A comment on the properties of the matter flow through the first Langrangian point

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ABSTRACT
We analyse properties of the mass outflow from the Roche-lobe filling component of a semi-detached binary system. We follow the approaches published by Paczyński & Sienkiewicz and by Lubow & Shu, which we compare with other simplified approaches. We find that the density of the flow at $L_1$ is orders of magnitude lower than the density on the same equipotential but away from $L_1$. Furthermore, the effective cross section of the flow, after averaging over its profile of the momentum density, is much lower than some published estimates done without accounting for the averaging. Thus, the use of some simplified formulae for the density and the flow cross section can lead to overestimates of the accretion rate and of the mass contained in the $L_1$ regions by very large factors unless they are supported by simultaneous integrations of the equations of stellar structure for the outer layers of the donor.

Key words: stars: general – stars: evolution – stars: binary – binaries: mass transfer

1 INTRODUCTION
The knowledge of the mass transfer rate from the donor to accretor, $\dot{M}$, in accreting stellar systems is of major importance for their understanding. By comparing the transfer rate with the accretion rate we can assess whether the mass transfer is conservative or associated with mass loss, as well as examine an effect of irradiation on the mass flow. The transfer rate then determines the past and future evolution of accreting systems. Observationally, $\beta M$ can be estimated from a long-term average of the luminosity in an X-ray binary, e.g., Coriat, Fender & Dubus (2012), Zdziarski, Ziolkowski & Mikołajewska (2019), where $\beta \leq 1$ is the fraction of the transferred mass accreted. However, for understanding of many aspects of these systems, it is important to estimate $\dot{M}$ theoretically, and compare the two estimates. In semi-detached binaries, such estimates can be done based on considering the evolutionary changes of the stellar radius compared to the changes of the Roche lobe, e.g., Webbink, Rappaport & Savonije (1983). If the mass transfer is driven by nuclear expansion of the donor, $\dot{M}$ is connected to the time scale of this expansion, see equation (4.16) of Frank, King & Raine (2002). Using both approaches, we can relatively reliably determine the rate of the mass transfer, $\dot{M}$.

On the other hand, sometimes it is useful to consider the dynamics of the flow through the vicinity of the inner Lagrangian point of the Roche lobe, $L_1$. This is the case if we need to consider the conditions near the $L_1$ point, e.g., in order to investigate the effects of irradiation. Such analysis is helped by approximate formulae relating the radius excess, $\Delta R$, i.e., the difference between the stellar radius and the radius of the Roche lobe, to the rate of the mass transfer $\dot{M}$ (Paczyński & Sienkiewicz 1972, hereafter PS72, who used in part unpublished results of Jędrzejec 1969; Savonije 1978, 1979, 1983).

2 ESTIMATES OF THE FLOW PARAMETERS
2.1 The approach of PS72
Here, we recall and extend some of the results of PS72 and compare them to other expressions in literature. We use the coordinate system in the co-rotating frame as shown by PS72 in their fig. A1. The $x$ axis connects the stellar centres, and it is given in units of the semi-major axis (equal to the separation between the stars, $A$, for a circular orbit). The orbital motion is neglected, and thus the system has an axial symmetry around $x$, and the $y$ axis is perpendicular to $x$ in any direction. We will also neglect a small difference between the cross sections at $x = 0$ of the star overflowing its Roche lobe, $y_1$, and of the equipotential surface which the star fills, $y$. This results in a factor of slightly less than unity by which the true $M$ should be multiplied (see table A1 in PS72).

The dimensionless radius of the flow cross section at $x = 0$ is then
given by (using equation A15 of PS72 at $y = 0$)

$$y_i \equiv \left(\frac{2A}{R} \frac{\Delta R}{R} \frac{\mu^{1/4}(1 - \mu)^{1/4}}{1 + 2\sqrt{\mu(1 - \mu)}}\right)^{1/2},$$  

(1)

where $R$ is the donor radius, $\Delta R$ is the radius excess, i.e., the difference between the stellar radius and the radius of the Roche lobe, $\mu \equiv M/(M + M_X)$, $M$ and $M_X$ are the donor and accretor masses, respectively, $A$ is given by the Kepler law,

$$A^3 = \frac{P^2GM}{4\pi^2\mu},$$  

(2)

and $P$ is the binary period. Equation (1) is based on purely geometrical considerations and as such is not subject to serious errors (other than neglecting the orbital motion and approximating $y_i$ by $y_i$). The total cross section of the flow at $L_i$ is given by $\Sigma = \pi(Ay_i)^2$.

Then, PS72 assume that pressure, $p$, is given by the polytropic relation,

$$p = K\rho^{1 + 1/n},$$  

(3)

where $K$ is a constant. With that assumption, they find the distribution of the momentum density, $\rho v$, as a function of the distance from the $L_i$ point along the $y$ axis,

$$\rho v \approx \begin{cases} C(y_i^2 - y^2)^{n + 1/2}, & y \leq y_i; \\ 0, & y > y_i, \end{cases}$$  

(4)

where

$$C(\frac{\Omega_1 h - \Omega_0}{n + 1/2})^{n + 1/2} = [K(1 + 1/n)]^{-n}, \quad \frac{\Omega_0}{\Omega_1} = \frac{GM}{A^2}, \quad \frac{h}{\mu(1 - \mu)} = \frac{1 + 2\sqrt{\mu(1 - \mu)}}{2\sqrt{\mu(1 - \mu)}},$$  

(5-7)

We note here that $\rho v$ decreases quickly with the distance along $y$ away from $L_i$, and its average value is given by

$$\langle \rho v \rangle = \frac{2}{2n + 3} \rho_{L_i} y_{L_i} y_{L_i},$$  

(8)

$$\rho_{L_i} y_{L_i} = \left[ \frac{\Delta R GM}{R^2 R(1/2 + n)} \right]^{1/(2n)} [K(1 + 1/n)]^{-n},$$  

(9)

where the subscript $L_i$ denotes a quantity measured at $L_i$.

We can also write in general

$$M = \Sigma_{coll} \rho_{L_i},$$  

(10)

where $\Sigma_{coll}$ is the effective cross section taking into account the averaging of equation (8),

$$\Sigma_{coll} = \frac{\Delta R GM}{R^2 R} \frac{\sqrt{\mu(1 - \mu)}}{\pi(2n + 3) [1 + 2\sqrt{\mu(1 - \mu)}]},$$  

(11)

If $M \leq 0.6M_X$, we can use the approximation to the donor’s Roche-lobe radius by Paczyński (1967), $R = (2GM)^{1/3}(P/9\pi)^{1/3}$, which yields

$$\Sigma_{coll} = R\Delta R \frac{3\pi\mu(1 - \mu)}{2(2n + 3) [1 + 2\sqrt{\mu(1 - \mu)}]},$$  

(12)

Thus $\Sigma_{coll} \sim R\Delta R$ (as noted by Savonije 1979), which is much larger than a very simple, but incorrect, estimate of $\sim (\Delta R)^2$. The $\Sigma_{coll}$ of equation (11) can be compared with an estimate of $\Sigma$ in Zdziarski, Wen & Gierliński (2007), which is an approximation to a formula of Savonije (1983),

$$\Sigma = \frac{\Delta R GM P^2}{R^2 R} \frac{\mu^{1/2}}{2\pi},$$  

(13)

This does not include the averaging and mass-ratio terms, and is usually an overestimate.

Using $M = \Sigma_{coll} \rho_{L_i}$ together with equations (9) and (11), we obtain

$$M = \left[ \frac{\Delta R GM}{R^2 R} \right]^{n + 3/2} \frac{\mu^{1/2}}{2\pi} \left[ 1 + 2\sqrt{\mu(1 - \mu)} \right] C_n,$$  

(14)

$$C_n = \frac{1}{\pi(2n + 3)(n + 1/2)} \frac{\mu^{1/2}}{(1 + 1/n)^n},$$  

(15)

which is in complete agreement with equations (A21), (A22) and (A17) of PS72 (who expressed $M$ in terms of $A$ instead of $P$). Note that the first and second term in equation (14) are given by very large and very small numbers, respectively, especially for a high $n$, and a care is needed in their numerical calculation.

We note that Savonije (1978) repeated the analysis of PS72 in a more precise way. In particular, he took into account the orbital motion of the matter in the binary system. His formulae are more complicated, but the general results are similar to those of PS72. In particular, he obtained the same functional form of the relations given here by equations (4) and (14); specifically, with the same power exponents.

Equation (14) can give us a reasonably accurate estimate of $\Delta R$ if we know $M$. This formula takes into account the physical state of the outflowing matter through the polytrope constant, $K$, and the polytropic exponent, $n$. Thus, in order to determine the values of these parameters we need a model of the outer layers of the donor. Importantly, we definitely need these values at the $L_i$ point rather than at the depth $\Delta R$ below the surface of the star far from $L_i$ (as practised by some simplified formulae users). Also, if we use directly equation (10) with estimates of $\Sigma$, we need the values of density and pressure at $L_i$ rather than away from it at the depth $\Delta R$.

Furthermore, we need an estimate of the velocity of the flow. The most natural assumption appears that the velocity of the matter passing through the $x = 0$ plane is similar to the sonic velocity, $c_s$. Arguments in favour of that were given by LS75 and Savonije (1978). If we assume that the matter crossing the $x = 0$ plane has at each point the velocity equal to the speed of sound, then for a polytrope

$$v^2 = c_s^2 = (1 + 1/n)K\rho^{1/n},$$  

(16)

We can then obtain the density profile at $x = 0$ with equation (4) to be given by

$$\rho(y) = \frac{C_{\text{in}}}{[K(1 + 1/n)]^{1/2n}}, \quad y \leq y_i,$$  

(17)

and $\rho_{L_i} = \rho(0)$.

### 2.2 The approach of LS75

LS75 used a different approach than PS72. Their description of the flow through $L_i$ point is more physical and covers much larger range of effects than that of PS72. On the other hand, their formulæ are less quantitative. They show that the dimensional flow radius is

$$r_c \approx c_s/\Omega,$$  

(18)

where $c_s$ is the gas sound speed at $L_i$ and $\Omega = 2\pi/P$ is the binary orbital frequency. By comparing with equation (1), we see that the
two become similar (apart from the \( \mu \)-dependent term of the order of unity) if \( \Delta R \sim H_\ast \), where \( H_\ast \) is the stellar scale height (since it is defined by \( c_s^2 = (H_\ast/R)(GM/R) \)). We note that this fixes \( \Delta R \) independently of the rate of the mass outflow and of the physical state of the outflowing matter (with the exception of its temperature), which does not seem to be generally valid. Still, at least in some cases (Section 3) it gives results that are surprisingly close to the results obtained using eq. (9) and assuming sonic velocity of gas at \( L_1 \).

As a consequence of equation (18), LS75 further argue that

\[
\rho_1 \approx \frac{M}{4\pi c_s^4} \frac{4\pi^2}{P^2 c_s^2}.
\]

The above equation is presented in the discussion of their equation (11). Using equation (10) with \( \nu_1 = c_s \), we have

\[
\Sigma_{eff} \approx \frac{v_1}{c_s} = \frac{c_s^2}{4\pi^2}.
\]

We note that the same formulae (with the accuracy to a factor of \( \pi \)) are used by Frank et al. (2002) (p. 352).

Also, LS75 estimated the ratio of the density at the Roche equipotential inside the star far from \( L_1 \), which we will denote as \( \rho_0 \), to that at \( L_1 \), as

\[
\frac{\rho_0}{\rho_{L_1}} = \frac{\Omega A}{c_s}.
\]

(see discussion after their equation 64).

### 3 COMPARISON WITH A MODEL OF GX 339–4

As an example, we consider a model of the binary system GX 339–4. This system accretes from an evolved low-mass star onto a black hole, and its binary period is \( P = 1.75877 \) d. Specifically, we consider a model slightly modified with respect to the evolutionary model D obtained by Zdziarski et al. (2019), and assume \( M = 6.1 \times 10^{-6} \) g s\(^{-1}\).

This \( M \) is equal to the value obtained from the evolutionary model by matching the rate of Roche-lobe expansion to the stellar expansion. In that model, we have \( M = 1M_\odot R = 2.83R_\odot \), the accretor mass of \( M_K = 0.8M_\odot \), giving \( \mu = 1/9 \), and the radius excess of \( \Delta R \approx 1.22 \times 10^5 \) cm (a slightly corrected value, following from equation 14). The stellar density at the depth \( \Delta R \) at the equipotential describing the Roche lobe away from \( L_1 \) was found to be \( \rho_0 = 3.1 \times 10^{-6} \) g cm\(^{-3}\) and the temperature was \( T_\odot = 1.56 \times 10^4 \) K. The local value of the polytropic exponent, \( n \), estimated at that depth, is \( n = 6.43 \). The matter there is almost fully ionized, and \( X = 0.74 \) and \( Z = 0.014 \) was assumed, which implies the mean molecular weight of 0.598. Using equation (3) and assuming ideal gas we obtain the entropy parameter of \( K \approx 1.55 \times 10^{13} \) (cgs), and the sound speed of \( c_{so} \approx 1.6 \times 10^7 \) cm s\(^{-1}\). These values of \( K \) and \( n \) are then assumed to apply to the flow through \( L_1 \) and be constant within it.

For the above parameters, we can find the velocity and density at \( L_1 \) from equations (16) and (17), respectively. They are \( v_{L_1} \approx 7.8 \times 10^5 \) cm s\(^{-1}\) and \( \rho_{L_1} \approx 3.4 \times 10^{-10} \) g cm\(^{-3}\). Thus, the density at \( L_1 \) is as much as \( \approx 10^4 \) times smaller than the stellar density at the depth \( \Delta R \) (while the sound speed is by a factor of two smaller).

The geometrical cross section of the flow can be calculated using equation (1). We get \( y \approx 0.027 \), which translates into the flow radius of \( 2.67 \times 10^{10} \) cm and the total cross section area of \( 2.24 \times 10^5 \) cm\(^2\). We note that either \( \rho v \) decreases fast with increasing \( y \), especially for a high value of \( n \), which decrease is taken into account in our formulae for either \( \rho v \) or \( \Sigma_{eff} \), equations (8), (11), respectively. For the example considered here, \( \rho v \) drops to \( 10^{-3} \) of its value at the point \( L_1 \) at \( y \approx 0.0215 \).

Now, let us compare the values of different parameters of the flow obtained with the help of different formulae used in literature.

We shall start with the geometrical cross section of the flow. As was shown above, the analysis of PS72 leads to the radius of the flow equal \( 2.67 \times 10^{10} \) cm. Using estimate of LS75 given by equation (18) we get \( r_1 \approx 1.89 \times 10^{10} \) cm, i.e., a value smaller by a factor of only \( \approx 1.4 \). Taking into account the approximate character of both estimates, the agreement is excellent.

We have to remember, however, that using equation (18) requires the knowledge of the sound speed at the \( L_1 \) point. In our case, this knowledge was based on the analysis following the approach of PS72 and on the simultaneous integration of the equations of stellar structure for the outer layers of the donor.

Now, let us compare the values of the flow effective (as opposed to the geometrical one) cross section area obtained with the different formulae listed in Section 2. The value following from the analysis of PS72 is given by equation (11). For the considered example, \( \Sigma_{eff} \approx 2.3 \times 10^{20} \) cm\(^2\). Then, the approximation neglecting the averaging of the flow over the cross section proposed by Zdziarski et al. (2007), of equation (13), yields \( \Sigma \approx 1.5 \times 10^{22} \) cm\(^2\), i.e., almost two orders of magnitude too much. The estimate by LS75, as given by our equation (20), leads to the value \( \Sigma_0 \approx 3.56 \times 10^{20} \) cm\(^2\). This value is only by a factor \( \approx 1.5 \) greater than the value obtained by us with equation (11), which again means that the agreement between the two approaches is excellent. Still, it requires the knowledge of the sound speed at \( L_1 \).

We next compare the values of the density at \( L_1 \) obtained with the different formulae listed in Section 2. The value following from the analysis of PS72 is given by equation (17), which in our case leads to \( \rho_{L_1} \approx 3.4 \times 10^{-10} \) g cm\(^{-3}\). The estimate by LS75, as given by our equation (19), leads to \( \rho_{L_1} \approx 2.20 \times 10^{-10} \) g cm\(^{-3}\). This is by a factor of only \( \approx 1.5 \) smaller than the value obtained by us with equation (17). This is the same factor 1.5 as found by us while comparing the different estimates of the effective cross section, which identity is a consequence of the equivalence of equations (19) and (20).

The value of the density of the gas at \( L_1 \) is certainly one of the important parameters of the flow. The fact that the values of this parameter obtained with two different approaches agree so well is certainly encouraging. It also supports the claim that this estimate is not far from the true value.

Discussing the approximate formulae present in the literature, we should devote attention to popular formulae equivalent to our equation (10) but used in improper way. As an example we may recall equation (9) of Zdziarski et al. (2007). Those authors use equation (10) but they replace the effective cross section area with the geometrical one and the density of the gas at \( L_1 \) with the density on the Roche-lobe equipotential away from \( L_1 \). Since, as we have seen, the effective cross section area might be overestimated this way by two orders of magnitude and the density of the gas even by four orders of magnitude, such use of equation (10) might lead to an error reaching even six orders of magnitude. It is therefore crucial, while using formulae of this type, to use the proper value of the gas density.

Another example may be found in Zdziarski et al. (2019). While those authors properly estimated \( M \) from their evolutionary model, they still used the above simplified treatment to estimate the mass stored within the \( L_1 \) flow, and thus the emptying time of that region, which was then used to estimate the variability time scale related to changing conditions at \( L_1 \). This lead to a major overestimate of the expected variability time scale. Using the proper description of the flow, we can calculate the mass column density, \( \sigma \), perpendicular to
the flow (i.e., the stored in a 1 cm thick slice around \(L_1\)). Integrating equation (17), we obtain

\[
\sigma = \pi A^2 \rho L_1 \frac{y_1^2}{n + 1}.
\]

(22)

For the considered example, \(y_1 \approx 0.027\), and \(\sigma \approx 8.4 \times 10^{10} \, \text{g cm}^{-1}\).

On the other hand, Zdziarski et al. (2019) used \(\sigma = \rho \Delta y_0\), which is \(4.8 \times 10^{10} \, \text{g cm}^{-1}\), a factor of \(\approx 5 \times 10^3\) too much. This then lead to a corresponding overestimate of the emptying time, \(\Delta t\). Those authors also used the \(M\) following from their evolutionary model and approximated the length of that region as \(\Delta R\), yielding \(\Delta t \approx 24\) yr. If we make the same assumptions as them but use the correct \(\sigma\), we obtain \(\Delta t \approx 1300\) s. Even if we change the assumption about the length of the \(L_1\) element made in Zdziarski et al. (2019) to a more realistic value, we can increase the above \(\Delta t\) by at most two orders of magnitude. Thus, the conclusion of them that \(\Delta t\) can account for the observed long-term changes of the accretion rate average over outburst is incorrect, and another mechanism is needed.

Finally, we should comment on the discrepancy between the density of the gas at the Roche equipotential inside the star far from \(L_1\) and the density obtained with the prescription of LS75, equation (21). Using it, we find the ratio of \(\approx 52\). As we saw in our analysis above, that ratio was about four orders of magnitude \((3.1 \times 10^{-4} \text{ vs } 3.4 \times 10^{-10} \, \text{g cm}^{-3})\).

This is a serious discrepancy, by more than two orders of magnitude. However, we believe that our estimate is closer to the real situation than that of LS75. The reason is the following one. We started with the determination of the radius excess, \(\Delta R\). We used for that purpose equation (14). Fortunately, the sensitivity of \(\Delta R\) to \(M\) is very weak. As demonstrated by Zdziarski et al. (2019) (see their fig. 3), increasing \(M\) by four orders of magnitude leads to increase of \(\Delta R\) by a factor of only two. Since we have a relatively reliable estimate of \(M\) based on evolutionary considerations, we may trust that \(\Delta R\) is determined with rather good precision. Knowing the depth of the Roche equipotential below the stellar surface, we may integrate the stellar structure equations (including the heat transfer equation) from the stellar surface inward. At the depth \(\Delta R\) we determine, among others, the density of the gas and the sound speed. This procedure is rather straightforward and should not raise serious doubts. On the other hand, the LS75 estimate is based on rather qualitative analysis of a very complicated flow in the outer stellar layers.

Finally, we should stress that all the considered descriptions are still quite approximate. Hydrodynamic simulations are needed to get more accurate descriptions.

4 CONCLUSIONS

We have overviewed and extended the treatment of the flow through \(L_1\) of PS72, and compared it with other, simplified, methods. We have stressed that the flow density at \(L_1\) can be several orders of magnitude lower than that at the same equipotential but away from \(L_1\).

Furthermore, one of the approximate estimates of the flow effective cross section used in literature overestimates the actual one by two orders of magnitude. The other one, given by LS75, leads to result that remains in excellent agreement with our estimate. The same is true about the estimate of the gas density at \(L_1\) given by LS75 (but to calculate it correctly we have to know the structure of the outer layers of the donor).

However, we have found that the estimate of the ratio of the den-