ASYMPTOTIC BEHAVIOR OF NEURAL FIELDS IN AN UNBOUNDED DOMAIN

SEVERINO HORÁCIO DA SILVA AND MICHEL BARROS SILVA

Abstract. In this paper, we prove the existence of a compact global attractor for the flow generated by equation
\[
\frac{\partial u}{\partial t}(x, t) + u(x, t) = \int_{\mathbb{R}^N} J(x-y)(f(u(y, t)))dy + h, \quad h > 0, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}_+
\]
in the weight space $L^p(\mathbb{R}^N, \rho)$. We also give uniform estimates on the size of the attractor and we exhibit a Lyapunov functional to the flow generated by this equation.

1. Introduction

In this work we consider the non local evolution equation
\[
\frac{\partial u}{\partial t}(x, t) = -u(x, t) + J \ast (f \circ u)(x, t) + h, \quad h > 0,
\]
where $u(x, t)$ is a real-valued function on $\mathbb{R}^N \times \mathbb{R}_+$, $h$ is a positive constant, $J \in C^1(\mathbb{R}^N)$ is a non negative even function supported in the ball of center at the origin and radius 1, and, $f$ is a non negative nondecreasing function. The $\ast$ above denotes convolution product in $\mathbb{R}^N$, namely:
\[
(J \ast u)(x) = \int_{\mathbb{R}^N} J(x-y)u(y)dy.
\]

The function $u(x, t)$ denotes the mean membrane potential of a patch of tissue located at position $x \in \mathbb{R}^N$ at time $t \geq 0$. The connection function $J(x)$ determines the coupling between the elements at position $x$ and position $y$. The non negative nondecreasing function $f(u)$ gives the neural firing rate, or averages rate at which spikes are generated, corresponding to an activity level $u$. The neurons at a point $x$ are said to be active if $f(u(x, t)) > 0$. The parameter $h$ denotes a constant external stimulus applied uniformly to the entire neural field.

For the particular case, where $N = 1$, there are already in the literature several works dedicated to the analysis of this model, (see, for example, [1], [4], [6], [11], [12], [14], [18], [19], [20], [21], [22], [23], [25] and [26]). Also there are some works for this model with $N > 1$, (see for example [7] and [14]).

In this paper we extend, for $L^p(\mathbb{R}^N, \rho)$, $N \geq 1$ and $1 < p < \infty$, the results (on global attractors) obtained in [20] in the phase space $L^p(\mathbb{R}, \rho)$. Furthermore, we exhibit a Lyapunov functional to the flow generated by (1.1).

2000 Mathematics Subject Classification. 45J05, 45M05, 34D45.
Key words and phrases. Well-posedness; global attractor; upper semicontinuity of attractors.

1Supported by CAPES/CNPq-Brazil.
2Supported by CNPq-Brazil.
This paper is organized as follows. In Section 2 we prove that, in the phase space $L^p(\mathbb{R}^N, \rho) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^N) : \int |u|^p \rho(x) dx < +\infty \}$, the Cauchy problem for (1.1) is well posed with globally defined solutions. In Section 3 we prove that the system is dissipative in the sense of [9], that is, it has a global compact attractor, generalizing Theorem 3.5 of [20]. In our proof, we use the Sobolev’s compact embedding $W^{-1,p}(B(0, l)) \hookrightarrow L^p(B(0, l))$ and the same techniques used in [15] and [20] (see also [2], [16] and [17] for related work). In Section 4, we prove some estimates for the attractor and finally, in Section 5, using ideas from [8], [13] and [22], we exhibit a Lyapunov function for the flow generated by (1.1).

2. Well-posedness

In this section we consider the flow generated by (1.1) in the space $L^p(\mathbb{R}^N, \rho)$ defined by

$$L^p(\mathbb{R}^N, \rho) = \{ u \in L^1_{\text{loc}}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^p \rho(x) dx < +\infty \},$$

with norm $\|u\|_{L^p(\mathbb{R}^N, \rho)} = \left( \int_{\mathbb{R}^N} |u(x)|^p \rho(x) dx \right)^{1/p}$. Note that, in this space, the constant function equal to 1 has norm 1.

As similarly assumed in [20], we assume here the following hypotheses on the functions $f$ and $\rho$:

(H1) the function $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, that is, there exists $k_1 > 0$ such that

$$|f(x) - f(y)| \leq k_1|x - y|, \quad \forall x, y \in \mathbb{R}, \quad (2.1)$$

(H2) $\rho : \mathbb{R}^N \to \mathbb{R}$ is an integrable positive even function with $\int_{\mathbb{R}^N} \rho(x) dx = 1$ and there exists constant $K > 0$ such that

$$\sup\{\rho(x) : x \in \mathbb{R}^N, |x - y| \leq 1\} \leq K \rho(y), \quad \forall y \in \mathbb{R}^N.$$

The corresponding higher-order Sobolev space $W^{k,p}(\mathbb{R}^N, \rho)$ is the space of functions $u \in L^p(\mathbb{R}^N, \rho)$ whose distributional derivatives up to order $k$ are also in $L^p(\mathbb{R}^N, \rho)$, with norm

$$\|u\|_{W^{k,p}(\mathbb{R}^N, \rho)} = \left( \sum_{i=1}^{k} \left\| \frac{\partial^i u}{\partial x^i} \right\|_{L^p(\mathbb{R}^N, \rho)}^p \right)^{1/p}.$$

Lemma 2.1. Suppose that (H2) holds. Then

$$\|J \star u\|_{L^p(\mathbb{R}^N, \rho)} \leq K^{1/p} \|J\|_{L^1} \|u\|_{L^p(\mathbb{R}^N, \rho)}.$$

Proof. Since $J$ is bounded and compact supported, $(J \star u)(x)$ is well defined for $u \in L^1_{\text{loc}}(\mathbb{R}^N)$. Thus,

$$\|J \star u\|_{L^p(\mathbb{R}^N, \rho)}^p = \int_{\mathbb{R}^N} |(J \star u)(x)|^p \rho(x) dx$$

$$= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} J(x - y)u(y) dy \right)^p \rho(x) dx$$

$$\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |J(x - y)||u(y)| dy \right)^p \rho(x) dx.$$

Using

$$\frac{p - 1}{p} + \frac{1}{p} = 1,$$

(2.2)
we obtain
\[ \|J * u\|_{L^p(B(y,1),\rho)}^p \leq \int_{\mathbb{R}^N} (\int_{\mathbb{R}^N} |J(x-y)|^{(p-1)/p} |J(x-y)|^{1/p} |u(y)| dy)^p \rho(x) dx. \]

By Holder’s inequality (see [3]), we have
\[ \|J * u\|_{L^p(B(y,1),\rho)}^p \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |J(x-y)| dy \right)^{(p-1)/p} \left( \int_{\mathbb{R}^N} |J(x-y)||u(y)| \rho(x) dx \right)^p \left( \int_{\mathbb{R}^N} |u(y)|^p dy \right) \rho(x) dx \]
\[ = \int_{\mathbb{R}^N} |||J|||_{L^1}^{-1} \left( \int_{\mathbb{R}^N} |J(x-y)||u(y)| \rho(x) dx \right)^p \left( \int_{\mathbb{R}^N} |u(y)|^p dy \right) \rho(x) dx. \]

Denoting the closed ball of center y and radius 1 by B[y,1] and using (1.2) and (H2), follows that
\[ \|J * u\|_{L^p(B(y,1),\rho)}^p \leq \int_{\mathbb{R}^N} \left( \int_{B[y,1]} J(x) \rho(x) dx \right) |u(y)|^p dy \]
\[ \leq \|J\|_{L^1}^{-1} \int_{\mathbb{R}^N} \left( K \rho(y) \int_{B[y,1]} J(x) dx \right) |u(y)|^p dy \]
\[ \leq K \|J\|_{L^1} \int_{\mathbb{R}^N} |u(y)|^p \rho(y) dy \]
\[ = K \|J\|_{L^1} \|u\|_{L^p(B(y,1),\rho)}^p. \]

It concludes the result.

**Proposition 2.2.** Suppose that the hypotheses (H1) and (H2) hold. Then the function

\[ F(u) = -u + J * (f \circ u) + h \]

is globally Lipschitz in L^p(\mathbb{R}^N, \rho).

**Proof.** From triangle inequality and Lemma 2.1, it follows that
\[ \|F(u) - F(v)\|_{L^p(\mathbb{R}^N, \rho)} \leq \|v - u\|_{L^p(\mathbb{R}^N, \rho)} + \|J * (f \circ u) - J * (f \circ v)\|_{L^p(\mathbb{R}^N, \rho)} \]
\[ \leq \|v - u\|_{L^p(\mathbb{R}^N, \rho)} + K^{1/p} \|J\|_{L^1} \|(f \circ u) - (f \circ v)\|_{L^p(\mathbb{R}^N, \rho)}. \]

We have
\[ \|(f \circ u) - (f \circ v)\|_{L^p(\mathbb{R}^N, \rho)} \leq \int_{\mathbb{R}^N} k_1^p |u(x) - v(x)|^p \rho(x) dx = k_1^p \|u - v\|_{L^p(\mathbb{R}^N, \rho)}. \]

Then
\[ \|F(u) - F(v)\|_{L^p(\mathbb{R}^N, \rho)} \leq (1 + K^{1/p} \|J\|_{L^1} k_1) \|u - v\|_{L^p(\mathbb{R}^N, \rho)}. \]

Therefore, F is globally Lipschitz in L^p(\mathbb{R}^N, \rho).

**Remark 2.3.** From Proposition 2.2 and standard results of ODEs in Banach spaces (see [4]), follows that the Cauchy problem for (1.1) is well posed in L^p(\mathbb{R}^N, \rho) with globally defined solutions.
3. Existence of a global attractor

In this section, we prove the existence of a global maximal invariant compact set \( \mathcal{A} \subset L^p(\mathbb{R}^N, \rho) \) for the flow of (1.1), which attracts each bounded set of \( L^p(\mathbb{R}^N, \rho) \) (the global attractor, see [9] and [24]), generalizing Theorem 3.3 in [20]. For this, beyond (H1) and (H2) we assume the following additional hypotheses:

(H3) there exists \( a > 0 \) such that \(|f(x)| \leq a, \quad \forall x \in \mathbb{R}^N\);

(H4) the non negative, symmetric bounded function \( J \) has bounded derivative with

\[
\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \partial_x J(x - y) dy \leq S \quad \text{and} \quad \sup_{y \in \mathbb{R}^N} \int_{\mathbb{R}^N} \partial_y J(x - y) dx \leq S,
\]

for some constant \( 0 < S < \infty \).

From now on we denote by \( S(t) \) the flow generated by (1.1).

We recall that a set \( B \subset L^p(\mathbb{R}^N, \rho) \) is an absorbing set for the flow \( S(t) \) in \( L^p(\mathbb{R}^N, \rho) \) if, for any bounded set \( B \subset L^p(\mathbb{R}^N, \rho) \), there is a \( t_1 > 0 \) such that \( S(t)B \subset B \) for any \( t \geq t_1 \), (see [24]).

**Lemma 3.1.** Suppose that the hypotheses (H1), (H2) and (H3) hold and let \( R = aK^{1/p}\|J\|_{L^1} + \varepsilon \) is an absorbing set for the flow \( S(t) \) in \( L^p(\mathbb{R}^N, \rho) \) for any \( \varepsilon > 0 \).

**Proof.** Let \( u(x, t) \) be the solution of (1.1) with initial condition \( u(\cdot, 0) \in L^p(\mathbb{R}^N, \rho) \). Then, by the variation of constants formula,

\[
u(x, t) = e^{-t}u(x, 0) + \int_0^t e^{s-t}J \ast (f \circ u)(x, s) + h) ds.
\]

Hence

\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^N, \rho)} \leq \|e^{-t}u(\cdot, 0)\|_{L^p(\mathbb{R}^N, \rho)} + \int_0^t e^{s-t}\|J \ast (f \circ u)(\cdot, s) + h\|_{L^p(\mathbb{R}^N, \rho)} ds
\]

\[
\leq e^{-t}\|u(\cdot, 0)\|_{L^p(\mathbb{R}^N, \rho)} + \int_0^t e^{s-t}\|J \ast (f \circ u)(\cdot, s)\|_{L^p(\mathbb{R}^N, \rho)} + h) ds.
\]

Then, using Lemma 2.1 it follows that

\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^N, \rho)} \leq e^{-t}\|u(\cdot, 0)\|_{L^p(\mathbb{R}^N, \rho)} + \int_0^t e^{s-t}[K^{1/p}\|J\|_{L^1}\|f(u(\cdot, 0))\|_{L^p(\mathbb{R}^N, \rho)} + h) ds
\]

Now, from (H3), we have

\[
\|f(u(\cdot, s))\|_{L^p(\mathbb{R}^N, \rho)}^p = \int_{\mathbb{R}^N} |f(u(x, s))|^p \rho(x) dx
\]

\[
\leq a^p \int_{\mathbb{R}^N} \rho(x) dx = a^p.
\]
Thus
\[
\|u(\cdot, t)\|_{L^p(\mathbb{R}^N, \rho)} \leq e^{-t}\|u(\cdot, 0)\|_{L^p(\mathbb{R}^N, \rho)} + \int_0^t e^{s-t} aK^{1/p} \|J\|_{L^1} + h \, ds
\]
\[
= e^{-t}\|u(\cdot, 0)\|_{L^p(\mathbb{R}^N, \rho)} + R.
\]
Therefore, for any \( t > \ln \left( \frac{\|u(\cdot, 0)\|_{L^p(\mathbb{R}^N, \rho)}}{\varepsilon} \right) \), we have \( \|u(\cdot, t)\|_{L^p(\mathbb{R}^N, \rho)} < \varepsilon + R \), and the proof is complete. \( \Box \)

**Lemma 3.2.** Suppose that the hypotheses \((H1)-(H4)\) hold. Then, for any \( \eta > 0 \), there exists \( t_\eta \) such that \( S(t_\eta)B(0, R+\varepsilon) \) has a finite covering by balls of \( L^p(\mathbb{R}^N, \rho) \) with radius smaller than \( \eta \).

**Proof.** From Lemma 3.1 it follows that \( B(0, R+\varepsilon) \) is invariant. Now, the solution of \((1.1)\) with initial condition \( u_0 \in B(0, R+\varepsilon) \) is given, by the variation of constant formula, by
\[
u(x, t) = e^{-t}u_0(x) \text{ and } w(x, t) = \int_0^t e^{s-t}[(J * (f \circ u))(x, s) + h]ds.
\]
Write
\[
v(x, t) = e^{-t}u_0(x) \text{ and } w(x, t) = \int_0^t e^{s-t}[(J * (f \circ u))(x, s) + h]ds.
\]
Let \( \eta > 0 \) given. We may find \( t(\eta) \) such that if \( t \geq t(\eta) \), then \( \|v(\cdot, t)\|_{L^p(\mathbb{R}^N, \rho)} \leq \frac{\eta}{2} \).

Now, using (H3), we obtain
\[
\|J * (f \circ u)(\cdot, s)\|_{L^p(\mathbb{R}^N, \rho)}^p = \int_{\mathbb{R}^N} |J * (f \circ u)(x, s)|^p \rho(x)dx
\]
\[
= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} J(x-y)f(u(y))dy \right|^p \rho(x)dx
\]
\[
\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} J(x-y)|f(u(y))|dy \right)^p \rho(x)dx
\]
\[
\leq \int_{\mathbb{R}^N} \left( a \int_{\mathbb{R}^N} J(x-y)dy \right)^p \rho(x)dx
\]
\[
= \int_{\mathbb{R}^N} (a\|J\|_{L^1})^p \rho(x)dx
\]
\[
= (a\|J\|_{L^1})^p \int_{\mathbb{R}^N} \rho(x)dx
\]
\[
= (a\|J\|_{L^1})^p.
\]
Thus,
\[
\|J * (f \circ u)(\cdot, s)\|_{L^p(\mathbb{R}^N, \rho)} \leq a\|J\|_{L^1}.
\]
Hence
\[
\|w(\cdot, t)\|_{L^p(\mathbb{R}^N, \rho)} \leq \int_0^t e^{-(t-s)}(\|J * (f \circ u)(\cdot, s)\|_{L^p(\mathbb{R}^N, \rho)} + \|h\|_{L^p(\mathbb{R}^N, \rho)})ds
\]
\[
\leq \int_0^t e^{-(t-s)}(a\|J\|_{L^1} + h)ds
\]
\[
= a\|J\|_{L^1} + h.
\]
(3.1)
On the other hand, by (H3), we have
\[
|w(x,t)| \leq \int_0^t e^{-(t-s)}\left[|J \ast (f \circ u)(x,s)| + h\right]ds
\]
\[
= \int_0^t e^{-(t-s)} \left| \int_{\mathbb{R}^N} J(x-y) f(u(y,t))dy \right| + h \ ds
\]
\[
\leq \int_0^t e^{-(t-s)} \left( \int_{\mathbb{R}^N} J(x-y) \|f(u(y,t))\|dy + h \right) ds
\]
\[
\leq \int_0^t e^{-(t-s)} \left( a \int_{\mathbb{R}^N} J(x-y)dy + h \right) ds
\]
\[
= \int_0^t e^{-(t-s)} (a\|J\|_{L^1} + h) ds
\]
\[
= a\|J\|_{L^1} + h. \quad (3.2)
\]
Furthermore, differentiating with respect to \(x_i\), for \(t \geq 0\), we have
\[
\frac{\partial w}{\partial x_i}(x,t) = \int_0^t e^{-(t-s)} \frac{\partial}{\partial x_i} J \ast (f \circ u)(x,s)ds, \quad i = 1, \ldots, N.
\]
Thus
\[
\left| \frac{\partial w}{\partial x_i}(x,t) \right| \leq \int_0^t e^{-(t-s)} \left| \frac{\partial}{\partial x_i} J \ast (f \circ u)(x,s) \right| ds.
\]
But, using (H4), obtain
\[
|\frac{\partial}{\partial x_i} J \ast (f \circ u)(x,s)| \leq \int_{\mathbb{R}^N} a |\frac{\partial}{\partial x_i} J(x-y)| dy
\]
\[
\leq aS < \infty,
\]
it follows that
\[
\left| \frac{\partial w}{\partial x_i}(x,t) \right| \leq \int_0^t e^{-(t-s)} aS ds \leq aS < \infty. \quad (3.3)
\]
Now, let \(l > 0\) be chosen such that
\[
(a\|J\|_{L^1} + h) \left( \int_{\mathbb{R}^N} (1 - \chi_{B[0,l]})^{p/2}/(p-1) \rho(x)dx \right)^{(p-1)/p^2} \leq \frac{\eta}{4}, \quad (3.4)
\]
where \(\chi_{B[0,l]}\) denotes the characteristic function of the ball \(B[0,l]\). Then, using (3.1), (3.2) and (3.4), we obtain
\[
\| (1 - \chi_{B[0,l]}(\cdot) w(\cdot, t) \|_{L^p(\mathbb{R}^N, \rho)}^p = \int_{\mathbb{R}^N} |(1 - \chi_{B[0,l]}(x))w(x,t)|^p \rho(x)dx
\]
\[
= \int_{\mathbb{R}^N} |(1 - \chi_{B[0,l]}(x))^p|w(x,t)|^p \rho(x)dx.
\]
Using (2.2) and Holder’s inequality, follows that
\[ \| (1 - \chi_{B(0,l)})(\cdot)w(\cdot, t) \|_{L^p(R^N, \rho)}^p = \]
\[ = \left( \int_{R^N} |w(x, t)|r(x)|1/p|(1 - \chi_{B(0,l)}(x))|w(x, t)|^{p-1}\rho(x)^{(p-1)/p}dx \right)^{p/(p-1)} \]
\[ \leq \left( \int_{R^N} |w(x, t)|r(x)dx \right)^{1/p} \left( \int_{R^N} |(1 - \chi_{B(0,l)}(x))|w(x, t)|^{p-1}\rho(x)dx \right)^{(p-1)/p} \]
\[ = \|w(\cdot, t)\|_{L^p(R^N, \rho)} \left( \int_{R^N} |(1 - \chi_{B(0,l)}(x))|w(x, t)|^{p-1}\rho(x)dx \right)^{(p-1)/p} \]
\[ \leq \left( a\|J\|_{L^1} + h \right) \left( \int_{R^N} |(1 - \chi_{B(0,l)}(x))|w(x, t)|^{p-1}\rho(x)dx \right)^{(p-1)/p} \]
\[ = \left( a\|J\|_{L^1} + h \right) \left( \int_{R^N} |(1 - \chi_{B(0,l)}(x))|w(x, t)|^{p-1}\rho(x)dx \right)^{(p-1)/p} \]
\[ < \frac{\eta}{4} \]

Also, by (3.2) and (3.3), the restriction of \(w(\cdot, t)\) to the ball \(B(0,l)\) is bounded in \(W^{1,p}(B(0,l))\) (by a constant independent of \(u_0 \in B(0, R + \varepsilon)\) and of \(t\)), and therefore the set \(\{\chi_{B(0,l)}w(\cdot, t)\} \) with \(w(\cdot, 0) \in B(0, R + \varepsilon)\) is relatively compact subset of \(L^p(R^N, \rho)\) for any \(t > 0\) and, hence, it can be covered by a finite number of balls with radius smaller than \(\frac{\eta}{4}\).

Therefore, since
\[ u(\cdot, t) = v(\cdot, t) + \chi_{B(0,l)}w(\cdot, t) + (1 - \chi_{B(0,l)})w(\cdot, t), \]
it follows that \(S(t\eta)B(0, R + \varepsilon)\) has a finite covering by balls of \(L^p(R^N, \rho)\) with radius smaller than \(\eta\), and the result is proved.

Denoting by \(\omega(C)\) the \(\omega\)-limit of a set \(C\), we obtain the result below, whose proof is omitted because it is very similar to Theorem 3.3 in [20].

**Theorem 3.3.** Assume the same hypotheses of Lemma 3.2. Then \(A = \omega(B(0, R + \varepsilon))\), is a global attractor for the flow \(S(t)\) generated by (1.1) in \(L^p(R^N, \rho)\) which is contained in the ball of radius \(R\).

4. BOUNDEDNESS RESULTS

In this section we prove uniform estimates for the attractor whose existence was given in the Theorem 3.3.

**Theorem 4.1.** Assume the same hypotheses of Lemma 3.2. Then the attractor \(A\) belongs to the ball \(\| \cdot \|_{L^\infty(R^N)} \leq r\), where \(r = a\|J\|_{L^1} + h\).

**Proof.** Let \(u(x, t)\) be a solution of (1.1) in \(A\). Then, as we see in (1.1),
\[ u(x, t) = \int_{-\infty}^{t} e^{-(t-s)}[J * (f \circ u)(x, s) + h]ds, \]
where the equality above is in the sense of $L^p(\mathbb{R}^N, \rho)$. Thus, using (H3), obtain
\[ |u(x, t)| \leq \int_{-\infty}^{t} e^{-(t-s)|J \ast (f \circ u)(x, s)| + h} ds \]
\[ \leq \int_{-\infty}^{t} (a\|J\|_{L^1} + h)e^{-(t-s)} ds \]
\[ = \int_{-\infty}^{t} r e^{-(t-s)} ds = r. \]

Proceeding as in [20], replacing $\|\cdot\|_{L^2(\mathbb{R}, \rho)}$ by $\|\cdot\|_{L^p(\mathbb{R}^N, \rho)}$, we obtain the following result.

**Theorem 4.2.** Assume the same hypotheses as in Lemma 3.2. Then, fixed $J_0$, for $J$ close to $J_0$, the family of attractors $\{A_J\}$ satisfies:
\[ \cup_J A_J \subset B[0, R], \]
and furthermore, it is upper semicontinuous with respect to $J$ at $J_0$, that is
\[ \sup_{x \in A_J} \inf_{y \in A_{J_0}} \|x - y\|_{L^p(\mathbb{R}^N, \rho)} \to 0, \quad \text{as } J \to J_0. \]

5. **Existence of energy functional**

In this section, we exhibit a energy functional for the flow of (1.1), which decreases along of solutions (1.1). For this, beyond hypotheses (H1)-(H4), we assume the following additional hypothesis on $f$:

(H5) the nondecreasing function $f$ takes values between 0 and $a$ and satisfying, for $0 \leq s \leq a$
\[ \left| \int_0^s f^{-1}(r) dr \right| < L < \infty. \quad (5.1) \]

(H6) $f$ satisfies
\[ \int_{\mathbb{R}^N} |f(u(x)) - f(u_0)| dx < \infty. \]

**Remark 5.1.** The hypothesis (H6) always occurs, for example, in fields with finite excited region, which tend to resting state, when $|x| \to \infty$.

Motivated by energy functionals from [7], [8], [13] and [22], we define $F : L^p(\mathbb{R}^N, \rho) \to \mathbb{R}$ by
\[ F(u) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} f(u(x)) \int_{\mathbb{R}^N} J(x - y) f(u(y)) dy + \int_{\mathbb{R}^N} f^{-1}(r) dr - h f(u(x)) \right] dx. \quad (5.2) \]

**Remark 5.2.** The similar functional given in [22] is well defined in whole phase space. Unfortunately this does not occur here, because the functional given in (5.2) can take values $\pm \infty$. An example where this occurs is when whole field is at homogeneous resting state with constant membrane potential $u_0$. In this case, the external stimulus applied, $h$, satisfies $h = u_0 - \|J\|_{L^1} f(u_0)$. 
Let \( u_0 \) be an equilibrium solution for (1.1), which is given implicitly by equation
\[
\|J\|_{L^1} f(u_0) + h.
\]
Write \( U = u - u_0 \) and \( g(U) = f(U + u_0) - f(u_0) \). Then the equation (1.1) can be write as
\[
\frac{\partial U}{\partial t}(x, t) = -U(x, t) + J \ast (g \circ U)(x, t).
\] (5.3)
For equation (5.3), we define the functional
\[
G(U) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} g(U(x)) \int_{\mathbb{R}^N} J(x - y)g(U(y))dy + \int_0^{g(U(x))} g^{-1}(r)dr \right] dx. \quad (5.4)
\]
Thus we obtain the following result:

**Theorem 5.3.** Let \( U(\cdot, t) \) be a solution of (5.3). Then, under the hypotheses, (H3), (H5) and (H6), we have
\[
G(U) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} [f(u(x)) - f(u_0)] \int_{\mathbb{R}^N} J(x - y)[f(u(y)) - f(u_0)]dy 
+ \int_{f(u_0)}^{f(u(x))} f^{-1}(r)dr \right] dx < \infty \quad (5.5)
\]
and
\[
\frac{d}{dt} G(U(x, t)) = -\int_{\mathbb{R}^N} f'(u(x, t)) \left( \frac{\partial u}{\partial x} (x, t) \right)^2 dx \leq 0. \quad (5.6)
\]

**Proof.** Since \( g(U) = f(U + u_0) - f(u_0) \), from equation (5.4), we obtain
\[
G(U) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} [f(U(x) + u_0) - f(u_0)] \int_{\mathbb{R}^N} J(x - y)[f(U(y) + u_0) - f(u_0)]dy 
+ \int_0^{g(U(x))} g^{-1}(r)dr \right] dx.
\]
Now, using that \( U = u - u_0, g(0) = 0 \) and the fact that \( f^{-1} \) and \( g^{-1} \) differ only by translation, which is an isometry, follows that
\[
\int_{f(u_0)}^{f(u(x))} f^{-1}(r)dr = \int_0^{f(U(x) + u_0) - f(u_0)} f^{-1}(r)dr = \int_0^{g(U(x))} g^{-1}(r)dr.
\]
Hence
\[
G(U) = \int_{\mathbb{R}^N} \left[ -\frac{1}{2} [f(u(x)) - f(u_0)] \int_{\mathbb{R}^N} J(x - y)[f(u(y)) - f(u_0)]dy 
+ \int_{f(u_0)}^{f(u(x))} f^{-1}(r)dr \right] dx.
\]
From hypotheses (H5) and (H6), it follows that \(|G(U)| < \infty\).
Furthermore, proceeding as in the Theorem 4.4 of [22], it is easy to verify that
\[
\frac{d}{dt} G(U(x, t)) = -\int_{\mathbb{R}^N} g(U(x, t)) \left( \frac{\partial U}{\partial t} (x, t) \right)^2 dx.
\]
Hence
\[
\frac{d}{dt} G(U(x,t)) = - \int_{\mathbb{R}^N} g'(U(x,t)) \left( \frac{\partial U}{\partial t}(x,t) \right)^2 dx
\]
\[
= - \int_{\mathbb{R}^N} \left[ f'(U(x,t) + u_0) - \frac{d}{dt}(f(u_0)) \right] \left( \frac{\partial u}{\partial t}(x,t) - \frac{\partial u_0}{\partial t} \right)^2 dx
\]
\[
= - \int_{\mathbb{R}^N} \left[ f'(u(x,t)) \right] \left( \frac{\partial u}{\partial t}(x,t) \right)^2 dx.
\]
From hypothesis (H3) the result follows. □

**Remark 5.4.** From Theorem 5.3, it follows that the functional given in (5.4) is actually a Lyapunov functional for the flow generated by equation (5.3).

6. **Concluding Remarks**

In this paper we extend results on global dynamical of the neural fields equation considering fields in \( x \in \mathbb{R}^N \) and more abstracts phase spaces. Although realistically, \( N \) should be equal to 1, 2 or 3, we do not restrict the calculations to this case, because all estimates also are valid with \( N > 3 \). Furthermore, motivated by energy functional existing in the literature, we exhibit one functional energy (type Lyapunov functional), which is well defined throughout phase space, which it is a lot important for studying existence and stability of solutions of equilibria of equations neural fields.

**Acknowledgments.** The authors would like to thank the professors Antonio L. Pereira (USP), and Flank D. M. Bezerra (UFPB) for their suggestions for this work.

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Severino Horácio da Silva
Unidade Acadêmica de Matemática e Estatística UAME/CCT/UFCG, Rua Aprígio Veloso, 882, Bairro Universitário CEP 58429-900, Campina Grande-PB, Brasil.
E-mail address: horacio@dme.ufcg.edu.br

Michel Barros Silva
Unidade Acadêmica de Matemática e Estatística UAME/CCT/UFCG, Rua Aprígio Veloso, 882, Bairro Universitário CEP 58429-900, Campina Grande-PB, Brasil.
E-mail address: michel@dme.ufcg.edu.br