Smallest eigenvalue distribution of the fixed-trace Laguerre beta-ensemble

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Abstract
In this paper we study the entanglement of the reduced density matrix of a bipartite quantum system in a random pure state. It transpires that this involves the computation of the smallest eigenvalue distribution of the fixed-trace Laguerre ensemble of \(N \times N\) random matrices. We showed that for finite \(N\) the smallest eigenvalue distribution may be expressed in terms of Jack polynomials. Furthermore, based on the exact results, we found a limiting distribution when the smallest eigenvalue is suitably scaled with \(N\) followed by a large \(N\) limit. Our results turn out to be the same as the smallest eigenvalue distribution of the classical Laguerre ensembles without the fixed-trace constraint. This suggests in a broad sense, the global constraint does not influence local correlations, at least, in the large \(N\) limit. Consequently, we have solved an open problem: the determination of the smallest eigenvalue distribution of the reduced density matrix—obtained by tracing out the environmental degrees of freedom—for a bipartite quantum system of unequal dimensions.

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1. Introduction

A bipartite quantum system is a composite system which can be described as a product state of two subsystems: one which we would like to study and the other describes the environment such as noise or heat baths.

Due to its central role in quantum information and quantum computation, which is considered as an indispensable resource, the entanglement of a bipartite quantum system has recently generated a flurry of activities and has been studied extensively [15].
There are extensive literature on this topic; here we list those which are of immediate relevance to our paper, see for examples [7, 8, 10, 12, 14, 16, 17]. The statistical properties of such random states are also important in the characterization of quantum chaotic systems, see [8, 10, 12] and references therein.

A bipartite quantum system consists of a system \( (A) \) and its environment \( (B) \). Explicitly, we consider a composite system which is described by an \((NM)\)-dimensional Hilbert space \( \mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)} \). Let \( |e_A^i\rangle \) and \( |e_B^j\rangle \) be the complete orthogonal basis for the subsystems \( A \) and \( B \), respectively. Without loss of generality, we assume that \( N \leq M \). Any quantum state \( |\Phi\rangle \) in the Hilbert space \( \mathcal{H}^{(NM)} \) can be expressed as the linear combination of \( |e_A^i\rangle \otimes |e_B^j\rangle \) as follows:

\[
|\Phi\rangle = \sum_{i=1}^{N} \sum_{j=1}^{M} X_{ij} |e_A^i\rangle \otimes |e_B^j\rangle,
\]

where the coefficients \( X_{ij} \in \mathbb{C} \) form a rectangular \( N \times M \) complex matrix \( X = [X_{ij}] \).

By a random state we understand that \( X_{ij} \) are random variables, so that the resulting density matrix of the subsystem \( A \), say, is a random matrix. See (1.2) below for a definition.

The composite state \( |\Phi\rangle \) is fully unentangled or separable if \( |\Phi\rangle \) can be expressed as a direct product of two states \( |\Phi^A\rangle \) and \( |\Phi^B\rangle \) drawn from the Hilbert space of \( A \) and \( B \) respectively, that is,

\[
|\Phi\rangle = |\Phi^A\rangle \otimes |\Phi^B\rangle;
\]

otherwise, it is referred to as an entangled state.

We say that \( |\Phi\rangle \) is a normalized pure state if and only if the associated density matrix defined by

\[
\rho = |\Phi\rangle \langle \Phi|
\]

satisfies

\[
\text{tr} \rho = 1.
\]

The reduced density matrix of the subsystem \( A \) obtained by tracing out the states of subsystem \( B \) is found to be, after an easy computation [2, 12],

\[
\rho_A := \text{tr}_B[\rho] = \sum_{j=1}^{M} \langle e_B^j | \rho | e_B^j \rangle = \sum_{i,j=1}^{N} W_{ij} |e_A^i\rangle \langle e_A^i| e_B^j\rangle,
\]

where \( W_{ij} \) are the entries of the \( N \times N \) square matrix \( W := XX^\dagger \)

with \( \text{tr} W = 1 \) implied by the normalization condition that \( \text{tr} \rho = 1 \). The reduced density matrix of the subsystem \( B \) is similarly defined, \( \rho_B = \text{tr}_A[\rho] \).

The fact that \( \text{tr} W = 1 \) implies the fixed-trace ensemble to be introduced below.

Because \( XX^\dagger \) and \( X^\dagger X \) have the same non-negative eigenvalues, it is not difficult to see that the reduced density matrices \( \rho_A \) and \( \rho_B \) have the same set of non-negative eigenvalues

\[
\{ \lambda_i \}_{i=1}^{N}
\]

and satisfy the fixed-trace condition

\[
\sum_{i=1}^{N} \lambda_i = 1.
\]
Let \( v_i^A \) be the eigenvector of the square matrix \( W \) corresponding to the eigenvalue \( \lambda_i \). Then the density matrix of the subsystem \( A \) can be expressed as

\[
\rho_A = \sum_{i=1}^{N} \lambda_i |v_i^A \rangle \langle v_i^A|.
\]

A similar representation holds for \( \rho_B \). The composite state \( |\Phi\rangle \) has the well-known Schmidt spectral decomposition [2]

\[
|\Phi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |v_i^A \rangle \otimes |v_i^B \rangle.
\]

As we have discussed above, a pure state is random if the coefficients \( X_{ij} \) are random variables. The simplest and most common choice is to take \( X_{ij} \) to be independent and identically distributed Gaussian random variables, see [12] for detailed exposition. Therefore, the eigenvalues of the reduced density matrix \( \rho_A \) are distributed according to the joint probability density function (jpdf) of the \( N \times N \) complex Wishart matrix whose trace is constrained to unity; this corresponds to the case where the ‘symmetry’ parameter \( \beta \) is 2. By the method of random matrix theory [13], it is easily shown that the jpdf of eigenvalues is given by, for \( \lambda_j \geq 0, j = 1, 2, \ldots, N \),

\[
P_\delta(\lambda_1, \lambda_2, \ldots, \lambda_N) = C_{N,M} \delta \left( N \sum_{i=1}^{N} \lambda_i - 1 \right) \prod_{i=1}^{N} \lambda_i^{\frac{\alpha}{2}} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta},
\]

where \( \alpha = M - N + 1 - \frac{2}{\beta} > -\frac{2}{\beta}, \beta > 0 \). In the situations where \( \beta = 1, 2, 4 \) we have the real, complex and quaternion fixed-trace Wishart random matrix ensembles.

For general \( \beta \), the jpdf in (1.4) can be realized by a tri-diagonal real symmetric matrices (with random entries) of the form (see [4])

\[
L_\beta = B_\beta B_\beta^T
\]

subject to \( \text{tr}[L_\beta] = 1 \), where \( B_\beta \) is a bi-diagonal random matrix of the form

\[
B_\beta \sim \begin{pmatrix}
\chi_{\beta M} & \chi_{\beta M-\beta} & \chi_{\beta M-2\beta} & \cdots & \chi_{\beta M-(N-1)\beta}
\end{pmatrix}
\]

and the random variable \( \chi_a \) has the density

\[
\frac{2^{1-a/2}}{\Gamma(a/2)} x^{a-1} e^{-x^2/2}, 
\]

\( x > 0 \).

The normalization constant, which can be computed in the closed form, is given by

\[
C_{N,M} = \frac{\Gamma\left(\frac{\beta}{2} MN\right) \left[\Gamma\left(1 + \frac{\beta}{2}\right)\right]^N}{\prod_{j=0}^{N-1} \Gamma\left(\frac{\beta}{2} (M - j)\right) \Gamma\left(1 + \frac{\beta}{2} (N - j)\right)}.
\]

See [21] for a derivation of \( C_{N,M} \).

The jpdf given in (1.4), for \( \beta = 2 \), can be traced to the work of Lloyd and Pagels [11] and of Page [16].

When \( M = N \), namely, the two subsystems Hilbert spaces are of equal dimension, and when \( \beta = 2 \), the jpdf is referred to in [17] as the ensemble of random density matrices with
respect to the Hilbert–Schmidt metric in the set of all density matrices of size $N$. The cases where $\beta = 1$ and $\beta = 4$ are also important because they describe systems with time-reversal invariance and rotational symmetry, respectively [13].

We assume $\alpha > -\frac{2}{\beta}$, so that $M - N > -1$. This is chosen so that the normalization constant (1.6) exists.

The study of the eigenvalues of the reduced density matrix $\rho_A = W$ is crucial for the understanding and utilization of entanglement. In principle, all the information about the spectral properties of the subsystem $A$, including its degree of entanglement, is encoded in the jpdf given by (1.4). For example, one classical measure of entanglement is the von Neumann entropy defined by

$$S = -\text{tr} \rho_A \ln \rho_A = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i.$$ 

The average entropy $\langle S \rangle$—with respect to the jpdf of (1.4)—was computed by Page [16] for large $N$ with $M \geq N$. It was found that

$$\langle S \rangle \approx \ln N - \frac{N}{2M},$$

which shows that a pure subsystem due to its coupling to the environment is more or less completely ‘randomized’.

Another important measure of entanglement is the smallest eigenvalue

$$\lambda_{\text{min}} = \min(\lambda_1, \lambda_2, \ldots, \lambda_N),$$

and its distribution function.

This provides, in addition to understanding the nature of entanglement, important information about the degree with which the effective dimension of the Hilbert space of the subsystem $A$ can be reduced [12]. Indeed, the average value $\langle \lambda_{\text{min}} \rangle$ of the smallest eigenvalue was studied by Znidaric [20] for the case $N = M$. Based on the exact $\langle \lambda_{\text{min}} \rangle$ for a small values of $N$, Znidaric conjectured that $\langle \lambda_{\text{min}} \rangle = 1/N^3$ for all $N$ in the complex case ($\beta = 2$). This conjecture was proved in [12] for $N = M$, both for the complex ($\beta = 2$) and the real ($\beta = 1$) cases. The problem of computing the distribution of $\lambda_{\text{min}}$ for unequal dimensions ($M > N$) was posed in [12], which remains open and it is this problem that we address.

In this paper we compute the distribution of the smallest eigenvalue $\lambda_{\text{min}}$ from the joint probability density function of (1.4) for general $\beta$. In particular, for the most interesting cases where $\beta = 1, 2, 4$, we will calculate the distribution of the smallest eigenvalue at finite $N$ and in a large $N$ limit to be described later. In the situation where $\beta = 1$, we assume $M - N$ is odd.

Let us fix the notations to be used throughout the paper.

Let $Q_{N,M}(x)$ be the probability that $\lambda_{\text{min}} \geq x$, that is,

$$Q_{N,M}(x) := \text{Prob} [\lambda_{\text{min}} \geq x] = \text{Prob} [\lambda_1 \geq x, \lambda_2 \geq x, \ldots, \lambda_N \geq x],$$

and the probability density function of the smallest eigenvalues is of course

$$P_{N,M}(x) = -\frac{d}{dx} Q_{N,M}(x).$$

In the situations where $\beta = 2, \alpha = 0$ ($M = N$), and $\beta = 1, \alpha = -\frac{1}{\beta}$ ($M = N$), closed form expressions of $P_{N,M}(x)$ and the moments of $\lambda_{\text{min}}$ were obtained in [12]. We will extend the above results and solve a problem posed in [12], namely, the determination of the probability distribution of $\lambda_{\text{min}}$ in the case of unequal dimensions where $M > N$.

Our results are summarized as follows.

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In this paper we extend the known results mentioned above to the cases where
\[ \frac{\beta}{2\alpha} = m \geq 0, \quad m \in \mathbb{N} \quad \text{and} \quad \beta > 0. \]  
(1.9)

We will see that \( Q_{N,M}(x) \) and \( P_{N,M}(x) \) are expressed in terms of the Jack symmetric polynomials.

Furthermore, after a scaling
\[
x = \frac{y}{4N^3},
\]
followed by the limit \( N \to \infty \), we show that, for a fixed \( m \),
\[
Q(y) := \lim_{N \to \infty} Q_{N,M} \left( \frac{y}{4N^3} \right) = \exp \left( -\beta \frac{y^8}{8} \right) {_0F_1}^{(\beta/2)} (2m/\beta; y_1, \ldots, y_m) |_{y_1 = y_2 = \cdots = y_m = \frac{y}{4N^3}} \quad (1.10)
\]
and
\[
P(y) := \lim_{N \to \infty} \frac{1}{4N^3} P_{N,M} \left( \frac{y}{4N^3} \right) = A_{m,\beta} \frac{y^m e^{-\beta y^8/8} \Gamma(1 + \beta/2)}{\Gamma(1 + m)\Gamma(1 + m + \beta/2)} \quad (1.11)
\]
where
\[
A_{m,\beta} = 4^m \frac{(\beta/2)^{\beta/2 + 2m + 1}}{\Gamma(1 + \beta/2)} \frac{\Gamma(1 + \beta/2)}{\Gamma(1 + m)\Gamma(1 + m + \beta/2)}.
\]
(1.12)

Here
\[
_0F_1 \left( \frac{2m}{\beta}; y_1, \ldots, y_m \right)
\]
is a generalized hypergeometric function.

Therefore the limiting distributions given by (1.10) and (1.11) are the same as the corresponding results for the Laguerre beta-ensemble, obtained in [3, 5, 6]. See also the relevant references in [3].

In the next section, we introduce a multiple integral from the jpdf which ultimately determines the smallest eigenvalue distribution for \( \beta > 0 \). In section 3, by using an alternative method of Mehta [13] we compute the smallest eigenvalue distribution for \( \beta = 2 \), and \( \alpha = M - N \) a non-negative integer, which we later specialize to \( \alpha = 2 \). The conclusion can be found in section 4.

2. Distribution of the smallest eigenvalue

For convenience let us initially take the trace to be \( t \), where \( t > 0 \). We define a function of \( x \) and \( t \) as:
\[
I(x, t) := \int_{[1,\infty]^N} A \left( \sum_{i=1}^{N} \lambda_i - t \right) \prod_{i=1}^{N} \lambda_i^{-\frac{\alpha}{2}} \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^\beta d^N \lambda,
\]
(2.1)

where \( d^N \lambda := d\lambda_1 \cdots d\lambda_N \). It clear that the distribution \( Q_{N,M}(x) \) is given by
\[
Q_{N,M}(x) = \frac{I(x, 1)}{I(0, 1)} \quad (2.2)
\]
With the parameter \( t \) we may now take a Laplace transform of \( I(x, t) \) with respect to \( t \).

The numerator of (2.2) is obtained by a Laplace transform of \( I(x, t) \) with respect to \( t \) and followed by setting \( t = 1 \) after a Laplace inversion. The denominator of (2.2) is \( 1/C_{N,M} \), with \( C_{N,M} \) as in (1.6).
To proceed, we first take the Laplace transform \( I(x, t) = \int_0^\infty I(x, t) e^{-st} dt \) with respect to \( t \) and compare the results to those of Laguerre \( \beta \)-ensemble, which has been well studied. Thus, 
\[
\int_0^\infty I(x, t) e^{-st} dt = \int_{[x, \infty)^N} e^{-x \sum_{i=1}^N \lambda_i} \prod_{1 \leq j < k \leq N} |\lambda_j - \lambda_k|^\beta d^N \lambda.
\]
(2.3)
After a linear shift followed a scale change, 
\[
zi = 2s(\lambda_i - x) \quad \text{or} \quad \lambda_i = x + \frac{\beta}{2s} z_i, \quad i = 1, \ldots, N,
\]
we find, 
\[
\int_0^\infty I(x, t) e^{-st} dt = \left( \frac{\beta}{2s} \right)^{2MN} e^{-sN} f \left( \frac{2x}{\beta}, \frac{\beta}{2} \right)
\]
(2.4)
where 
\[
J(x, \gamma) := \int_{[0, \infty)^N} \prod_{i=1}^N (x + zi) \gamma e^{-\frac{2}{\beta} zi} \prod_{1 \leq j < k \leq N} |z_j - z_k|^\beta d^N z.
\]
(2.5)
An inverse Laplace transform leaves 
\[
I(x, t) = L^{-1} \left[ \left( \frac{\beta}{2s} \right)^{2MN} e^{-sN} f \left( \frac{2x}{\beta}, \frac{\beta}{2} \right) \right](t).
\]
(2.6)
Hence, 
\[
Q_{N,M}(x) = \frac{I(x, 1)}{I(0, 1)} = L^{-1} \left[ \left( \frac{\beta}{2s} \right)^{2MN} e^{-sN} f \left( \frac{2x}{\beta}, \frac{\beta}{2} \right) \right](1)
\]
\[
= \Gamma \left( \frac{\beta}{2} MN \right) L^{-1} \left[ \frac{e^{-sN} f \left( \frac{2x}{\beta}, \frac{\beta}{2} \right)}{s^{2MN} f \left( 0, \frac{\beta}{2} \alpha \right)} \right](1)
\]
(2.7)
where we have used the following properties of the inverse Laplace transform: 
\[
L^{-1}[s^{-\alpha} \theta(t)] = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \theta(t), \quad \Re(\alpha) > 0,
\]
(2.9)
and 
\[
L^{-1}[f(s) e^{\sigma}](t) = L^{-1}[f(s)](\sigma + t).
\]
(2.10)
In (2.9) the Heaviside function \( \theta(t) \) is 1 for \( t > 0 \) and \( \theta(t) \) is 0, for \( t < 0 \).
We should like to mention that the numerator of 
\[
\frac{J(x, \frac{\beta}{2} \alpha)}{J(0, \frac{\beta}{2} \alpha)}
\]
(2.11)
is an important but difficult multiple integral, while the denominator can be evaluated as a particular Selberg integral.
Observe that (2.11) can be interpreted as the moment of order \( \frac{q}{2} \alpha \) of the characteristic polynomial of the equivalent tri-diagonal matrix model of Laguerre \( \beta \)-ensembles [4]. In particular, for \( \beta = 1, 2, 4 \), \( J(x, \alpha \beta /2) \) can be evaluated in closed form in terms of Laguerre polynomials, see chapter 22 of [13].

We recall for the reader the definition of the generalized hypergeometric function with positive parameter \( v \). This arises in an extensive investigation by Kaneko [9] on the multi-variable version of Amoto’s generalization of the Selberg integral. See also [3, 6].

The generalized hypergeometric function of \( m \) variables is as follows:

\[
p_{F_q}^{(v)}(a_1, \ldots, a_p; b_1, \ldots, b_q; x_1, \ldots, x_m) := \sum_{k=0}^{\infty} \sum_{|k| \leq k} \frac{[a_1]^{(v)} \cdots [a_p]^{(v)} [b_1]^{(v)} \cdots [b_q]^{(v)}}{[k]^{(v)} \cdots [k]^{(v)}} C^v_k (x_1, \ldots, x_m),
\]

where the sum over \( |k| = k \) in (2.12) is the sum over all partitions \( (\kappa_1, \ldots, \kappa_m) \) of non-negative integers such that

\[
\kappa_1 \geq \cdots \geq \kappa_m, \quad \sum_{j=1}^{m} \kappa_j = k.
\]

The generalized factorial function \( [a]^{(v)}_k \) is defined by

\[
[a]^{(v)}_k := \prod_{j=1}^{m} \left( a - \frac{1}{v} (j - 1) \right)^{\kappa_j},
\]

with

\[
(a)_k := a(a + 1) \cdots (a + k - 1).
\]

The \( m \)-variable function \( C^v_k(x_1, \ldots, x_m) \) is a homogeneous symmetric polynomial of degree \( k \) and it is proportional to the so-called Jack symmetric polynomials, see [9]. A classic reference for Jack symmetric polynomials is [18].

In the one-variable case, where \( m = 1 \), \( C^v_1(x) = x^k \), the generalized hypergeometric function reduces the one-variable hypergeometric function \( p_{F_q}(x) \).

For the non-negative integer \( m \), Forrester [6] showed that the quotient of (2.11) can be expressed as a terminating generalized hypergeometric function

\[
\frac{J(x, m)}{J(0, m)} = \text{i} F_1^{1/2}(-N; 2m/\beta; x_1, \ldots, x_m)_{|x_1=x_2=\cdots=x_m=-x},
\]

where \( m = \frac{q}{2} \alpha \) is a non-negative integer. Using the above expression, we obtain from (2.8) that

\[
Q_{N,M}(x) = \Gamma \left( \frac{\beta}{2} MN \right) \left[ \frac{1}{\sqrt{\beta} MN} F_1^{1/2}(-N; 2m/\beta; x_1, \ldots, x_m)_{|x_1=\cdots=x_m=-2x/\beta} \right] (1 - Nx)
\]

\[
= \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|k| \leq k} \frac{[-N]^k_{1/2} \left( 2m/\beta \right)}{\left( MNk \right)_{1/2} \left( MNk - k \right)} \frac{\left( 2x/\beta \right)}{\beta^{k/2}} \right] (1 - Nx)
\]

\[
= \sum_{k=0}^{\infty} \sum_{|k| \leq k} \frac{\Gamma(\frac{\beta}{2} MN)}{\Gamma(\frac{\beta}{2} MN - k)} \left[ \frac{[-N]^k_{1/2} \left( 2m/\beta \right)}{\left( MNk \right)_{1/2} \left( MNk - k \right)} \right] \left( \frac{2x}{\beta} \right)^k \frac{\Gamma(\frac{\beta}{2} k(1-m))}{k!} (1 - Nx)
\]

\[
\times \left( 1 - Nx \right)^{\frac{q}{2} MN - k - 1} \theta(1 - Nx)
\]
and compute all the moments of \( \lambda \).

\[
\mu_p := \langle \lambda^{p}_\text{min} \rangle = -\int_0^\infty x^p \, dQ_{N,M}(x) = p \int_0^\infty x^{p-1} \, Q_{N,M}(x) \, dx
= p \sum_{k=0}^\infty \sum_{|a|=k} \left( -\frac{2}{\beta} \right)^k \frac{\Gamma(\frac{\beta}{p+1})}{\Gamma(\frac{\beta}{MN-k})} \frac{[-N]^{(\beta/2)}_k}{[2m/\beta]^{(\beta/2)}_k} \frac{C^{(\beta/2)}_k(1) }{k!} \frac{\lambda^k}{(1-N_x)^{\frac{\beta}{2}}} \theta(1-N_x).
\] (2.17)

Here \( 1^m = (1, \ldots, 1) \) where the number of variables is \( m \). We emphasize that (2.17) is a finite sum.

Therefore, for finite \( N \), we obtained an expression of the smallest eigenvalue in terms of Jack polynomials evaluated at \((1, 1, \ldots, 1)\) that all the \( m \) entries are unity.

Next, we compute the moments of \( \lambda_{\text{min}} \) and its limiting distribution after scaling with \( N \).

**Moments of \( \lambda_{\text{min}} \).** From the explicit expression of the distribution in (2.17), one can easily compute all the moments of \( \lambda_{\text{min}} \). For the \( p \)th moment, \( p \geq 1 \), we have that

\[
\mu_p := \langle \lambda^{p}_\text{min} \rangle = -\int_0^\infty x^p \, dQ_{N,M}(x) = p \int_0^\infty x^{p-1} \, Q_{N,M}(x) \, dx
= p \sum_{k=0}^\infty \sum_{|a|=k} \left( -\frac{2}{\beta} \right)^k \frac{\Gamma(\frac{\beta}{p+1})}{\Gamma(\frac{\beta}{MN-k})} \frac{[-N]^{(\beta/2)}_k}{[2m/\beta]^{(\beta/2)}_k} \frac{C^{(\beta/2)}_k(1) }{k!} \frac{\lambda^k}{(1-N_x)^{\frac{\beta}{2}}} \theta(1-N_x).
\] (2.18)

**Limit distribution of \( \lambda_{\text{min}} \).** We now re-scale \( x \) so that \( x = \frac{y}{4N^2} \). Consider a \( \kappa \). If \( m \) is fixed, that is, \( M-N \) does not depend on \( N \), a simple computation produces two useful formulas:

\[
\frac{\Gamma(\frac{\beta}{p+1})}{\Gamma(\frac{\beta}{MN-k})} \frac{[-N]^{(\beta/2)}_k}{N^\beta} \to \left( -\frac{\beta}{2} \right)^k, \quad N \to \infty \tag{2.19}
\]

and

\[
\left( 1 - \frac{y}{4N^2} \right)^{\frac{\beta}{2}k} \to e^{-\frac{\beta}{4}y^2}, \quad N \to \infty. \tag{2.20}
\]

With (2.19) and (2.20), we find

\[
Q(y) = \lim_{N \to \infty} Q_{N,M} \left( \frac{y}{4N^2} \right)
= \sum_{k=0}^\infty \sum_{|a|=k} \frac{1}{[2m/\beta]^{(\beta/2)}_k} \frac{C^{(\beta/2)}_k(1) }{k!} \left( \frac{y}{4} \right)^k e^{-\frac{\beta}{4}y^2}
= e^{-\frac{\beta}{4}y^2} F_1^{(\beta/2)}(2m/\beta; \gamma_1, \ldots, \gamma_m | y_1 = \ldots = y_m = \frac{\beta}{2}). \tag{2.21}
\]

We see that our smallest eigenvalue distribution with the fixed-trace constraint, namely (2.21), is the same as that in [6] for the Laguerre \( \beta \)-ensemble without the fixed-trace constraint. Our result seems to indicate that global constraint does not influence the local eigenvalue distribution at least after suitable scaling followed by a \( N \to \infty \) limit.

Performing a similar computation, we find

\[
P(y) = \lim_{N \to \infty} \frac{1}{4N^2} P_{N,M} \left( \frac{y}{4N^2} \right) = -Q'(y)
= A_{m,\beta} y^m e^{-\frac{\beta}{4}y^2} \frac{F_1^{(\beta/2)}(2m/\beta+2; \gamma_1, \ldots, \gamma_m | y_1 = \ldots = y_m = \frac{\beta}{2})}{\Gamma(1/2)} \tag{2.22}
\]

where

\[
A_{m,\beta} = 4^m (\beta/2)^{\beta/2+2m+1} \frac{\Gamma(1+\beta/2)}{\Gamma(1+m)\Gamma(1+m+\beta/2)}. \tag{2.23}
\]
Note that since $P(y) = -Q'(y)$, (2.22) follows from the relation between the exact expressions of $P(y)$ and $Q(y)$ in [6].

In particular, when $m = 0$, we find $Q(y) = e^{-\beta y}$. When $m = 1$, we have

$$Q(y) = 2^{-1+\frac{1}{\beta}} \Gamma\left(\frac{2}{\beta}\right) e^{-\beta y} y^{\frac{1}{\beta}-1} I_{\frac{1}{\beta}-1}(\sqrt{y}).$$

Here $I_{\rho}(x)$ denotes the modified Bessel function of the first kind [1].

3. Another exact expression at $\beta = 2$

In this section, we assume that $\beta = 2$ and $\alpha = M - N$ is a non-negative integer. For $\beta = 1$ or 4, similar but more sophisticated results in chapter 22 of [13] can be used to obtain the corresponding results in this section.

Our main result is stated below:

$$Q_{N,M}(x) = \Gamma(MN)x^{MN-1} \mathcal{L}^{-1}\left[\frac{1}{x^{MN}} \det[L_{N+k-l}^{(j)}(-s)]_{k,l=0}^{a-1}\left(\frac{1-Nx}{x}\right)\right].$$

(3.1)

Here $L_N^{(p)}(x)$ is a Laguerre polynomial given by

$$L_N^{(p)}(x) = \sum_{j=0}^{N} \frac{(N + p)_j}{N - j} \frac{(-x)^j}{j!}.$$  

(3.2)

See the standard reference [19] for Laguerre polynomials.

Again we begin with equation (2.8), but with $\beta = 2$. We find

$$Q_{N,M}(x) = \Gamma(MN)x^{MN-1} \mathcal{L}^{-1}\left[\frac{1}{x^{MN}} \frac{J(sx, \alpha)}{J(0, \alpha)}\right] (1-Nx)$$

$$= \Gamma(MN) \frac{J(0,0)}{J(0, \alpha)} x^{MN-1} \mathcal{L}^{-1}\left[\frac{1}{x^{MN}} \frac{J(sx, \alpha)}{J(0, 0)}\right] (1-Nx).$$

(3.3)

The integral (equation (17.6.5), p 321, [13]) implies

$$J(0, \alpha) = \prod_{j=1}^{N} \Gamma(1+j) \Gamma(\alpha + j),$$

(3.4)

and hence

$$\frac{J(0, 0)}{J(0, \alpha)} = \prod_{j=1}^{N} \frac{\Gamma(j)}{\Gamma(\alpha + j)} = \left(\prod_{j=1}^{N} (j)_{a}\right)^{-1}. $$

(3.5)

On the other hand, it follows from equation (22.2.28), p 416, [13], that

$$(-1)^a \mathcal{N} J(-x, \alpha) \frac{J(0,0)}{J(0, \alpha)} = \left(\prod_{j=0}^{a-1} \Gamma(j)\right)^{-1} \det\left[\frac{d^i}{dx^i} C_{N+k}(x)\right]_{k,l=0}^{a-1},$$

(3.6)

where $C_j(x)$ is the monic polynomial related to the Laguerre polynomials as follows:

$$C_j(x) = (-1)^j j! L_j^{(0)}(x).$$

By an elementary differential-difference relation satisfied by the Laguerre polynomials,

$$\frac{d}{dx} L_j^{(p)}(x) = - L_{j-1}^{(p+1)}(x),$$

(3.8)
we obtain

\[
(-1)^N J(-x, \alpha) = J(0, 0) = \left( \prod_{l=0}^{a-1} \frac{1}{a-l} \right) \prod_{l=0}^{a-1} \left( 1 + (-1)^{N+l} N! \right) \left[ \frac{d^k}{dx^k} L_N^{(0)}(x) \right]^{a-1}_{k,l=0} \\
= (-1)^N \prod_{l=0}^{a-1} \left( \frac{N+l}{l!} \right) \det \left[ L_N^{(i)}(x) \right]^{a-1}_{k,l=0} \\
= (-1)^N \prod_{l=0}^{a-1} \left( \frac{l+1}{l!} \right) \det \left[ L_N^{(i)}(x) \right]^{a-1}_{k,l=0}. \tag{3.9}
\]

Note that

\[
\prod_{j=1}^{N} (j)_{\alpha} = \prod_{l=0}^{a-1} (d + 1)^N. \tag{3.10}
\]

Combining equations (3.5), (3.9) and (3.10), we obtain

\[
J(-x, \alpha) = \det \left[ L_N^{(i)}(x) \right]^{a-1}_{k,l=0}. \tag{3.11}
\]

Furthermore, from (3.3), we arrive at the following expression:

\[
Q_{N,M}(x) = \Gamma(MN) L^{-1} \left[ \frac{1}{x^{MN}} \det \left[ L_N^{(i)}((-sx)) \right]^{a-1}_{k,l=0} \right] (1 - Nx) \\
= \Gamma(MN) x^{MN} L^{-1} \left[ \frac{1}{x} \det \left[ L_N^{(i)}((-sx)) \right]^{a-1}_{k,l=0} \right] (1 - Nx) \\
= \Gamma(MN) x^{MN} L^{-1} \left[ \frac{1}{x^{MN}} \det \left[ L_N^{(i)}((-x)) \right]^{a-1}_{k,l=0} \right] \left( \frac{1 - Nx}{x} \right). \tag{3.12}
\]

where we have used a property of inverse Laplace transform:

\[
L^{-1} \left[ \frac{1}{B} f \left( \frac{B}{x} \right) \right](t) = \mathcal{L}^{-1} \left[ f(s) \right](bt).
\]

Now we focus on the case \( \alpha = M - N = 2 \). Note that since

\[
L_N^{(\alpha)}(-s) = \frac{(\rho + 1)N}{N!} \sum_{j=0}^{N} \frac{(-N)_j}{(\rho + 1)_j} \frac{(-s)^j}{j!}, \tag{3.13}
\]

we have

\[
\det \left[ L_N^{(i)}((-x)) \right]^{a-1}_{k,l=0} = L_N^{(0)}(-s) L_N^{(1)}(-s) - L_N^{(0)}(-s) L_N^{(1)}(-s) \\
= (N + 1) \sum_{i=0}^{N} \frac{(-N)_i}{i!} \frac{(-s)^i}{(2)_j} \frac{(-s)^j}{j!} \\
- N \sum_{i=0}^{N+1} \frac{(-N - 1)_i}{i!} \frac{(-s)^i}{(2)_j} \frac{(-s)^j}{j!} \\
= \sum_{i} \sum_{j} \frac{(-s)^{i+j}}{i! j!} \frac{(N + 1) (-N)_i (-N)_j}{(1)_i (2)_j} \\
- \sum_{i} \sum_{j} \frac{(-s)^{i+j}}{i! j!} \frac{(-N)_i (-N)_j}{N + 1} \\
= \sum_{i} \sum_{j} \frac{(-s)^{i+j}}{i! j!} \frac{(N + 1)(1 + j - i)}{N + 1 - i}. \tag{3.14}
\]
Here we have used the fact

\[(N + 1)(-N)_i(-N)_j \rightarrow -N(-N - 1)_i(-N + 1)_j = (-N)_i(-N)_j \frac{(N + 1)(1 + j - i)}{N + 1 - i}.
\]

In addition, since we are only interested in the case of very large \(N\), we can always write the sum in the form of (3.14) for fixed \(i, j\). Thus, it follows from equation (3.12) that

\[
Q_{N,M}(x) = \Gamma(MN)x^{MN-1} \sum_{i,j} \frac{(-1)^{i+j}}{i!j!} \frac{(-N)_i(-N)_j}{(1)_i(2)_j} \frac{(N + 1)(1 + j - i)}{N + 1 - i}
\]

\[
\times \left[ \frac{1}{s^{MN-i-j}} \right] \left( \frac{1 - Ns}{x} \right)^{s^{MN-i-j}}
\]

\[
\times \frac{\theta \left( \frac{1-Nx}{x} \right)}{\Gamma(MN-i-j)} \left( \frac{1 - Ns}{x} \right)^{s^{MN-i-j}}
\]

\[
= (1 - Nx)^{MN-1} \sum_{i,j} \frac{(-1)^{i+j}}{i!j!} \frac{(-N)_i(-N)_j}{(1)_i(2)_j} \frac{(N + 1)(1 + j - i)}{N + 1 - i}
\]

\[
\times \frac{\Gamma(MN)}{\Gamma(MN-i-j)} \left( \frac{1 - Ns}{x} \right)^{s^{MN-i-j}}.
\]

Setting \(s = \frac{y}{2N} \). Consider a fixed \(i, j\). We now deal with the limit of the individual factors on the right-hand side of equation (3.15) as \(N \rightarrow \infty\).

It is easily seen that (with \(x = \frac{y}{2N} \)), as \(N \rightarrow \infty\),

\[
(1 - Nx)^{MN-1} \rightarrow e^{-\frac{y}{4}}.
\]

\[
(1 - Nx)^{+j} \rightarrow 1,
\]

\[
\frac{(-N)_i(-N)_j}{N^{i+j}} \rightarrow (-1)^{i+j},
\]

\[
\frac{\Gamma(MN)}{\Gamma(MN-i-j)} N^{2i+2j} \rightarrow 1.
\]

Hence,

\[
\lim_{N \rightarrow \infty} Q_{N,M} \left( \frac{y}{4N^2} \right) = e^{-\frac{y}{4}} \sum_{i,j=0}^{\infty} \frac{(y/4)^{i+j}}{i!j!} \frac{1 + j - i}{(1)(2)} \left( \frac{1}{i!j!} \right) \left( \frac{1}{(i+1)(j+1)} \right)
\]

\[
= e^{-\frac{y}{4}} \sum_{i,j=0}^{\infty} \frac{(y/4)^{i+j}}{i!j!} \left( \frac{1}{i!j!} \right) \left( \frac{1}{(i+1)(j+1)} \right)
\]

\[
= e^{-\frac{y}{4}} \sum_{i=0}^{\infty} \frac{(y/4)^i}{i!} \sum_{j=0}^{\infty} \frac{(y/4)^j}{j!} - \frac{y}{4} e^{-\frac{y}{4}} \sum_{i=1}^{\infty} \frac{(y/4)^{i-1}}{(i-1)!} \sum_{j=0}^{\infty} \frac{(y/4)^j}{j!(j+1)!}
\]

\[
= e^{-\frac{y}{4}} \left( I_0^2(\sqrt{y}) - I_1^2(\sqrt{y}) \right),
\]

where the modified Bessel function of the first kind is given by

\[
I_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu + 1)} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\Gamma(\nu + k + 1)}.
\]
4. Conclusion

In this paper we have studied the exact probability distribution of the smallest eigenvalue of the density matrix of an entangled random pure state. This turns out to be the same as the smallest eigenvalue distribution of the fixed-trace Laguerre random matrix ensemble. We obtained, for a bipartite quantum system composed of two subsystems whose Hilbert spaces have dimensions $M$ and $N$ respectively, the exact smallest eigenvalue distribution for all $M \geq N$ (in the $\beta = 1$ case we assume $M - N$ is an odd integer). The distributions are expressed as the evaluation of a certain hypergeometric functions at special points and solve an open problem in quantum entanglement.

Our results not only provide important information on the entanglement of a bipartite quantum system in a random pure state, but also demonstrate the intimate relations between the entanglement of bipartite quantum systems and the fixed-trace Laguerre ensemble. Based on our results we may conclude that in a broad sense the global constraint does not influence local correlations at least in a certain large $N$ limit. An indication that this may have wider validity can be found in [10] where the authors studied the kernel and the $n$-point correlation functions of the fixed and bounded trace Laguerre ensembles. There it was found that the suitably scaled kernel in the large $N$ limit converges to the unrestricted kernel in the bulk, hard and soft edges, according to the terminology of Tracy and Widom. However, we would like to mention that the ‘matching up’ of the kernels in the constrained and the unrestricted ensembles does not imply that the ‘matching up’ of the corresponding distribution functions (in this instance the smallest eigenvalue distributions). Only a rather elaborate computation shows that this is the case.

Although there is currently no obviously effective characterization of the degree of entanglement, the von Neumann entropy, however, is considered to be useful as a measurement of entanglement in bipartite quantum systems. The distribution function of the von Neumann entropy may be successfully tackled using our approach.

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