Relaxation time of non-conformal plasma

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Abstract

We study effective relaxation time of viscous hydrodynamics of strongly coupled non-conformal gauge theory plasma using gauge theory/string theory correspondence. We compute leading corrections to the conformal plasma relaxation time from the relevant deformations due to dim-2 and dim-3 operators. We discuss in details the relaxation time $\tau_{\text{eff}}$ of $\mathcal{N} = 2^*$ plasma. For a certain choice of masses this theory undergoes a phase transition with divergent specific heat $c_V \sim |1 - T_c/T|^{-1/2}$. Although the bulk viscosity remains finite all the way to the critical temperature, we find that $\tau_{\text{eff}}$ diverges near the critical point as $\tau_{\text{eff}} \sim |1 - T_c/T|^{-1/2}$.

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Understanding the properties of strongly coupled quark-gluon plasma (sQGP) produced at RHIC [1] is an active research area. Of particular interest is the development of viscous hydrodynamic codes describing the evolution of the plasma ball at early stages of heavy ion collisions [2–4]. These hydrodynamic simulations require input of the phenomenological parameters, such as the shear $\eta$ and the bulk $\zeta$ viscosities, as well as higher order transport coefficients necessary to maintain causality of hydrodynamic description [5–8] (and typically, numerical stability of the codes). Currently, reliable computation of transport coefficients of sQGP is not available. As a result, one resorts to analysis of exactly soluble models of strongly coupled gauge theory plasma. Transport properties of a large class of strongly coupled gauge theory plasmas can be studied in a framework of gauge theory/string theory correspondence [9, 10]. Here, one discovers universal relations for the (first order) transport coefficients [11–14]

$$\frac{\eta}{s} = \frac{1}{4\pi}, \quad \frac{\zeta}{\eta} \geq 2 \left(\frac{1}{3} - c_s^2\right),$$

(1.1)

where $s$ is the entropy density and $c_s$ is the speed of sound in the plasma. The hope is that it is generic relations of the type (1.1) that might provide an input into realistic simulations of sQGP.

In this letter we study a specific second-order transport coefficient — the effective relaxation time $\tau_{eff}$ — in strongly coupled non-conformal gauge theory plasma. The motivation for our analysis is the recent work of Song and Heinz [15] where it was suggested that the relaxation time near the phase transition can be much larger due
to “critical slowing down”. Thus the sQGP produced in heavy ion collisions at RHIC might have strong sensitivity to the initial conditions for the bulk viscous pressure.

We begin in the next section with introducing the notion of the effective relaxation time. The simplest way to obtain non-conformal viscous 4-dimensional hydrodynamics is to compactify a conformal hydrodynamics in \((4 + k)\)-dimensions on a flat \(k\)-dimensional torus \([14]\). We discuss effective relaxation time of such compactified plasma. In section 3 we compute corrections to the effective relaxation time in the vicinity of the renormalization group (RG) fixed point induced by relevant operators (the mass terms). In section 4 we present results of the detailed analysis of the effective relaxation time in \(\mathcal{N} = 2^*\) plasma. We focus on a specific mass deformation of \(\mathcal{N} = 4\) supersymmetric Yang-Mills (SYM) plasma which leads to a phase transition in the infrared. We summarize our results in section 5.

## 2 \(\tau_{\text{eff}}\) from conformal hydrodynamics

The most general causal viscous relativistic hydrodynamics has many second order transport coefficients: 5 - in the case of conformal hydrodynamics \([7]\), and 13 - in the case of non-conformal hydrodynamics \([16]\). It is impractical to simulate hydrodynamic evolution in the full multiparameter phenomenological space of these transport coefficients. Thus, one typically limits discussion to a single 'effective' second order transport coefficient. Since different groups use different approximations of the general viscous hydrodynamics, it is necessary to related the analysis through some physical observable. We propose to use a dispersion relation of the linearized sound waves in plasma to define a common effective relaxation time \(\tau_{\text{eff}}\):

\[
\omega = \pm c_s k - i \Gamma k^2 \pm \frac{\Gamma}{c_s} \left( c_s^2 \tau_{\text{eff}} - \frac{\Gamma}{2} \right) k^3 + \mathcal{O}(k^4),
\]

where \(c_s\) is the speed of the sound waves (obtained from the equation of state), and \(\Gamma\) is the sound wave attenuation (determined by the shear and the bulk viscosities)

\[
c^2_s = \frac{\partial P}{\partial E}, \quad \Gamma = \left( \frac{2}{3} \frac{\eta}{E + P} + \frac{1}{2} \frac{\zeta}{E + P} \right). \tag{2.2}
\]

The relaxation time \(\tau_{\text{eff}}\) is the relaxation time of Müller-Israel-Stewart hydrodynamics \([5, 6]\); it is the relaxation time of relativistic conformal hydrodynamics as defined in \([7]\). In general non-conformal hydrodynamics \([16]\), \(\tau_{\text{eff}}\) is a combination of particular
second-order transport coefficients\textsuperscript{1}:

\[ \tau_{\text{eff}} = \tau_\pi + \frac{3 \xi}{4 \eta} \tau_\Pi \frac{1}{1 + \frac{3 \xi}{4 \eta}}. \]  

(2.3)

In \( \mathcal{N} = 4 \) SYM plasma at infinite 't Hooft coupling and in the planar limit, the relaxation time is \[7, 8\] \textsuperscript{2}

\[ \tau_{\text{eff}} T \equiv \tau_\pi^* T = \frac{2 - \ln 2}{2\pi}. \]

(2.4)

Starting from the conformal hydrodynamics in \( d = (4 + k) \)-dimensions we can obtain non-conformal hydrodynamics by compactification on a flat \( k \)-torus \[14\]. Since compactification can not modify the long-wavelength dispersion relation, the hydrodynamic dispersion relation for the sound waves (2.1) must be the same. In other words, if \( \{c_s^{(d)}, \Gamma^{(d)}, \tau_\pi^{(d)}\} \) are the hydrodynamic coefficients of the \( d \)-dimensional conformal plasma, then

\[ c_s = c_s^{(d)} = \frac{1}{\sqrt{d-1}}, \quad \Gamma = \Gamma^{(d)} = \frac{1}{4\pi T} \frac{d-2}{d-1}, \quad \tau_{\text{eff}} = \tau_\pi^{(d)}, \]  

(2.5)

where we used an explicit expression for the attenuation in \( d \)-dimensional conformal hydrodynamics \[7\]

\[ \Gamma^{(d)} = \frac{d-2}{d-1} \eta^{(d)} \eta^{(d)} = \frac{1}{\mathcal{E}^{(d)} + \mathcal{P}^{(d)}}, \]

(2.6)

the CFT equation of state \( \mathcal{E}^{(d)} = (d-1)\mathcal{P}^{(d)} \), and the universality of the shear viscosity to the entropy density ratio in strongly coupled plasmas \[11-13\]. Given (2.2), it is easy to see that the bulk viscosity bound proposed in \[14\] is saturated in such models:

\[ \frac{\zeta}{\eta} = 2 \left( \frac{1}{3} - c_s^2 \right). \]

(2.7)

Notice that the effective relaxation time does not change upon compactification. This fact, the bulk viscosity ratio (2.7), as well as similar relations for all the other second-order transport coefficients of non-conformal hydrodynamics of \[16\] can be obtained by a direct reduction of \( d \)-dimensional conformal hydrodynamics of \[7\] on a flat \( k \)-torus. For example, it can be shown that the “bulk” \( \tau_\Pi \) and the “shear” \( \tau_\pi \) relaxation times are the same (see also \[16\])

\[ \tau_\Pi = \tau_\pi = \tau_\pi^{(d)}. \]

(2.8)

\textsuperscript{1}See \[16\] for definition of \( \{\tau_\pi, \tau_\Pi\} \).

\textsuperscript{2}Finite coupling/non-planar corrections to the relaxation time of conformal plasmas are discussed in \[17, 18\].
The relaxation time of the $d = 5$ holographic CFT plasma was computed in [19], and of the $d = 6$ holographic CFT plasma in [20]:

$$
\tau_\pi^{(5)} T = \frac{5}{8\pi} \left( 2 - \frac{\pi}{5} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{\sqrt{5}} \coth^{-1} \sqrt{5} - \frac{1}{2} \ln 5 \right),
$$

$$
\tau_\pi^{(6)} T = \frac{3}{4\pi} \left( 2 - \frac{\pi}{6\sqrt{3}} - \frac{1}{2} \ln 3 \right).
$$

(2.9)

Notice that in both cases $d = \{5, 6\}$,

$$
\frac{\tau_\pi^{(d)}}{\tau_{\text{eff}}} > 1.
$$

(2.10)

Eq. (2.10) is our first indication that the effective relaxation time in non-conformal strongly coupled plasmas is larger than that of the four-dimensional conformal plasma. In the following sections we encounter more examples of this phenomenon.

3 \ \tau_{\text{eff}} \text{ in the vicinity of the renormalization group fixed point}

An RG fixed point is a conformal theory. The gauge/gravity correspondence [9] not only allows us to study a non-trivial (strongly interactive) fixed points (such as $\mathcal{N} = 4$ SYM at large 't Hooft coupling), but also provides tools of exploring the deformations of such fixed points by relevant (in the infrared) operators (see [21] for a pedagogical discussion). In this section we discuss the relaxation time of the strongly coupled holographic plasma in the vicinity of the fixed point perturbed by dimension-2 and dimension-3 operators. Controllable fixed points that are realized in the holographic framework are those of the supersymmetric gauge theories. In these cases we can think of the deformation operators as mass terms for the gauge theory bosons and fermions. Thermodynamic and hydrodynamic properties of mass deformed conformal gauge theories (in the holographic setting) were extensively studied in [14, 22–26]. In particular, it is straightforward to extend analysis of [24] and compute the $\mathcal{O}(k^3)$ contribution to the sound wave dispersion relation (2.1). We can then extract the effective relaxation time $\tau_{\text{eff}}$. It is convenient to parametrize the result in terms of the deviation parameter $\delta$,

$$
\delta \equiv \frac{1}{3} - c_s^2,
$$

(3.1)

away from the fixed point:

$$
\tau_{\text{eff}} = \tau_\pi^* \left( 1 + \beta_{[d]} \delta + \mathcal{O}(\delta^2) \right),
$$

(3.2)
where $\tau^*_\pi$ is the universal relaxation time of the holographic conformal hydrodynamics at (infinitely) strong coupling (2.4), and $\beta_{[p]}$ is the correction induced by the operator with $\text{dim}[O] = p$. Explicitly we find,

$$
\beta_{[p]} = \begin{cases} 
2.2837(0), & p = 3, \\
6.3016(8), & p = 2.
\end{cases}
$$

Again, in both cases the effective relaxation time is larger than that of the conformal plasma.

4 $\tau_{eff}$ in $\mathcal{N} = 2^*$ plasma

We would like now to study the effective relaxation time in a holographic model with strongly broken scale invariance. Our choice is the $\mathcal{N} = 2^*$ gauge theory at strong coupling. Recall that $\mathcal{N} = 2^*$ gauge theory is obtained as a supersymmetric mass deformation of $\mathcal{N} = 4$ SYM theory. The low-energy effective description of this theory is exactly soluble [27]; furthermore, for a specific point on its Coulomb branch one can construct an explicit holographic dual [28] and verify complete agreement between the field-theoretic and the dual gravitational descriptions [29,30]. At a finite temperature, one has additional freedom of assigning different masses $m_b \neq m_f$ to the bosonic and the fermionic components of the $\mathcal{N} = 2$ hypermultiplet [22]. The thermodynamics of this theory was studied extensively in [25], and the first order transport coefficients in [26]. Here, we extend the analysis of [26] for the dispersion relation of the sound waves in $\mathcal{N} = 2^*$ plasma to order $O(k^3)$. Using (2.1), we can extract the corresponding $\tau_{eff}$.

In what follows, we focus on a specific $\mathcal{N} = 2^*$ mass deformation of $\mathcal{N} = 4$ SYM, namely the one with $m_f = 0$, $m_b \neq 0$. This gauge theory undergoes an interesting phase transition for $T < T_c \approx m_b/2.29(9)$, where it becomes unstable with respect to energy density fluctuations [25]. Specifically, precisely at $T = T_c$ the speed of sound waves squared $c^2_s$ vanishes. A perturbative instability of this type is a defining feature of a second order phase transition. Since $c^2_s \propto (T - T_c)^{1/2}$, the specific heat $c_V$ diverges as $|1 - T_c/T|^{-1/2}$, suggesting that such a critical point is in the universality class of the mean-field tricritical point$^3$. Much like real QCD [32], the $\mathcal{N} = 2^*$ plasma has a

$^3$An identical critical point was recently found in the strongly coupled cascading gauge theory plasma [31].
Figure 1: (color online) Effective relaxation time $\tau_{\text{eff}}$ of $\mathcal{N} = 2^*$ strongly coupled plasma. The vertical red line indicates a phase transition with vanishing speed of sound.

Plateau in the reduced energy density $\frac{\xi}{\pi^2}$ which extends almost to the phase transition - at $T \sim \frac{1}{2} m_b$ the $\mathcal{N} = 2^*$ equation of state deviates from the conformal equation of state by less than 3% [25]. $\mathcal{N} = 2^*$ plasma has a rapidly growing bulk viscosity in the vicinity of the phase transition: at the transition point [14, 26],

$$\left. \frac{\zeta}{\eta} \right|_{T = T_c} = 6.65(3), \quad \text{or} \quad \left. \frac{\zeta}{s} \right|_{T = T_c} = 0.52(9),$$

(4.1)

which is close to the peak value $\frac{\zeta}{s} \sim 0.7$ extracted from lattice QCD [33].

Techniques for determining the sound wave dispersion relation in strongly coupled $\mathcal{N} = 2^*$ plasma has been developed in [24, 26]. We refer the reader for computational details to the original work. There is one subtlety though: in analyzing the differential equations describing the graviton wave function at order $\mathcal{O}(k^2)$ in the momentum (and correspondingly its dispersion relation at order $\mathcal{O}(k^3)$), we observe that these equations become singular in the limit $c_s^2 \to 0$. The latter singularity is removed if instead of directly extracting $\tau_{\text{eff}}$ we compute $c_s^2 \tau_{\text{eff}}$ (along with appropriate rescaling of the graviton wave function). Upon such rescaling\footnote{A similar procedure has been employed in [14, 26].}, all the quantities remain finite in the
vicinity of small $c_s^2$. The results of the computations are presented in Figure 1. Given that the speed of sound in $\mathcal{N} = 2^*$ plasma is always less than the conformal value [25], i.e., $\frac{1}{3}$, we see that for all temperatures down to $T_c$

$$
\tau_{\text{eff}} > \tau_{\pi}^*.
$$

Moreover, since the quantity $c_s^2 \tau_{\text{eff}}$ remains finite at $T = T_c$ while the speed of sound vanishes as $c_s^2 \propto (T - T_c)^{1/2}$, we conclude that $\tau_{\text{eff}}$ actually diverges as $T \to T_c$

$$
\tau_{\text{eff}} T_c \propto |1 - T_c/T|^{-1/2}, \quad T \to T_c + 0.
$$

5 Conclusion

In this letter we studied effective relaxation time in non-conformal strongly coupled plasmas using holographic techniques. We discussed theories obtained by compactification of higher-dimensional conformal plasmas on flat tori, theories obtained by deforming a conformal theory with a relevant operator. Finally, using $\mathcal{N} = 2^*$ strongly coupled plasma as an example, we explored the effective relaxation time in the regime with strongly broken scale invariance. In all cases we observed that effective relaxation time is longer (and often much longer) than the relaxation time of the $\mathcal{N} = 4$ SYM plasma in the planar limit and at infinite ’t Hooft coupling [7, 8]. In fact, in the $\mathcal{N} = 2^*$ model we find that the relaxation time diverges near the phase transition with vanishing speed of sound\(^5\). This example suggests that the critical slowing down in sQGP is a distinct possibility [15]. Further studies of more realistic models of QCD are necessary to settle this issue. From the holographic perspective, the relaxation time in the cascading gauge theory [34] near the deconfinement phase transition should provide a useful estimate.

In the derivation of the second order viscous hydrodynamics from Boltzmann equations one finds that the effective relaxation time is [36]

$$
\tau_{\text{Boltzmann}} = \frac{3\eta T}{2P} = 6 \frac{\eta}{s} \gtrsim \frac{3}{2\pi},
$$

which is much larger than $\tau_{\pi}^* T$ since the ratio of the shear viscosity to the entropy density at weak coupling substantially exceed the KSS viscosity bound [12]. This fact,

\(^5\)The same general features of the effective relaxation time can be extracted from the phenomenological models of holographic gauge/gravity correspondence studied in [35]. I would like to thank Todd Springer for sharing the results of his analysis.
supplemented with the analysis of the relaxation time at strong coupling presented in this letter, might lead to the speculation that $\tau^\star_\pi$ is the universal lower bound on the relaxation time. Unfortunately, much like it is the case with the KSS bound [37–39], this is not so: even though the finite 't Hooft coupling corrections tend to increase the relaxation time [17], the non-planar corrections would generically decrease the relaxation time [18]. Moreover, in a class of conformal gauge theory plasmas discussed in [40], the microscopic causality of the theory imposes the bound

$$\tau_{\text{eff}} \geq 0.56(1) \, \tau^\star_\pi.$$  \hspace{1cm} (5.2)

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**References**

[1] E. V. Shuryak, Nucl. Phys. A 750, 64 (2005) [arXiv:hep-ph/0405066].

[2] P. Romatschke and U. Romatschke, Phys. Rev. Lett. 99, 172301 (2007) [arXiv:0706.1522 [nucl-th]].

[3] H. Song and U. W. Heinz, Phys. Lett. B 658, 279 (2008) [arXiv:0709.0742 [nucl-th]].

[4] K. Dusling and D. Teaney, Phys. Rev. C 77, 034905 (2008) [arXiv:0710.5932 [nucl-th]].

[5] I. Muller, Z. Phys. 198 (1967) 329.

[6] W. Israel and J. M. Stewart, Annals Phys. 118 (1979) 341.

[7] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP 0804, 100 (2008) [arXiv:0712.2451 [hep-th]]
[8] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, JHEP **0802**, 045 (2008) [arXiv:0712.2456 [hep-th]].

[9] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[10] D. T. Son and A. O. Starinets, Ann. Rev. Nucl. Part. Sci. **57**, 95 (2007) [arXiv:0704.0240 [hep-th]].

[11] A. Buchel and J. T. Liu, Phys. Rev. Lett. **93**, 090602 (2004) [arXiv:hep-th/0311175].

[12] P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231].

[13] P. Benincasa, A. Buchel and R. Naryshkin, Phys. Lett. B **645**, 309 (2007) [arXiv:hep-th/0610145].

[14] A. Buchel, Phys. Lett. B **663**, 286 (2008) [arXiv:0708.3459 [hep-th]].

[15] H. Song and U. W. Heinz, arXiv:0907.2262 [nucl-th].

[16] P. Romatschke, arXiv:0906.4787 [hep-th].

[17] A. Buchel and M. Paulos, Nucl. Phys. B **805**, 59 (2008) [arXiv:0806.0788 [hep-th]].

[18] A. Buchel, M. P. Heller and R. C. Myers, “sQGP as hCFT,” arXiv:0908.2802 [hep-th].

[19] M. Haack and A. Yarom, JHEP **0810**, 063 (2008) [arXiv:0806.4602 [hep-th]].

[20] M. Natsuume and T. Okamura, Phys. Rev. D **77**, 066014 (2008) [Erratum-ibid. D **78**, 089902 (2008)] [arXiv:0712.2916 [hep-th]].

[21] J. Polchinski and M. J. Strassler, “The string dual of a confining four-dimensional gauge theory,” arXiv:hep-th/0003136.

[22] A. Buchel and J. T. Liu, JHEP **0311**, 031 (2003) [arXiv:hep-th/0305064].

[23] A. Buchel, Nucl. Phys. B **708**, 451 (2005) [arXiv:hep-th/0406200].
[24] P. Benincasa, A. Buchel and A. O. Starinets, Nucl. Phys. B 733, 160 (2006) [arXiv:hep-th/0507026].

[25] A. Buchel, S. Deakin, P. Kerner and J. T. Liu, Nucl. Phys. B 784, 72 (2007) [arXiv:hep-th/0701142].

[26] A. Buchel and C. Pagnutti, Nucl. Phys. B 816, 62 (2009) [arXiv:0812.3623 [hep-th]].

[27] R. Donagi and E. Witten, Nucl. Phys. B 460, 299 (1996) [arXiv:hep-th/9510101].

[28] K. Pilch and N. P. Warner, Nucl. Phys. B 594, 209 (2001) [arXiv:hep-th/0004063].

[29] A. Buchel, A. W. Peet and J. Polchinski, Phys. Rev. D 63, 044009 (2001) [arXiv:hep-th/0008076].

[30] N. J. Evans, C. V. Johnson and M. Petrini, JHEP 0010, 022 (2000) [arXiv:hep-th/0008081].

[31] A. Buchel, Nucl. Phys. B 820, 385 (2009) [arXiv:0903.3605 [hep-th]].

[32] F. Karsch and E. Laermann, “Thermodynamics and in-medium hadron properties from lattice QCD,” arXiv:hep-lat/0305025.

[33] H. B. Meyer, Phys. Rev. Lett. 100, 162001 (2008) [arXiv:0710.3717 [hep-lat]].

[34] I. R. Klebanov and M. J. Strassler, JHEP 0008, 052 (2000) [arXiv:hep-th/0007191].

[35] T. Springer, Phys. Rev. D 79, 086003 (2009) [arXiv:0902.2566 [hep-th]].

[36] R. Baier, P. Romatschke and U. A. Wiedemann, Phys. Rev. C 73, 064903 (2006) [arXiv:hep-ph/0602249].

[37] Y. Kats and P. Petrov, JHEP 0901, 044 (2009) [arXiv:0712.0743 [hep-th]].

[38] M. Brigante, H. Liu, R. C. Myers, S. Shenker and S. Yaida, Phys. Rev. D 77, 126006 (2008) [arXiv:0712.0805 [hep-th]].

[39] A. Buchel, R. C. Myers and A. Sinha, JHEP 0903, 084 (2009) [arXiv:0812.2521 [hep-th]].
[40] A. Buchel and R. C. Myers, “Causality of Holographic Hydrodynamics,” arXiv:0906.2922 [hep-th].