I. INTRODUCTION

Rapidly expanding field of interdisciplinary physics has witnessed the desire of statistical physicists to apply their knowledge to other areas of analytical sciences which till recently used to fall outside the traditional domain of physics. A broad class of systems exhibiting complex dynamics, associated with the presence of many interacting constituents, has shown to be successfully investigated with the concepts and techniques of statistical physics. Protein folding, financial markets, flow of vehicular traffic and granular material, surface growth of interfaces are a few examples of numerous applications of physical methods to the paradigm of inter-disciplinary researches. In particular, economic systems has remarkably spurred the interest of statistical physics community. Specifically, stock markets has been the main attractor of these interests and nowadays there is relatively a rich amount of results, both analytical and numerical on modelling of financial markets. Large number of traders with a diversity of trading strategies, conflicting interests and heterogeneous anticipations are generic aspects of financial market leading to unpredictable price fluctuations and other out of equilibrium phenomena which have challenged physicists as well as economists for explanation. Nevertheless, stock markets constitute only a part of economy and there are other major financial systems which can be studied in the context of statistical physics.

In this paper, we present both analytical and numerical investigations of one such economic systems: an insurance market. Human beings are continuously confronted to different types of dangers which threaten his/her life in many aspects. Illness, earthquake, car accident, burglary, death and many others are simple examples of inevitable dangers to which the human life is exposed. The occurrence of these adverse events is random in nature and frequently it is impossible to predict even the probability of their occurrence. Insurance is way to provide guarantee of compensation for the losses of these damages. The purpose of insurance is to indemnify policy holders against the occurrence of adverse events. There is tremendously variety in the events that are covered by insurance. Common to all insurance systems, there exists a risk i.e., a condition in which there is a possibility of an adverse deviation from an expected outcome. Some or all of the risk is transferred from the insured to the insurer with an expectation that through the pooling of risks, the insurer will improve the estimation of expected total losses. In this paper we aim to model the performance of a virtual insurance company in order to find a better insight into the stochastic nature of the problem.

II. FORMULATION OF THE MODEL

We now present our model which simulates the performance of an insurance company. In this paper we restrict ourselves to a single insurance class the so-called automobile insurance. In principle, a number of clients (policy holders) insure their cars against car accidents. The company issues the corresponding insurance policies (policy holders) insure their cars against car accidents. Each loss event i.e.; damages to cars due to accidents, is incorporated with a loss amount which should be regarded as the amount of loss confirmed by company for each accident. The loss size is itself a random variable ranging from tiny to large amounts. Upon occurring a loss event, the company pays the claim amount to the policy holder which reduces the company’s capital.
the other hand, the premium income which is injected to the company raises its capital. It is the competition of these two factors which determines the financial status of the company. A low frequency of loss events together with a high premium income makes the company profitable while a low premium income and high frequency of loss events gives rise to a ruin state. Evidently the amount of premium plays a dominant role in insurance industry. The task of premium calculations and the related strategies have been challenging and controversial subjects since insurance companies and management drastically affect the financial status of the company.

In a competitive economy, there are plenty of insurance companies each of which offer their own premium. If a company increases the premium, the number of clients willing to insure their cars decreases (due to competition of the other companies offering lower premiums). In other words, the company’s attempt to raise its income through increasing the premium may fail due to decrease of the number of insured which obviously reduces the premium income. Therefore, increasing the premium is a highly risky action and it is of prime importance for the company’s managers to have an estimation of the risk amount. Thus it is indispensable for a company to have a quantified knowledge on how the premium variations affect the long-term profit. To be more specific, in each day, a random number of clients insure their car in the company. We denote this number by \( I_t \) for \( t \)-th day. We assume that the number of daily policy issued obeys a simple statistics with premium-dependent characteristics such as mean, standard deviation etc. In this paper we assume that \( I_t \) is uniformly drawn from the interval \([a, b]\) with \( b = \frac{\tau}{T} \) and \( a = \frac{\tau}{T} \) such that \( I_t = \bar{I} \) for all \( t \). As a simple choice, we take an exponential form \( \bar{I}(p) = I_0 e^{-\frac{p}{M^\tau}} \) for the average number of daily policy holders. \( p_0 \) is an arbitrary reference point, \( I_0 \) is average number of insured for \( p = p_0 \) and the parameter \( \tau \) measures the predicted fall of the number of policy holders upon increasing the premium. For instance \( \tau = 0.45 \) corresponds to a 20 percent fall upon a 10 percent premium increase over \( p_0 \). We now discuss on the distribution of the daily number of loss events in \( t \)-th day which is denoted by \( A_t \). Evidently it implicitly depends on the previous number of issued policies \( \sum_{t' < t} I_{t'} \). However, for the sake of simplicity, we make a simple assumption such that \( A_t = \theta \bar{I} \) where constant \( \theta \) measures the ratio of loss events to number of issued policies at an average level. We recall that \( \theta \) can be interpreted as the probability of occurrence of loss event (per car). We note that \( \theta \) can be estimated via empirical data of realistic insurance companies. Analogous to \( I_t \), it is assumed that \( A_t \) is uniformly distributed in the interval \([a\theta, b\theta]\) such that \( A_t = \theta \bar{I} \) for all \( t \). The loss amount itself should be considered as a stochastic variable. In this paper we adopt Erlang’s model [1] for statistical description of loss amounts. According to this model, the distribution of loss amount obeys an exponential function of the form \( e^{-\frac{\xi}{\bar{I}}} \) where \( \xi \) denotes the average loss.

### III. STATISTICAL DESCRIPTION

In this section we present the analytical results which give the annual capital of the company in terms of insurance characteristics i.e., premium, statistics of the number of insured, statistics of loss events etc. For this purpose, in each working day we randomly draw two integer number \( N \) and \( A \) as the numbers of issued policies and loss events. Correspondingly, for each loss event, we attribute a random loss amount (drawn from Erlang distribution function). The sum of loss amounts gives us the daily loss which is denoted by \( L \). Denoting the number of annual working days by \( \mathcal{N} \) and the capital at the end of \( t \)-th day by \( C_t \), we simply have

\[
C_t = C_{t-1} + p I_t - L_t
\]

where \( t \) denotes the day number. Focusing our attention on the long-term profit, we now evaluate the average long-term capital of the company for a period of \( \mathcal{N} \) working days. From the above recursive relation, one simply finds by iteration:

\[
C_N = C_0 + p \sum_{t=1}^{\mathcal{N}} I_t - \sum_{t=1}^{\mathcal{N}} L_t
\]

which determines the long-term capital in terms of aggregate loss and premium income. In order to predict the capital, we now average the above relation over many virtual realisations of future each of which corresponds to \( \mathcal{N} \) working days. Let us first define such predictive averaging process (hereafter referred to averaging) for a generic variable \( \psi_t \) which for instance could be the number of accidents or the number of issued policies in \( t \)-th day. We define the average of \( \psi_t \) over \( M \) virtual runs as follows:

\[
< \psi_t > = \frac{1}{M} [\psi^{(1)}_t + \cdots + \psi^{(M)}_t]
\]

Where each \( \psi^{(i)}_t \) is a random number drawn from a specified distribution function. It can easily be verified that for the quantities \( I_t \) and \( A_t \) the averaging does not depend on day number \( t \). To see this explicitly let us evaluate the average of the number of issued policies in the \( t \)-th day. According to the above definition we have:

\[
< I_t > = \frac{1}{M} [I^{(1)}_t + \cdots + I^{(M)}_t]
\]

The term in the bracket is the sum of \( M \) realisations of the stochastic variable \( I_t \) and provided \( M \) is large enough, the sum simply converges to the average of the distribution function which is \( \bar{I} \) for all days. Concerning this fact, we now average over the long-term capital and obtain the following relation where we have dropped the initial capital \( C_0 \) for simplicity.
We define the critical value of premium \( p_c \) at which the average long-term profit equals zero \( <C_N> = 0 \). It is worth obtaining \( <\mathcal{L}> \), in terms of statistics of loss events. The loss amount in the \( t \)-th day is an extended random variable. By “extended” we mean that there are two sources of randomness. Firstly the number of loss events and secondly the loss amount for each loss event. In the appendix, it is shown, in details, that the average loss amount for each single loss event, multiplied by the average number of daily loss events i.e., \( <\mathcal{L}_t> = \xi \bar{A} \). Concerning the above considerations one obtains the following expression for the averaged long term capital:

\[
<\mathcal{C}> = N\mathcal{I}_0 e^{-\frac{p-p_0}{\tau p_0}} (p - \theta \xi)
\]

In our study, the numerical values of insurance parameters have been set from the empirical data taken from Iranian Insurance Industry for the year 2000. Specifically, we analysed the data taken from Iran Insurance Company the largest company operating in Iran. The empirical data were: 1,819,935 insurance policy issued, 348,961 loss events leading to 739.7 billion Rls loss. The premium amount was 378000 Rls on average. Based on the above data, we set our parameter as: \( \theta = 0.19, \xi = 2.12 \) million Rls and \( \mathcal{I}_0 = 6070 \). Also we take \( N = 300 \) active days. The following graph shows the behaviour of \( <\mathcal{C}> \) for different values of \( \tau \).

![Graph showing averaged long term profit of the virtual company](image)

\( \tau \): 0.45, 0.35 and 0.28 which correspondingly refer to 20, 25 and 30 percent fall in the number of issued policies upon increasing the premium by 10 percent.

Accordingly, two distinct regimes are identified: profit and loss. There is crossover premium \( p_c = \xi \theta \) above which the company profits. If the premium is below \( p_c \), the company is loss-making and in the vicinity of the crossover point, the company is profitless. The above graph gives us another useful knowledge. It determines the optimal premium \( p_{opt} \) at which the company’s capital is maximized. However, we should note that the aforementioned conclusions are based on mean field approach and in principle the company can not rely on these average-based arguments. They need a quantified insight into the risk of premium variations. A simple yet practical quantity which can give us an approximate measurement of the risk is the variance of long term profit in the predictive averaging process. At this stage, we try to evaluate the variance of the long-term capital \( \mathcal{C} = <(\mathcal{C} - \bar{C})^2> \) where for simplicity we have dropped the index \( \mathcal{N} \) from the capital index. After some straightforward mathematics we reach to the following formula:

\[
<\mathcal{C} - \bar{C}> = p^2 \sum_{t,t'}^N <\mathcal{I}_t \mathcal{I}_{t'}> - p^2 N^2 \bar{I}^2 + 2p N^2 \bar{I}^2 \xi \theta - 2p \sum_{t,t'}^N <\mathcal{I}_t \mathcal{I}_{t'}> + \sum_{t,t'}^N <\mathcal{I}_t \mathcal{I}_{t'}> - N^2 \bar{I}^2 \xi ^2 \theta^2
\]

It should be noted that even in the case where our stochastic variables have simple distribution functions, it is not yet an easy task to evaluate the correlation functions in the above formula. Recall the definition of \( <\mathcal{I}_t \mathcal{I}_{t'}> \) one simply finds:

\[
<\mathcal{I}_t \mathcal{I}_{t'}> = \frac{1}{M}[\mathcal{I}_1^{(1)} \mathcal{I}_{t'}^{(1)} + \cdots + \mathcal{I}_M^{(M)} \mathcal{I}_{t'}^{(M)}]
\]

which obviously differs from \( <\mathcal{I}_t \mathcal{I}_{t'}> \). To proceed further, we separate the terms \( t = t' \) in the above relation and use the mean-field approximation in the remaining terms \( t \neq t' \) to substitute all the terms such as \( <\mathcal{I}_t \mathcal{I}_{t'}> \) by \( <\mathcal{I}_t> <\mathcal{I}_{t'}> \). We thus obtain the following relation for variance of long term capital:

\[
<\mathcal{C} - \bar{C}> = N^2 (p^2 <\bar{I}^2> - \bar{I}^2) - 2p <\mathcal{I} \mathcal{L}> - \bar{I}^2 \xi \theta + N^2 \bar{I}^2 \xi ^2 \theta^2
\]

The stochastic variable \( \mathcal{I} \) has a uniform distribution function therefore its variance is simply obtained via the relation \( \sigma^2 = \frac{1}{12}(b-a)^2 \) where \( [a, b] \) denotes the interval over which the variable is distributed. We recall that \( b-a = \bar{I} \) for \( \mathcal{I} \). One can simply show that the variance of daily loss is \( \theta \) times the variance of \( \mathcal{I} \). Recalling the definition of the covariance between two arbitrary stochastic variables \( X \) and \( Y \):

\[
Cov(X,Y) = \frac{<XY> - <X><Y>}{\sigma_X \sigma_Y}
\]

It can be shown that inequality \(-1 \leq Cov(X,Y) \leq 1\) holds. We thus write the remaining term in the above
equation as \( \omega \sigma_T \sigma_L \) where \(-1 \leq \omega \leq 1\) denoted the covariance between \( \mathcal{I} \) and \( \mathcal{L} \). Putting everything together we simply arrive at the following approximate equation for the variance of long term capital:

\[
< (C - \bar{C})^2 > = \frac{N^2 \gamma^2}{12} [p^2 - 2 \rho \omega \xi \theta + \xi^2 \theta^2] \tag{11}
\]

It is seen that risk value has a quadratic dependence on premium \( p \) and is proportional to the number of working days \( N \).

The following graph depicts the behaviour of standard deviation as a function of premium for some values of \( \tau \):

![Graph showing the relationship between standard deviation, premium, \( \tau \), and \( \omega \).](image)

Fig. 2: mean-field standard deviation of annual capital for same three values of \( \tau \) as in Fig. 1. \( N = 300, \xi = 2.12 \) million Rs. and \( \omega = 0.8 \).

It is observed that risk value is an increasing function of \( \tau \). This indicates that for larger values of \( \tau \) which correspond to sharper decrease of insured number, we have a less-valued variance in long term capital. This means that company makes less profit but with more certainty, a situation which is desirable for cautious managers who avoid risky decisions. It would also be advantageous to look at the dependence of capital variance on loss size average \( \xi \). We recall that there are two principal sources of uncertainty in the in-flow/out-flow of the company’s income. The first one is related to the predictive number of insured, approximated by the parameter \( \tau \) in our model, which is the main in-flow portion of the company’s capital. The second one is incorporated with out-flow portion which is dominated by loss size average \( \xi \). The following graph exhibits the long term capital variance for various values of \( \xi \).

![Graph showing the relationship between standard deviation, premium, \( \xi \), and \( \omega \).](image)

Fig. 3: mean-field standard deviation of annual capital for some values of \( \xi \). \( N = 300 \) and \( \omega = 0.3 \) and \( \tau = 0.45 \).

The next concept we deal with is the bankruptcy. A useful quantity in risk management is the company’s ruin probability. This can be expressed as the probability that the long term capital falls below zero i.e., \( P(C < 0) \). To estimate the ruin probability, we use the so-called Tchebycheff inequality. The Tchebycheff inequality establishes a relation between the variance and the probability that a stochastic variable, with finite average and variance, can deviate by an arbitrary amount \( \epsilon (\epsilon > 0) \) from the mean value:

\[
P(|z - \mu | \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \tag{12}
\]

where \( \sigma_Z \) denotes the standard deviation of the stochastic variable \( Z \). Now let us estimate the probability that for a prescribed \( p \), the long term profit equals \(-\lambda \bar{C}(p)\) where the dimension-less parameter \( \lambda > 0 \) denotes the depth of ruin. According to Tchebycheff inequality if we take \( Z = \mathcal{C} \) one simply finds:

\[
P(C = -\lambda \bar{C}) \approx \frac{1}{2} P(|C - \bar{C}| \geq \bar{C}(1 + \lambda)) = \frac{\sigma^2}{2C^2(1 + \lambda)^2} \tag{13}
\]

Using equations (9) and (6) for \( \sigma \) and \( \bar{C} \) one obtains the upper limit of ruin probability in terms of \( p \) and \( \lambda \):

\[
P(C = -\lambda \bar{C}) \approx \frac{p^2 - 2 \rho \omega \theta \xi + \theta^2 \xi^2}{24N(p - \xi \theta)^2(1 + \lambda)^2} \tag{14}
\]

The following graph exhibits the ruin probability dependence on premium for various ruin depths.
Fig. 4: estimated ruin probability according to Tchebycheff inequality. \( \lambda \) determines the ruin depth. Here \( \lambda = 0.1 \). The rest of the parameter are remained unchanged as in fig. 2

In order to find a deeper insight into the stochastic nature of the problem, we have carried out simulations to obtain the variance of \( C \). We have simulated the company’s performance for a year and obtain the annual income \( C(N) \). Figure five exhibits the simulated variance of annual capital as a function of premium.

Fig. 5: simulated standard deviation of annual capital for the same values of \( \tau \). \( N = 300, \xi = 2.12 \) million Rls.

The simulation results show a notable deviation from those of mean-field approach and gives \( \omega \approx 0.5 \) as the best fitting value for the covariance parameter \( \omega \).

IV. SUMMARY AND CONCLUDING REMARKS

In this paper, we have tried to present, in some detail, a set of statistical facts emerging from the study of a virtual insurance company. The financial operations of an insurer can be viewed in terms of a series of cash inflows and outflows. Premiums and income from investments, together with certain other income, are added to the reservoir of the assets, while the reservoir is depleted by claim payments, expenses of running the business, taxes and possibly other items of outflow. From a practical point of view, it is of prime importance for every insurance company to have a quantified measurement of the risk of insuring. In this work, we have modelled the performance of an insurance company both analytically and numerically. Our results illustrates the dependence of long-term profit in terms of insurance parameters especially the premium. We have managed to give a quantified estimation for the risk of premium increasing.

Finally we should point out some issues we have not discussed here. The occurrence of loss events and its statistical features is the prominent factor affecting the company’s long term profit. Throughout the paper we have assumed that the loss size distribution obeys a simple statistics i.e. an exponential distribution function \( e^{-\xi} \). This is evidently a localized distribution function with finite mean and variance \( \xi \) and \( \xi^2 \) respectively. However, the main source of ruin in insurance industry is the occurrence of catastrophic events corresponding to overwhelming loss sizes. Obviously ordinary distribution functions such as Erlang fail to model these extreme events. Investigation on this issue needs more exploration.

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VI. APPENDIX

In the section we obtain the relation $\langle L_i \rangle = \xi \bar{A}$. In order to numerically calculate the average of loss amount in $t$-th day, we draw $l$ random numbers $n_1^{(1)}, \ldots, n_l^{(1)}$ from a uniform distribution function where $n_i^{(1)}$ denotes the number of car accidents in the $i$-th virtual realisation of $t$-th day. Let $z_{i,j}^{(1)}$ denote the loss amount due to the $i$-th accident in the $j$-th virtual realisation of $t$-th day. Accordingly we have:

$$\langle L \rangle = \frac{z_1^{(1)} + \cdots + z_{n_1^{(1)}}^{(1)} + \cdots + z_1^{(l)} + \cdots + z_{n_l^{(l)}}^{(l)}}{l}$$

(15)

where for simplicity we have dropped the day index $t$. The above relation can be rearranged as:

$$\frac{n_1^{(1)} z_1^{(1)} + \cdots + z_{n_1^{(1)}}^{(1)}}{l} + \cdots + \frac{n_l^{(l)} z_1^{(l)} + \cdots + z_{n_l^{(l)}}^{(l)}}{l}$$

(16)

Concerning the fact that each $n_i$ is considerably large, each sum converges to $\langle Z \rangle = \xi$ therefore we have:

$$\langle L \rangle = \xi \left[ \frac{n_1^{(1)} + \cdots + n_l^{(l)}}{l} \right]$$

(17)

Now the term in the bracket is simply the average of $A$ therefore we have the relation $\langle L \rangle = \xi \bar{A}$