RATIONAL MAPS FROM PUNCTUAL HILBERT SCHEMES OF K3 SURFACES

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Abstract. The purpose of this article is to study dominant rational maps from punctual Hilbert schemes of length \( k \geq 2 \) of projective K3 surfaces \( S \) containing infinitely many rational curves. Precisely, we prove that their image is necessarily rationally connected if this rational map is not generically finite. As an application, we simplify the proof of C. Voisin’s [21] of the fact that symplectic involutions of any projective K3 surface \( S \) act trivially on \( \text{CH}_0(S) \).

In this paper, we will work throughout over the field of complex numbers \( \mathbb{C} \).

1. Introduction

Recall that a K3 surface \( S \) is by definition a smooth projective surface with trivial canonical bundle \( K_S = \Omega_S^2 \) and vanishing \( H^1(S, \Theta_S) \). The Hilbert scheme of zero-dimensional subschemes of length \( k \geq 2 \) on the K3 surface \( S \) will be denoted by \( S^[[k]] \).

Recall that a proper variety \( X \) is said to be uniruled (resp. rationally connected) if a general point \( x \in X \) (resp. two general points \( x, y \in X \)) is contained in the image of a non-constant map \( \mathbb{P}^1 \to X \). These are obviously birationally invariant properties. It is also clear that \( \text{CH}_0(X) = \mathbb{Z} \) for any rationally connected variety \( X \). When \( X \) is smooth, rational connectedness is equivalent to the a priori weaker condition that two general points can be joined by a chain of rational curves.

The following is the main result we obtain in this article:

**Theorem 1.1.** Let \( f : S^[[k]] \to B \) be a dominant rational map to any variety \( B \) with \( \dim B < \dim S^[[k]] \). Suppose \( S \) contains infinitely many (singular) rational curves, then either \( B \) is a point or rationally connected.

We remark that if such a dominant map exists, then \( B \) is necessarily irreducible as \( S^[[k]] \) is smooth. Using the natural map \( S^k \to S^[[k]] \), instead of dealing with \( S^[[k]] \to B \) it is equivalent to look at the maps \( S^k \to B \) which are invariant under the action of \( \mathfrak{S}_k \) on \( S^k \) by permutation. This gives the following reformulation of Theorem 1.1.

**Theorem 1.2.** Let \( f : S^k \to B \) be a dominant rational map which is symmetric, that is invariant under the action of the symmetric group \( \mathfrak{S}_k \) with \( \dim B < \dim S^k \). Suppose \( S \) contains infinitely many (singular) rational curves, then either \( B \) is a point or rationally connected.

Using results of Graber-Harris-Starr in [9], Theorem 1.1 is in fact equivalent to the following a priori weaker

**Theorem 1.3.** With the same notations and hypotheses in Theorem 1.1, \( B \) is either a point or uniruled.

We will illustrate the equivalence of Theorem 1.1 and Theorem 1.3 in Section 2 and prove Theorem 1.3 in Section 3.

As an application of Theorem 1.1, we will give an alternative proof of C. Voisin’s main result in [21].

**Theorem 1.4.** [C. Voisin] Suppose \( S \) is a projective K3 surface and \( \tau \) is a symplectic involution acting on \( S \), then \( \tau \) acts as identity on \( \text{CH}_0(S) \).
The motivation of the statement of Theorem 1.4 comes from Bloch and Beilinson’s ambitious conjecture, whose aim is to understand the still-mysterious relations between Chow groups of a smooth projective variety $X$ and the Hodge structure of its Betti cohomology groups. More precisely, their conjecture predicts the existence of a decreasing filtration $F$ on the torsion-free part of $H^1(X)$ satisfying $F^{i+1}H^i(S) = 0$ such that the graded parts of $H^i(X) \otimes \mathbb{Q}$ is controlled by $H^i(X)$ in some sense to be precise. The reader is referred to [19, Partie VII] for a general discussion of Bloch’s conjecture and the interplay between Chow groups and Hodge levels of Betti cohomology groups.

For smooth projective surfaces, since $CH_2(S) = \mathbb{Z}$ and $CH_1(S) = Pic(X)$ are well understood, only the group $CH_0(S)$ need to be studied in Bloch-Beilinson conjecture. The conjectural decreasing filtration on $CH_0(S)$ satisfying $F^kCH_0(S) = 0$ for $k > 2$ would be:

(i) $F^0CH_0(S) = CH_0(S)$;
(ii) $F^1CH_0(S) = CH_0(S)_{hom}$, the group of zero-cycles of degree zero modulo rational equivalence;
(iii) $F^2CH_0(S) = CH_0(S)_{alb} = Ker(alb_S : CH_0(S)_{hom} \to Alb(S))$.

The corresponding special case of Bloch-Beilinson conjecture, also known as generalized Bloch conjecture for surfaces is stated as follows:

**Conjecture 1.5.** Let $\Gamma \in CH^2(S \times X)$ be a correspondence between a smooth projective variety $X$ and a smooth projective surface $S$ such that the cohomological correspondences $\Gamma^* : H^0(X, \Omega^2_X) \to H^0(S, \Omega^2_S)$ vanish for $i > 0$, then $\Gamma^* : CH_0(X)_{alb} \to CH_0(S)_{alb}$ vanishes as well.

If we take $X = S$ and $\Gamma = \Delta_{S \times S}$, the graph of the identity map, we get the usual Bloch conjecture. The conjecture is known in this case for surfaces which are not of general type [3], for Godeaux surfaces, for Catanese and Barlow surfaces [20], etc. .

Another special case, which is more related to the present article, concerns surfaces $S$ with irregularity $q = 0$. If we take $X = S$ and $\Gamma = \Delta_{S \times S} - \Gamma_f$, where $\Gamma_f$ is the graph of a finite order automorphism $f : S \to S$ which acts as identity on $H^0(S, \Omega^2_S)$ via the pullback morphism, one obtains the following

**Conjecture 1.6.** If $S$ is a surface with $q = h^{0,1} = 0$ and $f : S \to S$ is an automorphism of finite order acting trivially on $H^0(S, \Omega^2_S)$, then the induced map $f^*$ acts as identity on $CH_0(S)$.

A series of examples of surfaces with $q = 0$ is provided by $K3$ surfaces $S$. Such a surface has one-dimensional $H^0(S, \Omega^2_S)$ generated by a non-degenerated holomorphic two-form $\eta$. An automorphism $f : S \to S$ such that $f^*\eta = \eta$ is called a symplectic automorphism.

A recent new advance of Conjecture 1.6 for $K3$ surfaces was made by D. Huybrechts and M. Kemeny in [11]. They worked with invariant elliptic curves and solved Conjecture 1.6 for $K3$ surfaces with symplectic involutions $f$ in one of the three series in the classification introduced by van Geemen and Sarti [1]. In [21], C. Voisin showed in general that symplectic involutions act trivially on $CH_0(S)$ for any projective $K3$ surface $S$. The general statement of Conjecture 1.6 for $K3$ surfaces is proved soon after in [10] by D. Huybrechts.

**Theorem 1.7.** [M. Kemeny, D. Huybrechts, C. Voisin] Let $S$ be a projective $K3$ surface, $\eta$ be a non-zero holomorphic two-form on $S$, and $f : S \to S$ be an automorphism of finite order on $S$. Suppose $f^*\eta = \eta$, then $f$ acts trivially on $CH_0(S)$.

As was shown by Nikulin in [15], the only possible orders of $f$ range from one to eight. In Huybrechts’ proof, he studied case by case according to these finitely many possible orders using derived technique and Garbagnati and Sarti’s classification results [8] on lattices of the invariant part $H^2(X, \mathbb{Z})^f$ of action of symplectic automorphism $f$ with prime order.
The main construction in Voisin’s paper is the factorization

$$
\begin{array}{ccc}
\mathcal{S}[\mathcal{O}] & \xrightarrow{\gamma} & \mathcal{H}_0(S) \\
\gamma & \downarrow & \gamma_0 \\
\mathcal{P}_{(S,\mathcal{H})}(\mathcal{E}/\mathcal{C}) & \xrightarrow{\gamma_0} & \mathcal{P}_{(S,\mathcal{H})}(\mathcal{E}/\mathcal{C})
\end{array}
$$

(1.1)

where $\mathcal{P}_{(S,\mathcal{H})}(\mathcal{E}/\mathcal{C})$ is the universal Prym variety associated to complete linear system of curves of genus $g$ in the quotient surface $S/\mathcal{I}$. Our main application of Theorem 1.2 is the following

**Theorem 1.8.** Any smooth projective compactification of $\mathcal{P}_{(S,\mathcal{H})}(\mathcal{E}/\mathcal{C})$ is rationally connected.

In Section 4, we will explain how this theorem gives an alternative proof of Theorem 1.4.

The organization of this paper is as follows. In Section 2, we will recall some well-known facts concerning rationally connected varieties that we need in this article. The proof of the equivalence between Theorem 1.1 and Theorem 1.3 using argument of maximally rationally connected fibration will also be done in Section 2. After proving this equivalence, we will prove Theorem 1.3 in Section 3.

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2. Remarks on rationally connected varieties

In this section, we first recall the definition of MRC-fibrations and introduce the main theorem of Graber-Harris-Starr in [9]. Then we will use them to prove the equivalence between Theorem 1.1 and Theorem 1.3. In the end, we will explain the well-known fact that rational connectedness is an open property, which will be used in the last section.

Recall from [12] that for any variety $X$, there exists a rational map $\phi : X \to B$ unique up to birational equivalence characterized by the following properties:

(i) general fibers of $\phi$ is rationally connected;

(ii) for a general point $b \in B$, any rational curve passing through $X_b := \phi^{-1}(b)$ is actually contained in $X_b$.

The map $\phi : X \to B$ is called the maximal rationally connected (or MRC for short) fibration of $X$. The fundamental question whether the base $B$ of an MRC-fibration is uniruled remained open for a while [13, Conjecture IV.5.6]. It was finally answered by T. Graber, J. Harris, and J. Starr as a direct corollary of their main theorem in their paper [9]:

**Theorem 2.1.** [Graber-Harris-Starr] Let $g : X \to C$ be a proper morphism between complex varieties where $C$ is a smooth curve. If the general fiber of $g$ is rationally connected, then $g$ has a section.

**Remark 2.2.** This theorem was latter generalized by Starr and de Jong to varieties defined over an arbitrary algebraically closed field: any proper morphism from a smooth variety to a smooth curve whose general fibers are smooth and separably rationally connected, has a section [5].

In particular, Theorem 2.1 implies:

**Corollary 2.3.** [Graber-Harris-Starr] Let $g : X \to Z$ be a maximal rationally connected fibration where $X$ is an irreducible variety, then $Z$ is not uniruled.

Let us recall for completeness the proof of the above corollary.
Proof. After resolving the rational map \( g \) and singularities of \( X \), we can assume that \( g \) is a morphism and \( X \) is smooth. Suppose that \( Z \) were uniruled, then there exists a rational curve \( C \) passing through a general point of \( Z \) and we can suppose that \( g \) is dominant on \( C \). Denote by \( \widetilde{C} \) the normalization of \( C \). Up to replacing \( X \times_{Z} \widetilde{C} \) by its desingularization, the map \( \gamma_{C} : X \times_{Z} \widetilde{C} \rightarrow \widetilde{C} \) has a section \( D \) by Theorem 2.1, which is also a rational curve. Thus if \( y \in D \), the rational curve \( D \) passes through \( y \) and is not contracted by \( g \), which is absurd because \( g : X \rightarrow Z \) is an MRC-fibration. \( \square \)

**Remark 2.4.** An equivalent formulation of Corollary 2.3 is the following: if \( f : X \rightarrow B \) is a dominant map such that both the general fibers of \( f \) and the base \( B \) are rationally connected, then \( X \) is rationally connected as well [13, Proposition IV.5.6.3].

Thanks to Corollary 2.3, we can easily show that

**Corollary 2.5.** Theorem 1.1 and Theorem 1.3 are equivalent.

**Proof.** Only the converse direction needs to be proved. Suppose \( B \) is not a point; let \( B \rightarrow B' \) be an MRC fibration of \( B \). Since \( B' \) is not uniruled by Corollary 2.3, \( B' \) is a point by Theorem 1.3. Hence \( B \) is rationally connected. \( \square \)

To end up this section, the last property we will need about rational connectedness in this article is its openness property:

**Lemma 2.6.** If \( f : X \rightarrow B \) is a smooth projective morphism, then the set

\[
B_{rc} := \{ b \in B \mid X_{b} = f^{-1}(b) \text{ is rationally connected} \}
\]

is Zariski open.

**Proof.** Recall that a smooth projective variety \( Y \) is rationally connected if and only if it contains a “very free” curve, that is, a rational curve \( f : C = \mathbb{P}^{1} \rightarrow Y \) such that \( f^{*}T_{Y} \otimes \mathcal{O}_{\mathbb{P}^{1}}(-1) \) is generated by its global section [6 Corollary 4.17]. It is not hard to see that equivalently,

\[
H^{1}(C, f^{*}T_{Y} \otimes \mathcal{O}_{\mathbb{P}^{1}}(-2)) = 0.
\]

Suppose \( X_{b} \) is rationally connected and \( C \) is a very free curve in \( X_{b} \). Since \( C \) has ample normal bundle \( N_{f} = f^{*}T_{X}/T_{C} \), one has \( H^{1}(C, N_{f}) = 0 \), so the morphism \( C \rightarrow X_{b} \) can be deformed with \( X_{b} \). We conclude that \( H^{1}(C, T_{X} \otimes \mathcal{O}_{\mathbb{P}^{1}}(-2)) = 0 \) for \( b' \in B \) near \( b \) by semi-continuity. \( \square \)

**Remark 2.7.** In fact \( B_{rc} \) is also a closed set, but we will not use this fact.

3. Proof of Theorem 1.3

Let us recall for convenience the statement of Theorem 1.3 that we will prove in this section:

**Theorem.** Let \( f : S^{k} \rightarrow B \) be a symmetric dominant rational map with \( \dim B < \dim S^{k} \). Suppose \( S \) is a projective K3 surface containing infinitely many (singular) rational curves, then either \( B \) is a point or uniruled.

**Remark 3.1.** Before we start the proof, we remark that any birational modification of the rational map \( f : S^{[k]} \rightarrow B \) and the base \( B \) will not affect the hypotheses and the conclusion in Theorem 1.3. For instance up to desingularization of \( B \), we can always suppose that \( B \) is smooth and complete. This kind of modifications will be repeatedly used in the proof of Theorem 1.3.

Let us first prove the following general result.

**Lemma 3.2.** Let \( f : X \rightarrow B \) be a rational dominant map between smooth projective varieties such that \( \dim X > \dim B \). If \( D \) is an ample divisor on \( X \), then the restriction map \( f|_{D} \) is still dominant provided \( B \) is not uniruled.
Proof. Let \( f : \widetilde{X} \rightarrow B \) be a resolution of \( f \) after a sequence of blow-ups \( \pi : \widetilde{X} \rightarrow X \) of \( X \). Denote by \( \widetilde{D} \) the proper transform of \( D \) and \( D' := \pi^{-1}(D) \). Since \( D \) is ample and \([D'] = \pi^*[D] \) in \( \text{Pic}(\widetilde{X}) \), one deduces that \( D' \) is a big divisor. Suppose \( k = \dim X - \dim B \) and denoted by \( H \in NS(\widetilde{X}) \) the class of an ample divisor of \( \widetilde{X} \). As \( f \) is dominant, the class \( \widetilde{f}^*[x] \cdot H^{k-1} \in NS(X) \) where \( x \) is a point of \( X \), is a class of a movable curve. So \([D'] \cdot \widetilde{f}^*[x] \cdot H^{k-1} > 0 \) [18] Corollary 2.5, hence the restriction \( \tilde{f}_{|D'} \) is dominant.

Now let \( y \) be a very general point of \( B \) such that there is no rational curve passing through \( y \). We know that there exists \( x \in D' \) such that \( y = f(x) \). As \( \pi(x) \in D \), the fiber \( E_x := \pi^{-1}(\pi(x)) \) intersects \( D \). On the other hand, this fiber is connected and rationally connected. As there is no rational curve passing through \( y \), the fiber \( E_x \) is contracted to \( y \) by \( f \). As \( E_x \) meets \( \widetilde{D} \), we conclude that \( \tilde{f}^{-1}(y) \cap \widetilde{D} \neq \emptyset \), so \( \tilde{f}_{|D} \) (hence \( f_{|D} \)) is dominant. \( \Box \)

Proof of Theorem 1.3 Let \( D \) be a rational curve lying in an ample linear system of \( S \) (whose existence is due to Bogomolov-Mumford [18]). Since \( \Theta(D)^k \) is ample on \( S^k \), we deduce by Lemma 3.2 that the restriction of \( f \) on \( \cup_{j=0}^{k-1} S^j \times S^{k-j} \) (with \( S^0 \times D \times S^{k-1} = D \times S^{k-1} \) and \( S^{k-1} \times D \times S^0 = S^{k-1} \times D \)) is dominant. As this union is finite, we can suppose without loss of generality that the restriction on \( D \times S^{k-1} \) of \( f \) is dominant. (In fact, since \( f \) is symmetric, this is always the case.)

Assuming \( B \) is not uniruled, we want to show that \( B \) is a point; since \( D \times S^{k-1} \rightarrow B \) is dominant, the rational curve \( D \times z \) is contracted to a point by \( f \) (whenever defined) for a general point \( z \in S^{k-1} \). Up to birational equivalence of \( B \), we can suppose that \( D \times z \) is contracted to a point for every point \( z \in S^{k-1} \) by Remark 3.1. Since \( D \times z \) is ample in \( S \times z \), the fibers of the restriction map \( f_z := f_{S \times z} \) have positive dimension (which is a consequence of Hodge index theorem). So either \( S \times z \) is contracted to a curve \( C_z \) or to a point.

Next, we show that if \( S \times z \) is contracted to a curve \( C_z \), then \( C_z \) is necessarily a rational curve. Indeed, let \( \{C_i\} \) be the set of rational curves in \( S \) and suppose \( f(C_i \times z) = p_i \in C_z \) for all \( C_i \). Since there are infinitely many rational curves in \( S \) and \( S \) is not uniruled, one would have \( \sup_{p_i} p_i \cdot D = \infty \), which is absurd as \( \sup_{C_i} \pi^* t \cdot D < \infty \).

Set \[ U := \bigcup_{S \times z \text{ contracted to a curve}} S \times z \in S \times S^{k-1}. \]

Since \( B \) is not supposed to be uniruled, the restriction of \( f \) on \( U \) is not dominant. Therefore up to birational modification of \( B \), we can suppose that \( f \) contracts \( S \times z \) to a point for any \( z \in S^{k-1} \). As \( f \) is symmetric, we deduce that for all \( 0 < i \leq k \) the map \( f \) contracts \( z \times S \times z' \) to a point for any \( z \in S^{k-1} \) and \( z' \in S^{k-i} \). Therefore the image of \( f \) is a point, and we are done. \( \Box \)

4. Triviality of symplectic involution actions on \( \text{CH}_0(S) \)

The aim of this section is to give an alternative proof of Theorem 1.4 that symplectic involutions of a K3 surface \( S \) act as identity map on \( \text{CH}_0(S) \) using Theorem 3.1. Before we start the proof, let us recall the definition of Prym varieties, which are used in an essential way both in Voisin’s proof and ours.

4.1. Prym varieties. Let \( \pi : \widetilde{C} \rightarrow C \) be an étale double cover of a smooth curve \( C \) and consider the involution \( i : \widetilde{C} \rightarrow \widetilde{C} \) that interchanges the preimages of any point \( p \in C \). This involution \( i \) induces an endomorphism on the Jacobian of \( \widetilde{C} \) denoted \( i : J(\widetilde{C}) \rightarrow J(\widetilde{C}) \), and the Prym variety of \( \widetilde{C} \rightarrow C \) is defined as \[ \text{Prym}_{\widetilde{C}/C} = \text{Im} (\text{Id} - i) = \text{Ker} (\text{Id} + i)^n. \]
It is an abelian variety carrying a principal polarization and it is easy to see that \( \text{Prym}_{C/\mathbb{C}} \) is also isomorphic to \( (\text{Ker}\pi)^{\circ} \), where \( \pi \) is the norm map \( \pi : J(C) \to J(C) \). Since \( \pi \) is surjective, we deduce that \( \dim \text{Prym}_{C/\mathbb{C}} = g - 1 \) where \( g \) is the genus of \( C \) by Riemann-Hurwitz formula.

4.2. Sketch of Voisin’s proof of Theorem 1.4. Let us outline Voisin’s proof of Theorem 1.4. First of all, she showed that for any correspondence \( \Gamma \in \text{CH}^2(X \times Y) \) between smooth varieties \( X \) and \( Y \) with \( \dim Y = d \), if there exist

(i) some point \( x \in X \) such that \( \Gamma_*[x] = 0 \) in \( \text{CH}_0(Y) \);

(ii) some \( g > 1 \) such that

\[
\Gamma_*\left( X^g \right) := \left\{ \sum_{i=1}^g \Gamma_*[x_i] \in \text{CH}_0(Y) \mid (x_1, \ldots, x_g) \in X^g \right\} = \Gamma_*\left( X^{g-1} \right),
\]

then \( \text{Im}(\Gamma_*) \) is finite dimensional in the sense of Roitman [16], which means that there exists a smooth projective variety \( V \), not necessarily connected, and a correspondence \( \Gamma' \subset V \times X \) such that

\[
\text{Im}(\Gamma_*) \subset \{ [\Gamma'_*[x] \mid x \in V] \}.
\]

In particular, if the conclusion of the above result holds for \( X = Y = S \) and \( \Gamma = \Delta_S - \Gamma' \in \text{CH}^2(S \times S) \) where \( \Gamma' \in S \times S \) is the graph of \( \iota \), by a generalization of Roitman’s theorem [21] (Theorem 2.3), the morphism \( \Gamma' : \text{CH}_0(S)_{\text{hom}} \to \text{CH}_0(S) \) factors through the Albanese morphism

\[
\text{CH}_0(S)_{\text{hom}} \xrightarrow{\Gamma_*} \text{CH}_0(S) \xrightarrow{\text{alb}} \text{Alb}(S) = 0
\]

(4.1)

Since \( \text{CH}_0(S) \) is torsion-free [17] and the restriction of \( \Gamma_* \) on the anti-invariant part of \( \iota \)

\[
\text{CH}_0(S)^\perp := \{ z \in \text{CH}_0(S) \mid \iota(z) = -z \} \subset \text{CH}_0(S)_{\text{hom}}
\]

is multiplication by two, one deduces that \( \text{CH}_0(S)^\perp = 0 \), hence \( \Gamma_* = \text{Id} \) on \( \text{CH}_0(S) \).

It is clear that the zero-cycle \( 0_S \in \text{CH}_0(S) \) introduced in [2] defined as the class of any point sitting in a rational curve in \( S \) satisfies condition (i). To see that there exists \( g > 0 \) such that \( \Gamma_*\left( X^g \right) = \Gamma_*\left( X^{g-1} \right) \), one uses the following construction. Let \( \Sigma := S/1 \) be the (singular) quotient surface of \( S \) by the involution \( \iota \). Choose a very ample line bundle \( H \in \text{Pic}(\Sigma) \) and assume that \( c_1(H)^2 = 2g - 2 \). Since the canonical line bundle \( K_{\Sigma} \) is trivial, the genus of smooth curves in \( |H| \) is \( g \) and \( h^0(H) = g + 1 \). Now let \( s = (s_1, \ldots, s_g) \in S^g \) be a general point and let \( \overline{s} = (\overline{s}_1, \ldots, \overline{s}_g) \) be its image in \( \Sigma^g \), then there exists a unique curve \( C_s \) in \( |H| \) passing through \( \overline{s}_1, \ldots, \overline{s}_g \). Since \( C_s \) is general in \( |H| \), it is a smooth curve so the inverse image \( \overline{C}_s \subset S \) is smooth, connected, and is an étale cover of \( C_s \), which contains \( s_1, \ldots, s_g \). One notices that \( \Gamma_*([s]) = \sum_i ([s_i] - \iota([s_i])) \in \text{CH}_0(S) \) does not depend on \( \text{alb}_s \sum_i ([s_i] - \iota([s_i])) \in \text{Prym}_{C_s/\mathbb{C}} \), thus we obtain the following factorization of \( \Gamma_* : S^g \to \text{CH}_0(S)^\perp \):

\[
S^g \xrightarrow{\Gamma_*} \text{CH}_0(S)^\perp \xrightarrow{p} \mathcal{P}(S^g_{/\mathfrak{U}})(\overline{U}/\mathfrak{G})
\]

(4.2)

where \( \mathfrak{U} \to U \subset |H| \) (resp. \( \overline{U} \to U \)) is the universal smooth curve over the Zariski open set \( U \) of \( |H| \) parametrizing smooth curves (resp. universal family of double coverings over \( U \)), \( \mathcal{P}(S^g_{/\mathfrak{U}})(\overline{U}/\mathfrak{G}) \) is the
corresponding universal Prym varieties over \( U \), and \( \gamma_5 \) is defined as

\[
\gamma_5(s) = \text{alb}_{\Sigma} \left( \sum_i (\mathbb{L}_i - \mathbb{L}[s]) \right).
\]

Using the above factorization and the fact that Prym varieties of \( \emptyset \)leal double covers of curves of genus \( g \) are of dimension \( g - 1 \), she showed that \( \Gamma_s(X^g) = \Gamma_s(X^{g-1}) \), which concludes the proof.

4.3. Moduli space of polarized K3 surfaces with symplectic involution. The main references for this subsection are [13] and [7]. Let \( S \) be a K3 surface and \( \iota : S \to S \) be a symplectic involution. The action of \( \iota \) on \( S \) has eight fixed points. If we blow up the eight singularities in the quotient surface \( \Sigma = \frac{S}{\iota} \), we will get another (smooth) K3 surface \( \Sigma \). The K3 surfaces obtained in this way are called Nikulin surfaces.

Let us denote by \( E_1, \ldots, E_8 \) the eight exceptional divisors of \( \Sigma \), which are all disjoint \((-2\)\)-curves. The coarse moduli space of polarized K3 surfaces of genus \( 2g - 1 \) with symplectic involution can be described by some parameter space of Nikulin surfaces "of genus \( g \)" with extra datum. More precisely, it can be described as

\[
\mathcal{M}^{2g-1}_{\text{syminv}} := \left\{ \left( \Sigma, H, \mathcal{O}_\Sigma(E_1 + \cdots + E_8) \right) \mid \Sigma \text{ a K3 surface, } H \text{ a nef divisor of } \Sigma \text{ with } c_1(H)^2 = 2g - 2, \right. \\
\left. E_1 \cdot E_j = -2h_{ij}, \ H \cdot E_i = 0, \ 1/2 \, \mathcal{O}(E_1 + \cdots + E_8) \in \text{Pic}(\Sigma) \right\}
\]

This is an irreducible variety of dimension 11. As the divisor \( H \) deforms with \( \Sigma \) and \( h^0(H) = g + 1 \), we can consider the \( \mathbb{P}^g \)-bundle \( \mathcal{V} \) over \( \mathcal{M}^{2g-1}_{\text{syminv}} \) defined by

\[
\mathcal{V} := \left\{ (\Sigma, H, \mathcal{O}(E), C) \mid C \in [H] \right\}.
\]

**Remark 4.1.** The construction of the universal Prym variety \( \mathcal{P}(S,H)(\mathcal{L}/\mathcal{E}) \) varies in family in the coarse moduli space \( \mathcal{M}^{2g-1}_{\text{syminv}} \). More precisely, let \( \mathcal{V}^{\text{sing}} \subset \mathcal{V} \) be the locus where \( C \in [H] \) is singular. We can define the Jacobian fibration \( \mathcal{J} \) over \( \mathcal{U} = \mathcal{V} \setminus \mathcal{V}^{\text{sing}} \) such that the fiber over \( C \in [H] \) is \( \mathcal{J}(\bar{C}) \) where \( \bar{C} \subset S \) is the pre-image of \( C \subset S/\iota \). As the action \( \iota_\ast : \mathcal{J}(\bar{C}) \to \mathcal{J}(\bar{C}) \) extends to \( \iota_\ast : \mathcal{J} \to \mathcal{J} \), the universal family of \( \mathcal{P}(S,H)(\mathcal{L}/\mathcal{E}) \) is just \( \text{Im}(\text{Id} - \iota_\ast) \subset \mathcal{J} \).

4.4. Proof of Theorem 4.4 Let \( S \) be any K3 surface admitting a non-trivial symplectic involution \( \iota \) and \( H \) any polarization of \( S/\iota \). Let \( \mathcal{M}_{\text{syminv},H} \subset \mathcal{M}^{2g-1}_{\text{syminv}} \) be the moduli space of deformations of the pair \((S,H)\), where \( g \) is the genus of smooth members of \([H]\) on \( S/\iota \). For such a pair, let \( \mathcal{F} \) be a smooth projective compactification of \( \mathcal{P}(S,H)(\mathcal{L}/\mathcal{E}) \). From the existence of the rational map \( \gamma_5 : S[\mathcal{L}] \to \mathcal{P}(S,H)(\mathcal{L}/\mathcal{E}) \) and Theorem 4.4 one now deduces

**Corollary 4.2.** Let \((S,H,\iota)\) be as above and be parametrized by a general point in \( \mathcal{M}_{\text{syminv},H} \); then \( \mathcal{F} \) is rationally connected.

**Proof.** From [11], we know that the Picard rank of a K3 surface parametrized by a very general point in \( \mathcal{M}_{\text{syminv},H} \) is odd (equal to nine). In particular, for a very general point in \( \mathcal{M}_{\text{syminv},H} \), the corresponding K3 surface \( S \) has infinitely many (singular) rational curves [14]. We have the dominant rational map \( \gamma_5 : S[\mathcal{L}] \to \mathcal{F} \) and \( \dim S[\mathcal{L}] = 2g + 2g - 1 = \dim \mathcal{F} \), and thus \( \mathcal{F} \) is in fact rationally connected by Theorem 4.4 for our very general polarized K3 surface \( S \). As rational connectedness is an open property and the construction of \( \mathcal{P}(S,H)(\mathcal{L}/\mathcal{E}) \) and \( \mathcal{F} \) can be done in family over a nonempty Zariski open set of \( \mathcal{M}_{\text{syminv},H} \), we conclude that for general K3 surface \((S,H)\) admitting a non-trivial symplectic involution \( \iota \), \( \mathcal{F} \) is rationally connected. \( \square \)
Proof of Theorem 1.4. Let us first make the factorization of the incidence correspondence. Notice that \( \Gamma_r : \mathcal{S}^{(d)} \to \text{CH}_0(\mathcal{S}) \) factors through

\[
\Gamma_r : \text{CH}_0(\mathcal{S}^{(d)}) \to \text{CH}_0(\mathcal{S}),
\]

where \( \Gamma := I - (\text{Id}, i)(I) \in \text{CH}^2(\mathcal{S}^{(d)} \times \mathcal{S}) \).

**Lemma 4.3.** There exists a codimension 2 correspondence \( \Gamma' \in \text{CH}^2(\mathcal{F} \times \mathcal{S}) \) such that

\[
\Gamma_r \circ \gamma_s = \Gamma' : \text{CH}_0(\mathcal{S}^{(d)}) \to \text{CH}_0(\mathcal{S}).
\]

**Proof.** From the definition of \( \gamma_s \), it suffices to show that there exists \( \Gamma' \in \text{CH}^2(\mathcal{F} \times \mathcal{S}) \) such that the morphism \( p : \mathcal{S}(\mathcal{E}/\mathcal{E}) \to \text{CH}_0(\mathcal{S}) \) introduced in 4.2 factors through \( \Gamma' : \text{CH}_0(\mathcal{F}) \to \text{CH}_0(\mathcal{S}) \). Let \( \mathcal{U} \) and \( \mathcal{C} \) be some smooth projective compactifications of \( \mathcal{U} \) and \( \mathcal{C} \). Let \( \mathcal{F} \) be the restriction to \( \mathcal{S}(\mathcal{E}/\mathcal{E}) \times \mathcal{U} \) of the universal Poincaré divisor (unique up to linear equivalence) in \( \mathcal{F} \times \mathcal{U} \). We may assume that the morphism \( \mathcal{S}(\mathcal{E}/\mathcal{E}) \to \mathcal{U} \) extends to a morphism \( \mathcal{F} \to \mathcal{U} \) and we denote by \( \mathcal{C} \) the closure of \( \mathcal{F} \) in \( \mathcal{F} \times \mathcal{U} \). The inclusions of each fiber of \( \mathcal{C} \to \mathcal{U} \) in \( \mathcal{S} \) define a map \( \mathcal{S}(\mathcal{E}/\mathcal{E}) \times \mathcal{S} \to \mathcal{S}(\mathcal{E}/\mathcal{E}) \times \mathcal{U} \), hence a rational map \( \phi : \mathcal{F} \times \mathcal{U} \to \mathcal{F} \times \mathcal{S} \). Let

\[
\begin{array}{ccc}
\mathcal{F} \times \mathcal{U} & \xrightarrow{q} & \mathcal{F} \times \mathcal{S} \\
\vert & & \vert \\
\mathcal{C} & \xrightarrow{r} & \mathcal{F} \times \mathcal{S}
\end{array}
\]

be a resolution of \( \phi \) and set \( \Gamma' = r \circ q \circ \mathcal{C} \). Since the Chow groups of 0-cycles are invariant by birational modification, the induced correspondence \( \Gamma' : \text{CH}_0(\mathcal{F}) \to \text{CH}_0(\mathcal{S}) \) is independent of choice of all compactifications appeared in the proof and resolutions of \( \phi \), which gives a factorization of \( p : \mathcal{S}(\mathcal{E}/\mathcal{E}) \to \text{CH}_0(\mathcal{S}) \) by construction.

\( \square \)

**Corollary 4.4.** The morphism \( \Gamma_r : \text{CH}_0(\mathcal{S}^{(d)}) \to \text{CH}_0(\mathcal{S}) \) is identically zero.

**Corollary 4.4** implies Theorem 1.4 by the following factorization

\[
\begin{array}{ccc}
\mathcal{S}^{(d)} & \xrightarrow{\Gamma_r} & \text{CH}_0(\mathcal{S}) \\
\downarrow & & \downarrow \gamma_s \\
\text{CH}_0(\mathcal{S}^{(d)}) & \xrightarrow{0} & \text{CH}_0(\mathcal{F})
\end{array}
\]

using the fact that for a point \( z \in \mathcal{S}^{(d)} \) corresponding to a subscheme \( Z \) of \( S \), the classes \([z] \in \text{CH}_0(\mathcal{S}^{(d)})\) and \([Z] \in \text{CH}_0(\mathcal{F})\) satisfy \( \Gamma_r[z] = [Z] - r_*[Z] \).

\( \square \)

**Proof of Corollary 4.4.** Using Lemma 4.3, \( \Gamma_r \) factors through \( \text{CH}_0(\mathcal{F}) \). As \( \mathcal{F} \) is rationally connected by Corollary 4.2 one has \( \text{CH}_0(\mathcal{F}) = \mathbb{Z} \). Hence \( \Gamma_r[z] \) is independent of \( z \in \mathcal{S}^{(d)} \) where \( z \) again denotes the class of \( z \) in \( \text{CH}_0(\mathcal{S}^{(d)}) \). By choosing for \( z \) an \( i \)-invariant 0-cycle, we conclude that \( \Gamma_r[z] = 0 \) for any \( z \).

\( \square \)

**Remark 4.5.** It is tempting to ask whether one could apply this method to symplectic automorphisms \( \sigma \) with arbitrary finite order \( d > 2 \) instead of symplectic involutions. Unfortunately, this method fails to generalize for the reason of dimension. Indeed, choosing a very ample line bundle \( H \) in the quotient K3 surface \( S/(f) \) such that \( c_1(H)^2 = 2g - 2 \), then exactly as above, for a general point \( s = (s_1, \ldots, s_d) \in S^d \) there exists a unique smooth curve \( C_s \in |H| \) of genus \( g \) such that its inverse image \( \tilde{C}_s \) in \( S \) contains \( s_1, \ldots, s_d \).
Taking
\[ \Gamma_\ast(s) = \sum_i (s_i - f_\ast[s_i]), \]
one could again construct the factorization
\[ S[g] \xrightarrow{\Gamma_\ast} \mathcal{P}_{(S,H)}(\tilde{C}/C) \]
with this time, the fiber of the universal Prym variety \( \mathcal{P}_{(S,H)}(\tilde{C}/C) \to U \) over a smooth curve \( C \in U \) is the Prym variety of the étale cyclic covering \( \pi : \tilde{C} \to C \) induced by the quotient map \( S \to S/\langle f \rangle \) and is defined by
\[ \text{Prym}_{\tilde{C}/C} = \text{Im}(\text{Id} - \sigma) = \text{Ker}(\text{Id} + \sigma + \cdots + \sigma^{d-1}), \]
where \( \sigma : f(\tilde{C}) \to f(\tilde{C}) \) is the induced map on the Jacobian. Hence
\[ \dim \mathcal{P}_{(S,H)}(\tilde{C}/C) = (d-1)(g-1) + g \geq 2g = \dim S[g], \]
so we cannot apply Theorem [1.1]

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