GLOBAL SMOOTH SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH MAGNETIC EFFECT

DAIWEN HUANG
Institute of Applied Physics and Computational Mathematics
P. O. Box 8009, Beijing 100088, China

JINGJUN ZHANG*
College of Mathematics, Physics and Information Engineering
Jiaxing University, Zhejiang 314001, China

Dedicated to Professor Boling Guo on the occasion of his 80th birthday

Abstract. We consider the Cauchy problem of the nonlinear Schrödinger equation with magnetic effect, and prove global existence of smooth solutions and decay estimates for suitably small initial data. The key step in our analysis is to exploit the null structures for the phases, which allow us to close our argument in the framework of space-time resonance method.

1. Introduction. In this paper, we are concerned with the global existence of smooth solutions for the Cauchy problem of a set equations arising from plasma physics. The equations under study read

\[
\begin{align*}
\frac{\partial E}{\partial t} + \Delta E + |E|^2E + iE \times B &= 0, \\
\gamma^{-2}B_{tt} + \Delta^2 B &= i \Delta \nabla \times (\nabla \times (E \times \bar{E}))
\end{align*}
\]

with initial data

\[E(0, x) = E_0(x), \quad (B(0, x), B_t(0, x)) = (B_0(x), B_1(x)).\]  

Here, \(E : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{C}^3\) is the slowly varying amplitude of the high-frequency electric field (\(\overline{E}\) denotes the complex conjugate of \(E\)), and the function \(B : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3\) is the self-generated magnetic field. \(\gamma\) is the speed of electron, and the notation \(\times\) means the cross product for \(\mathbb{R}^3\) or \(\mathbb{C}^3\) valued vectors.

System (1) is a simplified model in plasma physics. It describes the nonlinear interaction between plasma-wave and particles [22], especially when the phase speed of plasma wave is much less than the speed of ions so that the fluctuation of the density satisfies a stationary equation

\[n = -\frac{|E|^2}{8\pi(T_i + T_e)}.\]  

It also exhibits that the self-generated magnetic field influences the high-frequency electric field directly by the coupled term \(iE \times B\), which in turn affects the density

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* Corresponding author: Jingjun Zhang.
of particles in a indirect way through the equation of \( n \). Therefore, system (1) can also be regarded as a magnetic type Zakharov system with \( n \) satisfying (3).

Omitting the effect of the magnetic field \( B \), the classical Zakharov system [31] plays an important role in plasmas, and there are a lot of mathematical results on the global existence of weak solutions or smooth solutions, local well-posedness and scattering theory for this system; cf. [1, 2, 3, 4, 7, 12, 13, 14, 15, 21, 28, 30] and the references therein. In the presence of magnetic field, we refer to [22, 26] for the background of physical importance and [8, 16, 17, 20, 25, 27, 32] for the mathematical theories of the magnetic type Zakharov systems.

For the Schrödinger equation with magnetic effect, we refer to the work [9, 10] in the case that the magnetic field satisfies

\[
\Delta B - i\alpha \nabla \times (\nabla \times (E \times E)) + \beta B = 0, \quad \alpha > 0, \quad \beta \leq 0.
\]

Note that system (1) reduces to the cubic nonlinear Schrödinger equation if we ignore the effect of \( B \) in (1) or let \( \gamma \to \infty \) in the equation of \( B \). Such equation has been studied by many researchers, see for example [6, 18, 19, 24]. However, as far as we know, there are no results on the global existence of the solution for system (1). Hence, in this work, we are interested in the mathematical theories (especially, the global dynamics) for this system.

For \( s \in \mathbb{R} \), we denote \( H^s \) (or \( \dot{H}^s \)) the inhomogeneous (or homogeneous) Sobolev spaces, equipped with the norm

\[
\|u\|_{H^s} := \| (I - \Delta)^{s/2} u \|_{L^2} = \| (1 + |\xi|^2)^{s/2} \hat{u} \|_{L^2},
\]

(or \( \|u\|_{\dot{H}^s} := \| (-\Delta)^{s/2} u \|_{L^2} = \| |\xi|^s \hat{u} \|_{L^2} \)),

where \( \hat{u} = \hat{u}(\xi) \) is the Fourier transform of \( u \), namely,

\[
\hat{u}(\xi) := \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x) dx.
\]

Now we state the main result of this work.

**Theorem 1.1.** Assume \( N \geq 100 \), \( \gamma > 0 \) and \( \gamma \neq 1 \). Suppose the initial data satisfies

\[
\| E_0 \|_{H^N} + \| xE_0 \|_{L^2} + \| x^2 E_0 \|_{L^2} \leq \epsilon_0,
\]

\[
\| (B_0, (-\Delta)^{-1} B_1) \|_{H^{N-1} \cap \dot{H}^{-1}} + \| x(B_0, (-\Delta)^{-1} B_1) \|_{L^2} + \| x^2(B_0, (-\Delta)^{-1} B_1) \|_{L^2} \leq \epsilon_0,
\]

where \( 0 < \epsilon_0 \ll 1 \). Then the Cauchy problem (1)–(2) has a unique global solution such that

\[
\| E(t) \|_{L^\infty} \lesssim \frac{\epsilon_0}{(1+t)^{5/4}}, \quad \| B(t) \|_{L^\infty} \lesssim \frac{\epsilon_0}{(1+t)^{4/3}}.
\]

This paper is organized as follows. In Section 2, we write system (1) into an integral system by using the profiles, and introduce the work space and the linear decay estimates. Section 3 is devoted to dealing with the energy estimate. The weighted estimates for the magnetic field and the Schrödinger component are given in Section 4 and Section 5, respectively. In Section 6, we present the proof of our main result.
2. Preliminaries. In order to prove Theorem 1.1, we rewrite system (1) into a first order system. To this end, we set

\[ M := B + i\gamma^{-1}(-\Delta)^{-1}B, \]  

(4)

then Cauchy problem (1)–(2) is reduced to a set of unknowns \((E, M)\)

\[
\begin{aligned}
    iE_t + \Delta E &= -|E|^2 E - \frac{1}{2} iE \times M - \frac{1}{2} iE \times \overline{M}, \\
    iM_t + \gamma\Delta M &= i\gamma \nabla(\nabla \cdot (E \times \overline{E})) - i\gamma\Delta(E \times \overline{E})
\end{aligned}
\]  

(5)

with initial data

\[ E(0, x) = E_0(x), \quad M(0, x) = M_0(x), \]  

(6)

where we have used the identity

\[ \nabla(\nabla \cdot u) = \Delta u + \nabla \times (\nabla \times u). \]

Now we restate Theorem 1.1 in terms of \((E, M)\).

**Theorem 2.1.** Let \(N \geq 100\), \(\gamma > 0\) and \(\gamma \neq 1\). Then there exists a positive constant \(\epsilon_0 \ll 1\) such that if the initial data satisfies

\[
\begin{align*}
    \|E_0\|_{H^N} + \|xE_0\|_{L^2} + \|x^2E_0\|_{L^2} &\leq \epsilon_0, \\
    \|M_0\|_{H^{N-1} \cap \dot{H}^{-1}} + \|xM_0\|_{L^2} + \|x^2M_0\|_{L^2} &\leq \epsilon_0,
\end{align*}
\]  

(7, 8)

then the Cauchy problem (5)–(6) admits a unique global solution \((E, M)\) satisfying

\[
\begin{align*}
    E &\in C([0, \infty); H^N), \\
    M &\in C([0, \infty); H^{N-1} \cap \dot{H}^{-1}), \\
    \|E(t)\|_{L^\infty} &\lesssim \frac{\epsilon_0}{(1 + t)^{5/4}}, \\
    \|M(t)\|_{L^\infty} &\lesssim \frac{\epsilon_0}{(1 + t)^{4/3}}.
\end{align*}
\]  

(9)

In view of (4), Theorem 1.1 follows immediately from the above theorem. Hence, from now on, we mainly focus on the proof of Theorem 2.1. To prove this result, we work on the framework of space-time resonance method \([11, 21, 29]\). Define the profiles

\[ f = e^{-it\Delta} E, \quad g = e^{-it\gamma\Delta} M, \]  

(10)

from system (5), \((f, g)\) satisfies

\[
\begin{align*}
    f_t &= e^{-it\Delta}(E_t - i\Delta E) = e^{-it\Delta}(|E|^2 E + \frac{1}{2} iE \times M + \frac{1}{2} iE \times \overline{M}), \\
    g_t &= e^{-it\gamma\Delta}(M_t - i\gamma\Delta M) = e^{-it\gamma\Delta}(\nabla(\nabla \cdot (E \times \overline{E})) - \Delta(E \times \overline{E})).
\end{align*}
\]  

(11)

Therefore, we have

\[
\begin{align*}
    \hat{f}(t, \xi) &= \hat{f}(0, \xi) + \frac{i}{(2\pi)^3} \int_0^t \int_{\mathbb{R}^3} e^{is\varphi(\xi, \eta, \sigma)} \hat{f}(s, \xi - \eta) \hat{f}(s, \eta - \sigma) d\varphi d\eta ds ds \\
    &- \frac{1}{2(2\pi)^{3/2}} \int_0^t \int_{\mathbb{R}^3} e^{i\xi \varphi(\xi, \eta, \sigma)} \hat{f}(s, \xi - \eta) \times \hat{g}(s, \eta) d\eta ds \\
    &- \frac{1}{2(2\pi)^{3/2}} \int_0^t \int_{\mathbb{R}^3} e^{i\xi \varphi(\xi, \eta, \sigma)} \hat{f}(s, \xi - \eta) \times \hat{g}(s, \eta) d\eta ds,
\end{align*}
\]  

(12)
and
\[ \hat{g}(t, \xi) = \hat{g}(0, \xi) + \frac{\gamma}{(2\pi)^{3/2}} \int_0^t \int_{\mathbb{R}^3} e^{i\xi \phi(s, \eta)} |\xi|^2 (\hat{f}(s, \xi - \eta) \times \hat{f}(s, \eta)) d\eta ds \]
\[ - \frac{\gamma}{(2\pi)^{3/2}} \int_0^t \int_{\mathbb{R}^3} e^{i\xi \phi(s, \eta)} (\xi \cdot (\hat{f}(s, \xi - \eta) \times \hat{f}(s, \eta))) d\eta ds, \]
(13)
where the phases \( \phi, \psi_\pm \) and \( \phi \) are
\[
\phi(\xi, \eta, \sigma) := |\xi|^2 - |\xi - \eta|^2 - |\eta - \sigma|^2 + |\sigma|^2 = 2\xi \cdot \eta + 2\eta \cdot \sigma - 2|\eta|^2,
\]
\[
\psi_+(\xi, \eta) := |\xi|^2 - |\xi - \eta|^2 - \gamma|\eta|^2 = 2\xi \cdot \eta - (\gamma + 1)|\eta|^2,
\]
\[
\psi_- (\xi, \eta) := |\xi|^2 - |\xi - \eta|^2 + \gamma|\eta|^2 = 2\xi \cdot \eta + (\gamma - 1)|\eta|^2,
\]
\[
\phi(\xi, \eta) := \gamma|\xi|^2 - |\xi - \eta|^2 + |\eta|^2 = (\gamma - 1)|\xi|^2 + 2\xi \cdot \eta.
\]
(14)
The integral identities (12)–(13) are the main equations that we will discuss later. Inspired by the work [21, 29], the important fact we observe is that there are several implicit relations for the phases
\[ \nabla_\xi \phi = \nabla_\sigma \phi, \]
\[ \nabla_\xi \psi_+ = \frac{2\eta}{\gamma + 1} |\eta|^2 \psi_+ - \frac{2\eta}{\gamma + 1} \eta \cdot \nabla_\eta \psi_+, \]
\[ \nabla_\xi \psi_- = \frac{2\eta}{(\gamma - 1)} |\eta|^2 \psi_- + \frac{2\eta}{(\gamma - 1)} \eta \cdot \nabla_\eta \psi_-), \]
\[ 2\xi = \nabla_\eta \phi. \]
(15)
The above identities show some null conditions of the phases, hence, we may regard the resonances for these four phases are null.

Now for any \( T > 0 \), we define the norm associated to our work space
\[ \|E\|_{X_T} := \sup_{t \in [0, T)} \left( \|E(t)\|_{H^{N}} + \|xf(t)\|_{L^2} + (1 + t)^{-1/2} |||x|^2f(t)||_{L^2} \right), \]
\[ \|M\|_{Y_T} := \sup_{t \in [0, T)} \left( \|M(t)\|_{H^{N-1} \cap H^{-1}} + \|xg(t)\|_{L^2} + (1 + t)^{-1/3} |||x|^2g(t)||_{L^2} \right). \]
Moreover, we set
\[ A_T := \|E\|_{X_T} + \|M\|_{Y_T}. \]
(16)

One of the basic ingredients in applying the method of space-time resonance is the linear dispersive estimate. For the Schrödinger operator \( e^{it\Delta} \), it is known that for \( p \in [2, +\infty] \),
\[ \|e^{it\Delta} f\|_{L^p(\mathbb{R}^3)} \lesssim \frac{1}{t^{\frac{3}{2}(1/2-1/p)}} \|f\|_{L^{p'}(\mathbb{R}^3)}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \]
\[ \|e^{it\Delta} f\|_{L^p(\mathbb{R}^3)} \lesssim \frac{1}{t} \|xf\|_{L^2(\mathbb{R}^3)}, \]
which can be found, for example, in [5]. Combining (16) and the inequalities
\[ \|f\|_{L^1(\mathbb{R}^3)} \lesssim \|xf\|_{L^2(\mathbb{R}^3)}^{1/2} |||x|^2f||_{L^2(\mathbb{R}^3)}^{1/2}, \]
\[ \|f\|_{L^{4/3}(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)}^{1/4} |||x|^4f||_{L^2(\mathbb{R}^3)}^{3/4}, \]
we obtain the following linear decay estimates

\[ \|e^{it\Delta}f\|_{L^\infty} \lesssim \frac{1}{(1 + t)^{5/4}} A_T, \]
\[ \|e^{it\Delta}f\|_{L^6} \lesssim \frac{1}{1 + t} A_T, \quad \|e^{it\Delta}xf\|_{L^6} \lesssim \frac{1}{(1 + t)^{1/2}} A_T, \]
\[ \|e^{it\Delta}f\|_{L^4} \lesssim \frac{1}{(1 + t)^{3/4}} A_T, \quad \|e^{it\Delta}xf\|_{L^4} \lesssim \frac{1}{(1 + t)^{3/8}} A_T. \]

Similarly, we have

\[ \|e^{it\gamma\Delta}g\|_{L^\infty} \lesssim \frac{1}{(1 + t)^{4/3}} A_T, \]
\[ \|e^{it\gamma\Delta}g\|_{L^6} \lesssim \frac{1}{1 + t} A_T, \quad \|e^{it\gamma\Delta}xg\|_{L^6} \lesssim \frac{1}{(1 + t)^{2/3}} A_T, \]
\[ \|e^{it\gamma\Delta}g\|_{L^4} \lesssim \frac{1}{(1 + t)^{3/4}} A_T, \quad \|e^{it\gamma\Delta}xg\|_{L^4} \lesssim \frac{1}{(1 + t)^{1/2}} A_T. \]

The estimates (17)–(22) are important in our succeeding analysis.

The main aim of Sections 3–5 is to derive the \textit{a priori} bound of the solution to (5) in the spaces \(X_T\) and \(Y_T\), and obtain the following type estimate

\[ A_T \lesssim \epsilon_0 + A_T^2, \]

then Theorem 2.1 follows by a standard continuation argument. In the estimates for the energy norm, since the regularity of \(E\) and \(M\) is not at the same level and the nonlinear term contains two order derivatives, the usual energy method will introduce one order loss of derivative. To overcome this difficulty, we use the idea of [21] to exploit the following positive properties

\[ |\psi_+| = |\eta||2\xi - (\gamma + 1)\eta| \gtrsim |\eta|^2, \]
\[ |\phi| = |\xi|(\gamma - 1)|\xi + 2\eta| \gtrsim |\xi|^2 \]

when \(|\xi| \sim |\eta| \gg |\xi - \eta|\), then we apply normal form transformation to recover the loss of derivatives (see the next section). In the weighted estimates of the solution, we will mainly deal with the terms which contain the growth factor \(s\) (or \(s^2\)). For the magnetic field, thanks to the derivative nonlinear structure, we can use the null resonance relation (46) to integrate by parts in \(\eta\) and thus close the estimates. For the electric filed, we exploit the key implicit conditions for the phases (see (71) and (94)), then we perform integration by parts both in \(s\) and \(\eta\) to estimate the problematic terms. The weighted estimates are presented in Sections 4-5. Here, we remark that the growth bound \((1 + t)^{1/2}\) for \(\|x|^2f\|_{L^2}\) which comes from (111) and (112) seems to be sharp since the decay rate of \(\|E\|_{L^4}\) is optimal.

3. Energy estimate. In this section, we prove the energy estimate for system (5). For simplicity, we rewrite (12)–(13) as

\[ f(t) = f(0) + \frac{i}{(2\pi)^3} F_1(t) - \frac{1}{2(2\pi)^{3/2}} F_2(t) - \frac{1}{2(2\pi)^{3/2}} F_3(t), \]
\[ g(t) = g(0) + \frac{\gamma}{(2\pi)^{3/2}} G_1(t) - \frac{\gamma}{(2\pi)^{3/2}} G_2(t), \]
where
\[
\begin{align*}
\hat{F}_1(t, \xi) &= \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{i x \psi(\xi, \eta, \sigma)} \hat{f}(\xi - \eta) \hat{f}(\eta - \sigma) \hat{f}(\sigma) d\eta d\sigma ds, \\
\hat{F}_2(t, \xi) &= \int_0^t \int_{\mathbb{R}^3} e^{i x \psi_+ (\xi, \eta)} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds, \\
\hat{F}_3(t, \xi) &= \int_0^t \int_{\mathbb{R}^3} e^{i x \psi_- (\xi, \eta)} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds, \\
\hat{G}_1(t, \xi) &= \int_0^t \int_{\mathbb{R}^3} e^{i x \phi(\xi, \eta)} |\xi|^2 (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds, \\
\hat{G}_2(t, \xi) &= \int_0^t \int_{\mathbb{R}^3} e^{i x \phi(\xi, \eta)} \xi \cdot (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds.
\end{align*}
\]

In the following contents of the paper, for \( a > 0 \) we denote by \( P_{\leq a} \) the frequency projection operator defined by
\[
\hat{P}_{\leq a} \hat{u} = \theta(\xi/a) \hat{u}(\xi),
\]
where \( \theta \) is a radial, smooth function satisfying \( 0 \leq \theta \leq 1, \ \theta(x) = 1 \) for \( |x| \leq 5/4 \) and \( \text{supp} \theta \subset B_{5/3}(0) \). Also, we define \( P_{> a} := 1 - P_{\leq a} \). The main estimate of this section is stated in the following proposition.

**Proposition 1.** Assume \((E, M)\) is a smooth solution of system (5) on \([0, T) \times \mathbb{R}^3\) such that \( A_T \ll 1 \), where \( A_T \) is defined by (16). Then we have
\[
\sup_{t \in [0, T)} (\|E(t)\|_{H^N} + \|M(t)\|_{H^{N-1} \cap H^{-1}}) \leq \|E(0)\|_{H^N} + \|M(0)\|_{H^{N-1} \cap H^{-1}} + CA_T^2,
\]
where the constant \( C \) is independent of \( T \).

**Proof.** Recall the definition (10), we know
\[
\|E(t)\|_{H^N} = \|f(t)\|_{H^N}, \quad \|M(t)\|_{H^{N-1} \cap H^{-1}} = \|g(t)\|_{H^{N-1} \cap H^{-1}}.
\]

Using the fact \( E \times E \) is purely imaginary, we see that the \( L^2 \) norm of \( E \) is conserved for all time as long as the solution exists. According to (23)–(24) and the conservation of the \( L^2 \) norm of \( E \), in order to prove the bound (25), it suffices to show
\[
\|F_1(t)\|_{H^N} + \|F_2(t)\|_{H^N} + \|F_3(t)\|_{H^N} \lesssim A_T^2,
\]
\[
\|G_1(t)\|_{H^{N-1} \cap H^{-1}} + \|G_2(t)\|_{H^{N-1} \cap H^{-1}} \lesssim A_T^2
\]
for all \( t \in [0, T) \).

By (17), the term \( F_1 \) can be directly estimated as
\[
\|F_1\|_{H^N} \lesssim \int_0^t \|e^{i \Delta f}\|_{H^N} \|e^{i \Delta f}\|_{L^\infty} \|e^{-i \Delta f}\|_{L^\infty} ds \lesssim A_T^3 \int_0^t \frac{1}{(1 + s)^{5/2}} ds \lesssim A_T^3.
\]

To estimate \( F_2 \), we split it into \( F_2 = F_2^{(1)} + F_2^{(2)} \) with
\[
\hat{F}_2^{(1)}(t, \xi) := \int_0^t \int_{\mathbb{R}^3} e^{i x \psi_+ (\xi, \eta)} (1 - \chi_1(\xi, \eta)) \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds,
\]
\[
\hat{F}_2^{(2)}(t, \xi) := \int_0^t \int_{\mathbb{R}^3} e^{i x \psi_+ (\xi, \eta)} \chi_1(\xi, \eta) \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds,
\]
where \( \chi_1(\xi, \eta) \) is a \( C^\infty \) function satisfying
\[
\chi_1(\xi, \eta) = 1 \text{ if } \max\left(\frac{8}{|\gamma - 1|}, 4\right)|\xi - \eta| \leq |\eta|,
\]
\[
\chi_1(\xi, \eta) = 0 \text{ if } \max\left(\frac{4}{|\gamma - 1|}, 2\right)|\xi - \eta| \geq |\eta|,
\]
\[
|\nabla^\beta \chi_1(\xi, \eta)| \lesssim_{\alpha, \beta} |\eta|^{-\alpha - \beta}.
\]

We first estimate the \( \dot{H}^N \) norm of \( F_2^{(1)} \). On the support of \( 1 - \chi_1 \), there is \( |\eta| \lesssim |\xi - \eta| \), so the \( N \) derivatives can be put on \( E = e^{it\Delta} f \), then by (20),
\[
\|F_2^{(1)}\|_{\dot{H}^N} \lesssim \int_0^t \|e^{it\Delta} f\|_{\dot{H}^N} \|e^{is\gamma \Delta} g\|_{L^\infty} ds \lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{\frac{3}{2} - \frac{N}{2}}} ds \lesssim A_T^2.
\]

For the term \( F_2^{(2)} \), we may assume \( |\eta| \geq 1 \) since the case \( |\eta| \leq 1 \) can be easily treated by using Hölder’s inequality and the fact \( \|P_{\leq 1} u\|_{\dot{H}^s} \lesssim \|u\|_{L^2} \) \( (s \geq 0) \). As the frequency of \( g \) is higher than \( f \), so if all the derivatives fall directly on the function \( M = e^{is\gamma \Delta} g \), it will appear a loss of derivative. To recover the loss of derivatives, we should exploit the nonvanishing property of the phase \( \psi_+ (\xi, \eta) \). Note that on the support of \( \chi_1 \), there hold \( |\xi| \sim |\eta| \) and
\[
|\psi_+| = |\eta||2\xi - (\gamma + 1)\eta| = |\eta||2(\xi - \eta) - (\gamma - 1)\eta| \geq |\eta||2|\xi - \eta| - |\gamma - 1|||\eta| \gtrsim |\eta|^2.
\]

This bound allows us to estimate \( F_2^{(2)} \) through integrating by parts in \( s \). Define
\[
\tilde{\chi}_1 := \frac{|\xi|^N \chi_1}{|\eta|^{N - 2} \psi_+},
\]
then
\[
|\xi|^N F_2^{(2)}(t, \xi) = \int_0^t \int_{\mathbb{R}^3} (e^{is\psi_+})_s \tilde{\chi}_1 \tilde{f}(\xi - \eta) \times |\eta|^{N - 2} \tilde{g}(\eta) d\eta ds
\]
\[
= \int_{\mathbb{R}^3} e^{it\psi_+} \tilde{\chi}_1 \tilde{f}(t, \xi - \eta) \times |\eta|^{N - 2} \tilde{g}(t, \eta) d\eta - \int_{\mathbb{R}^3} \tilde{\chi}_1 \tilde{f}(0, \xi - \eta) \times |\eta|^{N - 2} \tilde{g}(0, \eta) d\eta
\]
\[
- \int_0^t \int_{\mathbb{R}^3} e^{is\psi_+} \tilde{\chi}_1 \partial_s \tilde{f}(\xi - \eta) \times |\eta|^{N - 2} \tilde{g}(\eta) d\eta ds
\]
\[
- \int_0^t \int_{\mathbb{R}^3} e^{is\psi_+} \tilde{\chi}_1 \tilde{f}(\xi - \eta) \times |\eta|^{N - 2} \partial_s \tilde{g}(\eta) d\eta ds.
\]

The lower bound (30) yields \( |\tilde{\chi}_1| \lesssim 1 \). Moreover, by the property of \( \chi_1 \), we can see the symbol \( \tilde{\chi}_1 \) satisfies Coifman-Meyer bound. Using the classical Coifman-Meyer multiplier theorem, we obtain
\[
\| (31) \|_{L^2} \lesssim \| e^{it\Delta} f \|_{L^\infty} \| \nabla \| e^{it\gamma \Delta} g \|_{L^2} = \| E \|_{L^\infty} \| M \|_{\dot{H}^{N - 2}} \lesssim A_T^2.
\]

Similarly, the term (32) can be estimated as
\[
\| (32) \|_{L^2} \lesssim \| (E(0))_{\dot{H}^s} \|_{\dot{H}^{N - 2}} \lesssim A_T^2.
\]
From (11), we have
\[ \| e^{is\Delta} \partial_s f \|_{L^\infty} \leq \| E \|_{L^\infty}^3 + \| E \|_{L^\infty} \| M \|_{L^\infty} \lesssim A_T^2 (1 + s)^{-31/12}, \]
\[ \| \eta^{N-2} e^{is\gamma \Delta} \partial_s g \|_{L^2} \leq \| E \|_{H^N} \| E \|_{L^\infty} \lesssim A_T^2 (1 + s)^{-5/4}, \]
hence, it is easy to see
\[ \| (33) \|_{L^2} \lesssim \int_0^t \| e^{is\Delta} \partial_s f \|_{L^\infty} \| \nabla | \nabla^{-2} e^{is\gamma \Delta} g \|_{L^2} ds \lesssim A_T^3 \int_0^t \frac{1}{(1 + s)^{31/12}} ds \lesssim A_T^3, \]
\[ \| (34) \|_{L^2} \lesssim \int_0^t \| e^{is\Delta} f \|_{L^\infty} \| \nabla | \nabla^{-2} e^{is\gamma \Delta} \partial_s g \|_{L^2} ds \lesssim A_T^3 \int_0^t \frac{1}{(1 + s)^{5/2}} ds \lesssim A_T^3. \]
Therefore, there holds
\[ \| F_2^{(2)} \|_{H^N} \leq \| (31) \|_{L^2} + \| (32) \|_{L^2} + \| (33) \|_{L^2} + \| (34) \|_{L^2} \lesssim A_T^2. \] (35)
Following similar arguments as $F_2$, we can get
\[ \| F_3 \|_{H^N} \lesssim A_T^2. \] (36)
Thus, the desired bound (26) follows from (28), (29), (35) and (36).

It remains to show the bound (27). Since the arguments for $G_1$ and $G_2$ are similar, we only consider in detail the estimate for $G_1$. Due to the structure of derivative nonlinearity, the part of $H^{-1}$ norm can be estimated directly as
\[ \| \nabla^{-1} G_1 \|_{L^2} \lesssim \int_0^t \| \nabla E \|_{L^2} \| E \|_{L^\infty} ds \lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{5/4}} ds \lesssim A_T^2. \] (37)
Also, the part of $L^2$ norm can be easily treated as
\[ \| G_1 \|_{L^2} \lesssim \int_0^t \| \Delta E \|_{L^2} \| E \|_{L^\infty} ds \lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{5/4}} ds \lesssim A_T^2. \] (38)
Now we estimate $H^{-1 \infty}$ norm of $G_1$. As the nonlinear term contains two order derivatives, it will again produce the loss of derivatives. To close our argument, we introduce a smooth cut-off function $\chi_2(\xi, \eta)$ satisfying
\[ \chi_2(\xi, \eta) = 1 \text{ if } 4 \max(\frac{\gamma - 1}{\gamma + 1}, 1)|\xi - \eta| \leq |\eta|, \]
\[ \chi_2(\xi, \eta) = 0 \text{ if } 2 \max(\frac{\gamma - 1}{\gamma + 1}, 1)|\xi - \eta| \geq |\eta|, \]
\[ |\nabla_\xi \nabla_\eta \chi_2(\xi, \eta)| \lesssim_{\alpha, \beta} |\eta|^{-\alpha - \beta}, \]
then we decompose $G_1$ into $G_1^{(1)} + G_1^{(2)} + G_1^{(3)}$ with
\[
\hat{G}_1^{(1)}(t, \xi) := \int_0^t \int_{\mathbb{R}^3} \hat{\chi}_2(\xi, \eta)e^{is\phi(\xi, \eta)}|\xi|^2(\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds,
\]
\[
\hat{G}_1^{(2)}(t, \xi) := \int_0^t \int_{\mathbb{R}^3} \hat{\chi}_2(\xi, \xi - \eta)e^{is\phi(\xi, \eta)}|\xi|^2(\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds,
\]
\[
\hat{G}_1^{(3)}(t, \xi) := \int_0^t \int_{\mathbb{R}^3} [1 - \chi_2(\xi, \eta) - \chi_2(\xi, \xi - \eta)]e^{is\phi(\xi, \eta)}|\xi|^2(\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds.
\]
On the support of \(1 - \chi_2(\xi, \eta) - \chi_2(\xi, \xi - \eta)\), we have \(|\xi - \eta| \sim |\eta|\). Hence the term \(G_1(3)\) is estimated by

\[
\|G_1(3)\|_{H^{N-1}} \leq \int_0^t \left\| e^{is\Delta} f \right\|_{H^N} \left\| e^{-is\Delta} \mathcal{F}\|L_\infty\| ds \leq A_T^2 \int_0^t \frac{1}{(1 + s)^{9/8}} ds \lesssim A_T^2 \tag{39}
\]

where we have used the following bounds in the last step:

\[
\|P_{>(1+s)^{1/8}} \nabla E\|_{L_\infty} \lesssim (1 + s)^{-(N-3)/8} \|P_{>(1+s)^{1/8}} \nabla |N-2 E\|_{L_\infty} \lesssim (1 + s)^{-(N-3)/8} A_T, \tag{40}
\]

\[
\|P_{\leq(1+s)^{1/8}} \nabla E\|_{L_\infty} \lesssim (1 + s)^{1/8} \|E\|_{L_\infty} \lesssim (1 + s)^{-9/8} A_T.
\]

Note that the terms \(G_1^{(1)}\) and \(G_1^{(2)}\) correspond to \(|\xi - \eta| \ll |\eta| \sim |\xi|\) and \(|\eta| \ll |\xi - \eta| \sim |\xi|\), respectively, so by symmetry, it is sufficient to estimate \(G_1^{(1)}\). On the support of \(\chi_2(\xi, \eta)\), we have

\[
|\phi| = |\xi|(|\gamma - 1|\xi + 2\eta| = |\xi||(\gamma - 1)(\xi - \eta) + (\gamma + 1)\eta| \gtrsim |\xi||\eta| \sim |\xi|^2,
\]

which implies

\[
|\tilde{\chi}_2| : = \frac{|\xi|N-1|\xi|^2 \chi_2(\xi, \eta)|}{\xi|\eta|N-1} \lesssim 1.
\]

Integrating by parts in time gives

\[
|\xi|^{N-1} \tilde{G}_1^{(1)}(t, \xi) := \int_0^t \int_{\mathbb{R}^3} (e^{i\phi(\xi, \eta)})_s \tilde{\chi}_2(\tilde{f}(\xi - \eta) \times |\eta|^{N-1}\tilde{f}(\eta)) d\eta ds
\]

\[
= \int_{\mathbb{R}^3} e^{it\phi(\xi, \eta)} \tilde{\chi}_2(\tilde{f}(t, \xi - \eta) \times |\eta|^{N-1}\tilde{f}(t, \eta)) d\eta - \int_{\mathbb{R}^3} \tilde{\chi}_2(\tilde{f}(0, \xi - \eta) \times |\eta|^{N-1}\tilde{f}(0, \eta)) d\eta \tag{41}
\]

\[
- \int_0^t \int_{\mathbb{R}^3} e^{i\phi(\xi, \eta)} \tilde{\chi}_2(\partial_s \tilde{f}(\xi - \eta) \times |\eta|^{N-1}\tilde{f}(\eta)) d\eta ds \tag{42}
\]

\[
- \int_0^t \int_{\mathbb{R}^3} e^{i\phi(\xi, \eta)} \tilde{\chi}_2(\tilde{f}(\xi - \eta) \times |\eta|^{N-1}\partial_s \tilde{f}(\eta)) d\eta ds \tag{43}
\]

With similar arguments as the terms (31)–(34), we can obtain

\[
\|G_1^{(1)}\|_{H^{N-1}} \leq \left( \|(31)\|_{L^2} + \|(42)\|_{L^2} + \|(43)\|_{L^2} + \|(44)\|_{L^2} \right) \lesssim A_T^2. \tag{45}
\]

Therefore, combining (37), (38), (39) and (45) yields (27) as desired. This ends the proof of Proposition 1.

4. **Weighted estimate for the magnetic field.** This section is concerned on the weighted estimate for the magnetic component. Note that

\[
\phi(\xi, \eta) = \gamma |\xi|^2 - |\xi - \eta|^2 + |\eta|^2 = (\gamma - 1)|\xi|^2 + 2\xi \cdot \eta,
\]

so we have

\[
|\xi| - \frac{1}{2} \nabla_\eta \phi, \text{ or } |\xi| = \frac{\xi}{2|\xi|} \cdot \nabla_\eta \phi. \tag{46}
\]

Thanks to the derivative structure of the nonlinear term, we can use the relation (46) to exclude the phenomenon of time-space resonances.
Proposition 2. Assume \((E, M)\) satisfies system (5) on \([0, T) \times \mathbb{R}^3\) with \(T > 0\). If \(A_T \ll 1\), then

\[
\sup_{t \in [0, T)} (\|xg(t)\|_{L^2} + (1 + t)^{-1/3}\|x^2g(t)\|_{L^2}) \leq \|xM(0)\|_{L^2} + \|\|x^2M(0)\|_{L^2} + CA_T^2,
\]

(47)

where \(C\) is independent of \(T\).

Proof. In order to prove (47), we know from (23) and (24) that it suffices to show

\[
\|xG_j(t)\|_{L^2} + (1 + t)^{-1/3}\|x^2G_j(t)\|_{L^2} \lesssim A_T^2,
\]

(48)

for any \(t \in [0, T)\) and \(j = 1, 2\). As the essential structure of \(G_1\) and \(G_2\) is the same, for simplicity, we only concentrate on the estimate for \(G_1\) in this proof.

Recall

\[
\widehat{G_1}(t, \xi) = \int_0^t \int_{\mathbb{R}^3} e^{i\sigma(\xi, \eta)}|\xi|^2(\hat{f}(\xi - \eta) \times \hat{f}(\eta))d\eta ds.
\]

(49)

Without loss of generality, we may assume \(|\xi - \eta| \lesssim |\eta|\) in (49), otherwise we can make a change of variable \(\xi - \eta \to \tilde{\eta}\). Applying \(\nabla_\xi\) to \(G_1\) gives

\[
\nabla_\xi \widehat{G_1}(t, \xi) = \int_0^t \int_{\mathbb{R}^3} e^{i\sigma(\xi, \eta)}|\xi|^2(\nabla_\xi \hat{f}(\xi - \eta) \times \hat{f}(\eta))d\eta ds + \int_0^t \int_{\mathbb{R}^3} e^{i\sigma(\xi, \eta)}2\xi(\hat{f}(\xi - \eta) \times \hat{f}(\eta))d\eta ds + \int_0^t \int_{\mathbb{R}^3} is\nabla_\xi \phi e^{i\sigma(\xi, \eta)}|\xi|^2(\hat{f}(\xi - \eta) \times \hat{f}(\eta))d\eta ds.
\]

(50)

For (50), we put the derivatives on the function \(e^{-i\sigma\Delta}\). Using the bounds

\[
\|P_{> (1+s)^{1/4}} E\|_{L^\infty} \lesssim (1 + s)^{-(N-4)/16}\|\nabla\|_{L^\infty} E\|_{L^\infty} \lesssim (1 + s)^{1-(N-4)/16}A_T,
\]

\[
\|P_{\leq (1+s)^{1/4}} E\|_{L^\infty} \lesssim (1 + s)^{1/8}\|E\|_{L^\infty} \lesssim (1 + s)^{9/8}A_T,
\]

we obtain

\[
\|\nabla_\xi \widehat{G_1}\|_{L^2} \lesssim \int_0^t \|e^{i\sigma\Delta}xf\|_{L^2} \|e^{-i\sigma\Delta}\Delta f\|_{L^\infty} ds \lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{3/8}} ds \lesssim A_T^2.
\]

(53)

For (51), it is easy to see

\[
\|\nabla_\xi \widehat{G_1}\|_{L^2} \lesssim \int_0^t \|e^{i\sigma\Delta}f\|_{L^\infty} \|e^{-i\sigma\Delta}\Delta f\|_{L^2} ds \lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{3/4}} ds \lesssim A_T^2.
\]

(54)

Before estimating (52), note that when \(N \geq 100\) and \(0 < \delta \leq 1/30\),

\[
\|P_{> (1+s)^{1/4}} E\|_{H^3} \lesssim (1 + s)^{-(N-3)\delta}\|E\|_{H^N} \lesssim (1 + s)^{-3}A_T.
\]

This shows that the low order \(L^2\) norm of the high frequency part can produce a good decay factor, so when estimating (52), one can reduce our argument to the case

\[
|\xi - \eta|, |\eta|, |\xi| \lesssim (1 + s)^\delta.
\]

(55)

Namely, due to the good decay rate and high order energy, the low order derivative can be converted into a small growth factor. Now we use (46) to get

\[
is\nabla_\xi \phi e^{i\sigma\phi}|\xi|^2 = is(2(\gamma - 1)\xi + 2\eta)e^{i\sigma\phi}(\frac{\xi}{2|\xi|} \cdot \nabla_\eta \phi)|\xi| = ((\gamma - 1)\xi + \eta)(\xi \cdot \nabla_\eta e^{i\sigma\phi}),
\]
then integrating by parts in $\eta$ gives

\begin{align}
(52) &= -\int_0^t \int_{\mathbb{R}^3} \xi e^{ix\phi} (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds,
\end{align}

\begin{align}
(56) &= -\int_0^t \int_{\mathbb{R}^3} (\gamma - 1) \xi + \eta e^{ix\phi} \xi \cdot (\nabla_\eta \hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds,
\end{align}

\begin{align}
(57) &= -\int_0^t \int_{\mathbb{R}^3} (\gamma - 1) \xi + \eta e^{ix\phi} \xi \cdot (\hat{f}(\xi - \eta) \times \nabla_\eta \hat{f}(\eta)) d\eta ds.
\end{align}

Estimate for (56) is the same as (51). Using the assumption (55), we have

\begin{align}
\| (57) \|_{L^2} + \| (58) \|_{L^2} &\lesssim \int_0^t (1 + s)^{2\delta} \|xf\|_{L^2} \|E\|_{L^4} ds
\lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{5/4 - 2\delta}} ds \lesssim A_T^2.
\end{align}

From (53), (54) and (59), we thus get $\|xG_1\|_{L^2} \lesssim A_T^2$.

Applying $\Delta_\xi$ to (49), we can get after a direct computation

\begin{align}
\Delta_\xi \hat{G}_1 &= \int_0^t \int_{\mathbb{R}^3} e^{ix\phi} |\xi|^2 (\Delta_\xi \hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds
\end{align}

\begin{align}
&+ 6 \int_0^t \int_{\mathbb{R}^3} e^{ix\phi} (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds
\end{align}

\begin{align}
&+ \int_0^t \int_{\mathbb{R}^3} (is)^2 \nabla_\xi \phi |e^{ix\phi}| |\xi|^2 (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds
\end{align}

\begin{align}
&+ 6(\gamma - 1) \int_0^t \int_{\mathbb{R}^3} (is)e^{ix\phi} |\xi|^2 (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds
\end{align}

\begin{align}
&+ 4 \int_0^t \int_{\mathbb{R}^3} (is) \nabla_\xi \phi \cdot \xi e^{ix\phi} (\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds
\end{align}

\begin{align}
&+ 4 \int_0^t \int_{\mathbb{R}^3} e^{ix\phi} (\nabla_\xi \hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds
\end{align}

\begin{align}
&+ 2 \int_0^t \int_{\mathbb{R}^3} (is) \nabla_\xi \phi \cdot e^{ix\phi} |\xi|^2 (\nabla_\xi \hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds.
\end{align}

The term (60) is estimated similarly as (50). Applying the $L^2 \times L^\infty$ estimate, the treatment for (61) is trivial. The terms (63) and (64) can be estimated in the same way as (52). The estimate for (65) is essentially the same as (57). For (66), using the identity (46) to integrate by parts in $\eta$, we can obtain analogous terms as (60), (65) plus the term

\begin{align}
\int_0^t \int_{\mathbb{R}^3} (\gamma + 1) \xi + \eta e^{ix\phi} \xi \cdot (\nabla_\xi \hat{f}(\xi - \eta) \times \nabla_\eta \hat{f}(\eta)) d\eta ds.
\end{align}

From the estimate (19), one gets

\begin{align}
\| (67) \|_{L^2} &\lesssim \int_0^t (1 + s)^{2\delta} \|e^{ix\Delta_\xi f}\|_{L^4} \|e^{-is\Delta_\xi \hat{f}}\|_{L^4} ds
\lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{3/4 - 2\delta}} ds
\end{align}
Weighted estimate for the Schrödinger component.

Here, we point out that the growth factor \((1 + t)^{1/3}\) can also be replaced by \((1 + t)^{\alpha}\) with \(\alpha = 1/4^+\). However, for simplicity, we will not take this general bound in our arguments. Now, we are left to deal with the term (62). To eliminate the factor \(s^2\), we use (46) to integrate by parts in \(\eta\) twice and obtain the following contributions

\[
(62) = \int_0^t \int_{\mathbb{R}^3} m_1(\xi, \eta) e^{is\phi}(\hat{f}(\xi - \eta) \times \hat{f}(\eta)) d\eta ds \\
+ \int_0^t \int_{\mathbb{R}^3} m_2(\xi, \eta) e^{is\phi}(\nabla_\eta \hat{f}(\xi - \eta) \times \hat{f}(\eta)) + \hat{f}(\xi - \eta) \times \nabla_\eta \hat{f}(\eta)) d\eta ds \\
+ \int_0^t \int_{\mathbb{R}^3} m_3(\xi, \eta) e^{is\phi}(\Delta_\eta \hat{f}(\xi - \eta) \times \hat{f}(\eta)) + \hat{f}(\xi - \eta) \times \Delta_\eta \hat{f}(\eta)) d\eta ds \\
+ \int_0^t \int_{\mathbb{R}^3} m_4(\xi, \eta) e^{is\phi}(\nabla_\eta \hat{f}(\xi - \eta) \times \nabla_\eta \hat{f}(\eta)) d\eta ds,
\]

where \(m_1, \ldots, m_4\) are homogenous symbols of order no more than 2. All the above terms can be bounded by applying previous arguments. Therefore, we get

\[
\| |x|^2 G_2 |_{L^2} = \| \Delta_\xi \hat{G}_2 |_{L^2} \lesssim (1 + t)^{1/3} A_T^2.
\]

The proof of Proposition 2 is completed. \(\square\)

5. **Weighted estimate for the Schrödinger component.** In this section, we want to prove the following proposition.

**Proposition 3.** Let \((E, M)\) be the solution of system (5) on \([0, T] \times \mathbb{R}^3\) with \(T > 0\). If \(A_T \ll 1\), then

\[
\sup_{t \in [0, T]} \left(\| x f(t) \|_{L^2} + (1 + t)^{-1/2} \| |x|^2 f(t) \|_{L^2} \right) \leq \| x E(0) \|_{L^2} + \| |x|^2 E(0) \|_{L^2} + CA_T^2.
\]

(68)

The bound (68) follows immediately from (23), (24), (70) and (90). Hence, in the remaining parts of this section, it is sufficient for us to show Lemmas 5.1 and 5.4 below.

5.1. **Estimate for the cubic term.** Recall the definition \(F_1\) in (24), namely,

\[
\hat{F}_1(t, \xi) := \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{is\phi(\xi, \eta, \sigma)} \hat{f}(s, \xi - \eta) \hat{f}(s, \eta - \sigma) \hat{f}(s, \sigma) d\eta d\sigma ds,
\]

(69)

where

\[
\varphi(\xi, \eta, \sigma) = |\xi|^2 - |\xi - \eta|^2 - |\eta - \sigma|^2 + |\sigma|^2 = 2\xi \cdot \eta + 2\eta \cdot \sigma - 2|\eta|^2.
\]

**Lemma 5.1.** With the same assumptions as Proposition 3, we have

\[
\sup_{t \in [0, T]} \left(\| x F_1(t) \|_{L^2} + \| |x|^2 F_1(t) \|_{L^2} \right) \lesssim A_T^2.
\]

(70)

**Proof.** Taking \(\nabla_\xi\) to (69) and using the fact

\[
\nabla_\xi \varphi = 2\eta = \nabla_\sigma \varphi,
\]

(71)
we obtain
\[
\nabla_x \tilde{F}_1 = \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \nabla_x \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
+ \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} is \nabla_x \psi e^{ix\varphi} \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
= \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \nabla_x \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
- \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \tilde{f}(s, \xi - \eta) \nabla_\sigma \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
- \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \nabla_\sigma \tilde{f}(s, \sigma) d\eta d\sigma ds.
\]

Hence, there holds
\[
\|(72)\|_{L^2} + \|(73)\|_{L^2} + \|(74)\|_{L^2} \lesssim \int_0^t \|xf\|_{L^2} \|E\|_{L^\infty} ds \\
\lesssim A T \int_0^t (1 + s)^{-5/2} ds \lesssim A T. \tag{75}
\]

By (71) and a direct computation, we see
\[
\Delta_x \tilde{F}_1 = \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \Delta_x \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
+ \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \tilde{f}(s, \xi - \eta) \Delta_\sigma \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
+ \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \Delta_\sigma \tilde{f}(s, \sigma) d\eta d\sigma ds \\
+ 2 \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \tilde{f}(s, \xi - \eta) \nabla_\sigma \tilde{f}(s, \eta - \sigma) \nabla_\sigma \tilde{f}(s, \sigma) d\eta d\sigma ds \\
- 2 \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \nabla_x \tilde{f}(s, \xi - \eta) \nabla_\sigma \tilde{f}(s, \eta - \sigma) \tilde{f}(s, \sigma) d\eta d\sigma ds \\
- 2 \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{ix\varphi} \nabla_x \tilde{f}(s, \xi - \eta) \tilde{f}(s, \eta - \sigma) \nabla_\sigma \tilde{f}(s, \sigma) d\eta d\sigma ds.
\]

Therefore, we have
\[
\|(76)\|_{L^2} + \|(77)\|_{L^2} + \|(78)\|_{L^2} \lesssim \int_0^t \|x\|_{L^2} \|f\|_{L^\infty} ds \lesssim A T, \tag{82}
\]
and by (19),
\[
\|(79)\|_{L^2} + \|(80)\|_{L^2} + \|(81)\|_{L^2} \lesssim \int_0^t \|E\|_{L^\infty} \|e^{ix\Delta}(xf)\|_{L^2} ds \lesssim A T. \tag{83}
\]

Now, combining (75), (82) and (83), we thus get (70).

5.2. **Estimate for the quadratic term.** The aim of this subsection is to prove Lemma 5.4 below. To this aim, we need the following two lemmas.
Lemma 5.2. With the same assumptions as Proposition 3, we have
\[ e^{is\Delta}\partial_s(xf) = h_1 + h_2, \]
\[ h_1 := i(|E|^2 e^{is\Delta}(xf) + \frac{1}{2}ie^{is\Delta}(xf) \times M + \frac{1}{2}ie^{is\Delta}(xf) \times \overline{M}), \]
\[ h_2 := 2s(|E|^2 \nabla E + \frac{1}{2}i\nabla E \times M + \frac{1}{2}i\nabla E \times \overline{M}) \]
\[ - 2s\nabla(|E|^2 E + \frac{1}{2}iE \times M + \frac{1}{2}iE \times \overline{M}), \] \hspace{1cm} (84)
where \( h_1 \) and \( h_2 \) satisfy
\[ \|h_1\|_{L^8} \lesssim (1 + s)^{-11/6}A_T^2, \quad \|h_2\|_{L^8} \lesssim (1 + s)^{-3/2}A_T^2. \] \hspace{1cm} (85)

Proof. Note that for any \( \alpha \in \mathbb{R} \),
\[ x(e^{is\alpha u}) = e^{is\alpha}(xu) - 2\alpha is\nabla e^{is\alpha u}. \] \hspace{1cm} (86)
From this identity and (11), one can compute
\[ e^{is\Delta}\partial_s(xf) = i(|E|^2 xE + \frac{1}{2}ixE \times M + \frac{1}{2}ixE \times \overline{M}) \]
\[ - 2s\nabla(|E|^2 E + \frac{1}{2}iE \times M + \frac{1}{2}iE \times \overline{M}). \]
Moreover, (86) also gives
\[ xE = xe^{is\Delta}f = e^{is\Delta}(xf) - 2is\nabla E. \] \hspace{1cm} (87)
Hence, (84) follows from the above two equalities. Then by (17), (18) and (20),
\[ \|h_1\|_{L^8} \lesssim \|e^{is\Delta}(xf)\|_{L^8}(|E|^2L^\infty + \|M\|_{L^\infty}) \lesssim (1 + s)^{-11/6}A_T^2. \]

Similar to (40), we can convert the derivative \( \nabla \) into \((1 + s)^{1/12}\) due to the a priori energy bound on \( E \), so
\[ \|h_2\|_{L^\infty} \lesssim s(1 + s)^{1/12}\|E\|_{L^\infty}(\|E\|^2_{L^\infty} + \|M\|_{L^\infty}) \lesssim (1 + s)^{-3/2}A_T^2. \]
Thus (85) follows. The proof of Lemma 5.2 is finished. \hfill \Box

Lemma 5.3. Under the same assumptions as Proposition 3, we have
\[ |\nabla|^{-1}e^{is\gamma}\partial_s(xg) = \tilde{h}_1 + \tilde{h}_2, \]
\[ \tilde{h}_1 := -\gamma|\nabla|^{-1}\Delta((e^{is\Delta}xf) \times \overline{E}) + \gamma|\nabla|^{-1}\nabla \cdot ((e^{is\Delta}xf) \times \overline{E}), \]
\[ \tilde{h}_2 := 2\gamma is|\nabla|^{-1}\Delta(\nabla E \times \overline{E}) + 2\gamma|\nabla|^{-1}\nabla(\nabla E \times \overline{E}) \]
\[ - 2is\gamma|\nabla|^{-1}\nabla \cdot (\nabla E \times \overline{E}) - \gamma|\nabla|^{-1}\nabla(\nabla E \times \overline{E}) \]
\[ + 2is\gamma^2|\nabla|^{-1}\nabla(-\Delta(\nabla E \times \overline{E}) + \nabla \cdot (E \times \overline{E})) - \gamma|\nabla|^{-1}\nabla \cdot (E \times \overline{E})I_3, \] \hspace{1cm} (88)
where \( I_3 \) is a \( 3 \times 3 \) unit matrix. Moreover, \( \tilde{h}_1 \) and \( \tilde{h}_2 \) satisfy
\[ \|\tilde{h}_1\|_{L^2} \lesssim (1 + s)^{-13/12}A_T^2, \quad \|\tilde{h}_2\|_{L^3} \lesssim (1 + s)^{-11/12}A_T^2. \] \hspace{1cm} (89)

Proof. Using (11) and (86), one gets
\[ e^{is\gamma\Delta}\partial_s(xg) = -\gamma x\Delta(\nabla \cdot \overline{E}) + \gamma x\nabla \cdot (E \times \overline{E}) + 2is\gamma^2\nabla(\nabla \cdot (E \times \overline{E}) - \Delta(\nabla \cdot E)). \]
Furthermore, by (87) and the following identities
\[ x\Delta u = \Delta(xu) - 2\nabla u, \]
\[ x\nabla \cdot u = \nabla \cdot (xu) - \nabla u - (\nabla \cdot u)I_3, \]
we can obtain the decomposition (88) by a straightforward computation. To show (89), as explained in (55), we first convert each derivative into a factor \((1 + s)^{1/24}\), then by (18)–(19),

\[
\|\tilde{h}_1\|_{L^2} \lesssim (1 + s)^{1/24}\|e^{is\Delta}(xf)\|_{L^1}\|E\|_{L^4} \lesssim (1 + s)^{-13/12}A_T^2,
\]

\[
\|\tilde{h}_2\|_{L^3} \lesssim s(1 + s)^{1/12}\|E\|_{L^6} \lesssim (1 + s)^{-11/12}A_T^2.
\]

Hence, we obtain (89), which ends the proof of this lemma.

**Lemma 5.4.** With the same assumptions as Proposition 3, we have

\[
\sup_{t \in [0, T)} \|xF_j(t)\|_{L^2} + (1 + t)^{-1/2}\|xF_j(t)\|_{L^3} \lesssim A_T^2, \quad j = 2, 3,
\]

where the expressions of \(F_2\) and \(F_3\) are given by (24).

**Proof.** We first show for all \(t \in [0, T)\), there holds

\[
\|xF_2(t)\|_{L^2} \lesssim A_T^2.
\]

By (24), we see

\[
\nabla_{\xi} \hat{F}_2(t, \xi) = \int_0^t \int_{\mathbb{R}^3} e^{ix\psi^+} \nabla_{\xi} \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta \, ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} is \nabla_{\psi^+} e^{ix\psi^+} \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta \, ds,
\]

where

\[
\psi^+(\xi, \eta) = |\xi|^2 - |\xi - \eta|^2 - \gamma|\eta|^2 = 2\xi \cdot \eta - (\gamma + 1)|\eta|^2.
\]

With a direct \(L^2 \times L^\infty\) estimate, the bound for (92) is trivial. To estimate (93), the key observation is

\[
\nabla_{\psi^+} = \frac{2}{\gamma + 1} \frac{\eta}{|\eta|^2} \psi^+ - \frac{2}{\gamma + 1} \frac{\eta}{|\eta|^2} (\eta \cdot \nabla_{\psi^+}).
\]

Using this identity, we have

\[
(93) = \frac{2}{\gamma + 1} \int_0^t \int_{\mathbb{R}^3} is \frac{\eta}{|\eta|^2} e^{ix\psi^+} \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta \, ds
\]

\[
- \frac{2}{\gamma + 1} \int_0^t \int_{\mathbb{R}^3} \frac{s}{|\eta|^2} (\eta \cdot \nabla_{\psi^+}) e^{ix\psi^+} \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta \, ds.
\]

For (95), we integrate by parts in time and obtain

\[
\frac{\gamma + 1}{2} \cdot (95) = t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{ix\psi^+} \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta
\]

\[
- \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{ix\psi^+} \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta \, ds
\]

\[
- \int_0^t \int_{\mathbb{R}^3} s \frac{\eta}{|\eta|^2} e^{ix\psi^+} \partial_s \hat{f}(\xi - \eta) \times \hat{g}(\eta) \, d\eta \, ds
\]

\[
- \int_0^t \int_{\mathbb{R}^3} s \frac{\eta}{|\eta|^2} e^{ix\psi^+} \hat{f}(\xi - \eta) \times \partial_s \hat{g}(\eta) \, d\eta \, ds.
\]

In virtue of \(\dot{H}^{-1}\) norm of \(M\), we get

\[
\| (97) \|_{L^2} \lesssim t \| e^{it\Delta} f \|_{L^\infty} \| e^{it\gamma \Delta} g \|_{\dot{H}^{-1}} \lesssim A_T^2.
\]
The term (98) can be estimated similarly. For (99) and (100), we use (11) to obtain
\[ \| e^{is\Delta} \partial_s f \|_{L^\infty} \lesssim \| E \|_{L^\infty}^3 + \| E \|_{L^\infty} \| M \|_{L^\infty} \lesssim (1 + s)^{-31/12} A_T^2, \]
\[ \| |\nabla|^{-1} e^{is\gamma \Delta} \partial_s g \|_{L^2} \lesssim \| \nabla E \|_{L^2} \| E \|_{L^\infty} \lesssim (1 + s)^{-5/4} A_T^2, \]
hence, there hold
\[ \| (99) \|_{L^2} \lesssim \int_0^t s \| e^{is\Delta} \partial_s f \|_{L^\infty} \| |\nabla|^{-1} e^{is\gamma \Delta} g \|_{L^2} ds \lesssim A_T^2 \int_0^t (1 + s)^{-19/12} ds \lesssim A_T^2, \]
\[ \| (100) \|_{L^2} \lesssim \int_0^t s \| e^{is\Delta} f \|_{L^\infty} \| |\nabla|^{-1} e^{is\gamma \Delta} \partial_s g \|_{L^2} ds \lesssim A_T^2 \int_0^t (1 + s)^{-3/2} ds \lesssim A_T^2. \]
Therefore, the desired bound for (95) is obtained. For (96), we integrate by parts in \( \eta \) and get
\[
\frac{\gamma + 1}{2} \cdot (96) = \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi + \eta} \cdot (\nabla_\eta \hat{f}(\xi - \eta) \times \hat{g}(\eta)) d\eta ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi + \eta} \cdot (\hat{f}(\xi - \eta) \times \nabla_\eta \hat{g}(\eta)) d\eta ds,
\]
which gives
\[
\| (96) \|_{L^2} \lesssim \int_0^t \| e^{is\Delta} f \|_{L^\infty} \| |\nabla|^{-1} e^{is\gamma \Delta} g \|_{L^2} ds
\]
\[
+ \int_0^t \left( \| e^{is\Delta} x f \|_{L^2} \| e^{is\gamma \Delta} g \|_{L^\infty} + \| e^{is\Delta} f \|_{L^\infty} \| e^{is\gamma \Delta} x g \|_{L^2} \right) ds
\]
\[
\lesssim A_T^2 \int_0^t (1 + s)^{-5/4} ds \lesssim A_T^2.
\]
From the above estimates, we see (91) holds.
Next, we prove that for all \( t \in [0, T) \),
\[ \| |x|^2 F_2(t) \|_{L^2} \lesssim (1 + t)^{1/2} A_T^2. \]
Notice that
\[
\Delta_\xi \hat{F}_2(t, \xi) = \int_0^t \int_{\mathbb{R}^3} e^{is\psi} \Delta_\xi \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} 2i s \nabla_\xi \psi e^{is\psi} \nabla_\xi \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} (is \nabla_\xi \psi e^{is\psi} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds.
\]
The term (102) can be estimated easily by using a direct \( L^2 \times L^\infty \) estimate. Using the relation (94) to integrate by parts in \( s \) and \( \eta \), we rewrite (103) as
\[
\frac{\gamma + 1}{4} \cdot (103) = \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi} \nabla_\xi \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi} \nabla_\xi \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds
\]
Using this identity to integrate by parts in time and frequency, we obtain

\begin{align}
- \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi^+} \partial_\xi \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds & \quad (107) \\
- \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi^+} \nabla_\xi \hat{f}(\xi - \eta) \times \partial_\eta \hat{g}(\eta) d\eta ds & \quad (108) \\
+ \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi^+} \eta \cdot (\nabla_\eta \nabla_\xi \hat{f}(\xi - \eta) \times \hat{g}(\eta)) d\eta ds & \quad (109) \\
+ \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi^+} \eta \cdot (\nabla_\xi \hat{f}(\xi - \eta) \times \nabla_\eta \hat{g}(\eta)) d\eta ds. & \quad (110)
\end{align}

The estimates for (105) and (106) are similar, and we take (105) as an example. Indeed, we use the fact \( \|g\|_{\dot{H}^{-1/2}} \lesssim \|g\|_{\dot{H}^{-1}} + \|g\|_{L^2} \lesssim A_T \) and (18) to get

\begin{align*}
\|(105)\|_{L^2} & \lesssim \int_0^t \|e^{i\xi^\gamma A} f\|_{L^6} \|\nabla^{-1} e^{i\xi^\gamma A} g\|_{L^3} ds \\
& \lesssim (1 + t)^{-1/2} A_T \|\nabla^{-1/2} e^{i\xi^\gamma A} g\|_{L^2} \\
& \lesssim (1 + t)^{1/2} A_T^2.
\end{align*}

For (107), we use Lemma 5.2 to obtain

\begin{align*}
\|(107)\|_{L^2} & \lesssim \int_0^t \|e^{i\xi^\gamma A} \|_{L^6} \|\nabla^{-1} e^{i\xi^\gamma A} g\|_{L^3} + \|\partial_t g\|_{L^2} + \|\nabla^{-1} e^{i\xi^\gamma A} g\|_{L^2} ds \\
& \lesssim A_T^2 \int_0^t \left( \frac{1}{(1 + s)^{11/6}} + \frac{1}{(1 + s)^{3/2}} \right) ds \lesssim (1 + t)^{1/2} A_T^3.
\end{align*}

For (108), note that by (11),

\[ \|\nabla^{-1} e^{i\xi^\gamma A} \partial_\eta g\|_{L^3} \lesssim \|e^{i\xi^\gamma A} \partial_\eta g\|_{\dot{H}^{-1/2}} \lesssim \|E\|_{L^\infty} \|E\|_{\dot{H}^{3/2}} \lesssim (1 + s)^{-5/4} A_T^2. \]

This bound together with (18) yields

\begin{align*}
\|(108)\|_{L^2} & \lesssim \int_0^t \|e^{i\xi^\gamma A} f\|_{L^6} \|\nabla^{-1} e^{i\xi^\gamma A} \partial_\eta g\|_{L^3} ds \\
& \lesssim A_T^2 \int_0^t \left( \frac{1}{(1 + s)^{11/2}} + \frac{1}{(1 + s)^{5/4}} \right) ds \lesssim (1 + t)^{1/4} A_T^3.
\end{align*}

The argument for (109) is essentially the same as (102). For the term (110), we use (19) and (22) to obtain

\begin{align*}
\|(110)\|_{L^2} & \lesssim \int_0^t \|e^{i\xi^\gamma A} f\|_{L^4} \|e^{i\xi^\gamma A} xg\|_{L^4} ds \\
& \lesssim A_T^2 \int_0^t \left( \frac{1}{(1 + s)^{7/8}} \right) ds \lesssim (1 + t)^{1/8} A_T^2.
\end{align*}

It remains to estimate (104). By (94), one sees

\[ |\nabla_\xi \psi^+|^2 = \nabla_\xi \psi^+ \cdot 2\eta = \frac{4}{\gamma + 1} \psi^+ - \frac{4}{\gamma + 1} (\eta \cdot \nabla_\eta \psi^+). \]

Using this identity to integrate by parts in time and frequency, we obtain

\begin{align}
\gamma + 1 \quad \text{(104)} & = i\int_{\mathbb{R}^3} e^{i\xi^\gamma A} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta \\
& \quad + i \int_0^t \int_{\mathbb{R}^3} s e^{i\xi^\gamma A} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds \\
& \quad + i \int_0^t \int_{\mathbb{R}^3} s e^{i\xi^\gamma A} \hat{f}(\xi - \eta) \times \hat{g}(\eta) d\eta ds.
\end{align}

\[
- \int_0^t \int_{\mathbb{R}^3} s^2 e^{is\psi} \partial_x \hat{f}(\xi - \eta) \times \nabla_\eta \tilde{g}(\eta) d\eta ds \\
(\xi \cdot \nabla_\eta \tilde{g}(\eta))
\]

In virtue of (19) and (22), the term (111) is estimated as

\[
\| (111) \|_{L^2} \lesssim t^2 \| e^{it\Delta} f \|_{L^\infty} \| e^{it\gamma \Delta} g \|_{L^4} \lesssim (1 + t)^{1/2} A_T^2.
\]

Similar to (111), we can deal with the term (112). Using (11), we have

\[
\| (113) \|_{L^2} \lesssim \int_0^t s^2 \| e^{is\Delta} \partial_x f \|_{L^2} \| e^{is\gamma \Delta} g \|_{L^\infty} ds \\
\lesssim \int_0^t s^2 \| E \|_{L^2} (\| E \|_{L^\infty}^2 + \| M \|_{L^\infty}) \| M \|_{L^\infty} ds \lesssim (1 + t)^{1/3} A_T^3,
\]

and

\[
\| (114) \|_{L^2} \lesssim \int_0^t s^2 \| e^{is\Delta} f \|_{L^\infty} \| e^{is\gamma \Delta} \partial_x g \|_{L^2} ds \\
\lesssim \int_0^t s^2 \| E \|_{L^\infty}^2 \| E \|_{H^2} ds \lesssim (1 + t)^{1/2} A_T^3.
\]

The term (115) is analogous to (103), so we can apply the same strategy used in (103) to obtain the desired bound. For the last term (116), we again use the relation (94) to get

\[
(\gamma + 1) \cdot (116) = t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi} \frac{\hat{f}(\xi - \eta)}{\eta} \times \nabla_\eta \tilde{g}(\eta) d\eta \\
+ \int_0^t \int_{\mathbb{R}^3} \frac{\eta}{|\eta|^2} e^{is\psi} \frac{\hat{f}(\xi - \eta)}{\eta} \times \nabla_\eta \tilde{g}(\eta) d\eta ds
\]

In order to estimate (117)–(119), note first that

\[
\| e^{is\gamma \Delta} (xg) \|_{L^2} \lesssim \| e^{is\gamma \Delta} (xg) \|_{H^{-1/2}} \lesssim \| xg \|_{L^{3/2}} \\
\lesssim \| xg \|_{L^3}^{1/2} \| x^2 g \|_{L^2}^{1/2} \lesssim (1 + s)^{1/6} A_T,
\]

then by (17), (18), (20) and (11), we have

\[
\| (117) \|_{L^2} \lesssim t \| e^{it\Delta} f \|_{L^6} \| \nabla^{-1} e^{it\gamma \Delta} (xg) \|_{L^3} \lesssim (1 + t)^{1/6} A_T^2.
\]
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\[ \| \langle 118 \rangle \|_{L^2} \lesssim \int_0^t \| e^{ix\Delta} f \|_{L^6} \| \nabla^{-1} e^{ix\gamma \Delta} (xg) \|_{L^2} ds \lesssim (1 + t)^{1/6} A_T^2, \]

\[ \| \langle 119 \rangle \|_{L^2} \lesssim \int_0^t \| e^{ix\Delta} f \|_{L^6} \| \nabla^{-1} e^{ix\gamma \Delta} (xg) \|_{L^2} ds \lesssim A_T^2 \int_0^t s \left( \frac{1}{1 + s} \right) \left( \frac{1}{(1 + s)^{5/2}} + \frac{1}{(1 + s)^{3/2}} \right) (1 + s)^{1/2} ds \lesssim A_T^2. \]

For \( \langle 120 \rangle \), it follows from Lemma 5.3 that

\[ \| \langle 120 \rangle \|_{L^2} \lesssim \int_0^t s(\| e^{ix\Delta} f \|_{L^\infty} \| \tilde{h}_1 \|_{L^2} + \| e^{ix\Delta} f \|_{L^6} \| \tilde{h}_2 \|_{L^3}) ds \lesssim A_T^2 \int_0^t s \left( \frac{1}{(1 + s)^{7/3}} + \frac{1}{(1 + s)^{23/12}} \right) ds \lesssim (1 + t)^{1/12} A_T^2. \]

The term \( \langle 121 \rangle \) is similar to \( \langle 110 \rangle \), so we omit it. The last term \( \langle 122 \rangle \) is estimated by

\[ \| \langle 122 \rangle \|_{L^2} \lesssim \int_0^t \| e^{ix\Delta} f \|_{L^\infty} \| e^{ix\gamma \Delta} |x|^2 g \|_{L^2} ds \lesssim A_T^2 \int_0^t \frac{1}{(1 + s)^{3/4}} (1 + s)^{1/3} ds \lesssim (1 + t)^{1/12} A_T^2. \]

Therefore, combining the above estimates, we obtain \( \langle 101 \rangle \) as desired.

Finally, we should prove for all \( t \in [0, T] \), there holds

\[ \| x F_3(t) \|_{L^2} + (1 + t)^{-1/2} \| |x|^2 F_3(t) \|_{L^2} \lesssim A_T^2, \]

where \( F_3 \) is given in \( (24) \). Note that the phase \( \psi_- \) has similar null structure as \( \psi_+ \) (see \( (14)-(15) \)), one can apply analogous arguments as \( (91) \) and \( (101) \) to obtain the above bound. Since the proof is similar, we omit further details. This finishes the proof of Lemma 5.4.

6. Proof of Theorem 2.1. Now, by using the \( a \) priori estimates obtained in Propositions 1, 2 and 3, we now present the proof of Theorem 2.1.

Proof of Theorem 2.1. Let

\[ (E(t), M(t)) \in C([0, T^*); H^N \times (H^{N-1} \cap \dot{H}^{-1})] \]

be the local solution of system (5), which can be obtained by applying the argument of [16]. Here, \( T^* \) is the maximal existence time of the solution. Moreover, following the argument of [23], we also have \( x f, |x|^2 f, xg, |x|^2 g \in C([0, T^*); L^2) \). Hence, in order to obtain Theorem 2.1, we shall show \( T^* = +\infty \) if the initial data is small enough.

By the condition \( (7)-(8) \) and the continuity of the solution, there exists a time \( T > 0 \) such that \( A_T \leq 4\epsilon_0 \). Let \( T' \) be the supremum of \( T \) satisfying \( A_T \leq 4\epsilon_0 \), then from Proposition 1, Proposition 2 and Proposition 3, we have

\[ A_{T'} \leq 2\epsilon_0 + CA_T^2, \]

where \( C \) is independent of \( T' \). Then we can show \( T' = T^* \) provided that \( \epsilon_0 \leq (16C)^{-1} \). Indeed, if \( T' < T^* \), the above inequality gives \( 4\epsilon_0 \leq 2\epsilon_0 + 16C\epsilon_0^2 \), which is a contradiction for sufficiently small \( \epsilon_0 \). Therefore, we conclude that if \( \epsilon_0 \leq (16C)^{-1} \), then \( A_{T'} \leq 4\epsilon_0 \), which in turn implies \( T^* = +\infty \). Then the decay estimates in \( (9) \) follow easily from \( (17) \) and \( (20) \). This completes the proof of Theorem 2.1. \( \square \)
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E-mail address: huang_daiwen@iapcm.ac.cn

E-mail address: zjj_math@aliyun.com