Unicyclic Ramsey \((P_3, P_n)\)-minimal graphs obtained from trees in the same class

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Abstract. If \(G\), \(H\) and \(F\) are finite and simple graphs, notation \(F \rightarrow (G, H)\) means that for any red-blue coloring of the edges of \(F\), there is either a red subgraph isomorphic to \(G\) or a blue subgraph isomorphic to \(H\). A graph \(F\) is a Ramsey \((G, H)\)-minimal graph if \(F \rightarrow (G, H)\) and for any edge \(e \in E(F)\), graph \(F - e \not\rightarrow (G, H)\). The class of all Ramsey \((G, H)\)-minimal graphs (up to isomorphism) will be denoted by \(R(G, H)\). The characterization of all graphs in the infinite class \(R(P_3, P_n)\) is still open, for any \(n \geq 4\). In this paper, we find an infinite families of trees in \(R(P_3, P_5)\). We determine how to construct unicyclic graphs in \(R(P_3, P_n)\), for any \(n \geq 5\) from trees in the same class. Further, we give some properties for the unicyclic graphs constructed from trees in \(R(P_3, P_n)\), for any \(n \geq 5\).

1. Introduction
Some notations and terminology that are not mentioned specifically, we follow [4]. If \(F\), \(G\) and \(H\) are finite and simple graphs, the notation \(F \rightarrow (G, H)\) means that for any red-blue coloring of the edges of \(F\), there is either a red subgraph isomorphic to \(G\) or a blue subgraph isomorphic to \(H\). A graph \(F\) is a Ramsey \((G, H)\)-minimal graph if \(F \rightarrow (G, H)\) and for any edge \(e \in E(F)\), graph \(F - e \not\rightarrow (G, H)\). The class of all Ramsey \((G, H)\)-minimal graphs is denoted by \(R(G, H)\).

The class \(R(G, H)\) will be called Ramsey-finite or Ramsey-infinite depending upon whether \(R(G, H)\) is finite or infinite, respectively. Here, we denote a path on \(n\) vertices by \(P_n\) and denote a cycle on \(n\) vertices by \(C_n\). A connected graph \(G\) is called unicyclic graph if it contains exactly one cycle \(C_n\). We call \(G\) as an even unicyclic graph if \(G\) contain a cycle \(C_n\) for even \(n\), otherwise we call an odd unicyclic graph.

The main problem in the study of Ramsey \((G, H)\)-minimal graphs is how to characterize all graphs in \(R(G, H)\). Besides that, it is also interesting to determine whether \(R(G, H)\) is either Ramsey-finite or Ramsey-infinite. There are some results
related to Ramsey minimal graphs for pair paths. The trivially pair \((P_2, H)\) is Ramsey-finite where \(\mathcal{R}(P_2, H) = \{H\}\). Burr et al.\(^2\) characterized \(\mathcal{R}(P_3, P_3) = \{K_{1,3}, C_{2n+1} \text{ for } n \geq 1\}\), this means that class \(\mathcal{R}(P_3, P_3)\) is Ramsey-infinite. Then, Burr et al.\(^3\) showed that the pair \((P_m, P_n)\) is Ramsey-infinite, for \(n \geq m \geq 3\).

Yulianti et al.\(^7\) found some graphs and classes of graphs in \(\mathcal{R}(P_3, P_4)\). Recently, Rahmadani et al.\(^5\)-\(^6\) found some infinite families of trees in \(\mathcal{R}(P_3, P_n)\), for each \(n \geq 6\). Furthermore, in \(^1\) they also found an infinite family of unicyclic graphs containing cycle \(C_4\) in \(\mathcal{R}(P_3, P_n)\), for any \(n \geq 6\). In this paper, we find an infinite families of trees in \(\mathcal{R}(P_3, P_5)\). We determine how to construct unicyclic graphs in \(\mathcal{R}(P_3, P_n)\), for any \(n \geq 5\) from trees in the same class. Further, we give some properties for the unicyclic graphs constructed from trees in \(\mathcal{R}(P_3, P_n)\), for any \(n \geq 5\).

2. Main Results

In this section, we give our results. We find the results analytically. Firstly, we find an infinite family of trees in \(\mathcal{R}(P_3, P_5)\).

Consider a tree \(H\) with root \(r\) as in Figure 1.

![Figure 1. H with root r](image)

The following definition needed to construct trees in \(\mathcal{R}(P_3, P_5)\).

**Definition 2.1.** Let \(i\) be a positive integer. For any \(i \geq 3\), the graph \(P_{2i+1}^*\) is a tree constructed from a path with vertices \(v_1, v_2, \ldots, v_{2i+1}\) such that every vertex \(v_j\) for \(j = 1 \pmod{4}\) is connected to a center of \(K_{1,2}\) where \(j \geq 5\) and \(j \neq 2i+1\).

Now, we define trees \(T_5(i)\) for \(i \geq 0\) as follows.

**Definition 2.2.** Let \(T_5\) be an infinite family of trees which contains.

1. Trees \(T_5(0)\), \(T_5(1)\) and \(T_5(2)\) as in Figure 2.

![Figure 2. (a). T_5(0) (b). T_5(1) (c). T_5(2)](image)
(2) Trees $T_5(i)$, for any $i \geq 3$. This tree consists of two copies of $H$ connected by $P_{2i+1}$ by attaching each $r$ to $v_1$ and $v_{2i+1}$.

As illustration, consider $T_5(4)$ as in Figure 3.

![Figure 3. T_5(4)](image)

Next, we prove that trees constructed as in Definition 2.2 are in $\mathcal{R}(P_3, P_5)$.

**Theorem 2.3.** Let $G$ be any graph in $T_5$. Then $G \in \mathcal{R}(P_3, P_5)$.

**Proof.** It is easy to verify that $T_i$ for $i = 0, 1, 2$ belong to $\mathcal{R}(P_3, P_5)$. Let $G$ be any $T_i$, for $i \geq 3$ First, we show that $G \rightarrow (P_3, P_5)$. Consider any red-blue coloring of all edges of $G$. Suppose that we avoid red $P_3$ and blue $P_5$ in $G$. Consider maximal red colorings of $T_5(i)$ starting on $x_i$ to $y_i$ as the following cases. First case, if $x_iv_1$ is red and to avoid a blue $P_5$ then the remaining edges on the backbone path have color red and blue alternately. Second case, if $x_iv_1$ is blue and to avoid a blue $P_5$ then an edge of $K_{1,2}$ incident to $v_1$ must be red. Then, all edges of the longest blue path are in the backbone path. Observe that there are similar blue paths of length 3. However, one of edges incident to $y_i$ is also blue. Then, there will be a blue $P_5$ in $T_5(i)$. Hence, $T_5(i) \rightarrow (P_3, P_5)$.

Second, we show that for any $e \in E(G)$, graph $G - e \nrightarrow (P_3, P_5)$. Define subtrees of $T_5(i) \setminus \{v_1v_2, v_2v_{2i+1}\}$ as subtree $A$ containing $v_1$, subtree $B$ containing $v_{2i+1}$ and subtree $C$ containing $v_2$ and $v_{2i}$. Consider edges of subtree $C$ except of the backbone path as edges of a rooted tree with roots $v_{4j+1}$ for $j \in [1, \lfloor \frac{i-1}{2} \rfloor]$. (a) If $e$ is an edge in $A$ then color $v_iv_{i+1}$ for odd $i \in [1, 2i + 1]$ by red.

(b) If $e$ is an edge in $B$ then color $v_iv_{i+1}$ for even $i \in [2, 2i]$ by red.

(c) If $e = v_1v_{i+1}$, for $i \in [1, 2i + 1]$ then color $xv_1, v_{2i+1}x, v_jv_{j+1}, v_kv_{k+1}$ for even $j < i$ and for odd $k > i$ by red.

(d) If $e$ is an edge of a rooted tree with root $v_{4i+1}$ then color $xv_1, v_{2i-1}y_i, v_jv_{j+1}, v_kv_{k+1}$ for even $j < 4i + 1$ and for odd $k > 4i + 1$ by red.

In addition for each case, if $e$ is a pendant edge then color a pendant edge having a common vertex with $e$ by red. Each of this colorings, color the remaining edges by blue.

We obtain that $T_5(i) - e \nrightarrow (P_3, P_5)$ for any edge $e$ in $E(T_5(i))$. □

Next, we give trees in $\mathcal{R}(P_3, P_n)$ for any $n \geq 6$. 

Definition 2.4. Consider the following definitions.

- Let $i$ be a positive integer. For any $i \geq 2$, the graph $P_{2i+1}^{**}$ is a tree constructed from a path with vertices $v_1, v_2, \ldots, v_{2i+1}$ such that every vertex $v_j$ for $j = 1 \pmod{2}$ is connected to a pendant vertex and for $j = 1 \pmod{4}$ is connected to a center of $K_{1,2}$ where $j \geq 5$ and $j \neq 2i+1$.

- Let $i$ be a positive integer. For any $i \geq 2$, the graph $P_{2i+1}^{***}$ is a tree constructed from a path with vertices $v_1, v_2, \ldots, v_{2i+1}$ such that every vertex $v_j$ for $j = 1 \pmod{2}$ is connected to a pendant vertex and for $j = 1 \pmod{4}$ is connected to a center of $J$ where $j \geq 5$ and $j \neq 2i+1$.

Let $\mathcal{T}_6$ be an infinite family of trees which contains.

1. Trees $T_6(0)$ dan $T_6(1)$ as in Figure 4.

![Figure 4](attachment:figure4.png)

Figure 4. (a). $T_6(0)$  (b). $T_6(1)$

2. Trees $T_6(i)$, for any $i \geq 2$. This tree is constructed from two copies of $H_2$ and a $P_{2i}^{**}$ where each end vertex of $P_{2i}^{**}$ is identified to $v_1$ and $v_{2i+1}$.

Let $\mathcal{T}_7$ be an infinite family of trees which contains.

1. Trees $T_7(0)$ dan $T_7(1)$ as in Figure 5.

![Figure 5](attachment:figure5.png)

Figure 5. (a). $T_7(0)$  (b). $T_7(1)$

2. Trees $T_7(i)$, for any $i \geq 2$. This tree is constructed from two copies of $H_2$ and a $P_{2i+1}^{***}$ where each end vertex of $P_{2i+1}^{***}$ is identified to $v_1$ and $v_{2i+1}$.

Theorem 2.5. \[ The trees $T_6(i) \in \mathcal{R}(P_3, P_6)$ and $T_7(i) \in \mathcal{R}(P_3, P_7)$, for any $i \geq 1$. \]
Definition 2.6. [8] Consider the following definitions.

1. For some \( n \in [6, 7] \) and any \( k \geq 1 \), \( T_n(k) \) is a tree obtained from Theorem 2.5.
2. Graph \( T_{n+2}(k) \) is a tree constructed from \( T_n(k) \) by attaching two pendant edges to every pendant vertex of \( T_n(k) \).

From Definition 2.6, we obtain some infinite families of trees \( T_{n+2}(k) \), for \( n \geq 6 \). This process will be done recursively with initial graphs are trees \( T_n(k) \), for \( n \in [6, 7] \). In [6], Rahmadani et al. proved that trees \( T_{n+2}(k) \) for \( n \geq 6 \) are in \( \mathcal{R}(P_3, P_{n+2}) \).

Theorem 2.7. [6] For any \( k \geq 1 \) and \( n \geq 6 \), if \( T_n(k) \in \mathcal{R}(P_3, P_n) \) then \( T_{n+2}(k) \in \mathcal{R}(P_3, P_{n+2}) \).

By the previous theorems, we can observe that degree of each vertex of \( T_n(k) \) are at most three. Note that, by identifying two non adjacent vertices of a tree \( T \), we will obtain a unicyclic graph of order \( m - 1 \).

Motivated by this, we construct some infinite families of unicyclic graphs obtained from trees \( T_n(k) \) by identifying its two non adjacent vertices. Furthermore, we prove that some of the unicyclic graphs are in \( \mathcal{R}(P_3, P_n) \).

Theorem 2.8. [8] Let \( G \) be any unicyclic graph in \( \mathcal{R}(P_3, P_n) \), for \( n \geq 4 \) then the maximum degree of cycle vertices of \( G \) is either three or four. Moreover, the number of cycle vertices of degree four is at most two.

From Theorem 2.8, we only consider the unicyclic graphs constructed from \( T_n(k) \) having degree at most four.

Definition 2.9. Let \( C_m \) be a cycle of length \( m \geq 3 \). For any \( k \geq 1 \) and \( n \geq 5 \), graph \( U_m(T_n(k)) \) is a unicyclic graph containing a cycle \( C_m \) constructed from \( T_n(k) \) by identifying its two vertices, say \( u \) and \( v \). Denote a new vertex and a new edge obtained by identifying \( u \) and \( v \) in \( U_m(T_n(k)) \) by \( u^v \) (\( v^u \)) and \( uv^* \) respectively.

Observation 2.10. If \( d(u^v) = 4 \) then \( d(u) = 1 \) and \( d(v) = 3 \) (conversely).

By Definition 2.9, we have the following theorems.

Theorem 2.11. Let \( C_m \) be a cycle of length \( m \) in \( U_m(T_n(k)) \). Let \( k \) be the length of shortest path without \( uv^* \) connecting \( u^v \) to a pendant vertex. If \( d(u^v) = 4 \) and \( m + k - 1 \leq n - 1 \) then \( U_m(T_n(k)) \leftrightarrow (P_3, P_n) \).

Proof. To prove \( U_m(T_n(k)) \leftrightarrow (P_3, P_n) \), define a red-blue coloring on the edges of \( U_m(T_n(k)) \) as follows.

- (a) Color each one edge (no cycle edge) incident with cycle vertices by red.
- (b) Color other edges of the backbone path by red such that there is a red matching.
- (c) Color the remaining edges by blue.

By this coloring, since there is only one vertex of degree four then the blue path in the cycle can be only extended to a blue path with root \( u^v \). However, the length of the blue path is at most \( n - 2 \). Then, there is neither red \( P_3 \) nor blue \( P_n \) in \( U_m(T_n(k)) \). Therefore, \( U_m(T_n(k)) \leftrightarrow (P_3, P_n) \). \( \square \)
Theorem 2.12. Let $C_m$ be a cycle of length $m$ in $U_m(T_n(k))$. Let $k$ be the length of shortest path without $uv^*$ connecting $u^*$ to a pendant vertex. If $d(u^*) = 4$ and $m + k - 1 \geq n$ then $U_m(T_n(k)) - e \rightarrow (P_3, P_n)$, where $e$ is either $uv^*$ or an edge incident to $u^*$ and contained in $P_{uv^*}$.

Proof. Consider any red-blue coloring of the edges of $U_m(T_n(k))$. At most one of four edges incident to $u^*$ in $U_m(T_n(k))$ can be red. Observe that, there are three blue $P_n$ containing $u^*$ in $U_m(T_n(k))$. Now, consider any red-blue coloring of the edges of $U_m(T_n(k)) - e$ such that there is no red $P_3$. Consequently, if we delete $e$ of $U_m(T_n(k))$ then we have another blue path of length at least $n$. Then, $U_m(T_n(k)) - e \rightarrow (P_3, P_n)$. 

From previous theorems, if the maximum degree of $G$ is more than four then $G \not\in \mathcal{R}(P_3, P_n)$. Now, consider the unicyclic graphs obtained by identifying two vertices $u$ and $v$ of $T_n(i)$ such that $d(u) + d(v) \leq 3$. It means we have two cases.

Case 1. If $d(u) = 1$ and $d(v) = 2$ (conversely).

Case 2. If $d(u) = d(v) = 1$.

However, we show that there is no unicyclic graph in $\mathcal{R}(P_3, P_n)$ obtained from Case 1 above as in Theorem 2.15.

Definition 2.13. Define $V(T_n(k)) = X_k \cup Y_k \cup W_k$ with $X_k = \{x_k, x_{ks}|s = 1, 2\}$ and $Y_k = \{y_k, y_{ks}|s = 1, 2\}$ where $x_{ks}$ and $y_{ks}$ be vertices of degree one adjacent to $x_k$ and $y_k$, respectively.

From Definition 2.13, it is clear that every vertex of degree two is in $W_i$.

Theorem 2.14. If $d(u^*) = 3$ where $u \in X_k(Y_k)$ and $v \in W_k$ then there exists a pendant edge $e$ of $X_k(Y_k)$ such that $U_m(T_n(k)) - e \rightarrow (P_3, P_n)$.

Proof. By using similar proof to Theorem 2.12, consider any red-blue coloring of the edges of $U_m(T_n(k))$. At most one of three edges incident to $u^*$ in $U_m(T_n(k))$ can be red. Observe that, there are three blue $P_n$ containing $u^*$ in $U_m(T_n(k))$. Now, consider any red-blue coloring of the edges of $U_m(T_n(k)) - e$ such that there is no red $P_3$. Consequently, if we delete $e$ of $U_m(T_n(k))$ then we have another blue path of length at least $n$. Then, $U_m(T_n(k)) - e \rightarrow (P_3, P_n)$.

Conjecture 2.15. If $d(u^*) = 3$ where $u, v \in W_k$ then.

(i) $U_m(T_n(k)) \rightarrow (P_3, P_n)$, for any $n \geq 5$ and odd $m$.
(ii) $U_m(T_n(k)) \rightarrow (P_3, P_n)$, for odd $n$ and even $m$.
(iii) there exists a pendant edge $e$ of $X_k(Y_k)$ such that $U_m(T_n(k)) - e \rightarrow (P_3, P_n)$, for even $n, m$.

Now, we determine the unicyclic graphs belonging to $\mathcal{R}(P_3, P_n)$ constructed from trees $T_n(k)$ by identifying two pendant vertices.

Definition 2.16. For any $k \geq 1$, let $\mathcal{H}$ be a family of unicyclic graphs containing.

(1) Graph $H^1_n(k)$ is graph constructed from $T_n(k)$ by identifying a pendant vertex of $X_k$ with a pendant vertex of $Y_k$. 

(2) Graph $H_n^2(k)$ is graph constructed from $T_n(k)$ by identifying a pendant vertex of $X_k$ or $Y_k$ with a pendant vertex of $W_k$.

(3) Graph $H_n^3(k)$ is graph constructed from $T_n(k)$ by identifying two pendant vertices of $W_k$.

By Definition 2.16 we propose the following conjecture.

**Conjecture 2.17.** For any $k \geq 1$ and $i \in \{1, 2, 3\}$, all unicyclic graphs $H_n^i(k) \in \mathcal{R}(P_3, P_n)$ are in $\mathcal{R}(P_3, P_{n+2})$.

3. Conclusion

Rahmadani et al. [5]-[6] found some infinite families of trees in $\mathcal{R}(P_3, P_n)$, for any $n \geq 6$. First, we find an infinite families of trees for $\mathcal{R}(P_3, P_5)$. Then, Rahmadani et al. [1] gave some unicyclic graphs containing cycle $C_4$ in $\mathcal{R}(P_3, P_n)$, for $n \geq 4$. From the results, we determine how to construct unicyclic graphs in $\mathcal{R}(P_3, P_n)$, for $n \geq 5$ from trees in the same class. Further, we give some properties about the unicyclic graphs. We conjecture that by attaching two certain pendant vertices of trees in $\mathcal{R}(P_3, P_n)$, for any $n \geq 5$, we will obtain unicyclic graphs in the same class. In future, we will determine the sufficient and necessary conditions for all graphs in $\mathcal{R}(P_3, P_n)$. Furthermore, we will create algorithms to construct graphs in $\mathcal{R}(P_3, P_n)$ explicitly, for any $n \geq 5$.

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