Bott-Samelson Varieties
and Configuration Spaces

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Abstract

The Bott-Samelson varieties $Z$ are a powerful tool in the representation theory and geometry of a reductive group $G$. We give a new construction of $Z$ as the closure of a $B$-orbit in a product of flag varieties $(G/B)^l$. This also gives an embedding of the projective coordinate ring of the variety into the function ring of a Borel subgroup: $\mathbb{C}[Z] \subset \mathbb{C}[B]$.

In the case of the general linear group $G = GL(n)$, this identifies $Z$ as a configuration variety of multiple flags subject to certain inclusion conditions, controlled by the combinatorics of braid diagrams and generalized Young diagrams. The natural mapping $Z \to G/B$ compactifies the matrix factorizations of Berenstein, Fomin and Zelevinsky [2]. As an application, we give a geometric proof of the theorem of Kraskiewicz and Pragacz [12] that Schubert polynomials are characters of Schubert modules.

Our work leads on the one hand to a Demazure character formula for Schubert polynomials and other generalized Schur functions, and on the other hand to a Standard Monomial Theory for Bott-Samelson varieties. All our results remain valid in arbitrary characteristic and over $\mathbb{Z}$.

Introduction

The Bott-Samelson varieties are an important geometric tool in the theory of a reductive algebraic group (or complex Lie group) $G$. Defined in [4], they were exploited by Demazure [5] to analyze the flag variety $G/B$, its singular cohomology ring $H^\ast(G/B, \mathbb{C})$ (the Schubert calculus), and its projective coordinate ring $\mathbb{C}[G/B]$. Since the irreducible representations of $G$ are embedded in the coordinate ring, Demazure was able to obtain an iterative character formula [6] for these representations.

Bott-Samelson varieties are so useful because they “factor” the flag variety into a “product” of projective lines. More precisely, they are iterated $\mathbb{P}^1$-fibrations and each has a natural, birational map to $G/B$. The Schubert subvarieties themselves lift to iterated $\mathbb{P}^1$-fibrations under this map. The combinatorics of Weyl groups enters the picture because a given $G/B$ can be
“factored” in many ways, indexed by sequences \( i = (i_1, i_2, \ldots, i_N) \) such that \( w_0 = s_{i_1} s_{i_2} \cdots s_{i_N} \) is a reduced decomposition of the longest Weyl group element \( w_0 \) into simple reflections.

The Bott-Samelson variety \( Z_i \) is usually defined as a quotient:

\[
Z_i \overset{\text{def}}{=} (P_{i_1} \times \cdots \times P_{i_N})/B^N,
\]

where \( P_i \) are minimal parabolic subgroups, \( B \subset P_i \subset G \), and \( B^N \) acts freely on the right of \( P_{i_1} \times \cdots \times P_{i_N} \) by

\[
(p_1, \ldots, p_N) \cdot (b_1, \ldots, b_N) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{N-1}^{-1} p_N b_N).
\]

The natural map to the flag variety is given by multiplication: \( (p_1, \ldots, p_N) \mapsto p_1 p_2 \cdots p_N B \in G/B \).

In this paper, we first give a dual construction of \( Z_i \) as a subvariety rather than a quotient. It is the closure of a \( B \)-orbit inside a product of flag varieties:

\[
Z_i \cong B \cdot (s_{i_1} B, s_{i_1}s_{i_2} B, \ldots, w_0 B) \subset (G/B)^N,
\]

where \( B \) acts diagonally on \((G/B)^N\). Our constructions are partly inspired by Fulton’s work [8], Ch. 10.3.

In the case \( G = GL(n) \) or \( SL(n) \), this translates into an expression for \( Z_i \) as a “multiple Schubert variety”: configurations of many linear spaces in \( \mathbb{C}^n \) subject to certain inclusions involving a test flag. For example, for \( G = GL(3) \), \( i = 212 \), and the test flag \( \mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \), we get

\[
Z_i = \{ (V_1, V_2, V_2') \in \text{Gr}(1, \mathbb{C}^3) \times \text{Gr}(2, \mathbb{C}^3)^2 \mid V_2 \supset V_1 \subset V_2' \supset \mathbb{C}^1 \}.
\]

The natural birational map onto the flag variety is given by the projection \( (V_1, V_2, V_2') \mapsto (V_1, V_2) \). For \( GL(n) \), the combinatorics of such configuration varieties is controlled by certain generalized Young diagrams [8], [9], [22], [23]; or equivalently by the wiring diagrams and chamber sets of Berenstein, Fomin, and Zelevinsky [4], [17].

Secondly, we study more general configuration varieties, which are also closures of \( B \)-orbits in products of \( G/B \). These varieties are governed by similar combinatorics, are desingularized by the Bott-Samelson varieties, and include the flag and Schubert varieties.

Thirdly, we turn to the Borel-Weil theory of Bott-Samelson varieties. Our embedding of \( Z_i \) leads to an embedding of its projective coordinate ring into the regular functions on a Borel subgroup:

\[
\mathbb{C}[Z_i] \subset \mathbb{C}[B].
\]

That is, the space of sections of effective line bundles on \( Z_i \) can be realized in terms of certain polynomials on \( B \). (Here we use a vanishing theorem of W. van der Kallen [8].)
For $G = GL(n)$, the space of sections becomes a certain generalized Schur module ([1], [26], [22], [24], [25]) spanned by products of minors in the polynomial ring $\mathbb{C}[x_{ij}]_{i<j}$. Here, the bitableaux of Desarmenien, Kung, and Rota [7] (c.f. [15]), give the appropriate combinatorial formalism. A result of our construction is a Demazure character formula for these generalized Schur modules. Conversely, we get a standard monomial basis for the space of sections, which we pursue in our paper [13].

Fourthly, we apply our results to the Schubert modules of Kraskiewicz and Pragacz [12]. The characters of these modules are the Schubert polynomials, special algebraic representatives of the Schubert classes in the singular cohomology ring of $G/B$. Why the Schubert polynomials should appear as characters of $B$-modules remains a mystery, but our theory does lead (as suggested by a manuscript of V. Reiner and M. Shimozono) to a new proof of Kraskiewicz and Pragacz’s theorem. Our Demazure formula applies to these polynomials, and is basically different from the usual recurrence defining them. The combinatorics of this formula are examined in our paper [19].

To avoid intimidating terminology, we work over the base field $\mathbb{C}$ of complex numbers. The alert reader will note, however, that all our arguments remain valid without change over an algebraically closed field of arbitrary characteristic and over the integers.

Note. The geometry of a general reductive $G$ is largely confined to Sec 1 and 2. Those interested mainly in the combinatorial applications associated to $G = GL(n)$ may begin reading at Sec 3.

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1 Bott-Samelson varieties

1.1 Three constructions

In this section, $G$ is a reductive algebraic group. Our constructions are all valid over an arbitrary field, or over the integers, but we will use the complex numbers $\mathbb{C}$ for convenience.

Let $W$ denote the Weyl group generated by simple reflections $s_1, \ldots, s_r$, where $r$ is the rank of $G$. For $w \in W$, $\ell(w)$ denotes the length of a reduced
(i.e. minimal) decomposition $w = s_{i_1} \ldots s_{i_l}$, and $w_0$ is the element of maximal length.

We let $B$ be a Borel subgroup, $T \subset B$ a maximal torus (Cartan subgroup), and $U_\alpha \subset B$ the one-dimensional unipotent subgroup associated to the root $\alpha$. Let $P_k \supset B$ be the minimal parabolic associated to the simple reflection $s_k$, so that $P_k/B \cong \mathbb{P}^1$, the projective line. Also, take $\hat{P}_k \supset B$ to be the maximal parabolic associated to the reflections $s_1, \ldots, \hat{s}_k, \ldots, s_r$. Finally, we have the Schubert variety as a $B$-orbit closure inside the flag variety:

$$X_w = BwB \subset G/B$$

For what follows, we fix a reduced decomposition of some $w \in W$,

$$w = s_{i_1} \ldots s_{i_l},$$

and we denote $i = (i_1, \ldots, i_l)$.

Now let $P \supset B$ be any parabolic subgroup of $G$, and $X$ any space with $B$-action. Then the induced $P$-space is the quotient

$$P^B \times X \overset{\text{def}}{=} (P \times X)/B$$

where the quotient is by the free action of $B$ on $P \times X$ given by $(p, x) \cdot b = (pb, b^{-1}x)$. (Thus $(pb, x) = (p, bx)$ in the quotient.) The key property of this construction is that

$$X \rightarrow P^B \times X \downarrow$$

$$P/B$$

is a fiber bundle with fiber $X$ and base $P/B$. We can iterate this construction for a sequence of parabolics $P, P', \ldots$,

$$P^B \times P'^B \times \cdots \overset{\text{def}}{=} P^B \times (P'^B \times \cdots).$$

Then the quotient Bott-Samelson variety of the reduced word $i$ is

$$Z_i^{\text{quo}} \overset{\text{def}}{=} P_{i_1}^B \times \cdots \times P_{i_l}/B.$$

Because of the fiber-bundle property of induction, $Z_i^{\text{quo}}$ is clearly a smooth, irreducible variety of dimension $l$. It is a subvariety of

$$X_l \overset{\text{def}}{=} G^B \times \cdots \times G^B / B,$$

$i$ factors

$B$ acts on these spaces by multiplying the first coordinate:

$$b \cdot (p_1, p_2, \ldots, p_l) \overset{\text{def}}{=} (bp_1, p_2, \ldots, p_l).$$
The original purpose of the Bott-Samelson variety was to desingularize the Schubert variety $X_w$ via the multiplication map:

$$Z_i^{\text{quo}} \to X_w \subset G/B$$

$$(p_1, \ldots, p_l) \mapsto p_1 p_2 \cdots p_l B,$$

a birational morphism.

Next, consider the fiber product

$$G/B \times_{G/P} G/B \overset{\text{def}}{=} \{(g_1, g_2) \in (G/B)^2 \mid g_1 P = g_2 P\}.$$

We may define the fiber product Bott-Samelson variety

$$Z_i^{\text{fib}} \overset{\text{def}}{=} eB \times_{G/P_{i_1}} G/B \times_{G/P_{i_2}} \cdots \times_{G/P_{i_l}} G/B \subset (G/B)^{l+1}.$$

We let $B$ act diagonally on $(G/B)^{l+1}$; that is, simultaneously on each factor:

$$b \cdot (g_0 B, g_1 B, \ldots, g_l B) \overset{\text{def}}{=} (bg_0 B, bg_1 B, \ldots, bg_l B).$$

This action restricts to $Z_i^{\text{fib}}$. The natural map to the flag variety is the projection to the last coordinate:

$$Z_i^{\text{fib}} \to G/B$$

$$(eB, g_1 B, \ldots, g_l B) \mapsto g_l B$$

Finally, let us define the $B$-orbit Bott-Samelson variety as the closure (in either the Zariski or analytic topologies) of the orbit of a point $z_i$:

$$Z_i^{\text{orb}} \overset{\text{def}}{=} B \cdot z_i \subset G/\hat{P}_{i_1} \times \cdots \times G/\hat{P}_{i_l},$$

where

$$z_i = (s_{i_1}, \hat{P}_{i_1}, s_{i_2}, \hat{P}_{i_2}, \ldots, s_{i_l}, \cdot \cdot \cdot, s_{i_l}, \hat{P}_{i_l})$$

Again, $B$ acts diagonally. In this case the map to $G/B$ is more difficult to describe, but see Sec. 3.3.

1.2 Isomorphism theorem

The three types of Bott-Samelson variety are isomorphic.

**Theorem 1** (i) Let

$$\phi : X_i \to (G/B)^{l+1}$$

$$(g_1, g_2, \ldots, g_l) \mapsto (c, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_l),$$
where $\overline{\mathcal{P}}$ means the coset of $g$. Then $\phi$ restricts to an isomorphism of $B$-varieties

$$\phi : Z_i^{\text{quo}} \rightarrow Z_i^{\text{fib}}.$$ 

(ii) Let

$$\psi : X_l \rightarrow G/\tilde{P}_{i_1} \times G/\tilde{P}_{i_2} \times \cdots \times G/\tilde{P}_{i_l}$$

$$(g_0, g_1, \ldots, g_l) \mapsto \left( \frac{g_0}{g_1}, \frac{g_0 g_2}{g_1}, \ldots, \frac{g_0 g_2 \cdots g_l}{g_1} \right),$$

where $\overline{\mathcal{P}}$ means the coset of $g$. Then $\psi$ restricts to an isomorphism of $B$-varieties

$$\psi : Z_i^{\text{quo}} \rightarrow Z_i^{\text{orb}}.$$ 

Proof. (i) It is trivial to verify that $\phi$ is a $B$-equivariant isomorphism from $X_l$ to $eB \times (G/B)^l$ and that $\phi(Z_i^{\text{quo}}) \subset Z_i^{\text{fib}}$, so it suffices to show the reverse inclusion. Suppose

$$z_f = (eB, g_1 B, \ldots, g_l B) \in Z_i^{\text{fib}}.$$ 

Then

$$z_q = \varphi^{-1}(z_f) = (g_1, g_1^{-1} g_2, g_2^{-1} g_3, \ldots) \in X_l.$$ 

By definition, $eP_{i_1} = g_1 P_{i_1}$, so $g_1 \in P_{i_1}$. Also $g_1 P_{i_2} = g_2 P_{i_2}$, so $g_1^{-1} g_2 \in P_{i_2}$, and similarly $g_{k-1}^{-1} g_k \in P_{i_k}$. Hence $z_q \in Z_i^{\text{quo}}$, and $\phi(z_q) = z_f$.

(ii) First let us show that $\psi$ is injective on $Z_i^{\text{quo}}$. Suppose $\psi(p_1, \ldots, p_l) = \psi(q_1, \ldots, q_l)$ for $p_k, q_k \in P_{i_k}$. Then $p_1 \tilde{P}_{i_1} = q_1 \tilde{P}_{i_1}$, so that $p_1^{-1} q_1 \in \tilde{P}_{i_1} \cap P_{i_1} = B$. Thus $q_1 = p_1 b_1$ for $b_1 \in B$. Next, we have

$$p_1 p_2 \tilde{P}_{i_2} = q_1 q_2 \tilde{P}_{i_2} = p_1 b_1 q_2 \tilde{P}_{i_2},$$

so that $p_2^{-1} b_1 q_2 \in \tilde{P}_{i_2} \cap P_{i_2} = B$, and $q_2 = b_1^{-1} p_2 b_2$ for $b_2 \in B$. Continuing in this way, we find that

$$(q_1, q_2, \ldots, q_l) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_l^{-1} p_l b_l) = (p_1, p_2, \ldots, p_l) \in X_l.$$ 

Thus $\psi$ is injective on $Z_i^{\text{quo}}$.

Since we are working with algebraic morphisms, we must also check that $\psi$ is injective on tangent vectors of $Z_i^{\text{quo}}$. Now, the degeneracy locus

$$\{ z \in Z_i^{\text{quo}} \mid \text{Ker } d\psi_z \neq 0 \}$$

is a $B$-invariant, closed subvariety of $Z_i^{\text{quo}}$, and by Borel's Fixed Point Theorem it must contain a $B$-fixed point. But it is easily seen that the degeneracy point

$$z_0 = (e, \ldots, e) \in X_l$$

is the only fixed point of $Z_i^{\text{quo}}$. Thus if $d\psi$ is injective at $z_0$, then the degeneracy locus is empty, and $d\psi$ is injective on each tangent space. The injectivity
at \( z_0 \) is easily shown by an argument completely analogous to that for global injectivity given above, but written additively in terms of Lie algebras instead of multiplicatively with Lie groups.

Thus it remains to show surjectivity: that \( \psi \) takes \( Z_i^{quo} \) onto \( Z_i^{orb} \). Consider

\[
z_i^{quo} = (s_{i_1}, \ldots, s_{i_l}) \in X_i,
\]
a well-defined point in \( Z_i^{quo} \). Then

\[
\psi(z_i^{quo}) = z_1 = (s_{i_1} \hat{P}_{i_1}, s_{i_1} s_{i_2} \hat{P}_{i_2}, \ldots),
\]
and \( \psi \) is \( B \)-equivariant, so that \( \psi(Z_i^{quo}) \supset \overline{\psi(B \cdot z_i^{quo})} = B \cdot z_i^{orb} \).

Now we need only show that \( \psi(Z_i^{quo}) \subset Z_i^{orb} \), which results from the following:

**Lemma 2** \( B \cdot z_i^{quo} \) is an open dense orbit in \( Z_i^{quo} \).

**Proof.** Since \( Z_i^{quo} \) is irreducible of dimension \( l \), it suffices to show that the orbit has (at least) the same dimension. We may see this by determining \( \text{Stab}_B(z_i) \).

Suppose

\[
(b_{s_{i_1}, \ldots, s_{i_l}}) = (s_{i_1} b_1, b_1^{-1} s_{i_2} b_2, \ldots, b_1^{-1} s_{i_l} b_l) \in Z_i^{quo}.
\]

Then \( s_{i_l} = b_l^{-1} s_{i_l} b_1 \), and \( b_l^{-1} \in B \cap s_{i_l} B s_{i_l} \). Repeating this calculation leftward, we find that \( b \in B \cap wBw^{-1} \), so that \( \text{Stab}_B(z_i) \subset B \cap wBw^{-1} \). (Recall \( w = s_{i_1} \ldots s_{i_l} \).) Thus, using some well-known facts (see [27]) we have:

\[
\dim(B \cdot z_i^{quo}) = \dim(B) - \dim(\text{Stab}_B(z_i)) \\
\geq \dim(B) - \dim(B \cap wBw^{-1}) \\
= \dim(B) - (\dim(B) - \ell(w)) \\
= \ell(w) = l.
\]

Since the orbit can have dimension no bigger than \( l \), we must have equality. Thus the Lemma and the Theorem both follow. \( \bullet \)

**Corollary 3** For \( w = s_{i_1} \ldots s_{i_l} \), we have

\[
\text{Stab}_B(z_i \in Z_i) = \text{Stab}_B(wB \in G/B) = B \cap wBw^{-1}.
\]

### 1.3 Open cells

In view of the Theorem, we will let \( Z_i \) denote the abstract Bott-Samelson variety defined by any of our three versions. It contains the degenerate \( B \)-fixed point \( z_0 \) defined by:

\[
z_0 = (e, e, \ldots) \in Z_i^{quo} \\
= (eB, eB, \ldots) \in Z_i^{fib} \\
= (e \hat{P}_{i_1}, e \hat{P}_{i_2}, \ldots) \in Z_i^{orb}
\]
as well as the generating $T$-fixed point whose $B$-orbit is dense in $Z_i$:

$$z_i = (s_{i_1}, s_{i_2}, s_{i_3}, \ldots) \in Z_i^{quo}$$

$$= (eB, s_{i_1}B, s_{i_1} s_{i_2}B, \ldots) \in Z_i^{f^b}$$

$$= (s_{i_1} \hat{P}_{i_1}, s_{i_1} s_{i_2} \hat{P}_{i_2}, \ldots) \in Z_i^{pr^b}$$

We may parametrize the dense orbit $B \cdot z_i \subset Z_i$ by an affine cell. Consider the normal ordering of the positive roots associated to the reduced word $i$. That is, let

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \beta_3 = s_{i_1} s_{i_2}(\alpha_{i_3}), \quad \cdots$$

Recall that $U_{B_k}$ is the one-dimensional unipotent subgroup of $B$ corresponding to the positive root $\beta_k$. Then we have a direct product:

$$B = U_{\beta_1} \cdots U_{\beta_l} \cdot (B \cap wBw^{-1})$$

so that the multiplication map

$$U_{\beta_1} \times \cdots \times U_{\beta_l} \rightarrow B \cdot z_i$$

$$(u_1, \ldots, u_l) \mapsto u_1 \cdots u_l \cdot z_i$$

is injective, and an isomorphism of varieties. The left-hand side is isomorphic to an affine space $C^l$.

$Z_i$ also contains an opposite big cell centered at $z_0$ which is not the orbit of a group. Consider the one-dimensional unipotent subgroups $U_{-\alpha_i}$ corresponding to the negative simple roots $-\alpha_i$. The map

$$C^l \cong U_{-\alpha_{i_1}} \times \cdots \times U_{-\alpha_{i_l}} \rightarrow Z_i^{quo}$$

$$(u_1, \ldots, u_l) \mapsto (u_1, \ldots, u_l)$$

is an open embedding.

In the case of $G = GL(n)$, $B =$ upper triangular matrices, we may write an element of $U_{-\alpha_{i_k}}$ as $u_k = I + t_k e_k$, where $I$ is the identity matrix, $e_k$ is the sub-diagonal coordinate matrix $e_{(i_k+1,i_k)}$, and $t_k \in C$. If we further map $Z_i^{quo}$ to $G/B$ via the natural multiplication map, we get

$$(t_1, \ldots, t_l) \mapsto (I + t_1 e_1) \cdots (I + t_l e_l)$$

$$C^l \rightarrow N_-$$

$$\cap$$

$$Z_i^{quo} \rightarrow G/B$$

$$(p_1, \ldots, p_l) \mapsto p_1 \cdots p_l B$$

where $N_-$ denotes the unipotent lower triangular matrices (mod $B$). Thus the multiplication on the bottom is a compactification of the matrix factorizations studied by Berenstein, Fomin, and Zelevinsky [2].
2 Configuration varieties

We define a class of varieties (more general than the Schubert varieties) which are desingularized by Bott-Samelson varieties.

2.1 Definitions

We continue with the case of a general reductive group $G$. Given a sequence of Weyl group elements $w = (w_1, \ldots, w_k)$ and a sequence of indices $j = (j_1, \ldots, j_k)$, we consider the $T$-fixed point

$$z_{wj} = (w_1^j \cdot P_{j_1}, \ldots, w_k^j \cdot P_{j_k}) \in G/P_{j_1} \times \cdots \times G/P_{j_k},$$

and we define the configuration variety as the $G$-orbit closure

$$\mathcal{F}_{wj} \overset{\text{def}}{=} G \cdot z_{wj} \subset G/P_{j_1} \times \cdots \times G/P_{j_k}.$$

$G$ acts on this variety by multiplying each factor simultaneously (the diagonal action).

We may define a “flagged” version of this construction by replacing $G$ with $B$. The flagged configuration variety is the $B$-orbit closure

$$\mathcal{F}^B_{wj} \overset{\text{def}}{=} B \cdot z_{wj} \subset G/P_{j_1} \times \cdots \times G/P_{j_k}.$$

Again, $B$ acts diagonally.

Examples. (a) Take $w = (w, w, \ldots, w)$ for any $w \in W$ and $j = (1, 2, \ldots, r)$ (where $r = \text{rank } G$). Then the configuration variety is isomorphic to the flag variety of $G$, and the flagged configuration variety is isomorphic to the Schubert variety of $w$:

$$\mathcal{F}_{wj} \cong G/B \quad \quad \mathcal{F}^B_{wj} \cong X_w.$$

(b) For $j = i = (i_1, i_2, \ldots)$, a reduced word, and $w = (s_{i_1}, s_{i_1} s_{i_2}, \ldots)$, the flagged configuration variety is exactly our orbit version of the Bott-Samelson variety: $\mathcal{F}^B_{wj} = Z_i^{\text{orb}} = Z_i$. •

Remark. For a given $G$, there are only finitely many configuration varieties up to isomorphism. In fact, suppose a list $(w, j)$ has repetitions of some element of $w$ with identical corresponding entries in $j$. Then we may remove the repetitions and the configuration variety will not change (up to $G$-equivariant isomorphism), only the embedding. Thus, all configuration varieties are projections of a maximal variety. This holds for the flagged and unflagged cases.

Example. The maximal configuration variety for $G = GL(3)$ is the space of triangles \[20\], and corresponds to

$$w = (e, e, s_1, s_2, s_2 s_1, s_1 s_2) \quad \quad j = (1, 2, 1, 2, 1, 2).$$
Further entries would be redundant: for example, $s_1 \tilde{P}_2 = e \tilde{P}_2$. All other configuration varieties are obtained by omitting some entries of $w$ and the corresponding entries of $j$. Hence there are at most $2^k$ configuration varieties for $G$.

One might attempt to broaden the definition of configuration varieties by replacing the minimal homogeneous spaces $G/\tilde{P}_j$ by $G/P$ for arbitrary parabolics $P \supset B$. This gives the same class of varieties, however, since any $G/P$ can be embedded equivariantly inside a product of $G/\tilde{P}_j$’s, resulting in isomorphic orbit closures. Once again, this changes only the embeddings, not the varieties.

Varieties similar to our $F_{wj}$ are defined and some small cases are analyzed in Langlands’ paper [14].

2.2 Desingularization

Very little is known about general configuration varieties. However, certain of them are well understood because they can be desingularized by Bott-Samelson varieties.

Recall that a sequence $w = (w_1, \ldots, w_K)$ of Weyl group elements is increasing in the weak order on $W$ if there exist $u_1, u_2, \ldots, u_K$ such that $w_k = u_1 u_2 \cdots u_k$ and $\ell(w_k) = \ell(w_{k-1}) + \ell(u_k)$ for all $k$.

For $w = (w_1, \ldots, w_K)$ and $j = (j_1, \ldots, j_K)$, let $w^+ = (e, \ldots, e, w_1, \ldots, w_K)$ with $r$ added entries of $e$, and $j^+ = (1, 2, \ldots, r, j_1, \ldots, j_K)$. Clearly

$$F_{w, j} \cong F_{w^+, j^+}.$$ 

**Proposition 4** If $w$ is increasing in the weak order and $j$ is arbitrary, then the flagged configuration variety $F_{w, j}^B$ can be desingularized by a Bott-Samelson variety. That is, there exists a reduced word $i$ and a regular birational morphism

$$\pi : Z_i \to F_{w, j}^B.$$ 

Furthermore, the unflagged configuration variety $F_{w^+, j^+}$ is desingularized by the composite map

$$G \times Z_i \xrightarrow{id \times \pi} G \times F_{w, j}^B \cong G \times F_{w^+, j^+} \xrightarrow{\mu} F_{w^+, j^+},$$

where $id \times \pi$ is the map induced from $\pi$, and $\mu$ is the multiplication map $(g, v) \mapsto g \cdot v$.

**Remark.** The map

$$G \times Z_i \to G \times F_{w, j}^B \to F_{wj}$$

is a surjection from a smooth space to $F_{wj}$, but it is not birational in general. We will see in Sec 4 that for the purposes of Borel-Weil theory, this map can
I claim \( \phi \) variety: Define a projection map from the Bott-Samelson variety to the configuration

\[
\text{order. In fact, if } w \text{ is, for each } k \text{ closure).}
\]

\[ B \text{ defined. It is clearly } y \text{ for some } w \text{ have a unique factorization } w = wW. \]

\[ \text{Lemma 5 (a) For any } w \in W \text{ and parabolic } P \text{ with Weyl group } W(P), \text{ we have a unique factorization } w = \bar{w}y, \text{ where } y \in W(P), \bar{w} \text{ has minimal length in } \bar{w}W(P), \text{ and } \ell(w) = \ell(\bar{w}) + \ell(y). \]

(b) Suppose \( w \in W \) has minimum length in the coset \( wW(P) \), and consider the points \( wP \in G/P \) and \( wB \in G/B \). Then \( \text{Stab}_B(wP) = \text{Stab}_B(wB) \).

**Proof of Lemma.** (a) Well-known (see [10], [9]).

(b) The \( \supset \) containment is clear, so we prove the other. Let \( \Delta \) denote the set of roots of \( G, \Delta_+ \) the positive roots, \( \Delta(P) \) the roots of \( P \), etc. From considering the corresponding Lie algebras we obtain:

\[
\begin{align*}
\dim \text{Stab}_B(wB) &= |\Delta_+ \cap w(\Delta_+)| \\
\dim \text{Stab}_B(wP) &= |\Delta_+ \cap w(\Delta_+ \cup \Delta(P))|.
\end{align*}
\]

But the two sets on the right are identical. In fact, if \( w \) is minimal in \( wP \), then \( \Delta_+ \cap w(\Delta_+(P)) = \emptyset \). (See [10], 5.5, 5.7.)

**Proof of Proposition.** Denote \( W_k \overset{\text{def}}{=} W(\hat{P}_{jk}) \), a parabolic subgroup of the Weyl group. Given \( w \) and \( j \), we define a new sequence \( \bar{w} = (\bar{w}_1, \ldots, \bar{w}_K) \). Take \( \bar{w}_k \) to be the minimum-length coset representative in \( w_kW_k \), so that \( w_k = \bar{w}_ky_k \) for some \( y_k \in W_k \). I claim the new sequence \( \bar{w} \) is still increasing in the weak order. In fact, if \( w_k = u_1 \cdots u_k \) and \( \bar{u}_k \) is minimal in \( u_kW_k \), then \( \bar{w}_k = \bar{w}_{k-1}y_k\bar{u}_k \) and \( \ell(\bar{w}_k) = \ell(\bar{w}_{k-1}) + \ell(y_k) + \ell(\bar{u}_k) \). Note that it is possible that \( \bar{u}_k = e \), and \( \bar{w}_k = \bar{w}_{k+1} \).

Now let \( i \) be any reduced decomposition of the increasing sequence \( \bar{w} \): that is, for each \( k \) we have a reduced decomposition \( \bar{w}_k = s_{i_1}s_{i_2} \cdots s_{i_{l(k)}} \), where \( l(k) = \ell(\bar{w}_k) \), so that \( 0 \leq l(1) \leq l(2) \cdots \leq l(K) = l \). Also, \( i_{l(k)} = j_k \) for all \( k \).

Define a projection map from the Bott-Samelson variety to the configuration variety:

\[
\phi : Z_i = Z_{\bar{w}i}^{\text{orb}} \rightarrow (g_1\hat{P}_{i_1}, \ldots, g_l\hat{P}_{i_l}) \mapsto (g_{l(1)}\hat{P}_{j_1}, \ldots, g_{l(K)}\hat{P}_{j_K}).
\]

I claim \( \phi \) is well-defined, \( B \)-equivariant, onto, regular, and birational.

Now, \( \bar{w}_k \) and \( w_k \) are equal modulo \( W_k \), so \( \bar{w}_k\hat{P}_{jk} = w_k\hat{P}_{jk} \), and thus

\[
\phi(z_i) = z_{wj} = z_{wj} \in \mathcal{F}_{wj}^B.
\]

Since \( Z_i = B \cdot z_i \), this implies that the image of \( \phi \) lies inside \( \mathcal{F}_{wj} \), and \( \phi \) is well-defined. It is clearly \( B \)-equivariant and therefore onto (since \( \mathcal{F}_{wj}^B \) is a \( B \)-orbit closure).
The map is regular, and to show it is birational we need only check that it is a bijection between the big $B$-orbits in the domain and image. That is, we must show equality of the stabilizers

$$\text{Stab}_B(z_i) = \text{Stab}_B(z_{\tilde{w}^j}).$$

By the corollary in Section 1.2, we have $\text{Stab}_B(z_i) = \text{Stab}_B(wB \in G/B)$ for $w = s_{i_1} \cdots s_{i_{(K)}} = \tilde{w}_K$.

Now we use induction on the length of the sequence $w$. If the length $K = 1$, we have immediately that $\text{Stab}_B(z_{\tilde{w}^j}) = \text{Stab}_B(\tilde{w}_K \hat{P}_j) = \text{Stab}_B(\tilde{w}_K B)$ by the above Lemma. Assuming the assertion for $w' = (w_1, \ldots, w_{K-1})$ and using the Lemma, we have

$$\text{Stab}_B(z_{\tilde{w}^j}) = \text{Stab}_B(z_{\tilde{w}^j}) \cap \text{Stab}_B(\tilde{w}_K \hat{P}_j) = \text{Stab}_B(\tilde{w}_{K-1} B) \cap \text{Stab}_B(\tilde{w}_K B) = \text{Stab}_B(\tilde{w}_K B).$$

The remaining assertions about the unflagged $F_{w^j}$ follow easily. That is, the map of fiber bundles

$$G^B \times Z_i \to G^B \times F_{w^j}$$

is $G$-equivariant, onto, and regular and birational by our results above, and so is the multiplication map

$$G^B \times F_{w^j} \to F_{w^j}$$

since $\text{Stab}_G(z_{w^j}) = \text{Stab}_B(z_{w^j})$.

3 The Case of $GL(n)$

We begin again, restating many of our results more explicitly for the general linear group $G = GL(n, \mathbb{C})$. In this case $B = \text{upper triangular matrices}$, $T = \text{diagonal matrices}$, $r = n - 1$,

$$P_k = \{(x_{ij}) \in GL(n) \mid x_{ij} = 0 \text{ if } i > j \text{ and } (i, j) \neq (k + 1, k)\},$$

$$\hat{P}_k = \{(x_{ij}) \in GL(n) \mid x_{ij} = 0 \text{ if } i > k \geq j\},$$

and $G/\hat{P}_k \cong \text{Gr}(k, \mathbb{C}^n)$, the Grassmannian of $k$-dimensional subspaces of complex $n$-space.

Also $W = \text{permutation matrices}$, $\ell(w)$ = the number of inversions of a permutation $w$, $s_i = \text{the transposition } (i, i+1)$, and the longest permutation is $w_0 = n, n - 1, \ldots, 2, 1$. We will frequently use the notation

$$[k] = \{1, 2, 3, \ldots, k\}.$$
3.1 Subset families

First, we introduce some combinatorics. Define a subset family to be a collection
\[ D = \{ C_1, C_2, \ldots \} \]
of subsets \( C_k \subset [n] \). The order of the subsets is irrelevant in
the family, and we do not allow subsets to be repeated.

This relates to the previous sections as follows. To a list of permutations
\( w = (w_1, \ldots, w_K) \), \( w_k \in W \), and a list of indices
\( j = (j_1, \ldots, j_K) \), \( 1 \leq j_k \leq n \), we associate a subset family:
\[ D = D_{wj} \overset{\text{def}}{=} \{ w_1[j_1], \ldots, w_K[j_K] \} . \]
Here \( w[j] = \{ w(1), w(2), \ldots, w(j) \} \).

Now suppose the list of indices \( i = (i_1, i_2, \ldots, i_l) \) encodes a reduced decom-
position \( w = s_{i_1}s_{i_2} \cdots s_{i_l} \) of a permutation into a minimal number of simple
transpositions. We let \( w = (s_{i_1}, s_{i_2}, \ldots, w) \) and \( j = i \), and we define the
reduced chamber family \( D_i \overset{\text{def}}{=} D_{wj} \).

Further, define the full chamber family
\[ D_i^+ \overset{\text{def}}{=} \{ [1], [2], \ldots, [n] \} \cup D_i , \]
(which is \( D_{w+j} \) in our previous notation).

We tentatively connect these structures with geometry. Let \( \mathbb{C}^n \) have the
standard basis \( e_1, \ldots, e_n \). For any subset \( C = \{ j_1, \ldots, j_k \} \subset [n] \), the coordinate
subspace
\[ E_C = \text{Span}_\mathbb{C}\{ e_{j_1}, \ldots, e_{j_k} \} \in \text{Gr}(k) \]
is a \( T \)-fixed point in a Grassmannian. A subset family corresponds to a \( T \)-fixed
point in a product of Grassmannians
\[ z_D = (E_{C_1}, E_{C_2}, \ldots) \in \text{Gr}(D) \overset{\text{def}}{=} \text{Gr}(|C_1|) \times \text{Gr}(|C_2|) \times \ldots . \]
This is consistent with our previous notation for an arbitrary \( G \): for \( D = D_{wj} \), we have \( z_D = z_{wj} \). We defined configuration varieties and Bott-Samelson
varieties as orbit closures of such points (see also below, Sec 3.3).

Examples. For \( n = 3 \), \( G = \text{GL}(3) \), \( i = j = 121 \), we have
\( w = (s_1, s_2s_2, s_1s_2s_1) \), and the reduced chamber family
\[ D_{121} = \{ s_1[1], s_1s_2[2], s_1s_2s_1[1] \} \]
\[ = \{ \{ 2 \}, \{ 2, 3 \}, \{ 3 \} \} \]
\[ = \{ 2, 23, 3 \} \]
The full chamber family is \( D_{121}^+ = \{ 1, 12, 123, 2, 23, 3 \} \). The chamber family of
the other reduced word \( i = 212 \) is \( D_{212} = \{ 13, 3, 23 \} \).
For \( n = 4 \), let \( w = (e, s_1, s_1, s_3 s_2, s_1) \), \( j = (2, 1, 3, 1, 1) \). Then we have the subset family
\[
D_{wj} = \{ e[2], s_1[1], s_1[3], s_3 s_2[1], s_1[1] \} = \{ 12, 2, 123, 3, 2 \} = \{ 12, 123, 2, 3 \}
\]
Note that we remove repetitions in \( D \). The associated \( T \)-fixed configuration is
\[
z_D = (E_{12}, E_{123}, E_2, E_3) \in \text{Gr}(D) = \text{Gr}(2) \times \text{Gr}(3) \times \text{Gr}(1) \times \text{Gr}(1).
\]

3.2 Chamber families

Chamber families have a rich structure. (See \cite{17}, \cite{19}, \cite{25}.) Given a full chamber family \( D_1^+ \), we may omit some of its elements to get a subfamily \( D \subset D_1^+ \). The resulting \textit{chamber subfamilies} can be characterized as follows.

For two sets \( S, S' \subset [n] \), we say \( S \) is \textit{elementwise less than} \( S' \), \( \text{Selt} < S' \), if \( s < s' \) for all \( s \in S, s' \in S' \). Now, a pair of subsets \( C, C' \subset [n] \) is \textit{strongly separated} if
\[
(C \setminus C') \text{elt} < (C' \setminus C) \quad \text{or} \quad (C' \setminus C) \text{elt} < (C \setminus C'),
\]
where \( C \setminus C' \) denotes the complement of \( C' \) in \( C \). A family of subsets is called strongly separated if each pair of subsets in it is strongly separated.

\textbf{Proposition 6} (Le Clerc-Zelevinsky \cite{17}) A family \( D \) of subsets of \([n]\) is a \textit{chamber subfamily}, \( D \subset D_1^+ \) for some \( i \), if and only if \( D \) is strongly separated.

\textbf{Remarks.} (a) Reiner and Shimozono \cite{25} give an equivalent description of strongly separated families. Place the subsets of the family into lexicographic order. Then \( D = (C_1 \leq_{xy} C_2 \leq_{xy} \cdots) \) is strongly separated if and only if it is \textit{“\%-avoiding”}: that is, if \( i_1 \in C_{j_1}, i_2 \in C_{j_2} \) with \( i_1 > i_2, j_1 < j_2 \), then \( i_1 \in C_{j_2} \) or \( i_2 \in C_{j_1} \).

(b) If \( i = (i_1, \ldots, i_l) \) is an initial subword of \( i' = (i_1, \ldots, i_l, \ldots, i_N) \), then \( D_i \subset D_{i'} \). Thus the chamber families associated to decompositions of the longest permutation \( w_0 \) are the maximal strongly separated families.

(c) In \cite{19}, we describe the “orthodontia” algorithm to determine a reduced decomposition \( i \) associated to a given strongly separated family. See also \cite{22}.

\textbf{Examples.} (a) For \( n = 3 \), the chamber families \( D_{121}^+ = \{ 1, 12, 123, 2, 23, 3 \} \) and \( D_{212}^+ = \{ 1, 12, 123, 13, 3, 23 \} \) are the only maximal strongly separated families. The sets 13 and 2 are the only pair not strongly separated from each other.

(b) For \( n = 4 \), the strongly separated family \( D = \{ 24, 34, 4 \} \) is contained in the chamber sets of the reduced words \( i = 312132 \) and \( i = 123212 \). •
Chamber families can be represented pictorially in several ways, one of the most natural being due to Berenstein, Fomin, and Zelevinsky [2]. The wiring diagram or braid diagram of the permutation $w$ with respect to the reduced word $i$ is best defined via an example.

Let $G = GL(4)$, $w = w_0$ (the longest permutation), and $i = 312132$. On the left and right ends of the wiring diagram are the points 1, 2, 3, 4 in two columns. Each point $i$ on the left is connected to the point $w(i)$ on the right by a curve which is horizontal and disjoint from the other curves except for certain crossings. The crossings, read left to right, correspond to the entries of $i$. The first entry $i_1 = 3$ corresponds to a crossing of the curve on level 3 with the one on level 4. (The other curves continue horizontally.) The second entry $i_2 = 1$ crosses the curves on level 1 and 2, and so on.

**FIGURE 1**

If we add crossings only up to the $l$th step, we obtain the wiring diagram of the truncated word $s_{i_1} s_{i_2} \cdots s_{i_l}$.

Now we may construct the chamber family $D_i^+ = (1, 12, 123, 1234, 124, 2, 24, 4, 234, 34)$ as follows. Label each of the curves of the wiring diagram by its point of origin on the left. Into each of the connected regions between the curves, write the numbers of those curves which pass above the region. Then the sets of numbers inscribed in these chambers are the members of the family $D_i^+$. If we list the chambers from left to right, we recover the natural order in which these subsets appear in $D_i^+$.

Another way to picture a chamber family, or any subset family, is as follows. We may consider a subset $C = \{j_1, j_2, \ldots\} \subset [n]$ as a column of $k$ squares in the rows $j_1, j_2, \ldots$. For each subset $C_k$ in the chamber family, form the column associated to it, and place these columns next to each other. The result is an array of squares in the plane called a generalized Young diagram.

For our word $i = 312132$, we draw the (reduced) chamber family as:

$$D_i = \begin{array}{cccc}
1 & \square & & \\
2 & \square & \square & \square \\
3 & \square & & \\
4 & \square & \square & \square & \square \\
\end{array}$$

where the numbers on the left of the diagram indicate the level. See [23], [18], [19].

### 3.3 Varieties and defining equations

To any subset family $D$ we have associated a $T$-fixed point in a product of Grassmannians, $z_D \in \text{Gr}(D)$, and we may define as before the configuration
variety of $D$ to be the closure of the $G$-orbit of $z_D$:

$$\mathcal{F}_D = \overline{G \cdot z_D} \subset \text{Gr}(D);$$

and the flagged configuration variety to be the closure of its $B$-orbit:

$$\mathcal{F}^B_D = \overline{B \cdot z_D} \subset \text{Gr}(D).$$

Furthermore, if $D = D_1$, a chamber family, then the Bott-Samelson variety is the flagged configuration variety of $D_1$:

$$Z_1 = Z^{orb}_1 = F^B_{D_1}.$$

(We could also use the full chamber family $D_1^+$, since the extra coordinates correspond to the standard flag fixed under the $B$-action.)

Thus $\mathcal{F}_D$, $\mathcal{F}^B_D$, and $Z_1$ can be considered as varieties of configurations of subspaces in $C^n$, like the flag and Schubert varieties. We will give defining equations for the Bott-Samelson varieties analogous to those for Schubert varieties.

For a family $D$, define the flagged inclusion variety

$$\mathcal{I}^B_D = \left\{ (V_C)_{C \in D} \in \text{Gr}(D) \mid \forall C, C' \in D, C \subset C' \Rightarrow V_C \subset V_{C'} \right. \text{ and } \forall [i] \in D, V_{[i]} = C^i \left\};

B$ acts diagonally on $\mathcal{I}^B_D$.

**Example.** For $n = 4$, $i = 312132$, we may use the picture in the above example to write the inclusion variety $\mathcal{I}^B_{D_1^+}$ as the set of all 10-tuples of subspaces of $C^4$

$$(V_1, V_{12}, V_{123}, V_{1234}, V_{124}, V_2, V_{24}, V_4, V_{234}, V_{34})$$

with $\dim(V_C) = |C|$ and satisfying the following inclusions:

$$\begin{align*}
0 & \leftarrow \downarrow \leftarrow V_1 \downarrow \leftarrow V_2 \downarrow \leftarrow V_4 \\
C^1 = V_1 & \downarrow \leftarrow V_{12} \downarrow \leftarrow V_{24} \downarrow \leftarrow V_{34} \\
C^2 = V_{12} & \downarrow \leftarrow V_{123} \downarrow \leftarrow V_{124} \downarrow \leftarrow V_{234} \\
C^3 = V_{123} & \leftarrow V_{1234} = C^4
\end{align*}$$

where the arrows indicate inclusion of subspaces.

**Theorem 7** For every reduced word $i$, we have $Z_i \cong \mathcal{I}^B_{D_1^+}$. 16
Proof. Note that the generating point $z_{D_i}$ lies in $\mathcal{I}^B_{D_i}$, and $\mathcal{I}^B_{D_i}$ is $B$-equivariant, so $Z_i \subset \mathcal{I}^B_{D_i}$.

To show the reverse inclusion, we use our previous characterization

$$Z_i \cong Z_i^{fib} = e \times_{G/P_{i_1}} G/B \times_{G/P_{i_2}} G/B \times_{G/P_{i_3}} \cdots \times_{G/P_{i_l}} G/B.$$ 

We may write this variety as the $(l+1)$-tuples of flags $(V^{(k)}_1 \subset V^{(k)}_2 \subset \cdots \subset C^n)$, $k = 0,1,\ldots,l$, such that: $V^{(k)}_i = V^{(k+1)}_i$ for all $k$ and all $i \neq i_k$; and $V^{(0)}_i = C_i$ for all $i$.

Consider the map

$$\theta : Z_i^{fib} \rightarrow \text{Gr}(D)$$

We have seen in Theorem that $Z_i = Z_i^{orb} = \text{Im}(\theta)$, since $\theta = \psi \circ \phi^{-1}$. It remains to show that $\mathcal{I}^B_{D_i} \subset \text{Im}(\theta)$.

For each $k$, define $k^- = \max\{m \mid m < k, i_m = i_k + 1\}$ and $k^+ = \min\{m \mid m > k, i_m = i_k + 1\}$. Then it is easily seen that a configuration $(V_1, V_2, \ldots) \in \text{Gr}(D)$ lies in $\text{Im}(\theta)$ exactly when:

(i) for each $k$, we have $V_k \subset V_{k^-}$ and $V_k \subset V_{k^+}$ provided $k^-$ or $k^+$ is defined;

(ii) for each $k$, if $k^-$ is not defined, then $V_k \subset C_{i_k+1}$; and

(iii) for each $i$, if $k = \min\{m \mid i_m = i+1\}$, then $C_i \subset V_k$.

Note that for any $k$, the $k$th subset of $D_i$ is

$$C_k = s_{i_k} \cdots s_{i_k} [i_k] = s_{i_k} \cdots s_{i_k} [i_k] + [i_k + 1] \subset s_{i_k} \cdots s_{i_k} s_{i_k+1} [i_k + 1] = s_{i_k} \cdots s_{i_k} s_{i_k+1} [i_k+1] = C_{i_k+1}.$$ 

We can write similar inclusions of subsets for the other conditions (i)-(iii). This shows that the inclusions defining $\mathcal{I}^B_{D_i}$ do indeed imply those defining $\text{Im}(\theta)$, Q.E.D. •

Conjecture 8 For any subset family $D$, a configuration $(V_C)_{C \in D} \in \text{Gr}(D)$ lies in $\mathcal{F}_D$ exactly if, for every subfamily $D' \subset D$,

$$\dim(\bigcap_{C \in D'} V_C) \geq \mid \bigcap_{C \in D'} C \mid$$

$$\dim(\bigcup_{C \in D'} V_C) \leq \mid \bigcup_{C \in D'} C \mid$$

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Note that a configuration \((V_1, \ldots, V_l) \in \text{Gr}(D)\) lies in the flagged configuration variety \(F^B_D\) if and only if \((C^1, \ldots, C^n, V_1, \ldots, V_l)\) lies in the unflagged variety \(F^+_{D^+}\) of the augmented diagram \(D^+ \overset{\text{def}}{=} \{1\}, \{2\}, \ldots, \{n\} \cup D\). Hence the above conjecture gives conditions defining flagged configuration varieties as well as unflagged.

**Examples.** (a) If \(D = D_i\) is a chamber family, the conjecture reduces to the previous Theorem.

(b) The conjecture is known if \(D\) satisfies the “northwest condition” (see [18]): that is, the elements of \(D\) can be arranged in an order \(C_1, C_2, \ldots\) such that if \(i_1 \in C_{j_1}, i_2 \in C_{j_2}\), then \(\min(i_1, i_2) \in C_{\min(j_1, j_2)}\). In fact, it suffices in this case to consider only the intersection conditions of the conjecture.

It would be interesting to know whether the determinantal equations implied by the conditions of the above Theorem and Conjecture define \(F_D \subset \text{Gr}(D)\) scheme-theoretically.

Now, let \(D\) be a strongly separated family. We know by Proposition [2] that \(D\) is part of some chamber family \(D_i\), and by Theorem [3] we may take \(i\) so that the projection map \(Z_i = F^B_{D_i} \rightarrow F^B_D\) is birational.

**Example.** Let \(n = 7\), and consider the family \(D\) consisting of the single subset \(C = 12457\). Its configuration variety is the Grassmannian \(F_D = \text{Gr}(5, C^7)\), and its flagged configuration variety is the Schubert variety

\[
F^B_D = X_{211} = \{V \in \text{Gr}(5) \mid C^2 \subset V, \dim(C^5 \cap V) \geq 4\}.
\]

By the orthodontia algorithm [19], we find that this is desingularized by the reduced word \(i = 3465\), for which \(D_i = \{124, 1245, 123457, 12457\}\) and

\[
Z_i = \left\{ (V_{124}, V_{1245}, V_{123457}, V_{12457}) \in \text{Gr}(3) \times \text{Gr}(4) \times \text{Gr}(6) \times \text{Gr}(5) \mid \text{such that} \begin{array}{c} C^2 \subset V_{124} \subset C^4 \subset V_{123457} \subset V_{1245} \subset C^5, \end{array} \right\}.
\]

The desingularization map is the projection

\[
\pi : (V_{124}, V_{1245}, V_{123457}, V_{12457}) \mapsto V_{12457}.
\]

In [3] and Zelevinsky’s work [28], there are given several other desingularizations of Schubert varieties, all of them expressible as configuration varieties.

**4 Schur and Weyl modules**

We relate generalized Schur and Weyl modules for \(GL(n)\), which are defined in completely elementary terms, to the sections of line bundles on configuration varieties, and hence to the coordinate rings of these varieties.
One the one hand, this yields an unexpected Demazure character formula for the Schur modules, including the skew Schur functions and Schubert polynomials. On the other hand, it gives an elementary construction for line-bundle sections on Bott-Samelson varieties.

4.1 Definitions

We have associated to any subset family $D = \{C_1, \ldots, C_k\}$ a configuration variety $\mathcal{F}_D$ with $G$-action, and a flagged configuration variety $\mathcal{F}_D^B$ with $B$-action. Now, assign an integer multiplicity $m(C) \geq 0$ to each subset $C \in D$. For each pair $(D, m)$, we define a $G$-module and a $B$-module, which will turn out to sections of a line bundle on $\mathcal{F}_D$ and $\mathcal{F}_D^B$.

In the spirit of DeRuyts and Desarmenien-Kung-Rota, we construct these “Weyl modules” $M_{D, m}$ inside the coordinate ring of $n \times n$ matrices, and their flagged versions $M_{D, m}^B$ inside the coordinate ring of upper-triangular matrices. (I am grateful to Mark Shifman for pointing out this form of the definition.)

Let $C[x_{ij}]$ (resp. $C[x_{ij}]_{i \leq j}$) denote the polynomial functions in the variables $x_{ij}$ with $i, j \in [n]$ (resp. $x_{ij}$ with $1 \leq i \leq j \leq n$). For $R, C \subset [n]$ with $|R| = |C|$, let

$$\Delta^R_C = \det(x_{ij})_{(i \in R, j \in C)} \in C[x_{ij}]$$

be the minor determinant of the matrix $x = (x_{ij})$ on the rows $R$ and the columns $C$. Further, let

$$\tilde{\Delta}^R_C = \Delta^R_C|_{x_{ij} = 0, \forall i > j} \in C[x_{ij}]_{i \leq j}$$

be the same minor evaluated on an upper triangular matrix of variables.

Now, for a subset family $D = \{C_1, \ldots, C_l\}$, $m = (m_1, \ldots, m_l)$, define the Weyl module

$$M_{D, m} = \text{Span}_C \left\{ \Delta^R_{C_1} \cdots \Delta^R_{C_k} \Delta^R_{C_2} \cdots \Delta^R_{C_l} \biggm| \forall k, m \ R_{km} \subset [n] \text{ and } |R_{km}| = |C_k| \right\}.$$

That is, a spanning vector is a product of minors with column indices equal to the elements of $D$ and row indices taken arbitrarily.

For two sets $R = \{i_1, \ldots, i_c\}$, $C = \{j_1, \ldots, j_c\}$ we say $R \leq_{\text{cmp}} C$ (component-wise inequality) if $i_1 \leq j_1$, $i_2 \leq j_2$, ….. Define the flagged Weyl module

$$M_{D, m}^B = \text{Span}_C \left\{ \tilde{\Delta}^R_{C_1} \cdots \tilde{\Delta}^R_{C_k} \tilde{\Delta}^R_{C_2} \cdots \tilde{\Delta}^R_{C_l} \biggm| \forall k, m \ R_{km} \subset [n] \text{ and } |R_{km}| = |C_k|, R_{km} \leq_{\text{cmp}} C_k \right\}.$$

For $f(x) \in C[x_{ij}]$, a matrix $g \in G$ acts by left translation, $(g \cdot f)(x) = f(g^{-1}x)$. It is easily seen that this restricts to a $G$-action on $M_{D, m}$ and similarly we get a $B$-action on $M_{D, m}^B$.
We clearly have the diagram of $B$-modules:

\[
\begin{align*}
M_{D,m} & \subset \mathbb{C}[x_{ij}] \\
\downarrow & \\
M_{B,m} & \subset \mathbb{C}[x_{ij}]_{i\leq j}
\end{align*}
\]

where the vertical maps ($x_{ij} \mapsto 0$ for $i > j$) are surjective. That is, $M_{B,m}^B$ is a quotient of $M_{D,m}$.

The Schur modules are defined to be the duals

\[
S_{D,m} \overset{\text{def}}{=} (M_{D,m})^* \quad S_{B,m}^B \overset{\text{def}}{=} (M_{B,m}^B)^*.
\]

We will deal mostly with the Weyl modules, but everything we say will of course also apply to their duals.

**Example.** We adopt the “Young diagram” method for picturing subset families. (See Sec 3.2.) Let $n = 4$, $D = \{234, 34, 4\}$, $m = (2, 0, 3)$. (That is, $m(234) = 2$, $m(34) = 0$, $m(4) = 3$.) We picture this by writing each column repeatedly, according to its multiplicity. Zero multiplicity means we omit the column. Thus

\[
(D, m) = \begin{array}{cccc}
2 & \square & \square & \\
3 & \square & \\
4 & \square & \square & \square & \square
\end{array}
\]

\[\tau = \begin{array}{c}
1 \\
2 & 1 \\
3 & 2 \\
4 & 3 & 2 & 4 & 3
\end{array}
\]

The spanning vectors for $M_{D,m}$ correspond to all column-strict fillings of this diagram by indices in $[n]$. For example, the filling $\tau$ above corresponds to

\[
\begin{pmatrix}
\Delta_{134}^{13} & \cdot & \Delta_{23}^{23} & \cdot & \Delta_{3}^{3} & \Delta_{4}^{4} & \Delta_{5}^{5}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cdot & \cdot & x_{12} & x_{13} & x_{14} \\
\cdot & \cdot & x_{22} & x_{23} & x_{24} \\
\cdot & \cdot & x_{32} & x_{33} & x_{34}
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 1 & 2 & 2 \\
3 & 2 & 3 & 3 \\
4 & 3 & 2 & 4 & 4 & 4 & 4
\end{pmatrix}
\]

The last expression is in the letter-place notation of Rota et al [7].

A basis may be extracted from this spanning set by considering only the row-decreasing fillings (a normalization of the semi-standard tableaux), and in fact the Weyl module is the dual of the classical Schur module $S_{\lambda}$ associated to the shape $D$ considered as the Young diagram $\lambda = (5, 2, 2, 0)$.

The spanning elements of the flagged Weyl module $M_{D,m}^B$ correspond to the “flagged” fillings of the diagram: those for which the number $i$ does not appear above the $i^{th}$ level. For the diagram above, all the column-strict fillings are flagged, and $M_{D,m} \cong M_{D,m}^B$. 

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However, for 

\[(D', m) = (\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 0 \\
3 & 0 & 0 & 0 \\
4 & 0 & 0 & 0
\end{array})
\]

the filling \(\tau_1\) is not flagged, since 4 appears on the 3rd level, but \(\tau_2\) is flagged, and corresponds to the spanning element

\[\Delta_2^{34} \Delta_3^{124} \Delta_3^{23} \Delta_3^{2} = \begin{vmatrix}
x_{22} & x_{23} & x_{24} \\
x_{33} & x_{34} & 0 \\
0 & x_{44} & 0
\end{vmatrix} \cdot \begin{vmatrix}
x_{12} & x_{13} & x_{14} \\
x_{22} & x_{23} & x_{24} \\
0 & 0 & x_{44}
\end{vmatrix} \cdot x_{33} \cdot x_{23} \cdot x_{33}.\]

We have \(M_{D,m} \cong M'_{D,m} \cong M_{D,m}^* \cong S^*(5,2,2,0)^*\), the dual of a classical (irreducible) Schur module for \(GL(4)\), and \(M_{D,m}^* \cong S^*(2,5,2,0)\), the dual of the Demazure module with lowest weight \((0,2,5,2)\) and highest weight \((5,2,2,0)\). Cf. [22], [19].

Remarks. (a) In [13] we make a general definition of “standard tableaux” giving bases of the Weyl modules for strongly separated families.
(b) We briefly indicate the equivalence between our definition of the Weyl modules and the tensor product definition given in [1], [22], [18].

Let \(Y = Y_{D,m} \subset N \times N\) be the generalized Young diagram of squares in the plane associated to \((D, m)\) as in the above examples, and let \(U = (C^n)^*\). One defines \(M_Y^\text{tensor} = U^\otimes Y \gamma_Y\), where \(\gamma_Y\) is a generalized Young symmetrizer. The spanning vectors \(\Delta_\tau\) of \(M_{D,m}\) correspond to the fillings \(\tau : Y \to [n]\). Then the map

\[M_{D,m} \rightarrow M_{D,m}^\text{tensor} \quad \Delta_\tau \mapsto \left(\bigotimes_{(i,j) \in Y} e^*_{\tau(i,j)}\right) \gamma_Y\]

is a well-defined isomorphism of \(G\)-modules, and similarly for the flagged versions. This is easily seen from the definitions, and also follows from the Borel-Weil theorems proved below and in [18].

4.2 Borel-Weil theory

A configuration variety \(\mathcal{F}_D \subset \text{Gr}(D)\) has a natural family of line bundles defined by restricting the determinant or Plucker bundles on the factors of \(\text{Gr}(D)\). For
$D = (C_1, C_2, \ldots)$, and multiplicities $m = (m_1, m_2, \ldots)$, we define

$$\mathcal{L}_m \subset \bigoplus_i \mathcal{O}(m_1, m_2, \ldots)$$

$$\mathcal{F}_D \subset \text{Gr}(D) = \text{Gr}(|C_1|) \times \text{Gr}(|C_2|) \times \cdots$$

We denote by the same symbol $\mathcal{L}_m$ this line bundle restricted to $\mathcal{F}_B$. Note that in the case of a Bott-Samelson variety $\mathcal{F}_D = Z_i$, this is the well-known line bundle $\mathcal{L}_m \cong P_{i_1} \times \cdots \times P_{i_l} \times C$

$$(p_1, \ldots, p_l, v) \cdot (b_1, \ldots, b_l) \overset{\text{def}}{=} (p_1 b_1, \ldots, p_l b_l, \varpi_i(b_1^{-1})^{m_1} \cdots \varpi_i(b_l^{-1})^{m_l}, v),$$

$\varpi_i$ denoting the fundamental weight $\varpi_i(\text{diag}(x_1, \ldots, x_n)) = x_1 x_2 \cdots x_i$. Note that if $m_k \geq 0$ for all $k$ (resp. $m_k > 0$ for all $k$) then $\mathcal{L}_m$ is effective (resp. very ample). However, $\mathcal{L}_m$ may be effective even if some $m_k < 0$. See [13].

**Proposition 9** Let $(D, m)$ be a strongly separated subset family with multiplicity. Then we have

(i) $M_{D,m} \cong H^0(\mathcal{F}_D, \mathcal{L}_m)$

and $H^i(\mathcal{F}_D, \mathcal{L}_m) = 0$ for $i > 0$.

(ii) $M^B_{D,m} \cong H^0(\mathcal{F}^B_D, \mathcal{L}_m)$

and $H^i(\mathcal{F}^B_D, \mathcal{L}_m) = 0$ for $i > 0$.

(iii) $\mathcal{F}_D$ and $\mathcal{F}^B_D$ are normal varieties, projectively normal with respect to $\mathcal{L}_m$, and have rational singularities.

**Proof.** First, recall that we can identify the sections of a bundle over a single Grassmannian, $\mathcal{O}(1) \rightarrow \text{Gr}(i)$, with linear combinations of minors in the homogeneous Stiefel coordinates

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1i} \\ \vdots & \ddots & \vdots \\ x_{nj} & \cdots & x_{ni} \end{pmatrix} \in \text{Gr}(i),$$

namely the $i \times i$ minors $\Delta^R(x)$ on the rows $R \subset [n], \ |R| = i$. Thus, a typical spanning element of $H^0(\text{Gr}(D), \mathcal{O}(m))$ is the section

$$\Delta^{R_1}(x^{(1)}) \cdots \Delta^{R_{i_1}}(x^{(1)}) \Delta^{R_2}(x^{(2)}) \cdots \Delta^{R_{i_l}}(x^{(l)}),$$

where $x^{(k)}$ represents the homogeneous coordinates on each factor $\text{Gr}(|C_k|)$ of $\text{Gr}(D)$, and $R_{km}$ are arbitrary subsets with $|R_{km}| = i_k$.

Now, restrict this section to $\mathcal{F}_D \subset \text{Gr}(D)$ and then further to the dense $G$-orbit $G \cdot z_D \subset \mathcal{F}_D$. Parametrizing the orbit by $g \rightarrow g \cdot z_D$, we pull back
the resulting sections of \( H^0(\mathcal{F}_D, \mathcal{L}_m) \) to certain functions on \( G \subset \text{Mat}_{n \times n}(\mathbb{C}) \), which are precisely the products of minors defining the spanning set of \( M_{D,m} \).

This shows that

\[
M_{D,m} \cong \text{Im} \left[ H^0(\text{Gr}(D), \mathcal{O}(m)) \xrightarrow{\text{rest}} H^0(\mathcal{F}_D, \mathcal{L}_m) \right].
\]

Similarly for \( B \)-orbits, we have

\[
M_{D,m}^B \cong \text{Im} \left[ H^0(\text{Gr}(D), \mathcal{O}(m)) \xrightarrow{\text{rest}} H^0(\mathcal{F}_D^B, \mathcal{L}_m) \right].
\]

Now we invoke the key vanishing result, [18] Prop. 28 (due to W. van der Kallen and S.P. Inamdar, based on the work of O. Mathieu [21], P. Polo, et.al.) The conditions (\( \alpha \)) and (\( \beta \)) of that Proposition apply to \( \mathcal{F}_D \) because \( D \) is contained in a chamber family \( D^+_1 \) (Prop. 6 above). Furthermore, the proof of [18] Prop. 28 goes through identically with \( \mathcal{F}_D^B \) in place of \( \mathcal{F}_D \), merely replacing \( \mathcal{F}_{w_0, u_1, \ldots, u_r} \) by \( \mathcal{F}_{e, u_1, \ldots, u_r} \).

All of the assertions of our Proposition now follow immediately from the corresponding parts of [18], Prop. 28.

**Proposition 10** Suppose \((D, m), (\tilde{D}, \tilde{m})\) are strongly separated subset families with \( D \subset \tilde{D}, \tilde{m}(C) = m(C) \) for \( C \in D, m(C) = 0 \) otherwise. Then the natural projection \( \pi : \text{Gr}(\tilde{D}) \to \text{Gr}(D) \) restricts to a surjection \( \pi : \mathcal{F}_{\tilde{D}} \to \mathcal{F}_D \), and induces an isomorphism

\[
\pi^* : H^0(\mathcal{F}_D, \mathcal{L}_m) \xrightarrow{\sim} H^0(\mathcal{F}_{\tilde{D}}, \mathcal{L}_{\tilde{m}}),
\]

and similarly for the flagged case.

**Proof.** For the unflagged case, this follows immediately from [18], Prop. 28. Again, the argument given there goes through for the flagged case as well.

**Remarks.** (a) Note that the proposition holds even if \( \dim \mathcal{F}_{\tilde{D}} > \dim \mathcal{F}_D \).

(b) The Proposition allows us to reduce Weyl modules for strongly separated families to those for maximal strongly separated families, that is chamber families.

We may conjecture that the results of this section hold not only in the strongly separated case, but for all subset families and configuration varieties.

### 4.3 Demazure’s character formula

We now examine how the iterative structure of Bott-Samelson varieties influences the associated Weyl modules.

Define Demazure’s isobaric divided difference operator \( \Lambda_i : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n] \),

\[
\Lambda_i f = \frac{x_if - x_{i+1}s_if}{x_i - x_{i+1}}.
\]
For example for \( f(x_1, x_2, x_3) = x_1^2 x_2 x_3 \),

\[
\Lambda_2 f(x_1, x_2, x_3) = \frac{x_2(x_1^2 x_2 x_3) - x_3(x_1^2 x_2^2)}{x_2^2 - x_3} = x_1^2 x_2 x_3 (x_2 + x_3).
\]

For any permutation with a reduced decomposition \( \omega = s_{i_1} \ldots s_{i_l} \), define

\[
\Lambda_\omega \overset{\text{def}}{=} \Lambda_{i_1} \ldots \Lambda_{i_l},
\]

which is known to be independent of the reduced decomposition chosen.

By the (dual) character of a \( G \)- or \( B \)-module \( M \), we mean

\[
\text{char}^* M = \text{tr} (\text{diag}(x_1, \ldots, x_n) | M^*) \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].
\]

(We must take duals to get polynomial functions as characters.) Let \( \omega_i \) denote the \( i \)th fundamental weight, the multiplicative character of \( B \) defined by \( \omega_i (\text{diag}(x_1, \ldots, x_n)) = x_1 x_2 \ldots x_i \).

**Proposition 11** Suppose \((D, m)\) is strongly separated, and

\[
D \subset D^+_1 = \{[1], \ldots, [n], C_1, \ldots, C_l\},
\]

for some reduced word \( i = (i_1, \ldots, i_l) \). Define \( \mbar = (k_1, \ldots, k_n, m_1, \ldots, m_l) \) by \( \mbar(C) = m(C) \) for \( C \in D \), \( \mbar(C) = 0 \) otherwise. Then

\[
\text{char}^* M^B_{D, \mbar} = \omega_i^{k_1} \ldots \omega_i^{k_n} \Lambda_{i_1} \omega_i^{m_1} \ldots \Lambda_{i_l} \omega_i^{m_l}.
\]

Furthermore,

\[
\text{char}^* M_{D, \mbar} = \Lambda_w \text{ char}^* M^B_{D, \mbar},
\]

where \( w_0 \) denotes the longest permutation.

**Remark.** We explain in [16] how one can recursively generate the standard tableaux for \( M^B_D \) (in [13]) by “quantizing” this character formula. See also [19].

We devote the rest of this section to proving the Proposition.

For a subset \( C = \{j_1, j_2, \ldots\} \subset [n] \), and a permutation \( w \), let \( wC = \{w(j_1), w(j_2), \ldots\} \), and for a subset family \( D = \{C_1, C_2, \ldots\} \), let \( wD = \{wC_1, wC_2, \ldots\} \).

Now, for \( i \in [n - 1] \), let

\[
\Lambda_i D \overset{\text{def}}{=} \{s_i[i]\} \cup s_i D,
\]

where \( s_i[i] = \{1, 2, \ldots, i - 1, i + 1\} \). We say that \( D \) is \( i \)-free for \( i \in [n] \) if for every \( C \in D \), we have \( C \cap \{i, i + 1\} \neq \{i + 1\} \).

**Lemma 12** Suppose \((D, m)\) is strongly separated and \( i \)-free.

\((i)\) \( \mathcal{F}^B_{\Lambda_i D} \cong P_i \times^B \mathcal{F}^B_D \).

\((ii)\) \( \mathcal{F}^B_{s_i D} \cong P_i \cdot \mathcal{F}^B_D \subset \text{Gr}(D) \).

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(iii) The projection $F_{\Lambda,D}^B \to F_{s_i,D}^B$ is regular, surjective, and birational.

(iv) Let $\tilde{m}$ be the multiplicity on $\Lambda_iD$ defined by $\tilde{m}(s_iC) = m(C) = m_i(C)$ for $C \in D$, $\tilde{m}(s_i[\iota]) = m_0$. The bundle $L_m \to F_{\Lambda,D}^B$ is isomorphic to

$$L \cong P_i \times ((\varpi_i^{m_0})\circ \otimes L_m),$$

where $(\varpi_i^{m_0})^* \otimes L_m$ indicates the bundle $L_m \to F_{\Lambda,D}^B$ with its $B$-action twisted by the multiplicative character $(\varpi_i^{m_0})^* = \varpi_i^{-m_0}$.

Proof. (i) Since $D$ is $i$-free, we have $U_i z_D = z_D$, where $U_i$ is the one-dimensional unipotent subgroup corresponding to the simple root $\alpha_i$. We may factor $B$ into a direct product of subgroups, $B = U_i B' = B' U_i$. Then

$$F_{D}^B = B \cdot z_D = B' \cdot z_D.$$

Hence the $T$-fixed point $(s_i, z_D) \in P_i \times B F_{D}^B$ has a dense $B$-orbit:

$$B \cdot (s_i, z_D) = \overline{(U_i B' s_i, z_D)} = \overline{(U_i s_i, B' \cdot z_D)} = P_i \times B F_{D}^B.$$

Clearly, the injective map

$$\psi: P_i \times B Gr(D) \to Gr(i) \times Gr(D),
\begin{array}{l}
p(V) \\
\end{array}
\mapsto
\begin{array}{l}
(pC^i, pV)
\end{array}$$

takes $\psi(s_i, z_D) = z_{\Lambda,D}$, the $B$-generating point of $F_{\Lambda,D}^B$. Thus $\psi: P_i \times B F_{D}^B \to F_{\Lambda,D}^B$ is an isomorphism.

(ii+iii) By the above, the projection is a bijection on the open $B$-orbit, and hence is birational. The image of the projection is $P_i \times B F_{D}^B$, which must be closed since $P_i \times B F_{D}^B$ is a proper (i.e. compact variety).

(iv) Clear from the definitions. 

Lemma 13 Let $(D, m)$ be a strongly separated family and $i \in [n−1]$. Let

$$F' = P_i \times B F_{D}^B,
L' = P_i \times B L_m,$$

so that $L' \to F'$ is a line bundle. Then

$$\text{char}^* H^0(F', L') = \Lambda_i \text{ char}^* H^0(F_{D}^B, L_m).$$

Proof. By Demazure’s analysis of induction to $P_i$ (see [5], “construction élémentaire”) we have

$$\Lambda_i \text{ char}^* H^0(F_{D}^B, L_m) = \text{ char}^* H^0(F', L') - \text{ char}^* H^1(P_i/B, H^1(F_{D}^B, L_m)).$$

However, we know by [13], Prop.28 that $H^0(F_{D}^B, L_m)$ has a good filtration, so that the $H^1$ term above is zero. 

Corollary 14 If \((D, m)\) is strongly separated and \(i\)-free, and \((\Lambda_i D, \tilde{m})\) is a diagram with multiplicities \(\tilde{m}(s_i C) = m(C)\) for \(C \in D\), \(\tilde{m}(s_i[i]) = m_0\), then
\[
\text{char}^* M^B_{\Lambda_i D, \tilde{m}} = \Lambda_i \omega_i^{m_0} \text{ char}^* M^B_{D, m}.
\]

If \(m_0 = 0\), then
\[
\text{char}^* M^B_{s_i D, m} = \text{ char}^* M^B_{\Lambda_i D, \tilde{m}} = \Lambda_i \text{ char}^* M^B_{D, m}
\]

This follows immediately from the above Lemmas and Proposition 10.

Proof of Proposition. The first formula of the Proposition now follows from the above Lemmas and Prop 10. The second statement follows from Demazure’s character formula, combined with the vanishing result of \([18]\) Prop.28. •.

5 Schubert polynomials

In this section, we again work with \(G = GL(n)\). As a general reference, see Fulton [8].

There are two classical computations of the singular cohomology ring \(H^*(G/B, \mathbb{C})\) of the flag variety. That of Borel identifies the cohomology with a coinvariant algebra
\[
c : H^*(G/B, \mathbb{C}) \sim \mathbb{C}[x_1, \ldots, x_n]/I_+,
\]
where \(I_+\) is the the ideal generated by the non-constant symmetric polynomials. The map \(c\) is an isomorphism of graded \(\mathbb{C}\)-algebras, and the generator \(x_i\) represents the Chern class of the \(i\)th quotient of the tautological flag bundle. (This is not the dual of an effective divisor.)

The alternative picture of Schubert gives as a linear basis for \(H^*(G/B, \mathbb{C})\) the Schubert classes \(\sigma_w = [X_{w_0 w}]\), the Poincare duals of the Schubert varieties.

The isomorphism between these pictures was defined by Bernstein-Gelfand-Gelfand [3] and by Demazure [4], and given a precise combinatorial form by Lascoux and Schutzenberger [16]. It identifies certain Schubert polynomials \(S(w) \in \mathbb{C}[x_1, \ldots, x_n]\) with \(c(\sigma_w) = S(w) \pmod{I_+}\), and enjoying many remarkable properties.

They can be defined combinatorially by a descending recurrence, starting with the representative of the fundamental class of \(G/B\). For any permutation \(w\) with \(w s_i < w\) in the Bruhat order, and \(w_0\) the longest permutation, we have
\[
S(w_0) = x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}
\]
\[
S(ws_i) = \partial_i S(w),
\]

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where we use the divided difference operator \( \partial_i : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n] \),

\[
\partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}.
\]

(Note that \( A_i = \partial_i x_i \). This is special to the root system of type \( A_{n-1} \).)

**Example.** For \( G = GL(3) \), we have \( S(w_0) = x_1^2x_2, S(s_1s_2) = x_1x_2, S(s_2s_1) = x_1^2, S(s_2) = x_1 + x_2, S(s_1) = x_1, S(e) = 1 \). 

To compute any \( S(w) \), we write \( w_0 = ws_{i_1} \cdots s_{i_r} \), for some reduced word \( s_{i_1} \cdots s_{i_r} \), and we have

\[
S(w) = \partial_{i_1} \cdots \partial_{i_r} (x_1^{n-1}x_2^{n-2} \cdots x_{n-1}).
\]

In particular, we may take \( i_k \) to be the first ascent of \( w_k = ws_{i_1} \cdots s_{i_k-1} \); that is, \( i_k \) is the smallest \( i \) such that \( w_k(i+1) > w_k(i) \).

We now give a completely different geometric interpretation of the polynomials \( S(w) \) in terms of configuration varieties and Weyl modules. For a permutation \( w \) define the inversion family \( I(w) = \{ C_1(w), \ldots, C_{n-1}(w) \} \) with

\[
C_j(w) = \{ i \in [n] \mid i < j, w(i) > w(j) \}
\]

We may write this in our usual form \( (D, \mathbf{m}) \) by dropping any of the \( C_j(w) \) which are empty, and counting identical sets with multiplicity. We use the same symbol \( I(w) \) to denote this multiset \( (D, \mathbf{m}) \), so that \( I(w) - C \) means we decrease by one the multiplicity of the element \( C \in I(w) \). It is well-known that \( I(w) \) is strongly separated. (In fact, it is northwest. See \([22], [24], [18]\))

**Theorem 15** (Kruskeiwicz-Pragacz [14])

\[
\text{char}^* M^B_{I(w)} = S(w).
\]

**Proof.** (Magyar-Reiner-Shimozono) Let \( \chi(w) = \text{char}^* M^B_{I(w)} \). We must show that \( \chi(w) \) satisfies the defining relations of \( S(w) \).

First, \( I(w_0) = \{ [1], \ldots, [n-1] \} \),

\[
M^B_{I(w_0)} = C \cdot \tilde{\Delta}^1_1 \tilde{\Delta}^2_2 \cdots \tilde{\Delta}^{n-1}_{[n-1]},
\]

a one-dimensional \( B \)-module, and \( \chi(w_0) = x_1^{n-1}x_2^{n-2} \cdots x_{n-1} \).

Now, suppose \( ws_i < w \), and \( i \) is the first ascent of \( ws_i \). Then the \( w(i) \)th element of \( I(w) \) is \( C_{w(i)}(w) = [i] \). Letting

\[
I'(w) \overset{\text{def}}{=} I(w) - \{ [i] \},
\]

it is easily seen that:

(i) \( I'(w) \) is \( i \)-free,
(ii) \( I(w) = I'(w) \cup \{[i]\} \), and
(iii) \( I(ws_i) = s_iI'(w) \cup \{[i-1]\} \).
(Set \([0] = \emptyset\).)

Hence we obtain trivially:

\[
\chi(w) = x_1 \cdots x_i \text{ char}^* M^B_{I'(w)}
\]

\[
\chi(ws_i) = x_1 \cdots x_{i-1} \text{ char}^* M^B_{s_iI'(w)}.
\]

Since \( I'(w) \) is strongly separated and \( i \)-free, Cor 14 implies that

\[
\text{char}^* M^B_{s_iI'(w)} = \Lambda_i \text{ char}^* M^B_{I'(w)}.
\]

This is the key step of the proof.

Thus we have

\[
\chi(ws_i) = (x_1 \cdots x_{i-1}) \Lambda_i \text{ char}^* M^B_{I'(w)}
\]

\[
= \Lambda_i x_i^{-1} (x_1 \cdots x_i) \text{ char}^* M^B_{I'(w)}
\]

\[
= \Lambda_i x_i^{-1} \chi(w)
\]

\[
= \partial_i \chi(w).
\]

But now, using the first-ascent sequence to write \( w_0 = ws_i \cdots s_r \), we compute

\[
\chi(w) = \partial_1 \cdots \partial_r (x_1^{n-1}x_2^{n-2} \cdots x_{n-1}) = S(w).
\]

Our Demazure character formula (Prop 11) now allows us to compute Schubert polynomials by a completely different recursion from the usual one. In particular, the defining recursion goes from higher to lower degree, whereas our Demazure formula goes from lower to higher.

**Example.** For the permutation \( w = 24153 \) in \( GL(5) \), we have \( I(w) = \{12, 24\} \) (neglecting the empty set). Then the first-ascent sequence gives us:

\[
S(w) = \partial_1 \partial_3 \partial_2 \partial_1 \partial_4 \partial_3 (x_1^4 x_2^2 x_3^2 x_4).
\]

However, it is easier to compute that \( I(w) \subseteq D_i^+ \) for a chamber family with \( i = 132 \) (= reduced word \( s_1 s_3 s_2 \)), so that \( D_i = \{2, 124, 24\} \) and

\[
D_i^+ = \{1, 12, 123, 1234, 12345, 2, 124, 24\}
\]

\[
m = (0, 1, 0, 0, 0, 0, 0, 1)
\]

\[
S(w) = x_1x_2 \Lambda_1 \Lambda_3 \Lambda_2 (x_1x_2)
\]

\[
= x_1x_2 (x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4)
\]

See [19] for more examples of such computations.

•
6 Appendix: Non-reduced words

Let $G$ again be an arbitrary reductive group of rank $r$.

For future reference, we note that many of our results hold when the decomposition $w = s_{i_1} \cdots s_{i_l}$ is not of minimal length (that is, $\ell(w) < l$). We call the resulting $i = (i_1, \ldots, i_l)$ (with $i_k \in \{1, \ldots, r\}$) a non-reduced word.

In this case the quotient and fiber product definitions of the Bott-Samelson variety apply without change, and we still have $Z_{i} \cong Z_{i}^{\text{quo}} \cong Z_{i}^{\text{fib}}$, as shown in Thm 1(i). However, $Z_i$ is no longer the $B$-orbit closure of a $T$-fixed point, so we can no longer define $Z_{i}^{\text{orb}}$. Nevertheless, the map

$$\psi : X_l \to \text{Gr}_G(i) \overset{\text{def}}{=} G/\widehat{P}_{i_1} \times \cdots \times G/\widehat{P}_{i_l}$$

of Thm 1(ii) is still injective on $Z_{i}^{\text{quo}} \subset X_l$ (the first part of the proof of Thm 1(ii) is unchanged). Thus we may define an “embedded” version of $Z_i$,

$$Z_{i}^{\text{emb}} \overset{\text{def}}{=} \psi(Z_{i}^{\text{quo}}) \subset \text{Gr}_G(i),$$

so that $Z_{i}^{\text{emb}} = Z_{i}^{\text{orb}}$ if $i$ is reduced.

We can also define analogues of Weyl modules for a general $G$ and $i$. We once again have the minimal-degree line bundles $O(1)$ over the $G/\widehat{P}_i$, and hence $O(m) = O(m_{i_1}, \ldots, m_{i_l})$ over $\text{Gr}_G(i)$. Let $\mathcal{L}_m$ be the restriction of $O(m)$ to $Z_{i}^{\text{emb}}$. Then define

$$M_{i,m}^{B} \overset{\text{def}}{=} H^0(Z_i, \mathcal{L}_m).$$

These modules no longer embed in $\mathbb{C}[B]$, but they do have a spanning set of Plücker coordinates, the restrictions of sections from the ambient space $\text{Gr}_G(i)$:

**Proposition 16** Let $i = (i_1, \ldots, i_l)$ be an arbitrary word (not necessarily reduced), and $m = (m_{i_1}, \ldots, m_{i_l})$ with $m_j \geq 0$ for all $j$.

Then the restriction map

$$H^0(\text{Gr}_G(i), O(m)) \to H^0(Z_i, \mathcal{L}_m)$$

is surjective. Furthermore, $H^i(Z_i, \mathcal{L}_m) = 0$ for $i > 0$, and the Demazure character formula also holds:

$$\text{char}^* M_{i,m}^{B} = \Lambda_{i_1} \varpi_{i_1}^{m_{i_1}} \cdots \Lambda_{i_l} \varpi_{i_l}^{m_{i_l}},$$

$\varpi_i$ being the (multiplicative) fundamental weights and $\Lambda_i$ the Demazure operators on the ring of characters of $T$.

Once again the proof goes through as before, making appeal to the arguments of [18], Prop 28.
In the case of $G = GL(n)$, the $Z_i$ for non-reduced $i$ again have an explicit interpretation as configuration varieties. This is clear from the fiber-product realization $Z_i \cong Z_i^{fib}$; each extra factor in the Bott-Samelson variety corresponds to one new space in the data of the configuration variety.

For example, for $G = GL(3)$ and $i = 2112$, the Bott-Samelson variety is:

$$Z_i \cong \left\{ (V_2, V_1, V_2', V_1') \in \text{Gr}(2) \times \text{Gr}(1) \times \text{Gr}(1) \times \text{Gr}(2) \mid \text{with } C^1 \subset V_2 \supset V_1 \text{ and } V_2 \supset V_1' \subset V_2' \right\}.$$

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