Thermoacoustic tomography with an arbitrary elliptic operator

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Abstract
Thermoacoustic tomography is a term for the inverse problem of determining of one of the initial conditions of a hyperbolic equation from boundary measurements. In past publications, both stability estimates and convergent numerical methods for this problem were obtained only under some restrictive conditions imposed on the principal part of the elliptic operator. In this paper, logarithmic stability estimates are obtained for an arbitrary variable principal part of that operator. Convergence of the quasi-reversibility method to the exact solution is also established for this case. Both complete and incomplete data collection cases are considered.

1. Introduction
The goal of this paper is to show that logarithmic stability estimates as well as convergent numerical methods for the inverse problem of determining an initial condition in a general hyperbolic PDE of the second order can be obtained without any restrictions on its coefficients, except of some natural ones. In all previous publications on this topic, the principal part of the elliptic operator was subjected to some restrictive conditions. Naturally, our stability estimates imply uniqueness. Both complete and incomplete data collection cases are considered. We assume here that the data are given on the infinite time interval $t \in (0, \infty)$. Point 3 of remark 3.1 (subsection 3.1) justifies this assumption. For brevity, we are not concerned here with finest assumptions, such as, e.g., minimal smoothness, etc.

In thermoacoustic tomography (TAT), a short radio frequency pulse is sent into a biological tissue [1, 9]. Some energy is absorbed. It is well known that malignant lesions absorb more energy than healthy ones. Then the tissue expands and radiates a pressure wave, which is the solution of the following Cauchy problem:

$$u_{tt} = c^2(x)\Delta u, \quad x \in \mathbb{R}^3, \ t > 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0.$$
The function \( u(x, t) \) is measured by transducers at certain locations either at the boundary of the medium of interest or outside this medium. The function \( f(x) \) characterizes the absorption of the medium. Hence, if one knew the function \( f(x) \), then one would know the locations of malignant spots. The inverse problem consists in determining \( f(x) \) using those measurements.

First, we apply a well-known analogue of the Laplace transform to obtain a similar inverse problem for a parabolic PDE. Next, previous results of the author [17, 18] are used. In the complete data case, the logarithmic stability estimate follows from [17]. In the case when the data are given on a hyperplane, we significantly modify the proof of theorem 1 of [18]. More precisely, we prove our logarithmic stability estimate for an integral inequality rather than for the parabolic PDE. We need this generalization to establish the convergence rate of our numerical method. Results of both publications [17, 18] were obtained via Carleman estimates. In particular, a quite technical non-standard Carleman estimate was derived in [18], see lemma 3.1 in subsection 3.3. We refer to [33] for another logarithmic stability estimate of the initial condition of a parabolic equation with the self-adjoint operator \( L \) in a finite domain. A Carleman estimate was also used in this reference. An interesting feature of [33] is that observations are performed on an internal subdomain for times \( t \in (\tau, T) \) where \( \tau > 0 \). In addition, a numerical method was developed in [33].

1.1. Statements of inverse problems

Below \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). Let \( \Omega \subset \{x_1 > 0\} \) be a bounded domain with the boundary \( \partial \Omega \in C^4 \). For an arbitrary number \( T > 0 \), denote
\[
Q_T = \Omega \times (0, T), \quad S_T = \partial \Omega \times (0, T), \quad P = \{x_1 = 0\}, \quad P_T = P \times (0, T).
\]

Let \( m \geq 1 \) and \( k \geq 0 \) be integers and \( \alpha \in (0, 1) \). Below \( H^m, H^{2m, m} \) are Sobolev spaces, \( C^{k+\alpha}, C^{2k+\alpha, k+\alpha/2} \) are Hölder spaces and \( C^k, C^{2k} \) are spaces of continuously differentiable functions with standard norms. Traditionally (although not always) spaces \( C^{k+\alpha} \) and \( C^k \) are defined for bounded domains [25], whereas spaces \( C^{2k+\alpha, k+\alpha/2} \) are defined for both bounded and unbounded domains of the form \( G \times (0, T) \), where \( G \subset \mathbb{R}^n \) is an arbitrary domain [26]. Hence, for the sake of definiteness only, we define the space \( C^k(G) \) for an arbitrary domain \( G \subset \mathbb{R}^n \) as
\[
C^k(G) = \left\{ u : \|u\|_{C^k(G)} = \sum_{|\beta| \leq k} \sup_{G} |D^\beta u| < \infty \right\},
\]
where \( \beta \) is the multi-index. For convenience of notations, we set below \( C^k(\mathbb{R}^n) := C^k(\mathbb{R}^n) \).

Also, we need the space \( C^{k+2}(\mathbb{R}^n \times [0, T]) \),
\[
C^{k+2}(\mathbb{R}^n \times [0, T]) = \left\{ u(x, t) : \|u\|_{C^{k+2}(\mathbb{R}^n \times [0, T])} = \sum_{|\beta| \leq 2} \sup_{\mathbb{R}^n \times (0, T)} |D_t^\beta D_x^\gamma u| < \infty \right\}.
\]
We have
\[
C^{k+\alpha} \subset C^{k+1}, \quad C^{k+2}(\mathbb{R}^n \times [0, T]) \subset C^{2k+\alpha, 1+\alpha/2}(\mathbb{R}^n \times [0, T]),
\]
and
\[
\|u\|_{C^{2k+\alpha, 1+\alpha/2}(\mathbb{R}^n \times [0, T])} \leq \|u\|_{C^{k+2}(\mathbb{R}^n \times [0, T])}, \quad \forall u \in C^{k+2}(\mathbb{R}^n \times [0, T]), \forall \alpha \in (0, 1).
\]

Here, the constant \( C = C(\alpha) > 0 \) depends only on the number \( \alpha \in (0, 1) \).

Denote
\[
p := p(n) := \begin{cases} \frac{n + 1}{2} & \text{if } n \text{ is even,} \\ \frac{n}{2} & \text{if } n \text{ is odd.} \end{cases}
\]
Consider the elliptic operator $L$ of the second order with its principal part $L_0$,
\begin{equation}
Lu = \sum_{i,j=1}^{n} a_{i,j}(x)u_{x_i x_j} + \sum_{j=1}^{n} b_j(x)u_{x_j} + c(x)u, \quad x \in \mathbb{R}^n, \tag{1.6}
\end{equation}
\begin{equation}
L_0 u = \sum_{i,j=1}^{n} a_{i,j}(x)u_{x_i x_j}, \quad x \in \mathbb{R}^n. \tag{1.7}
\end{equation}

\begin{equation}
\mu_1 |\eta|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\eta_i \eta_j \leq \mu_2 |\eta|^2, \quad \forall x \in \mathbb{R}^n, \forall \eta \in \mathbb{R}^n; \mu_1, \mu_2 = \text{const.} > 0. \tag{1.8}
\end{equation}

$a_{i,j}, b_j, c \in C^{p+3}(\mathbb{R}^n)$. \tag{1.9}

Let the function $f(x)$ be such that
\begin{equation}
f \in H^{p+5}(\mathbb{R}^n), \quad f(x) = 0, \quad x \in \mathbb{R}^n \setminus \Omega, \tag{1.10}
\end{equation}
where the number $p$ is defined in (1.5).

Consider the following Cauchy problem:
\begin{equation}
\begin{aligned}
&u_{tt} = Lu, \quad x \in \mathbb{R}^n, \quad t \in (0, \infty), \quad (1.11) \\
&u(x, 0) = f(x), \quad u_t(x, 0) = 0. \quad (1.12)
\end{aligned}
\end{equation}

Existence and uniqueness of the solution $u \in H^2(\mathbb{R}^n \times (0, T)), \forall T > 0$ of this problem is guaranteed by theorem 2.2 (section 2). We consider the following two inverse problems.

**Inverse problem 1 (IP1).** Suppose that conditions (1.5)–(1.10) hold. Let the function $u \in H^2(\mathbb{R}^n \times (0, T)), \forall T > 0$ be the unique solution of problem (1.11), (1.12). Assume that the function $f(x)$ is unknown. Determine this function, assuming that the following function $\varphi_1(x, t)$ is known:
\begin{equation}
|_{S_\infty} = \varphi_1(x, t). \tag{1.13}
\end{equation}

**Inverse problem 2 (IP2).** Suppose that conditions (1.5)–(1.10) hold. Let the function $u \in H^2(\mathbb{R}^n \times (0, T)), \forall T > 0$ be the unique solution of problem (1.11), (1.12). Assume that the function $f(x)$ is unknown. Determine this function, assuming that the following function $\varphi_2(x, t)$ is known
\begin{equation}
|_{\partial P_\infty} = \varphi_2(x, t). \tag{1.14}
\end{equation}

IP1 has complete data collection, since the function $\varphi_1$ is known at the entire boundary of the domain of interest $\Omega$. On the other hand, IP2 represents a special case of incomplete data collection, since $\Omega \subset \{x_1 > 0\}$.

### 1.2. Brief overview of published results

TAT has attracted significant interest in the past several years. We now provide a brief overview of published mathematical results for TAT. We refer to [21] for a review paper. Neither stability estimates nor convergent numerical methods for an arbitrary time-independent principal part $L_0$ in (1.7) were obtained in the past. Except for the case $c(x) \equiv 1$ in [22], convergent numerical methods are known only for the case of complete data collection, i.e. when boundary measurements are given at the entire boundary of the domain of interest. Explicit formulas for the reconstruction of the function $f(x)$ for IP1 in the case when in (1.1) $c \equiv 1$ are derived in a
number of publications, see, e.g. [7–9, 21, 22]. These formulas lead to some stability estimates as well as to numerical methods with good performances.

Another approach to IP1 and IP2 is to consider the case when both Dirichlet and Neumann boundary conditions are given at $S_T$ for IP1 and at $P_T$ for IP2. An elementary, well known and stable procedure of deriving the Neumann boundary condition from the given Dirichlet boundary condition for both IP1 and IP2 is described in subsection 3.1 for the parabolic PDE. A very similar procedure takes place in the hyperbolic case. Consider now IP1. Since a certain norm of the Neumann boundary condition at $S_T$ can be estimated from the above by another norm of the data $\varphi_1(x, t)$ for $(x, t) \in S_T$, then the problem of estimating the initial condition $f(x)$ can be reformulated in a slightly different setting as the Cauchy problem for equation (1.11) with the lateral Dirichlet and Neumann data at $S_T$. This problem consists in estimating the function $u(x, t)$ inside the time cylinder $Q_T$.

We now comment on the Lipschitz stability estimate for that Cauchy problem with the lateral data for the particular case when initial conditions are as in (1.12). Consider the even extension of the function $u(x, t)$ with respect to $t$ and do not change notations for brevity, $u(x, -t) \equiv u(x, t)$, $t \in (0, T)$. Let $Q_T^+ = \Omega \times (-T, T)$, $S_T^+ = \partial \Omega \times (-T, T)$. Obviously, $\|u\|_{H^1(S_T)} = 2\|u\|_{L^2(\Omega)}$ and $\|\partial_t u\|_{L^2(S_T^+)} = 2\|\partial_\nu u\|_{L^2(\partial \Omega)}$, where $\partial_\nu$ means the normal derivative. The Lipschitz stability estimate for the Cauchy problem with the lateral data is

$$\|u\|_{H^1(Q_T^+)} \leq C [\|u\|_{L^2(\Omega)} + \|\partial_t u\|_{L^2(S_T^+)}] \quad (1.15)$$

with a constant $C > 0$ independent of the function $u$. Hence, the trace theorem implies the Lipschitz stability estimate for the function $f$ with a different constant $C$,

$$\|u(x, 0)\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)} \leq C [\|u\|_{H^1(S_T)} + \|\partial_t u\|_{L^2(S_T^+)}].$$

Estimate (1.15) is important in the control theory, since it is used for proofs of exact controllability theorems. For the first time, (1.15) was proved in 1986 in [34] for equation (1.1) with $c \equiv 1$ with the aim of applying to the control theory. However, the method of multipliers, which was proposed in [34], can handle neither variable lower order terms of the operator $L$ nor a variable coefficient $c(x)$. On the other hand, Carleman estimates are not sensitive to lower order terms of PDE operators and also can handle the case of a variable coefficient $c(x)$.

For the first time, the idea of using Carleman estimates for obtaining (1.15) was proposed in [14]. In this reference, (1.15) was proved for the case of the hyperbolic equation (1.11) with $L = \Delta + (\text{variable lower order terms})$. Next, the result of [14] was extended in [13, 16] to a more general case of the hyperbolic inequality

$$|u_t - \Delta u| \leq A |\nabla u| + |u_t| + |u| + |f| \quad \text{in } Q_T,$$

(1.16)

where $A = \text{const.} > 0$ and $f \in L^2(Q_T)$. Although in [13, 14, 16] $c \equiv 1$, it is clear from them that the key idea is in applying the Carleman estimate, while a specific form of the principal part of the hyperbolic operator should be such that the Carleman estimate would be valid. This thought is reflected in the proof of theorem 3.4.8 of [12]. Thus, the Lipschitz stability estimate (1.15) for the variable coefficient $c(x)$ was obtained in section 2.4 of [19] as well as in [5]. In particular, in [19] the hyperbolic inequality (1.16) was considered, in which $|u_t - \Delta u|$ was replaced with $|c^{-2}(x)u_t - \Delta u|$. The idea of [13] was used in the control theory in, e.g., [29, 30].

In the case of parabolic and elliptic operators, Carleman estimates are known for rather arbitrary variable principal parts [12, 19, 32]. On the other hand, it is well known that in the hyperbolic case the Carleman estimate can be effectively analytically verified for a generic operator $\partial_t^2 - L_0$ only if $L_0 = c^2(x)\Delta$, and a condition like

$$\langle x - x_0, \nabla (c^{-2}(x)) \rangle \geq 0, \quad \forall x \in \overline{\Omega}$$

(1.17)
holds. In (1.17), \( x_0 \) is a certain point and \( (\cdot, \cdot) \) is the scalar product in \( \mathbb{R}^n \). This is the reason why the above-mentioned Lipschitz stability estimates were established only using assumptions such as the one in (1.17). Clearly, (1.17) holds for \( c \equiv \text{const} \neq 0 \). See, e.g., theorem 1.10.2 in [4] for the proof of the Carleman estimate with condition (1.17). A more general case of condition (1.17) can be found in theorem 3.4.1 of [12]. The second way of proving Lipschitz stability estimates is via imposing some conditions on the Riemannian geometry of coefficients of the operator \( L_0 \) [3, 28, 36–38]. Publications [28, 36, 37] use Carleman estimates. In particular, the case of a hyperbolic inequality was considered in [36]. Unlike (1.17), conditions of the Riemannian geometry cannot be effectively analytically verified for an operator \( L_0 \) with generic coefficients, e.g. \( L_0 = c^2(x)\Delta \). A slight variation of (1.17) guarantees the non-trapping condition, see formula (3.24) in [35]. Uniqueness theorems for TAT were also obtained in [1, 9, 38] for the case (1.1), (1.2).

In addition to the Lipschitz stability, the quasi-reversibility method (QRM) for the above-mentioned Cauchy problem with the lateral data was developed in [14] and numerically tested in [5, 15, 20]. The convergence of the QRM solution to the exact solution was proven on the basis of the above Lipschitz stability results. Numerical testing has consistently demonstrated a high degree of robustness of QRM. In particular, accurate results were obtained in [20] with up to 50% noise in the data. We refer to the book [31] for the originating work on QRM. Some other numerical methods were proposed in [1, 38]. Convergence of all numerical methods mentioned in this section was proven only for the complete data collection case of IP1 with \( L_0 = c^2(x)\Delta \) and under some restrictive conditions imposed on the function \( c(x) \).

2. \( C^4 \)-smoothness of the solution of the Cauchy problem (1.11), (1.12)

First, we need to establish the \( C^4 \)-smoothness of the solution of the Cauchy problem (1.11), (1.12). This is done in theorem 2.2 of the current section. Proofs of all three results of this section are rather standard. In fact, these results are either explicitly formulated in classical books [10, 11, 27] or can be obtained by some modifications of proofs of these books. For this reason as well as for brevity, we only briefly outline the proof of theorem 2.2 here: for the convenience of the reader. We recall that we are not concerned in this paper with minimal smoothness requirements.

It is well known that the space \( H^k(\mathbb{R}^{n+1}) \) can be defined as (see, e.g., chapter 2 of [11])

\[
H^k(\mathbb{R}^{n+1}) = \left\{ u(x) : \|u\|_{H^k(\mathbb{R}^{n+1})} := \left( \int_{\mathbb{R}^{n+1}} |\tilde{u}(\xi)|^2 \sum_{|\beta| \leq k} |\xi|^{\beta} d\xi \right)^{1/2} < \infty \right\},
\]

where \( \tilde{u}(\xi) \) is the Fourier transform of the function \( u(x) \), \( x \in \mathbb{R}^{n+1} \). It is also well known that lemma 2.1 can be easily derived from (2.1).

**Lemma 2.1** (embedding). Let \( k > p + m = [(n + 1)/2] + m. Then \( H^k(\mathbb{R}^{n+1}) \subset C^m(\mathbb{R}^{n+1}) \) and there exists a constant \( K = K(n, m, k) > 0 \) depending only on listed parameters, such that

\[
\|u\|_{C^m(\mathbb{R}^{n+1})} \leq K\|u\|_{H^k(\mathbb{R}^{n+1})}, \quad \forall u \in H^k(\mathbb{R}^{n+1}).
\]

In particular, considering \( \mathbb{R}^n \) instead of \( \mathbb{R}^{n+1} \), and combining this lemma with (1.10), we obtain that

\[
f \in C^4(\mathbb{R}^n).
\]

Theorem 2.1 follows immediately from theorem 8.12 of chapter 8 of [10]. It is important for us in theorem 2.1 that the constant \( b \) is independent of the domain \( G \).
Theorem 2.1. Let $G \subset \mathbb{R}^n$ be a bounded domain. Let $L$ be the elliptic operator defined in (1.6)--(1.9). Let the function $u \in H^2(G)$ be such that $u(x) = 0$ near $\partial G$. Let $Lu \in H^1(G)$, $k \in [0, p + 3]$. Then $u \in H^{k+2}(G)$ and

$$
\|u\|_{H^{k+2}(G)} \leq b(\|Lu\|_{H^1(G)} + \|u\|_{L^2(G)}),
$$

where the constant $b = b(\mu_1, n, N_1)$ depends only on numbers $\mu_1, n, N_1$, where $\mu_1$ and $N_1$ were defined in (1.8) and (2.3), respectively.

Theorem 2.2. Let conditions (1.5)--(1.10) be satisfied. Denote.

$$
N_1 = \max \left\{ \max_{i,j=1,...,n} \|a_{i,j}\|_{C^{r+1}(\mathbb{R}^n)}, \max_{j=1,...,n} \|b_j\|_{C^r(\mathbb{R}^n)}, \|c\|_{C^{r+1}(\mathbb{R}^n)} \right\}. \quad (2.3)
$$

Then there exists unique solution $u \in H^2(\mathbb{R}^n \times (0, T)), \forall T > 0$ of problem (1.11), (1.12). Furthermore,

$$
u \in H^{p+5}(\mathbb{R}^n \times (0, T)) \subset C^4(\mathbb{R}^n \times [0, T])
$$

and the following estimate holds:

$$
\|u\|_{C^4(\mathbb{R}^n \times [0, T])} \leq B \exp(rT^2)\|f\|_{H^{p+1}(G)}, \quad \forall T > 0,
$$

where the number $p = p(n)$ is defined in (1.5), $r > 0$ is an arbitrary number and the constant $B = B(\mu_1, n, N_1, r)$ depends only on numbers $\mu_1, n, N_1, r$, where the number $\mu_1$ is defined in (1.8).

Remark 2.1. Most likely the term $\exp(rT^2)$ in (2.5) can be replaced with $\exp(dt)$, where the constant $d = d(\mu_1, n, N_1) > 0$ depends only on these numbers and $B$ is independent of $r$ in this case. However, such a replacement would require a more complicated proof, which would be distractive for our main goal: to investigate above inverse problems. The main point is that estimate (2.5) is sufficient for this goal.

Outline of the proof of theorem 2.2. In this proof, $Z = Z(\mu_1, n, N_1)$ and $d = d(\mu_1, n, N_1)$ denote different positive constants depending only on numbers $\mu_1, n, N_1$. Existence and uniqueness of the solution $u \in H^2(\mathbb{R}^n \times (0, T))$ of the problem (1.11), (1.12) follows from corollary 4.2 of chapter 4 of [27]. Since the function $f(x) = 0$ outside of the domain $\Omega$, then the property of the finite speed of propagation (theorem 2.2 of chapter 4 of [27]) implies that there exists a domain $G(T) \subset \mathbb{R}^n$ such that $u(x, t) = 0$ for all $t \in (0, T)$, for all points $x \in \mathbb{R}^n \setminus G(T)$ and for all points $x \in G(T)$ which are located in a small neighborhood of the boundary $\partial G(T)$ of $G(T)$.

For any $t \in (0, T)$, denote

$$
G_t(T) = \{(x, \tau) : x \in G(T), \tau = t\}, \quad \mathbb{R}_n = \{(x, \tau) : x \in \mathbb{R}^n, \tau = t\}.
$$

Consider the function $u^{(1)}(x, t) = u_t(x, t)$. Then this function is the weak $H^1(G(T) \times (0, T))$-solution of the following initial boundary value problem:

$$
u_t^{(1)} = Lu^{(1)}, \quad (x, t) \in G(T) \times (0, T),
$$

$$
u^{(1)}|_{t=0} = 0, \quad u_t^{(1)}|_{\partial G(T) \times (0, T)} = 0.
$$

Besides, $u^{(1)}(x, t) = 0$ for those points $x \in G(T)$ which are located in a small neighborhood of the boundary $\partial G(T)$. We now refer to theorems 3.2 and 4.1 as well as to standard energy estimates (2.15) and (3.3) of chapter 4 of [27]. These results combined with Gronwall’s
Hence, we consider the function \( u = u_1, u_2 = u_3 \), \( u_3 = u_4 \) for almost all \( t \in (0, T) \) and the following estimate holds for these values of \( t \):

\[
\|u\|_{L^2(G(T))} + \|\nabla u\|_{L^2(G(T))} + \|u_t\|_{L^2(G(T))} + \|u_{tt}\|_{L^2(G(T))} + \|u_{ttt}\|_{L^2(G(T))} \leq Z \|f\|_{H^1(\Omega)}.
\]

(2.7)

Let \( u'(x) = u(x, t) \) in the case when \( t \) is considered as a parameter. It follows from the proof of corollary 4.1 of chapter 4 of \([27]\) that the function \( u'(x) \) is the weak \( H^1(G(T)) \)-solution of the following Dirichlet boundary value problem for the elliptic equation for almost all \( t \in (0, T) \):

\[
L(u'(x)) = u_t(x, t), \quad x \in G(T),
\]

\[
\frac{\partial u'(x)}{\partial \eta} |_{\partial G(T)} = 0.
\]

(2.8)

The proof of corollary 4.1 of chapter 4 of \([27]\) implies that actually \( u'(x) \in H^2(G(T)) \) for almost all \( t \in (0, T) \). Next, using theorem 2.1 and (2.7), we obtain

\[
\|u'(t)\|_{H^2(G(T))} \leq Z \|f\|_{H^1(\Omega)}
\]

for almost all \( t \in (0, T) \). Hence, the function \( u \in H^2(G(T) \times (0, T)) \) and estimate (2.7) can be replaced with the following estimate for almost all \( t \in (0, T) \):

\[
\|\partial_{\beta}^\mu u(x, t)\|_{L^2(G(T))} \leq Z \|f\|_{H^1(\Omega)}, \quad |\beta| \leq 2.
\]

(2.9)

Recalling that the function \( u(x, t) = 0 \) for \( (x, t) \in (\mathbb{R}^n \setminus G(T)) \times (0, T) \), we obtain from (2.9)

\[
\|\partial_{\beta}^\mu u(x, t)\|_{L^2(\mathbb{R}^n)} \leq Z \|f\|_{H^1(\Omega)}, \quad |\beta| \leq 2.
\]

(2.10)

for almost all \( t \in (0, T) \). We continue this process via considering functions \( u^{(k)}(x, t) = \partial_{\beta}^\mu u(x, t), k \in [2, p + 4] \) and corresponding analogues of the initial boundary value problem (2.6) and the boundary value problem (2.8). Finally, obtain similarly with (2.10) that

\[
u \in H^{p+5}(\mathbb{R}^n \times (0, T)), \quad \forall T > 0,
\]

(2.11)

\[
\|\partial_{\beta}^\mu u(x, t)\|_{L^2(\mathbb{R}^n)} \leq Z \|f\|_{H^{p+5}(\Omega)}, \quad |\beta| \leq p + 5.
\]

(2.12)

for almost all \( t \in (0, T) \).

Consider now the even extension of the function \( u(x, t) \) with respect to \( t \) in \([t < 0], \) i.e. \( u(x, -t) := u(x, t), t > 0 \) and do not change notations for brevity. Then (1.11), (1.12), (2.11) and (2.12) imply that

\[
u \in H^{p+5}(\mathbb{R}^n \times (-T, T)), \quad \forall T > 0,
\]

(2.13)

\[
\|\partial_{\beta}^\mu u(x, t)\|_{L^2(\mathbb{R}^n)} \leq Z \exp(d|\beta|)|f|_{H^{p+5}(\Omega)}, \quad |\beta| \leq p + 5,
\]

(2.14)

for almost all \( t \in (-T, T) \).

To apply lemma 2.1, we would need to consider the function \( \tilde{u} := u \exp(-d|t|) \) and to prove that \( \tilde{u} \in H^{p+5}(\mathbb{R}^{n+1}) \). However, the function \( \exp(-d|t|) \) is non-differentiable at \( t = 0 \). Hence, we consider the function

\[
\tilde{u}(x, t) = u(x, t)e^{-st^2},
\]

(2.15)

where \( s > 0 \) is a certain number which we will choose later. Since \( \lim_{t \to \pm \infty} \exp(-s t^2 + d|t|) = 0 \), then (2.13) and (2.14) imply that

\[
\tilde{\Pi} \in H^{p+5}(\mathbb{R}^{n+1}), \quad \|\tilde{\Pi}\|_{H^{p+5}(\mathbb{R}^{n+1})} \leq B \|f\|_{H^{p+5}(\Omega)}.
\]

Hence, by (1.5) and lemma 2.1

\[
\tilde{\Pi} \in C^1(\mathbb{R}^{n+1}), \quad \|\Pi\|_{C^1(\mathbb{R}^{n+1})} \leq B \|f\|_{H^{p+1}(\Omega)}.
\]

(2.16)

Returning to the function \( u \) via (2.15) and using (2.16), we easily obtain that this function satisfies condition (2.4) and also

\[
\|u\|_{C^0(\mathbb{R}^n \times (0, T))} \leq B \exp(\gamma s T^2) \|f\|_{H^{p+1}(\Omega)},
\]

(2.17)

where \( \gamma = \gamma(p) > 0 \) is a certain number depending only on the number \( p \) in (1.5). Choose \( s = s(p, r) := r/\gamma \). Then, (2.17) implies (2.5).
3. Logarithmic stability

3.1. Transformation

For an integer, \( m \geq 0 \) denote

\[ K_m = \{ g \in C^m[0, \infty) : |g^{(j)}(t)| \leq A_\varepsilon \exp(k_\varepsilon t^2), j \in [0, m] \}, \]

where \( A_\varepsilon \) and \( k_\varepsilon \) are positive constants depending on the function \( g \). Consider the following well-known transformation \([19, 32]\), which transforms the hyperbolic Cauchy problem into a similar parabolic Cauchy problem,

\[ Lg = \mathcal{G}(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty \exp \left( -\frac{\tau^2}{4t} \right) g(\tau) \, d\tau, \quad \forall g \in K_m, t \in (0, (4k_\varepsilon)^{-1}). \quad (3.1) \]

Changing variables in (3.1) \( \tau \leftrightarrow z := \tau/(2\sqrt{t}) \), we obtain

\[ \lim_{t \to 0} \mathcal{G}(t) = g(0). \quad (3.2) \]

The transformation (3.1) is an analogue of the Laplace transform, and it is one-to-one. Let

\[ W(\tau, t) = \frac{1}{\sqrt{\pi t}} \exp \left( -\frac{\tau^2}{4t} \right). \]

Obviously

\[ W_\varepsilon(\tau, t) = \frac{1}{\sqrt{\pi t}} \exp \left( -\frac{\tau^2}{4t} \right). \quad (3.3) \]

Assume now that conditions (1.5)–(1.10) hold. It follows from theorem 2.2 that the solution \( u \in H^2(\mathbb{R}^n \times [0, T)) \), \( \forall T > 0 \) of the problem (1.11), (1.12) is such that \( u(x, t) \in K_\varepsilon \), \( \forall x \in \mathbb{R}^n \) as the function of \( t \) and any number \( k_\varepsilon \) > 0 can be chosen. For the sake of definiteness, we choose \( k_\varepsilon := 1/8 \). Hence, we can take \( t \in (0, 2) \) in (3.1) for \( g_\varepsilon(t) := u(x, t) \) as the function of \( t \). Let \( v(x, t) = (Lu)(x, t) \) for \( t \in (0, 2) \). Since \( \partial_\varepsilon u(x, 0) = \partial_\varepsilon^2 u(x, 0) = 0 \), then using (2.5), (3.3) and the integration by parts in integrals \( L(u_\varepsilon), L(\partial_\varepsilon^2 u) \), we obtain

\[ D_\varepsilon^j D_\varepsilon^k \psi = \mathcal{L}(D_\varepsilon^j D_\varepsilon^k u), \quad 2s + |\beta| \leq 4, \quad t \in (0, 2). \quad (3.4) \]

Hence, \( v \in C^{2,2}(\mathbb{R}^n \times [0, 1]) \). Hence, (1.3) and (1.4) imply that

\[ v \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^n \times [0, 1]), \quad \forall \alpha \in (0, 1). \quad (3.5) \]

By (1.11), (1.12), (3.2), (3.4) and (3.5), the function \( v(x, t) \) satisfies the following conditions:

\[ v_\varepsilon = Lu, \quad x \in \mathbb{R}^n, \quad t \in (0, 1), \]

\[ v(x, 0) = f(x). \quad (3.6) \]

\[ v(x, 0) = f(x). \quad (3.7) \]

We refer here to the well-known existence and uniqueness result for the solution \( v \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^n \times [0, T]), \forall T > 0 \) of the problem (3.6), (3.7) [26].

Below we work only with the function \( v(x, t) \) for \( t \in (0, 1) \). Denote

\[ \mathcal{L} \psi_1 = \mathcal{F}_1(x, t) = v|_{S_1}, \quad \mathcal{L} \psi_2 = \mathcal{F}_2(x, t) = v|_{P_1}. \quad (3.8) \]

By (3.5)

\[ \mathcal{F}_1 \in C^{2+\alpha,1+\alpha/2}(S_1), \quad \mathcal{F}_2 \in C^{2+\alpha,1+\alpha/2}(P_1). \quad (3.9) \]

Let

\[ \psi_1(x, t) = \partial_\varepsilon v|_{S_1}, \quad \psi_2(x, t) = \partial_\varepsilon v|_{P_1}. \quad (3.10) \]

By (3.5), \( \psi_1 \in C^{1+\alpha,a/2}(S_1), \psi_2 \in C^{1+\alpha,a/2}(P_1) \). We now describe an elementary, stable and well-known procedure for finding the normal derivative of the function \( v \) either at \( S_1 \) (in the
case of IP1) or at $P_1$ (in the case of IP2). In the case of IP1, we solve the initial boundary value problem for equation \(3.6\) for \((x, t) \in (\mathbb{R}^n \setminus \Omega) \times (0, 1)\) with the zero initial condition in \(\mathbb{R}^n \setminus \Omega\) (because of \((1.10)\)) and the Dirichlet boundary condition \(v|_\delta = \psi_1\). Hence, we uniquely find the function \(\psi_1\). Similarly, in the case of IP2, we uniquely find the function \(\psi_2\). Let \(V := V(x) \in \mathbb{R}^{n+1}, \forall x \in \mathbb{R}^n\) be the vector function of coefficients of the operator \(L\).

\[
V := V(x) = (a_{11}(x), a_{12}(x), \ldots, a_{n,n}(x), b_1(x), \ldots, b_n(x), c(x)).
\]

Using the estimate \((5.10)\) of theorem 5.2 of chapter 4 of \([26]\), \((3.8)\)–\((3.10)\) and \((3.11)\), we obtain that there exist numbers \(C(\Omega, n, V) > 0\) and \(C(n, V) > 0\) depending only on listed items such that

\[
\|\psi_1\|_{C^{1+\omega/2}(\overline{\Omega}_1)} \leq C(\Omega, n, V)\|\psi_1\|_{C^{1+\omega/2}(\Omega_1)},
\]

\[
\|\psi_2\|_{C^{1+\omega/2}(\overline{\Omega}_1)} \leq C(n, V)\|\psi_2\|_{C^{1+\omega/2}(\Omega_1)},
\]

also see point 1 of remark 3.1. Estimates \((3.12)\) and \((3.13)\) ensure the stability of this procedure.

Therefore, each problem IP1, IP2 is now replaced with the Cauchy problem for the parabolic PDE \((3.6)\) with the lateral data. These data are given at \(S_1\) for IP1 and at \(P_1\) for IP2. Uniqueness of the solution of these Cauchy problems follows from standard theorems about uniqueness of the continuation of solutions of parabolic PDEs with the lateral Cauchy data \([12, 19, 32]\).

In stability estimates, one is usually interested to see how the solution varies for a small variation of the input data. Since we took in \((3.1)\) \(k_\omega := 1/8\), which corresponds to \(r = 1/8\) in \((2.5)\), then we assume that in \((1.13)\)

\[
\|\varphi_1\|_{C^{1}(\overline{\Omega}_1)} \leq \delta \exp(T^2/8), \quad \forall T > 0
\]

and in \((1.14)\)

\[
\|\varphi_2\|_{C^{1}(\overline{\Omega}_1)} \leq \delta \exp(T^2/8), \quad \forall T > 0,
\]

where \(\delta \in (0, 1)\) is a sufficiently small number. Note that it is not necessary that \(\delta = B\), where \(B\) is the number from \((2.5)\). Indeed, the number \(B\) in \((2.5)\) is not assumed to be sufficiently small and it is involved in the estimate of the norm \(\|u\|_{C^{1}(\mathbb{R}^n \times [0, T])}\). On the other hand, the number \(\delta\) is a part of the estimate of the norm of the boundary data for either of the above inverse problems. Using \((3.1)\), \((3.4)\) and \((3.8)\)–\((3.15)\), we obtain with different constants \(C(\Omega, n, V), C(n, V) > 0\)

\[
\|\psi_1\|_{C^{1+\omega/2}(\overline{\Omega}_1)} + \|\psi_1\|_{C^{1+\omega/2}(\Omega_1)} \leq C(\Omega, n, V)\delta,
\]

\[
\|\psi_2\|_{C^{1+\omega/2}(\overline{\Omega}_1)} + \|\psi_2\|_{C^{1+\omega/2}(\Omega_1)} \leq C(n, V)\delta.
\]

It follows from \((3.16)\) that the following estimate holds with another constant \(\overline{C} := \overline{C}(\Omega, n, V) > 0\) depending only on \(\Omega, n, V\)

\[
\|\psi_1\|_{H^2(S_1)} + \|\psi_1\|_{L^2(S_1)} \leq \overline{C}(\Omega, n, V)\delta.
\]

Remark 3.1.

(1) The constant \(c > 0\) in the above-mentioned estimate \((5.10)\) of theorem 5.2 of chapter 4 of \([26]\) depends on some items. These items are not specified in \([26]\). On the other hand, \(c\) defines constants in estimates \((3.12), (3.13), (3.16)–(3.18)\). In this respect, the only thing which is clear from \([26]\) is that \(c = c(\Omega, n, V) := C(\Omega, n, V)\) in \((3.12)\) and \((3.16)\), \(c = c(n, V) = C(n, V)\) in \((3.13)\) and \((3.17)\) and \(c = c(\Omega, n, V) := \overline{C}(\Omega, n, V)\) in \((3.18)\).

A more detailed specification of items on which the constant \(c\) depends would require lengthy modifications of proofs of some theorems of chapter 4 of \([26]\). The latter is outside
the scope of this paper. This is the reason why constants in theorems 3.2 and 3.4 depend on \( \Omega, n, V \) rather than on \( \Omega, n, \mu_1, N_1 \). At the same time, corresponding constants of theorems 3.1, 3.3, 4.1, lemmas 3.1 and 3.2 depend on \( \Omega, n \) and some norms of coefficients of the operator \( L \). Also, see remark 3.3.

(2) The number \( \delta \) in (3.14) and (3.15) can be viewed as an upper estimate of the level of the error in the data \( \varphi_1, \varphi_2 \). Hence, theorems 3.2 and 3.4 below address the question of estimating variations of the solution \( f \) of either IP1 or IP2 via the upper estimate of the level of the error in the data.

(3) As to the use of the infinite time interval in IP1 and IP2, the author refers to his recent experience of working with time resolved real data for wave propagation processes, see chapter 5 and section 6.9 in [4]. The author has learned that almost all time resolved experimental data for wave propagation processes in non-attenuating media are highly oscillatory due to some unknown processes in measurement devices, see graphs of those data in [4]. Because of high oscillations, these data are not governed by a hyperbolic PDE even for the case of free space, where the wave equation is supposed to work. Therefore, the first step to make the inverse algorithm work was to preprocess the experimental data via a new data preprocessing procedure. This procedure uses only a small portion of the real data and immerses it in specially processed data for the uniform medium. Since the case of the uniform medium can be solved analytically, then there is no problem to know the immersed data for all \( t \in (0, \infty) \). Since accurate imaging results were obtained in [4] for the case of blind experimental data, then that data pre-processing procedure was unbiased.

3.2. Logarithmic stability estimate for inverse problem 1

To prove convergence of the QRM (theorem 4.1 in section 4), we need to consider a parabolic inequality in the integral form, which is more general than equation (3.6). Consider the function \( w \in C^{2,1}(\overline{Q}_1) \) satisfying the following inequality:

\[
\int_{Q_1} (w_t - Lw)^2 \, dx \, dt \leq K^2, \quad K = \text{const} \geq 0.
\]  

(3.19)

**Theorem 3.1.** Let conditions (1.6)–(1.8) be fulfilled, where in (1.6) and (1.8), \( x \in \mathbb{R}^n \) is replaced with \( x \in \overline{\Omega} \). Also, let condition (1.9) be replaced with the following two conditions:

\[
a_{i,j} \in C^1(\overline{\Omega}), \quad i, j = 1, \ldots, n,
\]

(3.20)

\[
b_j, c \in L_\infty(\Omega).
\]

(3.21)

Denote

\[
N_2 = \max \left( \max_{i,j=1,\ldots,n} \|a_{i,j}\|_{C^1(\overline{\Omega})}, \max_{j=1,\ldots,n} \|b_j\|_{L_\infty(\Omega)}, \|c\|_{L_\infty(\Omega)} \right).
\]

(3.22)

Let the function \( w \in C^{2,1}(\overline{Q}_1) \) satisfies inequality (3.19). For \( x \in \Omega \) let \( g(x) = w(x,0) \). Denote

\[
\beta_0(x,t) = w|_{S_t}, \quad \beta_1(x,t) = \partial_t w|_{S_t},
\]

\[
F = \|\beta_0\|_{H^1(S_t)} + \|\beta_1\|_{L^2(S_t)} + K.
\]

(3.23)

Assume that an upper bound \( C_1 = \text{const} > 0 \) of the norm \( \|\nabla g\|_{L^2(\Omega)} \) is given,

\[
\left( \sum_{i=1}^n \|g_{t_i}\|_{L^2(\Omega)}^2 \right)^{1/2} := \|\nabla g\|_{L^2(\Omega)} \leq C_1.
\]

(3.24)
Then there exist a constant $M_1 = M_1(\Omega, \mu_1, N_2) > 0$ and a sufficiently small number $\delta_1 = \delta_1(\Omega, \mu_1, N_2, C_1) \in (0, 1)$, both depending only on listed items, such that if $F \in (0, \delta_1)$, then the following logarithmic stability estimate is valid

$$
\|g\|_{L^2(\Omega)} \leq \frac{M_1 C_1}{\sqrt{\ln(F^{-1})}}.
$$

(3.25)

**Proof.** In this proof, $M_1 = M_1(\Omega, \mu_1, N_2) > 0$ denotes different positive constants depending only on $\Omega, \mu_1, N_2$. Let $\alpha \in (0, 1)$ be an arbitrary number. Then theorem 2 of [17] implies that there exists a constant $a = a(\Omega, \mu_1, N_2, \alpha) \in (0, 1)$ dependent only on listed items, such that

$$
\|g\|_{L^2(\Omega)} \leq \frac{M_1 C_1}{\alpha \sqrt{\ln[(aF)^{-1}]} + M_1 \left(\frac{1}{a}\right)^{\alpha} F^{1-\alpha}},
$$

as long as $F \in (0, 1)$. We can fix $\alpha$ via, e.g., setting $\alpha := 1/2$. It is clear therefore that there exists a sufficiently small number $\delta_1 = \delta_1(\Omega, \mu_1, N_2, C_1) \in (0, 1)$ such that if $F \in (0, \delta_1)$, then (3.26) implies (3.25).

□

**Remark 3.2.** It is of interest to figure out the qualitative behavior of the constant $M_1 = M_1(\Omega, \mu_1, N_2) > 0$ when numbers $\mu_1$ and $N_2$ increase. To do this, one needs to analyze the proof of theorem 2 of [17] as well as proofs of some other results of [17], which lead to theorem 2. This is because the proof of theorem 3.1 is based on theorem 2 of [17]. In addition, since some proofs of [17] use the Carleman estimate of section 1 of chapter 4 of [32] for the general parabolic operator of the second order, then one also needs to analyze the proof of the latter. This resulting analysis conducted by the author shows that $M_1$ decreases when $\mu_1$ increases and $M_1$ increases when $N_2$ increases.

While theorem 3.1 is valid in the purely parabolic framework, theorem 3.2 is concerned with IP1. It is natural, therefore that, unlike theorem 3.1, we now require condition (1.9), since this condition ensures the validity theorem 2.2. As shown in subsection 3.1, theorem 2.2 in turn leads to (3.5)–(3.7).

**Theorem 3.2.** Consider IP1. Let conditions (1.5)–(1.10) and (3.14) be valid. In addition, assume that the upper bound $C_2$ of the norm $\|\nabla f\|_{L^2(\Omega)}$ is given,

$$
\|\nabla f\|_{L^2(\Omega)} \leq C_2.
$$

(3.27)

Let $\overline{C} = \overline{C}(\Omega, n, V) > 0$ be the number in (3.18), where the vector function $V$ is defined in (3.11). Then there exists a constant $M_2 = M_2(\Omega, n, V) > 0$ and a sufficiently small number $\delta_2 = \delta_2(\Omega, n, V, C_1) \in (0, 1)$, both depending only on listed items, such that if the number $\delta$ in (3.14) is so small that $\overline{C} \delta \in (0, \delta_2)$, then the following logarithmic stability estimate is valid:

$$
\|f\|_{L^2(\Omega)} \leq \frac{M_2 C_2}{\sqrt{\ln[(\overline{C} \delta)^{-1}]}},
$$

(3.28)

**Proof.** It follows from (3.6) that (3.19) holds for the function $w := v$ with $K = 0$. As shown above, (3.18) follows from (3.14). Hence, using (3.18), we obtain that in (3.23) $F \leq \overline{C} \delta$. The application of theorem 3.1 finishes the proof.

□
Remark 3.3. As to the dependences $C = \overline{C}(\Omega, n, V), M_2 = M_2(\Omega, n, V), \delta_2 = \delta_2(\Omega, n, V, C_2)$ in this theorem, see point 1 of remark 3.1. Estimates (3.25) and (3.28) are the so-called conditional stability estimates, which is often the case in ill-posed problems [2, 4, 6, 32, 39]. As an example, we refer to Hölder stability estimates for solutions of some ill-posed problems for PDEs, see, e.g., [12, 19, 32]. Knowledge of the upper bound of the gradient in (3.24) and (3.27) corresponds well with the Tikhonov concept of compact sets as standard Carleman estimate for the parabolic operator [4, 19, 32], the integration domain of the Carleman weight function (CWF) in the Carleman estimate of lemma 3.1. The main pointwise inequality

\[ |\psi(t) - L_0\psi| \leq A(|\nabla \psi| + |\psi|), \quad A = \text{const} > 0, \]

(3.29)

where the operator $L_0$ is defined in (1.7). However, to prove the convergence of the QR RM (section 4), we need to estimate the initial condition for the case of the integral inequality, like the one in (3.19). The Carleman estimate of [18] is not a standard one. Indeed, unlike the standard Carleman estimate for the parabolic operator [4, 19, 32], the integration domain of [18] is a part of the strip $|t - \varepsilon| < \tau \varepsilon$, $\tau \in (0, 1)$, and the Carleman estimate does not break when $\varepsilon \to 0^+$. There are two main differences between theorem 3.3 and theorem 1 of [18]. First, we work now with the integral inequality instead of the pointwise inequality (3.29) of [18]. Second, it is assumed in [18] that the inequality (3.29) is valid in $\Theta \times (0, T)$, where $\Theta \subseteq \mathbb{R}^n$ is an unbounded domain. Knowledge of the Dirichlet boundary condition $\psi|_{\partial \Omega \times (0, T)} = 0$ is also assumed in [18], which is not the case for theorem 3.3.

Denote $\overline{\Omega} = (x_2, \ldots, x_n)$. Below we specify numbers $1/8, 1/4, 1/2, 5/8, 3/4$ for brevity only. In fact, some other numbers, respectively $\eta_1 < \eta_2 < \eta_3 < \eta_4 < \eta_5 < 1$, from the interval $(0, 1)$ can be used. Changing variables $(\xi', \tau') = (\sqrt{\sigma}x, \sigma t)$ with an appropriate constant $\sigma > 0$ and keeping the same notations for new variables for brevity, we obtain that

\[ \Omega \subset \left\{ x_1 + |\xi|^2 < \frac{1}{8}, x_1 > 0 \right\}. \]

(3.30)

Let $\varepsilon \in (0, 1)$ be a sufficiently small number. Consider the following functions $\psi(x, t), \varphi(x, t)$,

\[ \psi(x, t) = x_1 + |\xi|^2 + \left( \frac{t - \varepsilon}{\varepsilon^2} + \frac{1}{4} \right), \]

\[ \varphi(x, t) = \exp \left( \frac{\psi - \varepsilon}{\varepsilon} \right), \]

(3.31)

(3.32)

where $\nu > 1$ is a large parameter which will be defined later. The function $\varphi(x, t)$ is the Carleman weight function (CWF) in the Carleman estimate of lemma 3.1. The main difference between $\varphi(x, t)$ in (3.32) and the standard CWF for the parabolic operator [32] is that the small parameter $\varepsilon$ is involved in both functions $\psi(x, t)$ and $\varphi(x, t)$. For any domain $H \subset \{ (x, t) : x \in \mathbb{R}^n, t \in (0, 1) \}$, denote $RH$ its orthogonal projection on the hyperplane $\{ t = 0 \}$. Also, denote

\[ G_{3/4} = \{ (x, t) : \psi(x, t) < \frac{3}{4}, x_1 > 0 \}. \]

(3.33)
Using (3.30)–(3.34), we obtain
\[
G_{1/2} = \{(x, t) : \psi(x, t) < \frac{1}{2}, x_1 > 0\}. \tag{3.34}
\]

G_{1/2} \subseteq G_{3/4}, \psi^2(x, t) \geq \exp \left( \frac{2^{v+1}}{\varepsilon} \right) \text{ in } G_{1/2}. \tag{3.35}
\]

G_{3/4} \subseteq \left\{ |t - \varepsilon| < \frac{\varepsilon}{\sqrt{2}} \right\} \subseteq \{ t \in (0, 1) \}. \tag{3.36}
\]

Ω \subseteq RG_{1/2} \subseteq RG_{3/4}. \tag{3.37}
\]

∂G_{3/4} = \partial_1 G_{3/4} \cup \partial_2 G_{3/4}, \tag{3.38}
\]

∂G_{3/4} = \{ x_1 = 0 \} \cap \overline{G}_{3/4}, \tag{3.39}
\]

∂_2 G_{3/4} = \{ \psi(x, t) = \frac{3}{4}, x_1 > 0 \}. \tag{3.40}
\]

In addition, denote
\[
\Phi = \{(x, t) : x_1 \in (0, 1), \overline{x} \in (-1, 1)^{n-1}, t \in (0, 1)\}, \tag{3.41}
\]

∂_1 \Phi = \overline{\Phi} \cap P = \{(x, t) : x_1 = 0, \overline{x} \in (-1, 1)^{n-1}, t \in (0, 1)\}. \tag{3.42}
\]

By (3.31), (3.33), (3.36), (3.39), (3.41) and (3.42)
\[
\partial_1 G_{3/4} \subseteq \partial_1 \Phi. \tag{3.43}
\]

Recall that (3.15) implies (3.17). Hence, assume now that (3.15) holds. Then (3.17) and (3.42) imply that there exists a constant \( \tilde{C} = \tilde{C}(n, V, \Phi) > 0 \) such that
\[
\| \overline{\psi}_2 \|_{H^1(\partial_1 \Phi)} + \| \overline{\psi}_2 \|_{L^2(\partial_1 \Phi)} \leq \tilde{C} \delta. \tag{3.44}
\]

Lemma 3.1 follows immediately from theorem 2 of [18] and (3.38)–(3.40).

**Lemma 3.1.** Let coefficients of the operator \( L_0 \) in (1.7) satisfy conditions (1.8) and (3.20), where both \( \mathbb{R}^n \) in (1.8) and \( \Omega \) in (3.20) are replaced with \( R \Phi \). Denote
\[
N_3 = \max_{i,j=1,\ldots,n} \| a_{ij} \|_{C_0(\overline{\Phi})}. \tag{3.45}
\]

Then there exist a sufficiently large constant \( \delta_0 = \delta_0(\delta_1, N_3, \Phi) > 1 \), a sufficiently small number \( \varepsilon_0 = \varepsilon_0(\delta_1, N_3, \Phi) \in (0, 1) \) and a number \( C = C(\delta_1, N_3, \Phi) > 0 \), all three depending only on \( \delta_1, N_3, \Phi \), such that the following Carleman estimate holds:
\[
\int_{G_{3/4}} (u_t - L_0 u)^2 \psi^2(x, t) \, dx \, dt + \frac{C_3}{\varepsilon^3} \exp \left( \frac{2}{\varepsilon} \right) \int_{G_{3/4}} \left( u_t^2 + |\nabla u|^2 + u_t^2 \right) \, dx \, dt
\]
\[
+ \frac{C_3}{\varepsilon^3} \left( \left[ \frac{2^v}{\varepsilon} \right] + \left[ \frac{2^v(4^v}{\varepsilon^3} \right] \right) \int_{\partial_2 G_{3/4}} \left( u_t^2 + |\nabla u|^2 + u_t^2 \right) \, dS
\]
\[
\geq C \int_{G_{3/4}} \left( \frac{\varepsilon}{\varepsilon} |\nabla u|^2 + \frac{\varepsilon}{\varepsilon} \psi^{-2v} u_t^2 \right) \psi^2(x, t) \, dx \, dt,
\]
\[
\forall \varepsilon \geq \delta_0, \forall \varepsilon \in (0, \varepsilon_0), \forall u \in C^{2,1}(\overline{G}_{3/4}).
\]
It follows from (3.36) that lemma 3.1 provides the Carleman estimate in the narrow strip \(|t - \varepsilon| < \varepsilon/\sqrt{2}\). At the same time, it is also important in numerical studies of the QRM to estimate its solution in a non-narrow strip. This can be done via the standard Carleman estimate. Therefore, we introduce now notations, which are similar with (3.31)–(3.40), except that a narrow strip with respect to \(t\) is not used. Let

\[
\theta(x, t) = x_1 + |x|^2 + (t - \frac{1}{2})^2 + \frac{1}{2},
\]

(3.46)

\[
\xi(x, t) = \exp(\lambda \theta^{-n}),
\]

(3.47)

where \(\lambda > 1\) is a large parameter which is chosen later. Denote

\[
D_{3/4} = \{(x, t) : \theta(x, t) < \frac{1}{2}, x_1 > 0\},
\]

(3.48)

\[
D_{5/8} = \{(x, t) : \theta(x, t) < \frac{5}{8}, x_1 > 0\},
\]

(3.49)

\[
\partial D_{3/4} = \partial_1 D_{3/4} \cup \partial_2 D_{3/4},
\]

(3.50)

\[
\partial_1 D_{3/4} = \{x_1 = 0\} \cap \overline{D_{3/4}},
\]

(3.51)

\[
\partial_2 D_{3/4} = \{\theta(x, t) = \frac{1}{4}, x_1 > 0\},
\]

(3.52)

Using (3.30), (3.37), (3.42), (3.46), (3.48) and (3.49), we obtain

\[
\Omega \subset RD_{5/8} \subset RD_{3/4} = RG_{3/4},
\]

(3.53)

\[
D_{3/4} \subset \{(t - \frac{1}{2}) < \frac{1}{2}\} \subset \{t \in (0, 1)\},
\]

(3.54)

\[
D_{5/8} \subset D_{3/4} \subset \Phi.
\]

(3.55)

Lemma 3.2 follows from the above-mentioned Carleman estimate of section 1 of chapter 4 of [32] for the general parabolic operator of the second order as well as from (3.47)–(3.55).

**Lemma 3.2.** Let coefficients of the operator \(L_0\) in (1.7) satisfy conditions of lemma 3.1. Let \(N_1\) be the number defined in (3.45). Then there exist sufficiently large constants \(v_0 = v_0(\mu_1, N_1, \Phi) > 1, \lambda_0 = \lambda_0(\mu_1, N_1, \Phi) > 1\) and a constant \(C = C(\mu_1, N_1, \Phi) > 0\), all three depending only on \(\mu_1, N_1, \Phi\), such that the following Carleman estimate holds:

\[
\int_{D_{3/4}} (u - L_0 u)^2 |x|^2 (x, t) \, dx \, dt + C \lambda^3 v^3 \exp(2\lambda \cdot 4^v) \int_{\partial_1 D_{3/4}} (u^2 + |\nabla u|^2 + u_t^2) \, dS
\]

\[
+ C \lambda^3 v^3 \left(\frac{4}{3}\right)^{2v} \exp \left[2\lambda \left(\frac{4}{3}\right)^{2v}\right] \int_{\partial_2 D_{3/4}} (u^2 + |\nabla u|^2 + u_t^2) \, dS
\]

\[
\geq C \int_{D_{3/4}} (\lambda v |\nabla u|^2 + \lambda^3 v^3 \psi^{-2v} u^2) |x|^2 (x, t) \, dx \, dt, \forall v \geq v_0,
\]

\(\forall \lambda \geq \lambda_0, \forall u \in C^{2,1}(D_{3/4}).

**Theorem 3.3.** Let conditions (1.6)–(1.8) be valid, where in (1.6) and (1.8) ‘\(x \in \mathbb{R}^n\)’ is replaced with ‘\(x \in \mathbb{R}^\Phi\)’. Also, let conditions (3.20) and (3.21) hold, where \(\Omega\) is replaced with \(\Phi\). In addition, assume that condition (3.30) is in place. Suppose that the function \(w \in C^{2,1}(\Phi)\) satisfies the following integral inequality:

\[
\int_{\Phi} (w_t - Lw)^2 \, dx \, dt \leq K^2, \quad K = \text{const} \geq 0.
\]

(3.56)
For $x \in R\Phi$, let $g(x) = w(x, 0)$. Denote
\[ \beta_0(x, t) = w \mid_{\eta, \Phi}, \quad \beta_1(x, t) = \partial_t w \mid_{\eta, \Phi}. \]
Also, denote
\[ F = \|\beta_0\|_{H^\alpha(\eta, \Phi)} + \|\beta_1\|_{L^2(\eta, \Phi)} + K. \] (3.57)
Assume that an upper bound $C_3 = \text{const} > 0$ of the norm $\|w\|_{C_1(\Phi)}$ is given,
\[ \|w\|_{C_1(\Phi)} \leq C_3. \] (3.58)
Let $N_2$ be the number defined in (3.22), where $\Omega$ is replaced with $R\Phi$. Then there exist a constant $M_3 = M_3(\mu_1, N_2, \Phi) > 0$ and a sufficiently small number $\delta_3 = \delta_3(\mu_1, N_2, \Phi, C_3) \in (0, 1)$, both depending only on listed items, such that if $F \in (0, \delta_3)$, then the following logarithmic stability estimate is valid:
\[ \|g\|_{L^2(\Omega)} \leq \frac{M_3 C_3}{\sqrt{\ln (F^{-1})}}. \] (3.59)
In addition, there exists a number $\rho = \rho(\mu_1, N_2, \Phi, C_3) \in (0, 1/2)$ depending only on $\mu_1, N_2, \Phi, C_3$, such that if $F \in (0, \delta_3)$, then the following Hölder stability estimate is valid:
\[ \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} \leq M_3 C_3 F^\rho. \] (3.60)

**Proof.** Note that $N_1 \leq N_2$, where the number $N_1$ is defined in (3.45). Therefore, we replace in this proof the dependence of some numbers on $N_1$ in lemmata 3.1 and 3.2 with the dependence on $N_2$. In this proof, $\varepsilon_0 = \varepsilon_0(\mu_1, N_2, \Phi) \in (0, 1)$ denotes different sufficiently small numbers associated with lemma 3.1 and $M_1 = M_1(\mu_1, N_2, \Phi) > 0$ denotes different positive constants, both depend only on $\mu_1, N_2, \Phi$. By (3.30), (3.34), (3.37) and (3.41)
\[ \Omega \subset RG_{1/2} \subset R\Phi. \] (6.1)
Using (3.31), (3.32) and (3.56), we obtain
\[ \int_{G_{1/4}} (w_t - Lw)^2 \psi^2 \, dx \, dt \leq K^2 \exp \left( \frac{2 \cdot 4^v}{\varepsilon} \right), \forall \varepsilon \in (0, \varepsilon_0). \] (6.2)
Let $v_0 = v_0(\mu_1, N_2, \Phi) > 1$ and $C = C(\mu_1, N_2, \Phi) > 0$ be numbers of lemma 3.1. Using this lemma, (3.22), (3.43) and (3.57), we obtain for all $v \geq v_0, \varepsilon \in (0, \varepsilon_0)$
\[ \int_{G_{1/4}} (w_t - Lw)^2 \psi^2 \, dx \, dt \geq \int_{G_{1/4}} (w_t - L_0 w)^2 \psi^2 \, dx \, dt - M_3 \int_{G_{1/4}} (|w|^2 + |w|^2) \psi^2 \, dx \, dt \]
\[ \geq C \int_{G_{1/4}} \left( \frac{\varepsilon}{\varepsilon} |\nabla w|^2 + \frac{\varepsilon^5}{\varepsilon^2} \psi^{-2\varepsilon} w^2 \right) \psi^2 (x, t) \, dx \, dt \]
\[ - M_3 \int_{G_{1/4}} (|w|^2 + w^2) \psi^2 \, dx \, dt \]
\[ - \frac{C_0^3}{\varepsilon^3} \exp \left( \frac{2 \cdot 4^v}{\varepsilon} \right) \left( \|\beta_0\|_{L^2(\eta, \Phi)}^2 + \|\beta_1\|_{L^2(\eta, \Phi)}^2 \right) \]
\[ - \frac{C_0^3}{\varepsilon^3} \left( \frac{4}{3} \right)^2 \exp \left( \frac{2 \cdot 4^v}{\varepsilon} \right) \int_{G_{1/4}} (w^2 + |\nabla w|^2 + w^2) \, dS. \]

Fix a number $v \geq v_0$ such that
\[ \left( \frac{5}{6} \right)^v < \frac{1}{2}. \] (6.4)
Hence, decrease $\varepsilon_0 = \varepsilon_0(\mu_1, N_2, \Phi) \in (0, 1)$, so that $M_3 < C_\nu/(2\varepsilon), \forall \varepsilon \in (0, \varepsilon_0)$. Then (3.63) leads to the following estimate:

$$\int_{G_{1/4}} (w - Lu)^2 \varphi^2 \, dx \, dt \geq \frac{C}{\varepsilon} \int_{G_{1/4}} (|\nabla w|^2 + w^2)\varphi^2(x, t) \, dx \, dt - \frac{C}{\varepsilon^3} \exp \left( \frac{2}{\varepsilon} \frac{4^\nu}{3} \right) \left( \|\beta_0\|_{H^1(\Omega, \Phi)} + \|\beta_1\|_{L^2(\Omega, \Phi)} \right) \int_{\partial G_{1/4}} (u^2 + |\nabla u|^2 + u^2) \, dS, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Combining this with (3.57), (3.58) and (3.62) and decreasing $\varepsilon_0$, if necessary, we obtain

$$\int_{G_{1/4}} (|\nabla w|^2 + w^2)\varphi^2(x, t) \, dx \, dt \leq C \exp \left( \frac{2 \cdot 5^\nu}{\varepsilon} \right) F^2 + CC_3^2 \exp \left[ \frac{2}{\varepsilon} \left( \frac{5}{3} \right) \right].$$

(3.65)

On the other hand, (3.35) implies that

$$\int_{G_{1/2}} (|\nabla w|^2 + w^2)\varphi^2(x, t) \, dx \, dt \geq \int_{G_{1/2}} (|\nabla w|^2 + w^2)\varphi^2(x, t) \, dx \, dt \geq \exp \left[ \frac{2^{p+1}}{\varepsilon} \right] \int_{G_{1/2}} (|\nabla w|^2 + w^2) \, dx \, dt, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Combining this with (3.65), we obtain

$$\int_{G_{1/2}} (|\nabla w|^2 + w^2) \, dx \, dt \leq C \exp \left( \frac{2 \cdot 5^\nu}{\varepsilon} \right) F^2 + CC_3^2 \exp \left[ \frac{2}{\varepsilon} \left( \frac{5}{3} \right) \right],$$

(3.66)

Hence, using (3.64), we obtain

$$\int_{G_{1/2}} (|\nabla w|^2 + w^2) \, dx \, dt \leq C \exp \left( \frac{2 \cdot 5^\nu}{\varepsilon} \right) F^2 + CC_3^2 \exp \left( - \frac{2^\nu}{\varepsilon} \right), \quad \forall \varepsilon \in (0, \varepsilon_0).$$

By (3.31) and (3.34) $G_{1/2} \subset \{ t \in (\varepsilon/2, 3\varepsilon/2) \}$. Hence, the mean value theorem, (3.61) and (3.66) imply that there exists a number $t^* = t^*(\varepsilon) \in (\varepsilon/2, 3\varepsilon/2)$ such that for all $\varepsilon \in (0, \varepsilon_0)$

$$\|w(x, t^*(\varepsilon))\|_{L^2(\Omega)}^2 \leq \frac{C}{\varepsilon} \exp \left( \frac{2 \cdot 5^{p+1}}{\varepsilon} \right) F^2 + CC_3^2 \exp \left( - \frac{2^\nu}{\varepsilon} \right).$$

(3.67)

We have

$$w(x, t^*(\varepsilon)) = g(x) + \int_0^{t^*(\varepsilon)} u_\varepsilon(x, \tau) \, d\tau.$$

Hence, using (3.58), we obtain

$$\|w(x, t^*(\varepsilon))\|_{L^2(\Omega)}^2 \geq \frac{1}{\varepsilon} \|g\|_{L^2(\Omega)}^2 - \varepsilon^2 \|w_i\|_{L^2(\Phi)}^2 \geq \frac{1}{\varepsilon} \|g\|_{L^2(\Omega)}^2 - M_3 C_3^2 \varepsilon^2.$$

Combining this with (3.67), we obtain

$$\|g\|_{L^2(\Omega)}^2 \leq M_3 C^2 \varepsilon^2 + M_3 \exp \left( \frac{2 \cdot 5^{p+1}}{\varepsilon} \right) F^2 + M_3 C_3^2 \exp \left( - \frac{2^\nu}{\varepsilon} \right), \quad \forall \varepsilon \in (0, \varepsilon_0).$$

(3.68)

Choose $\varepsilon = \varepsilon(F)$ such that

$$\exp \left( \frac{2 \cdot 5^{p+1}}{\varepsilon} \right) F^2 = \exp \left( - \frac{2^\nu}{\varepsilon} \right).$$

(3.69)

Hence,

$$\varepsilon = \frac{1}{\ln(F^{-2/\nu})}, \quad q = 2 \cdot 5^{p+1} + 2^\nu - 1.$$
To ensure that \( \varepsilon \) is sufficiently small, i.e. \( \varepsilon \in (0, \varepsilon_0) \), we need to choose \( F \) so small that
\[
0 < F < \exp\left(-\frac{q}{2\varepsilon_0}\right).
\]
Hence, we choose \( \delta_1 = \exp[-q/(2\varepsilon_0)] \). Hence, (3.68), (3.69) and (3.70) lead to
\[
\|g\|_{L^2(\Omega)} \leq \frac{M_3C_4^2}{\ln(F^{-1/4})} + M_3\left(1 + C_2^2\right)(F^{2^\bullet})^{1/4} = \frac{M_3C_4^2}{\ln(F^{-1})} + M_3\left(1 + C_2^2\right)(F^{2^\bullet})^{1/4},
\]
as long as \( F \in (0, \delta_1) \). Decreasing, if necessary, \( \delta_3 = \delta_3(\mu_1, N_2, \Phi, C_3) \), we obtain (3.59) from (3.71).

Finally, we prove (3.60). Using (3.46)–(3.55) as well as lemma 3.2, we obtain similarly with (3.66)
\[
\int_{D_{\delta_3}} (|\nabla u|^2 + u^2) \, dx \, dt \leq C \exp(2\lambda \cdot 5^\bullet)F^2 + CC_2^2 \exp(-2^\nu+1\lambda), \quad \forall \lambda \geq \lambda_1,
\]
where \( \lambda_1 = \lambda_1(\mu_1, N_2, \Phi) > 1 \) is a sufficiently large number. Hence, similarly with (3.69), we choose \( \lambda = \lambda(F) \geq \lambda_1 \) such that
\[
\exp(2\lambda \cdot 5^\bullet)F^2 = \exp(-2^\nu+1\lambda).
\]
Similarly with (3.71), we obtain from (3.72) and (3.73)
\[
\int_{D_{\delta_3}} (|\nabla u|^2 + u^2) \, dx \, dt \leq M_3C_2^2F^{2\rho}. \quad \square
\]

**Remark 3.4.** To see the qualitative behavior of the constant \( M_3 = M_3(\mu_1, N_2, \Phi) > 0 \) when numbers \( \mu_1 \) and \( N_2 \) increase, one needs to analyze proofs of Carleman estimates of theorem 2 of [18] and of section 1 of chapter 4 of [32]. This is because lemmata 3.1 and 3.2 are derived from Carleman estimates of [18] and [32], respectively. Next, this analysis should be combined with the analysis of the proof of theorem 3.3. The author has done these. The resulting conclusion of the author is that \( M_3 \) decreases when \( \mu_1 \) increases and \( M_3 \) increases when \( N_2 \) increases.

**Theorem 3.4.** Consider IP2. Let conditions (1.5)–(1.10), (3.15) and (3.30) be valid. Also, assume that for a certain \( \alpha \in (0, 1) \) the upper bound \( C_4 \) of the norm \( \| f \|_{C^{\alpha+1}(\Omega)} \) is given, i.e. \( \| f \|_{C^{\alpha+1}(\Omega)} \leq C_4 \). Let \( \widetilde{C} = C(n, V, \Phi) > 0 \) be the number defined in (3.44), where \( V \) is defined in (3.11). Then there exists a constant \( M_3 = M_3(n, V, \Phi) > 0 \) and a sufficiently small number \( \delta_3 = \delta_3(n, V, \Phi, C_4) \in (0, 1) \), both depending only on listed parameters, such that if the number \( \delta \) in (3.15) is so small that \( \delta \delta < (0, \delta_3) \), then
\[
\| f \|_{L^2(\Omega)} \leq \frac{M_3C_4}{\sqrt{\ln(C\delta)^{-\alpha}}}. \quad (3.74)
\]

**Proof.** By theorem 2.2 there exists a unique solution \( u \in H^2(\mathbb{R}^n \times (0, T)) \), \( \forall T > 0 \) of the problem (1.11), (1.12), and the function \( u \) satisfies condition (2.4). Also, it was established in subsection 3.1 that the function \( v(x, t) = (Cu)(x, t) \) is defined for \( t \in (0, 2) \). Furthermore, the function \( v(x, t) \) is the unique solution of the problem (3.6), (3.7) satisfying (3.5). Recall that by (1.10) \( f(x) = 0 \) in \( \mathbb{R}^n \setminus \Omega \) and by (2.2) \( f \in C^4(\mathbb{R}^n) \). Hence, for any bounded domain \( \Omega' \subset \mathbb{R}^n \) such that \( \Omega \subset \Omega' \) we have \( f \in C^{2+\alpha}(\Omega') \), \( \forall \alpha \in (0, 1) \) and \( \| f \|_{C^{2+\alpha}(\Omega')} = \| f \|_{C^{2+\alpha}(\Omega)} \). Hence, the estimate (14.6) of section 14 of chapter 4 of [26] of the solution of the Cauchy problem for the parabolic equation implies that
\[
\| v \|_{C^{2+\alpha+1}(\mathbb{R}^n \times [0, 1])} \leq M_4\| f \|_{C^{2+\alpha}(\Omega)}. \quad (3.75)
\]
We have
\[ \|v\|_{C^1(\overline{\mathcal{F}})} \leq \|v\|_{C^{2, \alpha, 1+\alpha/2}(\mathcal{F})} \leq \|v\|_{C^{2, \alpha, 1+\alpha/2}(\mathbb{R}^n \times (0,1))}. \]  
(3.76)

Using (3.75) and (3.76), we obtain
\[ \|v\|_{C^1(\overline{\mathcal{F}})} \leq M_4 c_4. \]  
(3.77)

Equation (3.6) implies that we now can set in (3.56) \( K = 0 \). Recall that (3.15) implies (3.44). Hence, we obtain for the new number \( F \) in (3.57)
\[ F := \|v_2\|_{L^2(\partial_t \Phi)} + \|\overline{v}_2\|_{L^2(\partial_t \Phi)} \leq \tilde{C} \delta. \]  
(3.78)
Thus, theorem 3.3 implies that (3.74) follows from (3.77) and (3.78). \( \square \)

**Remark 3.5.** The dependence of the constant \( c > 0 \) in the above-mentioned estimate (14.6) of section 14 of chapter 4 of [26] is not specified in [26]. On the other hand, since we use this estimate in (3.75) with \( c := M_4 \), then, similarly with point 1 of remark 3.1, we can only say that in (3.75) \( M_4 = M_4(n, V) \). Next, since we use theorem 3.3 in the proof of theorem 3.4, then we should also assume the dependence of \( M_4 \) on the domain \( \Phi \), i.e. \( M_4 = M_4(n, V, \Phi) \).

4. The quasi-reversibility method

We construct the QRM only for the more difficult case of IP2. The case of IP1 is similar, and it can be derived from [17]. Also, we work in this section only in 3D, keeping the same notations as above. The construction in the 2D case is similar. We assume in this section that conditions (1.6)–(1.8) and (3.20), (3.21) are valid \( n = 3 \), where \( R\Phi \) replaces \( \mathbb{R}^n \) and \( \Omega \), respectively. Recall that the domain \( \Phi \) is defined in (3.41) and \( R\Phi \) is its orthogonal projection on the hyperplane \( \{t = 0\} \). Since it was described in subsection 3.1 how to stably obtain the Neumann boundary condition for both IP1 and IP2 after applying the transformation (3.1) to the function \( u(x, t) \), we assume now that we have both Dirichlet and Neumann boundary conditions at \( \partial_1 \Phi \):
\[ v \mid_{\partial_t \Phi} = \overline{v}_2(x, t), \quad \partial_3 v \mid_{\partial_t \Phi} = \overline{v}_2(x, t). \]  
(4.1)

In our case, the QRM means the minimization of the following Tikhonov functional
\[ J_\gamma(v) = \|v - Lv\|_{L^2(\overline{\mathcal{F}})}^2 + \gamma \|v\|_{H^1(\overline{\mathcal{F}})}^2, \]  
(4.2)
subject to the boundary conditions (4.1). In (4.2), \( \gamma > 0 \) is the regularization parameter, which should be chosen in accordance with the level of the error in the data.

The requirement \( v \in H^4(\Phi) \) is an over-smoothness. This condition is imposed to ensure that the function \( v \in C^1(\overline{\mathcal{F}}) \) because of (3.58) and the embedding theorem. However, the author’s numerical experience with the QRM has consistently demonstrated that one can significantly relax the required smoothness in practical computation, see [20, 23, 24] and section 6.8 of [4]. This is likely because one is not using an overly small grid step size in finite differences when computing via the QRM. Hence, one effectively works with a finite-dimensional space with not too many dimensions. This means that one can rely in this case on the equivalence of all norms in finite-dimensional spaces. Thus, most likely one can replace in real computations \( \gamma \|v\|_{H^1(\overline{\mathcal{F}})}^2 \) with \( \gamma \|v\|_{H^2(\overline{\mathcal{F}})}^2 \).

While (4.2) is good for computations, to prove the convergence of the QRM, we need to obtain zero boundary conditions at \( \partial_1 \Phi \). Assume that both functions \( \overline{v}_2, \overline{v}_2 \in H^{2,1}(\partial_1 \Phi) \). Denote
\[ s(x, t) = \overline{v}_2(\overline{x}, t) + x_1 \overline{v}_2(\overline{x}, t), \]
\[ s(x, t) = v(x, t) - s(x, t), \]
\[ h(x, t) = -(s_t - Ls)(x, t), \]
\[ \hat{f}(x) = \hat{v}(x, 0) = f(x) - s(x, 0). \]
Using (3.6), (3.7) and (4.1), we obtain
\[
\hat{v}_t - L\hat{v} = h(x, t), \quad (x, t) \in \Phi,
\]
(4.3)
\[
\hat{v} |_{\partial \Omega} = 0, \quad \hat{v}_t |_{\partial \Omega} = 0.
\]
(4.4)
Thus, we have obtained IP3.

**Inverse problem 3 (IP3).** Find the function \(\hat{f}(x)\) for \(x \in \Omega\) from conditions (4.3) and (4.4).

To solve IP3 via the QRM, we minimize the following analogue of the functional (4.2):
\[
\tilde{J}_\gamma (\hat{v}) = \|\hat{v}_t - L\hat{v} - h\|^2_{L^2(\Omega)} + \gamma \|\hat{v}\|^2_{H^1(\Phi)}, \quad \hat{v} \in H^1_0(\Phi),
\]
(4.5)
Let \(\cdot, \cdot\) and \([\cdot, \cdot]\) be scalar products in \(L^2(\Omega)\) and \(H^1(\Phi)\), respectively. Let the function \(\hat{v}_\gamma \in H^1_0(\Phi)\) be a minimizer of the functional (4.5). Then the variational principle implies that
\[
(\hat{v}_\gamma, \partial_t \hat{v}_\gamma - L\hat{v}_\gamma, \partial_t w - Lw) + \gamma [\hat{v}_\gamma, w] = (h, \partial_t w - Lw), \quad \forall w \in H^1_0(\Phi).
\]
(4.6)
Lemma 4.1 follows immediately from the Riesz theorem and (4.6).

**Lemma 4.1.** For every function \(h \in L^2(\Phi)\) and every \(\gamma > 0\), there exists unique minimizer \(v_\gamma = v_\gamma(h) \in H^1_0(\Phi)\) of the functional (4.5). Furthermore, the following estimate holds:
\[
\|v_\gamma\|_{H^1(\Phi)} \leq \frac{1}{\sqrt{2\gamma}} \|h\|_{L^2(\Phi)}.
\]

The idea now is that if \(v_\gamma(x, t) \in H^1_0(\Phi)\) is the minimizer mentioned in lemma 4.1, then the approximate solution \(\hat{f}_\gamma(x)\) of IP3 can be defined as
\[
\hat{f}_\gamma(x) = v_\gamma(x, 0).
\]
(4.7)
The question of convergence of minimizers of \(\tilde{J}_\gamma\) to the exact solution is more difficult than the question of existence of lemma 4.1. To establish convergence, we need to introduce the exact solution as well as the error in the data, just as this is always done in the regularization theory [2, 4, 6, 32, 39]. We assume that there exists ‘ideal’ noiseless data \(h^* \in L^2(\Phi)\). We also assume that there exists the ideal noiseless solution \(\hat{v}^* \in H^1_0(\Phi)\) of the following problem:
\[
\hat{v}^*_t - L\hat{v}^* = h^*(x, t), \quad (x, t) \in \Phi,
\]
(4.8)
\[
\hat{v}^* |_{\partial \Omega} = 0, \quad \hat{v}^*_t |_{\partial \Omega} = 0.
\]
(4.9)
Let \(\omega \in (0, 1)\) be a small number, which we regard as the level of the error in the data. We assume that
\[
\|h - h^*\|_{L^2(\Phi)} \leq \omega.
\]
(4.10)

**Remark 4.1.** For brevity, we work in this section with the parabolic IP3. Still, theorem 4.1 can be easily linked with the original hyperbolic IP2. Theorem 4.1 establishes the convergence rate of the QRM. Note that an upper estimate of the exact solution is often assumed to be known in the regularization theory, also see remark 3.2.

**Theorem 4.1.** Let conditions (3.30) and (4.10) be satisfied and the regularization parameter \(\gamma\) in (4.5) is chosen such that \(\gamma = \gamma(\omega) = \omega \in (0, 1)\). Let the function \(v_\gamma(\omega) \in H^1_0(\Phi)\) be the unique minimizer of the functional (4.5), which is guaranteed by lemma 4.1. Let \(\hat{v}^* \in H^1_0(\Phi)\) be the exact solution of the problem (4.8), (4.9). Let \(\|\hat{v}^*\|_{H^1(\Phi)} \leq Y\), where the upper estimate
\[ Y = \text{const} \geq 1 \] is given. Let \( N_2 \) be the number defined in (3.22), where \( \Omega \) is replaced with \( R \Phi \). Then there exist a constant \( M_5 = M_5(\mu_1, N_2, \Phi) > 0 \) and a sufficiently small number \( \omega_0 = \omega_0(\mu_1, N_2, \Phi, Y) \in (0, 1) \), both depending only on listed items, such that if \( \omega \) is so small that \( (Y^2 + 1)\omega \in (0, \omega_0) \), then the following logarithmic convergence rate takes place:

\[
\| f_{y(\omega)} - \hat{v}^{*}(x,0) \|_{L_2(\Omega)} \leq \frac{M_5 Y}{\sqrt{\ln(\omega^{-1})}},
\]

(4.11)

where the function \( f_{y(\omega)}(x) \) is defined in (4.7). In addition, there exists a number \( \rho = \rho(\mu_1, N_2, \Phi, Y) \in (0, 1/4) \) depending only on \( \mu_1, N_2, \Phi, Y \), such that

\[
\| \hat{v} - v_{y(\omega)} \|_{L_2(D_{h\lambda})} + \| \nabla \hat{v} - \nabla v_{y(\omega)} \|_{L_2(D_{h\lambda})} \leq M_3 Y \omega^\rho, \quad \forall \omega \in (0, \omega_0).
\]

(4.12)

**Proof.** It follows from (4.8) and (4.9) that the function \( \hat{v}^{*} \) satisfies the following analogue of (4.6):

\[
(\hat{v}^{*} - L\hat{v}^{*}, w_t - Lw) + \gamma[\hat{v}^{*}, w] = (h^{*}, w_t - Lw) + \gamma[\hat{v}^{*}, w], \quad \forall w \in H_0^1(\Phi).
\]

(4.13)

Let \( \tilde{v} = v_{y(\omega)} - \hat{v}^{*} \in H_0^1(\Phi) \) and \( \tilde{h} = h - h^{*} \in L_2(\Phi) \). Subtracting (4.13) from (4.6), we obtain

\[
(\tilde{v} - L\tilde{v}, w_t - Lw) + \gamma[\tilde{v}, w] = (\tilde{h}, w_t - Lw) - \gamma[\tilde{v}^{*}, w], \quad \forall w \in H_0^1(\Phi).
\]

Setting here \( w := \tilde{v} \) and using Cauchy–Schwarz inequality and (4.10), we obtain

\[
\int_{\Phi} (\tilde{v} - L\tilde{v})^2 \, dx \, dt \leq \gamma \| \tilde{v} \|_{H_1(\Phi)}^2 \leq \omega^2 + \gamma \| \tilde{v} \|_{H_1(\Phi)}^2 \leq \omega^2 + \gamma Y^2.
\]

(4.14)

Since \( \gamma(\omega) = \omega \in (0, 1) \), then (4.14) implies that \( \| \tilde{v} \|_{H_1(\Phi)} \leq Y + 1 \). Hence, using again (4.14) as well as embedding theorem, we obtain

\[
t \| \tilde{v} \|_{C^1(\overline{\Phi})} \leq \xi Y.
\]

(4.15)

\[
\int_{\Phi} (\tilde{v} - L\tilde{v})^2 \, dx \, dt \leq (Y^2 + 1)\omega,
\]

(4.16)

where the number \( \xi = \xi(\Phi) > 0 \) depends only on the domain \( \Phi \). We now apply theorem 3.3. Comparing (4.16) with (3.56) and (4.15) with (3.58), we set

\[
K := F := \sqrt{\omega(Y^2 + 1)}, \quad C_3 := \xi Y.
\]

(4.17)

Therefore, (4.11) and (4.12) follow from (3.57), (3.58), (3.59), (3.60) and (4.17). \( \square \)

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