Nakedness and curvature strength of shell-focusing singularity in the spherically symmetric space-time with vanishing radial pressure

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It was recently shown that the metric functions which describe a spherically symmetric space-time with vanishing radial pressure can be explicitly integrated. We investigate the nakedness and curvature strength of the shell-focusing singularity in that space-time. If the singularity is naked, the relation between the circumferential radius and the Misner-Sharp mass is given by $R \approx 2y_0^\beta m$ with $1/3 < \beta \leq 1$ along the first radial null geodesic from the singularity. The $\beta$ is closely related to the curvature strength of the naked singularity. For example, for the outgoing or ingoing null geodesic, if the strong curvature condition (SCC) by Tipler holds, then $\beta$ must be equal to 1. We define the “gravity dominance condition” (GDC) for a geodesic. If GDC is satisfied for the null geodesic, both SCC and the limiting focusing condition (LFC) by Królak hold for $\beta = 1$ and $y_0 \neq 1$, not SCC but only LFC holds for $1/2 \leq \beta < 1$, and neither holds for $1/3 < \beta < 1/2$, for the null geodesic. On the other hand, if GDC is satisfied for the timelike geodesic $r = 0$, both SCC and LFC are satisfied for the timelike geodesic, irrespective of the value of $\beta$. Several examples are also discussed.

PACS numbers: 04.20.Dw

I. INTRODUCTION

The cosmic censorship hypothesis (CCH) is one of the most important open problems in classical gravity (Penrose 1979). The CCH roughly says that the physically reasonable space-time contains no naked singularity. Since the CCH asserts the future predictability of the space-time, it is so helpful that several theorems on black holes have been proved under the assumption of CCH (Hawking and Ellis 1973). In spite of all the effort, the censorship has not yet been proved. In fact, there have been discovered several solutions which have naked singularities with matter content that satisfies energy conditions. Then the curvature strength of singularities was defined in a hope that weak convergence would reveal the extendibility of the space-time in a distributional sense. In this context, Tipler (1977) defined the strong curvature condition (SCC), while Królak (1987) defined weaker condition, which we call the limiting focusing condition (LFC).

One of the most important examples which have naked singularities is the Lemaître-Tolman-Bondi (LTB) solution. This solution describes the spherical collapse of an inhomogeneous dust ball. It has been proved that this solution has naked singularities from generic initial data. The naked singularities in this solution are classified to “shell-crossing” (Yodzis, Seifert and Müller zum Hagen 1973) and “shell-focusing” (Eardley and Smarr 1979, Christodoulou 1984) singularities. Newman (1986) showed that, for a null geodesic from the shell-crossing singularity, neither SCC nor LFC is satisfied. It is widely believed that the shell-crossing singularities would be harmless because they would be dealt with in some distributional sense. Newman (1986) also showed that, for a null geodesic from the shell-focusing singularity which results from generic smooth initial data, not SCC but LFC is satisfied. Hence, shell-focusing singularities will be more serious to CCH than shell-crossing singularities. It is important that the strength of the
singularity is determined by the curvature divergence not only on the null geodesic but also on the timelike geodesic. Recently, Deshingkar, Joshi and Dwivedi (1999) showed that both SCC and LFC are satisfied for the timelike geodesics which terminate at the shell-focusing singularity.

It might be thought that, since the dust is a pressureless fluid, there appears naked singularity which satisfies LFC. As an extension of the LTB solution, we will consider a spherically symmetric space-time with a fluid that has only tangential pressure. Magli (1997, 1998) solved an explicit solution with the mass-area coordinates. Here we give a formalism to examine the existence of naked central singularity and its curvature strength. Next we proceed further by defining the “gravity dominance condition”. After that we discuss several examples.

We follow the sign conventions of the textbook by Misner, Thorne and Wheeler (1973) about the metric, Riemann and Einstein tensors. We use the units with \( c = G = 1 \).

II. METRIC FUNCTIONS AND OCCURRENCE OF NAKED SINGULARITY

In a spherically symmetric space-time, the line element is written in the diagonal form as

\[
 ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + R^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2) .
\]  
(2.1)

Using the comoving coordinates, the stress-energy tensor \( T^\mu_\nu \) with vanishing radial pressure is of the following form:

\[
 T^\mu_\nu = \begin{pmatrix}
 -\epsilon & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & \Pi & 0 \\
 0 & 0 & 0 & \Pi 
\end{pmatrix},
\]  
(2.2)

where \( \epsilon(t, r) \) and \( \Pi(t, r) \) are the energy density and the tangential pressure, respectively. From the Einstein equation and the equation of motion for the matter, we obtain

\[
 m = F, \quad \epsilon = \frac{F'}{4\pi R^2 R'}, \quad \epsilon^{2\lambda} = R'^2 h^2, \quad \nu' = -\frac{1}{h} h R R', \quad \dot{R}^2 e^{-2\nu} = -1 + \frac{2F}{R} + \frac{1}{h'^2},
\]  
(2.3-2.7)

where an arbitrary function \( F = F(r) \) is the conserved Misner-Sharp mass (Misner and Sharp 1973). The dot and prime denote the partial derivatives with respect to \( t \) and \( r \), respectively. We have introduced a function \( h = h(r, R) \geq 0 \) as

\[
 \Pi = -\frac{R}{2h} h R \epsilon,
\]  
(2.8)

where the comma denotes the partial derivative. We should note that the definition of \( h \) is slightly different from Magli (1997, 1998)'s notation. The dust limit is given by \( h = h(r) \). Eqs. (2.6) and (2.7) are coupled and cannot be integrated explicitly.

We can express the metric functions explicitly by introducing the mass-area coordinate system

\[
 ds^2 = -A(m, R) dm^2 - 2B(m, R) dm dR - C(m, R) dR^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]  
(2.9)

which was introduced by Ori (1990). Since the derivation of the explicit solution was described in Magli (1998), here we only present the results:

\[
 A = H \left( 1 - \frac{2m}{R} \right), \quad B = \frac{\sqrt{H}}{|h| u}, \quad C = \frac{1}{u'^2},
\]  
(2.10-2.12)
where

\[
\sqrt{H} = \frac{R_0^0 h(m, R^0(m))}{|u^0|} + \int_{x^0}^R \frac{h}{x} \left(1 + \frac{x}{2} \left(\frac{1}{h^2}\right)_m\right) \left(1 + \frac{2m}{x} + \frac{1}{h^2}\right)^{-3/2} \, dx,
\]

(2.13)

\[
u \equiv \frac{dR}{dr} = \pm \sqrt{1 + \frac{2m}{R} + \frac{1}{h^2}},
\]

(2.14)

\[u^0(m) \equiv \pm \sqrt{1 + \frac{2m}{R^0(m)} + \frac{1}{h^2(m, R^0(m))}},
\]

(2.15)

\[R^0(m) \equiv R(0, F^{-1}(m)),
\]

(2.16)

the energy density is given as

\[
\epsilon = \frac{h}{4\pi R^2 |u| \sqrt{H}},
\]

(2.17)

and we have assumed \(\epsilon \geq 0\).

The shell-crossing singularity is the one characterized by \(R' = 0\) and \(R > 0\), while the shell-focusing singularity is the one characterized by \(R = 0\). Newman (1986) showed that the shell-crossing singularities do not satisfy even LFC for a null geodesic. Christodoulou (1984) showed that noncentral \((r > 0\) or \(m > 0)\) shell-focusing singularities are not naked. Therefore we concentrate on central \((r = 0\) or \(m = 0)\) shell-focusing singularities.

If and only if the singularity is naked, there exists an outgoing null geodesic which emanates from the singularity. In the mass-area coordinates, we can derive the root equation which probes the existence of such a geodesic as follows. The radial null rays are determined by the equation

\[
\frac{dR}{dm} = -B \mp \sqrt{H} = \sqrt{H} |u| \left(\frac{1}{h} \mp |u|\right).
\]

(2.18)

We should note that the upper sign refers to an outgoing null ray in a collapsing phase and an ingoing null ray in an expanding phase at the same time. Similarly, the lower sign refers to an ingoing null ray in a collapsing phase and an outgoing null ray in an expanding phase at the same time. Hereafter we mainly concentrate on a collapsing phase.

Here we define

\[
y \equiv \frac{R}{2m^\beta},
\]

(2.19)

where \(\beta\) is determined by requiring that \(y\) has a positive finite limit \(y_0\) along the null geodesic. Then, the regular center corresponds to \(\beta \leq 1/3\). If we assume that the energy density at the center is positive, the regular center corresponds to \(\beta = 1/3\). Note that we will only consider naked singularities with such \(\beta > 1/3\). It is noted that we will assume the existence of every limit through this paper in a sense including \(\pm \infty\). Then, from the l’Hospital’s rule,

\[
y_0 = \lim_{m \to 0} \frac{R}{2m^\beta} = \lim_{m \to 0} \frac{m^{1-\beta} dR}{2\beta \, dm} = \lim_{m \to 0} \frac{m^{1-\beta} \, \sqrt{H} |u| \left(\frac{1}{h} \mp |u|\right)}{|R = 2y_0m^\beta|}.
\]

(2.20)

Therefore, we obtain the root equation for the existence of the null geodesic from the central singularity

\[
y_0 = \frac{1}{2\beta} \lim_{m \to 0} \left[m^{3(1-\beta)/2} \sqrt{H} \left(-1 + \frac{1}{h^2}\right) m^{-(1-\beta)} + \frac{1}{y_0} \left(\frac{1}{h} \mp \sqrt{\frac{m^{1-\beta}}{y_0} - 1 + \frac{1}{h^2}}\right)\right],
\]

(2.21)

where the limit is taken along \(R = 2y_0m^\beta\). This equation was first derived by Magli (1998). As seen in this equation, the existence of a future-directed outgoing null ray from the singularity in a collapsing phase requires

\[
\frac{1}{3} < \beta \leq 1.
\]

(2.22)

From \(u^2 \geq 0\) and \(0 < y_0 < \infty\),

\[
\lim_{m \to 0} h \leq \begin{cases} 
1 & \text{for } \beta < 1, \\
(1 - y_0^{-1})^{-1/2} & \text{for } \beta = 1 \end{cases}
\]

(2.23)
where the limit is taken along the null ray which emanates from the singularity. Here we should note that, if the singularity is critically naked, i.e., if

\[ \lim_{m \to 0} \frac{2m}{R} = 1 \quad (2.24) \]

holds along the null ray, higher order analysis is needed.

### III. CURVATURE CONDITION ALONG NULL GEODESIC

We consider a radial null geodesic which emanates from or terminates at the naked singularity. We prepare a parallely propagated tetrad \( E_i \): \( i = 1, 2, 3, 4 \) with \( E_1 \cdot E_1 = E_2 \cdot E_2 = -E_3 \cdot E_4 = -E_4 \cdot E_3 = 1 \), all other products vanish and \( E_4 \) is equal to the tangent vector \( k^\mu \) of the null geodesic. In a spherically symmetric space-time, for the tetrad components of the Weyl tensor,

\[ C^m_{4n4} = 0 \quad (3.1) \]

holds for \( m, n = 1, 2 \). Define

\[ p = \lim_{\lambda \to +0} \lambda^\alpha R_{44}, \quad (3.2) \]

where \( R_{44} \) is defined by

\[ R_{44} \equiv R_{\mu\nu}k^\mu k^\nu \quad (3.3) \]

and \( \lambda \) is the affine parameter such that \( \lambda \to +0 \) corresponds to an approach to the singularity. Then, from Clarke and Królak (1985) and Clarke (1993),

**Lemma 1** For the radial null geodesic which emanates from or terminates at the singularity in the spherically symmetric space-time: SCC is satisfied if \( p \) is positive for \( \alpha = 2 \), and not satisfied if \( p \) is equal to 0 for \( \alpha < 2 \); LFC is satisfied if \( p \) is positive for \( \alpha = 1 \), and not satisfied if \( p \) is equal to 0 for \( \alpha < 1 \).

Since the null geodesic is given as

\[ k^R = \frac{-B \mp \sqrt{\Pi}}{C} k^m, \quad (3.4) \]

we obtain, from the form of the stress-energy tensor,

\[ R_{44} = 8\pi u^2 H(k^m)^2. \quad (3.5) \]

Then,

\[ \lambda^2 R_{44} = \frac{1}{2} |u| h \sqrt{H} \left( \frac{2m}{R} \right)^2 \left( \frac{\lambda}{m} \frac{dm}{d\lambda} \right)^2 \approx \frac{\beta q^2}{y_0} m^{1-\beta} \frac{h^2}{1 \mp h|u|} \quad (3.6) \]

holds, where \( q \) is defined as

\[ q = \lim_{m \to 0} \frac{d\ln m}{d\ln \lambda} \quad (3.7) \]

and we have used Eq. (2.20).

We should note that, for \( (\beta, y_0) \neq (1, 1) \),

\[ 0 < \lim_{m \to 0} \frac{h^2}{1 - h|u|} < \infty \quad (3.8) \]

and

\[ 0 \leq \lim_{m \to 0} \frac{h^2}{1 + h|u|} < \infty \quad (3.9) \]
hold, where the equality holds only when
\[ \lim_{m \to 0} h = 0 \] (3.10)
is satisfied. Therefore,
\[ R_{44} \propto \lambda^{(1 - \beta) - 2} \] (3.11)
holds for the outgoing null geodesic with \( 0 < q < \infty \) and \( (\beta, y_0) \neq (1, 1) \), and for the ingoing null geodesic with \( 0 < q < \infty \), \( (\beta, y_0) \neq (1, 1) \) and \( \lim_{m \to 0} h \neq 0 \). In summary, we present the following theorems:

**Theorem 1** For the outgoing radial null geodesic which emanates from the noncritically naked singularity with \( 0 < q < \infty \) if and only if \( 1 / 3 < \beta < 1 \) and \((1 - \beta)^{-1} < q < \infty \) are satisfied, neither SCC nor LFC holds, if and only if \( 1 / 3 < \beta < 1 \) and \( 0 < q \leq (1 - \beta)^{-1} \) are satisfied, not SCC but only LFC holds, and if and only if \( \beta = 1 \) is satisfied, both SCC and LFC hold.

**Theorem 2** For the ingoing radial null geodesic which terminates at the noncritically naked singularity with \( 0 < q < \infty \) and \( \lim_{m \to 0} h \neq 0 \): if and only if \( 1 / 3 < \beta < 1 \) and \((1 - \beta)^{-1} < q < \infty \) are satisfied, neither SCC nor LFC holds, if \( 1 / 3 < \beta < 1 \) and \( 0 < q \leq (1 - \beta)^{-1} \) are satisfied, not SCC but only LFC holds, and if and only if \( \beta = 1 \) is satisfied, both SCC and LFC hold.

In order to estimate \( q \), we must solve the null geodesic equation
\[ \frac{d}{d\lambda} \left( \sqrt{H} k^m \right) = \pm \frac{1}{2} \left[ A_{,R} + 2 B_{,R} |u| \sqrt{H} \left( \frac{1}{h} \pm \frac{1}{|u|} \right) + C_{,R} u^2 H \left( \frac{1}{h} \mp \frac{1}{|u|} \right)^2 \right] (k^m)^2 = 0, \] (3.12)
where we have used the null condition (3.4). From Eq. (3.13), we obtain
\[ \sqrt{H}_{,R} = \frac{h}{2|u|^3} \left( \frac{2}{R} \left( \frac{1}{h^2} \right)_{,m} \right). \] (3.13)
Using this, the following expressions are derived.
\[ A_{,R} = \frac{2m}{R^2} H + \frac{\sqrt{H}}{|u|^3} h \left( \frac{1}{h} \pm \frac{2m}{R} \right) \left( \frac{2}{R} \left( \frac{1}{h^2} \right)_{,m} \right), \] (3.14)
\[ B_{,R} = -\frac{1}{2|u|^3} \left( \frac{2}{R} \left( \frac{1}{h^2} \right)_{,m} \right) - \frac{\sqrt{H}}{|u|^3} h \left[ \left( 1 - \frac{2m}{R} \right) \frac{h_{,R}}{h} + \frac{m}{R^2} \right], \] (3.15)
\[ C_{,R} = \frac{2}{u^2} \left( \frac{m}{R^2} + \frac{1}{h^2} \frac{h_{,R}}{h} \right). \] (3.16)
Using these expressions, we finally obtain the radial null geodesic equation for \( m \to 0 \) in the explicit form
\[ \frac{d^2 m}{d\lambda^2} = \left[ 1 - \beta - \frac{\sqrt{H}}{2|u|} \left( \frac{2m}{R} + \frac{d}{d\ln m} \frac{1}{h^2} \right) \right] \frac{1}{m} \left( \frac{dm}{d\lambda} \right)^2, \] (3.17)
where the ordinary derivative is taken along \( R = 2y_0m^3 \). In evaluating the right hand side of Eq. (3.17), we have used
\[ R \approx 2y_0m^3, \] (3.18)
\[ |u| \approx m^{(1 - \beta)/2} \sqrt{\frac{1}{y_0} + m^{-(1 - \beta)} \left( \frac{1}{h^2} - 1 \right)}, \] (3.19)
\[ \sqrt{H} \approx \frac{\beta}{m |u| (1 \mp h|u|)}. \] (3.20)
On the other hand, if \( m \) is proportional to \( \lambda^q \) along the null geodesic, the following equation holds:
\[ \frac{d^2 m}{d\lambda^2} = \left( 1 - \frac{1}{q} \right) \frac{1}{m} \left( \frac{dm}{d\lambda} \right)^2. \] (3.21)
Comparing Eqs. (3.17) and (3.21), we can determine \( q \). Therefore the curvature divergence along the null geodesic is determined only from \( \beta \) and \( h \) along the geodesic.
Here we define the gravity dominance condition (GDC) and the gravity-dominated singularity as follows:

**Definition 1 (Gravity Dominance Condition)** For the geodesic which emanates from or terminates at the singularity, we have

$$\lim_{m \to 0} \frac{R}{2m} \left( \frac{1}{h^2} - 1 \right) = 0. \quad (4.1)$$

**Definition 2 (Gravity-Dominated Singularity)** A singularity is said to be gravity-dominated if and only if GDC is satisfied for every causal geodesic which emanates from or terminates at the singularity.

GDC is satisfied for the geodesic which emanates from or terminates at a very wide class of naked singularities. Furthermore, if the gravitational collapse of physical matter from regular initial data results in the central naked singularity formation, GDC is satisfied for the null geodesic, at least within our knowledge. The important example is the central singularity in the collapse of the spherical cluster of counterrotating particles which will be discussed in Sec. VI.

If GDC is satisfied for the null geodesic, the collapse is induced dominantly by the gravitational potential (see Eq. (2.7) or (2.14)) and the null geodesic equation is controlled only by the gravitational potential (see Eq. (3.17)). The latter can be shown by the following proposition.

**Proposition 1** If GDC is satisfied, then

$$\lim_{m \to 0} \frac{R}{2m} \frac{d}{d\ln m} \frac{1}{h^2} = 0 \quad (4.2)$$

**Proof.** We use $R \approx 2y_0 m^\beta$ along the null geodesic. Then, for $\beta < 1$, the l’Hospital’s rule applies because we have assumed the existence of the limit. From condition (4.1), the proposition holds. For $\beta = 1$, we set $f \equiv h^{-2} - 1$. Then condition (4.1) implies $\lim_{x \to 0} f(x) = 0$. From the mean value theorem, there exists $c \in (0, x)$ for any $x > 0$ such that

$$|c f'(c)| = \left| \frac{c f(x)}{x} \right| \leq |f(x)|.$$

Because we have assumed the existence of the limit, it must be zero. □

If GDC is satisfied for the null geodesic, Eqs. (3.19) and (3.20) become

$$|u| \approx y_0^{-1/2} m^{(1-\beta)/2}, \quad (4.3)$$

and

$$\sqrt{H} \approx \begin{cases} 
2\beta y_0^{3/2} m^{-3(1-\beta)/2}, & \text{for } \beta < 1 \\
\frac{2y_0}{y_0^{1/2} + 1}, & \text{for } \beta = 1 \text{ and } y_0 \neq 1
\end{cases}. \quad (4.4)$$

For $\beta < 1$, since Eq. (3.17) becomes

$$\frac{d^2 m}{d\lambda^2} = (1 - \beta) \frac{1}{m} \left( \frac{dm}{d\lambda} \right)^2, \quad (4.5)$$

we obtain

$$q = \frac{1}{\beta}. \quad (4.6)$$

Then, $1 < q < 3$ and $R \propto \lambda$ hold. Eq. (2.11) becomes
\[ R_{44} \propto \lambda^{-3+\beta^{-1}}. \]  

(4.7)

Therefore SCC is not satisfied. LFC is satisfied for \( 1/2 \leq \beta < 1 \), while LFC is not satisfied for \( 1/3 < \beta < 1/2 \).

For \( \beta = 1 \) and \( y_0 \neq 1 \), Eq. (3.17) becomes

\[ \frac{d^2 m}{d\lambda^2} = \frac{1}{2(y_0^{1/2} \pm 1)y_0^{1/2}} \frac{1}{m} \left( \frac{dm}{d\lambda} \right)^2. \]  

(4.8)

Therefore we obtain

\[ q = \frac{2(y_0^{1/2} \pm 1)y_0^{1/2}}{2(y_0^{1/2} \pm 1)y_0^{1/2} + 1}. \]  

(4.9)

Then, \( 0 < q < 1 \) and \( R \propto \lambda^q \) hold. Eq. (3.11) becomes

\[ R_{44} \propto \lambda^{-2}. \]  

(4.10)

Therefore both SCC and LFC are satisfied.

In summary, we present the following theorems:

**Theorem 3** Suppose that GDC is satisfied for a radial null geodesic which emanates from or terminates at the non-critically naked singularity. If and only if \( 1/3 < \beta < 1/2 \) is satisfied, neither SCC nor LFC holds, if and only if \( 1/2 \leq \beta < 1 \) is satisfied, not SCC but only LFC holds, and if and only if \( \beta = 1 \) is satisfied, then both SCC and LFC hold, for the radial null geodesic which emanates from or terminates at the singularity.

**Theorem 4** Suppose that GDC is satisfied for a radial null geodesic which emanates from or terminates at the non-critically naked singularity. Along the radial null geodesic,

\[ \lim_{m \to 0} \frac{R}{\lambda} \]  

is nonzero finite value or positive infinity. If and only if the limit converges, SCC does not hold, and if and only if the limit diverges, both SCC and LFC hold, for the null geodesic.

**V. CURVATURE CONDITION ALONG TIMELIKE GEODESIC**

Here we consider a timelike geodesic which terminates at the singularity. We prepare a parallely propagated tetrad \( E_i \) with \( E_1 \cdot E_1 = E_2 \cdot E_2 = E_3 \cdot E_3 = -E_4 \cdot E_4 = 1 \), all other products vanish and \( E_4 \) is equal to the tangent vector \( k^\mu \) of the timelike geodesic. We can define \( p \) and \( R_{44} \) by Eqs. (3.2) and (3.3), respectively, also for the timelike geodesic.

From Clarke and Królak (1985) and Clarke (1993), the following lemma holds.

**Lemma 2** For the timelike and null geodesic which emanate from or terminate at the singularity: SCC is satisfied if \( p \) is positive for \( \alpha = 2 \); LFC is satisfied if \( p \) is positive for \( \alpha = 1 \).

It seems to be cumbersome to examine the curvature divergence along all possible timelike geodesics. Then, we consider the simplest timelike geodesic, i.e., \( r = 0 \). It is easy to find that \( r = 0 \) is a timelike geodesic when the center is regular. As a matter of convenience, we adopt the coordinate system (2.1). Along \( r = 0 \), the \( R_{44} \) is calculated as

\[ R_{44} = 4\pi \epsilon = \frac{F'}{R^2 R'}, \]  

(5.1)

where we have used \( \Pi = 0 \) at the regular center which will be seen in Sec. VII.

We will consider the situation in which the central singularity occurs at \( t = 0 \) from the regular initial data at \( t = t_0 < 0 \). We choose the radial coordinate \( r \) as \( r = R(t_0, r) \). From regularity of the center, we obtain, for \( t_0 \leq t < 0 \),

\[ F(r) = F_3 r^3 + \cdots, \]  

(5.2)

\[ R(t, r) = R_1(t) r + \cdots, \]  

(5.3)

\[ \nu(t, r) = \nu_0(t) + \cdots, \]  

(5.4)
where “···” means the higher order terms with respect to \( r \). As we assume the positivity of the energy density at the center at \( t = t_0 \), \( F_3 > 0 \) must be satisfied. We set \( \nu_0(\tau) = 0 \) by using the scaling freedom of time coordinate. From this choice, the time coordinate \( t \) can coincide with the proper time \( \tau \) at the center. Substituting into Eq. (5.1), the value of \( R_{44} \) at the center is written as

\[
R_{44} = \frac{3F_3}{R_1^2}. \tag{5.5}
\]

Note that \( R_1 = 0 \) corresponds to the occurrence of the central singularity.

From the equation of motion of each mass shell (2.7), it is required that

\[
\frac{1}{R^2} - 1 = h_1(t)r^2 + \cdots. \tag{5.6}
\]

The lowest order of Eq. (2.7) becomes

\[
\dot{R}_1^2 = \frac{2F_3}{R_1} + h_1. \tag{5.7}
\]

Here we assume that GDC is satisfied for the timelike geodesic \( r = 0 \), where it should be noted that the value of

\[
\frac{R}{2m} \left( \frac{1}{R^2} - 1 \right)
\]

at \( r = 0 \) is understood as the limit of \( r \to 0 \). Then it is found that

\[
\lim_{t \to 0} \frac{R_1 h_1}{F_3} = 0. \tag{5.9}
\]

Hence, Eq. (5.7) becomes

\[
\dot{R}_1^2 \approx \frac{2F_3}{R_1} \tag{5.10}
\]

in the limit of \( t \to 0 \). This is integrable as

\[
R_1 \approx \left( \frac{9F_3}{2} \right)^{1/3} (-t)^{2/3} = \left( \frac{9F_3}{2} \right)^{1/3} (-\tau)^{2/3}. \tag{5.11}
\]

Eq. (5.5) becomes

\[
R_{44} \approx \frac{2}{3} \frac{1}{(-\tau)^2}. \tag{5.12}
\]

Therefore, for the timelike geodesic \( r = 0 \), both SCC and LFC are satisfied.

**Theorem 5** If GDC is satisfied for the timelike geodesic \( r = 0 \) which terminates at the singularity, then both SCC and LFC are satisfied for the timelike geodesic.

**VI. EXAMPLES**

**A. dust collapse**

The spherically symmetric dust collapse has been analyzed in the context of naked singularities by Eardley and Smarr (1979), Christodoulou (1984), Newman (1986), Joshi and Dwivedi (1993), Singh and Joshi (1996), and Jhingan, Joshi and Singh (1996). The stability of the Cauchy horizon against nonspherical perturbation was recently discussed by Iguchi, Nakao and Harada (1998) and Iguchi, Harada and Nakao (1998).

The dust fluid is given by \( h(r, R) = h(r) \). For simplicity we restrict our attention to the marginally bound collapse which is given by \( h = 1 \). It is trivial that the singularity is gravity-dominated. The space-time is given by the LTB
solution. The solution in the mass-area coordinates is given by Ori (1990) and Magli (1998). The solution contains an arbitrary function \( F(r) \). Here we choose the comoving radial coordinate \( r \) as \( r = R(t = t_0, r) \), i.e., \( R(m) = F^{-1}(m) \).

First we give the function \( F(r) \) as

\[
F(r) = F_3 r^3 + F_5 r^5 + \cdots,
\]
(6.1)

which corresponds to generic smooth initial data. For \( F_3 > 0 \) and \( F_5 < 0 \), Eq. (2.21) has a finite positive root

\[
y_0 = \left(\frac{-F_5}{4\sqrt{2} F_3^{13/6}}\right)^{2/3},
\]
(6.2)

with \( \beta = 7/9 \). From the results in Sec. [IV], not SCC but only LFC is satisfied for the radial null geodesic which emanates from the singularity.

Next, if we give \( F(r) \) as

\[
F(r) = F_3 r^3 + F_6 r^6 + \cdots,
\]
(6.3)

which corresponds to nongeneric regular initial data. For \( F_3 > 0 \) and \( F_6 < -(26\sqrt{2} + 15\sqrt{6}) F_3^{5/2} \), Eq. (2.21) has a finite positive root \( y_0 \) with \( \beta = 1 \), where \( y_0 > 1 \) is expressed using the root of some quartic equation. Then, from the results in Sec. [IV], both SCC and LFC are satisfied for the outgoing radial null geodesic which emanates from the singularity.

For the above two cases, the curvature strength is exactly the same for the ingoing radial null geodesic which terminates at the singularity.

On the other hand, both SCC and LFC are satisfied for the timelike geodesic \( r = 0 \). This fact was already shown by Deshingkar, Joshi and Dwivedi (1999). This can be confirmed by the result of Sec. V since GDC is also satisfied for the timelike geodesic. This is the case not only for marginally bound collapse but also for nonmarginally bound collapse because GDC is satisfied for the timelike geodesic.

B. cluster of counterrotating particles

The dynamical spherical cluster of counterrotating particles was introduced and analyzed by Datta (1970), Bondi (1971) and Evans (1976). The explicit solution for the metric functions was derived by Harada, Iguchi and Nakao (1998). They also examined the occurrence of naked singularity.

We again restrict our attention to the marginally bound collapse. Then, the model is given by

\[
h^2 = 1 + \frac{L^2}{R^2},
\]
(6.4)

where \( L = L(m) \) is the specific angular momentum. We give \( F(r) \) as in Eq. (6.1). If \( L(m) \) is given by \( L = 4m \), the metric functions are expressed by elementary functions. For this case, Harada, Iguchi and Nakao (1998) showed that Eq. (2.21) has a finite positive root

\[
y_0 = \left(\frac{24F_3^{2} - F_5}{4\sqrt{2} F_3^{13/6}}\right)^{2/3},
\]
(6.5)

for \( F_5 < 24F_3^2 \) with \( \beta = 7/9 \). Note that \( F_5 < 24F_3^2 \) is the same as the requirement of no shell-crossing singularity. GDC is satisfied for the null geodesic. From the results in Sec. [IV], not SCC but only LFC is satisfied for the radial null geodesic which emanates from or terminates at the singularity.

On the other hand, it is found that GDC is also satisfied for the timelike geodesic \( r = 0 \). From the result of Sec. V, both SCC and LFC are satisfied for this timelike geodesic. This is the case not only for marginally bound collapse but also for nonmarginally bound collapse because GDC is satisfied for the timelike geodesic.
C. $\Pi = k\epsilon$

We consider the equation of state

$$\Pi = k\epsilon,$$  \hspace{1cm} (6.6)

where $k$ is a constant. This will be the simplest nontrivial equation of state for tangential pressure. Singh and Witten (1997) examined the motion of a fluid with this equation of state. From Eqs. (2.6) and (2.8),

$$\nu' = 2k \frac{R'}{R}$$  \hspace{1cm} (6.7)

holds. Since regularity requires

$$\nu = \nu_0(t) + O(r^2),$$  \hspace{1cm} (6.8)
$$R = R_1(t)r + O(r^3),$$  \hspace{1cm} (6.9)

it is impossible to set regular initial data for $k \neq 0$. Therefore this model is not appropriate for a probe of CCH.

VII. CONCLUDING REMARKS

The nakedness and curvature strength of shell-focusing singularity in the spherically symmetric collapse of a fluid with vanishing radial pressure has been investigated. Along the first radial null ray from the naked singularity, $R \approx 2y_0m^\beta (1/3 < \beta \leq 1)$ is satisfied. The $y_0$ and $\beta$ are determined by some root equation. The $\beta$ is closely related to the curvature strength of the singularity for the null geodesic which emanates from or terminates at the singularity. Roughly speaking, $\beta = 1$ means SCC and vice versa.

Then, we have defined GDC for the geodesic which emanates from or terminates at the singularity. Suppose that GDC is satisfied for the null geodesic. For this class of noncritically naked singularities, if and only if $1/3 < \beta < 1/2$ is satisfied, neither SCC nor LFC holds, if and only if $1/2 < \beta < 1$ is satisfied, not SCC but only LFC holds, and if and only if $\beta = 1$ is satisfied, both SCC and LFC hold. Furthermore, for this class of noncritically naked singularities, if and only if $\lim_{m \to 0} \lambda^{-1}R$ diverges, SCC is satisfied.

We also have examined whether or not the curvature divergence condition is satisfied for a timelike geodesic. Suppose that GDC is satisfied for the timelike geodesic $r = 0$ which terminates at the singularity. Then, we have found that both SCC and LFC are satisfied for the timelike geodesic.

We have applied this formalism to the dust collapse and the collapse of counterrotating particles. It is noted that, with vanishing radial pressure, only if the ratio of the tangential pressure to the energy density vanishes at the center, it is possible to set regular initial data which is important ingredient when we consider physical situations.

Even if we include the tangential pressure, nakedness and curvature strength of the singularity are very similar to those of the dust model if the singularity is gravity-dominated. On the other hand, if the singularity is not gravity-dominated, then we may expect that the tangential pressure plays a crucial role in the nakedness of the singularity and the extendibility of the space-time beyond the singularity.

ACKNOWLEDGMENTS

We are grateful to T. Nakamura and M. Siino for helpful discussions. We are also grateful to H. Sato for his continuous encouragement. This work was partly supported by the Grant-in-Aid for Scientific Research (No. 9204) and for Creative Basic Research (No. 09NP0801) from the Japanese Ministry of Education, Science, Sports and Culture.

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