Abstract. It has recently recently been suggested that a relativistic Bose gas of some type may play a role in issues such as dark matter, dark energy, and in some other cosmological problems. In this article, we investigate one known exactly solvable model of a three-dimensional statistical-mechanical model of relativistic Bose gas which takes into account the existence of both particles and antiparticles. We derive exact expressions for the behavior of the Casimir force for a system subjected to film geometry under periodic boundary conditions. We show that the Casimir force between the plates is attractive, monotonic as a function of the temperature scaling variable, and has a scaling function that, at low temperatures, approaches a universal negative constant equal to the corresponding one for two-component three-dimensional Gaussian system. The force decays with the distance in a power-law near and below the bulk critical temperature $T_c$ of the Bose condensate, and exponentially above $T_c$. We obtain a closed-form exact expression for the Casimir amplitude $\Delta_{\text{Cas}}^{\text{RGB}} = -4\zeta(3)/(5\pi)$. We establish the precise correspondence of the scaling function of the free energy of the model with the scaling functions of two other well-known models of statistical mechanics: the spherical model, and the imperfect Bose gas model.

Keywords: solvable lattice models, exact results, critical exponents and amplitudes, classical phase transitions, finite-size scaling, Casimir effect
1. Introduction

The current article is devoted to the Casimir effect in relativistic Bose gas. The role of the onset of Bose-Einstein Condensation (BEC) in a variety of limiting situations has recently been examined in such diverse areas as: relativistic superfluids and quark matter [1], a relativistic pion gas [2], relativistic Bose gas trapped in a generic power-law potential [3], possible BEC of relativistic scalar field dark matter [4] (see also [5]), relativistic Bose gas at finite chemical potential and its relation to the sign problem in QCD [6], and particle number fluctuations in relativistic Bose and Fermi gases [7]. The possibility that, due to their superfluid properties, some compact astrophysical objects may contain a significant part of their matter in the form of a Bose-Einstein condensate is envisaged in [8]. In reference [9], the relativistic BEC is considered as a new system for an analogue model of gravity. A relativistic boson gas of particles and antiparticles in the Einstein Universe at high temperatures and densities is studied in [10]. In [11], a cosmological model is proposed in which dark energy is identified with the BEC of some boson fields. We note that, according to [12], a rapidly expanding BEC may be considered as a laboratory model of an expanding universe. Superfluidity in atomic Fermi gases, and the role Bardeen-Cooper-Schrieffer (BCS) -BEC crossover plays in this area is reviewed in [13] (see also [14]). The above list of references is far from exhaustive.
In the current article we study the behavior of Bose gas in one well known model of statistical physics for a relativistic Bose gas [15–17]. We hope that our article clarifies some questions concerning the types of interaction between bodies immersed in a relativistic Bose condensate. We will demonstrate that if two parallel plates are immersed in such a gas then they are attracted to one another. The force of attraction turns out to be long-ranged, i.e. to decay in a power law, based on temperature levels at or below the bulk critical temperature $T_c$ of the condensate. It is short-ranged, i.e. it decays exponentially, only above $T_c$. We determine the scaling function of this force, customarily referred to as a Casimir force, in terms of the appropriate scaling variables, and explicitly determine its value, known as the Casimir amplitude, at the bulk critical point of the system.

The Casimir effect is named for the Dutch physicist H B G Casimir. In 1948 [18], following a discussion with Niels Bohr [19], he realized that the zero-point fluctuations of the electromagnetic field in a vacuum lead to a force of attraction between two perfectly conducting parallel plates, and calculated this force. In 1978, Fisher and De Gennes [20] pointed out that a very similar effect exists in fluids with the fluctuating field being the field of its order parameter, in which the interactions in the system are mediated, not by photons, but by different types of massless excitation, such as critical fluctuations or Goldstone bosons (spin waves). Nowadays one usually refers to the corresponding Casimir effect as the critical or thermodynamic Casimir effect [21].

Currently Casimir, and Casimir-like, effects are objects of study in the fields such as quantum electrodynamics, quantum chromodynamics, cosmology, condensed matter physics, biology and, to some extent, nano-technology. The interested reader may consult an impressive number of reviews on the subject; see e.g., references [21–48]. So far the critical Casimir effect has enjoyed only two general reviews [21, 42] and a few relating to specific aspects of it [43–49].

The critical Casimir effect has been already directly observed, by means of light scattering measurements, in the interaction of a colloid spherical particle with a plate [50], both of which are immersed in a critical binary liquid mixture. Very recently, the nonadditivity of critical Casimir forces has been experimentally demonstrated in [51]. Indirectly, as a balancing force that determines the thickness of a wetting film in the vicinity of its bulk critical point, the Casimir force has been also studied in $^4$He [52, 53], as well as in $^3$He–$^4$He mixtures [54]. In [55, 56] measurements of Casimir force in thin wetting films composed of binary liquid mixture have also been performed. Studies in this field have also attracted considerable theoretical attention. Reviews on the corresponding results can be found in [43–48].

In recent years, the subject of the Casimir effect, and the corresponding Casimir force in Bose systems, have begun to gain attention [57–69]. Casimir force in ideal Bose gas with a film geometry has been studied in [58, 59, 61]. In reference [58], the case of periodic, Dirichlet, and Neumann boundary conditions is addressed for the first time. Reference [59] demonstrates that the problem may be reduced to calculating the force within a Gaussian model of properly defined $O(n)$ models, a subject considered in an earlier study [70]. Reference [61] extended the studies of the ideal Bose gas to...
include Robin boundary conditions, together with explicit expressions for the scaling function of the force under periodic, antiperiodic, Dirichlet, and Neumann boundary conditions. The case of the imperfect Bose gas, with a mean-field like interaction term, was investigated in [61–64]. The main conclusion of these studies is that the bulk system is characterized by the critical exponents of the spherical model [21, 71, 72], and that under periodic and Dirichlet boundary conditions [61] the model with a film geometry is equivalent to the properly defined interacting Bose gas with 2\( n \) internal degrees of freedom in the limit \( n \to \infty \), i.e. the ‘spherical model limit’. In a short-hand notation, one might refer to this model as the ‘\( O(2n) \)’ model for \( n \to \infty \), see reference [61]. According to the universality hypothesis [73], all of these models would be expected to possess the same scaling function of the free energy and the Casimir force. Only the names of the quantities involved and, therefore, the corresponding physical meaning, are different. This last turns out to be correct, as demonstrated in [61–64]. In the current article we study the Casimir effect in relativistic Bose gas. The bulk critical behavior of the model for general space dimension \( d \) has been considered in reference [15]. It has been demonstrated there that the critical exponents of this model are also equal to those of the spherical model. Thus, one expects, on the basis of universality, that the scaling functions of free energy and Casimir force in terms of properly defined scaling variables will be equal to those of the imperfect Bose gas, and of the spherical model. We will derive explicit exact results for the scaling function of the force under periodic boundary conditions, and will demonstrate that universality is obeyed. We will show that the force is attractive across the whole range of thermodynamic parameters under consideration. A closed form expression for the Casimir amplitude will be also obtained. Finally, we will discuss the precise mapping of the relativistic Bose gas model onto the mean spherical model [74], and the imperfect Bose gas [62] of classical systems. Prior to this, for the sake of completeness, we note that, in addition to the ideal Bose gas and the imperfect Bose gas, some results are available for the fluctuation-induced interaction between two impurities in a weakly-interacting one-dimensional Bose gas [66, 68] and, more generally, in quantum liquids [57, 60]. BEC mixtures have been objects of study in [65, 67, 69]. Furthermore, measurement of the Casimir–Polder force via the center-of-mass oscillations of a BEC has been reported in reference [75].

As stated above, in the current article we study the Casimir effect in relativistic Bose gas in space dimension \( d = 3 \). Let us begin by recalling that Bose–Einstein condensation can only occur when the particle number is conserved [76]. Thus, in any discussion of Bose–Einstein condensation for a relativistic Bose gas composed of particles with nonzero rest mass \( m \), at temperatures such that \( k_B T = O(mc^2) \text{ or greater} \), the possibility of particle–antiparticle pair production cannot be ignored, and must be taken into account [15, 16, 77–80]. In section 2 we will formulate the corresponding model by the method used in [16]. The results obtained in this way contain some of the required expressions for free energy in a film geometry, which can be used as a starting point for deriving the corresponding results relating to Casimir force. Next, in section 3, we derive exact results for the behavior of the scaling function of free energy in a film geometry: section 3.1 examines excess free energy, and Casimir force and Casimir amplitude are investigated in section 3.2. The technical details needed to
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry clarify the precise mapping of the relativistic Bose gas onto the spherical model and the imperfect Bose gas are given in the appendices—see A.1 and A.3 for the spherical model, and A.2 and A.4 for the imperfect Bose gas. The article concludes in section 4, where we discuss several points connected to the relations between the models, and some issues tacitly implied by these models, in relation to the effective interactions within the system.

2. The model

In [16] the authors consider an ideal Bose gas composed of \( N_1 \) particles and \( N_2 \) antiparticles, each of mass \( m \), confined to a three-dimensional cuboid cavity of sides \( L_1, L_2 \) and \( L_3 \) under periodic boundary conditions. Since particles and antiparticles are created only in pairs, the system is governed by the conservation of the number \( Q = N_1 - N_2 \), which may be looked upon as a kind of generalized ‘charge’. Thus, in equilibrium, the chemical potentials of the two species are equal and opposite, i.e., \( \mu_1 = -\mu_2 = \mu \). With respect to the occupation numbers \( N_1 \) and \( N_2 \) this results in

\[
N_1 = \sum_{\varepsilon(k)} \left[ e^{\beta(\varepsilon - \mu)} - 1 \right]^{-1}, \quad N_2 = \sum_{\varepsilon(k)} \left[ e^{\beta(\varepsilon + \mu)} - 1 \right]^{-1}, \tag{2.1}
\]

where

\[
\varepsilon(k) = \sqrt{k^2 + m^2}. \tag{2.2}
\]

In equation (2.1), equation (2.2), and thereafter, we use the units \( \hbar = c = k_B = 1 \), thus \( \beta = 1/T \), and let \( L_i, i = 1, 2, 3 \) are measured in terms of microscopic length scale, i.e., \( L_i, i = 1, 2, 3 \) are dimensionless. Then, under periodic boundary conditions, the eigenvalues \( k_i, (i = 1, 2, 3) \) of the wave vector \( k \) are given by \( k_i = (2\pi/L_i)n_i \), where \( n_i = 0, \pm 1, \pm 2, \ldots \). Let us stress that here both \( \varepsilon \) and \( \mu \) include the rest energy \( m \) of the particle, or of the antiparticle. The condition \( |\mu| \leq m \) ensures that the mean occupation numbers in the various states are positive definite. Obviously, one has two symmetric cases \( \mu > 0 \) and \( \mu < 0 \). If, for definiteness, one assumes \( \mu > 0 \) it follows that \( Q > 0 \), i.e., \( N_1 > N_2 \). In view of the conservation of \( Q \), \( \mu \) shall keep its sign. Thus, for definiteness in what follows, we assume \( \mu > 0 \).

3. On the finite size behavior of the model in film geometry

The pressure \( P \) in the grand canonical ensemble [16, 81] may then be written as

\[
P = -\frac{1}{\beta V} \sum_{\varepsilon_n} \left[ \ln \left( 1 - e^{-\beta(\varepsilon_n - \mu)} \right) \right] + \left[ \ln \left( 1 - e^{-\beta(\varepsilon_n + \mu)} \right) \right], \tag{3.3}
\]
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

where \( n = \{n_1, n_2, n_3\} \), and \( V = L_1L_2L_3 \). In accordance with standard thermodynamic relation, for the charge density we have

\[
\rho = \frac{Q}{V} = \left( \frac{\partial P}{\partial \mu} \right)_T.
\]  

(3.4)

Using the identity

\[
\sum_{j=1}^{\infty} \frac{\cosh(ja) \exp(-jb)}{j} = \frac{1}{2} \left\{ \log \left( 1 - e^{-(b-a)} \right) + \log \left( 1 - e^{-(b+a)} \right) \right\},
\]  

(3.5)

the expression in equation (3.3) can be reorganized in the more convenient form

\[
P(\beta, \mu, m|L_1, L_2, L_3) = \frac{2}{\beta V} \sum_{j=1}^{\infty} \cosh(j \beta \mu) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \exp \left[ -j \beta m \sqrt{1 + \frac{4\pi^2}{m^2} \sum_{i=1}^{3} \left( \frac{n_i}{L_i} \right)^2} \right].
\]  

(3.6)

In references [79, 80] specific techniques for dealing with sums of the above type have been developed. With their help, in reference [16] some results for the scaling function of the free energy in (a) fully finite, (b) square channel, and (c) film geometry have been reported. More specifically, three dimensional systems with periodic boundary conditions in geometry of : (a) a cube, i.e., \( L_1 = L_2 = L_3 = L \), (b) a square channel, i.e., \( L_1 \to \infty, L_2 = L_3 = L \), and (c) a film, i.e., \( L_1, L_2 \to \infty, L_3 = L \) were considered. For the purposes of the current article, we are mainly interested in a system with a film geometry with a finite thickness \( L \). In reference [16], explicit results for this case are only presented for the low and high-temperature asymptotic of the scaling function. For the intermediate region \( L(T - T_c)/T_c = O(1) \), only a numerical evaluation of the scaling function at the bulk critical point \( T = T_c \) is given [[16], p 1822]. In the current article we will fully cover this region, obtaining explicit results for the behavior of free energy, Casimir force, and Casimir amplitude.

The most general expressions are, of course, pertinent to the fully finite system. The remainder may be obtained by taking the appropriate limits, as specified above. In order to be specific, and to introduce the required notations, let us present the following expressions [16]:

\[
P = \frac{m^4}{2\pi^2} X(\beta, \mu) + \frac{1}{2\pi \beta} \left[ \sqrt{m^2 - \mu^2} H_2(\mu) + H_3(\mu) \right],
\]  

(3.7)

where

\[
X(\beta, \mu) = 2 \sum_{j=1}^{\infty} \cosh(j \beta \mu) \frac{K_2(j \beta m)}{(j \beta m)^2},
\]  

(3.8)
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

and

$$H_n(\mu) = \sum_{\mathbf{q}} \frac{\exp\left[-\sqrt{m^2 - \mu^2}\gamma(\mathbf{q})\right]}{\gamma^n(\mathbf{q})}, \quad (3.9)$$

with

$$\gamma(\mathbf{q}) = \sqrt{\sum_{i=1}^{3} q_i^2 L_i^2}, \quad \text{where } \mathbf{q} = \{q_1, q_2, q_3\}, \quad q_i = 0, \pm 1, \pm 2, \ldots \quad (3.10)$$

In equation (3.8) $K_2$ is the modified Bessel function of the second kind. In equation (3.9) the prime means that the term with $\mathbf{q} = \mathbf{0}$ is omitted and, therefore, $\gamma(\mathbf{q}) > 0$.

### 3.1. On the finite size behavior of free energy

In equation (3.7) the function $X(\beta, \mu)$ clearly reflects bulk behavior, while the terms in the quadratic brackets take into account the effects relating to the finite extensions of the system. Now, taking the limits $L_1, L_2 \to \infty$ and setting $L_3 = L$, we obtain the corresponding basic result for film geometry. Equations (3.7) and (3.8) remain formally the same, except that equation (3.9) simplifies to

$$H_n(\mu) = 2 \sum_{q=1}^{\infty} \frac{\exp\left[-q\sqrt{m^2 - \mu^2}L\right]}{(qL)^n} = 2(m^2 - \mu^2)^{n/2} (2y_L)^{-n} \text{Li}_n\left(e^{-2y_L}\right), \quad (3.11)$$

where we have introduced the parameter

$$y_L = \frac{1}{2} \sqrt{m^2 - \mu^2} L. \quad (3.12)$$

Thus, equation (3.7) becomes

$$P = \frac{m^4}{2\pi^2} X + L^{-3} \frac{1}{\pi\beta} \left[2y_L \text{Li}_2\left(e^{-2y_L}\right) + \text{Li}_3\left(e^{-2y_L}\right)\right]. \quad (3.13)$$

In equation (3.11), $\text{Li}_n(z)$ is the polylogarithm function, also known as the Jonquièrè’s function

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}. \quad (3.14)$$

$\text{Li}_n(z)$ relate directly to the Bose–Einstein functions [17]

$$g_{\nu}(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} \, dx}{z^{-1} e^x - 1}. \quad (3.15)$$

It is easy to show [17] that $\text{Li}_n(z) = g_{\nu}(z), 0 \leq z < 1$. Let us also note that sometimes $\text{Li}_n(z)$ are denoted as $F(z, n)$, or $F_n(z)$ [82] functions. Due to the aforementioned
diversity in notations, one may encounter results for Bose gas formulated in terms of different but otherwise equivalent functions. In the current article we will use formulations in terms of polylogarithm functions \( \text{Li}_n(z) \). As we will see later, technically this is an important moment, because the available identities for these functions will allow us to obtain a closed form explicit solution for the Casimir amplitude of the model.

Based on equations (3.4) and (3.13), for the case of a film geometry we have:

\[
\rho = \frac{m^3}{2\pi^2} W(\beta, \mu) - \frac{\mu^2}{2\pi^2} \sqrt{m^2 - \mu^2} \frac{\log (1 - e^{-2\beta \mu})}{y_L} \\
= \frac{m^3}{2\pi^2} W(\beta, \mu) + \frac{1}{L \pi \beta} [y_L - \log(2 \sinh y_L)].
\]  

(3.16)

Here

\[
W(\beta, \mu) = m \left( \frac{\partial X}{\partial \mu} \right)_\beta = 2 \sum_{j=1}^{\infty} \sinh(j \beta \mu) \frac{K_2(j \beta m)}{j \beta m}.
\]  

(3.17)

From equations (3.13) and (3.16) for the ‘thermal’, (see reference [16]) free energy density of the system, i.e., the temperature dependent aspect of the free energy, one obtains

\[
\bar{f} = \frac{\bar{F}}{V} = \frac{F - mQ}{V} = (\mu - m) \rho - P = -\frac{m^4}{2\pi^2} \left[ X(\beta, \mu) + \frac{m - \mu}{m} W(\beta, \mu) \right] \\
- L^{-3} \frac{1}{\pi \beta} \left[ 2y_L \text{Li}_2(e^{-2\beta \mu}) + \text{Li}_3(e^{-2\beta \mu}) \right] - \frac{1}{L} \frac{\mu(m - \mu)}{\pi \beta} [y_L - \log(2 \sinh y_L)].
\]  

(3.18)

When \( \mu \approx m \) the following expansions are valid:

\[
W(\beta, \mu) = W(\beta, m) - \frac{\pi \mu}{\beta m^3} \sqrt{m^2 - \mu^2} + \mathcal{O}(m^2 - \mu^2),
\]  

(3.19)

and

\[
X(\beta, \mu) = X(\beta, m) - \frac{m - \mu}{m} W(\beta, m) + \frac{\pi}{3 \beta m^4} (m^2 - \mu^2)^{3/2} + \mathcal{O}(m^2 - \mu^2)^2
\]  

(3.20)

or, in terms of \( y_L \),

\[
W(\beta, \mu) = W(\beta, m) - \frac{1}{L} \frac{2\pi \mu}{\beta m^3} y_L + \mathcal{O}(y_L^2),
\]  

(3.21)

and

\[
X(\beta, \mu) = X(\beta, m) - \frac{m - \mu}{m} W(\beta, m) + \frac{1}{L^3} \frac{8\pi}{3 \beta m^4} y_L^3 + \mathcal{O}(y_L^4).
\]  

(3.22)
With their help, equations (3.16) and (3.18) become
\[ \rho = \frac{m^3}{2\pi^2} W(\beta, m) - \frac{1}{L \pi \beta} \log(2 \sinh y_L), \tag{3.23} \]
and
\[ \bar{f} = \frac{m^4}{2\pi^2} X(\beta, m) + L^{-3} \frac{1}{\pi \beta} \left\{ \frac{2}{3} y_L^3 - 2 y_L \text{ Li}_2(e^{-2y_L}) \right. \\
- \left. \text{ Li}_3(e^{-2y_L}) - 2 y_L^2 [y_L - \log(2 \sinh y_L)] \right\}. \tag{3.24} \]
Thus, keeping in mind that \(X(\beta, m)\) is a regular term, in full accord with reference [16], the singular aspect, i.e. that possessing scaling behavior, of the free-energy density is given as:
\[ f^{(s)}(T; L) \equiv \frac{\beta \bar{F}^{(s)}}{V} = L^{-3} \frac{1}{\pi} \left\{ \frac{2}{3} y_L^3 - 2 y_L \text{ Li}_2(e^{-2y_L}) \right. \\
- \left. \text{ Li}_3(e^{-2y_L}) - 2 y_L^2 [y_L - \log(2 \sinh y_L)] \right\}. \tag{3.25} \]
Recalling now that the bulk critical point \(\beta_c\) is determined by the condition
\[ \rho = \frac{m^3}{2\pi^2} W(\beta_c, m), \tag{3.26} \]
(see equation (3.16)) with \(\mu(\beta_c) = m\), equation (3.23) then becomes
\[ W(\beta, m) - W(\beta_c, m) = \frac{1}{L \beta_c m^2} 2\pi \log(2 \sinh y_L). \tag{3.27} \]
Expanding the above in relation to \(\beta_c\), and introducing the notation \(x_T\), we obtain
\[ x_T \equiv \beta_c m^2 L [W(\beta, m) - W(\beta_c, m)] \\
\simeq \left( \beta_c^2 m^2 \left| \frac{\partial W}{\partial \beta} \right|_{\beta=\beta_c} \right) L T, \quad \text{with } T = \frac{T - T_c}{T_c}, \tag{3.28} \]
i.e., equation (3.27) now reads
\[ x_T = 2\pi \log(2 \sinh y_L). \tag{3.29} \]
Thus, for the singular aspect of the free energy density one has
\[ \bar{f} = L^{-3} X_i(x_T), \tag{3.30} \]
where \(X_i(x_T)\) is determined (see equation (3.25)) by the expression
\[ X_i(y_L) = \frac{1}{\pi} \left\{ \frac{2}{3} y_L^3 - 2 y_L \text{ Li}_2(e^{-2y_L}) - \text{ Li}_3(e^{-2y_L}) - 2 y_L^2 [y_L - \log(2 \sinh y_L)] \right\}, \tag{3.31} \]
and where \(y_L\), according to equation (3.29), is
\[ y_L(x_T) = \text{arcsinh} \left( \frac{1}{2} \exp \left( \frac{x_T}{2\pi} \right) \right). \tag{3.32} \]
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

It is therefore simple to verify that \( y_L(x_\tau) \) is a monotonically increasing function of \( x_\tau \). Then, since

\[
\frac{dX_i[y_L(x_\tau)]}{dx_\tau} = \frac{1}{\pi^2} \text{csch}^{-1}\left[2 \exp\left(-\frac{x_\tau}{2\pi}\right)\right]^2,
\]

we conclude that \( X_i \) is, as expected, a monotonically increasing function of \( x_\tau \).

In reference [16], the only specific results reported for \( X_i \) are those giving the values of \( X_i \) (see equation (67)) for low temperatures, i.e., for \( t < 0 \) when \( L \to \infty \), and in the opposite case of high temperatures, i.e., when \( t > 0 \) and \( L \to \infty \). One finds that

\[
X_i(x_\tau) \simeq \frac{1}{12\pi^4} x_\tau^3, \quad x_\tau \to \infty
\]

which is, in fact, the bulk result \( X^{(b)}_i(x_\tau) \), and

\[
X_i(x_\tau) = -\frac{1}{\pi} \zeta(3), \quad x_\tau \to -\infty.
\]

These results are obviously easily reproducible from equations (3.31) and (3.32), taking into account that \( y_L \to \infty \) when \( x_\tau \to \infty \), and \( y_L \to 0 \) when \( x_\tau \to -\infty \). Furthermore, from equation (3.32) with \( x_\tau = 0 \) one obtains \( y_L(x_\tau = 0) = \text{arcsinh}[1/2] \approx 0.481212 \) which is basically the value of 0.48 reported in [16], p 1822.

3.2. On the behavior of the excess free energy and the Casimir force

From equations (3.30), (3.31), and (3.34), it is easy to obtain the excess free energy normalized per unit area

\[
\beta f_{ex}(x_\tau) = L^{-(d-1)} X_{ex}(x_\tau), \quad \text{with } d = 3,
\]

where

\[
X_{ex}(x_\tau) = X_i(x_\tau) - X^{(b)}_i(x_\tau),
\]

i.e., the amount of free energy in excess of bulk energy. Explicitly, one has

\[
X_{ex}(x_\tau) = \frac{1}{\pi} \left[\frac{2}{3} (y_L^3 - y_\infty^3) + 2y_L^2 \ln \left(1 - e^{-2y_L}\right) - 2y_L \ln \left(e^{-2y_L}\right) - \text{Li}_2\left(e^{-2y_L}\right) - \text{Li}_3\left(e^{-2y_L}\right)\right],
\]

where \( y_L \) is given by equation (3.32), and

\[
y_\infty = \begin{cases} x_\tau/(2\pi), & x_\tau \geq 0 \\ 0, & x_\tau \leq 0. \end{cases}
\]

From the excess free energy, we can derive the corresponding expression for the Casimir force. By definition [21]:

\[
\beta F_{Cas} = -\frac{\partial \left[\beta f_{ex}(x_\tau)\right]}{\partial L}.
\]
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

From the above definitions, it follows that

$$\beta F_{\text{Cas}} = L^{-d} X_{\text{Cas}}(x_\tau), \quad d = 3,$$

(3.41)

where the scaling function of the Casimir force is

$$X_{\text{Cas}}(x_\tau) = (d - 1) X_{\text{ex}}(x_\tau) - \frac{1}{\nu} x_\tau \frac{\partial X_{\text{ex}}(x_\tau)}{\partial x_\tau}. \quad (3.42)$$

In the system considered here $d = 3$. From equation (3.28), taking into account that, according to the general theory [21], one shall have $x_\tau = C t L^{1/\nu}$ with $C$ being a given system dependent constant, and $\nu$ the critical exponent of the correlation length, we conclude that $\nu = 1$. Performing the calculations, from equation (3.42) one derives

$$X_{\text{Cas}}(x_\tau) = \frac{2}{\pi} \left[ \frac{1}{3} (y_L^3 - y_\infty^3) + 2 y_L \log (1 - e^{-2y_L}) + \text{Li}_2 (e^{-2y_L}) - y_L^2 \log (1 - e^{-2y_L}) \right]. \quad (3.43)$$

It is easy to check that $y_L \geq y_\infty$ and, thus, all terms in quadratic brackets are positive, i.e. $X_{\text{Cas}}(x_\tau) \leq 0$. This last implies that within relativistic Bose gas, the Casimir force is always attractive.

The Casimir amplitude can be derived from equations (3.38), or (3.43). From equation (3.39) one obtains $y_\infty(x_\tau = 0) = 0$, and from equation (3.32) that $y_L(x_\tau = 0) = \text{arcsinh}(1/2)$. Plugging these values into, for example, equation (3.38), following some manipulations based on identities for polylogarithmic functions (see below), one obtains

$$X_{\text{ex}}(x_\tau = 0) = \frac{1}{2} X_{\text{Cas}}(x_\tau = 0) \equiv \Delta_{\text{Cas}}^{\text{RBG}} = -\frac{4}{5\pi} \zeta(3). \quad (3.44)$$

Let us briefly elucidate the procedure leading to the above explicit result. First, let us note that

$$\text{arcsinh} \left( \frac{1}{2} \right) = \log \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \equiv \log \varphi, \quad (3.45)$$

where $\varphi$ is the so-called golden ratio

$$\varphi = \frac{1}{2} + \frac{\sqrt{5}}{2}. \quad (3.46)$$

Then

$$\exp[-2y_L(x_\tau = 0)] = \varphi^{-2} = 2 - \varphi, \quad (3.47)$$

and equation (3.38) takes the form

$$X_{\text{ex}}(x_\tau = 0) = \frac{1}{\pi} \left[ \frac{1}{6} \log^3(2 - \varphi) + \log(2 - \varphi) \text{Li}_2 (2 - \varphi) - \text{Li}_3 (2 - \varphi) \right], \quad (3.48)$$

https://doi.org/10.1088/1742-5468/ab900a
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry i.e., it can be expressed entirely in terms of the golden ratio. Next, using the identity \[83\] (see also \[84\]):

\[
\log(2 - \varphi) \operatorname{Li}_2(2 - \varphi) - \operatorname{Li}_3(2 - \varphi) = -\frac{1}{6} \log^3(2 - \varphi) - \frac{4\zeta(3)}{5}, \tag{3.49}
\]

we arrive at the result reported in equation (3.44). In fact, it can be shown \[83, 84\] that at \(z = 2 - \varphi\) both \(\operatorname{Li}_3(z)\) and \(\operatorname{Li}_2(z)\) may be expressed in terms of elementary functions:

\[
\begin{align*}
\operatorname{Li}_3(2 - \varphi) &= \frac{4}{5} \zeta(3) + \frac{1}{15} \pi^2 \log(2 - \varphi) - \frac{1}{12} \log^3(2 - \varphi), \\
\operatorname{Li}_2(2 - \varphi) &= \frac{1}{15} \pi^2 - \frac{1}{4} \log^2(2 - \varphi).
\end{align*} \tag{3.50}
\]

4. Concluding remarks and discussion

In this article we have derived exact analytical expressions for the singular aspect of free energy (see equation (3.31)), the excess free energy scaling function (see equation (3.38)), and Casimir force (see equation (3.43)) relating to relativistic Bose gas. We have determined explicit expressions for the Casimir amplitude of the model (see equation (3.44)), and have shown that Casimir force is an attractive force, monotonically increasing with temperature (see figure 1). We have also determined the low temperature asymptotic of free energy, i.e., of the excess free energy, in equation (3.34). Let us recall that the Casimir amplitude \[70\] for the \(n\)-component Gaussian model with \(d \in (2, 4)\) is

\[
\Delta_{\text{Cas}}^{\text{GM}}(n, d) = -n \frac{\Gamma(d/2)}{\pi^{d/2}} \zeta(d). \tag{4.51}
\]

With \(n = 2\) and \(d = 3\) this leads to \(\Delta_{\text{Cas}}^{\text{GM}}(2, 3) = -\zeta(3)/\pi\). Thus, the low temperature asymptotic of the finite-size aspect of free energy is equal to that one of the two-component Gaussian model.

The model considered here is characterized with \(d = 3\) and, as we have seen, \(\nu = 1\). As stated in the introduction, the bulk critical behavior of the model for general \(d\) has been considered in \[15\]. For \(d \in (2, 4)\) one obtains

\[
\begin{align*}
\nu &= \frac{1}{d - 2}, & \beta &= \frac{1}{2}, & \eta &= 0, & \alpha &= \frac{d - 4}{d - 2}, & \gamma &= \frac{2}{d - 2}.
\end{align*} \tag{4.52}
\]

These critical exponents coincide with those of the spherical model \[21, 71, 72\], and the imperfect non-relativistic Bose gas. Next, according to the universality hypothesis \[73\], all these models should possess the same scaling function of free energy and Casimir force, with only the names of the quantities involved and, therefore, the corresponding physical meaning, being different. On examining the result presented in equation (3.44), and the corresponding result for the spherical model (see equation (30) in \[74\]), we find that the Casimir amplitude for the relativistic Bose gas is exactly twice as large.
Figure 1. The behavior of the Casimir force within the relativistic Bose gas. We observe that the scaling function is negative, monotonic, and approaches an universal negative constant for low temperatures.

as for the spherical model. A careful comparison (see appendix) of the expressions for the scaling variables and excess free energy scaling functions reveals the possibility of mapping one into the other, with the result that

\[ 2X_{\text{ex}}^{(\text{SM})} = X_{\text{ex}}^{(\text{Bose})}, \]  

where the upper-scripts indicate to which model the corresponding scaling function belongs. In the same appendix the result for the scaling function of the excess free energy \( X_{\text{ex}}^{(\text{IB})} \) for \( T < T_c(\mu) \) for imperfect Bose gas with a mean-field-like interaction term, is also presented. A comparison with the results for the spherical model again shows that

\[ X_{\text{ex}}^{(\text{IB})} = 2X_{\text{ex}}^{(\text{SM})}. \]  

Thus, for \( T < T_c \), where \( T_c \) is the critical temperature for the corresponding model,

\[ X_{\text{ex}}^{(\text{Bose})} = X_{\text{ex}}^{(\text{IB})}. \]  

One might wonder how a model of ideal relativistic Bose gas is mathematically equivalent to a spherical model with short-range nearest neighbor interaction, and to that for an imperfect Bose gas. Indeed, the latter implies that the model considered is not really ‘purely non-interacting’. An examination of the conditions imposed on the model leads one to the conclusion that some sort of an effective interaction arises from the requirement that the density of ‘charge’ is fixed. This ‘interaction’ obviously acts uniformly across all particles in the system.

Let us note that in the cases of relativistic Bose gas, imperfect Bose gas, and of the spherical model, the value of the critical temperature \( \beta_c \), and the corresponding temperature dependence of the free energy around it, follows from one self-consistent equation—the equation for having a fixed length in terms of the spins in the spherical model (‘spherical field’ equation—see equation (14) in reference [85]), the ‘stationary-point equation’ (see equation (9) in reference [62]) for the imperfect Bose gas, or for the charge density in the case of relativistic Bose gas (see equation (3.16) above). Note that when fixing the charge density \( \rho \) one does not determine \( \mu(\rho) \), as in the usual

https://doi.org/10.1088/1742-5468/ab900a
ensemble transformation, but $\beta_c(\rho, m)$. This additional self-consistent equation is what associates the models mentioned above with the universality class of $O(n)$ models in the case of $n \to \infty$, and not to the Gaussian model. This is also the main difference with the ideal Bose gas model behavior considered in reference [59]—it is equivalent to the Gaussian model description within the $O(n)$ models’ formulation. To simplify: the spherical model, the imperfect Bose gas, and the relativistic model are equivalent to the Gaussian model in which the parameters satisfy one self-consistent equation. Actually, there is generally a mathematical difficulty in solving this equation. The fact that the models are equivalent to the Gaussian model with one additional equation requiring to be satisfied, leads to the result that the scaling function of free energy formally looks like that of the Gaussian model (compare, e.g., equation (3.13) with equation (3.38) for $d = 3$ in [86]; equation (17) in [74]; equation (3.10a) in [61], or with what follows from equation (2) in [59]), but the meaning of the parameters is different—they have to satisfy that additional equation. This last point leads to critical exponents and Casimir amplitudes, as well as the temperature dependence of the scaling functions, differing from those of the Gaussian model, and also renders the model with well defined free energy below $T_c$.

The consideration presented in this article relates to periodic boundary conditions. One might expect Dirichlet–Dirichlet boundary conditions to be much more realistic. If the analogy with the spherical model is then further preserved, which seems plausible, one might expect that the Casimir force is again attractive. It will, however, no longer be monotonic as a function of the temperature scaling variable, but will possess a deep minimum below the bulk critical temperature—see references [88, 89].

Finally, let us also make some comments regarding how the type of dispersion relation changes the properties of the considered system. In reference [82], Bose–Einstein condensation with single-particle energy spectrum $\varepsilon(k) \sim |k|^\sigma$, with $0 < \sigma \leq 2$ was examined. The results showed that systems with $\sigma = 1$ and $\sigma = 2$ belong to different universality classes for given values of dimensionality $d$ of the system. Only when the spectrum of the relativistic system, as explained in reference [15, 17], takes the form considered in the current article, and, as suggested by Haber and Weldon in reference [77, 78] the possibility of particle–antiparticle pair production in the system is taken into account, do the relativistic and non-relativistic Bose gasses belong to the same universality class. The effect of changes in dispersion relation on the value of the critical temperature, which is a non-universal quantity and thus model dependent, and the relation of $T_c$ with number density, are studied in reference [90]. The fact that the change of spectrum from $\varepsilon(k) \sim |k|$ to $\varepsilon(k) \sim |k|^2$ leads to different universality classes can be easily understood if one takes into account mapping onto the spherical model. While the $\varepsilon(k) \sim |k|^2$ spectrum corresponds to a spherical model with short-ranged interaction, that in which $\varepsilon(k) \sim |k|^\sigma$, and $0 < \sigma < 2$ is a model with a power-law decaying interaction decaying as $r^{-d-\sigma}$ with distance [21]. The critical exponents in the latter case do continuously depend on $\sigma$ for $\sigma < d < 2\sigma$ [17, 21]. For $d > 2\sigma$, the system is characterized by the mean-field critical exponents. Clearly, the system with $d = 3$ and $\sigma = 1$ does belong to the latter case. Regarding the critical behavior of the system, as is well known, the wavelength asymptotic of the spectrum is, i.e., the limit $|k| \to 0$ is important. Thus, for any fixed
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

In the introduction of this article, we discussed several topics in which the BEC of relativistic Bose gas is of essential importance. We hope that our considerations will be of use in some of them.

Acknowledgments

The author gratefully acknowledges the financial support via contract DN 02/8 from the Bulgarian National Science Fund.

Appendix. On the relation between the relativistic Bose gas model with two other models

In this appendix, we demonstrate the existence of a simple relation between the excess free energy scaling functions of the relativistic Bose gas model with those of the spherical model, and of imperfect Bose gas with mean-field type interaction. In order to introduce the notations, we first briefly reiterate the definitions of the mean spherical model and the imperfect Bose gas and then, using results reported in the literature, demonstrate that the scaling functions of the free energy in all three models do indeed coincide up to a factor of 2 in the case of relativistic and the imperfect Bose gas. In fact, reference [61] showed the equivalence of imperfect Bose gas with an interacting Bose gas with $2n$ internal degrees of freedom in the limit $n \to \infty$, i.e. the ‘spherical model limit’. In brief, one could term this model the ‘$O(2n)$’ model for $n \to \infty$ (see reference [61]). Note that the standard spherical model involves only a physically reasonably short-ranged pair potential.

A.1. A short definition of the spherical model

We consider a model embedded on a $d$-dimensional hypercubic lattice $\mathcal{L} \in \mathbb{Z}^d$, where $\mathcal{L} = L_1 \times L_2 \times \cdots \times L_d$. Let $L_i = N_i a_i$, $i = 1, \ldots, d$, where $N_i$ is the number of spins and $a_i$ is the lattice constant along the axis $i$ with $e_i$ being a unit vector along that axis, i.e., $e_i e_j = \delta_{ij}$. With each lattice site $r$ one associates a real-valued spin variable $S_r \in \mathbb{R}$ which obeys the constraint

$$\langle S_r^2 \rangle = 1, \quad \text{for all } r \in \mathcal{L}. \tag{A.1}$$

The average in equation (A.1) is with respect to the Hamiltonian of the model

$$\beta \mathcal{H} = -\frac{1}{2} \beta \sum_{r,r'} S_r J(r,r') S_{r'} - \sum_r h_r S_r + \sum_r \lambda_r (S_r^2 - 1), \tag{A.2}$$

where the Lagrange multipliers $\lambda_r$, called spherical fields, are determined so that equation (A.1) is fulfilled for all $r \in \mathcal{L}$. Equations (A.1) and (A.2) represent the most

https://doi.org/10.1088/1742-5468/ab900a
general definition of the so-called mean spherical model \[91–93\]. Its main difference from the standard Ising model is that equation (A.1) is fulfilled only in average and not for any state of the system. Obviously, a system with a translational invariance only requires a single spherical field equation, i.e. \( \lambda_r = \lambda \) for all \( r \in L \).

### A.2. A short definition of the imperfect Bose gas

Let us consider a model of an interacting Bose gas. We will only deal with the type of model in which the repulsive pair interaction between identical bosons is described by associating each pair of particles with some mean energy \( (a/V) \), where \( a > 0 \), and \( V \) denotes the volume occupied by the system. The Hamiltonian of such an imperfect Bose gas \[94\] composed of \( N \) particles is defined as

\[
H = H_0 + H_{mf},
\]

i.e. the sum of the kinetic energy

\[
H_0 = \sum_k \frac{\hbar^2 k^2}{2m} \hat{n}_k,
\]

and the term representing the mean-field approximation to the interparticle interaction

\[
H_{mf} = \frac{a}{V} \frac{N^2}{2}.
\]

The symbols \( \{ \hat{n}_k \} \) denote the particle number operators and the summation is over one-particle states \( \{ k \} \).

### A.3. Results for the scaling function of excess free energy for the spherical model

The results for this case have been reported in references \[85\] and \[74\]. There, the behavior of the model is investigated as a function of both temperature and magnetic field. Here, we are interested only in behavior related to temperature. Thus, the scaling function of excess free energy, is expressed as:

\[
X_{ex}^{(SM)}(x_r) = -\frac{1}{2\pi} \left[ \frac{1}{6} \left( \frac{y_L^{3/2}}{y_\infty} - y_\infty^{3/2} \right) + \sqrt{y_L} \Li_2 \left( e^{-\sqrt{y_L}} \right) + \Li_3 \left( \exp \left( -\sqrt{y_L} \right) \right) \right] - \frac{1}{8\pi} x_r \left( y_\infty - y_L \right),
\]

where

\[
x_r = 4\pi K_c \tau L.
\]

In a system with isotropic short-ranged interaction \( J \), and critical coupling \( K_c \), where \( K = \beta J \), is given in reference \[95\] as

\[
K_c = \frac{(\sqrt{3} - 1) \Gamma(1/24)^2 \Gamma(11/24)^2}{192\pi^3} \approx 0.252731.
\]
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

In equation (A.6) \(y_L \equiv y_L(x_r)\) and \(y_\infty \equiv y_\infty(x_r)\) are to be determined from equations

\[-x_r = -2 \ln \left[ 2 \sinh \left( \frac{1}{2} \sqrt{y_L} \right) \right],\]

for the finite system, and

\[x_r = \sqrt{y_\infty}\]

for the infinite system, when \(x_r \geq 0\). Where \(x_r < 0\), one obtains \(y_\infty = 0\).

Making the identifications

\[\frac{1}{2} \sqrt{y^{(SM)}_L} = y^{(Bose)}_L, \quad \frac{1}{2} \sqrt{y^{(SM)}_\infty} = y^{(Bose)}_\infty,\]

one concludes that

\[2X^{(SM)}_{ex} = X^{(Bose)}_{ex}.\]

The latter indicates that relativistic Bose gas is mathematically equivalent to the ‘two component’ spherical model. We recall that the same is also true in the case of imperfect Bose gas [61, 62]. This is demonstrated below.

A.4. Results for the scaling function of the free energy of the imperfect Bose gas

Below, we present some explicit expressions for the model of the imperfect Bose gas reported in [62]. For \(d \in (2, 4)\), and below the bulk condensation temperature \(T < T_c(\mu)\) and \(\tilde{\mu} \geq 0\), the scaling function \(X_{ex,IB}(x|d)\) takes the form

\[-X_{ex,IB}^{IB}(x|d) = \frac{\zeta(d/2)}{4\pi} x \left[ \sigma(x) \right]^2 + \frac{\Gamma(-d/2)}{2^d \pi^{d/2}} \left[ \sigma(x) \right]^d + \frac{2^{2-d/2}}{\pi^{d/2}} \sum_{n=1}^\infty \left[ \frac{\sigma(x)}{n} \right]^{d/2} K_{d/2}[n \sigma(x)],\]

(A.13)

with \(\sigma(x)\) obtained as a solution of

\[x \zeta \left( \frac{d}{2} \right) \pi^{d/2-1} - \frac{\Gamma(1 - d/2)}{2^{d-2}} \left[ \sigma(x) \right]^{d-2} = 2^{3-d/2} \left[ \sigma(x) \right]^{d/2-1} \sum_{n=1}^\infty n^{-(d/2-1)} K_{d/2-1}[n \sigma(x)].\]

(A.14)

Here

\[x = \tilde{\mu}(L/\lambda)^{d-2}, \quad \tilde{\mu} = (\mu - \mu_c)/\mu_c,\]

(A.15)

with

\[\mu_c(T) = \text{Li}_{d/2}(1) \left[ a/\lambda^d \right] = \zeta(d/2) \left[ a/\lambda^d \right].\]

(A.16)

Performing the identifications

\[\sigma = \sqrt{y_L}, \quad \text{and} \quad -x_r = x \zeta(d/2),\]

(A.17)
having set $d = 3$, one obtains, in comparison with equation (4.53), that

$$X^{(IB)}_{ex} = 2X^{(SM)}_{ex}. \quad (A.18)$$

The latter once again indicates that relativistic Bose gas is mathematically equivalent to the ‘two component’ spherical model.

References

[1] Nishida Y and Abuki H 2005 Phys. Rev. D 72 096004
[2] Begun V V and Gorenstein M I 2008 Phys. Rev. C 77 064903
[3] Su G, Chen J and Chen L 2006 J. Phys. A: Math. Gen. 39 4935–44
[4] Ureña-López L A 2009 J. Cosmol. Astropart. Phys. 014
[5] Böhmer C G and Harko T 2007 J. Cosmol. Astropart. Phys. 025
[6] Aarts G 2009 Phys. Rev. Lett. 102 131601
[7] Begun V V and Gorenstein M I 2006 Phys. Rev. C 73 054904
[8] Chavanis P H and Harko T 2012 Phys. Rev. D 86 064011
[9] Fagnocchi S, Finazzi S, Liberati S, Kormos M and Trombettoni A 2010 New J. Phys. 12 005012
[10] Parker L and Zhang Y 1991 Phys. Rev. D 44 2421–31
[11] Fukuyama T and Morikawa M 2006 J. Phys.: Conf. Ser. 31 139–42
[12] Eckel S, Kumar A, Jacobson T, Spielman I B and Campbell G K 2018 Phys. Rev. X 8 021021
[13] Yu Y and Chen Q 2010 Proc. of the 9th Int. Conf. on Materials and Mechanisms of Superconductivity (Physica C: Superconductivity and its Applications vol 470) S900–3
[14] Strinati G C, Pieri P, Röpke G, Schuck P and Urban M 2018 The BCS–BEC crossover: from ultra-cold Fermi gases to nuclear systems Phys. Rep. 738 1–76
[15] Singh S and Pandita P N 1983 Phys. Rev. A 28 1752–61
[16] Singh S and Pathria R K 1985 Phys. Rev. A 31 1816–24
[17] Pathria R K and Beale P D 2011 Statistical Mechanics 3rd edn (New York: Elsevier)
[18] Casimir H B 1948 Proc. K. Ned. Akad. Wet. 51 793
[19] Casimir H B G 1999 Some remarks on the history of the so called Casimir effect The Casimir Effect 50 Years Later Proc. of the 4th Workshop on Quantum Field Theory under the Influence of External Conditions ed M Bordag (Leipzig: World Scientific) 3–9
[20] Fisher M E and de Gennes P G 1978 C. R. Seances Acad. Sci. Paris Ser. B 287 207
[21] Brankov J G, Dantchev D M and Tonchev N S 2000 The Theory of Critical Phenomena in Finite-Size Systems-Scaling and Quantum Effects (Singapore: World Scientific)
[22] Pumien G, Müller B and Greiner W 1986 Phys. Rep. 134 87–193
[23] Mostepanenko V M and Trunov N N 1988 Sov. Phys. - Usp. 31 965
[24] Levin F S and Micha D A ed 1993 Long-Range Casimir Forces Finite Systems and Multiparticle Dynamics (Berlin: Springer)
[25] Mostepanenko V M and Trunov N N 1997 The Casimir Effect and its Applications (Moscow: Energoatomizdat)
[26] Milonni P W 1994 The Quantum Vacuum (San Diego: Academic)
[27] Kardar M and Golestanian R 1999 Rev. Mod. Phys. 71 1233–45
[28] Bordag M ed 1999 The Casimir effect 50 years later (World Scientific) Proc. of the Fourth Workshop on Quantum Field Theory under the Influence of External Conditions
[29] Bordag M, Mohideen U and Mostepanenko V M 2001 Phys. Rep. 353 1–205
[30] Milton K A 2001 The Casimir Effect: Physical Manifestations of Zero-Point Energy (Singapore: World Scientific)
[31] Milton K A 2004 J. Phys. A: Math. Gen. 37 R209–77
[32] Lamoreaux S K 2005 Rev. Prog. Phys. 68 201–36
[33] Bordag M, Klimchitskaya G L, Mohideen U and Mostepanenko V M 2009 Advances in the Casimir Effect (Oxford: Oxford University Press)
[34] Klimchitskaya G L, Mohideen U and Mostepanenko V M 2009 Rev. Mod. Phys. 81 1827–85
[35] French R H et al 2010 Rev. Mod. Phys. 82 1887–944
[36] Sergey D O, Sáez-Gómez D and Xambó-Desclamps S ed 2011 Cosmology, Quantum Vacuum and Zeta Functions (Springer Proc. in Physics vol 137) (Berlin: Springer)
[37] Dalvit D, Milonni P, Roberts D and da Rosa F E ed 2011 Casimir Physics 1st edn (Lecture Notes in Physics vol 834) (Berlin: Springer)

https://doi.org/10.1088/1742-5468/ab900a 18
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

[38] Milton K A, Abalo E K, Parashar P, Pourtolami N, Brevik I and Ellingsen S A 2012 J. Phys. A: Math. Gen. 45 374006

[39] Brevik I 2012 J. Phys. A: Math. Theor. 45 374003

[40] Woods L M, Dalvit D A R, Tkatchenko A, Rodriguez-Lopez P, Rodriguez A W and Podgornik R 2016 Rev. Mod. Phys. 88 045003

[41] Zhao R, Luo Y and Pendry J B 2016 Sci. Bull. 61 59–67

[42] Krech M 1994 Casimir Effect in Critical Systems (Singapore: World Scientific)

[43] Krech M 1999 J. Phys.: Condens. Matter 11 R391

[44] Gambassi A 2009 J. Phys.: Conf. Ser. 161 012037

[45] Toldin F P and Dietrich S 2010 J. Stat. Mech. 2010 P11003

[46] Gambassi A and Dietrich S 2011 Soft Matter 7 1247–53

[47] Dean D S 2012 Phys. Scr. 86 058502

[48] Vasilyev O A 2015 Monte Carlo Simulation of Critical Casimir Forces (Singapore: World Scientific) pp 55–110

[49] Martin P A and Zagrebnov V A 2006 Europhys. Lett. 74 754

[50] Fuchs J N, Recati A and Zwerger W 2007 Phys. Rev. A 75 043615

[51] Napiorkowski M and Piasecki J 2011 Phys. Rev. E 84 061105

[52] Dantchev D M 1998 Phys. Rev. E 58 1455–62

[53] Harber D M, Obrecht J M, McGuirk J M and Cornell E A 2005 Phys. Rev. A 72 033610

[54] Huang K 1987 Statistical Mechanics 2nd edn (New York: Wiley)

[55] Haber H E and Weldon H A 1981 Phys. Rev. Lett. 46 1497–500

[56] Haber H E and Weldon H A 1982 Phys. Rev. D 25 502–25

[57] Singh S and Pathria R K 1984 Phys. Rev. A 30 442–9

[58] Pisarski R and Stringari S 2016 Bose-Einstein Condensation and Superfluidity vol 164 (Oxford: Oxford University Press)

[59] Gunton J D and Buckingham M J 1968 Phys. Rev. 166 152–8

[60] Lewin L 1981 Polylogarithms and Associated Functions (Amsterdam: Elsevier)

[61] Sachdev S 1993 Phys. Lett. B 309 285

[62] Dantchev D 1996 Phys. Rev. E 53 2104–9

[63] Dantchev D and Krech M 2004 Phys. Rev. E 69 046119

[64] Diehl H W 1986 Field-Theoretical Approach to Critical Behavior of Surfaces Phase Transitions and Critical Phenomena vol 10 ed C Domb and J L Lebowitz (New York: Academic) p 76

https://doi.org/10.1088/1742-5468/ab900a
Exact results for the Casimir force of a three-dimensional model of relativistic Bose gas in a film geometry

[88] Diehl H W, Grüneberg D, Hasenbusch M, Hucht A, Rutkevich S B and Schmidt F M 2012 Europhys. Lett. 100 10004
[89] Dantchev D, Bergknoff J and Rudnick J 2014 Phys. Rev. E 89 042116
[90] Grether M, de Llano M and Baker G A 2007 Phys. Rev. Lett. 99 200406
[91] Berlin T H and Kac M 1952 Phys. Rev. 86 821–35
[92] Lewis H W and Wannier G H 1952 Phys. Rev. 88 682–3
[93] Knops H J F 1973 J. Math. Phys. 14 1918–20
[94] Davies E B 1972 Commun. Math. Phys. 28 69–86
[95] Joyce G S and Zucker I J 2001 J. Phys. A: Math. Gen. 34 7349

https://doi.org/10.1088/1742-5468/ab900a