Discrete $Z^γ$: embedded circle patterns with the combinatorics of the square grid and discrete Painlevé equations

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Abstract

A discrete analogue of the holomorphic map $z^γ$ is studied. It is given by Schramm’s circle pattern with the combinatorics of the square grid. It is shown that the corresponding circle patterns are embedded and described by special separatrix solutions of discrete Painlevé equations. Global properties of these solutions, as well as of the discrete $z^γ$, are established.

1 Introduction

The theory of circle patterns is a fast developing field of research on the border of complex analysis and discrete geometry. Recent progress in this area has its origin in Thurston’s idea [20] about approximating the Riemann mapping by circle packings. By now some aspects of the theory of circle patterns, as discrete analogs of conformal mappings, are well understood, while the others are still waiting to be clarified. Classical circle packings comprised of disjoint open disks were later generalized to circle patterns where the disks may overlap (see for example [12]). Different underlying combinatorics were considered. Schramm introduced a class of circle patterns with the combinatorics of the square grid [19]; hexagonal circle patterns with constant multi-ratios were studied by Bobenko, Hoffman and Suris in [4]; a new rich class of hexagonal patterns with constant intersection angles was investigated in [5].

The convergence of discrete conformal maps represented by circle packings was proven by Rodin and Sullivan [18]. For a prescribed regular combinatorics this result was refined. He and Schramm [11] showed that for hexagonal packings the convergence is $C^∞$. The uniform convergence for circle patterns with the combinatorics of the square grid and orthogonal neighboring circles was established by Schramm [19].

Other facts underlining the striking analogy between circle patterns and the classical theory are the uniformization theorem concerning circle packing realizations of cell complexes of a prescribed combinatorics [2], discrete maximum principle, Schwarz’s lemma and rigidity properties [15,12], discrete Dirichlet principle [10].

It turned out that an effective approach to the description and the construction of circle patterns with overlapping circles is given by the theory of integrable systems (see [7,4,5]). In particular, Schramm’s circle patterns studied in this paper are governed by a difference equation
which is the stationary Hirota equation (see [19]). This equation is an example of an integrable difference equation. It appeared first in a different branch of mathematics – the theory of integrable systems (see [21] for a good survey).

On the other hand, not very much is known about analogs of standard holomorphic functions, although computer experiments give evidence for their existence [9]. For circle packings with the hexagonal combinatorics the only explicitly described examples are Doyle spirals, which are discrete analogs of exponential maps, [8] and conformally symmetric packings, which are analogs of a quotient of Airy functions [3]. For patterns with overlapping circles more explicit examples are known: discrete versions of \( \exp(x) \), \( \text{erf}(x) \) [19], \( z^\gamma \), \( \log(x) \) [1] are constructed for patterns with underlying combinatorics of the square grid; \( z^\gamma \), \( \log(x) \) are also described for hexagonal patterns with both multi-ratio [4] and constant angle [5] properties.

Whereas computer experiments reveal a regular behavior of the circle patterns corresponding to discrete \( z^\gamma \) and \( \log(x) \), only the local property of immersionness was proved for the Schramm’s patterns [1]. This property turned out to be connected with special solutions of discrete Painlevé II equations, thus giving geometrical interpretation thereof. The aim of this paper is to prove the global property of embeddedness for the square grid circle patterns corresponding to \( z^\gamma \) (which was conjectured in [1]).

To visualize the analogy between Schramm’s circle patterns and conformal maps, consider regular patterns composed of unit circles and suppose that the radii are being deformed so as to preserve the orthogonality of neighboring circles and the tangency of half-neighboring ones. Discrete maps taking intersection points of the unit circles of the standard regular patterns to the respective points of the deformed patterns mimic classical holomorphic functions, the deformed radii being analogous to \( |f'(z)| \) (see Fig. 1).

Figure 1: Schramm’s circle patterns as discrete conformal map. Shown is the discrete version of the holomorphic mapping \( z^{3/2} \).

It is easy to show that the lattice comprised of the centers of circles of Schramm’s pattern and their intersection points is a special discrete conformal mapping (see Definition 1 below). The latter were introduced in [6] in the frames of discrete integrable geometry, originally without any relation to circle patterns.

**Definition 1** A map \( f : \mathbb{Z}^2 \to \mathbb{R}^2 = \mathbb{C} \) is called a discrete conformal map if all its elemen-
tary quadrilaterals are conformal squares, i.e., their cross-ratios are equal to -1:

\[ q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1 \] (1)

This definition is motivated by the following properties:
1) it is Möbius invariant,
2) a smooth map \( f : D \subset \mathbb{C} \to \mathbb{C} \) is conformal (holomorphic or antiholomorphic) if and only if

\[ \lim_{\epsilon \to 0} q(f(x, y), f(x + \epsilon, y), f(x, y + \epsilon), f(x + \epsilon, y + \epsilon)) = -1 \]

for all \((x, y) \in D\). For some examples see [6], [13]. A naive method to construct a discrete analogue of the function \( f(z) = z^\gamma \) is to start with \( f_{n,0} = n^\gamma, \ n \geq 0, \ f_{0,m} = (im)^\gamma, \ m \geq 0 \) and then compute \( f_{n,m} \) for any \( n, m > 0 \) using equation (1). But so determined map has a behavior which is far from that of usual holomorphic maps. Different elementary quadrilaterals overlap (see the left lattice in Fig. 2).

Figure 2: Two discrete conformal maps with close initial data \( n = 0, m = 0 \). The second lattice describes a discrete version of the holomorphic mapping \( z^{2/3} \).

**Definition 2** A discrete conformal map \( f_{n,m} \) is called embedded if inner parts of different elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) do not intersect.

The condition for discrete conformal map to be embedded can be relaxed as follows.

**Definition 3** A discrete conformal map \( f_{n,m} \) is called an immersion if inner parts of adjacent elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) are disjoint.

To construct an embedded discrete analogue of \( z^\gamma \), which is the right lattice presented in Fig. 2 a more complicated approach is needed. Equation (1) can be supplemented with the following nonautonomous constraint:

\[ \gamma f_{n,m} = 2n \frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} + 2m \frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}, \] (2)
which plays a crucial role in this paper. This constraint, as well as its compatibility with
\[ (1), \] is derived from some monodromy problem (see \[ (1) \]). Let us assume \( 0 < \gamma < 2 \) and denote
\[ \mathbb{Z}_+^2 = \{(n, m) \in \mathbb{Z}^2 : n, m \geq 0\}. \] Motivated by the asymptotics of the constraint \[ (2) \] at \( n, m \to \infty \) and the properties
\[ z^\gamma(\mathbb{R}_+) \in \mathbb{R}_+, \quad z^\gamma(i\mathbb{R}_+) \in e^{\gamma \pi i/2} \mathbb{R}_+ \]
of the holomorphic mapping \( z^\gamma \) we use the following definition \[ (7) \] of the ”discrete” \( z^\gamma \).

**Definition 4** The discrete conformal map \( Z_+^\gamma : \mathbb{Z}_+^2 \to \mathbb{C} \), \( 0 < \gamma < 2 \) is the solution of
\[ (1), (2) \] with the initial conditions
\[ Z_+^\gamma(0, 0) = 0, \quad Z_+^\gamma(1, 0) = 1, \quad Z_+^\gamma(0, 1) = e^{\gamma \pi i/2}. \] (3)

Obviously, \( Z_+^\gamma(n, 0) \in \mathbb{R}_+ \) and \( Z_+^\gamma(0, m) \in e^{\gamma \pi i/2}(\mathbb{R}_+) \) for any \( n, m \in \mathbb{N} \).

Fig. 2 suggests that \( Z_+^\gamma \) is embedded. The corresponding theorem is the main result of this paper.

**Theorem 1** The discrete map \( Z_+^\gamma \) for \( 0 < \gamma < 2 \) is embedded.

The proof is based on the analysis of geometric and algebraic properties of the corresponding
lattices. In Section 2 a brief review of the necessary results from \[ (1) \] is given. It is shown that
\( Z_+^\gamma \) corresponds to a circle pattern of Schramm’s type. (The circle pattern corresponding to
\( Z_{2/3}^2 \) is presented in Fig. 3) Next, analyzing the equations for radii of the circles we show
that in order to prove that \( Z_+^\gamma \) is embedded it is enough to establish a special property of
the separatrix solutions \( P_{N,M}, Q_{N,M} \) of the following infinite sequence of systems of ordinary
difference equations of Painlevé type \( (N \) labels the system):

\[
Q_{N,M+1} = \frac{(M + N)P_{N,M}(P_{N,M} - Q_{N,M}^2) - (M - N)Q_{N,M}^2(1 + P_{N,M})}{Q_{N,M}((M + N)(Q_{N,M}^2 - P_{N,M}) - (M - N)P_{N,M}(1 + P_{N,M}))}
\]

\[
P_{N,M+1} = \frac{(2M + \gamma)P_{N,M} + (2N + \gamma)Q_{N,M}Q_{N,M+1}}{(2(N + 1) - \gamma)P_{N,M} + (2(M + 1) - \gamma)Q_{N,M}Q_{N,M+1}}
\]

Figure 3: Schramm’s circle pattern corresponding to \( Z_{2/3}^2 \)
Namely, it is shown that $Z^\gamma$ is embedded if the infinite sequence of solutions $Q_{N,M}, P_{N,M}$ with special initial data $Q_{N,N}, P_{N,N}$ is subject to

$$(\gamma - 1)(Q^2_{N,M} - P_{N,M}) \geq 0, \quad Q_{N,M} > 0, \quad P_{N,M} > 0$$

The existence and uniqueness of the corresponding initial data are given in Section 3.

For $N = 0$ the system for $Q_{N,M}, P_{N,M}$ reduces to the special case of discrete Painlevé equation dPII

$$(n+1)(x_n^2 - 1)\left(\frac{x_{n+1} - ix_n}{i + x_n x_{n+1}}\right) - n(x_{n+1} + 1)\left(\frac{x_{n-1} + ix_n}{i + x_{n-1} x_n}\right) = \gamma x_n$$ (4)

(see [10], [17] for more examples). In [1] this equation was the main tool to prove the immersion of $Z^\gamma$: it was shown that $Z^\gamma$ is immersed if the unitary solution $x_n = e^{i\alpha_n}$ of this equation with $x_0 = e^{i\gamma\pi/4}$ lies in the sector $0 < \alpha_n < \pi/2$.

Similar problems have been studied in the frames of the isomonodromic deformation method [13]. In particular, connection formulas were derived. These formulas describe the asymptotics of solutions for $n \to \infty$ as a function of initial conditions (see in particular [10]). These methods seem to be insufficient for our purposes since we need to control the behavior of solutions for finite $n$'s as well. The geometric origin of our equations permits us to prove the abovementioned properties by purely geometric methods. To illustrate the difference between the immersed and embedded discrete conformal maps, let us imagine that the elementary quadrilaterals of the map are made of elastic inextensible material and glued along the corresponding edges to produce a surface with a border. If this surface is immersed it is locally flat. Being dropped down it will not have folds. At first sight it seems to be sufficient to give embeddedness, provided $Z_{n,0} \to \infty$ and $Z_{0,m} \to \infty$ as $n \to \infty$ (which follows from $Z^\gamma_{n,0} = \frac{2(\gamma)}{\gamma} \left(\frac{n}{2}\right)^\gamma \left(1 + O\left(\frac{1}{n^2}\right)\right)$, $n \to \infty$, see [1]). But a surface with such properties still may have some limit curve with self-intersections thus giving overlapping quadrilaterals. Hypothetical example of such a surface is shown in Fig. 4.

![Figure 4: Surface glued of quadrilaterals of immersed but non-embedded discrete map.](image)

2 Circle patterns and $Z^\gamma$

Let us recall the necessary results from [1]. $Z^\gamma$ of Definition 4 determines a special case of circle patterns with the combinatorics of the square grid as defined by Schramm in [19]. Indeed, discrete
If \( n \) is even, \( f_n \) is the center of a circle with the radius \( 1 \) that is tangent to \( f_{n+1} \) and \( f_{n+2} \). If \( n \) is odd, \( f_n \) is the center of a circle with the radius \( 1 \) that is tangent to \( f_{n+1} \) and \( f_{n+2} \). Given initial \( f_0, f_1 \) the constraint (2) allows one to compute \( f_{n+2} \) for any \( n, m \geq 1 \). Now using equation (1) one can successively compute \( f_{n,m} \) for any \( n, m \in \mathbb{N} \). Observe that if \( (f_{n+1,m} - f_{n,m}) = (f_{n,m+1} - f_{n,m}) \) then the quadrilateral \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) is of the kite form – it is inscribed in a circle and is symmetric with respect to the diameter of the circle \([f_{n,m}, f_{n+1,m+1}]\). If the angle at the vertex \( f_{n,m} \) is \( \pi/2 \) then the quadrilateral \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) is of the kite form too. In this case the quadrilateral is symmetric with respect to its diagonal \([f_{n,m+1}, f_{n+1,m}]\).

**Proposition 1** \([1]\) Let \( f_{n,m} \) satisfy (1) and (2) with initial data \( f_{0,0} = 0, f_{1,0} = 1, f_{0,1} = e^{i\alpha} \). Then all the elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) are of the kite form. All edges at the vertex \( f_{n,m} \) with \( n + m = 0 \) (mod 2) are of the same length

\[
|f_{n+1,m} - f_{n,m}| = |f_{n,m+1} - f_{n,m}| = |f_{n-1,m} - f_{n,m}| = |f_{n,m-1} - f_{n,m}|
\]

All angles between the neighboring edges at the vertex \( f_{n,m} \) with \( n + m = 1 \) (mod 2) are equal to \( \pi/2 \).

**Proposition 1** implies that for any \( n, m \) : \( n + m = 0 \) (mod 2) the points \( f_{n+1,m}, f_{n,m+1}, f_{n-1,m}, f_{n,m-1} \) lie on a circle with the center \( f_{n,m} \).

**Corollary 1** \([1]\) The circumscribed circles of the quadrilaterals \((f_{n-1,m}, f_{n,m-1}, f_{n+1,m}, f_{n,m+1})\) with \( n + m = 0 \) (mod 2) form a circle pattern of Schramm type (see \([2]\)), i.e. the circles of neighboring quadrilaterals intersect orthogonally and the circles of half-neighboring quadrilaterals with common vertex are tangent (see Fig. 3).

Consider the sublattice \( \{n, m : n + m = 0 \text{ (mod 2)}\} \) and denote by \( V \) its quadrant

\[
V = \{z = N + iM : N, M \in \mathbb{Z}^2, M \geq |N|\}
\]

where

\[
N = (n - m)/2, \quad M = (n + m)/2.
\]

We will use complex labels \( z = N + iM \) for this sublattice. Denote by \( C(z) \) the circle of the radius

\[
R(z) = |f_{n,m} - f_{n+1,m}| = |f_{n,m} - f_{n,m+1}| = |f_{n,m} - f_{n-1,m}| = |f_{n,m} - f_{n,m-1}| \quad (5)
\]

with the center at \( f_{N+M, M-N} = f_{n,m} \). From Proposition 1 it follows that any two circles \( C(z), C(z') \) with \( |z - z'| = 1 \) intersect orthogonally and any two circles \( C(z), C(z') \) with \( |z - z'| = \sqrt{2} \) are tangent.

Let \( \{C(z)\}, z \in V \) be a circle pattern of Schramm type on the complex plane. Define \( f_{n,m} : \mathbb{Z}_+^2 \to C \) as follows:

a) if \( n + m = 0 \) (mod 2) then \( f_{n,m} \) is the center of \( C(\frac{n-m}{2} + i\frac{n+m}{2}) \),

b) if \( n + m = 1 \) (mod 2) then \( f_{n,m} := C(\frac{n-m-1}{2} + i\frac{n+m-1}{2}) \cap C(\frac{n-m+1}{2} + i\frac{n+m+1}{2}) = C(\frac{n-m+1}{2} + i\frac{n+m-1}{2}) \cap C(\frac{n-m-1}{2} + i\frac{n+m+1}{2}) \).

Since all elementary quadrilaterals \((f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})\) are of the kite form equation (1) is satisfied automatically. In what follows the function \( f_{n,m} \), defined as above by a) and b) is called a discrete conformal map corresponding to the circle pattern \( \{C(z)\} \).
Theorem 1 \[1\] Let \( R \) be an immersion, then the radii \( R \) defined by (5) satisfies the following equations:

\[
R(z)R(z+1)(-2M - \gamma) + R(z+1)R(z+1+i)(2(N+1) - \gamma) + R(z+1+i)R(z+i)(2(M+1) - \gamma) + R(z+i)R(z)(-2N - \gamma) = 0, \tag{6}
\]
for \( z \in V_{l} := V \cup \{-N + i(N-1) \mid N \in \mathbb{N}\} \) and

\[
(N + M)(R(z)^2 - R(z+i))(R(z+i) + R(z+1) + (M - N)(R(z)^2 - R(z+i))R(z+1))(R(z+1) + R(z-i)) = 0, \tag{7}
\]
for \( z \in V_{int} := V \setminus \{\pm N + iN \mid N \in \mathbb{N}\} \).

Conversely let \( R(z) : V \rightarrow \mathbb{R}_+ \) satisfy (6) for \( z \in V_{l} \) and (7) for \( z \in V_{int} \). Then \( R(z) \) define an immersed circle packing with the combinatorics of the square grid. The corresponding discrete conformal map \( f_{n,m} \) is an immersion and satisfies (2).

From the initial condition (3) we have

\[
R(0) = 1, \quad R(i) = \tan \frac{\gamma \pi}{4}. \tag{8}
\]

Equation (6) at \( z = N + iN \) and \( z = -N + i(N-1), \ N \in \mathbb{N} \) reads as

\[
R(\pm(N + 1) + i(N+1)) = \frac{2N + \gamma}{2(N+1) - \gamma} R(\pm(N + iN)) \tag{9}
\]
and defines \( R(\pm N + iN) \) for all \( N \in \mathbb{N} \). Now equations (6,7) determine \( R(z) \) for any other \( z \in V \). Besides \( R(z) \) satisfy (7) at \( z = -N + iN, \ N \in \mathbb{N} \) which reads

\[
R(-N + 1 + iN)R(-N + i(N+1)) = R^2(-N + iN). \]

By symmetry one gets

\[
R(N - 1 + iN)R(N + i(N+1)) = R^2(N + iN). \]

This equation allows to compute \( R(N + i(N + 1)) \). (Moreover, it implies that the center \( O \) of \( C(N+iN) \) and the points \( A = C(N+iN) \cap C(N-1+iN) \) and \( B = C(N+iN) \cap C(N+i(N+1)) \) and \( B = C(N+iN) \cap C(N+i(N+1)) \cap C(N-1+iN) \cap C(N+i(N+1)) \) are collinear so the points \( f_{n,0} \) lie on a straight line.) To prove Theorem \[1\] we consider more general initial condition:

\[
R(0) = 1, \quad R(i) = a \tan \frac{\gamma \pi}{4}. \tag{10}
\]

Now the radii \( R(N + iN), R(N + i(N + 1)) \) can be represented in terms of \( \Gamma \)-function:

\[
R(N + iN) = c(\gamma) \frac{\Gamma(N + \gamma/2)}{\Gamma(N + 1 - \gamma/2)}, \quad \text{where } c(\gamma) = \frac{\gamma \Gamma(1 - \gamma/2)}{2 \Gamma(1 + \gamma/2)}, \tag{11}
\]

\[
R(N + i(N+1)) = \left(a \tan \frac{\gamma \pi}{4}\right)^{-1} \left(\frac{(2N-1 + \gamma)(2(N-3) + \gamma)(2(N-5) + \gamma)...}{(2N-\gamma)(2(N-2) - \gamma)(2(N-4) - \gamma)...}\right)^2. \tag{12}
\]

Theorem \[2\] allows us to reformulate the property of the circle lattice to be immersed completely in terms of the system (6,7). Namely to prove that \( Z^\gamma \) is an immersion one should show that the solution of the system (6,7) with initial data (3) is positive for all \( z \in V \) (see \[1\]). To prove Theorem \[1\] one need more subtle property of this solution.
Theorem 3 If for a solution \( R(z) \) of (6, 7) with \( \gamma \neq 1 \) and initial conditions 5 holds
\[
R(z) > 0, \quad (\gamma - 1)(R(z)^2 - R(z - i)R(z + 1)) \geq 0
\]
in \( V_{int} \), then the corresponding discrete conformal map is embedded.

Proof: Since \( R(z) > 0 \) the corresponding discrete conformal map is immersion (see 19). Consider piecewise linear curve \( \Gamma_n \) formed by segments \([f_{n,m}, f_{n,m+1}]\) where \( n > 0 \) and \( 0 \leq m \leq n - 1 \) and the vector \( v_n(m) = (f_{n,m} - f_{n,m+1}) \) along this curve. Due to Proposition 1 this vector rotates only in vertices with \( n + m = 0 \) (mod 2) as \( m \) increases along the curve. The sign of the rotation angle \( \theta_n(m) \), where \( -\pi < \theta_n(m) < \pi \), \( 0 < m < n \) is defined by the sign of expression \( R(z)^2 - R(z + i)R(z + 1) \) (note that there is no rotation if this expression vanishes), where \( z = (n - m)/2 + i(n + m)/2 \) is a label for the circle with the center in \( f_{n,m} \). If \( n + m = 1 \) (mod 2) define \( \theta_n(m) = 0 \). Now the theorem hypothesis and equation 7 imply that the vector \( v_n(m) \) rotates with increasing \( m \) in the same direction for all \( n \), and namely, clockwise for \( \gamma < 1 \) and counterclockwise for \( \gamma > 1 \). Consider the sector \( B := \{z = re^{i\varphi} : r \geq 0, \ 0 \leq \varphi \leq \gamma \pi/4\} \). The terminal points of the curves \( \Gamma_n \) lie on the sector border.

Lemma 1 For the curve \( \Gamma_n \) holds:
\[
\left| \sum_{m=1}^{n-1} \theta_n(m) \right| < \frac{\pi}{4}(1 + |1 - \gamma|)
\]  

Proof of Lemma 1 Let us prove the inequality 14 for \( 1 < \gamma < 2 \) by induction for \( n \). For \( n = 1 \) the inequality is obviously true since the curve \( \Gamma_1 \) is a segment perpendicular to \( R_+ \). Define the angle \( \alpha_n(m) \) between \( iR_+ \) and the vector \( v_n(m) \) by \( f_{n,m+1} - f_{n,m} = e^{i(\alpha_n(m) + \pi/2)}|f_{n,m+1} - f_{n,m}| \), where \( 0 \leq \alpha_n(m) < 2\pi \), \( 0 \leq m < n \). Then \( \sum_{m=1}^l \theta_n(m) = \alpha_n(l) - \alpha_n(0) + 2\pi k_n(l) \) for some positive integer \( k_n(l) \) increasing with \( l \). Note that \( \alpha_n(0) < \pi/2 \), which easily follows from 112, and \( \alpha_n(n - 1) < (\frac{\gamma \pi}{4} + \frac{\pi}{2}) - \frac{\pi}{2} = \frac{\gamma \pi}{4} \) since for immersed \( Z \) the angle between the vector \( v_n(n - 1) \) and \( e^{i\gamma \pi/4}R_+ \) is less then \( \frac{\gamma \pi}{4} \). Let 14 holds for \( n > 1 \): \( \left| \sum_{m=1}^{n-1} \theta_n(m) \right| = \sum_{m=1}^{n-1} \theta_n(m) < \frac{\pi}{4} \) (all \( \theta_n(m) \) are positive for \( 1 < \gamma < 2 \)). That implies \( k_n(l) = 0 \), since \( k_n(l) = \sum_{m=1}^l \theta_n(m) - \alpha_n(l) + \alpha_n(0))/2\pi \leq \sum_{m=1}^l \theta_n(m) + |\alpha_n(l)| + |\alpha_n(0)))/2\pi < (\gamma \pi/4 + \gamma \pi/4 + \pi/2)/2\pi < 1 \) and \( k_n(l) \) is integer. Let \( \alpha_{n+1}(l) = \alpha_n(l) + \sigma_n(l) \). All elementary quadrilaterals are of the kite form therefore \( |\sigma_n(l)| < \pi/2 \). Let us prove, that \( k_{n+1}(l) = 0 \) for \( 0 \leq m \leq n + 1 \). Obviously \( k_{n+1}(0) = 0 \). Assume \( k_{n+1}(l) = 0 \) but \( k_{n+1}(l + 1) > 0 \). The increment of l.h.s. of
\[
\sum_{m=1}^l \theta_n(m) = \alpha_{n+1}(l) - \alpha_n(0) + 2\pi k_{n+1}(l)
\]
as \( l \to l+1 \) is \( \theta_{n+1}(l+1) < \pi \). The increment of r.h.s. is no less than \( 2\pi + \alpha_{n+1}(l+1) - \alpha_{n+1}(l) \geq 2\pi - \alpha_{n+1}(l) \geq 2\pi - \alpha_{n}(l) - |\sigma(l)| > 2\pi - \gamma \pi/4 - \pi/2 > \pi \). The obtained contradiction gives \( k_{n+1}(l) = 0 \) and \( \sum_{m=1}^n \theta_n(m) = \alpha_{n+1}(n) - \alpha_{n+1}(0) \leq \alpha_{n+1}(n) < \frac{\gamma \pi}{4} \). Lemma 4 is proved.

The obvious corollary of Lemma 4 is that the curve \( \Gamma_n \) has no self-intersection and lies in the sector \( B \) since the rotation of the vector \( v_n(m) \) along the curve is less then \( \gamma \pi/4 < \pi/2 \). Each such curve cuts the sector \( B \) into a finite part and an infinite part. Since the curve \( \Gamma_n \) is convex and the borders of all elementary quadrilaterals \( (f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) \) for embedded
have the positive orientation the segments of the curve $\Gamma_{n+1}$ lie in the infinite part. Now the induction in $n$ completes the proof of Theorem 3 for $1 < \gamma < 2$. The proof for $0 < \gamma < 1$ is similar. The differences are that $\theta_n(m)$ is not positive, the angle $\alpha$ is naturally defined as negative: $-2\pi < \alpha_n(m) < 0$, so that $-\pi/2 < \alpha_n(0) \leq 0$ and \( \frac{\pi}{2} (2 - \gamma) < \alpha_n(n - 1) < 0 \). Details are left to the reader.

**Corollary 2**

\[
\lim_{n \to \infty} Z_{n,m}^\gamma = \infty, \quad \lim_{m \to \infty} Z_{n,m}^\gamma = \infty.
\]

Since the terminal points of the curves $\Gamma_n$ lie on the sector border the proof easily follows from convexity of the curves $\Gamma_n$, inequality (14) and from

\[
\lim_{n \to \infty} Z_{n,0}^\gamma = \infty.
\]

### 3 $Z^\gamma$ and discrete Painlevé equations

Let $R(z)$ be a solution of \((6,7)\) with initial condition \((8)\). For $z \in V_{\text{int}}$ define $P_{N,M} = P(z) = \frac{R(z+1)}{R(z-i)}$, $Q_{N,M} = Q(z) = \frac{R(z)}{R(z-i)}$. Then equations \((6,7)\) are rewritten as follows

\[
Q_{N,M+1} = \frac{(M+N)P_{N,M}(P_{N,M} - Q_{N,M}^2) - (M-N)Q_{N,M}^2(1 + P_{N,M})}{Q_{N,M}((M+N)(Q_{N,M}^2 - P_{N,M}) - (M-N)P_{N,M}(1 + P_{N,M}))}
\]

\[
P_{N,M+1} = \frac{(2M + \gamma)P_{N,M} + (2N + \gamma)Q_{N,M}Q_{N,M+1}}{(2(N+1) - \gamma)P_{N,M} + (2(M+1) - \gamma)Q_{N,M}Q_{N,M+1}}
\]

The property (13) for (15,16) reads as

\[
(\gamma - 1)(Q_{N,M}^2 - P_{N,M}) \geq 0, \quad Q_{N,M} > 0, \quad P_{N,M} > 0.
\]

Equations (15,16) can be considered as a dynamical system for variable $M$, the expressions \((11,12)\) defining initial conditions $P_{N,N+1}, Q_{N,N+1}$ for (15,16) as functions of $a, N$, only $Q_{N,N+1}$ being dependent on $a$.

**Theorem 4** There exists such $a > 0$ that for the solutions $R(z)$ of \((6,7)\) with initial conditions \((10)\) holds \((13)\).

**Proof:** Due to the following Lemma it is sufficient to prove \((13)\) only for $0 < \gamma < 1$.

**Lemma 2** If $R(z)$ is a solution of \((6,7)\) for $\gamma$ then $1/R(z)$ is a solution of \((6,7)\) for $\tilde{\gamma} = 2 - \gamma$.

Lemma is proved by straightforward computation.

Let $0 < \gamma < 1$ and $(P_{N,M}, Q_{N,M})$ correspond to the solution of \((6,7)\) defined by initial conditions \((10)\). Define the real function $F(P)$ for $P \in R_+$:

\[
F(P) = \sqrt{P} \text{ for } 0 \leq P \leq 1, \quad F(P) = 1 \text{ for } 1 \leq P.
\]

Designate

\[
\begin{align*}
D_u & := \{(P,Q) : P > 0, Q > F(P)\}, \quad D_d := \{(P,Q) : Q < 0\}, \\
D_0 & := \{(P,Q) : P > 0, 0 \leq Q \leq F(P)\}, \quad D_f := \{(P,Q) : P \leq 0, Q \geq 0\}.
\end{align*}
\]
Now define the infinite sequences \( \{q_n\}, \{p_n\}, n \in \mathbb{N} \) as follows:

\[
\{q_n(a)\} := \{Q_{0,1}, Q_{0,2}, Q_{1,2}, Q_{1,3}, Q_{1,2,3}, \ldots, Q_{0,M}, Q_{1,M}, \ldots, Q_{M-1,M}, \ldots\},
\]

\[
\{p_n(a)\} := \{P_{0,1}, P_{0,2}, P_{1,2}, P_{1,3}, P_{1,2,3}, \ldots, P_{0,M}, P_{1,M}, \ldots, P_{M-1,M}, \ldots\}.
\]

and the sets

\[
A_u(n) := \{a \in \mathbb{R}_+ : (p_n(a), q_n(a)) \in D_u, (p_k(a), q_k(a)) \in D_0 \ \forall \ 0 < k < n\},
\]

\[
A_d(n) := \{a \in \mathbb{R}_+ : (p_n(a), q_n(a)) \in D_d, (p_k(a), q_k(a)) \in D_0 \ \forall \ 0 < k < n\}.
\]

\( A_u(n) \) and \( A_d(n) \) are open sets since the denominators of (15,16) do not vanish in \( D_0 \). Indeed, the curve \( Q^2 = P + \frac{(M-N)}{(M+N)}P(P + 1) \) lies outside \( D_0 \). Moreover, direct computation shows that

\[
A_u(1) \neq \emptyset \quad \text{and} \quad A_d(2) \neq \emptyset,
\]

therefore the sets

\[
A_u := \cup A_u(k), \quad A_d := \cup A_d(k)
\]

are not empty. Finally, define

\[
A_0 := \{a \in \mathbb{R}_+ : (p_n(a), q_n(a)) \in D_0, \ \forall \ n \in \mathbb{N}\}.
\]

Note that \( A_0, A_u, A_d \) are mutually disjoint and the sequences \( \{p_n\}, \{q_n\} \) is so constructed that

\[
\mathbb{R}_+ = A_0 \cup A_u \cup A_d. \quad (18)
\]

Indeed \( (P_{N,M}, Q_{N,M}) \) can not jump from \( D_0 \) into \( D_f \) in one step \( M \to M + 1 \) since \( P_{N,M+1} \) is positive for positive \( P_{N,M}, Q_{N,M}, Q_{N,M+1} \). The relation (18) would be impossible for \( A_0 = \emptyset \), since the connected set \( \mathbb{R}_+ \) can not be covered by two open disjoint nonempty subsets \( A_u \) and \( A_d \), therefore \( A_0 \neq \emptyset \). Q.E.D.

**Proposition 2** The set \( A_0 \) consists of only one element, namely, \( A_0 = \{1\} \).

**Proof:** A positive solution \( R(z) \) of (17) provides an immersed discrete conformal map corresponding to the circle patterns with radii \( R(z) \). It was proven in [1] that the initial value \( 10 \) for immersed \( Z^\gamma \) consists of only one element \( a = 1 \). Q.E.D.

**Proof of Theorem 1**: combining Theorems 3 and 4 with Proposition 2 one easily deduces Theorem 1.

### 4 Concluding remarks

The approach suggested in this paper can be applied to prove the embeddedness of the circle patterns corresponding to discrete \( Z^2 \) and Log. (These patterns were proved to be immersed [1].) Slightly modified, it seems to be also applicable to show global properties of the discrete \( Z^K \) proposed in [1] for natural \( K > 2 \). Details will be discussed elsewhere.

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