POLARIZATIONS OF POWERS OF GRADED MAXIMAL IDEALS

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Abstract. We give a complete combinatorial characterization of all possible polarizations of powers of the graded maximal ideal \((x_1, x_2, \ldots, x_m)^n\) of a polynomial ring in \(m\) variables. We also give a combinatorial description of the Alexander duals of such polarizations. In the three variable case \(m = 3\) and also in the power two case \(n = 2\) the descriptions are easily visualized and we show that every polarization defines a (shellable) simplicial ball. We conjecture that any polarization of an Artinian monomial ideal defines a simplicial ball.

Introduction

For a monomial ideal \(I\) in a polynomial ring \(k[x_1, \ldots, x_m]\), where \(k\) is any field, one has the construction of polarization to a squarefree monomial ideal \(J\) in a larger polynomial ring.

For example, the ideal

\[(1) \quad I = (x_1, x_2)^2 = (x_1^2, x_1x_2, x_2^2) \subseteq k[x_1, x_2] \]

polarizes to the ideal

\[J = (x_{11}x_{12}, x_{11}x_{21}, x_{21}x_{22}) \subseteq k[x_{11}, x_{12}, x_{21}, x_{22}].\]

The quotient ring \(k[x_1, x_2]/I\) then comes from \(k[x_{11}, x_{12}, x_{21}, x_{22}]/J\) by cutting down by a regular sequence of variable differences

\[x_{11} - x_{12}, x_{21} - x_{22}.\]

These two graded rings have the same homological properties, such as codimension, codepth, and the same graded Betti numbers.

In general, for a monomial ideal \(I\) one gets the polarization \(J\) by taking each minimal generator of \(I\)

\[(2) \quad x_1^{a_1}x_2^{a_2}\cdots x_m^{a_m}\]

and making a minimal generator

\[(x_{11}x_{12}\cdots x_{1a_1}) \cdot (x_{21}x_{22}\cdots x_{2a_2}) \cdots \cdot (x_{m1}\cdots x_{ma_m})\]

of \(J\). We call this the standard polarization.
However, for a monomial ideal \( I \) in \( k[x_1, \ldots, x_m] \), there may be many other ways to get a squarefree monomial ideal \( J \) in a larger polynomial ring \( k[x_{ij}] \) such that \( k[x_i]/J \) comes from \( k[x_{ij}]/J \) by dividing out by a regular sequence of variable differences \( x_{i,j} - x_{i,j'} \).

For instance, if \( I \) is a strongly stable ideal one has the “b-polarization”, \([19],[23]\) which from the ideal \((1)\) constructs the ideal

\[
J^b = (x_{11}x_{12}, x_{11}x_{22}, x_{21}x_{22}) \subseteq k[x_{11}, x_{12}, x_{21}, x_{22}].
\]

(In this special case the ideals \( J \) and \( J^b \) are isomorphic, but this is not so in general.)

The \( b \)-polarization takes the minimal generator \((2)\) and makes a minimal generator

\[
(x_{1,1} \cdots x_{1,a_1}) \cdot (x_{2,a_1+1} \cdots x_{2,a_1+a_2}) \cdots \cdot (x_{m,a_1+\cdots+a_{m-1}+1} \cdots x_{m,a_1+\cdots+a_m})
\]

so the second index runs through the integers from 1 to \( a_1 + \cdots + a_m \).

A third example of a large class of polarizations are the letterplace ideals associated to poset ideals of \( P \)-partitions, \([11]\) and \([12]\). Although polarizations in various forms have been considered in the literature there has not been a systematic focused investigation into the variety of such ideals, and of the characterizing properties of such ideals. The purpose of this article is to undertake this. Along our investigations we put forward several conjectures concerning topological properties and relating to algebraic geometry.

**Polarizations of powers of the graded maximal ideal.** We combinatorially describe all possible polarizations of powers of the graded maximal ideal

\[
I = (x_1, \ldots, x_m)^n \subseteq k[x_1, \ldots, x_m].
\]

These maximal ideal powers are the ideals in \( k[x_1, \ldots, x_m] \) with maximal symmetry. They are \( GL(n) \)-invariant. A polarization is somehow a way of breaking this symmetry, but still keeping the homological properties. Thus we classify all symmetry breaks of this ideal. (Let us also mention that powers of graded maximal ideals have been studied from another combinatorial perspective in \([2]\), using discrete Morse theory to make cellular resolutions.)

Let \( \Delta_m(n) \) be the lattice simplex consisting of all \( a = (a_1, \ldots, a_m) \in \mathbb{N}_0^m \) with \( \sum_{i=1}^{m} a_i = n \). Its vertices are in one-one correspondence with minimal generators of \( (x_1, \ldots, x_m)^n \). See Figure \( 1 \) for \( \Delta_3(3) \). It is divided into smaller triangles, and of these triangles, three are pointing down; we refer to these as down-triangles. By going upwards along the edges in this diagram, Figure \( 1 \) we get a partial order \( \geq_1 \) on \( \Delta_m(n) \). If the monomial \((2)\) polarizes to a monomial

\[
m(a) = m_1(a) \cdots m_m(a)
\]

where \( m_i(a) \) maps to \( x_i^{a_i} \), let \( X_i(a) \) be the variables in \( m_i(a) \). Then \( X_i \) can be considered as a map from \( \Delta_m(n) \) to the boolean poset of subsets of the \( x_{ij} \)-variables. Similarly, we get maps \( X_i \) for each \( i = 1, \ldots, m \). For any polarization \( J \) of \( I \) in \((3)\), it turns out that each \( X_i \) is an isotope map for a partial order \( \geq_i \) on \( \Delta_m(n) \). Conversely, given the maps \( \{X_i\} \), we can construct monomial generators \( m(a) \) of an ideal \( J \). When is this a polarization of \( I \)?

For a given edge in \( \Delta_m(n) \) between \( a \) and \( b \), we call the edge a linear syzygy edge if there is a linear syzygy between the monomials \( m(a) \) and \( m(b) \). Our main result, Theorem \([4],[3]\) says that the maps \( \{X_i\} \) determine a polarization of \( I \) if and only if the
linear syzygy edges for these maps contains a spanning tree of the edge graph (this is a complete graph) of each (higher-dimensional) down-triangle of $\Delta_m(n)$.

Two special cases are worth attention due to the easy visualization of the various polarizations. First the polarizations of $(x_1, x_2, x_3)^n$ which are easily visualized in terms of Figure 1 see Theorem 3.2 and Example 3.3. Secondly the polarizations of $(x_1, x_2, \ldots, x_m)^2$, which are in one-to-one correspondence with trees on $(m+1)$ vertices, Theorem 5.1 and Examples 5.2 and 5.3.

**Polarizations of Artinian monomial ideals.** More generally, we discuss polarizations of Artinian monomial ideals. We conjecture (Conjecture 2.4) that any such polarization defines a simplicial ball by the Stanley-Reisner correspondence. Moreover, there should be a simple natural description of the Stanley-Reisner ideal of the boundary of this ball, a simplicial sphere. We show that all polarizations of $(x_1, x_2, x_3)^n$ and of $(x_1, \ldots, x_m)^2$ give simplicial balls, by showing that the Alexander duals of all such polarizations have linear quotients. Conjecture 2.4 is known for the standard polarization by S.Murai [18], and for letterplace ideals by the second author et.al. in [6]. We also conjecture, Conjecture 2.12 that all first order deformations of such ideals lift to global deformations. In particular they are always smooth points on any Hilbert scheme.

**Alexander duals of polarizations.** If $J$ is any polarization of an Artinian monomial ideal $I$ of $k[x_1, \ldots, x_m]$, then its Alexander dual $J^\vee$ will be generated by “colored” monomials of degree $m$, monomials of the form

$$x_{1j_1}x_{2j_2} \cdots x_{mj_m}$$

where for each $i = 1, \ldots, m$, we have one variable $x_{i,j_i}$ from the class of $i$-variables (color $i$). We call these rainbow monomials. The class of ideals generated by rainbow monomials and with $m$-linear resolution is precisely the class which is Alexander dual to the class of polarizations of Artinian monomial ideals in $m$ variables, Proposition 2.1. A concise criterion for ideals generated by rainbow monomials to have linear resolution is given by A.Nematbakhsh in [20], and we recall it in Theorem 2.8.
We describe the Alexander dual of any polarization of the maximal ideal power $(x_1, \ldots, x_m)^n$, in terms of the isotone maps $X_i$. As it turns out, the argument involves only a baby-form $\chi_i$ of the isotone maps $X_i$, where $\chi_i$ is an isotone map from $\Delta_m(n)$ to the poset $\{0 < 1\}$. This description of the Alexander dual, however, leaves something to be desired concerning transparency; for instance, it is not obvious that the number of generators is actually always $\binom{n+m-1}{m}$. We discuss problems concerning ideals generated by rainbow monomials in Section 2.

Organization of the paper. In Section 1 we recall the notions of separations, separated models and polarizations. We show some basic results on polarizations of artinian monomial ideals and introduce the isotone maps $X_i$. In Section 2 we discuss various conjectures and problems that have come up during our investigations. Section 3 describes polarizations in the three variable case: polarizations of $(x_1, x_2, x_3)^n$. In this case there is also a simple description of the isotone maps $X_i$. In Section 4 we give our main result concerning the complete combinatorial classification of all polarizations of $(x_1, \ldots, x_m)^n$. In Section 5 we show that polarizations of $(x_1, x_2, x_3)^m$ are shellable, which implies that they define simplicial balls by the Stanley-Reisner correspondence.

Note. The results in Section 3 and in Section 5 are essentially found in the unpublished preprint [17] by the third author.

1. Separations of monomial ideals

We recall the basic notions of separation of a monomial ideals and separated models, as introduced in [12]. We also define a polarization of a monomial ideal as a separation which is a squarefree monomial ideal. We consider artinian monomial ideals and show that for these the notion of polarization and separated models are the same. We introduce the isotone maps $X_i$ from the lattice simplex $\Delta_m(n)$ to the Boolean poset $B(n)$, which are our main gadgets to classify all the polarizations of maximal ideal powers.

If $R$ is a set, let $k[x_R]$ be the polynomial ring in the variables $x_r$ where $r \in R$. If $S \to R$ is a map of sets, it induces a $k$-algebra homomorphism $k[x_S] \to k[x_R]$ by mapping $x_s$ to $x_r$ if $s \mapsto r$.

1.1. Separations and polarizations.

Definition 1.1. Let $R' \xrightarrow{p} R$ be a surjection of finite sets with the cardinality of $R'$ one more than that of $R$. Let $r_1$ and $r_2$ be the two distinct elements of $R'$ which map to a single element $r$ in $R$. Let $I$ be a monomial ideal in the polynomial ring $k[x_R]$ and $J$ a monomial ideal in $k[x_{R'}]$. We say $J$ is a simple separation of $I$ if the following holds:

i. The monomial ideal $I$ is the image of $J$ by the map $k[x_{R'}] \to k[x_R]$.
ii. Both the variables $x_{r_1}$ and $x_{r_2}$ occur in some minimal generators of $J$ (usually in distinct generators).
iii. The variable difference \( x_{r_1} - x_{r_2} \) is a non-zero divisor in the quotient ring \( k[x_{R'}]/J \).

More generally, if \( R' \xrightarrow{p} R \) is a surjection of finite sets and \( I \subseteq k[x_R] \) and \( J \subseteq k[x_{R'}] \) are monomial ideals such that \( J \) is obtained by a succession of simple separations of \( I \), \( J \) is a separation of \( I \). If \( J \) has no further separation, we call \( J \) a separated model (of \( I \)).

In [1] it is shown that simple separations may be considered as deformations of the ideal \( I \).

Any monomial ideal may be separated to its standard polarization. So clearly any separated model is a squarefree monomial ideal. The standard polarization may, however, be further separable, so it may not be a separated model.

**Example 1.2.** Consider \( I = (x^2y^2, x^2z^2, y^2z^2) \) in \( k[x, y, z] \). The standard polarization is

\[
\tilde{I} = (x_0x_1y_0y_1, x_0x_1z_0z_1, y_0y_1z_0z_1).
\]

This may be further separated to

\[
J = (x_0x_1y_0y_1, x_0'x_1'z_0z_1, y_0y_1z_0z_1).
\]

**Definition 1.3.** Let \( I \subseteq k[x_R] \) be a monomial ideal and \( R' \to R \) be a surjection of finite sets. An ideal \( J \subseteq k[x_{R'}] \) is a polarization of \( I \) if \( J \) is squarefree and a separation of \( I \).

This general notion of polarization is likely first defined in [23]. By the example above it is not true that any polarization is a separated model. However, we shall see in Corollary [4] that for Artinian monomial ideals, these notions are equivalent.

We state a general lemma which will be useful later.

**Lemma 1.4.** Let \( I \) be a monomial ideal in \( k[x_0, x_1, \ldots, x_m] \) such that each generator of \( I \) is squarefree in the \( x_0 \)-variable. Then if \( (x_0 - x_1) \cdot f \) is in \( I \), then for every monomial \( m \) in \( f \) we have that \( x_0m \) and \( x_1m \) are in \( I \).

**Proof.** Let \( f = x_0^a f_a + x_0^{a-1} f_{a-1} + \cdots + f_0 \). Then if \( (x_0 - x_1)f \) is in \( I \), the only terms with \( x_0^{a+1} \) are the terms in \( x_0^{a+1} f_a \), and so these are in \( I \) since we are in a \( \mathbb{Z}^m \)-graded setting. But since \( I \) is squarefree in \( x_0 \), we have \( x_0f_a \) in \( I \) and so \( x_0^a f_a \) in \( I \). In this way we may peel off and get that all terms \( x_0^p f_p \) are in \( I \) for \( p \geq 1 \).

Then in \( (x_0 - x_1)f_0 \), the terms with \( x_0 \) are those in \( x_0f_0 \). Hence \( x_0f_0 \) is in \( I \) and so each monomial term \( x_0m \) is in \( I \). We also get \( x_1f \in I \) and each \( x_1m \in I \). \( \square \)

1.2. Polarizations of Artinian monomial ideals. We consider an Artinian monomial ideal \( I \subseteq k[x_1, \ldots, x_m] \). This is simply a monomial ideal such that for every index \( i \), some power \( x_i^{n_i} \) is a minimal generator of \( I \). Let \( \hat{X}_i = \{x_{i1}, x_{i2}, \ldots, x_{im_i}\} \) be a set of variables. We get a polynomial ring whose variables are those in the union of all these variables, and a homomorphism

\[
\pi : k[\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_m] \to k[x_1, \ldots, x_m],
\]

by mapping every variable in \( \hat{X}_i \) to \( x_i \).
In a polarization $J \subseteq k[\bar{X}_1, \ldots, \bar{X}_m]$ of $I$ we separate each monomial generator $x^a = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ to squarefree monomials

\begin{equation}
  m(a) = m_1(a) \cdot m_2(a) \cdots m_m(a),
\end{equation}

where $m_i(a)$ is a squarefree monomial of degree $a_i$ in variables from $\bar{X}_i$. Considering $x_i^{n_i}$ we see that we must have $n'_i \geq n_i$. We shall shortly show, Remark 1.6, that we may take $n'_i = n_i$.

Starting from $K[\bar{X}_1, \ldots, \bar{X}_m]/J$ we get the quotient ring $k[x_1, \ldots, x_m]/I$ be dividing out by a regular sequence consisting of variable differences $x_{ip} - x_{iq}$. For each $i$ we choose $(n_i - 1)$ linearly independent such variable differences. Any such sequence of variable differences in any order will do.

We may get intermediate separations of $I$ as follows. Choose surjections $p_i : \bar{X}_i \rightarrow \bar{X}'_i$. We get a map of polynomial rings

$$k[\bar{X}_1, \ldots, \bar{X}_m] \rightarrow k[\bar{X}'_1, \ldots, \bar{X}'_m].$$

The image of the polarization $J$ is an ideal $I'$ in $k[\bar{X}'_1, \ldots, \bar{X}'_m]$ and $I'$ is a separation of $I$. We then get from $k[\bar{X}_1, \ldots, \bar{X}_m]/J$ to $k[\bar{X}'_1, \ldots, \bar{X}'_m]/I$ by dividing out by a regular sequence of variable differences $x_{ia} - x_{ib}$ where for each $i$, $x_{ia}$ and $x_{ib}$ are in the same fiber $p_i^{-1}(x')$ of $p_i$, and we have $|p_i^{-1}(x')| - 1$ linearly independent such variable differences for each fiber.

The following lemma has some key consequences.

Lemma 1.5. Let $x^b$ and $x^a$ be minimal generators of the Artinian monomial ideal $I$ and $m(b)$ and $m(a)$ the corresponding generators in a polarization of $I$.

Fix an index $i$. If $a_i \geq b_i$ and $a_j \leq b_j$ for every $j \neq i$, then the $i$'th part $m_i(b)$ divides $m_i(a)$.

Proof. We shall use induction on $d = a_i - b_i$. If $d = 0$ then clearly $a = b$ and there is nothing to prove. We may also assume that $b_i \geq 1$, since else there is nothing to prove. We suppose $m_i(b)$ does not divide $m_i(a)$ and we shall derive a contradiction. If it does not divide we may factor $m_i(a)$ as $t_i(a) \cdot n_i(a)$ where $t_i(a)$ has degree $d + 1$ and has no common variable with $m_i(b)$. (We are of course using here that $m_i(a)$ is squarefree.) For simplicity we may re-index variables so that $t_i(a) = x_{i_1} x_{i_2} \cdots x_{i,d+1}$. We now in $k[\bar{X}_1, \ldots, \bar{X}_m]/J$ divide out by all the variable differences involving $\bar{X}_j$-variables where $j \neq i$, and by all variable differences $x_{ia} - x_{i,a+1}$ for $i = d+2, \ldots, n_i - 1$. Thus we are collapsing all the $\bar{X}_j$-variables into the single variable $x_j$ and the variables $x_{i,d+2}, \ldots, x_{i,n_i}$ into a single variable $x_i$. We get a quotient ring

\begin{equation}
  k[x_{i_1}, \ldots, x_{i,d+1}, x_1, \ldots, x_m]/I',
\end{equation}

where $I'$ is a separation of $I$. Note that $m(b)$ collapses to $x^b$ in $I'$. 
Consider now the variable difference \( x_{i,d+1} - x_i \) in the polynomial ring above. We see that

\[
(x_{i,d+1} - x_i)x_{i1} \cdots x_{id} \cdot x_i^{b_i-1} \prod_{j \neq i} x_j^{b_j} = x_{i1} \cdots x_{i,d+1} \cdot x_i^{b_i-1} \prod_{j \neq i} x_j^{b_j} - x_{i1} \cdots x_{i,d} \cdot x_i^{b_i} \prod_{j \neq i} x_j^{b_j}
\]

vanishes in the quotient ring \( \mathbb{Z}^m \), since the first term is divisible by the image of \( m_i(a) \) in \( I' \) and the second term is divisible by the image of \( m_i(b) \). Since \( x_{i,d+1} - x_i \) is not a zero divisor (it belongs to a regular sequence), we get from (6) that

\[
n = x_{i1} \cdots x_{id} \cdot x_i^{b_i-1} \prod_{j \neq i} x_j^{b_j}
\]

is in \( I' \). Now if \( d = 1 \), this monomial has \( \mathbb{Z}^m \)-degree \( \mathbf{b} \). But the monomial \( x^\mathbf{b} \) is in \( I' \), with the same degree. Since these are the \( \mathbb{Z}^m \)-degree of a generator of \( I \) there can only be a single monomial in \( I' \) with this \( \mathbb{Z}^m \)-degree. We get a contradiction. Now suppose \( d \geq 2 \). Then \( n \) is divisible by a generator \( m_i'(c) \) in \( I' \) which can not be \( x^\mathbf{b} \). We will have each \( c_j \leq b_j \) for \( j \neq i \), and \( c_i > b_i \). Furthermore we have \( a_i > c_i \) since \( n \) in (7) has \( i \)-degree \( d + b_i - 1 = a_i - 1 \). By induction on \( d \), considering the polarized ideal \( J \), then the generator \( m_i(b) \) here divides the generator \( m_i'(c) \). But then going to \( I' \) then \( x_i^{b_i} \) divides the image of \( m_i'(c) \), and so \( x_i^{b_i} \) would divide \( n \) of (7), a contradiction. \( \square \)

**Remark 1.6.** If \( m(a) \) is a minimal generator of \( J \), by the lemma \( m_i(a) \) will divide \( m_i(0, \ldots, n_i, \ldots, 0) \) which of course is just \( m(0, \ldots, n_i, \ldots, 0) \). This if the polarization of \( x_i^{n_i} \) is \( x_{i1}x_{i2} \cdots x_{i,m_i} \), then every \( x_i \)-variable occurring in the minimal generators of \( J \) are among these variables, and so we may take \( \tilde{X}_i = \{ x_{i1}, \ldots, x_{i,m_i} \} \).

The following is quite particular for Artinian monomial ideals, note Example 1.2.

**Corollary 1.7.** Every polarization of an Artinian monomial ideal \( I \) is a separated model for \( I \).

**Proof.** If the polarization \( J \) was not a separated model, then let \( J' \) be a further simple separation. Since \( I \) in \( k[x_1, \ldots, x_m] \) is an Artinian monomial ideal, every variable \( x_i \) of course occurs in a minimal generator of \( I \), in fact \( x_i^{n_i} \) is a minimal generator. Then if \( J' \) is in \( k[\tilde{X}_1', \ldots, \tilde{X}_m'] \) then every variable in this polynomial ring must also occur in a generator of \( J' \), by the definition of a separation. By the above Lemma 1.5 and Remark 1.6 if \( x_i^{n_i} \) polarizes to \( x_{i1} \cdots x_{i,m_i} \) then \( \tilde{X}'_i = \{ x_{i1}, \ldots, x_{i,m_i} \} \). But \( J \) is obtained by from \( J' \) by dividing out by a variable difference \( x_{ia} - x_{ib} \). Then the image of \( x_{i1} \cdots x_{i,m_i} \) in \( J \) would not be squarefree, a contradiction. \( \square \)

### 1.3. Polarizations of powers of the graded maximal ideal

We now consider powers of the maximal ideals

\[ M = (x_1, x_2, \ldots, x_m)^n \subseteq k[x_1, \ldots, x_m]. \]

A polarization of this ideal may by Remark 1.6 be taken to live in polynomial ring \( k[\tilde{X}_1, \ldots, \tilde{X}_m] \) where \( \tilde{X}_i = \{ x_{i1}, \ldots, x_{i,m_i} \} \). Our goal is to combinatorially classify all possible polarizations of \( M \) in this polynomial ring.
The generators of the monomial ideal $M$ are all monomials $x_1^{b_1} \ldots x_m^{b_m}$ with $b_1 + \ldots + b_m = n$.

**Definition 1.8.** $\Delta_m(n)$ is the subset of $\mathbb{N}_0^m$ of all tuples $b = (b_1, \ldots, b_m)$ of non-negative integers with $b_1 + \cdots + b_m = n$. For a given $b$, its support $\text{Supp} \ b$ is the set of all $i$ such that $b_i \geq 1$.

In a polarization $J$ of $M$ we have one minimal generator of $J$, $m(b)$ for every $b$ in $\Delta_m(n)$. Now fix an index $1 \leq i \leq m$. Then $\Delta_m(n)$ may be given a partial order $\geq_i$ by letting $b \geq_i a$ if $b_i \geq a_i$ and $b_j \leq a_j$ for $j \neq i$. Thus there is one maximal element $(0, \ldots, 0, n, 0, \ldots, 0)$ where $n$ is in position $i$, and it has minimal elements all $b$ with $b_i = 0$. This is a graded partial order with $b$ of rank $b_i$.

Now given any $b \in \Delta_m(n)$ we get from the polarization $J$ a squarefree monomial $m_i(b)$, see (4). The variables of this monomial is a subset of $\hat{X}_i$ which we denote as $X_i(b)$. Let $B(\hat{X}_i)$ be the Boolean poset on $\hat{X}_i$, a Boolean poset on a set of $n$ elements. We get a function

$$X_i : \Delta_m(n) \to B(\hat{X}_i)$$

$$b \mapsto X_i(b).$$

The following is an immediate consequence of Lemma 1.5.

**Corollary 1.9.** Let $J$ be a polarization of $M$. Then $X_i$ is an isotone rank-preserving map when $\Delta_m(n)$ has the ordering $\geq_i$.

**Remark 1.10.** Since $\hat{X}_i = \{x_1, \ldots, x_n\}$, the group $S_n$ acts on $\hat{X}_i$. Also the group $S_m$ acts on $k[x_1, \ldots, x_m]$ by permutation of variables and hence on the set of maps $\{X_i\}$. In all there is an action of a semi-direct product $S_m \ltimes (S_n)^m$ on $k[\hat{X}_1, \ldots, \hat{X}_m]$ compatible with the action of $S_m$ on $k[x_1, \ldots, x_m]$. Since $(x_1, \ldots, x_m)^n$ is equivariant for the group action, the isomorphism classes of polarizations of this maximal ideal power are precisely the orbits of $S_m \ltimes (S_n)^m$ on the set of polarizations.

The set $\Delta_m(n)$ can be considered as a lattice simplex in $\mathbb{N}_0^m$. We shall however only need the graph struture it induces. Given a point $c$ in $\Delta_m(n+1)$ and $i, j$ in the support of $c$. Let $e_i$ and $e_j$ be the unit coordinate vectors. Then we get an edge between the points $c - e_i$ and $c - e_j$ in $\Delta_m(n)$, denoted $(c; i, j)$. Every edge in $\Delta_m(n)$ is of this form for unique $c$, $i$, and $j$. A point $c$ of $\Delta_m(n+1)$ induces a subgraph of $\Delta_m(n)$, the complete down-graph $D(c)$ on the points $c - e_i$ for $i \in \text{Supp} \ c$. When $\text{Supp} \ c$ has cardinality three we call this a **down-triangle**. In Figure 2 we have three down-triangles.

Let $\Delta_m^+(n+1)$ be the subset of $\Delta_m(n+1)$ consisting of $c$ with $c_i \geq 1$ for every $i$. The complete down-graph $D(c)$ has induced simplex of full dimension $m - 1$ iff $c \in \Delta_m^+(n+1)$. Note that $\Delta_m^+(n+1)$ is in one-one correspondence with $\Delta_m(n+1 - m)$ by sending $c$ to $c - 1$ where $1 = (1, 1, \ldots, 1)$. In Figure 2 there are three full-dimensional complete down-graphs, or in this case down-triangles, corresponding to the three elements of $\Delta_3(1)$, or equivalently of $\Delta_3^+(4)$.

Each $a$ in $\Delta_m(n-1)$ also determines a subgraph of $\Delta_m(n)$, the complete up-graph $U(a)$ consisting of the points $a + e_i$ for $i = 1, \ldots, m$ and with edges $(a + e_i + e_j; i, j)$ for $i \neq j$. For each $a$ in $\Delta_m(n-1)$ the induced simplex of the up-graph $U(a)$ has full
When $m = 3$ we call this an up-triangle. In Figure 2 there are six up-triangles.

2. Conjectures and problems

Before embarking on the main results of the paper we here discuss conjectures and problems on Artinian monomial ideals in general that have come up during our investigations.

Recall that for a squarefree monomial ideal $I$ in a polynomial ring $S$, we say that $J$ is the Alexander dual of $I$ if the monomials in $J$ are precisely the monomials in $S$ which have nontrivial common divisor with every monomial in $I$, or equivalently, every generator of $I$.

We may consider each set of variables $\tilde{X}_i$, $i = 1, \ldots, m$ as a color class of monomials. A monomial $x_{1i}x_{2i} \cdots x_{mi}$ with one variable of each color is a rainbow monomial.

**Proposition 2.1.** The class of ideals generated by rainbow monomials and with $m$-linear resolution is precisely the class which is Alexander dual to the class of polarizations of Artinian monomial ideals in $m$ variables:

a. Let $J$ be a polarization of an Artinian monomial ideal $I$ in $k[x_1, \ldots, x_m]$. The Alexander dual ideal of $J$ is generated by rainbow monomials and has $m$-linear resolution.

b. If an ideal $J'$ is generated by rainbow monomials and has $m$-linear resolution (and every variable in the ambient ring occurs in some generator of the ideal), then its Alexander dual $J$ is a polarization of an artinian monomial ideal in $m$ variables.

**Proof.** a. Since $I$ is Cohen-Macaulay of codimension $m$, the same is true for $J$. Then the Alexander dual of $J$ is generated in degree $m$ and has $m$-linear resolution [7]. But if $m$ is a generator for this Alexander dual, it has a common variable with $x_{1i}x_{2i} \cdots x_{mi}$ (the polarization of $x_i^n$) for every $i = 1, \ldots, m$. Hence $m$ must have a variable of each of the $m$ colors.
Remark 2.2. Considering the $X_i$ as color classes, both the Artinian ideal $I$, the polarization $J$ and its Alexander dual are generated by colored monomials. Such ideals and the associated simplicial complexes have been considered in various settings, like balanced simplicial complexes by Stanley [22], relating to the colorful topological Helly theorem by Kalai and Meshulam, [16], and resolutions of such ideals by the second author [10].

Lemma 2.3. Let $\Delta(J)$ be the simplicial complex associated to the polarization $J$ of an Artinian monomial ideal $I$. Then every codimension one face of $\Delta(J)$ is contained in one or two facets. If $I$ is not a complete intersection, then at least once there is a codimension one face contained in exactly one facet.

Proof. Let $x_i^{w_i}$ for $i = 1, \ldots, m$ be contained in the minimal generators of $I$, and $N = \{(i, j) | i = 1, \ldots, m, j = 1, \ldots, n\}$. The facets of $\Delta(J)$ are the complements of subsets $A$ of $N$ where

$$A = \{(1, i_1), \ldots, (m, i_m)\}$$

is the index set of the rainbow generators $x_{1i_1}x_{2i_2} \cdots x_{mi_m}$ of the Alexander dual of $J$. A codimension one face is then the complement of $A \cup \{(p, j_p)\}$ for some pair $(p, j_p)$ of $N$. But this can be on at most two facets: The complement of $A$ or the complement of

$$A' = \{(1, i_1), \ldots, (p - 1, i_{p-1}); (p, j_p); (p + 1, i_{p+1}), \ldots, (m, i_m)\}.$$  

Note that in the case that $I$ is a complete intersection, $A'$ always corresponds to a monomial in the Alexander dual of $J$ and therefore the complement of $A'$ is also a facet of $\Delta(J)$. So in this case, we always have that every codimension 1 face is contained in exactly two facets.

In the case that $I$ is an Artinian monomial ideal that is not a complete intersection, it happens at least once that there is a codimension one face contained in exactly one facet. To see this, suppose $I$ has a generator $w = x_1^{a_1} \cdots x_m^{a_m}$ with $a_i < n_i$ for all $i$. Then there is at least one unused $x_{i,k_i}$ for all $i$ in the polarization of the monomial $w$. Then $n = x_{1,k_1}x_{2,k_2} \cdots x_{m,k_m}$ is a rainbow monomial not in the Alexander dual of $J$. On the other hand let $m$ be a rainbow monomial which is in the Alexander dual.

If there is only one color $p$ such that the $p$'th variable in $m$ and $n$ are different, then we are in the situation given in the first paragraph: Let $A$ correspond to the index set of the variables of $m$. Then $A'$ corresponds to the index set of $n$, and its complement is not a facet of $\Delta(J)$. The codimension one face corresponding to the complement of $A \cup (p, k_p)$, where $x_{p,k_p}$ is in $n$, is contained in exactly one facet: the facet corresponding to the complement of $A$.
If there are two or more colors with different variables in \( n \) and \( m \), make a new monomial \( n' \) by taking out one of those variables from \( n \) and take in the corresponding variable from \( m \). If \( n' \) is in the Alexander dual, we are in the situation of the previous paragraph. If \( n' \) is not in the Alexander dual, we are in the same situation as originally but we are "closer" to \( m \). In this way we may continue.

\[ \square \]

2.1. **Balls and spheres.** By a result of Björner [4, Thm.11.4] a constructible simplicial complex with the property of Lemma 2.3 above is a simplicial ball or a simplicial sphere (the latter when every codimension one face is on exactly two facets).

**Conjecture 2.4.** The simplicial complex \( \Delta(J) \) associated to a polarization \( J \) of an Artinian monomial ideal \( I \), is a simplicial ball, save for the case when \( I \) is a complete intersection, when it is a simplicial sphere.

By the result of Björner [4] a positive answer to the following question would settle the above conjecture.

**Question 2.5.** Do polarizations of Artinian monomial ideals have constructible (for instance shellable) simplicial complexes?

That the standard polarization of an Artinian monomial ideal is shellable seems first to have been shown by A.Soleyman Jahn in [21]. In [18] S.Murai uses this to conclude that the standard polarizations give simplicial balls. More generally it is shown that letterplace ideals define simplicial balls, [6], by showing that these simplicial complexes are shellable. Letterplace ideals are introduced in [12] and are polarizations of Artinian monomial ideals. The article [11] discusses such Artinian monomial ideals more in depth.

In our last Section 7 we show in the case of three variables that the Alexander dual of any polarization \( J \) has linear quotients, see [15, Sec.8.2, Cor.8.2.4] for this notion. In Section 5 we show when the power of the maximal ideal (in any number of variables) is two, then the Alexander dual has linear quotients. Thus in these cases the simplicial complex \( \Delta(J) \) is shellable and hence a simplicial ball.

For a letterplace ideal the second author et.al. in [6] get an explicit simple description of the Stanley-Reisner ideal of the boundary of the simplicial ball defined by the letterplace ideal. In fact a general result, see [5, Section 5], says that the canonical module of a Stanley-Reisner ring \( k[\Delta] \) identifies as a multigraded proper ideal of this ring \( k[\Delta] \) if and only if \( \Delta \) is a homology ball. Then the ideal defines the boundary of this homology ball, a homology sphere. In [6] en explicit description of this canonical module is given.

**Conjecture 2.6.** For polarizations of Artinian monomial ideals the canonical module identifies (in a simply described natural way) as a multigraded ideal of the Stanley-Reisner ring of the polarization.

**Consequence 2.7.** With Conjecture 2.4 this would give an explicit description of the Stanley-Reisner ring of the boundary of the simplicial ball, defining a simplicial sphere.
2.2. Rainbow monomial ideals with linear resolution. In Section 6 we describe the Alexander dual of any polarization $J$ of a maximal ideal power. It is generated by rainbow monomials of degree $m$. But while the description of the generators of $J$ is rather direct, Sections 3, 4, and 5 the description the Alexander dual is more subtle. For instance it is not obvious from the description that there is actually always \((\frac{n+m-1}{m})\) generators of the Alexander dual.

In [20] A. Nematbakhsh gives a precise description of when an ideal generated by rainbow monomials has linear resolution. His terminology for rainbow monomials of $d$ colors is edge monomials of $d$-partite $d$-uniform clutters. He is able to give a characterization through a remarkable connection to the article [8] where they give a characterization of when a point set in the multiprojective space \((\mathbb{P}^1)^n\) is arithmetically Cohen-Macaulay (meaning that the associated multihomogeneous coordinate ring is a Cohen-Macaulay ring). The characterization given in [20] by translating the one in [8] is the following.

**Theorem 2.8.** Let $I$ be generated by rainbow monomials in $d$ colors. Then $I$ has a $d$-linear resolution iff:

1. Whenever $m_1$ and $m_2$ are two rainbow monomials in $I$ (i.e. generators of degree $d$) with $\text{lcm}(m_1, m_2)$ of degree $\geq d + 2$, there is a third distinct rainbow monomial $m_3$ in $I$ dividing this least common multiple.

2. Whenever $m_1$ and $m_2$ are two rainbow monomials not in $I$ with $\text{lcm}(m_1, m_2)$ of degree $\geq d + 2$, there is a third distinct rainbow monomial $m_3$ not in $I$ dividing this least common multiple.

So this says that a subset $A$ of $\tilde{X}_1 \cdot \tilde{X}_2 \cdot \cdot \cdot \tilde{X}_d$ gives an ideal with $d$-linear resolution iff both $A$ and its complement are in some sense convex.

Fröberg’s theorem [14] characterizes when a monomial ideal generated in degree two has linear resolution. It is easily seen that both this theorem and the theorem above gives the following criterion when we have rainbow monomials with two colors: If $x_ay_b$ and $x_a'y_b'$ are in $I$, then either $x_ay_b'$ or $x_a'y_b$ is in $I$. Many attempts have been done to generalize Fröberg’s theorem to higher degrees, but none fully successful. For rainbow monomials however the above gives such a generalization.

**Example 2.9.** Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$ be three color classes and let $I$ be the ideal generated by the six monomials

$$x_1y_1z_2, x_1y_2z_1, x_2y_1z_1, x_2y_2z_1, x_2y_1z_2, x_1y_2z_2.$$ 

Then $I$ does not have linear resolution since $x_1y_1z_1$ and $x_2y_2z_2$ are not in $I$ and their least common multiple is not divided by any distinct rainbow monomial not in $I$.

If we remove $x_1y_1z_2$ we also do not have linear resolution, since now the least common multiple of $x_2y_1z_2$ and $x_1y_2z_2$ is not divided by any distinct rainbow monomial in $I$. If we remove both $x_1y_1z_2$ and $x_1y_2z_2$ then we do get a monomial ideal with linear resolution.

**Problem 2.10.** Consider an ideal $J'$ generated by rainbow monomials. Is there a direct criterion on this ideal to tell if its Alexander dual is a polarization of a power of a graded maximal ideal?
Since an Artinian monomial ideal has linear resolution iff it is a power of the graded maximal ideal, this is the same as asking for a criterion for the ideal $J'$ to be bi-Cohen-Macaulay: The ideal has both linear resolution and is Cohen-Macaulay.

Such a description would maybe involve reconstructing the isotone maps $X_i$ given in (8), from $J'$. In [3] a cellular resolution is computed when $J'$ is the Alexander dual of the standard polarization $J$ of any Artinian monomial ideal.

**Remark 2.11.** If we for each color class $i$ have only two variables $\{x_{i0}, x_{i1}\}$, then a rainbow monomial of degree $m$ may be identified with a binary string, say if $m = 6$ then 101011 corresponds to $x_{11}x_{20}x_{31}x_{40}x_{51}x_{61}$. Thus investigating homological properties of ideals generated by rainbow monomials with two variables of each color, corresponds to investigating algebraic and topological properties of sets of binary words.

2.3. **Deformations of polarizations.** In [13] the second author and A.Nematbakhsh showed that the letterplace ideals $L(2, P)$ (which are polarizations of quadratic Artinian monomial ideals) have unobstructed deformations when the Hasse diagram is a tree. Moreover we computed the full deformation family of these ideals. Together with G.Scattareggia we have also verified that all deformations of various polarizations of quadratic powers $(x_1, x_2, \ldots, x_m)^2$ lift to global deformations for $m = 3, 4$ by computing a full global family. We have also verified this for the letterplace ideal $L(3, 5)$ (introduced in [12]), a cubic ideal.

**Conjecture 2.12.** Every first order deformation of a polarization of an Artinian monomial ideal (regardless of whether the deformation is homogeneous for the standard grading) lifts to a global deformation. In particular, whenever such a polarization is on a (multigraded) Hilbert scheme, it is a smooth point.

If the homogeneous ideal $I$ in the polynomial ring $S$ corresponds to a point on a Hilbert scheme of subschemes of a projective space, then the tangent space of the Hilbert scheme at this point is the degree zero part $\text{Hom}_S(I, S/I)_0$. By explicit computation with Macaulay 2, one can verify that the dimension of this space varies quite much for different polarizations. For instance for the polarizations of $(x_1, x_2, x_3)^3$, the lowest dimension of the tangent space occurs for the standard polarization, with dimension 69. The largest dimension occurs for the b-polarization, giving a dimension of 105 (this ideal is a smooth point on the Hilbert scheme component of ideals of maximal minors of $3 \times 5$ matrices of linear forms). Furthermore there are also many values between 69 and 105. Thus if they are all smooth points, they would be on many different components of the Hilbert scheme.

3. **The case of three variables**

We consider the case of three variables $m = 3$. Instead of $x_1, x_2, x_3$ we write $x, y, z$ and for the sets of variables $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ we write $\tilde{X}, \tilde{Y}, \tilde{Z}$. We describe completely all polarizations of $(x, y, z)^3$. In this case there is a particularly nice easily visualized description of such polarizations in terms of the plane diagram $\Delta_3(n)$ of Figure 2, see Example 3.3.
3.1. **Linear and quadratic syzygy edges.** Consider a down-triangle in $\Delta_3(n)$ given by $(a + 1, b + 1, c + 1)$ where $(a, b, c)$ is in $\Delta_3(n - 2)$, see Figure 3.

The edge $((a + 1, b + 1, c + 1); 2, 3)$ may be called a $yz$-edge. It is the horizontal edge in Figure 3. Similarly on the right is an $xy$-edge and the left an $xz$-edge.

Let

$$X = m_x(a + 1, b + 1, c + 1), \quad Y = m_y(a + 1, b, c + 1), \quad Z = m_z(a + 1, b + 1, c),$$

see Figure 4. Then by Lemma 1.5

$$m_x(a + 1, b, c + 1) = X \cdot x_i, \quad m_z(a + 1, b, c + 1) = Z \cdot z_p$$

for some $x_i$ and $z_p$, and

$$m_x(a + 1, b + 1, c) = X \cdot x_j, \quad m_y(a + 1, b + 1, c) = Y \cdot y_q$$

for some $x_j$ and $y_q$.

Thus

$$m(a + 1, b, c + 1) = X \cdot Y \cdot Z \cdot x_i \cdot z_p,$$

$$m(a + 1, b + 1, c) = X \cdot Y \cdot Z \cdot x_j \cdot y_q.$$ 

We see that there is a linear syzygy between these monomials iff $x_i = x_j$ or equivalently iff $X(a + 1, b + 1, c)$ (the variables in $X \cdot x_j$) equals $X(a + 1, b, c + 1)$ (the variables in $X \cdot x_i$). Then we call the $yz$-edge $((a + 1, b + 1, c + 1); 2, 3)$ a **linear syzygy edge** (LS-edge).

If $x_i \neq x_j$ or equivalently $X(a + 1, b + 1, c) \neq X(a + 1, b, c + 1)$ we call this a **quadratic syzygy edge** (QS-edge). We have of course the same notions for the $xy$ and $xz$-edges of the down-triangle $D(a + 1, b + 1, c + 1)$.
3.2. Classification of the isotone maps. The symmetric group $S_n$ acts on each of the sets $X, Y, Z$ and each may be identified with $\{1, \ldots, n\}$. Thus $S_n$ acts on the isotone maps in $\mathfrak{S}$ which we now denote as

$$X, Y, Z : \Delta_m(n) \to B(n),$$

by acting on the Boolean poset $B(n)$. The following completely describes the combinatorics of such isotone maps. This is quite nice for the 3-variables case. For the $m$-variables case it seems harder to see a nice description.

**Proposition 3.1.** Let $m = 3$. The orbits under the action of $S_n$ of rank-preserving isotone maps $X : \Delta_3(n) \to B(n)$ are in one to one correspondence with subsets $Q \subseteq \Delta_3(n - 2)$, corresponding to the $yz$-edges of $\Delta_3(n)$ which are quadratic syzygy edges for $X$. The number of such orbits is then $2^3$.

**Proof.** In each orbit there is exactly one isotone map where the sets

$$X(0, 0, n) \subseteq X(1, 0, n - 1) \subseteq X(2, 0, n - 2) \subseteq \cdots \subseteq X(n, 0, 0)$$

are

$$(9) \quad \emptyset \subseteq \{1\} \subseteq \{1, 2\} \subseteq \cdots \subseteq \{1, 2, \ldots, n\}$$

so we need only consider isotone maps which fulfill this condition and show that they are in one-one correspondence with subsets $Q$ of $\Delta_3(n - 2)$.

Given such an isotone map $X$ we get a subset $Q$ of QS-edges, those $(a, b, c)$ such that $X(a + 1, b, c + 1) \neq X(a + 1, b + 1, c)$. Conversely given a subset $Q$ of $\Delta_3(n - 2)$ we show that this determines uniquely an isotone $X$ whose associated set of QS-edges is $Q$. The essential idea is the following simple observation:

Let $A$ be a set of cardinality $p - 1$, $B, C$ sets containing $A$ of cardinality $p$, and $D$ a set of cardinality $p + 1$ containing $B$ and $C$. For fixed $A, B, D$, then either $C = B$ or $C = A \cup (D \setminus B)$. In the latter case note that $A = B \cap C$ and $D = B \cup C$.

Let now $A, B, C, D$ be the four sets of cardinalities respectively $p - 1, p, p$ and $p + 1$:

$$X(p - 1, 1, n - p), X(p, 1, n - p - 1), X(p, 0, n - p), X(p + 1, 0, n - p - 1).$$

The last two are known by (9). By induction on $p$ the first is known. Then the second is completely determined by whether the edge $((p, 1, n - p); 2, 3)$ is a LS- or QS-edge. Thus given $Q$ we may determine all the $X(p, 1, n - p - 1)$ for $p = 0, \ldots, n - 1$. Then we may continue and similarly determine all $X(p, 2, n - p - 2)$ for $p = 0, \ldots, n - 2$ and so on. $\square$

3.3. Polarizations in three variables. We consider the case of $m = 3$ variables. A polarization of $M = (x, y, z)^n$ is given by three degree-preserving isotone maps

$$(10) \quad X, Y, Z : \Delta_3(n) \to B(n)$$

where the partial orders on $\Delta_3(n)$ are respectively $\geq_x, \geq_y$ and $\geq_z$. When given such isotone maps we get for $b \in \Delta_3(n)$ variables $X(b)$ and monomial $m_x(b) = \prod_{x_i \in X(b)} x_i$. Similarly we get $m_y(b)$ from $Y$ and $m_z(b)$ from $Z$. Let $J$ be the ideal generated by all monomials

$$m(b) = m_x(b) \cdot m_y(b) \cdot m_z(b)$$
as \( b \) varies over \( \Delta_3(n) \).

**Theorem 3.2.** The isotone maps \( X, Y, Z \) of (10) give a polarization of \((x, y, z)^a\) if and only if each down-triangle in \( \Delta_3(n) \) has at most one QS-edge.

**Example 3.3.** Consider the ideal
\[
I = (x_0x_1x_2, x_0x_1y_1, x_0x_1z_1, x_1y_0y_1, x_0y_0z_1, y_0y_1y_2, y_0y_1z_1, y_0z_1z_2, z_0z_1z_2)
\]
which is a polarization of \((x, y, z)^3 \subset k[x, y, z]\). In Figure 5 we denote quadratic syzygy edges by dashed lines. We see, confer Theorem 3.2, that each of the three down-triangles has exactly one quadratic syzygy edge.

**Remark 3.4.** In [17] the third author also gives a cellular resolution of these ideals. It is given by Figure 5 by removing the dotted lines (the QS-edges).

**Proof of Theorem 3.2.** Part 1. We assume that we have a polarization and show that each down-triangle has at most one QS-edge. So look at the down-triangle in Figure 3 given by 
\[
D(a + 1, b + 1, c + 1) \in \Delta_3(n + 1).
\]
By the arguments in Subsection 3.1 the monomials at the vertices of this down-triangle are those in Figure 4.

Suppose the \( yz \)-edge is a QS-edge, meaning that \( x_i \) and \( x_j \) are different. Then \( (x_i - x_j)XYZy_0z_p \) is zero in \( k[X, Y, Z]/J \). Then \( XYZy_0y_qz_p \) must be zero in this quotient ring and so this monomial is in \( J \). But it has the same \( Z^3 \)-degree as the monomial generator \( XYZy_0z_a \). Hence it must be equal to this and so \( q = b \) and \( z_p = z_a \), and the \( xy \)- resp. \( xz \)-edges are not QS-edges.

Part 2. We now assume that every down-triangle has at most one QS-edge, and we show that we get a polarization. Our sets of variables are
\[
\tilde{X} = \{x_1, \ldots, x_n\}, \quad \tilde{Y} = \{y_1, \ldots, y_n\}, \quad \tilde{Z} = \{z_1, \ldots, z_n\}.
\]
Let \( \tilde{X}^a = \{x_1, x_2, \ldots, x_{a-1}, x\} \). We have the natural surjection \( \tilde{X}^{pa} \to \tilde{X}^a \) sending \( x_i \) to \( x \) for \( i \geq a \). Consider the natural surjections \( \tilde{X}^{pa} \to \tilde{X}^a \to \tilde{X}^{a-1} \). We assume that we have maps \( p_x, p_y \) and \( p_z \) such that the ideal \( J \subseteq k[X, Y, Z] \) constructed from the isotone maps \( X, Y, Z : \Delta_3(n) \to B(n) \) is a separation of the image ideal \( I \subseteq k[X^a, Y^b, Z^c] \).
We must show that $x_{a-1} - x$ is a nonzero divisor of $k[\bar{X}^a, \bar{Y}^b, \bar{Z}^c]/I$. This will give the argument by descending induction on $a, b$ and $c$.

So suppose that $(x_{a-1} - x)f \equiv 0$ in $k[\bar{X}^a, \bar{Y}^b, \bar{Z}^c]/I$. By Lemma 1.4 both $x_{a-1}m$ and $xm$ are $\equiv 0$ for every term $m$ in $f$. We want to show that this gives $m \equiv 0$. Suppose not. Since $x_{a-1}m \equiv 0$, the monomial $x_{a-1}m$ in $k[\bar{X}^a, \bar{Y}^b, \bar{Z}^c]$ must be divisible by a generator $x_{a-1} = x_{a-1}a_1a_2a_3$ of $I$. Let $(a_0 + 1, a_1, a_2)$ be its degree. Similarly $xm$ is divisible by a generator $xb = xb_1b_2b_3$ of $I$. Let $(b_0 + 1, b_1, b_2)$ be its degree. We do the case $a_0 \geq b_0$ (the case $b_0 \geq a_0$ is analogous) and may consider three cases (recall that $\sum a_i = \sum b_i$):

1. $a_0 \geq b_0, a_1 \leq b_1, a_2 \leq b_2$
2. $a_0 \geq b_0, a_1 \geq b_1, a_2 \leq b_2$
3. $a_0 \geq b_0, a_1 \leq b_1, a_2 \geq b_2$

We do cases (1) and (2). The last is similar.

Case 1. Since $X$ is isotope on $\Delta_3(n)$ for the order $\geq x$, $xb$ divides $x_{a-1}a_x$. Then $xb_1b_2b_3$ divides $x_{a-1}m$ also (since $xb_1, b_2$ and $b_3$ are relatively prime and the latter two do divide $m$). Hence it divides the greatest common divisor of $x_{a-1}m$ and $xm$, which is $m$ and so $m \equiv 0$.

Case 2. We argue by induction on $|a_2 - b_2|$ that $m \equiv 0$. If $a_2 = b_2$ we are in Case 1. above (note then that $a_i = b_i$ for all three $i$) so $m \equiv 0$. So let $a_2 < b_2$. We may also suppose $a_1 > b_1$, otherwise we are in Case 1. Recall that $I$ has $x_{a-1}a_1a_2a_3$ in position $(a_0 + 1, a_1, a_2)$.

Case 2a. Suppose the $yz$-edge of $D(a_0 + 1, a_1, a_2 + 1)$ is a linear syzygy LS-edge. Since

$$(a_0 + 1, a_1, a_2) \leq_z (a_0 + 1, a_1 - 1, a_2 + 1) \leq_z (b_0 + 1, b_1, b_2),$$

by Lemma 1.3 the monomial in position $(a_0 + 1, a_1 - 1, a_2 + 1)$ is $x_{a-1}a_1a_2a_3(a_y/y')a_zz'$ where $a_zz'$ divides $b_2$. Then we may replace $x_{a-1}a_1a_2a_3a_y(y')a_zz'$, it still divides $x_{a-1}m$, and by our induction hypothesis get that we have $m \equiv 0$.

Case 2b. Suppose the $yz$-edge in $D(a_0 + 1, a_1, a_2 + 1)$ is a QS-edge. Then the $xz$-edge is an LS-edge and so in position $(a_0, a_1, a_2 + 1)$ we have either a monomial

$$(11) \quad i. \ a_1a_2a_3a_y(a_zz'), \quad \text{or} \quad ii. \ a_{a-1}(a_x/x')a_y(a_zz').$$

Since the $xy$-edge is also an LS-edge, in position $(a_0 + 1, a_1 - 1, a_2 + 1)$ we in any case have $z$-part equal to $a_zz'$, and by Lemma 1.3 applied to the triples $(a_0 + 1, a_1 - 1, a_2 + 1)$ and $(b_0 + 1, b_1, b_2)$, $a_zz'$ will divide $b_z$.

The upshot is that in case i. of (11) this generator divides $m$ so $m \equiv 0$, a contradiction, or in case ii., this monomial divides $x_{a-1}m$ and we have reduced our induction parameter $|a_2 - b_2|$. Induction gives that we must have $m \equiv 0$. \hfill \Box

4. Polarizations and linear syzygy edges

We here give our main result, the complete combinatorial description, Theorem 4.3, of all polarizations of powers of maximal ideals $\{x_1, \ldots, x_m\}^a$. Write $[m] = \{1, 2, \ldots, m\}$.

The essential objects are the rank-preserving isotone maps $X_i : \Delta_m(n) \to B(n)$ for $i = 1, \ldots, m$ and conditions on them. For each $b \in \Delta_m(n)$ we get a monomial
$m_i(b) = \prod_{j \in X_i(b)} x_{ij}$ in the variables $\hat{X}_i = \{x_{i1}, \ldots, x_{in}\}$, see (4). To the vertex $b$ we now associate the monomial

$$m(b) = \prod_{i=1}^{m} m_i(b)$$

and let $J$ be the ideal in $k[\hat{X}_1, \ldots, \hat{X}_m]$ generated by the $m(b)$.

If $B$ is a subset of $[m]$, denote by $1_B$ the $m$-tuple $\sum_{i \in B} e_i$. For instance, if $B = [m]$, then $1_B = (1, 1, \ldots, 1)$. The following is a generalization of the discussion in Subsection 3.1 and Figure 4.

**Lemma 4.1.** Let $c \in \Delta_m(n+1)$ have support $C \subseteq \{1, 2, \ldots, m\}$. The monomials associated to the vertices in the down-graph $D(c)$ have a common factor of degree $c - 1_C$. This common factor is $\prod_{i \in C} m_i(c - e_i)$.

**Proof.** Fix an element $j \in C$. For the order $\geq k$ we have $c - e_j \geq k c - e_k$ for every $k \in C$. Hence $X_k(c - e_k)$ is contained in $X_k(c - e_j)$ for every $k \in C$. Thus $m(c - e_j)$ has $m_k(c - e_k)$ as a factor for each $k \in C$. \hfill $\square$

**Definition 4.2.** An edge $(c; i, j)$ in $\Delta_m(n)$ (where $c \in \Delta_m(n+1)$) is a linear syzygy edge (LS-edge) if there is a monomial $m$ of degree $n - 1$ such that

$$m(c - e_i) = x_{jr} \cdot m, \quad m(c - e_j) = x_{is} \cdot m$$

for suitable variables $x_{jr} \in X_j$ and $x_{is} \in X_i$. So this edge gives a linear syzygy between the monomials $m(c - e_i)$ and $m(c - e_j)$. Equivalently

$$m_p(c - e_i) = m_p(c - e_j)$$

for every $p \neq i, j$; note that both $m_i(c - e_i)$ and $m_j(c - e_j)$ are common factors of $m(c - e_i)$ and $m(c - e_j)$. For given $c \in \Delta_m(n+1)$ let $LS(c)$ be the set of linear syzygy edges in the complete down-graph $D(c)$.

Here is the main theorem of this article.

**Theorem 4.3.** The isotone maps $X_1, \ldots, X_m$ determine a polarization of the ideal $(x_1, \ldots, x_m)^n$ if and only if for every $c \in \Delta_m(n+1)$, the linear syzygy edges $LS(c)$ contain a spanning tree for the down-graph $D(c)$.

We prove this towards the end of this section.

**Remark 4.4.** For every $c \in \Delta_m(n+1)$ which has support $\{i, j\}$ of cardinality 2, it is automatic by the condition that the $X_p$ are rank-preserving and isotone that the edge $(c; i, j)$ is a linear syzygy edge in $LS(c)$. So the conditions in Theorem 4.3 is automatically fulfilled for the $c$ with support of cardinality 2.

**Example 4.5.** Consider the case $m = 3$. A $c \in \Delta_3(n+1)$ has support of cardinality 1, 2 or 3. If it is 3, we have a down-triangle, and so at least two of the three edges must be linear syzygy edges. If the cardinality is 2, the edge is on the boundary of $\Delta_3(n)$ and it is a linear syzygy edge by the above remark. Hence the conditions of Theorem 4.3 correspond precisely to that of Theorem 3.2.
4.1. Spanning trees of down-graphs. Let \( R \subseteq [m] \) and \( c \in \Delta_m(n+1) \) with \( R \) contained in the support of \( c \). Let \( r, s \in R \). We say \((c; r, s)\) is an \( R\)-linear syzygy edge if
\[
X_p(c - e_r) = X_p(c - e_s)
\]
for \( p \in R \setminus \{r, s\} \). Note that we in any case have
\[
X_r(c - e_r) \subseteq X_r(c - e_s), \quad X_s(c - e_s) \subseteq X_s(c - e_r).
\]
Let \( D_R(c) \) be the complete graph with edges \((c; r, s)\) for \( r, s \in R \).

**Lemma 4.6.** Suppose for each \( c \in \Delta_m(n+1) \) the set of linear syzygy edges in \( \text{LS}(c) \) contains a spanning tree for \( D(c) \). Then for each \( R \subseteq \text{Supp} \, c \), the set of \( R\)-linear syzygy edges contains a spanning tree for \( D_R(c) \).

**Proof.** Let \( Q \) be the complement of \( R \) in \( \text{Supp} \, c \). Let \( r \) and \( s \) be two vertices in \( R \). There is a path from \( r \) to \( s \) in \( D(c) \) consisting of linear syzygy edges. It may be broken up into smaller paths: From \( r = r_0 \) to \( r_1 \), from \( r_1 \) to \( r_2 \), ..., from \( r_{p-1} \) to \( r_p = s \) where on the path from \( r_{i-1} \) to \( r_i \) the only vertices in \( R \) are the end vertices \( r_{i-1} \) and \( r_i \) while the in between vertices are all in \( Q \). We claim that each edge from \( r_{i-1} \) to \( r_i \) is an \( R\)-linear syzygy edge. This will prove the lemma.

Let the path from \( r_{i-1} \) to \( r_i \) be
\[
r_{i-1} = q_0, q_1, \ldots, q_t = r_i
\]
where \( q_1, \ldots, q_{t-1} \) are all in \( Q \). We must show that
\[
(12) \quad X_p(r_{i-1}) = X_p(r_i) \text{ for } p \in R \setminus \{r_{i-1}, r_i\}.
\]
But since the edges on the path are linear syzygy edges we have
\[
X_p(c - q_{j-1}) = X_p(c - q_j) \text{ for } p \not\in \{r_{i-1}, q_1, \ldots, q_{t-1}, r_i\}.
\]
This shows (12). \( \square \)

Given two \( m \)-tuples \( a = (a_1, \ldots, a_m) \) and \( b = (b_1, \ldots, b_m) \) in \( \Delta_m(n) \). Let \( [m] = A \cup B \) be the disjoint partition such that \( a_i \geq b_i \) for \( i \in A \) and \( a_i < b_i \) for \( i \in B \). We let
\[
D = \sum_{i \in B} (b_i - a_i) = \sum_{i \in A} (a_i - b_i)
\]
be a measure for the distance between \( b \) and \( a \). Let the least upper bound for \( a \) and \( b \) be
\[
a \vee b = (\max\{a_1, b_1\}, \ldots, \max\{a_m, b_m\}).
\]
Recall that \( m(a) \) and \( m(b) \) are the monomials in positions \( a \) and \( b \) respectively. The following is the main and crucial ingredient in the proof of Theorem 4.3.

**Proposition 4.7.** Given \( a, b \in \Delta_m(n) \). Suppose for every \( c \in \Delta_m(n+1) \) the linear syzygy edges \( \text{LS}(c) \) contains a spanning tree for the down-graph \( D(c) \). Then there is a path
\[
a = b_0, b_1, \ldots, b_N = b
\]
in \( \Delta_m(n) \) such that:

1. Every \( b_i \leq a \vee b \),
Given distinct \( a \) and \( b \). Let further \( T \) be a spanning tree for \( \Delta_m(n) \). Then there is a unique path from \( a \) to \( b \) in \( T \). We show that for any \( m(u) \) on this path then \( m(u) \) divides \( \text{lcm}(m(a), m(b)) \). It is enough to show for any \( k \in \text{Supp}(c) \) that any \( x_k \)-variable in \( m(u) \) is contained in each of the \( x_k \)-variables of \( m(a) \) or the \( x_k \)-variables of \( m(b) \).

**Case 1.** If the path from \( a \) to \( b \) does not contain \( c - e_k \), then since all edges on the path are linear syzygy edges, \( X_k(b_{i-1}) = X_k(b_i) \) for every \( i \).

**Case 2.** If the path from \( a \) to \( b \) contains \( c - e_k \), say this is \( b_t \), then:

- \( X_k(b_{i-1}) = X_k(b_i) \) for \( i < t \) and these are all equal to \( X_k(a) \),
- \( X_k(b_i) = X_k(b_{i+1}) \) for \( i > t \) and these are all equal to \( X_k(b) \),
- \( X_k(b_t) = X_k(c - e_k) \) is contained in both \( X_k(a) \) and \( X_k(b) \) by Lemma 4.6

**Lemma 4.8.** When the distance between \( a \) and \( b \) is one, there is a path from \( m(a) \) to \( m(b) \) fulfilling the conditions (1)-(3) of Proposition 4.7.

**Proof.** Choose \( c \in \Delta_m(n+1) \) such that \( a = c - e_i \) and \( b = c - e_j \). Let \( T \) be a spanning tree for \( D(c) \). Then there is a unique path from \( a \) to \( b \) in \( T \). We show that for any \( m(u) \) on this path then \( m(u) \) divides \( \text{lcm}(m(a), m(b)) \). It is enough to show for any \( k \in \text{Supp}(c) \) that any \( x_k \)-variable in \( m(u) \) is contained in each of the \( x_k \)-variables of \( m(a) \) or the \( x_k \)-variables of \( m(b) \).

**Proof of Proposition 4.7.** Given distinct \( a \) and \( b \) in \( \Delta_m(n) \), let

\[
B = \{ i \mid b_i > a_i \}, \quad A = \{ i \mid a_i \geq b_i \}.
\]

Let further \( A = A_1 \cup A_0 \) where \( A_1 \) consists of those \( j \in A \) such that there is some \( b' \) with

- \( b'_i = b_i \) for \( i \in B \),
- \( b'_i \leq a_i \) for \( i \in A \),
- \( b'_j = b_j \),
- There is some LS-path from \( b' \) to \( b \) where the vertices \( u \) on the path satisfy
  - \( u \leq a \cup b \), and
  - \( m(u) \) divides \( \text{lcm}(m(a), m(b)) \).

In particular note:

- \( A_1 \) is not empty: \( B \neq \emptyset \) and so there must be some \( j \in A \) with \( b_j < a_j \). Then we may take \( b' = b \).
- For \( i \in A_0 \) we have \( a_i = b'_i \geq b_i \).
- The distance \( d(a, b') = d(a, b) \) for any such \( b' \).

Choose \( \beta \in B \) and let \( R = A_1 \cup \{ \beta \} \). Consider the down-graph \( D_R(a + e_\beta) \). By Lemma 4.6 there is an \( R \)-linear syzygy edge \( (a + e_\beta; \beta, \alpha) \) for some \( \alpha \) in \( A_1 \). This is an edge from \( a \) to \( a + e_\beta - e_\alpha \). Then \( m_t(a + e_\beta - e_\alpha) \) divides \( m_t(a) \) for every \( t \in A_1 \), while for each \( t \in A_0 \cup B \) then \( m_t(a + e_\beta - e_\alpha) \) may include one variable which is not in \( \text{lcm}(m(a), m(b)) \) (by Lemma 4.6).
Since \( \alpha \in A_1 \) there is a \( b' \) as above with \( b'_\alpha < a_\alpha \). The distance between \( a + e_\beta - e_\alpha \) and \( b' \) is one smaller than the distance between \( a \) and \( b \). By induction there is an LS-path

\[
a + e_\beta - e_\alpha = b^0, b^1, \ldots, b^N = b',
\]

where each \( m(b^p) \) divides \( \text{lcm}(m(a + e_\beta - e_\alpha), m(b')) \). Let \( b^p \) be the last element on this path for which \( b^p_i \neq b_i' \) for some \( i \in A_0 \cup B \). Since going from \( b^p \) to \( b^{p+1} \) is a linear syzygy edge we must have, confer also the bullet points above, \( b^p_i = b_i' - 1 \) and \( b^p_i = b_i' \) for every \( i \) in \( (A_0 \cup B) \setminus \{i_0\} \). Now we make the following three observations.

1. Then \( m(b^p) \) divides \( \text{lcm}(m(a), m(b)) \) for the following reason: Each \( m_t(b^p) \) where \( t \in A \) divides this \( \text{lcm} \), since there is an \( R \)-linear syzygy between \( m(a) \) and \( m(a + e_\beta - e_\alpha) \) and so \( m_t(a) = m_t(a + e_\beta - e_\alpha) \) for \( t \neq \alpha \). For \( t = \alpha \) then \( m_t(a + e_\beta - e_\alpha) \) divides \( m_\alpha(a) \). When \( t \in A \cup B \), then since the edges on the path from \( b^p \) to \( b' \) are LS-edges we will have \( m_t(b^p) = m_t(b') \) for every \( t \in (A_0 \cup B) \setminus \{i_0\} \) and when \( t = i_0 \) then \( m_{i_0}(b^p) \) divides \( m_{i_0}(b') \).

2. We cannot have \( i_0 \in A_0 \) by the existence of \( b^p \): Then \( b^p_i = b_i' - 1 < a_{i_0} \) and so \( i_0 \) would be in \( A_1 \) by definition of \( A_1 \). Therefore \( i_0 \in B \).

3. Looking at \( a \) and \( b^p \) we see that
   - \( a_i \geq b_i^p \) for \( i \in A_1 \cup A_0 \),
   - \( a_i \leq b_i^p \) for \( i \in B \).

Since \( b^p_i = b_i' - 1 = b_{i_0} - 1 \) and \( b^p_i = b_i' \) for \( i \in B \setminus \{i_0\} \) the distance \( d(a, b^p) < d(a, b) \). By induction there is an LS-path from \( a \) to \( b^p \). Then we may splice this path with the path from \( b^p \) to \( b' \) and then further with the path from \( b' \) to \( b \). So we get an LS-path from \( a \) to \( b \) fulfilling the criteria of Proposition 4.7.

\[ \square \]

4.2. **Proofs of the main Theorem 4.3.** We show first Part a., that if the isotone maps \( \{X_i\} \) give a polarization, then for each \( c \in A_\Delta m(n+1) \) the linear syzygy edges \( LS(c) \) of the down-graph \( D(c) \) contain a spanning tree for this down-graph.

**Proof of Theorem 4.3.** Part a. We assume that the isotone maps \( \{X_i\} \) give an ideal \( J \) which is a polarization. We shall prove that every down-graph \( D(c) \) contains a spanning tree of linear syzygy edges. For simplicity we shall assume \( \text{Supp}(c) \) has full support \( \{m\} = \{1, 2, \ldots, m\} \). The arguments work just as well in the general case. Since by Lemma 4.1

\[
m = \prod_{i=1}^{m} m_i(c - e_i)
\]

of degree \( c - 1 \) is a divisor of \( m(c - e_v) \) for any \( c - e_v \) in \( D(c) \), we may write \( m(c - e_v) = m \cdot n(c - e_v) \) where \( n(c - e_v) \) has degree \( 1 - e_v \). For two distinct vertices \( c - e_v \) and \( c - e_w \) in \( D(c) \) we define the distance \( d(m(c - e_v), m(c - e_w)) \) to be the number of \( k \in [m] \) such that the either:

- The (unique) \( x_k \)-variables of \( n(c - e_v) \) and of \( n(c - e_w) \) are distinct,
- \( k = v \) (then \( n(c - e_v) \) has no \( x_v \)-variable),
- \( k = w \) (then \( n(c - e_w) \) has no \( x_w \)-variable),
Note that if the distance between $m(c - e_v)$ and $m(c - e_w)$ is 2, then the set of $k$'s is \{v, w\} and there is a linear syzygy between these monomials. Suppose now the vertices of $D(c)$ can be divided into two distinct subsets $V_1$ and $V_2$ such that there is no linear syzygy edge between a vertex in $V_1$ and a vertex in $V_2$.

Let $c - e_v$ in $V_1$ and $c - e_w$ in $W$ be such that the distance $d$ between $m(c - e_v)$ and $m(c - e_w)$ is minimal. We must have $d \geq 3$ and the number of vertices $m \geq 3$. For simplicity we may assume $v = 1$ and $w = 2$ and that we may write

$$n(c - e_2) = x_{1i_1}x_{3i_3} \cdots x_{m_{im}}, \quad n(c - e_1) = x_{2j_2}x_{3j_3} \cdots x_{m_{jm}},$$

where $x_{pi} \neq x_{pj}$ for $p = 3, \ldots, d$ and $x_{pi} = x_{pj}$ for $p > d$ where $d \geq 3$.

Consider the graded ring $k[\tilde{X}_1, \ldots, \tilde{X}_m]/J$ and divide out by the regular sequence $x_{pi} - x_{pj}$ for $p = 4, \ldots, d$. This is a regular sequence, since we assume we have a polarization. We get a quotient algebra $k[\tilde{X}_1, \ldots, \tilde{X}_m]/I$ and denote by $x_p$ the class $\overline{x_{pi}} = \overline{x_{pj}}$ for $p \geq 4$. In $I$ we have generators

$$\overline{n(c - e_2)} = \overline{m} \cdot \overline{n(c - e_2)}, \quad \overline{n(c - e_1)} = \overline{m} \cdot \overline{n(c - e_1)}.$$ 

Now $x_{3i_3} - x_{3j_3}$ is a non-zero divisor of $k[\tilde{X}_1, \ldots, \tilde{X}_m]/I$. Consider

$$(x_{3i_3} - x_{3j_3})x_{1i_1}x_{2j_2}x_{4} \cdots x_{m} \cdot \overline{m}.$$ 

It is zero in this quotient ring, and so

$$\overline{m} = x_{1i_1}x_{2j_2}x_{4} \cdots x_{m} \cdot \overline{m}$$

is zero in this quotient ring and so must be a generator of $I$ of degree $c - e_3$. But then the generator of this degree in the polarization $J$ must be

$$\overline{m}' = x_{1i_1}x_{2j_2}x_{4}k_{4} \cdots x_{m_{km}} \cdot \overline{m}$$

where each $k_p$ is either $i_p$ or $j_p$. Hence all $k_p = i_p = j_p$ for $p > d$. But then we see that the distance between $\overline{m}'$ and $m(c - e_2)$ is $\leq d - 1$ and similarly the distance between $\overline{m}'$ and $m(c - e_1)$ is $\leq d - 1$. Whether $\overline{m}'$ is now in $V_1$ or in $V_2$ we see that this contradicts $d$ being the minimal distance.

Proof of Theorem 4.3 Part b. We shall now prove that if each down-graph $D(c)$ contains a spanning tree of linear syzygy edges, then $J$ will be a polarization. Order the variables in each $\tilde{X}_i$ in a sequence $x_{i_1}, x_{i_2}, \ldots, x_{im}$. Let $\tilde{X}_i'$ consist of $x_{i_1}, \ldots, x_{ip_i}, x_i$ so we have a surjection $\tilde{X}_i \rightarrow \tilde{X}_i'$ for each $i$ sending $x_{ij}$ to itself for $j \leq p_i$, and to $x_i$ for $j > p_i$. Denote the image of $J$ in $k[\tilde{X}_1', \ldots, \tilde{X}_m']$ by $J'$ and the image of $m(a)$ by $m'(a)$. The quotient ring $k[\tilde{X}_1', \ldots, \tilde{X}_m']/J'$ is obtained from $k[\tilde{X}_1, \ldots, \tilde{X}_m]/J$ by dividing out by variable differences $x_{ij} - x_{i,j+1}$ for $i = 1, \ldots, m$ and $j > p_i$. We assume this is a regular sequence. We show that if we now divide out by $x_{ip_i} - x_{ip_i+1}$ this is a non-zero divisor of $k[\tilde{X}_1', \ldots, \tilde{X}_m']/J'$. By continuing as above we get eventually that $k[x_1, \ldots, x_m]/(x_1, \ldots, x_m)^n$ is a regular quotient of $k[\tilde{X}_1, \ldots, \tilde{X}_m]/J$ and so $J$ is a polarization of $(x_1, \ldots, x_m)^n$.

We write $x'_i = x_{ip_i}$. Suppose $(x'_i - x_i) \cdot f = 0$ in $k[\tilde{X}_1', \ldots, \tilde{X}_m']/J'$ where $f$ is a polynomial. By Lemma 1.4 (with $x'_i = x_{ip_i}$ taking the place of $x_0$), we must show that
if \( m \) is a monomial such that \( x_i' \cdot m = 0 \) and \( x_i \cdot m = 0 \), then \( m = 0 \) in the quotient ring. So some generator \( m'(a) \) of \( J' \) divides \( x_i' \cdot m \) and some generator \( m'(b) \) divides \( x_i \cdot m \).

By Proposition 4.7 there is a path from \( m(a) \) to \( m(b) \) consisting of linear syzygy edges and such that each \( m(u) \) on this path divides \( \text{lcm}(m(a), m(b)) \). The image \( m'(u) \) then divides \( x_i' \cdot m_i \), and \( u \leq a \lor b \). We will show by induction on the length of the path that some monomial \( m'(u_0) \) on this path divides \( m \), and so \( m \) is zero in the quotient ring \( k[\dot{X}_1, \ldots, \dot{X}_m]/J' \).

If the path has length one, there is a linear syzygy edge between \( m(a) \) and \( m(b) \). Write

\[
x_i' \cdot m = m'(a) \cdot n^0(a), \quad x_i \cdot m = m'(b) \cdot n^0(b),
\]

Write also \( m = (x_i')^p(x_i)^q \cdot n \) where \( n \) does not contain \( x_i' \) or \( x_i \). If none of \( m'(a) \) or \( m'(b) \) divides \( m \), then

\[
m'(a) = (x_i')^{p+1}(x_i)^q \cdot n^1(a), \quad m'(b) = (x_i')^p(x_i)^{q+1} \cdot n^1(b),
\]

where \( p' \leq p \) and \( q' \leq q \) (and \( n^1(a) \) and \( n^1(b) \) do not contain \( x_i' \) or \( x_i \)). But since the edge from \( a \) to \( b \) is a linear syzygy edge, we must have \( p' = p, q' = q \). But a linear syzygy edge involves variables of distinct \( x_i \)-type, which is not so here. Thus one of \( m'(a) \) or \( m'(b) \) must divide \( m \).

Suppose now the path has length \( \geq 2 \). We may assume \( a_i \geq b_i \). Let \( a \) to \( a' \) be the first edge along the path. Then the coordinate \( a'_i \leq a_i \).

If the coordinates \( a_i \) and \( a'_i \) are equal, the \( x_i \)-variables of \( m(a) \) and \( m(a') \) are the same, since this is a linear syzygy edge. Since \( m'(a) \) divides \( x_i \cdot m \) we get that \( m'_i(a') \) divides \( x_i' \cdot m \). If \( a'_i < a_i \) then when going from \( m(a) \) to \( m(a') \) some \( x_i \)-variable drops out by isotonicity of \( X_i \) and so also \( m_i(a') \) divides \( x_i' \cdot m \). For \( j \neq i \), since \( m'_j(a') \) divides \( x_j' \cdot m \), it must divide \( m \) since \( m'_j(a') \) contains no \( x_i \)-type variable. Hence \( m'(a') \) divides \( x_j' \cdot m \). By induction on path length, some \( m'(u) \) along the path divides \( m \).

\[
\square
\]

5. Degree two case: Polarizations of \((x_1, x_2, \ldots, x_m)^2\)

We show that polarizations of the second power \((x_1, x_2, \ldots, x_m)^2\) of the maximal ideal are in one-to-one correspondence with oriented trees with edges labels 1, 2, \ldots, \( m \). Moreover we show that the isomorphism classes of polarizations are in one-to-one correspondence with trees on \((m + 1)\) vertices.

First we recall a construction given in [9]. Given a directed tree \( T \) (the edges are directed) with edges labelled 1, 2, \ldots, \( m \). The label of an edge \( e \) is denoted \( l(e) \). Let \( \dot{X}_i = \{x_{i0}, x_{i1}\} \) for \( i = 1, \ldots, m \). We construct a monomial ideal \( J(T) \) in \( k[\dot{X}_1, \ldots, \dot{X}_m] \) generated by \((m + 1)\) monomials, one for each vertex of \( T \). Consider a vertex \( v \) of the tree \( T \). If an edge \( e \) in \( T \) is pointing in the direction towards \( v \) let \( e(v) = 1 \) and if \( e \) is pointing in the opposite direction, let \( e(v) = 0 \). To the vertex \( v \) of the tree \( T \) associate a monomial

\[
m_v = \prod_{e \in T} x_{l(e), e(v)},
\]
and let $J(T)$ be the ideal in $k[\bar{X}_1, \ldots, \bar{X}_m]$ generated by these monomials. We see that the $m_v$ are rainbow monomials. By [9] this is a Cohen-Macaulay monomial ideal of projective dimension one with $m$-linear resolution.

We now construct another monomial ideal $I(T)$ in this polynomial ring as follows. For each pair of distinct vertices $v, w$ of $T$ there is a unique path (forgetting the direction of the edges) between $v$ and $w$. Let $e$ be the edge on this path incident to $v$ and $f$ the edge on this path incident to $w$. Define the monomial

$$m_{v,w} = x_{l(e),e(v)}x_{l(f),f(v)},$$

and let $I(T)$ be the ideal generated by all these monomials.

**Theorem 5.1.** Let $T$ be a directed tree with edges labelled by $1, 2, \ldots, m$.

a. The ideals $I(T)$ and $J(T)$ are Alexander duals.

b. The ideal $I(T)$ is a polarization of $(x_1, \ldots, x_m)^2$, and every polarization of the latter ideal is of this form.

c. Two polarizations $I(T)$ and $I(T')$ are isomorphic if and only if the underlying (unlabelled, undirected) trees of $T$ and $T'$ are isomorphic.

**Example 5.2.** The two trees in Figure 6 give the non-isomorphic polarizations of $(x_1, x_2, x_3)^2$. The first tree gives the standard polarization

$$(x_{10}x_{11}, x_{10}x_{20}, x_{20}x_{21}, x_{10}x_{30}, x_{20}x_{30}, x_{30}x_{31}).$$

The second tree gives the $b$-polarization

$$(x_{10}x_{11}, x_{10}x_{21}, x_{20}x_{21}, x_{10}x_{31}, x_{20}x_{31}, x_{30}x_{31}),$$

which is the letterplace ideal $L(2, 3)$ of [12].

**Example 5.3.** The trees with five vertices, Figure 7, decorated with direction and labelling, give the three non-isomorphic polarizations of $(x_1, x_2, x_3, x_4)^2$.

The first tree gives the standard polarization. The second tree gives the polarization:

$$(x_{10}x_{11}, x_{20}x_{21}, x_{30}x_{31}, x_{40}x_{41}, x_{10}x_{21}, x_{10}x_{31}, x_{10}x_{41}, x_{20}x_{31}, x_{20}x_{41}, x_{31}x_{40}).$$
The third tree gives the b-polarization:
\[(x_{10}x_{11}, x_{20}x_{21}, x_{30}x_{31}, x_{40}x_{41}, x_{10}x_{20}, x_{10}x_{30}, x_{10}x_{40}, x_{20}x_{30}, x_{20}x_{40}, x_{30}x_{41}),\]
which is the letterplace ideal \(L(2, 4)\).

**Proof of Theorem 5.1.**

a. Consider the generator \(m_{v,w} = x_{l(e),e(w)}x_{l(f),f(v)}\) of \(I(T)\). Let \(u\) be another vertex of \(T\). If the first variable is not in the monomial \(m_u\) then \(w\) and \(u\) are in distinct directions from \(v\). If the second variable is not in \(m_u\), then \(v\) and \(u\) are in distinct directions from \(w\), but this situation is not possible in a tree.

Hence all the \(\binom{m+1}{2}\) monomials \(m_{v,w}\) are in the Alexander dual of \(J(T)\). But \(J(T)\) has linear resolution and so has multiplicity \(\binom{m+1}{2}\). Hence the simplicial complex defined by \(J(T)\) has this number of facets, each of cardinality \(2m - 2\). The Alexander dual must then be generated by \(\binom{m+1}{2}\) quadratic monomials and so these must be precisely the generators of \(J(T)\) making it the Alexander dual.

b. The ideal \(I(T)\) has codimension \(m\), since \(J(T)\) has \(m\)-linear resolution. Dividing out by the variable differences \(x_{10} - x_{11}\) we easily see that we get the ideal \((x_1, \ldots, x_m)^2\) which also has codimension \(m\). Hence this sequence of variable differences is a regular sequence for \(k[\tilde{X}_1, \ldots, \tilde{X}_m]/I(T)\), and so \(I(T)\) is a polarization of \((x_1, \ldots, x_m)^2\).

Conversely let \(I\) be a polarization of \((x_1, \ldots, x_m)^2\). Then \(I\) is Cohen-Macaulay of codimension \(m\) with 2-linear resolution. The Alexander dual \(J\) of \(I\) will then be Cohen-Macaulay of codimension 2 with \(m\)-linear resolution. By [9, Prop.2.4] there is a tree \(T\) such that that \(J\) is the image of the \(J(T)\) by a map of polynomial rings \(k[\tilde{X}_1, \ldots, \tilde{X}_m]\) (the ring where \(J(T)\) lives) to \(k[\tilde{X}_1, \ldots, \tilde{X}_m]\) (the ring where \(J\) lives), sending the variables in the former ring to monomials in the latter ring. But since the generators of \(J\) and \(J(T)\) have the same degree, we must map the variables to variables. Hence \(J\) is isomorphic to \(J(T)\), and \(J\) is obtained from \(J(T)\) by the action of an element of \(S_m \ltimes (\mathbb{Z}_2)^m\) (see Remark 1.10).

c. If the underlying trees are the same then clearly \(T\) and \(T'\) are related by the action of an element of \(S_m \ltimes (\mathbb{Z}_2)^m\), and so also \(I(T)\) and \(I(T')\). If the underlying trees are different, then \(J(T)\) and \(J(T')\) are not isomorphic: Their linear syzygies are given precisely by the edges of the trees. Any isomorphism between \(J(T)\) and \(J(T')\) would induce a bijection of monomial generators, corresponding to a bijection of vertices of the trees, such that the corresponding linear syzygies between generators, would correspond to a bijection between the edges of the trees.

Together with the discussion in Subsection 2.1 the following shows that polarizations of \((x_1, \ldots, x_m)^2\) define simplicial balls.

**Proposition 5.4.** The ideal \(J(T)\) has linear quotients.

**Proof.** Let \(r\) be any vertex in \(T\) and orient all the arrows away from \(r\). This gives a partial order on the vertices of the tree. Take a linear extension of this partial order and let \(I\) be generated by the monomials in an initial segment for this order. Let \(m_u\) be the subsequent monomial. We show that \(I : m_u\) is generated by variables. So let \(m \in I : m_u\). Then \(m \cdot m_u\) is divisible by some \(m_w\). Each of \(m_u\) and \(m_v\) have one variable for each edge in \(T\). Only the edges on the path between \(u\) and \(v\) give distinct variables in \(m_u\) and \(m_v\). Starting from \(u\) let \(e\) be the first edge on the path to \(v\). Let \(w\) be the
other end point of e. Of the variables in $m_u$ and $m_w$, only the e-variable is different and so $I : m_u$ contains the e-variable $x_{e,w}$ occurring in $m_w$. But then this e-variable is also in $m_v$ and so this variable divides $m$. □

6. Alexander Duals

Recall that for a squarefree monomial ideal $I$ in a polynomial ring $S$, we say that $J$ is the Alexander dual of $I$ if the monomials in $J$ are precisely the monomials in $S$ which have nontrivial common divisor with every monomial in $I$, or equivalently, every generator of $I$.

Let $I \subset k[\tilde{X}_1, \ldots, \tilde{X}_m]$ be a polarization of the ideal $(x_1, \ldots, x_m)^n$ in $k[x_1, \ldots, x_m]$. For any $a \in \Delta_m(n-1)$, consider the product set of monomials

$$\mathcal{M}(a) = \prod_{j=1}^{m} X_j(a + e_j).$$

It consists of monomials $x_{i_1} x_{i_2} \cdots x_{i_m}$ where $i$ is in $X_j(a + e_j)$. Let $J$ be the ideal generated by the monomials in the union of all the $\mathcal{M}(a)$ for $a \in \Delta_m(n-1)$.

**Theorem 6.1.** The ideal $J$ is the Alexander dual of $I$.

**Example 6.2.** Consider again the polarization $I$ from Example 3.3. Its graph of linear syzygies is again given in Figure 8 this time labeling each of the up-triangles in the graph. The up-triangle $i$ corresponds to an element $a_i \in \Delta_3(2)$, where $a_1 = (2, 0, 0), a_2 = (1, 1, 0), a_3 = (1, 0, 1), a_4 = (2, 0, 0), a_5 = (0, 1, 1)$, and $a_6 = (0, 0, 2)$. Applying Theorem 6.1 we list the set of monomials $\mathcal{M}(a_i)$:
\[ M(a_1) = \{ x_0 y_1 z_1, x_1 y_1 z_1, x_2 y_1 z_1 \} \]
\[ M(a_2) = \{ x_0 y_0 z_1, x_0 y_1 z_1, x_1 y_0 z_1, x_1 y_1, z_1 \} \]
\[ M(a_3) = \{ x_0 y_0 z_0, x_0 y_0 z_1, x_1 y_0 z_0, x_1 y_0 z_1 \} \]
\[ M(a_4) = \{ x y_0 z_1, x_1 y_1 z_1, x_1 y_2 z_1 \} \]
\[ M(a_5) = \{ x_0 y_0 z_1, x_0 y_0 z_2, x_0 y_1 z_1, x_0 y_1 z_2 \} \]
\[ M(a_6) = \{ x_0 y_0 z_0, x_0 y_0 z_1, x_0 y_0 z_2 \} \]

The boldface monomials are the ten distinct monomials we find from this process, which in fact generate the Alexander dual of \( I \).

We shall go through several steps in proving the above theorem. It turns out that we will be able to abstract the situation so our arguments will only involve a collection of isotone maps

\begin{equation}
\chi_i : \Delta_m(n) \rightarrow \{ 0 < 1 \}
\end{equation}

such that \( \chi_i(b) = 0 \) whenever \( b_i = 0 \). First we establish some notation. For a monomial \( w \in k[X_1, \ldots, X_m] \), define maps

\[ \chi_{i,w} : \Delta_m(n) \rightarrow \{ 0 < 1 \} \]

\[
\begin{cases}
0, & \text{no variable of } X_i(b) \text{ is in } w. \\
1, & \text{some variable of } X_i(b) \text{ is in } w.
\end{cases}
\]

We note some properties of \( \chi_{i,w} \) that follow directly from properties of the maps \( X_i \):

- If \( \chi_{i,w}(b) = 0 \), then \( \chi_{i,w}(b') = 0 \) for all \( b' \leq_i b \).
- If \( \chi_{i,w}(b) = 1 \), then \( \chi_{i,w}(b') = 1 \) for all \( b' \geq_i b \).
- If \( (c, j, k) \) is a linear syzygy edge where \( c \in \Delta_m(n + 1) \), then \( \chi_{i,w}(c - e_j) = \chi_{i,w}(c - e_k) \) for every \( i \neq \{j,k\} \).

Furthermore:

- The monomial \( w \in k[X_1, \ldots, X_m] \) is in the Alexander dual of \( I \) if and only if for every \( b \in \Delta_m(n) \) there is some \( i = 1, \ldots, m \) with \( \chi_{i,w}(b) = 1 \).
- The monomial \( w \) is in \( J \) if and only if there is some \( a \in \Delta_m(n - 1) \) such that \( \chi_{i,w}(a + e_i) = 1 \) for every \( i \).

We now abstract the situation and consider isotone maps as in (13).

**Definition 6.3.** A multidegree \( b \in \Delta_m(n) \) is a full zero-point for the collection \( \{ \chi_i \} \) if \( \chi_i(b) = 0 \) for every \( i = 1, \ldots, m \). An up-simplex \( U(a) \) of \( \Delta_m(n) \) has a zero if \( \chi_i(a + e_i) = 0 \) for some \( i \).

An edge \( (c; i, j) \) of \( \Delta_m(n) \) is a linear syzygy edge for the collection \( \{ \chi_i \} \) if \( \chi_p(c - e_i) = \chi_p(c - e_j) \) for every \( p \neq i, j \).

We shall prove the following.
Theorem 6.4. Given the collection of isotone maps \( \{ \chi_i \} \) such that for every down-graph of \( \Delta_m(n) \) the linear syzygy edges for \( \{ \chi_i \} \) contains a spanning tree. Then \( \{ \chi_i \} \) has a full zero-point in \( \Delta_m(n) \) if and only if every up-graph of \( \Delta_m(n) \) has a zero.

As a consequence we get Theorem 6.1.

Proof of Theorem 6.1. That \( w \notin J \) means that every up-graph in \( \Delta_m(n) \) has a zero for the \( \chi_{i,w} \)'s. That \( w \) is not in the Alexander dual of \( I \) means that it has a full zero-point for the \( \chi_{i,w} \)'s. Hence by Theorem 6.4 \( J \) will be the Alexander dual of \( I \). \( \square \)

6.1. Alexander Duals in Three Variables. We first prove Theorem 6.1 or rather Theorem 6.4 in the three variable case. The arguments are then easier to grasp and visualize. We prove separately the two directions of the equivalence of Theorem 6.4.

Proposition 6.5. Suppose every up-triangle in \( \Delta_3(n) \) has a zero for \( \chi_1, \chi_2 \) and \( \chi_3 \). Then some element of \( \Delta_3(n) \) is a full zero-point for these \( \chi_i \).

Proof. We proceed by induction the first coordinate \( b_1 \) of \( b \) and show the following for \( k = 1, \ldots, n \). Either there is a \( b \) with \( b_1 > n - k \) which is a full zero, i.e. \( \chi_i(b) = 0 \) for \( i = 1, 2, 3 \), or there is a \( b \) with \( b_1 = n - k \) and \( \chi_2(b) = \chi_3(b) = 0 \). When we come to \( k = n \) the statement will be proven, since then automatically also \( \chi_1(b) = 0 \).

For the base case \( k = 1 \), we need only consider the vertices of the up-triangle corresponding to \( a = (n-1,0,0) \). If it is an up-triangle with some zero, then either the vertex \((n,0,0)\) is a full zero-point, or one of the other two vertices \((n-1,1,0)\) or \((n-1,0,1)\) contains a zero and therefore both \( \chi_2 \) and \( \chi_3 \) are zero at this vertex.

Now let \( k \geq 2 \). By induction either there is a full zero-point \( b \) with \( b_1 > n - k + 1 \), or there is a \( b \) with \( b_1 = n - k + 1 \) and \( \chi_2(b) = \chi_3(b) = 0 \). We wish to show that there is some \( c \) such that either \( c_1 > n - k \) and \( c \) is full zero-point, or \( c_1 = n - k \) and \( \chi_2(c) = \chi_3(c) = 0 \). If there exists some \( b \) such that \( b_1 > n - k + 1 \) which is a full zero-point, we set \( c = b \) and we are done.

Now consider instead the case where we have a point \( b \) with \( b_1 = n - k + 1 \) and \( \chi_2(b) = \chi_3(b) = 0 \). Consider the up-triangle corresponding to \( a = b - e_1 \in \Delta_3(n-1) \). By assumption \( a \) has a zero. If the \( x_1 \)-corner is a zero, then we are done, as then \( \chi_1(a + e_1) = 0 \). So suppose this is not the case, and that one of the bottom two corners of \( a \) has a zero; without loss of generality, suppose that \( \chi_2(a + e_2) = 0 \). If \( \chi_3(a + e_2) = 0 \), then set \( c = a + e_2 \) and we are done. So suppose instead that \( \chi_3(a + e_2) = 1 \). Then we are in the situation of Figure 9.

Since \( \chi_3(a + e_2) \neq \chi_3(a + e_1) \), the edge from \( a + e_2 \) to \( b = a + e_1 \) is not a linear syzygy edge. Therefore the other two edges in the down-triangle \( D(a + e_1 + e_2) \) are linear syzygy edges. We have, confer Figure 10.

- The horizontal edge \( (a + e_1 + e_2; 2,3) \) going left from \( b \) is a linear syzygy edge. Setting \( b' = b + e_2 - e_3 \), this is the edge from \( b \) to \( b' \). We thus have: \( \chi_1(b') = \chi_1(b) = 1 \).
- The edge \( (a + e_1 + e_2; 1,3) \) is a linear syzygy edge. Hence \( \chi_2(b') = \chi_2(a + e_2) = 0 \).
- Additionally, \( b' \leq b \), and so \( \chi_3(b) = 0 \) implies that \( \chi_3(b') = 0 \).

Since every up-triangle in our graph of linear syzygies has a zero, now the only place to have this zero in the up-triangle \( U(a') \) where \( a' = b' - e_1 \) (the up-triangle to the left
of $U(a)$, is in the $y$-corner. We are then again in the same situation before: Either $\chi_3(a' + e_2) = 0$ and we set $c = a' + e_2$ and we are done, or $\chi_3(a' + e_2) = 1$ and we repeat this analysis for the next triangle.

Continuing, either we get some $a'' = a + t(e_2 - e_3)$ with $\chi_2(a'' + e_2)$ and $\chi_3(a'' + e_3)$ both zero, or we can increase $t$. Eventually we reach the border up-triangle where $a_3'' = 0$ and $\chi_2(a'' + e_2) = 0$. But since $a_3'' = 0$ we automatically have $\chi_3(a'' + e_2) = 0$. So we have our point $c = a'' + e_2$. □

We now prove the other direction of the if and only if statement in Lemma 6.4.

**Lemma 6.6.** Suppose there is a $b \in \Delta_3(n)$ with $\chi_2(b) = \chi_3(b) = 0$. Then every up-triangle $U(a)$ with $a_1 \geq b_1$ has a zero.

**Proof.** We proceed by induction on $k = n - b_1$. When $k = 1$, there is only one up-triangle to consider, given by $a = (n - 1, 0, 0)$. Clearly if $\chi_2(b) = \chi_3(b) = 0$, this up-triangle contains a zero.

Now let $k \geq 2$. We wish to show that if we have a $b \in \Delta_3(n)$ with $b_1 = n - k$ and $\chi_2(b) = \chi_3(b) = 0$, then all up-triangles $a$ with $a_1 \geq n - k$ has a zero.

Consider an up-triangle $a$ with $a_1 = n - k$. Either $a_2 < b_2$ (and so $a_3 \geq b_3$) and so $b \geq a + e_2$. This gives $\chi_2(a + e_2) = 0$. Otherwise $a_3 < b_3$ and we conclude similarly that $\chi_3(a + e_3) = 0$, so $U(a)$ has a zero.

Now consider the down-triangle $D(b + e_1)$. One of the edges $(b + e_1; 1, 2)$ or $(b + e_1; 1, 3)$ is a linear syzygy edge. Without loss of generality, suppose it is the first one. Then $\chi_3(b + e_1 - e_2) = \chi_3(b) = 0$. Also $b + e_1 - e_2 \leq_2 b$ and so $\chi_2(b + e_1 - e_2) = \chi_2(b) = 0$. 


Proof. We show this by induction on $\chi$. When the up-simplex given by $(n)$ maps Lemma 6.8.

6.4. The following is the crucial insight.

Proposition 6.5 and Corollary 6.7 proves Theorem 6.4 and so Theorem 6.1, in three variables.

6.2. Alexander Duals in $m$ variables. We now do the general case of $m$ variables and show Theorem 6.1 by proving the equivalent statement of Theorem 6.4.

Given isotone maps $\chi_i : \Delta_m(n) \to \{0 < 1\}$ for $i = 1, \ldots, m$ such that $\chi_i(b) = 0$ when $b_i = 0$, and every down-graph in the lattice simplex $\Delta_m(n)$ has a spanning tree of syzygy edges. We prove separately the two directions of the equivalence of Theorem 6.4. The following is the crucial insight.

Lemma 6.8. Let $b \in \Delta_m(n)$ have $\chi_i(b) = 0$ for every $i \neq 1$. Then every up-graph $U(a)$ with $a_1 \geq b_1$ has a zero.

Proof. We show this by induction on $m$, the number of variables, and the number $k$, where $b_1 = n - k$. We have already shown this for the three variable case above Lemma 6.6, the base case.

For the base case $k = 1$, we have exactly one up-simplex $a$ with $a_1 \geq n - k$. This is the up-simplex given by $(n - 1, 0, \ldots, 0)$. Then $b = a + e_j$ for some $j \neq 1$. But then $\chi_j(a + e_j) = 0$, so $a$ has a zero.

Now suppose $k \geq 2$ and we have some $b \in \Delta_m(n)$ with $b_1 = n - k$ such that $\chi_j(b) = 0$ for every $j \neq 1$. Consider the sublattice $\Delta_m^{n-k}(n) \subseteq \Delta_m(n)$ given by fixing the first coordinate to be $n - k$. This is a level set in $\Delta_m(n)$. Then $\Delta_m^{n-k}(n)$ is isomorphic to $\Delta_{m-1}(k)$. Restricting the maps $\chi_\ell : \Delta_m(n) \to \{0 < 1\}$ this isomorphism induces maps $\overline{\chi}_\ell : \Delta_{m-1}(k) \to \{0 < 1\}$ for $\ell = 2, \ldots, m$. Clearly $b$ is a full zero-point for the $\overline{\chi}_\ell$’s in $\Delta_m^{n-k}(n)$. By the induction hypothesis on the number of variables, there is a zero in every up-graph $U(\overline{a})$ of the sublattice $\Delta_m^{n-k}(n)$. But this implies that there is therefore a zero in every up-graph $U(a)$ with $a_1 = n - k$, in the original lattice $\Delta_m(n)$.

Now consider the down-graph $D(b + e_1)$. By assumption on the maps $\chi_i$, at least one edge $(b + e_1, 1, i)$ is a linear syzygy edge, so $\chi_j(b) = \chi_j(b + e_1 - e_i)$ for every $j \neq 1, i$. But $b + e_1 - e_i \leq b$, so we also know that $\chi_i(b + e_1 - e_i) = \chi_i(b)$. Therefore $\chi_j(b + e_1 - e_i) = 0$ for every $j \neq 1$, and $b + e_1 - e_i$ has first coordinate $n - k + 1$. Applying the induction hypothesis on $k$, every up-graph $U(a)$ with $a_1 \geq n - (k - 1)$ also has a zero.

Corollary 6.9. Suppose the maps $\chi_i$ have a full zero-point in $\Delta_m(n)$. Then every up-graph in $\Delta_m(n)$ has a zero.

Proof. Let $b$ be a full zero-point. Given an up-triangle $U(a)$ then at least for one $i$ we have $a_i \geq b_i$. We may then apply Lemma 6.8 to conclude that this $U(a)$ is an up-simplex with a zero.
6.3. Zeros in all up-simplices implies a zero point. We now prove the other direction of Theorem 6.4.

**Proposition 6.10.** Suppose every up-graph in $\Delta_m(n)$ has a zero for the $\chi_i$’s. Then there is an element of $\Delta_m(n)$ which is a full zero-point.

**Proof.** We suppose every up-graph in the $(m - 1)$-dimensional lattice simplex $\Delta_m(n)$ has a zero. Look at the lattice sub-simplex $\Delta_m^{i+}(n)$ of the same dimension $m - 1$ but size one less consisting of all the points $b$ where $b_1 \geq 1$. This sub-simplex will also fulfill the hypothesis with $\chi_1$ modified to $\chi_1'$ with $\chi_1'(b) = 0$ when $b_1 = 1$. (Note that this modification has no bearing on which up-graphs in $\Delta_i^{i+}(n)$ which have a zero since for such $U(a)$ we have $a_1 \geq 1$.) Hence by induction there is a point $b$ which is a zero point.

If $b_1 \geq 2$ we are done since then $\chi_i(b) = 0$ for every $i = 1, \ldots, m$. So suppose then $b_1 = 1$ so we are at the bottom layer of $\Delta_i^{i+}(n)$. There we only know that $\chi_i(b) = 0$ for $i > 1$. If also $\chi_1(b) = 0$ we are done, so assume then that $\chi_1(b) = 1$.

Look at the lattice sub-simplex $\Delta_{m-1}^{bot}(n)$ of dimension one less consisting of all points $b$ with $b_1 = 0$. This is the bottom layer of $\Delta_m(n)$. We shall show that there is some point here which is a full zero-point for $\Delta_m(n)$ (which is the same as being a full zero-point for $\Delta_{m-1}^{bot}(n)$). Suppose $\Delta_{m-1}^{bot}(n)$ has no full zero-point. Then by induction we know that $\Delta_{m-1}^{bot}(n)$ has an up-simplex $U(\bar{a})$ with 1 in every corner, where $\bar{a}$ is in $\Delta_{m-1}^{bot}(n - 1)$. It may be considered as a point $a$ in $\Delta_m(n - 1)$ with first coordinate $a_1 = 0$. Note that $U(\bar{a})$ has dimension $m - 2$, one less than the dimension of $\Delta_m(n)$.

We now have two locations:

- A point $b$ in $\Delta_m(n)$ with $b_1 = 1$ which has $\chi_i(b) = 0$ for every $i > 1$ but $\chi_1(b) = 1$.
- An up-graph $U(\bar{a})$ in the bottom layer sub-simplex $\Delta_{m-1}^{bot}(n)$ with $a_1 = 0$ and 1 at all the corners $i = 2, 3, \ldots, n$.

- In addition we have the hypothesis that every up-graph of $\Delta_m(n)$ has some zero at a corner.

We shall show that these three cannot occur together and thus conclude that the bottom layer $\Delta_m^{bot}(n)$ will have a full zero point. Note that this is true if $n = 1$ above.

If there is some $i \geq 2$ such that both $a_i \geq 1$ and $b_i \geq 1$, then look at the lattice sub-simplex $\Delta_{m}^{i+}(n)$ of $\Delta_m(n)$ consisting of all points $c$ in $\Delta_m(n)$ with $c_i \geq 1$. Every up-graph in $\Delta_{m}^{i+}(n)$ still has a zero for the isotone maps $\chi_j$. Also $b$ and the up-graph $U(a)$ can be considered to be in $\Delta_m^{i+}(n)$. By induction on the size $n$ we get that this situation cannot occur.

Suppose now that $a$ and $b$ have disjoint support. Let $i$ be in the support of $a$ and look at the down-graph $D(b + e_i)$. At least one edge $(b; i, j)$ for $j$ in $\text{Supp } b$ is a linear syzygy edge. There are two cases, according to whether $j = 1$ or $j \neq 1$.

Case 1. Suppose this edge is $(b + e_i; i, 1)$. Then $\chi_j(b + e_i - e_1) = \chi_j(b)$ for $j \neq 1, i$. Consider the bottom layer lattice simplex $\Delta_{m-1}^{bot}(n)$, and the sub lattice simplex $\Delta_{m-1}^{i+bot}(n)$ consisting of points $c$ with $c_i \geq 1$ (and $c_1 = 0$). By induction on dimension $\Delta_{m-1}^{i+bot}(n)$ satisfies the statement of the proposition. But note now:
• The up-graph $U(\overline{a})$ also identifies as the up-graph $U(\overline{a})$ in the lattice sub-simplex $\Delta_{m-1}^{i+\bot}(n)$. It is an up-graph with 1 at every corner $i \geq 2$.

• The point $b' = b + e_i - e_1$ is in $\Delta_{m-1}^{i+\bot}(n)$. It has $\chi_j(b') = 0$ for $j \neq 1, i$. It is therefore a full zero point in $\Delta_{m-1}^{i+\bot}(n)$. (Note that $b'$ is on the boundary of this since $b'_i = 1$. Thus $\chi_i$ is modified to $\chi'_i$ so that $\chi'(b) = 0$).

But this gives a contradiction since we by induction on dimension cannot both have a full zero-point and an up-graph with 1’s at all corners in the bottom lattice simplex $\Delta_{m-1}^{i+\bot}(n)$.

Case 2. Suppose the edge is $(b + e_i; i, j)$ where $j \neq 1$. Then $\chi_p(b + e_i - e_j) = \chi_p(b)$ for $p \neq i, j$. In particular $\chi_p(b + e_i - e_j)$ is:

- 0 for $p \neq 1, i, j$,
- 0 for $p = j$: Since $\chi_j(b) = 0$ and since $b \geq_j b + e_i - e_j$ we also have $\chi_j(b + e_i - e_j) = 0$,
- 1 when $p = 1$.

So for $b' = b + e_i - e_j$ we have $\chi_p(b') = 0$ for $p \neq 1, i$ and $\chi_1(b') = 1$. Then consider the lattice sub-simplex $\Delta_m^{i+}(n)$ of $\Delta_m(n)$ consisting of the points $c$ where $c_i \geq 1$. Modify $\chi_i$ to $\chi'_i$ such that $\chi'_i(c) = 0$ for those $c$ with $c_i = 1$. Now we induct on the size of this lattice simplex. For this lattice simplex we now have the same situation as in the first three bullet points of this proof:

- The point $b'$ has $\chi_p(b') = 0$ for $p \neq 1, i$. Also the modified $\chi'_p(b') = 0$ since $b'$ is on the boundary (has $b'_i = 1$) of $\Delta_m^{i+}(n)$. Furthermore $\chi_1(b') = 1$.
- The up-simplex $U(\overline{a})$ is on the bottom layer of $\Delta_m^{i+}(n)$ and has 1 at all corners 2, 3, . . . , $m$.
- Every up-simplex of $\Delta_m^{i+}(n)$ has some zero at a corner.

The $(m - 1)$-dimensional lattice complex $\Delta_m^{i+}(n)$ has size one less than $\Delta_m(n)$ and by induction on the three first bullet point of this proof, this cannot occur. We are thus left with the conclusion that the bottom layer $\Delta_{m-1}^{\bot}(n)$ of $\Delta_m(n)$, must have a full zero-point.

\[\square\]

7. Polarizations define shellable simplicial complexes

In this section we show that any polarization of the power $(x, y, z)^m$, is a monomial ideal with linear quotients. This is equivalent to its Alexander dual being a shellable simplicial complex. By a result of Björner [4, Thm.11.4] this immediately implies that these polarizations define simplicial balls, see Lemma 2.3 and Subsection 2.1.

An element $(a, b, c)$ in $\Delta_3(n - 1)$ corresponds to an up-triangle in $\Delta_3(n)$. If $x_\alpha \in X(a + 1, b, c)$ we say that $x_\alpha$ (or just $\alpha$) is an $x$-variable belonging to the up-triangle $U(a, b, c)$. Similarly if $y_\beta \in Y(a, b + 1, c)$ and $z_\gamma \in Z(a, b, c + 1)$. We also say the monomial $x_\alpha y_\beta z_\gamma$ (or just $\alpha \beta \gamma$) belongs to $(a, b, c)$.

Lemma 7.1. Suppose $\alpha \beta \gamma$ belongs to the up-triangle $U(a + 1, b, c)$ in $\Delta_3(n - 1)$, see Figure [17].
a. Then either the up-triangle $U(a, b + 1, c)$ or the up-triangle $U(a, b, c + 1)$ has a monomial $\alpha'\beta\gamma$ belonging to them.

b. If $\alpha$ either belongs to the up-triangle $U(a, b + 1, c)$ or to $U(a, b, c + 1)$, then $\alpha\beta\gamma$ will belong to one of the up-triangles $U(a, b, c + 1)$ or in $U(a, b + 1, c)$.

Proof. Look at the up-triangles in Figure 11. In the middle we have a down-triangle $D(a + 1, b + 1, c + 1) \in \Delta_3(n)$. Note that since $Y$ is isotone, $\beta$ will be in both the up-triangles $U(a + 1, b, c)$ and $U(a, b + 1, c)$ and since $Z$ isotone $\gamma$ in both $U(a + 1, b, c)$ and $U(a, b, c + 1)$.

a. If the edge $((a + 1, b + 1, c + 1); 1, 2)$ is a linear syzygy edge, then also $\gamma$ belongs to $U(a, b + 1, c)$, and if $((a + 1, b + 1, c + 1); 1, 3)$ is a linear syzygy edge then $\beta$ belongs to $U(a, b, c + 1)$. Since at least one of them is a linear syzygy edge we are done.

b. If $((a, b + 1, c + 1); 2, 3)$ is a linear syzygy edge, $\alpha$ is either in none or in both the two lower up-triangles. In then follows by part a. that $\alpha\beta\gamma$ belongs to one of these up-triangles.

If $((a, b + 1, c + 1); 2, 3)$ is not a linear syzygy edge, the two other edges are linear syzygy edges. By the argument in a. both the lower up-triangles contains $\beta$ and $\gamma$ and so at least on of them contains $\alpha\beta\gamma$. □

Let $\tilde{X}$ be a set of $x$-variables (with various indices) and $\tilde{Y}$ and $\tilde{Z}$ be sets of $y$- and $z$-variables.

**Lemma 7.2.** Let $I$ be an ideal generated by a subset of monomials in the product set $\tilde{X} \cdot \tilde{Y} \cdot \tilde{Z}$. Let $x_\alpha y_\beta z_\gamma$ be in $\tilde{X} \cdot \tilde{Y} \cdot \tilde{Z}$ but not in $I$. Then $I : x_\alpha y_\beta z_\gamma$ is generated by variables iff for every $x'_\alpha y'_\beta z'_\gamma \in I$ one of the variables $x'_\alpha, y'_\beta$ or $z'_\gamma$ is in the colon ideal.

Proof. Note that by the construction of $I$ and definition of $x_\alpha y_\beta z_\gamma$, none of the variables $x_\alpha, y_\beta$ or $z_\gamma$ can be in $I : x_\alpha y_\beta z_\gamma$.

That the first assertions implies the second is easy. Assume the second assertion holds. Then, if say $y'_\beta z'_\gamma$ is in the colon ideal, then $x_\alpha y_\beta y'_\beta z_\gamma z_\gamma'$ is in $I$. So at least some $x_\alpha y_\beta z_\gamma$ is in $I$, where $\tilde{\beta} = \beta'$ or $\tilde{\gamma} = \gamma'$. But by assumption then either $y_\beta'$ or $z_\gamma'$ is in the colon ideal. This implies the colon ideal is generated by variables. □

We now consider the monomials $x_\alpha y_\beta z_\gamma$ belonging to the up-triangles $U(a, b, c) \in \Delta_3(n - 1)$ and shall provide a total order on these monomials. First consider the partial order on triples where $(a, b, c) \geq (a', b', c')$ if $a \geq a'$ and take any linear extension on this to get a total order $>$ on triples.
Now for each up-triangle $U(a, b, c)$ we shall make a total order on the (degree 3) monomials belonging to it. For each $X(a + 1, b, c)$ choose any total order of the $x$-variables. For the $y$-variables we have an ascending chain:

$$Y(a, 1, n - a - 1) \subseteq Y(a, 2, n - a - 2) \subseteq \cdots \subseteq Y(a, n - a, 0).$$

We order the variables such that each new variable popping up in the chain is less than the foregoing variables. Similarly for the $z$-variables we have a chain

$$Z(a, n - a - 1, 1) \subseteq Z(a, n - a - 2, 2) \subseteq \cdots \subseteq Z(a, 0, n - a),$$

and we order the variables such that each new variables popping up in the chain is less than the foregoing variables. The monomials belonging to $U(a, b, c)$ correspond to

$$X(a + 1, b, c) \times Y(a, b + 1, c) \times Z(a, b, c + 1).$$

We get the partial product order on this and take a linear extension of this partial order.

We now order the monomials associated to the up-triangles in $\Delta_3(n)$ as follows. If $\alpha' \beta' \gamma'$ occurs first in $(a', b', c')$ and $\alpha \beta \gamma$ occurs first in $(a, b, c)$, then

(14) $\alpha' \beta' \gamma' > \alpha \beta \gamma$

if $(a', b', c') > (a, b, c)$, or if $(a', b', c') = (a, b, c)$ and the order of (14) is given by the order on the monomials belonging to the up-triangle $U(a, b, c)$.

**Proposition 7.3.** The ideal generated by all the variables belonging to the up-triangles of $\Delta_3(n)$, has linear quotients given by the ordering of the monomials above.

**Proof.** Let $\alpha \beta \gamma$ occur for the first time in the up-triangle $(u, b, c)$ and let $I$ be the ideal generated by all the foregoing monomials. We shall show that $I : x_\alpha y_\beta z_\gamma$ is generated by variables and use Lemma 7.2.

1. Let $\alpha' \beta' \gamma'$ be in $(u, b', c')$ where $(u, b', c') > (u, b, c)$. Suppose $c' \geq c$, see Figure 12 and so $\gamma$ belongs to $(u, b', c')$, since the map $Z : \Delta_3(n) \rightarrow B(n)$ is isotone.

a. If $\beta \geq \beta'$ then $\beta$ will be in $(u, b'', c'')$ where $b'' \leq b'$ so $c'' \geq c'$. Then $\beta$ will also be in $(u, b', c')$ and since $\gamma$ is in $(u, b', c')$ we will have $\alpha' \beta' \gamma'$ belonging to $(u, b', c')$. If $\alpha \neq \alpha'$ this gives $\alpha'$ in the colon ideal. If $\alpha = \alpha'$, then $\alpha \beta' \gamma'$ belongs to $(u, b', c')$ and so if $\beta \neq \beta'$ then $\beta'$ is in the colon ideal. Finally if $\gamma \neq \gamma'$ (but $\alpha = \alpha'$ and $\beta = \beta'$), we see that $\gamma'$ is in the colon ideal.

b. Assume now that $\beta < \beta'$. Note that since $b' \leq b$ we have $\beta'$ belonging to $(u, b, c)$. Then $\alpha \beta' \gamma$ is already in $I$ by the ordering on the monomials belonging to $(u, b, c)$, and hence $\beta'$ is in the colon ideal.
2. A symmetric argument works when $\alpha'\beta'\gamma'$ is in $(u, b', c')$ and $b' \geq b$.

3. Assume now that $\alpha'\beta'\gamma'$ belongs to the up-triangle $U(u + 1, b', c')$ where the sum of these coordinates is $n - 1$. Either $b' \geq b$ or $c' \geq c$. Suppose the latter, see Figure 13. Then $\beta'$ belongs to the up-triangle $U(u, b, c)$ due to $Z$ being isotone.

a. If $\beta' > \beta$ in the $u$-order, then $\alpha'\beta' > \alpha\beta\gamma$ and so the former belongs to $I$ and $\beta'$ is in the colon ideal $I : x_ay_\beta z_\gamma$.

b. If $\beta' = \beta$, then by Lemma 7.2 (applied in the $y$-direction, not $x$-direction) either $\alpha\beta\gamma$ is in $U(u + 1, b, c')$ or in $U(u, b - 1, c + 1)$. The latter must be the case since $\alpha\beta\gamma$ first occurs in $U(a, b, c)$.

c. If $\beta' < \beta$, then since $\beta'$ belongs to $U(u, b, c)$, $\beta$ must belong to $U(u, b - 1, c + 1)$ and by Lemma 5.1 $\alpha\beta\gamma$ will belong to $U(u, b - 1, c + 1)$.

In case b. and c. we may continue like this and push $\alpha\beta\gamma$ stepwise to the right, until we get to $(u, b - r, c + r)$ where $c + r = c' + 1$, and $(u, b - r, c + r) = (u, b', c' + 1)$, so $\alpha\beta\gamma$ is in both $U(u, b', c')$ and $U(u, b', c' + 1)$. Note that by $X$ being isotone $\alpha$ belongs to $(u + 1, b', c')$. We shall show that $\alpha'\beta'\gamma'$ or $\alpha'\beta'\gamma$ is in $(u + 1, b', c')$. This will show that either $z_{\gamma'}$ or $y_{\beta'}$ is in the colon ideal.

i. If $\beta = \beta'$ and $\gamma = \gamma'$ then $\alpha \neq \alpha'$ since $\alpha\beta\gamma$ first occurs in $U(a, b, c)$. So $\alpha'\beta'\gamma$ belongs to $U(u + 1, b', c')$ and so $x_{\alpha'}$ is in the colon ideal.

ii. If $\beta \neq \beta'$ and $\gamma = \gamma'$ then $\alpha'\beta'\gamma$ belongs to $U(u + 1, b', c')$ and so $x_{\beta'}$ is in the colon ideal.

iii. If $\beta = \beta'$ and $\gamma \neq \gamma'$ then $\alpha'\beta'\gamma$ belongs to $U(u + 1, b', c')$ and so $x_{\gamma'}$ is in the colon ideal.

iv. Suppose that $\beta \neq \beta'$ and $\gamma \neq \gamma'$. If the edge $((u + 1, b' + 1, c' + 1); 1, 2)$ is a linear syzygy edge then $\gamma$ is in $Z(u + 1, b', c' + 1)$ and so $\alpha'\beta'\gamma$ belongs to $U(u + 1, b', c')$. If $((u + 1, b' + 1, c' + 1); 1, 3)$ is a linear syzygy edge then $\beta \in Y(u + 1, b' + 1, c')$ and so $\alpha'\beta'\gamma'$ is belongs to $U(u + 1, b', c')$.

4. Suppose then that $\alpha'\beta'\gamma'$ is in $U(u + r, b', c')$ where $r \geq 2$.

a. If $b' \leq b$ and $c' \leq c$ (then at least one inequality is strict) then $\alpha'\beta'\gamma'$ is in $U(u + r, b', c')$ since the map $X$ is isotone, and $\alpha$ also belongs to either $U(u + r - 1, b' + 1, c')$ or $U(u + r - 1, b', c' + 1)$. Hence by Lemma 7.3 $\alpha'\beta'\gamma'$ is in one of these up-triangles. We may continue until either $u + r - 1 = u + 1$, treated in Case 3., or until $b' > b$ or $c' > c$. Assume $c' > c$. 

\[ \begin{array}{c}
\alpha' \\
\beta' \\
\gamma'
\end{array} \]

\[ \begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array} \]

\( (u, b, c) \)
b. We then assume $\alpha'\beta'\gamma'$ is in $(u+r,b',c')$ where $r \geq 2$ and $c' > c$. Note that by $Y$ being isotone and $b' < b$, $\beta'$ will belong to $U(u, b, c)$ and to $U(u, b - 1, c + 1)$ by Lemma 1.5.

b1. If $\beta' > \beta$, then $\alpha'\beta'\gamma > \alpha\beta\gamma$ and so $\beta'$ is in the colon ideal.

b2. If $\beta = \beta'$ then $\beta$ belongs to $U(u + r, b', c')$. By $Y$ being isotone, $\beta$ belongs to $U(u, b - 1, c + 1)$.

b3. If $\beta' < \beta$, then $\beta$ is in the up-triangle $U(u, b - 1, c + 1)$. In both cases b2. and b3. by Lemma 7.1 $\alpha'\beta'\gamma$ is either in up-triangle $U(u + 1, b - 1, c, 1)$, not possible, or in $U(u, b - 1, c + 1)$.

In this way we may continue going rightwards until we get to $(u, b - t, c + t)$ with $c + t = c'$. Then $(u, b + r, c')$ contains $\alpha\beta\gamma$ and so $\alpha$ is in $(u + r, b', c')$ and $(u + r - 1, b' + 1, c')$. Then $\alpha'\beta'\gamma'$ is in $(u + r, b', c')$ and since $\alpha\beta\gamma$ occurs first in $U(a, b, c)$ this is not equal to $\alpha'\beta'\gamma'$. By Lemma 7.1 we may push it down to level $u + r - 1$. In this way we can continue until we get $\alpha'\beta'\gamma'$ on level $m + 1$ which is treated in Case 3. □

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