Cosmological constraints on the curvaton web parameters

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We consider the mixed inflaton-curvaton scenario in which quantum fluctuations of the curvaton field during inflation lead to a relatively large curvature perturbation spectrum at small scales. We use the model of chaotic inflation with quadratic potential including supergravity corrections leading to a large positive tilt in the power spectrum of the curvaton field. The model is characterized by the strongly inhomogeneous curvaton field in the Universe and large non-Gaussianity of curvature perturbations at small scales. We obtained the constraints on the model parameters considering the process of primordial black hole (PBH) production in radiation era.

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I. INTRODUCTION

Curvaton mechanism which has been suggested ~15 years ago [1–5] now is the object of intense study. It is assumed, in the standard implementation of the curvaton model, that not the inflaton field perturbations are responsible for the primordial density fluctuations and for the cosmic microwave background fluctuations, but instead the (isocurvature) perturbations of the curvaton field $\sigma$. It is assumed that this curvaton field is subdominant during inflation but in post-inflationary epoch when Hubble constant becomes small, $H \sim m$ (where $m$ is the curvaton mass), curvaton starts oscillating in its potential and behaves as nonrelativistic matter. The energy density of the curvaton decreases as $\sim a^{-3}$ (a is the scale factor) whereas the energy density of radiation produced by the inflaton decay decreases as $a^{-4}$. As a result the curvaton energy density grows relative to radiation energy density until the curvaton contribution becomes significant. If it happens before the curvaton decay one can say that curvaton mechanism is “effective”, in a sense that just the curvaton (rather than inflaton) field perturbations during inflation determine the resulting (adiabatic) curvature perturbations at cosmological scales.

In scenarios with the “effective” curvaton there is the strong constraint on a value of the curvaton mass: it must be much smaller than the Hubble constant during inflation, $H_i$, otherwise the primordial density perturbations have too large spectral tilt. Moreover, if the ratio $m^2/H_i^2$ is not small, the coherent length of the curvaton field (i.e., the characteristic size of the region inside of which the field is approximately homogeneous) is also too small and, in particular, smaller than the current horizon size. In the latter case, the primordial perturbation spectrum is strongly non-Gaussian, in contradiction with observations.

The condition $m^2/H_i^2 \ll 1$ is too restrictive and prohibits an use, for a description of the curvaton, particle physics models predicting large ratios $m^2/H_i^2$ at inflation (e.g., some variants of supersymmetric theories). In this connection it is reasonable to consider also the mixed curvaton-inflaton scenarios [6, 7] in which the curvaton perturbations are additional to the usual perturbations produced by the inflaton. Combining two contributions, one can obtain the primordial perturbation spectrum which is in agreement with data at cosmological scales. At the same time, the prediction for smaller scales may be quite unusual: the spectrum can be, e.g., very blue (i.e., the spectral tilt is large and positive) and, besides, the perturbations can be strongly non-Gaussian. In particular, large value of the tilt arises due to nonrenormalizable and supergravity corrections to the Lagrangian of some supersymmetric theories inducing mass terms of the order $H^2 \ll \langle \sigma^2 \rangle$ [8–12].

In most curvaton scenarios it is assumed that the curvaton field in the Universe is highly homogeneous and, as a result, the non-Gaussianity is relatively small. According to the alternative hypothesis, after the long inflationary expansion, the average value of the curvaton field is close to zero, and the local value of the field has a Gaussian probability distribution, variance of which is given by the formula [13, 14]

$$\langle \sigma^2(x) \rangle = \frac{3H_i^4}{8\pi^2 m_*^2} = \left( \frac{H_i}{2\pi} \right)^2 \frac{1}{\ell_\sigma}. \quad (1)$$

Here, $m_*$ is the effective curvaton mass which differs from the true curvaton mass $m$ [15]. The corresponding coherent length is

$$\ell_c \sim \frac{1}{H_i} \exp \left( \frac{3H_i^2}{2m_*^2} \right) = \frac{1}{H_i} \exp \left( \frac{1}{\ell_\sigma} \right). \quad (2)$$

In Eqs. 1 and 2, $\ell_\sigma$ is the spectral tilt of the perturbation spectrum of the curvaton field, $\ell_\sigma = d \ln P_\sigma/d \ln k$. The assumption that $\dot{\sigma} = 0$ will have real sense if the scale of interest, $\ell_R = a_\xi/k_R$, will be larger than $\ell_c$ (both scales are calculated at the end of inflation). The value

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folds after the scale \( k \) leaves the horizon. The condition
\[
\ell_c \lesssim \ell_R \lesssim \frac{a_i}{H_0} \quad (4)
\]
leads to the inequality \( N \gg 1/t_\sigma \). It means that if \( t_\sigma \) is not small \((t_\sigma \sim 1)\), and the coherent length \( \ell_c \) is small, one anticipates the blue curvature spectrum (the curvaton contribution) and large non-Gaussianity at small scales. In this case, the data at cosmological scales are described by the inflaton fluctuations only. In the opposite case, if \( t_\sigma \) is very small, the number of e-folds \( N \), which is necessary for the fulfillment of the condition \( \ell_R \gg \ell_c \) becomes large, \( N \to N_{infl} \sim 60 \). In particular, if \( t_\sigma \approx 1/60 \), one has, instead of the inequality \( 4 \),
\[
\ell_c \sim \ell_R \sim \frac{a_i}{H_0} \quad (5)
\]
Traditionally, predictions for the primordial curvature perturbation spectrum in a region of small scales are constrained with a help of primordial black holes (PBHs). PBHs are produced in the early Universe, e.g., in radiation era, due to collapses of primordial density inhomogeneities \( 10, 22 \). Experimental limits from PBH overproduction had been studied in many articles, beginning from pioneering works \( 23, 24 \); for the latest reviews, see \( 25, 26 \).

In the concrete case of the curvaton model, the idea of PBH constraining at small scales was suggested in \( 27 \) and was considered, in more detail, in \( 28 \).

In the present work we consider the predictions of the mixed curvaton-inflaton scenario just for the case which is most relevant for the PBH constraining: we assume that \( i) \) the average value of the curvaton field in the Universe is zero, and the Eq. \( 11 \) holds, and \( ii) \) the spectral tilt \( t_\sigma \) is relatively large \((t_\sigma \sim 1)\) and positive. In this case adiabatic perturbations at small scales are produced mostly by the curvaton, resulting in a blue curvature spectrum. Large non-Gaussianity follows in this scenario from the quadratic dependence of the curvature on the curvaton field value. In this case, the typical size of the “curvaton domain” \( 24 \) is relatively small, it is smaller than the horizon size at the moment of the formation of PBH with a given mass.

Recently, the PBH formation in a curvaton scenario was studied in \( 30, 31 \). In contrast with the present work, authors of \( 30, 31 \) do not use the assumption about a long period of inflation happened well before the observable Universe left the horizon. They assume, instead, that the curvaton field is nearly homogeneous in the whole Universe. The possibility of an essential PBH production at small scales in such models depends on the concrete inflationary scenario used. The authors of \( 30 \) use for a curvaton field a variant of the axion model suggested in \( 32 \) which predicts extremely blue spectrum of curvature fluctuations, while the authors of \( 31 \) used the model with a convex potential [as the concrete realization of a “hilltop curvaton” scenario (see, e.g., \( 33 \))], in which strong scale dependence of the curvature power spectrum arises due to tachyonic enhancement effects.

The plan of the paper is as follows. In the next Section we derive the basic formula for the curvature perturbation spectrum used in the concrete calculations. In Sec. \( III \) the process of PBH production in our curvaton model is considered. The last Section contains the results of the calculation and conclusions. The technical details concerning the calculation of a probability density function (PDF) of the smoothed curvature field are discussed in the Appendix.

## II. CURVATURE PERTURBATION SPECTRUM FORMULA

Calculations of primordial curvature power spectra in mixed curvaton-inflaton scenario are carried out, in most cases, using the separated universe assumption and \( \delta N \)-formalism \( 34, 41 \). It had been shown, in particular \( 41 \), that the nonlinear curvature perturbation on an uniform energy density hypersurface, given by the formula
\[
\zeta(x) = \psi(t, x) + \frac{1}{3} \int_{\tilde{\rho}(t)}^{\rho(t)} \frac{d\tilde{\rho}}{P + \tilde{\rho}},
\]
is conserved on superhorizon scales, for a fluid with an equation of state \( P = P(\rho) \). In Eq. \( 6 \), \( \psi \) is the “nonlinear curvature perturbation” entering the expression for the locally defined scale factor
\[
a(x, t) = a(t)e^{\psi(t, x)},
\]
in our case there are two (non-interacting) fluids, radiation from an inflaton decay and an oscillating curvaton which we consider as pressureless matter field. Assuming that the curvaton decays on an uniform total density hypersurface, one has \( \psi = \zeta \) on this surface, and, from Eq. \( 6 \), one has
\[
\zeta_\tau = \zeta + \frac{1}{4} \ln \frac{\rho_\tau}{\tilde{\rho}_\tau},
\]
\[
\zeta_\sigma = \zeta + \frac{1}{3} \ln \frac{\rho_\sigma}{\tilde{\rho}_\sigma}.
\]
From here, one has for the fluid densities
\[
\rho_\sigma = \tilde{\rho}_\sigma e^{3(\zeta - \zeta_\tau)}, \quad \rho_\tau = \tilde{\rho}_\tau e^{4(\zeta - \zeta_\sigma)}.
\]
In the sudden decay approximation \( 3, 42, 43 \), the sum of densities is, on the decay hypersurface, equal to \( \tilde{\rho}(t_{dec}) \)
The early Universe follows the scenario considered in [13]. Eq. (14) is written for perturbations on superhorizon scales, where the gradient term \( \sim k^2/\Omega^2 \) is negligible. For a quadratic potential, \( V \), a fractional perturbation, \( \delta \sigma/\bar{\sigma} \), remains constant during the evolution.

As is pointed out in the Introduction, we assume that the early Universe follows the scenario considered in [13] ("the Bunch-Davies case"). In this scenario, \( \bar{\sigma} \) is close to zero. As for the \( \delta \sigma \), one can neglect its evolution during inflation. When the curvaton field is close to a minimum of the potential, then, due to a competition between the random walk and a (slow) roll, the typical value of the field, as can be easily shown, is \( \sim H^2/2\pi \), which is consistent with Eq. (1).

After an end of inflation, the evolution of the total curvaton field takes place (of the average value as well as of the perturbation). Following Ref. [45], we denote this evolution introducing the notation

\[ \delta \sigma_{osc} = g(\bar{\sigma}_e), \quad \delta \sigma_{osc} = g(\delta \sigma_e), \quad \delta \sigma_e = \bar{\sigma}_e - \frac{1}{2} N t, \quad \delta \sigma_e = \bar{\sigma}_e - \frac{1}{2} N t, \]

where \( \bar{\sigma}_e \) is the average value of the curvaton field at the end of inflation.

\[ g(\delta \sigma_e) = g'(\delta \sigma_e), \quad \delta \sigma_e = \bar{\sigma}_e - \frac{1}{2} N t. \]

The following steps are straightforward (see, e.g., [46]). The entropy perturbation \( S_{\sigma r} \) is obtained from Eq. (11), expanding left and right sides of it up to second order,

\[ S_{\sigma r} = 2g'\delta \sigma_e - \frac{g'^2}{g^2}(\delta \sigma_e)^2. \]

Further, expanding exponents in Eq. (11) up to second order, one obtains, using the connection of \( S_{\sigma r} \) with \( \zeta_e \), \( \zeta_{r} \) :

\[ \zeta = \zeta_r + \frac{2g'}{3g} R_{\sigma, dec} \delta \sigma_e + \frac{2}{9} \left[ \frac{3}{2} R_{\sigma, dec} - 2R_{\sigma, dec}^2 - R_{\sigma, dec}^3 \right] \left( \frac{g'}{g} \right)^2 \delta \sigma_e^2. \]

Here, \( R_{\sigma, dec} \) is given by the formula

\[ R_{\sigma, dec} = \frac{3\Omega_{\sigma, dec}}{4 - \Omega_{\sigma, dec}}. \]

Since, according to the definition of the \( R_{\sigma, dec} \), there is the proportionality \( R_{\sigma, dec} \sim \bar{\rho}_\sigma = m^2 \delta \sigma_{osc} \) (the proportionality coefficient is derived below; in Sec. III-A), and since \( \bar{\sigma}_e = \bar{\sigma}_{osc} \), it follows from Eq. (20) that only the term proportional to \( R_{\sigma, dec}/\bar{g}^2 \) survives in this Equation in the limit \( \bar{\sigma}_{osc} \rightarrow 0 \). It leads to the simple formula for

\[ 1 - \Omega_{\sigma, dec} e^{4(\zeta_r - \zeta)} + \Omega_{\sigma, dec} e^{3(\zeta_r - \zeta)} = 1, \]

\[ \Omega_{\sigma, dec} = \frac{\bar{\rho}_\sigma}{\bar{\rho}_\sigma + \bar{\rho}_{\pi, dec}}. \]
the curvaton-generated part of the total curvature perturbation:

\[ \zeta - \zeta_r \equiv \zeta(\sigma) = \frac{1}{3} R_{\sigma,dec} \left( \frac{g^*}{g} \right)^2 (\delta\sigma_*)^2. \] (28)

Everywhere below we will use for \( \zeta(\sigma) \) the notation \( \zeta_\sigma \), dropping the brackets in the index.

The power spectrum of \((\delta\sigma_*))^2\) is expressed through the power spectrum of the curvaton field perturbation \(P_{\zeta,dec}\) [27],

\[ P_{\delta\sigma_*}^{1/2} = \left( \frac{4}{t_\sigma} P_{\zeta,dec} \right)^{1/2}, \] (29)

and the power spectrum of the curvaton field is

\[ P_{\sigma_*} = \left( \frac{H_i}{2\pi} \right)^2 \left( \frac{k}{k_R} \right)^{t_\sigma} = \left( \frac{H_i}{2\pi} \right)^2 e^{-\left( N_{\text{obs}} - N \right) t_\sigma} \left( \frac{k}{H_0} \right)^{t_\sigma}. \] (30)

The spectral tilt \(t_\sigma\) is simply connected with a value of the effective mass of the curvaton field, \(m_*\):

\[ t_\sigma = \frac{2m_*^2}{3H_i^2}. \] (31)

The difference \(N_{\text{obs}} - N\) is the number of e-folds of “relevant inflation” [27], i.e., the number of e-folds passed from the moment when the observable Universe leaves horizon up to the moment when the scale \(k_R^{-1}\) leaves horizon. The scale \(k_R^{-1}\) enters horizon at the radiation era, just when the curvaton perturbation \(\zeta_\sigma\) is created. The value of \(k_R\) determines the value of horizon mass \(M_h\), and, correspondingly, the order of magnitude value of PBH mass that can be produced at this moment.

Finally, we obtain for the curvature spectrum the expression

\[ P_{\zeta,dec}^{1/2} = \frac{2}{3} R_{\sigma,dec} \frac{g^*}{g} \frac{1}{\sqrt{\pi}} \left( \frac{H_i^2}{k_R^2} \right)^{t_\sigma}. \] (32)

For calculations using this formula, one needs the relation \(R_{\sigma,dec}/g^*\). It is derived in the next Section, for the concrete choice of the potential [see Eq. (41)].

### III. PBH PRODUCTION IN THE CURVATON MODEL

#### A. Curvaton potential

Recently, a variety of models of chaotic inflation in supergravity, in connection with the curvaton scenario and curvaton web problem, had been introduced and studied [48]. Their models and conclusions, however, can not be used in our work straightforwardly because in our curvaton scenario: i) there is no degeneracy of masses of the inflaton and curvaton fields, and ii) our curvaton field is a real, single component field, rather than the radial component of a complex field, as in [48]. Both these features are not inconsistent with the general theory of chaotic inflation in supergravity [49, 50]: for example, the curvaton field can be imaginary part of the complex scalar field [50].

We consider the model with the simple phenomenological potential of the form

\[ V(\sigma) = \frac{\alpha}{2} \left( m^2 + \alpha H^2(t) \right). \] (33)

The corresponding effective mass of the curvaton field is \(m_*^2 = m^2 + \alpha H^2\) and the spectral tilt is given by

\[ t_\sigma = \frac{2}{3} \left( \alpha + \frac{m^2}{H_i^2} \right) \approx \frac{2}{3} \alpha. \] (34)

The evolution equation for the curvaton field \(\delta\sigma\) is given above [see Eq. (19)]. The calculation of \(\delta\sigma(t)\) starts at moment \(t = 0\) corresponding to an end of inflation and the beginning of the radiation-dominated era (the reheating is assumed to be instant).

The derivative \(g'\) is calculated numerically, and the initial conditions are:

\[ \delta\sigma(t = 0) = \delta\sigma_* \quad \text{and} \quad \delta\sigma'(t = 0) = 0. \] (35)

In our case, because the potential (33) is quadratic, \(g' = \delta\sigma_{\text{osc}}/\delta\sigma_*\). For the value of \(\delta\sigma_{\text{osc}}\), we take \(\delta\sigma_{\text{osc}} \equiv \delta\sigma(t_{\text{osc}})\), and the moment of time when oscillations start, \(t_{\text{osc}}\), is determined by the condition [51]

\[ \left| \frac{\delta\sigma}{\delta\sigma'} \right|_{t_{\text{osc}}} = H(t_{\text{osc}})^{-1}. \] (36)

According to this condition, after an onset of the oscillation the time scale of a change of the curvaton field is smaller that the expansion time \(H^{-1}\).

The example of the solution of Eq. (19) for the particular set of parameters, \(m/H_i = 0.1, \alpha = 1\), is shown in Fig. 1. The corresponding value of the derivative \(g'\) is equal to 0.62.
According to (33), the energy density of the (average) curvaton field at the moment $t_{osc}$ is $[H_{osc} \equiv H(t_{osc})]$

$$\bar{\rho}_{\sigma,osc} = \frac{\sigma_{osc}^2}{2} (m^2 + \alpha H_{osc}^2). \quad (37)$$

After the moment $t = t_{osc}$, and until the curvaton’s decay at $t = t_{dec}$, the curvaton is assumed to behave like a pressureless matter, so a value of the curvaton density at decay time is $[a(t_{osc}) \equiv a_{osc}, a(t_{dec}) \equiv a_{dec}]$

$$\bar{\rho}_{\sigma,dec} = \bar{\rho}_{\sigma,osc} \left(\frac{a_{osc}}{a_{dec}}\right)^3. \quad (38)$$

The radiation density at the moment $t_{dec}$ can be related to $H(t_{dec}) \equiv H_{dec}$ by using the Friedmann equation,

$$H_{dec}^2 = \frac{8\pi}{3m_{Pl}^2} \bar{\rho}_{r,dec} \quad (39)$$

(here and below we neglect $\bar{\rho}_{\sigma,dec}$ compared to $\bar{\rho}_{r,dec}$). From Eqs. (37, 38, 39) one obtains

$$\frac{\Omega_{\sigma,dec}}{\bar{\rho}_{r,dec}} = \frac{4\pi}{3m_{Pl}^2} \sigma_{osc} \left(\alpha + \frac{m^2}{H_{osc}^2}\right) \frac{a_{dec}}{a_{osc}} \quad (40)$$

Now, from Eqs. (27, 40), taking into account that $\Omega_{\sigma,dec} \ll 1$ and using relations $a \sim t^{1/2} \sim H^{-1/2}$, we obtain the final formula used in our calculations,

$$\frac{R_{\sigma,dec}}{\sigma^2} = \frac{\pi}{m_{Pl}^2 \sqrt{2\Gamma_{tosc}}} \left(\alpha + \frac{m^2}{H(t_{osc})^2}\right). \quad (41)$$

In this Equation, we used the equality $\alpha_{tosc} = H_{dec} = \Gamma_{\sigma}$, to obtain $t_{dec}$, while $t_{osc}$ is calculated numerically from the condition given by Eq. (36).

Note also that in a case when $\alpha = 0$ and $H_{osc} = m$, one obtains from Eq. (41)

$$\frac{\Omega_{\sigma,dec}}{\bar{\rho}_{r,dec}} = \frac{1}{6} \left(\frac{\sigma_{osc}}{M_P}\right)^2 \sqrt{\frac{m}{\Gamma_{\sigma}}} \quad (42)$$

($M_P = m_{Pl}/\sqrt{8\pi}$), which corresponds to a well-known result (see, e.g., [3]).

B. PDF for the curvature perturbation $\zeta$

It is generally assumed that the perturbations of the curvaton field at Hubble exit during inflation can be well described by a Gaussian random field (correspondingly, the equation (17) for $\sigma_{\tau}$ contains, in its right-hand side, no higher-order terms). In our curvaton model, the curvature perturbation $\zeta$ depends on the curvaton field quadratically. In this case, the field $\zeta$ is chi-squared distributed, so the probability density function for $\zeta$ perturbations is strongly non-Gaussian.

A formula for the PDF in the case of chi-square distribution of $\zeta$-field perturbations, i.e., in the case when

$$\zeta(x) = A \left[\chi(x)^2 - \langle \chi^2 \rangle\right] \quad (43)$$

is well known [52] (in our notations, $\chi \equiv \delta \sigma_\tau$; in contrast with the analogous formula (28) in Sec. IV in Eq. (43) the subtraction of $\langle \chi^2 \rangle$ is performed, to provide the condition $\langle \zeta \rangle = 0$).

For applications in PBH production calculations (with using the Press-Schechter formalism [53] one must derive the PDF for the smoothed field $\zeta$. This problem is thoroughly discussed in the Appendix. It is argued there that the PDF for the smoothed $\zeta$ field can be approximately written in the form

$$p_{\zeta,R}(\zeta_R) \approx \frac{1}{\sigma_{\zeta}(R)} p(\tilde{\nu}), \quad \tilde{\nu} = \frac{\zeta_R}{\sigma_{\zeta}(R)}. \quad (44)$$

Here, $\sigma_{\zeta}(R)$ is the variance of the smoothed $\zeta$ field [it is given by Eq. (A3)] and the function $p(\tilde{\nu})$ is given by Eq. (A10). Effects of the smoothing operation enter, in Eq. (A11), only through the variance, while the function $p(\tilde{\nu})$ is the same in smoothing and non-smoothing cases.

C. PBH mass spectrum and constraints

The PBH constraints are obtained using the Press and Schechter formalism generalized for a case of non-Gaussian PDFs. We will follow the Refs. [54, 57] working with the curvature perturbation $\zeta_R$ rather than with the density contrast. The basic formula in the Press and
threshold value for the PBH formation in the radiation era, for the density contrast, by the relation \[ \zeta_{c,R} = \frac{\rho_i}{M_i} \int n_{BH}(M_{BH})dM_{BH} = \int \frac{p_{\zeta,R}(\zeta_R)d\zeta_R}{\zeta_c} = P(\zeta > \zeta_c, R(M), t_i). \] (45)

In this Equation, \( P \) is the probability that in a region of comoving size \( R \) one has \( \zeta_R > \zeta_c \), where \( \zeta_c \) is the threshold value for the PBH formation in the radiation era, \( n(M) \) is the mass spectrum of the collapsed objects, \( \rho_i \) is the initial energy density. We will use the value of \( \zeta_c = 0.75 \) corresponding to the PBH formation criterion for the density contrast, \( \sigma_c = 1/3 \).

The PBH mass \( M_{BH} \) is connected with the mass of the fluctuation \( M \) by the relation \[ M_{BH} \approx f_h M_h = f_h M_{i}^{1/3} M_{c}^{1/3}, \] (46)
where \( M_h \) is the horizon mass corresponding to the time when the fluctuation of mass \( M \) crosses horizon in radiation era, \( M_i \) is the horizon mass at the start of the radiation era, \( t = t_i \). For the constant \( f_h \) we will use the value \( f_h = (1/3)^{1/2} \) \[ \text{[58, 59]} \]. In the approximation of the fast reheating, \( t_i \) coincides with the time of the end of inflation.

Using Eqs. (45) and (46) one obtains the formula for the PBH number density (mass spectrum) \[ n_{BH}(M_{BH}) = \left( \frac{4\pi}{3} \right)^{-1/3} \frac{\partial P}{\partial R} \left| f_h \rho_i^{2/3} M_i^{1/3} a_i M_{BH}^2 \right|, \] (47)
where \( a_i \) is the scale factor at the end of inflation,
\[ a_i = \frac{a_{eq}}{\sqrt{2H_i}t_{eq}^{1/2}}. \] (48)

and \( a_{eq}, t_{eq} \) are scale factor and time at matter-radiation equality, respectively. The derivative \( \partial P/\partial R \) is given by the expression
\[ \frac{\partial P}{\partial R} = \frac{\zeta_c}{\sigma_c(R)} \frac{d\sigma_c}{dR} p_{\zeta,R}(\zeta_c). \] (49)

This expression is obtained with using the formula \[ \text{[44]} \) for the non-Gaussian PDF. The dependence of the PBH number density on the curvature perturbation power spectrum \( P_\zeta \) arises just through the derivative \( \partial P/\partial R \).

If PBHs form at \( t = t_e \), one can calculate the energy density fraction of the Universe contained in PBHs at the time of formation (at this time, the horizon mass is equal to \( M_h(t_e) \equiv M_h^{\text{min}} \) \[ \text{[54]} \)):
\[ \Omega_{PBH}(M_h^{\text{min}}) \approx \frac{1}{\rho_i} \left( \frac{M_h^{\text{min}}}{M_i} \right)^{1/2} \int n_{BH}(M_{BH}) M_{BH}^2 d\ln M_{BH} \approx \frac{(M_h^{\text{min}})^{1/2}}{\rho_i M_i^{1/2} n_{BH}(M_{BH})} |_{M_{BH}=M_h^{\text{min}}}. \] (50)

In this formula, \( M_h^{\text{min}} \) is the minimum mass of the PBH mass spectrum, \( n_{BH}(M_{BH}) \approx f_h M_{i}^{1/2} \). The PBH mass spectrum is very steep, so, with high accuracy one has
\[ \Omega_{PBH}(M_h^{\text{min}}) \approx \beta_{PBH}(M_h^{\text{min}}), \] (51)
where \( \beta_{PBH} \) is, by definition (see, e.g., \[ \text{[26]} \)), the fraction of the Universe’s mass in PBHs at their formation time,
\[ \beta_{PBH}(M_h^{\text{min}}) \equiv \frac{\rho_{PBH}(t_e)}{\rho(t_e)}. \] (52)

Now, having Eqs. (50, 51), one can use the experimental limits on the value of \( \beta_{PBH} \) \[ \text{[20]} \) to constrain parameters of models used for PBH production predictions.

**IV. RESULTS AND DISCUSSION**

The examples of curvaton-generated curvature perturbation power spectra are shown in Fig. 2 and some examples of the PBH mass spectra calculations are given in Fig. 3. For each curve shown in Figs. 2, 3, the model parameter \( \Gamma_\sigma \) is chosen so that the predicted PBH abundance is of the same order of magnitude as the currently available limits \[ \text{[20]} \) on the parameter \( \beta_{PBH} \) in the corresponding PBH mass range. On the vertical axis of Fig. 3 the combination \( M_i^{1/2} \rho_i^{-1} n_{BH}(M_{BH}) \) is shown; just this combination is approximately equal to \( \beta_{PBH} \), as it follows from Eq. (50).

The following connection between the comoving scale \( k_R \) and horizon mass \( M_h \) (which is approximately equal to PBH mass) is used in Fig. 2 \[ \text{[20]} \):
\[ k_R \approx \frac{2 \times 10^{23}}{\sqrt{M_h/1\text{g}}} \text{Mpc}^{-1}. \] (53)
It is seen from Fig. 3 that for smaller values of \( \alpha \), the PBH mass spectra become more wide. The low mass cutoff of the curves shown is determined by the fact that no PBHs are formed before the curvaton decays at \( t = t_{\text{dec}} \), so the minimal PBH mass is \( M_{\text{PBH}}^{\text{min}} = f_h M_h(t_{\text{dec}}) \).

For the constraining of the curvaton model parameters, we used the limits for \( \beta_{\text{PBH}}(M_{\text{BH}}) \) from the review work 26. Demanding that PBHs are not overproduced, i.e., the value of \( \beta_{\text{PBH}}(M_{\text{BH}}) \) does not exceed the available limits 26, one may obtain the corresponding constraints on the parameters of the considered cosmological model. Such constraints are shown in Fig. 4 for the case of \( \alpha = 0.4 \) and in Fig. 5 for \( \alpha = 1 \).

In particular, in Figs. 4a and 5a we show the limits on the combination of parameters \( \Gamma_{\sigma}/m \) while in Figs. 4b and 5b - on the value of \( \Gamma_{\sigma} \) itself. The prohibited (by PBH overproduction) parameter ranges lie below the corresponding lines.

In the sudden decay approximation, there is a very simple approximate connection between \( \Gamma_{\sigma} \) and the PBH mass produced. It follows from the relations

\[
\frac{M_{\text{h}}(t_{\text{dec}})}{M_i} = \frac{t_{\text{dec}}}{t_i} = \frac{H_i}{H_{\text{dec}}} = \frac{H_i}{\Gamma_{\sigma}},
\]

\[
M_{\text{BH}} \approx f_h M_h(t_{\text{dec}}) = \frac{f_h m^2_{\text{Pl}}}{16 \Gamma_{\sigma}}.
\]

Thus, constraints on \( \Gamma_{\sigma} \) (see Figs. 4b, 5b) are at the same time constraints on the mass of PBHs that can be produced in this model (this is reflected on the vertical axis of the Figures 4, 5): the relation between \( M_{\text{BH}} \) and \( \Gamma_{\sigma} \) is given by Eq. (55).

Deriving the constraints, we use the condition

\[
\mathcal{P}_{\zeta, \sigma} < 2.4 \times 10^{-9} \quad \text{for} \quad k < k_c \approx 1 \text{Mpc}^{-1}
\]

in order not to contradict with the data on the cosmological scales.

One must note that the characteristic values of \( \mathcal{P}_{\zeta} \) which determine the constraints on the model parameters shown in Figs. 4, 5 are of order of \( \sim 10^{-13.5} \). This is consistent with the PBH constraints on \( \mathcal{P}_{\zeta} \) (for non-Gaussian \( \zeta \)-perturbations) obtained in our previous work 61 (see also 55).

In Figs. 4, 5 we show also the number of e-folds after the scale \( k_R \) leaves horizon,

\[
N = \log \frac{k_{\text{end}}}{k_R} = \log \frac{a_i H_i}{a_{\text{dec}} H_{\text{dec}}} = \frac{1}{2} \log \frac{H_i}{\Gamma_{\sigma}},
\]

as a function of the constrained model parameters. It is seen that for all cases corresponding to the obtained PBH constraints, \( N \gg 1 \). As pointed out in the Introduction, this is needed for the validity of the considered model.

One can see from the resulting Figs. 4, 5 that, generally, PBH constraints are very weak. The forbidden region contains too small values of \( \Gamma_{\sigma}/m \) (although the nucleosynthesis limit, \( \Gamma_{\sigma} \gtrsim (1\text{MeV})^2/M_P \),

![FIG. 4: a), b) The resulting constraints on the values of model parameters obtained for the curvaton model considered in this paper (for \( \alpha = 0.4 \)). Regions below the lines correspond to the sets of parameters that are prohibited by PBH overproduction. c) The values of \( \mathcal{N} \) corresponding to the constraints, as functions of Hubble parameter during inflation.](image-url)
allows such values). The PBH constraint works only for very high values of Hubble constant during inflation, $H_i \gtrsim 10^{11.5} \text{ GeV}$, and for very large values of curvaton masses, $m \gtrsim (10^{-4} \pm 10^{-1}) H_i$. For other values of parameters, the spectrum amplitude, $P_{\zeta}$, is too small and cannot be constrained. For illustrative purposes we show in Figs. 4, 5 constraints for a large interval of $H_i$ values, up to $10^{15}$ GeV, although there is a well-known upper bound on the Hubble parameter during inflation (according to the recent results of Planck collaboration, $H_i/M_P < 3.7 \times 10^{-5}$). One must note also that in the forbidden region the reheating temperatures are rather high ($T_{RH} \sim \sqrt{H_i M_P}$) and, in standard supersymmetric models, gravitinos are overproduced.

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Appendix A: Moments of PDF of $\zeta$-field

It follows from Eq. (28) that in our model the curvature perturbation depends on the Gaussian curvaton field $\delta \sigma$, quadratically,

$$\zeta = A(\delta \sigma^2 - \langle \delta \sigma^2 \rangle), \quad \text{(A1)}$$

$$A \equiv \frac{1}{3} R_{\sigma, dec} \frac{\sigma^2}{g^*}. \quad \text{(A2)}$$

In Eq. (A1) the constant term $A\langle \delta \sigma^2 \rangle$ is subtracted such that now $\langle \zeta \rangle = 0$, and $\zeta$ is the “overcurvature”. Introducing the notation $\delta \sigma \equiv \chi$, one has

$$\zeta = A(\chi^2 - \langle \chi^2 \rangle), \quad \text{(A3)}$$

and the PDF of the $\chi$ field is

$$p_\chi(\chi) = \frac{1}{\sigma_\chi \sqrt{2\pi}} e^{-\frac{\chi^2}{2\sigma_\chi^2}}, \quad \sigma_\chi^2 \equiv \langle \chi^2 \rangle. \quad \text{(A4)}$$

PDF of the $\zeta$ field is obtained from the PDF of the $\chi$ field using the Chapman-Kolmogorov equation,

$$p_\zeta(\zeta) = \int d\chi p_\chi(\chi) \delta_D \left[ \zeta - A(\chi^2 - \langle \chi^2 \rangle) \right] = \int d\chi p_\chi(\chi) \sum_i \left[ \delta_D (\chi - \chi_i) \frac{1}{|\mathbf{D}_{\chi_i}|} \right]. \quad \text{(A5)}$$

Here, $\chi_i$ are roots of the equation

$$A\chi^2 - A\langle \chi^2 \rangle - \zeta = 0. \quad \text{(A6)}$$

The final expression for the PDF of the $\zeta$ field is

$$p_\zeta(\zeta) = \frac{1}{A \sqrt{\frac{\zeta}{A} + \langle \chi^2 \rangle}} p_\chi \left( \sqrt{\frac{\zeta}{A} + \langle \chi^2 \rangle} \right). \quad \text{(A7)}$$

FIG. 5:  a), b) The resulting constraints on the values of model parameters obtained for the curvaton model considered in this paper (for $\alpha = 1$). Regions below the lines correspond to the sets of parameters that are prohibited by PBH overproduction. c) The values of $N$ corresponding to the constraints, as functions of Hubble parameter during inflation.
The variance of the $p_\zeta(\zeta)$ is

$$\langle \zeta^2 \rangle = \int_{\zeta_{min}}^{\infty} \zeta^2 p_\zeta(\zeta) d\zeta = 2\zeta_{min}^2 = 2A^2(\chi^2)^2. \quad (A8)$$

Using this equation, the distribution \[A7\] can be written in the form:

$$p_\zeta(\zeta) = \frac{1}{(\zeta^2)^{1/2}} p(\nu), \quad (A9)$$

$$p(\nu) = \frac{1}{\sqrt{1 + \nu^2}} e^{-\frac{1}{2}(1+\nu^2)}. \quad (A10)$$

In this equation, the notation $\nu = \zeta/(\zeta^2)^{1/2}$ is introduced. Note, that the product $p_\zeta(\zeta)d\zeta$ doesn’t depend on $\zeta$ and $(\zeta^2)^{1/2}$ separately, i.e.,

$$p_\zeta(\zeta)d\zeta = p(\nu)dv. \quad (A11)$$

The first (central) moments of the $p_\zeta$ are

$$\langle \zeta^3 \rangle = 8A^3(\chi^2)^3, \quad \langle \zeta^4 \rangle = 60A^4(\chi^2)^4, \quad (A12)$$

and the first cumulants, $\langle \zeta^n \rangle_c$, are given by the relations (see, e.g., [62])

$$\langle \zeta^2 \rangle_c = \langle \zeta^2 \rangle, \quad \langle \zeta^3 \rangle_c = \langle \zeta^3 \rangle, \quad \langle \zeta^4 \rangle_c = \langle \zeta^4 \rangle - 3\langle \zeta^2 \rangle^2, \quad \langle \zeta^5 \rangle_c = \langle \zeta^5 \rangle - 10\langle \zeta^2 \rangle\langle \zeta^3 \rangle. \quad (A13)$$

The reduced cumulants are defined by the relation (see, e.g., [64])

$$D_n = \frac{\langle \zeta^n \rangle_c}{(\zeta^2)^{n/2}}, \quad (A14)$$

For the first non-trivial reduced cumulants, skewness and kurtosis, one has, respectively,

$$D_3 = \frac{8A^3(\chi^2)^3}{[2A^2(\chi^2)^2]^{3/2}} = \sqrt{8}, \quad (A15)$$

$$D_4 = \frac{48A^4(\chi^2)^4}{[2A^2(\chi^2)^2]^{4/2}} = 12. \quad (A16)$$

The general formula for $D_n$ is remarkably simple,

$$D_n = 2^{n-1}(n - 1)! \quad (A17)$$

To find the PDF of the smoothed curvature fluctuations one must use the smoothed $\zeta$ field,

$$\zeta_R(x) = A \int d^3y W(|x - y|/R)\chi^2(y) - A(\chi^2) \int d^3y W(|x - y|/R). \quad (A18)$$

Here, $W(x/R)$ is the window function. In the present paper we use the Gaussian window function, defined by the equations

$$W(x/R) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad V = (2\pi)^{3/2}R^3. \quad (A19)$$

The general expressions for the cumulants of the PDF of the smoothed $\zeta$ field have been derived in [52] using the path integral formalism. In this formalism, authors of [52] expressed cumulants through the integrals in $k$-space,

$$\langle \zeta^4 \rangle_c = 2^{n-1}(n - 1)!A^n \prod \frac{d^3k_1}{(2\pi)^3} \ldots \frac{d^3k_n}{(2\pi)^3} P_\chi(k_1)\ldots P_\chi(k_n) \times$$

$$\times \hat{W}(|k_1 - k_2|/R)\ldots \hat{W}(|k_{n-1} - k_n|/R)\hat{W}(|k_n - k_1|/R). \quad (A20)$$

Here, $W(kR)$ is the window function in $k$-space, $W(kR) = e^{-k^2R^2/2}$, $P_\chi(k)$ is the power spectrum of the $\chi$ field,

$$P_\chi(k) = \frac{2\pi^2}{k^3} P_\chi(k). \quad (A21)$$

As one can see from Eq. (A20), values of the cumulants depend on the $k$-dependence of the power spectrum of the $\chi$ field and on the window size $R$. To study qualitatively the $R$-dependence of the cumulants it is more convenient to use the expressions for $\langle \zeta^4 \rangle_c$ through the integrals in real (configuration) space \[A22\]. The corresponding expression is

$$\langle \zeta^4 \rangle_c = \int W(|x - x_1|/R)W(|x - x_2|/R)\ldots W(|x - x_n|/R)\langle \zeta(x_1)\zeta(x_2)\ldots\zeta(x_n) \rangle_c d^3x_1 d^3x_2\ldots d^3x_n. \quad (A22)$$
Here, the connected $n$-point function in real space is given by the product of two-point correlation functions of $\chi$ field,

$$\langle \zeta(x_1) \cdots \zeta(x_n) \rangle_c \sim \xi_n(x_{12}) \cdots \xi_1(x_{n1}),$$  \hspace{1cm} (A23)

$$\xi_n(x_{ij}) = \int_0^\infty \mathcal{P}_n(k) \sin(kx_{ij}) \frac{dk}{k}$$  \hspace{1cm} (A24)

In Eqs. (A23, A24) we use the notation $x_{ij} = |x_i - x_j|$. If the power spectrum of the $\chi$ field has a form

$$\mathcal{P}_\chi \sim k^{\xi}, \quad \xi > 0,$$  \hspace{1cm} (A25)

it follows from Eq. (A24) that $\xi_n(x_{ij}) \sim x_{ij}^{-\xi}$, and

$$\langle \zeta(x_1) \cdots \zeta(x_n) \rangle_c \sim (x_{12} x_{23} \cdots x_{n1})^{-\xi}.$$  \hspace{1cm} (A26)

Integrals in Eq. (A22) converge, if $0 < \chi < 2.5$, and scale with the window size $R$. Therefore, there is the proportionality $\langle \zeta^2 \rangle_c \sim R^{-n_2}$, and, as a result, the reduced cumulants almost don’t depend on the smoothing scale,

$$D_{n,R} = \frac{\langle \zeta^2 \rangle_c}{\langle \zeta^2 \rangle_R^{n/2}} \sim R^{-n_2} \sim R^0.$$  \hspace{1cm} (A27)

The weak dependence of the reduced cumulants on $R$ suggests that the PDF of the smoothed $\zeta$ field can be written in the form analogous to Eq. (A19)

$$p_{\zeta,R}(\zeta(R)) = \frac{1}{\langle \zeta^2 \rangle_R^{1/2}} \tilde{p}(\tilde{\zeta}) = \frac{1}{\langle \zeta^2 \rangle_R^{1/2}} \tilde{p}(\nu),$$  \hspace{1cm} (A28)

$$\nu \equiv \frac{\zeta(R)}{\langle \zeta^2 \rangle_R^{1/2}}.$$  \hspace{1cm} Indeed, the reduced central moments for this PDF, which are given by the relation

$$\frac{\langle \zeta^2 \rangle_c}{\langle \zeta^2 \rangle_R^{n/2}} = \int \frac{\zeta^n}{\langle \zeta^2 \rangle_R^{n/2}} \tilde{p}(\nu) d\nu = \int \nu^n \tilde{p}(\nu) d\nu,$$  \hspace{1cm} (A29)

have a form which is independent on the smoothing scale, in accordance with Eq. (A27).

Evidently, the reduced cumulants which are connected with the reduced central moments by a relation analogous to (A13) also have this property.

Quantitative values of $D_{n,R}$ are different for different values of the power spectrum index $t_\chi$ (even if $D_{n,R}$ almost do not depend on $R$). One can expect, however, that if the positive tilt of the $\chi$-spectrum is not too large, $t_\chi \lesssim 1$, the approximate equality

$$D_{n,R} \approx D_n$$  \hspace{1cm} (A30)

takes place. This problem had been studied for the case $n = 3$, in [66], and, for the case $n = 4$, in [67]. It had been shown in [66, 67] that, really, if $t_\chi$ is not small enough (e.g., if $t_\chi = 2$) the cumulants $D_{n,R}$ are comparatively small, $D_{n,R} \ll D_n$, but they are close to $D_n$ in the limit $t_\chi \lesssim 1$ (just this limit is of interest for us in the present work).

Assuming that Eq. (A30) holds for all $n$ (i.e., that the reduced cumulants are the same in cases with smoothing and without smoothing), one can use for the PDF of the smoothed $\zeta$ field the expression

$$p_{\zeta,R}(\zeta(R)) = \frac{1}{\langle \zeta^2 \rangle_R^{1/2}} \tilde{p}(\nu),$$  \hspace{1cm} (A31)

where $\tilde{p}(\nu)$ is given by Eq. (A10), with a substitution $\nu \rightarrow \nu$. In this approximation, the effects of the smoothing come only through the variance $\langle \zeta^2 \rangle_R^{1/2}$ while the shape of the PDF is the same as in the non-smoothing case.

The variance, $\langle \zeta^2 \rangle_R^{1/2} = \sigma(\zeta(R))$, is given by the expression followed from the general formula (A22):

$$\langle \zeta^2 \rangle_R = \frac{2A^2}{(2\pi)^6} \int dkdk' \mathcal{P}_\chi(k)\mathcal{P}_\chi(k') \tilde{W}(|k-k'|R)^2.$$  \hspace{1cm} (A32)

Note, for completeness, that moments of the PDF of the $\zeta$ field are simply connected with polyspectra of the $\zeta$ field. In particular, using the definition

$$\langle \zeta(k_1)\zeta(k_2) \rangle = (2\pi)^3 \delta_D(k_1 + k_2) P_\zeta(k_1),$$  \hspace{1cm} (A33)

one can obtain from Eq. (A32) the simple formula for the variance:

$$\langle \zeta^2 \rangle_R = \sigma^2(\zeta(R)) = \int_0^\infty \tilde{W}^2(kR) \mathcal{P}_\zeta(k) \frac{dk}{k}.$$  \hspace{1cm} (A34)

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