One Loop Effects In Various Dimensions And D-Branes

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Abstract

We calculate some one loop corrections to the effective action of theories in $d$ dimensions that arise on the dimensional reduction of a Weyl fermion in $D$ dimensions. The terms that we are interested in are of a topological nature. Special attention is given to the effective actions of the super Yang Mills theories that arise on dimensional reduction of the $N = 1$ theory in six dimensions or on the dimensional reduction of the $N = 1$ theory in ten dimensions. In the latter case we suggest an interpretation of the quantum effect as a coupling of the gauge field on the brane to a relative background gauge field.

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1 Introduction

The low energy excitations of (d-1)-branes are governed by world volume super Yang-Mills theories that are obtained on dimensional reduction to \(d\) dimensions. For \(d \geq 5\) these theories are non-renormalizable and only make sense up to some scale. From the string theory point of view one needs to introduce \(\alpha'\) corrections, precisely those terms that were ignored in restricting one’s attention to the super Yang-Mills theory in the first place. One can ask, however, if there are field theoretic quantities that can be calculated that do not depend on the cutoff that one sets on the theory (alternatively that do not require \(\alpha'\) corrections)? Natural candidates are topological terms, of Chern-Simons type, that depend neither on the metric nor on the coupling. One knows that such terms are protected, perturbatively at least, without the need to invoke any supersymmetric non-renormalization theorems. They only pick up one loop corrections. Furthermore, these terms do not depend on the cutoff (and so will not depend on \(\alpha'\)).

The terms that we will obtain are, then, of Chern-Simons type, that is they are like secondary characteristic classes. Such terms arise in non-supersymmetric theories, however, for concreteness, we will presume in most of the text that we are given a supersymmetric system. Suppose, then, that we start with \(N = 1\) super Yang-Mills theory in \(D\) dimensions, where \(D = 3, 4, 6\) or 10. The resulting theory in \(d\) dimensions, obtained on reduction, will contain one vector \(A\) and \(D - d\) scalars \(\phi^a\). We presume that the theory is in the Coulomb branch, that is, all the massless fields are in the Cartan sub-algebra \(T\) of the gauge group.

The terms we are interested in will involve the normalized scalar field \(\hat{\phi} = \phi/\sqrt{|\phi|^2}\) not \(\phi\), where \(\phi \in T\), and all contractions are with respect to the \(D\) dimensional epsilon symbol suitably factorised. How does a term of the required form arise? The presence of the epsilon symbols means that they can only come from a fermion loop\(^1\), for even \(D\) with Weyl fermions one has

\[
\text{Tr} \frac{(1 + \Gamma_{D+1})}{2} \Gamma_{\mu_1} \cdots \Gamma_{\mu_d} \Gamma_{a_1} \cdots \Gamma_{a_{D-d}} \sim \epsilon_{\mu_1 \ldots \mu_d a_1 \ldots a_{D-d}}.
\]  

(1.1)

To contribute to the effective action the fermions must be charged with respect to the maximal torus (and hence massive). The origin of these terms is much like that of Chern-Simons terms that appear in odd dimensions associated with parity non-invariance [1].

\(^1\)Unless there are anti-symmetric tensors of particular rank present.
With the conditions specified above what type of terms can we have? We need to integrate a top form and so the number of times the gauge field, in the Cartan sub-algebra, appears without a derivative (call it \( \alpha \)), plus twice the number of factors of the field strength (\( \gamma \)) together with the number of derivatives (\( \beta \)) on \( \phi \) must equal the dimension; \( \alpha + 2\gamma + \beta = d \). To soak up the internal labels of the epsilon symbol there must be \( (D - d) \) factors of \( \hat{\phi} \). Because of the epsilon symbol there will be one \( \hat{\phi} \) appearing without a derivative. Let \( \sigma^n \) denote the pull back of the unit volume form for the n-sphere \( S^n \):

\[
\sigma^n \sim \epsilon^{a_1 \ldots a_{n+1}} d\hat{\phi}_{a_1} \ldots d\hat{\phi}_{a_n} \hat{\phi}_{a_{n+1}}, \quad d\sigma^n = 0.
\]

The scalars will enter in the combination \( \sigma^{D-d-1} \) so that \( \beta = D - d - 1 \). Here \( \sigma^0 = \text{sign}(\phi) \). Likewise at most one factor of the gauge field without a derivative can appear, i.e. \( \alpha = 0, 1 \). Consequently, \( \alpha + 2\gamma = 2d - D + 1 \) and if \( D \) is even \( \alpha = 1 \), while if \( D = 3 \) we have \( \alpha = 0 \). Let us consider \( D = 3 \) and \( D = 4 \) separately. The terms of interest are potentially

| \( d \) | \( D=3 \) | \( D=4 \) |
|------|------|------|
| 3    | \( \int \hat{A} \sigma^0 \) |          |
| 2    | \( \int F \sigma^0 \) | \( \int A \sigma^1 \) |
| 1    | \( \int \sigma^1 \) |          |

For \( D = 4 \) we see that the putative terms are odd under the Weyl group and so cannot arise from the vector multiplet. In section 4 we will present an argument that extends this conclusion to \( D = 4m \). For \( D = 3 \) the trace of 3 gamma matrices will give the sought for epsilon symbols and those terms that arise from reductions of the \( D = 3 \) theory are Weyl even and so could, in principle, appear. However, as can be seen explicitly, they do not arise in the one loop calculation. Indeed, once more in section 4, we will argue that for \( D \) odd a massless Dirac fermion coupled only to a gauge field cannot yield these topological terms upon reduction.

For those theories that arise on the reduction from the \( N = 1 \) SYM in six and ten dimensions, one finds two and four derivatives terms respectively:

| \( d \) | \( D=6 \) |
|------|------|
| 5    | \( \int \hat{A} \hat{F} \sigma^0 \) |
| 4    | \( \int \hat{A} \hat{F} \sigma^1 \) |
| 3    | \( \int \hat{A} \sigma^2 \) |

| \( d \) | \( D=10 \) |
|------|------|
| 9    | \( \int \hat{A} \hat{F} \sigma^0 \) |
| 8    | \( \int \hat{A} \hat{F} \sigma^1 \) |
| 7    | \( \int \hat{A} \hat{F} \sigma^2 \) |
| 6    | \( \int \hat{A} \sigma^3 \) |
| 5    | \( \int \hat{A} \sigma^4 \) | (1.3)
There is a pattern in these tables. For both types of reduction (from $D = 6$ or $D = 10$) in the highest dimension ($d = 5$ or $d = 9$) one obtains a conventional Chern-Simons type term. For each dimension one descends a curvature form $F$ is replaced by the derivative of a scalar, so that the degree of the volume form ascends, while keeping the number of derivatives fixed. Also one can read from these tables the analogues of the ‘peculiar’ symmetry found by Seiberg [2], namely there are conserved currents

$$j = \ast \left( F^{d-D/2} \sigma^{D-d-1} \right)$$

(1.4)
coupled to the gauge field itself.

So the ‘topological’ terms that we are interested in involve only the epsilon symbol and do not depend on the metric. In the supersymmetric field theories under consideration such terms can only arise from fermion loops and so we look for them in the fermion determinant contribution to the effective action,

$$\text{Tr} \ln \mathcal{D} = i \Gamma_{1\text{-loop}}.$$  

(1.5)

We perform a one loop computation, there are various non-renormalization arguments, that will be recalled below, which justify this at the perturbative level. Non-perturbative effects, can and do contribute.

We begin by considering theories that arise on dimensional reduction of the $N = 1$ theory in $D = 6$. These are perhaps the best studied theories and much is known about them. Consequently for $d = 4$ and $d = 5$ the results we find are already well known. The, minor, novelty for $d = 4$ is that the pre-potential is determined, perturbatively from the one-loop correction to the $\theta$ angle rather than to the $\beta$ function, these two ways of proceeding being equivalent once one assumes the pre-potential is holomorphic. For $d = 3$ it was suggested in [3, 4] that a one loop calculation should reproduce an index calculation for determining the metric on the moduli space and here we see that this is true.

We are rather more brief about the results obtained from the reduction of the $N = 1$ theory in $D = 10$. The problem here is that the topological terms are four derivative terms and so their direct geometric significance for the moduli space is less clear. In terms of branes, the topological terms correspond to an electric coupling of background Ramond-Ramond fields to the gauge field. These are not the Wess-Zumino terms, determined on consistency grounds for T-duality, that are added to the D-brane action.
Rather, though they have a similar form, they are induced by the world volume theory itself.

We would have liked to give a more detailed description of these topological objects in terms of branes (which is where our motivation comes from) but, unfortunately, we have not succeeded in finding a completely satisfactory description. Instead we have to content ourselves with the above facts and that these are of an interesting structure purely in field theory terms. On that note, we show how these may arise starting with some Weyl fermion theory in arbitrary dimension $D = 4m + 2$.

One comment about the notation. All the theories under consideration arise from the dimensional reduction of a theory in $D$ dimensions involving a Weyl fermion\(^2\). Throughout we will make use of the $D$ dimensional gamma matrix algebra and, since the spinors are chiral, we include a projection $(1 + \Gamma_{D+1})/2$ in gamma matrix traces. For supersymmetric theories for the vector multiplets we insert $(1 + \Gamma_{D+1})/2$ and a projection $(1 - \Gamma_{D+1})/2$ for the hyper-multiplets. For the $N = 1$ theory in $D = 10$ as the spinors are also Majorana we need to divide by a further factor of 2. Our conventions are that the Lie algebra valued fields appearing are real, the generators $T$ are hermitian so that

$$D_M = \partial_M - iA_M,$$

where $A_M = A_M T$, and $A_M^\dagger = A_M$. The metric in $D$ dimensions is $\eta_{MN} = \eta_{\mu\nu} \oplus -\delta_{ab}$, however, when the $D - d$ dimensional labels, $a, b, \ldots$, appear explicitly contraction will be with respect to $\delta_{ab}$. This means that $\Gamma_M = (\Gamma_\mu, \Gamma_a)$ and $\Gamma^M = (\Gamma^\mu, -\Gamma^a)$, where $\Gamma^a = \delta^{ab} \Gamma_b$.

2 Reduction From Six Dimensions

The $N = 1$ theory in six dimensions is composed of a vector $A_M$ and a Weyl spinor $\Psi$. This theory has an $SU(2)_R$ $R$-symmetry group that acts only on the fermion. When we reduce to $d$ dimensions that Lorentz group decomposes as $SO(5,1) \supset SO(d-1,1) \otimes SO(6-d)$. The vector decomposes as $A_M = A_\mu \oplus \phi_a$, with $\mu = 0, \ldots, d-1$ and $a = 1, \ldots, 6-d$. The $R$ symmetry is enhanced to $SO(6-d) \otimes SO(3)_R$ but which now also acts on the scalars $\phi$. The theories that we are considering here have an interpretation as configurations of various intersecting branes.

\(^2\)When $D$ is odd we will see that we do not find the terms we are looking for.
2.1 The Calculation

The variation of the one loop effective action, with $N_f = 0$, with respect to the gauge field is

$$i\delta \Gamma_{\text{1-loop}} = -i \int d^d x \ Tr \ ad(\delta A) < x | \frac{1}{\Gamma^M D_M} | x > .$$

(2.1)

We need to make use of the Lichnerowicz formula for the square of the Dirac operator in six dimensions,

$$\mathcal{D}^2 = D^M D_M - \frac{i}{2} \Gamma^M \Gamma^N \ ad(F_{MN})$$

$$= D^M D_M - \frac{i}{2} \Gamma^\mu \Gamma^\nu \ ad(F_{\mu \nu}) + i\Gamma^\mu \Gamma^a \partial_\mu \ ad(\phi_a) ,$$

(2.2)

where

$$D^M D_M = D^\mu D_\mu + \ ad(\phi_a) \ ad(\phi^a) .$$

(2.3)

Since every field appearing is taken to live in the Cartan sub-algebra the commutators $[\phi, \phi]$ and $[A_\mu, \phi]$ are zero. We wish to pick out of

$$i\delta \Gamma_{\text{1-loop}} = -i \int d^d x \ Tr \ ad(\delta A) . < x | \frac{\mathcal{D}}{\mathcal{D}^2} | x > ,$$

(2.4)

the term proportional to the epsilon symbol and not involving the metric. It should be clear from (2.3) that, in this context, we can safely replace $D^\mu D_\mu$ with $\partial^\mu \partial_\mu$. To obtain an epsilon symbol we should exactly saturate the number of $\Gamma$ matrices that appear in $\Gamma_7$. From the numerator $A$ cannot contribute since this would mean that the gauge field would appear twice without derivatives and such a term would vanish when contracted with the epsilon symbol. Likewise, the $\partial^\mu$ term in the numerator cannot contribute since, by an integration by parts, this would lead to a contribution where the gauge field always appears in the form of a field strength and a glance at the first table of (1.3) shows us that we are in search of terms that involve the gauge field without derivatives. Consequently, we may as well focus on

$$i\delta \Gamma_{\text{1-loop}} = \int d^d x \ Tr \ ad(\delta A) . \ ad(\Gamma^a \phi_a) . < x | \frac{1}{\mathcal{D}^2} | x > + . . . .$$

(2.5)

$^3$The trace here can extend over the entire Lie algebra as the Cartan components will not contribute to (2.1) in any case.
One must now expand the denominator so as to pick up a term with the field strength raised to the \(d - 3\) and \(\partial_\mu \phi_a\) raised to the power of \(5 - d\). A further simplification is that one does not need to worry about operator ordering since any error introduced will be of a higher order in derivatives or/and will involve the metric. Hence,

\[
\frac{1}{p^2} = \sum_{n=0}^{\infty} \frac{i^n \left( \frac{1}{2} \Gamma^\mu \Gamma^\nu \text{ad} (F_{\mu\nu}) - \Gamma^\alpha \partial_\alpha \text{ad} (\phi_a) \right)^n}{\left( \partial^2 + \text{ad} (\phi)^2 \right)^{n+1}}
\]

\[
= \ldots + (-1)^{d/2} \frac{(\Gamma^\mu \Gamma^\nu \text{ad} (F_{\mu\nu}))^d}{2^{d-3} (5 - d)! (d - 3)!} \frac{(\Gamma^\alpha \partial_\alpha \text{ad} (\phi_a))^{5-d}}{\left( \partial_{\phi}^2 \right)^3} + \ldots ,
\]

(2.6)

where

\[
\partial_{\phi}^2 = \partial^2 + (\text{ad}(\phi))^2 .
\]

(2.7)

The error in passing the \(F\) terms through the \(d\phi\) involves anti-commutators of gamma matrices and so metric pieces which we can safely ignore. The error in passing these operators through the Greens functions \(\partial_{\phi}^{-2}\) involves higher order derivatives which, to the order that we are working, can also be ignored.

We see from (2.6) that as \(d\) increases we exchange the field strength \(F_{\mu\nu}\) for the derivative of the scalar \(\partial \phi\). What is being held fixed is, from the six dimensional point of view, the term in the Taylor series proportional to \((\Gamma^M \Gamma^N F_{MN})^2\).

The only integral that needs to be evaluated is

\[
<x| \frac{1}{\partial_{\phi}^2} |x> = - \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - (\text{ad}(\phi))^2} = \frac{i}{2} \frac{\Gamma(3 - \frac{d}{2})}{(2\sqrt{\pi})^d} \frac{1}{|\text{ad}(\phi)|^{6-d}} .
\]

(2.8)

One may put the pieces together to determine the contribution to the effective action. We do this for the individual cases in the following subsections. In order to obtain the 1-loop effective action contribution of the hyper-multiplets one simply exchanges the ad representation in the formulae above with the appropriate one and multiplies by the ubiquitous \(-1\).
2.2 \( N = 4 \) SYM In \( d = 3 \)

Seiberg began the study of the Coulomb branch of the \( N = 4 \) theories in three dimensions [3]. He suggested, that for \( U(1) \) and \( SU(2) \) gauge groups that it would be possible to determine the metric on the moduli space by performing a one loop computation. This was re-iterated in a work by Seiberg and Witten [4], however, these authors determined the metric on the moduli space, for the two groups above, by counting zero modes in a monopole field. They found that the metric corresponds to that of a hyper Kähler manifold which, at infinity, is \( (\text{a quotient of}) \) the Lens space \( \mathbb{L}_{-4} \). This space, for \( SU(2) \), is the classical moduli space of BPS monopoles in the \( SU(2) \) theory.

For the metric on this space to be hyper-Kähler it is necessary that the coupling constant, and hence the metric on \( \mathbb{R}^3 \) is corrected to

\[
\frac{1}{e^2} \to \frac{1}{e^2} - \frac{s}{|x|}. \tag{2.9}
\]

That this is indeed the case has been established explicitly in [7]. The Coulomb branch of the \( SU(n) \) theories was investigated by Chalmers and Hanany [8] and they suggested that the moduli space corresponds to the centered moduli space of \( n \) monopoles. Once more the metric was determined by the use of the Callias index formula [9]. The index turns out not to depend on \( n \) and one finds once more that \( s_3 = -4 \). This is consistent with the perturbative formula that we have obtained above as we will now see.

The term, in the effective action, that we are after is\(^4\)

\[
i\Gamma_{1\text{-loop}} = -\sum_{\mathbf{R}} \frac{i s_3^R}{32\pi} \epsilon^{\mu\nu\lambda} \epsilon^{abc} \text{Tr}_{\mathbf{R}} \int d^3 x \ A_{\mu} \partial_\nu \hat{\phi}_a \partial_\lambda \hat{\phi}_b \hat{\phi}_c
\]

\[
= \sum_{\mathbf{R}} s_3^R \mathcal{I}_\mathbf{R}, \tag{2.10}
\]

where \( \mathbf{R} \) is the representation of the fermions. The perturbative calculation yields,

\[
i\delta\Gamma^{\text{adj}}_{1\text{-loop}} = \frac{i}{8\pi} \epsilon^{\lambda\mu\nu} \epsilon^{abc} \text{Tr} \int d^3 x \ \text{ad}(\delta A_\lambda) \ \text{ad}(\hat{\phi}_c) \ \partial_\nu \text{ad}(\hat{\phi}_a) \ \partial_\lambda \text{ad}(\hat{\phi}_b) + \ldots. \tag{2.11}
\]

\(^4\)The normalisation here is chosen so that \( s_3^{\text{adj}} = 1 \) corresponds to the degree one monopole bundle over the two sphere.
or
\[ i \Gamma^{\text{adj}}_{1-\text{loop}} = \frac{i}{8\pi} \epsilon^{\lambda\mu\nu} \epsilon_{abc} \text{Tr} \int d^3x \, \text{ad}(A_\lambda) \, \partial_\mu \text{ad}(\hat{\phi}_a) \, \partial_\nu \text{ad}(\hat{\phi}_b) \, \text{ad}(\hat{\phi}_c) + \ldots \] (2.12)

The theory with no hyper-multiplets has for its only non-zero contribution \( s_3^{\text{adj}} = -4 \). The hyper-multiplets all contribute \( s_R = 4 \). For \( SU(2) \) and a theory with \( N_f \) hyper-multiplets in the fundamental representation we find that there is a universal factor \( s_3 = N_f s_3^{\text{fund}}/2 + s_3^{\text{adj}} = 2N_f - 4 \) times the adjoint space factor \( I_{\text{adj}} \). This is in agreement with the index calculation for the number of zero modes in the presence of monopoles. The relative factor of 2 can be understood from the following considerations. One can scale \( \phi \) freely in (2.10) so the only thing we need to declare is the charge pre-factor of the gauge field. The charge in the adjoint representation is twice that in the fundamental. Looking forward a bit we see that this argument tells us that in four dimensions (where the gauge field appears twice) one obtains a factor of \((8 - 2N_f)\) and in five dimensions it is \((16 - 2N_f)\).

Apart from non-perturbative effects it is argued in [4] that one loop is exact. So we need look only at the term in the effective action that arises on integrating out the massive (and charged) fermions. For the theory with \( N_f = 0 \), this means that we are interested in (1.5) with \( \mathcal{D} \) acting on adjoint valued fermions.

Let us be explicit about the formula for the vector multiplet. Let \( \alpha \in \Delta^+ \) be the positive roots of the algebra. We have
\[
\frac{i}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} \text{Tr} \int d^3x \, \text{ad}(A_\mu) \, \partial_\nu \text{ad}(\hat{\phi}_a) \, \partial_\lambda \text{ad}(\hat{\phi}_b) \, \text{ad}(\hat{\phi}_c) \]
\[
= \frac{i}{4\pi} \epsilon^{\mu\nu\lambda} \epsilon_{abc} \sum_{\alpha \in \Delta^+} \int d^3x \, \alpha(A_\mu) \, \partial_\nu \alpha(\hat{\phi}_a) \, \partial_\lambda \alpha(\hat{\phi}_b) \, \alpha(\hat{\phi}_c), \quad (2.13)
\]
with
\[
\alpha(\hat{\phi}_a) = \frac{\alpha(\phi_a)}{\sqrt{\alpha(\phi_b)\alpha(\phi_c)}} \quad (2.14)
\]
The fields, \( \alpha(\hat{\phi}_i) \), that appear define \( S^2 \)'s in the moduli space. Notice that
\[
\sigma_\alpha = \frac{1}{8\pi} \epsilon^{abc} d\alpha(\hat{\phi}_a) \, d\alpha(\hat{\phi}_b) \, \alpha(\hat{\phi}_c) = (\sigma_\alpha)_{\mu\nu} \, dx^\mu \, dx^\nu \quad (2.15)
\]
is the unit volume form for the two sphere defined by $\alpha(\hat{\phi}_a)$.

There are exactly a positive number of roots of such spheres\(^5\), i.e. there are

$$\frac{1}{2}(\text{dim } G - \text{rank } G) \quad (2.16)$$

such spheres. For $SU(n)$ this is $n(n-1)/2$. These will play a role subsequently.

In three dimensions each vector is dual to a compact scalar. The moduli space is classically, before quotienting with the Weyl group, $(\mathbb{R}^3 \times S^1)^r$.

This is a $4r$ dimensional manifold with an $SO(3)$ action on it. It is naturally a hyper-Kähler manifold. The one-loop result that we have just calculated shows that the moduli space no longer has a product structure. Rather we saw that a number of spheres are appearing and the end result is that at “infinity” one finds not spheres times circles, but rather circle bundles over the spheres. To see this we sketch how the dualisation is to be performed.

One considers the effective action up to one loop order as a function of a ranks worth of arbitrary two forms $F^i$ and one adds Lagrange multipliers $\theta_i$ imposing the constraints that the $F^i$ are closed\(^6\). Explicitly the action is,

$$\int d^3 x \ g_{ij} \left( -\frac{1}{4} F^i_{\mu \nu} F^j_{\mu \nu} + \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^j \right) + \frac{i}{8\pi} \epsilon^{\mu \nu \lambda} F^i_{\mu \nu} \left( \partial_\lambda \theta_i - \beta_{ij}^a \partial_\lambda \hat{\phi}_a \right). \quad (2.17)$$

If one integrates out $\theta_i$ one finds the required constraint $dF^i = 0$ whose solution is taken to be $F^i = dA^i$. Once one substitutes this back into the action (2.17) one re-obtains the one loop corrected effective action. The $g_{ij}$ represent the coupling constants up to one loop, which we have not calculated, see [7], while the term involving $\beta_{ij}^a$ is, by definition,

$$\int d^3 x \ i \epsilon^{\mu \nu \lambda} F^i_{\mu \nu} \beta_{ij}^a \partial_\lambda \hat{\phi}_a = - \sum_{\mathbb{R}} is_3^R \epsilon^{\mu \nu \rho} \epsilon^{abc} \text{Tr} \int d^3 x \ A_\mu \partial_\nu \hat{\phi}_a \partial_\rho \hat{\phi}_b \hat{\phi} \quad (2.18)$$

Now, integrating out the fields $F^i$ give rise to a dual action, from which we obtain the metric. This is

$$\int d^3 x \ \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + 2g^{ij} \left( \partial_\lambda \theta_i - \beta_{ik}^a \partial_\lambda \hat{\phi}_a \right) \left( \partial^\lambda \theta_j - \beta_{jl}^b \partial^\lambda \hat{\phi}_b \right), \quad (2.19)$$

\(^5\)Since the negative roots just double the contribution of the positive roots we do not need to consider them independently.

\(^6\)The $\theta_i$ are also taken to be periodic, so that the harmonic parts of $F^i$ are integral.
where $g^{ij}$ is proportional to the inverse matrix of $g_{ij}$. From (2.19) we see that the product structure with the circles has been turned into a monopole bundle.

While the formulae above are quite general we would now like to make a brief comparison with the metric of [8]. These authors determined the metric for the $SU(n)$ moduli space. If one denotes the moduli space by $\mathcal{M}_{SU(n)}$, they show that $H_2(\mathcal{M}_{SU(n)}, \mathbb{Z}) = n(n-1)/2$ and, furthermore, they determined that $s_3^{adj} = -4$ by computing the Callias index in the presence of a monopole. The spheres that appear are the same two spheres that we see in the one-loop expression. Furthermore, the one loop computation also tells us that $s_3^{adj} = -4$. Put another way; one does not need to feed in information about the moduli space, rather the one-loop correction “knows” about the structure of $\mathcal{M}_{SU(n)}$.

2.3 $N = 2$ SYM In $d = 4$

The one loop contribution to the topological terms is

$$i\Gamma_{1\text{-loop}} = \sum_{R} \frac{s_4^{R}}{64\pi^2} \epsilon^{\mu\nu\lambda\rho} \epsilon^{ab} \text{Tr} \int d^4x \; \text{ad}(A_\mu) \partial_\nu \text{ad}(A_\lambda) \partial_\rho \text{ad}(\hat{\phi}_a) \text{ad}(\hat{\phi}_b) \text{ad}(\hat{\phi}_b) \text{ad}(\hat{\phi}_b) + \ldots .$$

(2.20)

From the perturbative calculation we have found that

$$i\delta\Gamma_{1\text{-loop}}^{adj} = \frac{i}{8\pi^2} \epsilon^{\mu\nu\rho\lambda} \epsilon^{ab} \text{Tr} \int d^4x \; \text{ad}(\delta A_\mu) \text{ad}(F_{\nu\rho}) \text{ad}(\partial_\lambda \hat{\phi}_a) \text{ad}(\hat{\phi}_b) + \ldots .$$

(2.21)

or

$$i\Gamma_{1\text{-loop}}^{adj} = \frac{i}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} \epsilon^{ab} \text{Tr} \int d^4x \; \text{ad}(A_\mu) \text{ad}(F_{\nu\rho}) \text{ad}(\partial_\lambda \hat{\phi}_a) \text{ad}(\hat{\phi}_b) + \ldots .$$

(2.22)

From this we see that $s_4^{adj} = 8$ and the hyper-multiplets give $s_4^{R} = -8$. Let us fix on $SU(2)$ and write $\hat{\phi}_a = (\cos \varphi, \sin \varphi)$ then we have

$$i\Gamma_{0+1\text{-loop}}^{adj} = \frac{i}{32\pi} \epsilon^{\mu\nu\lambda\rho} \int d^4x \; \left( \frac{\theta}{2\pi} - \frac{2\varphi}{\pi} \right) F_{\mu\nu} F_{\lambda\rho} + \ldots .$$

(2.23)

This agrees with the exact perturbative result found long ago [10, 11, 12]. To see this we recall that the one loop correction to the pre-potential, in the notation of [13], is

$$\mathcal{F}_{1\text{-loop}} = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}.$$

(2.24)
Consequently the one loop correction to $\theta/2\pi$ is

$$\text{Re} \frac{\partial^2 F_{1\text{-loop}}}{\partial a^2} = \text{Re} \frac{2i}{\pi} \ln \frac{a}{\Lambda}, \quad (2.25)$$

and, as $a \sim e^{i\varphi}$, one sees that this agrees with (2.23). While here we have only determined the one loop correction to the theta term, using supersymmetry and holomorphy one may deduce the complete form of the one-loop corrected pre-potential.

For a general group we may re-write (2.22) as

$$i \Gamma_{1\text{-loop}} = -\frac{i}{16\pi^2} \varepsilon^{\mu\nu\rho\lambda} \sum_{\alpha \in \Delta^+} \int d^4x \alpha (F_{\mu\nu}) \alpha (F_{\rho\lambda}) \varphi_{\alpha}, \quad (2.26)$$

where $\alpha(\hat{\phi}_a) = (\cos \varphi_a, \sin \varphi_a)$. From this we may deduce that

$$\text{Re} \frac{\partial^2 F_{1\text{-loop}}}{\partial a^i \partial a^j} = -\frac{2}{\pi} \sum_{\alpha \in \Delta^+} \alpha_i \alpha_j \varphi_{\alpha}, \quad (2.27)$$

or that

$$F_{1\text{-loop}} = \frac{i}{2\pi} \sum_{\alpha \in \Delta^+} \alpha(a)^2 \ln \frac{\alpha(a)^2}{\Lambda^2}. \quad (2.28)$$

Once more we have not at all controlled non-perturbative effects.

### 2.4 SYM in $d = 5$

Seiberg [2] initiated a study of non-renormalizable field theories in five (and six) dimensions (see also [14, 15]). As they are non-renormalizable these theories must be understood to come with some regularization, say a cutoff or dimensional regularization. The term that we are interested in is, in any case, finite so we do not really have to specify which except to say that it is a gauge invariant regularization. The “topological" term that can be generated in five dimensions is of the form

$$i \Gamma_{1\text{-loop}} =$$

$$-\sum_{R} \frac{ig^R}{48\pi^2} \varepsilon^{\mu\nu\lambda\rho\sigma} \text{Tr}_R \int d^5x \text{ad}(A_{\mu}) \text{ad}(\partial_{\nu}A_{\lambda}) \text{ad}(\partial_{\rho}A_{\sigma}) \text{sign(ad(\phi))} + \ldots. \quad (2.29)$$
In the case of $U(1)$, Witten [16] has performed the calculation. The result is $s_5(U(1)) = -N_f$. For a non-Abelian group $G$ with $N_f = 0$ we have

$$s_5^{\text{adj}}(G) = 1.$$  \hspace{1cm} (2.30)

Some remarks are in order. As explained in [16] the sign of the mass term in the $U(1)$ theory is observable. Hence, different signs correspond to different theories. On the other-hand, in the non-Abelian theory (without $U(1)$ factors) that sign is not observable. Rather, $\text{sign}(\text{ad}(\phi))$ appears in (2.29) as a consequence of gauge invariance. After restricting to the Cartan sub-algebra the Weyl group acts by permutation and (2.29) is indeed Weyl invariant. As an example, for $SU(2)$ the Weyl group acts by sending the Cartan generator to minus itself so that $A_\mu \rightarrow -A_\mu$ and $\Gamma_{1-\text{loop}}$ is invariant since $\text{sign}(\text{ad}(\phi))$ also changes sign. Now since the theory is Weyl invariant we can focus on the moduli space $\mathbf{R}^{\text{rank}}/\text{Weyl}$, which in the case of $SU(2)$ is $\mathbf{R}^+$. 

3 Reduction From Ten Dimensions

The $N = 1$ theory in ten dimensions is made up of a vector $A_M$ and a Majorana-Weyl spinor $\Psi$ and consequently there is no $R$ symmetry here. On reduction to $d$ dimensions the vector decomposes as $A_M = A_\mu \oplus \phi_a$ where $\mu = 0, \ldots, d-1$ and $a = 1 \ldots 10 - d$. The Lorentz group decomposes as $SO(9,1) \supset SO(d-1,1) \otimes SO(10-d)$ and $SO(10-d)$ is the $R$ symmetry group acting both on the spinor and on the scalars.

The interpretation of these theories in terms of branes is quite direct. The $U(n)$ theory in $d$ dimensions arises as $n$ $d-1$ branes come together. The broken theory, where the scalars have non-zero expectation values, correspond to the branes being separated. If one ignores the center of mass motion the theory of $n$ separated branes has a world volume $SU(n)$ gauge symmetry. Of course in what follows any gauge group may be chosen.

3.1 The Calculation in $D$ dimensions

We will perform the calculation in a little more generality than is required in this section as it will prove useful later. The dimension $D$ is taken to be even but otherwise arbitrary. Nevertheless, the calculation is almost identical to
the one we performed before so we can be brief about it. As before

\[ i\delta \Gamma_{1\text{-loop}}^{\text{adj}} = \text{Tr} \int d^d x \text{ad}(\delta A) \text{ad}(\Gamma^a \phi_a) <x| \frac{1}{p^2} |x>, \quad (3.1) \]

and one expands \( \mathcal{D}^{-2} \) out to the following order, once again ignoring questions of ordering,

\[ \frac{1}{p^2} = \ldots + \frac{i^{D/2-1}}{2} \Gamma^\mu \Gamma^\nu F_{\mu\nu} - \Gamma^\mu \Gamma^a \text{ad}(\partial_\mu \phi_a) \right)^{D/2-1} + \ldots \]

\[ = \ldots + \left( \frac{D/2 - 1}{d - D/2} \right) \frac{(\partial^2_\phi)^{D/2}}{(\partial^2_\phi)^{D/2}} \frac{(i^{D/2-1})^{D/2-1}}{(\partial^2_\phi)^{D/2}} \frac{1}{(2\sqrt{\pi})^d} |\text{ad}(\phi)|^{D-d}. \quad (3.2) \]

The integral that needs to be performed is

\[ <x| \frac{1}{(\partial^2_\phi)^{D/2}} |x> = (-1)^{D/2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - (\text{ad}(\phi))^2)^{D/2}} \]

\[ = i \frac{i}{\Gamma(D/2)} \frac{\Gamma(D - d/2)}{(2\sqrt{\pi})^d} \frac{1}{|\text{ad}(\phi)|^{D-d}}. \quad (3.3) \]

Chirality is taken to be defined with respect to the projector \((1 + \Gamma_{D+1})/2\) where,

\[ \Gamma_{D+1} = (-i)^{(D/2-1)} \Gamma_0 \Gamma_1 \ldots \Gamma_{D-1}. \quad (3.4) \]

It is convenient also to let \( d = 2k \) if \( d \) is even and \( d = 2k + 1 \) if odd.

For a Weyl fermion in \( D \) dimensions the one loop effective action in \( d \) dimensions is

\[ \Gamma_{1\text{-loop}}^{\text{adj}}(D, d) = iC(D,d) \text{Tr} \int d^d x G(A)^{\rho_1 \ldots \rho_{D-d-1}} \sigma(\hat{\phi})^{\rho_1 \ldots \rho_{D-d-1}} \quad (3.5) \]

where

\[ G(A)^{\rho_1 \ldots \rho_{D-d-1}} = \epsilon^{\lambda \mu_1 \nu_1 \ldots \mu_{d-D/2} \nu_{d-D/2} \rho_1 \ldots \rho_{D-d-1}}, \]

\[ \text{ad}(A_\lambda) \text{ad}(F_{\mu_1 \nu_1}) \ldots \text{ad}(F_{\mu_{d-D/2} \nu_{d-D/2}}), \quad (3.6) \]

\[ \sigma(\hat{\phi})^{\rho_1 \ldots \rho_{D-d-1}} = \epsilon^{\rho_1 \ldots \rho_{D-d-1} d} \partial_{\rho_1} \text{ad}(\hat{\phi}^{\rho_1}) \ldots \partial_{\rho_{D-d-1}} \text{ad}(\hat{\phi}^{\rho_{D-d-1}}) \text{ad}(\hat{\phi}^d). \quad (3.7) \]
and the co-efficient $C_{(D,d)}$ is
\[
C_{(D,d)} = \frac{(-1)^{k-1}D-d-1}{(d+1-D/2)} \left( \frac{D/2 - 1}{d-D/2} \right) \frac{\Gamma(D/2 - d/2)}{(2\sqrt{\pi})^d\Gamma(D/2)}. \tag{3.8}
\]

With $D = 6$ the results of the previous section are reproduced.

### 3.2 Interpretation

Now we set $D = 10$ to arrive at the one loop effective action in $d$ dimensions
\[
\Gamma_{1-\text{loop}}^{\text{adj}} = iC_d \text{Tr} \int d^d x G(A) \rho_1 \ldots \rho_{9-d} \sigma(\hat{\phi}) \rho_1 \ldots \rho_{9-d} \tag{3.9}
\]
where the co-efficient $C_d$ is
\[
C_d = \frac{1}{2} C_{(10,d)} \tag{3.10}
\]

since the $N = 1$ fermions in 10 dimensions are also Majorana. For $SU(2)$, as we discussed before, the relative contribution of the adjoint representation to the fundamental goes like $(2^d - 3 - 2N_f)$. In order to give a possible interpretation of the result we recall some facts about $p$-branes. To a single $p$-brane (of $D$ dimensional super-gravity) one can associate an electric charge, $Q_E$, with respect to a background gauge field, $C_D^{p+1}$. For a $p$-brane one has
\[
Q_E = \int_{S^{D-p-2}} \ast dC_D^{p+1}. \tag{3.11}
\]

At large transverse distance, $r$, to the brane the form behaves as
\[
C_D^{p+1} \sim Q_E r^{p+3-D} \omega_{p+1}, \tag{3.12}
\]
where $\omega_{p+1}$ is the volume form on the brane. Hence,
\[
\ast dC_D^{p+1} = Q_E \sigma^{D-p-2}. \tag{3.13}
\]

Now consider the situation of two well separated $(d-1)$ branes. We recall that the expectation values of the scalar fields, on dimensional reduction, measure the separation of the branes. They represent, therefore, directions normal to the branes. Consequently the unit spheres, with volume forms $\sigma^p$,
are orthogonal to the branes. Up to normalization, the one loop effective action has the form

\[ \Gamma_{\text{1-loop}} \sim \int A F^{d-D/2} \sigma^{D-d-1} \]
\[ \sim \int A F^{d-D/2} \ast d C^D_d \]
\[ \sim \int F^{d+1-D/2} \tilde{C}^{D-2-d}_D, \quad (3.14) \]

where \( \tilde{C}^{D-2-d}_D \) is the magnetic dual of \( C^d_D \), \( \ast d C^d_D = d \tilde{C}^{D-2-d}_D \), and the volume forms are now understood to be the pull backs to the brane. So we have found a coupling between the gauge field on the brane and a ‘relative’ Ramond-Ramond background gauge field \( \tilde{C}^{D-2-d}_D(\hat{\phi})_D \). When there are \( n \) well separated branes, the one loop effective action will involve couplings between the \( n-1 \), \( U(1) \) gauge fields and the \( n(n-1)/2 \) relative Ramond-Ramond background gauge fields \( \tilde{C}^{D-2-d}_D \).

While we have not proven that this picture comes out of string theory, we note that the numerology at least comes out right. The type II A 4-brane carries a magnetic charge under the three-form \( C \), so for separated 4-branes we can expect a relative three form background field. The type IIA 6-brane is, likewise, magnetically charged under the type IIA one-form potential. The type IIB 5-brane has a magnetic charge under the Ramond-Ramond two form.

Notice that we may also dualise the five dimensional ‘photon’. Ignoring all other quantum effects one obtains a metric on the moduli space, at infinity, of the form

\[ (d\theta - \beta)^2, \quad (3.15) \]

where \( \theta \) is a compact two-form, dual to the photon, and \( d\beta = \sigma_4 \).

4 Concluding Remarks

Throughout the text we have concentrated on the reduction of supersymmetric gauge theories. However, one may also consider non-supersymmetric theories even though these may not make sense as full fledged field theories for \( d > 4 \). The conditions that we saw in the introduction still need to be
met, namely that

\[ \alpha + 2\gamma + \beta = d \]
\[ \beta = D - d - 1, \quad (4.1) \]

but we must augment these with the condition that the action be Weyl group even (indeed more generally Weyl invariant). The latter implies that we always have an even number of fields appearing in the effective action,

\[ \alpha + \gamma + \beta + 1 = 2m \quad (4.2) \]

for some \( m \). Let us take \( D \) to be even, which implies that \( \alpha = 1 \). Then manipulation of (4.1) yields \( D = 4m + 2 \). So up to 10 dimensions the only one we overlooked in the text was \( D = 2 \) with \( d = 1 \). The generic formulae for Weyl fermions were given in the preceding section (3.5). What of \( D \) odd? In such cases the fermions are not Weyl, however, the trace of \( D \) gamma matrices will give the sought for epsilon symbols. But, from the way the calculation proceeds, as in (2.4), the trace is always over an even number of gamma matrices and so the topological terms never arise.

With a small variation on the theme one may recover the conventional Chern-Simons actions. For example, in \( d = 3 \) (\( D = 4 \)) such a term was precluded by Weyl reflections. But suppose that the scalar field is not in the adjoint of the Lie algebra but rather is proportional to the identity. In this case \( \phi \) would be a mass term, \( \sigma^0 \) would not be Weyl odd and the Chern-Simons action would survive the Weyl reflection. The co-efficient that one gets, \( C_{(4,3)} = i/16\pi \) agrees with that found for the \( U(1) \) theory in [1]. A similar story holds for other \( d = 4m - 1 \) (\( D = 4m \)).

The reductions that we have been considering have all been spatial. To obtain Euclidean field theories it makes sense to perform a reduction in the time direction [17]. Such reductions are especially useful in understanding how topological field theories arise [18, 19, 20, 21, 17, 22]. The fundamental difference with the spatial reduction is that now the scalars, \( \phi^a \), transform as a vector of \( SO(D - d - 1, 1) \). The normalised scalars \( \hat{\phi} \) do not define the sphere, \( S^{D-d-1} \subset \mathbb{R}^{D-d} \), but rather the hyperboloid, \( H^{D-d-1} \subset \mathbb{R}^{(D-d-1,1)} \).

More generally, one may also ask what happens when one wraps D-branes around non-trivial cycles of some Calabi-Yau manifold? This has been answered in [18]; some of the scalars become sections of the normal bundle to the cycle in the Calabi-Yau manifold. For us, this means that when computing one loop corrections we should further decompose the 10 dimensional
epsilon symbol (corresponding to the decomposition of the Lorentz group). The sections may be treated in the same manner as the gauge fields so that they do not gain vevs. Consequently, new one loop topological contributions can arise in these situations as well.

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