A Faithful Discretization of the Verbose Persistent Homology Transform

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Abstract

The persistent homology transform (PHT) represents a shape with a multiset of persistence diagrams parameterized by the sphere of directions in the ambient space. In this work, we describe a finite set of diagrams that discretize the PHT such that it faithfully represents the underlying shape. We provide a discretization that is exponential in the dimension of the shape. Moreover, we show that this discretization is stable with respect to various perturbations and we provide an algorithm for computing the discretization. Our approach relies only on knowing the heights and dimensions of topological events, which means that it can be adapted to provide discretizations of other dimension-returning topological transforms, including the Betti function transform. With mild alterations, we also adapt our methods to faithfully discretize the Euler characteristic function transform.

Keywords: immersed simplicial complexes, persistence diagrams, reconstruction, shape representation

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1 Introduction

Collections of topological descriptors are both empirically and theoretically useful for analysing data and differentiating shapes [1–8]. In [9], Turner et al. define the persistent homology transform (PHT) and Euler characteristic transform (which we refer to as the Euler characteristic function transform (ECFT)), which map a geometric simplicial complex in \( \mathbb{R}^d \) to a family of directional persistence diagrams and Euler characteristic functions indexed or parameterized by the directions in \( S^{d-1} \). They show that these infinite parameterized families are faithful representations of shape, meaning that no two distinct shapes have the same family of directional persistence diagrams or Euler characteristic functions. A few years later, several research groups independently observed that there exist finite faithful representations for various types of simplicial and cubical complexes [10–15]. Meanwhile, researchers are already applying the PHT and the closely related verbose PHT (VPHT) to represent various types of data sets in machine learning and statistical pipelines [9, 12, 16–19].

Motivated by the need for a provably faithful discretization of the VPHT in applications (rather than the current standard of using uniform sampling, which may or may not be faithful), Belton et al. gave a faithful discretization of the VPHT using only \( \Theta(n_0^2) \) verbose persistence diagrams for embedded plane graphs with \( n_0 \) vertices [11, Theorems 15 and 16]. In [20, Theorem 10], we remove the planar condition and improve this bound to \( O(d + n_1 \log n_0) \), where \( n_1 \) is the number of edges in the graph. To show that these discretizations are indeed faithful, both [11] and [20] use the proof method of reconstructing the shape from the set of diagrams. If a shape can be unambiguously reconstructed from a set of diagrams, then the set of diagrams is representative of the shape, i.e., the discretized PHT is faithful. This work takes the natural next step of using reconstruction to give the first explicit faithful discretization of the PHT for general simplicial complexes. This method also discretizes topological transforms in a larger family, including the verbose Betti function transform (VBFT), as well as the verbose Euler characteristic function transform (VECFT).
**Our Contribution**

We provide sufficient conditions for a faithful discretization of the VPHT (and other related topological transforms) in Theorem 19. While discretizations emerge as natural extensions to (V)PHT injectivity proofs, those discretizations are at least exponential in the size of the ambient space and based on geometric and topological properties of the shape. In this paper, we use the sufficient conditions of Theorem 19 to develop Theorem 39, leading to a discretization exponential in the dimension of the simplicial complex and an algorithm that reports the discretization in time dependant on the size of the of the simplicial complex. Moreover, in Section 7, we show that the discretization is stable with respect to multiple types of perturbations. This is the first study of producing a faithful discretization of the VPHT of an arbitrary simplicial complex in arbitrary dimension and of studying the stability of that discretization.

**2 Background Definitions**

We assume that the reader is familiar with homology groups (denoted $H_*$) and their Betti numbers (denoted $\beta_*$). For a more complete discussion on foundational computational topology, we refer the reader to [21, 22].

### 2.1 Lower-Star Filtrations and Persistence

We first review the building blocks for the VPHT: simplicial complexes, lower-star filtrations, and (verbose) persistence diagrams.

**General Position**

In what follows, when we say that a point set $V \subset \mathbb{R}^d$ is in *general position* we mean that, for all $k$ with $1 \leq k \leq d + 1$, every subset of $V$ of size $k$ is affinely independent. We denote the affine space spanned by $V$ by $\text{aff}(V)$.

**Simplices and Simplicial Complexes**

Let $k, d \in \mathbb{N}$. A *(geometric)* $k$-simplex $\sigma$ is the convex hull of a set of $k + 1$ affinely independent points in $\mathbb{R}^d$, denoted $\sigma = [v_0, v_1, \ldots, v_k]$. Each of these points is called a *vertex*, and we denote the vertex set of $\sigma$ by $\text{verts}(\sigma)$; at times, we use $\sigma$ in place of $\text{verts}(\sigma)$ in places where using $\text{verts}(\sigma)$ would make equations cumbersome to read. We call $k$ the *dimension* of $\sigma$, and write $\text{dim}(\sigma) := k$. For another simplex $\tau$, we say that $\tau$ is a *face* of $\sigma$ and $\sigma$ is a coface of $\tau$ if $\emptyset \neq \text{verts}(\tau) \subseteq \text{verts}(\sigma)$; we denote this relation by $\tau \preceq \sigma$. If $\tau \preceq \sigma$ but $\tau \neq \sigma$, then $\tau$ is called a proper face of $\sigma$, denoted $\tau \prec \sigma$.

A *GP-immersed simplicial complex* $K$ is a finite set of geometric simplices such that the vertex set of $K$ is in general position.\(^1\) We topologize $K$ with the Alexandroff topology. We denote the set of $k$-simplices in $K$ by $K_k$ and the

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\(^1\)Note that this is a stronger condition than the usual notions of immersions; as long as the vertices are in general position, self-intersections of a GP-immersed complex need not be transverse.
number of simplices in \( K_k \) by \( n_k \). We let \( n = \sum_{k \in \mathbb{N}} n_k \). The degree of \( v \in K_0 \) is the number of one-simplices (edges) that are cofaces of \( v \), and we denote this as \( \deg(v) \).

**Filtrations and Persistent Homology**

Let \( K \) be a simplicial complex and let \( f : K \to \mathbb{R} \) be a monotonic function, i.e., for each pair of simplices \( \tau \prec \sigma \in K \), we have \( f(\tau) \leq f(\sigma) \). Thus, each sublevel set \( F_t := f^{-1}(-\infty, t] \) with \( t \in \mathbb{R} \) is a simplicial complex. The parameterized sequence of subcomplexes \( \{F_t\}_{t \in \mathbb{R}} \), along with inclusions \( F_t \subseteq F_{t'} \) for all \( t \leq t' \), is the sublevel-set filtration of \( K \) with respect to \( f \). This filtration realizes at most \( n + 1 \) distinct complexes: the empty set and \( F_{f(\sigma)} \) for each \( \sigma \in K \). More generally, a filtration of \( K \) is a family of subcomplexes of \( K \) indexed by a poset such that inclusions of the subcomplexes follow the poset order.

One type of filtration of interest is an index filtration. This type of filtration realizes exactly \( n + 1 \) distinct complexes, meaning each subcomplex of \( K \) includes one more simplex than the previous subcomplex in the filtration. Thus, we see that an index filtration is equivalent to a total order on the simplices of \( K \).

Another filtration of interest is the lower-star filtration of a simplicial complex \( K \) GP-immersed in \( \mathbb{R}^d \) with respect to a direction \( s \in S^{d-1} \). For each vertex \( v \in K_0 \), the height of \( v \) in direction \( s \) is given by the dot product, \( s \cdot v \). The lower-star filter function in direction \( s \), denoted \( h_s : K \to \mathbb{R} \), defines a “height” of each simplex in \( K \) where \( h_s(\sigma) = \max\{s \cdot v \mid v \in \text{verts}(\sigma)\} \), i.e., \( h_s(\sigma) \) is the height of the highest vertex in \( \sigma \) with respect to \( s \). The lower-star filtration is the sublevel-set filtration of \( K \) with respect to \( h_s \). Notice that, for \( r, t \in \mathbb{R} \) such that \( r \leq t \), we have \( h_s^{-1}(-\infty, r] = h_s^{-1}(-\infty, t] \) if and only if no vertex has height in the interval \( (r, t] \). If all vertices are at distinct heights, there are \( n_0 \) distinct distinct subcomplexes, and there exists an ordering of the vertices \( \{v_{i_1}', v_{i_2}', \ldots, v_{i_{n_0}}'\} \) such that the complexes realized in the sublevel-set filtration are:

\[
0 \subset h_s^{-1}(-\infty, s \cdot v_{i_1}') \subset h_s^{-1}(-\infty, s \cdot v_{i_2}') \subset \cdots \subset h_s^{-1}(-\infty, s \cdot v_{i_{n_0}}').
\]

Let \( k \in \mathbb{N} \). Applying the homology functor to a filtration \( \{F_t\}_{t \in \mathbb{R}} \), we obtain the persistence module \( \{H_k(F_t)\}_{t \in \mathbb{R}} \). Here, we assume that homology is computed using field coefficients (e.g., \( \mathbb{Z}_2 \)). Then, each \( H_k(F_t) \) is a vector space, and, for each \( t \leq t' \), the inclusion of simplicial complexes \( F_t \subseteq F_{t'} \) induces a linear map \( f_k^{t'} : H_k(F_t) \to H_k(F_{t'}) \). Letting \( \beta_k^{x,y} \) denote the rank of \( f_k^{x,y} \),

\[
\mu_k(x, y) = \beta_k^{x,y-1} - \beta_k^{x,y} - \beta_k^{x-1,y-1} + \beta_k^{x-1,y}
\]

where \((a, b)^m\) denotes \( m \) copies of the point \((a, b)\), and \( \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\} \), we define the \( k \)-dimensional persistence diagram, as the following multiset:

\[
\mathcal{D}_k(f) := \{(x, y) \mu_k(x, y) \}_{(x, y) \in \overline{\mathbb{R}}^2}.
\]
In other words, each \((x, y) \in D_k(f)\) represents a \(k\)-dimensional homological generator \(\alpha\) that is born at \(x\) (that is, \([\alpha] \in H(F_x)\) but \([\alpha] \notin \text{im}(f_{k-\varepsilon}^{x-\varepsilon})\) for any \(\varepsilon > 0\)) and dies going into \(y\) (that is, \(y\) is the smallest index such that there exists \([\alpha'] \neq [\alpha] \in H(F_x)\) with \([\alpha] = [\alpha']\) in \(H(F_y)\)). The persistence diagram is the union of all \(k\)-dimensional diagrams: \(D(f) := \bigcup_{k \in \mathbb{Z}} D_k(f)\).

**Verbose Persistence Diagram**

Since simplices can have the same height in a general filtration, it is possible that the birth and the death of a \(k\)-cycle happen simultaneously, in which case, that cycle is not represented in the persistence diagram. However, having every simplex “appear” in the persistence diagram is helpful (in particular, the information we use from the persistence diagram is the height of each simplex). Thus, we introduce *verbose persistence diagrams* (VPDs). Note that verbose descriptors are sometimes also called *augmented* descriptors in the literature.

To define VPDs for the sublevel-set filtration of \(f : K \to \mathbb{R}\): \(\{\sigma_i \mid i \leq j\}\), the corresponding index filtration \(F_j^\prime \) is the nested sequence of subspaces of \(K\):

\[
\emptyset = F_0^\prime \subset F_1^\prime \subset F_2^\prime \subset \ldots \subset F_n^\prime = K.
\]

**Definition 1** (Verbose Persistence Diagram). Given a filter \(f : K \to \mathbb{R}\), let \(f_*\) be a compatible index filtration. For \(k \in \mathbb{N}\), the \(k\)-dimensional verbose persistence diagram is the following multiset:

\[
\hat{D}_k(f) := \{(f(f_*^{-1}(i)), f(f_*^{-1}(j))) \mid (i, j) \in D_k(f_*)\},
\]

where \(f(\emptyset) := \infty\). The verbose persistence diagram (VPD) of \(f\) is the union of all \(k\)-dimensional verbose persistence diagrams: \(\hat{D}(f) := \bigcup_{k \in \mathbb{Z}} \hat{D}_k(f)\).

Since the filter function \(f_*\) in the definition above need not be unique, it is not immediately clear that this definition is well-defined. A proof that VPDs are, in fact, well-defined can be found in Appendix B.

The VPD carries more information than the persistence diagram (as the height of each simplex is a coordinate of a persistence point). Yet, algorithms for computing PDs compute VPDs as an intermediate step; see, e.g., [21, Chapter VII.2]). The VPD has been used in several contexts already. For example, the definition of PD in McCleary and Patel [23] is the same as our definition of VPD. Usher and Zhang define verbose barcodes barcodes via the lens of filtered chain complexes in [24]. This perspective is also taken in, e.g., [25, 26], where instantaneous or length-zero bars are referred to as “ephemeral.” In
the literature, “verbose” is sometimes replaced by the word “augmented,” and “concise” by “non-augmented.”

We only use verbose persistence diagrams in this paper, so we use “diagram” as shorthand for “VPD.”

**Definition 2** (Verbose Persistent Homology Transform). Given a simplicial complex \( K \) \( \text{GP-immersed in } \mathbb{R}^d \), the \textit{verbose persistent homology transform} \((\text{VPHT})\) of \( K \), denoted \( \text{VPHT}(K) \), is the parameterized set of all directional diagrams; that is, \( \text{VPHT}(K) := \{(s, \hat{D}(h_s))\}_{s \in S^{d-1}} \).

Given a set of directions, \( S \subset S^{d-1} \), the restriction of \( \text{VPHT}(K) \) to \( S \), is:

\[
\hat{D}[K, S] := \{(s, \hat{D}(h_s))\}_{s \in S} \subset \text{VPHT}(K).
\]

We call \( \hat{D}[K, S] \) a \textit{discretization} of the VPHT. With \( S \) large enough (and selected wisely), the hope is that \( \hat{D}[K, S] \) caries the same information as \( \text{VPHT}(K) = \hat{D}[K, S^{d-1}] \). In other words, we hope that it is \textit{faithful}, as we define next:

**Definition 3** (Faithful Discretization of \( \text{VPHT}(K) \)). Given a simplicial complex \( K \) \( \text{GP-immersed in } \mathbb{R}^d \) and a finite set \( S \subset S^{d-1} \), we say that \( \hat{D}[K, S] \) is a faithful discretization of \( \text{VPHT}(K) \) if, for any simplicial complex \( K' \neq K \) \( \text{GP-immersed in } \mathbb{R}^d \), there exists an \( s \in S \) such that \( \hat{D}(h_s) \neq \hat{D}(h'_s) \), where \( h_s \) and \( h'_s \) are the lower-star filter functions of \( K \) and \( K' \) with respect to direction \( s \), respectively. That is, no other simplicial complex \( \text{GP-immersed in } \mathbb{R}^d \) has the same parameterized set of directional diagrams.

Although this paper is largely written in the language of the VPHT, the results hold for a large class of topological transforms. Important examples include the verbose Betti function transform (VBFT) and the verbose Euler characteristic function transform (VECFT). Note that Betti functions and Euler characteristic functions are often called Betti curves and Euler characteristic curves in the literature.

**Betti Functions**

The \( k \text{th} \) Betti number of a simplicial complex \( K \) is the rank of the \( k \)-dimensional homology group of \( K \), and is denoted \( \beta_k(K) = \text{rank}(H_k(K)) \). Measuring this quantity as a filtration parameter changes gives rise to the \( k \text{th} \) Betti function, and the set of \( k \text{th} \) Betti functions for all \( k \in \mathbb{Z} \) is collectively referred to as the \textit{Betti function}.

**Definition 4** (Betti Function (BF) and Verbose BF). Given a filter \( f : K \to \mathbb{R} \), let \( f' \) be a compatible index filtration. Let \( T = \{t_1, t_2, \ldots, t_\eta\} \) be the ordered set of filter parameters of \( f \) that witness a change in homology. The \( k \text{th} \) Betti
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function ($k^{th}$ BF) is a step function $\beta_{f,k} : \mathbb{R} \to \mathbb{Z}$ defined by

$$\beta_{f,k} := \{ \beta_k(p), p \in [t_i, t_{i+1}) \}.$$ 

Furthermore, we write $\sigma_i \in K_k$ to be the simplex such that $f'(\sigma_i) = i$. The $k^{th}$ 
verbose Betti function ($k^{th}$ VBF) is the decorated step function $\hat{\beta}_{f,k} : \mathbb{R} \to \mathbb{Z}$ 
defined by

$$\hat{\beta}_{f,k} = \{ \beta_k(p), p \in [f(\sigma_i), f(\sigma_j)) \},$$

where $(i, j]$ is an interval with constant value in the function $\beta_{f',k}$.

Note that some $[f(\sigma_i), f(\sigma_j))$ are empty, so the recorded Betti number takes 
place at a single point rather than an interval with positive measure. It is these 
extra points that “decorate” the otherwise regularly defined step function. 
Then, the VBF represents the BF as a parameterized count of $k$-simplices.

We can read off the BF from the PD by observing the following equality:

$$\hat{\beta}_{f,k}(p) = \left| \left\{ (a, b) \in \hat{D}_k(f) \text{ s.t. } a \leq p \text{ and } b \geq p \right\} \right|$$

In other words, the BF is a weaker invariant than the PD. Similarly, the 
VBF is a weaker invariant than the VPD; we further explore the relationship 
between these and other descriptors in [27]. Next, we observe that VBFs are 
dimension-returning.

**Corollary 5** (Properties of VBFs). Let $f : K \to \mathbb{R}$ be a monotonic func-
tion. For each $\sigma \in K$, the collection of functions $\{ \hat{\beta}_{f,k} \}_{k \in \mathbb{Z}}$ records $f(\sigma)$ and 
dimension of $\sigma$.

Next, we provide an explicit definition of the VBFT and make our final 
observation.

**Definition 6** (Verbose Betti Function Transform). Given a geometric simpli-
cial complex $K$ in $\mathbb{R}^d$, the verbose Betti function transform (VBFT) of $K$ is 
the set of all directional VBFs of lower-star filtrations over $K$, parameterized 
by the sphere of directions, $S^{d-1}$.

Parallel to Definition 3, we consider faithful discretizations of the VBFT.

**Euler Characteristic Functions**

The Euler characteristic of a simplicial complex is the alternating sum of the 
number of simplices of different dimensions: $\chi(K) = \sum_{i=0}^{\kappa} (-1)^i n_i$, where we 
recall that $n_i$ is the number of $i$-dimensional simplices in $K$ and $\kappa = \dim(K)$. 
Given a filtration, the Euler characteristic with respect to the filtration 
parameter is known as the Euler characteristic function (ECF). Formally,
Definition 7 (Euler Characteristic Function (ECF) and Verbose ECF). Given a filter \( f : K \rightarrow \mathbb{R} \), let \( f' \) be a compatible index filtration. Let \( \{F_i := f^{-1}(-\infty, t_i]\}_{i=1}^{n} \) be the filtration of \( K \) corresponding to \( f \). The Euler characteristic function is a step function \( \chi_f : \mathbb{R} \rightarrow \mathbb{Z} \) defined by

\[
\chi_f(p) := \sum_{k=0}^{\infty} (-1)^k n_k^{(i)},
\]

where \( p \in [t_i, t_{i+1}) \) and \( n_k^{(i)} \) is the number of \( k \)-simplices in \( F_i \). Furthermore, the verbose Euler characteristic function (VECF) is the function \( \hat{\chi}_f : \mathbb{R} \rightarrow \mathbb{Z}^2 \) defined by

\[
\hat{\chi}_f(p) = \left( \sum_{k=0}^{\infty} n_{2k}^{(i)}, \sum_{k=0}^{\infty} n_{2k+1}^{(i)} \right).
\]

In other words, the VECF represents the ECF as a parameterized count of positive (even parity) and negative (odd parity) simplices.

We can read off the ECF from the PD, and the VECF from the VPD. In other words, the ECF is a weaker invariant than the PD and the VECF is a weaker invariant than the VPD; see [27] for more on the relationships between these and other descriptors.

Remark 8 (From Persistence Diagrams to ECFs). Let \( f : K \rightarrow \mathbb{R} \) be a monotonic function. Let \( \{t_i\}_{i=1}^{\eta} \) be the set of event times, and let \( p \in [t_i, t_{i+1}) \). Then, the following holds:

\[
\chi_f(p) = \sum_{k \in \mathbb{Z}} \sum_{(a,b) \in D_k(f) \atop a \leq p < b} (-1)^k.
\]

For each \( p \in \mathbb{R} \), let

\[
A_k(p) := \{(a,b) \in \tilde{D}_{k-1}(f) \text{ s.t. } b \leq p\} \cup \{(a,b) \in \tilde{D}_k(f) \text{ s.t. } a \leq p\}.
\]

Then,

\[
\hat{\chi}_f(p) = \left( \sum_{k \in \mathbb{Z} \atop k \text{ even}} |A_k(p)|, \sum_{k \in \mathbb{Z} \atop k \text{ odd}} |A_k(p)| \right).
\]

Corollary 9 (Properties of VECFs). Let \( f : K \rightarrow \mathbb{R} \) be a monotonic function. For each \( \sigma \in K \), the function \( \hat{\chi}_f \) records \( f(\sigma) \) and the parity of \( \dim(\sigma) \).

Definition 10 (Verbose Euler Characteristic Function Transform). Given a geometric simplicial complex \( K \) in \( \mathbb{R}^d \), the verbose Euler characteristic function transform (VECFT) of \( K \) is the set of all directional ECFs of lower-star filtrations over \( K \), parameterized by the sphere of directions, \( \mathbb{S}^{d-1} \).
Parallel to Definition 3, we consider faithful discretizations of the VECFT.

2.2 Tools for Building a Faithful Discretization

We now give a lemma that relates simplices to points in an VPD, discuss the general position assumption, and define a tool used in our proofs of faithfulness called filtration hyperplanes. We prove the following lemma in Appendix A.

Lemma 11 (Simplex Count). Let \( K \) be a simplicial complex, \( k \in \mathbb{N} \) and \( c \in \mathbb{R} \). Let \( f : K \to \mathbb{R} \) be a monotonic function. Then, the \( k \)-dimensional simplices of \( K \) with a function value of \( c \) are in one-to-one correspondence with the points in the following multiset:

\[
\left\{(a, b) \in \mathcal{D}_k(f) \text{ s.t. } a = c\right\} \cup \left\{(a, b) \in \mathcal{D}_{k-1}(f) \text{ s.t. } b = c\right\}. \tag{5}
\]

Next, we define a structure that helps build a geometric intuition for several of the proofs that follow.

Definition 12 (Filtration Hyperplane). Let \( s \in \mathbb{S}^{d-1} \) be a unit vector, and let \( c \in \mathbb{R} \). Let \( H(s, c) \) be the \((d-1)\)-dimensional hyperplane that passes through the point \( cs \in \mathbb{R}^d \) and is perpendicular to \( s \). We define the closed half-spaces above and below this hyperplane with respect to direction \( s \) by \( H^+(s, c) \) and \( H^{-}(s, c) \), respectively.

Let \( V \) be a finite set of vertices in \( \mathbb{R}^d \) and let \( h_s : V \to \mathbb{R} \) be the lower-star filter function with respect to the direction \( s \). The filtration hyperplanes of \( V \) are the set of hyperplanes

\[
\mathcal{H}(s, V) := \{H(s, h_s(v))\}_{v \in V}.
\]

All hyperplanes in \( \mathcal{H}(s, V) \) are parallel to each other and perpendicular to the direction \( s \). Let \( K \) be a simplicial complex GP-immersed in \( \mathbb{R}^d \). Since the births in \( \mathcal{D}_0(h_s) \) are in one-to-one correspondence with the vertices of \( K \) by Lemma 11, there is a filtration hyperplane at every height at which a vertex lies in direction \( s \). In directions where vertices are at the same height, the filtration hyperplanes are not unique.

By observing intersections of a sufficient number of linearly independent filtration hyperplanes, we can form a grid of points of intersections, which will be a crucial tool in our vertex-reconstruction arguments.

Definition 13 (Filtration Grid). For \( n \geq d \), let \( s_1, s_2, \ldots, s_n \in \mathbb{S}^{d-1} \) be linearly independent and let \( P \in \mathbb{R}^d \) be a pointset. We define the filtration grid of \( P \) with respect to \( \{s_1, s_2, \ldots, s_n\} \) to be the grid of points, \( A \), arising from choosing one hyperplane in each set \( \mathcal{H}(s_i, K_0) \) for \( 1 \leq i \leq n \). That is, the filtration grid is the collection of \( n \)-way intersections of filtration hyperplanes. Note that \( P \subseteq A \) and \(|A| \leq |P|^d\).
Finally, we define the specific types of directions that are the building blocks of our faithful discretization. We begin with directions that are perpendicular to a simplex in a specific way.

**Definition 14 (P-Perpendicular).** Let $V \subset P \subset \mathbb{R}^d$ such that $P$ is in general position. We say that a direction $s \in S^{d-1}$ is $P$-perpendicular to $V$ if for all $u \in P$ and $v \in V$, $s \cdot u = s \cdot v$ if and only if $u \in V$. In other words, $s$ is perpendicular to $\text{aff}(V)$ and no other vertex in $P$ is at the same height as the vertices in $V$ with respect to $s$.

The next definition describes when a direction is a slight tilt of an initial direction that is $P$-perpendicular to some $V$, so that a specified subset $W$ (and no other points) pops above $V \setminus W$.

**Definition 15 ((P, V, W, s)-Perturbation).** Let $W \subset V \subset P \subset \mathbb{R}^d$ such that $P$ is in general position. Let $s \in S^{d-1}$ be $P$-perpendicular to $V$. Then, we say a direction $s_\ast \in S^{d-1}$ is a $(P, V, W, s)$-perturbation if the following hold:

1. The direction $s_\ast$ is $P$-perpendicular to $\text{aff}(V \setminus W)$.
2. The points in $W$ are above $V \setminus W$ with respect to $s_\ast$.
3. For all $p \in P \setminus V$, $p$ is strictly above (below) the height of $V \setminus W$ with respect to $s_\ast$ if and only if it is strictly above (below, respectively) $V$ with respect to $s$.

If $V$ defines the vertices of a simplex $\sigma$, then we may use $\sigma$ in place of $\text{verts}(\sigma)$ in the definition and construction above. Likewise for $W$.

### 3 The Main Result: A Faithful Discretization of the VPHT

In this section, we give sufficient properties for a set of diagrams to faithfully discretize the VPHT of a given simplicial complex, the proof of which is given in the remainder of the paper. To give a concrete construction, we also provide an explicit set of directions that satisfies these properties.

**Definition 16 (Vertex-Isolating).** We say that the pair $(K, S)$ is vertex-isolating if $\widehat{D}[K, S]$ has $d$ linearly-independent directions, denoted $\{s_1, s_2, \ldots, s_d\}$, and one additional direction that uniquely orders the filtration grid of $K_0$ with respect to $\{s_1, s_2, \ldots, s_d\}$. We may also say that $S$ is vertex-isolating with respect to $K$.

**Definition 17 (Simplex-Isolating).** Let $\sigma \in K \setminus K_0$. We say that the pair $(K, S)$ is $\sigma$-isolating if $S$ includes a direction $s_\sigma$ such that:

1. $s_\sigma$ is $K_0$-perpendicular to $\sigma$, and
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Fig. 1 The filtration hyperplanes (shaded in pink) corresponding to a pair of \([v_1, v_2, v_8]\)-isolating directions, where we have \(V = \{v_1, v_2, v_8\}\) and \(W = \{v_8\}\). One hyperplane corresponds to a direction \(s_V\) that is \(K_0\)-perpendicular to \(V = [v_1, v_2, v_8]\). The other corresponds to a direction that is a \((K, V, W, s_V)\)-perturbation) which, by pivoting \(s_V\) around \(V \setminus W\), “pops” the vertex of \(W\) above the filtration hyperplane.

2. for each \(\emptyset \neq W \subsetneq V = \text{verts}(\sigma)\), \(S\) includes a direction that is a \((K_0, V, W, s_\sigma)\)-perturbation.

If the pair \((K, S)\) is \(\sigma\)-isolating for every simplex \(\sigma \in K\), we say that \((K, S)\) is simplex-isolating. See Fig. 1.

Next, we make the observation that if there are directions that are \(\sigma\)-isolating for all maximal simplices of a complex, then the set of directions is simplex-isolating. Namely, there are directions that are \(\sigma\)-isolating for all simplices of a complex, regardless if the simplices are maximal.

Lemma 18 (Recursive Nature of \(\sigma\)-Isolating Directions). Let \(K\) be a simplicial complex \(GP\)-immersed in \(\mathbb{R}^d\), and let \(S\) be a set of directions such that \((K, S)\) is \(\sigma\)-isolating for every maximal simplex \(\sigma\). Then, \((K, S)\) is \(\tau\)-isolating for every simplex \(\tau \in K\). That is, \((K, S)\) is simplex-isolating.

Proof Let \(\tau \in K\). Then, either \(\tau\) is a maximal simplex or \(\tau\) is a proper face of a maximal simplex. If \(\tau\) is maximal, the claim follows immediately by assumption. Suppose, then, that \(\tau\) is a proper face of a maximal simplex \(\tau_m\). Denote the vertex sets of \(\tau\) and \(\tau_m\) by \(T\) and \(T_m\), respectively.

First, we will show \(S\) contains a direction that is \(K_0\)-perpendicular to \(\tau\). Since \(S\) satisfies Statement (1) of Definition 17, there exists a direction \(s_m \in S\) that is \(K_0\)-perpendicular to \(T_m\). Then, since \(S\) also satisfies Statement (2), and in particular when \(V = T_m\) and \(W = T_m \setminus T\), there is a direction \(s_\tau \in S\) that is a \((K_0, T_m, T_m \setminus T, s_m)\)-perturbation; \(s_\tau\) is \(K_0\)-perpendicular to \(T_m \setminus (T_m \setminus T) = T\), as desired.

Next, we will show that \(S\) contains a direction that is a \((K_0, \tau, \tau', s_\tau)\)-perturbation for every \(\tau' \leq \tau\). If \(\tau\) is a vertex, the claim is vacuously true. We therefore proceed assuming there exists some \(\tau' < \tau\) and let \(T' = \text{verts}(\tau')\). Since \(T_m \setminus (T \setminus T') \subset T_m\), and since \(S\) satisfies Statement (2) of Definition 17, there exists a direction \(s_p \in S\) that is a \((K_0, T_m, T_m \setminus (T \setminus T'), s_m)\)-perturbation. We
show that \( s_p \) satisfies the three properties of Definition 15. By definition, \( s_p \) is \( K_0 \)-perpendicular to \( T_m \setminus (T_m \setminus (T \setminus T')) = T \setminus T' \). Hence, \( s_p \) satisfies Statement (1) of Definition 15. Also by definition, the points of \( T_m \setminus (T \setminus T') \) are above \( T \setminus T' \) with respect to \( s_p \). Since \( T' \subseteq T_m \setminus (T \setminus T') \), we conclude that the points in \( T' \) are above \( T \setminus T' \) with respect to \( s_p \). Hence, \( s_p \) satisfies Statement (2) of Definition 15. Finally, let \( p \in K_0 \setminus T \). If \( p \in K_0 \setminus T_m \), then \( p \) is strictly above (below) the height of \( \tau' \) with respect to \( s_p \) if and only if it is strictly above (below) the height of \( \tau \setminus \tau' \) with respect to \( s_m \) by definition. Furthermore, this means \( p \) is above (below, respectively) \( \tau \setminus \tau' \) if and only if it is above (below) with respect to \( s_\tau \). If, instead, \( p \in K_0 \setminus (T_m \setminus T) \), then \( p \in \text{verts}(\sigma) \setminus (T \setminus T') \), so \( p \) is necessarily above \( \tau \setminus \tau' \) with respect to \( s_\tau \) and \( s_p \). Hence, \( s_p \) satisfies Statement (3) of Definition 15 and \( s_p \) is a \((K_0, \tau, \tau', s_\tau)\)-perturbation, as desired. \( \square \)

The remainder of this paper focuses on proving the following theorem:

**Theorem 19** (Sufficient Conditions for Faithful Discretization). Let \( K \) be a simplicial complex GP-immersed in \( \mathbb{R}^d \) such that \( \dim(K) = \kappa < d \), and let \( S \subseteq \mathbb{S}^{d-1} \) such that \((K, S)\) is vertex- and simplex-isolating. Then, \( \mathcal{D}[K, S] \) is a faithful discretization of \( VPT(K) \).

In Section 6.2, we present algorithms that compute a set of directions that is vertex- and simplex-isolating with respect to a simplicial complex \( K \). That is, we arrive at an explicit faithful discretization of the verbose PHT. However, before computing explicit directions, we first prove Theorem 19 for general sets satisfying the conditions of being vertex- and simplex-isolating.

## 4 Reconstruction of Simplicial Complexes in \( \mathbb{R}^d \)

In the following section, we prove Theorem 19 (Sufficient Conditions for Faithful Discretization) by reconstructing a GP-immersed simplicial complex. Our method first finds all zero-simplices, then all one-simplices, and so on. In what follows, let \( K \) be a simplicial complex GP-immersed in \( \mathbb{R}^d \) and let \( S \) be a set of directions satisfying the conditions of Theorem 19. We use \( \mathcal{D}[K, S] \) to reconstruct \( K \).

### 4.1 Vertex Reconstruction

Our method for finding zero-simplices is a straightforward generalization of the method of [11]. We briefly state the result here.

**Lemma 20** (Vertex Reconstruction, [11, Theorem 9]). Let \( K \) be a simplicial complex GP-immersed in \( \mathbb{R}^d \). Then, given a set of directions \( S \) that is vertex-isolating, \( \mathcal{D}[K, S] \) can reconstruct \( K_0 \).

We refer the reader to [11] for full details, but provide a proof sketch here. Since \( S \) is vertex-isolating, it contains a set of linearly-independent directions \( \{s_1, s_2, \ldots, s_d\} \). Let \( A \) denote the filtration grid of \( K_0 \) with respect
to \( \{s_1, s_2, \ldots, s_d\} \). Since \( S \) is vertex-isolating, it also contains some direction \( s_{d+1} \) that uniquely orders points of \( A \), leading to exactly \( n_0 \) pairwise intersections with \( H(s_{d+1}, K_0) \) and \( A \). It is then a simple matter of checking for the locations of these intersections, for example, with a brute force algorithm. See Fig. 2 for an example of how a vertex-isolating set may be used for vertex reconstruction.

![Fig. 2](image)

Fig. 2 The vertex set \( P \) (large black points) defines the filtration grid of \( P \) with respect to \( \{e_1, e_2\} \), denoted \( A \) (grey and black dots in the left figure). The direction \( s \), indicated on the right, uniquely orders the points of \( A \). Thus, since \( e_1 \) and \( e_2 \) are linearly independent and since the direction \( s \) uniquely orders the points of the \( A \), the set \( \{e_1, e_2, s\} \) is vertex-isolating for the given vertex set. To locate the vertices of the set, we simply need to identify all intersections of \( H(s, P) \) (diagonal pink dashed lines) with \( A \).

### 4.2 Higher-Dimensional Simplex Reconstruction

In this section, we focus on methods for finding higher-dimensional simplices, assuming that zero-dimensional simplices have already been found.

#### 4.2.1 Computing k-Indegree

The key to determining whether a simplex exists is the \( k \)-indegree of a potential simplex, \( \sigma \), which is the count of \( k \)-dimensional cofaces of \( \sigma \) contained in \( K \) occurring at the same height as \( \sigma \) in a particular direction. Importantly, these cofaces need not be proper, so for the particular case when \( k = \dim(\sigma) \), we count 1 if \( \sigma \in K \), and 0 if \( \sigma \not\in K \).

**Definition 21 (k-Indegree for Simplex).** Let \( K \) be a simplicial complex GP-immersed in \( \mathbb{R}^d \) and let \( \sigma \subset \mathbb{R}^d \) be a simplex such that \( \text{verts}(\sigma) \subseteq K_0 \). Furthermore, let \( s \in S^{d-1} \) be a direction \( K_0 \)-perpendicular to \( \text{aff}(\sigma) \). Then, the \( k \)-indegree of \( \sigma \) in direction \( s \) is the number of \( k \)-dimensional cofaces of \( \sigma \) contained in \( K \) that have the same height as \( \sigma \) in direction \( s \).

If a direction \( s \) is \( K_0 \)-perpendicular to \( \text{aff}(\sigma) \), all zero-simplices of \( \sigma \) are at the same height in direction \( s \). However, not all \( k \)-simplices at this height contribute to the \( k \)-indegree of \( \sigma \), as shown in Fig. 3. Thus, if we only use diagrams that are \( K_0 \)-perpendicular to a simplex, we may overcount \( k \)-indegree.
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Fig. 3  Computing the three-indegree for a two-simplex (triangle) in $\mathbb{R}^4$. The simplex $\sigma$ is shown in dark gray. The direction $s \in S^3$ is orthogonal to $\operatorname{aff}(\sigma)$ such that all other vertices shown are below $\sigma$ (note that $s = s_\sigma$ from Definition 17(1)). Although the three-indegree of $\sigma$ is one, the VPD in direction $s$ sees three tetrahedron at the same height as $\sigma$. Recursively using the three-indegree of all faces of $\sigma$ in tilted directions (that is, directions given in Definition 17(2)), the three-indegree of $\sigma$ can be defined by subtracting the indegrees of faces of $\sigma$ in specific directions ($3 - 1 - 1 = 1$) given in Equation (9).

To correctly compute the $k$-indegree of a simplex $\sigma$, we combine information from $\sigma$-isolating directions (Definition 17). Directions that are $\sigma$-isolating are slight perturbations of $s$, isolating faces of $\sigma$, so that we can check for $k$-simplices at the height of $\sigma$ that are “attached” to $\sigma$, but are not actually cofaces of $\sigma$ (see Fig. 3 for an illustration of this idea).

Algorithm 1  ComputeIndeg($\sigma, s, k, \hat{D}[K, S], T = \{\}$)

Input: $\sigma$, a simplex with $\operatorname{verts}(\sigma) \subseteq K_0$, $s \in S$ s.t. $s$ is $K_0$-perpendicular to $\sigma$, $k \in \mathbb{N}$ s.t. $k \geq \dim(\sigma)$, $\hat{D}[K, S]$, where $S$ satisfies the assumptions of Theorem 19, and a table $T$ for memoization.

Output: the $k$-indegree for $\sigma$

1: if $S$ is not simplex-isolating for $\sigma$ then
2:    return 0
3: else
4:  $c \leftarrow$ height of $\sigma$ in direction $s$
5:  $\text{numDeaths} \leftarrow$ number of $(k-1)$-dimensional deaths in $\hat{D}(h_s)$ at height $c$
6:  $\text{numBirths} \leftarrow$ number of $k$-dimensional births in $\hat{D}(h_s)$ at height $c$
7:  $\text{doubleCounts} \leftarrow 0$
8:  for $\tau \prec \sigma$ in non-descending order by dimension do
9:    $W \leftarrow \operatorname{verts}(\sigma) \setminus \operatorname{verts}(\tau)$
10:   $s_\tau \leftarrow$ a $(K_0, \operatorname{verts}(\sigma), W, s)$-perturbation
11:   if $T[\tau]$ was not computed yet then
12:      $T[\tau] \leftarrow \text{ComputeIndeg}(\tau, s_\tau, k, \hat{D}[K, S], T)$
13:   end if
14:   $\text{doubleCounts} \leftarrow \text{doubleCounts} + T[\tau]$
15: end for
16: return $\text{numDeaths} + \text{numBirths} - \text{doubleCounts}$
17: end if
To prove the correctness of Algorithm 1, we make the following observation.

**Lemma 22** (Indegree Contributors). Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$. Let $\sigma$ and $\tau$ be potential simplices with $\text{verts}(\tau) \subset \text{verts}(\sigma) \subseteq K_0$, and let $\sigma' \in K$. Let $k \in \mathbb{R}$ such that $k = \dim(\sigma')$. Let $s$ be $K_0$-perpendicular to $\sigma$ and let $s_{\tau}$ be a $(K_0, \sigma, \sigma \setminus \tau, s)$-perturbation. Then, $\sigma'$ contributes to the $k$-indegree of $\tau$ in direction $s_{\tau}$ if and only if $\sigma'$ is at the same height as $\sigma$ in direction $s$ and $\tau = \sigma \cap \sigma'$.

**Proof** Let $f: K \to \mathbb{R}$ ($f_{\tau}: K \to \mathbb{R}$, respectively) be the filter function for direction $s$ ($s_{\tau}$, respectively).

$(\Rightarrow)$ Suppose that $\sigma'$ contributes to the $k$-indegree of $\tau$ in direction $s_{\tau}$. Then, by the definition of $k$-indegree, $\tau \preceq \sigma'$ and $\sigma'$ is at the same height as $\tau$ with respect to direction $s_{\tau}$. Since $s_{\tau}$ is a $(K_0, \sigma, \sigma \setminus \tau, s)$-perturbation, this means $\sigma'$ is at the same height as $\tau$ in direction $s$, and therefore also at the same height as $\sigma$ in direction $s$. Since $\tau \prec \sigma$ by assumption, we have $\tau \preceq \sigma \cap \sigma'$. We must now show that $\sigma \cap \sigma' \subset \tau$.

By contradiction, suppose that $\sigma \cap \sigma' \not\subset \tau$. Then, there exists a vertex $v \in \sigma \cap \sigma'$ such that $v \notin \tau$. Thus, $v \in \sigma \setminus \text{verts}(\tau)$. Since $s_{\tau}$ is a $(K_0, \sigma, \sigma \setminus \tau, s)$-perturbation, we know that $s_{\tau} \cdot v > s_{\tau} \cdot w$ for all $w \in \text{verts}(\tau)$, a contradiction to the claim that $\sigma'$ contributes to the $k$-indegree of $\tau$ in direction $s_{\tau}$. Therefore, $\sigma \cap \sigma' \subset \tau$ as required.

$(\Leftarrow)$ Suppose that $\sigma'$ is at the same height as $\sigma$ in direction $s$ and that $\tau = \sigma \cap \sigma'$. If $\tau = \sigma'$, the claim follows from the definition of $k$-indegree. Then, suppose $\tau \subset \sigma'$. Denote the dimension of $\tau$ by $j$. Since $\sigma'$ is a $k$-simplex, $\tau$ is a $j$-simplex, and $\tau \prec \sigma'$, we can write $\tau = [v_0, v_1, \ldots, v_j]$ and $\sigma' = [v_0, v_1, \ldots, v_k]$ where $v_i \in K_0$. Then,

$$f_{\tau}(\sigma') = \max_{i=0}^{k} f_{\tau}(v_i) = \max \left( \max_{i=0}^{j} f_{\tau}(v_i), \max_{i=j+1}^{k} f_{\tau}(v_i) \right) = \max \left( f_{\tau}(\tau), \max_{i=j+1}^{k} f_{\tau}(v_i) \right).$$

(6)

Since $\sigma'$ is at the same height as $\sigma$ in direction $s$ and $\tau \subset \sigma'$ is also at the same height as $\tau$ in direction $s$, meaning that $f(v_i) \leq f(\tau)$ for all $0 \leq i \leq k$. Since $v_i$ is not in $\tau$ for $i > j$, it must also be the case that $f(v_i) < f(\tau)$ for $i > j$. Since $s_{\tau}$ is a $(K_0, \sigma, \sigma \setminus \tau, s)$-perturbation, by Statement (3) of Definition 15, any vertex below $\tau$ in direction $s$ is also below $\tau$ in direction $s_{\tau}$. Thus, $f_{\tau}(v_i) < f_{\tau}(\tau)$ for all $j < i \leq k$ and

$$\left( \max_{i=j+1}^{k} f_{\tau}(v_i) \right) < f_{\tau}(\tau).$$

Then, by Equation (6), $f_{\tau}(\sigma') = f_{\tau}(\tau)$. This taken together with $\tau \subset \sigma'$ shows that $\sigma'$ contributes to the $k$-indegree of $\tau$ in direction $s_{\tau}$. \qed

Although Lemma 22 allows us to state formally which cofaces contribute to a simplex’s $k$-indegree, the computation of $k$-indegree requires more than a single diagram (see Fig. 3). We use an inclusion-exclusion style argument to compute the $k$-indegree in Algorithm 1. The first time this algorithm is called, we have not yet computed any entries of $T$, a table that keeps track of any contributers to $k$-indegree that would be double counted. We prove the correctness of Algorithm 1 in the following theorem.

**Theorem 23** (Computing $k$-Indegree). Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$. Let $\sigma$ be a potential simplex with $\text{verts}(\sigma) \subseteq K_0$ and
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$s \in \mathbb{S}^{d-1}$ such that $s$ is $K_0$-perpendicular to $\sigma$. Then, for $k \geq \dim(\sigma)$, ComputeIndeg$(\sigma, s, k, \hat{D}[K, S])$ returns the $k$-indegree of $\sigma$ in direction $s$.

**Proof** First, we note that since $S$ is assumed to be simplex-isolating for $K$, if $S$ is not simplex-isolating for $\sigma$, then $\sigma$ is not a simplex of $K$ and thus must have $k$-indegree 0, which is the value returned on Line 2.

Suppose then that $S$ is simplex isolating for $\sigma$. We prove the claim inductively on $j = \dim(\sigma)$. For the base case $j = 0$, consider the zero-simplex $[v]$. Let $h_s : K \rightarrow \mathbb{R}$ be the filter function for direction $s$. We note that this base case is a generalization of [11, Lemma 11]. However, unlike in [11, Lemma 11], we are only making an argument for the $k$-indegree at a single vertex and not all vertices. As such, we can relax the requirement that no two vertices in $K_0$ have the same height in direction $s$ and just require that no vertices in $K_0 \setminus \{v\}$ have the same height in direction $s$ as $v$.

Thus, we have that $k$-indegree of $\sigma$ is equal to the number of $k$-simplices that have height $h_s(v)$, which, by Lemma 11, is

$$|f^{-1}(f(v))| = |\{(a, b) \in \hat{D}_{k-1}(h_s) \text{ s.t. } b = f(v)\}| + |\{(a, b) \in \hat{D}_k(h_s) \text{ s.t. } a = f(v)\}|.$$  

(7)

In Algorithm 1, notice that if $\sigma$ is a single vertex, we do not enter the loop that starts on Line 7. Thus, the return value is exactly the number given in Equation (7).

For the inductive assumption, let $j \geq 0$. We assume that Algorithm 1 returns the $k'$-indegree of $\tau$ in direction $s$, for all $\tau \in K_j$ and all $k' \geq j$.

For the inductive step, let $\dim(\sigma) = j + 1$. Let $k \geq j + 1$. Now, we compute the $k$-indegree of $\sigma$ in direction $s$. Using Lemma 11, we know that the number of $k$-simplices with height $h_s(\sigma)$ in direction $s$ is:

$$\delta := |\{(a, b) \in \hat{D}_{k-1}(h_s) \text{ s.t. } b = f(\sigma)\}| + |\{(a, b) \in \hat{D}_k(h_s) \text{ s.t. } a = f(\sigma)\}|.$$  

(8)

Let $F_\sigma$ denote this set of simplices, let $\sigma' \in F_\sigma$, and let $\tau \prec \sigma$. Suppose that $s_\tau$ is a $(K_0, \text{verts}(\sigma), \text{verts}(\sigma \setminus \tau), s)$-perturbation. By Lemma 22, the $k$-simplex $\sigma'$ contributes to the $k$-indegree of $\tau$ in direction $s_\tau$ if and only if $\tau = \sigma \cap \sigma'$.

Then, we can isolate each face $\tau$ of $\sigma$ and compute the $k$-indegree of $\tau$ using Equation (8), then add or subtract it from the $k$-indegree of $\sigma$, alternating by dimension of $\tau$. This ensures that no coface of $\tau \prec \sigma$ adds to the $k$-indegree of $\sigma$. Formally, this is seen in the equation for the $k$-indegree of $\sigma$:

$$\delta - \sum_{\tau \prec \sigma} \delta_\tau,$$  

(9)

where $\delta_\tau$ is the $k$-indegree of $\tau$ in the corresponding tilted directions. In Algorithm 1, numDeaths + numBirths is equal to $\delta$, and the values $\delta_\tau$ are computed in Line 11 of Algorithm 1. Thus, the return value matches Equation (9). \qed

4.3 Proof of Theorem 19 and Its Corollaries

Using the results from the previous subsection, we arrive at Algorithm 2 that fully reconstructs a GP-immersed simplicial complex.

**Theorem 24** (Simplicial Complex Reconstruction). Let $K$ be a $\kappa$-dimensional simplicial complex GP-immersed in $\mathbb{R}^d$, such that $\kappa \leq d - 1$. Let $S \subset \mathbb{S}^{d-1}$ satisfy the assumptions of Theorem 19 (that is, $(K, S)$ is vertex- and simplex-isolating). Then, Algorithm 2 reconstructs $K$. 
Algorithm 2 ReconstructComplex($\tilde{D}[K,S]$)

**Input:** $\tilde{D}[K,S]$, where $S$ is vertex- and simplex-isolating (Definitions 16 and 17).

**Output:** simplicial complex $K$.

1. $K_0 \leftarrow$ vertices of $K$, as found using the methods of [11, Theorem 9]
2. for $V \subseteq K_0$ with $1 < |V| \leq d$ and in non-decreasing size of $V$ do
3. \hspace{1em} $k \leftarrow |V| - 1$
4. \hspace{1em} if $V \setminus \{v_i\} \in K_{k-1}$ for all $v_i \in V$ and there exists a direction $s \in S$ that is $K_0$-perpendicular to $\sigma$ then
5. \hspace{2em} if ComputeIndeg($V, s, k, \tilde{D}[K,S], T = \emptyset$) = 1 then
6. \hspace{3em} Add $V$ to $K_k$
7. \hspace{2em} end if
8. \hspace{1em} end if
9. end for
10. return $K_0 \cup K_1 \cup \ldots \cup K_κ$

**Proof** We begin by reconstructing $K_0$ on Line 1 using the methods of [11, Theorem 9]. Algorithm 2 then iterates over all subsets of vertices $V \subset K_0$. We do not yet know if the simplex defined by $V$ is in $K$. Since sets are included in non-decreasing size, $K_{k-1}$ is finalized by the time $V$ is considered. The condition that the boundary of the simplex defined by $V$ is contained in $K_{k-1}$ is checked on Line 4. Since $\tilde{D}[K,S]$ is simplex-isolating, if $V$ defines a $k$-simplex of $K$, the set $S$ will contain a direction $s$ that is $K_0$-perpendicular to aff($V$). Thus, if there is no such direction, we know $V$ does not define a simplex of $K$. If there is such a direction, by Theorem 23, ComputeIndeg($V, s, k, \tilde{D}[K,S], T = \emptyset$) (Algorithm 1) returns the number of $k$-simplices at the height of aff($V$) that contain the simplex defined by $V$ as a face; since $k = |V| - 1$, this is either 0 if $V$ does not define a $k$-simplex of $K$, or 1 if $V$ does define a $k$-simplex of $K$. In the latter case, we add $V$ to $K_k$. Since we iterate over all subsets of $K_0$, the algorithm eventually finds all simplices. □

This theorem concludes the proof of Theorem 19. An immediate corollary is as follows:

**Corollary 25** (Subcomplexes are Represented). Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$. Let $S \subset \mathbb{S}^{d-1}$ satisfy the assumptions of Theorem 19. Let $K'$ be any subcomplex of $K$. Then $\tilde{D}[K',S]$ is a faithful discretization of $VPHT(K')$.

### 5 Faithful Discretizations of Other Transforms

In Section 4, we showed that a set of PDs that is vertex- and simplex-isolating for some simplicial complex $K$ is a faithful discretization of $VPHT(K)$. Here, we show that the properties of being vertex- and simplex-isolating are also sufficient to form faithful discretizations of other topological transforms, namely, any verbose dimension-returning transform (such as the VBFT), as well as the
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The extension of Section 4 to the former is nearly immediate. The extension to the VECFT is also reasonably straightforward, but requires more careful treatment.

### 5.1 Faithful Discretization of the VBFT and Other Verbose Dimension-Returning Transforms

The results of Section 4 rely on the heights and dimension of simplices only.\(^2\) Crucially, this means the same set of directions that faithfully discretize the VPHT can faithfully discretize any topological transform that contains the heights and dimension of simplices (e.g., the VBFT). We call such transforms *verbose dimension-returning transforms*.

**Corollary 26** (A Faithful Discretization of the VBFT and Other Verbose Dimension-Returning Transforms). Let \(K\) be a simplicial complex GP-immersed in \(\mathbb{R}^d\). Let \(S \subset \mathbb{S}^{d-1}\) satisfy the assumptions of Theorem 19. Then, the set \(S\) parameterizes a faithful discretization of the VBFT and other dimension-returning transforms for \(K\).

With small adaptations, our results can also be extended to the VECFT, despite the fact that the VECFT is not dimension-returning. We include details of this adaptation in the following subsection.

### 5.2 Faithful Discretization of the VECFT

In practice, verbose Euler characteristic functions (VECFs) are often preferred to VPDs due to the existence of faster algorithms for computing the functions. For instance, [28] gives an algorithm to compute the ECF in linear time. This faster computation time makes the (V)ECFT a good candidate for processing large amounts of data. For example, in [29], CT scans of barley seeds and spikes are considered in discrete “slices” with respect to 158 directions, creating a sampling of the associated Euler characteristic functions and therefore also sampling the ECFT. Through this sampling, they are able to use machine learning to distinguish between 28 barley phenotypes, agreeing with a biologically based classification.

In Corollary 26, we observed that a direction set \(S\) that is vertex- and simplex-isolating for a simplicial complex \(K\) not only yields a faithful discretization of the VPHT, but also of the VBFT as well as other verbose dimension-returning transforms. However, unlike VPDs or VBFs, VECFs are not dimension-returning. In particular, this means that we cannot directly compute \(k\)-indegree as in Algorithm 1 when using VECFs. However, we can adapt the notion of \(k\)-indegree to a notion of even- and odd-indegree. We

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\(^2\)In fact, if the vertex locations are known, then we are only using the total order of the vertices (with respect to the directions in \(S\)). Specifically, for each direction \(s\) and each vertex \(v \in K_0\), we define an equivalence relation on \(K\) such that the equivalence class for \(v\) is: \([v]_s := \{\sigma \in K \text{ s.t. } h_s(\sigma) = h_s(v)\}\). The set of equivalence classes is totally ordered by the height in direction \(s\).
will then show how to use even- and odd-indegree to reconstruct all higher-dimensional simplices, therefore showing that a set of directions that is vertex- and simplex-isolating gives us a faithful discretization of the VECFT. That is, we will show that the set of diagrams described in Section 3 that faithfully discretize the VPHT also faithfully discretize the VECFT.

The remainder of this section serves to prove this claim (Theorem 30). Our main tool is the following adaptation of the definition of $k$-indegree to even/odd-indegree, where, rather than counting simplices of a particular dimension, we count even-dimensional (or odd-dimensional) simplices.

**Definition 27 (Even/Odd-Indegree for Simplex).** Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$ and let $\sigma \subseteq \mathbb{R}^d$ be a simplex such that $\text{verts}(\sigma) \subseteq K_0$. Let $s \in S^{d-1}$ be a direction $K_0$-perpendicular to $\text{aff}(\sigma)$. Then, the even-indegree of $\sigma$ in direction $s$ (respectively, odd-indegree of $\sigma$ in direction $s$) is the number of even-dimensional (respectively, odd-dimensional) cofaces of $\sigma$ contained in $K$ that have the same height as $\sigma$ with respect to the direction $s$.

Just as we did with the notion of $k$-indegree, we need to keep track of how many times we double-count faces of a simplex in order to correctly find the even/odd-indegree of that simplex. The following lemma helps us prove the correctness of this calculation.

**Lemma 28 (Even/Odd-Indegree Contributors).** Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$. Let $\tau$ and $\sigma$ be potential simplices with $\text{verts}(\tau) \subseteq \text{verts}(\sigma) \subseteq K_0$, and let $\sigma' \in K$. Suppose $\dim(\sigma')$ is even (respectively, odd). Let $s$ be a direction that is $K_0$-perpendicular to $\sigma$ and let $s_\tau$ be a direction that is a $(K_0, \sigma, \sigma' \setminus \tau, s)$-perturbation. Then, $\sigma'$ contributes to the even-indegree (respectively, odd-indegree) of $\tau$ in direction $s_\tau$ if and only if $\sigma'$ is at the same height as $\sigma$ in direction $s$ and $\tau = \sigma \cap \sigma'$.

We omit a proof, as it is identical to the proof of Lemma 22. We now present Algorithm 3, which computes even-indegree. The case for computing odd-indegree is nearly identical. We assert the correctness Algorithm 3 in the following theorem.

**Theorem 29 (Computing Even/Odd-Indegree).** Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$. Let $\sigma$ be a simplex with $\text{verts}(\sigma) \subseteq K_0$ and $s \in S^{d-1}$ such that $s$ is $K_0$-perpendicular to $\sigma$. Then, for $k \geq \dim(\sigma)$, $\text{EvenIndeg}(\sigma, s, \tilde{\chi}(K, S))$ returns the even-indegree of $\sigma$ in direction $s$.

**Proof** The proof is similar to the proof of Theorem 23. Since $S$ is assumed to be simplex-isolating for $K$, if $S$ is not simplex-isolating for $\sigma$, then $\sigma$ is not a simplex of $K$ and thus must have even-indegree 0, which is the value returned on Line 2.

Suppose then that $S$ is simplex isolating for $\sigma$. We prove the claim inductively on $j = \dim(\sigma)$ and let $h_s : K \rightarrow \mathbb{R}$ denote the filter function for direction $s$. For the
Algorithm 3 EvenIndeg($\sigma, s, \chi[K, S], T = \{\})$

**Input:** $\sigma$ a simplex with $\text{verts}(\sigma) \subseteq K_0$, $s \in S$ s.t. $s$ is $K_0$-perpendicular to $\sigma$, $\chi[K, S]$, where $S$ satisfies the assumptions of Theorem 19, and a table $T$ for memoization.

**Output:** the even-indegree for $\sigma$

1. if $S$ is not simplex-isolating for $\sigma$ then
2. return 0
3. else
4. $c \leftarrow$ height of $\sigma$ in direction $s$
5. allEven $\leftarrow$ number of even-dimensional events in $\chi(h_s)$ at height $c$
6. doubleCounts $\leftarrow 0$
7. for $\tau \prec \sigma$ in non-descending order by dimension do
8. $W \leftarrow \text{verts}(\sigma) \setminus \text{verts}(\tau)$
9. $s_\tau \leftarrow$ a $(K_0, \text{verts}(\sigma), W, s)$-perturbation
10. if $T[\tau]$ was not computed yet then
11. $T[\tau] \leftarrow \text{EvenIndeg}(\tau, s_\tau, \chi[K, S], T)$
end if
12. doubleCounts $\leftarrow$ doubleCounts $+ T[\tau]$
end for
13. return allEven $-$ doubleCounts
end if

base case ($j = 0$), consider $\sigma = [v]$, a single zero-simplex. Since $s$ is $K_0$-perpendicular to $\sigma$, the even-indegree of $v$ is exactly equal to the number of even-simplices that have height $s \cdot v$, which can immediately be read off of VECF($s$). In Algorithm 3, notice that since the vertex $v$ does not have any proper faces, we do not enter the loop that starts on Line 7. Thus, the return value is exactly the number of even simplices with height $s \cdot v$.

For the inductive step, let $j \geq 0$. We assume that Algorithm 3 returns the even-indegree of $\tau$ in direction $s$, for all $\tau \in K_j$. For the inductive step, let $\dim(\sigma) = j + 1$. Now, we compute the even-indegree of $\sigma$ in direction $s$. Let $F_\sigma$ denote the set of even-dimensional simplices with height $h_s(\sigma)$ in direction $s$, whose cardinality is reported in VECF($s$). Let $\sigma' \in F_\sigma$, and let $\tau \prec \sigma$. Suppose that $s_\tau$ is a $(K_0, \text{verts}(\sigma), \text{verts}(\sigma \setminus \tau), s)$-perturbation. By Lemma 28, $\sigma'$ contributes to the even-indegree of $\tau$ in direction $s_\tau$ if and only if $\tau = \sigma \cap \sigma'$.

Then, we can isolate each face of $\tau \prec \sigma$, identify the even-indegree of $\tau$ by the inductive assumption, and add or subtract it from the even-indegree of $\sigma$, alternating by dimension of $\tau$. This ensures that no coface of $\tau \prec \sigma$ adds to the even-indegree of $\sigma$. Formally, this is seen in the equation for the even-indegree of $\sigma$

$$\delta - \sum_{\tau \prec \sigma} \delta_\tau,$$  \hspace{1cm} (10)

where $\delta_\tau$ is the even-indegree of $\tau$ in the corresponding tilted directions. In Algorithm 3, allEven is equal to $\delta$, and the values $\delta_\tau$ are computed in Line 11. Thus, the return value of Algorithm 3 matches the value of Equation (10).

The last step of the full process of reconstruction is to note that vertex-reconstruction only relied on the presence of events, not their dimension.
Therefore, vertex-isolating directions can be used with any verbose descriptor to reconstruct $K_0$. Finally, we can give a reconstruction algorithm and state the following main result.

**Algorithm 4** VECFReconstructComplex($\chi[K,S]$)

**Input:** $\chi[K,S]$, where $S$ is vertex- and simplex-isolating (Definitions 16 and 17).

**Output:** simplicial complex $K$.

1. $K_0 \leftarrow$ vertices of $K$, as found using the methods of [11, Theorem 9]
2. for $V \subseteq K_0$ with $1 < |V| \leq d$ and in non-decreasing size of $V$ do
3. 
4. if $V \setminus \{v_i\} \in K_{k-1}$ for all $v_i \in V$ and there exists a direction $s \in S$ that is $K_0$-perpendicular to $\sigma$ then
5. 
6. if $|V| - 1$ is even then
7. 
8. else
9. end if
10. if $x \geq 1$ then
11. Add $V$ to $K_k$
12. end if
13. end if
14. end for
15. return $K_0 \cup K_1 \cup \cdots \cup K_k$

**Theorem 30** (Sufficient Conditions for Faithful Discretization). Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$ such that $\dim(K) = \kappa < d$, and let $S \subset S^{d-1}$ such that $(K,S)$ is vertex- and simplex-isolating. Then, Algorithm 4 reconstructs $K$, so $\chi[K,S]$ is a faithful discretization of $\text{VECFT}(K)$.

**Proof** The initial steps of the proof follow the reasoning given in Theorem 24; we reconstruct $K_0$, and when we consider some $V$ with $|V| - 1 = k$, we have already reconstructed $K_{k-1}$. First, suppose $k$ is even. Since $\hat{\mathcal{D}}[K,S]$ is simplex-isolating, if $V$ defines a simplex of $K$, the set $S$ contains a direction $s$ that is $K_0$-perpendicular to $\text{aff}(V)$. Thus, if there is no such direction, we know $V$ does not define a simplex of $K$. If there is such a direction, by Theorem 29, $\text{EvenIndeg}(\sigma,s,\chi[K,S],T = \{\})$ (Algorithm 3) returns the number of even-simplices at the height of $\text{aff}(V)$ that contain $\sigma$ as a face; if this value is 0, we know $V$ is not a simplex of $K$. If this value is 1 or greater, we know $V$ does define a simplex of $K$, and we add $V$ to $K_k$. A nearly identical argument holds for the case that $k$ is odd. Since we iterate over all subsets of $K_0$, the algorithm eventually finds all simplices.

Finally, since we have shown VECFReconstructComplex($\chi[K,S]$) reconstructs $K$, we know $\chi[K,S]$ is a faithful discretization of $\text{VECFT}(K)$. □
6 Explicitly Building a Faithful Set

In this section, given a simplicial complex \( K \) GP-immersed in \( \mathbb{R}^d \), we provide algorithms to explicitly construct a set of directions \( S \subseteq \mathbb{S}^{d-1} \) so that \( (K, S) \) is vertex- and simplex-isolating.

6.1 Auxiliary Constructions

This section describes the methods that are used to compute the directions in Theorem 39: how to use Gram-Schmidt to find a direction orthogonal to a given plane, how to find a set of points that span a given affine subspace, how to “tilt” one direction towards another direction while controlling the order of a given set of points, and how to use tilt in order to “pop” a subset of vertices off of a given hyperplane.

**Computing a Perpendicular Direction**

We often compute directions orthogonal to the affine plane spanned by some set of \( k + 1 \) points. To do so, we use Gram-Schmidt orthogonalization to first find \( k \) vectors in the space spanned by the points, and then perform Gram-Schmidt orthogonalization on the standard basis vectors until we find one that is not in the space spanned by the points. The result is the orthogonal vector that we desire. While intuitive, the explicit formulation that we need is not easily accessible from the literature, so we derive the formulation here.

**Algorithm 5 Perp\((P)\)**

**Input:** \( P = \{p_0, p_1, \ldots, p_k\} \subset \mathbb{R}^d \), a set of affinely independent points such that \( k < d \).

**Output:** \( s \in \mathbb{S}^{d-1} \), a direction that is orthogonal to \( \text{aff}(P) \).

1. \( A \leftarrow \) the \( d \times k \) matrix containing \( p_i - p_0 \) in column \( i \) for \( 1 \leq i \leq k - 1 \)
2. \( \{b_i\}_{i=1}^k \leftarrow \) basis vectors for the space spanned by \( A \)
   \( \triangleright \) QR-decomposition
3. \( i \leftarrow 0 \)
4. \( s \leftarrow 0 \)
5. while \( s = 0 \) do
6. \( s \leftarrow \) the output of Gram-Schmidt on \( \{b_i\}_{i=1}^k \) with \( e_i \)
7. \( i \leftarrow i + 1 \)
8. end while
9. return \( s \)

**Lemma 31 (Computing a Perpendicular Direction).** Let \( P \subset \mathbb{R}^d \) be a point set in general position with \( |P| \leq d \). Then, the output of Algorithm 5 is a direction \( s \in \mathbb{S}^{d-1} \) orthogonal to \( \text{aff}(P) \) and runs in \( \Theta(d^3) \) time.
Proof Denote the points of $P$ by $p_0, p_1, \ldots, p_k$. In Line 1, we define $A$ as the $d \times k$ matrix consisting of $p_i - p_0$ in column $i$. By Theorem 7.1 in [30], $A$ has a reduced QR-factorization in which the columns of $Q$ are an orthonormal basis for the space spanned by $A$, or aff($P$). We find $Q$ using $k$ iterations of Gram-Schmidt on Line 2.

Then, in the loop at Lines 5–8, we iteratively test the result of performing Gram-Schmidt on the columns of $Q$ and a basis vector. The result is zero if the basis vector is contained in aff($P$). As soon as the result is nonzero, (which has to happen since aff($P$) has positive codimension, so can’t be spanned by all $e_i$s), we have found a vector orthogonal to aff($P$), and the value returned on Line 9 is as desired.

Building the matrix $A$ and normalizing $s$ takes constant time. Finding the reduced QR-factorization takes $k$ iterations of Gram-Schmidt, (Theorem 8.1, [30]) and finding the final vector that is orthogonal to the space spanned by $P$ requires at most $d$ iterations of Gram-Schmidt. Since each iteration of Gram-Schmidt takes $O(dk)$ time, we see that the total runtime is $\Theta(d^3)$.

□

The ability to find a direction perpendicular to some set of points using Algorithm 5 is used in Algorithm 8. Before introducing Algorithm 8, we first provide two additional algorithms that are utilized in Algorithm 8.

**Plane Filling**

Given a point set $P \subset \mathbb{R}^d$ of $k$ affinely independent points and a direction $s$, Algorithm 6 finds a complementary set of points $P'$ such that aff($P' \cup P$) has only two perpendicular directions, $s$ and $s'$. In other words, we find enough points in $\mathbb{R}^d$ so that they, along with the original point set $P$, “fill” the plane.

We do so by first considering a matrix $A$ that describes the equation of the $(d - 1)$-plane orthogonal to $s$ containing the points of $P$. Recall that the left null space of $A$ is the space of all vectors $n$ such that $n^TA = 0$. Thus, for such a vector $n$, the points $n - p_0$ are also in the plane described by $A$. We are able to find $d - |P|$ of these vectors and corresponding points by computing a basis of the left null space, since this space is $(d - |P|)$-dimensional.

**Lemma 32** (Plane Filling). Algorithm 6 is correct and runs in $\Theta(d^3)$ time.

Proof We first prove the runtime of this algorithm by analyzing what is done in each line of the algorithm. First, we initialize $P'$ to the empty set. Then, we construct a matrix $A$ (Line 2) containing $p_i - p_0$ in column $i$ for the first $d - k$ columns, and the vector $s$ in the last column, which takes $\Theta(kd)$ time. As the points in $P$ are affinely independent and $s$ is orthogonal to aff($P$), the dimension of the column space of the matrix $A$ (Line 2) is $k$, which means that there are $d - k$ vectors in its nullspace. We find the nullspace by first finding a basis for the space spanned by $A$ in Line 3, via a full QR-decomposition [30], and then by using Gram-Schmidt in Lines 5–13, taking $\Theta(d^3)$ time. Line 7 always has at least $d - k$ nonzero outputs, and we stop once we have $d - k$ directions perpendicular to aff($P$). We then iteratively add points to $P'$ in Line 9. Computing the full QR-decomposition and performing Gram-Schmidt dominates the algorithm; hence, the running time for Algorithm 6 is $\Theta(d^3)$.

Finally, for correctness, we prove that the output of Algorithm 6 has the desired properties. By construction, we have $|P'| = d - |P|$, i.e., dim(aff($P' \cup P$)) = $d - 1$. To
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Algorithm 6 PlaneFill\((P, s)\)

**Input:** \(P = \{p_0, p_1, \ldots, p_k\} \subset \mathbb{R}^d\), a set of affinely independent points such that \(k < d\); \(s \in S^{d-1}\), a direction that is orthogonal to aff\((P)\)

**Output:** a set \(P' \subset \mathbb{R}^d\) of \(d - |P|\) points at the same height as \(P\) with respect to direction \(s\) such that \(\text{dim(aff}(P' \cup P)) = d - 1\).

1: \(P' \leftarrow \emptyset\)
2: \(A \leftarrow \) the \(d \times k\) matrix containing \(p_i - p_0\) in column \(i\) for \(1 \leq i \leq k - 1\) and \(s\) in the \(k\)th column
3: \(\{b_i\}_{i=1}^k \leftarrow \) basis vectors for the space spanned by \(A\)
4: \(j \leftarrow 0\)
5: for \(i\) from 1 to \(d\) do
6:   while \(j < d - k\) do
7:     \(s_j \leftarrow \) the output of Gram-Schmidt on \(\{b_i\}_{i=1}^k\) with \(e_i\)
8:     if \(s_j \neq 0\) then
9:       \(P' \leftarrow P' \cup \{s_j + p_0\}\)
10:      \(j \leftarrow j + 1\)
11:   end if
12: end while
13: end for
14: return \(P'\)

show each point in \(P'\) is at the same height as \(P\), it suffices to show that \(s \cdot p_0 = s \cdot p_i'\) for all \(p_i' \in V'\). Since each \(s_j\) considered in Line 9 is in the basis for the left null space of \(A\) and \(s\) is a column of \(A\), all such vectors \(s_j\) are orthogonal to \(s\). Then, indeed, for all \(p_i' \in V'\), we have \(s \cdot p_i' = s \cdot (s_j + p_0) = s \cdot p_0\). □

**Tilting**

In the Algorithm 7, we find a direction that is a slight tilt of one input direction towards another, so that no vertex orders change. First, we explain the geometric intuition of the algorithm. Let \(S\) be the set of \(n\) line segments in \(\mathbb{R}^2\):

\[
S := \left\{ (0, s \cdot p), (1, s' \cdot p) \right\}_{p \in P}.
\]

Each line segment in \(S\) represents a linear interpolation between the points \((0, s \cdot p)\) and \((1, s' \cdot p)\), which correspond to the heights in directions \(s\) and \(s'\) of each point in \(P\). Moreover, we can parameterize each line segment in \(S\) as \((1-t)s \cdot p + ts' \cdot p\) for \(t \in [0, 1]\) (see the grey lines in Fig. 4). Then, vertical cross sections record vertex heights with respect to some direction that is an interpolation of \(s\) and \(s'\). We want to identify a particular \(t_0 > 0\) such that the ordering of the heights of points in direction \(s_t = (1-t_0)s + t_0s'\) is consistent with the ordering of the points in direction \(s\). Notice that no swapping of point order can occur before the intersection of black lines in Fig. 4; thus, we choose
Fig. 4 The solid grey lines in the figure above indicate the changing heights of points as we swing direction $s$ towards $s'$. Although we do not explicitly compute the grey lines, we know by simple geometry that no intersection of grey lines (and, in particular, no swapping of point orders) occurs before the $t$-value $t_*$, which corresponds to the intersection of the closest pairwise heights of points on the left and the extremal heights of points on the right, as indicated by the black lines. Since there are no crossings of line segments before $\frac{1}{2}t_*$, there is therefore no change in the order of points with respect to direction $s_t = (1 - \frac{1}{2}t_*)s + \frac{1}{2}t_*s'$.

Algorithm 7 Tilt$(s, s', P)$

Input: $P \subset \mathbb{R}$ finite; $s, s' \in \mathbb{S}^{d-1}$.
Output: $s_t \in \mathbb{S}^{d-1}$, a direction satisfying Statements (1)-(3) of Lemma 33.

1. $h_1, h_2 \leftarrow$ heights of points that are closest with respect to $s$ such that $h_1 \neq h_2$
2. $h'_1, h'_2 \leftarrow$ minimum and maximum heights of points with respect to $s'$
3. $t_* \leftarrow$ the solution to $(1 - t)h_1 + th'_2 = (1 - t)h_2 + th'_1$
4. return $(1 - \frac{1}{2}t_*)s + \frac{1}{2}t_*s'$

Lemma 33 (Tilt). Let $P \subset \mathbb{R}^d$ be a finite set. Let $s, s' \in \mathbb{S}^{d-1}$. Then, using Algorithm 7, we can compute a third direction $s_t = \text{Tilt}(s, s', P)$ in $\Theta(|P| \log |P| + d)$ time such that the following properties holds for all points $p_1, p_2 \in P$:

1. If $p_1$ is strictly above (below) $p_2$ with respect to direction $s$, then $p_1$ is strictly above (below, respectively) $p_2$ with respect to direction $s_t$.
2. If $p_1$ and $p_2$ are at the same height with respect to direction $s$ and $p_1$ is strictly above (below) $p_2$ with respect to direction $s'$, then $p_1$ is strictly above (respectively, below) $p_2$ with respect to direction $s_t$.
3. If $p_1$ is at the same height as $p_2$ with respect to both directions $s$ and $s'$, then $p_1$ and $p_2$ are at the same height with respect to direction $s_t$.

Proof First let $h_1, h_2$ be the heights of points in $P$ with the closest unequal heights in direction $s$, and let $h'_1, h'_2$ be the heights of points in $P$ with the extreme heights in
direction $s'$, as in Lines 1 and 2. Consider the lines between segments connecting $h_1$ to $h_2'$ and $h_2$ to $h_1'$; they intersect at some point $i$ that is at least as close to the left as the leftmost non-zero intersection of all linear interpolations of point heights with respect to directions $s$ and $s'$ (see Fig. 4). Let $t_*$ denote the first coordinate of $i$ as on Line 3. Since $\frac{1}{2}t_* < t_*$, segments in the interval $(0, \frac{1}{2}t_*)$ of linear interpolations of point heights must not have any crossings, and so the ordering of points that have unique heights with respect to $s$ is maintained, i.e., we see that $s_t$ satisfies Statement (1). Furthermore, if points have the same height with respect to direction $s$, they correspond to an intersection at zero of heights in the linear interpolation of point heights, meaning that $s_t$ orders points equivalently to how $s'$ orders the points, satisfying Statement (2) and Statement (3).

To find $h_1, h_2, h_1', h_2'$, we can sort the heights of points in directions $s$ and $s'$ in $\Theta(|P| \log |P|)$ time. Finding the intersection $t_*$ from the resulting two segments takes constant time, and returning $s_t = (1 + \frac{1}{2}t_*)s + \frac{1}{2}t_*s'$ takes $\Theta(d)$ time. Thus, the total runtime is $\Theta(|P| \log |P| + d)$. □

**Tilting to Pop**

Given a point set $P \subset \mathbb{R}^d$ in general position, two sets $W \subseteq V \subseteq P$, and a direction $s$ that is $P$-perpendicular to $V$, Algorithm 8 calculates a direction that is close to $s$ that “pops” all of the vertices in $W$ either entirely above or entirely below the vertices in $V \setminus W$.

Algorithm 8 TiltToPop($P, V, W, s$)

**Input:** $P, V, W$ point sets in $\mathbb{R}^d$ such that $P$ is in general position, $W \subseteq V \subseteq P$, and $|V| \leq d$; $s$, a direction $P$-perpendicular to $V$.

**Output:** $s_* \in S^{d-1}$, a direction that is a $(P, V, W, s)$-perturbation.

1: $V' \leftarrow \text{PlaneFill}(V, s)$ ▷ Algorithm 6
2: $s' \leftarrow \text{Perp}(\text{aff}(V' \cup (V \setminus W) \cup \{W - s\}))$ ▷ Algorithm 5
3: if $W$ is below $V \setminus W$ with respect to $s'$ then
4:    $s' \leftarrow -s'$
5: end if
6: return $s_* = \text{Tilt}(s, s', P \cup V')$ ▷ Algorithm 7

The algorithm begins in Line 1 by finding a set of points $V' \subset \mathbb{R}^d$ such that $\text{aff}(V' \cup V)$ is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$ in $\Theta(d^3)$ time by Lemma 32. The additional points help us control which way to tilt $s$. In particular, the direction $s'$ (computed on Line 2 in $\Theta(d^3)$ by Lemma 31) is perpendicular to $\text{aff}(V' \cup (V \setminus W))$, and is the direction towards which we can tilt in order to “pop” $W$ off of the plane orthogonal to $s$ at height $V \setminus W$. Since there are two choices for $s'$ in Line 2, the if statement in Lines 3 and 4 ensures that the direction is the one such that $W$ is above $V \setminus W$, and this is completed in $\Theta(d)$ time. Finally, we return $s_*$ on Line 6 using Algorithm 7, taking $\Theta(|P| \log |P| + d)$ by Lemma 33.
Lemma 34 (Tilting to Pop). Let $P$ be a finite point set in $\mathbb{R}^d$ in general position. Let $W \subseteq V \subseteq P$ and $s \in S^{d-1}$ such that $s$ is perpendicular to aff($V$). Then, Algorithm 8 calculates $\text{TiltToPop}(P, V, W, s)$ in $\Theta(|P| \log |P| + d^3)$ time, and the output is a $(P, V, W, s)$-perturbation.

Proof. The runtime was justified in the paragraph above detailing the algorithm. Recall what it means for the returned direction to be a $(P, V, W, s)$-perturbation:

1. The points in $W$ are above $V \setminus W$ with respect to direction returned.
2. For all $p \in P \setminus V$, $p$ is strictly above (below) the height of $V \setminus W$ with respect to the direction returned if and only if it is strictly above (below, respectively) $V$ with respect to $s$.
3. The direction returned is $P$-perpendicular to $V \setminus W$.

Before proving Statements (1)-(3), we establish three properties of $s'$ returned on Line 2. First, since aff($V' \cup (V \setminus W)$) has codimension one and the points are in general position, $s'$ is automatically $P$-perpendicular to this space.

Next, we show that all vertices of $W$ have the same height with respect to $s'$. Suppose $w_1, w_2 \in W$. Since $(w_1 - s), (w_2 - s) \in \text{aff}(V' \cup (V \setminus W) \cup \{W - s\})$, we have $s' \cdot (w_1 - s) = s' \cdot (w_2 - s)$. Then by adding $s' \cdot s$ to both sides, we obtain $s' \cdot w_1 = s' \cdot w_2$, i.e., all points of $W$ are at the same height with respect to $s'$.

Finally, we show $W \not\subseteq \text{aff}(V' \cup (V \setminus W) \cup \{W - s\})$. Suppose not. Then, since both $W$ and $W - s$ are in $\text{aff}(V' \cup (V \setminus W) \cup \{W - s\})$, the vector $s$ is parallel to $\text{aff}(V' \cup (V \setminus W) \cup \{W - s\})$. Furthermore, since both planes contain the same linearly independent set of $(d - 1)$ points, $V' \cup V$, we have $\text{aff}(V' \cup (V \setminus W) \cup \{W - s\}) = \text{aff}(V' \cup V)$. But then $s$ is parallel to $\text{aff}(V' \cup V)$, contradicting the property that $s$ is $P$-perpendicular to $\text{aff}(V' \cup V)$. Thus, we see that $s'$ orders all vertices of $W$ on the same side of $V \setminus W$; on Line 4, we ensure that all vertices of $W$ are specifically above $V \setminus W$ with respect to $s'$, therefore, by Lemma 33(2), all points of $W$ are above the points of $V \setminus W$ with respect to the direction $s_*$ returned on Line 6 and we have shown $s_*$ satisfies Statement (1).

We are now ready to show $s_*$ satisfies Statements (2) and (3). Since both $s$ and $s'$ are perpendicular to aff($V \setminus W$), the output $\text{Tilt}(s, s', P \cup V') = s_*$ is perpendicular to aff($V \setminus W$) by Lemma 33(3). To show that no other vertex of $P$ is at the same height of vertices in $V \setminus W$ with respect to $s_*$, let $p \in P \setminus (V \setminus W)$. If $p \in P \setminus V$, since $p$ is strictly above or below $V \setminus W$ with respect to direction $s$, then by Lemma 33(1), $p$ is strictly above or below $V \setminus W$ with respect to $s_*$, showing Statement (2). If $p \in W$, then it is at the same height as $V \setminus W$ in direction $s$ and above $V \setminus W$ in direction $s'$, thus, by Lemma 33(2), $p$ is above $V \setminus W$ in direction $s_*$ and we have shown that $s_*$ is $P$-perpendicular to $V \setminus W$, so Statement (3) of the current lemma is satisfied, concluding our proof. \hfill \Box

6.2 Building an Explicit Set

Next, we use the algorithms of Section 6.1 to construct directions in Theorem 39 for a given simplicial complex.
Construction 35 (Constructing a Faithful Set). Let $K$ be a simplicial complex $GP$-immersed in $\mathbb{R}^d$ such that $\dim(K) < d$, and let $S$ be the following set of directions constructed iteratively as follows:

1. Initially, let $S$ be the standard basis vectors $(e_1, e_2, \ldots, e_d)$, plus the additional direction $\text{PointIso}(K_0)$.
2. For every maximal $\sigma \in K$,
   
   (a) Let $s = \text{Perp}(\sigma)$ and let $H$ be the set of all vertices with the same height as $\sigma$ in direction $s$, and let $W = \text{PlaneFill}(H, s) \cup (H \setminus \sigma)$ and $V = W \cup \sigma$. Then, add the direction $s_{\sigma} := \text{TiltToPop}(K_0 \cup W, V, W, s)$ to $S$.
   
   (b) For each $\tau \in K$ such that $\tau \prec \sigma$, add the direction $\text{TiltToPop}(K_0, \sigma, \tau, s_{\sigma})$ to $S$.

The remainder of this section shows that Construction 35 forms a faithful discretization.

6.2.1 Directions for Vertices

Constructing a set of directions that faithfully represents a vertex set has been explored in previous work. By [11, Lemma 7], it suffices to construct a set of $d$ linearly independent directions, plus one additional direction so that there are exactly $n_0$ intersections of size $d + 1$ among all associated filtration hyperplanes. However, the construction of this final direction given in [11, Lemma 8] requires stricter general position assumptions to construct the set, namely, that no two vertices share any $e_i$-coordinate for $1 \leq i \leq d$. Here, we provide an algorithm to produce such a $(d + 1)$st direction when our pointset satisfies only the mild general position assumptions described in the “General Position” paragraph of Section 2.1.

Algorithm 9 $\text{PointIso}(P)$

Input: $P \subset \mathbb{R}^d$, a point set in general position
Output: $s \in S^{d-1}$, a direction that uniquely orders the filtration grid of $P$

1: $A \leftarrow$ the filtration grid of $P$ with respect to $\{e_1, e_2, \ldots, e_d\}$
2: $s \leftarrow e_1$
3: for $i$ from 2 to $d$ do
4:   $s \leftarrow \text{Tilt}(s, e_i, A)$
5: end for
6: return $s$

The next lemma proves correctness of Algorithm 9.

Lemma 36 (Correctness of Algorithm 9). Let $P$ be a finite point set in $\mathbb{R}^d$ in general position. Then, $\text{PointIso}(P)$ returns a direction that uniquely
orders the filtration grid of $P$ with respect to $\{e_1, e_2, \ldots, e_d\}$ and runs in $\Theta(d^2|P|^d \log |P|)$ time.

Proof First, we analyze runtime. On Line 4, we call Algorithm 7, which by Lemma 33 takes $\Theta(|A| \log |A| + d)$ time. Since this is called $d-1$ times during the loop on Lines 3–5 and since $|A| = \Theta(|P|^d)$, the total runtime of Algorithm 9 is $\Theta(d(|P|^d \log |P|^d + d)) = \Theta(d^2|P|^d \log |P|)$.

Next, we show Algorithm 9 is correct. Let $\pi_i$ be the standard projection map onto the $(e_1, e_2, \ldots, e_i)$-plane. As on Line 1, let $A$ be the filtration grid of $P$ with respect to $\{e_1, e_2, \ldots, e_d\}$ and note that $A$ is a grid of at most $|P|^d$ points. Let $j$ be the number of times the loop has been completed. We use the loop invariant that, Lines 3–5, $s$ totally orders the points of $A$. We first show this is true before entering the loop. We initialize $s = e_1$ on Line 2. Thus, since $e_1$ totally orders the points of $\pi_1(A)$, the loop invariant is satisfied.

Let $s^i$ denote the $i$th value of $s$ in Algorithm 9 (so $s^1$ is the initial direction defined in Line 2, $s^2$ is the direction updated by tilting towards $e_2$ the first time we encounter Line 3, etc. Note that this means $j = i - 1$.)

Suppose that the loop invariant is true going into the for loop of Lines 3–5. Recall by Lemma 33 that $\text{Tilt}(s^{i-1}, e_i, A)$ produces a direction $s^i$ so that, for all $a_1, a_2 \in A$,

1. If $a_1$ is strictly above (below) $a_2$ with respect to direction $s^{i-1}$, then $a_1$ is strictly above (below, respectively) $a_2$ with respect to direction $s^i$.
2. If $a_1$ and $a_2$ are at the same height with respect to direction $s^{i-1}$ and $a_1$ is strictly above (below) $a_2$ with respect to direction $e_i$, then $a_1$ is strictly above (respectively, below) $a_2$ with respect to direction $s^i$.
3. If $a_1$ is at the same height as $a_2$ with respect to both directions $s^{i-1}$ and $e_i$, then $a_1$ and $a_2$ are at the same height with respect to direction $s^i$.

Since $s^{i-1}$ provided a total order of $\pi_{i-1}(A)$ by assumption and given the statements above, we conclude that $s^i$ totally orders $\pi_i(A)$. Suppose that after the loop terminates, $s^d = \text{PointIso}(P)$, totally orders the points of $\pi_d(A)$. Then, since $\pi_d(A) = A$, the final direction totally orders the points of $A$. Finally, by the runtime analysis, the loop terminates and thus, Algorithm 9 is correct.

Using the previous lemma, we are now able to construct a set of directions that represent the vertex set of a simplicial complex. This is a generalization of [11, Lemma 7 and Theorem 9] and we give a brief restatement of the main idea of the proof.

Lemma 37 (Construction of Step 1 Directions and Vertex Reconstruction). Let $K \subset \mathbb{R}^d$ be a GP-immersed simplicial complex. Then the basis directions $e_1, e_2, \ldots, e_d$, along with $s = \text{PointIso}(K_0)$, are vertex-isolating.

Proof Let $A$ denote the filtration grid of $K_0$ with respect to $\{e_1, e_2, \ldots, e_d\}$. Since $s$ orders the points of $A$ uniquely by Lemma 36, we know by [11, Lemma 7] that the vertices $K_0$ are in one-to-one correspondence with the points $\mathbb{H}(s, K_0) \cap A$. Briefly, this is because $\mathbb{H}(s, K_0)$ has a unique filtration hyperplane passing through each point
of $K_0$. Then, since each point of $K_0$ lies on some point of $A$, and since no hyperplane of $\mathcal{H}(s, K_0)$ passes more than one point of $A$, we have $\mathcal{H}(s, K_0) \cap A = K_0$. □

6.3 Directions for Higher-Dimensional Simplices

Next, we show how auxiliary constructions of Section 6.1 can be used to construct sets of directions that faithfully represent all higher-dimensional simplices. If a simplex $\sigma$ is less than $(d - 1)$-dimensional, the direction returned by $\Perp(\sigma)$ is not guaranteed to be $K_0$-perpendicular to $\sigma$. That is, other vertices may have the same height as $\sigma$ with respect to this direction. Thus, to ensure we have a direction that places $\sigma$ at a unique height, we “pop” off any extra vertices using $\TiltToPop$, returning a tilted direction that is guaranteed to be $K_0$-perpendicular to $\sigma$.

**Lemma 38** (Construction of Step 2(a) Directions). Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$. Let $\sigma$ be a maximal simplex of $K$. Furthermore, let $s = \Perp(\sigma)$, $H$ denote the set of all vertices with the same height as $\sigma$ in direction $s$, $W = \PlaneFill(H, s) \cup (H \setminus \sigma)$, and $V = W \cup \sigma$. Then $\TiltToPop(K_0 \cup W, V, W, s)$ is $K_0$-perpendicular to $\sigma$. Furthermore, this direction can be computed in $O((n_0 + d) \log(n_0 + d) + d^3)$.

*Proof* We first assert that the inputs of $\TiltToPop(K_0 \cup W, V, W, s)$ are valid; note that, by construction, $V$ contains $d - 1$ points in general position, meaning that no other points have the same height as $V$ with respect to $s$. Thus, $s$ is $(K_0 \cup W)$-perpendicular to $V$. Also by construction, we have $W \subseteq V \subseteq K_0 \cup W$. Then, by Lemma 34, the direction returned by $\TiltToPop(K_0, V, W, s)$ is a $(K_0, V, W, s)$-perturbation. In particular, by Statement (1) of Definition 15, this means the direction is $K_0$-perpendicular to $V \setminus W = \sigma$, as desired.

Finding the set $H$ takes $\Theta(n_0)$ time. We compute $s = \Perp(\sigma)$ in $\Theta(d^3)$ time by Lemma 31. We compute $W = \PlaneFill(H, s) \cup (H \setminus \sigma)$, in $\Theta(d^3)$ time by Lemma 32. Finally, we compute $\TiltToPop(K_0 \cup W, V, W, s)$. $\TiltToPop$ has runtime $\Theta((n_0 + |W|) \log(n_0 + |W|) + d^3)$ by Lemma 34. Since $W$ has size $O(d)$, all these operations in total take $O((n_0 + d) \log(n_0 + d) + d^3)$ time. □

Finally, we show that Construction 35 indeed forms a faithful discretization and analyze the size and time complexity. Notice that we have a set of requirements that leads to a set of directions. It is possible that the set of directions has a smaller cardinality than the set of requirements, namely, if a direction satisfies two or more requirements. However, since we are considering an upper bound, we count directions assuming each direction satisfies just one requirement.

**Theorem 39** (Explicit Faithful Discretization). Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$ such that $\dim(K') < d$, and let $S$ be the set of directions from Construction 35: Then, $\tilde{D}[K, S]$ is a faithful discretization of size $O(n2^\kappa + d)$, and $S$ can be computed in $O(\log(n_0 + d)(d^2n_0^d + nd + n\kappa2^\kappa) + n^2d^3)$ time.
Proof By Lemma 36, the directions added to $S$ in Step 1 are vertex-isolating (Definition 16). By Lemma 38, the directions added to $S$ in Step 2(a) are $K_0$-perpendicular to every maximal simplex, satisfying the first condition of being simplex-isolating (Definition 17(1)). By Lemma 34, the directions added to $S$ in Step 2(b) are $(P, V, W, s)$-perturbations, satisfying the second condition of being simplex-isolating (Definition 17(2)). Thus, the directions of Step 2 are simplex-isolating. Since $(K, S)$ is both vertex- and simplex-isolating, by Theorem 19, $(K, S)$ is a faithful discretization.

Now, we analyze size and time bounds. By Lemma 36, the $\Theta(d)$ directions added to $S$ in Step 1 can be computed in time $\Theta(d^2 n_0^d \log n_0)$.

Next, we give bounds for Step 2(a). By Lemma 38, for each maximal $\sigma \in K$, the direction $s_\sigma$ in Step 2(a) can be computed in time $O((n_0 + d) \log (n_0 + d) + d^3)$. Since the total number of maximal simplices is $O(n)$, the total time computing directions in Step 2(a) is $O\left(n((n_0 + d) \log (n_0 + d) + d^3)\right)$.

Given $s_\sigma$ and $\tau \prec \sigma$, by Lemma 34, a single direction in Step 2(b) can be computed in time $\Theta(n_0 \log n_0 + d^3)$. Since for every maximal $i$-simplex of $K$, we compute one direction for each of its proper faces, each $i$-simplex adds a total of $2^{i+1} - 2$ directions in Step 2(b). Letting $\kappa = \dim K$, the number of directions in Step 2(b) is $\Theta(n 2^\kappa)$. Hence, the total time to compute directions in Step 2(b) is $O(n 2^\kappa(n_0 \log n_0 + d^3))$.

Thus, the set $S$ has $O(n 2^\kappa + d)$ directions and can be computed in time $O\left(\log(n_0 + d)(d^2 n_0^d + nd + n n_0 2^\kappa) + n 2^\kappa d^3\right)$. \qed

7 Stability Results for Faithful Discretizations

In this section, we make observations about the stability of our discretization. All proofs are given with reference to the VPHT; stability results for the VBFT and other verbose dimension-returning transforms, as well as stability results for the VECFT are given as corollaries.

Many important observations related to stability can be defined in terms of a stratification of the sphere induced by topological transforms, as has been noted in related work, e.g., [31, 32]. Additionally, the strata have nice combinatorial properties that allow for a sheaf- or cosheaf-theoretic interpretation. The sheaf/cosheaf viewpoint has been championed by many, e.g., [32–34]. The stability of the entire PHT (using height filtrations) is stated in [34, Theorem 4.2] in terms of interleaving distance between sheaves. Roughly, the stratification is defined by dividing the sphere of directions into regions (strata) where all directions within a particular stratum induce the same partial order on vertices (ordered by their height with respect to that direction) (see Fig. 5).

We begin by defining a strata-preserving map, which allows for a clean way to describe important relationships between sets of directions.

**Definition 40 (Strata-Preserving Map).** Given a set $S \subseteq S^{d-1}$ and a simplicial complex $K \subset \mathbb{R}^d$, we say that the map $f : S^{d-1} \to S^{d-1}$ is strata-preserving with respect to $K$ if, for every $s \in S$, the partial ordering of vertices induced by their heights with respect to direction $s$ and $f(s)$ is the same.
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Fig. 5 A two-simplex in $\mathbb{R}^3$ (left) stratifies $S^2$ where each stratum is a region containing all directions that define the same partial order on vertices. Notice that the set of directions perpendicular to any pair of vertices forms a great circle and the two directions perpendicular to $[v_1, v_2, v_3]$ correspond to the two three-way intersections of these great circles.

Observing that Theorem 19 gives us flexibility for choosing which directions parameterize a faithful discretization, the first stability result shows if we perturb those directions in a strata-preserving way, we wind up with another faithful discretization.

**Theorem 41 (Discretization is Robust to Perturbing Parameterization).** Let $K$ be a simplicial complex GP-immersed in $\mathbb{R}^d$, and let $S \subset S^{d-1}$ be such that $(K, S)$ is vertex- and simplex-isolating. If a function $f : S^{d-1} \rightarrow S^{d-1}$ satisfies the following conditions:

1. there are directions $s_1, s_2, \ldots, s_d, s_{d+1} \in S$ so that $f(s_1), f(s_2), \ldots, f(s_d)$ are linearly independent and so that $f(s_{d+1})$ uniquely orders the filtration grid of $K_0$ with respect to $\{f(s_1), f(s_2), \ldots, f(s_d)\}$; and
2. the map $f$ is strata-preserving,

then $\widehat{D}[K, f(S)]$ is a faithful discretization of $VPHT(K)$.

**Proof** The directions ensured by Statement (1) mean that $f(S)$ is vertex-isolating.

Next, we show that $f(S)$ is simplex-isolating. Let $\sigma$ be a maximal simplex. Since $(K, S)$ is $\sigma$-isolating, $S$ contains a direction $s_\sigma$ that is $K_0$-perpendicular to $\sigma$.

Since $f$ is strata-preserving by Statement (2), it preserves vertex order, so $f(s_\sigma)$ is also $K_0$-perpendicular to $\sigma$. Moreover, if $s'$ is a $(P, V, W, s_\sigma)$-perturbation, then $f(s')$ is a $(P, V, W, f(s_\sigma))$-perturbation. Hence, $(K, f(S))$ is $\sigma$-isolating.

Since $(K, f(S))$ is vertex- and simplex-isolating, we have met the assumption of Theorem 19; hence, $\widehat{D}[K, f(S)]$ is a faithful discretization of $VPHT(K)$. \qed

The results of Section 5 give us the following corollary.

**Corollary 42.** Given a map $f : S^{d-1} \rightarrow S^{d-1}$ as in Theorem 41, if $\widehat{D}$ is any verbose dimension-returning transform, then $\widehat{D}[K, f(S)]$ is a faithful discretization of the corresponding topological transform. Notably, $\widehat{\beta}[K, f(S)]$ is
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There exists a set \( S \) that is the same order of vertex heights in \( s \), so that, for every \( s \in S \), the order of vertex heights in \( L_0 \) with respect to \( s \) is the same order of vertex heights in \( K_0 \) with respect to \( s \). Then, there exists a set \( S' \subset \mathbb{S}^{d-1} \) that differs from \( S \) by at most a single direction such that \( \mathcal{D}[L, S'] \) is a faithful discretization of \( \text{VPHT}(L) \).

**Theorem 43** (Parameterization is Robust to Vertex Perturbations). Let \( K \) be a simplicial complex \( GP \)-immersed in \( \mathbb{R}^d \), and let \( S \) be as in Construction 35. Let \( L \) be a complex that arises by perturbing the vertices of \( K \) in such a way so that, for every \( s \in S \), the order of vertex heights in \( L_0 \) with respect to \( s \) is the same order of vertex heights in \( K_0 \) with respect to \( s \). Then, there exists a set \( S' \subset \mathbb{S}^{d-1} \) that differs from \( S \) by at most a single direction such that \( \mathcal{D}[L, S'] \) is a faithful discretization of \( \text{VPHT}(L) \).

**Proof** Let \( g: K \to L \) be the simplicial map determined by the perturbation described in the theorem. Note that this map is a homeomorphism (and hence a bijection).

The original direction set \( S \) is simplex-isolating for \( L \). By construction, \((K, S)\) is simplex-isolating. Now, let \( \sigma \) be a maximal simplex. Since \((K, S)\) is \( \sigma \)-isolating, there is a direction \( s_\sigma \in S \) that is \( K_0 \)-perpendicular to \( \sigma \). Since \( K \) and \( L \) have the same vertex order with respect to direction \( s_\sigma \), we know that \( s_\sigma \) is \( L_0 \)-perpendicular to \( g(\sigma) \). Moreover, if \( s' \) is a \((P, V, W, s_\sigma)\)-perturbation, then \( s' \) is also a \((g(P), g(V), g(W), s_\sigma)\)-perturbation. Hence, \((g(K), S) = (L, S)\) is \( g(\sigma) \)-isolating. Since \( g \) is a bijection, \((L, S)\) is simplex-isolating.

It is not guaranteed that the direction set \( S \) is vertex-isolating for \( L \). Suppose that \( s_1, s_2, \ldots, s_{d+1} \in S \) are vertex-isolating directions for \( K \), so that \( s_1, s_2, \ldots, s_d \) are linearly independent, and \( s_{d+1} \) orders the filtration grid of \( K \) with respect to \( \{s_1, s_2, \ldots, s_d\} \) uniquely. Notice that the filtration grid of \( L \) with respect to \( \{s_1, s_2, \ldots, s_d\} \) is distinct from the filtration grid of \( K \); it is no longer guaranteed that \( s_{d+1} \) will order this new filtration grid uniquely. Thus, we may need to replace \( s_{d+1} \) with a new direction \( s'_{d+1} \) that does order the filtration grid of \( L \) with respect to \( \{s_1, s_2, \ldots, s_d\} \) uniquely.

Thus, denoting the new direction set we obtain by modifying \( S \) with this single replacement \( S' \), we find that \( \mathcal{D}[L, S'] \) faithfully discretizes \( \text{VPHT}(L) \), as desired.

**Corollary 44.** Given a map \( g: K \to L \) as in Theorem 43, if \( \mathcal{D} \) is any verbose dimension-returning transform, then there exists a set \( S' \subset \mathbb{S}^{d-1} \) that differs from \( S \) by at most a single direction such that \( \mathcal{D}(L, S') \) is a faithful discretization of the corresponding topological transform. Notably, \( \mathcal{B}[L, S'] \) is a faithful discretization of \( \text{VBFT}(L) \). Furthermore, \( \hat{\chi}[L, S'] \) is a faithful discretization of \( \text{VECFT}(L) \).
Note that in the previous theorem and corollary, the set of simplicial complexes $L$ for which the original direction set $S$ does not lead to a faithful discretization (that is, the change of the single direction is required) has measure zero.\footnote{with respect to Lebesgue measure.}

8 Example of Building a Faithful Set

Here, we walk through a small example of building a faithful discretization of the VPHT, following Construction 35. Suppose we are given the simplicial complex $K$ in $\mathbb{R}^3$, shown in Fig. 6. Specifically, $K$ comprises: five vertices, $v_0 = (1,0,0)$, $v_1 = (0,1,0)$, $v_2 = (0,0,1)$, $v_3 = (0,2,0)$, and $v_4 = (0,2,2)$; four edges, $[v_0,v_1]$, $[v_1,v_2]$, $[v_0,v_2]$, and $[v_0,v_3]$; and a two-simplex $[v_0,v_1,v_2]$. There are three maximal simplices, $v_4$, $\sigma_1 = [v_0,v_3]$, and $\sigma_2 = [v_0,v_1,v_2]$.

Fig. 6 The simplicial complex $K$ used as an example in this section. We first find vertex-isolating directions: the basis vectors, along with a fourth direction that orders the filtration grid of $K_0$ with respect to the basis vectors uniquely. Next, for $\sigma_1$ and $\sigma_2$, we find simplex-isolating directions; directions that are $K_0$-perpendicular to $\sigma_1$ and $\sigma_2$, as well as directions that tilt these perpendicular directions so that proper faces of $\sigma_1$ (and $\sigma_2$) are “popped” above $\sigma_1$ (respectively, $\sigma_2$).

8.1 Tilt Examples

Since many algorithms used in Construction 35 make calls to Algorithm 7 (Tilt), we walk through two explicit examples in this section, using inputs that are relevant for later subsections.

We first describe the details of the call Tilt($s, s', P$) where $s = (1,0,0)$, $s' = (0,1,0)$ and $P = \{(0,1,2)\}^3 \subset \mathbb{R}^3$. We begin by finding the heights of points in $P$ with respect to the direction $s$ that are closest but distinct (Line 1 of Algorithm 7). There are only three distinct heights of points of $P$ with respect to the direction $s$, namely 0, 1, and 2. Choosing the first pair that satisfies our desired property, we have $h_1 = 0$ and $h_2 = 1$. Next, on Line 2, we find heights of points of $P$ in the direction $s'$ that are extremal, which are $h'_1 = 0$ and $h'_2 = 2$.\footnote{with respect to Lebesgue measure.}
Then, on Line 3, we compute the solution to \((1 - t)h_1 + th'_2 = (1 - t)h_2 + th'_1\), which is \(t_\ast = 1/3\). Finally, we return the tilted direction

\[
s_t = \left(1 - \frac{1}{2}t_\ast\right) e_1 + \frac{1}{2}t_\ast e_2 = \left(\frac{5}{6}, \frac{1}{6}, 0\right) \approx (0.8333, 0.1667, 0).
\]

Next, we detail a more involved instance of \(\text{Tilt}\), using \(s = (2\sqrt{5}/5, \sqrt{5}/5, 0)\), \(s' = (0.8, 0.4, 0.4472)\), and \(P = K_0\). First, on Line 1, we identify two heights of vertices with respect to \(s\) that are closest. The direction \(s\) orders vertices in an evenly spaced way \((v_0, v_3, v_4)\) all have height \(2\sqrt{5}/5\), \(v_1\) has height \(\sqrt{5}/5\), and \(v_2\) has height 0). Going through the sorted list, we first encounter \(h_1 = 0\) and \(h_2 = \sqrt{5}/5\). On Line 2, we find the minimum and maximum heights with respect to \(s'\). These are \(h'_1 = 0.4\) (the height of \(v_1\) with respect to \(s'\)) and \(h'_2 \approx 1.6944\) (the height of \(v_4\) with respect to \(s'\)). Finally, on Line 3 of Algorithm 7, we find the solution to \((1 - t)h_1 + th'_2 = (1 - t)h_2 + th'_1\), which is \(t_\ast \approx 0.2568\) This gives us the final output of \(\text{Tilt}(s, s', K_0)\),

\[
s_t = \left(1 - \frac{1}{2}t_\ast\right) s + \frac{1}{2}t_\ast s' \approx (0.8823, 0.4412, 0.0574).
\]

### 8.2 Vertex Isolating Directions

First, we find the directions of Step 1 of Construction 35. Specifically, we find directions that are vertex-isolating with respect to \(K\). By Lemma 37, the standard basis directions, \(e_1, e_2, e_3\), as well as a fourth direction as computed in \(\text{PointIso}(K_0)\) (Algorithm 9) are vertex-isolating with respect to \(K\). Thus, we walk through the computation of Algorithm 9. First, on Line 1, we begin by defining \(A\), the filtration grid of \(K_0\) with respect to \(\{e_1, e_2, \ldots, e_d\}\). In our example, \(A\) is a set of 27 gridpoints, \(\{0,1,2\}^3 \subset \mathbb{R}^3\). In Line 2, we initialize \(s = (1, 0, 0)\), and enter the loop on Lines 3–5. The first iteration calls \(\text{Tilt}((1,0,0),(0,1,0),A)\) (Algorithm 7), tilting \(e_1\) towards \(e_2\). As detailed in Section 8.1, this gives us \(s'^2 = (1 - \frac{1}{2}t_\ast)e_1 + \frac{1}{2}t_\ast e_2 \approx (0.8333, 0.1667, 0)\). We are then ready for the next and final iteration of the loop in Algorithm 9, finding the fourth direction in our set of vertex-isolating directions, which is

\[
s = \text{Tilt}(s'^2, (0,0,1), A) \approx (0.8013, 0.1603, 0.0385).
\]

### 8.3 Perpendicular Directions

Next, we compute directions as described in Step 2(a) of Construction 35 (equivalently, directions satisfying Statement (1) of Definition 17). In the language of Definition 17, for \(\sigma_i\) \((i = 1, 2)\) we need a direction \(s_{\sigma_i}\) that is \(K_0\)-perpendicular to \(\sigma_i\). We begin with the two-simplex, \(\sigma_2\), the more straightforward computation. First, we compute \(\text{Perp}(\sigma_2)\) (Algorithm 5). In Line 1 of Algorithm 5, we find a matrix \(A\) with \(v_i - v_0\) in the \(i\)th column,
which, for $\sigma_2$ is

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

Next, in Line 2, we use a QR-decomposition to find the basis vectors for the space spanned by $A$, which are, $b_1 = (-\sqrt{2}/2, \sqrt{2}/2, 0)$ and $b_2 = (-\sqrt{6}/6, -\sqrt{6}/6, \sqrt{6}/3)$. Then, in Lines 5–8, we perform Gram-Schmidt on $b_1$ and $b_2$ with each basis vector until the result is non-zero. The loop terminates after one iteration and we find $s'_{\sigma_2} = \text{Perp}(\sigma_2) = (\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3) \approx (0.5774, 0.5774, 0.5774)$.

The direction $s_{\sigma_2}$ is already $K_0$-perpendicular to $\sigma_2$ since $\sigma_2$ has codimension zero; we confirm this by completing the procedure suggested in Step 2(a) of Construction 35. Let $H$ be the set of vertices with the same height as $\sigma_2$ in direction $s_{\sigma_2}$, meaning that $H = \emptyset$. Then, we have $W = \text{PlaneFill}(H, s) \cup (H \setminus \sigma_2) = \emptyset$ and $V = W \cup \sigma_2 = \sigma_2$, so $\text{TiltToPop}(K_0, V, W, s_{\sigma_2}) = \text{TiltToPop}(K_0, \sigma_2, \emptyset, s_{\sigma_2}) = s_{\sigma_2}$ (this is highlighting the fact that, since the output of $\text{Perp}(\sigma_2)$ is already $K_0$-perpendicular to $\sigma_2$, we do not need to adapt it in any way).

Next, we shift focus to $\sigma_1$. To construct the direction described by Step 2(a) of Construction 35 for the edge $\sigma_1$, we again begin by finding $\text{Perp}(\sigma_1)$. This begins with the column matrix containing $v_3 - v_0$,

$$A_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$  

Then, we use QR-decomposition to find the basis for the space spanned by $A_1$, which consists of the single vector $b = (-\sqrt{5}/5, 2\sqrt{5}/5, 0)$. Performing Gram-Schmidt on $b$ and $e_1$ yields $s = \text{Perp}(\sigma_1) = (2\sqrt{5}/5, \sqrt{5}/5, 0)$. However, note that although $s$ is perpendicular to $\sigma_1$, it is not $K_0$-perpendicular, since the vertex $v_4$ is at the same height as $\sigma_1$ with respect to $s$. Thus, we correct $s$ by following the procedure suggested in Construction 35, Step 1. First, let $H_1$ be the set of all vertices with the same height as $\sigma_1$ in direction $s$, i.e., $H_1 = \{v_0, v_3, v_4\}$. Since aff($H_1$) has codimension one with $\mathbb{R}^3$, we find $\text{PlaneFill}(H_1, s) = \emptyset$, so that

$$W_1 = \text{PlaneFill}(H_1, s) \cup (H_1 \setminus \sigma_1)$$
$$= \emptyset \cup (\{v_0, v_3, v_4\} \setminus \{v_0, v_3\})$$
$$= \{v_4\}.$$  

Finally, setting $V_1 = W \cup \sigma_1 = \{v_0, v_3, v_4\}$, we use $\text{TiltToPop}(K_0 \cup W_1, V_1, W_1, s)$ (Algorithm 8) to find a direction $K_0$-perpendicular to $\sigma_1$. In
Line 1, we find $V' = \text{PlaneFill}(V_1, s)$ but since $\text{aff}(V_1)$ has codimension one with $\mathbb{R}^3$, we simply have $V' = \emptyset$. Next, in Line 2, we compute

$$s' = \text{Perp}(\text{aff}(V' \cup (V_1 \setminus W_1) \cup \{W_1 - s\})) = \text{Perp}(\text{aff}(v_0, v_3, v_4 - s)) = \text{Perp}(\text{aff}(v_0, v_3, (-2\sqrt{5}/5, 2 - \sqrt{5}/5, -2))) \\ \approx (0.8, 0.4, 0.4472).$$

Although for this example, $s'$ happens to be already $K_0$ perpendicular to $\sigma_1$, this is not generally the case. Thus, we continue on Line 6 and compute

$$s_{\sigma_1} = \text{Tilt}(s, s', K_0 \cup V') = \text{Tilt}(s, s', K_0) \approx (0.8823, 0.4412, 0.0574).$$

(Algorithm 7, see Section 8.1 for details), which is therefore also the output of $\text{TiltToPop}(K_0 \cup W, V, W, s_{\sigma_2})$, and a direction guaranteed to be $K_0$-perpendicular to $\sigma_1$.

### 8.4 Simplex-Isolating Directions

Next, we find the directions of Step 2(b) in Construction 35 (directions satisfying Statement (2) of Definition 17). That is, for every nonempty proper face of $\sigma_1$ and $\sigma_2$, we need a direction that pops the face above the remaining vertices of the maximal simplex. We walk through the procedure for a single face of $\sigma_2$; the remaining computations are similar and are summarized in Table 1.

Consider the vertex $[v_0] \prec \sigma_2$. By Theorem 39 Step 2, we can use the output of $\text{TiltToPop}(K_0, \sigma_2, v_0, s_{\sigma_2})$ (Algorithm 8) to find a direction that pops $v_0$ above $\sigma_2 \setminus v_0 = [v_1, v_2]$. First, in Line 1, we find $V' = \text{PlaneFill}(\sigma_2, s_{\sigma_2})$ using Algorithm 6. Since $\sigma_2$ has codimension one with $\mathbb{R}^3$, this is simply $V' = \emptyset$. Next, on Line 2 of Algorithm 8 we compute

$$s' = \text{Perp}(\text{aff}(V' \cup (\sigma_2 \setminus v_0) \cup \{v_2 - s_{\sigma_2}\})) = \text{Perp}(\text{aff}(\{v_1, v_2\} \cup \{v_0 - s_{\sigma_2}\})) \\ \approx (0.9636, 0.1890, 0.1890).$$

Since $v_0$ is above $[v_1, v_2]$ with respect to $s'$, we do not enter the IF loop on Line 3. Finally, we use Algorithm 7 and compute

$$s = \text{Tilt}(s_{\sigma_2}, s', K_0) \approx (0.6598, 0.4944, 0.4944),$$

which is therefore also the output to $\text{TiltToPop}(K_0, \sigma_2, v_0, s_{\sigma_2})$.

### Summary of Computations

Below is the complete list of computed directions, including those that were discussed in detail previously in this section.
Table 1 Directions that are vertex- and simplex-isolating for the simplicial complex of Fig. 6, computed as described in Construction 35

| Property | Description | Computed Directions (rounded to four decimal places) |
|----------|-------------|-----------------------------------------------------|
| Definition 16 | vertex-isolating | (1,0,0) (0,1,0) (0,0,1) (0.8013, 0.1603, 0.0385) |
| Statement (1) of Definition 17 | $K_0$-perpendicular to $\sigma_2$ | $s_{\sigma_2} = (0.5774, 0.5774, 0.5774)$ |
| Statement (2) of Definition 17 | $(K_0, \sigma_2, [v_0], s_{\sigma_2})$-perturbation | (0.6598, 0.4944, 0.4944) |
| | $(K_0, \sigma_2, [v_1], s_{\sigma_2})$-perturbation | (0.5357, 0.6187, 0.5357) |
| | $(K_0, \sigma_2, [v_0, v_1], s_{\sigma_2})$-perturbation | (0.5953, 0.5953, 0.4866) |
| | $(K_0, \sigma_2, [v_0, v_2], s_{\sigma_2})$-perturbation | (0.5959, 0.4835, 0.5959) |
| | $(K_0, \sigma_2, [v_1, v_2], s_{\sigma_2})$-perturbation | (0.5235, 0.5880, 0.5880) |
| | $(K_0, \sigma_2, [v_0, v_3], s_{\sigma_2})$-perturbation | (0.5561, 0.5560, 0.5866) |
| | $(K_0, \sigma_2, [v_1, v_3], s_{\sigma_2})$-perturbation | (0.5649, 0.5720, 0.5844) |
| | $(K_0, \sigma_1, s_{\sigma_2})$-perturbation | (0.5959, 0.4835, 0.5959) |

9 Discussion

In this work, we provide sufficient conditions for a faithful discretization of the VPHT. For a simplicial complex $\gamma P$-immersed in $\mathbb{R}^d$, with $n_0$ vertices, $n$ simplices, and dimension $\kappa$, the discretization is of size $O(n_0^2 \kappa + d)$ and it can be computed in $O(\log(n_0 + d)(d^2 n_0^d + nd + n n_0^2 \kappa) + n^2 \kappa^d)$. We show that the discretization is stable with respect to multiple types of perturbations. And, since only the presence and dimension of filtration events are used (and not birth/death pairing information), the techniques presented in the paper immediately apply to any dimension-returning transform, such as the verbose Betti function transform.

We show that, under mild adaptations, the methods of this paper also yield a faithful discretization of the verbose Euler Characteristic function transform. This is particularly important since, in practice, VECFs are often preferred to VPDs due to the existence of faster algorithms for computing the functions.

While the explicit faithful discretizations computed in this paper are the first of their kind, they generally contain more topological descriptors than are strictly necessary to form a faithful discretization. In ongoing work, we hope to better understand computing faithful discretizations with minimal cardinality.

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A Proof of Simplex Count Lemma

In this appendix, we provide the proof of Lemma 11. We first recall the statement of this well-known result.

**Lemma 11 (Simplex Count).** Let $K$ be a simplicial complex, $k \in \mathbb{N}$ and $c \in \mathbb{R}$. Let $f : K \to \mathbb{R}$ be a monotonic function. Then, the $k$-dimensional simplices of $K$ with a function value of $c$ are in one-to-one correspondence with the points in the following multiset:

$$\{ (a, b) \in \tilde{D}_k(f) \text{ s.t. } a = c \} \cup \{ (a, b) \in \tilde{D}_{k-1}(f) \text{ s.t. } b = c \}.$$  \hspace{1cm} (5)

**Proof** Let $f' : K \to \mathbb{N}$ be an index filter function compatible with $f$, and let $\{F'_i\}_{i \in \mathbb{N}}$ be the corresponding index filtration, where $F'_i := f'^{-1}(-\infty, i]$. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the ordering of simplices in $K$ such that $f'(\sigma_i) = i$. Consider the sets

$$C_B := \{(i, j) \in D_k(f') \text{ s.t. } f(\sigma_i) = c\} \quad C_D := \{(i, j) \in D_{k-1}(f') \text{ s.t. } f(\sigma_j) = c\}$$

and let $C = C_B \cup C_D$.

We start by defining a bijection $\phi : f^{-1}(c) \to C$. Let $\sigma_i \in K_k$ such that $f(\sigma_i) = c$. Since adding a single $k$-simplex either increases $\beta_k$ by one or decreases $\beta_{k-1}$ (see, e.g., [21, pp. 120–121]), either $\beta_k(F'_i) = \beta_k(F'_{i-1}) + 1$ or $\beta_{k-1}(F'_i) = \beta_{k-1}(F'_{i-1}) - 1$, but not both.

**Case 1:** ($\beta_k$ increases). There exists a unique birth at index $i$ in $D_k(f')$. Thus, let $j \in \mathbb{R}$ be such that $(i, j) \in D_k(f')$. Then, $(f'(\sigma_i), f'(\sigma_j)) = (i, j) \in C_B$. Define $\phi(\sigma_i) = (i, j)$.

**Case 2:** ($\beta_{k-1}$ decreases). There is a unique death at index $i$ in $D_k(f')$. Thus, let $j \in \mathbb{R}$ be such that $(j, i) \in D_k(f')$. Then, $(f'(\sigma_j), f'(\sigma_i)) = (j, i) \in C_D$. Define $\phi(\sigma_i) = (f'(\sigma_j), f'(\sigma_i)) = (j, i)$.

In other words, each $\sigma_i$ is mapped to the persistence pair containing $f'(\sigma_i)$. If $\phi(\sigma) = \phi(\tau) = (i, j)$, then, by construction, both $\sigma$ and $\tau$ are the same type of event. WLOG, assume that they are birth events. Then, $f'(\sigma) = f'(\tau)$, which, by injectivity of $f'$ means that $\sigma = \tau$. Thus, we have shown that $\phi$ is an injection.

We show that $\phi$ is a surjection by contradiction. Suppose there exists $(i, j) \in C$ such that there does not exist $\sigma \in f^{-1}(c)$ with $\phi(\sigma) = (i, j)$. Since $(i, j) \in C$, then either $f(\sigma_i) = c$ or $f(\sigma_j) = c$ (or both). WLOG, suppose $f(\sigma_i) = c$. Then, we find $\sigma_i \in f^{-1}(c)$ and, by construction, $\phi(\sigma_i) = (i, j) \in C_B$.

By Equation (3) and since $\tilde{D}(f)$ is well-defined (see Lemma 52), we know that the map $\psi : D_k(f') \to \tilde{D}_k(f)$ defined by $\psi(i, j) = (f(\sigma_i), f(\sigma_j))$ is a bijection. Since $C \subset D_k(f')$ and both functions are bijections, the composition $\psi \circ \phi$ is the one-to-one correspondence that we sought. \hfill $\square$

Thus, if some step in a filtration includes a single additional simplex, there must be a single additional point in the diagram.
Corollary 45 (Add One Simplex). Let \( K_i \) and \( K_{i+1} \) be two simplicial complexes such that \( K_{i+1} \setminus K_i \) is a single \( k \)-simplex. Let \( \beta_k^{(i)} \) and \( \beta_k^{(i+1)} \) denote the \( k \)th Betti numbers of \( K_i \) and \( K_{i+1} \), respectively. Then \( |\beta_k^{(i)} - \beta_k^{(i+1)}| = 1 \).

B Verbose Descriptors are Well-Defined

The verbose persistence diagram (VPD), verbose Betti function (VBF), and verbose Euler characteristic function (VECF) are each well-defined. We begin by showing VPDs are well defined. This boils down to proving that Definition 1 is independent of the choice of the index filtration. To prove this claim, we will make use of the following definition, which organizes collections of \( f \)-compatible filter functions.

Definition 46 (Compatible Transposition Sequence). Let \( K \) be a simplicial complex and let \( f: K \to \mathbb{R} \) be a filter function. Let \( f_1, f_2, \ldots, f_m \) be \( f \)-compatible index filtrations, where each adjacent pair \( f_j \) and \( f_{j+1} \) differ only by a single transposition. That is, for some \( i \in \mathbb{N} \), we have the equalities \( f_j^{-1}(i) = f_{j+1}^{-1}(i+1) \) and \( f_j^{-1}(i+1) = f_{j+1}^{-1}(i) \), but for every other \( i' \notin \{i, i+1\} \), we have \( f_j^{-1}(i') = f_{j+1}^{-1}(i') \). This sequence of such index filtrations is called an \( f \)-compatible transposition sequence.

We begin by considering transposition sequences with just two elements.

Lemma 47 (Commutative Diagrams). Let \( K \) be a simplicial complex and \( f: K \to \mathbb{R} \) be a filter function. Let \( f_1, f_2 \) be an \( f \)-compatible transposition sequence, where the transposition between \( f_1 \) and \( f_2 \) occurs at indices \( i \) and \( i + 1 \). Let \( \{K_j^{(1)}\}_{j \in \mathbb{N}} \) and \( \{K_j^{(2)}\}_{j \in \mathbb{N}} \) denote the corresponding filtrations. Then, the following diagram of inclusions commutes:

\[
\begin{align*}
K_i^{(1)} & \to K_{i-1}^{(1)} & \to K_{i-2}^{(1)} & \cdots & \cdots & \to K_{i+1}^{(1)} & \to K_{i+2}^{(1)} \\
K_i^{(2)} & \to K_{i-1}^{(2)} & \to K_{i-2}^{(2)} & \cdots & \cdots & \to K_{i+1}^{(2)} & \to K_{i+2}^{(2)}
\end{align*}
\]

(11)

Proof The proof follows from the fact that all maps are inclusion maps.

Applying homology to Equation (11), we obtain the following corollary.
Corollary 48 \textbf{(Commutative Diagrams on Homology).} The following diagram commutes:

\[
\begin{array}{ccccccc}
\vdots & \rightarrow & H(K_{i-2}^{(1)}) & \rightarrow & H(K_{i-1}^{(1)}) & \rightarrow & H(K_{i+1}^{(1)}) & \rightarrow & \vdots \\
\rightarrow & H(K_{i-2}^{(2)}) & \rightarrow & H(K_{i-1}^{(2)}) & \rightarrow & H(K_{i+1}^{(2)}) & \rightarrow & \rightarrow \\
\end{array}
\]

Suppose that \( f_1 \) and \( f_2 \) are \( f \)-compatible index filtrations that differ by a single transposition. We show that it does not matter if we compute \( f \) using \( f_1 \) or \( f_2 \). Following [35, p. 122], there are four cases to consider, namely, whether births or deaths occur at index \( i \) and \( i + 1 \). We first consider the case that \( i \) and \( i + 1 \) both correspond to birth events; the other cases are similar.

Lemma 49 \textbf{(Transpose Births).} Let \( K \) be a simplicial complex and \( f: K \rightarrow \mathbb{R} \) be a filter function. Let \( f_1 \) and \( f_2 \) be two \( f \)-compatible index filters that differ by exactly one transposition at index \( i \) and \( i + 1 \). If both \( i \) and \( i + 1 \) are birth coordinates in \( D(f_1) \), then

\[
\{(f(f_1^{-1}(i)), f(f_1^{-1}(j))) \}_{(i,j) \in D(f_1)} = \{(f(f_2^{-1}(i)), f(f_2^{-1}(j))) \}_{(i,j) \in D(f_1)},
\]

with \( f(\emptyset) := \infty \) and where \( = \) denotes equality as a multi-set.

Proof For \( j \in \{1, 2, \ldots, n\} \), let \( \sigma_j := f_1^{-1}(j) \); in other words, \( \sigma_1, \sigma_2, \ldots, \sigma_n \) is the ordering of the simplices such that \( f_1(\sigma_j) = j \). Since the transposition is at indices \( i \) and \( i + 1 \), we have \( f_1(\sigma_i) = f_2(\sigma_{i+1}) = i \) and \( f_2(\sigma_{i+1}) = f_1(\sigma_{i+1}) = i + 1 \). For all other \( \sigma' \in K \), we have \( f_1(\sigma') = f_2(\sigma') \).

By Corollary 45 and since \( i \) is a birth coordinate in \( D(f_1) \), we know that \( \text{rk}(H(K_i^{(1)})) = \text{rk}(H(K_{i+1}^{(1)})) + 1 \). Likewise, \( \text{rk}(H(K_{i+1}^{(1)})) = \text{rk}(H(K_i^{(1)})) + 1 \). Thus,

\[
\text{rk}(H(K_{i+1}^{(2)})) = \text{rk}(H(K_{i+1}^{(1)})) = \text{rk}(H(K_i^{(1)})) + 2 = \text{rk}(H(K_{i-1}^{(1)})) + 2.
\] (13)

Therefore, again by Corollary 45, and since \( K_{i-1}^{(2)} \) and \( K_{i+1}^{(2)} \) differ by two simplices, we know that both \( i \) and \( i + 1 \) are also birth coordinates in \( D(f_2) \).

For \( s, t \in \mathbb{R} \) with \( s \leq t \), the map \( i_{s,t}^{(1)}: K_s^{(1)} \rightarrow K_t^{(1)} \) induces a map on homology \( H(K_s^{(1)}) \rightarrow H(K_t^{(1)}) \). For \( k \in \mathbb{N} \), let \( \beta_{k,s,t}^{(1)} \) be the rank of this map restricted to the \( k \)-th homology group:

\[
\beta_{k,s,t}^{(1)} := \text{rk} \left( H_k(K_s^{(1)}) \rightarrow H_k(K_t^{(1)}) \right).
\] (14)

We define \( \beta_{k,s,t}^{(2)} \) analogously. By using \( f_* = f_1 \) in Definition 1 (and recalling the definition of a persistence diagram in Equation (2)), the multiplicity of \((s', t')\)
Similarly, let \( \mu_k^{(2)} : \mathbb{R}^2 \to \mathbb{Z} \) map \((s', t')\) to the multiplicity of \((s', t')\) in \( \widehat{D}_k(f) \), as defined using \( f_* = f_2 \) in Definition 1. Now, we must show that \( \mu_k^{(1)} = \mu_k^{(2)} \) for all \( k \).

We investigate the values of \( \beta_{k,s,t}^{(1)} \) for different values of \( s, t \). Since \( \beta_{k,s,t}^{(1)} = \beta_{k,[s],[t]}^{(1)} \), it is sufficient to only consider integer values of \( s, t \). First suppose \( s \neq i \) and \( t \neq i \). Then, since Diagram 12 commutes, we know:

\[
\beta_{k,s,t}^{(1)} = \text{rk} \left( H_k(K_s^{(1)}) \to H_k(K_t^{(1)}) \right) = \text{rk} \left( H_k(K_s^{(2)}) \to H_k(K_t^{(2)}) \right) = \beta_{k,s,t}^{(2)}.
\]

Next, consider the case \( s = i \). Since \( s = i \) is a birth parameter, there exists \( d_i \in \mathbb{R} \) such that \( (i, d_i) \in D(f_1) \). Similarly, let \( d_{i+1} \in \mathbb{R} \) such that \( (i+1, d_{i+1}) \in D(f_1) \). Then, if \( t \geq \max\{d_i, d_{i+1}\} \), or if \( t < \min(d_i, d_{i+1}) \), or if \( k \) does not equal the dimension of death at \( \max\{d_i, d_{i+1}\} \). Then, loosely speaking, \( K_i^{(1)} \to K_t^{(1)} \) and \( K_i^{(2)} \to K_t^{(2)} \) either do not include the deaths at \( d_i \) and \( d_{i+1} \), or they include both deaths. Thus,

\[
\beta_{k,s,t}^{(1)} = \beta_{k,i,t}^{(1)} \quad \text{by substitution}
\]

\[
= \text{rk} \left( H_k(K_i^{(1)}) \to H_k(K_t^{(1)}) \right) \quad \text{by Equation (14)}
\]

\[
= \text{rk} \left( H_k(K_i^{(2)}) \to H_k(K_t^{(2)}) \right) = \beta_{k,s,t}^{(2)}.
\]

Now, consider the case that \( k \) equals the dimension of death at \( \max\{d_i, d_{i+1}\} \), and a value \( t \in \mathbb{N} \) such that \( t \in [\min\{d_i, d_{i+1}\}, \max\{d_i, d_{i+1}\}] \). First, suppose \( d_i < d_{i+1} \), so that, loosely speaking, \( K_i^{(1)} \to K_t^{(1)} \) includes the death at \( d_i \) (but not at \( d_{i+1} \)), but \( K_i^{(2)} \to K_t^{(2)} \) does not include the death at \( d_i \) (nor at \( d_{i+1} \)). Then,

\[
\beta_{k,i,t}^{(1)} = \text{rk} \left( H_k(K_i^{(1)}) \to H_k(K_t^{(1)}) \right) = \text{rk} \left( H_k(K_i^{(2)}) \to H_k(K_t^{(2)}) \right) + 1 = \beta_{k,i,t}^{(2)} + 1
\]

If \( k \) equals the dimension of death at \( \max\{d_i, d_{i+1}\} \), and \( t \in \mathbb{N} \) such that \( t \in [\min\{d_i, d_{i+1}\}, \max\{d_i, d_{i+1}\}] \), but instead, we have \( d_i > d_{i+1} \), a symmetric argument shows

\[
\beta_{k,i,t}^{(1)} = \beta_{k,i,t}^{(2)} - 1.
\]

Now that we have determined all values of \( \beta_{k,s,t}^{(1)} \) and \( \beta_{k,s,t}^{(2)} \), we evaluate the sum in Equation (15). Since \( \beta_{k,s,t}^{(1)} = \beta_{k,s,t}^{(2)} \) whenever \( k \) does not equal the dimension of
death at \( \max\{d_i, d_{i+1}\} \), we see \( \mu_k^{(1)}(s', t') = \mu_k^{(2)}(s', t') \) for such choices of \( k \) and for all \( (s', t') \in \mathbb{R}^2 \).

Suppose instead that \( k \) equals the dimension of death at \( \max\{d_i, d_{i+1}\} \). Note that since \( f_1 \) and \( f_2 \) are compatible with \( f \), we have \( f(f_1^{-1}) = f(f_2^{-1}) \). Furthermore, the sets \( S = \{ s \text{ s.t. } f(f_1^{-1}(s)) = f(f_2^{-1}(s)) = s' \} \) and \( T = \{ t \text{ s.t. } f(f_1^{-1}(t)) = f(f_2^{-1}(t)) = t' \} \) are each sets of consecutive integers (with the exception of the possible set \( T = \{ \infty \} \)). We label elements \( S = \{ s_1, s_2, \ldots, s_r \} \), and write \( s_0 = s_1 - 1 \). By a telescoping argument, for any fixed \( t \in T \), we find

\[
\sum_{s \in S} \left( \beta_{k,s,t-1}^{(1)} - \beta_{k,s,t}^{(1)} - \beta_{k,s-1,t-1}^{(1)} + \beta_{k,s-1,t}^{(1)} \right) = \left( \beta_{k,s_1,t-1}^{(1)} - \beta_{k,s_1,t}^{(1)} - \beta_{k,s_0,t-1}^{(1)} + \beta_{k,s_0,t}^{(1)} \right) + \left( \beta_{k,s_2,t-1}^{(1)} - \beta_{k,s_2,t}^{(1)} - \beta_{k,s_1,t-1}^{(1)} + \beta_{k,s_1,t}^{(1)} \right) + \cdots + \left( \beta_{k,s_{r-1},t-1}^{(1)} - \beta_{k,s_{r-1},t}^{(1)} - \beta_{k,s_{r-2},t-1}^{(1)} + \beta_{k,s_{r-2},t}^{(1)} \right) = -\beta_{k,s_0,t-1}^{(1)} + \beta_{k,s_0,t}^{(1)} + \beta_{k,s,t-1}^{(1)} - \beta_{k,s,t}^{(1)}
\]

and similarly for \( \beta_{k,s,t}^{(2)} \) when considering \( f_2 \). We will show that these sums for \( f_1 \) and \( f_2 \) agree for any choice of \( t \).

Suppose for the fixed \( t \) chosen above, \( t \geq \max\{d_i, d_{i+1}\} \) or \( t < \min\{d_i, d_{i+1}\} \) and \( t - 1 \geq \max\{d_i, d_{i+1}\} \) or \( t - 1 < \min\{d_i, d_{i+1}\} \). For these values of \( t \), we know

\[
-\beta_{k,s_0,t-1}^{(1)} + \beta_{k,s_0,t}^{(1)} + \beta_{k,s_0,t-1}^{(1)} - \beta_{k,s_0,t}^{(1)} = -\beta_{k,s_0,t-1}^{(2)} + \beta_{k,s_0,t}^{(2)} + \beta_{k,s_0,t-1}^{(2)} - \beta_{k,s_0,t}^{(2)}
\]

Suppose for the fixed \( t \) chosen above, both \( t, t - 1 \in [\min\{d_i, d_{i+1}\}, \max\{d_i, d_{i+1}\}] \) and \( d_i < d_{i+1} \). Then we know

\[
-\beta_{k,s_0,t-1}^{(1)} + \beta_{k,s_0,t}^{(1)} + \beta_{k,s_0,t-1}^{(1)} - \beta_{k,s_0,t}^{(1)} = -\beta_{k,s_0,t-1}^{(2)} + \beta_{k,s_0,t}^{(2)} + \beta_{k,s_0,t-1}^{(2)} - \beta_{k,s_0,t}^{(2)}
\]

For the same \( t \) and \( t - 1 \), if \( d_i > d_{i+1} \), a nearly identical argument holds.

If \( t = \min\{d_i, d_{i+1}\} \) and \( d_i < d_{i+1} \), then we have

\[
-\beta_{k,s_0,t-1}^{(1)} + \beta_{k,s_0,t}^{(1)} + \beta_{k,s_0,t-1}^{(1)} - \beta_{k,s_0,t}^{(1)} = -\beta_{k,s_0,t-1}^{(2)} + \beta_{k,s_0,t}^{(2)} + \beta_{k,s_0,t-1}^{(2)} - \beta_{k,s_0,t}^{(2)}
\]

Again, for the same \( t \) and \( t - 1 \), if \( d_i > d_{i+1} \), a nearly identical argument holds.

Note that since we cannot have \( t - 1 \in [\min\{d_i, d_{i+1}\}, \max\{d_i, d_{i+1}\}] \) without \( t \) also being in this set, we have evaluated all possible cases of \( t \) and \( t - 1 \). In each case, the term of the sum corresponding to Equation (15) agreed between \( \mu^{(1)}(s', t') \) and \( \mu^{(2)}(s', t') \). This, combined with our previous arguments, means we have shown the equality \( \mu^{(1)}(s', t') = \mu^{(2)}(s', t') \), as desired. \( \square \)
We have, so far, only considered the case that $i$ and $i+1$ correspond to birth events. The remaining three cases (i.e., that they correspond to death events, a birth and death event, or a death and birth event) are similar. All cases taken together give us the following corollary.

**Corollary 50 (Single Transposition).** Let $K$ be a simplicial complex, $m \in \mathbb{N}$, and $f: K \to \mathbb{R}$ a filter function. Let $f_1$ and $f_2$ be two $f$-compatible index functions that differ by exactly one transposition. Then:

$$\{(f(f_2^{-1}(i)), f_1^{-1}(j))\}_{(i,j) \in \mathcal{D}(f_t)} = \{(f(f_1^{-1}(i)), f_2^{-1}(j))\}_{(i,j) \in \mathcal{D}(f_t)},$$

with $f(\emptyset) := \infty$ and where $=$ denotes equality as a multi-set.

Next, we move to transposition sequences of arbitrary finite length, and show that all index filtrations in a transposition sequence produce the same verbose persistence diagram.

**Lemma 51 (Transposition).** Let $K$ be a simplicial complex, $m \in \mathbb{N}$, and $f: K \to \mathbb{R}$ a filter function. Let $\{f_1, f_2, \ldots, f_m\}$ be an $f$-compatible transposition sequence. Then, for all $t \in \{1, 2, \ldots, m\}$,

$$\{(f(f_t^{-1}(i)), f_t^{-1}(j))\}_{(i,j) \in \mathcal{D}_k(f_t)} = \{(f(f_1^{-1}(i)), f_1^{-1}(j))\}_{(i,j) \in \mathcal{D}_k(f_t)},$$

with $f(\emptyset) := \infty$ and where $=$ denotes equality as a multi-set.

**Proof** For $t \in \{1, 2, \ldots m\}$, we write

$$A_t := \{(f(f_t^{-1}(i)), f_t^{-1}(j))\}_{(i,j) \in \mathcal{D}(f_t)}.$$  \hfill (16)

We prove the general claim by induction on $m$, the length of $f$-compatible transition sequence. The base case is straightforward: Let $f_1$ be an $f$-compatible index filtration. We immediately have $A_1 = A_2$.

For the inductive assumption, let $m \geq 1$, and suppose for any $f$-compatible transposition sequence, $\{f_1, f_2, \ldots, f_m\}$, we have $A_{m'} = A_1$ for all $m' \in \{1, 2, \ldots, m\}$.

Now, consider $\{f_1, f_2, \ldots, f_{m+1}\}$, an $f$-compatible transposition sequence of length $m+1$. By the inductive assumption, we know that $A_2 = A_3 = \ldots = A_{m+1}$, so it suffices to show that $A_1 = A_2$.

Recall that $f_1$ and $f_2$ differ by one transposition, so let $\sigma$ and $\tau$ be the simplices transposed and suppose the transposition occurs at indexes $i$ and $i+1$. Indeed, by Corollary 50, we have $A_1 = A_2$. Thus, we have shown the claim. \hfill \square

Using the preceeding lemma, we now show all index filtrations compatible with a fixed underlying filtration all produce the same VPD, meaning VPDs are well-defined.

**Lemma 52 (VPD is Well-Defined).** The verbose persistence diagram is well-defined.
Proof. Let $K$ be a simplicial complex, and let $f: K \to \mathbb{R}$ be a monotonic function. Let $f', f'': K \to \mathbb{R}$ be two index filters compatible with $f$. To show that the VPD is well-defined, we show that $f'$ and $f''$ produce the same VPD.

Since $f'$ and $f''$ are compatible with $f$ and produce a total order on the simplices of $K$, there exists a sequence of filters $\{f'_1, f'_2, \ldots, f'_m\}$ such that $f'_1 = f'$, $f'_m = f''$, and consecutive filter functions agree except for one $f$-compatible transposition. As in Equation (3) of Definition 1, define the following multisets:

$$A_t := \{(f(\sigma_i), f(\sigma_j))\}_{(i,j) \in \mathcal{D}_k(f'_t)}.$$  

By Lemma 51, we know that $A_t = A_1$ for all $t$. In particular, $A_m = A_1$. Thus, both $f'$ and $f''$ produce the same verbose persistence diagram, which means that VPDs are well-defined.

To show that VECFs are well defined, we first observe the following relationship between descriptors, arising from the connection between homology and Euler characteristic.

**Observation 53.** Given the verbose persistence diagram $\hat{D}(f)$ for some simplicial complex $K$ and monotonic function $f: K \to \mathbb{R}$, we know the Betti numbers at every sublevel set of the filtration and can thus construct the verbose Betti function $\hat{\beta}_f$. Furthermore, we can construct the verbose Euler characteristic function $\hat{C}_f$ by taking the alternating sum of these Betti numbers.

Then, well definedness of both the VBF and VECF are straightforward corollaries of Lemma 52; since two distinct index filters compatible with the same monotonic function produce the same VPD, they produce the same VBF and the same VECF.

**Corollary 54 (VBF is Well-Defined).** Let $K$ be a simplicial complex, and let $f: K \to \mathbb{R}$ be a monotonic function. Then, the verbose Betti function $\hat{\beta}_f$ is well-defined.

**Corollary 55 (VECF is Well-Defined).** Let $K$ be a simplicial complex, and let $f: K \to \mathbb{R}$ be a monotonic function. Then, the verbose Euler characteristic function $\hat{C}_f$ is well-defined.