figure 8, and smoothing the resulting vector field on the resulting manifold \( M := M_2 \cup M'_1 \), the resulting manifold \( M \) is a three-dimensional closed manifold and the resulting vector field \( X \) is a \( C^\infty \) Morse-Smale vector field with connecting orbit sets which have infinitely many connected components such that the \( \Omega \)-limit set is the union of closed orbits which consists of one sink, three saddles, one limit cycle \( \gamma \), and two sources, because the unstable manifold \( W^u(s_2) \) and the stable manifold \( W^s(s_1) \) intersect transversally on the boundary \( \partial M_2 = \partial M'_1 \) and the vector field \( X \) is transverse to \( \partial M_2 = \partial M'_1 \).

On the other hand, as mentioned above, it is known that the set of the unstable manifolds of singular points of a Morse flow on a closed manifold is a finite CW decomposition (cf. [1, Theorem 1]). Therefore the author would like to know whether any Morse flow on a closed manifold \( M \) whose abstract weak orbit space \( M/\![v]\) with the partial order \( \leq_v \) is a finite abstract multi-graph. In other words, does any connecting orbit set between distinct Morse sets of a Morse flow on a closed manifold consists of finitely many connected components? Equivalently, one would like to know an answer to the following question.

**Question 1.** Does the intersection of the unstable manifold of any singular point and of the stable manifold of any singular point of a Morse flow on a closed manifold consist of finitely many connected components?
4. Refinements of the Morse graphs. In this section, we obtain the Morse graph of a flow as a quotient space of the abstract orbit space with pre-orders $\leq_v$, $\leq_\alpha$, and $\leq_\omega$.

4.1. Orders for flows. We define orders for flows to reduce the Morse graph from the abstract orbit space with such orders.

4.1.1. Specialization orders for abstract weak orbit spaces. We define specialization orders $\leq_v$ on the abstract weak orbit space $X/\left[v\right]$ and the abstract orbit space $X/\left\langle v \right\rangle$ and for a flow on a topological space $X$ as follows:

$$[x] \leq_v [y] \text{ if } [x] \cap \bigcup_{y_1 \in [y]} O(y_1) \neq \emptyset$$

$$\left\langle x \right\rangle \leq_v \left\langle y \right\rangle \text{ if } \left\langle x \right\rangle \cap \bigcup_{y_1 \in (y)} O(y_1) \neq \emptyset$$

Notice that the height of the abstract weak orbit space of the flow need not be finite. Indeed, we have the following infinite property.

**Proposition 1.** For any Anosov diffeomorphism on a torus, the height of the abstract weak orbit space of the suspension flow is infinite.

**Proof.** Let $v_f$ be the suspension flow on a closed 3-manifold $M$ of an Anosov diffeomorphism $f$ on a torus $T^2$. We claim that the binary relation $\leq_{v_f}$ on the abstract weak orbit space $M/[v_f] \cong T^2/[f]$ has infinite height. Indeed, the Anosov diffeomorphism $f$ is topological conjugate to a toral automorphism and so is semi-conjugate to a shift map on a shift space given by finite symbols. Therefore there is a sequence $(x_n)_{n \in \mathbb{Z}_{\geq 0}}$ such that $\omega(x_n) \subsetneq \omega(x_{n+1})$. This implies the infiniteness of the height. \qed

We have the following observations to describe transitivities of orders of flows.

**Lemma 4.1.** Let $v$ be a flow on a topological space $X$. The following statements hold for any point $x \in X$ and for any point $y \in [x] \cup \left\langle x \right\rangle$:

1. If $x \in P(v)$, then $\alpha(x) = \alpha(y) = \alpha'(x) = \alpha'(y)$ and $\omega(x) = \omega(y) = \omega'(x) = \omega'(y)$.
2. If $X$ is Hausdorff and $x \notin R(v)$, then $\alpha'(x) = \alpha'(y)$ and $\omega'(x) = \omega'(y)$. 
Proof. Fix any point \( x \in X \). Suppose that \( x \in P(v) \). Then \( [x] = (x) \). By definition of the abstract weak orbit, we obtain \( \alpha(x) = \alpha(y) \). The non-recurrence implies that \( \alpha'(x) = \alpha(x) = \alpha(y) = \alpha'(y) \). By time reversion, we have \( \omega'(x) = \omega(x) = \omega(y) = \omega'(y) \).

Suppose that \( X \) is Hausdorff and that \( x \notin R(v) \). Thus we may assume that \( x \notin P(v) \). Then \( x \in Cl(v) \). Then \( [x] = (x) \). The Hausdorff separation axiom implies that the orbit \( O(x) \) is a closed subset and so \( \alpha'(x) = \emptyset = \alpha'(y) \). By time reversion, we have \( \omega'(x) = \emptyset = \omega'(y) \).

### 4.1.2. Order structures for flows.

Define pre-orders \( \leq_\alpha \) and \( \leq_\omega \) on \( X \) as follows:

\[
\begin{align*}
&x \leq_\alpha y \text{ if } x \in O(y) \cup \alpha(y) \quad \text{(i.e. } O(x) \cap (O(y) \cup \alpha(y)) \neq \emptyset) \\
&x \leq_\omega y \text{ if } x \in O(y) \cup \omega(y) \quad \text{(i.e. } O(x) \cap (O(y) \cup \omega(y)) \neq \emptyset) 
\end{align*}
\]

These orders induce the following pre-orders on the abstract weak orbit space \( X/\langle v \rangle \) and on the abstract orbit space \( X/\langle v \rangle \), by abuse of terminology, which are denoted by the same symbols:

\[
\begin{align*}
&[x] \leq_\alpha [y] \text{ if } [x] \cap ([y] \cup \alpha([y])) \neq \emptyset \\
&[x] \leq_\omega [y] \text{ if } [x] \cap ([y] \cup \omega([y])) \neq \emptyset \\
&(x) \leq_\alpha (y) \text{ if } (x) \cap ((y) \cup \alpha((y))) \neq \emptyset \\
&(x) \leq_\omega (y) \text{ if } (x) \cap ((y) \cup \omega((y))) \neq \emptyset 
\end{align*}
\]

We show that these binary relations \( \leq_\alpha \) and \( \leq_\omega \) on the abstract weak orbit space \( X/\langle v \rangle \) and on the abstract orbit space \( X/\langle v \rangle \) are pre-orders.

**Lemma 4.2.** The binary relations \( \leq_\alpha \) and \( \leq_\omega \) on the abstract weak orbit space \( X/\langle v \rangle \) of a flow \( v \) on a Hausdorff space \( X \) are pre-orders.

**Proof.** By definitions, reflexivity holds for the relations. Therefore it suffices to show transitivity. Fix points \( x, y, z \in X \) with \( [x] \leq_\alpha [y] \), \( [y] \leq_\alpha [z] \), \( [x] \neq [y] \) and \( [y] \neq [z] \). Then \( [x] \cap \alpha([y]) \neq \emptyset \) and \( [y] \cap \alpha([z]) \neq \emptyset \). This means that there are points \( x_1 \in [x] \), \( y_1, y_2 \in [y] \) and \( z_2 \in [z] \) such that \( x_1 \in \alpha(y_1) \) and \( y_2 \in \alpha(z_2) \). Suppose that \( y \notin R(v) \). Lemma 4.1 implies that \( \alpha'(y_1) = \alpha'(y_2) \). By \( O(x_1) \neq O(y_1) \), since the closedness of the \( \alpha \)-limit set, we have \( x_1 \in \alpha(y_1) \). The definition of the abstract weak orbit \( [y] \) implies that \( y_2 \in [y] \subseteq R(v) \). By Lemma 4.1, if \( y_1 \in \alpha(y_1) \) then the closedness of the \( \alpha(y_2) \) implies that \( x_1 \in \alpha(y_1) = \alpha(y_2) \subseteq \alpha(z_2) \) and so that \( [x] \cap \alpha([z]) \neq \emptyset \). Thus we may assume that \( y \in \omega(y_1) \). The closedness of the \( \omega(y_2) \) implies that \( x_1 \in \alpha(y_1) \subseteq O(y_1) = \omega(y_1) = \omega(y_2) \subseteq \alpha(z_2) \) and so that \( [x] \cap \alpha([z]) \neq \emptyset \). By the time reversion, symmetry implies that the binary relation \( \leq_\omega \) is a pre-order.

**Lemma 4.3.** The binary relations \( \leq_\alpha \) and \( \leq_\omega \) on the abstract orbit space \( X/\langle v \rangle \) of a flow \( v \) on a Hausdorff space \( X \) are pre-orders.

**Proof.** By definitions, reflexivity holds for the relations. Therefore it suffices to show transitivity. Fix points \( x, y, z \in X \) with \( (x) \leq_\alpha (y) \), \( (y) \leq_\alpha (x) \), \( (x) \neq (y) \) and \( (y) \neq (z) \). Suppose that \( y \notin R(v) \). Lemma 4.1 implies that \( \alpha'(y) = \alpha'(y_1) \) for any \( y_1 \in (y) \). Then \( \alpha'(y) \subseteq \alpha'(z_2) \). By \( (x) \cap (y) = \emptyset \), since the closedness of the \( \alpha \)-limit set, we have \( \emptyset \neq (x) \cap \alpha'(y) \subseteq (x) \cap (z_2) \). Therefore we may assume that \( y \in R(v) \). Then \( (y) = O(y) \) and so \( \emptyset \neq [x] \cap \alpha([y]) \subseteq O(y) \subseteq \alpha([z]) \). By the time reversion, symmetry implies that the binary relation \( \leq_\omega \) is a pre-order.
4.2. Reduction to the Morse graph. The Morse graph of a flow can be obtained as a quotient space of the abstract orbit space as follows.

Theorem 4.4. Let \( v \) be a flow on a compact metric space \( X \). Then the Morse graph \( G_v \) of a flow is a quotient space of the abstract orbit space \( X/\{v\} \) with pre-orders \( \leq_\alpha \) and \( \leq_\omega \). In particular, the underlying space of the Morse graph \( G_v \) of a flow is a quotient space of the abstract orbit space \( X/\{v\} \).

Proof. Since the direction of edges are determined by pre-orders \( \leq_\alpha \) and \( \leq_\omega \), it suffices to show that the underlying space of the Morse graph of a flow is a quotient space of the abstract orbit space. Recall that any \( \omega \)-limit set and any \( \alpha \)-limit set of a point in the compact metric space \( X \) is non-empty and connected, and the closure of a connected subset is connected, the closure of any Morse set is a union of abstract orbits, so is the complement of any Morse set. Thus any Morse set is a union of abstract orbits. Since the union of connected components of \( \text{Morse}(v) \) is closed and invariant and contains the non-wandering set \( \Omega(v) \). By the decomposition \( X = \text{Cl}(v) \cup \text{P}(v) \cup \text{R}(v) \), we have \( X - \text{P}(v) = \text{R}(v) \), since any recurrent point is non-wandering, we have that \( \text{Cl}(v) \cup \text{R}(v) = X - \text{P}(v) = \text{R}(v) \leq \Omega(v) \leq \text{Morse}(v) = \bigcup_i M_i \). Moreover [11, Lemma 3.1B] implies that connected components of \( \text{Morse}(v) \) are equivalence classes of the relation \( \approx_{\text{Morse}} \) on \( \text{Morse}(v) \), where \( x \approx_{\text{Morse}} y \) if \( x \sim_{\text{Morse}} y \) and \( y \sim_{\text{Morse}} x \). Since Morse sets are connected components of \( \text{Morse}(v) \), Morse sets are equivalence classes of the relation \( \approx_{\text{Morse}} \) on \( \text{Morse}(v) \). We show that any Morse set is a union of abstract orbits. Indeed, fix a point \( x \in X \). If \( x \in \text{Sing}(v) \), then the abstract orbit \( \langle x \rangle \) is the connected component of \( \text{Sing}(v) \) (resp. \( \text{Per}(v) \)) containing \( x \) and so is contained in some Morse set \( M_i \), because Morse sets are connected components of \( \text{Morse}(v) \) and \( \text{Cl}(v) \subseteq \text{Morse}(v) \). Suppose that \( x \in \text{R}(v) \). Then \( \text{O}(x) \subseteq \langle x \rangle = \hat{\text{O}}(x) \subseteq \overline{\text{O}(x)} \cap \text{R}(v) \subseteq \text{R}(v) \subseteq \text{Morse}(v) = \bigcup_i M_i \). The closedness and invariance of \( \text{Morse}(v) \) imply that \( \overline{\text{O}(x)} \subseteq \text{Morse}(v) \). Since the closure of a connected subset is connected, the closure \( \overline{\text{O}(x)} \) is connected and so is contained in some Morse set \( M_i \) for some \( i \). Then \( \langle x \rangle \subseteq \overline{\text{O}(x)} \subseteq M_i \). Suppose that \( x \in \text{P}(v) \cap \bigcup_i M_i \). Since \( \bigcup_i M_i = \text{Morse}(v) \), we have \( x \in \text{P}(v) \cap \text{Morse}(v) \). The closedness and invariance of \( \text{Morse}(v) \) imply that \( \overline{\text{O}(x)} \subseteq \text{Morse}(v) \). Since the closure of a connected subset is connected, the closure \( \overline{\text{O}(x)} \) is connected and so is contained in some Morse set \( M_i \). Then \( \alpha'(x) \cup \omega'(x) = \alpha(x) \cup \omega(x) \subseteq \overline{\text{O}(x)} \subseteq M_i \). By definition, we have that \( \langle x \rangle \subseteq \{ y \in \text{P}(v) \mid \alpha(x) = \alpha(y) \}, \omega(x) = \omega(y) \} \). Fix a point \( y \in \langle x \rangle \). For any \( \varepsilon > 0 \) and \( T > 0 \), there are a point \( a \in \alpha(x) = \alpha(y) \subseteq M_i \), a point \( \omega \in \omega(x) = \omega(y) \subseteq M_i \), an \( (\varepsilon,T) \)-chain from \( a \) to \( y \), and an \( (\varepsilon,T) \)-chain from \( y \) to \( \omega \). In other words, we obtain \( x \sim_{\text{Morse}} y \). By [11, Theorem 3.3C], we have \( w \sim_{\text{Morse}} z \) and \( z \sim_{\text{Morse}} w \) for any points \( w,z \in M_i \). Since \( \alpha, x, \omega \in M_i \), we obtain \( x \sim_{\text{Morse}} \alpha \) and \( \omega \sim_{\text{Morse}} x \). The transitivity of \( \sim_{\text{Morse}} \) implies that \( x \sim_{\text{Morse}} y \) and \( y \sim_{\text{Morse}} x \). In particular, we obtain \( \overline{\text{O}(y)} \subseteq \text{Morse}(v) \). Since \( \alpha(y) = \alpha(x) \subseteq M_i \) is connected, and since \( M_i \) is a connected component of \( \text{Morse}(v) \), we have \( \overline{\text{O}(y)} \subseteq M_i \). This means that \( \langle x \rangle \subseteq M_i \). Thus any Morse set is a union of abstract orbits. Since the union \( \bigcup_i M_i \) of Morse sets is a union of abstract orbits, so is the complement \( \bigcup_{j,k} D_{j,k} = X - \bigcup_i M_i \). By \( X - \text{P}(v) = \text{Cl}(v) \cup \text{R}(v) \subseteq \text{R}(v) \subseteq \text{Morse}(v) = \bigcup_i M_i = X - \bigcup_{j,k} D_{j,k} \), we have \( \text{P}(v) \setminus \text{Morse}(v) = \bigcup_{j,k} D_{j,k} = X - \bigcup_i M_i \). We show that any \( D_{j,k} \) consists of abstract orbits. Indeed, fix a point \( x \in D_{j,k} \). Then \( \langle x \rangle \subseteq \bigcup_{j,k} D_{j,k} = \text{P}(v) \setminus \text{Morse}(v) \),
Figure 9. The list of singular points appeared in quasi-regular flows.

\[ \alpha(x) \subseteq M_j, \text{ and } \omega(x) \subseteq M_k. \] Therefore \( \langle x \rangle \subseteq \{ y \in P(v) \setminus CR(v) \mid \alpha(x) = \alpha(y) \subseteq M_j, \omega(x) = \omega(y) \subseteq M_k \} \subseteq D_{j,k}. \) This means that \( D_{j,k} \) consists of abstract orbits. Therefore the Morse graph of \( v \) is a quotient space of the abstract orbit space \( X/\langle v \rangle \).

The previous theorem and Corollary 3 imply the following reduction.

**Corollary 5.** Let \( v \) be a flow on a compact metric space \( X \). Then the Morse graph \( G_v \) of a flow is a quotient space of the abstract weak orbit space \( X/\langle v \rangle \) and of the orbit class space \( X/\hat{\langle v \rangle} \) with pre-orders \( \leq_\alpha \) and \( \leq_\omega \).

5. **Refinement of the Reeb graphs of Hamiltonian flows on surfaces.** In this section, we show that the Reeb graphs of a Hamiltonian flow with finitely many singular points on a compact surface is a quotient space of the abstract weak orbit space. In fact, such a Reeb graph corresponds to its extended weak orbit space as an abstract graph.

5.1. **Notion of surface flows.** Recall several concepts to state properties of surface flows.

5.1.1. **Flow of finite type on a surface.** Let \( w \) be a flow on a surface \( S \). A singular point is nondegenerate if it is either a saddle, a \( \partial \)-saddle, a sink, a \( \partial \)-sink, a source, a \( \partial \)-source, or a center. In other words, a singular point is nondegenerate if and only if it is locally topologically equivalent to an isolated singular point \( p \) of a flow generated by a \( C^2 \) vector field \( X \) such that the determinant of the Hesse matrix \( (X_{ij}) \) is non-zero (i.e. \( X_{11}X_{22} - X_{12}X_{21} \neq 0 \)), where \( (x_1, x_2) \) is a local coordinate system and \( X_{ij} := \frac{\partial^2 X(p)}{\partial x_i \partial x_j} \). Recall that a point \( x \in X \) is locally topologically equivalent to a point \( y \in Y \) if there are neighborhoods \( U_x \) and \( U_y \) of \( x \) and \( y \) respectively, and a homeomorphism \( h: U_x \to U_y \) such that the images of connected components of the intersection of an orbit and \( U_x \) are those of the intersection of an orbit and \( U_y \) (i.e. \( h(C_p) = C_{h(p)} \) for any point \( p \in U_x \), where \( C_p \) is the connected component of \( O(p) \cap U_x \) containing \( p \) and \( C_{h(p)} \) is the connected component of \( O(h(p)) \cap U_y \) containing \( h(p) \)), and that \( h \) preserves the direction of the orbits. A flow is quasi-regular if any singular point either is locally topologically equivalent to a nondegenerate singular point or is a multi-saddle as in Figure 9. Here a multi-saddle is an isolated singular point with finitely many separatrices as in Figure 9. A flow \( v \) on a surface \( S \) is a flow of finite type on a surface if it satisfies the following conditions:

1. The flow \( v \) is quasi-regular.
2. There are at most finitely many limit cycles.
3. Any recurrent orbit is closed (i.e. \( S = Cl(w) \cup P(w) \)).
5.1.2. **Fundamental structures of flows on surfaces.** Let \( w \) be a flow on a surface \( S \). A flow box is homeomorphic to a rectangle \((0,1)^2\), \((0,1) \times [0,1]\), or \((0,1) \times [0,1]\) such that any open orbit arc is of the form \((0,1) \times \{y\}\) for some \( y \). By the flow box theorem for a continuous flow on a surface (cf. [3, Theorem 1.1, p.45]), for any point, there is its open neighborhood which is a flow box. A flow box is a trivial flow box if it is a saturated disk to which the orbit space of the restriction of the flow is an interval as on the left of Figure 10. An open saturated annulus is a transverse annulus if it consists of non-recurrent orbit which is topologically equivalent to a flow as on the right of Figure 10. A limit circuit \( \gamma \) is semi-repelling (resp. semi-attracting) if there are a neighborhood \( U \) of \( \gamma \) and a connected component \( A \) of \( U - \gamma \) such that \( \alpha(x) = \gamma \) (resp. \( \omega(x) = \gamma \)) for any \( x \in A \). A non-singular orbit of \( w \) is a semi-multi-saddle separatrix if it is a separatrix from or to a multi-saddle. A semi-multi-saddle separatrix of \( w \) is a multi-saddle separatrix if it is a separatrix between multi-saddles. A saturated subset of the flow \( w \) of finite type is an ss-component if it is either a sink, a \( \partial \)-sink, a source, a \( \partial \)-source, or a limit circuit. In other words, a saturated subset is an ss-component if and only if it is either a semi-repelling \( \alpha \)-limit set or a semi-attracting \( \omega \)-limit set. Here an \( \omega \)-limit set is semi-attracting if it is either a source, a \( \partial \)-source, or a semi-attracting limit circuit. An \( \alpha \)-limit set is semi-repelling if it is either a sink, a \( \partial \)-sink, or a semi-repelling limit circuit.

The union of multi-saddles, semi-multi-saddle separatrices, and ss-components of the flow \( w \) of finite type is called the multi-saddle connection diagram and denote by \( D(v) \). A connected component of the multi-saddle connection diagram is called a multi-saddle connection. Note that the multi-saddle connection diagram of a Hamiltonian vector field is also called the ss-multi-saddle connection diagram and denote by \( D_{ss}(v) \) in [45]. A subset of \( S \) which is either a torus, a Klein bottle, an open annulus, or an open Möbius band is periodic if it consists of periodic orbits. In the same way, an \( \omega' \)-limit set is semi-attracting if it is either a source, a \( \partial \)-source, or a semi-repelling limit circuit. An \( \alpha' \)-limit set is semi-repelling if it is either a sink, a \( \partial \)-sink, or a semi-attracting limit circuit. A separatrix from or to a saddle is self-connected if each separatrix from and to the saddle. A separatrix from or to a \( \partial \)-saddle is self-connected if each separatrix connecting two \( \partial \)-saddles on a boundary component of the surface (see Figure 11).
Extended (weak) orbit spaces of flows on surfaces. An extended orbit of a flow on a surface is an equivalence class of an equivalence relation \( \sim_{\text{ex}} \) defined by \( x \sim_{\text{ex}} y \) if they are contained in either an orbit or a multi-saddle connection, and that the extended orbit space of a flow \( w \) on a surface \( S \) is a quotient space \( S/\sim_{\text{ex}} \) and is denoted by \( S/\sim_{\text{ex}} \). Note that an extended orbit is an analogous concept of “demi-caractéristique” in the sense of Poincaré [35]. In particular, an extended orbit for a Hamiltonian flow on a compact surface corresponds to a “demi-caractéristique” in the sense of Poincaré. Moreover, the concept of “extended positive orbit” for a flow of finite type on a surface corresponds to one of “demi-caractéristique” in the sense of Poincaré. We also define a generalization of extended orbits, called extended weak orbits as follows. A closed connected invariant subset \( S \) consisting of finitely many abstract weak orbits is a quasi-saddle if (1) there are points \( x_\alpha, x_\omega \notin S \) such that \( \alpha(x_\alpha) \subseteq S \) and \( \omega(x_\omega) \subseteq S \), and (2) \( |\{x \mid \alpha(x) \subseteq S \text{ or } \omega(x) \subseteq S\}| < \infty \). Then such an abstract weak orbit \( [x] \) in the definition of quasi-saddle are called a quasi-saddle-separatrix, and the union of quasi-saddles and quasi-saddle-separatrices is called the quasi-saddle connection diagram. The connected component of the quasi-saddle connection diagram is called a quasi-saddle connection. By definition, the condition (1) in the definition of quasi-saddle is equivalent to the following condition: (1)' there are abstract weak orbits \( [x_\alpha] \) and \( [x_\omega] \) outside of \( S \) such that \( \alpha([x_\alpha]) = \bigcup_{y \in [x_\alpha]} \alpha(y) \subseteq S \) and \( \omega([x_\omega]) = \bigcup_{z \in [x_\omega]} \omega(z) \subseteq S \). We define the extended weak orbit \( [x]_{\text{ex}} \) of a point \( x \) by the equivalence class of \( \sim_{[w]_{\text{ex}}} \) containing \( x \), where \( \sim_{[w]_{\text{ex}}} \) is an equivalence class defined by \( y \sim_{[w]_{\text{ex}}} z \) if either \( [y] = [z] \) or there is a quasi-saddle connection which is closed and contains both \( [y] \) and \( [z] \). Then the quotient space \( X/\sim_{[w]_{\text{ex}}} \) is denoted by \( X/[w]_{\text{ex}} \) and is called the extended weak orbit space of a flow \( w \) on a topological space \( X \). Note that the extended weak orbit space of a flow on a topological space is a quotient space of the abstract weak orbit space \( X/[w] \). Moreover, we will show that the extended weak orbit space of a Hamiltonian flow with finitely many singular points on a compact surface is a quotient space of the extended orbit space (see Lemma 5.3).

Properties of Hamiltonian flows on surfaces. Recall several concepts for Hamiltonian flows on surfaces to state a reduction of the abstract weak orbit space of a Hamiltonian flow into the Reeb graph.

Fundamental notion of Hamiltonian flows on a compact surface. A \( C^r \) vector field \( X \) for any \( r \in \mathbb{Z}_{\geq 0} \) on an orientable surface \( S \) is Hamiltonian if there is a \( C^{r+1} \) function \( H : S \to \mathbb{R} \) such that \( dH = \omega(X, \cdot) \) as a one-form, where \( \omega \) is a volume form.
form of $S$. In other words, locally the Hamiltonian vector field $X$ is defined by $X = (\partial H/\partial x_2, -\partial H/\partial x_1)$ for any local coordinate system $(x_1, x_2)$ of a point $p \in S$. Note that a volume form on an orientable surface is a symplectic form. It is known that a $C^r$ ($r \geq 1$) Hamiltonian vector field on a compact surface is structurally stable with respect to the set of $C^r$ Hamiltonian vector fields if and only if both each singular point is nondegenerate and each separatrix is self-connected (see [20, Theorem 2.3.8, p. 74]). A flow is Hamiltonian if it is topologically equivalent to a flow generated by a Hamiltonian vector field. Note that the multi-saddle connection diagram $D(v)$ of a Hamiltonian flow $v$ on a compact surface is the union of multi-saddles and separatrices between them. We show the following observations.

**Lemma 5.1.** The following statements hold for a structurally stable Hamiltonian vector field $v$ on a compact connected surface $S$:

1. If the multi-saddle connection $D(v)$ is empty, then the abstract weak orbit space $S/[v]$ consists of at most three elements.
2. $|S/[v]| = 1$ if and only if $S$ is a closed periodic annulus.
3. $|S/[v]| = 2$ if and only if $S$ is a closed center disk.
4. $|S/[v]| = 3$ if and only if $S$ is a rotating sphere (i.e. consists of two centers and one open periodic annulus).

**Proof.** Let $v$ be a flow generated by a structurally stable Hamiltonian vector field on a compact surface $S$. [20, Theorem 2.3.8, p. 74] implies that each singular point is nondegenerate and each separatrix is self-connected. By definition of Hamiltonian, the flow $v$ is topologically equivalent to a flow generated by a Hamiltonian vector field. Since the whole space $S$ is two dimensional and any connected closed 1-manifold is a circle, the existence of the Hamiltonian with nondegenerate critical points implies each orbit is either a closed orbit or a multi-saddle separatrix. Then $S = \text{Cl}(v) \sqcup P(v)$. This implies that $v$ has neither sinks, $\partial$-sinks, sources, nor $\partial$-sources, and so that each singular point is either a center, a saddle, or a $\partial$-saddle. Therefore each boundary component which is not a periodic orbit contains at least two $\partial$-saddles and two multi-saddle separatrices, and so contains at least four abstract weak orbits. Moreover, since $S$ is a surface, the flow $v$ contains at least one periodic annulus. Therefore if the multi-saddle connection $D(v)$ is empty, then the abstract weak orbit space $S/[v]$ consists of one periodic annulus and at most two centers. If $S$ is a closed periodic annulus $U$ (resp. closed center disk, rotating sphere), then $|S/[v]| = 1$ (resp. 2, 3). Conversely, suppose that $|S/[v]| \leq 3$. Then each boundary component is a periodic orbit. This implies that each singular point is either a center or a saddle. Since the existence of saddle implies the existence of two self-conneced separatrices, there are no saddles. Therefore each singular point is a center. We claim that $P(v) = \emptyset$. Indeed, fix a point $x \in P(v)$. The generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points (cf. [30, Theorem 2.6.1] and [43, Corollary 6.6]) implies that $\alpha(x)$ is either a singular point, a periodic orbit, a limit circuit, or a Q-set. Here a Q-set is the closure of a non-closed recurrent orbit. The non-existence of non-closed recurrent orbits implies that one of Q-sets. Since any Hamiltonian vector field on a surface is volume-preserving and so has no limit circuits, the $\alpha$-limit set $\alpha(x)$ is a singular point, which contradicts that each singular point is a center. Thus $S = \text{Cl}(v)$. If $|S/[v]| = 1$, then $S = U$ is a closed periodic annulus. If $|S/[v]| = 2$ (resp. 3), then $\overline{U} - U$ consists of a singular point (resp. two singular points) and so one abstract weak orbit (resp. two abstract weak orbits).  \[\square\]
We have the following properties.

**Lemma 5.2.** Any Hamiltonian flow \( v \) on a compact connected surface \( S \) has at most finitely many singular points if and only if it is of finite type. In any case, the following statements hold:

1. Each singular point is either a center or a multi-saddle.
2. The multi-saddle connection diagram \( D(v) \) is a finite union of non-recurrent orbits and of multi-saddles.
3. The complement \( S - (\text{Sing}(v) \cup D(v)) = \text{Per}(v) \) consists of finitely many periodic annuli each of which is an abstract weak orbit.
4. The binary relation \( \leq_v \) for the abstract weak orbit space \( S/\![v] \) is a partial order and the specialization order.
5. The quotient map \( S \rightarrow S/\![v] \) is a finite poset-stratification with respect to \( \leq_v \).
6. If there is a multi-saddle, then the filtration \( \emptyset \subseteq S_{\leq 0} \subseteq S_{\leq 1} \subseteq S_{\leq 2} \) is a stratification with respect to \( \leq_v \), where \( S_{\leq 0} = \text{Sing}(v) \), \( S_{\leq 1} = \text{Sing}(v) \cup D(v) \), and \( S_{\leq 2} - S_{\leq 1} = \text{Per}(v) \).

**Proof.** Suppose that there are no multi-saddles. Then \([20, \text{Theorem 2.3.8, p. 74}]\) implies that \( v \) is topologically equivalent to a flow generated by a structurally stable Hamiltonian vector field. Then Lemma 5.1 implies that \( S \) is either a periodic annulus, a closed center disk, or a rotating sphere. This implies that all the assertions hold. Thus we may assume that there is a multi-saddle. By definition, a flow of finite type has at most finitely many singular points. Suppose that a Hamiltonian flow on a compact surface has at most finitely many singular points. Since a Hamiltonian flow on a compact surface is topologically equivalent to a divergence-free flow, divergence-free property implies the non-existence of limit cycles. Divergence-free flows on a compact manifold have no non-wandering domains and so are non-wandering. By \([10, \text{Theorem 3}]\), each singular point of a non-wandering flow with finitely many singular points on a compact surface is either a center or a multi-saddle. The generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points implies that \( \alpha(x) \) is either a singular point, a periodic orbit, a limit circuit, or a Q-set. The existence of Hamiltonian implies one of non-closed recurrent orbits. This means that the non-existence of Q-sets. Since any Hamiltonian vector field on a surface is volume-preserving and so has no limit circuits, the \( \alpha \)-limit set \( \alpha(x) \) is a singular point, which is either a center or a multi-saddle. Therefore each orbit is either a closed orbit or a multi-saddle separatrix. This implies that the flow is quasi-regular and so of finite type. Moreover, the multi-saddle connection diagram \( D(v) \) contains the union \( P(v) \) of non-recurrent orbits and consists of finitely many orbits. Therefore the complement \( S - (\text{Sing}(v) \cup D(v)) = \text{Per}(v) \) consists of finitely many periodic annuli each of which is an abstract weak orbit (see Figure 10). Then its boundary as a subset consists of centers, multi-saddles, and separatrices. Fix a connected component \( A \) of \( \text{Per}(v) \). Then \( A \) is a periodic annulus. The existence of multi-saddles implies that the boundary \( \partial A \) contains multi-saddles or separatrices. Since any separatrix is a multi-saddle separatrix, the boundary \( \partial A \) contains multi-saddles and multi-saddle separatrices. Then \( S_{\leq 0} = \text{Sing}(v) \) is the union of height zero abstract weak orbits with respect to \( \leq_v \), a downset \( S_{\leq 1} = \text{Sing}(v) \cup D(v) \) is the union of abstract weak orbits of height at most one with respect to \( \leq_v \), and a difference \( S_{\leq 2} - S_{\leq 1} = \text{Per}(v) \) is the union of height two abstract weak orbits with respect to \( \leq_v \). Therefore, the binary relation \( \leq_v \) on \( S/\![v] \) is a partial order and also the specialization order of
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\[ S/v \xrightarrow{\sim_{\text{Reeb}}} S/v_{\text{ex}} = S/\sim_{\text{Reeb}} \]
\[ S/[v] = S/\langle v \rangle \xrightarrow{\text{ex}} S/[v]_{\text{ex}} \]

**Figure 12.** Commutative diagram consisting of canonical projections among an orbit space, a Reeb graph, an abstract weak orbit space, and an extended weak orbit space of a Hamiltonian flow \( v \) with finitely many singular points on a compact surface \( S \).

\( S/[v] \). The finiteness of \( S/[v] \) implies that the quotient map \( S \to S/[v] \) is a finite poset-stratification with respect to \( \leq_{v} \).

We have the following reduction by collapsing open periodic annuli into singletons

**Lemma 5.3.** The extended weak orbit space of a Hamiltonian flow with finitely many singular points on a compact surface is a quotient space of the extended orbit space. Moreover, the extended weak orbit space corresponds to the abstract graph of the extended orbit space.

**Proof.** Let \( v \) be a Hamiltonian flow with finitely many singular points on a compact surface \( S \). Lemma 5.2 implies that each singular point is either a center or a multi-saddle, and that the complement of the multi-saddle connection diagram \( D(v) \) consists of periodic annuli. Then such periodic annuli are abstract weak orbits. Therefore it suffices to show that any multi-saddle connection is both an extended orbit and an extended weak orbit. Indeed, the multi-saddle connection diagram consists of finitely many closed multi-saddle connections. Therefore any quasi-saddle is a multi-saddle and so the quasi-saddle connection diagram corresponds to the multi-saddle connection diagram. Since any multi-saddle connections are closed, any extended weak orbit corresponds to an extended orbit. Then the assertion holds, by collapsing connected components of \( S - D(v) \), which are periodic annuli, into singletons.

We show that the Reeb graph of a Hamiltonian flow with finitely many singular points on a compact surface is a reduction of the abstract weak orbit space as an abstract graph as in Figure 12.

**Theorem 5.4.** The Reeb graph of a Hamiltonian flow with finitely many singular points on a compact surface corresponds to its extended weak orbit space as an abstract graph. Moreover, the Reeb graph corresponds to extended orbit space.

**Proof.** Let \( v \) be a Hamiltonian flow with finitely many singular points on a compact surface \( S \) and \( H \) the Hamiltonian generating \( v \). Lemma 5.2 implies that each singular point is either a center or a multi-saddle. Then the multi-saddle connection diagram consists of finitely many orbits and any multi-saddle connections are closed. This means that a connected component of the inverse image \( H^{-1}(c) \) for any \( c \in \mathbb{R} \) is either a periodic orbit or a closed multi-saddle connection. Therefore the Reeb graph of \( v \) corresponds to the extended orbit space \( S/v_{\text{ex}} \) as a graph. On the other hand, Lemma 5.3 implies that the extended weak orbit space \( S/[v]_{\text{ex}} \) is the abstract
6. Reconstructions of orbit spaces and abstract weak orbit spaces of flows. We recall fundamental notion of homeomorphisms to state reconstructions of (abstract weak) orbit spaces of flows from their time-one maps.

6.1. Fundamental notion of homeomorphisms and transversality of flows.

6.1.1. Suspension flows of homeomorphisms. For a homeomorphism \( f: X \to X \) on a topological space \( X \), consider a quotient space \( X_f := (X \times \mathbb{R}) / \sim_{\text{susp}} \) defined by \((x, t) \sim_{\text{susp}} (f^n(x), t-n)\) for any \( t \in \mathbb{R} \) and \( n \in \mathbb{Z} \), and define a flow \( v_f: \mathbb{R} \times X_f \to X_f \) by \( v_f(t, (x, s)) / \sim_{\text{susp}} = (x, t+s) / \sim_{\text{susp}} \), where \((x, s) / \sim_{\text{susp}}\) is the equivalence class of a point \((x, s) \in X \times \mathbb{R}\). Then \( v_f \) is called the suspension flow of \( f \) and \( X_f \) is called the mapping torus of \( f \). We have the following observation.

Lemma 6.1. Let \( f: X \to X \) be a homeomorphism on a topological space \( X \) and \( v_f \) the suspension flow of \( f \) on the mapping torus \( X_f \). The orbit spaces \( X/f \) and \( X_f/v_f \) are homeomorphic.

Proof. Let \( \tilde{v}_f \) be a flow on \( X \times \mathbb{R} \) by \( \tilde{v}_f(t, (x, s)) := (x, t+s) \) and \( F: X \times \mathbb{R} \to X \times \mathbb{R} \) a homeomorphism defined by \( F(x, t) = (f(x), t-1) \). Identify \( X \) with the orbit space \((X \times \mathbb{R}) / \tilde{v}_f = (X / \sim_{\text{susp}}) / (\mathbb{R} / \sim_{\text{susp}}) \approx (x, t) \). The mapping torus \( X_f = (X \times \mathbb{R}) / (x, t) \sim (f^n(x), t-n) \) equals to the orbit space \((X / \sim_{\text{susp}}) / (\mathbb{R} / \sim_{\text{susp}}) \approx X/f \), where \( \approx \) denotes the homeomorphic relation and \((\tilde{v}_f, F) \cong \mathbb{R} \times \mathbb{Z} \) is the abelian group generated by abelian groups \( v_f \cong \mathbb{R} \) and \( F \cong \mathbb{Z} \).

6.1.2. Notion of homeomorphisms. Let \( f \) be a homeomorphism on a topological space \( X \). An orbit \( O(x) \) of \( x \) is \( \{f^n(x) \mid n \in \mathbb{Z}\} \). A point \( x \in X \) is fixed if \( f(x) = x \), and is periodic if there is a positive integer \( k \) such that \( f^k(x) = x \). The \( \omega \)-limit set of a point \( x \in X \) is \( \omega(x) := \bigcap_{n \in \mathbb{Z}} \{f^m(x) \mid m > n\} \), and that the \( \alpha \)-limit set of \( x \) is \( \alpha(x) := \bigcap_{n \in \mathbb{Z}} \{f^m(x) \mid m < n\} \). Moreover, put \( \alpha'(x) := \alpha(x) \setminus O(x) \) and \( \omega'(x) := \omega(x) \setminus O(x) \). A point \( x \) is recurrent if \( x \in \alpha(x) \cup \omega(x) \). Denote by \( \text{Fix}(f) \) (resp. \( \text{Per}(f), \text{R}(f), \text{P}(f) \)) the set of fixed points (resp. periodic points, non-periodic recurrent points, non-recurrent points). Then \( X = \text{Per}(f) \cup \text{R}(f) \cup \text{P}(f) \). For any point \( x \in X \), define the orbit class \( \hat{O}(x) \) as follows: \( \hat{O}(x) := \{y \in X \mid \overline{O(x)} = \overline{O(y)}\} \subseteq \overline{O}(x) \). For a homeomorphism \( f \), define an abstract orbit, an abstract weak orbit, and so on using the suspension flow \( v_f \) via the canonical homeomorphism \( h_f: X_f/v_f \to X/f \). Indeed, let \( \pi_f: X \to X/f \) be the canonical quotient map and \( \pi_{v_f}: X_f \to X_f/v_f \) be the canonical quotient map as in Figure 13. The abstract weak orbit \([x]_f \) of a point \( x \in X \) by \( f \) is defined by \([x]_f := \pi_{f}^{-1}(h_f \circ \pi_{v_f}((x, 0) / \sim_{\text{susp}}), v_f)) \), where \((x, 0) / \sim_{\text{susp}}, v_f \) is the abstract weak orbit of the point \((x, 0) / \sim_{\text{susp}} \in X_f \) by the suspension flow \( v_f \). Similarly, the abstract orbit \((x)_f \) of a point \( x \in X \) by \( f \) is defined by \((x)_f := \pi_{f}^{-1}(h_f \circ \pi_{v_f}((x, 0) / \sim_{\text{susp}}, v_f)) \), where \((x, 0) / \sim_{\text{susp}}, v_f \) is the abstract orbit of the point \((x, 0) / \sim_{\text{susp}} \in X_f \) by the suspension flow \( v_f \). The abstract weak orbit space \( X/[f] \) (resp. abstract orbit space \( X/(f) \)) of a homeomorphism is the decomposition \( \{[x]_f \mid x \in X \} \) (resp. \( \{(x)_f \mid x \in X \} \)) of \( X \). By definitions, the abstract weak orbit spaces \( X/[f] \) and \( X_f/v_f \) are homeomorphic, and so are the abstract orbit spaces \( X/(f) \) and \( X_f/v_f \).
6.1.3. Transversality for continuous flows on surfaces. Notice that we can define transversality using tangential spaces, because each flow on a compact surface is topologically equivalent to a $C^1$-flow by the Gutiérrez’s smoothing theorem. However, we define immediately transversality to reparameterize flows modifying transverse arcs as follows. Recall that a curve (or arc) on a surface $S$ is a continuous mapping $C : I \to S$ where $I$ is a nondegenerate connected subset of a circle $S^1$. An orbit arc is an arc contained in an orbit. A curve is simple if it is injective. We also denote by $C$ the image of a curve $C$. Denote by $\partial C := C(\partial I)$ the boundary of a curve $C$, where $\partial I$ is the boundary of $I \subset S^1$. Put $\text{int} C := C \setminus \partial C$. A simple curve is a simple closed curve if its domain is $S^1$ (i.e. $I = S^1$). A simple closed curve is also called a loop. A curve $C$ is transverse to $v$ at a point $p \in \text{int} C$ if there are a small neighborhood $U$ of $p$ and a homeomorphism $h : U \to [-1,1]^2$ with $h(p) = 0$ such that $h^{-1}([-1,1] \times \{t\})$ for any $t \in [-1,1]$ is an orbit arc and $h^{-1}(\{0\} \times [-1,1]) = C \cap U$. A curve $C$ is transverse to $v$ at a point $p \in \partial C \cap \partial S$ (resp. $p \in \partial C \setminus \partial S$) if there are a small neighborhood $U$ of $p$ and a homeomorphism $h : U \to [-1,1] \times [0,1]$ (resp. $h : U \to [-1,1]^2$) with $h(p) = 0$ such that $h^{-1}([-1,1] \times \{t\})$ for any $t \in [0,1]$ (resp. $t \in [-1,1]$) is an orbit arc and $h^{-1}(\{0\} \times [0,1]) = C \cap U$. A simple curve $C$ is transverse to $v$ if so is it at any point in $C$. A simple curve $C$ which is transverse to $v$ is called a transverse arc.

6.2. Detection of original flows from the time-one maps. A function on a one-dimensional manifold is piecewise non-constant linear if there is a cover of nondegenerate closed intervals to which the restriction of the function is linear but not constant. We have the following property to reconstruct orbit spaces.

**Lemma 6.2.** Let $v$ be a Hamiltonian flow with finitely many singular points on a compact surface $S$. If the period function $p : \text{Per}(v)/v \to \mathbb{R}$ is piecewise non-constant linear, then the orbit space of $v$ is homeomorphic to the abstract weak orbit space of (the suspension flow of) the time-one map of $v$.

**Proof.** Lemma 5.2 implies that $v$ is of finite type and that $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup P(v)$. Moreover, the surface $S$ is a finite union of abstract weak orbits which are either centers, semi-multi-saddle separatrices, or periodic annuli. Then $\text{Per}(v)/v$ consists of finitely many intervals. The flow box theorem (cf. [3, Theorem 1.1, p.45]) implies that the period function $p : \text{Per}(v)/v \to \mathbb{R}$ is continuous. Suppose that $p : \text{Per}(v)/v \to \mathbb{R}$ is piecewise non-constant linear. Then the subset of periodic orbits with irrational periods and one with rational period are dense subsets of $\text{Per}(v)$. Let $f$ be the time-one map of $v$ and $v_f$ the suspension flow of $f$. Then
P(v) = P(f), Sing(v) ⊂ Fix(f), and Per(v) ∩ Per(f) = Per(v) ∩ R(f) = S. For a point \( x \in Per(v) \) with irrational period with respect to v, the abstract weak orbit of x by v is a torus and so the abstract weak orbit of x by f is \( O_v(x) \). For a point \( x \in Per(v) \) with rational period with respect to v, the abstract weak orbit of x by v is the connected component of Per(v) containing x which is a torus, and so the abstract weak orbit of x by f is \( O_v(x) \). For a point \( x \in Sing(v) \), the abstract weak orbit of x by v is the periodic orbit which is the connected component of Per(v) containing x, and so the abstract weak orbit of x by f is the fixed point. For a point \( x \in P(v) \), the abstract weak orbit of x by f is the connected component of P(f) containing x which is the saturation \( v_f(O_v(x)) \) of \( O_v(x) \) by v and so the abstract weak orbit of x by f is \( O_v(x) \). This means that then the abstract weak orbit spaces of v and of (the suspension flow of) the time-one map of v are homeomorphic. □

We show that the orbit space of a Hamiltonian flow with finitely many singular points on a compact surface is homeomorphic to the abstract weak orbit space of the time-one map by taking an arbitrarily small reparametrization. Precisely, we have the following reconstructions of topologies of flows.

**Theorem 6.3.** For any Hamiltonian flow \( v \) with finitely many singular points on a compact surface, there is an arbitrarily small reparametrization \( w \) of \( v \) with the compact-open topology such that the orbit space of \( v \) is homeomorphic to the abstract weak orbit space of the time-one map of the reparametrization \( w \).

**Proof.** Let \( S \) be a compact surface. Since \( S \) consists of finitely many connected components, we may assume that \( S \) is connected. Let \( \mathcal{H}(S) \) be the set of Hamiltonian flows with finitely many singular points on a compact surface \( S \) with the compact-open topology. Fix a Riemannian metric \( d \) on the compact surface \( S \). Since \( \mathcal{H}(S) \) is a complete metrizable space (cf. [26, Theorem 2.4]), fix a metric \( d_{\mathcal{H}(S)}(v,w) := \sup \{ d(v(t,x), w(t,x)) : (t,x) \in [0,1] \times S \} \) for any flows \( v, w \in \mathcal{H}(S) \). Fix a Hamiltonian flow \( v \) in \( \mathcal{H}(S) \) and a positive number \( \varepsilon \in (0,1) \). Lemma 5.2 implies that \( v \) is of finite type. By Gutierrez’s smoothing theorem, we may assume that \( v \) is \( C^\infty \). Moreover, the surface \( S \) is a finite union of abstract weak orbits of \( v \) which are either centers, semi-multi-saddle separatrices, or periodic annuli. Note that the orbit space of such a periodic annulus is an interval. Lemma 5.2 implies that there is a filtration \( \emptyset = S_{\leq 0} \subset S_{\leq 1} \subset S_{\leq 2} \) is a stratification with respect to \( \leq v \) such that \( S_{\leq 0} = Sing(v) \), \( S_{\leq 1} = Sing(v) \cup D(v) \), and \( S_{\leq 2} - S_{\leq 1} = Per(v) \) unless \( D(v) \) is empty. Identify \( S \) with the slice \( (S \times \{0\})/\sim_{\text{susp}} \subset S_{w_1} \) of the mapping torus \( S_{w_1} \), where \( w_1 \) is the time-one map of \( v \). We claim that there is a reparametrization \( w \) of \( v \) with \( d_{\mathcal{H}(S)}(v,w) < \varepsilon \) such that, for any periodic annulus \( A \) which is an abstract weak orbit of \( w \), the periodic annulus \( A \) is also a union of abstract weak orbits of \( w_1 \) and \( A/w = A_{w_1}/[v_{w_1}] \), where \( v_{w_1} \) is the suspension flow of \( w_1 \). Indeed, fix a periodic annulus \( A \) which is an abstract weak orbit of \( v \). Therefore any boundary component of \( A \) is either a center or a non-trivial circuit. Suppose that \( A \cap \partial S = \emptyset \). Then there is a transverse open arc \( T \) such that the two end points in \( T - T \) are singular points. The flow box theorem implies that for any point \( x \in T \) there is an open flow box \( V_x \) containing \( x \). Identify \( T \) with \( \{0\} \times \mathbb{R} \subset \mathbb{R}^2 \). Since any bounded closed interval is compact, we may assume that, for any \( n \in \mathbb{Z} \), there are positive numbers \( \varepsilon_n \in (0,1) \) with \( \varepsilon_n > \varepsilon_{|n|+1} = \varepsilon_{-(|n|+1)} \) and \( \lim_{n \to \infty} \varepsilon_n = \lim_{n \to -\infty} \varepsilon_n = 0 \) such that \( U_n := [-\varepsilon_n, \varepsilon_n] \times [n, n+1] \subset \bigcup_{x \in T} V_x \) is a closed flow box in which any closed orbit arc is of the form \( [-\varepsilon_n, \varepsilon_n] \times \{y\} \) for some \( y \). Then the union \( \bigcup_{n \in \mathbb{Z}} U_n \) is a neighborhood of \( T \). By a small perturbation
of the parameter of \( v \) in \( \bigcup_{n \in \mathbb{Z}} U'_n \), we can obtain a reparametrization \( \nu' \) of \( v \) such that there are positive numbers \( \varepsilon'_n \in (0, \varepsilon_n) \) and a continuous function \( \delta: T \to \mathbb{R}_{>0} \) such that \( d_{\mathcal{H}(S)}(v, \nu') < \varepsilon/2 \) and that there are trapezia \( U'_n \) as in Figure 14 defined by

\[
U'_n := \{(x, y) \in [-\varepsilon'_n, \varepsilon'_n] \times [n, n + 1] \mid (\varepsilon'_{n+1} - \varepsilon'_n)(y - n) \geq |x| - \varepsilon'_n \} \subset U_n
\]

such that \( v|_{S - \bigcup_{n \in \mathbb{Z}} U'_n} = v'|_{S - \bigcup_{n \in \mathbb{Z}} U'_n} \) and that \( dv'/dt|_{\nu'}(x, y) = (\delta((0, y)), 0) \) for any \((x, y) \in \text{int} U'_n\). Put \( U' := \bigcup_{n \in \mathbb{Z}} U'_n \), \( \partial^\alpha U'_n := (U'_n \cap \partial U_n') \cap ([\varepsilon'_n - \varepsilon'_n, 0] \times [n, n + 1]) \), and \( \partial^\perp U'_n := (U'_n \cap \partial U_n') \cap ([0, 0] \times [n, n + 1]) \). Then the restriction \( v'|_{U'} \) is a horizontal flow with constant speeds on all horizontal arc of forms \( I_y := [-\varepsilon'(y), \varepsilon'(y)] \times \{y\} \), where the point \( \varepsilon'(y) \) is determined by \( \{\varepsilon'(y)'\} = (\mathbb{R} \times \{y\}) \cap \partial^\perp U'_n \). Let \( p: T \to \mathbb{R} \) be the period function of their orbits with respect to \( \nu' \). Define a continuous function \( p': T \to (0, 1) \) by \( p'((0, y)) := t_+(y) - t_-(y) \), where \( t_-(y) \) is the negative hitting time from \( y \) to the backward vertical boundary \( \partial^\alpha U'_n \) of \( U'_n \) with respect to \( \nu' \) (i.e. \( t_-(y) \) is the negative largest number with \( v'_{\nu', (y)}(y) = (\varepsilon'(y), y) \)), and \( t_+(y) \) is the positive hitting time from \( y \) to the forward vertical boundary \( \partial^\perp U'_n \) of \( U'_n \) with respect to \( \nu' \) (i.e. \( t_+(y) \) is the positive smallest number with \( v'_{\nu', (y)}(y) = (\varepsilon'(y), y) \)). Put \( p'_{n, \min} := \min_{q \in T \cap U'_n} p'(q) \) and \( \delta_{n, \max} := \max_{q \in T \cap U'_n} \delta(q) \). Since the saturation \( \nu'(T) \) is the periodic annulus \( \mathbb{A} \), the functions \( p \) and \( p' \) are continuous. Since the restriction \( p|_{T \cap U'_n} \) is uniformly continuous, by a small perturbation of the parameter of \( v \) in \( \bigcup_{n \in \mathbb{Z}} U'_n \), for any positive integer \( m \in \mathbb{Z}_{>0} \), there is a piecewise non-constant linear continuous function \( p_m: T \to \mathbb{R} \) such that \( 0 < p(q) - p_m(q) < \min\{1/m, p'_{n, \min}/m, p'_{n, \min}/(m\delta_{n, \max})\} \) for any \( n \in \mathbb{Z} \) and any \( q \in \{0\} \times [n, n + 1] = T \cap U'_n \). Then \( \sup\{p(q) - p_m(q) \mid q \in T\} \leq 1/m \) and so \( \lim_{m \to \infty} \sup\{p(q) - p_m(q) \mid q \in T\} = 0 \). On the other hand, for any \( q = (0, y) \in T \), replacing \( v'|_{U'} \) such that the vector field \( dv'/dt|_{U', (x, y)} = (\delta(q), 0) \) becomes \( (\delta(q) + a, 0) \) for any \( a > 0 \), the periods of the orbit of \( q \) with respect to the resulting flows for any numbers \( a > 0 \) form \( (p(q) - p'(q), p(q)) \subset \mathbb{R}_{>0} \). Therefore there are numbers \( a_m(q) > 0 \) with \( \lim_{m \to \infty} a_m(q) = 0 \) such that the periods of the orbits of \( q \) with respect to the resulting flow, by choosing the function \( a_m \), are \( p_m(q) \), because \( p_m(q) \in (p(q) - p'_{n, \min}, p(q)) \subset (p(q) - p'(q), p(q)) \) for any \( n \in \mathbb{Z} \) and any \( p \in \{0\} \times [n, n + 1] = T \cap U'_n \).
We show that $a_m$ is a continuous positive function. Indeed, since $p'(q)δ(q)$ is the length of the interval $I_q$ on which the speed $δ(q)$ is replaced by the constant $δ(q) + a_m(q)$, we have

$$p(q) - p_m(q) = p'(q)δ(q)/(q) - p'(q)δ(q)/(δ(q) + a_m(q)) = \frac{p'(q)δ(q)}{(δ(q) + a_m(q))} = \frac{p'(q)\alpha_m(q)}{(δ(q) + a_m(q))}$$

and so $(p(q) - p_m(q))/p'(q) = a_m(q)/(δ(q) + a_m(q))$. Since functions $p - p_m$ and $p'(q)$ are positive and continuous, by $0 < p(q) - p_m(q) < p'(q)$, the function $C_m : T \to \mathbb{R}_{\geq 0}$ defined by $C_m(q) := (p(q) - p_m(q))/p'(q)$ is continuous and $0 < C_m(q) < \min\{1, 1/m, 1/(mδ_{\text{max}})\}$. Then $C_m(q)(q)/(δ(q) + a_m(q)) = a_m(q)$ and so $a_m(q) = δ(q)C_m(q)/(1 - C_m(q)) \in (0, (1 - m^{-1})^{-1}m^{-1})$. This implies that the function $a_m : T \to \mathbb{R}$ is continuous. The resulting flow $v'_m$ from $v'$ by replacing the vector field $dv'/dt|_{v'}(x, y) = (δ(q), 0)$ by $(δ(q) + a_m(q), 0)$ is continuous such that the piecewise non-constant linear continuous function $p_m : T \to \mathbb{R}$ is the period function with respect to $v'_m$. Since $0 < a_m(q) < (1 - m^{-1})^{-1}m^{-1}$ and so $\lim_{m \to \infty} a_m(q) = 0$, there is a large integer $M > 0$ such that $d_{\mathcal{H}(S)}(v', v'_{M}) < \varepsilon /2$. Then $d_{\mathcal{H}(S)}(v, v'_{M}) < \varepsilon$. Suppose that $A \cap \partial S \neq \emptyset$. Identifying $T$ with either $\{0\} \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$ or $\{0\} \times \{0, 1\} \subset \mathbb{R}^2$, the same argument implies the existence of a desired reparametrization $w$ of $v$ with $d_{\mathcal{H}(S)}(v, w) < \varepsilon$ and $A/w = A/w_{1}/[w_{1}]$. This completes the claim. Applying the above operations to all periodic annuli which are the connected components of the complement $S - D(v)$, we obtain the resulting vector field $w$ with $d_{\mathcal{H}(S)}(v, w) < \varepsilon$ and the period function $p : \text{Per}(v) \to \mathbb{R}$ is piecewise non-constant linear. Lemma 6.2 implies that the orbit space $S/v$ is homeomorphic to the abstract weak orbit space $S/[w_{1}]$ of the time-one map of the reparametrization $w$.

6.3. Coincidence of abstract weak orbit spaces of flows and the time-one maps. To state reconstructions of Morse-Smale flows, we introduce several concepts.

6.3.1. Higher abstract weak orbits. Let $X$ be a topological space with a flow. Put $[x]_{k} := [x]$ for any point $x \in X$. Define the $(k + 1)$-th abstract weak orbit $[x]_{k+1}$ of a point $x$ of $X$ for any $k \in \mathbb{Z}_{>0}$ as follows:

$$[x]_{k+1} := \left\{ \begin{array}{ll}
[x] & \text{if } x \in X - P(v) \\
\{y \in P(v) \mid [α(y)]_{k} = [α(x)]_{k}, [ω(y)]_{k} = [ω(x)]_{k}\} & \text{if } x \in P(v)
\end{array} \right.$$ 

where $[A]_{k} := \bigcup_{a \in A}[a]_{k}$. Define the $k$-th abstract weak orbit space $X/[v]_{k}$ as a quotient space $X/\sim_{[v]_{k}}$ defined by $x \sim_{[v]_{k}} y$ if $[x]_{k} = [y]_{k}$.

For a homeomorphism $f$, define $f$-th abstract weak orbits using the suspension flow $v_{f}$ via the canonical homeomorphism $h_{f} : X_{f}/v_{f} \to X/f$. Indeed, let $\pi_{f} : X \to X/f$ be the quotient map and $\pi_{v_{f}} : X_{f} \to X_{f}/v_{f}$ be the quotient map. Then the $k$-th abstract weak orbit $[x]_{f,k}$ of a point $x \in X$ by $f$ is defined by $[x]_{f,k} := \pi_{v_{f}}^{-1}(h_{f} \circ \pi_{f}((x, 0) \sim_{\text{Susp}}[v_{f}], k)) \subseteq X$. The $k$-th abstract weak orbit space $X/[f]_{k}$ of a homeomorphism is the decomposition $\{[x]_{f,k} \mid x \in X\}$ with the quotient topology.

6.3.2. Morse-Smale flows on compact manifolds. Under three generic conditions for differentials to guarantee structural stability, Labarca and Pacífico defined a Morse-Smale vector field on a compact manifold as follows [18]. A $C^{\infty}$ vector field $X$ on a compact manifold is Morse-Smale if (MS1) the non-wandering set $Ω(X)$ consists of finitely many hyperbolic closed orbits; (MS2) the restriction $X|_{∂M}$ is Morse-Smale; (MS3) for any orbits $O, O' \subset Ω(X)$ and for any non-transversal point $x \in W^{s}(O) \cap W^{u}(O')$, we have $x \in ∂M$ and either $O$ or $O'$ is singular with respect
to $X$ (i.e. $O \subseteq \text{Sing}(X)$ or $O' \subseteq \text{Sing}(X)$). Therefore we say that a flow is Morse-Smale if it is topologically equivalent to a flow generated by a vector field satisfying conditions (MS1)–(MS3). A flow is Morse if it is a Morse-Smale flow without limit cycles. Note that the three generic conditions for differentials form an open dense subset of the set of $C^\infty$ vector fields and that they are stated as follows: any closed orbit is $C^2$ linearizable, the weakest contraction at any closed orbit is defined, and the weakest expansion at any closed orbit is defined. Here the weakest contraction at a singular (resp. periodic) point $p$ is defined if the contractive eigenvalue with biggest real part among the contractive eigenvalues of $DX(p)$ (resp. $DX_f(p)$, where $X_f$ is the Poincaré map) is simple. Dually we can define that the weakest expansion at $p$ is defined. We observe the following properties.

Lemma 6.4. Any Morse-Smale flow on a compact surface satisfies the $C^2$ linearizable condition and the eigenvalue conditions up to topological equivalence.

Proof. Any hyperbolic closed orbit is either a sink, $\partial$-sink, a source, a $\partial$-source, a saddle, a $\partial$-saddle, and a limit cycle which is either attracting or repelling. Therefore such a limit cycle can be perturbed into a $C^2$ linearizable cycle. Any small neighborhood of a hyperbolic singular point can be identified with one of a hyperbolic singular points for a gradient flow and so can be perturbed into a $C^2$ linearizable singular point satisfying the eigenvalue conditions.

The structurally stability holds for Morse-Smale vector fields on compact manifolds under the $C^2$ linearizable condition and the eigenvalue conditions [18].

6.3.3. Finiteness of Morse-Smale flows. Recall that a finite filtration $\emptyset = M_{-1} \subseteq M_0 \subseteq \cdots \subseteq M_n = M$ of closed subsets of an $n$-dimensional manifold $M$ is a stratification if the difference $S_i := M_i - M_{i-1}$ is either the emptyset or an $i$-dimensional submanifold with $\overline{S_i} - S_i \subseteq \bigcup_{j<i} S_j$ for any $i \geq 0$. We have the following finiteness of Morse-Smale flows.

Proposition 2. The following properties hold for a Morse-Smale flow $v$ on a compact manifold $M$:

1. $M/[v] = M/\langle v \rangle$.
2. The abstract orbit space $M/\langle v \rangle$ with the partial order $\leq_v$ is an abstract multi-graph with finite vertices such that $M_0 = \text{Cl}(v)$ and $M_1 = \text{P}(v)$, where $M_i$ is the set of point of height $i$ with respect to the partial order $\leq_v$.
3. The abstract multi-graph $M/\langle v \rangle$ is finite if and only if the flow $v$ is of finite type.
4. If the flow $v$ is $C^\infty$, then any abstract orbits are embedded submanifolds.
5. If the flow $v$ is $C^\infty$ and is of finite type, the abstract orbit space $M/\langle v \rangle$ has a stratification $\emptyset = S_{-1} \subseteq S_0 \subseteq \cdots \subseteq S_n = M$ such that $S_i$ is the finite union of abstract orbits whose dimensions are less than $i + 1$.

Proof. Since the $\alpha$-limit sets and $\omega$-limit set of any point are hyperbolic closed orbits, each recurrent orbit is closed. By definitions of abstract weak orbit and abstract orbit, we obtain $M/[v] = M/\langle v \rangle$. Notice that the $\alpha$-limit set and the $\omega$-limit set of any point are contained in the non-wandering set $\Omega(v) = \text{Cl}(v)$ which is the finite union of hyperbolic orbits. The connectivity of the $\alpha$-limit set and the $\omega$-limit set implies that the $\alpha$-limit set and the $\omega$-limit set of any point are hyperbolic closed orbits. This implies that both the finite union of stable manifolds of closed orbits and the finite union of unstable manifolds of closed orbits are the
whole manifold. Moreover, there are at most finitely many \( \omega \)-limit sets and \( \alpha \)-limit sets. Fix a point \( x \in P(v) \). Then the abstract orbit \( (x) = [x] \) is the connected component of the connecting orbit set \( W^u(\alpha(x)) \cap W^s(\omega(x)) \) containing \( x \). Since unstable manifolds and stable manifolds are immersed submanifolds, the intersection \( W^u(\alpha(x)) \cap W^s(\omega(x)) \) is transverse at any point in the interior \( M - \partial M \), and the restriction \( (W^u(\alpha(x)) \cap W^s(\omega(x))) \cap \partial M \) is transverse in the boundary \( \partial M \). Therefore the intersection \( W^u(\alpha(x)) \cap W^s(\omega(x)) \) is also an immersed submanifold. Since the intersection \( W^u(\alpha(x)) \cap W^s(\omega(x)) \) is the connecting orbit set from the Morse set \( W^u(\alpha(x)) \) to the Morse set \( W^s(\omega(x)) \), the abstract orbit \( (x) \) corresponds to a connected component of the connecting orbit set from the Morse set \( W^u(\alpha(x)) \) to the Morse set \( W^s(\omega(x)) \). This means that the abstract orbit space \( M/\langle v \rangle \) with the partial order \( \mathcal{L}_v \) is an abstract multi-graph with finite vertices such that \( M_0 = \text{Cl}(v) \) and \( M_1 = P(v) \). Since the finite union of stable manifolds of closed orbits is the whole manifold \( M \), if \( v \) is of finite type, then the saturation of the union of neighborhoods of closed orbits is \( M \) and so the quasi-nondegeneracy implies that \( M/\langle v \rangle \) is finite. Conversely, since any recurrent orbit is closed, the non-existence of limit cycles implies that \( v \) is of finite type.

Suppose that \( v \) is \( C^\infty \). [23, Theorem 1] implies the existence of a Lyapunov function of \( v \). Therefore unstable manifolds and stable manifolds are embedded. Since the intersections of unstable manifolds and stable manifolds are also immersed submanifolds, the intersections are embedded. This means any abstract orbits are submanifolds. Moreover, suppose that \( v \) is of finite type. Then \( M \) consists of finitely many abstract orbits. Let \( S_i \) be the union of abstract orbits whose dimensions are less than \( i + 1 \). Then a filtration \( \emptyset = S_{-1} \subseteq S_0 \subseteq \cdots \subseteq S_n = M \) is a stratification. \( \square \)

### 6.3.4. Correspondence between the abstract weak orbit spaces of Morse-Smale flows and the second abstract weak orbit spaces of the time-one maps.

We show coincidences of the second abstract weak orbit spaces of a Morse-Smale flow on a compact manifold and of the time-one map.

**Theorem 6.5.** The (second) abstract weak orbit space of a Morse-Smale flow on a compact manifold is homeomorphic to the second abstract weak orbit space of the time-one map. Moreover, all periodic orbits of the flow has an irrational period if and only if the abstract weak orbit spaces of the time-one map are homeomorphic.

**Proof.** Let \( v \) be a Morse-Smale flow on a compact manifold \( M \), \( f := v_1: M \rightarrow M \) the time-one map of \( v \), and \( v_f \) the suspension flow of \( f \) on the mapping torus \( M_f \). Since the \( \omega \)-limit set and \( \alpha \)-limit set of any point by \( v \) are closed orbits, the abstract weak orbit space and the second abstract weak orbit space of \( v \) correspond to each other. Identify \( M \) with the slice \( (M \times \{0\})/\sim_{\text{susp}} \subseteq M_f \). Proposition 2 implies that \( M = \text{Cl}(v) \sqcup P(v) \) and each abstract weak orbit \( [x]_v \) by \( v \) is either a closed orbit \( O(x) \) or the connected component containing \( x \) of the connecting orbit set \( W^u(\alpha_v(x)) \cap W^s(\omega_v(x)) \) from the hyperbolic closed orbit \( W^u(\alpha_v(x)) \) to the hyperbolic closed orbit \( W^s(\omega_v(x)) \), where \( W^u(A) \) (resp. \( W^s(A) \), \( \alpha_v(x) \), \( \omega_v(x) \)) is the unstable manifold (resp. stable manifold, \( \alpha \)-limit set, \( \omega \)-limit set) with respect to \( v \). Then \( \text{Cl}(v) = \text{Per}(f) \sqcup \text{R}(f) \) and \( P(v) = P(f) \). Fix a point \( x \in M \). Suppose that \( x \in \text{Sing}(v) \). By isolatedness of closed orbits of \( v \), the orbit \( O_{v_f}(x) \) is a connected component of \( \text{Per}(v_f) \) and so \( O_{v_f}(x) = [x]_{v_f} \), where \( [x]_{v_f} \) is the abstract weak orbit of \( x = (x, 0) / \sim_{\text{susp}} \subseteq M_f \). This implies that the abstract weak orbit \( [x]_f \) is the singleton \( \{x\} = [x]_v \). Suppose that \( x \in \text{Per}(v) \). Assume that \( x \in \text{R}(f) \). Then
The abstract weak orbit space is a limit cycle of $f$ such that $\overline{O_f(x)}$ is a connected component of $R(f) \subseteq \text{Per}(v)$. Therefore the closure $\overline{O_v(x)}$ is a minimal torus which is a connected component of $R(v_f) \subseteq \text{Per}(v_f)$, and so $\overline{O_v(x)} = \overline{O_v(x)} \cap R(v_f)$. This means that $[x]_f = \overline{O_f(x)} = O_v(x) = [x]_v$. Thus we may assume that $x \in \text{Per}(f)$. Then $O_v(x)$ is the connected component of $\text{Per}(f)$ containing $x$. Therefore $v_f(O_v(x))$ is an invariant torus which is a connected component of $\text{Per}(v_f)$ and so is the abstract weak orbit $[x]_{v_f}$. This implies that $[x]_f = O_v(x) = [x]_v$. Thus we may assume that $x \in P(v)$. Then $x \in P(f)$ and so $x = (x,0) \sim_{\text{susp}} \in P(v_f)$. By definition of time-one map, the $\alpha$-limit set $\alpha_f(x)$ by $f$ is an invariant subset of the closed orbit $\alpha_v(x)$ and the $\omega$-limit set $\omega_f(x)$ by $f$ is an invariant subset of the closed orbit $\omega_v(x)$. In particular, if $\alpha_f(x) \subseteq R(f)$ (i.e the period of $\alpha_v(x)$ for $v$ is irrational) (resp. $\omega_f(x) \subseteq R(f)$ (i.e the period of $\omega_v(x)$ for $v$ is irrational)), then $\alpha_f(x) = \alpha_v(x)$ (resp. $\omega_f(x) = \omega_v(x)$) is a closed orbit of $v$. If $\alpha_f(x) \subseteq \text{Per}(f)$ (resp. $\omega_f(x) \subseteq \text{Per}(f)$), then $\alpha_f(x) \subseteq \alpha_v(x)$ (resp. $\omega_f(x) \subseteq \omega_v(x)$) is a finite subset contained in the closed orbit $\alpha_v(x)$ (resp. $\omega_v(x)$) of $v$, and so $[x]_f = [x]_v$. This means that $[x]_f \subseteq [x]_v$ if either $\alpha_f(x) \subseteq \text{Per}(f)$ or $\omega_f(x) \subseteq \text{Per}(f)$. Therefore the following statements are equivalent: (1) The limit sets $\alpha_f(x)$ and $\omega_f(x)$ are closed orbits of $v$; (2) $[x]_f = [x]_v$; (3) Each of $\alpha_f(x)$ and $\omega_f(x)$ of $v$ is either a singular point of $v$ or a periodic orbit of $v$ whose period is irrational. This means that all periodic orbits of the flow has an irrational period if and only if the abstract weak orbit spaces $M/[v]$ and $M/[f]$ are homeomorphic. Since $\bigcup_{p \in \text{Cl}(v)} W^u(p) = \bigcup_{p \in \text{Cl}(v)} W^s(p) = M$, the $\alpha$-limit set and $\omega$-limit set of any point with respect to $v$ is a closed orbit of $v$. For any $y \in [x]_v$, since any hyperbolic closed orbit of $v$ is the abstract weak orbit of $f$ and $f$, the invariant subset $[\alpha_f(y)]_f = \bigcup_{z \in \alpha_f(y)} [z]_f = \alpha_v(y) = \alpha_v(x)$ is a hyperbolic closed orbit of $v$ and so is $[\omega_f(y)]_f = \omega_v(x)$. Since $\alpha_v(x)$ and $\omega_v(x)$ are closed orbits of $v$, we obtain

\[
[x]_f = \{ y \in P(v) \mid [\alpha_f(y)]_f, [\omega_f(y)]_f \}
\]

Therefore the abstract weak orbit space $M/[v] = M/[v]_2$ of $v$ is homeomorphic to the second abstract weak orbit space $M/[v]_2$.

The non-existence of periodic orbits implies the following coincidence.

**Corollary 6.** The abstract weak orbit space of a Morse flow on a compact manifold is homeomorphic to the abstract weak orbit space of the time-one map.

7. **Triviality, non-triviality, and incompleteness of abstract weak orbit spaces.**

7.1. **Triviality.** We have the following characterization of a trivial abstract (weak) orbit space.

**Lemma 7.1.** The following statements are equivalent for a flow $v$ on a compact connected Hausdorff space $X$:

1. The abstract weak orbit space $X/[v]$ is a singleton.
(2) The abstract orbit space \( X/\langle v \rangle \) is a singleton.
(3) One of the following conditions holds:
   (i) The flow \( v \) is identical (i.e. \( X = \text{Sing}(v) \)).
   (ii) The flow \( v \) is pointwise periodic (i.e. \( X = \text{Per}(v) \)).
   (iii) The flow \( v \) is minimal.

In the case (iii), if \( v \) is neither identical nor pointwise periodic, then \( X = \text{R}(v) \).

Proof. Since the abstract orbit space \( X/\langle v \rangle \) is a quotient space of the abstract weak orbit space \( X/[v] \), if \( X/[v] \) is a singleton, then so is \( X/\langle v \rangle \). Suppose that the abstract orbit space \( X/\langle v \rangle \) is a singleton. Fix a point \( x \in X \). By definition of abstract weak orbit, the existence of minimal sets implies that either \( X = \text{Sing}(v) \), \( X = \text{Per}(v) \), or \( X = \text{R}(v) \). Thus we may assume that \( X = \text{R}(v) \). Then \( X = \langle x \rangle = \{ y \in \text{R}(v) \mid \overline{O(x)} = \overline{O(x)} \} \). This means that each minimal set is the whole space \( X \) and so \( X = \langle y \rangle = \overline{O(y)} \) for any \( y \in X \).

Suppose that \( X = \text{Sing}(v) \). Then \( [x] = \langle x \rangle \) for any point \( x \in X \) is the connected component of \( \text{Sing}(v) \) containing \( x \) and so \( X \). Suppose that \( X = \text{Per}(v) \). Then \( [x] = \langle x \rangle \) for any point \( x \in X \) is the connected component of \( \text{Per}(v) \) containing \( x \) and so \( X \). Suppose that \( v \) is minimal. Then each nonempty invariant set is \( X \) and so \( X = \alpha(x) \). \( x = \overline{O(x)} \) for any point \( x \in X \). This means that \( X = \langle x \rangle = \langle x \rangle \) for any point \( x \in X \).

We state an example of a periodic homeomorphism to show triviality of abstract weak orbit space for virtually trivial dynamics.

Example 1. Let \( g \) be a periodic homeomorphism on a compact orientable surface \( S \) and \( v_g \) the suspension flow of \( g \) on the mapping torus 3-manifold \( M \). Then the orbit space \( M/v_g \equiv M/\hat{v}_g \equiv S/g \) is an orbifold but the abstract (weak) orbit space \( M/[v_g] = M/\langle v_g \rangle \) and the Morse graph are singletons.

7.2. Recurrent dynamics. The Morse graphs of all examples in this subsection are singletons but the abstract weak orbit spaces are not singletons.

Example 2. Let \( D \) be a unit disk with the polar coordinate system \((r, \theta)\) and \( w \) a rotation defined by \( w(t, (r, \theta)) = (r, 2\pi rt + \theta) \). Denote by \( f_w \) the time-one map and by \( v_{f_w} \), the suspension flow on the resulting solid torus \( M \). Then each orbit of \( v_{f_w} \) is periodic or non-closed recurrent, and each minimal set of \( v_{f_w} \) is a periodic orbit or a torus. Therefore the orbit space \( M/v_{f_w} \) is not \( T_0 \), the orbit class space \( M/\hat{v}_{f_w} \) is \( T_1 \) but not \( T_2 \), and the abstract (weak) orbit space \( M/[v_{f_w}] = M/\langle v_{f_w} \rangle \) is \( D/v \) is a closed interval and so \( T_2 \).

To describe structures of abstract orbit weak spaces and extended weak orbit space, we define binary relation \( \leq_n \) on the abstract orbit weak class space \( X/[v] \) as follows: for any points \( x, y \in X \),
\[
[x] \leq_n [y] \text{ if } [x] \cap ([y] \cup \partial_n [y]) \neq \emptyset
\]
where the transverse boundary \( \partial_n \) is defined as follows: for any saturated subset \( A \subseteq X \),
\[
\partial_n A := (\overline{A} - A) \setminus \left( \bigcup_{z \in A} (\alpha(z) \cup \omega(z)) \right) = (\overline{A} - A) \setminus (\alpha(A) \cup \omega(A))
\]

Example 3. Let \( v \) be a Hamiltonian flow on a closed disk \( D_1 \) with a periodic orbit \( O = \partial D_1 \) with two centers \( c_1, c_2 \), one homoclinic saddle connection \( \{s\} \cap O_1 \cup O_2 \)
and three open periodic annuli $A_1, A_2, A_3$ (see Figure 15). Then $D_1 = \{c_1, c_2, s\} \sqcup O_1 \sqcup O_2 \sqcup A_1 \sqcup A_2 \sqcup A_3$ and $D_1/\{v\} = \{\{c_1\}, \{c_2\}, \{s\}, O_1, O_2, A_1, A_2, A_3\}$. Moreover, the abstract graph of the Reeb graph is the extended weak orbit space $D_1/\{v\}_{ex}$ which is the quotient space of the abstract weak orbit space $D_1/\{v\}$ by collapsing the homoclinic saddle connection $\{s\} \sqcup O_1 \sqcup O_2$ into a singleton.

**Example 4.** Let $v$ be a volume-preserving flow on a connected compact manifold which is neither identical, pointwise periodic, nor minimal. Lemma 7.1 implies that the abstract orbit space of $v$ is not a singleton. Since any volume-preserving flows on a compact manifold have no wandering domains and so are non-wandering and the $\Omega$-limit set is contained in the chain recurrent set $\text{CR}(v)$, we have $X = \Omega(v) = \text{CR}(v)$. This implies that the Morse graph is a singleton.

Recall that Nielsen-Thurston theorem [41] as follows: Let $S$ be a compact connected orientable surface and $f : S \to S$ a homeomorphism. Then there is a map $g$ isotopic to $f$ such that at least one of the following holds: (1) $g$ is periodic (i.e. $g^k = 1_S$ for some positive integer $k$); (2) $g$ preserves some finite union of disjoint simple closed curves on $S$, called $g$ reducible; or (3) $g$ is pseudo-Anosov. Recall that a topological space $(X, \tau)$ is Alexandroff if $\tau$ is the set of upsets with respect to the specialization order $\leq_\tau$ (i.e. $\tau = \{\uparrow A \mid A \subseteq X\}$). In other words, a topological space $(X, \tau)$ is Alexandroff if and only if the intersection of any family of open subsets is open. We state an example of a homeomorphism which is isotopic to a periodic homeomorphism.

**Example 5.** Let $v_g$ be the suspension flow on a closed 3-manifold $T^2_g$ of a homeomorphism $g$ on a torus $T^2$ whose minimal sets consist of a semi-attracting limit cycle $\gamma_1$, one semi-attracting and semi-repelling limit cycle $\gamma_2$, one semi-repelling limit cycle, and cycles as on the left of Figure 16. Then $T^2 = [\gamma_1] \sqcup [\gamma_2] \sqcup U_1 \sqcup U_2$ and the abstract weak orbit space $T^2_2/[v_g] \cong T^2/[g] = \{[\gamma_1], [\gamma_2], U_1, U_2\}$ with the pre-order $\leq_v$ is a poset with height one which consists of four points with $U_1 <_\alpha [\gamma_1] >_\omega U_2$ and $U_1 <_\omega [\gamma_2] >_\alpha U_2$, where $[\gamma_1]$ is a closed annulus consisting of cycles, the open annuli $U_1$ and $U_2$ are connected components of $T^2 - ([\gamma_1] \sqcup [\gamma_2])$. Note that the poset $T^2_2/[v_g]$ as an Alexandroff space is weakly homotopy equivalent to a circle.

We also state an example of a homeomorphism which is isotopic to a periodic homeomorphism and which is not topologically equivalent to the flow $g$ in the previous example but whose abstract weak orbit space is homeomorphic to one of $g$. 

![Figure 15](image-url)
Example 6. Let \( v_h \) be the suspension flow on a closed 3-manifold \( \mathbb{T}^2 \) of a homeomorphism \( h \) on a torus \( \mathbb{T}^2 \) whose minimal sets consist of a semi-attracting limit cycle \( \gamma_3 \), one semi-attracting and semi-repelling limit cycle \( \gamma_4 \), one semi-repelling limit cycle, and cycles as on the right of Figure 16. Then the abstract weak orbit space \( \mathbb{T}^2/\{v_h\} \cong \mathbb{T}^2/\{g\} \cong \mathbb{T}^2/\{w\} \) is homeomorphic to the abstract weak orbit space \( \mathbb{T}^2/\{v_h\} \cong \mathbb{T}^2/\{g\} \cong \mathbb{T}^2/\{w\} \) of the flow \( g \) in the previous example.

We state examples of Pseudo-Anosov homeomorphisms.

Example 7. Let \( v_g \) be the suspension flow on a closed 3-manifold \( M \) of a pseudo-Anosov homeomorphism \( g \) on a compact connected orientable surface \( S \). Then the abstract weak orbit space \( M/\{v_g\} \cong S/\{g\} \) contains abstract weak orbits contained in \( \text{Per}(v_g) \) and abstract weak orbits contained in dense orbits. In particular, Lemma 7.1 implies that the abstract weak orbit space \( M/\{v_g\} \cong S/\{g\} \) is not a singleton.

We state a flow \( v \) whose abstract weak orbit space with infinite height with respect to the binary relation \( \leq_v \) as in the proof of Proposition 1.

Example 8. Let \( v_f \) be the suspension flow of an Anosov diffeomorphism \( f \) on a closed manifold as in the proof of Proposition 1. Then the binary relation \( \leq_{v_f} \) on the abstract weak orbit space has infinite height.

Recall that a flow is topologically transitive if, for any pair of nonempty open subsets \( U \) and \( V \), there is a number \( t > 1 \) such that \( v_t(U) \cap V \neq \emptyset \). A topologically transitive flow \( w \) on a compact metric space \( M \) with \( \text{Cl}(w) = M \) is chaotic in the sense of Devaney if it is sensitive to initial conditions (i.e. there is a positive number \( \delta > 0 \) such that for any \( x \in M \) and any neighborhood \( U \) of \( x \), there are \( y \in U \) and \( T > 0 \) such that \( d(w^T(x), w^T(y)) > \delta \).

Example 9. Let \( w \) be a flow on a compact metric space \( M \) which is chaotic in the sense of Devaney. Then the abstract weak orbit space \( M/\{w\} \) contains abstract weak orbits contained in \( \text{Cl}(w) \) and abstract weak orbits contained in dense orbits. In particular, Lemma 7.1 implies that the abstract weak orbit space \( M/\{w\} \) is not a singleton.

7.3. Incompleteness of abstract weak orbit spaces and of Reeb graphs. Roughly speaking, the abstract weak orbit space of a flow has no information of spiral directions around limit circuits. Therefore abstract weak orbit space is not a complete invariant of Morse-Smale flows with limit cycles. In fact, there are
two Morse-Smale flows on a torus which are not topologically equivalent, but the abstract weak orbit spaces are isomorphic as in Figure 17. Moreover, an abstract weak orbit space has no information of cyclic orders around multi-saddles outside of the boundary of the surface. Therefore the abstract weak orbit space is not a complete invariant of Hamiltonian flows with finitely many singular points. Indeed, there are two Hamiltonian flows $v$ and $w$ on a disk $\mathbb{D}^2$ as in Figure 18 that are not topologically equivalent but whose abstract weak orbit spaces are isomorphic to each other. Moreover, the Reeb graphs of the Hamiltonian flows are also not complete. Indeed, let $v$ and $w$ be the flows as above in Figure 18. Then the abstract weak orbit spaces $\mathbb{D}/[v]$ and $\mathbb{D}/[w]$ are isomorphic. Since Reeb graphs of Hamiltonian flows on compact surfaces as abstract graphs corresponds to the extended weak orbit spaces $\mathbb{D}/[v]_{ex}$ and $\mathbb{D}/[w]_{ex}$ which are quotient spaces of abstract weak orbit spaces $\mathbb{D}/[v]$ and $\mathbb{D}/[w]$, the incompleteness of the abstract weak orbit spaces implies that one of the Reeb graphs.

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Figure 17. Two flows generated by Morse-Smale vector fields on a torus are not topologically equivalent but the abstract weak orbit spaces are isomorphic.
Figure 18. Two flows $v$ and $w$ on a disk $\mathbb{D}$ that are not topologically equivalent and whose multi-saddle connection diagrams are isomorphic as abstract multi-graphs but not isomorphic as plane graphs. In the diagram, dotted (resp. up, down) directed lines correspond to $\leq_{\phi}$ (resp. $\leq_{\alpha}$, $\leq_{\omega}$). In particular, the extended weak orbit spaces are isomorphic and are pre-ordered sets with the pre-orders $\leq_{v}$ each of which corresponds to the union of the pre-order $\leq_{v}$ and the binary relations $\leq_{\alpha}$, $\leq_{\omega}$ as a direct product.

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Received July 2021; revised October 2021; early access December 2021.

E-mail address: tomoo@gifu-u.ac.jp