Non-relativistic dynamics of the amplitude (Higgs) mode in superconductors

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(Dated: April 10, 2015)

Despite the formal analogy with the Higgs particle, the amplitude fluctuations of the order parameter in weakly-coupled superconductors do not identify a real mode with a Lorentz-invariant dynamics. Indeed, its resonance occurs at $2\Delta_0$, which coincides with the threshold $2E_{gap}$ for quasiparticle excitations, that spoil any relativistic dynamics. Here we investigate the fate of the Higgs mode in the unconventional case where $2E_{gap}$ becomes larger than $2\Delta_0$, as due to strong coupling or strong disorder. We show that also in this situation the amplitude fluctuations never identify a real mode at $2\Delta_0$, since such “bosonic” limit is always reached via a strong mixing with the phase fluctuations, which dominate the low-energy part of the spectrum. Our results have direct implications for the interpretation of the sub-gap optical absorption in disordered superconductors.

PACS numbers: 74.20.-z, 74.25.Gz, 74.62.En

In the BCS theory of superconductivity the formation of Cooper pairs and their condensation into the superfluid state are both consequences of the spontaneous breaking of the $U(1)$ gauge symmetry at $T_c[1]$. While the equilibrium value of the (complex) order parameter $\Delta_0$ is responsible for a gap in the single-particle excitation spectrum, the fluctuations of its phase represent the massless Goldstone mode that couples to the e.m. field leading to the Meissner effect. In addition, also energetically-costly collective amplitude fluctuations are possible, often named[2, 3] "Higgs mode" in analogy with the massive particle emerging in the Standard model from a similar mechanism of spontaneous symmetry breaking[4].

However, such analogy holds only in part, since the dynamics of the superconducting (SC) Higgs mode is not described in general by a Lorentz-invariant (LI) relativistic theory. Indeed, in conventional weakly-coupled superconductors the Higgs resonance occurs exactly at the threshold $2\Delta_0$ for single-particle excitations, that fully control its dynamics leading to a non-relativistic strongly overdamped mode[2, 5], whose experimental signature emerge usually only in out-of-equilibrium spectroscopy[6–9]. The situation can be different e.g. for lattice bosons at commensurate fillings, where fermionic quasiparticle excitations are absent. In this case the system has been described[6–8, 10, 12, 15, 16] by a SC-like LI $O(2)$ model:

$$S = \int dt dx \frac{1}{2} \left[ -|\partial_t \psi|^2 + c^2 |\nabla \psi|^2 + \frac{m^2}{4\Delta_0^2} (|\psi|^2 - \Delta_0^2)^2 \right]$$

(1)

where $\psi = (\Delta_0 + \Delta)e^{i\theta}$ represents the amplitude ($\Delta$) and phase ($\theta$) fluctuations of the SC order parameter. By retaining in Eq. (1) only Gaussian terms one has:

$$\langle |\Delta(q)|^2 \rangle = \frac{1}{-\omega^2 + m^2 + c^2 q^2}, \langle |\theta(q)|^2 \rangle = \frac{\Delta_0^{-2}}{-\omega^2 + c^2 q^2},$$

(2)

i.e. both the phase (Bogoliubov) and amplitude (Higgs) modes appear as elementary excitations with a LI dispersion. As it has been discussed in Refs. [8, 15, 16], the decay processes into phase modes at higher orders do not overdamp the Higgs resonance, that has been indeed probed in cold atoms by properly shaking the optical lattice[17, 18]. Motivated by this result, it has been recently suggested[7, 8, 19] that the $O(2)$ model (1) is also the correct paradigm for “bosonic” superconductors, i.e. fermionic systems where the energetic cost to create single-particle excitations $2E_{gap}$ is larger than twice the SC order parameter $2\Delta_0$. Indeed, if amplitude fluctuations still identified a resonance at $2\Delta_0$, it would be undamped by quasiparticle excitations, turning into a sharp resonance. This can happen for example in a charge-density-wave (CDW) superconductor, as it has been recently pointed out in Ref. [5]. Apart from this peculiar case, the separation between $E_{gap}$ and $\Delta_0$ can occur in homogeneous superconductors when the SC coupling strength increases[20, 21]. Moreover, it can be induced even in conventional (weakly-coupled) superconductors by strong disorder, as it has been observed experimentally[22–27] in materials like e.g. InO$_x$ and NbN, and understood theoretically as the effect of the localization of bosonic pairs with large $E_{gap}[28–30]$. However, this does not automatically imply the existence of a sharp Higgs mode at $2\Delta_0$, as it has argued recently[19] in the interpretation of the sub-gap optical absorption found experimentally in films of disordered superconductors[19, 31].

In this Letter we show that within a realistic fermionic model for superconductors the amplitude fluctuations computed at RPA level never identify a sharp LI mode at $2\Delta_0$, even in the bosonic limit where $E_{gap} > \Delta_0$. In the homogenous lattice case at strong coupling the separation $E_{gap} > \Delta_0$ can only be reached away from half filling, by moving the chemical potential outside the band edge. This implies however a strong violation of the particle-hole symmetry, that unavoidably mixes the...
amplitude and phase fluctuations and leaves at low energy only the phase mode, which maps in the Bogoliubov
sound of bosons, as already noticed in the past[2, 32–35, 37]. An analogous mechanism occurs in the presence
of the interaction term in mean-field approximation the Green’s function in the singular function (4) one recovers
the expected sound-like dispersion for phase fluctuations, while the vanishing of the $M_{11}^{BCS} \equiv \langle |\Delta|^2 \rangle$ element identifies the pole of the amplitude fluctuations. As one can easily see from Eq. (6), as $\omega \rightarrow 2\Delta_0$ is

$$F(\omega) \simeq \frac{N_F}{2\Delta_0 \sqrt{4\Delta_0^2 - \omega^2}}; \quad \omega < 2\Delta_0$$

so that the real part of the inverse Higgs propagator vanishes as $\sqrt{4\Delta_0^2 - \omega^2}$ at the Higgs mass $m = 2\Delta_0$, which coincides here with the threshold for single-particle excitations. In addition, $\text{Im} F(\omega) \sim (\omega^2 - 4\Delta_0^2)^{-1/2}$ at $\omega > 2\Delta_0$, leading to the well-known[40–42] overdamping of the Higgs spectral function in the BCS limit.

The BCS result (7) shows that even when the particle-hole symmetry leads to a vanishing of the first-order time derivatives[2], i.e. the off-diagonal terms of Eq. (4) and (7), the LI dynamics of the Higgs mode is prevented by the dynamical contribution of quasiparticles, encoded in the singular function $F(\omega)$. On the other hand, in the unconventional situation where the quasiparticle continuum $2E_{gap}$ moves away from $2\Delta_0$ one expects that $F(2\Delta_0) \simeq \infty$. Thus, within the diagonal structure (7) for the collective modes one would conclude that the Higgs mode should recover a LI dynamics, as recently suggested[7, 8, 19]. However, in the limit $2E_{gap} > 2\Delta_0$ the dynamics of collective modes is no more described by Eq. (7). To show this let us first consider the homogeneous strong-coupling limit $U \gg t$, where the SC order parameter and chemical potential for the model (S1) read[9, 33, 39]:

$$\Delta_0 \simeq \frac{U}{2} \sqrt{n(2-n)}; \quad \mu \simeq -\frac{U}{2}(1-n).$$

At $n \neq 1$, as soon as the chemical potential goes below the band edge $E_{gap} = \sqrt{4\Delta_0^2 + (4\mu^2)}$ becomes larger than $\Delta_0$. However, in this limit the particle-hole symmetry is strongly violated, leading to a large $\chi_{p\Delta}$ mixing between the amplitude and phase sectors in Eq. (4). This can be easily understood from Eq. (5) in the limit $t/U \simeq 0$, where $|\mu| \gg \Delta_0$ and $\chi_{p\Delta} \simeq 4\Delta_0\mu F(\omega)$, with
FIG. 1: Intensity map of the spectral function of the amplitude and phase modes in the homogeneous case at $U = 8t$ for $n = 0.1$ (upper panels) and $n = 1$ (lower panels). The solid red lines mark the value of $2\Delta_0$ and the dashed black lines the value of $2E_{gap}$. At $n = 0.1$ the bosonic limit $E_{gap} > \Delta_0$ is reached, but the amplitude mode is strongly mixed with the phase, which dominates any sub-gap structure of the Higgs (with zero spectral weight at $\omega = 0$). At $n = 1$ the system is always particle-hole symmetric so the amplitude and phase sectors remain decoupled. However, here $E_{gap} = \Delta_0$ and the Higgs mode has a broad spectral weight only above $E_{gap}$. The bending back of the phase mode at $Q = (\pi, \pi)$ (where its weight is zero) is a signature of the degenerate CDW instability present in the model (S1) at $n = 1$.

\[ F(\omega) \simeq 1/(E(4E^2 - \omega^2)) \quad \text{and} \quad E = \sqrt{\Delta_0^2 + \mu^2} \simeq U/2, \text{see Eq. (9).} \]  

In this situation the determinant of the matrix (4) at $q = 0$ is given by

\[ |\hat{M}| = \frac{\omega^2}{4} \hat{\chi}_{pp} \left[ (4\Delta_0^2 - \omega^2) F(\omega) - \frac{\chi_{p\Delta}}{\chi_{pp}} \right] \simeq \frac{\omega^2}{4} \hat{\chi}_{pp} F(\omega) [4(\Delta_0^2 + \mu^2) - \omega^2] \simeq \frac{\omega^2}{U^2} \]  

As one can see, the coupling $\chi_{p\Delta}$ between the amplitude and phase sectors removes completely any signature at $2\Delta_0$ and only the (Bogoliubov-like[2, 32–35, 37]) sound mode at $\omega = 0$ is left. By retaining a finite $t$ value in the evaluation of all the response functions (5)-(6) one can show[39] that $|\hat{M}|$ has a minimum at $2E_{gap}$. Thus, the spectral function of the Higgs mode still preserves some small spectral weight above $E_{gap}$, that is however strongly suppressed with respect to the weak-coupling case. These analytical estimates are confirmed by the numerical computation of the amplitude and phase spectral functions[39] at finite $\omega$ and $q$, shown in the upper panels of Fig. 1. As one can see, any sub-gap feature in the Higgs spectral function is present only at finite $q$ and it comes from the mixing to the phase, with no signature at the energy $2\Delta_0$, marked by the red line. On the other hand at half filling $n = 1$ one has $\mu = 0$ at all orders in $U$, so that it is always $E_{gap} = \Delta_0$. In addition, since the particle-hole symmetry is preserved, $\chi_{p\Delta} = 0$ and the amplitude and phase sectors remain decoupled, so that the Higgs mode shows only a weak spectral weight above $2E_{gap}$, see lower panels of Fig. 1. Notice that the absence of a low-energy signature at $n = 1$ in the amplitude sector at RPA level implies that even if some resonance emerged at higher order, as it happens e.g. in the half-filled Bose-Hubbard model very near the superfluid-insulator transition[7, 12, 15], it cannot be interpreted as an elementary (Gaussian) amplitude excitation.

FIG. 2: Spectral functions of the amplitude and phase modes at $q = 0$ for $n = 0.875$ obtained on a $20 \times 20$ lattice after average over 50 disorder configurations for two values of SC coupling ($U/t$) and disorder ($V_0/t$). The vertical dashed lines and green arrows mark the spectral gap $2E_{gap}$ and $2\Delta_0$, respectively, whose disorder dependence is shown in the insets. For comparison we also show $\sigma(\omega)$ (blue line) from Ref. [5].

A second possible route[29, 30] to achieve the separation between $E_{gap}$ and $\Delta_0$ in the model (S1) is by introducing disorder as a random on-site energy $V_i$ uniformly distributed in the interval $[-V_0, V_0]$. Indeed, already at the level of the mean-field inhomogeneous Bogoliubov-de-Gennes equations[29] one sees that while the average order parameter $\langle \Delta_0 \rangle$ decreases as $V_0$ increases, the spectral gap $2E_{gap}$, in the average density of states saturates to a finite value, see insets of Fig. 2, signalling the formation of local boson pairs[28–30]. However, once more this
does not imply that the Higgs mode emerges as a sharp resonance at \( \langle \Delta_0 \rangle \), but instead it acquires sub-gap spectral weight due to the mixing to the phase mode, that is induced in the disordered case even for weak SC coupling. In Fig. 2 we show the amplitude and phase spectral functions at \( q = 0 \) computed at RPA level\[39\] for two indicative values of SC coupling and disorder. As one can see, the Higgs mode acquires spectral weight below \( E_{\text{gap}} \) already for \( V_0/t = 0.5t \) (upper panels), where \( E_{\text{gap}} \) and \( \Delta_0 \) still coincide. For \( U/t = 5 \) this sub-gap feature closely follows the one seen in the phase sector, where it can be interpreted as a disorder-induced broadening of the sound mode, with a transfer of spectral weight from zero to finite frequency. Since the mixing between the amplitude and phase modes vanishes as \( \omega \to 0 \), see Eq. (4), this feature appears as a finite-energy maximum in the amplitude sector. At larger disorder \( V_0/t = 0.3 \) (lower panels) the mixing of the amplitude with the phase is stronger at all energies, and also in this case the finite spectral weight of the Higgs mode below \( 2E_{\text{gap}} \) can be hardly interpreted as a well-defined resonance at the typical scale \( \langle \Delta_0 \rangle \).

![Image](image_url)

**FIG. 3:** Dependence on the SC coupling strength \( U/t \) of the ratio \( \tilde{D}_s/D_s \) appearing in the definition (S22) of the current for the homogeneous model (S1) at various densities.

Let us finally comment on the relevance of these results for the experimental observation in disordered films of superconductors\[19, 31\] of a finite optical conductivity \( \sigma(\omega) \) below \( 2E_{\text{gap}} \), where quasiparticle excitations cannot contribute. One natural candidate are phase modes\[5, 44, 46\], since in the presence of disorder already one-phason processes lead to a finite-frequency optical response. In particular, in the disordered Hubbard model discussed above it has been shown\[5\] that all the processes exciting a single (amplitude, phase and charge) collective mode are optically active in the presence of disorder, but the largest sub-gap contribution comes from phase fluctuations. On the other hand, due to the non-trivial structure of the effective optical dipole, there is no simple correspondence between the spectral function of the phase mode and the optical conductivity, as one can see in Fig. 2 where we report for comparison \( \sigma(\omega) \) from Ref. [5]. An alternative interpretation\[19\] of the experiments invokes instead the relevance of higher-order processes, allowed already in the clean case. More specifically, in the bosonic model (1) the current is given in terms of the collective modes as\[6–8, 47\]:

\[
\mathbf{J} = -2e \left( D_s \nabla \theta + 2\tilde{D}_s \eta \nabla \theta \right), \quad \eta = \Delta/\Delta_0. \tag{11}
\]

where \( \tilde{D}_s \equiv D_s = e^2 \Delta_0^2 \) in the model (1). The second term in Eq. (S22) is responsible for the optical process involving the excitation of one phason plus one Higgs mode. Within the bosonic model (1), where the Higgs mode (2) has a sharp resonance at \( \omega = m \), this process\[6–8\] leads to a finite-frequency optical absorption with an edge exactly at \( m \). However, this result hardly applies to disordered films to infer the presence of an absorption at \( 2\Delta_0 < 2E_{\text{gap}} \), since, as we have seen, for fermions the spectral function of the Higgs mode does not identify a sub-gap LI resonance at \( 2\Delta_0 \). In addition, the equivalent of \( \tilde{D}_s \) can be estimated\[39\] in the fermionic model (S1) as:

\[
\tilde{D}_s = \frac{\Delta_0^2}{2N} \sum_{k,\nu} \frac{\partial^2 \epsilon_k}{\partial k^2} \frac{\xi_k - U \chi_{\nu} \Delta \chi_{\nu \rho \rho}}{2 \chi_{\nu \rho \rho}} \frac{\Delta_0^2}{2N} \sum_{k,\nu} \frac{\partial^2 \epsilon_k}{\partial k^2} \frac{1}{E_{\nu}^3}.
\tag{12}
\]

Thus, while \( \tilde{D}_s \approx D_s \) can be recovered at strong coupling, as shown in Fig. 3, in the BCS limit one easily sees that \( \tilde{D}_s \approx 0 \) for particle-hole symmetry, with a strong suppression of the optical processes involving the Higgs mode. Thus, even though these processes can be present in disordered films, their relevance for the sub-gap absorption in comparison to the lowest-order processes investigated in Ref. [5] is not yet fully understood, and it cannot be inferred by the analogy with the results obtained within the clean bosonic model (1).

In summary, we studied the evolution of the amplitude (Higgs) mode in a lattice model for fermions. We showed that even when \( 2\Delta_0 \) goes below the threshold \( 2E_{\text{gap}} \) for quasiparticle excitations the Higgs mode never identifies a sharp resonance at \( 2\Delta_0 \). Indeed, the separation between \( E_{\text{gap}} \) and \( \Delta_0 \), needed to remove the dynamical overscreening of quasiparticles to the Higgs mode, is only achieved by an explicit breaking of the particle-hole symmetry, which implies a strong mixing of the amplitude and phase fluctuations. Thus, any sub-gap feature in the Higgs mode does not correlate directly with the scale \( \Delta_0 \), but is a signature of the underlying phase mode. These results establish that the outcomes of the LI bosonic model (1) do not describe in general the physics of fermionic superconductors even in the "bosonic" limit, as it has been sometimes suggested\[7, 8, 19\]. On the other hand, the relativistic nature of the Higgs mode can be recovered in the BCS limit when a CDW gap contributes to \( E_{\text{gap}} \), as shown in Ref. [5]. An interesting open question is the possibility that a similar mechanism can hold also in systems like cuprate superconductors, where increasing experimental evidence\[48–51\] has been accumulating for the presence of a competing CDW order.
We acknowledge useful discussions with J. Lorenzana. This work has been supported by Italian MIUR under projects FIRB-HybridNanoDev-RBFR1236VV, PRINRIDEIRON-2012X3YFZ2 and Premiali-2012 AB-NANOTECH, and by the Deutsche Forschungsgemeinschaft under SE806/15-1.

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Supplemental Material

**EFFECTIVE ACTION FOR THE COLLECTIVE MODES: GENERAL FORMALISM**

Let us start from Eq. (3) of the manuscript, that we report here:

\[ H = -t \sum_{\langle i,j \rangle} (c_{i\sigma}^\dagger c_{j\sigma} + h.c.) - U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow} c_{i\downarrow}. \] (S1)

To investigate the physics of the collective modes around the mean field solution we follow the usual Hubbard-Stratonovich procedure\([S1]\), as implemented e.g. in Refs.\([S2, S3]\). We then introduce in the action for the fermions a bosonic complex field \(\psi_\Delta(\tau)\) which decouples the on-site interaction term of (S1) in the pairing channel. At \(T < T_c\) one can choose to represent the superconducting \((SC)\) fluctuations both in polar (amplitude and phase) or cartesian (real and imaginary parts) coordinates. In the former case the additional use of a Gauge transformation on the fermionic operators makes the dependence of the effective action on the time and spatial derivatives of the SC phase explicit, and it is then more convenient in the long-wavelength limit. The equivalence between the two approaches is guaranteed by the Ward identities, as discussed e.g. in Ref.\([S4]\). We will then decompose \(\psi_\Delta(\tau) = [\Delta_0 + \Delta_i(\tau)]e^{i\theta_i(\tau)}\), where \(\Delta_i(\tau)\) represent the amplitude fluctuations of \(\psi_\Delta\) around the mean-field value \(\Delta_0\) and \(\theta\) its phase fluctuations, which appears explicitly in the action after a Gauge transformation \(c_i \to c_i e^{i\phi_i/2}\). The interaction term of Eq. (S1) can also be decoupled\([S2, S3]\) in the particle-hole channel by means of a second (real) bosonic field \(\psi_\rho = \rho_0 + \rho\) which couples to the electronic density \(\Phi_{\rho,i} = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}\) and represents the density fluctuations \(\rho\) of the system around the mean-field value \(\rho_0\).

After the Hubbard-Stratonovich decoupling the action is quadratic in the fermionic fields that can then be integrated out leading to the effective action for the fields \(\Delta, \theta\) and \(\rho\):

\[ S_{eff}[\Delta, \theta, \rho] = S_{MF} + S_{FL}[\Delta, \theta, \rho], \] (S2)

where \(S_{MF} = \frac{N\Delta_0^2}{2} + \frac{N\rho^2}{2} - \text{Tr} \ln(-G^{-1}_0)\) is the mean field action, \(G^{-1}_0 = i\omega_n\sigma_0 - \xi_k\sigma_3 + \Delta_0\sigma_1\) is the BCS Green’s function and

\[ S_{FL} = \sum_{n \geq 1} \frac{\text{Tr}(G_0 \Sigma)^n}{n} \] (S3)

is the fluctuating one, with the trace acting both in spin and momentum space, where \(\Sigma_{kk'}\) denotes the self-energy for the fluctuating fields, which reads explicitly:

\[ \Sigma_{kk'} = -\frac{T}{N}\Delta(k - k')\sigma_1 - \frac{T}{N}\rho(k - k')\sigma_3 - \frac{T i}{N}\theta(k - k') \left[(k - k')_0\sigma_3 - (\xi_k - \xi_{k'})\sigma_0\right] - \frac{T}{2N} \sum_{q_1, q_2} \theta(q_1)\theta(q_2) \frac{\partial^2 \xi_k}{\partial k'^2} \sin \frac{q_1.\nu}{2} \sin \frac{q_2.\nu}{2} \sigma_3 \delta(q_1 + q_2 - k + k') + O(\theta^3), \] (S4)

with \(k = (i\Omega_n, k)\) and \(\Omega_n = 2\pi T n\) bosonic Matsubara frequencies. Notice that the last line of Eq. (S4) represents the transcription on the lattice of the usual \((\nabla\theta)^2\) term for a continuum model. In addition, in contrast to the continuum model, the lattice self-energy (S4) depends in principle\([S2, S3]\) on all higher-order powers of the \(\theta\) field, which are however irrelevant for the derivation of the Gaussian action.

To derive the Gaussian action for SC fluctuations we should retain the terms up to \(n = 2\) in Eq. (S3). The terms coming from an insertion of the \(\sigma_0\) term of Eq. (S4) describe the effects of a paramagnetic current, so that they lead for example to the depletion of the superfluid stiffness at finite temperature due to quasiparticle excitations. On the other hand, mixed terms containing a \(\sigma_0\) times a \(\sigma_1, \sigma_3\) matrix give higher-order contribution in \(q\). Thus, since we are interested in the \(T = 0\) and long-wavelength limit, in the clean case we can safely neglect these terms. With lengthy but straightforward calculations one can then show that at gaussian level \(S_{FL}\) can be written as:

\[ S_{FL}[\Delta, \theta, \rho] = \frac{1}{2} \sum_q \Psi^\dagger(q) M_{FL}(q) \Psi(q), \] (S5)
with $\Psi^T(q) = (\Delta(q) \ \theta(q) \ \rho(q))$ and:

$$M_{FL} = \begin{pmatrix} 2/U + \chi_{\Delta\Delta}(q) & -\frac{i\omega}{2} \chi_{\rho\Delta}(q) + \frac{D_\omega}{U} w(q) & \chi_{\rho\Delta}(q) \\ -\frac{i\omega}{2} \chi_{\rho\Delta}(q) & 2/U + \chi_{\rho\rho}(q) & 0 \\ \frac{i\omega}{2} \chi_{\rho\rho}(q) & 0 & 2/U + \chi_{\rho\rho}(q) \end{pmatrix},$$

(S6)

Here $\chi_{ij}(q) \equiv \frac{T}{N_s} \sum_k \text{Tr} [G_0(k+q)\sigma_i G_0(k)\sigma_j]$ are the response functions computed at BCS level. Since in Eq. (S4) the insertion of a $\hat{\sigma}_1$ or $\hat{\sigma}_3$ Pauli matrix corresponds to a term proportional to the amplitude or to the density/phase fluctuations, respectively, we made the correspondence $1 \rightarrow \Delta, 3 \rightarrow \rho$ in the notation for the BCS susceptibilities that appear as coefficients of the action (S5)-(S6). They are explicitly given at $T = 0$ by:

$$\chi_{\Delta\Delta}(q) = \frac{1}{N_s} \sum_k \frac{E_k^+ + E_k^-}{E_k^+ E_k^-} \cdot \frac{E_k^+ E_k^- + \xi_k^+ \xi_k^- - \Delta_0^2}{(\omega + i\delta)^2 - (E_k^+ + E_k^-)^2},$$

(S7a)

$$\chi_{\rho\Delta}(q) = \frac{\Delta_0}{N_s} \sum_k \frac{E_k^+ + E_k^-}{E_k^+ E_k^-} \cdot \frac{\xi_k^+ + \xi_k^-}{(\omega + i\delta)^2 - (E_k^+ + E_k^-)^2},$$

(S7b)

$$\chi_{\rho\rho}(q) = \frac{1}{N_s} \sum_k \frac{E_k^+ + E_k^-}{E_k^+ E_k^-} \cdot \frac{E_k^+ E_k^- - \xi_k^+ \xi_k^- + \Delta_0^2}{(\omega + i\delta)^2 - (E_k^+ + E_k^-)^2},$$

(S7c)

with $E_k^\pm = E_k \pm q/2$ and $\xi_k^\pm = \xi_k \pm q/2$.

The quantity $\tilde{D}_s$ (see also Eq. (S24) below) is the phase stiffness, that appears as coefficient of the term $w(q) \equiv 4 \sum_{\nu} \sin^2(q_\nu/2) \simeq q^2$, so that in the hydrodynamic limit the inverse of the bare phase-fluctuation propagator is: $4\tilde{D}_s^2(q) \approx -\kappa^2_0 + D_s q^2$, which defines the BCS sound velocity: $v_0^2 = \sqrt{\tilde{D}_s}/\kappa$, where $\kappa \equiv -\chi_{\rho\rho}(0) = \Delta_0^2 \sum_k E_k^{-3}$ is the bare compressibility. By means of the self-consistent equation for $\Delta_0$, i.e. $2/U = \sum_k 1/E_k$ one can also rewrite the $M_{\rho1}$ term as:

$$2/U + \chi_{\Delta\Delta}(\omega,0) = \frac{1}{N_s} \sum_k \frac{\omega^2 - 4\Delta_0^2}{\Delta_0^2} \left[ (\omega + i\delta)^2 - 4E_k^2 \right] = (4\Delta_0^2 - \omega^2)F(\omega),$$

(S8)

that corresponds to the expression used in Eq. (4) of the manuscript. To describe the fluctuations only in the SC sector we can integrate out the density fluctuations $\rho$ in Eq. (S6), so that the Gaussian action for SC fluctuations reads:

$$S_{FL}[\Delta, \theta] = \frac{1}{2} \sum_q \left( \Delta(-q) \theta(-q) \right) \hat{M}(q) \left( \Delta(q) \theta(q) \right),$$

(S9)

with $\hat{M}$ given by Eq. (4) of the manuscript, i.e.:

$$\hat{M}(q) = \begin{pmatrix} 2/U + \tilde{\chi}_{\Delta\Delta}(q) & -\frac{i\omega}{2} \tilde{\chi}_{\rho\Delta}(q) + \frac{D_\omega}{U} w(q) \\ -\frac{i\omega}{2} \tilde{\chi}_{\rho\Delta}(q) & 2/U + \tilde{\chi}_{\rho\rho}(q) \\ \frac{i\omega}{2} \tilde{\chi}_{\rho\rho}(q) & 0 & 2/U + \tilde{\chi}_{\rho\rho}(q) \end{pmatrix},$$

(S10)

The integration of the density field is equivalent as usual to the RPA dressing of the BCS susceptibilities. In particular, one has that $\tilde{\chi}_{ab} \equiv \chi_{ab} - \frac{\chi_{\rho\rho} \chi_{\Delta\Delta}}{\Delta_0^2}$.

**EXPANSION AROUND 2E_{\text{gap}} AT STRONG COUPLING**

Let us investigate in detail the amplitude fluctuations at $q = 0$. From Eq. (S10) we see that at $q = 0$ the RPA resummation of the bubbles in the density channel factorizes out, so that the spectral function of the Higgs $\rho_\Delta(\omega) \equiv \frac{1}{2} \text{Im}\{1/X_{\Delta\Delta}(\omega + i\delta, q = 0)\}$ is determined by the frequency behavior of the function:

$$X_{\Delta\Delta} \equiv \frac{2}{U} + \chi_{\Delta\Delta} - \frac{\chi_{\rho\Delta}^2}{\chi_{\rho\rho}}.$$

(L11)

Let us then investigate its behaviour from weak to strong coupling. As we discuss in the main text, at strong coupling one can reach the bosonic limit $E_{\text{gap}} \simeq U/2 \gg \Delta_0$ only away from half filling, by moving the chemical potential $\mu \approx -U(1 - n)/2$ below the band edge. In the following we will consider for instance the case where $\mu < 0$ goes below the band edge $-4t$ at strong coupling. Thus, by introducing $E_{\text{max}} = \sqrt{\Delta_0^2 + (4t + |\mu|)^2}$ and $E_{\text{min}} = \sqrt{\Delta_0^2 + (4t - |\mu|)^2}$ one has that at strong coupling $E_{\text{gap}} \approx E_{\text{min}}$. If we also approximate the momentum integration in the Eqs. (S7) with an energy integration over a constant density of stated $N_F$ we can put:

$$\frac{2}{U} + \chi_{\Delta\Delta} \approx (4\Delta_0^2 - \omega^2)I_0(\omega),$$

(S12a)

$$\chi_{\rho\rho}(\omega) \approx 4\Delta_0^2I_0(\omega),$$

(S12b)

$$\chi_{\rho\rho}(\omega) \approx 4\Delta_0I_1(\omega),$$

(S12c)

where the real and imaginary parts of the $I_{0,1}$ functions are given explicitly by:
\[
\text{Re} I_0(\omega) = -\rho \int_{-\mu-4t}^{\mu+4t} \frac{d\xi N_F}{\sqrt{\Delta_0^2 + \xi^2 (\omega^2 - 4\Delta_0^2 - 4\xi^2)}} = \tag{S13a}
\]

\[
= N_F \Theta(2\Delta_0 - \omega) \frac{\pi}{\omega \sqrt{4\Delta_0^2 - \omega^2}} \left[ \arctan \left( \frac{(4t + |\mu|)\omega}{\sqrt{4\Delta_0^2 - \omega^2} E_{\text{max}}} \right) + \arctan \left( \frac{(4t - |\mu|)\omega}{\sqrt{4\Delta_0^2 - \omega^2} E_{\text{max}}} \right) \right] \tag{S13b}
\]

\[
+ \frac{N_F}{2\omega \sqrt{\omega^2 - 4\Delta_0^2}} \ln \left( \frac{(4t + |\mu|)\omega + E_{\text{max}} \sqrt{\omega^2 - 4\Delta_0^2}}{(4t + |\mu|)\omega - E_{\text{max}} \sqrt{\omega^2 - 4\Delta_0^2}} \right), \quad (4t - |\mu|)\omega + E_{\text{min}} \sqrt{\omega^2 - 4\Delta_0^2} \tag{S13c}
\]

\[
\text{Im} I_0(\omega) = \begin{cases} 
\Theta(\omega - 2\Delta_0) \frac{\pi}{\omega \sqrt{\omega^2 - 4\Delta_0^2}}, & |\mu| < 4t, \\
\Theta(\omega - 2E_{\text{min}}) \frac{\pi}{2\omega \sqrt{\omega^2 - 4\Delta_0^2}}, & |\mu| > 4t 
\end{cases} \tag{S13d}
\]

\[
\text{Re} I_1(\omega) = \rho \int_{-\mu-4t}^{\mu+4t} \frac{d\xi N_F \xi}{\sqrt{\Delta_0^2 + \xi^2 (\omega^2 - 4\Delta_0^2 - 4\xi^2)}} = \frac{N_F}{4\omega} \ln \left| \frac{\omega + 2E_{\text{max}}}{\omega - 2E_{\text{min}}} \right| \tag{S14a}
\]

\[
\text{Im} I_1(\omega) = -\Theta(\omega - 2E_{\text{min}}) \frac{\pi}{4\omega}, \quad |\mu| > 4t \tag{S14b}
\]

so that e.g. \( I_0 \) is an approximated expression for the function \( F(\omega) \) introduced in Eq. (S8) above.

In the weak-coupling regime the chemical potential lies in general well inside the band \(|\mu| \ll 4t\), so that \( E_{\text{max}} \approx E_{\text{min}} \) and the integral (S14a) is approximately zero. This leads to the well-known decoupling of the amplitude and phase sectors due to the (approximate) particle-hole symmetry of the BCS solution, already mentioned in the main text. In addition, as far as \(|\mu| < 4t\) the two arctan in Eq. (S13b) have the same sign as \( \omega \to 2\Delta_0 \), so that they give a constant contribution and \( \text{Re} I_0(\omega) \) diverges as \( 1/\sqrt{4\Delta_0^2 - \omega^2} \), leading to \( \text{Re} X_{\Delta \Delta}(\omega) \approx (\omega^2 - 4\Delta_0^2) I_0(\omega) \propto \sqrt{4\Delta_0^2 - \omega^2} \) at the Higgs pole. At the same time \( \text{Re} I_0 \) is finite as \( \omega \to 2\Delta_0 \) since the logarithmic term in Eq. (S13c) vanishes exactly as \( \sqrt{\omega^2 - 4\Delta_0^2} \), so that \( \text{Re} X_{\Delta \Delta}(\omega) \approx (\omega^2 - 4\Delta_0^2) \) as \( \omega > 2\Delta_0 \). On the other hand when \( \omega > 2\Delta_0 \) the imaginary part \( \text{Im} I_0(\omega) \) diverges, see Eq. (S13d), so that \( \text{Im} X_{\Delta \Delta} \) grows as \( \sqrt{\omega^2 - 4\Delta_0^2} \) as \( \omega > 2\Delta_0 \), leading to the typical over-damped resonance at \( 2\Delta_0 \) in the spectral function \( \rho_\Delta \), as shown in Fig. S1a. It is worth noting that even retaining the small but finite value of \( \text{Re} I_1(\omega) \) does not change these results. Indeed, as far as \( |\mu| \) lies inside the band the function \( I_1(\omega) \) (and then \( \chi_{\rho \Delta} \)) remains finite around \( \omega = 2\Delta_0 \), see Eq. (S1a), while \( \chi_{\rho \Delta} \) is proportional to \( I_0 \), leading to the same results discussed so far in the case of perfect particle-hole symmetry.

At strong coupling (and away from half filling) as soon as \(|\mu| > 4t\) the optical gap moves from \( 2\Delta_0 \) to the higher value \( 2E_{\text{gap}} \approx 2E_{\text{min}} \). In this case the two arctan in Eq. (S13b) have opposite sign as \( \omega \to 2\Delta_0 \), so that they give overall a contribution proportional to \( 1/\sqrt{4\Delta_0^2 - \omega^2} \) that removes the divergence of \( \text{Re} I_0(\omega = 2\Delta_0) \). At the same time \( \text{Im} I_0 \) is only finite above \( 2E_{\text{gap}} \), see Eq. (S13d), so that the amplitude fluctuations described at BCS level by the single (S12a) term would recover a perfect Lorentz-invariant (LI) dynamics, with a sharp spectral function at \( 2\Delta_0 \). However, as soon as \( \mu \) moves outside the band edge the particle-hole symmetry is strongly violated and the coupling to the phase, dictated by the \( \chi_{\rho \Delta} \propto I_1 \) function, becomes large, moving the spectral weight of the Higgs fluctuations away from \( 2\Delta_0 \) towards the new optical gap \( 2E_{\text{gap}} \). In particular one can see from Eqs. (S13c) and (S14a) that both \( \text{Re} I_0 \) and \( \text{Re} I_1 \) diverge logarithmically as \( \omega \to 2E_{\text{min}} \equiv 2E_{\text{gap}} \), so that:

\[
\text{Re} I_0(\omega) \approx -\frac{N_F}{8E_{\text{gap}}(|\mu| - 4t)} \ln \left| 1 - \frac{\omega^2}{4E_{\text{gap}}^2} \right| + K_0 ; \tag{S15}
\]

\[
\text{Re} I_1(\omega) \approx -\frac{N_F}{8E_{\text{gap}}} \ln \left| 1 - \frac{\omega^2}{4E_{\text{gap}}^2} \right| + K_1 . \tag{S16}
\]

where \( K_0 = \frac{1}{8E_{\text{gap}}(|\mu| - 4t)} \ln \left| \frac{\Delta_0^2}{4(\mu + 4t)^2} E_{\text{gap}} \right| \) and \( K_1 = \frac{1}{8E_{\text{gap}}} \ln \left| \frac{4E_{\text{max}} - E_{\text{gap}}}{E_{\text{max}} + E_{\text{gap}}} \right| \). According to Eqs. (S15)-(S16) now all the bubbles entering the definition of the inverse Higgs propagator (S11) are singular at \( 2E_{\text{gap}} \), so that at \( \omega \leq 2E_{\text{gap}} \):

\[
X_{\Delta \Delta} = I_0(\omega) \left[ \omega^2 - 4\Delta_0^2 - 4 \left( \frac{I_1}{I_0} \right)^2 \right] \approx 8(\mu + 4t)^2 K_0 + 8(\mu - 4t)K_1 \tag{S17}
\]

where we used the fact that \( \omega^2 - 4\Delta_0^2 - 4(I_1/I_0)^2 \approx \omega^2 - 4E_{\text{gap}}^2 \) so that the divergence of \( I_0(\omega) \) as \( \omega \to 2E_{\text{gap}} \) is compensated by the quantity in square brackets in Eq. (S17), and only a finite value remains. This result coincides with the simplified expression computed
The numerical results for the amplitude fluctuations at $q = 0$ at $U = 8t$ for increasing value of the chemical potential. The numerical results are obtained with a broadening $\delta = 0.1\Delta_0$. The scaling factor $N_0$ represents the equivalent density of states needed to match the numerical result with the analytical expressions discussed in the text, and it accounts for the differences due to the change of doping. The inset shows the evolution of the SC order parameter $\Delta_0$. For comparison we show in panel (a) the spectral function of the BCS term (S12a) alone, that neglects the mixing to the phase sector. As one can see, as soon as the chemical potential moves outside the band edge ($|\mu| > 4t$) the BCS spectral function in panel (b) displays a strong sharpening, since the quasiparticle contribution encoded in the function $I_0(\omega)$ is no more diverging at $\omega = 2\Delta_0$, and the (bare) Higgs mode recovers a LI dynamics. However, as shown in panel (a), the coupling to the phase mode removes completely the signature at $2\Delta_0$ in the full Higgs spectral function and only a weak signature at $2E_{gap} > 2\Delta_0$ is left. The inset of panel (b) is a zoom around $2\Delta_0$.

SPECTRAL FUNCTION IN THE DISORDERED CASE

In the previous Sections we outlined the derivation of the effective action for the SC degrees of freedom by using a polar-coordinates representation for the HS field $\psi_\Delta$ further supplemented by a Gauge transformation. Indeed, this approach makes more transparent the structure of the collective modes in the long-wavelength limit, discussed in details above. On the other hand, for the computation of the finite-$q$ amplitude and phase spectral function, and in particular for the evaluation of them in the presence of disorder, it is more convenient to introduce the pairing operators in real space:

$$\delta \eta_i = c_i \bar{c}_i \tau - \langle c_i \bar{c}_i \tau \rangle, \quad (S18)$$

and to use a cartesian representation of local amplitude ($A$) and phase ($\Theta$) fluctuation operators

$$\delta A_i \equiv (\delta \eta_i + i\delta \Theta_i^\dagger) / \sqrt{2} \quad \delta \Theta_i \equiv i(\delta \eta_i - \delta \Theta_i^\dagger) / \sqrt{2}. \quad (S19)$$

Together with the local charge fluctuation

$$\delta \rho_i \equiv \sum_\sigma \left( c_i^\dagger c_i^\sigma - \langle c_i^\dagger c_i^\sigma \rangle \right),$$

one can set up a matrix of correlation functions

$$\chi_{nm}^{O,R}(\omega) = i \int dt e^{i\omega t} \langle \hat{T} \hat{O}_n(t) \hat{R}_m(0) \rangle \quad (S19)$$

which can be computed on the RPA level as described in Appendix A of Ref. [S5].

For the homogeneous system one can compute explicitly the correlation functions in momentum space. With respect to the bare susceptibilities defined in Eqs. (S7) the only difference is in the one coupling the amplitude and phase sector, that reads:

$$\chi_0^{A\Theta}(q) = -\frac{1}{2N} \sum_k \left[ \frac{\xi_k^+}{E_k^+} + \frac{\xi_k^-}{E_k^-} \right] \times \left[ \frac{1}{\omega + i\delta - E_k^+ - E_k^-} \right] \frac{1}{\omega + i\delta + E_k^+ + E_k^-} \quad (S20)$$

at $t = 0$ and quoted in Eq. (10) of the main manuscript. The finite value of Re$X_{A\Delta}$ as $\omega \to 2E_{gap}$, and the finite values of the imaginary parts of the $I_0, I_1$ functions at $2E_{gap}$, imply an even weaker resonance of the Higgs spectral function at the quasiparticle threshold $2E_{gap}$ when compared to the BCS case. This is clearly shown in Fig. S1, where we report the spectral function of the Higgs $\rho_\Delta(\omega)$ computed at $q = 0$ with a numerical computation of the full expression (S11) on the lattice model. As one can see, the Higgs spectral function is always overdamped and, as $|\mu|$ exceeds the value $4t$, its maximum moves away from $2\Delta_0$ towards slightly higher frequencies, with a further weakening and broadening of the optical-gap signature. For comparison we also show in Fig. S1(b) the BCS spectral function, i.e. the bare term (S12a) in the inverse Higgs propagator, $\rho_\Delta^{BCS}(\omega) \equiv \frac{1}{2\pi} \text{Im} \{1/(2U + \chi_{A\Delta}(\omega + i\delta, q = 0))\}$. Here as soon as $\mu$ goes below the band edge the quasiparticle continuum moves away from the Higgs pole leading to a sharp LI resonance. However, the unavoidable mixing to the phase removes completely this signature, leading back to the broad spectral function shown in panel (a).
As one can see, as \( q \to 0 \) one recovers \( \chi^A_0 = (\omega/\Delta_0) \chi_{p\Delta} \), see Eq. (S7b). On the other hand at large \( q \) the full amplitude \( \chi \) at RPA level the present formulation in cartesian coordinates is completely equivalent to the polar representation of SC fluctuations used in the previous sections, as one can show explicitly by means of the generalised Ward identities (cf. e.g. Ref. [S4]). The imaginary parts of the full amplitude \( \chi^A(\omega) \) and phase \( \chi^0(\omega) \) correlation functions are the quantities shown in Fig. 1 of the main paper.

To investigate the effect of disorder we add to Eq. S1 a random onsite potential, 
\[
H_{\text{dis}} = \sum_{i,\sigma} V_i c^\dagger_{i\sigma} c_{i\sigma} \tag{S21}
\]
with \(-V_0 \leq V_i \leq V_0\) being a random variable which is taken from a flat and normalized distribution. The correlation functions are then calculated on finite lattices (typically \( 20 \times 20 \)) and averaged over disorder configurations. The same approach can be used to calculate the optical conductivity in the presence of disorder (cf. Appendix A of Ref. [S5]) and the corresponding results are shown in Fig. 2 of the main manuscript.

**SUPERCONDUCTING COLLECTIVE-MODES CONTRIBUTION TO THE CURRENT**

Let us outline briefly the derivation of Eq. (11) and (12) of the main manuscript. In the \( O(2) \) model the current is easily obtained by means of the minimal-coupling substitution \(-i \nabla \rightarrow -i \nabla + 2eA\) as 
\[
J = -2e(2)(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = -2e(2) \Delta_0^2 (\nabla^2 \psi + 2\eta \nabla \psi) = -2e(2)(D_\nabla \nabla^2 \psi + 2\nabla^2 \eta \nabla \psi), \tag{S22}
\]
where \( D_\nabla = \tilde{D}_\nabla = e^2 \Delta_0^2 \eta \) and we put \( \psi = \Delta_0 (1 + \eta) e^{i\theta} \). Here we retained only leading-order terms in the amplitude \( \eta \) and phase fluctuations, in analogy with the discussion of Refs. [S6-S8]. The structure of the current in the lattice model (S1) differs in general from the expression (S22). Indeed, apart from the presence of the quasiparticle contribution to the current, absent in the bosonic \( O(2) \) model, the collective-modes contribution to the current does not display in general the local structure encoded in the form (S22). Nonetheless, since in general \( J = -\delta S_{FL}/\delta A \), a structure similar to Eq. (S22) can be derived by considering in the action (S3) all the terms of the form:
\[
S_{FL} \simeq \frac{1}{8} \int dx \left[ D_s + 2\tilde{D}_s \eta(x) \right] |\nabla \theta(x)|^2, \tag{S23}
\]
where \( D_s \) is the superfluid stiffness, defined as usual as:
\[
D_s = \frac{1}{2N_s} \sum_{k\nu} \frac{\partial^2 \xi_k}{\partial k^2_{\nu}} \left( 1 - \frac{\xi_k}{E_k} \right) \tag{S24}
\]
Let us then consider again the expansion (S3) and let us derive all the terms of third order in \( S_{FL}^{(3)} = \text{Tr} \left[ G_0 \Sigma^{(3)} G_0 \Sigma^{(1)} \right] \) containing \( (\nabla \theta)^2 \) times an other fluctuating field, that can be either the amplitude or the phase. By considering terms at second order \( n = 2 \) in the action (S3) arising from the convolution between \( \Sigma^{(2)} \simeq \theta^2 \Delta_3 \) and \( \Sigma^{(1)} \simeq \Delta \theta_1 + \rho \theta_3 \) we have:
\[
S_{FL}^{(3)} = \frac{1}{8} \int dx dy |\nabla \theta(x)|^2 (\eta(y) + \rho(y)) R(x - y) = \simeq \frac{1}{8} \int dx |\nabla \theta(x)|^2 [2D_{\theta \Delta} \eta(x) + 2D_{\theta \rho} \rho(x)] \tag{S25}
\]
where
\[
D_{\theta \Delta} = \frac{\Delta_0^2}{4N_s} \sum_{k\nu} \frac{\partial^2 \xi_k}{\partial k^2_{\nu}} \frac{\xi_k}{E_k^2} = \frac{\Delta_0^2}{2N_s} \sum_{k} \frac{\xi_k}{E_k^2} \tag{S26}
\]
\[
D_{\theta \rho} = \frac{\Delta_0^2}{4N_s} \sum_{k\nu} \frac{\partial^2 \xi_k}{\partial k^2_{\nu}} \frac{1}{E_k^2} \simeq \frac{\Delta_0^2}{2N_s} \sum_{k} \frac{\xi_k}{E_k^2} \tag{S27}
\]
As one can see, in the continuum limit where \( \xi_k \approx k^2/2m - \mu \) one has that \( D_s \approx n/m, D_{\theta \Delta} \approx -((\Delta_0/2m) \chi_{p\Delta}(\omega = 0)), \) and \( D_{\theta \rho} \approx -(\Delta_0/2m) \chi_{pp}(\omega = 0) \) where \( \chi_{p\Delta} \) and \( \chi_{pp} \) are given by Eqs. (S12c) and (S12b), respectively. As a consequence, the approximate particle-hole symmetry of the BCS solution also guarantees that at weak coupling the phase gradient decouples from the amplitude fluctuations, while \( D_{\theta \rho} \) is finite but usually smaller than \( D_s \), since \( D_{\theta \rho} \approx (\Delta_0/m) N_F \approx \Delta_0 \) while \( D_s \) is of order of the Fermi energy. In the opposite high-density limit, i.e. near half-filling where \( \mu \approx 0 \), \( D_{\theta \rho} \approx 0 \) while \( D_{\theta \Delta} \) is finite, but always much smaller than \( D_s \) since \( D_{\theta \rho} \sim \Delta_0/\Delta \) while \( D_s \sim \Delta \), with \( \Delta_0/\Delta \ll 1 \) at weak coupling. Thus, in the weak-coupling regime all the processes involving the excitation of one phonon plus one Higgs or density mode are suppressed in the fermionic model (S1).

In the strong-coupling regime, where \( \chi_{p\Delta} \) becomes as large as \( \chi_{pp} \) away from half-filling, we expect that the coefficients (S26)-(S27) become as large as \( D_s \), and a form similar to Eq. (S22) can be recovered. To account also for the contribution of the density field, absent in the \( O(2) \) bosonic model, we will make the very rough approximation to consider its effect only in the renormalisation of the \( D_{\theta \Delta} \) vertex of Eq. (S25), after integrating out the density field at RPA level. This procedure leads indeed to the form (S23) of the effective action, with the effective coefficient
\[
\tilde{D}_s = D_{\theta \Delta} - \frac{D_{\theta \rho} \chi_{p\Delta}(0)}{2U + \chi_{pp}(0)}, \tag{S28}
\]
that coincides with Eq. (12) of the manuscript. In the strong-coupling regime the quantity (S28) can be easily estimated at leading order in the small coupling $\alpha = (2t/U)$, see e.g. Ref. [S9]. In particular one has that:

$$\chi_{\rho\Delta} \simeq -\frac{2}{U}\delta \sqrt{1 - \delta^2}$$  \hspace{1cm} (S29)

$$\chi_{\rho\rho} \simeq -\frac{2}{U}(1 - \delta^2)$$  \hspace{1cm} (S30)

$$D_s \simeq U\alpha^2(1 - \delta^2)$$  \hspace{1cm} (S31)

$$D_{\theta\Delta} \simeq -\frac{U}{2}\alpha^2(1 - \delta^2)(1 - 3\delta^2)$$  \hspace{1cm} (S32)

$$D_{\theta\rho} \simeq \frac{3U}{2}\alpha^2\delta(1 - \delta^2)^{3/2}$$  \hspace{1cm} (S33)

where $\delta \equiv 1 - n$ and Eq. (S29) only holds for $\delta \neq 0$. As a consequence one immediately sees that $D_s/D_s \to 1$ for any finite doping away from half-filling, where the bosonic limit is recovered. On the other hand at half-filling $\chi_{\rho\Delta} = 0$ at all values of $U$ so that $D_s = D_{\theta\Delta}$ always and $\tilde{D}_s/D \to -1/2$ at strong coupling, as shown in Fig. 3 of the manuscript.

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