The quantum geometric limit

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Abstract

In Einstein's *gedankenexperiment* for measuring space and time, an ensemble of clocks moving through curved spacetime measures geometry by sending signals back and forth, as in the global positioning system (GPS). Combining well-known quantum limits to measurement with the requirement that the energy density of clocks and signals be no greater than the black hole density leads to the quantum geometric limit: the total number of ticks of clocks and clicks of detectors that can be contained in a four volume of spacetime of radius $r$ and temporal extent $t$ is less than or equal to $rt/\pi \ell_P t_P$, where $\ell_P$, $t_P$ are the Planck length and time. The quantum geometric limit suggests that each event or ‘op’ that takes place in a four-volume of spacetime is associated with a Planck-scale area. This paper shows that the quantum geometric limit can be used to derive general relativity: if each quantum event is associated with a Planck-scale area removed from two-dimensional surfaces in the volume in which the event takes place, then Einstein’s equations must hold.

The quantum geometric limit imposes a fundamental physical bound to the accuracy with which quantum systems can measure the geometry of spacetime [1-2]. This limit arises naturally from the combination of well-known quantum limits to measurement accuracy, together with the requirement that the apparatus used to measure space and time be no denser than a black hole. The quantum geometric limit connects previous limits to measuring spacetime [3-4], the physics of computation [5-7], holography [8-13], and quantum mechanics on curved spacetime [14-15, 17]. In particular, the holographic
principle encourages us to imagine each bit within a spacelike three volume as projected onto the two-dimensional surface of that volume at a density of no greater than the Planck scale. By contrast, the quantum geometric limit encourages us to imagine each event or ‘op’ that occurs within a spacetime four volume as projected onto the two-dimensional surfaces in that volume at a density of no greater than the Planck scale. In [15] Jacobson showed that the holographic area-entropy law can be used as a basis for deriving Einstein’s equations. This paper shows that the same is true for the quantum geometric limit: Einstein’s equations follow from the assumption that each quantum event or op removes a Planck-scale area from two-dimensional surfaces in the volume in which the event occurs.

1 Limits to measuring spacetime

The Margolus-Levitin theorem [5] states that the time $\Delta t$ it takes a quantum system such as a clock to go from one state to an orthogonal state is greater than or equal to $\frac{\pi \hbar}{2E}$, where $E$ is the expectation value of the energy of the system above the ground state energy. As an example of a system that saturates the Margolus-Levitin theorem, consider a two-level system or qubit with energy eigenstates $|0\rangle$ with energy 0 and $|1\rangle$ with energy $\hbar \omega$. The state $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\omega \Delta t}|1\rangle)$ then saturates the Margolus-Levitin theorem for $E = \frac{\hbar \omega}{2}$, and $\Delta t = \frac{\pi \hbar}{2E} = \frac{\pi}{\omega}$. The Margolus-Levitin theorem and its variants such as the quantum speed limit [7] can be combined with the fundamental physical limits to computation and measurement [1-2, 5-7] to put bounds on the accuracy to which spacetime geometry can be measured.

Consider Einstein’s seminal thought experiment for measuring spacetime geometry, in which spacetime is filled with a ‘swarm’ of clocks that map out the spacetime geometry by exchanging signals with the other clocks and measuring the signals’ times of arrival. This thought experiment is of course the basis for the global positioning system (GPS). The
clocks could be as large as GPS satellites, or as small as elementary particles. References [1-2] put bounds on how accurately this swarm of clocks can map out a sub-volume of spacetime with radius \( r \) over time \( t \). Every tick of a clock or click of a detector is an elementary event in which a system goes from a state to orthogonal state. Accordingly, the total number of ticks and clicks that can take place within the volume is limited by the Margolus-Levitin theorem: it is less than or equal to \( \# \equiv 2Et/\pi \hbar \), where \( E \) is the quantum expectation value of the energy of the clocks within the volume, measured from their ground state. Call \( \# \) the number of quantum ‘ops’ accumulated by the system. A quantum op occurs when the state of the system accrues an average phase \( Et = \pi/2 \) relative to the phase of the ground state, whether the system moves to an orthogonal state or not. For example, a thermal state of the system can be stationary but still accrue quantum ops.

If the clocks are packed too densely within some sub-volume, they will form a black hole and be unable to send signals to clocks outside their horizon. That is, clocks within a black hole cease to participate in global measurement of spacetime. (They may still measure spacetime within the horizon.) To prevent the formation of an horizon, the total of clocks and signals within spacelike regions of radius \( r \) must be less than \( c^4r/2G \). The Margolus-Levitin theorem together with the requirement that the clocks and signals participating in measuring spacetime be no greater than the black-hole density implies the quantum geometric limit [1,2]: the total number of elementary events and the number of ops that can occur in such a volume of spacetime are bounded by

\[
\# \leq \frac{c^4rt}{\pi \hbar G} = \frac{rt}{\pi l_p t_p}.
\]

The quantum geometric limit (1) was derived without any recourse to quantum gravity: the Planck scale makes its appearance simply from combining quantum limits to measurement with the requirement that sub-volumes of the GPS system be no denser than black holes.
Rotating and charged black holes possess a more complex Kerr-Newman structure [14,16,18] that could modify equation (1). Within a black hole, arbitrarily large matter densities can occur in the approach to the singularity, so clocks and signals could potentially violate equation (1). In a Schwarzschild black hole, the maximum proper time ticked out by a free-falling clock after it passes the event horizon at radius $R = \frac{2GE}{c^4}$ is $t = \frac{\pi R}{c}$ [18], suggesting that the quantum geometric limit still holds in the sense that the maximum number of ticks and clicks experienced by observers within the hole before they hit the singularity is $\leq \frac{R^2}{\ell^2_p}$. The ability of clocks and signals falling into a complex singularity structure such as in Kerr-Newman black holes which possess closed timelike curves [18] is an open question. In any case, the clocks and signals inside the black hole can not participate in the overall GPS system’s measurement of spacetime outside the hole.

Although straightforward to derive, the quantum geometric limit’s association of events with two-dimensional Planck scale areas is somewhat surprising. A priori, one might have thought that the number of events within a four volume would be limited by the measure of that volume divided by the Planck scale to the fourth power. Alternatively, holography could be taken to suggest that the number of events be proportional to the surface area of the volume times time divided by Planck scale cubed. The quantum geometric limit (1) shows that the concentration of possible events in spacetime is sparser than either of these guesses indicate.

2 Deriving Einstein’s equations from the quantum geometric limit

The holographic principle arose out of black-hole thermodynamics and quantum field theory on curved spacetime. In [15], Jacobson turned the argument around: he showed how Einstein’s equations can be derived from combining the holographic entropy-area
law with the fact that accelerated observers see horizon radiation with a temperature proportional to their acceleration. (See also the work of Verlinde [19] and Dreyer [20].)

The quantum geometric limit suggests each quantum event in spacetime is associated with a two-dimensional Planck-scale area. Now turn the argument around and derive Einstein’s equations from the quantum geometric limit. The basic idea is straightforward: our \textit{ansatz} is that each elementary quantum ‘op’ \textit{removes} a Planck scale area from the two-dimensional sections of the spatial three volume in which the event occurs. The removal of area from a flat two-dimensional section causes it to curve. This curvature is the curvature of spacetime, and as will now be shown, it induces the spacetime to obey Einstein’s equations.

Consider an inertial observer following a geodesic through spacetime. Consider a local region of spacetime in the vicinity of the observer sufficiently small that the energy-momentum and curvature tensors are effectively constant over this region, and the spacetime within the region is close to Minkowski space (i.e., $Kr^2 << 1$, where $r$ is the radius of the region and $K$ is the maximum curvature within the region). The observer describes this local region of spacetime by an orthonormal tetrad of vectors (a vierbein) \{${e^a}_\mu$\}. ${e^0}_\mu$ is the timelike tangent vector to her path, and \(e_j^\mu\), \(j = 1, 2, 3\) are an orthonormal triad of spacelike vectors. We have \(e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab}\), where \(\eta^{ab} = \text{diag}(-1, +1, +1, +1)\) is the Minkowski-space metric, and \(g^{\mu\nu}\) is the spacetime metric. Similarly, \(e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}\).

Latin indices of \(e^a_\mu\) are raised and lowered using \(\eta_{ab}\), and Greek indices are raised and lowered using \(g_{\mu\nu}\).

The observer maps out her local region of spacetime using GPS coordinates (figure 1). Consider the causal diamond $\Diamond(r, t)$ formed by the the intersection of the interiors of the forward light cone emitted at time $t - r/c$ and the backward light cone absorbed at time $t + r/c$. In Minkowski space the measure of the four-volume $\Diamond(r, t)$ is $2\pi r^4 / 3c$. The light cones intersect at a 2-sphere $S$. Define the spacelike geodesic 2-disc $D_{12}$ by extending
from the observer’s position at time $t$ the space-like geodesics initially tangent to the 12 plane. $D_{12}$ has circumference $C_{12}$ and Gaussian curvature $K_{12}$. Similarly, $D_{23}$ and $D_{31}$ are the geodesic 2-disks given by extending the spacelike geodesics initially tangent to the 23 and 31 planes. These 2-disks have circumferences $C_{23}$, $C_{31}$, and curvatures $K_{23}$, $K_{31}$.

A quantum ‘op’ corresponds to an average accumulation of phase $E \Delta t / \hbar = \pi / 2$ in the observer’s frame, where $E$ is the observed energy above the local ground state energy. When a clock ticks or a detector clicks, at least one quantum op is performed. The average energy density above the ground state energy within the light cones as measured by the observer is $T^{\mu \nu} e^0_\mu e^0_\nu$. The total number of ops recorded by the observer within the two light cones is therefore equal to

$$\# = (2 / \pi \hbar) T^{\mu \nu} e^0_\mu e^0_\nu (2 \pi / 3) r^4 / c. \quad (3)$$

Our ansatz is that each op within the volume $\diamondsuit(r,t)$ removes a Planck-scale area from the two-dimensional surfaces associated with the volume (figure 1). Compared with flat space, each op removes a net area $\Delta A = \alpha \ell_p^2$ from the three 2-disks $D_{12}$, $D_{23}$, $D_{31}$. Removing area from an initially flat 2-disk causes it to curve. In the limit of small $r$ the Bertrand-Diquet-Puiseux theorem implies that the Gaussian curvature $K_{ij}$ of the $ij$ disk of the disk is related to the deficit area $\Delta A_{ij}$ removed by $\Delta A_{ij} = \pi r^4 K_{ij} / 12$. Equivalently, the circumference of the $ij$ disk is reduced by $\Delta C_{ij} = \pi r^3 K_{ij} / 3$, and the area of the two-sphere $S$ is reduced by $4 \pi r^4 K / 9$, where $K = K_{12} + K_{23} + K_{31}$ is the half the curvature of the spacelike three volume orthogonal to the motion of the observer [21-22]. (I am indebted to T. Jacobson for pointing out the connection to the deficit area of the 2-sphere [23].)

To relate the area removed to the Riemann tensor, note that the Gaussian curvature of the geodesic 2-disk $D_{ij}$ is equal to the sectional curvature [21,24]:

$$K_{ij} = R^{\mu \nu \rho \sigma} e_i^\mu e_j^\nu e_\rho^\rho e_\sigma^\sigma$$

(4)
where $R^\mu_{\nu\rho\sigma}$ is the Riemann tensor. Removing a total area $\alpha \ell_p^2$ from the three 2-disks $D_{ij}$ then yields

$$
\alpha \ell_p^2 = (\pi/12)(K_{12} + K_{23} + K_{31})r^4
= (\pi/24)(R_{\mu\nu}e^0_\mu e^0_\nu + \sum_{j=1}^3 R_{\mu\nu}e^j_\mu e^j_\nu)r^4
= (\pi/12)(R_{\mu\nu} - (1/2)g_{\mu\nu}R)e^0_\mu e^0_\nu r^4. \tag{5}
$$

Combining equation (5) for the change in area together with equation (3) for the total number of ops yields

$$
(16/\pi \hbar)\alpha \ell_p t_P T_{\mu\nu}e^0_\mu e^0_\nu = (R_{\mu\nu} - (1/2)g_{\mu\nu}R)e^0_\mu e^0_\nu. \tag{6}
$$

Setting $\alpha = \pi^2/2$, and noting that $e^0_\mu$ can be any timelike unit vector – that is, equation (6) should hold for all observers – implies Einstein’s equations.

Equations (5-6) show that if each op that takes place removes an area $A = \pi^2 \ell_p^2$ from the initially flat two-dimensional spacelike 2-disks $D_{ij}$ contained within the causal diamond $\Diamond(r, t)$ of intersecting light cones in which the op takes place, inducing curvature, then the resulting curved spacetime obeys Einstein’s equations. Equivalently, Einstein’s equations arise if each op removes an area $8\pi^2 \ell_p^2/3$ from the 2-sphere defined by the intersection of the light cones. Our derivation effectively ‘de-dimensionalizes’ the observation [21-22] that in Einstein’s equations the local energy density as perceived by an observer is proportional to the curvature of the spacelike 3-volume orthogonal to her path. The derivation reveals the origin of the quantum geometric limit: there is only so much area one can remove from a 2-sphere.

Note that this derivation yields Einstein’s equations without an intrinsic cosmological constant. The lack of an intrinsic cosmological term arises from the requirement – enjoined by the quantum mechanics of measurement – that the number of events or ops be an observable quantity. Consequently, that number can depend only on the energy.
above the ground state energy. If only observable quantum phases contribute to the inter-
teraction between matter and geometry, then the vacuum energy does not contribute to
the gravitational energy. Of course, the equations do not rule out a form of matter that
corresponds to a cosmological term.

In certain circumstances – e.g., at the horizon of a black hole, at the mouth of a
wormhole, or between the plates of capacitor (the Casimir effect) – quantum field theory
allows the local energy density to be observably lower than the energy density of the
global vacuum [14]. Does the possibility of such negative energy densities mean that
there can be negative ops (‘nops’)? The definition of an op should be able reconcile, for
example, the number of ops measured by an observer falling into a black hole with that
measured by an observer far from the hole. A full quantum field theoretic treatment of
the quantum geometric limit lies outside of the scope of the current paper, however, and
will be undertaken elsewhere.

In general, the quantum state of the local matter is a mixture, e.g., a thermal state.
Even when the overall state of the matter is pure, the local state is typically mixed due to
entanglement: for example, vacuum entanglement is responsible for the entropy-area law
[13]. The Page-Geilker experiment [25] suggests that – at least in relatively macroscopic
situations – different components of the mixture yield different local geometries. For a
mixed state, the number of ops can be evaluated separately in each component of the
mixture. In an entangled state, the geometry induced by a particular component of the
local mixed state is correlated with the geometry induced by the corresponding state of
the matter elsewhere.

Before closing, it is worthwhile to compare the derivation of Einstein’s equations from
the quantum geometric limit with that of Jacobson [15]. Einstein’s equations relate energy
density to curvature. Jacobson relates energy density to a dimensionless quantity, entropy
– measured in bits – by introducing Planck’s constant $\hbar$ in the expression for the Unruh
temperature. The entropy-area correspondence [13] suggests that each bit of entropy is associated with a two-dimensional area $\eta$, which Jacobson takes to be an area added to a cross-section of the horizon. The Raychaudhuri equation then implies Einstein’s equations with gravitational constant $G = \eta/(4 \ln 2 \hbar)$, so that $\eta$ is indeed a Planck-scale area.

By comparison, the quantum geometric limit relates energy density to a dimensionless quantity – number of ops – by applying the Margolus-Levitin theorem, which introduces $\hbar$. The requirement that collections of clocks and signals within a four volume not exceed the black-hole density leads to the quantum geometric limit. This limit suggests that each op is associated a two-dimensional Planck scale area, which we take to be an area removed from two-dimensional surfaces within the four volume in which the op takes place. The introduction of the Planck length squared inserts the gravitational constant and removes $\hbar$. The relationship between Gaussian curvature and sectional curvature then implies Einstein’s equations.

The main difference between the two approaches, of course, is that in Jacobson’s case the fundamental dimensionless quantity is a bit, a unit of information, whereas here it is an op – a unit of action or change.

### 3 Conclusion

Intriguing connections between quantum information and gravity have been arising for decades [1-4, 6, 8-13, 15-27]. This paper attempted to elucidate those connections by applying fundamental quantum limits to measurement of space and time. As in [15], the result is not a theory of quantum gravity per se, but rather a quantum theory which gives rise to general relativity under simple assumptions. The quantum geometric limit states that the number of elementary events such as clock ticks, detector clicks, or bit flips that can be contained in a four volume of space time of covariant radius $r$ and spatial extent $t$ is limited by $rt/\pi l_p t_p$. By bounding the number of ops that can take place within a four
volume, this limit is complementary to the holographic limit, which bounds the number of bits associated with a spatial three volume by the surface area of the volume divided by the Planck length squared. Holography encourages us to imagine the bits of information characterizing the quantum state of the spatial three volume as projected onto the two-dimensional boundary of the volume at a density no greater than on the order of one bit per Planck length squared. The quantum geometric limit encourages us to imagine the elementary events or ‘ops’ that occur within a spacetime four volume as projected onto two-dimensional surfaces in that volume at a density no greater than on the order of one op per Planck length squared. If each op removes a Planck-scale area from those surfaces, then Einstein’s equations hold.

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Figure 1: An inertial clock falling through spacetime sets up a local coordinate system using GPS coordinates. The forward light cone of signals emitted by the clock at time $t - r/c$ intersects with the backward light cone of signals absorbed by the clock at time $t + r/c$ to form a covariant 2-sphere $S$ of radius $r$ at time $t$. If each elementary quantum operation that occurs within the light cones removes area $A = \frac{\pi^2 \ell_p^2}{2}$ from the two-dimensional disks $D$ in the interior of the two-sphere, and area $8\pi^2 \ell_p^2/3$ from the surface of the two-sphere, the resulting curvature makes Einstein’s equations hold.