The Spatial String Tension in High Temperature Lattice Gauge Theories

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Abstract

We develop some techniques which allow an analytic evaluation of space-like observables in high temperature lattice gauge theories. We show that such variables are described extremely well by dimensional reduction. In particular, by using results obtained in the context of “Induced QCD”, we evaluate the contributions to space-like observables coming from the Higgs sector of the dimensionally reduced action, we find that they are of higher order in the coupling constant compared to those coming from the space-like action and hence negligible near the continuum limit. In the case of SU(2) gauge theory our results agree with those obtained through Montecarlo simulations both in (2+1) and (3+1) dimensions and they also indicate a possible way of removing the gap between the two values of $g^2(T)$ recently appeared in the literature.

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1 Introduction

It is by now widely accepted that \((d+1)\) dimensional Lattice Gauge Theories drastically simplifies at high temperature and can be described by some effective \(d\)-dimensional model. This approach, which is usually called “dimensional reduction”, is rather non-trivial. For instance, it can be shown that the naive assumption of a complete decoupling of the “non-static modes” (namely, the degrees of freedom, in the compactified “temperature” direction) does not hold in general and that these non-static modes induce some well defined interaction in the static sector. The predictions of dimensional reduction have been successfully tested recently in the case of \((3+1)\) dimensional SU(2) \([1]\) and SU(3) \([2]\) LGT, by looking at “time-like” observables (correlations of Polyakov loops).

At the same time there has been a renewed interest, in the last year, in the study of space-like Wilson loops at high temperature, both in \((2+1)\) and \((3+1)\) dimensions and both for continuous (SU(2) and SU(3)) \([3, 4, 5]\) and discrete (\(Z_2\)) \([6]\) gauge groups. Even if the space-like string tension extracted from these Wilson loops has nothing to do with the interquark potential (as a matter of fact, it is different from zero, and actually increases with the temperature, also in the deconfined region) there are various reasons of interest in its behaviour. First, it can be used to predict various physical features of the theory, like the temperature dependence of quark-antiquark correlations in the space-like directions \([7]\). Second, it can be related to the thickness of the flux tube joining the quark-antiquark pair at zero temperature \([3, 5, 6]\). Third it can be used as a further independent check of the dimensional reduction approach in the case of space-like observables. This last point was addressed in \([4]\) (see also \([9]\) for a comprehensive review) where, by using a set of very precise data on SU(2), an impressive agreement between the space-like string tension in \((3+1)\) dimensions and the zero temperature string tension in 3 dimensions was found. The only missing point for a complete success of the dimensional reduction program was a misagreement (roughly of a factor of two) between the coupling constants quoted in \([1]\) and that of \([4]\).

In this paper we want to further pursue this analysis by using some recent results obtained in a completely different framework, namely the so called “induced QCD” \([10]\). We shall show in sect.2 and 3 that, within this framework, it is possible to define directly on the lattice a dimensional reduction approach for high temperature LGT’s \([11]\). Moreover we shall show that, as far as space-like observables are concerned, further simplifications occur, part of the dimensionally reduced effective action can be neglected, thus allowing in particular cases to make explicit, analytic calculations. We shall then test our predictions in the case of the SU(2) model in \((2+1)\) (sect.4) and \((3+1)\) (sect.5) dimensions. In the latter case we shall give some argument to remove the apparent discrepancy between the results of ref. \([1]\) and \([4]\).
2 General Setting and Notations

Let us consider a pure gauge theory with gauge group $SU(N)$ defined on a $d+1$ dimensional cubic lattice. In order to describe a finite temperature LGT we must take periodic boundary conditions in one direction (which we shall call from now on “time-like” direction), while the boundary conditions in the other $d$ direction (which we shall call “space-like”) can be chosen freely. Let us take a lattice of $N_t$ ($N_s$) spacings in the time (space) direction. The theory will contain only gauge fields described by the link variables $U_{n;i} \in SU(N)$ where $n \equiv (\vec{x}, t)$ denotes the space-time position of the link and $i$ its direction. It is useful to choose different couplings in the time and space directions. Let us call them $\beta_t$ and $\beta_s$ respectively. Let us take the simplest choice for the lattice gauge action, namely the Wilson action:

$$S_W = \sum_n \frac{1}{N} \text{Re} \left\{ \beta_t \sum_i \text{Tr}_f(U_{n,0;i}) + \beta_s \sum_{i<j} \text{Tr}_f(U_{n;i,j}) \right\}, \quad (1)$$

where $\text{Tr}_f$ denotes the trace in the fundamental representation and $U_{n,0;i}$ ($U_{n;i,j}$) are the time-like (space-like) plaquette variables, defined as usual:

$$U_{n;i,j} = U_{n;i} U_{n+i,j} U_{n+j,i}^{\dagger} U_{n;i,j}^{\dagger}. \quad (2)$$

In the following we shall call $S_s$ ($S_t$) the space-like (time-like) part of $S_W$. $\beta_s$ and $\beta_t$ are related to the (bare) coupling constant $g$ and the temperature $T$ of the gauge theory by the usual relations:

$$\frac{2N}{g^2} = a^{3-d} \sqrt{\beta_s \beta_t}, \quad T = \frac{1}{N_t a} \sqrt{\frac{\beta_t}{\beta_s}}. \quad (3)$$

where $a$ is the space-like lattice spacing while $\frac{1}{N_t a}$ is the time-like spacing. The two are related by the adimensional ratio $\epsilon \equiv \frac{1}{N_t a}$. We can solve the above equations in terms of $\epsilon$ as follows:

$$\beta_t = \frac{2N}{g^2} a^{d-3} \quad (4)$$

$$\beta_s = \frac{2N \epsilon}{g^2} a^{d-3}. \quad (5)$$

In a finite temperature discretization it is possible to define gauge invariant variables which are topologically non-trivial loops, closed due to the periodic boundary conditions in the time directions. The simplest choice is the Polyakov loop defined as follows:

$$P(\vec{x}) = \text{Tr} \prod_{t=1}^{N_t} (U_{\vec{x},t,0}) \quad (6)$$
where \( \vec{x} \) labels the space coordinates of the lattice sites. Moreover, an important feature of the finite temperature theory with respect to the zero temperature case is that it has a new global symmetry (independent from the gauge symmetry) with symmetry group the center \( C \) of the gauge group (in our case \( \mathbb{Z}_N \)). The Polyakov loop turns out to be a natural order parameter for this symmetry.

For \( d > 1 \) these theories admit a deconfinement transition at \( T = T_c \). In the following we shall be interested in the high temperature phase (\( T > T_c \)). It is possible to obtain, just from the definition itself of the model, some general properties of this high temperature regime (see for instance \( [12] \)). In this region the symmetry with respect of the center of the group is broken, the theory is deconfined, the Polyakov loop has a non-zero expectation value and, what is more important, it is an element of the center of the gauge group. In the following this \( \mathbb{Z}_N \) degeneracy will not play any role and we shall assume to have lifted it, by selecting the vacuum in which \( \langle P(\vec{x}) \rangle = 1 \).

The simplest approach to the description of the high temperature phase is the so called “complete dimensional reduction” according to which the degrees of freedom in the compactified time direction (the non-static modes) decouple \([13]\). The resulting theory is a \( d \)-dimensional gauge theory coupled to a scalar field in the adjoint representation, and the corresponding action is the sum of the purely space-like part \( S_s \) and a new term \( S_h \) which is the remnant of the \( S_t \) term:

\[
S_h(m_0) = \frac{\beta_h}{N} \text{Tr}_t \left( -m_0^2 \phi(\vec{x})^2 + \sum_{i=1}^{d} U_{\vec{x},i} \phi(\vec{x}) U_{\vec{x},i}^\dagger \phi(\vec{x} + \hat{i}) \right)
\]

where \( \phi(x) \) is an Hermitian \( N \times N \) matrix and \( m_0^2 = d \).

However, it is by now clear that such a complete dimensional reduction does not take place in general, that non-static modes cannot be neglected and that they induce some well defined interaction terms in the static sector. These induced interactions can be explicitly evaluated and taken into account (see \([1, 2]\), and references therein). They amount to a self-interaction term for the scalar field \( \phi \):

\[
S_{ns} = -\frac{\beta_h}{N} \sum_{\vec{x}} \left\{ \frac{h}{2} \text{Tr}_t \phi^2(\vec{x}) + \frac{k_1}{2} \text{Tr}_t \phi^4(\vec{x}) \right\}
\]

The detailed derivation of this term and the values of \( h \) and \( k \) in the \( N = 2, 3, d = 3 \) case, can be found in ref. \([1, 4]\) (for \( N > 3 \) a term of the type \( (\text{Tr}_t \phi^2(\vec{x}))^2 \) should be also taken into account).

Notice that the quadratic term can be absorbed in a redefinition of the mass term \( m^2 = m_0^2 + h/2 = d + h/2 \).

Following the literature on the subject \([1, 4]\), we shall simply call this procedure “dimensional reduction” and the resulting effective action: \( S_s + S_h + S_{ns} \) the “dimensionally reduced action”.

The \( S_h \) and \( S_{ns} \) contributions to the action are very important if one studies time-like observables, such as the correlations of Polyakov loops, but it turns out
that they are almost negligible if space-like observables are studied. This has been anticipated in [9] and will be also shown in the following.

3 The Strategy

Our starting point is the following assumption.

It is clear from equations (3,4,5) that we have a residual freedom in the choice of our lattice regularization, since several choices of \((N_t, \beta_t, \beta_s)\) correspond to the same set of \((g, T)\) values. We assume that there is a region in the \((g^2, T)\) plane in which we are allowed to fix this residual freedom by choosing \(N_t = 1, \ \beta_t = \frac{2NT}{g^2} a^{d-2}, \ \beta_s = \frac{2N}{g^2} a^{d-4}\).

The resulting action is exactly of the type one would obtain within the framework of a complete dimensional reduction (see ref. [11]). In fact \(S_s\) is not affected by the transformation (except for the rescaling of \(\beta_s\)), while \(S_t\) exactly becomes the \(S_h\) term defined above. In fact, since we are in the broken symmetry phase we can expand \(U_{\vec{x}:0}\) as follows (since \(N_t = 1\) we eliminate any reference to the time coordinate)

\[
U_{\vec{x}:0} \equiv e^{i\phi(\vec{x})} = 1 + i\frac{\phi(\vec{x})}{\sqrt{\beta_t}} - \frac{\phi^2(\vec{x})}{2\beta_t} + \cdots
\]

where \(\phi(x)\) is an Hermitian \(N \times N\) matrix. By inserting (9) in \(S_t\) we find:

\[
S_t(m_0) = \frac{\beta_t}{N} \text{tr} \left\{ d \Omega_x 1 + \frac{1}{\beta_t} \sum_{\vec{x}} \left( -m_0^2 \phi(\vec{x})^2 + \sum_{i=1}^{d} U_{\vec{x}:i} \phi(\vec{x}) U_{\vec{x}:i}^\dagger \phi(\vec{x} + i) \right) \right\}
\]

where \(m_0^2 = d, \ \Omega_x\) is the volume of the \(d\) dimensional space, and we used the periodic boundary conditions and the fact that \(N_t = 1\) to identify the two space-like \(U\) matrices.

Apart from an irrelevant constant, eq.(10) coincides with \(S_h(m_0)\) defined in the previous section. The coupling constant \(\beta_h\) has been conventionally fixed to 1 in eq.(10). It depends on the mean value of the Polyakov loops around its minimum (see [11] for the details), but its precise value turns out to be irrelevant, the only important quantity being the mass term \(m_0\).

There are some obvious cautions which one must have in rescaling the \((\beta_t, \beta_s, N_t)\) parameters:

i] One must be in the scaling regime (to be defined more precisely later) to extract the same physics from different lattice regularizations.

ii] The rescaling of \(N_t\) implies a change of the ratio \(\epsilon\) between time-like and space-like lattice spacings. In \(d = 3, g\) is adimensional, and such a rescaling requires a proper \(\Lambda(\epsilon)/\Lambda\) correction.
When $N_t$ is pushed up to the extreme value $N_t = 1$ spurious effects due to lattice artifacts must be expected. Actually, this is nothing but the lattice counterpart of the decoupling of non-static modes and reminds us that the naive limit $N_t = 1$ corresponds to the complete dimensional reduction. The problem is cured by adding to the action the $S_{ns}$ term thus obtaining the correct dimensional reduction.

We still have to fix the region in the $(g^2, T)$ plane where we expect our assumptions to hold. This is not difficult for $g^2$, which we must require to be in the region where scaling already holds in the zero temperature regime of the theory. On the contrary we have no general argument to set a threshold in $T$ which can only be estimated a posteriori. However it is by now widely accepted that in (3+1) non-abelian L.G.T., dimensional reduction holds for $T > 2T_c$. Teper's results \cite{3, 8} suggest that in the (2+1) dimensional case this threshold can be set already at $T > T_c$. This fact will play a major role in the comments of the last section. Let us stress that the most important consequence of our dimensional reduction scheme is that the space-time coupling constant, namely the coupling constant of the dimensionally reduced theory, is completely fixed to be

$$\beta_s = \frac{(2N_d - 4)}{(g^2 T)}$$

where $g^2$ is the coupling of the original $(d + 1)$ theory. This turns out to be a very stringent test of the whole approach, and it is the ultimate reason of its predictive power.

The second step is now to show that the contributions due to $S_h$ and $S_{ns}$ are negligible if one studies space-like observables. More precisely, we will show that their contributions are of order $1/\beta^2$ and are negligible in the continuum limit $\beta \rightarrow \infty$ We shall obtain this result in two steps: first we take into account only $S_h$, in which the field $\phi$ appears only quadratically. This allows us to integrate the field $\phi$ exactly and gives the so called “induced action” for the space-like gauge fields \cite{10, 11},

$$\int DU D\Phi exp(-S_t(m)) \sim \int DU exp(-S_{ind}[U])$$

with:

$$S_{ind}[U] = -\frac{1}{2} \sum_{\Gamma} \frac{|\text{Tr} U[\Gamma]|^2}{l[\Gamma](2m_0^2)l[\Gamma]}$$

where $l[\Gamma]$ is the length of the loop $\Gamma$, $U[\Gamma]$ is the ordered product of link matrices along $\Gamma$ and the summation is over all closed loops. It is possible to restrict this summation to the non-backtracking loops only, through a suitable renormalization of the mass term $m_0 \rightarrow m_R$. This can be done by “dressing” any non-backtracking closed loop with all possible backtracking paths starting from a generic site of the loop.

This induced action is very interesting and has been extensively studied in the context of the so called Kazakov-Migdal model \cite{10}. In particular it has been shown \cite{14} that, if no other gauge self-interaction term is present, near the continuum limit all the loops equally contribute, one is not allowed to truncate the sum to the smallest loops, and the resulting theory in the continuum is very different
from ordinary QCD. Moreover, as in the induced action all gauge fields are in the adjoint representation, the expectation value of a space-like Wilson loop is exactly zero. The vanishing of spatial Wilson loops in a pure induced action indicates that in our case such quantities are dominated by the space-like part of the Wilson action $S_s$ and that the induced action can be treated as a perturbation. We shall see in the next section by an explicit calculation in the $d = 2$ case, that all contributions of order $1/\beta_s$ coming from $S_{ind}$ cancel exactly and one is left with subleading $1/\beta_s^2$ contributions only. The reason of this result can be easily understood if one truncates $S_{ind}$ to its first term (we shall see in the next section that in $d = 2$ this truncation already gives a very good approximation of the whole series). This reduction is particularly interesting because the resulting action is exactly of the so-called “fundamental-adjoint” type. For instance in the $SU(2)$ case:

$$S_{f-a} = \sum_{n,i<j} \left\{ \frac{\beta_f}{2} \text{Tr}(U_{n;ij}) + \beta_a \text{Tr}_a(U_{n;ij}) \right\} , \quad \text{(13)}$$

where $\text{Tr}_a$ is the trace in the adjoint representation. In our case $\beta_f = \beta_s$ and $\beta_a = 1/(2m_R^2)^4$. This class of actions was carefully studied some years ago. It was shown that in 2 and 3 dimensions, near the continuum limit and for small values of $\beta_a$, the correction with respect to the usual Wilson action is completely encoded by a suitable rescaling of $\beta_f$, for instance in the $SU(2)$ case: $\beta_f \rightarrow (\beta_f + 8\beta_a)$. Since in the dimensionally reduced theory the coupling constant is dimensional, any dimensional quantity, say “Q”, must scale as $Q \sim c/\beta_s$ and the above rescaling implies $Q \sim c/(\beta_s + 8\beta_a) \sim c/\beta_s - 8c\beta_a/\beta_s^2$.

Finally, let us consider the effect of $S_{ns}$. The quartic term in $S_{ns}$ makes it impossible to integrate explicitly over the fields $\phi(x)$, however the properties of the induced action can equivalently be studied by integrating over the gauge fields $U_{\vec{x};i}$ in $S_t$. This can be done by using the well known Itzykson-Zuber formula:

$$I(\phi(x), \phi(y)) = \int DU \exp \left( N \text{tr}\phi(x)U\phi(y)U^\dagger \right) \propto \frac{\text{det}_{ij} \exp(N\lambda_i(x)\lambda_j(y))}{\Delta(\lambda(x))\Delta(\lambda(y))} \quad \text{(14)}$$

where $(x, y)$ are nearest neighbour links of the lattice, $\lambda_i(x)$ are the eigenvalues of the matrix $\phi(x)$ and

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \quad \text{(15)}$$

is the Vandermonde determinant.

Notice that in the $SU(2)$ case, which is the one relevant in the following section, there is only one field $\lambda(x)$. It has been shown that in this case, and for the range of parameter in which we are interested, the theory is very well described by a simple mean field approximation. Moreover it has been shown in [14, 17], how to deal with the unusual normalization of the adjoint term, which has been chosen for future convenience.
with the terms coming from the perturbative expansion of $S_s$, which give origin to the so called “filled Wilson loops” [7]. Their contribution can be precisely estimated within the mean-field approximation. The main outcome of this alternative approach is that it allows to study more general than quadratic potentials and, in particular, to take into account the contribution of $S_{ns}$. The important point is that (within the mean field approximation) all the results obtained above by integrating first on the $\phi$ field, and also all the steps of the strong coupling expansion that we shall describe in the next section remain unaltered. The only effect of the $S_{ns}$ term is to change the expansion parameter $m_R$, whose value can be evaluated within the mean field approximation as a function of $h$ and $k$.

4 SU(2) in (2+1) dimensions

The case $d = 2$ is particularly interesting because in the dimensionally reduced theory one can use strong coupling expansions to test our assumption and the various approximations. Moreover, very precise and careful estimates of all physical quantities relevant to our analysis have been recently published in the case of the SU(2) model [3, 8]. Let us summarize these results (see [3, 8] for details). In the following $\beta_s = \beta_t = \beta = 4/(a g^2)$ and all the stated results are intended to be valid near the continuum limit, namely for large values of $\beta$.

i] The zero temperature string tension $\sigma(0)$ behaves as follows:

$$a\sqrt{\sigma(0)} = \frac{1.336(10)}{\beta} + \frac{1.122}{\beta^2}$$  \hspace{1cm} (16)

ii] The deconfinement temperature is:

$$\frac{T_c}{\sqrt{\sigma(0)}} = 1.121(8)$$  \hspace{1cm} (17)

iii] In the deconfined phase $T > T_c$ the space-like string tension rises linearly with the temperature according to the law:

$$\sigma(aT) = l_0 \sigma(0) T$$  \hspace{1cm} (18)

where $l_0$ is a new physical length (related to the thickness of the chromoelectric flux tube at zero temperature, see [3, 3]) whose value is:

$$l_0 = \frac{1.22(4)}{T_c} = \frac{1.09(5)}{\sqrt{\sigma(0)}}.$$  \hspace{1cm} (19)

It is interesting to see that the behaviour of eq.(18) is present in the whole high temperature phase, starting from the critical point. To be precise, eq.(18) has been tested at $\beta = 9$ (where the critical temperature is between $N_t = 5$ and $N_t = 6$) in the whole range $N_t = 2 - 6$. 

8
Inserting eq. (19) in (18) we obtain:

\[ a \sqrt{\sigma(0)} = 0.92(5) \frac{a \sigma(aT)}{T} \]  \hspace{1cm} (20)

Let us now assume dimensional reduction, and rescale the above lattices to \( N_t = 1 \) as described in the previous section. The resulting, rescaled, and now asymmetric couplings are:

\[ \beta_s = \frac{\beta}{aT}, \beta_t = \beta aT \]

Let us first neglect the contribution coming, in the notations of the previous section, from \( S_h \) and \( S_{ns} \) and take only into account the two-dimensional theory given by \( S_s \). This is exactly solvable, and the string tension is (see for instance [18]):

\[ a^2 \sigma(aT) = -\log \left( \frac{I_2(\beta_s)}{I_1(\beta_s)} \right) \]  \hspace{1cm} (21)

where \( I_n(\beta) \) is the \( n^{th} \) modified Bessel function.

By using the large \( \beta \) expansion of the Bessel function we obtain:

\[ a^2 \sigma(aT) = -\frac{aT}{2\beta} + \cdots \]  \hspace{1cm} (22)

and by inserting this into eq.(20) we find:

\[ a\sqrt{\sigma(0)} = \frac{1.38(6)}{\beta} \]  \hspace{1cm} (23)

which is in remarkable agreement with the known value given in eq.(16). Notice that within our framework the linear rise of the space-like string tension with the temperature also has a natural explanation, since it simply encodes the \( T \)-dependence of \( \beta_s \) with respect to \( \beta \). The impressive agreement between (16) and (23) is due to the fact that the corrections coming from \( S_h \) and \( S_{ns} \) are of order \( 1/\beta^2 \) and do not affect the behaviour of eq.(23).

The fact that in \( d = 2 \) the strong coupling expansion converges up to the continuum limit allows us to give some more evidence of this statement, and more generally to show the validity of the approximations made in the last part of the previous section. In particular we shall show that \( S_h \) and \( S_{ns} \) give only corrections of order \( 1/\beta^2_s \) to the string tension and moreover that the coefficient of this \( 1/\beta^2_s \) term is very small and actually negligible with respect to other \( 1/\beta^2_s \) terms of the expansion. In order to show this let us first integrate over the field \( \phi \) and let us study the induced action \( S_{ind} \). Let us select one particular loop \( \Gamma \) in \( S_{ind} \). By using the fact that for SU(2) \( |Tr_U[\Gamma]|^2 = Tr_{su}[U[\Gamma]] \) and by taking into account that the same loop appears \( 2 l(\Gamma) \) times (two possible orientations and \( l(\Gamma) \) starting points), the corresponding Boltzmann factor can be written as \( \exp\{\lambda(\Gamma)Tr_{su}(U[\Gamma])\} \) with \( \lambda(\Gamma) = 1/(2m_R^2)^{(\Gamma)} \). Its character expansion is (setting \( \lambda(\Gamma) = \lambda \) for brevity):

\[ e^{\lambda} \chi_{su}^{(U[\Gamma])} = e^{\lambda} \left[ I_0(2\lambda) - I_1(2\lambda) \right] [1 + \sum_{j=1}^{\infty} \mu_j(\lambda)\chi_j(U[\Gamma])] \]  \hspace{1cm} (24)
where $\chi_j$ is the character of the $j^{th}$ irreducible representation and

$$
\mu_j(\lambda) = \frac{I_j(2\lambda) - I_{j+1}(2\lambda)}{I_0(2\lambda) - I_1(2\lambda)}
$$

so that

$$
\mu_j(\lambda) \sim \lambda^j \sim (2m^2_{R})^{-j/2}[\Gamma].
$$

For any loop $\Gamma$ (of length $l[\Gamma]$ and area $A$) and for any representation $j$ the first contribution is given by the insertion of the loop $\Gamma$ inside the Wilson loop (we assume to be in the limit of infinite area Wilson loops and neglect boundary corrections). One must take into account the two possible choices ($j - 1/2$ and $j + 1/2$) for the representation of the space-like plaquettes inside the loop $\Gamma$ and then subtract the “excluded vacuum” contribution. The result is:

$$
[\mu_j(\lambda)] \left\{ (j + 1) \left[ \frac{I_{2j+2}(\beta_s)}{I_2(\beta_s)} \right]^A + j \left[ \frac{I_{2j}(\beta_s)}{I_2(\beta_s)} \right]^A - (2j + 1) \left[ \frac{I_{2j+1}(\beta_s)}{I_1(\beta_s)} \right]^A \right\}
$$

$$
\sim [\mu_j(\lambda)] \frac{2A^2(j + 1)(2j + 1)}{\beta_s^2}
$$

As anticipated, both the constant and the $1/\beta_s$ terms vanish in eq.(27).

Diagrams in which several different loops $\Gamma$ are simultaneously present and overlap do not allow a similar compact expression, but also for them it can be shown, through an iterative elimination of all possible subdiagrams, that only $1/\beta_s^2$ terms survive. Let us call $B_{\text{ind}}(m^2_{R})/\beta_s^2$ the overall contribution (eq.(27) plus multiple insertions of $\Gamma$ loops) of $S_{\text{ind}}$ to the order $1/\beta_s^2$ to the string tension. Even if we cannot give the explicit value of $B_{\text{ind}}(m^2_{R})$, we can at least evaluate its order of magnitude.

A first, simple calculation can be made within the strong coupling expansion: we have checked by evaluating the first few orders that for $m_R > 1.5$ $B_{\text{ind}}$ is almost saturated by the terms of eq.(27). Even more, it can be seen that the sum is actually dominated by its first term, the single plaquette contribution. If we keep in the expansion only the simple plaquette ($l[\Gamma] = 4$, $A = 1$), and restrict ourselves to the $j = 1$ representation the contribution is $-12\mu_1(\lambda)/\beta_s^2 \sim -12/(2m_{R}^2)^4\beta_s^2$. This is exactly what one would find shifting $\beta_s \to (\beta_s + 8/(2m_{R}^2)^4)$ in eq.(22) as described in the last part of the previous section.

In order to test the order of magnitude of $B_{\text{ind}}(m^2_{R})$ beyond the first orders of the strong coupling expansion, let us look at its contribution to the space-like string tension. Taking also into account the next to leading order expansion of the Bessel functions, we have:

$$
a^2 \sigma = \frac{3}{2\beta_s} + \frac{3}{4\beta_s^2} + \frac{B_{\text{ind}}(m^2_{R})}{\beta_s^2}
$$

2Notice the different normalization with respect to the usual character expansions.
It is interesting to notice that the contribution coming from \( S_{\text{ind}} \) is opposite in sign to that due to the next to leading order expansion of the Bessel functions. This gives us a direct way (by simply looking to the sign of a possible \( T^2 \) term in \( \sigma(aT) \)) to test the order of magnitude of \( B_{\text{ind}}(m_{R}^2) \) with respect to the next to leading Bessel correction.

A \( T^2 \) dependence of \( \sigma(aT) \) implies a linear \( T \) dependence in \( l_0 \). A good parameter to test this behavior is \( \delta_l(aT) = l_0(aT)/l_0(aT_c) \), which we can estimate to be:

\[
\delta_{th}l_0(aT) = \left( 1 + \frac{a(T - T_c)}{2\beta} - \frac{2B_{\text{ind}}(m_{R}^2)a(T - T_c)}{3\beta} \right)
\]

(29)

where the second term comes from the next to leading order expansion of the Bessel function and the last term is contribution of \( S_{\text{ind}} \).

This behaviour can be tested with a set of very precise data on \( l_0(T) \) obtained by M. Teper at \( \beta = 9 \) \[19\]. The results are collected in Tab.1; \( \delta_{exp}l_0 \) is normalized at \( aT = 1/6 \). In order to make contact with eq. (19), one must use the fact that at \( \beta = 9, 1/(aT_c) = 5.65 \) \[3, 8\].

| \( N_t \) | \( l_0/a \) | \( \delta_{exp}l_0 \) |
|-----------|-------------|----------------|
| 2         | 7.21(12)    | 1.062(30)      |
| 3         | 7.32(7)     | 1.078(22)      |
| 4         | 7.08(6)     | 1.043(20)      |
| 5         | 6.92(8)     | 1.019(20)      |
| 6         | 6.79(8)     | 1.019(20)      |

Tab.I. \( l_0 \) and \( \delta l_0 \) as a function of the inverse temperature \( N_t = 1/aT \) in the \((2+1)\) dimensional \( SU(2) \) LGT at \( \beta = 9 \).

It is possible to see that the \( T \)-dependence of \( l_0 \) has exactly the same sign and order of magnitude of the next to leading term of the Bessel function and, as a consequence of this, that \( B_{\text{ind}}(m_{R}^2) \) must be very small and that it is well approximated by its first term in the strong coupling expansion even if all orders are taken into account.

5 \( SU(2) \) in \((3+1)\) dimensions

In \( d = 3 \) the validity of dimensional reduction has been already well established by G. Bali and collaborators in \[4\]. By using the set of very precise determinations of \( \sigma(aT) \) at \( \beta = 2.74 \) shown in Tab.II,
| $T/T_c$ | $\sqrt{\sigma}/T_c$ |
|---------|------------------|
| 2       | 1.97(3)          |
| 2.67    | 2.43(3)          |
| 4       | 3.28(2)          |
| 8       | 5.70(4)          |

**Tab.II.** Space-like string tension as a function of the temperature in the (3+1) dimensional SU(2) LGT at $\beta = 2.74$, taken from ref. [4].

they were able to show that the space-like string tension behaves, for $T > 2T_c$ as

$$\sqrt{\sigma(aT)} = (0.334 \pm 0.014) \ g^2(T)T \quad (30)$$

with

$$g^{-2}(T) = \frac{11}{12\pi^2} \log(T/\Lambda_T) \quad (31)$$

with $\Lambda_T = 0.050(10)T_c$. This result is in remarkable agreement with eq.(16) (remember that $\beta = 4/g^2$). This is indeed an impressive test of dimensional reduction. The only remaining problem is a discrepancy of a factor of two between the coupling constant $g^2(T)$, as extracted from (31), and the value quoted in ref. [1].

The only improvement that our analysis can add to this result is to clarify this last issue. The point is that, as we remarked earlier, P.Lacock and collaborators define their reduced theory exactly as if they had chosen $N_t = 1$. This means that if we want to compare the coupling constant $g^2(T)$ with that of ref. [4], we must extrapolate $N_t \to 1$. This can be achieved by using eqs.(30,31). One obtains

$$\sqrt{\sigma(aT = 1)/T_c} = 9.97(8)$$

which would correspond to $g^2 = 1.86$. This result is still rather far from the value $g^2 \sim 1.31$ quoted by P.Lacock and collaborators. But this is not the end of the story. If we want to make contact with the dimensionally reduced theory we must compare our data with the exact behaviour of the string tension of the three dimensional SU(2) model, namely we must also take into account the $1/\beta^2$ term in eq.(16), which, due to the fact that $\beta$ is not very large, turns out to be rather important. This is similar to what we did in the $d = 2$ case taking into account the next to leading order of the Bessel function. Inserting our value of $\sqrt{\sigma}$ in eq.(16), and solving with respect to $\beta$ we find $\beta = 2.79(6)$, namely $g^2 = 1.43(3)$, where the quoted error takes also into account the uncertainty in the second coefficient of eq.(16). The main reason of interest of this result is that it is a further, independent, cross-check of the proposed dimensional reduction framework,

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3We have chosen the one loop formula of ref. [4] so as to compare with the $\Lambda(\epsilon)/\Lambda$ correction (see next section) which is also a one loop result.

4Obtained by interpolating the two results at $\beta_4 = 2.80$ and $\beta_4 = 2.50$ (in their notations) quoted in [4].
since it coincides, within the errors, with the value of the coupling in the (3+1) dimensional model. Moreover it almost fills the gap with the value of $g^2$ found by P.Lacock and collaborators.

As a last remark let us notice that the corrections coming from $S_h$ and $S_{ns}$ are again very small and essentially negligible. Let us again truncate the induced action to its first term, then its contribution can be encoded in the shift $\beta \to \beta + 8/(2m_R^2)^4$.

In this case, thanks to [1], precise values of the constants $h$ and $k$ are known, and $m_R^2$ turns out to be $m_R^2 \sim 2.3$. Inserting this value it is easy to see that the the correction is smaller than the quoted error of $\beta$.

6 Conclusions and Speculations.

In this paper we have shown that a simple dimensional reduction scheme can describe rather well space-like observables at high temperature. We gave arguments to support the conjecture that only the space-like part of the original action (after a suitable rescaling of the couplings) is relevant for the result. All the results quoted in the previous sections were rather straightforward applications of well known ideas (dimensional reduction) and techniques (Migdal-Kazakov approach to induced actions).

Let us now conclude with two more speculative, although very suggestive considerations.

\[ \textbf{Determination of } \Lambda_T \]

The crucial difference between the $d = 2$ and $d = 3$ cases is that in $d = 3$, $g$ is adimensional, and the rescaling of $N_t$ requires a proper $\Lambda(\epsilon)/\Lambda$ correction. While in $d = 2$ we had that $\sqrt{\sigma(aT)/T}$ was constant, namely $\frac{\sqrt{\sigma(aT_1)}}{T_1} \frac{T_2}{\sqrt{\sigma(aT_2)}} = 1$, we now expect (assuming $T_1 > T_2$):

\[ R \left( \frac{T_1}{T_c}, \frac{T_2}{T_c} \right) \equiv \frac{\sqrt{\sigma(aT_1)}}{T_1} \frac{T_2}{\sqrt{\sigma(aT_2)}} = \frac{\Lambda(T_1)}{\Lambda(T_2)} = \frac{T_1}{T_2} \].

(32)

The $\Lambda(\epsilon)/\Lambda$ correction was studied by F. Karsch in [20]. He was able to obtain in the $(g \to 0, \epsilon$ fixed) limit the following result

\[ \log \left( \frac{\Lambda(\epsilon)}{\Lambda} \right) = -\frac{c_\sigma(\epsilon) + c_\tau(\epsilon)}{4b_0} \],

(33)

where $c_\sigma(\epsilon)$ and $c_\tau(\epsilon)$ are two known functions (see [20]) whose main property is that in the limit $g \to 0$:

\[ \left. \frac{\partial c_\sigma}{\partial \epsilon} \right|_{\epsilon=1} + \left. \frac{\partial c_\tau}{\partial \epsilon} \right|_{\epsilon=1} = b_0 \]

(34)
with \( b_0 = 11/24\pi^2 \). Some particular values of eq.(33) are: \( \Lambda(4/3)/\Lambda \sim 0.925 \), \( \Lambda(3/2)/\Lambda \sim 0.90 \), \( \Lambda(2)/\Lambda \sim 0.84 \).

By using the values of \( \sigma(T) \) listed above, we can construct the following ratios: \( R(4,2) = 0.832(17) \), \( R(8,4) = 0.869(12) \), \( R(2.67,2) = 0.924(20) \), \( R(4,2.67) = 0.901(15) \) which are in good agreement with the corresponding values of \( \Lambda(\epsilon)/\Lambda \).

We want to stress however that this agreement should be taken with some caution. The main point is that the two scales \( \Lambda_T \) and \( \Lambda(\epsilon) \) have a completely different origin. While eq.(31) is a perturbative result in \( \varphi^2 \), and as far as \( T \) is much larger than \( \Lambda_T \) it correctly describes the \( T \) dependence of \( \varphi \) and consequently of \( \sigma \) for any \( T \); our result is perturbative both in \( \varphi^2 \) and \( \varphi^2 \tau \) (in the notations of ref. [20]), namely in the space-like and time-like couplings. This means that in our case \( T \) must be as small as possible if we want to correctly describe the data.

This can be very explicitly seen if we try to extract from the known values of \( \Lambda(\epsilon)/\Lambda \) the scale \( \Lambda_T \).

This can be done by equating

\[
\Lambda(T_1)/\Lambda = \log \frac{T_2}{\Lambda_T}/\log \frac{T_1}{\Lambda_T}
\]

The values of \( \Lambda_T \) obtained in this way depend on \( T_1 \) and \( T_2 \). We obtain:

\[
\Lambda_T(2,2.67) = 0.057T_c, \quad \Lambda_T(2,4) = 0.053T_c, \quad \Lambda_T(2.67,4) = 0.070T_c, \quad \Lambda_T(4,8) = 0.105T_c.
\]

As expected these values of \( \Lambda_T \) are in good agreement with the one \( \Lambda_T = 0.050(10)T_c \) found by G. Bali and collaborators if small temperatures are chosen, but they rapidly grow as larger \( T \)'s are chosen. Notice, as a side remark, that the two loop value of \( \Lambda_T \) reported in ref. [4] is \( \Lambda_T = 0.076(13)T_c \), thus indicating that as higher perturbative terms are taken into account, the agreement between the two approaches can be extended at higher temperatures.

Let us conclude by noticing that the above result, namely the possibility to predict the value of the scale \( \Lambda_T \) from the known value of \( \Lambda(\epsilon)/\Lambda \) is quite general, it can be extended to any gauge group \( SU(N) \) and can be written in a rather elegant form in the \( g \to 0 \) limit. Making an expansion for small rescaling \( \epsilon = T_1/T_2 \sim 1 \) and choosing a reference temperature \( T_{ref} \), which should be the smallest possible value of \( T \) compatible with the various thresholds of the problem, we can use eq.(34) and we obtain \( \Lambda_T \sim e^{-4}T_{ref} \). Assuming, as reference temperature the threshold \( T_{ref} = 2T_c \) at which the scaling behaviour of eq.(31) is expected to hold, we find:

\[
\Lambda_T \sim 2e^{-4}T_c = 0.37T_c
\]

\( T_c \) and \( \sqrt{\sigma(0)} \) in three dimensions

In the (2+1) case dimensional reduction predicts that the space-like string tension must grow linearly with \( T \). It is rather interesting to notice that, in this case, this behaviour sets in already at \( T = T_c \), thus suggesting that the whole deconfined
phase could be described using dimensional reduction. This leads to the following speculation. Let us assume the idealized picture in which the space-like string tension is exactly constant as a function of $T$ in the region $T < T_c$, and then at $T = T_c$ sharply starts to rise linearly. This implies the relation:

$$\sigma(0) = aT_c \sigma(aT = 1) = \frac{3T_c}{2a\beta}$$

which can be easily generalized to the case of SU(N):

$$\sigma(0) = \frac{(N^2 - 1)T_c}{2a\beta}$$

Let us now assume the following adimensional relation

$$\sigma(0) = \frac{\pi T^2}{3}$$

which comes from the completely different context of the effective string approach to the interquark potential in LGT’s. This relation does not depend of the gauge group but only on the number of space-time dimensions of the model, hence it holds unchanged for all $N$’s. In the SU(2) case it is again verified only within 15% (see the comment at the end of [8] on this point). Combining (37) and (38) we can predict the value of the (zero temperature) string tension and the critical temperature for SU(N) gauge theories in (2+1) dimension as a function of the coupling constant $\beta$.

$$\sigma(0) = \frac{3(N^2 - 1)^2}{4\pi\beta^2}$$

$$T_c = \frac{3(N^2 - 1)}{2\pi\beta}$$

Up to our knowledge, there is only one numerical result (besides those on SU(2)) against which we can test these conjectures. It is the critical coupling at which the SU(3) model with $N_t = 2$ undergoes the deconfinement transition, which turns out to be $\beta_c \sim 8.17$ [22]. Setting $N = 3$ and $T_c = 2$ in eq. (40) we find $\beta = 7.64$ which is only 10% away from the numerical result.

It would be quite interesting to have some further independent checks of these predictions.

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5 It is easy to see looking at Teper’s data that this change of behaviour is not so sharp, notwithstanding this, the idealized picture described above is only 20% away from the real behaviour.
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