On a variant of Gagliardo-Nirenberg inequality deduced from Hardy*

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Abstract

We obtain new variants of weighted Gagliardo-Nirenberg interpolation inequalities in Orlicz spaces, as a consequence of weighted Hardy-type inequalities. The weights we consider need not be doubling.

1 Introduction

Gagliardo-Nirenberg interpolation inequalities have already a long history and several mathematicians investigated their numerous variants. Their rudiments can be found in the old papers of Landau (see e.g. [35]), and now they are often identified with their classical variant

\[
\|\nabla^{(k)} u\|_{L^q(\Omega)} \leq C_1 \|u\|_{L^r(\Omega)}^{1 - \frac{k}{m}} \|\nabla^{(m)} u\|_{L^p(\Omega)}^{\frac{k}{m}} + C_2 \|u\|_{L^r(\Omega)} \tag{1.1}
\]

where \(\Omega \subseteq \mathbb{R}^n\) is a domain with sufficiently smooth boundary, \(u : \Omega \to \mathbb{R}\) belongs to an appropriate Sobolev space on \(\Omega\), \(\frac{1}{q} = (1 - \frac{k}{m}) \frac{1}{r} + \frac{k}{m} \frac{1}{p}\), \(0 < k < m\). This inequality for \(\Omega = \mathbb{R}\) and \(p = q = r = \infty\) was obtained by Kolmogorov [31] (\(C_2 = 0\) there), whereas Gagliardo [14] and Nirenberg [41] independently proved its extensions to the form (1.1). We refer to the book [39] for an extensive description of their historical evolution.

Since the Gagliardo-Nirenberg inequalities involve two differential operators: \(\nabla^{(k)} u\) and \(\nabla^{(m)} u\), they are more difficult to analyse than Hardy-type inequalities, which involve one differential operator only. This is one of the reasons why many questions concerning the validity of the Gagliardo-Nirenberg inequalities remain unsolved so far. For example, one asks about Orlicz-space generalizations of (1.1), which can be further extended to Orlicz spaces \(L^M(\mu)\) with a Radon measure \(\mu\), especially when \(\mu\) is nondoubling.

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Interest in inequalities in Orlicz-space setting arise from linear and nonlinear PDEs, and calculus of variations, which in turn come from mathematical physics. See e.g. [1, 2, 12, 15, 16, 47], where many motivations for investigating degenerate PDE’s in Orlicz spaces can be found.

The purpose of this paper is to show that certain variants of weighted modular Hardy inequalities

\[ \int_{\Omega} P(|\nabla \varphi||u|)d\mu \leq K_1 \int_{\Omega} P(A|\nabla u|)d\mu + K_2 \int_{\Omega} M(|u|)d\mu, \quad u \in C_0^\infty(\Omega), \] (1.2)

where \( \mu(dx) = e^{-\varphi(x)}dx \), \( \varphi \) is locally Lipschitz, imply variants of Gagliardo-Nirenberg inequalities for modulars:

\[ \int_{\Omega} M(|\nabla u|)d\mu \leq L \int_{\Omega} P(|\nabla^{(2)} u|)d\mu + \int_{\Omega} Q(B|u|)d\mu, \quad \text{where } \theta > 0, \]

and for norms:

\[ \|\nabla u\|_{L^M(\Omega,\mu)} \leq L_1 \sqrt{\|\nabla^{(2)} u\|_{L^P(\Omega,\mu)} \|u\|_{L^Q(\Omega,\mu)}} + L_2 \|u\|_{L^Q(\Omega,\mu)}, \] (1.3)

valid with general (\( u \)-independent) constants \( A, K_1, K_2, L, L_1, L_2, B \). Our approach requires \( M \) to be an \( N \)-function satisfying the \( \Delta_2 \)-condition. Functions \( P \) and \( Q \) are tied with \( M \) by Young-type inequality:

\[ \frac{M(u)}{u^2} vw \leq M(u) + P(v) + Q(w). \]

For details, see Theorems 3.1, 3.2, and 3.3.

As opposed to previous works of Gutierrez and Wheeden [17], and also Chua [10, 11], Bang and coauthors ([3]-[5]), the measure \( \mu \) considered here needs not to be doubling. This allows for obtaining inequalities e.g. for measures with finite mass on unbounded domains, which were formerly excluded from investigation. In papers [17, 10, 11], Gagliardo-Nirenberg inequalities were deduced from local Poincaré inequalities. In present work, we work with global inequalities only, and show how global Hardy-type inequalities result in Gagliardo-Nirenberg inequalities.

Inequality (1.3), obtained here as a consequence of Hardy inequality, extends our former results from [27]. In that paper, nondoubling measures were considered as well, but the conditions for \( M, P, Q \) were different. In particular, the case \( M = P = Q \) was not permitted in [27], see Remark 4.4 in [27]. This situation is rectified by present approach.

It is our intention to focus on Hardy-type inequalities (1.2). They imply a big range of Gagliardo-Nirenberg inequalities. Since (1.2) is valid for a vast class of admissible measures, possibly nondoubling, our approach yields Gagliardo-Nirenberg inequalities which are often new also in the \( L^p \)-setting.

For other results concerning the Gagliardo-Nirenberg inequalities in Orlicz spaces, we refer to [3]-[5], [23]-[30].


2 Preliminaries

Notation
In the sequel we assume that $\Omega \subset \mathbb{R}^n$ is an open domain. By $C_0^\infty(\Omega)$ we denote smooth functions compactly supported in $\Omega$, we use the standard notation $W^{m,p}(\Omega)$ and $W^{m,p}_{loc}(\Omega)$ for global and local variants of Sobolev spaces. Lower-case symbol $c$ denotes a universal constant whose value is irrelevant. For important constants we use the upper-case letters.

Orlicz spaces
Let us report some basic information about Orlicz spaces, referring e.g. [32, 44] for details.

Suppose that $\mu$ is a positive Radon measure on $\mathbb{R}_+$ and let $M : [0, \infty) \to [0, \infty)$ be an $N-$function, i.e. a continuous convex function satisfying $\lim_{\lambda \to 0} \frac{M(\lambda)}{\lambda} = \lim_{\lambda \to \infty} \frac{\lambda}{M(\lambda)} = 0$. The symbol $M^*$ denotes the function complementary to the given $N-$function $M$, i.e. the function $M^*(y) = \sup_{x>0} [xy - M(x)]$, where $y \geq 0$. It is again an $N-$function and it satisfies the Young inequality:

$$xy \leq M(x) + M^*(y), \quad \text{for } x, y \geq 0.$$ 

Given two functions $M_1$ and $M_2$, we write $M_1 \asymp M_2$ if there exist two constants $c_1, c_2$ such that $c_1 M_2(\lambda) \leq M_1(\lambda) \leq c_2 M_2(\lambda)$, for every $\lambda > 0$ (or for every $\lambda$ from indicated range).

Let $\mu$ be a nonnegative Borel measure on $\Omega$. The weighted Orlicz space $L^M(\Omega, \mu)$ we deal with is by definition the space

$$L^M(\Omega, \mu) \overset{def}{=} \{ f : \Omega \to \mathbb{R} \text{ measurable} : \int_{\Omega} M\left(\frac{|f(x)|}{K}\right)d\mu(x) \leq 1 \text{ for some } K > 0 \},$$

equipped with the Luxemburg norm

$$\| f \|_{L^M(\Omega, \mu)} = \inf \{ K > 0 : \int_{\Omega} M\left(\frac{|f(x)|}{K}\right)d\mu(x) \leq 1 \}.$$ 

It is a Banach space. When $M(\lambda) = \lambda^p$, then we have $L^M(\Omega, \mu) = L^p(\mu(\Omega))$ – the classical $L^p$ space.

The function $M$ is said to fulfill the $\Delta_2-$condition if and only if for some constant $c > 0$ and every $\lambda > 0$, we have

$$M(2\lambda) \leq cM(\lambda). \quad (2.1)$$

In the class of differentiable convex functions the $\Delta_2-$condition is equivalent to:

$$\lambda M'(\lambda) \leq DM(\lambda), \quad (2.2)$$
satisfied for every $\lambda > 0$, with the constant $D$ being independent of $\lambda$ (see e.g. [32], Theorem 4.1). The smallest possible constant in (2.2) will be denoted by $D_M$. One considers also the condition

$$dM(\lambda) \leq \lambda M'(\lambda).$$

(2.3)

It holds true with some $d > 1$ when the dual function, $M^*$, satisfies the $\Delta_2$--condition (see e.g. [32], Theorem 4.3 or [26], Proposition 4.1). The biggest possible constant in (2.3) will be denoted by $d_M$. Constants $d_M$ and $D_M$ are called Simonenko lower and upper indices of $M$ (see e.g. [8], [45]).

When $M$ is such an $N$--function that both (2.2) and (2.3) are satisfied, then

$$M(a\lambda) \leq \max(a^{d_M}, a^{D_M})M(\lambda) =: \bar{c}(a)M(\lambda),$$

(2.4)

for every $\lambda > 0, a > 0$ (see e.g. [28], Lemma 4.1, part iii)).

We will need the following property of modulars:

$$\int_{\Omega} M\left(\frac{f(x)}{\|f\|_M(\alpha, \mu)}\right) d\mu(x) \leq 1.$$

(2.5)

When $M$ satisfies the $\Delta_2$--condition, then (2.5) becomes an equality.

**The assumptions**

The assumptions considered in the sequel are as follows.

(M) $M : [0, \infty) \rightarrow [0, \infty)$ is an $N$--function of class $C^1((0, \infty))$, satisfying the $\Delta_2$--condition and such that $\frac{M(\lambda)}{\lambda}$ is bounded next to zero;

(µ) $\mu(dx) = \exp(-\varphi(x))dx$ is a Radon measure on $\Omega$, where $\varphi : \Omega \rightarrow \mathbb{R}$ belongs to $W^{1, \infty}_{loc}(\Omega)$;

(Y) $P$ and $Q$ are two real nonnegative and nondecreasing measurable functions on $[0, \infty)$ with $P(0) = Q(0) = 0$, such that for any $u, v, w > 0$ the following Young-type inequality holds

$$\frac{M(u)}{u^2}vw \leq M(u) + P(v) + Q(w).$$

(2.6)

The inequality (Y) is fulfilled for example when the following condition holds (see [24], Cor. 4.1):

(MF) $M$ is an $N$--function satisfying the $\Delta_2$--condition and such that $M(\lambda)/\lambda^2$ is nondecreasing, $P(\lambda) = CM(F(\sqrt{\lambda}))$, $Q(\lambda) = CM(F^*(\sqrt{\lambda}))$, where $F$ is another $N$--function.

Remark 2.1 1. The choice of $F(\lambda) = \frac{\lambda^2}{2}$ in (MF) gives that (Y) holds with $P = Q = M$ (with $C = 1$).
2. Suppose that $M(\lambda) = \lambda^p$, $p \geq 2$. Choose $F(\lambda) = \frac{\lambda^s}{s}$ with $s = \frac{2q}{p}$ to obtain $P(\lambda) = C\lambda^q$, $Q(\lambda) = C\lambda^r$, where $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$ (the classical Gagliardo-Nirenberg triple).

3. When $M(\lambda) = \lambda^p \ln(2 + \lambda^\alpha), p \geq 2, \alpha > 0$, then the choice of $F(\lambda) = \lambda^s \ln(2 + \lambda^\mu), s = \frac{2q}{p}, \mu = \beta - \alpha p$ results in $P(\lambda) \approx \lambda^q \ln(2 + \lambda^\beta)$, $Q(\lambda) \approx \lambda^r \ln(2 + \lambda^\gamma)$, where the parameters are related through $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$ (the logarithmic Gagliardo-Nirenberg triple considered in [23] and [25]).

4. Inequality (2.6) can be obtained from the multiversion of the Young inequality due to Cooper:

$$\Pi_\nu a_\nu \leq \sum_\nu \int_0^{a_\nu} \frac{g_\nu(x)}{x} dx,$$

where $\Pi_\nu g_\nu^{-1}(x) = x$, $a_\nu \geq 0$ and $g_\nu$ are continuous strictly increasing functions such that $g_\nu([0, \infty)) = [0, \infty)$ (see Theorem 159 in [19] and Theorem 4.3 in [24]).

3 Main results

Our goal now is to show that certain Hardy-type inequalities imply the Gagliardo-Nirenberg ones. Let us start with the following result.

**Theorem 3.1** Let $M, P, Q$ be three functions on $[0, \infty)$ satisfying (M), (Y), and let $\mu$ be a Radon measure on $[0, \infty)$ satisfying (\mu). Suppose that the following Hardy-type inequality holds true:

\[ (H) \text{ for any } u \in C_0^\infty(\Omega) \]

\[ \int_\Omega P(|\nabla \varphi| |u|) d\mu \leq K \int_\Omega P(A|\nabla u|) d\mu \]  \hspace{1cm} (3.1)

with positive constants $K, A$ not depending on $u$.

Then we have:

1) there exist constants $L, B > 0$ such that for any $\theta > 0$ and any $u \in C_0^\infty(\Omega)$

\[ \int_\Omega M(|\nabla u|) d\mu \leq L \int_\Omega P(\theta|\nabla(2)u|) d\mu + \int_\Omega Q(\frac{B}{\theta} |u|) d\mu, \]  \hspace{1cm} (3.2)

2) if additionally $P$ and $Q$ are $N-$functions, then for any $u \in C_0^\infty(\Omega)$ we have

\[ \|\nabla u\|_{L^M(\Omega, \mu)} \leq \tilde{L} \sqrt{\|\nabla(2)u\|_{L^P(\Omega, \mu)} \|u\|_{L^Q(\Omega, \mu)}}, \]  \hspace{1cm} (3.3)

where $\tilde{L} = 2(L + 2)\sqrt{B}$, $L$ and $B$ are the same as in (3.2).
Remark 3.1 Under the assumptions of Theorem 3.1, when either \( P \) or \( Q \) satisfies the \( \Delta_2 \)-condition, then we have

\[
\int_{\Omega} M(|\nabla u|)d\mu \leq L_1 \int_{\Omega} P(|\nabla^{(2)} u|)d\mu + L_2 \int_{\Omega} Q(|u|)d\mu, \tag{3.4}
\]

with \( L_1, L_2 \) independent of \( u \).

Proof. We start with the following inequality (Lemma 3.1 of [27]), valid for any \( u \in C_0^\infty(\Omega) \):

\[
I = \int_{\Omega} M(|\nabla u|)d\mu \leq \int_{\Omega \cap \{\nabla u \neq 0\}} \frac{M(|\nabla u|)}{|\nabla u|^2} |\nabla^{(2)} u| |u| d\mu + \int_{\Omega \cap \{\nabla u \neq 0\}} \frac{M(|\nabla u|)}{|\nabla u|} |\nabla \varphi| |u| d\mu
\]

where \( \alpha_n \) depends on \( n \) only. In [27], the proof is given for \( \Omega = \mathbb{R}^n \), and \( \varphi \in C^1(\mathbb{R}^n) \). It requires only minor alterations to cover the present case.

To estimate \( I_1 \) and \( I_2 \), we use the assumption (2.6) twice. One has, for any given \( \epsilon, \theta > 0 \):

\[
I_1 = \epsilon \int_{\Omega \cap \{\nabla u \neq 0\}} \left( \frac{M(|\nabla u|)}{|\nabla u|^2} \right) \left( \theta |\nabla^{(2)} u| \right) \left( \frac{|u|}{\theta \epsilon} \right) d\mu
\]

\[
\leq \epsilon \int_{\Omega} M(|\nabla u|)d\mu + \epsilon \int_{\Omega} P(\theta |\nabla^{(2)} u|)d\mu + \epsilon \int_{\Omega} Q\left( \frac{|u|}{\theta \epsilon} \right) d\mu, \tag{3.6}
\]

and

\[
I_2 = \epsilon \int_{\Omega \cap \{\nabla u \neq 0\}} \left( \frac{M(|\nabla u|)}{|\nabla u|^2} \right) \left( |\nabla \varphi| \frac{\theta}{A} |\nabla u| \right) \left( \frac{|u|}{\theta \epsilon} \right) d\mu
\]

\[
\leq A \epsilon \int_{\Omega} M(|\nabla u|)d\mu + A \epsilon \int_{\Omega} P(|\nabla \varphi| \frac{\theta}{A} |\nabla u|)d\mu + A \epsilon \int_{\Omega} Q\left( \frac{|u|}{\theta \epsilon} \right) d\mu. \tag{3.7}
\]

To estimate the central term in (3.7), we apply inequality (3.1) to the function \( \frac{\theta}{A} f(x) \) where \( f(x) = |\nabla u(x)| \). Whenever \( \nabla u \neq 0 \), one has:

\[
\frac{\partial}{\partial x_i} f(x) = \langle \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\partial}{\partial x_i} u(x) \rangle,
\]

and so

\[
|\nabla f(x)|^2 = \sum_{i=1}^n \left| \left( \frac{\nabla u(x)}{|\nabla u(x)|}, \frac{\partial}{\partial x_i} u(x) \right) \right|^2 \leq \sum_{i=1}^n \left\| \frac{\partial}{\partial x_i} (\nabla u(x)) \right\|^2 = \|\nabla^{(2)} u(x)\|^2.
\]
Since $P$ is nondecreasing, we get from (H):

$$\int_{\Omega} P(|\nabla \varphi| \frac{\theta}{A} |\nabla u|) d\mu \leq K \int_{\Omega} P(\theta |\nabla (2) u|) d\mu.$$  

Using this fact, summing up estimates (3.6) and (3.7) we obtain:

$$I \leq \epsilon(\alpha_n + A) \int_{\Omega} P(\theta |\nabla (2) u|) d\mu + \epsilon(\alpha_n + A) \int_{\Omega} Q(\frac{|u|}{\theta}) d\mu.$$  

Choose $\epsilon = \frac{1}{2(\alpha_n + A)}$, which after rearranging gives

$$I \leq (K + 1) \int_{\Omega} P(\theta |\nabla (2) u|) d\mu + \int_{\Omega} Q \left( \frac{2(\alpha_n + A)}{\theta} |u| \right) d\mu.$$  

This proves statement 1).

To prove statement 2) we take an arbitrary $u \in C_0^\infty(\Omega)$ and apply (3.2) to

$$\tilde{u} := \frac{u}{a + b}, \quad \text{where} \quad a := \|\theta \nabla (2) u\|_{L^P(\Omega, \mu)}, \quad b := \|\frac{B}{\theta} u(x)\|_{L^Q(\Omega, \mu)}.$$  

Without loss of generality we can assume that both numbers $a$ and $b$ are not zero, because when either $a$ or $b$ equals to zero, then we must have $\nabla u(x) = 0$ a.e. and (3.3) follows trivially. Inequality (3.2) for $\tilde{u}$ reads

$$\int_{\Omega} M(|\nabla \tilde{u}|) d\mu \leq L \int_{\Omega} P\left(\frac{|\theta \nabla (2) u|}{a + b}\right) d\mu + \int_{\Omega} Q\left(\frac{\frac{B}{\theta} u}{a + b}\right) d\mu$$  

$$\leq L \int_{\Omega} P\left(\frac{|\theta \nabla (2) u|}{a}\right) d\mu + \int_{\Omega} Q\left(\frac{\frac{B}{\theta} u}{b}\right) d\mu$$  

$$= L \int_{\Omega} P\left(\frac{|\theta \nabla (2) u|}{\|\theta \nabla (2) u\|_{L^P(\Omega, \mu)}}\right) d\mu + \int_{\Omega} Q\left(\frac{\|B\theta u\|_{\|\theta \nabla (2) u\|_{L^Q(\Omega, \mu)}}}{\|\theta \nabla (2) u\|_{L^P(\Omega, \mu)}}\right) d\mu \leq L + 1.$$  

In the last inequality we have used the property $\int_{\Omega} R\left(\frac{w}{\|w\|_{L^R(\Omega, \mu)}}\right) d\mu \leq 1$ of modular functionals. Since for any $f \in L^M(\Omega, \mu)$ one has $\|f\|_{L^M(\Omega, \mu)} \leq \int_{\Omega} M(|f|) d\mu + 1$ (see (9.4) and (9.20) of [32]), this gives

$$\|\nabla \tilde{u}\|_{L^M(\Omega, \mu)} \leq L + 2.$$  

Consequently,  

$$\|\nabla u\|_{L^M(\Omega, \mu)} \leq (L + 2)(a + b) = (L + 2) \left(\theta \|\nabla (2) u\|_{L^P(\Omega, \mu)} + \frac{B}{\theta} \|u(x)\|_{L^Q(\Omega, \mu)}\right).$$  

Minimizing the right hand side with respect to $\theta$ gives the result. The proof is complete.  

□

Our next theorem covers the case when the Hardy inequality (H) does not hold, but it holds when an extra term, depending on $u$, is added to the right-hand side.
**Theorem 3.2** Suppose that the assumptions of Theorem 3.1 are satisfied, and the following Hardy-type inequality holds true:

\[(H1)\] for any \( u \in C^\infty_0(\Omega) \),

\[
\int_\Omega P(\|\nabla \varphi\|\|u\|)d\mu \leq K_1 \int_\Omega P(A\|\nabla u\|)d\mu + K_2 \int_\Omega M(\|u\|)d\mu \tag{3.8}
\]

with positive constants \( K_1, K_2, A \) not depending on \( u \).

Then we have:

1) there exist constants \( L, B > 0 \) such that for any \( \theta \in (0,1] \) and any \( u \in C^\infty_0(\Omega) \)

\[
\int_\Omega M(\|\nabla u\|)d\mu \leq L \int_\Omega P(\theta\|\nabla^{(2)} u\|)d\mu + \int_\Omega Q\left(\frac{B}{\theta}\|u\|\right)d\mu, \tag{3.9}
\]

2) if \( P \) and \( Q \) are \( N \)-functions, then for any \( u \in C^\infty_0(\Omega) \) we have

\[
\|\nabla u\|_{L^M(\Omega,\mu)} \leq \tilde{L} \sqrt{\|\nabla^{(2)} u\|_{L^P(\Omega,\mu)}\|u\|_{L^Q(\Omega,\mu)}} + L_2 \|u\|_{L^Q(\Omega,\mu)}, \tag{3.10}
\]

where \( L_1 = 2(L + 2)\sqrt{B}, L_2 = 2(L + 2)B, L \) and \( B \) are the same as in (3.9).

**Proof.**

1) We start with inequality (3.5) and repeat the arguments in the proof of Theorem 3.1 up to the formula (3.7). Now, instead of (3.1), we apply (3.8) to the function \( f(x) = \frac{\theta}{A}\|\nabla u(x)\| \). Since we have assumed \( \theta \leq 1 \), we get:

\[
\int_\Omega P(\|\nabla \varphi\|\theta\|\nabla u\|)d\mu \leq K_1 \int_\Omega P(\theta\|\nabla^{(2)} u\|)d\mu + \bar{K}_2 \int_\Omega M(\|\nabla u\|)d\mu,
\]

where \( \bar{K}_2 = \bar{c}(\frac{\theta}{A})K_2, \bar{c}(\cdot) \) comes from (2.4). This, (3.6) and (3.7) lead to the inequality

\[
I \leq \epsilon(\alpha_n + A + \bar{K}_2 A)I + \epsilon(\alpha_n + K_1 A) \int_\Omega P(\theta\|\nabla^{(2)} u\|)d\mu + \epsilon(\alpha_n + A) \int_\Omega Q\left(\frac{\|u\|}{\theta\epsilon}\right)d\mu.
\]

The choice of \( \epsilon = \left(2(\alpha_n + A + A\bar{K}_2)\right)^{-1} \) implies

\[
I \leq L \int_\Omega P(\theta\|\nabla^{(2)} u\|)d\mu + \int_\Omega Q\left(\frac{B}{\theta}\|u\|\right)d\mu, \tag{3.11}
\]

where \( L = K_1 + 1, B = 2(\alpha_n + A + A\bar{c}(\frac{\theta}{A})K_2) \). This completes the proof of part 1).

2) To prove the second part we observe that arguments similar to those in the proof of second part in Theorem 3.1 lead to the inequality

\[
\|\nabla u\|_{L^M(\Omega,\mu)} \leq \tilde{L} \left(\theta\|\nabla^{(2)} u\|_{L^P(\Omega,\mu)} + \frac{B}{\theta}\|u\|_{L^Q(\Omega,\mu)}\right),
\]
where $\tilde{L} = L + 2$, $L = K_1 + 1$ is the constant from (3.11), $\theta \in (0, 1]$ is arbitrary. Minimization of the inequality $a \leq \tilde{L}(\theta b + \frac{1}{\theta} c)$ with respect to $\theta \in (0, 1]$ gives the desired result. Indeed, when $\theta := \sqrt{\frac{c}{b}} \in (0, 1)$, then we get $a \leq 2\tilde{L}\sqrt{bc}$, while in the remaining case $c \geq b$ we have $a \leq 2\tilde{L}c$ (choose $\theta = 1$). In either case inequality $a \leq 2\tilde{L}\sqrt{bc}$ holds. This completes the proof of the theorem.

Since the condition (Y) is fulfilled for $P = Q = M$ (see Remark 2.1), as an immediate consequence we obtain the following:

**Theorem 3.3** Suppose that $M$ is an $N$–function satisfying condition (M), and let $\mu$ be a Radon measure on $[0, \infty)$ satisfying $(\mu)$. Moreover, assume that $M(\lambda)/\lambda^2$ is nondecreasing. If for every $u \in \mathcal{C}_0^\infty(\Omega)$ the following Hardy-type inequality holds true:

$$
\int_{\Omega} M(|\nabla \varphi| |u|) \, d\mu \leq K_1 \int_{\Omega} M(|\nabla u|) \, d\mu + K_2 \int_{\Omega} M(|u|) \, d\mu,
$$

then we have:

1) there exist positive constants $C_1, C_2$ such that for any $\theta \in (0, 1]$ and any $u \in \mathcal{C}_0^\infty(\Omega)$

$$
\int_{\Omega} M(|\nabla u|) \, d\mu \leq C_1 \int_{\Omega} M(\theta |\nabla^{(2)} u|) \, d\mu + C_2 \int_{\Omega} M(\frac{|u|}{\theta}) \, d\mu;
$$

2) there exist positive constants $\tilde{C}_1, \tilde{C}_2$ such that for any $u \in \mathcal{C}_0^\infty(\Omega)$

$$
\|\nabla u\|_{L^M(\Omega, \mu)} \leq \tilde{C}_1 \sqrt{\|\nabla^{(2)} u\|_{L^M(\Omega, \mu)}} \|u\|_{L^M(\Omega, \mu)} + \tilde{C}_2 \|u\|_{L^M(\Omega, \mu)}.
$$

**Remark 3.2** Constants $L, B$ in the inequality (3.2) are of the form: $L = K + 1, B = 2(\alpha_n + A)$, where $K, A$ are the same as in (3.1), $\alpha_n$ depends on the dimension (it appears in (3.5) the proof of Theorem 3.1). It is known that $\alpha_n \leq c\sqrt{n}$, with $c > 0$ independent of $n$ (see Lemma 3.1 in [27]). This is readily seen from the proof of Theorem 3.1.

**Remark 3.3** It can be deduced from the proof of Theorem 3.2 that constants $L, B$ in the inequality (3.9) are of the form: $L = K_1 + 1, B = 2(\alpha_n + A + \tilde{c}(\frac{1}{\theta})K_2)$, where $A, K_1, K_2$ are the same as in (3.8), $\tilde{c}$ is the same as in (2.4), $\alpha_n$ appears in (3.5) is such that $\alpha_n \leq c\sqrt{n}$, with $c > 0$ independent of $n$.

**Remark 3.4 (open question)** Condition $M \in \Delta_2$ in the assumptions of Theorem 3.1 is only needed to derive the formula (3.5). We do not know whether one can extend (3.2), (3.3) to functions $M$ for which the $\Delta_2$–conditions is not satisfied. Some results concerning the Gagliardo-Nirenberg inequalities (3.2), (3.3) hold true without the $\Delta_2$–condition imposed on $M$, but for a restricted family of measures, see e.g. [3],[5],[26].
4 Discussion and examples

Three theorems from the previous section reduce the question about the validity of the Gagliardo-Nirenberg inequality for given \( N \)-functions and measures to a question about the validity of Hardy-type inequalities. We will discuss it now.

4.1 The scope of Theorem 3.1

4.1.1 The case \( \Omega = \mathbb{R}_+ \), condition (H)

A necessary and sufficient condition for Radon measures \( \mu, \nu \) to obey the inequality

\[
\int_{\mathbb{R}_+} \left( \int_0^x f(t) dt \right)^p d\nu(x) \leq C \int_{\mathbb{R}_+} |f(x)|^p d\mu(x),
\]

(4.1)

was given by Muckenhoupt (see [37] or [38], Section 1.3, Theorem 1):

\[
sup_{r > 0} (\nu(r, \infty))^{1/p} \left( \int_0^r \left( \int_0^{\infty} \frac{d\mu^*(x)}{dx} \right)^{\frac{1}{p-1}} dx \right)^{(p-1)/p} < \infty,
\]

(4.2)

where \( \mu^* \) is the absolutely continuous part of \( \mu \). Since for \( u \in C_0^\infty(0, \infty) \) one has \( u(x) = \int_0^x u'(t) dt \), it follows that whenever \( \nu, \mu \) obey (4.2), then the inequality

\[
\int_{\mathbb{R}_+} |u(x)|^p d\nu(x) \leq C \int_{\mathbb{R}_+} |u'(x)|^p d\mu(x)
\]

(4.3)

holds for \( u \in C_0^\infty(\mathbb{R}_+) \). Observe that in the particular case of \( d\nu(x) = |\phi'(x)| p \exp(-\phi(x)) dx \), \( d\mu(x) = \exp(-\phi(x)) dx \), inequality (4.3) is nothing but our condition (H) for \( P(\lambda) = \lambda^p \). In this case, condition (4.2) reads:

\[
sup_{r > 0} \left( \int_0^\infty |\phi'(x)|^p \exp(-\phi(x)) dx \right)^{1/p} \left( \int_0^r \exp(-\phi(x))^{-\frac{1}{p-1}} dx \right)^{1-1/p} < \infty,
\]

(4.4)

and so when (4.4) holds true, then (H) is true for \( P(\lambda) = \lambda^p \), \( \Omega = \mathbb{R}_+ \). Therefore we obtain:

**Theorem 4.1** Let \( p > 1 \) be given, and let \( \mu(dx) = e^{-\varphi(x)} dx \) be a Radon measure on \([0, \infty)\) satisfying (\( \mu \)). Suppose that (4.4) holds true. Next, let \( M \) be an \( N \)-function satisfying condition (M), and let \( Q \) be another \( N \)-function, such that (Y) is satisfied for \( M, P = P(\lambda) = \lambda^p \), and \( Q \). Then for any \( u \in C_0^\infty(\mathbb{R}_+) \) one has

\[
\int_{\mathbb{R}_+} M(|u'|) d\mu(x) \leq K_1 \int_{\mathbb{R}_+} |u''|^p d\mu + K_2 \int_{\mathbb{R}_+} Q(|u|) d\mu,
\]

(4.5)

and also

\[
\|u'\|_{L^M(\mathbb{R}_+, \mu)} \leq K \|u''\|_{L^p(\mathbb{R}_+, \mu)} \|u\|_{L^Q(\mathbb{R}_+, \mu)},
\]

(4.6)

where the constants \( K_1, K_2, K \) do not depend on \( u \).
We illustrate this case with two examples.

**Example 4.1 [classical Hardy inequality]**

Consider the classical Hardy inequality, which deals with power weights [18], [19]:

\[
\int_{\mathbb{R}^+} |u(t)|^p t^\alpha dt \leq C \int_{\mathbb{R}^+} |u'(t)|^p t^\alpha dt,
\]

(4.7)

where \(C = \left( \frac{p}{\alpha - p + 1} \right)^p\), \(\alpha \neq p - 1\). In this case we have \(\mu(dt) = \exp(-\varphi(t))dt\), where \(\varphi(t) = -\alpha \ln t\). In particular \(\varphi'(t) = -\frac{\alpha}{t}\), and (4.7) reads

\[
\int_{\mathbb{R}^+} (|\varphi'(t)| |u(t)|)^p t^\alpha dt \leq \tilde{C} \int_{\mathbb{R}^+} |u'(t)|^p t^\alpha dt,
\]

where \(\tilde{C} = \left( \frac{p\alpha}{\alpha - p + 1} \right)^p\).

**Example 4.2 [Hardy inequality and power-exponential weights]**

We now consider measures on \((0, \infty)\) with power-exponential-type densities:

\[
\mu(dx) = x^\alpha e^{-x^\beta} dx = \exp(-\varphi(x))dx, \quad \alpha \geq 0, \beta > 0, \quad \varphi(x) = -\ln x^\alpha + x^\beta. \quad (4.8)
\]

This class of measures contains in particular the exponential distribution \((\alpha = 0, \beta = 1)\) and the Gaussian distribution \((\alpha = 0, \beta = 2)\).

As \(|\varphi'(x)| \asymp \frac{1}{x}\) for \(x\) being small and \(|\varphi'(x)| \asymp x^{\beta-1}\) for \(x\) close to \(\infty\), the Muckenhoupt condition (4.4) for the measure (4.8) is equivalent to:

\[
\sup_{r > 0} \left( \frac{A(r)}{B(r)} \right)^{\frac{1}{p}} < \infty,
\]

where

\[
A(r) := \left( \int_r^1 x^{-p+\alpha} e^{-x^\beta} dx \right) \chi_{r < 1} + \left( \int_r^\infty x^{(\beta-1)p+\alpha} e^{-x^\beta} dx \right) \chi_{r \geq 1},
\]

\[
B(r) := \int_0^r \frac{x^\beta}{x^p} dx. \quad (4.9)
\]

We observe that \(A(r)\) is finite for all choices of \(\alpha \geq 0, \beta > 0\), whereas \(B(r)\) is finite if and only if \(\alpha < p - 1\). Both functions \(A, B\) are locally bounded and continuous next to 0 and \(\infty\). Moreover, for \(r\) close to 0, we have

\[
A(r)^{\frac{1}{p}} (B(r))^{1-\frac{1}{p}} \asymp \left( r^{-1+\frac{\beta}{p}+\frac{\beta}{p}} + C \right) \cdot \left( r^{-\frac{\alpha}{p}+1-\frac{1}{p}} \right) < \text{Const}.
\]

Therefore (4.9) holds true if and only if \(\lim_{r \to \infty} A(r) B(r)^{p-1} < \infty\).

By a direct application of the de l’Hospital rule we see that for \(a \in \mathbb{R}, b > 0\)

\[
\lim_{r \to \infty} \frac{\int_r^\infty x^a e^{-x^b} dx}{r^{a+1-b} e^{-r^b}} = \lim_{r \to \infty} \frac{x^a e^{-r^b}}{br^a e^{-r^b} - (a+1-b)r^{a-b} e^{-r^b}} = \frac{1}{b},
\]

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and for $a < 1, C > 0$

$$\lim_{r \to \infty} \frac{\int_{r}^{C} \frac{e^{C \beta}}{x^{a{(b-1)}}} dx}{r^{a \beta}} = \lim_{r \to \infty} \frac{bC e^{C \beta}}{r^{a \beta}} - \frac{(a + b - 1)e^{C \beta}}{r^{a \beta + 1}} = \frac{1}{bC}.$$ 

Therefore, for $r$ large, we have

$$A(r) \approx r^{(\beta - 1)(p - 1) + \alpha} e^{-r^p} \quad \text{and} \quad B(r) \approx r^{-(p - 1)(\beta - 1)} e^{-r^p},$$

and so $A(r)B(r)^{p-1} \propto \text{Const}$ for large values of $r$. Therefore (4.9) is fulfilled whenever $0 \leq \alpha < p - 1, \beta > 0$.

We end up with the following theorem.

**Theorem 4.2** Let $p > 1$ and let $\mu(dx) = x^{\alpha} e^{-x^p} dx$, where $\alpha \neq p - 1, \beta = 0$, or $0 \leq \alpha < p - 1, \beta > 0$. Suppose that $M$ is a an $N$–function satisfying condition (M), and $Q$ is another $N$–function, such that

$$\frac{M(t)}{t^{2r}} rs \leq M(t) + cr^p + Q(s), \quad \text{for every } t, r, s > 0.$$

Then inequalities (4.5) and (4.6) hold for any $u \in C^1_0(\Omega)$, with constants $K_1, K_2, K$ independent of $u$.

**Remark 4.1** As to the validity of Orlicz-space counterparts of (4.1), which would then yield (H), we refer to the papers of Bloom-Kerman [6, 7], Lai [36], Heinig-Maligranda [21], Bloom-Kerman [7], Heinig-Lai [20], their references and also to the authors’ paper [28], Section 3.3, where another type of sufficient conditions for (H) to hold on $\mathbb{R}_+$ is given.

### 4.1.2 Multidimensional Hardy inequalities

**Inequalities on bounded domains**

The multidimensional Hardy inequalities of the form

$$\int_{\Omega} \left( \frac{1}{\delta(x)} |u(x)| \right)^q q^q \delta(x)^a dx \leq C \int_{\Omega} |\nabla u(x)|^q \delta(x)^a dx, \quad u \in C_0^1(\Omega), \quad (4.10)$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with sufficiently regular boundary, $a < q - 1, 1 < q < \infty, \delta(x) = \text{dist}(x, \partial \Omega)$, were first obtained by Necas [40] (for bounded domains with Lipschitz boundary) and extended further by Kufner ([33], Theorem 8.4) and Wannebo [48] to Hölder domains.

As a direct consequence of of Theorem 3.1, we obtain Gagliargo-Nirenberg-type inequalities within $L^p$-spaces with the distance from the boundary, which can be stated as follows.
Theorem 4.3 Suppose that $\Omega$ is a bounded Lipschitz domain and $q > 1$, $a > q - 1$. Then we have

i) if $p \geq 2$, $r > 1$, $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$ then for every $u \in C^\infty_0(\Omega)$ one has:

$$
\left( \int_\Omega |\nabla u(x)|^p \delta(x)^a \, dx \right)^{\frac{2}{p}} \leq c \left( \int_\Omega |\nabla^2 u(x)|^q \delta(x)^a \, dx \right)^{\frac{1}{q}} \cdot \left( \int_\Omega |u(x)|^r \delta(x)^a \, dx \right)^{\frac{1}{r}},
$$

with a constant $c > 0$ independent of $u$.

ii) if $M$ and $Q$ are $N-$functions such that

$$
\frac{M(u)}{u^2} vw \leq M(u) + cv^q + Q(w),
$$

and $M$ satisfies condition (M), then for every $u \in C^\infty_0(\Omega)$ one has:

$$
\left( \int_\Omega M(|\nabla u(x)|) \delta(x)^a \, dx \right) \leq c \left( \int_\Omega |\nabla^2 u(x)|^q \delta(x)^a \, dx + \int_\Omega Q(|u(x)|) \delta(x)^a \, dx \right),
$$

and

$$
\|\nabla u\|^2_{L^M(\Omega,\mu)} \leq c \|\nabla^2 u\|_{L^q(\Omega,\mu)} \|u\|_{L^Q(\Omega,\mu)},
$$

with constants independent on $u$.

Proof. i) Obviously, $M(\lambda) = \lambda^p$ satisfies condition (M) and $P(\lambda) = \lambda^q, Q(\lambda) = \lambda^r$ satisfy (Y) due to Remark 2.1. Moreover, the measure $\mu(dx) = \exp(-\varphi(x)) dx$ where $\varphi(x) = -a \ln \delta(x)$ satisfies ($\mu$). It is not hard to verify that $|\nabla \varphi(x)| \asymp \frac{1}{\delta(x)}$ (note that $|\nabla \delta| \asymp \text{Const}$). This together with (4.10) implies (H). Now it suffices to apply Theorem 3.1. Part ii) is proven similarly.

Remark 4.2 Note that $\delta(x)^a$ is an $A_p$-weight when $-1 < a < q - 1$ (see e.g. [46] for definition of $A_p$ weights introduced by Muckenhoupt). Gagliardo-Nirenberg inequalities with $A_p$-weights within homogeneous spaces were earlier obtained in [11], [26] by different methods. Those results also covered the case $1 < p < 2$.

Remark 4.3 Counterparts of inequality (4.10) in Orlicz norms were obtained by Cianchi in [13].

Inequalities on $\mathbb{R}^n$

Hardy inequality on $\mathbb{R}^n$ with power weights (see e.g. [9], [38], page 70 in [34], and their references)

$$
|||x|^{q-1}|u||_{L^q} \leq C|||x|^a |\nabla u||_{L^q},
$$

where $u \in C^\infty_0(\mathbb{R}^n), \frac{1}{q} + \frac{a-1}{n} > 0, q > 1$, give rise to the Gagliardo-Nirenberg inequalities on $\mathbb{R}^n$ with power weights $|x|^a$ and $N-$functions $M, P = P(\lambda) = \lambda^q, Q$, satisfying (Y). The result, obtained directly from Theorem 3.1 reads as follows.
Theorem 4.4  Suppose that \( \frac{1}{q} + \frac{α - 1}{n} > 0, q > 1 \). Then we have

i) if \( p \geq 2, r > 1, \frac{2}{p} = \frac{1}{q} + \frac{1}{r} \) then for every \( u \in C^∞_0(Ω) \) one has:

\[
\left( \int_Ω |\nabla u(x)|^p |x|^α dx \right)^{\frac{2}{p}} \leq c \left( \int_Ω |\nabla u(x)|^q |x|^α dx \right)^{\frac{1}{q}} \cdot \left( \int_Ω |u(x)|^r |x|^α dx \right)^{\frac{1}{r}},
\]

with a constant \( c > 0 \) independent of \( u \).

ii) if \( M \) and \( Q \) are \( N \)–functions such that

\[
M(u) \frac{M(u)}{u^2} vw \leq M(u) + cv^q + Q(w),
\]

and \( M \) satisfies condition \( (M) \), then for every \( u \in C^∞_0(Ω) \) one has:

\[
\left( \int_Ω M(|\nabla u(x)|)|x|^α dx \right) \leq c \left( \int_Ω |\nabla u(x)|^q |x|^α dx + \int_Ω Q(|u(x)|)|x|^α dx \right),
\]

and

\[
\|\nabla u\|_{L^M(Ω,μ)}^2 \leq c\|\nabla (2) u\|_{L^q(Ω,μ)}\|u\|_{L^q(Ω,μ)},
\]

with constants independent on \( u \).

4.2 The scope of Theorem 3.2

We are now to discuss the validity of Theorem 3.2. To abbreviate the discussion we only focus on the condition \( (H1) \) in its assumptions. Contrarily to the Hardy inequality \( (H) \) the condition \( (H1) \) barely seems to be investigated.

4.2.1 Inequalities on bounded domains

(A) Result by Oinarov. Condition \( (H1) \) and its special variant \( (3.12) \) has drawn less attention than the ‘classical’ Hardy inequality \( (H) \). In his paper [42], Oinarov considered inequalities

\[
\left( \int_a^b |ωu|^q dr \right)^{\frac{1}{q}} \leq C \left( \int_a^b |pu'|^p dr + \int_a^b |v|u|^p dr \right)^{\frac{1}{p}},
\]

for general weights \( ω, v, ρ \) and derived necessary and sufficient conditions needed for \( (4.17) \) to hold for all \( u \in C^∞_0(a,b) \). Our condition \( (3.12) \) with \( μ(dx) = e^{-ϕ} e^{-\frac{ϕ}{p}} dx \) is exactly \( (4.17) \), under the choice of \( ω(r) = ϕ'(r)e^{-\frac{ϕ(r)}{p}}, v(r) = ρ(r) = e^{-\frac{ϕ(r)}{p}}, p = q \).

(B) Result by Cianchi. Orlicz-norms counterpart of \( (H1) \):

\[
\|\frac{u}{d^{1+α}}\|_{L^B(G)} \leq C \left( \|\frac{u}{d^{α}}\|_{L^A(G)} + \|\nabla u\|_{L^A(G)} \right)
\]
has been established by Cianchi in [13]. Here $G \subset \mathbb{R}^n$ is a sufficiently regular domain, $A$ and $B$ are $N$-functions related by a certain domination condition (in particular it is possible to have $A = B$), $d(x) = \text{dist}(x, \partial G)$, and the measure considered is the Lebesgue measure. In our work, we need modular versions of (4.18); in general they do not come as its direct consequence. Note that in the case of homogeneous $N$-functions (4.18) is an extension of (4.10).

(C) Authors’ approach. Inequalities on intervals. In the forthcoming paper [29], the authors work out condition concerning $M, \varphi$ which ensure the validity of (3.12) on intervals $(a, b) \subset \mathbb{R}$, not excluding the cases $a = -\infty$ or $b = \infty$. See also the paper [43] devoted solely to the Hardy and Gagliardo-Nirenberg inequalities for Gaussian measure on $\mathbb{R}^n$.

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