The Helmholtz boundary element method does not suffer from the pollution effect

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Abstract

In $d$ dimensions, approximating an arbitrary function oscillating with frequency $\lesssim k$ requires $\sim k^d$ degrees of freedom. A numerical method for solving the Helmholtz equation (with wavenumber $k$ and in $d$ dimensions) suffers from the pollution effect if, as $k \to \infty$, the total number of degrees of freedom needed to maintain accuracy grows faster than this natural threshold (i.e., faster than $k^d$ for domain-based formulations, such as finite element methods, and $k^{d-1}$ for boundary-based formulations, such as boundary element methods).

It is well known that the $h$-version of the finite element method (FEM) (where accuracy is increased by decreasing the meshwidth $h$ and keeping the polynomial degree $p$ fixed) suffers from the pollution effect, and research over the last $\sim 30$ years has resulted in a near-complete rigorous understanding of how quickly the number of degrees of freedom must grow with $k$ (and how this depends on both $p$ and properties of the scatterer).

In contrast to the $h$-FEM, at least empirically, the $h$-version of the boundary element method (BEM) does not suffer from the pollution effect (recall that in the boundary element method the scattering problem is reformulated as an integral equation on the boundary of the scatterer, with this integral equation then solved numerically using a finite-element-type approximation space). However, the current best results in the literature on how quickly the number of degrees of freedom for the $h$-BEM must grow with $k$ fall short of proving this.

In this paper, we prove that the $h$-version of the Galerkin method applied to the standard second-kind boundary integral equations for solving the Helmholtz exterior Dirichlet problem does not suffer from the pollution effect when the obstacle is smooth and nontrapping; see Theorem 2.1 below.

Keywords: Helmholtz equation, scattering, high frequency, boundary integral equation, boundary element method, pollution effect.

AMS subject classifications: 65N38, 65R20, 35J05

1 Introduction

The boundary element method is a popular way of computing approximations to solutions of scattering problems involving the Helmholtz equation. It has long been observed, but not yet proved, that this method does not suffer from the pollution effect (in contrast to the finite element method [9]). The main result of this paper is that the $h$-version of the Helmholtz boundary element method, using the standard second-kind boundary integral equations, does not suffer from the pollution effect when the obstacle has Dirichlet boundary conditions and is smooth and nontrapping; see Theorem 2.1 below.

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In this introduction, we give a recap of the concepts needed to understand this result, namely the Helmholtz scattering problem and the concept of nontrapping (§1.1), a precise definition of the pollution effect (§1.2), our current understanding of the pollution effect for finite- and boundary-element methods (§1.3-1.4), and the definition of the boundary element-method (§1.5-1.6). The main result is then stated in §2, the ideas behind the result are discussed in §3, and the result is proved in §4-6. An alternative proof of the main result in the special case when the obstacle is a 2-d ball using only Fourier series and asymptotics of Hankel and Bessel functions in given in §7.

1.1 The Helmholtz scattering problem

The Helmholtz equation

\[ \Delta u + k^2 u = 0 \]  

(1.1)

with wavenumber \( k > 0 \) is arguably the simplest possible model of wave propagation. For example, if we look for solutions of the wave equation

\[ \frac{\partial^2 U}{\partial t^2} - c^2 \Delta U = 0 \]  

(1.2)

in the form

\[ U(x, t) = u(x)e^{\pm i\omega t}, \]  

(1.3)

then the function \( u(x) \) satisfies the Helmholtz equation (1.1) with \( k = \omega/c \). Assuming a similar dependence on time reduces the Maxwell equations to the so-called time-harmonic Maxwell equations, which, in certain situations, can be further reduced to the Helmholtz equation. Similarly, the time-harmonic elastic wave equation (often called the Navier equation) also reduces to the Helmholtz equation in certain circumstances.

Because the Helmholtz equation is at the heart of linear wave propagation, much effort has gone into both studying the properties of its solutions (for example their asymptotic behaviour as \( k \to \infty \)) and designing methods for computing the solutions efficiently; see, e.g., the SIAM Review articles [49, 96, 73, 81, 97, 79, 34, 80, 9, 35, 94, 52, 114, 112, 44, 63].

The main results of this paper concern the classic scattering problem of the Helmholtz equation posed in the exterior of an obstacle with Dirichlet boundary conditions. For simplicity, we consider scattering of a plane-wave by an obstacle with zero Dirichlet boundary conditions – the so-called “plane-wave sound-soft scattering problem”. In Remark 2.4 below we outline how our results carry over to the general Dirichlet problem (i.e., the Helmholtz equation posed in the exterior of an obstacle with a general Dirichlet boundary condition).

Let \( \Omega \subset \mathbb{R}^d, \ d \geq 2 \) be a bounded open set – the “scatterer” or “obstacle” – such that its open complement \( \Omega^+ := \mathbb{R}^d \setminus \overline{\Omega} \) is connected. Let \( \Gamma := \partial \Omega^- \); our main result requires that \( \Gamma \) is smooth (i.e., \( C^\infty \)), although the scattering problem is well-defined for Lipschitz \( \Gamma \). Recall that weak solutions of second-order linear elliptic PDEs posed on a bounded domain \( D \) naturally live in the Sobolev space \( H^1(D) \), where a function is in \( H^1(D) \) if both the function and its (weak) derivative are in \( L^2(D) \). Here \( \Omega^+ \) is unbounded, and so we work in \( H^1_{\text{loc}}(\Omega^+) \), i.e., the space of functions that are in \( H^1(D) \) for every bounded \( D \subset \Omega^+ \).

**Definition 1.1 (Plane-wave sound-soft scattering problem)** Given \( k > 0 \) and the incident plane wave \( u^I(x) := \exp(ikx \cdot \hat{a}) \) for \( \hat{a} \in \mathbb{R}^d \) with \( |\hat{a}| = 1 \), find the total field \( u \in H^1_{\text{loc}}(\Omega^+) \) satisfying \( \Delta u + k^2 u = 0 \) in \( \Omega^+ \),

\[ u = 0 \quad \text{on} \ \Gamma, \]

and, where \( u^S := u - u^I \) is called the scattered field and satisfies

\[ \partial_r u^S -iku^S = o(r^{1-d/2}) \quad \text{as} \quad r := |x| \to \infty, \quad \text{uniformly in} \ x/r. \]

(1.4)

It is well-known that the solution of the sound-soft plane-wave scattering problem exists and is unique; see, e.g., [36, Theorem 3.13], [24, Theorem 2.12 and Corollary 2.13].

The condition (1.4) is the Sommerfeld radiation condition, and expresses mathematically that, with the choice \( e^{-i\omega t} \) in (1.3), the scattered wave moves away from the obstacle towards infinity. Indeed, one can show that (1.4) implies that

\[ u^S(x) = \frac{e^{ikr}}{r^{(d-1)/2}} \left( f(x/r) + O \left( \frac{1}{r} \right) \right) \quad \text{as} \quad r \to \infty, \]
for some function $f(x/r)$ (i.e., a function of the non-radial variables). Recalling that $k = \omega/c$, we find that the corresponding solution $U^S(x, t) := u^S(x)e^{-i\omega t}$ of the wave equation (1.2) satisfies

$$U^S(x, t) = \frac{e^{ik(r-ct)}}{r^{(d-1)/2}} \left( f(x/r) + O \left( \frac{1}{r} \right) \right),$$

and we recognise $e^{ik(r-ct)}$ as a wave travelling in the positive $r$ direction (i.e., away from the obstacle) as time $t$ increases. \(^1\)

The key geometric condition that governs the behaviour of Helmholtz solutions with $k$ large is that of *trapping/nontrapping* (see, e.g., [89, Epilogue §1], [137]).

**Definition 1.2 (Nontrapping)** The obstacle $\Omega^- \subset \mathbb{R}^d$ is nontrapping if $\Gamma$ is $C^\infty$ and, given $R$ such that $\Omega^- \subset B_R(0)$, there exists a $T(R) < \infty$ such that all the billiard trajectories that start in $\Omega^+ \cap B_R(0)$ at time zero leave $\Omega^+ \cap B_R(0)$ by time $T(R)$.

If $\Omega^-$ is not nontrapping, then we say that it is *trapping*. See Figure 1.1 for an example of a non-trapping obstacle and a trapping obstacle. The requirement in Definition 1.2 that $\Gamma$ is $C^\infty$ is imposed so that when the billiard trajectories hit $\Gamma$, their reflection according to the law of geometric optics (“angle of incidence = angle of reflection”) is well-defined (see [109]). There has been much rigorous study of the reflection of high-frequency waves from non-smooth obstacles (see, e.g., [125, 54, 33, 32, 131, 110, 111]), but this does not impact the results of the present paper since we assume that $\Gamma$ is smooth (see §3 for a discussion of why we make this assumption).

Our main results are proved under the assumption that $\Omega^-$ is nontrapping, but in §3 we discuss to what extent they hold in the trapping case.

### 1.2 What is the pollution effect?

**Informal definition.** A numerical method for solving the Helmholtz equation (with wavenumber $k$) suffers from the pollution effect if, as $k \to \infty$, *the total number of degrees of freedom needed to maintain accuracy grows faster than $k^n$*, where $n$ is the dimension of the physical domain in which the problem is formulated. Having number of degrees of freedom growing like $k^n$ is the natural threshold for the problem since an oscillatory function with frequency $\leq k$ can be approximated by piecewise polynomials with $k^n$ degrees of freedom (this is expected from the Nyquist–Shannon–Whittaker sampling theorem [132, 124]; see, e.g., [10, Theorem 5.21.1]).

**Abstract framework covering both BEM and FEM.** Let $V$ be a Hilbert space and let $A : V \to V'$ be a continuous, invertible linear operator, where $V'$ is the dual space of $V$. Given $f \in V'$, let $v \in V$ be the solution of $Av = f$; i.e., $v = A^{-1}f$.

Let $(V_N)_{N \geq 0}$ be an increasing sequence of finite-dimensional subspaces of $V$ with dimension $N$ (i.e., total number of degrees of freedom $N$), such that $V_N$ are asymptotically dense in $V$, in the sense that, for all $w \in V$, the best approximation error $\min_{u_N \in V_N} \| w - u_N \|_V \to 0$ as $N \to \infty$.

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\(^1\)If we seek solutions of the wave equation as $U(x, t) = u(x)e^{i\omega t}$, then the radiation condition corresponding to outgoing waves is $\partial_r w^3 + iku^3 = o(r^{(1-d)/2})$ as $r \to \infty$. 

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Figure 1.1: On the left, a non-trapping object, and on the right, a trapping obstacle and one of its trapped rays.
Let \( v_N \) be the computed approximation in \( V_N \) to \( v \); we write this as \( v_N = (A^{-1})_N f \), so that \( (A^{-1})_N : V' \to V_N \) is the approximation of the solution operator.

For the finite-element method \( v \) is the restriction to the computational domain of the solution \( u \) of the sound-soft scattering problem (modulo any error incurred by this restriction), \( V \) is the space \( H^1 \), and \( n = d \). For the boundary-element methods we consider below, \( v \) is a function on \( \Gamma \) (possibly the normal derivative of \( u \)), \( V \) is \( L^2(\Gamma) \) (i.e., square integrable functions on \( \Gamma \)), and \( n = d - 1 \), since the boundary \( \Gamma \) is \((d - 1)\) - dimensional.

**Quasi-optimality.** A fundamental property one seeks to prove about a sequence of approximate solutions \((v_N)_{N>0}\) is that they are asymptotically quasi-optimal; i.e., there exists \( N_0 > 0 \) and \( C_{qo} > 0 \) such that
\[
\|v - v_N\|_V \leq C_{qo} \min_{w_N \in V_N} \|v - w_N\|_V \quad \text{for all } N \geq N_0, \quad \text{where } v = A^{-1}f \text{ and } v_N = (A^{-1})_N f.
\]
(1.5)

The approximate solutions \((v_N)_{N>0}\) would be optimal if \( \|v - v_N\|_V = \min_{w_N \in V_N} \|w - w_N\|_V \); “quasi-optimality” is then optimality up to a constant factor, and “asymptotically” refers to the fact that (1.5) holds for sufficiently large \( N \).

The standard analysis of finite- and boundary-element methods for the Helmholtz equation proves that, for fixed \( k \), the computed solutions are asymptotically quasi-optimal (see, e.g., [19] for FEM and [127, 121] for BEM), i.e., for each \( k > 0 \) there exists \( N_0 = N_0(k) \), depending on \( k \) in some unspecified way, such that (1.5) holds.

**Precise definition of the pollution effect.** The pollution effect is when, for any choice of \( N \) bounded by a constant multiple of \( k^\alpha \) (i.e., \( N \leq \Lambda k^\alpha \) for some \( \Lambda > 0 \), and any choice of data \((f \in V')\), the smallest possible \( C_{qo} \) in (1.5) is unbounded in \( k \). That is, if
\[
\inf_{\Lambda > 0} \limsup_{k \to \infty} \inf_{N \leq \Lambda k^\alpha} \sup_{f \in V'} \left\{ C_{qo} : \|A^{-1}f - (A^{-1})_N f\|_V \leq C_{qo} \min_{w_N \in V_N} \|A^{-1}f - w_N\|_V \right\} = \infty;
\]
(1.6)
see, e.g., [9, Definition 2.1].

When the meshes in the FEM or BEM are quasi-uniform (informally, all the mesh elements are of comparable size; see [121, Definition 4.1.13] for a precise definition), then the total number of degrees of freedom \( N \sim (p/h)^n \), where \( h \) is the mesh-width and \( p \) the polynomial degree.

In the \( h \)-version of the FEM or BEM accuracy is increased by decreasing \( h \) and keeping \( p \) fixed, and thus \( N \sim k^\alpha \) corresponds to \( hk \sim 1 \). For these methods, the \( \inf_{N \leq \Lambda k^\alpha} \) in the definition of the pollution effect (1.6) can then be replaced by \( \inf_{\Lambda \leq h^k} \).

### 1.3 The pollution effect for finite-element methods is well understood

Empirically, the \( h \)-version of the FEM applied to the Helmholtz equation suffers from the pollution effect. Furthermore [9] proved that in two or higher dimensions the pollution effect is unavoidable for the \( h \)-FEM; more precisely, [9] worked in the framework of “generalised FEMs” introduced in [8] and proved that, in two or higher dimensions, any method with fixed polynomial degree \( p \) (or, more generally, a fixed stencil) suffers from the pollution effect; see [9, Theorem 4.6].

Given that the \( h \)-FEM suffers from the pollution effect, two natural questions are the following.

Q1. How must \( h \) depend on \( k \) for the quasi-optimal error estimate (1.5) to hold with \( C_{qo} \) independent of \( k \)?

In engineering applications, the most-commonly used measure of error is the relative error
\[
\frac{\|v - v_N\|_V}{\|v\|_V}.
\]
(1.7)

However, the relative error can only be small when restricting attention to a subclass of data. Indeed, since \( A \) is assumed to be invertible, given \( V_N \), we can choose \( v \in V \) orthogonal to \( V_N \), let \( f := Av \), and let \( v_N := (A^{-1})_N f \). Then
\[
\|v - v_N\|_V^2 = \|v\|_V^2 + \|v_N\|_V^2 \geq \|v\|_V^2,
\]
and thus the relative error cannot be small for all possible data.

Q2 For a physically-relevant class of data \( \tilde{V} \subset V' \) (such as that coming from an incident plane wave as in Definition 1.1), how must \( h \) depend on \( k \) for the relative error to be controllably small? I.e., given \( \varepsilon > 0 \) and \( V' \), how must \( h \) depend on \( k \) and \( \varepsilon \) such that for all \( f \in \tilde{V} \) the relative error (1.7) is \( \leq \varepsilon \)?

For the \( h \)-FEM applied to non-trapping problems, the answer to Q1 is that \( hpk^{p+1} \) must be sufficiently small, and the answer to Q2 is that \( h^{2p}k^{2p+1} \) must be sufficiently small for data oscillating at scale \( k^{-1} \).

These answers were first obtained for 1-d Helmholtz problems by [7, 78, 77] (see also [75, Chapter 4]). Obtaining the multi-dimensional analogues of these results for a range of different FEMs (including discontinuous Galerkin and interior penalty methods) and different ways of truncating the infinite domain \( \Omega^+ \) (absorbing boundary conditions, perfectly-matched layers), remains a very active research area; see [104, 120, 50, 107, 108, 47, 106, 133, 136, 45, 28, 13, 29, 134, 93, 30, 60, 66, 84, 135, 85, 87, 61].

There has been much research on designing FEMs that mitigate against the pollution effect; we now briefly discuss research on this in four directions.

- The answers to Q1 and Q2 show that the pollution effect is less pronounced for high-order methods, i.e., methods with larger \( p \). It was proved for a variety of constant-coefficient Helmholtz problems in [107, 108] (where any boundaries are analytic) and variable-coefficient problems in [84] that there exists \( C_1, C_2 > 0 \) such that the \( hp \)-FEM is quasi-optimal with \( C_{qo} \) independent of \( k \) if

\[
\frac{hk}{p} \leq C_1 \quad \text{and} \quad p \geq C_2 \log k. \tag{1.8}
\]

Since \( N \sim (p/h)^d \), choosing \( h \) and \( p \) such that the inequalities (1.8) hold with equality implies that the \( hp \)-FEM does not suffer from the pollution effect.

- Trefftz methods seek to use non-polynomial basis functions that are solutions of the Helmholtz equation \( \Delta u + k^2 u = 0 \); the simplest example of these are plane waves \( e^{ik\hat{a} \cdot x} \), where one then needs to choose the directions \( \hat{a}_j \). For most Trefftz spaces, one cannot enforce continuity across interfaces separating mesh elements, and thus the standard variational formulation of the Helmholtz equation (the basis of the FEM) cannot be used, and one must work in a least-squares or discontinuous Galerkin setting. Although the pollution effect is less pronounced, the \( h \)-versions of these methods still suffer from it, and they also suffer from conditioning issues; see the review [72] (in particular [72, §5]) and the references therein.

- The multiscale method of [62, 118, 20] (see also the recent work [70, 53]), involving special pre-computed test functions, does not suffer from the pollution effect provided that (a) a certain oversampling parameter grows logarithmically with \( k \), and (b) the problems defining the test functions, posed on subsets of the computational domain, are solved accurately on a scale smaller than \( k^{-1} \). Alternative multiscale methods for solving the Helmholtz equation include [115, 31, 98].

- The “discontinuous Petrov Galerkin (DPG)” method of [41] can be thought of as a least-squares method in a nonstandard inner product. This method suffers from the pollution effect to a lesser extent than the standard \( h \)-FEM, but is not pollution free; see [71] for a recent numerical investigation.

\[2\]We note that the pollution effect for Helmholtz finite element and finite difference methods can also be heuristically studied via so-called “dispersion analysis” [69, 76, 78, 42, 1]. Here finite-element or finite-difference schemes are studied on an infinite uniform mesh for problems where an exact solution is \( u(x) = e^{ikx} \), and one seeks the “discrete wavenumber” \( \tilde{k} \) such that a numerical solution is \( u_N(x_j) = e^{i\tilde{k}x_j} \), where \( x_j \) are the nodes. The condition \( -h^{2p}k^{2p+1} \) sufficiently small” (i.e., the answer to Q2) arises as the condition for \( |\tilde{k} - k| \) to be controllably small; see [77, Theorem 3.2], [75, Theorem 4.22].
1.4 The pollution effect for boundary-element methods is not yet rigorously understood

The situation for the BEM is well-summarised by the following quotation from [12].

It is generally admitted that Boundary Integral Equations (BIE) lead to less “pollution effect” than FEMs even if to our knowledge, no formal study has confirmed such a property.

Indeed, it is completely standard in the numerical-analysis and engineering communities to compute approximations to Helmholtz scattering problems via boundary integral equations using a fixed number of degrees of freedom per wavelength, i.e., \( N \sim k^{d-1} \), both for Galerkin [51, 16] and collocation [99, 100] BEMs, and also for Nyström methods [21, 88, 68].

Numerical experiments indicate that, at least for obstacles without strong trapping, the h-BEM is quasioptimal (with constant independent of \( k \)) if \( hk \) is sufficiently small; see [95, §4], [65, §5]. However, in existing theoretical investigations [22, 11, 95, 105, 65, 59], the best result is that the h-BEM is quasi-optimal (with constant independent of \( k \)) if \( hk^{4/3} \) is sufficiently small and the scatterer is smooth and convex [59, Theorem 1.10(c)] (the current best results for more general domains, which are also in [59], involve higher powers of \( k \)).

We recalled in §1.3 how the hp-FEM does not suffer from the pollution effect if \( h \) and \( p \) satisfy (1.8), and the results of [95, 105] show the analogous result for the hp-BEM; i.e., if \( \Gamma \) is analytic the hp-BEM is quasioptimal with \( C_{\text{eq}} \) independent of \( k \) if (1.8). The abstract to [95] remarks that

Numerical examples . . . even suggest that in many cases quasi-optimality is given under the weaker condition that \( kh/p \) is sufficiently small [with \( p \) fixed].

In this paper we rigorously explain this observation, showing that the h-BEM does not suffer from the pollution effect if \( hk \) is sufficiently small and the obstacle is non-trapping (note that, in contrast to [95, 105], our analysis is not explicit in \( p \), since we do not impose the condition that \( \Gamma \) is analytic).

1.5 The Helmholtz plane-wave sound-soft scattering problem solved via boundary integral equations

The standard second-kind boundary integral equations for solving the plane-wave sound-soft scattering problem of Definition 1.1 can be expressed in terms of the solution of boundary integral equations involving the operators

\[
A_k := \frac{1}{2}I + D_k - ikS_k, \quad \text{and} \quad A'_k := \frac{1}{2}I + D'_k - ikS_k
\]  

(1.9)

where \( S_k, D_k, \) and \( D'_k \) are the single-, double-, and adjoint-double-layer operators defined in (1.12) and (1.13) below. The ’ notation is used since \( A_k \) and \( A'_k \) are adjoint with respect to the real-valued \( L^2(\Gamma) \) inner product.

There are a variety of spaces in which one can pose equations involving \( A_k \) and \( A'_k \). The most natural space for solving such equations with the Galerkin method is \( L^2(\Gamma) \) (since the inner product is local). When \( \Gamma \) is \( C^1 \), \( S_k, D_k, \) and \( D'_k \) are compact on \( L^2(\Gamma) \), and thus \( A_k \) and \( A'_k \) are compact perturbations of a multiple of the identity. Such integral operators fall into the class of “second-kind” operators – see [5, §1.1.4] – and the solvability of integral equations involving these operators is covered by Fredholm theory \( ^4 \). One can then show that \( A_k \) and \( A'_k \) are bounded and invertible operators from \( L^2(\Gamma) \) to itself when \( \Gamma \) is smooth [36, Theorem 3.33] (indeed, even when \( \Gamma \) is only Lipschitz; see [25, Theorem 2.7], [24, Theorem 2.27]).

\(^3\)Intriguingly however, [101, 15, 102] recently identified a loss of accuracy similar to the pollution effect in the collocation BEM applied to interior Helmholtz problems.

\(^4\)Indeed, the study of analogous second-kind boundary integral equations for Laplace’s equation was the main motivation for the development of this abstract functional-analysis theory; see, e.g., [103, §1]).
How the boundary integral equations (1.9) are obtained. Let \( \Phi_k(x, y) \) be the fundamental solution of the Helmholtz equation

\[
\Phi_k(x, y) := \frac{i}{4} \left( \frac{k}{2\pi|x-y|} \right)^{(d-2)/2} H^{(1)}_{(d-2)/2}(k|x-y|) = \begin{cases} \frac{i}{4} H^{(1)}_0(k|x-y|), & d = 2, \\ \frac{e^{ik|x-y|}}{4\pi|x-y|}, & d = 3, \end{cases}
\]

where \( H^{(1)}_m \) denotes the Hankel function of the first kind of order \( m \). The single- and double-layer potentials, \( S_k \) and \( D_k \) respectively, are defined for \( k \in \mathbb{C} \), \( \phi \in L^2(\Gamma) \), and \( x \in \mathbb{R}^d \setminus \Gamma \) by

\[
S_k \phi(x) = \int_\Gamma \Phi_k(x,y)\phi(y) \, ds(y) \quad \text{and} \quad D_k \phi(x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} \phi(y) \, ds(y).
\]

The standard single-layer, double-layer, and adjoint-double-layer, operators are defined for \( k \in \mathbb{C} \), \( \phi \in L^2(\Gamma) \), and \( x \in \Gamma \) by

\[
S_k \phi(x) := \int_\Gamma \Phi_k(x,y)\phi(y) \, ds(y), \quad D_k \phi(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} \phi(y) \, ds(y), \quad D'_k \phi(x) := \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial \nu(x)} \phi(y) \, ds(y).
\]

We now state the result expressing the solution of the plane-wave sound-soft scattering problem of Definition 1.1 in terms of boundary integral equations involving \( A_k \) and \( A'_k \), and discuss the ideas behind its proof.

**Theorem 1.3 (The plane-wave sound-soft scattering problem formulated in terms of boundary integral equations.)**

(i) If \( u \) is solution of the plane-wave sound-soft scattering problem of Definition 1.1, then

\[
A'_k \partial^+_\nu u = \partial^+_\nu u^I - iku^I, \quad \text{and} \quad u = u^I - S_k(\partial^+_\nu u).
\]

(ii) If \( v \in L^2(\Gamma) \) is the solution to

\[
A_k v = -u^I, \quad \text{then} \quad u = u^I + (D_k - i k S_k)v
\]

is the solution of the plane-wave sound-soft scattering problem of Definition 1.1.

**References for the proof.** Part (i) is proved in, e.g., [24, Theorem 2.46]. Part (ii) is proved in, e.g., [24, Equations 2.70-2.72].

As the two parts of Theorem 1.3 indicate, there are two ways to formulate the solution of the plane-wave sound-soft scattering problem of Definition 1.1 (and scattering problems with other boundary conditions) in terms of boundary integral equations. The first is called the direct method, and is based on Green’s integral representation, which expresses \( u^S \) in terms of the single- and double-layer potentials \( S_k \) and \( D_k \). Indeed,

\[
u^S(x) = D_k(u^S)(x) - S_k(\partial^+_\nu u^S)(x) \quad \text{for} \quad x \in \Omega^+,
\]

where \( \partial^+_\nu u^S \) is the normal derivative of \( u^S \) on \( \Gamma \) (where the normal \( \nu \) is taken pointing outwards of \( \Omega^- \)); see, e.g., [36, Theorem 3.3], [103, Theorems 7.5 and 9.6]. Since \( u^I \) is a solution of the Helmholtz equation in \( \Omega^- \), Green’s integral representation also implies that

\[
0 = D_k(u^I)(x) - S_k(\partial^+_\nu u^I)(x) \quad \text{for} \quad x \in \Omega^-;
\]

see, e.g., [36, Theorem 3.1]. Adding (1.16) and (1.17), using that \( u := u^I + u^S \) and the boundary condition \( u = 0 \) on \( \Gamma \), we find that

\[
u(x) = u^I(x) - S_k(\partial^+_\nu u)(x) \quad \text{for} \quad x \in \Omega^+.
\]

Taking the limits of both (1.18) and its normal derivative as \( x \) approaches \( \Gamma \) from \( \Omega^+ \), multiplying the former by \(-ik\) and adding it to the latter, we obtain the integral equation (1.14). The reason
we take a linear combination of these two equations is that the combination is well-posed (i.e., $A'_k$ is invertible), but the individual equations on their own are not; this idea of taking a linear combination goes back to [18, 90, 117].

The second method, the *indirect method*, poses the ansatz that

$$u^S(x) = (D_k - ikS_k)v(x)$$

(1.19)

for some unknown density $v$. Taking the limit of this ansatz as $x$ approaches $\Gamma$ from $\Omega^+$ and imposing the boundary condition that $w^S = -u'$ on $\Gamma$, we obtain the integral equation (1.15).

While in the direct method, the unknown of the integral equation has immediate physical relevance (it is $\partial_{\nu}u$, the unknown in the indirect method has a less immediate physical interpretation; see [121, §3.4.3] for a discussion of how this difference affects the relative merits of the direct and indirect methods.

1.6 The Galerkin method and assumptions on the boundary-element space

We consider solving the boundary integral equation $Av = f$ in $L^2(\Gamma)$ with the Galerkin method: given a finite-dimensional subspace $V_N \subset L^2(\Gamma)$,

$$\text{find } v_N \in V_N \text{ such that } \langle Av_N, w_N \rangle_{L^2(\Gamma)} = \langle f, w_N \rangle_{L^2(\Gamma)} \text{ for all } w_N \in V_N. \tag{1.20}$$

The abstract framework in §1.2 involved the operator $(A^{-1})_N$ mapping the data to the approximate solution; we show in §4 below (see (4.4)) that, for the Galerkin method, $(A^{-1})_N = (P_NA)^{-1}P_N$, where $P_N$ is the orthogonal projection from $V$ to $V_N$ and $P_NA$ is considered as an operator from $V_N$ to $V_N$ (after using the fact that $V$ is a Hilbert space to identify $V$ and $V'$).

The $h$-version of the boundary element method uses a sequence of approximation spaces $(V_{h,k})_{k>0}$ given by piecewise polynomials of degree $p$ for some fixed $p \geq 0$ on a sequence of meshes of diameter $h > 0$; for ease of notation we let $(V_h)_{h>0} := (V_{h,k})_{k>0}$. It is well-known that, when the meshes are additionally *shape-regular* (for each element, its width divided by the diameter of the largest inscribed ball is uniformly bounded; see [121, Definition 4.1.12]), these subspaces satisfy the following assumption.

**Assumption 1.4** $(V_h)_{h>0}$ is a sequence of finite-dimensional subspaces of $L^2(\Gamma)$ such that, for all $h > 0$,

$$\min_{w_h \in V_h} \|w - w_h\|_{L^2(\Gamma)} \leq C_{\text{approx}}h \|w\|_{H^1(\Gamma)} \text{ for all } w \in H^1(\Gamma). \tag{1.21}$$

(Recall that $\|w\|^2_{H^1(\Gamma)} := \|\nabla w\|^2_{L^2(\Gamma)} + \|w\|^2_{L^2(\Gamma)}$, where $\nabla$ is the surface gradient operator, defined in terms of a parametrisation of the boundary by, e.g., [24, Equation A.14].)

Indeed, piecewise-polynomial subspaces satisfying Assumption 1.4 are described in [121, Chapter 4], with [121, Theorem 4.3.23] showing that the spaces of continuous boundary-element functions denoted by $S^0_v$ [121, Definition 4.1.36] satisfy Assumption 1.4 and [121, Theorem 4.3.19] showing that the spaces of discontinuous boundary-element functions denoted by $S^{p-1}_v$ [121, Definition 4.1.17] satisfy Assumption 1.4. Note that, in these cases, the constant $C_{\text{approx}}$ depends on $p$.

2 The main result: the $h$-BEM does not suffer from the pollution effect

**Theorem 2.1 (Quasi-optimal error estimate for $hk$ sufficiently small)** Suppose that $\Omega$ is nontrapping, $A$ is either $A_k$ or $A'_k$, and $(V_h)_{h>0}$ satisfies Assumption 1.4.

For all $k_0 > 0$, there exists $C_{\text{ppw}} > 0$ and $C_{\text{qq}} > 0$ such that if

$$hk \leq C_{\text{ppw}} \quad \text{and} \quad k \geq k_0, \tag{2.1}$$

then, for all $f \in L^2(\Gamma)$, the Galerkin solution $v_N$ to (1.20) exists, is unique, and satisfies

$$\|v - v_N\|_{L^2(\Gamma)} \leq C_{\text{qq}} \min_{w_N \in V_h} \|v - w_N\|_{L^2(\Gamma)}. \tag{2.2}$$
If the spaces \((V_h)_h>0\) are quasi-uniform, then \(N \sim h^{-d}\), and thus Theorem 2.1 shows that the Galerkin method is quasi-optimal (with constant independent of \(k\)) when the total number of degrees of freedom is a multiple of \(k^d\); i.e., the \(h\)-BEM does not suffer from the pollution effect.

Theorem 2.1 covers the Galerkin method applied to \(Av = f\) for general \(f \in L^2(\Gamma)\). We now restrict to the case when the data comes from the plane-wave sound-soft scattering problem (i.e., the right-hand side \(f\) is as described in Theorem 1.3), and bound the relative error. To do this, we use in the bound (2.2) the bound (1.21) from Assumption 1.4 and the following lemma (proved in [58, Lemma 1.3]), describing the oscillatory character of the solution \(v\) in this case.

**Lemma 2.2 (Bound on the unknown \(v\) in the BIEs for the sound-soft scattering problem)**
Suppose that \(A\) is one of \(A_k, A'_k\) and \(v\) is the solution to \(Av = f\) where the right-hand side \(f\) is as described in Theorem 1.3. Then given \(k_0 > 0\) there exists \(C_{\text{rel}} > 0\) such that
\[
\|v\|_{H^1(\Gamma)} \leq C_{\text{rel}}k \|v\|_{L^2(\Gamma)} \quad \text{for all } k \geq k_0. 
\tag{2.3}
\]

**Corollary 2.3 (Bound on the relative error for \(hk\) sufficiently small)**
Suppose that \(\Omega^-\) is nontrapping, \(A\) is either \(A_k\) or \(A'_k\), and \((V_h)_h>0\) satisfies Assumption 1.4. For all \(k_0 > 0\), there exists \(C_{\text{ppw}} > 0\) and \(C_{\text{qo}} > 0\) such that if (2.1) holds, then for all data \(f\) coming from the plane-wave sound-soft scattering problem the Galerkin solution \(v_N\) to (1.20) exists, is unique, and satisfies
\[
\|v - v_N\|_{L^2(\Gamma)} \leq C_{\text{qo}}C_{\text{rel}}hk \|v\|_{L^2(\Gamma)}. 
\tag{2.4}
\]

The bound (2.4) shows that a prescribed relative error can be achieved with a choice of \(h\) such that \(hk \sim 1\). Indeed, given \(\varepsilon > 0\), if
\[
hk \leq \min \left\{\varepsilon (C_{\text{qo}}C_{\text{rel}})^{-1}, C_{\text{ppw}}\right\},
\]
then \(\|v - v_N\|_{L^2(\Gamma)}/\|v\|_{L^2(\Gamma)} \leq \varepsilon\).

**Remark 2.4 (General Dirichlet boundary conditions)**
The general exterior Dirichlet problem is: given \(k > 0\) and \(g_D \in H^{1/2}(\Gamma)\), find \(u^S \in H^1_{\text{loc}}(\Omega^+)\) such that \(\Delta u^S + k^2u^S = 0\) in \(\Omega^+, u^S = g_D\) on \(\Gamma\), and \(u^S\) satisfies the radiation condition (1.4).

For the indirect method, we pose the ansatz (1.19) and take the limit of this as \(x\) approaches \(\Gamma\) from \(\Omega^+\) to obtain the equation \(A_k v = g_D\). Since \(g_D \in H^{1/2}(\Gamma)\), this is a priori an equation in \(H^{1/2}(\Gamma)\); however, since \(A_k\) is bounded and invertible as an operator from \(H^s(\Gamma)\) to itself for \(0 \leq s \leq 1\) [24, Theorem 2.27], and \(H^{1/2}(\Gamma) \subset L^2(\Gamma)\), we can consider this equation in \(L^2(\Gamma)\), and solve it using the Galerkin method as in §1.6. In contrast, the exterior Dirichlet problem can only be solved by the direct method with the integral equation posed in \(L^2(\Gamma)\) when \(g_D \in H^1(\Gamma);\) see [24, Section 2.6].

3 Discussions of the ideas behind the proof of Theorem 2.1

The proof of Theorem 2.1 consists of three ingredients.

1. A slight modification of a standard condition for quasi-optimality of the Galerkin method applied to operators that are a perturbation of the identity (see (4.6) in Theorem 4.2 below), with this condition based on writing the Galerkin method as a projection method and using the result that if \(\|T\| < 1\) then \(I + T\) is invertible with \(\|(I + T)^{-1}\| \leq (1 - \|T\|)^{-1}\).

2. Bounds on the components the boundary integral operators \(S_k, D_k,\) and \(D_k\) that have frequencies \(\geq k\) (see Theorem 5.1) where the statement that a function has “frequencies \(\geq k\)” is understood by expanding the function in terms of eigenfunctions of the surface Laplacian on \(\Gamma\) (see §5).

We see in §6 that these two ingredients prove the following result.
Lemma 3.1 Suppose $\mathcal{A}$ is either $A_k$ or $A_k'$, and $(V_h)_{h>0}$ satisfies Assumption 1.4. For all $k_0 > 0$, there exists $C_1 > 0$ such that if $k \geq k_0$ and

$$h k (1 + \|A^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)}) \leq C_1$$

then, for all $f \in L^2(\Gamma)$, the Galerkin solution $v_N$ to (1.20) exists, is unique, and satisfies

$$\|v - v_N\|_{L^2(\Gamma)} \leq 2 \|A^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \min_{w_N \in V_N} \|v - w_N\|_{L^2(\Gamma)}.$$  

The result of Theorem 2.1 then follows from the third ingredient.

3. If $\Omega^-$ is non-trapping then, given $k_0 > 0$, there exists $C > 0$ such that

$$\|A^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C \quad \text{for all } k \geq k_0.$$  

Discussion of Point 1. It is perhaps surprising that the simple condition from Theorem 4.2, combined with Points 2 and 3, gives a better result for the Galerkin method applied to $\mathcal{A}$ (at least when $\Omega^-$ is nontrapping) than more-sophisticated conditions for quasi-optimality used in [22, 11, 95, 59], which are all ultimately based on the ideas in the “Schatz argument” in the finite-element setting; see [122, 120].

Discussion of Point 2. The bounds on the high-frequency components of $S_k, D_k,$ and $D_k'$ in Theorem 5.1 come from viewing these boundary integral operators as semiclassical pseudodifferential operators. We do not need any of the details of these operators in this paper, but it is instructive to discuss briefly here how, on the one hand, using pseudodifferential operators to study boundary integral equations is completely standard, but, on the other hand, the full potential of these operators for studying Helmholtz problems with large $k$ has not been fully exploited.

Recall that the theory of standard pseudodifferential operators on a smooth surface $\Gamma$ can be viewed as a generalisation of Fourier analysis on the circle. The use of pseudodifferential properties in both the analysis and numerical analysis of boundary integral equations is well established, see, e.g., [38, 128, 39, 37, 123, 91, 119, 92, 3, 74], and current [40, 17, 67, 6, 2, 4, 64, 23].

A class of pseudodifferential operators exists that is tailor-made for studying problems where oscillations happen at a large frequency $k$; these are precisely semiclassical pseudodifferential operators [138], Appendix E. The adjective “semiclassical” essentially means “high frequency”, and comes from the origin of this theory in the study of how classical dynamics arise from quantum mechanics in the high-energy limit (see, e.g., [138, §1]).

Whereas $S_k, D_k,$ and $D_k'$ are standard pseudodifferential operators (of order $-1$; see, e.g., [74, §9.2.2], [119, §7], [129, Chapter 7, Section 11]), they are not semiclassical pseudodifferential operators. Instead, each is the sum of a semiclassical pseudodifferential operator and an operator acting only on frequencies $\leq k$ that transports mass between points on the boundary connected by rays; this decomposition was recently established in [55, Chapter 4], with [55, Lemma 4.27] explicitly writing out the decomposition when $\Gamma$ is curved. The estimates on boundary layer operators at high frequency in Theorem 5.1 were then proved using the ideas from [55, Chapter 4] in [57, Theorem 4.3].

Finally, we note that the assumption in §1.1 that $\Gamma$ is smooth is because the theory of pseudodifferential operators is simplest on smooth domains. In principle, Lemma 3.1 holds when $\Gamma$ is $C^M$ for some $M > 0$; one could go through the arguments to determine a sufficiently-large value of $M$; alternatively one could use more sophisticated pseudodifferential techniques to lower the regularity further; see, e.g., [130, Chapter 13].

Discussion of Point 3. The estimate (3.3) is proved in [14, Theorem 1.13] using the following decompositions of $A_k^{-1}$ and $(A_k')^{-1}$ [24, Theorem 2.33],

$$A_k^{-1} = I - (\text{ItD})^{-1} [(\text{DtN})^+ - ik] \quad \text{and} \quad (A_k')^{-1} = I - [(\text{DtN})^+ - ik](\text{ItD})^{-1}.$$  

Here, $(\text{DtN})^+$ is the Dirichlet-to-Neumann map for the Helmholtz equation $\Delta u^S + k^2 u^S = 0$ in $\Omega^+$ satisfying the Sommerfeld radiation condition (1.4) and $(\text{ItD})^{-1}$ is the map $g \mapsto u|_\Gamma$ where, given $g \in L^2(\Gamma), u \in H^1(\Omega^-)$ is the solution of the interior impedance problem

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega^-, \quad \partial_n u - iku = g \quad \text{on } \Gamma.$$  

(3.5)
The decompositions in (3.4) imply that bounds on $A_k^{-1}$ and $(A'_k)^{-1}$ can be obtained from $k$-explicit bounds on $(\text{DtN})^+$ and $(\text{ItD})^−$. These estimates are obtained in [14] for non-trapping $\Omega^−$ (following the proof in [26, Theorem 4.3] of the analogous bound for $\Omega^−$ that are star-shaped with respect to a ball).

The presence of $(\text{DtN})^+$ in (3.4) is expected since $(\text{DtN})^+$ is essentially the solution operator for the problem (and we are using the Galerkin method applied to $A_k$ or $A'_k$ to approximate this solution operator). The map $(\text{ItD})^−$ appears in (3.4) since $A_k$ and $A'_k$ can also be used to solve the interior impedance problem; see, e.g., [24, Theorem 2.30].

We highlight that proving $k$-explicit bounds on exterior Helmholtz solution operators is a classic problem considered since the 1960’s, with interest in the interior impedance problem (3.5) arising more recently both from this problem’s role in determining the behaviour of $A_k$ and $A'_k$ and because this problem is often used as a model problem in the numerical analysis of FEMs; see the literature reviews in [27], [86] (for exterior problems) [126, §1.2], [14, §1.2] (for both exterior and interior problems) and [56, Sections 1.1 and 1.4] (for interior problems).

When $\Omega^−$ is trapping, $\|A^−\|_{L^2(Γ)→L^2(Γ)}$ grows with $k$ (see [27, 86]). Thus, although (3.1), (3.2) give a result about convergence of the $h$-BEM for $\Omega^−$ trapping, this result does not show that the $h$-BEM does not suffer from the pollution effect. Results about the $k$-independent quasi-optimality of the $h$-BEM in the case when $\Omega^−$ is trapping can be found in [58], although these also do not show that the $h$-BEM does not suffer from the pollution effect. The experiments in [65, Figure 2] show that, at least for a certain form of mild trapping (so-called parabolic trapping), the $h$-BEM does not suffer from the pollution effect, although proving this remains open.

### 4 Formulation of the Galerkin method as a projection method and an abstract condition for quasi-optimality

As in §1.2, $V$ is a Hilbert space with dual $V'$, and we let $B : V → V'$ be a continuous, invertible, linear operator. Later we restrict attention to the case when $B$ is a perturbation of the identity, i.e., $B = I + K$, and we apply these results with $B = 2A$, with $A$ one of $A_k$ and $A'_k$ (since $A_k$ and $A'_k$ are perturbations of $\square I$ (1.9)).

Given $f ∈ V'$, let $v$ be the solution of the variational problem

\[
\text{find } v ∈ V \text{ such that } ⟨Bv, w⟩_{V'×V} = ⟨f, w⟩_{V'×V} \quad \text{for all } w ∈ V, \tag{4.1}
\]

i.e. $v = B^{-1}f$. Then, given $V_N ⊂ V$ closed, the Galerkin approximation to $v$ with respect to $V_N$, $v_N := (B^{-1})_N f$, is defined as the solution of the Galerkin equations

\[
\text{find } v_N ∈ V_N \text{ such that } ⟨Bv_N, w⟩_{V'×V} = ⟨f, w⟩_{V'×V} \quad \text{for all } w ∈ V_N. \tag{4.2}
\]

We now rewrite the equations (4.2) using the orthogonal projection operator $P_N : V → V_N$. Then, $(I − P_N)$ is the orthogonal projection onto the orthogonal complement of $V_N$ and, in particular,

\[
\|(I − P_N)w\|_V = \min_{w_N ∈ V_N} \|w − w_N\|_V. \tag{4.3}
\]

The Galerkin equations (4.2) are then equivalent to the operator equation

\[
P_NBv_N = P_N f, \quad v_N ∈ V_N, \tag{4.4}
\]

where we have used that $V$ is a Hilbert space to identify $V$ and $V'$ when applying $P_N$ to $B$ on the left. If $B = I + K$, then, since $v_N ∈ V_N$, (4.4) simplifies to

\[
(I + P_NK)v_N = P_N f; \tag{4.5}
\]

see, e.g., [5, §3.1.3], [83]. Despite the fact that formally (4.5) is posed on $V_N$, the operator $I + P_NK$ maps $V_N → V_N$ and hence we can study the operator $(I + P_NK)$ as a mapping $V → V$.

**Lemma 4.1.** (Quasi-optimality in terms of the norm of the discrete inverse) If

$I + P_NK : V → V$ is invertible, then the Galerkin solution, $v_N$, solving (4.2) exists, is unique, and satisfies

\[
\|v − v_N\|_V ≤ \|(I + P_NK)^{-1}\|_{V → V} \|(I − P_N)v\|_V.
\]
Proof. Since \( I + P_N K : V \to V \) is invertible and \( I + P_N K : V_N \to V_N \), the solution \( v_N \) to (4.5) exists, lies in \( V_N \), and is unique as an element of \( V \). Then, by (4.1) and (4.2),

\[
(I + P_N K)(v - v_N) = (I + P_N K)v - P_N f = v + P_N K v - P_N ((I + K)v) = (I - P_N) v.
\]

\[\blacksquare\]

**Theorem 4.2 (Sufficient condition for quasi-optimality)** Let \( \delta > 0 \). If \( B = I + K \) and

\[
\|(I - P_N)K(I + K)^{-1}\|_{V \to V} \leq 1 - \delta,
\]

then the Galerkin solution \( v_N \), solving (4.5), exists, is unique, and satisfies

\[
\|v - v_N\|_V \leq \delta^{-1} \|(I + K)^{-1}\|_{V \to V} \|(I - P_N) v\|_V.
\]

**Proof.** The basis of the proof of (4.7) is Lemma 4.1 and the result that if \( \|T\| < 1 \) then \( I + T \) is invertible with \( \|(I + T)^{-1}\| \leq (1 - \|T\|)^{-1} \). Indeed, since

\[
I + P_N K = I + K - (I - P_N)K = (I - (I - P_N)K(I + K)^{-1})(I + K).
\]

Therefore, if (4.6) holds, then

\[
(I + P_N K)^{-1} = (I + K)^{-1}(I - (I - P_N)K(I + K)^{-1})^{-1}.
\]

Thus, by (4.6),

\[
\|(I + P_N K^{-1})\|_{V \to V} \leq \delta^{-1} \|(I + K)^{-1}\|_{V \to V}.
\]

and the result (4.7) follows from applying Lemma 4.1

\[\blacksquare\]

**Remark 4.3** An analogous result to Theorem 4.2 under the condition

\[
\|(I + K)^{-1}(I - P_N)K\|_{V \to V} < 1,
\]

is stated in, e.g., [83, Theorem 10.1]; [5, Theorem 3.1.1]; this result was used in the \( h \)-BEM context in [65], [59, Lemma 3.3]. Here we factor out \( I + K \) from the right in (4.8), rather than the left, leading to (4.6) rather than (4.9).

5 The high-frequency behaviour of the boundary integral operators \( S_k, D_k, \) and \( D'_k \)

**Functions of the surface Laplacian defined via eigenfunction expansion.** Let \( \lambda_j \) be the eigenvalues of the surface Laplacian (a.k.a. the Laplace-Beltrami operator) \( -\Delta_G \), and let \( \{u_{\lambda_j}\}_{j=1}^\infty \) be an orthonormal basis for \( L^2(\Gamma) \) of eigenfunctions; i.e.,

\[
(-\Delta_G - \lambda_j)u_{\lambda_j} = 0 \quad \text{and} \quad \|u_{\lambda_j}\|_{L^2(\Gamma)} = 1;
\]

when \( \Gamma \) is the unit circle, \( \{u_{\lambda_j}\}_{j=1}^\infty \) can be taken to be \( \{\frac{1}{\sqrt{2\pi}} e^{ijt}\}_{j=-\infty}^\infty \); see §7 below.

We then define functions of \( -\Delta_G \) using expansions in this basis. Precisely, for a function \( f \in L^\infty(\mathbb{R}) \) and \( v \in L^2(\Gamma) \),

\[
f(-\Delta_G)v := \sum_{j=1}^\infty f(\lambda_j)\langle v, u_{\lambda_j}\rangle_{L^2(\Gamma)} u_{\lambda_j}.
\]

(5.1)

By taking norms and using orthonormality of the basis, we see that

\[
\|f(-\Delta_G)v\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \|f\|_{L^\infty(\mathbb{R})}.
\]

(5.2)
**Frequency cut-offs defined as functions of the surface Laplacian.** With \( u_{\lambda_j} \) defined above, we say that “a function \( v \) has frequency \( \geq M \)” if, for some \( a_{\lambda_j} \in \mathbb{C} \)

\[
v = \sum_{\lambda_j \geq M} a_{\lambda_j} u_{\lambda_j}.
\]

For \( \chi \in C_c^\infty(\mathbb{R}) \) with \( \chi \equiv 1 \) on \( U \subset \mathbb{R} \), the operator \((1 - \chi(-k^{-2}\Delta_f))\) therefore restricts to functions with frequencies outside the set \( kU \). In particular, if \( \chi \equiv 1 \) near \([-1,1] \), then \((1 - \chi(-k^{-2}\Delta_f))\) restricts to functions with frequencies \( \geq k \).

**Theorem 5.1 (The high-frequency behaviour of \( S_k, D_k, \) and \( D'_k \))** Suppose \( \chi \in C_c^\infty(\mathbb{R}) \) with \( \chi \equiv 1 \) in a neighborhood of \([-1,1] \). Then for all \( k > 0 \) there exists \( C > 0 \) such that for all \( k \geq k_0 \)

\[
\|(1 - \chi(-k^{-2}\Delta_f))D_k\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} + \|(1 - \chi(-k^{-2}\Delta_f))D'_k\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} \leq Ck,
\]

\[
\|(1 - \chi(-k^{-2}\Delta_f))S_k\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} \leq C.
\]

(5.3)

By the discussion above, we see that the bounds in (5.3) are bounds on the outputs of \( D_k, D'_k, \) and \( S_k \) with frequencies \( \geq k \).

**References for the proof of Theorem 5.1.** This is proved in [57, Theorem 4.4]; see also [58, Theorem 3.1, Remark 4.2]; we note that the key ingredient [57, Lemma 3.10] is a simplified version of [55, Lemma 4.27], and the semiclassical analogue of [129, Chapter 7, §11] and [74, Theorem 8.4.3].

**Lemma 5.2 (Smoothing property of compactly-supported functions of \(-k^{-2}\Delta_f\))** Suppose that \( f \in L^\infty_c(\mathbb{R}) \). Then for all \( s \geq 0 \) there exists \( C_s, f > 0 \) such that

\[
\|f(-k^{-2}\Delta_f)\|_{L^2(\Gamma) \rightarrow H^s(\Gamma)} \leq C_s f^k\|f\|_{L^\infty(\Gamma)}
\]

(5.4)

**Proof.** By elliptic regularity, given \( \ell > 0 \) there exists \( C_\ell \) such that for all \( v \)

\[
\|v\|_{H^\ell(\Gamma)} \leq C_\ell \left( \|(-\Delta_f)^{\ell/2}v\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)} \right);
\]

this follows from interior regularity for second-order elliptic operators with variable coefficients; see [48, Section 6.3.1]. Thus

\[
\|f(-k^{-2}\Delta_f)v\|_{H^\ell(\Gamma)} \leq C_\ell \left( \|(-\Delta_f)^{\ell/2}f(-k^{-2}\Delta_f)v\|_{L^2(\Gamma)} + \|f(-k^{-2}\Delta_f)v\|_{L^2(\Gamma)} \right).
\]

(5.5)

By (5.2), the last term on the right-hand side of (5.5) is bounded by \( C\|v\|_{L^2(\Gamma)} \) for \( C \) depending on \( f \) but independent of \( k \). For the first term on the right-hand side of (5.5) we use that fact that \( s^f f(s) \in L^\infty \) (since \( f \) has compact support) to see that

\[
\|(-\Delta_f)^{\ell/2}f(-k^{-2}\Delta_f)v\|_{L^2(\Gamma)} = k^{2\ell} \|(-\Delta_f)^{\ell/2}f(-k^{-2}\Delta_f)v\|_{L^2(\Gamma)} \leq k^{2\ell} \|s^f f(s)\|_{L^\infty} \leq \tilde{C}_\ell k^{2\ell},
\]

for some \( \tilde{C}_\ell > 0 \). Using these bounds in (5.5) we obtain the bound (5.4) for even \( s \). The bound for odd \( s \) then follows by interpolation (see, e.g., [103, Theorems B.2]) using the fact that \( H^s(\Gamma) \) is an interpolation scale (see, e.g., [103, Theorem B.11]).

6 Proof of Theorem 2.1

It is sufficient to prove Lemma 3.1, since Theorem 2.1 then follows from the bound (3.3).

As described in §3, we use Theorems 4.2 and 5.1. We apply the former with \( B = 2A \), so that \( K = 2A - I \), and \( \delta = 1/2 \). Thus, we only need to prove that there exists \( C_1 > 0 \) such that if (3.1) holds then

\[
\|(I - P_N)(2A - I)(2A)^{-1}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq \frac{1}{2}.
\]
By the bound (1.21) from Assumption 1.4, it is sufficient to show that there exists $C_1 > 0$ such that if (3.1) holds then
\[ hC_{\text{approx}} \|(2A - I)(2A)^{-1}\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} \leq \frac{1}{2}. \]
We therefore only need to show that
\[ \|(2A - I)(2A)^{-1}\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} \leq C_2 k(1 + \|A^{-1}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}), \] (6.1)
and then the result holds with $C := (2C_{\text{approx}}C_2)^{-1}$.
To prove (6.1), let $\chi \in C^\infty_c(\mathbb{R})$ with $\chi \equiv 1$ in a neighborhood of $[-1, 1]$. Since $1 = \chi + (1 - \chi)$,
\[
(2A - I)(2A)^{-1} = \chi(-k^{-2}\Delta_I)(2A - I)(2A)^{-1} + (1 - \chi(-k^{-2}\Delta_I))(2A - I)(2A)^{-1}
= \chi(-k^{-2}\Delta_I)(I - (2A)^{-1}) + (1 - \chi(-k^{-2}\Delta_I))(2A - I)(2A)^{-1}.
\] (6.2)
To deal with the first term on the right-hand side of (6.2), we use Lemma 5.2 applied with $f = \chi$ to find that
\[
\|\chi(-k^{-2}\Delta_I)(I - (2A)^{-1})\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} \leq \|\chi(-k^{-2}\Delta_I)\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} (1 + \|(2A)^{-1}\|_{L^2 \rightarrow L^2})
\leq C_3 k(1 + \|A^{-1}\|_{L^2 \rightarrow L^2}),
\] (6.3)
for some $C_3 > 0$ (independent of $k$). We now consider the second term on the right-hand side of (6.2) when $A = A_k$; the proof when $A = A_k'$ follows in exactly the same way, just replacing $D_k$ by $D_k'$. By the definition of $A_k$ (1.9) and Theorem 5.1,
\[
\|(1 - \chi(-k^{-2}\Delta_I))(2A - I)(2A)^{-1}\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} = \|(1 - \chi(-k^{-2}\Delta_I))(D_k - ikS_k)(2A_k)^{-1}\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)}
\leq \|(1 - \chi(-k^{-2}\Delta_I))(D_k - ikS_k)\|_{L^2(\Gamma) \rightarrow H^1(\Gamma)} \frac{1}{2} \|A_k^{-1}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}
\leq C_4 k \|A_k^{-1}\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)},
\] (6.4)
for some $C_4 > 0$ (independent of $k$). Combining (6.3) and (6.4) we obtain (6.1), and the proof is complete.

7 A simple proof of Theorem 2.1 when $\Gamma$ is the unit circle

Ultimately the most flexible tools to study the large $k$ behaviour of Helmholtz boundary integral operators come from semiclassical analysis. Nevertheless, in the special case when $\Gamma$ is the unit circle, Theorem 2.1 can be proved using only results about Fourier series and the asymptotics of Bessel and Hankel functions. The advantage of the latter proof, therefore, is that it only uses classical tools of applied mathematics; furthermore, since we write this proof mirroring the general proof in §6, we hope it makes the ideas in §6 clearer.

7.1 Recap of Fourier-series results.

Suppose $\Gamma$ is the circle, with parametrisation $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi)$. With this parametrisation, $L^2(\Gamma)$ is isometrically isomorphic to $L^2(0, 2\pi)$. Given $v \in L^2(0, 2\pi)$, define the $n$th Fourier coefficient of $v$ by
\[
\hat{v}_n := \frac{1}{\sqrt{2\pi}} \langle v, e^{in} \rangle_{L^2(0, 2\pi)} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-int} v(t) \, dt, \quad \text{so that} \quad v(t) = \sum_{n=-\infty}^{\infty} \hat{v}_n e^{int} \sqrt{2\pi} \] (7.1)
as an $L^2$ function. Parseval’s theorem states that
\[
\|v\|^2_{L^2(0, 1)} = \sum_{n=-\infty}^{\infty} |\hat{v}_n|^2.
\] (7.2)
7.2 Results about the eigenvalues of $2A_k$.

When $\Gamma$ is a circle, $A_k = A_k'$ since $D_k = D_k'$; this follows from the definitions of $D_k$ and $D_k'$ and the geometric property that $(x - y) \cdot \nu(y) = (x - y) \cdot \nu(x)$ for $x, y$ on a circle.

Lemma 7.1 (Expression for eigenvalues of $2A_k$ in terms of Bessel and Hankel functions) If
\[
\lambda_m(k) := \pi k H_{|m|}^{(1)}(k) (iJ_{|m|}'(k) + J_{|m|}(k)),
\] then
\[
(2A_k v)(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \lambda_m(k) \hat{e}_m e^{int}.
\]

References for the proof. See, e.g., [82, §4 (in particular Equation 4.4)] or [43, Lemma 4.1].

Theorem 7.2 (Sign property of eigenvalues of $2A_k$ on unit circle) If $\Gamma$ is the unit circle, then there exists $k_0 > 0$ such that, for all $m$ and for all $k \geq k_0$,
\[
\Re \lambda_m(k) \geq 1.
\]

Reference for the proof. This is proved in [43, Theorem 4.2] using asymptotics of Bessel and Hankel functions.

The only other rigorous result about the eigenvalues $\lambda_m(k)$ that we need is the following.

Lemma 7.3 (Asymptotics of $\lambda_m(k)$ as $m \to \infty$ with $m > k$) Let $z := k/m$. Then, for all $\delta > 0$ there exists $C > 0$ such that for $0 < z < 1 - \delta$,\[
|\lambda_m(k) - 1| \leq C z.
\]

Proof. We now review some standard facts about uniform asymptotics for the Bessel functions $J_m(mz)$ and $H_m^{(1)}(mz)$ [116], [113, Section 10.20], where $m \geq 0$ and $z < 1 - \delta$. We define the decreasing bijection $(0, 1) \ni z \mapsto \zeta(z) \in (0, \infty)$ by
\[
\zeta := \frac{3}{2} \left( \int_{z}^{1} t^{-1} (1 - t^2)^{1/2} dt \right)^{2/3},
\]
and recall the definition of the Airy function, $\text{Ai}$,
\[
\text{Ai}(x) := \frac{1}{\pi} \int_{0}^{\infty} \cos \left( \frac{t^3}{3} + xt \right) dt.
\]

By [113, Section 9.7], for $|\arg(x)| < \pi - \delta$,
\[
\text{Ai}(x) = \exp \left( - \frac{2}{3} x^{3/2} \right) \left( \frac{1}{2\sqrt{\pi}} x^{-1/4} + O(x^{-7/4}) \right),
\]
\[
\text{Ai}'(x) = \exp \left( - \frac{2}{3} x^{3/2} \right) \left( - \frac{1}{2\sqrt{\pi}} x^{1/4} + O(x^{-5/4}) \right),
\]
where the branch cut is taken on $x \in (-\infty, 0)$. Moreover, by [113, Section 9.9], $|\text{Ai}(x)|, |\text{Ai}'(x)| > 0$ for $x \notin (\infty, 0)$.

Then, by [113, Section 10.20], uniformly for $m \geq 1$ and $0 < z < 1$,
\[
J_m(mz) = \left( \frac{4\zeta}{1 - z^2} \right)^{1/4} \left( m^{-1/3} \text{Ai}(m^{2/3} \zeta) + O \left( m^{-5/3} \zeta^{-1/2} \text{Ai}'(m^{2/3} \zeta) \right) \right),
\]
\[
J_m'(mz) = - \frac{2}{z} \left( \frac{1 - z^2}{4\zeta} \right)^{1/4} \left( m^{-2/3} \text{Ai}'(m^{2/3} \zeta) + O \left( m^{-4/3} \zeta^{1/2} \text{Ai}(m^{2/3} \zeta) \right) \right),
\]
\[
H_m^{(1)}(mz) = 2e^{-\pi i/3} \left( \frac{4\zeta}{1 - z^2} \right)^{1/4} \left( m^{-1/3} \text{Ai}(e^{2\pi i/3} m^{2/3} \zeta) + O \left( m^{-5/3} \zeta^{-1/2} \text{Ai}'(e^{2\pi i/3} m^{2/3} \zeta) \right) \right),
\]
\[
(7.9)
\]
where \( \zeta := (1 + |\zeta|^2)^{1/2} \). Next, note that when \( 0 < z < 1 - \delta \), there exists \( c_\delta > 0 \) such that \( \zeta \geq c_\delta \) and thus we can use the asymptotics for Airy functions (7.8). Putting these asymptotics in (7.9) and using the definition of \( \lambda_m(k) \) (7.3), we obtain that for any \( \delta > 0 \), there exists \( C > 0 \) such that
\[
\left| \lambda_m(k) - 1 \right| = \left| \pi k H^{(1)}_{|m|}(k) \left( iJ'_{|m|}(k) + J_{|m|}(k) \right) - 1 \right| \leq C \frac{k}{m}, \quad \text{for } m > (1 + \delta)k,
\]
as claimed.

### 7.3 Proof of Theorem 2.1 when \( \Gamma \) is the unit circle

Observe that in the case of the circle, the functional calculus for the surface Laplacian reviewed in Section 5 is simply the theory of Fourier multipliers; i.e. the collection \( \{ \frac{1}{\sqrt{2\pi}} e^{int} \}_{m = -\infty}^{\infty} \) is an orthonormal basis of eigenfunctions of \(-\Delta_{\Gamma}\) satisfying
\[
(-\Delta_{\Gamma} - m^2) \frac{1}{\sqrt{2\pi}} e^{int} = 0, \quad \left\| \frac{1}{\sqrt{2\pi}} e^{int} \right\|_{L^2(\Gamma)} = 1.
\]
Thus (5.1) becomes
\[
f(-\Delta_{\Gamma})v := \frac{1}{\sqrt{2\pi}} \sum_{m = -\infty}^{\infty} f(m^2) \hat{v}_m e^{int}.
\]
(7.10)

To prove Theorem 2.1, we only need to check the conditions of Theorem 4.2 with \( I + K = 2A = 2A_k \). Using Assumption 1.4 as in the beginning of §6, we see that we only need to prove the bound (6.1). The expansion (7.4) implies that
\[
((2A_k)^{-1}v)(t) = \frac{1}{\sqrt{2\pi}} \sum_{m = -\infty}^{\infty} (\lambda_m(k))^{-1} \hat{v}_m e^{int}.
\]
By Theorem 7.2, and the fact that \( |\lambda_m| \geq |\Re \lambda_m| \geq 1 \),
\[
\sup_{m} |\lambda_m(k)|^{-1} \leq 1.
\]
(7.11)

Therefore, by taking \( L^2 \) norms and using orthonormality (in a similar way to how (5.2) is obtained), we obtain the bound (3.3) in this setting
\[
\|((2A_k)^{-1}v)\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \sup_{m} |\lambda_m(k)|^{-1} \leq 1.
\]
(7.12)

To prove the bound (6.1), we therefore only need to show that
\[
\|((2A_k - I)(2A_k)^{-1})\|_{L^2(\Gamma) \to H^1(\Gamma)} \leq Ck;
\]
(7.13)
we do this using the splitting (6.2) with \( \chi \in C^\infty_{\text{comp}}(\mathbb{R}; [0, 1]) \) with \( \chi \equiv 1 \) on \([-1 - \varepsilon, 1 + \varepsilon]\). To deal the first term on the right-hand side of (6.1), we observe that, by (7.12)
\[
\|I - (2A_k)^{-1}\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq 2.
\]
(7.14)
Next, recall that the \( H^1(\Gamma) \) norm is defined in terms of Fourier coefficients by
\[
\|v\|_{H^1(\Gamma)}^2 := \sum_{m = -\infty}^{\infty} (1 + m^2) |\hat{v}_m|^2.
\]
(7.15)
This definition, along with the compact support of \( \chi \), implies the following analogue of Lemma 5.2 with \( s = 1 \)
\[
\|\chi(-k^2\Delta_{\Gamma})\|_{L^2(\Gamma) \to H^1(\Gamma)} \leq \sup_{m} \left( (1 + m^2) |\chi(k^2m^2)| \right) \leq Ck^2.
\]
(7.16)
Combining (7.14) and (7.16), we obtain the following bound the first term of the right-hand side of (6.2)
\[
\|\chi(-k^2\Delta_{\Gamma})(I - (2A_k)^{-1})\|_{L^2(\Gamma) \to H^1(\Gamma)} \leq Ck.
\]
(7.17)
To deal the second term on the right-hand side of (6.2), we observe that
\[
(1 - \chi(-k^2 \Delta_{\Gamma}))(2A_k - I)(2A_k)^{-1} e^{imt} = (1 - \chi(-k^2 m^2)) \frac{\lambda_m(k) - 1}{\lambda_m(k)} e^{imt}.
\]
Thus, using the Fourier representation of \((1 - \chi(-k^2 \Delta_{\Gamma}))(2A_k - I)(2A_k)^{-1}\) and the definition of the \(H^1(\Gamma)\) norm (7.15), we find that
\[
\| (1 - \chi(-k^2 \Delta_{\Gamma}))(2A_k - I)(2A_k)^{-1} \|_{L^2(\Gamma) \rightarrow H^1(\Gamma)}^2 \leq \sup_m \left( (1 + m^2) \left( 1 - \chi(k^2 m^2) \right) \frac{|\lambda_m(k) - 1|}{|\lambda_m(k)|} \right).
\]
By the definition of \(\chi\), \((1 - \chi(k^2 m^2)) = 0\) when \(m^2 \leq (1 + \varepsilon) k^2\), and \((1 - \chi(k^2 m^2)) \leq 1\) for all \(m\); therefore
\[
\| (1 - \chi(-k^2 \Delta_{\Gamma}))(2A_k - I)(2A_k)^{-1} \|_{L^2(\Gamma) \rightarrow H^1(\Gamma)}^2 \leq \sup_{m^2 \geq (1 + \varepsilon) k^2} \left( (1 + m^2) \frac{|\lambda_m(k) - 1|}{|\lambda_m(k)|} \right).
\]
Observe from (7.3) that \(\lambda_m(k) = \lambda_m'(k)\); the regime \(m^2 \geq (1 + \varepsilon) k^2\) is therefore exactly that covered by Lemma 7.3 (with \((1 + \varepsilon)^{-1/2} = 1 - \delta\)). Using Lemma 7.3 along with (7.11), we obtain that
\[
\| (1 - \chi(-k^2 \Delta_{\Gamma}))(2A_k - I)(2A_k)^{-1} \|_{L^2(\Gamma) \rightarrow H^1(\Gamma)}^2 \leq \sup_{m^2 \geq (1 + \varepsilon) k^2} \left( (1 + m^2) \frac{|\lambda_m(k) - 1|}{|\lambda_m(k)|} \right)
\leq C^2 \sup_{m^2 \geq (1 + \varepsilon) k^2} \left( (1 + m^2) \frac{k^2}{m^2} \right) \leq C' k^2.
\]
Combining the bounds (7.17) and (7.18), we obtain (7.13), and the proof is complete.

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