We consider a model where a light scalar field (with mass $\lesssim 30\,\text{eV}$), conjectured to be dark matter, has a non-minimal coupling to gravity. In the non-relativistic limit, this new coupling introduces a self-interaction term in the scalar-field equation of motion, and modifies the source term for the gravitational field. Moreover, in the small-coupling limit justified by the observed dark-matter density, the system further reduces to the Gross-Pitaevskii-Poisson equations, which remarkably also arise from a self-gravitating and self-interacting Bose-Einstein condensate system. We derive predictions of our model on linear and non-linear structure formation by exploiting this unexpected connection.

We first write down the general theory and derive the minimal coupling term is crucial to the renormalizability of a scalar-field theory in curved space-time \cite{7-9}. This coupling will change the dynamical behavior of dark matter and can potentially have observable effects in structure formation.

Motivated by various particle physics considerations, there are many flavors of these wave dark matter models, most of which consist of a scalar field that is minimally coupled to gravity. However, the scalar field could, or some \cite{10-16} would argue in general has to, be non-minimally coupled to gravity. It has been shown that a non-minimal coupling will naturally arise as quantum corrections to a minimally coupled classical theory \cite{6}. Moreover, a non-minimal coupling term is crucial to the renormalizability of a scalar-field theory in curved space-time \cite{7-9}. This coupling will change the dynamical behavior of dark matter and can potentially have observable effects in structure formation.

Here, we consider a non-minimal coupling of the form $\phi^2 R$, where $\phi$ is the scalar field and $R$ is the Ricci scalar. We first write down the general theory and derive the equations of motion in the non-relativistic limit. We point out the difference between this theory and the minimally coupled theory of wave dark matter. We then discuss the small-coupling limit valid in most practical cases, while making connections to self-gravitating and self-interacting Bose-Einstein condensate \cite{10-16}. Following these connections, we present predictions of this model on linear and non-linear structure formation. We end with several concluding remarks.

We consider the theory $S = S_{\text{EH}} + S_\phi$, where

$$S_{\text{EH}} = \int d^4x \sqrt{-g} \frac{R}{16\pi G}$$

is the familiar Einstein-Hilbert action, and

$$S_\phi = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} (\nabla^\mu \phi)(\nabla_\mu \phi) - V(\phi) - \frac{1}{2} \xi R \phi^2 \right]$$

is the scalar-field action with a non-minimal coupling to gravity. Here, $g \equiv \det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$; $V(\phi)$ is the potential of the scalar field; and $\xi$ is a dimensionless coupling constant. The equation of motion for the scalar field is determined by $\delta S/\delta \phi = 0$, which in this case gives

$$\Box \phi - \xi R \phi - V'(\phi) = 0.$$  \hspace{1cm} (3)

Here, $\Box \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is the d’Alembertian with $\nabla_\mu$ being the covariant derivative, and $V'(\phi) \equiv dV(\phi)/d\phi$. In the rest of this paper, we assume $\phi = 0$ is a local minimum of the potential, and the excursion of $\phi$ is small enough so that we can write $V(\phi) = m^2 \phi^2/2$. We define the energy-momentum tensor of the scalar field as $T^\phi_{\mu\nu} = -(2/\sqrt{-g}) \delta S/\delta g^{\mu\nu}$ and obtain

$$T^\phi_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - g_{\mu\nu} \left[ \frac{1}{2} (\nabla_\alpha \phi)(\nabla^\alpha \phi) + V(\phi) \right] + \xi \left[ G_{\mu\nu} \phi^2 + g_{\mu\nu} \Box(\phi^2) - \nabla_\mu \nabla_\nu (\phi^2) \right].$$

\hspace{1cm} (4)

It is worth noting that the variation of the last term in Eq. (2) is subject to the Leibniz rule, and thus more complex than the variation of the Ricci scalar $R$ in Eq. (1). This explains the three terms proportional to $\xi$ in Eq. (4). The Einstein equation $\delta S/\delta g^{\mu\nu} = 0$ can then be written as

$$G_{\mu\nu} = 8\pi GT^\phi_{\mu\nu}.$$ 

\hspace{1cm} (5)

Here, $G_{\mu\nu} \equiv R_{\mu\nu} - R g_{\mu\nu}/2$ is the Einstein tensor. Note that, due to the non-minimal coupling, the right-hand side of Eq. (5) also contains the geometric quantity $G_{\mu\nu}$.

We consider the weak-gravity limit in the Newtonian gauge with only the scalar metric perturbation $\Psi$, where the line element is

$$ds^2 = -(1 + 2\Psi)dt^2 + (1 - 2\Psi)dx^i dx^j.$$ 

\hspace{1cm} (6)

Note that we ignore the cosmic expansion for simplicity. The effect of cosmic expansion can be simply restored by considering “Newtonian cosmology” \cite{11}, provided that the pressure is negligible in comparison with the energy.
density. We also neglect any anisotropic stress so that the other metric perturbation variable $\Phi$ is equal to $\Psi$, and the gravity sector is described by only one variable $\Psi$. This will be consistent with the non-relativistic limit we are about to take later in this work. From now on, we assume $|\Psi| \ll 1$ and only work to leading order in $\Psi$.

This space-time geometry implies some formulas that will be useful later. For a scalar quantity $f = f(\vec{x}, t)$, the d’Alembertian is

$$\Box f = (1 + 2\Psi)\nabla^2 f - (1 - 2\Psi)\partial_\mu \partial^\mu f + 4(\partial_t \Psi)(\partial_t f); \quad (7)$$

the $(0, 0)$ component of the Hessian is

$$\nabla^0 \nabla_0 f = \nabla \Psi \cdot \nabla f + (\partial_t \Psi)(\partial_t f) - (1 - 2\Psi)\partial_t^2 f; \quad (8)$$

the Ricci scalar is

$$R = 2\nabla^2 \Psi - 6\partial_t^2 \Psi; \quad (9)$$

and the $(0, 0)$ component of the Einstein tensor $G^\mu_\nu$ is

$$G^0_0 = -2\nabla^2 \Psi. \quad (10)$$

Note that here $\nabla$ without the subscript is the flat spatial gradient operator, and should not be confused with $\nabla_\mu$.

In the non-relativistic limit we will discuss, only two equations are important. The first equation is the scalar-field equation of motion. By inserting Eq. (7) (with $f = \phi$) and Eq. (9) into Eq. (3), we have

$$(1 + 2\Psi)\nabla^2 \phi - (1 - 2\Psi)\partial_t^2 \phi - m^2 \phi + 4(\partial_t \Psi)(\partial_t \phi) - \xi(2\nabla^2 \Psi - 6\partial_t^2 \Psi)\phi = 0. \quad (11)$$

The second equation is the $(0, 0)$ component of the Einstein equation $G^0_0 = 8\pi G T^0_0$. By inserting Eqs. (7) and (8) (with $f = \phi^2$) plus Eq. (10) into Eq. (5), we have

$$\nabla^2 \Psi = 4\pi G \rho_\phi, \quad (12)$$

where the energy density $\rho_\phi \equiv -T^0_0$ of the scalar field is

$$\rho_\phi = \frac{1}{2}(1 - 2\Psi)(\partial_t \phi)^2 + \frac{1}{2}(1 + 2\Psi)(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2 + \xi[2(\nabla^2 \phi)\phi^2 + (\nabla \Psi)(\nabla \phi)^2 - (1 + 2\Psi)(\nabla^2 \phi^2) - 3(\partial_t \Psi)(\partial_t \phi^2)]. \quad (13)$$

Note that due to the non-minimal coupling, $\rho_\phi$ itself now contains $\nabla^2 \Psi$. This implies that the metric perturbation $\Psi$ is not sourced by $\rho_\phi$, but a slightly more complicated term. Later, in the non-relativistic limit, we will show exactly how the source term is modified.

We now work out the non-relativistic limit of the equation of motion, Eq. (11), and the Einstein equation, Eq. (12), by factoring out the fast-varying oscillation $e^{-imt}$ in $\phi$ as

$$\phi = \frac{1}{\sqrt{2m}}(\psi e^{-imt} + \psi^* e^{imt}). \quad (14)$$

The newly defined complex scalar $\psi$ is then slowly varying (i.e. $\partial_t \ll m$ when acting on everything other than $\phi$). The non-relativistic limit also implies that the gradient of $\phi$ (and $\psi$) is small (i.e. $\nabla \ll m$ when acting on $\phi$ or $\psi$). Now, we proceed by working in the appropriate orders of $\Psi, \partial_t/m$, and $\nabla/m$. We shall also average out the fast-varying contribution to the energy density $\rho_\phi$. Since these standard procedures have been detailed for a minimally coupled scalar field in Ref. [18], we only explain in detail how we approximate terms proportional to $\xi$. In Eq. (11), we only keep $\nabla^2 \Psi$ but neglect $\partial_t^2 \Psi$. This is justified by the smallness of the spatial part of the Einstein equation (i.e. $|G^j_\nu| \ll |G^0_0|$) in the non-relativistic limit. In Eq. (12), we only keep $(\nabla^2 \Psi)\phi^2$. The terms $(\nabla \Psi)(\nabla \phi^2)$ and $(1 + 2\Psi)(\nabla^2 \phi^2)$ are neglected due to the smallness of $\nabla \phi$; the term $(\partial_t \Psi)(\partial_t \phi^2)$ is approximately $(\partial_t \Psi)(\partial_t |\psi|^2/m)$ after averaging out the fast-varying contribution, and is then neglected due to the smallness of $\partial_t \psi$.

Doing so, Eqs. (11) and (12) then become the coupled differential equations

$$i\partial_t \psi = -\frac{\nabla^2 \psi + m \Psi \psi + \xi \nabla(\psi^*) \psi}{2m}, \quad (15)$$

$$\nabla^2 \Psi = m \nabla(\psi^*), \quad (16)$$

where we define the effective potential,

$$\mathcal{V}(\psi) = \frac{4\pi G m |\psi|^2}{m - 8\pi \xi G |\psi|^2}. \quad (17)$$

It is obvious that without non-minimal coupling ($\xi = 0$), the system further reduces to the Schrödinger-Poisson equations. In the presence of non-minimal coupling, a self-interaction term is added to the Schrödinger equation, and the source term of the Poisson equation is modified. As will become clear later, we are practically always in the small-non-minimal-coupling limit $\xi \ll m/(8\pi G |\psi|^2)$. This implies $\mathcal{V}(\psi) \approx 4\pi G |\psi|^2$, and the system reduces to the Gross-Pitaevskii-Poisson equations

$$i\partial_t \psi = -\frac{\nabla^2 \psi + m \Psi \psi + 4\pi G m |\psi|^2 \psi}{2m}, \quad (18)$$

$$\nabla^2 \Psi = 4\pi G m |\psi|^2. \quad (19)$$

From this familiar form, we can then interpret $m |\psi|^2$ as the dark-matter density $\rho$ (not to be confused with $\rho_\phi$). So, retrospectively, the small coupling limit is valid when

$$\xi \ll \frac{m^2}{8\pi G \rho} \approx 8 \times 10^{15} \left(\frac{m}{10^{-22} \text{eV}}\right)^2 \left(\frac{1 \text{GeV/cm}^3}{\rho}\right), \quad (20)$$

which is easily satisfied if $\xi$ is not unnaturally large. We proceed with the small-coupling limit from now on.

Interestingly, the Gross-Pitaevskii-Poisson equations also arise in the description of self-gravitating and self-interacting Bose-Einstein condensate [10][12]. Although
the equations are equivalent, the physical picture is completely different. In the Bose-Einstein condensate scenario, the self-interaction term $\propto |\psi|^2 \psi$ is the direct result of the explicitly introduced contact interaction between the bosons, while in our case it arises from the scalar field’s non-minimal coupling to gravity. This unexpected correspondence allows us to benefit from the results of previous work. For instance, we provide the translation from Ref. [12] to our paper,

$$h \to 1, \quad N \to 1, \quad g \to \frac{4\pi G}{m^2} \xi. \quad (21)$$

The last one can be substituted by $a \to Gm\xi$. On the left-hand side, $h$ is the reduced Planck constant; $N$ is the number of particles in the condensate; and $g$ is the contact-interaction strength, with the s-wave scattering length $a$ being an equivalent representation. We emphasize that the duality established here is only in the appropriate limits of both theories (non-relativistic and small-coupling limit for the theory presented in this work, and non-relativistic limit for the Bose-Einstein condensate). While the duality likely does not exist in the full theory, relativistic studies of the self-interacting Bose-Einstein condensate (for instance, Refs. [13, 14]), once reduced to the non-relativistic limit, should re-obtain this duality.

The Gross-Pitaevskii-Poisson equations, Eqs. (18) and (19), have a fluid description. We define the fluid variables

$$\rho \equiv m|\psi|^2, \quad \text{and} \quad \vec{v} \equiv \frac{1}{m} \nabla \arg \psi. \quad (22)$$

It can be shown that Eqs. (18) and (19) are equivalent to the following fluid equations

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \quad (23)$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \Psi - \frac{\nabla \vec{P}_q}{\rho} - \frac{\nabla \cdot \vec{P}_q}{\rho}, \quad (24)$$

$$\nabla^2 \Psi = 4\pi G \rho. \quad (25)$$

These equations can also be obtained by translating Eqs. (10), (14), and (7) in Ref. [12] using our Eq. (21). Here we define the quantum pressure tensor and the pressure caused by non-minimal coupling as

$$P_{ij} = -\frac{\rho}{4m^2} \partial_i \partial_j \ln \rho, \quad \text{and} \quad P_{\xi} = \frac{2\xi G}{m^2} \rho^2, \quad (26)$$

respectively. While we have a fully classical system here, we still choose to follow the widely accepted name of $P_Q$ as the quantum pressure. Note that the non-minimal coupling amounts to an isotropic pressure term, whereas the quantum pressure is anisotropic. We refer to $P_{\xi}$ as the “$\xi$-pressure” from now on.

The Jeans scale $k_J$ of this theory is crucial to linear structure formation. Intuitively, gravity pulls the dark matter together and makes structure grow, while the quantum pressure and the $\xi$-pressure resist this effect. Quantitatively, we examine the divergence of the right-hand side of Eq. (24) by setting $\rho = \bar{\rho}(1 + \delta)$ and working to leading order in $\delta$, which gives (in the original sequence of terms)

$$-4\pi G \bar{\rho} \delta - \frac{4\pi \xi G \bar{\rho}}{m^2} \nabla^2 \delta + \frac{1}{4m^2} \nabla^4 \delta. \quad (27)$$

Here, in the first term, we use the Poisson equation, Eq. (24), with only the perturbation $\bar{\rho}$, without the background $\bar{\rho}$, as the source (i.e. the “Jeans swindle” [19]). Substituting $\nabla$ with $i\vec{k}$, we see that a given $k$-mode of density perturbation $\delta$ will grow normally (as cold dark matter) at low $k$ when gravity wins, but this growth is suppressed at high $k$ when the quantum pressure and the $\xi$-pressure win. The transition, namely the Jeans scale $k_J$, follows from solving the resulting quadratic equation in $k^2$. Doing so, we find,

$$k_J = (16\pi G \bar{\rho})^{\frac{1}{2}} m^{\frac{3}{2}} \left( \sqrt{1 + \frac{4\pi G \bar{\rho}}{m^2} \xi^2} - \sqrt{\frac{4\pi G \bar{\rho}}{m^2} \xi^2} \right)^{\frac{1}{2}}. \quad (28)$$

Note that this formula can also be translated by translating Eq. (138) in Ref. [12] using our Eq. (21). The dimensionless quantity

$$\frac{4\pi G \bar{\rho}}{m^2} \xi^2 \simeq 8 \times 10^{-25} \left( \frac{\xi}{0.1} \right)^2 \left( \frac{10^{-22} \text{eV}}{m} \right)^2 \times \left( \frac{\bar{\rho}}{1.3 \times 10^{-6} \text{GeV/cm}^3} \right) \quad (29)$$

determines $k_J$’s deviation from the Jeans scale $k_{J0} = (16\pi G \bar{\rho})^{\frac{1}{2}} m^{\frac{3}{2}}$ of the minimally coupled theory, and is likely a small number if $\xi$ is not unnaturally large. This can be understood, in a different way, by considering the scales $k_\xi$ and $k_Q$ at which the quantum pressure and the $\xi$-pressure, respectively, become comparable to gravity. By comparing the second and the third terms to the first term in Eq. (27), we find

$$k_\xi = \frac{m}{\sqrt{\xi}}, \quad \text{and} \quad k_Q = (16\pi G \bar{\rho})^{\frac{1}{2}} m^{\frac{3}{2}} = k_{J0}. \quad (30)$$

Taking the same anchored values as in Eq. (29), we see that $k_\xi^{-1} \sim 0.02 \text{pc}$ is far smaller than $k_Q^{-1} \sim 0.014 \text{Mpc}$, so the new $\xi$-pressure is not important in modifying the Jeans scale $k_{J0} = (k_Q)$ of the minimally coupled theory. But, we need to point out that, in principle, it is still possible for a large enough $\xi$, without violating the small-coupling limit in Eq. (20), to make Eq. (29), hence the deviation of $k_J$ from $k_{J0}$, large.

However, the $\xi$-pressure will almost always be important at small scales, implying noticeable change in nonlinear structure formation. In wave dark matter models, numerical simulations show that the dark-matter halo will host an enormous soliton — as massive as $10^8 M_\odot$
for a $10^{10} M_\odot$ halo with $m \sim 10^{-22}$ eV \cite{20,21}. The profile of a soliton for a given mass can be obtained by solving the hydrostatic version of Eqs. (23), (24), and (25) \cite{13}. It is also possible to obtain approximate analytical results by assuming a Gaussian density profile $\rho(r) = M(\pi R^2)^{-1/2} \exp(-r^2/R^2)$ of the soliton, and minimize the energy functional with respect to the characteristic soliton size $R$ for a given soliton mass \cite{12}. The two approaches are found to yield similar results \cite{13}. Here, we translate the Bose-Einstein-condensate result, Eq. (92) in Ref. \cite{12}, using our Eq. (21), giving the mass-radius relation

$$R \sim \frac{(9\pi/2)^{1/2}}{GMm^2} \left[ \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{8\xi}{3\pi}(GMm)^2} \right].$$

The “±” only takes the plus sign for $\xi > 0$, but takes the plus and the minus signs for $\xi < 0$. We emphasize that the minus-sign branch when $\xi < 0$ is dynamically unstable \cite{12} (i.e. a local maximum of the energy functional). Representative evaluations of this formula are given in Fig. 1. We see that for the dynamically stable branches, a positive $\xi$ increases the radii of large-mass solitons, whereas a negative $\xi$ decreases them. We also see that, for a negative $\xi$, there is a maximum mass and a minimum radius for a stable soliton. In addition to this limit, the soliton cannot become too compact (i.e. $R \sim GM$), at which point our non-relativistic treatment will become insufficient, and the soliton might collapse into a black hole \cite{22,24}.

Now, the mass-radius relationship, Eq. (31), can be used to interpret observational results. We only present two preliminary examples here, and leave a full-fledged study to future work. First, a recent dynamical analysis of the Galactic center suggests a $10^9 M_\odot$ solitonic core with a size of 0.1 kpc \cite{23}. This would only be compatible, in the minimally coupled theory, with $m \approx 10^{-22}$ eV. When allowing non-minimal coupling, interpretations with different $m$ emerge when $|\xi|$ becomes large ($\gtrsim 10^4$, but still within the small-coupling limit). Second, the minimally coupled theory predicts the mass-radius scaling $R \sim M^{-1}$, in tension with the observed constant core surface density for various low-mass galaxies \cite{20} (i.e. $M/R^2 \approx \text{const.}$, implying $R \sim M^{1/2}$). Although the non-minimal coupling cannot fully resolve the problem, it will alleviate it by providing a $R \sim M^0$ scaling via the plateau part of Eq. (31) at large soliton mass.

Before closing, we identify some similarities between the theory presented here, in the small-coupling limit, and other models of wave dark matter. A minimally coupled ($\xi = 0$) theory with an explicit self-interaction $\lambda \phi^4 \in V(\phi)$ in Eq. (2) will also yield similar Gross-Pitaevskii-Poisson equations, Eqs. (18) and (19). \cite{27}. This can be understood by rewriting the non-minimal coupling $\phi^2 R$ in the non-relativistic limit $-\phi^2 R \sim \phi^2 \nabla^2 \Psi \sim \phi^4$. Here, the first relation is implied by Eq. (9), and the second by the leading contribution from time averaging the Poisson equation, Eq. (19). This also means that our theory can be described by the non-relativistic effective field theory for scalar dark matter, detailed recently in Ref. \cite{28}.

In this work, we discuss a theory of wave dark matter that has a non-minimal coupling $\phi^2 R$ to gravity. We derive the equations of motion for this theory in the non-relativistic and small-coupling limit and present an equivalent fluid description with the Gross-Pitaevskii-Poisson equations. From that, we also point out a connection between this theory and previous research on self-gravitating and self-interacting Bose-Einstein condensate. We proceed to discuss some phenomenology of linear and non-linear structure formation. Future work may explore the next-to-leading-order effect in $\xi$; the cosmological matter power spectrum incorporating the full expansion history of the Universe; the production process in the early universe; numerical simulations of halo formation, etc. It should also be interesting to see the consequences of other forms of non-minimal coupling.

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[1] G. Bertone, D. Hooper and J. Silk, “Particle dark matter: Evidence, candidates and constraints,” Phys. Rept. 405, 279-390 (2005) [arXiv:hep-ph/0404175 [hep-ph]].

[2] W. Hu, R. Barkana and A. Gruzinov, “Cold and fuzzy dark matter,” Phys. Rev. Lett. 85, 1158-1161 (2000) [arXiv:astro-ph/0003365 [astro-ph]].

[3] L. Hui, J. P. Ostriker, S. Tremaine and E. Witten, “Ultralight scalars as cosmological dark matter,” Phys. Rev. D 95, no.4, 043541 (2017) [arXiv:1610.08297 [astro-ph.CO]].

[4] L. Hui, “Wave Dark Matter,” [arXiv:2101.11735 [astro-ph.CO]].

[5] V. Faraoni, “Inflation and quintessence with nonminimal coupling,” Phys. Rev. D 62, 023504 (2000) [arXiv:gr-qc/0002091 [gr-qc]].

[6] N. D. Birrell and P. C. W. Davies, “Quantum Fields in Curved Space,”

[7] C. G. Callan, Jr., S. R. Coleman and R. Jackiw, “A New improved energy - momentum tensor,” Annals Phys. 59, 42-73 (1970)

[8] D. Z. Freedman and E. J. Weinberg, “The Energy-Momentum Tensor in Scalar and Gauge Field Theories,” Annals Phys. 87, 42-73 (1970)

[9] D. Z. Freedman, I. J. Muzinich and E. J. Weinberg, “On the Energy-Momentum Tensor in Gauge Field Theories,” Annals Phys. 87, 95 (1974)

[10] P. H. Chavanis, “BEC dark matter, Zeldovich approximation and generalized Burgers equation,” Phys. Rev. D 84, 063518 (2011) [arXiv:1103.3219 [astro-ph.CO]].

[11] P. H. Chavanis, “Growth of perturbations in an expanding universe with Bose-Einstein condensate dark matter,” Astron. Astrophys. 537, A127 (2012) [arXiv:1103.2069 [astro-ph.CO]].

[12] P. H. Chavanis, “Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates with short-range interactions: I. Analytical results,” Phys. Rev. D 84, 043531 (2011) [arXiv:1103.2050 [astro-ph.CO]].

[13] P. H. Chavanis and L. Delfini, “Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates with short-range interactions: II. Numerical results,” Phys. Rev. D 84, 043532 (2011) [arXiv:1103.2054 [astro-ph.CO]].

[14] L. Visinelli, S. Baum, J. Redondo, K. Freese and F. Wilczek, “Dilute and dense axion stars,” Phys. Lett. B 777, 64-72 (2018) [arXiv:1710.08910 [astro-ph.CO]].

[15] M. Colpi, S. L. Shapiro and I. Wasserman, “Boson Stars: Gravitational Equilibria of Selfinteracting Scalar Fields,” Phys. Rev. Lett. 57, 2485-2488 (1986)

[16] J. Y. Widdicombe, T. Helfer, D. J. E. Marsh and E. A. Lim, “Formation of Relativistic Axion Stars,” JCAP 10, 005 (2018) [arXiv:1806.09367 [astro-ph.CO]].

[17] H. Kodama and M. Sasaki, “Cosmological Perturbation Theory,” Prog. Theor. Phys. Suppl. 78, 1-166 (1984)

[18] D. J. E. Marsh, “Axion Cosmology,” Phys. Rept. 643, 1-79 (2016) [arXiv:1510.07633 [astro-ph.CO]].

[19] M. Falco, S. H. Hansen, R. Wojtak and G. A. Mamon, “Why does the Jeans Swindle work?,” Mon. Not. Roy. Astron. Soc. 431, 6 (2013) [arXiv:1210.3363 [astro-ph.CO]].

[20] H. Y. Schive, T. Chiuhe and T. Broadhurst, “Cosmic Structure as the Quantum Interference of a Coherent Dark Wave,” Nature Phys. 10, 496-499 (2014) [arXiv:1406.6586 [astro-ph.GA]].

[21] H. Y. Schive, M. H. Liao, T. P. Woo, S. K. Wong, T. Chiuhe, T. Broadhurst and W. Y. P. Hwang, “Understanding the Core-Halo Relation of Quantum Wave Dark Matter from 3D Simulations,” Phys. Rev. Lett. 113, no.26, 261302 (2014) [arXiv:1407.7762 [astro-ph.GA]].

[22] Z. Nazari, M. Cicolì, K. Clough and F. Muia, “Oscillon collapse to black holes,” JCAP 05, 027 (2021) [arXiv:2010.05933 [gr-qc]].

[23] F. Muia, M. Cicolì, K. Clough, F. Pedro, F. Quevedo and G. P. Vacca, “The Fate of Dense Scalar Stars,” JCAP 07, 044 (2019) [arXiv:1906.09346 [gr-qc]].

[24] T. Helfer, D. J. E. Marsh, K. Clough, M. Fairbairn, E. A. Lim and R. Becerril, “Black hole formation from axion stars,” JCAP 03, 055 (2017) [arXiv:1609.04724 [astro-ph.CO]].

[25] I. De Martino, T. Broadhurst, S. H. H. Tye, T. Chiuhe and H. Y. Schive, “Dynamical Evidence of a Solitonic Core of 10^9 M_⊙ in the Milky Way,” Phys. Dark Univ. 28, 100503 (2020) [arXiv:1807.08153 [astro-ph.GA]].

[26] A. Burkert, “Fuzzy Dark Matter and Dark Matter Halo Cores,” Astrophys. J. 904, no.2, 161 (2020) [arXiv:2006.11111 [astro-ph.GA]].

[27] E. G. M. Ferreira, “Ultra-Light Dark Matter,” [arXiv:2005.03254 [astro-ph.CO]].

[28] B. Salehian, H. Y. Zhang, M. A. Amin, D. I. Kaiser and M. H. Namjoo, “Beyond Schrödinger-Poisson: Nonrelativistic Effective Field Theory for Scalar Dark Matter,” [arXiv:2104.10128 [astro-ph.CO]].