On the extension of Jackiw’s scalar theory to
(2+1)-dimensional gravity

V. B. Bezerra¹, J. Spinelly² and C. Romero³

Departamento de Física, Universidade Federal da Paraíba
Caixa Postal 5008, 58051-970 João Pessoa,PB,Brazil

Abstract

We study some aspects of three-dimensional gravity by extending Jackiw’s scalar theory to (2 + 1)-dimensions and find a black hole solution. We show that in general this theory does not possess a Newtonian limit except for special metric configurations.
PACS nos. 04.20-q, 41.20-q.

¹e-mail: valdir@fisica.ufpb.br
²e-mail: spinelly@fisica.ufpb.br
³e-mail: cromero@fisica.ufpb.br
1 Introduction

Three-dimensional Einstein gravity has recently developed into an area of active research. In this framework it was showed that, although the presence of mass can not induce curvature (locally curvature vanishes everywhere except at the sources), it does affect the space around the particle. The geometry around a point particle is locally flat, but conical in form, with a deficit angle proportional to the particle’s mass. Thus, a system of gravitating point particle sources only affects geometry globally rather than locally and, as a consequence, local dynamics is replaced by global effects. Quantities such as the total energy-momentum and angular momentum for a system of point particles are defined by global geometric properties of the space-time surrounding the sources. In other words, curvature is created by sources, but only locally at their positions; elsewhere space-time remains flat and for this reason there can exist no interaction between sources. As there are no effects of gravity outside matter, light emitted from the surface of a star will always escape to infinity, and therefore, black holes do not exist in three-dimensional Einstein gravity. Moreover gravity does not propagate outside matter: test particles placed in vacuum do not experience any acceleration. On the other hand, Newton’s gravity theory in this dimension predicts a logarithmic gravitational potential outside matter. As a consequence, a test particle placed in vacuum always accelerates in Newton’s theory. For this reason Einstein’s theory in (2 + 1)-dimensional space-time cannot reduce to Newton’s theory by means of linearization.

In order to overcome the problem concerning the non-existence of New-
tonian limit in this dimension some ideas have been presented, such as the construction of a teleparallel theory or weakening Einstein equations in the same way as did Jackiw in his formulation of gravity in (1 + 1)-dimensions. In two dimensions, Einstein’s theory does not exist because in this case the Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \) vanishes identically and the Einstein-Hilbert action \( \int d^2x \sqrt{g}R \) is a surface term - it is the Euler topological invariant and does not lead to equations of motion.

In the middle of eighties, Jackiw and Teitelboim suggested that a way to obtain a non-trivial dynamics in (1 + 1)-dimensions is to introduce an additional gravitational variable \( \eta \) (scalar Lagrange multiplier) and construct the following non-trivial action

\[
\int d^2x \sqrt{g} \eta (R - \Lambda),
\]

where \( R \) is the Ricci scalar and \( \Lambda \) is the cosmological constant. In this way we can obtain a non-trivial theory of gravity in two dimensions with field equations

\[
R - \Lambda = T,
\]

where \( T \) is the trace of the matter stress-energy tensor.

2 Extending Jackiw’s scalar theory

As assumed by Romero and Dahia, we will consider that the eq.(2) with \( \Lambda = 0 \) describes gravity in (2 + 1)-dimensions. Thus for source free regions we have
\[ R = 0. \] 

(3)

Now, let us consider the problem of finding the motion of a test particle under the influence of a static, circularly symmetrical matter distribution.

Differently from Romero and Dahia\cite{3}, who considered a conformally flat metric that solves eq. (3), let us choose the static, circularly symmetrical line element given by

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2d\theta^2, \] 

(4)

with \( A \) and \( B \) being functions of \( r \) only. Of course, if \( A \) and \( B \) are independent, then they cannot be determined by eq. (3) alone. To overcome this difficulty let us reduce the number of degrees of freedom of the geometry by choosing a metric tensor with only one degree of freedom. This can be done if we take \( B = A^{-1} \). In this case, the line element (4) becomes

\[ ds^2 = -Adt^2 + (A)^{-1}dr^2 + r^2d\theta^2. \] 

(5)

Putting eq. (5) into (3), leads to

\[ \frac{d^2A}{dr^2} + \frac{2}{r} \frac{dA}{dr} = 0, \]

(6)

We can immediately write down the solution of eq. (5), which is given by

\[ A = a + \frac{b}{r}, \]

(7)

where \( a \) and \( b \) are constants.
Therefore, by requiring asymptotical flatness the line element in the present case can be written as

\[ ds^2 = - \left( 1 - \frac{2\beta}{r} \right) dt^2 + \left( 1 - \frac{2\beta}{r} \right)^{-1} dr^2 + r^2 d\theta^2, \]  

where we have chosen \( a = 1 \) and \( b = -2\beta \).

Surely, (8) represents the (2+1)-dimensional analogue of the Schwarzschild space-time. A simple look at the light cones structure reveals that the surface \( r = 2\beta \) acts as an event horizon. At the value \( r = 2\beta \) we have a removable coordinate singularity. On the other hand \( r = 0 \) represents an essential singularity, as can be seen directly from the Riemann tensor scalar invariant \( R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{24\beta^2}{r^6} \).

Therefore, the extension of Jackiw’s scalar theory to (2 + 1)-dimensions allows the existence of black hole solutions, which are not predicted by three-dimensional Einstein gravity, except for the case of non-vanishing cosmological constant\(^7\).

For the metric (8), the geodesic equations of motion are

\[ \frac{d}{d\lambda} \left[ \left( -1 + \frac{2\beta}{r} \right) \frac{dt}{d\lambda} \right] = 0, \]  

\[ \frac{d}{d\lambda} \left[ r^2 \frac{d\theta}{d\lambda} \right] = 0 \]  

and

\[ - \left( 1 - \frac{2\beta}{r} \right) \left( \frac{dt}{d\lambda} \right)^2 + \left( 1 - \frac{2\beta}{r} \right)^{-1} \left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\theta}{d\lambda} \right)^2 = -\varepsilon. \]
where $\varepsilon$ is a constant that takes the values $-1,0,1$, for space-like, null and time-like curves, respectively.

Integrating (9) and (10) yields

\[
\left(1 - \frac{2\beta}{r}\right) \frac{dt}{d\lambda} = E \tag{12}
\]

and

\[
r^2 \frac{d\theta}{d\lambda} = L, \tag{13}
\]

where $E$ and $L$ are integration constants.

For $\varepsilon = 1$ and $\lambda = \tau$ ($\tau$ being the proper time as measured by a particle following geodesic motion) it follows from eqs. (11), (12) and (13) that

\[
\frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{E^2}{2}. \tag{14}
\]

The above equation might be loosely interpreted as the equivalent of energy conservation in Newtonian gravity with $V(r)$ playing the role of an ”effective potential” given by

\[
V(r) = \frac{\varepsilon^2}{2} - \frac{\beta \varepsilon}{r} + \frac{L^2}{2r^2} - \frac{\beta L^2}{r^3}. \tag{15}
\]

By the same token $L$ would be looked upon as the angular momentum of the particle per unit mass.

The critical points of the ”effective potential” (15) can be obtained by solving

\[
\frac{\beta \varepsilon}{r^2} - \frac{L^2}{r^3} + \frac{3\beta L^2}{r^4} = 0. \tag{16}
\]
For null geodesics ($\varepsilon = 0$) it is easily seen from (14) and (16) that photons can follow the circular orbits $r = 3\beta$ provided that $E^2 = \frac{L^2}{2\gamma\beta^2}$. These orbits are highly unstable as $r = 3\beta$ is an isolated maximum for $V(r)$. If $E^2 > \frac{L^2}{2\gamma\beta^2}$, then either the photon will fall towards the singularity ($r < 3\beta$) or it will recede into infinity ($r > 3\beta$).

For massive particles ($\varepsilon = 1$) one or two circular orbits with radii $r_{\pm} = \frac{L^2 \pm (L^2 - 12\beta^2 L^2)^{1/2}}{2\beta}$ are possible according to $L^2 = 12\beta^2$, or $L^2 > 12\beta^2$, respectively. If $L^2 < 12\beta^2$, then it is not hard to see that the particle is forced to move towards the singularity at $r = 0$.

3 Newtonian limit of Jackiw’s scalar theory

Given that Einstein’s (2 + 1)-dimensional gravity exhibits such a drastic departure from the corresponding Newtonian gravity, one would wonder whether or not the present extension of Jackiw’s scalar gravity does have a Newtonian limit. This question is better examined if one considers Galilean coordinates, the ones in which

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

(17)

with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $|h_{\mu\nu}| << 1$. It is a well-known fact that for any metric theory of gravity small departures from Minkowski flat space-time lead to geodesic equation which, for a non-relativistic particle, perfectly mimic Newton’s equation of motion in a classical gravitational field, as long as the metric tensor is time-independent. On the other hand one expects
that with the same kind of approximation the field equations should reduce to Poisson’s equation for the classical field. When these two conditions are consistently fulfilled, then one would say that the theory has a Newtonian limit.

Let us first briefly recall how the effect of a gravitational field of force can be obtained by using the geodesic equations of motions under the circumstances described above. Thus, consider the line element of a nearly Minkowskian metric tensor given by

\[ ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + h_{\mu\nu} dx^\mu dx^\nu, \]  

(18)

with \( x^0 = ct, \mu, \nu = 0,1,2 \). If the geodesic curve is parametrized by the coordinate time \( t \), then we have

\[
\left( \frac{ds}{dt} \right)^2 = -c^2 \left( 1 - \beta^2 - \frac{h_{\mu\nu}}{c^2} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right),
\]

(19)

where \( \beta = \frac{v}{c} \) and \( v \) denotes the velocity of the particle along the geodesic. For non-relativistic motion \( \beta \) is small and, in our approximation, only first-order terms in \( \beta \) and \( h_{\mu\nu} \) will be retained. Thus, to first order in \( h_{\mu\nu} \) and \( \beta \), (19) becomes

\[
\left( \frac{ds}{dt} \right)^2 \simeq c^2 (h_{00} - 1).
\]

(20)

Now, let us consider the geodesic equations

\[
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0.
\]

(21)
Again, keeping only first-order terms in \( h_{\mu\nu} \) and \( \beta \), it can be easily verified that (21) becomes

\[
\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{00} \left( \frac{dx^0}{ds} \right)^2 = 0.
\]

(22)

On the other hand, in this approximation, we have

\[
\Gamma^i_{00} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^i},
\]

(23)

for \( i = 1, 2 \). From (22) and (23) one gets

\[
\frac{d^2 x^i}{dt^2} = \frac{\partial}{\partial x^i} \left( -\frac{h_{00}}{c^2} \right),
\]

(24)

which looks like Newton’s equation of motion for a particle in a classical gravitational field provided that we identify the scalar gravitational field as being

\[
\phi = \frac{c^2}{2} h_{00},
\]

(25)

and requiring that \( h_{00} \) and \( \phi \) vanish at infinity. At this point let us linearize the field equation of Jackiw’s gravity theory by assuming the weak-field approximation (17). If matter is present Jackiw’s field equation is given by

\[
R = kT,
\]

(26)

where \( T = T^\mu_\mu \) denotes the trace of the energy-momentum \( T_{\mu\nu} \) and \( k \) is constant. In the non-relativistic regime \( \frac{|T_{ij}|}{T_{00}} \ll 1 \), i.e., all stress are small compared to the density of energy \( T_{00} = \rho \). Thus, we have \( T \simeq \rho \). It remains
to find the expression for the linearized $R$. This can be done simply by noting that to first order in $h_{\mu\nu}$ we have

$$R_{\mu\nu} = -\frac{1}{2} \left( \Box^2 h_{\mu\nu} - \frac{\partial^2}{\partial x^\lambda \partial x^\mu} h^{\lambda \nu} - \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^{\lambda \mu} + \frac{\partial^2}{\partial x^\mu \partial x^\nu} h \right),$$

(27)

where $h = h^{\mu \mu}$. Recalling that in this approximation indices are lowered and raised with the Minkowski metric tensor, from (9) one readily obtains

$$R = -\Box^2 h + \frac{\partial^2}{\partial x^\lambda \partial x^\nu} h^{\lambda \nu}.$$  

(28)

This equation can be simplified if we choose to work in the so-called harmonic coordinate system, that is, the one for which $\frac{\partial h_{\lambda \nu}}{\partial x^\lambda} = \frac{1}{2} \frac{\partial h_{\lambda \lambda}}{\partial x^\nu}$.

Finally, if one assumes that the metric is not time-dependent, Jackiw field equation (26) turns into

$$\nabla^2 \phi = kc^2 \rho - \frac{c^2}{2} \nabla^2 (h_{11} + h_{22}),$$

(29)

where use has been made of the equation (23). Given that the second term of the right-hand side of equation above does not vanish in general, (29) is not identical to Poisson’s equation for the classical gravitational field. It is worth mentioning, however, that in the static conformally flat case $h_{\mu\nu} = \epsilon \eta_{\mu\nu}$, hence

$$R = \frac{3}{2} \nabla^2 h_{00},$$

(30)

and a Newtonian limit can be defined, a result which was previously obtained by Cornish and Frenkel [8]. Therefore, we conclude that the extension of
Jackiw’s scalar theory to (2+1)-dimensional gravity does not lead to a theory with proper Newtonian limit, except for a few special metric configurations.

**Acknowledgment**

This work was partially supported by CNPq and CAPES. We are indebted to F. Dahia for enlightening discussions.

**References**

[1] Deser S, Jackiw R and ’tHooft, G (1984), Ann. Phys. 152, 220.

[2] Kawai, T (1993), Phys. Rev. D 48, 5668.

[3] Romero, C and Dahia, F (1994), Int. J. Theor. Phys. 33, 2091.

[4] Jackiw, R (1985), Nucl. Phys. B 252, 343.

[5] Jackiw, R and Teitelboim, C (1984) in *Quantum Theory of Gravity*, Christensen, S ed. (Adam Hilger, Bristol).

[6] Brown, J D, Henneaux, M and Teitelboim, C (1986), Phys. Rev. D 33, 319.

[7] Mann, R B (1995), *Lower dimensional black holes: inside and out*, gr-qc/9501038.

[8] Cornish, N J and Frankel, N E (1991), Phys. Rev. D 43, 2555.