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Diagrammatic bounds on the lace-expansion coefficients for oriented percolation

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In this note, we provide a complete proof of [2, Proposition 3.3]. For notational convenience, we use bold letters to denote vertices in \( \mathbb{Z}^{d+1} \), e.g., \( o \equiv (o, 0) \) and \( x \); if necessary, we denote the spatial and temporal components of a given vertex \( v \) by \( \sigma_v \) and \( \tau_v \) respectively: \( v = (\sigma_v, \tau_v) \). To identify the starting and terminal points, we write, e.g., \( \varphi_p(v; x) = \mathbb{P}_p(v \rightarrow x) \) and abbreviate it to \( \varphi_p(x) \) if \( v = o \); in particular, \( \varphi_p(v; x) = \varphi_p(x - v) \) if the model is translation-invariant. Let \( \text{piv}(v; x) \) denote the (random) set of pivotal bonds for \( \{v \rightarrow x\} \).

1 Bounds in terms of two-point functions

In this section, we prove bounds on \( \pi_p^{(N)}(x) \) and \( \Pi_p^{(N)}(x) \), for fixed \( x \), in terms of two-point functions. To prove these bounds, we do not have to assume translation-invariance.

Recall that the lace-expansion coefficients \( \pi_p^{(N)}(x) \) and \( \Pi_p^{(N)}(x) \) for \( N \geq 1 \) are defined in terms of the event

\[
\tilde{E}_{b_N}^{(N)}(x) = \{o \Rightarrow b_1\} \cap \bigcap_{i=1}^{N} E(b_i, b_{i+1}; \tilde{C}^{b_i}(b_{i-1})),
\]

where \( \tilde{b}_N = (b_1, \ldots, b_N) \) is an ordered set of bonds and

\[
E(b, x; C) = \{b \rightarrow x \in C\} \cap \{b' \in \text{piv}(b, x) \text{ satisfying } b' \in C\},
\]

\[
\tilde{C}^b(v) = \{x \in \mathbb{Z}^{d+1} : v \rightarrow x \text{ without using } b\}.
\]

Lemma 1.

\[
\pi_p^{(0)}(x) \equiv \mathbb{P}_p(o \Rightarrow x) \leq \delta_{x,o} + (q_p * \varphi_p)(x)^2,
\]

and, for \( N \geq 1 \),

\[
\pi_p^{(N)}(x) \equiv \sum_{b_N} \mathbb{P}_p(\tilde{E}_{b_N}^{(N)}(x)) \leq \sum_{u_1, \ldots, u_{N+1}} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p(v_1) \prod_{i=1}^{N} \mathbb{P}_p(u_i, v_i; u_{i+1}, v_{i+1}),
\]

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where

$$
\Xi_p(u, v; u', v') = \left( \xi^1_p(u, v; u', v') + \xi^2_p(u, v; u', v') \right) \varphi_p(u', v')/2^{d_{u', v'}}.
$$

(1.6)

Then, by (1.10) and the BK inequality and using the Markov property, we obtain

$$
\left\{ \begin{array}{ll}
\hat{\xi}^1_p(u, v; u', v') = (q_p \ast \varphi_p)(u; u') (q_p \ast \varphi_p)(v; v'), \\
\hat{\xi}^2_p(u, v; u', v') = (q_p \ast \varphi_p)(u; v') (q_p \ast \varphi_p)(v; u').
\end{array} \right.
$$

(1.7)

Proof. Since (1.4) is already proved in [2, (3.18)], it remains to show (1.5). By definition, we can easily see that

$$
E(b, x; \mathcal{C}^b(y)) \subset \{ y \to x \} \circ \{ b \to x \},
$$

(1.8)

where $E_1 \circ E_2$ is the event that $E_1$ and $E_2$ occur bond-disjointly (i.e., $E_1$ occurs on some bond set $B$ and $E_2$ occurs on $B^c$). Similarly,

$$
\{ o \to v \} \cap \{ o \to x \} \subset \bigcup_u \left\{ \{ o \to u \to v \} \circ \{ o \to v \} \circ \{ u \to x \} \right\},
$$

(1.9)

$$
E(b, v; \mathcal{C}^b(y)) \cap \{ b \to x \} \subset \bigcup_{u: u \to v > \tau^b_k} \left\{ \{ y \to u \to v \} \circ \{ b \to v \} \circ \{ u \to x \} \right\}
$$

\begin{equation}
\cup \left\{ \{ y \to v \} \circ \{ b \to u \to v \} \circ \{ u \to x \} \right\}.
\end{equation}

(1.10)

To prove (1.5), we use (1.8)–(1.10) and the BK inequality and pay attention to which event depends on which time interval. For example, by (1.8),

$$
\tilde{E}^{(N)}_{b_{N-1}}(x) \subset \tilde{E}^{(N-1)}_{b_{N-1}}(x) \cap \{ \tilde{b}_{N-1} \to x \} \circ \{ b_N \to x \}.
$$

(1.11)

Since $\tilde{E}^{(N-1)}_{b_{N-1}}(x)$ depends only on bonds before time $\tau^b_{2N}$, we can use the BK inequality to obtain

$$
\sum_{b_N} \mathbb{P}_p(\tilde{E}^{(N)}_{b_{N}}(x)) \leq \sum_{v_N} \mathbb{P}_p(\tilde{E}^{(N-1)}_{b_{N-1}}(v_N) \cap \{ \tilde{b}_{N-1} \to x \})(q_p \ast \varphi_p)(v_N; x).
$$

(1.12)

Then, by (1.10) and the BK inequality and using the Markov property, we obtain

$$
\sum_{b_{N-1}} \mathbb{P}_p(\tilde{E}^{(N-1)}_{b_{N-1}}(v_N) \cap \{ \tilde{b}_{N-1} \to x \})
$$

$$
\leq \sum_{v_{N-1}, u_N} \left( \mathbb{P}_p\left(\tilde{E}^{(N-2)}_{b_{N-2}}(v_{N-1}) \cap \{ \tilde{b}_{N-2} \to u_N \}\right)(q_p \ast \varphi_p)(v_{N-1}; v_N) \right.
$$

$$
+ \mathbb{P}_p\left(\tilde{E}^{(N-2)}_{b_{N-2}}(v_{N-1}) \cap \{ \tilde{b}_{N-2} \to v_N \}\right)(q_p \ast \varphi_p)(v_{N-1}; u_N) \bigg) \varphi_p(u_N; v_N) \varphi_p(u_N; x).
$$

(1.13)

Since $\tau_{u_N} \leq \tau_{v_N} < \tau_{2x}$ (due to $(q_p \ast \varphi_p)(v_N; x)$ in (1.12) and $\varphi_p(u_N; v_N)$ in (1.13)), we can replace the last term in (1.13) by $(q_p \ast \varphi_p)(u_N; x)$, using the trivial inequality

$$
\varphi_p(u; x) \leq (q_p \ast \varphi_p)(u; x) \quad (u \neq x).
$$

(1.14)
Summarizing these bounds, we have

\[
\sum_{b_{N-1}b_N} \mathbb{P}_p(\tilde{E}^{(N)}_{b_N}(x)) \leq \sum_{u_{N-1}u_N,v_N} \left( \mathbb{P}_p \left( \tilde{E}^{(N-2)}_{b_{N-2}}(v_{N-1}) \right) - \right. \\
+ \mathbb{P}_p \left( \tilde{E}^{(N-2)}_{b_{N-2}}(v_{N-1}) \right) (q_p \ast \varphi_p)(v_{N-1};v_N) \\
\times \varphi_p(u_N;v_N) \Xi_p(u_N,v_N;x,x). \tag{1.15}
\]

Using (1.13)–(1.14) again, but with different variables, we obtain

\[
\sum_{b_{N-2}b_{N-1}b_N} \mathbb{P}_p(\tilde{E}^{(N)}_{b_N}(x)) \leq \sum_{u_{N-1}u_N,v_N} \left( \mathbb{P}_p \left( \tilde{E}^{(N-3)}_{b_{N-3}}(v_{N-2}) \right) - \right. \\
+ \mathbb{P}_p \left( \tilde{E}^{(N-3)}_{b_{N-3}}(v_{N-2}) \right) (q_p \ast \varphi_p)(v_{N-2};v_N) \\
\times \varphi_p(u_{N-1};v_{N-1}) \Xi_p(u_{N-1},v_{N-1};u_N,v_N) \Xi_p(u_N,v_N;x,x). \tag{1.16}
\]

We repeat this procedure until we arrive at

\[
\sum_{\tilde{b}_N} \mathbb{P}_p(\tilde{E}^{(N)}_{\tilde{b}_N}(x)) \leq \sum_{u_2,u_N,v_1,v_N} \left( \mathbb{P}_p \left( \{ \emptyset \Rightarrow v_1 \} \cap \{ \emptyset \Rightarrow u_2 \} \right) (q_p \ast \varphi_p)(v_1,v_2) \\
+ \mathbb{P}_p \left( \{ \emptyset \Rightarrow v_1 \} \cap \{ \emptyset \Rightarrow v_2 \} \right) (q_p \ast \varphi_p)(v_1,u_2) \varphi_p(u_2,v_2) \\
\times \prod_{i=2}^{N-1} \Xi_p(u_i,v_i;u_{i+1},v_{i+1}) \Xi_p(u_N,v_N;x,x). \tag{1.17}
\]

By (1.9) and the BK inequality and using the Markov property and (1.14) under the restriction \( \tau_{v_1} < \tau_{u_2} \), we obtain (1.5).

**Lemma 2.** For \( N \geq 1 \),

\[
\Pi_p^{(N)}(x) \equiv \sum_{b_N} \sum_{j=1}^{N} \mathbb{P}_p(\tilde{E}^{(N)}_{b_N}(x) \cap \{ b = b_j \ \text{or} \ b \in \text{piv}(\tilde{b}_j, \tilde{b}_{j+1}) \} ) \\
\leq \sum_{u_1,\ldots,u_{N+1}} \varphi_p(u_1) \varphi_p(u_1;v_1) \varphi_p(v_1) \sum_{j=1}^{N} \Xi_p(u_j,v_j;u_{j+1},v_{j+1}) \\
\times \left( \Xi_p(u_j,v_j;u_{j+1},v_{j+1}) + \Theta_p(u_j,v_j;u_{j+1},v_{j+1}) + \Theta'_p(u_j,v_j;u_{j+1},v_{j+1}) \right), \tag{1.18}
\]
where the empty product \( \prod_{i \neq j} \Xi_p(u_i, v_i; u_{i+1}, v_{i+1}) \) for the case of \( N = 1 \) is 1 by convention, and

\[
\Theta_p(u, v; u', v') = \left( \theta_p(u, v; u', v') + \theta_p'(u, v; u', v') \right) \varphi_p(u'; v')/2^{\delta_{u', v'}},
\]

(1.19)

\[
\begin{align*}
\theta_p(u, v; u', v') &= (q_p * \varphi_p)(u; u') (q_p * \varphi_p)(v; v'), \\
\theta_p'(u, v; u', v') &= (q_p * \varphi_p)(u; u') (q_p * \varphi_p)(v; u'), \\
\Theta_p(u, v; u', v') &= (q_p * \varphi_p)(u; v') (q_p * \varphi_p)(v; u') (q_p * \varphi_p)(u'; v').
\end{align*}
\]

(1.20) (1.21)

**Proof.** Since

\[
\sum_{b_N, b} \mathbb{P}_p \left( \tilde{E}^{(N)}_{b_N}(x) \cap \{ b = b_j \ \text{or} \ b \in \text{piv}(\tilde{b}_j, \tilde{b}_{j+1}) \} \right) = \pi^{(N)}_p(x) + \sum_{b_N, b} \mathbb{P}_p \left( \tilde{E}^{(N)}_{b_N}(x) \cap \{ b \in \text{piv}(\tilde{b}_j, \tilde{b}_{j+1}) \} \right),
\]

(1.22)

it suffices to investigate the sum on the right-hand side. To do so, we use the following relations that are similar to (1.18) and (1.10):

\[
E(b', x; \bar{C}^N(y)) \cap \{ b \in \text{piv}(\bar{b}, x) \} \subset \{ y \to x \} \circ \{ b' \to b \to x \},
\]

(1.23)

\[
E(b', v; \bar{C}^N(y)) \cap \{ b \in \text{piv}(\bar{b}, v) \} \cap \{ \bar{v} \to x \} \subset \bigcup_{u: u > \bar{y}} \left\{ \{ y \to v \} \circ \{ b' \to b \to v \} \circ \{ u \to x \} \right\}
\]

\[
\bigcup \left\{ \{ y \to v \} \circ \{ b' \to b \to u \to v \} \circ \{ u \to x \} \right\}
\]

\[
\bigcup \left\{ \{ y \to v \} \circ \{ b' \to u \to b \to v \} \circ \{ u \to x \} \right\}. \tag{1.24}
\]

First we let \( j = N \). By (1.23) and using the BK inequality and the Markov property, we obtain

\[
\sum_{b_N, b} \mathbb{P}_p \left( \tilde{E}^{(N)}_{b_N}(x) \cap \{ b \in \text{piv}(b_N, x) \} \right) \leq \sum_{u_N} \mathbb{P}_p \left( \tilde{E}^{(N-1)}_{b_{N-1}}(v_N) \cap \{ b_{N-1} \to x \} \right) (q_p * \varphi_p * q_p * \varphi_p)(v_N; x),
\]

(1.25)

which is equivalent to (1.12), except for the last term \((q_p * \varphi_p * q_p * \varphi_p)(v_N; x)\). Therefore, by following the same line as in (1.13) – (1.17), we obtain

\[
\sum_{b_N, b} \mathbb{P}_p \left( \tilde{E}^{(N)}_{b_N}(x) \cap \{ b \in \text{piv}(b_N, x) \} \right) \leq \sum_{u_1, \ldots, u_N} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p(v_1) \prod_{i=1}^{N-1} \Xi_p(u_i, v_i; u_{i+1}, v_{i+1}) \\
\times \Theta_p(u_N, v_N; x, x). \tag{1.26}
\]

Applying (1.5) to \( \pi^{(N)}_p(x) \) in (1.22) and using \( \Theta'_p(u, v; u', v') = 0 \) for \( u' = v' \) (since \((\varphi_p * q_p * \varphi_p)(u'; v')\) in (1.21) is zero if \( u' = v' \)), we obtain the term for \( j = N \) in (1.18).

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Next we let \( j < N \). Following the same line as in (1.12)–(1.16), we obtain

\[
\sum_{b_j, b} \mathbb{P}_p \left( \tilde{E}_{b_j}^{(j)}(x) \cap \{ b \in \text{piv}(\tilde{b}_j, b_{j+1}) \} \right)
\leq \sum_{\substack{\tau_{j+1} > \tau_{u_j} \\
u_{j+1}, \ldots, \nu_N}} \left( \mathbb{P}_p \left( \tilde{E}_{b_{j+1}}^{(j)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow \nu_{j+1} \} \right) \right) 
\]

\[
= \sum_{\substack{\tau_{j+1} > \tau_{u_j} \\
u_{j+1}, \ldots, \nu_N}} \left( \mathbb{P}_p \left( \tilde{E}_{b_{j+1}}^{(j)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow \nu_{j+1} \} \right) \right) 
\]

\[
	imes \mathbb{P}_p \left( \tilde{E}_{b_j}^{(j)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow v_{j+1} \} \right) 
\]

Then, by (1.24) and using the BK inequality and the Markov property,

\[
\sum_{b_j, b} \mathbb{P}_p \left( \tilde{E}_{b_j}^{(j)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow u_{j+2} \} \right) 
\]

\[
\leq \sum_{\substack{\tau_{j+1} > \tau_{u_j} \\
u_{j+1}, \ldots, \nu_N}} \left( \mathbb{P}_p \left( \tilde{E}_{b_{j+1}}^{(j-1)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow u_{j+1} \} \right) \right) 
\]

\[
\times \mathbb{P}_p \left( \tilde{E}_{b_j}^{(j-1)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow v_{j+1} \} \right) 
\]

\[
\varphi_p(u_{j+1}; v_{j+1}) \varphi_p(u_{j+1}; v_{j+2}); 
\]

where the last term can be replaced by \((q_p * \varphi_p)(u_{j+1}; u_{j+2})\), because \(\tau_{u_{j+1}} \leq \tau_{v_{j+1}} < \tau_{u_{j+2}}\) (due to the restriction in (1.27) and the factors \(\varphi_p(u_{j+1}; v_{j+1})\) and \((q_p * \varphi_p)(u_{j+1}; v_{j+1})\) in (1.28)). Using (1.28) as well as that with \(u_{j+2}\) replaced by \(v_{j+2}\), we obtain

\[
(1.27) \quad \leq \sum_{\substack{\tau_{j+1} > \tau_{u_j} \\
u_{j+1}, \ldots, \nu_N}} \left( \mathbb{P}_p \left( \tilde{E}_{b_{j+1}}^{(j-1)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow u_{j+1} \} \right) \right) 
\]

\[
\times \mathbb{P}_p \left( \tilde{E}_{b_j}^{(j-1)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow v_{j+1} \} \right) 
\]

\[
\times \mathbb{P}_p \left( \tilde{E}_{b_j}^{(j-1)}(v_{j+1}) \cap \{ b \in \text{piv}(\tilde{b}_j, v_{j+1}) \} \cap \{ \tilde{b}_j \rightarrow u_{j+2} \} \right) 
\]

\[
|_{i=j+1}^{N-1} \mathbb{P}_p(u_i; v_i; u_{i+1}; v_{i+1}) \mathbb{P}_p(u_N; v_N; x, x). 
\]
Repeatedly using (1.13)–(1.14) with different variables, we finally arrive at

\[(1.29) \leq \sum_{u_j, \ldots, u_{N+1}} \sum_{\tilde{b}_{j-2}} \left( \mathbb{P}_p \left( \tilde{E}_{\tilde{b}_{j-2}}^{(j-2)} (v_{j-1}) \cap \{ \tilde{b}_{j-2} \rightarrow u_j \} \right) \left( q_p \ast \varphi_p \ast q_p \ast \varphi_p \right) (v_{j-1}; v_j) \right) \]

\[\quad + \mathbb{P}_p \left( \tilde{E}_{\tilde{b}_{j-2}}^{(j-2)} (v_{j-1}) \cap \{ \tilde{b}_{j-2} \rightarrow v_j \} \right) \left( q_p \ast \varphi_p \ast q_p \ast \varphi_p \right) (v_{j-1}; u_j) \varphi_p (u_j; v_j) \]

\[\times \left( \Theta_p (u_j, v_j; u_{j+1}, v_{j+1}) + \Theta'_p (u_j, v_j; u_{j+1}, v_{j+1}) \right) \prod_{i=j+1}^{N} \Xi_p (u_i, v_i; u_{i+1}, v_{i+1}) \]

\[\vdots \]

\[\leq \sum_{u_1, \ldots, u_{N+1}} \varphi_p (u_1) \varphi_p (v_1 - u_1) \prod_{i \neq j} \Xi_p (u_i, v_i; u_{i+1}, v_{i+1}) \]

\[\times \left( \Theta_p (u_j, v_j; u_{j+1}, v_{j+1}) + \Theta'_p (u_j, v_j; u_{j+1}, v_{j+1}) \right). \quad (1.30)\]

Combining this with the bound (1.5) on \( \pi_p^N (x) \) in (1.22), we obtain the term for \( j < N \) in (1.18).

The proof of (1.18) is completed by summing the above bounds over \( j = 1, \ldots, N \). 

2 Proof of [2, Proposition 3.3]

In this section, we prove [2, Proposition 3.3] using Lemmas 1-2 and assuming translation-invariance.

Let \( \varphi_p^{(m)}(v; x) = \varphi_p(v; x) m^{r_x - r_v} \). Recall that the weighted bubble \( W_p^{(m)}(k) \) and the triangles \( T_p^{(m)} \) and \( \tilde{T}_p \) are defined as

\[ W_p^{(m)}(k) = \sup_{x \in \mathbb{Z}^{d+1}} \sum_v (1 - \cos (k \cdot \sigma_v)) \times \begin{cases} (q_p \ast \varphi_p)(v) (m q_p \ast \varphi_p^{(m)})(x; v) & (m < 1), \\ (m q_p \ast \varphi_p^{(m)})(v) (q_p \ast \varphi_p)(x; v) & (m \geq 1), \end{cases} \]

\[ T_p^{(m)} = \sup_{x \in \mathbb{Z}^{d+1}} \sum_v (q_p \ast \varphi_p \ast q_p \ast \varphi_p)(v) (m q_p \ast \varphi_p^{(m)})(x; v), \]

\[ \tilde{T}_p = \sup_{x \in \mathbb{Z}^{d+1}} \sum_v (q_p \ast \varphi_p \ast q_p \ast \varphi_p)(v) (q_p \ast \varphi_p)(x; v), \]

and that the square \( S_p^{(m)} \) and the H-shaped diagram \( H_p \) are defined as (cf., [2, Figure 2])

\[ S_p^{(m)} = \sup_{x \in \mathbb{Z}^{d+1}} \sum_v (q_p \ast \varphi_p \ast \varphi_p \ast \varphi_p)(v) (m q_p \ast \varphi_p^{(m)})(x; v), \]

\[ H_p = \sup_{x, y \in \mathbb{Z}^{d+1}} \sum_{u, v, w} (q_p \ast \varphi_p)(u) (\varphi_p \ast q_p \ast \varphi_p)(u; v) (q_p \ast \varphi_p)(x; v) (q_p \ast \varphi_p)(u; w) (q_p \ast \varphi_p)(v; y + w). \]

[2 Proposition 3.3] is an immediate consequence of the following lemma:
Lemma 3. (i) For \( N \geq 0 \) and \( \ell = 0, 1, 2 \),
\[
\sum_{x \in \mathbb{Z}^d \times \mathbb{Z}_+} \tau^\ell_p \pi_p^{(N)}(x)m^{T_p} \leq (N + 1)^\ell (1 + 2T_p^{(m)}) (2T_p^{(m)})^{(N - 1)\nu_0} \times \begin{cases} T_p^{(m)} & (\ell \leq 1), \\ S_p^{(m)} & (\ell = 2), \end{cases} 
\tag{2.6}
\]
\[
\sum_{x \in \mathbb{Z}^d \times \mathbb{Z}_+} (1 - \cos(k \cdot \sigma_x)) \pi_p^{(N)}(x)m^{T_p} \leq 3(N + 1)^2 (1 + 2T_p^{(m)}) (2T_p^{(m)})^{(N - 1)\nu_0} W_p^{(m)}(k). 
\tag{2.7}
\]

(ii) For \( N \geq 1 \),
\[
\sum_{x \in \mathbb{Z}^d \times \mathbb{Z}_+} \Pi_p^{(N)}(x) \leq N(1 + 2T_p^{(1)}) (T_p^{(1)} + \tilde{T}_p^{(1)})^{N - 1} + H_p(2T_p^{(1)})^{(N - 2)\nu_0}. 
\tag{2.8}
\]

Proof of Lemma 3(i). First we prove \((2.6)-(2.7)\) for \( N = 0 \). By \((1.4)\), we readily obtain
\[
\sum_{x \in \mathbb{Z}^d \times \mathbb{Z}_+} \tau^\ell_p \pi_p^{(0)}(x)m^{T_p} \leq \sum_{x} \tau^\ell_p (q_p \ast \varphi_p)(x) (mq_p \ast \varphi_p^{(m)})(x) \leq \begin{cases} T_p^{(m)} & (\ell \leq 1), \\ S_p^{(m)} & (\ell = 2), \end{cases} 
\tag{2.9}
\]
where we have used (cf., \([3 \ (5.17)]\))
\[
\tau_x (q_p \ast \varphi_p)(x) = \sum_{t=1}^{t_x} (q_p \ast \varphi_p)(x) \leq \sum_{t=1}^{t_x} \sum_{v; \tau \equiv \ell} (q_p \ast \varphi_p)(v) \varphi_p(v; x) = (q_p \ast \varphi_p \ast \varphi_p)(x), 
\tag{2.10}
\]
\[
\tau_x^2 (q_p \ast \varphi_p)(x) \leq \tau_x (q_p \ast \varphi_p \ast \varphi_p)(x) \leq (q_p \ast \varphi_p \ast \varphi_p \ast \varphi_p)(x). 
\tag{2.11}
\]
We note that we have multiplied one of the two diagram lines (i.e., \((q_p \ast \varphi_p)(x)\)) by \( \tau^\ell_x \) and the other by \( m^{T_p} \). If we multiply either \((q_p \ast \varphi_p)(x)\) or \((mq_p \ast \varphi_p^{(m)})(x)\) (depending on whether \( m < 1 \) or \( m \geq 1 \)) by \( 1 - \cos(k \cdot \sigma_x) \) instead of \( \tau^\ell_x \), we obtain
\[
\sum_{x} (1 - \cos(k \cdot \sigma_x)) \pi_p^{(0)}(x)m^{T_p} \leq W_p^{(m)}(k), 
\tag{2.12}
\]
as required.

Next we prove \((2.6)\) for \( N \geq 1 \) and \( \ell = 0 \). We note that, as in the \( N = 0 \) case above, there are two “external” diagram lines from \( \mathbf{o} \) to \( \mathbf{x} \) in each of the \( 2^{N - 1} \) bounding diagrams in \((1.5)\). Each line looks like
\[
\varphi_p(y_1) \prod_{i=1}^{N-1} (q_p \ast \varphi_p)(y_i; y_{i+1}) (q_p \ast \varphi_p)(y_N; x), 
\tag{2.13}
\]
where each \( y_i \) is either \( u_i \) or \( v_i \) in \((1.5)\); denote the line with \( y_1 = v_1 \) by \( \omega_{v_1} \equiv (\mathbf{o}, v_1, \ldots, \mathbf{x}) \) and the other by \( \omega_{u_1} \equiv (\mathbf{o}, u_1, \ldots, \mathbf{x}) \). Multiplying \( \omega_{v_1} \) by \( m^{T_p} \) and using
\[
2 \sum_{x} \varphi_p(y; x; \mathbf{x})m^{T_p} 
\sum_{u, u} (\xi^1_p(o, y; u, v)m^{T_p} + \xi^x_p(o, y; u, v)m^{T_p}) \varphi_p(u; v) 
\leq 2T_p^{(m)} \quad (y \in \mathbb{Z}^{d+1}), 
\tag{2.14}
\]
\[
\sum_{u, v} (\xi^1_p(o, u; u, v)m^{T_p} + \xi^x_p(o, u; u, v)m^{T_p}) \varphi_p(u; v) 
\]
and
\[ \sum_{u,v} \varphi_p(u) \varphi_p(u;v) \varphi_p(v) m^{\tau u} \leq 1 + \sum_{v \neq o} (\varphi_p \ast \varphi_p)(v) (mq_p \ast \varphi_p^{(m)})(v) \leq 1 + 2T_p^{(m)}, \tag{2.15} \]
we obtain
\[ \sum_{x} \pi_p^{(N)}(x)m^{\tau x} \leq (1 + 2T_p^{(m)})(2T_p^{(m)})^{N-1}T_p^{(m)} \quad (N \geq 1), \tag{2.16} \]
as required.

Before proceeding the proof, we define (cf., (1.6))
\[ \tilde{\Xi}_p(u, v; \omega) = \varphi_p(u; v) \left( \xi_p^1(u, v, u', v') + \xi_p^x(u, v, u', v') \right)/2^{S_{u, v}}, \tag{2.17} \]
which satisfies similar bounds to (2.14), due to translation-invariance. We note that, by using (2.17), the bound in (1.5) can be reorganized as
\[ \sum_{u_1, \ldots, u_{N+1}, v_1, \ldots, v_{N+1}} \varphi_p(u_1) \varphi_p(v_1) \prod_{i=1}^{N} \tilde{\Xi}_p(u_i; v_i; u_{i+1}, v_{i+1}), \tag{2.18} \]
or, for \( j = 1, \ldots, N \), as
\[ \sum_{u_1, \ldots, u_{N+1}, v_1, \ldots, v_{N+1}} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p(v_1) \left( \prod_{i=1}^{j-1} \tilde{\Xi}_p(u_i; v_i; u_{i+1}, v_{i+1}) \right) \]
\[ \times \left( \xi_p^1(u_j, v_j; u_{j+1}, v_{j+1}) + \xi_p^x(u_j, v_j; u_{j+1}, v_{j+1}) \right) \left( \prod_{i=j+1}^{N} \tilde{\Xi}_p(u_i; v_i; u_{i+1}, v_{i+1}) \right). \tag{2.19} \]

Now we prove (2.6) for \( N \geq 1 \) and \( \ell = 1, 2 \). To do so, we multiply \( \omega_{v_1} \) by \( m^{\tau x} \) as before, and multiply \( \omega_{u_1} = (\omega^{(i)}_{u_1}, \ldots, \omega^{(N+1)}_{u_1}) \), where \( \omega^{(i)}_{u_1} = u_i, \omega^{(i)}_{u_1} \in \{u_i, v_i\} \) for \( i = 2, \ldots, N \) and \( \omega^{(N+1)}_{u_1} = \omega \), by \( \tau^x_{u_1} \), using the decomposition
\[ \tau \chi = \tau_{u_1} + \sum_{j=1}^{N} (\tau_{\omega^{(j)}_{u_1}} - \tau_{\omega^{(j)}_{u_1}}). \tag{2.20} \]
Consider, e.g., the bounding diagram with \( \omega^{(i)}_{u_1} = u_i \) for all \( i = 2, \ldots, N \); we denote this diagram by \( U(x) \) for convenience. Then, by (2.18) and (2.10)–(2.11), the contribution from \( \tau^x_{u_1} \) is bounded as
\[ \sum_{u_1, \ldots, u_{N+1}, v_1, \ldots, v_{N+1}} \tau^{\ell}_{u_1} \varphi_p(u_1) \varphi_p^{(m)}(v_1) \prod_{i=1}^{N} \varphi_p(u_i; v_i) \xi_p^1(u_i; v_i; u_{i+1}, v_{i+1}) m^{\tau_{u_{i+1}} - \tau_{v_i}} \leq (T_p^{(m)})^N \times \begin{cases} T_p^{(m)} & (\ell = 1), \\ \varphi_p^{(m)} & (\ell = 2), \end{cases} \tag{2.21} \]
and, by (2.19) and (2.20)–(2.21) and using (2.13), the contribution from each \((\tau_{u_{j+1}} - \tau_{u_j})\) is bounded as

\[
\sum_{u_1, \ldots, u_{N+1}; v_1, \ldots, v_{N+1}} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p^{(m)}(v_1) \left( \prod_{i=1}^{j-1} \xi_p^{(i)}(u_i; v_i; u_{i+1}, v_{i+1}) m^{\tau_{u_{i+1}} - \tau_{u_i}} \varphi_p(u_{i+1}; v_{i+1}) \right) \\
\times (\tau_{u_{j+1}} - \tau_{u_j})^{\ell} \xi_p^{(j)}(u_j; v_j; u_{j+1}, v_{j+1}) m^{\tau_{v_{j+1}} - \tau_{v_j}} \\
\times \left( \prod_{i=j+1}^{N} \varphi_p(u_i; v_i) \xi_p^{(i)}(u_i; v_i; u_{i+1}, v_{i+1}) m^{\tau_{v_{i+1}} - \tau_{v_i}} \right) \\
\leq (1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} \times \begin{cases} 
T_p^{(m)} & (\ell = 1), \\
S_p^{(m)} & (\ell = 2).
\end{cases}
\]  

(2.22)

Therefore, for \(\ell = 1\),

\[
\sum_{x} \tau_x U(x) m^{\tau_x} \leq N(1 + 2T_p^{(m)})(T_p^{(m)})^{N} + (T_p^{(m)})^{N+1} \leq (N + 1)(1 + 2T_p^{(m)})(T_p^{(m)})^{N}.
\]  

(2.23)

The other \(2N - 1\) bounding diagrams obey the same bound. This completes the proof of (2.6) for \(\ell \leq 1\).

The cross terms for \(\ell = 2\) can also be bounded similarly. For example, the contribution from \((\tau_{u_{j'+1}} - \tau_{u_{j'}})(\tau_{u_{j+1}} - \tau_{u_j})\) with \(j' < j\) is bounded, by using (2.19) (cf., (2.22)), by

\[
(T_p^{(m)})^{N-j'} \sum_{u_1, \ldots, u_{j'+1}; v_1, \ldots, v_{j'+1}} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p^{(m)}(v_1) \left( \prod_{i=1}^{j'-1} \xi_p^{(i)}(u_i; v_i; u_{i+1}, v_{i+1}) m^{\tau_{v_{i+1}} - \tau_{v_i}} \varphi_p(u_{i+1}; v_{i+1}) \right) \\
\times (\tau_{u_{j'+1}} - \tau_{u_{j'}}) \xi_p^{(j)}(u_{j'}, v_{j'}; u_{j'+1}, v_{j'+1}) m^{\tau_{v_{j'+1}} - \tau_{v_{j'}}} \varphi_p(u_{j'+1}; v_{j'+1}) \\
\leq (1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} S_p^{(m)}.
\]  

(2.24)

There are \(N(N-1) - 1\) more cross terms that obey the same bound. There are \(2N\) cross terms remaining, each of which is bounded by \((T_p^{(m)})^{N} S_p^{(m)}\). Therefore,

\[
\sum_{x} \tau_x^2 U(x) m^{\tau_x} \leq N^2(1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} S_p^{(m)} + (2N + 1)(T_p^{(m)})^{N} S_p^{(m)} \\
\leq (N + 1)^2(1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} S_p^{(m)}.
\]  

(2.25)

The other \(2N - 1\) bounding diagrams than \(U(x)\) obey the same bound. This completes the proof of (2.6).

Finally we prove (2.7) for \(N \geq 1\). If \(m < 1\), then we multiply \(\omega_{v_1}\) by \(m^{\tau_x}\) as before, and multiply \(\omega_{u_1}\) by \(1 - \cos(k \cdot \sigma_x)\) and use the decomposition (cf., [4 (4.50)])

\[
1 - \cos(k \cdot \sigma_x) \leq (2N + 3) \left( 1 - \cos(k \cdot \sigma_{u_1}) + \sum_{j=1}^{N} \left( 1 - \cos \left( k \cdot (\sigma_{\omega_{u_1}} - \sigma_{\omega_{u_1}}) \right) \right) \right).
\]  

(2.26)
For example, consider the bounding diagram $U(x)$ again, where $\omega^{(i)}_{u_i} = u_i$ for all $i = 2, \ldots, N$. Similarly to (2.21)–(2.22), we have

$$
\sum_{u_1, \ldots, u_N+1 \atop v_1, \ldots, v_N+1 \atop (u_{N+1} = v_{N+1})} (1 - \cos(k \cdot \sigma_{u_i})) \varphi_p(u_1) \varphi^{(m)}_p(v_1) \prod_{i=1}^{N} \varphi_p(u_i, v_i) \xi^l_p(u_i, v_i; u_{i+1}, v_{i+1}) m^{\tau_{e_i+1} - \tau_{e_i}} 
\leq W_p^{(m)}(k)(T_p^{(m)})^N, \quad (2.27)
$$

and

$$
\sum_{u_1, \ldots, u_N+1 \atop v_1, \ldots, v_N+1 \atop (u_{N+1} = v_{N+1})} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi^{(m)}_p(v_1) \left( \prod_{i=1}^{j-1} \xi^l_p(u_i, v_i; u_{i+1}, v_{i+1}) m^{\tau_{e_i+1} - \tau_{e_i}} \varphi_p(u_{i+1}; v_{i+1}) \right)
\times \left(1 - \cos(k \cdot (\sigma_{u_{j+1}} - \sigma_{u_j})) \right) \xi^l_p(u_j, v_j; u_{j+1}, v_{j+1}) m^{\tau_{e_{j+1}} - \tau_{e_j}}
\times \left( \prod_{i=j+1}^{N} \varphi_p(u_i; v_i) \xi^l_p(u_i, v_i; u_{i+1}, v_{i+1}) m^{\tau_{e_i+1} - \tau_{e_i}} \right)
\leq (1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} W_p^{(m)}(k). \quad (2.28)
$$

Therefore,

$$
\sum_x (1 - \cos(k \cdot \sigma_x)) U(x) m^{\tau_x} \leq (2N + 3) (T_p^{(m)})^N W_p^{(m)}(k) + N(1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} W_p^{(m)}(k)
\leq 3(N + 1)^2(1 + 2T_p^{(m)})(T_p^{(m)})^{N-1} W_p^{(m)}(k). \quad (2.29)
$$

The other $2^{N-1} - 1$ bounding diagrams than $U(x)$ obey the same bound.

If $m \geq 1$, then we multiply $\omega^{(i)}_{u_i}$ by $(1 - \cos(k \cdot \sigma_x)) m^{\tau_x}$ and use the decomposition (2.26). The rest is the same. This completes the proof of (2.7) for $N \geq 1$.

**Proof of Lemma 3(ii).** First we recall (1.18). Since we have the bound (2.16) on the contribution from $\Xi_p(u_j, v_j; u_{j+1}, v_{j+1})$, it thus remains to investigate the contributions from $\Theta_p(u_j, v_j; u_{j+1}, v_{j+1})$ and $\Theta'_p(u_j, v_j; u_{j+1}, v_{j+1})$. However, since (cf., (2.19))

$$
\sum_{u_1, \ldots, u_N+1 \atop v_1, \ldots, v_N+1 \atop (u_{N+1} = v_{N+1})} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p(v_1) \left( \prod_{i=1}^{j-1} \Xi_p(u_i, v_i; u_{i+1}, v_{i+1}) \right) \left( \prod_{i=j+1}^{N} \Xi_p(u_i, v_i; u_{i+1}, v_{i+1}) \right)
\times \left( \theta_p(u_j, v_j; u_{j+1}, v_{j+1}) + \theta^\times_p(u_j, v_j; u_{j+1}, v_{j+1}) \right) \leq (1 + 2T_p^{(1)})(2T_p^{(1)})^{N-1} \hat{T}_p, \quad (2.30)
$$

and, for $j < N$,

$$
\sum_{u_1, \ldots, u_N+1 \atop v_1, \ldots, v_N+1 \atop (u_{N+1} = v_{N+1})} \varphi_p(u_1) \varphi_p(u_1; v_1) \varphi_p(v_1) \left( \prod_{i=1}^{j-1} \Xi_p(u_i, v_i; u_{i+1}, v_{i+1}) \right) \left( \prod_{i=j+2}^{N} \Xi_p(u_i, v_i; u_{i+1}, v_{i+1}) \right)
\times \Theta'_p(u_j, v_j; u_{j+1}, v_{j+1}) \left( \xi^l_p(u_{j+1}, v_{j+1}; u_{j+2}, v_{j+2}) \right) \xi^l_p(u_{j+1}, v_{j+1}; u_{j+2}, v_{j+2}) \leq (1 + 2T_p^{(1)})(2T_p^{(1)})^{N-2} H_p, \quad (2.31)
$$
we obtain
\[
\sum_{x} \Pi_p^{(N)}(x) \leq (1 + 2T_p^{(1)}) \left( N(T_p^{(1)} + \tilde{T}_p)(2T_p^{(1)})^{N-1} + (N - 1)H_p(2T_p^{(1)})^{N-2} \right)
\]
\[
\leq N(1 + 2T_p^{(1)}) \left( (T_p^{(1)} + \tilde{T}_p)(2T_p^{(1)})^{N-1} + H_p(2T_p^{(1)})^{(N-2)^v_0} \right).
\]  (2.32)

This completes the proof of Lemma 3(ii).

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