BELL'S THEOREM WITHOUT HIDDEN VARIABLES

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Introduction As stated by Prof. d'Espagnat at this Conference, hidden variables theories are not the only theories in trouble with locality. Following ideas expressed in Refs. 1 and 2, I will demonstrate the CHSH inequality\(^3\) from locality alone without using either determinism or the concept of hidden variables. Then, I will comment about the violation of this inequality by quantum theory.

Let us consider two measuring apparatus located in two different places A and B. There is a knob a on apparatus A and a knob b on apparatus B. Since A and B are separated in space, it is natural to think that what will happen in A is independent of the setting of the knob b and vice versa. The principles of relativity seem to impose this point of view if the time at which the knobs are set and the time of the measurements are so close that, in the time laps, no light signal can travel from A to B or vice versa. Then, no signal can inform a measurement apparatus of what the knob setting of the other is. However, there are cases where the predictions of quantum theory make that independence assumption impossible. If quantum theory is true, there are cases where the results of the measurements in A will depend on the setting of knob b and/or the results of the measurements in B will depend on the setting of knob a.

Definitions A simple experiment consists of the recording of \(N\) events and each one of these events involves a measurement in both apparatus A and B. There are only two possibilities for the outcome of the measurement in each apparatus
and we label these measurement results as +1 and −1. The j'th event corresponds to a response $\alpha_j$ in A and $\beta_j$ in B. Each term $\alpha_j$ and $\beta_j$ is either +1 or −1. We define the correlation by a statistical mean.

$$c = \frac{1}{N} \sum_j \alpha_j \beta_j = \langle \alpha_j \beta_j \rangle$$

(1)

where the symbol $\langle \rangle$ around a quantity designates the statistical mean of that quantity over j. $C$ is equal to the fraction of events where $\alpha_j$ and $\beta_j$ have the same signs minus the fraction where they have opposite signs.

There are two positions $a^{(1)}$ and $a^{(2)}$ of the knob a and two possible positions $b^{(1)}$ and $b^{(2)}$ of the knob b. $C$ will depend on a and b. We define

$$c^{(1,1)} = \langle \alpha_j \beta_j \rangle \text{ if } a = a^{(1)}, \ b = b^{(1)}$$

$$c^{(2,1)} = \langle \alpha_j \beta_j \rangle \text{ if } a = a^{(2)}, \ b = b^{(1)}$$

$$c^{(1,2)} = \langle \alpha_j \beta_j \rangle \text{ if } a = a^{(1)}, \ b = b^{(2)}$$

$$c^{(2,2)} = \langle \alpha_j \beta_j \rangle \text{ if } a = a^{(2)}, \ b = b^{(2)}$$

(2)

The Locality Assumption We first have to suppose that it makes sense to consider the different results of a future experiment for different settings of the knobs a and b, although only one setting, at most, will actually be chosen for the actual experiment. Then, for a given position of b, we can write that $\alpha_j$ will be equal to $\alpha_j^{(1)}$ if $a = a^{(1)}$ and $\alpha_j$ will be equal to $\alpha_j^{(2)}$ if $a = a^{(2)}$, for the same event j but different settings of knob a. Similarly, for a given value of a, we define the values $\beta_j^{(1)}$ and $\beta_j^{(2)}$ of $\beta_j$ if $b = b^{(1)}$ or $b^{(2)}$. Let's suppose, in addition, that the measurement $\alpha_j$ in A will not depend on b, and the measurement $\beta_j$ in B will not depend on a. Then, for each event j, we can write

$$\alpha_j = \alpha_j^{(1)} \text{ if } a = a^{(1)} \text{ whatever } b \text{ is}$$

$$\alpha_j = \alpha_j^{(2)} \text{ if } a = a^{(2)}$$

$$\beta_j = \beta_j^{(1)} \text{ if } b = b^{(1)} \text{ whatever } a \text{ is}$$

$$\beta_j = \beta_j^{(2)} \text{ if } b = b^{(2)}$$

(3)
We refer to this assumption as the locality assumption.

**THE CHSH Inequality**  Now we can define the following mathematical expression for event \( \#j \)

\[
\gamma_j = \alpha_j (1) \beta_j (1) + \alpha_j (2) \beta_j (1) + \alpha_j (1) \beta_j (2) - \alpha_j (2) \beta_j (2)
\]  (4)

Each product \( \alpha_j \beta_j \) is equal to \( \pm 1 \), therefore \( \gamma_j \) is an even integer. Furthermore, \( \gamma_j \) cannot be equal to \( 4 \) because \( \alpha_j (2) \beta_j (2) \) is equal to the product of the first three terms of the right hand side of equation (4), and it is equal to \( +1 \) if the first three terms are positive. Therefore,

\[ \gamma_j \leq 2 \]  (5)

It follows that

\[
\frac{1}{N} \sum_{j} \gamma_j = <\gamma_j> = <\alpha_j (1) \beta_j (1)> + <\alpha_j (2) \beta_j (1)> + <\alpha_j (1) \beta_j (2)> - <\alpha_j (2) \beta_j (2)>
\]  (6)

From equations (2), (3) and (6), we derive

\[ c(1,1) + c(2,1) + c(1,2) - c(2,2) \leq 2 \]  (7)

Equation (7) is the CHSH inequality\(^3\) that is a generalization of Bell's inequality.\(^5\)

**Quantum Theoretical Predictions**  Let's consider an idealized experiment where each event consists of the detection in A and B of two photons emitted in the direction of A and B by an atom located between A and B. The detection is made with the help of a polarization analyzer based on an idealized birefringent prism so that the polarization of the photon is measured with respect to the axis of the analyzer. We count \( \alpha_j = +1 \) when the photon is measured with one polarization in A and \( \alpha_j = -1 \) with the other. We count \( \beta_j = \pm 1 \) similarly, depending on the polarization of the photon measured in B.
Let $a$ and $b$ be the angles of the analyzers in A and B. The efficiencies are all 100%.

There is a 100% correlation between the planes of polarization of the two photons at emission.

Quantum theory makes a prediction for the quantity $C$ of Eq. (1):

if the number $N$ of events is large enough

$$C \equiv \cos(2a - 2b)$$

in almost all cases. (The sign $\equiv$ means approximately equal.)

Then, for

$$\begin{cases}
a^{(1)} = 45^\circ \\
a^{(2)} = 0^\circ \\
b^{(1)} = 22.5^\circ \\
b^{(2)} = 67.5^\circ
\end{cases}$$

quantum theory predicts

$$C^{(1,1)} + C^{(2,1)} - C^{(1,2)} - C^{(2,2)} \approx 2\sqrt{2} > 2$$

in almost all cases. This prediction is in contradiction with (7). Therefore, the predictions of quantum theory are incompatible with our locality assumptions.

**Conclusions** If quantum theory is correct, it is not allowed to think about the options of the future as resulting in a sequence $\alpha_j$ depending on the choice made for knob a only and a sequence $\beta_j$ depending on the choice of knob b alone. The $\alpha_j$'s will be different for different settings of b and/or the $\beta_j$'s will be different for different settings of a.

This demonstration has been performed without invoking either the concept of hidden variables, or the concept of determinism. We used only the results $\alpha_j$ and $\beta_j$'s of the measurements, the hypothetical values $\alpha_j^{(1)}$, $\alpha_j^{(2)}$, $\beta_j^{(1)}$ and $\beta_j^{(2)}$ that $\alpha_j$ and $\beta_j$ would take for different knob settings and the
locality concepts as defined by Eqs. (3). The theorem affects more theories
then just the hidden variable theories.

Alternate Formulation There is an alternate way to formulate this theorem without using the concept of hypothetical results of experiments that will not be performed, but involving only the results of actual experiments. Let's consider the following thought experiment. The two photon experiment with N events is repeated a lot of times in all the four combinations of a and b defined by Eq. (9). What is the probability that one can find a set of four experiments, i.e., one for each of the four knob combinations, such that Eqs. (3) are valid?

We have to select experiment for different settings of knob b such that the lists of \( a_j \)'s are identical from \( j = 1 \) to \( j = N \). Then we have to make a similar selection for \( \beta_j \) and knob a. N is finite. The probability to find a set of experiments associated to such sequences \( a_j(1), a_j(2), \beta_j(1) \) and \( \beta_j(2) \) is very small but, because of all the possible statistical fluctuations, it is not zero. However, if in addition we require that each correlation defined by Eq. (2) differs from the expectation value predicted by quantum theory by less than the quantity \( \sqrt{2} - 1 \), then the probability to find such a set of four experiments is an absolute zero. Indeed, any set of sequences \( a_j(1), a_j(2), \beta_j(1) \) and \( \beta_j(2) \) that one introduces in Eqs. (3) will generate correlations \( C \) that satisfy inequality (7), and inequality (7) cannot be satisfied if all the correlations are different from \( \cos(2a - 2b) \) by less than \( \frac{\sqrt{2} - 1}{2} \).

If we perform a lot of experiments with the different knob settings of Eqs. (9) and look for experiments that have identical \( a_j \)'s for different settings of knob b, and the same \( \beta_j \)'s for different settings of knob a, the only set of experiments one may find will contain experiments whose results are very different from the prediction of quantum theory. Actually, the differences between the
statistical mean $C$ of Eq. (1) and the quantum theoretical predictions (Eq. (8)), in absolute value, will add up to at least $2(\sqrt{2}-1)$.

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2. J.S. Bell, Res. Th. 2053 - CERN The Theory of Local Beables, communication at the 6th Gift conference, Jaca, 2-7 June 1975; J. F. Clauser and M. A. Horne, Phys. Rev. D 10, 526 (1974); N. Herbert Am. Jour. Phys. 43, 315 (1975).

3. CHSH stands for J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett 23, 880 (1969).

4. That consideration is related to Stapp's concept of "contrafactual definiteness" (See Ref. 1). It would not make sense to talk about non-actual experiments if, for instance, we believed that there is no free will, and the choice by an experimenter is determined anyway. Otherwise, I believe that considering different possible results of a future experiment, depending on one's decision, is a rather natural way of thinking.

5. J. S. Bell, Physics (N.Y.), 1, 195 (1964).

6. Note that the conventional formalism of quantum theory also involves non-local evolutions of the wave function when the wave function collapses.
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