The role of zero-mode contributions
in the matching for the twist-3 PDFs $e(x)$ and $h_L(x)$

Shohini Bhattacharya\textsuperscript{(a)}, Krzysztof Cichy\textsuperscript{(b)}, Martha Constantinou\textsuperscript{(a)},
Andreas Metz\textsuperscript{(a)}, Aurora Scapellato\textsuperscript{(b)}, Fernanda Steffens\textsuperscript{(c)}

\textsuperscript{(a)} Temple University, 1925 N. 12th Street, Philadelphia, PA 19122-1801, USA
\textsuperscript{(b)} Faculty of Physics, Adam Mickiewicz University, Uniwersytetu Poznańskiego 2, 61-614 Poznań, Poland
\textsuperscript{(c)} Institut für Strahlen- und Kernphysik, Rheinische Friedrich-Wilhelms-Universität Bonn,
Nussallee 14-16, 53115 Bonn, Germany

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The perturbative procedure of matching was proposed to connect parton quasi-distributions that
are calculable in lattice QCD to the corresponding light-cone distributions which enter physical
processes. Such a matching procedure has so far been limited to the twist-2 distributions. Recently,
we addressed the matching for the twist-3 PDF $g_T(x)$. In this work, we extend our perturbative
calculations to the remaining twist-3 PDFs, $e(x)$ and $h_L(x)$. Unlike the case of twist-2 and $g_T(x)$,
we find that the light-cone and quasi-distributions do not fully agree in the infrared, which indicates
a breakdown of matching. We identify singular zero-mode contributions as the source of this issue.
Whether a meaningful matching is still feasible remains to be understood.
I. INTRODUCTION

The twist-3 parton distribution functions (PDFs) \( e(x) \) and \( h_L(x) \) were introduced some 30 years ago. They complement the twist-3 PDF \( g_T(x) \), which enters the cross section of polarized deep-inelastic lepton-nucleon scattering (DIS). Twist-3 PDFs are of general interest as they contain information about quark-gluon-quark correlations in the nucleon. Moreover, a semi-classical relation between the function \( e(x) \) and the (transverse) force acting on transversely polarized quarks in an unpolarized nucleon has been reported in Ref. \([5]\). Recently, the role of \( e(x) \) has also been discussed in relation to the mass structure of hadrons \([6]\) (see, also Ref. \([7]\)). Unlike \( g_T(x) \), both \( e(x) \) and \( h_L(x) \) are chiral odd and hence can only show up in observables with other chiral-odd functions. This feature makes it challenging to extract information on these functions from experiment. In Ref. \([2]\), it was argued that \( e(x) \) can be accessed in an unpolarized Drell-Yan process, but only at the level of twist-4. Soon after, it was shown that \( h_L(x) \) can be used to address information on \( h_L(x) \) from the CLAS collaboration, based on preliminary data from the CLAS collaboration, can be found in Ref. \([14]\). A twist-3 double-spin asymmetry in the Drell-Yan process could be used to address \( h_L(x) \) \([11, 12]\), and other final states in polarized hadronic collisions could in principle be considered as well — see, for instance, the discussion in Refs. \([17, 18]\). But so far no information exists on \( h_L(x) \) from the experimental side.

A. Delta function singularities in \( e(x) \) and \( h_L(x) \)

An interesting and sometimes controversially-discussed feature of \( e(x) \) and \( h_L(x) \) regards the possible existence of singular zero-mode \((x = 0)\) contributions, that is, delta-function singularities \((\delta(x))\), and their implication on sum rules. For the sake of this discussion, we summarize below the sum rules for the lowest moments of \( e(x) \) and \( h_L(x) \). By definition, the lowest moment of the the flavor-singlet combination of \( e(x) \) gives the pion-nucleon sigma term \( \sigma_{\pi N} \),

\[
\int_{-1}^{1} dx (e^u(x) + e^d(x)) = \frac{\sigma_{\pi N}}{m},
\]

where,

\[
\sigma_{\pi N} = \frac{m}{2M_N} \langle P | \bar{\psi}^u(0)\psi^u(0) + \bar{\psi}^d(0)\psi^d(0) | P \rangle , \quad m = \frac{1}{2}(m_u + m_d),
\]

and \( M_N \) is the nucleon mass. On the basis of rotational invariance, it was shown that the lowest moments of \( h_L(x) \) and the twist-2 transversity \( h_1(x) \) \([15, 19]\) are connected as

\[
\int_{-1}^{1} dx h_L(x) = \int_{-1}^{1} dx h_1(x),
\]

which is the counterpart of the Burkhardt-Cottingham sum rule that relates \( g_T(x) \) and the (twist-2) helicity distribution \( g_1(x) \) \([20]\).

As mentioned above, there has been discussion on whether one can get around the presence of the zero-modes. Refs. \([21, 23]\) emphasized that a \( \delta(x) \) singularity in \( e(x) \) is a consequence of the QCD equation of motion (EOM). Specifically, one can split \( e(x) \) as

\[
e^q(x) = \frac{\delta(x)}{2M_N} \langle P | \bar{\psi}^q(0)\psi^q(0) | P \rangle + \bar{e}^q(x) + e_m^q(x),
\]

where \( \bar{e} \) is a “pure” twist-3 term (which encodes quark-gluon-quark interactions) and \( e_m \) is a current-mass term. Using the decomposition of Eq. \([4]\) in the above mentioned sum rule, one finds

\[
\int dx \bar{e}^q(x) = 0, \quad \int dx e_m^q(x) = 0,
\]

which implies that the first moment of \( e(x) \) entirely receives contribution from the \( \delta(x) \) term. Very recently, it was argued, again on the grounds of EOM approach, that the coefficient of \( \delta(x) \) is zero \([24]\). A critique on that work was
drawn in Ref. [6], ruling out the possibility of a cancellation of \(\delta(x)\) in \(e(x)\). By reconstructing \(h_L(x)\) from its operator product expanded (OPE) form, Ref. [2] showed that \(h_L(x)\) comprised of three terms: a twist-2 term, a “pure” twist-3 term, and a current-mass term. Through a foreseeable discontinuity in the integral relation between \(h_L(x)\) and the mass term, Ref. [25] indicated the existence of a possible \(\delta(x)\) in \(h_L(x)\). The need for such a singularity was also justified for a compliance with the sum rule mentioned in Eq. (3) as the twist-2 part, \(h_1(x)\), is continuous at \(x = 0\).

The first attempt to calculate \(e(x)\) and \(h_L(x)\) was made in the MIT bag model [2,26]. However, no \(\delta(x)\) singularity was found. Calculations in diquark spectator models, with form factors, did also not indicate such singularities [27]. A recent study in the same (spectator) model [28], using a cut-off for the transverse momentum integration instead was found. Calculations in diquark spectator models, with form factors, did also not indicate such singularities [27].

One-loop perturbative calculations of \(e(x)\) and \(h_L(x)\) in quark target models [25,28] also indicated the presence of \(\delta(x)\). Interestingly, in calculations employing the light-front Hamiltonian approach instead of the Feynman-diagram approach, as in Refs. [25,28], no such singularities were observed in \(e(x)\) [33] and \(h_L(x)\) [34], which can well be due to an insufficiency of the used approach to deal with zero modes. Generally, it is accepted that sum rules like in Eq. (3) are violated if \(\delta(x)\) contributions are not included in the twist-3 PDFs [25,28,34]. We note in passing that zero-mode contributions can also generate discontinuities for higher-twist generalized parton distributions [28,35], thus endangering factorization of certain observables in hard exclusive reactions. This point is closely related to the main result of the present work about potential issues in relating higher-twist quasi-PDFs and light-cone PDFs through a factorization-type formula.

### B. Accessing PDFs from lattice QCD

By now, we already see that there are various theoretical statements available in the literature about the \(\delta(x)\) singularities, with some of them being contradictory. Lattice QCD calculations with appropriate lattice parameters close to the continuum limit and with large volumes, may be able to offer some insights on the above matter in the future. However, the explicit time-dependence of the light-cone PDFs prohibits their direct calculation on Euclidean lattices. In 2013, there was a breakthrough proposal by Ji to calculate instead parton quasi-distributions (quasi-PDFs) [36,37]. Quasi-PDFs are defined in terms of spatial correlation functions of fast-moving hadrons, and therefore can be directly calculated on Euclidean lattices. At large, but finite, momentum, such correlation functions can be matched to their respective light-cone PDFs prior to the UV renormalization. On the lattice, one is constrained to apply the UV renormalization before taking the infinite momentum limit. The issue of the limits leads to differences in the UV behavior between the light-cone PDFs and the quasi-PDFs. The key underlying idea of this approach is that the non-perturbative physics should be the same for the light-cone and the quasi-PDFs. The differences in the UV behavior can be calculated and rectified perturbatively in Large Momentum Effective Theory (LaMET), through a procedure known as matching [38,40]. Apart from the quasi-PDF approach as a way to directly access the \(x\)-dependence of the PDFs in lattice QCD, several other ideas have been put forth [41,52].

In the last few years, there has been significant advances, both in theory and in lattice QCD. This includes the proof of renormalizability [53-55], the development of a renormalization prescription [56,57], which was extensively implemented on the lattice [58,65]. A plethora of other aspects regarding quasi-PDFs and Euclidean correlators in general have also been extensively studied [66-93]. The first lattice results for quasi-PDFs and other related quantities constitute an important development in this field [48,57,58,61,94-120]. Additionally, the verification of convergence of quasi-PDFs to their light-cone counterparts in model calculations further substantiate these quasi-distributions to be reliable tools to study the light-cone PDFs [121-139]. We refer to [134-137] for an up-to-date compendium of progress in the field of studying light-cone PDFs through Euclidean correlators in lattice QCD.

The procedure of matching has largely been explored for the twist-2 distribution functions [38,40,56-60,71,80,80,138,144]. Recently, we computed the first ever one-loop matching equations for the twist-3 PDF \(g_T(x)\) [143], which we implemented on lattice data in Ref. [140]. Here, we extend our work, for the case of \(e(x)\) and \(h_L(x)\). Specifically, we calculate the light-cone PDFs \(e(x)\) and \(h_L(x)\), and their quasi-PDF counterparts, \(e_Q\) and \(h_{L,Q}\), in a quark target...
to one-loop order in perturbative QCD (pQCD). The ultimate goal of this work is to obtain the appropriate matching equations. As argued in this work, this is a highly nontrivial task. The main challenge lies in the IR difference between the light-cone and quasi-PDF results. This mismatch in the IR region, which is observed in the \( e(x) \) and \( h_L(x) \) for the first time, suggests limitations in the extraction of the matching procedure. Within the present work, we identify the infamous zero-modes as the cause for this mismatch. Formally, this implies that one cannot extract the matching in the same straightforward manner as in the case of twist-2 and \( g_T(x) \).

We organize the manuscript as follows: In Sec. II we provide the definition of the light-cone PDFs \( e(x) \) and \( h_L(x) \), and of the corresponding quasi-PDFs \( e_Q(x) \) and \( h_{LQ}(x) \). In Sec. III we present one-loop pQCD results for \( e(x) \) (\( e_Q(x) \)) and \( h_L(x) \) (\( h_{LQ}(x) \)) in the Feynman gauge with three different IR regulators: nonzero gluon mass, nonzero quark mass and dimensional regularization (DR). Sec. IV addresses the problem of the IR mismatch between the light-cone and quasi-PDF results. We summarize our results in Sec. V.

II. DEFINITIONS

We start by recalling the definition of twist-3 light-cone PDFs \( e(x) \) and \( h_L(x) \) for quarks. Generally, light-cone PDFs are defined through the correlation function

\[
\Phi^{[\Gamma]}(x,S) = \frac{1}{Z} \int \frac{d^4 z}{(2\pi)^3} e^{ik \cdot z} \langle P,S|\bar{\psi}(\frac{x}{2}) \Gamma \Psi(\frac{x}{2})|P,S\rangle e^{iP \cdot x}  
\]

where \( \Gamma \) denotes a generic gamma matrix. Color gauge invariance of this bi-local quark-quark correlator is enforced by the Wilson line

\[
W(-\frac{x}{2},\frac{x}{2})|_{z^+ = 0, \vec{z} = 0} = \mathcal{P} \exp \left( -ig_s \int_{\frac{x}{2}}^{\frac{x}{2}} dy^- A^+(0^+,y^-,\vec{0}_\perp) \right),
\]

where \( \mathcal{P} \) is a path-ordered exponential depending on the plus-component of the gluon field. The hadron is characterized by its 4-momentum \( P \) and a covariant spin vector \( S \) which can be written as

\[
S^\mu = (S^+,S^-,\vec{S}_\perp) = \left( \frac{\lambda P^+}{M}, -\frac{\lambda M}{2P^+}, \vec{S}_\perp \right),
\]

where \( \lambda \) is the helicity of the hadron and \( M \) is its mass. The spin vector satisfies the relation \( P \cdot S = 0 \) by definition. The twist-3 light-cone PDFs \( e(x) \) and \( h_L(x) \) are then defined as

\[
\Phi^{[\Gamma]} = \frac{1}{2P^+} \bar{u}(P,S) \Gamma u(P,S) e(x) = \frac{M}{P^+} e(x),
\]

\[
\Phi^{[i\sigma^+ - \gamma_5]} = \frac{1}{2P^+} \bar{u}(P,S) i\sigma^+ - \gamma_5 u(P,S) h_L(x) = \frac{M}{P^+} \lambda h_L(x),
\]

where \( u(P,S) \) (\( \bar{u}(P,S) \)) is the spinor for the incoming (outgoing) hadron, \( \sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \) and \( \gamma_5 \) is the usual matrix which anti-commutes with any other Dirac matrix.

We now turn to the quasi-PDFs which are defined through the spatial correlation function

\[
\Phi^{[\Gamma]}_{Q}(x,S;P_3) = \frac{1}{Z} \int \frac{d^3 \vec{z}}{2\pi} e^{ik \cdot z} \langle P,S|\bar{\psi}(\frac{x}{2}) \Gamma \Psi(\frac{x}{2})|P,S\rangle e^{iP_3 \cdot \vec{z}}  
\]

1 For a generic four-vector \( v \) we denote the Minkowski components by \((v^0, v^1, v^2, v^3)\) and the light-cone components by \((v^+, v^-, \vec{v}_\perp)\), with \( v^+ = \frac{1}{\sqrt{2}} (v^0 + v^3) \), \( v^- = \frac{1}{\sqrt{2}} (v^0 - v^3) \) and \( \vec{v}_\perp = (v^1, v^2) \).
with the Wilson
\[
W_Q\left(\frac{\hat{z}}{2}, \frac{\hat{y}}{2}\right)\bigg|_{\vec{z}^0=0, \vec{z}_\perp=0} = \mathcal{P} \exp \left(-ig_s \int_{-\frac{z^3}{3}}^{\frac{z^3}{3}} dy^3 A^3(0, \vec{0}_\perp, y^3)\right).
\] (12)

The spin vector in this case is written as
\[
S^\mu = (S^0, \vec{S}_\perp, S^3) = \left(\lambda \frac{P^3}{M}, \vec{S}_\perp, \lambda \frac{P^0}{M}\right).
\] (13)

The quasi-PDFs of interest are then defined as
\[
\Phi_Q^{[1]} = \frac{M}{P^3} e_Q(x; P_3), \quad \Phi_Q^{[\sigma^0 \gamma_5]} = \frac{M}{P^3} \lambda h_{L,Q}(x; P_3).
\] (14)

The definitions of the quasi-PDFs are such that their lowest moments are $P^3$ independent [129],
\[
\int dx e_Q(x; P^3) = \int dx e(x), \quad \int dx h_{L,Q}(x; P^3) = \int dx h_L(x).
\] (15)

### III. ONE-LOOP RESULTS

In this section, we calculate the perturbative corrections to the light-cone PDFs and the quasi-PDFs to one-loop order. In principle, one can do these calculations in any gauge and the final result should be independent of the gauge. Here, we choose to work in the Feynman gauge for which the contributing real and virtual diagrams are shown in Fig. 1 and Fig. 2 respectively. We regulate the infrared (IR) divergences by making use of 3 different schemes: non-zero parton mass regulations ($m_g \neq 0$ for gluon mass and $m_q \neq 0$ for quark mass) and dimensional regularization (DR). The ultraviolet (UV) divergences in the problem have consistently been tackled with DR. The individual diagrams have additional divergences at $x = 1$. However, the combination of real and virtual corrections (which are proportional to $\delta(1-x)$) is well-defined. Since our computations are at the level of the partons (these results are prior to embedding them into a full correlator picture), we use $m_q$ and $p (= xP)$ as the mass and 4-momentum for the (quark) target.
A. Results for $e(x)$

In this subsection, we focus on the light-cone PDF $e(x)$ and its corresponding quasi-PDF $e_Q(x)$.

1. Light-cone PDF

Let us discuss first the computation of the real diagrams. The one-loop correction for Fig. (1a) is calculated as

$$\frac{m_q}{p^+} e^{(1a)}(x) = -\frac{ig^2 C_F \mu^2 g_{\mu\nu}}{4} \int_{-\infty}^{\infty} d^n k \, Tr. \left[ (p + m_q) \gamma^\nu \left( \frac{k}{m_q} + m_q \right) \frac{i(\gamma^\mu)}{2} \right] \delta \left( x - \frac{k^+}{p^+} \right) \frac{1}{p^+}, \quad (16)$$

where $g$ denotes the coupling for the quark-gluon-quark vertex and $C_F = 4/3$ is the color factor. The integrals in Eq. (16) have been analytically continued to $n = 4 - 2\epsilon$ dimensions to regularize the divergences present otherwise. Here $\epsilon$ is the DR regulator. If $\epsilon$ is used for the UV divergences, then $\epsilon \to \epsilon_{UV} > 0$ (and the corresponding subtraction scale is $\mu \to \mu_{UV} > 0$), while if it is used for the IR divergences then $\epsilon \to \epsilon_{IR} < 0$ (and $\mu \to \mu_{IR} > 0$). Trace algebra simplifies Eq. (16) to

$$e^{(1a)}(x) = -\frac{ig^2 C_F \mu^2}{(2\pi)^n} p^+ \int_{-\infty}^{\infty} d^{n-2}k_\perp dk^- dk^+ \frac{(2 - n)2 p \cdot k + n(k^2 + m_q^2)}{(k^2 - m_q^2 + i\epsilon)^2((p - k)^2 - m_q^2 + i\epsilon)} \delta \left( x - \frac{k^+}{p^+} \right) \frac{1}{p^+}. \quad (17)$$

We will use the following abbreviation to present our one-loop results

$$P_{UV} = \frac{1}{\epsilon_{UV}} + \ln 4\pi - \gamma_E,$$

and similarly $P_{IR}$ for the IR. After regulating UV and IR divergences in the $k_\perp$ integrals, Eq. (17) for $m_g \neq 0$ case can be written as

$$e^{(1a)}(x) \bigg|_{m_g} = e^{(1a)}(x) \bigg|_{m_g}, \quad (18)$$

where the “singular” part of the light-cone PDF $e(x)$ (denoted as $e_{(s)}$) is given by

$$e^{(1a)}(x) \bigg|_{m_g} = \frac{\alpha_s C_F}{2\pi} \delta(x) \left( P_{UV} + \ln \frac{\mu_{UV}^2}{m_q^2} - 1 \right), \quad (19)$$

and the “canonical” (or the regular) part of the light-cone PDF $e(x)$ (denoted as $e_{(c)}$) is given by

$$e^{(1a)}(x) \bigg|_{m_g} = \frac{\alpha_s C_F}{2\pi} \delta(x) \left( P_{UV} + \ln \frac{\mu_{UV}^2}{\mu_{IR}^2} \right), \quad (20)$$

It is interesting to discuss the above results. We divided the result into two distinct parts: (a) singular, and (b) canonical. The singular part of the PDF has a zero-mode $\delta(x)$ contribution. Such a singularity originates from a term proportional to $p \cdot k$ (see the first term in Eq. (17)), which can be used to cancel the gluon propagator leading to

$$\int dk^- \frac{1}{(k^2 - m_q^2 + i\epsilon)^2} = (2 - n) \frac{i\pi}{(k^2 + m_q^2)^{2/3}} \delta(x). \quad (21)$$

The $k_\perp$ integral in Eq. (21) has a UV divergence which is regulated by DR, and the coefficient of this integral is such that the UV pole $1/\epsilon_{UV}$ allows for a $\delta(x)$ contribution in Eq. (19). For $m_g \neq 0$, one should in principle set the quark mass term in Eq. (21) to zero. In doing so, we confront an IR divergence in the limit $k_\perp \to 0$. As we pointed out in Ref. [145], this IR divergence is left unattended when one works with a nonzero gluon mass, and this is a new feature.
appearing at the level of twist-3. In fact, this insufficiency of the gluon mass as an IR regulator is only confined to this specific singular zero-mode term present in Fig. (1a). For practical reasons, we suggest(ed) to handle the IR divergence by either retaining the quark mass term in Eq. (21) or by using DR. For $g_T$, the two methods lead to two (qualitatively) different answers, namely, the $\delta(x)$ drops out when using DR [135]. For $e(x)$, as well as for $h_L(x)$, the coefficient of the $k_\perp$ integral in Eq. (21) is such that, regardless of the IR scheme, the $\delta(x)$ term survives. There is another crucial difference between the $\delta(x)$ appearing here versus those in $g_T$. The $\delta(x)$ for $e(x)$ and $h_L(x)$ comes in with an prefactor that has an explicit dependence on the IR pole. On the other hand, the prefactor of $\delta(x)$ for $g_T$ is IR-finite. We shall see later that this feature creates a major complication for the matching. Note that the two results for the singular part of $e(x)$ in Eq. (19) correspond to the two options of working with either $m_q \neq 0$, or DR for the $k_\perp$ integral in Eq. (21). For the canonical part of $e(x)$, $m_q \neq 0$ is sufficient to regulate the IR divergences and, therefore, we have a unique result in Eq. (20). With $m_q \neq 0$ and DR for the IR, one obtains

$$e^{(1a)}(x)\bigg|_{m_q} = e^{(1a)}_{(s)}(x)\bigg|_{m_q} + e^{(1a)}_{(c)}(x)\bigg|_{m_q} = \frac{\alpha_s C_F}{2\pi} \delta(x) \left( P_{\text{UV}} + \ln \frac{\mu^2_{\text{UV}}}{m_q^2} - 1 \right) + \frac{\alpha_s C_F}{2\pi} \left( P_{\text{UV}} + \ln \frac{\mu^2_{\text{UV}}}{(1-x)^2 m_q^2} - \frac{2}{1-x} \right),$$

$$e^{(1a)}_{(s)}(x)\bigg|_{\epsilon_{\text{IR}}} = e^{(1a)}_{(s)}(x)\bigg|_{\epsilon_{\text{IR}}} + e^{(1a)}_{(c)}(x)\bigg|_{\epsilon_{\text{IR}}} = \frac{\alpha_s C_F}{2\pi} \delta(x) \left( P_{\text{UV}} - P_{\text{IR}} + \ln \frac{\mu^2_{\text{UV}}}{\mu_{\text{IR}}^2} \right) + \frac{\alpha_s C_F}{2\pi} \left( P_{\text{UV}} - P_{\text{IR}} + \ln \frac{\mu^2_{\text{UV}}}{\mu_{\text{IR}}^2} \right).$$

(22)

Therefore, for all three IR regulators, the $\delta(x)$ contributes.

The diagram of Fig. (1b) is calculated as

$$\frac{m_q}{p^+} e^{(1b)}(x) = -ig^2 C_F 2 \epsilon_{\text{uv}} v^\nu \int_{-\infty}^{\infty} d^n k \frac{1}{(2\pi)^n} \text{Tr.} \left[ (\not{p} + m_q) \frac{1}{2} (\not{k} + m_q) \gamma^\mu \right] \delta \left( x - \frac{k^+}{p^+} \right) \frac{1}{p^+}.$$ (23)

Here, $v$ is defined such that $v^2 = 0$ and $v \cdot a = a^+$ for any four-vector $a^\mu$. The results for the three IR regulators are

$$e^{(1b)}_{(s)}(x)\bigg|_{m_q} = \frac{\alpha_s C_F}{2\pi} \frac{1 + x}{2(1-x)} \left( P_{\text{UV}} + \ln \frac{\mu^2_{\text{UV}}}{x m_q^2} \right),$$

(24)

$$e^{(1b)}_{(s)}(x)\bigg|_{\epsilon_{\text{IR}}} = \frac{\alpha_s C_F}{2\pi} \frac{1 + x}{2(1-x)} \left( P_{\text{UV}} + \ln \frac{\mu^2_{\text{UV}}}{(1-x)^2 m_q^2} \right),$$

(25)

$$e^{(1b)}_{(c)}(x)\bigg|_{\epsilon_{\text{IR}}} = \frac{\alpha_s C_F}{2\pi} \frac{1 + x}{2(1-x)} \left( P_{\text{UV}} - P_{\text{IR}} + \ln \frac{\mu^2_{\text{UV}}}{\mu_{\text{IR}}^2} \right).$$

(26)

The diagram of Fig. (1c) gives the same result as the one of Fig. (1b). For the light-cone PDFs, the diagram of Fig. (1d) drops out because the results are proportional to $v^2$.

We now proceed with the computation of the virtual diagrams. The quark self-energy diagram in Fig. (2a) is independent of the Dirac structure and we presented it in Re. [135]. We quote the results here for the sake completeness,

$$e^{(2a)}(x)\bigg|_{m_q} = -\frac{\alpha_s C_F}{2\pi} \int_0^1 dy \left( P_{\text{UV}} + \ln \frac{\mu^2_{\text{UV}}}{y m_q^2} - 1 \right),$$

(27)

$$e^{(2a)}(x)\bigg|_{m_q} = -\frac{\alpha_s C_F}{2\pi} \int_0^1 dy (1-y) \left( P_{\text{UV}} + \ln \frac{\mu^2_{\text{UV}}}{(1-y)^2 m_q^2} - \frac{1+y^2}{(1-y)^2} \right),$$

(28)

$$e^{(2a)}_{(s)}(x)\bigg|_{\epsilon_{\text{IR}}} = -\frac{\alpha_s C_F}{2\pi} \int_0^1 dy \left( P_{\text{UV}} - P_{\text{IR}} + \ln \frac{\mu^2_{\text{UV}}}{\mu_{\text{IR}}^2} \right),$$

(29)

where $y$ is the (integrated) loop momentum fraction.
The initial expression for the diagrams of Fig. (2b) and Fig. (2c), is the same as the ones of Fig. (1b) and Fig. (1c), respectively, modulo an overall sign (see Re. [145]). Therefore the results are

\[ e^{(2b)}_{\text{m}_g} = -\frac{\alpha_s C_F}{2\pi} \int_0^1 dy \frac{1 + y}{2(1 - y)} \left( \mathcal{P}_{UV} + \ln \frac{\mu_{\text{UV}}^2}{y m_g^2} \right), \]  

(30)

\[ e^{(2b)}_{\text{m}_q} = -\frac{\alpha_s C_F}{2\pi} \int_0^1 dy \frac{1 + y}{2(1 - y)} \left( \mathcal{P}_{UV} + \ln \frac{\mu_{\text{UV}}^2}{(1 - y)^2 m_q^2} \right), \]  

(31)

\[ e^{(2b)}_{\text{m}_q} = -\frac{\alpha_s C_F}{2\pi} \int_0^1 dy \frac{1 + y}{2(1 - y)} \left( \mathcal{P}_{UV} - \mathcal{P}_{\text{IR}} + \ln \frac{\mu_{\text{UV}}^2}{\mu_{\text{IR}}^2} \right). \]  

(32)

Finally, the diagram of Fig. (2d) does not contribute, similar to the corresponding real diagram of Fig. (1d). All these results for the virtual diagrams are to be understood with an overall prefactor of \( \delta(1 - x) \) which we have left out for simplicity.

2. Quasi-PDF

We have outlined the procedure of calculating the real and virtual diagrams for the quasi-PDFs in Ref. [145]. Here, we only quote the final results, and refer to Ref. [145] for more details. For the quasi-PDF \( e_{Q} \), the diagram of Fig. (1a) gives\(^2\)

\[ e_{Q(1a)}^{(1a)}(x) \bigg|_{m_g} = e_{Q(1a)}^{(1a)}(x) + e_{Q(1c)}^{(1a)}(x) \bigg|_{m_g} \]

\[ \begin{align*}
  &\frac{1}{x} \\
  &0 < x < 1 \\
  &-\frac{1}{x} \\
\end{align*} \]

\[ \begin{align*}
  &\left\{ \begin{array}{ll}
  \ln \frac{x}{x-1} & x > 1 \\
  \ln \frac{4(1-x)p_3^2}{m_g^2} - \frac{1-x}{x} & 0 < x < 1 \\
  \ln \frac{x-1}{x} & x < 0,
  \end{array} \right. \]  

(33)

\[ e_{Q(1a)}^{(1a)}(x) \bigg|_{m_q} = e_{Q(1a)}^{(1a)}(x) + e_{Q(1c)}^{(1a)}(x) \bigg|_{m_q} \]

\[ \begin{align*}
  &\frac{1}{x} \\
  &0 < x < 1 \\
  &-\frac{1}{x} \\
\end{align*} \]

\[ \begin{align*}
  &\left\{ \begin{array}{ll}
  \ln \frac{x}{x-1} & x > 1 \\
  \ln \frac{4x p_3^2}{(1-x)m_q^2} - \frac{2}{1-x} & 0 < x < 1 \\
  \ln \frac{x-1}{x} & x < 0,
  \end{array} \right. \]  

(34)

\[ e_{Q(1a)}^{(1a)}(x) \bigg|_{\text{IR}} = e_{Q(1a)}^{(1a)}(x) + e_{Q(1c)}^{(1a)}(x) \bigg|_{\text{IR}} \]

\[ \begin{align*}
  &\frac{1}{x} \\
  &0 < x < 1 \\
  &-\frac{1}{x} \\
\end{align*} \]

\[ \begin{align*}
  &\left\{ \begin{array}{ll}
  \ln \frac{x}{x-1} & x > 1 \\
  -\mathcal{P}_{\text{IR}} + \ln \frac{4x(1-x)p_3^2}{\mu_{\text{IR}}^2} & 0 < x < 1 \\
  \ln \frac{x-1}{x} & x < 0.
  \end{array} \right. \]  

(35)

\(^2\) For convenience of notation, in our results we use that \( p_3^2 = (p^3)^2 \).
for the three IR regulators. Once again, the result for Fig. (1a) can be divided into a singular and a canonical part. The term which generates a $\delta(x)$ in the light-cone PDF $e(x)$ gives rise to a $1/x$ pole, as $x \to 0$, in the quasi-PDF $e_Q$. Comparing this with the quasi-g_T (which we denote as $g_{T,Q}$), we find that the singular terms in $g_{T,Q}$ have an overall prefactor of $\epsilon$. Due to the fact that the accompanying $k_\perp$ integrals are both UV and IR finite, the $1/x$ poles vanish in the limit $\epsilon \to 0$. Such $1/x$ poles survive in $e_Q$ and $h_{L,Q}$ because they are accompanied by a prefactor of $(1 - \epsilon)$ (see the first term in Eq. (17)). Note that the canonical part of $e_Q$ (and $h_{L,Q}$) may also have a $1/x$ pole depending on the IR regulator (see $m_g \neq 0$ result in Eq. (38)).

The contribution from the diagram of Fig. (1b) is given by

$$e_Q^{(1b)}(x) \bigg|_{m_g} = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\ln \frac{x}{x-1} & x > 1 \\
\ln \frac{4(1-x)p^2_3}{m_g^2} & 0 < x < 1 \\
\ln \frac{x-1}{x} & x < 0,
\end{array} \right. \tag{36}$$

$$e_Q^{(1b)}(x) \bigg|_{m_q} = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\ln \frac{x}{x-1} & x > 1 \\
\ln \frac{4xp^2_3}{(1-x)m_q^2} & 0 < x < 1 \\
\ln \frac{x-1}{x} & x < 0,
\end{array} \right. \tag{37}$$

$$e_Q^{(1b)}(x) \bigg|_{\epsilon_{IR}} = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\ln \frac{x}{x-1} & x > 1 \\
-\mathcal{P}_{IR} + \ln \frac{4(1-x)p^2_3}{\mu^2_{IR}} & 0 < x < 1 \\
\ln \frac{x-1}{x} & x < 0.
\end{array} \right. \tag{38}$$

The diagram of Fig. (1c) gives the same result as above. Unlike the case of light-cone PDFs, the diagram of Fig. (1d) is non-vanishing for the quasi-PDFs and the result is given by

$$e_Q^{(1d)}(x) = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\frac{1}{1-x} & x > 1 \\
\frac{1}{x-1} & 0 < x < 1 \\
\frac{1}{x-1} & x < 0
\end{array} \right. \tag{39}$$

with the result being independent of the IR regulator.

We now take up the virtual diagrams. The quark self-energy diagram, which has been computed in our previous work [145], is given by

$$e_Q^{(2a)} \bigg|_{m_g} = -\frac{\alpha_s C_F}{2\pi} (1 - \epsilon_{UV}) C(\epsilon_{UV}) \left( \frac{p^3}{\mu_{UV}} \right)^{-2\epsilon_{UV}} \int dy \left\{ \begin{array}{ll}
y^{-2\epsilon_{UV}} \left( y \ln \frac{y}{y-1} - 1 \right) & y > 1 \\
y^{-2\epsilon_{UV}} \left( y \ln \frac{4(1-y)p^2}{m_g^2} + 1 - 2y \right) & 0 < y < 1 \\
(-y)^{-2\epsilon_{UV}} \left( y \ln \frac{y-1}{y} + 1 \right) & y < 0
\end{array} \right. \tag{40}$$
self-energy results can be found in Ref. [145].

\[ e^{(2a)}_{\nu} \Big|_{m_q} = -\frac{\alpha_s C_F}{2\pi} C(\epsilon_{\text{UV}}) \left( \frac{p^3}{\mu_{\text{UV}}} \right)^{-2\epsilon_{\text{UV}}} \int dy \left\{ \begin{array}{ll}
\frac{(1 - \epsilon_{\text{UV}})}{y} & y > 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y} & 0 < y < 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & y < 0,
\end{array} \right. \]

\[ e^{(2a)}_{\nu} \Big|_{\epsilon_{\text{IR}}} = -\frac{\alpha_s C_F}{2\pi} (1 - \epsilon_{\text{UV}}) C(\epsilon_{\text{UV}}) \left( \frac{p^3}{\mu_{\text{UV}}} \right)^{-2\epsilon_{\text{UV}}} \int dy \left\{ \begin{array}{ll}
\frac{y^{-2\epsilon_{\text{UV}}}}{y} & y > 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & 0 < y < 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & y < 0,
\end{array} \right. \]

where

\[ C(\epsilon_{\text{UV}}) = \frac{\pi^{1/2 - \epsilon_{\text{UV}}}}{(2\pi)^{-2\epsilon_{\text{UV}}} \Gamma[1/2 - \epsilon_{\text{UV}}]} \] (43)

The (integrated) loop momentum fraction \( y \) is defined through the relation \( k^3 = y p^3 \). A detailed discussion of these self-energy results can be found in Ref. [145].

For the diagram of Fig. (2b) we find

\[ e^{(2b)}_{\nu} \Big|_{m_q} = -\frac{\alpha_s C_F}{2\pi} C(\epsilon_{\text{UV}}) \left( \frac{p^3}{\mu_{\text{UV}}} \right)^{-2\epsilon_{\text{UV}}} \int dy \left\{ \begin{array}{ll}
\frac{y^{-2\epsilon_{\text{UV}}}}{y} & y > 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & 0 < y < 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & y < 0,
\end{array} \right. \] (44)

\[ e^{(2b)}_{\nu} \Big|_{\epsilon_{\text{IR}}} = -\frac{\alpha_s C_F}{2\pi} C(\epsilon_{\text{UV}}) \left( \frac{p^3}{\mu_{\text{UV}}} \right)^{-2\epsilon_{\text{UV}}} \int dy \left\{ \begin{array}{ll}
\frac{y^{-2\epsilon_{\text{UV}}}}{y} & y > 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & 0 < y < 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & y < 0,
\end{array} \right. \] (45)

\[ e^{(2b)}_{\nu} \Big|_{\epsilon_{\text{IR}}} = -\frac{\alpha_s C_F}{2\pi} C(\epsilon_{\text{UV}}) \left( \frac{p^3}{\mu_{\text{UV}}} \right)^{-2\epsilon_{\text{UV}}} \int dy \left\{ \begin{array}{ll}
\frac{y^{-2\epsilon_{\text{UV}}}}{y} & y > 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & 0 < y < 1 \\
\frac{y^{-2\epsilon_{\text{UV}}}}{y^2 - 5y + 1} & y < 0,
\end{array} \right. \]

and the diagram in Fig. (2c) gives the exact same result.
Finally, we find the following for the diagram in Fig. (2d)

\[
\epsilon^{(2d)}_Q = -\frac{\alpha_s C_F}{2\pi} C(\epsilon_{UV}) \left( \frac{p^3}{\mu^2_{UV}} \right)^{-2\epsilon_{UV}} \int dy \begin{cases} 
\frac{y^{-2\epsilon_{UV}}}{1-y} & y > 1 \\
\frac{y^{-2\epsilon_{UV}}}{y-1} & 0 < y < 1 \\
(-y)^{-2\epsilon_{UV}} \frac{1}{y-1} & y < 0 .
\end{cases}
\]

(47)

All of the \( y \) integrals appearing in the virtual diagrams are logarithmically divergent. These UV divergences can be renormalized in the MS scheme.

### B. Results for \( h_L \)

In this subsection, we present results for the light-cone PDF \( h_L(x) \) and the quasi-PDF \( h_{L,Q}(x) \).

#### 1. Light-cone PDF

The contribution from the diagram of Fig. (1a) can be obtained by making the replacement of \( 1 \rightarrow i\sigma^+ - \gamma_5 \) in Eq. (16). The resulting expressions with the three IR regulators are shown below.

For \( m_q = 0 \):

\[
h_L^{(1a)}(x) \bigg|_{m_q} = h_{L(0)}^{(1a)}(x) + h_{L(c)}^{(1a)}(x) \bigg|_{m_q}
\]

(48)

where the singular part of the light-cone PDF \( h_L(x) \) is

\[
h_{L(0)}^{(1a)}(x) = \begin{cases} 
h_{L(0)}^{(1a)}(x) \bigg|_{m_q} = -\frac{\alpha_s C_F}{2\pi} \delta(x) \left( \mathcal{P}_{UV} + \ln \frac{\mu^2_{UV}}{m_q^2} - 1 \right) \\
h_{L(0)}^{(1a)}(x) \bigg|_{\epsilon_{IR}} = -\frac{\alpha_s C_F}{2\pi} \delta(x) \left( \mathcal{P}_{UV} - \mathcal{P}_{IR} + \ln \frac{\mu^2_{UV}}{\mu^2_{IR}} \right)
\end{cases}
\]

(49)

and the canonical part of the light-cone PDF \( h_L(x) \) is

\[
h_{L(c)}^{(1a)}(x) \bigg|_{m_q} = \frac{\alpha_s C_F}{2\pi} \left( \mathcal{P}_{UV} + \ln \frac{\mu^2_{UV}}{x m_q^2} + \frac{(1-x)(1-2x)}{x} \right).
\]

(50)

For \( m_q \neq 0 \) and DR for the IR:

\[
h_L^{(1a)}(x) \bigg|_{m_q} = h_{L(0)}^{(1a)}(x) \bigg|_{m_q} + h_{L(c)}^{(1a)}(x) \bigg|_{m_q}
\]

\[
= -\frac{\alpha_s C_F}{2\pi} \delta(x) \left( \mathcal{P}_{UV} + \ln \frac{\mu^2_{UV}}{m_q^2} - 1 \right) + \frac{\alpha_s C_F}{2\pi} \left( \mathcal{P}_{UV} + \ln \frac{\mu^2_{UV}}{(1-x)^2 m_q^2} + 2x - 3 - \frac{1+x}{1-x} \right),
\]

\[
h_L^{(1a)}(x) \bigg|_{\epsilon_{IR}} = h_{L(0)}^{(1a)}(x) \bigg|_{\epsilon_{IR}} + h_{L(c)}^{(1a)}(x) \bigg|_{\epsilon_{IR}}
\]

\[
= -\frac{\alpha_s C_F}{2\pi} \delta(x) \left( \mathcal{P}_{UV} - \mathcal{P}_{IR} + \ln \frac{\mu^2_{UV}}{\mu^2_{IR}} \right) + \frac{\alpha_s C_F}{2\pi} \left( \mathcal{P}_{UV} - \mathcal{P}_{IR} + \ln \frac{\mu^2_{UV}}{\mu^2_{IR}} \right).
\]

(51)
The discussions for the diagram in Fig. (1a) made in the context of \( e(x) \) carries over to \( h_L(x) \). Note that the singular terms for \( h_{L,Q} \) and \( e(x) \) \( (e_Q) \) are the same except for an overall sign. After making the replacement of \( 1 \to i\sigma^+\gamma_5 \) in Eq. (23), we find that the results for the diagrams shown in Figs. (1b) and (1c) are the same as that of \( e(x) \). This is due to the relevant trace algebra. As a consequence, the results of all the virtual diagrams for \( h_L(x) \) are the same as that of \( e(x) \).

2. Quasi-PDF

For the quasi-PDF \( h_{L,Q} \) we find for the diagram in Fig. (1a):

\[
\left. h_{L,Q}^{(1a)}(x) \right|_{m_q} = \left. h_{L,Q(s)}^{(1a)}(x) + h_{L,Q(c)}^{(1a)}(x) \right|_{m_q}
\]

\[
= \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
-\frac{1}{x} & x > 1 \\
-\frac{1}{x} & 0 < x < 1 \\
\frac{1}{x} & x < 0
\end{array} \right. + \left\{ \begin{array}{ll}
\ln \frac{x}{x-1} & x > 1 \\
\frac{4(1-x)p_3^2}{m_q^2} + \frac{1-x}{x} & 0 < x < 1 \\
\frac{x-1}{x} & x < 0
\end{array} \right.
\]

(52)

\[
\left. h_{L,Q}^{(1a)}(x) \right|_{\epsilon_{ir}} = \left. h_{L,Q(s)}^{(1a)}(x) + h_{L,Q(c)}^{(1a)}(x) \right|_{\epsilon_{ir}}
\]

\[
= \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
-\frac{1}{x} & x > 1 \\
-\frac{1}{x} & 0 < x < 1 \\
\frac{1}{x} & x < 0
\end{array} \right. + \left\{ \begin{array}{ll}
\ln \frac{x}{x-1} & x > 1 \\
\frac{4xp^2_3}{(1-x)\mu_{ir}^2} - \frac{2}{1-x} & 0 < x < 1 \\
\frac{x-1}{x} & x < 0
\end{array} \right.
\]

(53)

The other diagrams yield the same results as \( e_Q \) (see corresponding comment in previous sub-section).

IV. THE PROBLEM OF AN UNCANCELLED IR DIVERGENCE

Schematically, the relation between light-cone and quasi-PDFs is expressed through the following factorization theorem up to power corrections that are suppressed with respect to the hadron momentum,

\[
\tilde{q}(x; P^3) = \int_{-1}^{+1} \frac{dy}{|y|} C \left( \frac{x}{y} \right) q(y) + \mathcal{O} \left( \frac{1}{P^3} \right).
\]

(55)

In Eq. (55), the symbol \( \tilde{q}(q) \) stands for a quasi-PDF (light-cone PDF) of a parton inside a hadron, while \( C \) denotes the matching coefficient. For twist-3 PDFs, we expect mixing between different operators, even for the quark non-singlet
case. This point, however, should be irrelevant for the main result discussed in this section. The key feature of the factorization-type formula in (55) is the IR-finiteness of the matching coefficient \( C \). To derive the first order correction to the matching coefficient, one applies a perturbative expansion of Eq. (55) in powers of \( \alpha_s \), leading to

\[
C(x) = \delta(1-x) + \frac{\alpha_s C_F}{2\pi} \left[ \frac{\tilde{\Gamma}(x) - \Gamma(x)}{1} + \frac{\alpha_s C_F}{2\pi} (1-x) \tilde{\Pi} - \Pi \right].
\]

In Eq. (56), \( \tilde{\Gamma} \) (\( \tilde{\Pi} \)) and \( \Pi \) (\( \tilde{\Pi} \)) are the real corrections and the virtual corrections for the light-cone (quasi-) PDFs, respectively. Eq. (56) implies that the matching coefficient, at the lowest nontrivial order in perturbation theory, is given by the difference between one-loop results for the quasi-PDFs and the light-cone PDFs. Matching, in conjunction with proper renormalization, corrects for the different UV behavior between the light-cone and quasi-distributions such that in the limit of \( P^3 \rightarrow \infty \) one is able to recover the light-cone distributions.

As we explained in Sec. [13], the formalism of the matching relies on the fact that the IR behavior of the light-cone and quasi distributions are the same. Previous papers on matching calculations for the twist-2 distributions have confirmed this [38–40, 53, 66, 79, 80, 90, 138–144]. In Ref. [145], where we addressed the one-loop matching formula for \( g_T \), we also found an agreement between the IR behavior of the light-cone PDF \( g_T \) and the quasi-PDF \( g_{T,Q} \). However, this no longer holds for \( e(x) \) and \( h_L(x) \).

We find that the prefactors of the \( \delta(x) \) terms in the light-cone PDFs \( e(x) \) and \( h_L(x) \) have an IR pole. The quasi-PDFs \( e_Q \) and \( h_{L,Q} \) exhibit a \( 1/x \) term and therefore also a singularity at \( x = 0 \). However, the functional forms of the singular terms in the light-cone and quasi-PDFs do not match. This can be seen in the comparison of the singular parts of the light-cone and quasi-PDFs of the diagram in Fig. (1a) (see, for instance, Eq. (19) and Eq. (33)). On the other hand, for the canonical parts of Fig. (1a), for all the other diagrams and for all three regulators, we find an exact match in the IR pole between the light-cone and the quasi-PDFs. Therefore, we infer an IR agreement everywhere except at \( x = 0 \). To the best of our knowledge, the issue we find here for the twist-3 functions \( e(x) \) and \( h_L(x) \) is the first clear indication that agreement between quasi-PDFs and light-cone PDFs in the IR region cannot be taken for granted. Strictly speaking, this issue of the IR mismatch, stemming from the singular zero-mode contribution, leads to a breakdown of matching. In other words, for \( x = 0 \) we do not have a perturbatively calculable matching coefficient. Based on the general structure of the matching formula, this result can affect the light-cone PDFs beyond the specific point \( x = 0 \). Moreover, the aforementioned \( 1/x \) singularity in the quasi-PDFs \( e_Q \) and \( h_{L,Q} \) can cause an additional problem for the numerical evaluation of the matching formula.

It is interesting, and in view of the moment relations in Eq. (15), actually expected, that the singular terms in the light-cone PDFs and the quasi-PDFs provide the same result upon integration over \( x \). In particular, one would expect to get the same IR divergent terms in both cases. This is indeed true, but the analysis requires some care. We need to remember that the above one-loop results for the quasi-PDFs are based on a Taylor expansion to extract the leading-twist contribution. Prior to this expansion, one finds the following singular parts of \( e_Q \) and \( h_{L,Q} \),

\[
e_Q(s) \Big|_{m_q} = -h_{L,Q}(s) \Big|_{m_q} = \frac{\alpha_s C_F}{2\pi} \frac{p^3}{\sqrt{x^2 p^2_q + m_q^2}} \quad -\infty < x < \infty,
\]

\[
e_Q(s) \Big|_{e_{IR}} = -h_{L,Q}(s) \Big|_{e_{IR}} = \frac{\alpha_s C_F}{2\pi} \frac{p^3}{\sqrt{x^2 p^2_q}} \quad -\infty < x < \infty.
\]

Obviously, if DR is used as IR regulator, there is no difference between the non-expanded and the expanded results. For a nonzero quark mass, however, the expressions before and after the Taylor expansion are different. In that case one needs to start from the non-expanded expression in order to do the \( x \) integration, while the expanded result can’t be integrated due to a divergence. The \( x \) integrals of the terms in (57), (58) are both UV and IR divergent. Using DR to isolate the UV divergences, we find

\[
\int dx \, e_Q(s)(x) \Big|_{m_q} = - \int dx \, h_{L,Q}(s)(x) \Big|_{m_q} = \frac{\alpha_s C_F}{2\pi} \begin{cases} 
\frac{1}{2} P_{UV} + \frac{1}{2} \ln \frac{\mu^2}{p^2_q} - \frac{1}{2} + \frac{1}{4} \ln 1 & x > 1 \\
\frac{1}{2} P_{UV} + \frac{1}{2} \ln \frac{\mu^2}{p^2_q} - \frac{1}{2} + \frac{1}{4} \ln 1 & 0 < x < 1
\end{cases}
\]

(59)
The renormalized expressions of the light-cone results. As for quasi-PDFs, we renormalize the MS scheme can be represented in the compact form. Generally, our work shows that a statement like “quasi-PDFs and light-cone PDFs contain the same non-perturbative physics” is actually nontrivial. In fact, it is possible that for other (higher-twist) correlation functions issues similar to the one reported here may exist. As far as the matching coefficient for \( x \) matching formula can be established, despite the fact that we may not find a perturbatively calculable matching coefficient for which we have to look at results prior to the IR divergences present in the singular terms of the distributions. The difference between integrals of the above terms in the same way as the IR poles that appear in these \( x \) integrals are exactly the same poles that are seen in the \( \delta(x) \) terms in the light-cone PDFs. In fact, the full results for the lowest moments of the singular terms of the quasi-PDFs and light-cone PDFs agree, that is, they satisfy Eq. (15). (For the proper check of the lowest moment, one has to do the \( x \) integrals of the above terms in the same way as the \( k_\perp \) integrals are handled for the light-cone PDFs.) This can be considered a consistency check of this part of the calculation. On the other hand, this outcome does, a priori, not help to solve the issue of the matching for which we have to look at results prior to the \( x \) integration.

In the following, we take a brief look at the difference between one-loop results for the quasi-PDFs and the light-cone PDFs. As discussed above, this procedure gives the matching coefficient, provided that matching exists. Here, we will refrain from calling this difference a “matching coefficient” due to the issues discussed above. For the purpose of this discussion, we take the \( \overline{\text{MS}} \) renormalized expressions of the light-cone results. As for quasi-PDFs, we renormalize the virtual diagram results in the same scheme, leaving the real diagram results as it is. The basics steps to do this exercise have been outlined in Ref. [145]. The difference between one-loop results for \( e_Q \) (\( h_{L,Q} \)) and \( e(x) \) (\( h_L(x) \)) in the \( \overline{\text{MS}} \) scheme can be represented in the compact form

\[
C_{\overline{\text{MS}}} \left( \xi, \frac{\mu^2}{p_T^3} \right) = \delta(1 - \xi) + C_{\overline{\text{MS}}}^{(c)} \left( \xi, \frac{\mu^2}{p_T^3} \right) + C_{\overline{\text{MS}}}^{(s)} \left( \xi, \frac{\mu^2}{p_T^3} \right),
\]

where the first term corresponds to the tree-level distributions, while the second and the third terms are the differences from the singular and canonical parts of the distributions, respectively.

The difference between \( e_Q(x) \) and \( e(x) \) from the singular terms is IR-dependent and it reads

\[
\left. C_{\overline{\text{MS}}}^{(s)} \left( \xi, \frac{\mu^2}{p_T^3} \right) \right|_{m_q \neq 0} = \left. C_{\text{MS}}^{(s)} \left( \xi, \frac{\mu^2}{p_T^3} \right) \right|_{\epsilon_{\text{IR}}} = \left. C_{\text{MS}}^{(s)} \left( \xi, \frac{\mu^2}{p_T^3} \right) \right|_{\epsilon_{\text{IR}} = 0} = \alpha_s C_F \left( \frac{1}{\xi} \right) \delta(\xi) \left( \ln \frac{\mu^2}{m_q^2} - 1 \right) 0 < \xi < 1
\]

Here we have done the change of variable \( x \to \xi \), in order to reserve \( x \) as the variable signifying the momentum fraction carried by quarks inside the hadrons, that is \( p_T^3 = xP^3 \). The two results correspond to working with either \( m_q \) or DR for the IR divergences present in the singular terms of the distributions. The difference between \( e_Q \) and
$e(x)$ from the canonical terms is

\[
C^{(c)}_{\text{MS}}(\xi, \mu^2_{p_3}) = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\left[ \frac{2}{1-\xi} \ln \frac{\xi}{\xi-1} + \frac{1}{1-\xi} + \frac{1}{\xi} \right] - \frac{1}{\xi} & \xi > 1 \\
\left[ \frac{2}{1-\xi} \ln \frac{4(1-\xi)\mu^2_{p_3}}{\mu^2} - \frac{1}{1-\xi} \right] + 0 & 0 < \xi < 1 \\
\left[ \frac{2}{1-\xi} \ln \frac{\xi-1}{\xi} - \frac{1}{1-\xi} + \frac{1}{1-\xi} \right] + \frac{1}{1-\xi} & \xi < 0
\end{array} \right.
\]

where the plus-prescription $[...]+$ has been defined at $\xi = 1$. The above equation reaffirms that there is an exact agreement in the IR poles for the canonical terms of the distributions. This result is the same for all three IR regulators. This finding is in agreement with previous studies reporting that the matching coefficient is regulator-independent. We have discussed this point in Ref. [145] in the context of $g_T$.

We now turn our attention to the difference between one-loop results for $h_{L,Q}(x)$ and $h_L(x)$ in the $\overline{\text{MS}}$. For the IR-dependent singular term we obtain

\[
C^{(s)}_{\text{MS}}(\xi, \mu^2_{p_3}) = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
C^{(s)}_{\text{MS}}(\xi, \mu^2_{p_3}) \bigg|_{m_q \neq 0} = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\left[ \frac{1}{\xi} \right] - \frac{1}{\xi} & \xi > 1 \\
\left[ 1 \right] - \frac{\ln \frac{\mu^2_{m_q}}{\mu^2} - 1 \right] & 0 < \xi < 1 \\
\left[ \frac{1}{\xi} \right] + \frac{\delta(\xi)}{\xi - 1} + \delta(\xi) & \xi < 0
\end{array} \right. & \xi > 1 \\
C^{(s)}_{\text{MS}}(\xi, \mu^2_{p_3}) \bigg|_{\epsilon_{IR}} = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\left[ \frac{1}{\xi} \right] - \frac{\ln \frac{\mu^2_{m_q}}{\mu^2} - 1 \right] & 0 < \xi < 1 \\
\left[ \frac{1}{\xi} \right] + \frac{\delta(\xi)(1 + \ln 4\pi - \gamma_E)}{\epsilon_{IR}} & \xi < 0
\end{array} \right. & \xi > 1
\end{array} \right.
\]

and for the canonical term, which is also independent of the IR regulator, we get

\[
C^{(c)}_{\text{MS}}(\xi, \mu^2_{p_3}) = \frac{\alpha_s C_F}{2\pi} \left\{ \begin{array}{ll}
\left[ \frac{2}{1-\xi} \ln \frac{\xi}{\xi-1} + \frac{1}{1-\xi} + \frac{1}{\xi} \right] - \frac{1}{\xi} & \xi > 1 \\
\left[ \frac{2}{1-\xi} \ln \frac{4(1-\xi)\mu^2_{p_3}}{\mu^2} - 2(1-\xi) \right] + 0 & 0 < \xi < 1 \\
\left[ \frac{2}{1-\xi} \ln \frac{\xi-1}{\xi} - \frac{1}{1-\xi} + \frac{1}{1-\xi} \right] + \frac{1}{1-\xi} & \xi < 0
\end{array} \right.
\]

+ \frac{\alpha_s C_F}{2\pi} \delta(1-\xi) \left( 1 + \ln \frac{\mu^2_{m_q}}{4\mu^2_{p_3}} \right).
\]
In this paper, we present a calculation of the twist-3 light-cone PDFs $e(x)$ and $h_L(x)$ and their quasi-PDF counterparts $e_Q(x)$ and $h_{L,Q}(x)$ for a quark target to one-loop order in perturbation theory. We have regulated the IR divergences in 3 different ways: non-zero parton mass regulations, that is $m_g \neq 0$ and $m_q \neq 0$, and DR. The UV divergences are regulated using DR.

Throughout our work, we point out the main differences between these results and the ones from our previous work on $g_T(x)$ \[145\]. Specifically, we discuss the role played by singular zero-mode contributions in the matching for $e(x)$ and $h_L(x)$. While a $\delta(x)$ may or may not arise in $g_T$ depending upon the IR scheme, it is bound to be present in $e(x)$ and $h_L(x)$. Even more importantly, the $\delta(x)$ in $e(x)$ and $h_L(x)$ is accompanied by prefactors that exhibit an IR divergence. The ($x$-dependent) quasi-PDFs $e_Q$ and $h_{L,Q}$ have a different pole structure at $x = 0$. As a result, at present we don't have a matching coefficient for $x = 0$, which may prevent one from obtaining reliable numerical results for the light-cone PDFs in an extended $x$ range. Put differently, it may be impossible to extract the light-cone PDFs $e(x)$ and $h_L(x)$ from lattice QCD calculations via the quasi-PDF approach in the same way as one extracts the twist-2 PDFs. Complete matching equations for twist-3 PDFs may involve operator mixing, which we did not consider in the present work. On the other hand, such a mixing should not affect the reported mismatch between light-cone and quasi-PDFs in the IR region.

The question whether the presence of the zero-mode contributions can prevent the extraction of $e(x)$ and $h_L(x)$ from lattice QCD remains open at the moment. But we hope that the analytical results presented here will stimulate further theoretical investigations. They may also be useful in order to find a candidate for a meaningful matching formula for the twist-3 PDFs $e(x)$ and $h_L(x)$.

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