OPTIMAL RELATIONS BETWEEN $L^p$-NORMS FOR THE HARDY OPERATOR AND ITS DUAL

V.I. KOLYADA

Abstract. We obtain sharp two-sided inequalities between $L^p$-norms ($1 < p < \infty$) of functions $Hf$ and $H^*f$, where $H$ is the Hardy operator, $H^*$ is its dual, and $f$ is a nonnegative measurable function on $(0, \infty)$. In an equivalent form, it gives sharp constants in the two-sided relations between $L^p$-norms of functions $H\varphi - \varphi$ and $\varphi$, where $\varphi$ is a nonnegative nonincreasing function on $(0, +\infty)$ with $\varphi(+\infty) = 0$. In particular, it provides an alternative proof of a result obtained by N. Kruglyak and E. Setterqvist (2008) for $p = 2k$ ($k \in \mathbb{N}$) and by S. Boza and J. Soria (2011) for all $p \geq 2$, and gives a sharp version of this result for $1 < p < 2$.

1. Introduction and main results

Denote by $\mathcal{M}^+(\mathbb{R}_+)$ the class of all nonnegative measurable functions on $\mathbb{R}_+ \equiv (0, +\infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}_+)$. Set

$$Hf(x) = \frac{1}{x} \int_0^x f(t) \, dt$$

and

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t} \, dt.$$ 

These equalities define the classical Hardy operator $H$ and its dual operator $H^*$. By Hardy’s inequalities [5, Ch. 9], these operators are bounded in $L^p(\mathbb{R}_+)$ for any $1 < p < \infty$. Furthermore, it is easy to show that for any $f \in \mathcal{M}^+(\mathbb{R}_+)$ and any $1 < p < \infty$ the $L^p$-norms of $Hf$ and $H^*f$ are equivalent. Indeed, let $f \in \mathcal{M}^+(\mathbb{R}_+)$. By Fubini’s theorem,

$$Hf(x) = \frac{1}{x} \int_0^x dt \int_t^x \frac{f(u)}{u} \, du \leq \frac{1}{x} \int_0^x H^*f(t) \, dt.$$
On the other hand, Fubini’s theorem gives that
\[ H^* f(x) = \int_x^\infty \frac{du}{u^2} \int_x^u f(t) \, dt \leq \int_x^\infty \frac{Hf(u)}{u} \, du. \]

Using these estimates and applying Hardy’s inequalities [5, p. 240, 244], we obtain that
\[ \frac{1}{p'} ||Hf||_p \leq ||H^* f||_p \leq p ||Hf||_p \quad \text{for} \quad 1 < p < \infty \quad (1.1) \]
(as usual, \( p' = \frac{p}{p-1} \)).

However, the constants in (1.1) are not optimal. The objective of this paper is to find optimal constants. Our main result is the following theorem.

**Theorem 1.1.** Let \( f \in \mathcal{M}^+(\mathbb{R}_+) \) and let \( 1 < p < \infty \). Then
\[ (p - 1)||Hf||_p \leq ||H^* f||_p \leq (p - 1)^{1/p} ||Hf||_p \quad (1.2) \]
if \( 1 < p \leq 2 \), and
\[ (p - 1)^{1/p} ||Hf||_p \leq ||H^* f||_p \leq (p - 1)||Hf||_p \quad (1.3) \]
if \( 2 \leq p < \infty \). All constants in (1.2) and (1.3) are the best possible.

Clearly, the problem on relations between various norms of Hardy operator and its dual is of independent interest (cf. [4]). At the same time, this problem has an equivalent formulation in terms of the difference operator \( H\varphi - \varphi \).

Let \( \varphi \) be a nonincreasing and nonnegative function on \( \mathbb{R}_+ \) such that \( \varphi(\infty) = 0 \). The quantity \( H\varphi - \varphi \) plays an important role in Analysis (see [2], [3], [4], [6], [7] and references therein). It is well known that the norms \( ||H\varphi - \varphi||_p \) and \( ||\varphi||_p \) \((1 < p < \infty)\) are equivalent (see [1] p. 384). However, the *sharp* constant is known only in the following inequality.

Let \( \varphi \) be a nonincreasing and nonnegative function on \( \mathbb{R}_+ \). Then for any \( p \geq 2 \)
\[ ||H\varphi - \varphi||_p \leq (p - 1)^{-1/p} ||\varphi||_p, \quad (1.4) \]
and the constant is optimal.

This result was obtained in [7] for \( p = 2k \) \((k \in \mathbb{N})\) and in [2] for all \( p \geq 2 \) (we observe that (1.4) is a special case of the inequality proved in [2] for weighted \( L^p \)-norms).

We shall show that inequality (1.4) is equivalent to the first inequality in (1.3):
\[ ||Hf||_p \leq (p - 1)^{-1/p} ||H^* f||_p, \quad 2 \leq p < \infty. \quad (1.5) \]
Thus, (1.5) can be derived from (1.4). However, below we give a simple direct proof of (1.5). Moreover, Theorem 1.1 has the following equivalent form.

**Theorem 1.2.** Let \( \varphi \) be a nonincreasing and nonnegative function on \( \mathbb{R}_+ \) such that \( \varphi(+\infty) = 0 \) and let \( 1 < p < \infty \). Then

\[
(p - 1)||H\varphi - \varphi||_p \leq ||\varphi||_p \leq (p - 1)^{1/p}||H\varphi - \varphi||_p \quad (1.6)
\]

if \( 1 < p \leq 2 \), and

\[
(p - 1)^{1/p}||H\varphi - \varphi||_p \leq ||\varphi||_p \leq (p - 1)||H\varphi - \varphi||_p \quad (1.7)
\]

if \( 2 \leq p < \infty \). All constants in (1.6) and (1.7) are the best possible.

2. Proofs of main results

**Proof of Theorem 1.1.** Taking into account (1.1), we may assume that \( Hf \) and \( H^*f \) belong to \( L^p(\mathbb{R}_+) \). We may also assume that \( f(x) > 0 \) for all \( x \in \mathbb{R}_+ \). Denote

\[
I_p = \int_0^\infty \left( \frac{1}{x} \int_0^x f(t) \, dt \right)^p \, dx.
\]

Since \( Hf \in L^p(\mathbb{R}_+) \), we have

\[
Hf(x) = o(x^{-1/p}) \quad \text{as} \quad x \to 0^+ \quad \text{or} \quad x \to +\infty.
\]

Thus, integrating by parts, we obtain

\[
I_p = p' \int_0^\infty x^{1-p} f(x) \left( \int_0^x f(t) \, dt \right)^{p-1} \, dx. \quad (2.1)
\]

Further, set

\[
I_p^* = \int_0^\infty \left( \int_t^\infty \frac{f(x)}{x} \, dx \right)^p \, dt. \quad (2.2)
\]

First we shall prove that

\[
(p - 1)I_p \leq I_p^* \quad \text{if} \quad 2 \leq p < \infty \quad (2.3)
\]

and

\[
I_p^* \leq (p - 1)I_p \quad \text{if} \quad 1 < p \leq 2. \quad (2.4)
\]

Set

\[
\Phi(t, x) = \int_t^x \frac{f(u)}{u} \, du, \quad 0 < t \leq x,
\]

and \( G(t, x) = \Phi(t, x)^p \). Since \( G(t, t) = 0 \), we have

\[
\left( \int_t^\infty \frac{f(x)}{x} \, dx \right)^p = \int_t^\infty G'_x(t, x) \, dx = p \int_t^\infty \frac{f(x)}{x} \Phi(t, x)^{p-1} \, dx.
\]
Thus, by Fubini’s theorem,

\[ I_p^* = p \int_0^\infty \int_t^\infty \frac{f(x)}{x} \Phi(t, x)^{p-1} \, dx \, dt \]

\[ = p \int_0^\infty \frac{f(x)}{x} \int_0^x \Phi(t, x)^{p-1} \, dt \, dx. \quad (2.5) \]

On the other hand, Fubini’s theorem gives that

\[ \int_0^x f(t) \, dt = \int_0^x \Phi(t, x) \, dt. \]

Hence, by (2.1),

\[ I_p = p' \int_0^\infty x^{1-p} f(x) \left( \int_0^x \Phi(t, x) \, dt \right)^{p-1} \, dx. \quad (2.6) \]

Comparing (2.1) with (2.2), we see that \( I_2 = I_2^* \). In what follows we assume that \( p \neq 2 \).

Let \( p > 2 \). Then by Hölder’s inequality

\[ \left( \int_0^x \Phi(t, x) \, dt \right)^{p-1} \leq x^{p-2} \int_0^x \Phi(t, x)^{p-1} \, dt. \]

Thus, by (2.5) and (2.6),

\[ I_p \leq p' \int_0^\infty \frac{f(x)}{x} \int_0^x \Phi(t, x)^{p-1} \, dt \, dx = \frac{I_p^*}{p-1}, \]

and we obtain (2.3).

Let now \( 1 < p < 2 \). Applying Hölder’s inequality, we get

\[ \int_0^x \Phi(t, x)^{p-1} \, dt \leq x^{2-p} \left( \int_0^x \Phi(t, x) \, dt \right)^{p-1}. \]

Thus, by (2.5) and (2.6),

\[ I_p^* \leq p \int_0^\infty x^{1-p} f(x) \left( \int_0^x \Phi(t, x) \, dt \right)^{p-1} \, dx = (p-1)I_p, \]

and we obtain (2.4).

Inequalities (2.3) and (2.4) imply the first inequality in (1.3) and the second inequality in (1.2), respectively.

Now we shall show that

\[ I_p^* \leq (p-1)^p I_p \quad \text{if} \quad 2 < p < \infty \quad (2.7) \]

and

\[ (p-1)^p I_p \leq I_p^* \quad \text{if} \quad 1 < p < 2. \quad (2.8) \]
Observe that by our assumption ($f > 0$ and $H^* f \in L^p(\mathbb{R}^*)$),
\[ 0 < \int_t^\infty \frac{f(x)}{x} \, dx < \infty \quad \text{for all} \quad t > 0. \]
Thus, for any $q > 0$ we have
\[ \left( \int_t^\infty \frac{f(x)}{x} \, dx \right)^q = q \int_t^\infty \frac{f(x)}{x} \left( \int_x^\infty \frac{f(u)}{u} \, du \right)^{q-1} \, dx. \quad (2.9) \]
Applying this equality with $q = p$ in (2.2) and using Fubini’s theorem, we obtain
\[ I_p^* = p \int_0^\infty f(x) \left( \int_x^\infty \frac{f(u)}{u} \, du \right)^{p-1} \, dx. \quad (2.10) \]
Further, apply (2.9) for $q = p - 1$ and use again Fubini’s theorem. This gives
\[
I_p^* = p(p-1) \int_0^\infty f(x) \int_x^\infty \frac{f(u)}{u} \left( \int_u^\infty \frac{f(v)}{v} \, dv \right)^{p-2} \, du \, dx
\]
\[ = p(p-1) \int_0^\infty \frac{f(u)}{u} \left( \int_u^\infty \frac{f(v)}{v} \, dv \right)^{p-2} \int_0^u f(x) \, dx \, du. \]
Set
\[ \varphi(u) = \frac{f(u)^{1/(p-1)}}{u} \int_0^u f(x) \, dx \]
and
\[ \psi(u) = f(u)^{(p-2)/(p-1)} \left( \int_u^\infty \frac{f(x)}{x} \, dx \right)^{p-2} \]
(recall that $f > 0$). Then we have
\[ I_p^* = p(p-1) \int_0^\infty \varphi(u) \psi(u) \, du. \quad (2.11) \]
Furthermore, by (2.1),
\[ \int_0^\infty \varphi(u)^{p-1} \, du = \int_0^\infty \frac{f(u)}{u^{p-1}} \left( \int_0^u f(x) \, dx \right)^{p-1} \, du = \frac{I_p}{p}, \quad (2.12) \]
and by (2.10),
\[ \int_0^\infty \psi(u)^{(p-1)/(p-2)} \, du = \int_0^\infty f(u) \left( \int_u^\infty \frac{f(x)}{x} \, dx \right)^{p-1} \, du = \frac{I_p^*}{p} \quad (2.13) \]
for any $p > 1$, $p \neq 2$. 

Let $p > 2$. Applying in (2.11) Hölder’s inequality with the exponent $p - 1$ and taking into account equalities (2.12) and (2.13), we obtain

$$I_p^* \leq p(p - 1) \left( \frac{I_p}{p} \right)^{(p-2)/(p-1)} \left( \frac{I_p^*}{p} \right).$$

This implies (2.7), which is the second inequality in (1.3).

Let now $1 < p < 2$. Applying in (2.11) Hölder’s inequality with the exponent $p - 1 \in (0, 1)$ (see [5, p. 140]), and using equalities (2.12) and (2.13), we get

$$I_p^* \geq p(p - 1) \left( \frac{I_p}{p} \right)^{(p-2)/(p-1)} \left( \frac{I_p^*}{p} \right).$$

Thus,

$$(I_p^*)^{1/(p-1)} \geq (p - 1)^{p/(p-1)} I_p^{1/(p-1)}.$$

This implies (2.8), which is the first inequality in (1.2).

It remains to show that the constants in (1.2) and (1.3) are optimal.

First, set $f_\varepsilon(x) = \chi_{[1, 1+\varepsilon]}(x)$ ($\varepsilon > 0$). Then

$$\|H f_\varepsilon\|_p^p = \int_1^{1+\varepsilon} x^{-p}(x - 1)^p \, dx + \varepsilon \int_{1+\varepsilon}^{\infty} x^{-p} \, dx.$$

Thus,

$$\frac{\varepsilon^p(1 + \varepsilon)^{1-p}}{p - 1} \leq \|H f_\varepsilon\|_p \leq \frac{\varepsilon^p(1 + \varepsilon)^{1-p}}{p - 1} + \varepsilon^{p+1}.$$

Further,

$$\|H^* f_\varepsilon\|_p^p = \int_0^1 \left( \int_1^{1+\varepsilon} \frac{dt}{t} \right)^p \, dx + \int_1^{1+\varepsilon} \left( \int_x^{1+\varepsilon} \frac{dt}{t} \right)^p \, dx$$

$$= (\ln(1 + \varepsilon))^p + \int_1^{1+\varepsilon} \left( \ln \frac{1 + \varepsilon}{x} \right)^p \, dx.$$

Thus,

$$(\ln(1 + \varepsilon))^p \leq \|H^* f_\varepsilon\|_p^p \leq (\ln(1 + \varepsilon))^p (1 + \varepsilon).$$

Using these estimates, we obtain that

$$\lim_{\varepsilon \to 0^+} \frac{\|H f_\varepsilon\|_p}{\|H^* f_\varepsilon\|_p^p} = (p - 1)^{-1/p}.$$

It follows that the constants in the right-hand side of (1.2) and the left-hand side of (1.3) cannot be improved.

Let $1 < p < 2$. Set $f_\varepsilon(x) = x^{p-1/p} \chi_{[0,1]}(x)$ ($0 < \varepsilon < 1/p$). Then

$$\|H f_\varepsilon\|_p^p \geq \int_0^1 \left( \frac{1}{x} \int_{\varepsilon}^{x} t^{p-1/p} \, dt \right)^p \, dx = \frac{p^p}{\varepsilon^p(p - 1 + \varepsilon)^p}.$$
On the other hand, 

\[ \|H^* f_\varepsilon\|_p \leq \left( \frac{1}{p} - \varepsilon \right)^{-p} \int_0^1 x^{(\varepsilon-1/p)p} \, dx = \frac{p^p}{\varepsilon p(1 - \varepsilon p)^p}. \]

Hence, 

\[ \lim_{\varepsilon \to 0^+} \frac{\|H f_\varepsilon\|_p}{\|H^* f_\varepsilon\|_p} \geq \frac{1}{p - 1}. \]

This implies that the constant in the left-hand side of (1.2) is optimal.

Let now \( p > 2 \). Set \( f_\varepsilon(x) = x^{-\varepsilon-1/p} \chi_{[1, +\infty)}(x) \) \((0 < \varepsilon < 1/p')\). Then

\[ \|H^* f_\varepsilon\|_p^p \geq \int_1^\infty \left( \int_x^\infty \frac{dt}{t^{1+1/p+\varepsilon}} \right)^p \, dx = \frac{p^p}{\varepsilon p(1 + \varepsilon)^p}. \]

and

\[ \|H f_\varepsilon\|_p^p \leq \int_1^\infty \left( \frac{1}{x} \int_0^x \frac{dt}{t^{1/p+\varepsilon}} \right)^p \, dx = \frac{p^p}{\varepsilon p(p - 1 - \varepsilon p)^p}. \]

Thus,

\[ \lim_{\varepsilon \to 0^+} \frac{\|H^* f_\varepsilon\|_p}{\|H f_\varepsilon\|_p} \geq p - 1. \]

This shows that the constant in the right-hand side of (1.3) is the best possible. The proof is completed.

**Remark 2.1.** We emphasize that in Theorem 1.1 we do not assume that \( f \) belongs to \( L^p(\mathbb{R}^+) \). It is clear that the condition \( H f \in L^p(\mathbb{R}^+) \) does not imply that \( f \in L^p(\mathbb{R}^+) \). For example, let \( f(x) = |x-1|^{-1/p}_\varepsilon \chi_{[1, 2]}(x) \), \( p > 1 \). Then

\[ H f(x) = 0 \quad \text{for} \quad x \in [0, 1] \quad \text{and} \quad H f(x) \leq \frac{p'}{x} \quad \text{for} \quad x \geq 1. \]

Thus, \( H f \in L^p(\mathbb{R}^+) \), but \( f \not\in L^p(\mathbb{R}^+) \).

Now we shall show that Theorems 1.1 and 1.2 are equivalent. First we observe that without loss of generality we may assume that a function \( \varphi \) in Theorem 1.2 is locally absolutely continuous on \( \mathbb{R}^+ \). Indeed, let \( \varphi \) be a nonincreasing and nonnegative function on \( \mathbb{R}^+ \) such that \( \varphi(+\infty) = 0 \). Set

\[ \varphi_n(x) = n \int_x^{x+1/n} \varphi(t) \, dt \quad (n \in \mathbb{N}). \]

Then functions \( \varphi_n \) are nonincreasing, nonnegative, and locally absolutely continuous on \( \mathbb{R}^+ \). Besides, the sequence \( \{\varphi_n(x)\} \) increases for any \( x \in \mathbb{R}^+ \) and converges to \( \varphi(x) \) at every point of continuity of \( \varphi \).

By the monotone convergence theorem, \( H \varphi_n(x) \to H \varphi(x) \) as \( n \to \infty \) for any \( x \in \mathbb{R}^+ \), and \( \|\varphi_n\|_p \to \|\varphi\|_p \). Furthermore, in Theorem 1.2 we
may assume that \( \varphi \in L^p(\mathbb{R}_+) \) (in conditions of this theorem the norms \( \|H\varphi - \varphi\|_p \) and \( \|\varphi\|_p \) are equivalent \[1\] p. 384). Using this assumption, Hardy’s inequality, and the dominated convergence theorem, we obtain that \( \|H\varphi_n - \varphi_n\|_p \to \|H\varphi - \varphi\|_p \).

Let \( \varphi \) be a nonincreasing, nonnegative, and locally absolutely continuous function on \( \mathbb{R}_+ \) such that \( \varphi(+\infty) = 0 \). Then

\[
H\varphi(x) - \varphi(x) = \frac{1}{x} \int_0^x [\varphi(t) - \varphi(x)] \, dt
\]

\[
= \frac{1}{x} \int_0^x \int_t^x |\varphi'(u)| \, du \, dt = \frac{1}{x} \int_0^x u |\varphi'(u)| \, du.
\]

Set \( u |\varphi'(u)| = f(u) \). Since \( \varphi(+\infty) = 0 \), we have

\[
\varphi(x) = \int_x^\infty |\varphi'(u)| \, du = \int_x^\infty \frac{f(u)}{u} \, du.
\]

Thus,

\[
H\varphi(x) - \varphi(x) = \frac{1}{x} \int_0^x f(u) \, du = Hf(x)
\]

and

\[
\varphi(x) = \int_x^\infty \frac{f(u)}{u} \, du = \mathcal{H}^*f(x).
\]

Conversely, if \( f \in \mathcal{M}^+(\mathbb{R}_+) \) and

\[
\int_0^x f(u) \, du < \infty \quad \text{for any} \quad x > 0,
\]

we define \( \varphi \) by \(2.15\) and then we have equality \(2.14\). These arguments show the equivalence of Theorems 1.1 and 1.2.

References

[1] C. Bennett and R. Sharpley, Interpolation of operators, Academic Press, Boston 1988.
[2] S. Boza and J. Soria, Solution to a conjecture on the norm of the Hardy operator minus the identity, J. Funct. Anal. 260 (2011), 1020 – 1028.
[3] M. Carro, A. Gogatishvili, J. Martín and L. Pick, Functional properties on rearrangement invariant spaces defined in terms of oscillations, J. Funct. Anal. 229 (2005), 375 – 404.
[4] M. Carro, A. Gogatishvili, J. Martín and L. Pick, Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces, J. Operator Theory 59 (2008), 309 – 332.
[5] G.H. Hardy, J.E. Littlewood, and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, Cambridge, 1967.
[6] V.I. Kolyada, On embedding theorems, in: Nonlinear Analysis, Function Spaces and Applications, vol. 8 (Proceedings of the Spring School held in Prague, 2006), Prague, 2007, 35 - 94.
[7] N. Kruglyak and E. Setterqvist, *Sharp estimates for the identity minus Hardy operator on the cone of decreasing functions*, Proc. Amer. Math. Soc. **136** (2008), 2005 – 2013.

DEPARTMENT OF MATHEMATICS, KARLSTAD UNIVERSITY, UNIVERSITETSGATAN 1, 651 88 KARLSTAD, SWEDEN

*E-mail address: viktor.kolyada@kau.se*