On Spin $L$-Functions for $GSO_{10}$

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Abstract

In this paper we construct a Rankin-Selberg integral which represents the Spin$_{10} \times$ St $L$-function attached to the group $GSO_{10} \times PGL_2$. We use this integral representation to give some equivalent conditions for a generic cuspidal representation on $GSO_{10}$ to be a functorial lift from the group $G_2 \times PGL_2$.

1 Introduction

In this paper we construct a new Rankin-Selberg integral which represents an $L$ function corresponding to the group $GSO_{10} \times PGL_2$. More precisely, let $\pi$ denote a generic cuspidal representation defined on the group $GSO_{10}(A)$, and let $\tau$ denote a cuspidal representation defined on $PGL_2(A)$. For simplicity, we shall assume that $\pi$ has a trivial central character. The $L$ group of $GSO_{10}(A) \times PGL_2(A)$ is the group $GSpin_{10}(C) \times SL_2(C)$. Consider the 32 dimensional irreducible representation of this group given by Spin$_{10} \times St$. Here Spin$_{10}$ is the 16 dimensional irreducible Spin representation of GSpin$_{10}(C)$ and St denotes the standard representation of the group SL$_2(C)$. To this irreducible representation, one can attach the 32 degree partial $L$ function denoted by $L^S(Spin_{10} \times St, \pi \times \tau, s)$.

To study this $L$ function we construct a global integral which is given by

$$\int_{Z(A)GSO_{10}(F) \backslash GSO_{10}(A)} \varphi_\pi(g)\theta_\tau(g)E(g,s)dg. \quad (1)$$

Here $\varphi_\pi$ is a vector in the space of the representation $\pi$, $Z$ is the center of $GSO_{10}$ and $E(g,s)$ is an Eisenstein series which described in the beginning of section 2. The interesting representation in integral (1) is the representation $\theta_\tau$. This representation
is constructed as a residue of an Eisenstein series defined on the group $GSO_{10}(\mathbb{A})$ as described fully in section 3. The main two properties of this representation are first, its dependence on the cuspidal representation $\tau$, and second, its smallness. In fact, as we prove in section 3, this representation is attached to the unipotent orbit $(3^31)$.

After showing that this integral is Eulerian, and that we obtain the above $L$ function, we then study the poles of this $L$ function. We show that it can have at most a simple pole at $s = 1$. It is well known, see [K], that the stabilizer of a generic point in the space of the representation $Spin_{10}(\mathbb{C}) \times SL_2(\mathbb{C})$, is the group $G_2(\mathbb{C}) \times SL_2(\mathbb{C})$. Hence one expects that the above $L$ function will have a simple pole at $s = 1$ if and only if the cuspidal representation $\pi$ is the functorial lift from $G_2 \times PGL_2$. In fact this is exactly what we prove. Indeed, the main result of this paper is given by the following theorem, which is stated and proved in section 6. (See section 6 for precise notations).

Main Theorem: Let $\pi$ be an irreducible generic cuspidal representation of the group $GSO_{10}(\mathbb{A})$ which has a trivial central character. Then the following are equivalent:

1) The partial $L$ function $L^S(Spin_{10} \times St, \pi \times \tau, 2s - 1/2)$ has a simple pole at $s = 3/4$.

2) The period integral

$$\int_{SO_{10}(F) \backslash SO_{10}(\mathbb{A})} \varphi_\pi(g)\theta_\tau(g)\theta(g)dg$$

is nonzero for some choice of data.

3) There exists a generic cuspidal representation $\sigma$ of the exceptional group $G_2(\mathbb{A})$ such that $\pi$ is the weak lift from the representation $\sigma \times \tau$ of the group $G_2(\mathbb{A}) \times PGL_2(\mathbb{A})$.

We now describe the content of the paper. In section two we introduce the global integral we consider, and show that it is Eulerian. We try to do it in an abstract way. Indeed, we assume the existence of a representation $\theta$ defined on the group $GSO_{10}(\mathbb{A})$ which is attached to the unipotent orbit $(3^31)$. Using that data, we prove in theorem 1, that the global integral is Eulerian. This method has the advantage that it will work for any representation which will satisfy the properties listed in the theorem. This means, that for any such representation the integral will be Eulerian. In section 3 we construct an example of such a representation, which we denote by $\theta_\tau$, by means of a residue of an Eisenstein series. We are well aware of the existence of other such representations as well.

Section 4 is devoted to the unramified computations. The local integral and $L$-function are each expressed as a power series in $q^{-2s+1/2}$, $\chi$, and $\chi^{-1}$, where $\chi$ is the character from which $\tau$ is induced. The coefficients are traces of irreducible representations of $Spin_{10}(\mathbb{C})$. The power series are simplified by multiplying them by a certain polynomial. The desired equality is equivalent to a formula for tensor products of representations of “rectangular shape” which is due to Okada.

Sections 5 and 6 are devoted to the main theorem. The idea of the proof is as follows. The hard part in the theorem is to prove that part 2 implies part 3. To do that we first show that the cuspidal representation $\pi$ is an endoscopic lifting from a cuspidal representation $\rho \times \tau$ defined on the group $Sp_6(\mathbb{A}) \times SL_2(\mathbb{A})$. This part is not trivial. To do it we need to use a new construction of a lifting which was announced in [G2] and is a work in progress in [G3]. In section 5 we give the precise details we need.
We don’t give all the proofs, they will appear in [G3], but we give enough details for the reader to get the whole picture. Then, assuming part two of the above theorem, we show that the representation $\epsilon$ defined on $Sp_6(\mathbb{A})$ is actually a functorial lift from a generic representation $\sigma$ defined on the exceptional group $G_2(\mathbb{A})$.

Finally, we wish to remark that the representation $\theta_r$ seems to occur in other constructions of Rankin-Selberg integrals. Recently, we constructed some other global integrals, using this representation, which we proved to be Eulerian. It is our full intention to try to find the $L$ functions they represent.

2 A Global Integral

Let $G$ denote the similitude orthogonal group $GSO_{10}$. Let $\pi$ denote a generic irreducible cuspidal representation on the group $G(\mathbb{A})$. We shall assume that it has a trivial central character. Let $P$ denote the standard maximal parabolic subgroup of $G$ whose Levi part is $GL_1 \times GL_5$ which contains the standard Borel subgroup consisting of upper unipotent matrices. We shall denote by $U(P)$ the unipotent radical subgroup of $P$. Let $E(g, s)$ denote the Eisenstein series defined on the group $G(\mathbb{A})$ which is associated with the induced representation $Ind_{P(\mathbb{A})}^{G(\mathbb{A})} \delta_P$.

Let $\theta$ denote any automorphic representation defined on the group $G(\mathbb{A})$. Assume that it has a trivial central character. For a cusp form $\varphi_\pi$ in the space of $\pi$ we consider the integral

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \varphi_\pi(g)\theta(g)E(g, s)dg$$

(3)

Here $Z$ is the center of the group $G$. We are mainly interested in understanding what conditions we need to impose on the representation $\theta$ so that integral (3) will be Eulerian with the Whittaker function defined on the representation $\pi$.

In terms of matrices we consider the group $G$ relative to the form defined by the matrix $J_{10}$. Here and elsewhere, the matrix $J_n$ is the $n \times n$ matrix with ones on the other diagonal and zeros elsewhere. For $1 \leq i \leq 5$, let $\alpha_i$ denote the five simple roots of the group $G$. Let $x_{\alpha_i}(r)$ denote the one dimensional unipotent subgroup corresponding to the root $\alpha_i$. We label the roots such that

$$x_{\alpha_1}(r) = I + re'_{1,2} \quad x_{\alpha_2}(r) = I + re'_{2,3} \quad x_{\alpha_3}(r) = I + re'_{3,4} \quad x_{\alpha_4}(r) = I + re'_{4,5} \quad x_{\alpha_5}(r) = I + re'_{4,6}$$

Here $I$ is the $10 \times 10$ identity matrix and $e'_{i,j} = e_{i,j} - e_{11-j,11-i}$. For $1 \leq i \leq 5$ let $w[i]$ denote the simple reflection corresponding to the simple root $\alpha_i$. We shall write $w[i_1 i_2 \ldots i_r]$ for $w[i_1]w[i_2] \ldots w[i_r]$.

Let $\psi$ denote a nontrivial character on the group $F \backslash \mathbb{A}$. Let $U$ denote the maximal unipotent subgroup of $G$ which consists of upper triangular matrices. We define two characters on the group $U$. For an automorphic form $\phi$ defined on $G(\mathbb{A})$ we define its Whittaker model as

$$W_\phi(g) = \int_{U(F)\backslash U(\mathbb{A})} \phi(ug)\psi_U^{-1}(u)du$$
Here, for \( u = (u_{i,j}) \in U \), we define \( \psi'_U(u) = \psi(u_{1,2} + u_{2,3} + u_{3,4} + u_{4,5} + u_{4,6}) \). Similarly, we define

\[
\phi^{U,\psi_U}(g) = \int_{U(F) \backslash U(A)} \phi(ug)\psi_U(u)du
\]

where now \( \psi_U(u) = \psi(u_{1,2} + u_{2,3} + u_{4,5} + u_{4,6}) \).

In \([G-R-S1]\) it is explained how to associate with a unipotent class of a classical group a set of Fourier coefficients. We also use the notation \( O \) as explained there. Roughly speaking, if \( \sigma \) is an automorphic representation of the group \( G \) one defines \( O_G(\sigma) \) as follows. It is defined to be the set of all unipotent classes of \( G \) such that for all \( O' \) with the property that if \( O' \) is greater or not related to a member in \( O_G(\sigma) \), then \( \sigma \) has no nontrivial Fourier coefficient corresponding to the unipotent class \( O' \). Also, the representation \( \sigma \) has a nonzero Fourier coefficient corresponding to any unipotent class \( O \) in \( O_G(\sigma) \).

We are now ready to prove the following

**Theorem 1**: We keep the above notations. Let \( \theta \) be an automorphic representation of the group \( G(A) \) which satisfies:

1) \( O_G(\theta) = (3^31) \).
2) The integral \( \theta^{U,\psi_U}(g) \) is not zero for some choice of data.

Then, integral \( (3) \) is Eulerian and for \( Re(s) \) large it equals

\[
\int_{Z(A)U(A) \backslash G(A)} \int_{\mathbb{A}^2} W_\pi(g)\theta^{U,\psi_U}(g)f_s(w[53]x_{\alpha_3}(r_1)x_{\alpha_3+\alpha_5}(r_2)g)dr_1dr_2dg
\]

\[
(4)
\]

**Proof**: For \( Re(s) \) large we unfold integral \( (3) \) and we obtain

\[
\int_{Z(A)P(F) \backslash G(A)} \varphi_\pi(g)\theta(g)f_s(g)dg
\]

\[
(5)
\]

Next we expand \( \theta(g) \) along the unipotent radical of \( P \). The Levi part of \( P \) acts on \( U(P) \) with three orbits. Indeed, we may identify \( U(P) \) with all matrices in \( G \) of the form \( \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \). Thus we may parameterize the orbits by the rank of the matrices \( X \), which must be even. It follows from the cuspidality of \( \pi \) that the rank zero and rank two orbits contribute zero to the integral. Denote

\[
\theta^{U(P),\psi}(g) = \int_{X(F) \backslash X(A)} \theta \left( \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \right) \psi(x_{2,1} + x_{3,2})dX.
\]

The stabilizer of this character inside the Levi part of \( P \) is the group \( GSp_4 \). Thus \( (5) \) equals

\[
\int_{Z(A)GSp_4(F)U(P)(F) \backslash G(A)} \varphi_\pi(g)\theta^{U(P),\psi}(g)f_s(g)dg
\]

\[
(6)
\]

Consider the unipotent group \( L \) which is generated by the matrices \( I_{10} + r e_{1,2}^i; \ I_{10} + r e_{1,3}; \ I_{10} + r e_{1,4}; \ I_{10} + r e_{1,5}. \) We expand \( \theta^{U(P),\psi}(g) \) along the group \( L(F) \backslash L(A) \).
The group $GSp_4$ acts on $L$ with two orbits. By the cuspidality of $\pi$, the trivial orbit contributes zero to the integral. Denote $V = L \cdot U(P)$, and define a character $\psi$ of $V$ as follows. For $v \in V$ write $v = lu$ with $l = (l_{i,j}) \in L$ and $u = \begin{pmatrix} I & X \\ I & I \end{pmatrix} \in U(P)$. We define $\psi(v) = \psi(l_{1,2} + x_{2,1} + x_{3,2})$. Thus we are left with the open orbit and hence integral \(7\) equals

$$
\int_{Z(A)R(F)V(A)\backslash G(A)} \varphi_{V,\psi}^{-1}(g)\theta_{V,\psi}(g)f_s(g)dg
$$

where

$$
\theta_{V,\psi}(g) = \int_{V(F)\backslash V(A)} \theta(vg)\psi(v)dv
$$

and similarly we define $\varphi_{V,\psi}^{-1}(g)$. Also, the group $R = GL_2 \cdot Y$ where

$$
Y = \{(I_{10} + y_1 e_{2,3})(I_{10} - y_1 e_{2,5})(I_{10} + y_2 e_{2,4})(I_{10} + y_2 e_{3,5})(I_{10} + y_3 e_{2,5})\}
$$

The group $GL_2$ is embedded in $G$ as all matrices of the form $\text{diag}(|h|, |h|, 1, |h|, h, 1, 1)$ where $h \in GL_2$ and $|h| = \text{deth}$.

Next we consider the Fourier expansion

$$
\theta_{V,\psi}(g) = \int_{F \backslash A} \theta_{V,\psi}((I_{10} + y_3 e_{2,5})g)dy_3 + \sum_{\alpha \in F \backslash A} \int_{F \backslash A} \theta_{V,\psi}((I_{10} + y_3 e_{2,5})g)\psi(\alpha y_3)dy_3
$$

We claim that each summand in the summation over $\alpha$ on the right hand side is zero. Indeed, since $\alpha \neq 0$ it follows that the integration over $y_3$ together with the integration over $V$, produces a Fourier coefficient which corresponds to the unipotent class $(52^11)$. By assumption 1) in the theorem it follows that these Fourier coefficients are all zero. Thus we are left only with the constant term.

Denote $w = w[35]$. Let $Y_c = I_{10} + y_3 e_{2,5}$ and denote $V_1 = wV_c w^{-1}$. To describe the group $V_1$ in term of matrices, let $U_1$ denote the standard unipotent radical of the maximal parabolic subgroup of $G$ whose Levi part is $GL_1 \times GSO_8$. We consider the character $\psi_U$ as a character of the group $U_1$ by restriction. Next we consider the unipotent subgroup of $U$ defined by all matrices of the form

$$
l(r_1, r_2, r_3, r_4, r_5, r_6, r_7) = I_{10} + r_1 e_{2,3} + r_2 e_{2,4} + r_3 e_{2,5} + r_4 e_{2,6} + r_5 e_{2,7} + r_6 e_{2,8} + r_7 e_{4,5}
$$

Finally, we consider the group of all unipotent matrices of the form $z(m_1, m_2) = I_{10} + m_1 e_{4,3} + m_2 e_{6,3}$. A matrix multiplication implies that we have the factorization $v_1 = u_1 l(r_1, 0, r_3, 0, r_5, r_6, r_7) z(m_1, m_2)$ where $v_1 \in V_1$ and $u_1 \in U_1$. We also consider the character $\psi_U$ as a character of the group $V_1$ by restriction. Thus, for the above factorization, we have $\psi_U(v_1) = \psi_U(u_1)\psi(r_1 + r_7)$. From all this we obtain that \(7\) equals

$$
\int_{Z(A)GL_2(F)Y(F)\backslash G(A)\backslash V(A)\backslash G(A)} \varphi_{V,\psi}^{-1}(wg)\theta_{V,\psi}(wg)f_s(g)dg
$$

\(8\)
We proceed with the following Fourier expansion

$$\theta^{V_1, \psi_U}(wg) = \sum_{\delta_i \in F(F \setminus A)^2} \int \theta^{V_1, \psi_U}(l(0, r_2, 0, r_4, 0, 0, 0)wg)\psi(\delta_1 r_2 + \delta_2 r_4)dr_2dr_4$$

Collapsing summation with integration this also equals

$$\int \theta^{V_2, \psi_U}(z(m_1, m_2)wg)dm_1dm_2$$

Here $V_2$ is the unipotent subgroup of $U$ generated by the group $U_1$ and by the unipotent group consisting of all matrices $l(r_1, r_2, r_3, r_4, r_5, r_6, r_7)$. We view $\psi_U$ as a character of $V_2$ by restriction. Performing the same expansion for $\varphi_\pi$, integral (8) equals

$$\int \int \varphi_\pi^{V_2, \psi_U^{-1}}(z(m_3, m_4)wg)\theta^{V_2, \psi_U}(z(m_1, m_2)wg)f_s(g)dm_1dg \tag{9}$$

where the $g$ integration is as in (8). Next we expand $\theta^{V_2, \psi_U}$ along the unipotent subgroup of $U$ generated by all matrices of the form $t(r_1, r_2, r_3, r_4) = I_{10} + r_1e'_{3,4} + r_2e'_{3,5} + r_3e'_{3,6} + r_4e'_{3,7}$. It is not hard to check that the only nonzero contribution to the expansion comes from the constant term. Indeed, all other terms in the expansion will produce Fourier coefficients which correspond to unipotent classes which are greater than $(3^31)$. By assumption 1) these Fourier coefficients are zero. Let $V_3$ denote the unipotent subgroup of $U$ generated by $V_2$ and all matrices of the form $t(r_1, r_2, r_3, r_4)$. The above discussion implies that $\theta^{V_2, \psi_U} = \theta^{V_3, \psi_U}$. Factoring the integration over the group $Y$, integral (9) equals

$$\int \int \int \varphi_\pi^{V_2, \psi_U^{-1}}(t(0, r_2, 0, r_4)z(m_3, m_4)wg)\theta^{V_3, \psi_U}(z(m_1, m_2)wg)f_s(g)dr_2dm_3dg \tag{10}$$

Here the $g$ variable is integrated over $Z(A)GL_2(F)Y(A)V(A)\setminus G(A)$.

We now expand the function $\varphi_\pi^{V_2, \psi_U^{-1}}$ in the above integral along the matrices $t(r_1, 0, r_3, 0)$ with points in $F \setminus A$. The group $GL_2$ acts on this expansion with two orbits. The trivial one contributes zero to the integral because of the cuspidality of $\pi$. Thus (10) equals

$$\int \int \int \varphi_\pi^{V_3, \psi_U^{-1}}(z(m_3, m_4)wg)\theta^{V_3, \psi_U}(z(m_1, m_2)wg)f_s(g)dm_3dg \tag{11}$$

Here $N'$ is the maximal unipotent subgroup of $GL_2$ which, when embedded inside $G$, is a subgroup of $U$. The group $GL_1$ consists of all diagonal matrices inside the group $G$ of the form $\text{diag}(a, a, a, 1, 1, a, a, 1, 1, 1)$. The character $\psi_U'$, which was defined before the theorem, is viewed as a character of $V_3$ by restriction. Next we factor the integration in (11) over the group $N$. The group $GL_1$ acts on this group with two orbits. The
trivial one contributes zero by cuspidality. Thus, (11) equals
\[
\int_{Z(\mathbf{A})U_0(\mathbf{A}) \backslash G(\mathbf{A})} \int_{\mathbf{A}^4} W_\pi(z(m_3, m_4)wg)\theta^{U,\psi_U}(z(m_1, m_2)wg) f_s(g) dm_j dg
\]  
(12)

Here \(U_0\) is the group generated by \(V, Y\) and \(N\). Next we conjugate the matrix \(z(m_3, m_4)\) across \(w\) and collapse summation with integration. We also change variables \(g \mapsto w^{-1}g\). Thus integral (12) equals
\[
\int_{Z(\mathbf{A})U_1(\mathbf{A}) \backslash G(\mathbf{A})} \int_{\mathbf{A}^2} W_\pi(g)\theta^{U,\psi_U}(z(m_1, m_2)wg) f_s(w[3]g) dm_j dg
\]  
(13)

where \(U_1\) is the subgroup of \(U\) defined as follows. Let \(U'_0\) be the subgroup of \(U_0\) where we omit the one dimensional unipotent subgroups corresponding to the roots \(\alpha_5\) and \(\alpha_3 + \alpha_5\). Then \(U_1 = w^{-1}U'_0w\). Factoring the integration \(U_1(\mathbf{A}) \backslash U(\mathbf{A})\), we obtain integral [4].

3 A Construction of a Small Representation

In this section we construct a representation of the group \(G(\mathbf{A})\) and show that it satisfies the assumptions stated in Theorem 1. This representation will depend on a choice of a cuspidal representation of \(GL_2(\mathbf{A})\).

Let \(\tau\) denote an irreducible cuspidal representation of the group \(GL_2(\mathbf{A})\). We will assume that it has a trivial central character. Let \(\sigma(\tau)\) denote the symmetric square lift of \(\tau\) to \(GL_3(\mathbf{A})\). This lift was constructed by Gelbart and Jacquet in [G-J]. Let \(\mu(\tau) = \tau \otimes \tau\) denote the tensor product representation of \(GSO_4(\mathbf{A})\). Denote by \(Q\) the standard maximal parabolic subgroup of \(G\) whose Levi part is \(GL_3 \times GSO_4\). We shall denote by \(U(Q)\) its unipotent radical. Let \(E_\tau(g, s)\) denote the Eisenstein series defined on the group \(G(\mathbf{A})\) which is associated to the induced representation \(\text{Ind}_{Q(\mathbf{A})}^{G(\mathbf{A})} (\sigma(\tau) \otimes \mu(\tau))\delta^s_Q\).

From the Langlands theory the poles of this Eisenstein series are determined by the poles of the constant terms. Since we induce from cuspidal data, it follows that we only need to consider the constant term along \(U(Q)\). This is easily computed and hence the poles of \(E_\tau(g, s)\) are determined by
\[
\frac{L^S(\sigma(\tau) \times \mu(\tau), 6s - 3) L^S(\sigma(\tau), 12s - 6)}{L^S(\sigma(\tau) \times \mu(\tau), 6s - 2) L^S(\sigma(\tau), 12s - 5)}
\]
where \(S\) is a finite set of places, including the archimedean ones, such that outside of \(S\) all data is unramified. The above \(L\) function is equal to
\[
\frac{L^S(\sigma(\tau) \times \sigma(\tau), 6s - 3) L^S(\sigma(\tau), 6s - 3) L^S(\sigma(\tau), 12s - 6)}{L^S(\sigma(\tau) \times \sigma(\tau), 6s - 2) L^S(\sigma(\tau), 6s - 2) L^S(\sigma(\tau), 12s - 5)}
\]
from which we deduce that the Eisenstein series has a simple pole at \(s = 2/3\). We denote \(\theta_\tau(g) = \text{Res}_{s=2/3} E_\tau(g, s)\). In the rest of this section we will show that this representation does satisfy the assumptions of Theorem 1.
Since $\sigma(\tau)$ and $\mu(\tau)$ are both generic, the integral $\theta^{L,\psi\psi}(g)$, which was defined at the beginning of section 2, is not zero for some choice of data. Since this integration corresponds to a unipotent orbit of the type $(3^31)$, all we need to verify is that $\theta_\tau$ has no nonzero Fourier coefficient which corresponds to any unipotent class which is greater than or not related to $(3^31)$.

To do that we use the same method as in [G-R-S2] section two. We start by studying the unramified local representation corresponding to $\theta_\tau$. Let $F$ be a local nonarchimedean field where $\tau$ is unramified. Assume that $\tau$ is a constituent of the induced representation $Ind_{B_2}^{GL_2} \chi^{31/2}_{B_2}$ where $\chi$ is an unramified character of $F^*$ and $B_2$ is the Borel subgroup of $GL_2$. By the definition of the symmetric square lift, the unramified local representation corresponding to $\sigma(\tau)$ is $Ind_{B_3}^{GL_3} \chi^{31/2}_{B_3}$. Here $B_3$ is the Borel subgroup of $GL_3$, and $\chi_1$ is defined as $\chi_1(diag(a,b,c)) = \chi^2(ac^{-1})$. In a similar way we have $\mu(\tau) = Ind_{B_4}^{GSO_4} \chi^{31/2}_{B_4}$. Here $B_4$ is the Borel subgroup of $GSO_4$ and $\chi_2(diag(abr, ar, a^{-1}, a^{-1}b^{-1})) = \chi^2(ab)\chi(r)$. Here the factor $\chi(r)$ occurs because we assume that $\mu(\tau)$ has a trivial central character.

From all this we deduce

**Lemma 2:** Let $\theta'_\tau$ denote the unramified constituent of $\theta_\tau$ at a nonarchimedean place. Then $\theta'_\tau$ is a sub-quotient of $Ind_{P}^{G} \chi^{31/2}_{Q}$. Here, for all $g \in GL_3$ and $h \in GSO_4$ we define $\chi_3((g,h)) = \chi^2(detg)\chi^3(\lambda(h))$ where $\lambda(h)$ denotes the similitude factor of the matrix $h$.

**Proof:** Let $B$ denote the Borel subgroup of $G$. The unramified representation $\theta'_\tau$ is a constituent of the induced representation $Ind_{B}^{G} \chi^{31/2}_{B}$ where $\chi_4(t) = \chi^2(a_1a_3^{-1}a_4)\chi(r)$ and $t = diag(ra_1, ra_2, ra_3, ra_4, r, 1, a_4^{-1}, a_3^{-1}, a_2^{-1}, a_1^{-1})$. Let $w_0$ denote the Weyl element of $G$ which has a one at the entries

$$(1, 1); (2, 4); (3, 8); (4, 2); (5, 6); (6, 5); (7, 9); (8, 3); (9, 7); (10, 10)$$

and zero elsewhere. Then $Ind_{B}^{G} \chi^{31/2}_{B}$ is isomorphic to $Ind_{B}^{G} \chi^{w_0}_{B}$ where $\chi^{w_0}(t) = \chi_4((w_0^{-1})t)$. This can be written as $Ind_{B}^{G}(\chi_5^{31/2}_{B}, \chi^{31/2}_{B})/^{1/2}_{Q} \delta^{1/2}_{Q}$ where $\chi_5(t) = \chi^2(a_1a_2a_3)$ and $\chi_6(t) = \chi^3(r)$. From this it follows that $\theta'_\tau$ is a sub-quotient of $Ind_{P}^{G} \chi^{31/2}_{Q}$.

We return to the global situation. We need to prove that $\theta_\tau$ has no nonzero Fourier coefficient which corresponds to any unipotent class which is greater than or not related to $(3^31)$. As mentioned in section two, in [G-R-S1] it is explained how to associate with a unipotent class of a group $G$ a set of Fourier coefficients. It is clear that $\theta_\tau$ is not generic. Hence it has no nonzero Fourier coefficient with respect to the unipotent class (91). Arguing as in [G-R-S1] lemma 2.6 we deduce that if $\theta_\tau$ has a nonzero Fourier coefficient corresponding to the unipotent class (73) then it has a nonzero Fourier coefficient which corresponds to the unipotent class (713). Similarly, if $\theta_\tau$ has a nonzero Fourier coefficient which corresponds to the unipotent class (5r_1\ldots r_k) then it has a nonzero Fourier coefficient which corresponds to the unipotent class (51^5). Arguing as in [G-R-S2] lemma 3, we deduce that if $\theta_\tau$ has no nonzero Fourier coefficients which corresponds to the unipotent class (51^5) then it has no nonzero Fourier coefficient corresponding to the unipotent class (71^3). We should remark that these last statements are true for any automorphic representation of $G$ and not only for $\theta_\tau$. Hence it is enough to show that $\theta_\tau$ has no nonzero Fourier coefficients which
correspond to the unipotent classes $(51^5)$ and to $(4^21^2)$. We now describe the families of Fourier coefficients which correspond to these two unipotent classes.

Let $V$ denote the unipotent radical subgroup of the standard parabolic subgroup of $G$ whose Levi part is $GL^2_3 \times GSO_6$. We view $V$ as a subgroup of the maximal unipotent subgroup $U$ of $G$ which consists of upper triangular matrices. Let $a \in F^*$. We define a character $\psi_{V,a}$ of the group $V$ as follows. For $v = (v_{i,j}) \in V$, set $\psi_{V,a}(v) = \psi(v_{1,2} + v_{2,5} + av_{2,6})$. Thus the Fourier coefficient of the representation $\theta_\tau$ which corresponds to the unipotent class $(51^5)$ is given by

$$\int_{V(F) \backslash V(A)} \theta_\tau(vg)\psi_{V,a}(v)dv$$

Next we describe the Fourier coefficient which corresponds to the unipotent class $(4^21^2)$. To do that, let $R$ denote the subgroup of $U$ defined as the group generated by all one dimensional unipotent subgroup associated with the positive roots of $G$ where we omit the roots $\alpha_1, \alpha_3, \alpha_4$ and $\alpha_5$. Thus, the dimension of $R$ is 16. For $r = (r_{i,j}) \in R$ we define $\psi_R(r) = \psi(r_{1,3} + r_{2,4} + r_{3,7})$. The Fourier coefficient which corresponds to the unipotent class $(4^21^2)$ is given by

$$\int_{R(F) \backslash R(A)} \theta_\tau(rg)\psi_R(r)dr$$

To prove our result we will show that the local unramified component $\theta'_\tau$ of $\theta_\tau$ does not support a local functional of the above type. More precisely, we prove

**Lemma 3:** Let $O$ denote one of the unipotent orbits $(51^5)$ or $(4^21^2)$. Then $\theta'_\tau$ has no nonzero linear functional $l_{O,a}$ which satisfies $l_{O,a}(\rho(v)x) = \psi_{V,a}^{-1}(v)l_{O,a}(x)$ and no nonzero linear functional $l_O$ which satisfies $l_O(\rho(r)x) = \psi_R^{-1}(r)l_O(x)$. This is for all $v \in V$, $r \in R$ and $x$ is a vector in the space of $\theta'_\tau$. Here we denoted by $\rho$ the action of the representation $\theta'_\tau$.

**Proof:** Arguing as in [G-R-S2] section 2 lemma 2 we have to show that the representation $Ind_{Q}^{G} \chi_{30}^{1/2}$ does not support any of these functionals. From the Bruhat theory this reduces to the problem of showing that $Ind_{Q}^{G} \chi_{30}^{1/2}$ has no admissible double coset in the space $Q \backslash G/L$ where $L$ is either $V$ or $R$. By that we mean that for any $g \in Q \backslash G/L$ there is $l \in L$ such that $glg^{-1} \in Q$ and that $\psi_L(l) \neq 1$. Here $\psi_L$ is either $\psi_{V,a}$ or $\psi_{R}$. From the Bruhat decomposition we obtain that each element $g \in Q \backslash G/L$ can be written as $g = wu$ where $w$ is a Weyl element of $G$ and $u \in U$. As in [G-R-S2] section 2 lemma 2 we deduce that $wu$ is not admissible if and only if $w$ is not admissible. Let $U(Q)^-$ denote the transpose group of $U(Q)$. Thus $U(Q)^-$ consists of lower unipotent matrices. Let $x_1(r_1) = I_{10} + r_1e_{1,2}^2$, $x_2(r_2) = I_{10} + r_2e_{2,5}$ and $x_3(r_3) = I_{10} + r_3e_{2,6}$. These are precisely the three one dimensional unipotent groups on which the character $\psi_{V,a}$ is not trivial. To prove that $w$ is not admissible it is enough to show that for some $i$, we have $wx_i(r_i)w^{-1} \in Q$. Assume not. This means that $wx_i(r_i)w^{-1} \in U(Q)^-$ for all $i$. Since $w$ is in $G$ we have that if $w_{i,j} = 1$ then $w_{11-i,11-j} = 1$. This means that if $wx_2(r_2)w^{-1} = I_{10} + r_2e_{i,j}'$ then $wx_3(r_3)w^{-1} = I_{10} + r_3e_{i,11-j}'$. Hence if $wx_2(r_2)w^{-1} \in [U(Q)^-,U(Q)^-]$ then
\(wx_3(r_3)w^{-1} \in Q\). Similarly, if \(wx_3(r_3)w^{-1} \in [U(Q)^-, U(Q)^-] \) then \(wx_2(r_2)w^{-1} \in Q\). From this it follows that both \(wx_2(r_2)w^{-1}, wx_3(r_3)w^{-1} \in U(Q)^-/[U(Q)^-, U(Q)^-] \). Since \(x_1(r_1)\) does not commute with both \(x_2(r_2)\) and \(x_3(r_3)\), it follows the same for \(wx_1(r_1)w^{-1}\). Hence, also \(wx_1(r_1)w^{-1} \in U(Q)^-/[U(Q)^-, U(Q)^-]\). However, there is no one dimensional unipotent subgroup of \(G\) which corresponds to a root in \(G\), which has the property that it is in \(U(Q)^-/[U(Q)^-, U(Q)^-]\) and which will commute with both \(wx_2(r_2)w^{-1}\) and \(wx_3(r_3)w^{-1}\). Thus every \(w\) is not admissible and the lemma is proved for the functional \(l_C, a\). A similar proof holds for the group \(R\) and the functional \(l_O\). ■

4 The Unramified Computation

In this section we carry out the unramified computation of the local integral which corresponds to integral \([14]\). Let \(F\) denote a nonarchimedean field, and assume that all groups are defined over \(F\). Let \(\pi\) denote an unramified irreducible generic representation of the group \(G\). We assume that it has a trivial central character. We denote by \(I(s)\) the local induced representation \(Ind_{B}^{\pi}\). Let \(\tau\) denote an unramified irreducible representation of \(GL_2\) and assume that it has a trivial central character. As in section 3 lemma 2 we denote by \(\theta^s_{\tau}\) the local unramified constituent of the global representation \(\theta_{\tau}\). The integral we consider is

\[
\int_{Z^U \setminus G/F^2} W_{\pi}(g)\theta_{\tau}^{U,\psi_U}(g)f_s(w[53]x_{a_3}(r_1)x_{a_3+a_5}(r_2)g)\psi(r_1)dr_1dr_2dg (14)
\]

Assume that all functions are the \(K\) fixed vectors in their space of representation, where \(K\) is the maximal compact subgroup of \(G\). Let \(Spin_{10}\) denote the 16 dimensional irreducible Spin representation of \(GSpin_{10}(\mathbb{C})\), the \(L\) group of \(G\). The corresponding \(L\) function was defined in \([G1]\). We denote by \(L(Spin_{10} \times St, \pi \times \tau, s)\) the 32 dimensional \(L\) function which consists of the tensor product of these two representations. In this section we prove

**Proposition 4:** For all unramified data, and for \(Re(s)\) large, integral \([14]\) equals

\[
\frac{L(Spin_{10} \times St, \pi \times \tau, 2s - 1/2)}{\zeta(8s)\zeta(8s - 2)} (15)
\]

**Proof:** Let \(T\) denote the maximal torus of \(G\). Using the Iwasawa decomposition, integral \([14]\) equals

\[
\int_{Z \setminus T} W_{\pi}(t)\theta_{\tau}^{U,\psi_U}(t)\delta_B^{-1}(t) f_s(w[53]x_{a_3}(r_1)x_{a_3+a_5}(r_2)t)\psi(r_1)dr_1dr_2dt
\]

where \(B\) is the Borel subgroup of \(G\). We parameterize an element \(t\) in \(Z \setminus T\) as \(t = diag(ab_1, ab_2, ab_3, ab_4, a, 1, b_4^{-1}, b_3^{-1}, b_2^{-1}, b_1^{-1})\). For this parameterization \(\delta_B^{-1}(t) = |a|^{-10}|b_1|^{-8}|b_2|^{-6}|b_3|^{-4}|b_4|^{-2}\).

Next we compute the inner integration along the two dimensional unipotent subgroup. This is done exactly as in \([G1]\) section 3 right after identity (11). We thus
obtain
\[
\int_{F^2} f_s(w[53]x_{\alpha_3}(r_1)x_{\alpha_3+\alpha_5}(r_2)t)\psi(r_1)dr_1dr_2 =
\]
\[
\zeta(8s-2) \zeta(8s)/(1-|b_3b_4^{-1}|8s-2q^{-8s+2}|a|^{2s+1}|b_1b_2|^4|b_3|^{-4s+2}|b_4|^{4s-1})
\]

Here we used the fact that \(\delta_\tau(t) = |a|^{10s}|b_1b_2b_3b_4|^{4s}\). Also, we denote \(q = |p|\) where \(p\) is a generator of the maximal ideal of the ring of integers in the field \(F\).

Recall, from section 3, that \(\theta_{\tau,\psi_U}^G\) is the local component of a residue of an Eisenstein series at the point \(2/3\). Hence we have

\[
\theta_{\tau,\psi_U}^G(t) = \delta_{Q}^{1/3}(t)W_\tau \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} W_{\tau \times \tau} \begin{pmatrix} ab_1 & a \\ b^{-1} & b^{-1}_4 \end{pmatrix}
\]

Here \(W_\tau\) is the unramified Whittaker function corresponding to the induced representation \(Ind_{B_3}^{GL_3} \chi_1\delta_{B_3}^{1/2}\) of \(GL_3\) which was defined in section 3. Similarly, the function \(W_{\tau \times \tau}\) is the unramified Whittaker function corresponding to the induced representation \(Ind_{B_4}^{GSO_4} \chi_2\delta_{B_4}^{1/2}\). Denote \(K_{\tau}(t) = W_\tau(t)\delta_B^{-1/2}(t)\) and also

\[
K_{\tau} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} K_{\tau \times \tau} \begin{pmatrix} ab_1 & a \\ b^{-1} & b^{-1}_4 \end{pmatrix} =
\]

\[
W_\tau \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} W_{\tau \times \tau} \begin{pmatrix} ab_1 & a \\ b^{-1} & b^{-1}_4 \end{pmatrix} |b_1^{-1}b_3||a|^{-1/2}|b_4|^{-1}
\]

Using the fact that \(\delta_{Q}^{1/3}(t) = |a|^{3}|b_1b_2b_3|^2\) we obtain that integral \([43]\) equals

\[
\zeta(8s-2) \zeta(8s) \int_{Z\setminus T} K_{\tau}(t)K_{\tau} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} K_{\tau \times \tau} \begin{pmatrix} ab_1 & a \\ b^{-1} & b^{-1}_4 \end{pmatrix} \times
\]

\[
(1-|b_3b_4^{-1}|8s-2q^{-8s+2}|a|^{2s-1/2}|b_1b_2b_3b_4|^{4s-1}dt
\]

We consider the following change of variables. Set \(a \mapsto t_1t_5^{-1}, b_1 \mapsto t_2t_3t_4t_5, b_2 \mapsto t_3t_4t_5, b_3 \mapsto t_4t_5\) and \(b_4 \mapsto t_5\). With this change of variables the torus \(Z\setminus T\) is parameterized as \(t = \text{diag}(t_1t_2t_3t_4t_5, t_1t_3t_4t_5, t_1t_4t_5, t_1t_5, t_1, t_5, 1, t_1^{-1}, t_2^{-1}, t_3^{-1}, t_4^{-1})\). Thus the above integral equals

\[
\zeta(8s-2) \zeta(8s) \int_{Z\setminus T} K_{\tau}(t)K_{\tau} \begin{pmatrix} t_2t_3 \\ t_1 \end{pmatrix} K_{\tau \times \tau} \begin{pmatrix} t_1t_5 \\ t_1 \\ t_5 \end{pmatrix} \times
\]

\[
(1-|b_3b_4^{-1}|8s-2q^{-8s+2}|a|^{2s-1/2}|b_1b_2b_3b_4|^{4s-1}dt
\]
Let \( P = \sum m_i \omega_i \) evaluated at the semi-simple conjugacy class of \( G_{\text{Spin}}(C) \) associated with \( \pi \). Here \( \omega_i \) is the i-th fundamental representation of \( G_{\text{Spin}}(C) \). It follows from the Casselman-Shalika formula \([C-S]\) that \( K_\pi(t) = \langle n_2, n_3, n_4, n_5, n_1 \rangle \). Similarly, we denote by \( (m) \) the trace of the irreducible representation \( m \omega \) evaluated at the semi-simple conjugacy class of \( GL_2(C) \) associated with \( \tau \). Using again the Casselman-Shalika formula we obtain

\[
K_\tau \left( \begin{array}{ccc} t_2 t_3 & & \\ t_2 & t_1 & \\ & & 1 \end{array} \right) K_{\tau \times \tau} \left( \begin{array}{ccc} t_1 t_5 & & \\ t_1 & t_5 & \\ & & 1 \end{array} \right) = l(n_2, n_3) \otimes (n_1) \otimes (n_5)
\]

where \( l(n_2, n_3) \) denotes the restriction of the irreducible representation with highest weight \([n_2, n_3]\) of \( GL_3(C) \) to the group \( SO_3(C) \).

Thus, to prove the proposition we are reduced to prove the identity

\[
\sum_{n_i}^\infty \langle n_2, n_3, n_4, n_5, n_1 \rangle k(n_1, n_2, n_3, n_5) y^{n_1 + 2n_2 + 4n_3 + 2n_4 + 3n_5} (1 - y^{4(n_4 + 1)}) = (1 - y^4)^2 L(Spin_{10} \times St, \pi \times \tau, 2s - 1/2)
\]

where \( k(n_1, n_2, n_3, n_5) = l(n_2, n_3) \otimes (n_1) \otimes (n_5) \). For \( \chi \) as in section 3, we have

\[
L(Spin_{10} \times St, \pi \times \tau, 2s - 1/2) = L(Spin_{10}, \pi \otimes \chi, 2s - 1/2)L(Spin_{10}, \pi \otimes \chi^{-1}, 2s - 1/2).
\]

By the Poincaré identity, and Brion’s decomposition of the symmetric algebra of \( Spin_{10} \) (see [Br]) we have

\[
L(Spin_{10}, \pi \otimes \chi, 2s - 1/2) = \sum_{m, \ell = 0}^\infty (m, 0, 0, \ell) \chi^{2m+\ell} y^{2m+\ell}.
\]

Let

\[
P(\chi, y) = (1 - \chi^8 y^8) + (\chi^6 y^6 - \chi^2 y^2)(1, 0, 0, 0, 0) \chi^2 y^2 + \chi^3 y^3(0, 0, 0, 1, 0) - \chi^5 y^5(0, 0, 0, 0, 1).
\]

Then, using again the Casselman-Shalika formula, we obtain

\[
P(\chi, y) \sum_{m, \ell = 0}^\infty (m, 0, 0, \ell) \chi^{2m+\ell} y^{2m+\ell} = \sum_{\ell = 0}^\infty (0, 0, 0, \ell) \chi^\ell y^\ell.
\]

So, our identity reduces to

\[
P(\chi, y) P(\chi^{-1}, y) \sum_{n_i}^\infty \langle n_2, n_3, n_4, n_5, n_1 \rangle k(n_1, n_2, n_3, n_5) y^{n_1 + 2n_2 + 4n_3 + 2n_4 + 3n_5} (1 - y^{4(n_4 + 1)})
\]
Here and henceforth, by abuse of notation, we write \((n_2, n_3, n_4, n_5, n_1)\) for the representation itself, as well as the value of the trace at the conjugacy class associated to \(\pi\).

**Lemma 5:** We have

\[
(0, 0, 0, 0, \ell) \otimes (0, 0, 0, 0, m) = \sum_{a+b \leq \text{min}(m,\ell)} (a, 0, b, 0, m + \ell - 2a - 2b).
\]

**Proof:** This result is due to Okada [10] see also [11].

What remains is to check that

\[
P(\chi, y)P(\chi^{-1}, y) \sum_{n_i} (n_2, n_3, n_4, n_5, n_1) k(n_1, n_2, n_3, n_5) y^{n_1+2n_2+4n_3+2n_4+3n_5} (1-y^{4(n_4+1)})
\]

\[
= (1 - y^4)^2 \sum_{n_2+n_4 \leq \text{min}(m,\ell)} \left( \sum_{n_2+n_4 \leq \text{min}(m,\ell)} (n_2, 0, n_4, 0, m + \ell - 2n_2 - 2n_4) \right) \chi^{m-\ell} y^{m+\ell} \tag{16}
\]

\[
= (1 - y^4)^2 \sum_{n_2, n_4, n_1 = 0} (n_2, 0, n_4, 0, n_1) y^{2n_2+2n_4+n_1} \chi^{-n_1} \frac{1 - \chi^{2(n_1+1)}}{1 - \chi^2}
\]

We check (16) in two stages. First, we show that

\[
P(\chi, y) \sum_{n_i} (n_2, n_3, n_4, n_5, n_1) k(n_1, n_2, n_3, n_5) y^{n_1+2n_2+4n_3+2n_4+3n_5} (1-y^{4(n_4+1)}) \tag{17}
\]

\[
= (1 - y^4)^2 \sum_{n_i} \frac{1 - \chi^{2(n_1+1)}}{1 - \chi^2} \frac{1 - \chi^{2(n_2+1)}}{1 - \chi^2} \chi^{-(n_1+2n_2+2n_3+n_5)} y^{n_1+2n_2+4n_3+2n_4+3n_5}
\]

Then, we check that \(P(\chi^{-1}, y)\) times this sum is the desired sum. First, using the Casselman-Shalika formula, we note that \(k(n_1, n_2, n_3, n_5)\) is equal to

\[
\frac{(1 - \chi^{2(n_2+1)})(1 - \chi^{2(n_3+1)})(1 - \chi^{2(n_2+n_3+2)})(1 - \chi^{2(n_1+1)})(1 - \chi^{2(n_5+1)})}{(1 - \chi^2)^4(1 - \chi^4)} \chi^{-(n_1+2n_2+2n_3+n_5)}
\]

To check (17), we shall verify that the coefficient of \((n_2, n_3, n_4, n_5, n_1)\) is the same on both sides. Let

\[
g_{[n_i]}(\chi, y) = \frac{(1 - \chi^{2(n_2+1)})(1 - \chi^{2(n_3+1)})(1 - \chi^{2(n_2+n_3+2)})(1 - \chi^{2(n_1+1)})(1 - \chi^{2(n_5+1)})}{(1 - \chi^2)^4(1 - \chi^4)} \chi^{-(n_1+2n_2+2n_3+n_5)} y^{n_1+2n_2+4n_3+2n_4+3n_5} (1 - y^{4(n_4+1)}).
\]
Let $\Gamma_1$ denote the set of weights of $(1, 0, 0, 0, 0)$, $\Gamma_4$ those of $(0, 0, 0, 1, 0)$, and $\Gamma_5$ those of $(0, 0, 0, 0, 1)$. Then, we claim that

$$P(\chi, y) \sum_{n_i}^\infty (n_2, n_3, n_4, n_5, n_1) g_{[n_i]}(\chi, y) = \sum_{n_i}^\infty (n_2, n_3, n_4, n_5, n_1) G_{(\nu)}(\chi, y)$$

where

$$G_{(\nu)}(\chi, y) = (1 - \chi^6y^6)g_{[n_i]}(\chi, y) + (\chi^6y^6 - \chi^2y^2) \sum_{w \in \Gamma_1} g_{[n_i-w_i]}(\chi, y) + \chi^3y^3 \sum_{w \in \Gamma_4} g_{[n_i-w_i]}(\chi, y) - \chi^5y^5 \sum_{w \in \Gamma_5} g_{[n_i-w_i]}(\chi, y).$$

Indeed, Brauer proved in general that if $\chi_\lambda$ denotes the character of the irreducible representation of highest weight $\lambda$, then

$$\chi_\lambda \chi_\mu = \sum_\nu s_\nu \chi_{|\lambda + \nu + \rho| - \rho},$$

where the sum is over the weights of the representation with highest weight $\mu$, with multiplicity, $|\tau|$ denotes the dominant member of the Weyl orbit of $\tau$, and $s_\nu$ is equal to $(-1)^w$ if $w(\lambda + \nu + \rho) = |\lambda + \nu + \rho|$ for a unique $w$, and zero if $\lambda + \nu + \rho$ has a stabilizer in the Weyl group. See [13] p. 171 and exercises. In our case, no element of $\Gamma_1 \cup \Gamma_4 \cup \Gamma_5$ has an entry less than $-1$, so when $\lambda + \nu + \rho$ is not dominant, it has a stabilizer in the Weyl group. This proves [13] when no $n_i$ is zero. If any of the $n_i$ is zero, there are extra terms on the right hand side corresponding to $n_i - w_i = -1$. But referring back to the formula for $g_{[n_i]}$, we see that all these terms vanish anyway.

Hence, to check [17], we now need to check that $G_{(\nu)}$ is equal to the coefficient of $(n_2, n_3, n_4, n_5, n_1)$ on the right side of [17]. Replacing $\chi^{2(n_1+1)}$ by $X_i$ and $y^{4n_4}$ by $Y_4$ we obtain an identity of polynomials in seven variables which is straightforward to verify by computer.

Now, we need to check that

$$P(\chi^{-1}, y) \sum_{n_i}^\infty \frac{1 - \chi^{2(n_1+1)}}{1 - \chi^2} \frac{1 - \chi^{2(n_2+1)}}{1 - \chi^2} \chi^{-(n_1+2n_2+2n_3+n_5)} y^{n_1+2n_2+4n_3+2n_4+3n_5}$$

$$= \sum_{n_2, n_4, n_1=0}^\infty (n_2, 0, n_4, 0, n_1) y^{2n_2+2n_4+n_1} \chi^{-n_1} \frac{1 - \chi^{2(n_1+1)}}{1 - \chi^2}.$$

Let

$$h_{[n_i]}(\chi, y) = \frac{1 - \chi^{2(n_1+1)}}{1 - \chi^2} \frac{1 - \chi^{2(n_2+1)}}{1 - \chi^2} \chi^{-(n_1+2n_2+2n_3+n_5)} y^{n_1+2n_2+4n_3+2n_4+3n_5}$$

when all $n_i$ are nonnegative. Note that this expression is equal to zero if $n_1$ or $n_2$ is equal to $-1$. We extend the definition of $h_{[n_i]}$ to be zero if $\min(n_3, n_4, n_5) = -1$. Then the left hand side is equal to

$$\sum_{n_i}^\infty (n_2, n_3, n_4, n_5, n_1) H_{(\nu)}(\chi, y),$$
where

\[ H_{(n_i)}(\chi, y) = (1 - \chi^{-8} y^8) h_{[n_i]}(\chi, y) + (\chi^{-6} y^6 - \chi^{-2} y^2) \sum_{w \in \Gamma_1} h_{[n_i-w]}(\chi, y) + \]

\[ \chi^{-3} y^3 \sum_{w \in \Gamma_4} h_{[n_i-w]}(\chi, y) - \chi^{-5} y^5 \sum_{w \in \Gamma_5} h_{[n_i-w]}(\chi, y). \]

We claim that this is equal to

\[ y^{2n_2+2n_4+n_1} \chi^{-n_1} \frac{1 - \chi^{2(n_1+1)}}{1 - \chi^2}, \]

when \( n_3 = n_5 = 0 \), and zero otherwise. To show this, we partition each \( \Gamma_i \) into eight sets, according to whether the \( w_3, w_4, \) and \( w_5 \) are positive or nonpositive. This determines in which \( H_{(n_i)} \) the corresponding term is zero by convention.

We replace \( \chi^{2(n_1+1)} \) and \( \chi^{2(n_2+1)} \) by \( X_1 \) and \( X_2 \) throughout. A factor of

\[ \chi^{-(n_1+2n_2+2n_3+n_5)} y^{n_1+2n_2+4n_3+2n_4+3n_5} (1 - \chi^2)^{-2} \]

may be factored out of the sum and moved to the other side, which then becomes

\[ R(\chi, X_1, X_2) = \frac{X_2}{\chi^2} (1 - X_1)(1 - \chi^2), \]

when \( n_3 = n_5 = 0 \). On the other side of the equation we have a polynomial obtained by summing

\[ (1 - X_1 \chi^{-2w_1})(1 - X_2 \chi^{-2w_2}) \chi^{w_1+2w_2+2w_3+w_5} y^{-(w_1+2w_2+4w_3+2w_4+3w_5)} \]

over the weights, with the correct coefficients from \( P \). We break this into eight pieces.

Let \( Q_{nnn} \) denote the sum over weights where \( w_3, w_4, w_5 \) are all nonpositive, \( Q_{ppp} \) the sum where they are all positive, \( Q_{npn} \) the sum over weights such that \( w_3 \) and \( w_5 \) are nonpositive, and \( w_4 \) is positive, and so on. Since \( (1 - \chi^8 y^8) h_{[n_i]}(\chi, y) \) is there for every \([n_i] \), it is included in \( Q_{nnn} \). The answers are:

\[ Q_{ppp} = Q_{pnn} = Q_{npp} = Q_{nnn} = 0 \]

\[ Q_{nnn} = Q_{pnp} = R \]

\[ Q_{nnn} = Q_{pnn} = -R. \]

From here it is an easy application of the inclusion-exclusion principle. (It is easy to check by hand that \( Q_{ppp} = Q_{pnn} = Q_{npp} = 0 \), since in all three cases the set of roots satisfying the desired condition is empty.)
5 A construction of a certain lifting

In this section we shall use a certain small representation, defined on the group $SO_{22}(A)$, in order to construct the endoscopic lifting from $PGL_2 \times Sp_6$ to the group $SO_{10}$. Then, in the next section we will use this construction to prove our main result.

This construction is a special case of a more general set of constructions which are defined on classical groups. It is a part of a work in progress of the first named author and was partly announced in [G2]. To make things clear, we shall now sketch this construction and just state the main results. The details of the proofs will appear in [G3].

Let $\tau$ denote a cuspidal representation of $PGL_2(A)$. In section 3 we constructed a small representation $\theta_{\tau}$, defined on the group $GSO_{10}(A)$. Clearly the construction there is also valid if we consider the group $SO_{10}$ instead. Recall that $\sigma(\tau)$ is the cuspidal representation of $GL_3(A)$ obtained by the symmetric square lift of $\tau$, as constructed in [G2]. Let $E_{\tau}(g,s)$ denote the Eisenstein series defined on $GL_6(A)$ associated with the induced representation $Ind_{L(A)}^G(\sigma(\tau) \otimes \sigma(\tau))\delta_L$. Here $L$ is the maximal parabolic subgroup of $GL_6$ whose Levi part is $GL_3 \times GL_3$. It is well known, see for example [J], that this representation has a unique simple pole and its residue, which we shall denote by $E_{\tau}$, is the well known Speh representation. Let $Q$ denote the maximal parabolic subgroup of the split orthogonal group $SO_{22}$, whose Levi part is $GL_6 \times SO_{10}$. Let $\tilde{E}_{\tau}(m,s)$ denote the Eisenstein series defined on $SO_{22}(A)$ associated with the induced representation $Ind_{Q(A)}^{SO_{22}(A)}(E_{\tau} \otimes \theta_{\tau})\delta_Q^s$.

The following two facts about this Eisenstein series are proved in [G3].

a) In the domain $Re(s) > 1/2$, the Eisenstein series $\tilde{E}_{\tau}(m,s)$ has a unique simple pole. We shall denote the residue representation which is obtained by $\tilde{\theta}_{\tau}$. By abuse of notations we shall denote by $\tilde{\theta}_{\tau}(m)$ a vector in this representation when realized in the space of automorphic forms.

b) Using the notations introduced in the beginning of section 2, we have $\mathcal{O}_{SO_{22}}(\tilde{\theta}_{\tau}) = (3^71)$. This means that $\tilde{\theta}_{\tau}$ has a nonzero Fourier coefficient which corresponds to the unipotent orbit $(3^71)$. Also, for any unipotent orbit $\mathcal{O}$ which is greater than or not related to $(3^71)$, the representation $\tilde{\theta}_{\tau}$ has no nonzero Fourier coefficient corresponding to the orbit $\mathcal{O}$.

At this point we are ready to introduce the global lifting we need for our result. Let $\epsilon_1$ denote a cuspidal representation defined on the group $Sp_6(A)$. Let $\theta_\phi$ denote the theta representation defined on the group $\tilde{Sp}_{60}(A)$. Here $\tilde{Sp}$ denotes the double cover of the symplectic group and $\phi$ denotes a Schwartz function defined on $A^{30}$. Let $U$ denote the unipotent radical subgroup of $Q$ and let $H_{61}$ denote the Heisenberg group with 61 variables. We shall write an element $h \in H_{61}$ as $h = (X,Y,z)$ where $X,Y \in Mat_{3 \times 10}$ and $z \in Mat_{1 \times 1}$. We define a homomorphism $l : U \to H_{31}$ as follows. In terms of matrices we can identify $U$ with the group of all matrices

$$ u = \begin{pmatrix} I_3 & Y & R_1 & R_2 \\ I_3 & X & R_3 & R_1^* \\ I_{10} & X^* & Y^* \end{pmatrix} \begin{pmatrix} I_3 \\ I_3 \end{pmatrix} $$

(19)
Here $X, Y \in \text{Mat}_{3\times 10}$, $R_1 \in \text{Mat}_{3\times 3}$ and $R_2, R_3 \in \text{Mat}_{3\times 3}^0 = \{ R \in \text{Mat}_{3\times 3} : R^tJ_3 + J_3R = 0 \}$. The matrix $J_3$ was defined in the beginning of section 2. For all $u \in U$ as above, if $R_1 = (r_{i,j})$ we define $l(u) = (X, Y, r_{1,1} + r_{2,2} + r_{3,3})$. This map is clearly onto.

We now consider

$$f_1(g) = \int_{\text{Sp}_6(F) \setminus \text{Sp}_6(A)/U(F) \setminus U(A)} \int_{\text{SO}_{10}(F) \setminus \text{SO}_{10}(A)/U(F) \setminus U(A)} \varphi_{\epsilon_1}(h) \tilde{\theta}_\phi(l(u)(h, g)) \tilde{\theta}_\tau(u(h, g)) dudh$$  \hspace{1cm} (20)

Here $g \in \text{SO}_{10}(A)$, and the embedding of the groups $\text{Sp}_6$ and $\text{SO}_{10}$ in $\text{Sp}_{60}$ and $\text{SO}_{22}$ are as follows. First, inside $\text{Sp}_{60}$ we embed these two groups via the tensor product representation. In $\text{SO}_{22}$, we embed them in the obvious way inside the Levi part of $Q$ which is $\text{GL}_6 \times \text{SO}_{10}$.

Let $\pi_1$ denote the automorphic representation on $\text{SO}_{10}(A)$ defined by all right translations of the functions $f_1(g)$ defined in (20). We summarize some of the properties of this representation. The proofs are given in [G3].

1) The representation $\pi_1$ is a cuspidal representation provided a certain integral is zero. This is the tower property, and the certain integral we refer to is the lifting in the previous stage. Since we will not need this point we shall not discuss it any further.

2) The representation $\pi_1$ is generic if and only if $\epsilon_1$ is a generic cuspidal representation of $\text{Sp}_6(A)$.

It is important to us that we can interchange the representations $\pi_1$ and $\epsilon_1$ in integral (20). More precisely, let now $\pi$ denote a cuspidal representation defined on $\text{SO}_{10}(A)$. Then we can form the space of automorphic functions defined by

$$f(h) = \int_{\text{SO}_{10}(F) \setminus \text{SO}_{10}(A)/U(F) \setminus U(A)} \int_{\text{SO}_{10}(F) \setminus \text{SO}_{10}(A)/U(F) \setminus U(A)} \varphi_\pi(g) \tilde{\theta}_\psi(l(u)(h, g)) \tilde{\theta}_\tau(u(h, g)) dudg$$  \hspace{1cm} (21)

Let $\epsilon$ denote the automorphic representation of $\text{Sp}_6(A)$ defined by right translations of the above functions $f(h)$. The last fact that we need, and which will be proved in [G3], is the relations at the unramified components of each of these representations. More precisely, let $\pi, \epsilon$ and $\tau$ be as above. Suppose that the integral

$$\int_{\text{Sp}_6(F) \setminus \text{Sp}_6(A)/\text{SO}_{10}(F) \setminus \text{SO}_{10}(A)/U(F) \setminus U(A)} \int_{\text{Sp}_6(F) \setminus \text{Sp}_6(A)/\text{SO}_{10}(F) \setminus \text{SO}_{10}(A)/U(F) \setminus U(A)} f_\epsilon(h) \varphi_\pi(g) \tilde{\theta}_\psi(l(u)(h, g)) \tilde{\theta}_\tau(u(h, g)) dudgdh$$  \hspace{1cm} (22)

is not zero for some choice of data. Then we have a "Howe duality" type of result. That is,

3) Suppose that integral (22) is not zero for some choice of data. Let $\nu$ be a finite nonarchimedean place where all the data is unramified. Let $\theta_{\tau, \nu}, \epsilon_\nu, \pi_\nu$ and $\omega_{\psi, \nu}$ denote the local representations corresponding to the global ones which appear in integral (22). Then, that integral induces an element in the space

$$\text{Hom}_{\text{Sp}_6 \times \text{SO}_{10}}((\theta_{\tau, \nu} \otimes \omega_{\psi, \nu})_U, \epsilon_\nu \otimes \pi_\nu)$$

where $\ldots U$ is the corresponding Jacquet module. As in [G2] section 6, it will be proved in [G3], that any two of the representations $\tau_\nu, \epsilon_\nu, \pi_\nu$ determines the third one uniquely.
From this we deduce that the representation $\pi_1$, as defined by integral (20), if nonzero, is the weak functorial lift from $\epsilon_1$ and $\tau$. Similarly, if nonzero, integral (21) defines an automorphic representation $\epsilon$ on $Sp_6(A)$, such that the representation $\pi$ is the weak lift from $\epsilon$ and $\tau$. In the following Lemma, we will prove that $\epsilon$, as defined by integral (21), is cuspidal. We don’t know if the image of the lift is irreducible, however, it is clear that all irreducible summands are nearly equivalent.

We start by proving

**Lemma 6:** The representation $\epsilon$ is cuspidal.

**Proof:** Let $V$ be a standard unipotent subgroup of $Sp_6$. By standard we mean that $V$ consists of upper unipotent matrices. We shall use the form $\begin{pmatrix} -J_3 & J_3 \\ J_3 & J_3 \end{pmatrix}$ to represent the symplectic group $Sp_6$ in terms of matrices. Write

$$\tilde{\varphi}(l(u)(v,g)) = \sum_{\xi \in Mat_{3 \times 10}(F)} \omega(l(u)(v,g))\phi(\xi) = \sum_{\xi \in Mat_{3 \times 10}(F)} \omega((X+\xi,Y,z)(v,g))\phi(0)$$

Here $\omega$ is the Weil representation and the above equalities are obtained from the well known formulas for the Weil representation (see for example [G-R-S3]). Plugging this identity into (21), collapsing the summation and the integration over the $X$ variable in $U$, we deduce that the integral $\int_{V(F) \setminus V(A)} f(v)dv$ is zero for all choice of data, if and only if the integral

$$\int_{SO_{10}(F) \setminus SO_{10}(A)} \int_{V(F) \setminus V(A)} \int_{U_1(F) \setminus U_1(A)} \varphi_\pi(g)\tilde{\varphi}_\tau(u_1(v,g))\psi_1(R_1)du_1dvdg \quad (23)$$

is zero for all choice of data. Here $U_1$ is the subgroup of $U$ which consists of all matrices as in (19) such that $X = 0$. The character $\psi_1$ is defined as follows. Write $u_1$ in the coordinates as given in (19). If $R_1 = (r_{i,j}) \in Mat_{3 \times 3}$, then $\psi_1(R_1) = \psi(r_{1,1} + r_{2,2} + r_{3,3})$.

At this point we need to consider the three nonconjugate maximal unipotent radicals of $Sp_6$. We shall work out the details in the case where $V$ is the unipotent radical of the Siegel parabolic. The other two cases are treated similarly. Let

$$V = \left\{ v = \begin{pmatrix} I_3 & Z \\ I_3 \end{pmatrix} : Z \in Mat_{3 \times 3}; Z^tJ_3 = J_3Z \right\}$$

Thus (24) is equal to

$$\int \varphi_\pi(g)\tilde{\varphi}_\tau \begin{pmatrix} I_3 & Z & Y & R_1 & R_2 \\ I_3 & R_3 & R_1^* & Y^* \\ I_{10} & Y^* & Y^* & Z^* \\ I_3 & Z^* & Z^* & I_{10} \end{pmatrix} (1,g) \psi_1(R_1)du_1dZdg \quad (24)$$

Here $g$ and $u_1$ are integrated as in (23) and $Z$ is integrated over $V(F) \setminus V(A)$. If $Z = (z_{i,j})$ we consider the Fourier expansion of the above integral along the variables $z_{2,3}, z_{3,2}$ and $z_{3,3}$ each integrated over $F \setminus A$. Conjugating by a suitable discrete matrix
in $R_3$ we collapse summation and integration, and deduce that the vanishing of (24) for all choice of data is equivalent to the vanishing of

$$\int \varphi_\pi(g)\tilde{\theta}_\tau \left[ \begin{pmatrix} I_3 & Z & Y & R_1 & R_2 \\ I_3 & R_1^* & Y^* & I_3 \\ I_10 & & I_3 \\ I_3 & Z^* & I_3 \end{pmatrix} (1,g) \right] \psi_1(R_1)du_1dZdg$$

(25)

for all choice of data. Now $Z$ is integrated over $\text{Mat}_{3\times3}(F)\setminus\text{Mat}_{3\times3}(A)$. Let $U_2$ denote the unipotent subgroup of $SO_{22}$ which consists of all matrices of the form

$$u = \begin{pmatrix} I_3 & Z & Y & R_1 & R_2 \\ I_3 & R_1^* & Y^* & I_3 \\ I_10 & & I_3 \\ I_3 & Z^* & I_3 \end{pmatrix}$$

(26)

Conjugating integral (25) by a suitable Weyl element, it is then enough to show that the integral

$$\int \varphi_\pi(g)\tilde{\theta}_\tau \left[ \begin{pmatrix} I_3 & R_1 & Y & Z & R_2 \\ I_3 & Z^* & I_10 & I_3 \\ R_3 & Y^* & I_3 & R_1^* \\ I_3 & I_3 \end{pmatrix} (1,g) \right] \psi_1(R_1)du_2dg$$

(27)

is zero for all choice of data. Here $u_2$ is integrated over $U_2(F)\setminus U_2(A)$. Next we consider the Fourier expansion of (27) along the unipotent group

$$\begin{pmatrix} I_3 \\ I_3 \\ I_10 \\ I_3 \end{pmatrix} R_3 \in \text{Mat}_{3\times3}^0$$

with points in $F\setminus A$. Under the action of $GL_3(F)$ embedded as $\text{diag}(m, m, I_{10}, m^*, m^*)$, there are two orbits to consider in this expansion. The first one is given by a sum of integrals of the form

$$\int \varphi_\pi(g)\tilde{\theta}_\tau \left[ \begin{pmatrix} I_3 & R_1 & Y & Z & R_2 \\ I_3 & R_3 & Z^* & I_3 \\ I_10 & & I_3 \\ I_3 & Y^* & R_1^* & I_3 \\ I_3 \end{pmatrix} (1,g) \right] \psi_1(R_1)\psi_2(R_3)du_2dR_3dg$$

(28)

where $\psi_2(R_3) = \psi(r_{1,2})$ for all $R_3 = (r_{i,j})$. After some further tedious expansions and suitable conjugations one can show that the above integral is a sum of Fourier
coefficients where each one of them corresponds to a unipotent orbit which is bigger than or equal to \((41^{18})\). Since \(O_{SO_{22}}(\tilde{\theta}_\pi) = (3^71)\) it follows that \(\varphi_\pi\) is zero for all choice of data. Thus we are left with the trivial orbit. That is, we need to show that for all choice of data the integral

\[
\int \varphi_\pi(g)\tilde{\theta}_\pi \left[ \begin{array}{cccc}
I_3 & R_1 & Y & Z \\
I_3 & X & R_3 & Z^* \\
I_{10} & X^* & Y^* \\
I_3 & R_1^* & I_3 & I_3
\end{array} \right] (1, g) \psi_1(R_1)du_2dR_3dg
\]

is zero. Next we expand integral \(29\) along the unipotent set

\[
\left( \begin{array}{c} I_3 \\
I_3 \\
I_{10} \\
I_3
\end{array} \right) = \left( \begin{array}{c} X \\
X^* \\
I_3 \\
I_3
\end{array} \right) \quad X \in Mat_{3 \times 10}
\]

with points in \(F \setminus A\). The group \(GL_3(F) \times SO_{10}(F)\) acts on this expansion with various orbits. The orbits can be identified with vectors \(\xi_i \in F^{10}\), for \(1 \leq i \leq 3\), according to the length of these vectors. If at least one of these vectors has nonzero length, then the corresponding Fourier coefficient contains as inner integration a Fourier coefficient which corresponds to the unipotent orbit \((51^{17})\). By the smallness of \(\tilde{\theta}_\pi\) this Fourier coefficient is zero. Thus we are left with the cases where the length of all vectors is zero. Hence we can write the above Fourier expansion as a sum of integrals of the form

\[
\int \varphi_\pi(g)\tilde{\theta}_\pi \left[ \begin{array}{cccc}
I_3 & R_1 & Y & Z \\
I_3 & X & R_3 & Z^* \\
I_{10} & X^* & Y^* \\
I_3 & R_1^* & I_3 & I_3
\end{array} \right] (1, g) \psi_1(R_1)\psi_\nu(X)X_1dXdu_2dR_3dg
\]

Here \(\nu = (\nu_1, \nu_2, \nu_3)\) is one of the vectors \((0, 0, 0), (1, 0, 0), (1, 1, 0)\) or \((1, 1, 1)\). The characters \(\psi_\nu\) are defined as follows. Given \(X = (x_{i,j}) \in Mat_{3 \times 10}\) we define \(\psi_\nu(X) = \psi(\nu_1x_{1,1} + \nu_2x_{2,2} + \nu_3x_{3,3})\) where \(\nu\) is any one of the above four vectors. The variable \(g\) is integrated over \(SO_{10-2(\nu_1+\nu_2+\nu_3)}(F)\)\(U_\nu \setminus SO_{10}(A)\), where \(U_\nu\) is the unipotent radical of the maximal parabolic subgroup of \(SO_{10}\) whose Levi part is \(GL_{\nu_1+\nu_2+\nu_3} \times SO_{10-2(\nu_1+\nu_2+\nu_3)}\). If \(\nu = (0, 0, 0)\) then from the definition of \(\tilde{\theta}_\pi\) we obtain the integral \(\int \varphi_\pi(g)\tilde{\theta}_\pi(g)dg\) as inner integration. Here \(g\) is integrated over \(SO_{10}(F)\)\(\setminus SO_{10}(A)\). Hence by the cuspidality of \(\pi\) this integral is zero. If \(\nu\) is any one of the other three cases, we continue by expanding integral \(30\) along the group \(U_\nu(F)\)\(\setminus U_\nu(A)\). If this group is not abelian, we first expand along its center. By the smallness of \(\tilde{\theta}_\pi\) we obtain that all orbits in the various expansions are zero except the constant term along \(U_\nu\). This follows from the fact that each nontrivial Fourier coefficient will contain, as inner integration, a Fourier coefficient corresponding to a unipotent orbit which is greater than or not related to \((3^71)\). Hence it will give zero contribution. The constant term will also vanish. Indeed, in this case after factoring the integral, we will obtain
the integral \( \int \varphi_\pi(ug)du \) as inner integration. Here, the variable \( u \) is integrated over \( \mathcal{U}_\nu(F) \backslash \mathcal{U}_\nu(A) \). Hence we get zero by the cuspidality of \( \pi \). This completes the proof of the vanishing for the unipotent radical of the Siegel parabolic. We still have to prove the same for the other two maximal unipotent radicals. This is done in a similar way and we shall omit the details. ■

6 The Main Theorem

We recall the basic notations we used in the previous sections. Let \( \pi \) denote a generic cuspidal representation of \( GSO_{10}(A) \). We shall assume that \( \pi \) has a trivial central character. Let \( \tau \) denote a cuspidal representation of \( PGL_2(A) \). In section two we introduced the global integral

\[
\int_{Z(A)G(F) \backslash G(A)} \varphi_\pi(g)\theta_\tau(g)E(g,s)dg
\]

(31)

Here \( G = GSO_{10} \), and \( Z \) is the center of \( G \). The representation \( \theta_\tau \) was introduced in section three, and the Eisenstein series \( E(g,s) \) was defined at the beginning of section two. It follows from sections 2 and 3 that (31) unfolds to a Whittaker integral, and hence is Eulerian. From Proposition 4 in section 4 we deduce that at the non-archimedean unramified places this integral represents the \( L \)-function \( L(Spin_{10} \times St, \pi \times \tau, 2s - 1/2) \). Let \( S \) denote a finite number of places in \( F \) which includes all archimedean places, such that outside of \( S \) all data is unramified. We shall denote by \( L_S(Spin_{10} \times St, \pi \times \tau, 2s - 1/2) \) the corresponding partial \( L \)-function.

It follows from [K-R] that the Eisenstein series \( E(g,s) \) can have at most a simple pole at the points \( s = 1 \) and \( s = 3/4 \). The residue at \( s = 1 \) of \( E(g,s) \) is the trivial representation, and hence the residue of integral (31) at the other point. We are interested in the residue of (31) at the other point. In this section only, we shall denote by \( \theta \) the residual representation of \( E(g,s) \) at \( s = 3/4 \). In other words, we denote \( \theta(g) = \text{Res}_{s=3/4}E(g,s) \). It follows from [G-R-S4] section 6 that \( \theta \) is the irreducible minimal representation of \( G \). We are now ready to state and prove our main result

**Main Theorem:** Let \( \pi \) be an irreducible generic cuspidal representation of the group \( G(A) \) which has a trivial central character. Then the following are equivalent:

1) The partial \( L \)-function \( L_S(Spin_{10} \times St, \pi \times \tau, 2s - 1/2) \) has a simple pole at \( s = 3/4 \).

2) The period integral

\[
\int_{SO_{10}(F) \backslash SO_{10}(A)} \varphi_\pi(g)\theta_\tau(g)\theta(g)dg
\]

(32)

is nonzero for some choice of data.

3) There exists a generic cuspidal representation \( \sigma \) of the exceptional group \( G_2(A) \) such that \( \pi \) is the weak lift from the representation \( \sigma \times \tau \) of the group \( G_2(A) \times PGL_2(A) \).

**Proof:** Two parts are not hard to prove. Suppose 1) holds. Arguing as in [Ga-S] section 7, one can show that for any place in the global field \( F \), given a value of \( s \in \mathbb{C} \), one
can choose data such that the local integral \( \int_{Z(A)GSO_{10}(F)\backslash GSO_{10}(A) \times GSO_{10}(A)} \varphi_s(g)\theta_x(g) \theta(g) dg \) is nonzero for some choice of data. Factoring the similitude in (33), we deduce that if the partial \( L \) has a pole at \( s = 3/4 \), then integral (32) is nonzero for some choice of data. Thus 1) implies 2).

The implication 3) implies 1) is also not hard. Suppose that \( \pi \) is the weak lift from \( \sigma \times \tau \). Then branching from \( SO_{10}(C) \) to \( G_2(C) \times SL_2(C) \) we obtain that \( L^S(Spin_{10} \times St, \pi \times \tau, 2s - 1/2) \) is equal to

\[
L^S(G_2 \times Sym^2, \sigma \times \tau, 2s - 1/2)L^S(G_2, \sigma, 2s - 1/2)L^S(Sym^2, \tau, 2s - 1/2)\zeta^S(2s - 1/2)
\]

Here \( G_2 \) is the standard seven dimensional representation of \( G_2(C) \) and \( Sym^2 \) is the symmetric square representation of \( SL_2(C) \). Also, \( \zeta^S(2s - 1/2) \) denotes the partial global \( \zeta \) function. Since \( \sigma \) is generic it follows from \([G-R-S5]\) that \( \sigma \) has a nontrivial lift to a generic cuspidal representation of \( Sp_6(A) \) or to a cuspidal representation on \( PGL_3(A) \). Using the lifting of generic cuspidal representations from Classical groups to \( GL_n \) as proved in \([C-K-P-S-S]\), we deduce that the above product of \( L \) functions is actually a product of partial \( L \) functions associated with tensor product representations on certain \( GL \)’s. Hence, at \( s = 3/4 \) these \( L \) functions cannot vanish. On the other hand, \( \zeta^S(2s - 1/2) \) has a simple pole at that point. Hence 3) implies 1).

Finally we prove that 2) implies 3). Before providing the details, let us first explain the idea of the proof. Assuming (32) is nonzero for some choice of data, we need to construct a generic cuspidal representation \( \sigma \) of the group \( G_2(A) \) and prove that \( \pi \) is the weak lift of \( \sigma \times \tau \). Suppose that there exists a cuspidal representation \( \epsilon \) of \( Sp_6(A) \) such that the integral

\[
f^{R,\psi}(h) = \int_{SL_2(A) \backslash SL_2(F) \times V(F) \backslash V(A)} f(vm)\psi_V(v) dv dm
\]

is nonzero for some choice of data. Here \( f \) is a vector in the space of \( \epsilon \) and the group \( R = V \cdot SL_2 \) is defined as follows. Let \( V \) denote the unipotent radical of the maximal parabolic subgroup of \( Sp_6 \) whose levi part is \( GL_2 \times SL_2 \). Thus in matrices we have

\[
V \cdot SL_2 = \left\{ \begin{pmatrix} I_2 & x & y \\ I_2 & x^* & y \\ I_2 & m & m \end{pmatrix} : x \in Mat_{2 \times 2}; y \in Mat_{2 \times 2}^2; m \in SL_2 \right\}
\]

Here \( Mat_{2 \times 2}^2 = \{ y \in Mat_{2 \times 2} : y^tJ_2 = J_2y \} \). Also, we define \( \psi_V(v) = \psi(trx) \). It is also not hard to prove that if \( \epsilon \) is such that integral (31) is nonzero for some choice of data, then \( \epsilon \) must be generic. Assuming all this, it follows from \([G-J]\) (see also \([G-H]\) the discussion before and after equation (27)) that \( \epsilon \) is the weak functorial lift from a cuspidal representation \( \sigma \) on \( G_2(A) \).

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Let $\epsilon$ denote the representation defined by right translations of all functions given by integral (21). As proved in section 5 we know that $\epsilon$ is a cuspidal representation of $Sp_6(A)$. Thus if we can prove that $f^{R,\psi}(h)$ is nonzero for some choice of function $f(h)$ as in (21) it will follow from section 5 that $\pi$ is a weak endoscopic lift from $\epsilon \times \tau$. From the above discussion it will follow that $\epsilon$ is a lift from a generic cuspidal representation $\sigma$ of $G_2(A)$. This will prove that 2) implies 3).

Thus, to conclude the proof of the theorem we will show that if (32) is not zero for some choice of data, then the integral

$$\int_{SL_2(F) \backslash SL_2(A) \backslash V(F) \backslash V(A)} \int_{U(F) \backslash U(A)} \varphi_\tau(g) \tilde{\theta}_\phi(l(u)(vm, g)) \tilde{\theta}_\tau(u(vm, g)) \psi_V(v) dudgdm$$

is not zero for some choice of data. Here $g$ is integrated over $SO_10(F) \backslash SO_10(A)$. Since the inner integrations, over the variables $u$ and $g$ produce a cusp form on $Sp_6(A)$, the integral over $SL_2(F) \backslash SL_2(A)$ converges. At some point we will need to interchange the order of the integration. The justification for this is given in [K-R] Proposition 5.3.1. In other words, one needs to add the action of the Casimir element for $SL_2$ inside the Schwartz function inside the theta representation. Doing that, the theta representation becomes rapidly decreasing as a function of the $SL_2$. Because of [G-R-S4] Theorem 6.9 this will not harm the generality of our argument.

We start by unfolding the theta series. As in section 5 we have

$$\tilde{\theta}_\phi(l(u)(vm, g)) = \sum_{\eta, \xi} \omega_\psi(l(u)(vm, g)) \phi(\eta, \xi) = \sum_{\eta, \xi} \omega_\psi(l(\eta)(l(u)(vm, g)) \phi(0, \xi).$$

Here $\eta \in Mat_{2 \times 10}$ and $\xi \in F^{10}$. Plugging this into (35) we collapse the summation over $\eta$ with the suitable integration inside $U$. We then consider the Fourier expansion along the unipotent subgroup of $SO_{22}$ defined by $I_{22} + r_1 e'_{1,3} + r_2 e'_{1,4} + r_3 e'_{2,3} + r_4 e'_{2,4} + r_5 e'_{2,6}$. Here $e'_{i,j} = e_{i,j} - e_{23-j,23-i}$ and $e_{i,j}$ is the matrix which has one at the $(i, j)$ position and zero elsewhere. Once again, collapsing summation and integration, integral (35) equals

$$\int \varphi_\tau(g) \sum_{\xi} \omega_\psi(l(u_2)(m, g)l(u_1)) \phi(0, \xi) \tilde{\theta}_\tau(u_3u_2(m, g)u_1) \psi_{U_3}(u_3)d(...)$$

Here, the variables $m$ and $g$ are integrated as in (35). In term of matrices we have

$$u_1 = \begin{pmatrix} I_2 & I_2 & I_2 & I_2 \\ I_2 & z_1 & z_2 & z_1^* \\ I_10 & z_3 & z_3^* & I_2 \\ I_2 & I_2 & I_2 & I_2 \end{pmatrix} \quad u_2 = \begin{pmatrix} I_2 & y_1 & y_2 & I_2 \\ I_2 & y_1 & y_2 & I_2 \\ I_10 & y_1^* & y_2^* & I_2 \\ I_2 & I_2 & I_2 & I_2 \end{pmatrix}$$

(37)

Here, the variable $z_1$ is integrated over $Mat_{2 \times 2}(A)$, the variable $z_2$ is integrated over $Mat^0_{2 \times 2}(A)$ (see the definition right after (19)) and $z_3$ is integrated over $Mat_{2 \times 10}(A)$. 23
The variable $y_1$ is integrated over $\text{Mat}_{2\times 10}(F) \backslash \text{Mat}_{2\times 10}(A)$ and $y_2$ is integrated over $\text{Mat}^0_{2\times 2}(F) \backslash \text{Mat}^0_{2\times 2}(A)$. The variable $u_3$ is defined as

$$u_3 = u_3(x_1, x_2, x_3, r_1, r_2, r_3, r_4) = \begin{pmatrix} I_2 & x_1 & x_2 & r_1 & r_2 & r_3 & r_4 \\ I_2 & x_3 & r_3 & * & * & * & * \\ I_2 & r_2 & * & * & * & * & * \\ I_2 & & & & & & * \\ I_2 & & & & & & * \\ I_2 & & & & & & * \\ I_2 & & & & & & * \\ I_2 & & & & & & * \end{pmatrix}$$ (38)

Here all variables are integrated over $\text{Mat}_{a\times b}$, for suitable choice of $a$ and $b$ with points in $F \setminus A$, except $r_4$ which is integrated over $\text{Mat}^0_{2\times 2}$. Finally, the character $\psi_{U_3}$ is defined as follows. For $r_2 = (r_{i,j})$ and $x_3 = (x_{i,j})$ as described in (35), we set $\psi_{U_3}(u_3) = \psi(r_{1,1} + r_{2,2} + x_{1,1} + x_{2,2})$. Notice that the variable $v$ which appeared inside the Weil representation one equation before does not appear inside $\omega_\psi$ in (36).

This follows from the fact that $\omega_\psi((v,1))\phi(0,\xi) = \phi(0,\xi)$.

The group $u_3(0, r_4)$ is the one dimensional unipotent group which is the center of the maximal unipotent subgroup of $SO_{22}$. Let $N_2$ denote the standard unipotent radical of the maximal parabolic subgroup of $SO_{22}$ whose Levi part is $GL_2 \times SO_{18}$. We have

$$\int_{F \setminus A} \tilde{\theta}_\tau(u_3(0, r_4))dr_4 = \sum_{\alpha \in \text{Mat}_{2\times 18}(F)} \int_{\text{Mat}_{2\times 18}(A)} \tilde{\theta}_\tau(n_2)\psi(\alpha \cdot n_2)dn_2$$

where the sum is over all $\alpha \in \text{Mat}_{2\times 18}(F)$. The group $SO_{18}(F)$ acts on this expansion. If $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ where $\alpha_i \in F^{18}$, then the various orbits can be parameterized by the rank of the matrix $\alpha$ and by the length of $\alpha_i$.

Plugging this expansion into (36), we claim that only one orbit contributes a nonzero term. Indeed, after plugging in (36), we conjugate the matrix $u_3(x_1, x_2, 0, r_1, r_2, r_3, 0)$ from right to left. Because of the character $\psi_{U_3}$ we get zero contribution unless the rank of $\alpha$ is two and both vectors $\alpha_i$ are nonzero and have zero length. From this we deduce that (36) equals

$$\int_{F \setminus A} \varphi_\pi(g) \sum_\xi \omega_\psi(l(u_2)(m, g)l(u_1))\phi(0, \xi) \sum_\gamma \tilde{\theta}_\tau^{N_2, \psi}(\gamma u_3)u_2(m, g)u_1)\psi_{U_3}(u_3)\sigma(...)$$ (39)

Here $\gamma \in P^0_2(SO_{14})(F) \backslash SO_{18}(F)$ where $P^0_2(SO_{14})$ is the subgroup of $SO_{18}$ which consists of all matrices of the form

$$\begin{pmatrix} I_2 & * & * \\ k & * & * \\ I_2 & * & * \end{pmatrix} \quad k \in SO_{14}$$

Also, we denote

$$\tilde{\theta}_\tau^{N_2, \psi}(k) = \int_{N_2(F) \backslash N_2(A)} \tilde{\theta}_\tau(n_2k)\psi_{N_2}(n_2)dn_2 \quad k \in SO_{22}$$
where $\psi_{N_2}$ is defined as follows. Identifying $N_2$ with the group of all matrices of the form $n_2 = u_3(x_1, x_2, 0, r_1, r_2, r_3, r_4)$, we set $\psi_{N_2}(n_2) = \psi(trx_1)$.

Next we consider the double coset space $P^0_2(SO_{14}) \backslash SO_{18}/P_2(SO_{14})$. Here $P_2(SO_{14})$ is the standard parabolic subgroup of $SO_{18}$ whose levi part is $GL_2 \times SO_{14}$. Thus $P_2(SO_{14})$ contains $P^0_2(SO_{14})$. Considering the various orbits and arguing as above, one can show that only one orbit, which we shall describe below, contributes a nonzero term to (39). Let

$$w_1 = \begin{pmatrix} I_2 & I_2 \\ I_2 & I_{10} \\ I_2 & I_2 \\ I_2 & I_2 \end{pmatrix}$$

$$t(\delta) = \begin{pmatrix} I_2 & \delta \\ I_2 & I_{10} \\ I_2 & \delta^* \\ I_2 & I_2 \end{pmatrix}$$

where $\delta \in Mat_{2 \times 2}$. Then integral (39) equals

$$\int \phi_\pi(g) \sum_\xi \omega_\psi(l(u_2)(m, g)l(u_1)) \phi(0, \xi) \sum_{\gamma, \delta} \thetau_{N_2, \psi}(w_1 t(\delta) u_3 \gamma u_2 m g u_1) \psi_{N_2}(u_3) dx(...)

Here $\gamma \in P^0_2(SO_{10})(F) \backslash SO_{14}(F)$ where $P^0_2(SO_{10})$ is the subgroup of $SO_{14}$ which consists of all matrices in of the form

$$\begin{pmatrix} I_2 & * & * \\ k & * \\ I_2 \end{pmatrix} \quad k \in SO_{10}$$

Also, we have $\delta \in Mat_{2 \times 2}(F)$. In (40) we conjugate the matrix $u_3(x_1, 0)$ across $t(\delta)$. We obtain from the commutation relations the matrix $u_3(0, x_1, 0)$. When we conjugate this matrix across $w_1$ and change variables in $N_2$, we obtain the integral

$$\int \psi(tr(x_1 \delta)) dx_1$$

as inner integration. Clearly this integral is zero unless $\delta = 0$. Thus, if $\delta \neq 0$ in (40) we get zero contribution.

Next we consider the space $P^0_2(SO_{10}) \backslash SO_{14}/P_2(SO_{10})$. Here $P_2(SO_{10})$ is the standard parabolic subgroup of $SO_{14}$ whose levi part is $GL_2 \times SO_{10}$. Thus $P_2(SO_{10})$ contains $P^0_2(SO_{10})$. As before one can check that all orbits contribute zero except one. Let

$$w_2 = \begin{pmatrix} I_2 & I_2 \\ I_2 & I_{10} \\ I_2 & I_2 \\ I_2 & I_2 \end{pmatrix}$$

$$t(\delta, \mu) = \begin{pmatrix} I_2 & \delta \\ I_2 & I_{10} \\ I_2 & \delta^* \\ I_2 & I_2 \end{pmatrix}$$

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where $\delta \in Mat_{2\times 10}$ and $\mu \in Mat_{2\times 2}^0$. Conjugating by the matrices $u_3(0, x_2, 0, r_1, 0)$ across from left to right, as we did before with $u_3(x_1, 0)$, we obtain that only when $\delta = 0$ and $\mu = 0$ then we get a nonzero contribution. Thus, integral (40) equals

$$\int \varphi_\pi(g) \sum_\xi \omega_\psi(l(u_2)(m, g)) l(u_1)) \phi(0, \xi) \tilde\theta_\tau^{N_2, \psi}(w_1 w_2 u_3(x_3) u_2(m, g) u_1) \psi(tr x_3) d(...)$$

where we wrote $u_3(x_3)$ for $u_3(0, 0, x_3, 0)$.

Next, we consider the Fourier expansion

$$\tilde\theta_\tau^{N_2, \psi}(k) = \sum_{\beta \in F_{F_\mathbf{A}}} \int \tilde\theta_\tau^{N_2, \psi}((I_{22} + r e_{3, 19}) k) \psi(\beta r) dr$$

One can check that if $\beta \neq 0$ then the corresponding summand is zero. Indeed, in this case we obtain as inner integration, Fourier coefficients of $\tilde\theta_\tau$ corresponding to unipotent orbits which are greater or equal to (41$^8$). By section 5 point b) it follows that all these Fourier coefficients are zero. Thus we are left with the summand corresponding to $\beta = 0$. This we can further expand along the set

$$\begin{pmatrix} I_2 & x & x & x^* \\ I_{12} & x^* \\ I_2 & I_2 \\ I_2 & I_2 \end{pmatrix}$$

where $x \in Mat_{2\times 12}$. The group $SO_{12}$ acts on this expansion, and as before we can parameterize the various orbits by matrices $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \in Mat_{2\times 12}(F)$ where the invariants are the rank of $\alpha$ and the length of $\alpha_i$. Once again we are left with the contribution of one orbit which corresponds to rank two matrices such that both $\alpha_i$ are nonzero vectors of length zero. Thus, integral (40) is equal to

$$\int \varphi_\pi(g) \sum_\xi \omega_\psi(l(u_2)(m, g)) l(u_1)) \phi(0, \xi) \sum_\gamma \tilde\theta_\tau^{N_4, \psi}(\gamma w_1 w_2 u_3(x_3) u_2(m, g) u_1) \psi(tr x_3) d(...)$$

Here, the group $N_4$ consists of all unipotent matrices in $SO_{22}$ of the form

$$\begin{pmatrix} I_2 & r_1 & * & * & * & * \\ I_2 & r_2 & * & * & * & * \\ I_2 & * & * & * \\ I_2 & * & * & * \\ I_2 & r_1^* & * \\ I_2 & r_2^* \end{pmatrix}$$

and

$$\tilde\theta_\tau^{N_4, \psi}(k) = \int_{N_4(F) \setminus N_4(A)} \tilde\theta_\tau(n_4 k) \psi_{N_4}(n_4) d n_4$$
where \( \psi_{N_4}(n_4) = \psi(tr(r_1 + r_2)) \). Also, we have \( \gamma \in P_2^0(SO_{10})(F) \setminus SO_{14}(F) \). Continuing this process of Fourier expansions it follows from the smallness properties of \( \tilde{\theta}_\tau \) that

\[
\tilde{\theta}_\tau^{N_6,\psi}(k) = \int_{N_6(F) \setminus N_6(A)} \tilde{\theta}_\tau(n_6 k) \psi_{N_6}(n_6) dn_6
\]

Here \( N_6 \) is the unipotent subgroup of \( SO_{22} \) which consists of all matrices of the form

\[
\begin{pmatrix}
I_2 & r_1 & * & * & * & * \\
I_2 & r_2 & * & * & * & * \\
I_2 & r_3 & * & * & * \\
I_10 & r_3^* & * \\
I_2 & r_2^* & * \\
I_2 & r_1^* \\
I_2
\end{pmatrix}
\]

and \( \psi_{N_6} \) is the character \( \psi_{N_4} \) extended trivially from \( N_4 \) to \( N_6 \).

As before we consider the space \( P_2^0(SO_{10}) \setminus SO_{14}/P_2(SO_{10}) \). We obtain a nonzero contribution from one term. Thus, integral \( \sum \) equals

\[
\int \varphi_\pi(g) \sum_\xi \omega_\psi(l(u_2) \phi(0, \xi) \sum_{\delta, \mu} \tilde{\theta}_\tau^{N_6,\psi}(w_2 l(\delta, \mu) w_1 u_2)(m, g) u_1) du_1 du_2 dg dm
\]

Conjugating \( w_2 w_1 l(\delta, \mu) w_1 w_2 \) we see that as a group, this group of matrices coincides with the group \( U_2 \). Thus we may collapse summation with integration. Hence

\[
(43)
\]

equal to

\[
\int \int_{(U_1 U_2)(A)} \varphi_\pi(g) \sum_\xi \omega_\psi((m, g) l(u_2 u_1)) \phi(0, \xi) \tilde{\theta}_\tau^{N_6,\psi}(w_2 w_1 u_2)(m, g) u_2 u_1) du_1 du_2 dg dm
\]

Here the variables \( m \) and \( g \) are integrated as before. That is, the variable \( m \) is integrated over \( SL_2(F) \setminus SL_2(A) \), and \( g \) is integrated over \( SO_{10}(F) \setminus SO_{10}(A) \).

Notice that \( N_6 = U V_6 \) where \( U \) is the unipotent radical of the parabolic subgroup \( Q \) as defined in \( 19 \). The group \( V_6 \) is the group of all unipotent matrices of the form

\[
V_6 = \left\{ \begin{pmatrix}
I_2 & r_1 & r_2 \\
I_2 & r_3 \\
I_2 \\
I_10 & r_3^* & r_2^* \\
I_2 & r_2^* \\
I_2 & r_1^* \\
I_2
\end{pmatrix} \right\}
\]

To complete the proof of the theorem, we assume that part 2) in the statement of the Theorem holds, and assume that integral \( (44) \) is zero for all choice of data. We shall
derive a contradiction. First, using arguments similar as to those in [Ga-S] section 7, we may deduce that if indeed (44) is zero for all choice of data, then so is the integral

$$\int_{SL_2(F) \backslash SL_2(A)} \int_{SO_{10}(F) \backslash SO_{10}(A)} \varphi(g) \sum_{\xi} \omega_{\psi}((m, g)) \phi_2(\xi) \tilde{\theta}_{\tau}^{N_6, \psi}(w_2 w_1 w_2(m, g)) dg dm$$

(45)

Here we wrote $\varphi = \phi_1 \otimes \phi_2$ where $\phi_1$ is a Schwartz function of $Mat_{2 \times 10}(A)$ and $\phi_2$ is a Schwartz function of $A^{10}$. Since $w_2 w_1 w_2$ normalizes the group of matrices $(m, g)$ as above, we conjugate it to the right and may ignore it.

Next, we claim that $\tilde{\theta}_{\tau}^{N_6, \psi}((m, g)) = \tilde{\theta}_{\tau}^{N_6, \psi}((1, g))$ for all $m \in SL_2(A)$ embedded in $SO_{22}(A)$ as in (45). We will prove it in Lemma 7 below. Assuming that, we deduce that if (45) is zero for all choice of data, then so is the integral

$$\int_{SO_{10}(F) \backslash SO_{10}(A)} \varphi(g) f(g) \tilde{\theta}_{\tau}^{N_6, \psi}((1, g)) dg$$

(46)

where we denoted

$$f(g) = \int_{SL_2(F) \backslash SL_2(A)} \sum_{\xi} \omega_{\psi}((m, g)) \phi_2(\xi) dm$$

Notice that the inner summation is in fact the theta representation defined on the group $\tilde{Sp}_{20}(A)$. As explained in the beginning of the proof, after possible adjustments of the Schwartz function, it follows from [G-R-S4] Theorem 6.9, that $f(g)$ is actually the minimal representation of $SO_{10}$ which we denoted by $\theta$. Hence, from (46) and the above discussion, we may assume that the integral

$$\int_{SO_{10}(F) \backslash SO_{10}(A)} \varphi(g) \theta(g) \tilde{\theta}_{\tau}^{N_6, \psi}((1, g)) dg$$

(47)

is zero for all choice of data. Recall that $N_6 = UV_6$. Also the character $\psi_{N_6}$ is trivial on $U_6$. In Lemma 7 below, we shall prove that

$$\int_{V_6(F) \backslash V_6(A)} E_{\tau}(vh) \psi_{V_6}(v) dv$$

(48)

is nonzero for some choice of data. Here $E_{\tau}$ is the Speh representation as was defined at the beginning of section 5, and by abuse of notations, we view $V_6$ as a subgroup of $GL_6$. The character $\psi_{V_6}$ is the restriction of $\psi_{N_4}$ to the group $V_6$.

As a function of $g \in SO_{10}(A)$, the constant term $\tilde{\theta}_{\tau}^{N_6, \psi}((1, g))$, is realized in the representation space of $\theta_{\tau}$. In other words, in view of the nonvanishing property of (48), it follows that the vanishing for all data of (47) implies that (44) is zero for all choice of data. But this contradicts our assumption. Thus 2) implies 3). This completes the proof of the main theorem. ❑
We still need to prove

**Lemma 7:** Let $E_r$ denote the Speh representation on $GL_6(\mathbb{A})$ as defined at the beginning of section 5. Then the integral

$$\mathcal{F}(h) = \int_{(\text{Mat}_{2 \times 2}(F) \setminus \text{Mat}_{2 \times 2}(\mathbb{A}))^3} E_r \left( \begin{pmatrix} I_2 & x & z \\ I_2 & y & \tau \end{pmatrix} h \right) \psi(\text{tr}(x+y)) \, dx dy dz \quad (49)$$

is nonzero for some choice of data. Moreover, if $h = \text{diag}(m, m, m)$, where $m \in SL_2(\mathbb{A})$, then the $\mathcal{F}(h) = \mathcal{F}(e)$.

**Proof:** It follows from [G4] Proposition 5.3 that $\mathcal{O}(E_r) = (3^2)$. This means that $E_r$ has no nonzero Fourier coefficients for any unipotent orbit which is bigger than or not related to $(3^2)$. Let $t(r) = \text{diag}(\begin{pmatrix} 1 & r \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & r \\ 1 & 1 \end{pmatrix})$. We expand $\mathcal{F}(h)$ along the group of matrices $t(r)$ with points in $F \setminus \mathbb{A}$. We have

$$\mathcal{F}(h) = \sum_{\alpha \in F \setminus \mathbb{A}} \int \mathcal{F}(t(r)h) \psi(\alpha r) \, dr$$

If $\alpha$ is not zero then the corresponding expansion consists of Fourier coefficients for $E_r$ which correspond to unipotent orbits which are greater than or not related to $(3^2)$. Hence, if $\alpha \neq 0$, we get zero contribution which means that $\mathcal{F}(t(r)h) = \mathcal{F}(h)$ for all $r \in \mathbb{A}$. Let $\nu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Denote $w = \text{diag}(\nu, \nu, \nu)$. Then clearly $\mathcal{F}(wh) = \mathcal{F}(h)$. Since $w$ and $t(r)$ generate $SL_2(\mathbb{A})$, the second assertion follows.

As for the first part, we shall assume that $\mathcal{F}(h)$ is zero for all choice of data and derive a contradiction. Let $w$ denote the Weyl element of $GL_6$ with one at positions $(1, 1); (2, 3); (3, 5); (4, 2); (5, 4); (6, 6)$ and zero elsewhere. Conjugating $w$ from left to right we deduce that the integral

$$\int_{(F \setminus \mathbb{A})^{12}} E_r \left( \begin{pmatrix} 1 & x_1 & x_2 & r_1 & r_2 \\ 1 & x_3 & r_3 & 1 \\ 1 & y_1 & y_2 & z_1 \\ 1 & y_2 & z_2 \end{pmatrix} \right) \psi(x_1 + x_3 + y_1 + y_3) \, d(...) \quad (50)$$

is zero for all choice of data. Expand [50] along the unipotent group consisting of all matrices $I_6 + r_4 e_{1,4} + r_5 e_{2,5} + r_6 e_{3,6}$ where $r_i \in F \setminus \mathbb{A}$. Conjugating by a suitable discrete matrix and collapsing summations with integrations, [50] is equal to

$$\int E_r \left( \begin{pmatrix} 1 & x_1 & x_2 & r_4 & r_1 & r_2 \\ 1 & x_3 & r_5 & r_3 & 1 \\ 1 & y_1 & y_2 & r_6 & z_1 \\ 1 & y_2 & z_2 & z_3 & 1 \end{pmatrix} \right) \psi(x_1 + x_3 + y_1 + y_3) \, d(...) \quad (51)$$

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where the variables \( z_i \) are integrated over \( A \) and all the rest over \( F \setminus A \). Thus we conclude that integral (51) is zero for all choice of data. As in [Ga-S] section 7, it follows that we may ignore the integration over the \( z_i \) variables. That is, the inner integration is zero for all choice of data. Expand the inner integration in (51) along \( I_6 + r_7 e_{2,4} + r_8 e_{3,5} \) where \( r_i \in F \setminus A \). We claim that except the constant term, all other terms contribute zero to the expansion. Indeed, this follows from the fact that each other term produces a Fourier coefficient for \( E_\tau \) which corresponds to a unipotent orbit which is greater than or not related to \((3^2)\). Next, expanding along \( I + r_9 e_{3,4} \) with \( r_9 \in F \setminus A \), all terms, except the constant term, contribute zero. Thus we may deduce that the integral

\[
\int_{(F \setminus A)^{12}} E_\tau \left( \begin{pmatrix} 1 & x_1 & x_2 & r_4 & r_1 & r_2 \\ 1 & x_3 & r_7 & r_5 & r_3 & r_0 \\ 1 & r_9 & r_8 & r_6 & r_1 \\ 1 & y_1 & y_2 & 1 & y_2 \\ 1 & y_1 + y_3 \end{pmatrix} \right) \psi(x_1 + x_3 + y_1 + y_3) d(...) \tag{52}
\]

is zero for all choice of data. However, from the definition of the Eisenstein series \( E_\tau(g, s) \), see beginning of section 5, it follows that this last integral cannot be zero for all choice of data. Thus we obtained a contradiction. ■

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