On the back reaction of gravitational and particle emission and absorption from straight thick cosmic strings: A toy model

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The emission and absorption of gravitational waves and massless particles of an infinitely long straight cosmic string with finite thickness are studied. It is shown in a general term that the back reaction of the emission and absorption always makes the symmetry axis of the string singular. The singularity is a scalar singularity and cannot be removed.

I. INTRODUCTION

Topological defects that may have been formed in the early Universe have been studied extensively since the pioneering work of Kibble [1]. This is mainly due to their implications to the formation of the large-scale structure of the Universe and the formation of galaxies [2]. During the past twenty years, the scenario of the structure and galaxies formations from topological defects has experienced several revolutionary stages [2]. In particular, it was shown in 1997 [3] that the models were seriously in conflict with the observational data from COBE, while a year later [4] it was found that, after adding the cosmological constant into the models and by properly tuning some relevant parameters, consistent predictions can be achieved.

In this Letter, we shall not be concerned with the above mentioned problem, but stress another aspect that has been ignored in most numerical simulations [2], that is, the back reaction of gravitational waves and particles emitted by the defects. As shown in [5] by studying the full coupled Einstein-Maxwell-Higgs equations, cosmic strings always emit gravitational waves and particles due to their contractions. The radiation continues until the strings settle into their static configurations.

The study of the back reaction of the radiation is not trivial [2]. To have it attackable, in this Letter we shall consider a very ideal situation, that is, an infinitely long straight cosmic string with its energy-momentum tensor (EMT) being given by

\[ T_{\mu\nu}^{cs} = \sigma(t, r) \{ u_\mu u_\nu - z_\mu z_\nu \}, \]

where \( u^\lambda u_\lambda = -z^\lambda z_\lambda = +1 \). It may be argued that the model is too artificial to have anything to do with a realistic cosmic string. However, we believe that it does give some hint on how an important role the back reaction of gravitational waves and particles emitted by cosmic strings may play. Moreover, it has been shown by several authors that the EMT given by Eq. (1) for a straight cosmic string is valid at least to the first-order approximation [2]. Therefore, the results obtained here are expected to be valid to the same order, too.

The gravitational and particle radiation from an infinitely long thick cosmic string was studied recently by Wang and Santos, and they found that the back reaction of the radiation always turns the symmetry axis of the string singular. The singularity is a scalar singularity and cannot be removed.

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into a spacetime singularity \[1\]. In the above considerations, cosmic strings were assumed only emitting gravitational and particle radiation. However, since the spacetime outside of the string is curved, the outgoing radiation is always expected to be backscattered by the spacetime curvature, so the ingoing radiation in general also exists \[1\]. Yet, the useful cosmic strings usually are assumed to be formed at the temperature \( T \approx 10^{15} \text{GeV} \). With such a high temperature, it would be expected that the Universe was filled with various kinds of radiation, gravitational and particle-like, with very high velocities. Such, as a first-order approximation, we may consider them as null dust. Therefore, for a more realistic model both outgoing and ingoing radiations should be included. Then, a natural question is: What roles do the interaction between the ingoing radiation and cosmic string and the interaction between the ingoing and outgoing radiations play? Using the general results obtained by Letelier and Wang \[8\], we shall study these interactions. In particular, we shall show that the non-linear interaction between the ingoing and outgoing radiations makes the singularity on the symmetry axis stronger.

II. THE EMISSION AND ABSORPTION OF COSMIC STRINGS

To start with, let us consider the metric for the spacetimes with cylindrical symmetry \[2\]

\[
\tag{2} ds^2 = e^{2(K-U)}(dt^2 - dr^2) - e^{2U} dz^2 - e^{-2U} W^2 d\phi^2,
\]

where \( K, U, \) and \( W \) are functions of \( t \) and \( r \) only, and \( \{t, r, z, \phi\} \) are the usual cylindrical coordinates. The existence of an axis and the local-flatness condition at the axis require

\[
|\partial \phi|^2 = -g_{\phi\phi} = -e^{-2U} W^2 \to O(r^2), \tag{3}
\]

as \( r \to 0 \), where we have taken \( r = 0 \) as the location of the axis. Taking Eq.(3) as the source of the Einstein field equations \( G_{\mu\nu} = T^c_{\mu\nu} \), Shaver and Lake \[10\] showed that the metric coefficient \( K \) is a function of \( t \) only, i.e., \( K(t, r) = K^{cs}(t) \), and that

\[
\tag{4} \sigma = e^{2(U-K)} \frac{W_{tt} - W_{rr}}{W},
\]

where \( (\_)_x \equiv \partial(\_)/\partial x \), and

\[
\tag{5} u_\mu = (g_{tt})^{1/2} \delta^t_\mu, \quad z_\mu = (-g_{zz})^{1/2} \delta^z_\mu.
\]

If we assume that the energy density of the string is finite as \( r \to 0 \), it can be shown that this condition together with Eq.(3) require

\[
\tag{6} W(t, r) \sim w_1(t)r + w_2(t)r^3 + O(r^4),
\]

\[
e^{U(t,r)} \sim w_1(t) + a_1(t)r + O(r^2),
\]

as \( r \to 0 \). Eq.(6) and \( K = K^{cs}(t) \) are sufficient to ensure that the axis is free of spacetime singularities. To show this, let us first choose a null tetrad,

\[
L^\mu = \frac{e^{U-K}}{\sqrt{2}} (\delta^t_\mu - \delta^r_\mu), \quad N^\mu = \frac{e^{U-K}}{\sqrt{2}} (\delta^t_\mu + \delta^r_\mu),
\]

\[
M^\mu = \frac{1}{\sqrt{2}} (e^{-U} \delta_\mu^z + \frac{i}{W} \delta_\mu^\phi), \quad \bar{M}^\mu = \frac{1}{\sqrt{2}} (e^{-U} \delta_\mu^z - \frac{i}{W} \delta_\mu^\phi).
\]

Then, it can be shown that the non-vanishing components of the Ricci tensor \( R_{\mu\nu} \) and the Weyl tensor \( C_{\mu\nu\lambda\delta} \) are given by

\[1\] It should be noted that the backscattering argument for the existence of the ingoing radiation is not superficial. In fact, in black hole physics it is exactly this argument that leads to the existence of both ingoing and outgoing radiations, and the interaction of them is the crucial point that turns Cauchy horizons into spacetime singularities and whereby the predictability of the Einstein theory is preserved. For the details on this aspect, we refer readers to \[5\].
\[ \Phi_{00} = \frac{1}{2} S_{\mu\nu} L^\mu L^\nu = \frac{1}{4} \sigma, \quad \Phi_{02} = \frac{1}{2} S_{\mu\nu} M^\mu M^\nu = -\frac{1}{4} \sigma, \]
\[ \Psi_0 = -C_{\mu\nu\lambda\delta} L^\mu M^\nu L^\lambda M^\delta \]
\[ = \frac{1}{4} \sigma - \frac{1}{2} e^{2(U-K)} [U_{tt} + U_{rr} - 2U_{tr} + 3(U_t - U_r)^2 - 2K_t(U_t - U_r)] , \]
\[ \Psi_2 = -\frac{1}{2} C_{\mu\nu\lambda\delta} [L^\mu N^\nu L^\lambda N^\delta - L^\mu N^\nu M^\lambda M^\delta] \]
\[ = \frac{1}{12} \sigma + \frac{1}{2} e^{2(U-K)} [U_{tt} + U_{rr} + U_t^2 - U_r^2] , \]
\[ \Psi_4 = -C_{\mu\nu\lambda\delta} N^\mu M^\nu N^\lambda M^\delta \]
\[ = -\frac{1}{4} \sigma - \frac{1}{2} e^{2(U-K)} [U_{tt} + U_{rr} + 2U_{tr} + 3(U_t + U_r)^2 - 2K_t(U_t + U_r)] , \]

where \( \sigma \) is given by Eq. \((4)\), and \( S_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu} R/4 \). From Eqs. \((4)\) \textendash \((9)\) we can see that all the non-vanishing components, \( \Phi_{AB} \) and \( \Psi_A \), are regular at the axis. On the other hand, the fourteen scalars built from the Riemann tensor are the combinations of these quantities \([11]\). From there we can see that if these components are not singular at the axis, so do the fourteen scalars. This completes the proof to the above claim.

To consider the emission and absorption of gravitational waves and null dust fluids, from Eq.\((2.9)\) of \([8]\), we have

\[ T^\text{out}_{\mu\nu} = \rho^\text{out} N_\mu N_\nu, \quad T^\text{in}_{\mu\nu} = \rho^\text{in} L_\mu L_\nu, \]  

(10)

we find that these null dust fluids have only contribution to the metric coefficients \( g_{tt} \) and \( g_{rr} \), where the two null vectors \( L^\mu \) and \( N^\mu \) given by Eq.\((8)\) define, respectively, the ingoing and outgoing null geodesic congruence. In particular, if

\[ \{ K^{cs}(t, r), U^{cs}(t, r), W^{cs}(t, r) \} \]

(11)

is a solution of the Einstein field equations \( G^{cs}_{\mu\nu} = T^{cs}_{\mu\nu} \), then

\[ \{ K, U, W \} = \{ K^{cs} + a(u) + b(v), U^{cs}, W^{cs} \}, \]

(12)

is a solution of the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{cs}_{\mu\nu} + T^\text{out}_{\mu\nu} + T^\text{in}_{\mu\nu} , \]

(13)

where \( T^{cs}_{\mu\nu}, T^\text{out}_{\mu\nu} \) and \( T^\text{in}_{\mu\nu} \) are defined, respectively, by Eqs. \((8)\) \textendash \((11)\), with

\[ \rho^\text{in} = - \frac{a'(u)(W_{tt} + W_{rr})}{\sqrt{2}W}, \quad \rho^\text{out} = - \frac{b'(v)(W_{tt} - W_{rr})}{\sqrt{2}W}, \]

\[ \sigma = e^{2(U-K)r} + a(u) + b(v) \frac{(W_{tt} - W_{rr})}{W}, \]

(14)

where \( a(u) \) and \( b(v) \) are arbitrary functions, \( u \equiv (t + r)/\sqrt{2}, \ v \equiv (t - r)/\sqrt{2} \), and a prime denotes the ordinary derivative with respect to the indicated argument. Eq. \((13)\) shows that the solution of Eq. \((12)\) indeed represents a cosmic string accompanied by two null dust fluids, one is outgoing, and the other is ingoing. Note that it is always possible by properly choosing the two functions \( a(u) \) and \( b(v) \) to ensure the energy densities of the two null fluids to be non-negative. In the following, we always assume that this is the case. To show that the outgoing null dust fluid is emitted by the string, let us consider the conservation law \( T_{\mu\nu,\lambda} g^{\nu\lambda} = 0 \), which can be written in the form

\[ T^{cs}_{\mu\nu,\lambda} g^{\nu\lambda} = J^\text{out}_{\mu} + J^\text{in}_{\mu}, \]

(15)

where
\begin{align}
J^\text{out}_\mu &= - \left( \rho^\text{out} N_\mu N_\nu \right)_\lambda g^{\nu \lambda} = \frac{1}{2} \sigma b'(v) N_\mu, \\
J^\text{in}_\mu &= - \left( \rho^\text{in} L_\mu L_\nu \right)_\lambda g^{\nu \lambda} = \frac{1}{2} \sigma a'(u) L_\mu.
\end{align}

These expressions show clearly that the outgoing null dust fluid is indeed emitted by the string, while the ingoing one is absorbed by it.

To obtain a cosmic string with a finite thickness, we may follow [6] to cut the spacetime along a hypersurface, say, \( r = r_0(t) \), and then join the part with \( r \leq r_0(t) \) with a spacetime where only the two null fluids exist. However, since here we are mainly interested in the properties of the spacetime near the axis, we shall not consider such a junction, and simply assume, without loss of generality, that this is always possible.

It should be noted that the solution of Eq.(12) does not only represent the emission and absorption of null dust fluids of cosmic strings, but also the emission and absorption of gravitational waves. To see this, let us calculate the corresponding Weyl tensor, which is thought of as representing the pure gravitational field \([12,13]\). It can be shown that it has only three non-vanishing components, \( \Psi_0^A, (A = 0, 2, 4) \), the definitions of them are given by Eq.(4), and each of them is given by

\begin{align}
\Psi_0^A &= e^{a(u)+b(v)} \Psi_0^c + \Psi_0^n, \\
\Psi_2^A &= e^{a(u)+b(v)} \Psi_2^c, \\
\Psi_4^A &= e^{a(u)+b(v)} \Psi_4^c + \Psi_4^n, \tag{17}
\end{align}

where \( \Psi_0^c \)'s are given by Eq.(4), and

\begin{align}
\Psi_0^n &= - \frac{b'(v)e^\Omega}{2\sqrt{2}} \left[ 2(U_t - U_r) - \frac{W_t - W_r}{W} \right], \\
\Psi_4^n &= - \frac{a'(u)e^\Omega}{2\sqrt{2}} \left[ 2(U_t + U_r) - \frac{W_t + W_r}{W} \right]. \tag{18}
\end{align}

with \( \Omega \equiv 2(U - K) + a(u) + b(v) \). The \( \Psi_0^c \)'s represent the gravitational field of the cosmic string, \( \Psi_0^n \) represents the outgoing gravitational wave emitted by the string, and \( \Psi_4^n \) the ingoing gravitational wave absorbed by the string \([12,13]\).

### III. DISCUSSIONS

The back reaction of the gravitational and particle radiation to the spacetime can be studied by considering the Kretschmann scalar, which can be written in the form \([8]\)

\begin{align}
\mathcal{R} &= R^\mu_\nu\lambda\sigma \, P^\mu_\nu_\lambda\sigma \\
&= e^{2(a+b)} \mathcal{R}^c_s + 2e^\Omega \sigma \left( \rho^\text{out} + \rho^\text{in} \right) + 4e^{2\Omega} \rho^\text{out} \rho^\text{in} \\
&+ 16 \left[ \Psi_0^c \Psi_4^c + \Psi_4^c \Psi_0^n \right] + 16 \Psi_0^n \Psi_4^n, \tag{19}
\end{align}

where \( \mathcal{R}^c_s \) is the corresponding Kretschmann scalar for the pure cosmic string, given by

\begin{align}
\mathcal{R}^c_s &= 16 \left[ \Psi_0^c \Psi_4^c + 3(\Psi_2^c)^2 + (\Phi_{02}^c)^2 + 2(\Phi_{11}^c)^2 \\
&+ \Phi_{00}^c \Phi_{22}^c + 6(\Lambda^c)^2 \right], \tag{20}
\end{align}

which is regular at the axis \( r = 0 \), as can be seen from Eqs. (5), (6) and (8). The Kretschmann scalar contains five terms, each of which has the following physical interpretation \([8]\): The first term represents the contribution of the string, and, as shown above, it is always finite at the axis. The second term represents the interaction between the string and the two null dust fluids, while the third term represents the interaction between the two null dust fluids. The fourth term represents the gravitational interaction between the gravitational field of the string and the outgoing and ingoing gravitational waves, while the last term represents the interaction between the two outgoing and ingoing gravitational waves. From Eqs.(4), (14) and (18) we find that

\begin{align}
\rho^\text{out} \sim \frac{b'(t)}{r}, \\
\rho^\text{in} \sim \frac{a'(t)}{r}, \\
\Psi_0^n \sim \frac{b'(t)}{r} e^{\Omega(t)}, \\
\Psi_4^n \sim \frac{a'(t)}{r} e^{\Omega(t)}, \tag{21}
\end{align}
Combining Eqs. (19) and (21), we find that the interaction between the cosmic string and the two null dust fluids and the interaction between the gravitational field of the string and the two gravitational waves make the Kretschmann scalar diverge like $r^{-1}$ at the axis for generic choice of the functions $a(u)$ and $b(v)$, while the interaction between the two null dust fluids and the interaction between the outgoing and ingoing gravitational waves make the Kretschmann scalar diverge like $r^{-2}$.

Therefore, it is concluded that the back reaction of emission and absorption of gravitational and particle radiation of a cosmic string always makes the symmetry axis of the string singular. The singularity is a scalar one and cannot be removed. Our considerations do not refer to any particular solutions, and consequently are general.

It should be noted that the existence of the gravitational waves or null dust fluids in our model may be considered as incidental, in the sense that we can not switch off one of them while maintaining the other. However, it seems that the conclusion about the formation of the spacetime singularities on the axis will not be changed dramatically, even in the models where they are separable. This can be partially seen from Eqs. (19). Of course, it would be very interesting to construct models where they are separable and then study their effects independently. Moreover, instead of taking the results about the formation of spacetime singularities on the axis as something against the scenario of the structure and galaxies formations from topological defects, we would like to consider them as the indication of the important role that the back reaction of gravitational and particle radiation may play, and more realistic models, such as, strings with finite length and non-cylindrical symmetry, may avoid the formation of these singularities.

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[1] T.W.B. Kibble, J. Phys. A9, 1387 (1976).
[2] A. Vilenkin and E.P.S. Shellard, Cosmic Strings and other Topological Defects, (Cambridge University Press, Cambridge, 1994); M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. 58, 477 (1995).
[3] U.-L. Pen, U. Seljak, and N. Turok, Phys. Rev. Lett. 79, 1611 (1997); A. Albrecht, R.A. Battye, and J. Robinson, ibid., 79, 4736 (1997).
[4] R.A. Battye, J. Robinson, and A. Albrecht, Phys. Rev. Lett. 80, 4847 (1998); P.P. Avelino, E.P.S. Shellard, J.H.P. Wu, and B. Allen, ibid., 81, 2008 (1998).
[5] J.A. Stein-Schabes and A.B. Burd, Phys. Rev. D37, 1401 (1988); V. Moncrief, ibid., D39, 429 (1989); R. Gregory, ibid., D39, 2108 (1989).
[6] A.Z. Wang and N.O. Santos, Class. Quantum Grav. 13, 715 (1996).
[7] E. Poisson and W. Israel, Phys. Rev. D41, 1796 (1990); W. Isreal, Inter. J. Mod. Phys. D3, 71 (1994).
[8] P.S. Letelier and A.Z. Wang, Phys. Rev. D49, 5105 (1994).
[9] Kip S. Thorne, Phys. Rev. B138, 251 (1965).
[10] E. Shaver and K. Lake, Phys. Rev. D40, 3287 (1989); 41, 3865(E) (1990).
[11] S.J. Campbell and J. Wainwright, Gen. Relativ. Grav. 12, 987 (1977).
[12] P. Szekeres, J. Math. Phys. 6, 1387 (1965); ibid. 7, 751 (1966).
[13] A.Z. Wang, Phys. Rev. D44, 1120 (1991).