A primer on computational group homology and cohomology∗

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These are expanded lecture notes of a series of expository talks surveying basic aspects of group cohomology and homology. They were written for someone who has had a first course in graduate algebra but no background in cohomology. You should know the definition of a (left) module over a (non-commutative) ring, what $\mathbb{Z}[G]$ is (where $G$ is a group written multiplicatively and $\mathbb{Z}$ denotes the integers), and some ring theory and group theory. However, an attempt has been made to (a) keep the presentation as simple as possible, (b) either provide an explicit reference of proof of everything.

Several computer algebra packages are used to illustrate the computations, though for various reasons we have focused on the free, open source packages. In particular, Graham Ellis generously allowed extensive use of his HAP\cite{E1} documentation (which is sometimes copied almost verbatim) in the presentation below.

1 Introduction

First, some words of motivation.

Let $G$ be a group and $A$ a $G$-module\footnote{We call an abelian group $A$ (written additively) which is a left $\mathbb{Z}[G]$-module a $G$-module.}.

Let $A^G$ denote the largest submodule of $A$ on which $G$ acts trivially. Let us begin by asking ourselves the following natural question.
**Question:** Suppose $A$ is a submodule of a $G$-module $B$ and $x$ is an arbitrary $G$-fixed element of $B/A$. Is there an element $b$ in $B$, also fixed by $G$, which maps onto $x$ under the quotient map?

The answer to this question can be formulated in terms of group cohomology. ("Yes", if $H^1(G, A) = 0"). The details, given below, will help motivate the introduction of group cohomology.

Let $A_G$ is the largest quotient module of $A$ on which $G$ acts trivially. Next, we ask ourselves the following analogous question.

**Question:** Suppose $A$ is a submodule of a $G$-module $B$ and $b$ is an arbitrary element of $B_G$ which maps to 0 under the natural map $B_G \rightarrow (B/A)_G$. Is there an element $a$ in $a_G$ which maps onto $b$ under the inclusion map?

The answer to this question can be formulated in terms of group homology. ("Yes", if $H_1(G, A) = 0"). The details, given below, will help motivate the introduction of group homology.

Group cohomology arises as the right higher derived functor for $A \mapsto -A_G$. The cohomology groups of $G$ with coefficients in $A$ are defined by

$$H^n(G, A) = \text{Ext}^n_{ZG}(Z, A).$$

(See §4 below for more details.) These groups were first introduced in 1943 by S. Eilenberg and S. MacLane [EM]. The functor $A \mapsto A_G$ on the category of left $G$-modules is additive and left exact. This implies that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is an exact sequence of $G$-modules then we have a **long exact sequence of cohomology**

$$0 \rightarrow A^G \rightarrow B^G \rightarrow C^G \rightarrow H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C) \rightarrow H^2(G, A) \rightarrow \ldots$$  \hspace{1cm} (1)

Similarly, group homology arises as the left higher derived functor for $A \mapsto A_G$. The homology groups of $G$ with coefficients in $A$ are defined by

$$H_n(G, A) = \text{Tor}^n_{ZG}(Z, A).$$

(See §5 below for more details.) The functor $A \mapsto A_G$ on the category of left $G$-modules is additive and right exact. This implies that if
is an exact sequence of $G$-modules then we have a **long exact sequence of homology**

$$
\cdots \rightarrow H_2(G, C) \rightarrow H_1(G, A) \rightarrow H_1(G, B) \rightarrow H_1(G, C) \rightarrow A_G \rightarrow B_G \rightarrow C_G \rightarrow 0.
$$

Here we will define both cohomology $H^n(G, A)$ and homology $H_n(G, A)$ using projective resolutions and the higher derived functors $Ext^n$ and $Tor_n$. We “compute” these when $G$ is a finite cyclic group. We also give various functorial properties, such as corestriction, inflation, restriction, and transfer. Since some of these cohomology groups can be computed with the help of computer algebra systems, we also include some discussion of how to use computers to compute them. We include several applications to group theory.

One can also define $H^1(G, A)$, $H^2(G, A)$, \ldots, by explicitly constructing cocycles and coboundaries. Similarly, one can also define $H_1(G, A)$, $H_2(G, A)$, \ldots, by explicitly constructing cycles and boundaries. For the proof that these constructions yield the same groups, see Rotman [R], chapter 10.

For the general outline, we follow §7 in chapter 10 of [R] on homology. For some details, we follow Brown [B], Serre [S] or Weiss [W].

For a recent expository account of this topic, see for example Adem [A]. Another good reference is Brown [B].

## 2 Differential groups

In this section cohomology and homology are viewed in the same framework. This “differential groups” idea was introduced by Cartan and Eilenberg [CE], chapter IV, and developed in R. Godement [G], chapitre 1, §2. However, we shall follow Weiss [W], chapter 1.

### 2.1 Definitions

A **differential group** is a pair $(L, d)$, $L$ an abelian group and $d : L \rightarrow L$ a homomorphism such that $d^2 = 0$. We call $d$ a **differential operator**. The group
\[ H(L) = \text{Kernel } (d) / \text{Image } (d) \]
is the derived group of \((L, d)\). If
\[ L = \bigoplus_{n=-\infty}^{\infty} L_n \]
then we call \(L\) graded. Suppose \(d\) (more precisely, \(d|_{L_n}\)) satisfies, in addition, for some fixed \(r \neq 0\),
\[ d : L_n \rightarrow L_{n+r}, \quad n \in \mathbb{Z}. \]
We say \(d\) is compatible with the grading provided \(r = \pm 1\). In this case, we call \((L, d, r)\) a graded differential group. As we shall see, the case \(r = 1\) corresponds to cohomology and the case \(r = -1\) corresponds to homology. Indeed, if \(r = 1\) then we call \((L, d, r)\) a (differential) group of cohomology type and if \(r = -1\) then we call \((L, d, r)\) a group of homology type. Note that if \(L = \bigoplus_{n=-\infty}^{\infty} L_n\) is a group of cohomology type then \(L' = \bigoplus_{n=-\infty}^{\infty} L'_n\) is a group of homology type, where \(L'_n = L_{-n}\), for all \(n \in \mathbb{Z}\).

For the impatient: For cohomology, we shall eventually take \(L = \bigoplus_n \text{Hom}_G(X_n, A)\), where the \(X_n\) form a chain complex (with +1 grading) determined by a certain type of resolution. The group \(H(L)\) is an abbreviation for \(\bigoplus_n \text{Ext}^1_{\mathbb{Z}[G]} (\mathbb{Z}, A)\). For homology, we shall eventually take \(L = \bigoplus_n \mathbb{Z} \otimes_{\mathbb{Z}[G]} X_n\), where the \(X_n\) form a chain complex (with -1 grading) determined by a certain type of resolution. The group \(H(L)\) is an abbreviation for \(\bigoplus_n \text{Tor}^1_{\mathbb{Z}[G]} (\mathbb{Z}, A)\).

Let \((L, d) = (L, d_L)\) and \((M, d) = (M, d_M)\) be differential groups\(^2\). A homomorphism \(f : L \rightarrow M\) satisfying \(d \circ f = f \circ d\) will be called admissible. For any \(n \in \mathbb{Z}\), we define \(n f : L \rightarrow M\) by \((n f)(x) = n \cdot f(x) = f(x) + \cdots + f(x)\)

\(^2\)Of course, to be more precise, we should use different symbols for the differential operators of \(L\) and \(M\). For notational simplicity, we use the same symbol and hope the context removes any ambiguity.
If $f$ is admissible then so is $nf$, for any $n \in \mathbb{Z}$. An admissible map $f$ gives rise to a map of derived groups: define the map $f_* : H(L) \to H(M)$, by $f_*(x + dL) = f(x) + dM$, for all $x \in L$.

### 2.2 Properties

Let $f$ be an admissible map as above.

1. The map $f_* : H(L) \to H(M)$ is a homomorphism.

2. If $f : L \to M$ and $g : L \to M$ are admissible, then so is $f + g$ and we have $(f + g)_* = f_* + g_*$. 

3. If $f : L \to M$ and $g : M \to N$ are admissible then so is $g \circ f : L \to N$ and we have $(g \circ f)_* = g_* \circ f_*$. 

4. If

$$0 \to L \xrightarrow{i} M \xrightarrow{j} N \to 0 \tag{3}$$

is an exact sequence of differential groups with admissible maps $i, j$ then there is a homomorphism $d_* : H(N) \to H(L)$ for which the following triangle is exact:

\[
\begin{array}{ccc}
H(N) & \xrightarrow{j_*} & H(M) \\
\downarrow{i_*} & & \downarrow{} \\
H(L) & \xrightarrow{d_*} & H(N)
\end{array}
\]

This diagram $\text{3}$ encodes both the long exact sequence of cohomology $\text{11}$

---

$^3$This is a special case of Théorème 2.1.1 in [G].
and the long exact sequence of homology [2].

Here is the construction of $d_*$:

Recall $H(N) = \text{Kernel}(d)/\text{Image}(d)$, so any $x \in H(N)$ is represented by an $n \in N$ with $dn = 0$. Since $j$ is surjective, there is an $m \in M$ such that $j(m) = n$. Since $j$ is admissible and the sequence is exact, $j(dm) = d(j(m)) = dn = 0$, so $dm \in \text{Kernel}(j) = \text{Image}(i)$. Therefore, there is an $\ell \in L$ such that $dm = i(\ell)$. Define $d_*(x)$ to be the class of $\ell$ in $H(L)$, i.e., $d_*(x) = \ell + dL$.

Here’s the verification that $d_*$ is well-defined:

We must show that if we defined instead $d_*(x) = \ell' + dL$, some $\ell' \in L$, then $\ell' - \ell \in dL$. Pull back the above $n \in N$ with $dn = 0$ to an $m \in M$ such that $j(m) = n$. As above, there is an $\ell \in L$ such that $dm = i(\ell)$. Represent $x \in H(N)$ by an $n' \in N$, so $x = n' + dN$ and $dn' = 0$. Pull back this $n'$ to an $m' \in M$ such that $j(m') = n'$. As above, there is an $\ell' \in L$ such that $dm' = i(\ell')$. We know $n' - n \in dN$, so $n' - n = dn''$, some $n'' \in N$. Let $j(m'') = n''$, some $m'' \in M$, so $j(m' - m - dm'') = n' = n - j(dm'') = n' - n - dj(m'') = n' - n - dn'' = 0$.

Since the sequence $L - M - N$ is exact, this implies there is an $\ell_0 \in L$ such that $i(\ell_0) = m' - m - dm''$. But $di(\ell_0) = i(d\ell_0) = dm' - dm = i(\ell' - \ell) = i(\ell' - \ell)$, so $\ell' - \ell \in dL$.

5. If $M = L \oplus N$ then $H(M) = H(L) \oplus H(N)$.

**proof:** To avoid ambiguity, for the moment, let $d_X$ denote the differential operator on $X$, where $X \in \{L, M, N\}$. In the notation of (3), $j$ is projection and $i$ is inclusion. Since both are admissible, we know that $d_{M|L} = d_L$ and $d_{M|N} = d_N$. Note that $H(X) \subset X$, for any differential group $X$, so $H(M) = H(M) \cap L \oplus H(M) \cap N \subset H(L) \oplus H(N)$. It follows from this that that $d_* = 0$. From the exactness of the triangle (4), it therefore follows that this inclusion is an equality.

\[ \square \]

6. Let $L, L', M, M', N, N'$ be differential groups. If

\[
\begin{array}{cccccc}
0 & \longrightarrow & L & \overset{i}{\longrightarrow} & M & \overset{j}{\longrightarrow} & N & \longrightarrow & 0 \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow & \\
0 & \longrightarrow & L' & \overset{i'}{\longrightarrow} & M' & \overset{j'}{\longrightarrow} & N' & \longrightarrow & 0 \\
\end{array}
\]  

(5)

\[
0 \longrightarrow L \overset{i}{\longrightarrow} M \overset{j}{\longrightarrow} N \longrightarrow 0
\]
is a commutative diagram of exact sequences with $i, i', j, j', f, g, h$ all admissible then

\[
\begin{array}{c}
H(L) \xrightarrow{i_*} H(M) \\
f_* \downarrow \quad \quad g_* \downarrow \\
H(L') \xrightarrow{i'_*} H(M')
\end{array}
\]

commutes,

\[
\begin{array}{c}
H(M) \xrightarrow{j_*} H(N) \\
g_* \downarrow \quad \quad h_* \downarrow \\
H(M') \xrightarrow{i'_*} H(N')
\end{array}
\]

commutes, and

\[
\begin{array}{c}
H(N) \xrightarrow{d_*} H(L) \\
h_* \downarrow \quad \quad f_* \downarrow \\
H(N') \xrightarrow{d_*} H(L')
\end{array}
\]

commutes.

This is a case of Theorem 1.1.3 in [W] and of Théorème 2.1.1 in [G].

The proofs that the first two squares commute are similar, so we only verify one and leave the other to the reader. By assumption, (5) commutes and all the maps are admissible. Representing $x \in H(M)$ by $x = m + dM$, we have

\[
h_* j_*(x) = h_* (j(m) + dN) = hj(m) + dN' = gi'(m) + dN'
= g_* (i'(m) + dM') = g_* i'_*(m + dM) = g_* i'_*(x),
\]
as desired.

The proof that the last square commutes is a little different than this, so we prove this too. Represent $x \in H(N)$ by $x = n + dN$ with $dn = 0$ and recall that $d_* (x) = \ell + dL$, where $dm = i(\ell), \ell \in L$, where $j(m) = n$, for $m \in M$. We have
\[ f_* d_* (x) = f_*(\ell + dL) = f(\ell) + dL'. \]

On the other hand,

\[ d_* h_*(x) = d_*(h(n) + dN') = \ell' + dL', \]

for some \( \ell' \in L' \). Since \( h(n) \in N' \), by the commutativity of (5) and the definition of \( d_* \), \( \ell' \in L' \) is an element such that \( i'(\ell') = g_! (\ell) \). Since \( i' \) is injective, this condition on \( \ell' \) determines it uniquely mod \( dL' \). By the commutativity of (5), we may take \( \ell' = f(\ell) \).

7. Let \( L, L', M, M', N, N' \) be differential graded groups with grading +1 (i.e., of “cohomology type”). Suppose that we have a commutative diagram, with all maps admissible and all rows exact as in (5). Then the following diagram is commutative and has exact rows:

\[
\begin{array}{cccccccc}
... & H_{n-1}(N) & \xrightarrow{d_*} & H_n(L) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(N) & \xrightarrow{d_*} & H_{n+1}(L) & ... \\
& h_* & & f_* & & g_* & & h_* & & f_* & \\
... & H_{n-1}(N') & \xrightarrow{d_*} & H_n(L') & \xrightarrow{i'_*} & H_n(M') & \xrightarrow{j'_*} & H_n(N') & \xrightarrow{d_*} & H_{n+1}(L') & ...
\end{array}
\]

This is Proposition 1.1.4 in [W]. As pointed out there, it is an immediate consequence of the properties, 1-6 above.

Compare this with Proposition 10.69 in [R].

8. Let \( L, L', M, M', N, N' \) be differential graded groups with grading −1 (i.e., of “homology type”). Suppose that we have a commutative diagram, with all maps admissible and all rows exact, as in (5). Then the following diagram is commutative and has exact rows:

\[
\begin{array}{cccccccc}
... & H_{n+1}(N) & \xrightarrow{d_*} & H_n(L) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(N) & \xrightarrow{d_*} & H_{n-1}(L) & ... \\
& h_* & & f_* & & g_* & & h_* & & f_* & \\
... & H_{n+1}(N') & \xrightarrow{d_*} & H_n(L') & \xrightarrow{i'_*} & H_n(M') & \xrightarrow{j'_*} & H_n(N') & \xrightarrow{d_*} & H_{n-1}(L') & ...
\end{array}
\]

This is the analog of the previous property and is proven similarly.

Compare this with Proposition 10.58 in [R].
9. Let \((L, d)\) be a differential graded group with grading \(r\). If \(d_n = d|_{L_n}\) then \(d_{n+r} \circ d_n = 0\) and

\[
\cdots \rightarrow L_{n-r} \xrightarrow{d_{n-r}} L_n \xrightarrow{d_n} L_{n+r} \xrightarrow{d_n} L_{n+2r} \rightarrow \cdots
\]

is exact.

10. If \(\{L_n \mid n \in \mathbb{Z}\}\) is a sequence of abelian groups with homomorphisms \(d_n\) satisfying (6) then \((L, d)\) is a differential group, where \(L = \bigoplus_n L_n\) and \(d = \bigoplus d_n\).

### 2.3 Homology and cohomology

When \(r = 1\), we call \(L_n\) the **group of \(n\)-cochains**, \(Z_n = L_n \cap \text{Kernel}(d_n)\) the group of \(n\)-**cocycles**, and \(B_n = L_n \cap \text{d}(L_{n-1})\) the group of \(n\)-**coboundaries**.

We call \(H_n(L) = Z_n / B_n\) the \(n^\text{th}\) **cohomology group**. When \(r = -1\), we call \(L_n\) the **group of \(n\)-chains**, \(Z_n = L_n \cap \text{Kernel}(d_n)\) the group of \(n\)-**cycles**, and \(B_n = L_n \cap d_{n+1}(L_{n+1})\) the group of \(n\)-**boundaries**. We call \(H_n(L) = Z_n / B_n\) the \(n^\text{th}\) **homology group**.

### 3 Complexes

We introduce complexes in order to define explicit differential groups which will then be used to construct group (co)homology.

#### 3.1 Definitions

Let \(R\) be a non-commutative ring, for example \(R = \mathbb{Z}[G]\).

We shall define a “finite free, acyclic, augmented chain complex” of left \(R\)-modules.

A **complex** (or chain complex or \(R\)-complex with a negative grading) is a sequence of maps

\[
\cdots \rightarrow X_{n+1} \xrightarrow{\partial_{n+1}} X_n \xrightarrow{\partial_n} X_{n-1} \xrightarrow{\partial_{n-1}} X_{n-2} \rightarrow \cdots
\]

for which \(\partial_n \partial_{n+1} = 0\), for all \(n\). If each \(X_n\) is a free \(R\)-module with a finite basis over \(R\) (so is \(\cong R^k\), for some \(k\)) then the complex is called **finite free**. If this sequence is exact then it is called an **acyclic complex**. The complex
is **augmented** if there is a surjective $R$-module homomorphism $\epsilon : X_0 \to Z$ and an injective $R$-module homomorphism $\mu : Z \to X_{-1}$ such that $\partial_0 = \mu \circ \epsilon$, where (as usual) $Z$ is regarded as a trivial $R$-module.

The **standard diagram** for such an $R$-complex is

\[\cdots \to X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\partial_0} X_{-1} \xrightarrow{\partial_{-1}} X_{-2} \xrightarrow{\partial_{-2}} X_{-3} \to \cdots \]

Such an acyclic augmented complex can be broken up into the **positive part**

\[\cdots \to X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\epsilon} Z \to 0,\]

and the **negative part**

\[0 \to Z \xrightarrow{\mu} X_{-1} \xrightarrow{\partial_{-1}} X_{-2} \xrightarrow{\partial_{-2}} X_{-3} \to \cdots .\]

Conversely, given a positive part and a negative part, they can be combined into a standard diagram by taking $\partial_0 = \mu \circ \epsilon$.

If $X$ is any left $R$-module, let $X^* = \text{Hom}_R(X, Z)$ be the dual $R$-module, where $Z$ is regarded as a trivial $R$-module. Associated to any $f \in \text{Hom}_R(X, Y)$ is the pull-back $f^* \in \text{Hom}_R(Y^*, X^*)$. (If $y^* \in Y^*$ then define $f^*(y^*)$ to be $y^* \circ f : X \to Z$.) Since “dualizing” reverses the direction of the maps, if you dualize the entire complex with a $-1$ grading, you will get a complex with a $+1$ grading. This is the **dual complex**.

When $R = \mathbb{Z}[G]$ then we call a finite free, acyclic, augmented chain complex of left $R$-modules, a **$G$-resolution**. The maps $\partial_i : X_i \to X_{i-1}$ are sometimes called **boundary maps**.

**Remark 1** Using the command `BoundaryMap` in the GAP CRIME package of Marcus Bishop, one can easily compute the boundary maps of a cohomology object associated to a $G$-module. However, $G$ must be a $p$-group.

**Example 1** We use the package HAP [E1] to illustrate some of these concepts more concretely. Let $G$ be a finite group, whose elements we have ordered in some way: $G = \{g_1, \ldots, g_n\}$. 

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Since a $G$-resolution $X_*$ determines a sequence of finitely generated free $\mathbb{Z}[G]$-modules, to concretely describe $X_*$ we must be able to concretely describe a finite free $\mathbb{Z}[G]$-module. In order to represent a word $w$ in a free $\mathbb{Z}[G]$-module $M$ of rank $n$, we use a list of integer pairs $w = \{[i_1, e_1], [i_2, e_2], \ldots, [i_k, e_k]\}$. The integers $i_j$ lie in the range $\{-n, \ldots, n\}$ and correspond to the free $\mathbb{Z}[G]$-generators of $M$ and their additive inverses. The integers $e_j$ are positive (but not necessarily distinct) and correspond to the group element $g_{e_j}$.

Let’s begin with a HAP computation.

```
gap> LoadPackage("hap");
true
gap> G:=Group([[(1,2,3),(1,2)]]);;
    
gap> R:=ResolutionFiniteGroup(G, 4);;

This computes the first 5 terms of the $G$-resolution ($G = S_3$)

$$X_4 \xrightarrow{\delta_4} X_3 \xrightarrow{\delta_3} X_2 \xrightarrow{\delta_2} X_1 \xrightarrow{\delta_1} X_0 \rightarrow \mathbb{Z} \rightarrow 0.$$ 

The boundary maps $\delta_i$ are determined from the boundary component of the GAP record $R$. This record has (among others) the following components:

- $R!.dimension(k)$ – the $\mathbb{Z}[G]$-rank of the module $X_k$,
- $R!.boundary(k, j)$ – the image in $X_{k-1}$ of the $j$-th free generator of $X_k$,
- $R!.elts$ – the elements in $G$,
- $R!.group$ is the group in question.

Here is an illustration:

```
gap> R!.group;
Group([ (1,2), (1,2,3) ])
    
gap> R!.elts;
[ ( ), (2,3), (1,2), (1,2,3), (1,3,2), (1,3) ]
    
gap> R!.dimension(3);
4
    
gap> R!.boundary(3,1);
[ [ 1, 2 ], [ -1, 1 ] ]
```
\[ \text{gap} \backslash R^! \text{.boundary}(3,2); \]
\[ \begin{bmatrix} 2 & 2 \\ -2 & 4 \end{bmatrix} \]
\[ \text{gap} \backslash R^! \text{.boundary}(3,3); \]
\[ \begin{bmatrix} 3 & 4 \\ 1 & 3 \\ -3 & 1 \\ -1 & 1 \end{bmatrix} \]
\[ \text{gap} \backslash R^! \text{.boundary}(3,4); \]
\[ \begin{bmatrix} 2 & 5 \\ -3 & 3 \\ 2 & 4 \\ -1 & 4 \\ 2 & 1 \\ -3 & 1 \end{bmatrix} \]

In other words, \( X_3 \) is rank 4 as a \( G \)-module, with generators \( \{ f_1, f_2, f_3, f_4 \} \) say, and

\[
\begin{align*}
\delta_3(f_1) &= f_1g_2 - f_1g_1, \\
\delta_3(f_2) &= f_2g_2 - f_2g_4, \\
\delta_3(f_3) &= f_3g_4 - f_3g_1 + f_1g_3 - f_1g_1, \\
\delta_3(f_4) &= f_2(g_1 + g_3 + g_5) - f_3g_3 + f_1g_4 - f_3g_1.
\end{align*}
\]

Now, let us create another resolution and compute the equivariant chain map between them. Below is the complete GAP session:

\[ \text{gap} \backslash G_1 := \text{Group([[(1,2,3),(1,2)]]);} \]
\[ \text{Group([ (1,2,3), (1,2) ]) \} } \]
\[ \text{gap} \backslash G_2 := \text{Group([[(1,2,3),(2,3)]]);} \]
\[ \text{Group([ (1,2,3), (2,3) ]) } \]
\[ \text{gap} \backslash \phi := \text{GroupHomomorphismByImages}(G_1, G_2, \{(1,2,3), (1,2)\}, \{(1,2,3), (2,3)\}); \]
\[ \begin{bmatrix} (1,2,3), (1,2) \end{bmatrix} \rightarrow \begin{bmatrix} (1,2,3), (2,3) \end{bmatrix} \]
\[ \text{gap} \backslash R_1 := \text{ResolutionFiniteGroup}(G_1, 4); \]
\[ \text{Resolution of length 4 in characteristic 0 for Group([ (1,2), (1,2,3) ]).} \]
\[ \text{gap} \backslash R_2 := \text{ResolutionFiniteGroup}(G_2, 4); \]
\[ \text{Resolution of length 4 in characteristic 0 for Group([ (2,3), (1,2,3) ]).} \]
\[ \text{gap} \backslash \text{ZP}\_\text{map} := \text{EquivariantChainMap}(R_1, R_2, \phi); \]
\[ \text{Equivariant Chain Map between resolutions of length 4.} \]
\[ \text{gap} \backslash \text{map} := \text{TensorWithIntegers( ZP\_map);} \]
\[ \text{Chain Map between complexes of length 4.} \]
\[ \text{gap} \backslash \text{Hphi} := \text{Homology( map, 3);} \]
\[ f_1, f_2, f_3 \] \rightarrow [ f_2, f_2*f_3, f_1*f_2^2 ]

gap> AbelianInvariants(Image(Hphi));
\[ 2, 3 \]

gap>
gap> GroupHomology(G1,3);
\[ 6 \]

gap> GroupHomology(G2,3);
\[ 6 \]

In other words, \( H(\phi) \) is an isomorphism (as it should be, since the homology is independent of the resolution choosen).

### 3.2 Constructions

Let \( R = \mathbb{Z}[G] \).

#### 3.2.1 Bar resolution

This section follows §1.3 in [W].

Define a symbol \([.]\) and call it the **empty cell**. Let \( X_0 = R[.] \), so \( X_0 \) is a finite free (left) \( R \)-module whose basis has only 1 element. For \( n > 0 \), let \( g_1, \ldots, g_n \in G \) and define an \( n \)-**cell** to be the symbol \([g_1, \ldots, g_n] \). Let

\[ X_n = \bigoplus_{(g_1, \ldots, g_n) \in G^n} R[g_1, \ldots, g_n], \]

where the sum runs over all ordered \( n \)-tuples in \( G^n \).

Define the differential operators \( d_n \) and the augmentation \( \epsilon \), as \( G \)-module maps, by

\[
\begin{align*}
\epsilon([g]) &= 1, \quad g \in G \\
d_1([g]) &= g[.] - [.], \\
d_2([g_1, g_2]) &= g_1[g_2] - [g_1g_2] + [g_1], \\
&\vdots \\
d_n([g_1, \ldots, g_n]) &= g_1[g_2, \ldots, g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_n] \\
&\quad + (-1)^n [g_1, \ldots, g_{n-1}],
\end{align*}
\]
for $n \geq 1$. Note that the condition $\epsilon(g[\cdot]) = 1$ for all $g \in G$ is equivalent to saying $\epsilon([\cdot]) = 1$. This is because $\epsilon$ is a $G$-module homomorphism and $\mathbb{Z}$ is a trivial $G$-module, so $\epsilon(g[\cdot]) = g \epsilon([\cdot]) = g \cdot 1 = 1$, where the (trivial) $G$-action on $\mathbb{Z}$ is denoted by a ·.

The $X_n$ are finite free $G$-modules, with the set of all $n$-cells serving as a basis.

**Proposition 2** With these definitions, the sequence

$$
\cdots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,
$$

is a free $G$-resolution.

Sometimes this resolution is called the **bar resolution**[^1]. There are two other resolutions we shall consider. One is the closely related “homogeneous resolution” and the other is the “normalized bar resolution”.

This simple-looking proposition is not so simple to prove. First, we shall show it is a complex, i.e., $d^2 = 0$. Then, and this is the most non-trivial part of the proof, we show that the sequence is exact.

First, we need some definitions and a lemma.

Let $f : L \to M$ and $g : L \to M$ be +1-graded admissible maps. We say $f$ is **homotopic** to $g$ if there is a homomorphism $D : L \to M$, called a **homotopy**, such that

- $D_n = D|_{L_n} : L_n \to M_{n+1}$,
- $f - g = Dd + dD$.

If $L = M$ and the identity map $1 : L \to L$ is homotopic to the zero map $0 : L \to L$ then the homotopy is called a **contracting homotopy** for $L$.

**Lemma 3** If $L$ has a contracting homotopy then $H(L) = 0$.

**proof**: Represent $x \in H(L)$ by $\ell \in L$ with $d\ell = 0$. But $\ell = 1(\ell) - 0(\ell) = dD(\ell) + Dd(\ell) = dD(\ell)$. Since $D : L \to L$, this shows $\ell \in dL$, so $x = 0$ in $H(L)$. □

Next, we construct a contracting homotopy for the complex $X_*$ in Proposition with differential operator $d$. Actually, we shall **temporarily** let $X_{-1} = \mathbb{Z}$, $X_{-n} = 0$ and $d_{-n} = 0$ for $n > 1$, so that that the complex is infinite in both directions. We must define $D : X \to X$ such that

[^1]: This resolution is not the same as the resolution computed by HAP in Example[^2].

[^2]: Example
\[ D_{-1} = D|_Z : Z \to X_0, \]
\[ D_n = D|_{X_n} : X_n \to X_{n+1}, \]
\[ \epsilon D_{-1} = 1 \text{ on } Z, \]
\[ d_1 D_0 + D_{-1} \epsilon = 1 \text{ on } X_0, \]
\[ d_{n+1} D_n + D_{n-1} d_n = 1 \text{ in } X_n, \text{ for } n \geq 1. \]

Define

\[
\begin{align*}
D_{-n} &= 0, \quad n > 1, \\
D_{-1}(1) &= [.] , \\
D_0([g]) &= [g], \\
D_n([g_1, \ldots, g_n]) &= [g, g_1, \ldots, g_n], \quad n > 0,
\end{align*}
\]

and extend to a \( \mathbb{Z} \)-basis linearly.

Now we must verify the desired properties.

By definition, for \( m \in \mathbb{Z} \), \( \epsilon D_{-1}(m) = \epsilon(m[]) = m\epsilon([]) = m. \) Therefore, \( \epsilon D_{-1} \) is the identity map on \( \mathbb{Z}. \)

Similarly,

\[
(d_1 D_0 + D_{-1} \epsilon)([g]) = d_1([g]) + D_{-1}(1) \\
= [g] - [[]] + D_{-1}(1) = [g] - [[]] + [[]] = [g].
\]

For the last property, we compute

\[
d_{n+1} D_n([g_1, \ldots, g_n]) = d_{n+1}([g, g_1, \ldots, g_n]) \\
= g[g_1, \ldots, g_n] - [gg_1, \ldots, g_n] \\
+ \sum_{i=1}^{n-1} (-1)^{i-1}[g, g_1, \ldots, g_i, g_{i+2}, \ldots, g_n] \\
+ (-1)^n [g, g_1, \ldots, g_n],
\]

and
\[ D_{n-1} d_n (g[g_1, \ldots, g_n]) \]
\[ = D_{n-1} (g d_n ([g_1, \ldots, g_n])) \]
\[ = D_{n-1} (g g_1 [g_2, \ldots, g_n] \]
\[ + \sum_{i=1}^{n-1} (-1)^i [g_1, \ldots, g_i-1, g_i g_{i+1}, g_{i+2}, \ldots, g_n] \]
\[ + (-1)^n [g_1, \ldots, g_{n-1}] \]
\[ = [g g_1, g_2, \ldots, g_n] \]
\[ + \sum_{i=1}^{n-1} (-1)^i [g, g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n] \]
\[ + (-1)^n [g, g_1, \ldots, g_{n-1}]. \]

All the terms but one cancels, verifying that \(d_{n+1} D_n + D_{n-1} d_n = 1\) in \(X_n\), for \(n \geq 1\).

Now we show \(d^2 = 0\). One verifies \(d_1 d_2 = 0\) directly (which is left to the reader). Multiply \(d_k D_{k-1} + D_{k-2} d_{k-1} = 1\) on the right by \(d_k\) and \(d_{k+1} D_k + D_{k-1} d_k = 1\) on the left by \(d_k\):

\[ d_k D_{k-1} d_k + D_{k-2} d_{k-1} d_k = d_k = d_k d_{k+1} D_k + d_k D_{k-1} d_k. \]

Cancelling like terms, the induction hypothesis \(d_{k-1} d_k = 0\) implies \(d_k d_{k+1} = 0\). This shows \(d^2 = 0\) and hence that the sequence in Proposition 2 is exact. This completes the proof of Proposition 2. □

The above complex can be “dualized” in the sense of §3.1. This dualized complex is of the form

\[ 0 \to \mathbb{Z} \xrightarrow{\mu} X_{-1} \xrightarrow{d_{-1}} X_{-2} \xrightarrow{d_{-2}} X_{-3} \to \ldots . \]

The standard \(G\)-resolution is obtained by splicing these together.

### 3.2.2 Normalized bar resolution

Define the normalized cells by

\[ [g_1, \ldots, g_n]^* = \left\{ \begin{array}{ll}
[g_1, \ldots, g_n], & \text{if all } g_i \neq 1, \\
0, & \text{if some } g_i = 1.
\end{array} \right. \]
Let \( X_0 = \mathbb{R}[.] \) and
\[
X_n = \bigoplus_{(g_1, \ldots, g_n) \in G^n} \mathbb{R}[g_1, \ldots, g_n]^* , \quad n \geq 1,
\]
where the sum runs over all ordered \( n \)-tuples in \( G^n \). Define the differential operators \( d_n \) and the augmentation map exactly as for the bar resolution.

**Proposition 4** With these definitions, the sequence
\[
\cdots \to X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,
\]
is a free \( G \)-resolution.

Sometimes this resolution is called the normalized bar resolution.

**proof:** See Theorem 10.117 in [R]. \( \square \)

### 3.2.3 Homogeneous resolution

Let \( X_0 = \mathbb{R} \), so \( X_0 \) is a finite free (left) \( \mathbb{R} \)-module whose basis has only 1 element. For \( n > 0 \), let \( X_n \) denote the \( \mathbb{Z} \)-module generated by all \((n + 1)\)-tuples \((g_0, \ldots, g_n)\). Make \( X_i \) into a \( G \)-module by defining the action by
\[
g : X_n \to X_n \text{ by } g : (g_0, \ldots, g_n) \mapsto (gg_0, \ldots, gg_n), \quad g \in G.
\]

Define the differential operators \( \partial_n \) and the augmentation \( \epsilon \), as \( G \)-module maps, by
\[
\epsilon(g) = 1,
\]
\[
\partial_n(g_0, \ldots, g_n) = \sum_{i=0}^{n-1} (-1)^i (g_0, \ldots, \hat{g}_i, g_{i+1}, \ldots, g_n),
\]
for \( n \geq 1 \).

**Proposition 5** With these definitions, the sequence
\[
\cdots \to X_2 \xrightarrow{\partial_2} X_1 \xrightarrow{\partial_1} X_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0,
\]
is a \( G \)-resolution.
Sometimes this resolution is called the **homogeneous resolution**.
Of the three resolutions presented here, this one is the most straightforward to deal with.

**proof**: See Lemma 10.114, Proposition 10.115, and Proposition 10.116 in [R]. □

## 4 Definition of \( H^n(G, A) \)

For convenience, we briefly recall the definition of \( \text{Ext}^n \). Let \( A \) be a left \( R \)-module, where \( R = \mathbb{Z}[G] \), and let \((X_i)\) be a \( G \)-resolution of \( \mathbb{Z} \). We define

\[
\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Kernel}(d^*_{n+1})/\text{Image}(d^*_n),
\]

where

\[
d^*_n : \text{Hom}(X_{n-1}, A) \to \text{Hom}(X_n, A),
\]

is defined by sending \( f : X_{n-1} \to A \) to \( fd_n : X_n \to A \). It is known that this is, up to isomorphism, independent of the resolution chosen. Recall \( \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A) \) is the right-derived functors of the right-exact functor \( A \mapsto A^G = \text{Hom}_G(\mathbb{Z}, A) \) from the category of \( G \)-modules to the category of abelian groups. We define

\[
H^n(G, A) = \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A),
\]

When we wish to emphasize the dependence on the resolution chosen, we write \( H^n(G, A, X_\ast) \).

For example, let \( X_\ast \) denote the bar resolution in \( \$3.2.1 \) above. Call \( C^n = C^n(G, A) = \text{Hom}_G(X_n, A) \) the **group of \( n \)-cochains of \( G \) in \( A \)**, \( Z^n = Z^n(G, A) = C^n \cap \text{Kernel}(\partial) \) the group of **\( n \)-cocycles**, and \( B^n = B^n(G, A) = \partial(C^{n-1}) \) the group of **\( n \)-coboundaries**. We call \( H^n(G, A) = Z^n/B^n \) the **\( n \)-th cohomology group of \( G \) in \( A \)**. This is an abelian group.

We call also define the cohomology group using some other resolution, the normalized bar resolution or the homogeneous resolution for example. If we wish to express the dependence on the resolution \( X_\ast \) used, we write \( H^n(G, A, X_\ast) \). Later we shall see that, up to isomorphism, this abelian group is independent of the resolution.

The group \( H^2(G, \mathbb{Z}) \) (or \( H^2(G, \mathbb{C}) \)) is sometimes called the **Schur multiplier** of \( G \). Here \( \mathbb{C} \) denotes the field of complex numbers.
We say that the group $G$ has **cohomological dimension** $n$, written $\text{cd}(G) = n$, if $H^{n+1}(H, A) = 0$ for all $G$-modules $A$ and all subgroups $H$ of $G$, but $H^n(H, A) \neq 0$ for some such $A$ and $H$.

**Remark 2**

- If $\text{cd}(G) < \infty$ then $G$ is torsion-free.$^5$
- If $G$ is a free abelian group of finite rank then $\text{cd}(G) = \text{rank}(G)$.
- If $\text{cd}(G) = 1$ then $G$ is free. This is a result of Stallings and Swan (see for example [R], page 885).

### 4.1 Computations

We briefly discuss computer programs which compute cohomology and some examples of known computations.

#### 4.1.1 Computer computations of cohomology

GAP [Gap], MAGMA [MAGMA], and Macaulay 2 [Mac] can compute some cohomology groups.

For example, GAP will compute the Schur multiplier $H^2(G, \mathbb{C})$ using the `AbelianInvariantsMultiplier` command. To find $H^2(A_5, \mathbb{C})$, where $A_5$ is the alternating group on 5 letters, type

```gap
gap> A5:=AlternatingGroup(5);
Alt( [ 1 .. 5 ] )
gap> AbelianInvariantsMultiplier(A5);
[ 2 ]
```

So, $H^2(A_5, \mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$.

GAP will do other cohomology computations as well (see §37.22 of the GAP manual, D. Holt’s package cohom, M. Bishop’s package CRIME for cohomology of $p$-groups, or G. Ellis’ package HAP for group homology and cohomology of finite or (certain) infinite groups, all described at [Gap]).

**Example 6** This example uses MAGMA. Recall $GF(q)$ denotes the finite field with $q$ elements.

$^5$This follows from the fact that if $G$ is a cyclic group then $H^n(G, \mathbb{Z}) \neq 0$, discussed below.
\begin{verbatim}
> G:=CyclicGroup(5);
> M := TrivialModule(G, Integers());
> X := CohomologyModule(G, M);
> CohomologyGroup(X, 1);

Full Quotient RSpace of degree 0 over Integer Ring
Column moduli:
  [ ]
> CohomologyGroup(X, 2);

Full Quotient RSpace of degree 1 over Integer Ring
Column moduli:
  [ 5 ]
>
> G := Alt(5);
> M := TrivialModule(G, GF(7));
> X := CohomologyModule(G, M);
> CohomologyGroup(X, 1);

Full Vector space of degree 0 over GF(7)
> CohomologyGroup(X, 2);

Full Vector space of degree 0 over GF(7)

This implies $H^1(C_5, \mathbb{Z}) = H^1(A_5, GF(7)) = H^2(A_5, GF(7)) = 0$.
We also refer to the MAGMA manual [MAGMA] and to the paper [EK].
\end{verbatim}

**Example 7** This example uses SAGE, which wraps several of the HAP functions.

```
sage: G = AlternatingGroup(5)
sage: G.cohomology(1,7)
Trivial Abelian Group
```

This implies $H^1(A_5, GF(7)) = H^2(A_5, GF(7)) = 0$.

For Macaulay, we refer to the Macaulay 2 manual [Mac].

### 4.1.2 Examples

Some example computations.
1. $H^0(G, A) = A^G$. This is by definition.

2. Let $L/K$ denote a Galois extension with finite Galois group $G$. We have $H^1(G, L^\times) = 1$. This is often called Hilbert’s Theorem 90. See Theorem 1.5.4 in [W] or Proposition 2 in §X.1 of [S].

3. Let $G$ be a finite cyclic group and $A$ a trivial torsion-free $G$-module. Then $H^1(G, A) = 0$. This is a consequence of properties given in the next section.

4. If $G$ is a finite cyclic group of order $m$ and $A$ is a trivial $G$-module then

$$H^2(G, A) = A/mA$$

This is a consequence of properties given below. For example, $H^2(GF(q)\times, \mathbb{C}) = 0$.

5. If $|G| = m$, $rA = 0$ and $gcd(r, m) = 1$, then $H^n(G, A) = 0$, for all $n \geq 1$.

This is Corollary 3.1.7 in [W]. For example, $H^1(A_5, \mathbb{Z}/7\mathbb{Z}) = 0$.

5 Definition of $H_n(G, A)$

We say $A$ is projective if the functor $B \mapsto \text{Hom}_G(A, B)$ (from the category of $G$-modules to the category of abelian groups) is exact. Recall, if

$$P_Z = \cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

is a projective resolution of $\mathbb{Z}$ then

$$\text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Kernel} (d_n \otimes 1_A)/\text{Image} (d_{n+1} \otimes 1_A).$$

It is known that this is, up to isomorphism, independent of the resolution chosen. Recall $\text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A)$ are the right-derived functors of the right-exact functor $A \mapsto A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$ from the category of $G$-modules to the category of abelian groups. We define
When we wish to emphasize the dependence on the resolution, we write \( H_n(G, A, P_Z) \).

**Remark 3** If \( G \) is a \( p \)-group, then using the command `ProjectiveResolution` in GAP’s CRIME package, one can easily compute the minimal projective resolution of a \( G \)-module, which can be either trivial or given as a MeatAxe\(^6\) module.

Since we can identify the functor \( A \mapsto A_G \) with \( A \mapsto A \otimes_{\mathbb{Z}[G]} \mathbb{Z} \) (where \( \mathbb{Z} \) is considered as a trivial \( \mathbb{Z}[G] \)-module), the following is another way to formulate this definition.

If \( \mathbb{Z} \) is considered as a trivial \( \mathbb{Z}[G] \)-module, then a free \( \mathbb{Z}[G] \)-resolution of \( \mathbb{Z} \) is a sequence of \( \mathbb{Z}[G] \)-module homomorphisms

\[
... \rightarrow M_n \rightarrow M_{n-1} \rightarrow ... \rightarrow M_1 \rightarrow M_0
\]

satisfying:

- **(Freeness)** Each \( M_n \) is a free \( \mathbb{Z}[G] \)-module.
- **(Exactness)** The image of \( M_{n+1} \rightarrow M_n \) equals the kernel of \( M_n \rightarrow M_{n-1} \) for all \( n > 0 \).
- **(Augmentation)** The cokernel of \( M_1 \rightarrow M_0 \) is isomorphic to the trivial \( \mathbb{Z}[G] \)-module \( \mathbb{Z} \).

The maps \( M_n \rightarrow M_{n-1} \) are the boundary homomorphisms of the resolution. Setting \( TM_n \) equal to the abelian group \( M_n / G \) obtained from \( M_n \) by killing the \( G \)-action, we get an induced sequence of abelian group homomorphisms

\[
... \rightarrow TM_n \rightarrow TM_{n-1} \rightarrow ... \rightarrow TM_1 \rightarrow TM_0
\]

This sequence will generally not satisfy the above exactness condition, and one defines the integral homology of \( G \) to be

\[
H_n(G, \mathbb{Z}) = \text{Kernel} (TM_n \rightarrow TM_{n-1}) / \text{Image} (TM_{n+1} \rightarrow TM_n)
\]

for all \( n > 0 \).

\(^6\)Available from [http://www.math.rwth-aachen.de/~MTX/](http://www.math.rwth-aachen.de/~MTX/)
5.1 Computations

We briefly discuss computer programs which compute homology and some examples of known computations.

5.1.1 Computer computations of homology

Example 8 Here is an example using HAP.

```
gap> LoadPackage("hap");
true
gap> G := SymmetricGroup(5);
  Sym([ 1 .. 5 ]);  
  gap> GroupHomology(G,2);
  [ 2 ]
gap> GroupHomology(G,1);
  [ 2 ]
gap> GroupHomology(G,3);
  [ 2, 4, 3 ]
gap> GroupHomology(G,4);
  [ 2 ]
gap> GroupHomology(G,5);
  [ 2, 2, 2 ]
gap> GroupHomology(G,6);
  [ 2, 2 ]
gap> GroupHomology(G,7);
  [ 2, 2, 4, 3, 5 ]
```

The last one means that

\[ H_7(S_5, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}). \]

Using HAP, one can also compute the Poincaré series of \( G \) mod \( p \), i.e., a quotient of polynomials \( P(x)/Q(x) \) whose power series coefficient of \( x^k \) equals the rank of the \( \mathbb{Z}/p\mathbb{Z} \)-vector space \( H_k(G, \mathbb{Z}/p\mathbb{Z}) \), for \( k \geq 0 \).

```
gap> G:=Group([[1,2,3],[1,2]]);
Group([ [ 1, 2, 3 ], [ 1, 2 ] ])
```

\(^7\)The a fairly high degree of certainty - HAP does not guarantee that the output is correct.
The Poincaré series of $H_*(S_3, \mathbb{Z}/2\mathbb{Z})$ is $(1 - x)^{-1}$ and the Poincaré series of $H_*(S_4, \mathbb{Z}/2\mathbb{Z})$ is $(1 + x^2)(1 - x - x^3 + x^4)^{-1}$.

**Example 9** This example uses SAGE, which wraps several of the HAP functions.

```
sage: G = AlternatingGroup(5)
sage: G.homology(1)
Trivial Abelian Group
sage: G.homology(1,7)
Trivial Abelian Group
sage: G.homology(2,7)
Trivial Abelian Group
```

This implies $H_1(A_5, \mathbb{Z}) = H_1(A_5, GF(7)) = H_2(A_5, GF(7)) = 0$.

### 5.1.2 Examples

Some example computations.

1. If $A$ is a $G$-module then $Tor_0^Z[G](Z, A) = H_0(G, A) = A_G \cong A/DA$.

   **proof:** We need some lemmas.

   Let $\epsilon : Z[G] \to Z$ be the augmentation map. This is a ring homomorphism (but not a $G$-module homomorphism). Let $D = \text{Kernel}(\epsilon)$ denote its kernel, the **augmentation ideal**. This is a $G$-module.

   **Lemma 10** As an abelian group, $D$ is free abelian generated by $G - 1 = \{g - 1 \mid g \in G\}$. 

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We write this as \( D = \mathbb{Z}(G - 1) \).

**proof of lemma:** If \( d \in D \) then \( d = \sum_{g \in G} m_g g \), where \( m_g \in \mathbb{Z} \) and \( \sum_{g \in G} m_g = 0 \). Thus, \( d = \sum_{g \in G} m_g(g - 1) \), so \( D \subset \mathbb{Z}(G - 1) \). To show \( D \) is free: If \( \sum_{g \in G} m_g(g - 1) = 0 \) then \( \sum_{g \in G} m_g g - \sum_{g \in G} m_g = 0 \) in \( \mathbb{Z}[G] \). But \( \mathbb{Z}[G] \) is a free abelian group with basis \( G \), so \( m_g = 0 \) for all \( g \in G \). \( \square \)

**Lemma 11** \( \mathbb{Z} \otimes_{\mathbb{Z}[G]} A = A/DA \), where \( DA \) is generated by elements of the form \( ga - a \), \( g \in G \) and \( a \in A \).

Recall \( A_G \) denotes the largest quotient of \( A \) on which \( G \) acts trivially\(^8\).

**proof of lemma:** Consider the \( G \)-module map, \( A \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \), given by \( a \mapsto 1 \otimes a \). Since \( \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \) is a trivial \( G \)-module, it must factor through \( A_G \). The previous lemma implies \( A_G \cong A/DA \). (In fact, the quotient map \( q : A \to A_G \) satisfies \( q(ga - a) = 0 \) for all \( g \in G \) and \( a \in A \), so \( DA \subset \text{Kernel}(q) \). By maximality of \( A_G \), \( DA = \text{Kernel}(q) \). QED) So, we have maps \( A \to A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \). By the definition of tensor products, the map \( \mathbb{Z} \times A \to A_G \), \( 1 \times a \mapsto 1 \cdot aDA \), corresponds to a map \( \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \to A_G \) for which the composition \( A_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \to A_G \) is the identity. This forces \( A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \). \( \square \)

See also \# 11 in §6.

2. If \( G \) is a finite group then \( H_0(G, \mathbb{Z}) = \mathbb{Z} \).

This is a special case of the example above (taking \( A = \mathbb{Z} \), as a trivial \( G \)-module).

3. \( H_1(G, \mathbb{Z}) \cong G/[G, G] \), where \([G, G]\) is the commutator subgroup of \( G \).

This is Proposition 10.110 in \[R\], §10.7.

**proof:** First, we claim: \( D/D^2 \cong G/[G, G] \), where \( D \) is as in Lemma. To prove this, define \( \theta : G \to D/D^2 \) by \( g \mapsto (g - 1) + D^2 \). Since \( gh - 1 - (g - 1)(h - 1) = (g - 1)(h - 1) \), it follows that \( \theta(gh) = \theta(g)\theta(h) \), so \( \theta \) is a homomorphism. Since \( D/D^2 \) is abelian and \( G/[G, G] \) is the maximal abelian quotient of \( G \), we must have \( \text{Kernel}(\theta) \subset [G, G] \). Therefore, \( \theta \) factors through \( \theta' : G/[G, G] \to D/D^2 \), \( g[G, G] \mapsto (g - 1) + D^2 \). Now,\(^8\)Implicit in the words “largest quotient” is a universal property which we leave to the reader for formulate precisely.

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we construct an inverse. Define $\tau : D \to G/[G, G]$ by $g - 1 \mapsto g[G, G]$. Since $\tau(g - 1 + h - 1) = g[G, G] \cdot h[G, G] = gh[G, G]$, it is not hard to see that this is a homomorphism. We would be essentially done (with the construction of the inverse of $\theta'$, hence the proof of the claim) if we knew $D^2 \subset \text{Kernel} (\tau)$. (The inverse would be the composition of the quotient $D/D^2 \to D/\text{Kernel} (\tau)$ with the map induced from $\tau$, $D/\text{Kernel} (\tau) \to G/[G, G]$.) This follows from the fact that any $x \in D^2$ can be written as $x = \left(\sum g m_g (g - 1)\right) \left(\sum h m'_h (h - 1)\right) = \left(\sum g, h m_g m'_h (g - 1) (h - 1)\right)$, so $\tau(x) = \prod g, h (gh g^{-1} h^{-1})^{m_g m'_h} [G, G] = [G, G]$. QED (claim)

Next, we show $H_1(G, Z) \cong D/D^2$. From the short exact sequence $0 \to D \to Z[G] \to Z \to 0$, we obtain the long exact sequence of homology

$$\cdots \to H_1(G, D) \to H_1(G, Z[G]) \to H_1(G, Z) \xrightarrow{\partial} H_0(G, D) \xrightarrow{f} H_0(G, Z[G]) \xrightarrow{\epsilon_*} H_0(G, Z) \to 0.$$  \hspace{1cm} (11)

Since $Z[G]$ is a free $Z[G]$-module, $H_1(G, Z[G]) = 0$. Therefore $\partial$ is injective. By item # 1 above (i.e., $H_0(G, A) \cong A/DA \cong A_G$, we have $H_0(G, Z) \cong Z_G = Z$ and $H_0(G, Z[G]) \cong Z[G]/D \cong Z$. By $\text{(1)}$, $\epsilon_*$ is surjective. Combining the last two statements, we find $Z/\text{Kernel} (\epsilon_*) \cong Z$. This forces $\epsilon_*$ to be injective. This, and $\text{(1)}$, together imply $f$ must be 0. Since this forces $\partial$ to be an isomorphism, we are done. \hspace{1cm} $\square$

4. Let $G = F/R$ be a presentation of $G$, where $F$ is a free group and $R$ is a normal subgroup of relations. Hopf’s formula states: $H_2(G, Z) \cong (F \cap R)/[F, R]$, where $[F, R]$ is the commutator subgroup of $G$.

See $[\text{R}], \S 10.7$.

The group $H_2(G, Z)$ is sometimes called the Schur multiplier of $G$.

6 **Basic properties of $H^n(G, A)$, $H_n(G, A)$**

Let $R$ be a (possibly non-commutative) ring and $A$ be an $R$-module. We say $A$ is injective if the functor $B \mapsto \text{Hom}_G(B, A)$ (from the category of $G$-modules to the category of abelian groups) is exact. (Recall $A$ is projective
if the functor \( B \mapsto \text{Hom}_G(A, B) \) is exact.) We say \( A \) is **co-induced** if it has the form \( \text{Hom}_Z(R, B) \) for some abelian group \( B \). We say \( A \) is **relatively injective** if it is a direct factor of a co-induced \( R \)-module. We say \( A \) is **relatively projective** if

\[
\pi : \mathbb{Z}[G] \otimes_Z A \to A,
\]

\[
x \otimes a \mapsto xa,
\]

maps a direct factor of \( \mathbb{Z}[G] \otimes_Z A \) isomorphically onto \( A \). These are the \( G \)-modules \( A \) which are isomorphic to a direct factor of the induced module \( \mathbb{Z}[G] \otimes_Z A \). When \( G \) is finite, the notions of relatively injective and relatively projective coincide.

1. The definition of \( H^n(G, A) \) does not depend on the \( G \)-resolution \( X_* \) of \( \mathbb{Z} \) used.

2. If \( A \) is an projective \( \mathbb{Z}[G] \)-module then \( H^n(G, A) = 0 \), for all \( n \geq 1 \).

   This follows immediately from the definitions.

3. If \( A \) is an injective \( \mathbb{Z}[G] \)-module then \( H_n(G, A) = 0 \), for all \( n \geq 1 \).

   See also \([S]\), §VII.2.

4. If \( A \) is a relatively injective \( \mathbb{Z}[G] \)-module then \( H^n(G, A) = 0 \), for all \( n \geq 1 \).

   This is Proposition 1 in \([S]\), §VII.2.

5. If \( A \) is a relatively projective \( \mathbb{Z}[G] \)-module then \( H^n(G, A) = 0 \), for all \( n \geq 1 \).

   This is Proposition 2 in \([S]\), §VII.4.

6. If \( A = A' \oplus A'' \) then \( H^n(G, A) = H^n(G, A') \oplus H^n(G, A'') \), for all \( n \geq 0 \). More generally, if \( I \) is any indexing family and \( A = \oplus_{i \in I} A_i \), then \( H^n(G, A) = \oplus_{i \in I} H^n(G, A_i) \), for all \( n \geq 0 \).

   This follows from Proposition 10.81 in §10.6 of Rotman \([R]\).

---

\( ^9 \)These notions were introduced by Hochschild \([Ho]\).
7. If

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]

is an exact sequence of \( G \)-modules then we have a long exact sequence of cohomology \((1)\). See \([S], \S VII.2\), and properties of the \( \text{ext} \) functor \([R], \S 10.6\).

8. \( A \rightarrow H^n(G, A) \) is the higher right derived functor associated to \( A \rightarrow A^G = \text{Hom}_G(A, \mathbb{Z}) \) from the category of \( G \)-modules to the category of abelian groups.

This is by definition. See \([S], \S VII.2\), or \([R], \S 10.7\).

9. If

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]

is an exact sequence of \( G \)-modules then we have a long exact sequence of homology \((2)\). In the case of a finite group, see \([S], \S VIII.1\). In general, see \([S], \S VII.4\), and properties of the \( \text{Tor} \) functor in \([R], \S 10.6\).

10. \( A \rightarrow H_n(G, A) \) is the higher left derived functor associated to \( A \rightarrow A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \) on the category of \( G \)-modules.

This is by definition. See \([S], \S VII.4\), or \([R], \S 10.7\).

11. If \( G \) is a finite cyclic group then

\[
H_0(G, A) = A_G, \\
H_{2n-1}(G, A) = A^G / NA, \\
H_{2n}(G, A) = \text{Kernel (N)}/DA,
\]

for all \( n \geq 1 \).

To prove this, we need a lemma.

**Lemma 12** Let \( G = \langle g \rangle \) be acyclic group of order \( k \). Let \( M = g - 1 \) and \( N = 1 + g + g^2 + \ldots + g^{k-1} \). Then
\[ \cdots \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{M} \mathbb{Z}[G] \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{M} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0, \]

is a free \( G \)-resolution.

**Proof of Lemma:** It is clearly free. Since \( MN = NM = (g - 1)(1 + g + g^2 + \ldots + g^{k-1}) = g^k - 1 = 0 \), it is a complex. It remains to prove exactness. Since \( \text{Kernel}(\epsilon) = D = \text{Image}(M) \), by Lemma 10, this stage is exact.

To show \( \text{Kernel}(M) = \text{Image}(N) \), let \( x = \sum_{j=0}^{k-1} m_j g^j \in \text{Kernel}(M) \). Since \((g - 1)x = 0\), we must have \( m_0 = m_1 = \ldots = m_{k-1} \). This forces \( x = m_0 N \in \text{Image}(N) \). Thus \( \text{Kernel}(M) \subset \text{Image}(N) \). Clearly \( MN = 0 \) implies \( \text{Image}(N) \subset \text{Kernel}(M) \), so \( \text{Kernel}(M) = \text{Image}(N) \).

To show \( \text{Kernel}(N) = \text{Image}(M) \), let \( x = \sum_{j=0}^{k-1} m_j g^j \in \text{Kernel}(N) \). Since \( Nx = 0 \), we have \( 0 = \epsilon(Nx) = \epsilon(N)\epsilon(x) = k\epsilon(x) \), so \( \sum_{j=0}^{k-1} m_j = 0 \). Observe that

\[
x = m_0 \cdot 1 + m_1 g + m_2 g^2 + \ldots + m_{k-1} g^{k-1} \\
= (m_0 - m_0 g) + (m_0 + m_1) g + m_2 g^2 + \ldots + m_{k-1} g^{k-1} \\
= (m_0 - m_0 g) + (m_0 + m_1) g - (m_0 + m_1) g^2 \\
+ (m_0 + m_1 + m_2) g^2 - (m_0 + m_1 + m_2) g^3 + \ldots \\
+ (m_0 + \ldots + m_{k-1}) g^{k-1} = (m_0 + \ldots + m_{k-1}) g^k.
\]

where the last two terms are actually 0. This implies \( x = -M(m_0 + (m_0 + m_1) g + (m_0 + m_1 + m_2) g^2 + \ldots + (m_0 + \ldots + m_{k-1}) g^{k-1}) \in \text{Image}(M) \). Thus \( \text{Kernel}(N) \subset \text{Image}(M) \). Clearly \( NM = 0 \) implies \( \text{Image}(M) \subset \text{Kernel}(N) \), so \( \text{Kernel}(N) = \text{Image}(M) \).

This proves exactness at every stage. \( \square \)

Now we can prove the claimed property. By property 1 in §5.1.2, it suffices to assume \( n > 0 \). Tensor the complex in Lemma 12 on the right with \( A \):

\[
\cdots \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{N_*} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{M_*} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{N_*} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \xrightarrow{\epsilon} \mathbb{Z} \otimes \mathbb{Z}[G] A \to 0,
\]

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where the new maps are distinguished from the old maps by adding an asterisk. By definition, $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \cong A$, and by property 1 in §5.1.2, $\mathbb{Z} \otimes_{\mathbb{Z}[G]} A \cong A/DA$. The above sequence becomes

$$
\cdots \to A \xrightarrow{N} A \xrightarrow{M} A \xrightarrow{N} A \xrightarrow{M} A \xrightarrow{\xi} A/DA \to 0.
$$

This implies, by definition of Tor,

$$
Tor_{2n-1}^{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Kernel}(M_{\ast})/\text{Image}(N_{\ast}) = A^G/NA,
$$

and

$$
Tor_{2n}^{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Kernel}(N_{\ast})/\text{Image}(M_{\ast}) = A[N]/DA.
$$

See also [S], §VIII.4.1 and the Corollary in §VIII.4.

12. The group $H^2(G, A)$ classifies group extensions of $A$ by $G$.

This is Theorem 5.1.2 in [W]. See also §10.2 in [R].

13. If $G$ is a finite group of order $m = |G|$ then $mH^n(G, A) = 0$, for all $n \geq 1$.

This is Proposition 10.119 in [R].

14. If $G$ is a finite group and $A$ is a finitely-generated $G$-module then $H^n(G, A)$ is finite, for all $n \geq 1$.

This is Proposition 3.1.9 in [W] and Corollary 10.120 in [R].

15. The group $H^1(G, A)$ constructed using resolutions is the same as the group constructed using 1-cocycles. The group $H^2(G, A)$ constructed using resolutions is the same as the group constructed using 2-cocycles.

This is Corollary 10.118 in [R].

16. If $G$ is a finite cyclic group then

$$
H^0(G, A) = A^G,
$$

$$
H^{2n-1}(G, A) = \text{Kernel} N/DA,
$$

$$
H^{2n}(G, A) = A^G/NA,
$$

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for all \( n \geq 1 \). Here \( N : A \to A \) is the norm map \( Na = \sum_{g \in G} ga \) and \( DA \) is the augmentation ideal defined above (generated by elements of the form \( ga - a \)).

**proof:** The case \( n = 0 \): By definition, \( H^0(G, A) = \text{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Hom}_G(\mathbb{Z}, A) \). Define \( \tau : \text{Hom}_G(\mathbb{Z}, A) \to A^G \) by sending \( f \mapsto f(1) \). It is easy to see that this is well-defined and, in fact, injective. For each \( a \in A^G \), define \( f_a = f \in \text{Hom}_G(\mathbb{Z}, A) \) by \( f(m) = ma \). This shows \( \tau \) is surjective as well, so case \( n = 0 \) is proven.

Case \( n > 0 \): Applying the functor \( \text{Hom}_G(\ast, A) \) to the \( G \)-resolution in Lemma 12 to get

\[
\cdots \leftarrow \text{Hom}_G(\mathbb{Z}[G], A) \overset{N_*}{\leftarrow} \text{Hom}_G(\mathbb{Z}[G], A) \overset{M_*}{\leftarrow} \text{Hom}_G(\mathbb{Z}[G], A) \overset{\varepsilon_*}{\leftarrow} \text{Hom}_G(\mathbb{Z}, A) \leftarrow 0.
\]

It is known that \( \text{Hom}_G(\mathbb{Z}[G], A) \cong A \) (see Proposition 8.85 on page 583 of [R]). It follows that

\[
\cdots \leftarrow A \overset{N_*}{\leftarrow} A \overset{M_*}{\leftarrow} A \overset{\varepsilon_*}{\leftarrow} A^G \leftarrow 0.
\]

By definition of \( \text{Ext} \), for \( n > 0 \) we have

\[
\text{Ext}^{2n}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Kernel} (M_*)/\text{Image} (N_*) = A^G/NA,
\]

and

\[
\text{Ext}^{2n-1}_{\mathbb{Z}[G]}(\mathbb{Z}, A) = \text{Kernel} (N_*)/\text{Image} (M_*) = \text{Kernel} (N)/(g-1)A,
\]

where \( g \) is a generator of \( G \) as in Lemma 12. □

See also [S], §VIII.4.1 and the Corollary in §VIII.4.

17. If \( G \) is a finite cyclic group of order \( m \) and \( A \) is a trivial \( G \)-module then

\[
H^0(G, A) = A^G,
\]

\[
H^{2n-1}(G, A) \cong A[m],
\]

\[
H^{2n}(G, A) \cong A/mA,
\]

for all \( n \geq 1 \).

This is a consequence of the previous property.
7 Functorial properties

In this section, we investigate some of the ways in which $H^n(G, A)$ depends on $G$.

One way to construct all these in a common framework is to introduce the notion of a “homomorphism of pairs”. Let $G, H$ be groups. Let $A$ be a $G$-module and $B$ an $H$-module. If $\alpha : H \to G$ is a homomorphism of groups and $\beta : A \to B$ is a homomorphism of $H$-modules (using $\alpha$ to regard $B$ as an $H$-module) then we call $(\alpha, \beta)$ a homomorphism of pairs, written

$$(\alpha, \beta) : (G, A) \to (H, B).$$

Let $G \subset H$ be groups and $A$ an $H$-module (so, by restriction, a $G$-module). We say a map

$$f_{G,H} : H^n(G, A) \to H^n(H, A),$$

is transitive if $f_{G_2,G_3}f_{G_1,G_2} = f_{G_1,G_3}$, for all subgroups $G_1 \subset G_2 \subset G_3$.

Let $X_\ast$ be a $G$-resolution and $X_\ast'$ a $H$-resolution, each with a $-1$ grading. Associated to a homomorphism of groups $\alpha : H \to G$ is a sequence of $H$-homomorphisms

$$A_n : X'_n \to X_n,$$  

$n \geq 0$, such that $d_{n+1}A_{n+1} = A_n d'_{n+1}$ and $\epsilon A_0 = \epsilon'$.

**Theorem 13**

1. If $(\alpha, \beta) : (G, A) \to (G', A')$ and $(\alpha', \beta') : (G', A') \to (G'', A'')$ are homomorphisms of pairs then so is $(\alpha' \circ \alpha, \beta' \circ \beta) : (G, A) \to (G'', A'')$.

2. Suppose $(\alpha, \beta) : (G, A) \to (G', A')$ is homomorphism of pairs, $X_\ast$ is a $G$-resolution, and $X_\ast'$ is a $G'$-resolution (each infinite in both directions, with a $-1$ grading). Let $H^n(G, A, X_\ast)$ denote the derived groups associated to the differential groups $\text{Hom}_G(X_\ast, A)$ with $+1$ grading. There is a homomorphism

$$(\alpha, \beta)_{X_\ast,X'_\ast} : H^n(G, A, X_\ast) \to H^n(G', A', X'_\ast)$$

satisfying the following properties.

(a) If $G = G'$, $A = A'$, $X = X'$, $\alpha = 1$ and $\beta = 1$ then $(1, 1)_{X_\ast,X'_\ast} = 1.$
(b) If \((\alpha', \beta') : (G', A') \to (G'', A'')\) is homomorphism of pairs, \(X''_x\) is a \(G''\)-resolution then
\[
(\alpha' \circ \alpha, \beta' \circ \beta)_x = (\alpha', \beta')_x \circ (\alpha, \beta)_x.
\]

(c) If \((\alpha, \gamma) : (G, A) \to (G', A')\) is homomorphism of pairs then
\[
(\alpha, \beta + \gamma)_x = (\alpha, \beta)_x + (\alpha, \gamma)_x.
\]

Remark 4 For an analogous result for homology, see §§III.8 in Brown [B].

proof: We sketch the proof, following Weiss, [W], Theorem 2.1.8, pp 52-53.

(1): This is “obvious”.

(2): Let \((\alpha, \beta) : (G, A) \to (G', A')\) be a homomorphism of pairs. Using \([12]\), we have an associated chain map
\[
\alpha^* : \text{Hom}_G(X_*, A) \to \text{Hom}_{G'}(X'_*, A')
\]
of differential groups (Brown §III.8 in [B]). The homomorphism of cohomology groups induced by \(\alpha^*\) is denoted
\[
\alpha^*_{n, X, X'} : H^n(G, A, X_*) \to H^n(G', A', X'_*)
\]
Properties (a)-(c) follow from \([22]\) and the corresponding properties of \(\alpha^*\).

As the cohomology groups are independent of the resolution used, the map \((\alpha, \beta)_x : H^n(G, A, X_*) \to H^n(G', A', X'_*)\) is sometimes simply denoted by
\[
(\alpha, \beta)_x : H^n(G, A) \to H^n(G', A'). \tag{13}
\]

7.1 Restriction

Let \(X_\star = X_\star(G)\) denote the bar resolution.

If \(H\) is a subgroup of \(G\) then the cycles on \(G\), \(C^n(G, A) = \text{Hom}_G(X_n(G), A)\), can be restricted to \(H\): \(C^n(H, A) = \text{Hom}_H(X_n(H), A)\). The restriction map \(C^n(G, A) \to C^n(H, A)\) leads to a map of cohomology classes:
\[
\text{Res} : H^n(G, A) \to H^n(H, A).
\]

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In this case, the homomorphism of pairs is given by the inclusion map \( \alpha : H \to G \) and the identity map \( \beta : A \to A \). The map \( \text{Res} \) is the induced map defined by (13). By the properties of this induced map, we see that \( \text{Res}_{H,G} \) is transitive: if \( G \subset G' \subset G'' \) then

\[
\text{Res}_{G',G} \circ \text{Res}_{G'',G'} = \text{Res}_{G'',G}.
\]

A particularly nice feature of the restriction map is the following fact.

**Theorem 14** If \( G \) is a finite group and \( G_p \) is a \( p \)-Sylow subgroup and if \( H^n(G, A)_p \) is the \( p \)-primary component of \( H^n(G, A) \) then

(a) there is a canonical isomorphism \( H^n(G, A) \cong \oplus_p H^n(G, A)_p \), and

(b) \( \text{Res} : H^n(G, A) \to H^n(G_p, A) \) restricted to \( H^n(G, A)_p \) (identified with a subgroup of \( H^n(G, A) \) via (a)) is injective.

**proof:** See Weiss, [W], Theorem 3.1.15. \( \square \)

**Example 15** Homology is a functor. That is, for any \( n > 0 \) and group homomorphism \( f : G \to G' \) there is an induced homomorphism \( H_n(f) : H_n(G, \mathbb{Z}) \to H_n(G', \mathbb{Z}) \) satisfying

- \( H_n(gf) = H_n(g)H_n(f) \) for group homomorphisms \( f : G \to G' \) \( g : G' \to G'' \),

- \( H_n(f) \) is the identity homomorphism if \( f \) is the identity.

The following commands compute \( H_3(f) : H_3(P, \mathbb{Z}) \to H_3(S_5, \mathbb{Z}) \) for the inclusion \( f : P \hookrightarrow S_5 \) into the symmetric group \( S_5 \) of its Sylow 2-subgroup. They also show that the image of the induced homomorphism \( H_3(f) \) is precisely the Sylow 2-subgroup of \( H_3(S_5, \mathbb{Z}) \).

\[
\text{gap> } \text{S}_5:=\text{SymmetricGroup}(5);; \text{P:=SylowSubgroup(S}_5,2);;
\text{gap> } \text{f:=GroupHomomorphismByFunction(P,S}_5, x->x);;
\text{gap> } \text{R:=ResolutionFiniteGroup(P,4)};;
\text{gap> } \text{S:=ResolutionFiniteGroup(S}_5,4);;
\text{gap> } \text{ZP\_map:=EquivariantChainMap(R,S,f)};;
\text{gap> } \text{map:=TensorWithIntegers(ZP\_map)};;
\]

\( ^{10} \) There is an analog of the restriction for homology which also satisfies this transitive property (Proposition 9.5 in Brown [B]).
If $H$ is a subgroup of finite index in $G$ then there is an analogous restriction map in group homology (see for example Brown [B, §III.9]).

**Example 16** Let $m = dk$ be a product with $d > 1$, $k > 1$, and let $G = \mathbb{Z}/m\mathbb{Z}$, $H = \mathbb{Z}/d\mathbb{Z}$. We can and do identify $H$ with $(m/d)\mathbb{Z}/m\mathbb{Z}$. By §6, we have

$$H^{2n}(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z}, \quad H^{2n+1}(\mathbb{Z}/\ell\mathbb{Z}, \mathbb{Z}) = 0,$$

for $n > 0$. The map

$$\text{Res} : H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \to H^{2n}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z})$$

is simply the multiplication by $m/d$ map:

$$\text{Res} : \mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} (m/d)\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

### 7.2 Inflation

Let $X_\ast$ denote the bar resolution of $G$. Recall

$$X_n = \oplus_{(g_1, \ldots, g_n) \in G^n} R[g_1, \ldots, g_n],$$

where the sum runs over all ordered $n$-tuples in $G^n$. If $H$ is a subgroup of $G$, let $X^H_n$ denote the complex defined by

$$X^H_n = \oplus_{(g_1H, \ldots, g_nH) \in (G/H)^n} R[g_1H, \ldots, g_nH].$$

This is a resolution, and we have a chain map defined on $n$-cells by $[g_1, \ldots, g_n] \mapsto [g_1H, \ldots, g_nH]$.

Suppose that $H$ is a normal subgroup of $G$ and $A$ is a $G$-module. We may view $A^H$ as a $G/H$-module. In this case, the homomorphism of pairs is given by the quotient map $\alpha : G \to G/H$ and the inclusion map $\beta : A^H \to A$. The **inflation** map $\text{Inf}$ is the induced map defined by (13), denoted
The inflation-restriction sequence in dimension $n$ is

$$0 \rightarrow H^n(G/H, A^H) \xrightarrow{\text{Inf}} H^n(G, A) \xrightarrow{\text{Res}} H^n(H, A).$$

For a proof, see Weiss, [W], §3.4.

There an analog of this inflation-restriction sequence for homology.

Example 17  Let $H \subset G$ be as in Example 16 above. We have $G/H \cong \mathbb{Z} / (m/d) \mathbb{Z}$. The map

$$\text{Inf} : H^{2n}((\mathbb{Z}/m\mathbb{Z}) / (\mathbb{Z}/d\mathbb{Z}), \mathbb{Z}) \rightarrow H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$$

is simply the mod $m$ map:

$$\text{Inf} : \mathbb{Z} / (m/d) \mathbb{Z} \xrightarrow{\text{mod } m} \mathbb{Z} / m\mathbb{Z}.$$

Note that, by Example 16, the composition map

$$\text{Res} \circ \text{Inf} : H^{2n}((\mathbb{Z}/m\mathbb{Z}) / (\mathbb{Z}/d\mathbb{Z}), \mathbb{Z}) \rightarrow H^{2n}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z})$$

is the mod $m$ map followed by the multiplication by $m/d$ map. The kernel of $\mathbb{Z}/m\mathbb{Z} \xrightarrow{m/d} (m/d)\mathbb{Z} / m\mathbb{Z} \cong \mathbb{Z} / d\mathbb{Z}$ is $\cong (\mathbb{Z}/m\mathbb{Z}) / (\mathbb{Z}/d\mathbb{Z})$, so $\text{Res} \circ \text{Inf} = 0$ map, as the inflation-restriction sequence predicts.

7.3 Transfer

This map is also called corestriction.

Let $H$ be a subgroup of $G$ of finite index and let $G = \cup_{i=1}^{m} g_i H$, where $S = \{g_i \mid 1 \leq i \leq m\}$ denotes a complete set of coset representatives.

For any $G$-module $A$, define the trace $T = T_{G/H} : A \rightarrow A$ by

$$T(a) = \sum_{s \in S} s(a), \quad a \in A.$$

Properties of the trace:

1. $T$ is independent of the set $S$ of representatives choosen (so $T$ only depends on $G/H$),
2. $T$ is transitive: if $G \subset G' \subset G''$ then 
\[ T_{G'/G} \circ T_{G''/G'} = T_{G''/G}. \]

3. If $H = 1$ then $T : A \to A^G$ is a $G$-module homomorphism.

4. $T_{G/H}$ induces a homomorphism $T^*_{G/H} : \text{Hom}_H(A, B) \to \text{Hom}_G(A, B)$.

Let $X_*$ denote the bar resolution of $G$. Then $X_*$ is also a resolution of $\mathbb{Z}$ as $H$-modules. The trace map induces chain maps $T^*_{G/H,n} : \text{Hom}_H(X_n, A) \to \text{Hom}_G(X_n, A)$ (as in the proof of Theorem 13). This induces a map on cohomology denoted
\[ \text{Cor} : H^n(H, A) \to H^n(G, A). \]

This is the corestriction or transfer map on cohomology.

There an analog of this for homology (Brown [B], §II.9).

**Remark 5** According to Brown §III.9 in [B], this does not arise from a homomorphism of pairs.

This transfer map is also transitive. Moreover, it commutes with restriction and inflation (see Propositions 2.4.5 and 2.4.6 in Weiss [W] for the precise statements).

**Lemma 18** If $H$ has index $m$ in $G$ then 
\[ \text{Cor} \circ \text{Res} : H^n(G, A) \to H^n(G, A) \]

is the multiplication by $m$ map.

**proof:** See Corollary 2.4.9 on page 76, in §2.4 of Weiss [W]. □

There is an analog of this property for homology (see §§III.8-III.9 in Brown [B]).

**Example 19** Let $H \subset G$ be as in Example 16 above. The map
\[ \text{Cor} : H^{2n}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}) \to H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \]

is simply the multiplication by $m/d$ map:
\[ \text{Inf} : \mathbb{Z}/d\mathbb{Z} \xrightarrow{m/d} \mathbb{Z}/m\mathbb{Z}. \]

Note that, by Example 16, the composition map

\[ \text{Cor} \circ \text{Res} : H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \to H^{2n}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) \]

is the multiplication by \( m/d \) map followed by the mod \( m \) map. So, \( \text{Cor} \circ \text{Res} = m/d \), as the above lemma predicts.

**Example 20** The function `PrimePartDerivedFunctor(G,R,T,n)` uses this description of the kernel to compute the abelian invariants of \( H_n(G,\mathbb{Z})(p) \) starting from:

- the group \( G \),
- a \( \mathbb{Z}[P] \)-resolution \( R \) for a Sylow \( p \)-subgroup \( P \subset G \),
- the functor \( T = \text{TensorWithIntegers} \),
- the integer \( n > 0 \).

The following commands show that the Mathieu group \( M_{23} \) has third integral homology with 2-part \( H_3(M_{23},\mathbb{Z})(2) = 0 \).

```gap
gap> M_23:=MathieuGroup(23);;
gap> gens:=GeneratorsOfGroup(SylowSubgroup(M_23,2));;
gap> gensP:=[gens[4],gens[6],gens[2]];;
   #This list generates a Sylow 2-subgroup of M_23.
gap> R:=ResolutionFiniteGroup(gensP,4);;
gap> T:=TensorWithIntegers;;
gap> PrimePartDerivedFunctor(M_23,R,T,3);
[ ]
```

Similar commands can be used to show that \( H_n(M_{23},\mathbb{Z}) = 0 \), for \( n = 1, 2, 3, 4 \). The triviality of the first four integral homology groups of \( M_{23} \) was first proved in 2000 by J. Milgram. A group \( G \) is said to be **\( k \)-connected** if \( H_n(G,\mathbb{Z}) = 0 \), for \( n = 1, 2, \ldots, k \). No example of a 5-connected finite group is yet known!
7.4 Shapiro’s lemma

Let $H$ be a subgroup of $G$ and let $A$ be an $H$-module. Let

$$\text{Ind}_H^G(A) = \{ f : G \to A \mid f(hg) = h \cdot f(g), \; h \in H, \; g \in G \}.$$ 

This is the $G$-module $A$ induced from $H$ to $G$. The action of $G$ on $\text{Ind}_H^G(A)$ is by right-translation: $g : f(x) \mapsto f(xg)$.

In this case, the homomorphism of pairs is given by the inclusion map $\alpha : H \to G$ and the map $\beta : \text{Ind}_H^G(A) \to A$ defined by $\beta(f) = f(1)$. The map

$$(\alpha, \beta)_* : H^n(G, \text{Ind}_H^G(A)) \to H^n(H, A)$$

is the induced map defined by (13).

**Theorem 21** (Shapiro’s lemma) This map $(\alpha, \beta)_*$ is an isomorphism.

**proof:** See Weiss, [W], §3.7.14. □

There is an analog of Shapiro’s lemma for homology (Proposition 9.5 in Brown [B]).

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