The Complexity of Finding Temporal Separators under Waiting Time Constraints *

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Abstract

In this work, we investigate the computational complexity of Restless Temporal \((s, z)\)-Separation, where we are asked whether it is possible to destroy all restless temporal paths between two distinct vertices \(s\) and \(z\) by deleting at most \(k\) vertices from a temporal graph. A temporal graph has a fixed vertex but the edges have (discrete) time stamps. A restless temporal path uses edges with non-decreasing time stamps and the time spent at each vertex must not exceed a given duration \(\Delta\).

Restless Temporal \((s, z)\)-Separation naturally generalizes the NP-hard Temporal \((s, z)\)-Separation problem. We show that Restless Temporal \((s, z)\)-Separation is complete for \(\Sigma_2^P\), a complexity class located in the second level of the polynomial time hierarchy. We further provide some insights in the parameterized complexity of Restless Temporal \((s, z)\)-Separation parameterized by the separator size \(k\).

1 Introduction

Capturing dynamic changes in a network plays an increasingly important role in network analysis and algorithmics and temporal graphs are a popular model that is able to represent such changes over time [6, 14, 16, 19, 21]. Especially the notion of connectivity is much more intricate in the temporal setting and path-related problems were among the first ones studied on temporal graphs [3, 17]. In particular, vertex separators are NP-hard to find in temporal graphs [17] whereas it is possible to find them in polynomial time in static graphs [1, Theorem 6.8].

In this work we study the computational complexity of finding vertex separators in a temporal graph that destroy all temporal paths that obey certain waiting time constraints. We call such paths restless temporal paths and their “waiting” or “pausing” time at a vertex is restricted to some prescribed duration \(\Delta\). Restless temporal paths naturally model infection transmission chains of diseases that grant immunity upon recovery [15]. Such transmission routes are captured by the well-established SIR-model (Susceptible-Infected-Recovered) [3, 15, 22]. This also motivates the search of vertex separators since they naturally model breaking infection transmission by vaccinations. This work is focused on the analysis of the computational complexity of Restless Temporal \((s, z)\)-Separation, the problem of deciding whether a temporal graph admits a separator of size at most \(k\) that destroys all restless temporal paths between two designated vertices \(s\) and \(z\) (a formal problem definition is given in Section 2).

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Related Work. The problem of finding minimum temporal separators was first studied by Kempe et al. [17] and they proved it is NP-hard. In contrast, Berman [5] proved earlier that destroying all temporal path between two designated vertices by deleting a minimum number of edges instead of vertex can be done in polynomial time. Zschoche et al. [28] and Fluschnik et al. [12] further studied the computational complexity of finding temporal separators of bounded size and provide (parameterized) algorithms as well as hardness result for several restricted cases. Maack et al. [20] studied the problem on specific temporal graph classes.

The computation of restless temporal paths has been studied by Casteigts et al. [7]. They show that deciding whether a restless temporal path exists between two vertices is NP-complete and they give several further (parameterized) hardness and algorithmic results. Notably, finding restless temporal walks between two vertices in known to be polynomial-time solvable [4].

Our Contributions. We analyze the computational complexity of RESTLESS TEMPORAL \((s, z)\)-SEPARATION and show that this problem is \(\Sigma_p^2\)-complete. We further give some insights on the parameterized complexity of RESTLESS TEMPORAL \((s, z)\)-SEPARATION parameterized by the separator size.

2 Preliminaries and Basic Observations

In this section, we formally introduce the most important concepts related to restless temporal separators and give the formal problem definition of RESTLESS TEMPORAL \((s, z)\)-SEPARATION. We further discuss some basic observations.

Computational Complexity. In this work we show that the problem under consideration is complete for the complexity class \(\Sigma_p^2\) [2, 26]. This complexity class is located in the second level of the polynomial time hierarchy and contains both NP and coNP. It is closed under polynomial-time many-one reductions and, intuitively, contains all problems that are at most as hard as the problem \(\exists \forall\)-SAT [2, 26], where we are given a Boolean formula \(\phi\) in conjunctive normal form and the variables of \(\phi\) are partitioned into two sets \(X\) and \(Y\), and we are asked to decide whether there exists an assignment for all variables in \(X\) such that for all possible assignments for the variables in \(Y\), the formula \(\phi\) evaluates to true. The problem \(\exists \forall\)-SAT is complete for \(\Sigma_p^2\) [2, 26]. The class \(\Sigma_p^2\) can also be characterized as the set of all problems that can be solved by an NP-machine that has oracle access to an NP-complete problem [2, 26].

We use standard notation and terminology from parameterized complexity theory [8, 10, 11, 24] and give here a brief overview of the most important concepts that are used in this paper. A parameterized problem is a language \(L \subseteq \Sigma^* \times \mathbb{N}\), where \(\Sigma\) is a finite alphabet. We call the second component the parameter of the problem. A parameterized problem is in the complexity class XP if there is an algorithm that solves each instance \((I, r)\) in \(|I|^{f(r)}\) time, for some computable function \(f\). A parameterized problem is fixed-parameter tractable (i.e., in the complexity class FPT) if there is an algorithm that solves each instance \((I, r)\) in \(|I|^{O(1)}\) time, for some computable function \(f\). A decidable parameterized problem \(L\) admits a polynomial kernel if there is a polynomial-time algorithm that transforms each instance \((I, r)\) into an instance \((I', r')\) such that \((I, r) \in L\) if and only if \((I', r') \in L\) and \(|(I', r')| \in r^{O(1)}\). If a parameterized problem is hard for the parameterized complexity class \(W[1]\) or \(W[2]\) then it is (presumably) not in FPT. The complexity classes \(W[1]\) and \(W[2]\) is closed under parameterized reductions, which may run in FPT-time and additionally set the new parameter to a value that exclusively depends on the old parameter. They are part of the W-hierarchy and it is known that \(\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \ldots \subseteq \text{XP}\). If a parameterized problem is NP-hard (resp. coNP-hard) for constant parameter values, then the problem is para-NP-hard (resp. para-coNP-hard).
Temporal Graphs, Paths, and Separators. An (undirected, simple) temporal graph is a tuple $G = (V, E_1, E_2, \ldots, E_t)$ or $G = (V, (E_i)_{i \in [t]})$ for short, with $E_i \subseteq \binom{V}{2}$ for all $i \in [t]$. We call $\ell(G) := \ell$ the lifetime of $G$. We call the graph $G_i(G) = (V, E_i(G))$ the layer $i$ of $G$ where $E_i(G) := E_i$. If $E_i = \emptyset$, then $G_i$ is a trivial layer. We call layers $G_i$ and $G_{i+1}$ consecutive. We call $i$ a time step. If an edge $e$ is present at time $i$, that is, $e \in E_i$, we say that $e$ has time stamp $i$ and call the pair $(e, i)$ a time edge. We further denote $V(G) := V$.

A restless temporal path is not allowed to wait an arbitrary amount of time in a vertex, but has to leave any vertex it visits within the next $\Delta$-window, for some given value for $\Delta$. Formally, they are defined as follows.

**Definition 1** (Restless Temporal Walk / Restless Temporal Path). A $\Delta$-restless temporal walk of length $n$ from vertex $s$ to vertex $z$ in a temporal graph $G = (V, (E_i)_{i \in [t]})$ is a sequence $P = ((s = v_0, v_1, t_1), (v_1, v_2, t_2), \ldots, (v_{n-1}, v_n = z, t_n))$ such that for all $i \in [n]$ we have that $(v_{i-1}, v_i) \in E_i$ and for all $i \in [n-1]$ we have that $t_i \leq t_{i+1} \leq t_i + \Delta$. Moreover, we call $P$ a $\Delta$-restless temporal path of length $n$ if $v_i \neq v_j$ for all $i, j \in \{0, \ldots, n\}$ with $i \neq j$. We say that $P$ respects the maximum waiting time $\Delta$.

We call the problem of checking whether there exists a restless temporal $(s, z)$-path in a given temporal graph for a given $\Delta$ value “RESTLESS TEMPORAL $(s, z)$-PATH”. This problem is known to be NP-hard and has been thoroughly investigated by Casteigts et al. [7].

Now we are ready to give the formal definition of a restless temporal $(s, z)$-separator, which should destroy all restless temporal $(s, z)$-paths in a given temporal graph.

**Definition 2** ($\Delta$-Restless Temporal $(s, z)$-Separator). Let $G = (V, (E_i)_{i \in [t]})$ be a temporal graph with $s, z \in V$. Let $\Delta \leq \ell$. A vertex set $S \subseteq V \setminus \{s, z\}$ is a $\Delta$-restless temporal $(s, z)$-separator for $G$ if there is no $\Delta$-restless temporal $(s, z)$-path in $G - S$.

We can now formally define the (decision) problem of finding a $\Delta$-restless temporal $(s, z)$-separator in a given temporal graph $G$ with two distinct vertices $s$ and $z$.

| RESTLESS TEMPORAL $(s, z)$-SEPARATION |
|--------------------------------------|
| **Input:** A temporal graph $G = (V, (E_i)_{i \in [t]})$, two distinct vertices $s, z \in V$, and two integers $k \in \mathbb{N}$ and $\Delta \leq \ell$. |
| **Question:** Does $G$ admit a $\Delta$-restless temporal $(s, z)$-separator of size at most $k$? |

**Basic Observations.** Since RESTLESS TEMPORAL $(s, z)$-SEPARATION generalizes the NP-hard TEMPORAL $(s, z)$-SEPARATION problem [12, 14, 28], we can observe that the problem is clearly NP-hard. However, we can also observe that we presumably cannot verify in polynomial time whether a vertex set $S$ is a $\Delta$-restless temporal $(s, z)$-separator for a given temporal graph $G$. Casteigts et al. [7] showed checking whether there is a $\Delta$-restless temporal path from $s$ to $z$ in $G - S$ is NP-hard. Note that RESTLESS TEMPORAL $(s, z)$-SEPARATION with $k = 0$ is the complement of RESTLESS TEMPORAL $(s, z)$-PATH. Hence we can observe the following.

**Observation 3.** RESTLESS TEMPORAL $(s, z)$-SEPARATION is coNP-hard for all $\Delta \geq 1$ even if $k = 0$.

This implies that RESTLESS TEMPORAL $(s, z)$-SEPARATION is presumably not contained in NP.

Furthermore, it is easy to observe that computational hardness of RESTLESS TEMPORAL $(s, z)$-PATH for some fixed value of $\Delta$ implies hardness for all larger values of $\Delta$. This allows us to construct hardness reductions for small fixed values of $\Delta$ and still obtain general hardness.
results. The proof of this observation is essentially the same as the proof of an analogous result for Restless Temporal \((s, z)\)-Path by Casteigts et al. [7].

**Observation 4.** For every fixed \(\Delta\), Restless Temporal \((s, z)\)-Separation on instances \((G, s, z, k, \Delta + 1)\) is computationally at least as hard as Restless Temporal \((s, z)\)-Separation on instances \((G, s, z, k, \Delta)\).

*Proof.* The result immediately follows from the observation that a temporal graph \(G\) contains a \(\Delta\)-restless temporal \((s, z)\)-path if and only if the temporal graph \(G'\) contains a \((\Delta + 1)\)-restless temporal \((s, z)\)-path, where \(G'\) is obtained from \(G\) by inserting one trivial (that is, edgeless) layer after every \(\Delta\) consecutive layers.

Finally, we can also observe that Restless Temporal \((s, z)\)-Separation is fixed-parameter tractable when parameterized by the number \(|V|\) of vertices. We can check for each vertex set size \(k\) whether it is \(\Delta\)-restless temporal \((s, z)\)-separator. We remove it from the input temporal graph and then use an FPT-algorithm for Restless Temporal \((s, z)\)-Path when parameterized by the number \(|V|\) of vertices [2] to verify whether there is no \(\Delta\)-restless temporal \((s, z)\)-path.

**Observation 5.** Restless Temporal \((s, z)\)-Separation parameterized by the number \(|V|\) of vertices is fixed-parameter tractable.

However, we presumably cannot obtain a polynomial kernel for the parameter number \(|V|\) of vertices since we can observe the following kernelization lower bound for Restless Temporal \((s, z)\)-Separation.

**Observation 6.** Restless Temporal \((s, z)\)-Separation parameterized by the number \(|V|\) of vertices does not admit a polynomial kernel for all \(\Delta \geq 1\) unless \(NP \subseteq coNP/poly\).

This follows directly from the result by Casteigts et al. [7] that Restless Temporal \((s, z)\)-Path does not admit a polynomial kernel when parameterized by the number \(|V|\) of vertices for all \(\Delta \geq 1\) unless \(NP \subseteq coNP/poly\). This follows from the observation that Restless Temporal \((s, z)\)-Separation with \(k = 0\) is the complement of Restless Temporal \((s, z)\)-Path and hence a polynomial kernel for Restless Temporal \((s, z)\)-Separation parameterized by \(|V|\) would also be a polynomial kernel for Restless Temporal \((s, z)\)-Path parameterized by \(|V|\).

### 3 Computational Complexity of Restless Temporal Separators

In this section we investigate the computational complexity of Restless Temporal \((s, z)\)-Separation. The fact that the problem is both NP-hard and coNP-hard as observed in the previous section already suggests that Restless Temporal \((s, z)\)-Separation is located somewhere higher in the polynomial time hierarchy. Indeed we can show that Restless Temporal \((s, z)\)-Separation is \(\Sigma_2^P\)-complete. This implies, for example, that we presumably cannot use SAT-solvers or ILP-solvers to compute \(\Delta\)-restless temporal \((s, z)\)-separators. To show \(\Sigma_2^P\)-hardness, we give a reduction from \(\exists\forall\)-SAT, where we are given a Boolean formula \(\phi\) in conjunctive normal form and a partition of variables of \(\phi\) into two sets \(X\) and \(Y\). Then we are asked to decide whether there exists an assignment for all variables in \(X\) such that for all possible assignments for the variables in \(Y\), the formula \(\phi\) evaluates to true. The very rough idea of our reduction is that the vertices selected for the separator correspond to an assignment for the variables in \(X\) and if there is an assignment for the variables in \(Y\) such that \(\phi\) evaluates to false, then the temporal graph should still contain a \(\Delta\)-restless temporal \((s, z)\)-path after the separator vertices are removed.

**Theorem 7.** Restless Temporal \((s, z)\)-Separation is \(\Sigma_2^P\)-complete for all \(\Delta \geq 1\) even if every edge has only one time stamp.
Proof. We present a polynomial-time reduction from the $\Sigma_2^P$-complete problem $\exists \forall$-SAT to $2\exists \forall$-SAT, where we are given a Boolean formula $\phi$ in conjunctive normal form and the variables of $\phi$ are partitioned into two sets $X$ and $Y$, and we are asked to decide whether there exists an assignment for all variables in $X$ such that for all possible assignments for the variables in $Y$, the formula $\phi$ evaluates to true.

Let $\phi(X, Y)$ denote an instance of $\exists \forall$-SAT, let $n_X = |X|$, $n_Y = |Y|$, and let $m$ be the number of clauses in $\phi$. We assume that no clause of $\phi$ contains a variable several times. We construct a temporal graph $\mathcal{G} = (V, (E_i)_{i \in [m]})$ with $\ell = 2m + 1$, consisting of three gadgets. We start with the "exists gadget" in which we have to select the vertices of the $\Delta$-restless temporal $(s, z)$-separator. Intuitively, this chooses an assignment for the variables in $X$. The next gadget is the "forall gadget" which must be passed by every $\Delta$-restless temporal $(s, z)$-path. This gadget can be traversed in $2^{n_Y}$ ways which, intuitively, represent all possible assignments for the variables in $Y$. The last gadget is the clause gadget which, intuitively, a $\Delta$-restless temporal $(s, z)$-path can only pass if there is a clause that is not satisfied. We set $\Delta = 1$ and $k = n_X$. We next give formal descriptions of the gadgets.

**Exists Gadget.** We start by creating two vertices $s$ and $z$. For every variable $x_i \in X$ we create two vertices $x_i^{(T)}$ and $x_i^{(F)}$ and we add edges $\{s, x_i^{(T)}\}$, $\{x_i^{(T)}, x_i^{(F)}\}$, and $\{x_i^{(F)}, z\}$ to $E_1$. This already completes the construction of the exists gadget. We can see that we created $n_X$ internally vertex-disjoint $\Delta$-restless temporal $(s, z)$-paths. Since we set $k = n_X$, we have that every $\Delta$-restless temporal $(s, z)$-separator has to contain one vertex from each of these paths.

**Forall Gadget.** For every variable $y_i \in Y$ we create two vertices $y_i^{(T)}$ and $y_i^{(F)}$. We further create $n_Y + 1$ vertices $s_1, s_2, \ldots, s_{n_Y+1}$. For all $i \in [n_Y]$ we add edges $\{s_i, y_i^{(T)}\}$, $\{s_i, y_i^{(F)}\}$, $\{y_i^{(T)}, s_{i+1}\}$, and $\{y_i^{(F)}, s_{i+1}\}$ to $E_1$. We further add edge $\{s, s_1\}$ to $E_1$. This completes the construction of the forall gadget. We can see that there are $2^{n_Y}$ different $\Delta$-restless temporal paths from $s_1$ to $s_{n_Y+1}$. Intuitively, each one of these represents an assignment for the variables in $Y$.

**Clause Gadget.** For every clause $c_i$ of $\phi$ we create two vertices $c_i^{(1)}$ and $c_i^{(2)}$. For every $i \in [m]$ we add edge $\{c_i^{(1)}, c_i^{(2)}\}$ to $E_{2i+1}$ and if $i < m$, then we add edge $\{c_i^{(2)}, c_{i+1}^{(1)}\}$ to $E_{2i+2}$. We further add edge $\{s_{n_Y+1}, c_1^{(1)}\}$ to $E_2$. We call this part of the gadget the clause selection path.

Let $c_1$ be a clause for some $i \in [m]$. Without loss of generality let $c_1$ contain variables $x_1, \ldots, x_j$ and $y_1, \ldots, y_{j_2}$. Then we add the following edges to $E_{2i+1}$:

- If $x_1$ appears non-negated in $c_1$, then we add edge $\{c_1^{(2)}, x_1^{(F)}\}$, otherwise we add edge $\{c_1^{(2)}, x_1^{(T)}\}$.
- For all $j \in [j_1 - 1]$, if $x_j$ appears non-negated in $c_1$, then set $v_j = x_j^{(F)}$, otherwise set $v_j = x_j^{(T)}$. If $x_{j+1}$ appears non-negated in $c_1$, then set $v_{j+1} = x_{j+1}^{(F)}$, otherwise set $v_{j+1} = x_{j+1}^{(T)}$. We add edge $\{v_j, v_{j+1}\}$.
- If $x_{j_1}$ appears non-negated in $c_1$, then set $v = x_{j_1}^{(F)}$, otherwise set $v = x_{j_1}^{(T)}$. If $y_1$ appears non-negated in $c_1$, then set $w = y_1^{(F)}$, otherwise set $w = y_1^{(T)}$. We add edge $\{v, w\}$.
- For all $j \in [j_2 - 1]$, if $y_j$ appears non-negated in $c_1$, then set $v_j = y_j^{(F)}$, otherwise set $v_j = y_j^{(T)}$. If $y_{j+1}$ appears non-negated in $c_1$, then set $v_{j+1} = y_{j+1}^{(F)}$, otherwise set $v_{j+1} = y_{j+1}^{(T)}$. We add edge $\{v_j, v_{j+1}\}$.
- If $y_{j_2}$ appears non-negated in $c_1$, then we add edge $\{y_{j_2}^{(F)}, z\}$, otherwise we add edge $\{y_{j_2}^{(T)}, z\}$.

We do this for all clauses in $\phi$. This completes the construction of the clause gadget. Intuitively, a $\Delta$-restless temporal $(s, z)$-path should only be able to traverse the clause gadget if there is a clause that is not satisfied.
This finishes the construction of $G = (V, (E_i)_{i \in [g]})$. The construction is illustrated in Figure 1. Recall that $\Delta = 1$ and $k = n_X$. It is easy to check that $G$ can be constructed in polynomial time and that every edge has at most one time stamp.

**Correctness.** Now we show that $G$ admits a $\Delta$-restless temporal $(s, z)$-separator of size at most $k$ if and only if $\phi$ is a YES-instance of $\exists \forall$-SAT.

$(\Rightarrow)$: Assume that there is an assignment for the variables in $X$ such that for all assignments for the variables of $Y$ we have that $\phi$ evaluates to true. We construct a $\Delta$-restless temporal $(s, z)$-separator $S$ for $G$ as follows. For each $i \in [n_X]$, if variable $x_i$ is assigned the value true, then we add the vertex $x_i^{(T)}$ to $S$, otherwise we add $x_i^{(F)}$ to $S$. Clearly, we have that $|S| = n_X = k$. In the following, we show that $S$ is a $\Delta$-restless temporal $(s, z)$-separator for $G$.

Assume for contradiction that there is a $\Delta$-restless temporal $(s, z)$-path $P$ in $G - S$. It is easy to see that all $\Delta$-restless temporal $(s, z)$-paths in $G$ that only use edges from the exists gadget are destroyed in $G - S$ since from every such path, we put one vertex into $S$. Observe that all time edges adjacent to $z$ that are not part of the exists gadget have a time stamp of three or larger. Hence, to reach a time edge with time step two, $P$ has to pass the forall gadget to reach time edge $\{s_{n_Y+1}, c_i^{(1)}\}$, which is the only time edge with time stamp two. From this it follows that for every $i \in [n_Y]$ we have that either $y_i^{(T)} \in V(P)$ or $y_i^{(F)} \in V(P)$. Then the path $P$ enters the clause selection path of the clause gadget. To reach $z$, the path $P$ has to leave this path at some vertex $c_j^{(2)}$ for some $j \in [m]$ (meaning that $c_j^{(2)} \in V(P)$ and $c_j^{(1)} \notin V(P)$). We claim that this implies that clause $c_j$ is not satisfied if the variables from $Y$ are assigned the following truth values: for each $i \in [n_Y]$, if $y_i^{(T)} \in V(P)$, then we set $y_i$ to true, otherwise we set $y_i$ to false. Assuming that $c_j^{(2)} \in V(P)$ and $c_j^{(1)} \notin V(P)$, the only way to reach $z$ from $c_j^{(2)}$ is through vertices that correspond to the variables appearing in clause $c_j$, since the time stamps from all paths from the clause selection path to $z$ differ by at least two. More specifically, for each variable $x_i$ ($y_i$) appearing in $c_j$, we have that $V(P)$ contains vertex $x_i^{(F)}$ ($y_i^{(F)}$) if $x_i$ ($y_i$) appears non-negated in $c_j$ and $V(P)$ contains vertex $x_i^{(T)}$ ($y_i^{(T)}$) otherwise. This means for
the variables \( x_i \) that are set to truth values that do not satisfy clause \( c_j \), otherwise the corresponding vertices would be contained in the separator \( S \). For the variables \( y_i \) this means the assignment we constructed earlier also sets them to truth values that do not satisfy clause \( c_j \), otherwise the corresponding vertices would have been used by \( P \) when the path was passing the forall gadget at time step one. Hence, we have found an assignment for the variables in \( Y \) such that together with the given assignment for the variables in \( X \), the formula \( \phi \) evaluates to false—a contradiction.

\( \left(\leftarrow\right) \): Let \( S \subseteq V \setminus \{s, z\} \) with \( |S| \leq k \) be a \( \Delta \)-restless temporal \((s, z)\)-separator for \( G \). Let us first look at the exists gadget of \( G \). It consists of \( n_X \) internally vertex-disjoint \( \Delta \)-restless temporal \((s, z)\)-paths, each one visiting four vertices: \( s, x_i^{(T)}, x_i^{(F)}, \) and \( z \) for some \( i \in [n_X] \). Of each of these \( \Delta \)-restless temporal \((s, z)\)-paths, one vertex other than \( s \) or \( z \) has to be contained in \( S \), otherwise \( S \) would not be a \( \Delta \)-restless temporal \((s, z)\)-separator. It follows that for all \( i \in [n_X] \) either \( x_i^{(T)} \) or \( x_i^{(F)} \) is contained in \( S \) (and also no other vertices are contained in \( S \) since \( k = n_X \)). This lets us construct an assignment for the variables in \( X \) as follows. For every \( i \in [n_X] \), if \( x_i^{(T)} \in S \), then we set \( x_i \) to true, otherwise we set \( x_i \) to false. We claim that using this assignment for the variables in \( X \), we have that for all assignments for the variables in \( Y \) the formula \( \phi \) evaluates to true.

Assume for the sake of contradiction that there is an assignment for the variables in \( Y \) such that together with the constructed assignment for the variables in \( X \), the formula \( \phi \) evaluates to false. Then we can construct a \( \Delta \)-restless temporal \((s, z)\)-path in \( G - S \) as follows. Starting from \( s \) we traverse the forall gadget as follows. Starting with \( i = 1 \) to \( n_Y \) we visit \( s_i \) and then \( y_i^{(T)} \) if \( y_i \) is set to true, and \( y_i^{(F)} \) otherwise. Then we visit \( s_{n_Y + 1} \). Up until this point, the path only uses time edges with time stamp one. Since \( \phi \) evaluates to false, there is at least one clause in \( \phi \) that is not satisfied. Let \( c_j \) be that clause. We continue our \( \Delta \)-restless temporal path from \( s_{n_Y + 1} \) to \( c_j^{(2)} \). Since \( c_j \) evaluates to false, the vertices corresponding to the variables in \( X \) appearing in \( c_j \) are not contained in \( S \), otherwise, by construction, the clause \( c_j \) would evaluate to true. Similarly, the vertices corresponding to the variables in \( Y \) appearing in \( c_j \) have not been visited by the path when traversing the forall gadget. Hence, we can continue the \( \Delta \)-restless temporal path from \( c_j^{(2)} \) to \( z \)—a contradiction.

**Containment in \( \Sigma^P_\Delta \).** Our proof so far shows that Restless Temporal \((s, z)\)-Separation is \( \Sigma^P_\Delta \)-hard. To show that the problem is \( \Sigma^P_\Delta \)-complete, we show that it is contained in \( \Sigma^P_\Delta \). Recall that \( \Sigma^P_\Delta \) contains all problems that can be solved by an NP-machine that has oracle access to an NP-complete problem [24, 26]. We can solve an instance \( G = (V, (E_i)_{i \in [\ell]}, s, z, k, \Delta) \) of Restless Temporal \((s, z)\)-Separation with such a machine as follows. We non-deterministically guess a set \( S \subseteq V \) of size at most \( k \) and then produce an instance \((G - S, s, z, \Delta)\) of Restless Temporal \((s, z)\)-Path. Since Restless Temporal \((s, z)\)-Path is contained in NP, we can reduce it to the NP-complete problem to which we have oracle access. We use the reduction to produce an equivalent instance of the NP-complete problem we have oracle access to and query the oracle with this instance. If the oracle answers NO, then we have found a \( \Delta \)-restless temporal \((s, z)\)-separator of size at most \( k \) for \( G \) and can answer YES. It is easy to see that the described machine has an accepting path if and only if the Restless Temporal \((s, z)\)-Separation instance is a YES-instance.

From a parameterized complexity perspective we can make one rather straightforward observation. Since Restless Temporal \((s, z)\)-Separation generalizes Temporal \((s, z)\)-Separation, we know that Restless Temporal \((s, z)\)-Separation parameterized by the separator size \( k \) is \( W[1] \)-hard [28]. However, we can observe that Restless Temporal \((s, z)\)-Separation is even \( W[2] \)-hard when parameterized by the separator size \( k \) by a straightforward
reduction from Hitting Set, where we model each element of the universe with a vertex and each set by a path through the corresponding vertices. The waiting time $\Delta$ allows us to obtain a one-to-one correspondence between restless temporal paths in the constructed temporal graph and sets in the Hitting Set instance. We remark that the reduction we use to show this result has been used in a very similar way by Zschoche [27] to show that finding temporal separators of bounded size that destroy all $\Delta$-restless temporal paths from $s$ to $z$ is W[2]-hard when parameterized by the bound on the separator size.

**Proposition 8.** Restless Temporal $(s, z)$-Separation parameterized by the separator size $k$ is W[2]-hard for all $\Delta \geq 1$.

**Proof.** We present a parameterized polynomial-time reduction from Hitting Set, where we are given a universe set $U$, a family of sets $S_1, \ldots, S_m \subseteq U$, and an integer $h$, and are asked whether there is a hitting set $S^* \subseteq U$ with $|S^*| \leq h$ such that for all $i \in [m]$ we have that $S^* \cap S_i \neq \emptyset$.

Hitting Set is W[2]-complete when parameterized by $h$ [3, 25].

Given an instance $(U, (S_i)_{i \in [m]}, h)$ of Hitting Set, we construct a temporal graph $G = (V, (E_i)_{i \in [\ell]})$ with $\ell = 2m$ as follows. We set $V = U \cup \{s, z\}$ and for each set $S_i$ with $i \in [m]$ we create two layers $G_{2i-1}$ and $G_{2i}$. In layer $G_{2i-1}$ we create a path from $s$ to $z$ that visits all vertices in $S_i$ in an arbitrary order. The layer $G_{2i}$ is trivial. We set $\Delta = 1$ and $k = h$. This finishes the construction. It is easy to check that this can be done in polynomial time.

**Correctness.** The correctness is straightforward to see. A $\Delta$-restless temporal $(s, z)$-separator for $G$ has to contain at least one vertex from each set $S_i$ with $i \in [m]$, otherwise there would be a layer that contains a $\Delta$-restless temporal $(s, z)$-path. It follows that a $\Delta$-restless temporal $(s, z)$-separator for $G$ is a hitting set for $(U, (S_i)_{i \in [m]})$. For the other direction, one has to observe that due to the waiting time restriction $\Delta$ and the trivial layers that are present in $G$, each $\Delta$-restless temporal $(s, z)$-path in $G$ corresponds to a set $S_i$ for some $i \in [m]$ of the Hitting Set instance. It follows that a hitting set contains at least one vertex from each $\Delta$-restless temporal $(s, z)$-path in $G$.

We remark that it is open whether Temporal $(s, z)$-Separation parameterized by the separator size $k$ is contained in W[1]. Hence, Proposition 8 does not necessarily imply that Restless Temporal $(s, z)$-Separation parameterized by the separator size $k$ is harder than Temporal $(s, z)$-Separation parameterized by the separator size $k$. We remark that containment of Restless Temporal $(s, z)$-Separation parameterized by the separator size $k$ in W[2] is unlikely since Observation 3 shows that Restless Temporal $(s, z)$-Separation parameterized by the separator size $k$ is also para-coNP-hard. We conjecture that Restless Temporal $(s, z)$-Separation parameterized by the separator size $k$ is complete for the parameterized complexity class $\Sigma^p_2[k^*]$ [13].

4 Conclusion

In this work we studied the computational complexity of deciding whether a temporal graph admits separators of bounded size under the restless temporal path model studied by Castegits et al. [7]. We established that Restless Temporal $(s, z)$-Separation is complete for $\Sigma^p_2$, a complexity class that is located in the second level of the polynomial time hierarchy. This implies, for example, that we presumably cannot use SAT-solvers or ILP-solvers to compute restless temporal separators. We also provide some preliminary results for the parameterized complexity of Restless Temporal $(s, z)$-Separation parameterized by the separator size $k$. We show that the parameterized problem is hard for para-coNP and hard for W[2]. We conjecture
that and leave as an open question for future research whether the parameterized problem is complete for $\Sigma^p_2[k^\ast]$.

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