A COMPUTATIONAL FRAMEWORK FOR TWO-DIMENSIONAL RANDOM WALKS WITH RESTARTS

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Abstract. The treatment of two-dimensional random walks in the quarter plane leads to Markov processes which involve semi-infinite matrices having Toeplitz or block Toeplitz structure plus a low-rank correction. Finding the steady state probability distribution of the process requires to perform operations involving these structured matrices. Recently, a computational framework for handling such problems without truncating the dimension of the state space and the size of the infinite matrices has been proposed in [5]. The goal is achieved by approximating this kind of matrices as the sum of a banded semi-infinite Toeplitz and of a low-rank correction with finite support. We propose an extension of this framework which allows to deal with more general situations such as processes involving restart events. This is motivated by the need for modeling processes that can incur in unexpected failures like computer system reboots. Algebraically, this gives rise to corrections with infinite support that cannot be treated using the tools currently available in the literature. We present a theoretical analysis of an enriched Banach algebra that, combined with appropriate algorithms, enables the numerical treatment of these problems. The results are applied to the solution of bidimensional Quasi-Birth-Death processes with infinitely many phases which model random walks in the quarter plane, relying on the matrix analytic approach. This methodology reduces the problem to solving a quadratic matrix equation with coefficients of infinite size. We provide conditions on the transition probabilities which ensure that the solution of interest of the matrix equation belongs to the enriched algebra. The reliability of our approach is confirmed by extensive numerical experimentation on some case studies.

1. Introduction. The treatment of the infinite data structures arising from Markov processes usually relies on the assumption that jumps between states become unlikely when their mutual distance increases. For instance, this is natural when considering random walks on lattices where the particle is forced to move to nearby positions at each step. However, there are models that incorporate a global communication with a certain subset of states. A rich source of case studies comes from random walks with restart. This topic has been analyzed under different perspectives [13, 23, 27, 28]. Including resetting events is required in various applications such as modeling computer system reboots [27], intermittent searches involved in relocation phases of foraging animals [10, 13, 22] and computing network indices [1, 19]. Another example arises in computing return probabilities in certain double Quasi-Birth-Death processes: as shown in [6, Section 5.2], it can happen that the probability of going back to a certain state, in finite time, is positive independently of the starting position. An analogous situation is encountered in [35] in the case of an M/T-SPH/1 queue system.

In many queueing models, transition probabilities only depend on the mutual distances between the states. In this case it is possible to handle systems with infinite state space by means of a finite number of parameters. Moreover, this feature translates in considering semi-infinite matrices which have a Toeplitz structure, i.e., matrices $T(a) = (t_{i,j})_{i,j \in \mathbb{Z}^+}$ such that $t_{i,j} = a_{j-i}$ for some given sequence $a = \{a_k\}_{k \in \mathbb{Z}}$, where the indices $i, j$ range in the set $\mathbb{Z}^+$ of positive integers. Indeed, Toeplitz matrices, finite or infinite, are almost ubiquitous in mathematical models where shift invariance properties are satisfied by some function.

Computing the invariant probability measure $\pi$ of random walks in the quarter plane is a non trivial computational task due to the infinite nature of the problem. In [11] and [16], the problem

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is faced by looking for representation of $\pi$ given in terms of countably infinite sums of geometric terms. This strategy restricts the applicability of this technique to a limited number of problems which exclude certain transitions. On the other hand, the matrix analytic approach of [30] allows to reduce the problem to solving a suitable quadratic matrix equation under no restriction on the allowed transitions. In [5], [6], a framework has been introduced to handle such equations in the case of matrix coefficients of infinite size, making it possible to compute an arbitrary number of components of $\pi$ in a finite number of arithmetic operations. However, this approach does not allow to deal with models where some restart condition is involved. In fact, in [5] the authors introduce the class $QT$ of semi-infinite matrices which can be approximated by the sum of a semi-infinite Toeplitz matrix and a correction with finite support, i.e., with a finite number of nonzero entries. But this class cannot deal with models involving long-distance jumps, like the one occurring in restarts, as well as in double QBDs in the cases where the probability of going back to a certain state, in finite time, is positive as in the example of [6, Section 5.2].

In this paper, we propose a generalization of the class $QT$ which includes corrections defining bounded linear operators in $\ell^\infty$ with possibly unbounded support. The only restriction is that the values of the entries stabilize when moving along each column. We show that this is a suitable framework for studying models with restarts and allows to weaken some assumptions made in [5], simplifying the underlying theory. Then we present an application to the analysis of Quasi–Birth–Death processes modeling random walks in the quarter plane.

More specifically, we introduce the sets $QT^d_\infty$ and $EQT$ of semi-infinite matrices with bounded infinity norm. The former is made by matrices representable as a sum of a Toeplitz matrix and a compact correction with columns having entries which decay to zero. The latter is formed by matrices in $QT^d_\infty$ plus a further correction of the kind $ev^T$ for $e = (1, 1, \ldots, v_i)_{i \in \mathbb{Z}^+}$ such that $\sum_{i=1}^{\infty} |v_i| < \infty$. We prove that $QT^d_\infty$ and $EQT$ are Banach algebras, i.e., they are Banach spaces with the infinity norm, closed under matrix multiplication. Moreover, matrices in both classes can be approximated up to any precision by a finite number of parameters. This allows to handle these classes computationally and to apply numerical algorithms valid for finite matrices to the case of infinite matrices. We also show the way of modifying the Matlab Toolbox cqt-toolbox of [7] in order to include and operate with these extended classes.

The introduction of the new classes $QT^d_\infty$ and $EQT$ allows to handle cases which were not treatable with the available classes, typically when restart is implicitly or explicitly involved in the model as in the cases of [6, Section 5.2] and [35]. Relying on the above classes we derive some properties of the minimal nonnegative solution $G$ of the matrix equation.

\begin{equation}
A_1 X^2 + A_0 X + A_{-1} = X,
\end{equation}

associated with double QBDs [21] describing random walks in the quarter plane where the coefficients $A_i$ are nonnegative matrices in $QT^d_\infty$ whose Toeplitz component is tridiagonal. This class of problems covers a wide variety of two-queue models with various service policies as non-preemptive priority, $K$-limited service, server vacation and server setup [31]. Models of this kind concern, for instance, bi-lingual call centers [34], generalized two-node Jackson networks [32], two-demand models [15], two-stage inventory queues [17], and more. Computing the minimal nonnegative solution $G$ of this matrix equation is a fundamental step to solve the QBD by means of the matrix analytic approach of [30]. We refer the reader to the books [3], [21] for more details to this regard. In particular, we provide general conditions on the transition probabilities of the random walk in order that $G \in QT^d_\infty$ or $G \in EQT$. 

The paper is organized as follows. In Section 2 we report some examples of mathematical models which motivate our analysis. Section 3 concerns the introduction and analysis of the classes $\mathcal{QT}_d$ and $\mathcal{EQT}$. It is divided in some subsections, where we introduce notations, briefly recall the definition and some properties of $\mathcal{QT}$ matrices from [5], summarize a few known results of functional analysis used throughout the paper, introduce and analyze the classes $\mathcal{QT}_p$, $\mathcal{QT}_\infty$, $\mathcal{EQT}$, prove that $\mathcal{QT}_\infty$, $\mathcal{EQT}$ are Banach algebras, and outline some implementation issues. In Section 4 we analyze double QBDs which model random walks in the quarter plane where the matrices $A_i$ for $i = -1, 0, 1$ are tridiagonal Toeplitz with subdiagonal, diagonal, and superdiagonal entries $a_{i,j}$, for $j = -1, 0, 1$, respectively, except in the first row where the entries are $y_{i,0}$ and $y_{i,1}$. The values $a_{i,j}$ and $y_{i,j}$ are the probabilities of transition from position $(r,s)$ to position $(r+j,s+i)$ for $r \neq 0$ and for $r = 0$, respectively. Relying on the classes $\mathcal{QT}_\infty$ and $\mathcal{EQT}$, we prove that the minimal nonnegative solution $G$ of equation (1.1) can be written as $G = T(g) + E_g$ where $E_g$ has bounded infinity norm and $T(g)$ is the Toeplitz matrix associated with the function $g(z)$ which solves a suitable scalar quadratic equation. We show that if the overall probability to move down is higher than the overall probability to move up, then the solution $G$ belongs to $\mathcal{QT}_\infty$. Therefore, one can plug known available algorithms — valid for finite matrices — into our proposed computational framework, to approximate $G$. We show that if the above condition is not satisfied and $G$ is strongly ergodic, then $G \in \mathcal{EQT}$ so that, once again, $G$ can be computed by the available algorithms. Finally, in Section 5 we test the computational framework on some representative examples, and in Section 6 we draw the conclusions.

2. Some motivating examples. Let us consider some examples arising from matrix analytic methods for stochastic processes, which require the extended algebras $\mathcal{QT}_\infty$ and $\mathcal{EQT}$.

2.1. 1D random walk with reset. Recently some interests has been raised by models that incorporate exogenous drastic events. Examples might include catastrophes, rebooting of a computer or a strike causing a shutdown in the transportation system. In its simplest form we can think about a random walk on $\mathbb{Z}^+$ whose transitions allow to reach an initial state from every state. Here, we consider a discrete time Markov chain on the set of states $\mathbb{Z}^+$, whose probability of left/right jumps and the restart are independent of the current state with the only exception of the boundary condition. In this setting the transition probability matrix $P$ belongs to $\mathcal{EQT}$ and takes the form

$$P = \begin{bmatrix}
  b_0 & a_0 \\
  c + r & b & a \\
  r & c & b & a \\
  \vdots & \ddots & \ddots & \ddots \\
\end{bmatrix},$$

where the parameters are nonnegative and such that $a + b + c + r = 1$ and $b_0 + a_0 = 1$. This transition probability matrix is a particular case of the class of GI/M/1 Markov chains where matrices are in (block) lower Hessenberg form [30], [20]. A usual quantity of interest for these models is the steady state vector $\pi$ such that $\pi^T P = \pi^T$, $\pi^T e = 1$. For semi-infinite Toeplitz matrices plus a finite support correction, efficient techniques for computing the steady state vector have been developed [3]. In our case, we can decompose $P = T + ev^T$, where $T \in \mathcal{QT}_\infty^d$ is semi-infinite quasi-Toeplitz and $v^T = (r, 0, \ldots)$, and get the relation

$$\pi^T = \pi^T P = \pi^T T + (\pi^T e) v^T = \pi^T T + v^T.$$

This yields $\pi^T (I - T) = v^T$ that enables to retrieve $\pi^T$ by solving a linear system with the matrix $I - T$ in $\mathcal{QT}_\infty^d$. Note that, in this case, the class $\mathcal{QT}_\infty^d$ is used both in the formulation of the
problem and in the algorithmic procedure which is simply reduced to the application of the Matlab backslash command available in the extended cqt-toolbox \[7\].

A different algorithmic approach, which exploits the computational properties of the class \(\mathcal{EQT}\), is to apply the power method implemented by means of the repeated squaring technique to generate the sequence \(P_{k+1} = P_k^2\) starting with \(P_0 = P\), which converges quadratically to the limit \(e \pi^T\) under ergodicity assumptions. In this case, since \(\mathcal{EQT}\) is an algebra, all the matrices \(P_k\) belong to \(\mathcal{EQT}\) and can be computed by means of the command \(\text{P} = \text{P}^* \text{P}\); available in the extended arithmetic of the cqt-toolbox, see Section 3.4.

2.2. Two-node Jackson network with reset. Here, we consider the Two-node Jackson network of \[29\] modified by allowing a reset. This model, represented by a continuous time Markov chain, is described in Figure 2.1 and consists of two queues \(Q_1\) and \(Q_2\) with buffers of infinite capacity. Customers arrive at \(Q_1\) and \(Q_2\) according to two independent Poisson processes with rates \(\lambda_1, \lambda_2\). Customers are served at \(Q_1\) and \(Q_2\) with independent service times exponentially distributed with rates \(\mu_1, \mu_2\), respectively. On leaving \(Q_1\), two events may occur: either there is a reset of the queue where all the customers waiting to be served in \(Q_1\) leave the system, this happens with probability \(1 - \gamma\) for \(0 < \gamma < 1\); or, with probability \(\gamma\), one customer exits from \(Q_1\). The latter enters \(Q_2\) with probability \(p\) or leaves the system with probability \(1 - p\), where \(0 < p < 1\).

After completing service at \(Q_2\), the customer may enter again \(Q_1\) with probability \(q\) or may leave the system with probability \(1 - q\), where \(0 < q < 1\).

The probability matrix, obtained after uniformization from the generator matrix encoding the transition rates \[21\], is given by

\[
(2.1) \quad P = \begin{bmatrix}
B_0 & B_1 & A_1 \\
A_{-1} & A_0 & A_1 \\
& & \ddots & \ddots & \ddots
\end{bmatrix},
\]

where:

\[
A_1 = \frac{1}{\theta} \begin{bmatrix}
\lambda_2 & \gamma \mu_1 & \lambda_2 \\
\gamma \mu_1 & \lambda_1 & \lambda_1 \\
& & \ddots & \ddots & \ddots
\end{bmatrix}, \quad A_{-1} = \frac{1}{\theta} \begin{bmatrix}
(1 - q) \mu_2 & \mu_2 \\
(1 - q) \mu_2 & \mu_2 \\
& & \ddots & \ddots & \ddots
\end{bmatrix},
\]

\[
A_0 = \frac{1}{\theta} \begin{bmatrix}
\gamma (1 - p) \mu_1 & \lambda_1 & 0 \\
\lambda_1 & \gamma (1 - p) \mu_1 & \lambda_1 \\
& & \ddots & \ddots & \ddots
\end{bmatrix} + \frac{1 - \gamma}{\theta} e \pi^T, \quad B_0 = A_0 + \frac{\mu_2}{\theta} I, \quad B_1 = A_1,
\]

and \(\theta = 1 - \gamma + \gamma \mu_1 + \mu_2 + \lambda_1 + \lambda_2\).

Several generalizations of this model are possible. For instance, we may allow different reset levels or we may allow reset also in the second queue \(Q_2\). In that case we would obtain a GI/M/1 Markov chain with semi-infinite blocks as those analyzed in \[20\].

Computing the steady state vector for this process can be recasted to finding the minimal non-negative solution \(G\) to the matrix equation \(1.1\), see for instance \[3\], \[21\] according to the matrix analytic approach of \[30\]. Solution methods, when \(A_1, A_0\) and \(A_{-1}\) are finite matrices, are analyzed in \[3\] and \[21\]. If \(A_1, A_0, A_{-1}\), and \(G\) belong to a Banach algebra whose arithmetic operations can be carried out on a computer, these algorithms can be applied to the case of infinitely many states.

In this example we have \(A_{-1} \in \mathcal{QT}^d_\infty\) and \(A_0 \in \mathcal{EQT}\). In general, it is not always true that \(G \in \mathcal{QT}^d_\infty\). The next section provides an example in this regard.
Fig. 2.1. Pictorial description of the transitions for the two node Jackson network with reset. The queue $Q_1$ is on the left, the queue $Q_2$ is on the right. The square denoted by $R$ indicates the reset event which is triggered with probability $1 - \gamma$ after service at the queue $Q_1$.

Fig. 2.2. Pictorial description of a random walk in the quarter plane, the point of the grid which have integer coordinates $(r, s)$, correspond to the states of the Markov chain.

2.3. A Quasi-Birth-and-Death problem. Finally, we report two examples from [6] and from [35] where the coefficient matrices of (1.1) belong to $\mathcal{QT}_\infty$ but $G$ does not.

Consider a discrete-time Markov chain with state space $\mathbb{N}^2$ which describes a random walk in the quarter plane where states are described by pairs $(r, s)$ for $r, s \in \mathbb{N}$. The allowed transitions are:

(i) if $r, s > 0$, $(r, s) \rightarrow (r + 1, s)$ with probability 1/2; $(r, s) \rightarrow (r, s - 1)$ with probability 1/2;

(ii) if $s \geq 0$, $(0, s) \rightarrow (0, s + 1)$ with probability 1/2; $(0, s) \rightarrow (1, s)$ with probability 1/2;

(iii) if $r > 0$, $(r, 0) \rightarrow (r - 1, 0)$ with probability 1.

Pictorially, these rules can be described as shown in Figure 2.2, where the $r$ coordinate is the abscissa, and $s$ is the ordinate. In Figure 2.2, the numbers on the arrows denote the probability associated with different transitions; the three dots represent the possible cases (i), (ii), and (iii) in which the particle might be. This is encoded in the transition probability matrix (2.1) where
$A_{-1}, A_0, A_1, B_0,$ and $B_1$ are defined as follows:

$$a_1 = \frac{1}{2}(I-e_1e_1^T), \quad A_0 = \frac{1}{2}Z, \quad A_{-1} = e_1e_1^T, \quad B_0 = \frac{1}{2}Z^T, \quad B_1 = \frac{1}{2}I.$$  

Here $Z = (z_{i,j})$ is the shift matrix such that $z_{i+1,i} = 1$, $z_{i,j} = 0$ elsewhere and $e_1 = (1, 0, \ldots)^T$. All these matrices belong to $QT_{\infty}$, so $P$ is a semi-infinite block matrix with semi-infinite blocks.

The minimal nonnegative solution of (1.1) is $G = ee_1^T$ [6]. Despite the coefficients $A_{-1}, A_0, A_1$ belong to $QT_{\infty}^d$, the solution $G$, does not belong to $QT_{\infty}^d$. In particular, any approximation $\hat{G}$ of $G$ in $QT_{\infty}^d$ will be affected by an error $\|\hat{G} - G\|_{\infty} \geq 1$. On the other hand, $G \in EQT$.

A similar example is considered in [35] where the continuous time model analyzed by the authors, defined by the parameters $a, b, \lambda > 0, \beta = (a + b + \lambda)^{-1}$, leads to the matrices $A_{-1} = b\theta e_1e_1^T, A_0 = b\theta Z + a\theta Z^T, A_1 = \lambda \theta I$, and the minimal nonnegative solution of (1.1) is $G = ee_1^T$. Moreover, the minimal nonnegative solution $R$ of the equation $R = R^2A_{-1} + RA_0 + A_1$ has inverse given by the perturbed tridiagonal matrix $\text{tridiag}(-\theta b, 1, -\theta a) - \theta \lambda ee_1^T$, compare [35, eq. (8)], thus $R, R^{-1} \in EQT \setminus QT_{\infty}^d$.

3. $QT$ matrices. We denote by $\ell^p$, with $1 \leq p \leq \infty$, the usual Banach space of $p$-summable sequences $x = (x_j)_{j \in \mathbb{Z}^+}$, with the norms

$$\|x\|_p := \begin{cases} \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} & p < \infty \\ \sup_{j} |x_j| & p = \infty \end{cases}$$

and by $B(\ell^p)$ the set of bounded linear operators from $\ell^p$ into itself with the operator norm $\|A\|_p = \sup\|Ax\|_p = \sup\|x\|_p = 1. A$ sequence $x$ will be also referred to as a semi-infinite vector, or simply a vector. Moreover, we denote by $K(\ell^p) \subset B(\ell^p)$ the subset formed by compact operators, and by $e = (1, 1, \ldots)^T \in \ell^\infty$ the vector of all ones. Throughout this work, we will only consider operators that can be represented as matrices with respect to the standard basis $\{e_i\}_{i \in \mathbb{N}}$. This restrict the focus on operators that act on (and whose image is contained in) the closure of such set, which is smaller than the entire space when $p = \infty$, since $\ell^\infty$ is not separable.

The Wiener class $W$ is the set of Laurent series $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$ such that $\|a\|_W := \sum_{i \in \mathbb{Z}} |a_i|$ is finite. This set, which contains complex valued functions defined on the unit circle, is a Banach algebra [9] with the norm $\|\cdot\|_W$. The map that associates a function $a(z) \in W$, called symbol, with the semi-infinite Toeplitz matrix $T(a) = (t_{i,j})_{i,j \in \mathbb{Z}^+}$, $t_{i,j} = a_{i-j}$, is a bijection between $W$ and the set of bounded Toeplitz operators on $B(\ell^p)$ for $1 \leq p \leq \infty$.

In [5], a new class of semi-infinite matrices is introduced, denoted by $QT$, and is defined as the set of matrices that can be written as the sum of a (semi-infinite) Toeplitz matrix $T(a)$ such that $a'(z) = \sum_{i \in \mathbb{Z}} i a_i z^i \in W$ and a correction $E$ such that $\sum_{i,j \in \mathbb{Z}^+} |E_{i,j}|$ is finite. The class $QT$ is endowed with an appropriate norm, which makes it a Banach algebra. This norm is denoted by $\|\cdot\|_{QT}$ and is defined as follows:

$$\|T(a) + E\|_{QT} = \|a\|_W + \|a'\|_W + \|E\|_F, \quad \|E\|_F := \sum_{i,j \in \mathbb{Z}^+} |E_{i,j}|.$$  

This framework has shown to be very effective in the development of numerical algorithms that treat the infinite dimensional case “directly”, without the need of truncating matrices to finite size. It provides a practical tool for solving computational problems like computing matrix functions.
and solving matrix equations where the input is given by $QT$ matrices. We refer the reader to 
\cite{2,4,6-8,33} for some examples where this arithmetic has been used numerically to solve various kinds of tasks.

However, several aspects of the theory are not yet completely satisfactory. For instance, the requirement that the symbol $a'(z)$ lives in $W$ is stronger than simply requiring $a(z) \in W$, and seems artificial. Moreover, there are cases in the setting of Markov chains that fit very naturally in the requirement that the symbol $a'(z)$ lives in $W$ is stronger than simply requiring $a(z) \in W$, and seems artificial. Moreover, there are cases in the setting of Markov chains that fit very naturally in the set of low-rank perturbations of semi-infinite Toeplitz matrices, but cannot be described under this framework because the correction $E$ does not have finite norm when considering $\|\cdot\|_p$. A couple of examples are given in Section 2. The aim of this section is introducing a superset of $QT$ that allows to treat such cases maintaining the features needed to establish a computational framework. Let us first introduce some notation.

Given $a(z) \in W$ define $a^+(z) = \sum_{i \in \mathbb{Z}^+} a_i z^i$, $a^-(z) = \sum_{i \in \mathbb{Z}^+} a_{-i} z^i$ so that $a(z) = a_0 + a^-(z^{-1}) + a^+(z)$, and associate with $a^\pm(z)$ the following semi-infinite Hankel matrices

$$H(a^+) = (H^+_{i,j}), \quad H^+_{i,j} = a_{i+j-1},$$
$$H(a^-) = (H^-_{i,j}), \quad H^-_{i,j} = a_{-i-j+1}.$$

The following result from \cite[Proposition 1.3]{9} links semi-infinite Toeplitz and Hankel matrices.

**Theorem 3.1** (Gohberg-Feldman). If $a(z) \in W$ then $\|T(a)\|_p \leq \|a\|_W$, $\|H(a^-)\|_p \leq \|a\|_W$, $\|H(a^+)\|_p \leq \|a\|_W$. If $c(z) = a(z)b(z)$ where $a(z), b(z) \in W$, then

$$T(a)T(b) = T(c) - H(a^-)H(b^+).$$

The Hankel matrices $H(a^-)$ and $H(b^+)$ are compact operators in $B(\ell^p)$ for every $1 \leq p \leq \infty$ \cite[Proposition 1.2]{9}.

**3.1. The class of $QT_p$ matrices.** A more general approach for defining the set of quasi-Toeplitz matrices is avoiding the norm $\|\cdot\|_Q$ and keeping the induced operator norm $\|\cdot\|_p$.

**Definition 3.2.** Given an integer $p$, $1 \leq p \leq \infty$, we say that the semi-infinite matrix $A$ is $p$-Quasi-Toeplitz if it can be written in the form

$$A = T(a) + E,$$ where $a(z) \in W$, and $E$ defines a compact operator in $B(\ell_p)$. We refer to $T(a)$ as the Toeplitz part of $A$, and to $E$ as the correction. We denote the set of $p$-Quasi-Toeplitz matrices as $QT_p$.

The set $QT_p$ is closed under product. In fact, denoting $A = T(a) + E_a$, $B = T(b) + E_b$ in $QT_p$ one has

$$C = AB = T(a)T(b) + T(a)E_b + E_aT(b) + E_aE_b.$$ Moreover, denoting $c(z) = a(z)b(z)$, since in view of Theorem 3.1 we have $T(a)T(b) = T(c) - H(a^-)H(b^+)$, then it follows that

$$C = T(c) + E_c,$$
$$E_c = -H(a^-)H(b^+) + T(a)E_b + E_aT(b) + E_aE_b.$$ The matrix $E_c$ is compact in $B(\ell^p)$ since each summand is compact in $B(\ell^p)$ being the product of two compact operators in $B(\ell^p)$. This proves that $QT_p$ is closed under matrix multiplication, and being a subspace of $B(\ell^p)$, we have the following.
Theorem 3.3. The class $\mathcal{QT}_p$ for any integer $p$, $1 \leq p \leq \infty$ is an algebra in $\mathcal{B}(\ell^p)$.

Remark 3.4. The set $\mathcal{QT}_p$ is not necessarily topologically closed for $1 < p < \infty$; for instance, for $p = 2$ it is known that $\|T(a)\|_2 = \|a\|_\infty$ [9], where $\|a\|_\infty$ is intended as the sup-norm of continuous function over $\mathbb{T}$. By the Du Bois-Reymond theorem [12] there exists a continuous function $a$ whose Fourier series is not summable. The latter could be approximated uniformly with polynomials in view of Weierstrass’ theorem, and this produces a sequence of operators $T(a_n) \to T(a)$ in the 2-norm — but whose limit has symbol outside the Wiener class. In Section 3.3 we show that for the case $p = \infty$, which is the one of interest for our applications, the set $\mathcal{QT}_\infty$ is a (closed) Banach algebra.

The following result ensures that the set $\mathcal{QT}_p$ extends $\mathcal{QT}$.

Lemma 3.5. For any integer $1 \leq p \leq \infty$, it holds $\mathcal{QT} \subset \mathcal{QT}_p$.

Proof. Let $A = T(a) + E$ be such that $\|E\|_p \leq 1$. Without loss of generality we can consider the case $\|E\|_p = 1$ so that $\|E_{ij}\|_{\ell^p} \leq 1$ for all $i, j$. In fact, if $\|E\|_p = \theta \neq 1$, the condition $\|E\|_p \leq \|E\|_p$ is equivalent to $\|\theta^{-1}E\|_p \leq \|\theta^{-1}E\|_p$, that is, $\|E\|_p \leq \|\tilde{E}\|_p$ where $\tilde{E} = \theta^{-1}E$ is such that $\|\tilde{E}\|_p = 1$.

Let $x$ be such that $\|x\|_p = 1$, $y = Ex$ so that $\|y\|_p \leq \|E\|_p$. Observe that $\|x_i\| \leq 1$ for all $i$ so that

$$|y_i| \leq \sum_{j \geq 1} |E_{ij}x_j| \leq \sum_{j \geq 1} |E_{ij}| \leq 1.$$

Since $p \geq 1$, then

$$|y_i|^p \leq |y_i| \leq \sum_{j \geq 1} |E_{ij}| \implies \|y\|_p \leq \|E\|_p^{1/p} = \|E\|_p,$$

where the last equality holds since $\|E\|_p = 1$. This way, $\|E\|_p = \sup_{\|x\|_p = 1} \|Ex\|_p \leq \|E\|_p$. \qed

The inclusion is strict; in fact the following three semi-infinite matrices defined by $A = T(a) + E$ with $T(a) = 0$ and $E = (E_{ij})$ such that

$$E_{ij} = i^{-2}, \quad E_{ij} = \begin{cases} i^{-\frac{p}{2}} & j = 1 \\ 0 & \text{otherwise} \end{cases}, \quad E_{ij} = j^{-2},$$

belong to $\mathcal{QT}_p \setminus \mathcal{QT}$. For $p = 1$, $E = \mathcal{QT}$, and $p = \infty$, respectively.

Matrices in the $\mathcal{QT}_p$ class, for $p \neq 1, \infty$, can be approximated to any arbitrary precision by using a finite number of parameters, in the following sense.

Lemma 3.6. Let $A = T(a) + E$ be such that $\|E\|_p$, for some integer $p \in (1, \infty)$, then, for any $\epsilon > 0$ there exist $\tilde{E} \in \mathcal{K}(\ell^p)$ with finite support and a Laurent polynomial $\tilde{a}(z)$ such that $\|A - \tilde{A}\|_p \leq \epsilon$ where $\tilde{A} = T(\tilde{a}) + \tilde{E}$.

Proof. Since $a(z) \in \mathcal{W}$, there exists a Laurent polynomial $\tilde{a}(z)$ such that $\|a - \tilde{a}\|_W \leq \frac{\epsilon}{2}$, and therefore $\|T(a) - T(\tilde{a})\|_p \leq \|a - \tilde{a}\|_W \leq \frac{\epsilon}{2}$. Since $E$ is compact and since $\ell^p$ for $1 \leq p < \infty$ admits a Schauder basis, finite rank operators are dense in $\mathcal{K}(\ell^p)$, see [24, Theorem 4.1.33]. Therefore we can find $\tilde{E}$ of finite rank $k$ such that $\|E - \tilde{E}\|_p \leq \frac{\epsilon}{2}$. Thus, we can write $\tilde{E} = \sum_{j=1}^k u_j v_j^T$, with $u_j \in \ell^p$ and $v_j \in \ell^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, and $p, q > 1$. This implies that each $u_j, v_j$ can be approximated arbitrarily well with vectors of finite support $\tilde{u}_j, \tilde{v}_j$ such that $\|u_j v_j^T - \tilde{u}_j \tilde{v}_j^T\| \leq \frac{\epsilon}{2k}$. Setting $\tilde{E} := \sum_{j=1}^k \tilde{u}_j \tilde{v}_j^T$, which has finite support, concludes the proof. \qed
3.2. The class $\mathcal{QT}_d^\ell$. Observe that Lemma 3.6 does not hold for $p = 1$ and for $p = \infty$. In fact, for any random vector with components in modulus less than 1 we have $v e_1^T \in \mathcal{QT}_\infty$ and $e_1 v^T \in \mathcal{QT}_1$. On the other hand, $v$ cannot be approximated to any precision with a finite number of parameters. This limitation is a serious drawback from the computational point of view especially for $p = \infty$ since the $\ell^\infty$ environment is the natural setting for Markov chains.

For this reason, we introduce a slightly different definition for the case $p = \infty$; the case $p = 1$ can be treated by considering the transpose matrix$^1$ of elements in $\mathcal{QT}_\infty$.

Definition 3.7. A matrix $E \in \mathcal{B}(\ell^\infty)$ has the decay property if the vector $w = |E| e$, $w = (w_i)$ is such that $\lim_{i \to \infty} w_i = 0$, where $|E| = (|E_{i,j}|)$.

Definition 3.8. We define $\mathcal{QT}_d^\ell$ the class of all the matrices which can be written in the form $A = T(a) + E$, where $a(z) \in W$ and $E \in \mathcal{B}(\ell^\infty)$ has the decay property. The superscript “$d$” denotes “decay”.

The decay property allows to state an approximability result in the same spirit of Lemma 3.6 for matrices in $\mathcal{B}(\ell^\infty)$.

Lemma 3.9. Let $E \in \mathcal{B}(\ell^\infty)$, and let $E^{(k)}$ be the matrix that coincides with $E$ in the leading principal $k \times k$ submatrix and is zero elsewhere. Then, the following are equivalent:

(i) $E$ has the decay property;
(ii) $\lim_{k \to \infty} \|E - E^{(k)}\|_\infty = 0$.

In particular, if $E$ has the decay property then it represents a compact operator in $\mathcal{B}(\ell^\infty)$.

Proof. We first prove (i) $\implies$ (ii). Since $w = |E| e$ is such that $\lim_{i \to \infty} w_i = 0$, then for any $\epsilon > 0$ there exists $m$ such that $w_i \leq \epsilon$ for any $i > m$. Therefore, the matrix $E^{(m)}$ is such that the vector $v = |E - E^{(m)}| e$ has components $v_i \leq \epsilon$ for $i > m$. On the other hand, since $|E| \in \mathcal{B}(\ell^\infty)$, then each row $v^{(i)} = e_i^T |E|$ has sum of its entries finite, therefore there exists $n_i$ such that $\sum_{j=n_i+1}^{\infty} r^{(i)}_j \leq \epsilon$. Setting $n = \max\{m, n_1, n_2, \ldots, n_m\}$ yields $\|E - E^{(k)}\|_\infty \leq \epsilon$ for any $k \geq n$.

Concerning (ii) $\implies$ (i), we consider $v^{(k)} = |E - E^{(k)}| e$, and $w = |E| e$. Observe that, since $E_{i,j}^{(k)} = 0$ for $i > k$ or for $j > k$, then $v_{i,j}^{(k)} = w_i$ for $i > k$. Moreover, since $\|v^{(k)}\|_\infty = \|E - E^{(k)}\|_\infty$, then $\lim_{k \to \infty} \|v^{(k)}\|_\infty = \lim_{k \to \infty} \|E - E^{(k)}\|_\infty = 0$ so that for any $\epsilon > 0$ there exists $k_0$ such that $\|v^{(k)}\|_\infty \leq \epsilon$ for any $k \geq k_0$, whence $v_i^{(k_0)} \leq \epsilon$ for any $i$. In particular, $v_i^{(k_0)} \leq \epsilon$ for any $i$. Thus, since $w_i = v_i^{(k_0)}$ for any $i > k_0$, then $w_i \leq \epsilon$ for any $i > k_0$. Finally, since $E$ is the limit of compact operators it is compact.

An immediate consequence of Lemma 3.9 is that any $A \in \mathcal{QT}_\infty^d$ can be approximated by a finitely representable matrix in $\mathcal{QT}_\infty^d$.

Corollary 3.10. Let $A = T(a) + E \in \mathcal{QT}_\infty^d$. Then, for every $\epsilon > 0$ there exists a Laurent polynomial $a(z)$ and an integer $k$ such that $\|A - T(a) - E^{(k)}\|_\infty \leq \epsilon$.

The class of matrices having the decay property is closed as specified by the following

Theorem 3.11. Let $E_k \in \mathcal{B}(\ell^\infty)$, for $k \in \mathbb{Z}^+$, have the decay property. Assume that there exists $E \in \mathcal{B}(\ell^\infty)$ such that $\lim_{k \to \infty} \|E_k - E\|_\infty = 0$. Then $E$ has the decay property as well.

Proof. It is enough to prove that $\lim_{i} v_i = 0$ for $v = |E| e$. Denote $v^{(k)} = |E_k| e$. From $|E_k - E| \geq \|E_k - |E|\|$ we deduce that $|E_k - E| e \geq \|E_k - |E|\| e \geq |v^{(k)} - v|$. Whence $\|E_k - E\|_\infty = 1$. Note that, even if $\ell^d$ is much smaller of the dual of $\ell^\infty$, the additional constrain of considering operators representable as matrices over the canonical basis, implies $\mathcal{QT}_1 = (\mathcal{QT}_\infty)^*$.
∥Ek − E∥∞ ≥ ∥v(k) − v∥∞. This implies that \( \lim_k \sup_i |v_i^{(k)} - v_i| = 0 \). We now deduce that \( \lim_i v_i = 0 \). From the condition \( \lim_k \sup_i |v_i^{(k)} - v_i| = 0 \) we find that for any \( \epsilon > 0 \) there exists \( k_0 \) such that \( \sup_i |v_i^{(k)} - v_i| \leq \epsilon \) for any \( k \geq k_0 \), that is \( |v_i^{(k)} - v_i| \leq \epsilon \) for any \( i \) and for \( k \geq k_0 \). Therefore \( v_i \in [v_i^{(k)} - \epsilon, v_i^{(k)} + \epsilon] \) for any \( i \) and for any \( k \geq k_0 \). On the other hand from the condition \( \lim_i v_i^{(k)} = 0 \) for any \( k \) we deduce that for any \( \epsilon > 0 \) and for any \( k \) there exists \( i_k \) such that \( |v_i^{(k)}| \leq \epsilon \) for any \( i \geq i_k \). Combining the two properties yields \( v_i \in [-2\epsilon, 2\epsilon] \) for any \( i \geq i_{k_0} \). That is \( \lim_i v_i = 0 \).

We consider the quotient space of \( B(\ell^\infty) \) under the equivalence relation: \( A \equiv B \) if and only if \( A - B \) has the decay property. If \( A \equiv B \) is representable with a finite number of parameters, then, in light of Lemma 3.9, every \( B \) such that \( A \equiv B \) is also representable using a finite number of parameters. Matrices with the decay property form a right ideal.

**Lemma 3.12.** Let \( A, B \in B(\ell^\infty) \) such that \( A \equiv 0 \). Then

(i) if \( B \equiv 0 \) then \( A + B \equiv 0 \),

(ii) \( AB \equiv 0 \),

(iii) if \( B = T(b) \) with \( b \in W \) then \( BA \equiv 0 \).

**Proof.** Claim (i) easily follows applying the definition. Concerning (ii), we notice that \( |AB|e \leq |A||B|e \leq ∥B∥_∞|A|e \) which is an infinitesimal vector. Let \( w = |A|e \) with entries \( w_i \) such that \( \lim_{i \to \infty} w_i = 0 \). In order to prove (iii), let us start by considering \( B = T(b) \) where the symbol \( b \) has finite support, more precisely \( b_j = 0 \) whenever \( |j| > k \), for some \( k \in \mathbb{N} \). Then we have \( |BA|e \leq |B|w = q \) whose entries \( g_i \) verify \( g_i = \sum_{j=i-k}^{i+k} |b_{j-i}| |w_j| \) for \( i > k \). Therefore \( g_i \to 0 \). If \( b \) has not finite support we consider \( b_k \) the Laurent polynomial obtained by truncating \( b \) with coefficients in the exponent range \([-k, k] \); clearly \( ∥T(b) - T(b_k)∥_∞ \to 0 \) which implies \( ∥T(b)A - T(b_k)A∥_∞ \to 0 \). Hence, the claim follows applying Theorem 3.11.

Note that, \( A \equiv 0 \) \( \iff \) \( BA \equiv 0 \); indeed consider \( A = e_1e_1^T \) and \( B = ee_1^T \) as a counterexample.

We shall now prove that the Hankel matrices arising in Theorem 3.1 have the decay property.

**Lemma 3.13.** Let \( a(z) \in W \) then \( H(a^-) \equiv 0 \) and \( H(a^+) \equiv 0 \).

**Proof.** Consider the vector \( w = |H(a^-)|e \); it holds that \( w_i = \sum_{j=-i}^{\infty} |a_j| \), whence \( \lim w_i = 0 \), i.e., \( H(a^-) \equiv 0 \). The same holds for \( H(a^+) \).

This, combined with Lemma 3.12 yields the following Corollary.

**Corollary 3.14.** Let \( a, b \in W \) then

\begin{equation}
T(a)T(b) \equiv T(ab) \equiv T(b)T(a),
\end{equation}

\begin{equation}
T(a)T(a^{-1}) \equiv I, \quad \text{if } a(z) \neq 0 \text{ for } |z| = 1.
\end{equation}

The next result will be crucial for proving the closedness of \( QT^d_\infty \).

**Lemma 3.15.** If \( A \in QT^d_\infty \), \( A = T(a) + E \), then \( ∥A∥_∞ \geq ∥a∥_W \).

**Proof.** We prove that for any \( \epsilon > 0 \) there exists \( i_0 \) such that for any \( i \geq i_0 \) we have \( e_i^T|A|e \geq ∥a∥_W - 2\epsilon \). Since \( ∥A∥_∞ = \sup_i e_i^T|A|e \), then from the latter inequality it follows that \( ∥A∥_∞ \geq ∥a∥_W \). In order to prove the claim, we observe that since \( |A| \geq ∥T(a) - |E| \) we have

\[ e_i^T|A|e \geq e_i^T|T(a)|e - \epsilon_i|E|e. \]
From the decay property of $E$ we have that there exists $h_0$ such that for any $i \geq h_0$ we have $e_i/E|e| \leq \epsilon$. On the other hand, since $e_i^T |T(a)| = \sum_{j=-\infty}^{|a_j|} = \|a\|_\infty - \sum_{j=-\infty}^{-|a_j|},$ and since $a(z) \in W$ then there exists $k_0$ such that $e_i^T |T(a)| = \|a\|_\infty - \epsilon_i,$ where $|\epsilon_i| \leq \epsilon$ for any $i \geq k_0$. Thus for any $i \geq i_0 = \max\{h_0, k_0\}$ we have

$$e_i^T |A| e \geq \|a\|_\infty - |\epsilon_i| - \epsilon \geq \|a\|_\infty - 2\epsilon.$$ 

\[ \square \]

**Theorem 3.16.** The class $\mathcal{QT}_\infty^d$ is a Banach algebra with the infinity norm.

**Proof.** For the property of algebra it is enough to show that if $A = T(a) + E_a, B = T(b) + E_b$ are in $\mathcal{QT}_\infty^d,$ then also $A + B, \alpha A$ and $AB$ are in $\mathcal{QT}_\infty^d$. For the first two matrices the property is trivial since $aE_a$ and $E_a + E_b$ have the decay property. For the third condition, Lemma 3.12 and Corollary 3.14 imply $AB = T(ab)$.

It remains to prove that $\mathcal{QT}_\infty^d$ is complete. If $X_k = T(x_k) + E_k \in \mathcal{QT}_\infty^d$ is a Cauchy sequence with the infinity norm, then, since $\mathcal{B}(\ell^\infty)$ is a Banach space there exists $X \in \mathcal{B}(\ell^\infty)$ such that $\lim_k \|X_k - X\|_\infty = 0$. We have to prove that $X \in \mathcal{QT}_\infty^d, \text{i.e., } X = T(x) + E$ for some $x(z) \in W$ and $E \in \mathcal{B}(\ell^\infty)$ with the decay property. From Lemma 3.15 we have $\|X_k - X_h\|_\infty \geq \|x_k - x_h\|_\infty$ therefore, since $\{X_k\}$ is Cauchy, then also $\{x_k(z)\}$ is Cauchy with the Wiener norm. Thus, since $W$ is a Banach space then there exists $x(z) \in W$ such that $\lim_k \|x_k(z) - x(z)\|_\infty = 0$. Now consider $E_k - E_h$, Since $E_k - E_h = X_k - X_h + T(x_k - x_h)$ we have $\|E_k - E_h\|_\infty \leq \|X_k - X_h\|_\infty + \|x_k - x_h\|_\infty$, whence $E_k$ is Cauchy in $\mathcal{B}(\ell^\infty)$ therefore there exists $E \in \mathcal{B}(\ell^\infty)$ such that $\lim_k \|E_k - E\|_\infty = 0$. It remains to prove that $E$ has the decay property. This follows from Theorem 3.11.\[ \square \]

**3.3. The class $\mathcal{EQT}$.** The matrices involved in the examples of Section 2 modeling stochastic processes with restarts do not belong to $\mathcal{QT}_\infty^d$. Indeed, their correction part has columns which do not decay to 0, but instead converge to a certain limit. In particular, the correction does not have the decay property but it is still (approximately) representable by a finite set of parameters. In this section we introduce an appropriate extension of $\mathcal{QT}_\infty^d$ that includes these matrices.

**Definition 3.17.** We say that the semi-infinite matrix $A$ is extended-quasi-Toeplitz if it can be written in the form

$$A = T(a) + E + ev^T,$$

where $a(z) \in W, E = 0$ and $v \in \ell^1$. We denote the set of extended-quasi-Toeplitz matrices with the symbol $\mathcal{EQT}$.

Clearly, $\mathcal{QT}_\infty^d \subset \mathcal{EQT} \subset \mathcal{B}(\ell^\infty)$, and in view of Corollary 3.10 the matrices in these classes are representable with a finite number of parameters within a given error bound $\epsilon$. Indeed, the term $ev^T$ in (3.2) can be approximated — in the $\infty$-norm — by truncating $v \in \ell^1$ to a vector of finite support. Similarly to $\mathcal{QT}_\infty^d$, the set $\mathcal{EQT}$ is a Banach algebra. It is immediate to check that $A, B \in \mathcal{EQT} \implies A + B \in \mathcal{EQT}$. Multiplication requires some explicit computations.

**Lemma 3.18.** Let $A = T(a) + E_a + ev_a^T$ and $B = T(b) + E_b + ev_b^T$ be matrices in $\mathcal{EQT}$. Then $C = AB \in \mathcal{EQT}$ and $C = T(c) + E_c + ev_c^T$ where $c = ab, v_c = \left(\sum_{j \in \mathbb{Z}} a_j\right) v_b + b^Tv_a$ and

$$E_c = T(a)E_b + E_aT(b) - H(a^-)H(b^+) + E_aE_b + (E_a - H(a^-))ev_a^T.$$ 

**Proof.** The result follows via a direct computation using the relation $T(a)e = \left(\sum_{j \in \mathbb{Z}} a_j\right) e - H(a^-)e$. Note that, $E_c \equiv 0$ in view of Lemma 3.12.\[ \square \]
In order to state the main result of this section, we need the following generalization of Lemma 3.15.

**Lemma 3.19.** If \( A \in \mathcal{EQT}, A = T(a) + E + ev^T \), then \( \|A\|_\infty \geq \|a\|_{\infty} + \|v\|_1 \).

**Proof.** We prove that for any \( \epsilon > 0 \) there exists \( k \) such that \( \|e_k^TA\|_1 \geq \|a\|_{\infty} + \|v\|_1 - 5\epsilon \) so that the claim follows from the inequality \( \|A\|_\infty \geq \|e_k^TA\|_1 \) and by the arbitrariness of \( \epsilon \). To this end, given \( \epsilon \), it is sufficient to choose \( k = 2p + 1 \) where \( p \) is large enough so that \( \sum_{i=p+1}^{\infty} |a_i| \leq \epsilon \), \( \sum_{i=-\infty}^{-p-1} |a_i| \leq \epsilon \) and \( w_k \leq \epsilon \) where \( w = |E|_e \). This way the \( k \)th row of \( A \) is \( r_k = e_k^TA = v^T + u^T + s^T \) where \( u^T = [a_{-2p}, a_{-2p+1}, \ldots], s^T = e_k^TE \). Observe that \( \|s\|_1 = w_k \leq \epsilon \) so that

\[
(3.3) \quad \|r_k\|_1 \geq \|v + u\|_1 - \epsilon.
\]

In order to estimate \( \|v + u\|_1 \), decompose \( v \) as \( v = \tilde{v} + \hat{v} \) where \( \tilde{v} = [v_1, \ldots, v_p, 0, \ldots]^T, \hat{v} = [0, \ldots, 0, v_{p+1}, \ldots]^T \). Do the same with \( u = \tilde{u} + \hat{u} \). Since \( \tilde{v} \) and \( \hat{v} \) have disjoint supports, then \( \|\tilde{v} + \tilde{u}\|_1 = \|\tilde{v}\|_1 + \|\tilde{u}\|_1, \) moreover, thanks to the choice of \( p \), we have \( \|\tilde{v} + \tilde{u}\|_1 \leq 2\epsilon \). Thus, we deduce that

\[
(3.4) \quad \|v + u\|_1 \geq \|\tilde{v} + \hat{u}\|_1 - \|\hat{v} + \tilde{u}\|_1 \geq \|\tilde{v}\|_1 + \|\tilde{u}\|_1 - 2\epsilon.
\]

Finally, since \( \tilde{v} = v - \hat{v} \) we deduce that \( \|\tilde{v}\|_1 \geq \|v\|_1 - \epsilon \), and similarly, \( \|\tilde{u}\|_1 \geq \|u\|_1 - \epsilon \). Combining the latter two inequalities with (3.3) and (3.4), yields

\[
\|r_k\|_1 \geq \|v + u\|_1 - \epsilon \geq \|\tilde{v} + \hat{u}\|_1 - 5\epsilon
\]

which completes the proof. \( \square \)

**Remark 3.20.** Lemma 3.19 allows to easily show the uniqueness of the decomposition of an element in \( \mathcal{EQT} \). Indeed, suppose there exist two different representations of the same matrix \( A = T(a) + E_a + ev_a^T = T(a') + E_{a'} + ev_{a'}^T \). Then

\[
0 = \|A - A\|_\infty \geq \|a - a'\|_{\infty} + \|v_a - v_{a'}\|_1 \implies a \equiv a', \quad v_a = v_{a'}.
\]

By difference, we finally get \( E_a = E_{a'} \).

**Theorem 3.21.** The class \( \mathcal{EQT} \) is a Banach algebra with the infinity norm.

**Proof.** The class is clearly closed under addition and multiplication by a scalar. Moreover, it is closed under multiplication in view of Lemma 3.18. In order to prove that it is a Banach space, it is sufficient to follow the same argument used in the proof of Theorem 3.16 relying on Lemma 3.19. \( \square \)

### 3.4. Extended cqt-toolbox

Here, we describe how the computational framework for \( \mathcal{EQT} \) has been implemented on top of cqt-toolbox \cite{7}. The latest release of the software includes this tool.

A matrix \( A \in \mathcal{EQT} \) is represented relying on the unique decomposition (see Remark 3.20) \( A = T(a) + E + ev^T \). The terms \( T(a) \) and \( E \) are represented using the same data structures as the \( QT_\infty \) class. This is possible because the entries of \( E \) \( \div 0 \) allows to truncate it to its top-left corner. The format is extended by storing a truncation \( \tilde{v} \) of the vector \( v \in \ell^1 \). This is performed by requiring \( \|v - \tilde{v}\|_1 \leq \epsilon \|A\|_\infty \). As illustrative example, we report the Matlab code that define the matrix \( A_0 \) of the Jackson network with reset introduced in Section 2.2.
>> E = γ * μ₁ + γ - 1;
>> pos = [0 λ₁];
>> neg = [0 γ * (1 - p) * μ₁];
>> v = 1 - γ;
>> A₀ = cqt('extended', neg, pos, E, v);

We conclude the section by summarizing the relations that link the parameters defining the input of a matrix operations to those of its outcome. Some of them have been already presented in Section 3.3, the others can be verified via a direct computation. In what follows we consider two \( \mathcal{EQT} \) matrices \( A = T(a) + E_a + ev_a^T \) and \( B = T(b) + E_b + ev_b^T \).

**Addition** If \( C = A + B \) then

\[
C = T(a + b) + E_c + ev_{a + b}^T, \quad E_c = E_a + E_b.
\]

**Multiplication** If \( C = AB \) then

\[
C = T(ab) + E_c + e(s_a v_b + B^T v_a)^T,
E_c = T(a)E_b + E_a T(b) - H(a^-)H(b^+) + E_a E_b + (E_a - H(a^-))ev_b^T
\]

\[
s_a = \sum_{j \in \mathbb{Z}} a_j.
\]

**Inversion** The inversion formula is obtained by means of the Woodbury identity, considering a \( \mathcal{EQT} \) matrix as a rank one correction of a \( \mathcal{QT}^d \) matrix. If \( C = A^{-1} \) then

\[
C = (T(a) + E_a)^{-1} - (T(a) + E_a)^{-1} ev_a^T (T(a) + E_a)^{-1} / (1 + v_a^T (T(a) + E_a)^{-1} e).
\]

In the description of the inversion, although the terms are not separated as in the other expressions, all the operations involved are performed with the addition and multiplication formulas for the \( \mathcal{QT}^d \) class.

**4. Double QBDs and related random walks in the quarter plane.** In this section we give conditions under which the minimal nonnegative solution \( G \) of equation (1.1), which we rewrite as

\[
(4.1) \quad X = A_{-1} + A_0 X + A_1 X^2,
\]

belongs to \( \mathcal{QT}^d \) or to \( \mathcal{EQT} \) when the coefficients \( A_{-1}, A_0, A_1 \in \mathcal{QT}^d \) originate from a random walk in the quarter plane governed by a discrete time Markov chain. The Markov chain describes the dynamics of a particle \( p \) which can occupy the points of a grid in the quarter plane of integer coordinates \( (r, s) \), for \( r, s \geq 0 \). If \( p \) occupies an inner position, i.e., if \( r, s > 0 \), then at each instant of time it can move to \((r + j, s + i)\) with given probabilities \( a_{i,j} \) for \( i, j = -1, 0, 1 \). If the particle is along the \( y \) axis, i.e., if \( r = 0 \) and \( s > 0 \), then it can move to \((j, s + i)\) with given probability \( y_{i,j} \) for \( i = -1, 0, 1 \), \( j = 0, 1 \). Similarly, if the particle is along the \( x \) axis, i.e., if \( r > 0 \) and \( s = 0 \), then it can move to \((r + j, i)\) with probability \( x_{i,j} \) for \( i = 0, 1 \), \( j = -1, 0, 1 \). Finally, if \( p \) is in the origin, it can move to the position \((j, i)\) with probability \( a_{i,j} \) for \( i, j = 0, 1 \). Figure 2.2 pictorially describes an example of random walk in the quarter plane.

The Markov chain which describes this model is defined by the double infinite set of states \((r, s), r, s \geq 0,\) and by the transition probability matrix \( P \) whose entry with row index \((r, s)\) and
column index \( (r', s') \) provides the probability of transition from state \( (r, s) \) to state \( (r', s') \) in one time unit. Due to the double indices, the matrix \( P \) has a block structure and can take a different form according to the kind of lexicographical order which is used to sort the pairs \( (r, s) \). Ordering the states column-wise according to the ordering \( (r, s) \), \( s = 0, 1, \ldots, r = 0, 1, \ldots \), yields

\[
P = \text{qtoep}(B_0, B_1; A_{-1}, A_0, A_1), \quad \text{with}
\]

\[
A_i = \text{qtoep}(y_{i,0}, y_{i,1}; a_{i,-1}, a_{i,0}, a_{i,1}), \quad B_i = \text{qtoep}(a_{i,0}, a_{i,1}; x_{i,-1}, x_{i,0}, x_{i,1}),
\]

where \( \text{qtoep}(b_0, b_1; a_{-1}, a_0, a_1) \) denotes the quasi Toeplitz matrix with symbol \( a_{-1}z^{-1} + a_0 + a_1z \) and with correction \( E = e_1(b_0 - a_0, b_1 - a_1, 0, \ldots) \). Similarly we define the block quasi Toeplitz matrix \( \text{qtoep}(B_0, B_1; A_{-1}, A_0, A_1) \). More specifically we have

\[
P = \begin{bmatrix}
B_0 & B_1 & A_{-1} & A_0 & A_1 \\
A_{-1} & A_0 & A_1 & & \\
& & \ddots & \ddots & \ddots \\
& & & & 
\end{bmatrix}.
\]

Ordering the states row-wise as \( (r, s) \), \( r = 0, 1, \ldots, s = 0, 1, \ldots \), yields

\[
\hat{P} = \text{qtoep}(\hat{B}_0, \hat{B}_1; \hat{A}_{-1}, \hat{A}_0, \hat{A}_1), \quad \text{with}
\]

\[
\hat{A}_j = \text{qtoep}(x_{0,j}, x_{1,j}; a_{-1,j}, a_0,j, a_{1,j}), \quad \hat{B}_j = \text{qtoep}(a_{0,j}, a_{1,j}; y_{-1,j}, y_{0,j}, y_{1,j}).
\]

The matrix (4.2) defines a Double Quasi–Birth-Death (DQBD) process [25], [21], which leads to the matrix equation (4.1). We have a similar equation if the row-wise ordering of the states is adopted. We refer to the row-wise representation as the flipped version which is obtained by exchanging the roles of the axes.

It is useful to denote

\[
\begin{align*}
x_{i,:}(z) &= x_{i,-1}z^{-1} + x_{i,0} + x_{i,1}z, \quad i = 0, 1, & x_{j,:}(w) &= x_{0,j} + x_{1,j}w, \quad j = -1, 0, 1, \\
y_{i,:}(z) &= y_{i,0} + y_{i,1}z, \quad i = -1, 0, 1, & y_{j,:}(w) &= y_{-1,j}w^{-1} + y_{0,j} + y_{1,j}w, \quad j = 0, 1, \\
a_{i,:}(z) &= a_{i,-1}z^{-1} + a_{i,0} + a_{i,1}z, & a_{j,:}(w) &= a_{-1,j}w^{-1} + a_{0,j} + a_{1,j}w, \quad i, j = -1, 0, 1.
\end{align*}
\]

For the sake of notational simplicity, if not differently specified, we write \( a_{i,:}(z) \) in place of \( a_{i,:}(z) \). Since \( a_{i,j} \) are probabilities we have \( a_{i,j} \geq 0 \), \( \sum_{i,j} a_{i,j} = 1 \), that is, \( a_{-1}(1) + a_0(1) + a_1(1) = 1 \). Similarly for \( x_{i,j}, y_{i,j} \) and \( a_{i,j} \). Moreover, we introduce the following notation

\[
\begin{align*}
d_1 &= a_{1,:}(1) - a_{-1,:}(1), & d_2 &= a_{1,:}(1) - a_{-1,:}(1), \\
s_1 &= y_{1,:}(1) - y_{-1,:}(1), & s_2 &= x_{1,:}(1) - x_{-1,:}(1), \\
r_1 &= d_2x_{1,:}(1) - d_1s_2, & r_2 &= d_1y_{1,:}(1) - d_2s_1.
\end{align*}
\]

The following result of [14, Theorem 1.2.1] and [26, Lemma 6.4] provides a necessary and sufficient condition for the positive recurrence of the random walk in terms of the values of the probabilities \( a_{i,j}, x_{i,j}, y_{i,j} \).

**Lemma 4.1.** Assume that \( (d_1, d_2) \neq (0, 0) \). The DQBD process is positive recurrent if and only if one of the following conditions holds:

1. \( d_1 < 0, \ d_2 < 0, \ r_1 < 0, \ r_2 < 0; \)
2. \( d_1 \geq 0, \ d_2 < 0, \ r_2 < 0, \) and \( s_2 < 0 \) for \( x_{1,:}(1) = 0 \);
3. $d_1 < 0$, $d_2 \geq 0$, $r_1 < 0$ and $s_1 < 0$ for $y(-1) = 0$.

In the following, we will consider the inequalities $A_{-1}e > A_1e$ or $A_{-1}e \geq A_1e > 0$. For the structure of the matrices $A_1$ and $A_{-1}$, this set of infinitely many inequalities reduces just to a pair of inequalities. For instance, the condition $A_{-1}e > A_1e$ is equivalent to $a_{-1}(1) > a_1(1)$, $y_{-1}(1) > y_1(1)$, while $A_{-1}e \geq A_1e > 0$ is equivalent to $a_{-1}(1) \geq a_1(1) > 0$, $y_{-1}(1) \geq y_1(1) > 0$. From the probabilistic point of view, the above inequalities say that the overall probability that the particle moves down is greater than the overall probability that the particle moves up. We observe that, according to Lemma 4.1 if $A_{-1}e > A_1e$ and $A_{-1}e > A_1e$ then condition 1 holds. Moreover, if the DQBD is positive recurrent, then at least one of the conditions $a_{-1}(1) > a_1(1)$, $a_{-1}(1) > a_1(1)$ is satisfied.

Now, we are ready to prove the following result which gives sufficient conditions for the stochasticity of $G$.

**Theorem 4.2.** If $A_{-1}e > A_1e$ the minimal nonnegative solution $G$ of the matrix equation (4.1) is stochastic, i.e., $Gz = e$.

**Proof.** Observe that $G$ is independent of the values $x_{i,j}$ defining $B_0$ and $B_1$. Therefore, it is sufficient to choose the probabilities $x_{i,j}$, $i = 0, 1$, $j = -1, 0, 1$ in such a way that the DQBD (4.2) defined by the matrices $A_{-1}, A_0, A_1$ and by the boundary conditions $B_0, B_1$ is positive recurrent. In light of Theorem 7.1.1 of [21], this implies that $Ge = e$. To this end, consider the DQBD (4.2) defined by the matrices $A_{-1}, A_0, A_1$ and by the boundary conditions $B_0, B_1$ to be suitably chosen. The assumption $A_{-1}e > A_1e$ implies that $d_1 < 0$. If $d_2 \geq 0$, then we choose $x_{i,j}$ such that $r_1 < 0$. This way, in view of part 3 of Lemma 4.1, the DQBD is positive recurrent. On the other hand if $d_2 < 0$, since $s_1 < 0$, then $r_2 > 0$. Concerning $r_1$, we choose $x_{i,j}$ such that $r_1 < 0$, so that, in view of part 1 of Lemma 4.1, the DQBD is positive recurrent. 

Consider the sequence $\{G_k\}$ defined by

\[
G_k = 0 \\
G_{k+1} = A_1G_k^2 + A_0G_k + A_{-1}, \quad k = 0, 1, \ldots
\]

Since $A_{-1}, A_0, A_1, G_0 \in QT^d_\infty$ and since $QT^d_\infty$ is an algebra, then all the matrices $G_k$ belong to $QT^d_\infty$ so that they can be written as $G_k = T(g_k) + E_k$. Moreover, from (4.3) it follows that $g_k(z) \in W$ is a Laurent polynomial and $E_k$ has a finite support. Observe also that, by construction, the symbols $g_k(z)$ are such that

\[
g_{k+1}(z) = a_{-1}(z) + a_0(z)g_k(z) + a_1(z)g_k(z)^2, \quad g_0(z) = 0.
\]

Equation (4.4) can be viewed as a functional relation between Laurent polynomials in the variable $z$, and also as a point-wise equation valid for any complex value $\zeta$ of the variable of $z$ such that $|\zeta| = 1$.

It is well known [21] that $\{G_k\}$ is an increasing sequence which converges point-wise to the minimal nonnegative solution $G$ of the matrix equation (4.1). Our aim is to provide sufficient conditions under which the sequence $\{G_k\}$ converges in the infinity norm and the limit $G$ can be written in the form $G = T(g) + E_g$. We split this analysis into two parts: the analysis of the sequence $g_k(z)$ and that of the correction $E_k$.

**4.1. A scalar equation.** In this section we prove that the sequence $\{g_k(z)\}$ of Laurent polynomials defined in (4.4) converges in the Wiener norm to a fixed point $g(z) \in W$ of (4.4), we show
that $g(z)$ has nonnegative coefficients, is such that $g(1) \leq 1$ and for any $z \in \mathbb{C}$ of modulus 1, $g(z)$ is the solution of minimum modulus of the scalar equation $a_1(z)\lambda^2 + (a_0(z) - 1)\lambda + a_{-1}(z) = 0$.

We need the following notation. Given two functions $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$, $b(z) = \sum_{i \in \mathbb{Z}} b_i z^i$, $a(z), b(z) \in W$ we write $a(z) \leq_{cw} b(z)$ if the inequality holds coefficient-wise, i.e., if $a_i \leq b_i$ for $i \in \mathbb{Z}$. We have the following result.

**Theorem 4.3.** Under the assumption $a_{i,j} \geq 0$, $\sum_{i,j=-1}^1 a_{i,j} = 1$, there exists $g(z) \in W$ such that $\lim_{k} \|g - g_k\|_W = 0$, where $g_k(z)$ is defined in (4.4). Moreover $g(1) \leq 1$, $0 \leq_{cw} g_k(z) \leq_{cw} g_{k+1}(z)$ for $k = 0, 1, \ldots$, and for any $\zeta$ such that $|\zeta| = 1$, $g(\zeta)$ solves the equation in $\lambda$

$$(4.5) \quad a_1(\zeta)\lambda^2 + (a_0(\zeta) - 1)\lambda + a_{-1}(\zeta) = 0,$$

for $z = \zeta$, and $|g(\zeta)| \leq 1$. Moreover, $g(1) = 1$ if and only if $a_{-1}(1) \geq a_1(1)$; if $a_{-1}(1) < a_1(1)$, then $g(1) = a_{-1}(1)/a_1(1)$.

**Proof.** Let us prove by induction on $k$ that $0 \leq_{cw} g_k(z) \leq_{cw} g_{k+1}(z)$ and that $g_k(1) \leq g_{k+1}(1) \leq 1$. For $k = 0$ we have $g_0(z) = 0$ and $g_1(z) = a_{-1}(z)$ so that $0 \leq_{cw} g_0(z) \leq_{cw} g_1(z)$, moreover $g_0(1) = 0 \leq g_1(1) = 1$. For the inductive step, assume $0 \leq_{cw} g_{k-1}(z) \leq_{cw} g_k(z)$, $g_{k-1}(1) \leq g_k(1) \leq 1$ and prove that $0 \leq_{cw} g_k(z) \leq_{cw} g_{k+1}(z)$ and $g_k(1) \leq g_{k+1}(1) \leq 1$. Since $a_0(z) \geq 0$, by the inductive assumption we have $g_{k+1}(z) = a_{-1}(z) + a_0(z)g_k(z) + a_1(z)g_k(z)^2 \geq_{cw} a_{-1}(z) + a_0(z)g_{k-1}(z) + a_1(z)g_{k-1}(z)^2 = g_k(z) \leq_{cw} g_{k+1}(z)$ and $g_k(1) \leq g_{k+1}(1) \leq 1$. Since $a_0(z) \geq 0$, by the inductive assumption we have $g_{k+1}(z) = a_{-1}(z) + a_0(z)g_k(z) + a_1(z)g_k(z)^2 \geq_{cw} a_{-1}(z) + a_0(z)g_{k-1}(z) + a_1(z)g_{k-1}(z)^2 = g_k(z) \leq_{cw} g_{k+1}(z)$ and $g_k(1) \leq g_{k+1}(1) \leq 1$. Now we prove that the sequence $g_k(z)$ is a Cauchy sequence in the norm $\|\cdot\|_W$. For $k > h$, since $g_h(z) \geq_{cw} g_k(z) \geq_{cw} 0$ we have

$$(4.6) \quad \left\| g_k - g_h \right\|_W = g_k(1) - g_h(1).$$

Since the sequence $g_k(1)$ is nondecreasing and bounded from above, then it converges, thus it is a Cauchy sequence so that, in view of (4.6) also $g_k(z)$ is a Cauchy sequence in the norm $\|\cdot\|_W$. Since $W$ is a Banach algebra, then $g_k(z)$ converges in norm to $g(z) \in W$ and $g(1) \leq 1$. Finally, for any given $\zeta$ such that $|\zeta| = 1$, we have $g(\zeta) = \lim_{k} g_k(\zeta)$ so that, by a continuity argument and in view of (4.4), $g(\zeta)$ solves equation (4.5). Moreover, since $g(z) \geq_{cw} 0$ then $|g(z)| \leq g(1) \leq 1$ for $|z| = 1$. If $\zeta = 1$, the solutions of equation (4.5) are $1$ and $a_{-1}(1)/a_1(1)$ (if $a_1(1) \neq 0$). Since $g(1) \leq 1$, then $g(1) = 1$ if and only if $a_{-1}(1) \geq a_1(1)$.

We prove that for any $\zeta$ of modulus 1, the value $g(\zeta)$ is the solution of minimum modulus of the equation (4.5) where $g(z)$ is the function of Theorem 4.3. This can be shown by using the following result and Lemma 4.4, which weaken the assumptions of Theorem 5.1 of [5].

**Lemma 4.4.** Assume that there exists $i \in \{-1,0,1\}$ such that $|a_i(z)| < a_1(1)$ for any $z \neq 1$ with $|z| = 1$. Then for any $\zeta \neq 1$ with $|\zeta| = 1$, equation (4.5) has a solution of modulus less than 1 and a solution of modulus greater than 1.

**Proof.** Let us prove that for any $\zeta \neq 1$ such that $|\zeta| = 1$ there are no solutions $\lambda$ of (4.5) of modulus 1. By contradiction, if $|\lambda| = 1$ then $1 = |\lambda| = |a_{-1}(\lambda) + a_0(\zeta) + a_1(\zeta)\lambda^2| \leq |a_{-1}(\zeta)| + |a_0(\zeta)| + |a_1(\zeta)| \leq |a_{-1}(1)| + |a_0(1)| + |a_1(1)| = 1$ which is a contradiction. Now, define $f(x) = x(1 - a_0(\zeta))$ and $g(x) = x^2 a_1(\zeta) + a_{-1}(\zeta)$ and observe that for $|x| = 1$

$$|f(x)| = |1 - a_0(\zeta)| \geq 1 - |a_0(\zeta)| \geq 1 - a_0(1) = a_{-1}(1) + a_1(1),$$
$$|g(x)| \leq |a_1(\zeta)| + |a_{-1}(\zeta)| \leq a_1(1) + a_{-1}(1).$$
Therefore $|f(x)| \geq |g(x)|$, moreover, the inequality is strict in view of the assumption $|a_i(z)| < a_i(1)$ for at least an index $i$. By applying Rouche theorem [18, Theorem 4.10b], it follows that $f(x)$ and $f(x) + g(x) = x^2a_1(\zeta) + (a_0(\zeta) - 1)x + a_{-1}(\zeta)$ have the same number of roots in the open unit circle. On the other hand the function $f(x)$ has the only root $x = 0$ since $1 - a_0(\zeta) \neq 0$ for any $\zeta \neq 1$, $|\zeta| = 1$.

Remark 4.5. Observe that the condition $|a_i(z)| < a_i(1)$ can be equivalently rewritten as $a_{i,j} = 0$ for at most one value of $j$ so that the cases not covered by the above theorem are the ones where $a_i(z) = a_i z^k i$ for $k \in \{-1, 0, 1\}$ and $\alpha_{-1}, \alpha_0, \alpha_1 \geq 0$, $\alpha_{-1} + \alpha_0 + \alpha_1 = 1$. For instance, if $\alpha_i = 1/3$ and $k_j = i$, $i = -1, 0, 1$, then the quadratic equation has the double solution $\lambda = \zeta^{-1}$ of modulus $1$.

The following result characterizes the case where equation (4.5) has two solutions with the same modulus.

**Lemma 4.6.** Assume that $a_{i,j} \geq 0$, $\sum_{i,j=-1}^1 a_{i,j} = 1$, and $a_1(z) \neq 0$. If for a given $\zeta$, $|\zeta| = 1$, equation (4.5) has two solutions $\lambda_1$, $\lambda_2$ such that $|\lambda_1| = |\lambda_2|$ then there exists $k \in \{-1, 0, 1\}$ such that $\lambda_1 = \lambda_2 = \zeta^k$.

**Proof.** We use a continuity argument. Since $a_1(z) \neq 0$, we assume for simplicity that $a_{1,1} \neq 0$. Choose $0 < \epsilon < a_{1,1}$ replace $a_{1,1}$ with $a_{1,1} - \epsilon$ and replace $a_{1,-1}$ with $a_{1,-1} + \epsilon$. The new values of $a_{i,j}$ satisfy the assumption of Lemma 4.4. Therefore there exist two solutions $\lambda_1(\epsilon)$, $\lambda_2(\epsilon)$ such that $|\lambda_1(\epsilon)| < 1 < |\lambda_2(\epsilon)|$. By letting $\epsilon \rightarrow 0$ and setting $\lambda_i := \lim_{\epsilon \rightarrow 0} \lambda_i(\epsilon)$, then by continuity $|\lambda_1| \leq 1 \leq |\lambda_2|$, so that $\lambda_1$ is still a, possibly non-unique, solution of minimum modulus of (4.5). On the other hand if $|\lambda_1| = |\lambda_2|$ then necessarily $|\lambda_1| = |\lambda_2| = 1$. If the assumption of Lemma 4.4 are satisfied then $\zeta = 1$ and $\lambda_1 = \lambda_2$. If not, in view of Remark 4.5, there exist $\alpha_i \geq 0$, $k_i \in \{-1, 0, 1\}$, $i = -1, 0, 1$, such that $\alpha_{-1} + \alpha_0 + \alpha_1 = 1$ and $a_1(z) = a_i z^{k_i}$, $i = -1, 0, 1$. On the other hand since $|\lambda_1| = |\lambda_2| = 1$, and $\lambda_1 \lambda_2 = (\alpha_{-1} \zeta^{k_i-1})/(\alpha_1 \zeta^{k_i})$, then $\alpha_{-1} = \alpha_1$ so that $\alpha_0 = 1 - 2\alpha_1$, $\alpha_1 \leq 1/2$. Thus, $\lambda_1, \lambda_2$ solve the equation

$$\zeta^{k_i} \lambda^2 + (\alpha_0 \zeta^{k_i-1}/\alpha_1 + \lambda) + \zeta^{k_i-1} = 0.$$ 

Since $|\lambda_1| = |\lambda_2| = 1$ then $|\alpha_0 \zeta^{k_i-1}/\alpha_1| = |\lambda_1 + \lambda_2| \leq 2$, that is, $|\alpha_0 \zeta^{k_i-1}/\alpha_1| = 1 - \alpha_0$. Setting $\zeta^{k_i} = \cos \theta + \sin \theta$ the latter inequality turns into $\alpha_0 \leq \alpha_0 \cos \theta$. This is possible if and only if $\theta = 0$ or $\alpha_0 = 0$. In the former case we have either $\zeta = 1$ or $k_0 = 0$. If $\zeta = 1$ then $\lambda_1 = \lambda_2 = 1$. If $k_0 = 0$ then $\lambda_1$ and $\lambda_2$ solve the equation $\zeta^{k_i} \lambda^2 + (\alpha_0 - 1)/\alpha_1 \lambda + \zeta^{k_i-1} = 0$ that is $\zeta^{k_i} \lambda^2 - 2\lambda + \zeta^{k_i-1} = 0$. The sum of the solutions is $\lambda_1 + \lambda_2 = 2/\zeta^{k_i}$ so that $|\lambda_1 + \lambda_2| = 2$. Thus, necessarily we have $\lambda_1 = \lambda_2 = \zeta^{-k_i}$. In the remaining case $\alpha_0 = 0$, we deduce that $\alpha_1 = \alpha_{-1} = 1/2$, so that the quadratic equation is $\zeta^{k_i} \lambda^2 - 2\lambda + \zeta^{k_i-1} = 0$ and the same analysis applies.

We may conclude with the following

**Theorem 4.7.** Under the assumption $a_{i,j} \geq 0$, $\sum_{i,j=-1}^1 a_{i,j} = 1$, for any $\zeta$ such that $|\zeta| = 1$, the value $\theta = \lim_k g_k(\zeta)$ is the solution of minimum modulus of (4.5). Moreover, $\theta = g(\zeta)$ where $g$ is the function defined in Theorem 4.3.

**Proof.** In the case where $a_1(z) \equiv 0$ the equation has only one solution which is the one of minimum modulus. If $a_1(z) \neq 0$ Lemma 4.6 guarantees the existence of the minimal solution of (4.5). The claim follows from Theorem 4.3. Since $g_k(z)$ converges in the Wiener norm to $g(z)$, then $\lim_k g_k(\zeta) = g(\zeta)$.

We will refer to the function $g(z)$ as to the minimal solution of (4.5).
4.2. Conditions for the compactness of $E_g$. In view of the results of the previous section, under the only assumption $a_{i,j} \geq 0$, $\sum_{i,j=1}^{1} a_{i,j} = 1$, we may write

\begin{equation}
G = T(g) + E_g
\end{equation}

where $E_g := G - T(g)$, and $\|E_g\|_\infty \leq \|G\|_\infty + \|T(g)\|_\infty \leq 1 + g(1) \leq 2$ so that $E_g \in B(\ell^\infty)$, moreover we have $|E_g|e \leq Ge + T(g)e \leq 2e$. If $G \in B(\ell^p)$ then $\|E_g\|_p \leq \|T(g)\|_p + \|G\|_p \leq \|g\|_\infty + \|G\|_p$. We may synthesize this property in the following

**Theorem 4.8.** The minimal nonnegative solution $G$ of the matrix equation in (4.1) can be written as $G = T(g) + E_g$ where $g(z) \in \mathcal{W}$ is such that $g(1) \leq 1$ and $g(z)$ is the solution of minimum modulus of equation (4.5). Moreover, $E_g \in B(\ell^\infty)$ is such that $\|E_g\|_\infty \leq 1 + g(1)$ and $|E_g|e \leq 2e$. Finally, if $G \in B(\ell^p)$ then $E_g \in B(\ell^p)$.

Now we are ready to provide conditions under which $G$ belongs to $QT^d_\infty$ or to $EQT$. In [21] it is proven that the sequence $(G_k)$ generated by (4.3) converges monotonically and point-wise to $G$. In general, monotonic point-wise convergence does not imply convergence in norm, as shown in the following example. Let $v^{(k)}(i) = (v_i^{(k)})$, where $v_i^{(k)} = \frac{1}{(k+1)^\ell}$ for $k \geq 1$, is such that $v^{(k)} \in \ell^1$, $\lim_k v_i^{(k)} = 0$ monotonically but $\|v^{(k)}\|_1 = \frac{k}{k+1}$ so that $\lim_k \|v^{(k)}\|_1 = 1$. The example can be adjusted to the $p$ norm and extended to the case of matrices. In fact, the sequence $A_k = v^{(k)}e^T$ is a sequence of compact operators in $B(\ell^1)$ such that $\lim_k A_k = 0$ where convergence is point-wise and monotonic, but $\lim_k \|A_k\|_1 = 1$.

Under the assumption $A_{-1}e > A_1e$, it is shown in Theorem 4.2 of [8] that the sequence $(G_k)$ generated by (4.3) converges in the infinity norm to $G$. The following result slightly weakens the assumptions and is the basis to prove that in this case $G \in QT^d_\infty$.

**Theorem 4.9.** If $A_{-1}e > A_1e$, or if $A_{-1}e \geq A_1e > 0$, then for the sequence $G_k$ generated by (4.3) we have $\lim_k \|G_k - G\|_\infty = 0$.

**Proof.** Subtracting the equation $G_{k+1} = A_{-1} + A_0G_k + A_1G_k^2$ from the equation $G = A_{-1} + A_0G + A_1G^2$ and setting $E_k = G - G_k$, we get

\begin{equation}
E_{k+1} = A_0E_k + A_1(E_kG + G_kE_k).
\end{equation}

By proceeding similarly to the proof of Theorem 4.2 of [8], we may show that $E_k \geq 0$, so that $\|E_k\|_\infty = \|v_k\|_\infty$ where $v_k = E_k e$. Thus,

$$v_{k+1} = A_0v_k + A_1(E_kGe + G_kv_k) \leq (A_0 + A_1 + A_1G_k)v_k,$$

where we have used the property $Ge \leq e$. Whence we get $\|v_{k+1}\|_\infty \leq \|v_k\|_\infty \gamma_k$, for $\gamma_k = \|A_0 + A_1 + A_1G_k\|_\infty$. On the other hand, since $0 \leq (A_0 + A_1 + A_1G_k)e = (I - (A_{-1} - A_1G_k))e$, where we used the identity $e = (A_{-1} + A_0 + A_1)e$, and since $\|A_0 + A_1 + A_1G_k\|_\infty = \|(A_0 + A_1 + A_1G_k)e\|_\infty$, we have $\gamma_k = \|(I - (A_{-1} - A_1G_k))e\|_\infty$. Therefore $\gamma_k < 1$ if and only if the vector $w_k := (A_{-1} - A_1G_k)e$ has positive components which do not decay to zero. Since $G_k$ has finite support, the vector $A_1G_k e$ has finite support so that the condition $a_{-1}(1) \neq 0$ implies that the components of $w_k$ do not decay to zero. Thus, it is enough to prove that $w_k > 0$. Since $G \geq G_k$, then $Ge \geq G_ke$ so that $(A_{-1} - A_1G_k)e \geq (A_{-1} - A_1)e$. Whence the condition $(A_{-1} - A_1)e > 0$ implies that the vector $w_k$ has positive components. In the case where $(A_{-1} - A_1)e \geq 0$ and $A_1e > 0$, we may prove by induction that $G_ke < e$. In fact, for $k = 0$ the property holds since $G_0 = 0$. For the implication
\( k \to k + 1 \) we have

\[
G_{k+1}e = (A_{-1} + A_0 G_k + A_1 G_k^2)e \leq (A_{-1} + A_0 + A_1 G_k)e < (A_{-1} + A_0 + A_1)e = e,
\]

where we used the fact that \( A_1 G_k e < A_1 e \) since \( G_k e < e \) and \( A_1 \) has at least a nonzero entry in each row since by assumption \( A_1 e > 0 \). From the property \( G_k e < e \) we get \( A_1 G_k e < A_1 e \) so that \( w_k = (A_{-1} - A_1 G_k)e > (A_{-1} - A_1)e \geq 0 \).

Remark 4.10. Recall that the condition \( A_{-1}e > A_1e \) implies that \( a_{-1}(1) > a_1(1) \) while the condition \( A_{-1}e \geq A_1e \) implies that \( a_{-1}(1) \geq a_1(1) \). In both cases the quadratic equation \( a_1(1)\lambda^2 + (a_0(1) - 1)\lambda + a_{-1}(1) = 0 \) has two real solutions \( \lambda_1 = 1 \) and \( \lambda_2 = a_{-1}(1)/a_1(1) \). Moreover \( \lambda_1 = 1 \) is the minimal solution. In particular, in view of Theorem 4.3, we have \( g(1) = 1 \). Conversely, if \( g(1) = 1 \) is the minimal solution of the above quadratic equation then, for Theorem 4.3, \( a_{-1}(1) \geq a_1(1) \).

The convergence properties of the sequence \( \{G_k\} \) stated by Theorem 4.9 allow to provide sufficient conditions under which \( G \in \mathcal{Q}\mathcal{T}_d \).

**Theorem 4.11.** If \( \lim_k \|G_k - G\|_\infty = 0 \) then the minimal nonnegative solution \( G \) of the matrix equation (4.1) can be written as \( G = T(g) + E_g \) where \( g(z) \in \mathcal{W} \) is the minimal solution of (4.5), and \( E_g \in \mathcal{B}(\ell^\infty) \) has the decay property.

**Proof.** Consider the sequence \( G_k = T(g_k) + E_k \in \mathcal{Q}\mathcal{T}_\infty \) generated by (4.3), where \( g_k(z) \in \mathcal{W} \) and \( E_k \) has finite support. Concerning the first part, we observe that \( \|E_k - E_g\|_\infty \leq \|G_k - G\|_\infty + \|T(g_k) - T(g)\|_\infty \). Thus, since \( \|T(g_k) - T(g)\|_\infty = \|T(g - g_k)\|_\infty = \|g - g_k\|_{\mathcal{W}} \), in view of Theorem 4.3 we have \( \lim_k \|T(g_k) - T(g)\|_\infty = 0 \). Since \( \lim_k \|G_k - G\|_\infty = 0 \), we conclude that \( \lim_k \|E_k - E_g\|_\infty = 0 \). Since \( E_k \) has finite support then it has the decay property so that, for Theorem 3.11, \( E_g \) has the decay property as well.

From Theorem 4.9 the condition \( A_{-1}e > A_1e \), which is equivalent to \( a_{-1}(1) > a_1(1) \) and \( y_{-1}(1) > y_1(1) \), implies \( \lim_k \|G_k - G\|_\infty = 0 \). We will weaken the assumptions of Theorem 4.9 by removing the boundary condition \( y_{-1}(1) > y_1(1) \). To this aim, consider the correction \( E_g = G - T(g) \in \mathcal{B}(\ell^\infty) \), where \( G \) is the minimal nonnegative solution to the equation (4.1) and \( g(z) \) is the solution of minimum modulus to the equation (4.5) which exists under the assumptions of Theorem 4.7. Observe that if \( E_g \) has not the decay property then \( w = |E_g|e \) is such that \( \|w\|_\infty < \infty \) but \( \lim_{k \to \infty} w_k \), if it exists, is not zero.

The following lemma is needed to prove the main result of this section. The only assumption needed is that \( a_1(1) + a_{-1}(1) > 0 \). This condition is very mild since it excludes only the case where \( a_{i,j} = 0 \) for \( i = 1, -1 \) and for any \( j \).

**Lemma 4.12.** Assume that \( a_1(1) + a_{-1}(1) > 0 \) and define

\[
\psi(z) = \frac{a_1(z)}{1 - a_0(z) - a_1(z)g(z)}, \quad \text{for } |z| = 1.
\]

Then \( \psi(z) \in \mathcal{W} \), \( \psi(z) \geq cw_0 \), \( \|\psi\|_{\mathcal{W}} = \psi(1) \) and for \( G = T(g) + E_g \) we have

\[
E_g = T(\psi^k)E_g G^k, \quad k = 0, 1, 2, \ldots.
\]

**Proof.** We show that the function \( \varphi(z) = 1 - \gamma(z) \), \( \gamma(z) = a_0(z) + a_1(z)g(z) \), is such that \( \varphi(z) \neq 0 \) for \( |z| = 1 \). Since \( \gamma(z) \geq cw_0 \), then \( |\gamma(z)| \leq \gamma(1) \), so that it is sufficient to prove that \( \gamma(1) < 1 \). We have

\[
\gamma(1) = a_0(1) + a_1(1)g(1) \leq a_0(1) + a_1(1) = 1 - a_{-1}(1).
\]
Therefore, if $a_{-1}(1) > 0$ then $\gamma(1) < 1$. On the other hand, if $a_{-1}(1) = 0$ then $g(1) = 0$ and $a_1(1) > 0$ since, by assumption, $a_1(1) + a_{-1}(1) > 0$, so that

$$\gamma(1) = a_0(1) = 1 - a_1(1) < 1.$$  

This way, $\psi(z) = a_1(z)/\varphi(z) \in W$. Moreover, since $\sum_{k=0}^{\infty} \gamma(1)^k = 1/(1 - \gamma(1)) < \infty$, and $\gamma(z) \geq cw_0$, then $\sum_{k=0}^{\infty} \gamma(z)^k \in W$ and coincides with $1/\varphi(z)$. Moreover, since $\gamma(z) \geq cw_0$ then $1/\varphi(z) \geq cw_0$ and $\psi(z) \geq cw_0$. From the condition $A_1 G^2 + (A_0 - I)G + A_{-1} = 0$, relying on Lemma 3.12 and Corollary 3.14, we obtain

$$T(a_1)E_gG = T(1 - a_0 - a_1g)E_g.$$  

By multiplying to the left both sides of (4.9) by $T(1/\varphi(z))$, in view of (3.1), we get

$$T(\psi)E_gG = E_g, \quad \psi(z) = \frac{a_1(z)}{1 - a_0(z) - a_1(z)g(z)}.$$  

Finally, by multiplying the above equation to the left by $T(\psi)$ and to the right by $G$, by means of the induction argument, we get (4.8)  

$$\psi(1) = 1.$$  

It is interesting to point out that if $a_{-1}(1) \neq 0$ then the function $\psi(z)$ can be written in a simpler form as $\psi(z) = g(z) \frac{a_1(z)}{a_{-1}(z)}$.

We are ready to prove the main theorem of this section which provides conditions under which $G \in QT_\infty$ or $G \in EQT$.

**Theorem 4.13.** Assume that $a_{-1}(1) + a_1(1) > 0$. Let $G$ be the minimal nonnegative solution of (4.1) decomposed as $G = T(g) + E_g$, where $g(z)$ is the minimal solution of (4.5) and $E_g := G - T(g)$. Then the following properties hold:

1. If $a_{-1}(1) > a_1(1)$, then $E_g$ has the decay property.
2. If $a_{-1}(1) < a_1(1)$ and $\lim_k \|G^k\|_\infty = 0$ then $E_g$ has the decay property.
3. If $a_{-1}(1) < a_1(1)$, $G$ is stochastic and strongly ergodic, that is $\lim_k \|G^k - e\pi_g^T\|_\infty = 0$, then $E_g = (1 - g(1))e\pi_g^T + S_g$, where $S_g$ has the decay property, and $\pi_g^TG = \pi_g^Te = 1$.
4. If $G$ is stochastic and $E_g$ has the decay property, then $a_{-1}(1) \geq a_1(1)$ and $g(1) = 1$.

**Proof.** The proof of properties 1–3 relies on equation (4.8) and on the limit for $k \to \infty$ of its right-hand side. This limit depends on the value of $\|\psi\|_w = \psi(1)$, where $\psi(z)$ is defined in Lemma 4.12. Therefore we show that either $\psi(1) = 1$ or $\psi(1) < 1$ and we deduce the properties of $E_g$ accordingly. Observe that if $a_{-1}(z) = 0$ then $a_0(1) + a_1(1) = 1$ and $g(1) = 0$ so that $\psi(1) = 1$. If $a_{-1}(z) \neq 0$, for Theorem 4.3 we may distinguish two cases: the case where $a_{-1}(1)/a_1(1) > 1$ and the case $a_{-1}(1)/a_1(1) < 1$. In the first case $g(1) = 1$ so that $\psi(1) = a_1(1)/a_{-1}(1) < 1$. In the second case $g(1) = a_{-1}(1)/a_1(1)$ so that $\psi(1) = 1$. Consider the case $a_{-1}(1) > a_1(1)$. Since $g(1) = 1$ then $\psi(1) = a_1(1)/a_{-1}(1) < 1$. Moreover, since $\psi(\epsilon) \geq cw_0$ then $\|\psi(\epsilon)^k\|_w = \psi(1)^k$, whence $\lim_k \|\psi^k\|_w = \lim_k \psi(1)^k = 0$. Therefore $\lim_k \|T(\psi^k)\|_\infty = 0$. On the other hand, since $G \geq e$ and $G \geq 0$ then $\|G^k\|_\infty \leq 1$. Whence, since $E_g \in B(\ell^\infty)$ then from equation (4.8) in Lemma 4.12 we have $\lim_k \|T(\psi^k)E_gG^k\|_\infty \leq \lim_k \|T(\psi^k)\|_\infty \|E_g\|_\infty \|G^k\|_\infty = 0$. That is, the sequence $\{F_k\}$, $F_k = E_g - T(\psi^k)E_gG^k$, is such that $F_k \to 0$ and $\lim_k \|E_g - F_k\|_\infty = 0$. In view of Theorem 3.11, applied to the sequence $\{F_k\}$, we conclude that $E_g \equiv 0$ so that $E_g$ fulfills the decay property. Now, consider the case $a_{-1}(1) < a_1(1)$. Observe that since $\psi(1) = 1$ then $\|\psi\|_w = \psi(1) = 1$ and $\|\psi^k\|_w = \psi(1)^k = 1$, therefore $\|T(\psi^k)\|_\infty = \psi(1)^k = 1$. If $\lim_k \|G^k\|_\infty = 0$ then taking the limit
in (4.8), in view of Theorem 3.11 applied to the sequence \( \{ F_k \} \) we deduce that \( E_g \) has the decay property. On the other hand, if the Markov chain associated with the matrix \( G \) is strongly ergodic, that is, \( \lim_k \| G^k - e \pi^T \|_\infty = 0 \), we have \( G^k = e \pi^T + R_k \) where \( \lim_k \| R_k \|_\infty = 0 \). Therefore

\[
E_g - T(\psi^k)E_g e \pi^T \hat{\psi} = \hat{E}_k, \quad \hat{E}_k = T(\psi^k)E_g R_k.
\]

Since \( \| \hat{E}_k \|_\infty \leq ||T(\psi^k)||_\infty \| E_g \|_\infty \| R_k \|_\infty = \| E_g \|_\infty \| R_k \|_\infty \), then \( \lim_k \| \hat{E}_k \|_\infty = 0 \). Now define \( A = E_g - (1 - g(1))e \pi^T \), and \( A_k = A - \hat{E}_k \). Since \( E_g e = Ge - T(g)e \hat{\psi} \), then \( (1 - g(1))e \parallel T(\psi^k)E_g e \hat{\psi} \) whence

\[
A_k = E_g - (1 - g(1))e \pi^T - \hat{E}_k \equiv E_g - T(\psi^k)E_g e \pi^T - T(\psi^k)E_g R_k
\]

\[
= E_g - T(\psi^k)E_g (e \pi^T + R_k) = E_g - T(\psi^k)E_g G^k \equiv 0
\]

in view of (4.8), thus \( A_k = 0 \). Since \( \lim_k \| A - A_k \|_\infty = 0 \) we may apply Theorem 3.11 and conclude that \( A \equiv 0 \), that is \( E_g \equiv (1 - g(1))e \pi^T + S_g \), in other words \( E_g = (1 - g(1))e \pi^T + S_g \) where \( S_g \) has the decay property. Concerning the last property, consider \( w := |E_g|e = |G - T(g)|e \geq |G e - T(g)|e = |e - T(g)|e \). Since by assumption, \( \lim_k w_i = 0 \) then \( \lim_i (T(g)|e)_i = 1 \). On the other hand, since \( g(z) \in W \) has nonnegative coefficients then \( \lim_i (T(g)|e)_i = g(1) \), so that \( g(1) = 1 \). Since \( g(1) = 1 \) is the minimal nonnegative solution of the scalar equation \( a_1(1)\lambda^2 + (a_0(1) - 1)\lambda + a_{-1}(1) = 0 \), in view of Theorem 4.3, it follows that \( a_{-1}(1) \geq a_1(1) \).

5. Numerical results. This section is devoted to validate the computational framework on the examples discussed in Section 2. The experiments are carried out on a Laptop with the dual-core Intel Core i7-7500U 2.70 GHz CPU and 16 GB of RAM. The implementation relies on the cqt-toolbox [7], tested under MATLAB2019a. We have used the tolerance \( 5 \cdot 10^{-15} \) for truncation and compression in the cqt-toolbox.

5.1. 1D random walk with reset. As a first example we consider the computation of the steady state vector \( \pi \) for the random walk with reset described in Section 2.1. We assume the following configuration for the transition probabilities: \( a = 0.2, b = 0.3, c = 0.4 \) and \( r = 0.1 \). The algorithms described in Section 2.1 provide similar results: the computational times are in between \( 10^{-3} \) and \( 10^{-2} \) and the residuals \( ||\pi TP - \pi^T||_1 \) are of the order \( 10^{-15} \). In Figure 5.1 we plot the first (non negligible) entries of \( \pi \) and the component-wise absolute differences between the exact solution \( \pi \) and the corresponding approximations computed by means of the two approaches.

5.2. Tandem Jackson queue with restart. Let us test the problem in Section 2.2 by solving equation (1.1) with coefficients defined as in (2.2) by means of the iteration (4.3) with \( G_0 = ee^T \in QT \). The parameters are set as follows: \( \lambda_1 = 2, \mu_1 = 3, \lambda_2 = 1, \mu_2 = 2, p = 0.3, q = 0.2, \gamma = 0.9 \). This makes \( \{ G_k \}_{k \geq 0} \) a sequence of stochastic matrices. As stopping criterion we set \( ||G_{k+1} - G_k||_\infty < 10^{-14} \). Note that \( ||G_{k+1} - G_k||_\infty \) coincides with the residual error \( ||A_1G_k^2 + A_0G_k + A_{-1} - G_k||_\infty \). The number of iterations, the residual error and the CPU time are reported in Table 5.1 in the row labeled as Tandem.

5.3. Quasi–Birth-and-Death problems. Consider the quadratic equation (1.1) with coefficients defined as in (2.3). Once again we employ the functional iteration (4.3), setting the starting guess \( G_0 = ee^T \) where \( v \) is a random non negative vector such that \( ||v||_1 = 1 \). In our implementation we have chosen \( v \) with 10 nonzero entries. Any other choice for the support of \( v \) is valid; we just want to avoid \( G_0 = ee^T \) because this is already the minimal non negative solution that would make
Fig. 5.1. On the left, the entries of the steady state vector $\pi$ for the Random walk with reset. On the right, the component-wise errors of the approximations computed with the two methods.

| Case study | Iterations | Residual Error | Approximation Error | CPU Time |
|------------|------------|----------------|---------------------|----------|
| Tandem     | 280        | $5.11 \cdot 10^{-14}$ | —                   | 5.30 s   |
| QBD 1      | 51         | $1.67 \cdot 10^{-14}$ | $2.61 \cdot 10^{-14}$ | 0.15 s   |
| QBD 2      | 159        | $4.63 \cdot 10^{-14}$ | —                   | 3.90 s   |

Table 5.1

Numerical results for the case studies “Tandem” and “Quasi-Birth-and-Death”. The stopping tolerance has been set to $5 \cdot 10^{-14}$ and the compression tolerance of the cqt-toolbox to $5 \cdot 10^{-15}$.

The algorithm converge in one step. Since in this case the exact solution is known in closed form ($G = ee_1^T$), we have also computed the approximation error, i.e., the distance from the solution in the $\mathcal{EQT}$ norm. The number of iterations, the residual error, the approximation error and the CPU time are reported in Table 5.1 in the row labeled as QBD 1.

As a second test, we consider a case covered by Theorem 4.13 where $a_{-1}(1) < a_1(1)$. The exact solution is not known a-priori, has a non-zero Toeplitz part and belongs to $\mathcal{EQT} \setminus \mathcal{QT}_\infty^d$. More specifically, in the random walk we allow the particle of coordinates $(i,j)$ to move down when $i,j > 0$ with probability $\frac{1}{6}$, and to move up with probability $\frac{1}{3}$. The probability of moving left is left unchanged to $\frac{1}{2}$. The matrices $A_{-1}, A_0, A_1, B_0, B_1$ are as follows:

$$A_1 = \frac{1}{3}(I - e_1 e_1^T), \quad A_0 = \frac{1}{2} Z, \quad A_{-1} = \frac{1}{6} I + \frac{5}{6} e_1 e_1^T, \quad B_0 = \frac{1}{2} Z^T, \quad B_1 = \frac{1}{2} I.$$  

We observe that, with any choice of $G_0 \in \mathcal{QT}_\infty^d$, the sequence generated by (4.3) remains in $\mathcal{QT}_\infty^d$, and cannot converge in the infinity norm to the solution $G \in \mathcal{EQT} \setminus \mathcal{QT}_\infty^d$. Pointwise convergence still occurs, but the support of the correction of $G_k$ grows to infinity and dramatically slows down the computation. Therefore, as starting guess we choose $G_0 = ee_1^T$. The numerical results are reported in Table 5.1 in the line marked as QBD 2. The solution can be written as $G = T(g) + E_g + \frac{1}{2} ee_1^T$ where $\|g\|_W = \frac{1}{2}$ and $\|E_g\|_\infty \approx 0.3229$.

6. Conclusions. We have introduced a computational framework for handling classes of structured semi-infinite matrices encountered in the analysis of random walks in the quarter plane which include rare events as reset and catastrophes. This framework consists of two matrix classes $\mathcal{QT}_\infty^d$ and $\mathcal{EQT}$ which extend the quasi Toeplitz matrices introduced in [5] and [6]. We proved that both classes are Banach algebras, that matrices in these classes can be approximated to any arbitrary precision in the infinite norm with a finite number of parameters and that a finite arithmetic can
be designed and implemented by extending the cqt-toolbox of [7]. In particular the computation of the invariant probability measure, performed by means of the matrix analytic approach of [30] can be achieved by solving a quadratic matrix equation with coefficients in the classes \(QT^d_\infty\) or \(EQT\). We have given conditions on the probabilities of the random walk under which the minimal nonnegative solution \(G\) of such quadratic matrix equations belongs either to \(QT^d_\infty\) or to \(EQT\). Examples of algorithms for computing \(G\) are given. Numerical experiments show the effectiveness of our approach.

Some issues are still left to investigate. Namely, the analysis of the more general case where the coefficients \(A_i = T(a_i) + E_i\) have a banded structure, that is \(a_i(z)\) is a general Laurent polynomial; the study of the specific features of the solution \(G\) when \(a_{-1}(-1) = a_1(1)\); and the challenging case of multidimensional random walks with more than two coordinates where the matrix coefficients \(A_i\) have a multilevel structure, say, they are block quasi Toeplitz matrices with quasi Toeplitz blocks.

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