A nonlinearity in permanent-magnet systems used in watt balances

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Abstract
Watt balances are used to measure the Planck constant and will be used in the future to realize mass at the kilogram level. They increasingly rely on permanent magnet systems to generate the magnetic flux. It has been known that the weighing current might affect the magnetization state of the permanent-magnet system used in these systems, causing a systematic bias that can lead to an error in the result if not accounted for. In this article, a simple model explaining the effect of the weighing current on the yoke of the magnet is developed. This model leads to a nonlinear dependence of the magnetic flux density in the gap that is proportional to the squared value of the coil current. The effect arises from changing the reluctance of the yoke by the additional field produced by the coil. Our analysis shows that the effect depends on the width of the air gap, the magnetic flux density in the air gap, and the BH curve of the yoke material.

Suggestions to reduce the nonlinear effect are discussed.

Keywords: watt balance, permanent magnet, magnetic nonlinearity

(Some figures may appear in colour only in the online journal)

1. Introduction
The watt balance was originally proposed by B P Kibble in 1975 [1] and is an instrument that is used to precisely measure the Planck constant $h$. In the new International System of Units (SI) [2], it will constitute one method to realize the unit of mass at the kilogram level. Currently, several national metrology institutes (NMIs) are in the process of building a watt balance, since it is seen as an ideal apparatus to realize and maintain the unit of mass in the new SI. A review on watt balance experiments is given in [3].

A watt balance is typically operated alternatively in two separate modes: weighing and velocity. In weighing mode, a magnetic force is generated by passing a dc current $I$ through a coil in an area with a magnetic flux density $B$. The magnetic force is balanced by the gravitational force acting on a test mass $m$, i.e., $BLI = mg$, where $L$ is the wire length in the coil and $g$ the gravitational acceleration. In velocity mode, the geometric factor $BL$ is calibrated by moving the coil in the same magnetic field with a velocity $v$ while measuring an induced voltage $U$ across the coil, i.e., $U = BLv$. The combination of the two measurement modes allows a virtual comparison of electrical power to mechanical power. The Planck constant can be obtained since electrical power can be measured as the product of two frequencies and $h$ by the virtue of the Josephson effect [4] and the quantum Hall effect [5].

For the watt balance experiment to work, it is assumed that $BL$ is the same in the two modes. However, in weighing mode the current in the coil produces a magnetic field that could cause a change in the magnetic flux density $B$. The magnetic flux density $B$ is in general an unknown function of the weighing current $I$. The functional form of this relationship is typically approximated using the expression [6]:

$$ (BL)_w \approx (BL)_v (1 + \alpha I + \beta I^2), \quad (1) $$

where $(BL)_w$ and $(BL)_v$ are the so-called geometric factors in weighing mode and velocity mode, respectively. Here, $\alpha$ and $\beta$ denote the linear and quadratic coefficients. The weighing mode is typically carried out in a fashion that the linear term is eliminated: two measurements, mass-off and mass-on, are performed during weighing mode [7]. The currents in mass-off and mass-on measurements are nearly equal and opposite, cancelling any effect caused by $\alpha$. The quadratic...
term, however, cannot be eliminated and can lead to a bias in the measurement.

The nonlinear effect caused by the parallel component of the weighing flux has been studied in [8] and $\beta$ was determined by considering the magnetic reluctance change in upper and lower yokes. It was found that the main part of the nonlinear error from the parallel component is canceled by averaging the upper and lower yokes. In the end, the size of the bias in the measurement introduced by this component is negligible compared to the desired accuracy of the watt balance, which is typically a few parts in $10^5$. Recently, a different mechanism that can produce a quadratic dependence of $BL$ on the current was found while investigating the NIST-4 magnet [9, 10] at National Institute of Standards and Technology, USA. The quadratic term arises due to a change in reluctance of the yoke near the coil caused by the perpendicular component of the additional magnetic field $H$ created by the weighing current. In this article, we consider the origin of this nonlinear effect, estimate its order of magnitude, and discuss strategies to reduce or even remove this error by design improvements, active compensation, or corrections.

2. Magnetic error analysis

Recently, yoke-based permanent-magnet systems have become the preferred choice in watt balances [9, 11–13, 15]. Compared to electro magnets, these systems benefit from a stronger magnetic field, lower operating cost, and better magnetic self-shielding. Figure 1 shows two typical designs for such magnet systems. The two-permanent-magnet, one-coil structure as shown in figure 1(a) is employed by the BIPM watt balance [11] at the Bureau International des Poids et Mesures, METAS-2 [12] at the Federal Institute of Metrology, Switzerland, and NIST-4 [9]. The one-permanent-magnet, two-coil structure as shown in figure 1(b) is built into the NPL-NRC watt balance originally constructed by the National Physical Laboratory, UK and currently operated by

![Diagram](image-url)
Two different, but related, effects of the weighing current on the magnetic flux in the gap aspects are investigated: First, the variation of the total flux through the gap due to the change of the reluctance of the yoke due to magnetization from the coil current is considered. Second, the redistribution of flux around the coil in the air gap is investigated. This occurs because the reluctance of the iron closer to the coil changes differently than that of the iron further away from the coil.

2.1. Total magnetic flux change

The two magnetic circuits most common in watt balances are shown on the right of equation (2), the total reluctance of the system is a sum of three parts: the reluctances of the air gap, the permanent magnets and the yoke. The reluctances of the iron closer to the coil changes differently than that of the iron further away from the coil.

The symbols \( F \) and \( F_w \) denote the magnetic reluctance lengths, \( \mu \) the permeability, and \( S \) the magnetic reluctance areas.

In weighing mode, the current in the coil generates an additional magnetic field, and clearly part of this field must penetrate the yoke. As a result, the magnetic reluctance in some areas of the yoke will change as the soft iron material is magnetized, and hence the reluctance of the complete magnetic circuit will change. Therefore, the BL in weighing mode will slightly differ from its value in velocity mode.

In velocity mode, the equation of the magnetic circuit can be written as

\[
F = R_v \phi_v \quad \text{with} \quad R_v = \left( \frac{l_0}{\mu_0 S_0} + \frac{l_m}{\mu_m S_m} + \frac{l_y}{\mu_y S_y} \right),
\]

where \( R_v \) denotes the reluctance of the magnetic circuit in velocity mode, \( \phi_v \) the flux through the magnetic system and \( F \) the magnetomotive force of the permanent magnets. As shown on the right of equation (2), the total reluctance of the system is a sum of three parts: the reluctances of the air gap, the permanent magnet and the yoke. Here, \( l_0, l_m, l_y \) denote the magnetic reluctance lengths, \( S_0, S_m, S_y \) the magnetic reluctance areas, and \( \mu_0, \mu_m, \mu_y \) the permeability of the air gap, the permanent magnets and the yoke. The reluctances of various magnetic paths depend on the exact geometries, which can be difficult to evaluate. In this article, all values for the areas and lengths of different flux paths are equivalent, i.e., average, values. In equation (2), \( \mu_m \approx \mu_0 \ll \mu_y \), thus the total magnetic reluctance is dominated by the contributions of the permanent magnet and the air gap.

In weighing mode, the current in the coil generates additional fields in the yoke. The additional fields are separated into two components: parallel (subscript \( \parallel \) ) and perpendicular (subscript \( \perp \) ) to the flux generated by the permanent magnet. The magnetic equation in weighing mode can be expressed as

\[
F = R_w \phi_w \quad \text{with} \quad R_w = \left( \frac{l_0}{\mu_0 S_0} + \frac{l_m}{\mu_m S_m} + \frac{l_{\parallel}}{\mu_{v\parallel} S_{\parallel}} + \frac{l_{\perp}}{\mu_{v\perp} S_{\perp}} \right),
\]

where \( l_{\parallel} \) and \( l_{\perp} \) denote the magnetic reluctance length; \( S_{\parallel} \) and \( S_{\perp} \) the magnetic reluctance areas; and \( \mu_{v\parallel} \) and \( \mu_{v\perp} \) the permeability of the regions of the yoke where the field generated by the weighing current is parallel and perpendicular to the original magnetic flux. From equations (2) and (3), the relative magnetic field change can be calculated as

\[
\frac{\phi_w}{\phi_v} - 1 = \frac{R_v}{R_w} - 1 = \frac{R_v - R_w}{R_w} \approx \frac{R_v - R_w}{R_v}.
\]

In the last approximation \( R_w \) in the denominator was replaced by \( R_v \), since these two terms differ very little from each other.

In the three equations above, it is tacitly assumed that the magnetomotive force is independent of the current in the coil, i.e., \( F = F_v = F_w \). In reality, this is not the case, since the magnetic field produced by the coil during weighing mode will change the working point of the permanent magnet along the demagnetization curve. However, this effect depends linearly on the weighing current and will cancel by current reversal (mass-on versus mass-off).

To simplify the analysis, we split the reluctance of the yoke during velocity mode in the same two regions as in the weighing mode, yielding

\[
\frac{l_{\parallel}}{\mu_{v\parallel} S_{\parallel}} = \frac{l_{\parallel}}{\mu_{v\parallel} S_{\parallel}} + \frac{l_{\perp}}{\mu_{v\perp} S_{\perp}},
\]

where \( \mu_{v\parallel} \) and \( \mu_{v\perp} \) are the permeabilities of two regions in velocity mode. Since there is no current in the coil during velocity mode, the symbols \( \parallel \) and \( \perp \) only denote the yoke locations. As shown in figure 1, a watt balance magnet typically exhibits up-down symmetry. Hence the parallel component of the magnetic field of the coil will increase the field in one half of the yoke and decrease the field in the other half by the same amount, \( \Delta H_j \). In a small range of the yoke \( BH \) curve, the \( \mu_v(H) \) function can be considered to be linear, leading to

\[
\frac{l_{\parallel}/2}{(\mu_{v\parallel} + \mu(H)\Delta H_j) S_{\parallel}} + \frac{l_{\perp}/2}{(\mu_{v\perp} - \mu(H)\Delta H_j) S_{\perp}} \approx \frac{l_{\parallel}}{\mu_{v\parallel} S_{\parallel}}.
\]

Here, \( \chi \) is the derivative of \( \mu(H) \) with respect to \( H \) at the working point of the yoke, i.e. \( \chi = \partial \mu / \partial H |_{H=H_w} \). Equation (6) shows that the reluctance of the yoke parts, at which the field from the weighing current is parallel to the flux from the permanent magnet does not change between weighing mode and velocity mode in a symmetric structure. This is because the two components cancel each other. The higher order terms in equation (6) are negligible compared to the watt balance uncertainty goal [8].

The areas of the yoke, where the field from the weighing current is perpendicular to the flux from the permanent magnets are located around the coil. In these areas, the field generated by the weighing current is much larger than in the areas where the field is parallel to the flux. In addition the cross sections of the former areas are smaller than those of the latter areas. The magnetic field strength increases from \( H_v \) in velocity mode to \( H_w \) in weighing mode by

\[
H_w^2 = H_v^2 + (\Delta H_j)^2 \implies H_w \approx H_v + (\Delta H_j)^2 / 2H_v
\]

where \( \Delta H_j \) is the increment of the magnetic field strength due to the perpendicular component of the field produced by the coil. The permeability in this area is given by

\[
\mu_{v\perp} = \mu_{v\perp} + (\Delta H_j)^2 / 2H_v \partial \mu / \partial H |_{H=H_v}.
\]
Two parts contribute to the reluctance of each segment: the reluctance of the air gap and that of the yoke. In weighing mode, the reluctances $R_{uw}$ and $R_{cw}$ can be written as

$$R_{uw} = \frac{l_u}{\mu_w S_u} + \frac{l_u}{\mu_w S_u} \quad \text{and} \quad R_{cw} = \frac{l_c}{\mu_c S_c} + \frac{l_o}{\mu_w S_w},$$

(10)

where $l_u$, $l_c$, and $l_o$ are the yoke lengths between surfaces $A$ and $B$; $\mu_w$ and $\mu_c$ the permeability of the yoke for upper/lower and center segments. Note that these segments have the same geometrical parameters and the areas should be one third of the total, i.e., $l_u = l_c = S_y / 3$; $S_u = S_o = S_i / 3$. The flux through the center circuit $\phi_{cw}$ can be written as

$$\phi_{cw} = \frac{1}{R_{cw}} \left(2 \frac{R_{cw}}{R_{cw}} + 1\right)^{-1} \phi_w = \left(2 \frac{R_{cw}}{R_{cw}} + 1\right)^{-1} \phi_w.$$  

(11)

The relative change of the geometrical factor $BL$ at the weighing position (the center segment) in two modes is calculated as

$$\frac{(BL)_w}{(BL)_v} = 1 - \phi_{cw} - 1 = \phi_{uw} - 3 - 1 = (1 + \xi_1)(1 + \xi_2) - 1 \approx \xi_1 + \xi_2.$$  

(12)

Here, $1 + \xi_1 = \phi_{uw} / \phi_v$ and $1 + \xi_2 = 3/(2R_{cw}/R_{uw} + 1)$. An expression for $\xi_1$ is given in equation (9), therefore only a calculation for $\xi_2$ is required. Similar to the discussion in section 2.1, $\xi_2$ is solved as

$$\xi_2 = \frac{3}{2R_{cw} + 1} - 1 \approx \frac{2}{3} \left(1 - \frac{R_{cw}}{R_{uw}}\right) \approx \frac{2S_y l_u}{3S_y l_0} \frac{1 - \mu_w}{\mu_u}.$$  

(13)

Analogous to (8), $\mu_u$ and $\mu_v$ can be obtained using

$$\mu_u = \mu_v + \frac{(\Delta H_u)^2}{2H_u} \frac{\partial \mu}{\partial H} \bigg|_{H=H_u}$$

and

$$\mu_c = \mu_v + \frac{(\Delta H_c)^2}{2H_c} \frac{\partial \mu}{\partial H} \bigg|_{H=H_c},$$

(14)

where $\Delta H_u$ is the perpendicular magnetic field change in upper/lower segment and $\Delta H_c$ is in the middle segment.

Substituting equation (14) into (13) yields

$$\xi_2 = \frac{2\mu_0 S_{olc} (\Delta H_u^2 - \Delta H_c^2)}{3\mu L_i S_y l_0} \frac{\partial \mu}{\partial H} \bigg|_{H=H_c}.  

(15)

By adding $\xi_1$ in (9) to $\xi_2$ in equation (15), the total bias can be calculated as

$$\xi = \xi_1 + \xi_2 \approx \left(\frac{l_u}{\mu_w S_u} + \frac{l_v}{\mu_w S_u} \right) \frac{2\mu}{\mu_v S_w} \left(k_1^2 - k_2^2\right) \frac{\partial \mu}{\partial H} \bigg|_{H=H_c},$$

(16)

where $k_1 = \Delta H_u / \Delta H_c$ and $k_2 = \Delta H_u / \Delta H_c$ are two magnetic field ratios. As $\mu_u$ and $\mu_v$ have similar values, it is reasonable to assume $\mu_v \approx \mu_u$.

The bias depends on the squared values of $\Delta H_u$, $\Delta H_c$, $\Delta H_u$ and $\Delta H_c$ and hence quadratically on the current in the coil. Besides the current, the bias depends on parameters of the magnet system, most importantly at the working point of the yoke at $H = H_c$. The bias can be eliminated by choosing parameters

Figure 2. Three-reluctance model of the magnet in weighing mode. Up-down symmetry about the center is assumed.

It can be seen from equation (7) that the magnetic field would increase independent of the current direction. Combining (3), (6), and (8) allows one to rewrite (4) as

$$\phi_{uw} / \phi_v - 1 \approx \frac{l_u}{\mu_v S_u} \frac{(\Delta H_u)^2}{2H_u} \frac{\partial \mu}{\partial H} \bigg|_{H=H_u}.$$  

(9)

In this section, it was assumed that the relative distribution of the flux in the air gap remains the same, i.e., is independent of the weighing current. In the next section, the effects of a flux redistribution in the air gap are considered.

2.2. Redistribution of the magnetic flux density in the air gap due to the weighing current

The weighing current in the coil produces an additional magnetic field which needs to be added to the already existing field produced by the permanent magnet system. The magnetic field produced by the magnet system in the yoke, near the gap, is in general uniform along the vertical axis. The magnetic field produced by the coil is largest at the coil position. Hence the reluctance of the yoke will change more at the coil position than above and below it. This nonuniform reluctance along the z axis of the yoke will lead to a redistribution of the magnetic flux density in the gap. This redistribution causes the flux integral during the weighing mode, $(BL)_w$, to be different from the flux integral during velocity mode, $(BL)_v$.

Figure 2 shows a simple model that can be used to evaluate this effect. A and B is a schematic representation of two vertical surfaces with the same magnetic potential, one in the inner yoke, the other in the outer yoke. The flux flows perpendicular through these two surfaces, such that in each mode, the total flux through the two planes is calculated as

$$\phi_{uw} = \phi_{cv} = \phi_v / 3.$$
such that the yoke is at its maximum relative permeability, i.e.,

\[ \frac{\partial \mu}{\partial H} \bigg|_{H=H_0} = 0. \]  

To model magnet systems that differ from ideal systems described above, we introduce a new variable,

\[ \delta = H_v - H_m \]

with \( H_m \) such that \( \frac{\partial \mu}{\partial H} \bigg|_{H=H_m} = 0 \)

in the next section.

3. Evaluation and discussion

In this section, the magnetic bias is calculated for typical parameters of a watt balance. To keep the analysis simple, we assume perfect up-down symmetry and that the position of the coil in the weighing mode is at the symmetry plane. Thus an average magnetic field change in the yokes along the central horizontal axis \( r \) could be used for calculating the \( \Delta H_\perp \) value, i.e.,

\[ \Delta H_\perp = \int_0^l \Delta H(r, z = 0) dr. \]  

We further assume that in weighing mode the coil produces a force of \( F = mg \approx 5 \text{N} \), which is typical for a 1 kg watt balance. In this case, the product of the coil current and the number of windings is given by a scalar form of the weighing equation as

\[ NI = \frac{F}{2\pi r_0 B_0} = \frac{mg}{2\pi r_0 B_0}. \]  

where \( r_0 \) is the mean radius of the coil and \( B_0 \) the mean value of the magnetic flux density at the coil position. The flux density contributed by the weighing current in the coil is calculated using the following approximations: The permeability of the yoke is set to the value at the working point, \( \mu_v = \mu(H_0) \) and the magnetomotive force of both magnets are set to zero. Since all flux produced by the coil flux in the yoke is perpendicular to the \( r \) axis in the central plane (\( z = 0 \)) and the additional magnetic density is continuous along the flux lines, the additional magnetic flux change in the yokes, \( \Delta B_\perp = \mu_v \Delta H_\perp \), can be considered to be equal to the flux in the yoke-air boundary. By Ampere’s law, we have

\[ 2l_0 \frac{\Delta B_\perp}{\mu_0} + l_y \frac{\Delta B_\perp}{\mu_v} = NI, \]  

where \( l_y \) is the total length of the magnetic field through the yoke and \( l_0 \) the width of the air gap. Since \( \mu_v \gg \mu_0 \), the second term can be neglected and \( \Delta H_\perp \) is given by

\[ \Delta H_\perp = \frac{\Delta B_\perp}{\mu_v} = \frac{NI\mu_0}{2l_0\mu_v}. \]  

To verify equation (22), calculations based on the finite element method (FEM) were performed. For these FEM calculations, an air gap width of \( l_0 = 30 \text{ mm} \), a relative permeability of the yoke of \( \mu_v/\mu_0 = 1000 \), and a magnetomotive force of the coil of \( NI = 8 \text{ A turns} \) is assumed. Figure 3 shows the magnetic field in an area around the coil. Figure 4 shows the field in the plane of the coil as a function of radius. Both figures show that the magnetic field decreases rapidly with increasing distance from the coil. The FEM calculated mean magnetic field change in the yoke, i.e., \( \Delta H_\perp \), is 0.16 A m\(^{-1} \) which agrees with 0.13 A m\(^{-1} \) calculated using the approximation (22). FEM calculations with different yoke permeabilities and air gap widths were performed and compared to equation (22), see figure 5. The model agreed reasonably with the simulation for all 15 combinations. The agreement is better for smaller gap widths and larger relative permeabilities.

Substituting equation (20) and equation (22) into equation (9), we obtain

\[ \xi_1 \approx \frac{l_0}{l_0 + l_m} \frac{m^2 g^2 \mu_0^3}{32\pi^2 r_0^3 B_0^2 l_0^2 \mu_v^3 H_0} \frac{\partial \mu}{\partial H} \bigg|_{H=H_0}. \]  

It can be seen from equation (16) that \( \xi_2 \) has a similar expression as \( \xi_1 \) and their ratio depends only on factors...
describing the magnet’s geometry and a coefficient $\kappa_2^2 - \kappa_1^2$, i.e., $\xi_2$ is solved as

$$
\xi_2 \approx \frac{2l_c}{S_0} m^2 \mu_0 \mu_a \left( \kappa_2^2 - \kappa_1^2 \right) \frac{\partial \mu}{\partial H} \bigg|_{H=H_c}. \tag{24}
$$

In order to obtain the value of $\xi_2$, two magnetic field ratios $\kappa_1 = \Delta H_0/\Delta H_1$ and $\kappa_2 = \Delta H_0/\Delta H_2$ need to be calculated. Note that in equation (24) $\Delta H_1$, $\Delta H_2$, and $\Delta H_0$ are different integral quantities in the same magnetic field, hence both $\kappa_1$ and $\kappa_2$ are considered as constants. Here the two ratios are determined by FEM simulation with $l_0 = 30$ mm, $\mu_\perp / \mu_\parallel = 1000$. The distances between reference surfaces (A and B) and the air gap are 60 mm and 40 mm. The calculated perpendicular components of the magnetic field along the vertical axis $z$ are shown in figure 6. It can be calculated from the simulation that $\kappa_1 = 0.16/0.16 = 1$ and $\kappa_2 = 0.27/0.16 = 1.7$.

Equations (23) and (24) determine the total bias $\xi$ as

$$
\xi \approx \left( \frac{l_c}{S_0} + \frac{2l_c}{S_0} \right) \frac{m^2 \mu_0}{\mu_a} \frac{\partial \mu}{\partial H} \bigg|_{H=H_c}. \tag{25}
$$

It can be seen from equation (25) that the bias is mainly related to three parameters: the magnetic flux density $B_0$ in the air gap, the gap width $l_0$ and the dependence $\mu(H)$ of the yoke. In the evaluation, the $\mu H$ curve of AISI 1021 steel, which was used in building the NIST-4 magnet, is assumed (shown in figure 7). The maximum relative permeability is 1137 at $H_m = 464$ A m$^{-1}$. Some geometrical factors are assumed as shown in table 1.

In order to demonstrate the bias as a function of the magnetic field offset $\delta$, two different scenarios were considered. In the first scenario, the magnetic flux density in the gap remained the same $B_0 = 0.5$ T while the width of the air gap was changed. In the second scenario, the width remained the same $l_0 = 10$ mm and the flux density was changed. The results were expressed as the relative error of the Planck constant (the bias) as functions of the magnetic field strength offset $\delta$ and are shown in figure 8.

As shown in figure 8, the bias has the opposite sign as the magnetic field offset $\delta$. Further, the slope of the bias for negative offsets is larger than for positive offsets. Moreover, equation (25) shows that the bias is (1) inverse proportional to $B_0^2$; (2) inverse proportional to $\mu_0^3$; (3) and depends critically on $l_0$ (inverse to $l_0^n$, $2 < n < 3$). A magnet design with a narrow air gap benefits from a stronger magnetic field, but increases the bias error. During the design process for a permanent magnet system for a watt balance, all parameters should be carefully optimized.

To verify the three-reluctance model for calculating $\xi_2$ in section 2.2, another FEM calculation was performed. A multi-yoke structure at the weighing position is designed as shown in figure 9 according to the coil flux contribution and all layers are set to different permeabilities where ($\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$)
Figure 8. Relative error for the Planck constant as a function of \( \delta \). Here \( \delta = H_v - H_m \) with \( H_m \) such that \( \frac{\partial \mu}{\partial H} \bigg|_{H=H_m} = 0 \).

Figure 9. Relative error for the Planck constant as a function of the magnetic field strength offset from the maximum permeability point. \( \Delta B_a \) denotes relative permeabilities of the yokes numbered 1, 2, 3, and 4. In order to obtain enough resolution, the contrast in permeabilities was exaggerated. The numbers (960, 970, 980, 1000) were used, which have a maximum difference in relative permeability of 40, about \( 4 \times 10^5 \) larger than in reality. The simulation result is shown in figure 9. A second parameter set with (996, 997, 998, 1000), with a maximum difference in relative permeability of 4 is also calculated. Its effect is about 10 times smaller than the first set. The result shows the nonlinearity is less than 7%. Thus a relative change of the magnetic field at the weighing position can be estimated using \( \frac{15 \times 10^{-6}}{4 \times 10^5} = 3.8 \times 10^{-11} \) where the first value \( 15 \times 10^{-6} \) is read off the blue dashed line of figure 9 at \( z = 0 \) and the \( 4 \times 10^5 \) is a scale factor assumed to scale the FEM simulation back to the range of permeability expected in reality. The FEM simulation agrees with the result obtained using equation (24).

Note, \( \delta \) is not the average magnetic field difference of the whole yoke but the areas of the yoke adjacent to the coil in the weighing position. In reality, \( \delta \) can be quite large, e.g., several hundreds A m\(^{-1}\). Table 2 gives a summary of the parameters (magnetic flux density in the air gap \( B_a \), air gap width \( l_0 \), and \( \delta \) of watt balances built at different laboratories around the world. The parameter \( \delta \) is calculated using the given value of \( B_a \), \( l_0 \), the mean radius of the air gap, and the \( BH \) curve of the AISI 1021 steel. The latter is a convenient assumption. In reality, different materials for yokes are employed. Hence, the numbers in the table are only an estimate. The results show the bias amplitude from the magnetic nonlinearity is less than \( 1 \times 10^{-9} \), which is negligible with respect to the uncertainty goals of these watt balances.

All the evaluation and discussion are based on the analysis without considering the yoke \( BH \) hysteresis. The hysteresis of the yoke may partly reduce this error, because the magnetic flux density in the weighing mode will remain for a while in the velocity mode. But the hysteresis effect, e.g., systematic effect from the non-symmetry of the minor \( BH \) hysteresis loops, is complex and should be studied in the future.

4. Suggestions

In this section, some suggestions are provided to reduce this nonlinear error.

The first conclusion is to make the working point for the yoke near weighing position approach the maximum permeability as much as possible, i.e., \( \delta = 0 \). Based on equation (25), the best working point of the yoke near the weighing position is the zero crossing point of the error curve shown in figure 8. As in the air gap, the magnetic flux density drops following a \( 1/r \) function (\( r \) is the radius), the magnetic field for the inner yoke \( H_{in} \) is different from that of the outer yoke \( H_{out} \). From the calculation in figure 3, a 50% weight of magnetic field change can be applied for both inner and outer yokes, thus the design should meet

\[
\frac{H_{in} + H_{out}}{2} = H_m
\]
To establish equation (26), an idea is to make adjustable magnetic compensations for the yoke around the weighing position. For example, current carrying compensation coils can be considered to generate opposite additional flux during the weighing mode. Also, small compensation permanent magnets can also shift the BH working point of the yoke.

The second conclusion is that the bias error is inverse to the product $B_{c_n}^2$, $\mu_n^3$, and $l_0^2(2 < n < 3)$. Thus strong magnetic field $B_{c_n}$, large air gap width $l_0$ and high permeability yoke are recommended for building a watt balance.

A third suggestion is to measure the amplitude of this effect in order to make possible corrections for the Planck constant value. Several watt balance experiments have published measured values of the Planck constant obtained by weighing different mass values. By analysing the obtained values of $h$ as a function of weight, the nonlinear effects of the magnet system can be measured. Unfortunately, at present, not enough information is available to perform a thorough analysis. We would like to encourage the experimenters to perform measurements over a larger range of masses, include more mass points, and collect more statistics on each point. Such a data set would allow verification of this and other models of the magnetic circuit and also allow the measured values to be extrapolated to a location where the magnet effects vanish.

The magnetic error is due to the division of the watt balance experiment in two modes, weighing and velocity mode. Future ideas and experiments to combine the moving and velocity modes should be encouraged [17, 18].

5. Conclusion

A nonlinear magnetic error in watt balance operation, which arises from the magnetic reluctance change of the yoke near the weighing position, is investigated. This error is proportional to the squared value of the coil current. The analysis shows that this error can be optimized by making the yoke around the weighing position work at the maximum permeability point of the BH curve. Further study evaluates the possible amplitude of the error as a function of the magnetic flux density difference between the actual and maximum-permeability points for the yoke near the weighing position. The result shows this nonlinearity is typically less than 1 part in 10^9 which is negligible compared to a watt balance uncertainty of several parts in 10^8. Therefore, at least in present stage, this nonlinear effect is not a limitation for watt balances.

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