Convergence in $L_p([0,T])$ of wavelet expansions of \( \varphi \)-sub-Gaussian random processes

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Abstract The article presents new results on convergence in $L_p([0,T])$ of wavelet expansions of \( \varphi \)-sub-Gaussian random processes. The convergence rate of the expansions is obtained. Specifications of the obtained results are discussed.

Keywords Convergence rate · Convergence in probability · \( \varphi \)-sub-Gaussian random process · Wavelets

Mathematics Subject Classification (2000) 60G10 · 60G15 · 42C40

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1 Introduction

Multiresolution analysis of deterministic signals by the use of wavelets has been extensively studied in recent years. However, in the context of stochastic processes, general wavelet approximations has not yet been fully investigated. In the majority of cases developed deterministic methods and used error measures and metrics may not be appropriate to investigate wavelet representations of stochastic processes. It indicates the necessity of elaborating special stochastic techniques.

From a practical point of view, multiresolution analysis provides an efficient framework for the decomposition of random processes. Wavelet representations could be used to convert the problem of analyzing a continuous-time random process to that of analyzing a random sequence, which is much simpler. This approach is widely used in statistics to estimate a curve given observations of the curve plus some noise, in time series analysis for smoothing functional data, in simulation studies of various functionals defined on realizations of a random process, etc.

Recently, a considerable attention was given to wavelet orthonormal series representations of stochastic processes. Numerous results, applications, and references on convergence of wavelet expansions of random processes in various spaces can be found in Atto, Berthoumieu 2010, Bardet, Tudor 2010, Cambanis, Masry 1994, Clausel et al 2012, Didier, Pipiras 2008, Kurbanmuradov, Sabelfeld 2008, Kozachenko et al 2011, 2013, Kozachenko, Polosmak 2008, Zhang, Waiter 1994, just to mention a few.

Figures 1 and 2 illustrate wavelet expansions of stochastic processes. A simulated realization of the process $X(t)$ and its two wavelet reconstructions with different numbers of terms are plotted in Figure 1. Figure 2 displays boxplots of mean-square approximation errors for 500 simulated realizations of $X(t)$ for each reconstruction. Figure 2 suggests that empirical probabilities of large errors become smaller when the number of terms in the wavelet expansion increases.

Although the mentioned effect is well-known for deterministic functions, it has to be established theoretically for different stochastic processes and probability metrics. Numerical simulation results need to be confirmed by theoretical analysis. It is also important to obtain theoretical estimations of the rate of convergence for various stochastic wavelet expansions.

Our focus in this paper is on convergence in $L_p([0,T])$ of wavelet expansions of $\varphi$-sub-Gaussian random processes. The paper extends the recent results on uniform convergence in the papers Kozachenko et al. 2011, 2013, Kozachenko, Polosmak 2008 to new classes of stochastic processes and probability metrics.

The analysis and the approach presented in the paper contribute to the investigations of wavelet expansions of random processes in the former literature. The approach is of a special interest if $p > 2$, as it extends the available $L_2$ results. In that sense, Theorems 4-7 are of special importance. The results are obtained under simple assumptions which can be easily verified. The paper deals with the most general class of wavelet expansions in comparison with particular cases considered by different authors, see, for example, Cambanis, Masry 1994, Kurbanmuradov, Sabelfeld 2008.

\footnote{The figures have been generated by the R packages geoR and wmtsa. The Daubechies D8 wavelet basis and resolution levels 4 and 6 were used.}
Wavelet expansions of $\varphi$-sub-Gaussian processes

The organization of the article is the following. In the second section we introduce the necessary background from the theory of $\varphi$-sub-Gaussian random variables and processes. Section 3 discusses wavelet expansions and approximations of stochastic processes. In §4 we present the main results on convergence in $L_p([0,T])$ of wavelet expansions of $\varphi$-sub-Gaussian random processes. In this section we also obtain the rate of convergence of the wavelet expansions and discuss some specifications for which the assumptions in the theorems can be easily verified.

2 $\varphi$-Sub-Gaussian random processes

In this section, we review the definition of $\varphi$-sub-Gaussian random processes and their relevant properties.

The space of $\varphi$-sub-Gaussian random variables was first introduced in the paper [Kozachenko, Ostrovskyi 1985]. More information about the space of $\varphi$-sub-Gaussian random variables and processes can be found in Buldygin, Kozachenko 2000, Giuliano Antonini et al. 2003, Kozachenko, Kamenshchikova 2009.

Definition 1 (Buldygin, Kozachenko 2000) A continuous even convex function $\varphi(x)$, $x \in \mathbb{R}$, is called an Orlicz N-function, if it is monotonically increasing for $x > 0$, $\varphi(0) = 0$, $\varphi(x)/x \to 0$, when $x \to 0$, and $\varphi(x)/x \to \infty$, when $x \to \infty$.

Assumption Q Let $\varphi(\cdot)$ be an Orlicz N-function and $\lim_{x \to 0} \varphi(x)/x^2 = c > 0$.

Remark 1 The constant $c$ can be equal to $+\infty$. 

Fig. 1 Stochastic process and its two wavelet reconstructions
Proposition 1 (Buldygin, Kozachenko 2000) Every N-function $\varphi(\cdot)$ can be represented as $\varphi(u) = \int_0^{|u|} f(v) \, dv$, where $f(\cdot)$ is a monotonically nondecreasing, right-continuous function, such that $f(0) = 0$ and $f(x) \to \infty$, when $x \to \infty$. The function $f(\cdot)$ is called a density of $\varphi(\cdot)$.

Definition 2 (Buldygin, Kozachenko 2000) Let $\varphi(\cdot), x \in \mathbb{R}$, be an Orlicz N-function. The function $\varphi^*(\cdot), x \in \mathbb{R}$, defined by the formula $\varphi^*(x) := \sup_{y \in \mathbb{R}} (xy - \varphi(y))$ is called the Young–Fenchel transform of $\varphi(\cdot)$.

Let $\{\Omega, \mathcal{B}, \mathbb{P}\}$ be a standard probability space and $L_p(\Omega)$ denote a space of random variables having finite $p$-th absolute moments.

Definition 3 (Giuliano Antonini et al. 2003) Let $\varphi(\cdot)$ be an Orlicz N-function satisfying the assumption $Q$. A zero mean random variable $\xi$ belongs to the space $\text{Sub}_{\varphi}(\Omega)$ (the space of $\varphi$-sub-Gaussian random variables), if there exists a constant $r_\xi \geq 0$ such that the inequality $E \exp(\lambda \xi) \leq \exp(\varphi(r_\xi \lambda))$ holds for all $\lambda \in \mathbb{R}$.

Proposition 2 (Kozachenko, Ostrovskyi 1985) The space $\text{Sub}_{\varphi}(\Omega)$ is a Banach space with respect to the norm

$$\tau_{\varphi}(\xi) := \inf\{a \geq 0 : E \exp(\lambda \xi) \leq \exp(\varphi(a \lambda)), \lambda \in \mathbb{R}\}.$$ 

Remark 2 A Gaussian centered random variable $\xi$ belongs to the space $\text{Sub}_{\varphi}(\Omega)$, where $\varphi(x) = x^2/2$ and $\tau^2_{\varphi}(\xi) = E\xi^2$.

Definition 4 (Buldygin, Kozachenko 2000) Let $T$ be a parametric space. A random process $X(t), t \in T$, belongs to the space $\text{Sub}_{\varphi}(\Omega)$ if $X(t) \in \text{Sub}_{\varphi}(\Omega)$ for all $t \in T$ and $\sup_{t \in T} \tau_{\varphi}(X(t)) < \infty$. 

Fig. 2 Boxplots of reconstruction errors for two approximations
Let us denote $\tau_\varphi(t) := \tau_\varphi(X(t))$.

**Definition 5** [Kozachenko, Kovalchuk 1985] A family $\mathcal{F}$ of random variables $\xi \in \text{Sub}_\varphi(\Omega)$ is called strictly $\text{Sub}_\varphi(\Omega)$ if there exists a constant $C_{\mathcal{F}} > 0$ such that for any finite set $I$, $\xi_i \in \mathcal{F}$, $i \in I$, and for any arbitrary $\lambda_i \in \mathbb{R}$, $i \in I$:

$$
\tau_\varphi \left( \sum_{i \in I} \lambda_i \xi_i \right) \leq C_{\mathcal{F}} \left( \mathbb{E} \left( \sum_{i \in I} \lambda_i \xi_i \right)^2 \right)^{1/2}.
$$

$C_{\mathcal{F}}$ is called a determinative constant.

**Definition 6** [Kozachenko, Kovalchuk 1985] $\varphi$-sub-Gaussian random process $X(t)$, $t \in T$, is called strictly $\text{Sub}_\varphi(\Omega)$ if the family of random variables $\{X(t), t \in T\}$ is strictly $\text{Sub}_\varphi(\Omega)$. The determinative constant of this family is called a determinative constant of the process and denoted by $C_X$.

**Remark 3** Gaussian centered random process $X(t)$, $t \in T$, is a strictly $\text{Sub}_\varphi(\Omega)$ process, where $\varphi(x) = x^2/2$ and the determinative constant $C_X = 1$.

**Theorem 1** [Kozachenko, Kamenshchikova 2009] Let $\{T, \Lambda, \mu\}$ be a measurable space and $X(t)$, $t \in T$, be a random process from the space $\text{Sub}_\varphi(\Omega)$.

If $\int_T (\tau_\varphi(t))^p \, d\mu(t) < \infty$ for some $p \geq 1$, then the integral $\int_T |X(t)|^p \, d\mu(t)$ exists with probability 1 and the following inequality holds:

$$
P \left\{ \int_T |X(t)|^p \, d\mu(t) > \varepsilon \right\} \leq 2 \exp \left\{ -\varphi^*(\varepsilon/c)^{1/p} \right\},
$$

for each non-negative

$$
\varepsilon > c \cdot \left( f \left( p(c/\varepsilon)^{1/p} \right) \right)^p,
$$

where $c := \int_T (\tau_\varphi(t))^p \, d\mu(t)$ and $f(\cdot)$ is a density of $\varphi(\cdot)$.

**Example 1** If $\varphi(x) = |x|^{\alpha}/\alpha$, $1 < \alpha \leq 2$, then $f(x) = x^{\alpha-1}$ and $\varphi^*(x) = |x|^\beta/\beta$, where $\beta > 1$ and $1/\alpha + 1/\beta = 1$. Hence, $c \left( f \left( p(c/\varepsilon)^{1/p} \right) \right)^p = c^\alpha p^{\alpha-1} \varepsilon^{1-\alpha}$. Therefore, inequality (1) holds if $\varepsilon > c \cdot p^{-\alpha \cdot \alpha'}$.  

**Remark 4** If $X(t)$, $t \in T$, is a Gaussian centered random process and $\sigma(t) := \left( \mathbb{E}(X(t))^2 \right)^{1/2}$, then inequality (1) holds true for $\varepsilon > \tilde{c} p^{\alpha}/\alpha$, where $\tilde{c} := \int_T (\sigma(t))^p \, d\mu(t)$. For such $\varepsilon$ the following estimate is valid

$$
P \left\{ \int_T |X(t)|^p \, d\mu(t) > \varepsilon \right\} \leq 2 \exp \left\{ -\frac{1}{2} \left( \frac{\varepsilon}{\tilde{c}} \right)^{2/p} \right\}.
$$

**Remark 5** If $X(t)$, $t \in T$, is a strictly $\text{Sub}_\varphi(\Omega)$ random process, then

$$
c \leq C_X^p \int_T \left( \mathbb{E}(X(t))^2 \right)^{\frac{p}{2}} \, d\mu(t).$$
3 Wavelet representation of random processes

In this section we introduce wavelet representations and approximations of non-random functions and stochastic processes.

Let $\phi(x), x \in \mathbb{R}$, be a function from the space $L_2(\mathbb{R})$ such that $\hat{\phi}(0) \neq 0$ and $\hat{\phi}(y)$ is continuous at 0, where $\phi(y) = \int_{\mathbb{R}} e^{-iyx} \phi(x) \, dx$ is the Fourier transform of $\phi$.

Suppose that the following assumption holds true: $\sum_{k \in \mathbb{Z}} |\hat{\phi}(y+2\pi k)|^2 = 1$ (a.e.)

There exists a function $m_0(x) \in L_2([0,2\pi])$, such that $m_0(x)$ has the period $2\pi$ and

$$\hat{\phi}(y) = m_0(y/2) \hat{\phi}(y/2) \text{ (a.e.)}$$

In this case the function $\phi(\cdot)$ is called the f-wavelet.

Let $\psi(\cdot)$ be the inverse Fourier transform of the function

$$\hat{\psi}(y) := m_0(\frac{y}{2} + \pi) \cdot \exp \left\{ -i \frac{y}{2} \cdot \hat{\phi}(\frac{y}{2}) \right\}.$$

Then the function $\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixy} \hat{\psi}(y) \, dy$ is called the m-wavelet.

Let

$$\phi_{jk}(x) := 2^{j/2} \phi(2^j x - k), \quad \psi_{jk}(x) := 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

It is known that the family of functions $\{\phi_{jk}, \psi_{jk} : j \in \mathbb{N}_0, k \in \mathbb{Z}\}$ is an orthonormal basis in $L_2(\mathbb{R})$ (see, for example, [Daubechies 1992]).

An arbitrary function $f(x) \in L_2(\mathbb{R})$ can be represented in the form

$$f(x) = \sum_{k \in \mathbb{Z}} \alpha_{0k} \phi_{0k}(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \beta_{jk} \psi_{jk}(x), \quad (2)$$

$$\alpha_{0k} := \int_{\mathbb{R}} f(x) \phi_{0k}(x) \, dx, \quad \beta_{jk} := \int_{\mathbb{R}} f(x) \psi_{jk}(x) \, dx.$$

The representation (2) is called a wavelet representation.

The series (2) converges in $L_2(\mathbb{R})$ i.e. $\sum_{k \in \mathbb{Z}} |\alpha_{0k}|^2 + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} |\beta_{jk}|^2 < \infty$.

The integrals $\alpha_{0k}$ and $\beta_{jk}$ may also exist for functions from $L_0(\mathbb{R})$ and other function spaces. Therefore it is possible to obtain the representation (2) for function classes which are wider than $L_2(\mathbb{R})$ (see, for example, [Jaffard 2001, Triebel 2008]).

**Assumption S** [Hardle et al. 1998] For the function $\phi(\cdot)$ there exists a decreasing function $\Phi(x), x \geq 0$, such that $\Phi(0) < \infty$, $|\phi(x)| \leq \Phi(|x|)$ (a.e.), and the integral $\int_{\mathbb{R}} \Phi(|x|) \, dx$ is finite.

Let $X(t), t \in \mathbb{R}$, be a random process such that $E X(t) = 0$ for all $t \in \mathbb{R}$. If sample trajectories of this process are in the space $L_2(\mathbb{R})$ with probability one, then it is possible to obtain the representation (wavelet representation)

$$X(t) = \sum_{k \in \mathbb{Z}} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t), \quad (3)$$
with mean-square integrals
\[ \xi_{nk} := \int_{\mathbb{R}} X(t) \phi_{nk}(t) dt, \quad \eta_{jk} := \int_{\mathbb{R}} X(t) \psi_{jk}(t) dt. \]

However, the majority of random processes does not possess the required property. For example, sample paths of stationary processes are not in the space \( L_2(\mathbb{R}) \) (a.s.). However, in many cases it is possible to construct a representation of type (3) for \( X(t) \).

Consider the approximants of \( X(t) \) defined by
\[ X_{n,k_n}(t) := \sum_{|k| \leq k'_0} \xi_{nk} \phi_{nk}(t) + \sum_{j=0}^{n-1} \sum_{|k| \leq k_j} \eta_{jk} \psi_{jk}(t), \]
where \( k_n := (k'_0(n), k_0(n), \ldots, k_{n-1}(n)) \).

For the sake of simplicity, we will omit the index \((n)\) in \((k'_0(n), k_0(n), \ldots, k_{n-1}(n))\) and will use the notation \( k_n \to \infty \) to denote that \( n \to \infty, k'_0 \to \infty, \) and \( k_j \to \infty \) for all \( j \in \mathbb{N}_0 \).

Theorem 2 below guarantees the mean-square convergence of \( X_{n,k_n}(t) \) to \( X(t) \).

**Theorem 2** \([\text{Kozachenko, Polosmak 2008}]\) Let \( X(t), t \in \mathbb{R} \), be a random process such that \( \mathbb{E} X(t) = 0, \mathbb{E}|X(t)|^2 < \infty \) for all \( t \in \mathbb{R} \), and its covariance function \( R(t,s) \) is continuous. Let the \( f \)-wavelet \( \phi \) and the \( m \)-wavelet \( \psi \) be continuous functions and the assumption \( S \) hold true for both \( \phi \) and \( \psi \). Suppose that there exist a function \( A : (0, \infty) \to (0, \infty) \) and \( x_0 \in \mathbb{R} \) such that \( c(ax) \leq c(x) \cdot A(a), \) for all \( x \geq x_0 \).

Assume that there exists an even non-decreasing on \([0, \infty)\) function \( c(x) \), \( x \in \mathbb{R} \), with \( c(0) > 1 \) such that \( \int_0^\infty c(x) \Phi(|x|) dx < \infty \) and \( |R(t,t)|^{1/2} \leq c(t) \) for all \( t \in \mathbb{R} \). Then

1. \( X_{n,k_n}(t) \in L_2(\Omega) \);
2. \( X_{n,k_n}(t) \to X(t) \) in mean square when \( k_n \to \infty \).

**Remark 6** It was shown that the integrals in (4) exist if \( X(t) \) satisfies the assumptions of Theorem 2, see \([\text{Kozachenko, Polosmak 2008}]\). Though there are other sufficient conditions in the literature which guarantee the existence of the integrals in (4) and the convergence in the mean-square sense in Theorem 2, the assumptions in terms of the functions \( c(\cdot) \) and \( \Phi(\cdot) \) can be easily verified in many practical examples.

4 Convergence of wavelet expansions of \( \varphi \)-sub-Gaussian random processes

In this section we present the main results on convergence in \( L_p(T) \), \( T = [0,T] \), \( T > 0 \), of the wavelet expansions of \( \varphi \)-sub-Gaussian random processes. The rate of convergence in the space \( L_p([0,T]) \) is obtained. We also present some specifications of the general results for which the assumptions can be easily verified.

In \([\text{Giuliano Antonini et al. 2003}]\), it was shown that there always exists a constant \( c_\varphi > 0 \) such that \( \mathbb{E}|\xi|^2 \leq c_\varphi \tau_\varphi^2(\xi) \). Therefore, random processes from the space \( \text{Sub}_p(\Omega) \) belong to the space \( L_2(\Omega) \). The following theorem is a corollary of Theorem 2 and the estimate \( (R(t,t))^{1/2} \leq \sqrt{c_\varphi} \tau_\varphi(t) \).
Theorem 3 Let $X(t), t \in \mathbb{R}$, be a random process such that $X(\cdot) \in \text{Sub}_\varphi(\Omega)$ and $X(t)$ is continuous in the norm $\tau_\varphi(\cdot)$. Let the functions $\phi(\cdot), \psi(\cdot), \Phi(\cdot), A(\cdot)$, and $c(\cdot)$ satisfy the assumptions of Theorem 2.

If $\int_0^\infty c(x)\Phi(|x|) \, dx < \infty$ and $\tau_\varphi(t) \leq c(t)$ for all $t \in \mathbb{R}$, then

1. $X_n,k_n(t) \in L_2(\Omega)$;
2. $X_n,k_n(t) \rightarrow X(t)$ in mean square when $k_n \rightarrow \infty$.

Definition 7 A random process $X(t), t \in \mathbb{R}$, belongs to the space $L_p([0,T])$ if

$$\mathbb{P} \left\{ \left( \int_0^T |X(t)|^p \, dt \right)^{1/p} < +\infty \right\} = 1.$$

Theorem 4 Let $X(t), t \in \mathbb{R}$, be a random process such that $X(\cdot) \in \text{Sub}_\varphi(\Omega)$ and $X(t)$ is measurable and continuous in the norm $\tau_\varphi(\cdot)$. Let the assumptions of Theorem 3 are satisfied. Suppose that

$$\int_0^T (\tau_\varphi(t))^p \, dt < \infty$$

for some $T > 0$ and $p \geq 1$.

If

$$\int_0^T (\tau_\varphi(X_{n,k_n}(t) - X_{m,k_m}(t)))^p \, dt \rightarrow 0,$$

when $k_n, k_m \rightarrow \infty$, then $X_{n,k_n}(t) \rightarrow X(t)$ in the space $L_p([0,T])$ in probability, when $k_n \rightarrow \infty$, i.e. for all $\varepsilon > 0$

$$\mathbb{P} \left\{ \left( \int_0^T |X_{n,k_n}(t) - X(t)|^p \, dt \right)^{1/p} > \varepsilon \right\} \rightarrow 0,$$

when $k_n \rightarrow \infty$.

Furthermore, the following estimate holds

$$\mathbb{P} \left\{ \int_0^T \left| X_{n,k_n}(t) - X(t) \right|^p \, dt > \varepsilon \right\} \leq 2 \exp \left\{ -\varphi^* \left( \frac{\varepsilon}{c_n^\infty} \right)^{1/p} \right\},$$

where $c_n^\infty := \int_0^T (\tau_\varphi(X_{n,k_n}(t) - X(t)))^p \, dt$ and $\varepsilon > c_n^\infty \left( f \left( p \left( c_n^\infty / \varepsilon \right)^{1/p} \right) \right)^p$.

Proof. First, we will show that the random variables $\xi_{ok}$ and $\eta_{jk}$ are from the space $\text{Sub}_\varphi(\Omega)$. We prove it only for $\eta_{jk}$. The case of $\xi_{ok}$ can be dealt with similarly.

We will need the following generalization of the Minkowski inequality.

Lemma 1 If $X(t), t \in \mathbb{R}$, is a measurable $\varphi$-sub-Gaussian random processes, then

$$\tau_\varphi \left( \int_\mathbb{R} X(t) \, dt \right) \leq \int_\mathbb{R} \tau_\varphi(t) \, dt.$$
Proof. Without loss of generality we may assume that $\tau_\varphi(t) > 0$ for all $t \in \mathbb{R}$ and $\int_\mathbb{R} \tau_\varphi(t) \, dt < \infty$.

Let us denote

$$I := \mathbb{E} \left( \exp \left\{ \lambda \int_\mathbb{R} \mathbf{X}(t) \, dt : \left( \int_\mathbb{R} \tau_\varphi(s) \, ds \right)^{-1} \right\} \right)$$

$$= \mathbb{E} \left( \exp \left\{ \lambda \int_\mathbb{R} \frac{\mathbf{X}(t)}{\tau_\varphi(t)} \cdot \frac{\tau_\varphi(t)}{\int_\mathbb{R} \tau_\varphi(s) \, ds} \, dt \right\} \right), \quad \lambda \in \mathbb{R}. $$

Notice that $\exp(\cdot)$ is a convex function and

$$\int_\mathbb{R} \tau_\varphi(t) \cdot \left( \int_\mathbb{R} \tau_\varphi(s) \, ds \right)^{-1} \, dt = 1. $$

Hence, by Jensen’s inequality, we obtain

$$\exp \left\{ \lambda \int_\mathbb{R} \frac{\mathbf{X}(t)}{\tau_\varphi(t)} \cdot \frac{\tau_\varphi(t)}{\int_\mathbb{R} \tau_\varphi(s) \, ds} \, dt \right\} \leq \int_\mathbb{R} \exp \left\{ \lambda \frac{\mathbf{X}(t)}{\tau_\varphi(t)} \right\} \cdot \frac{\tau_\varphi(t)}{\int_\mathbb{R} \tau_\varphi(s) \, ds} \, dt. $$

Therefore,

$$I \leq \int_\mathbb{R} \mathbb{E} \left( \exp \left\{ \lambda \frac{\mathbf{X}(t)}{\tau_\varphi(t)} \right\} \right) \cdot \frac{\tau_\varphi(t)}{\int_\mathbb{R} \tau_\varphi(s) \, ds} \, dt. $$

By the definition of $\tau_\varphi(\cdot)$ we get

$$\mathbb{E} \left( \exp \left\{ \lambda \frac{\mathbf{X}(t)}{\tau_\varphi(t)} \right\} \right) \leq \exp\{\varphi(\lambda)\}. $$

By inequalities (5) and (9) we obtain $I \leq \exp\{\varphi(\lambda)\}$. Finally, the statement of the lemma follows from the definition of $\tau_\varphi(\cdot)$. \qed

By Lemma 1, we get

$$\tau_\varphi(\eta_{jk}) \leq \int_\mathbb{R} \tau_\varphi(t) \left| \psi_{jk}(t) \right| \, dt \leq \int_\mathbb{R} \tau_\varphi(t) 2^{j/2} \left| \psi(2^j t - k) \right| \, dt$$

$$\leq 2^{j/2} \int_\mathbb{R} c(t) \left| \psi(2^j t - k) \right| \, dt = 2^{-j/2} \int_\mathbb{R} c \left( \frac{|u + k|}{2^j} \right) \psi(u) \, du$$

$$\leq 2^{-j/2} \int_\mathbb{R} c \left( \frac{|u + k|}{2^j} \right) \phi(|u|) \, du < +\infty,$$

since $c(|u + k|) \leq c(2u) \leq c(u)A(2)$ for sufficiently large $u$.

Thus, the processes $\mathbf{X}_{n,k_n}(t)$ belongs to the space $Sub_\varphi(\Omega)$.

To prove the theorem, it is enough to show that for all $\varepsilon > 0$

$$\mathbb{P} \left\{ \int_0^T \left| \mathbf{X}_{n,k_n}(t) - \mathbf{X}_{m,k_m}(t) \right|^p \, dt > \varepsilon \right\} \rightarrow 0,$$

when $k_n$ and $k_m \rightarrow +\infty$.

Indeed, by Theorem 1 and 5, $\mathbf{X}(t) \in L_p([0,T])$ with probability 1.

If the condition (10) is satisfied then there exists a process $\mathbf{Y}(t) \in L_p([0,T])$ such that

$$\mathbb{P} \left\{ \int_0^T \left| \mathbf{X}_{n,k_n}(t) - \mathbf{Y}(t) \right|^p \, dt > \varepsilon \right\} \rightarrow 0.$$
when \( k_n \to +\infty \), and by Theorem 3 we get \( Y(t) = X(t) \) with probability 1.

To prove (11) we denote

\[
e^{m_n} := \int_0^T \left( \tau_\varphi \left( X_{n,k_n}(t) - X_{m,k_m}(t) \right) \right)^p dt.
\]

By Theorem 1 the following inequality holds

\[
P \left\{ \int_T \left| X_{n,k_n}(t) - X_{m,k_m}(t) \right|^p dt > \varepsilon \right\} \leq 2 \exp \left\{ -\varphi^* \left( \varepsilon / c^{m_n} \right)^{1/p} \right\}
\]

for \( \varepsilon > c^{m_n} \cdot \left( \int_0^T \left( \tau_\varphi \left( X_{n,k_n}(t) - X_{m,k_m}(t) \right) \right)^p dt \right)^p \).

The condition (4) implies that \( c^{m_n} \to 0 \) as \( n, m \to \infty \). The function \( f(x) \) is right-continuous and \( f(0) = 0 \). Hence \( f \left( \int_0^T \left( \tau_\varphi \left( X_{n,k_n}(t) - X_{m,k_m}(t) \right) \right)^p dt \right)^p \to 0 \), \( n, m \to \infty \). Therefore, for arbitrary \( \varepsilon > 0 \) and sufficiently large \( n, m \) inequality (11) holds true. By definitions 1 and 2 the right-hand side of the inequality vanishes when \( c^{m_n} \to 0 \), which implies (10).

Notice, that \( X_{n,k_n}(t) - X_{m,k_m}(t) \to X_{n,k_n}(t) - X(t) \) in \( L_p([0,T]) \) with probability 1 and \( c^{m_n} \to c^\infty \), when \( k_m \to +\infty \). Hence, we obtain inequality (7) if we let \( k_m \to 0 \) in (10).

**Corollary 1** Let \( X(t), t \in \mathbb{R} \), be a strictly Sub\( \varphi \) random process with a deterministic constant \( C_X \). Then the condition (5) holds true if the integral \( \int_0^T \left( \mathbb{E}(X(t))^2 \right)^{p/2} dt \) is convergent.

If

\[
\int_0^T \left( \mathbb{E} \left( X_{n,k_n}(t) - X_{m,k_m}(t) \right)^2 \right)^{p/2} dt \to 0,
\]

then the condition (5) holds when \( k_n, k_m \to \infty \). The inequality (7) is valid for

\[
e^{\infty} = C_X^p \int_0^T \left( \mathbb{E} \left( X_{n,k_n}(t) - X(t) \right)^2 \right)^{p/2} dt.
\]

**Proof.** The statement of Corollary 1 follows from Theorem 3, Definition 6, and the fact that a linear closure of a family of strictly Sub\( \varphi \) random variables is a family of strictly Sub\( \varphi \), see Kozachenko, Kovalchuk 1985.

**Corollary 2** The statement of Corollary 1 is valid if instead of (12) the series

\[
\sum_{k \in \mathbb{Z}} \left( \mathbb{E} \xi_{0k}^2 \right)^{1/2} |\phi_{0k}(t)| + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} \left( \mathbb{E} \eta_{jk}^2 \right)^{1/2} |\psi_{jk}(t)|
\]

converges uniformly on the interval \([0,T]\). Then the inequality (7) holds for

\[
e^{\infty} = C_X^p \sup_{0 \leq t \leq T} \left( \sum_{|k| \geq k_0} \left( \mathbb{E} \xi_{0k}^2 \right)^{1/2} |\phi_{0k}(t)| + \sum_{j=0}^{n-1} \sum_{|k| \geq k_j} \left( \mathbb{E} \eta_{jk}^2 \right)^{1/2} |\psi_{jk}(t)|
\]

\[
+ \sum_{j=n}^{\infty} \sum_{k \in \mathbb{Z}} \left( \mathbb{E} \eta_{jk}^2 \right)^{1/2} |\psi_{jk}(t)| \right)^p.
\]
Proof. The statement of Corollary 2 follows from (13) and the estimate
\[
\left( \mathbb{E}(X_{n,k_n}(t) - X(t))^2 \right)^{1/2} = \left( \mathbb{E} \left( \sum_{|k| \geq k_0} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{|k| \geq k_j} \eta_{jk} \psi_{jk}(t) \right) \right)^{1/2}
+ \sum_{j=n}^{\infty} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t) \right)^{1/2} \leq \sum_{|k| \geq k_0} \left( \mathbb{E} \xi_{0k}^2 \right)^{1/2} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{|k| \geq k_j} \left( \mathbb{E} \eta_{jk}^2 \right)^{1/2}
\times |\psi_{jk}(t)| + \sum_{j=n}^{\infty} \sum_{k \in \mathbb{Z}} \left( \mathbb{E} \eta_{jk}^2 \right)^{1/2} |\psi_{jk}(t)|.
\]
\[
\left( \mathbb{E}(X_{n,k_n}(t) - X(t))^2 \right)^{1/2} \leq \left( \mathbb{E} \left( \sum_{|k| \geq k_0} \xi_{0k} \phi_{0k}(t) + \sum_{j=0}^{n-1} \sum_{|k| \geq k_j} \eta_{jk} \psi_{jk}(t) \right) \right)^{1/2}
+ \sum_{j=n}^{\infty} \sum_{k \in \mathbb{Z}} \eta_{jk} \psi_{jk}(t).}
\]
\[
\sup_{x \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\delta(x - k)| \leq 3\Phi(0) + 4 \int_{1/2}^{\infty} \Phi(t) \, dt =: C_\delta \tag{14}
\]
and
\[
\sup_{|x| \leq T} \sum_{|k| \geq k_1} |\delta(x - k)| \leq \int_{k_1 - T - 1}^{\infty} \Phi(t) \, dt + \int_{1}^{\infty} \Phi(t) \, dt =: C_\delta(T, k_1) \tag{15}
\]
for \( k_1 \geq T + 1. \)

Proof. Inequality (14) is a simple modification of an inequality from Hardle et al. 1998. Therefore we only prove (15).

By the assumption S,
\[
\sum_{|k| \geq k_1} |\delta(x - k)| \leq \sum_{k \geq k_1} (\Phi(|x + k|) + \Phi(|x - k|)) =: z_{k_1}(x),
\]
where \( z_{k_1}(x) \) is an even function. As \( \Phi(\cdot) \) is a decreasing function on \([0, +\infty)\) and \( k_1 \geq T + 1, \) we obtain
\[
\sup_{|x| \leq T} \sum_{|k| \geq k_1} |\delta(x - k)| \leq \sup_{0 \leq x \leq T} z_{k_1}(x) \leq \sum_{k \geq k_1} (\Phi(|k - T|) + \Phi(|k|))
\leq \sum_{k \geq k_1} \left( \int_{k - 1}^{k_T} \Phi(t) \, dt + \int_{k - 1}^{T} \Phi(t) \, dt \right) \leq \int_{k_1 - T - 1}^{\infty} \Phi(t) \, dt + \int_{1}^{\infty} \Phi(t) \, dt.
\]

Theorem 5 Let \( X(t), t \in \mathbb{R}, \) be a strictly \( \text{Sub}_\phi(\Omega) \) random process for which the assumptions of Theorem 2 hold true and \( \int_0^T (\mathbb{E}(X(t))^2)^{p/2} \, dt < +\infty. \)

If
\[
\sup_{k \in \mathbb{Z}} (\mathbb{E}|\xi_{0k}|^2)^{1/2} C_\phi + \sum_{j=0}^{\infty} \sup_{k \in \mathbb{Z}} (\mathbb{E}|\eta_{jk}|^2)^{1/2} 2^{j/2} C_\psi < \infty, \tag{16}
\]
then \( X_{n,k_n}(t) \to X(t) \) in the space \( L_p(0,T) \) in probability, when \( k_n \to \infty. \)
Furthermore, for $\varepsilon > c_{n}^{\infty} \left( \frac{p}{p} \right)^{1/p} \left( \frac{c_{n}^{\infty}}{\varepsilon} \right)^{1/p} \right)^{p}$ inequality (1) is valid when

$$c_{n}^{\infty} = C_{X}^{p} \left( \sup_{k \in \mathbb{Z}} \left( \mathbb{E}|\xi_{0k}|^{2} \right) \right)^{1/2} \mathbb{C}(T, k_{0}) + \sum_{j=0}^{J-1} \sup_{k \in \mathbb{Z}} \left( \mathbb{E}|\eta_{jk}|^{2} \right)^{1/2} \times 2^{j/2} \mathbb{C}(T, k_{j}) + \sum_{j=J}^{\infty} \sup_{k \in \mathbb{Z}} \left( \mathbb{E}|\eta_{jk}|^{2} \right)^{1/2} \cdot 2^{j/2} \mathbb{C}_{\phi}^{2},$$

where $J := \min \{ n, \min \{ j \in \mathbb{N}_{0} : k_{j} < 2^{j}T + 1 \} \}$.

**Proof.** Notice that $J \rightarrow \infty$, when $k_{n} \rightarrow \infty$. By the choice of $J$ and (16) we get $c_{n}^{\infty} \rightarrow 0$, when $k_{n} \rightarrow \infty$. The statement of the theorem follows from Theorem 4, Corollary 2, Lemma 2, and the fact that

$$\sum_{k \in \mathbb{Z}} |\phi_{jk}(t)| = 2^{j/2} \sum_{k \in \mathbb{Z}} \phi(2^{j}t - k).$$

**Theorem 6**  Let $X(t), t \in \mathbb{R}$, be a strictly Sub-$\varphi$ stationary short-memory random process with the covariance function $R(t - s) := \mathbb{E}(X(t)X(s))$. Let the assumptions of Theorem 2 are satisfied and $\int_{0}^{\infty} \left( \mathbb{E}|X(t)|^{2} \right)^{p/2} dt < +\infty$. Suppose that there exists the Fourier transform $\hat{R}(z) = \int_{\mathbb{R}} e^{-izt} R(t) dt$ and for some $\alpha > 0$ :

$$\int_{\mathbb{R}} \left| \hat{R}(z) \right| \cdot |z|^\alpha \cdot d\mathbb{R} < \infty.$$

If $\hat{\psi}(\cdot)$ is a Lipschitz function of order $\alpha/2$, then (16) holds.

**Proof.** By Parseval’s theorem and the representation $\hat{\psi}_{jk}(z) = 2^{-j/2} e^{-iz} \hat{\psi} \left( \frac{z}{2^{j}} \right)$ , we deduce

$$\mathbb{E}|\eta_{jk}|^{2} = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{R}(u - v) \hat{\psi}_{jk}(u) du \hat{\psi}_{jk}(v) dv \right| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-izx}}{2\pi} \hat{R}(z) \hat{\psi}_{jk}(z) d\mathbb{R} \hat{\psi}_{jk}(v) dv \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \hat{R}(z) \right| \left| \hat{\psi}_{jk}(z) \right| \left| \hat{\psi}_{jk}(z) \right| dz = \frac{1}{2^{j+1} \pi} \int_{\mathbb{R}} \left| \hat{R}(z) \right| \cdot |\hat{\psi}(\frac{z}{2^{j}})|^{2} dz. \quad (18)$$

By the Lipschitz conditions, (18), and $\hat{\psi}(0) = 0$, we obtain that for all $k \in \mathbb{Z}$:

$$\left| \mathbb{E} \eta_{jk} \right|^{2} \leq \frac{C}{2^{j+1} \pi} \int_{\mathbb{R}} \left| \hat{R}(z) \right| \cdot \frac{|z|^\alpha}{2^{j}} \right|^{2} \int_{\mathbb{R}} \left| \hat{R}(z) \right| \cdot |z|^\alpha d\mathbb{R}, \quad (19)$$

where $C > 0$ is the Lipschitz constant.

Similarly to (18) we deduce

$$\mathbb{E}|\xi_{0k}|^{2} \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left| \hat{R}(z) \right| \cdot \left| \hat{\phi}(z) \right|^{2} dz < \infty. \quad (20)$$

The integral in (20) is finite because the Assumption S implies $\phi(\cdot) \in L_{1}(R)$ and therefore $\hat{\phi}(\cdot)$ is bounded.

By estimates (19) and (20) the series in (16) is convergent.
Theorem 7 Let $X(t), t \in \mathbb{R}$, be a strictly Sub-$\phi$($\Omega$) random process with the continuous covariance function $R(t,s) := \mathbb{E}X(t)\overline{X(s)}$. Suppose that $\int_0^T \left( \mathbb{E}(X(t))^2 \right)^{1/2} dt < +\infty$, there exist $\hat{R}_2(z,w) := \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-izu}e^{-iwv}R(u,v)\,du\,dv$, and for some $\alpha > 0$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{R}_2(z,w)| \cdot |z|^\alpha \cdot |w|^\alpha \,dz\,dw < \infty.$$  

If the assumptions of Theorem 6 are satisfied and $\hat{\psi}(\cdot)$ is a Lipschitz function of order $\alpha$, then (16) holds true.

Proof. By Parseval’s theorem,

$$\mathbb{E}|\eta_{jk}|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}X(u)\overline{X(v)} \psi_{jk}(u)\overline{\psi_{jk}(v)} \,du\,dv$$
$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{R}_2(z,w) \psi_{jk}(z) \overline{\hat{\psi}_{jk}(w)} \,dz\,dw$$
$$\leq \frac{1}{2(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \hat{R}_2(z,w) \cdot |\psi \left( \frac{z}{2^j} \right)| \cdot |\hat{\psi} \left( \frac{w}{2^j} \right)| \right| \,dz\,dw. \quad (21)$$

By properties of the m-wavelet $\psi(\cdot)$ we have $\hat{\psi}(0) = 0$. Therefore, using the Lipschitz conditions, we obtain

$$\mathbb{E}|\eta_{jk}|^2 \leq \frac{C^2}{(2\pi)^2 \cdot 2^{(1+2\alpha)}} \int_{\mathbb{R}} \left| \hat{R}_2(z,w) \right| \cdot |z|^\alpha \cdot |w|^\alpha \,dz\,dw,$$

where $C > 0$ is the Lipschitz constant.

Analogously to (21) we deduce that

$$\mathbb{E}|\xi_{0k}|^2 \leq \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \hat{R}_2(z,w) \right| \cdot |\hat{\phi}(z)| \cdot |\hat{\phi}(w)| \,dz\,dw < \infty$$

due to the boundedness of $\hat{\phi}(\cdot)$.

An application of the above estimates gives the convergence of (16). $\square$

Remark 7 Conditions of Theorems 6 and 7 on the random process $X(t)$ are formulated in terms of its spectral density. These conditions are related to the behavior of the high-frequency part of the spectrum. Such assumptions are standard in the convergence studies of stochastic approximations.

Remark 8 If the assumptions of Theorem 6 or 7 are satisfied, then (7) holds true for $c_n^\infty$ given by (17). All the terms in (17) can be easily computed in practice for specific stochastic processes and wavelet bases.

5 Conclusions

The obtained results may have various practical applications for the approximation and simulation of random processes. The analysis of the rate of convergence provides a constructive algorithm for determining the number of terms in the wavelet expansions to ensure the approximation of stochastic processes with given accuracy.
The developed methodology and results are important extensions of the recent findings in the wavelet approximation theory of stochastic processes to the space $L_p([0,T])$ and the class of $\varphi$-sub-Gaussian random processes. This class plays an important role in generalizations of various theoretical properties of Gaussian processes. In addition to classical applications of $\varphi$-sub-Gaussian random processes in signal processing, the results can also be used in new areas, like compressed sensing and actuarial modelling, consult, for example, Labate et al. 2013, Vershynin 2012, Yamnenko 2006.

It would be of interest

- to adopt and specify the results for different wavelet bases, which satisfy the assumptions of the theorems, for example, Daubechies, Battle-Lemarie and Meyer wavelets;
- to examine the tightness of the estimates by simulations.

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