Some prospective criteria for non-unique solutions of ordinary differential equations

Juxia Xiong, Jinzhao Wu, Ying Zhang and Qinggeng Jin

ABSTRACT
Ordinary differential equations (ODEs) have a wide range of potential applications in science and engineering with regard to nonlinear dynamic systems. Frequently, there is a focus upon locating unique solutions to ODEs, with non-unique solutions being viewed as potentially problematic. However, some have recognized the importance of examining the character of non-unique solutions as well in order to properly understand the behaviour of physical systems. In some areas of engineering, notably control theory, the latter concern has become pressing. In this paper, by studying the asymptotic stability of second-order ordinary differential systems, we present a theorem creatively and prove it strictly by two lemmas. Using the criteria for non-unique solutions of first-order ordinary differential equations at points of equilibrium, we can solve engineering problems effectively. The applicability of this novel approach to the solution of engineering problems is provided through an example relating to the optimization of finite time controllers.

ARTICLE HISTORY
Received 22 February 2019
Accepted 29 May 2019

KEYWORDS
Ordinary differential equations; non-unique solutions; control theory; finite time; asymptotic stability; equilibrium

1. Introduction
Ordinary differential equations (ODEs) have many important applications across fields as diverse as automatic control (Golnaraghi & Kuo, 2010), the design of electrical systems (Bellen, Guglielmi, & Ruehli, 1999), the calculation of trajectories (Betts, 1998), the stability of aircraft (Cook, 2012), missile flight-planning (Kuo, Soetanto, & Chiou, 2015), and chemical reaction processes (Gillespie, 1976). For all of these domains, a variety of engineering and scientific problems can be expressed as ODEs for which solutions can then be sought mathematically (Strogatz, 2015). Over recent decades, there has been a great deal of progress in the theoretical handling of ODEs, especially in relation to the determination of unique solutions (Cid, 2003; Flandoli, 2009; Pan, Wang, & Yan, 2012). However, research regarding non-unique solutions to ODEs, their characteristics, and their potential application has been much-neglected by comparison. Historically, there have been certain flurries of interest in non-unique solutions to ODEs over the years. Newton himself engaged in some discussion of them in Chapter 2 of his Methodus fluxionum et Serierum Infinitarum (Newton, 1671). In the 1920s and 1930s, a range of scholars took an interest in non-unique solutions for first-order ODEs (summarized and developed in Hamilton (1938)). In the 1960s and 1970s, a handful of researchers explored their relevance to problems regarding fluid dynamics (Zandbergen & Dijkstra, 1977), the topological properties of systems (Bernfeld, 1972), and the character of unstable manifolds (Kelley, 1966). And in the 1980s, a series of related papers (Harrau & Weissler, 1982; Weissler, 1986) explored certain aspects of non-uniqueness in relation to heat equations from the perspective of asymptotic stability. In Agarwal and Lakshmikantham (1993), there is a specific discussion of uniqueness and non-uniqueness in ODEs. However, it turns out here that the focus is primarily upon criteria for uniqueness, not non-uniqueness. Thus, as a body of research, studies of non-uniqueness are, overall, both scant and inconclusive.

Some researchers have made a compelling case that is essential to understand the characteristics of non-unique solutions to ODEs because, without this understanding, it is not possible to fully model the behaviour of physical and dynamic systems (Gifford & Tomlinson, 1989; Yodovich, 2004). However, despite this recognition of their potential importance, theoretical understanding remains limited. As a result, existing theories are still a long way from being able to service an increasingly large number of requirements for this kind of understanding in practical engineering.
Generally speaking, when ODEs are Lipschitz continuous, there exists a unique solution to any initial value problem. However, by the same token, when ODEs are non-Lipschitz continuous, it is possible for there to be no unique solutions. Recently, in the field of control theory, there has been a burgeoning interest in finite time stability, and within this domain, the prospective properties of non-unique solutions at equilibrium points (i.e. the origin) (Bhat & Bernstein, 2002; Coron, 2006; Hong, Huang, & Xu, 1999). In relation to this interest, this paper presents a new way of arriving at the sufficiency conditions for a specific kind of ODEs with non-unique solutions. The method is framed around the asymptotic stability of second-order system equilibrium points.

In order to define the problem, this paper is seeking to address, let us take the following ordinary differential equation:

\[ q(z) \frac{dw(q(z))}{dz} = g(z, w(q(z))) \]

\[ q(0) = 0 \]

and the second-order system

\[ x_1 = f(x_2) \]

\[ x_2 = g(x_1, x_2) \]

where \( f \) and \( g \) are both continuous functions and \( w \) is the inverse function of \( f \).

Assuming local conditions, let us now consider the continuous solutions of Equation (2) and examine the following hypotheses:

H1: If the second-order system in Equation (2) has a unique equilibrium point \((0, 0)\) and is asymptotically stable, then, when \( x_2 \neq 0 \), \( f \) and \( g \) will be Lipschitz continuous.

H2: \( f(x_2) \) increases strictly monotonically.

H3: When \( x_1 \neq 0 \), then \( x_1 g(x_1, 0) < 0 \).

H4: The ordinary Equation (1) has a solution \( q(z) \).

We will verify these hypotheses in the next steps of this paper to prove that Equation (1) is generative with non-unique solutions. In turn, this will enable us to examine some of the characteristics of those solutions.

2. Theorem and proofs

**Theorem:** According to the hypotheses H1–H4, the ordinary equation (1) will have an infinite number of solutions.

Before the main proof of this theorem, we need to prove two lemmas. The lemmas also make clear how the non-unique solutions arise and some of their characteristics.

These lemmas provide the grounds upon which the overall theorem can be proved.

**Lemma 2.1:** Taking H1–H4 as conditions, let us suppose that \((x_1(t), x_2(t))\) is one of the possible solutions of the second-order system (2), in this case, setting time by \( x_1(T) = x_2(T) = 0 \). There will then exist an \( S \) such that \( x_1(t)x_2(t) < 0 \) for \( S < t < T \), where \( 0 < S < T \).

**Proof of Lemma 2.1:** Taking the Cartesian coordinate system \( x_1Ox_2 \), let us examine the tendency of the trajectories for the solution \((x_1(t), x_2(t))\). We shall do this by looking at a series of cases and explicating the particular workings of each case.

Case 1: The tendency of the trajectories in the first quadrant

Supposing that \( t_1 > 0 \) and \( x_1(t_1) > 0, x_2(t_1) > 0 \), so \( x_1(t_1) = f(x_2(t_1)) > 0 \). In that case, there will exist a \( \delta > 0 \), \( \delta > 0 \) when \( t_1 < t < t_2 + \delta, x_1(t) > x_1(t_1) > 0 \). There will then exist a \( T_1 > t_1 \) such that \( x_2(T_1) = 0 \) because \( \lim_{t \to -\infty} x_1(t), x_2(t) = 0 \), which implies that the trajectories of \((x_1(t), x_2(t))\) intersect the \( x_1 \) axis. Let \( T_1 = \min\{t | t > t_1, x_2(t) = 0\} \), then \( x_1(T_1) > x_1(t_1), x_2(T_1) = 0, x_2(T_1) = g(x_1(T_1), 0) < 0 \). So, the trajectories must enter the fourth quadrant from the first quadrant.

Case 2: The tendency of the trajectories in the fourth quadrant

C2.1: When \( x_1 > 0 \) then \( g(x_1, 0) < 0 \). This being so, the trajectories cannot pass along the positive \( x_1 \) axis and enter the first quadrant from the fourth quadrant.

C2.2: When a trajectory tends towards the origin, it cannot actually pass through the origin because the origin is an equilibrium point.

C2.3: When the trajectories pass along the negative \( x_2 \) axis to enter the third quadrant, then it will be the same as in Case 1 and the trajectories will enter the second quadrant.

Case 3: The tendency of the trajectories in the second quadrant

This is the same as Case 2. Here:

C3.1: A trajectory cannot pass along the negative \( x_1 \) axis to enter the third quadrant from the second quadrant.

C3.2: If a trajectory tends towards the origin, it cannot pass through the origin because of the origin being an equilibrium point.

C3.3: If the trajectories pass through the positive \( x_2 \) axis, then they will enter the first quadrant.
**Summarizing the cases**

Summing up across all of these cases, the tendency of the trajectories display two phenomena:

1. A tendency towards the origin in the second or fourth quadrant.
2. A tendency towards the origin by spiraling around the origin.

In the case of the second tendency, there is a sequence $t_i, i = 1, 2, \ldots$, where $t_i + 1 > t_i, \lim_{t \to \infty} t_i = \infty$, $x_1(t_i) = 0$, $x_1(t_i)x_1(t_i+1) = f(x_2(t_i))f(x_2(t_i+1)) < 0$, and we know $q(0) = 0$ so

$$\dot{x}_1(t_i)\dot{x}_1(t_i+1) = [x_1(t_i) - q(x_1(t_i))][\dot{x}_1(t_i+1) - q(x_1(t_i+1))] < 0 \quad (3)$$

If we now consider the function $H(x_1, x_1) = \dot{x}_1 - q(x_1)$, point $(x_1, \dot{x}_1)$ is along a trajectory for Equation (2) that passes through the origin. It is clear, in that case, that $H(x_1, \dot{x}_1)$ is a continuous function and that there exists a point $(x_1(s), \dot{x}_1(s))$ such that $H(x_1(s), \dot{x}_1(s)) = 0$, where $t_i < s < t_i+1$.

Let us examine this case further:

(i) If $x_1(s) = 0$ then $q(x_1(s)) = \dot{x}_1 = 0$. So, $g(\dot{x}_1(s), w(q(x_1(s)))) = 0 = g(x_1(s), x_2(s))$. This means that $(x_1(s), x_2(s)) \neq (0, 0)$ is the equilibrium point of second-order system (2). The contradiction here confirms hypothesis H1.

(ii) When $\dot{x}_1(s) \neq 0$, point $(x_1(s), \dot{x}_1(s))$ is on the trajectory of (2). In that case we can assume $\dot{x}_1(t_i)) > 0, x_1(s) > 0$. If we now suppose $\dot{x}_1(t_i) = q(x_1(t_i))$, then

$$x_2 = w(q(x_1(t_i)))$$

$$\dot{x}_2 = \frac{dw(q(x_1(t_i)))}{dx_1} \dot{x}_1 = q(x_1(t_i)) \frac{dw(q(x_1(t_i)))}{dx_1} = g(x_1, x_2)$$

Now we can say that $\dot{x}_1 = q(x_1(t))$ will suffice for the second-order system (2), which passes through the point $(x_1(s), \dot{x}_1(s))$.

According to H1, the solution that passes through $(x_1(s), \dot{x}_1(s))$ will be unique. Let us suppose that this trajectory intersects $\dot{x}_1 = 0$ at $(x_1(t_i'), \dot{x}_1(t_i'))$, such that $\dot{x}_1(t_i) > 0, x_1(t_i) > 0$. This means that $H(x_1(t_i'), \dot{x}_1(t_i'))$, $x_1(t_i')$ is a continuous function. So, $\dot{x}_1(t_i') = 0$, and as with (i), this contradiction again confirms that hypothesis H1 holds.

Given (i) and (ii), if $H(x_1, \dot{x}_1)$ is evaluated along the trajectory of (2), it cannot change sign. In that case the trajectory must be aligned with the first tendency, which means there is an $S$ where $0 < S < T$, such that $x_1(t)f(\dot{x}_2(t)) < 0, x_1(t)x_1(t) < 0$, where $S < t < T$.

Thus Lemma 2.1 can be considered to be proved.

**Lemma 2.2:** According to Lemma 2.1, there is a $r \in R^+$, a function $h$ that is continuous and that moves evenly away from 0 such that $x_1 = h(x_1), h(0) = 0$ for $r < t < T$.

**Proof of Lemma 2.2:** On the basis of Lemma 2.1 we know there is an $S$ satisfying $0 < S < T$ such that $x_1(t)x_1(t) < 0$ for $S < t < T$. It is clear that $x_1, \dot{x}_1$ cannot change sign when $t$ is used in this way. So, if $x_1(t)$ is positive, it will decrease as a function of $t$ and will increase if it is negative. Thus, no value of $x_1$ can be attained twice along the trajectory $(x_1(t), x_2(t))$ for $t > S$. This implies that $x_1(t)$ can be expressed as a function of $x_1(t) : \dot{x}_1(t) = h(x_1(t))$ for $t > S$. $h(x_1)$ here is smooth, except perhaps at 0, because $\dot{x}_1 = h(x_1)$, so $\dot{x}_1 = (dh/dx_1)x_1$ or $dh/dx_1 = \dot{x}_1/x_1$, and $x_1(t)/\dot{x}_1(t)$ is continuous (except perhaps at $x_1 = 0$). This being so $x_1 = 0$ is on the trajectory only when $x_1 = 0$. Thus, $h$ is continuous at zero and Lemma 2.2 is proved.

**Proof of the Theorem:** With the preceding lemmas in hand we shall now prove the principal theorem:

When $(x_1, x_2) \neq (0, 0)$, then the solution of system (2), $(x_1(t), x_2(t))$ will suffice where $0 < S < t < T \leq \infty$ and thus $x_1(t) = h(x_1(t))$, where $x_1(t)$ increases or decreases monotonically to zero.

So we have

$$x_2 = w(x_1) = w(h(x_1))$$

$$\dot{x}_2 = \frac{dw(h(x_1))}{dx_1} \dot{x}_1 = h(x_1) \frac{dh(h(x_1))}{dx_1} = g(x_1, w(h(x_1)))$$

If we suppose $x_1 = z$, then

$$h(z) \frac{dh(z)}{dz} = g(z, w(h(z))).$$

There are, then, different initial values for different solutions of the second-order system (2) that can be used to arrive at the different functions for $h(z)$. In that case, if we set it so that $q(z) = h(z)$, the solution of the ordinary differential equation (1) will have non-unique solutions.

In order to test the theorem, in the next section, we shall apply it to a specific example relating to finite time control.

**3. Example**

The following ODE is taken from Haimo (1986), where it was used to demonstrate how continuous finite time differential equations can be used as effective controllers for
where \( r_1, r_2 > 0, 0 < \sigma_1 < \frac{1}{2} \).

Although Equation (4) has an infinite number of possible solutions, there was no mention of this fact in the original publication. This reflects the issue raised at the beginning of this paper regarding a tendency to focus on unique rather than non-unique solutions to ODEs and the general lack of published work that discusses the specific grounds that may give rise to them.

First of all, we are going to look at the following second-order system associated with Equation (4):

\[
\begin{align*}
\dot{x}_1(t) &= x_2 \\
\dot{x}_2 &= -k_1 x_1 - k_2 \text{sign}(x_2)^2
\end{align*}
\]  

(5)

where \( k_1 = \frac{1}{2} r_1, k_2 = \frac{1}{2} r_2, \sigma_2 = 2\sigma_1 \). We can define \( \text{sign}^2 z = |z|^2 \text{sgn}(z) \) for \( z \) and \( a \) in \( \mathbb{R} \). If we now construct the Lyapunov function,

\[
v = \left( \frac{k_1}{2} \right) x_1^2 + \frac{1}{2} x_2^2,
\]

\[
\dot{v} |_{z(2)} = k_1 x_1 x_2 + x_2 ( - k_1 x_1 - k_2 \text{sign}(x_2)^2 ) = - k_2 \text{sign}(x_2)^2 \leq 0
\]

\[
\dot{v} |_{z(2)} = 0 \Rightarrow x_2 = 0 \Rightarrow x_1 = 0.
\]

In the theoretical treatment of asymptotic stability in Liao (1999), the second-order system (5) was proposed to be asymptotically stable. Thus, (5) should meet the conditions expressed in H1. Further inspection reveals that system (5) also satisfies the conditions expressed in H2 and H3.

On the basis of (5), it is possible to arrive at an ODE that is similar in form to (1):

\[
q(z) \frac{dw(q(z))}{dz} = -k_1 z - k_2 |q(z)|^2 \text{sgn}(q(z))
\]

(6)

\[q(0) = 0\]

If we now let \( q(z) = -\sqrt{p(z)} \), then \( \frac{1}{2} (dp(z)/dz) = -k_1 z + k_2 |p(z)|^{2/2} \).

In that case

\[
\frac{dp(z)}{dz} = -r_1 z + r_2 |p(z)|^{\sigma_1}
\]

(7)

\[p(0) = 0\]

So, if \( |p(z)| \) is small enough then \( -r_1 z + r_2 |p(z)|^{\sigma_1} \geq -r_1 z + \tilde{r}_2 |p(z)|^{1/2} \), where \( \tilde{r}_2 : \Delta = \tilde{r}_2^2 - 8r_1 > 0 \). Thus, if we take the following ODE:

\[
\frac{dp(z)}{dz} = -r_1 z + \tilde{r}_2 |p(z)|^{1/2}
\]

(8)

\[p(0) = 0\]

this quite clearly has the following solution: \( p(z) = k^2 z^2 \), where \( k = (\tilde{r}_2 + \sqrt{\Delta}) / 4 \).

If we compare the theorems presented in Liao (1999), it can be found that, for the maximum solution \( \tilde{p}(z) \) of the ODE (7), it is possible for \( \tilde{p}(z) \geq p(z) \geq 0 \), where \( z > 0 \).

Thus, for ODE (6) above, there is a solution: \( q(z) = -\tilde{(p(z))}^{1/2} \). In that case, we can now say that system (5) satisfies H4.

If we now apply the theorem presented in part 2 of this paper to this example, it will be seen that ODE (6) has an infinite number of possible solutions. By extension, this also indicates that (4) has an infinite number of possible solutions.

4. Conclusion

In this paper, we have analysed the asymptotic stability of second-order ordinary systems. We looked at an initial candidate ODE and an associated second-order system and formulated four hypotheses upon which we could found an investigation of non-uniqueness. We presented a core theorem expressing the view that the initial ODE would have an infinite number of possible solutions. This theorem was then proved by way of first of all proving two corollary lemmas. The validity of the theorem was further tested by working through an actual example of asymptotic stability taken from a published investigation relating to finite time control. On the basis of this, we have been able to provide clear criteria for non-uniqueness in first order ordinary differential equations. The given proof can, in turn, be used to solve a specific set of mathematical problems in engineering as well as in other domains. This examination of the nature of non-unique solutions to ODEs and theoretical expression of some of their properties serves to redress in some small part the notable lack of studies of this kind in the literature. Clearly, there is a need for significantly more work yet to put non-uniqueness on an equal theoretical footing to uniqueness, so this remains a project to be developed and sustained in future research.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This paper was supported by the National Natural Science Foundation of China [grant numbers 11461006 and 61772006], the
Science and Technology Program of Guangxi [grant number AB17129012], the Science and Technology Major Project of Guangxi [grant number AA17204096], Special Fund for Scientific and Technological Bases and Talents of Guangxi [grant number AD16380076], the Special Fund for Bagui Scholars of Guangxi, the Teaching Reform Project of Guangxi Higher Education Undergraduate [grant number 2017JGB192], the promotion project of basic faculties for young and middle-aged college teachers in Guangxi [grant numbers 2017KY0174, 2018KY0164 and 2018KY0166], and the Degree and Graduate Education Reform Project of Guangxi University for for Nationalities [grant number gxun-chxjg201603].

ORCID

Juxia Xiong  http://orcid.org/0000-0003-1279-6002

References

Agarwal, R. P., & Lakshmikantham, V. (1993). Uniqueness and nonuniqueness criteria for ordinary differential equations. Series in real analysis: Vol. 6. Singapore: World Scientific.

Bellen, A., Guglielmi, N., & Ruehli, A. E. (1999). Methods for linear systems of circuit delay differential equations of neutral type. IEEE Transactions on Circuits & Systems I Fundamental Theory & Applications, 46(1), 212–215.

Bernfeld, S. R. (1972). Non-unique critical points of ordinary differential equations. Theory of Computing Systems, 6(12), 60–71.

Betts, J. T. (1998). Survey of numerical methods for trajectory optimization. Journal of Guidance Control & Dynamics, 21(2), 193–207.

Bhat, S. P., & Bernstein, D. S. (2002). Continuous finite-time stabilization of the translational and rotational double integrator. IEEE Transactions on Automatic Control, 43(5), 678–682.

Cid, J. N. (2003). On uniqueness criteria for systems of ordinary differential equations. Journal of Mathematical Analysis & Applications, 281(1), 264–275.

Cook, M. V. (2012). Flight dynamics principles: A linear systems approach to aircraft stability and control. Oxford: Butterworth-Heinemann.

Coron, J. M. (2006). On the stabilization in finite time of locally controllable systems by means of continuous time-varying feedback law. SIAM Journal on Control & Optimization, 33(3), 804–833.

Flandoli, F. (2009). Remarks on uniqueness and strong solutions to deterministic and stochastic differential equations. Metrika, 69(2–3), 101–123.

Gifford, S. J., & Tomlinson, G. R. (1989). Recent advances in the application of functional series to non-linear structures. Journal of Sound & Vibration, 135(2), 289–317.

Gillespie, D. T. (1976). A general method for numerically simulating the stochastic time evolution of coupled chemical reactions. Journal of Computational Physics, 22(4), 403–434.

Golnaraghi, F., & Kuo, B. (2010). Design of control systems. In Automatic control systems (9th ed., pp. 664–835). Reading: Wiley.

Haimo, V. T. (1986). Finite time controllers. SIAM Journal on Control and Optimization, 24, 760–770.

Hamilton, O. H. (1938). Non-unique solutions of systems of first order ordinary differential equations. The Annals of Mathematics, 39(4), 786–793.

Haraux, A., & Weissler, F. B. (1982). Non-uniqueness for a semilinear initial value problem. Indiana University Mathematics Journal, 31(2), 167–189.

Hong, Y. G., Huang, J., & Xu, Y. S. (1999). On an output feedback finite-time stabilisation problem. Proceedings of the 38th IEEE conference on decision and control (Cat. No.99CH36304). Arizona, United States.

Kelley, A. (1966). The stable, center-stable, center-unstable, unstable manifolds. Journal of Differential Equations, 3(4), 546–570.

Kuo, C. Y., Soetanto, D., & Chiou, Y. C. (2015). Geometric analysis of flight control command for tactical missile guidance. IEEE Transactions on Control Systems Technology, 9(2), 234–243.

Liao, X. (1999). Stability theory, methods and applications (in Chinese). Wuhan: Huazhong University of Science and Technology Press.

Newton, I. (1671). Methodus fluxionum et serierum infinitarum. The mathematical papers of Isaac Newton: Vol. 7 (2008). Cambridge: Cambridge University Press.

Pan, Y., Wang, M., & Yan, Y. (2012). Uniqueness theorem for ordinary differential equations with Hölder continuity. Pacific Journal of Mathematics, 2, 453–473.

Strogatz, S. H. (2015). Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering. Computers in Physics, 8(3), 532–532.

Weissler, F. B. (1986). Asymptotic analysis of an ordinary differential equation and non-uniqueness for a semilinear partial differential equation. Archive for Rational Mechanics & Analysis, 91, 231–245.

Yodovitch, VI (2004). Mathematical models for natural sciences. Lecture course. Moscow.

Zandbergen, P. J., & Dijkstra, D. (1977). Non-unique solutions of the navier-stokes equations for the Karman swirling flow. Journal of Engineering Mathematics, 11(2), 167–188.