Operator-valued rational functions

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Abstract. In this paper we show that every inner divisor of the operator-valued coordinate function, $zI_E$, is a Blaschke-Potapov factor. We also introduce a notion of operator-valued “rational” function and then show that $\Delta$ is two-sided inner and rational if and only if it can be represented as a finite Blaschke-Potapov product; this extends to operator-valued functions the well-known result proved by V.P. Potapov for matrix-valued functions.

1 Introduction

Many properties of matrix-valued functions may not be transferred to operator-valued functions, since some properties of finite matrices are destined to fail for infinite matrices. For example, if $A$ is an $n \times n$ matrix of $H^2$-functions and $B$ is an $n \times n$ diagonal constant inner function, i.e., $B = \text{diag}(\theta, \ldots, \theta)$ (for $\theta$ an inner function), then $A$ and $B$ are left coprime if and only if $A$ and $B$ are right coprime; in other words, left-coprimeness and right-coprimeness coincide for $A$ and $B$ (cf. [CHL4, Lemma C.14]). However this is no longer true for operator-valued functions. For example, if $A(z) := S$ (the shift on $\ell^2$) and $B(z) := \theta I$ (where $I$ is the identity operator on $\ell^2$), then $A$ and $B$ are right coprime, but not left coprime (cf. [CHL4, Example C.12]).

In this paper we consider a question on left inner divisors of the “operator-valued coordinate” function $zI_E$ (where $E$ is a Hilbert space). We consider the well-known result, proved by V.P. Potapov [Po], that every rational inner $n \times n$ matrix-valued function can be written as a finite Blaschke-Potapov product. We extend this result to the case of operator-valued functions.

Let $X$ be a complex Banach space and $T$ denote the unit circle in the complex plane $\mathbb{C}$. For $1 \leq p < \infty$, let $L^p(T, X)$ be the linear space of all (equivalence classes of) strongly measurable functions $f : T \to X$ for which

$$\int_T ||f||^p dm < \infty,$$

where $m$ is the normalized Lebesgue measure on $T$. We define $L^\infty(T, X)$ as the linear space of all (equivalence classes of) strongly measurable functions $f : T \to X$ for which

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there exists \( r > 0 \) such that \( m(\{ z \in \mathbb{T} : ||f(z)|| > r \}) = 0 \). Endowed with the norms
\[
||f||_{L^p(\mathbb{T}, X)} := \left( \int_{\mathbb{T}} ||f||^p dm \right)^{\frac{1}{p}}
\]
and
\[
||f||_{L^\infty(\mathbb{T}, X)} := \inf \{ r > 0 : m(\{ z \in \mathbb{T} : ||f(z)|| > r \}) = 0 \},
\]
the spaces \( L^p(\mathbb{T}, X) \) are complex Banach spaces \( (1 \leq p \leq \infty) \). For \( f \in L^1(\mathbb{T}, X) \), the \( n \)-th Fourier coefficient of \( f \), denoted by \( \hat{f}(n) \), is defined by
\[
\hat{f}(n) := \int_{\mathbb{T}} T^n f(z) \, dm(z) \quad \text{for each } n \in \mathbb{Z},
\]
where the integral is understood in the sense of the Bochner integral. For \( 1 \leq p \leq \infty \), define
\[
H^p(\mathbb{T}, X) := \{ f \in L^p(\mathbb{T}, X) : \hat{f}(n) = 0 \text{ for } n < 0 \}.
\]
Let \( \mathcal{B}(D, E) \) denote the set of all bounded linear operators between complex Hilbert spaces \( D \) and \( E \) and abbreviate \( B(E, E) \) to \( B(E) \). For \( 1 \leq p \leq \infty \), let \( L^p(\mathbb{T}, \mathcal{B}(D, E)) \) be the set of all (SOT-measurable) \( \mathcal{B}(D, E) \)-valued functions \( \Phi \) on \( \mathbb{T} \) such that \( \Phi(\cdot)x \in L^p(\mathbb{T}, E) \) for each \( x \in D \). A function \( \Phi \in L^p(\mathbb{T}, \mathcal{B}(D, E)) \) is called a strong \( L^p \)-function. We can see that every function in \( L^p(\mathbb{T}, \mathcal{B}(D, E)) \) is a strong \( L^p \)-function. The notion of strong \( L^2 \)-function was introduced by V. Peller in [P4]; the formal theory of strong \( L^2 \)-functions was developed in [CHL4].

If \( \Phi \in L^1(\mathbb{T}, \mathcal{B}(D, E)) \) and \( x \in D \), then \( \Phi(\cdot)x \in L^1_E \). The \( n \)-th Fourier coefficient of \( \Phi \in L^1(\mathbb{T}, \mathcal{B}(D, E)) \), denoted by \( \hat{\Phi}(n) \), is given by
\[
\hat{\Phi}(n)x := \Phi(\cdot)x(n) \quad (n \in \mathbb{Z}, \ x \in D).
\]
We define
\[
H^1(\mathbb{T}, \mathcal{B}(D, E)) := \{ \Phi \in L^1(\mathbb{T}, \mathcal{B}(D, E)) : \hat{\Phi}(n) = 0 \text{ for } n < 0 \}.
\]
Let \( L^\infty(\mathbb{T}, \mathcal{B}(D, E)) \) be the space of all bounded SOT-measurable \( \mathcal{B}(D, E) \)-valued functions on \( \mathbb{T} \). Then we can easily see that
\[
L^\infty(\mathbb{T}, \mathcal{B}(D, E)) = L^\infty_\mathcal{B}(\mathbb{T}, \mathcal{B}(D, E)).
\]
Indeed, obviously \( L^\infty(\mathbb{T}, \mathcal{B}(D, E)) \subseteq L^\infty_\mathcal{B}(\mathbb{T}, \mathcal{B}(D, E)) \). For the reverse inclusion, suppose \( \Phi \in L^\infty_\mathcal{B}(\mathbb{T}, \mathcal{B}(D, E)) \). Then \( \Lambda : x \mapsto \Phi(\cdot)x \) is a closed linear transform from \( X \) into \( L^\infty(\mathbb{T}, E) \), so that, by the closed graph theorem, it is bounded. Thus for almost all \( z \in \mathbb{T} \),
\[
||\Phi(z)||_{\mathcal{B}(D, E)} = \sup_{||x|| = 1} ||\Phi(z)x||_E \leq ||\Lambda||,
\]
which implies \( \Phi \in L^\infty(\mathbb{T}, \mathcal{B}(D, E)) \). This proves (1). We will also write \( H^\infty(\mathbb{T}, \mathcal{B}(D, E)) \equiv H^\infty_\mathcal{B}(\mathbb{T}, \mathcal{B}(D, E)) \).
Write $I_E$ for the identity operator acting on $E$. Write $M_{m \times n}$ for the set of $m \times n$ complex matrices and abbreviate $M_{n \times n}$ to $M_n$. Also we abbreviate $I_{M_n}$ to $I_n$. We say that a function $\Delta \in H^\infty(\mathbb{T}, B(D, E))$ is an inner function if

$$\Delta^* \Delta = I_D \text{ a.e. on } \mathbb{T}$$

and that $\Delta$ is a two-sided inner function if $\Delta \Delta^* = I_E$ a.e. on $\mathbb{T}$ and $\Delta^* \Delta = I_D$ a.e. on $\mathbb{T}$. For a function $\Phi \in H^\infty(\mathbb{T}, B(D, E))$, we say that an inner function $\Delta \in H^\infty(\mathbb{T}, B(D', E))$ is a left inner divisor of $\Phi$ if $\Phi = \Delta A$ for some $A \in H^\infty(\mathbb{T}, B(D, D'))$ and that $\Omega \in H^\infty(\mathbb{T}, B(D, E'))$ is an right inner divisor of $\Phi$ if $\Phi = B\Omega$ for some $B \in H^\infty(\mathbb{T}, B(E', E))$. A function $\Delta$ is an inner divisor of $\Phi$ if it is both a left and a right inner divisor of $\Phi$. As customarily done, we say that two inner functions $A, B \in H^\infty(\mathbb{T}, B(E))$ are equal if they are equal up to a unitary constant right factor, i.e., there exists a unitary (constant) operator $V \in B(E)$ such that $A = BV$.

Note that if $V$ is a unitary operator in $B(E)$, then for every $\Phi \in H^\infty(\mathbb{T}, B(E))$,

$$\Phi = V(V^*\Phi) = (\Phi V^*)V,$$

which implies that $V$ is an inner divisor of $\Phi$.

For a function $\Phi \in H^\infty(\mathbb{T}, B(E))$, we say that a function $\Delta \in H^\infty(\mathbb{T}, B(E))$ is a nontrivial left (resp. right) inner divisor of $\Phi$ if $\Delta$ is a non-unitary operator and is a left (right, resp.) inner divisor of $\Phi$.

For $\alpha \in \mathbb{D}$, write

$$b_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha}z},$$

which is called a Blaschke factor. If $M$ is a closed subspace of a Hilbert space $E$, then a function of the form

$$b_m P_M + (I_E - P_M)$$

is called a (operator-valued) Blaschke-Potapov factor, where $P_M$ is the orthogonal projection of $E$ onto $M$. A function $D$ is called a (operator-valued) finite Blaschke-Potapov product if $D$ is of the form

$$D = V \prod_{m=1}^{M} \left( b_m P_m + (I - P_m) \right),$$

where $V$ is a unitary operator, $b_m$ is a Blaschke factor, and $P_m$ is an orthogonal projection in $E$ for each $m = 1, \ldots, M$. In particular, a scalar-valued function $D$ reduces to a finite Blaschke product $D = \nu \prod_{m=1}^{M} b_m$, where $\nu = e^{i\omega}$. It is known (cf. [Po]) that an $n \times n$ matrix function $D$ is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product.

On the other hand, we may ask a question: What is a left inner divisor of $zI_n$? For this question, we may guess that each left inner divisor of $zI_n$ is a Blaschke-Potapov
factor. More specifically, we wonder if a left inner divisor of \( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \equiv zI_2 \) should be of the following form up to a unitary constant right factor (also up to unitary equivalence):

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}, \quad \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}.
\]

For example, \( A \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix} \) is a left inner divisor of \( \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} \equiv zI_2 \): indeed,

\[
A \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} z & z \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}.
\]

In this case, if we take a unitary operator \( V := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \), then

\[
\begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} = V \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ 1 & z \end{bmatrix} = V \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -z \\ 1 & z \end{bmatrix} \cdot V^* \cdot V.
\]

In fact, it was shown in \([\text{CHL1}, \text{Lemma 2.5}]\) that every left inner divisor of \( zI_n \in H^\infty(\mathbb{T}, M_n) \) is a Blaschke-Potapov factor. This fact is useful for the study of coprime-ness of functions (cf. \([\text{CHL3}]\)). In \([\text{CHL4}, \text{p.23}]\), the authors asked:

**Question 1.1.** Is the statement in (2) still true for operator-valued functions?

We will call a function of the form \( zI_E \) the operator-valued coordinate function. This allows us to rephrase Question \([\text{CHL4}]\) as follows: Is every left inner divisor of the operator-valued coordinate function a Blaschke-Potapov factor? In Section 2, we give an affirmative answer to this question. In Section 3, we introduce a notion of operator-valued “rational” function and then show that \( \Delta \) is two-sided inner and rational if and only if it can be represented as a finite Blaschke-Potapov product, which extends the well-known result for the matrix-valued functions due to V.P. Potapov \([\text{Po}]\). In Section 4, we consider coprime operator-valued rational functions. Lastly, in Section 5, we take a glance at right coprime-ness and subnormality of Toeplitz operators, aiming at shedding new light on the differences between matrix-valued functions and operator-valued functions.

To proceed, we give an elementary observation.

If \( \dim E < \infty \) and \( \Theta \in H^\infty(\mathbb{T}, B(E)) \) is a two-sided inner function, then any left inner divisor of \( \Theta \) is two-sided inner (cf. \([\text{CHL3}, \text{Lemma 4.10}]\)). We can say more:

**Lemma 1.2.** (cf. \([\text{CHL4}, \text{Lemma 2.2}]\)) Let \( \Theta \in H^\infty(\mathbb{T}, B(E)) \) be a two-sided inner function. If \( \Delta \in H^\infty(\mathbb{T}, B(E)) \) is a left inner divisor of \( \Theta \), then \( \Delta \) is two-sided inner.

**Proof.** If \( \Delta \) is a left inner divisor of \( \Theta \), we may write

\[
\Theta = \Delta \Omega \quad \text{for some } \Omega \in H^\infty(\mathbb{T}, B(E)).
\]

Thus \( \Delta \) is surjective, and hence unitary a.e. on \( \mathbb{T} \). Thus \( \Delta \) is two-sided inner. \( \square \)
In particular, if $\Theta := \theta I_E$ (for $\theta$ a scalar inner function) then every left inner divisor of $\Theta$ is an inner divisor of $\Theta$. However, in general, a left inner divisor of a two-sided inner function need not be its right inner divisor. To see this, we first observe:

**Lemma 1.3.** Let $\Theta \in H^\infty(T, B(E))$ be a two-sided inner function and $\Delta \in H^\infty(T, B(E))$ be a left inner divisor of $\Theta$. Then $\Delta$ is an inner divisor of $\Theta$ if and only if $\Theta \Delta^* \in H^\infty(T, B(E))$.

**Proof.** Let $\Theta \in H^\infty(T, B(E))$ be a two-sided inner function and $\Delta \in H^\infty(T, B(E))$ be a left inner divisor of $\Theta$. Then, by Lemma 1.2, $\Delta$ is two-sided inner and we may write $\Theta = \Delta \Omega$ ($\Omega \in H^\infty(T, B(E))$).

Suppose that $\Delta$ is an inner divisor of $\Theta$. Then we can also write $\Theta = \Psi \Delta$ for some $\Psi \in H^\infty(T, B(E))$. Thus we have that $\Theta \Delta^* = \Psi \Delta \Delta^* = \Psi \in H^\infty(T, B(E))$. The converse is obvious.

We then have:

**Example 1.4.** Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical orthonormal basis for $L^2(T)$. Define $\Delta$ and $\Theta$ in $H^\infty(T, B(L^2(T)))$ by

$$
\Delta(z)e_n := \begin{cases} 
  e_{n+1}z & \text{if } n \geq 0 \\
  e_{n+1} & \text{if } n < 0
\end{cases}
$$

and

$$
\Theta(z)e_n := \begin{cases} 
  e_{-n+1}z^2 & \text{if } n \leq 1 \\
  e_{-n+1} & \text{if } n > 1
\end{cases}
$$

Then $\Theta$ and $\Delta$ are two-sided inner. Observe that

$$
\Delta^*(z)e_n = \begin{cases} 
  e_{n-1}z^{-1} & \text{if } n \geq 1 \\
  e_{n-1} & \text{if } n < 1
\end{cases}
$$

and hence

$$
\Delta^*(z)\Theta(z)e_n = \begin{cases} 
  e_{-n}z & \text{if } n < 1 \\
  e_{-1}z^2 & \text{if } n = 1 \\
  e_{-n} & \text{if } n > 1
\end{cases}
$$

Thus $\Delta$ is a left inner divisor of $\Theta$. Since $\Theta(z)\Delta^*(z)e_3 = e_{-1}z^{-1}$, it follows from Lemma 1.3 that $\Delta$ is not a right inner divisor of $\Theta$.

On the other hand, Lemma 1.2 may fail if “left” is replaced by “right.”

**Example 1.5.** Let $S$ be the shift operator on $H^2(T)$ defined by

$$(Sf)(z) := zf(z) \quad (f \in H^2(T), \ z \in T)$$

and let $\Delta(z) := S \in H^\infty(T, B(H^2(T)))$. Then

$$
\Delta(z)^* \Delta(z) = S^* S = I,
$$

which implies that $\Delta$ is a right inner divisor of (a two-sided inner function) $I$. But $\Delta$ is not two-sided inner. By Lemma 1.2, $\Delta$ is not a left inner divisor of $I$. 
2 Inner divisors of the operator-valued coordinate functions

For a complex Banach space $X$ and an open subset $G$ of $\mathbb{C}$, a function $A : G \to X$ is called holomorphic if, in any sufficiently small disc $D(\lambda, r) = \{ \zeta : |\zeta - \lambda| < r \} \subset G$, $A$ is the sum of convergent power series

$$A(\zeta) = \sum_{n=0}^{\infty} \hat{A}(n)(\zeta - \lambda)^n \quad (\hat{A}(n) \in X).$$

Denote by $\text{Hol}(G, X)$ the space of all holomorphic functions $A : G \to X$. Let us associate to any function $f$ on $D$ a family of function $f_r$ on $T$, defined by

$$f_r(z) := f(rz) \quad (0 \leq r < 1).$$

For $1 \leq p < \infty$, let $H^p(D, X)$ be the set of all functions $f \in \text{Hol}(D, X)$ satisfying

$$||f||_{H^p(D, X)} := \left( \sup_{0<r<1} \int_T ||f_r||^p dm \right)^{\frac{1}{p}} < \infty.$$ 

We define $H^\infty(D, X)$ be the set of all bounded functions $f \in \text{Hol}(D, X)$.

If $\Phi \in L^\infty(T, \mathcal{B}(D, E))$ and $\zeta = re^{i\theta} \in D$, let $P[\Phi]$ denote the Poisson integral of $f$ defined by

$$P[\Phi](\zeta)x := \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\Phi(e^{it})x \, dt \quad (x \in D; 0 \leq r < 1).$$

It is known (cf. [GHKL, Lemma 2.1]) that if $\Phi \in H^\infty(T, \mathcal{B}(D, E))$ and $\Psi \in H^\infty(T, \mathcal{B}(D', E))$, then

$$P[\Phi \Psi] = P[\Phi]P[\Psi].$$

We now consider Question 1.1. We actually wish to study a more general case, and we first observe that if $A$ is an inner divisor of $z^NI_E$, then there exists a function $\Omega \in H^\infty(T, \mathcal{B}(E))$ such that

$$A\Omega = \Omega A = z^NI_E \quad \text{a.e. on } T,$$

so that

$$A^* z^NI_E = \Omega \in H^\infty(T, \mathcal{B}(E)),$$

which implies that $A$ is a polynomial of degree at most $N$.

We are ready for:

**Theorem 2.1.** Let $A$ be a polynomial of degree $N$. Then the following are equivalent:

(a) $A$ is an inner divisor of $z^NI_E$;
(b) \( \hat{A}(n) \) and \( \hat{A}(n)^* \) are partial isometries for each \( n = 0, 1, \cdots, N \), and

\[
E = \bigoplus_{n=0}^{N} \text{ran} \hat{A}(n) = \bigoplus_{n=0}^{N} \text{ran} \hat{A}(n)^*;
\]

(c) \( A \) is a finite Blaschke-Potapov product of the form

\[
A(z) = V \prod_{m=1}^{N} (zP_m + (I - P_m)),
\]

where \( P_m \) is the orthogonal projection from \( E \) onto \( \bigoplus_{n=m}^{N} \text{ran} \hat{A}(n)^* \), and \( V := \text{diag}(\hat{A}(0)_{\text{ran} \hat{A}(0)^*}, \hat{A}(1)_{\text{ran} \hat{A}(1)^*}, \cdots, \hat{A}(N)_{\text{ran} \hat{A}(N)^*}) \).

**Proof.** (a) \( \Rightarrow \) (b): Suppose that \( A \) is an inner divisor of \( z^N I_E \). Without loss of generality we may assume that \( A_0 \) and \( A_N \) are nonzero. Write

\[
A^{(1)}(z) := \sum_{n=1}^{N} \hat{A}(n) z^{n-1}.
\]

Then

\[
A(z) = \hat{A}(0) + z A^{(1)}(z).
\]

By Lemma 1.2 \( A \) is two-sided inner. Thus, for almost all \( z \in T \),

\[
A^{(1)}(z) \hat{A}(0) = 0 \quad \text{and} \quad \hat{A}(0) A^{(1)}(z) + A^{(1)}(z)^* A^{(1)}(z) = I_E,
\]

so that

\[
\hat{A}(0) A^{(1)}(z) \hat{A}(0) = \hat{A}(0) \quad \text{and} \quad A^{(1)}(z) A^{(1)}(z)^* A^{(1)}(z) = A^{(1)}(z),
\]

which implies that \( \hat{A}(0) \) and \( A^{(1)}(z) \) are partial isometries for almost all \( z \in T \) (cf. [Ha2]). Since \( A \) is inner, it follows from (5) that, for almost all \( z \in T \),

\[
\ker \hat{A}(0) \bigcap \ker A^{(1)}(z) = \{0\}.
\]

Since \( (H \cap K)^\perp = H^\perp \bigcup K^\perp \) for closed subspaces \( H, K \) of \( E \), it follows that

\[
\text{ran} \hat{A}(0)^* \bigcup \text{ran} A^{(1)}(z)^* = E.
\]

On the other hand, it follows from (5) that \( \text{ran} \hat{A}(0)^* \subseteq (\text{cl ran} A^{(1)}(z)^*)^\perp \) and, by (7), we have that

\[
\text{ran} \hat{A}(0)^* \bigoplus (\text{ran} A^{(1)}(z)^*)^\perp = E.
\]

Thus, for almost all \( z \in T \),

\[
(\ker A^{(1)}(z))^\perp = \text{ran} A^{(1)}(z)^* = \ker \hat{A}(0).
\]
Similarly, we can show that, for almost all \( z \in T \),
\[
\text{ran} \ A^{(1)}(z) = \ker \ A(0)^* = (\text{ran} \ A(0))^\perp. \tag{8}
\]
Since \( A^{(1)}(z) \) is a partial isometry, \( A^{(1)}|_{\ker \ A(0)} : T \to B(\ker \ A(0), (\text{ran} \ A(0))^\perp) \) is two-sided inner. Now we will show that
\[
\text{ran} \ A(1) \subseteq (\text{ran} \ A(0))^\perp. \tag{9}
\]
For \( x \in E \), it follows from (8) that
\[
A^{(1)}(z)x = \sum_{n=1}^{N} \hat{A}(n)xz^{n-1} \in (\text{ran} \ A(0))^\perp \text{ for almost all } z \in T
\]
\[
\Rightarrow \langle \hat{A}(n)x, \hat{A}(0)y \rangle = 0 \text{ for all } y \in E
\]
\[
\Rightarrow P[A^{(1)}](\zeta)x = \sum_{n=1}^{N} \hat{A}(n)x\zeta^{n-1} \in (\text{ran} \ A(0))^\perp \text{ for all } \zeta \in \mathbb{D}
\]
\[
\Rightarrow \hat{A}(1)x = P[A^{(1)}|(0)]x \in (\text{ran} \ A(0))^\perp,
\]
which proves (9). Put \( A^{(2)}(z) := \sum_{n=2}^{N} \hat{A}(n)z^{n-2} \). Then, by the same argument, we have that
\[(i) \ \hat{A}(1) \text{ is a partial isometry;}
\]
\[(ii) \ (\ker A^{(2)}(z))^\perp = \ker \hat{A}(1) \text{ and } \text{ran} \ A^{(2)}(z) = (\text{ran} \ A(0) \bigoplus \text{ran} \ A(1))^\perp;
\]
\[(iii) \ A^{(2)}|_{\ker \hat{A}(1)} : T \to B(\ker \hat{A}(1), (\text{ran} \ A(1))^\perp) \text{ is two-sided inner;}
\]
\[(iv) \ \text{ran} \ A(2) \subseteq (\text{ran} \ A(0) \bigoplus \text{ran} \ A(1))^\perp.
\]
Continuing this process, we have that \( \hat{A}(n) \) is a partial isometry for each \( n = 0, 1, \ldots, N \), and
\[
\text{ran} \ A(N) \subseteq \left( \bigoplus_{n=0}^{N-1} \text{ran} \ A(n) \right)^\perp.
\]
But since \( A(z) \) is unitary for almost all \( z \in T \), we have that
\[
\bigoplus_{n=0}^{N} \text{ran} \ A(n) = E.
\]
Similarly, we also have that \( \hat{A}(n)^* \) is a partial isometry for each \( n = 0, 1, \ldots, N \), and
\[
E = \bigoplus_{n=0}^{N} \text{ran} \ A(n)^*.
(b) ⇒ (c): Suppose \( \hat{A}(n) \) and \( \hat{A}(n)^* \) are partial isometries for each \( n = 0, 1, \cdots, N \), and
\[
E = \bigoplus_{n=0}^{N} \text{ran} \hat{A}(n) = \bigoplus_{n=0}^{N} \text{ran} \hat{A}(n)^*.
\]
Write \( E_n := \text{ran} \hat{A}(n) \) and \( F_n := \text{ran} \hat{A}(n)^* \). Then we can write
\[
A(z) = \begin{bmatrix}
\hat{A}(0)|_{F_0} & 0 & 0 & \cdots & 0 \\
0 & (\hat{A}(1)|_{F_1})z & 0 & \cdots & 0 \\
\vdots & 0 & (\hat{A}(2)|_{F_2})z^2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & (\hat{A}(N)|_{F_N})z^N
\end{bmatrix} : \bigoplus_{n=0}^{N} F_n \to \bigoplus_{n=0}^{N} E_n.
\]
Let \( P_m \) be the orthogonal projection from \( E \) onto \( \bigoplus_{k=m}^{N} F_k \) (\( m = 1, 2, \cdots, N \)). Then
\[
A(z) = V \prod_{m=1}^{N} \left( zP_m + (I - P_m) \right),
\]
where \( V := \text{diag}(\hat{A}(0), \hat{A}(1), \cdots, \hat{A}(N)) \) is unitary.

(c) ⇒ (a): Obvious.

This completes the proof. \( \square \)

The following corollary gives an affirmative answer to Question 1.1.

**Corollary 2.2.** If \( \Delta \in H^\infty(T, \mathcal{B}(E)) \) is a left inner divisor of \( zI_E \), then \( \Delta \) is a Blaschke-Potapov factor.

**Proof.** Immediate from Theorem 2.1 \( \square \)

We recall:

**Lemma 2.3.** \([N2] \text{ Theorem 3.11.10} \) Let \( A \in H^\infty(\mathbb{D}, \mathcal{B}(D, E)) \). Then the nontangential SOT limit
\[
\lim_{r \to 1} A(rz) \equiv bA(z)
\]
exists for almost all \( z \in \mathbb{T} \), such that \( bA \in H^\infty(\mathbb{T}, \mathcal{B}(D, E)) \) and
\[
A(\zeta)x = P[bA](\zeta)x,
\]
for \( x \in D \) and \( \zeta \in \mathbb{D} \). The Taylor coefficients of \( A \) coincide with the nonnegatively-indexed Fourier coefficients of \( bA \). Moreover, the mapping \( b : A \to bA \) is an isometric bijection from \( H^\infty(\mathbb{D}, \mathcal{B}(D, E)) \) onto \( H^\infty(\mathbb{T}, \mathcal{B}(D, E)) \).
For $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(E))$ and $\alpha \in \mathbb{D}$, write
\[ \Phi_\alpha := \Phi \circ b_\alpha. \]
Then we can easily check the following (cf. [GHKL]):
(a) $\Phi_\alpha \in H^\infty(\mathbb{T}, \mathcal{B}(E))$;
(b) If $\Delta$ is an inner function with values in $\mathcal{B}(E)$, then so is $\Delta_\alpha$.

We then have:

**Corollary 2.4.** Let $A \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. Then the following are equivalent:
(a) $A$ is an inner divisor of $b^N \mathbb{I}_E$;
(b) $A$ is a polynomial in $b_\alpha$ of degree at most $N$, of the form
\[ A = \sum_{n=0}^{N} A_n b^n_\alpha, \]
where $A_n$ and $A^*_n$ are partial isometries for each $n = 0, 1, \cdots, N$, and
\[ E = \bigoplus_{n=0}^{N} \text{ran} A_n = \bigoplus_{n=0}^{N} \text{ran} A^*_n; \]
(c) $A$ is a finite Blaschke-Potapov product of the form
\[ A = V \prod_{m=1}^{N} \left( b_\alpha P_m + (I - P_m) \right), \]
where $P_m$ is the orthogonal projection from $E$ onto $\bigoplus_{n=m}^{N} \text{ran} A^*_n$, and $V := \text{diag}(A_0|\text{ran} A^*_0, A_1|\text{ran} A^*_1, \cdots, A_N|\text{ran} A^*_N)$.

**Proof.** Observe that
(i) $A(z) = \sum_{n=0}^{N} A_n b_\alpha(z)^n \iff A_{-\alpha}(z) = \sum_{n=0}^{N} A_n z^n$;
(ii) $A$ is an inner divisor of $b^N \mathbb{I}_E \iff A_{-\alpha}$ is an inner divisor of $z^N \mathbb{I}_E$;
(iii) $A(z) = V \prod_{m=1}^{N} \left( b_\alpha P_m + (I - P_m) \right) \iff A_{-\alpha}(z) = V \prod_{m=1}^{N} \left( z P_m + (I - P_m) \right)$. 
Thus the result follows at once from Theorem 2.1.

**Corollary 2.5.** Let $A \in H^\infty(\mathbb{T}, \mathcal{B}(E))$. If $A$ is a nontrivial inner divisor of $b^N \mathbb{I}_E$, then $\hat{A}_{-\alpha}(n) \neq 0$ for some $n = 1, 2, \cdots, N$.

**Proof.** Immediate from Corollary 2.4. □
3 Operator-valued rational functions

In this section we will introduce the notion of operator-valued “rational” function. Recall that a matrix-valued function is rational if its entries are rational functions. But this definition is not appropriate for operator-valued functions, in particular $H^\infty$-functions, even though the terminology of “entry” may be properly interpreted. Thus, the new idea should capture and encapsulate a definition of operator-valued rational function which is equivalent to the condition that each entry is rational when the function is matrix-valued. In the sequel, we give a formal definition of operator-valued rational function.

To do so, we first recall Toeplitz and Hankel operators. For $\Phi \in L^\infty(T, B(D, E))$, the Hankel operator $H_\Phi : H^2(T, D) \to H^2(T, E)$ is defined by

$$H_\Phi f := JP_- (\Phi f) \quad (f \in H^2(T, D)),$$

where $J$ denotes the unitary operator from $L^2(T, E)$ to $L^2(T, E)$ given by $(Jg)(z) := zg(z)$ for $g \in L^2(T, E)$ and $P_-$ is the orthogonal projection from $L^2(T, E)$ onto $L^2(T, E) \ominus H^2(T, E)$. Also a Toeplitz operator $T_\Phi : H^2(T, D) \to H^2(T, E)$ is defined by

$$T_\Phi f := P_+ (\Phi f) \quad (f \in H^2(T, D)),$$

where $P_+$ is the orthogonal projection from $L^2(T, E)$ onto $H^2(T, E)$.

For a $B(D, E)$-valued function $\Phi$, write

$$\tilde{\Phi}(z) := \Phi(z) \quad \text{and} \quad \bar{\Phi} := \tilde{\Phi}^*.$$

As usual, a shift operator $S_E$ on $H^2(T, E)$ is defined by

$$(S_E f)(z) := zf(z) \quad \text{for each } f \in H^2(T, E).$$

The following theorem is a fundamental result in modern operator theory.

The Beurling-Lax-Halmos Theorem. [CHL4], [Ha2], [FF], [Pe] A subspace $M$ of $H^2(T, E)$ is invariant for the shift operator $S_E$ on $H^2(T, E)$ if and only if

$$M = \Delta H^2(T, E'),$$

where $E'$ is a subspace of $E$ and $\Delta$ is an inner function with values in $B(E', E)$. Furthermore, $\Delta$ is unique up to a unitary constant right factor, i.e., if $M = \Theta H^2(T, E'')$, where $\Theta$ is an inner function with values in $B(E'', E)$, then $\Delta = \Theta V$, where $V$ is a unitary operator from $E'$ onto $E''$.

If $\Phi \in L^\infty(T, B(D, E))$ then, by the Beurling-Lax-Halmos Theorem,

$$\ker H_{\Phi^*} = \Delta H^2(T, E')$$

for some inner function $\Delta$ with values in $B(E', E)$. We note that $E'$ may be the zero space and $\Delta$ need not be two-sided inner.

We recall:
**Definition 3.1.** [CHL4] A function \( \Phi \in L^2(T, B(D, E)) \) is said to be of bounded type if \( \ker H_\Phi^* = \Theta H^2_T \) for some two-sided inner function \( \Theta \) with values in \( B(E) \).

It is known that if \( \phi \equiv (\phi_{ij}) \) is a matrix-valued function of bounded type then each entry \( \phi_{ij} \) is of bounded type, i.e., \( \phi_{ij} \) is a quotient of two bounded analytic functions. In particular, it is also known ([Ab], [CHL1]) that if \( \phi \in L^2 \) is of bounded type then \( \phi \) can be written as

\[
\phi = \theta a \quad (\text{where } \theta \text{ is an inner function and } a \in H^2). \tag{10}
\]

For \( \Phi, \Psi \in H^\infty(T, B(D_1, E)) \) and \( \Delta \) is a two-sided inner function with values in \( B(E) \), we say that \( \Phi \) and \( \Psi \) are left coprime if \( \Phi \) and \( \Psi \) has no common nontrivial left inner divisor. Also, we say that \( \Phi \) and \( \Psi \) are right coprime if \( \tilde{\Phi} \) and \( \tilde{\Psi} \) are left coprime.

**Lemma 3.2.** [CHL4, Lemma 2.4] If \( \Phi \in L^\infty(T, B(D, E)) \) and \( \Delta \) is a two-sided inner function with values in \( B(E) \), then the following are equivalent:

(a) \( \tilde{\Phi} \) is of bounded type, i.e., \( \ker H_{\Phi^*} = \Delta H^2(T, E) \);

(b) \( \Phi = \Delta A^* \), where \( A \in H^\infty(T, B(E, D)) \) is such that \( \Delta \) and \( A \) are right coprime.

We now introduce:

**Definition 3.3.** A function \( \Phi \in H^\infty(T, B(D, E)) \) is said to be rational if

\[
\theta H^2(T, E) \subseteq \ker H_{\Phi^*}. \tag{11}
\]

for some finite Blaschke product \( \theta \).

Observe that if \( \Phi \in H^\infty(T, B(D, E)) \), then

\[
\Phi \text{ is rational } \iff \tilde{\Phi} \text{ is of bounded type.} \tag{12}
\]

To see this, suppose \( \Phi \) is rational. By definition and the Beurling-Lax-Halmos Theorem there exist a finite Blaschke product \( \theta \) and an inner function \( \Delta \in H^\infty(T, B(E', E')) \) such that

\[
\theta H^2(T, E) \subseteq \ker H_{\Phi^*} = \Delta H^2(T, E'),
\]

which implies that \( \Delta \) is a left inner divisor of \( \theta I_E \). Thus \( \Delta \) is two-sided inner, so that by Lemma 3.2, \( \Phi \) is of bounded type, which proves (12).

Also, if \( \Phi \equiv (\phi_{ij}) \in H^\infty(T, M_{m \times n}) \) is a rational function in the sense of Definition 3.3 then each entry \( \phi_{ij} \) is rational. To see this suppose a matrix-valued function \( \Phi \) satisfies the condition (11). Put \( A := \theta \Phi^* \). Then \( A \in H^\infty(T, M_{n \times m}) \) and \( \Phi = \theta A^* \). Thus \( \phi_{ij} \) can be written as

\[
\phi_{ij} = \theta a_{ij} \quad (a_{ij} \in H^\infty).
\]

Via the Kronecker’s Lemma [Ni1, p.183], we can see that

\[
\phi_{ij} \text{ is rational } \iff \phi_{ij} = \theta a_{ij} \text{ with a finite Blaschke product } \theta,
\]
which says that each $\phi_{ij}$ is rational.

In particular, if $\theta = z^n$ in (11), $\Phi$ becomes an operator-valued polynomial.

In 1955, V.P. Potapov [Po] proved that an $n \times n$ matrix-valued function $\Phi$ is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product. In this section, we extend this result to operator-valued functions. To do so, we first observe:

**Lemma 3.4.** Suppose that $\theta$ is a finite Blaschke product of the form

$$\theta = \prod_{n=1}^{M} b_{\alpha_n} \quad (\alpha_n \in \mathbb{D})$$

and $\Delta$ is an inner divisor of $\Theta = \theta I_E$. Let $\Omega := \Theta \Delta^*$ and $P_n$ be the orthogonal projection of $E$ onto $\text{cl ran } P[\Omega](\alpha_n)$. Then $\Delta(b_{\alpha_n} P_n + (I_E - P_n))^* = \Delta$ is an inner divisor of $\Theta_n := \theta b_{\alpha_n} I_E$ for each $n = 1, 2, \cdots, M$.

Proof. Write

$$A := P[\Delta] \quad \text{and} \quad C := P[\Omega].$$

Then it follows from (11) that

$$P[\Theta] = AC = CA.$$ 

Since $0 = P[\Theta](\alpha_n) = A(\alpha_n) C(\alpha_n)$, we have that $\text{cl ran } C(\alpha_n) \subseteq \ker A(\alpha_n)$. Thus we may write

$$A(\alpha_n) = A(\alpha_n) b_{\alpha_n} P_n + (I_E - P_n) \quad \text{for some } A_n \in H^\infty(\mathbb{D}, \mathcal{B}(E)).$$

It thus follows from (13) that

$$A = A(\alpha_n) b_{\alpha_n} P_n + (I_E - P_n) + A_n(P_n + b_{\alpha_n}(I_E - P_n))(b_{\alpha_n} P_n + (I_E - P_n)) = \left[A(\alpha_n) + A_n(P_n + b_{\alpha_n}(I_E - P_n))\right](b_{\alpha_n} P_n + (I_E - P_n)).$$

Since $A(\alpha_n) + A_n(P_n + b_{\alpha_n}(I_E - P_n)) \in H^\infty(\mathbb{D}, \mathcal{B}(E))$, it follows from Lemma 2.3 that

$$\Delta_n := b(A(\alpha_n) + A_n(P_n + b_{\alpha_n}(I_E - P_n))) \in H^\infty(T, \mathcal{B}(E))$$

and by (12),

$$\Delta = b(A) = \Delta_n(b_{\alpha_n} P_n + (I_E - P_n)). \quad (14)$$

Now write $B_n := b_{\alpha_n} P_n + (I_E - P_n)$. Then $(B_n C)(\alpha_n) = (I_E - P_n) C(\alpha_n) = 0$. Thus we can write

$$B_n C = b_{\alpha_n} I_E b(F) \quad \text{for some } F \in H^\infty(\mathbb{D}, \mathcal{B}(E)).$$

Thus $P[\Theta] = AC = AB_n^*(b_{\alpha_n} I_E) b(F)$ and hence, by (14) and (14), we have

$$\Theta = \Delta_n(b_{\alpha_n} I_E)b(F),$$

$$\Theta = \Delta_n(b_{\alpha_n} I_E)b(F),$$
so that
\[ \theta b_{\alpha_n} I_E = \Delta_n b(E). \]

This completes the proof. \(\square\)

We then have:

**Theorem 3.5.** Let \( \theta \) be a finite Blaschke product. If \( \Delta \) is an inner divisor of \( \Theta = \theta I_E \), then \( \Delta \) is a finite Blaschke-Potapov product.

**Proof.** Suppose \( \theta \) is a finite Blaschke product of the form
\[ \theta = \prod_{n=1}^{M} b_{\alpha_n}. \]

If \( \Delta \) is an inner divisor of \( \Theta = \theta I_E \), then \( \Omega := \Theta \Delta^* \) is also inner divisor of \( \Theta \). Let \( P_n \) be the orthogonal projection of \( E \) onto \( \text{cl ran} P[\Omega](\alpha_n) \). Then it follows from Lemma 3.4 that \( \Delta_1 := \Delta (b_{\alpha_M} P_M + (I_E - P_M))^* \) is an inner divisor of \( \theta b_{\alpha_M} I_E \). By the same argument we have that
\[ \Delta_2 := \Delta_1 (b_{\alpha_{M-1}} P_{M-1} + (I_E - P_{M-1}))^* = \Delta (b_{\alpha_M} P_M + (I_E - P_M))^* (b_{\alpha_{M-1}} P_{M-1} + (I_E - P_{M-1}))^* \]
is an inner divisor of \( \theta b_{\alpha_{M-1}} b_{\alpha_M} I_E \). Continuing this process, we have that
\[ \Delta_M := \Delta \prod_{n=0}^{M-1} (b_{\alpha_{M-n}} P_{M-n} + (I_E - P_{M-n}))^* \]
is an inner divisor of \( I_E \). Thus \( V \equiv \Delta_M \) is a unitary operator, and hence
\[ \Delta = V \prod_{n=1}^{M} (b_{\alpha_n} P_n + (I_E - P_n)). \]
is a finite Blaschke-Potapov product. This completes the proof. \(\square\)

**Corollary 3.6.** A function \( \Phi \in H^\infty(T, B(D, E)) \) is rational if and only if
\[ \Phi = \Delta A^*, \]
where \( \Delta \) is a finite Blaschke-Potapov product and \( A \in H^\infty(T, B(E, D)) \) is such that \( \Delta \) and \( A \) are right coprime.

**Proof.** This follows from Lemma 3.2 (12) and Theorem 3.5 \(\square\)

We are ready for:
Corollary 3.7. A two-sided inner function $\Phi \in H^{\infty}(T,B(E))$ is rational if and only if it can be represented as a finite Blaschke-Potapov product.

Proof. Suppose that $\Phi$ is rational and two-sided inner. Then it follows from Corollary 3.6 that

$$\Phi = \Delta A^*, \quad (15)$$

where $\Delta$ is a finite Blaschke-Potapov product and $A \in H^{\infty}(T,B(E))$. Since $\Phi$ and $\Delta$ are two-sided inner, so is $A$. Thus, by (15), $\Phi$ is a left inner divisor of $\Delta$, and hence the result follows from Theorem 3.5. The converse is clear. This completes the proof.

4 Coprime operator-valued rational functions

In this section we consider coprime operator-valued rational functions.

Lemma 4.1. Let $\Phi \in H^{\infty}(T,B(E))$. If $P[\Phi](\alpha)$ has no dense range for $\alpha \in \mathbb{D}$, then

$$P := b_{\alpha} P_M + (I - P_M) \quad (M := \ker P[\Phi](\alpha)^*)$$

is a nontrivial left inner divisor of $\Phi$.

Proof. Write $A := P[\Phi]$; that is, $A$ is the Poisson integral of $\Phi$, defined by (3). Suppose that the range of $A(\alpha)$ is not dense. Then $M := \ker A(\alpha)^* = (\text{cl ran } A(\alpha))^\perp \neq \{0\}$. Put $P := b_{\alpha} P_M + (I - P_M)$. Then $(P^* b_{\alpha} I_E A)(\alpha) = 0$, and hence we can write

$$P^* b_{\alpha} I_E A = b_{\alpha} I_E A_1 \quad \text{for some } A_1 \in H^{\infty}(\mathbb{D},B(E)),$$

which implies that $A = PA_1$. This completes the proof.

For an inner function $\theta$, let $Z(\theta)$ be the set of all zeros of $\theta$. Then we have:

Theorem 4.2. Let $\Phi \in H^{\infty}(T,B(E))$ and $\Theta := \theta I_E$ with a finite Blaschke product $\theta$. Then the following statements are equivalent:

(a) $P[\Phi](\alpha)$ has dense range for each $\alpha \in Z(\theta)$;

(b) $\Phi$ and $\Theta$ are left coprime.

Proof. (a) $\Rightarrow$ (b): Suppose that $\Phi$ and $\Theta$ are not left coprime. Then by Theorem 3.5, there exist $\alpha_0 \in Z(\theta)$ and a nonzero subspace $M$ of $E$ such that

$$\Phi = (b_{\alpha_0} P_M + (I - P_M))\Omega,$$

where $\Omega \in H^{\infty}(T,B(E))$. Thus $\text{cl ran } P[\Phi](\alpha_0) \subseteq M^\perp \neq E$.

(b) $\Rightarrow$ (a): This follows from at once form Lemma 4.1.
Lemma 4.3. If $\Phi \in \mathcal{H}^\infty(T, B(D, E))$, then $\tilde{\Phi} \in \mathcal{H}^\infty(T, B(E, D))$. In this case,

$$P[\tilde{\Phi}] = \tilde{P[\Phi]}$$

Proof. Since $\tilde{\Phi}(n) = \Phi(n)^*$ for all $n = 0, 1, 2, \cdots$, it follows that

$$P[\tilde{\Phi}] (\zeta) = \sum_{n=0}^{\infty} \tilde{\Phi}(n)^* \zeta^n = \tilde{P[\Phi]} (\zeta)$$

Corollary 4.4. Let $\Phi \in \mathcal{H}^\infty(T, B(E))$. If $P[\Phi](\alpha)$ is not injective for $\alpha \in \mathbb{D}$, then

$$P := b_\alpha P_M + (I - P_M) \quad (M := \ker P[\Phi](\alpha))$$

is a nontrivial right inner divisor of $\Phi$.

Proof. Suppose that $P[\Phi](\alpha)$ is not injective. Then, by Lemma 4.3, $(P[\Phi](\alpha))^* = P[\tilde{\Phi}](\alpha)$ has no dense range. Let

$$Q := b_{\alpha} P_M + (I - P_M),$$

where $M := \ker P[\Phi](\alpha) = \ker P[\tilde{\Phi}](\alpha)^* \neq \{0\}$. Then it follow from Lemma 4.1 that $Q$ is a nontrivial left inner divisor of $\Phi$. But since $Q$ is two-sided inner, it follows that $P = \tilde{Q}$ is a nontrivial right inner divisor of $\Phi$. This completes the proof.

We also have:

Corollary 4.5. Let $\Phi \in \mathcal{H}^\infty(T, B(E))$ and $\Theta := \theta I_E$ with a finite Blaschke product $\theta$. Then the following statements are equivalent:

(a) $P[\Phi](\alpha)$ is injective for each $\alpha \in \mathcal{Z}(\theta)$;

(b) $\Phi$ and $\Theta$ are right coprime.

Proof. Immediate from Theorem 4.2 and Lemma 4.3

Corollary 4.6. Let $\Phi \in \mathcal{H}^\infty(T, M_n)$ and $\Theta := \theta I_n$ with a finite Blaschke product $\theta$. Then the following statements are equivalent:

(a) $P[\Phi](\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$;

(b) $\Phi$ and $\Theta$ are right coprime;

(c) $\Phi$ and $\Theta$ are left coprime.

Proof. The equivalence (a) $\leftrightarrow$ (b) follows from Theorem 4.2 and Corollary 4.5 together with matrix theory. The equivalence (b) $\leftrightarrow$ (c) comes from [CHL2, Lemma 3.3].

The equivalence (b) $\leftrightarrow$ (c) of Corollary 4.6 may fail for operator-valued functions. For example, if we take $E = \ell^2(\mathbb{Z}^+)$, then $S_E$ and $\zeta I_E$ are right coprime, but not left coprime (cf. [CHL4]).
5 Miscellany

In this section, we establish some key differences between matrix-valued functions and operator-valued functions.

5.1 A glance at right coprime-ness

If $\Phi$ and $\Psi$ are not left coprime then there exists a common nontrivial left inner divisor $\Delta$ of both $\Phi$ and $\Psi$. However we don’t guarantee that this is still true for right coprime-ness. In other words, if $\Phi$ and $\Psi$ are not right coprime then by definition $\Phi = A\Delta$ and $\Psi = B\Delta$ for some nontrivial inner function $\tilde{\Delta}$. However we need not expect that $\Delta$ is inner.

We here give a sufficient condition for the existence of a common nontrivial right inner divisor of two functions when they are not right coprime.

To see this, we first recall that a function $F \in H^\infty(T, \mathcal{B}(E',E))$ is called outer if $\cl F H^2(T, E) = H^2(T, E)$. We then have an analogue of the scalar factorization theorem (called the inner-outer factorization):

**The inner-outer factorization** [Nil]. If $A \in H^\infty(T, \mathcal{B}(D, E))$, then we can write $A = A^i A^e$ (inner-outer factorization), where $E'$ is a subspace of $E$, $A^i \in H^\infty(T, \mathcal{B}(E',E))$ is an inner function, and $A^e \in H^\infty(T, \mathcal{B}(D, E'))$ is an outer function.

The following lemma is a characterization of functions of bounded type.

**Lemma 5.1.** [CHL4 Corollary 2.25.] Let $\Omega$ be an inner function with values in $\mathcal{B}(D, E)$. Then $\tilde{\Omega}$ is of bounded type $\iff [\Omega, \Omega_c]$ is two-sided inner, where $\Omega_c$ is the complementary factor of $\Omega$, i.e., $\ker H_\Omega = [\Omega, \Omega_c] H^2_{D \oplus D'}$ for some Hilbert space $D'$, and $[\Omega, \Omega_c]$ denotes the $1 \times 2$ matrix whose entries are $\Omega$ and $\Omega_c$.

We then have:

**Theorem 5.2.** Suppose that $\Phi \in H^\infty(T, \mathcal{B}(D, E_1))$ and $\Psi \in H^\infty(T, \mathcal{B}(D, E_2))$ are not right coprime. If there exists a nontrivial left inner divisor $\Omega$ of $\Delta := \text{left-g.c.d.}(\Phi, \Psi)$ and $\tilde{\Omega}$ is of bounded type, then $[\Omega, \Omega_c]$ is a common nontrivial right inner divisor of both $\Phi$ and $\Psi$.

**Proof.** Since $\Phi \in H^\infty(T, \mathcal{B}(D, E_1))$ and $\Psi \in H^\infty(T, \mathcal{B}(D, E_2))$ are not right coprime, $\Delta := \text{left-g.c.d.}(\Phi, \Psi) \in H^\infty(T, \mathcal{B}(D_1, D))$ is not a unitary operator and we can write

$$\Phi = \Delta \Phi_1 \quad \text{and} \quad \Psi = \Delta \Psi_1,$$

(16)
where $\tilde{\Phi}_1 \in H^\infty(\mathbb{T}, \mathcal{B}(E_1, D_1))$ and $\tilde{\Psi}_1 \in H^\infty(\mathbb{T}, \mathcal{B}(E_2, D_1))$. Since $\Omega$ is a left inner divisor of $\Delta$, we can write

$$\Delta = \Omega \Delta_1 \quad (\Delta_1 \in H^\infty(\mathbb{T}, \mathcal{B}(D_1, D_2)) \quad (\Omega \in H^\infty(\mathbb{T}, \mathcal{B}(D_2, D))) (17)$$

Since $\tilde{\Omega}$ is not a unitary operator and is of bounded type, by Lemma 5.1 $\Omega_0 \equiv [\Omega, \Omega_c]$ is not a unitary operator and a two-sided inner function. Note that $\Omega_c$ is an inner function with values in $\mathcal{B}(D_3, D)$. Thus we can write

$$\Delta = \Omega \Delta_1 = [\Omega, \Omega_c] \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} = \Omega_0 \Delta_0 \quad \text{(where } 0 : D_4 \to D_3 \text{).}$$

It thus follows from (16) and (17) that

$$\Phi = \tilde{\Phi}_1 \Delta_0 \tilde{\Omega}_0 \quad \text{and} \quad \Psi = \tilde{\Psi}_1 \Delta_0 \tilde{\Omega}_0.$$ 

But since $\Omega_0$ is two-sided inner, we have that $\tilde{\Omega}_0$ is (two-sided) inner. This completes the proof.

**Corollary 5.3.** Let $\Phi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_1))$, $\Psi \in H^\infty(\mathbb{T}, \mathcal{B}(D, E_2))$ and $\Delta := \text{left-g.c.d.}(\tilde{\Phi}, \tilde{\Psi})$. If $\Delta$ is of bounded type, then $[\Delta, \Delta_c]$ is a common nontrivial right inner divisor of both $\Phi$ and $\Psi$.

**Proof.** Immediate from Theorem 5.2.

### 5.2 Subnormality of Toeplitz operators

In 1970, P.R. Halmos posed the following problem, listed as Problem 5 in his series of lectures, “Ten problems in Hilbert space” [Ha1]:

Is every subnormal Toeplitz operator either normal or analytic?

Halmos’ Problem 5 has been partially answered in the affirmative by many authors. However, in 1984, Halmos’ Problem 5 was answered in the negative by C. Cowen and J. Long [CoL]. Despite considerable efforts, to date researchers have been unable to characterize subnormal Toeplitz operators in terms of their symbols. Thus we have:

**Halmos’ Problem 5 reformulated.** Which Toeplitz operators are subnormal?

For cases of matrix-valued symbols, the subnormality of Toeplitz operators was studied in [CHL2], in which it was shown that if the matrix-valued symbol $\Phi$ satisfies a general condition on coprime factorization and $T_\Phi$ is subnormal then it is either normal or analytic. Also in [CHKL], it was conjectured that every subnormal Toeplitz operator with matrix-valued rational symbol is unitarily equivalent to a direct sum of a normal operator and a Toeplitz operator with analytic symbol. In fact, if an $n \times n$ matrix-valued function $\Phi$ is analytic then the normal extension of $T_\Phi$ is the multiplication operator $M_\Phi$, so clearly
\( T_\Phi \) is subnormal. However, this is not the case for the operator-valued symbols. In this section we will give an example (see Example 5.5 below). On the other hand, if \( \Phi \) is matrix-valued and \( T_\Phi \) is subnormal (even hyponormal), then \( \Phi \) should be normal, i.e., \( \Phi^* \Phi = \Phi \Phi^* \) a.e. on \( \mathbb{T} \) (cf. \cite{GHR}). However, this may also fail for operator-valued symbols.

**Example 5.4.** Let \( S := T_z \) on \( H^2(\mathbb{T}) \) and \( \Phi(z) = Sz^n \in H^\infty(\mathbb{T}, B(H^2(\mathbb{T}))) \ (n \geq 0) \). Then
\[
T_\Phi^* T_\Phi = T_S^* S = I_{H^2(\mathbb{T}, H^2(\mathbb{T}))},
\]
so that \( T_\Phi \) is quasinormal and hence subnormal. However,
\[
\Phi(z) \Phi^*(z) = SS^* \neq S^* S = \Phi^*(z) \Phi(z) \quad \text{for all } z \in \mathbb{T},
\]
which implies that \( \Phi \) is not normal. Here we don’t need to expect that the multiplication operator \( M_\Phi : L^2(\mathbb{T}, B(H^2(\mathbb{T}))) \rightarrow L^2(\mathbb{T}, B(H^2(\mathbb{T}))) \) is a normal extension of \( T_\Phi \). Indeed, it is easy to show that \( M_\Phi \) is not normal, and hence \( M_\Phi \) can never be a normal extension of \( T_\Phi \). What is a normal extension of \( T_\Phi \)? Let \( B := M_z \) on \( L^2(\mathbb{T}) \) and \( \Psi(z) := Bz^n \in H^\infty(\mathbb{T}, B(L^2(\mathbb{T}))) \). Then a straightforward calculation shows that the multiplication operator \( M_\Psi : L^2(\mathbb{T}, B(L^2(\mathbb{T}))) \rightarrow L^2(\mathbb{T}, B(L^2(\mathbb{T}))) \) is a normal extension of \( T_\Phi \).

The following simple example shows that analytic Toeplitz operators with operator-valued symbols need not be even hyponormal.

**Example 5.5.** Let \( \Phi(z) = S^* \in H^\infty(\mathbb{T}, B(H^2(\mathbb{T}))) \) and \( e_0 \) be the constant function \( 1 \in H^2(\mathbb{T}) \). If \( f(z) = e_0 z \), then
\[
\langle (T_\Phi^* T_\Phi - T_\Phi T_\Phi^*) f, f \rangle = \langle -e_0 z, e_0 z \rangle = -1 < 0,
\]
which implies that \( T_\Phi \) is not hyponormal and hence not subnormal even though \( \Phi \) is analytic.

We would like to pose:

**Question 5.6.** Which analytic Toeplitz operators with operator-valued symbols are subnormal?

For a sufficient condition, one may be tempted to conjecture that if \( \Phi \in H^\infty(\mathbb{T}, B(H^2(\mathbb{T}))) \) and if \( \Phi(z) \) is subnormal for almost all \( z \in \mathbb{T} \), then \( T_\Phi \) is subnormal. We have not been able to decide whether this is true.

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