Coupled dynamics in gluon mass generation
and the impact of the three-gluon vertex

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Abstract

We present a detailed study of the subtle interplay transpiring at the level of two integral equations that are instrumental for the dynamical generation of a gluon mass in pure Yang-Mills theories. The main novelty is the joint treatment of the Schwinger-Dyson equation governing the infrared behaviour of the gluon propagator and of the integral equation that controls the formation of massless bound-state excitations, whose inclusion is instrumental for obtaining massive solutions from the former equation. The self-consistency of the entire approach imposes the requirement of using a single value for the gauge coupling entering in the two key equations; its fulfillment depends crucially on the details of the three-gluon vertex, which contributes to both of them, but with different weight. In particular, the characteristic suppression of this vertex at intermediate and low energies enables the convergence of the iteration procedure to a single gauge coupling, whose value is reasonably close to that extracted from related lattice simulations.

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I. INTRODUCTION

The nonperturbative aspects of the gluon propagator, $\Delta_{\mu\nu}^{ab}(q)$, are considered to be especially relevant for the qualitative and quantitative understanding of a wide range of important physical phenomena, such as confinement, chiral symmetry breaking, and bound-state formation. A particularly interesting feature, which manifests itself both in the Landau gauge and away from it, is the saturation of its scalar form factor, $\Delta(q^2)$, in the deep infrared (IR), i.e., $\Delta(0) = c_0 > 0$. This special behavior, which is believed to be intimately connected with the emergence of a fundamental mass scale, was firmly established in a variety of SU(2) [1–3] and SU(3) [4–8] large-volume lattice simulations, and has been extensively studied in the continuum within diverse theoretical frameworks [9–41].

In one of these approaches, the Schwinger-Dyson equation (SDE) that controls the evolution of the gluon propagator has been shown to yield an infrared finite (“massive”) solution. We emphasize that the relevant SDE was formulated within the framework developed through the merging of the pinch-technique (PT) [9, 42–46] with the background-field method (BFM) [47], to be referred to as “PT-BFM scheme” [15, 48, 49]. Inherent to this scheme is the distinction between background ($B$) and quantum ($Q$) gluons, and the proliferation of the possible Green’s functions that one may form with them. Particularly relevant for what follows is the distinction between the $QQ$ and $QB$ gluon self-energies, and the $Q^3$ and $BQ^2$ three-gluon vertices, to be denoted by $\Gamma$ and $\tilde{\Gamma}$, respectively.

An indispensable ingredient for the realization of this scenario is the presence of massless poles of the type $1/q^2$ in the vertices with one $B$ leg, which enter into the $QB$ gluon self-energy [50–53] and implement the well-known Schwinger mechanism for gauge-boson mass generation [54–59]. The origin of these poles is dynamical, owing to the formation of colored bound-state excitations, which are massless due to the strong binding induced by the Yang-Mills interactions. The integral equations that govern their formation constitute a system of homogeneous linear Bethe-Salpeter equations (BSEs), which determines the derivatives of the corresponding “bound-state wave functions”. In the present work we will simplify the degree of complexity by restricting the possibility of pole formation only in $\tilde{\Gamma}$, thus reducing the aforementioned system into a single BSE, which determines the corresponding derivative, to be denoted by $\tilde{C}_1'(k^2)$.

Evidently, the self-consistent implementation of the dynamical picture described above
hinges on the subtle interplay between the BSE and SDE, and the compatibility of the various field-theoretic ingredients that enter in them. The purpose of the present work is to focus on a particularly pivotal aspect of this interplay, and elucidate the decisive impact not only of $\tilde{\Gamma}$, whose $1/q^2$ pole enforces the desired infrared finiteness of $\Delta(q^2)$, but especially of $\Gamma$, whose infrared structure affects both the kernel of the BSE and a crucial two-loop component of the SDE.

In order to appreciate the circumstances described above in some detail, let us first observe that the pole BSE and the gluon SDE are tightly intertwined mainly because $(i)$ the SDE expresses the value of $\Delta^{-1}(0)$ as an integral that involves $\tilde{C}'_1(k^2)$ [52], while, at the same time, $(ii)$ $\tilde{C}'_1(k^2)$ is known to be proportional to $d m^2(k^2)/dk^2$ [50]. Thus, once obtained from the BSE, it provides, upon integration, the running gluon mass $m^2(k^2)$, a notion that dates back to the pioneering work of [9]. This dual role of $\tilde{C}'_1(k^2)$, coupled to the obvious requirement that $\Delta^{-1}(0) = m^2(0)$, imposes finally a stringent constraint on the strong coupling $\alpha_s = g^2/4\pi$; specifically, the value of $\alpha_s$ used in $(i)$, to be denoted by $\alpha_{s}^{\text{SDE}}$, ought to coincide with that employed in $(ii)$, to be denoted by $\alpha_{s}^{\text{BSE}}$.

As advocated above, the nonperturbative behavior of the vertex $\Gamma$ becomes relevant when trying to enforce the equality $\alpha_{s}^{\text{SDE}} = \alpha_{s}^{\text{BSE}}$. Note in particular that $(a)$ $\Gamma$ enters linearly in the SDE-derived expression that determines the value of $\Delta^{-1}(0)$ and quadratically in the kernel of the BSE, rendering it renormalization group invariant (RGI), and $(b)$ below 1 GeV the vertex $\Gamma$ is suppressed with respect to its tree-level value, reversing its sign around 100 MeV, and finally diverging logarithmically at the origin.

It turns out that, when the tree-level expression of $\Gamma$ is used in the evaluation of $(i)$ and $(ii)$, the resulting values for $\alpha_{s}^{\text{SDE}}$ and $\alpha_{s}^{\text{BSE}}$ do not coincide. Instead, if one employs a standard nonperturbative Ansatz for $\Gamma$, which encodes the features mentioned in $(b)$, one finds that, indeed, $\alpha_{s}^{\text{SDE}} = \alpha_{s}^{\text{BSE}}$. The common value is given by $\alpha_s = 0.45$, when the momentum subtraction (MOM) renormalization is implemented at $\mu = 4.3\text{GeV}$. This particular value for $\alpha_s$ is to be contrasted with the one obtained (for the same $\mu$) from the lattice simulation of the three-gluon vertex $\Gamma$ in [60], namely $\alpha_s = 0.32$. This discrepancy appears to be more than acceptable given the approximations implemented when deriving both the SDE and the BSE, and, in particular, the simplifications applied in the renormalization of the former, and the truncations imposed when constructing the kernel of the latter.
II. SCHWINGER MECHANISM AND VERTICES WITH MASSLESS POLES

Throughout this work, we consider a SU(3) pure Yang-Mills theory (no dynamical quarks). In the Landau gauge, the gluon propagator $\Delta_{\mu\nu}^{ab}(q) = \delta_{ab} \Delta_{\mu\nu}(q)$ has the form

$$\Delta_{\mu\nu}(q) = -i \Delta(q^2) P_{\mu\nu}(q); \quad P_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_{\mu} q_{\nu}}{q^2},$$

(2.1)

where $\Delta(q^2)$ is related to the form factor of the gluon self-energy $\Pi_{\mu\nu}(q) = P_{\mu\nu}(q) \Pi(q^2)$, through $\Delta^{-1}(q^2) = q^2 + i \Pi(q^2)$. Lattice data for this (quenched) quantity, renormalized at $\mu = 4.3$ GeV, are shown in Fig. 1 and will serve as the main input in the ensuing analysis. In addition, the ghost propagator $D_{\mu\nu}^{ab}(q^2) = i \delta_{ab} D(q^2)$ furnishes the dressing function, $F(q^2)$, defined as $F(q^2) = q^2 D(q^2)$; in the Landau gauge (again at $\mu = 4.3$ GeV), $F(0) \approx 2.9$.

In the PT-BFM framework, the SDE of $\Delta(q^2)$ is expressed in terms of the QB self-energy $\tilde{\Pi}_{\mu\nu}(q)$, namely (see Fig. 2)

$$\Delta^{-1}(q^2) P_{\mu\nu}(q) = \frac{q^2 P_{\mu\nu}(q) + i \tilde{\Pi}_{\mu\nu}(q)}{1 + G(q^2)},$$

(2.2)

where $G(q^2)$ is the $g_{\mu\nu}$ component of a special two-point function $63$. In the Landau gauge only, the important relation $1 + G(0) = F^{-1}(0)$ holds exactly $23, 64$. 

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FIG. 1. Lattice data for the quenched Landau gauge gluon propagator obtained from a tree-level Symanzik (tlSym) improved gauge action $[60, 61]$ (calibrated following the procedure described in $[62]$), compared with the corresponding data obtained from a Wilson gauge action $[5]$. The momentum axis is linear on the left of the vertical dashed line and logarithmic on the right, an artifice that clearly exposes the existence of a saturation point at IR momenta.
FIG. 2. The procedure leading to the dynamical gluon mass generation within the PT-BFM framework.

The main advantage of expressing the gluon SDE in terms of $\tilde{\Pi}_{\mu\nu}(q)$ rather than $\Pi_{\mu\nu}(q)$ arises from the fact that, when contracted from the side of the $B$-gluon, each fully dressed vertex satisfies a linear (Abelian-like) Slavnov-Taylor identity (STI). In particular, the $BQ^2$ vertex $\tilde{\Gamma}_{\mu\alpha\beta}$ satisfies (color omitted)

$$q^\mu\tilde{\Gamma}_{\mu\alpha\beta}(q, r, p) = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p), \quad (2.3)$$

Let us now turn to the infrared dynamics described by Eq. (2.2). As has been shown recently [52], if the vertices carrying the $B$ leg do not contain massless poles of the type $1/q^2$, then the $\Delta(q^2)$ governed by Eq. (2.2) remains rigorously massless. The demonstration relies on the subtle interplay between the Ward-Takahashi identities (WTIs), satisfied by the vertices as $q \to 0$, and an integral relation known as the “seagull identity” [52, 65]. The basic steps that lead to this result may be exemplified in terms of the $BQ^2$ vertex $\tilde{\Gamma}_{\mu\alpha\beta}$; the inclusion of the remaining vertices is conceptually straightforward [52].

To that end, consider the limit of the STI (2.3) as $q \to 0$, assuming that $\tilde{\Gamma}_{\mu\alpha\beta}$ does not contain $1/q^2$ terms. Then, the Taylor expansion of both sides generates the corresponding WTI

$$\tilde{\Gamma}_{\mu\alpha\beta}(0, r, -r) = -i \frac{\partial}{\partial r^\mu} \Delta^{-1}_{\alpha\beta}(r), \quad (2.4)$$
FIG. 3. The BSE satisfied by the bound-state wave function $\tilde C_{\alpha\beta}$ (upper line) and the simplified four gluon kernel used.

which, when used in the evaluation of the gluon SDE, yields

$$\Delta^{-1}(0) = \int \frac{\partial}{\partial k^\mu} F_\mu(k) = 0; \quad F_\mu(k) = k_\mu F(k^2) \quad (2.5)$$

where $F(k^2) = \Delta(k^2)[c_1 + c_2 Y(k^2)]$, with $c_1, c_2 \neq 0$, and

$$Y(k^2) = \frac{1}{(d-1)} \frac{k_\alpha}{k^2} \int_\ell \Delta^{\alpha\rho}(\ell) \Delta^{\beta\sigma}(\ell + k) \Gamma_{\sigma\rho\beta}(-\ell - k, \ell, k). \quad (2.6)$$

Note that we have introduced the dimensional regularization integral measure $\int_k \equiv \frac{\mu^\epsilon}{(2\pi)^d} \int d^d k$, with $d = 4 - \epsilon$ the space-time dimension, and $\mu$ the 't Hooft mass scale.

In order to circumvent the result of Eq. (2.5), one must allow $\tilde \Gamma_{\mu\alpha\beta}$ to contain longitudinally coupled $1/q^2$ poles; their inclusion, in turn, triggers the Schwinger mechanism [54, 55], finally enabling the generation of a gauge boson mass [56–59]. More specifically, we have

$$\tilde \Gamma_{\mu\alpha\beta}(q, r, p) = \tilde \Gamma_{\mu\alpha\beta}^{np}(q, r, p) + \frac{q_\mu}{q^2} \tilde C_{\alpha\beta}(q, r, p), \quad (2.7)$$

where the superscript “np” indicates the “no-pole” part, and $\tilde C_{\alpha\beta}$ is the aforementioned bound-state wave function. Evidently, the Bose-symmetry of the vertex under the exchange $(\alpha, r) \leftrightarrow (\beta, p)$ imposes the relation $\tilde C_{\alpha\beta}(0, r, -r) = 0$.

Next, in order to preserve the BRST symmetry of the theory, we demand that all STIs maintain their exact form in the presence of these poles; therefore, Eq. (2.3) will now read

$$q^\mu \tilde \Gamma_{\mu\alpha\beta}^{np}(q, r, p) + \tilde C_{\alpha\beta}(q, r, p) = i\Delta_{\alpha\beta}^{-1}(r) - i\Delta_{\alpha\beta}^{-1}(p). \quad (2.8)$$
Taking the limit of Eq. (2.8) as $q \to 0$ and matching the lowest order terms in $q$, the corresponding WTI becomes

$$
\tilde{\Gamma}^{\text{np}}_{\mu \alpha \beta}(0, r, -r) = -i \frac{\partial}{\partial r^\mu} \Delta^{-1}_\alpha(r) - \left\{ \frac{\partial}{\partial q^\mu} \tilde{C}_{\alpha \beta}(q, r, -r) \right\}_{q=0}.
$$

(2.9)

The presence of the second term on the r.h.s. of Eq. (2.9) has far-reaching consequences for the infrared behavior of $\Delta$. Specifically, a repetition of the steps leading to Eq. (2.5) reveals that, whereas the first term on the r.h.s. of Eq. (2.9) reproduces again Eq. (2.5) (and its contribution thus vanishes), the second term survives, giving

$$
\Delta^{-1}(0) = \frac{3}{2} g^2 C_F(0) \int_k k^2 \Delta^2(k^2) \left[ 1 - \frac{3}{2} g^2 C_A Y(k^2) \right] \tilde{C}_1'(k^2),
$$

(2.10)

where $\tilde{C}_1(q, -k - q, k)$ is the form factor of $g_{\alpha \beta}$ in the tensorial decomposition of $\tilde{C}_{\alpha \beta}$,

$$
\tilde{C}_1'(k^2) = \lim_{q \to 0} \left\{ \frac{\partial \tilde{C}_1(q, k, -k - q)}{\partial (k + q)^2} \right\},
$$

(2.11)

and $C_A$ is the Casimir eigenvalue of the adjoint representation [$N$ for SU($N$)]. Note that the one- and two-loop dressed contributions enter into the mass condition (2.10) with a different relative sign, a fact that is crucial for the ensuing analysis.

III. BSE FOR THE MASSLESS BOUND-STATES

The dynamical equation that governs $\tilde{C}_1(k^2)$ may be derived from the SDE satisfied $\tilde{\Gamma}_{\mu \alpha \beta}(q, r, p)$, as $q \to 0$. In this limit, the derivative term becomes the leading contribution, given that $\tilde{C}_{\alpha \beta}(0, r, -r) = 0$, and the resulting homogeneous equation assumes the form of a BSE (see Fig. 3), given by [53]

$$
f^{\text{ann}} \lim_{q \to 0} \tilde{C}_{\alpha \beta}(q, r, p) = f^{abc} \lim_{q \to 0} \left\{ \int_k \tilde{C}_{\gamma \delta}(q, k, -k - q) \Delta^\gamma(k) \times \Delta^\delta(k + q) K_{\rho \alpha \beta \sigma}(-k, r, p, k + q) \right\}.
$$

(3.1)

To proceed further, we will approximate the four-gluon BS kernel $K$ by the lowest-order set of diagrams appearing in its skeleton expansion, given by the diagrams $(b_1)$, $(b_2)$, and $(b_3)$, shown in the second line of Fig. 3. It turns out that, if we use the tree-level four-gluon vertex in the evaluation of $(b_1)$, its contribution in the above kinematic limit vanishes. Diagrams
(b_2) and (b_3), which carry a statistical factor of 1/2, are considered to contain fully dressed gluon propagators and three gluon vertices \( \Gamma \) (note that all gluons are of the \( Q \)-type). As a consequence, the resulting BSE does not depend on the value of the MOM subtraction point \( \mu \), because the two graphs composing its kernel may be written as the “square” of the formally RGI combination

\[
R^{\alpha \beta}(q_1, q_2, q_3) = g\Delta(q_1)\Delta^{1/2}(q_2)\Gamma^{\alpha \beta}(q_1, q_2, q_3),
\]

namely, setting \( q = 0 \), \( (b_2) \sim R(-k, k - r, r)R(k, r - k, -r) \) and \( (b_3) \sim R(-k, k + r, -r)R(k, -r - k, r) \).

The vertex \( \Gamma \) contains 14 form factors \([66]\), whose nonperturbative structure, albeit subject of various studies \([67–74]\), is only partially known. Therefore, for the purposes of the present work, we will consider the simple Ansatz

\[
\Gamma_{\alpha \beta}(q, r, p) = f(r)\Gamma^{(0)}_{\alpha \beta}(q, r, p),
\]

where \( \Gamma^{(0)} \) is the standard tree-level expression of the vertex, and the form factor \( f(r) \) is considered to be a function of a single kinematic variable. Then, using Eq. (3.3) into Eq. (3.1), we arrive at the final equation

\[
\tilde{C}_1'(q^2) = \frac{8\pi}{3}\alpha_s C_A \int_k \tilde{C}_1'(k^2) \frac{(q\cdot k)[q^2k^2 - (q\cdot k)^2]}{q^4k^2(k + q)^2} \Delta^2(k)\Delta(k + q) \times f^2(k + r) \left[ 8q^2k^2 + 6(q\cdot k)(q^2 + k^2) + 3(q^4 + k^4) + (q\cdot k)^2 \right].
\]

The functional form we will employ for \( f(r) \) is motivated by a considerable number of lattice simulations and studies in the continuum. In particular, for certain characteristic kinematic configurations (such as the symmetric and the soft gluon limits), the vertex is suppressed with respect to its tree-level value, reverses its sign for relatively small momenta (an effect known as “zero crossing”), and finally diverges at the origin \([67–74]\). The reason for this particular behavior may be traced back to the delicate balance between contributions originating from gluon loops, which are “protected” by the corresponding gluon mass, and the “unprotected” logarithms coming from the ghost loops that contain massless ghosts. Early lattice indication for a zero crossing in SU(2) Yang-Mills theories can be found in \([75, 76]\), whereas the effect has been recently confirmed to be present also in the case of SU(3) theories \([60, 61, 77]\). A compilation of the lattice data of \([60, 61]\), properly normalized by dividing out the coupling \([ g = 2 \text{ at } \mu = 4.3 \text{ GeV} \text{ for the set at hand, corresponding to } \alpha_s = 0.32 ] \), is shown in Fig. 4.
FIG. 4. Compilation of SU(3) lattice data (evaluated with various $\beta$, volumes and actions) for the form factor $f$ in the symmetric configuration [60, 61].

IV. RUNNING GLUON MASS FROM THE BSE

In the absence of poles, the validity of Eq. (2.5) suggests that $\Delta^{-1}(q^2) = q^2 J(q^2)$, where the function $J(q^2)$ captures the perturbative contributions and diverges as $\ln q^2$ at the origin. Instead, the infrared saturation of $\Delta^{-1}(q^2)$ motivates the physical parametrization

$$\Delta^{-1}(q^2) = q^2 J(q^2) + m^2(q^2), \quad (4.1)$$

with $m^2(0) \neq 0$. Note that $J(q^2)$ is also affected by the presence of the mass, since most of its logarithms are now “protected”.

If we now introduce Eq. (4.1) in the rhs of Eq. (2.8), it is natural to associate the $J$ terms with the $q^\mu \tilde{\Gamma}^{\mu np}_{\rho s}$ on the l.h.s, and, correspondingly,

$$\tilde{C}_{\alpha\beta}(q, r, p) = m^2(p^2) P_{\alpha\beta}(p) - m^2(r^2) P_{\alpha\beta}(r). \quad (4.2)$$

Focusing on the $g_{\alpha\beta}$ components of Eq. (4.2), we obtain

$$\tilde{C}_1(q, r, p) = m^2(r^2) - m^2(p^2), \quad (4.3)$$

which, in the limit $q \to 0$, leads to the important result [50]

$$\tilde{C}_1'(r^2) = \frac{dm^2(r^2)}{dr^2}. \quad (4.4)$$

Then, upon integration,

$$m^2(x) = \Delta^{-1}(0) + \int_0^x dy \tilde{C}_1(y), \quad (4.5)$$
where \( x = q^2 \) and \( y = r^2 \). Eq. (4.5) establishes thus a possible link between the solution of the BSE (3.4) and what has been identified in the literature with the dynamically generated gluon mass [78]. However, in order for the quantity \( m^2(q^2) \) to admit a running mass interpretation in the sense familiar from the quark case, it needs to: (i) be a monotonically decreasing function of \( q^2 \); (ii) vanish in the UV, \( i.e., satisfy m^2(\infty) = 0 \).

To explore the implications of these requirements, let \( S' \) be a general solution of the BSE (3.4) corresponding to a certain (eigen)value of the strong coupling, \( \alpha_s = \alpha_s^{\text{BSE}} \). The typical shape of such solutions is shown in Fig. 5. Then, one has \( \tilde{C}'_1(x) = cS'(x) \), where \( c \) is a normalization constant that needs to be determined. To this end, observe that, with the kernel used, \( S' \) is positive definite; then, the requirement of a monotonically decreasing \( m^2(x) \) forces \( c \) to be negative: \( c = -|c| \). Furthermore, the condition \( m^2(\infty) = 0 \) fixes its
modulus, since Eq. (4.5) implies
\[ \Delta^{-1}(0) = |c| \int_0^\infty dy S'(y), \] (4.6)
so that \(|c| = 0.0076\). Substitution of Eq. (4.6) into Eq. (4.5) yields
\[ m^2(x) = |c| \int_x^\infty dy S'(y), \] (4.7)
which, upon integration, gives rise to the squared running mass shown in Fig. 5; it may be accurately fitted by
\[ m^2(q^2) = m^2(0)/[1 + (q^2/m_1^2)^{1+p}], \] (4.8)
with \(m_1 = 0.36\) GeV and \(p = 0.1\), in excellent agreement with the behavior found in [79].

V. BSE/SDE CONSISTENCY CONDITION

Let us now return to Eq. (2.10), whose derivation was carried out before renormalization. Its renormalization may be carried out by introducing the standard renormalization constants for the propagators, vertices, and the coupling. Then, using the constraints that the various STIs impose on these constants, all quantities entering into Eq. (2.10) can be converted into renormalized ones, and the replacement
\[ 1 - \frac{3}{2} g^2 C_A Y(k^2) \rightarrow Z_3 - \frac{3}{2} Z_4 g^2 \alpha_s C_A Y(k^2) \] (5.1)
must be implemented on its r.h.s., with \(Z_3\) and \(Z_4\) the renormalization constants of the \(Q^3\) and \(Q^4\) vertices, respectively.

The presence of \(Z_3\) and \(Z_4\) converts the computation of the rhs of Eq. (2.10) into a highly nontrivial exercise, which requires, among other things, the detailed knowledge of the structure of the \(Q^3\) and \(Q^4\) vertices. Therefore, as is common in this type of analysis, we will simplify the situation by setting \(Z_3 = Z_4 = 1\).

Then, substituting Eq. (4.6) into Eq. (2.10), we obtain a second order algebraic equation for \(\alpha_s\), given by
\[ A\alpha_s^2 + B\alpha_s + C = 0, \] (5.2)
where, passing to Euclidean space and using spherical coordinates,

\[ A = \frac{3C_A^2}{32\pi^3} F(0) \int_0^\infty dy y^2 \Delta^2(y) Y(y) S'(y), \]

\[ B = -\frac{3C_A}{8\pi} F(0) \int_0^\infty dy y^2 \Delta^2(y) S'(y), \]

\[ C = -\int_0^\infty dy S'(y). \]  

(5.3)

with \( A > 0 \) and \( B, C < 0 \). The unique positive solution of Eq. (5.2) is given by

\[ \alpha_s^{SDE} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \]

(5.4)

which shows how the existence of a positive coupling relies on a delicate interplay between the strength of the one- and two-loop dressed contributions in the gluon SDE.

We will now perform a numerical analysis in order to establish if the equality \( \alpha_s^{SDE} = \alpha_s^{BSE} \) can be indeed realized, and, if so, at what value of the strong coupling \( \alpha_s \).

In order to fully appreciate the importance of employing a nontrivial \( f \) in this context, let us set \( f = 1 \) both in Eq. (2.6) and Eq. (3.4). Then, a straightforward calculation yields the rather disparate set of values \( \alpha_s^{SDE} = 0.42 \) and \( \alpha_s^{SDE} = 0.27 \). As we will see, the effect of using a physically motivated \( f \) will be a slight increase in \( \alpha_s^{SDE} \) combined with a considerable increase in \( \alpha_s^{BSE} \).

Let us choose for \( f \) a fit to the data of Fig. 4 given by [60]

\[ f(q^2) = \lambda \left[ 1 + b \ln \frac{q^2 + M^2}{\mu^2} + c \ln \frac{q^2}{\mu^2} + e \frac{M^2(q^2 - \mu^2)}{(q^2 + M^2)(\mu^2 + M^2)} \right], \]

(5.5)

with \( \mu = 4.3 \) GeV the renormalization scale. We set \( b = e = -5.30, c = 5.40, M = 0.124 \) GeV, but leave the scale factor \( \lambda \) undetermined for the moment.

Next, using the same three-gluon vertex approximation [3.3], Eq. (2.6) yields (in spherical coordinates\(^1\) and \( d = 4 \))

\[ iY(y) = \frac{1}{24\pi^3} \int_0^\infty dt \int_0^\pi d\omega \sin^2 \omega \left[ 5 + (y + \sqrt{yt} \cos \omega)/u \right] \]

\[ \times f(u) \Delta(t) \Delta(u). \]  

(5.6)

\(^1\) Here we set \( t = \ell^2 \) and \( u = (k+\ell)^2 \) and \( k \cdot \ell = \sqrt{yt} \cos \omega \).
We emphasize that \( Y \) is computed for the first time using full gluon propagators and a nonperturbative Ansatz for the three-gluon vertex; this is a major improvement, given that all previous treatments of this quantity were purely perturbative (one loop) \[78\].

We then proceed as follows. To begin with, both in the evaluation of (3.4) and Eq. (5.4) we use as input for \( \Delta(k^2) \) and \( F(0) \) the lattice data of \[5\]. Then, we set in Eq. (5.5) the convenient starting value \( \lambda_0 = 1 \), and determine the value of the coupling \( \alpha_{\text{BSE}} = \alpha_0 \) for which the BSE (3.4) yields the nontrivial solution \( S' \); specifically, we find that \( \alpha_0 = 0.61 \).

Next, we substitute \( S' \) into Eq. (5.3) and compute the coefficients \( A_0, B_0 \) and \( C_0 \) of Eq. (5.3), whose values are (all in GeV\(^2\)) \( A_0 = 156.2, B_0 = -40.7 \) and \( C_0 = -18.5 \). Substituting them into Eq. (5.4), one obtains \( \alpha_{\text{SDE}} = 0.5 \); evidently, \( \alpha_{\text{SDE}} \neq \alpha_0 \). In order to achieve the desired equality \( \alpha_{\text{SDE}} = \alpha_{\text{BSE}} \), note that, if \( \lambda \) is moved from \( \lambda_0 = 1 \), the BSE will yield precisely the same solution as before provided that its coupling is rescaled to \( \alpha_{\text{BSE}} = \alpha_0/\lambda^2 \) (recall that the BSE is quadratic in \( f \)). In addition, since \( Y \) is linear in \( f \), we will simply have that \( A \to \lambda A_0 \), while \( B \) and \( C \) remain at their initial values. Therefore, imposing the condition \( \alpha_{\text{SDE}} = \alpha_{\text{BSE}} \) implies that the scale factor \( \lambda \) has to be such that

\[
-B_0 + \frac{B_0^2 - 4\lambda A_0 C_0}{2\lambda A_0} = \frac{\alpha_0}{\lambda^2},
\]

or, equivalently,

\[
C_0 \lambda^3 + \alpha_0 B_0 \lambda + \alpha_0^2 A_0 = 0,
\]

whose only real solution is \( \lambda \approx 1.16 \). A shown in Fig. 6, the \( f \) obtained from Eq. (5.5) using this special value for \( \lambda \) fits particularly well the lattice data. Thus, the two couplings converge to the single value \( \alpha_{\text{BSE}} = \alpha_{\text{SDE}} = 0.45 \), corresponding to \( g = 2.4 \) at \( \mu^2 = 4.3 \) GeV, which is 20\% off the value used for \( g \) in the lattice simulations mentioned above.

As a final possibility, let us assign to the SDE and the BSE different forms of \( f \), by setting \( \lambda_{\text{BSE}} \neq \lambda_{\text{SDE}} \). This difference may be considered as a simple way of accounting for the fact that, while in the SDE all arguments of \( f(x, y, z) \) are integrated over (being virtual), in the BSE the third argument is associated with the external momentum \( p \); this, in turn, may modify slightly the corresponding integrated strengths. Then, a straightforward repetition of the iteration procedure described above reveals that one may obtain \( \alpha_{\text{BSE}} = \alpha_{\text{SDE}} = 0.32 \) by choosing \( \lambda_{\text{BSE}} = 1.37 \) and \( \lambda_{\text{SDE}} = 1.96 \); the corresponding \( f \) are shown in Fig. 6.
FIG. 6. The three-gluon vertex form factor $f(q^2)$ at $\lambda = 1.16$ which leads to the equality $\alpha_s^{\text{SDE}} = \alpha_s^{\text{BSE}} = 0.45$ (red continuous curve). The dashed (light blue) lines show the different form factors needed in the BSE (dashed) and SDE (dot-dashed) to force the equality at $\alpha_s^{\text{SDE}} = \alpha_s^{\text{BSE}} = 0.32$.

VI. CONCLUSIONS

We have carried out an extensive analysis of the interlocked dynamics between the SDE of the gluon propagator $\Delta(q^2)$ and a BSE that generates massless bound state poles. These poles constitute an indispensable ingredient of the particular realization of the Schwinger mechanism employed in a series of works in order to obtain infrared finite (massive) solutions for $\Delta(q^2)$. The notion of coupling the two equations is novel, and its possible ramifications for the overall self-consistency of the entire formalism have not been explored before in the relevant literature.

Our three main results may be summarized as follows. First, we have obtained a running gluon mass, displaying all expected physical features, directly from the solution of the BSE. This possibility was envisaged in earlier works [51], but the two conditions discussed after Eq. (4.5), which are crucial for obtaining a positive-definite and monotonically decreasing gluon mass, were not fully appreciated. Second, we have carried out a nonperturbative computation of the quantity $Y$, whose role is crucial for obtaining from the SDE a positive-definite gluon mass. Third, we have demonstrated that the inclusion of the three-gluon vertex is of paramount importance for the fulfillment of a basic self-consistency requirement. In particular, the nontrivial infrared dynamics of this vertex compensate the original discrepancy in the value of $\alpha_s$ used in the SDE and the BSE sectors, allowing finally for a
single common value, $\alpha_s = 0.45$.

The deviation from the $\alpha_s = 0.32$ estimated from the lattice simulations of [60, 61] may be attributed to a variety of reasons.

To begin with, the skeleton expansion of the BSE kernel has been truncated at the lowest order, shown in Fig. 3. It would be very important to verify the impact of the next order corrections (“one-loop” fully dressed). In fact, even the impact of graph $(b_1)$, whose vanishing seems to be an accident of setting the four-gluon vertex at tree level, ought to be reconsidered, using a more complete structure for this vertex [80, 81].

In addition, the transition from Eq. (5.1) to Eq. (5.2) was implemented by setting into the former $Z_3 = Z_4 = 1$. A more complete treatment of this issue has been given in [79]; the resulting kernel, however, is substantially more difficult to calculate, and only Ansätze have been studied thus far. Unfortunately, the complicated nature of this problem makes progress in this direction rather slow.

Turning to $f$, it is clear that the form of Eq. (3.3) is rather restrictive, given that the full tensorial basis for expanding $\Gamma$ consists of 14 elements. In addition, $f$ has been considered to be a function of a single variable (symmetric configuration : $q^2 = r^2 = p^2$). Clearly, a more complete integration over all available momenta and angles could shift the coincidence value of $\alpha_s$ closer to $\alpha_s = 0.32$, as exemplified in the last part of Section 5 by employing $\lambda_{BSE} \neq \lambda_{DSE}$.

Last but not least, the assumption that only the vertex $\Gamma$ develops a massless pole may have to be revisited, allowing the remaining vertices, and especially the ghost-gluon vertex, to form part of a more complex BSE system.

We hope to return to some of the issues mentioned above in the near future.

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