Abstract: In this work, we introduce an efficient scheme for the numerical solution of some Boundary and Initial Value Problems (BVPs-IVPs). By using an operational matrix, which was obtained from the first kind of Chebyshev polynomials, we construct the algebraic equivalent representation of the problem. We will show that this representation of BVPs and IVPs can be represented by a sparse matrix with sufficient precision. Sparse matrices that store data containing a large number of zero-valued elements have several advantages, such as saving a significant amount of memory and speeding up the processing of that data. In addition, we provide the convergence analysis and the error estimation of the suggested scheme. Finally, some numerical results are utilized to demonstrate the validity and applicability of the proposed technique, and also the presented algorithm is applied to solve an engineering problem which is used in a beam on elastic foundation.

Keywords: boundary and initial value problems; chebyshev-spectral method; elastic foundation; numerical treatment

MSC: 35A24; 80M22
with
\[ \beta_{k0}u(-1) + \sum_{j=1}^{n-1} \beta_{kj}u^{(j)}(1) = l_k, \quad (k = 0, 1, 2, \ldots, n - 1), \] (2)
and also the following IVPs,
\[ \sum_{i=0}^{n} a_i D^{i}u(x) = R(x), \quad x \in [0,1], \] (3)
with
\[ \sum_{j=0}^{n-1} \beta_{kj}u^{(j)}(0) = l_k, \quad (k = 0, 1, 2, \ldots, n - 1), \] (4)
where \( a_i, \beta_{kj} \in \mathbb{R} \) are constants, \( (i = 0, 1, \ldots, n) \), \( (j, k = 0, 1, \ldots, n - 1) \), and \( R(x) \) and \( u(x) \) are given and unknown functions, respectively.

Among numerical approaches, spectral methods are very successful and powerful tools for the numerical solution of differential and integral equations involving derivatives of integer and non-integer order. Effective properties that have encouraged many experts in numerical analysis to use spectral methods for different kinds of mathematical problems are spectral accuracy and the ease of applying these methods. One of the most important advantages of these methods is their high accuracy, the so-called convergence of “infinite order”. Therefore, when the exact solution is infinitely differentiable, the numerical approximation converges faster than \( M^{-n} \), where \( M \) is the order of approximation and \( n > 0 \) is constant [12,13].

Pseudospectral, Galerkin, and Tau methods are several kinds of spectral methods which can be obtained from a weighted residual method [12]. The Tau method, which can be explained as a special case of spectral methods, was invented by Lanczos [14]. It has become increasingly popular with applications in many disciplines such as porous media, viscoelasticity, electrochemistry, and other problems where high accuracy is desired. Further detailed information of this procedure can be found in [15–17].

In the literature, there are several approaches to obtain numerical methods to solve boundary value problems and other related problems such as Chebyshev-type, Runge–Kutta type, Wavelet–Galerkin method, and Shannon approximation [5,6,18–23]. Wolf [24] has been concerned with the numerical solution of non-singular integral and integro-differential equations by iteration with Chebyshev series. In [25], a Chebyshev series were used for nonlinear differential equations. Various works, such as the works in [26,27], introduced and discussed a Chebyshev technique for solving nonlinear optimal control problems. Mezzadri et al. [28] have been concerned with nonlinear programming methods for the solution of optimal control problems via a Chebyshev technique.

The proposed approach in this work is referred to as operational in the sense that it makes it possible to transform a given differential equation into an algebraic equation. An efficient algorithm of the Tau approximation based on Chebyshev polynomials is presented to numerically solve the boundary and initial value problems. One of the main advantages of this methodology is that the matrices generated of BVPs and IVPs are sparse. The proposed procedure can be very attractive to users who are concerned with memory conservation, much less computational cost and much more computational speed (see Remark 2). Furthermore, the simplicity of the scheme and low algorithm run time are among its other advantages. Ultimately, to demonstrate the validity and applicability of the method, an engineering problem of the fourth-order ODE which is arising in elastic foundation is solved.

The organization of this paper is as follows. In Section 2, we present the basic properties of Chebyshev polynomials. Converting boundary value problem (1) and initial value problem (3) to a matrix form based on the new proposed algorithm is shown and the convergence analysis for the first kind of Chebyshev expansion is given in Section 3. The last part of this work is concerned with four numerical examples to illustrate the method.
2. Some Preliminaries

Now, we remind some basic properties and essentials of Chebyshev polynomials, which are applied further in this work [1].

**Definition 1.** The first kind of Chebyshev polynomials of order \( n \) is described as
\[
T_n(x) = \cos[n \cos^{-1} x], \quad x \in [-1, 1], \quad n = 0, 1, 2, \ldots \tag{5}
\]

Furthermore, due to (5), we can write
\[
T_n(\cos \theta) = \cos(n \theta), \quad \theta \in [0, \pi], \quad n = 0, 1, 2, \ldots
\]

**Theorem 1.** The Chebyshev polynomials \( T_n(x) \) satisfy the following relation,
\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \ldots \tag{6}
\]

**Remark 1.** Using simple computation the following relations with derivatives of Chebyshev polynomials can be obtained,
\[
\begin{align*}
T_0(x) &= T_1'(x), \\
T_1(x) &= \frac{1}{4}T_2'(x), \\
T_n(x) &= \left[ \frac{T_{n+1}'(x)}{2(n+1)} - \frac{T_{n-1}'(x)}{2(n-1)} \right], \quad n \geq 2. \tag{7}
\end{align*}
\]

**Proposition 1.** Chebyshev polynomials are orthogonal functions:
\[
\int_{-1}^{1} T_n(x)T_m(x)(1 - x^2)^{-1/2}dx = \frac{\pi}{2}c_n\delta_{nm}, \tag{8}
\]

where \((1 - x^2)^{-1/2}\) is weighting function and
\[
c_n = \begin{cases} 
2 & n = 0, \\
1 & n > 0,
\end{cases}
\quad \text{and} \quad \delta_{nm} = \begin{cases} 
0 & n \neq m, \\
1 & n = m.
\end{cases} \tag{9}
\]

The Chebyshev polynomials \( T_n(x) \) are defined on the interval \([-1, 1]\). In order to use these polynomials on the interval \([a, b]\), we introduce \( x = \frac{b-a}{2} t + \frac{b+a}{2} \).

3. Outline of the Method for Boundary and Initial Value Problems

Here, Chebyshev–Tau method is expressed to solve boundary value problem (1) and initial value problem (3). The first part is concerned with the matrix structure produced by proposed method, the convergence, and the error analysis for the mentioned polynomials. The second part will be discussed with the numerical solvability of the system of algebraic equations arising in the presented method.

3.1. Operational Matrix of Derivative

Let functions \( u(x) \) and \( Du(x) \) be expanded by the Chebyshev polynomials in \([-1, 1]\) as
\[
u(x) = \sum_{n=0}^{\infty} v_n T_n(x) = T_x V^T, \tag{10}\]
\[ Du(x) = \sum_{n=0}^{\infty} \alpha_{1,n} T_n(x) = T_x W_1^T, \] (11)

where \( V = (v_0, v_1, v_2, \ldots) \), \( W_1 = (\omega_{1,0}, \omega_{1,1}, \omega_{1,2}, \ldots) \) and \( T_x = (T_0(x), T_1(x), T_2(x), \ldots) \). Moreover, for any integrable function \( u(x) \) on \([-1, 1]\), we define \( \nu_n \) as follows,

\[ \nu_n = \langle u(x), T_n(x) \rangle = \frac{2}{\pi c_n} \int_{-1}^{1} u(x) T_n(x) \omega(x) \, dx. \]

Taking the derivative of (10), we have

\[ Du(x) = \sum_{n=1}^{\infty} \nu_n T'_n(x). \] (12)

Due to (11) and (12), we obtain

\[ \sum_{n=1}^{\infty} \nu_n T'_n(x) = \omega_{1,0} T_0(x) + \omega_{1,1} T_1(x) + \sum_{n=2}^{\infty} \omega_{1,n} T_n(x), \]

by using Equation (7), we can rewrite the last equation as follows,

\[ \Rightarrow \nu_1 T'_1(x) + \nu_2 T'_2(x) + \nu_3 T'_3(x) + \ldots = \omega_{1,0} T'_0(x) + \frac{\omega_{1,1}}{4} T'_2(x) + \sum_{n=2}^{\infty} \omega_{1,n} \left[ T'_{n+1}(x) - \frac{T'_n(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} \right], \]

then we conclude that

\[
\begin{aligned}
\nu_1 &= \frac{1}{2} [2\omega_{1,0} - \omega_{1,2}], \\
\nu_2 &= \frac{1}{4} [\omega_{1,1} - \omega_{1,3}], \\
\nu_3 &= \frac{1}{6} [\omega_{1,2} - \omega_{1,4}], \\
&\quad \vdots \\
\nu_{N-2} &= \frac{1}{2(N-2)} [\omega_{1,N-3} - \omega_{1,N-1}], \\
&\quad \vdots
\end{aligned}
\] (13)

or equivalently

\[
\begin{aligned}
\omega_{1,0} &= \nu_1 + 3\nu_3 + 5\nu_5 + \ldots + \frac{\tau_0\nu_{N-1}}{2} + \frac{\tau_0\nu_N}{2} + \ldots, \\
\omega_{1,i} &= \sum_{n=i+1}^{\infty} \nu_n \tau_{i,n}, \quad i \geq 1,
\end{aligned}
\] (14)

where

\[ \tau_{i,n} = \begin{cases} 0, & i + n \text{ even} \\ 2n, & \text{otherwise}. \end{cases} \]
Let us introduce matrix $E$ as a coefficient of matrix $V$, then from (14) we have $W_1^T = EV^T$. The upper triangular matrix $E$, via easy calculations, is as follows,

$$
E = \begin{bmatrix}
0 & 1 & 0 & 3 & 0 & 5 & \ldots \\
0 & 0 & 4 & 0 & 8 & 0 & \ldots \\
0 & 0 & 0 & 6 & 0 & 10 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}.
$$

**Theorem 2.** Let $D^k u(x) = \sum_{n=0}^{\infty} \omega_{k,n} T_n(x) = T_x W_k^T$ be a Chebyshev polynomial with $W_k = [\omega_{k,0}, \omega_{k,1}, \omega_{k,2}, \ldots]$, $T_x = [T_0(x), T_1(x), T_2(x), \ldots]$, $V = [v_0, v_1, v_2, \ldots]$, then for any $k \in \mathbb{N}$, we obtain

$$
D^k u(x) = T_x E^k V^T.
$$

**Proof.** As we pointed out previously, Equation (14) can be written to the following matrix structure,

$$
W_1 = (EV^T)^T = VE^T.
$$

Furthermore, due to Equation (11), we have

$$
Du(x) = T_x EV^T.
$$

From Equation (7), we can write

$$
T_n'(x) = \frac{1}{2} \left[ \frac{T''_{n+1}(x)}{n+1} - \frac{T''_{n-1}(x)}{n-1} \right].
$$

Given the assumption, it follows that

$$
D^2 u(x) = \sum_{n=0}^{\infty} \omega_{2,n} T_n(x) = T_x W_2^T.
$$

Using the process of obtaining Equation (15), we conclude

$$
W_2^T = EW_1^T.
$$

Due to (15) and the last equation, we get $W_2^T = E^2 V^T$. Therefore, by repeating this scheme, it follows that $W_k^T = E^k V^T$, and we can write

$$
D^k u(x) = T_x E^k V^T.
$$

□

The following theorem is concerned with the convergence and the error analysis for the Chebyshev expansion.

**Theorem 3.** Suppose $u(x)$ is a function that satisfies $\int_{-1}^{1} |u^{(2)}(x)|^2 dx < \infty$ and $|u^{(2)}(x)| \leq K$ for some constant $K$. Then, we have

$u(x)$ can be expanded as $\sum_{n=0}^{\infty} v_n T_n(x)$ and the series converges to $u(x)$ uniformly, in other words

$$
u(x) = \sum_{n=0}^{\infty} v_n T_n(x),$$
where \( \nu_n = \langle u(x), T_n(x) \rangle \).

Moreover, for an error estimation the following inequality holds,

\[
\varepsilon \leq \left( 2K^2 \pi \sum_{n=N+1}^{\infty} \frac{1}{(n-1)^2} \right) \frac{1}{2},
\]

where \( \varepsilon = \int_{-1}^{1} |u(x) - \sum_{n=0}^{N} \nu_n T_n(x)|^2 \omega(x)dx \).

**Proof.** By showing the absolute convergence of the series \( \sum_{n=0}^{\infty} |\nu_n| \), it can be concluded that the series \( \sum_{n=0}^{\infty} \nu_n T_n(x) \) converges to \( u(x) \) uniformly. According to definition of \( \nu_n \) we have

\[
\nu_n = \langle u(x), T_n(x) \rangle = \frac{2}{\pi c_n} \int_{-1}^{1} u(x) T_n(x) \omega(x)dx,
\]

substituting \( \omega(x) = (1 - x^2)^{-1/2} \) and \( x = \cos \theta \) yields

\[
\nu_n = \frac{2}{\pi c_n} \int_{0}^{\pi} u' \cos \theta T_n \cos \theta (1 - \cos^2 \theta)^{-1/2} \sin \theta d\theta
\]

\[
= \frac{2}{\pi c_n} \int_{0}^{\pi} u \cos \theta \cos n\theta d\theta.
\]

Using the integration, we have:

\[
\nu_n = \frac{2}{\pi c_n n} \int_{0}^{\pi} u' \cos \theta \sin n\theta \sin \theta d\theta
\]

\[
= \frac{1}{\pi c_n n} \int_{0}^{\pi} u' \cos \theta (\cos(n-1)\theta - \cos(n+1)\theta) d\theta
\]

\[
= \frac{1}{\pi c_n n} \left[ \int_{0}^{\pi} u' \cos \theta \cos(n-1)\theta d\theta - \int_{0}^{\pi} u' \cos \theta \cos(n+1)\theta d\theta \right],
\]

with integration again, \( \nu_n \) can be obtained as

\[
\nu_n = \frac{1}{\pi c_n n} \left[ \int_{0}^{\pi} u'' \cos \theta \sin(n-1)\theta \sin \theta d\theta - \int_{0}^{\pi} u'' \cos \theta \sin(n+1)\theta \sin \theta d\theta \right],
\]

thus for \( n > 1 \), (for \( n = 0, c_n = 1 \)), we have

\[
|\nu_n| \leq \frac{1}{\pi n} \int_{0}^{\pi} |u'' \cos \theta| \left[ \frac{1}{n-1} - \frac{1}{n+1} \right].
\]

Furthermore, due to assumption and by simple computation, we can obtain

\[
|\nu_n| \leq \frac{2K}{n^2 - 1} \leq \frac{2K}{|n-1|^2}.
\]

Finally, the last inequality shows that the series \( \sum_{n=0}^{\infty} |\nu_n| \) is absolutely convergent. Let us set

\[
e = \int_{-1}^{1} |u(x) - \sum_{n=0}^{N} \nu_n T_n(x)|^2 \omega(x)dx
\]

\[
e = \int_{-1}^{1} \left| \sum_{n=0}^{\infty} \nu_n T_n(x) - \sum_{n=0}^{N} \nu_n T_n(x) \right|^2 \omega(x)dx
\]
\[
\int_{-1}^{1} | \sum_{n=N+1}^{\infty} v_n T_n(x)^2 \omega(x) dx,
\]
due to orthogonality of \(T_n(x)\) and value of \(|v_n|\), we have:
\[
e \leq \sum_{n=N+1}^{\infty} |v_n| \frac{2\pi}{2} \leq 2K^2\pi \sum_{n=N+1}^{\infty} \frac{1}{|n-1|^4}.
\]

3.2. Description of the Proposed Method for BVPs and IVPs

The main object of this section is applying the obtained consequence for constructing the Tau approximate solution with Chebyshev polynomials of the boundary value problem (1).

We suppose that
\[
u_N(x) = \sum_{n=0}^{N} v_n T_n(x) = T_N, V_T,
\]
\[
D^n \nu_N(x) = T_N, E_{T_N} V_T,
\]
\[
R(x) = \sum_{n=0}^{N} b_n T_n(x) = T_N, b_T,
\]
where \(V_N = (v_0, v_1, \ldots, v_N), T_N, x = (T_0(x), T_1(x), \ldots, T_N(x)), b_N = (b_0, b_1, \ldots, b_N)\) and \(E_N\) is a finite form of \(E\).

Using the above relations, (1) can be written as
\[
a_n T_N, E_N v_n T_N + a_{n-1} T_N, E_N v_{n-1} T_N + \ldots + a_1 T_N, E_N v_1 T_N + a_0 T_N, b_T = T_N, b_T.
\]

According to the linear independence of Chebyshev polynomials, we have
\[
(a_n E_N^n + a_{n-1} E_N^{n-1} + \ldots + a_1 E_N + a_0 I_{N+1}) V_T = b_T.
\]

Let us set
\[
X = a_n E_N^n + a_{n-1} E_N^{n-1} + \ldots + a_1 E_N + a_0 I_{N+1},
\]
similarly, for the Equation (2) we set
\[
\beta k_0 \sum_{n=0}^{N} v_n T_n(-1) + \sum_{j=1}^{n-1} \beta k_j \sum_{n=0}^{N} v_n T_n^{(j)}(1) = l_k, \quad (k = 0, 1, 2, \ldots, n - 1).
\]

Due to (22) and (23), the desired matrix can be written as
\[
\begin{bmatrix}
X & \begin{bmatrix}
v_0 \\
v_1 \\
\vdots \\
v_N
\end{bmatrix}
\end{bmatrix} = \begin{bmatrix}
b_0 \\
\vdots \\
b_{N-u} \\
l_0 \\
\vdots \\
l_{n-1}
\end{bmatrix},
\]
where $X_n$ is obtained by eliminating the last $n$ row of matrix $X$ and

$$
B = \begin{bmatrix}
\beta_{00} & \beta_{01} & \cdots & \beta_{0,n-1} \\
\beta_{10} & \beta_{11} & \cdots & \beta_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n-1,0} & \beta_{n-1,1} & \cdots & \beta_{n-1,n-1}
\end{bmatrix}
\begin{bmatrix}
T_0(-1) & T_1(-1) & \cdots & T_n(-1) \\
0 & T'_1(1) & \cdots & T'_n(1) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_{n-1}^{(n-1)}(1)
\end{bmatrix}.
$$

(25)

By solving a sparse system of above algebraic equation, we can obtained unknown $v_i$, $(i = 0, 1, \ldots, N)$ and finally, $u_N(x)$, the desired approximation, can be computed from the following relation,

$$
u_N(x) = T_{N,x}V^T_N.
$$

The following Algorithm 1 summarizes our proposed method.

**Algorithm 1:** The construction of proposed method for the boundary value problems (BVPs)

**Step 1.** Input:

$N, n, c_{N,i}, a_i, l_j, (i = 0, \ldots, n, j = 0, \ldots, n-1)$;

$R(x)$;

**Step 2.** Compute:

$$b_k = \frac{2c_k}{\pi} \int_{-1}^1 R(x)T_k(x)(1-x^2)^{-\frac{1}{2}}dx;$$

for $k = 0, \ldots, N - n$.

**Step 3.** Compute the matrix $E_N$ from matrix $E$.

**Step 4.** Compute:

4.1. $E_i^j$ from step 3, $(i=2, \ldots, n)$.

4.2. $X_n$ based on $X$ from (22).

**Step 5.** Set:

$$B = \begin{bmatrix}
\beta_{00} & \beta_{01} & \cdots & \beta_{0,n-1} \\
\beta_{10} & \beta_{11} & \cdots & \beta_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{n-1,0} & \beta_{n-1,1} & \cdots & \beta_{n-1,n-1}
\end{bmatrix}
\begin{bmatrix}
T_0(-1) & T_1(-1) & \cdots & T_n(-1) \\
0 & T'_1(1) & \cdots & T'_n(1) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_{n-1}^{(n-1)}(1)
\end{bmatrix}.
$$

**Step 6.** Obtain the solution $V_N$ from the system of (24) and set:

$$u_N(x) = T_{N,x}V^T_N.$$

Construction of the Proposed Method for IVPs

Here, we consider the initial value problems (3) as

$$\sum_{i=0}^n a_i D^i u(x) = R(x), \quad x \in [0, 1],$$

with

$$\sum_{j=0}^{n-1} \beta_{kj}u^{(j)}(0) = l_k, \quad (k = 0, 1, 2, \ldots, n - 1).$$

It is clear that the given scheme for BVPs (1) and Algorithm 1, with small changes, can be easily applied to these problems, thus we refrain from going into details. Only Step 5 in Algorithm 1 will be changed, and we have
therefore, we can obtain \( V_N \) from the system of (24) and finally the desired solution can be achieved from \( u_N(x) = T_N, x V_N \).

**Remark 2.** As a consequence of the described method, a significant advantage for increasing the computation speed is to the sparsity of matrices \( E_N, E_N^2, \ldots, E_N^n \). If we consider matrix \( E_N \) with an odd dimension, then the number of non-zero matrix elements is \( \frac{N(N+2)}{4} \). Moreover, the number of non-zero matrix elements for matrix \( E_N \) with an even dimension is \( \left[ \frac{N+1}{2} \right]^2 \). Furthermore, the number of non-zero matrix elements for matrix \( E_N^2 \) with an even and odd dimension are \( \frac{N^2 - 1}{4} \) and \( \frac{N^2}{4} \), respectively and so it goes. Figures 1–3 indicate that as the derivative order rise, the sparsity of matrices and computational speed also increase.

**Figure 1.** The sparsity structure of \( E_N \) for even value of \( N \).

**Figure 2.** The sparsity structure of \( E_N^2 \) for even value of \( N \).
4. Numerical Results

In this section, the numerical results of four test problems are reported. We solve the examples using proposed Tau approximation method based on Chebyshev basis functions. In the tables presented here, the maximal differences between exact and approximation solution has been shown by “Maximal Errors”. We compare our obtained results with in the some existing numerical methods. The numerical results of the given method have been shown in Figures 4–7.

Figure 4. The Tau approximation of order 3 and \(N = 7\) for Example 1 using Chebyshev basis.

Figure 5. The Tau approximation of order 1 and \(N = 5\) for Example 2 using Chebyshev basis.
Example 1. Consider the third-order boundary value problem

$$u'''(x) - 6u''(x) + 11u'(x) - 6u(x) = \sin x,$$  \hspace{1cm} (27)

with the conditions

$$u(-1) = \frac{1}{e^1} + \frac{1}{e^2} + \frac{1}{e^3} - 0.1 \cos(-1),$$
$$u'(1) = e^1 + 2e^2 + 3e^3 + 0.1 \sin(1),$$
$$u''(1) = e^1 + 4e^2 + 9e^3 + 0.1 \cos(1),$$

and the exact solution of Example 1 is $u(x) = e^x + e^{2x} + e^{3x} - 0.1 \cos x$.

We take $N = 7$, for computational details of the described technique in Section 3. By applying mentioned algorithm, the following matrices will be obtained.

$$E_7 = \begin{bmatrix}
0 & 1 & 0 & 3 & 0 & 5 & 0 & 7 \\
0 & 0 & 4 & 0 & 8 & 0 & 12 & 0 \\
0 & 0 & 0 & 6 & 0 & 10 & 0 & 14 \\
0 & 0 & 0 & 0 & 8 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 14
\end{bmatrix},
E_7^2 = \begin{bmatrix}
0 & 0 & 4 & 0 & 32 & 0 & 108 & 0 \\
0 & 0 & 0 & 24 & 0 & 120 & 0 & 336 \\
0 & 0 & 0 & 0 & 48 & 0 & 192 & 0 \\
0 & 0 & 0 & 0 & 0 & 80 & 0 & 280 \\
0 & 0 & 0 & 0 & 0 & 0 & 120 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 168 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$
\[
E_7^3 = \begin{bmatrix}
0 & 0 & 24 & 0 & 360 & 0 & 2016 \\
0 & 0 & 0 & 192 & 0 & 1728 & 0 \\
0 & 0 & 0 & 6 & 0 & 480 & 0 & 3360 \\
0 & 0 & 0 & 0 & 0 & 960 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

The left and right hand-side of (24) are, respectively,

\[
\begin{bmatrix}
X_3 \\
B
\end{bmatrix} = \begin{bmatrix}
-6 & 11 & -24 & 57 & -192 & 415 & -648 & 2093 \\
0 & -6 & 44 & -144 & 280 & -720 & 1860 & -2016 \\
0 & 0 & -6 & 66 & -288 & 590 & -1152 & 3514 \\
0 & 0 & 0 & -6 & 88 & -480 & 1092 & -1680 \\
0 & 0 & 0 & 0 & -6 & 110 & -720 & 1834 \\
0 & 0 & 4 & 24 & 80 & 200 & 420 & 784 \\
0 & 1 & 4 & 9 & 16 & 25 & 36 & 49 \\
1 & -1 & 1 & -1 & 1 & -1 & -1
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
b_3 \\
b_4 \\
l_0 \\
l_1 \\
l_2
\end{bmatrix} = \begin{bmatrix}
0 \\
0.880101 \\
0 \\
-0.0391267 \\
0 \\
213.098 \\
77.8372 \\
0.498972
\end{bmatrix}.
\]

By solving the linear system (24), we obtain vector \( V_7 \) as follows,

\[
V_7 = [8.40476, 12.2529, 6.15946, 2.41236, 0.744749, 0.187294, 0.0555411, 0.0130075].
\]

By putting these values in Equation (18), an approximate solution will be obtained. The numerical results are given in Table 1.

| N  | Maximal Errors         | Presented Method |
|----|------------------------|------------------|
| 7  | \(9.12 \times 10^{-2}\) |                  |
| 8  | \(5.74 \times 10^{-3}\) |                  |
| 10 | \(1.22 \times 10^{-4}\) |                  |
| 12 | \(1.58 \times 10^{-6}\) |                  |
| 15 | \(1.49 \times 10^{-9}\) |                  |

Example 2. Consider the first-order initial value problem:

\[
0.25u'(x) + u(x) = 1.
\]

(28)
The exact solution of this problem under the initial condition \( u(0) = 0 \) is given by

\[
u(x) = 1 - e^{-4x}.
\] (29)

This problem was also solved in [4, 21] by a method based on Legendre and Chebyshev polynomials. The best reported maximum error is \( O(10^{-4}) \). Table 2 shows that we can attain good numerical results compare to the numerical results in [4, 21] for \( n \geq 10 \).

**Table 2.** Results of Example 2, using presented method.

| \( N \) | Present Method | Method in [3] | Method in [9] |
|-------|----------------|---------------|---------------|
| 6     | \( 1.17 \times 10^{-2} \) | \( 1.45 \times 10^{-2} \) | \( 2.7 \times 10^{-2} \) |
| 8     | \( 9.09 \times 10^{-3} \) | \( 2.1 \times 10^{-3} \) | \( 3.8 \times 10^{-3} \) |
| 11    | \( 4.48 \times 10^{-5} \) | \( 2.03 \times 10^{-4} \) | \( 4.11 \times 10^{-4} \) |
| 15    | \( 1.66 \times 10^{-8} \) | – | – |
| 18    | \( 4.93 \times 10^{-12} \) | – | – |

**Example 3.** Consider an engineering problem of the fourth-order initial value problem which is arising in elastic foundation [20] as follows,

\[
u^{(4)}(x) + u(x) = 1, \quad 0 < x < 1, \quad u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 0, \quad u'''(0) = 0,
\] (30)

with the exact solution \( u(x) = 1 - \frac{1}{2} e^{\frac{x}{\sqrt{2}}} (1 + e^{\sqrt{2}x}) \cos(\frac{x}{\sqrt{2}}) \).

**Example 4.** Consider the following fifth-order boundary value problem,

\[
u^{(5)}(x) - 3u^{(4)}(x) - 16u'(x) + 48u(x) = 48,
\] (31)

with

\[
u(-1) = e^{-2} + e^{2} + e^{-3} + \cos 2 - \sin 2 + 1,
\]
\[
u'(1) = 2e^{2} - 2e^{-2} + 3e^{3} + 2 \cos 2 - 2 \sin 2,
\]
\[
u''(1) = 4e^{2} + 4e^{-2} + 9e^{3} - 4 \cos 2 - 4 \sin 2,
\]
\[
u'''(1) = 8e^{2} - 8e^{-2} + 27e^{3} - 8 \cos 2 + 8 \sin 2,
\]
\[
u^{(4)}(1) = 16e^{2} + 16e^{-2} + 81e^{3} + 16 \cos 2 + 16 \sin 2.
\]

and the exact solution \( u(x) = e^{2x} + e^{-2x} + e^{3x} + \cos 2x + \sin 2x + 1 \).

To obtain the approximate solution of Examples 3 and 4, we apply the proposed Tau method. The obtained numerical results in Table 3, show that the desired accuracy is obtained.

**Table 3.** Results of Examples 3 and 4, using presented method.

| \( N \) | Exp. 3 | Exp. 4 |
|-------|-------|-------|
| 11    | \( 1.54 \times 10^{-9} \) | \( 2.63 \times 10^{-3} \) |
| 13    | \( 5.85 \times 10^{-14} \) | \( 4.11 \times 10^{-5} \) |
| 15    | \( 3.64 \times 10^{-15} \) | \( 5.15 \times 10^{-7} \) |
| 17    | \( 1.27 \times 10^{-16} \) | \( 5.01 \times 10^{-9} \) |
5. Conclusions

In this paper, an efficient and accurate numerical algorithm has been applied to solve boundary and initial value problems by using the Chebyshev–Tau method. Due to the wide application of boundary value problems in many different fields, an engineering problem which is arising in elastic foundation is chosen as test example. By utilizing proposed method, the sparsity of the obtained derivative operational matrix makes us able to solve the linear system with much less computational cost and much more computational speed. The reported results indicated that the proposed scheme can obtain appropriate approximate solutions in comparing with the exact solution. Moreover, by increasing the order of Chebyshev polynomial, a better approximation was achieved.

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