Naked singularity formation for higher dimensional inhomogeneous dust collapse

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We investigate the occurrence and nature of a naked singularity in the gravitational collapse of an inhomogeneous dust cloud described by higher dimensional Tolman-Bondi space-time for non-marginally bound case. The naked singularities are found to be gravitationally strong.

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I. INTRODUCTION

The Cosmic Censorship Conjecture (CCC)[1,2] states that the space-time singularity produced by gravitational collapse must be covered by the horizon. The CCC is yet one of the unresolved problem in General Relativity. In fact the conjecture has not yet any precise mathematical proof. However the singularity theorems as such do not state anything about the visibility of the singularity to an outside observer. Several models related to the gravitational collapse of matter has so far been constructed where one encounters a naked singularity [3-8].

It is known that the Tolman-Bondi metric admits both naked and covered singularities depending upon the choice of initial data and that there is a smooth transition from one phase to the other. Moreover, according to the strong version of the CCC, such singularities are not even locally naked, i.e., no non-spacelike curve can emerge from such singularities (see [2] for reviews of the CCC). Consequently, examples that appear to violate the CCC are important and they are an important tools to study this important issue.

It is our purpose here to show that the occurrence of a strong curvature naked singularity is not confined to self-similar space-times or null dust only by pointing out a wide class of Tolman-Bondi models which are non-self-similar in general where such a singularity forms. In Sec.II, we review 5D Tolman-Bondi solutions in non-marginally bound case and in Sec.III, we discuss the nature and existence of central shell focusing naked singularity. Finally in Sec.IV, we show that gravitational collapse of 5D space-times gives rise to naked singularities which are gravitationally strong.

II. FIVE DIMENSIONAL TOLMAN-BONDI SOLUTION

The five dimensional Tolman-Bondi metric in co-moving co-ordinates is given by

\[ ds^2 = e^\nu dt^2 - e^{\lambda} dr^2 - R^2 d\Omega^2_3 \]  

where \( \nu, \lambda, R \) are functions of the radial co-ordinate \( r \) and time \( t \) and \( d\Omega^2_3 \) represents the metric on the 3-sphere. Since we assume the matter in the form of dust, the motion of particles will be geodesic allowing us to write \( e^\nu = 1 \). Using comoving co-ordinates one can in view of the field equations [9], arrive at the following relations in 5 dimensional space-time

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\[ e^\lambda = \frac{R^2}{1 + f(r)} \]  

(2)

and

\[ \dot{R}^2 = f(r) + \frac{F(r)}{R^2} \]  

(3)

where, \( f(r) \) and \( F(r) \) are arbitrary functions of radial co-ordinate \( r \) alone with the restriction \( 1 + f(r) > 0 \) for obvious reasons. The functions \( F(r) \) and \( f(r) \) in fact refer to the mass function and the binding energy function respectively.

The energy density \( \rho(t, r) \) is therefore given by

\[ \rho(t, r) = \frac{3F'(r)}{2R^3R'} \]  

(4)

As we are concerned with the gravitational collapse dust we require \( \dot{R}(t, r) < 0 \) and without loss of generality we rescale \( R \) such that

\[ R(0, r) = r \]  

(5)

Integrating eq.(3) and using the relation (5), we have the solution

\[ R^2 = r^2 + ft^2 - 2t\sqrt{F + fr^2} \]  

(6)

The central singularity occurs at \( r = 0 \), the corresponding time being \( t = 0 \). We denote by \( \rho(r) \) the initial density:

\[ \rho(r) \equiv \rho(0, r) = \frac{3F'}{2r^3} \Rightarrow F(r) = \frac{2}{3} \int \rho(r)r^3dr \]  

(7)

In 4D case, the mass function \( F(r) \) involves the integral \( \int \rho(r)r^2dr \) [2]. It can be seen from eq.(4) that the density diverges faster in 5D as compared to 4D. Thus given a regular initial surface, the time for the occurrence of the central shell focusing singularity for the collapse developing from that surface is reduced as compared to the 4D case. The reason for this stems from the form of the mass function in eq.(7). In a ball of radius 0 to \( r \), for any given initial density profile \( \rho(r) \), the total mass contained in the ball is greater than in the corresponding 4D case. Hence there is relatively more mass-energy collapsing in the space-time as assumed overall positivity of mass-energy (energy condition).

### III. EXISTENCE AND NATURE OF NAKED SINGULARITY

In the context of Tolman-Bondi space-times, shell crossing singularities are defined by \( R' = 0, R > 0 \) and they can be naked. It has been shown [10] that shell crossing singularities are gravitationally weak and hence such singularities can not be considered seriously in the context of the CCC. On the other hand, central shell focusing singularities (characterized by \( r = 0 \) and \( R = 0 \)) are also naked and gravitationally strong as well. Thus, unlike shell crossing singularities, shell focusing singularities do not admit any metric extension through them. Here we wish to investigate a similar situation in our 5D space-time. Christodoulou [11] pointed out in the 4D case that the non-central singularities are not naked. Hence we shall confine our discussion to the central shell focusing singularity.
Now $t = t_s(r)$ is the instant of shell focusing singularity occurring at $r$ i.e., $R(t_s(r), r) = 0$. So eq.(6) yields to

$$t_s(r) = \frac{\sqrt{fr^2 + F} - \sqrt{F}}{f}$$ (8)

Let us assume [12]

$$F(r) = r^2 \lambda(r)$$
$$\alpha = \alpha(r) = \frac{\alpha'}{f}$$
$$\beta = \beta(r) = \frac{\beta'}{r}$$
$$R(t, r) = r P(t, r)$$ (9)

So using equations (3), (6) and (9), we have the following expressions

$$R' = \frac{1}{2P} \left[ 2 + \alpha \left( P^2 - 1 + \frac{2t}{r} \sqrt{f + \lambda} \right) - \frac{t}{r} \frac{(\lambda \beta + f \alpha + 2f)}{\sqrt{f + \lambda}} \right]$$ (10)

and

$$\dot{R}' = \frac{1}{2r \sqrt{\lambda + fP^2}} \left[ \frac{\lambda}{P^3} \left\{ 2 + \alpha \left( P^2 - 1 + \frac{2t}{r} \sqrt{f + \lambda} \right) - \frac{t}{r} \frac{(\lambda \beta + f \alpha + 2f)}{\sqrt{f + \lambda}} \right\} - Pf \alpha - \frac{\lambda \beta}{P} \right]$$ (11)

When \(\lambda(r) = \text{constant}\) and \(f(r) = \text{constant}\), then space-time becomes self-similar. Now we restrict ourselves to functions \(f(r)\) and \(\lambda(r)\) which are analytic at \(r = 0\), such that \(\lambda(0) > 0\) and this implies that \(t_s(0) = 0\). It follows that the point \(r = 0, t = 0\) corresponds to the central singularity on the hypersurface \(t = 0\). From eq.(4) it is seen that the density at the centre \((r = 0)\) behaves with time as \(\rho = \frac{3}{2r^2}\). This means that the density is finite at any time \(t\), but becomes singular at \(t = 0\).

We wish to investigate if the singularity, when the central shell collapses to the centre \(r = 0\), is naked. The singularity is naked if and only if there exists a null geodesic that emanates from the singularity. Let \(K^a = \frac{dx^a}{d\mu}\) be the tangent vector to the radial null geodesic, where \(\mu\) is the affine parameter. Then we derive the following equations:

$$\frac{dK^t}{d\mu} + \frac{\dot{R}'}{\sqrt{1 + f}} K^r K^t = 0$$ (12)

$$\frac{dt}{dr} = K^t K^r = \frac{R'}{\sqrt{1 + f(r)}}$$ (13)

Now we introduce a new variable \(X = \frac{t}{r}\), then the function \(P(t, r) = P(X, r)\) is given, with the help of (6) and (8), by

$$[f(X - \Theta) - \sqrt{\lambda}]^2 = \lambda + fP^2$$ (14)

where we have put \(t_s(r) = r\Theta(r)\) with

$$P^2 = 1 + fX^2 - 2X \sqrt{f + \lambda}$$ (15)
The nature of the singularity (a naked singularity or a black hole) can be characterized by the existence of radial null geodesics emerging from the singularity. The singularity is at least locally naked if there exist such geodesics and if no such geodesics exist it is a black hole. If the singularity is naked, then there exists a real and positive value of $X_0$ as a solution to the equation

$$X_0 = \lim_{t \to 0} \frac{1}{r} = \lim_{t \to 0} \frac{dt}{dr} = \lim_{t \to 0} \frac{\sqrt{1+f}}{r}$$ \hspace{1cm} (16)$$

Define $\lambda_0 = \lambda(0), \alpha_0 = \alpha(0), f_0 = f(0), \Theta_0 = \Theta(0)$ and $Q = Q(X) = P(X, 0)$, so equations (14) and (15) reduces to

$$[f_0(X - \Theta_0) - \sqrt{\lambda_0}]^2 = \lambda_0 + f_0Q^2$$ \hspace{1cm} (17)

and

$$Q^2 = 1 + f_0X^2 - 2X\sqrt{f_0 + \lambda_0}$$ \hspace{1cm} (18)

Now from eq.(9) it is to be seen that $\beta(0) = 2$. We would denote $Q_0 = Q(X_0)$, the equation (16) simplifies to

$$G(X_0) = 0$$ \hspace{1cm} (19)

where

$$G(X) = \frac{1}{2Q} \left[ 2 + \alpha_0 \left( Q^2 - 1 + 2X\sqrt{f_0 + \lambda_0} \right) - \frac{(2\lambda_0 + f_0\alpha_0 + 2f_0)}{\sqrt{f_0 + \lambda_0}} X - 2XQ\sqrt{1+f_0} \right]$$ \hspace{1cm} (20)

If the equation $G(X) = 0$ has a real positive root, the singularity could be naked. If no real positive root is found the singularity $t = 0, r = 0$ is obviously black hole.

From equation (13), we have

$$\frac{dX}{dr} = \frac{G(X)}{\sqrt{f_0 + \lambda_0}} + \frac{Y(X, r)}{r}$$ \hspace{1cm} (21)

where $Y(X, r)$ is function of $X$ and $r$ such that at $r = 0, Y(X, 0) = 0$. Since $X_0$ is a root of $G(X) = 0$ (see, eq.(19)), so we can express $G(X)$ in the following form

$$G(X) = q_0\sqrt{1 + f_0} (X - X_0) + O(X - X_0)^2$$ \hspace{1cm} (22)

where

$$q_0 = \frac{1}{Q_0\sqrt{1 + f_0}} \left[ f_0\alpha_0X_0 - \frac{(2\lambda_0 + f_0\alpha_0 + 2f_0)}{2\sqrt{f_0 + \lambda_0}} \right] + \left( \sqrt{f_0 + \lambda_0} - f_0X_0 \right) \frac{X_0}{Q_0^2} - 1$$

Substituting (22) in (21) we have

$$\frac{dX}{dr} - (X - X_0) \frac{q_0}{r} = \frac{H}{r}$$ \hspace{1cm} (23)

where $H = H(X, r) = Y(X, r) + O(X - X_0)$ such that $H(X_0, 0) = 0$.

Integrating (23) we have
\[ X - X_0 = C r^{q_0} + r^{q_0} \int H r^{-q_0-1} dr \]  

(24)

where \( C \) is the constant of integration which labels different geodesics. From eq.(24) it can be shown that \( X \to X_0 \) as \( r \to 0 \) for \( q_0 > 0 \) or \( < 0 \). Therefore the single null geodesic described by \( C = 0 \) always terminates at the singularity \( t = 0, r = 0 \) with \( X = X_0 \). For \( q_0 > 0 \) there are infinitely many integral curves (characterized by the different values of \( C \)) terminate at the singularity. But for \( q_0 < 0 \) there is only one singular geodesic (characterized by \( C = 0 \)) terminates at the singularity.

From (8) we have

\[ \Theta_0 = \frac{1}{f_0}(\sqrt{f_0 + \lambda_0} - \sqrt{\lambda_0}) \]  

(25)

Using (17) to (20) and (25) we have the algebraic equation for \( X_0 \),

\[ aX_0^4 + bX_0^3 + cX_0^2 + dX_0 + 4 = 0 \]  

(26)

where

\[ a = f_0\{f_0(\alpha_0^2 - 4) - 4\}, \]

\[ b = 8(1 + f_0)\sqrt{f_0 + \lambda_0} - \frac{2\alpha_0 f_0(2f_0 + \alpha_0 f_0 + 2\lambda_0)}{\sqrt{f_0 + \lambda_0}}, \]

\[ c = -4(1 + f_0) + 4\alpha_0 f_0 + \frac{(2f_0 + \alpha_0 f_0 + 2\lambda_0)^2}{f_0 + \lambda_0}, \]

\[ d = -\frac{4(2f_0 + \alpha_0 f_0 + 2\lambda_0)}{\sqrt{f_0 + \lambda_0}}. \]

This algebraic equation gives us information about the behavior of the tangent near the singularity. In fact, the central shell focusing singularity is at least locally naked if the above eq.(26) has at least one positive root. The tangents to the escaping geodesics near the singularity are determined by the roots of the equation. The smallest root (say, \( X_0^s \)) of \( X_0 \) is called the Cauchy horizon of the space-time as it indicates the earliest ray escaping from the singularity. Thus no solution is possible in the region \( X_0^s < X_0^s \). Further, for non existence of any positive root, it is not possible to have any future directed radial null geodesic emanating from the singularity i.e., the singularity is fully covered by trapped surface and we have only black hole solution.

We shall now discuss the nature of the roots of the fourth degree equation in \( X_0 \). We note that if \( a < 0 \) then equation (26) has at least one positive and at least one negative root provided \( \alpha_0^2 < 4 \left(1 + \frac{1}{f_0}\right)\). For example, if we choose \( f_0 = 1, \lambda_0 = 1.1 \) and \( \alpha_0 = 3 \) then \( X_0 = 354393 \) will correspond to Cauchy horizon. If we choose \( a = 0 \) then the algebraic equation reduces to a cubic equation which has at least one positive root provided \( f_0 > 0 \). However, for \(-1 < f_0 < 0 \) (note that we must have \( 1 + f_0 > 0 \), see eq.(2)) at least one positive root is possible if \( f_0 \) is close to zero, otherwise, for \( f_0 \) close to \(-1 \) we may or may not have positive real root. The table shows the numerical results for different choices of the parameters. Therefore, we conclude that it is possible to have a naked singularity for various values of parameters \( f_0, \lambda_0 \) and \( \alpha_0 \) as shown in the table.
TABLE: Roots of equation (26) for various values of the parameters $f_0, \lambda_0$ and $\alpha_0$.

| $f_0$ | $\lambda_0$ | $\alpha_0$ | Roots ($X_0$)          |
|-------|-------------|-------------|-------------------------|
| -0.03 | 0.034       | 0.2         | 4.32936, 1.16321        |
| -1.0  | 0.11        | 0.2         | 1.72737, 1.48602        |
| -1.0  | 0.11        | 0.5         | -                       |
| -0.5  | 0.51        | 0.001       | -                       |
| 1.0   | 0.1         | 3.0         | 10.3982, 1.48206, 0.354393 |
| 3.0   | 0.001       | 1.0         | 0.557735, 0.481183      |
| 9.0   | 0.01        | 5.0         | 0.329341                |
| 9.0   | 0.01        | 5.0         | 0.093828                |

IV. STRENGTH OF NAKED SINGULARITY

Finally we need to determine curvature strength of the naked singularity which is an important aspect of a singularity [13]. There has been attempt to relate the strength of a singularity to its stability [14]. A singularity is gravitationally strong or simply strong if it destroys by crushing or stretching any objects that fall into it and weak if no object that falls into the singularity is destroyed in this way. Clarke and Krolak [15] have shown that a sufficient condition for strong curvature singularity defined by Tipler [16] is that for at least one non-spacelike geodesic with affine parameter $\mu$, in the limiting approach to the singularity, we must have

$$\lim_{\mu \to 0} \mu^2 \psi = \lim_{\mu \to 0} \mu^2 R_{ab} K^a K^b > 0$$  \hspace{1cm} (27)

where $R_{ab}$ is the Ricci tensor. Our purpose here is to investigate the above condition along future directed radial null geodesics that emanates from the naked singularity. Now equation (27) can be expressed as

$$\lim_{\mu \to 0} \mu^2 \psi = \lim_{\mu \to 0} \frac{3F'_{\frac{r}{R}}}{2rP_{\frac{R}{r}}} \left(\frac{\mu K^t}{R}\right)^2$$  \hspace{1cm} (28)

Using L’Hospital’s rule and using equations (10) and (11), the equation (28) can be written as

$$\lim_{\mu \to 0} \mu^2 \psi = \frac{12\lambda_0 Q_0 X_0 (\lambda_0 + f_0 Q_0^2) \sqrt{1 + f_0}}{[2\lambda_0 X_0 \sqrt{1 + f_0} - 2\lambda_0 Q_0 - f_0 \alpha_0 Q_0^2]^2} > 0$$  \hspace{1cm} (29)

Thus along radial null geodesics coming out of a singularity $\lim_{\mu \to 0} \mu^2 \psi$ is finite and hence the strong curvature condition is satisfied.
V. CONCLUDING REMARKS

In this paper, the 5D Tolman-Bondi space-time have been studied for the formation of naked singularities in spherical dust collapse in non-marginally bound case \((f \neq 0)\). The nature of the singularity depends on the value of the curvature parameter \(f\) at \(r = 0\) (i.e. \(f_0\)). In fact for \(f_0 > 0\) or \(f_0 < 0\) (but close to zero) the singularity may be naked but it depends also on the choice of the other two parameters involved. The strong curvature condition for naked singularity is satisfied for this 5D inhomogeneous dust collapse. The natural questions that arise how the extra dimension influence the formation of a naked singularity and what is the role of the curvature parameter in classifying the singularity. These questions will be addressed in a subsequent paper. For future work, it will be interesting to study the exact restrictions for formation of naked singularity on the parameters involved and their physical implications.

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