A MACAULAY2 PACKAGE FOR STANLEY SIMPLICIAL POSET IDEALS

NATHAN NICHOLS
UNIVERSITY OF MINNESOTA (TWIN CITIES)

Abstract. We give a description of a new Macaulay2 package called SimplicialPosets. This package provides functions for working with simplicial posets and calculating their generalized Stanley-Reisner ideals. For practical purposes, we also introduce of a new random model for a class of simplicial posets which generalizes existing models for random simplicial complexes such as the Kahle model.

1. Introduction

A simplicial poset \( P \) is a poset with a unique minimum element \( \hat{0} \) such that for any vertex \( v \in P \), the lower set \( L_v = [\hat{0}, v] \) is a boolean lattice. Simplicial posets were originally introduced by Stanley in [6], who defined an associated ring \( A_P \) that generalizes the Stanley-Reisner ring of a simplicial complex. In his original paper, Stanley shows that certain properties of simplicial posets are easier to prove than their analogs for simplicial complexes.

The purpose of the package SimplicialPosets is to add features for working with Stanley simplicial poset rings to Macaulay2. Previously, Macaulay2 only supported calculations with simplicial complexes and Stanley-Reisner rings. This package also tries to address the problem of coming up with examples of simplicial posets.

2. Basic operations

The most basic capability of this package is to calculate whether or not a poset is a simplicial poset. This is useful in many situations.

\begin{verbatim}
i2 : B := booleanLattice 4;
i3 : isSimplicial B
\end{verbatim}

\( o3 = \text{true} \)

The next example illustrates that Stanley simplicial poset ideals are a generalization of Stanley-Reisner ideals. This will require functions from the standard library packages SimplicialComplexes and Graphs.

First, we will generate a random simplicial complex by taking the flag complex of an Erdős-Rényi random graph. This algorithm is known as the Kahle model (see [3]). The most relevant property of this model is that every simplicial complex on \( n \) points has a nonzero probability for \( 0 < p < 1 \).

\begin{verbatim}
i2 : n = 5; p = 0.5;
i4 : R := QQ[vars(0..n)];
i5 : E := select(edges completeGraph(R,n), (e -> random(1.0) < p));
i6 : G := graph(R,E);
i7 : C := cliqueComplex(G);
\end{verbatim}

Now that a random simplicial complex has been generated, one may compare its Stanley-Reisner ideal to the Stanley simplicial poset ideal of its face poset:

\begin{verbatim}
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2010 Mathematics Subject Classification. Primary 05E40; Secondary 05-04.
\end{verbatim}
i8 : P := facePoset C;
i9 : I1 := minimalPresentation stanleyPosetIdeal P;
i10 : I2 := ideal(C);
i11 : M := map(ring(I2), ring(I1), vars(ring(I2)));
i12 : I2 == M(I1)
o12 = true

As one would expect, that the package *SimplicialPosets* is fully consistent with the package *SimplicialComplexes*. When given the face poset of a simplicial complex, *stanleyPosetIdeal* outputs its Stanley-Reisner ideal.

The function *fromFVector* implements Stanley’s construction of a simplicial poset with a given f-vector (see Theorem 2.1 in [6].)
i6 : fromFVector({1,3,3})
o6 = Relation Matrix: | 1 1 1 1 1 1 1 |
     | 0 1 0 1 0 1 1 |
     | 0 0 1 1 0 1 1 |
     | 0 0 0 1 0 0 0 |
     | 0 0 0 0 1 0 0 |
     | 0 0 0 0 0 1 0 |
     | 0 0 0 0 0 0 1 |

The primary intention of the function *fromFVector* is to serve as a quick way to check any kind of f-vector related conjecture. It is worth noting that Stanley’s construction is not typically unique for a given f-vector. The output of *fromFVector* is only guaranteed to be one possible realization of Stanley’s construction.

3. Defining simplicial posets

It is more difficult to manually specify a simplicial poset than an abstract simplicial complex because a simplicial poset is not fully determined by its maximal vertices. Also unlike the case for simplicial complexes, there are currently no known models for random simplicial posets. It is recommended that readers who are interested in the topic of random simplicial complexes refer to the work of De Loera et al [3].

If this package is to be useful as a research tool, users must have some source of non-trivial simplicial posets to work with. To prevent everybody from having to come up with this source themselves, *SimplicialPosets* includes a function called *randSimplicialPoset*.

The idea behind the model implemented by the function *randSimplicialPoset* is to define a deterministic function $\Theta(\Delta_1, \Delta_2)$ that takes two abstract simplicial complexes $\Delta_1$ and $\Delta_2$ on the same set of points and produces a simplicial poset. An induced distribution on simplicial posets is obtained by setting $\Delta_1$ and $\Delta_2$ to random variables. In this implementation, $\Delta_1$ and $\Delta_2$ come from the Kahle model.

The next section is devoted to defining the mathematical function $\Theta$. It is important that certain properties of this function are proven because the sample space of this distribution is limited to a certain class of simplicial posets.

4. Random simplicial posets

Simplicial posets arise as the posets of cells of a CW complex ordered by inclusion. Topological constructions on CW complexes such as taking a quotient space can be thought of in terms of the combinatorial operations that they induce on the CW complex’s simplicial poset of cells. In this section, we will frequently refer to the operation of taking a quotient space. It will make more sense to work directly with the combinatorial definitions instead of their topological counterparts.
This construction is restated from a paper due to Björner [1] but was originally defined by Garsia and Stanton [4]. \( L_a \) denotes the lower set of the vertex \( a \).

**Definition 1.** Let \( P \) be the face poset of a simplicial complex \( \Delta \). An equivalence relation \( \sim \) on \( P \) is called a **gluing relation** if it has the following properties:

1. \( \tau \sim \sigma \) and \( \sigma \neq \tau \) implies that \( \tau \) and \( \sigma \) are incomparable with \( \text{rank}(\tau) = \text{rank}(\sigma) \). Also, \( \tau \) and \( \sigma \) must have no common upper bound.
2. If \( \tau \sim \sigma \), then every element of \( L_\tau \) must be related to some element of \( L_\sigma \).

A poset \( P/\sim \) is defined as the set of equivalence classes of \( \sim \) ordered by the following rule:

\[
C_1 \leq C_2 \iff \exists v \in C_1 \text{ s.t. } v \leq w \text{ for some } w \in C_2.
\]

In [1], Björner states that any simplicial poset can be constructed as a quotient by a gluing relation on some simplicial complex. For the purposes of defining the function \( \Theta \), we will prove a stronger version of this fact.

**Definition 2.** Let \( Q \) be an arbitrary simplicial poset. Let \( F \) be the set of maximal vertices of \( Q \). Define a simplicial poset \( P \) called the **separation** of \( Q \) as follows:

\[
P = \bigsqcup_{x \in F} (L_x - 0) \cup \hat{0}
\]

The element \( \hat{0} \) is subtracted from each term in the disjoint union in order to ensure that the minimum element of \( P \) is unique.

The following theorem explicitly constructs a simplicial poset \( P' \) from which any simplicial poset \( P \) can be obtained as a quotient by a gluing relation:

**Theorem 1.** Let \( P' \) be the separation of simplicial poset \( P \). Then, \( P \) can be realized (up to isomorphism) as a gluing of \( P' \).

*Proof.* Vertices of \( v' \in P' \) are of the form \( v' = (i, v) \) for some \( i \in \mathbb{N} \) and \( v \in P \). Let \( \sigma \) be the function defined by \((i, v) \mapsto v\). Next, define a relation \( \sim \) on \( P' \):

\[
v \sim w \iff \sigma(v) = \sigma(w)
\]

Verify that \( \sim \) is a gluing relation and that \( P'/\sim \) is canonically isomorphic to \( P \). \( \square \)

Let \( \Delta_1 \) and \( \Delta_2 \) be simplicial complexes on the same set of points with face posets \( P_1 \) and \( P_2 \). Let \( P_1' \) be the separation of \( P_1 \) and \( \sigma : P_1' \to P_1 \) be defined as above. Construct another relation \( \doteq \) on \( P_1' \) as follows:

\[
a \doteq b \iff \sigma(a) = \sigma(b) \text{ and } \sigma(a) \in P_1 \cap P_2
\]

It is true that \( \doteq \) is a gluing relation which refines \( \sim \). The function \( \Theta(\Delta_1, \Delta_2) \) can now be defined.

**Definition 3.** For simplicial posets \( \Delta_1, \Delta_2 \) on the same set of points, \( \Theta \) is defined as follows:

\[
\Theta(\Delta_1, \Delta_2) := P_1'/\doteq
\]

We will now prepare to prove the main theorem of this section.

**Definition 4.** Let \( P \) be a simplicial poset. For a maximal vertex \( v \in P \), let \( A(v) \) denote the set of atoms of \( P \) below \( v \). Analogously, define \( A(P) \) as the following subset family:

\[
A(P) := \{A(v) : v \text{ maximal in } P\}
\]

\( A(P) \) will be called the **atom family** of \( P \).

**Definition 5.** Let \( P \) be a simplicial poset with set \( F \) of maximal vertices. Define the **meet poset** \( M(P) \) as follows:

\[
M(P) = \bigcup_{A,B \in F, A \neq B} (L_A \cap L_B)
\]
Theorem 2. Let $P$ be a simplicial poset. Consider the following propositions:

(1) As a subset family of the atoms of $P$, $A(P)$ is an antichain.
(2) $M(P)$ is isomorphic to the face poset of some simplicial complex.
(3) $P$ is equal to $\Theta(\Delta_1, \Delta_2)$ (up to isomorphism) for some simplicial complexes $\Delta_1, \Delta_2$.

Claim: (3) holds if and only if (1) and (2) hold.

Proof. First, assume (3). We must show (1) and (2).

(1): Let $f : P' \to P'/\simeq$ be the canonical quotient map. For each maximal vertex $v \in P'$, verify that $f(L_v)$ is a boolean lattice with the same rank as $L_v$. The atoms of $f(L_v)$ are in bijection with the atoms of $L_v$ since the relations between atoms under $\simeq$ and $\sim$ are the same. This is because every atom is always contained in the intersection $P_1 \cap P_2$.

(2): The construction of $\simeq$ is such that the $M(P'/\simeq) \cong P_1 \cap P_2$, the right hand side being a simplicial complex since it is the intersection of two simplicial complexes. To see this, observe that property (1) of Definition only allows for incomparable vertices with no common upper bound to be related under a gluing relation. If two vertices are related, then they must be below more than one maximal vertex. Under $\simeq$, the intersection $P_1 \cap P_2$ determines which vertices are related.

Now, assume that (1) and (2) hold. We will construct $\Delta_1$ and $\Delta_2$ such that (3) holds.

Let $\Delta_1$ be the simplicial complex whose facets are the sets in $A(P)$. Let $\Delta_2$ be the simplicial complex $M(P)$. Notice that $\Delta_1$ and $\Delta_2$ share the same set of points. Our choice of $\Delta_1$ and $\Delta_2$ is such that $P_1 \cap P_2 = M(P)$. It is easy to see that this implies $\simeq$ and $\sim$ are equivalent relations. By Theorem there is a canonical isomorphism $\Theta(\Delta_1, \Delta_2) \to P$. The proof is now complete.

Here are two examples that show properties (1) and (2) are independent of eachother.

Example 1. Stanley’s construction of a simplicial poset with $f$-vector $< 1, 2, 2 >$ satisfies (2) but not (1).

Example 2. For an example satisfying (1) but not (2), let $P$ be any simplicial poset such that (2) does not hold. For a maximal vertex $v \in P$, adjoin a new maximal vertex $v'$ to $P$ defined by the following properties:

(1) $\text{rank}(v') = \text{rank}(v) + 1$
(2) $v \leq v'$
(3) $A(v') = A(v) \cup a$ where $a$ is a new, unique atom.

Perform this operation twice on each maximal $v \in P$ to obtain a poset $P^+$ such that $M(P^+) = P$. $A(P^+)$ is an antichain because every maximal vertex is above an atom exclusive to that vertex.

The previous construction of a simplicial poset with a given meet poset is implemented in SimplicialPosets as fromMeetPoset. We also have implemented meetPoset and atomFamily. See the documentation for more information.

5. More examples

We now return from the theoretical description to give a demo of the implementation. The parameters to randSimplicialPoset are the number of points and two Kahle model probability parameters.

```
i1 : P = randSimplicialPoset(5, 0.5, 0.5)
o1 = Poset

i2 : getFVector(P)
o2 = {1, 5, 5}
o2 : List
```
Notice that the return type of `randSimplicialPoset` is a `Poset`, which is a type that comes from the package `Posets`. Generally, there is no `SimplicialPoset` type. The return values of functions from `SimplicialPosets` can be easily used with functions from `Posets`. For example:

```i3 : displayPoset(P, SuppressLabels => false);
```

The previous line of code will try to open a PDF file with a Hasse diagram of P. In this case, here is what it looks like:

```
6 7 8 9 10

0 2 3 4 5
```

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