Adaptive Dynamic Programming and Data-Driven Cooperative Optimal Output Regulation with Adaptive Observers

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Abstract—In this paper, a novel adaptive optimal control strategy is proposed to achieve the cooperative optimal output regulation of continuous-time linear multi-agent systems based on adaptive dynamic programming (ADP). The proposed method is different from those in the existing literature of ADP and cooperative output regulation in the sense that the knowledge of the exosystem dynamics is not required in the design of the exostate observers for those agents with no direct access to the exosystem. Moreover, an optimal control policy is obtained without the prior knowledge of the modeling information of any agent while achieving the cooperative output regulation. Instead, we use the state/input information along the trajectories of the underlying dynamical systems and the estimated exostates to learn the optimal control policy. Simulation results show the efficacy of the proposed algorithm, where both estimation errors of exosystem matrix and exostates, and the tracking errors converge to zero in an optimal sense, which solves the cooperative optimal output regulation problem.

Index Terms—Optimal control, reinforcement learning, adaptive dynamic programming, cooperative optimal control, cooperative optimal output regulation.

I. INTRODUCTION

In the past decade, cooperative control of multi-agent systems (MASs) has gained numerous attentions due to its importance in real-world applications. The cooperative output regulation problem (CORP) is mainly concerned in designing distributed controllers to achieve asymptotic tracking of a class of reference inputs, in addition to rejecting the disturbances in leader-follower MASs. The problem is usually formulated as leader tracking and disturbance rejection, wherein the subsystems (followers) are split into two groups. The first group of followers have direct access to the signal of the exosystem (leader), and the second group consists of those who do not have a direct access to it. The designed controller ensures the stability of the closed-loop system of the whole MASs.

Related Works

Due to its massive impact and effectiveness in engineering applications, the CORP has been widely investigated for both continuous-time linear systems [1]–[7], and discrete-time linear systems [8]–[11]. Such applications include connected and autonomous vehicles, cooperative robot reconnaissance, and satellite clustering [12]–[15]. In addition, the output regulation has been considered for nonlinear systems, see [16]–[20] and references therein.

Generally speaking, the CORPs are solved using either the feedback-feedforward control strategy, or the internal model principle. Moreover, the cooperative control problem is mainly solved in a distributed way, such that when not all systems in the network have a direct access to the exosystem, those systems reconstruct the exosystem signals through their communication channels with their direct neighbors [21]. For instance, a distributed observer is designed in [6] to estimate the exogenous signals for the agents with no direct access to exosystems.

Besides the issues of accessibility and maintaining the asymptotic tracking, having the full knowledge of the dynamics of each agent is a difficult task or impossible practically. Moreover, the complexity of modeling a dynamical system increases dramatically as the number of agents and their states increase. In order to address this challenge, the authors in [22] have developed an indirect adaptive control approach to solve the CORP with unknown system dynamics. However, the designed control policy may be far from being optimal, whether in its transient or steady state since the main objective of to achieve the closed-loop stability with rejecting disturbances. These gaps have been filled up in [23] and [24] where a solution to the cooperative output regulation problem (COORP) was proposed, such that a data-driven optimal controller is designed to approximate the feedback and feedforward control gains without the knowledge of MASs’ dynamics using the online state/input information collected along the trajectories of each subsystem. Using reinforcement learning (RL) and Bellman’s principle of optimality [25], adaptive dynamic programming (ADP) methods [26]–[39] are developed such that each agent can learn towards the optimal control policy by interacting with its unknown environment. With this learning framework, one can develop an adaptive optimal controller which behaves optimally on a long term without the knowledge of the system matrices. Differential game theory has also been considered with output regulation problems in [40]. It studies how systems interact with each other and considers them as players in a game, and provides them with utility functions and learning rules to achieve a collective goal [41].

Main Contributions

In this work, we propose an innovative adaptive optimal control design algorithm to obtain an approximated optimal feedback-feedforward controller by means of ADP in parallel with the exosystem estimator [22] so that each agent can achieve asymptotic tracking while rejecting disturbances without previous knowledge of the agents’ and the exosystem’s dynamics. First, our proposed algorithm is different from those presented in [13]–[15], [22], in the sense that the designed feedback and feedforward gains in our work are optimal and are achieved adaptively. Second, comparing to our previous work [23], [24], the prior knowledge of the exosystem dynamics and the frequencies of the exostates is not required. Therefore, this work is the first of its kind to solve the COORPs with adaptive observer using the feedback-feedforward strategy, wherein the designed feedback-feedforward gains are optimal. Third, neither the knowledge of the agents’ dynamics nor that of the exosystem is required in the proposed approach with guarantees convergence analysis of the proposed algorithm provided. Last but not least, the cooperative output regulation is achieved with rigorously stability analysis such that the tracking error along with the exostate estimation errors converge to zero asymptotically.
Structure
The rest of this paper is organized as follows. Section II formulates the problem and covers the preliminaries. The main results of the ADP approach along with the distributed exosystem estimator are presented in Section III. In Section IV, simulation results are given to illustrate the efficacy of the proposed algorithm. Last but not least, the conclusion is drawn in Section V.

Notations
The operator \( | \cdot | \) represents the Euclidean norm for vectors and the induced norm for matrices. \( \mathbb{Z}_+ \) denotes the set of nonnegative integers. The Kronecker product is represented by \( \otimes \), and the block diagonal matrix operator is denoted by \( \text{diag} \). \( I_n \) denotes the identity matrix of dimension \( n \) and \( 0_{n \times m} \) denotes a \( n \times m \) zero matrix. vec(\( A \)) = \begin{bmatrix} a_1^T \ a_2^T \ldots \ a_m^T \end{bmatrix}^T, \) where \( a_i \in \mathbb{R}^n \) is the \( i \)th column of \( A \in \mathbb{R}^{n \times m} \). For a symmetric matrix \( P = P^T \in \mathbb{R}^{m \times m} \), vecs(\( P \)) = \begin{bmatrix} p_{11} \ 2p_{12} \ldots \ 2p_{1m} \ p_{22} \ 2p_{23} \ldots \ 2p_{2m} \ldots \ p_{m-1,m} \ p_{mm} \end{bmatrix}^T \in \mathbb{R}_{+}^{m(m+1)} \). \( \mathbb{R}_{+}^{m(m+1)} \) is the set of positive definite \( m \times m \) symmetric matrices. \( \eta_i \) denotes the \( i \)th subsystem if and only if \( A_i = B_i \) is Hurwitz.

Remark 1: Unlike [22], where the matrix \( F \) has to be in the form \( \{0, 1, 2, 0\} \), such that the block \{0, 1\} is repeated \( q/2 \) times, the ADP-based approach in our work is more flexible and has no restrictions on the structure of the matrix \( F \).

Remark 2: Assumption 2 guarantees that at least one follower has access to the exogenous signals, and that all subsystems have access to at least one of their neighbors’ exosystem estimation \( \eta_j \), \( \forall j \in N_i \) so that \( \epsilon_i \neq 0 \), \( \forall \epsilon_i \in \mathbb{V} \). If the exostate is accessible by all the followers, the CORP can be solved by a decentralized control policy in the form of

\[ u_i = -K_i x_i + L_i v, \quad i \in \mathcal{F}, \]

where \( K_i \) and \( L_i \) are the feedback and forward gain matrices, respectively.

Definition 1: For any \( i \in \mathcal{F} \), a control feedback gain matrix \( K_i \in \mathbb{R}^{m \times m} \) is called stabilizing for the \( i \)th subsystem if and only if \( A_i - B_i K_i \) is Hurwitz.

Note that the decentralized controller (5) is not applicable according to the Assumption 2. If the exosystem matrix \( E \) is known to all followers, then one may design a distributed observer [6], [23] to estimate the exostate \( v \). However, if the exact knowledge of the exosystem matrix is not available, it is impossible to use the distributed observer proposed in [6], [23] to estimate the exostates signals. This barrier has been removed by proposing a distributed adaptive estimator in [22], in which the estimation does not require a prior knowledge of the exosystem matrix. We will take the advantage of this result to develop a new ADP algorithm to solve the COORP with adaptive observer.

III. SOLVING COORP WITH ADAPTIVE OBSERVER
In this section, an ADP-based approach with distributed adaptive observer is proposed with stability and convergence analysis provided. To begin with, we consider observing the unknown exosystem states. The local observation error for the agent \( i \) is defined as follows.

\[ \epsilon_i = \sum_{j=1}^{N} a_{ij} (\eta_j - \eta_i) + m_{ij} (\eta_i - v), \]

such that \( \eta_i \) and \( \epsilon_i \) are vectors in the form of

\[ \eta_i = [\eta_{i,1} \ldots \eta_{i,q}]^T, \quad \epsilon_i = [\epsilon_{i,1} \ldots \epsilon_{i,q}]^T. \]

By having \( \epsilon_i \to 0 \), \( \forall \epsilon_i \in \mathbb{V} \), it is then achievable that \( \eta_i \to v, \forall v \in \mathbb{V} \). This enables us to reconstruct the exostate \( v \) for non-target nodes that do not have access to it. The distributed adaptive observer is as follows.

\[ \dot{\eta}_i = \dot{E}_i \eta_i + (\dot{\Delta}_m - \dot{E}_i) \epsilon_i, \]

with \( \dot{\Delta}_m \in \mathbb{R}^{q \times q} \) a Hurwitz diagonal matrix defined as

\[ \dot{\Delta}_m = -\text{diag}(a_r I_2)_{q/2}, \quad a_r > 0, \quad r = 1, \ldots, q/2. \]

The estimation of the exosystem matrix \( E \) for the \( i \)th follower is

\[ \tilde{E}_i = \text{diag} \left[ \begin{bmatrix} 0 & (\dot{\Delta}_i)_{i} \end{bmatrix} \right]_{q/2}, \]
where \( (\hat{w}_i) \) is the estimate of the frequencies of the leader system for the \( i \)-th subsystem described by the following dynamical equation.
\[
\left( \begin{array}{c}
\dot{w}_i \\
\end{array} \right) = \kappa_i \left( \bar{\eta}_i(2r-1)E_i(2r) - \bar{\eta}_i(2r)E_i(2r-1) \right),
\]
with \( \kappa_i > 0 \) being a constant design gain.

**Lemma 1** ([22]): Under Assumption 2, by considering (8)-(11) the adaptation law (11) guarantee that \( \bar{\eta}_i \to \eta_i \) and \( \bar{E}_i \to E \) as \( t \to \infty \), \( \forall i \) in \( \mathcal{V} \).

With the estimated exosystem matrix and exostate, we will introduce the ADP strategy to compute the optimal feedback-feedforward control policy to solve the COORP.

The COORP studied considers both transient and steady state responses of each subsystem. The formulation follows the traditional linear optimal control output regulation problem [23] that the optimal distributed output regulation problem needs to solve the following two problems besides the CORP.

**Problem 1:**
\[
\min_{(X_i^T, U_i)} \text{Tr}(X_i^T\hat{Q}_iX_i + U_i^T\bar{R}_iU_i),
\]
subject to \( X_iE = A_iX_i + B_iU_i + D_i \),
\[
0 = C_iX_i + F_i,
\]
where \( \hat{Q}_i = (\hat{Q}_i)^T > 0 \) and \( \bar{R}_i = (\bar{R}_i)^T > 0 \), for all \( i \) in \( \mathcal{F} \).

Based on Assumption 3, the solvability of the regulator equations defined by (13)-(14) is guaranteed and the pairs \((X_i, U_i)\) exist for any matrices \( D_i \) and \( F_i \), \( \forall i \) in \( \mathcal{F} \); see [43]. Additionally, the solution to Problem 1, i.e., \((X_i^T, U_i)\) is unique.

**Problem 2:**
\[
\min_{u_i} \int_0^\infty \left[ |c_i|q_i + |\bar{u}_i|r_i \right] dt,
\]
subject to \( \dot{x}_i = A_i\bar{x}_i + B_iu_i \),
\[
e_i = C_i\bar{x}_i,
\]
where \( q_i = (q_i)^T \geq 0 \), \( r_i = (r_i)^T \geq 0 \), with \( A_i, \sqrt{Q}C_i \) being observable for all \( i \) in \( \mathcal{F} \). The equations (16)-(17) form the error system with \( \bar{x}_i := x_i - X_iv \) and \( \bar{u}_i := u_i - U_iv \).

Note that if the followers’ dynamics in (2) are known, one can develop a distributed optimal control mechanism in the following form.
\[
u_i^*(K^*_i, L^*_i) = -K^*_i x_i + L^*_i x_i,
\]
where \( K_i^* = R_i^{-1}B_i^TP_i^* \), and \( P_i^* \) is the unique solution of the following algebraic Riccati equation (ARE)
\[
A_i^TP_i^* + P_i^*A_i + C_i^TQ_iC_i - P_i^*B_iR_i^{-1}B_i^TP_i^* = 0.
\]

The solutions to the regulator equations (13)-(14), i.e., \((X_i, U_i)\), form the optimal feedforward gain matrix such that
\[
L^*_i = U_i + K^*_i X_i, \quad \forall i \in \mathcal{F}.
\]

It is remarkable that equation (19) is nonlinear in \( P^*_i \). Therefore, in this paper we consider an iterative method to solve \( P^*_i \), i.e., ADP. In particular, we use PI since the rate of convergence of PI is quadratic, since it is a Newton-Raphson based method. In PI, the iterative process to find the optimal control policy is done by alternating two stages, i.e., policy evaluation, and policy improvement. The following lemma shows the convergence of (19) in the sense of the PI method.

**Lemma 2** ([42]): Let \( K_{i,0} \in \mathbb{R}^{n_i \times n_i} \) be a stabilizing feedback gain matrix \( \forall i \in \mathcal{F} \) the matrix \( P_{i,k} = (P_{i,k})^T > 0 \) be the solution of the following gain equation
\[
P_{i,k}(A_i - B_iK_{i,k-1}) + (A_i - B_iK_{i,k-1})^TP_{i,k} + C_i^TQ_iC_i
+ K_{i,k-1}^TR_iK_{i,k-1} = 0,
\]
and the control gain matrix \( K_{i,k} \), with \( k = 1, 2, \ldots \), are defined recursively by
\[
K_{i,k} = R_i^{-1}B_i^TP_{i,k-1}.
\]

Then the following properties hold for any \( k \in \mathbb{Z}_+, i \in \mathcal{F} \)
1) The matrix \( A_i - B_iK_{i,k} \) is Hurwitz.
2) \( P_i^* \preceq P_{i,k} \preceq P_{i,k-1} \).
3) \( \lim_{k \to \infty} K_{i,k} = K^*_i \), \( \lim_{k \to \infty} P_{i,k} = P_i^* \).

Since the unavailability of the system and exosystem dynamics is considered in our approach, the states, inputs and exostates information collected along the trajectories of the underlying dynamical systems are used to learn the optimal feedback and feedforward gain matrices \( K^*_i \) and \( L^*_i \) for all \( i \) in \( \mathcal{F} \). By solving \( U_i^* \) and \( X_i^* \) from (13)-(14), one is able to solve for \( L_i^* \) from (20).

Now we are ready to introduce the variables \( \tilde{x}_{ij} = x_i - X_jv \), \( j = 0, 1, 2, \ldots, h_i + 1 \) with \( h_i = (n_i + p_i)q \) being the dimension of the null space of \((I_i \otimes C_i)\), where the following is met: \( X_i0 = 0_{n_i \times q}, X_i1 \in \mathbb{R}^{n_i \times q} \) such that \( C_iX_i1 + F_i = 0 \), and \( X_ij \in \mathbb{R}^{n_i \times q} \), \( \forall j \in 2, 3, \ldots, h_i + 1 \), such that all \( vec(X_{ij}) \) form a basis for \( ker(I_i \otimes C_i) \), where \( ker() \) denotes the null space.

The definition of \( \tilde{x}_{ij} \) enables us to solve the Sylvester map of trail matrices \( X_{ij} \) which is in itself crucial for solving the regulator equations to finally approximate \( L^* \) and \( K^* \) without previous knowledge of the systems’ matrices \( (A_i, B_i, D_i) \). By taking the time derivative along the trajectories of \( \tilde{x}_{ij} \) we have
\[
\dot{\tilde{x}}_{ij} = \tilde{x}_i - X_jv
= A_i\tilde{x}_i + B_iu_i + D_i v - X_j Ev
= A_{i,k}\tilde{x}_{ij} + B_i(\tilde{K}_{i,k}\tilde{x}_{ij} + u_i) + (D_i - S_i(X_{ij}))v,
\]
where \( A_{i,k} = A_i - B_i\tilde{K}_{i,k} \) and \( S_i(X) = XE - A_iX \) is a Sylvester map, \( S_i : \mathbb{R}^{n_i \times q} \to \mathbb{R}^{n_i \times q} \). By taking the integration over the time interval \([t, t + \delta t], \delta t > 0\), we obtain
\[
|\tilde{x}_{ij}(t + \delta t)|^2p_{i,k} - |\tilde{x}_{ij}(t)|^2p_{i,k}
\]
\[
= \int_t^{t + \delta t} \left[|\tilde{x}_{ij}(t')|^2(A_i^TP_{i,k} + P_{i,k}A_i) + 2(u_i + K_{i,k}\tilde{x}_{ij})^TP_{i,k}\tilde{x}_{ij}
+ 2v^T(D_i - S_i(X_{ij}))^TP_{i,k}\tilde{x}_{ij}\right]dt
\]
\[
= \int_t^{t + \delta t} \left[-|\tilde{x}_{ij}(t')|^2Q_{i,k}^TP_{i,k}\tilde{x}_{ij}
+ 2(u_i + K_{i,k}\tilde{x}_{ij})^TR_{i,k}\tilde{x}_{ij} + 2v^T(D_i - S_i(X_{ij}))^TP_{i,k}\tilde{x}_{ij}\right]dt.
\]

Using Kronecker product properties, (24) can be written in a compact form as follows, wherein the approximated values of \( P_{i,k} \) and \( K_{i,k+1} \) can be solved in the sense of least square errors
\[
\Psi_{ijk} \left[ \begin{array}{c}
vec(P_{i,k}) \\
vec(K_{i,k+1})
\end{array} \right] = \Phi_{ijk},
\]
where
\[
\Phi_{ijk} = \left[ \delta_{i,j+1} \tilde{x}_{ij}, -2\Gamma_{x_{ij}}x_{ij}(I_i \otimes K_{i,k}^T R_i) - 2\Gamma_{x_{ij}}u_i(I_i \otimes R_i),
- 2\Gamma_{x_{ij}}v \right]
\]
\[
\Psi_{ijk} = \left[ vec(a(t_1)) - vec(a(t_2)), \ldots, vec(a(t_{l-1})) - vec(a(t_l)) \right]^T.
\]
\[
\Gamma_{a,b} = \left[ \int_{t_0}^{t_1} a \otimes b d\tau, \int_{t_1}^{t_2} a \otimes b d\tau, \ldots, \int_{t_{l-2}}^{t_{l-1}} a \otimes b d\tau \right]^T.
\]
The time sequence \( \{ t_l \} \) is a strictly increasing sequence. The uniqueness of the solution to equation (25) is guaranteed when the following rank condition is met:

\[
\text{rank} \left( \sum_{i,j} \Gamma_{j,i} \tilde{x}_j(t_l), \Gamma_{j,i} n_i \right) = \frac{n_i (n_i + 1)}{2} + (m_i + q_i) n_i, \quad (26)
\]

**Remark 3:** In order to satisfy the condition in (26), exploration noise is added to the applied input during the learning phase. The exploration noise is usually random noise, random sinusoidal signals, or summation of sinusoidal signals with different frequencies.

The general solution to the regulator equations (13)-(14) is obtained from the following.

\[
X_i = X_{i1} + \sum_{j=2}^{h_i+1} \alpha_{ij} X_{ij}, \quad \alpha_{ij} \in \mathbb{R},
\]

\[
S_i(X_i) = S_i(X_{i1}) + \sum_{j=2}^{h_i+1} \alpha_{ij} S_i(X_{ij}) = B_i U_i + D_i. \quad (27)
\]

In matrix form, the equation in (27) can be written as

\[
A_i \dot{X}_i = b_i,
\]

where

\[
A_i = [A_1 \quad A_2],
\]

\[
A_{i1} = \begin{bmatrix} \text{vec}(S_i(X_{i2})) & \ldots & \text{vec}(S_i(X_{i,h+1})) \end{bmatrix},
\]

\[
A_{i2} = \begin{bmatrix} 0 & -I_q \otimes (P_{i,k}^{-1} K_i k R_i) \n \n -I_{n_i,q} \n 0 \n \end{bmatrix},
\]

\[
X_i = [x_{i2}, \ldots, x_{i,h+1}, \text{vec}(X_i)^T, \text{vec}(U_i)^T]^T,
\]

\[
b_i = \begin{bmatrix} \text{vec}(-S_i(X_{i1}) + D_i) \n \n -\text{vec}(X_{i1}) \end{bmatrix}.
\]

In Theorem 1, we show that although the estimations of the exostates are used instead of their actual values, the cooperative output regulation can be achievable. Furthermore, the ADP algorithm to solve the COORP with the adaptive observer is shown in Algorithm 1 with proof of convergence shown in Theorem 2.

**Theorem 1:** Given Assumptions 1-3, and under the system described by (1)-(11), if \( \hat{A}_i = A_i - B_i K_i \) is Hurwitz \( \forall i \in \mathcal{F} \), then the closed-loop state-feedback controller \( u_i = u_i^*(K_i, \hat{L}_i) \) achieves the cooperative output regulation, where \( \hat{L}_i = K_i X_i + \hat{U}_i \), and the pairs \((X_i, \hat{U}_i)\) are the solutions of the following regulator equations.

\[
\dot{X}_i = \dot{\hat{X}}_i = A_i \hat{X}_i + B_i \hat{U}_i + D_i,
\]

\[
0 = C_i \hat{X}_i + F_i, \quad \forall i \in \mathcal{F}.
\]

**Proof:** Let \( \tilde{x}_i = x_i - \hat{x}_i, \tilde{u}_i = u_i - \hat{U}_i \), and \( \tilde{v} = v - \hat{v} \). Taking the time derivative of \( \tilde{x}_i \) the following equation is obtained.

\[
\dot{\tilde{x}}_i = \dot{\hat{x}}_i - \dot{\hat{x}}_i \hat{t}_i = A_i \tilde{x}_i + B_i \tilde{u}_i + D_i \tilde{v}_i - \dot{\hat{x}}_i \hat{t}_i \hat{t}_i = A_i \tilde{x}_i + B_i \tilde{u}_i + K_i \tilde{x}_i + D_i \tilde{v}_i - \dot{\hat{x}}_i \hat{t}_i \hat{t}_i = A_i \tilde{x}_i + B_i \tilde{u}_i + K_i \tilde{x}_i - \hat{L}_i \hat{t}_i + (D_i + B_i \hat{L}_i) \hat{t}_i - \dot{\hat{x}}_i \hat{t}_i \hat{t}_i = A_i \tilde{x}_i + B_i \tilde{u}_i + (K_i \tilde{x}_i - \hat{L}_i \hat{t}_i) + (D_i + B_i \hat{L}_i) \hat{t}_i - \dot{\hat{x}}_i \hat{t}_i \hat{t}_i.
\]

By (18), we have \( \tilde{u}_i = -K_i \tilde{x}_i + \hat{L}_i \hat{t}_i \). In addition, since \( v \) is bounded, so is \( \hat{t}_i \). Given \( A_i \), is Hurwitz, we have (29) is input state stable with \( -X_i (\hat{\alpha}_m - \hat{\hat{\alpha}}) \hat{t}_i \) and \( \hat{t}_i \) as inputs. In other words, there exist a function \( \beta \) of class \( \mathcal{KL} \) and a function \( \gamma \) of class \( \mathcal{K} \) such that

\[
|\tilde{x}_i(t)| \leq \beta(|\tilde{x}_i(0)|, t) + \gamma \left( \sup_{0 \leq \tau \leq t} \{|\hat{t}_i(\tau)|, |\hat{Q}_i(\tau)|\} \right), \quad (30)
\]

where \( \hat{Q}_i = \hat{X}_i (\hat{\alpha}_m - \hat{\hat{\alpha}}) \hat{t}_i \). Therefore, with the existence of the estimate of \( v, \tilde{x} \) remains bounded. In addition, on the basis of [22], we have \( \lim \hat{t}_i = 0 \) and \( \lim \hat{v}_i = 0 \) \( \forall i \in \mathcal{F} \). Hence, it is concluded that \( \lim \tilde{x}_i = 0 \) and \( \lim \hat{t}_i = 0 \). The proof is thus completed.

**Remark 4:** If the rank condition in (26) is satisfied, then for any small constant \( c > 0 \) there exist constants \( \kappa_c > c \), \( r = 1, 2, \ldots, q/2 \), and \( k^* \in \mathbb{Z}_+ \) such that the sequences \( \{P_{i,k}\}_{k=0}^{\infty} \) and \( \{K_{i,k}\}_{k=1}^{\infty} \) learned from Algorithm 1 satisfy the inequalities \( |P_{i,k}^{(k^*)} - P_{i}^{(k)}| < c \) and \( |K_{i,k}^{(k^*)} - K_{i}^{(k)}| < c \), respectively.

**Proof:** The condition (26) ensures that (25) has a unique solution. As the convergence of steps 6-12 has been shown in [23], we can always find a small constant \( c > 0 \) such that the pairs \( (P_{i,k}^{(k^*)}, K_{i,k}^{(k^*)}) \) is close enough to \( (P_{i,k}, K_{i,k}) \) solved from (21)-(22) satisfying the inequalities \( |P_{i,k}^{(k^*)} - P_{i}^{(k)}| < c \) and \( |K_{i,k}^{(k^*)} - K_{i}^{(k)}| < c \), for every \( k \in \mathbb{Z}_+ \) and \( i \in \mathcal{F} \). Hence, from Lemma 1 and Theorem 1, it is always guaranteed that there exists a constant \( c_2 > 0 \) such that \( |P_{i,k}^{(k^*)} - P_{i}^{(k)}| < c_2 \) and \( |K_{i,k}^{(k^*)} - K_{i}^{(k)}| < c_2 \), for every \( k \in \mathbb{Z}_+ \) and \( i \in \mathcal{F} \). Since \( \hat{v}_i \), defined in (11) is uniformly bounded [22, Theorem 1]. Using the triangular inequality, we can find an iteration index \( k^* \) and a small constant \( c > 0 \) such that the inequalities \( |P_{i,k}^{(k^*)} - P_{i}^{(k)}| < c \) and \( |K_{i,k}^{(k^*)} - K_{i}^{(k)}| < c \) are satisfied. The proof is thus completed.

**Remark 5:** It is worth mentioning that the steps 10-16 in Algorithm 1 are PI-based. A value iteration (VI) based method, similar to the one developed in our previous work in [44], can be used to replace these steps. In the VI-based framework, an initial stabilizing policy is not required. However, its convergence rate is slower than the quadratic convergence rate of PI used in Algorithm 1.

**Algorithm 1** The proposed Algorithm 1 is an off-policy learning method. Each follower learns its own optimal policy independently, which makes it more practical, especially for large scale systems.

**IV. SIMULATION AND RESULTS**

In this section, we illustrate the efficacy of the proposed Algorithm 1 in an example, in which the system consists of four followers and a leader as depicted in Fig. 1. The exosystem (#1) is a harmonic oscillator described by the matrix \( E \) and the followers (#1-4) are described by the matrices \( A_i, B_i, C_i, D_i, \) and \( F_i \).
In this example we assume that there is no prior knowledge of the dynamics of the system \((A_i, B_i, \text{ and } D_i)\), or the exosystem dynamics \((E)\). The system matrices for the system described in (1)-(3) are shown below for simulation purposes.

\[
A_i = \begin{bmatrix} 1 & 1 + i & 0 \\ 0 & 2 & -0.5i \\ 1 & 0 & 1 + i \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, \quad C_i = \begin{bmatrix} 1/i & 0 & 0 \end{bmatrix},
\]

\[
D_i = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1.5i \\ 0 & 1 & 0 \end{bmatrix}, \quad F_i = \begin{bmatrix} -0.75i & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
\]

and \(E = \text{bdiag} \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -0.75 & 0 \end{bmatrix} \right) \).

The weighting matrices of the cost function are \(Q = I_3\) and \(R = 1\), the initial values are \(v(0) = [0 \ 1 \ 0 \ 0.5]\), \((\bar{v}_i(0) = 0)\), and the rest of the parameters are \(\varepsilon_i = 10^{-4}\), \(\kappa_r = [40 \ 40]\) and \(\alpha_r = [15 \ 15] \), \(\gamma_i = 1, 2, 3, 4\). During the time interval \(0 \leq t \leq 8s\), an essentially bounded exploration noise \(\zeta_i\) is added to the applied initial control policy. Using Algorithm 1, first \(\hat{E}_i\) is estimated, then the approximations of the optimal feedback and feedforward control gain matrices \(K_i^*\) and \(L_i^*\) are calculated, respectively. Fig. 2 depicts that \(P_{i,k}\) obtained by Algorithm 1 converge to their optimal values \(P_i^*\) obtained by solving directly from (19), and the convergence is achieved in less than or equal to 19 iterations. The optimal solution to the regulator equations obtained is used to calculate the feedforward gains, which are shown with their corresponding actual ones for the sake of comparison.

\[
L_{14}^{(14)} = \begin{bmatrix} 2.8801 & -11.9485 & 16.4917 & 12.4644 \\ 2.8801 & -11.9484 & 16.4918 & 12.4641 \\ 1.0720 & -6.2090 & 15.1043 & 7.4341 \\ 1.0721 & -6.2089 & 15.1043 & 7.4340 \end{bmatrix},
\]

\[
L_{1}^{*} = \begin{bmatrix} 2.8801 & -11.9485 & 16.4917 & 12.4644 \\ 2.8801 & -11.9484 & 16.4918 & 12.4641 \\ 1.0720 & -6.2090 & 15.1043 & 7.4341 \\ 1.0721 & -6.2089 & 15.1043 & 7.4340 \end{bmatrix},
\]

\[
L_{1}^{*} = \begin{bmatrix} -3.1127 & -7.3517 & 13.5064 & 5.2960 \\ -3.1117 & -7.3508 & 13.5063 & 5.2923 \\ -8.5758 & -9.4777 & 13.3007 & 4.4879 \\ -8.5725 & -9.4729 & 13.3089 & 4.4654 \end{bmatrix}.
\]

From Fig. 2 and the above-mentioned feedforward gain matrices, it can be noticed that the approximated control policy converges to the optimal policy, while neither the system dynamics nor that of the exosystem are known. Moreover, Fig. 3 demonstrates the convergence of the tracking error and the estimation error. Finally, one can observe from Fig. 4 that all the followers can achieve asymptotic tracking while rejecting nonvanishing disturbance.

V. CONCLUSION

This paper studies the cooperative output regulation problem of a class of continuous-time linear multi-agent systems with unknown system dynamics. A distributed control policy is derived by first estimating the exosystem dynamics for each follower, then adaptive dynamic programming (ADP) is used to approximate the optimal solution to the regulator equations. The effectiveness of the proposed algorithm and the ability to achieve asymptotic tracking while rejecting nonvanishing disturbances are demonstrated by both theoretical analysis and performed simulation. Future work includes extending this work to a class of nonlinear systems with robust analysis subject to external disturbances.

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