A COMPLETE INVARIANT FOR CLOSED SURFACES IN THE THREE-SPHERE

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Abstract. In this paper we use diagrams in categories to construct a complete invariant, the fundamental tree, for closed surfaces in the (based) 3-sphere, which generalizes the knot group and its peripheral system. From the fundamental tree, we derive some computable invariants that are capable to distinguish inequivalent handlebody links with homeomorphic complements. To prove the completeness of the fundamental tree, we generalize the Kneser conjecture to 3-manifolds with boundary, a topic interesting in its own right.

1. Introduction

The knot group is one of the most influential and effective invariants of knots. It differentiates all prime knots up to mirror image [20], and furthermore, coupled with the peripheral system of the knot, it gives a complete invariant [19], [5]. This complete invariant has been recently generalized to connected closed surfaces in the oriented $S^3$ with a basepoint $\infty$ in [2], where it is shown that, given a connected closed embedded surface $\Sigma \subset S^3 \setminus \infty$, its ambient isotopy type is determined by the span of fundamental groups

$$\pi_1(E) \leftarrow \pi_1(\Sigma) \rightarrow \pi_1(F)$$

plus the intersection form on $H_1(\Sigma)$, the abelianization of $\pi_1(\Sigma)$, where the oriented 3-manifolds $E$ and $F$ are the closures of connected components of the complement $S^3 \setminus \Sigma$ with $F$ containing $\infty$ and the outward normal of $\Sigma$ pointing toward $F$.

The aim of the present paper is to further generalize this complete invariant to closed embedded surface $\Sigma$ in $S^3 \setminus \infty$, where $\Sigma$ is not necessarily connected. We denote such an embedding by the pair $S = (S^3, \Sigma)$. Two pairs $S = (S^3, \Sigma)$ and $S' = (S^3, \Sigma')$ are equivalent if $\Sigma$ and $\Sigma'$ are ambient isotopic by a basepoint-preserving ambient isotopy. Suppose $\Sigma$ consists of $n$ connected components $\Sigma_i$, $i = 1, \ldots, n$. Then the closure of connected components of the complement $S^3 \setminus \Sigma$ are $n + 1$ oriented connected 3-manifolds $F_j$, $j = 0, \ldots, n$. We call each $F_j$ a solid part of $S = (S^3, \Sigma)$ and assume by convention that $F_0$ contains $\infty$. For the sake of simplicity, we shall abbreviate connected components to components.

Each $\Sigma_i$ is the intersection of exactly two solid parts of $S$, so if we think of $F_j$ as a node and $\Sigma_i = F_j \cap F_k$ as an edge between the nodes representing $F_j$ and $F_k$, then we get a based tree $\Lambda_S$ with the base node representing $F_0$. Intuitively, the based tree $\Lambda_S$ indicates "how far" each solid part $F_j$ is from $\infty$, or more precisely, how many components of $\Sigma$ sit in between $F_j$ and $\infty$. Regarding $\Lambda_S$ as an unordered based 1-dimensional simplicial complex, we can consider its subdivision $sd \Lambda_S$, which comes with a natural partial order on its vertices, and hence can be considered as a based category, a category with a selected base object.

\begin{equation}
F_1 \xleftarrow{\Sigma_1} F_0 \xrightarrow{\Sigma_2} F_2 \xrightarrow{\Sigma_3} F_3
\end{equation}

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Taking into account the inclusions $\Sigma_i \hookrightarrow F_j$ and $\Sigma_i \hookrightarrow F_k$, where $\Sigma_i = F_j \cap F_k$, we can think of a pair $S = (S^3, \Sigma)$ as a sd $\Lambda$-diagram of oriented manifolds (1.1), namely a based functor $MT(S)$ from $sd \Lambda_S$ to $Mfd$, the category of oriented manifolds, which sends each node $\alpha$ in $\Lambda_S$ to a solid part $F_j$, the barycenter $\hat{\alpha}\beta$ of two nodes $\alpha$ and $\beta$ in $\Lambda_S$ to the intersection $\Sigma_i = F_j \cap F_k$, and each span $\alpha \rightarrow \hat{\alpha}\beta \rightarrow \beta$ to $F_j \leftarrow \Sigma_i \rightarrow F_k$, where $F_j$ and $F_k$ are images of $\alpha$ and $\beta$ under $MT(S)$, respectively. By convention we orient $\Sigma_i$ such that its normal points toward the side containing $\infty$.

It is not difficult to see that two pairs $S = (S^3, \Sigma)$ and $S' = (S^3, \Sigma')$ are equivalent if and only if their induced diagrams of manifolds are equivalent. By two induced diagrams of manifolds are equivalent we understand there is an equivalence of based categories
\[
\mathcal{E} : sd \Lambda_S \Rightarrow sd \Lambda_{S'}
\]
and a natural transformation $\Phi_M$ between $MT(S)$ and $MT(S') \circ \mathcal{E}$ such that $\Phi_M(\bullet)$ is an orientation-preserving (= o.p.) homeomorphism for each node $\bullet \in sd \Lambda_S$:

\[\begin{array}{ccc}
sd \Lambda_S & \xrightarrow{MT(S)} & Mfd \\
\mathcal{E} \downarrow & & \downarrow \Phi_M \\
sd \Lambda_{S'} & \xrightarrow{MT(S')} & Mfd
\end{array}\]

(1.2)

Applying the fundamental group functor to $MT(S)$ and $MT(S')$, we get two based diagrams of groups, denoted by
\[
FT^u(S) : sd \Lambda_S \Rightarrow \text{Grp}_f \\
FT^u(S') : sd \Lambda_{S'} \Rightarrow \text{Grp}_f,
\]
where $\text{Grp}_f$ is the category of finitely generated groups with homomorphisms modulo conjugation. The question thus arises as to whether or not an equivalence between the induced based diagrams of groups implies an equivalence between $S$ and $S'$. Due to the presence of chiral objects, such as trefoil knots, the question does not have an affirmative solution in general. However, if we integrate the orientation information of $S = (S^3, \Sigma)$ into the functor $FT^u(S)$, where the superscript $u$ stands for unoriented, then we get a complete invariant of $S$. More precisely, we consider a new functor (fundamental tree) $FT(S)$, which is the functor $FT^u(S)$ decorated with an intersection form on the abelianization of $FT^u(S)(\hat{\alpha}\beta)$, for each barycenter $\hat{\alpha}\beta$, namely the bi-linear map on homology groups
\[
I : H_1(\Sigma) \times H_1(\Sigma) \rightarrow \mathbb{Z},
\]
where $\Sigma$ is the component of $\Sigma$ corresponding to $\hat{\alpha}\beta$. The completeness of the fundamental tree $FT(S)$ is the main result of the paper.

**Theorem 1.1.** Two pairs $S, S'$ are equivalent if and only if $FT(S)$ and $FT(S')$ are equivalent in the sense that there exist a based equivalence of categories
\[
\mathcal{E} : sd \Lambda_S \rightarrow sd \Lambda_{S'}
\]
and a natural isomorphism
\[
\Phi : FT(S) \Rightarrow FT(S') \circ \mathcal{E}
\]
such that, for each barycenter $\hat{\alpha}\beta$ in $sd \Lambda_S$, the isomorphism on homology induced from $\Phi(\hat{\alpha}\beta)$ preserves intersection forms.
Theorem 1.1 implies a complete invariant for unbased pairs \((S^3, \Sigma)_u\), which are surfaces in an unbased \(S^3\). Two unbased pairs \((S^3, \Sigma)_u, (S^3, \Sigma')_u\) are equivalent if there exists an o.p. self-homeomorphism of \(S^3\) sending \(\Sigma\) to \(\Sigma'\).

Corollary 1.2. The unbased pairs \((S^3, \Sigma)_u\) and \((S^3, \Sigma')_u\) are equivalent if and only if there exist points \(* \in S^3 \setminus \Sigma\) and \(*' \in S^3 \setminus \Sigma'\) such that \(\mathcal{FT}(S)\) and \(\mathcal{FT}(S')\) are equivalent, where \(S = (S^3, \Sigma)\) and \(S' = (S^3, \Sigma')\) are “based” pairs induced from \(\ast\) and \(\ast'\), respectively.

There are inequivalent pairs which are equivalent as unbased pairs. For instance, the boundary of a toric shell \(T \subset \mathbb{R}^3 \subset S^3\) and the boundary of a tubular neighborhood of a Hopf link \(H \subset \mathbb{R}^3 \subset S^3\) are equivalent as embeddings in an unbased \(S^3\) but not in a based \(S^3\).

Another variant of \((S^3, \Sigma)\) is labeled pairs \((S^3, \Sigma)_l\), which are labeled surfaces \(\Sigma = \{\Sigma_1, \ldots, \Sigma_n\}\) in a based \(S^3\). Two labeled pairs \((S^3, \Sigma)_l, (S^3, \Sigma')_l\) are equivalent if there exists an o.p. homeomorphism \(f: S^3 \to S^3\) with \(f(\Sigma) = \Sigma'\) and \(f(\Sigma_i) = \Sigma'_i\), for every label \(i\). Theorem 1.1 entails the following complete invariant of \((S^3, \Sigma)_l\).

Corollary 1.3. The pairs \((S^3, \Sigma)_l\) and \((S^3, \Sigma')_l\) are equivalent if and only if there exist an equivalence \(\mathcal{E}: \text{sd} \Lambda_S \to \text{sd} \Lambda_{S'}\) and a natural isomorphism \(\Phi: \mathcal{FT}(S) \to \mathcal{FT}(S') \circ \mathcal{E}\) such that \(\mathcal{E}\) sends the node representing \(\Sigma_i\) to the node representing \(\Sigma'_i\), and for each barycenter \(\alpha \beta\) in \(\text{sd} \Lambda_S\), the isomorphism on homology induced from \(\Phi(\alpha \beta)\) preserves intersection forms.

The notion of labeled pairs comes in handy when we discuss surface links in Section 6, given a pair \(S = (S^3, \Sigma)\), the surface link associated to a node \(\alpha \in \Lambda_S\) is the pair \((S^3, \partial F_\alpha)\) given by keeping only components of \(\Sigma\) that are in the boundary of the solid part \(F_\alpha\) corresponding to \(\alpha\). The topology of a pair \((S^3, \Sigma)\) is determined by its associated surface links in the following sense.

Theorem 1.4. Two pairs \(S, S'\) are equivalent if and only if there exists an equivalence

\[ \mathcal{E}: \text{sd} \Lambda_S \to \text{sd} \Lambda_{S'} \]

such that surface links associated to \(\alpha\) and \(\mathcal{E}(\alpha)\) are equivalent as labeled pairs, for every \(\alpha \in \Lambda_S\), where we select an arbitrary labeling on \(S\) and let the labeling on \(S'\) be the induced labeling via \(\mathcal{E}\).

A crucial ingredient in the proof of Main Theorem 1.1 is a generalized Kneser conjecture as stated below (Lemma 4.4).

Lemma 1.5 (Generalized Kneser conjecture). Let \(M\) be a compact, connected 3-manifold, and suppose there exists an isomorphism

\[ \pi_1(M, *_M) \overset{\phi}{\to} A_1 * A_2, \]

with \(A_1 * A_2\) the free product of two groups \(A_1, A_2\), such that for any component \(\Sigma\) of \(\partial M\), the composition

\[ \pi_1(\Sigma) \to \pi_1(M) \overset{\phi}{\to} A_1 * A_2 \]

factors through either \(A_1 \to A_1 * A_2\) or \(A_2 \to A_1 * A_2\). Then there exists a connected sum decompositon

\[ M \simeq M_1 \# M_2 \]

such that \(\phi\) induces an isomorphism \(\pi_1(M_i) \simeq A_i\) and \(\Sigma \subset \partial M_i\) if \(i\) factors through \(A_i\).
Lemma 1.5 generalizes the Kneser conjecture [6], [7], as explained in Remark 4.2, the condition (1.3) is satisfied when $M$ is $\partial$-irreducible. The proof of the lemma employs Stallings’ binding ties [14], [6], [7]. For ease of presentation, we have simplified the result of the lemma; it can be shown that the connecting arc between the base points $\ast_{\Sigma}$ and $\ast_{M}$ respects the prime decomposition (see Lemma 4.4 and Remarks 4.1-4.2 for more details, and 4. below for why we need this).

Outline of Proof. The proof of Theorem 1.1 occupies entire Section 5, and contains details on how to choose connecting arcs between base points. The ideas behind the proof, outline below, are not complicated, however.

Recall that an equivalence between $\mathcal{FT}(S)$ and $\mathcal{FT}(S')$ consists of isomorphisms

$$
\phi_{\Sigma_i} : \pi_1(\Sigma_i) \to \pi_1(\Sigma'_i)
$$
$$
\phi_{F_j} : \pi_1(F_j) \to \pi_1(F'_j)
$$

induced by $\Phi$, for $i = 0, \ldots, n$ and $j = 1, \ldots, n$, such that the diagram

$$
\begin{array}{ccc}
\pi_1(F_i) & \xrightarrow{\phi_{F_j}} & \pi_1(F'_j) \\
\downarrow & & \downarrow \\
\pi_1(\Sigma_i) & \xrightarrow{\phi_{\Sigma_i}} & \pi_1(\Sigma'_i)
\end{array}
$$

(1.4)

commutes, up to conjugation, for every $\Sigma_i \subset \partial F_j$, where

$$
\pi_1(\Sigma_i) = \mathcal{FT}(S)(\alpha \beta), \quad \pi_1(\Sigma'_i) = \mathcal{FT}(S') \circ E(\alpha \beta),
$$
$$
\pi_1(F_j) = \mathcal{FT}(S)(\alpha), \quad \pi_1(F'_j) = \mathcal{FT}(S') \circ E(\alpha),
$$

for some node $\alpha, \beta \in \Lambda_S$. By the Dehn-Nielsen-Baer theorem, there exists an o.p. homeomorphism $f_{\Sigma} : \Sigma \to \Sigma'$ realizing each isomorphism $\phi_{\Sigma_i}$. The plan is then to extend $f_{\Sigma}$ over each solid part $F_j$ of $(S^3, \Sigma)$ to get an o.p. self-homeomorphism $f : S^3 \to S^3$ with $f(\Sigma) = \Sigma'$. In other words, it amounts to solving the problem:

**Problem 1.6.** Let $F$ and $F'$ be 3-submanifolds in $S^3$. Suppose there is an o.p. homeomorphism

$$
f_{\partial F} : \partial F \to \partial F'
$$

and an isomorphism $\phi_{F} : \pi_1(F) \to \pi_1(F')$ such that the diagram

$$
\begin{array}{ccc}
\pi_1(F) & \xrightarrow{\phi_{F}} & \pi_1(F') \\
\downarrow & & \downarrow \\
\pi_1(\Sigma) & \xrightarrow{f_{\partial F}} & \pi_1(\Sigma')
\end{array}
$$

(1.5)

commutes, up to conjugation, for every component $\Sigma$ of $\partial F$, where $\Sigma' = f_{\partial F}(\Sigma)$. Then there exists an o.p. homeomorphism $f_{F} : F \to F''$ extending $f_{\partial F}$.

One key tool for constructing $f_{F}$ is Waldhausen’s theory on Haken manifolds [19] (see Lemma 4.6). In Waldhausen’s theorems, 3-manifolds are required to be irreducible and $\partial$-irreducible, but in general $F$ is neither, so we need to decompose $F$. We start with the prime connected sum decomposition of $F$,

$$
F \simeq M_1 \# \ldots \# M_p,
$$

(1.6)
This decomposition induces a(n) (algebraic) factorization of the diagram of groups $\mathcal{FT}(S)$ at $\pi_1(F)$. Fig. 1.7 illustrates the case $p = 3$,

$$
\begin{array}{c}
\pi_1(F) \\
\pi_1(\Sigma_1) \quad \pi_1(\Sigma_2) \quad \pi_1(\Sigma_3) \quad \pi_1(M_1) \quad \pi_1(M_2) \quad \pi_1(M_3) \\
\pi_1(\Sigma_4) \quad \pi_1(\Sigma_5) \quad \pi_1(\Sigma_6) \quad \pi_1(\Sigma_7) \quad \pi_1(\Sigma_8) \\
\end{array}
$$

(1.7)

where $\Sigma_i$, $i = 1, \ldots, 4$, are components of $\Sigma$ with $\partial M_1 = \Sigma_1$, $\partial M_2 = \Sigma_2 \cup \Sigma_3$, and $\partial M_3 = \Sigma_4$.

Via isomorphisms $\phi_F$ and $\phi_{\Sigma} = f_{\partial F}$, the factorization of $\mathcal{FT}(S)$ induces an algebraic factorization of $\mathcal{FT}(S')$ at $\pi_1(F')$, and we want to show that this algebraic factorization can be realized topologically by the prime decomposition of $F'$. This is the crucial step leading us to the generalized Kneser conjecture (Lemma 1.5), which implies that the algebraic factorization of $\mathcal{FT}(S')$ is indeed induced from the prime decomposition of $F'$:

$$
F' \simeq M'_1 \# \ldots \# M'_p,
$$

(1.8)

and furthermore $\phi_F$ induces an isomorphism $\phi_{M_i}$ which fits into the commutative diagram

$$
\begin{array}{c}
\pi_1(M_i) \\
\phi_F \quad \phi_{\Sigma} \\
\pi_1(\Sigma) \\
\end{array}
$$

(1.9)

where $f(\Sigma) = \Sigma' \subset \partial M'_i$ and $\Sigma$ is a component of $\partial M_i$ (see Fig. 1.10 for the case $p = 2$).

This way, we reduce Problem 1.6 to the special case where $F$ is prime, an assumption we shall make from now on till the end of the sketch. Note that any 3-submanifold of $S^3$ is prime if and only if it is irreducible.

To apply Waldhausen’s theorems, we need to decompose $F$ into even simpler pieces; to this aim, we consider the $\partial$-prime decomposition of $F$,

$$
F \simeq E_1 \# \ldots \# E_m.
$$

(1.11)

Unlike the prime decomposition (1.6), decomposition (1.11) affects the boundary $\partial F$. For instance, if $\Sigma$ is a component of $\partial F$, then (1.11) induces a connected sum decomposition (not necessarily prime):

$$
\Sigma \simeq \Theta_1 \# \ldots \# \Theta_q.
$$

(1.12)
Together with the homeomorphism \( f_{\partial F} : \Sigma \to \Sigma' \) and Dehn’s lemma, the \( \partial \)-prime decomposition \((1.11)\) of \( F \) induces a decomposition of \( F' \),
\[
F' \simeq E'_1 \# \ldots \# E'_m;
\]
a priori, we do not know if \((1.13)\) is \( \partial \)-prime. At this stage, we extend \( f_{\partial F} : \Sigma \to \Sigma' \) over the separating disks in \((1.11)\) so that \( f_{\partial F} \) induces an isomorphism
\[
\pi_1(\Theta_i) \xrightarrow{\phi_{\partial F}} \pi_1(\Theta'_i),
\]
where \( \Theta_i \) is a factor in \((1.12)\) and \( \Theta'_i = f_{\partial F}(\Theta_i) \).

Employing the Kurosh subgroup theorem, we see that \( \phi_F \) respects the free product decompositions of \( \pi_1(F) \) and \( \pi_1(F') \) induced from \((1.11)\) and \((1.13)\), respectively, in the sense that \( \phi_F \) induces an isomorphism \( \phi_{E_j} \) that fits in the commutative diagram
\[
\begin{array}{ccc}
\pi_1(E_j) & \xrightarrow{\phi_{E_j}} & \pi_1(E'_j) \\
\pi_i(\Theta) & \xrightarrow{\phi} & \pi_i(\Theta')
\end{array}
\]
for every \( \Theta \subset \partial E_j \) and \( f_{\partial F}(\Theta) = \Theta' \subset \partial E'_j \) and every \( j \). Since \( E_j \) is a \( \partial \)-prime, prime 3-submanifold of \( S^3 \), it is either a solid torus or \( \partial \)-irreducible (Lemma \[4.2\]); therefore it has an indecomposable fundamental group by Kneser’s conjecture; on the other hand, any 3-manifold with indecomposable fundamental group is \( \partial \)-prime; this proves that \((1.13)\) is indeed \( \partial \)-prime.

In this way, we reduce Problem \[1.6\] further to finding an \( \alpha.p. \) homeomorphism \( f_{E_j} \) that extends \( f_{\partial E_j} \), where \( f_{\partial E_j} \) is the restriction of \( f_{\partial F} \) on \( \partial E_j \).

If \( E_j \) is \( \partial \)-irreducible, we apply a variant of Waldhausen’s Theorem \[19\] Theorem 7.1 (Lemma \[4.6\]) to \((1.14)\) to get an \( \alpha.p. \) homeomorphism \( f_{E_j} \) that extends \( f_{\partial E_j} \); otherwise \( E_j \) is a solid torus, and \( f_{E_j} \) is easy to construct in this case.

Gluing \( f_{E_j} \), \( j = 1, \ldots, m \), together along separating disks in \((1.11)\), we obtain a homeomorphism \( f_F \) that extends \( f_{\partial F} \), and hence solves Problem \[1.6\]. Repeat the same construction for each solid part \( F_i \), \( i = 0, \ldots, n \), of \( (S^3, \Sigma) \), to get an \( \alpha.p. \) homeomorphism \( f_{F_i} \), for each \( i \), and glue \( f_F \) together along \( \Sigma \). Then we get the required equivalence between \((S^3, \Sigma)\) and \((S^3, \Sigma')\).

**Base points and connecting arcs:** In the above outline, to make main ideas of the proof stand out, we have left out some subtle details about how to choose connecting arcs between the base point of \( F \) and base points of components of \( \partial F \). Having appropriate systems of connecting arcs and base points is vital for our proof; the issue consists of four closely related parts.

1. We need to verify that \( \mathcal{FT}(S) \) does not depend on the choice of connecting arcs. This is not difficult to see, since changing the connecting arc between base points of a component \( \Sigma \) of \( \partial F \) and \( F \) does not change the conjugate class of the induced homomorphism
\[
\pi_1(\Sigma) \to \pi_1(F).
\]
This is an easy but useful observation as it allows us to choose connecting arcs appropriate to different situations.

2. By properly choosing connecting arcs, we can modify
\[
\pi_1(\Sigma') \to \pi_1(F')
\]
in the diagram \((1.5)\) such that it commutes strictly. This is again a simple but essential step because, as we shall explain in \(4\), it ensures that connecting arcs in \((S^3, \Sigma)\) and \((S^3, \Sigma')\) are similar in kind.

3. The system of connecting arcs in \( S \) should respect decompositions \((1.6)\) and \((1.11)\).
**Definition 1.1.** A system of connecting arcs between base points of \( F \) and each component of \( \partial F \) is good with respect to (1.6) if there exist connecting arcs between base points of \( M_i \) and each component \( \Sigma \subset \partial M_i \) and between base points of \( M \) and \( M_i \) such that the diagram below commutes, for each \( i \).

\[
\begin{array}{ccc}
\pi_1(M) & \rightarrow & \pi_1(M) \\
\downarrow & & \downarrow \\
\pi_1(\Sigma) & \rightarrow & \gamma_*
\end{array}
\]

Note that \( \gamma_* \) is the homomorphism induced from the system of connecting arcs.

Such a good system of connecting arcs can be obtained by constructing first connecting arcs between a selected base point \( *_{M_i} \) of \( M_i \) and its boundary component and then connecting arcs between \( *_{M_i} \) and the base point \( *_F \) of \( F \). The situation with the \( \partial \)-prime decomposition (1.11), on the contrary, is more involved.

**Definition 1.2.** A system of connecting arcs between base points of \( F \) and each component of \( \partial F \) is good with respect to (1.11) if there exist connecting arcs between base points of \( E_j \) and each component \( \Theta_i \) of \( \partial E_j \), between base points of \( E_j \) and \( F \), and base points of \( \Theta_i \) and \( \Sigma \) such that the diagram below commutes, for every \( i,j \).

\[
\begin{array}{ccc}
\pi_1(E_j) & \rightarrow & \pi_1(F) \\
\downarrow & & \downarrow \\
\pi_1(\Theta_i) & \rightarrow & \pi_1(\Sigma)
\end{array}
\] (1.15)

Fig. 1.1 illustrates a system that is not good, where \( *_{i} \) are base points of components of \( \partial F \), and the shadowed region is the \( \partial \)-prime factor \( E_1 \). As we can see there exist no connecting arcs between base points of \( E_1 \) and \( \Theta_i \) such that the diagram (1.15) commutes for \( i = 1, 2 \) however we choose them. There do exist a good system, however, and a construction is given in Section 5.

Unlike in \( S \), we are not free to choose connecting arcs between base points in \( S' \); they are already chosen in 2. to make diagram (1.5) commutes. More specifically, we choose a good system of connecting arcs in \( S \) first and then use 2. to modify connecting arcs in \( S' \) to make the diagram (1.5) commute strictly. A priori, we do not know whether the system of connecting arcs in \( S' \) respects (1.8) and (1.13). This leads us to the forth point.

4. The strict commutativity of the diagram (1.5) and the system of connecting arcs in \( S \) being good imply that, up to homotopy, the system of connecting arcs in \( S' \) is also good with respect to (1.8) and (1.13). The case of prime decomposition is considered in Lemma 4.4, and the case of \( \partial \)-prime decomposition is proved in Section 5. This assertion is crucial for the argument as it allows us to safely decompose solid parts of (\( S^3, \Sigma \)) and (\( S^3, \Sigma' \)) into simpler pieces (1.9), (1.14) till we reach the situation where Waldhausen’s theory (Lemma 4.6) applies.

**Structure.** The paper is organized as follows: Basic definitions and convention are reviewed in Section 2. Section 3 discusses the depth tree and the graft decomposition of a pair (\( S^3, \Sigma \)); the graft decomposition provides a convenient way to
decompose \((S^3, \Sigma)\) into non-splittable pairs and make the presentation neater. By a non-splittable pair \((S^3, \Sigma)\) we understand a pair \((S^3, \Sigma)\) whose solid parts are prime 3-manifolds. In Section 3, we summarize 3-manifold topology needed in the proof of Theorem 1.1; there the generalized Kneser conjecture is proved in details. The proof of Theorem 1.1 occupies Section 5. Section 6 explains how a pair \((S^3, \Sigma)\) can be decomposed in terms of surface links, a generalization of handlebody links; there computable invariants derived from the fundamental tree \(\mathcal{F}(\Sigma)\) are also introduced and used to distinguish inequivalent handlebody links with homeomorphic complements.

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\section{Preliminaries}

Throughout the paper, we work in the PL category; manifolds and maps are assumed to be piecewise linear [11]. Unless otherwise specified, \(S^3\) denotes a 3-sphere with a base point \(\infty\), and 3-manifolds are connected and compact.

\begin{definition}[Embedding] We use the symbol \(\Sigma = (S^3, \Sigma)\) or simply the pair \((S^3, \Sigma)\) to denote an embedding of a closed, not necessarily connected, surface \(\Sigma\) in \(S^3 \setminus \infty\).
\end{definition}

\begin{definition}[Equivalence] Two pairs \((S^3, \Sigma), (S^3, \Sigma')\) are equivalent if there exists an ambient isotopy \(F_t : S^3 \to S^3\) fixing \(\infty\) such that \(F_1(\Sigma) = \Sigma'\). The set of equivalence classes of \((S^3, \Sigma)\) is denoted by \(\text{Sur}\).
\end{definition}

Two pairs \((S^3, \Sigma), (S^3, \Sigma')\) are equivalent if and only if there exists an o.p. self-homeomorphism of \(S^3\) preserving \(\infty\) and sending \(\Sigma\) to \(\Sigma'\) [11].

\begin{definition}[Components] Connected components of a space \(X\) are abbreviated to components of \(X\); non-bold letters \(X\) are reserved for components of \(X\). For instance, given a pair \((S^3, \Sigma)\), \(\Sigma\) denotes a component of \(\Sigma\).
\end{definition}

\begin{definition}[Solid parts] Given a pair \((S^3, \Sigma)\), if \(\Sigma\) has \(n\) components \(\Sigma_i, i = 1, \ldots, n\), then the complement \(S^3 \setminus \Sigma\) consists of \(n + 1\) components. The closures of these components are denoted by \(F_j, j = 0, \ldots, n\). By convention we assume \(\infty \in F_0\), and call \(F_j\) a solid part of \((S^3, \Sigma)\).
\end{definition}

\begin{lemma} Every two solid parts of \((S^3, \Sigma)\) intersect at no more than one component of \(\Sigma\), and every component of \(\Sigma\) is the intersection of exactly two solid parts of \((S^3, \Sigma)\).
\end{lemma}

\begin{proof} It follows from the fact that every connected closed surface divides \(S^3\) into two connected components. \qed
\end{proof}

Thinking of \(F_j, j = 0, \ldots, n\), as nodes and \(\Sigma_i = F_j \cap F_k\) as edges connecting nodes representing \(F_j\) and \(F_k\), we obtain a based graph with the base node being the node representing \(F_0\). There can be no loops in the based graph. If there is a loop, then we select a node in the loop and remove components of \(\Sigma\) representing edges of the loop that are not adjacent to the selected node. This way, we get a new pair \((S^3, \Sigma')\) which has two solid parts intersecting at more than one component of \(\Sigma'\) and hence a contradiction, so the based graph is in fact a based tree.
Definition 2.5 (Depth tree). The based tree constructed above is called the depth tree of $S = (S^3, \Sigma)$ and denoted by $\Lambda_S$. A solid part $F_j$ has depth $k \in \mathbb{N}$ if the node in $\Lambda_S$ representing $F_j$ is connected to $F_0$ by $k$ edges. In particular, $F_0$ has depth 0.

Definition 2.6 (Barycentric diagram). Given a (based) finite graph $\Gamma$, the associated (based) barycentric diagram $sd\Gamma$ is a (based) diagram obtained by replacing each edge $j \rightarrow k$ by the span $j \leftarrow \hat{jk} \rightarrow k$.

Alternatively, $sd\Gamma$ can be viewed as the barycentric subdivision of $\Gamma$, where $\hat{jk}$ is the barycenter of the edge $j \rightarrow k$. On the other hand, as a small diagram, $sd\Gamma$ can also be considered as a small category.

Definition 2.7 (Equivalence). Two (based) finite graphs $\Gamma$ and $\Gamma'$ are equivalent if there exists an equivalence of (based) categories $E: sd\Gamma \rightarrow sd\Gamma'$.

The above definition is equivalent to saying $\Gamma$ and $\Gamma'$ are isomorphic as based graphs. In our setting, it is more convenient to use the categorical definition, which allows us to translate the geometric description of $(S^3, \Sigma)$ into a more categorical one. Note that equivalent pairs have equivalent depth trees. The notion of depth tree comes in handy when we discuss the graft decomposition of $(S^3, \Sigma)$ in Section 3.

Definition 2.8 (Barycentric diagram in a category). Given a (based) finite graph $\Gamma$ and a category $C$, a (based) $sd\Gamma$-diagram in $C$ (or a (based) barycentric diagram in $C$ of type $sd\Gamma$) is a functor $\mathcal{F}$ from $sd\Gamma$ to $C$.

Definition 2.9 (Equivalence). Let $\mathcal{F}$ be a (based) $sd\Gamma$-diagram in $C$ and $\mathcal{F}'$ a (based) $sd\Gamma'$-diagram in $C$. Then they are equivalent if there exists an equivalence of (based) categories $E: sd\Gamma \rightarrow sd\Gamma'$ and a natural isomorphism $\Phi : \mathcal{F} \Rightarrow \mathcal{F}' \circ E$.

The set of equivalence class of barycentric diagrams in $C$ is denoted by $C_{BD}$.

One main example is the $sd\Lambda_S$-diagram $MT(\bullet)$ in the category of oriented compact manifolds $Mfd$ associated to a pair $S = (S^3, \Sigma)$; as explained in Introduction, we have the following lemma.

Lemma 2.2. The functor $MT(\bullet)$ induces an injective mapping $MT : Sur \rightarrow Mfd_{BD}$.

Composing $MT$ with the fundamental group functor $\pi_1(\bullet)$ gives a $sd\Lambda_S$-diagram $FT^\pi(\bullet)$ in the category of finitely-generated groups $Grp_f$ with homomorphisms modulo conjugation. The orientation information gets lost during the passage, and the induced mapping $FT^\pi : Sur \rightarrow Grp^f_{BD}$ is no longer injective.

3. Grafting decomposition

3.1. Geometric graft decomposition.

Definition 3.1 (Separating/nonseparating sphere). Given a pair $(S^3, \Sigma)$, a 2-sphere $S$ in $S^3$ with $S \cap \Sigma = \emptyset$ is a separating sphere of $(S^3, \Sigma)$ if both connected components of $S^3 \setminus S$ have non-empty intersection with $\Sigma$; otherwise, $S$ is a non-separating sphere of $(S^3, \Sigma)$. 
Without loss of generality, we may assume that $\mathcal{S}$ does not contain the base point $\infty$. If $\mathcal{S}$ is non-separating, then there is a 3-ball $B$ in $S^3$ bounded by $\mathcal{S}$ such that $B \cap \Sigma = \emptyset$, and if $\mathcal{S}$ is in the solid part $F$ of $(S^3, \Sigma)$, then $B$ must be in $F$ as well; up to ambient isotopy, we may assume $\infty \notin B$.

**Definition 3.2 (Splittable/non-splittable pair).** A pair $(S^3, \Sigma)$ is splittable if it admits a separating sphere; otherwise, it is non-splittable.

If $\Sigma$ contains only one component, then $(S^3, \Sigma)$ is non-splittable; the converse is not true in general.

Given two pairs $(S^3, \Sigma)$ and $(S^3, \Sigma')$, we can construct a new pair by the following gluing operation: Select two non-separating spheres $\mathcal{S} \subset F$ and $\mathcal{S}' \subset F'$, where $F$ (resp. $F'$) is a solid part of $(S^3, \Sigma)$ (resp. $(S^3, \Sigma')$). Then remove the 3-balls bounded by $\mathcal{S}$ and $\mathcal{S}'$ that contain no components of $\Sigma$ and $\Sigma'$, respectively, and glue $\mathcal{S}$ and $\mathcal{S}'$ together via an orientation-reversing homeomorphism. The resulting new pair $(S^3, \Sigma \cup \Sigma')$ is always splittable.

**Corollary 3.1 (Independence).** The above gluing operation does not depend on the choice of non-separating spheres in solid parts $F$ and $F'$.

**Proof.** Suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ are two non-separating spheres of $(S^3, \Sigma)$ in $F$. By the innermost circle argument, we may assume $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. Let $B_1$ and $B_2$ be 3-balls in $F$ bounded by $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Then either $B_1 \cap B_2 = \emptyset$ or one 3-ball contains the other. If $B_1 \cap B_2 = \emptyset$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are isotopic in $F$. If either $B_1 \subset B_2$ or $B_2 \subset B_1$, then the annulus theorem implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ are isotopic in $F$. The same arguments apply to non-separating spheres of $(S^3, \Sigma')$ in $F'$. The corollary then follows from the fact that gluing along isotopic non-separating 2-spheres results in equivalent pairs. $\square$

In this paper we shall focus mainly on a special case of the gluing operation.

**Definition 3.3 (Grafting).** By grafting a pair $S' = (S^3, \Sigma')$ onto another pair $\mathcal{S} = (S^3, \Sigma)$ at a solid part $F_i$ of $(S^3, \Sigma)$ we understand performing the gluing operation between the solid part $F_0$ of $(S^3, \Sigma')$ which contains the base point, and the solid part $F_i$ of $(S^3, \Sigma)$. The resulting pair is denoted by $(S^3, \Sigma) \leftarrow F_i (S^3, \Sigma')$ or simply $\mathcal{S} \leftarrow F_i \mathcal{S}'$.

A pair $\mathcal{S} = (S^3, \Sigma)$ is said to be obtained by performing grafting operations finitely many times if

$$
\mathcal{S} \simeq \mathcal{S}_1 \leftarrow F_{i_1}^{(1)} \mathcal{S}_2 \leftarrow \cdots \leftarrow \mathcal{S}_k, \tag{3.1}
$$

where $F_{i_j}^{(j)}$ is a solid part of the pair $\mathcal{S}_j = (S^3, \Sigma_j)$, $j = 1, \ldots, k-1$. (3.1) is called a graft decomposition of $(S^3, \Sigma)$, and $k$ is the length of the graft decomposition. If $(S^3, \Sigma_j)$ is non-splittable, $j = 1, \ldots, k$, then (3.1) is a non-splittable graft decomposition of $(S^3, \Sigma)$.

We drop $F_{i_j}^{(j)}$ from the notation when there is no need to specify the solid parts.

**Proposition 3.2 (Non-splittable graft decomposition).** Every pair $\mathcal{S}$ admits a non-splittable graft decomposition. Furthermore, suppose

$$
S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_m \quad \text{and} \quad \mathcal{S}_1 \leftarrow S_1' \leftarrow \cdots \leftarrow S_p'
$$

are two non-splittable graft decompositions of $\mathcal{S}$. Then $m = p$, and after reindexing if necessary, $S_i$ and $S_i'$ are equivalent, for every $i$. 

Proof. First we index solid parts $F_j$, $j = 0, \ldots, n$, of $\mathcal{S} = (S^3, \Sigma)$ in such a way that, if $j \geq i$, then $F_j$ has the same depth as $F_i$ or greater. We prove the existence and uniqueness by induction on $n$.

**Existence:** Consider the set

$$\{j \mid F_j \text{ is reducible} \}.$$  \hspace{1cm} (3.3)

If the set (3.3) is empty, for instance, when $n = 0$ and $n = 1$, then $\mathcal{S}$ is non-splittable, and there is nothing to prove.

Suppose (3.3) is non-empty. Then let $k$ be the maximum of the set (3.3). Since $F_k$ is reducible, there exists a 3-ball $B$ in $S^3 \setminus \infty$ with $\partial B \subset F_k$, $\partial B \cap \partial F_k = \emptyset$, and $B \cap F_k$ a prime 3-manifold, where $B \cap F_k$ is obtained by capping off the spherical component $\partial B$ of $\partial(B \cap F_k)$ with a 3-ball.

Ignoring components of $\Sigma$ that are in $B$, we obtain a new pair $\tilde{\mathcal{S}}$ with less solid parts. By induction $\tilde{\mathcal{S}}$ admits a non-splittable graft decomposition:

$$\tilde{\mathcal{S}} \simeq S_1 \leftarrow \cdots \leftarrow S_q.$$  

On the other hand, considering only components of $\Sigma$ that are in $B$, we get another pair $\mathcal{S} = (S^3, \Sigma)$, which is non-splittable. Since $\mathcal{S}$ can be obtained by grafting $\mathcal{S}$ onto $\tilde{\mathcal{S}}$ at the solid part $F_k \cup B$ of $\tilde{\mathcal{S}}$, we have the non-splittable graft decomposition of $\mathcal{S}$:

$$\mathcal{S} \simeq S_1 \leftarrow \cdots \leftarrow S_q \leftarrow \mathcal{S}.$$  

**Uniqueness:** Let $k$, $\tilde{\mathcal{S}}$, and $\mathcal{S}$ be as above. Then we observe that $\mathcal{S}$ can be identified with one of $S_i$, $i = 1 \ldots m$ (resp. $S'_i$, $i = 1 \ldots p$) in (3.2). Up to reindexing, we may assume

$$\mathcal{S} \simeq S_m \simeq S'_p,$$

and thus, (3.2) induces two non-splittable graft decompositions of $\tilde{\mathcal{S}}$:

$$S_1 \leftarrow \cdots \leftarrow S_{m-1} \text{ and } S'_1 \leftarrow \cdots \leftarrow S'_{p-1}.$$  

$\tilde{\mathcal{S}}$ has fewer solid parts than $\mathcal{S}$, so by the induction hypothesis, $m - 1 = p - 1$ and $S_j \simeq S'_j$, after reindexing if necessary, for $j = 1 \ldots m - 1$. The proof is complete. $\square$

**A detour: edge-labeled trees and trivial pairs.** A (based) edge-labeled tree is a (based) tree with a non-negative integer assigned to each edge. Given a pair $\mathcal{S} = (S^3, \Sigma)$, we label each edge of the depth tree $\Lambda_\mathcal{S}$ by assigning the genus of the component of $\Sigma$ the edge represents to the edge. The resulting tree is called edge-labeled depth tree and denoted by $\Lambda^*_\mathcal{S}$.

**Definition 3.4 (Isomorphism of marked trees).** Two (based) edge-labeled trees $\Lambda^*_1$ and $\Lambda^*_2$ are isomorphic if there exists an isomorphism of (based) trees between $\Lambda^*_1$ and $\Lambda^*_2$ such that the labels of corresponding edges are identical.

**Definition 3.5.** A pair $\mathcal{S}$ has the type of a based edge-labeled tree $\Lambda^*$ if its edge-labeled depth tree $\Lambda^*_\mathcal{S}$ is isomorphic to $\Lambda^*$.

The simplest (based) non-degenerate tree is a (based) 1-simplex; a based 1-simplex with a label $g$ is denoted by $\Lambda^0_g$. A pair $(S^3, \Sigma)$ of type $\Lambda^0_g$ is a connected surface $\Sigma = \Sigma$ embedded in $S^3$.

**Definition 3.6 (Trivial pair).** A pair $\mathcal{S} = (S^3, \Sigma)$ of type $\Lambda^0_g$ is trivial if $\Sigma = \Sigma$ is trivially embedded in $S^3$. A pair $\mathcal{S} = (S^3, \Sigma)$ of type $\Lambda^*$ is trivial if each $\Sigma_i = (S^3, \Sigma_i)$ in the non-splittable decomposition of $\mathcal{S}$

$$\mathcal{S} \simeq S_1 \leftarrow \cdots \leftarrow S_n,$$

is a trivial pair of type $\Lambda^0_{g_i}$, where $g_i$ is the genus of $\Sigma_i$. 
Corollary 3.3 (Equivalence of trivial pairs). Any two trivial pairs $S$ and $S'$ of type $\Lambda^*$ are equivalent.

Proof. We shall prove a stronger statement: If $E$ (resp. $E'$) is the isomorphism between $\Lambda^*_f$ and $\Lambda^*$ (resp. $\Lambda^*_g$ and $\Lambda^*$), then the equivalence between $S$ and $S'$ can be chosen to respect $E$ and $E'$.

We prove it by induction on the number of the nodes in $\Lambda^*$. If $\Lambda^*$ has only one node, then $\Sigma$ is empty and the assertion holds trivially.

Suppose the statement is true for any based edge-labeled tree with less than $m > 1$ nodes and $\Lambda^*$ has $m$ nodes. Then there exists a component $\Sigma_i = F_i \cap F_k$ of $\Sigma$ such that the solid part $F_k$ does not contain the base point and any other components of $\Sigma$. Via the equivalence $E$, we may assume $\tilde{m}$, $j$, and $k$ are corresponding edge and nodes in $\Lambda^*$. Also via $E'$, we let $\Sigma'_i$, $F'_j$ and $F'_k$ are the component of $\Sigma'$ and solid parts of $(\tilde{S}', \Sigma')$ corresponding to $\tilde{m}$, $j$, and $k$ in $\Lambda^*$. Now, if

\[ S \simeq S_1 \circ S_2 \circ \cdots \circ S_m \quad \text{and} \quad S' \simeq S'_1 \circ S'_2 \circ \cdots \circ S'_m \]

are the non-splitable graft decompositions of $S$ and $S'$, then after reindexing if necessary, we have $S_m = (S^0, \Sigma_i)$ and $S_m' = (S'^0, \Sigma_i')$. In other words, $S$ and $S'$ can be obtained by grafting $S_m$ and $S_m'$ onto

\[ \tilde{S} := S_1 \circ S_2 \circ \cdots \circ S_m \quad \text{and} \quad \tilde{S}' := S'_1 \circ S'_2 \circ \cdots \circ S'_m \]

at the solid parts of $\tilde{S}$ and $\tilde{S}'$ containing $F_j \cup F_k$ and $F'_j \cup F'_k$, respectively. Since $S_m$ and $S_m'$ are trivial pairs of type $\Lambda^*_f$, $S_m$ and $S_m'$ are equivalent by [18]. On the other hand, $\tilde{S}$ and $\tilde{S}'$ is of type $\Lambda^*$, which is $\Lambda^*$ with the node $k$ removed, and the isomorphism $E$ (resp. $E'$) induces an isomorphism between $\Lambda^*_g$ and $\Lambda^*$ (resp. $\Lambda^*_g'$ and $\Lambda^*$). By induction, there is an equivalence between $\tilde{S}$ and $\tilde{S}'$ sending the solid part containing $F_j \cup F_k$ to the solid part containing $F'_j \cup F'_k$. Gluing this equivalence and the equivalence between $S_m$ and $S'_m$ together, we get an equivalence between $S$ and $S'$.

\[ \square \]

3.2. Algebraic graft decomposition. In Definition 2.8 we introduce barycentric diagrams in a category $C$; in this subsection, we shall focus on the case $C = \text{Grp}_f$.

Definition 3.7 (Barycentric diagrams in $\text{Grp}_f$ with pairing). A (based) $\text{sl} \Gamma$-diagram in $\text{Grp}_f$ with pairing is a (based) $\text{sl} \Gamma$-diagram $G$ in $\text{Grp}_f$ together with a non-degenerate pairing

\[ I : V_{\alpha \beta} \times V_{\alpha \beta} \to \mathbb{Z}, \]

for every barycenter $\alpha \beta$ in $\text{sl} \Gamma$, where $V_{\alpha \beta}$ is the free abelian group given by the free part of the abelianization of $G(\alpha \beta)$.

Definition 3.8 (equivalence). Given a (based) $\text{sl} \Gamma$-diagram $G$ with pairing and a $\text{sl} \Gamma'$-diagram $G'$ with pairing, they are equivalent if there exists a (based) equivalence $E : \text{sl} \Gamma \to \text{sl} \Gamma'$ and a natural isomorphism $\Phi : G \Rightarrow G' \circ E$ such that $\Phi(\alpha \beta)$ preserves the pairings, for every barycenter $\alpha \beta$.

We often refer to a $\text{sl} \Gamma$-diagram in $\text{Grp}_f$ with pairing as a barycentric diagram in $\text{Grp}_f$ with pairing when the type $\text{sl} \Gamma$ is irrelevant in discussions. The set of equivalence classes of all barycentric diagrams in $\text{Grp}_f$ with pairing is denoted by $\text{Grp}^{\text{dp}}_{f \cdot f'}$.

Definition 3.9 (Join of two finite graphs). Let $\Gamma$ and $\Gamma'$ be two finite graphs with base nodes $*$ and $*$', respectively, and $i$ be a selected node in $\Gamma$. The join $\Gamma \vee_i \Gamma'$ is a based graph, with the base node, obtained by identifying $*' \in \Gamma'$ with $i \in \Gamma$. 

The barycentric subdivision of $\Gamma \vee \Gamma'$ can be identified with a pushout of
$$\text{sd} \Gamma \leftarrow \mathbf{1} \rightarrow \text{sd} \Gamma',$$
where $\mathbf{1} = \{1\}$ is the trivial category, and $i$ and $i'$ are functors sending $1$ to $i \in \Gamma$ and $1$ to $i' \in \Gamma'$, respectively.

**Definition 3.10 (Grafting a barycentric diagram to the other).** Suppose $G : \text{sd} \Gamma \rightarrow \text{Grp}_f$ and $G' : \text{sd} \Gamma' \rightarrow \text{Grp}_f$ are two barycentric diagrams in $\text{Grp}_f$ with pairing. Then the barycentric diagram in $\text{Grp}_f$ obtained by grafting $G'$ onto $G$ at $G(i)$ is the functor
$$G \mathbin{\leftarrow i} G' : \text{sd}(\Gamma \vee i \Gamma') \rightarrow \text{Grp}_f$$
given by the assignment:
- $v \mapsto G(v)$ for $v \in \text{sd} \Gamma \setminus \{i\}$
- $w \mapsto G'(w)$ for $w \in \text{sd} \Gamma' \setminus \{i'\}$
- $u \mapsto G(i) * G'(i')$ for $u = [i'] = [i]$, where by $A * B$ we understand the free product of two groups $A$ and $B$.

**Definition 3.11 (Algebraic graft decomposition).** Let $G : \text{sd} \Gamma \rightarrow \text{Grp}_f$ be a barycentric diagram in $\text{Grp}_f$ with pairing. If $\Gamma = \Gamma_1 \vee i_1 \Gamma_2 \vee i_2 \cdots \vee i_{n-1} \Gamma_n$ and
$$G \simeq G_1 \mathbin{i_1} G_2 \mathbin{i_2} \cdots \mathbin{i_{n-1}} G_n,$$  \quad (3.4)
where $i_k \in \Gamma_k$ and $G_k : \text{sd} \Gamma_k \rightarrow \text{Grp}_f$, $k = 1 \ldots n$, are barycentric diagrams in $\text{Grp}_f$ with pairing, then we say (3.4) is a graft decomposition of $G$.

Recall that, given a pair $\mathcal{S} = (\mathcal{S}^3, \Sigma)$, for any component $\Sigma_i$ of $\Sigma$, we orient $\Sigma_i$ in such a way that its normal vectors point toward the component of $\mathcal{S}^3 \setminus \Sigma_i$ containing $\infty$; the orientation induces an intersection form on $H_1(\Sigma_i)$, which is the abelianization of $\mathcal{F}T^u(S)(\hat{j}\hat{k})$ when $\Sigma_i = F_j \cap F_k$.

**Definition 3.12 (Fundamental tree).** The fundamental tree of a pair $\mathcal{S} = (\mathcal{S}^3, \Sigma)$ is a barycentric diagram in $\text{Grp}_f$ with pairing given by the functor $\mathcal{F}T^u(S)$ together with the intersection form on $H_1(\Sigma_i)$, for each $i$. $\mathcal{F}T(S)$ denotes the fundamental tree of $\mathcal{S}$.

The fundamental tree induces a mapping
$$\mathcal{F}T : \text{Sur} \rightarrow \text{Grp}_{f,p}^\text{BD},$$
and our main theorem asserts that $\mathcal{F}T(\mathcal{S})$ is a complete invariant of $\mathcal{S}$.

**Theorem 3.4.** The mapping $\mathcal{F}T$ is injective.

The proof of Theorem 3.4 is given in Section 5 and lemmas needed in the proof are discussed in Section 4.

4. **Surface and 3-manifold topology**

Some tools from low-dimensional topology are collected in this section; in particular, we give a detailed proof of a generalized Kneser conjecture (Lemma 4.4), which is of independent interest. In this section we specify base points in fundamental groups as some constructions depend heavily on the choice of base points and connecting arcs between them.

We begin with a corollary of the well-known Dehn-Neilsen-Baer theorem.

**Lemma 4.1.** Let $\Sigma$ and $\Sigma'$ be two closed oriented surfaces. Then
(1) Every isomorphism
\[ \phi : \pi_1(\Sigma, *) \to \pi_1(\Sigma', *) \]
that preserves intersection forms on \( H_1(\Sigma) \) and \( H_1(\Sigma') \) can be realized by an o.p. homeomorphism \( f : \Sigma \to \Sigma' \) (i.e. \( f(*) = \phi(*) = *' \)).

(2) Given two o.p. homeomorphisms
\[ f, g : (\Sigma, *) \to (\Sigma', *'), \]
if \( f_* \) and \( g_* \) are conjugate, then \( f \) and \( g \) are isotopic.

**Proof.** See [16, Theorem 4.1].

**Lemma 4.2.** Let \( F \) be an irreducible 3-manifold embedded in \( S^3 \). Suppose \( F \) is \( \partial \)-prime but not \( \partial \)-irreducible. Then \( F \) is homeomorphic either to a solid torus or to a 3-ball.

**Proof.** If \( \partial F \) contains a spherical component, then \( F \) must be a 3-ball by the irreducibility. If \( \partial F \) contains no 2-spheres, then the \( \partial \)-irreducibility of \( F \) implies that there exists an essential loop on \( \partial F \) which bounds a disk \( D \) in \( F \). If \( \partial D \) separates \( \partial F \), then \( D \) must separate \( F \) since every connected surface in \( S^3 \) separates \( S^3 \) into two connected parts, but that would contradict the \( \partial \)-primeness of \( F \).

Therefore, \( \partial D \) cannot separate \( \partial F \), and hence there exists a loop on \( \partial F \) intersecting \( \partial D \) at exactly one point. Using the tubular neighborhood of the loop and \( D \), we obtain a \( \partial \)-prime decomposition
\[ F \simeq (D^2 \times S^1) \#_3 F'. \]

But \( F \) is \( \partial \)-prime, so \( F' \) is a 3-ball.

**Lemma 4.3.** Every compact 3-manifold admits a \( \partial \)-prime decomposition.

**Proof.** See [3, Section 8.1].

**Convention:** Given a subspace \( A \) in \( X \) and two selected points \( *_A \) and \( *_X \), by an arc \( \gamma \) connecting \( *_A \) to \( *_X \) we understand an oriented arc starting from \( *_A \) to \( *_X \); its induced homomorphism
\[ \pi_1(A, *_A) \xrightarrow{\gamma_*} \pi_1(X, *_X) \]

sends the homotopy class of a loop \( l \) in \( A \) to the homotopy class of \( \gamma \circ t \circ \gamma^{-1} \) in \( X \), where the operation \( \alpha \ast \beta \) of two paths \( \alpha \) and \( \beta \) with \( \alpha(0) = \beta(1) \) is defined by
\[ \alpha \ast \beta(t) = \beta(2t) \quad \text{if} \quad 0 \leq t \leq \frac{1}{2}, \quad \alpha \ast \beta(t) = \alpha(2t - 1) \quad \text{if} \quad \frac{1}{2} \leq t \leq 1. \]

The next lemma concerns connected sum of 3-manifolds and generalizes the classical Kneser’s conjecture [6, 7, Theorem 7.1].

**Lemma 4.4.** Let \( M \) be a 3-manifold with no spherical components in \( \partial M \), and \( \Sigma_1 \) and \( \Sigma_2 \) be disjoint surfaces with \( \Sigma_1 \cup \Sigma_2 = \partial M \). Suppose there exists an isomorphism
\[ \pi_1(\Sigma_1, *_{\Sigma_1}) \xrightarrow{\delta_{ij}} \pi_1(\Sigma_2, *_{\Sigma_2}) \xrightarrow{\delta_{ij}} \pi_1(M, *_M) \]
with \( A_1 \ast A_2 \) the free product of two groups \( A_1, A_2 \), and for each component \( \Sigma_{ij} \) of \( \Sigma_1 \), there exists an arc \( \delta_{ij} \) connecting the base point \( *_{ij} \) of \( \Sigma_{ij} \) to the base point \( *_M \) of \( M \) such that the composition
\[ \pi_1(\Sigma_{ij}, *_{ij}) \xrightarrow{\delta_{ij}} \pi_1(M, *_M) \xrightarrow{\delta_{ij}} \pi_1(M, *_M) \]
factors through \( A_1 \ast A_2 \). Then there exists a separating 2-sphere \( S_0 \) in \( M \) which gives a connected sum decomposition of \( M \):
\[ M \simeq M_1 \# M_2 \quad \text{with} \quad \partial M_i = \Sigma_i, \]
and $\phi$ induces an isomorphism $\pi_1(M_i, *_{M_i}) \simeq A_i$, $i = 1, 2$, where $M_i$ is obtained by capping off $\tilde{M}_i$, the closure of a connected component of $M \setminus \Sigma_\phi$, by a 3-ball.

Furthermore, one can find arcs $\delta_i$ connecting $*_{M_i}$ to $*_M$ and arcs $\epsilon_{ij}$ in $\tilde{M}_i$ connecting $*_{ij}$ to $*_{M_i}$ such that the following diagram commutes:

$$
\begin{array}{c}
A_1 \ast A_2 \\
\sim
\end{array}
\begin{array}{c}
\sim
\phi
\end{array}
\begin{array}{c}
\pi_1(M, *_M)
\delta_{i*}
\end{array}
\begin{array}{c}
\pi_1(M_i, *_{M_i})
\epsilon_{ij*}
\end{array}
\begin{array}{c}
\sim
\pi_1(\Sigma_{ij}, *_{ij})
\end{array}

(4.1)

**Proof.** Firstly, note that if one of $A_i$, $i = 1, 2$, say $A_1$, is trivial, then $\Sigma_1$ must be empty. To see this, we recall the exact sequence

$$H_2(M, \partial M; R) \to H_1(\partial M; R) \to H_1(M; R),$$

where $R = \mathbb{Z}$ when $\partial M$ is orientable; otherwise, $R = \mathbb{Z}_2$. This implies that any two loops in $\partial M$ coming from $H_2(M, \partial M; R)$ have null intersection number. Hence, if $\Sigma_i \neq \emptyset$, then the induced homomorphism

$$H_1(\Sigma_i; R) \to H_1(M; R)$$

and therefore the induced homomorphism

$$\pi_1(\Sigma_{ij}, *_{ij}) \to \pi_1(M, *_M)$$

are non-trivial, for every $j$. This would imply $A_1$ is non-trivial because $\phi$ is an isomorphism and hence contradict the assumption. Therefore $\Sigma_i = \emptyset$. Now, let the separating sphere $\Sigma_\phi$ be any 2-sphere that bounds a 3-ball away from all connecting arcs. Then the desired properties follow.

From now on, we assume both $A_1$ and $A_2$ are non-trivial, the proof for this case is divided into two three steps. The separating 2-sphere $\Sigma_\phi$ in $M$ is constructed in Step 1, $\partial M_i = \Sigma_i$ is proved in Step 2, and the commutative diagram (4.1) is examined in Step 3.

**Step 1: separating 2-sphere $\Sigma_\phi$.** Following methods in [14], [6] Lemma, and [17] Chap.7, we consider two aspherical CW-complexes $K_1$ and $K_2$ with $\pi_1(K_i, *) \simeq A_i$. Connecting $K_1$ and $K_2$ with a one-simplex $I = [0, 1]$ by gluing $0, 1 \in I$ to base points of $K_1, K_2$, respectively, we obtain a new CW-complex $K$. Let $* := \frac{1}{2} \in I$ be the base point of $K$. Then there is an obvious isomorphism $\pi_1(K, *) \simeq A_1 \ast A_2$, and hence $\phi$ can be viewed as an isomorphism from $\pi_1(M, *_M)$ to $\pi_1(K, *)$. Since $K$ is aspherical, the isomorphism can be realized by a map $h : M \to K$.

By [17] Lemma 1.1, we may assume i) $h$ transverse to $*$ (i.e. $h^{-1}(I')$ has a product structure $h^{-1}(*) \times I'$ on which $h$ restricts to the projection onto $I'$, where $I' = [\frac{1}{2}, \frac{3}{4}]$ is a subinterval of $I'$), and ii) $h^{-1}(*)$ consists of incompressible surfaces. Since $h_* = \phi$ is an isomorphism, a component in $h^{-1}(*)$ is either a disk or a 2-sphere. We let $(n_d, n_s)$ denotes the numbers of disks and spheres in $h^{-1}(*)$, and define a linear order $\preceq$ on the set of pairs of non-negative integers by declaring $(a, b) \preceq (c, d)$ if either $a < c$ or $a = c; b < d$. We assume $h$ is chosen such that it satisfies conditions i) and ii) and minimizes $(n_d, n_s)$.

The goal is to show $(n_d, n_s)$ of $h$, a minimizer, is $(0, 1)$, and $h^{-1}(*)$ is the required 2-sphere $\Sigma_\phi$.

**Disks:** Observe that $\{\partial D_k\}_{k=1}^{n_d}$ separates $\partial M$ into several components, and crossing through a disk $\partial D_k$ means going from one component to the other. If we
think of each $\partial D_k$ as an edge and the closure of each component of the complement 
\[ \partial M \setminus \bigcup_{k=1}^{n_d} \partial D_k \]
as a node, we get a graph $G$ (Fig. 4.1).

**Claim 1:** There is no loop in $G$. Suppose there is a loop in $G$; without loss of generality, we may assume the boundary of $D := D_1$ represents an edge in the loop, and $\partial D$ is in the component $\Sigma := \Sigma_{11}$ of $\Sigma_1$. Then there exists an embedded loop $l$ transversal to $\bigcup_{k=1}^{n_d} \partial D_k$ in $\Sigma$ such that $l \cap D$ is a point and $l \cap D_k$ contains no more than one point, for every $k = 2, \ldots, n_d$. In other words, $l$ is dual to $\partial D$ in $\Sigma$ and therefore essential in $M$ by (4.2).

Now, $h^{-1}(\ast)$ divides $l$ into $2n$ arcs $l_1, \ldots, l_{2n}$ with end points of each $l_i$ lying in different disks. Up to reindexing $l_1, \ldots, l_{2n}$, we may assume $h \circ l_1$ is a loop in $K_1$ when $i$ is odd and in $K_2$ when $i$ is even. Since the composition (4.3) sends $l$ to an element $y \cdot x_1 \cdot x_2 \ldots x_{2n-1} \cdot x_{2n} \cdot y^{-1} \in A_1$, where $x_i = [h(l_i)]$ in $A_j$, $i \equiv j \pmod{2}$, some $x_i$ must be trivial, say $x_1$. Suppose $\partial l_1 = p, q$ are in disks $D_i$ and $D_j$, respectively. Then we homotopy a neighborhood $l_1$ in $\Sigma$, using a homotopy similar to the one in [17, p.507] (Fig. 4.2) such that $D_i, D_j \subset h^{-1}(*)$ is replaced by a disk obtained by performing a $\nabla$-move along $l_1$ ([15, Sec. 3]). But, this contradicts the minimality of $\#h^{-1}(\ast)$, and hence $G$ is a union of trees.

**Claim 2:** There is at most one end with genus larger than 0 in each component of $G$. Suppose there are two ends with genus larger than 0 in a component $G$ of $G$; we homotopy a neighborhood $l_1$ in $\Sigma$, using a homotopy similar to the one in [17, p.507] (Fig. 4.2) such that $D_i, D_j \subset h^{-1}(\ast)$ is replaced by a disk obtained by performing a $\nabla$-move along $l_1$. A 1-valent node in $G$ is called an end of $G$; an end of $G$ corresponds to a component of $\partial M \setminus \bigcup_{k=1}^{n_d} \partial D_k$, which is adjacent to only another component. The genus of an end is the genus of the corresponding component.

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factors through $A_1$. Thus, we obtain $x = [h(l)] \neq 1 \in A_2$ and, at the same time, $\phi \circ \gamma_i(l') = yx_i^{-1} \in A_1$, for some $y \in \pi_1(M, \ast M)$; this is possible only when $x = 1$. Hence, we get a contradiction. A similar argument shows that $h(\Sigma')$ cannot be in the component of $K \setminus \ast$ containing $K_2$.

Suppose that both $h(\Sigma')$ and $h(\Sigma'')$ are in the component of $K \setminus \ast$ containing $K_1$. Then we connect the loop $l'$ (resp. $l''$) to $\ast \Sigma$ with an arc $\alpha'$ (resp. $\alpha''$) in $\Sigma$ to get a connecting arc $\beta' := \delta \ast \alpha'$ (resp. $\beta'' := \delta \ast \alpha''$) from the loop $l'$ (resp. $l''$) to $\ast_M$. The intersection $\beta' \cap h^{-1}(\ast)$ (resp. $\beta'' \cap h^{-1}(\ast)$) separates $\beta'$ (resp. $\beta''$) into connected subarcs $\beta'_1, \ldots, \beta'_m$ (resp. $\beta''_1, \ldots, \beta''_m$).

At least one of $\delta \phi_i(\alpha^i \ast l' \ast \alpha'^{-1})$ and $\delta \phi_i(\alpha'' \ast l'' \ast \alpha''^{-1})$, say $\delta \phi_i(\alpha^i \ast l' \ast \alpha'^{-1})$, has the form $y_m \cdots y_1 \ast x \ast y_1^{-1} \cdots y_m^{-1} \in A_1$ with $m > 0$, where the element $x = [h(l')]$ is in $A_1$, and the element $y_i$, induced from $\beta'_i$, is in $A_j$, $i \equiv j + 1 \pmod{2}$. This implies some $h \circ \beta'_i$, say $h \circ \beta'_i$, is non-essential. Thus, we may perform the operation in the proof of Claim 1 to merge disks containing the endpoints of $\beta'_i$ using $\beta'_i$ (Fig. 4.2) and get a contradiction. This completes the proof of Claim 2.

Now, we observe that if $G$ contains a non-degenerate tree, a tree having at least one edge, then by Claim 2 it has at least one end with genus 0. This would imply that there exists a disk $D$ in $h^{-1}(\ast) \cap \Sigma$ with $\partial D$ cutting off a disk $D'$ from $\Sigma$. In this case, we push $D \cup D'$ away from $\partial M$ and get a 2-sphere $\mathcal{S}$ in the interior of $M$. Since $\pi_2(K, \ast)$ is trivial, we can deform $h$ such that the disk $D$ in $h^{-1}(\ast)$ is replaced with the 2-sphere $\mathcal{S}$ without affecting other components in $h^{-1}(\ast)$, but this contradicts the minimality of $h$ (see [7, p.66] for more details).

In conclusion, $G$ must be a union of degenerate trees; that is a collection of nodes with no edges. So, there is no disk in $h^{-1}(\ast)$ and $n_d = 0$.

**Claim 3:** There are no more than one 2-sphere. This follows from the standard binding tie argument ([14], [6], [7, p.67]). For the sake of completeness, we outline its proof below. If $h^{-1}(\ast)$ contains more than one 2-sphere, then we consider arcs with two ends lying in different components of $h^{-1}(\ast)$ and mapped to non-essential loops under $h$. Let $\alpha$ be such an arc that minimizes $\# h^{-1}(\ast) \cap \alpha$, and $\mathcal{S}_1$ and $\mathcal{S}_2$ be the 2-spheres in $h^{-1}(\ast)$ connected by $\alpha$. Then one can show that the interior of $\alpha$ must have trivial intersection with $h^{-1}(\ast)$ ([7, p.67]), and thus we can homotopy $h$ such that $\mathcal{S}_1$ and $\mathcal{S}_2$ are replaced by the 2-sphere given by the union

$$\partial \mathcal{N}(\alpha) \setminus (\mathcal{S}_1 \cup \mathcal{S}_2) \cup (\mathcal{S}_1 \cup \mathcal{S}_2) \setminus N(\alpha),$$

with other components in $h^{-1}(\ast)$ intact, where $\mathcal{N}(\alpha)$ is the closure of a tubular neighborhood $N(\alpha)$ of $\alpha$. This contradicts the minimality of $\# h^{-1}(\ast)$.

**Consequences of the results in Step 1:** $h^{-1}(\ast)$ contains only one connected component and it is a 2-sphere $\mathcal{S}_\phi$, which separates $M$ into two connected components; we denote their closures by $\tilde{M}_1$ and $\tilde{M}_2$. This induces a decomposition $M \simeq M_1 \# M_2$, where $M_i$ is obtained by capping off the spherical component in $\partial \tilde{M}_i$ with a 3-ball $B^3$. Furthermore, we may choose the base point $\ast_{M_i}$ to be in a tubular neighborhood of $\mathcal{S}_\phi$ such that $h(\ast_{M_i}) = \frac{1}{2} \in I \subset K$. Then there are natural arcs $\delta_i$ connecting $\ast_{M_i}$ to $\ast_{M_i}$ in $\tilde{M}_i$ and isomorphisms $\pi_1(M_i, \ast_{M_i}) \simeq A_i$ such that the following diagram commutes

$$A_1 \ast A_2 \xrightarrow{\phi} \pi_1(M, \ast M) \xrightarrow{\delta_i} \pi_1(M_i, \ast_{M_i}).$$

(4.4)
Step 2: Locate $\Sigma_1$ and $\Sigma_2$. In this step, we show that $\partial M_i = \Sigma_i$, $i = 1, 2$. Suppose it is not the case. Without loss of generality, we may assume that a component $\Sigma$ of $\Sigma_1$ is mapped into the connected component of $K \setminus \ast$ containing $K_2$ under $h$. As we shall see, this would lead to a contradiction as in the first part of the proof of Claim 2. Firstly, we choose an essential loop $l$ in $\Sigma$ that is also essential in $M$ by (4.2), and connect it to $\ast M$ in $M_2$; we denote the resulting loop by $\tilde{l}$, and observe that $x = [h \circ l]$ is non-trivial in $A_2$. On the other hand, the composition $[4.3]$ implies $y \cdot x \cdot y^{-1} \in A_1$ for some $y \in A$, but this is possible only when $x = 1$. So, every component of $\Sigma_1$ must be in $\partial M_1$. Similarly, we have $\Sigma_2 = \partial M_2$.

Step 3: Homotopy $\delta_{ij}$. Consider a connected component $\Sigma_{ij}$ of $\Sigma_i$, and recall that $\delta_{ij}$ is the connecting arc from $*_{ij} \ast M$ in the lemma and $\delta_i$ the connecting arc from $*_{M_i}$ to $* M$ in diagram (4.4). We want to show that $\delta_{ij}^{-1} \delta_{ij}$ can be homotoped with endpoints fixed to an arc in $M_i$. To see this, we first choose an arc $\gamma$ connecting $*_{ij}$ to $*_{M_i}$ in $M_i$, and let $\alpha = \delta_i \ast \gamma$. Next, we pick an element $[l] \in \pi_1(\Sigma_{ij}, *_{ij})$ not in the kernel of $\delta_i$, and observe that the elements $x = \phi \circ \alpha \circ [l]$ and $z = \phi \circ \delta_i \ast [l]$ both are in $A_i$. Because $z = y \cdot x \cdot y^{-1}$ with $x$ and $z$ in $A_i$, the element $y$, the homotopy class of $h(\delta_{ij} \ast \alpha^{-1})$, must be in $A_1$ as well. In particular, there exists a loop $\beta$ in $M_i$ such that $\delta_i \ast \beta \ast \delta_i^{-1}$ and $\delta_{ij} \ast \alpha^{-1} = \delta_i \ast \gamma^{-1} \ast \delta_i^{-1}$ represent the same element in $\pi_1(M, *_{M_i})$. This implies $\delta_{ij}^{-1} \ast \delta_{ij}$ is homotopic to $\beta \ast \gamma$, a loop entirely in $M_i$ (right figure in (4.5)).

Therefore, we may assume $\delta_{ij}$ is obtained by $\delta_i$ and an arc $\epsilon_{ij} = \beta \ast \gamma$ connecting $*_{ij}$ to $*_{M_i}$ and is entirely in $M_i$. As a result, the induced homomorphism $\epsilon_{ij}$

$$\pi_1(\Sigma_{ij}, *_{ij}) \xrightarrow{\epsilon_{ij} \ast} \pi_1(M_i, *_{M_i})$$

fits in the commutative diagram (left) below:

![Diagram](4.5)

Remark 4.1. For simplicity’s sake, we assume $\partial M$ contains no spherical components in Lemma [4.4], but it is actually not necessary. To see this, we first cap off spherical components of $\partial M$ by some 3-balls to get a new 3-manifold $M'$ without spherical boundary components. Then, applying Lemma [4.4] to $M'$, we obtain a separating sphere $\Sigma_\phi$ such that the induced connected sum decomposition $M' \simeq M'_1 \# M'_2$ has the property: Every component $\Sigma_{ij}$ of $\partial M$ with positive genus is in $M'_1$. Without loss of generality, we may assume $\Sigma_\phi$ is disjoint from those attached 3-balls. Now, if $\Sigma$ is a 2-sphere on the wrong side of $\Sigma_\phi$, we modify $\Sigma_\phi$ as follows. Firstly, connect $\Sigma_\phi$ to $\Sigma$ via an embedded 1-handle $h(D^2 \times I)$ in the interior of $M'$ with $h(D \times I)$ away from the attached 3-balls, and $h(D \times \{0\}) = \Sigma_\phi \cap h(D^2 \times I)$ and $h(D \times \{1\}) = \Sigma \cap h(D^2 \times I)$. Secondly, consider the union

$$\Sigma_\phi \setminus h(D \times \{0\}) \cup h(D^2 \times I) \cup \Sigma \setminus h(D^2 \times \{1\}),$$

which is a 2-sphere, and push it into the interior of $M$ to get a new separating 2-sphere. This way, we get $\Sigma$ on the correct side of the new separating 2-sphere with other components unaffected by the operation. Applying this procedure to all spherical components of $\partial M$ that are on the wrong side of $\Sigma_\phi$, and then removing
the attached 3-balls, we get a separating 2-sphere in $M$ that induces the desired connected sum decomposition of $M$.

**Remark 4.2.** Lemma 4.4 implies the classical Kneser’s conjecture [6, 7 Chap. 7], where $M$ is assumed to be $\partial$-irreducible without additional assumptions on the induced homomorphism $\pi_1(\Sigma) \to \pi_1(M)$, where $\Sigma$ is a component of $\partial M$ with positive genus.

In general, given a connecting arc $\delta$ from $*_\Sigma$ to $*_M$, the composition

$$\pi_1(\Sigma, *_\Sigma) \xrightarrow{\delta} \pi_1(M, *_M) \xrightarrow{\phi} A_1 * A_2$$

needs not factor through $A_1$ or $A_2$.

However, if $M$ is $\partial$-irreducible, then $\pi_1(\Sigma, *_\Sigma)$ can be identified with an indecomposable subgroup in $g \cdot A_1 \cdot g^{-1}$, for some $g \in A_1 * A_2$. Replacing the connecting arc $\delta$ with $\gamma * \delta$, where $\gamma$ is a loop representing $\phi^{-1}(g)$ in $M$, we obtain that the composition

$$\pi_1(\Sigma, *_\Sigma) \xrightarrow{(\gamma * \delta)_*} \pi_1(M, *_M) \xrightarrow{\phi} A_1 * A_2$$

factors through $A_i$. Applying this construction to every component of $\partial M$, we see the assumption of the classical Kneser conjecture imply conditions in Lemma 4.4.

The following corollary is a consequence of Lemma 4.4: it plays an important role in the proof of the main theorem as it allows us to decompose an isomorphism between fundamental groups of two 3-manifolds $M, M'$ into isomorphisms between fundamental groups of their prime factors.

**Corollary 4.5.** Given two 3-manifolds $M$ and $M'$, suppose $M \simeq M_1 \# M_2$ and there exist isomorphisms

$$\phi_M : \pi_1(M, *_M) \to \pi_1(M', *_M'), \quad \phi_{ij} : \pi_1(\Sigma_{ij}, *_{ij}) \to \pi_1(\Sigma_{ij}', *_{ij}')$$

such that the following diagram commutes

$$\begin{array}{ccc}
\pi_1(M, *_M) & \xrightarrow{\phi_M} & \pi_1(M', *_M') \\
\downarrow{\delta_{ij,*}} & & \downarrow{\delta'_{ij,*}} \\
\pi_1(\Sigma_{ij}, *_{ij}) & \xrightarrow{\phi_{ij}} & \pi_1(\Sigma_{ij}', *_{ij}')
\end{array}$$

for every connected component $\Sigma_{ij} \subset \partial M$, where

$$\bigcup_{i,j} \Sigma_{ij} = \partial M_i \quad \text{and} \quad \bigcup_{i,j} \Sigma_{ij}' = \partial M',$$

and $\delta_{ij}$ (resp. $\delta'_{ij}$) is a connecting arc from $*_ij$ to $*_M$ (resp. from $*_ij'$ to $*_M'$).

Furthermore, suppose that the arc $\delta_{ij}$ respects the decomposition $M \simeq M_1 \# M_2$ in the sense that it can be decomposed as $\delta_{ij} = \delta_i * \epsilon_{ij}$, with $\delta_i$ and $\epsilon_{ij}$ connecting arcs from $*_M$ to $*_M$ and from $*_ij$ to $*_M$, respectively. Then there exist $M_1', M_2'$ such that $M' \simeq M_1' \# M_2'$, $\partial M_i' = \bigcup_{i,j} \Sigma_{ij}'$, $i = 1, 2$, and isomorphisms

$$\psi_i : \pi_1(M_i, *_M) \to \pi_1(M_i', *_M')$$

such that the following diagram commutes

$$\begin{array}{ccc}
\pi_1(M_i, *_M) & \xrightarrow{\psi_i} & \pi_1(M_i', *_M') \\
\downarrow{\epsilon_{ij,*}} & & \downarrow{\epsilon'_{ij,*}} \\
\pi_1(\Sigma_{ij}, *_{ij}) & \xrightarrow{\phi_{ij}} & \pi_1(\Sigma_{ij}', *_{ij}')
\end{array}$$

commutes, where $\epsilon'_{ij}$ is an arc connecting $*_ij'$ to $*_M'$ induced by $\epsilon_{ij}, \delta_i$, and $\delta'_{ij}$.
The next lemma is a corollary of Waldhausen’s theorem [19] Theorem 7.1; Corollary 6.5.

**Lemma 4.6.** Let $M, M'$ be oriented irreducible and $\partial$-irreducible 3-manifolds. Suppose there exist an o.p. homeomorphism $f : \partial M \to \partial M'$ and an isomorphism $\pi_1(M, *_M) \to \pi_1(M', *_{M'})$ and, for every component $\Sigma_i$ of $\partial M$ and $\Sigma'_i = f(\Sigma_i) \subset \partial M'$, there is an arc $\gamma_i$ (resp. $\gamma'_i$) connecting the base point $*_i \in \Sigma_i$ (resp. $*_i' \in \Sigma'_i$) to the base point $*_M \in M$ (resp. $*_M' \in M'$) such that the diagram

$$
\begin{array}{ccc}
\pi_1(M, *_M) & \xrightarrow{\phi} & \pi_1(M', *_M') \\
\gamma_i & \downarrow & \gamma'_i \\
\pi_1(\Sigma_i, *_i) & \xrightarrow{f_*} & \pi_1(\Sigma'_i, *_i')
\end{array}
$$

(4.6)

commutes, up to conjugation, where $f_*$ is the restriction of $f$ on $\Sigma_i$. Then there exists an o.p. homeomorphism $h : M \to M'$ realizing $\phi$ and restricting to $f$ on $\partial M$.

**Proof.** We argue by induction on $n$, the number of the components of $\partial M$.

Without loss of generality, we assume that the base point $*_M = *_0 \in \Sigma_0$ (resp. $*_M' = *'_0 \in \Sigma'_0$). Let $\gamma_i$ (resp. $\gamma'_i$), $i = 1, \ldots, n$, be connecting arcs between $*_i$ and $*_0$ (resp. $*_i'$ and $*_0'$). Modifying $\gamma'_i$, we may assume diagram (4.6) commutes strictly. Then by Waldhausen’s theorem [19] Corollary 6.5], there exists an o.p. homeomorphism $h_0$ which realizes $\phi$ and extends $f_0$.

Now, suppose there exists an o.p. homeomorphism $h_{k-1}$ that realizes $\phi$ and extends $f_i$, $i = 1, \ldots, k - 1$. Then we want to show that $h_{k-1}$ is isotopic, relative to $\bigcup_{i=1}^{k-1} \Sigma_i$, to an o.p. homeomorphism $h_k$ that realizes $\phi$ and restricts to $h_k$ on $\Sigma_k$.

To construct $h_k$, we first move the base point $*_M$ (resp. $*_M'$) to $*_k$ (resp. $*_k'$) along $\gamma_k$ (resp. $\gamma'_k$). This gives us the commutative diagram below:

$$
\begin{array}{ccc}
\pi_1(M, *_M) & \xrightarrow{\phi} & \pi_1(M', *_{M'}) \\
\gamma_k & \downarrow & \gamma'_k \\
\pi_1(\Sigma_k, *_k) & \xrightarrow{f_*} & \pi_1(\Sigma'_k, *'_k)
\end{array}
$$

(4.7)

Applying Waldhausen’s theorem [19] Corollary 6.5] to diagram (4.7), we obtain an o.p homeomorphism $h'_k : (M, *_k) \to (M', *'_k)$ realizing the isomorphism $\gamma_k^{-1} \circ \phi \circ \gamma_k$ and restricting to the homeomorphism $f_k$ on $\Sigma_k$. Isotopying $h'_k$, we can further assume $h'_k(*_M) = *_{M'}$.

To compare $\phi$ with the induced homomorphism

$$h'_k : \pi_1(M, *_M) \to \pi_1(M', *_{M'})$$

we observe the commutative diagrams below:

$$
\begin{array}{ccc}
\pi_1(M, *_M) & \xrightarrow{\phi} & \pi_1(M', *_{M'}) \\
\gamma_k & \downarrow & \gamma'_k \\
\pi_1(\Sigma_k, *_k) & \xrightarrow{f_*} & \pi_1(\Sigma'_k, *'_k)
\end{array}
$$

Diagram (4.8) implies that

$$\sigma_* \circ \phi = h'_k : \pi_1(M, *_M) \to \pi_1(M', *_{M'})$$
where \( \sigma \) is the path \( h_k'(\gamma_k)\gamma_k'^{-1} \). In particular, \( h_{k-1} \), which realizes \( \phi \), is isotopic to \( h_k' \) [10] Theorem 7.1, and hence the restriction of \( h_{k-1} \) on \( \Sigma_k \) is isotopic to \( f_k \). Using the collar neighborhood of \( \Sigma_k \), we isotopy \( h_{k-1} \), relative to \( \bigcup_{i=1}^{k-1} \Sigma_i \), to an o.p. homeomorphism \( h_k \) that restricts to \( f_k \) on \( \Sigma_k \).

\[ \square \]

5. Proof of the main theorem

**Proof of Theorem 3.4.** Suppose the pairs \( S = (S^3, \Sigma) \) and \( S' = (S^3, \Sigma') \) have equivalent fundamental trees. Then after reindexing \( \Sigma_i', F_j' \) if necessary, we have isomorphisms

\[
\begin{align*}
\phi_{\Sigma_i}: \pi_1(\Sigma_i) &\to \pi_1(\Sigma_i'), \\
\phi_{F_j}: \pi_1(F_j) &\to \pi_1(F_j'),
\end{align*}
\]

such that whenever \( \Sigma_i \subseteq F_j, \Sigma_i' \subseteq F_j' \) the diagram

\[
\begin{array}{ccc}
\pi_1(F_j) & \xrightarrow{\phi_{\Sigma_i}} & \pi_1(\Sigma_i) \\
\downarrow{\phi_{F_j}} & & \downarrow{\phi_{\Sigma_i}} \\
\pi_1(F_j') & \xrightarrow{\phi_{\Sigma_i}} & \pi_1(\Sigma_i')
\end{array}
\]

(5.1)

commutes, up to conjugation, for some connecting arcs \( \gamma_i \) (resp. \( \gamma_i' \)) from the base point of \( \Sigma_i \) (resp. \( \Sigma_i' \)) to the base point of \( F_j \) (resp. \( F_j' \)). By modifying \( \gamma_i' \) properly, we may assume that the diagram (5.1) commutes strictly.

The proof consists of two steps: We first prove the theorem for the special case where \( S \) and therefore \( S' \) are non-splittable. Then we reduce the general case to the non-splittable case via the non-splittable graft decomposition of \( S \) and Lemma 4.3.

**Case 1. Non-splittable pair \( S = (S^3, \Sigma) \).** The construction of the equivalence between \( S \) and \( S' \) is in essence similar to the one in [2]: We first observe that there is an o.p. homeomorphism \( f: \Sigma \to \Sigma' \) realizing \( \phi_{\Sigma_i} \), for \( i = 1, \ldots, n \), by Lemma 4.3. Then we extend \( f|_{\partial F_j} \) over \( F_j \), for each \( j \), to get the desired equivalence. To construct the extension of \( f|_{\partial F_j} \), as outlined in Introduction, we first decompose \( F_j \) into \( \partial \)-prime factors and then apply Lemma 4.6.

For the sake of simplicity, we let \( F = F_j \) be a solid part of \( (S^3, \Sigma) \). We consider the \( \partial \)-prime decomposition of \( F \):

\[
F \simeq E_0 \#_b \ldots \#_b E_m.
\]

(5.2)

and denote separating disks in (5.2) by \( D_i \), \( i = 1, \ldots, m \). Observe that each \( D_i \) in \( F \) separates \( F \) into two connected parts, and thus if we think of \( E_j \), \( j = 0, \ldots, m \), as nodes and each \( D_i \) as an edge between \( E_j \) and \( E_k \) when \( E_j \cap E_k = D_i \), then we get a based tree \( \Pi \) with \( E_0 \) corresponding to the base node.

The depth of a node of \( \Pi \) or of the 3-manifold it represents is defined as the number of edges between each node and the base node. Without loss of generality, we may assume \( D_i \) is the separating disk in \( \partial E_i \) that separates \( E_i \) from \( *_F \) when \( i \neq 0 \); in other words, \( D_i \) is represented by the first edge from the node representing \( E_i \) to the base node in \( \Pi \). Given a component \( \Sigma \) of \( \partial F \), the closest part of \( \Sigma \) to \( E_0 \) with respect to (5.2) is the component of \( \partial E_s \) that has non-trivial intersection with \( \Sigma \), where \( E_s \) has the shallowest depth among members of the set

\[
\{ E_i | E_i \cap \Sigma \neq \emptyset \}.
\]

**• A good system of connecting arcs with respect to (5.2).** We first select a point on each \( \partial D_i, i = 1, \ldots, m \), and let the base point \( *_F \) of \( F \) be on a component of \( \partial E_0 \cap \partial F \). Secondly, for each component \( \Sigma \) of \( \partial F \), if \( *_F \notin \Sigma \), the base point \( *_{\Sigma} \)
is chosen to be on the closest part of Σ to $E_0$ with respect to (5.2); if $*F \in \Sigma$, then we let $*\Sigma = *F$.

Next, we construct arcs between these points, which are built from smaller arcs starting from base points of components of $\partial F$ and selected points on $\partial D_i$.

**Arcs starting from the base point of $\Sigma$.** Let $\Sigma$ be a component of $\partial F$. If its base point $*\Sigma \neq *F$ is in $\partial E_0$, then we connect it to $*F$ by an arc in $E_0$; if $*\Sigma = *F$, then we use the constant path. If $*\Sigma$ is in $\partial E_i$, for some $i \neq 0$, then we select an arc in $E_i$ connecting $*\Sigma$ to the selected point on $\partial D_i$ in $E_i$. Notice that $\partial D_i$ is necessarily not in $\Sigma$ in this case.

**Arcs starting from the selected point in $\partial D_i$ ($i \neq 0$).** Let $\Sigma$ be the component of $\partial F$ containing $\partial D_i$. If $D_i$ is between $E_i$ and $E_h$ and the base point $*\Sigma$ is in $\partial E_h$, then we connect the selected point in $\partial D_i$ to $*\Sigma$ via an arc in $\Sigma \cap E_h$. If $D_i$ is between $E_i$ and $E_h$, but the base point $*\Sigma$ is not in $\partial E_h$, then we connect the selected point in $\partial D_i$ to the selected point on $\partial D_h$ via an arc in $\Sigma \cap E_h$.

**Embedded tree $\Upsilon$.** These connecting arcs together induce an embedded tree $\Upsilon \subset F$. The nodes of $\Upsilon$ are selected points on $\partial D_i$, $i = 1, \ldots, m$, and base points of components of $\partial F$ and $F$, and its edges are connecting arcs between them. To see $\Upsilon$ is an embedded tree, we define a partial ordering on its nodes in the following manner: The base point $*F$ is of order 0, and $*\Sigma \neq *F$ is of order 2k + 1 if $*\Sigma$ is in $E_i$, and there are $k$ edges in $\Upsilon$ between nodes corresponding to $E_i$ and $E_0$. The selected point in $\partial D_i$ is of order 2k if there are $k$ edges in $\Upsilon$ between nodes corresponding to $E_i$ and $E_0$. From the construction of $\Upsilon$, each node is connected by exactly one edge to a unique node with smaller order, so there can be no loop in $\Upsilon$. This also implies $\Upsilon$ is connected, for every node eventually connects to $*F$.

Here we explain how this embedded tree $\Upsilon$ gives rise to natural base points of $E_i$ and each component of $\partial E_i$, $i = 0, \ldots, m$, and connecting arcs between them. We let $*F$ be the base points of $E_0$ and the selected point in $\partial D_i$ the base point of $E_i$, $i \neq 0$. We let $*\Sigma$ be the base point of the component of $\partial E_i$ containing $\Sigma \cap E_i$ if $*\Sigma \in E_i$, and let the selected point in $\partial D_i$ be the base point of the component of $\partial E_i$ containing $D_i$, $i \neq 0$. Connecting arcs between them are unique paths between them in the embedded tree $\Upsilon$.

Given a component $\Theta$ of $\partial E_i$, the system of base points and connecting arcs induces the commutative diagram

$$
\begin{array}{ccc}
\pi_1(F) & \xleftarrow{\iota_i} & \pi_1(E_i) \\
\downarrow & & \downarrow \\
\pi_1(\Sigma) & \xleftarrow{\iota_i} & \pi_1(\Theta)
\end{array}
$$

(5.3)

where $\Sigma$ is the component of $\partial F$ having non-trivial intersection with $\Theta$, and $\iota_i$ is the unique path in $\Upsilon$ connecting $*E_i$ and $*F$; other homomorphisms in (5.3) are also induced from connecting arcs in $\Upsilon$.

According to how base points are chosen, we separate components $\Theta$ of $\partial E_i$ into two categories: one consists of those with base point $*\Theta = *E_i$, and the other comprises those having $*\Theta = *\Sigma$, where $\Sigma \cap \Theta \neq \emptyset$. We denote the first kind by $\Theta_1^*$ and the second by $\Theta_2^*$.

Observe that connecting arcs $\iota_i$, $i = 1, \ldots, m$, induce a free product decomposition of $\pi_1(F)$:

$$
\pi_1(F) \simeq \pi_1(E_0) * \cdots * \pi_1(E_m).
$$

(5.4)

Since $E_i$ is either a solid torus or a $\partial$-irreducible manifold by Lemma 4.2, the decomposition (5.4) is a free product decomposition with indecomposable factors.

- **Induced decomposition of $F'$.** Recall that $f$ is the $o.p.$ homeomorphism realizing $\phi_{\Sigma_i}$ in the diagram (5.1) and sending base points of components of $\partial F$ to
base points of components of $\partial F'$ and that, in our assumption, $*F \in \partial F$. Without loss of generality, we may assume the base point of $F'$ is $f(*F) \in \partial F'$; this can be done by moving the original base point of $F'$ to $f(*F)$ along the connecting arc between them. Note that $f(*F)$ is a base point of a component of $\partial F'$ because $*F \sim *\Sigma$, for some $\Sigma$. This modification does not change the strict commutativity of \((5.1)\).

By Dehn’s lemma, the loop $f(\partial D_i)$ in $F'$ bounds a disk $D'_i$ in $F'$, for each $i$. Since $F'$ is irreducible, by the innermost circle argument and induction, we may assume $D'_i$, $i = 1, \ldots, m$, are disjoint, and we extend the homeomorphism $f: \Sigma \to \Sigma'$ over $\bigcup_{i=1}^m D_i$. Also, since $F'$ is in $S^3$, disks $D'_i$, $i = 1, \ldots, m$, separate $F'$ into $m + 1$ components, and induce a boundary connected sum decomposition of $F'$:

$$F' \simeq E'_0 \# b \ldots \#_b E'_{m}, \quad (5.5)$$

where factors are indexed in such a way that $*F' \in E'_0$ and $D'_i \subset \partial E'_i$, $i \neq 0$, separates $E'_i$ from $*F'$.

There are natural base points $*_{E'_i}$ of $E'_i$ and $*\Theta'$ of every component $\Theta'$ of $\partial E'_i$, $i = 0, \ldots, m$, given by the images $f(*_{E'_i})$ and $f(*_{F'}(\Theta'))$, respectively. Also, $f$ induces a connecting arc from $*\Theta'$ to $*\Sigma'$, which is the image of the arc from $*_{F'}(\Theta')$ to $*\Sigma'$, where $\Sigma' = f(\Sigma)$, and $\Sigma'$ is the component of $\partial F'$ with $\Sigma' \cap E'_i \subset \Theta'$.

Among components of $\partial E'_i$, the special component that contains $*_{E'_i}$ is denoted by $\Theta_i'$. A priori, we do not know if $f^{-1}(\Theta_i') = \Theta_i'$; in fact, it is not clear at this stage, given a component $\Theta$ of $E_i$, if $f(\Theta) =: \Theta'$ is a component of $E'_i$.

\textbullet\ Compatibility between \((5.2)\) and \((5.5)\). To unfold relations between $\Theta'_i$ and $\Theta_i'$ and between $E_i$ and $E'_i$, we consider the associated tree $\Pi'$ of \((5.5)\), whose nodes are $E'_i$, $i = 0, \cdots, m$, and edge between $E'_i$ and $E'_k \supset D_i$ is the disk $D_i = E'_k \cap E'_i$.

Claim 1: The assignment

$$D_i \mapsto D'_i, \quad E_i \mapsto E'_i \quad (5.6)$$

induces an isomorphism of based trees between $\Pi$ and $\Pi'$. Furthermore, we have $f(\Theta_i') = \Theta_i'$, and $\phi_F$ induces an injection $\pi_1(E_i) \to \pi_1(E'_i)$, for each $i$.

Firstly, we observe that, for the special component $\Theta_i'$, there is a natural connecting arc from $*\Theta_i'$ to $*_{E'_i}$ given by the images of the connecting arc from $*_{F'}(\Theta_i')$ to $*_{E'_i}$. Joining together the inverse of the connecting arc, the arc from $*\Theta_i'$ to $*\Sigma'$, where $\Sigma'$ is the component with $\Sigma' \cap \Theta_i' \neq \emptyset$, and the arc from $*\Sigma'$ to $*F'$, we get an arc $\iota'_i$ from $*_{E'_i}$ to $*F'$, for each $i$. As with \((5.4)\), $\iota'_i$, $i = 0, \cdots, m$, induce a free product decomposition of $\pi_1(F')$:

$$\pi_1(F') \simeq \pi_1(E'_0) \ast \cdots \ast \pi_1(E'_m). \quad (5.7)$$

and the commutative diagram in Fig. 5.1a

\(\text{(A) Special component } \Theta_i'\)

\(\text{(B) Other component } \Theta'\)

**Figure 5.1.** Connecting arcs in $E'_i$

If $\Theta' \subset \partial E'_i$ is a component that does not contain $*_{E'_i}$. Then there is no canonical choice of a connecting arc between $*\Theta'$ and $*_{E'_i}$, so we simply pick an arbitrary arc $\tau_i, \Theta'$ in $E'_i$ that connects $*\Theta'$ to $*_{E'_i}$. Joining $\tau_i, \Theta'$ with the connecting arcs from
π₁(Θ₀)

We prove this by contradiction. Suppose D′₀ = E′₀ ∩ E′₀′ with k ≠ i, and let Σ_j (resp. Σ_j′) be the component of ∂E_i (resp. ∂E_i′) containing D_j (resp. D_j′). Then Σ_j ∩ ∂E_i is sent either to Σ_j′ ∩ ∂E_i′ or to Σ_j′′ ∩ ∂E_j′. In either case, we have the diagram below

\[
\begin{array}{c}
\pi_1(E_0) \ast \cdots \ast \pi_1(E_m) \xrightarrow{\sim} \pi_1(F) \xrightarrow{\phi_F} \pi_1(F') \xrightarrow{\sim} \pi_1(E_0') \ast \cdots \ast \pi_1(E_m') \\
\pi_1(\Theta_0) \xrightarrow{\sim} \pi_1(\Sigma) \xrightarrow{\sim} \pi_1(\Sigma') \xrightarrow{\sim} \pi_1(\square)
\end{array}
\]
In conclusion, for any $E'_l$ with depth $l$, if $D_j = E_l \cap E'_j$, then $D'_j = E'_l \cap E'_j$, and hence the assignment (5.6) induces an isomorphism between the subtrees of $H$ and $H'$ consisting of nodes with depth $l$ or less and edges between them.

The above argument also implies $f(\Theta'_j) = \Theta'_j$ and the commutative diagram:

$$
\begin{array}{ccc}
\pi_1(E_0) \cdots \pi_1(E_m) & \sim & \pi_1(F) \\
\pi_1(E_i) & \phi_f & \pi_1(E'_i) \\
\pi_1(\Theta_j) & \sim & \pi_1(\Theta'_j)
\end{array}
$$

(5.12)

where $\Sigma$ (resp. $\Sigma'$) is the component with $\Theta^*_j \cap \Sigma \neq \emptyset$ (resp. $\Theta^*_j \cap \Sigma' \neq \emptyset$).

As with the case of (5.10), the diagram (5.12) implies that $\pi_1(E_j) \to \pi_1(E'_j)$ is injective, and the inductive step is completed.

Now, since $\phi_f$ is an isomorphism, the induced injection $\phi_{E'_i} : \pi_1(E_i) \to \pi_1(E'_i)$ is in fact an isomorphism, for every $i$, and it fits into the commutative diagram:

$$
\begin{array}{ccc}
\pi_1(E_i) & \phi_{E'_i} & \pi_1(E'_i) \\
\pi_1(\Theta'_j) & & \pi_1(\Theta'_j)
\end{array}
$$

(5.13)

We have shown that $f(\Theta'_j) = \Theta'_j$. We shall see that $f$ sends other components of $\partial E_i$ to $\partial E'_i$ as well.

**Claim 2:** If $\Theta$ is a component of $\partial E_i$, $f(\Theta)$ is a component of $\partial E'_i$.

It suffices to consider the case where $\Theta$ does not contain $D_i$. Suppose $\Theta' := f(\Theta)$ is a component of $\partial E'_i$. Then we have the following diagram:

$$
\begin{array}{ccc}
\pi_1(E_0) \cdots \pi_1(E_m) & \sim & \pi_1(F) \\
\pi_1(E_i) & \phi_f & \pi_1(E'_i) \\
\pi_1(\Theta) & \sim & \pi_1(\Theta')
\end{array}
$$

(5.14)

where all squares commute strictly except for the right square, which commutes up to conjugation due to (5.9), and $\Sigma, \Sigma'$ are components with non-trivial intersection with $\Theta, \Theta'$, respectively. The diagram (5.14) implies that the intersection

$$
g \cdot \pi_1(E'_j) \cdot g^{-1} \cap \pi_1(E'_i)
$$

(5.15)

contains the image of $\pi_1(\Theta')$ under the composition

$$
\pi_1(\Theta') \to \pi_1(\Sigma') \to \pi_1(F'),
$$

for some $g \in \pi_1(F)$, and thus (5.15) is non-trivial. This implies $j = i$, for if $j \neq i$, it would contradicts that $\pi_1(E_i)$ and $\pi_1(E'_j)$ are factors in (5.7). This proves that $f$ restricts to a homeomorphism between $\partial E_i$ and $\partial E'_i$.

- **Modify $\tau_{i, \Theta}$ such that $g$ in (5.9) is the identity.** Recall that $\tau_{i, \Theta}$ is the randomly selected arc from $*_{E'_i}$ to $*_{E'_i}$ in Fig. 5.1 and it induces an injection

$$
\varphi'_{i,*} : \pi_1(E'_i) \to \pi_1(F'),
$$

which satisfies (5.9).

To find out what $g$ in (5.9) is, we first observe that commutative diagrams in Fig. 5.1 entail commutative diagrams in Fig. 5.2 respectively.
for some $g$ in (5.7) and hence (5.18) entails

$$\alpha$$ where is a loop representing $g \cdot \ldots \cdot \pi_1$. Then diagrams in Fig. 5.2 imply that

$$(\Lambda) \phi_{E_i} \text{ and } \iota'$$

**Figure 5.2.** Comparing $\vartheta'_i$ and $\iota'_i$

Denote the composition

$$\pi_1(\Theta) \to \pi_1(E_i) \xrightarrow{\phi_{E_i}} \pi_1(E'_i)$$

in (5.2a) by $\lambda$ and the composition

$$\pi_1(\Theta) \to \pi_1(\Theta') \xrightarrow{\tau_{i, \Theta'}} \pi_1(E'_i)$$

in (5.2b) by $\kappa$. Then diagrams in Fig. 5.2 imply that

$$\vartheta'_i \circ \lambda = \iota'_i \circ \lambda. \quad (5.16)$$

On the other hand, by (5.9) we know that

$$\vartheta'_i = g \cdot \iota'_i \cdot g^{-1}, \quad (5.17)$$

for some $g \in \pi_1(F')$. This, along with (5.16), implies that

$$\iota'_i(\lambda(x)) = \vartheta'_i(\kappa(x)) = g \cdot \iota'_i(\kappa(x)) \cdot g^{-1}. \quad (5.18)$$

Via the identification (5.7), we may assume $\iota'_i$ is the inclusion into the factor $\pi_1(E'_i)$ in (5.7) and hence (5.18) entails

$$\pi_1(E'_i) \cap g \cdot \pi_1(E'_i) \cdot g^{-1} \neq \emptyset,$$

and therefore $g \in \pi_1(E'_i)$. Replacing the connecting arc $\tau_{i, \Theta'}$ with $\delta_{i, \Theta'} = \alpha \ast \tau_{i, \Theta'}$, where $\alpha$ is a loop representing $g$, we obtain a new composition

$$\pi_1(\Theta) \to \pi_1(\Theta') \xrightarrow{\delta_{i, \Theta'}} \pi_1(E'_i)$$

which is $g \cdot \kappa \cdot g^{-1}$ and hence identical to $\lambda$. As with (5.8), $\delta_{i, \Theta'}$ induces an injection

$$\pi_1(E'_i) \to \pi_1(F')$$

which is identical to $g^{-1} \cdot \vartheta'_i \cdot g$ and hence $\iota'_i$. So, the connecting arc $\delta_{i, \Theta'}$ is what we are looking for; it makes the cube below commute

$$\begin{array}{c}
\pi_1(F) \xrightarrow{\pi_1(\Theta)} \pi_1(\Sigma) \\
\pi_1(E_i) \xrightarrow{\phi_{E_i}} \pi_1(E'_i) \\
\pi_1(F) \xrightarrow{\pi_1(\Theta)} \pi_1(\Sigma) \\
\pi_1(E'_i) \xrightarrow{\tau_{i, \Theta'}} \pi_1(E''_i)
\end{array}$$

$$(5.19)$$
Consequently, the system of connecting arcs in $S'$ is also good and compatible with the one in $S$; In particular, we have the commutative diagram:

\[
\begin{array}{ccc}
\pi_1(E_i) & \xrightarrow{\phi_i} & \pi_1(E_i') \\
\downarrow & & \downarrow \\
\pi_1(\Theta) & \xrightarrow{f_i} & \pi_1(\Theta')
\end{array}
\] (5.20)

for every component $\Theta$ of $\partial E_i$.

Now, we are in a position to apply Lemma 4.6 to complete the proof for Case 1. Lemma 4.6 along with Lemma 4.2 implies that there exists a homeomorphism $f_{E_i} : E_i \to E_i'$, which realizes $\phi_{E_i}$ and restricts to $f_{\partial E_i}$ for every $i$. Gluing them together along $D_i$, $i = 1, \ldots, m$, we get a homeomorphism $f : F \to F'$, which realizes $\phi_F$ and restricts to $f_{\partial F}$. Applying the same procedure to every solid part $F_j$ in $(S^3, \Sigma)$, and then gluing homeomorphisms $f_{F_j} : F_j \to F_j'$, $j = 0, \ldots, n$, along their restrictions on boundaries, we obtain the equivalence between $(S^3, \Sigma)$ and $(S^3, \Sigma')$.

Case 2. General pair $S = (S^3, \Sigma)$. By Proposition 3.2, the pair $S$ admits a unique non-splittable graft decomposition:

\[ S \simeq S_1 \ast \cdots \ast S_m. \] (5.21)

Given a solid part $F$ of $S$, the graft decomposition (5.21) induces a prime decomposition of $F$

\[ F \simeq M_0 \# \cdots \# M_l. \] (5.22)

Let $S_i, i = 1, \ldots, l$, denote separating spheres in (5.22) and $\bar{M}_i, i = 0, \ldots, l$, closures of components of the complement

\[ F \setminus \bigcup_{i=1}^l S_i \]

Note that $\bar{M}_i$ is obtained by capping $M_i$ with a 3-ball.

• A good system of connecting arcs with respect to (5.22). Without loss of generality, we may assume $*_{\Sigma} \in \bar{M}_0$, and we select a point $*_{M_i} \in \bar{M}_i$, for each $i \neq 0$. Secondly, connect $*_{M_i}$ to $*_{\Sigma}$ via an arc in $F$, and for each component $\Sigma$ of $\partial M_i$, we choose an arc in $\bar{M}_i$ connecting the base point $*_{\Sigma}$ to $*_{M_i}$. Joining the connecting arc from $*_{\Sigma}$ to $*_{M_i}$ with the one from $*_{M_i}$ to $*_{\Sigma}$, we obtain a system of connecting arcs compatible with (5.22) (Definition 1.1). Repeating the procedure for every solid part $F$ of $S = (S^3, \Sigma)$, and then applying $\mathcal{F}\mathcal{T}$ to (5.21), we obtain a graft decomposition of the fundamental tree of $S$:

\[ \mathcal{F}\mathcal{T}(S) \simeq \mathcal{F}\mathcal{T}(S_1) \ast \cdots \ast \mathcal{F}\mathcal{T}(S_m). \] (5.23)

Since $\mathcal{F}\mathcal{T}(S)$ and $\mathcal{F}\mathcal{T}(S')$ are equivalent, (5.23) induces a graft decomposition of $\mathcal{F}\mathcal{T}(S')$:

\[ \mathcal{F}\mathcal{T}(S') \simeq T_1 \ast \cdots \ast T_m. \] (5.24)

• The “algebraic” graft decomposition (5.24) can be realized by the non-splittable graft decomposition of $S'$. In other words, we want to show that, after reindexing if necessary, the non-splittable graft decomposition of $S'$,

\[ S' \simeq S'_1 \ast \cdots \ast S'_m \]

satisfies $\mathcal{F}\mathcal{T}(S'_i) \simeq T_i, i = 1, \ldots, m.$
We prove the claim by induction on the length $m$ of the graft decomposition of $S$. When $m = 1$, $S$ and therefore $S'$ are non-splittable by Lemma 4.4. The claim follows trivially in this case.

Suppose $m > 1$. Then we index solid parts $F_j$, $j = 0, \ldots, n$, of $S$ such that $F_j$ has the same depth as $F_i$ or greater in $\Lambda_S$ if $j \geq i$, and we consider

$$k := \max\{j \mid F_j \text{ is reducible}\}.$$  \hfill (5.25)

**Subcase 1:** If $\partial F_k$ contains at least one spherical component $\mathfrak{S}$ that does not separate $F_k$ from $\infty$, then $\mathfrak{S}$ bounds a 3-ball $B$ with $B \cap \Sigma = \emptyset$. Up to reindexing (5.23), we may assume $S_m$ is the trivial pair of genus 0 containing $\mathfrak{S}$. The corresponding component $\mathfrak{S}'$ in $\Sigma'$ is also a 2-sphere, and $S'_m = (\mathfrak{S}', \mathfrak{S}')$ realizes $T_m$. Let $S'$ be the pair obtained by removing $\mathfrak{S}'$ from $\Sigma'$. Then we have the graft decomposition of $\mathcal{FT}(S')$:

$$\mathcal{FT}(S') \simeq T_1 \ast \cdots \ast T_{m-1}.$$ \hfill (5.26)

By induction hypothesis, the non-splittable graft decomposition of $S'$ realizes (5.26): the claim then follows from the fact that $S \simeq S' \ast \cdots \ast S'_m$.

**Subcase 2:** Suppose $\partial F_k$ contains only one spherical component $\mathfrak{S}$ and it separates $F_k$ from $\infty$. Let $S_j$ be the trivial pair of genus 0 induced from $\mathfrak{S}$. Then $S \simeq S \ast S_j \ast \mathfrak{S}$, where $S$ is the pair obtained by removing components of $\Sigma$ on the side of $\mathfrak{S}$ opposite to $\infty$, and $\mathfrak{S}$ is the pair obtained by removing components of $\Sigma$ on the same side of $\mathfrak{S}$ as $\infty$. This implies, up to reindexing, the graft decompositions:

$$\mathcal{FT}(S) \simeq \mathcal{FT}(S_1) \ast \cdots \ast \mathcal{FT}(S_{l-1})$$

$$\mathcal{FT}(S) \simeq \mathcal{FT}(S_{l+1}) \ast \cdots \ast \mathcal{FT}(S_m).$$

On the other hand, the corresponding 2-sphere $\mathfrak{S}'$ in $\Sigma'$ realizes $T_l$ and induces a graft decomposition of $S'$:

$$S' \simeq S' \ast \cdots \ast S'_l \ast \mathfrak{S}'$$

such that

$$\mathcal{FT}(S'_l) \simeq T_l$$

$$\mathcal{FT}(S') \simeq T_{l+1} \ast \cdots \ast T_m.$$ \hfill (5.28)

As with Subcase 1, the induction hypothesis implies the claim.

**Subcase 3:** Suppose $\partial F_k$ has no spherical components. Then there exists a 2-sphere $\mathfrak{S}$ separating $F_k$ into two 3-manifolds $F_k$ and $\overline{F}_k$ such that the one, say $F_k$, on the side of $\mathfrak{S}$ opposite to $\infty$ is non-splittable. Removing components of $\Sigma$ that are on the same side of $\mathfrak{S}$ as $\infty$, we get a non-splittable pair, and up to reindexing, we may assume it is $S_m$ in (5.21).

Let $B$ (resp. $A_m$) denote the fundamental group $\pi_1(F_k)$ (resp. $\pi_1(\overline{F}_k)$). Then $\pi_1(F)$ is isomorphic to a non-trivial free product $B \ast A_m$ such that, for any component $\Sigma$ of $\partial F_k$ (resp. of $\partial \overline{F}_k$), the homomorphism

$$\pi_1(\Sigma) \to \pi_1(F)$$

factors through $B$ (resp. $A_m$). Via equivalence (5.1) between $S$ and $S'$, the fundamental group of the solid part $F'_k$ of $S'$ corresponding to $F_k$ also satisfies

$$\pi_1(F'_k) \simeq B \ast A_m,$$

and for any $\pi_1(\Sigma')$ in $T_m$ (resp. in $T_1 \ast \cdots \ast T_{n-1}$), the homomorphism

$$\pi_1(\Sigma') \to \pi_1(F'_k) \simeq B \ast A_m.$$
factors through $A_m$ (resp. $B$). Applying Lemma 4.4 we obtain a decomposition

$$ F'_k \simeq \tilde{F}'_k \# \tilde{T}'_k \text{ with } \pi_1(\tilde{F}'_k) \simeq B, \quad \pi_1(\tilde{T}'_k) \simeq A_m $$

such that for any component $\Sigma'$ of $\tilde{F}'_k$ (resp. $\tilde{T}'_k$) the homomorphism

$$ \pi_1(\Sigma') \to \pi_1(F'_k) $$

factors through

$$ \pi_1(\Sigma') \to \pi_1(\tilde{F}'_k) \quad (\text{resp. } \pi_1(\Sigma') \to \pi_1(\tilde{T}'_k)). $$

This implies a graft decomposition

$$ S' \simeq \tilde{S}' \times \ldots \times \tilde{S}_m' \text{ with } \cal{FT}(\tilde{S}') \simeq T_1 \times \ldots \times T_{n-1}; \quad \cal{FT}(\tilde{S}_m') \simeq T_m, $$

where $\tilde{S}'$ is the pair consisting of components of $\Sigma$ that are in $\partial \tilde{F}'_k$, and $S_m'$ is the pair containing only components of $\partial \tilde{T}'_k$. By Lemma 4.4, $S_m'$ is necessarily non-splittable, and the claim follows from the induction hypothesis.

In this way, we see that any equivalence between $S$ and $S'$ induces an equivalence between $\cal{FT}(S)$ and $\cal{FT}(S')$, for every $i$; thus the problem is reduced to Case 1, for once an equivalence between $S_i$ and $S'_i$ is established, for every $i$, we can glue them together to get an equivalence between $S$ and $S'$.

\[ \square \]

6. Surface links

Given a pair $S = (S^3, \Sigma)$, we say it has a star-shaped depth tree $\Lambda_S$ if every node except for the base node in $\Lambda_S$ connects to the base node by exactly one edge.

**Definition 6.1 (Surface link).** A pair $S = (S^3, \Sigma)$ is called a surface link if its depth tree is star-shaped.

For surface links, we adopt the convention that the solid part $F_i$ is separated from $\infty$ by $\Sigma_i$, for every $i \neq 0$. Also, recall that given a pair $S = (S^3, \Sigma)$ the associated surface link corresponding to a node $\alpha$ in $\Lambda_S$ is the pair given by forgetting components of $\Sigma$ that are not in the boundary of the corresponding solid part $F_\alpha$ and letting the new base point be in $F_\alpha$. We are now in the position to prove Theorem 4.1, which asserts the topology of $S$ is essentially determined by its associated surface links.

**Proof of Theorem 4.1.** Let $\alpha$ and $\beta$ be two adjacent nodes in $\Lambda_S$ and $F_\alpha = \cal{MT}(\alpha)$, $F_\beta = \cal{MT}(\beta)$, $F'_\alpha = \cal{MT}'(\alpha)$, and $F'_\beta = \cal{MT}'(\beta)$.

By assumption, the labeled surface links associated to $\alpha$ (resp. $\beta$) and $\cal{E}(\alpha)$ (resp. $\cal{E}(\beta)$) are equivalent, so we have o.p. homeomorphisms $f_\alpha : S^3 \to S^3$ and $f_\beta : S^3 \to S^3$ sending $F_\alpha$ to $F'_\alpha$, and $F_\beta$ to $F'_\beta$, respectively. Now, we construct an o.p. homeomorphism $f : S^3 \to S^3$ such that $f(F_\alpha \cap F_\beta) = F'_\alpha \cap F'_\beta$ and $f(F_\alpha \cup F_\beta) = F'_\alpha \cup F'_\beta$.

Let $\Sigma$ (resp. $\Sigma'$) be the intersection $F_\alpha \cap F_\beta$ (resp. $F'_\alpha \cap F'_\beta$) and $\tilde{F}_\alpha$ and $\tilde{F}_\beta$ (resp. $\tilde{F}'_\alpha$ and $\tilde{F}'_\beta$) be the closures of the complements of $S^3 \setminus \Sigma$ (resp. $S^3 \setminus \Sigma'$) that contain $F_\alpha$ and $F_\beta$ (resp. $F'_\alpha$ and $F'_\beta$), respectively.

Let $N(\Sigma)$ (resp. $N(\Sigma')$) be a tubular neighborhood of $\Sigma$ (resp. $\Sigma'$) in $S^3$. Then there are homeomorphisms

$$ S^3 \simeq \tilde{F}_\alpha \cup \Sigma \times I \cup \tilde{F}_\beta \quad \text{and} \quad S^3 \simeq \tilde{F}'_\alpha \cup \Sigma' \times I \cup \tilde{F}'_\beta, $$

sending $N(\Sigma)$ to $\Sigma \times I$, (resp. $N(\Sigma')$ to $\Sigma' \times I$) where $i_\alpha$ (resp. $i'_\alpha$) and $i_\beta$ (resp. $i'_\beta$) are inclusions $\Sigma \hookrightarrow \partial \tilde{F}_\alpha$ and $\Sigma \hookrightarrow \partial \tilde{F}_\beta$. (resp. $\Sigma' \hookrightarrow \partial \tilde{F}'_\alpha$ and $\Sigma' \hookrightarrow \partial \tilde{F}'_\beta$).
Because \( f_\alpha \) and \( f_\beta \) are isotopic homeomorphisms, there is a homeomorphism
\[
\Psi : S^3 \times I \to S^3 \times I
\]
such that \( \Psi(\cdot, 0) = f_\alpha(\cdot) \) and \( \Psi(\cdot, 1) = f_\beta(\cdot) \). It restricts to a homeomorphism
\[
\Psi|_{\Sigma \times I} : \Sigma \times I \to \Psi(\Sigma \times I) =: W \subset S^3 \times I,
\]
which implies the following commutative diagram of groups, up to conjugation:

\[
\begin{array}{ccc}
\pi_1(\Sigma) & \xrightarrow{(f_\beta|\Sigma)_*} & \pi_1(\Sigma') \\
\downarrow i_{1*} & & \downarrow i_{1*}' \\
\pi_1(\Sigma \times I) & \xrightarrow{(\Psi|_{\Sigma \times I})_*} & \pi_1(W) \\
\downarrow i_{0*} & & \downarrow i_{0*}' \\
\pi_1(\Sigma) & \xrightarrow{(f_\alpha|\Sigma)_*} & \pi_1(\Sigma')
\end{array}
\quad (6.1)
\]

where \( i_{k*} \) are isomorphisms induced from the inclusion \( i_k : \Sigma \hookrightarrow \Sigma \times \{k\} \subset \Sigma \times I \) (resp. \( i'_k : \Sigma' \hookrightarrow \Sigma' \times \{k\} \subset W \), \( k = 0, 1 \).

The diagram \((6.1)\) implies that the homomorphisms \( f_\beta|_{\Sigma} \) and \( f_\alpha|_{\Sigma} \) are equivalent, up to conjugation: By Lemma \((4.1)\), \( f_\alpha|_{\Sigma} \) and \( f_\beta|_{\Sigma} \) are isotopic, and hence there is a homeomorphism
\[
\Psi_\Sigma : \Sigma \times I \to \Sigma' \times I
\]
such that \( \Psi(\cdot, 1)\Sigma = f_\beta|\Sigma(\cdot) \) and \( \Psi(\cdot, 0)\Sigma = f_\alpha|\Sigma(\cdot) \).

Gluing the three homeomorphisms \( f_\alpha \), \( f_\beta \) and \( \Psi_\Sigma \) together, we get a homeomorphism
\[
S^3 \simeq \tilde{F}_\alpha \bigcup_{i_\alpha} \Sigma \times I \bigcup_{i_\beta} \tilde{F}_\beta \xrightarrow{f_\alpha \cup \Psi \cup f_\beta} \tilde{F}_\alpha' \bigcup_{i_\alpha} \Sigma' \times I \bigcup_{i_\beta} \tilde{F}_\beta' \simeq S^3,
\]
which sends \( F_\alpha \cup F_\beta \) to \( F'_\alpha \cup F'_\beta \) and \( F_\alpha \cap F_\beta = \Sigma \) to \( F'_\alpha \cap F'_\beta = \Sigma' \).

Applying the above construction, we glue homeomorphisms between surface links associated to every adjacent nodes to obtain an o.p. homeomorphism
\[
f : S^3 \to S^3
\]
that sends \( \Sigma \) to \( \Sigma' \).

In view of Theorem \((1.3)\) and the fact that \( \text{sd} \Lambda_S \) and \( \text{sd} \Lambda_{S'} \) are equivalent as (based) categories if and only if \( \Lambda_S \) and \( \Lambda_{S'} \) are isomorphic as (based) graphs, to differentiate two pairs \( S \) and \( S' \), we can first compare their depth trees \( \Lambda_S \) and \( \Lambda_{S'} \) and if they are isomorphic, then we analyze surface links associated to corresponding nodes in \( \Lambda_S \) and \( \Lambda_{S'} \).

**Remark 6.1**. The existence of an equivalence \( E : \text{sd} \Lambda_S \to \text{sd} \Lambda_{S'} \) is necessary in Theorem \((1.3)\). Both pairs \( S \) and \( S' \) in Fig. \((6.1a)\) have two trivial knots, two Hopf links and one Whitehead link as associated surface links, but no equivalence between \( \text{sd} \Lambda_S \) and \( \text{sd} \Lambda_{S'} \) induces a 1-1 correspondence between their associated surface links.

The use of labeled pairs is also essential in Theorem \((1.3)\). For pairs \( S \) and \( S' \) in Fig. \((6.1l)\), the assignment sending Node \( i \) to Node \( i \) gives a (unique) equivalence between \( \text{sd} \Lambda_S \) and \( \text{sd} \Lambda_{S'} \) such that surface links associated to corresponding nodes are equivalent as unlabeled pairs. For instance, surface links corresponding to Node 1 are equivalent handlebody links (Fig. \((6.1l)\)). But, since \( F_0 \cap F_1 = \Sigma \) in \( S \), whereas \( F_0 \cap F_1 = \Sigma \) in \( S' \), they are not equivalent as labeled pairs.

**Definition 6.2.** A handlebody link \((S^3, \Sigma)\) is a surface link with the solid part \( F_i \) a handlebody, for \( i \neq 0 \).

We derive some invariants for handlebody links employing Theorem \((3.4)\) and homomorphisms of \( \pi_1(F_0) \) to a finite group \( G \).
Definition 6.3. Let $S = (S^3, \Sigma)$ be a handlebody link, and $\Sigma_i, i = 1, \ldots, n$, be the components of $\Sigma$. Then $\mathcal{H}(S)$ denotes the set of homomorphisms from $\pi_1(F_0)$ to $G$, up to automorphisms of $G$.

An element $x$ in $\mathcal{H}(S)$ is called proper with respect to $\Sigma_i$ if representing homomorphisms of $x$ are surjective, but become non-surjective after precomposing with the homomorphism $\pi_1(\Sigma_i) \rightarrow \pi_1(F_0)$.

An element $x$ in $\mathcal{H}(S)$ is proper with respect to a subset $A$ of $\{\Sigma_i\}_{i=1}^n$ if $x$ is proper with respect to every member in $A$. The set of proper elements in $\mathcal{H}(S)$ with respect to $A$ is denoted by $PH(S)_A$.

Definition 6.4. Given $A \subset \{\Sigma_i\}_{i=1}^n, |A| = k$, the $G$-image of a handlebody link $S = (S^3, \Sigma)$ with respect to $A$ is a set of unordered $k$-tuples of subgroups of $G$, up to automorphism, indexed by elements in $PH(S)_A$ defined as follows:

$$G\text{-im}(S)_A := \{(H_1, H_2, \ldots, H_k)_x \mid A = \{\Sigma_{i_1}, \ldots, \Sigma_{i_k}\}, x \in PH(S)_A\}, \quad (6.2)$$

where $H_i$ in a $k$-tuple $(\cdot \cdot)_x$ is the image of the homomorphism

$$\operatorname{Ker}(\pi_1(\Sigma_i) \rightarrow \pi_1(F_i)) < \pi_1(\Sigma_i) \rightarrow \pi_1(F_0) \phi \rightarrow G,$$

and $\phi$ is a representative of $x$.

The $k$-fold $G$-image of $S$ is defined by

$$G\text{-im}(S)^k := \{G\text{-im}(S)_A \mid A \subset \{\Sigma_i\}_{i=1}^n, |A| = k\}. \quad (6.3)$$

When $k = 1$, we call it the individual $G$-image of $S$, and omit the superscript $1$. The following corollary of Theorem 3.4 implies the $k$-fold $G$-image is an invariant of handlebody links.

**Theorem 6.1.** Let $S = (S^3, \Sigma), S' = (S^3, \Sigma')$ be handlebody links with $\Sigma, \Sigma'$ both having $n$ components. Then $S$ and $S'$ are equivalent if and only if there exists a permutation $\sigma$ on $\{1, \ldots, n\}$ and isomorphisms $\phi_0$ and $\phi_i$ such that the diagram

$$\begin{array}{ccc}
\pi_1(F_0) & \xrightarrow{\phi_0} & \pi_1(F'_0) \\
\pi_1(\Sigma_i) & \xrightarrow{\phi_i} & \pi_1(\Sigma'_{\sigma(i)})
\end{array}$$
commutes, up to conjugation, subgroups $\phi_i(\text{Ker}(i_1))$, $\text{Ker}(i'_\sigma(i_1))$ are conjugate in $\pi_1(\Sigma'_\sigma(i_1))$, and the induced isomorphism from $\phi_i$ on homology preserves intersection forms. In particular, the only if part implies

$$G \cdot \text{im}(S)_A = G \cdot \text{im}(S'_{\sigma(A)}),$$

for any $A = \{\Sigma_1, \cdots, \Sigma_{i_1}\} \subset \{\Sigma_i\}_{i=1}^n$, where $\sigma(A) = \{\Sigma'_\sigma(i_1), \cdots, \Sigma'_{\sigma(i_1)}\}$.

Example 6.1. We compute individual $G$-images of handlebody links in Fig. 6.3. Note that they have homeomorphic complements as HL2 and HL3 can be obtained by twisting HL1 along some annuli (see [9], [12], or [2, Sect. 4] for the twist construction); having different individual $A_4$-images, they are not equivalent though.

In Fig. 6.3 $\Sigma_1$ denotes the component in bold and $\Sigma_2$ the other component. Using the program Appcontour [13], we find that there are 33 proper homomorphisms with respect to $\Sigma_1$ and 33 with respect to $\Sigma_2$. 18 among them have that the image of $\pi_1(\Sigma_i)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ in $A_4$, 12 among them $\mathbb{Z}_3$, and 3 among them $\mathbb{Z}_2$. Their individual $A_4$-images are recorded in Table 1, where we can also see that no ambient isotopy can swap the two components of HL1 or of HL3.

|       | Image of $\pi_1(\Sigma_i)$ in $A_4$ |
|-------|--------------------------------------|
|       | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 18$ | $\mathbb{Z}_2 : 12$ | $\mathbb{Z}_2 : 3$ |
| HL1   | $\Sigma_1$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 8$ | $\mathbb{Z}_3 : 9$ | $\mathbb{Z}_2 : 2$ |
|       | $\Sigma_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 10$ | $\mathbb{Z}_3 : 9$ | $\mathbb{Z}_2 : 3$ |
| HL2   | $\Sigma_1$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 8$ | $\mathbb{Z}_3 : 9$ | $\mathbb{Z}_2 : 3$ |
|       | $\Sigma_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 9$ | $\mathbb{Z}_3 : 12$ | $\mathbb{Z}_2 : 3$ |
| HL3   | $\Sigma_1$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 8$ | $\mathbb{Z}_3 : 9$ | $\mathbb{Z}_2 : 3$ |
|       | $\Sigma_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 10$ | $\mathbb{Z}_3 : 12$ | $\mathbb{Z}_2 : 2$ |

There are only three proper homomorphisms with respect to $\{\Sigma_1, \Sigma_2\}$, and the 2-fold $A_4$ images of HL1, HL2 and HL3 are recorded in Table 2 which shows that the 2-fold $A_4$-image is unable to differentiate HL1 and HL3.
Table 2. The 2-fold $A_4$-images

|     | $(\Sigma_1, \Sigma_2)$ |
|-----|-----------------------|
| HL 1| $\{(\mathbb{Z}_2, \mathbb{Z}_3), (\mathbb{Z}_2 \times \mathbb{Z}_2), (\mathbb{Z}_3, \mathbb{Z}_3)\}$ |
| HL 2| $\{(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3), (\mathbb{Z}_2 \times \mathbb{Z}_2), (\mathbb{Z}_3, \mathbb{Z}_3)\}$ |
| HL 3| $\{(\mathbb{Z}_3, \mathbb{Z}_2), (\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3), (\mathbb{Z}_3, \mathbb{Z}_3)\}$ |

Table 3. The $A_4$-image of the link in Fig. 6.2

|     | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_2$ | $0$ |
|-----|---------------------|-------------|-------------|-----|
| $\Sigma$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 32$ | $\mathbb{Z}_3 : 36$ | $\mathbb{Z}_2 : 12$ | $0 : 4$ |
| $\Sigma'$ | $\mathbb{Z}_2 \times \mathbb{Z}_2 : 32$ | $\mathbb{Z}_3 : 52$ | $\mathbb{Z}_2 : 24$ | $0 : 0$ |

In the above examples, the numbers of proper homomorphisms with respect to $\Sigma_1$ or $\Sigma_2$ are not only the same but the images of $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$ are also identical. This is not true in general.

Example 6.2. The 2-component handlebody link in Fig. 6.2 is an example showing that numbers of proper homomorphisms with respect to different components could be quite different even if their knot types are the same individually. The link in Fig. 6.2 has 120 proper homomorphisms with respect to $\Sigma$ and 136 proper homomorphisms with respect to $\Sigma'$. Table 3 displays the individual $A_4$-image of the link.

References

[1] A. Hatcher, A proof of the Smale Conjecture. Ann. of Math. 117 (1983), 553–607.
[2] G. Bellettini, M. Paolini, Y.-S. Wang, On closed oriented surfaces in the 3-sphere, arXiv:1902.05030[math.GT].
[3] B. Farb, D. Margalit, A Primer on Mapping Class Groups, Princeton University Press, (2011).
[4] R. H. Fox, On the imbedding of polyhedra in 3-space, Ann. of Math. 2(49) (1948), 462–470.
[5] C. Gordon, J. Luecke, Knots are determined by their complements J. Amer. Math. Soc. 2 (1989), 371–415.
[6] W. Heil, On Kneser’s conjecture for bounded 3-manifolds, Proc. Comb. Phil. Soc. 71 (1972), 71–90.
[7] J. Hempel, 3-manifolds, AMS Chelsea Publishing, Providence, RI, (2004).
[8] A. G. Kurosh, The theory of groups Translated from the Russian and edited by K. A. Hirsch. 2nd English ed. 2 volumes Chelsea Publishing Co., New York (1960) Vol. 1: 272 pp. Vol. 2: 308 pp.
[9] J. H. Lee, S. Lee, Inequivalent handle-body-knots with homeomorphic complements, Algebr. Geom. Topol. 12 (2012), 1059-1079.
[10] E. E. Moise Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung. Annals of Mathematics, Second Series, 56 (1953), 96–114.
[11] E. E. Moise, Geometric Topology in Dimensions 2 and 3, Springer-Verlag, Grad. Texts in Math. 47, (1977).
[12] M. Motto, Inequivalent genus two handlebodies in $S^3$ with homeomorphic complements, Topology Appl. 36(3) (1990), 283–290.
[13] M. Paolini, Appcontour. Computer software. Vers. 2.5.3. Apparent contour. (2018) http://appcontour.sourceforge.net/.
[14] J. R. Stallings, A topological proof of Grushko’s theorem on free products Math. Z. 90 (1965), 1–8.
[15] S. Suzuki, On surfaces in 3-sphere: prime decompositions, Hokkaido Math. J. 4 (1975), 179–195.
[16] G. A. Swarup, Some properties of 3-manifolds with boundary, Quart. J. Math. Oxford 21 (1970), 1–23.
[17] F. Waldhausen, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*, Topology *6* (1967), 505–517.
[18] F. Waldhausen, *Heegaard-Zerlegungen der 3-Sphäre*, Topology *7* (1968), 195–203.
[19] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. *87* (1968), 56–88.
[20] W. Whitten, *Knot complements and groups*, Topology *26*, Issue 1, (1987), 41–44.

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