A note on electrical and thermodynamic properties of Isolated Horizon

Gerui Chen
Institute of Theoretical Physics, Beijing University of Technology, Beijing 100124, China

Xiaoning Wu
Institute of Mathematics, Academy of Mathematics and System Science, Chinese Academy of Sciences, Beijing 100190, China
Hua Loo-Keng Key Laboratory of Mathematics, CAS, Beijing 100190, China
State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Sijie Gao
Department of Physics, Beijing Normal University, Beijing 100875, China

Abstract

The electrical laws and Carnot cycle of Isolated Horizon (IH) are investigated in this paper. We establish the Ohm’s law and Joule’s law of an Isolated Horizon, and find that the conceptual picture of black holes (Membrane Paradigm) can also apply to this kind of quasi-local black holes. We also investigate the geometrical properties near a non-rotating IH, and find that under the first-order approximation of $r$, there exist a Killing vector and a Hamiltonian conjugate to it, so this vector is a physical observer. We calculate the energy as measured at infinity of a particle at rest outside a non-rotating IH, and use this result to construct a reversible Carnot cycle with the Isolated Horizon as a cold reservoir, which confirms the thermodynamic nature of Isolated Horizon.

Keywords: Isolated Horizon, membrane paradigm, electrical laws, black hole thermodynamics, reversible Carnot cycle

*Electronic address: chengerui@emails.bjut.edu.cn
†Electronic address: wuxn@amss.ac.cn
‡Electronic address: sijie@bnu.edu.cn
I. INTRODUCTION

Since the first exact solution of Einstein equation was found out, studying properties of black holes has become an important part of gravitational physics. Black hole physics causes deep, unsuspected connections among classical general relativity, quantum physics and statistical mechanics. However, the classical definition of a black hole \cite{1, 2}, which formalizes the notion of spacetime region from which nothing can ever escape, is too global: it requires knowledge of the entire future of the spacetime, and can not satisfy the requirement for practical research \cite{8}, so alternate notions of black hole \cite{3, 4} were devised which possess a quasi-local description formalism. In recent years, a new, quasi-local framework was introduced by Ashtekar and his collaborators to analyze different facets of black holes in a unified way \cite{5–8}. Compared with the event horizon, this framework doesn’t need the knowledge of overall spacetime, and only involves quasi-local conditions, so it accords with the practical physical process. In this framework, black holes in equilibrium are described by (Weakly) Isolated Horizons (WIH). This paradigm leads to significant generalization of several results in black hole physics and obtains considerable success.

In 1978, Damour found that instead of the conventional view of a black hole as simply an empty region in spacetime, the black hole horizon could be viewed as a physical membrane endowed with specified mechanical and electromagnetic properties; for example, the membrane of Kerr black hole behaves as a metallic shell whose surface resistivity is $4\pi \, [9]$. Such idea was investigated by other researchers later and was called “membrane paradigm of black hole” \cite{10, 11}. This paradigm was reconsidered in the framework of AdS/CFT correspondence more than twenty years later. There are similar properties between the fluid/gravity duality in the AdS/CFT framework and black-hole membrane \cite{12} and this kind of universality was interpreted as the Wilson renormalization group flow in the AdS/CFT framework \cite{13}. More developments in this area can be found in Refs. \cite{14}. This paper is concerned with the electrical properties of Isolated Horizons. The electrical laws of Kerr black hole \cite{9} are only related to the quantities on the horizon, so it is natural to think that the laws are suitable for Isolated Horizon.

Hawking radiation of Weakly Isolated Horizon was investigated recently \cite{15, 16}, which confirms one aspect of thermodynamic nature of Isolated Horizon. Considering the importance of Carnot cycle near a black hole to black hole thermodynamics \cite{17–19}, we design a
reversible Carnot cycle outside a non-rotating Isolated Horizon to give a further confirmation of the thermodynamic nature of Isolated Horizon.

The organization of this paper is as follows. In Section 2, we briefly review the definition and near-horizon geometry of (Weakly) Isolated Horizon. In Section 3, we establish the electrical laws of an Isolated Horizon by following Damour’s method. In Section 4, we investigate the properties near a non-rotating Isolated Horizon under the first-order approximation of $r$, and construct a reversible Carnot cycle near the horizon to confirm the thermodynamical nature of Isolated Horizon. Finally, we make some discussions and conclusions.

II. ISOLATED HORIZON AND ITS NEAR-HORIZON GEOMETRY

In this section we briefly review the definition and geometric properties of (Weakly) Isolated Horizon. According to the works by Ashtekar and his collaborators [5–8], Weakly Isolated Horizon (WIH) is defined by

Definition Let $(M, g)$ be a spacetime. $H$ is a 3-dim null hyper-surface in $M$ and $l^a$ is the tangent vector field of the generator of $H$. $H$ is said to be a Weakly Isolated Horizon (WIH), if

1) $H$ has the topology of $S^2 \times \mathbb{R}$;
2) The expansion of the null generator of $H$ is zero, i.e. $\Theta_l = 0$ on $H$;
3) $T_{ab}v^b$ is future causal for any future causal vector $v^b$ and Einstein equation holds in the neighborhood of $H$;
4) $[\mathcal{L}_l, D_a]l^b = 0$ on $H$, where $D_a$ is the induced covariant derivative on $H$.

A Weakly Isolated Horizon is said to constitute an Isolated Horizon (IH) if $[\mathcal{L}_l, D_a] = 0$ on $H$ [7]. From the definition, Isolated Horizon is the special case of Weakly Isolated Horizon, so IH has all the properties of WIH.

It is convenient to introduce Bondi-like coordinates $(u, r, \theta, \phi)$ in the neighborhood of the horizon $H$ in the following way [20, 21]. First, denote the tangent vector of null generator of $H$ as $l^a$ and another real null vector field as $n^a$. The foliation of $H$ gives us the natural coordinates $(\theta, \varphi)$. Lie dragging $(\theta, \phi)$ along each generator of $H$ together with the parameter $u$ of $l^a$ forms the coordinates $(u, \theta, \varphi)$ on $H$. Finally, choose the affine parameter $r$ of $n^a$ as the fourth coordinate, then we obtain the Bondi-like coordinates $(u, r, \theta, \phi)$ near the horizon.
With the Bondi-like coordinate system in hand, we construct the null tetrad as

\[ l^a = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X \frac{\partial}{\partial \varsigma} + \overline{X} \frac{\partial}{\partial \overline{\varsigma}}, \]

\[ n^a = -\frac{\partial}{\partial r}, \]

\[ m^a = \omega \frac{\partial}{\partial r} + \xi_3 \frac{\partial}{\partial \varsigma} + \xi_4 \frac{\partial}{\partial \overline{\varsigma}}, \]

\[ \overline{m}^a = \overline{\omega} \frac{\partial}{\partial r} + \overline{\xi}_3 \frac{\partial}{\partial \varsigma} + \overline{\xi}_4 \frac{\partial}{\partial \overline{\varsigma}}, \]

where \( U \equiv X \equiv \omega \equiv 0 \) on \( H \) (following the notation in Ref. \[8\], equalities restricted to \( H \) are denoted by \( \equiv \)), and \( \varsigma = e^{i\phi} \cot \theta \). Note that \( n^a \) and \( l^a \) are future directed.

We take the spacetime metric \( g_{ab} \) to have a signature \((-+,++,+)\), so the metric can be expressed as

\[ g^{ab} = m^a m^b + \overline{m}^a m^b - n^a n^b. \]

The matrix form of the metric is

\[ g^{\mu\nu} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 2(U + |\omega|^2) & X + (\overline{\omega} \xi_3 + \omega \overline{\xi}_4) & \overline{X} + (\overline{\omega} \xi_4 + \omega \overline{\xi}_3) \\
0 & X + (\overline{\omega} \xi_3 + \omega \overline{\xi}_4) & 2\xi_3 \overline{\xi}_4 & |\xi_3|^2 + |\xi_4|^2 \\
0 & \overline{X} + (\overline{\omega} \xi_4 + \omega \overline{\xi}_3) & |\xi_3|^2 + |\xi_4|^2 & 2\overline{\xi}_3 \xi_4
\end{pmatrix}, \]

and, on the horizon, the metric reduces to

\[ g^{\mu\nu} \equiv \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 2\xi_3(0) \xi_4(0) & |\xi_3(0)|^2 + |\xi_4(0)|^2 \\
0 & |\xi_3(0)|^2 + |\xi_4(0)|^2 & 2\overline{\xi}_3(0) \xi_4(0)
\end{pmatrix}, \]

where \( \xi_3(0) \) means the value of \( \xi_3 \) on the horizon. The dual basis of the tetrad (1) can be calculated as

\[ m_a = \frac{-\overline{\xi}_3}{|\xi_4|^2 - |\xi_3|^2} d\varsigma + \frac{\overline{\xi}_4}{|\xi_4|^2 - |\xi_3|^2} d\overline{\varsigma} + \frac{\overline{\xi}_3 X - \overline{\xi}_4 X}{|\xi_4|^2 - |\xi_3|^2} du, \]

\[ m_a = \frac{\xi_4}{|\xi_4|^2 - |\xi_3|^2} d\varsigma + \frac{-\xi_3}{|\xi_4|^2 - |\xi_3|^2} d\overline{\varsigma} + \frac{\xi_3 \overline{X} - \xi_4 X}{|\xi_4|^2 - |\xi_3|^2} du, \]

\[ -l_a = (U - \frac{\xi_4 \overline{\omega} - \overline{\xi}_3 \omega}{|\xi_4|^2 - |\xi_3|^2} X - \frac{\overline{\xi}_3 \omega - \xi_4 \overline{\omega}}{|\xi_4|^2 - |\xi_3|^2} \overline{X}) du - dr \\
+ \frac{\xi_4 \overline{\omega} - \overline{\xi}_3 \omega}{|\xi_4|^2 - |\xi_3|^2} d\varsigma + \frac{\overline{\xi}_3 \omega - \xi_4 \overline{\omega}}{|\xi_4|^2 - |\xi_3|^2} d\overline{\varsigma}, \]

\[ -n_a = du. \]
The commutators $[-l^a, n^a]$ and $[m^a, n^a]$ tell us that

\[
\begin{align*}
\frac{\partial U}{\partial r} &= (\varepsilon + \bar{\varepsilon}) + \pi \omega + \pi \omega, \\
\frac{\partial X}{\partial r} &= \pi \xi_4 + \pi \xi_3, \\
\frac{\partial \omega}{\partial r} &= \pi + \lambda \omega + \mu \omega, \\
\frac{\partial \xi_3}{\partial r} &= \lambda \xi_4 + \mu \xi_3, \\
\frac{\partial \xi_4}{\partial r} &= \lambda \xi_3 + \mu \xi_4.
\end{align*}
\] (6)

In the Newman-Penrose formalism, we can require the following gauge conditions:

\[
\begin{align*}
\nu &= \tau = \gamma = \alpha + \beta = \mu = \pi = 0, \\
\varepsilon - \bar{\varepsilon} &= \lambda = \mu = 0,
\end{align*}
\] (7)

which mean that the tetrad vectors are parallelly transported along $n^a$ in spacetime.

The fourth requirement in the definition of WIH implies that there exists a one form $\omega_a$ on $H$ such that $D_a l^b \omega_a l^b$ and $\mathcal{L}_l \omega^a \equiv 0$. In terms of the Newman-Penrose formalism, $\omega_a$ can be expressed as

\[
\omega_a = -(\varepsilon + \bar{\varepsilon}) n_a + (\alpha + \beta) m_a + (\bar{\alpha} + \beta) m_a = -(\varepsilon + \bar{\varepsilon}) n_a + \pi m_a + \bar{\pi} m_a,
\] (8)

where $\varepsilon + \bar{\varepsilon}$ is constant on $H$ [8]. The definition of WIH also implies

\[
\rho \equiv \sigma \equiv 0.
\] (9)

Based on Ref. [8], not any choice of time direction can give a Hamiltonian evolution, and only some suitably chosen time direction can lead to a well-defined horizon mass. In Ref. [8], A. Ashtekar and B. Krishnan gave a canonical way to choose the time direction $t^a$ for a WIH, and the restriction of $t^a$ to $H$ should be a linear combination of the null normal $l^a$ and the axisymmetric vector $\psi^a$,

\[
t^a \equiv B_l l^a - \Omega l \psi^a,
\] (10)

where $B_l$ and $\Omega_l$ are constant on the horizon. Compared with the Schwarzschild case, the parameter of $t^a$ takes the place of the Killing time. With the canonical time direction $t^a$, Ref. [8] established the zeroth and the first law of WIH. By definition, the surface gravity of $H$ is $\kappa_t := B_l l^a \omega_a = B_l (\varepsilon + \bar{\varepsilon})$, which is constant on $H$, so the zeroth law of black hole mechanics is valid for WIH. The first law is expressed as

\[
\delta M_H(t) = \frac{\kappa_t}{8\pi} \delta a_H + \Omega_t \delta J_H,
\] (11)

where $M_H(t)$ is the horizon mass, $a_H$ is the area of the cross section of WIH, $\Omega_t$ is the angular velocity of the horizon, and $J_H = -\frac{1}{8\pi} \oint_S (\omega_a \psi^a) dS$ is the angular momentum. The first law...
of WIH is the generalization of the first law of stationary black hole. Since Hawking radiation of WIH was investigated in Refs. [15, 16], the laws of WIH mechanics were upgraded to the laws of WIH thermodynamics.

III. THE ELECTRICAL PROPERTIES OF ISOLATED HORIZON

In this section, we investigate the electrical laws of an Isolated Horizon following Damour’s method [9]. Let us choose a set of real Bondi-like coordinates \((u, r, \theta, \varphi)\). According to Eq. (4), and the fact that the topological structure of an Isolated Horizons is \(S^2 \times R\), the metric on the horizon can be also expressed as

\[
g_{\mu\nu} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & f^2(\theta) & 0 \\
0 & 0 & 0 & f^2(\theta) \sin^2 \theta
\end{pmatrix}.
\] (12)

The physical observer should be the canonical time \(t^a\). According to the relationship:

\[
(\frac{\partial}{\partial t})^a \equiv B_t (\frac{\partial}{\partial u})^a - \Omega (\frac{\partial}{\partial \varphi})^a,
\] (13)

we introduce a new set of coordinates \((t, \tilde{r}, \tilde{\theta}, \tilde{\varphi})\). We have substituted \(\Omega\) for \(\Omega_t\) for simplicity. The relationship between Bondi-like coordinates \((u, r, \theta, \varphi)\) and the new ones on the horizon is

\[
t \equiv \frac{1}{B_t} u, \quad \tilde{r} \equiv r, \quad \tilde{\theta} \equiv \theta, \quad \tilde{\varphi} \equiv \frac{\Omega}{B_t} u + \varphi,
\] (14)

so we have

\[
\begin{align*}
\frac{\partial}{\partial t} & \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial \tilde{r}} + \frac{\partial}{\partial \tilde{\theta}} \frac{\partial}{\partial \tilde{\varphi}} + \frac{\partial}{\partial \tilde{\varphi}} \frac{\partial}{\partial \tilde{\varphi}} \equiv B_t \frac{\partial}{\partial u} - \Omega \frac{\partial}{\partial \varphi}, \\
\frac{\partial}{\partial \tilde{r}} & \equiv \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \tilde{\theta}} \equiv \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \tilde{\varphi}} \equiv \frac{\partial}{\partial \varphi}, \\
dt & \equiv \frac{1}{B_t} du, \quad d\tilde{r} \equiv dr, \quad d\tilde{\theta} \equiv d\theta, \quad d\tilde{\varphi} \equiv d\varphi + \frac{\Omega}{B_t} du.
\end{align*}
\] (15)

The metric (12) in the new coordinates can be expressed as

\[
g_{\mu\nu} \equiv \begin{pmatrix}
0 & \frac{1}{B_t} & 0 & 0 \\
\frac{1}{B_t} & 0 & 0 & \frac{\Omega}{B_t} \\
0 & 0 & f^2(\tilde{\theta}) & 0 \\
0 & \frac{\Omega}{B_t} & 0 & f^2(\tilde{\theta}) \sin^2 \tilde{\theta}
\end{pmatrix}.
\] (16)
then the intrinsic geometry of cross section \( t = \text{const} \) and \( \tilde{r} = 0 \) is

\[
ds^2 = f^2(\tilde{\theta})(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\varphi}^2).
\]  

(17)

The basis of the cross section is

\[
e_{(\tilde{\theta})} = \frac{1}{f} \frac{\partial}{\partial \tilde{\theta}}, \quad e_{(\tilde{\varphi})} = \frac{1}{f \sin \tilde{\theta}} \frac{\partial}{\partial \tilde{\varphi}},
\]

and the corresponding dual basis is

\[
\omega_{(\tilde{\theta})} = f d\tilde{\theta}, \quad \omega_{(\tilde{\varphi})} = f \sin \tilde{\theta} d\tilde{\varphi}.
\]  

(19)

Given an electromagnetic test field \( F_{ab} \) regular on the horizon \( H \) we define the tangential electric field and the normal magnetic induction by restricting the form \( F = \frac{1}{2} F_{\rho\sigma}(d\rho)_a \wedge (d\sigma)_b \) to the horizon \( \tilde{r} = 0 \). Namely,

\[
F_{ab} = \frac{1}{2} F_{\rho\sigma}(dt)_a \wedge (d\sigma)_b + \frac{1}{2} F_{\tilde{r}\sigma}(d\tilde{r})_a \wedge (d\sigma)_b + \frac{1}{2} F_{\tilde{\theta}\sigma}(d\tilde{\theta})_a \wedge (d\sigma)_b + \frac{1}{2} F_{\tilde{\varphi}\sigma}(d\tilde{\varphi})_a \wedge (d\sigma)_b = \left[ F_{\tilde{\theta}t} d\tilde{\theta} + F_{\tilde{\varphi}t} d\tilde{\varphi} \right] \wedge dt + F_{\tilde{\theta}\tilde{\varphi}} d\tilde{\theta} \wedge d\tilde{\varphi} = (E_{(\tilde{\theta})}\omega^{(\tilde{\theta})} + E_{(\tilde{\varphi})}\omega^{(\tilde{\varphi})}) \wedge dt + B_\perp \omega^{(\tilde{\theta})} \wedge \omega^{(\tilde{\varphi})},
\]

(20)

where

\[
E_{(\tilde{\theta})} = F_{(\tilde{\theta})} = \frac{1}{f} F_{\tilde{\theta}t} = \frac{1}{f} (B_t F_{\theta u} - \Omega F_{\theta \varphi})
\]

\[
E_{(\tilde{\varphi})} = F_{(\tilde{\varphi})} = \frac{1}{f \sin \tilde{\theta}} F_{\tilde{\varphi}t} = \frac{1}{f \sin \tilde{\theta}} B_t F_{\varphi u} = \frac{B_t}{f \sin \theta} F_{\varphi u},
\]

\[
B_\perp = F_{(\tilde{\theta})} = \frac{1}{f^2 \sin \theta} F_{\theta \varphi} = \frac{1}{f^2 \sin \theta} F_{\theta \varphi}.
\]  

(21)

By following Damour’s method \cite{9}, we can endow charges and currents on the horizon to keep the conservation of four-current outside the black hole since we do not want to consider what happens inside the black hole. There exists a four-current \( J(u, r, \theta, \varphi) \) which is defined and conserved all over spacetime, \( \nabla_a J^a = 0 \). We find a complementary current \( j^a \) with support on \( r = 0 \) such that \( J^a Y(r) + j^a \) is conserved, where \( Y \) is the Heaviside function. By replacing \( F^{ab} \) with \( F^{ab} Y(r) \) in the equation \( J^a = \frac{1}{4\pi} \nabla_b F^{ab} \), we get the conserved current \( J^a Y(r) + j^a \) where \( j^a = (4\pi)^{-1} F^{ar} \delta(r) \).
Next we calculate the Dirac distribution $\delta_H$ on the horizon. For the spacetime, the volume element is

$$
\varepsilon_{abcd} = \varepsilon_{1234} m_a \wedge m_b \wedge l_c \wedge n_d = \frac{1}{\big|\xi_4|^{2} - |\xi_3|^{2} \sin^{2} \frac{\theta}{2}} \coth \frac{\theta}{2} du \wedge dr \wedge d\theta \wedge d\phi,
$$

(22)

and the volume element of the cross section is

$$
\varepsilon_{ab} = \frac{1}{|\xi_4(0)|^{2} - |\xi_3(0)|^{2} \sin^{2} \frac{\theta}{2}} \coth \frac{\theta}{2} d\theta \wedge d\phi.
$$

(23)

From the equation,

$$
\int f(u, r, \theta, \phi) \delta_{H} \delta(u - u_0) \varepsilon_{abcd} = \int f(u_0, 0, \theta, \phi) \varepsilon_{ab},
$$

we easily find

$$
\delta_{H} = \delta(r).
$$

(24)

(25)

Now we can write the complementary current $j^a$, with support on the horizon, as

$$
\dot{j}^a = K^a \delta_{H},
$$

(26)

with

$$
K^a = \frac{1}{4\pi} F^a_{tr},
$$

(27)

which is the surface four-current density and can be decomposed into a surface charge density $\sigma$ and the geometrical components of a surface current density $\vec{K}$,

$$
\sigma = K^a (dt)_a = \frac{1}{4\pi B_t} F^{ur},
\quad
K^\theta = \frac{1}{4\pi} F^{ar} \omega(\theta) = \frac{1}{4\pi f} F_{\theta u},
\quad
K^\phi = \frac{1}{4\pi} F^{ar} \omega(\phi) = \frac{F_{\psi u}}{4\pi f \sin \theta} + V \sigma.
$$

(28)

It’s not hard to find the following two relationships,

$$
\frac{1}{f} (B_t F_{\theta u} - \Omega F_{\theta \phi}) + V(\phi) \frac{1}{f^2 \sin \theta} F_{\theta \phi} = 4\pi B_t \cdot \frac{1}{4\pi f} F_{\theta u},
$$

(29)

$$
\frac{1}{f \sin \theta} B_t F_{\phi u} = 4\pi B_t \cdot \frac{F_{\psi u}}{4\pi f \sin \theta}.
$$

(30)
where $V(\tilde{\varphi}) = f \sin \theta \Omega$ can be interpreted as the velocity of the horizon. From Eqs. (21, 28), we have

$$E(\tilde{\varphi}) + V(\tilde{\varphi}) B_\perp = 4\pi B_t K^{(\tilde{\varphi})},$$

$$E(\tilde{\varphi}) = 4\pi B_t [K^{(\tilde{\varphi})} - \sigma V].$$

(31)

In vector form,

$$\vec{E} + \vec{V} \times \vec{B}_\perp = 4\pi B_t (\vec{K} - \sigma \vec{V}),$$

(32)

which is the Ohm’s law of Isolated Horizon, and the surface resistivity is $4\pi B_t$. For Kerr black holes, $l^a \equiv \partial_\alpha + \Omega \partial_{\varphi}$, we have $B_t = 1$, so the surface resistivity is $4\pi$, which returns to Damour’s result [9].

It is useful to introduce the notions of the surface conduction current $\vec{C}$ (the total current $\vec{K}$ minus the convection current $\sigma \vec{V}$) and the “dragged-along” electric field $\vec{E}^*$,

$$\vec{C} = \vec{K} - \sigma \vec{V}, \quad \vec{E}^* = \vec{E} + \vec{V} \times \vec{B}_\perp.$$  

(33)

So the Ohm’s law (32) can be rewritten as

$$\vec{E}^* = 4\pi B_t \vec{C}.$$  

(34)

Next we use the surface conduction current to express the Joule’s law. We can define the heat $dQ$ dissipated in the hole as [17]

$$dQ = dM - \Omega dS_z,$$

(35)

where $\Omega$ is the angular velocity and $dA, dM, dS_z$ are the increases in, respectively, area, mass, and angular momentum of IH. The total energy flux into the hole is given by an integral on the horizon [22],

$$\dot{M} = \int T^a_{\alpha\beta} l^\alpha dA = \int T_{ab} l^a dA,$$

(36)

where $T_{ab}$ is the test energy-momentum tensor at the horizon. The angular momentum flux is

$$\dot{S}_z = - \int T_{ab} l^a \left( \frac{\partial}{\partial \varphi} \right)^b dA = - \int T_{ab} l^a \left( \frac{\partial}{\partial \varphi} \right)^b dA = - \int T_{a\varphi} l^a dA.$$

(37)
So we get the heat production as

\[ \dot{Q} = \dot{M} - \Omega \dot{S}_z = \int (T_{at} + \Omega T_{a\varphi}) l^a dA = B_t \int T_{ab} l^a l^b dA. \] (38)

In the case of an electromagnetic field we have on the horizon,

\[ T_{ab} l^a l^b = (4\pi)^{-1} F_{ac} F_b c l^a l^b = (4\pi)^{-1} F_{uc} F_u c \]
\[ = (4\pi)^{-1} (F_{ur} F_u r + F_{u\theta} F_u \theta + F_{u\varphi} F_u \varphi) \]
\[ = (4\pi)^{-1} (g^{\theta\theta} F_{u\theta} F_{u\theta} + g^{\varphi\varphi} F_{u\varphi} F_{u\varphi}) \]
\[ = (4\pi)^{-1} (g^{\theta\theta} F_{\theta u} F_{\theta u} + g^{\varphi\varphi} F_{\varphi u} F_{\varphi u}), \] (39)

and from Eqs. (21, 28), we have

\[ F_{\theta u} = \frac{1}{B_t} \left( f E_{(\tilde{\theta})} + \Omega F_{\theta\varphi} \right) = \frac{f}{B_t} (E_{(\tilde{\theta})} + VB_\perp), \]
\[ F_{\theta u} = 4\pi f K^{(\tilde{\theta})}, \]
\[ F_{\varphi u} = \frac{f \sin \theta}{B_t} E_{(\tilde{\varphi})} = (K^{(\tilde{\varphi})} - \sigma V) 4\pi f \sin \theta. \] (40)

By putting Eqs. (40) into Eq. (39), we obtain

\[ T_{ab} l^a l^b = \frac{1}{4\pi} \left[ \frac{1}{f^2} 4\pi f K^{(\tilde{\theta})} \frac{f}{B_t} (E_{(\tilde{\theta})} + VB_\perp) \right. \]
\[ + \left. \frac{1}{f^2 \sin^2 \theta} \frac{f \sin \theta}{B_t} E_{(\tilde{\varphi})} 4\pi f \sin \theta (K^{(\tilde{\varphi})} - \sigma V) \right] \]
\[ = \frac{1}{B_t} \left[ K^{(\tilde{\theta})}(E_{(\tilde{\theta})} + VB_\perp) + E_{(\tilde{\varphi})}(K^{(\tilde{\varphi})} - \sigma V) \right] \]
\[ = \frac{1}{B_t} \left( \vec{E} + \vec{V} \times \vec{B}_\perp \right) \cdot (\vec{K} - \sigma \vec{V}) \]
\[ = \frac{1}{B_t} \vec{E}^* \cdot \vec{C}, \] (41)

where we have used Eqs. (33) in the last equality.

Hence we get the Joule’s law,

\[ \dot{Q} = B_t \int T_{ab} l^a l^b dA = \int 4\pi B_t |\vec{C}|^2 dA. \] (42)

This is the Joule’s law of Isolated Horizon, and the surface resistivity is also $4\pi B_t$. From the two laws we see that IH can be thought as a metal shell whose surface resistivity is $4\pi B_t$. We do not use all the conditions of Isolated Horizon, so the results are very general.
IV. CARNOT CYCLE NEAR A NON-ROTATING ISOLATED HORIZON

In this section, we firstly investigate some properties near a non-rotating IH, and find that under the first-order approximation of $r$, $\frac{\partial}{\partial u}$ is a Killing vector and there exists a Hamiltonian conjugate to it, so $\frac{\partial}{\partial u}$ is a physical observer. Then we construct a reversible Carnot cycle with the Isolated Horizon as a cold reservoir to confirm the thermodynamic nature of IH.

For a non-rotating Isolated Horizon, the angular momentum is zero, that is, $J_H = 0$. According to the definition of angular momentum of IH and Eq. (8), we find that

$$\pi \hat{=} 0,$$

so we have from the Eqs. (6),

$$\frac{\partial U}{\partial r} \hat{=} (\varepsilon + \varpi), \quad \frac{\partial X}{\partial r} \hat{=} 0, \quad \frac{\partial \omega}{\partial r} \hat{=} 0. \quad (44)$$

The tetrad condition $\mathcal{L}_i m^a \hat{=} 0$ means that

$$\frac{\partial \xi_3}{\partial u} \hat{=} 0, \quad \frac{\partial \xi_4}{\partial u} \hat{=} 0. \quad (45)$$

For IH, $[\mathcal{L}_i, D_a] \hat{=} 0$, so we have $[\mathcal{L}_i, D_a] n_b \hat{=} 0$. Considering $\mathcal{L}_i m_a \hat{=} 0$, we obtain that

$$\mathcal{L}_i (D_a n_b) \equiv \mathcal{L}_i S_{ab} \hat{=} 0. \quad (46)$$

From Eqs. (B19, B20) in Ref. [7],

$$\tilde{S}_{ab} := \tilde{q}_a q_b D_c n_d \equiv \mu (m_a m_b + \overline{m}_a m_b) + \lambda m_a m_b + \overline{\lambda m}_a m_b, \quad (47)$$

$$\mathcal{L}_i \tilde{S}_{ab} \equiv (D\mu)(m_a \overline{m}_b + \overline{m}_a m_b) + (D\lambda) m_a m_b + (D\overline{\lambda}) \overline{m}_a m_b, \quad (48)$$

where $\tilde{q}_b \equiv m_b \overline{m}^a + \overline{m}_b m^a$, we find that

$$\mathcal{L}_i \tilde{S}_{ab} \equiv \tilde{q}_a q_b D_c S_{cd} \hat{=} 0, \quad \mathcal{L}_i S_{cd} \hat{=} 0. \quad (49)$$

so from the Eq. (48), we have

$$D\mu \hat{=} 0, \quad D\lambda \hat{=} 0. \quad (50)$$

These two conditions above are given by Isolated Horizon. So we find that $\frac{\partial U}{\partial r}, \frac{\partial X}{\partial r}, \frac{\partial \omega}{\partial r}, \frac{\partial \xi_3}{\partial r}, \frac{\partial \xi_4}{\partial r}$ are not related to $u$ on the horizon, that is to say, the Taylor series of $U, \omega, X, \xi_3, \xi_4$ to the first-order approximation of $r$ are independent of $u$. From Eqs. (77,
78, 79) in Ref. [21] and Eq. (3) the metric and its inverse to the first-order approximation of \( r \) are not related to \( u \), so \( \frac{\partial}{\partial u} \) can be thought as a Killing vector approximatively.

Based on Ref. [23], the necessary and sufficient condition for the existence of a Hamiltonian conjugate to \( \eta^a \) on slice \( \Sigma \) is

\[
\int_{\partial \Sigma} \eta \cdot \omega(g_{ab}, \delta_1 g_{ab}, \delta_2 g_{ab}) = 0,
\]

where \( \omega \) is the presymplectic current 3-form. In general relativity, the presymplectic current 3-form is

\[
\omega_{abc} = \frac{1}{16\pi} \varepsilon_{abc} w^d,
\]

where

\[
w^a = P^{abcdef} [\gamma_{2bc} \nabla_d \gamma_{1ef} - \gamma_{1bc} \nabla_d \gamma_{2ef}],
\]

with

\[
P^{abcdef} = g^{ae} g^{fb} g^{cd} - \frac{1}{2} g^{ad} g^{be} g^{cf} - \frac{1}{2} g^{bc} g^{ae} g^{df} - \frac{1}{2} g^{bc} g^{ad} g^{ef} + \frac{1}{2} g^{ab} g^{cd} g^{ef}.
\]

In our case it is straightforward to find that

\[
\int \frac{\partial}{\partial u} \cdot \omega(g_{ab}, \delta_1 g_{ab}, \delta_2 g_{ab}) = \int \left( \frac{\partial}{\partial u} \right)^l \omega_{ljk} = \int \left( \frac{\partial}{\partial u} \right)^l \varepsilon_{aljk} \frac{w^a}{16\pi} = \int \left( \frac{\partial}{\partial u} \right)^l \omega_{ljk} \sim O(r^2).
\]

In the last equality, we use \( \left( \frac{\partial}{\partial u} \right)^l m_l \sim O(r^2) \). So there exists a Hamiltonian conjugate to \( \frac{\partial}{\partial u} \) near the horizon, and \( \frac{\partial}{\partial u} \) can be thought as a physical observer.

Next we will compute the energy as measured at infinity of a particle near the horizon. The Taylor series of the coefficients to the first-order approximation of \( r \) are

\[
U = (\varepsilon + \pi) r + O(r^2),
\]

\[
X = O(r^2),
\]

\[
\omega = O(r^2),
\]

\[
\xi_3 = \xi_3(0) + \xi_3^{(1)}(0)r + O(r^2),
\]

\[
\xi_4 = \xi_4(0) + \xi_4^{(1)}(0)r + O(r^2),
\]

(56)
where \( \xi_3^{(1)} \) is the first-order derivative of \( r \). Putting expressions (56) into the null tetrad (1), we get

\[
\begin{align*}
l^a &= \frac{\partial}{\partial u} + \left[ (\varepsilon + \bar{\varepsilon}) r + O(r^2) \right] \frac{\partial}{\partial r}, \\
n^a &= -\frac{\partial}{\partial r}, \\
m^a &= O(r^2) \frac{\partial}{\partial r} + \left[ \xi_3(0) + \xi_3^{(1)}(0) r + O(r^2) \right] \frac{\partial}{\partial \zeta} \\
&\quad + \left[ \xi_4(0) + \xi_4^{(1)}(0) r + O(r^2) \right] \frac{\partial}{\partial \xi}, \\
\overline{m}^a &= O(r^2) \frac{\partial}{\partial r} + \left[ \overline{\xi}_3(0) + \overline{\xi}_3^{(1)}(0) r + O(r^2) \right] \frac{\partial}{\partial \overline{\zeta}} \\
&\quad + \left[ \overline{\xi}_4(0) + \overline{\xi}_4^{(1)}(0) r + O(r^2) \right] \frac{\partial}{\partial \overline{\xi}}.
\end{align*}
\]

Introduce new coordinates in the neighborhood of IH as

\[
t := \frac{1}{B_t} u - r_s, R := r,
\]

where \( dr_s = \frac{dr}{f(r)} \) and \( f(r) = 2B_t(\varepsilon + \bar{\varepsilon}) r \). It is easy to find that

\[
\begin{align*}
\frac{\partial}{\partial u} &= \frac{1}{B_t} \frac{\partial}{\partial t} - \frac{\partial}{\partial r}, \\
\frac{\partial}{\partial r} &= \frac{1}{f} \frac{\partial}{\partial t} + \frac{\partial}{\partial R}, \\
du &= B_t dt + B_t \frac{dR}{f(R)}, \\
\end{align*}
\]

We use Eqs. (59) to express the null tetrad (57) in the new coordinates as

\[
\begin{align*}
l^a &= \frac{1}{2B_t} \frac{\partial}{\partial t} + \left[ (\varepsilon + \bar{\varepsilon}) R + O(R^2) \right] \frac{\partial}{\partial R}, \\
n^a &= \frac{1}{f} \frac{\partial}{\partial t} - \frac{\partial}{\partial R}, \\
m^a &= O(R^2) \left[ -\frac{1}{f^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial R} \right] + \left[ \xi_3(0) + \xi_3^{(1)}(0) R + O(R^2) \right] \frac{\partial}{\partial \zeta} \\
&\quad + \left[ \xi_4(0) + \xi_4^{(1)}(0) R + O(R^2) \right] \frac{\partial}{\partial \xi}, \\
\overline{m}^a &= O(R^2) \left[ -\frac{1}{f^2} \frac{\partial}{\partial t} + \frac{\partial}{\partial R} \right] + \left[ \overline{\xi}_3(0) + \overline{\xi}_3^{(1)}(0) R + O(R^2) \right] \frac{\partial}{\partial \overline{\zeta}} \\
&\quad + \left[ \overline{\xi}_4(0) + \overline{\xi}_4^{(1)}(0) R + O(R^2) \right] \frac{\partial}{\partial \overline{\xi}}.
\end{align*}
\]

From the expressions above we get easily

\[
g^{tt} = -\frac{1}{B_t f(R)}. \tag{62}
\]
Putting Eqs. (60) into Eqs. (5) and ignoring the higher order terms, we obtain

$$g_{RR} = -\frac{2B^2_t}{f^2(R)}(U - \frac{\xi_4 \omega - \xi_3 \omega}{|\xi_4|^2 - |\xi_3|^2} X - \frac{\xi_4 \omega - \xi_3 \omega}{|\xi_4|^2 - |\xi_3|^2} X)$$

$$+ \frac{2B^2_t}{f^2(R)} \frac{(|\xi_4|^2 - |\xi_3|^2)^2}{(|\xi_4|^2 - |\xi_3|^2)^2} + \frac{2B_t}{f(R)}$$

$$\dot{g}_{tR} \doteq 0,$$

$$g_{tt} \doteq -B_t f(R). \quad (63)$$

We will use these results later.

The Hamilton-Jacobi equation in curved spacetime is

$$g^{\mu\nu} \partial_\mu S \partial_\nu S + m^2 = 0, \quad (64)$$

where $m$ is the rest mass, and $S$ is the Hamilton principal function. Because $\frac{\partial}{\partial t} = B_t \frac{\partial}{\partial u}$ is a Killing vector, we can separate $S$ as

$$S = -V(t) + W(R, x^A). \quad (65)$$

The components of generalized momentum $P_\mu = \frac{\partial S}{\partial x_\mu}$ are

$$P_t = \frac{\partial S}{\partial t} = -\frac{\partial V(t)}{\partial t} = -\dot{V}(t),$$

$$P_R = \frac{\partial S}{\partial R} = \frac{\partial W(R, x^A)}{\partial R},$$

$$P_A = \frac{\partial W(R, x^A)}{\partial x^A}, \quad (66)$$

where $\dot{V}(t)$ is the energy of the particle. We consider a particle at rest outside a non-rotating Isolated Horizon, so for this particle, $R$ and $x^A$ are constant, which means

$$P_R = 0, \quad P_A = 0. \quad (67)$$

Then Eq. (64) becomes

$$g^{tt}(\dot{V})^2 + m^2 = 0, \quad (68)$$

and it is easy to find that

$$\dot{V} = \sqrt{B_t f m^2}. \quad (69)$$
This is the energy as measured at infinity of a particle at rest outside the horizon. When a particle is located at $R = \delta$ which is the radial coordinate distance away from the horizon, we have

$$f = 2B_t(\varepsilon + \bar{\varepsilon})\delta. \quad (70)$$

Putting this expression into Eq. (69), we obtain

$$\dot{V} = \sqrt{2(\varepsilon + \bar{\varepsilon})}\delta^{\frac{3}{2}}mB_t. \quad (71)$$

Now let us change the coordinate distance $\delta$ to the proper distance $l$. The general expression is

$$dl^2 = \gamma_{ij}dx^i dx^j = (g_{ij} - \frac{g_{ij}g_{0j}}{g_{00}})dx^i dx^j. \quad (72)$$

We consider the radial distance, so

$$dl^2 = g_{RR}dR^2 = \frac{B_t}{f(R)}dR^2, \quad (73)$$

that is,

$$dl = \left[\frac{B_t}{f(R)}\right]^{\frac{1}{2}}dR. \quad (74)$$

So the proper distance $l$ corresponding to coordinate distance $\delta$ is

$$l = \int^{\delta}_0 \left[\frac{B_t}{f(R)}\right]^{\frac{1}{2}}dR = \int^{\delta}_0 \left[\frac{1}{2(\varepsilon + \bar{\varepsilon})R}\right]^{\frac{1}{2}}dR
= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{R}}\right]^{\frac{1}{2}}dR = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2(\varepsilon + \bar{\varepsilon})}}\right]^{\frac{1}{2}}\int^{\delta}_0 R^{-\frac{1}{2}}dR
= \frac{\sqrt{2}}{(\varepsilon + \bar{\varepsilon})^{\frac{1}{2}}}\delta^{\frac{1}{2}}. \quad (75)$$

Putting the result above into Eq. (71), we have

$$\dot{V} = mlB_t(\varepsilon + \bar{\varepsilon}). \quad (76)$$

The surface gravity of IH is $\kappa = B_t(\varepsilon + \bar{\varepsilon})$, so the energy can be rewritten as

$$\dot{V} = m \cdot l \cdot \kappa. \quad (77)$$
This is the energy as measured at infinity of a particle at rest outside the horizon with the radial proper distance $l$ away from the horizon.

The red-shift factor is defined by

$$\chi = \sqrt{-g_{00}}, \quad (78)$$

and we have

$$\frac{d\chi}{dl} = \frac{d\chi}{dR} \frac{1}{\sqrt{g_{11}}} = B_t(\varepsilon + \overline{\varepsilon}) = \kappa, \quad (79)$$

so the energy as measured at infinity of the particle can be expressed as

$$\dot{V} = m\chi. \quad (80)$$

This is the important result we will use in the following discussion.

Hawking radiation of Isolated Horizon was confirmed recently \cite{15, 16}, however, can it really be regarded as a thermodynamical system? We will analyze its thermodynamical property in another way. Following Ref. \cite{24}, we design a reversible Carnot cycle with a non-rotating Isolated Horizon being the cold reservoir. The thermodynamic nature of Isolated Horizon is confirmed further in the framework of Carnot cycle.

We divide the total process into the following four steps (see Fig. 1):

**Step (a) A hot reservoir near an Isolated Horizon is filled with thermal radiation of temperature $T_1$.** Like Ref. \cite{24} we require the chemical potential of the thermal radiation vanish. The Isolated Horizon is used as the cold reservoir, so we need $T_1 > T_H$ ($T_H$ is the temperature of IH). The initial state of the working substance is an empty box and the mass of the empty box is negligible. For designing a reversible process, we replace one side of the box with a piston in our model. When we pull the piston slowly down to the other side of the box such that the box finally is full of thermal radiation of temperature $T_1$, there is some work done by the thermal radiation (see Fig. 1). After the process above, we have a box of substance with energy $E_1$. Since the process step (a) is quasi-static and isothermal, we can use the first law of thermodynamics in the following form,

$$Q'_1 = T_1S_1 = E_1 + W'_1, \quad (81)$$

where $Q'_1$ is the heat flowing into the box, $S_1$ is the entropy of the matter in the box, and $W'_1$ is the work done by the thermal radiation. Please note that all the quantities are locally
measured, so the corresponding quantities measured at infinity are

\[ Q_1 = \chi_1 T_1 S_1 = \chi_1 E_1 + \chi_1 W'_1 = \chi_1 E_1 + W_1, \]

where \( \chi_1 \) is the red-shift factor at the hot reservoir.

\[ (82) \]

**FIG. 1:** a reversible Carnot Cycle outside a non-rotating Isolated Horizon

**Step (b) adiabatic process.** This process actually consists of two sub-steps.

**Sub-step (b1)** The box is lowered slowly down to some point near the black hole with the red-shift factor \( \chi_2 \). In this step, the substance inside does not change, so all locally measured quantities, such as energy, entropy, temperature, etc, remain unchanged. However, the energy measured by the agent holding the string at infinity becomes \( \chi_2 E_1 \), which means the work done in the lowering process is

\[ W_2 = E_1 (\chi_1 - \chi_2). \]

\[ (83) \]

Please note that this is not the thermodynamic work done by the substance in the box since the volume of the substance is unchanged, and the work is done by gravity in a pure
mechanical process. An analog of the adiabatic process in an ordinary Carnot cycle is given in sub-step (b2) below.

**Sub-step (b2)** This sub-step plays an essential role in our model. In a typical Carnot cycle, the working substance should experience an adiabatic expansion to cool down to temperature $T_2$, the temperature of the cold reservoir. However, the adiabatic lowering process–sub-step (b1) does not change the temperature of the substance, so it is necessary to add another adiabatic process to cool down the substance from $T_1$ to $T_2$. But the temperature at any point outside the black hole is not well-defined since the box is not in direct contact with the Isolated Horizon. We can use a plausible criterion, i.e. the conservation of the total entropy, to specify $T_2$ as follows. Suppose that the box cools down to $T_2$ via an adiabatic expansion. The entropy of box remains $S_1 = \frac{Q_1}{\chi_1 T_1}$ (see Eq. (82)). Then the box gives up all its energy to the black hole which is shown in step (c) below. The energy that the black hole finally absorbs is $Q_2 = \chi_2 T_2 S_1$ (see Eq. (88)). Hence the change in the total entropy is

$$\Delta S = -S_1 + \frac{\chi_2 T_2 S_1}{T_H}. \quad (84)$$

Because for a reversible adiabatic process, the total entropy should not change, that is, $\Delta S = 0$, we find immediately from Eq. (84),

$$T_2 = \frac{T_H}{\chi_2}. \quad (85)$$

Different from sub-step (b1), sub-step (b2) changes the parameter set of the box from $(E_1, T_1)$ to $(E_2, T_2)$. The entropy $S_1$ keeps unchanged and the adiabatic expansion takes place in the same location, so the work measured from infinity is given by

$$W_3 = \chi_2 (E_1 - E_2). \quad (86)$$

**Step (c)** Release the substances from the box into the black hole. We use the piston in the opposite way to push out the thermal radiation in a quasi-static and isothermal manner. We have

$$Q'_2 = T_2 S_1 = E_2 - W'_4, \quad (87)$$

where prime is used to label locally measured quantity. The corresponding quantities measured at infinity are

$$Q_2 = \chi_2 Q'_2 = \chi_2 T_2 S_1, \quad (88)$$

$$W_4 = \chi_2 W'_4 = \chi_2 (E_2 - Q'_2). \quad (89)$$
Note that $Q'_2$ is the energy released from the box seen by a local observer while the energy absorbed by the black hole is $Q_2$, instead of $Q'_2$.

**Step (d)** Lift the empty box back and return to its initial state. Nothing happens in this process because the mass of the empty box is negligible.

Now let us compute the thermal efficiency for the complete reversible Carnot cycle. It is easy to check that the total work $W \equiv W_1 + W_2 + W_3 + W_4$ satisfies the familiar relation in a typical Carnot cycle,

$$W = Q_1 - Q_2. \quad (90)$$

Hence the efficiency is

$$\eta = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\chi_2 T_2}{\chi_1 T_1} = 1 - \frac{T_H}{\chi_1 T_1}. \quad (91)$$

This is the desired efficiency for a reversible Carnot engine operating between two heat sources with temperatures $T_1$ and $T_H$ respectively. Therefore the thermodynamic nature of Isolated Horizon is confirmed in the framework of Carnot cycle and the efficiency of this cycle confirms that Isolated Horizon behaves as a thermodynamic object with Hawking temperature $T_H$.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we investigate the electrical and thermodynamical properties of Isolated Horizon. By following Damour’s method, we establish the Ohm’s law and Joule’s law of an Isolated Horizon, so we generalize Damour’s results. From the calculation, we find that the results are very general, since we do not use all the conditions of Isolated Horizon. We investigate the geometry in the vicinity of a non-rotating Isolated Horizon, and find that under the first-order approximation of $r$, $\frac{\partial}{\partial u}$ is a Killing vector and there exists a Hamiltonian conjugate to it, so $\frac{\partial}{\partial u}$ is a physical observer. We calculate the energy as measured at infinity of a particle at rest outside the horizon and construct a reversible Carnot Cycle with an Isolated Horizon as a cold reservoir, which gives a further confirmation of the thermodynamic nature of Isolated Horizon.
Acknowledgments

This research was supported by NSFC Grants No. 11175245, 11075206, 11235003, 11375026 and NCET-12-0054.

[1] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime, Cambridge University Press, 1973.
[2] R. M. Wald, General Relativity, University of Chicago Press, Chicago, 1984.
[3] D. Pejerski and E. Newman, J. Math. Phys. 9(1971)1929.
[4] S. A. Hayward, Class. Quant. Grav. 10(1993)137.
[5] A. Ashtekar, C. Beetle, S. Fairhurst, Class. Quant. Grav. 16(1999)L1-L7.
[6] A. Ashtekar, et al., Phys. Rev. Lett, 85(2000)3564.
[7] A. Ashtekar, C. Beetle, J. Lewandowski, Class. Quant. Grav. 19(2002)1195.
[8] A. Ashtekar, B. Krishnan, Living Rev. Rel. 7(2004)10.
[9] T. Damour, Phys. Rev. D 18(1978)3598.
[10] K. Thorne, R. Price, and D. Macdonald, Black Holes: The Membrane Paradigm, Yale University Press (1986); R. H. Price and K. S. Thorne, Phys. Rev. D 33(1986)915.
[11] M. K. Parikh, F. Wilczek, Phys. Rev. D 58(1998)064011.
[12] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 87(2001)081601; G. Policastro, D. T. Son and A. O. Starinets, JHEP 0209(2002)043.
[13] N. Iqbal and H. Liu, Phys. Rev. D 79(2009)025023; I. Bredberg, C. Keeler, V. Lysov and A. Strominger, JHEP 1103(2011)141.
[14] R.-G. Cai, L. Li and Y.-L. Zhang, JHEP 1107(2011)027; C. Niu, Y. Tian, X.-N. Wu and Y. Ling, Phys. Lett. B 711(2012)411; V. Lysov and A. Strominger, PhD dissertation, “From Petrov-Einstein to Navier-Stokes”, Harvard University, 2014.
[15] Xiaoning Wu, Sijie Gao, Phys. Rev. D 75(2007)044027.
[16] Xiaoning Wu, Chao-Guang Huang, Jia-Rui Sun, Phys. Rev. D 77(2008)124023.
[17] J. D. Bekenstein, Phys. Rev. D 7(1973)2333.
[18] J. D. Bekenstein, Phys. Rev. D 23(1981)287.
[19] W. G. Unruh and R. M. Wald, Phys. Rev. D 25(1982)942.
[20] H. Friedrich, Proc. Roy. Soc. Lond. A 378(1981)169-184, 401-421.

[21] Badri Krishnan, Class. Quantum Grav. 29(2012)205006.

[22] S. W. Hawking, in Black holes, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1973).

[23] R. M. Wald, A. Zoupas, Phys. Rev. D 61(2000)084027.

[24] Deng Xi-Hao, Gao Si-Jie, Chinese Physics B 18(2009)927.