Finite groups with systems of $K$-$\mathcal{F}$-subnormal subgroups

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Abstract

Let $\mathcal{F}$ be a class of group. A subgroup $A$ of a finite group $G$ is said to be $K$-$\mathcal{F}$-subnormal in $G$ if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \unlhd A_i$ or $A_i/(A_{i-1})A_i \in \mathcal{F}$ for all $i = 1, \ldots, n$. A formation $\mathcal{F}$ is said to be $K$-lattice provided in every finite group $G$ the set of all its $K$-$\mathcal{F}$-subnormal subgroups forms a sublattice of the lattice of all subgroups of $G$.

In this paper we consider some new applications of the theory of $K$-lattice formations. In particular, we prove the following

**Theorem A.** Let $\mathcal{F}$ be a hereditary $K$-lattice saturated formation containing all nilpotent groups.

(i) If every $\mathcal{F}$-critical subgroup $H$ of $G$ is $K$-$\mathcal{F}$-subnormal in $G$ with $H/F(H) \in \mathcal{F}$, then $G/F(G) \in \mathcal{F}$.

(ii) If every Schmidt subgroup of $G$ is $K$-$\mathcal{F}$-subnormal in $G$, then $G/G_\mathcal{F}$ is abelian.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathcal{F}$ is a non-empty class of group, and if $1 \in \mathcal{F}$, then $G_\mathcal{F}$ denotes the intersection of all normal subgroups $N$ of $G$ with $G/N \in \mathcal{F}$; $G_\mathcal{F}$ is the product of all normal subgroups $N$ of $G$ with $N \in \mathcal{F}$.

For any equivalence $\pi$ on the set of all primes $\mathbb{P}$, we write $\text{part}(\pi)$ to denote the partition of $\mathbb{P}$ defined by $\pi$. On the other hand, for any partition $\sigma$ of $\mathbb{P}$, we write $\text{eq}(\sigma)$ to denote the equivalence on $\mathbb{P}$ defined by $\sigma$.

If $\sigma = \{\sigma_i | i \in I\}$ is a partition of $\mathbb{P}$ (that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$), then $G$ is said to be: $\sigma$-primary [1] if $G$ is a $\sigma_i$-group for some $i$; $\sigma$-decomposable (Shemetkov [2]) or $\sigma$-nilpotent (Guo and Skiba [3]) if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$; $\sigma$-soluble [1] if every chief factor of $G$ is $\sigma$-primary. We use $\mathcal{R}_\sigma$ to denote the class of all $\sigma$-nilpotent groups.

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If \( \sigma^0 = \{ \sigma^j \mid j \in J \} \) is another partition of \( \mathbb{P} \), then we write \( \sigma^0 \leq \sigma \) provided \( eq(\sigma^0) \subseteq eq(\sigma) \), that is, for each \( j \in J \) there is \( i \in I \) such that \( \sigma^j_i \subseteq \sigma_i \). It is clear that if \( G \) is \( \sigma^0 \)-nilpotent (respectively \( \sigma^0 \)-soluble), then \( G \) is \( \sigma \)-nilpotent (respectively \( \sigma \)-soluble).

We say that \( \mathcal{F} \) is \( \sigma \)-nilpotent (respectively \( \sigma \)-soluble) if every group in \( \mathcal{F} \) is \( \sigma \)-nilpotent (respectively \( \sigma \)-soluble).

For any set \( \{ \sigma^i \mid i \in I \} \) of partitions \( \sigma^i \) of \( \mathbb{P} \) we put

\[
\bigcap_{i \in I} \sigma^i = \text{part} \left( \bigcap_{i \in I} eq(\sigma^i) \right).
\]

If \( \{ \sigma^i \mid i \in I \} \) is the set of all partitions \( \sigma^i \) of \( \mathbb{P} \) such that \( \mathcal{F} \) is \( \sigma^i \)-nilpotent (respectively \( \sigma^i \)-soluble) for all \( i \), then write \( \Sigma_n(\mathcal{F}) \) (respectively \( \Sigma_n^*(\mathcal{F}) \)) to denote the partition \( \bigcap_{i \in I} \sigma^i \). It is clear that \( \Sigma_n(\mathcal{F}) \subseteq \{ \sigma^i \mid i \in I \} \) (respectively \( \Sigma_n^*(\mathcal{F}) \subseteq \{ \sigma^i \mid i \in I \} \)), and \( \Sigma_n(\mathcal{F}) \) (respectively \( \Sigma_n^*(\mathcal{F}) \)) is the smallest element in \( \{ \sigma^i \mid i \in I \} \), that is, \( \Sigma_n(\mathcal{F}) \leq \sigma^i \) (respectively \( \Sigma_n^*(\mathcal{F}) \leq \sigma^i \)) for all \( i \).

Recall that a class of groups \( 1 \in \mathcal{F} \) is a formation if for every group \( G \) every homomorphic image of \( G/G^\mathcal{F} \) belongs to \( \mathcal{F} \). The formation \( \mathcal{F} \) is said to be: saturated if \( G \in \mathcal{F} \) whenever \( \Phi(G) \leq G^\mathcal{F} \); hereditary if \( H \in \mathcal{F} \) whenever \( H \subseteq G \in \mathcal{F} \).

A subgroup \( A \) of \( G \) is said to be \( \mathcal{F} \)-subnormal in \( G \) in the sense of Kegel [6] or \( K \)-\( \mathcal{F} \)-subnormal in \( G \) [7, 6.1.4] if there is a subgroup chain

\[
A = A_0 \leq A_1 \leq \cdots \leq A_n = G
\]

such that either \( A_{i-1} \leq A_i \) or \( A_i/(A_{i-1})_{A_i} \in \mathcal{F} \) for all \( i = 1, \ldots, n \). In particular, \( A \) of \( G \) is said to be \( \sigma \)-subnormal in \( G \) [1] provided \( A \) is \( K \)-\( \mathcal{F}_\sigma \)-subnormal in \( G \), that is, there is a subgroup chain

\[
A = A_0 \leq A_1 \leq \cdots \leq A_n = G
\]

such that either \( A_{i-1} \leq A_i \) or \( A_i/(A_{i-1})_{A_i} \) is \( \sigma \)-primary for all \( i = 1, \ldots, n \).

The set \( \mathcal{L}_{K\mathcal{F}}(G) \) of all \( K \)-\( \mathcal{F} \)-subnormal subgroups of \( G \) is partially ordered with respect to set inclusion. Moreover, \( \mathcal{L}_{K\mathcal{F}}(G) \) is a lattice since \( G \in \mathcal{L}_{K\mathcal{F}}(G) \) and, by [7, Lemma 6.1.7], for any \( A_1, \ldots, A_n \in \mathcal{L}_{K\mathcal{F}}(G) \) the subgroup \( A_1 \cap \cdots \cap A_n \in \mathcal{L}_{K\mathcal{F}}(G) \), so this intersection is the greatest lower bound for \( \{ A_1, \ldots, A_n \} \) in \( \mathcal{L}_{K\mathcal{F}}(G) \).

The formation \( \mathcal{F} \) is called \( K \)-lattice [7] if in every group \( G \) the lattice \( \mathcal{L}_{K\mathcal{F}}(G) \) is a sublattice of the lattice \( \mathcal{L}(G) \) of all subgroups of \( G \).

The full classification of hereditary \( K \)-lattice saturated formations were given in the papers [12, 13] (see also Ch. 6 in [7]). The formations of such kind were useful in the study of many problems in the theory of finite groups (see, in particular, the recent papers [?, 8, 9] and Ch. 6 in [7]).

In the given paper, we consider two new applications of hereditary \( K \)-lattice saturated formations.
Recall that \( G \) is said to be \( \mathfrak{F} \)-critical if \( G \) is not in \( \mathfrak{F} \) but all proper subgroups of \( G \) are in \( \mathfrak{F} \) [10, p. 517]; \( G \) is said to be a Schmodt group provided \( G \) is \( \mathfrak{N} \)-critical, where \( \mathfrak{N} \) is the class of all nilpotent groups.

A large number of publications are related to the study of the influence on the structure of the group of its critical subgroups, in particular, Schmidt subgroups. It was proved, for example, that if every Schmidt subgroup of \( G \) is subnormal, then \( G' \leq F(G) \) [15, 16]. Later, this result was generalized in the paper [17], where it was proved that if every Schmidt subgroup of \( G \) is \( \sigma \)-subnormal in \( G \), then \( G' \leq F_{\sigma}(G) \) (here \( F_{\sigma}(G) = G_{\mathfrak{N}_{\sigma}} \) is the \( \sigma \)-Fitting subgroup of \( G \), that is, the product of all normal \( \sigma \)-nilpotent subgroups of \( G \)).

Our first observation is the following generalization of these results.

**Theorem A.** Let \( \mathfrak{F} \) be a hereditary \( K \)-lattice saturated formation containing all nilpotent groups.

(i) If every \( \mathfrak{F} \)-critical subgroup \( H \) of \( G \) is \( K-\mathfrak{F} \)-subnormal in \( G \) with \( H/F(H) \in \mathfrak{F} \), then \( G/F(G) \in \mathfrak{F} \).

(ii) If every Schmidt subgroup of \( G \) is \( K-\mathfrak{F} \)-subnormal in \( G \), then \( G/G_{\mathfrak{F}} \) is abelian.

Note that if \( \mathfrak{F} = \mathfrak{N} \) is the formation of all nilpotent groups, then a subgroup \( A \) of \( G \) is \( K-\mathfrak{F} \)-subnormal in \( G \) if and only if \( A \) is subnormal in \( G \). Hence we get from Theorem A(i) the following two known results.

**Corollary 1.1** (Semenchuk [15]). If every Schmidt subgroup of \( G \) is subnormal in \( G \), then \( G \) is metanilpotent.

**Corollary 1.2** (Monakhov and Knyagina [16]). If every Schmidt subgroup of \( G \) is subnormal in \( G \), then \( G/F(G) \) is abelian.

From Theorem A(ii) we get the following

**Corollary 1.3** (Al-Sharo, Skiba [17]). If every Schmidt subgroup of \( G \) is \( \sigma \)-subnormal in \( G \), then \( G/F_{\sigma}(G) \) is abelian.

Recall that if \( M_n < M_{n-1} < \ldots < M_1 < M_0 = G \) (*), where \( M_i \) is a maximal subgroup of \( M_{i-1} \) for all \( i = 1, \ldots, n \), then the chain (*) is said to be a maximal chain of \( G \) of length \( n \) and \( M_n \) \((n > 0)\), is an \( n \)-maximal subgroup of \( G \).

If \( \mathfrak{F} \) is a saturated formation containing all nilpotent groups, then \( G \in \mathfrak{F} \) if and only if every maximal subgroup of \( G \) is \( K-\mathfrak{F} \)-subnormal in \( G \). But when we deal with hereditary \( K \)-lattice saturated formations, the following result is true.

**Theorem B.** Let \( \mathfrak{F} \) be a hereditary \( K \)-lattice saturated formation containing all nilpotent groups and \( \sigma = \Sigma_{\mathfrak{F}}(\mathfrak{F}) \). Then the following statements hold:

(i) Every maximal chain of \( G \) of length 2 includes a proper \( K-\mathfrak{F} \)-subnormal subgroup of \( G \) if and only if either \( G \in \mathfrak{F} \) or \( G \notin \mathfrak{F} \) is a Schmidt group with abelian Sylow subgroups.
(ii) If every maximal chain of $G$ of length 3 includes a proper $K$-$\mathfrak{F}$-subnormal subgroup of $G$, then $G$ is $\sigma$-soluble.

In the case $\mathfrak{F} = \mathfrak{N}$ we get from Theorem B the following two known results.

**Corollary 1.4** (Spencer [18]). If every maximal chain of length 3 includes a proper subnormal subgroup of $G$, then $G$ is soluble.

**Corollary 1.5** (Spencer [18]). If every maximal chain of a non-nilpotent group $G$ of length 2 includes a proper subnormal subgroup of $G$, then $G$ is a Schmidt group with abelian Sylow subgroups.

In the next two results, $\pi$ is a non-empty set of primes.

**Corollary 1.6.** Suppose that $\mathfrak{F} = \mathfrak{F}_{\pi}$ is the class of all $\pi'$-groups. If every maximal chain of $G$ of length 3 includes a proper $K$-$\mathfrak{F}$-subnormal subgroup of $G$, then $G$ is $\pi$-soluble.

**Corollary 1.7.** Suppose that $\mathfrak{F} = \mathfrak{N}_{\sigma}$, where $\sigma = \{\pi, \pi'\}$. If every maximal chain of $G$ of length 3 includes a proper $K$-$\mathfrak{F}$-subnormal subgroup of $G$, then $G$ is $\pi$-separable.

2 Proof of Theorem A

In view of Proposition 2.2.8 in [7], we have the following

**Lemma 2.1.** Let $\mathfrak{F}$ be a non-empty formation. If $N$ and $U$ are subgroups of $G$ such that $N$ is normal in $G$ and $G = NU$, then:

(i) $(G/N)^\mathfrak{F} = G^\mathfrak{F}N/N$, and

(ii) $G^\mathfrak{F}N = U^\mathfrak{F}N$.

**Lemma 2.2** (See Corollary 4.2.1 in [2]). If $\mathfrak{F}$ is a saturated formation containing all nilpotent groups and $E$ is a normal subgroup of $G$ such that $E/E \cap \Phi(G) \in \mathfrak{F}$, then $E \in \mathfrak{F}$.

Let $\mathfrak{F}$ is a hereditary formation. Then a subgroup $A$ of $G$ is said to be $\mathfrak{F}$-subnormal in $G$ if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that $A_i/(A_{i-1})A_i \in \mathfrak{F}$ for all $i = 1, \ldots, n$. A formation $\mathfrak{F}$ is said to be lattice [7] provided in every group $G$ the set of all its $\mathfrak{F}$-subnormal subgroups forms a sublattice of the lattice of all subgroups of $G$.

**Lemma 2.3.** If $\mathfrak{F}$ is a hereditary $K$-lattice saturated formation containing all nilpotent groups and $G$ is an $\mathfrak{F}$-critical group with $G/F(G) \in \mathfrak{F}$, then $G$ is a Schmidt group.

**Proof.** Suppose that this lemma is false. By [7] Theorem 6.3.15, $\mathfrak{F}$ is lattice. Hence by Lemma 13 in [13], $(G/\Phi(G))^\mathfrak{F}$ is a non-abelian minimal normal subgroup of $G/\Phi(G)$. But Lemma 2.1 implies that

$$(G/\Phi(G))^\mathfrak{F} = G^\mathfrak{F}\Phi(G)/\Phi(G) \simeq G^\mathfrak{F}/(G^\mathfrak{F} \cap \Phi(G))$$
is nilpotent. Hence \((G/\Phi(G))^{\mathfrak F}\) is nilpotent by Lemma 2.2, so it is abelian. This contradiction completes the proof of the lemma.

**Lemma 2.4** (See [7, Theorems 6.3.3, 6.3.15]). Let \(\mathfrak F\) be a hereditary \(K\)-lattice saturated formation containing all nilpotent groups and \(\sigma = \{\sigma_i \mid i \in I\} = \Sigma_\alpha(\mathfrak F)\). Then the following statements hold:

(i) If \(\pi \subseteq \sigma_i\) for some \(i\), then each soluble \(\pi\)-group is contained in \(\mathfrak F\).

(ii) If \(A \in \mathfrak F\) and \(B \in \mathfrak F\) are \(K\)-\(\mathfrak F\)-subnormal subgroups of \(G\), then \(\langle A, B \rangle \in \mathfrak F\).

**Lemma 2.5** (See [7, Theorem A]). If \(G\) is \(\sigma\)-soluble, for some partition \(\sigma = \{\sigma_i \mid i \in I\}\) of \(\mathbb P\), then \(G\) possesses a Hall \(\sigma_i\)-subgroup for all \(i\).

We say that \(G\) is \(\sigma\)-metanilpotent if \(G/F_\sigma(G)\) is \(\sigma\)-nilpotent.

**Lemma 2.6** (See [7, Proposition 4.2]). If \(A\) is a \(\sigma\)-subnormal subgroup of \(G\), for some partition \(\sigma\) of \(\mathbb P\), and \(A\) is \(\sigma\)-metanilpotent (respectively \(\sigma\)-nilpotent), then \(A^G\) is \(\sigma\)-metanilpotent (respectively \(\sigma\)-nilpotent).

**Lemma 2.7** (see [1, Lemma 2.6]). If \(A\) is a \(\sigma\)-subnormal subgroup of \(G\), where \(\sigma = \{\sigma_i \mid i \in I\}\), and \(A\) is \(\sigma_i\)-group, then \(A \leq O_{\sigma_i}(G)\).

**Proof of Theorem A.** Suppose that this theorem is false and let \(G\) be a counterexample of minimal order. Then \(G \notin \mathfrak F\).

(*) If \(H\) is an \(\mathfrak F\)-critical subgroup of \(G\), then \(H^\mathfrak F \leq F(G)\). Hence \(G\) possesses an abelian minimal normal subgroup, \(R\) say.

Since \(G \notin \mathfrak F\), \(G\) has an \(\mathfrak F\)-critical subgroup, \(H\) say. The hypothesis implies that \(H/F(H) \in \mathfrak F\) and so \(H^\mathfrak F \leq F(H)\) since \(\mathfrak F\). Moreover, \(H^\mathfrak F\) is subnormal in \(G\) by [7, Lemma 6.1.9]. But then, by using [7, Theorem 6.3.3] in the case when \(\mathfrak F = \mathfrak A\), we get that \(1 < H^\mathfrak F \leq F(G)\). Hence we have (*).

(i) Assume that this is false.

(1) The hypothesis holds for every subgroup of \(G\). Hence \(E/F(E) \in \mathfrak F\) for every proper subgroup \(E\) of \(G\).

If \(E \in \mathfrak F\), it is clear. Now assume that \(E \notin \mathfrak F\) and let \(H\) be any \(\mathfrak F\)-critical subgroup of \(E\), then \(H\) is \(K\)-\(\mathfrak F\)-subnormal in \(G\) by hypothesis, so \(H\) is \(K\)-\(\mathfrak F\)-subnormal in \(E\) by [7, Lemma 6.1.7]. Therefore the hypothesis holds for \(E\), so the choice of \(G\) implies that we have (1).

(2) \((G/N)/F(G/N) \in \mathfrak F\) for each minimal normal subgroup \(N\) of \(G\).

In view of the choice of \(G\), it is enough to show that the hypothesis holds for \(G/N\). Suppose that this is false. Then \(G/R \notin \mathfrak F\). Let \(K/N\) be any \(\mathfrak F\)-critical subgroup of \(G/N\), and let \(L\) be any minimal supplement to \(N\) in \(K\). Then \(L \cap N \leq \Phi(L)\). Moreover, \(K/N = LN/N \cong L/L \cap N\) is an \(\mathfrak F\)-critical group, so \(L/\Phi(L)\) is an \(\mathfrak F\)-critical group. Now let \(A_1, \ldots, A_t\) be the set of all \(\mathfrak F\)-critical subgroups of \(L\) and \(V = \langle A_1, \ldots, A_t \rangle\). First we show that \(L = V\). It is clear that \(V\) is normal in \(L\). Suppose that
$V < L$. Then $V\Phi(L) < L$, so $V\Phi(L)/\Phi(L) < L/\Phi(L)$ and hence

$$V\Phi(L)/\Phi(L) \simeq V/V \cap \Phi(L) \in \mathfrak{F}$$

since $L/\Phi(L)$ is an $\mathfrak{F}$-critical group. But then $V \in \mathfrak{F}$ by Lemma 2.2, so $A_1 \in \mathfrak{F}$ since $\mathfrak{F}$ is hereditary by hypothesis. This contradiction shows that $V = L = \langle A_1, \ldots, A_t \rangle$. Since $\mathfrak{F}$ is a $K$-lattice formation, it follows that $L$ is $K$-$\mathfrak{F}$-subnormal in $G$. Hence $K/N = LN/N$ is $K$-$\mathfrak{F}$-subnormal in $G/N$ by [7, Lemma 6.1.6].

Finally, we show that $(K/N)/F(K/N) \in \mathfrak{F}$. Claim (*) implies that for each $i$ we have $A_i^\delta \leq F(G) \cap L \leq F(L)$. Hence

$$A_iF(L)/F(L) \simeq A_i/A_i \cap F(L) \simeq (A_i/A_i^\delta)/((A_i \cap F(L))/A_i^\delta) \in \mathfrak{F}.$$  

On the other hand, $A_iF(L)/F(L)$ is $K$-$\mathfrak{F}$-subnormal in $L/F(L)$ by [7, Lemma 6.1.6]. Hence

$$L/F(L) = \langle A_1F(L)/F(L), \ldots, A_tF(L)/F(L) \rangle \in \mathfrak{F}$$

by Lemma 2.4(ii). Thus $L^\delta \leq F(L)$. Therefore, by Lemma 2.1(ii), $N$

$$(K/N)^\delta = K^\delta N/N = L^\delta N/N \simeq L^\delta/L^\delta \cap N$$

is nilpotent. Therefore $(K/N)/F(K/N) \in \mathfrak{F}$, so the hypothesis holds for $G/N$.

(3) $R$ is the unique minimal normal subgroup of $G$ and $R \notin \Phi(G)$. Hence $R = F(G) = O_p(G) = C_G(R)$ for some prime $p$.

In view of Claim (2) and Lemma 2.1, we get that

$$(G/R)^\delta = DR/R \simeq D/D \cap R$$

is nilpotent. Hence the choice of $G$ and Lemma 2.2 imply that $R \notin \Phi(G)$. Finally, if $G$ has a minimal normal subgroup $N \neq R$, then $D \simeq D/1 = D/R \cap N$ is nilpotent and so $G/F(G) \in \mathfrak{F}$, contrary the choice of $G$. Hence $R$ is the unique minimal normal subgroup of $G$, so we have (3) by [10, A, 15.6].

(4) Final contradiction for (i).

Assume that $G/R \notin \mathfrak{F}$. Then, in view of Claim (3), for some maximal subgroup $M$ of $G$ we have $G = R \rtimes M$, where $M \simeq G/R \notin \mathfrak{F}$. Now let $H$ be any $\mathfrak{F}$-critical subgroup of $M$. Then $1 < H^\delta \leq F(G) \cap M$ by Claim (*). But $F(G) \cap M = R \cap M = 1$ by Claim (3). Therefore $H^\delta = 1$ and so $H \in \mathfrak{F}$, a contradiction. Thus Statement (i) is true.

(ii) Assume that this assertion is false.

(5) The hypothesis holds for every subgroup of $G$. Hence $E/E_\mathfrak{F}$ is abelian for every proper subgroup $E$ of $G$ (See Claim (1)).

(6) $(G/N)/(G/N)_\mathfrak{F}$ is abelian for each minimal normal subgroup $N$ of $G$ (See Claim (2)).
(7) \( R \) is the unique minimal normal subgroup of \( G \) and \( R \not\subseteq \Phi(G) \). Hence \( G = R \times M \) and \( R = F(G) = O_p(G) = C_G(R) \) for some prime \( p \) and some maximal subgroup \( M \) of \( G \).

First note that \( G' = G^\mathfrak{A} \), where \( \mathfrak{A} \) is the class of all abelian groups. Claim (6) and Lemma 2.1 imply that
\[
(G/R)' = G'R/R \cong G'/G' \cap R \in \mathfrak{F}.
\]
Hence the choice of \( G \) and Lemma 2.2 imply that \( R \not\subseteq \Phi(G) \). Finally, if \( G \) has a minimal normal subgroup \( N \neq R \), then \( G' \cong G'/1 = G'/R \cap N \in \mathfrak{F} \) and so \( G/G_{\mathfrak{F}} \) is abelian, contrary to the choice of \( G \). Hence we have (7) by \([10, \text{III}, 5.2]\).

(8) \( G/R \in \mathfrak{F} \). (See the proof of Claim (4) and use Claim (7)).

(9) If \( \sigma = \{\sigma_i | i \in I\} \), where \( \sigma = \Sigma_n(\mathfrak{F}) \), then \( G \) is \( \sigma \)-soluble and
\[
R = O_{\sigma_i}(G) = F_{\sigma}(G)
\]
is a Hall \( \sigma_i \)-group of \( G \) for some \( i \).

Since \( G/R \in \mathfrak{F} \) by Claim (8) and also every group in \( \mathfrak{F} \) is \( \sigma \)-nilpotent by definition (and so \( \sigma \)-soluble), Claim (7) implies that \( G \) is \( \sigma \)-soluble.

Claims (7), (8) and Lemma 2.5 imply that \( R \leq F_{\sigma}(G) = O_{\sigma_i}(G) \leq H \), where \( H \) is a Hall \( \sigma_i \)-group of \( G \) for some \( i \). Since \( R = G^\mathfrak{A} \) by Claim (8) and every group in \( \mathfrak{F} \) is \( \sigma \)-nilpotent, \( H/O_{\sigma_i}(G) \) is a normal subgroup of \( G/O_{\sigma_i}(G) \). Hence \( H \leq O_{\sigma_i}(G) \), so \( H = O_{\sigma_i}(G) \). Therefore \( F_{\sigma}(G) = H \).

Now assume that \( R < H \). By the Schur-Zassenhaus theorem, \( G \) has a \( \sigma_i \)-complement, \( W \) say. Then \( V = RW < G \), so Claim (5) implies that \( V/V_3 \) is abelian. Since \( R \) is a Hall \( \sigma_i \)-group of \( V \), \( V_3 = R \times (V_3 \cap W) \). But Claim (7) implies that \( C_G(R) \leq R \). Hence \( V_3 = R \) and so \( W \cong V/R \) is abelian.

Suppose that \( H \not\in \mathfrak{F} \) and let \( A \) be an \( \mathfrak{F} \)-critical subgroup of \( H \). Then, by Lemma 2.3 and \([11, \text{III}, 5.2]\), \( A = A_q \times A_r \) is a Schmidt group, where \( A_q \in \text{Sylow}_q(A) \), \( A_r \in \text{Sylow}_r(A) \) and \( q, r \in \sigma_i \). Hence \( A \not\in \mathfrak{F} \) by Lemma 2.4(i). This contradiction shows that \( H \in \mathfrak{F} \), so \( H \leq G_{\mathfrak{F}} \). Therefore \( G/G_{\mathfrak{F}} = (G/H)/(G_{\mathfrak{F}}/H) \) is abelian since \( G/H \cong W \) is abelian, contrary to the choice of \( G \). Thus \( R = H \), so we have (9).

(10) \( M \) is a Miller-Moreno group (that is, a \( \mathfrak{U} \)-critical group, where \( \mathfrak{U} \) is the class of all abelian groups). Moreover, \( M \) is a \( q \)-group for some prime \( q \neq p \).

First note that \( M \) is a Hall \( \sigma_i^e \)-subgroup \( M \) of \( G \) by Claims (7) and (9). Now, let \( S \) be any maximal subgroup of \( M \). Then \( RS/(RS)_G \) is abelian by Claim (5). In view of Claim (7), \( R = (RS)_G \) and hence \( S \cong RS/R \) is abelian. Therefore the choice of \( G \) and Claim (7) imply that \( M \) is a \( \mathfrak{U} \)-critical group. Therefore, \( M \) is either a Schmidt group or a minimal non-abelian group of prime power order \( q^e \). In the former case, by \([11, \text{III}, 5.2]\), \( M = Q \times V \), where \( Q = M^R \in \text{Sylow}_q(M) \), \( V \in \text{Sylow}_r(M) \) and \( q \neq r \). Since \( R = C_G(R) \) is a Hall \( \sigma_i \)-group of \( G \) by Claim (9), \( RQ \not\in \mathfrak{F} \), so \( RQ \) has an \( \mathfrak{F} \)-critical subgroup \( A \). By hypothesis, \( A \) is \( K-\mathfrak{F} \)-subnormal in \( G \), so it is \( \sigma \)-subnormal in \( G \). Therefore \( A^G \) is a
meta-$\sigma$-nilpotent group by Lemma 2.6. Similarly, $RV$ has an $\mathfrak{F}$-critical subgroup $B$ and $B^G$ is meta-$\sigma$-nilpotent. But then $G = A^G B^G$ is a meta-$\sigma$-nilpotent. Indeed, $F_\sigma(A)$ is a characteristic $\sigma$-nilpotent subgroup of $A$, so $F_\sigma(A)$ is a $\sigma$-nilpotent $\sigma$-subnormal subgroup of $G$. Hence $F_\sigma(A) \leq R = F_\sigma(G)$ and $F_\sigma(G) \cap A \leq F_\sigma(A)$, which implies that $AF_\sigma(G)/F_\sigma(G) \simeq A/A \cap F_\sigma(G)$ is $\sigma$-nilpotent $\sigma$-subnormal subgroup of $G/F_\sigma(G)$. Similarly, $BF_\sigma(G)/F_\sigma(G)$ is a $\sigma$-nilpotent $\sigma$-subnormal subgroup of $G/F_\sigma(G)$. Therefore $G/R = G/F_\sigma(G) \simeq M$ is $\sigma$-nilpotent by Lemma 2.6. Therefore, since $M$ is $K$-$\mathfrak{F}$-subnormal in $G$ by hypothesis, $M$ is $\sigma$-subnormal in $G$ and so $M \leq F_\sigma(G) = R$ by Lemma 2.6. This contradiction shows that we have the second case and so Claim (10) holds.

**Final contradiction for (ii).** From Claims (7), (9) and (10) we get that $G = R \times M$, where $R = G^\mathfrak{F}$ and $M$ is a $\mathfrak{F}$-critical $q$-group for some prime $p \in \sigma_i$ and $q \in \sigma_j$, where $i \neq j$. Let $Z$ be a group of order $q$ in $Z(M)$ and $E = RZ$. Then $E \not\in \mathfrak{F}$ by Claim (7) since all $\{p, q\}$-groups contained in $\mathfrak{F}$ are nilpotent. Let $A = A_p \times Z$ be an $\mathfrak{F}$-critical subgroup of $E$.

Note that $R = R_1 \times \cdots \times R_t$, where $R_k$ is a minimal normal subgroup of $E$ for all $k = 1, \ldots, t$ by the Mashke's theorem. Suppose that $A < E$. Then there is a proper subgroup $V$ of $E$ such that $A \leq V$ and either $E/V_E$ is a $p$-group or $V$ is normal in $E$. Then $Z \leq V_E < E$, so for some $k$ we have $R_k \not\leq V_E$. Hence $R_k \leq C_E(V_E)$, so $R_k \leq N_G(Z) = M$. Thus $Z$ is normal in $G$ since $M$ is a maximal subgroup of $G$ and hence $Z \leq C_G(R) = R$, a contradiction. Therefore $E = A$, so $R = P$ and $Z$ acts irreducibly on $R$.

It is clear that $Z \leq \Phi(M)$ and so every maximal subgroup of $W$ acts irreducibly on $R$, which implies that every maximal subgroup of $W$ is cyclic. Hence $q = 2$ and so $|R| = p$. It follows that $G/R = G/C_G(R) \simeq W$ is abelian, contrary to Claim (10). Thus Statement (ii) is true.

The theorem is proved.

### 3 Proof of Theorem B

If $G \in \mathfrak{F}$, then every subgroup of $G$ is clearly $K$-$\mathfrak{F}$-subnormal in $G$ since $\mathfrak{F}$ is a hereditary formation by hypothesis. Moreover, if $G$ is a Schmidt group with abelian Sylow subgroups, then every proper subgroup $H$ of $G$ is subnormal in $G$ and so it is $K$-$\mathfrak{F}$-subnormal in $G$.

Now assume that $G \not\in \mathfrak{F}$ and that every maximal chain of $G$ of length 2 includes a proper $K$-$\mathfrak{F}$-subnormal subgroup of $G$. We show that in this case $G$ is a Schmidt group with abelian Sylow subgroups. First note that $G \not\in \mathfrak{F}$ implies that for some maximal subgroup $M$ of $G$ we have $G/M_G \not\in \mathfrak{F}$ since $\mathfrak{F}$ is a saturated formation. Hence $M$ is not $K$-$\mathfrak{F}$-subnormal in $G$. Therefore every maximal subgroup of $M$ is $K$-$\mathfrak{F}$-subnormal in $G$ by hypothesis. Hence $M$ is a cyclic Sylow $p$-subgroup of $G$ because $\mathfrak{F}$ is a lattice formation. Therefore $G$ is soluble by the Deskins-Janko-Thompson theorem [11, 7.4, IV]. Suppose that $M_G \neq 1$ and let $R$ be a minimal normal subgroup of $G$ contained in $M_G$. Then $R \leq Z(G)$, since $M \leq C_G(R)$ and $M$ is a maximal subgroup of $G$ and it is evidently not normal in $G$. In view of [7, Lemma 6.1.6], the hypothesis holds for $G/R$ and, clearly, $G/R \not\in \mathfrak{F}$. Hence $G/R$
is a Schmidt group with abelian Sylow subgroups. Therefore, if $V/R$ is any maximal subgroup of $G/R$, then $V/R$ is nilpotent and so $V$ is nilpotent since $R \leq Z(G)$. Therefore $G$ is a Schmidt group. It is clear also that the Sylow subgroups of $G$ abelian.

Now assume that $M_G = V_G = 1$. Then $G = R \rtimes M$, where $R$ is a minimal normal subgroup of $G$ and $R$ is a Sylow $p$-subgroup of $G$ for some prime $q \neq p$. Let $V$ be the maximal subgroup of $M$. Then $V, R, M \in \mathfrak{F}$ since $\mathfrak{F}$ contains all nilpotent groups by hypothesis and so $G$ is an $\mathfrak{F}$-critical group because $RV \in \mathfrak{F}$ by Lemma 2.4(ii). But then $G$ is a Schmidt group by Lemma 2.3, and the Sylow subgroups $R$ and $M$ of $G$ are abelian. Thus Statement (i) is true.

(ii) Suppose that this assertion is false and let $G$ be a counterexample of minimal order. Then $G \notin \mathfrak{F}$, since otherwise $G$ is $\sigma$-soluble by definition of $\sigma$. Let $\Sigma_n(\mathfrak{F}) = \sigma^0 = \{\sigma^0 j | j \in J\}$. Then, evidently, $\sigma \subseteq \sigma^0$.

First note that $G/R$ is $\sigma$-soluble. Indeed, if $R$ is a maximal subgroup or a 2-maximal subgroup of $G$, it is clear. Otherwise, the hypothesis holds for $G/R$ by [7, Lemma 6.1.6], so the choice of $G$ implies that $G/R$ is $\sigma$-soluble. Hence $R$ is the unique minimal normal subgroup of $G$ and $R$ is not $\sigma$-soluble. Hence $R$ is not abelian and $R \leq G^\mathfrak{F}$.

Let $p$ be any odd prime dividing $|R|$ and $R_p$ a Sylow $p$-subgroup of $R$. The Frattini argument implies that there is a maximal subgroup $M$ of $G$ such that $N_G(R_p) \leq M$ and $G = RM$. It is clear that $M_G = 1$, so $M$ is not $K$-$\mathfrak{F}$-subnormal in $G$ since $G/M_G \simeq G$. Let $D = M \cap R$. Then $R_p$ is a Sylow $p$-subgroup of $D$.

(1) $D$ is not nilpotent. Hence $D \notin \Phi(M)$ and $D$ is not a $p$-group.

Assume that $D$ is a nilpotent. Then $R_p$ is normal in $M$. Hence $Z(J(R_p))$ is normal in $M$. Since $M_G = 1$, it follows that $N_G(Z(J(R_p))) = M$ and so $N_R(Z(J(R_p))) = D$ is nilpotent. This implies that $R$ is $p$-nilpotent by Glauberman-Thompson’s theorem on the normal $p$-complements. But then $R$ is a $p$-group, a contradiction. Hence we have (1).

(2) $R < G$.

Suppose that $R = G$ is a simple non-abelian group. Assume that some proper non-identity subgroup $A$ of $G$ is $K$-$\mathfrak{F}$-subnormal in $G$. Then there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \leq A_i$ or $A_i/(A_i - 1) A_i \in \mathfrak{F}$ for all $i = 1, \ldots , t$. Without loss of generality, we can assume that $M = A_{n-1} < G$. Then $M_G = 1$ since $G = R$ is simple, so $G \simeq G/1 \in \mathfrak{F}$, a contradiction. Hence every proper $K$-$\mathfrak{F}$-subnormal subgroup of $G$ is trivial.

Let $Q$ be a Sylow $q$-subgroup of $G$, where $q$ is the smallest prime dividing $|G|$, and let $L$ be a maximal subgroup of $G$ containing $Q$. Then, in view of [11, IV, 2.8], $|Q| > q$. Let $V$ be a maximal subgroup of $Q$. If $|V| = q$, then $Q$ is abelian, so $Q < L$ by [11, IV, 7.4]. Hence there is a 3-maximal subgroup $W$ of $G$ such that $V \leq W$. But then some proper non-identity subgroup of $G$ is $K$-$\mathfrak{F}$-subnormal in $G$ by hypothesis, a contradiction. Therefore $|V| > q$, which again implies that some proper non-identity subgroup of $G$ is $K$-$\mathfrak{F}$-subnormal in $G$. This contradiction shows that we have
(2).

(3) \(M\) is \(\sigma\)-soluble.

If every maximal subgroup of \(M\) has prime order, it is evident. Otherwise, let \(L < T < M\), where \(T\) is a maximal subgroup of \(M\) and \(L\) is a maximal subgroup of \(T\). Since \(M\) is not \(K\)\(-\mathfrak{F}\)subnormal in \(G\), either \(L\) or \(T\) is \(K\)-\(\mathfrak{F}\)subnormal in \(G\) and so it is \(K\)-\(\mathfrak{F}\)subnormal in \(M\) by [7, Lemma 6.1.7]. Hence \(M\) is \(\sigma\)-soluble by Part (i).

(4) \(M = D \rtimes T\), where \(T\) is a maximal subgroup of \(M\) of prime order.

In view of Claim (1), there is a maximal subgroup \(T\) of \(M\) such that \(M = DT\). Then \(G = RM = R(DT) = RT\) and so, in view of Claim (2), \(T \neq 1\). Hence \(G\) has no a proper subgroup \(V\) such that either \(V \leq G\) or \(V/V_G \in \mathfrak{F}\). Therefore \(T\) is not \(K\)-\(\mathfrak{F}\)-subnormal in \(G\).

Assume that \(|T|\) is not a prime and let \(V\) be a maximal subgroup of \(T\). Since \(M\) and \(T\) are not \(K\)-\(\mathfrak{F}\)-subnormal in \(G\), every maximal subgroup of \(T\) is \(K\)-\(\mathfrak{F}\)-subnormal in \(G\) and so it is also \(K\)-\(\mathfrak{F}\)-subnormal in \(T\). Then \(T, V \in \mathfrak{F}\). Moreover, since \(|T|\) is not a prime and \(V \neq 1\). Claim (3) implies that \(V\) is \(\sigma\)-soluble. Then \(V\) is \(\sigma^0\)-soluble, so for some \(i\) we have \(O_{\sigma^0_i}(V) \neq 1\). On the other hand, \(O_{\sigma^0_i}(V) \in \mathfrak{F}\) since \(V \in \mathfrak{F}\) and \(O_{\sigma^0_i}(V)\) \(K\)-\(\mathfrak{F}\)-subnormal in \(G\) since \(V\) is \(K\)-\(\mathfrak{F}\)-subnormal in \(G\).

Therefore \(R \leq \langle O_{\sigma^0_i}(V) \rangle^G \in \mathfrak{F}\) by Lemma 2.4(ii). Hence \(R\) is \(\sigma\)-soluble since every group in \(\mathfrak{F}\) is \(\sigma\)-soluble by definition. This contradiction completes the proof of (4).

Final contradiction for (ii). Since \(T\) is a maximal subgroup of \(M\) and it is cyclic, \(M\) is soluble and so \(|D|\) is a prime power, which contradicts (1). Thus Statement (ii) is true.

The theorem is proved.

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