COMBINATORIAL RICCI FLOW ON COMPACT 3-MANIFOLDS WITH BOUNDARY

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Abstract. Combinatorial Ricci flow on an ideally triangulated compact 3-manifold with boundary was introduced by Luo as a 3-dimensional analog of Chow-Luo’s combinatorial Ricci flow on a triangulated surface and conjectured to find algorithmically the complete hyperbolic metric on the compact 3-manifold with totally geodesic boundary. In this paper, we prove Luo’s conjecture affirmatively by extending the combinatorial Ricci flow through the singularities of the flow if the ideally triangulated compact 3-manifold with boundary admits such a metric.

Mathematics Subject Classification (2010). 53C44; 52C99.

Keywords. Combinatorial Ricci flow; 3-manifolds with boundary; Extension.

1. Introduction

Motivated by Chow-Luo’s combinatorial Ricci flow on triangulated surfaces [5] and Hamilton’s Ricci flow on three dimensional Riemannian manifolds [26], Luo [29] introduced combinatorial Ricci flow for hyper-ideal polyhedral metrics on ideally triangulated compact 3-manifolds with boundary consisting of surfaces of negative Euler characteristic. By Moise [34], every compact 3-manifold can be ideally triangulated. It is conceivable that one could use the combinatorial Ricci flow to give a new proof of Thurston’s geometrization theorem for these 3-manifolds. Luo [29] conjectured that the combinatorial Ricci flow could be used to find algorithmically the complete hyperbolic metric with totally geodesic boundary on a compact 3-manifold with boundary. In this paper, we prove Luo’s conjecture affirmatively if such a metric exists on the ideally triangulated compact 3-manifold with boundary.

The main tool used in the proof is the extension of dihedral angles of a hyper-ideal tetrahedron introduced by Luo-Yang [32]. For the combinatorial Ricci flow on an ideally triangulated 3-manifold with boundary, the hyper-ideal tetrahedra may degenerate along the flow, which corresponds to the singularities of the flow and brings the main difficulty for the analysis of the long time behavior of the flow. We overcome this difficulty by extending the flow through the singularities using Luo-Yang’s extension. It is shown that the solution of combinatorial Ricci flow can be uniquely extended for all time in this way. Combining with Luo-Yang’s $C^1$ smooth convex extension...
of co-volume function, we prove that for any initial hyper-ideal polyhedral metric on an ideally triangulated compact 3-manifold with boundary, the extended solution of combinatorial Ricci flow converges exponentially fast to a complete hyperbolic metric with totally geodesic boundary if such a metric exists on the ideally triangulated compact 3-manifold with boundary, which confirms Luo’s conjecture.

We state the results more precisely as follows. Suppose \((M, T)\) is an ideally triangulated compact 3-manifold with boundary. \(E, T\) represent the sets of edges and tetrahedra in the triangulation \(T\) respectively. Replacing each tetrahedron (truncated tetrahedron more precisely) in \(T\) by a hyper-ideal tetrahedron and gluing them isometrically along the hexagonal faces, we obtain a hyperbolic cone manifold with boundary. Here a hyper-ideal tetrahedron is a compact convex polyhedron in \(\mathbb{H}^3\) that is diffeomorphic to a truncated tetrahedron in \(E^3\), which has four right-angled hyperbolic hexagonal faces and four hyperbolic triangular faces. Any triangular face is required to be orthogonal to its three adjacent hexagonal faces. Note that, by the cosine laws for hyperbolic triangles and right-angled hyperbolic hexagons, the geometry of a hyper-ideal tetrahedron is uniquely determined by the lengths of six edges given by pairwise intersections of four hexagonal faces. Based on this observation, a hyper-ideal polyhedral metric on an ideally triangulated compact 3-manifold with boundary is defined as follows.

**Definition 1.1** \((29, 32)\). Suppose \((M, T)\) is an ideally triangulated compact 3-manifold with boundary. A hyper-ideal polyhedral metric on \((M, T)\) is a map \(l : E \to (0, +\infty)\) such that for any topological truncated tetrahedron in \(T\), \(l_{ij}, l_{ik}, l_{ih}, l_{jk}, l_{jh}, l_{kh}\) form the lengths of six edges given by pairwise intersections of four hexagonal faces in a hyper-ideal tetrahedron.

\(M\) can be taken as the hyperbolic cone manifold generated by gluing the hyper-ideal tetrahedra isometrically along the hexagonal faces and \(\partial M\) has a natural triangulation induced from the cell decomposition of \(M\). The boundary \(\partial M\) is required to be consisting of surfaces with negative Euler characteristic. For simplicity, we still call \(T\) a triangulation of \(M\). The set of the edges given by intersections of hexagonal faces is denoted by \(E\) in the following if there is no confusing in the context. Note that this is different from the edge set in the cell decomposition of \(M\). The space of hyper-ideal polyhedral metrics on \((M, T)\) is denoted by \(\mathcal{L}(M, T)\), which is an open subset of \(\mathbb{R}_{>0}^E\). The combinatorial Ricci curvature \(K_{ij} : \mathcal{L}(M, T) \to (-\infty, 2\pi)\) at the edge \(\{ij\} \in E\) is defined to be \(2\pi\) less the sum of dihedral angles at the edge. A hyper-ideal polyhedral metric on the triangulated manifold \((M, T)\) with zero combinatorial Ricci curvature corresponds to a complete hyperbolic metric on the manifold \(M\) with totally geodesic boundary.

Luo \([29]\) introduced the following combinatorial Ricci flow

\[
\frac{dl_{ij}}{dt} = K_{ij}
\]  

(1.1)
for hyper-ideal polyhedral metrics on ideally triangulated compact 3-manifolds with boundary consisting of surfaces of negative Euler characteristic and established some of the basic properties of the combinatorial Ricci flow (1.1). Luo further conjectured that the combinatorial Ricci flow (1.1) could be used to find algorithmically the complete hyperbolic metric on a compact 3-manifold with totally geodesic boundary.

The main purpose of this paper is to give an affirmative answer to Luo’s conjecture under the condition that there exists a hyper-ideal polyhedral metric with zero combinatorial Ricci curvature on the ideally triangulated compact 3-manifold with boundary. The main results are as follows.

**Theorem 1.1.** Suppose \((M, \mathcal{T})\) is an ideally triangulated compact 3-manifold with boundary. For any initial hyper-ideal polyhedral metric in \(L(M, \mathcal{T})\), the solution of combinatorial Ricci flow (1.1) can be extended to be a solution existing for all time.

**Theorem 1.2.** Suppose \((M, \mathcal{T})\) is an ideally triangulated compact 3-manifold with boundary. If there exists a hyper-ideal polyhedral metric \(l^* \in L(M, \mathcal{T})\) with zero combinatorial Ricci curvature, the extended solution of combinatorial Ricci flow converges exponentially fast to \(l^*\) for any initial hyper-ideal polyhedral metric in \(L(M, \mathcal{T})\).

The local convergence of combinatorial Ricci flow (1.1) was obtained by Luo [29]. Theorem 1.1 and Theorem 1.2 give the global convergence of the extended combinatorial Ricci flow. Theorem 1.1 and Theorem 1.2 imply an algorithm to find the hyper-ideal polyhedral metric with zero combinatorial Ricci curvature if such a metric exists on the ideally triangulated compact 3-manifold with boundary \((M, \mathcal{T})\). Note that a hyper-ideal polyhedral metric with zero combinatorial Ricci curvature on \((M, \mathcal{T})\) corresponds to a complete hyperbolic metric on \(M\) with totally geodesic boundary, the existence of which is necessary for an algorithm in Luo’s conjecture for the triangulated manifold \((M, \mathcal{T})\). Therefore, Theorem 1.1 and Theorem 1.2 confirm Luo’s conjecture for combinatorial Ricci flow on ideally triangulated compact 3-manifolds with boundary.

Suppose \((M, \mathcal{T})\) is an ideally triangulated compact 3-manifold with boundary, we use \(d_e\) to denote the number of tetrahedra adjacent to an edge \(e \in E\). As an application of Theorem 1.1 and Theorem 1.2, we have the following result.

**Corollary 1.1.** Suppose \((M, \mathcal{T})\) is an ideally triangulated compact 3-manifold with boundary.

1. If \(d_e \leq 6\) for any \(e \in E\), there exists no hyper-ideal polyhedral metric on \((M, \mathcal{T})\) with zero combinatorial Ricci curvature.
2. If \(d_e = d_{e'} = N > 6\) for any \(e, e' \in E\), then for any initial hyper-ideal polyhedral metric in \(L(M, \mathcal{T})\), the extended solution of combinatorial Ricci flow converges exponentially fast to a hyper-ideal
polyhedral metric in $L(M,T)$ with zero combinatorial Ricci curvature.

One can also modify the combinatorial Ricci flow (1.1) to find hyper-ideal polyhedral metrics with prescribed combinatorial Ricci curvatures and then use Luo-Yang’s extension to study the long time behavior of the modified combinatorial Ricci flow.

There have been many work on combinatorial curvature flows on three dimensional manifolds. See the work of Dai-Ge [6], Ge-Hua [10], Ge-Jiang-Shen [15], Ge-Ma [16], Ge-Xu [17, 18, 20], Ge-Xu-Zhang [21], Glickenstein [22, 23] and others. The difference between their work and ours is that we consider compact hyperbolic cone 3-manifolds with boundary generated by isometrically gluing hyper-ideal tetrahedra. The combinatorial Ricci flow for compact 3-manifolds with boundary have applications in engineering fields such as shape classification, see [44] for example. A work closely related to this work is the combinatorial Ricci flow for decorated hyperbolic polyhedral metrics on cusped 3-manifolds introduced by Yang [43] following Luo’s combinatorial Ricci flow for hyper-ideal polyhedral metrics on compact 3-manifolds with boundary. The author [41] recently extended Yang’s combinatorial Ricci flow on ideally triangulated cusped 3-manifolds using Luo-Yang’s extension [32] and proved that the existence of a complete hyperbolic metric on a cusped 3-manifold is equivalent to the convergence of the extended combinatorial Ricci flow, which gives a new characterization of existence of a complete hyperbolic metric on a cusped 3-manifold dual to Casson and Rivin’s programm. The extended combinatorial Ricci flow also provides an elegant and effective algorithm for finding complete hyperbolic metrics on cusped 3-manifolds. The author [42] further introduced the combinatorial Calabi flow on cusped 3-manifolds to find complete hyperbolic metrics on such manifolds. The basic properties of the combinatorial Calabi flow were established in [42]. There have been many work using the extension method to study the long time behavior of combinatorial curvature flows on low dimensional manifolds. See Ge-Hua [10], Ge-Jiang [11, 12, 13, 14], Ge-Jiang-Shen [15], Ge-Xu [19], Gu-Guo-Luo-Sun-Wu [24], Gu-Luo-Sun-Wu [25], Xu [39, 40, 41], Zhu-Xu [45] and others.

The paper is organized as follows. In Section 2 we recall some basic properties of hyper-ideal tetrahedra; In Section 3 we recall the extension of dihedral angles introduced by Luo-Yang [32]; In Section 4 we introduce the extension of the combinatorial Ricci flow (1.1) and prove a generalization of Theorem 1.1; In Section 5 we prove generalizations of Theorem 1.2 and Corollary 1.1; In Section 6 we study the modified combinatorial Ricci flow to find hyper-ideal polyhedral metrics with prescribed combinatorial Ricci curvatures. In Section 7 we give some remarks and propose some questions.
2. Preliminaries on hyper-ideal tetrahedra

In this section, we recall some basic properties of hyper-ideal tetrahedra and set up the notations used in the following of the paper. For more details on hyper-ideal tetrahedra, please refer to [1, 2, 7, 8, 29, 32, 36].

Suppose \( \sigma \) is a hyper-ideal tetrahedron, which is a compact convex polyhedron in hyperbolic space \( \mathbb{H}^3 \) diffeomorphic to a truncated tetrahedron in the Euclidean space \( \mathbb{E}^3 \) with four hexagonal faces and four triangular faces (see Figure 1). Denote the four triangular faces of \( \sigma \) as \( \Delta_i, i = 1, 2, 3, 4, \) which are hyperbolic triangles and called vertex triangles in the following. The edge joining \( \Delta_i \) and \( \Delta_j \) is denoted by \( e_{ij} \), the length of which is \( l_{ij} \). The edge joining \( e_{ij}, e_{jk}, e_{ik} \) is denoted by \( H_{ijk} \), which is a right-angled hyperbolic hexagon. The vertex triangle is required to be orthogonal to the three adjacent hexagonal faces. The intersection of two hexagonal faces is called an edge and the intersection of a hexagonal face and a vertex triangle is called a vertex edge. The dihedral angle at \( e_{ij} \) is the angle between the two hexagonal faces \( H_{ijk} \) and \( H_{ijh} \), which is denoted by \( a_{ij} \).

The length of the vertex edge \( \Delta_i \cap H_{ijk} \) is denoted by \( x_{jk} \). Let \( L \) be the set of vectors \( (l_{12}, \ldots, l_{34}) \in \mathbb{R}_{>0}^6 \) such that there exists a nondegenerate hyper-ideal tetrahedron with \( l_{ij} \) as the length of the edge \( \{ij\} \).

**Figure 1.** hyper-ideal tetrahedron (this figure is produced by Luo and Yang)

**Proposition 2.1** ([1, 2, 7, 8, 29]). \( L \) is a simply connected open subset of \( \mathbb{R}^6_{>0} \).

For any hyper-ideal polyhedral metric \( l \in L \) on \( \sigma \), one can define the volume function \( V \) and the co-volume function

\[
F_{\sigma} = 2V + \sum_{i<j} a_{ij}l_{ij}. \tag{2.1}
\]
By the Schlaffi formula \[4, 30, 33, 38\], we have

\[dF_{\sigma} = \sum_{i<j} a_{ij} dl_{ij},\]

which implies the matrix \(\Lambda_{\sigma} := \left[\frac{\partial a_{ij}}{\partial l_{kh}}\right]_{6 \times 6} = \text{Hess}_{l} F_{\sigma}\) is symmetric and the 1-form \(\sum_{i<j} a_{ij} dl_{ij}\) is closed on \(\mathcal{L}\). Combining with Proposition 2.1, \(\int \sum_{i<j} a_{ij} dl_{ij}\) is well-defined on \(\mathcal{L}\) and equals to the co-volume function \(F_{\sigma}\) up to a constant. Furthermore, we have the following property of the matrix \(\Lambda_{\sigma}\).

**Theorem 2.1** \((29, 36)\). For a hyper-ideal tetrahedron \(\sigma\) with dihedral angle \(a_{ij}\) and length \(l_{ij}\) at the edge \(\{ij\}\), the matrix \(\Lambda_{\sigma} = \left[\frac{\partial a_{ij}}{\partial l_{kh}}\right]\) is symmetric and strictly positive definite on \(\mathcal{L}\). In particular, the co-volume function \(F_{\sigma} = 2V + \sum_{i<j} a_{ij} l_{ij}\) is a locally strictly convex smooth function of the length variables \((l_{12}, \cdots, l_{34}) \in \mathcal{L}\) with \(\frac{\partial F_{\sigma}}{\partial l_{ij}} = a_{ij}\).

Luo \[29\] further introduced the following function

\[H(l) = \sum_{\sigma \in \mathcal{T}} F_{\sigma} - 2\pi \sum_{\{ij\} \in \mathcal{E}} l_{ij}, \quad (2.2)\]

for \(l \in \mathcal{L}(M, \mathcal{T})\) and proved the following result.

**Theorem 2.2** \((29)\). The combinatorial Ricci flow (1.1) is the negative gradient flow of the locally strictly convex function \(H\) defined on \(\mathcal{L}(M, \mathcal{T})\). Specially, the matrix \(\Lambda = \left[\frac{\partial K_{ij}}{\partial l_{kh}}\right]\) is symmetric and strictly negative definite on \(\mathcal{L}(M, \mathcal{T})\).

Based on Theorem 2.1 and Theorem 2.2 Luo \[29\] proved the local rigidity of hyper-ideal polyhedral metrics in \(\mathcal{L}(M, \mathcal{T})\) and the local convergence of combinatorial Ricci flow (1.1) on ideally triangulated compact 3-manifolds with boundary. However, the open set \(\mathcal{L}(M, \mathcal{T})\) may not be convex, which causes the main difficulty for the global rigidity of hyper-ideal polyhedral metrics and the global convergence of combinatorial Ricci flow (1.1). Luo-Yang \[32\] proved the global rigidity of hyper-ideal polyhedral metrics.

3. **LUO-YANG’S EXTENSION OF DIHEDRAL ANGLES**

To prove the global rigidity of hyper-ideal polyhedral metrics on compact 3-manifolds with boundary, Luo-Yang \[32\] carefully analysed the set of degenerate hyper-ideal polyhedral metrics on a hyper-ideal tetrahedron. Based on the characterization of the set of degenerate hyper-ideal polyhedral metrics on a hyper-ideal tetrahedron, they extended the dihedral angles of a hyper-ideal tetrahedron to be defined on \(\mathbb{R}^6\), from which they proved the global rigidity of hyper-ideal polyhedral metrics. We recall Luo-Yang’s extension in this section, which is the main tool we use to extend the combinatorial Ricci flow (1.1). For more details of the extension, please refer to \[32\].
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For \((l_{12}, \cdots, l_{34}) \in \mathbb{R}^6_0\) and \(\{i, j, k, h\} = \{1, 2, 3, 4\}\), let \(l_{ji} = l_{ij}\) for \(i \neq j\) and let
\[
x_{jk}^i = \cosh^{-1} \left( \frac{\cosh l_{ij} \cosh l_{ik} + \cosh l_{jk}}{\sinh l_{ij} \sinh l_{ik}} \right)
\]
and
\[
\phi_{kh}^i = \frac{\cosh x_{jk}^i \cosh x_{jh}^i - \cosh x_{kh}^i}{\sinh x_{jk}^i \sinh x_{jh}^i}.
\]
It is proved that \(\phi_{kh}^i = \phi_{kh}^j\) for any \(l \in \mathbb{R}^6_0\) (32, Lemma 4.3). Therefore, one can define \(\phi_{ij} : \mathbb{R}^6_{>0} \to \mathbb{R}\) by \(\phi_{ij}(l) = \phi_{kh}^i(l)\). If \((l_{12}, \cdots, l_{34}) \in \mathcal{L} \subset \mathbb{R}^6_0\) are lengths of edges in a hyper-ideal tetrahedron, then \(\phi_{ij} = \cos a_{ij}\), where \(a_{ij}\) is the dihedral angle along the edge \(\{ij\}\).

For \(\{i, j\} \subset \{1, 2, 3, 4\}\), let \(\Omega_{ij}^\pm = \{l \in \mathbb{R}^6_{>0} | \pm \phi_{ij}(l) \geq 1\}\) and \(X_{ij}^\pm = \{l \in \mathbb{R}^6_{>0} | \phi_{ij}(l) = \pm 1\}\). Luo-Yang proved the following property of \(\Omega_{ij}^\pm\) and \(X_{ij}^\pm\).

**Lemma 3.1** (32, Lemma 4.6). For \(\{i, j, k, h\} = \{1, 2, 3, 4\}\), we have

1. \((1)\): \(\Omega_{ij}^- \cap \Omega_{ik}^- = \emptyset\);
2. \((2)\): \(\Omega_{ij}^- = \Omega_{kh}^{-}\) and \(\Omega_{ij}^+ = \Omega_{ik}^- \cup \Omega_{ih}^-\);
3. \((3)\): \(X_{ij}^- = X_{kh}^-\) and \(X_{ij}^+ = X_{ik}^- \cup X_{ih}^-\).

Set \(\Omega_1 = \Omega_{12}^-, \Omega_2 = \Omega_{13}^-, \Omega_3 = \Omega_{14}^-, X_1 = X_{12}^-\), \(X_2 = X_{13}^-\), \(X_3 = X_{14}^-\). As a corollary of Lemma 3.1, Luo-Yang obtained the following characterization of \(\mathcal{L}\) and \(\mathbb{R}^6_{>0} \setminus \mathcal{L}\).

**Proposition 3.1** (32, Proposition 4.5). Let \(\partial \mathcal{L}\) be the frontier of \(\mathcal{L}\) in \(\mathbb{R}^6_{>0}\). Then \(\partial \mathcal{L} = X_1 \cup X_2 \cup X_3\), where \(X_i\), \(i = 1, 2, 3\) is a real analytic codimension-1 submanifold of \(\mathbb{R}^6_0\). The complement \(\mathbb{R}^6_0 \setminus \mathcal{L}\) is a disjoint union of three manifolds \(\Omega_i\) with boundary so that \(\Omega_i \cap \partial \mathcal{L} = X_i\), \(i = 1, 2, 3\).

Based on this characterization, Luo-Yang (32) introduced the following extension of the dihedral angle \(a_{ij}\). For \(i \neq j\), define \(a_{ij}|_{\Omega_{ij}^\pm} = 0\) and \(a_{ij}|_{\Omega_{ij}^-} = \pi\), where \(\Omega_{ij}^\pm\) is the closure of \(\Omega_{ij}^\pm\) in \(\mathbb{R}^6_{\geq 0}\). Then the dihedral angle \(a_{ij} : \mathcal{L} \to \mathbb{R}\) is continuously extended to be defined on \(\mathbb{R}^6_{\geq 0}\).

Luo-Yang further extended \(a_{ij}\) to be defined on \(\mathbb{R}^6\) as follows. For each \(l = (l_{12}, \cdots, l_{34}) \in \mathbb{R}^6\), let \(l^+ = (l_{12}^+, \cdots, l_{34}^+) \in \mathbb{R}^6_{\geq 0}\), where \(l_{ij}^+ = \max\{0, l_{ij}\}\).

For any \(l \in \mathbb{R}^6\), set
\[
\tilde{a}_{ij}(l) = a_{ij}(l^+),
\]
then \(\tilde{a}_{ij}\) is a continuous extension of \(a_{ij}\) defined on \(\mathbb{R}^6\).

Based on the extension theory of closed 1-forms and convex functions established in [3, 31], Luo-Yang (32) obtained the following extension of the co-volume function of \(F_\sigma\).

**Theorem 3.1** (32, Corollary 4.12). The function \(\tilde{F}_\sigma\) defined by
\[
\tilde{F}_\sigma = \int_{(0, \cdots, 0)}^{l^+} \sum_{i \neq j} \tilde{a}_{ij}(l) dl_{ij} + F_\sigma(0, \cdots, 0) \quad (3.1)
\]
is a well-defined $C^1$-smooth convex function defined on $\mathbb{R}^6$ extending the co-volume function $F_\sigma = 2V + \sum_{i<j} a_{ij}l_{ij}$ defined on $L \subset \mathbb{R}^6_>$, where $F_\sigma(0, \cdots, 0) = 16\Psi(\pi/4)$ with $\Psi$ being the Lobachevsky function.

**Remark 3.1.** By direct calculations, Luo-Yang [32] proved that $\frac{\partial \tilde{F}}{\partial l_{kl}} \neq 0$ for $l \in X_i$. This implies $\tilde{a}_{ij}$ is $C^0$ smooth and not $C^1$ smooth. As a corollary, $\tilde{F}_\sigma$ is not $C^2$ smooth on $\mathbb{R}^6$.

Using the extension of the co-volume function $F_\sigma$ in Theorem 3.1, Luo-Yang [32] proved the following global rigidity of hyper-ideal polyhedral metrics.

**Theorem 3.2** ([32], Theorem 1.2 (b)). For any ideally triangulated compact 3-manifold with boundary $(M, T)$, a hyper-ideal polyhedral metric on $(M, T)$ is determined by its combinatorial Ricci curvature, that is, the curvature map $K : L(M, T) \to \mathbb{R}^E$ is injective.

**Remark 3.2.** Luo-Yang [32] proved the global rigidity for hyper-ideal polyhedral metrics on triangulated compact pseudo 3-manifolds, which are generalizations of triangulated compact 3-manifolds. Please refer to [32] for more details.

Using the extension $\tilde{a}_{ij}$ of the dihedral angle $a_{ij}$, Luo-Yang [32] extended the combinatorial Ricci curvature to be defined on $\mathbb{R}^E$ by

$$\tilde{K}_{ij} = 2\pi - \sum_{\sigma \in T} \tilde{a}_{ij},$$

where the summation is taken with respect to the tetrahedra adjacent to the edge $\{ij\} \in E$. In the following, $l \in \mathbb{R}^E$ is called as a generalized hyper-ideal polyhedral metric and $\tilde{K}_{ij}(l)$ is called the generalized combinatorial Ricci curvature at the edge $\{ij\} \in E$ for the generalized hyper-ideal polyhedral metric $l \in \mathbb{R}^E$.

For the proof of the main results in this paper, we give the following result on rigidity of the generalized hyper-ideal polyhedral metrics, which is a slight generalization of Luo-Yang’s global rigidity in Theorem 3.2.

**Theorem 3.3.** For an ideally triangulated compact 3-manifold $(M, T)$, suppose $\bar{K} = K(\bar{l})$ for some $\bar{l} \in L(M, T)$. If there exists a generalized hyper-ideal polyhedral metric $l^* \in \mathbb{R}^E$ such that $\bar{K}(l^*) = \bar{K}$, then $l^* = l$.

**Proof.** The proof follows essentially Luo-Yang’s proof of Theorem 3.2 in [32]. For completeness, we give the proof here. Suppose $l^* \neq \bar{l}$. Define

$$\bar{F}(l) = \sum_{\sigma \in T} \bar{F}_\sigma(l)$$

for $l \in \mathbb{R}^E$. Then $\bar{F}$ is $C^1$ smooth and convex on $\mathbb{R}^E$ and smooth and locally strictly convex on $L(M, T)$ with $\nabla_{l_{ij}} \bar{F} = 2\pi - \bar{K}_{ij}$ by Theorem 2.1 and Theorem 3.1. Define

$$f(t) = \bar{F}(tl^* + (1-t)\bar{l})$$
for $t \in [0, 1]$. Then $f(t)$ is a $C^1$ smooth convex function on $[0, 1]$ and a smooth strictly convex function in a neighborhood $[0, \epsilon)$ for some positive number $\epsilon < 1$, which implies $f'(t)$ is a monotone nondecreasing function on $[0, 1]$ and strictly nondecreasing on $[0, \epsilon)$.

Note that $f'(t) = \sum_{(ij) \in E} \left[ 2\pi - \tilde{K}_{ij}(tl^* + (1-t)\tilde{l}) \right] \cdot (l_{ij}^* - l_{ij})$, we have $f'(0) = f'(1)$ by $\tilde{K}(l^*) = K(\tilde{l})$. This implies $f'(t)$ is a constant on $[0, 1]$, which contradicts that $f'(t)$ is strictly nondecreasing on $[0, \epsilon)$. Q.E.D.

4. Uniqueness and long time existence of the extended combinatorial Ricci flow

Using the extended combinatorial Ricci curvature $\tilde{K}_{ij}$ defined by (3.2), we can define a new flow as follows for $l \in \mathbb{R}^E$

$$\frac{dl_{ij}}{dt} = \tilde{K}_{ij}. \tag{4.1}$$

Note that $\tilde{K}_{ij}$ is continuous, therefore the solution of the new flow (4.1) exists for short time by the ODE theory (See [27] Theorem 2.1 for example). However, $\tilde{K}_{ij}$ is not locally Lipschitz continuous by Remark 3.1, the uniqueness of the solution for the new flow (4.1) can not be derived directly from the standard ODE theory. For the flow (4.1), we still have the uniqueness of the solution.

**Theorem 4.1.** The solution of the flow (4.1) is unique for any initial generalized hyper-ideal polyhedral metric $l_0 \in \mathbb{R}^E$.

**Proof.** The idea of the proof comes from Ge-Hua [10], but the proof is simpler for our case. We present the proof here for completeness. Suppose $l_1(t)$ and $l_2(t)$ are two solutions of the flow (4.1) on $[0, T]$ for the same initial value $l_0 \in \mathbb{R}^E$, where $T$ is some positive constant. To prove the uniqueness of the solution, we just need to prove $l_1(t) = l_2(t)$ for any $t \in [0, T]$.

By Theorem 3.1, the energy function $\tilde{H}$ defined by (2.2) could be extended to be a $C^1$ smooth convex function

$$\tilde{H}(l) = \sum_{\sigma \in T} \tilde{F}_{\sigma} - 2\pi \sum_{(ij) \in E} l_{ij} \tag{4.2}$$

defined on $\mathbb{R}^E$ with $\nabla \tilde{H} = -\tilde{K}$. We claim that

$$\left( \nabla \tilde{H}(l_1) - \nabla \tilde{H}(l_2) \right) \cdot (l_1 - l_2) \geq 0,$$

which is equivalent to

$$\left( \tilde{K}(l_1) - \tilde{K}(l_2) \right) \cdot (l_1 - l_2) \leq 0, \tag{4.3}$$

for any $l_1, l_2 \in \mathbb{R}^E$. 
To prove the claim, take \( \varphi_\epsilon(x) = \frac{1}{C e^{1-|x|^2}}, \ x \in \mathbb{R}^E, \) as the standard mollifier with
\[
\varphi(x) = \begin{cases} 
C e^{1-|x|^2}, & |x| < 1, \\
0, & |x| \geq 1,
\end{cases}
\]
where \( C \) is chosen to satisfy \( \int_{\mathbb{R}^E} \varphi(x) \, dx = 1. \) The \( \epsilon \)-mollifier \( \tilde{H}_\epsilon \) of \( \tilde{H} \) is defined to be
\[
\tilde{H}_\epsilon(l) = \tilde{H} * \varphi_\epsilon(l)
\]
for \( l \in \mathbb{R}^E. \) By the convexity and \( C^1 \) smoothness of \( \tilde{H}, \tilde{H}_\epsilon \) is a \( C^\infty \) smooth convex function on \( \mathbb{R}^E \) and \( \tilde{H}_\epsilon \to \tilde{H} \) in \( C^1_{\text{loc}}(\mathbb{R}^E) \). By the \( C^\infty \) smoothness and convexity of \( \tilde{H}_\epsilon \), we have
\[
\left( \nabla \tilde{H}_\epsilon(l_1) - \nabla \tilde{H}_\epsilon(l_2) \right) \cdot (l_1 - l_2) \geq 0
\]
for any \( l_1, l_2 \in \mathbb{R}^E. \) Let \( \epsilon \to 0, \) we have
\[
\left( \nabla \tilde{H}(l_1) - \nabla \tilde{H}(l_2) \right) \cdot (l_1 - l_2) \geq 0,
\]
which completes the proof of the claim.

Set \( f(t) = ||l_1(t) - l_2(t)||^2. \) Then we have \( f(t) \geq 0 \) for \( t \in [0, T] \) with \( f(0) = 0 \) by the initial condition \( l_1(0) = l_2(0). \) Furthermore,
\[
\frac{df}{dt} = (\frac{dl_1}{dt} - \frac{dl_2}{dt}) \cdot (l_1(t) - l_2(t)) = (\tilde{K}(l_1(t)) - \tilde{K}(l_2(t))) \cdot (l_1(t) - l_2(t)) \leq 0
\]
by (4.3), which implies \( f(t) \equiv 0 \) for \( t \in [0, T]. \) Therefore, \( l_1(t) = l_2(t) \) for any \( t \in [0, T], \) from which the uniqueness follows. Q.E.D.

**Remark 4.1.** Although the solution of the extended flow (4.1) is unique, there may exist some other different extensions of the solution of combinatorial Ricci flow (1.1) on \( \mathbb{R}^E. \) The key point is that the solution of the extended flow (4.1) depends on the extension of the combinatorial Ricci curvature. For example, one can also extend the combinatorial Ricci curvature by symmetry to \( \mathbb{R}^E, \) which is different from the extension used here. The author thanks Tian Yang for pointing this out to the author.

As a corollary of Theorem 4.1, we have the following result which shows that the solution of the new flow (4.1) extends the solution of the combinatorial Ricci flow (1.1) for any initial hyper-ideal polyhedral metric in \( L(M, T). \)

**Corollary 4.1.** For any initial hyper-ideal polyhedral metric \( l(0) \in L(M, T), \) denote the solutions of the combinatorial Ricci flow (1.1) and the flow (4.1) as \( l(t) \) and \( \tilde{l}(t) \) respectively. Then \( \tilde{l}(t) = l(t) \) whenever \( l(t) \) exists.

We call the flow (4.1) as the extended combinatorial Ricci flow in the following.
Theorem 4.2. The solution of extended combinatorial Ricci flow (4.1) exists for all time for any initial generalized hyper-ideal polyhedral metric \( l(0) \in \mathbb{R}^E \).

Proof. By definition, the extension \( \tilde{a}_{ij} \) of the dihedral angle function \( a_{ij} \) is bounded by \( \pi \). By the finiteness of the triangulation \( T \), the extended combinatorial Ricci curvature \( \tilde{K} \) is uniformly bounded by some constant \( C \) depending on the triangulation \( T \) of \( M \). If \( l(t) \) is the solution of extended combinatorial Ricci flow (4.1), then we have

\[
\left| \frac{dl_{ij}}{dt} \right| \leq C
\]

for any \( \{ij\} \in E \), which implies \( |l_{ij}(t)| \leq |l_{ij}(0)| + Ct \) for any edge \( \{ij\} \in E \) and \( t \in [0, +\infty) \). Therefore, the solution of extended combinatorial Ricci flow (4.1) does not go to infinity in finite time, which implies the solution of extended combinatorial Ricci flow (4.1) exists for all time by ODE theory. Q.E.D.

Theorem 4.1, Corollary 4.1 and Theorem 4.2 together imply Theorem 1.1.

5. Convergence of the extended combinatorial Ricci flow

Theorem 5.1. Suppose the solution \( l(t) \) of extended combinatorial Ricci flow (4.1) converges to a generalized hyper-ideal polyhedral metric \( l^* \in \mathbb{R}^E \), then \( K_{ij}(l^*) = 0 \) for any edge \( \{ij\} \in E \).

Proof. By the continuity of the generalized combinatorial Ricci curvature \( \tilde{K}_{ij}(l) \) in \( l \in \mathbb{R}^E \), we have

\[
\tilde{K}_{ij}(l^*) = \lim_{t \to +\infty} \tilde{K}_{ij}(l(t))
\]

for any edge \( \{ij\} \in E \). As \( \lim_{t \to +\infty} l(t) = l^* \), there exists \( \xi_n \in (n, n + 1) \) such that

\[
\frac{dl_{ij}}{dt} \big|_{t=\xi_n} = l_{ij}(n + 1) - l_{ij}(n) \to 0,
\]

as \( n \to +\infty \). By the extended combinatorial Ricci flow (4.1), we have

\[
\tilde{K}_{ij}(l(\xi_n)) = \left. \frac{dl_{ij}}{dt} \right|_{t=\xi_n} \to 0.
\]

Combining with (5.1), we have \( \tilde{K}_{ij}(l^*) = 0 \). Q.E.D.

We have the following generalization of Theorem 1.2 for the extended combinatorial Ricci flow (4.1).

Theorem 5.2. Suppose \((M, T)\) is an ideally triangulated compact 3-manifold with boundary composed of surfaces of negative Euler characteristic. If \((M, T)\) admits a hyper-ideal polyhedral metric \( l^* \in \mathcal{L}(M, T) \) with zero combinatorial Ricci curvature, then for any initial generalized hyper-ideal polyhedral metric \( l(0) \in \mathbb{R}^E \) on \((M, T)\), the solution \( l(t) \) of the extended combinatorial Ricci flow (4.1) converges exponentially fast to \( l^* \).
Proof. Recall that the function $H$ in (2.2) could be extended to be defined on $\mathbb{R}^E$ by
\[
\tilde{H}(l) = \sum_{\sigma \in T} \tilde{F}_\sigma - 2\pi \sum_{\{ij\} \in E} l_{ij},
\]
which is a $C^1$ smooth convex function on $\mathbb{R}^E$ with $\nabla \tilde{H} = -\tilde{K}$ by Theorem 3.1. By the condition that $K(l^*) = 0$ for $l^* \in \mathcal{L}(M, T)$, we have $\nabla \tilde{H}(l^*) = 0$ and $l^*$ is a minimal point of $\tilde{H}$ on $\mathbb{R}^E$. Furthermore, $\lim_{l \to \infty} \tilde{H}(l) = +\infty$ by the following property of convex functions, the proof of which could be found in [19] (Lemma 4.6). This implies that $\tilde{H}$ is a proper function on $\mathbb{R}^E$.

Lemma 5.1. Suppose $f(x)$ is a $C^1$ smooth convex function on $\mathbb{R}^n$ with $\nabla f(x_0) = 0$ for some $x_0 \in \mathbb{R}^n$, $f(x)$ is $C^2$ smooth and strictly convex in a neighborhood of $x_0$, then $\lim_{x \to \infty} f(x) = +\infty$.

Set $\phi(t) = \tilde{H}(l(t))$, where $l(t)$ is a solution of the extended combinatorial Ricci flow (4.1). Then $\phi(t)$ is a $C^1$ smooth function for $t \in [0, +\infty)$ with
\[
\frac{d\phi}{dt} = \nabla_{l(t)} \cdot \frac{dl}{dt} = -\sum_{\{ij\} \in E} \tilde{K}_{ij}^2 \leq 0,
\]
which implies $\phi(t)$ is uniformly bounded for $t \in [0, +\infty)$, i.e. $\tilde{H}$ is bounded along the solution $l(t)$ of the extended combinatorial Ricci flow (4.1). By the properness of $\tilde{H}$ on $\mathbb{R}^E$, we have the solution $l(t), t \in [0, +\infty)$, lies in a compact subset of $\mathbb{R}^E$.

By the boundedness of $\phi(t)$ on $[0, +\infty)$ and monotonicity of $\phi(t)$ from (5.2), we have $\lim_{t \to +\infty} \phi(t)$ exists. Therefore, there exists $\xi_n \in (n, n + 1)$ such that
\[
\phi(n + 1) - \phi(n) = \left. \frac{d\phi}{dt} \right|_{t = \xi_n} = -\sum_{\{ij\} \in E} \tilde{K}_{ij}^2(l(\xi_n)) \to 0,
\]
as $n \to +\infty$. By the boundedness of $l(t)$ for $t \in [0, +\infty)$, there exists a subsequence of $\xi_n$, still denoted by $\xi_n$ for simplicity, such that $l(\xi_n) \to \bar{l}$ for some $\bar{l} \in \mathbb{R}^E$ as $n \to +\infty$, which implies $\tilde{K}(\bar{l}) = 0$ by (5.3) and the continuity of the generalized combinatorial Ricci curvature $\tilde{K}$.

Note that $\tilde{K}(\bar{l}) = 0$ for $\bar{l} \in \mathbb{R}^E$ and $K(l^*) = 0$ for $l^* \in \mathcal{L}(M, T)$, we have $\bar{l} = l^*$ by Theorem 3.3. Therefore, there is a sequence $\xi_n \to +\infty$ such that $\lim_{n \to +\infty} l(\xi_n) = l^*$, which implies that for $n$ large enough, $l(\xi_n)$ will lie in a small enough neighborhood of $l^*$.

Note that $l^*$ is a local attractor of the extended combinatorial Ricci flow (4.1) by Theorem 2.2, which was also observed by Luo [29], we have the solution $l(t)$ of the flow (4.1) converges exponentially fast to $l^*$ by Lyapunov stability theorem [35, Chapter 5].

Q.E.D.

Remark 5.1. Combinatorial Calabi flow for hyper-ideal polyhedral metrics on ideally triangulated compact 3-manifolds with boundary was introduced
by Ge-Xu-Zhang [21], which was defined as
\[
\frac{dl_{ij}}{dt} = -\Delta K_{ij},
\]
(5.4)
where \(\Delta = \Lambda = \left(\frac{\partial K_{ij}}{\partial l_{kh}}\right)\) is the combinatorial Laplace operator for hyper-ideal polyhedral metrics introduced by Luo [29]. Combinatorial Calabi flow (5.4) is a negative gradient flow of the combinatorial Calabi energy \(C(l) = \sum_{\{ij\} \in E} K_{ij}^2\) [21]. As \(\bar{a}_{ij}\) is \(C^0\) and not \(C^1\) by Remark 3.1, the combinatorial Laplace operator \(\Delta = \left(\frac{\partial K_{ij}}{\partial l_{kh}}\right)\) can not be extended continuously to be defined on \(\mathbb{R}^E\) by Luo-Yang’s extension [32]. This causes that the extension method used for combinatorial Ricci flow (1.1) in this paper does not work for the combinatorial Calabi flow (5.4). Similar phenomenons happen for combinatorial Calabi flow for vertex scaling on closed surfaces [9, 45]. The convergence of the combinatorial Calabi flow for vertex scaling was proved in [45] using a different extension of combinatorial curvature introduced in [24, 25]. A key point of the proof in [45] is that the extension of combinatorial curvature introduced in [24, 25] is \(C^1\) smooth.

As an application of Theorem 5.2, we have the following result, which is a generalization of Corollary 1.1.

**Corollary 5.1.** Suppose \((M, T)\) is an ideally triangulated compact 3-manifold with boundary.

1. If \(d_e \leq 6\) for any \(e \in E\), there exists no hyper-ideal polyhedral metric in \(L(M, T)\) with zero combinatorial Ricci curvature.
2. If \(d_e = N > 6, \forall e \in E\), for some constant \(N\), the solution of extended combinatorial Ricci flow (4.1) converges to a hyper-ideal polyhedral metric in \(L(M, T)\) with zero combinatorial Ricci curvature for any initial generalized hyper-ideal polyhedral metric in \(\mathbb{R}^E\).

**Proof.** For the first part of the corollary, suppose there exists a hyper-ideal polyhedral metric \(l^* \in L(M, T)\) with zero combinatorial Ricci curvature. Suppose \(S\) is a component of the boundary \(\partial M\), \(|V|, |E|, |F|\) represent the number of vertices, edges and faces of \(S\). By the Gauss-Bonnet formula, we have \(|V| - |E| + |F| = \chi(S) < 0\). Note that \(3|F| = 2|E|\) and \(2|E| = \sum_{x \in V} d_x \leq 6|V|\) by the condition \(d_e \leq 6\), we have
\[
0 = \frac{1}{3}|E| - |E| + \frac{2}{3}|E| \leq |V| - |E| + |F| = \chi(S) < 0,
\]
which is impossible. Therefore, there exists no hyper-ideal polyhedral metric in \(L(M, T)\) with zero combinatorial Ricci curvature in this case.

For the second part of the corollary, we just need to prove that there exists a hyper-ideal polyhedral metric \(l^* \in L(M, T)\) with zero combinatorial Ricci curvature, then the result follows from Theorem 5.2. As the numbers of tetrahedra adjacent to the edges in \(E\) are all the same, we can assign each edge in \(E\) with the same length \(s \in (0, +\infty)\), which corresponds to a
hyper-ideal polyhedral metric in $L(M, T)$. In this case, the lengths of vertex edges $x^i_{jk}$ are all the same with
\[
\cosh x^i_{jk} = \frac{\cosh l_{ij} \cosh l_{ik} + \cosh l_{jk}}{\sinh l_{ij} \sinh l_{ik}} = \frac{\cosh s}{\cosh s - 1},
\]
which implies that
\[
\cos a_{ij} = \frac{\cosh x^i_{jk} \cosh x^i_{jh} - \cosh x^i_{kh}}{2 \cosh s - 1}.
\]
Therefore, the combinatorial Ricci curvature along any edge $\{ij\} \in E$ is
\[
K(s) = 2\pi - N \arccos \frac{\cosh s}{2 \cosh s - 1},
\]
which is a strictly decreasing function of $s \in (0, +\infty)$. Note that $\lim_{s \to 0} K(s) = 2\pi > 0$ and $\lim_{s \to +\infty} K(s) = \frac{\pi}{3} (6 - N) < 0$ by the condition $N > 6$. By the intermediate value theorem, there exists $s^* \in (0, +\infty)$ such that $K(s^*) = 0$. Therefore, $l^* = s^*(1, \cdots, 1)^T$ is a hyper-ideal polyhedral metric in $L(M, T)$ with zero combinatorial Ricci curvature. Q.E.D.

**Remark 5.2.** By the proof of Corollary 5.1 (1), the condition on the number of tetrahedra adjacent to an edge in $E$ could be changed to be the condition $d_x \leq 6$ on the degree of any vertex $x$ on the boundary surfaces. Furthermore, for the nonexistence result in Corollary 5.1 (1), we just need one component of the boundary satisfies the condition $d_x \leq 6$ for any vertex $x$ in the boundary component.

### 6. Combinatorial Ricci Flow for Prescribed Combinatorial Ricci Curvature

We can modify the combinatorial Ricci flow to find hyper-ideal polyhedral metrics with prescribed combinatorial Ricci curvature $K$ as follows
\[
\frac{dl_{ij}}{dt} = K_{ij} - \overline{K}_{ij},
\]
Using the generalized combinatorial Ricci curvature $\overline{K}_{ij}$, we can also extend the modified combinatorial Ricci flow (6.1) to the following form
\[
\frac{dl_{ij}}{dt} = \overline{K}_{ij} - \overline{K}_{ij},
\]
which is called the extended modified combinatorial Ricci flow in the following. The results for the modified combinatorial Ricci flow (6.1) and extended modified combinatorial Ricci flow (6.2) are paralleling to those of the combinatorial Ricci flow (1.1) and the extended combinatorial Ricci flow (4.1). We state the results as follows.

**Theorem 6.1.** Suppose $(M, T)$ is an ideally triangulated compact 3-manifold with boundary. The solution of the extended modified combinatorial Ricci flow (6.2) exists for all time for any initial generalized hyper-ideal polyhedral
metric in $\mathbb{R}^E$ and uniquely extends the solution of the modified combinatorial Ricci flow (6.1) for any initial hyper-ideal polyhedral metric in $\mathcal{L}(M, T)$.

**Theorem 6.2.** Suppose $(M, T)$ is an ideally triangulated compact 3-manifold with boundary. If $(M, T)$ admits a hyper-ideal polyhedral metric $\bar{l} \in \mathcal{L}(M, T)$ with combinatorial Ricci curvature $\bar{K}$, then the solution $l(t)$ of the extended modified combinatorial Ricci flow (6.2) converges exponentially fast to $\bar{l}$ for any initial generalized hyper-ideal polyhedral metric in $\mathbb{R}^E$.

The proof of Theorem 6.1 is paralleling to that of Theorem 1.1 and the proof of Theorem 6.2 is similar to that of Theorem 1.2 with the function $H(l)$ replaced by

$$
\Pi(t) = \sum_{\sigma \in T} \tilde{F}_{\sigma} - 2\pi \sum_{\{ij\} \in E} l_{ij} - \sum_{(ij) \in E} K_{ij} l_{ij}
$$

and the extension $\tilde{H}(l)$ replaced by

$$
\tilde{\Pi}(l) = \sum_{\sigma \in T} \tilde{F}_{\sigma} - 2\pi \sum_{\{ij\} \in E} l_{ij} - \sum_{(ij) \in E} K_{ij} l_{ij}.
$$

As the proofs for Theorem 6.1 and Theorem 6.2 are almost the same as those of Theorem 1.1 and Theorem 1.2 for the combinatorial Ricci flow (1.1) and the extended combinatorial Ricci flow (4.1), we omit the details of the proofs here.

7. Some remarks and questions

In this paper, we have proved the global convergence of the extended combinatorial Ricci flow (4.1) to a complete hyperbolic metric on $(M, T)$ with totally geodesic boundary under the condition that there exists a hyper-ideal polyhedral metric with zero combinatorial Ricci curvature on the ideally triangulated compact 3-manifold with boundary $(M, T)$. However, there may exist no hyper-ideal polyhedral metric with zero combinatorial Ricci curvature on an ideally triangulated compact 3-manifold with boundary $(M, T)$, while there exists a complete hyperbolic metric on the compact 3-manifold $M$ with totally geodesic boundary. In other words, there may exist combinatorial obstacles for the existence of hyper-ideal polyhedral metrics with zero combinatorial Ricci curvature on the ideally triangulated compact 3-manifold with boundary $(M, T)$.

An example of similar case is the vertex scaling for piecewise linear metric on surfaces [28], for which there are combinatorial obstacles for the existence of conformal factors with constant combinatorial curvature on a surface with a fixed triangulation. For surfaces with fixed triangulations, Ge-Jiang [11] used the extension introduced by Bobenko-Pinkall-Springborn [3] and Luo [31] to extend Luo’s combinatorial Yamabe flow for vertex scaling on a triangulated surface [28] and proved the convergence of the extended combinatorial Yamabe flow under the existence of a conformal factor with constant
combinatorial curvature on the surface with a fixed triangulation. The convergence of the extended combinatorial Yamabe flow depends on the triangulation of the surface and the existence of conformal factors with constant combinatorial curvature.

However, if one takes the piecewise constant curvature metric as a cone metric on the surface and does surgery on the Delaunay triangulations of the surface by edge flipping, it has been proved that there always exists a conformal factor defined on the vertices such that the induced polyhedral metric on the surface has constant combinatorial curvature \([24, 25]\). Furthermore, one can use the combinatorial Yamabe flow with surgery \([24, 25]\) and combinatorial Calabi flow with surgery \([15]\) to find polyhedral metrics with constant combinatorial curvature, the convergence of which do not depend on the initial triangulation of the surface and existence of conformal factors with constant combinatorial curvature.

For compact 3-manifolds with boundary, it is natural to ask the following question.

**Question:** Is there any way to do surgery on ideally triangulated compact 3-manifolds with boundary to ensure the long time existence and convergence of the combinatorial Ricci flow \((1.1)\)?

This is similar to the case of combinatorial Yamabe flow and combinatorial Calabi flow for vertex scaling on closed surfaces and similar to the case of Hamilton’s Ricci flow on 3-dimensional Riemannian manifolds. Similar questions were also asked by Luo in \([29]\). If one can give an affirmative answer to this question, it is conceive that one can give a solution to Luo’s conjecture without the assumption of existence of a hyper-ideal polyhedral metric with zero combinatorial Ricci curvature, a proof of the convergence of combinatorial Calabi flow \((5.4)\) and a new proof of Thurston’s geometrization theorem for compact 3-manifolds with boundary consisting of surfaces of negative Euler characteristic.

**Acknowledgements**

Part of the series of work, including \([41, 42]\) and this paper, was done when the author was visiting the Rutgers University. The author thanks Professor Feng Luo for his invitation to Rutgers University and communications and interesting on these work. The author thanks Professor Feng Luo and Professor Tian Yang for explaining the details of their joint work \([32]\) to the author. Part of the series of work was reported in an online seminar of Rutgers University in April 2020. The author thanks the participants in the seminar for communication and suggestions. The research of the author is supported by Fundamental Research Funds for the Central Universities and National Natural Science Foundation of China under grant no. 61772379.
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