Construction D’ Lattices for Power-Constrained Communications

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Abstract—Designs and methods for nested lattice codes using Construction D’ lattices for coding and convolutional code lattices for shaping are described. Two encoding methods and a decoding algorithm for Construction D’ coding lattices that can be used with shaping lattices for power-constrained channels are given. We construct nested lattice codes with good coding properties, a high shaping gain, and low-complexity encoding and decoding. Convolutional code generator polynomials for Construction A lattices with the greatest shaping gain are given, as a result of an extensive search. It is shown that rate 1/3 convolutional codes provide a more favorable performance-complexity trade-off than rate 1/2 convolutional codes. Tail-biting convolutional codes have higher shaping gain than that of zero-tailed convolutional codes. A design for quasi-cyclic low-density parity-check (QC-LDPC) codes to form Construction D’ lattices which have efficient encoding and indexing is presented. The resulting QC-LDPC Construction D’ lattices are evaluated using four shaping lattices: the E₈ lattice, the BW₁₆ lattice, the Leech lattice and our best-found convolutional code lattice, showing a shaping gain of approximately 0.65 dB, 0.86 dB, 1.03 dB and 1.25 dB at dimension 2304.

Index Terms—Construction D’ lattices, convolutional codes, nested lattice codes, QC-LDPC codes, shaping gain

I. INTRODUCTION

A. Motivation

The capacity of the additive white Gaussian noise (AWGN) channel cannot be achieved when equiprobable QAM signal constellations are used at high signal-to-noise ratio (SNR) [1], because they incur a πε/6 (1.53 dB) loss as the dimension n → ∞. This loss can be overcome using spherical constellations that produce Gaussian-like distributions, but decoding an n-sphere is impractical. Constellation-shaping techniques that produce Gaussian-like distributions with reasonable complexity are desirable.

Lattices are a natural fit for wireless communications because they provide reliable transmission using real-valued algebra and higher transmit power efficiency than conventional constellations at high SNR. Lattices also form an important component of compute-and-forward relaying [2], which provides high throughput and high spectral efficiency. Voronoi constellations [4], [5], also called nested lattice codes [6], constructed using a coding lattice Λc and a shaping lattice Λs, can be used for power-constrained communications. Erez and Zamir [6] showed that nested lattice codes can achieve the AWGN channel capacity, if Λc is channel-good and the Voronoi region of Λs is hyperspherical, using dithering and minimum mean-square error (MMSE) scaling techniques. For high code rates, dithering is not required [7] and the role of MMSE scaling becomes negligible [6].

Two lattices Λc and Λs are called self-similar if Λs is an integer-scaled version of Λc. The design of Λc and Λs has competing requirements, as Λc demands good coding properties and an efficient decoding algorithm while Λs needs good shaping gains and low-complexity quantization. The design of Λc and Λs can be separated, under the principle of the separation of coding gain and shaping gain [1]. Rectangular encoding and indexing for non-self-similar nested lattice codes were proposed in [8], and conditions on lattice constructions were given.

Low-density parity-check (LDPC) codes have been implemented in a wide variety of communications applications because of their capacity-achievability, efficient encoding, low-complexity decoding, and suitability for hardware implementation. For these reasons, LDPC codes are also suitable for constructing lattices. Lattices based on binary LDPC codes using Construction D’ were first introduced in [9]. Recently Branco da Silva and Silva [10] proposed efficient encoding and decoding for Construction D’ lattices, particularly for LDPC codes. A codeword and cosets of component linear codes are used to form systematic codewords for Construction D’ lattices. This encoding method naturally produces lattice points in a hypercube. However, a hypercube does not provide shaping gain. A shaping lattice Λs is needed to do so.

B. Contributions

In this paper we tackle the encoding and decoding problem of Construction D’ lattices to be used in power-constrained communications. To achieve shaping gain, convolutional code lattices formed by Construction A are used, with aspects of design, performance and complexity considered. The main contributions of this paper are as follows.

We propose two encoding methods and a decoding algorithm for Construction D’ lattices suitable for power-constrained channels. Encoding method A encodes integers with an approximate lower triangular (ALT) check matrix. Encoding method B shows how binary information bits are mapped to a lattice point using the check matrix of the underlying nested linear codes for a Construction D’ lattice. Multistage successive cancellation decoding algorithm employing binary decoders is used; the decoder uses re-encoding based on encoding method B; this method is distinct from
which was restricted to encoding/decoding Construction D’ lattices with hypercubical constellations and cannot achieve shaping gains in a power-constrained channel. A definition of Construction D’ using check-matrix perspective is also given, which is equivalent to the classical congruences definition [11, p. 235]. This is discussed in Section II.

A method to efficiently obtain triangular generator matrices for Construction A lattices is given, as a modification of the classical method [11, p. 183] (also [12, pp. 32–33]). Our method allows forming the generator matrix without swapping code bit positions for convolutional code lattices with underlying rate 1/2, 1/3, … binary convolutional codes. We show that convolutional code lattices using rate 1/3 codes provide a better tradeoff between shaping gain and quantization complexity than do rate 1/2 codes. The best-found shaping gains are given in Subsection V-B as a result of an exhaustive search over convolutional code generator polynomials.

When applied to high-dimensional nested lattice codes, the conventional indexing algorithm [8, Sec. IV-B] may encounter large-valued integers, which causes an integer overflow when implemented—it may fail to recover information even in the absence of noise. To solve this problem, we modified the algorithm to bound the values of integers that are used internally, without changing the solution; this is described in Subsection III-A.

We construct quasi-cyclic (QC)-LDPC codes to form Construction D’ lattices (termed QC-LDPC Construction D’ lattices), because QC-LDPC codes are widely used in recent wireless communication standards. We give a design of parity-check matrices for nested QC-LDPC codes that can be easily triangularized and thus efficient encoding and indexing is allowed. An existing Construction D’ lattice based on a QC-LDPC code and a single parity-check (SPC) product-like code [13] is not suitable for indexing/shaping because the SPC product-like code parity-check matrix is not necessarily triangularized. The design and the triangularization method is described in Section V. Numerical results of shaping QC-LDPC Construction D’ lattices using the $E_8$, $BW_{16}$, Leech and convolutional code lattice shaping are given in Section VI as well as the comparison with hypercube shaping.

C. Related Work

Lattices from linear codes have potential since the decoder for linear codes can be employed to find the nearest lattice point given a point. Well-known methods to build lattices from linear codes are Construction A and D/D’ [11, Ch. 5, 8]. Construction D/D’ generate lattices from multi-level nested binary linear codes. Binary Construction A lattices are the special case of one-level Construction D lattices. Unlike Construction A and D using generator matrices, Construction D’ describes lattices by check matrices and thus is suitable for LDPC codes.

Erez and ten Brink employed trellis shaping, constructing lattices based on convolutional codes and Construction A that were used for vector quantization in a dirty paper coding scheme [14]: four rate 1/2 convolutional codes and their shaping gains were given. Kudryashov and Yurkov found generator polynomials of rate 1/2 convolutional codes that provide the best asymptotic normalized second moment (equivalently, shaping gain) with respect to zero-tailed termination, and near-optimum shaping gain at low dimensions with respect to tail-biting termination in [15] and [16], respectively. Our results extend their work addressing the optimality of shaping gain to a wider range of dimensions and code rates, and additionally consider the shaping gain-complexity tradeoff.

Using self-similar nested lattice codes, a shaping gain of 0.4 dB was shown for low-density lattice codes (LDLCs) [17], and a shaping gain of 0.776 dB was claimed at $n = 60$ for Construction A lattices based on QC-LDPC codes [18]. A shaping gain of 0.65 dB and 0.86 dB was observed using the $E_8$ lattice and the $BW_{16}$ lattice for shaping LDLC lattices, respectively [19]. Leech lattice has 1.03 dB shaping gain, and was used to shape LDA lattices [20]. Convolutional code lattices to shape LDLC lattices [21] a shaping gain of 0.87 dB was preserved at $n = 36$. A shaping gain of 0.63 dB was found using the $E_8$ lattice for shaping BCH code-based Construction D lattices [22].

D. Notation

A tilde indicates a vector or matrix which has only 0s and 1s — $\tilde{x}$ and $\tilde{H}$ are binary while $x$ and $H$ are not necessarily so. Operations over the real numbers $\mathbb{R}$ are denoted $+,$ (the operator $\cdot$ is sometimes omitted) while operations over the binary field $\mathbb{F}_2$ are denoted $\oplus, \circ$. The matrix transpose is denoted $(\cdot)\,^\mathsf{T}$. Element-wise rounding to the nearest integer is denoted $\lfloor \cdot \rfloor$.

II. CONSTRUCTION D’ LATTICES

We first review the definition of lattices and nested binary codes, then give a definition of Construction D’ using check-matrix perspective which is equivalent to the congruences definition. Afterwards, how to form lattices from nested binary codes using Construction D’ is shown. Lastly we propose two equivalent encoding methods and a decoding algorithm for Construction D’ lattices to be used in power-constrained channels.

A. Preliminaries

1) Lattices

An $n$-dimensional lattice $\Lambda$ is a discrete additive subgroup of $\mathbb{R}^n$. Let a generator matrix of $\Lambda$ be $G$ with basis vectors in columns. For integers $b \in \mathbb{Z}^n$, a vector $x$ is a lattice point given by $x = G \cdot b$. We define the check matrix $H$ as $H = G^{-1}$ so that $H \cdot x = b$. For a lattice with check matrix $H$, $x$ is a lattice point if and only if $H \cdot x$ is a vector of integers. If $W$ is an $n$-by-$n$ unimodular matrix then $G \cdot W$ is also a generator matrix for $\Lambda$.

2)This is the definition used in low-density lattice codes [17], and is distinct from the definition of [9]. Note also that the check matrix of a Construction D’ lattice is related to, but distinct from, the parity-check matrices of the corresponding binary codes.
2) Nested Binary Linear Codes

**Definition 1:** Let the row vectors \( h_1, h_2, \ldots, h_n \) be a basis for \( \mathbb{F}_2^n \). For level \( a \geq 1 \), \( C_0 \subseteq C_1 \subseteq \cdots \subseteq C_a = \mathbb{F}_2^n \) are nested linear codes if \( h_{k_i+1}, \ldots, h_n \) are \( r_i = n - k_i \) parity-checks for \( C_i \), where \( k_i \) denotes the dimension of code \( C_i \), whose rate is \( R_i = k_i/n \). That is, a codeword \( \hat{x} \in C_i \) if and only if:

\[
h_j \odot \hat{x} = 0, \quad (1)
\]

for \( j = k_i + 1, \ldots, n \) and \( i = 0, 1, \ldots, a - 1 \).

The \( n \)-by-\( n \) matrix of row vectors is denoted

\[
\tilde{H} = \begin{bmatrix}
-h_1 & - & - \\
- & h_2 & - \\
- & - & h_3 \\
\end{bmatrix}. \quad (2)
\]

The matrix \( \tilde{H}_0 \) is the parity-check matrix for \( C_0 \), and consists of \( r_0 \) rows, from \( h_{k_0+1} \) to \( h_n \). The matrix \( \tilde{H}_1 \) is the parity-check matrix for \( C_1 \), and consists of \( r_1 \) rows, from \( h_{k_1+1} \) to \( h_n \), and so on. This illustrates that the parity-check matrix for \( C_0 \) contains the check matrices for the supercodes \( C_1, \ldots, C_{a-1} \). The basis vectors \( h_1 \to h_{k_0} \) do not contribute to the error-correction capability of the code, but are selected so that \( \tilde{H} \) is a unimodular matrix, as shown below.

3) Construction D’

Construction D’ converts a set of parity-checks defining nested binary linear codes into congruences for a lattice [23, p. 235]. A vector \( x \) satisfies a congruence \( h = [h_1, \ldots, h_n] \) with respect to a modulo value \( q \) if:

\[
h \cdot x \equiv 0 \quad (\text{mod } q). \quad (3)
\]

A congruence can be expressed in an equivalent way. Let \( h' = h/q \). Then \( x \) satisfies this congruence if and only if:

\[
h' \cdot x \quad \text{is an integer.} \quad (4)
\]

Any \( x \) satisfying (3) will also satisfy (4).

Two equivalent definitions of Construction D’ are given. The classical definition of Construction D’ uses congruences of parity-checks of nested binary codes.

**Definition 2 (Construction D’ (congruences)):** [23] Let \( C_0 \subseteq C_1 \subseteq \cdots \subseteq C_a = \mathbb{F}_2^n \) be nested linear codes. Let the dimension of \( C_i \) be \( k_i \). Let \( \tilde{H}_1, \ldots, \tilde{H}_a \) be a basis for \( \mathbb{F}_2^n \) such that \( C_i \) is defined by \( n - k_i \) parity-check vectors \( h_{k_{i+1} \ldots, h_n.} \). Then the Construction D’ lattice is the set of all vectors \( x \in \mathbb{Z}^n \) satisfying the congruences:

\[
h_j \cdot x \equiv 0 \quad (\text{mod } 2^{i+1}), \quad (5)
\]

for all \( i \in \{0, \ldots, a - 1\} \) and \( k_i + 1 \leq j \leq n \).

Instead of congruences, the following definition uses the check matrix earlier defined as \( H = G^{-1} \).

**Definition 3 (Construction D’ (check matrix)):** An \( n \)-by-\( n \) unimodular matrix \( H \) includes the parity-check matrices of nested linear codes \( C_0 \subseteq C_1 \subseteq \cdots \subseteq C_a = \mathbb{F}_2^n \). The dimension of \( C_i \) is \( k_i \) for \( i = 0, 1, \ldots, a \), and it has \( k_i < k_{i+1} \). Let \( D \) be a diagonal matrix with entries:

\[
d_{j,j} = 2^{-i}, \quad (6)
\]

for \( k_{i-1} < j \leq k_i \) where \( k_{-1} = 0 \) and \( k_n = n \). Then the Construction D’ lattice is the set of all vectors \( x \) satisfying:

\[
H \cdot x \quad \text{are integers,} \quad (7)
\]

where

\[
H = D \cdot \tilde{H} \quad (8)
\]

is the lattice check matrix.

The following proposition shows that the two definitions are equivalent.

**Proposition 1:** Let \( h_1, \ldots, h_n \) in Definition 2 be the rows of \( H \) in Definition 3. Then the lattice given by Definition 2 is identical to the lattice of Definition 3.

**Proof:** It should be clear that because the congruences in (3) can be expressed as (4), then relevant rows of check matrix \( H \) are an alternative form of the respective congruences. However, our definition of check matrix \( H \) does not include Definition 2’s restriction to \( x \in \mathbb{Z}^n \). To achieve this, it is required that \( \tilde{H} \) be unimodular, so that the Construction D’ lattice in Definition 3 satisfies \( \Lambda \subseteq \mathbb{Z}^n \). To see this, \( G = H^{-1} = \tilde{H}^{-1} \). Since \( \tilde{H} \) is unimodular, \( \tilde{H}^{-1} \) is an integer matrix. \( \Lambda \) also is a matrix of integers. Thus \( G \) is an integer matrix and \( \Lambda \subseteq \mathbb{Z}^n \).

As a matter of design, after \( \tilde{H}_0 \) to \( \tilde{H}_{a-1} \) are fixed, the upper rows of \( \tilde{H} \) should be chosen such that \( \tilde{H} \) is unimodular; it is also convenient to choose these upper rows so that \( \tilde{H} \) is ALT form.

**B. Encoding Construction D’ Lattices**

Two equivalent encoding methods are given. Encoding method A finds a lattice point \( x \) given \( b \in \mathbb{Z}^n \) using its check matrix \( H \) in the ALT form. Encoding method B describes explicitly how information bits \( u_i \) of the component binary linear code \( C_i \) are mapped to a vector of integers \( b \) and a lattice point. The two encoding methods can be applied to produce nonhypercubical constellations, which is distinct from the encoding in [10].

1) Encoding Method A

Near linear-time encoding of LDPC codes can be accomplished using a parity-check matrix in the ALT form [23]. This idea inspired us to implement encoding of Construction D’ lattice \( \Lambda \) with a similar procedure. The steps are distinct from [23] because check matrix \( H \) of \( \Lambda \) is a real-valued square matrix.

A vector \( b \) of integers is provided, which can be considered to be a message sequence. and the corresponding lattice point \( x \) is found by solving: \( H \cdot x = b \). If \( H \) is not too big, then \( x \) can be found by matrix inversion: \( x = H^{-1} \cdot b \). If \( H \) is large but is sparse and in the ALT form, as may be expected for Construction D’ lattices based on LDPC codes, then the following procedure can be used.

Suppose that \( H \) is in the ALT form, that is, it is partially lower triangular. Specifically, \( H \) can be written as:

\[
H = \begin{bmatrix}
B & A \\
X & C
\end{bmatrix}. \quad (9)
\]

where \( A \) is an \( s \)-by-\( s \) lower-triangular matrix with non-zero elements on the diagonal; \( X \) is a \( g \)-by-\( g \) square matrix. The
"gap" is $g$—the smaller the gap, the easier the encoding. Let
$\Delta = (X - CA^{-1}B)^{-1}$. The blockwise inverse $[24]$ of $H$ is:

$$H^{-1} = \left[ \begin{array}{c}
\Delta \\
A^{-1} + A^{-1}B\Delta C A^{-1} - A^{-1}B\Delta
\end{array} \right].$$

(10)

Using the block structure, $H \cdot x = b$ can be written as:

$$[B \ A] [x_1 \ldots x_g ] = \begin{bmatrix} b_1 \\ \vdots \\ b_g \end{bmatrix}.$$ 

(11)

To perform encoding, first $x_1, \ldots, x_g$ are found using $[10]$:

$$[x_1] = [-\Delta CA^{-1} \Delta] \cdot b.$$ 

(12)

Then, coordinates $x_{g+1}, \ldots, x_n$ are found sequentially by back-substitution, using the lower triangular structure of $H$ which has entry $h_{j,w}$ in row $j$, column $w$. For $w = g + 1, \ldots, n$:

$$x_w = \frac{1}{h_{j,w}} (b_j - \sum_{i=1}^{w-1} h_{j,i}x_i),$$

where $j = w - g$.

This method is efficient when $g$ is small and $H$ is sparse. It uses pre-computation and storage of the $g$-by-$n$ matrix in $[12]$. The sum in $[13]$ is performed over few non-zero terms in sparse $H$. If the check matrix $H$ is purely triangular, then encoding is simply performed by back-substitution.

**Example 1:** Consider a 10-dimensional Construction D' lattice $\Lambda$ generated by nested binary codes $C_0 \subset C_1$ with parity-check matrix $H_0$ and $H_1$, respectively. Let $\Lambda$ be described by a check matrix $H$ in the ALT form, expressed as:

$$H = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\
1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 \\
1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 & 0
\end{bmatrix}.$$ 

$$\frac{1}{2} \tilde{H}_0$$

\begin{equation}
\frac{1}{4} \tilde{H}_1
\end{equation}

where the block partition follows $[9]$.

Assume an arbitrary vector of integers $b = [1, 2, 0, 2, 4, 0, 2, 0, 2, 1]^T$. Using $[12]$ the first two positions of the lattice point $x$ are computed:

$$x_1 = -11, x_2 = 52.$$ Then applying $[13]$ we obtain $x = [-11, 52, 12, -10, 21, 2, 27, 9, -16, -47]^T$.

**2) Encoding Method B**

Encoding can also be performed by mapping the message sequence consisting of information vectors $u_i \in \mathbb{F}_2^{k_i}$ of $C_i$ for $i = 0, 1, \ldots, a - 1$ and an integer vector $z \in \mathbb{Z}^n$ to a lattice point $x$. In addition, we show explicitly how $u_i, z$ of method B correspond to $b$ of method A, with respect to a lattice point $x$, to establish the equivalence of method A and method B.

For clarity, consider $a = 3$. The integer vector $b$ is related to $u_0, u_1, u_2$ and $z$ as:

$$b_j = u_{0j} + 2u_{1j} + 4u_{2j} + 8z_j, \quad \text{for} \quad 1 \leq j \leq k_0,$$

$$b_j = u_{1j} + 2u_{2j} + 4z_j, \quad \text{for} \quad k_0 < j \leq k_1,$$

$$b_j = u_{2j} + 8z_j, \quad \text{for} \quad k_1 < j \leq k_2,$$

$$b_j = 2z_j, \quad \text{for} \quad k_2 < j \leq n.$$ Let $u'_i$ be the zero-padded version of $u_i$, to have $n$ components:

$$u'_i = [u_{i1}, u_{i2}, \ldots, u_{ik_i}, 0, \ldots, 0]^T.$$ 

(19)

Then, the integer vector $b$ is written as:

$$b = D \cdot (u'_0 + 2u'_1 + 4u'_2 + 8z),$$

(20)

where $D$ is given in Definition $[3]$. For Construction D', the lattice point $x$ may be decomposed as:

$$x = \sum_{i=0}^{a} 2^ix_i,$$

(21)

with components $x_i$, depending on $u_i$, expressed below; $x_i$ are not necessarily binary.

Now we describe how information bits are related to a lattice point, and show that recovering integers from a lattice point is possible. Using (6) and (20)–(21) we have

$$H \cdot x = b,$$

(22)

$$\tilde{H} \cdot x = b,$$

(23)

and the lattice components $x_i \in \mathbb{Z}^n$ satisfy:

$$\tilde{H} \cdot (x_0 + 2x_1 + \cdots + 2^ax_a) = u'_0 + 2u'_1 + \cdots + 2^az,$$

(24)

Note that the encoding performed using (25)–(26) is equivalent to encoding method A.
Algorithm 1: Decoding Construction D’ Lattices

Input: noisy input \( y \)

Output: estimated lattice point \( \hat{x} \)

\[
y_0 = y;
\]

\[
y_0' = \text{mod}_2 (y_0 + 1) - 1;
\]

\[
\hat{x}_0 = \text{Deco}(y_0');
\]

\[
\hat{u}_0' = \tilde{H} \odot \hat{x}_0 \text{ then solve } \tilde{H} \cdot \hat{x}_0 = \hat{u}_0';
\]

for \( i = 1, 2, \ldots, a - 1 \) do

\[
y_i = (y_{i-1} - \hat{x}_{a-1})/2;
\]

\[
y_i' = \text{mod}_2 (y_i + 1) - 1;
\]

\[
\hat{x}_i = \text{Dec}_i(y_i');
\]

\[
\hat{u}_i' = \tilde{H} \odot \hat{x}_i \text{ then solve } \tilde{H} \cdot \hat{x}_i = \hat{u}_i';
\]

end

\[
y_a = (y_{a-1} - \hat{x}_{a-1})/2;
\]

\[
\hat{x}_a = [y_a];
\]

\[
\hat{x} = \hat{x}_0 + 2 \hat{x}_1 + \cdots + 2^{a-1} \hat{x}_{a-1} + 2^a \hat{x}_a
\]

C. Decoding Construction D’ Lattices

Re-encoding using the generator matrix is typically needed for multistage decoding of Construction D lattices \cite{25}. To produce hypercubical constellations with Construction D’, multistage decoding may compute cosets instead of re-encoding \cite{10}. For Construction D’, we extend \cite{25} and perform re-encoding using the check matrix, and describe a multistage successive cancellation decoding algorithm for Construction D’ such that non-hypercubical constellations are allowed. In particular, this decoding algorithm is suitable for Construction D’ coding lattices to be used with shaping lattices, likewise employing a binary decoder \( \text{Dec}_i \) of \( C_i \), but we use re-encoding corresponding to encoding method B. The encoding and decoding scheme is shown in Fig. 1 where encoding method B is to demonstrate the validity of the decoding algorithm.

Proposition 2: For Construction D’, the lattice component \( x_i \) is congruent modulo 2 to a codeword \( \tilde{x}_i \in C_i \), for \( i = 0, \ldots, a - 1 \).

Proof: The lattice component \( x_i \) satisfies \( \tilde{H} \cdot x_i = u_i' \) and the codeword satisfies \( \tilde{H} \cdot \tilde{x}_i = 0 \). Recall the last \( n - k_i \) positions of \( u_i' \) are 0s. Row \( l \) of \( \tilde{H} \) is equal to row \( l + k_i \) of \( \tilde{H} \), call this row \( H_l \). By definition, \( H_l \cdot x_i = 0 \) and \( H_l \cdot \tilde{x}_i = 0 \) for \( l = 1, 2, \ldots, n - k_i \). Thus, \( x_i \) mod 2 = \( \tilde{x}_i \) and the proposition holds.

Consider a lattice point \( x \) transmitted over a channel and the received sequence is \( y_0 = x + w \), where \( w \) is noise. Decoding proceeds recursively for \( i = 0, 1, \ldots, a - 1 \). The decoding result at level \( i - 1 \) is used before beginning decoding at level \( i \). Each level receives \( y_i \in \mathbb{R}^n \) as input, which is mapped to a vector \( y_i' = \text{mod}_2 (y_i + 1) - 1 \) with each element \( y_i' \in [0, 1] \) for \( j = 1, 2, \ldots, n \). For binary decoders using log-likelihood ratio (LLR) as input, the bit LLR value \( \text{LLR} = \log \frac{P(x_j = 0|y_i')}{P(x_j = 1|y_i')} \) may be estimated as \( \frac{1 - 2y_i'}{2} \). The decoder \( \text{Dec}_i \) produces a binary codeword \( \tilde{x}_i \) closest to \( y_i' \), which is an estimate of \( \tilde{x}_i \). It is necessary to find \( \tilde{x}_i \). If \( \tilde{x}_i \) does not contain an embedded \( \hat{u}_i' \), first find \( \hat{u}_i' = \tilde{H} \odot \hat{x}_i \). Then re-encoding is performed to find \( \hat{x}_i \), that is, \( 25 \). This estimated component \( \hat{x}_i \) is subtracted from the input, and this is divided over reals by 2: \( y_{i+1} = (y_i - \hat{x}_i)/2 \) to form \( y_{i+1} \), which is passed as input to the next level. This process continues recursively, until \( y_a \) is obtained. The integers are estimated as \( \tilde{x}_a = [y_a] \). The estimated lattice point is written as \( \hat{x} = \hat{x}_a + 2 \hat{x}_1 + \cdots + 2^a \hat{x}_a \). This successive cancellation decoding is described in Algorithm 1.

Furthermore, a Construction D’ lattice point can also be generated without the need to use the zero-padded \( u_i \), but is written as \( x = 2^a z + \sum_{i=1}^{a-1} 2^i x_i \) and the lattice components \( x_i \) should be in a systematic form:

\[
x_i = [u_{i_1}, u_{i_2}, \ldots, u_{i_k}, x_{i_{k+1}}, \ldots, x_{i_n}]^T
\]

where \( x_{i_k+1}, \ldots, x_{i_n} \) are found to satisfy \( \tilde{H}_i \cdot x_i = 0 \). Note that \( x_i \) are not necessarily binary. Therefore decoding Construction D’ lattices can also be performed using \( \text{(27)} \) for re-encoding; this is distinct from Algorithm 1.

III. NESTED LATTICE CODES

A coding lattice \( \Lambda_c \) and a shaping lattice \( \Lambda_s \) are used to form a nested lattice code \( \mathcal{C} \). A practical self-similar \( \mathcal{C} \) in general does not provide both good coding and shaping properties because \( \Lambda_c \) and \( \Lambda_s \) have competitive design requirements. See \cite{17}, \cite{18}. A pair of distinct lattices \( \Lambda_c \) and \( \Lambda_s \) to form \( \mathcal{C} \) is desirable, where \( \Lambda_c \) has good coding properties and \( \Lambda_s \) has good shaping properties. This was used in past work: shaping LDLC lattices using the \( E_8 \) lattice and the \( BW_{16} \) lattice \cite{19}, convolutional code lattices \cite{21}, and shaping LDA lattices using the Leech lattice \cite{20}. These results show that \( \Lambda_c \) and \( \Lambda_s \) can be designed to provide both good coding properties and efficiently achievable shaping gains.

Encoding (mapping information to lattice codewords) and indexing (the inverse mapping) of nested lattice codes can be performed if the check matrix of \( \Lambda_c \) and the generator matrix of \( \Lambda_s \) are triangularizable. The matrices used for encoding and indexing are assumed lower triangular. The encoding method follows \cite{8} is briefly reviewed. Then we modify the indexing method in \cite{8}, Sec. IV-B so that bounding values for integers are found, thus overcoming the integer overflow problem for high-dimensional lattices. After that, the coding scheme used in this paper is described.

A. Encoding and Indexing of Nested Lattice Codes

1) Preliminaries

Let \( G_c \) and \( G_s \) be a generator matrix of \( \Lambda_c \) and \( \Lambda_s \) respectively. The check matrix of \( \Lambda_c \) is \( H_c = G_c^{-1} \). To build a nested lattice code, a coding lattice \( \Lambda_c \) and a shaping lattice \( \Lambda_s \) satisfy \( \Lambda_s \subseteq \Lambda_c \) (is referred to as the sublattice condition \cite[12, p. 179]{12}).

Lemma 1: \cite{8} Lemma 1 \( \Lambda_s \subseteq \Lambda_c \) if and only if \( H_c G_s \) is a matrix of integers.

A nested lattice code can be defined by \( \mathcal{C} = \Lambda_c \cap \mathcal{V} \) where \( \mathcal{V} \) is the zero-centered Voronoi region of the shaping lattice \( \Lambda_s \).

3) We have \( H \cdot x = D \cdot (\tilde{H} \cdot x_0 + \cdots + 2^{a-1} \tilde{H} \cdot x_{a-1} + 2^a z) \). Recognize that the vector \( \tilde{H} \cdot x \) is an integer in rows 1 to \( k_1 \) and is 0 in rows \( k_1 + 1 \) to \( n \). The product \( 2^D \cdot \tilde{H} \cdot x \) is also an integer vector. Thus, \( H \cdot x \) is an integer vector. So the decomposition of \( x \) is a lattice point.
The codebook is the set of coset leaders of a quotient group \( \Lambda_c/\Lambda_s \).

2) Encoding

The mapping from integers to a lattice codeword in \( \mathbb{C} \) is called encoding \([8]\). Assume that \( \mathbf{H}_c \) and \( \mathbf{G}_s \) are lower triangular. Let \( h_{i,i} \) and \( g_{r,i} \) be diagonal elements of \( \mathbf{H}_c \) and \( \mathbf{G}_s \) for \( i = 1, 2, \ldots, n \). It follows that \( M_i = h_{i,i}g_{r,i} \) is a positive integer. Let information be represented by a vector of integers \( \mathbf{b} \) where \( b_i \in \{0, 1, \ldots, M_i - 1\} \) and position \( i \) encodes \( \log_2 M_i \) bits. Encoding is bijectively mapping \( \mathbf{b} \) to \( \mathbf{x}' \in \mathbb{C} \), where the number of codewords is \( |\mathbb{C}| = \prod_{i=1}^{n} M_i \). The lattice codeword is given by:

\[
\mathbf{x}' = \mathbf{x} \mod \Lambda_s,
\]

where \( \mathbf{x} \in \Lambda_c \) can be found by solving \( \mathbf{H}_c \mathbf{x} = \mathbf{b} \) using the methods in Section IV. Here \( \mathbf{H}_c \) need not be lower triangular but needs to be triangularizable using a unimodular transformation. Note that dithering is omitted when discussing encoding and indexing for simplicity, and will be described in Subsection III-B.

Example 2: Let the shaping lattice \( \Lambda_s = 4\Lambda_A^{10} \) be described by a generator matrix \( \mathbf{G}_s \), which is the scaled-by-4 version of the matrix \( \mathbf{G}_{A,s}^{10} \), that will be given in Example 3 in Section IV. Let \( \mathbf{H}_c \) be the triangularized version of the lattice matrix \( \mathbf{H} \) in equation (13) with \( \mathbf{W} \cdot \mathbf{H} = \mathbf{H}_c \) for a unimodular lattice matrix \( \mathbf{W} \). The lower-triangular matrix

\[
\mathbf{H}_c = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/4 -1/4 -1/4 -1/4 -1/4 -1/4 -1/4 -1/4 -1/4 -1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

is used when encoding and indexing. The diagonal elements \( M_i \) of \( \mathbf{H}_c \mathbf{G}_s \) for \( i = 1, 2, \ldots, 10 \) are:

\( M_i \in \{4, 8, 4, 4, 2, 4, 2, 2, 2, 2\} \), which gives the range of information integers. Then the code rate is \( R = \frac{\log_2 |\mathbb{C}|}{\log_2 M_i} = 1.7 \) bits per dimension. Assume the information vector is: \( \mathbf{b} = [2, 4, 1, 2, 0, 0, 2, 1, 0, 0]^T \). By solving \( \mathbf{H}_c \mathbf{x} = \mathbf{b} \) using back-substitution a lattice point \( \mathbf{x} = [2, 4, 1, 8, -2, -9, 3, -7, -5, 2]^T \) is generated. The shaping operation \( \mathbf{x}' \) or equivalently \( \mathbf{b}' \) using \( 4\Lambda_A^{10} \) gives \( \mathbf{x}' = [-2, 0, -3, 0, -2, -1, -1, -1, -2]^T \). In this example, \( 4\Lambda_A^{10} \) has a shaping gain of 0.58 dB which is obtainable because any lattice codeword \( \mathbf{x}' \) lies in the zero-centered Voronoi region of \( 4\Lambda_A^{10} \)—this produces a nonhypercubical constellation.

3) Indexing

The inverse of encoding is called indexing that maps a lattice codeword \( \mathbf{x}' \in \mathbb{C} \) to the vector of integers \( \mathbf{b} \) used by the encoder. Note that \( \mathbf{x}' \) and \( \mathbf{G}_s \mathbf{b} \) are in the same coset, so when \( \mathbf{x} \neq \mathbf{x}' \), in general, using \( \mathbf{x} \mod \Lambda_s \) cannot recover \( \mathbf{b} \). and thus an indexing method is necessary. This can be done by a systematic procedure as suggested in [8] Sec. IV-B]. The modulo-\( \Lambda_s \) expression \( \mathbf{x}' = \mathbf{G}_s \mathbf{b} - Q_{\Lambda_s}(\mathbf{G}_s \mathbf{b}) \),

\[
\mathbf{b}' = \mathbf{b} - \mathbf{H}_c Q_{\Lambda_s}(\mathbf{G}_s \mathbf{b}),
\]

where \( Q_{\Lambda_s} \) is a lattice quantizer that finds the nearest lattice point in \( \Lambda_s \) given a point. Let \( \mathbf{b}' = \mathbf{H}_c \mathbf{x}' \). Multiply \( \mathbf{H}_c \) on the left of both sides of \( \mathbf{b}' = \mathbf{H}_c \mathbf{x}' \) so that \( \mathbf{b}' = \mathbf{H}_c \mathbf{x}' \). The indexing can be performed by finding \( t \in \mathbb{Z}^n \) that satisfies \( Q_{\Lambda_s}(\mathbf{G}_s \mathbf{b}) = \mathbf{G}_s t \) such that

\[
\mathbf{b}' = \mathbf{b} - \mathbf{H}_c \mathbf{G}_s t.
\]

The indexing algorithm was described in [8] Sec. IV-B]. Consider high-dimensional nested lattice codes. As the integers \( b_i, t_i \) are found sequentially, the values for \( t_i \) can become large which leads to an integer overflow problem in practical implementations, depending on the elements of \( \mathbf{H}_c \mathbf{G}_s \) and especially when \( \mathbf{G}_s \) has large scaling.

Now we propose a modified method suitable for indexing high-dimensional nested lattice codes. Instead of using (31) we introduce \( s \in \mathbb{Z}^n \) such that

\[
\mathbf{b}' = \mathbf{b} + \mathbf{H}_c \mathbf{G}_s s - \mathbf{H}_c \mathbf{G}_s e,
\]

where \( e = t + s \) will be shown to be bounded. The solution \( \mathbf{b} \) can be found without explicitly computing \( t \) and \( s \), thus the integer overflow problem can be avoided.

These equations are solved sequentially first for \( i = 1 \), then \( i = 2, \ldots, n \), using the triangular structure of \( \mathbf{H}_c \mathbf{G}_s \) which is expressed as

\[
\mathbf{H}_c \mathbf{G}_s = \begin{bmatrix}
\theta_{1,1} & 0 & \cdots & 0 \\
\theta_{2,1} & \theta_{2,2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{n,1} & \theta_{n,2} & \cdots & \theta_{nn}
\end{bmatrix},
\]

where \( \theta_{i,i} = M_i \). The first line of (32) is

\[
b'_1 = b_1 + M_1 s_1 - M_1 e_1.
\]

Then for \( i = 2, \ldots, n \):

\[
b'_i = b_i + \sum_{j=1}^{i-1} \theta_{i,j} s_j + M_i s_i - \sum_{j=1}^{i-1} \theta_{i,j} e_j - M_i e_i.
\]

Firstly, the solution of \( b_i \) is found as follows. To obtain \( e_i \) we write

\[
q_i = s_i + \sum_{j=1}^{i-1} \theta_{i,j} s_j,
\]

but \( s_i \) need not to be computed. Then \( q_i \) should be chosen such that \( e_i \) is bounded and after \( e_i \) is obtained as indexing proceeds, the value is used for \( i + 1, \ldots, n \). The solution \( b_i \) of (32) is the same as that of (31) by choosing \( q_i \) such that

\[
q_i / \text{lcm}(M_{i+1}, \ldots, M_n) \quad \text{is an integer}.
\]

The algorithm is given as follows. The solution of (34) is
b_1 and e_1 given by
\[ b_1 = b'_1 \mod M_1, \quad \text{and} \]
\[ e_1 = \frac{b_1 - b'_1}{M_1} \mod \text{lcm}(M_2, M_3, \ldots, M_n). \]

Then for \( i = 2, \ldots, n \), \[35\] has solution \( b_i \) and \( e_i \) given by
\[ b_i = b'_i + \sum_{j=1}^{i-1} \theta_{ij} e_j \mod M_i, \quad \text{and} \]
\[ e_i = \frac{b_i - b'_i - \sum_{j=1}^{i-1} \theta_{ij} e_j}{M_i} \mod \text{lcm}(M_{i+1}, \ldots, M_n), \]
where the integer \( 0 \leq e_i < \text{lcm}(M_{i+1}, \ldots, M_n) \) is thus bounded—which is practical.

Triangular \( H_c \) and \( G_e \) allow efficient encoding and indexing, where \( H_c \) and \( G_e \) can be obtained from triangularizable full-rank check matrix and generator matrix of \( \Lambda_c \) and \( \Lambda_e \) respectively. We have not yet found a straightforward method to index nested lattice codes using non-triangular matrices.

**B. Coding Scheme**

Erez and Zamir \[6\] proposed a coding scheme using nested lattice codes with dithering and MMSE scaling techniques that can achieve the capacity of the power-constrained AWGN channel, which is transformed into a modulo-lattice additive noise channel. We use a similar coding scheme, but additionally include the indexing. Since this paper considers primarily high rate codes in the high-SNR domain, the MMSE scaling is close to 1. As proven by di Pietro, Zémor, and Boutros \[7\], dithering is not mandatory because lattice points of \( \Lambda_c \) at high code rate fill well the Voronoi region of \( \Lambda_c \).

Let the dither \( \mathbf{U} \) be uniformly distributed in Voronoi region of \( \Lambda_c \), which is independent of the lattice point \( \mathbf{x} \) of \( \Lambda_c \). Instead of using \( \mathbf{28} \), a vector \( \mathbf{x''} = \mathbf{x} - \mathbf{U} \mod \Lambda_c \) is sent to the AWGN channel. The average transmitted power per symbol \( E_s = \frac{1}{2} E[|\mathbf{x''}|^2] = \frac{1}{2} E[|\mathbf{U}|^2] \) can also be represented by \( E_s = \text{NSM} \cdot V^{2/n}(\Lambda_c) \) where \( \Lambda_c \) has normalized second moment \( \text{NSM} \) \[25\] eq. (1) and volume \( V(\Lambda_c) \). The MMSE scaling coefficient \( \alpha \) is defined as \( \alpha = E_r/(E_s + \sigma^2) \) where \( 0 \leq \alpha \leq 1 \). The signal-to-noise ratio is defined as \( \text{SNR} = E_r/\sigma^2 \). Thus \( \alpha \) can also be expressed as \( \alpha = \text{SNR}/(1 + \text{SNR}) \). Given a received sequence \( \mathbf{y''} = \mathbf{x''} + \mathbf{w} \) where \( \mathbf{w} \) is noise, the decoder is computed to output \( \mathbf{y} = \alpha \mathbf{y''} + \mathbf{U} \). See \[6, 20\].

The rate of a nested lattice code \( \mathcal{C} \) is defined by
\[ R = \frac{1}{n} \log_2 |\mathcal{C}| = \frac{1}{n} \log_2 \left| \frac{\text{det}(G_e)}{\text{det}(G_c)} \right|. \]

The average transmitted power per bit can be computed \( E_b = E_s/R \). In this paper we measure the decoding error rate of nested lattice codes as a function of \( E_b/N_0 = E_b/2\sigma^2 = \text{SNR}/2R \). To observe the shaping gains, it is convenient to define the Shannon limit in terms of \( E_b/N_0 \) as \( 10 \log_{10}(2^{2R} - 1)/(2R) \) given in decibels.

**IV. CONVOLUTIONAL CODE LATTICES**

This section focuses on the design of convolutional code lattices which are Construction A lattices using convolutional codes. The zero-centered Voronoi region \( \mathcal{V} \) of a convolutional code lattice is used to construct the nested lattice code. The effectiveness of an \( n \)-dimensional lattice quantizer is measured by the shaping gain with respect to the normalized second moment of \( \mathcal{V} \) and that of the integer lattice \( \mathbb{Z}^n \). The shaping gain measures the signal power reduction, and the theoretic limit is 1.53 dB given by an \( n \)-sphere as \( n \to \infty \) \[1\].

The shaping gains of convolutional code lattices were studied in \[14–16, 21, 27\]. Convolutional code lattices have high shaping gain, flexibility of lattice dimension, and low-complexity quantization using the well-known Viterbi algorithm. For these reasons, convolutional code lattices are suitable as shaping lattices. We are interested in both shaping gain and quantization complexity of both zero-tailed convolutional codes and tail-biting convolutional codes is also studied.

**A. Triangular Matrix of Construction A Lattices**

Construction A with binary codes corresponds to the case of one-level Construction D. Triangular matrices provide convenient encoding and indexing, thus we discuss how to obtain a triangular generator matrix \( G_{\Lambda_A} \) for a Construction A lattice \( \Lambda_A \). The well-known methods in \[11, 12, 21, 27\] is applied to build convolutional code lattices from zero-tailed convolutional codes and tail-biting convolutional codes. An exhaustive search finding the convolutional code generator polynomial that provides best-found shaping gain is performed. The tradeoff between shaping gain and quantization complexity of both zero-tailed convolutional codes and tail-biting convolutional codes is also studied.
first one in position \( j \), then replace column \( j \) of \( 2\mathbf{I}_n \) with \( \mathbf{g}_i \) for all \( i \). As a Construction A lattice, the determinant is \( \det (\Lambda_A) = \det (G_{AA}) = 2^{n-k} \).

**Example 3:** Consider a generator matrix \( G' \) of a non-systematic feedforward zero-tailed convolutional code with generator polynomials represented as octal numbers \([7, 5] \), where the information sequence has length 3. Then apply Construction A to form a lattice \( \Lambda_{10}^A \) by replacing the 3 columns in \( 2\mathbf{I}_{10} \) using the columns in \( G' \), resulting in a lower-triangular generator matrix \( G_{10}^{AA} \). This is expressed as:

\[
G' = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\Rightarrow
G_{10}^{AA} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

This convolutional code lattice has a shaping gain of 0.58 dB, obtained using the Viterbi algorithm for quantization.

### B. Best-Found Convolutional Code Lattices

In this subsection, rate 1/2 and 1/3 convolutional codes with nonsystematic feedforward encoders are used to build Construction A lattices \( \Lambda_A \). Let \( m \) be the memory order of convolutional code encoders. The number of states is \( 2^m \).

The generator matrix of zero-tailed convolutional codes has the desired form described in the previous subsection, and thus is straightforward to find a lower-triangular generator matrix for convolutional code lattices. Let \( R_{\text{ZTCC}} \) be the code rate of a zero-tailed convolutional code. The information length is \( k = nR_{\text{ZTCC}} - m \). Therefore rate loss exists and it affects the code performance when \( n \) is small.

Tail-biting convolutional codes have excellent coding performance at short-to-medium block length, thus are suitable to form Construction A shaping lattices for low-to-moderate dimension. The information length is \( k = nR_{\text{TBC}} \).

A convolutional code lattice may be scaled by \( K = 2^2, 2^3, 2^4, \ldots \) to be used with a Construction D/D' coding lattice to form a nested code lattice, so as to satisfy Lemma [1]

Generator polynomials which give good coding properties for convolutional codes are well-known [29, Ch. 12]. However, it is not clear if such generator polynomials are the best choice for shaping lattices. We performed an exhaustive search of generator polynomials for rate 1/2 and 1/3 nonsystematic feedforward convolutional codes. For each one, the shaping gain of the resulting lattice was found by Monte Carlo integration using at least \( 10^7 \) samples.

For rate 1/2 convolutional codes, it is worthwhile to mention that the generator polynomials for zero-tailed codes we found for asymptotic shaping gain match those provided in [15], except for \( m = 5 \), where we found \((75, 57) \) provides 0.01 dB higher asymptotic shaping gain than \((61, 57) \). The shaping gain of tail-biting codes with short block length were also studied in [16], which is higher than that of the Leech lattice.

The greatest shaping gain we found for various \( m \) and \( n \) is shown in Fig. [3]. In general, tail-biting convolutional codes have higher shaping gains than zero-tailed convolutional codes, for a given dimension. For a range of dimensions, a generator polynomial with a shaping gain which is either the best-found shaping gain or within 0.01 dB to the best-found shaping gain is provided in Table I with exceptions as follows. An improvement for around 0.03–0.08 dB shaping gain can be obtained using generator polynomials \((77, 76, 73) \) at \( n = 18 \) and \((331, 257) \) at \( n = 24 \) for zero-tailed convolutional codes, and using generator polynomials \((31, 27), \) \((73, 25), \) \((144, 57), \) \((250, 67), \) \((37, 33, 25) \) and \((75, 45, 26) \) at \( n = 144 \) for tail-biting convolutional codes instead. The asymptotic shaping gain obtained at \( n = 2^{20} \) and \( n = 2^{20} + 2 \) for rate 1/2 and 1/3 convolutional codes respectively is also provided. It is observed that at moderate dimensions the shaping gain of convolutional code lattices using tail-biting convolutional polynomials [3] we found independently, and are grateful to the anonymous reviewer for pointing us to [15], [16]. The shaping gains shown in [15], [16] are slightly higher, but by no more than 0.0060 dB; we have no particular explanation for this discrepancy.

### TABLE I

| Convolutional code | \( n \) | Excluding Construction A lattice, \( 24 \leq n \leq 204 \) | \( 72 \leq n \leq 144 \) | \( n > 144 \) | Asymptotic shaping gain, dB |
|-------------------|------|-----------------|-----------------|--------|------------------|
| Rate 1/2, tail-biting | 2  | 7.3, 7.7 | 7.3, 7.7 | 7.7, 7.6 | - |
| Rate 1/3, tail-biting | 3  | 7.3, 7.7 | 7.7, 7.6 | 7.6, 7.5 | - |

We found these rate 1/2 code polynomials independently, and are grateful to the anonymous reviewer for pointing us to [15], [16]. The shaping gains shown in [15], [16] are slightly higher, but by no more than 0.0060 dB; we have no particular explanation for this discrepancy.
codes can achieve the asymptotic shaping gain.

C. Complexity of Quantization

In this subsection, we study the tradeoff between shaping gain and quantization complexity for convolutional code lattices, when the Viterbi algorithm is used. Construction A lattice quantization [11, p. 450] requires 5 operations per dimension to lift the binary codeword to a lattice point and the inverse. The Viterbi decoder uses $2^m$ comparisons at each trellis stage where the total number of trellis stages is $nR_{ZTCC}$. It is assumed that $n$ is much larger than $m$ so that the contribution of termination and initialization to complexity can be ignored. Thus the normalized time complexity is $5 + 2^m R_{ZTCC}$.

We also analyzed the complexity of quantizing convolutional code lattices based on tail-biting convolutional codes using the wrap-around Viterbi algorithm [50] with a maximum of $I$ iterations and an ad-hoc suboptimal scheme [31] that decodes repeated-$J$-times sequence using the Viterbi algorithm with zero termination, requiring $5 + 2^m R_{TBCC} I$ and $(5 + 2^m R_{TBCC}) J$ operations per dimension respectively. The results given in the previous subsection were obtained using $J \geq 16$ and $nJ \geq 1152$ for the ad-hoc decoding.

The normalized quantization complexity (or number of operations per dimension) is shown in Fig. 4 as a function of asymptotic shaping gain. Rate 1/3 convolutional codes outperform rate 1/2 convolutional codes for $m = 3$ and $m = 4$ in terms of shaping gain and quantization complexity, and convolutional code lattices based on rate 1/2 convolutional codes have the best shaping gain for a fixed memory order $m = 2$ and $m = 5$. Decoding tail-biting convolutional codes requires higher complexity than that of zero termination. In summary, using rate 1/3 convolutional codes produces a more favorable performance-complexity trade-off.

The quantization for convolutional code lattices is optimal by employing the Viterbi algorithm and is close-to-optimal for tail-biting codes. The best-found efficiently achievable shaping gain is 1.25 dB with code rate 1/2 and $m = 7$ which is higher than the 1.03 dB of the Leech lattice.

V. QC-LDPC Construction D’ Lattices

In this section, we consider two-level Construction D’ lattices. One approach of lattice construction can employ QC-LDPC codes and single parity-check product codes [13]. The first level code parity-check matrix consists of a top matrix that is modified from a QC-LDPC code [32, Table I] and bottom rows which contribute to parity checks for the product code. The second level code parity-check matrix is constructed using a submatrix of the previous level’s matrix. For this design, it is not clear how to obtain a triangular lattice quantization matrix for a Construction D’ lattice. This work motivates us to design Construction D’ lattices using only QC-LDPC codes where the second level code matrix H_1 can be generated using row operations on a submatrix of the first level code matrix H_0.

A design of QC-LDPC code C_0 with a parity-check matrix H_0 is presented, where the position of non-zero blocks is found by binary linear programming [33]. A subcode condition C_0 \subset C_1 must be satisfied to form a 2-level Construction D’ lattice, and this is not straightforward. In [10], H_0 was obtained from H_1 by performing check splitting or PEG-based check splitting. In contrast to [10] we design H_0 so that H_1 may be constructed using row operations, where H_0 and H_1 can be easily triangularized and thus efficient encoding and indexing is allowed. With this design, a straightforward method to find a triangular matrix for Construction D’ lattices is also given.

A. Design QC-LDPC Codes for Construction D’

The parity-check matrix H_0 of a QC-LDPC code C_0 can be expressed by

$$
H_0 = \begin{bmatrix}
P_{1,1} & P_{1,2} & \ldots & P_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
P_{M,1} & P_{M,2} & \ldots & P_{M,N}
\end{bmatrix},
$$

(44)

where P is a Z-by-Z right-shift cyclic-permutation matrix and $-1 \leq p_{i,j} < Z$ is an integer. For $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$, when $p_{i,j} = -1$, instead use the all-zeros matrix and P_0 is the identity matrix I_Z. The block length of C_0 is $n = ZN$.

Now we give a specific design of binary QC-LDPC codes C_0 and C_1 for 2-level Construction D’ lattices. The parity-check matrices H_0 and H_1 are designed such that: 1) $C_0 \subset C_1$ 2) H_0 and H_1 are of full rank 3) H_0 and H_1 can be easily triangularized 4) H_0 and H_1 have girth as high as possible. Property 1) allows C_0 and C_1 to form a Construction D’ lattice. It is convenient to generate a triangular check matrix of A using H_i with properties 2) and 3). Property 4) is designed subject to the error correction performance.

To meet the design requirements, binary linear programming can be used to find a binary matrix with $M = 12$ rows and
$N = 24$ columns whose element one represents a non-zero block in the prototype matrix of $H_0$ and a zero represents a 0 block [33]. Then the prototype matrix of $H_0$ is generated by choosing the $-1 < p_{i,j} < Z$ of non-zero blocks such that the check matrices $H_0$ and $H_1$ have girth as high as possible, where $H_1$ is constructed by the sum of the block rows of $H_0$ in sets $\{5, 7, 9, 11\}$ and $\{6, 8, 10, 12\}$ respectively. For more detail, see [33]. For circulant size $Z = 96$, the prototype matrix of $H_0$ and $H_1$ are given in Tables II and III. Note that we assigned a double circulant $p_{12,23}^* = p_{12,23}^{(2)}$. Let $p_{12,23}^*(1) = a$ and $p_{12,23}^*(2) = b$ be selected such that $Q = I_Z + P^a + P^b$ (45) is a triple circulant and full rank. The lower-triangular $H_1$ can be obtained as follows. Let $V$ be the block-wise sum of the two block rows of $H_1$ over GF(2). The twenty-third block column of $V$ is a square matrix $Q$ [45]. Using only row operations over GF(2), $Q$ can be transformed to triangular form $T$. Find a binary matrix $W$ such that $W \odot Q = T$. Replace the first block row of $H_1$ by $W \odot V$ then the resulting matrix is lower-triangular and denoted $H_1$. After that, $H_0$ is built by replacing the bottom two block rows of $H_0$ by $H_1$.

**B. Triangular Matrix of Construction D’ Lattices**

A lower-triangular check matrix $H$ for a 2-level Construction D’ lattice is used for encoding. This can be constructed if the parity-check matrices $H_0$ and $H_1$ for nested binary codes $C_0 \subset C_1$ are triangularizable. Transform $H_0$ and $H_1$ into lower-triangular form by performing block row operations in the binary field, resulting in $H_0$ and $H_1$ respectively. The triangular matrix $H_0$ must contain the basis vectors of $H_1$ such that they both satisfy Definition [3]. Then a lower-triangular check matrix of $H$ is built using Definition [3] in Section II.

- The design of parity-check matrices $H_0$ and $H_1$ for QC-LDPC codes given in the previous subsection allows a straightforward method to generate the lower-triangular check matrix $H_0$. Let $p_{12,23}^*(1) = a$ and $p_{12,23}^*(2) = b$ be selected such that $Q = I_Z + P^a + P^b$ (45) is a triple circulant and full rank. The lower-triangular $H_1$ can be obtained as follows. Let $V$ be the block-wise sum of the two block rows of $H_1$ over GF(2). The twenty-third block column of $V$ is a square matrix $Q$ [45]. Using only row operations over GF(2), $Q$ can be transformed to triangular form $T$. Find a binary matrix $W$ such that $W \odot Q = T$. Replace the first block row of $H_1$ by $W \odot V$ then the resulting matrix is lower-triangular and denoted $H_1$. After that, $H_0$ is built by replacing the bottom two block rows of $H_0$ by $H_1$.

**VI. NUMERICAL RESULTS**

Construction D’ lattices $\Lambda_c$ of dimension $n = 2304, 5016, 10008$ formed by QC-LDPC codes were evaluated in the power-constrained AWGN channel. At the decoder, the re-encoding implicitly assumes that method B of Section II is being used, which is equivalent to method A of Section II even for triangular Construction D’ matrices of Section V. The belief propagation decoder of LDPC codes ran maximum 50 iterations. The well-known low-dimensional $E_8$, $BW_{16}$ and Leech lattices were each used for shaping a 2304-dimensional coding lattice. We also used convolutional code lattices for shaping. A variety of zero-tailed convolutional codes were chosen based on the best-found generator polynomials and complexity analysis of quantization employing the Viterbi algorithm in Section IV for shaping $n = 2304, 5016, 10008$-dimensional Construction D’ lattices. The channel model follows Fig. 2 where the encoding and indexing are performed as shown in Subsection II-A.
For comparison we performed hypercube shaping\(^8\) where lattice points of an \(a\)-level Construction D’ lattice were transformed into a hypercube \(B = \{0, 1, \ldots, L - 1\}^n\) for an integer \(L\) being a multiple of \(2^a\). Hypercube shaping for Construction D’ can be performed as follows. Let a Construction D’ lattice \(\Lambda_c\) have a lower triangular check matrix \(H_c\) with diagonal elements \(h_{i,i}\) for \(i = 1, \ldots, n\), and let \(L I_n\) be a generator matrix of the “shaping lattice” \(\Lambda_s = LZ^n\) where \(I_n\) is an identity matrix of size \(n\). Choose \(L\) such that the product of \(H_c\) and \(L I_n\) is a matrix of integers. The information vector consists of integers in \(\{0, 1, \ldots, Lh_{i,i} - 1\}\). Performing modulo-\(L\) on a lattice point of \(\Lambda_c\) is the “shaping” operation reducing the lattice point in a hypercube \(B\). This is simpler than the sequential computations in (17) to (33) to transform a lattice point into \(B\). The code rate is \(R' = \frac{1}{n} \log_2 \left( \frac{L^n}{|\det(G_s)|} \right)\).

For \(a\)-level Construction D’ lattices with hypercube shaping, it is natural to use \(2^a\)-PAM signalling. For the shaped lattice codes in this paper, the lattice points \(x\) are integers due the use of Construction D’; however greater than \(2^a\) modulation levels are required. Construction D’ lattices with hypercube shaping can also use greater than \(2^a\) modulation levels, but no shaping gain is provided.

\(\text{A. } E_8, BW_{16} \text{ and Leech Lattice Shaping}\)

Well-known low-dimensional lattices were used for shaping high-dimensional lattices because they can provide good shaping gains and their decoding is well-studied. The \(E_8\) lattice, the \(BW_{16}\) lattice and the Leech lattice have optimal quantization algorithms [26], [34], [35]. The authors in [19] used the \(E_8\) and \(BW_{16}\) lattices for shaping LDLC lattices. At \(n = 24\) the Leech lattice has a shaping gain of 1.03 dB, which was used for shaping LDA lattices [20]. Following [19], [20] we built shaping lattices using direct sum of scaled copies of the \(E_8, BW_{16}\), and Leech lattices by a scale factor \(K\). Let \(H_c\) be the check matrix of an \(n\)-dimensional Construction D’ coding lattice, and \(G\) be the generator matrix of an \(n'\)-dimensional lattice where \(n/n'\) is an integer. The factor \(K\) is chosen such that \(H_c G_s \in Z^n\) where \(G_s\) is a block diagonal matrix of size \(n/n'\) with each block \(KG\). Rectangular encoding and its inverse indexing can be efficiently implemented due to the lower-triangular structure in matrix \(H_c\) and \(G_s\). By choosing various \(K\) we generated nested lattice codes with a variety of code rates \(R\).

For shaping the 2304-dimensional Construction D’ lattice, the same code rate for both the \(E_8\) lattice shaping and hypercube shaping can be easily achieved. Let \(K_{BW_{16}} = 280\sqrt{2}\) and \(K_{\text{Leech}} = 168\sqrt{8}\), then \(BW_{16}\) and Leech lattice shaping produce code rate approximately 8.2959 and 8.3090, respectively, close to \(R = R' = 8.2993\) of choosing \(K_{E_8} = K = 472\). The word error rate is given in Fig. 5 as a function of \(E_b/N_0\). If we take account of the code rate differences, a 0.65 dB, 0.86 dB and 1.03 dB shaping gain is preserved respectively, as the full shaping gain of the \(E_8, BW_{16}\) and Leech lattices.

\(\text{B. Convolutional Code Lattices for Shaping Construction D’ Lattices}\)

In this paper we consider high-dimensional Construction D’ lattices, thus zero-tailed convolutional codes are suitable for constructing convolutional code lattices for shaping. At \(n \geq 2304\), using zero-tailed convolutional codes provides comparable shaping gain and requires lower quantization complexity than that of tail-biting convolutional codes. A variety of convolutional code lattices based on rate 1/2, 1/3 zero-tailed convolutional codes selected from Table I\(^\text{IV}\) were also used for shaping the proposed QC-LDPC Construction D’ Lattices, where the smallest possible scale factor \(K = 4\) to satisfy Lemma I\(^\text{IV}\) can produce a code rate approximately 2.084 and 1.917 respectively.

Lattices are ideal at high code rate thus we chose \(K > 4\) for evaluation. The nested lattice code parameters in our

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\(^8\)The work in [10] can also produce a hypercubical constellation, but we perform hypercube shaping with respect to our proposed decoding algorithm.
simulations are listed in Table IV including the code rates, close to that of hypercube shaping for a fair comparison. The numerical results in terms of word error rate as a function of $E_b/N_0$ are shown in Fig. 6. Convolutional code lattice shaping using a rate 1/3 convolutional code with memory order $m = 5$ was performed for $n = 2304, 5016, 10008$, showing an improvement on the error-correction performance and the shaping gain as $n$ increases. For a fixed dimension $n = 2304$, we show that a higher shaping gain is achieved by increasing the memory order $m$. The numerical results of using rate 1/2 zero-tailed convolutional codes are also provided, where the code rate was chosen as close as possible to hypercube shaping. The resulting shaping gains are approximate to the estimated shaping gains listed in Table IV if we take account of the code rate differences.

We also compared the shaping gain of a convolutional code lattice (chose $\gamma_0$ to produce a high shaping gain) with that of the $E_b/N_0$ and Leech lattices as plotted in Fig. 5 for $n = 2304$. The shaping gain of 1.25 dB was preserved with convolutional code lattice shaping—this is the best-found shaping gain achieved by lattice shaping in the power-constrained channel, to the best of the authors’ knowledge. For the four shaping lattices: convolutional code lattice, the $E_8$ lattice, the $BW_{16}$ lattice and the Leech lattice, using a smallest possible scale factor $4, 4\sqrt{2}, 4\sqrt{3}$ respectively for shaping the proposed 2304-dimensional Construction D’ lattice, the integers solutions $e_i \in [0, \beta]$ [41] are bounded by $\beta = 8, 16, 16, 32$. The values of integers are bounded by $\beta = 944, 944, 1120, 1344$ for the results in Fig. 5. Regarding the distance to the Shannon limit, while the LDA lattice construction [20] has better performance, it requires nonbinary LDPC codes, whereas our construction uses lower-complexity binary LDPC codes. The LDLC construction [19] has similar performance, but higher decoding complexity than binary LDPC codes.

### Table IV

| Dimension | Convolutional code lattice shaping | Hypercube shaping |
|-----------|-----------------------------------|-------------------|
| $n$       | $\gamma_0$ dB                     | $K$   | $R$   | $L$   | $R^*$ |
| 2304      | 5 $\gamma_1$ 1.1731               | 20    | 4.4074 | 32    | 4.4167 |
| 5016      | 5 $\gamma_1$ 1.1772               | 20    | 4.4065 | 32    | 4.4167 |
| 10008     | 5 $\gamma_1$ 1.1790               | 20    | 4.4058 | 32    | 4.4167 |
| 2304      | 2 $\gamma_2$ 0.9022               | 20    | 4.4061 | 32    | 4.4167 |
| 2304      | 4 $\gamma_3$ 1.1259               | 20    | 4.4070 | 32    | 4.4167 |
| 2304      | 4 $\gamma_4$ 1.1186               | 24    | 4.5034 | 32    | 4.4167 |
| 2304      | 5 $\gamma_5$ 1.1756               | 24    | 4.5038 | 32    | 4.4167 |
| 2304      | 7 $\gamma_6$ 1.2500               | 332   | 8.2947 | 472   | 8.2993 |

point of view, they are also suitable for hardware implementations, as well-understood QC-LDPC codes and convolutional codes with the Viterbi algorithm are used. An open problem is to optimize the LDPC degree distributions using density evolution techniques, which help us design QC-LDPC codes with good error-correction performance, such that the resulting QC-LDPC Construction D’ lattices are optimized.

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