PERIODS OF AN ARRANGEMENT OF SIX LINES AND CAMPEDELLI SURFACES

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ABSTRACT. We define a period map for classical Campedelli surfaces, using a covering trick as in the case of Enriques surfaces: the period map is shown to come from a family of Enriques surfaces, obtained as quotients of the Campedelli surface by an involution.

The period map realises an isomorphism between a projective variety obtained by invariant theory, and the Baily-Borel compactification of an arithmetic quotient, in the same fashion as in the work of Matsumoto, Sasaki and Yoshida. The result is proved from scratch using traditional methods.

As another consequence we determine properties of the monodromy of Campedelli surfaces with a choice of double cover.

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INTRODUCTION

This work aims at investigating the periods of Campedelli surfaces, and their monodromy. Campedelli surfaces were among the first examples of general type surfaces with \( p_g = 0 \), meaning that there exists no holomorphic differential 2-form on them: they do not have periods in the traditional sense. However, they do have nonzero 2-forms with values in a non trivial local system. Much like in the case of Enriques surfaces, this allows to define a period vector, which corresponds to periods of an étale double cover.

Campedelli surfaces. The construction of Campedelli surfaces is very explicit: consider the projective space \( \mathbb{P}^6 \) and the linear system of quadrics generated by the forms \( x_i^2 \), where \( x_1, \ldots, x_7 \) are a system of projective coordinates. Selecting a subsystem generated by four general equations defines a surface in \( \mathbb{P}^6 \), which is a smooth complete intersection.

Consider also the finite group \( G = (\mathbb{Z}/2)^3 \) and its seven non-trivial characters with values in \( \{ \pm 1 \} \): it defines an action on \( \mathbb{P}^6 \), when given a mapping between these characters and the seven coordinates \( x_i \). In generic situations, the action of \( G \) is free, and Campedelli surfaces are obtained as quotients under this action. A less general choice of equations yields surfaces with rational double points: if the action of \( G \) is still free, the quotient surface is the canonical model of a smooth surface containing rational \((-2)\)-curves, which still satisfies \( p_g = q = 0 \), and can be considered an appropriate extension of the definition. In this construction, \( G \) can be identified with the topological fundamental group of Campedelli surfaces.

Campedelli surfaces have a natural projective parameter space [Miy77] which is the Grassmann variety \( \text{Gr}(4,7) \), corresponding to the choice of its equations inside the vector space generated by the \( x_i^2 \) polynomials. The parameter point of a Campedelli surface \( X \) is only defined up to a \( G \)-equivariant change of coordinates (action of the diagonal torus \( T \)). It also depends on an identification between \( G \) and \( (\mathbb{Z}/2)^3 \), which determines the labelling of the seven coordinates. It is thus natural to consider the variety \( (\text{Gr}(4,7) \sslash T) / \text{GL}_3(\mathbb{F}_2) \) as a (compactification of a) moduli variety for Campedelli surfaces. The finite group \( \text{GL}_3(\mathbb{F}_2) \) represents the coordinate permutations that do not change the resulting surface.

In order to study a period map, we are required to fix a non trivial character \( \kappa \) of \( G \) (in order to define the period vector): the natural coarse moduli variety for pairs \((X, \kappa)\) is then \( (\text{Gr}(4,7) \sslash T) / S_4 \). The finite group which appears here is the affine group of \((\mathbb{F}_2)^2\), which is isomorphic to \( S_4 \): it is the subgroup of \( \text{GL}_3(\mathbb{F}_2) \) which fixes the chosen character \( \kappa \). It also acts on the coordinates of \( \mathbb{P}^6 \), permuting the six other coordinates (in the same way as it permutes the six pairs of numbers among \( \{1,2,3,4\} \)).

Configurations of lines. The various involutions \( x_i \mapsto -x_i \) of \( \mathbb{P}^6 \) act on intersections of diagonal quadrics: in particular, each of them descends to an involution \( s_i \) of Campedelli surfaces as defined above, and generate a subgroup \((\mathbb{Z}/2)^6 \sslash G \simeq (\mathbb{Z}/2)^3\) of their automorphism group. A geometric way of realising this quotient is obtained by squaring all coordinates, which gives a finite morphism \( \mathbb{P}^6 \to \mathbb{P}^6 \). This morphism is also well-defined on a Campedelli surface, and the image of the induced morphism \( X \to \mathbb{P}^6 \) is a plane (since the squared coordinates of points of \( X \) satisfy linear relations). This exhibits \( X \) as an abelian cover of \( \mathbb{P}^2 \) ramified over a configuration of seven lines [AP09; Par91]. Another moduli variety for Campedelli surfaces is then the GIT quotient \( ((\mathbb{P}^2)^7 \sslash \text{PSL}_3) / \text{GL}_3(\mathbb{F}_2) \), where \( \text{GL}_3(\mathbb{F}_2) \) is seen as a subgroup of \( S_7 \).
We will find that the differential of the period map is never injective: indeed, an étale double cover of a Campedelli surface is a special case of a Todorov surface, whose period map is known not to be injective. More precisely, the period map of Campedelli surfaces can be factored through that of a family of lattice-polarised Enriques surfaces, whose natural parameter space is $\text{Gr}(3, 6)$, up to commuting actions of $\mathbb{Z}/2 ∋ \mathfrak{S}_3 ∼ \mathbb{Z}/2 × \mathfrak{S}_4 ⊂ \mathfrak{S}_6$ and an involution $Q$ which is described in section 3.2. The correspondence between Campedelli surfaces and Enriques surfaces is given by the rational map $\text{Gr}(4, 7) → \text{Gr}(3, 6)$ which maps a (generic) 4-dimensional subspace of $\mathbb{C}^7$ to its 3-dimensional intersection with a given hyperplane $\mathbb{C}^6 ⊂ \mathbb{C}^7$. In terms of configurations of seven lines, it translates to the fact that periods with values in the local system $\mathbb{Z}_\kappa$ do not depend on the position of the line labelled by $\kappa$.

This correspondence gives insight about the relationship between Campedelli surfaces and their periods: the isomorphism classes of Enriques surfaces are determined by their periods according to the work of Horikawa, and the correspondence between Campedelli surfaces and Enriques surfaces described above is given by geometry.

The subgroups of $\mathfrak{S}_6$ which appear are related in the following way: $\mathfrak{S}_4$ acts on the six 2-element subsets of $\{1, 2, 3, 4\}$. Since $\mathfrak{S}_4$ acts on the 3 decompositions of $\{1, 2, 3, 4\}$ as complementary pairs, we can write it as a semi-direct product $\mathfrak{V}_4 ⋊ \mathfrak{S}_3$ where $\mathfrak{V}_4$ is the group of double transpositions. We have inclusions $\mathfrak{S}_4 ∼ (\mathbb{Z}/2)^2 ⋊ \mathfrak{S}_3 ⊂ (\mathbb{Z}/2) ≍ \mathfrak{S}_3 ⊂ \mathfrak{S}_6$ where $(\mathbb{Z}/2) ≍ \mathfrak{S}_3$ is the wreath product of $\mathbb{Z}/2$ by $\mathfrak{S}_3$, which also acts naturally on three pairs of objects. All of these groups act on $\mathbb{C}^6$ by permutation of coordinates, and on $\mathbb{C}^7$ by adding a fixed coordinate.

**Enriques surfaces polarised by a $D_6$ lattice.** The Enriques surfaces appearing as quotients of Campedelli surfaces are also naturally parametrised by their period space: geometric arguments show that their $H^{1,1}(\mathbb{Z})$ contains a distinguished copy of the $D_6$ lattice. Their generic transcendental lattice (orthogonal complement to the $D_6$ lattice) is then isomorphic to $L = \mathbb{Z}^2(2) ⊕ \mathbb{Z}^4(-1)$, which defines a bounded symmetric domain $D_L$ and an arithmetic quotient $X_L = D_L/\mathcal{O}(L)$. The question is then: is it true that all (or almost all) such Enriques surfaces are obtained from Campedelli surfaces (and what are the geometric properties of the correspondence)?

We can obtain the following structure:

**Theorem.** The structure of the period map of Campedelli surfaces $\text{Gr}(4, 7) → X_L$ can be described by the following diagram

$$
\begin{array}{c}
\text{Gr}(4, 7) \to \text{Gr}(3, 6) \\
\downarrow \quad \downarrow \\
\text{Gr}(4, 7)/\mathfrak{S}_4 \to \text{Gr}(3, 6)/(\mathbb{Z}/2)^2 \times \mathfrak{S}_3 \to \text{Gr}(3, 6)/(\mathbb{Z}/2) ≍ \mathfrak{X}_L^{BB}
\end{array}
$$

where $\mathfrak{X}_L^{BB}$ is the Baily-Borel compactification of $X_L$. In the leftmost square, the various arrows are the rational maps coming from the geometric correspondences described earlier.

This statement includes the fact that the natural GIT moduli space for Enriques surfaces polarised by the lattice $D_6$, whose presentation is deduced from the moduli space we chose for Campedelli surfaces, is isomorphic to a Baily-Borel compactification by means of the
period map. The bulk of this paper consists in a proof of this fact, which is to be related to the work of Matsumoto, Sasaki, Yoshida [MSY92], who prove a similar statement for K3 surfaces which are double planes ramified over six lines, with different techniques. Since indeed, a $D_6$-polarised Enriques surfaces are bidouble covers of the plane, also ramified over six lines: from both the geometric and lattice-theoretic point of view, the moduli space appearing here is a 15-fold cover of theirs, but this aspect is not studied here.

Another way of summarising the present work is the following:

**Theorem.** The bidouble covers of the plane, ramified over three pairs of lines have a natural GIT-theoretic moduli space, given by $\text{Gr}(3, 6)/(\mathbb{Z}/2 \wr S_3)$. The period map of Enriques surfaces realises this space as a degree 2 cover of the natural Baily-Borel compactification of the associated period space.

The isomorphism classes of Enriques surfaces which appear are polarised by a lattice $D_6$, and correspond generically to exactly two nonequivalent configurations of lines, related by a Cremona transformation. Such a generic Enriques surface gives rise to four deformation types of Campedelli surfaces with choice of a local system.

Note that given a generic Enriques surface as above, there are two choices of linear systems (corresponding to the inequivalent polarisations of degree four which describes it as a bidouble cover of $\mathbb{P}^2$), such that a double cover, with ramification locus chosen in these linear systems is a Campedelli surface. The additional factor two arises from the fact that there are two ways of constructing double covers with given ramification locus, since an Enriques surface is not simply connected.

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**Notations.** In the table below we gather a list of common notations used throughout the paper.

| Notation | Description |
|----------|-------------|
| $X$      | a Campedelli surface or its canonical model |
| $\tilde{X}$ or $Y$ | the universal cover of $X$ |
| $G$      | the group $(\mathbb{Z}/2\mathbb{Z})^3$ |
| $\kappa$ | a character of $\pi_1(X)$ or $G$ |
| $\mathbb{Z}_\kappa$, $\mathbb{R}_\kappa$, $\mathbb{C}_\kappa$ | the local systems of $\mathbb{Z}$-modules, $\mathbb{R}$-vector spaces, $\mathbb{C}$-vector spaces attached to $\kappa$ |
| $\mathcal{L}_\kappa$ | the 2-torsion holomorphic line bundle associated to $\kappa$ |
| $X_\kappa$ | the étale double cover of $X$ associated to $\kappa$ |
| $s_\kappa$ | the involution of $X$ associated to $\kappa$ |
| $S_\kappa$ | the quotient of $X$ by $s_\kappa$ |
| $T_\kappa$ | the quotient of $X_\kappa$ by the natural lift of $s_\kappa$ |
| $S$      | an Enriques surface |
| $T$      | the K3 universal over of $S$ |
| $\mathbb{Z}^{p,q}$ | the odd unimodular integral lattice of signature $(p, q)$ |
| $D_{p,q}$ | the even sublattice of $\mathbb{Z}^{p,q}$ |
| $D_n$    | the even sublattice of the Euclidean lattice $\mathbb{Z}^n$ |
| $L$      | the lattice $\mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$ |
1. CAMPEDELLI SURFACES

Campedelli surfaces (and numerical Campedelli surfaces which share the same numerical invariants) have been thoroughly studied [Cam32, Miy77, CMLP08]. A number of useful results are contained in the unpublished manuscript of M. Reid [Rei].

In this section we give the basic definitions of Campedelli surfaces, and their first properties. We are mainly concerned about involutions that are induced by coordinate reflections in \( \mathbb{P}^6 \). In the generic case, the quotient of a Campedelli surface \( X \) by such a reflection is an Enriques surface whose periods determine those of \( X \) (proposition 1.9). This Enriques surface usually has six nodes corresponding to isolated fixed points on \( X \).

We then give local properties of the period map. Its differential has rank 4 at points of the moduli space parameterising smooth surfaces (proposition 1.14). We will define later a natural 4-dimensional period domain for Campedelli surfaces. We also prove in section 1.6, along the lines of [Voi86], a property which is needed later: when crossing the discriminant hypersurface, the period of the associated vanishing cycle has non-zero derivative, on the double cover ramified over the (corresponding irreducible component of the) discriminant.

1.1. Description and general properties.

Definition. A numerical Campedelli surface is a minimal smooth projective surface \( X \) with numerical invariants \( p_g = q = 0 \) and \( K_X^2 = 2 \).

A (classical) Campedelli surface is a numerical Campedelli surface whose fundamental group is isomorphic to \( (\mathbb{Z}/2)^3 \).

Many topological invariants of Campedelli surfaces can be calculated in terms of these numbers: since \( \chi(\mathcal{O}_X) = 1 \), by Noether’s formula, the topological Euler characteristic is \( e(X) = c_2(X) = 12 - K_X^2 = 10 \), thus the nonzero Hodge numbers of \( X \) are \( h^{0,0} = h^{2,2} = 1 \) and \( h^{1,1} = 8 \). The signature of \( X \) is \( \tau(X) = (K_X^2 - 2e(X))/3 = -6 \), hence the torsion-free quotient of \( H^2(X, \mathbb{Z}) \) (denoted by \( H^2(X, \mathbb{Z}_{\text{num}}) \)), which is a \( \mathbb{Z} \)-module of rank 8 with a unimodular quadratic form, is isomorphic to \( \mathbb{Z}^{1,7} \), the standard Lorentzian lattice.

Campedelli surfaces have a six-dimensional, unirational moduli variety. This is implied by the following structure theorem:

Theorem 1.1 (see [Miy76] or [Rei]). The universal cover \( \tilde{X} \) of a (classical) Campedelli surface \( X \) is birational to a complete intersection of 4 diagonal quadrics in \( \mathbb{P}^6 \), where \( \pi_1(X) \) acts by its 7 distinct nonzero characters. Moreover, this complete intersection is the canonical model of \( \tilde{X} \), and its quotient is the canonical model of \( X \).

Because of this simple description, we give the name of canonical Campedelli surface to the canonical model itself.

An abstract approach to this property is the fact that the action of \( G = \pi_1(X) \) on \( H^0(\tilde{X}, K_{\tilde{X}}) \) is decomposed as a sum of eigenspaces for each character \( \kappa \in \hat{\mathbb{G}} \), which can be identified with \( H^0(X, K_X \otimes L_\kappa) \), where \( L_\kappa \) is the (flat) line bundle arising from the representation of \( \pi_1(X) \) given by \( \kappa \). Each of these spaces has dimension one, except for \( \kappa = 0 \) (since \( p_g(X) = h^0(X, K_X) = 0 \)). Up to homothety, it is then possible to choose canonical coordinates \( x_\kappa \) (where \( 0 \neq \kappa \in \hat{\mathbb{G}} \)) for the embedding of \( \tilde{X} \) in \( |K_{\tilde{X}}|^8 \simeq \mathbb{P}^6 \).

Since diagonal quadratic equations in \( \mathbb{P}^6 \) form a vector space of rank 7, the choices of a linear system spanned by four elements can be identified with points in the Grassmann
variety $\text{Gr}(4, 7)$: this gives an identification between the moduli space of triples:

$$(X, f : G = (\mathbb{Z} / 2\mathbb{Z})^3 \simeq \pi_1(X), g : \mathbb{P}^6 \simeq |K_X|^*)$$

consisting of a Campedelli surface with a framing of its fundamental group and a $G$-equivariant linear isomorphism of $|K_X|^*$ with $\mathbb{P}^6$, and an open Zariski subset $\mathcal{M}_u$ of $\text{Gr}(4, 7)$ which parametrises linear systems of diagonal quadrics whose base locus is a normal surface with at worst ordinary double points as singularities.

The smoothness of such a complete intersection is easy to detect:

**Proposition 1.2.** A complete intersection of four diagonal quadrics in $\mathbb{P}^6$ is smooth if and only if its linear system does not contain a quadric of rank three. Under this assumption no point has more than two vanishing coordinates.

**Proof.** Let $\tilde{X} \subset \mathbb{P}^6$ denote the surface defined by the linear system. If it contains a quadric of rank three, e.g. $x_3^2 + x_5^2 + x_7^2$, there is a point of $\tilde{X}$ such that $x_3 = x_5 = x_7 = 0$. Such a point is indeed defined by three quadratic equations inside the space $\{x_3 = x_5 = x_7 = 0\} \simeq \mathbb{P}^3$. By the Jacobian criterion, the complete intersection cannot be smooth at this point, which is a vertex of $\{x_3^2 + x_5^2 + x_7^2 = 0\}$.

Conversely, if the quadrics have the form, $Q_j = \sum q_{ij} x_i^2$, the Jacobian matrix of the linear system is given by $\left(q_{ij} x_i^2\right)_{ij}$. Assume this matrix has not full rank at a point $R$. If $R$ has four nonzero coordinates, the corresponding minor should vanish, and this implies that some linear combination of the $Q_j$ has rank three. If $R$ has four zero coordinates (say $x_1 = \cdots = x_4 = 0$), since $R$ is solution to the equations, the matrix $(q_{ij})_{i=5,6,7}$ has rank at most two. Thus there is a two dimensional subsystem of $\langle Q_1 \rangle$ which is contained in $\langle x_1^2, \ldots, x_4^2 \rangle$ and one of them has rank three. \hfill \Box

**Definition.** A framed Campedelli surface is a pair $(X, \varphi : G \to \pi_1(X))$ where $X$ is a Campedelli surface, and $\varphi$ is an isomorphism between $G = (\mathbb{Z} / 2\mathbb{Z})^3$ and $\pi_1(X)$.

A natural moduli space for framed Campedelli surfaces is the GIT quotient of $\text{Gr}(4, 7)$ under action of the diagonal torus $T$ in $\text{SL}_7$. Indeed, the set of $G$-equivariant isomorphisms of $H^0(\tilde{X}, K_X)^\vee$ with $\mathbb{C}^7$ is a torsor under $T$: any such isomorphism must be compatible with the splitting of these spaces into a direct sum of one-dimensional eigenspaces.

Our main object of interest is the following:

**Definition.** A marked Campedelli surface is a pair $\kappa = (X, X_\kappa \to X)$ where $X$ is a Campedelli surface and $X_\kappa \to X$ is an étale connected double cover of $X$.

The datum of a marked Campedelli surface amounts to give, along with $X$, one of the following (equivalent) structures:

- a connected étale double cover $X_\kappa \to X$;
- a non-trivial rank one local system of integral coefficients $\mathbb{Z}_\kappa$ (whose square is the constant sheaf);
- a 2-torsion non trivial holomorphic line bundle on $X$, denoted by $\mathcal{L}_\kappa$;
- a non trivial character of $\pi_1(X), \kappa : \pi_1(X) \to \mathbb{Z} / 2\mathbb{Z}$;
- an effective divisor numerically equivalent to $K_X$ (which must have the form $H_\kappa \in |K_X + \mathcal{L}_\kappa|$).

**Proposition 1.3.** The moduli space of framed Campedelli surfaces is connected and can be written as a $S_4$-Galois cover of the moduli space of marked Campedelli surfaces.
Proof. The set of isomorphisms between $G = (\mathbb{Z}/2\mathbb{Z})^3$ and $\pi_1(X)$ mapping a chosen character $\kappa$ to the character $(b_1, b_2, b_3) \mapsto b_1$ is naturally equipped with a free transitive action of the affine linear group $GA_2(F_2)$ of the plane $\mathbb{A}^2(F_2)$, and $GA_2(F_2)$ is canonically isomorphic to the permutation group of the subset $\{g \in G$ such that $\kappa(g) = 1\}$. \hfill \Box

The periods of a marked Campedelli surface, with a given holomorphic 2-form with values in $\mathcal{C}_X$ (equivalently, an element $\omega \in H^0(K_X + L_\omega)$), are the various integrals of $\omega$ along cycles of $H_2(X, \mathbb{Z}_\kappa)$: they are fully determined by the associated element $\omega \in H^2(X, \mathbb{C}_\kappa)$.

**Proposition 1.4.** The lattice $H^2(X, \mathbb{Z}_\kappa)$ (modulo torsion) is equipped with the quadratic form defined by the cup product with values in $H^1(X, \mathbb{Z}_\kappa \otimes \mathbb{Z}_\kappa) \simeq \mathbb{Z}$. As such, it is unimodular and isomorphic to $\mathbb{Z}^{2,8}$, the standard odd quadratic lattice with signature $(2,8)$.

Proof. The cohomology of $\mathcal{C}_X$ has the same numerical invariants as the usual cohomology of $X$: $e(X) = 10$, and $\tau(X) = 6$. Its only nonzero Betti number is $b_2(X, \mathbb{Z}_\kappa) = 10$, and the value of $\tau(X)$ forces the signature to be $(2,8)$.

By Poincaré-Verdier duality, $H^2(X, \mathbb{Z}_\kappa)$ is unimodular, and since we know it is indefinite and odd (the signature is not a multiple of 8), its isomorphism class is uniquely determined (see [Ser77]). \hfill \Box

1.2. **Involutions of a Campedelli surface.** Let $X$ be a canonical Campedelli surface, and $G = (\mathbb{Z}/2\mathbb{Z})^3$ as before. Let $\tilde{X}$ be the universal cover of $X$. Then $X$ is isomorphic to $\tilde{X}/G$ and $\tilde{X}$ can be written as a complete intersection of 4 diagonal quadrics in $\mathbb{P}^6$.

Let $\Gamma$ be the projection in $PGL_7$ of the diagonal group $(\pm 1)^7$, acting on $\tilde{X}$: $G$ is naturally embedded in $\Gamma$, its action being given by its seven non-trivial characters.

Note that $\tilde{X}$ is a $\Gamma$-invariant subvariety of $\mathbb{P}^6$, and that squaring coordinates gives a Galois cover $sq: \mathbb{P}^6 \to \mathbb{P}^6$ with group $\Gamma$. Since $sq(\tilde{X})$ is defined in $\mathbb{P}^6$ by four linear equations, it is isomorphic to a plane, and the map $\tilde{X} \to sq(\tilde{X})$ is also a $\Gamma$-Galois cover. This proves the following proposition:

**Proposition 1.5.** A Campedelli surface is a Galois cover of $\mathbb{P}^2$ with group $\Gamma/G$. \hfill \Box

The group $\Gamma/G$ will be identified with a set of reflections in $\mathbb{P}^6$, giving a simple description of the group of automorphisms of a generic Campedelli surface. The elements of $\Gamma$ can be classified by weight: this notion will provide a convenient vocabulary for the rest of the paper.

**Definition.** The weight of an element of $\Gamma$ (represented by a diagonal matrix $g$ in $(\pm 1)^7 \subset GL_7$) is defined as $|\mathrm{Tr}g|$ (the difference between the number of +1 and −1 coefficients in $g$).

There are in $\Gamma$

- one element with weight 7: the identity;
- 7 elements with weight 5: the reflections;
- 21 elements with weight 3;
- 35 elements with weight 1, seven of them being the nonzero elements of $G$.

**Proposition 1.6.** The projection $\Gamma \to \Gamma/G$ induces a bijection between the elements $s_i$ (reflections across coordinate hyperplanes) and nonzero elements of $\Gamma/G$.

Moreover, if $\chi$ is a nonzero character of $G$, and $s_\chi$ is the reflection across the associated hyperplane, the map $\chi \mapsto [s_\chi] \in \Gamma/G$ is a group isomorphism.
Proof. Let $s_i$ and $s_j$ be different reflections in $\Gamma$, and suppose $s_i - s_j$ lies in $G$: then there would be an element of $G$ with weight 3, which is impossible. Thus $\Gamma \to \Gamma/G$ is injective on the $s_i$’s.

Moreover, let $i$, $j$, $k = i + j$ be nonzero elements in $\hat{G}$: then $G$ contains a unique nonzero element $g$ annihilated by $i$, $j$ and $k$. By definition, $\pm(s_is_jsk)$ is the element of $\Gamma$ associated to $g$, hence $s_k = s_i + s_j$ in $\Gamma/G$. \qed

The character group $\hat{G}$ is thus realised as a subgroup of $\text{Aut}(X)$: for a generic $X$, this inclusion is even an equality. This results from the fact that the bicanonical map must be equivariant under the action of $\text{Aut}(X)$, and from the fact that a general configuration of 7 lines in the plane has no automorphisms.

Another interpretation of these facts is that the quadratic form over $\left(\mathbb{Z}/2\mathbb{Z}\right)^7$ defined by

$$q(x_1, \ldots, x_7) = \sum_i x_i^2 + \sum_{i<j} x_ix_j$$

has a polar symplectic form

$$b(u, v) = \sum_{i \neq j} u_iv_j = q(u + v) - q(u) - q(v)$$

whose kernel is the vector

$$(1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)$$

It thus defines a non-degenerate quadratic form on $\Gamma$, for which nonzero isotropic vectors are vectors of weight 1, the weight being defined as the difference between the number of zero and nonzero coordinates. If $x$ has $n$ nonzero coordinates, $q(x) \equiv n(n + 1)/2$, which is zero iff $n = 0, 3, 4, 7$.

Then $G$ is an isotropic subspace of $\Gamma$ where the quadratic form vanishes, and $b$ defines a non-degenerate pairing between $G$ and $\Gamma/G$. Taking $G$ to be generated by the lines of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

for which coordinates are the 7 characters of $G$, it is easy to check that the pairings $b(e_i, \cdot)$ with the basis vectors coincide with the nonzero element of $\hat{G}$ given by the $i$-th column of the matrix.

1.3. Quotients of Campedelli surfaces under involutions. Let $(X, \kappa)$ be a (possibly nodal) marked Campedelli surface, and $s_\kappa$ the involution of $X$ canonically associated to the character $\kappa$, which is ramified over $D_\kappa$, the unique effective divisor in $|K_X + L_\kappa|$ (a curve of arithmetic genus 3). We refer to [CMLP08] for theoretical results about the possible quotients of numerical Campedelli surfaces under involutions, since we are dealing here with a particular case of those.

**Proposition 1.7.** When $X$ is smooth, the fixed point set of $s_\kappa$ consists of $D_\kappa$ and six isolated points.

Proof. A point $x \in X$ is fixed under the action of $s_\kappa$ if and only if there exists $g \in G$ such that $s_\kappa \cdot \tilde{x} = g \cdot \tilde{x}$ for some lifting of $x$ to $\tilde{X}$. If $g$ is the identity, this means that $x$ lies on $D_\kappa$.

Otherwise, if $g + s_\kappa$ has weight 1 ($\kappa(g) \neq 0$), $x$ has at least three vanishing coordinates: this is impossible.
If \( g + s_\kappa \) has weight 3, then \( \kappa(g) = 0 \) and \( x \) should have only two vanishing coordinates (corresponding to characters \( \chi, \chi + \kappa \) such that \( \chi(g) = 0 \) and \( \chi \neq \kappa, 0 \)). We find two fixed points for each \( \chi \) corresponding to \( D_\chi \cap D_{\chi + \kappa} \) so there are six isolated fixed points for \( s_\kappa \).

**Proposition 1.8.** If \( x \in X \) is a node of a Campedelli surface, \( x \) is the intersection of three divisors \( D_{\kappa_1}, D_{\kappa_2}, D_{\kappa_3} \), such that the \( \kappa_i \) generate \( \hat{G} \), and \( x \) is fixed by any involution in \( \Gamma/G \) (notably \( s_\kappa \)).

**Proof.** This follows from the description of the Campedelli surface as an octuple plane, and the classification of singularities which can be found at [AP09], see also proposition 1.2. □

In this description, if \( \kappa \) is one of the \( \kappa_i \)'s, the node lies on \( D_\kappa \), and is part of the fixed locus described in proposition 1.7. If \( \kappa \) is a sum of two \( \kappa_i \)'s, then the node is a fixed point of the type \( D_\chi \cap D_{\chi + \kappa} \) also described in prop. 1.7. Nodes that add new fixed points are defined by \( D_{\kappa_1} \cap D_{\kappa_2} \cap D_{\kappa_3} \) where \( \sum \kappa_i = \kappa \).

There are at most four such nodes: the number of bases of \( \hat{G} \) (up to permutation of vectors) whose sum is \( \kappa \) is 4 (there are \( 7 \cdot 6 \cdot 4/3! = 28 \) bases, and the number of sums is 7, giving four bases for each possible sum).

**Proposition 1.9.** Suppose \( X \) is smooth. The quotient of \( X \) by \( s_\kappa \) is an Enriques surface \( S_\kappa \) with six ordinary double points. If \( \hat{X} \) is the blow-up of the isolated fixed points, and \( \hat{S}_\kappa \) is the minimal resolution of \( S_\kappa \), the morphism \( \hat{X} \to \hat{S}_\kappa \) is a double cover ramified over \( D_\kappa + \sum E_i \) where \( E_i \) are the exceptional curves, where \( D_\kappa \) is now a genus 3 curve on \( S_\kappa \).

**Proof.** Let \( X_\kappa \) be the connected étale double cover of \( X \) associated to the character \( \kappa \). Since \( X_\kappa \) can be written as \( \hat{X}/H_\kappa \), where \( H_\kappa \subset G \) is the kernel of \( \kappa \), \( s_\kappa \) has a natural lift to \( X_\kappa \) associated to the reflection of \( \mathbb{P}^6 \) across the hyperplane \([x_\kappa = 0]\).

The previous argument still applies and shows that \( s_\kappa \) fixes the image of the hyperplane \( D_\kappa \) and 12 isolated points. Drawing a diagram for this situation (here \( T_\kappa = X_\kappa/s_\kappa \)),

\[
\begin{array}{ccc}
X_\kappa & \longrightarrow & X \\
\downarrow & & \downarrow \\
T_\kappa & \longrightarrow & S_\kappa
\end{array}
\]

we note that the unique nonzero section of \( K_X + L_\kappa \) lifts to a holomorphic 2-form on \( T_\kappa \), which vanishes only along \( D_\kappa \): considering an expression of this form in local coordinates along \( D_\kappa \) shows that it is necessarily invariant under \( s_\kappa \), and descends to a non-vanishing 2-form on \( T_\kappa \). It follows that \( T_\kappa \) is a K3 surface with 12 isolated double points.

It is now easy to check that the action of \( G/H_\kappa \), defining an involution of \( T_\kappa \), has no fixed point (it is represented by an element \( g \) of \( \Gamma \), with weight one, such that \( \kappa(g) = 1 \)). This proves that \( S_\kappa \) is an Enriques surface. □

Note that if \( s_\kappa \) has \( \nu \) fixed nodes outside the usual fixed locus, the quotient \( S_\kappa \) is a rational surface with \( K_X^2 = -\nu \). We have seen that \( \nu \leq 4 \).

1.4. **Infinitesimal variation of periods.** Let \( \kappa \in \hat{G} \) be a fixed character: following the results of Griffiths, the infinitesimal variation of \( \omega \in H^2(X, \mathbb{C}_\kappa) \) is described by the infinitesimal \( \kappa \)-periods map:

\[
H^1(X, TX) \to \text{Hom}(H^0(X, K_X + L_\kappa), H^1(X, \Omega^1_X(L_\kappa)))
\]
It is indeed related to the standard situation, in the following way: if \(X_κ\) is the étale cover of \(X\) associated to \(κ\), whose Galois group is identified with \(\mathbb{Z}/2\mathbb{Z}\), \(H^1(X, TX)\) is identified to the invariant part of \(H^1(X_κ, TX_κ)\), and \(H^0(κ)\), \(H^1(X_κ, \Omega^1_κ)\) can be split into the direct sum of an invariant and anti-invariant part. The infinitesimal variation of \(κ\)-periods is then identified to an eigenspace of the usual variation of Hodge structure of \(X_κ\) (which is \(\mathbb{Z}/2\mathbb{Z}\)-equivariant for all elements of \(H^1(X, TX)\)).

Since \(|K_X + L_κ|\) contains only \(D_κ\), we can use the isomorphism \(TX \simeq \Omega^1_Χ \otimes (−K_X)\) to identify the infinitesimal period map with the linear map
\[
H^1(X, TX) \rightarrow H^1(X, TX(D_κ))
\]
(recall that the first space has dimension 6 while the target space has dimension 8), induced by the product by a section vanishing on \(D_κ\).

Following the method of Konno [Kon91], it will be shown in the next section that the kernel of this map is the space of deformations such that the family of maps \(Q_t : X_t \rightarrow |2K_X|ammu \simeq \mathbb{P}^2\) moves the lines \(Q_t(D_κ)\) but not \(Q_t(D_χ)\) for \(χ \neq κ\). These deformations can be described as a deformation of coordinates
\[
x_κ^2 \sim x_κ^2 + \sum_{i \neq κ} ε_i x_i^2.
\]

Let \(T\) be the subsheaf of \(TX\) consisting of vector fields which are tangent to \(D\) (we drop \(κ\) subscripts starting from here) along \(D\). It is defined by a Cartesian square, which induces short exact sequences:
\[
\begin{array}{c}
\text{TX(−D)} \longrightarrow T \longrightarrow TD \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{TX(−D)} \longrightarrow TX \longrightarrow TX_D \\
\downarrow \quad \downarrow \quad \downarrow \\
N_D \longrightarrow N_D
\end{array}
\]

The long exact sequences arising from the previous diagram can be used to build a commutative diagram:
\[
\begin{array}{c}
H^0(D, TX(D)) \quad - \quad - \quad H^0(D, N_D(D)) \\
\downarrow \quad \downarrow \quad \downarrow \\
H^1(X, TX) \quad - \quad - \quad H^1(X, T(D)) \quad - \quad - \quad H^1(D, TD(D)) \\
\downarrow \quad \downarrow \quad \downarrow \\
H^1(X, TX) \quad - \quad - \quad H^1(X, TX(D)) \quad - \quad - \quad H^1(D, TX(D)) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H^1(D, N_D(D)) \quad - \quad - \quad H^1(D, N_D(D)) = 0
\end{array}
\]

where all lines, the 2nd column and the dashed arrows are exact sequences. Note that \(H^0(X, TX(D)) = 0\) since as before, \(TX(D)\) is isomorphic to \(\Omega^1_X \otimes L_κ\), whose sections are part of the Hodge decomposition of \(H^1(X, \mathbb{C}_κ) = 0\). This implies that the maps in the top left square are injective.
Notice that $N_D(D)$ is a line bundle on $D$ of degree $2g_D - 2 = 4$, with $h^0 = 2$ (since $N_D(D) = \mathcal{O}_D(2D)$, which is the line bundle corresponding to the linear system of quadrics).

**Proposition 1.10.** The map $H^0(D, TX(D)) \rightarrow H^0(D, N_D(D))$ is an isomorphism.

**Proof.** Since the (meromorphic) vector field $\partial / \partial x_k$ is tangent to $X$ along $D$ (all equations of $X$ have zero derivative at $x_k = 0$), any vector field

$$\sum_{i \neq k} \alpha_i x_i^2 \frac{1}{x_k} \partial / \partial x_k$$

is a regular section of $TX(D)$ over $D$. This shows that the composite

$$\mathcal{O}_D(2D) \rightarrow TX(D)|_D \rightarrow N_D(D)$$

where the first map is $s \mapsto \frac{s}{x_k} \partial / \partial x_k$

is an isomorphism. \hfill \qed

This implies that $H^0(D, N_D(D))$ lifts to a 2-dimensional subspace of $H^1(X, TX)$ which is the kernel of the infinitesimal period map. The corresponding deformations can be expressed informally by writing

$$dx_k = \frac{1}{x_k} \sum_{i \neq k} \varepsilon_i x_i^2 \leftrightarrow d(x_k^2) = 2 \sum_{i \neq k} \varepsilon_i x_i^2$$

1.5. **The Jacobian ring.** The classical theory of Griffiths [Gri69, Voi02] details how the infinitesimal variation of Hodge structure of a smooth hypersurface in $\mathbb{P}^n$ (inside the moduli space of hypersurfaces) can be recovered from the Jacobian ring of its equation. A similar construction describes the variation of periods for complete intersections. We refer to [Kon91] for a detailed exposition of the theory.

Let $Y$ be a complete intersection of four diagonal quadrics in $\mathbb{P}^6$, and $\Lambda = |3Y(2)| \simeq \mathbb{P}^3$ be the linear system of quadrics through $Y$. Then $\mathbb{P}^6 \times \Lambda$ carries a “universal” divisor $\tilde{Y}$, whose fibre over $\ell \in \Lambda$ is the quadric defined by $\ell$. If $\pi$ is the projection $\tilde{Y} \rightarrow \mathbb{P}^6 \times \Lambda \rightarrow \mathbb{P}^6$, the fibres of $\pi$ over $Y$ are projective spaces, and $\pi$ is a $\mathbb{P}^2$-bundle outside $\pi^{-1}(Y)$.

**Theorem 1.11** (Konno). The variation of Hodge structure of $H^8(\tilde{Y})$ is canonically identified (up to a shift in gradings) with the variation of Hodge structure of $H^2(Y)$, which is canonically embedded as $H^2(R^6\pi_*\mathcal{Z}_Y) \subset H^8(\hat{Y}, \mathbb{Z})$.

Consider the bigraded ring $S^{*,*} = \mathbb{C}[x_i; y_j]$ with 11 variables, which is generated by projective coordinates on $\mathbb{P}^6$ $(i = 1 \ldots 7)$ and $\Lambda$ $(j = 1, 2, 3, 4)$. Suppose

$$f = y_1 Q_1 + y_2 Q_2 + y_3 Q_3 + y_4 Q_4$$

(where the $Q_j$’s are diagonal quadratic forms $Q_j = \sum q_{ij} x_i^2$) is a parametrisation of the linear system $\Lambda$: it is an equation of bidegree $(2, 1)$ of $\tilde{Y} \subset \mathbb{P}^6 \times \mathbb{P}^3$. Then the variation of Hodge structure for $\tilde{Y}$ can be elegantly described by the Jacobian ring of $f$ [Gre85].

For the sake of consistency with [Kon91], we also introduce the notation

$$R_{p,q} = (S/J)^{2p-7q} \simeq H^0(\mathbb{P}^6 \times \Lambda, p\tilde{Y} + qK)$$

where $K$ is the canonical class of $\mathbb{P}^6 \times \Lambda$. 


The Jacobian ideal of $\tilde{Y}$ is
\[
J^{\bullet \bullet} = S(\partial_{x_i} f, \partial_{y_j} f) = S \left\langle \sum_j q_{ij} x_i y_j, Q_j \right\rangle.
\]

**Theorem 1.12 (Konno).** If $p + q = 8$, the space $H^p(\tilde{Y}, \Omega^q)_{\text{prim}}$ is isomorphic to
\[
R_{p+1,1} = (S/J)^{2p-5p-3} \simeq H^0((p+1)\tilde{Y} + K_{\mathbb{P}^6 \times \mathbb{A}}),
\]
using Griffiths's techniques, which associate to $P \in R_{p+1,1}$ the meromorphic volume form $P\Omega/f^{p+1}$, where $\Omega$ is a standard volume form on $\mathbb{P}^6 \times \mathbb{P}^3$ with values in $\mathcal{O}(7,4)$.

In this setting, the first-order deformations of the linear system associated to $f$ are given by the elements of $R_{1,0} = (S/J)^{2\cdot 1}$, while the infinitesimal period map of $Y$ is given by the natural morphism
\[
R_{1,0} \rightarrow \text{Hom}(R_{3+1,1}, R_{4+1,1})
\]
induced by multiplication in the ring $S/J$.

**Proposition 1.13.** The infinitesimal period map for $(X = Y/G, \kappa \in \hat{G})$, which the identification given by theorem [1.12] is proportional to the natural map given by multiplication
\[
R_{1,0}^{(\kappa)} \rightarrow \text{Hom}(R_{3+1,1}^{(\kappa)}, R_{4+1,1}^{(\kappa)})
\]
where $V^{(\kappa)}$ is the isotypic component of $V$ where $G$ acts by the character $\kappa$. Moreover, $R_{4,1}^{(\kappa)}$ is one-dimensional, generated by $x_\kappa$, so the map
\[
x_\kappa : R_{1,0}^{(\kappa)} \rightarrow R_{4+1,1}^{(\kappa)}
\]
also describes the infinitesimal $\kappa$-period map.

An explicit description of $R_{1,0}^{(\kappa)}$ is given by the $G$-invariant part of the vector space, generated by the monomials $x_i x_j y_k$ in $S/J$, which is isomorphic to
\[
\frac{\langle x_i^2 y_j \rangle}{\langle Q_k y_j, \sum_j q_{ij} x_i^2 y_j \rangle}
\]
which has dimension $6 = 28 - 16 - 7 + 1$, since the only non trivial relation is $\sum Q_j y_j = \sum_{i,j} q_{ij} x_i^2 y_j$. The brackets here denote the linear span of the elements they enclose.

The space $R_{3,1}^{(\kappa)} = (S/J)^{3\cdot 1}$ is
\[
\frac{\langle x_i x_j x_t y_j \rangle}{\langle x_\kappa Q_j y_j, \sum_j q_{ij} x_i x_j x_t y_j \rangle}
\]
where $p + q + r = \kappa$ and $i + s + t = \kappa$. The second space has dimension $4$ as we see below, and the second space has dimension $8 - 6 = 2$ (note there are two possible triples $(p, q, r)$). These dimensions coincide with the dimension of eigenspaces of $s_\kappa$ on $H^{1,1}(Y, \mathbb{C}_\kappa)$: the (twisted) differential form $\Omega/f^{p+1}$ is anti-invariant under $s_\kappa$, as well as $x_\kappa$. 

The image of multiplication by $x_\kappa$ lies inside the first space, and by simplifying out $x_\kappa$, we observe that the annihilator of $x_\kappa$ can be analysed by looking at the quotient morphism,

\[(1.a) \quad \frac{\langle x_i^2 y_j \rangle}{\langle Q_i y_j, \sum_j q_{ij} x_i^2 y_j \rangle} \rightarrow \frac{\langle x_i^2 y_j \rangle}{\langle Q_i y_j, \sum_j q_{ij} x_i^2 y_j, \sum_j q_{ij} x_i^2 y_j \rangle}\]

**Proposition 1.14.** If $Q$ is the matrix of a linear system defining a smooth (universal cover of) Campedelli surface, the target of the morphism in equation [1.a] has dimension 4.

The space of linear combinations $\sum a_{ij} x_i^2 y_j$ is identified with the space of matrices $(a_{ij})$ with size $7 \times 4$. The subspace $\langle Q_i y_j \rangle$ is then the space of products $MQ$ where $M \in \mathbb{C}^{4 \times 4}$. Since $Q$ defines a surface, it must have full rank, hence the space of products $MQ$ has dimension 16. We use an intermediate lemma to prove proposition [1.14].

**Lemma 1.15.** If $Q = (q_{ij})$ is the matrix of the linear system defining a smooth Campedelli surface, written in the standard basis $(x_i^2)_{i \in G}$, no set of four columns of $Q$ are linearly dependent.

**Proof.** If there were such a set, then some element of the linear system would be a quadric of rank three: then there would be three linearly dependent monomials $x_i^2$ in the bicanonical linear system. The configuration of lines describing the Campedelli surface has a triple point, which creates a singularity. \[\square\]

**Proof of proposition [1.14]** We work with the spaces of matrices described above: a matrix $MQ$ lies in $\langle \sum_j q_{ij} x_i^2 y_j, \sum_j q_{ij} x_i^2 y_j \rangle$, if $Mx$ is a linear combination of $x$ and $K$ for any column $x$ of $Q$ ($K$ being the column $\kappa$). Then $K$ is an eigenvector of $M$, and we need to analyse its action on $\mathbb{C}^4/K$.

The six columns of $Q$ (except $K$) should define 3 different eigenvectors of $M$ in $\mathbb{C}^4/K$. Using the previous lemma, we know that no three of them are linearly dependent. By standard arguments, this forces $M$ to act as a scalar multiplication on $\mathbb{C}^4/K$. Conversely, if $M$ acts as a scalar on $\mathbb{C}^4/K$, then $MQ$ lies in the given subspace: the space of such $MQ$ has dimension 5.

The dimension of $\langle Q_i y_j, \sum_j q_{ij} x_i^2 y_j, \sum_j q_{ij} x_i^2 y_j \rangle$ is then $16 + (7 + 7 - 1) - 5 = 24$, hence the result. \[\square\]

1.6. **The period map around Campedelli surfaces with double points.** We will need to examine the behaviour of the period map around a Campedelli surface with a double point, following the method described by C. Voisin in [Voi86]. We are interested in the following particular situation: let $\alpha, \beta, \gamma$ be a basis of $(\mathbb{Z}/2)^3$, and $\kappa = \alpha + \beta + \gamma$.

Let $X$ be a Campedelli surface (the associated configuration of 7 lines in the bicanonical plane should have at worst triple points) and assume that the lines $D_\alpha, D_\beta$ and $D_\gamma$ are concurrent in $|2X|$: then $x_\alpha^2 + x_\beta^2 + x_\gamma^2$ belongs to the defining linear system, and there exists a point $R$ such that $x_\alpha = x_\beta = x_\gamma = 0$. Note that there exists an element of $\pi_1(X)$ whose action on coordinates reverses the sign of $x_i$ for $i = \alpha, \beta, \gamma, \kappa$. This implies that $R$ is fixed under the involution $s_\alpha$ in the following, we denote $(\kappa, \alpha, \beta, \gamma)$ by numbers 4, 5, 6, 7.

Note that $R$ cannot have a fourth vanishing coordinate, since this would mean that a fourth branching line is concurrent with $D_\alpha, D_\beta, D_\gamma$.

By the Picard-Lefschetz formula, the monodromy of the Gauss-Manin connection around the hypersurface of configurations with three concurrent lines has order two, and there cannot exist (even locally) a continuous choice of trivialisation of the lattices $H^2(X_\kappa(t), \mathbb{Z})$ (hence neither of $H^2(X_t, \mathbb{Z})$) in any neighbourhood of $X$ in the moduli space.
However, taking the double cover ramified along this hypersurface trivialises the monodromy and allows to get a simultaneous resolution of singularities [Ati58], since the hypersurface parametrises surfaces acquiring double points. It becomes possible to define a local period map with values in a type IV domain $\mathcal{D}$, which is then holomorphic.

Let $t$ be a complex parameter and consider the deformation of $X$ given by the matrices

$$
\begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\
  b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\
  c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
  0 & 0 & 0 & -t^2 & 1 & 1 & 1
\end{pmatrix}
$$

which corresponds to the double cover of a one-parameter family which is transverse to the hypersurface defined by the minor of the first columns.

We make the following regularity hypothesis:

the determinant $\left| \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right|$ does not vanish.

**Lemma 1.16.** This hypothesis is equivalent to requiring that $R$ has no fourth vanishing coordinate, or to the assumption that no set of four lines are concurrent. $\square$

As before, the period map of this family can be identified (locally) with the period map of the associated family of hypersurfaces of bidegree $(1,2)$ in $\mathbb{P}^3 \times \mathbb{P}^6$, with equations $F_t = \sum u_j q_j$ where $q_1 = \sum a_i x_i^2$ and so on.

The singular locus of $F_t$ is defined by the equations $q_j$ and the vanishing of the Jacobian matrix of $(q_1, q_2, q_3, q_4)$ on $(u_i)$. For generic choices of $q_1, q_2, q_3$ (described by our regularity hypothesis), there is an isolated fixed point at $([0 : 0 : 0 : 1], R)$. Choosing $u_4 = 1, x_4 = 1$ as a local chart, the equation of the family is given by

$$u_1 q_1 + u_2 q_2 + u_3 q_3 - t^2 + x_5^2 + x_6^2 + x_7^2 = 0$$

which defines an isolated ordinary double point: in other words, $\{q_1, q_2, q_3, x_5, x_6, x_7\}$ is a set of local coordinates on $\mathbb{P}^6$. The action of $s_k$ lifts to these local coordinates by changing signs in $x_5, x_6, x_7$.

If $\tilde{X}_t$ is the complete intersection of quadrics associated to the parameter $t$, and $Y_t$ is the associated 8-fold in $\mathbb{P}^3 \times \mathbb{P}^6$, the Hodge structure on $H^8(Y_t)$ is described by $H^{3,5}(Y_t) = \mathbb{C} \tilde{\omega}$, where

$$\tilde{\omega} = \text{Res}_{Y_t} \frac{x_k \Omega}{F_t}$$

where $\Omega$ is a non-vanishing generator of $\Omega^9(4,7)$.

By the works of Griffiths and Konno, we know that the variation of Hodge structure along this family will be described by the following type $(4,4)$ form:

$$\tilde{\omega}' = \text{Res}_{Y_t} \frac{tu_4 x_3^2 x_k \Omega}{F_t^5}$$

up to some multiplicative constant.

In the local chart described above ($u_4 = x_4 = 1$), we can write

$$\tilde{\omega}' = \text{Res}_{Y_t} \frac{tx_k \Omega}{F_t^5}$$
where $\Omega$ is the canonical holomorphic volume form on $\mathbb{C}^3 \times \mathbb{C}^6$.

If $\kappa$ were not $\alpha + \beta + \gamma$, then $R$ would not be a fixed point of the involution: the regularity of the period map follows from a blow-up at the singular point of the family $Y_t$. In the affine chart where $x_i = tX_i$, $q_i = tQ_i$, $u_i = tU_i$ (note that $x_\kappa$ is nonzero):

\[
\hat{\omega}'(t) = \text{Res}_{Y_t} \frac{t^{10}x_\kappa \Omega_{bl}}{t^{10}(F^bl_t)^5}
\]

where $F^bl_t = U_1Q_1 + U_2Q_2 + U_3Q_3 + X_1^2 + X_2^2 + X_3^2 - 1$

A change of variables turns $F_t$ into $\sum Y_i - 1$, and the period over the corresponding vanishing sphere has a finite limit (since the given residue is the way of expressing the primitive cohomology of the quadric).

We now need to know how this differential form integrates in the case where $R$ is invariant under $s_k$: in this case some cohomology class is anti-invariant under $s_k$ and its period vanishes for $t = 0$. Such a situation has already been studied by Horikawa for Enriques surfaces [Hor78b].

The key fact is that the projective quadric in $\mathbb{P}^9$, which is the normal cone to the singular point in the 9-fold $\mathcal{Y} = \{Y_t\}$, has two base classes in $H^{4,4}$, corresponding to classes of maximal isotropic subspaces, $\sigma_1$ and $\sigma_2$, which form a hyperbolic plane. They are such that $h^4 = \sigma_1 + \sigma_2$ is the class of a 4-dimensional linear section, and $\rho = \sigma_1 - \sigma_2$ is the class of the real sphere $\mathbb{S}^8$. Note that $\rho^2 = -2$.

Since $s_k$ acts by changing signs of three coordinates, it exchanges $\sigma_1$ and $\sigma_2$ (it is not in $SO_{10}$), hence the real sphere is anti-invariant under this transformation ($s_k$ reverses the orientation of the associated real manifold), and by the same argument the associated period has a nonzero derivative at $t = 0$.

### 2. Todorov surfaces and double covers of Enriques surfaces

Todorov surfaces were introduced by Todorov to give examples of surfaces whose infinitesimal period map is nontrivial and not injective [Tod81]. A systematic study of these surfaces is done in [Mor88], which we quote for most of the properties stated below.

Campedelli surfaces are double covers of Enriques surfaces, and their étale double covers (whose periods are what we actually study) are Todorov surfaces. In section 2.2 we carry out a study of basic properties of double covers of Enriques surfaces, similar to [Mor88]. If $X \to S$ is such a double cover, the geometry of $S$ usually gives good information about the transcendental part of $H^2(S, \mathbb{Z}_\mathbb{C})$, and in section 2.3 we compute a formula giving the index of the embedding $H^2(S, \mathbb{Z}_\mathbb{C}) \to H^2(X, \mathbb{Z}_\mathbb{C})$ (theorem 2.9) between (twisted) cohomology lattices. It will be used to compare the global period maps for $S$ and $X$.

#### 2.1. Classical Todorov surfaces.

**Definition** (Todorov surface). A Todorov surface is a surface $Z$ with canonical singularities, and an involution $j$, such that $S = Z/j$ is a K3 surface with rational double points and $\chi(O_Z) = 2$.

If $\tilde{Z}$ is the minimal resolution of $Z$, the natural morphism $\tilde{j} : \tilde{Z} \times_{Z/j} Z \to \tilde{Z}$ lifts $j$ to an involution of $\tilde{Z}$. The quotient $\Sigma = \tilde{Z}/\tilde{j}$ is again a K3 surface, which is a partial desingularisation of $S$.

**Definition** (Fundamental invariants). Let $(Z, j)$ be a Todorov surface. Then $\Sigma$ as above is a nodal K3 surface: the lattice generated by its nodes has index $2^a$ inside its primitive saturation in
$H^2(\Sigma, \mathbb{Z})$. The number of nodes of $\Sigma$ is denoted by $k$. The fundamental invariants of $Z$ are the integers $(\alpha, k)$.

**Theorem 2.1** (Todorov). Several topological invariants of $\tilde{Z}$ can be calculated using the fundamental invariants. The order of the 2-torsion subgroup of $H^2(\tilde{Z}, \mathbb{Z})$ is $2^\alpha$. The divisorial part of the ramification locus $B \subset \Sigma$ satisfies $B^2 = 2k - 16$. The integer $k$ can also be expressed as $c_1(\tilde{Z})^2 + 8$.

**Proposition 2.2.** If $X$ is a Campedelli surface without $(-2)$-curves, $X_\kappa$ is a Todorov surface with invariants $(2, 12)$.

Proof. As seen in section 1.3, the involution $s_\kappa$ on $X$ lifts to $X_\kappa$, which is a projective surface with rational double points, and the quotient $X_\kappa/s_\kappa$ is the universal cover of the Enriques surface $X/s_\kappa$. We recover $\alpha = 2$ from the order of the 2-torsion subgroup of $H^2(X_\kappa, \mathbb{Z})$ (which is identified with the character group of $\tau_1(X_\kappa)$), and $k = 12$ from the number of double points of $T_\kappa = X_\kappa/s_\kappa$ (we know that $S_\kappa$ has six double points).

Another invariant of Todorov surfaces is a natural sublattice of the Picard group of the underlying K3 surface:

**Definition** (Todorov lattice [Mor88]). Let $Z$ be a smooth Todorov surface and $\Sigma = Z/j$ be the associated nodal K3 surface. The resolution of singularities of $\Sigma$ is denoted by $\hat{\Sigma}$: each double point of $\Sigma$ is resolved to a $(-2)$-curve $E_i$ on $\hat{\Sigma}$. The Todorov lattice associated to $Z$ is the primitive saturation $L_T(Z)$ of the sublattice $\langle B, E_i \rangle$ of $H^2(\hat{\Sigma}, \mathbb{Z})$ generated by the ramification divisor $B$ and the classes $E_i$.

The Todorov lattice is the Picard lattice of K3 surfaces associated to generic Todorov surfaces with given invariants $(\alpha, k)$. Such K3 surfaces are parametrised by a period domain: a moduli space for Todorov surfaces can thus be constructed using their period map [Mor88].

**2.2. Todorov-Enriques surfaces.** We define a class of surfaces inspired by the definition of Todorov surfaces. This class includes Keum-Naie surfaces [Nai94] and the construction by Mendes Lopes and Pardini [MLP04] of surfaces such that $p_g = q = 0$. We gather here a collection of results which fit Campedelli surfaces in this class.

**Definition.** A Todorov-Enriques surface is a pair $(X, j)$ where $X$ is a canonical surface with an involution $j : X \to X$, such that $\chi(O_X) = 1$, and $X/j$ is an Enriques surface with at worst rational double points.

Replacing if necessary $X$ by its minimal desingularisation, we will assume that $X$ is smooth, and that the double points of $X/j$ come from the isolated fixed points of $j$. The smooth Enriques surface obtained by blowing-up the double points of $X/j$ is denoted by $S$.

The double points of $X/j$ define $(-2)$-classes $E_1, \ldots, E_k$ in $\text{Pic}(S)$. The invariants of $(X, j)$ are $k$ (the number of fixed points of $j$, which is also the number of double points on $X/j$ and $\alpha$, which is the dimension of the kernel of the natural map

$$\mathbb{Z}/2\mathbb{Z} \langle E_1, \ldots, E_k \rangle \to \frac{\text{Pic} S}{2\text{Pic} S}$$

(the space of “even sets” made of $E_1, \ldots, E_k$).

**Proposition 2.3.** The ramification divisor $B \subset X/j$ has self-intersection $2K_X^2 = 2k - 8$, and $h^0(S, \mathcal{O}_S(B)) = k - 3$. 

which determine which expresses \( H \)

The 2-torsion subgroup of \( F \) fixed locus \( H \) subgroup of \( 3B^2/2 \) -3k and the identity \( B^2 = 2k - 8 \).

\[ \square \]

**Proposition 2.4.** The inequality \( 2\alpha \leq k \leq \alpha + 5 \) holds, or equivalently \( k - 5 \leq \alpha \leq k/2 \).

**Proof.** Let \( N_5 \) be the primitive saturation of the lattice generated by the nodal classes \( E_i \) in Pic(S) (which is a rank 10 unimodular lattice).

The double point lattice \( N_5 \) of \( S \) has rank \( k \) and discriminant \( 2^k - 2\alpha \) (hence \( k \geq 2\alpha \)), but its orthogonal complement has rank \( 10 - k \). Since the discriminant group of \( N_5 \) is a \( \mathbb{F}_2 \)-vector space of rank at most \( 10 - k \), which is isomorphic to the discriminant group of \( N_5 \), \( k - 2\alpha \leq 10 - k \).

Using this inequality (and the fact that \( K_{X} = k - 4 > 0 \)), the possible values of \( (\alpha, k) \) are

\[
(0, 5) \quad (1, 5) \quad (2, 5) \quad (1, 6) \quad (2, 6) \quad (3, 6) \quad (2, 7) \quad (3, 7) \quad (3, 8) \quad (4, 8)
\]

Remember that an even set of nodes on an Enriques surface is made of 4 or 8 nodes. If \( k = 5 \), the existence of two distinct even sets of nodes would imply the existence of an even set of two nodes, which is impossible.

If \( k = 8 \), the universal cover of \( S \) is necessarily a Kummer surface: the study of even sets on a Kummer surface (see [BHPvD04] for example) tells us that in this case \( \alpha = 4 \).

The pair \((3, 6)\) is also impossible: since any even set of nodes has four elements, two distinct even sets must share exactly two nodes, in other words, the complements of distinct even sets are disjoint, so there cannot be more than three non trivial even sets.

The possible values of \((\alpha, k)\) are now

\[
(0, 5) \quad (1, 5) \quad (1, 6) \quad (2, 6) \quad (2, 7) \quad (3, 7) \quad (4, 8)
\]

**Proposition 2.5.** The 2-torsion subgroup of Pic(X) is an extension of \((\mathbb{Z}/2\mathbb{Z})^\alpha\) by the 2-torsion subgroup of \( H^2(S, \mathbb{Z}) \) (which is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \)).

**Proof.** Let \( \pi : X_b \rightarrow S \) be the projection, and \( U \) be the complement in \( S \) of the image of the fixed locus \( F \subset X_b \) of \( j \), and \( u : U \hookrightarrow S \) be the standard inclusion. Consider the extensions of sheaves

\[
e_U : 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_*\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0
\]

\[
e_S : 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_*\mathbb{Z}/2\mathbb{Z} \rightarrow u_*\mathbb{Z}/2\mathbb{Z} \rightarrow 0
\]

\[
\chi : 0 \rightarrow u_*\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow i_*\mathbb{Z}/2\mathbb{Z} \rightarrow 0
\]

which determine \( e_U \in H^1(U, \mathbb{Z}/2\mathbb{Z}) \), \( e_S \in \text{Ext}_S^1(u_*\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \) and \( \chi \in \text{Ext}_S^1(i_*\mathbb{Z}/2\mathbb{Z}, u_*\mathbb{Z}/2\mathbb{Z}) \).

The exact sequence of sheaves

\[
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_*\mathbb{Z}/2\mathbb{Z} \rightarrow u_*\mathbb{Z}/2\mathbb{Z} \rightarrow 0
\]

induces an exact sequence

\[
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \simeq H^1(S, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(X_b, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1_c(S \setminus F, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^2(S, \mathbb{Z}/2\mathbb{Z}).
\]

which expresses \( H^1(X_b, \mathbb{Z}/2\mathbb{Z}) \) as an extension of \( \ker e_S \subset H^1_c(S \setminus F, \mathbb{Z}/2\mathbb{Z}) \) by \( \mathbb{Z}/2 \).

The relative cohomology exact sequence now tells us that

\[
H^0(S, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^0(F, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\chi} H^1_c(S \setminus F, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(S, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(F, \mathbb{Z}/2\mathbb{Z})
\]
is exact, but the last map being injective, $H^1_c(S \setminus F, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^f/e$, where $f$ is the number of components of $F$ and $e$ is the sum of elements of $F$. We must now compute

$$\ker e_S = \ker(e_S : H^0(F, \mathbb{Z}/2) \to H^2(S, \mathbb{Z}/2))/(e)$$

But $e_SX$ is an element of $\text{Ext}^2_\mathbb{Z}(i_*, \mathbb{Z}/2, \mathbb{Z}/2) \simeq H^0(\text{Ext}^2(i_*, \mathbb{Z}/2, \mathbb{Z}/2))$ which is totally determined by looking at the sections of $\text{Ext}^1(i_*, \mathbb{Z}/2, u_i\mathbb{Z}/2)$ and $\text{Ext}^1(u_i\mathbb{Z}/2, \mathbb{Z}/2)$ determined by $e_S$ and $\chi$. It is easy to check that it is exactly the Gysin map.

Its kernel is identified with the code of even sets of nodes in $(\mathbb{Z}/2\mathbb{Z})^k$. 

This shows that $(1, 5)$ is impossible, since it would give a numerical Godeaux surface such that $H^1(X, \mathbb{Z}/2\mathbb{Z})$ contains $(\mathbb{Z}/2\mathbb{Z})^2$, which is impossible [Miy76, sec. 3].

**Proposition 2.6.** A Todorov-Enriques surface admits a canonical étale double cover, induced by the universal cover of the quotient Enriques surface. This double cover is a Todorov surface.

Since Todorov surfaces are known to satisfy $q = 0$ [Mor88], Todorov-Enriques surfaces satisfy $p_g = q = 0$.

If $X$ is a Todorov-Enriques surface, and $(\tilde{\alpha}, \tilde{k})$ are the invariants of a Todorov surface $Y$ which is an étale double cover of $X$, then $\tilde{k} = 2k$, and we have the following possibilities.

\[
\begin{array}{cccccccc}
(\alpha, k) & (0, 5) & (1, 6) & (2, 6) & (2, 7) & (3, 7) & (4, 8) \\
(\tilde{\alpha}, \tilde{k}) & (0, 10); (1, 10); (1, 12); (2, 12); (2, 12); (3, 14); (3, 14); (5, 16) \\
\end{array}
\]

In [Nai94], D. Naie constructed examples of such surfaces for the invariants $(0, 5), (1, 6), (2, 6), (2, 7)$ and $(4, 8)$, starting with an Enriques surface with 8 nodes (the value of $\alpha$ can be deduced from the observation that the 2-torsion groups have order 2 for $k = 5, 4$ or 8 for $k = 6, 2^3$ for $k = 7$). The general description of surfaces of type $(2, 7)$ can be found in [MLP04]. The case $(3, 7)$ is actually impossible [MLP04, 4.4]: the Enriques surface would be realised as a surface with seven nodes in $\mathbb{P}^3$, the nodes being aligned like the seven points of $\mathbb{P}^2(\mathbb{F}_2)$, which is impossible.

**Proposition 2.7.** The Campedelli surfaces are examples of Todorov-Enriques surfaces with invariants $(\alpha, k) = (2, 6)$.

**Proof.** Let $(X, \kappa)$ be a marked (smooth) Campedelli surface and $S = X/s$ be the associated Enriques surface with six nodes. Here $B$ is the (reduced) image of $D_i$ in $S$ ($B^2 = 2k - 8 = 4$ by proposition 2.3). This gives $k = 6$, and explained after definition 2.2 $\alpha = 2$ means there are three sets of even nodes among the six nodes on $S$.

Note that any twisted canonical divisor $D_i \neq D_\kappa$ goes through the two fixed points $\{e_i, e'_i\} = D_i \cap D_{i+k}$ and no other fixed point (except those of $D_\kappa$); let $B_i$ be the image of $2D_i$ in $S$, which is a divisor with generic multiplicity two, which pulls back to $2K_X$, $B_i$ is an element of $|2B|$.

There are two rational curves $E_i, E'_i$ in the desingularisation $\tilde{S}$ corresponding to the points $e_i$ and $e'_i$. The decomposition of $B_i$ into irreducible components is $B'_i + S_i + S'_i$ (since $S_i$ pulls back to $2E_i$), and $B'_i = 2B - S_i - S'_i$ is divisible by two. This implies that the complementary set of four rational curves on $\tilde{S}$ is even, since the existence of the ramified double cover $X \to S$ requires that the sum of $B$ and the six rational $(-2)$-curves is divisible by two.

The three possible pairs of fixed points $D_i \cap D_{i+k}$ provide the three required even sets of nodes. 

$\square$
It should be noted that numerical Campedelli surfaces may be also constructed using an Enriques surface with invariants \((\alpha, k) = (1, 6)\), giving fundamental groups \(\mathbb{Z}/4 \times \mathbb{Z}/2\) or \(\mathbb{Z}/2\). We hope to study these families of Enriques surfaces in detail in a future work.

2.3. Embeddings of cohomology lattices for double covers. In order to compare the period map of Enriques surfaces with the actual period map of a family of covering Todorov-Enriques surfaces, we give a formula computing the index of the map \(H^2(S, \mathbb{Z}_-) \to H^2(X, \mathbb{Z}_k)^G_{\text{num}}\), when \(f : X \to S\) is a degree two cover of an Enriques surface, \(G\) is the group generated by the associated involution of \(X\), and \(\mathbb{Z}_k = \bar{f}^*\mathbb{Z}_-\). Then \(f^* : H^2(S, \mathbb{Z}_-) \to H^2(X, \mathbb{Z}_k)^\text{num}\) is a morphism of quadratic lattices, which multiplies the intersection form by two. The notation \(H_{\text{num}}\) denotes the quotient of an abelian group \(H\) by its torsion subgroup (cohomology classes up to numerical equivalence).

We make the following assumptions: the ramification locus of \(f\) is a disjoint union of \(\rho\) smooth curves (\(\rho\) being a positive integer), and any étale double cover \(X'\) of \(X\) is regular \((q(X') = 0)\). The only needed consequence of \(S\) being an Enriques surface is that \(H^3(S, \mathbb{Z}_-) = 0\).

The computation is done using spectral sequences for \(G\)-equivariant cohomology: here \(H^k_G(X, \bullet)\) can be understood as the \(k\)-th derived functor of \(\mathcal{F} \mapsto \Gamma(X, \mathcal{F})^G\). The Borel diagram for the action of \(G\) is

\[
\begin{array}{ccc}
\text{EG} & \xleftarrow{\pi} & X \\
\downarrow & & \downarrow \gamma \\
BG & \xleftarrow{f} & S \\
\end{array}
\]

where \(BG\) is the classifying space of \(G\) and \(EG\) the universal \(G\)-bundle over \(BG\). Then \(H^*_G(X, \mathbb{Z}_k) = H^*(X/G, \mathbb{Z}_k)\) can be calculated by the Leray spectral sequences for \(\pi\) and \(\gamma\).

\[
E^{p,q}_2 = H^p(G, H^q(X, \mathbb{Z}_k)) \Rightarrow H^{p+q}_G(X, \mathbb{Z}_k) \\
E^{p,q}_2 = H^p(S, R^q\gamma_\ast \mathbb{Z}_k) \Rightarrow H^{p+q}_G(X, \mathbb{Z}_k)
\]

The computation of sheaves \(R^q\gamma_\ast \mathbb{Z}_k\) corresponds to cohomology groups of \(\mathbb{Z}/2\mathbb{Z}\):

\[
\gamma_\ast \mathbb{Z}_k \cong \mathbb{Z}_- \quad R^{2k+1} \gamma_\ast \mathbb{Z}_k = 0 \quad \text{and} \quad R^{2k+2} \gamma_\ast \mathbb{Z}_k \cong (\mathbb{Z}/2\mathbb{Z})_F \quad (k \geq 0).
\]

The map \(f^*\) is decomposed as follows:

\[
f^* : H^2(S, \mathbb{Z}_-) \xrightarrow{\gamma^*} H^2_G(X, \mathbb{Z}_k) \xrightarrow{f^*} H^2(X, \mathbb{Z}_k)^G
\]

By eliminating torsion, we obtain two maps of free abelian groups

\[
H^2(S, \mathbb{Z}_-)^\text{num} \to H^2_G(X, \mathbb{Z}_k)^\text{num} \quad \text{(of index } 2^{N_1})
\]

and \(H^2_G(X, \mathbb{Z}_k)^\text{num} \to H^2(X, \mathbb{Z}_k)^G\) (of index \(2^{N_2}\)).

The index of \(f^*\) is thus expressed as the \((N_1 + N_2)\)-th power of two. We are actually going to prove the following formulas:

**Proposition 2.8.** The integer \(N_1\) (see proposition \[2.10\]) is equal to

\[
\rho - \ell_2(H^2_G(X, \mathbb{Z}_k)) + \ell_2(H^2(S, \mathbb{Z}_-)).
\]

The integer \(N_2\) (see propositions \[2.12\] and \[2.13\]) is equal to

\[
\ell_2(H^2_G(X, \mathbb{Z}_k)) - \ell_2(H^2(X, \mathbb{Z}_k)^G) - \varepsilon
\]
where \( \epsilon \) is 1 if \( Z_\kappa \) is trivial, 0 otherwise.

Here \( \ell_2 \) denotes the length of the 2-adic torsion subgroup.

**Theorem 2.9.** Under the hypotheses above, the map of lattices associated to \( f^* \) has index
\[
2^{\ell_2(H^2(S,\mathbb{Z}_-)) - \ell_2(H^2(X,\mathbb{Z}_\kappa)^G) + \rho}
\]
if \( Z_\kappa \) is non trivial,
\[
2^{\ell_2(H^2(S,\mathbb{Z}_-)) - \ell_2(H^2(X,\mathbb{Z}_\kappa)^G) + \rho - 1}
\]
otherwise.

2.3.1. **Second spectral sequence.** Since \( E_{p,q}^2 = 0 \) for any odd \( q \), the differential \( d_2 \) is zero. The remaining differential is \( d_3^{(1,2)} : H^0(F,\mathbb{Z}/2\mathbb{Z}) \to H^3(S,\mathbb{Z}_-) = 0 \). The resulting filtration on \( H^0_G(X,\mathbb{Z}_\kappa) \) is graded by
\[
Gr^2 = H^2(S,\mathbb{Z}_-), \quad Gr^1 = 0, \quad Gr^0 = H^0(F,\mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^\rho
\]

**Proposition 2.10.** The map \( H^2(S,\mathbb{Z}_-) \to H^2_G(X,\mathbb{Z}_\kappa) \) is injective, its cokernel is a \( \mathbb{Z}/2\mathbb{Z} \)-module of rank \( \rho \), and \( N_1 = \rho - \ell_2(H^0_G(X,\mathbb{Z}_\kappa)) + \ell_2(H^2(S,\mathbb{Z}_-)) \), as stated by formula 2.9.

**Proof.** Let \( T(S), T_G(X) \) be the torsion groups of \( H^2(S,\mathbb{Z}_-) \) and \( H^2_G(X,\mathbb{Z}_\kappa) \). The natural map \( H^2(S,\mathbb{Z}_-)/T(S) \to H^2_G(X,\mathbb{Z}_\kappa)/T(S) \) has again cokernel \( (\mathbb{Z}/2\mathbb{Z})^\rho \). This cokernel is an extension of the 2-torsion of the target (with length \( \ell_2(H^0_G(X,\mathbb{Z}_\kappa)) - \ell_2(H^2(S,\mathbb{Z}_-)) \)) and the contribution from the torsion-free part (a group of order \( 2^\kappa \)). We get the equation stated above. \( \square \)

2.3.2. **First spectral sequence.**

**Lemma 2.11.** Let \( Z_\kappa \) be a \( \mathbb{Z} \)-local system on an algebraic surface \( X \) with no irregular étale double cover. Then \( H^1(X,\mathbb{Z}_\kappa) = 0 \) if \( Z_\kappa = Z_X, \mathbb{Z}/2\mathbb{Z} \) if \( Z_\kappa \) is non trivial.

**Proof.** If \( Z_\kappa = Z_X \), then \( H^1(X,\mathbb{Z}) = \text{Hom}(H_1(X),\mathbb{Z}) \) by the universal coefficient theorem, and this is torsion-free, but since \( H^1(X,\mathbb{C}) = 0 \), it must be zero.

If \( Z_\kappa \) is non trivial, then \( Z_\kappa \) is given by a character \( \kappa : \pi_1(X) \to \mathbb{Z}/2\mathbb{Z} \). There is a canonical double cover \( X_\kappa \to X \) and a short exact sequence:
\[
0 \to \mathbb{Z}^2 \to \mathbb{Z} \to H^1(X,\mathbb{Z}_\kappa) \to H^1(X_\kappa,\mathbb{Z}) = 0.
\]

\( \square \)

Suppose that \( Z_\kappa = Z_X \). Then \( H^p(G, H^q(X,\mathbb{Z}_\kappa)) \) is zero for \( q = 1 \), and for \( (p,q) = (2p'+1,0) \). The graded parts \( Gr^p \) of \( H^2_G(X,\mathbb{Z}) \) are
\[
Gr^2 = H^2(G,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}
\]
\[
Gr^1 = 0
\]
\[
Gr^0 = H^0(G, H^2(X,\mathbb{Z}))
\]

**Proposition 2.12.** When \( Z_\kappa \) is trivial, the map \( H^2_G(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})^G \) is surjective and induces an isomorphism between the torsion-free quotients (i.e. \( N_2 = 0 \)). Moreover the following relation holds:
\[
\ell_2(H^2_G(X,\mathbb{Z})) = \ell_2(H^2(X,\mathbb{Z})^G) + 1.
\]
Suppose now that \( Z_\kappa \neq Z_X \): then \( H^0(X, Z_\kappa) = 0 \) and \( H^1(X, Z_\kappa) = \mathbb{Z}/2 \). The graded parts \( \text{Gr}^i \) of \( H^2_G(X, Z_\kappa) \) are

\[
\begin{align*}
\text{Gr}^2 &= 0 \\
\text{Gr}^1 &= H^1(G, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \\
\text{Gr}^0 &= \ker : H^0(G, H^2(X, Z_\kappa)) \to H^2(G, H^1(X, Z_\kappa))
\end{align*}
\]

hence there is an exact sequence

\[
0 \to \mathbb{Z}/2\mathbb{Z} \to H^2_G(X, Z_\kappa) \to H^2(X, Z_\kappa)^G \to \mathbb{Z}/2\mathbb{Z} \to 0.
\]

**Proposition 2.13.** When \( Z_\kappa \) is not trivial, \( N_2 = \ell_2(H^2_G(X, Z_\kappa)) - \ell_2(H^2(X, Z_\kappa)^G) \) as in formula 2.6.

3. **Enriques Surfaces with a \( D_{1,6} \) Polarisation**

The Enriques surfaces appearing as quotients of Campedelli surfaces have (in the generic case) six ordinary double points: these surfaces are exactly the Enriques surfaces \( S \) containing a \( D_6 \) sublattice in \( H^{1,1}(S, \mathbb{Z}) \). In other words their six nodes can form three even sets, as seen in proposition 2.7.

This allows to determine the generic transcendental part of \( H^2(S, \mathbb{Z}) \) (section 3.4), which is the orthogonal complement of the \( D_6 \) lattice. It is isomorphic to \( \mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1) \) (proposition 3.28).

These Enriques surfaces can be defined as bidouble covers of the plane (proposition 3.4), whose ramification locus consists of three pairs of lines. We will see that a Cremona transformation can map these pairs to another configuration (which is not projectively equivalent in generic cases), defining a birationally equivalent Enriques surface (proposition 3.9).

3.1. **Linear Systems and Geometry.** Let \( D_{1,6} \) be the index 2 sublattice of the standard Lorentzian lattice \( \mathbb{Z}^{1,6} \), containing vectors whose sum of coordinates is even. If \( \langle e_0; e_1, \ldots, e_6 \rangle \) is the canonical basis of \( \mathbb{Z}^{1,6} \), we distinguish a norm 4 vector \( 2e_0 \) and six mutually orthogonal \((-2)\)-vectors \( e_1 \pm e_2, e_3 \pm e_4, e_5 \pm e_6 \).

**Definition.** A \( D_{1,6} \)-polarised Enriques surface is an Enriques surface whose Picard group contains a primitively embedded copy of \( D_{1,6} \) such that the distinguished vectors described above correspond to 6 smooth rational curves \( R_i \) and a nef class \( H \) with \( H^2 = 4 \).

This is equivalent to the requirement that the \( R_i \)'s contain three even sets of rational curves, and \( H + \sum R_i \) is divisible by two.

In proposition 2.7, we proved essentially that the quotient of a smooth Campedelli surface by the involution induced by a reflection of the form \( s_\kappa \) is such an Enriques surface: the six nodes are images of the isolated fixed points of \( s_\kappa \) and form three even sets.

In this case, the Enriques surface can be written as the quotient of a complete intersection of three diagonal quadrics in \( \mathbb{P}^5 \) by a group \( G \simeq (\mathbb{Z}/2)^3 \): the linear system \( H \) is generated by \( G \)-invariants quadratic forms (the various \( x_i \)).

**Proposition 3.1.** On a \( D_{1,6} \)-Enriques surface \( S \), the linear system \(|H|\) is base point free and induces a map \( S \to \mathbb{P}^2 \) of degree 4.
Proof. According to [Cos85], since $H$ is nef, $|H|$ has no fixed components (nef divisors with fixed components have self-intersection 2). Moreover $|H|$ has base points if and only if it has the form $2E + F$ (where $E$ and $F$ are half-elliptic pencils with $EF = 1$) or $3E + R$ where $R$ is a smooth rational curve and $E$ is a half-pencil with $ER = 1$ [Cos85, 2.12].

Following [Mor88], we note that $E \cdot H = 1$, but

$$E \cdot (H + \sum R_i)$$

is an even integer, so $E$ should intersect one of the $R_i$'s, let $Q$ be this rational curve.

The case $H = 3E + R$. Then $0 = Q \cdot H = Q \cdot (3E + R)$, hence $Q \cdot R < 0$, hence $Q = R$, but then $0 = H \cdot Q = (3E + R) \cdot R = 1$, yielding a contradiction.

The case $H = 2E + F$. We note that $Q \cdot (2E + F)$ is zero, but since $Q \cdot E > 0$, $Q \cdot F < 0$, which is absurd.

It is also known that $H^1(O_S(H)) = 0$, see [Cos85].

This proposition is a particular case of the following analogue of [Mor88, Lemma 5.1]

**Proposition 3.2.** Let $S$ be a smooth Enriques surface with $\nu$ disjoint rational $(-2)$-curves $R_i$, and $H$ a nef divisor with $H^2 > 0$ such that $H \cdot R_i = 0$ and $H + \sum R_i$ is divisible by 2 in $\text{Pic}(S)$.

Then $|H|$ contracts the curves $R_i$, and is base point free, except in the special case: $H^2 = 2$ and there is one rational curve such that the others form an even set of nodes. In the special case, the special rational curve is a fixed component of $|H|$, and is not contracted by $|H|$.

**Proof.** According to [Cos85], if $H$ has a fixed component or base points, it can be written $kE + F$, where $E$ and $F$ are half-elliptic pencils with $EF = 1$, or $(k + 1)E + R$ where $E$ is a half-pencil, $R$ a rational nodal curve and $ER = 1$. In both cases $H^2 = 2k$, and $HE = 1$. We need to show that we are then precisely in the special case.

Since $E$ has an even intersection number with $H + \sum R_i$ (which is divisible by two), and $E \cdot H = 1$, for some $Q$ among the $R_i$'s, $E \cdot Q > 0$.

Now $Q \cdot H = 0$. In the case $H = (k + 1)E + R$, it follows that $QR < 0$, hence $Q = R$, and $HR = k + 1 + R^2 = 0$, which is a contradiction, except when $k = 1$. If $H = kE + F$, then $HQ = 0$ gives $FQ < 0$, which is also a contradiction.

Thus $k = 1$, and it follows that $2E = H - Q$ is divisible by two, but since $H + \sum R_i$ is also divisible by two, $\sum_{R_i \neq Q} R_i$ is an even set of nodal curves, hence we are in the special case. In other cases, $H$ is base point free and has no fixed component: since $H \cdot R_i = 0$, the curves $R_i$ are contracted by $|H|$.

Conversely, under the hypotheses of the special case, let $Q$ be the distinguished rational curve. Then $H + Q$ and $H - Q$ are divisible by two and effective. Let $H = 2E + Q$. Then $h^0(H) \geq h^0(2E) \geq 2$, and since $H$ is nef and big, $h^0(H) = \chi(H) = 2$, hence $Q$ is a fixed component of $H$.

**Proposition 3.3.** Let $S$ be a $D_{1,6}$-polarised Enriques surface, $H$ the distinguished positive class in $D_{1,6}$. Then $S \rightarrow |H|^\vee$ is a surjective morphism of degree 4, contracting the 6 rational curves and ramified over six lines.

**Proof.** As before, $H$ is base point free, has no fixed components, and $H \cdot R_i = 0$ so $|H|$ contracts all curves $R_i$. Consider the three elliptic pencils $2E_1 = H - R_1 - R_2$, $2E_2 = H - R_3 - R_4$, $2E_3 = H - R_5 - R_6$ (by construction they are 2-divisible). They have a natural interpretation as linear subsystems of $|H|$, so they are pulled back from pencils of lines in $|H|^\vee$. 

The corresponding half-pencils map to six lines in the plane. Each of them is part of the ramification locus, since pulling back one of these lines gives a multiplicity 2 divisor.

Now note that \( K_S = K_{p^2} + R \) (the formula is both valid for \( S \) and the nodal surface obtained by contracting the \( R_i' \) s) where \( R \) is the ramification locus of \( S \to \mathbb{P}^2 \). Then \( R \equiv 3H + K_S \) (since \( K_{p^2} \equiv -3H \)), and the definition of the pencils implies

\[
R \equiv \sum_{i=1,2,3} E_i + (E_i + K_S) + \sum_{i=1}^6 R_i.
\]

Since \( R \) is an effective divisor containing all half-pencils as well as the \( R_i' \) s, the linear equivalence above is an equality.

In the proof above, we see that \( E_1 \cdot R_1 = E_1 \cdot R_2 = 1 \): this indicates that the image point of \( R_1 \) and \( R_2 \) is actually the base point of the elliptic pencil \( |2E_1| \), which is the intersection of the images of \( E_1 \) and \( E_1' = E_1 + K_S \) in the plane.

**Proposition 3.4.** The above morphism is a Galois cover with Galois group \((\mathbb{Z}/2)^2\).

If \( X \) is a Campedelli surface, with an involution \( s_\kappa \), the Galois cover \( X \to \mathbb{P}^2 \) given by the bicanonical map (proposition 1.5) factors through the quotient \( X/\langle \kappa, s_\kappa \rangle \), and the Galois group of the cover \( X \to \mathbb{P}^2 \) is identified with the order 4 group \( \langle 1 \rangle \otimes \langle G, s_\kappa \rangle \).

**Proof of proposition 3.4.** Let \( E_i \) (\( i = 1, 2, 3 \)) be the three half-pencils as above, and consider the divisor \( M_i = E_j + E_k \), where \( \{i, j, k\} = \{1, 2, 3\} \).

Then \( M_i \) is a genus two linear system, and by [Cos83] 6.1 we know that \( |2M_i| \) defines a degree two morphism onto a degree 4 del Pezzo surface. Let \( s_1 \) be the associated involution. Note that \( |H| \) is a linear subsystem of \( |2M_i| \) by the map \( D \mapsto D + L_{23} \) where \( L_{23} \) is the unique element of \( |H - R_3 - R_4 - R_5 - R_6| \). Hence \( X \to |H|^\vee \) is \( s_1 \)-equivariant.

Such involutions determine the divisor class \( |2M_i| \) as the ramification divisor of \( X \to X/s_1 \).

Hence \( s_i \) (\( i = 1, 2, 3 \)) are distinct involutions, and since \( X \to |H|^\vee \) has degree four, the group \( \langle 1, s_1, s_2, s_3 \rangle \) is exactly the Galois group.

A Galois cover as above is called a bidouble cover: it is a special case of abelian covers which are studied in [Par91]. Such a cover has three involutions: for each of them, we consider the image of the (divisorial part of the) fixed locus in the base variety (here \( \mathbb{P}^2 \)). Write \( f, g, h \) for sections of line bundles defining them. By Galois theory, \( fgh \) can be written as a square \( s^2 \), where \( s \) is a section of some line bundle.

Then the bidouble cover can be defined by equations of the form

\[
u^2 = f \quad v^2 = g \quad w^2 = h \quad uvw = s
\]

(change signs of \( u, v, w \) gives the other component \( uvw = -s \)). In the case of \( D_{1,6} \)-Enriques surfaces, the divisors of \( f, g, h \) are \( e_i + e'_i + e_j + e'_j \) (denoting by \( e_i \) the image of \( E_i \) in the plane). This point of view highlights the fact that the data of three pairs of lines in the planes \( (e_i, e'_i) \) (\( i = 1, 2, 3 \)) determines uniquely a surface which is generically the blow-down of six rational curves on an Enriques surface.

**3.2. An involution of the moduli space.** Consider a \( D_{1,6} \)-polarised Enriques surface \( S \): we will now choose more symmetric notations. The six rational curves are denoted by \( R_i^+ \) and \( R_i^- \), the six half-pencils \( E_i^+ \) and \( E_i^- \), such that \( 2E_i^+ = 2E_i^- = H + R_i^+ + R_i^- \). The linear system \( |H| \) determines a finite Galois cover \( S_r \to \mathbb{P}^2 \), where \( S_r \) is obtained from \( S \) by contracting the \((-2)\)-curves \( R_i^\pm \).
Assume that $S$ is given by three pairs of lines in general position, whose vertices are denoted by $a_i$ (which are the images of the double points of $S$).

Let $\mathcal{P}$ be the blowup of the three vertices; the pencil of lines through $a_i$ corresponds to a linear system $|e_i|$ on $\mathcal{P}$ such that $e_i^2 = 0$, and $e_i e_j = 1$ if $i \neq j$. The proper transforms of the three pairs of lines define pairs of divisors $e_i^\pm$ in each $|e_i|$.

Let $S'$ be the (well-defined) bidouble cover of $\mathcal{P}$ ramified over the three pairs $(e_i^\pm)$. The linear system $e_i$ pulls back to the elliptic pencil $|2E_i|$. Note that $\mathcal{P}$ has $6$ $(-1)$-curves, given by the $a_i$ and $\ell_i$ (proper transform of the line $a_i a_i$). The inverse image of $a_i$ in $S'$ is made of two disjoint curves since $a_i$ is disjoint from $e_i^\pm$ and $e_i^\pm$ $(\{i, j, k\} = \{1, 2, 3\})$: it is actually the union of two $(-2)$-curves. It follows that $S$ and $S'$ are isomorphic, and that the inverse image of $a_i$ is identified with $R_i^+ + R_i^-$. The curves $a_i$ and $\ell_i$ play equivalent roles (they can be exchanged by a standard quadratic transformation). The intersection numbers on $\mathcal{P}$ are $a_i \cdot \ell_j = 1$ if $i \neq j$, zero otherwise. Using a similar proof, we show that $\ell_i$ pull back to two disjoint $(-2)$-curves $\Lambda_i^\pm$; $\ell_i$ is also disjoint from $e_i^\pm$ and $e_i^\pm$. If $h$ denotes the linear system of lines on $\mathbb{P}^2$ and its pull-back to $\mathcal{P}$, $h = e_i + a_i$ and $h = \ell_i + a_i + a_k$.

The $D_{1,6}$-polarisation on $S$ can be described by $3H = \sum \Lambda_i^\pm + 2\sum R_i^\pm$ and the six rational curves $R_i^\pm$ mapping to $a_i$. Let $h' = e_i + \ell_i = 2h - a_1 - a_2 - a_3$.

**Proposition 3.5.** The pull-back $H'$ of $h'$ to $S$ is an effective divisor such that $H'^2 = 4$, and $\Lambda_i^\pm$ define six disjoint $(-2)$-curves such that $H' + \sum \Lambda_i^\pm$ is divisible by two. In other words $(H', \Lambda_i^\pm)$ is another $D_{1,6}$-polarisation on $S$.

**Proof.** The pullback of $h'$ is $H' = 2E_i + \Lambda_i^+ + \Lambda_i^-$, which is an effective divisor and $(H')^2 = 4$. The formula

$$H' + \sum \Lambda_i^\pm = (3H - H) - \sum R_i^\pm + \sum \Lambda_i^\pm = 2(\sum \Lambda_i^\pm + \sum R_i^\pm) - (H + \sum R_i^\pm)$$

shows that it is also 2-divisible.

Moreover $H' - \Lambda_i^+ - \Lambda_i^-$ is the pull-back of $h' - \ell_i = e_i$, which is the elliptic pencil $2E_i$. This proves that the lattice generated by $H'$ and the $\Lambda_i^\pm$ is $D_{1,6}$. □

**Proposition 3.6.** The new $D_{1,6}$-polarisation defines another configuration of lines which can be obtained from the initial one by performing a Cremona quadratic transformation of the plane, with vertices $a_i$.

**Proposition 3.7.** The nonzero intersection numbers of the $R_i$’s and $\Lambda_i$’s are $R_i^\pm \cdot \Lambda_i^\pm = 1$ for $i \neq j$.

**Proposition 3.8.** A $D_{1,6}$-polarised Enriques surface $S$ defines an embedding of the root lattice $D_6 \subset H^2(S, \mathbb{Z})$, using the classes of the six rational curves $R_i^\pm$ as a standard orthogonal frame.

**Proof.** The curves $R_i^\pm$ and $\Lambda_i^\pm$ define classes in $H^2(S, \mathbb{Z})$ (up to a choice of sign), using their fundamental classes and the fact that $\mathbb{Z}$ restricts to a trivial local system on rational curves. Their self-intersection is $-2$: this gives an embedding of $\mathbb{Z}^6(-2)$ in $H^2(S, \mathbb{Z})$, since the $R_i^\pm$ do not intersect.

The intersection numbers show that the classes $[R_1^+], [\Lambda_1^+], [R_2^+], [\Lambda_2^+], [R_3^+]$ (up to a choice of signs) form a $D_6$ lattice: the associated curves define a dual graph which is the Dynkin diagram of $D_6$ (this is graph is simply connected, so there actually exists a suitable choice of sign).
By a small deformation, we can assume that the Hodge structure on $H^2(S, \mathbb{Z}_-)\) is generic (using the infinitesimal period map), and that the $D_6$ lattice is actually the whole $H^{1,1}(S, \mathbb{Z}_-)\). It is then easily checked that $[R_2^+] + [R_2^-] + [R_3^+] + [R_3^-]$ is divisible by 2: it has even pairing with any of the roots mentioned above. □

Configurations of six lines may be represented by elements of $\text{Gr}(3, 6)$: the coefficients of six equations of lines in $\mathbb{P}^2$ may be written in the 6 columns of a $3 \times 6$ matrix, whose lines define a three-dimensional subspace of $\mathbb{C}^6$ if the lines do not pass through a common point. A given point in $\text{Gr}(3, 6)$ defines the configuration only up to a projective transformation. We now prove the following fact:

**Proposition 3.9.** There is an explicit biregular involution $Q$ on the Grassmann variety $\text{Gr}(3, 6)$, such that for general $x \in \text{Gr}(3, 6)$, $x$ and $Q(x)$ represent conjugate configurations under the transformation described above: $Q(x)$ is equivalent to the Cremona transform of $x$ with vertices $x_1 \wedge x_2$, $x_3 \wedge x_4$, $x_5 \wedge x_6$.

**Proof.** Given a configuration of six lines in general position, there exists a choice of basis such that $A_1 = [1 : 0 : 0]$, etc. Let $M$ be a matrix of the configuration given by columns $M_1, \ldots, M_6$, and $N$ be the matrix whose lines are $M_1 \wedge M_2, M_3 \wedge M_4, M_5 \wedge M_6$ (which are coordinates of the vertices of each pair of lines).

Then $N$ is invertible and

$$NM = \begin{pmatrix} 0 & 0 & m_{123} & m_{124} & m_{125} & m_{126} \\ m_{341} & m_{342} & 0 & 0 & m_{345} & m_{346} \\ m_{561} & m_{562} & m_{563} & m_{564} & 0 & 0 \end{pmatrix}$$

where $m_{ijk} = \det(M_i, M_j, M_k)$.

The quadratic transformation with centres $A_1$ can be described by inverting each of the non-zero coefficients of $NM$, or equivalently, exchanging nonzero coefficients in each column (up to a dilation on columns).

Let $Q$ be the transformation of $\mathbb{P}^{18} \supset \text{Gr}(3, 6)$ which exchanges the 6 pairs of coordinates as above (e.g. $x_{123}$ and $x_{563}$) and leaves the others untouched. Then $Q$ is a well-defined involution of $\text{Gr}(3, 6)$, and it acts as specified. □

The description of the action of Coble’s association show that it differs from $Q$ by the transpositions of each pair of columns. In particular if $t$ is an element of the torus $T$, then $Q(t \cdot x) = t'Q(x)$, where $t'$ is obtained from $t^{-1}$ by exchanging the pairs of coefficients. It follows $Q$ also acts on the ring of $T$-invariants of $\text{Gr}(3, 6)$, and descends to an involution of $\text{Gr}(3, 6) \cong T$.

### 3.3. GIT stability for configurations of six lines.

The moduli space we are interested in is the space of configurations of six lines, which can be described as a GIT quotient $\mathbb{C}^{18} \sslash (\text{GL}_3 \times T)$ where $T$ is the diagonal torus in $\text{GL}_6$ and $\text{GL}_3 \times T$ acts on $\mathbb{C}^3 \otimes \mathbb{C}^6 \cong \mathbb{C}^{18}$. This space is also $\mathbb{P}^2 = \text{SL}_3$ whose coordinate ring is determined by the $\mathbb{G}_m$-equivariant line bundle $\mathcal{O}(1, \ldots, 1)$, or $\text{Gr}(3, 6) \sslash T$ where $\text{Gr}(3, 6)$ is the Grassmannian, with the linear action of $T$ on $\wedge^3 \mathbb{C}^6$ (which is the target of the Plücker embedding of the Grassmannian).

Semi-stability and instability can be given the following description: we say a configuration of points in the plane has type $ijk$ with respect to a flag $\mathbb{C}^3 = V_1 \supset V_2 \supset V_3$ of $\mathbb{C}^3$ iff the associated matrix has $i$ (resp. $j$, $k$) columns belonging to the first (resp. second, third) element of the flag (note that $i + j + k = 6$). The type of a configuration can be read...
on the Schubert cell it belongs to: configurations of type ijk corresponding to a Schubert cell $X_{i,i,j,i+j+k}$. Stability can be decided from the type, using Seshadri’s criterion. In this particular case, the conditions are very explicit:

**Proposition 3.10.** Given a flag of $\mathbb{C}^3$ and a 1-parameter subgroup of $\text{SL}_3$ $t \mapsto (t^a, t^b, t^c)$ where $a > b > c$, a point $p$ of $(\mathbb{P}^2)^6$ is unstable (resp. non stable) w.r.t. $(g_t)$ iff $p$ is described by an element of the Schubert cell $X_{i,i,j,i+j+k}$ where $ia + jb + kc < 0$ (resp. $\leq 0$).

**Proof:** In the Segre embedding, basic coordinates of $p$ are given by the product of coordinates from each points: the given formula is the biggest weight which can be obtained in this way. $\square$

It is enough to check stability against subgroup whose weights are $(2, -1, -1)$ and $(1, 1, -2)$: we need to check the sign of $2i - (j + k) = 3i - 6$ and $(i + j) - 2k = 6 - 3k$.

**Proposition 3.11.** If a configuration has type $0xx$ in some basis, it is unstable.

In the following we thus only consider configurations having type ijk where $i > 0$. The shape of configurations of given type is here described by interpreting them as configurations of lines, which is more directly related to the shape of corresponding surfaces.

**Proposition 3.12.** The unstable types are $1xx$ and $xx3$. The strictly semi-stable types are $231$ (4 concurrent lines), $312$ (2 identical lines), and $222$.

The notions of stability, semi-stability and polystability coincide for the GIT quotients $\text{Gr}(3, 6) \sslash T$ and $(\mathbb{P}^2)^6 \sslash \text{SL}_3$.

We obtain the following stratification of the Grassmannian:
- unstable 141 (codim. 3): five concurrent lines;
- unstable 213 (codim. 4): three identical lines;
- polystable 222 (codim. 6): three pairs of identical lines, the stabiliser has dimension 2, note there are three possible combinatorial types $(AA, BB, CC)$, $(AA, BC, BC)$ or $(AB, BC, CA)$ (where $A, B, C$ denote lines from each pair);
- semistable 222 (codim. 3): four concurrent lines, two of them being identical;
- polystable 231=312 (codim. 4): 2 identical and 4 concurrent; there are two ways, up to permutation, in which they can be arranged (either two lines in the same pairs coincide, or two lines from different pairs coincide), the stabiliser has dimension 1;
- semistable 231 (codim. 2): 4 concurrent lines;
- semistable 312 (codim. 2): 2 identical lines;
- type 321 (codim. 1): only one concurrent triple of lines;
- type 411 (open): lines linearly in general position (intersection of the “big cells”).

The description of the stability locus can be summarised as follows:

**Proposition 3.13.** A configuration of lines is stable if and only if it consists of six distinct lines and has at worst triple points.

### 3.4. The cohomology of a generic $D_{1,6}$-Enriques surfaces.

#### 3.4.1. Cohomology lattices of Enriques surfaces.

Let $S$ be a generic smooth Enriques surface of type $D_{1,6}$. Traditional presentations of the period map of Enriques surfaces use the anti-invariant cohomology lattice of the double cover $\pi : T \to S$, which is isomorphic to $H \oplus E_{10}(-2)$. For some uses, it may be more convenient to study the unique non-trivial
local system of rank one on \(S\), which we denote by \(\mathbb{Z}_-\). A lattice-theoretic version of this choice was described by Allcock \([\text{All00}]\).

**Proposition 3.14.** The cohomology groups of \(\mathbb{Z}_-\) are \(H^0 = 0\), \(H^1 = \mathbb{Z}/2\mathbb{Z}\) and \(H^2\) is torsion-free.

**Proof.** From the exact sequence of sheaves

\[
0 \to \mathbb{Z}_- \xrightarrow{\pi^{-1}} \pi_* \mathbb{Z}_T \xrightarrow{\pi_*} \mathbb{Z}_S \to 0
\]

we deduce

\[
0 = H^0(S, \mathbb{Z}_-) \to H^0(T, \mathbb{Z}) \to H^0(S, \mathbb{Z}) \to H^1(S, \mathbb{Z}_-) \to H^1(T, \mathbb{Z}) = 0
\]

which gives \(H^0\) and \(H^1\), since \(H^0(T, \mathbb{Z}) \to H^0(S, \mathbb{Z})\) is integration on fibres.

Since \(H^1(S, \mathbb{Z}) = 0\), the torsion subgroup of \(H^2(S, \mathbb{Z}_-\)) maps injectively to the torsion subgroup of \(H^2(T, \mathbb{Z})\), which is zero. \(\square\)

**Proposition 3.15.** The torsion subgroups of \(H^2(S, \mathbb{Z}_-\)) and \(H^3(S, \mathbb{Z}_-\)) are isomorphic.

**Proof.** Since \(\mathbb{Z}_-\) is Verdier self-dual, Poincaré-Verdier duality gives a (degenerate) spectral sequence

\[
\text{Ext}^p_q(H^{n-q}(S, \mathbb{Z}_-), \mathbb{Z}) \implies H^{p+q}(S, \mathbb{Z}_-)
\]

Hence there is a short exact sequence

\[
0 \to \text{Ext}^1(H^2(S, \mathbb{Z}_-), \mathbb{Z}) \to H^3(S, \mathbb{Z}_-) \to \text{Hom}(H^1(S, \mathbb{Z}_-), \mathbb{Z}) = 0
\]

And similarly

\[
0 \to \text{Ext}^1(H^3(S, \mathbb{Z}_-), \mathbb{Z}) \to H^2(S, \mathbb{Z}_-) \to \text{Hom}(H^2(S, \mathbb{Z}_-), \mathbb{Z}) \to 0
\]

\(\square\)

**Proposition 3.16.** The lattice \(H^2(S, \mathbb{Z}_-\)) is odd and unimodular, with signature \((2, 10)\).

**Proof.** The parity of the lattice follows from Wu’s formula: the Steenrod square \(Sq^2\) coincides with the cup product with the second Stiefel-Whitney class of \(S\). But the class \(w_2(S)\) can be seen as the reduction mod 2 of \(c_1(S) \in H^2(S, \mathbb{Z})\), which is nonzero. Since the reduction map \(H^2(S, \mathbb{Z}_-) \to H^2(S, \mathbb{Z}/2)\) is surjective (\(H^2(\mathbb{Z}_-) = 0\)), the intersection form is odd.

Unimodularity follows from Poincaré duality. Additionally, \(H^2(S, \mathbb{R})\) and \(H^2(S, \mathbb{R}_-)\) have the same signature, either by using index formula, or by identifying \(H^2(S, \mathbb{R}_-)\) with the anti-invariant subspace of \(H^2(T, \mathbb{R})\), and resorting to standard calculations \([\text{BHPVdV04}]\). \(\square\)

**Proposition 3.17.** Let \(H^2(T, \mathbb{Z}_-)\) be the sublattice of vectors in \(H^2(T, \mathbb{Z})\) which are anti-invariant under the involutive deck transformation of \(T \to S\). The pull-back map from the lattice \(H^2(S, \mathbb{Z}_-)\) to \(H^2(T, \mathbb{Z}_-)\) is an isometric embedding of index 2.

**Proof.** This is either deduced from our computations in section 2.3, or from the unimodularity of \(H^2(S, \mathbb{Z}_-)\) and the fact that \(H^2(T)_-(1/2)\) is isometric to \(H(1/2) \oplus E_{10}(-1)\) \([\text{BHPVdV04}]\), which has discriminant 1/4. \(\square\)

There is actually a unique odd unimodular sublattice of \(H(1/2) \oplus E_{10}(-1)\), as shown by Allcock in \([\text{All00}]\).

**Theorem 3.18** (Allcock). There is a unique odd unimodular sublattice of \(H^2(T)_-(1/2)\). The dual of \(H^2(T)_-(1/2)\) is characterised as the lattice of even vectors in this unimodular lattice.
Proof. Let \( L \) be the lattice \( H(1/2) \oplus E_{10}(-1) \) and \( L^\vee = H(2) \oplus E_{10}(-1) \) be the dual lattice. If \( M \) is a unimodular sublattice of \( L \), then \( L^\vee \subset M^\vee = M \subset L \).

Note that \( L/L^\vee = H(1/2)/H(2) \), which is isomorphic to \((\mathbb{Z}/2)^2\). The quadratic form on \( L/L^\vee \) is well defined mod 2, and takes values 0, 0, 1, on the three nonzero vectors. So among the three integral lattices between \( L \) and \( L^\vee \), there is only one which is odd. \( \square \)

### 3.4.2. The orthogonal complement of the polarising lattice

We are interested in the embedding \( D_6 \hookrightarrow \mathbb{Z}_{2,10} \) which represents the type \((1,1)\) part of the lattice \( H^2(S, \mathbb{Z}_-^n) \) for a generic \( D_{1,6} \)-Enriques surface \( S \). Its orthogonal complement (the “transcendental part”) will be denoted by \( L \). Note that \( D_6 \subset H^2(S, \mathbb{Z}_-^n) \) is primitive, since its pull-back to \( H^2(T, \mathbb{Z}_-^n) \) is also primitive.

First note that the dual of \( D_6 \subset \mathbb{Z}_6 \) (the embedding being the standard embedding of \( D_6 \) as the even sublattice of \( \mathbb{Z}_6 \)) is the lattice generated by \( \mathbb{Z}_6 \) and \( (1/2,...,1/2) \). The discriminant group of \( D_6 \) is then generated by a basis vector of \( \mathbb{Z}_6 \) and the half-integer vector. The matrix of the associated bilinear form \( b : \text{Sym}^2(D_6^*/D_6) \to \mathbb{Q}/\mathbb{Z} \) is

\[
\begin{pmatrix}
0 & 1/2 \\
1/2 & 1/2
\end{pmatrix}
\]

The quadratic form takes the basis vector to 1 (mod 2), the half-integer vector to \(-1/2\) (mod 2) and their sum to \(-1/2\).

The following proposition is classical:

**Proposition 3.19.** Let \( M \) be a primitive sublattice of a unimodular lattice \( L \). The inclusions \( M \oplus M^\perp \subset L \subset M^* \oplus (M^\perp)^* \) define maps \( L \to M^*/M \) and \( L \to (M^\perp)^*/M^\perp \) which induce a bijective correspondence between the discriminant groups.

This correspondence is an isometry for the associated bilinear forms \( -b_M \) and \( b_{M^\perp} \) with values in \( \mathbb{Q}/\mathbb{Z} \). If all lattices involved are even, it is also an isometry for the quadratic forms \( -q_M \) and \( q_{M^\perp} \), with values in \( \mathbb{Q}/2\mathbb{Z} \).

Conversely, given such an isometry of bilinear forms, the pull back of the graph along the map \((M \oplus M^\perp)^* \to D_M \oplus D_{M^\perp}\) defines a unimodular lattice.

**Proposition 3.20.** The orthogonal complement of \( D_6(-1) \) in \( \mathbb{Z}_{2,10}^2 \) is a lattice \( L \) of signature \((2,4)\) and discriminant 4. Its discriminant bilinear form has matrix \( \begin{pmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \).

**Lemma 3.21.** The lattice \( L \) is isomorphic to an index two sublattice of \( \mathbb{Z}_{2,4}^2 \).

**Proof.** The values of the discriminant bilinear form show that there is a lattice between \( L \) and \( L^* \) which is integral. Since it must be unimodular, it is isomorphic to \( \mathbb{Z}_{2,4}^2 \). \( \square \)

Note that any sublattice of index two in \( \mathbb{Z}_{2,4}^2 \) has the form \( \mathbb{Z}_{p,q} \oplus D_{r,s} \), where \( D_{r,s} \) is the lattice of vectors in \( \mathbb{Z}_{r,s}^2 \) whose coordinates have an even sum: such a sublattice is necessarily obtained as the kernel of a group homomorphism \( \mathbb{Z}_{2,4} \to \mathbb{Z}/2\mathbb{Z} \), which is the set of vectors whose sum of specified coordinates is even. The discriminant of \( D_{r,s} \) is 4.
Lemma 3.22. A sublattice \( \mathbb{Z}^p,q \oplus D_{r,s} \) of \( \mathbb{Z}^{2,4} \) has a 2-torsion discriminant group if and only if \( r + s \) is even. The index 2 sublattices of \( \mathbb{Z}^{2,4} \) with 2-torsion discriminant group are

\[
\begin{align*}
D_{2,4} &\simeq H \oplus H \oplus \mathbb{Z}^2(-2) \\
D_{0,4} \oplus \mathbb{Z}^2 &\simeq H(2) \oplus \mathbb{Z}^{1,3} \\
D_{2,2} \oplus \mathbb{Z}^{0,2} &\simeq H(2) \oplus H \oplus \mathbb{Z}^2(-1) \\
D_{2,0} \oplus \mathbb{Z}^{0,4} &\simeq \mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1) \\
D_{0,2} \oplus \mathbb{Z}^{2,2} &\simeq \mathbb{Z}^{2,2} \oplus \mathbb{Z}^2(-2) \\
D_{1,3} \oplus \mathbb{Z}^{1,1} &\simeq H \oplus \mathbb{Z}^2(-2) \oplus \mathbb{Z}^{1,1} \\
D_{1,1} \oplus \mathbb{Z}^{1,3} &\simeq H(2) \oplus \mathbb{Z}^{1,3}
\end{align*}
\]

A basis for \( H \subset D_{1,3} \) is given by \( (1,1,0,0) \) and \( (1;0,1,0) \). A basis for \( \mathbb{Z}^{0,3} \) in \( \mathbb{Z} \oplus D_{1}(-1) \) is given by \( (1;1,1,0,0), (1;0,1,1,0), (1;0,0,1,1) \). Note that

\[
D_{2,2} \oplus \mathbb{Z}^{0,2} \simeq D_{1,1} \oplus \mathbb{Z}^{1,3} \simeq D_{0,4} \oplus \mathbb{Z}^{2,0}
\]

and \( D_{2,0} \oplus \mathbb{Z}^{0,4} \simeq D_{1,3} \oplus \mathbb{Z}^{1,1} \simeq D_{0,2} \oplus \mathbb{Z}^{2,2} \).

Lemma 3.23. The discriminant bilinear forms of \( D_6 \) and \( D_{r,s} \) coincide if and only if \( r - s \equiv 6 \mod 4 \). This rules out \( (r, s) = (2, 2) \) and \( (r, s) = (1, 1) \) and \( (r, s) = (0, 4) \).

Lemma 3.24. The discriminant quadratic forms, with values in \( \mathbb{Q}/\mathbb{Z} \), of \( D_6 \) and \( D_{2,4} \) coincide. In particular, any unimodular lattice containing orthogonal primitive copies of \( -D_6 \) and \( D_{2,4} \) should be even.

Proof. Any isomorphism between the discriminant groups which is an isometry between the bilinear forms is automatically an isometry for the quadratic forms whenever they coincide. \( \square \)

There is only one case left.

Theorem 3.25. The lattice \( L \) is isomorphic to \( D_{2,0} \oplus \mathbb{Z}^{0,4} \simeq D_{1,3} \oplus \mathbb{Z}^{1,1} \simeq D_{0,2} \oplus \mathbb{Z}^{2,2} \). \( \square \)

Corollary 3.26. All primitive embeddings \( D_6(-1) \subset \mathbb{Z}^{2,10} \) are conjugate.

Any primitive embedding \( D_6(-1) \subset \mathbb{Z}^{2,10} \) can be factored as \( D_6(-1) \to \mathbb{Z}^{2,6} \to \mathbb{Z}^{2,6} \oplus \mathbb{Z}^{0,4} \) or \( D_6(-1) \to E_8(-1) \to \mathbb{Z}^{2,2} \oplus E_8(-1) \).

Proof. For any primitive embedding \( D_6(-1) \subset \mathbb{Z}^{2,10} \), \( D_\perp \) contains a copy of \( \mathbb{Z}^{0,4} \), hence the image of \( D_6(-1) \) is contained in the orthogonal complement of \( \mathbb{Z}^{0,4} \), which is isomorphic to \( \mathbb{Z}^{2,6} \). Similarly, \( D_6(-1) \) and \( D_{0,2} \) should span a copy of \( E_8(-1) \).

The fact that all primitive embeddings are conjugate follows from the construction of the unimodular lattice: suppose we are given two sublattices of the form \( D_6(-1) \oplus D_\perp \) inside \( \mathbb{Z}^{2,10} \), and an isomorphism between the copies of \( D_6(-1) \). Then there is a choice of isomorphism between the copies of \( D_\perp \) which is compatible with the correspondences between discriminant groups induced by \( \mathbb{Z}^{2,10} \); it thus extends to an automorphism of \( \mathbb{Z}^{2,10} \). \( \square \)

A particular embedding of \( D_6(-1) \) in \( \mathbb{Z}^{2,10} \) is given by the embedding of Dynkin diagrams between \( D_6 \) and \( E_8 \), and the fact that \( \mathbb{Z}^{2,10} \simeq \mathbb{Z}^{2,2} \oplus E_8(-1) \).
4. The period map

In this section we study the period map of $D_{1,6}$-polarised Enriques surfaces, which are parametrised by the space of line configurations in the plane: we have seen in section 3.1 how three pairs of lines gave rise to an Enriques surface as a bidouble cover of the plane.

The period map behaves well on the stable locus of the parameter space (which can be identified with $\mathbb{C}^{3 \times 6}$): since extra singularities appear when configurations of lines acquire triple points, the discriminant locus consists of the vanishing locus of Plücker coordinates (which map $\mathbb{C}^{3 \times 6}$ to the Grassmann variety $\text{Gr}(3, 6) \subset \mathbb{P}^{19}$). This is divisor with normal crossings (proposition 4.5), and the finite cover obtained by adding square roots to these coordinates produces a local uniformisation for the period map, with values in the space $\mathcal{X}_1 = \mathcal{D}(L)/\mathcal{O}(L)$.

The goal is to actually prove that denoting by $\mathcal{M}_{\text{GIT}}$ the quotient $\text{Gr}(3, 6) \sslash T$ (where the stable locus parametrises configurations having at worst triple points), the quotient $\mathcal{M}_{\text{GIT}}/\langle W_3 \times \langle Q \rangle \rangle$ (where $W_3$ is the wreath product $\mathbb{Z}/2 \wr \mathfrak{S}_3$ and $Q$ is the Cremona transformation of proposition 3.9) is actually isomorphic to $\mathcal{X}_1$ via the period map. Theorem 4.14 states that a suitably chosen point has only one preimage, and that the period map is a local isomorphism around it (using section 1.6).

A criterion of Looijenga and Swierstra, described in section 4.2.1, can be used to prove that the complement of $\mathcal{M}^s_{\text{GIT}}$ in $\mathcal{M}_{\text{GIT}}$ is mapped to the complement of $\mathcal{X}_1$ in $\mathcal{X}^\text{BB}_L$, i.e. that the map $\mathcal{M}^s_{\text{GIT}} \to \mathcal{X}_1$ is proper, and the previous remark show that it has degree one. It suffices to note that this map has finite fibres (since fibres correspond to a set of [almost] polarisations on some K3 surface), to obtained the required isomorphism.

The extension to the compactified moduli spaces follows from a Hartogs-type argument. However, we give an explicit calculation in section 4.2 which details the geometry of the singularities in the neighbourhood of one of the boundary strata.

4.1. The period mapping and its extension to the stable locus. Let $S$ be a $D_{1,6}$-polarised Enriques surface. Then $H^2(S, \mathbb{Z}_-)$ contains a distinguished $D_6$ sublattice as previously explained, and the line of (twisted) 2-forms $\mathcal{C} \omega$ lies in the complex vector space spanned by the orthogonal complement of the distinguished sublattice. If $\varphi$ is a choice of isometry between $D_6 \subset H^2(S, \mathbb{Z}_-)$ and $L$, $(S, \varphi)$ determines a point in $\mathbb{P}(L \otimes \mathbb{C})$ which is the image of the line $\mathcal{C} \omega$. Let $q$ be the quadratic form on $L$: for example, we can choose integral coordinates $(x_1, x_2; y_1, y_2, y_3, y_4)$ such that

$$q = 2x_1^2 + 2x_2^2 - y_1^2 - y_2^2 - y_3^2 - y_4^2.$$  

The associated symmetric bilinear form is denoted by $q(a, b)$.

Since $H^2(S, \mathbb{Z}_-)$ is a polarised Hodge structure, a representative $\omega$ must satisfy $q(\omega) = 0$ and $q(\omega, \bar{\omega}) > 0$. The period point of $(S, \varphi)$ is the element $[\omega]$ of the Hermitian symmetric
domain associated to the lattice \(L\) computed above:

\[
\mathcal{D}_L = \{[\omega] \in \mathbb{P}(L \otimes \mathbb{C}) \text{ such that } \langle \omega, \bar{\omega} \rangle > 0 \text{ and } \langle \omega, \omega \rangle = 0\}
\]

The period domain \(\mathcal{D}_L\) contains two connected components: the equations of \(\mathcal{D}_L\) induce the constraints

\[
2x_1^2 + 2x_2^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2
\]

\[
2|x_1|^2 + 2|x_2|^2 > |y_1|^2 + |y_2|^2 + |y_3|^2 + |y_4|^2
\]

which imply, for example, that \(\langle \omega, \bar{\omega} \rangle > 0\) and \(\langle \omega, \omega \rangle = 0\).

Let \(k\) forms by \(\gamma\) as elements of \(H^1\). Since \(\omega\) is a smooth field unique up to a scalar).

There is a well-defined holomorphic map \(U^\text{sm} \rightarrow \mathcal{X}_L\), which is locally liftable to \(\mathcal{D}_L\). The automorphic line bundle over \(\mathcal{X}_L\) lifts to the linearised line bundle \(\mathcal{O}(3)\) on \(\mathbb{P}(U^\text{sm})\).

**Theorem 4.1** (Griffiths [Voï02, chapitre 10]). Let \(S \rightarrow B\) be a holomorphic family of \(D_{1,6}\)-polarised Enriques surfaces, parametrised by a base \(B\), equipped with a continuous family of isomorphisms \(\phi(b)\) between the orthogonal complement of the \(D_6\) sublattice of \(H^2(S_b, Z_\sim)\) and \(L\).

Then the map \(B \rightarrow \mathcal{D}_L\) mapping a point \(b\) to the period point of \((S_b, \phi(b))\) is holomorphic.

Let \(\Gamma = O(L)\) and \(\mathcal{X}_L = \mathcal{D}_L/\Gamma\): note that elements of \(\Gamma\) can exchange the connected components of \(\mathcal{D}_L\). Our goal is to define a period map from the stable locus \(\mathcal{M}^s\) to \(\mathcal{X}_L\), and then to extend it to a morphism from the full GIT quotient \(\mathcal{M}_{\text{GIT}}\) to the Baily-Borel compactification \(\mathcal{X}^{BB}_L\) of \(\mathcal{X}_L\).

Let \(U^\text{sm}\) and \(U^s\) be the open subsets in \(C^{18}\) parametrising configurations of lines without triple points (resp. with at worst triple points).

**Proposition 4.2.** There is a well-defined holomorphic map \(U^\text{sm} \rightarrow \mathcal{X}_L\), which is locally liftable to \(\mathcal{D}_L\). The automorphic line bundle over \(\mathcal{X}_L\) lifts to the linearised line bundle \(\mathcal{O}(3)\) on \(\mathbb{P}(U^\text{sm})\).

**Proof.** Let \(T \rightarrow U^\text{sm}\) be the family of K3 surfaces with double points, defined in \(\mathbb{P}^5\) by the equations \(u^2 = f_t(x, y, z), v^2 = g_t(x, y, z)\) and \(w^2 = h_t(x, y, z)\), where \(f_t, g_t, h_t\) are products of two linear forms, depending on the parameter \(t \in U^\text{sm}\). There is an associated family \(\tilde{T}\) of smooth K3 surfaces over \(U^\text{sm}\): it is obtained as the corresponding cover of \(P\) (a family of rational surfaces such that \(P_t\) is the plane blown up at the vertices of \(f_t, g_t,\) and \(h_t\)).

Let \(dF, dG, dH\) be the differentials of these equations, which can be naturally interpreted as elements of \(H^0(T_t, N^*_{1,6/\mathbb{P}^5}(2))\), and \(\tau\) be a fixed section of \((\det T \otimes \mathbb{P}^5)(-6)\) on \(\mathbb{P}^5\) (which is unique up to a scalar).

Then the pairing of \(\tau\) with \(dF \wedge dG \wedge dH\) is a well-defined, nowhere degenerate bivector field \(w\) (section of the dual of \(\Omega^2_{T_t}\)) over the regular part of \(T_t\), which defines a symplectic form \(\omega \equiv w^{-1}\) on \(\tilde{T}_t\).

Since \(\tilde{T}\) is a locally trivial fibration, there is on the universal cover of \(U^\text{sm}\) a uniform choice of basis of twisted homology classes \(\gamma_i\) (\(i = 1...6\)), giving a period map from the universal cover of \(U^\text{sm}\) to \(\mathcal{D}_L\). This gives local lifts for the quotient map:

\[
U^\text{sm} = \frac{U^\text{sm}}{\pi_1(U^\text{sm})} \rightarrow \frac{\mathcal{D}_L}{\Gamma} = \mathcal{X}_L
\]

Let \(T_{ks}\) be the K3 surface obtained from a surface \(T_s\) by multiplying the matrix \(s\) of linear forms by \(k\). An isomorphism \(m_k : T_s \rightarrow T_{ks}\) is obtained by multiplying \(u, v,\) and \(w\), by \(k\). The differential forms transform as \((m_k)^* (dF dG dH)_{ks} = k^6 (dF dG dH)_s\), and \(\tau\) pulls back as
It results that $w_{ks} = k^3 w_s$ under the identification $m_k : T_s \simeq T_{ks}$ and that the periods of $T_{ks}$ are $k^{-3}$ times the periods of $T_s$. This proves that the automorphic line bundle pulls back as $O(3)$ on $\mathbb{P}(U^{sm})$.

The period map is of course equivariant under action of $GL_3$ and $T$. Note that the 2-form $\omega$ defined above can be written with the simpler formula:

$$\omega = \frac{z \, dx \wedge dy + x \, dy \wedge dz + y \, dz \wedge dx}{uvw}$$

since $\omega dF dG dH$ is a multiple of the standard 5-form on $\mathbb{P}^5$ with values in $O(6)$ (replace $dF$ by $udu$, etc.).

**Proposition 4.3.** The period map from the universal cover of $U^{sm}$ to $D_L$ is submersive. Consequently, the map $M_{sm}^\text{GIT} \to X_L$ can be locally lifted to an étale map with target $D_L$.

**Proof.** Since the two spaces have the same dimension, it is enough to prove that it is a submersion. But if $V^{sm}$ is the subspace of $\mathbb{C}^{4 \times 7}$ which parametrises smooth Campedelli surfaces, the forgetful map $V^{sm} \to U^{sm}$ is itself submersive.

Since the local period map $V^{sm} \to D_L$ is submersive (section 1.5), the result follows. \qed

We are going to use the following theorem of Borel

**Theorem 4.4** (Borel [Bor72, Thm. A]). Let $U$ be a polydisc in $\mathbb{C}^n$, and $U^*$ the complement of a standard normal crossing divisor in $U$. Let $D$ be a bounded symmetric domain, $\Gamma$ an arithmetic subgroup of the associated group $G$, and let $X = D/\Gamma$.

Suppose we are given a holomorphic map $f : U^* \to X$. Then if $f$ is locally liftable to $D$, then $f$ extends to a holomorphic map $U \to \overline{X}$ where $\overline{X}$ is the Baily-Borel compactification of $X$.

The extension to the whole stable locus can be carried out using this theorem and the following property:

**Proposition 4.5.** The complement of $M_{sm}^\text{GIT}$ in $M_{GIT}^s$ is made of coordinate hyperplane sections, with normal crossings.

The geometry of the variety $M_{GIT}$ is very well known and studied thoroughly, for example in [MSY92] in a similar context. We nevertheless provide a proof for the needed properties. Note the statement here concerns configurations of six ordered lines.

**Proof.** First observe that a stable configuration of six lines, having at worst standard triple points, can have at most four such points (proposition 3.13). Indeed, a given line cannot go through three triple points, hence five triple points would involve at least 8 lines. If there are four triple points, then any line goes through two of them. The graph having triple points as vertices and lines as edges is made of four vertices of valence three, which is only possible if it is a complete graph. This means that the combinatorics of the corresponding arrangement of lines (up to permutation in $S_6$) are fully determined (they form a complete quadrangle).

Then note that a stable configuration of lines always contains four lines which constitute a projective frame. If this were not the case, the 15 sets of four lines would contribute 5 triplets of concurrent lines (because each triplet is part of at most 3 sets of four lines), which is impossible by the previous remark.
Again up to a permutation, we can assume the stable configuration is described by a matrix of the form

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & a & b \\
0 & 1 & 0 & 1 & c & d \\
0 & 0 & 1 & 1 & e & f
\end{pmatrix}
\]

- For one triple point, we can assume that the vanishing minor is \( m_{125} = e \), since a projective frame can be made out of every 4-tuple of non-concurring lines. Neither \( a \) nor \( b \) can vanish, hence can be set to 1. The local structure of \( \mathcal{M}_{\text{GIT}} \) around this point is then described by the “étale slice”, and the locus of triple points is a smooth divisor \( e = 0 \).

In the case of two triple points, choosing two lines from each triple makes a projective frame: the vanishing minors are

- either \( m_{125} = m_{346} = e = b - d = 0 \); then \( a \) and \( b \) cannot vanish again, hence we set them to 1, the locus of triple points is the union of \( e = 0 \), \( d = 1 \) which meet transversely;
- either \( m_{125} = m_{345} = e = a - c = 0 \); then \( a \) and \( f \) cannot both vanish, the locus of triple points is a union of \( e = 0 \), \( c = 1 \).

- In the case of three triple points, two triples have a common line, say 125 and 345, up to permutation the third can be chosen to be 136. The corresponding equations are \( e = a - c = d = 0 \). Again \( a \) and \( b \) cannot vanish, hence can be chosen to be 1, and in the étale slice, the locus of triple points is defined by \( ed(c - 1) = 0 \), around \( e = d = c - 1 = 0 \).

- In the case of four triple points, as before we can assume that two triples are 125 and 345, then the others can be chosen to be 136 and 246, corresponding to the vanishing of \( e \), \( d \), \( b - f \) and \( a - c \). As before, \( a \), \( b \) can be chosen to be 1, and the locus of triple points is then the union of \( e = 0 \), \( d = 0 \), \( c = 1 \), \( f = 1 \) which is again a normal crossing divisor. \( \square \)

Triple points in the configuration of lines correspond to the appearance of rational double points in the K3 surfaces \( T_t \): each triple point creates a local monodromy or order two. These local monodromies are eliminated by the following process: let \( \widetilde{U}^s = \text{sq}^{-1}(U^s) \) be the finite cover obtained as the inverse image under the morphism \( \text{sq} : \mathbb{P}^{19} \to \mathbb{P}^{19} \) which takes each Plücker coordinate to its square. By the previous proposition, \( \widetilde{U}^s \) is the smooth locally closed subscheme of \( \mathbb{P}^{19} \), and the action of the diagonal torus \( T \) admits a lift to \( \widetilde{U}^s \) (and actually also to the inverse image of the Grassmannian, which is a singular variety).

**Proposition 4.6.** Let \( \widetilde{\mathcal{M}}^s_{\text{GIT}} \) be the GIT quotient \( \widetilde{U}^s / T \), which is a Galois cover of \( \mathcal{M}^s_{\text{GIT}} \). Then the period map \( \widetilde{\mathcal{M}}^s_{\text{GIT}} \to \mathcal{X}_l \) has local lifts to \( \mathcal{D}_L \).

**Proof.** Using the previous remark and the calculation carried in section \( \textbf{[L.6]} \), the finite cover \( \widetilde{\mathcal{M}}^s_{\text{GIT}} \to \mathcal{M}^s_{\text{GIT}} \) trivialises the monodromy: the period map extends regularly and has local lifts to \( \mathcal{D}_L \). \( \square \)

4.2. **Local structure around the one-dimensional \( \chi \) stratum.** Consider a polystable configuration in the one-dimensional stratum of \( \mathcal{M}_{\text{GIT}} \). It is made of four concurrent lines (whose cross-ratio is denoted by \( t \)), and two identical lines. We focus on the \( \chi \) case, where the identical lines belong to the same pair. Up to some permutation, we can assume that
the configuration is given by the matrix
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & t & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\]
with \( t \neq 0 \). Its stabiliser has dimension 1, it acts by multiplying lines by \((\lambda^{-1}, \lambda^{-1}, \lambda^2)\) and columns by \((\lambda, \lambda, \lambda, \lambda^{-2}, \lambda^{-2})\).

An “étale slice” of \( \mathcal{M}_{\text{GIT}} \) around this configuration given by the five parameter family:
\[
N(t, a, b, c, d) = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & c \\
0 & 1 & 1 & t & 0 & d \\
0 & 0 & a & b & 1 & 1
\end{pmatrix}
\]
where \( \mathbb{G}_m \) acts by \( \lambda \cdot (t, a, b, c, d) \mapsto (t, \lambda^2 a, \lambda^3 b, \lambda^{-3} c, \lambda^{-3} d) \) on the parameters (this action is equivalent to the action on lines and columns above). The étale slice itself is the quotient \( \mathbb{C} \langle t, a, b, c, d \rangle \cong \mathbb{G}_m \) whose coordinate ring is \( \mathbb{C}[t, ac, ad, bc, bd] \) (a cylinder over the quadratic cone with equation \((ac)(bd) = (ad)(bc))\).

Also recall that for a generic configuration \( \gamma \), the quadratic transformation associated to the vertices of the three pairs of lines defines a new configuration \( Q(\gamma) \) with isomorphic associated Enriques surface. Then \( Q \) defines a biregular involution of \( \mathcal{M}_{\text{GIT}} \), commuting with the action of \((\mathbb{Z}/2 \wr S_3)\).

**Proposition 4.7.** \( Q \) lifts to a biregular involution of the étale slice mentioned above, given by the composite of \((a, b) \leftrightarrow (c, d)\) and reversing the order of the four first columns. Therefore the period map is invariant under exchange of \((a, b)\) and \((c, d)\).

**Proof.** Recall that \( Q \) can be defined as follows on matrices:
\[
\begin{pmatrix}
0 & 0 & a_2 & b_2 & c_3 & d_3 \\
-1 & b_1 & c_2 & d_2 & 0 & 0 \\
0 & c_1 & -1 & b c & 0 & a_3 & b_3
\end{pmatrix} \mapsto \begin{pmatrix}
0 & 0 & c_2 & d_2 & a_3 & b_3 \\
-1 & c_1 & d_1 & a_2 & b_2 & 0 & 0 \\
0 & a_1 & b_1 & 0 & 0 & c_3 & d_3
\end{pmatrix}
\]

Given a configuration with matrix:
\[
N = \begin{pmatrix}
1 & 0 & 1 & 1 & 0 & c \\
0 & 1 & 1 & t & 0 & d \\
0 & 0 & a & b & 1 & 1
\end{pmatrix}
\]
its three vertices are given by the columns of the matrix
\[
M = \begin{pmatrix}
b - at & -d & 0 \\
a - b & c & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
and the associated matrix in this new basis is
\[
M^T N = \begin{pmatrix}
b - at & a - b & 0 & 0 & t - 1 & bc + ad - bd - act + t - 1 \\
-d & c & c - d & ct - d & 0 & 0 \\
0 & 0 & a & b & 1 & 1
\end{pmatrix}
\]
(write $f(a, b, c, d) = bc + ad - bd - act + t - 1 = f(c, d, a, b)$). The transformed configuration $Q(M^1N)$ has matrix

$$
\begin{pmatrix}
-d & c & 0 & 0 & 1 & 1 \\
 b - at & a - b & a & b & 0 & 0 \\
 0 & 0 & c - d & ct - d & t - 1 & f(a, b, c, d)
\end{pmatrix}
\equiv
\begin{pmatrix}
 ct - d & c - d & 0 & 0 & t - 1 & f(c, d, a, b) \\
 b & a & a - b & b - at & 0 & 0 \\
 0 & 0 & c & -d & 1 & 1
\end{pmatrix}
$$

(the second matrix is obtained by reversing the first four columns as well as the lines). Now applying the substitution

$$(a, b, c, d) \mapsto (c, d, a, b)$$

gives back (up to changes of signs in columns) the matrix $M^1N$.

Note that the lift to the étale slice is equivariant under the action of $\mathbb{C}^\times$. It is useful to define a similar étale slice for the ramified cover $\widetilde{\mathcal{M}}_{\text{GIT}}$: for this we need to formally add square roots to all minors of $N(t, a, b, c, d)$.

If $a, b, c, d$ are in a small neighbourhood of zero, and $t$ lies in a small disc $U$ avoiding 0 and 1, then the only minors of the configuration matrix which possibly vanish involve two columns from the first four and one of the last two. More explicitly: three lines become concurrent when either $a, b, a - b, at - b, c, d, c - d, ct - d$ vanishes.

Let $\mathcal{E}_i = (E_i)$ be two identical families of cones of elliptic curves with equation $u_i^2 = x_i^2 - y_i^2, v_i^2 = tx_i^2 - y_i^2$. Then $\mathcal{E}_1 \times U \mathcal{E}_2$ maps to a ramified cover of $\mathbb{C}^5$ (with coordinates $t \in U, a, b, c, d$):

$$a = x_1^2, \quad b = y_1^2, \quad c = x_2^2, \quad d = y_2^2$$

Each $\mathcal{E}_i$ is equipped with an action of $\mathbb{G}_m$, $(u_i, v_i, x_i, y_i) \mapsto (\lambda u_i, \lambda v_i, \lambda x_i, \lambda y_i)$.

**Proposition 4.8.** The action of $\mathbb{G}_m$ on the affine space lifts to the action of $\mathbb{G}_m$, $\mathcal{E}_1 \times U \mathcal{E}_2$ with weights $(6, -6)$, and $\mathcal{E}_1 \times U \mathcal{E}_2 \parallel \mathbb{G}_m$ is a local model for $\mathcal{M}_{\text{GIT}}$.

4.2.1. **The boundary period map.** To study the behaviour of the period map, we will not work on $\mathcal{M}_{\text{GIT}}$ itself, but on the étale slice of $\mathcal{M}_{\text{GIT}}$ defined earlier. We denote by $Z_i$ the central curve of $\mathcal{E}_i$ (defined by $u_i = v_i = x_i = y_i = 0$), and by $\tilde{Z}_i$ the blow-up of $\mathcal{E}_i$ along $Z_i$.

We already know that a period map is well-defined and locally liftable over $\mathcal{M}_{\text{GIT}}$:

**Proposition 4.9.** The period map from $\mathcal{E}_1 \times U \mathcal{E}_2$ to $\mathcal{X}_i$ is well-defined and locally liftable to $D_L$ outside of $Z_1 \times \mathcal{E}_2 \cup \mathcal{E}_1 \times Z_2$.

This observation is compatible with a classical description: if $\Gamma$ is a neat arithmetic subgroup of $O(L), D_L/\Gamma$ locally looks like a family over a disc $U$ of cones over a product of elliptic curves, near a point of a 1-dim. boundary component.

**Proposition 4.10.** The period map with source domain $\tilde{\mathcal{E}}_1 \times U \tilde{\mathcal{E}}_2$ is defined on the complement of a normal crossing divisor, which is the union of $\tilde{Z}_1 \times \tilde{\mathcal{E}}_2$ and $\tilde{\mathcal{E}}_2 \times \tilde{Z}_2$ (here $\tilde{Z}_i$ is the exceptional divisor in $\tilde{\mathcal{E}}_i$).
By Borel's extension theorem, the period map has a unique extension to this desingularisation, which takes values in the Baily-Borel compactification $\mathcal{X}_L^{BB}$.

We now want to prove that for a fixed $t$, the exceptional divisor is contracted to a point in $\mathcal{X}_L^{BB}$, which would prove that in some small neighbourhood of the semistable point, the period map $\mathcal{M}_{GIT} \to \mathcal{X}_L^{BB}$ is well-defined. Since $Q$ lifts to the ramified cover, and exchanges the two factors, it is enough to prove it for one of the components: we choose to study the case where $c = d = 0$ (double line).

**Proposition 4.11.** Let $\Delta \to \tilde{E}_1 \times \tilde{E}_2$ be a one-parameter family such that $c = d = 0$ at the central point. Then for generic values of $a$ and $b$, the limit of the period map only depends on $t$.

The proof of this proposition is delayed to the next section: for the other boundary strata, we rely on a weaker result. We are going to prove that the rational map given by the period map $\mathcal{M}_{GIT} \to \mathcal{X}_L^{BB}$ maps the special semistable strata (0-dimensional strata) to the boundary as well.

Consider a one-parameter degeneration with special fibre a configuration of three double lines: the three ramification divisors degenerate to $x^2$, $y^2$, $z^2$ (for given projective plane coordinates $[x : y : z]$). We are using the same criterion as Looijenga in [Loo09] and [LS08, sec. 3].

**Proposition 4.12** (see [LS08][Loo09]). Let $T_t$ be a one-parameter family of K3 surfaces whose special fibre is singular, and $\omega$ be a choice of twisted 2-forms on $(T_t)$ such that $\omega_{t=0}$ is a nonzero cohomology class in $H^2((T_0)_{reg}, \mathbb{C})$, but the $L^2$-norm of $\omega$ goes to infinity as $t \to 0$.

Then the limit of the period map of the family as $t \to 0$ belongs to $\mathbb{P} \ker : H^2(T_t, \mathbb{C}) \to H^2((T_0)_{reg}, \mathbb{C}))$. Moreover, the limit of the period map belongs to the boundary of the period domain.

On the special fibre, the chosen 2-form can be written as

$$\omega = \frac{xdydz + ydzdx + zdxdy}{xyz}$$

which has non zero periods on the cycle

$$\gamma : (\theta_1, \theta_2) \mapsto [\exp(i\theta_1) : \exp(i\theta_2) : 1]$$

since for $z = 1$, $\omega = dx/x \wedge dy/y$: which has period $(2i\pi)^2$.

Moreover $\omega \wedge \tilde{\omega} = dxzdydz/(xyz)^2$ is not integrable over $\mathbb{C}^2$: it follows that for any smoothing of this central fibre, the monodromy cannot have finite order. Now using Borel's extension theorem, we know that some birational model of $\mathcal{M}_{GIT}$, which is an isomorphism outside the non-stable locus, can be mapped regularly to $\mathcal{X}_L^{BB}$. The discussion above implies that the fibre above the special semistable point is mapped to the boundary. The discussion is similar for the remaining boundary strata, which correspond to similar line configurations, with various labellings.

### 4.3. The most special configuration and its isotropy group.

We are interested in a very special point of $\mathcal{X}_L$.

**Proposition 4.13.** There exists a unique period point $[\omega_0]$ such that $\omega_0^\perp$ in $L$ is isomorphic to $\mathbb{Z}^4(-1)$.

Since there are $(-1)$-classes in $\omega_0^\perp$, this $\omega_0$ is of course not the period point of an Enriques surface (but rather the one of a rational degeneration of Enriques surfaces).
Proof. Such period points are in bijection with isometry classes of decompositions \(L \simeq \mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)\). Indeed, any positive plane \(T\) such that \(T^\perp \simeq \mathbb{Z}^4(-1)\) gives such a decomposition. It is clear that any two such decompositions are conjugate under some isometry. \(\square\)

We want to prove the following theorem: for convenience, \(W_3\) denotes the wreath product \((\mathbb{Z}/2 \rtimes S_3)\).

**Theorem 4.14.** There is only one point which is mapped to \([\omega_0]\) under the period map \(P : \mathcal{M}_{\text{GIT}}/(W_3 \times \langle Q \rangle) \to \mathcal{X}_{\text{L}}^{\text{BB}}\), and \(P\) is a local isomorphism around these points.

In order to prove the theorem, we will describe how to uniformise the period map for each of these spaces: we will show that in a neighbourhood of the special point, uniformisation is obtained by performing a finite cover of degree \(24 \times 32\), both at the source and target, and that the uniformised period map is étale.

The stabiliser of the special period point is the group of automorphisms of \(\mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)\), preserving the direct sum decomposition (and the connected components of \(D_L\)): it is the quotient of \(SO_2(\mathbb{Z}) \times O_4(\mathbb{Z})\) by \(\pm \text{id}\), which is a group of order \(24 \times 32\). This settles the statement for the target space.

**Proposition 4.15.** A stable configuration of lines has period point \(\omega_0\) if and only if it is a complete quadrangle and the three pairs of lines are the opposite sides of the quadrangle.

**Proof.** The associated surface should be a K3 surface with involution, with 4 disjoint rational \((-2)\)-curves orthogonal to the polarisation \(H\). Each one defines a fixed rational double point, and the only stable configurations having these should have exactly four distinct triple points through which goes a line from each pair.

Of course no line can go through three triple points, and counting vertices and edges in the graph drawn by the lines, we see that each line goes through exactly two triple points, giving a complete quadrilateral. \(\square\)

**Proposition 4.16.** The GIT quotient \((\mathbb{P}^2)^6//\text{PSL}_3\) contains exactly two points representing complete quadrangles, whose triple points lie exactly on one line from each pair \(L_1L_2, L_3L_4, L_5L_6\).

**Proof.** First note that a complete quadrangle has trivial stabiliser in \(\text{PGL}_3\). Fixing one of the lines \(\ell_1\) the combinatorics of the arrangement are fully determined by the two pairs of lines meeting on \(\ell_1\): there are two possible inequivalent configurations (of six ordered lines). \(\square\)

A point representing a complete quadrangle lies at the intersection of four transverse hypersurfaces (one for each concuring triplet of lines).

The group \((\mathbb{Z}_2)^3 \rtimes S_3\) acts on these two points by the exact sequence

\[1 \to S_4 \to (\mathbb{Z}/2) \rtimes S_3 \to \mathbb{Z}/2 \to 1\]

where the first map is the action of \(S_4\) on the six pairs of vertices (\(S_4\) is a semi-direct product \((\mathbb{Z}/2)^2 \rtimes S_3\)), and the second one the signature map.

As before let \(\widetilde{U}^s\) be the ramified cover which adds a square root to each Plücker coordinate. The action of the torus \(T\) lifts to \(\widetilde{U}^s\) and defines a quotient \(\widetilde{\mathcal{M}}^s_{\text{GIT}}\).

**Theorem 4.17.** The period map \(\widetilde{U}^s \to \mathcal{X}_1\) is locally liftable to \(\mathcal{D}_L\) and the local lifts are submersive around the special points.
Proof. Consider the 4-parameter family of configurations:
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & a + b & 1 \\
1 & 1 & -1 & -1 & 1 & c + d \\
1 & -1 & 1 & -1 & a - b & c - d
\end{pmatrix}
\]
corresponding to the configuration of lines
\[
\begin{align*}
(x + y = \pm 1, x - y = \pm 1, y + a(x + 1) + b(x - 1) = 0, x + c(y + 1) + d(y - 1) = 0
\end{align*}
\]
The relevant minors or the matrix are
\[
m_{135} = 4b, m_{245} = -4a, m_{146} = 4d, m_{236} = -4c
\]
which define divisors with normal crossings. Hence adding a square root to each of
the variables \(a, b, c, d\) allows to lift the period map to \(\tilde{\mathcal{U}} \to D_L\) where \(\mathcal{U}\) is a small
neighbourhood of the origin. Write \(a = \alpha^2, b = \beta^2, c = \gamma^2, d = \delta^2\).
Similarly, write \(S_a, S_b, S_c, S_d\) for the four \((-1)\)-classes which are associated to the vanish-
ing of \(a, b, c, d\). They provide a basis of \(H^{1,1}(\tilde{S}, \mathbb{Z})\) at the special point, and they also give
local coordinates for \(D_L\) around the associated period point.

We now need to compute the partial derivatives of \(\int S^* \omega\), with respect to the four
variables: it is clear that the Jacobian matrix is diagonal, so it is enough to prove that
\[
\frac{\partial}{\partial \alpha}(\int S_a \omega) \text{ is nonzero. But this computation was done in section 1.6.}
\]
Note that since we are dealing with the lift of the period map, the integral structure is
no longer relevant, and we can again work with Campedelli surfaces (and their universal
cover), instead of Enriques surfaces, to determine the local structure of the period map. \(\square\)

The action of \(W_3 \times \mathbb{Z}_2\) lifts to \(\tilde{\mathcal{M}}^s_{\text{GIT}}\) in the following way: \(W_3\) acts by permutation matrices
on \(\bigwedge^3(C^6)\) (with coefficients \(\pm 1\)). Then its natural lift \(\tilde{W}_3 \simeq (\pm)^{19} \times W_3\) acts on the finite
cover \(\tilde{\mathcal{M}}^s_{\text{GIT}} \to \mathcal{M}^s_{\text{GIT}}\) by changing signs in the square roots of the Plücker coordinates and
by the action of \(W_3\) on the base.

**Proposition 4.18.** The stabiliser of a special configuration in \(\tilde{\mathcal{M}}^s_{\text{GIT}}\) inside \(\tilde{W}_3\) is an order \(24 \times 16\)
group.

**Proof.** Remember that the local structure of \(\tilde{\mathcal{M}}^s_{\text{GIT}} \to \mathcal{M}^s_{\text{GIT}}\) around a special configuration
is the same as the double cover of \(\mathbb{C}^4\) given by the formula \((a, b, c, d) \mapsto (a^2, b^2, c^2, d^2)\).
The number \(24 \times 16\) then accounts for the actions of \((\mathbb{Z}/2)^4\) and \(S_4\) on this space, which
stabilise the origin. \(\square\)

**Proposition 4.19.** The action of \(Q\) lifts to the ramified cover. As a consequence, the uniformisation
\(\tilde{\mathcal{M}}^s_{\text{GIT}} \to \mathcal{M}^s_{\text{GIT}}/W_3 \times (Q)\) has local degree \(24 \times 32\) on small neighbourhoods of the special points.

This proves the main result of this section:

**Theorem 4.20.** The period map \(\mathcal{M}_{\text{GIT}}/(W_3 \times \mathbb{Z}/2) \to \mathcal{X}_L^{BB}\) is birational.

**Proof.** The preimage is \(\mathcal{X}_L\) under the period map is exactly the stable locus, hence the period
map \(\mathcal{M}^s_{\text{GIT}} \to \mathcal{X}_L\) is proper, and finite (it lifts locally to an étale map).
The map $\mathcal{M}_{\text{GIT}}/(W_3 \times \mathbb{Z}/2) \to \mathcal{X}_L$ is finite and the preimage of $[\omega_0]$ consists of a single point $P$. Considering the diagram of germs around the special points

$$\begin{eqnarray*}
(\mathcal{M}_{\text{GIT}}^s, *) & \xrightarrow{\phi} & (D_1, [\omega_0]) \\
(\mathcal{M}_{\text{GIT}}^s/W_3 \times \langle Q \rangle, *) & \xrightarrow{\phi} & (\mathcal{X}_L, [\omega_0])
\end{eqnarray*}$$

we check that the period map is a local isomorphism around this point. But since the map is finite, the preimage of a small contractible neighbourhood of $[\omega_0]$ can be chosen to be a small contractible neighbourhood of $P$: this proves that a generic fibre consists of a single element, and that $\mathcal{M}_{\text{GIT}}/(W_3 \times \mathbb{Z}/2) \to \mathcal{X}_L$ is an isomorphism. □

The previous property can be refined in a genuine isomorphism.

**Corollary 4.21.** The rational map of the previous theorem is an isomorphism.

**Proof.** We already know that the natural map $\mathcal{M}_{\text{GIT}}/(W_3 \times \mathbb{Z}/2) \to \mathcal{X}_L$ is an isomorphism, and that the ample line bundles $\mathcal{O}(1)$ over $\mathcal{M}_{\text{GIT}}$ and the automorphic line bundle $\mathcal{L}$ over $\mathcal{X}_L$ are identified under this isomorphism, using the formula for the universal twisted 2-form. In particular, the sections of these line bundles on these Zariski open subsets can be identified.

The projective coordinate ring of the Baily-Borel compactification is by definition $\bigoplus H^0(\mathcal{X}_L, L^k)$ [BB66] while $\mathcal{M}_{\text{GIT}}/(W_3 \times \mathbb{Z}/2)$ can be obtained from the ring of invariants $\text{Proj} \bigoplus H^0(\mathcal{M}_{\text{GIT}}^s, \mathcal{O}(3k))_{\text{SL}_3 \times T \times W_3 \times \mathbb{Z}/2}$.

But sections over $\mathcal{M}_{\text{GIT}}^s$ are nothing more than invariant sections over the stable locus in the Grassmannian, which is the complement of a codimension 2 union of Schubert varieties: they extend to the whole Grassmannian, where invariant sections give, by definition, the coordinate ring of the GIT quotient. □

5. **Mixed Hodge structures and boundary configurations**

As an appendix to section 4.2, we determine more precisely how the Hodge structure on the twisted cohomology of Enriques surfaces $H^2(S, \mathbb{Z}_-)$ degenerates when it approaches the boundary component “of type $\chi$” which was studied there.

The asymptotic behaviour of the periods is characterised by an isotropic sublattice $I$ in the lattice $\mathbb{Z}^{2,10}$ (which is isomorphic to the generic $H^2(S, \mathbb{Z}_-)$. The main result is proposition 5.3 which states that $I^+ / I$ is isomorphic to $E_8(-1)$. A consequence is that the analogous computation in the smaller lattice $L$ (which is used for our period map) gives a quadratic lattice $I^+ / I \subset L / I$ isomorphic to $\mathbb{Z}^2(-2)$. This fact is used to determine the complete correspondence between boundary strata of the GIT moduli space and strata of the Baily-Borel compactification $\mathcal{X}_L^{BB}$ (see section 7).

5.1. **Geometric setup.** The type $\chi$ degeneration is a 1-parameter degeneration of Enriques surfaces, with central fibre a rational surface self-intersecting along an elliptic curve. It happens when two lines belonging to the same ramification divisor coincide.

The type $\chi$ degeneration admits the following description: consider the blowup $\mathcal{P}$ of $\mathbb{P}^2$ at three points $A, B, C$, and let $L_A, L_B, L_C$ be the $(-1)$-curves arising as the proper transforms...
of the sides BC, CA, AB. Then the linear systems $H_A = H - A$, $H_B$, $H_C$ are pencils of rational curves with self-intersection zero. Note that $K_P = -3H + A + B + C = -H_A - H_B - H_C$.

A $D_{1,6}$-Enriques surface can be constructed as the bidouble cover of $P$ with ramification divisors of the form $R_A = D_A + D'_A$ where $D_A$ and $D'_A$ are elements of $|H_A|$. A type I degeneration is obtained by making $R_C$ into a double line, and we additionally suppose that $R_A$ and $R_B$ remain generic (see figure 1).

![Figure 1. Type $\chi$ degeneration](image)

**Proposition 5.1.** The central fibre of a type $\chi$ degeneration of the K3 surfaces is a union of two isomorphic rational surfaces $V_i$, obtained as bidouble covers ramified over $R_A$ and $R_B$, and $K_V = -H_C$.

The central fibre of the corresponding degeneration of Enriques surfaces is a rational surface whose normalisation has $K^2 = 0$, having everywhere normal crossings along a smooth elliptic curve $E$.

Note that this degeneration can be obtained from the classical model of 6 lines in the plane by blowing up the four points lying over the vertex of the two collapsing lines.

**Proposition 5.2** (see Friedman, or details in the next section). A type $\chi$ degeneration is a semistable type II degeneration in the sense of Kulikov. The limit mixed Hodge structure of a type $\chi$ degeneration has $W_1 = H^1(E)$, $W_2/W_1 \simeq E^\perp/E$ (where $E$ is an element of $H^2(V_1 \sqcup V_2)$).

Let $V$ be the normalisation of the degenerate surface, which is a Del Pezzo surface of degree 4 blown-up at 4 points. Note that $E = -K_V$, hence the lattice $E^\perp$ is isomorphic to $-E_9 \simeq 2E \oplus -E_8$.

**Proposition 5.3.** The mixed Hodge structure of a type $\chi$ degeneration is characterised by a lattice $E^\perp/E \simeq E_8(-1)$, and only depends (up to isomorphism) on the isomorphism class of the elliptic curve $E$. 
Since the orthogonal complement of $D_6$ in $E_8$ is isomorphic to $\mathbb{Z}^2(2)$, the corresponding boundary component of the period space corresponds to isotropic sublattices of $L$ such that $I^+/I \simeq \mathbb{Z}^2(-2)$.

5.2. **The Clemens-Schmid exact sequence for cohomology with local coefficients.** Let $S \to \Delta$ be a type $\chi$ degeneration of Enriques surfaces, and write $S = T/\iota$ where $\iota$ is an involution without fixed points. As usual, we denote by $\mathbb{Z}_-$ the nontrivial local system of integers over $S$. The fibre of $S$ over $t \in \Delta$ is denoted by $S_t$ and $S^*$ is the fibre product $S \times_{\Delta} \Delta^*$.

**Proposition 5.4.** The homology groups of $S_0$ with coefficients in $\mathbb{Z}_-$ are $H_0 = \mathbb{Z}/2\mathbb{Z}$, $H_1 = 0$, $H_2 = \mathbb{Z}^{11}$, $H_3 = 0$, $H_4 = \mathbb{Z}$.

**Proof.** Remember that $S_0$ is obtained from its normalisation $V_0$ by replacing the double curve $\tilde{E}$ by the singular locus $E$. Note that $\mathbb{Z}_-$ pulls back to $\mathbb{Z}$ on $V_0$. Then by a Mayer-Vietoris type exact sequence

$$H_n(E, \mathbb{Z}) \to H_n(V_0, \mathbb{Z}) \to H_n(S_0, \mathbb{Z}_-) \to -1$$

We get $H_0 = \mathbb{Z}/2$, $H_1 = 0$, $H_2$ is an extension of $H_2(E)$ by $H_2(V_0, \mathbb{Z})/[E]$, $H_3 = 0$, $H_4 = \mathbb{Z}$. □

**Proposition 5.5.** The cohomology groups of $S_0$ with coefficients in $\mathbb{Z}_-$ are $H^0 = 0$, $H^1 = \mathbb{Z}/2$, $H^2 = \mathbb{Z}^{11}$, $H^3 = 0$, $H^4 = \mathbb{Z}$.

**Proof.** From the sequence of sheaves

$$0 \to (\mathbb{Z}_-)_{S_0} \to V_*\mathbb{Z}_{V_0} \to \mathbb{Z}_E \to 0$$

we derive the long exact sequence

$$H^n(S_0, \mathbb{Z}_-) \to H^n(V_0, \mathbb{Z}) \to H^n(E, \mathbb{Z}) \to +1$$

We get $H^0 = 0$, $H^1 = \mathbb{Z}/2$, $H^2$ is an extension of $H^2(V_0)$ by $H^1(E)$, $H^3 = 0$, $H^4 = \mathbb{Z}$. □

**Proposition 5.6.** The Clemens-Schmid exact sequence

$$H_3(S_0, \frac{1}{2}\mathbb{Z}_-) = Z\mathbb{E} \to H^2(S_0, \mathbb{Z}_-) \to H^2_{\operatorname{lim}}(S_t, \mathbb{Z}_-) \to N \to H^2_{\operatorname{lim}}(S_t, \mathbb{Z}_-)$$

is exact over the integers.

**Proof.** Proceeding as in [Fri84], we decompose the Clemens-Schmid exact sequence into a part of Wang’s long exact sequence

$$H^1(S_t, \mathbb{Z}_-) \to H^2(S^*, \mathbb{Z}_-) \to H^2(S_t, \mathbb{Z}_-) \xrightarrow{T-1} H^2(S_t, \mathbb{Z}_-)$$

and the relative cohomology exact sequence

$$H^1(S^*, \mathbb{Z}_-) \to H^2(S^*, \mathbb{Z}_-) \to H^2(S, \mathbb{Z}_-) \to H^2(S^*, \mathbb{Z}_-) \to H^3(S, \mathbb{Z}_-)$$

By Poincaré duality, $H^3(S, \mathbb{Z}_-) \simeq H_3(S_0, \mathbb{Z}_-) = 0$. Hence $H^2(S^*, \mathbb{Z}_-) = \operatorname{cokernel}$ of $H^2(S, \mathbb{Z}_-) \to H^2(S, \mathbb{Z}_-)$ which is also $H_4(S_0, \mathbb{Z}_-) \to H^2(S, \mathbb{Z}_-)$. Moreover, since the image of a generator of $H_4(S_0, \mathbb{Z}_-) = \pm 2E$, the 2-torsion element in $H^2(S^*, \mathbb{Z}_-)$ is $[E]$.

Now $S_t$ is a honest Enriques surface, hence $H^2(S_t, \mathbb{Z}_-)$ is torsion-free, and $H^1(S_t, \mathbb{Z}_-)$ is isomorphic to $\mathbb{Z}/2$, hence the kernel of $T-1$, or equivalently the monodromy operator $N$, is the quotient $H^2(S^*, \mathbb{Z}_-)/[E]$. □
As a conclusion, \( \text{Gr}_2^W H^2_{\text{lim}}(S_1, Z_-) \) is isomorphic to \( H^2(S_0, Z_-)/[E], H^1(E, Z) \). But
\[
H^2(S_0, Z_-)/H^1(E, Z) \cong E_+ \subset H^2(V_0, Z).
\]

6. THE MONODROMY OF MARKED CAMPEDELLI SURFACES

The results we obtained concerning Enriques surfaces can be summarised by the theorem stated in the introduction:

**Theorem.** The structure of the period map of Campedelli surfaces can be described by the following diagram

\[
\begin{array}{ccc}
\text{Gr}(4, 7) & \to & \text{Gr}(3, 6) \\
\downarrow & & \downarrow \\
\text{Gr}(4, 7)/\mathfrak{S}_4 & \to & \text{Gr}(3, 6)/\langle Z/2 \rangle^2 \times \mathfrak{S}_3 \\
\end{array}
\]

where \( \mathcal{X}^\text{BB}_L \) is the Baily-Borel compactification of \( \mathcal{X}_L \).

In particular, the fibres of the period map we defined have generically four connected components. It is natural to ask whether the same statement is true when considering the period map of the integral Hodge structure \( H^2(X, Z) \) (we call Campedelli lattice the underlying integral quadratic lattice), instead of \( H^2(S, Z_-) \) (whose lattice is isomorphic to \( L \)).

The main result of this section is corollary \[6.8\] the embedding of the lattice \( L \) in \( H^2(X, Z) \) is invariant under isometries of \( L \), meaning that any isometry of \( L \) induces an isometry of the overlattice \( H^2(X, Z) \) (it is not true, however, that \( L \) is invariant under any isometry of \( H^2(X, Z) \)). In particular, the isomorphism class of the Hodge structure \( H^2(S, Z_-) \) fully determines that of \( H^2(X, Z) \), meaning that the true period map of marked Campedelli surfaces also has disconnected fibres.

6.1. **The cohomology lattice of Campedelli surfaces.** Suppose \( X \) is a smooth Campedelli surface, with a chosen involution \( s_\kappa \) and \( X/s_\kappa = S \). By blowing up the six fixed points, we get a double cover \( \hat{X} \to \hat{S} \) ramified over six \((-2)\)-curves and a genus 3 curve, where \( S \) is the minimal resolution of \( S_\kappa \) and \( \hat{X} = X \) blown up at the six isolated fixed points of \( s_\kappa \). The results of section 2.3 apply, and the index of \( H^2(\hat{S}, Z_-)(2) \subset H^2(\hat{X}, Z_\kappa)^{s_\kappa} \) is \( 2^{6-2+7} = 2^5 \).

The notation \( H_{\text{num}} \) still denotes the quotient of \( H \) by its torsion subgroup. We will see in the next section that \( H^2(\hat{S}, Z_-) \) contains \( L \oplus D_6(-1) \) as an index 4 subgroup: consider the following maps
\[
L(2) \oplus D_6(-2) \to H^2(\hat{S}, Z_-)(2) \to H^2(X, Z_\kappa)^{s_\kappa} \oplus Z^6(-1)
\]
where the first one has index four, the second one index \( 2^5 \).

Then the equation
\[
4 \cdot 2^5 = \left[ Z^6(-1) : D_6(-2) \right] \times \left[ H^2(X, Z_\kappa)^{s_\kappa} : L(2) \right]
\]
gives that \( L(2) \subset H^2(X, Z_\kappa)^{s_\kappa} \) has index \( 2^3 \), and the lattice \( L_0 = H^2(X, Z_\kappa)^{s_\kappa} \) has also discriminant 4. Our goal is now to characterise \( L_0 \) as an integral overlattice of \( L(2) \): as such, it is completely determined by the image \( \Lambda_0 \) of \( L_0 \) in \( L(2)^*/L(2) = \Lambda \).
Lemma 6.1. Let $\Lambda = \mathbb{Z}/4(4)^2 \oplus \mathbb{Z}/2(-2)^4$ be the discriminant group of $L(2)$, with its fractional bilinear form

$$b(x, y) = \frac{x_1y_1 + x_2y_2}{4} - \frac{x_3y_3 + x_4y_4 + x_5y_5 + x_6y_6}{2} \pmod{1}.$$ 

An index $2^3$ integral overlattice of $L(2)$ correspond to an isotropic subgroup of order 8 of $\Lambda$. Elements with integral norm form the subgroup $\Lambda_0$ of order 64

$$\{(a, b, c, d, e, f) \text{ such that } a \equiv b \pmod{2}, s + c + d + e + f \equiv 0 \pmod{2}\}$$

where $s$ is the common parity of $a$ and $b$.

Proof. The decomposition $L(2) = \mathbb{Z}^2(4) \oplus \mathbb{Z}^4(-2)$ induces the description of $\Lambda$ as a product of cyclic groups, and the description of isotropic vectors results from the fact that $(x^2 + y^2)$ is even if and only if $x \equiv y \pmod{2}$. \qed

The determination of $\Lambda_0$ uses a strong symmetry property of the lattice $L_0$: from this point, we identify $L$ with the (limit of the) lattice $H^1_{\text{GIT}}(\mathbb{Z}_-)$ at the special point $[\omega_0]$ representing a configuration with four triple points. Consider a uniformising disk $\mathbb{D}$ around $[\omega_0]$ in the space $\mathcal{M}_{\text{GIT}}$. This disk supports a global identification of the twisted Picard lattice of a general member with $L$. The $S_4$-action on the configuration, extends to an action on the disk (which at first order is equivalent to the linear action of $S_4$ on four coordinates). As in section 4.3, the group $S_4$ acts on the six lines of the configuration by its embedding $S_4 \subset (\mathbb{Z}/2)^6 S_4$: this action corresponds to the permutation of coordinates which we used to define Campedelli surfaces and their quotients (via the isomorphism $S_4 \simeq \text{GA}_2(\mathbb{F}_2)$ with the group of affine transformations of $\mathbb{A}^2(\mathbb{F}_2)$).

Proposition 6.2. The $S_4$-action on $\mathbb{D}$ lifts to the family of Enriques surfaces supported by the complement of the triple point locus, hence to a well-defined automorphism of $L$. \qed

The action of $S_4$ on the four triple points of the configuration gives the following property of the action on the cohomology:

Proposition 6.3. The action of $S_4$ on the lattice induces the natural action of $S_4$ on the $\mathbb{Z}^4(-1)$ summand (up to choices of signs).

We are then looking for the particular isotropic subgroup $\Lambda_0$ corresponding to the overlattice of Campedelli surfaces, which is also globally defined on $\mathbb{D}$: there exists a family of Campedelli surfaces, corresponding to the choices of a seventh line, whose base maps to $\mathbb{D}$ with connected fibres, and to which the $S_4$-action can be lifted, using the same action on coordinates. This family defines a uniform choice of $\Lambda_0$ in $\Lambda$ over $\mathbb{D}$.

Proposition 6.4. The subgroup $\Lambda_0$ is invariant under the $S_4$ action. \qed

Proposition 6.5. The projection of $\Lambda_0$ of the second summand $(\mathbb{Z}/2)^4$ consists of even vectors (i.e. the sum of their coordinates is even).

Proof. A $S_4$-invariant subspace of $(\mathbb{Z}/2)^4$ contains all vectors of a given weight (number of nonzero coefficients). Since the projection of $\Lambda_0$ may only contain at most 8 elements, there cannot be a weight one vector (basis vector), and there cannot be a weight three vector either (the sum of three different weight 3 vectors is a weight 1 vector). The subgroup of even vectors contains exactly eight elements. \qed
In particular, \( \Lambda_0 \) only contains elements \((a, b, c, d, e, f)\) such that \(c + d + e + f\) is even, then by the characterisation of isotropic vectors given above, \(a\) and \(b\) are both even. But since the pairing of \((2a, 2b, 1, 1, 0, 0)\) and \((2a', 2b', 0, 1, 1, 0)\) is \(1/2\) (thus not an integer), \(\Lambda_0\) cannot actually contain even vectors except for the characteristic element \((1, 1, 1)\): it is the only non zero element of \(\Lambda\) which is orthogonal to the 2-torsion subgroup of \(\Lambda\).

**Proposition 6.6.** The group \(\Lambda_0\) is \(\langle 2a, 2b, c + d + e + f \rangle\) (where the letters stand for the standard generators of \(\Lambda\)).

**Proof.** The discussion has shown that \(\Lambda_0\) was a subgroup of \(\langle 2a, 2b, c + d + e + f \rangle\), which has order 8. \(\square\)

This implies that \(L_0\) is absolute in the sense that it is invariant under the action of \(O(L)\), in other words any element of \(O(L)\) naturally induces an element of \(O(L_0)\). This is because \(\Lambda_0\) is generated by the 2-divisible elements of \(\Lambda\), and by the characteristic vector \(c + d + e + f\). This leads to the following description:

**Theorem 6.7.** The cohomology lattice of a linear system \(\mathbb{Z}_\kappa\) on a Campedelli surface splits as an invariant part of signature \((2, 4)\) and an anti-invariant part which is negative definite.

The invariant part is isomorphic to \(\mathbb{Z}^2 \oplus D_4(-1)\), which is \(L(2) \simeq \mathbb{Z}^2(4) \oplus \mathbb{Z}^4(-2)\) is embedded in the most natural way (as computed above).

**Corollary 6.8.** The automorphism group \(\text{Aut}(L)\) is naturally embedded in the orthogonal group of the Campedelli lattice, i.e. any automorphism of \(L\) extends to an isometry of the Campedelli lattice. Two marked Campedelli surfaces with the same associated Enriques surface have the same invariant Hodge structure \(H^2(\mathbb{Z}_\kappa)\).

In particular marked Campedelli surfaces with the same periods form a disconnected family.

### 7. Details on the boundary period map

The list of boundary components in both the GIT moduli space and the Baily-Borel compactification of the period space are listed in figure 2. The description of the strata is given by the lines which coincide, the lines of each pair being denoted by \(L_A, L_A'\) for example. The entries in the table result from the discussion below: this section is dedicated to the classification of primitive isotropic sublattices of \(L\) up to isometries.

| GIT component | Dimension | Isotropic \(\ell\) | \(\ell^* / \ell\) |
|---------------|-----------|-------------------|----------------|
| \(L_A = L_A'\) (type \(\chi\)) | 1         | Odd plane         | \(\mathbb{Z}^2(-2)\) |
| \(L_A = L_B\) | 1         | Even plane        | \(\mathbb{Z}^2(-1)\) |
| \(L_A = L_A', L_B = L_B'\) | 0         | Odd of type 2     | \(H \oplus \mathbb{Z}^2(-2)\) |
| \(L_A = L_C', L_B = L_C\) | 0         | Odd of type 1     | \(\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2) = \mathbb{Z}^{1,1}(2) \oplus \mathbb{Z}^2\) |
| \(L_A = L_C', L_B = L_C'\) | 0         | Even              | \(\mathbb{Z}^{1,3}\) |

**Figure 2.** List of boundary components in GIT and Baily-Borel compactifications

#### 7.1. Boundary components and local structure

We give here an example of how the one-dimensional boundary components can be explicitly identified.

**Proposition 7.1.** The boundary period map \(\mathcal{M}_\chi / (\mathbb{Z}/2 \wr S_3) \to X(2A_1)\) is birational.
We know from the study of type $\chi$ degenerations that $M_\chi$ is mapped to $X(2A_1)$. In the next section, we prove that $X(2A_1)$ is the quotient of the upper half-plane under the action of an index three subgroup of $PSL_2(\mathbb{Z})$. An elementary argument can be used to reprove this using almost only lattice arithmetic.

**Proof.** For generic $\chi$ configurations, $M_\chi$ is isomorphic to $P1$, the isomorphism being given by the cross-ratio $t$ of the ordered four lines crossing the double line. The singular locus is an elliptic curve whose ramification points can be chosen to be $(\pm 1, \pm \sqrt{t})$: its $j$-invariant is a degree 6 rational function of $t$.

The group $(\mathbb{Z}/2 \wr S_3)$ acts on the cross-ratio with an order 4 kernel. Hence the $j$-invariants $M_\chi/(\mathbb{Z}/2 \wr S_3) \to P1$ and $X(2A_1) \to P1$ both have degree 3, hence the period map has degree one. □

### 7.2. Isotropic sublattices of the cohomology of $D_6$ Enriques surfaces.

Denote by $L$ the lattice $\mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$. Remember that $L$ embeds as an index two sublattice of $\mathbb{Z}^2,4$.

#### 7.2.1. Isotropic vectors.

**Lemma 7.2.** If $\ell$ is a primitive isotropic vector in $L$, the set of $\langle \ell, x \rangle$ for $x \in L$ is either $\mathbb{Z}$, either $2\mathbb{Z}$.

**Proof.** Let $d$ be the gcd of all $\langle \ell, x \rangle$. If $\ell = (a_1, a_2; b_1, b_2, b_3, b_4)$ in the coordinates where $L = \mathbb{Z}^2(2) \oplus \mathbb{Z}^4(-1)$, then $d$ is the gcd of $2a_i$ and $b_i$.

Since the gcd of $a_i$ and $b_i$ is one, the conclusion follows. □

**Lemma 7.3.** If $\ell$ is an even vector, there exists a decomposition $L = \mathbb{Z}^{1,1}(2) \oplus \mathbb{Z}^{1,3}$ such that $\ell$ belongs to the first summand. Up to automorphism of $L$, there is only one even primitive isotropic vector.

**Lemma 7.4.** If $\ell$ is an odd vector, then $\ell$ belongs to a unimodular sublattice of $L$ of signature $(1,1)$. The corresponding possible decompositions of $L$ are:

- $\mathbb{Z}^{1,1} \oplus (\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2))$
- $H \oplus (\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2))$
- $\mathbb{Z}^{1,1} \oplus D_{1,3}$

giving two orbits (the first and the second decompositions occur for the same vectors). We will say the first case is type 1, the last case being type 2. The orbits are characterised by $\ell^\perp/\ell$, which is either $\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$ or $H \oplus \mathbb{Z}^2(-2)$.

**Proof.** Let $(\ell, m_1, \ldots, m_4)$ be a basis of $\ell^\perp$ and $\nu$ be such that $\langle \ell, \nu \rangle$ is minimal. In the basis, $(\ell, \nu, m_*)$ the quadratic form has matrix

$$
\begin{pmatrix}
0 & \varepsilon & 0 \\
\varepsilon & \nu^2 & N \\
0 & N & A
\end{pmatrix}
$$

where $A$ is the matrix of the quadratic form on $\bigoplus \mathbb{Z}m_*$ and $N_*$ is $\langle \nu, m_* \rangle$.

Its determinant is $4 = -\varepsilon^2 \det A$. If $\varepsilon = 2$, i.e. $\ell$ is even, $A$ is unimodular, hence $m_*$ span a unimodular lattice whose orthogonal complement contains $\ell$ and has discriminant 4: its matrix should have the form $(\frac{0}{2} \frac{2}{n})$. Since it has the same discriminant form as $L$, it must be
an index two sublattice of $H$ or $\mathbb{Z}^{1,1}$. Then it is $\mathbb{Z}^{1,1}(2)$ (since $H(2)$ has two isotropic elements for its discriminant bilinear form).

If $\varepsilon = 1$, then $\mathbb{Z}\ell \oplus \mathbb{Z}\nu$ is unimodular, and $(m_*)$ should be a index 2 lattice in $\mathbb{Z}^{1,3}$, hence the decompositions mentioned in the statement.

**Proposition 7.5.** The lattice $L$ contains three orbits of primitive isotropic vectors. The corresponding vectors are even, odd of type 1, odd of type 2. They are distinguished by the fact that $\ell^{-1}/\ell$ is isomorphic to $\mathbb{Z}^{1,3}$ (resp. $\mathbb{Z}^{1,1} \oplus \mathbb{Z}^2(-2)$, $D_{1,3} = H \oplus \mathbb{Z}^2(-2)$).

7.2.2. *Isotropic planes.*

**Proposition 7.6.** The lattice $L$ contains two orbits of maximal isotropic sublattices, distinguished by the fact that they contain (or not) even isotropic vectors.

More precisely, if $\lambda$ is an odd isotropic lattice of rank 2, then there exists a decomposition $L = \mathbb{Z}^{2,2} \oplus \mathbb{Z}^2(-2)$ such that $\lambda$ is a maximal isotropic lattice of the first summand. Primitive isotropic vectors in $\mathbb{Z}^{2,2}$ are either odd of type 1, e.g. $(1,0;1,0)$, or odd of type 2, e.g. $(1,1;1,1)$ (whose orthogonal complement is even).

If $\lambda$ is even (i.e. contains even isotropic vectors), it is maximal isotropic in a $\mathbb{Z}^2 \oplus \mathbb{Z}^2(-2)$ summand of $L$, and $\lambda^{\perp}/\lambda$ is isomorphic to $\mathbb{Z}^2(-1)$. It contains primitive even vectors and odd vectors of type 1.

**Proof.** Let $\lambda$ be a rank 2 isotropic sublattice of $L$. As before, choose a basis 

$$(\lambda_1, \lambda_2, \nu_1, \nu_2, \mu_1, \mu_2)$$

such that $\nu_*$ span $\lambda^{\perp}/\lambda$. Then the matrix of the quadratic form of $L$ can be written with $(2 \times 2)$-sized blocks:

$$\begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ A^T & C^T & D \end{pmatrix}$$

and the discriminant is $4 = \det B (\det A)^2$. Then either $\det B = 1$ or $\det A = 1$.

If $\det A = 1$, then $\lambda$ is embedded in the unimodular lattice spanned by $\lambda$ and $\mu$, which is indefinite hence isomorphic to $H^\perp$ or $\mathbb{Z}^{2,2}$. But the case $H^\perp$ is impossible, since it would imply that $L$ is even. In this case $L$ contains no primitive even isotropic vectors (it would imply that $\det A$ is even).

If $\det B = 1$, then $\lambda \oplus \nu$ has signature $(2,2)$ and the same discriminant form as $L$, and $B$ is definite of rank 2, and unimodular, hence isomorphic to $\mathbb{Z}^2$. Since $\lambda \oplus \nu$ cannot be isomorphic to $D_{2,2}$, which has the wrong bilinear form, it must be isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}^2(-2)$. □

The study of isotropic sublattices of $L$ allows to determine the structure of the boundary components of the Baily-Borel-Satake compactification of the period space.

**Proposition 7.7.** The boundary of $D_L$ consists of 3 distinguished points $p_{\text{even}}, q_1, q_2$ and two rational curves $C_{\text{even}}$ going through $p_{\text{even}}$ and $q_1$, $C_{\text{odd}}$ going through $q_1$ and $q_2$.

**Proof.** The identification of dimension 1 strata results from the study of type $\chi$ degenerations. The identification between strata of dimension zero results from the incidence relations with the one-dimensional strata. □

7.3. *Modular curves at the boundary of the period space.* In the following, $\gamma$ will denote the index three subgroup of $\text{GL}_2(\mathbb{Z})$ of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a \equiv d$ and $b \equiv c \pmod{2}$. 
7.3.1. Boundary curve for odd isotropic lattices.

**Proposition 7.8.** Let $L$ be a primitive isotropic plane in $L \simeq \mathbb{Z}^2 \oplus \mathbb{Z}^2(-2)$, such that $I^\perp/I \simeq \mathbb{Z}^2(-2)$. Then up to isometry, we can assume that $I \subset \mathbb{Z}^2$.

The image of the stabiliser of $I$ in $\text{Aut}(L)$ is an index three subgroup $\gamma(I)$ of $\text{GL}_2(I)$, which is identified with $\gamma$ for a suitable choice of basis.

Before explaining the proof, we choose a basis of $L$ which gives bases for $I$, $I^\perp/I$, and $L/I^\perp \simeq I^\vee$ (and assume that the intersection matrix of $I$ and $I^\vee$ is given by the identity matrix). Then an isometry of $L$ preserving $I$ has a matrix of the following form

$$
\begin{pmatrix}
H & A & B \\
0 & G & C \\
0 & 0 & H^\dagger
\end{pmatrix}
$$

where $H^\dagger$ is the inverse transpose of $H$, and $G \in O_2(\mathbb{Z})$. In order to define an isometry, the matrix has to satisfy additionally

- $A^\dagger H^\dagger - 2G^\dagger C = 0$;
- $(H^\dagger)^T H^\dagger + B^\dagger H^\dagger + H^{-1}B - 2C^\dagger C = I$.

where $I$ is the identity matrix.

**Proof.** Choose a basis $(e_1, e_2; f_1, f_2)$ of $\mathbb{Z}^2$ such that $I$ is generated by $(e_1 + f_1, e_2 + f_2)$. Then choosing any orthogonal basis $(g_1, g_2)$ for $\mathbb{Z}^2(-2)$ and $(e_1, e_2)$ for a basis of $I^\vee$ gives a basis $(e_1 + f_1, e_2 + f_2, g_1, g_2, e_1, e_2)$ of $L$ satisfying the axioms above.

With the previous notations,

$$
(H^\dagger)^T H^\dagger = I + (H^{-1}B) + (H^{-1}B)^T \quad \text{(mod 2)}
$$

thus is either the identity matrix or $\begin{pmatrix}1 & 1 \\ 1 & 1 \end{pmatrix}$. Hence $H$ must map to an element of the orthogonal group $O_2(\mathbb{F}_2)$ which has index three in $\text{GL}_2(\mathbb{F}_2)$ ($H$ is an element of $\gamma$).

Conversely, if $H \pmod{2}$ is in $O_2(\mathbb{F}_2)$, then write $(H^\dagger)^T (H^\dagger) = I - 2N$ where $N$ is an integral symmetric matrix. Then $A = 0$, $B = HN$ and $C = 0$, $G = I$ defines an isometry of $L$, whose restriction to $I$ is given by $H$. \qed

Let $\Gamma_1$ be the stabiliser of $I$ in $\text{Aut} L$ (notations are borrowed from [Loo03]) and let

$$
1 \to N_1 \to \Gamma_1 \to \gamma_1 \times O_2(\mathbb{Z}) \to 1
$$

be its Levi decomposition. Note also that $N_1$ is a central extension

$$
1 \to \mathbb{Z} \to N_1 \to \text{Hom}(I^\vee, I^\perp/I) \to 0
$$

where $\mathbb{Z}$ is the subgroup of matrices

$$
\begin{pmatrix}
1 & 0 & B \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

where $B$ is integral and skew-symmetric (which is central in $N_1$).

Let $D_L$ be the period domain associated to the lattice $L$, and $D_L^{BB}$ the topological space defined by Satake, such that the Baily-Borel compactification $\mathcal{X}_L^{BB}$ is $D_L^{BB}/\text{Aut}(L)$. Then the boundary curve $\mathcal{X}_L(2A_1)$ corresponding to odd isotropic planes is isomorphic to $\mathfrak{h}/\gamma$, where $\gamma$ is the index three subgroup of $\text{SL}_2(\mathbb{Z})$ as above.
Proposition 7.9. Let $\Gamma^I$ be the kernel of $\Gamma_1 \to \gamma_1$. Then in a neighbourhood of a generic point of $\mathcal{X}_I(2A_1)$, $\mathcal{X}_{BB}^I$ is locally isomorphic to $D_{L}^{BB}/\Gamma^I$.

Following Looijenga [Loo03], let $\pi_W$ be the projection $L \to L/W$, and consider the morphisms
\[ D_L \to \pi_1 D_L \to \pi_1 \perp D_L \]
and the filtration
\[ 0 \to \mathbb{Z} \to N_1 \to \Gamma^I \to \Gamma_1 \]
with quotients $\mathbb{Z}$, $\text{Hom}(I^I/I, I)$, $O_2(\mathbb{Z})$, $\gamma_1$.

Note that $D_L \to \pi_1 D_L$ is a bundle of upper half-planes, acted on by the subgroup acting as the identity on $L/I$, which is $\mathbb{Z} \subset N_1$, acting by translations.

Proposition 7.10. $D_L/\mathbb{Z} \to \pi_1 D_L$ is a bundle of punctured disks.

Now consider $\pi_1 D_L \to \pi_1 \perp D_L$. Note that $\pi_1 D_L$ is the set of images of positive planes inside $\text{Gr}(2, L/I \otimes \mathbb{R})$ or $\text{Gr}(2, I/I \mathbb{R})$, which is a principal bundle over $\pi_1 D_L$ under action of $\text{Hom}_\mathbb{R}(I, I^I/I)$ (which is the stabiliser of $I$ in $\text{GL}(I^I)$, which acts by isometries).

Proposition 7.11. $\pi_1 D_L/N_1 \to \pi_1 \perp D_L$ is a bundle of abelian surfaces.

Here $\pi_1 \perp D_L$ is the upper half plane associated to $L/I^I \simeq \Gamma^I$ which is acted on by $\gamma_1$, it maps to the boundary curve $\mathcal{X}_I(2A_1)$.

Proposition 7.12. There are natural projections
\[ D_L/N_1 \to \pi_1 D_L/N_1 \to \pi_1 \perp D_L \]
such that the second morphism is a bundle of abelian surfaces, and the first morphism is a bundle of punctured disks, and $O_2(\mathbb{Z})$ acts properly on the first two spaces.

The local structure of the Baily-Borel compactification is obtained by extending $D_L/\Gamma_1^I$ into a disk bundle, and contracting central fibres, then taking the quotient under $O_2(\mathbb{Z})$.

7.3.2. Boundary curve for even isotropic lattices.

Proposition 7.13. Let $I$ be a primitive isotropic plane in $L \simeq \mathbb{Z}^2 \oplus \mathbb{Z}^2(-2) \oplus \mathbb{Z}^2(-1)$, such that $I^I/I \simeq \mathbb{Z}^2(-1)$. Then up to isometry, we can assume that $I \subset \mathbb{Z}^2 \oplus \mathbb{Z}^2(-2)$.

The image of the stabiliser of $I$ in $\text{Aut}(L)$ has index three in $\text{SL}_2(I)$.

Note that a primitive isotropic plane in $\mathbb{Z}^2 \oplus \mathbb{Z}^2(-2)$ (in a canonical basis $(e_1, e_2; f_1, f_2)$) is given by $I = \langle e_1 + e_2 + f_1, e_1 - e_2 + f_2 \rangle$. Let $g_1, g_2$ be an orthonormal basis of the remaining summand $\mathbb{Z}^2(-1)$, and consider the basis of $L$ given by
\[ e_1 + e_2 + f_1, e_1 - e_2 + f_2, g_1, g_2, e_1, e_2 \]
The matrix of the quadratic form is then
\[ \begin{pmatrix} 0 & 0 & M \\ 0 & -1 & 0 \\ M & 0 & 1 \end{pmatrix} \]
where $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. A matrix
\[ g = \begin{pmatrix} H & A & B \\ 0 & K & C \\ 0 & 0 & L \end{pmatrix} \]
then defines an isometry if and only if
• $K \in O_2(\mathbb{Z})$;
• $H^\top ML = M$;
• $A^\top ML - K^\top C = 0$;
• $B^\top ML + L^\top MB + L^\top L - 2C^\top C = I$.

**Lemma 7.14.** An explicit computation shows that if $X = MYM/2$, and $X, Y$ are elements of $GL_2(\mathbb{Z})$, then $X$ and $Y$ belong to $\gamma$.

**Proof.** Write $Y = \left(\begin{array}{cc}a & b \\ c & d\end{array}\right)$. Then
\[
X = \frac{1}{2} \left(\begin{array}{cc}a + b + c + d & a - b + c - d \\ a + b - c - d & a - b - c + d\end{array}\right)
\]
which implies that $a + b + c + d$ is even, that is $Y \in \gamma$. It is then easy to check that $X \in \gamma$. □

**Proof of the proposition.** From the above equations we get $H^\top = ML^{-1}M/2$, hence $H$ is an element of $\gamma$.

Conversely, given any matrix $H$ mapping to $O_2(F_2)$, i.e. $H \in \gamma$, there exists an isometry $g = \left(\begin{array}{ccc}MH^\dagger M/2 & 0 & B \\ 0 & 1 & 0 \\ 0 & 0 & H\end{array}\right)$ where $B^\top MH + H^\top MB + H^\top H = I$. Since $I - H^\top H = 2N$ for some integral symmetric matrix $N$. Note that diagonal coefficients of $N$ are even, hence either $N$ or $N' = N + \left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$ is even. Setting $B = MH^\dagger N/2$ or $MH^\dagger N'/2$ gives $g$. □

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