Liouville Theory: Ward Identities for Generating Functional and Modular Geometry

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Abstract

We continue the study of quantum Liouville theory through Polyakov’s functional integral [1, 2], started in [3]. We derive the perturbation expansion for Schwinger’s generating functional for connected multi-point correlation functions involving stress-energy tensor, give the “dynamical” proof of the Virasoro symmetry of the theory and compute the value of the central charge, confirming previous calculation in [3]. We show that conformal Ward identities for these correlation functions contain such basic facts from Kähler geometry of moduli spaces of Riemann surfaces, as relation between accessory parameters for the Fuchsian uniformization, Liouville action and Eichler integrals, Kähler potential for the Weil-Petersson metric, and local index theorem. These results affirm the fundamental role, that universal Ward identities for the generating functional play in Friedan-Shenker modular geometry [4].

1 According to [1, 2, 3], the correlation function of puncture operators in Liouville theory is given by the following functional integral

\[ \langle X \rangle = \int_{\mathcal{C}(X)} \mathcal{D}\phi \ e^{-(1/2\pi\hbar)S(\phi)}, \]

where \( X = \hat{\mathbb{C}} \setminus \{w_1, \ldots, w_n\}, \ w_i \neq w_j \text{ for } i \neq j \), is the \( n \)-punctured Riemann sphere. “Domain of integration” \( \mathcal{C}(X) \) consists of conformal metrics \( ds^2 = e^{\phi(w, \bar{w})}|dw|^2 \) on \( X \), satisfying the
following asymptotics near punctures $w_1, \ldots, w_{n-1}, w_n = \infty$,

$$e^\phi \approx \frac{1}{r_i^2 \log^2 r_i}, \ i = 1, \ldots, n,$$

where $r_i = |w - w_i|$, $i = 1, \ldots, n-1$, $r_n = |w|$, and $w$ is the global complex coordinate on $X$. Also $h > 0$ is a coupling constant and functional $S(\phi)$ is Liouville action \cite{5},

$$S(\phi) = \lim_{\epsilon \to 0} \{ \int_{X_\epsilon} (|\phi_w|^2 + e^\phi) d^2 w + 2\pi n \log \epsilon + 4\pi(n-2) \log |\log \epsilon| \},$$

where $X_\epsilon = X \setminus \bigcup_{i=1}^{n-1} \{|w-w_i| < \epsilon\} \cup \{|w| > 1/\epsilon\}$. Classical equations of motion $\delta S = 0$ yield Liouville equation—the equation for complete conformal metric on $X$ of constant negative curvature $-1$. It has a unique solution, called Poincaré, or hyperbolic metric, and is denoted by $\phi_{cl}$.

As in \cite{3}, we define $< X >$—the expectation value of the Riemann surface $X$—by its perturbation expansion around the classical solution $\phi = \phi_{cl}$ (thus making a choice of the “integration measure” in (\cite{3})). Corresponding propagator $G(w, w')$ is given by the Green’s function of operator $2\Delta + 1$, where

$$\Delta = -e^{-\phi_{cl}} \partial_{w\bar{w}}^2$$

is a hyperbolic Laplacian on $X$. Logarithmic divergence of $G$ at coincident points is renormalized in a reparametrization invariant way using the geodesic distance in the Poincaré metric (see, e.g., \cite{3}).

2 Conformal invariance of Liouville theory implies, in particular, that its stress-energy tensor is traceless. Namely, its $(2, 0)$-component $T(\phi)(w)$ is given by

$$T(\phi) = \frac{1}{h} (\phi_{ww} - \frac{1}{2} \phi^2_w),$$

is conserved on classical equations of motion

$$\partial_w T_{cl} = 0,$$

and has the transformation law of the projective connection (times $1/h$) under holomorphic change of coordinates, i.e.

$$\tilde{T}(\tilde{w}) = T(f(\tilde{w}))(f'(\tilde{w}))^2 + \frac{1}{h} S(f)(\tilde{w}), \ w = f(\tilde{w}).$$
Here $S$ stands for the Schwarzian derivative,

$$S(f) = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2.$$ 

Similarly, the $(0,2)$-component $\bar{T}(\phi)(w)$ of the stress-energy tensor is given by

$$\bar{T}(\phi) = \frac{1}{h} (\phi \bar{w} - \frac{1}{2} \phi_\bar{w}^2).$$

According to Belavin-Polyakov-Zamolodchikov [6], conformal symmetry manifests itself through the infinite sequence of conformal Ward identities, that relate correlation functions involving the stress-energy tensor components with correlation functions without them. Specifically, we define multi-point expectation values of the holomorphic and anti-holomorphic components of the stress-energy tensor as

$$\langle \prod_{i=1}^k T(u_i) \prod_{j=1}^l \bar{T}(\bar{v}_j) X \rangle = \int_{C(X)} D\phi \prod_{i=1}^k T(\phi)(u_i) \prod_{j=1}^l \bar{T}(\phi)(\bar{v}_j) e^{(-1/2\pi h)S(\phi)},$$

and denote by $\langle\langle \cdots \rangle\rangle$ their normalized connected forms. Thus, for example,

$$\langle\langle T(w)X \rangle\rangle = \frac{\langle T(w)X \rangle}{\langle X \rangle}, \quad \langle\langle \bar{T}(\bar{w})X \rangle\rangle = \frac{\langle \bar{T}(\bar{w})X \rangle}{\langle X \rangle},$$

$$\langle\langle T(u)T(v)X \rangle\rangle = \frac{\langle T(u)T(v)X \rangle}{\langle X \rangle} - \langle\langle T(u)X \rangle\rangle \langle\langle T(v)X \rangle\rangle,$$

$$\langle\langle T(u)\bar{T}(\bar{v})X \rangle\rangle = \frac{\langle T(u)\bar{T}(\bar{v})X \rangle}{\langle X \rangle} - \langle\langle T(u)X \rangle\rangle \langle\langle \bar{T}(\bar{v})X \rangle\rangle,$$

etc.

These correlation functions should satisfy conformal Ward identities of BPZ [6, formulas (3.10), (3.15)], that we present in the following succinct form (after fixing overall $\text{SL}(2, \mathbb{C})$-symmetry by normalizing $w_{n-2} = 0, w_{n-1} = 1, w_n = \infty$, and using [6, formulas (A.7)]).

Namely, denote by $\Delta(h) = \bar{\Delta}(h)$ conformal weights of the puncture operator, by $c$—the central charge of the Virasoro algebra, and by $L(w)$ and $\bar{L}(\bar{w})$—the following first order differential operators

$$L(w) = \sum_{i=1}^{n-3} R(w, w_i) \partial w_i, \quad \bar{L}(\bar{w}) = \sum_{i=1}^{n-3} R(\bar{w}, \bar{w}_i) \partial \bar{w}_i,$$

where

$$R(w, w_i) = \frac{1}{w - w_i} + \frac{w_i - 1}{w} - \frac{w_i}{w - 1} = \frac{w_i(w_i - 1)}{(w - w_i)w(w - 1)}.$$
We have

\[
\langle \langle T(w)X \rangle \rangle_0 = \langle \langle T(w)X \rangle \rangle - T_s(w) = \mathcal{L}(w) \log X,
\]

(2)

where

\[
T_s(w) = \sum_{i=1}^{n-1} \frac{\Delta(h)}{(w-w_i)^2} + \frac{(2-n)\Delta(h)}{w(w-1)},
\]

and analogous expression for \( \langle \langle \bar{T}(\bar{w})X \rangle \rangle \). The Ward identities for two-point correlation functions have the form

\[
\langle \langle T(u) T(v) \rangle \rangle = \frac{c/2}{(u-v)^4} + \{2R_v(u,v) + R(u,v) \partial_v + \mathcal{L}(u)\} \langle \langle T(v)X \rangle \rangle,
\]

(3)

\[
\langle \langle T(u) \bar{T}(\bar{v})X \rangle \rangle = \mathcal{L}(u) \langle \langle \bar{T}(\bar{v})X \rangle \rangle = \mathcal{L}(u) \bar{\mathcal{L}}(\bar{v}) \log X.
\]

(4)

Similar formulas can be obtained for the Ward identities for multi-point correlation functions (see Sect. 4). According to BPZ [3], relations (2)—(3) state, at the level of correlation functions, that puncture operators are primary fields and \( T(w), \bar{T}(\bar{w}) \) are generating functions of the holomorphic and anti-holomorphic Virasoro algebras, that mutually commute in virtue of (4).

Since we have defined expectation values through functional integrals, we need to affirm the validity of conformal Ward identities (2)—(4). This will be done by using the perturbation theory.

We start with formula (2), which, according to [3], encodes important information about modular geometry—a Kähler geometry of the modular space, in our case—moduli space \( \mathcal{M}_n \) of Riemann surfaces of genus 0 with \( n > 3 \) punctures. Validity of (2) at the tree level is equivalent to the relation between accessory parameters of the Fuchsian uniformization and the classical Liouville action, conjectured in [2] and proved in [5] (see [7] for review). In the one-loop approximation, validity of (2) yields an explicit formula for the first derivative of the Selberg zeta function (evaluated at the special point \( s = 2 \)) with respect to the moduli parameters proved in [8].

It is remarkable, that Ward identity (2) fits perfectly well into the general philosophy of Friedan-Shenker modular geometry [4], that interprets the expectation value \( \langle X \rangle \) as a Hermitian metric in a certain holomorphic line bundle over \( \mathcal{M}_n \), and quadratic differential \( \langle \langle T(w)X \rangle \rangle_0 \ dw^2 \)—as a \((1,0)\)-component of the canonical metric connection. Indeed,
according to [3, Section 2.6], quadratic differentials $-R(w, w_i)dw^2$, considered as a $(1, 0)$-forms on $\mathcal{M}_n$, are dual to Beltrami differentials, representing vector fields $\pi \partial w_i$, $i = 1, \ldots, n-3$, on $\mathcal{M}_n$. Thus, $\mathcal{L}(w)$ can be interpreted as $(1, 0)$-component $\partial \times \big(-1/\pi\big)$ of exterior differential $d = \partial + \bar{\partial}$ on $\mathcal{M}_n$. Therefore, formula (2) reads

$$<\langle T(w)X \rangle>_0 dw^2 = -\frac{1}{\pi} \partial \log <X >,$$

which should be compared with [4, formula (6)].

Next, consider connected two-point correlation function $<\langle T(u)T(v)X \rangle>$. At the tree level, we get

$$<\langle T(u)T(v)X \rangle>_{\text{tree}} = \frac{2\pi}{\hbar} (\partial^2_u - (\phi_{cl})_u \partial_u)(\partial^2_v - (\phi_{cl})_v \partial_v)G(u, v).$$

Expression (3) has $(6/\hbar)(u-v)^{-4}$ as leading singularity at the diagonal $u = v$, affirming that $c_{cl} = 12/\hbar$. Substituting (3) into the Ward identity (3) and using $<\langle T(v)X \rangle>_{\text{tree}} = T_{cl}(u)$, we see that validity of (3) at the tree level is equivalent to the relation between derivatives of accessory parameters and Eichler integrals (see, e.g., [3, Ch. 5]) proved in [3]. At the one-loop level, we get $2\pi^2 G^2_{uv}(u, v)$ as the leading singular term in $<\langle T(u)T(v)X \rangle>$, which produces the quantum correction $c_{\text{loop}} = 1$ to the central charge of the Virasoro algebra. Inspection of higher loops shows that they do not contain singular terms of order $(u - v)^{-4}$.

Thus, $c_{\text{Liouv}} = 1 + 12/\hbar$, as was calculated in [3]. Moreover, validity of the Ward identity (3) in all loops is equivalent to remarkable (and rather complicated!) relations between Green’s functions and their derivatives with respect to the moduli parameters, which can be verified directly. This provides a “dynamical proof” of the Virasoro algebra symmetry of the Liouville theory.

It is also instructive to examine the two-point correlation function $<\langle T(u)\bar{T}(\bar{v})X \rangle>$. Corresponding Ward identity (4) can be rewritten in the form

$$<\langle T(u)\bar{T}(\bar{v})X \rangle> du^2 d\bar{v}^2 = \frac{1}{\pi^2} \partial \bar{\partial} \log <X >,$$

which allows to interpret $<\langle T(u)\bar{T}(\bar{v})X \rangle>$ as a $(1, 1)$-component of the curvature form for the connection in the holomorphic line bundle over $\mathcal{M}_n$, associated with Hermitian metric $<X>$ (cf. [3, formula (15)]). At the tree level

$$<\langle T(u)\bar{T}(\bar{v})X \rangle>_{\text{tree}} = \frac{2\pi}{\hbar} (\partial^2_u - (\phi_{cl})_u \partial_u)(\partial^2_v - (\phi_{cl})_v \partial_v)G(u, v).$$

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This expression can be shown to coincide with Hermitian form of the Weil-Petersson metric on moduli space \( \mathcal{M}_n \). Since \( \log < X >_{\text{tree}} = - (1/2\pi h) S_{\text{cl}} \), we see that at the tree level, the Ward identity (7) yields the classical Liouville action as a (local) Kähler potential of the Weil-Petersson metric on \( \mathcal{M}_n \)! This fact was proved in [5]. Similarly, validity of (7) in the one-loop approximation is equivalent to the local index theorem for the families of punctured Riemann surfaces [8].

Examined cases show that conformal Ward identities encode important information about Kähler geometry of moduli spaces. Multi-point correlation functions provide further insight, affirming the profound role of Ward identities in modular geometry. In this respect we derive perturbative expansion for the generating functional and present the universal Ward identities it satisfies.

Following Schwinger [10], we consider a bounded Beltrami differential \( \mu d\bar{w}/dw \) on Riemann surface \( X \) as an external source and define the generating functional for normalized multi-point correlation functions of the stress-energy tensor by the following expression

\[
Z(\mu, \bar{\mu}; X) = \frac{Z(\mu, \bar{\mu}; X)}{< X >}, \tag{9}
\]

and

\[
Z(\mu, \bar{\mu}; X) = \int_{c(X)} D\phi \exp\{- \frac{1}{2\pi h} S(\phi) + \text{v.p.} \int_X (T(\phi)\mu + \bar{T}(\phi)\bar{\mu})d^2w\}. \tag{10}
\]

Here, since Riemann surface \( X \) has a global coordinate \( w \), we can consider \((0,2)\)-component of the stress-energy tensor \(((0,2)\)-component) as quadratic differential \( Tdw^2 \) (\( \bar{T}d\bar{w}^2 \)), so that \( T\mu dw \wedge d\bar{w} \) (\( \bar{T}\bar{\mu}d\bar{w} \wedge dw \)) is a \((1,1)\)-form on \( X \). In general case, one should choose a projective connection \( T_0 \) and use quadratic differential \( (T - T_0) dw^2 \) (cf. the difference \( << T(w)X >> - T_s(w) \) in (2)). The integral in (10) is the principal value integral in virtue of the second order poles of \( T \) and \( \bar{T} \) at the punctures.

Generating functional for the connected multi-point correlation functions is defined as

\[
\frac{1}{h} \mathcal{W}(\mu, \bar{\mu}; X) = \log Z(\mu, \bar{\mu}; X), \tag{11}
\]

so that

\[
h << \prod_{i=1}^k T(u_i) \prod_{j=1}^l \bar{T}(\bar{v}_j)X >> = \frac{\delta^{k+l} \mathcal{W}(\mu, \bar{\mu}; X)}{\delta \mu(u_1) \cdots \delta \mu(u_k) \delta \bar{\mu}(\bar{v}_1) \cdots \delta \bar{\mu}(\bar{v}_l)}|_{\mu = \bar{\mu} = 0}. \]
Expanding around the classical solution \( \phi = \phi_{cl} \), we derive the perturbation expansion for the functional integral \( \bar{\Omega} \). The final result reads

\[
Z(\mu, \bar{\mu}; X) = \exp\{\text{v.p.} \int_X (T_{cl} \mu + \bar{T}_{cl} \bar{\mu}) d^2 w - \frac{1}{2\pi \hbar} S_{\text{int}}(\frac{\delta}{\delta \chi})\} \times \exp\{\frac{\pi}{\hbar} \int_X \chi G(1 - 2\pi Ge^{-\phi_{cl}} \partial_w \mu \partial_w - 2\pi Ge^{-\phi_{cl}} \partial_{\bar{w}} \bar{\mu} \partial_{\bar{w}})^{-1}(\chi) d^2 w\} |_{\chi = f + \bar{f}} \times \text{Det}(1 - 2\pi Ge^{-\phi_{cl}} \partial_w \mu \partial_w - 2\pi Ge^{-\phi_{cl}} \partial_{\bar{w}} \bar{\mu} \partial_{\bar{w}})^{-1/2}.
\]

where \( G = (2\Delta + 1)^{-1}, f = e^{-\phi_{cl}(\mu_{ww} + ((\phi_{cl})_w)_w)} \) is a globally defined function on \( X \), and

\[
S_{\text{int}}(\psi) = \sum_{k=3}^{\infty} \frac{1}{k!} \int_X \psi^k e^{\phi_{cl}} d^2 w.
\]

In order to obtain perturbation expansion for the generating functional \( \mathcal{W} \), one should use the formula

\[
\log \text{Det}(1 - A) = \sum_{k=1}^{\infty} \frac{1}{k} \text{Tr} A^k,
\]

and the reparametrization invariant regularization procedure. Previously considered examples are special cases of this scheme.

Conformal Ward identities \( (2) - (4) \) (or, more precisely, corresponding operator product expansions) are equivalent to the universal Ward identities for the generating functional \( \mathcal{W} \). They formally coincide with the Ward identity in the light-cone gauge, derived by Polyakov \( \text{[11]} \) (see also \( \text{[4]} \) formulas \( (10)-(12) \) and \( \text{[12]} \) formula \( (2.9) \)). Namely, we have

\[
(\partial_{\bar{w}} + \pi\mu \partial_w + 2\pi \mu_w) \frac{\delta \mathcal{W}}{\delta \mu(w)}(\mu, \bar{\mu}; X) = -\frac{\pi \hbar C}{12} \partial_{\bar{w}}^3 \mu + h \partial_{\bar{w}} T_{s}(w) + \partial_{\bar{w}} \mathcal{L}(w) \{\mathcal{W}(\mu, \bar{\mu}; X) + \hbar \log <X>\},
\]

and

\[
(\partial_w + \pi \bar{\mu} \partial_{\bar{w}} + 2\pi \bar{\mu}_w) \frac{\delta \mathcal{W}}{\delta \bar{\mu}(w)}(\mu, \bar{\mu}; X) = -\frac{\pi \hbar C}{12} \partial_w^3 \bar{\mu} + h \partial_w \bar{T}_{s}(\bar{w}) + \partial_w \bar{\mathcal{L}}(\bar{w}) \{\mathcal{W}(\mu, \bar{\mu}; X) + \hbar \log <X>\}.
\]

Using the formula

\[
\partial_u R(u, v) = \pi \delta(u - v),
\]

where \( u, v \neq 0, 1 \), one immediately gets \( (2) - (4) \) from the universal Ward identities. We anticipate their fundamental role in the modular geometry, which will be discussed elsewhere.
We finally note, that our results can be generalized for compact Riemann surfaces of genus $> 1$. In this case, in order to formulate Ward identities and to define the generating functional for correlation functions, one needs to choose a projective connection $T_0$ and to consider the difference $T - T_0$ (cf. [13]).

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