On Hölder Continuity and Equivalent Formulation of Intrinsic Harnack Estimates for an Anisotropic Parabolic Degenerate Prototype Equation.

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Abstract

We give a proof of Hölder continuity for bounded local weak solutions to the equation

\[ u_t = \sum_{i=1}^{N} (|u_{x_i}|^{p_i-2}u_{x_i})_{x_i}, \quad \text{in } \Omega \times [0,T], \quad \Omega \subset \subset \mathbb{R}^N, \quad (0.1) \]

under the condition \(2 < p_i < \bar{p}(1 + 2/N)\) for each \(i = 1, \ldots, N\), being \(\bar{p}\) the harmonic mean of the \(p_i\)s, via recently discovered intrinsic Harnack estimates. Moreover we establish equivalent forms of these Harnack estimates within the proper intrinsic geometry.

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Dedicated to Prof. Francesco Altomare.

1 Introduction and main result.

Equations of the kind of (0.1) fall into the wide class of degenerate equations, because their coordinate modulus of ellipticity \(|u_{x_i}|^{p_i-2}\) vanishes as each coordinate derivative \(u_{x_i}\) goes to zero. This behaviour is classic in equations like the degenerate \(p\)-Laplacian

\[ u_t - \text{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad \text{in } \Omega \times (0,T), \quad \Omega \subset \subset \mathbb{R}^N, \quad (1.1) \]

as the classical modulus of ellipticity, \(|\nabla u|^{p-2}\nabla u\) vanishes when the gradient of the solution vanish itself. The theory of degenerate parabolic equations has been pioneered by the method of intrinsic scaling (see for instance [7]), which provides a correct interpretation of the evolution of the equation in a particular geometry dictated by the solution itself, hence the name. Nevertheless, the application of this method to the anisotropic case is not straightforward, as the degenerative behavior of the equation is purely directional i.e. some partial derivatives can vanish while some other may dictate the diffusion. When \(p_i \equiv p\) the equation (0.1) is a different equation from (1.1) and is named the orthotropic \(p\)-Laplacian. The prototype equation (0.1) reflects the modeling of many materials that reveal different diffusion rates along different directions,
such as liquid crystals, wood or earth’s crust (see [22]). The regularity of bounded local weak solutions to equations of the kind of (0.1) with measurable and bounded coefficients is still an open problem. The main difference with standard non linear regularity theory is the directional growth of the operator, usually referred to as non standard growth (see [1], [3]). This requires the definition of a new class of function spaces, called anisotropic Sobolev spaces (see Section 2), and whose study is still open and challenging.

1.1 The elliptic open problem.

Even in the elliptic case, the regularity theory for such equations requires a bound on the sparseness of the powers \( p_i \). For instance in the general case the weak solution can be unbounded, as proved in [14], [16]. Lipschitz bounds have been obtained by Marcellini in his pioneering work [18], supposing the coefficients are regular enough. However, the boundedness of solutions was proved in [3] under the assumption that

\[
P < N, \quad \max\{p_1, ..., p_N\} < \bar{p},
\]

where \( \bar{p} \) and \( \bar{p}^* \) are defined respectively in (2.1) and (2.2). Regularity properties are proved only on strong assumptions on the regularity of the coefficients (see [13], [18], [17]).

Even the elliptic case, when the coefficients are rough, Hölder continuity remains still nowadays an open problem. Indeed, continuity conditioned to boundedness has been proved in [11] by means of intrinsic scaling method, but with a condition of stability on the exponents \( p_i \) which is only qualitative. Removability of singularities has been considered in [23], where the idea of working with fundamental solutions in the anisotropic framework had yet been taken in consideration.

1.2 The parabolic open problem.

For what concerns the boundedness of local weak solutions to equations behaving as (0.1), it has been proved in [19], [12] that local weak solutions are bounded if

\[
p_i < \bar{p} \left(1 + \frac{2}{N}\right), \quad i = 1, ..., N.
\]

In [12] some useful \( L^\infty \) estimates are provided, together with finite speed of propagation and lower semi-continuity of solutions. These have been the starting point for the study of fundamental solutions to (0.1) (see for instance [4], [2]), and the behaviour of their support [5].

Recently an approach based on an expansion of positivity relying on the behaviour of fundamental solutions has brought the authors to prove in [5] the following Harnack inequality, properly structured in an intrinsic anisotropic geometry that we are about to describe. Fix a number \( \theta > 0 \) to be defined later, and define the anisotropic cubes

\[
\mathcal{K}_\rho(\theta) := \prod_{i=1}^{N} \left\{ x_i < \theta^{\frac{\overline{p}^*-\bar{p}}{\overline{p}^*}} \rho^{\overline{p}^*} \right\}.
\]

Next define the following centered, forward and backward anisotropic cylinders, for a generic point \((x_0, t_0)\):

\[
\begin{align*}
\text{centered cylinders:} \quad & (x_0, y_0) + Q_\rho(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times \{t_0 - \theta^{\overline{p}^*-\bar{p}} \rho^{\overline{p}^*}, t_0 + \theta^{2-\bar{p}} \rho \bar{p}\}; \\
\text{forward cylinders:} \quad & (x_0, y_0) + Q_\rho^+(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times \{t_0, t_0 + \theta^{2-\bar{p}} \rho \bar{p}\}; \\
\text{backward cylinders:} \quad & (x_0, y_0) + Q_\rho^-(\theta) = \{x_0 + \mathcal{K}_\rho(\theta)\} \times \{t_0 - \theta^{2-\bar{p}} \rho \bar{p}, t_0\}.
\end{align*}
\]

We use the following Theorem 1.1 as the essential tool to prove the Hölder continuity of solutions, as in the approach pioneered by J.Moser in [20] and successively developed in [8] for degenerate parabolic equations of \( p \)-Laplacian type with measurable coefficients.

**Theorem 1.1.** Let \( u \) be a non-negative local weak solution to (0.1) such that for some point \((x_0, t_0) \in \Omega_T\) we have \( u(x_0, t_0) > 0 \). There exist constants \( c, \gamma \) depending only on the data such that for all intrinsic cylinders \((x_0, t_0) + Q_{4\rho}^+(\theta)\) contained in \( \Omega_T \) as in (1.5) we have

\[
u(x_0, t_0) \leq \gamma \inf_{x_0 + \mathcal{K}_\rho(\theta)} u(x, t_0 + \theta^{2-\bar{p}} \rho \bar{p}), \quad \theta = \left(\frac{c}{u(x_0, t_0)}\right).
\]
implies local Hölder continuity of the solution to (2.1). When the elliptic counterpart is considered, we deal with stationary solutions to Remark 1.1.

and we define the Sobolev exponent of the harmonic mean

and deeply different from the isotropic case where the elliptic estimate holds in classic cubes, is the intrinsic geometry in which the expansion of positivity can be brought by means of the comparison principle (see [5]). In the present work we show that Theorem 1.1 implies local Hölder continuity of the solution to (0.1).

**Theorem 1.2.** Let \( u \) be a local weak solution to (0.1). Then \( u \) is locally Hölder continuous in \( \Omega_T \), i.e. there exist constants \( \gamma > 1, \alpha \in (0, 1) \) depending only on the data, such that for each compact set \( K \subset \subset \Omega_T \) we have

\[
|u(x, t) - u(y, s)| \leq \gamma ||u||_\infty \left( \sum_{i=1}^{N} |x_i - y_i|^{\frac{\beta - p_i}{\pi - p_i}} ||u||_\infty^{\frac{\beta - p_i}{\pi - p_i}} + |t - s| \frac{\gamma}{\pi} ||u||_\infty^{\frac{\beta - 2}{\pi}} \right)^{\alpha},
\]

for every pair of points \( (x, t), (y, s) \in K \), with

\[
\pi\text{-dist}(K, \partial \Omega_T) := \inf \left\{ \left( |x_i - y_i|^{\frac{\beta - p_i}{\pi - p_i}} \wedge |t - s| \frac{\gamma}{\pi} ||u||_\infty^{\frac{\beta - 2}{\pi}} \right) | (x, t) \in K, (y, s) \in \partial \Omega_T, i = 1..N \right\}
\]

Moreover with a similar approach to what is done for the case of isotropic operators in [9], we show that a re-formulation of it can be exhibited so that one recovers the classic Pini-Hadamard estimate (see [21] for the complete reference) when \( p_i \equiv 2 \) for all \( i = 1, .., N \):

\[
\gamma^{-1} \sup_{K_{\rho}(x_0)} u(\cdot, t_0 - \rho^2) \leq u(x_0, t_0) \leq \gamma \inf_{K_{\rho}(x_0)} u(\cdot, t_0 + \rho^2), \quad \gamma > 0,
\]

provided the parabolic cylinders \((x_0, t_0) + Q^+_{4\rho}\) are all contained in \( \Omega_T \). Indeed the following theorem can be shown to be consequence by the sole Theorem 1.1.

**Theorem 1.3.** Let \( u \) be a non-negative local weak solution to (0.1) such that for some point \((x_0, t_0) \in \Omega_T \) we have \( u(x_0, t_0) > 0 \). There exist constants \( c, \gamma \) depending only on the data such that for all intrinsic cylinders \((x_0, t_0) + Q^+_{4\rho}(\theta) \) contained in \( \Omega_T \) as in (1.5) we have

\[
\gamma^{-1} \sup_{x_0 + K_{\rho}(\theta)} u(x, t_0 - \theta^2 \rho^2) \leq u(x_0, t_0) \leq \gamma \inf_{x_0 + K_{\rho}(\theta)} u(x, t_0 + \theta^2 \rho^2), \quad \theta = \left( \frac{c}{u(x_0, t_0)} \right).
\]

**Remark 1.1.** When the elliptic counterpart is considered, we deal with stationary solutions to (0.1), so that the behavior at each time is always the same. In this context, easily deductible by (1.10), we get the usual sup-inf estimate with no need of a waiting time. What is essential (at least for the proof of (1.6) in [5]) and deeply different from the isotropic case where the elliptic estimate holds in classic cubes, is the intrinsic space geometry (1.4) of \( K_{\rho} \).

## 2 Preliminaries

Given \( p := (p_1, .., p_N) \), \( p > 1 \) with the usual meaning, we assume that the harmonic mean is smaller than the dimension of the space variables

\[
\overline{p} := \left( \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \right)^{-1} < N,
\]

and we define the Sobolev exponent of the harmonic mean \( \bar{p} \),

\[
\overline{p}^* := \frac{N\overline{p}}{N - \overline{p}}
\]

We will suppose without loss of generality along this note that the \( p_i \)'s are ordered increasingly. Next we introduce the natural parabolic anisotropic spaces. Given \( T > 0 \) and a bounded open set \( \Omega \subset \mathbb{R} \) we define

\[
W^{1,p}_0(\Omega) := \{ u \in W^{1,1}_0(\Omega) | D_i u \in L^p(\Omega) \}
\]
\[ W_{\text{loc}}^{1,p}(\Omega) := \{ u \in L^1_{\text{loc}}(\Omega) \mid D_i u \in L^p_{\text{loc}}(\Omega) \} \]

\[ L^p(0,T; W_{\text{loc}}^{1,p}(\Omega)) := \{ u \in L^1(0,T; W_{\text{loc}}^{1,p}(\Omega)) \mid D_i u \in L^p(0,T; L^p_{\text{loc}}(\Omega)) \} \]

By a local weak solution of (0.1) we understand a function

\[ u \in C^0(0,T; L^2_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0,T; W_{\text{loc}}^{1,p}(\Omega)) \]

such that for all \( 0 < t_1 < t_2 < T \) and any test function \( \varphi \in C^\infty(0,T; C^\infty(\Omega)) \) satisfies

\[ \int_\Omega u \varphi \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega (-u \varphi_t + \sum_{i=1}^{N} |u_{x_i}|^{p-2} u_{x_i} \varphi_{x_i}) \, dx \, dt = 0. \] (2.3)

By a density and approximation argument this actually holds for any test function of the kind

\[ \varphi \in W^{1,2}_{\text{loc}}(0,T; L^2_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0,T; W_{\text{loc}}^{1,p}(\Omega)) \]

for any \( p_i \)-semirectangular domain \( \Omega \subset \subset \mathbb{R}^N \), where traces can be properly defined (see Theorem 3 in [15]).

**Definition 2.1.** [15] If the set of the \( n \) elements of the vector \( (p_1, \ldots, p_N) \) consists of \( L \) distinct values, let us denote the multiplicity of each of the values by \( n_i, i=1,\ldots,L \) such that \( n_1 + \ldots + n_L = n \). We say that a bounded domain \( \Omega \subset \mathbb{R}^N \) satisfies the \( p_i \) semirectangular restriction related to the vector \( (p_1, \ldots, p_N) \), if there exists bounded Lipschitz domains \( \Omega_i \subset \mathbb{R}^n, i=1,\ldots,L, \) such that \( \Omega = \Omega_1 \times \ldots \times \Omega_L \).

### 3 Proof of Theorem 1.2.

We employ the intrinsic Harnack inequality of Theorem 1.6 to establish locally quantitative Hölder estimates for local, weak solutions \( u \) of (0.1), conditioned to the boundedness condition (1.3). Fix a point \( (x_0, t_0) \in \Omega_T \) which, up to translations, we will consider to be the origin in \( \mathbb{R}^{N+1} \), and for an initial radius \( \rho_0 > 0 \), consider the cylinder \( Q' = K_{\rho_0} \times (-\rho_0^2, 0] \), with vertex at \((0,0)\), and set

\[ M_0 = \sup_{Q'} u, \quad m_0 = \inf_{Q'} u, \quad \text{and} \quad \omega_0 = M_0 - m_0 = \text{osc}_u. \]

With \( \omega_0 \) at hand we can construct the initial cylinder for our purpose, intrinsically scaled:

\[ Q_0 = K_{\rho_0} \times (-\rho_0^2, 0], \]

and \( \omega_0 \) is to be determined later only in terms of the data and independent of \( u, \rho_0 \). The accommodation of degeneracy deals with the following fact: if \( \omega_0 > c\rho_0 \), then \( Q_0 \subset Q' \). Converse inequality would lead directly to continuity in \( Q_0 \).

**Proposition 3.1.** Either \( \omega_0 \leq c\rho_0 \), or there exist numbers \( \gamma > 1, \delta, \varepsilon \in (0,1) \), that can be quantitatively determined only in terms of the data and independent of \( u, \rho_0 \), such that if we set

\[ \omega_n = \delta \omega_{n-1}, \quad \theta_n = \left( \frac{c}{\omega_n} \right), \quad \theta_i,n = \left( \frac{c}{\omega_n} \right), \quad \rho_n = \varepsilon \rho_{n-1}, \quad \text{and} \]

\[ Q_n = Q_{\rho_n}(\theta_n) = \prod_{i=1}^{N} \left\{ |x_i| < \theta_i,n \rho_n \right\} \times (-\theta_i,n^2 \rho_n^2, 0], \quad \text{for} \quad n \in \mathbb{N}, \quad \text{holding} \quad Q_{n+1} \subset Q_n, \]

then the oscillation in \( Q_n \) can be controlled with the same constant by the oscillation in \( Q_{n-1} \), i.e.

\[ \text{osc}_u \leq \omega_n. \] (3.2)
Proof. The proof is by induction: we show constants \( \delta, \varepsilon, c \) depending only upon the data, such that if the statement holds for the \( n \)-th step, then it holds the \( n + 1 \)-th. Coherently with the accommodation of degeneracy, the first inductive step has already been stated. Assume now that \( Q_n \) has been constructed and that the statement holds up to \( n \). Set

\[
M_n = \sup_{Q_n} u, \quad m_n = \inf_{Q_n} u, \quad \text{and} \quad P_n = (0, \frac{1}{2} \theta_n \rho_n^\theta).
\]

The point \( P_n \) is roughly speaking the point whose coordinates are the mid-point of each coordinate of \( Q_n \). On a first glance, we observe that we can assume

\[
\omega_n \leq M_n - m_n = \text{osc}_n u,
\]

because otherwise there is nothing to prove. At least one of the two inequalities

\[
M_n - u(P_n) > \frac{1}{4} \omega_n, \quad \text{or} \quad u(P_n) - m_n > \frac{1}{4} \omega_n
\]

must hold. Indeed if it is not so, we arrive to the absurd

\[
\omega_n \leq M_n - m_n \leq \frac{1}{2} \omega_n.
\]

We assume that the first inequality holds true, the demonstration for the second case is similar. The function \((M_n - u)\) is a nonnegative weak solution of (0.1) in \(Q_n\), and satisfies the intrinsic Harnack inequality (1.6) with respect to \(P_n\), if its waiting time and space

\[
\left(\frac{c_1}{M_n - u(P_n)}\right)^{\bar{p} - 2} \leq \left(\frac{c_1}{M_n - u(P_n)}\right)^{\bar{p} - 1}
\]

are inside the respective time and space of \(Q_n\). To this aim, we define \(c\) to be greater than \(c_1\) so that

\[
\left(\frac{c_1}{M_n - u(P_n)}\right) \leq \left(\frac{c}{\omega_n}\right), \quad \text{as} \quad \omega_n \leq 4 \left(M_n - u(P_n)\right).
\]

Finally, we have the estimate

\[
\inf_{Q_{n+1}} (M_n - u) \geq \frac{1}{\gamma} (M_n - u(P_n)) > \frac{1}{4\gamma} \omega_n.
\]

This means, as \(\inf(-u) = -\sup u\), that

\[
M_n \geq \sup_{Q_{n+1}} u + \frac{1}{4\gamma} \omega_n \geq \sup_{Q_{n+1}} u + \frac{1}{4\gamma} \omega_n, \quad \text{if} \quad Q_{n+1} \subset Q_{\rho_n/4} \subset Q_{\rho_n},
\]

leading us, by subtracting \(\inf_{Q_{n+1}} u\) from both sides, to

\[
M_n - \inf_{Q_n} u \geq M_n - \inf_{Q_{n+1}} u \geq \text{osc}_{Q_n+1} u + \frac{1}{4\gamma}, \quad \text{if} \quad Q_{n+1} \subset Q_{\rho_n/4} \subset Q_{\rho_n},
\]

and thus to

\[
\text{osc}_{Q_{n+1}} u \leq \delta \omega_n = \omega_{n+1}, \quad \text{if} \quad Q_{n+1} \subset Q_{\rho_n/4} \subset Q_{\rho_n}.
\]

By choosing

\[
\delta = 1 - \frac{1}{4\gamma}, \quad \text{and} \quad \varepsilon = \frac{1}{4\gamma} \frac{\omega_{n+1}}{\omega_n}, \quad (3.3)
\]

we manage to have both the inclusion \(Q_{n+1} \subset Q_{\rho_n/4} \subset Q_{\rho_n}\) and the \((n + 1)\)-th conclusion of the iterative step (3.2). Indeed, by direct computation,

\[
\theta_{n+1}^\rho \rho_{n+1}^\theta = \left(\frac{c}{\omega_n+1}\right)^{\bar{p} - 2} \left(\frac{\rho_n^\theta}{(4\gamma - 1)\omega_{n+1}}\right)^{\bar{p} - 2} \left(\frac{\rho_n}{4}\right)^{\bar{p}} = \left(\frac{c}{\omega_n}\right)^{\bar{p} - 2} \left(\frac{\rho_n}{4}\right)^{\bar{p}} = \theta_n^\rho (\rho_n/4)^{\bar{p}},
\]
precisely, and for each \( i \in \{1, \ldots, N\} \) as \( p_i > 2 \) there holds

\[
\frac{\theta_{i,n+1} \theta_i}{\omega_{n+1}} = \left( \frac{c}{\omega_n} \right)^{\frac{\theta_{i,n}}{p_i}} \left( \frac{\rho_n}{4} \left( 1 - \frac{1}{4\gamma} \right)^{\frac{\theta_{i,n}}{p_i}} \right) \rho_n = \left( \frac{c}{(1 - \frac{1}{4\gamma})\omega_n} \right)^{\frac{\theta_{i,n}}{p_i}} \left( 1 - \frac{1}{4\gamma} \right)^{\frac{\theta_{i,n}}{p_i}} \rho_n \leq \frac{\theta_{i,n}}{\omega_n} \left( \frac{\rho_n}{4} \right)^{\frac{\theta_i}{p_i}}.
\]

3.1 Conclusion of the proof of Theorem 1.2.

From the previous proposition, we have the conclusion that on each such intrinsically scaled cylinder \( \mathcal{Q}_n \) there holds \( \text{osc}_{\mathcal{Q}_n} u \leq \omega_n \), so that by induction and the definition of \( \omega_n \) we have

\[
\text{osc}_{\mathcal{Q}_n} u \leq \delta^n \omega_0.
\]

Let now \( 0 < \rho < r \) be fixed, and observe that there exists a \( n \in \mathbb{Z} \) such that, by use of (3.3), we have

\[
e^{n+1} r \leq \rho \leq e^n r.
\]

This implies

\[
(n + 1) \geq \ln \left( \frac{\rho}{r} \right) = \ln \left( \frac{\rho}{r} \right) = \alpha \quad \text{with} \quad \alpha = \frac{\ln(\delta)}{|\ln(\epsilon)|},
\]

by an easy change of basis on the logarithm. Thus by (3.3) and (3.4) we get

\[
\text{osc}_{\mathcal{Q}_n} u \leq \text{osc}_{\mathcal{Q}_n} u \leq \omega_0 \left( \frac{\rho}{r} \right)^\alpha.
\]

Now finally we give Hölder conditions to each variable, irrespective to the others.

Fix \((x, t), (y, s) \in K, s > t, \) let \( R > 0 \) to be determined later, and construct the intrinsic cylinder \((y, s) + \mathcal{Q}_R(M)\), where \( M = \|u\|_{L^\infty(\Omega_T)} \). This cylinder is contained in \( \Omega_T \) if the variables satisfy for each \( i = 1, \ldots, N, \)

\[
M^{\frac{n-1}{p_i}} R^{\frac{\theta_i}{p_i}} \leq \inf \left\{ |x_i - y_i|, \quad \text{for} \quad x \in K, y \in \partial \Omega \right\} \quad \text{and} \quad M^{\frac{n-1}{2}} R \leq \inf_{t \in K} t^\frac{\theta_i}{2}.
\]

This is easily achieved if we set, for instance,

\[
2R = \pi \text{-dist}(K; \partial \Omega).
\]

To prove the Hölder continuity in the variable \( t \), we first assume that \((s - t) \leq M^{2-\bar{p}} R^{\bar{p}} \). Then \( \exists \rho < -0 \in (0, R) \) such that \((s - t)^\frac{1}{2} M^{\frac{n-2}{2}} = \rho < -0 \), and the oscillation (3.5) gives

\[
\text{osc}_{\mathcal{Q}_\rho} u \leq \gamma \omega_0 \left( \frac{\rho}{R} \right)^\alpha,
\]

implying

\[
|u(x, s) - u(x, t)| \leq \gamma M \left( \frac{M^{\frac{n-2}{2}} |s - t|^{\frac{1}{2}}}{\pi \text{-dist}(K; \partial \Omega)} \right)^\alpha,
\]

as claimed. If otherwise \( s - t \geq M^{2-\bar{p}} R^{\bar{p}} \) then, exploiting the fact that \( \rho_0^2 \leq 4R \), we have

\[
|u(x, s) - u(x, t)| \leq |u(x, s)| + |u(x, t)| \leq 2M \left( \frac{M^{\frac{n-2}{2}} |s - t|^{\frac{1}{2}}}{\pi \text{-dist}(K; \partial \Omega)} \right)^\alpha.
\]

About the space variables, we have for each \( i \)-th one the following alternative:
• If $|y_i - x_i| < M^{p_0 - p} R^\frac{p}{p_1}$, and then analogously $\exists \rho_0 \in (0, R)$ such that $\rho_0 = |y_i - x_i| M^{p_0 - p}$ and the oscillation reduction \((3.5)\) gives

$$\operatorname{osc} u \leq \operatorname{osc} u \leq \gamma \omega_0 \left(\frac{\rho_0}{R}\right)^\alpha, \quad \Rightarrow \quad |u(y_i, t) - u(x_i, t)| \leq M \left(\frac{|y_i - x_i| M^{p_0 - p}}{\pi \operatorname{dist}(k; \partial \Omega_T)}\right)^\alpha.$$ 

• If otherwise $|y_i - x_i| \geq M^{p_0 - p} R^\frac{p}{p_1}$ then similarly

$$|u(y_i, t) - u(x_i, t)| \leq 2M \leq 4M \left(\frac{|y_i - x_i| M^{p_0 - p}}{\pi \operatorname{dist}(k; \partial \Omega_T)}\right)^\alpha.$$ 

The proof is completed.

4 Proof of Theorem 1.10.

We take as hypothesis that for each radius $r > 0$ such that the intrinsic cylinder $Q_r(\theta)$ is contained in $\Omega_T$, the right-hand Harnack estimate \((1.6)\) holds and we show that the full Harnack estimate \((1.10)\) comes as a consequence.

4.1 Step 1.

Let us suppose that there exists a time $t_1 < t_0$ such that

$$u(x_0, t_1) = 2\gamma u(x_0, t_0), \quad (4.1)$$

where $\gamma, c > 0$ are the constants in \((1.6)\). For such a time there must hold

$$t_0 - t_1 > \theta_1^{p_0 - 2} r^{\bar{p}} := c u(x_0, t_1)^{2 - \bar{p}} r^{\bar{p}} = c \frac{u(x_0, t_0)^{2 - \bar{p}} r^{\bar{p}}}{(2\gamma)^{\bar{p} - 2} r^{\bar{p}}}, \quad (4.2)$$

owing last equality to \((4.1)\). Indeed if \((4.2)\) were violated then $t_0 \in [t_1, t_1 + \theta_1^{2 - \bar{p}} r^{\bar{p}}]$, and by applying \((1.6)\) evaluated in $(x_0, t_1)$ for a radius $r > 0$ small enough, we would incur a contradiction

$$u(x_0, t_1) \leq \gamma u(x_0, t_0) \quad \iff \quad 2\gamma u(x_0, t_0) \leq u(x_0, t_0).$$

So \((4.2)\) holds, and we set $t_2$ to be the time

$$t_2 = t_0 - \theta_1^{2 - \bar{p}} r^{\bar{p}}. \quad (4.3)$$

By \((4.2)\) we deduce that $t_1 < t_2 < t_0$ and again by the right-hand Harnack estimate \((1.6)\) we have that

$$u(x_0, t_0) = \frac{u(x_0, t_1)}{2\gamma} \leq u(x_0, t_2) < 2\gamma u(x_0, t_0), \quad (4.4)$$

where the last inequality comes from $t_1$ being the first time before $t_0$ respecting \((4.1)\). The contradiction of \((4.1)\) is, in our context $u(x_0, t_2) < 2\gamma u(x_0, t_0)$, because the converse inequality conflicts with our hypothesis \((1.6)\). Now, let $r > 0$ be fixed, as in \((1.6)\), and consider the vector $\xi \in \mathbb{R}^N$ whose components are

$$\xi_1 := \theta_1^{\frac{p_0 - p}{p_1}} r^{p_1}, \quad \theta = \left(\frac{c}{u(x_0, t_0)}\right). \quad (4.5)$$

Now for each vector of parameters $s \in [0, 1]^N$ define $\xi_s = (s_1 \xi_1, \ldots, s_N \xi_N)$. As $s$ varies in $[0, 1]^N$, the configuration $x_0 + \xi_s$ describes all points of $x_0 + K_r(\theta)$. Consider $s \in [0, 1]^N$ such that the vector $\xi_s$ satisfies $u(x_0 + \xi_s, t_2) = 2\gamma u(x_0, t_0)$. We claim that such an $s$ does not exist or that $s \geq 1$: in either case the conclusion is that

$$\sup_{x_0 + K_r(\theta)} u(\cdot, t_2) \leq 2\gamma u(x_0, t_0). \quad (4.6)$$
Thus to establish the claim, assume that such vector $\bar{s}$ exists and that $\bar{s} < 1$. Apply the estimate (1.6) in the point $(x_2, t_2)$ with $x_2 = x_0 + \xi \bar{s}$ to get
\[
u(x_2, t_2) \leq \gamma \inf_{x_2 + \mathcal{K}_r(\theta_{t_2})} u(\cdot, t_2 + \theta_{t_2}^{\bar{s}-2} r^{\bar{p}}) = \inf_{x_2 + \mathcal{K}_r(\theta_{t_2})} u(\cdot, t_0), \quad \text{being } \theta_{t_2} = \frac{c}{u(x_2, t_2)},
\]
where last equality holds because of
\[
t_2 + \theta_{t_2}^{\bar{s}-2} r^{\bar{p}} = t_0 - \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{p}-2} r^{\bar{p}} + \left(\frac{c}{2\gamma u(x_2, t_2)}\right)^{\bar{p}-2} r^{\bar{p}} = t_0,
\]
being $x_2$ the point for which holds $u(x_2, t_2) = u(x_0 + \xi \bar{s}, t_2) = 2\gamma u(x_0, t_0)$ by assumption. But since $\bar{s} < 1$, then $x_0 \in \{x_2 + \mathcal{K}_r(\theta_{t_2})\}$ and we arrive to the contradiction
\[
2\gamma u(x_0, t_0) = u(x_2, t_2) \leq \gamma \inf_{x_2 + \mathcal{K}_r(\theta_{t_2})} u(\cdot, t_0) \leq \gamma u(x_0, t_0).
\]
Finally, the contradiction implies that (4.6) holds, which means that for each $r > 0$ such that $Q_{4r}(\theta) \subseteq \Omega_T$ there holds
\[
\sup_{x_0 + \mathcal{K}_r(\theta)} u(\cdot, t_0 - \theta_{t_0}^{\bar{s}-2} r^{\bar{p}}) \leq 2\gamma^2 u(x_0, t_0).
\]
Let $\rho > 0$ be such that the right hand side of (1.10) holds, then by choosing $r = \rho(2\gamma)^{\frac{p-2}{\bar{p}}} \bar{p}$ we obtain, by suitably redefining the constants, the full estimate (1.10).

### 4.2 Step 2

Suppose on the contrary that such a time $t < t_0$ for which holds (4.1) does not exist. In this case there must hold
\[
u(x_0, t) < 2\gamma u(x_0, t_0), \quad \text{for all } t \in [t_0 - \theta(4r)^{\bar{p}}, t_0],
\]
because the converse inequality would be in conflict with the holding Harnack estimate. We establish by contradiction that this in turn implies
\[
\sup_{x_0 + \mathcal{K}_r(\theta)} u(\cdot, t_0 - \theta_{t_0}^{\bar{s}-2} r^{\bar{p}}) \leq 2\gamma^2 u(x_0, t_0).
\]
If not, there contemporally hold (4.8) and a fortiori
\[
\sup_{x_0 + \mathcal{K}_r(\theta)} u(\cdot, \bar{t}) > 2\gamma^2 u(x_0, t_0) > u(x_0, \bar{t}), \quad \text{for } \bar{t} = t_0 - \theta_{t_0}^{\bar{s}-2} r^{\bar{p}}.
\]
Thus by the proven continuity in space, there must exist by the intermediate value theorem a point $\bar{x} \in x_0 + \mathcal{K}_r(\theta)$ such that
\[
u(\bar{x}, \bar{t}) = 2\gamma u(x_0, t_0).
\]
We apply the Harnack estimate (1.6) centered in $(\bar{x}, \bar{t})$ to get
\[
u(\bar{x}, \bar{t}) \leq \gamma \inf_{\bar{x} + \mathcal{K}_r(\bar{t}_i)} u(\cdot, \bar{t} + \theta_{\bar{t}_i}^{\bar{s}-2} r^{\bar{p}}), \quad \text{where } \bar{t}_i = \frac{c}{u(\bar{x}, \bar{t})}.
\]
Now, as $\gamma > 1$ and $p_i > 2$ for each $i \in \{1, \ldots, N\}$ we have
\[
\begin{aligned}
\theta_{\bar{t}_i}^{\bar{s}-\bar{p}} r^{\bar{p}} &= \left(\frac{2}{2\gamma u(x_0, t_0)}\right)^{\bar{s}-\bar{p}} r^{\bar{p}} \geq \left(\frac{2}{2\gamma u(x_0, t_0)}\right)^{\bar{s}-\bar{p}} r^{\bar{p}} = \theta_{\bar{t}_i}^{\bar{s}-\bar{p}} r^{\bar{p}} \Rightarrow x_0 \in \{\bar{x} + \mathcal{K}_r(\bar{t}_i)\},
\
\bar{t}_i + \theta_{\bar{t}_i}^{\bar{s}-2} r^{\bar{p}} &= t_0 - \left(\frac{c}{u(x_0, t_0)}\right)^{\bar{s}-2} r^{\bar{p}} + \left(\frac{c}{2\gamma u(x_0, t_0)}\right)^{\bar{s}-2} r^{\bar{p}} < t_0.
\end{aligned}
\]
and thus, finally,
\[
2\gamma^2 u(x_0, t_0) = u(\bar{x}, \bar{t}) \leq \gamma u(x_0, \bar{t} + \theta_{\bar{t}_i}^{\bar{s}-2} r^{\bar{p}}) < 2\gamma^2 u(x_0, t_0),
\]
oweing last inequality to (4.10) and establishing (4.9) by contradiction. Finally estimate (4.9), by eventually redefining the constants, is the desired left-hand estimate of (1.10).
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