THE ACTION BY NATURAL TRANSFORMATIONS OF A GROUP ON A DIAGRAM OF SPACES

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Abstract. For $C$ a $G$-category, we give a condition on a diagram of simplicial sets indexed on $C$ that allows us to define a natural $G$-action on its homotopy colimit, and in some other simplicial sets and categories defined in terms of the diagram. Well-known theorems on homeomorphisms and homotopy equivalences are generalized to an equivariant version.

1. Introduction

Let $G$ be a group, and $C$ be any small category. Consider a $C$-diagram of simplicial sets, where the values of the diagram have a $G$-action. Then several structures defined in terms of the diagram, like the colimit and the homotopy colimit, have an induced structure of $G$-object. However, it is often the case that one has a diagram $F : C \to D$ where $C$ is a small $G$-category, $D$ is an arbitrary category and the values of $F$ do not necessarily have a $G$-action, however the homotopy colimit of $F$ does have it. This situation was considered in [3], and independently, by this author in his Ph. D. thesis [9], where the concept of an action of a group $G$ on a functor $F$ by natural transformations is introduced. Here we define it formally in section 3, after the basic definitions in section 2.

We show that there are induced $G$-actions on colimits, coends, and bar and Grothendieck constructions of functors on which $G$ acts by natural transformations. In section 4 we consider the homotopy colimit, and show some basic identities involving the constructions defined so far. In section 5 we prove the equivariant homotopy invariance of the bar construction. Finally, in section 6 we prove the equivariant versions of the four theorems listed in [1, p. 154] about the homotopy colimit. Some of them were noted in [3], however a mild additional hypothesis lets us obtain a more precise result.

Some of the proofs are as those in [2], adapted for the case of the group action. However we include more details than in the cited paper, given that homotopy colimit methods have recently been used by non-topologists, see for example [11].

2. Preliminaries

Let $G$ be a finite group. We will denote by $G$ the category with a single object $*$, in which $\text{hom}_G(*,*) = G$ and the composition corresponds to group multiplication. For $n \geq 0$, let $[n]$ be the category associated to the poset $\{0, 1, \ldots, n\}$ with the usual order.
Let $\text{SCat}$ be the category of small categories and $\Delta$ be the full subcategory of $\text{SCat}$ with objects $\{ [n] \mid n \geq 0 \}$. If $\mathcal{C}$ and $\mathcal{D}$ are categories, with $\mathcal{C}$ small, we denote by $\mathcal{D}^\mathcal{C}$ the category of functors $\mathcal{C} \to \mathcal{D}$ ([4, page 40]). The category of simplicial sets (see [5]), denoted $\text{sSet}$, is equal to $\text{Set}^{\Delta^{op}}$. The category of small $G$-categories is defined as $\text{SCat}^G$. We identify a small $G$-category with the image of the functor $G \to \text{SCat}$. From now on, $\mathcal{C}$ will denote a small $G$-category. Note that for each $g \in G$ we have a functor $g: \mathcal{C} \to \mathcal{C}$, and composition of functors correspond to group multiplication. We consider the nerve functor $N: \text{SCat} \to \text{sSet}$, given by $\mathcal{C} \mapsto (\text{hom}_{\text{SCat}}(-, \mathcal{C}): \Delta^{op} \to \text{Set})$. The nerve functor sends $G$-categories to $G$-simplicial sets. There is also a geometric realization functor $|\cdot|: \text{sSet} \to \text{Top}$, that sends $G$-simplicial sets to $G$-topological spaces. We denote $|N(\mathcal{C})|$ simply as $|\mathcal{C}|$.

If $\mathcal{D}$ is any category, an object $D$ in it is called a $G$-object if there is a collection of $\mathcal{D}$-maps $\{g: D \to D\}$, indexed by the elements of $G$, such that the map corresponding to the identity element is the identity, and composition of maps corresponds to group multiplication.

If $X$ and $Y$ are $G$-topological spaces, a $G$-homotopy from $X$ to $Y$ is a continuous map $H: X \times [0, 1] \to Y$ such that $H(gx, t) = gH(x, t)$ for all $g \in G$, $x \in X$ and $t \in [0, 1]$. Two $G$-maps $f_1, f_2: X \to Y$ are $G$-homotopic if there is a $G$-homotopy $H$ from $X$ to $Y$ such that $H(x, 0) = f_1(x)$ and $H(x, 1) = f_2(x)$. In this case we write $f_1 \simeq_G f_2$. The $G$-topological spaces $X$ and $Y$ are $G$-homotopy equivalent if there are $G$-maps $f: X \to Y$ and $f': Y \to X$ such that $ff' \simeq_G 1_X$ and $f'f \simeq_G 1_Y$.

We say that two $G$-categories $\mathcal{C}_1, \mathcal{C}_2$ are $G$-homotopy equivalent if the spaces $|\mathcal{C}_1|, |\mathcal{C}_2|$ are. It is not required that the map $|\mathcal{C}_1| \to |\mathcal{C}_2|$ defining the homotopy equivalence is induced from a functor $\mathcal{C}_1 \to \mathcal{C}_2$.

If we have a functor $F: \mathcal{C} \to \mathcal{D}$ together with natural transformations $\eta_g: F \to Fg$ such that $\eta_1$ is the identity and $\eta_{gg'} = \eta_g \eta_{g'}$, Jackowski and Słomińska call $F$ a right $G$-functor [3]. Independently, I defined and used the same concept in my Ph. D. thesis [9], and said that in such situation, $G$ acts on $F$ by natural transformations, or simply that $G$ acts on the functor $F$. In this paper, we will use both terms indistinctly.

In the case that $\mathcal{C}$ and $\mathcal{D}$ are small $G$-categories, and $F: \mathcal{C} \to \mathcal{D}$ is a functor such that $F(gC) = gF(C)$, $F(g\phi) = gF(\phi)$ for all $g \in G$, $C \in \text{obj} \mathcal{C}$ and all $\mathcal{C}$-morphisms $\phi$, we will say that $F$ is an equivariant functor.

3. Definition and Examples

We define now our main subject of study in detail:

**Definition 3.1.** Let $F: \mathcal{C} \to \mathcal{D}$ a functor, where $\mathcal{C}$ is a small $G$-category and $\mathcal{D}$ is an arbitrary category. Suppose that we are given a family of $\mathcal{D}$-maps $\eta = \{ \eta_g, X: F(X) \to F(gX) \}$ indexed by $g \in G$ and $X \in \text{obj} \mathcal{C}$ such that

1. $\eta_{1, X} = 1_{F(X)}$ for all $X \in \text{obj} \mathcal{C}$,
2. $\eta_{g_1, g_2, X} = \eta_{g_1, g_2, X}$ for any $X \in \text{obj} \mathcal{C}$, $g_1, g_2 \in G$,
3. $\eta_{g, X} \circ F(f) = F(gf) \circ \eta_{g, X}$ for any $g \in G$ and $f: X \to Y$ a map in $\mathcal{C}$.

Then, we will say that the family $\eta$ defines an action of $G$ on the functor $F$, or more succinctly, that $G$ acts on the functor $F$, or, following [3], that $F$ is a right $G$-functor.
Definition 3.2. Let $F_1, F_2 : C \to D$ be two functors on which $G$ acts, by $\eta^1, \eta^2$ respectively. A morphism of functors with $G$-action is a natural transformation $\epsilon : F_1 \to F_2$ such that $\eta^2_{g,X} \circ \epsilon_X = \epsilon_{gX} \circ \eta^1_{g,X}$ for all $g \in G, X \in \text{obj} C$.

As it is mentioned in [3] and [9], the usefulness of this concept lies on the fact that, when $G$ acts on $F$, there is a natural action of $G$ on the simplicial set $\text{hocolim} F$, and in several other structures defined in terms of $F$. On the other hand, it is often the case that we can derive a functor on which $G$ acts by natural transformations from a $G$-object. We show some examples.

Example 3.3. Let $C$ and $D$ be $G$-categories, and $F : C \to D$ an equivariant functor. Then, for $D, D' \in \text{obj} D$, we have a category $D \setminus F / D'$ with objects $\text{obj}(D \setminus F / D') = \{ (u,C,v) \mid C \in \text{obj} C, D \xrightarrow{u} FC \xrightarrow{\rho} D' \}$, and a morphism $p : (u,C,v) \to (u',C',v')$ given by a $C$-map $p : C \to C'$ such that $F(p) \circ u = u'$ and $v' \circ F(p) = v$.

There is a functor $D^{\text{op}} \times D \to \text{SCat}$ defined on objects by $(D,D') \mapsto D \setminus F / D'$. If $(\phi, \psi) : (D,D') \to (E,E')$ is a morphism in $D^{\text{op}} \times D$, the associated functor $D \setminus F / D' \to E \setminus F / E'$ sends $(u,C,v)$ to $(u\phi, C, \psi v)$.

Then $D^{\text{op}}$ and $D^{\text{op}} \times D$ have an obvious structure of $G$-categories, and there is an action of $G$ on the functor $D^{\text{op}} \times D \to \text{SCat}$ we just defined: for $g \in G$, set $\eta_g (D,D') = gD \setminus F / gD'$ as the functor $D \setminus F / D' \to gD \setminus F / gD'$ given by $(u,C,v) \mapsto (gu, gC, gv)$. Note, for example, that for $u : D \to F(C)$, we have that $gu : gD \to gF(C) = F(gC)$.

In this context, we can also define categories $D \setminus F$ and $F \setminus D$ with the obvious objects and morphisms, and obtain functors $D^{\text{op}} \to \text{SCat}$, $D \to \text{SCat}$ with a $G$-action. If $\nu : F_1 \to F_2$ is an equivariant natural transformation (i.e. a natural transformation such that $g\nu_C = \nu_{gC}$), then there is an induced morphism of right $G$-functors $\tilde{\nu} : - \setminus F_1 \to - \setminus F_2$, given by $\tilde{\nu}_D : D \setminus F_1 \to D \setminus F_2, (u,C) \mapsto (\nu_C u, C)$.

Example 3.4. Again, let $C$ and $D$ be $G$-categories, and $F : C \to D$ an equivariant functor. There is a functor $C^{\text{op}} \times C \to \text{Set}$ defined on objects by $(X,Y) \mapsto \text{hom}_D(FX,FY)$ and on morphisms by $(\phi, \psi) \mapsto (f \mapsto F\phi \circ f \circ F\psi)$. It has a $G$-action defined by $\eta_{g,(X,Y)} : \text{hom}_D(FX,FY) \to \text{hom}_D(gFX,gFY)$, $f \mapsto gf$.

Since any set $X$ can be considered as a simplicial set $Y$ such that $Y_n = X$ for all $n$ and all faces and degeneracies equal to the identity, we can as well consider the last function as taking values in the category of simplicial sets.

Example 3.5. Let $C$ be a $G$-category and $F : C \to D$ a right $G$-functor with $G$-action given by $\eta$. Assume that $F$ has a colimit, that is, there is an object $\text{colim} F$ in $D$ and a collection of $D$-maps $\{ \rho_X : FX \to \text{colim} F \}_{X \in \text{obj} C}$ that form a limiting cone from $F$ with base $\text{colim} F$ (see for example [4, p. 67]). Let $g \in G$. Then the natural transformation $F \to Fg$ induces a map $g : \text{colim} F \to \text{colim} F \cong \text{colim} F$ such that $\rho_{gX} \circ \eta_{g,X} = g \circ \rho_X$ for all $X \in \text{obj} C$. It can be shown that the collection of maps $\{ g : \text{colim} F \to \text{colim} F \}_{g \in G}$ give an structure of $G$-object on $\text{colim} F$. Furthermore, if $Z$ is any $G$-object in $D$ and there is a cone $\{ \sigma_X : FX \to Z \}$ from $F$ to $Z$ such that $\sigma_{gX} \circ \eta_{g,X} = g \circ \sigma_X$ for all $g \in G$ and all $X \in \text{obj} C$, then the map induced by the properties of the colimit $M : \text{colim} F \to Z$ is in fact equivariant.

For example, if $C$ is a discrete small $G$-category, then it can be identified with a $G$-set. A functor $F : C \to D$ corresponds to a collection of $D$-objects, indexed by the objects of $C$. If $F$ is a right $G$-functor, then $\text{colim} F = \coprod_{C \in \text{obj} C} F(C)$ is a $G$-object.

As a particular case, consider $H \leq G$ a subgroup, and let $C$ be the discrete small $G$-category with object set $G//H = \{ a_1 H, a_2 H, \ldots, a_n H \}$, that is, the set of left
cosets of \( H \) in \( G \) with the usual action by left translation, where \( a_1 = 1 \). Let \( Z \) be an \( H \)-simplicial set, and consider the constant functor \( F: C \to sSet \) with value \( Z \). We define a \( G \)-action \( \eta \) on \( F \) as follows: Let \( \eta_{g,H}: F(H) \to F(gH) \) be defined as \( z \mapsto hz \), where \( g = a_i h, h \in H \); and then \( \eta_{g,aH}(z) = \eta_{ga,H}(z) \). It is straightforward to check that this defines an action of \( G \) on \( F \), and so \( \text{colim} F \) is a \( G \)-simplicial set. This construction is usually known as the induced action from \( H \) to \( G \). We will denote \( \text{colim} F \) in this case as \( Z \uparrow^G_H \).

We also note that a morphism of right \( G \)-functors induced an equivariant map between the corresponding colimits of the functors.

**Example 3.6.** In a similar way, if \( Z: C \times C^{\text{op}} \to D \) is a right \( G \)-functor with a coend (see [4, p. 226]) with defining maps \( \alpha_C: Z(C,C) \to \text{coend} Z \), then \( \text{colim} Z \) becomes a \( G \)-object, with action satisfying \( \alpha_{gC} \circ \eta_{g,(C,C)} = g \circ \alpha_C \).

For example, let \( F: C \to sSet, T: C^{\text{op}} \to sSet \) be functors, with actions of \( G \) on both \( F \) and \( T \), given by \( \eta^F, \eta^T \). Then \( Z = F \times T \) is a right \( G \)-functor \( C \times C^{\text{op}} \to sSet \). Its coend is a \( G \)-simplicial set denoted by \( F \otimes_C T \).

As in the case of limits, a morphism of right \( G \)-functors induces an equivariant map between the corresponding coends.

**Example 3.7.** Let \( C \) be a \( G \)-category and \( Z: C \times C^{\text{op}} \to sSet \) a functor, with an action of \( G \) on \( Z \) given by \( \eta \). We have a simplicial set \( B(C,Z) \), called the (simplicial) **bar construction** (see [6]), such that

\[
B(C,Z)_n = \bigoplus_{X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} X_n \in N(C)_n} Z(X_0, X_n)_n
\]

with boundaries and degeneracies given by:

\[
d^i(\phi_1, \ldots, \phi_n; z) = \begin{cases} (\phi_2, \ldots, \phi_n; d^0(Z(\phi_1, 1_{X_n}(z))) & i = 0, \\
(\phi_1, \ldots, \phi_{i+1}; d^iZ (\phi_i, \phi_{i+1}, \ldots, \phi_n); d^iZ) & 1 \leq i \leq n - 1, \\
(\phi_1, \ldots, \phi_{n-1}; d^n(Z(1_{X_0}, \phi_n)(z))) & i = n \end{cases}
\]

\[
s^i(\phi_1, \ldots, \phi_n; z) = (\phi_1, \ldots, \phi_i, 1_{X_n}, \phi_{i+1}, \ldots, \phi_n; s^i z), & 0 \leq i \leq n.
\]

The action of \( G \) on \( Z \) gives a structure of \( G \)-simplicial set on \( B(C,Z) \), with action of \( g \in G \) defined as:

\[
g(\phi_1, \ldots, \phi_n; z) = (g\phi_1, \ldots, g\phi_n; \eta_{g,(X_0,X_n)}(z))
\]

If the functor \( Z \) is of the form \( F \times T \) as in the previous example, then we denote \( B(C,Z) \) as \( B(F,C,T) \).

**Example 3.8.** Let \( F: C \to SCat \) be a functor. We define a category \( C \uparrow F \) with objects the pairs \((X,a)\) with \( X \in \text{obj} C, a \in \text{obj} F(X) \). A map \((X,a) \to (Y,b)\) is given by a pair \((f,u)\) such that \( f: X \to Y \) is a map in \( C \) and \( u: F(f)(a) \to b \) is a map in the category \( F(Y) \). The category \( C \uparrow F \) is called the **Grothendieck Construction** on \( F \) (see [7]).

If \( F: C \to SCat \) is a right \( G \)-functor, then \( C \uparrow F \) is a small \( G \)-category with action on objects given by

\[
g(X,a) = (gX, \eta_{g,X}(a))
\]
and on maps by
\[ g((X, a) \mapsto (f, u), (Y, b)) = (gf, \eta_{g, Y}(u)) \]

We end this section by stating some basic and easily provable properties of right $G$-functors.

**Proposition 3.9.** If $F: C \to D$ is a functor with a $G$-action given by $\eta$ and $X$ is an object in $C$, then $FX$ is a $G_X$-object, where $G_X$ is the stabilizer of $X$ under the action of $G$ on $\text{obj} C$. The action is defined by the maps $\eta_{g, X}: FX \to FX$.

**Proposition 3.10.** ((2.3) from [3]) Let $F: C \to D$ be a right $G$-functor, $S: C' \to C$ an equivariant functor, and $T: D \to E$ any functor. Then both $F \circ S$ and $T \circ F$ have induced structures of right $G$-functors.

For example, for any $G$-category $C$, we have a right $G$-functor $N(-\setminus C): C \to sSet$.

## 4. The Homotopy Colimit

Let $C$ be a $G$-category and $Z: C \times C^\text{op} \to sSet$ a right $G$-functor. We start by noting the equivariant isomorphism:

\[ Z \otimes_{C \times C^\text{op}} N(-\setminus C/-) \cong_G B(C, Z), \]

which can be proven by showing that $B(C, Z)$ satisfies the definition of coend of the functor $Z \times N(-\setminus C/-) : (C \times C^\text{op}) \times (C \times C^\text{op})^\text{op} \to sSet$. In the case that $Z = F \times T$ with $F: C \to sSet$, $T: C^\text{op} \to sSet$ are right $G$-functors, and using Fubini’s theorem for coends [4, p. 230], this leads to

\[ F \otimes_C N(-\setminus C/-) \otimes_C T \cong_G B(F, C, T). \]

Using that, we can prove that for right $G$-functors $F: D \to sSet$, $T: C^\text{op} \times D \to sSet$, and $U: D^\text{op} \to sSet$, we have

\[ B(B(F, C, T), D, U) \cong_G B(F, C, B(T, D, U)), \]

whose non-equivariant version is 3.1.3 from [2].

If $C$ is any $G$-category, we will denote by $*$ the functor $C \to sSet$ that is constant with value the simplicial set with exactly one simplex in each dimension. It is clearly has a structure of right $G$-functor.

**Definition 4.1.** Let $F: C \to sSet$ a functor. Its **homotopy colimit** $\text{hocolim}_C F$ is defined as $F \otimes_C N(-\setminus C)$.

If $F$ is a right $G$-functor, then $Z = F \times N(-\setminus C)$ has a natural structure of right $G$-functor, so in this case $\text{hocolim}_C F = \text{coend } Z$ is a $G$-simplicial set.

Note that the map of right $G$-functors $N(-\setminus C) \to *$ induces an equivariant map

\[ \text{hocolim}_C F = F \otimes_C N(-\setminus C) \to F \otimes_C * = \text{colim } F, \]

and the map of right $G$-functors $F \to *$ induces an equivariant map

\[ \text{hocolim}_C F = F \otimes_C N(-\setminus C) \to N(-\setminus C) \otimes_C * = N(C). \]

One also can prove the isomorphism of right $G$-functors:

\[ N(-\setminus C/-) \otimes_C * \cong N(-\setminus C). \]
which leads to the equivariant isomorphism:
\[(14) \quad B(F, C, \ast) \cong \text{hocolim}_C F.\]

Finally, we note that just by categorical arguments, one obtains:

**Proposition 4.2.** Let \( S : D \to C \) be an equivariant functor between \( G \)-categories, and let \( F : C \to \text{sSet} \) a right \( G \)-functor. Then, with the induced right \( G \)-functor structure on \( F \circ S \), we have:

1. \( \hom_C(C, -) \otimes_D N(-, D) \cong B(\hom_C(C, -), D, \ast) \cong N(C\backslash C) \) as right \( G \)-functors on the argument \( C \).
2. \( (F \circ S)(D) \cong F \otimes_C \hom_C(-, SD) \cong B(F, C, \hom_C(-, SD)) \), as right \( G \)-functors on \( D \).

As a consequence of this proposition, if we take \( S = 1_C \) to be the identity functor, we obtain that
\[(15) \quad B(\hom_C(C, -), C, \ast) \cong N(C\backslash C) \cong \ast,\]
for all \( C \in \text{obj C} \), since \( C\backslash C \) has an initial object \( 1_C : C \to C \) fixed by \( G_C \) ([10, (4.3)]).

5. **The Homotopy Invariance Theorem**

The proofs of the theorems of the next section are based on this important theorem. The reader may refer to [8] for the properties of induced topological spaces.

**Theorem 5.1.** Let \( Z, Z' : C \times C^{\text{op}} \to \text{sSet} \) two right \( G \)-functors. Let \( \epsilon : Z \to Z' \) be a map of right \( G \)-functors such that \( \epsilon_{X,Y} : Z(X,Y) \to Z'(X,Y) \) is a \( G_{(X,Y)} \)-homotopy equivalence for all \( X \in \text{obj C}, Y \in \text{obj C}^{\text{op}} \). Then the map \( \bar{\epsilon} \) induced by \( \epsilon \):
\[(16) \quad \bar{\epsilon} : B(C, Z) \to B(C, Z')\]

is a \( G \)-homotopy equivalence.

**Proof.** From [6], we know that \( B(C, Z) \) is the diagonal of a bisimplicial set \( \tilde{B}(C, Z) \) with \( (m, n) \)-simplices the set
\[(17) \quad \coprod_{X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_m} X_m} Z(X_0, X_m)_n.\]

From the examples, we know that this coproduct has an action of \( G \) given by:
\[(18) \quad g(\phi_1, \ldots, \phi_m; z) = (g\phi_1, \ldots, g\phi_m; \eta_{g,(X_0, X_m)}(z)),\]
and this makes \( \tilde{B}(C, Z) \) a bisimplicial \( G \)-set. We have that \( \epsilon \) induces a map \( \bar{\epsilon} : B(C, Z) \to \tilde{B}(C, Z') \), sending
\[(19) \quad (\phi_1, \ldots, \phi_m; z) \mapsto (\phi_1, \ldots, \phi_m; \epsilon_{X_0, X_m}(z)),\]
The map \( \bar{\epsilon} \) is equivariant, and so if we define \( \bar{\epsilon} \) as diag \( \bar{\epsilon} \), then \( \bar{\epsilon} \) is equivariant as well.
Let us denote \( X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} X_m \in N(C)_m \) by \( \bar{X} \). According to Theorem (3.8) from [10], in order to prove that \( \bar{e} \) is a \( G \)-homotopy equivalence, it is sufficient to prove that
\[
(20) \quad \hat{e}_{m,-} : \prod_{\bar{X} \in N(C)_m} Z(X_0, X_m) \to \prod_{\bar{X} \in N(C)_m} Z'(X_0, X_m)
\]
is a \( G \)-homotopy equivalence for all \( m \). Taking geometric realization on both sides of (20), since geometric realization commutes with coproducts, we obtain:
\[
(21) \quad |\hat{e}_{m,-}| : \prod_{\bar{X} \in N(C)_m} |Z(X_0, X_m)| \to \prod_{\bar{X} \in N(C)_m} |Z'(X_0, X_m)|
\]
Let \( E_m \) be a set of representatives for the orbits of the action of \( G \) on \( N(C)_m \). Then the map in (21) can be written as:
\[
(22) \quad |\hat{e}_{m,-}| \uparrow_{G'}_{G} : \prod_{\bar{Y} \in E_m} |Z(X_0, X_m)| \uparrow_{G'}_{G} \to \prod_{\bar{Y} \in E_m} |Z'(X_0, X_m)| \uparrow_{G'}_{G}
\]
Since by hypothesis, each \( e_{(X_0, X_m)} \) is a \( G(X_0, X_m) \)-homotopy equivalence, given that \( G' \leq G(X_0, X_m) \), they are also \( G' \)-homotopy equivalences, and so each map \( |Z(X_0, X_m)| \uparrow_{G'}_{G} \to |Z'(X_0, X_m)| \uparrow_{G'}_{G} \) is a \( G \)-homotopy equivalence. Therefore the map in (22) is a coproduct of \( G \)-homotopy equivalences, hence a \( G \)-homotopy equivalence, as we wanted to prove. \( \square \)

6. Further Theorems

**Theorem 6.1.** (Equivariant Homotopy Invariance Of The Homotopy Colimit). Let \( F, F' : C \to \text{sSet} \) right \( G \)-functors, and \( \epsilon : F \to F' \) a map of right \( G \)-functors such that each \( \epsilon_X : FX \to F'X \) is a \( G_X \)-homotopy equivalence. Then the induced map \( \bar{\epsilon} : \text{hocolim}_C F \to \text{hocolim}_C F' \) is a \( G \)-homotopy equivalence.

**Proof.** Straightforward from Theorem 5.1, since the homotopy colimit is a special case of a bar construction. \( \square \)

**Theorem 6.2.** (Reduction Theorem) Let \( S : D \to C \) be an equivariant functor between \( G \)-categories, and let \( F : C \to \text{sSet} \) a right \( G \)-functor. Then we have the equivariant isomorphism
\[
(23) \quad \text{hocolim}_D F \circ S \cong_G F \otimes_C N(-\setminus S)
\]

**Proof.**
\[
\text{hocolim}_D F \circ S = (F \circ S) \otimes_D N(-\setminus D) \quad \text{Definition of hocolim}
\]
\[
\cong_G (F \otimes_C \text{hom}_C(-, SD)) \otimes_D N(-\setminus D) \quad \text{Proposition 4.2.2}
\]
\[
\cong_G F \otimes_C (\text{hom}_C(C, S- \otimes_D N(-\setminus D)) \quad \text{Fubini's theorem}
\]
\[
\cong_G F \otimes_C N(-\setminus S) \quad \text{Proposition 4.2.1} \quad \square
\]

In [3, (2.6)], this result is given as a homotopy equivalence. However, as noted in [2, 4.4], this is even an isomorphism, which in this case is equivariant.

**Theorem 6.3.** (Cofinality Theorem) Let \( S : D \to C \) be an equivariant functor between \( G \)-categories, and let \( F : C \to \text{sSet} \) a right \( G \)-functor. Consider the induced right \( G \)-functor structure on \( F \circ S \). If \( N(C \setminus S) \) is \( G_C \)-contractible for all objects \( C \) in \( C \), then \( \text{hocolim}_D F \circ S \cong_G \text{hocolim}_C F \).
Proof:

\[
hocolim_{\mathbf{D}} F \circ S = B(F \circ S, \mathbf{D}, \ast) = B(B(F, \mathbf{C}, \text{hom}_{\mathbf{C}}(-, SD)), \mathbf{D}, \ast) \]

Equation 14

\[
\cong_{G} B(F, \mathbf{C}, B(\text{hom}_{\mathbf{C}}(C, S-), \mathbf{D}, \ast)) \]

Equation 10

\[
\cong_{G} B(F, \mathbf{C}, N(- \setminus S)) \]

Proposition 4.2.1

\[
\simeq_{G} B(F, \mathbf{C}, \ast) = \text{hocolim}_{\mathbf{D}} F \]

Hypothesis

We note that the hypothesis about \( G_{C} \)-contractibility of the fiber \( N(C \setminus S) \) allows us to conclude the \( G \)-homotopy. Compare with [3, (2.7)], where this result is given as a homotopy equivalence not necessarily equivariant.

**Theorem 6.4.** (Homotopy Pushdown Theorem) Let \( S: \mathbf{D} \rightarrow \mathbf{C} \) be an equivariant functor and \( F: \mathbf{D} \rightarrow \mathbf{sSet} \) a right \( G \)-functor. Let \( S_{h_{0}}(F): \mathbf{C} \rightarrow \mathbf{sSet} \) the functor given by \( C \mapsto B(F, \mathbf{D}, \text{hom}_{\mathbf{C}}(S-, C)) \). Then \( S_{h_{0}}(F) \) is a right \( G \)-functor and \n
\[
hocolim_{\mathbf{C}} S_{h_{0}}(F) \simeq_{G} \text{hocolim}_{\mathbf{D}} F \]

\]

Proof:

\[
hocolim_{\mathbf{C}} S_{h_{0}}(F) = B(B(F, \mathbf{D}, \text{hom}_{\mathbf{C}}(S-, C)), \mathbf{C}, \ast) \]

Definition

\[
\cong_{G} B(F, \mathbf{D}, B(\text{hom}_{\mathbf{C}}(SD, -), C, \ast)) \]

Equation 10

\[
\simeq_{G} B(F, \mathbf{D}, \ast) = \text{hocolim}_{\mathbf{D}} F \]

Equation 15

\]

□

Note that we also used the equivariant homotopy invariance (Theorem 5.1) of the bar construction in the last step. Hence in [3, (2.5)] we do have a \( G \)-homotopy equivalence.

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