Schrödinger flows on Grassmannians

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Abstract

The geometric non-linear Schrödinger equation (GNLS) on the complex Grassmannian manifold $M$ is the evolution equation on the space $C(R, M)$ of paths on $M$:

$$J_t \gamma_t = \nabla_x \gamma_t,$$

where $\nabla$ is the Levi-Civita connection of the Kähler metric and $J$ is the complex structure. GNLS is the Hamiltonian equation for the energy functional on $C(R, M)$ with respect to the symplectic form induced from the Kähler form on $M$. It has a Lax pair that is gauge equivalent to the Lax pair of the matrix non-linear Schrödinger equation (MNLS). We construct via gauge transformations an isomorphism from $C(R, M)$ to the phase space of the MNLS equation so that the GNLS flow corresponds to the MNLS flow. The existence of global solutions to the Cauchy problem for GNLS and the hierarchy of commuting flows follows from the correspondence. Direct geometric constructions show the flows are given by geometric partial differential equations, and the space of conservation laws has a structure of a non-abelian Poisson group. We also construct a hierarchy of symplectic structures for GNLS. Under pullback, the known order $k$ symplectic structures correspond to the order $k - 2$ symplectic structures that we find. The shift by two is a surprise, and is due to the fact that the group structures depend on gauge choice.

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1. Introduction

Harmonic maps from one Riemannian manifold \( N \) to another \( M \) are critical points for the energy
\[
E(\gamma) = \frac{1}{2} \int_N |d\gamma|^2 d\mu_N.
\]
The Euler-Lagrange equation is
\[
\triangle_\gamma \gamma = \text{tr}(\nabla(d\gamma)) = 0.
\]
When \( N \) is \( S^2 \) or a 2-dimensional torus and \( M \) is a symmetric space, the moduli space of harmonic maps has been successfully studied by many authors using techniques from integrable systems ([U], [BFPP], [BG]).

The heat flow for harmonic maps from \( N \) to \( M \) is the gradient flow for the energy and has the form
\[
\gamma_t = \triangle_\gamma \gamma.
\]
If \( M \) is a Kähler manifold, we have a complex structure \( J : TM \to TM, J^2 = -\text{id} \), and the equation
\[
J_\gamma \gamma_t = \nabla_\gamma \gamma
\]
has the type of a non-linear Schrödinger equation. Very little is known about this equation except when \( \dim(N) = 1 \). See recent work by Chang, Shatah and Uhlenbeck for a discussion of the radially symmetric case in dimension 2 ([CSU]). For \( N = R^n, n = 1, 2, 3 \) and \( M = S^2 \), this equation is a simplification of the Landau-Lifshitz equation for a continuous anisotropic magnet. For \( N = R^1 \) and \( M = S^2 \), the GNLS is often referred to as the continuous isotropic Heisenberg ferromagnetic model ([FT]).

Let \( (M, g, J) \) be a Kähler manifold with metric \( g \) and complex structure \( J \). The one-dimensional geometric non-linear Schrödinger equation (GNLS) with target \( M \) is the evolution equation on the space \( C(R, M) \) of smooth paths from \( R \) to \( M \):
\[
J_\gamma \gamma_t = \nabla_\gamma \gamma_x,
\]
where \( \nabla \) is the Levi-Civita connection of \( g \). The symplectic form
\[
\tau_\gamma(v_1, v_2) = g(J_\gamma(v_1), v_2)
\]
on \( M \) induces a natural symplectic structure \( \hat{\tau} \) on \( C(R, M) \)
\[
\hat{\tau}_\gamma(\xi, \eta) = \int_{-\infty}^{\infty} g(J_\gamma(x)(\xi(x)), \eta(x)) dx.
\]
Equation (1.1) is the Hamiltonian equation of the energy functional with respect to \( \hat{\tau} \). The main goal of this paper is to study the Hamiltonian theory of the geometric non-linear Schrödinger equation with target the Grassmannians. The
theory can be extended to the more general case of compact Hermitian symmetric spaces.

The matrix non-linear Schrödinger equation (MNLS) is

\[ q_t = i(q_{xx} + 2q q^*) \],

(1.3)

where \( q \) is a map from \( R^2 \) to the space \( \mathcal{M}_{k \times (n-k)} \) of \( k \times (n-k) \) complex matrices. This equation was first studied by Fordy and Kulish in [FK] as a generalization of the NLS equation. Note that if \( q \) is a 1 \( \times \) 1 matrix, this is the NLS equation. When \( k = 1 \), \( q \) is a vector and this is known as the vector non-linear Schrödinger equation.

The GNLS with target a Grassmanian is known to be gauge equivalent to the matrix non-linear Schrödinger equation. This correspondence for \( S^2 = CP^1 \) is contained in the classical Hasimoto transform and is described in detail by Faddeev-Takhtajan ([FT]). We describe this gauge equivalence in general, and examine the behavior of the hierarchies of symplectic structures under the gauge equivalence. We say that a symplectic structure is of order \( k \) if the corresponding Poisson structure is given by an order \( k \) integro-differential operator. Each equation has a natural order zero symplectic structure that arises from a coadjoint orbit. Under the equivalence, the two order zero symplectic structures do not correspond to each other. In fact, we may regard the gauge change as naturally generating a hierarchy of symplectic structures, although we would only generate the even order structures in this fashion.

Hasimoto ([H]) showed that NLS is equivalent to the equation of da Rios ([dR]):

\[ \alpha_t = \alpha_x \times \alpha_{xx}, \]

which models the movement of a thin vortex filament in a viscous liquid. This equation preserves arc length and \( \gamma = \alpha_x \) satisfies the GNLS with target \( S^2 \). Langer and Perline ([LP]) generalize this to MNLS. They give a geometric realization of the MNLS as an arc length preserving curve evolution in \( R^{k(n-k)} \).

The MNLS is a second flow of the \( u(n) \)-hierarchy, and the Hamiltonian theory has been described by many authors. Let \( a \) denote the diagonal matrix with eigenvalues \( i/2 \) and \( -i/2 \) and multiplicities \( k \) and \( (n-k) \) respectively:

\[ a = \begin{pmatrix} i/2 I_k & 0 \\ 0 & -i/2 I_{n-k} \end{pmatrix} \].

Let \( u(n)_a \) denote the centralizer of \( a \), and \( u(n)^\perp_a \) the orthogonal complement of \( u(n)_a \) in \( u(n) \). We find

\[ u(n)_a = \{ y \in u(n) \mid [y, a] = 0 \} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \mid A_1 \in u(k), A_2 \in u(n-k) \right\}, \]

\[ u(n)^\perp_a = \{ y \in u(n) \mid < y, z > = 0 \text{ for all } z \in u(n)_a \} = \left\{ \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \mid q \in \mathcal{M}_{k \times (n-k)} \right\} \cong \mathcal{M}_{k \times (n-k)}. \]
Let $\mathcal{S}(\mathbb{R}, u(n)_{a}^{\perp})$ denote the space of smooth maps from $\mathbb{R}$ to $u(n)_{a}^{\perp}$ that are in the Schwartz class. Since $\text{ad}(a)$ is a skew-adjoint, linear isomorphism on $u(n)_{a}^{\perp}$, the 2-form defined by

$$w(v_1, v_2) = \int_{-\infty}^{\infty} - \text{ad}(a)^{-1}(v_1), v_2 > dx$$  \tag{1.4}$$

is symplectic on $\mathcal{S}(\mathbb{R}, u(n)_{a}^{\perp})$. The second flow in the AKNS $u(n)$-hierarchy is

$$u_t = (Q_2)_x + [u, Q_2],$$  \tag{1.5}$$

where

$$u = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} -iq^*q_x & iq_x \\ iq^*_x & iq^*q \end{pmatrix}.$$  \tag{1.6}$$

The second flow (1.5) has a Lax pair

$$\left[ \frac{\partial}{\partial x} + a\lambda + u, \frac{\partial}{\partial t} + a\lambda^2 + u\lambda + Q_2 \right] = 0,$$  \tag{1.7}$$

i.e., $u$ is a solution of (1.5) if and only if $u$ satisfies (1.7) for all $\lambda$. This is the Hamiltonian equation of the functional

$$F_2(u) = \int_{-\infty}^{\infty} -\frac{1}{4} \text{tr}(u^2_x) + \frac{1}{8} (\text{tr}(u^2))^2 dx$$  \tag{1.8}$$

on $\mathcal{S}(\mathbb{R}, u(n)_{a}^{\perp})$ with respect to $w$. When written in terms of $q$, equation (1.5) is the MNLS (1.3),

$$F_2(q) = \int_{-\infty}^{\infty} -\frac{1}{2} \|q_x\|^2 + \frac{1}{2} \|q\|^4 dx,$$

$$w(q_1, q_2) = \int_{-\infty}^{\infty} < -iq_1, q_2 > dx.$$

So we shall also refer equation (1.5) as the MNLS.

We list some of the known properties of the MNLS equation (1.5):

1. Beals and Coifman construct the inverse scattering theory for the first order system $d_x + a\lambda + u$ to solve the Cauchy problem globally with Schwartz class initial data ([BC]).

2. We constructed in [TU2] an action of the rational loop group on the space of solutions of the MNLS equation such that the action of a linear fractional transformations gives a Bäcklund or Darboux transformation. Moreover, the orbit of the rational loop group at the vacuum can be computed by explicit formulas.
(3) Fordy and Kulish showed in 1983 ([FK]) that the MNLS equation has a sequence of commuting Hamiltonians \( \{ F_j \} \) with respect to \( w \) such that the Hamiltonian equation for \( F_2 \) is the MNLS equation.

(4) There is a sequence of symplectic structures \( \{ w_k \} \) on \( \mathcal{S}(R, \mathcal{M}_{k \times (n-k)}) \):

\[
(w_k)_u(v_1, v_2) = \int_{-\infty}^{\infty} \text{tr}((W_k)_u^{-1}(v_1)v_2)dx,
\]

where \( W_0 = - \text{ad}(a) \) and \( (W_k)_u \) is an integro-differential operator of order \( k \) (cf. [Te]). Moreover, there is a Lenard-Magri relation:

\[
u_t = [\nabla F_j, a] = W_0(\nabla F_j) = W_1(\nabla F_{j-1}) = \cdots = (W_k)_u(\nabla F_{j-k}).
\]

Or equivalently, the Hamiltonian equation for \( F_j \) with respect to \( w_0 = w \) is the Hamiltonian equation for \( F_{j-k} \) with respect to \( w_k \).

(5) Let \( H_+ \) denote the group generated by holomorphic maps \( e_{b,j}(\lambda) = e^{b\lambda^j} \) with \( b \in u(n)_a \) and \( j \) a positive integer. We proved in [TU1] that there is an action of \( H_+ \) on \( (\mathcal{S}(R, u(n)^+_a), w) \) such that:

(i) the \( H_+ \)-action is Poisson.

(ii) Let \( \xi_{b,j} \) denote the vector field generated by the action of the one-parameter subgroup \( e_{b,t,j} \) of \( H_+ \). Then the flow generated by \( \xi_{a,2} \) is the MNLS flow, and \( [\xi_{b_1,j_1}, \xi_{b_2,j_2}] = \xi_{[b_1,b_2],j_1+j_2} \). In particular, \( \xi_{b,j} \) commutes with \( \xi_{a,2} \).

(iii) The flows generated by the one parameter subgroups \( e_{at,j} \) are the commuting Hamiltonian flows found by Fordy and Kulish.

The sequence of symplectic structures \( w_k \) on \( \mathcal{S}(R, u(n)^+_a) \) can be constructed using coadjoint orbits of certain loop algebra of an affine algebra. We will use a similar method to construct a sequence of symplectic structures \( \tau_k \) on \( C(R, \text{Gr}(k,n)) \) such that \( \tau_0 = \hat{\tau} \) (defined by (1.2)), \( \tau_k \) is of order \( k \) and the GNLS is Hamiltonian with respect to \( \tau_k \) for all \( k \). Here \( \text{Gr}(k,n) \) is isometrically embedded as the Adjoint orbit in \( u(n) \) at \( a \).

The gauge equivalence of Lax pairs of GNLS and MNLS equation gives rise to an isomorphism \( \Phi \) from \( C(R, \text{Gr}(k,n)) \) to \( \mathcal{S}(R, u(n)^+_a) \) such that the GNLS flow corresponds to the MNLS flow. If the isomorphism \( \Phi \) were a symplectic morphism from \( (C(R, \text{Gr}(k,n)), \hat{\tau}) \) to \( (\mathcal{S}(R, \mathcal{M}_{k \times (n-k)}), w) \), then we could translate the known Hamiltonian theory for MNLS to GNLS. But \( \Phi \) is not symplectic. However, we show that \( \Phi^*(w) \) is an order \( -2 \) symplectic form on \( C(R, \text{Gr}(k,n)) \). In general the pull back of the order \( k \) symplectic form \( w_k \) is the order \( k-2 \) symplectic form \( \tau_{k-2} \) under \( \Phi \). Therefore the above properties (1)–(5) hold for the GNLS with some minor changes.

Every compact Hermitian symmetric space \( G/K \) has a standard isometric embedding as an adjoint orbit in \( G \). So the above results should hold for any compact Hermitian symmetric space. The non-compact examples may give different interesting phenomena.
This paper is organized as follows. In section 2, we use a standard embedding of $Gr(k, n)$ in $u(n)$ to write down the GNLS and its Lax pair, and show that its Lax pair is gauge equivalent to that of the MNLS equation. In section 3, we construct the isomorphism $\Phi$ from the phase space of the GNLS to the phase space of MNLS and compute the differential of $\Phi$. In section 4, we show that the commuting flows of the MNLS equation give rise to a sequence of commuting, geometric, Hamiltonian flows on the path space of $Gr(k, n)$ with respect to $\hat{\tau}$. In section 5, we use the loop algebra of certain affine algebra to construct a sequence of symplectic structures $\hat{\tau}_k$ (of order $k$) on the phase space of the GNLS and show that they are the pull back of the known order $k + 2$ symplectic form on the phase space of the MNLS equation under $\Phi$. We also give a Lenard-Magri relation for the GNLS. We use the same method to construct the two standard symplectic forms for the KdV equation in section 6.

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2. Schrödinger flow on $Gr(k, n)$

We use a standard embedding of $Gr(k, n)$ to write down the geometric non-linear Schrödinger equation (GNLS) and its Lax pair. We show that the Lax pairs of GNLS and MNLS are gauge equivalent.

2.1 Proposition. Let $\langle x, y \rangle = -\text{tr}(xy)$ denote the invariant inner product on $u(n)$, and $M$ the Adjoint $U(n)$-orbit at $a = \begin{pmatrix} \frac{i}{2}I_k & 0 \\ 0 & -\frac{i}{2}I_{n-k} \end{pmatrix}$ in $u(n)$. Then $M$ equipped with the induced metric is the Hermitian symmetric space $Gr(k, C^n)$, and the GNLS with target $Gr(k, n)$ is

$$\gamma_t = [\gamma, \gamma_{xx}], \quad \gamma(x, t) \in M. \quad (2.1)$$

PROOF. Since the centralizer $U(n)_a = U(k) \times U(n-k)$, $M$ is diffeomorphic to $U(n)/(U(k) \times U(n-k)) \simeq Gr(k, C^n)$. Note that both the induced metric and the standard Kähler metric on $M$ are invariant under $U(n)$. A direct computation shows that they agree at $a$. Hence $M$ is the Hermitian symmetric space $Gr(k, n)$. The complex structure of $M$ at $y$ is given by $\text{ad}(y)$. Hence the symplectic form $\hat{\tau}$ on $C(R, M)$ is

$$\hat{\tau}_\gamma(\delta_1 \gamma, \delta_2 \gamma) = \int_{-\infty}^{\infty} \langle \text{ad}(\gamma)(\delta_1 \gamma), \delta_2 \gamma \rangle \, dx.$$
The gradient of the energy functional $E$ is $-\nabla \gamma_x \gamma_x$, where $\nabla$ is the covariant derivative. So the Hamiltonian equation for $E$ (the GNLS on $Gr(k, n)$) is

$$\gamma_t = [-\nabla \gamma_x \gamma_x, \gamma] = [\gamma, \nabla \gamma_x \gamma_x].$$

Let $TM$ and $\nu(M)$ denote the tangent and normal bundles of $M$ in $u(n)$, and $\pi_t$, $\pi_n$ the orthogonal projection onto $TM$ and $\nu(M)$ respectively. Since the metric on $M$ is the induced metric, $\nabla \gamma_x \gamma_x = \pi_t(\gamma_{xx}) = r_{xx} - \pi_n(\gamma_{xx})$.

But if $\gamma = gag^{-1}$, then $\nu(M)\gamma = gu(n)a^{-1}$. So $[\gamma, v] = 0$ for all $v \in \nu(M)\gamma$. This implies that $[\gamma, \pi_n(\gamma_{xx})] = 0$. Hence the GNLS is equation (2.1).

Next, we give a Lax pair for equation (2.1).

2.2 Proposition. Let $M$ be the adjoint orbit at $a$ as in Proposition 2.1. Then $\gamma$ satisfies equation (2.1) if and only if

$$\left[ \frac{\partial}{\partial x} + \gamma \lambda, \frac{\partial}{\partial t} + \gamma \lambda^2 + [\gamma, \gamma_x] \lambda \right] = 0 \text{ for all } \lambda \in C. \quad (2.2)$$

PROOF. Equation (2.2) is equivalent to

$$(\gamma \lambda)_t - (\gamma \lambda^2 + [\gamma, \gamma_x] \lambda)_x = [\gamma \lambda, \gamma \lambda^2 + [\gamma, \gamma_x] \lambda]. \quad (2.3)$$

Compare coefficient of $\lambda^j$ in equation (2.3) to give

$$\gamma_t - [\gamma, \gamma_x]_x = 0,$$

$$\gamma_x + \text{ad}(\gamma)^2(\gamma_x) = 0.$$

The first equation gives (2.1). The second equation is always true because $\text{ad}(\gamma)^2 = -\text{id}$ on $TM_\gamma$ for $\gamma \in M$.  

The MNLS (1.5) has a Lax pair

$$\left[ \frac{\partial}{\partial x} + a \lambda + u, \frac{\partial}{\partial t} + a \lambda^2 + u \lambda + Q_2 \right] = 0, \quad (2.4)$$

where $u, Q_2$ are given by (1.6).

2.3 Proposition. The Lax pairs (2.2) and (2.4) are gauge equivalent. In particular, if $u$ is a solution of the MNLS (1.5) then $\gamma = Ea^{-1}$ is a solution of the GNLS, where $E$ is a solution of

$$\begin{cases} E^{-1}E_x = u, \\ E^{-1}E_t = Q_2. \end{cases}$$
PROOF. Let \( u \) be a solution of the MNLS equation. Substitute \( \lambda = 0 \) in the corresponding Lax pair (2.4) to get 
\[
\left[ \frac{\partial}{\partial x} + u, \frac{\partial}{\partial t} + Q_2 \right] = 0.
\]
So there exists \( E \) satisfies
\[
E^{-1}E_x = u, \quad E^{-1}E_t = Q_2, \quad E(0,0) = I.
\]
Set \( \gamma = EaE^{-1} \). Apply gauge transformation of \( E \) to the Lax pair (2.4) to get
\[
E \left( \frac{\partial}{\partial x} + a\lambda + u \right) E^{-1} = \frac{\partial}{\partial x} + \gamma \lambda + EuE^{-1} - E_xE^{-1} = \frac{\partial}{\partial x} + \gamma \lambda,
\]
\[
E \left( \frac{\partial}{\partial t} + a\lambda^2 + u\lambda + Q_2 \right) E^{-1} = \frac{\partial}{\partial t} + \gamma \lambda^2 + EuE^{-1}\lambda + EQ_2E^{-1} - E_tE^{-1},
\]
\[
= \frac{\partial}{\partial t} + \gamma \lambda^2 + EuE^{-1}\lambda.
\]
Since \( \gamma_x = E[u,a]E^{-1} \), \( [\gamma, \gamma_x] = E[a, [u,a]]E^{-1} \). But \( \text{ad}(a)^2 \) is \(-\text{id}\) on \( u(n)^\perp \). Hence \( [\gamma, \gamma_x] = EuE^{-1} \). This proves that the gauge transformation of the Lax pair (2.4) by \( E \) is the Lax pair (2.2). 

3. Development map of an Adjoint orbit

Let \( a \in u(n) \) be a fixed diagonal matrix, and \( M \) the adjoint \( U(n) \)-orbit at \( a \) in \( u(n) \). So \( M \) is a flag manifold if \( a \) has distinct eigenvalues, and is a partial flag manifold if \( a \) has multiple roots. Let \( C_a(R,M) \) denote the space of smooth paths \( \gamma : R \to M \) such that \( \gamma(-\infty) = \lim_{x \to -\infty} \gamma(x) = a \) and \( \gamma_x : R \to u(n) \) is in the Schwartz class. We are motivated by the gauge equivalence given in Proposition 2.3 to construct an isomorphism \( \Phi \) from \( C_a(R,M) \) onto the linear space \( S(R, u(n)^\perp) \). We show that the GNLS and MNLS flows correspond under \( \Phi \). We also will use this isomorphism to study the relation between the hierarchies of symplectic structures for the GNLS and MNLS flows in section 5.

3.1 Theorem. Let \( M \) be the Adjoint orbit at \( a \in u(n) \), and
\[
\Psi : S(R, u(n)^\perp) \to C_a(R,M)
\]
the map defined by \( \Psi(u) = gag^{-1} \), where \( g \) is the solution of \( g^{-1}g_x = u \) such that \( g(-\infty) = I \). Then \( \Psi \) is an isomorphism.

PROOF. First we prove that \( \Psi \) is one to one. Assume \( \Psi(f) = \Psi(g) \). Then we have
\[
faf^{-1} = gag^{-1}, \quad f^{-1}f_x \in u(n)^\perp, \quad g^{-1}g_x \in u(n)^\perp
\]
\[
f(-\infty) = g(-\infty) = I.
\]
Set \( h = f^{-1}g \). Since \( hah^{-1} = a, h(x) \in U(n)_a \) for all \( x \). Note
\[
g^{-1}g_x = (fh)^{-1}(fh)_x = h^{-1}f^{-1}f_xh + h^{-1}h_x.
\]
But $h^{-1}u(n)_{a}^\perp h \subseteq u(n)_{a}^\perp$ and $g^{-1}g_x \in u(n)_{a}^\perp$ imply that $h^{-1}h_x$ lies in $u(n)_{a}^\perp$.

But $h^{-1}h_x$ also lies in $u(n)_a$. So $h^{-1}h_x = 0$, which implies that $h$ is constant.

But $h(-\infty) = I$. So $h = I$ and $f = g$. This proves $\Psi$ is one to one.

To prove $\Psi$ is onto, given $\gamma \in C_a(R, M)$, choose a smooth map $f : R \to U(n)$ such that $\gamma = faf^{-1}$ and $f(-\infty) = I$. Set $v = f^{-1}f_x$. Let $v_0$ and $v_1$ denote the projections of $v$ onto $u(n)_a$ and $u(n)_a^\perp$ respectively. Then

$$\gamma_x = f[f^{-1}f_x, a]f^{-1} = f[v, a]f^{-1} = f[v_1, a]f^{-1}.$$  

Since $\gamma_x$ is in the Schwartz class and the image of $f$ lies in the compact set $U(n)$, $v_1$ is in the Schwartz class. Now solve $h : R \to U(n)_a$ such that

$$h_x = -v_0h, \quad h(-\infty) = I.$$  

Set $g = fh$. Then $\gamma = faf^{-1} = fhah^{-1}f^{-1} = gag^{-1}$. But

$$g^{-1}g_x = h^{-1}f^{-1}f_xh + h^{-1}h_x = h^{-1}vh + h^{-1}h_x$$
$$= h^{-1}vh - h^{-1}v_0h = h^{-1}(v - v_0)h$$
$$= h^{-1}v_1h \quad \in h^{-1}u(n)_a^\perp h = u(n)_a^\perp.$$  

So $\Psi(g) = \gamma$ and $\Psi$ is onto.

It is not difficult to see that in the appropriate topologies, the map $\Psi$ is smooth. We explicitly compute the differential in Proposition 3.7.

\section*{3.2 Definition.}

The development map $\Phi$ on the Adjoint orbit $M$ of $u(n)$ at $a$ is defined to be the inverse of $\Psi$, i.e., $\Phi = \Psi^{-1} : C_a(R, M) \to S(R, u(n)_a^\perp)$.

As a consequence of the proof of Theorem 3.1, we have a description of $\Phi$:

\section*{3.3 Corollary.}

If $\gamma \in C_a(R, M)$, then there is a unique $g : R \to U(n)$ such that $\gamma = gag^{-1}$, $g(-\infty) = I$, and $g^{-1}g_x \in u(n)_a^\perp$. Moreover, $\Phi(\gamma) = g^{-1}g_x$.

We need the following Poisson operator defined in $[Te]$ to write down the differential of $\Phi$. A Poisson operator is a map on the tangent bundle of a manifold, which can be used to define a Poisson or symplectic structure.

\section*{3.4 Definition.}

Given $u \in S(R, u(n)_a^\perp)$, we introduce the integro-differential operator $P_u : S(R, u(n)_a^\perp) \to C^\infty (R, u(n)_a^\perp)$ defined by

$$P_u(v) = v_x + \pi_a^\perp([u, v]) - [u, Tu(v)],$$

where $Tu(v)(x) = \int_{-\infty}^{x} \pi_a([u(y), v(y)])dy$ and $\pi_a$, $\pi_a^\perp$ are the orthogonal projections of $u(n)$ onto $u(n)_a$ and $u(n)_a^\perp$ respectively.

Note that

$$P_u(v) = (v - Tu(v))_x + [u, v - Tu(v)]$$

and $P_u(v)$ lies in $u(n)_a^\perp$. So we get a characterization of $P_u$:
3.5 Proposition ([Te]). Let \( u, v \in S(R, u(n)_{\mathfrak{a}}^{\perp}) \). Then there exists a unique \( \tilde{v} : R \to u(n) \) satisfies the following conditions:

(i) \( \pi_{\mathfrak{a}}(\tilde{v}) = v \), where \( \pi_{\mathfrak{a}} \) is the projection onto \( u(n)_{\mathfrak{a}}^{\perp} \),
(ii) \( (\tilde{v})_x + [u, \tilde{v}] \in u(n)_{\mathfrak{a}}^{\perp} \),
(iii) \( \tilde{v}(-\infty) = 0 \).

Moreover, \( P_u(v) = [d_x + u, \tilde{v}] \).

3.6 Corollary ([Te]). The operator \( P_u \) is injective.

PROOF. Suppose \( P_u(v) = 0 \). Let \( \tilde{v} = v - T_u(v) \), where \( T_u \) is the operator given in Definition 3.4. So \( P_u(v) = [d_x + u, \tilde{v}] = 0 \), which implies that \( \tilde{v}(x) \) is conjugate to \( \tilde{v}(0) \) for all \( x \in R \). Since the conjugate class of \( v(0) \) in \( u(n) \) is compact and \( \tilde{v}(-\infty) = 0 \), \( \tilde{v} = 0 \).

Recall that \( \text{ad}(a) \) is a linear isomorphism on \( u(n)_{\mathfrak{a}}^{\perp} \).

3.7 Proposition. Let \( \gamma \in C_a(R, M) \), and \( g : R \to U(n) \) such that \( \gamma = gag^{-1}, g^{-1}g_x = \Phi(\gamma) = u \) and \( g(-\infty) = I \). If \( v \in S(R, u(n)_{\mathfrak{a}}^{\perp}) \), then
\[
d\Phi_{\gamma}(gvg^{-1}) = -P_u(ad(a)^{-1})(v).
\]

PROOF. Take variations of \( \gamma = gag^{-1} \) and \( g^{-1}g_x = u \) to get
\[
\delta\gamma = g[g^{-1}\delta g, a]g^{-1}, \quad \delta u = [d_x + u, g^{-1}\delta g].
\]
Set \( v = [g^{-1}\delta g, a] \). It follows from Proposition 3.5 and the formula for \( \delta u \) that
\[
d\Phi_{\gamma}(\delta\gamma) = \delta u = -P_u(ad(a)^{-1})(v).
\]

The following operators are needed later for the constructions of the symplectic structures on \( C_a(R, M) \).

3.8 Definition. Given \( \gamma \in C_a(R, M) \), let \( L_{\gamma} : TC_a(R, M)_{\gamma} \to TC_a(R, M)_{\gamma} \) denote the operator defined as follows: Write \( \gamma = gag^{-1} \) such that \( g^{-1}g_x = \Phi(\gamma) = u \) and \( g(-\infty) = I \). Then
\[
L_{\gamma}(gvg^{-1}) = gP_u(v)g^{-1}.
\]

We give a geometric description of this operator:

3.9 Proposition. Let \( M \) be the Adjoint orbit at \( a \) in \( u(n) \), and \( \eta \) a unique vector field along \( \gamma \in C_a(R, M) \) that is tangent to \( M \). Then there exists a unique vector field \( \zeta \) along \( \gamma \) normal to \( M \) such that \( (\eta + \zeta)_x \) is tangent to \( M \) and \( \zeta(-\infty) = 0 \). Moreover, \( L_{\gamma}(\eta) = (\eta + \zeta)_x \).
3.10 Theorem. Let \( a = \frac{i}{2} \left( \begin{array}{cc} I_k & 0 \\ 0 & -I_{n-k} \end{array} \right) \), and \( M \) the adjoint orbit at \( a \). Then the GNLS flow on \( C_a(R, M) \) corresponds to the MNLS flow on \( S(R, u(n)_a^\perp) \) under the development map \( \Phi \).

PROOF. Suppose \( \gamma(x, t) \) is a solution of the GNLS. Write \( \gamma = gag^{-1} \) such that \( g^{-1}g_x \in u(n)_a^\perp \) and \( \lim_{x \to -\infty} g(x, t) = I \) for all \( t \). Set \( u(x, t) = g(x, t)ag(x, t)^{-1} \), i.e.,

\[
\gamma(t, \cdot) = \Phi(\gamma(t, \cdot)).
\]

Then

\[
u_t = d\Phi_\gamma(\gamma_t).
\] (3.1)

A direct computation gives

\[
\gamma_x = g[g^{-1}g_x, a]g^{-1} = g[u, a]g^{-1},
\]

\[
\gamma_{xx} = g([u_x, a] + [u, [u, a]])g^{-1}.
\]

Recall that

\[
\mathcal{K} = u(n)_a = \{ y \in u(n) \mid [y, a] = 0 \}
\]

\[
= \left\{ \left( \begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right) \mid A_1 \in u(k), A_2 \in u(n-k) \right\},
\]

\[
\mathcal{P} = u(n)_a^\perp = \{ y \in u(n) \mid < y, z > = 0 \text{ for all } z \in u(n)_a \}
\]

\[
= \left\{ \left( \begin{array}{cc} 0 & q \\ -q^* & 0 \end{array} \right) \mid q \in \mathcal{M}_{k \times (n-k)} \right\}.
\]

It follows from a simple computation that

\[
[\mathcal{K}, \mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{P}, \mathcal{P}] \subset \mathcal{K}.
\]

(This is true for a Cartan decomposition of any symmetric space). Since \([a, \mathcal{K}] = 0 \) and \([u, [u, a]] \in [\mathcal{P}, [\mathcal{P}, \mathcal{K}]] \subset \mathcal{K} \), we have

\[
[g[a, [u_x, a] + [u, [u, a]]]g^{-1} = g[a, [u_x, a]]g^{-1}.
\]

But \( \text{ad}(a)^2 = -\text{id} \) on \( \mathcal{P} \). So we get \( [\gamma, \gamma_{xx}] = gu_xg^{-1} \). Substitute this into (3.1) to get

\[
u_t = d\Phi_\gamma(\gamma_t) = d\Phi_\gamma([\gamma, \gamma_{xx}]) = d\Phi_\gamma(gu_xg^{-1}).
\]

By Proposition 3.7, we conclude

\[
u_t = -P_u \text{ad}(a)^{-1}(u_x).
\]

A direct computation shows that the right hand side is equal to \( (Q_2)_x + [u, Q_2] \), i.e., \( u \) is a solution of the MNLS (1.5). \( \blacksquare \)
4. Commuting Hamiltonians for GNLS

We recall the definition of the orbit symplectic structure on an adjoint orbit of a compact Lie algebra \(U\), which is identified with \(U^*\) via the Killing form (trace). Given \(x \in U\), let \(\tilde{x}\) denote the infinitesimal vector field generated by \(x\) under the coadjoint action. Then the Kostant-Kirillov orbit symplectic form \(\tau\) on a coadjoint orbit \(O\) of \(U^*\) is given by

\[
\tau_\ell(\tilde{x}(\ell), \tilde{y}(\ell)) = \ell([x,y]), \quad \ell \in O, x, y \in U.
\]

Identify \(x \in U\) with \(\ell_x \in U^*\) defined by \(\ell_x(y) = \text{tr}(xy)\). Then adjoint orbits are identified as coadjoint orbits. Let \(a \in u(n)\) be a diagonal matrix, and \(M_a\) the adjoint orbit at \(a\). Then the orbit symplectic form on \(M_a\) is

\[
\tau_x(v_1, v_2) = \langle -\text{ad}(x)^{-1}(v_1), v_2 \rangle.
\]

It induces a symplectic form \(\hat{\tau}\) on the space \(C_a(R, M_a)\) of maps to \(M_a\):

\[
\hat{\tau}_\gamma(v_1, v_2) = \int_{-\infty}^{\infty} < -\text{ad}(\gamma)^{-1}(v_1), v_2 > dx.
\]

Recall that \(\Phi\) is the development map defined in section 3 and \(w\) is the symplectic form on \(S(R, u(n)_a^\perp)\) defined by

\[
w(v_1, v_2) = \int_{-\infty}^{\infty} < -\text{ad}(a)^{-1}(v_1), v_2 > dx. \quad (4.1)
\]

Then a direct computation we do later shows that \(\Phi^*(w)\) is not equal to \(\tau\). We now have two symplectic structures on \(C_a(R, M_a)\). The flows on \(C_a(R, M_a)\) pulled back from flows in the \(u(n)\)-hierarchy are of course Hamiltonian with respect to \(\Phi^*(w)\). But we will show that these flows are also Hamiltonian with respect to \(\hat{\tau}\).

Next we recall the construction of the hierarchy for the MNLS. Let \(Q_j(u)\) be the sequence of operators in \(u : R \to u(n)_a^\perp\) determined by the following recursive formula

\[
(Q_j(u))_x + [u, Q_j(u)] = [Q_{j+1}(u), a],
\]

\[
Q_0(u) = a, \quad Q_j(u)(-\infty) = 0. \quad (4.2)
\]

Sattinger proved in [Sa] (cf. also [TU1]) that \(Q_j(u)\) is an order \(j - 1\) polynomial differential operator in \(u\). The \(j\)-th flow in the \(u(n)\)-hierarchy on \(S(R, u(n)_a^\perp)\) is

\[
u_t = (Q_j(u))_x + [u, Q_j(u)] = [Q_{j+1}(u), a], \quad (4.3)
\]
which is the Hamiltonian flow for

\[ F_j(u) = -\frac{1}{j+1} \int_{-\infty}^{\infty} \text{tr}(Q_{j+2}(u)a)dx \]  

(4.4)

with respect to \( w \). Or in other words,

\[ \nabla F_j(u) = \pi_a^\perp(Q_{j+1}(u)), \]  

(4.5)

where \( \pi_a, \pi_a^\perp \) denote the orthogonal projections of \( u(n) \) onto \( u(n)_a \) and \( u(n)_{a}^\perp \) respectively. It follows from Proposition 3.5 and formula (4.2) that

\[ P_u(\pi_a^\perp(Q_j)) = [Q_{j+1}, a]. \]  

(4.6)

Or equivalently,

\[ \pi_a^\perp(Q_{j+1}) = -\text{ad}(a)^{-1}P_u(\pi_a^\perp(Q_j)). \]  

(4.7)

Set

\[ H_j(\gamma) = F_j(\Phi(\gamma)) = -\frac{1}{j+1} \int_{-\infty}^{\infty} \text{tr}(Q_{j+2}(\Phi(\gamma))a)dx. \]  

(4.8)

4.1 Proposition. Let \( \gamma = gag^{-1} \), \( g(-\infty) = I \), and \( g^{-1}g_x = u = \Phi(\gamma). \) Then

\[ \nabla H_j(\gamma) = g\pi_a^\perp(Q_{j+2}(u))g^{-1}. \]

PROOF. Write \( \delta \gamma = gvg^{-1}. \) Then

\[
\begin{align*}
dH_\gamma(\delta \gamma) &= \langle \nabla H_j(\gamma), \delta \gamma \rangle = \langle g^{-1} \nabla H_j(\gamma)g, v \rangle \\
&= d(F_j)_u(d\Phi_\gamma(\delta \gamma)) = \langle \nabla F_j(u), d\Phi_\gamma(\delta \gamma) \rangle, \text{ by (4.5)} \\
&= \langle \pi_a^\perp(Q_{j+1}(u)), d\Phi_\gamma(\delta \gamma) \rangle.
\end{align*}
\]

Proposition 3.7 implies that

\[ d\Phi_\gamma(gvg^{-1}) = -P_u(\text{ad}(a)^{-1}(v)). \]

Continue the above calculation, we get

\[
\begin{align*}
\langle \pi_a^\perp(Q_{j+1}(u)), d\Phi_\gamma(\delta \gamma) \rangle &= \langle \pi_a^\perp(Q_{j+1}(u)), -P_u(\text{ad}(a)^{-1}(v)) \rangle \\
&= \langle -\text{ad}(a)^{-1}P_u(\pi_a^\perp(Q_{j+1}(u))), v \rangle, \text{ by (4.7)} \\
&= \langle \pi_a^\perp(Q_{j+2}(u)), v \rangle = \langle g\pi_a^\perp(Q_{j+2}(u))g^{-1}, gvg^{-1} \rangle \\
&= \langle g\pi_a^\perp(Q_{j+2}(u))g^{-1}, \delta \gamma \rangle.
\end{align*}
\]

So \( \nabla H_j(\gamma) = g\pi_a^\perp(Q_{j+2}(u))g^{-1}. \)
4.2 Corollary. The Hamiltonian equation of $H_j = F_j \circ \Phi$ ($j \geq 0$) with respect to the symplectic structure $\hat{\tau}$ on $C_a(R, M)$ is

$$\gamma_t = g[Q_{j+2}(u), a]g^{-1}, \quad (4.9)$$

where $u = \Phi(\gamma)$ and $g$ satisfies $g^{-1}g_x = u$ and $g(-\infty) = I$.

**PROOF.** A direct computation gives

$$\gamma_t = [\nabla H_j(\gamma), \gamma] = [g^{\perp}_a(Q_{j+2}(u))g^{-1}, gag^{-1}] = g^{-1}[Q_{j+2}(u), a]g.$$  

4.3 Corollary. $\Phi$ maps the Hamiltonian flow of $H_j = F_j \circ \Phi$ with respect to $\hat{\tau}$ (equation (4.9)) to the Hamiltonian flow of $F_{j+2}$ with respect to $w$ (the $(j+2)$-th flow (4.3)).

**PROOF.** Let $\gamma(x, t)$ be a solution of (4.9), and $u(\cdot, t) = \Phi(\gamma(\cdot, t))$. Write $\gamma(x, t) = g(x, t)ag(x, t)^{-1}$ such that $g(-\infty, t) = I$, $g^{-1}g_x = u$. Then

$$u_t = d\Phi(\gamma_t) = d\Phi(g[Q_{j+2}(u), a]g^{-1}), \quad \text{by Proposition 3.7},$$

$$= -P_u ad(a)^{-1}([Q_{j+2}(u), a]) = -P_u(\pi_a(Q_{j+2}(u)), \quad \text{by (4.6)},$$

$$= [Q_{j+3}(u), a].$$

Next we show that when the adjoint orbit is the Grassman manifold both $H_j$ and its Hamiltonian flows are geometric, i.e., they only depend on the intrinsic geometry of $Gr(k, n)$. To show this, it suffices to show that $Q_j(\Phi(\gamma))$ can be expressed in terms of covariant derivatives and the complex structure.

4.4 Proposition. If $a = \frac{i}{2} \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$, then $Q_j(\Phi(\gamma))$ is a polynomial in

$$\{\nabla^k \gamma_x, J_{\gamma}((\nabla^k \gamma_x) \gamma_x) | 0 \leq k \leq j - 1\}.$$

**PROOF.** Since $Q_j(u)$ is a polynomial differential operator of order $j - 1$, there is a polynomial map $f_j$ such that

$$Q_j(u) = f_j(u, d^1_x u, d^2_x u, \cdots, d^{j-1}_x u),$$

where $d_x^k u = d^k u/dx^k$. But $\text{tr}(gBg^{-1}) = \text{tr}(B)$, and

$$gf_j(u, d^1_x u, \cdots, d^{j-1}_x u)g^{-1} = f_j(gug^{-1}, gd^1_x ug^{-1}, \cdots, gd^{j-1}_x ug^{-1}).$$
We claim that
\[ gd^k_x u g^{-1} = J_\gamma \nabla^k_{\gamma_x} (\gamma_x). \] (4.10)

To prove this, we use again the fact that \( u(n) = u(n)_a + u(n)_{a}^\perp = \mathcal{K} + \mathcal{P} \) is a Cartan decomposition and
\[ TM_{gag^{-1}} = g\mathcal{P} g^{-1}, \quad \nu(M)_{gag^{-1}} = g\mathcal{K} g^{-1}. \]

Suppose \( u = \Phi(\gamma) \), \( g^{-1} x u = u \) and \( g(-\infty) = I \). Then \( \gamma = gag^{-1} \) and \( \gamma_x = g[u, a] g^{-1} \). The complex structure on \( M \) at \( \gamma \) is \( J_\gamma = \text{ad}(\gamma) \). Since \( J_\gamma^2 = -I \), \( gug^{-1} = J_\gamma(\gamma_x) \). But \( (gug^{-1})_x = g(u_x + [u, u]) g^{-1} = gu_x g^{-1} \in TM_\gamma \) implies that \( gu_x g^{-1} = \nabla_{\gamma_x} J_\gamma(\gamma_x) \). Since \( M \) is Kähler, \( \nabla J = J \nabla \). So \( gu_x g^{-1} = J_\gamma(\nabla_{\gamma_x} \gamma_x) \). Then the claim follows from \([\mathcal{P}, \mathcal{P}] \subset \mathcal{K}\) and induction. ■

If \( a = \frac{i}{2} \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix} \), then the first few \( Q_j \)'s can be computed directly from the recursive formula (4.2):
\[
\begin{align*}
Q_1(u) &= u = \begin{pmatrix} 0 \\ -q^* \\ 0 \end{pmatrix}, \\
Q_2 &= \begin{pmatrix} -iq^* \\ iq_x^* \\ iq^* q \end{pmatrix}, \\
Q_3 &= \begin{pmatrix} -aq_x^* + q_x q^* \\ q_x^* + 2q^* q \\ q^* q - q^* q_x \end{pmatrix}.
\end{align*}
\] (4.11)

Note that the second flow is the MNLS flow. The functional \( H_j = F_j \circ \Phi \) is given by the following formulae:
\[
\begin{align*}
H_0(\gamma) &= \frac{1}{2} \int_{-\infty}^{\infty} \| \gamma_x \|^2 dx, \\
H_1(\gamma) &= \frac{1}{2} \int_{-\infty}^{\infty} < J_\gamma \nabla_{\gamma_x} \gamma_x, \gamma_x > dx, \\
H_2(\gamma) &= \int_{-\infty}^{\infty} \frac{1}{2} \| \gamma_x \|^2 - \frac{1}{4} \| \nabla_{\gamma_x} \gamma_x \|^4 dx.
\end{align*}
\]

The corresponding Hamiltonian flows on \( C_a(R, Gr(k, n)) \) with respect to \( \hat{\tau} \) are
\[
\begin{align*}
\gamma_t &= J_\gamma (\nabla_{\gamma_x} \gamma_x), \\
\gamma_t &= (\nabla_{\gamma_x}^2 \gamma_x + 2\gamma_x^2), \\
\gamma_t &= J_\gamma (\nabla_{\gamma_x}^3 \gamma_x - 6\gamma_x).\end{align*}
\]

Note that even powers of \( \gamma_x \) are not tensorial, but the odd powers of \( \gamma_x \) are for Hermitian symmetric spaces. Clearly the above flows contain only certain expressions in these odd tensor products and their covariant derivatives.
5. Symplectic structures for GNLS and MNLS

A two form \( \omega \) on \( M \) is called a weakly non-degenerate if the map \( Y: TM \to T^*M \) defined by \( Y(v_1)(v_2) = \omega(v_1, v_2) \) is injective, and \( w \) is a weak symplectic form if \( w \) is closed and weakly non-degenerate (cf. [CM]). A smooth function \( f \) on \( M \) defines a Hamiltonian vector field \( X_f \) if \( df_x \) lies in the image of \( Y_x \) for all \( x \in M \). Then

\[
X_f(x) = (Y_x)^{-1}(df_x).
\]

The Poisson bracket for two such functions \( f_1, f_2 \) is given by

\[
\{f_1, f_2\}(x) = \omega(X_{f_1}, X_{f_2}) = df_1(X_{f_2}).
\]

Note that when \( M \) is of finite dimension, a weak symplectic form is symplectic, but when \( M \) is of infinite dimension, this is not necessarily the case.

Given a manifold \((M, g)\), assume that we can find an embedding \( f \) from \( M \) to a co-adjoint orbit \( N \) in the dual \( G^* \) of a Lie algebra \( G \) such that \( f^*(\theta) \) is weakly non-degenerate, where \( \theta \) is the Kostant-Kirillov symplectic form on \( N \). Since \( f^*(\theta) \) is always closed, \( f^*(\theta) \) is a weak symplectic form on \( M \). Weak symplectic structures of soliton equations are often constructed this way. The procedures can be described as follows:

1. Identify the algebraic structure (i.e., the reality condition) of the Lax pair of the soliton equation, i.e., find a suitable real loop subalgebra of the affine Kac-Moody algebra defined by the reality condition.
2. Find natural embeddings of the phase space into coadjoint orbits of the above double loop algebra.
3. Prove the restrictions of orbit symplectic forms are weakly non-degenerate.

The main goal of this section is to construct a hierarchy of symplectic structures for the GNLS equation and show that the pull back of the order \( k \) symplectic form for the MNLS equation is the order \( k - 2 \) symplectic structure in the hierarchy.

We first give a short review of the general construction (cf. [Te], [TU1]). Let \( \mathcal{G} \) be a Lie algebra equipped with a non-degenerate bilinear form \( (\cdot, \cdot) \). A triple \((\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-)\) is a Manin-triple with respect to \( (\cdot, \cdot) \) if \( \mathcal{G}_+, \mathcal{G}_- \) are Lie subalgebras of \( \mathcal{G} \) such that \( \mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_- \) as vector spaces, \( \mathcal{G}^\perp = \mathcal{G}_+ \), and \( \mathcal{G}^\perp = \mathcal{G}_- \). Let \( C_s(R, \mathcal{G}) \) denote the Lie algebra of smooth maps \( A \) from \( R \) to \( \mathcal{G} \) such that \( A(x)(\lambda) \) decays as \( x \to \infty \) for each \( \lambda \). Then \( (\cdot, \cdot) \) induces an ad-invariant form on \( C_s(R, \mathcal{G}) \):

\[
((A, B)) = \int_{-\infty}^{\infty} (A(x), B(x)) dx.
\]

Let

\[
\tilde{C}(R, \mathcal{G}) = C_s(R, \mathcal{G}) + \mathcal{G} \tilde{c}
\]

denote the Lie algebra extension of \( C(R, \mathcal{G}) \) defined by

\[
[A, B]_0 = [A, B] + \rho(A, B)\tilde{c}.
\]
Here $\hat{c}$ is the generator of the center of the extension and $\rho$ defined by
\[
\rho(A, B) = \int_{-\infty}^{\infty} (A(x), B(x)) \, dx
\]
is a cocycle on $C_s(R, \mathcal{G})$. In order for the expression to be skew adjoint, we must have decay at infinity. This requirement will require later that we impose constraints to insure that the higher order symplectic forms are well-defined.

Let $C(R, \mathcal{G})$ denote the smooth maps from $R$ to $\mathcal{G}$ such that $A(x)(\lambda)$ is bounded in $x$ for each $\lambda$. Then the set $d_x + C(R, \mathcal{G})$ can be identified as a subset of the dual $\tilde{C}(R, \mathcal{G})^*$ via
\[
d_x(c) = 1, \quad d_x(B) = 0,
A(B) = \int_{-\infty}^{\infty} (A(x), B(x)) \, dx, \quad A(c) = 0,
\]
where $A \in C(R, \mathcal{G})$ and $B \in C_s(R, \mathcal{G})$. The coadjoint action $\ast$ of $g \in C_s(R, \mathcal{G})$ at $d_x + A$ is given by the gauge transformation:
\[
g \ast (d_x + A) = g(d_x + A)g^{-1} = d_x + gAg^{-1} - g_xg^{-1}.
\]
For more detail see [Ka] and [PS]. The set $d_x + C(R, \mathcal{G})$ is invariant under the coadjoint action. The phase space of a soliton equation often occurs as an coadjoint $\tilde{C}(R, \mathcal{G})$-orbit, which is equipped with the orbit symplectic form. The second symplectic form for the soliton equation is often obtained by finding a new embedding of the phase space in another coadjoint orbit via an ad-invariant bilinear form $(\cdot, \cdot)_{\Lambda}$ on $\mathcal{G}$ defined by some “Casimir” operator $\Lambda$. Here a “Casimir” operator on $\mathcal{G}$ is a linear isomorphism $\Lambda$ of $\mathcal{G}$ that satisfies the condition:
\[
\Lambda([x, y]) = [\Lambda(x), y] = [x, \Lambda(y)]
\]
for all $x, y \in \mathcal{G}$. Usually $\Lambda$ is multiplication by an element in the center of the enveloping algebra. Then the bilinear form $(\cdot, \cdot)_{\Lambda}$ defined by $\Lambda$,
\[
(x, y)_{\Lambda} = (\Lambda(x), y),
\]
is non-degenerate and ad-invariant.

We apply the abstract construction above to the $u(n)$-hierarchy. Let $\mathcal{G}$ be the Lie algebra of holomorphic maps $A$ from $\infty > |\lambda| > \frac{1}{e}$ to $GL(n, C)$ that satisfies the $u(n)$-reality condition
\[
A(\bar{\lambda})^* + A(\lambda) = 0.
\]
Let $\mathcal{G}_+$ denote the subalgebra of $A \in \mathcal{G}$ such that $A$ can be extended holomorphically to $C$, and $\mathcal{G}_-$ the subalgebra of $A \in \mathcal{G}$ such that $A$ can be extended
holomorphically to a neighborhood of $\infty$ in $S^2 = \mathbb{C} \cup \{\infty\}$ and $A(\infty) = 0$. Note that

$$A(\lambda) = \begin{cases} \sum_{j \geq 0} A_j \lambda^j, & \text{if } A \in \mathcal{G}_+, \\ \sum_{j < 0} A_j \lambda^j, & \text{if } A \in \mathcal{G}_-. \end{cases}$$

Let $\langle , \rangle$ denote the ad-invariant bi-linear form

$$\langle A, B \rangle = \oint \text{tr}(A(\lambda)B(\lambda))d\lambda$$
on $\mathcal{G}$. Or equivalently, if $A(\lambda) = \sum_j A_j \lambda^j$ and $B = \sum_k B_k \lambda^k$, then

$$\langle A, B \rangle = \sum_j \text{tr}(A_j B_{-j-1}).$$

Then $(\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-)$ is a Manin triple with respect to $\langle , \rangle$. Let $a$ be a fixed diagonal matrix in $u(n)$. A direct computation shows that the coadjoint $\tilde{C}(R, \mathcal{G}_-)$-orbit $S_0$ at $d_x + a\lambda$ is

$$S_0 = \{d_x + a\lambda + u \mid u \in S(R, u(n)_{1-a}^\perp)\},$$

and the orbit symplectic form $w_0$ is the zero order symplectic form $w$ defined by (4.1). The formula is

$$w_0(A, B) = \int_{-\infty}^{\infty} \langle -\text{ad}(a)^{-1}(A(x)), B(x) \rangle dx.$$

For an integer $k$, let $\Lambda_k : \mathcal{G} \to \mathcal{G}$ denote the operator defined by

$$\Lambda_k(A)(\lambda) = \lambda^{-k} A(\lambda).$$

Then $\Lambda_k$ is a Casimir operator, and

$$\langle A, B \rangle_{\Lambda_k} = \langle \Lambda_k(A), B \rangle = \sum_j \text{tr}(A_{i}B_{-i+k-1}). \quad (5.1)$$

For $k < 0$, the space $d_x + A(x)$ such that

$$A(x)(\lambda) = \sum_{j \geq k} \xi_j(x)\lambda^j$$

is identified as a subspace of the dual of $\tilde{C}(R, \mathcal{G}_-)$ via $\langle \langle \langle , \rangle \rangle_{\Lambda_k}$ and is invariant under the coadjoint action. The infinitesimal vector field generated by

$$\xi_- = \xi_{-1}(x)\lambda^{-1} + \cdots + \xi_{-j}(x)\lambda^{-j} + \cdots \in C_s(R, \mathcal{G}_-)$$
\[ \xi_-(A) = \pi_{k,\infty}([\xi_-, d_x + A]). \]

Here \(\pi_{k,\infty}\) is the projection

\[ \pi_{k,\infty}(\sum_j A_j \lambda^j) = \sum_{j \geq k} A_j \lambda^j. \]

Let

\[ S_k = \text{the coadjoint } \tilde{\mathcal{C}}(R, \mathcal{G}_-) \text{ orbit at } d_x + a\lambda. \]

The orbit symplectic form \(\theta_k\) on the coadjoint orbit \(S_k\) is

\[ \theta_k(\delta_1 u, \delta_2 u) = \langle \xi_-, \eta_-(A) \rangle_{\Lambda_k} = \int_{-\infty}^{\infty} \text{tr}(\xi_{k-1}(x)\delta_2 u(x))dx, \quad (5.2) \]

where \(\delta_1 u = \xi_-(A)\) and \(\delta_2 u = \eta_-(A)\).

We claim that \(S_k \cap S_0\) is a finite codimension submanifold of \(S_0\). If \(\delta u\) is tangent to \(S_k \cap S_0\) at \(d_x + a\lambda + u\), then there exists

\[ \xi_- = \xi_{-1}(x)\lambda^{-1} + \xi_{-2}(x)\lambda^{-2} + \cdots \in \mathcal{G}_- \]

such that

\[ \delta u = \xi_-(d_x + a\lambda + u) = \pi_{k,\infty}([\xi_-, d_x + a\lambda + u]) \]

\[ = [\xi_{-1}, a] + \lambda^{-1} ([\xi_{-2}, a] + [\xi_{-1}, d_x + u]) + \cdots + \lambda^k ([\xi_{k-1}, a] + [\xi_k, d_x + u]). \]

Compare coefficients of \(\lambda^j\) in the above equation to get

\[ \begin{cases} 
[\xi_{-1}, a] = \delta u, \\
(\xi_i)_x + [u, \xi_i] = [\xi_{i-1}, a], & \text{if } k \leq i \leq -1, \\
\xi_i(x) \in \mathcal{S}(R, u(n)), & \text{for all } k \leq i \leq -1. 
\end{cases} \quad (5.3) \]

Given \(\delta u \in \mathcal{S}(R, u(n)_{a^\perp})\), we have

\[ \pi_{a^\perp}(\xi_{-1}) = - \text{ad}(a)^{-1}(\delta u) \]

and

\[ \pi_a(\xi_{-1}(x)) = - \int_{-\infty}^{x} \pi_a([u, \pi_{a^\perp}(\xi_{-1})])dy. \]

So \(\xi_{-1}\) is in the Schwartz class if \(\int_{-\infty}^{\infty} [u, \text{ad}(a)^{-1}(\delta u)]dx = 0\). Continuing inductively, we get a formula for the \(u(n)_{a^\perp}\) part

\[ \pi_{a^\perp}(\xi_{i-1}) = - \text{ad}(a)^{-1}((\xi_i)_x + [u, \xi_i]). \quad (5.4) \]
For the $u(n)_a$ part, the integral formula

$$\pi_a(\xi_i) = -\int_{-\infty}^{\infty} \pi_a([u, \pi_a^\perp(\xi_i)])dy$$

leaves with constraints

$$\int_{-\infty}^{\infty} \pi_a([u, \pi_a^\perp(\xi_i)])dx = 0,$$

which are necessary for $\pi_a(\xi_i)$ in Schwartz class for $k \leq i \leq -1$. Hence $S_0 \cap S_k$ is of finite codimension in $S_0$.

Use the Poisson operator $P_u$ defined in 3.4 to write (5.4) as

$$\pi_a^\perp(\xi_{i-1}) = (-\text{ad}(a)^{-1}P_u)(\pi_a^\perp(\xi_1)).$$

(5.5)

By induction,

$$\pi_a^\perp(\xi_{k-1}) = (-\text{ad}(a)^{-1}P_u)^{-k}(-\text{ad}(a)^{-1}(\delta u)).$$

(5.6)

Let $w_k$ denote the restriction of the orbit symplectic form $\theta_k$ to $S_0 \cap S_k$. Substituting (5.6) to (5.2), we get for $k \leq 0$

$$(w_k)_u(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} (-1)^{-k+1} \text{tr} \left( ((\text{ad}(a)^{-1}P_u)^{-k} \text{ad}(a)^{-1}(\delta_1 u)) \delta_2 u \right) dx.$$

Since $\text{ad}(a)^{-1}$ is an isomorphism and $P_u$ is injective (Proposition 3.5), $w_k$ is a weak symplectic form on $S_0 \cap S_k$.

For $k = 1$, the space of $d_x + A(x)$ such that $A(x)(\lambda)$ is of the form $\sum_{j \leq 0} A_j(x)\lambda^j$ can be identified as a subset of $\tilde{C}(R,G_+)^*$ via $<,>_1$. The coadjoint orbit $S_1$ at $d_x$ is a submanifold of $d_x + S(R,u(n))$, and $\delta u$ is tangent to $S_1$ at $d_x + u$ if and only if there exists $\xi \in S(R,u(n))$ such that

$$\delta u = [d_x + u, \xi].$$

For example, $v$ is tangent to $S_1$ at $d_x$ if and only if $v = \xi_x$ for some $\xi$ in the Schwartz class. So $v$ must satisfies the condition $\int_{-\infty}^{\infty} v(x)dx = 0$. Let $w_1$ denote the restriction of the orbit symplectic form of $S_1$ to $S_0 \cap S_1$. A direct computation shows that

$$(w_1)_u(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} \text{tr}(P_u^{-1}(\delta_1 u)\delta_2 u)dx.$$

For $k > 1$, the space of $d_x + A(x)$ such that $A(x)(\lambda)$ is of the form $\sum_{j \leq k-1} A_j(x)\lambda^j$ can be identified as a subset of $\tilde{C}(R,G_+)$, and it is invariant under the coadjoint action. Let $S_k$ denote the coadjoint $\tilde{C}(R,G_+)$-orbit at
\[ d_x + a\lambda, \text{ and } \theta_k \text{ its orbit symplectic form. Note that } \delta u \text{ is tangent to } S_k \cap S_0 \text{ if there exist } \xi_0, \cdots, \xi_{k-1} \text{ such that } \]

\[
\begin{align*}
[d_x + u, \xi_0] &= -\delta u, \\
[d_x + u, \xi_i] &= [\xi_{i-1}, a], & \text{if } 0 \leq i \leq k - 1, \\
\lim_{x \to \pm \infty} \xi_i(x) &= 0, & \text{if } 0 \leq i \leq k - 1.
\end{align*}
\]

The ordinary differential equations can always be solved. Finite constraints appear from requiring \( \xi_i(\infty) = 0 \). Let \( w_k \) denote the restriction of \( \theta_k \) to \( S_0 \cap S_k \). The formula for \( w_k \) can be computed similarly as in the case \( k \leq 0 \). To summarize, we have

5.1 Theorem.

(i) \( S_k \cap S_0 \) is a finite dimensional submanifold of \( S_0 \).
(ii) \( w_k \) is an order \( k \) weak symplectic form on \( S_k \cap S_0 \), and

\[
(w_k)_u(\delta_1 u, \delta_2 u) = \int_{-\infty}^{\infty} (-1)^{-k+1} \text{tr}((-1)^{-k} \text{ad}(a)^{-1} P_u)^{-k} \text{ad}(a)^{-1} (\delta_1 u) \delta_2 u) dx.
\]

To construct symplectic structures for the GNLS flow, we proceed in the same way as for the MNLS flow, except that we use a different Manin triple. Let \( G \) be as above, and

\[
(A, B) = \sum_i \text{tr}(A_i B_{k+1-i}).
\]

Let \( G_{>0} \) denote the subalgebra of \( A \in G \) that can be holomorphically extended to \( C \) and \( A(0) = 0 \), and \( G_{<0} \) the subalgebra of \( A \in G \) that can be holomorphically extended to a neighborhood of \( \infty \) in \( S^2 = C \cup \{\infty\} \). Then:

(i) \( (G, G_{>0}, G_{<0}) \) is a Manin triple with respect to \( ( , ) \).
(ii) \( (A, B)_{\Lambda_k} = \sum_i \text{tr}(A_i B_{k+1-i}) \).

The space \( C(R, G_{>0}) \) can be identified as a subspace of \( \hat{C}(R, G_{<0})^* \) via \( ( , ) \) and is invariant under the coadjoint action. Let \( M_0 \) denote the coadjoint \( \hat{C}(R, G_{<0}) \)-orbit at \( a\lambda \). Then \( M_0 \) is the set of \( d_x + \gamma \lambda \), where \( \gamma(x) \) lies in the adjoint orbit of \( a \) in \( U(n) \), and the orbit symplectic form \( \tau_0 \) is \( \hat{\tau} \), i.e.,

\[
\hat{\tau}_\gamma(\delta_1 \gamma, \delta_2 \gamma) = \int_{-\infty}^{\infty} \text{tr}(-\text{ad}(\gamma)^{-1} (\delta_1 \gamma), \delta_2 \gamma) dx.
\]

For \( k < 0 \), the space of \( d_x + A(x) \) so that

\[
A(x)(\lambda) = \sum_{j \geq k+1} A_j(x) \lambda^j
\]
can be identified as a subset of $\tilde{C}(R, \mathcal{G}_{\leq 0})^*$ via the bilinear form $(\cdot, \cdot)_\Lambda$ and is invariant under the coadjoint action. Let $\mathcal{M}_k$ denote the coadjoint $\tilde{C}(R, \mathcal{G}_{\leq 0})$-orbit at $d_x + a\lambda$, $\theta_k$ the orbit symplectic form, and $\tau_k$ the restriction of $\theta_k$ to $\mathcal{M}_0 \cap \mathcal{M}_k$.

For $k > 0$, the space of $d_x + A(x)$ so that

\[ A(x)(\lambda) = \sum_{j \leq k} A_j(x)\lambda^j \]

can be identified as a subset of $\tilde{C}(R, \mathcal{G}_{> 0})^*$ via the bilinear form $(\cdot, \cdot)_\Lambda$ and is invariant under the coadjoint action. Let $\mathcal{M}_k$ denote the coadjoint $\tilde{C}(R, \mathcal{G}_{> 0})$-orbit at $d_x + a\lambda$, $\theta_k$ the orbit symplectic form, and $\tau_k$ the restriction of $\theta_k$ to $\mathcal{M}_0 \cap \mathcal{M}_k$. We go into less detail here, as the construction and proof are small modifications of the MNLS case.

5.2 Theorem. (i) $\mathcal{M}_0 \cap \mathcal{M}_k$ is a finite codimension submanifold of $C_a(R, M_a)$, (ii) $\tau_k$ is an order $k$ weak symplectic form on $\mathcal{M}_0 \cap \mathcal{M}_k$ and

\[ (\tau_k)_\gamma (\delta_1 \gamma, \delta_2 \gamma) = \int_{-\infty}^{\infty} (-1)^{-k+1} \text{tr}(((\text{ad}(\gamma)^{-1}L_\gamma)^{-k} \text{ad}(\gamma)^{-1}(\delta_1 \gamma))(\delta_2 \gamma))dx, \]

where $L_\gamma$ is the operator defined in Definition 3.8.

PROOF. We will give the computation for $k < 0$. The computation for $k > 0$ is similar. Note that $\delta \gamma$ lies in the tangent space of $\mathcal{M}_0 \cap \mathcal{M}_k$ if and only if there exist $\xi_-$ so that

\[ (\delta \gamma)\lambda = \pi_{k+1,\infty}(\xi_-, d_x + \gamma \lambda), \] (5.8)

where $\pi_{k+1,\infty}(\sum_j A_j \lambda^j) = \sum_{j \geq k+1} A_j \lambda^j$. The calculation below is entirely algebraic. Write

\[ \xi_- = \xi_0 + \xi_{-1} \lambda^{-1} + \xi_{-2} \lambda^{-2} + \cdots. \]

Equation (5.8) gives

\[ \left\{ \begin{array}{ll}
[\xi_0, \gamma] = \delta \gamma, \\
(\xi_j)_{x} - [\xi_{j-1}, \gamma] = 0, & \text{if } k + 1 \leq j \leq 0, \\
\lim_{x \to \pm \infty} \xi_j(x) = 0, & \text{if } k + 1 \leq j \leq 0.
\end{array} \right. \] (5.9)

Let $\xi_j^T$ denote the projection of the vector field $\xi_j$ along $\gamma$ to $TM_\gamma$. Proposition 3.9 implies that if $\xi_j$'s satisfy (5.9) then

\[ \xi_j^T (\delta \gamma) = (- \text{ad}(\gamma)^{-1}L_\gamma)^{-j}(- \text{ad}(\gamma)^{-1})(\delta \gamma). \] (5.10)

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Given two tangent vectors $\delta_1 \gamma, \delta_2 \gamma$, there exist $\xi, \eta \in C(R, -\infty, 0)$ such that
\[ \lambda \delta_1 \gamma = \pi_{k+1, \infty}((\xi, d_x + \gamma \lambda)), \quad \lambda \delta_2 \gamma = \pi_{k+1, \infty}((\eta, d_x + \gamma \lambda)). \]
Write $\xi = \sum_{j \leq 0} \xi_j \lambda^j$ and $\eta = \sum_{j \leq 0} \eta_j \lambda^j$. Then the restriction of the orbit symplectic form on $\mathcal{M}_k$ to $\mathcal{M}_k \cap \mathcal{M}_0$ is
\[ \tau_k(\delta_1 \gamma, \delta_2 \gamma) = (\langle \xi, (\delta_2 \gamma) \rangle)_{\Lambda_k} = \int_{-\infty}^{\infty} \text{tr}(\xi_k, \delta_2 \gamma)dx. \] (5.11)
Substitute (5.10) into (5.11) to get formula (5.7).

5.3 Theorem. $\Phi^*(w_k) = \tau_{k-2}$.

PROOF. Write $\gamma = gag^{-1}$ such that $g^{-1}g_x = \Phi(\gamma) = u$ and $g(-\infty) = I$.

Then
\[ \delta_i \gamma = [g^{-1}\delta_i g, a]g^{-1}. \]
Set
\[ v_i = [g^{-1}\delta_i g, a], \quad I_a = \text{ad}(a), \quad Y_\gamma = \text{ad}(\gamma). \]
Let $\langle \cdot, \cdot \rangle_o$ denote the $L^2$ inner product. We compute $\Phi^*(w_k)$.
\[
\langle \Phi^*(w_k) \rangle_o(\delta_1 \gamma, \delta_2 \gamma) = \langle w_k \rangle_o(d\Phi_\gamma(\delta_1 \gamma), d\Phi_\gamma(\delta_2 \gamma)), \quad \text{by Proposition 3.7}
\]
\[ = w_k(P_u I^{-1}_a(v_1), P_u I^{-1}_a(v_2)), \quad \text{by Theorem 5.1}
\]
\[ = (-1)^{-k+1} \langle (I_a^{-1} P_u)^{-k} I^{-1}_a P_u I^{-1}_a(v_1), P_u I^{-1}_a(v_2) \rangle_o
\]
\[ = (-1)^{-k+1} \langle (I_a^{-1} P_a)^{-k+1} I_a^{-1}(v_1), P_a I_a^{-1}(v_2) \rangle_o
\]
\[ = (-1)^{-k+1} \langle (I_a^{-1} P_u)^{-k+2} I_a^{-1}(v_1), v_2 \rangle_o
\]
\[ = (-1)^{-k+1} \langle g((I_a^{-1} P_u)^{-k+2} I_a^{-1}v_1)g^{-1}, g v_2 g^{-1} \rangle_o. \]
Note that
\[ g I_a(v)g^{-1} = g[a, v]g^{-1} = [gag^{-1}, gvg^{-1}] = [\gamma, gvg^{-1}] = Y_\gamma(gvg^{-1}). \]
The operator $L_\gamma$ is defined in 3.8 by $L_\gamma(gvg^{-1}) = gP_a(v)g^{-1}$. So we have
\[ Y_\gamma^{-1} L_\gamma(gvg^{-1}) = g(I_a^{-1} P_u(v))g^{-1}. \]
Hence
\[ \Phi^*(w_k) \rangle_o(\delta_1 \gamma, \delta_2 \gamma) = (-1)^{-k+1} \langle (Y_\gamma^{-1} L_\gamma)^{-k+2} Y_\gamma^{-1}(\delta_1 \gamma), \delta_2 \gamma \rangle_o, \]
which is equal to $(\tau_{k-2})_o(\delta_1 \gamma, \delta_2 \gamma)$.

5.4 Corollary. If $j \geq 1$, then the Hamiltonian equation for $H_j = F_j \circ \Phi$ with respect to $\tau_0 = \hat{\tau}$ satisfies the following Lenard-Magri relation
\[ \gamma_t = [\nabla H_j(\gamma), \gamma] = L_\gamma(\nabla H_{j-1}(\gamma)), \] (5.12)
where $L_\gamma$ is defined in 3.8. In other words, the Hamiltonian equation for $H_j$ with respect to $\tau_0$ is the Hamiltonian equation for $H_{j-1}$ with respect to $\tau_1$. 

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PROOF. The Hamiltonian equation for $H_j$ with respect to $\tau_0$ is $\gamma_t = [\nabla H_j(\gamma), \gamma]$. By Proposition 4.1, we have

\[
L_{\gamma}(\nabla H_{j-1}(\gamma)) = L_{\gamma}(g\pi_a^{-1}(Q_{j+1}(u))g^{-1}),
\]

by (4.6),

\[
g[Q_{j+2}(u), a]g^{-1} = [gQ_{j+2}(u)g^{-1}, \gamma] = [\nabla H_j(\gamma), \gamma].
\]

6. Symplectic structures for KdV

Since we have set up the machinery for constructing symplectic structures and applied it to two examples, we take this opportunity to show that two structures for KdV can be obtained in the same fashion.

The KdV equation,

\[
q_t = \frac{1}{4}(q_{xxx} - 6q_{xx}),
\]

has a Lax pair:

\[
\left[ \frac{\partial}{\partial x} + a\lambda + u, \frac{\partial}{\partial t} + a\lambda^3 + u\lambda^2 + Q_2\lambda + Q_3 \right] = 0,
\]

where

\[
a = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad u = \left( \begin{array}{cc} 0 & q \\ 1 & 0 \end{array} \right),
\]

\[
Q_2 = \left( \begin{array}{cc} -\frac{q}{2} & -\frac{q_x}{2} \\ 0 & \frac{q}{2} \end{array} \right), \quad Q_3 = \left( \begin{array}{cc} \frac{q_x}{4} & \frac{q_{xx} - 2q^2}{4} \\ -\frac{q}{4} & -\frac{q_x}{4} \end{array} \right).
\]

This Lax pair satisfies the following reality condition:

\[
\begin{cases}
A(\overline{\lambda}) = A(\lambda), \\
\phi(\lambda)^{-1}A(\lambda)\phi(\lambda) = \phi(-\lambda)^{-1}A(-\lambda)\phi(-\lambda), \quad \text{where} \quad \phi(\lambda) = \left( \begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right).
\end{cases}
\]

We call this the KdV reality condition (cf. [TU2]).

Let $\mathcal{G}, \mathcal{G}_+, \mathcal{G}_-, <, >, ^\Lambda_k$ be as in section 5, and $\mathcal{G}^{kdv}$ ($\mathcal{G}^{kdv}_\pm$ resp.) the space of all $A \in \mathcal{G}$ ($\mathcal{G}^{kdv}_\pm$ resp.) that satisfies the KdV-reality condition. Recall that if $\xi = \sum_i \xi_i \lambda^i$ and $\eta = \sum_j \eta_j \lambda^j$, then the bilinear form $<, >^\Lambda_k$ defined by (5.1) is

\[
< \xi, \eta >^\Lambda_k = \sum_i \text{tr}(\xi_i \eta_{-i+k-1}).
\]

Let $e_{ij} \in sl(2)$ denote the matrix whose entries are zero except the $ij$-th entry equals to 1.
6.1 Lemma. Let $\xi(\lambda) = \sum_j \xi_j \lambda^j$ with $\xi_j = \begin{pmatrix} A_j & B_j \\ C_j & -A_j \end{pmatrix} \in \text{sl}(2, \mathbb{R})$. Then $\xi$ satisfies the KdV-reality condition if and only if

$$\xi_{2j} = \begin{pmatrix} A_{2j} & B_{2j} \\ C_{2j} & -A_{2j} \end{pmatrix}, \quad \xi_{2j+1} = \begin{pmatrix} C_{2j} & -2A_{2j} \\ 0 & -C_{2j} \end{pmatrix}$$

for all $j$.

PROOF. $\xi$ satisfies the KdV-reality condition if and only if the coefficient of $\lambda^{2j+1}$ in $\phi(\lambda)^{-1} \xi_-(\lambda)\phi(\lambda)$ is zero for all $j$, i.e.,

$$\begin{cases} A_{2j+1} - C_{2j} = 0, \\ B_{2j+1} + 2A_{2j} - C_{2j-1} = 0, \\ C_{2j+1} = 0. \end{cases}$$

So we have

$$C_{2j+1} = 0, \quad A_{2j+1} = C_{2j}, \quad B_{2j+1} = -2A_{2j},$$

which proves the Lemma. $\blacksquare$

6.2 Proposition. The restriction of the bilinear form $<,>_\Lambda_k$ to $\mathcal{G}^{kdv}$ is degenerate if $k$ is even, and is non-degenerate if $k$ is odd.

PROOF. By Lemma 6.1, $be_{12} \in \mathcal{G}^{kdv}$. If $k$ is even, then

$$< be_{12}, \xi >_{\Lambda_k} = bC_{k-1}$$

for all $\xi \in \mathcal{G}^{kdv}$, where $\xi = \sum_i \xi_i \lambda^i$ and $\xi_i = \begin{pmatrix} A_i & B_i \\ C_i & -A_i \end{pmatrix}$. But $k-1$ is odd, so $C_{k-1} = 0$. This shows that $<,>_\Lambda_k$ is degenerate.

Next we prove that if $k$ is odd then $<,>_\Lambda_k$ is non-degenerate on $\mathcal{G}^{kdv}$. Let

$$\xi = \sum_i \xi_i \lambda^i, \eta = \sum_j \eta_j \lambda^j \in \mathcal{G}^{kdv}, \quad \xi_i = \begin{pmatrix} A_i & B_i \\ C_i & -A_i \end{pmatrix}, \quad \eta_i = \begin{pmatrix} A'_i & B'_i \\ C'_i & -A'_i \end{pmatrix}.$$  

Then

$$< \xi, \eta >_{\Lambda_k} = \sum_i \text{tr}(\xi_i \eta_{i+1}) = \sum_j \text{tr}(\xi_{2j} \eta_{2j+k-1}) + \text{tr}(\xi_{2j+1} \eta_{-2j+k-2}).$$

Note $-2j + k - 2$ is odd. By Lemma 6.1, we get

$$< \xi, \eta >_{\Lambda_k} = \sum_j 2A_{2j}A'_{-2j+k-1} + C_{2j}(B'_{-2j+k-1} + 2C'_{-2j+k-3}) + B_{2j}C'_{-2j+k-1}.$$
It follows that if \( < \xi, \eta >_{\Lambda_k} = 0 \) for all \( \eta \in \mathcal{G}^{kdv} \) then \( \xi = 0 \). This proves \( <, >_{\Lambda_k} \) is non-degenerate.

It follows from Lemma 6.1 that

\[
A_0 = a\lambda + e_{21}
\]

satisfies the KdV-reality condition. So it belongs to \( \mathcal{G}^{kdv}_+ \).

Let \( k \leq -1 \) be an odd integer. Then the set of \( d_x + A(x) \) such that

\[
A(x)(\lambda) = \sum_{j \geq k} A_j(x)\lambda^j
\]

can be identified as a subset of \( \tilde{C}(R, \mathcal{G}^{kdv}_-)^* \) via \( <, >_{\Lambda_k} \), and it is invariant under the coadjoint \( \tilde{C}(R, \mathcal{G}^{kdv}_-)^* \)-action. Set

\[
\begin{cases}
\Omega_k = \text{the coadjoint } \tilde{C}(R, \mathcal{G}^{kdv}_-)^*\text{-orbit at } d_x + a\lambda + e_{21}, \\
\sigma_k = \text{the orbit symplectic form on } \Omega_k.
\end{cases}
\]

The set of \( d_x + A(x) \in d_x + C(R, \mathcal{G}^{kdv}) \) such that \( A(x)(\lambda) = \sum_{i \leq 0} A_i(x)\lambda^i \)

can be identified as a subset of \( \tilde{C}(R, \mathcal{G}^{kdv}_+)^* \) via \( <, >_{\Lambda_1} \), and is invariant under the coadjoint \( \tilde{C}(R, \mathcal{G}^{kdv}_+)^* \)-action. Set

\[
\begin{cases}
\Omega_1 = \text{the coadjoint } \tilde{C}(R, \mathcal{G}^{kdv}_+)^*\text{-orbit at } d_x + e_{21}, \\
\sigma_1 = \text{the orbit symplectic form on } \Omega_1.
\end{cases}
\]

By Lemma 6.1, \( d_x + a\lambda + e_{21} + u \) satisfies the KdV-reality condition if and only if

\[
u = qe_{12} = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}.
\]

Now set

\[
\mathcal{N}_k = \begin{cases}
\{ q \in S(R, R) \mid (d_x + A_0 + qe_{12}) \in \Omega_k \} & \text{if } k \leq -1 \text{ is odd}, \\
\{ q \in S(R, R) \mid (d_x + e_{21} + qe_{12}) \in \Omega_0 \} & \text{if } k = 1.
\end{cases}
\]

Let \( \beta_k \) denote the restriction of the orbit symplectic form on \( \Omega_k \) to \( \mathcal{N}_k \) (here we identify \( \mathcal{N}_k \) as a subspace of \( \Omega_k \) via \( q \mapsto d_x + A_0 + qe_{12} \) if \( k \leq -1 \) is odd and \( q \mapsto q + e_{21} + qe_{12} \) if \( k = 1 \)). Write

\[
(\beta_k)_q(\delta_1 q, \delta_2 q) = \int_{-\infty}^{\infty} ((J_k)^{-1}(\delta_1 q))\delta_2 q dx.
\]

6.3 Theorem.

(i) \( (J_{-1})_q(v) = -2v_x \).

(ii) \( (J_1)_q(v) = \frac{1}{2}v_{xxx} - 2qv_x - q_x v \).
PROOF. (i) $\delta q$ lies in the tangent space of $\mathcal{N}_{-1}$ at $q$ if and only if there exists $\xi_- \in C(R, \mathcal{G}_{-1}^{dv})$ such that

$$e_{12} \delta q = \pi_{-1,\infty}([\xi_-, d_x + a\lambda + e_{21} + qe_{12}]),$$

where

$$\pi_{-1,\infty}(\sum_j A_j \lambda^j) = \sum_{j \geq -1} A_j \lambda^j.$$ 

This implies that

$$\begin{cases}
[\xi_{-1}, a] = e_{12} \delta q, \\
[d_x + e_{21} + qe_{12}, \xi_{-1}] = [\xi_{-2}, a], \\
\lim_{x \to \pm\infty} \xi_j(x) = 0, & \text{if } j = -1, -2.
\end{cases} \quad (6.6)$$

Let $\delta_1 u = (\delta_1 q)e_{12}, \delta_2 u = (\delta_2 q)e_{12}$ be two tangent vectors of $\mathcal{N}_{-1}$ at $q$. So there exist $\xi = \sum_{j \leq -1} \xi_j \lambda^j$ and $\eta = \sum_{j \leq -1} \eta_j \lambda^j$ satisfying equation (6.6).

Write

$$\xi_j = \begin{pmatrix} A_j & B_j \\ C_j & -A_j \end{pmatrix}.$$ 

By definition of the orbit symplectic form, we get

$$(\beta_{-1})_q(\delta_1 q, \delta_2 q) = \int_{-\infty}^{\infty} \text{tr}(\xi_{-2} \delta_2 qe_{12}) dx = \int_{-\infty}^{\infty} C_{-2} \delta_2 q dx.$$ 

Next we solve $\delta_1 q$ in terms of $C_{-2}$. The first equation in (6.6) implies that

$$\delta_1 q = -2B_{-1}. \quad (6.7)$$

The second equation in (6.6) gives

$$\begin{cases}
(A_{-1})_x - B_{-1} = 0, \\
(B_{-1})_x - 2qA_{-1} = -2B_{-2}, \\
2A_{-1} = 2C_{-2}.
\end{cases} \quad (6.8)$$

Substitute (6.7) into (6.8) to get $\delta_1 q = -2(C_{-2})_x$. This proves $J_{-1} = -2d_x$.

(ii) If $\delta q$ is tangent to $\mathcal{N}_1$ at $q$, then there exist $\xi \in C(R, \mathcal{G}_{+1}^{dv})$

$$\xi(\lambda) = \xi_0 + \xi_1 \lambda + \xi_2 \lambda^2 + \cdots$$

such that

$$\delta u = (\delta q)e_{12} = \pi_{-\infty,0}(\xi, d_x + e_{21} + qe_{12}) = [\xi_0, d_x + e_{21} + qe_{12}]. \quad (6.9)$$
Write $\xi_0 = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$. Then
\[
(\beta_1)_q(\delta_1 q, \delta_2 q) = \int_{-\infty}^{\infty} <\xi_0, \delta_2 q e_{12}> dx = \int_{-\infty}^{\infty} C\delta_2 q dx.
\]

We need to compute $\delta q$ in terms of $C$. To do this, we equate the entries of equation (6.9) to get
\[
\begin{cases}
A_x + q C - B = 0, \\
C_x + 2A = 0, \\
2qA - B_x = \delta q.
\end{cases}
\]
The second equation gives $A = -C_x/2$. Substitute this to the first equation to solve $B$ in terms of $C$. Then the last equation solves
\[
\delta q = \frac{1}{2} C_{xxx} - q_x C - 2q C_x,
\]
which gives the formula for $J_1$. ■

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