STRONG PEAK POINTS AND DENSENESS OF STRONG PEAK FUNCTIONS

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Abstract. Let $C_b(K)$ be the set of all bounded continuous (real or complex) functions on a complete metric space $K$ and $A$ a closed subspace of $C_b(K)$. Using the variational method, it is shown that the set of all strong peak functions in $A$ is dense if and only if the set of all strong peak points is a norming subset of $A$. As a corollary we show that if $X$ is a locally uniformly convex, complex Banach space, then the set of all strong peak functions in $\mathcal{A}(B_X)$ is a dense $G_\delta$ subset. Moreover if $X$ is separable, smooth and locally uniformly convex, then the set of all norm and numerical strong peak functions in $\mathcal{A}_u(B_X : X)$ is a dense $G_\delta$ subset. In case that a set of uniformly strongly exposed points of a (real or complex) Banach space $X$ is a norming subset of $\mathcal{P}(nX)$ for some $n \geq 1$, then the set of all strongly norm attaining elements in $\mathcal{P}(nX)$ is Fréchet differentiable is a dense $G_\delta$ subset.

1. Main Result

Let $K$ be a complete metric space and $C_b(K)$ the Banach space of all bounded (real or complex) continuous functions on $K$ with sup norm $\|f\| = \sup\{|f(t)| : t \in K\}$. A nonzero function $f \in A$ is said to be a strong peak function at $t \in K$ if whenever there is a sequence $\{t_n\}$ such that $\lim_k |f(t_k)| = \|f\|$, we get $\lim_k t_k = t$. The corresponding point $t \in K$ is said to be a strong peak point for $A$. We denote by $\rho A$ the set of all strong peak points for $A$.

Bishop’s theorem [3] says that if $A$ is a (complex) uniform algebra on a compact metric space $K$, then the set $\rho A$ is a norming subset for $A$. That is, $\|f\| = \sup_{t \in K} |f(t)|$ for each $f \in A$. It is observed [6] that the denseness of the set of all strong peak functions implies that $\rho A$ is a norming subset of $A$ and Bishop’s theorem is generalized there with applications to the existence of numerical boundaries.

In this paper, we prove that the converse holds. Precisely if $\rho A$ is a norming subset for $A$, then the set of all strong peak functions in $A$ is dense. We use the variational method similar to [7].

Theorem 1.1. Let $A$ be a closed subspace of $C_b(K)$, where $K$ is a complete metric space. The set $\rho A$ is a norming subset of $A$ if and only if the set of all strong peak functions in $A$ is a dense $G_\delta$ subset of $A$.

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Proof. Let $d$ be the complete metric on $K$. Fix $f \in A$ and $\epsilon > 0$. For each $n \geq 1$, set

$$U_n = \{ g \in A : \exists z \in \rho A \text{ with } |(f - g)(z)| > \sup \{(f - g)(x) : d(x, z) > 1/n \} \}.$$ 

Then $U_n$ is open and dense in $A$. Indeed, fix $h \in A$. Since $\rho A$ is a norming subset of $A$, there is a point $w \in \rho A$ such that

$$|(f - h)(w)| > \|f - h\| - \epsilon/2.$$ 

Put $g(x) = h(x) + \epsilon \cdot p(x)$, where $p$ is a strong peak function at $w$ such that $|p(x)| < 1/3$ for $\|x - w\| > 1/n$, $|p(w)| = 1$ and $|f(w) - h(w) - \epsilon p(w)| = |f(w) - h(w)| + \epsilon$. Then $\|g - h\| \leq \epsilon$ and

$$|(f - g)(w)| = |f(w) - h(w) - \epsilon p(w)| = |f(w) - h(w)| + \epsilon$$

$$> \|f - h\| + \epsilon/2$$

$$\geq \sup \{|(f - h)(x) - \epsilon p(x)| : d(x, w) > 1/n \}.$$ 

$$= \sup \{|(f - g)(x)| : d(x, w) > 1/n \}.$$ 

That is, $g \in U_n$.

By the Baire category theorem there is a $g \in \bigcap U_n$ with $\|g\| < \epsilon$, and we shall show that $f - g$ is a strong peak function. Indeed, $g \in U_n$ implies that there is $z_n \in X$ such that

$$|(f - g)(z_n)| > \sup \{|(f - g)(x)| : d(x, z_n) > 1/n \}.$$ 

Thus $d(z_p, z_n) \leq 1/n$ for every $p > n$, and hence $\{z_n\}$ converges to a point $z$, say. Suppose that there is another sequence $\{x_k\}$ in $B_X$ such that $\{|(f - g)(x_k)|\}$ converges to $\|f - g\|$. Then for each $n \geq 1$, there is $M_n \geq 1$ such that for every $m \geq M_n$,

$$|(f - g)(x_m)| > \sup \{|(f - g)(x)| : d(x, z_n) > 1/n \}.$$ 

Then $d(x_m, z_n) \leq 1/n$ for every $m \geq M_n$. Hence $\{x_m\}$ converges to $z$. This shows that $f - g$ is a strong peak function at $z$.

The direct argument shows that the set of all strong peak functions of $A$ is a $G_\delta$ subset of $A$ (cf. Proposition 2.19 in [11]). This proves the necessity.

Concerning the converse, it is shown [6] that if the set of all strong peak functions is dense in $A$, then the set of all strong peak points is a norming subset of $A$. \qed

Remark 1.2. Let $Y$ be a Banach space and $C_b(K : Y)$ the Banach space of all bounded continuous functions from a complete metric space $K$ into $Y$ with the sup norm $\|f\| = \sup \{|f(x)| : x \in K \}$ for each $f \in C_b(K : Y)$. Then Theorem [11] also holds for each closed subspace $A$ of $C_b(K : Y)$.

Let $B_X$ be the unit ball of the Banach space $X$. Recall that the point $x \in B_X$ is said to be a smooth point if there is a unique $x^* \in B_{X^*}$ such that $\text{Re} \ x^*(x) = 1$. We
denote by $\text{sm}(B_X)$ the set of all smooth points of $B_X$. We say that a Banach space is smooth if $\text{sm}(B_X)$ is the unit sphere $S_X$ of $X$. The following corollary shows that if $\rho A$ is a norming subset, then the set of smooth points of $B_A$ is dense in $S_A$.

**Corollary 1.3.** Suppose that $K$ is a complete metric space and $A$ is a subspace of $C_b(K)$. If $\rho A$ is a norming subset of $A$, then the set of all smooth points of $B_A$ contains a dense $G_\delta$ subset of $S_A$.

**Proof.** It is shown [6] that if $f \in A$ is a strong peak function and $\|f\| = 1$, then $f$ is a smooth point of $B_A$. Then Theorem [1,1] completes the proof. □

2. Denseness of strongly norm attaining polynomials

Let $X$ be a Banach space over a scalar field (real or complex) $F$ and $X^*$ the dual space of $X$. If $X$ and $Y$ are Banach spaces, an $N$-homogeneous polynomial $P$ from $X$ to $Y$ is a mapping such that there is an $N$-linear (bounded) mapping $L$ from $X \times \cdots \times X$ to $Y$ such that $P(x) = L(x, \ldots, x)$, $\forall x \in X$.

$P(\mathcal{N}X : Y)$ denote the Banach space of all $N$-homogeneous polynomials from $X$ to $Y$, endowed with the polynomial norm $\|P\| = \sup_{x \in B_X} \|P(x)\|$. When $Y = F$, $P(\mathcal{N}X : Y)$ is denoted by $P(\mathcal{N}X)$. We refer to [3] for background on polynomials. An $N$-homogeneous polynomial $P : X \to Y$ is said to strongly attain its norm at $z$ if whenever there is a sequence $\{x_n\}$ in $B_X$ such that $\lim_n \|P(x_n)\| = \|P\|$, we get a convergent subsequence of $\{x_n\}$ which converges to $\lambda z$ for some $\lambda \in S_C$.

An element $x \in B_X$ is said to be a strongly exposed point for $B_X$ if there is a linear functional $f \in B_{X^*}$ such that $f(x) = 1$ and whenever there is a sequence $\{x_n\}$ in $B_X$ satisfying $\lim_n \text{Re} f(x_n) = 1$, we get $\lim_n \|x_n - x\| = 0$. A set $\{x_\alpha\}$ of points on the unit sphere $S_X$ of $X$ is called uniformly strongly exposed (u.s.e.), if there are a function $\delta(\epsilon)$ with $\delta(\epsilon) > 0$ for every $\epsilon > 0$, and a set $\{f_\alpha\}$ of elements of norm 1 in $X^*$ such that for every $\alpha$, $f_\alpha(x_\alpha) = 1$, and for any $x$,

$$\|x\| \leq 1 \text{ and } \text{Re} f_\alpha(x) \geq 1 - \delta(\epsilon) \text{ imply } \|x - x_\alpha\| \leq \epsilon.$$ 

In this case we say that $\{f_\alpha\}$ uniformly strongly exposes $\{x_\alpha\}$. Lindenstrauss [12, Proposition 1] showed that if $S_X$ is the closed convex hull of a set of u.s.e. points, then $X$ has property A, that is, for every Banach space $Y$, the set of norm-attaining elements is dense in $\mathcal{L}(X, Y)$, the Banach space of all bounded operators of $X$ into $Y$.

The following theorem gives stronger result.
Recall that the norm $\| \|$ of a Banach space is said to be Fréchet differentiable at $x \in X$ if
\[ \lim_{\delta \to 0} \sup_{\|y\|=1} \frac{\|x + \delta y\| + \|x - \delta y\| - 2\|x\|}{\delta} = 0. \]
It is well-known that the set of Fréchet differentiable points in a Banach space is a $G_\delta$ subset [2, Proposition 4.16]. It is shown [9] that in a real Banach space $X$, the norm of $\mathcal{P}(n)X$ is Fréchet differentiable at $Q$ if and only if $Q$ strongly attains its norm.

**Theorem 2.1.** Let $X, Y$ be Banach spaces over $\mathbb{F}$ and $k \geq 1$. Suppose that the u.s.e. points $\{x_\alpha\}$ in $S_X$ is a norming subset of $\mathcal{P}(kX)$. Then the set of strongly norm attaining elements in $\mathcal{P}(kX : Y)$ is dense. In particular, the set of all points at which the norm of $\mathcal{P}(n)X$ is Fréchet differentiable is a dense $G_\delta$ subset.

**Proof.** Let $\{x_\alpha\}$ be a u.s.e. points and $\{x_\alpha^*\}$ be the corresponding functional which uniformly strongly exposes $\{x_\alpha\}$. Let $A = \mathcal{P}(kX : Y)$, $f \in A$ and $\epsilon > 0$. Fix $n \geq 1$ and set
\[ U_n = \{g \in A : \exists z \in pA \text{ with } \|(f - g)(z)\| > \sup\{(f - g)(x) : \inf_{|\lambda| = 1} \|x - \lambda z\| > 1/n\}\}. \]
Then $U_n$ is open and dense in $A$. Indeed, fix $h \in A$. Since $\{x_\alpha\}$ is a norming subset of $A$, there is a point $w \in \{x_\alpha\}$ such that
\[ |(f - h)(w)| > \|f - h\| - \delta(1/n)\epsilon. \]
Put $g(x) = h(x) - \epsilon \cdot p(x)^k \frac{f(w) - h(w)}{\|f(w) - h(w)\|}$, where $p$ is a strongly exposed functional at $w$ such that $|p(x)| < 1 - \delta(1/n)$ for $\inf_{|\lambda| = 1} \|x - \lambda w\| > 1/n$, $p(w) = 1$. Then $\|g - h\| \leq \epsilon$ and
\[
\|(f - g)(w)\| = \|f(w) - h(w) + \epsilon p(x)^k \frac{f(w) - h(w)}{\|f(w) - h(w)\|}\| = \|f(w) - h(w)\| + \epsilon \\
> \|f - h\| + \epsilon(1 - \delta(1/n)) \\
\geq \sup\{(|f - h)(x)| - \epsilon p(x)^k \frac{f(w) - h(w)}{\|f(w) - h(w)\|} : \inf_{|\lambda| = 1} \|x - \lambda w\| > 1/n\}. \\
= \sup\{|(f - g)(x)| : \inf_{|\lambda| = 1} \|x - \lambda w\| > 1/n\}. 
\]

That is, $g \in U_n$.

By the Baire category theorem there is a $g \in \bigcap U_n$ with $\|g\| < \epsilon$, and we shall show that $f - g$ is a strong peak function. Indeed, $g \in U_n$ implies that there is $z_n \in X$ such that
\[ \|(f - g)(z_n)\| > \sup\{|(f - g)(x)| : \inf_{|\lambda| = 1} \|x - \lambda z_n\| > 1/n\}. \]
Thus $\inf_{|\lambda| = 1} \|z_p - \lambda z_n\| \leq 1/n$ for every $p > n$, and $\inf_{|\lambda| = 1} |z_n^*(z_p) - \lambda| = 1 - |z_n^*(z_p)| \leq 1/n$ for every $p > n$. So $\lim_{n} \inf_{p>n} |z_n^*(z_p)| = 1$. Hence there is a subsequence of $\{z_n\}$ which converges to $z$, say by [1] Lemma 6]. Suppose that there is another sequence
\{x_k\} in \(B_X\) such that \(\|(f - g)(x_k)\|\) converges to \(\|f - g\|\). Then for each \(n \geq 1\), there is \(M_n\) such that \(M_n \geq n\) and for every \(m \geq M_n\),

\[
\|(f - g)(x_m)\| > \sup \{\|(f - g)(x)\| : \inf_{|\lambda| = 1} \|x - \lambda z_n\| > 1/n\}.
\]

Then \(\inf_{|\lambda| = 1} \|x_m - \lambda z_n\| \leq 1/n\) for every \(m \geq M_n\). So \(\inf_{|\lambda| = 1} \|x_m - \lambda z\| \leq \inf_{|\lambda| = 1} \|x_m - \lambda z_n\| + \|z - z_n\| \leq 2/n\) for every \(m \geq M_n\). Hence we get a convergent subsequence of \(x_n\) of which limit is \(\lambda z\) for some \(\lambda \in S_C\). This shows that \(f - g\) strongly norm attains at \(z\).

It is shown [6] that the norm is Fréchet differentiable at \(P\) if and only if whenever there are sequences \(\{t_n\}, \{s_n\}\) in \(B_X\) and scalars \(\alpha, \beta\) in \(S_F\) such that \(\lim_n \alpha P(t_n) = \lim_n \beta P(s_n) = \|P\|\), we get

\[(2.1) \quad \lim_n \sup_{\|Q\| = 1} (\alpha Q(t_n) - \beta Q(s_n)) = 0.
\]

We have only to show that every nonzero element \(P\) in \(A\) which strongly attains its norm satisfies the condition (2.1). Suppose that \(P\) strongly attains its norm at \(z\) and \(P \neq 0\).

For each \(Q \in A\), there is a \(k\)-linear form \(L\) such that \(Q(x) = L(x, \ldots, x)\) for each \(x \in X\). The polarization identity [3] shows that \(\|Q\| \leq \|L\| \leq (k^k/k!)\|Q\|\). Then for each \(x, y \in B_X\), \(\|Q(x) - Q(y)\| \leq n\|L\||x - y||\) and

\[
\|Q(x) - Q(y)\| \leq \frac{k^{k+1}}{k!}\|Q\||x - y|.
\]

Suppose that there are sequences \(\{t_n\}, \{s_n\}\) in \(B_X\) and scalars \(\alpha, \beta\) in \(S_F\) such that \(\lim_n \alpha P(t_n) = \lim_n \beta P(s_n) = \|P\|\), then for any subsequences \(\{s'_n\}\) of \(\{s_n\}\) and \(\{t'_n\}\) of \(\{t_n\}\), there are convergent further subsequences \(\{t''_n\}\) of \(\{t'_n\}\) and \(\{s''_n\}\) of \(\{s'_n\}\) and scalars \(\alpha''\) and \(\beta''\) in \(S_F\) such that \(\lim_n t''_n = \alpha''z\) and \(\lim_n s''_n = \beta''z\). Then \(\alpha P(\alpha''z) = \beta P(\beta''z) = 1\). So \(\alpha(\alpha'')^k = \beta(\beta'')^k\).

Then we get

\[
\lim_n \sup_{\|Q\| = 1} (\alpha Q(t''_n) - \beta Q(s''_n)) \leq \lim_n \sup_{\|Q\| = 1} (\alpha Q(t''_n) - \alpha Q(\alpha''z)) - (\beta Q(\beta''z) - \beta Q(s''_n))
\]

\[
\leq \lim_n \frac{k^{k+1}}{k!} (\|t''_n - \alpha''z\| + \|\beta''z - s''_n\|) = 0.
\]

This implies that \(\lim_n \sup_{\|Q\| = 1} (\alpha Q(t_n) - \beta Q(s_n)) = 0\). Therefore the norm is Fréchet differentiable at \(P\). This completes the proof. \(\square\)

**Remark 2.2.** Suppose that the \(B_X\) is the closed convex hull of a set of u.s.e points, then the set of u.s.e. points is a norming subset of \(X^* = P(1X)\). Hence the elements in \(X^*\) at which the norm of \(X^*\) is Fréchet differentiable is a dense \(G_\delta\) subset.
3. Denseness of strong peak holomorphic functions

Let \( X \) be a complex Banach space and \( B_X \) the unit ball of \( X \). We consider two Banach algebras as subspaces of complex \( C_b(B_X) \):

\[
A_b(B_X) = \{ f \in C_b(B_X) : f \text{ is holomorphic on the interior of } B_X \} \\
A_u(B_X) = \{ f \in A_b(B_X) : f \text{ is uniformly continuous on } B_X \}
\]

We shall denote by \( A(B_X) \) either \( A_b(B_X) \) or \( A_u(B_X) \).

Using the Bourgain-Stegall variational method \([3, 14]\), it is shown \([6]\) that if \( X \) is a complex Banach space with the Radon-Nikodým property, then the set of all strong peak functions in \( A(B_X) \) is dense. In case that \( X \) is locally uniformly convex, it is shown \([6]\) that the set of norm attaining elements is dense in \( A(B_X) \). That is, the set consisting of \( f \in A(B_X) \) with \( \| f \| = |f(x)| \) for some \( x \in B_X \) is dense in \( A(B_X) \).

The following corollary gives a stronger result. Notice that if \( X \) is locally uniformly convex, then every point of \( S_X \) is the strong peak point for \( A(B_X) \) [10]. Theorem 1.1 and Corollary 1.3 implies the following.

**Corollary 3.1.** Suppose that \( X \) is a locally uniformly convex, complex Banach space. Then the set of all strong peak functions in \( A(B_X) \) is a dense \( G_\delta \) subset of \( A(B_X) \). In particular, the set of all smooth points of \( B_{A(B_X)} \) contains a dense \( G_\delta \) subset of \( S_{A(B_X)} \).

Let \( X \) be a complex Banach space. Let \( A_b(B_X : X) \) to be the Banach space consisting of \( X \)-valued bounded continuous functions \( f \) from \( B_X \) to \( X \), which is also holomorphic on the interior of \( B_X \) with the sup norm

\[
\| f \| = \sup\{ \| f(x) \| : x \in B_X \}.
\]

The space \( A_u(B_X : X) \) is the subspace of \( A_b(B_X : X) \) consisting of all uniformly continuous functions on \( B_X \). We denote by \( A(B_X : X) \) either \( A_b(B_X : X) \) or \( A_u(B_X : X) \).

Let \( \Pi(X) = \{ (x, x^*) \in B_X \times B_{X^*} : x^*(x) = 1 \} \) be the topological subspace of \( B_X \times B_{X^*} \), where \( B_X \) and \( B_{X^*} \) is equipped with norm and weak-* compact topology respectively. For \( f \in A(B_X : X) \), the numerical radius \( v(f) \) of \( f \) is defined by \( v(f) = \sup\{ \| x^* f(x) \| : (x, x^*) \in \Pi(X) \} \).

The **numerical strong peak function** is introduced in [11] and the denseness of holomorphic numerical strong peak functions in \( A(B_X : X) \) is studied. The function \( f \in A(B_X : X) \) is said to be a numerical strong peak function if there is \((x, x^*)\) such that \( \lim_n |x^*_n f(x_n)| = v(f) \) for some \( \{ (x_n, x^*_n) \}_n \) in \( \Pi(X) \) implies that \((x_n, x^*_n)\) converges to \((x, x^*)\) in \( \Pi(X) \). The function \( f \in A(B_X : X) \) is said to be numerical radius attaining if there is \((x, x^*)\) in \( \Pi(X) \) such that \( v(f) = |x^* f(x)| \).
Proposition 3.2. Suppose that the set $\Pi(X)$ is complete metrizable and the set $\Gamma = \{(x, x^*) \in \Pi(X) : x \in \rho \mathcal{A}(B_X) \cap \text{sm}(B_X)\}$ is a numerical boundary. That is, $v(f) = \sup_{(x, x^*)} |x^*f(x)|$ for each $f \in \mathcal{A}(B_X : X)$. Then the set of all numerical strong peak functions in $\mathcal{A}(B_X : X)$ is a dense $G_\delta$ subset of $\mathcal{A}(B_X : X)$.

Proof. Let $A = \mathcal{A}(B_X : X)$ and let $d$ be a complete metric on $\Pi(X)$. In [11], it is shown that if $\Pi(X)$ is complete metrizable, then the set of all numerical peak functions in $A$ is a $G_\delta$ subset of $A$. We need prove the denseness. Let $f \in A$ and $\epsilon > 0$. Fix $n \geq 1$ and set
\[ U_n = \{g \in A : \exists (z, z^*) \in \Gamma \text{ with } |z^*(f - g)(z)| > \sup \{|x^*(f - g)(x)| : d((x, x^*), (z, z^*)) > 1/n\} \} \]
Then $U_n$ is open and dense in $A$. Indeed, fix $h \in A$. Since $\Gamma$ is a numerical boundary for $A$, there is a point $(w, w^*) \in \Gamma$ such that
\[ |w^*(f - h)(w)| > v(f - h) - \epsilon/2. \]
Notice that $d((x, x^*), (w, w^*)) > 1/n$ implies that there is $\delta_n > 0$ such that $\|x - w\| > \delta_n$. Choose a peak function $p \in \mathcal{A}(B_X)$ such that $\|p\| = 1 = |p(w)|$ and $|p(x)| < 1/3$ for $\|x - w\| > \delta_n$ and $|w^*(f - h)(w) - \epsilon p(w)| = |w^*(f - h)(w) - w^*h(w)| + \epsilon |p(w)| = |w^*(f - h)(w) - w^*h(w)| + \epsilon$.
Put $g(x) = h(x) + \epsilon \cdot p(x)w$, where $p$ is a peak function at $w$ such that $|p(x)| < 1/3$ for $\|x - w\| > \delta_n$,
\[ |w^*(f - g)(w)| = |w^*(f - h)(w) - \epsilon p(w)| = |w^*(f - h)(w) - w^*h(w)| + \epsilon \]
\[ > v(f - h) + \epsilon/2 \]
\[ \geq \sup \{|x^*(f - h)(x) - \epsilon p(x)x^*(w)| : d((x, x^*), (w, w^*)) > 1/n\} \]
\[ = \sup \{|x^*(f - g)(x)| : d((x, x^*), (w, w^*)) > 1/n\} \]
That is, $g \in U_n$.

By the Baire category theorem there is a $g \in \bigcap U_n$ with $\|g\| < \epsilon$, and we shall show that $f - g$ is a strong peak function. Indeed, $g \in U_n$ implies that there is $(z_n, z^*_n) \in \Gamma$ such that
\[ |z^*_n(f - g)(z_n)| > \sup \{|x^*(f - g)(x)| : d((x, x^*), (z_n, z^*_n)) > 1/n\} \]
Thus $d((z_p, z^*_p), (z_n, z^*_n)) \leq 1/n$ for every $p > n$, and hence $\{(z_n, z^*_n)\}$ converges to a point $(z, z^*)$, say. Suppose that there is another sequence $\{(x_k, x^*_k)\}$ in $\Pi(X)$ such that $\{|x^*_k(f - g)(x_k)|\}$ converges to $v(f - g)$. Then for each $n \geq 1$, there is $M_n \geq 1$ such that for every $m \geq M_n$,
\[ |x^*_m(f - g)(x_m)| > \sup \{|x^*(f - g)(x)| : d((x, x^*), (z_n, z^*_n)) > 1/n\} \]
Then \(d((x_m, x_m^*), (z_n, z_n^*)) \leq 1/n\) for every \(m \geq M_n\). Hence \(\{(x_m, x_m^*)\}\) converges to \((z, z^*)\). This shows that \(f - g\) is a numerical strong peak function at \((z, z^*)\). \(\square\)

It is shown \([11]\) that the set of all numerical radius attaining elements is dense in \(\mathcal{A}(B_X : X)\) if \(X\) is locally uniformly convex. In addition, if \(X\) is separable and smooth, we get the stronger result on \(\mathcal{A}_u(B_X : X)\).

**Corollary 3.3.** Suppose that \(X\) is separable, smooth and locally uniformly convex. Then the set of norm and numerical strong peak functions is a dense \(G_\delta\) subset of \(\mathcal{A}_u(B_X : X)\).

**Proof.** It is shown \([11]\) that if \(X\) is separable, \(\Pi(X)\) is complete metrizable. In view of \([13, \text{Theorem 2.5}]\), \(\Gamma\) is a numerical boundary for \(\mathcal{A}_u(B_X : X)\). Hence Proposition 3.2 shows that the set of all numerical strong peak functions is dense in \(\mathcal{A}_u(B_X : X)\). Finally, Corollary 3.1 implies that the set of all norm and numerical peak functions is a dense \(G_\delta\) subset of \(\mathcal{A}_u(B_X : X)\). \(\square\)

It is shown \([5]\) that the set of all strong peak points for \(\mathcal{A}(B_X)\) is dense in \(S_X\) if \(X\) is an order continuous locally uniformly \(c\)-convex sequence space. (For the definition see \([5]\)).

**Corollary 3.4.** Let \(X\) be an order continuous locally uniformly \(c\)-convex, smooth Banach sequence space. Then the set of all norm and numerical strong peak functions in \(\mathcal{A}_u(B_X : X)\) is a dense \(G_\delta\) subset of \(\mathcal{A}_u(B_X : X)\).

**Proof.** Notice that \(X\) is separable since \(X\) is order continuous. Hence the set of all smooth points of \(B_X\) is dense in \(S_X\) by the Mazur theorem and \(\Pi(X)\) is complete metrizable \([11]\). In view of \([13, \text{Theorem 2.5}]\), \(\Gamma\) is a numerical boundary for \(\mathcal{A}_u(B_X : X)\). Hence Proposition 3.2 shows that the set of all numerical strong peak functions is a dense \(G_\delta\) subset of \(\mathcal{A}_u(B_X : X)\). Theorem 1.1 also shows that the set of all strong peak functions is a dense \(G_\delta\) subset of \(\mathcal{A}_u(B_X : X)\). This completes the proof. \(\square\)

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