Corrected analytical solution of the generalized Woods–Saxon potential for arbitrary $\ell$ states

O Bayrak and E Aciksoz

Department of Physics, Akdeniz University, 07058 Antalya, Turkey

E-mail: bayrak@akdeniz.edu.tr

Received 20 August 2014, revised 26 September 2014
Accepted for publication 23 October 2014
Published 31 December 2014

Abstract

The bound state solution of the radial Schrödinger equation with the generalized Woods–Saxon potential is carefully examined using the Pekeris approximation for arbitrary $\ell$ states. The energy eigenvalues and the corresponding eigenfunctions are analytically obtained for different $n$ and $\ell$ quantum numbers. The closed forms obtained are applied to calculate the single particle energy levels of a neutron orbiting around $^{56}$Fe nucleus in order to check the consistency between the analytical and the Gamow code results. The analytical results are in good agreement with the results obtained using Gamow code for $\ell = 0$.

Keywords: Woods–Saxon potential, eigenvalues and eigenfunctions, analytical solution, Gamow code

(Some figures may appear in colour only in the online journal)

1. Introduction

The Woods–Saxon potential was first proposed by R D Woods and D S Saxon approximately sixty years ago [1] in order to explain the elastic scattering of 20 MeV protons by medium and heavy nuclei. Since then, the Woods–Saxon potential has attracted a great deal of interest over the years and has been one of the most useful models for determining the single particle energy levels of nuclei [2–4] and the nucleus–nucleus interactions [5–7]. The modified version of the Woods–Saxon potential consists of the Woods–Saxon and its derivative called the Woods–Saxon surface potential and is given by [8–10],

\[ V(r) = -\frac{V_0}{1 + e^{(\frac{r}{a})}} - \frac{W_0 e^{(\frac{-r}{a})}}{(1 + e^{(\frac{r}{a})})^2}, \]

where $V_0$ and $W_0$ represent the depths of the potential well. $R$ and $a$ are the radius of the potential and the width of the surface diffuseness, respectively. In figure 1, a form of the generalized Woods–Saxon potential (GWS) versus the inter-nuclear distance is shown for the given potential parameters with several $W_0$ values. The surface term in the generalized Woods–Saxon potential induces an extra potential pocket especially at the surface region of the potential, and this pocket is very important in order to explain the elastic scattering of some nuclear reactions [5, 6]. Moreover, the Woods–Saxon surface potential induces a potential barrier for $W_0 > 0$ so that it may be used in the explanation of the resonant states (quasi-bound states) in nuclei. There are some special cases of the generalized Woods–Saxon potential: the GWS potential is reduced to the standard Woods–Saxon form for $W_0 = 0$ and the square well potential for $W_0 = 0$ and $a \to 0$ [11]. Furthermore the GWS potential is reduced to the Rosen–Morse potential [12] for $R = 0$ [13].

The relativistic treatment of a Dirac particle in the Woods–Saxon potential field is examined in three dimensions for $\ell = 0$ [14]. Moreover, the transmission coefficient and bound state solutions of one dimensional Woods–Saxon potential are analytically studied [15]. The Klein–Gordon equation in the presence of a spatially one-dimensional Woods–Saxon potential is also examined. The scattering state solutions are obtained in terms of hypergeometric functions and the condition for the existence of transmission resonances is derived [16]. Furthermore, the s-wave solution of the Dirac equation for a particle moving in the spherically symmetric
Woods–Saxon potential under the conditions of the exact spin and the pseudospin symmetry limit is examined [17] and is discussed in [18, 19].

It is known that the exact analytical solutions of the wave equations (Schrödinger, Dirac, etc.) are very important since a closed form of the wave function is more convenient than the wave function obtained by numerical calculation in explaining the behavior of the system under consideration. Unfortunately, there are few potentials such as harmonic oscillators, Coulomb and Kratzer potentials etc [11] which have the exact analytical solution with a centrifugal term. In the literature, there are some efforts to calculate the single particle energy levels of nuclei. When we consider a two-body system with the reduced mass \( \mu \) moving under the generalized Woods–Saxon potential, the effective potential is

\[
V_{\text{eff}}(r) = V(r) + V_{1e}(r) = -\frac{V_0}{1 + e^{-r/a}} - \frac{W_0 e^{-r/\delta}}{1 + e^{-r/a}} + \frac{\ell(\ell + 1)\hbar^2}{2\mu r^2},
\]

where \( \mu = \frac{m_n m_A}{m_n + m_A} \), \( m_n \) and \( m_A \) are the atomic mass of the neutron and the core nucleus, respectively. There is no analytical solution of equation (2) due to the polynomial form of the centrifugal barrier term. Therefore, we have to use an approximation for the centrifugal term similar to other authors [26]. In the literature there are a few approximation procedures [20, 21]. One of them is the Pekeris approximation [20] based on an expansion of the centrifugal barrier depending on the internuclear separation up to second order [26].

2. The energy eigenvalues and eigenfunctions

The generalized Woods–Saxon potential or special forms of it are very useful in order to describe the interactions between systems, especially in nuclear physics. In order to explain the single particle energy levels or elastic scattering of nuclei, the Woods–Saxon potential is generally used. Since the interactions usually occur at the surface region of nuclei for both bound and continuum states, the form of the potential at the surface is crucially important. Therefore the surface term in equation (1) would be a very convenient model in order to calculate the single particle energy levels of nuclei. When we consider a two-body system with the reduced mass \( \mu \) moving under the generalized Woods–Saxon potential, the effective potential is

\[
V_{\text{eff}}(r) = V(r) + V_{1e}(r)
\]

where \( \mu = \frac{m_n m_A}{m_n + m_A} \), \( m_n \) and \( m_A \) are the atomic mass of the neutron and the core nucleus, respectively. There is no analytical solution of equation (2) due to the polynomial form of the centrifugal barrier term. Therefore, we have to use an approximation for the centrifugal term similar to other authors [26]. In the literature there are a few approximation procedures [20, 21]. One of them is the Pekeris approximation [20] based on an expansion of the centrifugal barrier depending on the internuclear separation up to second order [26].

The quasi-analytical solution of the effective potential in equation (2) with the Pekeris approximation [20] can be obtained within the framework of the Nikiforov–Uvarov approximation. The authors in [26, 27] have used the transformation and have obtained an \( R \)-dependent eigenvalue equation by using the Nikiforov–Uvarov method [26, 27]. However, we have realized that the Nikiforov–Uvarov method cannot take into account the boundary condition correctly since the Woods–Saxon potential has a different character close to \( r = R \). Therefore, in this article, we have carefully examined the radial Schrödinger equation with the generalized Woods–Saxon potential by using the Pekeris approximation, in terms of the correct boundary conditions. In the next section, we present the calculation procedure. Then section 3 is devoted to the summary and conclusion.
(NU) or asymptotic iteration (AIM) methods as follows,
\[ n_r (n_r + 1) - \beta^2 - \gamma_r^2 + (1 + 2n_r)e + 2\varepsilon \]
\[ + (1 + 2n_r + 2\varepsilon)\sqrt{\varepsilon^2 + \gamma_r^2 - \beta^2} = 0, \]
with the following definitions:
\[ -\varepsilon^2 = \frac{2\mu_{(E - \delta C_r)}}{h^2}, \]
\[ \beta^2 = \frac{2\mu_{(V_0 - \delta C_r)}}{h^2}, \]
\[ \gamma_1^2 = \frac{2\mu_{a^2c^2}}{h^2}, \]
\[ \gamma_2^2 = \frac{2\mu_{a^3W_0}}{h^2}, \]
\[ C_0 = 1 - \frac{4}{\alpha} + \frac{12}{\alpha^2}, \]
\[ C_1 = \frac{8}{\alpha} - \frac{48}{\alpha^2}, \]
\[ C_2 = \frac{48}{\alpha^2}, \]
\[ \alpha = \frac{R}{\varepsilon}, \]
\[ \delta = \frac{\ell (\ell + 1)h^2}{2\mu_0 R^2}. \]

In [13], the consistency of the analytical results of [10] with the Gamow code [31] has been checked by calculating the single particle energy levels of the neutrons moving around the \(^{56}\text{Fe}\) nucleus for the given potential parameters. As can be seen in [13], the results are inconsistent with the numerical calculations. We have also confirmed that there are inconsistencies between equation (3) and the Gamow code results for \(\ell = 0\). In equation (3), if one uses \(W_0 = 0\), the results of [26] can be obtained for arbitrary \(\ell\) states. Furthermore if we take \(\ell = 0\) in equation (3), we get the results of [10]. Therefore we should say that equation (3) determined by the NU method is incorrect. If we used any analytical solution methods such as the asymptotic iteration method (AIM) [28], supersymmetry (SUSY) [12], etc to solve the corresponding equations with the generalized Woods–Saxon potential, we would find the same results in equation (3).

In the literature, there are similar calculations for the analytical solution of the generalized Woods–Saxon potential with the relativistic or non-relativistic wave equations by using the analytical methods [29]. The origin of the problem is due to the boundary conditions so that the analytical methods cannot take them into account correctly since the Woods–Saxon potential has a different characteristic neighborhood \(r = R\). Therefore we have to carefully examine the boundary conditions. In order to get the asymptotic behavior of the wave function \(u_{ad}(z)\), we can use the Nikiforov–Uvarov method [23] and easily get
\[ \phi(z) = z^\varepsilon (1 - z)^{\ell + \beta^2 - \eta^2}. \]
As a result, the asymptotic solution of the wave function is
\[ u_{ad}(z) = z^\varepsilon (1 - z)^{\ell + \beta^2 - \eta^2}. \]
where \(z = \frac{1}{1 + e^{i2\varepsilon}}\) and \(\eta^2 = \varepsilon^2 - \beta^2 + \gamma_1^2\). The wave function in equation (5) satisfies the boundary conditions i.e., \(u_{ad}(r \to 0, z \to 1) \to 0\) and \(u_{ad}(r \to \infty, z \to 0) \to 0\). The Schrödinger equation becomes for equation (5),
\[ z(1 - z) \frac{d^2 u_{ad}(z)}{dz^2} + [1 + 2\varepsilon - (2 + 2\varepsilon + 2\eta)z] \frac{d u_{ad}(z)}{dz} \]
\[ - \left[ - (\gamma_1^2 + \gamma_2^2) + \varepsilon + \eta + (\varepsilon + \eta)^2 \right] u_{ad}(z) = 0. \]
It is known that the hypergeometric equation [30] is defined as
\[ z(1 - z) \frac{d^2 w(z)}{dz^2} + [c - (a + b + 1)z] \frac{d w(z)}{dz} \]
\[ - abw(z) = 0, \]
and one of the solutions is \(w(z) = F_1(a, b; c; z)\) [30]. In order to get \(a, b, c\) parameters we compare equation (6) with equation (7) and find
\[ a = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4\gamma_1^2 + 4\gamma_2^2 + 2\varepsilon + 2\eta} \right), \]
\[ b = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4\gamma_1^2 + 4\gamma_2^2 + 2\varepsilon + 2\eta} \right), \]
\[ c = 1 + 2\varepsilon. \]
Consequently we have \(u_{ad}(z) = z^\varepsilon (1 - z)^{\ell + \beta^2 - \eta^2} \)
\[ z F_1(a, b; c; z) \]
To study in the vicinity of \(z = 1\), we use the relation [11],
\[ z F_1(a, b; c; y) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b - 1)} \]
\[ \times z F_1(a, b; a + b - c + 1; 1 - y) + \frac{\Gamma(c)\Gamma(a + b - c)\Gamma(1 - y)^{c - a - b}}{\Gamma(a)\Gamma(b)} \]
\[ \times z F_1(c - a, c - b; c + a - b + 1; 1 - y). \]
\[ z F_1(a, b; c; 0) = \frac{\Gamma(a + b - c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b - 1)} \]
\[ \left( 1 + e^{i\pi a} \right)^{2\eta} = -1, \]
where \(\eta = i\lambda\) and \(\lambda = \sqrt{\beta^2 - \gamma_1^2 - \varepsilon^2}\). Evaluating equation (10) for the given \(a, b, c\) parameters in equation (8), we obtain
\[ \Gamma(2i\lambda) \frac{\Gamma(2) + (1 + \frac{1}{2}\sqrt{1 + 4\gamma_1^2 + 4\gamma_2^2 + \varepsilon - i\eta})}{\Gamma(-2i\lambda) \Gamma(2) + \Gamma\left( 1 - 2i\lambda \right) \frac{1}{2} \sqrt{1 + 4\gamma_1^2 + 4\gamma_2^2 + \varepsilon + i\eta}} \]
\[ \times \frac{\Gamma\left( \frac{1}{2} - i\lambda \right) \frac{1}{2} \sqrt{1 + 4\gamma_1^2 + 4\gamma_2^2 + \varepsilon - i\eta}}{\Gamma\left( \frac{1}{2} + i\lambda \right) \frac{1}{2} \sqrt{1 + 4\gamma_1^2 + 4\gamma_2^2 + \varepsilon + i\eta}} \]
\[ \times \left( 1 + e^{i\pi a} \right)^{2\eta} = -1. \]
In equation (11), \(e^{i\pi a} \approx 1\) for given realistic parameters. Therefore we can use an approximation \(1 + e^{i\pi a} \approx e^{i\pi a}\) with small errors in the eigenvalues. We can easily get the
following equation by using \(e^{-2i\arg f(z)} = \frac{f(z)}{f'(z)}\) relation,

\[
\exp\left[2i\left[\arg f(2i\ell) - \arg f'\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\eta_1^2 + 4\eta_2^2 + \varepsilon + i\eta}\right)\right.\right.
\]
\[
\left.\left.-\arg f'\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\eta_1^2 + 4\eta_2^2 + \varepsilon + i\eta}\right) + \frac{R\ell}{a}\right]\right]\right] = -1.
\]

Therefore, the corrected energy eigenvalue equation in a closed form becomes,

\[
\arg f(2i\ell) = \arg f'\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\eta_1^2 + 4\eta_2^2 + \varepsilon + i\eta}\right)
\]
\[
\arg f'\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\eta_1^2 + 4\eta_2^2 + \varepsilon + i\eta}\right) + \frac{R\ell}{a}
\]
\[
= -\left(n_\ell + \frac{1}{2}\right)\pi, \quad n_\ell = 0, \pm 1, \pm 2, \ldots,
\]

where \(n_\ell\) is the radial node number. The quantum number is \(n = n_\ell + 1\). In order to test the accuracy of equation (13), we calculate the single particle energy levels of the neutron rotating around a \(^{56}\text{Fe}\) nucleus by using the potential parameters given in [10]. The Woods–Saxon potential parameters are \(V_0 = 40.5 + 0.13A = 47.78\text{ MeV},\) \(R = 4.9162\) fm and \(a = 0.6\) fm. Here \(A\) is the atomic mass number of \(^{56}\text{Fe}\) nuclei. The reduced mass consists of neutron mass \(m_n = 1.00866u\), and \(^{56}\text{Fe}\) core mass \(m_A = 56u\). In Table 1, we show the agreement between the energy eigenvalue equation given by equation (13) and the numerical calculation obtained by Gamow code [31] for the neutron plus \(^{56}\text{Fe}\) nuclei system with several \(n_\ell\) quantum numbers and \(W_0\) parameters. There are small inaccuracies in the analytic and numeric results since we have made the approximation in equation (11). It might be seen that the neutron is unbound for \(n_\ell = 3, W_0 = 0\) and equation (13) gives \(E_{n_\ell} = 62.9775\) MeV, but this energy eigenvalue is not acceptable as physically, i.e., it cannot satisfy the boundary condition, and is only a mathematical result. There are similar situations for \(n_\ell = 3, W_0 = 50\) MeV; \(n_\ell = 2, W_0 = 100\) MeV; and \(n_\ell = 2, 3, W_0 = 50\) MeV. However there are the quasi-bound states (resonant states) for \(n_\ell = 2, W_0 = -100\) MeV since the nuclear potential has a potential barrier inducing the resonant states in figure 1. Our result for \(n_\ell = 2, W_0 = -100\) MeV is physically incorrect. In order to calculate the energy eigenvalues of the resonant states, the complex scaling method (CSM) can be used [32]. Another interesting point is that we cannot calculate the bound state energy eigenvalues by using equation (13) for \(n_\ell = 2, 3, W_0 = 100\) MeV since the right and left sides of equation (13) do not intersect a real line. It should be noted that the \(f\)-state solution of the generalized Woods–Saxon potential in terms of the Pekeris approximation is valid only for small \(\alpha\) values.

The radial wave function corresponding to the eigenvalue equation equation (13) in terms of equation (4) and equation (8) can be written in a closed form as follows,

\[
u_{nf}\ell(r) = N \left(\frac{1}{\alpha + e^{i\pi/\delta}}\right)^{\ell} \left(\frac{1 - 1}{1 + e^{i\pi/\delta}}\right)^{\ell} \times 2F_1\left(\frac{1}{2} - \frac{1}{2}\sqrt{1 + 4\eta_1^2 + 4\eta_2^2 + \varepsilon + i\eta}\right).
\]

| \(W_{00}\) (MeV) | \(n_\ell\) | \(E_{n_\ell}\) \(\text{Analytical}\) (MeV) | \(E_{n_\ell}\) \(\text{Gamow}\) (MeV) |
|-----------------|---------|-------------------------------|-------------------|
| 0               | 0       | -38.3004                      | -38.3002          |
| 0               | 1       | -18.2254                      | -18.2227          |
| 0               | 2       | -0.2678                       | -0.2663           |
| 0               | 3       | 62.9775                       | unbound           |
| 50              | 0       | -41.1965                      | -41.1964          |
| 50              | 1       | -23.8789                      | -23.8788          |
| 50              | 2       | -3.6472                       | -3.6471           |
| 50              | 3       | 52.0232                       | unbound           |
| 100             | 0       | -45.4453                      | -45.4446          |
| 100             | 1       | -29.1659                      | -29.1642          |
| 100             | 2       | undetermined                  | -7.8143           |
| 100             | 3       | undetermined                  | unbound           |
| -50             | 0       | -36.2136                      | -36.2168          |
| -50             | 1       | -12.8469                      | -12.8504          |
| -50             | 2       | 18.5701                       | unbound           |
| -50             | 3       | 64.2816                       | unbound           |
| -100            | 0       | -34.5902                      | -34.5956          |
| -100            | 1       | -8.0843                       | -8.0902           |
| -100            | 2       | 19.2098                       | 17.74 + i(-8.36)  |
corresponding eigenfunction in the closed form for any level of neutrons rotating around a $^{56}$Fe nucleus for the given potential parameters in order to check the consistency between the analytical and the Gamow code results. We have shown that the obtained analytical results in this study are in good agreement with the results obtained by the Gamow code for the $\ell = 0$ state. The resonant state solutions of the generalized Woods–Saxon potential in a closed form are in progress.

3. Conclusion

We have studied the approximate analytical solution of the Schrödinger equation in the presence of the generalized Woods–Saxon potential by using the Pekeris approximation and the properties of the hypergeometric functions for any $\ell$ state. We have seen that the Nikiforov–Uvarov method cannot take into account the correct boundary conditions for the generalized Woods–Saxon potential. Therefore we have carefully examined the asymptotic behavior of the wave function of the generalized Woods–Saxon potential and have obtained the corrected eigenvalue equation and the corresponding eigenfunction in the closed form for any $\ell$ state. We have also calculated the single particle energy level of neutrons rotating around a $^{56}$Fe nucleus for the given potential parameters in order to check the consistency between the analytical and the Gamow code results. We have shown that the obtained analytical results in this study are in good agreement with the results obtained by the Gamow code for the $\ell = 0$ state. The resonant state solutions of the generalized Woods–Saxon potential in a closed form are in progress.

Acknowledgements

The authors would like to thank TÜBİTAK and Akdeniz University for the financial supports as well as Dr A Soylu for useful comments on the manuscript.

References

[1] Woods R D and Saxon D S 1954 Phys. Rev. 95 577
[2] Bohr A and Mottelson B 1998 Nuclear Structure (Singapore: World Scientific)
[3] Gómez J M G, Kar K, Kota V K B, Molina R A and Retamosa J 2003 Phys. Lett. B 567 251
[4] Massen S E and Panos C P 1998 Phys. Lett. A 256 530
[5] Brandon M E and Satchler G R 1997 Phys. Reports 285 143
[6] Satchler G R 1991 Phys. Reports 199 147
[7] Khoa D T, Satchler G R and von Oertzen W 1997 Phys. Rev. C56 2
[8] Zaichenko A K and Ol’khovskii V S 1976 Theor. Math. Phys. 27 475
[9] Fakhri H and Sadeghi J 2004 Mod. Phys. Lett. A29 615
[10] Berkdemir C, Berkdemir A and Sever R 2005 Phys. Rev. C 72 027001
[11] Flügge S 1994 Practical Quantum Mechanics vol 1 (Berlin: Springer)
[12] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267
[13] Berkdemir C, Berkdemir A and Sever R 2006 Phys. Rev. C 74 039902(E)
[14] You G J, Zheng F X and Xin X F 2002 Phys. Rev. A 66 062105
[15] Kennedy P 2002 J. Phys. A: Math. Gen. 35 689
[16] Rojas C and Villalba V M 2005 Phys. Rev. A 71 052101
[17] Guo J Y and Sheng Z Q 2005 Phys. Lett. A 338 90
[18] Br’a H, Jakubský V and Znojil M 2006 Phys. Lett. A 350 421
[19] Guo J Y and Sheng Z Q 2006 Phys. Lett. A 350 425
[20] Pekeris C L 1933 Phys. Rev. 45 98
[21] Greene R L and Aldrich C 1976 Phys. Rev. A 14 2363
[22] Bayrak O, Kocak G and Boztosun I 2006 J. Phys. A: Math. Gen. 39 11521
[23] Nikiforov A F and Uvarov V B 1988 Special Functions of Mathematical Physics (Basel: Birkhäuser)
[24] Ikot A N and Akpan I O 2012 Chin. Phys. Lett. 29 090302
[25] Görnö B and Köksal K 2007 Phys. Scr. 76 565
[26] Badalov V H, Ahmadov H I and Ahmadov A I 2009 Int. J. Mod. Phys. E 18 631
[27] Badalov V H, Ahmadov H I and Badalov S V 2010 Int. J. Mod. Phys. E 19 1463
[28] Sahâ A, Das U and Talukdar B 2011 Phys. Scr. 83 065003
[29] Ikhdair S M and Sever R 2007 Ann. Phys., Lpz. 16 218
[30] Arda A and Sever R 2008 J. Int. Mod. Phys. C 19 763
[31] Aydogdu O and Sever R 2010 Eur. J. Phys. J. A 43 73
[32] Ikhdair S M and Sever R 2010 Cent. Eur. J. Phys. 8 652
[33] Oudi R, Hassanabadi S, Rajabi A A and Hasanabadi H 2012 Commun. Theor. Phys. 57 15
[34] Pahlavani M R and Alavi S A 2012 Commun. Theor. Phys. 58 739
[35] Ikhdair S M, Falaye B J and Hamzavi M 2013 Chinese Phys. Lett. 30 020305
[36] Abramowitz M and Stegun I A 1992 Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (New York: Dover Publications)
[37] Verte T, Pal K F and Balogh Z 1982 Comp. Phys. Comm. 27 309
[38] Aguilar J and Combes J M 1971 Commun. Math. Phys. 22 269
[39] Baslev E and Combes J M 1971 Commun. Math. Phys. 22 280