Stable sets of contracts in two-sided markets

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To Michel Balinski on the occasion of 87-th birthday.

Abstract

We revisit the problem of existence of stable systems of contracts with arbitrary sets of contracts. We show that stable sets of contracts exists if choices of agents satisfy path-independence. We call such choice functions Plott functions. Our proof is based on application of Zorn lemma to a special poset of semi-stable pairs. Moreover, we construct a dynamic process on the poset (generalizing algorithm Gale and Shapley) steady states of which are stable sets.

In Appendix we discuss Lehmann hyper-orders and establish a bijection between the set of Lehmann hyper-orders and the set of Plott functions.

Keywords: Plott choice functions, stable and semi-stable pairs, Zorn lemma, lattice, Noetherian order.

1 Introduction

There are many studies of stable sets of contracts in many-to-many setup (bilateral markets), beginning from the fundamental paper by Gale and Shapley [7]. Let us mention some of the key papers: Kelso and Crawford [9], Roth [15], Fleiner [6], Baiou and Balinski [2], Hatfield and Milgrom [8]. There are two sides of agents (men and women, students and colleges, doctors and hospitals, workers and firms, banks and clients), and agents of one of the sides may sign contracts with the agents of the opposite side. The amount of contracts which agents can sign is arbitrary. Baiou and Balinski [2] introduced the notion of schedule matching which made it possible to consider, as a part of the contract not only the hiring of a particular worker by a particular firm, but also the number of hours of employment of the worker in the firm.

Agents preferences over set of contracts are given by choice functions. Such a viewpoint on agents preferences dates back to Roth [15]. Roth also formulated a clever generalization of the maximization of usual preferences of Gale and Shapley [7]: for existence of stable contracts, the choice functions have to be Plott functions, that are functions satisfying path-independence due to Plott [14].

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Gale and Shapley’s concept of stability is a milestone of matching theory. There were two main approaches to establish existence of stable contracts: either, for a case of finite sets of contracts, to use a variant of the Gale-Shapley algorithm, or, for infinite sets of contracts, to use the Tarski fixed point theorem as Fleiner did in [6].

Here we propose a new approach to the existence problem using the Zorn lemma. We construct an auxiliary poset of semi-stable pairs, and maximal elements of this posets gives us stable sets of contracts. Because of the Zorn lemma, such maximal elements exist. Thus we do not care about the issue of finiteness of infiniteness.

Beside that, we construct a process defined on this poset of semi-stable pairs (transfinite in general), such that the steady states of this process give us stable sets of contracts. This more refined approach allows us not only to prove the existence of stable sets of contracts, but also the structure properties of those stables sets.

The article is organized as follows. In Section 2, we introduce the general concept of a stable contract system. Although any number of agents is allowed in the model, it can be reduced to the case when there are only two agents. In Section 3 we recall some of the properties of Plott functions, that we use in our study of stability. In Section 5 we reformulate the concept of a stable system in the form of a stable pair introduced by Fleiner [6]. Some weakening of this concept leads to semi-stable pairs. This is, in a sense, a key step, because the existence of stable pairs and stable contract systems is reduced to the existence of maximum elements in the poset of semi-stable pairs. In Section 6, we construct a process of “improving” semi-stable pairs (a variant of the Gale-Shapley algorithm), such that their steady states give us stable pairs and thus stable contract systems. This process has some good properties that allow us to establish the structural properties of a set of stable sets.

In the Appendix we give one more bijection between the set of Plott functions and the set of Lehmann hyper-orders.

## 2 Stable contract systems

There are two complementary sets $\mathcal{F}$ and $\mathcal{W}$ of agents (for example, firms and workers). For each pair of agents of opposite types $(f, w)$, $f \in \mathcal{F}$, $w \in \mathcal{W}$, there is a set of contracts $C(f, w)$ which they can sign. Thus, $C(f) = \bigsqcup_{w \in \mathcal{W}} C(f, w)$ is the set of contracts available for agent $f$.

Which subset of contracts will be signed, depends on preferences of the agents. Following accepted tradition in the literature, going back to Roth [15], preferences of agents are specified by the corresponding choice functions. This means the following: suppose a set of contracts $X \subseteq C(f)$ is available to sign for an agent $f$, then $f$ chooses a subset of $G_f(X) \subseteq X$ to sign. The preferences of agent $f$ are given by the choice function $G_f : 2^{C(f)} \to 2^{C(f)}$ that sends $X$ to $G_f(X) \subseteq X$, $X \subseteq C_f$.

Similarly, the preferences of agent $w$ are given by the choice function $F_w$ defined on the set $C(w) = \bigsqcup_{f \in \mathcal{F}} C(f, w)$.

Denote by $C = \bigsqcup_{f \in \mathcal{F}} C(f) = \bigsqcup_{w \in \mathcal{W}} C(w)$ the set of all possible contracts. For the given preferences of agents $G_f$, $f \in \mathcal{F}$, and $F_w$, $w \in \mathcal{W}$, which subset $S \subseteq C$ could be implemented? Gale and Shapley [7] proposed the stability concept for contract systems.
Namely, two requirements have to be satisfied for a stable set $S$ of contracts.

Firstly, no one of agents wish to abandon any of the contracts from the proposed set $S$. This means that, for any agent $f$, the equality holds $G_f(S \cap C(f)) = S \cap C(f)$, and, for any agent $w$, it holds $F_w(S \cap C(w)) = S \cap C(w)$.

Secondly, there is no pair $(f, w)$ which wants to conclude a new contract. That is if $c$ belongs to $C(f, w)$ and does not belong to $S$, then either $c$ does not belong to $G_f(S \cup c)$, or $c$ does not belongs to $F_w(S \cup c)$.

If both of these conditions are met, the contract system $S$ is called a stable set.

It is convenient to rewrite these conditions in more compact form by aggregating all workers into one Worker, and all firms into one Firm. The Worker’s preferences are represented by a choice function $F$ defined on the set $C$ being the union of individual ones,

$$F(X) = \prod w F_w(X \cap C(w)).$$

Similarly, the preferences of the Firm are given by the choice function $G$ on $C$,

$$G(X) = \prod f G_f(X \cap C(f)).$$

Then, the subset $S \subseteq C$ is stable if and only if

S1. $F(S) = S$, $G(S) = S$;

S2. If $c \notin S$, then either $c \notin F(S \cup c)$, or $c \notin G(S \cup c)$.

Here is a couple of examples.

**Example 1.** There are two agents. The set $C$ consists of six contracts (depicted below as circles). The choice of the Worker is defined by maximization of $u_1$ (if the utility is non-negative) and the choice of the Firm is defined by similar maximization of $u_2$.

Stable contracts are depicted by black circles.

**Example 2.** The set $C$ consists of two contracts, $a$ and $b$. The preferences of the Firm is the choice function $G$ and for the Worker it is $F$ that are specified as follows:

$$F(a) = a, F(b) = \emptyset, F(a, b) = \{a, b\};$$

$$G(a) = a, G(b) = b, G(a, b) = b.$$
We claim that there are no stable sets in such a case. In fact, the empty set \( \emptyset \) is unstable, since \( a \) is better for both. The singleton \( a \) is unstable due to the presence of a contract \( b \): \( F(a,b) \) contains \( b \), as well as \( G(a,b) \). The set \( C = \{a,b\} \) is unstable, because the Firm prefers to abandon \( a \). The singleton \( b \) is unstable, since the Worker refuses such a contract.

Example 2 shows that for the existence of a stable set, the choice functions \( F \) and \( G \) have to satisfy certain assumptions.

Choice functions, abstracted from choices of 'best variants', are of special interest in choice theory. There are two interconnected approaches to formalize the notion of a choice of 'best variants'. Due to the first one, we have to pick a preference structure on \( C \), and form, with help of it, a choice operator. Due to the second (or dual) one, we impose some requirements like 'rationality' or 'consistency' on choice operators and try to build a corresponding choice mechanisms (see, for example, Aizerman and Malishevsky [1]). Such requirements of 'rationality' are as follows.

- Suppose for a set of available contracts \( X \subseteq C \) to the Firm, some contract \( c \in X \) is not chosen from \( X \), \( c \notin G(X) \). Then for the set of available contracts \( X - c \), the choice remains the same as for \( X \), \( G(X - c) = G(X) \). This property of the choice functions is called independence from rejected alternatives, or the Outcast property.

- Now, let, for a set of available contracts \( X \subseteq C \) to the Firm, some contract \( c \in X \) is chosen from \( X \), \( c \in G(X) \). Suppose the set of available contracts diminished, for some reason, to a subset \( Y \subseteq X \), but \( c \) remains available in \( Y \), \( c \in Y \). Then \( c \) has to be chosen from \( Y \), \( c \in G(Y) \). This property is called the Heredity.

The second requirement means that contracts are substitutes for each other, they are interesting in themselves, and not because of their connection with another contracts.

Note that the second requirement of substitutability was violated in Example 2.

Roth [15] have shown that these two properties are sufficient for the existence of stable sets. Aizerman and Malishevsky [1] have shown that these two requirements on choice functions are equivalent to the so-called path independence, introduced for the first time by Plott [14]. Path independence means that the result of choice does not change when we divide a given set in two groups \( X \) and \( Y \) and then take the choice from a smaller set consisting of the group \( X \) and the 'winners' of the group \( Y \). Choice functions satisfying path independence are called Plott functions.

In the following section we collected results on Plott functions that will be needed further.

3 Plott functions

The Plott condition is expressed by the following compact formula:

\[
G(X \cup Y) = G(G(X) \cup Y).
\]

\[(PI)\]
An equivalent condition is \(G(X \cup Y) = G(G(X) \cup G(Y))\). Obviously, this implies the idempotency of \(G\): \(G(G(X)) = G(X)\) for any \(X \subset C\).

Here are some important properties of Plott functions which are useful for our analysis of stability.

### 3.1 Noetherian linear orders

The set of Plott functions is a huge set. They form a semi-lattice wrt union with the top element \(1(X) = X\), \(X \subset C\) and the minimal element is \(0(X) = \emptyset\), \(X \subset C\).

Namely, for a family \((G_i, i \in I)\) of Plott functions, the union \(\bigcup G_i\) is a Plott function as well. Let us prove this claim for two functions. If \(F\) and \(G\) are Plott functions on \(C\), then \(F \cup G\) is also a Plott function, which means that the following equality holds

\[
(F \cup G)(X \cup Y) = (F \cup G)((F \cup G)(X) \cup Y).
\]

The RHS is the union of \(F(FX \cup GX \cup Y)\) and \(G(FX \cup GX \cup Y)\). Because \(GX \subseteq X\), we have \(F(FX \cup GX \cup Y) = F(X \cup GX \cup Y) = F(X \cup Y)\). Similarly, \(G(FX \cup GX \cup Y) = G(X \cup Y)\). Hence by the definition \((F \cup G)(X \cup Y) = F(X \cup Y) \cup G(X \cup Y)\) and we get the claim proved.

This property of stability holds for any family of Plott functions. In particular, we have that aggregating Plott functions (see Section 2), yields a Plott function.

Single-valued Plott functions are in bijection with Noetherian linear orders.

**Definition.** A linear order \(<\) on \(C\) is called **Noetherian** if there are no infinite strictly increasing sequences \(c_1 < c_2 < \ldots\) of elements of \(C\). Equivalently, we can say that any non-empty subset of \(X \subseteq C\) has a maximum element.

For a Noetherian order \(<\), let us define a choice function \(G = \max_<\) by sending a set \(X\) to \(G(X)\) constituted from the maximum (relative to \(<\)) element in \(X\) (of course, \(G(\emptyset) = \emptyset\)). It is easy to see that such a choice function \(G = \max_<\) satisfies the Plott condition. Note that without the Noetherian condition, the statement is incorrect.

Let us note one characteristic property of the choice function \(G = \max_<\). Such a choice function is “single-valued”: \(G(X)\) contains exactly one element (if, of course, \(X \neq \emptyset\)). The converse is also true: if \(G\) is a single-valued Plott function, it is given by maximization of some Noetherian linear order \(<\). This order \(<\) can be given explicitly: for different \(a, b \in C\)

\[
a < b \iff G(a, b) = b.
\]

It is easy to understand that the relation \(<\) is transitive and complete, so that we get a linear order. As for the Noetherian property: suppose there is an infinite increasing sequence \(a_1 < a_2 < \ldots\). Let us form the set \(A = \{a_1, a_2, \ldots\}\). \(G(A)\) consists of a single element; let this element be \(a_i\). Due to the heredity property (see [3.2]) for \(\{a_n, a_{n+1}\} \subseteq A\), we get \(G(a_n, a_{n+1}) = G(A) = a_n\). On the other hand, from \(a_n < a_{n+1}\) we see that \(G(a_n, a_{n+1}) = a_{n+1}\). A contradiction.

According to the Zermelo theorem, there is a plethora of Noetherian linear orders on \(C\), so we get a plethora of Plott functions.
One can construct a non-empty valued Plott functions from Noetherian linear orders. For a Plott function $G$, let us take the set of Noetherian linear orders $<_i, i \in I$, which are inferior to $G$. That is, for any $X \subseteq C$, it holds $\max_{<_i}(X) \subseteq G(X), i \in I$. Then $G$ is equal to the union of Plott functions $\max_{<_i}, i \in I$. This construction of Plott functions generalizes the theorem of Aizerman and Malishevski, and was established in [5].

3.2 Heredity and Outcast

From the definition of a Plott function, the following inclusion holds

$$G(X \cup Y) \subseteq G(X) \cup G(Y).$$

(S)

For sets $A \subseteq B$, let us set $X = A$ and $Y = B \setminus A$. Then from (S) we get

$$G(B) \subseteq G(A) \cup G(B \setminus A),$$

that is nothing but the heredity requirement ([1]): for $A \subseteq B$

$$G(B) \cap A \subseteq G(A).$$

In other words, if an element $a$ belongs to a small set $A$ and is chosen from a bigger set $B$ then it has been chosen from the smaller set $A$.

Note that inclusion (S) is valid for any number of sets: for any family $(X_i, i \in I)$ of subsets of $C$, there is an inclusion:

$$G(\cup_i X_i) \subseteq \cup_i G(X_i).$$

The second important property of Plott functions is the outcast property: assume that $G(X) \subseteq Y \subseteq X$; then $G(Y) = G(X)$.

Indeed, $G(Y) = G(G(X) \cup Y) = G(G(X) \cup G(Y)) = G(X \cup Y) = G(X)$.

These two properties characterise Plott functions: if a choice function $G$ satisfy Heredity and Outcast, then $G$ is a Plott function.

3.3 Closure operator

Note that, for a Plott function, the property (PI) is valid not only for two sets, but for any number of them. More precisely, for any family $(X_i, i \in I)$ of subsets of $C$, the following equality holds

$$G(\cup_i X_i) = G(\cup_i G(X_i)).$$

Indeed, if $Y = \cup_i G(X_i)$ and $X = \cup_i X_i$, we have (see 3.2)

$$G(\cup_i X_i) \subseteq \cup_i G(X_i) \subseteq \cup_i X_i.$$

Due to Outcast, we have $G(\cup_i X_i) = G(\cup_i G(X_i))$. 

6
By applying the above equality to the collection \( I(X) = \{ Y \subseteq C, G(Y) = G(X) \} \), we get that this collection has the largest element, namely the union \( Z = \bigcup_{Y \in I(X)} Y \). Indeed,

\[
G(Z) = G(\bigcup_{Y \in I(X)} Y) = G(\bigcup_{Y \in I(X)} G(Y)) = G(G(X)) = G(X).
\]

We denote by \( G^*(X) \) this largest set. In other words, for any \( X \subseteq C \), the following two statements are equivalent:

1. \( G(Y) = G(X) \),
2. \( X \subseteq Y \subseteq G^*(X) \).

Thus, for a Plott function \( G \), we get an expanding operator \( G^* \). It can be shown that \( G^* \) is a closure operator. Moreover, for a Plott function \( G \), the closure \( G^*(X) \) is a kind of convex hull of \( X \) (for details see \[10\] for finite \( C \) and \[13\] in the general case). The inversion to the operator \( G^* \) is defined by the formula:

\[
G(X) = \{ x \in X, x \notin G^*(X - x) \}.
\]

In particular, when \( X = \emptyset \), we get that there is the largest subset of \( G^*(\emptyset) \), the choice from which is empty. It can be called Nil-set for a Plott function \( G \), \( Nil(G) \) (or Dummy set, the complement to it is the ‘support’ of \( G \)). This nil-set does not affect the choice in any way: for any \( X \), \( G(X) = G(X - Nil(G)) = G(X \cup Nil(G)) \) holds.

### 3.4 Blair hyper-order

To compare subsets in \( C \) in terms of their attractiveness for a G-agent with Plott function \( G \), Blair \[13\] proposed to use a natural (hyper)-relation \( \preceq_G \) on \( C \). Namely, for subsets \( A \) and \( B \) of \( C \), we set

\[
A \preceq_G B, \text{ if } G(A \cup B) \subseteq B.
\]

In this case, \( G(A \cup B) = G(G(A \cup B) \cup B) = G(B) \). In other words, \( A \preceq_G B \) is equivalent to the inclusion \( A \subseteq G^*(B) \) (as well as to the inclusion \( G^*(A) \subseteq G^*(B) \)). The relation \( A \preceq_G B \) means that adding \( A \) to \( B \) does not affect the preferred elements in \( B \), and therefore, adding \( A \) to \( B \) does not increase the attractiveness of \( B \). For any \( A \), it holds \( G^*(A) \preceq_G G^*(A) \).

**Lemma 3.1** The hyper-relation \( \preceq = \preceq_G \) is transitive.

**Proof.** Let \( A \preceq_G B \) and \( B \preceq_G D \). Then \( G(A \cup D) = G(A \cup G(D)) = G(A \cup G(B \cup D)) = G(G(A \cup B) \cup D) = G(G(B) \cup D) = G(B \cup D) = G(D) \). That is nothing but \( A \preceq_G D \). \( \Box \)

Another important property of this hyper-relation is its consistency with the unions.

**Lemma 3.2** Let \((X_i, i \in I)\) be a family of subsets in \( C \), and \( X_i \preceq_G Y \) for any \( i \in I \). Then \( \bigcup_i X_i \preceq_G Y \).

The consistency shows the “qualitative” character of the preorder \( \preceq_G \): the quantity can not bit the quality.

**Proof.** Let \( X = \bigcup_i X_i \). Then

\[
G(X \cup Y) = G((\bigcup_i X_i) \cup Y) = G(\bigcup_i (X_i \cup Y)) = G(\bigcup_i G(X_i \cup Y)) = G(\bigcup_i G(Y)) = G(Y).
\]
4 Properties of stable contract systems

From now on we suppose that the choice function $G$ of the Worker and the choice function $F$ of the Firm are two Plott functions defined on the set $C$ of contracts. For given Plott functions $F$ and $G$, let us denote the Worker as $F$-agent and the Firm as $G$-agent.

Recall that a stable contract system (or a stable set) is a subset $S \subseteq C$ such that $S_1$ and $S_2$ are satisfied. Note that $S_2$ can be rewritten as follows:

$$S_2'. \quad F^*(S) \cup G^*(S) = C.$$

To see that, let $c \in F(S \cup c)$; then $c \notin F^*(S)$. Symmetrically, $c \notin G^*(S)$. Which violates $S_2'$. Conversely, let $c$ not belong to $F^*(S)$. Then $c \in F(S \cup c)$, because otherwise from Outcast we would have $F(S \cup c) = S$ and $c \in F^*(S)$. Symmetrically, if $c$ does not belong to $G^*(S)$, then $c \in G(S \cup c)$, contrary to $S_2$.

Denote by $ST$ the set of stable subsets in $C$. We are interested in the structure of this set. Firstly, we prove that $ST$ is nonempty and, as a rule, contains several stable sets. Secondly, we show that if one stable system is better than another for the $F$-agent then it is vice versa for the $G$-agent. Namely, the following assertion is true.

**Proposition 4.1** The restriction of the Blair hyper-preorder $\preceq_G$ to the set $ST$ (as well as the restriction of $\preceq_F$) is an order (that is, an antisymmetric relation).

**Proof.** Let $S$ and $T$ be stable sets and $S \preceq_G T$. Then $G(S \cup T) = G(T) = T$. If $T \preceq_G S$ holds as well, then $G(S \cup T) = G(S) = S$, that is, the equality $S = T$ holds. □

A remarkable fact (3 15) is that the orders $\preceq_F$ and $\preceq_G$ on $ST$ are opposite to each other. That is

$$T \preceq_G S \text{ if and only if } S \preceq_F T.$$

Roth 15 called this as the polarization of the interests of the opposite parties. The proof is based on the following lemma (see [12, Lemma 20]).

**Lemma 4.2** Let $S$ be a stable set, and $T$ be an arbitrary subset in $C$. If $S \preceq_G T$, then $G(T) \preceq_F S$.

We give a more concise proof than [12, Lemma 20].

**Proof.** If $S \preceq_G T$, then $G(S \cup T) = G(T)$. Let $x$ be an arbitrary element of $G(T) - S$. We state that $x \preceq_F S$. From this, obviously, follows $G(T) \preceq_F S$.

Thus, we have to check the relation $x \preceq_F S$, that is $x \notin F(S \cup x)$. On the contrary, suppose $x \in F(S \cup x)$. Then, due to $S_2$, we have $x \notin G(S \cup x)$. Due the heredity of $G$, we get $x \notin G(S \cup T) = G(T)$, which contradicts the fact that $x \in G(T)$. □

Furthermore, we consider the set $ST$ as a poset with the partial order $\preceq=\preceq_G$. We will see below that this poset is a nonempty complete lattice. Knuth is credited for the discovery of this remarkable fact which established in full generality by Blair 3. In particular, there is a stable set that is least attractive for the $F$-agent and most attractive for the $G$-agent. We postpone the construction to Section 6, where we get this fact as the result of some natural process generalizing the Gale-Shapley algorithm.

8
5 Stable and semi-stable pairs

Since $F$ and $G$ are Plott functions, we can reformulate (see [6]) the stability condition.

**Definition.** A stable pair is a pair $(Y, Z)$ of subsets in $C$ such that the following properties are satisfied:

SP1 $Y \cup Z = C$,

SP2 $G(Y) = F(Z)$.

This means a division of competencies. Namely, the entire set of contracts $C$ is covered by two sets $Y$ and $Z$, the $G$-agent chooses from $Y$, and the $F$-agent chooses from $Z$. If their choices coincide, we can expect that they yield a stable set. The following Lemma states the equivalence of stable pairs and stable sets.

**Lemma 5.1** If $(Y, Z)$ is a stable pair, then $S = G(Y) = F(Z)$ is a stable set. Conversely, if $S \subseteq C$ is a stable set, then the pair $(G^*(S), F^*(S))$ is a stable pair.

**Proof.** Let $S$ be a stable set, $Y = G^*(S)$ and $Z = F^*(S)$. Then, since $F(Z) = F(F^*(S)) = F(S) = S$ and $G(Z) = G(G^*(S)) = G(S) = S$, we get the property SP2. The SP1 is nothing but S2’.

Conversely, let the pair $(Y, Z)$ be stable and $S = G(Y) = F(Z)$. The property S1 follows from the idempotence of $F$; indeed, $F(S) = F(F(Z)) = F(Z) = S$. Similarly, $S = G(S)$. The property S2’ follows from the fact that $Y \subseteq G^*(S)$ and $Z \subseteq F^*(S)$. □

If the pair $(Y, Z)$ is stable, we call $S = G(Y) = F(Z)$ a stable set corresponding to the pair $(Y, Z)$. But how to construct stable pairs? To do this, we will slightly weaken the concept of stability by introducing a degree of asymmetry of agents.

**Definition.** A pair $(Y, Z)$ of subsets in $C$ is called semi-stable if the following conditions hold

SSP1 $Y \cup Z = C$,

SSP2 $G(Y) \subseteq F(Z)$.

Of course, if SSP2 is satisfied as an equality, then we get validity of SP2 and, hence, $(Y, Z)$ is a stable pair. On the other hand, it is easier to find pairs which satisfy the semi-stability conditions; for example, the pair $(\emptyset, C)$ is semi-stable. Denote the SSP set of semi-stable pairs, and endow it with the following order relation:

$$(Y, Z) \leq (Y', Z')$$ if $Y \subseteq Y', Z' \subseteq Z$.

In other words, the first component increases, and the second decreases.

For a semi-stable pair $(Y, Z)$, define a new pair $\Phi(Y, Z) = (Y', Z')$, where

$$Y' = Y \cup F(Z), \quad Z' = Z - F(Z) \cup G(F(Z)). \quad (1)$$
Lemma 5.2 For a semi-stable pair $(Y, Z)$, the pair $(Y', Z')$ defined in (1) is also semi-stable.

Indeed, **SSP1** obviously holds, since we have $Y' \cup Z' = Y \cup Z \cup G(F(Z)) = C$. It remains to check that $G(Y')$ is contained in $F(Z')$. Because $G(Y) \subseteq F(Z)$, we have $G(Y') = G(Y \cup F(Z)) = G(G(Y) \cup F(Z)) = G(F(Z))$. Therefore, we have to check that $G(F(Z))$ is a subset of $F(Z')$. Due to the heredity of $F$, from the inclusion of $Z' \subseteq Z$, we obtain the inclusion of $F(Z) \cap Z' \subseteq F(Z')$. Since $G(F(Z))$ is a subset of $Z'$ and subset of $F(Z)$, we conclude $G(F(Z)) \subseteq F(Z) \cap Z' \subseteq F(Z')$. □

Corollary 5.3 Maximal elements of the poset (SSP, $\leq$) are stable pairs.

Proof. Let a pair $(Y, Z)$ be a maximal element of the poset (SSP, $\leq$). Since $(Y, Z) \leq (Y', Z')$, it holds that $Z = Z'$, and hence, by the definition $Z'$, we have $F(Z) = G(F(Z))$. Because $Y = Y'$, $F(Z) \subset Y$, and we get the inclusions $G(Y) \subset F(Z) \subset Y$. Therefore, from the outcast property for $G$, we get $G(Y) = G(F(Z))$. Thus $G(Y) = F(Z)$ and the pair $(Y, Z)$ is stable. □

We claim that the poset (SSP, $\leq$) is an inductive ordered set (or chain complete poset), which means that any chain in this set has an upper bound. Then maximal elements in the poset (SSP, $\leq$) exists by Zorn’s Lemma. Moreover, for any pair $(Y, Z)$ there is a maximal pair that is superior to $(Y, Z)$. In order to establish chain completeness of the poset (SSP, $\leq$) we prove a more general statement.

Lemma 5.4 Let $((Y_i, Z_i), i \in I)$ be a family of semi-stable pairs, $Y = \cup_i Y_i$, $Z = \cap_i Z_i$. Then the pair $(Y, Z)$ is semi-stable.

Proof. Because of the inclusion $Y \cup Z_i \supseteq Y_i \cup Z_i = C$ and the distributive law, we have

$$Y \cup Z = Y \cup (\cap_i Z_i) = \cap_i (Y \cup Z_i) = \cap_i C = C.$$  

To verify the inclusion $G(Y) \subset F(Z)$, firstly, let us establish the inclusion $G(Y) \subseteq Z$. Suppose $y \in G(Y)$ and $y \not\in Z$. In such a case, $y$ does not belong to some $Z_i$. From $Y_i \cup Z_i = C$ follows that $y \in Y_i$. From the heredity of $G$, we get that $y \in G(Y_i)$, hence $y \in F(Z_i) \subseteq Z_i$. This contradiction proves the inclusion $G(Y) \subseteq Z$.

Let $y$ be an element of $G(Y)$ and suppose that for some $i$, $y \in G(Y_i)$. Then, because $G(Y_i) \subseteq F(Z_i)$, we get $y \in F(Z_i)$. Since $y \in Z$, from the heredity of $F$ we get $y \in F(Z)$. □

Thus the poset (SSP, $\leq$) has a maximal element. Hence, due to Corollary 5.3, such an element is a stable pair and gives a stable set.

6 Process of sequential improvement

In the previous section, we have shown that, for any semi-stable pair $(Y, Z)$, there exists a stable set $S$ which is an upper bound to $(Y, Z)$. In fact, we can get more than just an
existence theorem. Namely, we can us the idea of the above proof to define a process of construction of a stable set that is not only an upper bound to the original pair, but also is the minimal upper bound (in the sense that will be defined later).

Specifically, for each semi-stable pair \((Y, Z)\), we defined by (1) a new semi-stable pair \(\Phi(Y, Z) = (Y', Z')\). Thus, on the set SSP of semi-stable pairs, the monotonic dynamics \(\Phi : SSP \rightarrow SSP\) is defined. The fixed points of this dynamics are stable sets. The transformation \((Y, Z)\) to \(\Phi(Y, Z) := (Y', Z')\) is nothing else but the Gale-Shapley algorithm: at the step \((Y, Z)\), the F-agent, makes an offer \(F(Z)\) to the G-agent, and than the G-agent accepts the part \(G(F(Z))\) of the offer and rejects the rest \(F(Z) - G(F(Z))\). The process continues at the step \((Y', Z')\) and so on.

Remark. Lehmann ([12]) defined the dynamics only in terms of \(Z, Z' = Z - (F(Z) - G(F(Z))\). Due to this approach, it is not very clear which \(Z\) we can take as the initial state. Lehmann showed that if we begin the process with \(Z = C\), it stabilizes and gives a stable set. Note that our process can be started with any semi-stable pair. This flexibility to choose the initial pair of the process allows us to get some interesting lattice properties of the set of stable pairs.

Let \(P_0 = (Y_0, Z_0)\) be a starting semi-stable pair. We get pairs \(P_1 = \Phi(P_0), P_2 = \Phi(P_1)\) and so on. However, this process may not lead to a stable pair in a finite number of steps, it may run indefinitely. In such a case, let us define \(P_\omega\) to be the pair \((\cup_k Y_k, \cap_k Z_k)\) and restart the process at the pair \(P_\omega\). That is we have to consider a transfinite process.

This means that, for any ordinal number \(\alpha\) (that is, for any well-ordered set \(\alpha\)), we must define a semi-stable pair \(P_\alpha = (Y_\alpha, Z_\alpha)\). This is done in two different ways, depending on whether the number \(\alpha\) is a limit or not. A number \(\alpha\) is called \(\text{limit}\) if it has no immediate predecessor, that is, it does not have the form \(\alpha = \beta + 1\). For a non-limit number \(\alpha = \beta + 1\), we set \(P_\alpha = \Phi(P_\beta)\), and, for a limit number \(\alpha\), we set \(P_\alpha = (\cup_{\beta<\alpha} Y_\beta, \cap_{\beta<\alpha} Z_\beta)\).

Because of lemma [5.4] for any ordinal \(\alpha\), the pair \(P_\alpha\) is semi-stable. Since the dynamics of \(\Phi\) is monotone, sooner or later the transfinite sequence of \(P_\alpha\) reaches the steady state. Let us check that the decreasing sequence \(Z_\alpha\) reaches the steady state (then the sequence \(Y_\alpha\) also stabilises). Indeed, consider the increasing sequence \(Z_\alpha := C - Z_\alpha\). If, for each step of transition from \(\alpha\) to \(\alpha + 1\), the set of \(Z_\alpha\) strictly increases, then the cardinality of \(Z_\alpha\) is not less than the cardinality of \(\alpha\). Hence we get a contradiction at the step when the cardinality of \(\alpha\) is greater than the cardinality of \(C\).

Thus, we begin the transformation process at an arbitrary semi-stable pair \(P_0\) and get a (transfinite) sequence \((P_\alpha)\), which has a steady state for large \(\alpha\). This steady state is a stable pair denoted by \(\Phi_\infty(P_0) = (Y_\infty, Z_\infty)\). According to the proof of Corollary [5.3] this pair is stable and gives the corresponding stable system of contracts, which we denote by \(\sigma(Y_0, Z_0) = S\). Since the process \(P_\alpha\) improves the position of the G-agent all the time (and the position of the F-agent is getting worse), we get that \(Y_0 \preceq_G S\) (respectively, \(S \preceq_F Z_0\)). So the result of the process is a stable set \(S\), which is no worse for the G-agent than the initial state \(Y_0\).

In fact, the resulting stable set \(S\) is minimal with the property of being better than \(Y_0\). This follows from
Theorem 6.1 Let \((Y, Z)\) be a semi-stable pair, \(T\) be a stable set, and \(Y \preceq_G T\). Then, for the limit stable set \(S = \sigma(Y, Z)\), it is true that \(S \preceq_G T\).

Proof. \(Y \preceq_G T\) is equivalent to the inclusion \(Y \subseteq G^*T\). We claim that \(T \subseteq Z\). To show this, suppose that some \(t \in T\), and \(t\) does not belong to \(Z\). Since \(Y \cup Z = C\), \(t \in Y\) and, hence, \(t \in G^*(T)\). Because \(T\) is a stable set, we have \(G(G^*(T)) = G(T) = T\), that implies the inclusion \(t \in G(G^*(T))\). From the heredity property of \(G\), due to \(t \in Y \subseteq G^*T\), we have \(t \in G(Y)\). Due to \(S2\) \(G(Y) \subseteq F(Z)\). Therefore, \(t \in Z\), a contradiction.

Therefore \(T \subseteq Z\), and hence \(T \preceq_G F(Z)\). Because of Lemma 1.2 we have \(F(Z) \preceq_G T\). Since \(Y \preceq_G T\), for \(Y' = F \cup F(Z)\), we get (see the proof of Proposition 5.3) that \(Y' \preceq_G T\). That is, for the pair \((Y', Z') = \Phi(Y, Z)\), we have the same relation \(Y' \preceq_G T\). Hence, for any pair \((Y_a, Z_a)\), we have \(Y_a \preceq_G T\). Therefore, for the limit pair \((Y_\infty, Z_\infty)\), we have \(Y_\infty \preceq_G T\). Since \(S \subseteq Y_\infty\), we conclude \(S \preceq_G T\). \(\square\)

7 Some consequences

Here are some consequences of our results.

Theorem 6.1 shows that, for given initial semi-stable pair \((Y, Z)\), the steady state \(S = \sigma(Y, Z)\) of the process \(\Phi\) is not only a stable set, but a minimal stable state among the set of stable sets which dominate \(Y\). In particular, for the initial semi-stable pair \((\emptyset, C)\), the stable set \(\sigma(\emptyset, C)\) is the worst for \(G\)-agent and the best for \(F\)-agent, a fact discovered in [7].

Moreover, the poset \((\text{ST}, \preceq_G)\) is a complete lattice. In other words, for any family \((S_i, i \in I)\) of stable sets, there is the least upper bound \(S = \bigvee_i S_i\). Namely, due to Lemma 5.4 for the pairs \((G^*(S_i), F^*(S_i))\), we get the semi-stability of the pair \((Y, Z)\), where \(Y = \bigcup_i G^*(S_i)\), \(Z = \bigcap_i F^*(S_i)\). Let \(S = \sigma(Y, Z)\) be the corresponding stable set, then from Theorem 6.1 for any \(i \in I\), it follows that \(S_i \preceq_G S\) holds. On the other hand, if \(T\) is a stable set and, for any \(i\), \(S_i \preceq_G T\), then \(G^*(S_i) \preceq_G T\), and due to Lemma 6.2 we get \(\bigcup_i G^*(S_i) \preceq_G T\). Therefore, due to Theorem 6.1 \(S \preceq_G T\). But this means exactly that \(S\) is the least upper bound for the family \((S_i)\).

Another consequences related to comparative statics. Suppose that \(S\) is a stable set with respect to agents with choice functions \(F\) and \(G\). Suppose that the preferences of the \(F\)-agent have changed to a new choice function \(F'\), such that \(F \leq F'\), that is, for any \(X \subseteq C\), \(F(X) \subseteq F'(X)\). In other words, the \(F\)-agent has “weakened” requirements for the best contracts. Note that if \(S \subseteq C\) is a stable set for agents with choice functions \(F\) and \(G\), \(S\) remains stable for agents with choice functions \(F'\) and \(G\).

For a stable set \(S\), consider a pair \((Y', Z')\), where \(Y' = G^*(S)\) and \(Z' = F'^*(F^*(S))\). \((Y', Z')\) is a semi-stable pair with respect to agents with choice functions \(F'\) and \(G\). It is easy to see that \(\text{SSP1}, Y' \cup Z' = C\), holds true, and \(\text{SSP2}\) follows as well,

\[ G(Y') = S = F(F^*(S)) \subseteq F'(F^*(S)) = F'(Z'). \]

Let \(S' = \sigma'(Y', Z')\) be the corresponding limit stable for the process \(\Phi\) defined with respect to to \(F'\) and \(G\). Then, according to Theorem 6.1 \(S'\) is not worse than the initial \(Y' = G^*(S)\), \(Y' \preceq_G S'\), that is, \(S \preceq_G S'\).
This defines a natural transition from the old stable sets \( S \) to the new \( S' \), \( S \preceq_G S' \). In other words, weakening the requirements of the \( F \)-agent improves the position of the \( G \)-agent. From Lemma 4.2, we see that \( S' = G(S') \preceq_F S \) holds. That is changing from \( F \) to an upper \( F' \), \( F \leq F' \), only worsening the outcome for the \( F \)-agent.

Appendix. Lehmann hyper-orders

With any Plott function \( G \), the Blair hyper-order \( \preceq_G \) is associated. Recall that it is defined by

\[
A \preceq_G B \text{ iff } G(A \cup B) \subseteq B.
\]

Lehmann [11] defined another (more strong) hyper-order \( \prec_G \) by

\[
A \prec_G B \text{ iff } G(B) \neq \emptyset \text{ and } G(A \cup B) \text{ does not intersect with } A.
\]

Note that in this case, \( G(A \cup B) \) is contained in \( B \) and moreover coincides with \( G(B) \). In fact, we have \( G(A \cup B) = G(G(A \cup B) \cup B) = G(B) \).

This ‘strict’ hyper-order is interesting because it is uniquely defines the initial Plott function \( G \). To do this, we need to recover the set \( G(B) \) for any \( B \subseteq C \) in terms of \( \prec_G \). Here it is convenient to introduce two notions related to the hyper-relation \( \prec \). Namely, a set \( B \) is called essential if \( \emptyset \prec B \), and insignificant otherwise.

If \( B \) is insignificant, then we have \( G(B) = \emptyset \).

If \( B \) is essential, then there is \( G(B) \neq \emptyset \), then \( G(B) = B - A \), where \( A = \{a \in B, a \prec_G B\} \). In fact, \( \{a\} \) does not intersect with \( G(a \cup B) = G(B) \), so \( a \) is not contained in \( G(B) \).

On the other hand, if \( b \in G(B) \), then the relation \( b \prec_G B \) does not performed.

Below we will get the necessary and sufficient conditions on a hyper-order \( \prec \), which has the form \( \prec_G \) for some Plott function \( G \).

The following are the important properties of the hyper-relation \( \prec_G \).

**Proposition A1.** The hyper-relation \( \triangleleft \prec_G \) has following properties:

L0. \( \prec \) irreflexive.

L1. Left weakening. If \( A' \subseteq A \prec B \), then \( A' \prec B \).

L2. Union. Let \( (A_i, i \in I) \) be a nonempty family of subsets in \( C \). If \( A_i \prec B \) for any \( i \in I \), then \( \bigcup_i A_i \prec B \).

L3. Right strengthening. If \( A \prec B \subseteq B' \), then \( A \prec B' \).

L4. Cancelation. If \( A \prec A \cup B \), then \( A \prec B \).

L5. Domination. Any essential set dominates (in the sense of \( \prec \) ) any non-essential one.

**Proof.** L0 is obvious.

Let’s check L1. We are given that \( G(A \cup B) \subseteq B \) and even \( G(A \cup B) = G(B) \). Then \( G(A' \cup B) = G(A' \cup G(B)) = G(A' \cup G(A \cup B)) = G(A' \cup A \cup B) = G(A \cup B) \) is contained in \( B \).

Check L2. \( G((\bigcup_i A_i) \cup B) = G(\bigcup_i (A_i \cup B)) = G(\bigcup_i G(A_i \cup B)) \) is contained in \( \bigcup_i G(A_i \cup B) \) and especially in \( B \).
Let’s check L3. \(G(A \cup B') = G(A \cup B \cup B') = G(G(A \cup B) \cup B') = G(G(B) \cup B')\) is contained in \(B'\), since \(G(B) \subseteq B \subseteq B'\).

Let’s check L4. \(G(A \cup B) = G((A \cup B) \cup B) = G(G(A \cup B) \cup B) = G(B) \cup B = G(B)\) and is nonempty.

Let’s check L5. Let \(A\) be essential, and \(B\) be insignificant. The essentiality of \(A\) means that \(G(A) \neq \emptyset\). The insignificance of \(B\) means that \(G(B) = \emptyset\). We have to check that \(G(A \cup B) = G(A \cup G(B)) = G(A)\) does not intersect with \(B\). Let’s assume that it intersects, and \(b \in B\) belongs to \(G(A)\). Then \(b\) is selected in the larger set \(A \cup B\) and belongs to the smaller one \(B\); from the heredity we have \(b \in G(B)\). This contradicts to the emptiness of \(G(B)\). \(\Box\)

**Definition.** The Lehmann hyper-order is a hyper-relation, which satisfies the properties L0-L5.

**Remark.** Lehmann \([11]\) called a qualitative measure a transitive hyper-relation which satisfies L0-L4 and such that the union of negligible sets is negligible. The next lemma shows that the transitivity is a redundant requirement.

**Lemma A1.** The Lehmann hyper-order is transitive.

*Proof.* Let \(A \prec B\) and \(B \prec C\). Then \(A \prec B \cup C\) (according to L3), as well as \(B \prec B \cup C\). Hence by L2, it holds true \(A \cup B \prec B \cup C\), and due to L3 we have \(A \cup B \prec A \cup B \cup C\). Then from L4 follows \(A \cup B \prec C\), and hence from L1 \(A \prec C\). \(\Box\)

This lemma justifies the use of the term hyper-order.

Now we construct a mapping from the set of Lehmann hyper-orders to the set of Plott functions.

Let \(\prec\) be a Lehmann hyper-order and \(D\) be the set of all negligible elements in \(C\). For a set \(A \subseteq C\), define

\[
L(A) = D \cup \{c \in C, c \prec A\}.
\]

Note that, for a negligible set \(A\), we have \(L(A) = D\). Because of the union property L2, for an essential set \(A\), it holds that \(L(A) \prec A\). Note that \(L(A)\) is the largest subset with the property \(L(A) \prec A\).

**Lemma A2.** If \(A \subseteq L(A) \cup B\), then \(L(A) \subseteq L(B)\).

*Proof.* For a negligible set \(A\), the statement is obviously true. Let \(A\) be essential, then \(L(A) \prec A\). Since \(A \subseteq L(A) \cup B\), then, from L3, we have \(L(A) \prec L(A) \cup B\). From L4, we get \(L(A) \prec B\). This implies that \(B\) is essential and \(L(A) \subseteq L(B)\). \(\Box\)

Now let’s us define a choice function \(T = T_\prec\) by the rule

\[
T(A) = A \setminus L(A).
\]

**Proposition A3.** \(T\) is a Plott function.

*Proof.* Let us verify that \(T\) satisfies the heritage and outcast properties.

For the heredity: let \(A \subseteq B\) and \(a \in T(B) \cap A\). Then, suppose on the contrary that \(a \notin T(A)\), that is, \(a \in L(A)\). If \(a \in D\), then \(a \in L(B)\), that contradicts to the fact that
$a \in T(B)$. If $a \notin D$, then $a \prec A$ and from L3 $a \prec B$, $a \in L(B)$, a contradiction to $a \in T(B)$.

For the outcast: let $T(A) \subseteq B \subseteq A$. Then for a negligible set $A$, we have that $B$ is negligible as well and $T(A) = T(B)$. For an essential set $A$, from $T(A) \subseteq B$ follows $A \subseteq L(A) \cup A \subseteq L(A) \cup B$. Then, by Lemma 3.2, $L(A) \subseteq L(B)$. By the same Lemma applied to the inclusion $B \subseteq A$, we have $L(B) \subseteq L(A)$. From that we get $L(A) = L(B)$ and $T(A) = T(B)$. □

Thus, the operations $G \mapsto \prec_G$ and $\prec \mapsto T$ are mutual inverse and establish a bijection between the set of Plott functions and the set of Lehmann hyper-orders.

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