A CURIOUS RESULT ON BREAKING TIES AMONG SAMPLE MEDIANs

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Abstract. It is well known that any sample median value (not necessarily unique) minimizes the empirical $L^1$ loss. Interestingly, we show that the minimizer of the $L^{1+\epsilon}$ loss exhibits a singular phenomenon that provides a unique definition for the sample median as $\epsilon \to 0$. This definition is the unique point among all candidate median values that balances the logarithmic moment of the empirical distribution. The result generalizes directly to breaking ties among sample quantiles.

1. Introduction

Given an empirical distribution $F_n(x)$, it is well known that its mean is the unique minimizer of the empirical squared loss $E_n(\theta - X)^2 = \int_{-\infty}^{\theta} (\theta - x)^2 dF_n + \int_{\theta}^{\infty} (x - \theta)^2 dF_n$. This follows from the first order condition

$$\int_{-\infty}^{\theta} (\theta - x)dF_n = \int_{\theta}^{\infty} (x - \theta)dF_n,$$

which can be seen as finding the point $\theta$ that balances the first moment of the distribution.

It is also well known that the median of an empirical distribution need not be unique, but can take on an interval of values if $n$ is even. If it is the absolute loss $E_n|\theta - X| = \int_{-\infty}^{\theta} (\theta - x)dF_n + \int_{\theta}^{\infty} (x - \theta)dF_n$ that one is interested in minimizing, then any median value satisfying $F_n(\theta) = 1/2$ is a solution to the first order condition

$$(1.1) \int_{-\infty}^{\theta} dF_n = \int_{\theta}^{\infty} dF_n,$$

which seeks any point that balances the zero-th moment of the distribution. Informally, the non-uniqueness of the median can be attributed to the fact that merely balancing the zero-th moment does not provide enough “discriminative” power, while balancing the first moment does.

In order to report a unique sample median, some method of breaking ties among candidate median values is necessary. Textbook treatments and software packages typically define the sample median as the midpoint of the interval [2]. Equivalent problems emerge in the calculations of sample quantiles in general. A variety of alternative estimators based on interpolation, linear combinations of order statistics, or smoothing-type approaches [1, 3, 4, 5, 6] have been proposed, typically under...
the assumption of IID samples from a population with a uniquely defined quantile (e.g., when the population distribution is continuous).

In this note, we show that balancing an ever so slightly higher order moment than the zero-th one leads to a way to tiebreak among the sample medians. Recalling that \( \log x \) is asymptotically dominated by \( x^p \) for any \( p > 0 \), consider balancing the logarithmic moment:

\[
\int_{-\infty}^{\theta} \log(\theta - x) dF_n = \int_{\theta}^{\infty} \log(x - \theta) dF_n.
\]

We show that this is equivalent to the minimization of \( E_n |\theta - X|^{1+\epsilon} \) in the limit \( \epsilon \downarrow 0 \):

\[\text{The unique minimizer of } E_n |\theta - X|^{1+\epsilon} \text{ converges to a candidate value for the median as } \epsilon \downarrow 0. \]

If there are multiple candidate values, then the one that balances (1.2) is the unique limit. This singular behaviour of the first order condition converging to (1.2) rather than (1.1) gives rise to an interesting way for defining the median uniquely. The same idea generalizes directly to defining unique sample quantiles \( q_\alpha \).

2. Result

Given \( \alpha \in (0, 1) \), we define a modified version of the weighted absolute loss for quantile regression as

\[
L_{\alpha, \epsilon}(x, q) = \begin{cases} 
(1 - \alpha)(q - x)^{1+\epsilon} & q > x, \\
\alpha(x - q)^{1+\epsilon} & x \geq q.
\end{cases}
\]

If \( \epsilon = 0 \) then we have the usual loss used in quantile regression, whose expectation with respect to an empirical distribution \( F_n(x) \) is minimized by any \( \alpha \)-quantile \( q_\alpha \) satisfying \( F_n(q_\alpha) = \alpha \). The median naturally corresponds to the case where \( \alpha = 1/2 \).

The expectation of \( L_{\alpha, \epsilon}(x, q) \) is

\[
E_n L_{\alpha, \epsilon}(x, q) = (1-\alpha) \int_{-\infty}^{q} (q - x)^{1+\epsilon} dF_n + \alpha \int_{q}^{\infty} (x - q)^{1+\epsilon} dF_n,
\]

and its derivative at \( q \) is

\[
(1 - \alpha) \int_{-\infty}^{q} (q - x)^{\epsilon} dF_n - \alpha \int_{q}^{\infty} (x - q)^{\epsilon} dF_n
\]

up to a constant \( 1 + \epsilon \).

When \( \epsilon > 0 \), \( E_n L_{\alpha, \epsilon}(x, q) \) has a unique minimizer because it is strongly convex in \( q \) (the integrals are sums). The minimizer balances the weighted \( \epsilon \)-th order moment in (2.3). Whereas for \( \epsilon = 0 \) the zero-th order moment is balanced by possibly many values. Lemma 1 below shows that the minimization of (2.1) as \( \epsilon \downarrow 0 \) is qualitatively very different from the \( \epsilon = 0 \) case.

**Lemma 1.** (Properties of the minimizer of \( E_n L_{\alpha, \epsilon}(x, q) \) as \( \epsilon \downarrow 0 \))

(i) Suppose there exists a unique \( \alpha \)-quantile \( q_\alpha \), i.e. \( F_n(q_\alpha -) < \alpha \) and \( F_n(q_\alpha) > \alpha \). Then \( q_\alpha \) minimizes \( E_n L_{\alpha, \epsilon}(x, q) \) for all sufficiently small \( \epsilon \).

(ii) If no unique \( \alpha \)-quantile exists, then \( F_n(q) = \alpha \) in some interval \([q^L_\alpha, q^H_\alpha]\). In the interior of this interval, the unique solution that balances the weighted log-moment

\[
(1 - \alpha) \int_{-\infty}^{q} \log(q - x) dF_n - \alpha \int_{q}^{\infty} \log(x - q) dF_n = 0.
\]
is the limit of the minimizer of $\mathbb{E}_n L_{\alpha, x}(x, q)$ as $\epsilon \downarrow 0$, i.e. $\lim_{\epsilon \downarrow 0} \arg \min_q \mathbb{E}_n L_{\alpha, x}(x, q)$.

The intuition for the result is simple but elegant: Perturbing $\epsilon$ about 0 yields approximations for the terms

$$
\int_{-\infty}^{q} (q - x)^\epsilon dF_n \approx F_n(q) + \epsilon \int_{-\infty}^{q} \log(q - x) dF_n,
$$

$$
\int_{q}^{\infty} (x - q)^\epsilon dF_n \approx 1 - F_n(q) + \epsilon \int_{q}^{\infty} \log(x - q) dF_n,
$$

hence the first order condition obtained from setting the derivative (2.3) to zero is

$$
F_n(q) - \alpha + \epsilon \left\{ (1 - \alpha) \int_{-\infty}^{q} \log(q - x) dF_n - \alpha \int_{q}^{\infty} \log(x - q) dF_n \right\} \approx 0.
$$

The dominant term above is $F_n(q) - \alpha$, so the limiting minimizer has to be an $\alpha$-quantile. In case (i) this is the unique $q_\alpha$. For case (ii), among $q \in [q_\alpha^L, q_\alpha^H)$, the term in the curly brackets also matter now, thus giving rise to the logarithmic moment condition \((2.4)\).

**Proof.** For case (i) where there is a unique $\alpha$-quantile $q_\alpha$, set $q = q_\alpha -$ and use Taylor’s theorem to obtain

$$
\int_{-\infty}^{q} (q - x)^\epsilon dF_n = F_n(q_\alpha -) + \mathcal{O}(\epsilon),
$$

$$
\int_{q}^{\infty} (x - q)^\epsilon dF_n = 1 - F_n(q_\alpha -) + \mathcal{O}(\epsilon).
$$

The derivative (2.3) at $q = q_\alpha -$ is then $F_n(q_\alpha -) - \alpha + \mathcal{O}(\epsilon) < 0$ for $\epsilon$ small enough. Likewise, the derivative at $q = q_\alpha +$ is $F_n(q_\alpha) - \alpha + \mathcal{O}(\epsilon) > 0$ for $\epsilon$ small enough. Hence $q_\alpha$ is the minimizer.

For case (ii), note that $F_n(x)$ has atoms at $x = q_\alpha^L, q_\alpha^H$ but none in $(q_\alpha^L, q_\alpha^H)$. Hence within this interval, the first integral in (2.4) is increasing in $q$ while the second one is decreasing. Moreover the left hand side of (2.4) approaches $-\infty$ as $q \downarrow q_\alpha^L$, and approaches $+\infty$ as $q \uparrow q_\alpha^H$. Hence (2.4) has a unique solution in $(q_\alpha^L, q_\alpha^H)$. Within this interval, applying Taylor’s theorem shows that

$$
\int_{-\infty}^{q} (q - x)^\epsilon dF_n = \alpha + \epsilon \int_{-\infty}^{q} \log(q - x) dF_n + \mathcal{O}(\epsilon^2),
$$

$$
\int_{q}^{\infty} (x - q)^\epsilon dF_n = 1 - \alpha + \epsilon \int_{q}^{\infty} \log(x - q) dF_n + \mathcal{O}(\epsilon^2),
$$

and so the first order condition (setting the derivative (2.3) to zero) is

$$
(1 - \alpha) \int_{-\infty}^{q} \log(q - x) dF_n - \alpha \int_{q}^{\infty} \log(x - q) dF_n = \mathcal{O}(\epsilon).
$$

If $q$ does not satisfy (2.4), then the left hand side above is $\mathcal{O}(1)$ in a neighbourhood of $q$ that does not contain the solution to (2.4). Hence the minimizer will eventually be outside of the neighbourhood as $\epsilon \downarrow 0$. \hfill \Box
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