Yang-Mills Instanton Sheaves with Higher Topological Charges

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Abstract

We explicitly construct $SL(2, C)$ Yang-Mills three and four instanton sheaves on $CP^3$. These results extend the previous construction of Yang-Mills instanton sheaves with topological charge two [18].
I. INTRODUCTION

One of the most important developments for the interplay of quantum field theory and algebraic geometry in 1970s was the discovery of Yang-Mills (YM) instantons. Historically, the first BPST $SU(2)$ 1-instanton solution with 5 moduli parameters was discovered in 1975. Soon later the CFTW $k$-instanton solutions with $5k$ moduli parameters were constructed, and then the so-called JNR $k$-instanton solutions with $5k + 4$ ($5, 13$ for $k = 1, 2$) parameters was proposed based on the $4D$ conformal symmetry group of YM equation. This important issue was finally solved in 1978 by ADHM using method in algebraic geometry. The complete solutions of finite action (anti)self-dual YM (SDYM) equation was found to contain $8k - 3$ moduli parameters for each $k$th homotopy class. Applications of YM instantons on quantum field theory, in particular QCD, can be found at

The original ADHM theory used the monad construction combining with the Penrose-Ward transform to construct the most general instanton solutions by establishing an one to one correspondence between (anti)self-dual $SU(2)$-connections on $S^4$ and global holomorphic vector bundles of rank two on $CP^3$. The latter one was much easier to identify and the explicit closed forms of the complete $SU(2)$ instanton solutions for $k \leq 3$ had been worked out in. From physicist point of view, the YM instantons can also be seen through the
lens of string theory. They can be embedded in Heterotic stringy soliton solutions \[8, 9\]. They can also be described in terms of worldsheet supersymmetric sigma-models \[10\]. On the other hand, one simple way to see ADHM instantons from physicist point of view was to use the brane constructions in type II string theory \[11–13\].

In a recent paper \[14\], the quaternion calculation of $SU(2)$ ADHM construction was generalized to the biquaternion calculation with biconjugation operation, and a class of non-compact $SL(2, C)$ YM instanton solutions with $16k - 6$ parameters for each $k$th homotopy class was constructed. The number of parameters $16k - 6$ was first conjectured by Frenkel and Jardim in \[15\] and was proved recently in \[16\] from the mathematical point of view.

These new $SL(2, C)$ instanton solutions include as a subset the previous $SU(2, C) (M, N)$ instantons constructed in 1984 \[17\].

One important motivation to study $SL(2, C)$ instanton solutions has been to understand, in addition to the holomorphic vector bundles on $CP^3$ in the ADHM construction which has been well studied in the $SU(2)$ instantons, the instanton sheaves on the projective space. One key hint for the existence of $SL(2, C)$ instanton sheaves on $CP^3$ was the discovery \[14\] of singularities for $SL(2, C)$ instanton solutions on $S^4$ which can not be gauged away as in the case of $SU(2)$ instantons.

The first YM instanton sheaves was constructed recently only for $SL(2, C)$ 2-instanton solutions \[18\]. It is thus of interest to see whether there exist general YM $k$-instanton sheaves with higher topological charges. Since it was shown that \[18\] the $SL(2, C)$ extended $(M, N)$ $k$-instanton solutions with $10k$ parameters on $S^4$ correspond to the locally free sheaves or holomorphic vector bundles on $CP^3$, one needs to consider the non-diagonal $k$-instanton solutions with $k \geq 3$ in order to get YM instanton sheaves.

In this paper, we will explicitly construct $SL(2, C)$ Yang-Mills three and four instanton sheaves on $CP^3$ which extend our previous construction of two instanton sheaves. For the case of three instanton sheaves, we make use of a set of $SU(2)$ ADHM three instanton data to do the construction. For the case of four instanton sheaves, in addition to the known explicit $SU(2)$ $k$-instanton solutions with $k \leq 3$, there existed in the literature the so-called $SU(2)$ ADHM *symmetric* four instanton \[19\] solutions. The motivation to explicitly calculate these ADHM instanton solutions with higher topological charges was to construct the approximate minimum energy skyrmion fields in the Skyrme model of low energy hadronic physics.

Indeed, it was suggested by Atiyah and Manton \[20\] that low energy skyrmion fields of
charges $k$ can be approximated by computing holonomy of $k$-instanton on $R^4$ along lines parallel to the Euclidean time direction $[21]$. In contrast to the Skymion fields construction, for our purpose in this paper instead, our motivation is to use these $SU(2)$ ADHM instanton data to construct $SL(2, C)$ instanton sheaves on $CP^3$ with higher topological charges. Surprisingly, we will see that there do exist instanton sheaf structures on these symmetric four instanton solutions. It is thus an interesting issue to understand the relationship between YM symmetric instantons $[19, 22, 23]$ on $S^4$ and YM instanton sheaves on $CP^3$ constructed in this paper.

This paper is organized as following. In section II, we review the biquaternion construction of $SL(2, C)$ YM instanton which was developed recently by the present authors $[14]$. We also introduce the complex ADHM equations $[24]$ and the monad construction which will be used in the follow-up sections. In section III, we discuss in details the construction of a class of $SL(2, C)$ YM three instanton sheaves. The construction was extended to the $SL(2, C)$ YM four instanton sheaves by using a class of $SU(2)$ symmetric four instanton solutions in section IV. A brief conclusion was given in section V.

II. Biquaternions and $SL(2, C)$ Yang-Mills Instanton Construction

We first briefly review the $SL(2, C)$ YM theory. There are two linearly independent choices of $SL(2, C)$ group metric $[25]$

$$g^a = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, g^b = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$  \hspace{1cm} (2.1)

where $I$ is the $3 \times 3$ unit matrix. In general, one can choose

$$g = \cos \theta g^a + \sin \theta g^b$$  \hspace{1cm} (2.2)

where $\theta$ is a real constant. It can be shown that $SL(2, C)$ is isomorphic to $S^3 \times R^3$, and one can calculate its third homotopy group $[17]$

$$\pi_3[SL(2, C)] = \pi_3[S^3 \times R^3] = \pi_3(S^3) \cdot \pi_3(R^3) = Z \cdot I = Z$$  \hspace{1cm} (2.3)

where $I$ is the identity group, and $Z$ is the integer group.
Wu and Yang \cite{25} have shown that complex $SU(2)$ gauge fields are related to the real $SL(2, C)$ gauge fields. Starting from $SU(2)$ complex gauge field formalism, one can write down the $SL(2, C)$ YM equations. For the complex gauge field

$$G^a_\mu = A^a_\mu + iB^a_\mu,$$  

the corresponding complex field strength is defined as ($g = 1$)

$$F^a_{\mu\nu} \equiv H^a_{\mu\nu} + iM^a_{\mu\nu}, a, b, c = 1, 2, 3$$

where

$$H^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}(A^b_\mu A^c_\nu - B^b_\mu B^c_\nu),$$

$$M^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \epsilon^{abc}(A^b_\mu B^c_\nu - A^b_\mu B^c_\nu).$$

The $SL(2, C)$ YM equation can then be written as

$$\partial_\mu H^a_{\mu\nu} + \epsilon^{abc}(A^b_\mu H^c_{\mu\nu} - B^b_\mu M^c_{\mu\nu}) = 0,$$

$$\partial_\mu M^a_{\mu\nu} + \epsilon^{abc}(A^b_\mu M^c_{\mu\nu} - B^b_\mu H^c_{\mu\nu}) = 0,$$

and the $SL(2, C)$ SDYM equations are

$$H^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}H_{\alpha\beta},$$

$$M^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}M_{\alpha\beta}.$$  

YM equation for the choice $\theta = 0$ in this paper can be derived from the following Lagrangian

$$L = \frac{1}{4}(H^a_{\mu\nu}H^a_{\mu\nu} - M^a_{\mu\nu}M^a_{\mu\nu}).$$

We now proceed to review the construction of $SL(2, C)$ YM instantons \cite{14, 17}. We will use the convention $\mu = 0, 1, 2, 3$ and $\epsilon_{0123} = 1$ for 4D Euclidean space. In contrast to the quaternion in the $Sp(1) (= SU(2))$ ADHM construction, the authors of \cite{14} used \textit{biquaternion} to construct $SL(2, C)$ YM instantons. A quaternion $x$ can be written as

$$x = x_\mu e_\mu, \ x_\mu \in R, \ e_0 = 1, e_1 = i, e_2 = j, e_3 = k$$

where $e_1, e_2$ and $e_3$ anticommute and obey

$$e_i \cdot e_j = -e_j \cdot e_i = \epsilon_{ijk}e_k; \ i, j, k = 1, 2, 3,$$

$$e_i^2 = -1, e_2^2 = -1, e_3^2 = -1.$$  

5
The conjugate quaternion is defined to be

\[ x^\dagger = x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3 \]  \hspace{1cm} (2.13)

so that the norm square of a quaternion is

\[ |x|^2 = x^\dagger x = x_0^2 + x_1^2 + x_2^2 + x_3^2. \]  \hspace{1cm} (2.14)

Occasionally the unit quaternions can be expressed as Pauli matrices

\[ e_0 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_i \rightarrow -i\sigma_i \ ; \ i = 1, 2, 3. \]  \hspace{1cm} (2.15)

A biquaternion (or complex-quaternion) \( z \) can be written as

\[ z = z_\mu e_\mu, \quad z_\mu \in C \]  \hspace{1cm} (2.16)

or

\[ z = x + yi \]  \hspace{1cm} (2.17)

where \( x \) and \( y \) are quaternions and \( i = \sqrt{-1} \). In the recent construction of \( SL(2, C) \) YM instantons [14], the biconjugation [26] of \( z \) is defined to be

\[ z^{\ast} = z_\mu e_\mu^\dagger = z_0e_0 - z_1e_1 - z_2e_2 - z_3e_3 = x^\dagger + y^\dagger i, \]  \hspace{1cm} (2.18)

in contrast to the complex conjugation

\[ z^* = z_\mu^* e_\mu = z_0^* e_0 + z_1^* e_1 + z_2^* e_2 + z_3^* e_3 = x - yi. \]  \hspace{1cm} (2.19)

The norm square of a biquaternion is defined to be

\[ |z|^2_c = z^{\ast} z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2, \]  \hspace{1cm} (2.20)

which is a complex number in general and a subscript \( c \) is used in the norm.

We now briefly review how to extend the quaternion construction of ADHM \( SU(2) \) instantons to the \( SL(2, C) \) YM instantons. The first step was to introduce the \((k+1) \times k\) biquaternion matrix

\[ \Delta(x) = A + Dx, \]  \hspace{1cm} (2.21)

which satisfies the quadratic condition

\[ \Delta(x)^\ast \Delta(x) = f^{-1} = \text{symmetric, non-singular } k \times k \text{ matrix for } x \notin J. \]  \hspace{1cm} (2.22)
Note that for \( x \in J \), \( \det(\Delta(x) \Delta(x)) = 0 \). The set \( J \) is called singular locus or jumping lines in the mathematical literature. There are no jumping lines for the case of \( SU(2) \) instantons on \( S^4 \). On the other hand, in the \( SU(2) \) quaternion case, the symmetric condition on \( f^{-1} \) implies \( f^{-1} \) is real; while for the \( SL(2, C) \) biquaternion case, it implies \( f^{-1} \) is complex which means \( [\Delta(x) \Delta(x)]_{ij}^\mu = 0 \) for \( \mu = 1, 2, 3 \).

To construct the self-dual gauge fields, one introduces a \((k + 1) \times 1\) dimensional biquaternion vector \( v(x) \) satisfying the two conditions

\[
\begin{align*}
  v^\circ(x) \Delta(x) &= 0, \\
  v^\circ(x) v(x) &= 1.
\end{align*}
\]

Finally one can calculate the gauge fields

\[
G_\mu(x) = v^\circ(x) \partial_\mu v(x),
\]

which is a \( 1 \times 1 \) biquaternion.

In the canonical form of the construction, one can set

\[
D = \begin{bmatrix} 0_{1 \times k} \\ I_{k \times k} \end{bmatrix},
A = \begin{bmatrix} \lambda_{1 \times k} \\ -y_{k \times k} \end{bmatrix}
\]

where \( \lambda \) and \( y \) are biquaternion matrices with orders \( 1 \times k \) and \( k \times k \) respectively, and \( y \) is symmetric \( y = y^T \). One can show that in the canonical form the constraints for the moduli parameters become

\[
A^\circ_{ci} A_{cj} = 0, \ i \neq j, \ \text{and} \ y_{ij} = y_{ji}.
\]

The total number of moduli parameters for \( k \)-instanton is \( 16k - 6 \). Note that for the \( SU(2) \) instantons, \( \lambda \) and \( y \) are the usual quaternion matrices.

There was another approach to solve \( SL(2, C) \) YM instantons through the complex ADHM equations

\[
\begin{align*}
  \begin{bmatrix} B_{11}, B_{12} \end{bmatrix} + I_1 J_1 &= 0, \tag{2.27a} \\
  \begin{bmatrix} B_{21}, B_{22} \end{bmatrix} + I_2 J_2 &= 0, \tag{2.27b} \\
  [B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1 J_2 + I_2 J_1 &= 0 \tag{2.27c}
\end{align*}
\]

where \( B_{ij} \) are \( k \times k \) complex matrices and \( J_i \) are \( 2 \times k \) complex matrices. They are the complex ADHM data \( (B_{lm}, I_m, J_m) \) with \( l, m = 1, 2 \). For the case of \( SU(2) \) ADHM instantons, we
impose the conditions \[18\]

\[ I_1 = J_1^\dagger, I_2 = -I, J_1 = I_1^\dagger, J_2 = J, \]
\[ B_{11} = B_2^\dagger, B_{12} = B_1^\dagger, B_{21} = -B_1, B_{22} = B_2 \]  

(2.28a)

to recover the real ADHM equations

\[ \begin{bmatrix} B_1, B_2 \end{bmatrix} + IJ = 0, \]  

(2.29a)

\[ \begin{bmatrix} B_1, B_1^\dagger \end{bmatrix} + \begin{bmatrix} B_2, B_2^\dagger \end{bmatrix} + II^\dagger - J^\dagger J = 0. \]  

(2.29b)

Indeed one can identify the ADHM data \((B_{lm}, I_m, J_m)\) from the moduli parameters in Eq.(2.25) \[18\]. The first step is to use Eq.(2.15) to transform the biquaternion \(A\) in Eq.(2.25) into the explicit matrix representation (EMR). For the \(k\)-instanton case, the EMR of the \((k + 1) \times k\) biquaternion matrix \(A\) in Eq.(2.25) can be written as a \(2(k + 1) \times 2k\) complex matrix \(A_E\). The next step is to use the following rearrangement rule for an element \(z_{ij}\) in \(A_E\) \[18\]

\[
\begin{align*}
&z_{2n-1,2m-1} \rightarrow z_{n,m}, \\
&z_{2n-1,2m} \rightarrow z_{n,k+m}, \\
&z_{2n,2m-1} \rightarrow z_{k+n,m}, \\
&z_{2n,2m} \rightarrow z_{k+n,k+m}.
\end{align*}
\]  

(2.30)

to obtain \(A_E^r\). Finally one can then do the following identification for the complex ADHM data

\[
A_E^r = \begin{bmatrix}
J_1 & J_2 \\
B_{11} & B_{21} \\
B_{12} & B_{22}
\end{bmatrix}.
\]  

(2.31)

Similar procedure can be perform on \(A^\odot\). With these identifications, one can show that the \(SL(2,C)\) YM instantons constructed previously in \[18\] are solutions of the complex ADHM equations in Eq.(2.27a) to Eq.(2.27c).

Finally, in the monad construction of holomorphic vector bundles on the projective space, one introduces the \(\alpha\) and \(\beta\) matrices as functions of homogeneous coordinates \([x : y : z : w]\)
of $CP^3$ and define
\[
\alpha = \begin{bmatrix} zB_{11} + wB_{21} + x \\ zB_{12} + wB_{22} + y \\ zJ_1 + wJ_2 \end{bmatrix}, \tag{2.32a}
\]
\[
\beta = \begin{bmatrix} -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2 \end{bmatrix}. \tag{2.32b}
\]

It can be shown that the condition
\[
\beta\alpha = 0 \tag{2.33}
\]
is satisfied if and only if the complex ADHM equations in Eq.(2.27a) to Eq.(2.27c) holds.

In this construction, Eq.(2.33) implies $\text{Im} \alpha$ is a subspace of $\text{Ker} \beta$ which allows one to consider the quotient vector space $\text{Ker} \beta / \text{Im} \alpha$ at each point of $CP^3$. For the $SU(2)$ ADHM instantons, the map $\beta$ is surjective and the map $\alpha$ is injective and $\dim(\text{Ker} \beta / \text{Im} \alpha) = k + 2 - k = 2$ on every points of $CP^3$, thus one can use the holomorphic vector bundles of rank 2 to describe $SU(2)$ instantons.

For the case of $SL(2, C)$ instantons, we will show that for some general $k$-instantons $\alpha$ may not be injective at some points of $CP^3$ for some ADHM data, so the dimension of the vector space $(\text{Ker} \beta / \text{Im} \alpha)$ may vary from point to point on $CP^3$, and one is led to use sheaf description for these $SL(2, C)$ YM instantons or "instanton sheaves" on $CP^3$.

In a recent publication, some $SL(2, C)$ 2-instanton sheaves were constructed in [18]. In the following sections, we will calculate a class of Yang-Mills $SL(2, C)$ $k$-instanton sheaves with $k = 3$ and $k = 4$.

\section*{III. THE $SL(2, C)$ THREE INSTANTON SHEAVES}

There existed complete construction of the $SU(2)$ $k$-instanton solutions for $k \leq 3$ in the literature. For the higher instanton solutions, there were the so-called $SU(2)$ ADHM symmetric 4-instanton [19] and 7-instanton [23] solutions. The motivation to explicitly calculate these ADHM symmetric higher instanton solutions was to construct the approximate minimum energy skyrmion fields in the Skyrme model of low energy hadronic physics. Our motivation in this paper is to use these $SU(2)$ ADHM data to construct $SL(2, C)$ instanton sheaves with higher topological charges.
We begin with a $SU(2)$ 3-instanton solution proposed in [22]

$$A = \begin{bmatrix}
e_1 & e_2 & e_3 \\
0 & e_3 & e_2 \\
e_3 & 0 & e_1 \\
e_2 & e_1 & 0
d\end{bmatrix},$$

(3.34)

which was tetrahedrally symmetric and was used to calculate the approximate 3-Skyrme field. There was a two parameter generalization of the ADHM data $A$ in Eq.(3.34). For our purpose here, we drop out the symmetric constraints and propose

$$A_3 = \begin{bmatrix}
ae_1 & be_2 & ce_3 \\
0 & ce_3 & be_2 \\
ce_3 & 0 & ae_1 \\
be_2 & ae_1 & 0
d\end{bmatrix}.$$

(3.35)

One can easily calculate

$$A_3^\otimes A_3 = \begin{bmatrix}
(a^2 + b^2 + c^2) e_0 & 0 & 0 \\
0 & (a^2 + b^2 + c^2) e_0 & 0 \\
0 & 0 & (a^2 + b^2 + c^2) e_0 
d\end{bmatrix}.$$

(3.36)

Thus $A_3$ proposed in Eq.(3.35) satisfies the constraints in Eq.(2.26) for arbitrary $a, b$ and $c \in C$ for the $SL(2, C)$ case, and represents a class of ADHM data. Note that for $a, b$ and $c \in R$, $A_3$ represents a class of $SU(2)$ 3-instanton solutions.

We are now ready to check whether there exists instanton sheaf structure for these 3-instanton solutions. We first calculate $A_{3E}^\otimes$, the EMR of $A_3$, and then do the rearrangement rule to obtain $A_{3E}^\otimes$ [18] and finally identify the corresponding ADHM data $(B_{lm}, I_m, J_m)$ to
be

\[
J_1 = \begin{bmatrix}
0 & 0 & -ic \\
-ia & b & 0
\end{bmatrix}, \quad J_2 = \begin{bmatrix}
-ia & -b & 0 \\
0 & 0 & ic
\end{bmatrix}, \quad (3.37)
\]

\[
B_{11} = \begin{bmatrix}
0 & -ic & 0 \\
-ia & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B_{21} = \begin{bmatrix}
0 & 0 & -b \\
0 & 0 & -ia \\
-b & -ia & 0
\end{bmatrix}, \quad (3.38)
\]

\[
B_{12} = \begin{bmatrix}
0 & 0 & b \\
0 & -ia & 0 \\
b & -ia & 0
\end{bmatrix}, \quad B_{22} = \begin{bmatrix}
0 & ic & 0 \\
0 & 0 & 0
\end{bmatrix}. \quad (3.39)
\]

The next step is to check the costability conditions [15]. Since it was shown that [18] for the \(SL(2, C)\) 2-instanton solutions constructed in [14] by the biquaternion method, the stability conditions and the costability conditions were equivalent, and the proof of this equivalence can be easily generalized to the \(k\)-instanton cases, so we will again check only the costability conditions in this paper.

We want to check whether there exists a common eigenvector \(v\) in the costable condition [15]

\[
(zB_{11} + wB_{21}) v = -x v, \quad (3.40a)
\]

\[
(zB_{12} + wB_{22}) v = -y v, \quad (3.40b)
\]

\[
(zJ_1 + wJ_2) v = 0 \quad (3.40c)
\]

where \([x : y : z : w]\) are homogeneous coordinates of \(CP^3\). If the common eigenvector \(v\) exists, then the dimension of the quotient space \((\text{Ker } \beta/ \text{Im } \alpha)\) in the monad construction will not be a constant since the map \(\alpha\) fail to be injective. In this case, the holomorphic vector bundle description on \(CP^3\) break down and one is led to use sheaf to describe instanton on \(CP^3\).

We first use Eq. (3.40c) to obtain

\[
v \sim \begin{bmatrix}
-ibc (w^2 - z^2) \\
-ac (z^2 + w^2) \\
-2iabzw
\end{bmatrix} . \quad (3.41)
\]
On the other hand, Eq. (3.40a) and Eq. (3.40b) give

\[
\begin{bmatrix}
x & -icz & -bw \\
-icz & x & -iaw \\
-bw & -iaw & x
\end{bmatrix} v = 0,
\]

(3.42)

\[
\begin{bmatrix}
y & icw & bz \\
icw & y & -iaz \\
bz & -iaz & y
\end{bmatrix} v = 0.
\]

(3.43)

For simplicity, let's choose

\[z \in \mathbb{C}, w = 0\]

(3.44)
on $CP^3$, then $v$ becomes

\[v \sim \begin{bmatrix} ib \\ -a \\ 0 \end{bmatrix}.
\]

(3.45)

The two characteristic equations corresponding to Eq. (3.42) and Eq. (3.43) become

\[x \left( x^2 + c^2 z^2 \right) = 0,\]

(3.46)

\[y \left[ y^2 + (a^2 - b^2) z^2 \right] = 0,\]

(3.47)

which give the solutions

\[x = 0 \text{ or } \pm icz,\]

(3.48)

\[y = 0 \text{ or } \pm i\sqrt{a^2 - b^2} z.\]

(3.49)

There are four cases for the choices of $x$ and $y$ above. For the first case we choose $x = 0$ and $y = 0$, then with Eq. (3.45), Eq. (3.42) and Eq. (3.43) become

\[
\begin{bmatrix} 0 & -icz & 0 \\
-icz & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix} ib \\ -a \\ 0 \end{bmatrix} = 0,
\]

(3.50)

\[
\begin{bmatrix} 0 & 0 & bz \\
0 & 0 & -iaz \\
bz & -iaz & 0 \\
\end{bmatrix} \begin{bmatrix} ib \\ -a \\ 0 \end{bmatrix} = 0,
\]

(3.51)
which give
\[ c = 0, a = \pm ib, b \neq 0. \] (3.52)

For the second case, we choose \( x = \pm icz \) and \( y = 0 \). In this case Eq.(3.42) and Eq.(3.43) become
\[
\begin{bmatrix}
\pm icz & -icz & 0 \\
-icz & \pm icz & 0 \\
0 & 0 & \pm icz
\end{bmatrix}
\begin{bmatrix}
ib \\
-a \\
0
\end{bmatrix} = 0,
\] (3.53)

\[
\begin{bmatrix}
0 & 0 & bz \\
0 & 0 & -iaz \\
bz & -iaz & 0
\end{bmatrix}
\begin{bmatrix}
ib \\
-a \\
0
\end{bmatrix} = 0,
\] (3.54)

which give
\[ c \in \mathbb{C}, a = \pm ib, b \neq 0. \] (3.55)

We conclude that for the ADHM data given at Eq.(3.55) and at the point \([x : y : z : w] = [\pm ic : 0 : 1 : 0]\) on \(CP^3\), the map \(\alpha\) fails to be injective, thus one is led to use sheaf description for these instanton sheaves on \(CP^3\). Note that for the \(c = 0\) case, Eq.(3.55) reduces to Eq.(3.52).

It is important to note that for the case of \(SU(2)\) 3-instanton, \(a\) and \(b\) are both real numbers which are inconsistent with Eq.(3.55). So the corresponding \(SU(2)\) 3-instanton solutions are locally free. This is consistent with the known vector bundle description of \(SU(2)\) 3-instanton on \(CP^3\).

For the third case, we choose \( x = 0 \) and \( y = \pm i\sqrt{a^2 - b^2}z \). In this case Eq.(3.42) and Eq.(3.43) become
\[
\begin{bmatrix}
0 & -icz & 0 \\
-icz & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
ib \\
-a \\
0
\end{bmatrix} = 0,
\] (3.56)

\[
\begin{bmatrix}
\pm i\sqrt{a^2 - b^2}z & 0 & bz \\
0 & \pm i\sqrt{a^2 - b^2}z & -iaz \\
0 & -iaz & \pm i\sqrt{a^2 - b^2}z
\end{bmatrix}
\begin{bmatrix}
ib \\
-a \\
0
\end{bmatrix} = 0,
\] (3.57)

which give
\[ c = 0, a = \pm ib, b \neq 0. \] (3.58)
For the fourth case, we choose \( x = \pm icz \) and \( y = \pm i\sqrt{a^2 - b^2} z \). In this case Eq.\((3.42)\) and Eq.\((3.43)\) become

\[
\begin{pmatrix}
\pm icz & -icz & 0 \\
-icz & \pm icz & 0 \\
0 & 0 & \pm icz \\
\end{pmatrix}
\begin{pmatrix}
ib \\
-a \\
0 \\
\end{pmatrix} = 0, \tag{3.59}
\]

\[
\begin{pmatrix}
\pm i\sqrt{a^2 - b^2} z & 0 & bz \\
0 & \pm i\sqrt{a^2 - b^2} z & -iaz \\
0 & 0 & \pm i\sqrt{a^2 - b^2} z \\
\end{pmatrix}
\begin{pmatrix}
ib \\
-a \\
0 \\
\end{pmatrix} = 0, \tag{3.60}
\]

which give

\[
\begin{aligned}
c \in \mathbb{C}, \\
a = \pm ib, \\
b \neq 0.
\end{aligned} \tag{3.61}
\]

We conclude that for the ADHM data given at Eq.\((3.61)\) and at the point \([ x : y : z : w ] = [ \pm ic : \mp\sqrt{2b} : 1 : 0 ]\) on \( CP^3 \), the map \( \alpha \) fails to be injective, thus one is led to use sheaf description for these instanton sheaves on \( CP^3 \). Note that for the \( c = 0 \) case, Eq.\((3.61)\) reduces to Eq.\((3.58)\). Again \( SU(2) \) instanton sheaf is not allowed for this case.

Note that in the above 3-instanton calculation, we have assumed \( w = 0 \) on \( CP^3 \) in Eq.\((3.44)\). We expect that other choices of points on \( CP^3 \) will give more 3-instanton sheaf structure for some other ADHM data. We conclude the discussion of 3-instanton sheaves. In the next section, we turn to discuss the 4-instanton sheaves.

### IV. THE FOUR INSTANTON SHEAVES

As has been well known, there are no complete \( SU(2) \) 4-instanton explicit solutions in the literature. In [19], a two parameter family of \( SU(2) \) 4-instanton ADHM data was constructed [19]

\[
A_4 = \begin{bmatrix}
\sqrt{2be_0} & \sqrt{2be_1} & \sqrt{2be_2} & \sqrt{2be_3} \\
a(e_1 + e_2 + e_3) & -b(e_2 + e_3) & -b(e_3 + e_1) & -b(e_1 + e_2) \\
-b(e_2 + e_3) & a(e_1 - e_2 - e_3) & b(e_2 - e_1) & b(e_1 - e_3) \\
-b(e_3 + e_1) & b(e_2 - e_1) & a(-e_1 + e_2 - e_3) & b(e_3 - e_2) \\
-b(e_1 + e_2) & b(e_1 - e_3) & b(e_3 - e_2) & a(-e_1 - e_2 + e_3)
\end{bmatrix}, \tag{4.62}
\]
which was used to calculate the approximate four Skyrme field. For this case, it was shown that $A^*_4 A_4 = (8b^2 + 3a^2)I_4$, thus $A_4$ in Eq. (4.62) does represent a class of $SU(2)$ ADHM 4-instanton data [19].

For our purpose here, we want to study whether the corresponding $SL(2, C)$ ADHM 4-instanton data contain the structure of instanton sheaves or not. To do the calculation, we first extend the parameters $a$ and $b$ in Eq. (4.62) to complex numbers, and replace the quaternion calculation by biquaternion calculation with biconjugation operation [14]. One easily gets $A^*_4 A_4 = (8b^2 + 3a^2)I_4$.

We then calculate $A_{4E}$, the EMR of $A_4$, and do the rearrangement rule to obtain $A^r_{4E}$ [18] and finally identify the corresponding ADHM data $(B_{lm}, I_m, J_m)$ to be

\[
J_1 = \begin{bmatrix} \sqrt{2} & 0 & 0 & -i\sqrt{2} \\ 0 & -i\sqrt{2} & \sqrt{2} & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & -i\sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 & i\sqrt{2} \end{bmatrix}, \quad (4.63)
\]

\[
B_{11} = \begin{bmatrix} -ia & i & 0 \\ i & ia & 0 \\ i & 0 & ia \\ 0 & i & -i \end{bmatrix}, \quad B_{21} = \begin{bmatrix} (-1 - i)a & 1 & i & 1 + i \\ 1 & (1 - i)a & -1 + i & -i \\ i & -1 + i & (-1 + i)a & 1 \\ 1 + i & -i & 1 & (1 + i)a \end{bmatrix}, \quad (4.64)
\]

\[
B_{12} = \begin{bmatrix} (1 - i)a & -1 & i & -1 + i \\ -1 & (1 - i)a & 1 + i & -i \\ i & 1 + i & (1 + i)a & -1 \\ -1 + i & -i & -1 & (-1 + i)a \end{bmatrix}, \quad B_{22} = \begin{bmatrix} ia & -i & -i & 0 \\ -i & -ia & 0 & -i \\ -i & 0 & -ia & i \\ 0 & -i & -ia & ia \end{bmatrix} \quad (4.65)
\]

where we have put $b = 1$. The reason is as following. First we want to restrict the two parameter $SL(2, C)$ ADHM data to be on $[a : b] \in CP^1$ and simplify the calculation. Moreover, a general result in the mathematics literature [27] claims to the effect that the moduli space of unframed rank $2n$ instanton bundles over $CP^{2n+1}$ is an affine variety. This result suggests that it is not the case that for all $[a : b]$ of $CP^1$, the above ADHM data with parameters on $[a : b]$ (hence on $CP^1$) gives only bundle solutions without exceptions, simply because it is well known that an affine variety cannot contain any projective subvarieties of positive dimension.

Indeed, we shall show below that for certain values of $[a : b]$, the above ADHM data gives the sheaf (non-bundle) solutions. Our result will be consistent with the mathematics result above.
The next step is to check whether there exists a common eigenvector $v$ in the costable condition \[15\]. For simplicity, we choose $w = 0$ and then put $z = 1$ on $CP^3$ without loss of generality

$$w = 0, z = 1. \quad (4.66)$$

For these choices, Eq.(3.40c) gives two possible eigenvectors

$$v_1 = \begin{bmatrix} i \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -i \\ 1 \\ 0 \end{bmatrix}. \quad (4.67)$$

On the other hand, in order to have nontrivial $v$ solutions, Eq.(3.40a) and Eq.(3.40b) give the characteristic equations

$$x^4 + 2x^2a^2 + 4x^2 + a^4 + 4a^2 + 4 = 0, \quad (4.68)$$

$$y^4 + 4a^4 + 16a^2 + 12 = 0 \quad (4.69)$$

respectively. The solutions for $x$ and $y$ are

$$x_1 = \sqrt{-a^2 - 2}, x_2 = -\sqrt{-a^2 - 2} \quad (4.70)$$

and

$$y_1 = (-4a^4 - 16a^2 - 12)^{1/4}, y_2 = i \left(-4a^4 - 16a^2 - 12\right)^{1/4}, \quad (4.71)$$

$$y_3 = -(-4a^4 - 16a^2 - 12)^{1/4}, y_4 = -i \left(-4a^4 - 16a^2 - 12\right)^{1/4}.\quad$$

We first choose $x = x_1 = \sqrt{-a^2 - 2}$, then Eq.(3.40a) becomes

$$\left(B_{11} + \sqrt{-a^2 - 2}\right) v = 0, \quad (4.72)$$

which gives two eigenvector solutions

$$v_3 = \begin{bmatrix} 2i \\ ia - \sqrt{-a^2 - 2} \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ a + i\sqrt{-a^2 - 2} \\ 0 \\ 1 \end{bmatrix}. \quad (4.73)$$
Since we need to have a common eigenvector to insure the instanton sheaf structure, we impose the condition that the linear system

\[ c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0, \quad (4.74) \]

or equivalently

\[
\begin{bmatrix}
i & 0 & \frac{2i}{i a - \sqrt{a^2 - 2}} & 1 \\
0 & -i & a + i \sqrt{-a^2 - 2} & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix} = 0,
\]

(4.75)

contains nontrivial \((c_1, c_2, c_3, c_4)\) solutions. Surprisingly the determinant of the coefficient matrix in Eq. (4.75) vanishes for any \(a\) ! Thus one can easily calculate the solution

\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix} = \begin{bmatrix}
-1 \\
\frac{-i}{2} (1 + i) \left(a + i \sqrt{-a^2 - 2}\right) \\
\frac{i}{2} (1 + i) \left(a + i \sqrt{-a^2 - 2}\right) \\
1
\end{bmatrix},
\]

(4.76)

which gives the common eigenvector

\[
v = c_1 v_1 + c_2 v_2 = \begin{bmatrix}
-i \\
\left(\frac{-i}{2}\right) (1 + i) \left(a + i \sqrt{-a^2 - 2}\right) \\
\left(\frac{i}{2}\right) (1 + i) \left(a + i \sqrt{-a^2 - 2}\right) \\
-1
\end{bmatrix}.
\]

(4.77)

Finally we need to check whether the common eigenvector \(v\) in Eq. (4.77) satisfies Eq. (3.40b).

For the first choice of \(y = y_1 = (-4a^4 - 16a^2 - 12)^{\frac{1}{2}}\) in Eq. (4.71), Eq. (3.40b) gives

\[
\left[B_{12} + (-4a^4 - 16a^2 - 12)^{\frac{1}{2}} I_1\right] v = 0,
\]

(4.78)

or explicitly

\[
-i \left[ (1 - i) a + \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} \right] + (1 + i) \left(a + i \sqrt{-a^2 - 2}\right) + 1 - i = 0,
\]

\[
2i \left[ (1 + i) \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} \right] \left(a + i \sqrt{-a^2 - 2}\right) + \left(a + i \sqrt{-a^2 - 2}\right) = 0,
\]

\[
2 - i \left(a + i \sqrt{-a^2 - 2}\right) + \left(\frac{1}{2} - \frac{i}{2}\right) \left[ (1 + i) a + \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} \right] \left(a + i \sqrt{-a^2 - 2}\right) = 0,
\]

\[
1 + i + (1 + i) \left(a + i \sqrt{-a^2 - 2}\right) + (1 - i) a - \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} = 0.
\]
The roots of these equations are

\[ \sqrt{3i}, -\sqrt{3i}; \]
\[ \sqrt{3i}, -\sqrt{3i}, \frac{1}{4}\sqrt{-26 + 2\sqrt{7}i}, -\frac{1}{4}\sqrt{-26 + 2\sqrt{7}i}; \]
\[ \sqrt{3i}, -\sqrt{3i}, \frac{1}{4}\sqrt{-26 - 2\sqrt{7}i}, \frac{1}{4}\sqrt{-26 - 2\sqrt{7}i}; \]
\[ \sqrt{3i}, -\sqrt{3i}; \]

which again surprisingly contain common roots

\[ a = \pm i\sqrt{3}. \]  

We conclude that for the case of choosing

\[ a = i\sqrt{3}, \]

the ADHM data are \( J_1 \) and \( J_2 \) in Eq.(4.63) and

\[ B_{11} = \begin{bmatrix} \sqrt{3} & i & i & 0 \\ i & -\sqrt{3} & 0 & i \\ i & 0 & -\sqrt{3} & -i \\ 0 & i & -i & \sqrt{3} \end{bmatrix}, B_{12} = \begin{bmatrix} \sqrt{3}(1+i) & -1 & i & -1+i \\ -1 & \sqrt{3}(1-i) & 1+i & -i \\ i & 1+i & \sqrt{3}(-1+i) & 1 \\ -1+i & -i & -1 & \sqrt{3}(-1-i) \end{bmatrix}, \]

\[ B_{21} = \begin{bmatrix} \sqrt{3}(1-i) & 1 & i & 1+i \\ 1 & \sqrt{3}(1+i) & -1+i & -i \\ i & -1+i & \sqrt{3}(-1-i) & 1 \\ 1+i & -i & 1 & \sqrt{3}(-1+i) \end{bmatrix}, B_{22} = \begin{bmatrix} -\sqrt{3} & -i & -i & 0 \\ -i & \sqrt{3} & 0 & -i \\ -i & 0 & \sqrt{3} & i \\ 0 & -i & i & -\sqrt{3} \end{bmatrix}. \]

There exists a common eigenvector of Eq.(3.40a), Eq.(3.40b) and Eq.(3.40c)

\[ v = \begin{bmatrix} \frac{1}{2}(1-i)(\sqrt{3}+1) \\ \frac{i}{2}(1-i)(\sqrt{3}+1) \\ i \\ -1 \end{bmatrix}. \]

The map \( \alpha \) fails to be injective at

\[ [z : w : x : y] = [1 : 0 : 1 : 0] \]
on $CP^3$, thus one is led to use sheaf description for this 4-instanton sheaves on $CP^3$. Moreover, since $a$ is not a real number, again $SU(2)$ instanton sheaf is not allowed for this case. This is consistent with the common wisdom.

Similarly, for the choice of
\[ a = -i\sqrt{3}, \]
the ADHM data are $J_1$ and $J_2$ in Eq. (4.63) and
\[
B_{11} = \begin{bmatrix}
-\sqrt{3} & i & 0 \\
i & \sqrt{3} & i \\
0 & -i & -\sqrt{3}
\end{bmatrix},
B_{12} = \begin{bmatrix}
\sqrt{3}(-1-i) & -1 & i & -1+i \\
i & \sqrt{3}(-1+i) & 1+i & -i \\
i & 1+i & \sqrt{3}(1-i) & -1
\end{bmatrix},
B_{21} = \begin{bmatrix}
\sqrt{3}(-1+i) & 1 & i & 1+i \\
i & \sqrt{3}(-1-i) & -1+i & -i \\
i & -1+i & \sqrt{3}(1+i) & 1 \\
1+i & -i & 1 & \sqrt{3}(1-i)
\end{bmatrix},
B_{22} = \begin{bmatrix}
\sqrt{3} & -i & -i & 0 \\
i & -\sqrt{3} & 0 & -i \\
i & 0 & -\sqrt{3} & i \\
0 & -i & i & \sqrt{3}
\end{bmatrix}
\]
and the common eigenvector is
\[
v = \begin{bmatrix}
-i \\
\frac{1}{2}(1+i)(\sqrt{3}-1) \\
\frac{1}{2}(-1+i)(\sqrt{3}-1) \\
-1
\end{bmatrix}.
\]
The map $\alpha$ fails to be injective at the same point in Eq. (4.84) on $CP^3$, and one ends up with another instanton sheaf case for these ADHM data.

For the second choice of $y = y_2 = i(-4a^4 - 16a^2 - 12)^{\frac{1}{2}}$ in Eq. (4.71), Eq. (3.40b) gives
\[
\left[ B_{12} + i(-4a^4 - 16a^2 - 12)^{\frac{1}{2}} I_4 \right] v = 0
\]
or explicitly
\[
-i \left[ (1-i) a + i \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} \right] + (1+i) \left( a + i\sqrt{-a^2 - 2} \right) + 1 - i = 0,
\]
\[
y \left( a + i\sqrt{-a^2 - 2} \right) + \left( a + i\sqrt{-a^2 - 2} \right) = 0,
\]
\[
2 - i \left( a + i\sqrt{-a^2 - 2} \right) + \left( \frac{1}{2} - \frac{i}{2} \right) \left[ (1+i) a + i \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} \right] \left( a + i\sqrt{-a^2 - 2} \right) = 0,
\]
\[
1 + i + (-1+i) \left( a + i\sqrt{-a^2 - 2} \right) + (1 - i) a - i \left(-4a^4 - 16a^2 - 12\right)^{\frac{1}{2}} = 0.
\]
The roots of these equations are

$$\sqrt{3}i, -\sqrt{3}i;$$  \hspace{1cm} (4.89)

$$\sqrt{3}i, -\sqrt{3}i;$$

$$\sqrt{3}i, -\sqrt{3}i, \frac{1}{4} \sqrt{-26 - 2\sqrt{7}i}, \frac{1}{4} \sqrt{-26 - 2\sqrt{7}i};$$

$$\sqrt{3}i, -\sqrt{3}i, \frac{1}{4} \sqrt{-26 + 2\sqrt{7}i}, -\frac{1}{4} \sqrt{-26 + 2\sqrt{7}i};$$

which again contain common roots

$$a = \pm i\sqrt{3}. \hspace{1cm} (4.90)$$

For the third choice of \(y = y_3 = -(-4a^4 - 16a^2 - 12)^{\frac{1}{4}}\) in Eq.(4.71), Eq.(3.40b) gives

$$\left[ B_{12} - (-4a^4 - 16a^2 - 12)^{\frac{1}{4}} I_4 \right] v = 0, \hspace{1cm} (4.91)$$

or explicitly

$$-i \left[ (1 - i) a - (-4a^4 - 16a^2 - 12)^{\frac{1}{4}} \right] + (1 + i) \left( a + i\sqrt{-a^2 - 2} \right) + 1 - i = 0,$$

$$2 + \left( \frac{1}{2} - \frac{i}{2} \right) (1 + i) \left[ (1 - i) a - (-4a^4 - 16a^2 - 12)^{\frac{1}{4}} \right] \left( a + i\sqrt{-a^2 - 2} \right) + \left( a + i\sqrt{-a^2 - 2} \right) = 0,$$

$$1 + i + (-1 + i) \left( a + i\sqrt{-a^2 - 2} \right) + (1 - i) a + (-4a^4 - 16a^2 - 12)^{\frac{1}{4}} = 0.$$

The roots of these equations are

$$\sqrt{3}i, -\sqrt{3}i, \frac{1}{4} \sqrt{-26 - 2\sqrt{7}i}, \frac{1}{4} \sqrt{-26 - 2\sqrt{7}i};$$  \hspace{1cm} (4.92)

$$\sqrt{3}i, -\sqrt{3}i;$$

$$\sqrt{3}i, -\sqrt{3}i, \frac{1}{4} \sqrt{-26 + 2\sqrt{7}i}, -\frac{1}{4} \sqrt{-26 + 2\sqrt{7}i};$$

$$\sqrt{3}i, -\sqrt{3}i;$$

which again contain common roots

$$a = \pm i\sqrt{3}. \hspace{1cm} (4.93)$$

For the fourth choice of \(y = y_4 = -i(-4a^4 - 16a^2 - 12)^{\frac{1}{4}}\) in Eq.(4.71), Eq.(3.40b) gives

$$\left[ B_{12} - i \left( -4a^4 - 16a^2 - 12 \right)^{\frac{1}{4}} I_4 \right] v = 0 \hspace{1cm} (4.94)$$
or explicitly

\[
- i \left[ \left( 1 - i \right) a - i \left( -4a^4 - 16a^2 - 12 \right)^{\frac{1}{2}} \right] + \left( 1 + i \right) \left( a + i \sqrt{-a^2 - 2} \right) + 1 - i = 0,
\]

\[
2i + \left( \frac{-1}{2} - \frac{i}{2} \right) \left( 1 + i \right) \left[ \left( -1 - i \right) a - i \left( -4a^4 - 16a^2 - 12 \right)^{\frac{1}{2}} \right] \left( a + i \sqrt{-a^2 - 2} \right) + \left( a + i \sqrt{-a^2 - 2} \right) = 0,
\]

\[
2 - i \left( a + i \sqrt{-a^2 - 2} \right) + \left( \frac{1}{2} - \frac{i}{2} \right) \left[ \left( 1 + i \right) a - i \left( -4a^4 - 16a^2 - 12 \right)^{\frac{1}{2}} \right] \left( a + i \sqrt{-a^2 - 2} \right) = 0,
\]

\[
1 + i + \left( -1 + i \right) \left( a + i \sqrt{-a^2 - 2} \right) + \left( -1 - i \right) a + i \left( -4a^4 - 16a^2 - 12 \right)^{\frac{1}{2}} = 0.
\]

The roots of these equations are

\[
\sqrt{3i}, -\sqrt{3i}, \frac{1}{4} \sqrt{-26 + 2\sqrt{7}i}, \frac{1}{4} \sqrt{-26 + 2\sqrt{7}i};
\]

\[
\sqrt{3i}, -\sqrt{3i}, \frac{1}{4} \sqrt{-26 - 2\sqrt{7}i}, \frac{1}{4} \sqrt{-26 - 2\sqrt{7}i};
\]

\[
\sqrt{3i}, -\sqrt{3i};
\]

\[
\sqrt{3i}, -\sqrt{3i};
\]

which again contain common roots

\[
a = \pm i \sqrt{3}.
\]

We conclude that for \( x = x_1 = \sqrt{-a^2 - 2} \) and all four choices of \( y \) in Eq. (4.71), the map \( \alpha \) fails to be injective at the same point \([z : w : x : y] = [1 : 0 : 1 : 0]\) on \( CP^3 \) for the ADHM data in Eq. (4.63), Eq. (4.82) and Eq. (4.86), and one is led to use sheaf description for these 4-instantons. Again \( SU(2) \) instanton sheaf is not allowed for all cases.

Finally, a moment of thought leads one to extend the above 4-instanton sheaf structure in Eq. (4.96) to more general ADHM data. We first note that for \( b \neq 0 \), Eq. (4.62) can be rewritten as

\[
A_4 = b \left[ \begin{array}{cccc}
\sqrt{2}e_0 & \sqrt{2}e_1 & \sqrt{2}e_2 & \sqrt{2}e_3 \\
\frac{a}{b} (e_1 + e_2 + e_3) & -(e_2 + e_3) & -(e_3 + e_1) & -(e_1 + e_2) \\
-(e_2 + e_3) & \frac{a}{b} (e_1 - e_2 - e_3) & (e_2 - e_1) & (e_1 - e_3) \\
-(e_3 + e_1) & (e_2 - e_1) & \frac{a}{b} (-e_1 + e_2 - e_3) & (e_3 - e_2) \\
-(e_1 + e_2) & (e_1 - e_3) & (e_3 - e_2) & \frac{a}{b} (-e_1 - e_2 + e_3)
\end{array} \right].
\]

Our previous results imply that for the ADHM data \( b = 1 \) and \( \frac{a}{b} = \pm i \sqrt{3} \), there exist sheaf structure. Now we can easily show that for the ADHM data

\[
a = \pm i \sqrt{3}b, \ b \neq 0, b \in C,
\]

(4.98)
there are sheaf structure for the 4-instanton. Indeed the same common eigenvectors in Eq. (4.83), Eq. (4.87) exist for these ADHM data at points

\[ [z : w : x : y] = [1 : 0 : b : 0] \] (4.99)
on \( CP^3 \). This result can be easily checked by using Eq. (3.40a), Eq. (3.40b) and Eq. (3.40c).

We have checked that for the second choice of \( x = x_2 = -\sqrt{-a^2 - 2} \), there is no common eigenvector \( v \) for the system and thus no sheaf structure of the YM 4-instantons.

In the above long calculation of searching YM 4-instanton sheaves, we have assumed \( w = 0 \) on \( CP^3 \) in Eq. (4.66). We expect that other choices of points on \( CP^3 \) will give more 4-instanton sheaf structure for some other ADHM data.

V. CONCLUSION

In this paper, we first construct a class of \( SL(2, C) \) Yang-Mills ADHM 3-instanton data, and then demonstrate the existence of YM 3-instanton sheaves on \( CP^3 \). We then use a class of two parameter ADHM symmetric 4-instanton data constructed in the literature \([19]\) to demonstrate the existence of YM 4-instanton sheaves. The results we obtained in this paper extend the recent construction of Yang-Mills 2-instanton sheaves \([18]\) to higher instanton sheaves. It is of interest to understand the relationship between YM symmetric instantons \([19, 22, 23]\) on \( S^4 \) and YM instanton sheaves on \( CP^3 \) constructed in this paper.

Since it is a nontrivial task to explicitly construct non-diagonal \([14]\) higher ADHM instanton data \([23]\), the explicit construction of the general higher \( k \)-instanton sheaves remains an open question. However, it is believed that this new YM instanton sheaf structure persists for arbitrary higher \( k \)-instanton, and is a common feature for non-compact SDYM theory which does not exist for the usual YM theory based on the compact Lie group.

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[1] A. Belavin, A. Polyakov, A. Schwartz, Y. Tyupkin, "Pseudo-particle solutions of the Yang-Mills equations", Phys. Lett. B 59 (1975) 85.

[2] E.F. Corrigan, D.B. Fairlie, Phys. Lett. 67B (1977) 69; G. 'tHooft, Phys. Rev. Lett., 37 (1976) 8; F. Wilczek, in "Quark Confinement and Field Theory", Ed. D. Stump and D. Weingarten, John Wiley and Sons, New York (1977).

[3] R. Jackiw, C. Rebbi, "Conformal properties of a Yang-Mills pseudoparticle", Phys. Rev. D 14 (1976) 517; R. Jackiw, C. Nohl and C. Rebbi, "Conformal properties of pseudoparticle configurations", Phys. Rev. D 15 (1977) 1642.

[4] M. Atiyah, V. Drinfeld, N. Hitchin, Yu. Manin, "Construction of instantons", Phys. Lett. A 65 (1978) 185.

[5] G. 't Hooft, "Computation of the quantum effects due to a four-dimensional pseudoparticle", Phys. Rev. D 14 (1976) 3432. G. 't Hooft, "Symmetry breaking through Bell-Jackiw anomalies", Phys. Rev. Lett. 37 (1976) 8.

[6] C. Callan Jr., R. Dashen, D. Gross, "The structure of the gauge theory vacuum", Phys. Lett. B 63 (1976) 334; "Toward a theory of the strong interactions", Phys. Rev. D 17 (1978) 2717. R. Jackiw, C. Rebbi, "Vacuum periodicity in a Yang-Mills quantum theory", Phys. Rev. Lett. 37 (1976) 172.

[7] N. H. Christ, E. J. Weinberg and N. K. Stanton, "General Self-Dual Yang-Mills Solutions", Phys. Rev. D 18 (1978) 2013. V. Korepin and S. Shatashvili, "Rational parametrization of the three instanton solutions of the Yang-Mills equations", Math. USSR Izversiya 24 (1985) 307.

[8] A. Strominger, "Heterotic Solitons," Nuclear Physics B343 (1990) 167.

[9] C. G. Callan, Jr., J. A. Harvey, and A. Strominger, "World-brane Actions For String Solitons," Nucl. Phys. B367 (1991) 60, and "Supersymmetric String Solitons," in the Proceedings, String Theory and Quantum Gravity '91 (Trieste, 1991).

[10] E. Witten, "Sigma models and the ADHM construction of instantons," J. Geom. Phys. 15, 215 (1995) [hep-th/9410052].

[11] M. R. Douglas, "Gauge fields and D-branes", J. Geom. Phys. 28, 255 (1998) [hep-th/9604198].

[12] D. Tong and K. Wong, "ADHM Revisited: Instantons and Wilson Lines", Phys. Rev. D 91,
[13] M. Billo, M. Frau, I. Pesando, F. Fucito, A. Lerda and A. Liccardo, “Classical gauge instantons from open strings”, JHEP 0302, 045 (2003) [hep-th/0211250]. K. Hashimoto and S. Terashima, “ADHM is tachyon condensation”, JHEP 0602, 018 (2006) [hep-th/0511297].

[14] S. H. Lai, J. C. Lee and I. H. Tsai, "Biquaternions and ADHM Construction of Non-Compact SL(2,C) Yang-Mills Instantons", Annals Phys. 361 (2015) 14.

[15] I. Frenkel and M. Jardim, "Complex ADHM equations and sheaves on \( P^3 \)”, Journal of Algebra 319 (2008) 2913-2937. J. Madore, J.L. Richard and R. Stora, "An Introduction to the Twistor Programme”, Phys. Rept. 49, No. 2 (1979) 113-130.

[16] M. Jardim and M. Verbitsky, "Trihyperkahler reduction and instanton bundles on \( CP^3 \)”, Compositio Math. 150 (2014) 1836.

[17] K. L. Chang and J. C. Lee, "On solutions of self-dual SL(2,C) gauge theory”, Chinese Journal of Phys. Vol. 44, No.4 (1984) 59. J.C. Lee and K. L. Chang, "SL(2,C) Yang-Mills Instantons”, Proc. Natl. Sci. Counc. ROC (A), Vol 9, No 4 (1985) 296.

[18] S. H. Lai, J. C. Lee and I. H. Tsai, "Yang-Mills Instanton Sheaves", arXiv: 1603.02860, to be published in Annals of Physics.

[19] R.A. Leese and N.S. Manton, "Stable instanton-generated Skyrme fields with baryon numbers three and four", Nuclear Physics A572 (1994) 575-599.

[20] M.F. Atiyah and N.S. Manton, Phys. Lett. B222, 438 (1989); Commun. Math. Phys.153, 391 (1993).

[21] For a review, see N.S. Manton and Paul Sutcliffe, "Topological Solitons”, (2004),Cambridge Monographs on Mathematical Physics.

[22] Conor J. Houghton, "Instanton vibrations of the 3-Skyrmion”, Phys.Rev. D60 (1999) 105003 [hep-th/9905009].

[23] Michael Singer and Paul Sutcliffe, "Symmetric Instantons and Skyrme Fields”, Nonlinearity 12 (1999) 987-1003 [hep-th/9901075].

[24] S. Donaldson, "Instantons and Geometric Invariant Theory”, Comm. Math. Phys. 93 (1984) 453–460.

[25] Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D12, 3843 (1975); Phys. Rev.D13, (1976) 3233.

[26] W. R. Hamilton, "Lectures on Quaternions", Macmillan & Co, Cornell University Library (1853).
[27] L. Costa, G. Ottaviani, "Nondegenerate multidimensional matrices and instanton bundles",
Trans. Amer. Math. Soc. 355 (2002), 49-55.