Abstract

We consider the hashing of a set $X \subseteq U$ with $|X| = m$ using a simple tabulation hash function $h : U \rightarrow [n] = \{0, \ldots, n - 1\}$ and analyse the number of non-empty bins, that is, the size of $h(X)$. We show that the expected size of $h(X)$ matches that with fully random hashing to within low-order terms. We also provide concentration bounds. The number of non-empty bins is a fundamental measure in the balls and bins paradigm, and it is critical in applications such as Bloom filters and Filter hashing. For example, normally Bloom filters are proportioned for a desired low false-positive probability assuming fully random hashing (see en.wikipedia.org/wiki/Bloom_filter). Our results imply that if we implement the hashing with simple tabulation, we obtain the same low false-positive probability for any possible input.
1 Introduction

We consider the balls and bins paradigm where a set $X \subseteq U$ of $|X| = m$ balls are distributed into a set of $n$ bins according to a hash function $h : U \rightarrow [n]$. We are interested in questions relating to the distribution of $|h(X)|$, for example: What is the expected number of non-empty bins? How well is $|h(X)|$ concentrated around its mean? And what is the probability that a query ball lands in an empty bin? These questions are critical in applications such as Bloom filters [3] and Filter hashing [7].

In the setting where $h$ is a fully random hash function, meaning that the random variables $(h(x))_{x \in U}$ are mutually independent and uniformly distributed in $[n]$, the situation is well understood. The random distribution process is equivalent to throwing $m$ balls sequentially into $n$ bins by for each ball choosing a bin uniformly at random and independently of the placements of the previous balls. The probability that a bin becomes empty is thus $(1 - 1/n)^m$; so the expected number of non-empty bins is exactly $\mu_0 := n(1 - (1 - 1/n)^m)$ and, unsurprisingly, the number of non-empty bins turns out to be sharply concentrated around $\mu_0$ (see for example Kamath et al. [8] for several such concentration results).

In practical applications fully random hashing is unrealistic and so it is desirable to replace the fully random hash functions with realistic and implementable hash functions that still provide at least some of the probabilistic guarantees that were available in the fully random setting. However, as the mutual independence of the keys is often a key ingredient in proving results in the fully random setting most of these proofs do not carry over. Often the results are simply no longer true and if they are one has to come up with alternative techniques for proving them.

In this paper, we study the number of non-empty bins when the hash function $h$ is chosen to be a simple tabulation hash function [13, 21]; which is very fast and easy to implement (see description below in Section 1.1). We provide estimates on the expected size of $|h(X)|$ which asymptotically match those with fully random hashing on any possible input. To get a similar match within the classic $k$-independence paradigm [20], we would generally need $k = \Omega((\log n)/(\log \log n))$. For comparison, simple tabulation is the fastest known 3-independent hash function [14]. We will also study how $|h(X)|$ is concentrated around its mean.

Our results complements those from [14], which show that with simple tabulation hashing, we get Chernoff-type concentration on the number of balls in a given bin when $m \gg n$. For example, the results from [14] imply that all bins are non-empty with high probability (whp) when $m = \omega(n \log n)$. More precisely, for any constant $\gamma > 0$, there exists a $C > 0$ such that if $m \geq C n \log n$, all bins are non-empty with probability $1 - O(n^{-\gamma})$. As a consequence, we only have to study $|h(X)|$ for $m = O(n \log n)$ below. On the other hand, [14] does not provide any good bounds on the probability that a bin is non-empty when, say, $m = n$. In this case, our results imply that a bin is non-empty with probability $1 - 1/e \pm o(1)$, as in the fully random case. The understanding we provide here is critical to applications such as Bloom filters [3] and Filter hashing [14], which we describe in section 2.1 and 2.2.

We want to emphasize the advantage of having many complementary results for simple tabulation hashing. An obvious advantage is that simple tabulation can be reused in many contexts, but there may also be applications that need several strong properties to work in tandem. If, for example, an application has to hash a mix of a few heavy balls and many light balls, and the hash function do not know which is which, then the results from [14] give us the Chernoff-style concentration of the number of light balls in a bin while the results of this paper give us the right probability that a bin contains a heavy ball. For another example where an interplay of properties becomes important see section 2.2 on Filter hashing. The reader is

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1Here we use “asymptotically” in the classic mathematical sense to mean equal to within low order terms, not just within a constant factor.
referred to [13] for a survey of results known for simple tabulation hashing, as well as examples where simple tabulation does not suffice and where slower more sophisticated hash functions are needed.

1.1 Simple tabulation hashing

Recall that a hash function $h$ is a map from a universe $U$ to a range $R$ chosen with respect to some probability distribution on the set of all such functions. If the distribution is uniform (equivalently the random variables $(h(x))_{x \in U}$ are mutually independent and uniformly distributed in $R$) we will say that $h$ is fully random.

Simple tabulation was introduced by Zobrist [21]. For simple tabulation $U = [u] = \{0, \ldots, u-1\}$ and $R = [2^r]$ for some $r \in \mathbb{N}$. The keys $x \in U$ are viewed as vectors $x = (x[0], \ldots, x[c-1])$ of $c = O(1)$ characters with each $x[i] \in \Sigma := [u^{1/c}]$. The simple tabulation hash function $h$ is defined by

$$h(x) = \bigoplus_{i \in [c]} h_i(x[i]),$$

where $h_0, \ldots, h_{c-1} : \Sigma \to R$ are independent fully random hash functions and where $\oplus$ denotes the bitwise XOR. What makes it fast is that the character domains of $h_0, \ldots, h_{c-1}$ are so small that they can be stored as tables in fast cache. Experiments in [14] found that the hashing of 32-bit keys divided into 4 8-bit characters was as fast as two 64-bit multiplications. Note that on machines with larger cache, it may be faster to use 16-bit characters. As useful computations normally involve data and hence cache, there is no commercial drive for developing processors that do multiplications much faster than cache look-ups. Therefore, on real-world processors, we always expect cache based simple tabulation to be at least comparable in speed to multiplication. The converse is not true, since many useful computations do not involve multiplications. Thus there is a drive to make cache faster even if it is too hard/expensive to speed up multiplication circuits.

Other important properties include that the $c$ character table lookups can be done in parallel and that when initialised the character tables are not changed. For applications such as Bloom filters where more than one hash function is needed another nice property of simple tabulation is that the output bits are mutually independent. Using $(kr)$-bit hash values is thus equivalent to using $k$ independent simple tabulation hash functions each with values in $[2^r]$. This means that we can get $k$ independent $r$-bit hash values using only $c$ lookups of $(kr)$-bit strings.

1.2 Main Results

We will now present our results on the number of non-empty bins with simple tabulation hashing.

The expected number of non-empty bins: Our first theorem compares the expected number of non-empty bins when using simple tabulation to that in the fully random setting. We denote by $p_0 = 1 - (1 - 1/n)^m < m/n$ the probability that a bin becomes non-empty and by $\mu_0 = np_0$ the expected number of non-empty bins when $m$ balls are distributed into $n$ bins using fully random hashing.

**Theorem 1.1.** Let $X \subseteq U$ be a fixed set of $|X| = m$ balls. Let $y \in [n]$ be any bin and suppose that $h : U \to [n]$ is a simple tabulation hash function. If $p$ denotes the probability that $y \in h(X)$ then

$$|p - p_0| \leq \frac{m^{2-1/c}}{n^2} \quad \text{and hence} \quad |E[|h(X)|] - \mu_0| \leq \frac{m^{2-1/c}}{n}.$$
If we let $y$ depend on the hash of a distinguished query ball $q \in U \setminus X$, e.g., $y = h(q)$, then the bound on $p$ above is replaced by the weaker $|p - p_0| \leq \frac{2m^{2-1/c}}{n^2}$.

The last statement of the theorem is important in the application to Bloom filters where we wish to upper bound the probability that $h(q) \in h(X)$ for a query ball $q \notin X$.

To show that the expected relative error $\mathbb{E}[|h(X)|] - \mu_0 / \mu_0$ is always small, we have to complement Theorem 1.1 with the result from [14] that all bins are full, whp, when $m \geq Cn \log n$ for some large enough constant $C$. In particular, this implies $\mathbb{E}[|h(X)|] - \mu_0 / \mu_0 \leq 1/n$ when $m \geq Cn \log n$. The relative error from Theorem 1.1 is maximized when the bounds from Corollary 1.2.

Let $X \subseteq U$ be a fixed sets of $|X| = m$ balls and let $h : U \to [n]$ be a simple tabulation hash function. Then $\mathbb{E}[|h(X)|] - \mu_0 / \mu_0 = \tilde{O}(n^{-1/c})$.

As discussed above, the high probability bound from [14] takes over when the bounds from Theorem 1.1 get weaker. This is because the analysis in this paper is of a very different nature than that in [14].

**Concentration of the number of non-empty bins:** We now consider the concentration of $|h(X)|$ around its mean. In the fully random setting it was shown by Kanna et al. [3] that the concentration of $|h(X)|$ around $\mu_0$ is sharp: For any $\lambda \geq 0$ it holds that

$$\Pr(|h(X)| - \mu_0 | \geq \lambda) \leq 2 \exp \left(- \frac{\lambda^2 (n - 1/2)}{\mu_0 (2n - \mu_0)} \right) \leq 2 \exp \left(- \frac{\lambda^2}{2 \mu_0} \right),$$

which for example yields that $|h(X)| = \mu_0 + \mathcal{O}(\sqrt{\mu_0 \log n})$ whp, that is, with probability $1 - O(n^{-\gamma})$ for any choice of $\gamma = O(1)$. Unfortunately we cannot hope to obtain such a good concentration using simple tabulation hashing. To see this, consider the set of keys $[2^\ell] \times [m/2^\ell]$ for any constant $\ell$, e.g. $\ell = 1$, and let $\mathcal{E}$ be the event that $h_i(0) = h_i(1)$ for $i = 0, \ldots, \ell - 1$. This event occurs with probability $1/n^\ell$. Now if $\mathcal{E}$ occurs then the keys of $X_i = [2^\ell] \times \{i\}$ all hash to the same value namely $h_0(0) \oplus \cdots \oplus h_{\ell-1}(0) \oplus h_{\ell}(i)$. Furthermore, these values are independently and uniformly distributed in $[n]$ for $i \in [m/2^\ell]$ so the distribution of $|h(X)|$ becomes identical to the distribution of non-empty bins when $m/2^\ell$ balls are thrown into $n$ bins using truly random hashing. This observation ruins the hope of obtaining a sharp concentration around $\mu_0$ and shows that the lower bound in the theorem below is best possible being the expected number of non-empty bins when $\Omega(m)$ balls are distributed into $n$ bins.

**Theorem 1.3.** Let $X \subseteq U$ be a fixed sets of $|X| = m$ keys. Let $h : U \to [n]$ be a simple tabulation hash function. Then whp

$$|h(X)| \geq n \left(1 - \left(1 - \frac{1}{n}\right)^{\Omega(m)}\right)$$

As argued above, the lower bound in Theorem 1.3 is optimal. Settling with a laxer requirement than high probability, it turns out however that $|h(X)|$ is somewhat concentrated around $\mu_0$. This is the content of the following theorem which also provides a high probability upper bound on $|h(X)|$.

**Theorem 1.4.** Let $X \subseteq U$ be a fixed sets of $|X| = m$ keys. Let $h : U \to [n]$ be a random simple tabulation hash function. For $t \geq 0$ it holds that

$$\Pr[|h(X)| \geq \mu_0 + 2t] = O\left(\exp\left(\frac{-t^2}{2m^{2-1/c}}\right)\right),$$

and

$$\Pr[|h(X)| \leq \mu_0 - 2t] = O\left(\exp\left(\frac{-t^2}{2m^{2-1/c}} + \frac{m^2}{nt^2}\right)\right).$$
The term $m^2/(nt^2)$ in the second bound in the theorem may be unexpected but it has to be there (at least when $m = O(n)$) as we will argue after proving the theorem.

Theorem 1.4 is proved using Azuma’s inequality (which we will state and describe later). It turns out that when $m \ll n$ one can obtain stronger concentration using a stronger martingale inequality. For intuition, the reader is encouraged to think of the fully random setting where $m$ balls are thrown sequentially into $n$ bins independently and uniformly at random: In this setting the allocation of a single ball can change the conditionally expected number of non-empty bins by much — for that to happen the ball has to hit a bin that is already non-empty, and the probability that this occurs is at most $m/n \ll 1$. Using a martingale inequality by McDiarmid [9], that takes the variance of our random variables into consideration, one can obtain the following result which is an improvement over Theorem 1.5 when $m \ll n$, and matches within $O$-notation when $m = \Theta(n)$.

**Theorem 1.5.** Let $X \subseteq U$ be a fixed sets of $|X| = m$ keys. Let $h : U \to [n]$ be a random simple tabulation hash function. Assume $m \leq n$. For $t \geq 0$ it holds that

$$
\Pr [|h(X)| \geq \mu_0 + t] = \exp \left(-\Omega \left( \min \left\{ \frac{t^2}{3^c}, \frac{t}{n} \right\} \right) \right), \quad \text{and} \quad \Pr [|h(X)| \leq \mu_0 - t] = \exp \left(-\Omega \left( \min \left\{ \frac{t^2}{3^c}, \frac{t}{n} \right\} \right) \right) + O \left( \frac{m^2}{nt^2} \right).
$$

(1.3)

(1.4)

The above bounds are unwieldy so let us disentangle them. First, one can show using simple calculus that when $2 \leq m \leq n$ then $\mu_0 = m - \Theta(m^{1/c})$. If $m^{1+1/c} = o(n)$ we thus have that $\mu_0 = m - o(m^{1/c})$. To get a non-trivial bound from (1.3) we have to let $t = \Omega(m^{1-1/c})$ and then $\mu_0 + t = m + \omega(m^{1/c})$. This means that (1.3) is trivial when $m^{1+1/c} = o(n)$ as we can never have more than $m$ non-empty bins. For comparison, (1.1) already becomes trivial when $m^{1+1/(2c)} = o(n)$.

Suppose now that $m^{1+1/c} = \Omega(n)$. For a given $\delta$ put

$$
t_0 = \eta \max \left\{ \sqrt{\frac{m^{3-1/c} \log 1}{n}}, m^{1-1/c} \log \frac{1}{\delta} \right\},
$$

for some sufficiently large $\eta = O(1)$. Then (1.3) gives that $\Pr [|h(X)| \geq \mu_0 + t_0] \leq \delta$. It remains to understand $t_0$: Assuming that $m^{1+1/c} \geq n \log \frac{1}{\delta}$, we have that $t_0 = O \left( \sqrt{\frac{m^{3-1/c} \log 1}{n}} \right)$. For comparison, to get the same guarantee on the probability using (1.1) we would have to put $t_0 = \Omega \left( \sqrt{\frac{m^{2-1/c} \log 1}{n}} \right)$, which is a factor of $\sqrt{n/m}$ larger.

Turning to (1.4), it will typically in applications be the term $O \left( \frac{m^2}{nt^2} \right)$ that dominates the bound. For a given $\delta$ we would choose $t = \max \{t_0, m/\sqrt{n\delta} \}$ to get $\Pr [|h(X)| \leq \mu_0 - t] = O(\delta)$.

### 1.3 Projecting into Arbitrary Ranges

Simple tabulation is an efficient hashing scheme for hashing into $r$-bit hash values. But what do we do if we want hash values in $[n]$ where $2^{r-1} < n < 2^r$, say $n = 3 \times 2^{r-2}$? Besides being of theoretical interest this is an important question in several practical applications. For example, when designing Bloom filters (which we will describe shortly), to minimize the false positive probability, we have to choose the size $n$ of the filters such that $n \approx m/\ln(2)$. When $n$ has to
be a power of two, we may be up to a factor of $\sqrt{2}$ off, and this significantly affects the false positive probability. Another example is cuckoo hashing \cite{MS90}, which was shown in \cite{HHK+17} to succeed with simple tabulation with probability $1 - O(n^{-1/3})$ when $2m(1 + \varepsilon) \leq n$. If $m = 2^r$ we have to choose $n$ as large as $2^{r+2} = 4m$ to apply this result, making it much less useful.

The way we remedy this is a standard trick, see e.g. \cite{ALMT00}. We choose $r$ such that $2^r \gg n$, and hash in the first step to $r$-bit strings with a simple tabulation hash function $h: U \to [2^r]$. Usually $2^r \geq n^2$ suffices and then the entries of the character tables only becomes twice as long.

Defining $s: [2^r] \to [n]$ by $s(y) = \lfloor yn/2^r \rfloor$ our combined hash function $U \to [n]$ is simply defined as $s \circ h$. Note that $s$ is very easy to compute since we do just one multiplication and since the division by $2^r$ is just an $r$-bit right shift. The only property we will use about $s$ is that it is most uniform meaning that for $z \in [n]$ either, $|s^{-1}\{z\}| = \lfloor 2^{r-1}/n \rfloor$ or $|s^{-1}\{z\}| = \lceil 2^{r-1}/n \rceil$. For example, we could also use $s': [2^r] \to [n]$ defined by $s'(y) = y \pmod{n}$, but $s$ is much faster to compute.

Note that if $2^r \geq n^2$, then $\left|\lfloor s^{-1}(\{z\})\rfloor - \frac{1}{m}\right| \leq 2^{-r} \leq n^{-2}$.

A priori it is not obvious that $s \circ h$ has the same good properties as “normal” simple tabulation. The set of bins can now be viewed as $\{s^{-1}\{z\} : z \in [n]\}$, so each bin consists of many “sub-bins”, and the result on the number of non-empty sub-bins does not translate directly to any useful result on the number of non-empty bins. Nonetheless, many proofs of simple tabulation do not need to be modified much in this new setting. For example, the simplified proof given by Aamand et al. \cite{AAK+13} of the result on cuckoo hashing from \cite{HHK+17} can be checked to carry over to the case where the hash functions are implemented as described above if $r$ is sufficiently large. We provide no details here.

For the present paper the relevant analogue to Theorem \ref{thm:main} is the following:

\textbf{Theorem 1.6.} Let $X \subseteq U$ be a fixed set of $|X| = m$ balls, and let $S \subseteq [2^r]$ with $|S|/2^r = \rho$. Suppose $h: U \to [2^r]$ is a simple tabulation hash function. Define $p_0' = 1 - (1 - \rho)^m$. If $p$ denotes the probability that $h(X) \cap S \neq \emptyset$, then

$$|p - p_0'| \leq m^{2-1/\epsilon} \rho^2$$

If we let $S$ (and hence $\rho$) depend on the hash of a distinguished query ball $q \in U \setminus X$, then the bound on $p$ above is replaced by the weaker $|p - p_0| \leq m^{2-1/\epsilon} \rho^2$.

If we assume $2^r \geq n^2$, say, and let $S = s^{-1}\{z\}$ be a bin of $S \subseteq [2^r]$ we obtain the following estimate on $p$:

$$|p - p_0| \leq |p - p_0'| + |p_0' - p_0| \leq m^{2-1/\epsilon} \left(\frac{1}{n} + \frac{1}{2^r}\right)^2 + \frac{m}{2^r} = \frac{m^{2-1/\epsilon}}{n^2}(1 + o(1))$$

This is very close to what is obtained from Theorem \ref{thm:main} and to make the difference smaller we can increase $r$ further.

There are also analogues of Theorem \ref{thm:main} \cite{HHK+17} and \ref{thm:main2} in which the bins are partitioned into groups of almost equal size and where the interest is in the number of groups that are hit by a ball. To avoid making this paper unnecessarily technical, we refrain from stating and proving these theorems, but in Section \ref{sec:alternatives} we will show how to modify the proof of Theorem \ref{thm:main} to obtain Theorem \ref{thm:main}.

\subsection{Alternatives}

One natural alternative to simple tabulation is to use $k$-independent hashing \cite{HA95}. Using an easy variation of an inclusion-exclusion based argument by Mitzenmacher and Vadhan \cite{MV98} one can
show that if $k$ is odd and if $m \leq n$ the probability $p$ that a given bin is non-empty satisfies
\[
p_0 - O\left(\left(\frac{m}{n}\right)^{k+1} \frac{1}{k!}\right) \leq p \leq p_0 + O\left(\left(\frac{m}{n}\right)^{k+1} \frac{1}{(k+1)!}\right),
\]
and this is optimal, at least when $k$ is not too large, say $k = o(\sqrt{m})$ — there exist two (different) $k$-independent families making respectively the upper and the lower bound tight for a certain set of $m$ keys. A similar result holds when $k$ is even. Although $p$ approaches $p_0$ when $k$ increases, for $k = O(1)$ and $m = \Omega(n)$, we have a deviation by an additive constant term. In contrast, the probability that a bin is non-empty when using simple tabulation is asymptotically the same as in the fully random setting.

Another alternative when studying the number of non-empty bins is to assume that the input comes with a certain amount of randomness. This was studied in [11] too and a slight variation\(^2\) of their argument shows that if the input $X \subseteq U$ has enough entropy the probability that a bin is empty is asymptotically the same as in the fully random setting even if we only use 2-independent hashing. This is essentially what we get with simple tabulation. However, our results have the advantage of holding for any input with no assumptions on its entropy. Now (1.5) also suggests the third alternative of looking for highly independent hash functions. For the expectation (1.5) shows that if $m \leq n$ we would need $k = \Omega(\log n / \log \log n)$ to get guarantees comparable to those obtained for simple tabulation. Such highly independent hash functions were first studied by Siegel [15], the most efficient known construction today being the double tabulation by Thorup [16] which gives independence $u^{\Omega(1/c^2)} \gg \log n$ using space $O(\epsilon u^{1/c})$ and time $O(\epsilon)$. While this space and time matches that of simple tabulation within constant factors, it is slower by at least an order of magnitude. As mentioned in [16], double tabulation with 32-bit keys divided into 16-bit characters requires 11 times as many character table lookups as with simple tabulation and we lose the same factor in space. The larger space of double tabulation means that tables may expand into much slower memory, possibly costing us another order of magnitude in speed.

There are several other types of hash functions that one could consider, e.g., those from [6, 12], but simple tabulation is unique in its speed (like two multiplications in the experiments from [14]) and ease of implementation, making it a great choice in practice. For a more thorough comparison of simple tabulation with other hashing schemes, the reader is referred to [14].

## 2 Applications

Before proving our main results we describe two almost immediate applications.

### 2.1 Bloom Filters

Bloom filters were introduced by Bloom [3]. We will only discuss them briefly here and argue which guarantees are provided when implementing them using simple tabulation. For a thorough introduction including many applications see the survey by Broder and Mitzenmacher [4]. A Bloom filter is a simple data structure which space efficiently represents a set $X \subseteq U$ and supports membership queries of the form “is $x$ in $X$?”. It uses $k$ independent hash functions $h_0, \ldots, h_{k-1} : U \rightarrow [n]$ and $k$ arrays $A_0, \ldots, A_{k-1}$ each of $n$ bits which are initially all 0. For each $x \in X$ we calculate $(h_i(x))_{i \in [k]}$ and set the $h_i(x)$’th bit of $A_i$ to 1 noting that a bit may

\(^2\)Mitzenmacher and Vadhan actually estimate the probability of getting a false positive when using $k$-independent hashing for Bloom filters, but this error probability is strongly related to the expected number of non-empty bins $E[||h(X)||]$ (in the fully random setting it is $E[||h(X)||]/n$). Thus only a slight modification of their proof is needed.
be set to 1 several times. To answer the query “is \( q \) in \( X \)” we check if the bits corresponding to \((h_i(q))_{i \in [k]}\) are all 1, outputting “yes” if so and “no” otherwise. If \( q \in X \) we will certainly output the correct answer but if \( q \notin X \) we potentially get a false positive in the case that all the bits corresponding to \((h_i(q))_{i \in [k]}\) are set to 1 by other keys in \( X \). In the case that \( q \notin X \) the probability of getting a false positive is
\[
\prod_{i=0}^{k-1} \Pr[h_i(q) \in h_i(X)],
\]
which with fully random hashing is \( p_0^k = (1 - (1 - 1/n)^m)^k \approx (1 - e^{-m/n})^k \).

It should be noted that Bloom filters are most commonly described in a related though not identical way. In this related setting we use a single \((kn)\)-bit array \( A \) and let \( h_1, \ldots, h_{k-1} : U \to [kn] \), setting the bits of \( A \) corresponding to \((h_i(x))_{i \in [k]} \) to 1 for each \( x \in X \). With fully random hashing the probability that a bit is set to 1 is then \( q_0 := 1 - (1 - \frac{1}{kn})^{mk} \) and the probability of a false positive is thus at most \( q_0^k = (1 - (1 - \frac{1}{kn})^{mk})^k \leq p_0^k \). Despite the difference, simple calculus shows that \( p_0 - q_0 = O(1/n) \) and so
\[
p_0^k - q_0^k = (p_0 - q_0) \sum_{i=0}^{k-1} p_0^i q_0^{k-i-1} = O \left( \frac{k p_0^{k-1}}{n} \right).
\]
In particular if \( p_0 = 1 - \Omega(1) \) or if the number of filters \( k \) is not too large (both being the case in practice) the failure probability in the two models are almost identical. We use the model with \( k \) different tables each of size \( n \) as this makes it very easy to estimate the error probability using Theorem \ref{thm:1.1} and the independence of the hash functions. We can in fact view \( h_i \) as a map from \( U \) to \([kn]\) but having image in \([(i+1)n]/[in]\) getting us to the model with just one array.

From Theorem \ref{thm:1.1} we immediately obtain the following corollary.

**Corollary 2.1.** Let \( X \subseteq U \) with \( |X| = m \) and \( y \in U \setminus X \). Suppose we represent \( X \) with a Bloom filter using \( k \) independent simple tabulation hash functions \( h_0, \ldots, h_{k-1} : U \to [n] \). The probability of getting a false positive when querying \( q \) is at most
\[
\left( p_0 + \frac{2m2^{-1/c}}{n^2} \right)^k.
\]

At this point one can play with the parameters. In the fully random setting one can show that if the number of balls \( m \) and the total number of bins \( kn \) are fixed one needs to choose \( k \) and \( n \) such that \( p_0 \approx 1/2 \) in order to minimise the error probability (see \ref{thm:1.5}). For this, one needs \( m \approx n \ln(2) \) and if \( n \) is chosen so, the probability above is at most \((p_0 + O(n^{-1/c}))^k \). In applications, \( k \) is normally a small number like 10 for a 0.1\% false positive probability. In particular, \( k = n^{o(1)} \), and then \((p_0 + O(n^{-1/c}))^k = p_0^k(1 + o(1)) \), asymptotically matching the fully random setting.

To resolve the issue that the range of a simple tabulation function has size \( 2^r \) but that we wish to choose \( n \approx m/\ln(2) \), we choose \( r \) such that \( 2^r \geq n^2 \) and use the combined hash function \( s o h : U \to [n] \) described in Section \ref{sec:combined}. Now appealing to Theorem \ref{thm:1.6} instead of Theorem \ref{thm:1.1} we can again drive the false positive probability down to \( p_0^k(1 + o(1)) \) when \( k = n^{o(1)} \).

**Alternatives:** The argument by Mitzenmacher and Vadhan \footnote{Mitzenmacher, M., and Vadhan, S. P. (2006). “The power of small fibers in Bloom filters.”} discussed in relation to \ref{thm:1.5} actually yields a tight bound on the probability of a false positive when using \( \ell \)-independent hashing for Bloom filters. We do not state their result here but mention that when \( \ell \) is constant
the error probability may again deviate by an additive constant from that of the fully random setting. It is also shown in [11] that if the input has enough entropy we can get the probability of a false positive to match that from the fully random setting asymptotically even using 2-independent hashing, yet it cannot be trusted for certain types of input.

Now, imagine you are a software engineer that wants to implement a Bloom filter, proportioning it for a desired low false-positive probability. You can go to a wikipedia page (en.wikipedia.org/wiki/Bloom_filter) or a textbook like [10] and read how to do it assuming full randomness. If you read [11], what do you do? Do you set \( \ell = 2 \) and cross your fingers, or do you pay the cost of a slower hash function with a larger \( \ell \), adjusting the false-positive probabilities accordingly? Which \( \ell \) do you pick?

With our result, there are now hard choices. The answer is simple. We just have to add that everything works as stated for any possible input if the hashing is implemented with simple tabulation hashing (en.wikipedia.org/wiki/Tabulation_hashing) which is both very fast and very easy to implement.

2.2 Filter Hashing

In Filter hashing, as introduced by Fotakis et al. [7], we wish to store as many elements as possible of a set \( X \subseteq U \) of size \( |X| = m = n \) in \( d \) hash tables \( (T_i)_{i \in [d]} \). The total number of entries in the tables is at most \( n \) and each entry can store just a single key. For \( i \in [d] \) we pick independent hash functions \( h_i : U \to [n] \) where \( n_i \) is the number of entries in \( T_i \). The keys are allocated as follows: We first greedily store a key from \( h_0^{-1}([y]) \) in \( T_0[y] \) for each \( y \in h_0(X) \). This lets us store exactly \( |h_0(X)| \) keys. Letting \( S_0 \) be the so stored keys and \( X_1 = X \setminus S_0 \) the remaining keys, we repeat the process, storing \( |h(X_1)| \) keys in \( T_1 \) using \( h_1 \) etc.

An alternative and in practice more relevant way to see this is to imagine that the keys arrive sequentially. When a new key \( x \) arrives we let \( i \) be the smallest index such that \( T_i[h_i(x)] \) is unmatched and store \( x \) in that entry. If no such \( i \) exists the key is not stored. The name Filter hashing comes from this view which prompts the picture of particles (the keys) passing through filters (the tables) being caught by a filter only if there is a vacant spot.

The question is for a given \( \varepsilon > 0 \) how few filters that are needed in order to store all but at most \( \varepsilon n \) keys with high probability. Note that the remaining \( \varepsilon n \) keys can be stored using any hashing scheme which uses linear space, for example Cuckoo hashing with simple tabulation [13] [14], to get a total space usage of \((1 + O(\varepsilon)) n\).

One can argue that with fully random hashing one needs \( \Omega(\log^2(1/\varepsilon)) \) filters to achieve that whp at least \( (1 - \varepsilon)n \) keys are stored. To see that we can achieve this bound with simple tabulation we essentially proceed as in [7]. Let \( \gamma > 0 \) be any constant and choose \( \delta > 0 \) according to Theorem 1.3 so that if \( X \subseteq U \) with \( |X| = m \) and \( h : U \to [n] \) is a simple tabulation hash function, then \( |h(X)| \geq n(1 - (1 - 1/n)^{\delta m}) \) with probability at least \( 1 - n^{-\gamma} \).

Let \( m_0 = n \). For \( i = 0, 1, \ldots \), we pick \( n_i \) to be the largest power of two below \( \delta m_i / \log(1/\varepsilon) \). We then set \( m_{i+1} = n - \sum_{j=0}^{i} n_j \), terminating when \( m_{i+1} \leq \varepsilon n \). Then \( T_i \) is indexed by \((\log_2 n_i)\)-bit strings — the range of a simple tabulation hash function \( h_i \). Letting \( d \) be minimal such that \( m_d \leq \varepsilon n \) we have that \( (1 - \varepsilon)n \leq \sum_{i \in [d]} n_i \leq n \) and as \( n_i \) decreases by at least a factor of \( \left(1 - \frac{\delta}{2 \log(1/\varepsilon)}\right) \) in each step, \( d \leq \lfloor 2 \log(1/\varepsilon)^2 / \delta \rfloor \).

How many bins of \( T_i \) get filled? Even if all bins from filters \( (T_j)_{j < i} \) are non-empty we have at least \( m_i \) balls left and so with probability \( 1 - O(n_i^{-\gamma}) \) the number of bins we hit is at least

\[
n_i (1 - (1 - 1/n_i)^{\delta m_i}) \geq n_i (1 - e^{-\delta m_i / n_i}) \geq n_i (1 - \varepsilon).
\]

Thus, with probability at least \( 1 - O(dn_i^{-\gamma}) \), for each \( i \in [d] \), filter \( i \) gets at least \( (1 - \varepsilon)n_i \) balls. Since \( \sum_{i \in [d]} n_i \geq (1 - \varepsilon)n \), the number of overflowing balls is at most \( 2\varepsilon n \) in this case.
Assuming for example that $\varepsilon = \Omega(n^{-1/2})$, as would be the case in most applications, we get that the fraction of balls not stored is $O(\varepsilon)$ with probability at least $1 - O(n^{-\gamma/2})$.

**Alternatives** The hashing scheme for Filter hashing described in [7] uses $(\lceil 12|\ln(4/\varepsilon) + 1| \rceil)$-independent polynomial hashing to achieve an overflow of at most $\varepsilon n$ balls. In particular the choice of hash functions depends on $\varepsilon$ and becomes more unrealistic the smaller $\varepsilon$ is. In contrast when using simple tabulation (which is only 3-independent) for Filter hashing we only need to change the number of filters, not the hashing, when $\varepsilon$ varies. It should be mentioned that only $|\ln(4/\varepsilon)^2|$ filters are needed for the result in [7] whereas we need a constant factor more. It can however be shown (we provide no details) that we can get down to $d = [2 \log(1/\varepsilon)^2]$ filters by applying (1.2) of Theorem 1.4 if we settle for an error probability of $O(n^{-1+\eta})$ for a given constant $\eta > 0$.

Taking a step back we see the merits of a hashing scheme giving many complementary probabilistic guarantees. As shown by Pătraşcu and Thorup [14], Cuckoo hashing [13] implemented with simple tabulation succeeds with probability $1 - O(n^{-1/3})$ (for a recent simpler proof of this result, see Aamand et al. [1]). More precisely, for a set $X'$ of $m'$ balls, let $n'$ be the least power of two bigger than $(1 + \Omega(1))m'$. Allocating tables $T_0', T_1'$ of size $n'$, and using simple tabulation hash functions $h_0', h_1' : U \rightarrow [n']$, with probability $1 - O(n^{-1/3})$ Cuckoo hashing succeeds in placing the keys such that every key $x \in X'$ is found in either $T_0'[h_0'(x)]$ or $T_1'[h_1'(x)]$. In case it fails, we just try again with new random $h_0', h_1'$.

We now use Cuckoo hashing to store the $n' = O(\varepsilon n)$ keys remaining after the filter hashing, appending the Cuckoo tables to the filter tables so that $T_{d+i} = T_i'$ and $h_{d+i} = h_i'$ for $i = 0, 1$. Then $x \in X$ if and only if for some $i \in [d + 2]$, we have $x = T_i[h_i(x)]$. We note that all these $d + 2$ lookups could be done in parallel. Moreover, as the output bits of simple tabulation are mutually independent, the $d + 2$ hash functions $h_i : U \rightarrow [2^{m_i}]$, $2^{m_i} = n_i$, can be implemented as a single simple tabulation hash function $h : U \rightarrow [2^{m_1 + \cdots + m_{d+2}}]$ and therefore all be calculated using just $c = O(1)$ look-ups in simple tabulation character tables.

### 3 Preliminaries

As in [16] we define a **position character** to be an element $(j,a) \in [c] \times \Sigma$. Simple tabulation hash functions are initially defined only on keys in $U$ but we can extend the definition to sets of position characters $S = \{(j,a_j) : j \in [k]\}$ by letting $h(S) = \bigoplus_{j \in [k]} h_{j}(a_j)$. This coincides with $h(x)$ when the key $x \in U = [\Sigma]^c$ is viewed as the set of position characters $\{(i,x[i]) : i \in [c]\}$.

We start by describing an ordering of the position characters, introduced by Pătraşcu and Thorup [14] in order to prove that the number of balls hashing to a specific bin is Chernoff concentrated when using simple tabulation. If $X \subseteq U$ is a set of keys and $\prec$ is any ordering of the position characters $[c] \times \Sigma$ we for $\alpha \in [c] \times \Sigma$ define $X_\alpha = \{x \in X \mid \forall \beta \in [c] \times \Sigma : \beta \ni x \Rightarrow \beta \ni \alpha\}$. Here we view the keys as sets of position characters. Further define $G_\alpha = X_\alpha \setminus (\bigcup_{\beta \prec \alpha} X_\beta)$ to be the set of keys in $X_\alpha$ containing $\alpha$ as a position character. Pătraşcu and Thorup argued that the ordering may be chosen such that the groups $G_\alpha$ are not too large.

**Lemma 3.1** (Pătraşcu and Thorup [14]). Let $X \subseteq U$ with $|X| = m$. There exists an ordering $\prec$ of the position characters such that $|G_\alpha| \leq mn^{-1/c}$ for all position characters $\alpha$. If $q$ is any (query) key in $X$ or outside $X$, we may choose the ordering such that the position characters of $q$ are first in the order and such that $|G_\alpha| \leq 2mn^{-1/c}$ for all position characters $\alpha$.

Let us throughout this section assume that $\prec$ is chosen as to satisfy the properties of Lemma 3.1. A set $Y \subseteq U$ is said to be $d$-bounded if $|h^{-1}\{z\} \cap Y| \leq d$ for all $z \in R$. In other words no bin gets more than $d$ balls from $Y$. 

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Lemma 3.2 (Pătraşcu and Thorup [14]). Assume that the number of bins $n$ is at least $m^{1-1/(2c)}$. For any constant $\gamma$, and $d = \min \{2c(3+\gamma)^c, 2^{2c(3+\gamma)}\}$ all groups $G_\alpha$ are $d$-bounded with probability at least $1 - n^{-\gamma}$.

Lemma 3.3 (Pătraşcu and Thorup [14]). Let $\varepsilon > 0$ be a fixed constant and assume that $m \leq n^{1-\varepsilon}$. For any constant $\gamma$ no bin gets more than $\min \left( ((1+\gamma)/\varepsilon)^c, 2^{(1+\gamma)/\varepsilon} \right) = O(1)$ balls with probability at least $1 - n^{-\gamma}$.

Let us describe heuristically why we are interested in the order $\prec$ and its properties. We will think of $h$ as being uncovered stepwise by fixing $h(\alpha)$ only when $(h(\beta))_{\beta < \alpha}$ has been fixed. At the point where $h(\alpha)$ is to be fixed the internal clustering of the keys in $G_\alpha$ has been settled and $h(\alpha)$ acts merely as a translation, that is, as a shift by an XOR with $h(\alpha)$. This viewpoint opens up for sequential analyses where for example it may be possible to calculate the probability of a bin becoming empty or to apply martingale concentration inequalities. The hurdle is that the internal clustering of the keys in the groups are not independent as the hash value of earlier position characters dictate how later groups cluster so we still have to come up with ways of dealing with these dependencies.

4 Proofs of main results

In order to pave the way for the proofs of our main results we start by stating two technical lemmas, namely Lemma 4.1 and 4.2 below. We provide proofs at the end of this section. Lemma 4.1 is hardly more than an observation. We include it as we will be using it repeatedly in the proofs of our main theorems.

**Lemma 4.1.** Assume $\alpha \geq 1$ and $m,m_0 \geq 0$ are real numbers. Further assume that $0 \leq g_1, \ldots, g_k \leq m_0$ and $\sum_{i=1}^{k} g_i = m$. Then

$$\sum_{i=1}^{k} g_i^\alpha \leq m_0^{\alpha-1} m. \quad (4.1)$$

If further $m_0 \leq n$ for some real $n$ then

$$\prod_{i=1}^{k} \left(1 - \frac{g_i}{n}\right) \geq \left(1 - \frac{m_0}{n}\right)^{m/m_0}. \quad (4.2)$$

In our applications of Lemma 4.1, $g_1, \ldots, g_k$ will be the sizes of the groups $G_\alpha$ described in Lemma 3.1 and $m_0$ will be the upper bound on the group sizes provided by the same lemma.

For the second lemma we assume that the set of keys $X$ has been partitioned into $k$ groups $(X_i)_{i \in [k]}$. Let $C_i$ denote the number of sets $\{x,y\} \subseteq X_i$ such that $x \neq y$ but $h(x) = h(y)$, that is, the number of pairs of colliding keys internal to $X_i$. Denote by $C = \sum_{i=1}^{k} C_i$ the total number of collisions internal in the groups. The second lemma bounds the expected value of $C$ as well as its variance in the case where the groups are not too large.

**Lemma 4.2.** Let $X \subseteq U$ with $|X| = m$ be partitioned as above. Suppose that there is an $m_0 \geq 1$ such that for all $i \in [k]$, $|X_i| \leq m_0$. Then

$$\mathbb{E}[C] \leq \frac{m \cdot m_0}{2n}, \quad \text{and} \quad (4.3)$$

$$\operatorname{Var}[C] \leq \frac{(3^c + 1)m^2}{n} + \frac{m \cdot m_0^2}{n^2}. \quad (4.4)$$
For a given query ball \( q \in U \setminus X \) and a bin \( z \in [n] \), the upper bound on \( \mathbb{E}[C] \) is also an upper bound on \( \mathbb{E}[C | h(q) = z] \). For the variance estimate note that if in particular \( m_0^2 = O(mn) \), then \( \text{Var}[C] = O(m^2/n) \).

We will apply this lemma when the \( X_i \) are the groups arising from the order \( < \) of Lemma 3.1.

With these results in hand we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let us first prove the theorem in the case where \( y \) is a fixed bin not chosen dependently on the hash value of a query ball. If \( m^{1-1/c} \geq n \) the result is trivial as then the stated upper bound is at least 1. Assume then that \( m^{1-1/c} \leq n \). Consider the ordering \( \alpha_1 < \cdots < \alpha_k \) of the position characters obtained from Lemma 3.1 such that all groups \( G_i := G_{\alpha_i} \) have size at most \( m^{1-1/c} \). We will denote by \( m_0 := m^{1-1/c} \) the maximal possible group size.

We randomly fix the \( h(\alpha_i) \) in the order obtained from \( < \) not fixing \( h(\alpha_j) \) before having fixed \( h(\alpha_j) \) for all \( j < i \). If \( x \in G_i \) then \( h(x) = h(\alpha_i) \oplus h(x \setminus \{\alpha_i\}) \) and since \( \beta \prec \alpha_i \) for all \( \beta \in x \setminus \{\alpha_i\} \) only \( h(\alpha_i) \) has to be fixed in order to settle \( h(x) \). The number of different bins hit by the keys of \( G_i \) when fixing \( h(\alpha_i) \) is thus exactly the size of the set \( \{h(x \setminus \{\alpha_i\}) : x \in G_i \} \) which is simply translated by an XOR with \( h(\alpha_i) \) and for \( x \in G_i \) we have that \( h(x) \) is uniform in its range when conditioned on the values \( (h(\alpha_j))_{j < i} \).

To make it easier to calculate the probability that \( y \in h(X) \) we introduce some dummy balls. At the point where we are to fix \( h(\alpha_i) \) we independently on \( (h(\alpha_j))_{j < i} \) in any deterministic way choose a set \( D_i \subseteq R = [n] \) of dummy balls, disjoint from \( \{h(x \setminus \{\alpha_i\}) : x \in G_i \} \), such that \( \{|h(x \setminus \{\alpha_i\}) : x \in G_i \} \cap D_i \) has size exactly \( |G_i| \). We will say that a bin \( z \) is hit if either \( z \in h(X) \) or there exists an \( i \) such that \( z = d \oplus h(\alpha_i) \) for some \( d \in D_i \). In the latter case we will say that \( z \) is hit by a dummy ball. This modified random process can be seen as ensuring that when we are to finally fix the hash values of the elements of \( G_i \) by the last translation with \( h(\alpha_i) \), we modify the group by adding dummy balls to ensure that exactly \( |G_i| \) bins are hit by either a ball in \( G_i \) or a dummy ball in \( D_i \). We let \( D = \sum_{i=1}^{k} |D_i| \) denote the total number of dummy balls.

Let \( \mathcal{H} \) denote the event that \( y \) is hit and \( D \) denote the event that \( y \) is hit by a dummy ball. With the presence of the dummy balls, \( \Pr[\mathcal{H}] \) is easy to calculate:

\[
\Pr[\mathcal{H}] = 1 - \prod_{i=1}^{k} \left( 1 - \frac{|G_i|}{n} \right) \geq 1 - \sum_{i=1}^{k} \left( 1 - \frac{1}{n} \right)^{|G_i|} = p_0.
\]

Clearly \( \Pr[y \in h(X)] \geq \Pr[\mathcal{H}] - \Pr[D] \) so for a lower bound on \( \Pr[y \in h(X)] \) it suffices to upper bound \( \Pr[D] \). Let \( D_i \) denote the event that \( y \) is hit by a dummy ball from \( D_i \). We can calculate

\[
\Pr[D_i] = \sum_{\ell=0}^{\infty} \Pr[D_i | |D_i| = \ell] \cdot \Pr[|D_i| = \ell].
\]

The conditional probability \( \Pr[D_i | |D_i| = \ell] \) is exactly \( \ell/n \) as the choice of \( D_i \) only depends on the hash values \( (h(\alpha_j))_{j < i} \) and when translated by an XOR with \( h(\alpha_i) \) the bin \( y \) is hit with probability \( |D_i|/n \). It follows that \( \Pr[D_i] = \mathbb{E}[|D_i|]/n \). Finally the total number of dummy balls is upper bounded by the number \( C \) of internal collisions in the groups, so Lemma 4.2 gives that \( \Pr[D] \leq \mathbb{E}[C]/n \leq \frac{m^{2-1/c}}{2n^2} \). This gives the desired lower bound on \( p \) (throwing away the factor of 1/2, in order to simplify the statement in the theorem).

For the upper bound note that \( \Pr[y \in h(X)] \leq \Pr[\mathcal{H}] \) so by Lemma 4.1

\[
p \leq \Pr[\mathcal{H}] \leq 1 - \left( 1 - \frac{m_0}{n} \right)^{m/m_0}.
\]

Using the inequality \( (1 + \frac{x}{n})^\ell \geq e^{\ell} \left( 1 - \frac{x^2}{n} \right) \) holding for \( \ell \geq 1 \) and \( |x| \leq \ell \) with \( x = -m/n \) and \( \ell = m/m_0 \) (note that \( |x| \leq \ell \) as we assumed that \( m^{1-1/c} \leq n \)) we obtain that

\[
p \leq 1 - e^{-m/n} \left( 1 - \frac{m \cdot m_0}{n^2} \right) \leq 1 - e^{-m/n} + \frac{m \cdot m_0}{n^2} \leq p_0 + \frac{m^{2-1/c}}{n^2},
\]

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as desired. The bound on $\mathbb{E}[h(X)]$ follows immediately as $\mathbb{E}[|h(X)|] = \sum_{y \in [n]} \Pr[y \in h(X)]$.

Finally consider the case where $y$ is chosen conditioned on $h(q) = z$ for a query ball $q \notin X$ and a bin $z$. Here we may assume that $2m1^{-1/c} \leq n$ as otherwise the claimed upper bound is at least 1. We choose the ordering $\prec$ such that the position characters of $q$ are first in the order and such that all groups have size at most $2m1^{-1/c}$ which is possible by Lemma 3.3. Let $m_0 = \min(m, 2m1^{-1/c})$ denote the maximal possible group size. Introducing dummy balls the same way as before and repeating the arguments, the probability of the event $\mathcal{H}$ that $y$ is hit satisfies

$$p_0 \leq \Pr[\mathcal{H} | h(q) = z] \leq 1 - \left(1 - \frac{m_0}{n}\right)^{m/m_0} \leq 1 - e^{-m/n} \left(1 - \frac{m \cdot m_0}{n^2}\right) \leq p_0 + \frac{2m2^{-1/c}}{n^2}.$$ 

The desired upper bound follows immediately as $\Pr[y \in h(X) | h(q) = z] \leq \Pr[\mathcal{H} | h(q) = z]$. For the lower bound we again let $\mathcal{D}$ denote the event that $y$ is hit by a dummy ball and $\mathcal{D}_i$ denote the event that $y$ is hit by a dummy ball from $D_i$. Then

$$\Pr[\mathcal{D}_i | h(q) = z] = \sum_{\ell = 0}^{\infty} \Pr[\mathcal{D}_i | h(q) = z \land |D_i| = \ell] \times \Pr[|D_i| = \ell | h(q) = z].$$

As before we have that $\Pr[\mathcal{D}_i | h(q) = z \land |D_i| = \ell] = \ell/n$ since the hash values of the position characters of $q$ are fixed before $h(\alpha_i)$. Thus,

$$\Pr[\mathcal{D}_i | h(q) = z] = \sum_{\ell = 0}^{\infty} \frac{\ell}{n} \Pr[|D_i| = \ell | h(q) = z] = \frac{\mathbb{E}[|D_i| | h(q) = z]}{n},$$

and another union bound gives that

$$\Pr[\mathcal{D} | h(q) = z] \leq \frac{\mathbb{E}[D | h(q) = z]}{n} \leq \frac{\mathbb{E}[C | h(q) = z]}{n} \leq \frac{m2^{-1/c}}{n^2},$$

where we in the last step used Lemma 4.2.

We are now going to prove Theorem 1.3. We start out by recalling Azuma’s inequality.

**Theorem 4.3 (Azuma’s inequality [2]).** Suppose $(X_i)_{i=0}^k$ is a martingale satisfying that $|X_i - X_{i-1}| \leq s_i$ almost surely for all $i = 1, \ldots, k$. Let $s = \sum_{i=1}^k s_i^2$. Then for any $t \geq 0$ it holds that

$$\Pr(X_k \geq X_0 + t) \leq \exp\left(-\frac{t^2}{2s}\right), \quad \text{and} \quad \Pr(X_k \leq X_0 - t) \leq \exp\left(-\frac{t^2}{2s}\right).$$

To apply Azuma’s inequality we need to recall a little measure theory. Suppose $(\Omega, \mathcal{F}, \Pr)$ is a finite measure space (that is $\Omega$ is finite), and that $Y : \Omega \to \mathbb{R}$ is an $\mathcal{F}$-measurable random variable. A sequence of $\sigma$-algebras $(\mathcal{F}_i)_{i=1}^k$ on $\Omega$ is called a filter of the $\sigma$-algebra $\mathcal{F}$ if $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_k = \mathcal{F}$. Defining $Y_i = \mathbb{E}[Y | \mathcal{F}_i]$, the sequence $(Y_i)_{i=0}^k$ becomes a martingale with $Y_0 = \mathbb{E}[Y]$ and $Y_k = Y$. It is for such martingales that we will apply Azuma’s inequality.

**Proof of Theorem 1.3**. By the result by Pătraşcu and Thorup [14] we may assume that $m \leq Cn \log n$ for some constant $C$ as otherwise all bins are full whp from which the results of the theorem immediately follow.

Let $G_1, \ldots, G_k$ be the groups described in Lemma 3.3 and $\alpha_1, \ldots, \alpha_k$ be the corresponding position characters. Again we think of the $h(\alpha_i)$ as being fixed sequentially. We let $(\Omega, \mathcal{F}, \Pr)$ be the underlying probability space when choosing $h$, that is, $\Omega$ is the set of all simple tabulation hash functions, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\Pr$ is the uniform probability measure on $\Omega$. For $i = 0, \ldots, k$ we
define $F_i = \sigma(h(\alpha_1), \ldots, h(\alpha_i))$ to be the $\sigma$-algebra generated by the hash values of the first $i$ position characters. Then $\{\emptyset, \Omega\} = F_0 \subseteq \cdots \subseteq F_k = F$ is a filter of $F$.

Ideally we would hope that for the martingale $(X_i)_{i=0}^k = (E[h(X_i) | F_i])_{i=0}^k$ we could effectively bound $|X_i - X_{i-1}|$ and thus apply Azuma’s inequality. This is however too much to hope for — the example with keys $[2] \times [m/2]$ shows that the hash value of a single position character can have a drastic effect on the conditionally expected number of non-empty bins. To remedy this we will again be using dummy balls but this time in a different way.

First of all, we let $\gamma > 0$ be any constant. Since $m \leq Cn \log n$, Lemma 3.2 gives that there exists a $d = d(\gamma) = O(1)$ such that all groups are $d$-bounded with probability at least $1 - n^{-\gamma}$. Here is how we use the dummy balls: After having fixed $(h(\alpha_j))_{j<i}$ we again look at the set $G'_i := \{h(x \setminus \{\alpha_i\}) : x \in G_i\}$ letting $I^- = \{i \in [k] : |G'_i| \geq \lceil |G_i|/d \rceil \}$ and $I^+ = \{i \in [k] : |G'_i| < \lceil |G_i|/d \rceil \}$. For $i \in I^-$ we dependently on $(h(\alpha_j))_{j<i}$ choose a set $D_i^{-} \subseteq G'_i$ such that $|G'_i \setminus D_i^-| = \lceil |G_i|/d \rceil$. Similarly we for $i \in I^+$ choose a set $D_i^+ \subseteq R$ disjoint from $G'_i$ such that $|G'_i \cup D_i^+| = \lceil |G_i|/d \rceil$. We say that bin $z$ is hit if there exists an $i$ such that either

1. $i \in I^-$ and $z = y \oplus h(\alpha_i)$ for some $y \in G_i \setminus D_i^-$, or
2. $i \in I^+$ and $z = y \oplus h(\alpha_i)$ for some $y \in G_i \cup D_i^+$.

This modified random process obtained by adding balls if $|G'_i|$ is too large and removing balls if it is too small can be seen as ensuring that when we are to finally fix the hash values of the elements of $G_i$ by the last translation by $h(\alpha_i)$ we first modify the group to ensure that we hit exactly $\lceil |G_i|/d \rceil$ bins.

Importantly, we observe that if $G_i$ is $d$-bounded then $|G'_i| \geq |G_i|/d$ and since $|G'_i|$ is integral $|G'_i| \geq \lceil |G_i|/d \rceil$. Thus if all groups are $d$-bounded $I^+ = \emptyset$, and no dummy balls are added.

Letting $H$ denote the number of bins hit, we have that

$$E[H] = n \left(1 - \prod_{i=1}^k \left(1 - \frac{|G'_i|/d}{n}\right)\right) \geq n \left(1 - \prod_{i=1}^k \left(1 - \frac{1}{n}\right)^{|G_i|/d}\right) \geq n \left(1 - \left(1 - \frac{1}{n}\right)^{m/d}\right).$$

We now wish to apply Azuma’s inequality to the martingale $(H_i)_{i=0}^k = (E[H | F_i])_{i=0}^k$. To do this we require a good upper bound on $|H_i - H_{i-1}|$ and we claim that in fact $|H_i - H_{i-1}| \leq |G_i|$. To see this, let the random variable $N_i$ denote the number of bins not hit when the hash values of the first $i$ position characters has been settled. Then $H_i = n - N_i \prod_{j>i} \left(1 - \frac{|G'_j|/d}{n}\right)$ and so

$$|H_i - H_{i-1}| = \prod_{j>i} \left(1 - \frac{|G'_j|/d}{n}\right) N_i \left(1 - \frac{|G'_i|/d}{n}\right) N_{i-1} \leq |N_i - N_{i-1} + \frac{|G'_i|/d \cdot N_{i-1}}{n}|.$$}

Now $N_{i-1} - |G'_i|/d \leq N_i \leq N_{i-1}$ as at least 0 and most $|G'_i|/d$ bins are hit after fixing $h(\alpha_i)$ and from this it follows that $|H_i - H_{i-1}| \leq |G'_i|/d \leq |G_i|$.

Letting $s_i = |G_i|$ we have that $\sum_{i=1}^k s_i^2 \leq m^2 \log n$ by Lemma 4.1 and thus we can apply Azuma’s inequality to obtain that

$$\Pr(H \leq E[H] - t) \leq \exp \left(-\frac{t^2}{2m^{2-1/c}}\right).$$

Putting $t = \sqrt{\gamma \cdot 2m^{2-1/c} \log n}$ we obtain that with probability at least $1 - n^{-\gamma}$

$$H \geq \ell(m, n) := n \left(1 - \left(1 - \frac{1}{n}\right)^{m/d}\right) - \sqrt{\gamma \cdot 2m^{2-1/c} \log n}.$$
As $I^+ = \emptyset$ with probability at least $1 - n^{-\gamma}$ and as we in this case have that $|h(X)| \geq H$ we have that $|h(X)| \geq \ell(m,n)$ with probability at least $1 - 2n^{-\gamma}$.

The remaining part of proof is just combining what we have together with a little calculus! We first consider the case $m \leq n$. In this case the lower bound simply states that $|h(X)| = \Omega(m)$. To see that this bound holds observe that if (for example) $m \leq n^{1/2}$ then by Lemma 3.3 no bin gets more than a constant number of balls with probability at least $1 - n^{-\gamma}$. In particular $|h(X)| = \Omega(m)$ with probability at least $1 - n^{-\gamma}$. If on the other hand $m \geq n^{1/2}$ then $\sqrt{\gamma \cdot 2m^{2-1/c}} \log n = o(m)$ and $\ell(m,n) = \Omega(m) - o(m) = \Omega(m)$ which again gives the desired result.

Finally suppose $m \geq n$. Let $\alpha := (1 - 1/n)^{n/(2d)} \leq e^{-1/(2d)}$ and let $\beta$ be a constant so large that $\beta \geq 2d$ and $n(1 - 1/n)^{m/\beta} \geq \frac{1}{1-\alpha} \sqrt{\gamma \cdot 2m^{2-1/c}} \log n$, the last requirement being possible as we assumed $m \leq Cn \log n$. Then

$$
\frac{\ell(m,n)}{n} \geq 1 - \left(1 - \frac{1}{n}\right)^{m/d} - (1 - \alpha) \left(1 - \frac{1}{n}\right)^{m/\beta} \\
\geq 1 - \left(1 - \frac{1}{n}\right)^{m/\beta} \left(1 - \frac{1}{n}\right)^{m/(2d)} + (1 - \alpha) \geq 1 - \left(1 - \frac{1}{n}\right)^{m/\beta}.
$$

Since $|h(X)| \geq \ell(m,n)$ with probability at least $1 - 2n^{-\gamma}$ this gives the desired result. 

We now prove Theorem 1.4.

**Proof of Theorem 1.4.** When $m^{-1/(2c)} \geq n$ the probability bounds of the theorem are trivial since they are $\Omega(1)$ when $t \leq n$. We therefore assume henceforth that $m^{-1/(2c)} \leq n$.

Again consider the order $<$ obtained from Lemma 3.3 such that for all $i$ we have $|G_i| \leq m^{-1/c}$. We again think of the hash values of the position characters as being fixed in the order obtained from $<$. We also introduce dummy balls in exactly the same way as we did in the proof of Theorem 1.1 using the same definition of a bin being hit.

Letting $H$ denote the number of bins hit (by an $x \in X$ or a dummy ball) we have that $E[H] = n \left(1 - \prod_{i=1}^{k} \left(1 - \frac{|G_i|}{n}\right)\right)$, like in the proof of Theorem 1.1 and $\mu_0 \leq E[H] \leq \mu_0 + \frac{m^{2-1/c}}{n}$.

Furthermore letting $\mathcal{F}_i = \sigma(h(\alpha_1), \ldots, h(\alpha_i))$ be the $\sigma$-algebra generated by $(h(\alpha_j))_{j \leq i}$, the same argument as in the proof of Theorem 1.3 gives that $H_i = E[H_{\mathcal{F}_i}]$ is a martingale satisfying that $|H_i - H_{i-1}| \leq |G_i|$ for all $i$. We can thus apply Azuma’s inequality with $s_i = |G_i|$ and $s = \sum_{i=1}^{k} s_i^2 \leq m^{2-1/c}$ (here we used Lemma 1.1) to obtain that

$$
Pr[H \geq E[H] + t] \leq \exp \left(\frac{-t^2}{2m^{2-1/c}}\right), \quad \text{and} \quad (4.5)
$$

$$
Pr[H \leq E[H] - t] \leq \exp \left(\frac{-t^2}{2m^{2-1/c}}\right). \quad (4.6)
$$

We now wish to translate this concentration result on the number of bins hit when the dummy balls are included to a concentration result on $|h(X)|$. We begin with the bound in (1.1). As $|h(X)| \leq H$ it suffices to bound the probability $Pr[H \geq \mu_0 + 2t]$. Since $E[H] \leq \mu_0 + \frac{m^{2-1/c}}{n}$,

$$
Pr[H \geq \mu_0 + 2t] \leq Pr \left[ H - E[H] \geq 2t - \frac{m^{2-1/c}}{n} \right],
$$
so when \( t > \frac{m^{2-1/c}}{n} \) the result follows immediately from (4.3). If on the other hand \( t < \frac{m^{2-1/c}}{n} \) then \( \frac{m^2}{n^2} < \frac{m^{2-1/c}}{n} \leq 1 \) and the result is trivial as the right hand size in (4.1) can be as large as \( \Omega(1) \) which is a valid upper bound on any probability.

We now turn to the proof of (1.2). Letting \( E \) denote the event that \( |h(X)| \leq \mu_0 - 2t \) and \( A \) the event that \( H \leq \mu_0 - t \) we have that

\[
\Pr[|h(X)| \leq \mu_0 - 2t] = \Pr[E] \leq \Pr[A] + \Pr[E \land \neg A].
\]

By (4.6) and since \( \mu_0 \leq \mathbb{E}[H] \) we can upper bound \( \Pr[A] \leq \exp \left( \frac{-t^2}{2m^2-1/t^2} \right) \). For the other term we note that \( E \land \neg A \) entails that at least \( t \) bins are hit by a dummy ball. In particular the number of dummy balls is at least \( t \). As the number \( C \) of internal collisions of the groups is an upper bound on the number of dummy balls this in turn implies, \( t \leq C \). We may assume that \( t \geq m^{1-1/(2c)} \) as otherwise (1.1) is trivial. As we assumed, \( m^{1-1/(2c)} \leq n \) it follows from Lemma 4.2 that \( \mathbb{E}[C] \leq \frac{m^{2-1/c}}{2n} \leq \frac{tm^{1-1/(2c)}}{2n} \leq t/2 \) and so \( t - \mathbb{E}[C] \geq t/2 \). Applying Chebychev’s inequality as well as (4.4) of Lemma 1.2 we thus obtain,

\[
\Pr[E \land \neg A] \leq \Pr[C \geq t] \leq \Pr[C - \mathbb{E}[C] \geq t/2] \leq \frac{4 \var{C}}{t^2} = O \left( \frac{m^2}{t^2 n} \right).
\]

Combining the two bounds completes the proof.

We promised to argue why we cannot dispose with the term \( m^2/(nt^2) \) in general. Suppose that \( m = O(n) \) and let \( t = m^{1/2+\alpha} \) for an \( \alpha \in (1/2, 1) \) such that \( m^{1/2+\alpha} \in [\sqrt{m}, m/2] \), and consider the set of keys \( [m/t] \times [t] \). With probability \( \Omega((m/t)^2/n) = \Omega(m^2/(t^2 n)) \) we have that \( h_0(a_0) = h_0(a_1) \) for two distinct \( a_0, a_1 \in [m/t] \). Conditioned on this event the expected number of non-empty bins is at most \( n \left( 1 - \left( 1 - \frac{m/t-1}{n} \right)^t \right) \) which can be shown to be \( \mu_0 - \Omega(t) \) by standard calculus. The additive term \( \Omega(t) \) comes from the fact that the \( t \) pairs of colliding keys \( \{(a_0, b), (a_0, b)\}_{b \in [t]} \) causes the expected number of non-empty bins to decrease by \( \Omega(t) \) when \( m = O(n) \). Thus the deviation by \( \Omega(t) \) from \( \mu_0 \) occurs with probability \( \Omega(m^2/(nt^2)) \).

We will now set the stage for the proof of Theorem 1.5. As mentioned in the introduction we require a stronger martingale inequality than that by Azuma. The one we use is due to Mcdiarmid [9]. Again assume that \( (\Omega, \mathcal{F}, \Pr) \) is a finite probability space, that \( X : \Omega \to \mathbb{R} \) is an \( \mathcal{F} \)-measurable random variable, that \( (\mathcal{F}_t)_{t=0}^k \) is a filter of \( \mathcal{F} \), and that \( X_t = \mathbb{E}[X \mid \mathcal{F}_t] \). Also recall the definition of conditional variance: If \( \mathcal{G} \subseteq \mathcal{F} \) is a \( \sigma \)-algebra, then \( \var{X \mid \mathcal{G}} = \mathbb{E}[\var{X \mid \mathcal{G}}] = \mathbb{E}[(X - \mathbb{E}[X \mid \mathcal{G}])^2 \mid \mathcal{G}] = \mathbb{E}[X^2 \mid \mathcal{G}] - \mathbb{E}[X \mid \mathcal{G}]^2 \).

**Theorem 4.4** (Mcdiarmid [9]). Assume that \( \var{X_i \mid \mathcal{F}_{i-1}} \leq \sigma_i^2 \) for \( i = 1, \ldots, k \) and further that \( X_i - X_{i-1} \leq M \) for \( i = 1, \ldots, k \). Then for \( t \geq 0 \),

\[
\Pr[X - \mathbb{E}[X] \geq t] \leq \exp \left( \frac{-t^2}{2 \sum_{i=1}^k \sigma_i^2 + Mt/3} \right).
\]

With this tool in hand we are ready to prove Theorem 1.5, the main technical challenge being to argue why we can apply Theorem 4.4.

**Proof of Theorem 1.5.** We introduce dummy balls exactly in the proof of Theorem 1.4 and consider the same martingale \( (H_t)_{t=0}^k = (\mathbb{E}[H \mid \mathcal{F}_t])_{t=0}^k \), where \( H \) is the number of bins hit (by either a dummy ball or a ball from \( X \)). We already saw that \( |H_i - H_{i-1}| \leq |G_i| \leq
\( m^{1-1/c} \), so we let \( M := m^{1-1/c} \). What remains is to upper bound \( \text{Var}[H_i \mid \mathcal{F}_{i-1}] \). First note that
\[
\text{Var}[H_i \mid \mathcal{F}_{i-1}] = \mathbb{E}[(H_i - \mathbb{E}[H_i \mid \mathcal{F}_{i-1}])^2 \mid \mathcal{F}_{i-1}]
= \mathbb{E}[(H_i - H_{i-1})^2 \mid \mathcal{F}_{i-1}].
\]

We denote by \( N_i \) the number of bins that are empty after the hash values of the first \( i \) position characters has been settled. Then, by the same reasoning as in the proof of Theorem 1.3 we have that
\[
|H_i - H_{i-1}| \leq N_i - N_{i-1} + \frac{|G_i| \cdot N_{i-1}}{n}.
\]

Now let \( T_i = |G_i| - N_{i-1} + N_i \) denote the number of bins hit in the \( i \)'th step that were already hit in the \((i-1)\)'st step. As \( \mathbb{E}[T_i \mid \mathcal{F}_{i-1}] = (n - N_{i-1})|G_i|/n \), the above inequality reads
\[
|H_i - H_{i-1}| \leq |T_i - \mathbb{E}[T_i \mid \mathcal{F}_{i-1}]|,
\]
and so,
\[
\text{Var}[H_i \mid \mathcal{F}_{i-1}] \leq \text{Var}[T_i \mid \mathcal{F}_{i-1}] \leq \mathbb{E}[T_i^2 \mid \mathcal{F}_{i-1}].
\]

Now, \( T_i^2 \) counts the number of 2-tuples \((y, z)\) with \( y, z \in \{h(x \setminus \{\alpha_i\}) : x \in G_i \} \cup D_i \) such that \( h(y) \) and \( h(z) \) are already hit after the \((i-1)\)'st step. Conditioned on \( \mathcal{F}_{i-1} \) the probability that this occurs for a given such pair is at most \( \frac{n - N_{i-1}}{n} \leq \frac{m}{n} \), and there are exactly \( |G_i|^2 \) such pairs. Hence
\[
\text{Var}[H_i \mid \mathcal{F}_{i-1}] \leq \frac{m}{n} |G_i|^2 := \sigma_i^2.
\]

By Lemma 1.2, \( \sum_{i=1}^{k} \sigma_i^2 \leq \frac{m^{3-1/c}}{n} \) so Theorem 1.4 gives that for \( t \geq 0 \)
\[
\Pr[H - \mathbb{E}[H] \geq t] \leq \exp \left( -t^2 \frac{2}{m^{3-1/c} + m^{1-1/c} / 3} \right)
\leq \exp \left( - \min \left\{ -t^2 / \left(4m^{3-1/c} / n\right), 3t / 2m^{1-1/c} \right\} \right).
\]

As \( \mathbb{E}[H] \leq \mu_0 + \frac{m^{2-1/c}}{n} \) this is also an upper bound on \( \Pr[|h(X)| \geq \mu_0 + t + \frac{m^{2-1/c}}{n}] \). Now the same argument as in the proof of Theorem 1.3 leads to the upper bound (1.3).

Finally, to prove (1.4) we use the same strategy as above but this time we define \( H' = -H \) and the martingale \((H'_i)_{i=0}^k = (\mathbb{E}[H' \mid \mathcal{F}_i])_{i=0}^k \). Then \( |H'_i - H'_{i-1}| = |H_i - H_{i-1}| \leq M \) and \( \text{Var}[H'_i \mid \mathcal{F}_{i-1}] = \text{Var}[H_i \mid \mathcal{F}_{i-1}] \) for \( i = 1, \ldots, k \), so we get a bound as in (1.7), but this time on \( \Pr[H' - \mathbb{E}[H'] \geq t] = \Pr[H - \mathbb{E}[H] \leq -t] \).

As in the proof of Theorem 1.3, the event \( |h(X)| \leq \mu_0 - t \) implies that either \( A: H - \mathbb{E}[H] \leq -t/2 \) or \( B: \) the number of internal collisions \( C \) is at least \( t/2 \). \( \Pr[A] \) is bounded using (1.7), giving us the first term of the bound in (1.4). For \( \Pr[B] \), note that we may assume that \( t \geq 4m^{1-1/c} \) as otherwise (1.7) is trivial. In that case \( \mathbb{E}[C] \leq m^{2-1/c} / n \leq \frac{m}{n} \leq \frac{t}{4} \), so \( t/2 - \mathbb{E}[C] \geq t/4 \). Lemma 1.2 thus gives that \( \Pr[B] = O(\frac{m^2}{nt}) \) — the second term in the bound (1.4). The proof is complete.\[ \blacksquare \]
4.1 Proofs of technical lemmas

For proving Lemma 4.2 and Lemma 4.1 we need to briefly discuss the independence of simple tabulation. In the notion of \( k \)-independence introduced by Wegman and Carter \[20\] simple tabulation is only 3-independent as shown by the set of keys \( S = \{(a_0, b_0), (a_0, b_1), (a_1, b_0)(a_1, b_1)\} \). Indeed \( \bigoplus_{x \in S} h(x) = 0 \) showing that the keys do not hash independently. The issue is that since each position character appears an even number of times in \( S \) the addition over \( \mathbb{Z}_2 \) causes the terms to cancel out. This property in a sense characterises dependencies of keys as shown by Thorup and Zhang \[19\].

**Lemma 4.5** (Thorup and Zhang \[19\]). The keys \( x_1, \ldots, x_k \in U \) are dependent if and only if there exists a non-empty subset \( I \subseteq \{1, \ldots, k\} \) such that each position character in \( (x_i)_{i \in I} \) appears an even number of times. In this case we have that \( \bigoplus_{i \in I} h(x_i) = 0 \).

For keys \( x, y \in U \) we write \( x \oplus y \) for the symmetric difference of \( x \) and \( y \) when viewed as sets of position characters. Then the property that each position character appearing an even number of times in \( (x_i)_{i \in I} \) can be written as \( \bigoplus_{i \in I} x_i = \emptyset \). As shown by Dahlgaard et al. \[5\] we can efficiently bound the number of such tuples \( (x_i)_{i \in I} \).

**Lemma 4.6** (Dahlgaard et al. \[5\]). Let \( A_1, \ldots, A_{2t} \subseteq U \). The number of \( 2t \)-tuples \( (x_1, \ldots, x_{2t}) \in A_1 \times \cdots \times A_{2t} \) such that \( x_1 \oplus \cdots \oplus x_{2t} = \emptyset \) is at most \( ((2t-1)!)^t \prod_{i=1}^{2t} \sqrt{|A_i|} \). Here \( a!! \) denotes the product of all the positive integers in \( \{1, \ldots, a\} \) having the same parity as \( a \).

We now provide the proofs of Lemmas 4.2 and 4.1. Since we need Lemma 4.1 in the proof of Lemma 4.2 we prove that first.

**Proof of Lemma 4.1** We prove the following more general statement: Let \( f : [0, m_0] \to \mathbb{R} \) be convex with \( f(0) = 0 \). Let \( 0 \leq g_1, \ldots, g_k \leq m_0 \) be such that \( m = \sum_{i=1}^{k} g_i \). Define \( S := \sum_{i=1}^{k} f(g_i) \). Then \( S \leq (m/m_0) f(m_0) \).

To see why the statement holds note that by convexity, \( f(x) + f(y) \leq f(x - t) + f(y + t) \) if \( 0 \leq t \leq x \leq y \leq m_0 - t \). To maximize \( S \) we thus have to set \( k = \lceil m/m_0 \rceil \), \( g_1 = \cdots = g_{k-1} = m_0 \) and \( g_k = m - \sum_{i=1}^{k-1} g_i = \varepsilon m_0 \), where \( \varepsilon \in [0, 1) \). It follows that

\[
S \leq \left( \frac{m}{m_0} - \varepsilon \right) f(m_0) + f(\varepsilon m_0).
\]

Finally \( f(\varepsilon m_0) \leq \varepsilon f(m_0) \) using convexity and that \( f(0) = 0 \), so \( S \leq (m/m_0) f(m_0) \) as desired.

The first inequality \[4.1\] of the lemma follows immediately from the above statement with \( f(x) = x^a \) which is convex since \( a \geq 1 \). For inequality \[4.2\] we may assume that \( n > m_0 \) as the result is trivial when \( n = m_0 \). We then define \( f(x) = -\log(1 - x/n) \) which is convex with \( f(0) = 0 \). Then

\[
S = -\sum_{i=1}^{k} \log \left( 1 - \frac{g_i}{n} \right) \leq -\frac{m}{m_0} \log \left( 1 - \frac{m}{n} \right),
\]

which upon exponentiation leads to inequality \[4.2\].

**Proof of Lemma 4.2** We define \( g_i = |X_i| \) for \( i \in [k] \). Now \[4.3\] is easily checked. Indeed, since simple tabulation is \( 2 \)-independent,

\[
\mathbb{E}[C] = \sum_{i=1}^{k} \left( \frac{g_i}{2} \right) \frac{1}{n} \leq \frac{1}{2n} \sum_{i=1}^{k} \frac{g_i^2}{2} \leq \frac{m \cdot m_0}{2n}.
\]
where in the last step we used Lemma 4.1. The last statement of the lemma concerning $\mathbb{E}[C \mid h(q) = z]$ follows from the same argument this time however using that simple tabulation is 3-independent.

We now turn to (4.4). Writing $\text{Var}[C] = \mathbb{E}[C^2] - (\mathbb{E}[C])^2$ our aim is to bound $\mathbb{E}[C^2]$. Note that $C^2$ counts the number of tuples $(\{x, y\}, \{z, w\})$ such that $x \neq y$ and $z \neq w$ but $h(x) = h(y)$ and $h(z) = h(w)$ and furthermore $x, y \in G_i$ and $z, w \in G_j$ for some $i, j \in [k]$. We denote the set of such tuples $T$ and for $\tau = (\{x, y\}, \{z, w\}) \in T$ we let $X_\tau$ be the indicator for the event that both $h(x) = h(y)$ and $h(z) = h(w)$. Then

$$\mathbb{E}[C^2] = \sum_{\tau \in T} \mathbb{P}(X_\tau = 1). \tag{4.8}$$

We now partition $T$ by letting

- $T_1$ be the elements of $T$ for which $\{x, y\} = \{z, w\}$.
- $T_2$ be the elements of $T$ for which $|\{x, y\} \cup \{z, w\}| = 3$.
- $T_3$ be the elements of $T$ for which $x, y, z, w$ are distinct and independent.
- $T_4$ be the elements of $T$ for which $x, y, z, w$ are distinct and dependent and there is an $i \in [k]$ such that $x, y, z, w \in G_i$.
- $T_5$ be the remaining elements of $T$, that is, those elements $(\{x, y\}, \{z, w\})$ such that $x, y, z, w$ are distinct and dependent and such that $\{x, y\} \subseteq G_i$ and $\{z, w\} \subseteq G_j$ for some distinct $i, j \in [k]$.

Putting $S_j = \sum_{\tau \in T_j} \mathbb{P}(X_\tau = 1)$ the sum in (4.8) can be written as $\sum_{j=1}^5 S_j$ and we can efficiently upper bound each of the inner sums as we now show. Clearly,

$$S_1 = \sum_{i=1}^k \left( \frac{g_i}{2} \right) \frac{1}{n} = \mathbb{E}[C].$$

For the second sum we use that simple tabulation is 3-independent and that $|\{x, y\} \cup \{z, w\}| = 3$ implies that $x, y, z, w$ belongs to the same group $G_i$ for some $i \in [k]$. Hence

$$S_2 = \sum_{i=1}^k \left( \frac{g_i}{2} \right) \cdot 2 \cdot \left( \frac{g_i - 2}{1} \right) \frac{1}{n^2} \leq \frac{1}{n^2} \sum_{i=1}^k g_i^2 \leq \frac{m \cdot m_0^2}{n^2},$$

again using Lemma 4.1 to bound the sum of cubes. Finally we upper bound $S_3$ as

$$S_3 \leq \frac{1}{n^2} \left( \sum_{i=1}^k \left( \frac{g_i}{2} \right) \left( \frac{g_i - 2}{2} \right) \right) + \sum_{i,j \in [k]: i \neq j} \left( \frac{g_i}{2} \right) \left( \frac{g_j}{2} \right) \leq \frac{1}{n^2} \left( \sum_{i=1}^k \left( \frac{g_i}{2} \right) \right)^2 = \mathbb{E}[C]^2.$$
Combining all this we find that

\[
\text{Var}[C] = \mathbb{E}[C^2] - \mathbb{E}[C]^2 \leq \mathbb{E}[C] + \frac{m \cdot m_0^2}{n^2} + 3' \frac{m^2}{n} \leq \frac{m \cdot m_0^2}{n^2} + \frac{(3' + 1)m^2}{n},
\]

as desired.

\section{Handling bins consisting of many subbins}

In this section we show how to modify the proof of Theorem 1.1 to obtain Theorem 1.6.

\textit{Proof of Theorem 1.6.} We may assume that \(\rho n^{1-1/c} \leq 1\) as otherwise the result is trivial.

As usual we consider the ordering on the position characters, \(\alpha_1 \prec \cdots \prec \alpha_k\), obtained from Lemma 3.1 and we fix the values \(h(\alpha_i)\) in this order. Suppose that \((h(\alpha_j))_{j<i}\) are fixed and let \(V_i = \{y \in [2^r] \mid \exists x \in G_i : h(x \setminus \{\alpha_i\}) + y \in S\}\) denote those hash values \(h(\alpha_j)\) that would cause \(S \cap h(G_i) \neq \emptyset\). Note that \(V_i\) is a random variable depending only on \((h(\alpha_j))_{j<i}\).

Let \(D_i \subseteq [2^r]\), \(V_i\) be a set of \textit{dummy hash values} chosen independently on \((h(\alpha_j))_{j<i}\) such that \((|D_i| + |V_i|)/2^r = \rho|G_i|\). As \(|G_i| \leq m^{1-1/c}\) and so \(\rho|G_i| \leq \rho m^{1-1/c} \leq 1\) this is in fact possible. We say that \(S\) is \textit{hit} if there exists and \(i \in \{1, \ldots, k\}\) such that \(h(\alpha_i) \in V_i \cup D_i\), and we denote this event \(H\). Defining \(m_0 = m^{1-1/c}\) we then have

\[
\Pr[H] = 1 - \prod_{i=1}^{k} (1 - |G_i|\rho) \leq 1 - (1 - m_0\rho)^{m/m_0} \leq 1 - e^{-m_0\rho}(1 - mm_0\rho^2) \leq p_0 + m_0^{-1/c} \rho^2,
\]

using the same inequality as in the proof of Theorem 1.1. This is clearly also an upper bound on \(p = \Pr[h(X) \cap S \neq \emptyset] = \Pr[\bigcup_{i=1}^{k} (h(\alpha_i) \in V_i)]\).

Now for the lower bound: For \(i \in \{1, \ldots, k\}\) we let \(D_i\) and \(R_i\) denote events that \(h(\alpha_i) \in D_i\) and that \(h(\alpha_i) \in V_i\) respectively. Then

\[
\Pr\left[\bigcup_{i=1}^{k} D_i\right] \leq \sum_{i=1}^{k} \Pr[D_i] = \sum_{i=1}^{k} \left(\Pr[R_i \cup D_i] - \Pr[R_i]\right) = \rho m - \sum_{i=1}^{k} \Pr[R_i].
\]

By the Bonferroni inequality, and 2-independence

\[
\Pr[R_i] = \Pr[h(G_i) \cap S \neq \emptyset] \geq |G_i|\rho - \left(\frac{|G_i|}{2}\right)\rho^2,
\]

so it follows that \(\Pr[\bigcup_{i=1}^{k} D_i] \leq \sum_{i=1}^{k} \left(\frac{|G_i|}{2}\right)\rho^2 \leq m^{2-1/c} \rho^2\). Finally

\[
p \geq \Pr[H] - \Pr\left[\bigcup_{i=1}^{k} D_i\right] \geq p_0 - m^{2-1/c} \rho^2,
\]

which completes the proof of the lower bound.

The case where we condition on the event \(E\) that \(h(q) = z\) for a \(z \in [2^r]\) is handled analogously but this time choosing the order \(\prec\) as described in the second part of Lemma 3.1. The upper bound on \(p\) then follows as before and for the lower bound we use 3-independence of simple tabulation when applying the Bonferroni inequality to lower bound \(\Pr[V_i \mid E]\).
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