GENERIC SINGULARITIES OF HOLOMORPHIC FOLIATIONS

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Singular behaviour of holomorphic dynamics has been of great interest to researchers of related areas. Baum has analyzed the behaviour of leaves of a holomorphic foliation around a singularity where the singular set has one less dimension than the leaf dimension, [2]. He then used this with Bott to describe their by now well known residue classes, under the same requirement about the dimension of the singular set, [3], see also [18]. Later Cenkl showed that Baum-Bott residue classes can be described also for foliations where the singular set has smaller dimension, [5]. This brought up the question about the structure of such singularities. In this article we study the structure of the singularity of a holomorphic foliation for which the dimension of the singular set is not necessarily one less than the leaf rank. We show that the structure of the leaves around a point on the singular set is determined by the vector space rank of the sheaf defining the foliation at this point. The generic singularity is then the one for which this rank is zero. We also show that in general the singularity of a holomorphic foliation is locally the pullback of a generic singularity.

In this article we also give examples of holomorphic foliations of rank \( k \), on an \( n \) dimensional manifold, with a singular set of dimension \( r \), where \( r \) is any natural number less than \( k \), subject only to the condition that \( r \geq n - k - 1 \) which is exactly a condition stated by Malgrange in [9]. Such examples are missing in the literature. The examples given here fill in this gap and also illustrate the claims of our main theorem.

We have utilized Baum’s techniques of [4] where, with the above notation, he has studied the case \( r = k - 1 \) which is also the case studied by Vishik in [18].

Many authors such as Verjovsky, Alexander and Thom has analyzed foliations of rank \( k \) with isolated singularities, which justifies our results, [1], [8], [17]. See also [7] where foliations with curves having isolated singularities are examined.

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version of this article. His suggestions improved the article considerably.

The set up and the problem

Throughout the article $M$ will denote a complex manifold of dimension $n$. $TM$ will denote its holomorphic tangent bundle and $\tau$ the sheaf of holomorphic sections of $TM$. We will use $\xi$ to denote an integrable full coherent subsheaf of $\tau$ of fixed rank $k$. (For the definition of “full” see [3].) $Z$ will denote a fixed connected component of the singular set $S$ of $\xi$. We will assume that $Z$ is irreducible of fixed dimension $r$. $\vartheta$ will denote the structure sheaf of $M$ and $\mu$ will be the maximal ideal of $\vartheta$.

For each point $x \in M$, $T_x(\xi)$ will denote the vector subspace of $T_xM$ defined as $\xi_x/\xi_x \otimes \mu_x$. For an equivalent definition of $T_x(\xi)$ using vector fields see either [3] or [2].

There is a natural filtration of $Z$ depending on the behaviour of $T_x(\xi)$ on $Z$;

$$Z = Z^1 \supset Z^2 \supset \cdots \supset Z^k$$

where

$$Z^i = \{ x \in Z \mid \dim T_x(\xi) \leq k - i \}.$$

Each $Z^i$ is a closed subvariety.

To formulate the results it will be convenient to introduce the following formalism: Let $\text{Fol}(n,k,r,s)$ denote the set of isomorphism classes of singular holomorphic foliations around the origin in $\mathbb{C}^n$ defined as follows: An element in $\text{Fol}(n,k,r,s)$ is represented by an equivalence class of pairs $< F, U >$ where

1) $U$ is an open neighbourhood of the origin in $\mathbb{C}^n$ and $F$ is an integrable full coherent subsheaf of the tangent sheaf of $\mathbb{C}^n$ on $U$.
2) $\text{rank} F = k$.
3) The singular set $Z$ of $F$ is connected, smooth, contains the origin and is of dimension $r$.
4) The dimension of $T_x(\xi)$ is $s$ for all $x$ in $Z$. i.e. we assume that $Z = Z^{k-s} - Z^{k-s+1}$. This corresponds to locally choosing $U$ small enough to exclude $Z^{k-s+1}$.

(For alternate definitions see [1], [5] and [4]).

Two elements $< F_1, U_1 >$ and $< F_2, U_2 >$ represent the same element if there is an invertible holomorphic map $f$ of $W \subset U_1 \cap U_2$ onto $W$ with $f^* F_1 = F_2$.

We are interested in the cases when $n > k > r \geq s \geq 0$.

With any element $\xi = < F, U >$ of $\text{Fol}(n,k,r,s)$ we use the notation $\xi \times \mathbb{C}^m$ to denote that element of $\text{Fol}(n+m, k+m, r+m, s+m)$ which represents the
foliation on $\mathbb{C}^n \times \mathbb{C}^m$ given by the geometric pull back of the leaves of $\xi$ on $\mathbb{C}^n$. The notation $\text{Fol}(n, k, r, s) \times \mathbb{C}^m$ then denotes the set of all $\xi \times \mathbb{C}^m$ where $\xi \in \text{Fol}(n, k, r, s)$.

We will need the following map

$$i_m : \text{Fol}(n, k, r, 0) \to \text{Fol}(n + m, k, r + m, 0)$$

where if $\alpha$ is in $\text{Fol}(n, k, r, 0)$ and singularly foliates $\mathbb{C}^n$ with leaves $\{L\}$ then the leaves of $i_m(\alpha)$ on $\mathbb{C}^n \times \mathbb{C}^m$ are all of the form $L \times \{t\}$ for $t \in \mathbb{C}^m$, i.e. each slice $\mathbb{C}^n \times \{t\}$ is foliated as prescribed by $\alpha$. We denote this phenomena by

$$i_m(\alpha) = \bigcup_{t \in \mathbb{C}^m} \alpha \times \{t\}.$$

Now we can state our main result:

**Main Theorem:** $\text{Fol}(n, k, r, s) = \text{Fol}(n-s, k-s, r-s, 0) \times \mathbb{C}^s$.

i.e. for every $\xi$ in $\text{Fol}(n, k, r, s)$ there is an $\eta_\xi$ in $\text{Fol}(n-s, k-s, r-s, 0)$ such that $\xi = \eta_\xi \times \mathbb{C}^s$.

Furthermore we will call $\xi$ in $\text{Fol}(n, k, r, s)$ “split” if the corresponding $\eta_\xi$, whose existence is established by the above theorem, is of the form $i_{r-s}(\alpha)$ for some $\alpha$ in $\text{Fol}(n-r, k-s, 0, 0)$. A necessary condition for $\xi$ to be split is $n-r > k-s$. A split foliation locally looks like a collection of foliations with isolated singularities; see the examples’ section.

**The proof of the main theorem**

Let $p$ be a smooth point of $Z$ and choose a local coordinate system $(z, U)$ centered at $p$ such that

$$Z \cap U = \{q \in U | z_{r+1}(q) = \cdots z_n(q) = 0 \}.$$ 

Baum’s tangency lemma, [2], forces $T_z(\xi)$ to stay in $T_x(Z)$ for all $x \in Z$. Since the rank of $T_z(\xi)$ is fixed throughout $Z$ and $\xi$ is coherent, we can choose vector fields $V_1, \ldots, V_s$ on $U$ such that

1) $V_i(x), \ldots, V_s(x)$ are in $T_x(\xi)$ for all $x \in U$.
2) $V_i(x) = \frac{\partial}{\partial z_i}|x$ for $x \in Z$, $i = 1, 2, \ldots, s$.
3) $V_1(x), \ldots, V_s(x), \frac{\partial}{\partial z_{r+1}}|x, \ldots, \frac{\partial}{\partial z_n}|x$ are linearly independent for all $x \in U$. 

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This suggests that we write $Z$ as $AZ \times BZ$ where

$$AZ = \{ x \in Z | z_{s+1}(x) = \cdots = z_n(x) = 0 \}$$

$$BZ = \{ x \in Z | z_1(x) = \cdots = z_s(x) = 0 \text{ and } z_{r+1}(x) = \cdots = z_n(x) = 0 \}.$$ 

Note that $\dim AZ = s$ and $\dim BZ = r - s$. We also define a disc $D$ in $U$ as

$$D = \{ x \in U | z_1(x) = \cdots = z_s(x) = 0 \}.$$ 

Then the dimension of $D$ is $n - s$ and $U$ can be realized as $AZ \times D$ in the usual sense.

The vector fields $V_1, \ldots, V_s$ generate a locally free subsheaf $\zeta$ of $\xi$ with rank $s$. At each point of $U$ the complement of $\zeta$ in $\xi$ is a $k - s$ dimensional vector space, except on $Z$ where the complement is zero. Thus $\xi$ can be realized as $\zeta \oplus \eta$, where $\eta$ is a coherent subsheaf of $\xi$ with rank $k - s$ on $U - Z$ and $0$ on $Z$. Note that since $\xi$ is coherent it is generated by $V_1, \ldots, V_s$ and some other sections $v_1, \ldots, v_d$ around $p$, and $\eta$ is then the sheaf generated by $v_1, \ldots, v_d$.

For all $x$ in $D$, the vector space corresponding to $\eta_x$, (the vector space generated by germs of $\eta_x$ evaluated at $x$), lies in the space generated by $\frac{\partial}{\partial z_{s+1}}|_x, \ldots, \frac{\partial}{\partial z_n}|_x$ because of the choice of $V_i$’s. Hence $\eta|D$ lies in the tangent sheaf of $D$, and consequently so does $[\eta|D, \eta|D]$ where $[,]$ denotes the lie bracket. Also $[\eta, \eta] \subset \xi$, and since $\xi|D$ and the tangent sheaf of $D$ have only $\eta|D$ in common we conclude that $\eta|D$ is integrally closed.

Hence $\eta| = \eta|D$ foliates $D$ with leaves of rank $k - s$ and with a singularity along $BZ$. Since $BZ$ is in $Z$ and the rank of $\eta$ is zero on $Z$, the rank of $\eta|_Z$ is zero along its singularity.

Thus $\eta|_Z$ is in $Fol(n - s, k - s, r - s, 0)$ around $p$.

Now we use Baum’s trick to carry the foliation of $D$ over $U$ along $AZ$. For this we define a vector field $V_i$ for each point $t = (t_1, \ldots, t_s, 0, \ldots, 0)$ of $AZ$;

$$V_i(x) = t_1V_1(x) + \cdots + t_sV_s(x), \quad x \in U.$$ 

We will move $D$ with the flow generated by $V_i$;

$$D_t := \exp(V_t(D)).$$ 

We assume that $U$ is shrunk so that all the $D_t$’s will lie in $U$, and will be disjoint for different $t$. Since $V_t(p)$ lies in the tangent space of $AZ$, $\exp(V_t(p))$ will also lie
in AZ. We expect $U$ to be small so that $exp(V_t(D))$ will not intersect AZ in any point other than $exp(V_t(p))$.

We will now find the intersection point of $exp(V_t(D))$ with AZ. Recall that $exp_{V_t(p)} : C \to U$ is the unique map $y \mapsto (e_1(y), ..., e_n(y))$ which sends 0 to $p$ and whose differential sends $\frac{dy}{dz}$ of $T_0C$ to $V_t(p)$ of $T_pU$. In particular

$$(d \exp_{V_t(p)})(\frac{dy}{dz})(z_i) = V_t(p)(z_i) = (t_1 \frac{\partial}{\partial z_1}|_p + \cdots + t_k \frac{\partial}{\partial z_k}|_p)(z_i) = \begin{cases} z_i & \text{if } 0 < i \leq s, \\ 0 & \text{if } s < i \leq n. \end{cases}$$

On the other hand

$$(d \exp_{V_t(p)})(\frac{dy}{dz})(z_i) = d \frac{dy}{dz} (z_i \circ (\exp_{V_t(p)}(y))) = d \frac{dy}{dz} (e_1(y), ..., e_n(y)) = d \frac{dy}{dz} e_i(y)$$

Hence, noting that 0 goes to $p$, we have

$$\exp_{V_t(p)}(y) = (yt_1, ..., yt_s, 0, ..., 0).$$

And finally,

$$\exp(V_t(p)) = \exp_{V_t(p)}(1) = (t_1, ..., t_s, 0, ..., 0) = t,$$

thus showing that

$$D_t \cap AZ = \{t\}.$$ 

Since $V_t(x)$ is in $T_x(\xi)$ for all $x$ in $U$, the flow passing through $x$, the integral curve of $V_t$, remains in the same leaf of $\xi$ as $x$.

For every $x \in U$ we can find a unique $t = t(x)$ in AZ and a unique $q = q(x)$ in $D$, choosing $U$ smaller if necessary, such that

$$x = \exp(V_t(q)).$$
Since \( t = (t_1(x), ..., t_s(x), 0, ..., 0) \) and \( q = (0, ..., 0, q_{s+1}(x), ..., q_n(x)) \) we can choose to denote \( x \) with the coordinates \((t_1(x), ..., t_s(x), q_{s+1}(x), ..., q_n(x))\). This makes \( U \) holomorphically equivalent to \( AZ \times D \) around \( p \) with respect to these new coordinates, which we agree to denote by \( Z \odot D \).

Let \( pr_2 : U \to D \) be the map which sees \( U \) as \( AZ \odot D \) and projects to the second component; \[ x \mapsto (q_{s+1}(x), ..., q_n(x)). \]

If \( D \) is foliated by \( \eta_\xi \) with leaves \( \{L\} \), as mentioned before, then any leaf \( L \) is \( k - s \) dimensional and is carried along the flow lines of \( V_t \) with the parameter \( t \) lying in an \( s \) dimensional space \( AZ \). Hence the images of \( L \) along these flow lines, all remaining in the same leaf of \( \xi \) as \( L \), fill out a \( k \) dimensional space, which must in turn be a whole leaf of \( \xi \) in \( U \). Thus \( U = AZ \odot D \) is foliated by leaves of the form \( L \times AZ \simeq L \times C^s \) with a singularity along \( BZ \), which means finally that \( \xi \) is of the form \( \eta_\xi \times C^s \) where \( \eta_\xi \) is in \( \text{Fol}(n-s, k-s, r-s, 0) \).

One interesting case is when the foliation \( \eta_\xi \) can further be decomposed; for this let \( K \) be the disc \[ K = \{x \in U | z_1(x) = \cdots = z_r(x) = 0 \}, \quad \dim K = n - r. \]

\( K \) is a disc in \( D \) perpendicular to \( BZ \) at \( p \). If \( K \) intersects the leaves of \( \xi \) transversally, then the leaves of \( \xi \) singularly foliate \( K \) with \( k - s \) dimensional leaves and with an isolated singularity. This foliation can be moved along \( AZ \) via \( V_t \) as before to foliate \( K \times AZ \) with a rank \( k \) foliation and with a singularity along \( AZ \). We then have to repeat this success at all the other points of \( BZ \), besides \( p \), (see example 3).

**Examples**

1) The lines through 0 foliate \( C^n \setminus 0 \) thus showing that \( \text{Fol}(n, 1, 0, 0) \) is not empty. Next consider the following foliation by divisors: for any \( \lambda \in C \) let \[ L_\lambda = \{(X_1, ..., X_n) \in C^n | X_1^2 + \cdots + X_n^2 = \lambda \}. \]

Each \( L_\lambda \) is smooth except when \( \lambda \) is zero, in which case \( L_\lambda \) has an isolated singularity at the origin. The collection \( \{L_\lambda\} \) foliates \( C^n \) with an isolated singularity at the origin. Thus \( \text{Fol}(n, n-1, 0, 0) \neq \emptyset \).

2) For each \( \lambda = (\lambda_1, \lambda_2) \in C^2 \) define \[ L_\lambda = \{(X_1, ..., X_n) \in C^n | X_1^2 + \cdots + X_{n-1}^2 = \lambda_1, \ X_n = \lambda_2 \}. \]
Each $L_\lambda$ is smooth except when $\lambda_1 = 0$, in which case $L_{(0, \lambda_2)}$ has an isolated singularity at $(0, \ldots, 0, \lambda_2)$. The set $\{L_\lambda\}$ foliates $\mathbb{C}^n$ with a singularity along the last coordinate. Thus $Fol(n, n - 2, 1, 0)$ is not empty. Since $s = 0$ we can check if the foliation is split. In this example $AZ$ is just the origin because each $L_{(\lambda_1, \lambda_2)}$ lies in the hyperplane $X_n = \lambda_2$, and the tangent vectors of the leaf do not have an $X_n$ component. The leaves hence intersect the hyperplane transversally. This means that each such hyperplane is foliated as in example 1. Let the foliation of the hyperplane be $\eta$, which belongs to $Fol(n - 1, n - 2, 0, 0)$. Then the foliation of this example is of the form $i_1(\alpha) \times \mathbb{C}^1$ and is therefore split.

3) We now give a less obvious example of a split foliation. We will foliate $\mathbb{C}^n$ with rank $k$ leaves and with a singularity along an $r$ dimensional subvariety. For this type of construction to work we must necessarily have $r \geq n - k - 1$, which turns out to be a bound given by Malgrange in \footnote{1}. For each $\lambda = (\lambda_1, \ldots, \lambda_{n-k}) \in \mathbb{C}^{n-k}$ we define the leaf $L_\lambda$ as

$$L_\lambda = \{(X_1, \ldots, X_n) \in \mathbb{C}^n | X_1^2 + \cdots + X_{n-r}^2 = \lambda_1, X_{n-r+1} = \lambda_2, \ldots, X_{2n-k-r-1} = \lambda_{n-k} \}.$$ 

Dimension of each leaf is $k$. When $\lambda_1 \neq 0$ then $L_\lambda$ is smooth and when $\lambda_1 = 0$ then $L_\lambda$ is singular along a $k + r + 1 - n$ dimensional subvariety. The set of singular leaves is $n - k - 1$ dimensional. Hence the dimension of the singularity is $r$. Let us call this foliation $\xi$. By examining the Jacobian of the system giving the foliation we can find that $s = k + r + 1 - n$. Hence

$$\xi \in Fol(n, k, r, k + r + 1 - n).$$

We note that

$$Z = \{(0, \ldots, 0, X_{n-r+1}, \ldots, X_n) \in \mathbb{C}^n \}, \quad \dim Z = r.$$

$$AZ = \{(0, \ldots, 0, X_{2n-k-r}, \ldots, X_n) \in \mathbb{C}^n \}, \quad \dim AZ = k + r + 1 - n.$$ 

$$BZ = \{(0, \ldots, 0, X_{n-r+1}, \ldots, X_{2n-k-r-1}, 0, \ldots, 0) \in \mathbb{C}^n \}, \quad \dim BZ = n - k - 1.$$

$$D = \{(X_1, \ldots, X_{2n-k-r-1}, 0, \ldots, 0) \in \mathbb{C}^n \}, \quad \dim D = 2n - k - r - 1.$$ 

$D$ is foliated by the same set of $n - k$ equations that foliate $\mathbb{C}^n$. Call this foliation $\eta$. The singular set of $\eta$ is precisely $BZ$. Hence

$$\eta \in Fol(\dim D, \dim D - (n - k), \dim BZ, 0)$$

$$\eta \in Fol(2n - k - r - 1, n - r - 1, n - k - 1, 0).$$
Note that the set of foliation germs to which $\eta_\xi$ belongs is of the form $\text{Fol}(n-s, k-s, r-s, 0)$.

Next $\eta_\xi$ can further be decomposed. Each slice
\[
\{(\ast, \ldots, \ast, \lambda_2, \ldots, \lambda_{n-k}, 0, \ldots, 0) \in \mathbb{C}^n\}
\]
of $D$ is foliated by the single equation $X_1^2 + \cdots + X_{n-r}^2 = \lambda_1$. Let this foliation be $\alpha$. Then
\[
\begin{align*}
\alpha & \in \text{Fol}(\dim \text{ of slice}, (\dim \text{ of slice}) - 1, 0, 0) \\
\alpha & \in \text{Fol}(n-r, n-r-1, 0, 0).
\end{align*}
\]
Hence finally $\eta_\xi = i_{n-k-1}(\alpha)$ and $\xi = \eta_\xi \times \mathbb{C}^{k+r+1-n}$, showing that $\xi$ is a split foliation.

**Remarks**

The problem discussed here is closely related to the classical extension problem of coherent sheaves, see [11] and [13]. For further geometry of singular holomorphic foliations, and their residues, see [12] and [14].

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