Interaction-enhanced ferromagnetic insulating state of the edge of a two-dimensional topological insulator

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The spin-related aspects of electron-electron interactions at the edge of a two-dimensional topological insulator are addressed. In the absence of the magnetic field, the interacting edge electrons form a helical Luttinger liquid with gapless collective excitations that carry both spin and charge. The edge exhibits in-plane XY ferromagnetic correlations but the long-range order is not formed and the edge is a perfect conductor. Applying the magnetic field orders the edge ferromagnetically and opens a gap $\Delta$ in the spectrum of collective spin-charge excitations, which can be substantially enhanced by the interactions. The exponent of the scaling dependence $\Delta \propto H^{1/(2-K)}$ of the gap on the magnetic field $H$ is controlled by the Luttinger liquid interaction parameter $K$, which can be extracted from the activation behavior $G_{xx} \sim \frac{e^2}{2\pi}\exp(-\Delta/T)$ of the longitudinal conductance at lower temperatures $T \ll \Delta$. This provides a simple way to probe the interactions experimentally.

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Introduction. Topological insulators [1–15] form a new class of materials with nontrivial band structure caused by spin-orbit interactions. The key feature that distinguishes a topological insulator from a conventional one is the presence of gapless single-particle surface or edge states. The edge of a two-dimensional (2D) topological insulator [1], [2], [3] supports two branches of gapless counter-propagating helical states with opposite spin projections on the axis perpendicular to the plane of the sample (Fig. 1). Protected by the time-reversal symmetry (TRS) against single-particle backscattering [14], [17], these edge modes serve as nearly ideal conducting channels that give rise to the quantum spin Hall (QSH) effect. So far, the 2D topological insulator was realized in HgTe-CdTe quantum wells [10–11], and shortly after confirmed experimentally [10–11].

The interactions between electrons in the counter-propagating states lead to a novel helical Luttinger liquid (LL) phase [16–23] with unusual spin-related behavior. It would be beneficial to detect the signatures of interactions in the experiment. However, in the critical LL state the collective excitations are gapless and the edge conductance remains essentially unaffected by the interactions [24]. Therefore, as long as TRS is preserved, the interactions do not reveal themselves in the longitudinal conductance $G_{xx} = \frac{e^2}{2\pi}\exp(-\Delta/xx)$ and probing them by transport measurement requires creating a tunneling setup, such as a point contact between two QSH edges [24–22].

The situation changes if one breaks TRS. Experimentally, this may easily be achieved by applying the external magnetic field [10–11]. The magnetic field couples the helical states, spin polarizes the edge in the plane of the sample, and opens a gap in the single-particle spectrum. As will be shown here, in the absence of the magnetic field the interactions in the helical liquid result in the tendency towards in-plane XY ferromagnetism manifested in the power-law decay of the spin correlations. Therefore, electron interactions will naturally facilitate the formation of the ferromagnetic state once the magnetic field is applied. This is revealed in the enhancement of the gap in the spectrum of collective excitations as compared to the noninteracting case. The gap leads to the insulating behavior of the edge at low temperatures. The suppression of the longitudinal conductance with the applied magnetic field was indeed observed experimentally in HgTe quantum wells [10–11].

The present paper is devoted to the theory of this ferromagnetic insulating state of the edge of a 2D topological insulator, induced by the magnetic field and enhanced by the interactions. The obtained scaling dependence of the transport gap $\Delta \propto H^{1/(2-K)}$ on the magnetic field $H$ allows one to directly extract the Luttinger liquid interaction parameter $K$ from the activation behavior $G_{xx} \sim \frac{e^2}{2\pi}\exp(-\Delta/T)$ of the longitudinal conductance. A device in the Hall-bar geometry with the applied magnetic field thus provides a minimal setup to probe the interactions and the predicted dependence could be tested.
already on the existing data \cite{10,11}.

Model and Hamiltonian. The effective low-energy Hamiltonian for the interacting edge electrons in the presence of the magnetic field \cite{11,18} may be written down in the helical basis of right-moving (with respect to the $x$ direction along the edge) spin-up ($\uparrow$) and left-moving spin-down ($\downarrow$) states as

\begin{equation}
\hat{H} = \hat{H}_0 + \hat{H}_m + \hat{H}_i, \quad \hat{H}_0 = \int dx \psi_\uparrow^\dagger(x) \sigma_z \psi(x),
\end{equation}

\begin{equation}
\hat{H}_m = -\Delta_0 \int dx \psi_\uparrow^\dagger(x) (\sigma_x \cos \varphi_0 + \sigma_y \sin \varphi_0) \psi(x),
\end{equation}

\begin{equation}
\hat{H}_i = \frac{1}{2} \int dx \, dx' \psi_\uparrow^\dagger(x) \psi_\uparrow^\dagger(x') V(x - x') \psi_\sigma(x) \psi_\sigma(x').
\end{equation}

Here, $\psi = (\psi_\uparrow, \psi_\downarrow)$ is the two-component fermionic field operator, $\hat{\rho} = -i \hbar \partial_x$, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices in the helical basis. The part $\hat{H}_m$ describes the effect of the external magnetic field. For the in-plane orientation, $\mathbf{H} = H (\cos \varphi_0, \sin \varphi_0, 0)$, only the Zeeman effect is present, whereas the orbital effect vanishes; the angle $\varphi_0$ correspond to the direction of the field in the $xy$ plane of the 2D sample and the gap is given by the Zeeman energy $\Delta_0 \parallel \approx \mu_B H$. In case of the perpendicular orientation of the field, $\mathbf{H} = (0, 0, H)$, the Zeeman effect does not affect the dynamics and only the orbital effect remains. The orbital effect of the perpendicular field is estimated \cite{11} to be stronger than the in-plane Zeeman effect, $\Delta_0 \perp \approx 10 \Delta_0 \parallel$: $\Delta_0 \parallel \approx 3 \text{K}$ and $\Delta_0 \perp \approx 30 \text{K}$ at $H = 1 \text{T}$. For arbitrary field-orientation, the single-particle gap $\Delta_0$ scales linearly with the magnetic field, $\Delta_0 \propto H$.

We consider the case of Coulomb interactions, $V(x) = e_x^2 / |x|$ in Eq. (3), possibly screened by the nearby metallic electrodes beyond some length $l_s$; the charge $e_x = e / \sqrt{\pi}$ incorporates the effects of screening by the dielectric environment. This allows us to consider both unscreened and screened interactions, the latter modeling practically any finite-range interactions. The short-scale spatial cutoff $\alpha$ of the theory [Eqs. (1), (2), and (3)] and of the potential $V(x)$ is set by the decay scale of the edge states into the bulk. For simplicity, it is assumed that the chemical potential is exactly at the branch crossing $\epsilon = 0$ of the unperturbed edge spectrum $\epsilon_p = \pm v_p$, where the correlation effects are strongest. This can be achieved by tuning the gate voltage to the minimum of the longitudinal conductance.

The Hamiltonian $\hat{H}$ [Eqs. (1), (2), and (3)] describes one-dimensional interacting Dirac fermions, which are massive in the presence of the magnetic field. For point interactions, this is known as the Thirring model \cite{28,29}. This model can be mapped to the sine-Gordon model by mean of the bosonization procedure \cite{28,29}. One relates the fermion fields $\psi_{\tau,\downarrow}(x)$ of the right and left movers to the bosonic ones $\phi_{\tau,\downarrow}(x)$ as

\begin{equation}
\psi_{\tau,\downarrow}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\pm i \phi_{\tau,\downarrow}(x)},
\end{equation}

where the Klein factors are omitted. The operators $\phi(x) = \frac{1}{2} [\phi_\uparrow(x) + \phi_\downarrow(x)]$ and $\theta(x) = \frac{1}{2} [\phi_\uparrow(x) - \phi_\downarrow(x)]$ satisfy the canonical (up to a coefficient) commutation relations $[\phi(x), \partial_x \phi(x')] = -i \pi \delta(x - x')$ and are related to the coordinate and momentum variables of the collective excitations. In terms of $\phi(x)$ and $\theta(x)$, the Hamiltonian $\hat{H}$ [Eqs. (1), (2), and (3)] can be expressed as

\begin{equation}
\hat{H}_0 = \frac{\hbar}{2\pi} \int dx \, v \left[ (\partial_x \phi)^2 + (\partial_x \phi')^2 \right],
\end{equation}

\begin{equation}
\hat{H}_m = -\frac{\Delta_0}{\pi \alpha} \int dx \, \cos[2 \phi(x) + \varphi_0],
\end{equation}

\begin{equation}
\hat{H}_i = \frac{1}{2\pi^2} \int dx \, dx' \partial_x \phi(x) V(x - x') \partial_x \phi(x').
\end{equation}

The sine-Gordon Hamiltonian, Eqs. (4), (7), and (8), describes the dynamics of the collective edge excitations of a 2D topological insulator in the presence of the magnetic field. Further, we proceed with the analysis of this Hamiltonian.

Collective spin-charge excitations. To visualize the collective excitations described by Eqs. (5), (6), and (7), let us link the fields $\phi(x)$ and $\theta(x)$ to the physical observables. From the relation (4), one obtains

\begin{equation}
\begin{pmatrix} s_x(x) \\ s_y(x) \end{pmatrix} = \frac{1}{2\pi \alpha} \begin{pmatrix} \cos(-2\phi(x)) \\ \sin(-2\phi(x)) \end{pmatrix}
\end{equation}

for the $x$ and $y$ components of the spin density operator $s(x) = \psi_\sigma^\dagger(x) \sigma_\sigma' \psi_{\sigma'}(x)$ (defined without 1/2 factor) and

\begin{equation}
\begin{pmatrix} s_z(x) \\ \rho(x) \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} \theta(x) \\ \phi(x) \end{pmatrix}
\end{equation}

for the $z$ component of the spin density and the particle density $\rho(x) = \psi_\sigma^\dagger(x) \psi_{\sigma}(x)$ operators. As seen from Eq. (5), the angle $-2\phi(x)$ corresponds to the direction of
the spin polarization in the \( xy \) plane and the field \( \varphi(x) \) is thus directly related to the spin degrees of freedom. At the same time, according to Eq. (9), the charge density is determined by the gradient of \( \varphi(x) \). Therefore, the collective excitations carry both charge and spin, which is a direct consequence of the coupling between the spin and orbital degrees of freedom in the single-particle states. A kink of height \( \pi \) in \( \varphi(x) \) rotates the spin polarization in the \( xy \) plane by \( 2\pi \) and simultaneously accumulates a unit charge in the region of variation of \( \varphi(x) \), Fig. 2. It was suggested in Ref. 18 to exploit this bonding of spin and charge degrees of freedom to observe charge fractionalization effects in domain-wall structures with inhomogeneous magnetization.

**Gapless helical liquid, \( H = 0 \).** Let us first consider the system in the absence of the magnetic field, \( H_m = 0 \), when the edge is in the helical LL phase, and obtain the excitation spectrum and basic correlations. The calculations can be conveniently performed in the Langrange finite-temperature formalism. From Eqs. (9) and (7), the action for the Fourier transformation \( \varphi(\omega_n, q) = \int_0^{kT} d\tau \int dx e^{i\omega_n\tau - iqx} \varphi(\tau, x) \) (\( \omega_n = 2\pi Tn, n \in \mathbb{Z} \)) of the phase field takes the form

\[
S_0[\varphi] + S_1[\varphi] = T \sum_{\omega_n} \frac{1}{2\pi} \left( \frac{1}{u_q \omega_n^2 + u_q q^2} \right) |\varphi(\omega_n, q)|^2 \frac{2\pi K_q}{2}.
\]

The momentum-dependent velocity \( u_q \) and LL interaction parameter \( K_q \) are given by

\[
u_q/v = 1/K_q = \sqrt{1 + V(q)/\pi hv} = \sqrt{r_s ln[1/(q_s \alpha_s)]},
\]

where \( V(q) = 2e^2 ln[1/(q_s \alpha_s)] \) is the Fourier transform of the potential \( V(x) \), \( r_s = 2e^2/\pi hv \) is the Coulomb parameter, \( q_s = \max(q, 1/l_s) \), and \( \alpha_s = \alpha e^{-1/r_s} \).

From Eqs. (10) and (11), one obtains the excitation spectrum \( \omega(q) = u_q |q| \) of the collective edge excitations of a 2D topological insulator. For unscreened Coulomb interactions \( V(q) = 2e^2 ln[1/(q_s \alpha_s)] \) at \( q_s \gtrsim 1 \), \( u_q \) and \( K_q \) depend logarithmically on \( q \) and the excitations have a 1D plasmon-type spectrum \( \omega(q) \propto q \sqrt{ln(1/q)} \). At spatial scales exceeding the screening length \( l_s \), \( q_s \lesssim 1 \), the interactions become effectively short-range with \( V(q) \) saturating to the value \( V(q) \lesssim 1/l_s \) = \( 2e^2 ln(l_s/\alpha_s) \). The velocity \( u_q = u \) and interaction parameter \( K_q = K \) become \( q \)-independent, \( \nu/v = 1/K = \sqrt{r_s ln(l_s/\alpha_s)} \), and the spectrum \( \omega(q) = u |q| \) linear. In the absence of the magnetic field the spectrum is gapless, but for unscreened Coulomb interactions the log-dependence of the velocity \( u_q \) signals of a strong tendency towards gap opening.

Let us now study the correlations. The operators that describe coupling between the counter-propagating helical modes are given by the “spin-flip” components \( s_{+}(x) = s_{-}(x) \pm i s_{y}(x) \) of the spin density \( s \),

\[
s_{+}(x) = \psi_{1}^{\dagger}(x) \psi_{-}(x) = \frac{e^{-2i\varphi(x)}}{2\pi \alpha}.
\]

The pairing tendency is thus directly related to the spin polarization in the \( xy \) plane of the 2D sample. Calculating the correlation function of \( s_{+}(x) \) with respect to the action \( (10) \) at zero temperature \( T = 0 \), we obtain

\[
\langle s_{+}(x) s_{-}(0) \rangle \propto \begin{cases} \exp \left[ -4 \sqrt{\ln \left( |x|/\alpha_s \right)/r_s \right], & |x| \lesssim l_s, \\ \left( l_s/|x| \right)^{2K}, & |x| \gtrsim l_s. \end{cases}
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\]

For screened Coulomb interactions at \( |x| \gtrsim l_s \) the correlation \( (13) \) of the in-plane spin density \( s_{x,y}(x) \) have a LL power-law decay. For unscreened Coulomb interactions at \( |x| \lesssim l_s \), the decay is slower than any power law. The interactions in the helical liquid thus result in the tendency towards in-plane \( XY \) ferromagnetic ordering. However, due to strong quantum fluctuations in a 1D system the long-range order is not formed, \( \langle s(x) \rangle = 0 \). For unscreened Coulomb interactions, the tendency towards ferromagnetism is as strong as that towards Wigner crystallization in a conventional one-dimensional electron system 24, 27. Note that numerical factors in the spectrum \( \omega(q) = u_q |q| \) [Eq. (11)] and correlation function \( (13) \) differ from those of Refs. 26, 27 because in our case electrons are single-flavored.

In the massless LL phase, the edge conductance \( G = e^2/\pi h \) is essentially unaffected by the interactions and the edge remains a perfect conducting channel 24, 25. Therefore, in the absence of perturbations that break TRS, the interactions do not reveal themselves in the transport measurement of the longitudinal conductance \( G_{xx} = 2G \) of the sample (factor 2 due to two edges).

**Gapped ferromagnetic phase, \( H > 0 \).** The situation changes, if the magnetic field is applied, \( \Delta_0 > 0 \) in \( H_m \) [Eq. (2)]. Even in the absence of interactions, the magnetic field couples the helical counter-propagating states \( \mathbb{1} \pm \mathbb{1} \) according to Eq. (2) and opens a gap \( \Delta_0 \) in the single-particle spectrum \( \epsilon_{p} = \pm \sqrt{(vp)^2 + \Delta_0^2} \) of the Hamiltonian \( \tilde{H} + H_m \), Fig. 3. In the ground state, the edge becomes spin polarized in the plane of the sample in the direction \( \varphi_0 \), \( \langle s(x) \rangle \propto (\cos \varphi_0, \sin \varphi_0, 0) \).
Opening of the single-particle gap \( \Delta_0 \) has a direct consequence on transport. For the noninteracting electrons, the edge conductance can be calculated using the Landauer formula and for long enough edge of length \( L \gg h\nu/\Delta_0 \) it is given by

\[
G(T) = \frac{e^2}{2\pi \hbar} \frac{2}{\exp(\Delta_0/T) + 1}.
\]

The presence of the gap makes the edge insulating at temperatures \( T \ll \Delta_0 \), where the conductance follows the Arrhenius activation law \( G(T) \approx \frac{e^2}{2\pi \hbar} \exp(-\Delta_0/T) \).

Let us now take the interactions into account. In terms of the collective excitations, the effect of the magnetic field is described by the cosine term \( \varphi(x) \) in the bosonized Hamiltonian. The fact that the ground state is spin polarized means that the phase field \( \varphi(x) \) is locked in the minimum of the cosine term, \( \varphi(x) = -\varphi_0/2 \). The collective excitations are now massive and for low energies described by the fluctuations of \( \varphi(x) \) around this minimum. Since even without the magnetic field the interactions tend to order the edge ferromagnetically, naturally, the gap \( \Delta \) in the spectrum of the collective excitations turns out to be enhanced compared to its bare single-particle value \( \Delta_0 \). For screened Coulomb interactions we obtain

\[
\Delta \sim \epsilon_0 \left( \frac{\Delta_0}{\epsilon_0} \right)^{\frac{1}{2\pi}} \propto H^{\frac{1}{2\pi}},
\]

up to a numerical factor \( \sim 1 \). Here \( \epsilon_0 \) is the bulk insulator gap, which determines the high energy cutoff of the edge spectrum and is assumed \( \epsilon_0 \gg \Delta_0 \). For HgTe quantum wells, it is estimated \( \epsilon_0 \sim 100K \). The result \((14)\) can be obtained by several means, e.g., using the self-consistent harmonic approximation [29].

The gap \((14)\) has a power-law dependence on the bare gap \( \Delta_0 \sim \mu \mu H \) and hence on the magnetic field \( H \). The exponent \((1/2 - K)\) of this dependence is controlled by the LL interaction parameter \( K \), which varies between \( K = 1 \) in the noninteracting case and \( K = 0 \) for infinitely strong finite-range interactions; these cases give the lowest \( \Delta_{\text{min}} = \Delta_0 \) and highest \( \Delta_{\text{max}} \sim \sqrt{\Delta_0 \epsilon_0} \times \sqrt{\hbar} \) possible values of the many-body gap \( \Delta \), respectively. Due to the long-range nature of the Coulomb forces, for unscreened interactions the gap appears to be close to \( \Delta_{\text{max}} \), even for moderate interaction strength \( r_s \sim 1 \). Performing the harmonic approximation \(29\), we obtain

\[
\Delta^2 \sim \Delta_0 \epsilon_0 \exp(-\sqrt{2 \ln(\epsilon_0/\Delta_0)}/r_s).
\]

The gap \((15)\) differs from the \( K = 0 \) limit \( \Delta_{\text{max}} \) of Eq. \((14)\) only by a function of \( \Delta_0/\epsilon_0 \) that varies slower than any power law. The result \((15)\) applies if the correlation length \( l_\Delta = h\nu/\Delta \) determined from Eq. \((15)\) does not exceed the screening length, \( l_\Delta \lesssim l_s \). Otherwise, what concerns the gap, the interactions are effectively screened and the gap is given by Eq. \((14)\). For unscreened Coulomb interactions, the enhancement of the gap could thus be quite substantial: for \( \Delta_0 \sim 1K \) and \( \epsilon_0 \sim 100K \) one gets \( \Delta_{\text{max}} \sim 10K \). The enhancement of the gap means, in particular, that interactions should favor observation of the effects predicted in Ref. [18].

**Experimental manifestation.** The main practical consequence of the interactions is that the many-body gap \( \Delta \) (Eq. \((13)\)) plays the role of the transport gap and enters the activation law \( G_{xx}(T) \approx \frac{e^2}{2\pi \hbar} \exp(-\Delta_{\text{max}}/T) \) for the longitudinal conductance at temperatures \( T \ll \Delta \). Therefore, the scaling dependence \((14)\) of the gap \( \Delta \) on the magnetic \( H \) should be directly accessible in the transport measurement allowing one to extract the Luttinger liquid interaction parameter \( K \). This provides a direct way to probe the interactions in the transport experiment and the predicted dependence could be tested already on the existing data [10, 11].

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