THE GENERALIZED SCHWARZ INEQUALITY FOR SEMI-HILBERTIAN SPACE OPERATORS AND SOME A-NUMERICAL RADIUS INEQUALITIES

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Abstract. In this work, the mixed Schwarz inequality for semi-Hilbertian space operators is proved. Namely, for every positive Hilbert space operator \( A \). If \( f \) and \( g \) are nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \ (t \geq 0) \), then

\[
\|Tx, y\|_A \leq \|f(T_x)\|_A \|g(T_y)\|_A
\]

for every Hilbert space operator \( T \) such that the range of \( T^*A \) is a subset in the range of \( A \), such that \( A \) commutes with \( T \), and for all vectors \( x, y \in \mathcal{H} \), where \( |T|_A = (AT^*A)^{1/2} \) such that \( T^*A = A^\dagger T^*A \), where \( A^\dagger \) is the Moore-Penrose inverse of \( A \). Based on that, some inequalities for the \( A \)-numerical radius are introduced.

1. Introduction

Let \( \mathcal{B}(\mathcal{H}) \) be the Banach algebra of all bounded linear operators defined on a complex Hilbert space \( \langle \mathcal{H}; \langle \cdot, \cdot \rangle \rangle \) with the identity operator \( 1_\mathcal{H} \) in \( \mathcal{B}(\mathcal{H}) \). For \( A \in \mathcal{B}(\mathcal{H}) \) we denote by \( R(A) \) and \( N(A) \) the range and the null space of \( A \), respectively. By \( \overline{R(A)} \) we denote the norm closure of \( R(A) \). Let \( T^* \) be the adjoint of \( T \). The cone of all positive (semidefinite) operators is given by

\[
\mathcal{B}^+ (\mathcal{H}) = \{ A \in \mathcal{B}(\mathcal{H}) : \langle Ax, x \rangle \geq 0, \forall x \in \mathcal{H} \}.
\]

Every \( A \in \mathcal{B}^+ (\mathcal{H}) \) defines the following positive semidefinite sesquilinear form:

\[
\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, (x, y) \mapsto \langle x, y \rangle_A = \langle Ax, y \rangle.
\]

The seminorm induced by this sesquilinear form is given by \( \|x\|_A = \sqrt{\langle x, x \rangle_A} \). It is well-known that \( \|x\|_A \) is a norm on \( \mathcal{H} \) if and only if \( A \) is injective and \( \langle \mathcal{H}, \| \cdot \|_A \rangle \) is complete if and only if \( R(A) \) is closed in \( \mathcal{H} \).

An operator \( S \in \mathcal{B}(\mathcal{H}) \) is said to be \( A \)-adjoint of an operator \( T \in \mathcal{B}(\mathcal{H}) \) if \( \langle Tx, y \rangle_A = \langle x, Sy \rangle_A \). In other words, \( S \) is an \( A \)-adjoint of \( T \) if and only if \( S \) is a solution of the equation \( AX = T^*A \) in \( \mathcal{B}(\mathcal{H}) \). For \( T \in \mathcal{B}(\mathcal{H}) \) the existence of an \( A \)-adjoint of \( T \) is not guaranteed. The set of all operators acting on \( \mathcal{H} \) that admit \( A \)-adjoints is denoted by \( \mathcal{B}_A(\mathcal{H}) \). The existence of such set of operators is guaranteed by Douglas theorem [11] that

\[
\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : R(T^*A) \subseteq R(A) \}.
\]

Moreover, if \( T \) is then the operator equation \( AX = T^*A \) has a unique solution, denoted by \( T^*A \), satisfying \( R(T^*A) \subseteq R(A) \) and \( N(T^*A) \subseteq N(T^*A) \). The distinguished \( A \)-adjoint operator of \( T \) or simply \( T^\dashv \) can be computed as \( T^\dashv A = A^\dagger T^*A^* \), and satisfy the equation \( AT^\dashv A = T^*A \), where \( A^\dagger \) is the Moore-Penrose inverse of \( A \) (see [3] and [4]).

Denotes \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \) the set of all operators \( T \in \mathcal{B}(\mathcal{H}) \) such that \( T \) is bounded induced by the semi-norm \( \|\cdot\|_A \). In other words,

\[
\mathcal{B}_{A^{1/2}}(\mathcal{H}) := \{ T \in \mathcal{B}(\mathcal{H}) : \|Tx\|_A \leq \lambda \|x\|_A, \text{ for some } \lambda > 0 \text{ and all } x \in \mathcal{H} \}.
\]

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Members of $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are called $A$-bounded operators. In fact, if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, then the $A$-operator seminorm is defined as:

$$\|T\|_A = \sup_{x \in \overline{\mathcal{R}(A)} \setminus \{0\}} \frac{\|Tx\|_A}{\|x\|_A} = \sup \{\|Tx\|_A : x \in \mathcal{H}, \|x\|_A = 1\}.$$ 

It is convenient to note that it may happen that $\|T\|_A = +\infty$ for some operator $T \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Also, we need to mention that the inclusions

$$\mathcal{B}_A(\mathcal{H}) \subset \mathcal{B}_{A^{1/2}}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$$

hold with equality if $A$ is injective and has closed range. But neither $\mathcal{B}_A(\mathcal{H})$ nor $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ is closed and dense in $\mathcal{B}(\mathcal{H})$.

In particular, we should note that if $T \in \mathcal{B}_A(\mathcal{H})$ then $T^{1/2} \in \mathcal{B}_A(\mathcal{H})$ and $(T^{1/2})^{1/2} = P_ATP_A$, where $P_A$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Moreover, we have

$$\|T^{1/2}\|_A = \|T\|_A.$$ 

An operator $T \in \mathcal{B}_A(\mathcal{H})$ is called $A$-selfadjoint if $AT$ is selfadjoint, i.e., $AT = T^*A$, or simply $\langle Tx, x \rangle_A \in \mathbb{R}$ for all $x \in \mathcal{H}$. Also, $T$ is called $A$-positive if $AT$ is positive $AT > 0$. Note that if $T$ is $A$-selfadjoint then $T \in \mathcal{B}_A(\mathcal{H})$. An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be $A$-normal if $TT^{1/2} = T^{1/2}T$. The fact that every selfadjoint operator is normal does not hold in this case; i.e., an $A$-selfadjoint operator is not necessarily $A$-normal (see [6, Example 5.1]). Indeed, this property holds if $T$ commutes with $A$. We note that, for any $T \in \mathcal{B}_A(\mathcal{H})$ the $A$-Cartesian decomposition is given by $T = \text{Re}_A(T) + i \text{Im}_A(T)$, where $\text{Re}_A(T) = \frac{T + T^*}{2}$ and $\text{Im}_A(T) = \frac{T - T^*}{2i}$. Moreover, $\text{Re}_A(T)$ and $\text{Im}_A(T)$ are $A$-selfadjoint operators.

In 2012, Saddi [28], introduced the definition of $A$-spectral radius as follows:

$$(1.1) \quad r_A(T) = \lim_{n \to \infty} \sup_n \|T^n\|_A^{1/n}.$$ 

But indeed, this formula was recently proved by Feki in [15], where he gave a counterexample showing that the definition of Saddi in [28] doesn’t guarantee that $r_A(T) < \infty$. The Feki definition of $A$-spectral radius reads:

$$(1.2) \quad r_A(T) = \inf_{n \in \mathbb{N}} \|T^n\|_A^{1/n}.$$ 

For $A$-bounded linear operator $T$ on a Hilbert space $\mathcal{H}$, the $A$-numerical range $W_A(T)$ is the image of the unit sphere of $\mathcal{H}$ under the positive semidefinite sesquilinear quadratic form $x \to \langle Tx, x \rangle_A$ associated with the operator. More precisely,

$$W_A(T) = \{\langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1\}.$$ 

Also, the $A$-numerical radius is defined to be

$$w_A(T) = \sup \{\|\lambda\| : \lambda \in W_A(T)\} = \sup_{\|x\|_A=1} |\langle Tx, x \rangle_A|.$$ 

For more about properties of $A$-numerical range and $A$-numerical radius, see [6]–[10], [13], [25], [27], and [32].

Recently, it was shown that the inequality [15] (see also [16])

$$r_A(T) \leq w_A(T) \leq \|T\|_A$$

for any $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Also, $\|\|_A$ and $w_A(T)$ are equivalent seminorm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ satisfying the inequality:

$$\frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A.$$ 

The first inequality becomes equality if $AT^2 = 0$ and the second inequality becomes equality if $T$ is $A$-normal (see [15]).

The Schwarz inequality for positive operators reads that if $A$ is a positive operator in $\mathcal{B}(\mathcal{H})$, then

$$(1.3) \quad |\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle$$
A setting of the Reid inequality. Recently by Feki in [20], Halmos presented his stronger version of the Reid inequality (1.4) by replacing $r(B)$ instead of $\|B\|$.

In 1952, Kato [21] introduced a companion inequality of (1.3), called the mixed Schwarz inequality, which asserts
\begin{equation}
\langle Ax, y \rangle^2 \leq \langle A^{2\alpha} x, x \rangle \langle A^{2(1-\alpha)} y, y \rangle, \quad 0 \leq \alpha \leq 1.
\end{equation}
for every operators $A \in B(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$, where $|A| = (A^*A)^{1/2}$.

In 1988, Kittaneh [23] proved a very interesting extension combining both the Halmos–Reid inequality (1.2) and the mixed Schwarz inequality (1.5). His result reads that
\begin{equation}
\langle ABx, y \rangle \leq r(B) \|f(\|A\|)x\|g(\|A^*\|)y\|
\end{equation}
for any vectors $x, y \in \mathcal{H}$, where $A, B \in B(\mathcal{H})$ such that $|A|B = B^*|A|$ and $f, g$ are nonnegative continuous functions defined on $[0, \infty)$ satisfying that $f(t)g(t) = t^2$ $(t \geq 0)$. Clearly, choose $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ with $B = 1_{\mathcal{H}}$, we refer to (1.5). Moreover, choosing $\alpha = \frac{1}{2}$ some manipulations refer to the Halmos version of the Reid inequality.

In this work, the corresponding version of the well-known Kittaneh inequality (1.6), which is also known as the mixed Schwarz inequality for the semi-Hilbertian space operators is introduced. Based on that, some inequalities for the $A$-numerical radius are proved. A generalization of the Euclidean operator $A$-radius with some basic properties are discussed and elaborated. A generalization of an important inequality proved recently by Feki in [14] for the generalized Euclidean operator $A$-radius is also considered.

2. Preliminaries and Lemmas

The corresponding version of Schwarz inequality for $A$-positive operators reads that if $T$ is $A$-positive operator in $B_A(\mathcal{H})$, then
\begin{equation}
\langle Tx, y \rangle_A^2 \leq \langle T x, x \rangle_A \langle T y, y \rangle_A
\end{equation}
for any vectors $x, y \in \mathcal{H}$. The proof of this result can be done using the same argument of the proof of the classical Schwarz inequality for positive operators taking into account that we use the semi-inner product induced by $A \in B^+(\mathcal{H})$.

In order to introduce the corresponding version of the mixed Schwarz inequality ($\langle .. \rangle$) for semi-Hilbertian space operators we need the following sequence of lemmas; which have been pointed out in ... but for general Hilbert space operators. We rewrite these lemmas in appropriate way for semi-Hilbertian space operators.

**Lemma 1.** Let $A \in B^+(\mathcal{H})$. Let $T, S$ and $R$ be operators in $B_A(\mathcal{H})$, where $T$ and $S$ are $A$-positive. Then
\[
\begin{bmatrix} T & R^2 A \\ R & S \end{bmatrix}
\]
is $A$-positive operator in $B_A(\mathcal{H} \oplus \mathcal{H})$ if and only if $\langle Rx, y \rangle_A^2 \leq \langle T x, x \rangle_A \langle S y, y \rangle_A$ for all $x, y \in \mathcal{H}$, where $A = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \in B^+(\mathcal{H} \oplus \mathcal{H})$.

**Proof.** The proof is straightforward by replacing the inner product $\langle .. \rangle$ by the semi-inner product $\langle .. \rangle_A$ and setting $A = T, B = S, C = R$ and $C^* = R^2 A$, in [23, Lemma 1].

**Lemma 2.** Let $A \in B^+(\mathcal{H})$. Let $T, S$ and $R$ be operators in $B_A(\mathcal{H})$, where $T$ and $S$ are $A$-positive and $SR = RT$. If
\[
\begin{bmatrix} T & R^2 A \\ R & S \end{bmatrix}
\]
is $A$-positive operator in $B_A(\mathcal{H} \oplus \mathcal{H})$, then
\[
\begin{bmatrix} f^2(T) & R^2 A \\ R & g^2(S) \end{bmatrix}
\]
is also $A$-positive, where where $f$ and $g$ are non-negative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, and $A = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \in B^+(\mathcal{H} \oplus \mathcal{H})$.\hfill\blacksquare
Proof. The proof is straightforward by replacing the inner product $\langle \cdot, \cdot \rangle$ by the semi-inner product $\langle \cdot, \cdot \rangle_A$ and setting $A = T$, $B = S$, $C = R$ and $C^* = R^{\sharp A}$, in [23, Lemma 2]. □

It is well known that for every selfadjoint operator $T \in \mathcal{B}(\mathcal{H})$ the inequality

$$|(Tx,x)| \leq ||T||_A^2 \langle x,x \rangle$$

holds for every vector $x \in \mathcal{H}$, where $|T| = (T^*T)^{1/2}$. To find out what are the appropriate conditions required for generalizing the previous inequality and in lighting of what was discussed previously, neither selfadjoint operators nor $A$-selfadjoint operator will be useful in this case. In fact, we need a new type of selfadjoint operators that covers the equality $T = T^\sharp A$; which holds if and only if $T$ is $A$-selfadjoint and $\mathcal{R}(T) \subseteq \mathcal{R}(A)$. From now on, we call this property a $\sharp_A$-selfadjointness property. In general, for $T \in \mathcal{B}_A(\mathcal{H})$, neither $T^{\sharp_A}T$ nor $TT^{\sharp_A}$ is positive. However, these operators are $A$-selfadjoint and $A$-positive. Thus, in viewing of these facts, we are able to define the $A$-absolute value operator of $T$, such as $|T|_A^2 = AT^{\sharp_A}T$, which is positive operator, and we write $|T|_A = (AT^{\sharp_A}T)^{1/2}$. This property is called the uniqueness of the square root of $A$-positive operators. We note that, the $A$-absolute value operator is selfadjoint if $T$ is $A$-selfadjoint and $A$ commutes with $T$. Moreover, we have

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2.$$

The following lemma plays a main role in the presentation of the mixed Schwarz inequality for semi-Hilbertian space operators.

**Lemma 3.** Let $A \in \mathcal{B}^+(\mathcal{H})$. If $T \in \mathcal{B}_A(\mathcal{H})$ is $A$-positive such that $AT = TA$, then $\begin{bmatrix} |T|_A & T^{\sharp_A} \\ T & |T^{\sharp_A}|_A \end{bmatrix}$ is positive operator in $\mathcal{B}_A(\mathcal{H} \oplus \mathcal{H})$, where $A = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \in \mathcal{B}^+(\mathcal{H} \oplus \mathcal{H})$.

**Proof.** Let $F = \begin{bmatrix} 0 & T^{\sharp_A} \\ T & 0 \end{bmatrix} \in \mathcal{B}_A(\mathcal{H} \oplus \mathcal{H})$. Since $T$ is $A$-positive and $AT^{\sharp_A} = T^{\sharp_A}A$, therefore it is easy to see that $F$ is $A$-positive, and $F^{\sharp_A} = \begin{bmatrix} 0 & T^{\sharp_A} \\ T^{\sharp_A} & 0 \end{bmatrix}$. Moreover, we have $AF^{\sharp_A} = F^{\sharp_A}A$, and

$$AF^{\sharp_A}F = \begin{bmatrix} AT^{\sharp_A}T & 0 \\ 0 & A(T^{\sharp_A})^{\sharp_A}T^{\sharp_A} \end{bmatrix} = \begin{bmatrix} AT^{\sharp_A}T & 0 \\ 0 & (T^{\sharp_A})^{\sharp_A}AT^{\sharp_A} \end{bmatrix} = \begin{bmatrix} AT^{\sharp_A}T & 0 \\ 0 & (T^{\sharp_A})^{\sharp_A}T^{\sharp_A}A \end{bmatrix} = \begin{bmatrix} |T|_A^2 & 0 \\ 0 & (T^{\sharp_A})|_A^2 \end{bmatrix} = |F|_A^2 \geq 0,$$

Therefore, the uniqueness of the square root of $A$-positive operators, implies that

$$|F|_A = (AF^{\sharp_A}F)^{1/2} = \begin{bmatrix} |T|_A & 0 \\ 0 & |T^{\sharp_A}|_A \end{bmatrix} = \begin{bmatrix} AT^{\sharp_A}T & 0 \\ 0 & (T^{\sharp_A})^{\sharp_A}AT^{\sharp_A} \end{bmatrix}.$$

Hence $F + |F|_A$ is $A$-positive; i.e., $\begin{bmatrix} |T|_A & T^{\sharp_A} \\ T & |T^{\sharp_A}|_A \end{bmatrix}$ is $A$-positive operator in $\mathcal{B}_A(\mathcal{H} \oplus \mathcal{H})$. □

Now, we are ready to state the corresponding new version of the mixed Schwarz inequality for semi-Hilbertian space operators.

**Theorem 1.** Let $A \in \mathcal{B}^+(\mathcal{H})$ be any positive operator. If $T \in \mathcal{B}_A(\mathcal{H})$ such that $AT = TA$. If $f$ and $g$ are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then

$$|(Tx,y)_A| \leq \|f(|T|_A x)\|_A \|g(|T^{\sharp_A}|_A y)\|_A,$$

for all vectors $x, y \in \mathcal{H}$. (2.2)
Proof. Since $T^2_A = A^1 T^* A$, then we observe that
\[
|T^2_A|^2_A T = (T^2_A)^{\frac{1}{2}} A T (A^1 T^* A)^{\frac{1}{2}} A^1 T^* A A^1 T^* A AT
\]
\[
= A(T^* A)^{\frac{1}{2}} A^1 T^* A AT
\]
\[
= AA^1 TAA^1 T^* AAT
\]
\[
= TT^* AT
\]
\[
= TAT^2 A
\]
\[
= T |T|^2_A.
\]
it follows that $T |T|^2_A = |T^2_A|^2_A T$. Thus, by Lemmas 2 and 3, we have
\[
\left[ \begin{array}{c}
\frac{f^2 (|T|^2_A)}{T} \\
\frac{T^2_A}{g^2 (|T^2_A|^2_A)}
\end{array} \right]
\]
is a positive operator in $B_A (\mathcal{H} \oplus \mathcal{H})$. The required inequality now follows from Lemma 1. 

Remark 1. Under the assumptions of Theorem 1. Choosing $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, we get
\[
|\langle Tx, y \rangle_A|^2 \leq \langle |T|^2_A x, x \rangle_A \langle |T^2_A|^2 A y, y \rangle_A, \quad 0 \leq \alpha \leq 1
\]
for all vectors $x, y \in \mathcal{H}$. Setting $\alpha = \frac{1}{2}$, we get
\[
|\langle Tx, y \rangle_A|^2 \leq \langle |T|^2_A x, x \rangle_A \langle |T^2_A|^2 A y, y \rangle_A
\]
for all vectors $x, y \in \mathcal{H}$.

Theorem 2. Let $A \in B^\pm (\mathcal{H})$ be any positive operator. If $T \in B_{A^{1/2}} (\mathcal{H})$ such that $A$ commutes with $T$ and $|T|^2 S = S|T|^2 S |T|$. If $f$ and $g$ are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ ($t \geq 0$), then
\[
|\langle Tx, y \rangle_A| \leq r_A (T) \|f (|T|^2_A x)\|_A \|g (|T^2_A|^2_A y)\|_A.
\]
for all vectors $x, y \in \mathcal{H}$.

Proof. The proof goes likewise the proof [23, Theorem 5] by rewriting the proof for the sem-inner product $\langle \cdot, \cdot \rangle_A$, taking into account Theorem 1.

Lemmas 4. Let $f$ be a non-negative convex function defined on a real interval $I$. Then for every positive operator $T \in B_A (\mathcal{H})$ whose $sp_A (T) \subseteq I$, we have
\[
f (\langle Tx, x \rangle_A) \leq f (\langle T x, x \rangle_A)
\]
for all vectors $x \in \mathcal{H}$. If $f$ is concave then the inequality is reversed.

Proof. Since $f$ is convex then for any $x, t \in I$ there is a $\lambda \in \mathbb{R}$ such that
\[
f (t) \geq f (s) + \lambda (t - s)
\]
Since $T$ is positive, then $T$ is selfadjoint operator. Using the functional calculus for sesquilinear form, thus we have
\[
f (T) \geq f (s) + \lambda (T - s)
\]
which is equivalent to write
\[
\langle f (T) x, x \rangle_A \geq f (s) 1_\mathcal{H} + \lambda \langle (T - s 1_\mathcal{H}) x, x \rangle_A
\]
for all vectors $x \in \mathcal{H}$. Setting $s = \langle Tx, x \rangle_A$, we have
\[
\langle f (T) x, x \rangle_A \geq f (\langle Tx, x \rangle_A) 1_\mathcal{H} + \lambda \langle (T - \langle Tx, x \rangle_A) x, x \rangle_A
\]
\[
= f (\langle Tx, x \rangle_A) + \lambda [(Tx, x) - \langle Tx, x \rangle_A]
\]
\[
= f (\langle Tx, x \rangle_A)
\]
for all vectors $x \in \mathcal{H}$, and this proves the required result.

The following version of Hölder–McCarty inequality holds for semi-Hilbertian operators.
Corollary 1. Let $T \in \mathcal{B}_A(H)$, such that $T$ is positive and $x \in \mathcal{H}$ be an $A$-unit vector. Then,
\begin{equation}
\langle Tx, x \rangle_A^r \leq \langle T^r x, x \rangle_A, \quad r \geq 1
\end{equation}
and
\begin{equation}
\langle T^r x, x \rangle_A \leq \langle Tx, x \rangle_A^r, \quad 0 \leq r \leq 1
\end{equation}

Proof. Let $f(t) = t^r$ $(r \geq 1)$ in Lemma 4. For the second inequality (2.6), apply the reversed version of (2.5) for the function $f(t) = t^r$ $(0 \leq r \leq 1)$.

By noting that, for $A$-positive operator $T$ we have
\begin{equation}
\langle Tx, x \rangle_A = \langle AT x, x \rangle^r \leq (\langle AT \rangle^r x, x) = \langle AT(\langle AT \rangle^{-1} x, x) = \langle T(\langle AT \rangle^{-1} x, x \rangle_A, \quad \forall r \geq 1
\end{equation}
which implies that the inequality
\begin{equation}
\langle Tx, x \rangle_A^r \leq \langle T(\langle AT \rangle^{-1} x, x \rangle_A, \quad r \geq 1
\end{equation}
holds if and only if $AT$ is positive, i.e., $T$ is $A$-positive; which indeed, the corresponding version of H"older–McCarty inequality for $A$-positive operators act on semi-Hilbertian spaces. Similarly, we have
\begin{equation}
\langle Tx, x \rangle_A^r \geq \langle T(\langle AT \rangle^{-1} x, x \rangle_A, \quad 0 \leq r \leq 1
\end{equation}
hold for $A$-positive operator $T$.

Corollary 2. Let $T \in \mathcal{B}_A(H)$, such that $T$ is $A$-selfadjoint and $x \in \mathcal{H}$ be an $A$-unit vector. Then,
\begin{equation}
\|\langle Tx, x \rangle_A^r \| \leq \langle |T| A x, x \rangle_A
\end{equation}

Proof. Since $T = T^\sharp$, by letting $y = x$ in (2.2) with $S = I$ we have the inequality (2.8).

In fact, one may establish a generalization of Theorem 1 to several operators, by letting $T_i, \in \mathcal{B}_A(H)$ $(i = 1, \cdots, n)$ such that
\begin{equation}
AT_i = T_i A \quad \text{and}
\end{equation}
If $f, g$ are as above, proceeding as in the proof of Theorem 1, then we have
\begin{equation}
\left\| \sum_{i=1}^{n} T_i \right\|_A \leq \left( \sum_{i=1}^{n} \| f(|T_i|_A) x \|_A \right) \left( \sum_{i=1}^{n} \| g(|T_i|_A^p) x \|_A \right)^{1/q}
\end{equation}

For all $x, u \in \mathcal{H}$, which follows by the Hölder–McCarty inequality, where $p, q$ are conjugate exponents, i.e., $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Thus, one may has the following norm inequality
\begin{equation}
\left\| \sum_{i=1}^{n} T_i \right\|_A \leq \left( \sum_{i=1}^{n} \| f(|T_i|_A) \|_A \right)^{1/p} \left( \sum_{i=1}^{n} \| g(|T_i|_A^p) \|_A \right)^{1/q}
\end{equation}

For instance, consider $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, one has from (2.9) that
\begin{equation}
\left\| \sum_{i=1}^{n} T_i \right\|_A \leq \left( \sum_{i=1}^{n} \| T_i \|_A^p \right)^{1/p} \left( \sum_{i=1}^{n} \| T_i \|_A^{1-\alpha} \right)^{1/q}
\end{equation}

Remark 2. In particular case for $n = 1$ (setting $T_1 = S$), then we have
\begin{equation}
\| S \|_A \leq \| S \|_A^\alpha \| S^{1-\alpha} \|_A, \quad 0 \leq \alpha \leq 1.
\end{equation}

Also, for $n = 2$ and $p = q = 2$ we get
\begin{equation}
\| T_1 + T_2 \|_A \leq \left( \| T_1 \|_A^2 + \| T_2 \|_A^2 \right)^{1/2} \left( \| T_1 \|_A^{1-\alpha} \|^2 + \| T_2 \|_A^{1-\alpha} \|^2 \right)^{1/2}
\end{equation}
for all $\alpha \in [0, 1]$.
The next result provides a new extension of the mixed Schwarz inequality (2.2) using A-Cartesian decomposition.

**Theorem 3.** Let $T \in \mathcal{B}(\mathcal{H})$ such that $AT = TA$, with the A-Cartesian decomposition $T = P + iQ$. If $f$ and $g$ are as in Theorem 1, then

$$
(2.11) \quad |\langle Tx, y \rangle_A| \leq \|f(|P|_A) x\|_A \|g(|P|^2 A) y\|_A + \|f(|Q|_A) x\|_A \|g(|Q|^2 A) y\|_A
$$

for all $x, y \in \mathcal{H}$.

**Proof.** Let $P + iQ$ be the A-Cartesian decomposition of $T$. Then

$$
|\langle Tx, y \rangle_A| = \left( |P^2 x, y\rangle_A + \langle Q^2 x, y\rangle_A \right)^{1/2}
$$

$$
\leq |P^2 x, y\rangle_A| + |Q^2 x, y\rangle_A|
$$

$$
\leq \|f(|P|_A) x\|_A \|g(|P|^2 A) y\|_A + \|f(|Q|_A) x\|_A \|g(|Q|^2 A) y\|_A
$$

for all $x, y \in \mathcal{H}$, where the last inequality follows form (2.2).

**Remark 3.** The above version of the mixed Schwarz inequality is a generalization of the main result in [2].

**Corollary 3.** Let $T \in \mathcal{B}(\mathcal{H})$ such that $AT = TA$, with the A-Cartesian decomposition $T = P + iQ$. Then

$$
(2.12) \quad |\langle Tx, y \rangle_A| \leq \left\{ \left| \langle P^2 x, y \rangle_A \|P^2 A\|^{2(1-\alpha)} \right|^2 + \left| \langle Q^2 x, y \rangle_A \|Q^2 A\|^{2(1-\alpha)} \right|^2 \right\}
$$

for all $x, y \in \mathcal{H}$.

**Proof.** Setting $f(t) = t^\alpha$ and $g(t) = t^1 - 0 \leq \alpha \leq 1, t \geq 0$ in Theorem 2 we get (2.10).

The A-Cartesian companion decomposition of the mixed Schwarz inequality (2.3) can be deduced as follows:

**Corollary 4.** Let $T \in \mathcal{B}(\mathcal{H})$ such that $AT = TA$, with the A-Cartesian decomposition $T = P + iQ$. Then

$$
(2.13) \quad |\langle Tx, y \rangle_A| \leq \frac{1}{2} \left( \left| \langle P^2 x, y \rangle_A \|P^2 A\|^{2(1-\alpha)} \right|^2 + \left| \langle Q^2 x, y \rangle_A \|Q^2 A\|^{2(1-\alpha)} \right|^2 \right)\frac{1}{2} \left( \left| \langle P^2 x, y \rangle_A \|P^2 A\|^{2(1-\alpha)} \right|^2 + \left| \langle Q^2 x, y \rangle_A \|Q^2 A\|^{2(1-\alpha)} \right|^2 \right)
$$

for all $x, y \in \mathcal{H}$ and any $0 \leq \alpha \leq 1$.

**Proof.** From (2.12) we have

$$
|\langle Tx, y \rangle_A| \leq \frac{1}{2} \left( \left| \langle P^2 x, y \rangle_A \|P^2 A\|^{2(1-\alpha)} \right|^2 + \left| \langle Q^2 x, y \rangle_A \|Q^2 A\|^{2(1-\alpha)} \right|^2 \right)\frac{1}{2} \left( \left| \langle P^2 x, y \rangle_A \|P^2 A\|^{2(1-\alpha)} \right|^2 + \left| \langle Q^2 x, y \rangle_A \|Q^2 A\|^{2(1-\alpha)} \right|^2 \right)
$$

which gives the required result.

### 3. A-numerical radius inequalities

In this section some inequalities for the A-numerical radius are presented, indeed the next two results generalizes the first two results in [17].

**Theorem 4.** Let $T \in \mathcal{B}_A(\mathcal{H})$, such that $AT = TA, 0 \leq \alpha \leq 1$ and $r \geq 1$. Then

$$
(3.1) \quad w_A^r(T) \leq \frac{1}{2} \left\{ \|T^{2r} A \|^{2(1-\alpha)} + \|T^{2r} A\|^{2(1-\alpha)} \right\}
$$
Proof. Let \( x \in \mathcal{H} \) be an A-unit vector, then

\[
|\langle Tx, x \rangle_A| \leq \left( \frac{\left\langle T^{2\alpha} A x, x \right\rangle_A}{\alpha} + \left\langle T^{4\alpha} A x, x \right\rangle_A^{2-\alpha} \right)^{1/2} \quad \text{(by (2.3))}
\]

\[
\leq \left( \frac{\left\langle T^{2\alpha} A x, x \right\rangle_A + \left\langle T^{4\alpha} A x, x \right\rangle_A^{2-\alpha} \right)^{1/2} \quad \text{by Power mean inequality)
\]

Therefore,

\[
|\langle Tx, x \rangle_A| \leq \frac{1}{2} \left( \left\langle |T|^{2\alpha} A + |T^{4\alpha} A|^{2-\alpha} \right\rangle A \right).
\]

Taking the supremum over all A-unit vector \( x \in \mathcal{H} \) we get the required result. \( \square \)

**Theorem 5.** Let \( T \in \mathcal{B}_A (\mathcal{H}) \), such that \( AT = TA, 0 \leq \alpha \leq 1 \) and \( r \geq 1 \). Then

\[
(3.2) \quad w^A_{2r} (T) \leq \left\| \alpha |T|^{2r} A + (1 - \alpha) |T^{4\alpha} A|^{2r} A \right\|
\]

Proof. Let \( x \in \mathcal{H} \) be an A-unit vector, then

\[
|\langle Tx, x \rangle_A|^2 \leq \left\langle T^{2\alpha} A x, x \right\rangle_A \left\langle T^{4\alpha} A x, x \right\rangle_A^{2-\alpha} \quad \text{by (2.3)}
\]

\[
\leq \left\langle T^{2\alpha} A x, x \right\rangle_A \left\langle T^{4\alpha} A x, x \right\rangle_A^{2-\alpha} \quad \text{by (2.7)}
\]

\[
\leq (\alpha \left\langle T^{2\alpha} A x, x \right\rangle_A + (1 - \alpha) \left\langle T^{4\alpha} A x, x \right\rangle_A^{2-\alpha} \right)^{1/r} \quad \text{by AM-GM inequality)
\]

\[
\leq (\alpha \left\langle T^{2\alpha} A x, x \right\rangle_A + (1 - \alpha) \left\langle T^{4\alpha} A x, x \right\rangle_A^{2-\alpha} \right)^{1/r} \quad \text{by (2.6)}
\]

Therefore,

\[
|\langle Tx, x \rangle_A|^{2r} \leq \left( \left\langle \alpha |T|^{2r} A + (1 - \alpha) |T^{4\alpha} A|^{2r} A \right\rangle A \right)^{1/r}.
\]

Taking the supremum over all A-unit vector \( x \in \mathcal{H} \) we get the required result. \( \square \)

**Theorem 6.** Let \( T \in \mathcal{B}_A (\mathcal{H}) \), such that \( AT = TA \), with the A-Cartesian decomposition \( T = P + iQ \). If \( f \) and \( g \) are as in Theorem 1. Then

\[
(3.3) \quad w_A (T) \leq \left\| f^p (|P|_A) + f^p (|Q|_A) \right\|^{1/p} \left\| g^q (|P^{4\alpha} A|_A) + g^q (|Q^{4\alpha} A|_A) \right\|^{1/q}
\]

\[
\leq \frac{1}{p} \left\| f^p (|P|_A) + f^p (|Q|_A) \right\| + \frac{1}{q} \left[ g^q (|P^{4\alpha} A|_A) + g^q (|Q^{4\alpha} A|_A) \right]
\]

for all \( p, q \geq 2 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. Letting $y = x$ in (2.11), then we have
\[
\langle Tx, y \rangle_A \leq \left\{ \| f (P^\alpha_A) x \|_A \| g (Q^\alpha_A) y \|_A + \| f (Q^\alpha_A) x \|_A \| g (P^\alpha_A) y \|_A \right\}
\]
\[
\leq \left( \| f (P^\alpha_A) x \|_A^p \| g (P^\alpha_A) y \|_A^q + \| f (Q^\alpha_A) x \|_A^p \| g (Q^\alpha_A) y \|_A^q \right)^{1/p} \text{ (by Hölder inequality)}
\]
\[
= \left( \langle f^2 (P^\alpha_A) x, x \rangle_A^{p/2} + \langle f^2 (Q^\alpha_A) x, x \rangle_A^{p/2} \right)^{1/p}
\times \left( \langle g^2 (P^\alpha_A) x, x \rangle_A^{q/2} + \langle g^2 (Q^\alpha_A) x, x \rangle_A^{q/2} \right)^{1/q}
\]
\[
\leq \left( \langle f^p (P^\alpha_A) x, x \rangle_A + \langle f^p (Q^\alpha_A) x, x \rangle_A \right)^{1/p}
\times \left( \langle g^q (P^\alpha_A) x, x \rangle_A + \langle g^q (Q^\alpha_A) x, x \rangle_A \right)^{1/q} \text{ (by (2.6))}
\]
\[
\leq \frac{1}{p} \left( \langle f^p (P^\alpha_A) x, x \rangle_A + \langle f^p (Q^\alpha_A) x, x \rangle_A \right)^{1/p} \langle g^q (P^\alpha_A) x, x \rangle_A^{1/q}
\]
\[
\leq \frac{1}{p} \left( \langle f^p (P^\alpha_A) x, x \rangle_A + \langle f^p (Q^\alpha_A) x, x \rangle_A \right)^{1/p} \frac{1}{q} \langle g^q (P^\alpha_A) x, x \rangle_A^{1/q}
\]
for all $p, q \geq 2$ with $\frac{1}{p} + \frac{1}{q} = 1$. Taking the supremum over all $A$-unit vector $x \in \mathcal{H}$ we get the desired result. 

\[\square\]

Corollary 5. Let $T \in \mathcal{B}_A (\mathcal{H})$, such that $AT = TA$, with the $A$-Cartesian decomposition $T = P + iQ$. If $f$ and $g$ are as in Theorem 1. Then
\[
w_A (T) \leq \sqrt{\| P^{2\alpha_A} + Q^{2\alpha_A} \| \sqrt{\| P^{2(1-\alpha)}_A + Q^{2(1-\alpha)}_A \|}}
\]\[
\leq \frac{1}{2} \left( \| P^{2\alpha_A} + Q^{2\alpha_A} \| + \| P^{2(1-\alpha)}_A + Q^{2(1-\alpha)}_A \| \right)
\]
for $0 \leq \alpha \leq 1$.

Proof. Take $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ $(0 \leq \alpha \leq 1)$, and setting $p = q = 2$ in Theorem 6. 

\[\square\]

Theorem 7. Let $T_i, i \in \mathcal{B}_A (\mathcal{H}) (i = 1, \cdots, n)$ such that $AT_i = T_i A$. Then,
\[
w_A^p \left( \sum_{i=1}^n T_i \right) \leq \frac{1}{2np-1} \left( \| T_i^{2\alpha_A} + T_i^{2p(1-\alpha)}_A \|_A \right)
\]
for all $p \geq 1$ and all $0 \leq \alpha \leq 1$.

Proof. Setting $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$ $(0 \leq \alpha \leq 1)$ in the first inequality (2.9), we have
\[
\left\langle \left( \sum_{i=1}^n T_i \right) x, x \right\rangle_A \leq \sum_{i=1}^n \left\langle T_i^{2\alpha_A} x, x \right\rangle_A \left\langle T_i^{2(1-\alpha)}_A x, x \right\rangle_A
\]
\[
\leq \sum_{i=1}^n \frac{1}{2} \left\langle T_i^{2\alpha_A} x, x \right\rangle_A^{1/2} \left\langle T_i^{2(1-\alpha)}_A x, x \right\rangle_A^{1/2}
\]

It follows that
\[
\left| \left\langle \sum_{i=1}^{n} T_i \right| x, x \right\rangle_A \right|^p \leq \left( \sum_{i=1}^{n} \left| \left\langle T_i \right|^{2\alpha} x, x \right\rangle_A \right)^\frac{p}{2} \left( \left\langle \left| T_i \right|^{2(1-\alpha)} x, x \right\rangle_A \right)^\frac{p}{2} \leq \frac{1}{n^{p-1}} \sum_{i=1}^{n} \left| \left\langle T_i \right|^{2\alpha} x, x \right\rangle_A \leq \frac{1}{n^{p-1}} \sum_{i=1}^{n} \left( \left| T_i \right|^{2\alpha} x, x \right\rangle_A + \left\langle T_i \right|^{2(1-\alpha)} x, x \right\rangle_A \leq \frac{1}{n^{p-1}} \sum_{i=1}^{n} \left( \left| T_i \right|^{2\alpha} + \left| T_i \right|^{2(1-\alpha)} \right) x, x \rangle_A \leq \frac{1}{n^{p-1}} \left( \sum_{i=1}^{n} \left| T_i \right|^{2\alpha} + \left| T_i \right|^{2(1-\alpha)} \right) x, x \rangle_A \leq \frac{1}{n^{p-1}} \left( \sum_{i=1}^{n} \left| T_i \right|^{2\alpha} + \left| T_i \right|^{2(1-\alpha)} \right) x, x \rangle_A \] (by AM-GM inequality)
for all \( p \geq 1 \), which proves the required result.

\[ \Box \]

**Corollary 6.** Let \( T_1, T_2 \in B_A (\mathcal{H}) \), such that \( AT_i = T_i A \) \((i = 1, 2)\). Then,
\[(3.5) \quad w_n \left( (T_1 + T_2) \right) \leq \frac{1}{2^p} \left\| T_1^{2\alpha} + T_1^{2(1-\alpha)} + T_2^{2\alpha} + T_2^{2(1-\alpha)} \right\|_A \]
for \( 0 \leq \alpha \leq 1 \).

**Proof.** Setting \( n = 2 \) in Theorem 7. \[ \Box \]

**Theorem 8.** Let \( B, T, C, E, S, F \in B_A (\mathcal{H}) \) such that \( A \) commutes with both \( T \) and \( S \). Then,
\[(3.6) \quad w_A (CTD + ESF) \leq \frac{1}{2} \left\| D^{\alpha} |T|^{2\alpha} D + C |T|^{2(1-\alpha)} C^{\alpha} + F |T|^{2\alpha} + E |T|^{2(1-\alpha)} E^{\alpha} \right\|_A \]
for \( 0 \leq \alpha \leq 1 \).

**Proof.** Employing the triangle inequality, the mixed Schwarz inequality (2.3), and then the AM-GM inequality, it follows that
\[
\left| \left\langle (CTD + ESF) x, x \right\rangle_A \right| \leq \left| \langle CT Dx, x \rangle_A \right| + \left| \langle ESFx, x \rangle_A \right| = \left| \langle TDx, C^{\alpha} x \rangle_A \right| + \left| \langle SFx, E^{\alpha} x \rangle_A \right| \leq \left\langle |T|^{2\alpha} Dx, Dx \right\rangle_A \frac{1}{2} \left( \left| T \right|^{2(1-\alpha)} C^{\alpha} x, C^{\alpha} x \right\rangle_A \frac{1}{2} \right. \\
+ \left| T \right|^{2(1-\alpha)} C^{\alpha} x, C^{\alpha} x \right\rangle_A \frac{1}{2} \\
\leq \frac{1}{2} \left[ \left| T \right|^{2\alpha} Dx, Dx \right\rangle_A + \left| T \right|^{2(1-\alpha)} C^{\alpha} x, C^{\alpha} x \right\rangle_A \\
+ \left| S \right|^{2\alpha} Fx, Fx \right\rangle_A + \left| S \right|^{2(1-\alpha)} E^{\alpha} x, E^{\alpha} x \right\rangle_A \left] \frac{1}{2} \right. \\
= \frac{1}{2} \left[ \left| D T \right|^{2\alpha} Dx, Dx \right\rangle_A + \left| C T \right|^{2(1-\alpha)} C^{\alpha} x, C^{\alpha} x \right\rangle_A \\
+ \left| F \right|^{2\alpha} |S|^{2\alpha} Fx, Fx \right\rangle_A + \left| F \right|^{2(1-\alpha)} E^{\alpha} x, E^{\alpha} x \right\rangle_A \left] \frac{1}{2} \right. \\
= \frac{1}{2} \left( \left| D T \right|^{2\alpha} D + C |T|^{2(1-\alpha)} C^{\alpha} + F |T|^{2\alpha} + E |T|^{2(1-\alpha)} E^{\alpha} \right) x, x \right\rangle_A \]
Taking the supremum over all \( A \)-unit vector \( x \in \mathcal{H} \) we get the desired result. \[ \Box \]
Inequality (3.6) yields several numerical radius inequalities as special cases. A sample of elementary inequalities is demonstrated in the following remarks.

Remark 4. Letting $T = I$ (the identity operator) and $S = 0$ in Theorem 2, it follows that $A^{2} = \{T^{2}\} = AT^{2}T = AI = A$, we obtain the inequality

$$w_{A}(CD) \leq \frac{1}{2} \left\| D^\alpha A^\alpha D + CA^{1-\alpha}C^\alpha \right\|_{A}.$$  

Remark 5. Letting $C = D = E = F = I$ in Theorem 2, we obtain the inequality

$$w_{A}(T + S) \leq \frac{1}{2} \left\| T_{2}^{2} A^{2} + |T_{2}^{2}\|_{A}^{2(1-\alpha)} + |S_{2}^{2|A} + |S_{2}^{2\alpha}_{A}|^{2(1-\alpha)} \right\|_{A}.$$  

Remark 6. Letting $T = S = I, E = D, F = \pm C$ in Theorem 2, we obtain the inequality

$$w_{A}(CD \pm DC) \leq \frac{1}{2} \left\| C^{\alpha} A^{\alpha} C + CA^{1-\alpha}C^\alpha + D^{\alpha} A^{\alpha} D + DA^{1-\alpha}D^\alpha \right\|_{A},$$

which gives an estimate for the numerical radius of the commutator $CD \pm DC$.

4. Generalized Euclidean $A$-numerical radius inequalities

In 2018, Baklouti et al. [6] introduced the concept of Euclidean operator $A$-radius of an $n$-tuple $T = (T_{1}, \cdots, T_{n}) \in B(\mathcal{H})^{n} := B(\mathcal{H}) \times \cdots \times B(\mathcal{H})$. Namely, for $T_{1}, \cdots, T_{n} \in B(\mathcal{H})$, we have

$$w_{e,A}(T_{1}, \cdots, T_{n}) := \sup_{\|x\|_{A} = 1} \left( \sum_{i=1}^{n} \|T_{i}x, x\|_{A} \right)^{1/2}.$$  

In the same work, the authors proved that

$$\frac{1}{2} \sqrt{n} \left\| \sum_{k=1}^{n} T_{k}T_{k}^{\alpha} \right\|_{A}^{1/2} \leq w_{A}(T_{1}, \cdots, T_{n}) \leq \left\| \sum_{k=1}^{n} T_{k}T_{k}^{\alpha} \right\|_{A}^{1/2}.$$  

As a direct consequence of (4.1), if $T = B + iC$ is the $A$-Cartesian decomposition of $A$, then

$$w_{e,A}^{2}(B, C) = \sup_{\|x\|_{A} = 1} \left\{ \|Bx, x\|_{A}^{2} + \|Cx, x\|_{A}^{2} \right\} = \sup_{\|x\|_{A} = 1} \|T_{x, x}\|_{A} = w_{A}^{2}(T).$$  

But since $T^{\alpha}T + TT^{\alpha} = 2(B^{2} + C^{2})$, then we obtain

$$\frac{1}{16} \left\| T^{\alpha}T + TT^{\alpha} \right\|_{A} \leq w_{A}^{2}(T) \leq \frac{1}{2} \left\| T^{\alpha}T + TT^{\alpha} \right\|_{A}.$$  

This inequality were improved by Bhunia et al. in [10] for $T \in B_{A^{1/2}}(\mathcal{H})$ and independently generalized by Feki in [14] for $T \in B_{A}(\mathcal{H})$, where they proved that

$$\frac{1}{4} \left\| TT^{\alpha} + T^{\alpha}T \right\|_{A} \leq w_{A}^{2}(T) \leq \frac{1}{2} \left\| TT^{\alpha} + T^{\alpha}T \right\|_{A}.$$  

The sharpness of (4.3) can be found in [14]. It should be noted that the inequality (4.3) is a kind of generalization of Kittaneh inequality [22]

$$\frac{1}{4} \left\| T^{*}T + TT^{*} \right\| \leq w_{A}^{2}(T) \leq \frac{1}{2} \left\| T^{*}T + TT^{*} \right\|$$

for Hilbert space operator $T \in B(\mathcal{H})$. These inequalities are sharp.

Let $T = B + iC$ be the $A$-Cartesian decomposition, then $B$ and $C$ are $A$-selfadjoint and $T^{\alpha}T + TT^{\alpha} = 2(B^{2} + C^{2})$. Therefore, (4.3) can be reformulated as

$$\frac{1}{2} \left\| B^{2} + C^{2} \right\|_{A} \leq w_{A}^{2}(T) \leq \left\| B^{2} + C^{2} \right\|_{A},$$

or equivalently as

$$\frac{1}{4} \left\| (B + C)^{2} + (B - C)^{2} \right\|_{A} \leq w_{A}^{2}(T) \leq \frac{1}{2} \left\| (B + C)^{2} + (B - C)^{2} \right\|_{A}.$$  

The purpose of this section is to generalize the Euclidean operator $A$-radius for $n$-tuple $n$-tuple $T = (T_{1}, \cdots, T_{n}) \in B(\mathcal{H})^{n} := B(\mathcal{H}) \times \cdots \times B(\mathcal{H})$, which extends (4.3) and also improves (4.1). In light
of the above $A$-Cartesian decomposition of the inequality (4.2), a generalization of the inequality (4.4) is given as well.

We may start this section with the following result.

**Theorem 9.** Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $AT = TA$, with the $A$-Cartesian decomposition $T = B + iC$, $0 \leq \alpha \leq 1$, and $r \geq 1$. Then

\[
\tag{4.5}
w_A^r(T) \leq \frac{1}{2} \left\| B \right\|^2_{ra} + \left\| B^2_{ra} \right\|^2_{1-\alpha} + \left\| C \right\|^2_{ra} + \left\| C^2_{ra} \right\|^2_{1-\alpha} \,.
\]

**Proof.** Since we have

\[
|\langle T x, x \rangle_A| = \left( \langle B x, x \rangle_A^2 + \langle C x, x \rangle_A^2 \right)^{1/2} = \left( \langle B x, x \rangle_A^r + \langle C x, x \rangle_A^r \right)^{1/2}
\]

\[
\leq \left( \left\| B \right\|^2_{ra} \langle x, x \rangle_A + \left\| B^2_{ra} \right\|^2_{1-\alpha} \langle x, x \rangle_A + \left\| C \right\|^2_{ra} \langle x, x \rangle_A + \left\| C^2_{ra} \right\|^2_{1-\alpha} \langle x, x \rangle_A \right)^{1/2}
\]

\[
= \frac{1}{2} \left( \left\| B \right\|^2_{ra} + \left\| B^2_{ra} \right\|^2_{1-\alpha} + \left\| C \right\|^2_{ra} + \left\| C^2_{ra} \right\|^2_{1-\alpha} \right)^{1/2},
\]

which implies that

\[
|\langle T x, x \rangle_A| \leq \frac{1}{2} \left( \left\| B \right\|^2_{ra} + \left\| B^2_{ra} \right\|^2_{1-\alpha} + \left\| C \right\|^2_{ra} + \left\| C^2_{ra} \right\|^2_{1-\alpha} \right)^{1/2}.
\]

Taking the supremum over all $A$-unit vector $x \in \mathcal{H}$, we get the required result. \( \square \)

The generalized Euclidean operator $A$-radius of $T_1, \ldots, T_n$ would be defined as

\[
w_{p,A}(T_1, \ldots, T_n) := \sup_{\|x\|_{A   }=1} \left( \sum_{i=1}^n |\langle T_i x, x \rangle_A|^p \right)^{1/p}, \quad p \geq 1.
\]

This generalizes the concepts of Euclidean operator radius of an $n$-tuple considered by Baklouti et al. [6]. If $p = 1$ then $w_{1,A}(T_1, \ldots, T_n)$ (also, it is denoted by $w_{R,A}(T_1, \ldots, T_n)$) is called the Rhombic $A$-numerical radius which have been studied in [5] but for operators in $\mathcal{B}(\mathcal{H})$. In an interesting case, $w_{1,A}(C, \ldots, C) = n \cdot w_A(C)$.

The $A$-Crawford number is defined to be

\[
c_A(T) = \inf \{|\lambda|: \lambda \in W_A(T)\} = \inf_{\|x\|_{A   }=1} |\langle T x, x \rangle_A|.
\]

Consequently, we define the generalized $A$-Crawford number as:

\[
c_{p,A}(T_1, \ldots, T_n) := \inf_{\|x\|_{A   }=1} \left( \sum_{i=1}^n |\langle T_i x, x \rangle_A|^p \right)^{1/p}, \quad p \geq 1.
\]

In case $p = 1$, the generalized Crawford number is called the Rhombic $A$-Crawford number and is denoted by $c_{R,A}(T_1, \ldots, T_n)$.

We note that in case $p = \infty$, the generalized Euclidean operator radius is defined as:

\[
w_{\infty,A}(T_1, \ldots, T_n) := \sup_{\|x\|_{A   }=1} \sum_{i=1}^n |\langle T_i x, x \rangle_A| - \inf_{\|x\|_{A   }=1} \sum_{i=1}^n |\langle T_i x, x \rangle_A| = w_{R,A}(T_1, \ldots, T_n) - c_{R,A}(T_1, \ldots, T_n).
\]

Thus, the inequality

\[
\tag{4.6}
w_{\infty,A}(T_1, \ldots, T_n) \leq w_{p,A}(T_1, \ldots, T_n) \leq w_{R,A}(T_1, \ldots, T_n)
\]
for all $p \in (1, \infty)$. This fact follows by Jensen’s inequality applied for the function $h(p) = w_{p,A} (T_1, \cdots, T_n)$, which is log-convex and decreasing for all $p > 1$.

On the other hand, by employing the Jensen’s inequality

$$
\left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \frac{1}{n} \sum_{k=1}^{n} a_k^p,
$$

which holds for every finite positive sequence of real numbers $(a_k)_{k=1}^{n}$ and $p \geq 1$; by setting $a_k = |\langle T_kx, x \rangle_A|$ for all $(k = 1, 2, \cdots, n)$, we get

$$
\sum_{k=1}^{n} |\langle T_kx, x \rangle_A| \leq n^{1 - \frac{1}{p}} \left( \sum_{k=1}^{n} |\langle T_kx, x \rangle_A|^p \right)^{\frac{1}{p}}.
$$

Taking the supremum over all $A$-unit vector $x \in \mathcal{H}$, one could get

$$
(4.7) \quad w_{R,A} (T_1, \cdots, T_n) \leq n^{1 - \frac{1}{p}} w_{p,A} (T_1, \cdots, T_n).
$$

Combining the inequalities (4.6) and (4.7) we get

$$
(4.8) \quad w_{\infty,A} (T_1, \cdots, T_n) \leq w_{p,A} (T_1, \cdots, T_n) \leq w_{R,A} (T_1, \cdots, T_n) \leq n^{1 - \frac{1}{p}} w_{p,A} (T_1, \cdots, T_n).
$$

More generally, in the power mean inequality

$$
\left( \frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{\frac{1}{p}} \leq \left( \frac{1}{n} \sum_{k=1}^{n} a_k^q \right)^{\frac{1}{q}}, \quad \forall p \leq q
$$

if one chooses $a_k = |\langle T_kx, x \rangle_A|$ for all $(k = 1, 2, \cdots, n)$, then we have

$$
\left( \frac{1}{n} \sum_{k=1}^{n} |\langle T_kx, x \rangle_A|^p \right)^{\frac{1}{p}} \leq \left( \frac{1}{n} \sum_{k=1}^{n} |\langle T_kx, x \rangle_A|^q \right)^{\frac{1}{q}}.
$$

Taking the supremum over all $A$-unit vector $x \in \mathcal{H}$, we get

$$
(4.9) \quad w_{p,A} (T_1, \cdots, T_n) \leq n^{\frac{p-1}{p}} w_{q,A} (T_1, \cdots, T_n), \quad \forall q \geq p \geq 1.
$$

Indeed, one can refine (4.8) by applying the Jensen’s inequality

$$
(4.10) \quad \left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^p \leq \frac{1}{n} \sum_{k=1}^{n} a_k^p - \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^{p} \geq 2
$$

which obtained from more general result for superquadratic functions [1].

Thus, by setting $a_k = |\langle T_kx, x \rangle_A|$ in (4.10) we get

$$
\left( \sum_{k=1}^{n} |\langle T_kx, x \rangle_A|^p \right)^{\frac{1}{p}} \leq n^{p-1} \sum_{k=1}^{n} |\langle T_kx, x \rangle_A|^p - n^{p-1} \sum_{k=1}^{n} \left| \langle T_kx, x \rangle_A \right| - \frac{1}{n} \sum_{j=1}^{n} \left| \langle T_jx, x \rangle_A \right|^{p}
$$

$$
\leq n^{p-1} \sum_{k=1}^{n} |\langle T_kx, x \rangle_A|^p - n^{p-1} \sum_{k=1}^{n} \left| \langle T_kx, x \rangle_A \right| - \frac{1}{n} \sup_{\|x\|=1} \sum_{j=1}^{n} |\langle T_jx, x \rangle_A|^p.
$$
Taking the supremum again over all $A$-unit vector $x \in \mathcal{H}$, we get

$$
\sup_{\|x\|_A = 1} \left( \sum_{k=1}^{n} \left| \langle T_k x, x \rangle_A \right|^p \right)
\leq \sup_{\|x\|_A = 1} \left\{ n^{p-1} \sum_{k=1}^{n} \left| \langle T_k x, x \rangle_A \right|^p - n^{p-1} \sup_{\|x\|_A = 1} n \sum_{k=1}^{n} \left| \langle T_j x, x \rangle_A \right|^p \right\}
\leq n^{p-1} \sup_{\|x\|_A = 1} \sum_{k=1}^{n} \left| \langle T_k x, x \rangle_A \right|^p - n^{p-1} \inf_{\|x\|_A = 1} \sum_{k=1}^{n} \left| \langle T_j x, x \rangle_A \right|^p
= n^{p-1} w_{p,A}^p (T_1, \cdots, T_n) - n^{p-1} \inf_{\|x\|_A = 1} \sum_{k=1}^{n} \left| \langle T_k x, x \rangle_A \right|^p - \frac{1}{n} w_{R,A} (T_1, \cdots, T_n)^p,
$$

which gives

$$
w_{R,A}^p (T_1, \cdots, T_n) \leq n^{p-1} w_{p,A}^p (T_1, \cdots, T_n) - n^{p-1} \inf_{\|x\|_A = 1} \sum_{k=1}^{n} \left| \langle T_k x, x \rangle_A \right|^p - \frac{1}{n} w_{R,A} (T_1, \cdots, T_n)^p.
$$

which refine the right hand side of (4.8). Clearly, all above mentioned inequalities generalize and refine some inequalities obtained in [24]. For recent inequalities, counterparts, refinements and other related properties concerning the generalized Euclidean operator radius the reader may refer to [5], [12], [18], [19], [29], [30], and [31].

Next, we give a generalization of (4.3) and refine (indeed improve) (4.2) (and thus (4.1)) to the generalized Euclidean operator radius.

**Theorem 10.** Let $T_k \in \mathcal{B}_A (\mathcal{H})$ $(k = 1, \cdots, n)$. Then

$$
\frac{1}{2^{p+1} n^{p-1}} \left\| \sum_{k=1}^{n} T_k^2 T_k + T_k T_k^* \right\|^p_A \leq w_{2p,A}^p (T_1, \cdots, T_n) \leq \frac{1}{2^p} \left\| \sum_{k=1}^{n} \left( T_k^* T_k + T_k T_k^* \right) \right\|^p_A
$$

for all $p \geq 1$.

**Proof.** Let $B_k + iC_k$ be the $A$-Cartesian decomposition of $T_k$ for all $k = 1, \cdots, n$. Then, we have

$$
\left| \langle T_k x, x \rangle_A \right| = \left| \langle B_k x, x \rangle_A + \langle C_k x, x \rangle_A \right| = \left| \langle B_k x, x \rangle_A \right|^2 + \left| \langle C_k x, x \rangle_A \right|^2\geq \frac{1}{2^p} \left( \left| \langle B_k x, x \rangle_A \right| + \left| \langle C_k x, x \rangle_A \right| \right)^2 \geq \frac{1}{2^p} \left| \langle B_k x, x \rangle_A \right| \left| \langle C_k x, x \rangle_A \right| \geq \frac{1}{2^p} \left| \langle B_k \pm C_k x, x \rangle_A \right|.
$$

Summing over $k$ and then taking the supremum over all $A$-unit vector $x \in \mathcal{H}$, we get

$$
w_{2p,A}^p (T_1, \cdots, T_n) \geq \frac{1}{2^p} \frac{1}{n^{p-1}} \sup_{\|x\|_A = 1} \left( \sum_{k=1}^{n} \left| \langle B_k \pm C_k x, x \rangle_A \right| \right)^p
\geq \frac{1}{2^p} \frac{1}{n^{p-1}} \sup_{\|x\|_A = 1} \left( \sum_{k=1}^{n} \left| \langle B_k \pm C_k x, x \rangle_A \right|^2 \right)^p \quad \text{(by Jensen’s inequality)}
= \frac{1}{2^p} \frac{1}{n^{p-1}} \left\| \sum_{k=1}^{n} (B_k \pm C_k)^2 \right\|^p_A.
$$
Thus,

\[ 2w_{2p,A}^p (T_1, \cdots, T_n) \geq \frac{1}{2p} \frac{1}{n^{p-1}} \left( \sum_{k=1}^n (B_k + C_k)^2 \right)^p + \frac{1}{2p} \frac{1}{n^{p-1}} \left( \sum_{k=1}^n (B_k - C_k)^2 \right)^p \]

\[ \geq \frac{1}{2p} \frac{1}{n^{p-1}} \left( \sum_{k=1}^n (B_k + C_k)^2 + \sum_{k=1}^n (B_k - C_k)^2 \right)^p \]

\[ = \frac{1}{2p} \frac{1}{n^{p-1}} \left( \sum_{k=1}^n \{ (B_k + C_k)^2 + (B_k - C_k)^2 \} \right)^p \]

\[ = \frac{1}{n^{p-1}} \left( \sum_{k=1}^n B_k^2 + C_k^2 \right)^p \]

\[ = \frac{1}{n^{p-1}} \left( \sum_{k=1}^n T^*_k T_k + T_k T^*_k \right)^p \]

\[ = \frac{1}{2p} \frac{1}{n^{p-1}} \left( \sum_{k=1}^n T^*_k T_k + T_k T^*_k \right)^p, \]

and hence,

\[ w_{2p} (T_1, \cdots, T_n) \geq \frac{1}{2p+1} \frac{1}{n^{p-1}} \left( \sum_{k=1}^n T^*_k T_k + T_k T^*_k \right)^p, \]

which proves the left hand side of the inequality in (4.11).

To prove the second inequality, for every \( A \)-unit vector \( x \in \mathcal{H} \) we have

\[ \sum_{k=1}^n |\langle T_k x, x \rangle_A|^2p = \sum_{k=1}^n \left( |\langle B_k x, x \rangle_A|^2 + |\langle C_k x, x \rangle_A|^2 \right)^p \]

\[ \leq \sum_{k=1}^n \left( |\langle B_k^2 x, x \rangle_A|^p + |\langle C_k^2 x, x \rangle_A|^p \right) \]

\[ = \sum_{k=1}^n \langle (B_k^2 + C_k^2) x, x \rangle_A^p, \]

which implies that

\[ \sup_{\|x\|_A=1} \left\| \sum_{k=1}^n |\langle T_k x, x \rangle_A|^2p = w_{2p,A}^p (T_1, \cdots, T_1) \leq \sup_{\|x\|_A=1} \sum_{k=1}^n \langle (B_k^2 + C_k^2) x, x \rangle_A^p \right\| \]

\[ = \left\| \sum_{k=1}^n (B_k^2 + C_k^2) \right\|_A \]

\[ = \frac{1}{2p} \left\| \sum_{k=1}^n \left( T^*_k T_k + T_k T^*_k \right) \right\|_A, \]

which proves the right hand side of (4.11).

\[ \square \]

**Remark 7.** Clearly, by setting \( n = 1 \) and \( p = 1 \) in (4.11) we recapture (4.2).

A very interesting case of (4.11) is considered in the following corollary.

**Corollary 7.** Let \( T, S \in \mathcal{B} (\mathcal{H}) \). Then

\[ \frac{1}{2p} \left\| T^* A T + T T^* + S^* A S + S S^* A \right\|_A^p \leq w_{2p,A}^p (T, S) \]

\[ \leq \frac{1}{2p} \left\| (T^*_k T + T T^*_k)^p + (S^*_k S + S S^*_k)^p \right\|_A, \]
for all $p \geq 1$.

**Proof.** Setting $n = 2$ in (4.11). \hfill \Box

**Remark 8.** In particular, setting $p = 1$ in (4.12) we get

$$\frac{1}{4} \left\| T \bar{\alpha} T + T T \bar{\alpha} + S \bar{\alpha} S + S S \bar{\alpha} \right\|_A \leq w_{c, A} (T, S) \leq \frac{1}{2} \left\| T \bar{\alpha} T + T T \bar{\alpha} + S \bar{\alpha} S + S S \bar{\alpha} \right\|_A.$$ 

Moreover, if we choose $T = S$, then

$$\frac{1}{2} \left\| T \bar{\alpha} T + T T \bar{\alpha} \right\|_A \leq w_{c, A} (T, T) \leq \left\| T \bar{\alpha} T + T T \bar{\alpha} \right\|_A.$$ 

**Remark 9.** A lower and upper bounds for the Rhombic numerical radius could be deduced as follows: In (4.8) the inequality holds for any $p \geq 1$. Setting $p = 2q$, then (4.8) reduces to

$$w_{2q, A} (T_1, \ldots, T_n) \leq w_{R, A} (T_1, \ldots, T_n) \leq n^{1 - \frac{q}{2q}} w_{2q, A} (T_1, \ldots, T_n).$$

which implies that

$$w_{2q, A}^q (T_1, \ldots, T_n) \leq w_{R, A}^q (T_1, \ldots, T_n) \leq n^{q - \frac{1}{2}} w_{2q, A}^q (T_1, \ldots, T_n).$$

Combining the inequalities (4.11) with (4.13) we get

$$\frac{1}{2^{2q+1} n^{q+1}} \left\| \sum_{k=1}^{n} T_k \bar{\alpha} T_k + T_k T_k \bar{\alpha} \right\|_A^q \leq w_{2q, A}^q (T_1, \ldots, T_n) \leq w_{R, A}^q (T_1, \ldots, T_n) \leq n^{q - \frac{1}{2}} w_{2q, A}^q (T_1, \ldots, T_n) \leq \frac{n^{q - \frac{1}{2}}}{2q} \left\| \sum_{k=1}^{n} \left( T_k \bar{\alpha} T_k + T_k T_k \bar{\alpha} \right) \right\|_A^q$$

for any $q \geq \frac{1}{2}$.

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