New $G_2$ Metric, D6-branes and Lattice Universe

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ABSTRACT

We construct a new (singular) cohomogeneity-three metric of $G_2$ holonomy. The solution can be viewed as a triple intersection of smeared Taub-NUTs. The metric comprises three non-compact radial-type coordinates, with the principal orbits being a $T^3$ bundle over $S^1$. We consider an M-theory vacuum $(\text{Minkowski})_4 \times \mathcal{M}_7$ where $\mathcal{M}_7$ is the $G_2$ manifold. Upon reduction on a circle in the $T^3$, we obtain the intersection of a D6-brane, a Taub-NUT and a 6-brane with R-R 2-form flux. Reducing the solution instead on the base space $S^1$, we obtain three intersecting 6-branes all carrying R-R 2-form flux. These two configurations can be viewed as a classical flop in the type IIA string theory. After reducing on the full principal orbits and the spatial world-volume, we obtain a four-dimensional metric describing a lattice universe, in which the three non-compact coordinates of the $G_2$ manifold are identified with the spatial coordinates of our universe.

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1 Introduction

Seven-dimensional manifolds of $G_2$ holonomy have long been known to exist. The construction of explicit non-compact $G_2$ metrics began ten years ago, when asymptotically conical metrics of cohomogeneity one were found [1, 2]. The physical interest of $G_2$ manifolds has increasingly significantly with the discovery of M-theory, because they are the most natural compactifying spaces from the eleven-dimensional point of view. It is expected that M-theory compactified on a $G_2$ manifold gives rise to an $\mathcal{N} = 1$ super Yang-Mills theory in $D = 4$ [3]. The $G_2$ manifold with principal orbits $S^3 \times S^3$ provides a geometrical demonstration of the classical flop of the type IIA superstring theory [4]. In [5], M-theory dynamics on a $G_2$ manifold were discussed.

Recently, a large class of new metrics of $G_2$ holonomy have been obtained [6]-[17], following the construction of the first examples of asymptotically locally conical spin(7) manifolds [18]. These examples have non-abelian isometry groups. $G_2$ metrics with nilpotent isometry groups were also constructed in [19], which can be obtained by taking the Heisenberg or Euclidean limits of the non-abelian examples. Whilst it is of great interest to construct regular $G_2$ metrics, physically, it is essential to have an appropriate singularity structure to give rise to chiral fermions in $D = 4$ [20, 21].

In section 2, we construct a new non-compact cohomogeneity three metric with $G_2$ holonomy. The metric has three non-compact radial-type coordinates, with the principal orbits being a $T^3$ bundle over $S^1$. The isometry group of the metric is a four-dimensional nilpotent Lie group. The metric has either power-law singularities or delta-function singularities. The solution can be viewed as the intersection of three smeared Taub-NUTs. When one of the Taub-NUT charges is set to zero, the metric describes a product of $S^1$ with a six-dimensional non-compact Calabi-Yau manifold.

In section 3, we consider an M-theory vacuum (Minkowski)$_4 \times \mathcal{M}_7$, where $\mathcal{M}_7$ is the $G_2$ manifold. We show that by dimensionally reducing the solution on one of the circles in the $T^3$, we obtain a type IIA configuration with one D6-brane, one Taub-NUT and one 6-brane with an R-R 2-form flux. On the other hand, if we reduce the solution on the base space $S^1$, we obtain an intersection of three 6-branes all carrying R-R 2-form flux. These two configurations can be viewed as the classical flop of type IIA string theory on a non-compact six-dimensional Kähler manifold with a nilpotent isometry group. The origin of the flop is that the $T^3$ bundle over $S^1$ principal orbits can also be viewed as $S^1$ bundle over $T^3$.

In section 4, we perform a Kaluza-Klein reduction on the full principal orbits and the
spatial world-volume. We obtain three perpendicularly intersecting membranes in $D = 4$, describing a lattice universe. In this picture, the three non-compact coordinates of the $G_2$ manifold are identified with the spatial coordinates of our universe. We conclude the letter in section 5.

## 2 New $G_2$ metric

The metric ansatz is given by

$$ds^2_i = H_1 \, dx_1^2 + H_2 \, dx_2^2 + H_3 \, dx_3^2 + H_1 \, H_2 \, H_3 \, dz_1^2 + H_1^{-1} \, (dz_1 + H_1' \, z_2 \, dz_4)^2$$
$$+ H_1 \, H_2^{-1} \, (dz_2 + H_2' \, z_3 \, dz_4)^2 + H_2 \, H_3^{-1} \, (dz_3 + H_3' \, z_1 \, dz_4)^2,$$

where $H_1$, $H_2$ and $H_3$ are functions of $x_1$, $x_2$ and $x_3$ respectively. The prime on $H_i'$ denotes a derivative with respect to the argument of $H_i$:

$$H'_1 = \partial_{x_1} H_1, \quad H'_2 = \partial_{x_2} H_2, \quad H'_3 = \partial_{x_3} H_3. \quad (2)$$

The natural vielbein basis is

$$e^0 = \sqrt{H_1 \, H_2 \, H_3} \, dz_4, \quad e^1 = \sqrt{H_1} \, dx_1, \quad e^2 = \sqrt{H_2} \, dx_2, \quad e^3 = \sqrt{H_3} \, dx_3,$$
$$e^4 = \sqrt{H_3 \, H_1^{-1}} \, (dz_1 + H_1' \, z_2 \, dz_4), \quad e^5 = \sqrt{H_1 \, H_2^{-1}} \, (dz_2 + H_2' \, z_3 \, dz_4),$$
$$e^6 = \sqrt{H_2 \, H_3^{-1}} \, (dz_3 + H_3' \, z_1 \, dz_4). \quad (3)$$

The associative 3-form in this basis is given by

$$\Phi = e^{016} + e^{024} + e^{035} + e^{125} - e^{134} + e^{236} - e^{456}, \quad (4)$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$. The metric (1) has $G_2$ holonomy if and only if $\Phi$ is closed and co-closed. We find that the closure and co-closure of $\Phi$ implies that

$$H''_i = 0, \quad i = 1, 2, 3,$$

implying that

$$H_1 = 1 + m_1 \, x_1, \quad H_2 = 1 + m_2 \, x_2, \quad H_3 = 1 + m_3 \, x_3. \quad (5)$$

Here the constant 1 is included so that $H_i$ does not vanish when $m_i = 0$. Clearly the metric has a power-law singularity whenever any of the $H_i$ vanishes. The metric can also be recast in a “co-moving” frame,

$$ds^2_i = dr_1^2 + dr_2^2 + dr_3^2 + \frac{9}{4} (m_1 \, m_2 \, m_3 \, r_1 \, r_2 \, r_3)^{2/3} \, dz_4^2 + \frac{(m_3 \, r_3)^{2/3}}{(m_1 \, r_1)} \, (dz_1 + m_1 \, z_2 \, dz_4)^2$$
$$+ \frac{(m_1 \, r_1)^{2/3}}{(m_2 \, r_2)} \, (dz_2 + m_2 \, z_3 \, dz_4)^2 + \frac{(m_2 \, r_2)^{2/3}}{(m_3 \, r_3)} \, (dz_3 + m_3 \, z_1 \, dz_4)^2. \quad (7)$$


The fibration in the \( z_1 \), \( z_2 \) and \( z_3 \) coordinates implies that the constants \( m_i \) are quantised, namely
\[
  m_1 = n_1 \frac{L_1}{L_2 L_4}, \quad m_2 = n_2 \frac{L_2}{L_3 L_4}, \quad m_3 = n_3 \frac{L_3}{L_1 L_4},
\]
where \( n_i \) are integers and \( L_i \) are the periods of the \( z_i \). For simplicity, we can set \( L_i = \ell_p \) where \( \ell_p \) is the Plank length, and then \( m_i = n_i/\ell_p \).

The metric has a power-law singularity when any of the \( H_i \) vanishes. This can be avoided by instead taking
\[
  H_i = 1 + \sum_\alpha m_i^\alpha |x_i^\alpha - x_i^\alpha|.
\]
such that \( H_i \) is positive definite. However, in doing so, we have introduced delta function singularities.

When all three of the \( H_i' \) are non-vanishing, the metric describes three intersecting Taub-NUTs with three independent non-vanishing smeared charges. The metric has three non-compact coordinates \( x_1 \), \( x_2 \) and \( x_3 \). The principal orbits are \( T^3 \) bundle over \( S^1 \); they are parameterised by the coordinates \((z_1, z_2, z_3)\) and \( z_4 \) respectively. The metric is of cohomogeneity three since it depends explicitly on the three non-compact coordinates \( x_i \). Despite the dependence on the \((z_1, z_2, z_3)\) coordinates, they, together with \( z_4 \), parameterise a four-dimensional nilpotent Lie group \( G \), which is the isometry group of the metric, and thus the four-dimensional principal orbits are homogeneous.

When two of the \( H_i' \) vanish, the metric describes a direct product of Euclidean 3-space and a smeared Taub-NUT. if instead only one of \( H_i' \) vanishes, in which case the metric was obtained in \( \mathbb{P}^3 \), it describes a product of an \( S^1 \) with a Calabi-Yau 6-manifold. To see this in detail, let us set \( H_3 = 1 \). The metric of the Calabi-Yau manifold is then given by
\[
  ds_6^2 = H_1 dx_1^2 + H_2 dx_2^2 + H_1 H_2 dz_4^2 + H_2 dz_3^2 \\
  + H_1^{-1} (dz_1 + H_1' z_2 dz_4)^2 + H_1 H_2^{-1} (dz_2 + H_2' z_3 dz_4)^2.
\]
and the Kähler form is given by
\[
  J = e^0 \wedge e^5 - e^1 \wedge e^4 + e^2 \wedge e^6,
\]
where the vielbein is given by \( (3) \) with \( H_3 = 1 \).
3 Intersecting D6-branes

Having obtained the new $G_2$ metric, one may consider an M-theory vacuum solution given by the direct product of Minkowski 4-spacetime and the $G_2$ manifold, namely

$$ds_{11}^2 = -dt^2 + dw_1^2 + dw_2^2 + dw_3^2 + ds_i^2.$$  \hfill (12)

The solution can be viewed as a triple intersection of smeared Taub-NUTs, with the metric represented by the diagram

|   | $t$ | $w_1$ | $w_2$ | $w_3$ | $x_1$ | $x_2$ | $x_3$ | $z_1$ | $z_2$ | $z_3$ | $z_4$ |
|---|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $H_1(x_1)$ | x | x | x | x | – | x | x | * | – | x | – |
| $H_2(x_2)$ | x | x | x | x | x | – | x | x | * | – | – |
| $H_3(x_3)$ | x | x | x | x | x | x | – | – | x | * | – |

Diagram 1. Triple intersections of Taub-NUTs. Here $\times, –$ and $*$ denote the world-volume, transverse space, and fibre coordinates respectively.

There is a $U(1)$ isometry for each of the $z_i$ coordinates, and so we can reduce the metric on any $z_i$ to obtain a solution in type IIA theory. The $z_i$ for $i = 1, 2, 3$ are equivalent, and hence the reduction can be discussed using $z_1$ as a representative. The resulting type IIA solution is given by

$$e^\Phi ds_{10}^2 = -dt^2 + dw_1^2 + dw_2^2 + dw_3^2 + H_1 dx_1^2 + H_2 dx_2^2 + H_3 dx_3^2 + H_1 H_2 H_3 dz_4^2 + H_1 H_2^{-1} (dz_2 + H_2^2 z_3 dz_4)^2 + H_2 H_3^{-1} dz_3^2 + H_3 H_1 (H_1^t z_2)^2 dz_4^2$$

$$-W^{-1} (H_3 H_1^{-1} H_1^t z_2 dz_4 - H_2 H_3^{-1} H_3^t z_4 dz_3)^2,$$

$$e^\phi = W^{-\frac{3}{4}}, \quad W = H_3 H_1^{-1} + H_2 H_3^{-1} (H_3^t z_4)^2,$$

$$A_{(1)} = W^{-1} (H_3 H_1^{-1} H_1^t z_2 dz_4 - H_2 H_3^{-1} H_3^t z_4 dz_3).$$  \hfill (13)

Note that before performing the Kaluza-Klein reduction, we have made a coordinate transformation $z_3 \to z_3 - H_3^t z_1 z_4$ in the metric (11). Clearly, the solution describes an intersection of three objects. The one parameterised by $H_1$ is a smeared D6-brane, and the one parameterised by $H_2$ is a Taub-NUT. The one associated with $H_3$ is a 6-brane carrying an R-R 2-form flux, but it differs from a standard D6-brane.

We can instead reduce the solution on the $z_4$ coordinate, giving the type IIA solution

$$e^{\Phi} ds_{10}^2 = -dt^2 + dw_1^2 + dw_2^2 + dw_3^2 + H_1 dx_1^2 + H_2 dx_2^2 + H_3 dx_3^2$$

$$= -dt^2 + dw_1^2 + dw_2^2 + dw_3^2 + H_1 dx_1^2 + H_2 dx_2^2 + H_3 dx_3^2.$$  \hfill (14)
The solution describes three intersecting 6-branes all carrying R-R 2-form flux. These 6-branes are different from the usual D6-brane coming from the reduction of the fibre coordinate of a Taub-NUT in $D = 11$.

The two configurations arising from the reduction on $z_1$ or $z_4$ can be viewed as a classical flop in the type IIA string theory on the non-compact Kähler manifold. The flop in $D = 10$ can be geometrically explained by the fact that the $T^3$ bundle over $S^1$ principal orbits of the four-dimensional nilpotent Lie group $G$ can also be described as an $S^1$ bundle over $T^3$. However, the two descriptions are somewhat different. In the latter case, the fibre is the circle group in $G$ generated by $\frac{\partial}{\partial z_4}$. In the former case, the fibre is not the orbit of a three-dimensional subgroup of $G$ because $\frac{\partial}{\partial z_i}$ for $i = 1, 2, 3$ are not themselves Killing vectors; we must add a multiple of $\frac{\partial}{\partial z_4}$. In fact the flop involves interchanging the fibre and base spaces of the $U(1)$ fibration. This is analogous to the flop discussed in [4].

4 Lattice universe

The new $G_2$ metric ($\mathbb{I}$) that we have obtained is in fact inspired by the four-dimensional intersecting membrane solution that describes the lattice universe [23]. There has been experimental evidence suggesting that the network of galaxy superclusters and voids seems to form a three-dimensional lattice with a spacing of about $120h^{-1}$ Mpc (where $h^{-1}$ is the Hubble constant in units of $100 km s^{-1} Mpc^{-1}$) [25, 24, 27]. In [23], an M-theory solution was constructed to describe such a lattice structure, which can be realised by considering non-standard brane intersections of two M5-branes and one Taub-NUT, or two Taub-NUTs and one M5-brane. In the latter case, turning off the M5-brane charge causes the solution to reduce to the product of 5-dimensional Minkowski spacetime and the non-compact Calabi-Yau manifold given in (10).

In section 2, we obtained the new $G_2$ metric ($\mathbb{I}$) by adding an extra fibration on the seventh coordinate. This procedure follows the general prescription of obtaining $G_2$ manifolds from six-dimensional Kähler manifolds, described in detail in [28].

\footnote{In [24], a triply quasi-periodic Gibbons-Hawking metric was obtained.}
If we reduce the M-theory solution on the world-volume spatial coordinates \( w_i \) and also on the \( T^3 \) bundle over \( S^1 \) principal orbits, we obtain three perpendicularly intersecting membranes in \( D = 4 \), with the metric

\[
 ds_4^2 = (H_1 H_2 H_3)^{1/2} (-dt^2 + H_1 dx_1^2 + H_2 dx_2^2 + H_3 dx_3^2). \tag{15}
\]

This metric was first obtained in [23], although it was supported by very different field strength. The functions of \( H_i \) in this case are given by (9) describing periodic arrays of intersecting membranes.

In this static cosmological model the spatial world-volume and the \( T^3 \) bundle over \( S^1 \) principal orbits are viewed as an internal space, whilst the three non-compact coordinates \( x_1, x_2 \) and \( x_3 \) of the \( G_2 \) manifold are identified with the spatial coordinates of our universe. This is rather natural since the principal orbits are clearly compact, and the spatial world-volume can be wrapped on a compact space such as \( T^3 \). Although it is not likely that the metric (15) describes our actual universe, since it preserves \( \mathcal{N} = 1 \) supersymmetry; it is nevertheless rather suggestive that the lattice structure should emerge from a metric with \( G_2 \) holonomy.

## 5 Conclusions

In this letter, we constructed a new cohomogeneity-three metric with \( G_2 \) holonomy. It has three radial-type coordinates, with the principle orbits being a \( T^3 \) bundle over \( S^1 \). The solution can be viewed as three intersecting Taub-NUTs. We performed Kaluza-Klein reduction on the \( S^1 \) and instead on a circle in the \( T^3 \). The two resulting type IIA configurations can be viewed as a classical flop in type IIA string theory on a non-compact Kähler six-manifold. Although the type IIA solutions do not describe the triply intersecting D6-branes advocated in [21, 29] for the realisation of chiral fermions, it is nevertheless of interest to investigate further if chirality could arise from the singularities of our \( G_2 \) metric. The metric provides a concrete example for studying such an issue, since it can be viewed as the lifting of a D6-brane configurations in \( D = 10 \).

If we perform a Kaluza-Klein reduction on the entire four-dimensional full principal orbits and the spatial world-volume, we obtain triply intersecting membranes in \( D = 4 \), describing a lattice universe. The construction takes full advantage of the non-compact nature of the manifold, in that the three non-compact coordinates are precisely identified with the spatial coordinates of our universe. It is of great interest to investigate further the significance of such a configuration arising from a \( G_2 \) manifold.
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