SOME INEQUALITIES FOR ALTERNATING KUREPA’S FUNCTION

Branko J. Malešević

In this paper we consider alternating Kurepa’s function $A(z)$ [4]. We give some recurrent relations for alternating Kurepa’s function via appropriate sequences of rational functions and gamma function. Also we give some inequalities for the real part of alternating Kurepa’s function $A(x)$ for values of argument $x > -2$. The obtained results are analogous to results from [5].

1. Alternating Kurepa’s function $A(z)$

R. Guy considered, in the book [3] (p. 100.), the function of alternating left factorial $A(n)$ as an alternating sum of factorials $A(n) = n! - (n-1)! + \ldots + (-1)^{n-1}1!$. Let us use the standard notation:

$$A(n) = \sum_{i=1}^{n} (-1)^{n-i}i!.$$  

(1)

Sum (1) corresponds to the sequence A005165 in [6]. An analytical extension of the function (1) over the set of complex numbers is determined by the integral:

$$A(z) = \int_{0}^{\infty} e^{-t} t^{z+1} - (-1)^{z} t \frac{dt}{t+1},$$

(2)

which converges for $\text{Re } z > 0$ [4]. For function $A(z)$ we use the term alternating Kurepa’s function. It is easily verified that alternating Kurepa’s function is a solution of the functional equation:

$$A(z) + A(z-1) = \Gamma(z+1).$$

(3)

Let us observe that since $A(z-1) = \Gamma(z+1) - A(z)$, it is possible to make the analytical continuation of alternating Kurepa’s function $A(z)$ for $\text{Re } z \leq 0$. In that way, the alternating Kurepa’s function $A(z)$ is a meromorphic function with simple poles at $z = -n$ ($n \geq 2$) [4].

Let us emphasize that in the following consideration, in the sections 2. and 3., it is sufficient to use only fact that function $A(z)$ is a solution of the functional equation (3). In section 4., we give some inequalities for the real part of alternating Kurepa’s function $A(x)$ for values of argument $x > -2$.

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2. Representation of the alternating Kurepa’s function
via sequences of polynomials and gamma function

Let us introduce a sequences of polynomials:

\[ p_n(z) = (z - n + 1)p_{n-1}(z) + (-1)^n, \]

with initial member \( p_0(z) = 1 \). Analogously to results from [2], the following statements are true:

**Lemma 2.1** For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \) we have explicitly:

\[ p_n(z) = (-1)^n \left( 1 + \sum_{j=0}^{n-1} \prod_{i=0}^{j} (-1)^{j-i}(z - n + i + 1) \right). \]

**Theorem 2.2** For each \( n \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus (\mathbb{Z}^- \cup \{0, 1, 2, \ldots, n-2\}) \) is valid:

\[ A(z) = (-1)^n A(z - n) + p_{n-1}(z) \cdot \Gamma(z-n+2). \]

3. Representation of the alternating Kurepa’s function
via sequences of rational functions and gamma function

Let us observe that on the basis of a functional equation for the gamma function
\( \Gamma(z + 1) = z\Gamma(z) \), it follows that the alternating Kurepa’s function is solution of the following functional equation:

\[ A(z + 1) - zA(z) - (z + 1)A(z - 1) = 0. \]

For \( z \in \mathbb{C} \setminus \{-1\} \), based on (7), we have:

\[ A(z - 1) = -\frac{z}{z + 1} A(z) + \frac{1}{z + 1} A(z + 1) = q_1(z) A(z) - r_1(z) A(z + 1), \]

for rational functions \( q_1(z) = -\frac{z}{z + 1}, r_1(z) = -\frac{1}{z + 1} \) over \( \mathbb{C} \setminus \{-1\} \). Next, for \( z \in \mathbb{C} \setminus \{-1, 0\} \), based on (7), we obtain:

\[ A(z - 2) = \frac{1}{z} A(z) - \frac{z-1}{z} A(z - 1) \]

\[ = \frac{1}{z} A(z) - \frac{z-1}{z} \left( -\frac{z}{z + 1} A(z) + \frac{1}{z + 1} A(z + 1) \right) \]

\[ = \frac{z^2 + 1}{z(z+1)} A(z) - \frac{z-1}{z(z+1)} A(z + 1) = q_2(z) A(z) - r_2(z) A(z + 1), \]

Letters \( p, q, r, g \) are printed in the funny italic \text{T}e\text{lX} font.
Some inequalities for alternating Kurepa's function

for rational functions $q_2(z) = \frac{z^2+1}{z(z+1)}$, $r_2(z) = \frac{z-1}{z(z+1)}$ over $\mathbb{C}\{-1,0\}$. Thus, for values $z \in \mathbb{C}\{-1,0,1,\ldots,n-2\}$, based on (7), by mathematical induction it is true:

$$A(z-n) = q_n(z)A(z) - r_n(z)A(z+1), \quad \text{(10)}$$

for rational functions $q_n(z), r_n(z)$ over $\mathbb{C}\{-1,0,1,\ldots,n-2\}$ which fulfill the same recurrent relations:

$$q_n(z) = \frac{z-n+1}{z-n+2} q_{n-1}(z) + \frac{1}{z-n+2} q_{n-2}(z) \quad \text{(11)}$$

and

$$r_n(z) = \frac{z-n+1}{z-n+2} r_{n-1}(z) + \frac{1}{z-n+2} r_{n-2}(z), \quad \text{(12)}$$

with different initial functions $q_{1,2}(z)$ and $r_{1,2}(z)$.

Based on the previous consideration we can conclude:

**Lemma 3.1** For each $n \in \mathbb{N}$ and $z \in \mathbb{C}\{-1,0,1,\ldots,n-2\}$ let the rational function $q_n(z)$ be determined by the recurrent relation (11) with initial functions $q_1(z) = -\frac{z}{z+1}$ and $q_2(z) = \frac{z^2+1}{z(z+1)}$. Thus the sequences $q_n(z)$ has an explicit form:

$$q_n(z) = (-1)^n \left( 1 + \sum_{j=1}^{n} \prod_{i=1}^{j} \frac{(-1)^i}{z+2-i} \right). \quad \text{(13)}$$

**Lemma 3.2** For each $n \in \mathbb{N}$ and $z \in \mathbb{C}\{-1,0,1,\ldots,n-2\}$ let the rational function $r_n(z)$ be determined by the recurrent relation (12) with initial functions $r_1(z) = -\frac{1}{z+1}$ and $r_2(z) = \frac{z-1}{z(z+1)}$. Thus the sequences $r_n(z)$ has an explicit form:

$$r_n(z) = (-1)^{n-1} \left( \sum_{j=1}^{n} \prod_{i=1}^{j} \frac{(-1)^i}{z+2-i} \right). \quad \text{(14)}$$

**Theorem 3.3** For each $n \in \mathbb{N}$ and $z \in \mathbb{C}\{-1,0,1,\ldots,n-2\}$ we have:

$$A(z) = (-1)^n \left( A(z-n) + ((-1)^n - q_n(z)) \cdot \Gamma(z+2) \right) \quad \text{(15)}$$

and

$$A(z) = (-1)^n \left( A(z-n) + r_n(z) \cdot \Gamma(z+2) \right). \quad \text{(16)}$$
4. Some inequalities for the real part of alternating Kurepa’s function

In this section we consider alternating Kurepa’s function $A(x)$, given by an integral representation (2), for values of argument $x > -2$. The real and imaginary parts of the function $A(x)$ are represented by:

$$\text{Re} \ A(x) = \int_0^\infty e^{-t} \frac{t^{x+1} - \cos(\pi x) t}{t+1} \ dt$$

and

$$\text{Im} \ A(x) = -\int_0^\infty e^{-t} \sin(\pi x) \frac{t}{t+1} \ dt.$$  

In this section we give some inequalities for the real part of alternating Kurepa’s function $A(x)$ for values of argument $x > -2$. The following statements are true:

**Lemma 4.1** The function:

$$\beta(x) = \int_0^\infty e^{-t} \frac{t^{x+1}}{t+1} \ dt,$$

over set $(-2, \infty)$ is positive, convex and fulfill an inequality:

$$\beta(x) \geq \beta(x_0) = 0.401\,855 \ldots$$

with equality in the point $x_0 = -0.108\,057 \ldots$.

**Proof.** For positive function $\beta(x) \in C^2(-2, \infty)$, on the basis of (19), the condition of convexity $\beta''(x) > 0$ is true. Next, based on (19), we can conclude $\lim_{x \to 0^+} \beta(-2 + \varepsilon) = +\infty$ and $\lim_{x \to +\infty} \beta(x) = +\infty$. Therefore, we can conclude that exists exactly one minimum $x_0 \in (-2, +\infty)$. Using standard numerical methods it is easily determined $x_0 = -0.108\,057 \ldots$ and $\beta(x_0) = 0.401\,855 \ldots$.

**Lemma 4.2** The function:

$$\gamma(x) = \int_0^\infty e^{-t} \frac{\cos(\pi x) \cdot t}{t+1} \ dt,$$

over set $(-2, \infty)$, is determined with:

$$\gamma(x) = (1 + e^{\text{Ei}(-1)}) \cdot \cos(\pi x) = 0.403\,652 \ldots \cdot \cos(\pi x).$$

where $\text{Ei}(t) = \int_{-\infty}^t \frac{e^u}{u} \ du \ (t < 0)$ is function of exponential integral ([1], 8.211-1).
**Lemma 4.3** The function \( \text{Re} \, A(x) \), over set \((-2, \infty)\), is determined as difference:

\[
\text{Re} \, A(x) = \beta(x) - \gamma(x).
\]

and has two roots \( x_1 = -0.015401\ldots \) and \( x_2 = 0 \). The function \( \text{Re} \, A(x) \) is positive over set:

\[
D_1 = (-2, x_1) \cup (0, \infty)
\]

and negative over set:

\[
D_2 = (x_1, 0).
\]

**Proof.** Let \( \beta(x) \) be function from lemma 4.1 and let \( \gamma(x) \) be function from lemma 4.2. For value \( x_2 = 0 \) it is true \( \beta(x_2) = \gamma(x_2) = 0.403652\ldots \), i.e. value \( x_2 = 0 \) is a root of function \( \text{Re} \, A(x) \). Let us prove that function \( \text{Re} \, A(x) \) has exactly one root \( x_1 \in (x_0, x_2) \), where \( x_0 = -0.108057\ldots \) is value from lemma 4.1. It is true \( \beta(x_0) = 0.401855\ldots > 0.380061\ldots = \gamma(x_0) \). Let us notice that \( \beta(x) \) is convex and increasing function over set \((x_0, x_2)\) and let us notice that \( \gamma(x) \) is concave and increasing function over same set \((x_0, x_2)\). Therefore, we can conclude that function \( \text{Re} \, A(x) \) has exactly one root \( x_1 \in (x_0, x_2) \). Using numerical methods we can determined \( x_1 = -0.015401\ldots \). On the basis of the graphs of the functions \( \beta(x) \) and \( \gamma(x) \) we can conclude that function \( \text{Re} \, A(x) \) has exactly two roots \( x_1 \) and \( x_2 \) over set \((-2, \infty)\). Hence, the sets \( D_1 \) and \( D_2 \) are correctly determined. \( \blacksquare \)

**Lemma 4.4** For \( x \in (-1, 1 + x_1] \cup [1, \infty) \) it is true:

\[
\Gamma(x + 1) \geq \text{Re} \, A(x),
\]

while the equality is true for \( x = 1 + x_1 \) or \( x = 1 \).

**Proof.** For \( x > -1 \) it is true:

\[
\Gamma(x + 1) \geq \text{Re} \, A(x) = \Gamma(x + 1) - \text{Re} \, A(x - 1) \iff \text{Re} \, A(x - 1) \geq 0.
\]

Right side of the previous equivalence is true for \( x - 1 \in (-2, x_1] \cup [0, \infty) \), i.e. \( x \in (-1, 1 + x_1] \cup [1, \infty) \). \( \blacksquare \)

In the following considerations let us denote \( E_a = (a, a + 2 + x_1] \cup [a + 2, \infty) \) for fixed \( a \geq -1 \).

**Corollary 4.5** For fixed \( k \in \mathbb{N} \) and values \( x \in E_k \) following inequality is true:

\[
\frac{\text{Re} \, A(x - k - 1)}{\Gamma(x - k)} \leq 1,
\]

while the equality is true for \( x = k + 2 + x_1 \) or \( x = k + 2 \).
In the next two proofs of theorems which follows we use the auxiliary sequences of functions:

\[ g_k(x) = \sum_{i=0}^{k-1} (-1)^{k+i} \Gamma(x + 1 - i) \quad (k \in \mathbb{N}), \]

for values \( x > k - 2 \). Let us notice that for \( x > k - 2 \) it is true:

\[ g_k(x) = \Gamma(x + 2) \cdot \tau_k(x). \]

Then, the following statements are true:

**Theorem 4.6** For fixed odd number \( k = 2n+1 \in \mathbb{N} \) and values \( x \geq k+1 \) the following double inequality is true:

\[ \frac{p_k(x)}{\frac{1}{P_k(x)} - 1} \cdot \left( -\tau_k(x) \right) \leq \frac{\text{Re} A(x)}{\Gamma(x + 2)} < \left( -\tau_k(x) \right), \]

while the equality is true for \( x = k+1 \).

**Proof.** Based on lemma 4.3, using theorem 3.3, the following inequality is true:

\[ \text{Re} A(x) \leq -g_{2n+1}(x), \]

for values \( x \in E_{k-2} \). On the other hand, based on (28), for values \( x \in E_{k-1} \) we can conclude:

\[
\begin{align*}
\frac{\text{Re} A(x)}{g_{2n+1}(x)} &= -1 + \frac{\text{Re} A(x - 2n - 1)}{\frac{1}{g_{2n+1}(x)} - 1} = -1 + \frac{\text{Re} A(x - 2n - 1)}{\Gamma(x - 2n)(g_{2n+1}(x) + 1)} \\
&= -1 + \frac{\text{Re} A(x - 2n - 1)/\Gamma(x - 2n)}{\frac{1}{g_{2n+1}(x)} - 1} \leq - \frac{p_{2n+1}(x)}{\frac{1}{g_{2n+1}(x)} - 1}.
\end{align*}
\]

From (32) and (33), using (30), the double inequality (31) follows for values \( x \geq k+1 \).

**Theorem 4.7** For fixed even number \( k = 2n \in \mathbb{N} \) and values \( x \geq k+1 \) the following double inequality is true:

\[ \tau_k(x) < \frac{\text{Re} A(x)}{\Gamma(x + 2)} \leq \frac{p_k(x)}{\frac{1}{P_k(x)} - 1} \cdot \tau_k(x), \]

while the equality is true for \( x = k+1 \).

**Proof.** Based on lemma 4.3, using theorem 3.3, the following inequality is true:

\[ \text{Re} A(x) \geq g_{2n}(x), \]
for values $x \in E_{k-2}$. On the other hand, based on (28), for values $x \in E_{k-1}$ we can conclude:

$$\frac{\text{Re} \ A(x)}{g_{2n}(x)} = 1 + \frac{\text{Re} \ A(x-2n)}{g_{2n}(x)} = 1 + \frac{\text{Re} \ A(x-2n)}{\Gamma(x-2n+1)(p_{2n}(x)-1)}$$

$$= 1 + \frac{\text{Re} \ A(x-2n)/\Gamma(x-2n+1)}{p_{2n}(x)-1} \leq \frac{p_{2n}(x)}{p_{2n}(x)-1}.$$

From (35) and (36), using (30), the double inequality (34) follows for values $x \geq k+1$.

**Corollary 4.8** For fixed number $k \in \mathbb{N}$ and values $x \geq k+1$ the following double inequality is true:

$$r_k(x) < (-1)^k \frac{\text{Re} \ A(x)}{\Gamma(x+2)} \leq \frac{p_k(x)}{p_k(x) - (-1)^k} \cdot r_k(x),$$

while the equality is true for $x = k+1$.

**Corollary 4.9** On the basis of theorems 4.6 and 4.7 we can conclude:

$$\lim_{x \to \infty} \frac{\text{Re} \ A(x)}{\Gamma(x+2)} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{\text{Re} \ A(x)}{\Gamma(x+1)} = 1.$$

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University of Belgrade, Faculty of Electrical Engineering, P.O.Box 35-54, 11120 Belgrade, Serbia & Montenegro
malesevic@etf.bg.ac.yu

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