POSITIVITY AND COMPLETENESS OF INVARIANT METRICS

TAEYONG AHN, HERVÉ GAUSSIER, KANG-TAE KIM

ABSTRACT. We present a method for constructing global holomorphic peak functions from local holomorphic support functions for broad classes of unbounded domains in $\mathbb{C}^n$. As an application, we establish a method for showing the positivity and completeness of invariant metrics including the Bergman metric mainly for the unbounded domains.

1. INTRODUCTION

The purpose of this article is twofold: (1) to present a method of obtaining global holomorphic peak functions at boundary points for unbounded domains in $\mathbb{C}^n$ from any local holomorphic support functions, and (2) to establish the positivity and completeness of invariant metrics, primarily of the Bergman metric, of certain unbounded domains in $\mathbb{C}^n$.

As a consequence, we shall demonstrate applications of these methods to broad collections of unbounded domains that include the Kohn-Nirenberg domains, the Fornæss domains, more generally those defined by (the “positive variations” of) weighted-homogeneous plurisubharmonic polynomial defining functions and some more. Notice that several of these domains are not known whether they can be biholomorphic to bounded domains.

2. PRELIMINARY NOTATION AND TERMINOLOGY

2.1. The ball. With \( \|(z_1, \ldots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2} \) we let \( B^n(p, r) = \{ z \in \mathbb{C}^n : \|z - p\| < r \} \). If \( p \) is the origin and \( r = 1 \), then we denote by \( B^n := B^n(0, 1) \).

2.2. Distance to the complement. \( \delta_U(z) := \min\{1, \text{dist} (z, \mathbb{C}^n \setminus U)\} \), where \( U \) is an open subset in \( \mathbb{C}^n \) and “dist” means the Euclidean distance.

2.3. Holomorphic peak and support functions. Let \( \Omega \) be an open set in \( \mathbb{C}^n \). Let \( \mathcal{O}(\Omega) := \{ h : \Omega \to \mathbb{C}, h: \text{holomorphic} \} \) and \( \mathcal{O}^*(\Omega) := \{ h : \Omega \to \mathbb{C}, h \neq 0 \text{ on } \Omega, h: \text{holomorphic} \} \).

For a boundary point \( p \in \partial \Omega \), a peak function at \( p \) for \( \mathcal{O}(\Omega) \) (or, a global holomorphic peak function for \( \Omega \) at \( p \)) is defined to be a holomorphic function \( f \in \mathcal{O}(\Omega) \) such that:

(i) \( \lim_{\Omega \ni z \to p} f(z) = 1 \), and

(ii) for every \( r > 0 \) there exists \( s > 0 \) satisfying \( |f(z)| < 1 - s \) for every \( z \in \Omega \setminus B^n(p, r) \).
The point \( p \) is called a peak point of \( \Omega \) for \( O(\Omega) \) (or, a global holomorphic peak point of \( \Omega \)) in such a case.

By a local holomorphic peak function at a boundary point, say \( q \in \partial \Omega \), we mean a peak function at \( q \) for \( O(V \cap \Omega) \) for some open neighborhood \( V \) of \( q \) in \( \mathbb{C}^n \). The point \( q \) is then called a local holomorphic peak point of \( \Omega \).

If a holomorphic function \( f \) is defined in an open neighborhood of the closure \( \overline{\Omega} \) of the open set \( \Omega \) in \( \mathbb{C}^n \) in such a way that it is also a peak function at \( p \) for \( O(\Omega) \), such \( f \) is called a global holomorphic support function of \( \Omega \) at \( p \). Local holomorphic support functions are defined likewise: a local holomorphic support function in \( \Omega \) is a peak function from a local holomorphic support function.

3. Technical theorem for unbounded domains

Here we present the main technical theorem on how to obtain a global holomorphic peak function from a local holomorphic support function.

If \( g \in O(V) \), then we denote by \( Z_g := \{ z \in V : g(z) = 1 \} \).

**Theorem 3.1.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \). If \( p \in \partial \Omega \) satisfies the following two properties:

\begin{enumerate}
  \item There exists an open neighborhood \( V \) of \( p \) in \( \mathbb{C}^n \) and a function \( g \in O(V) \) supporting \( V \cap \Omega \) at \( p \).
  \item There are constants \( r_1, r_2, r_3 \) with \( 0 < r_1 < r_2 < r_3 < 1 \) and \( B^n(p, r_3) \subset V \), and there exists a Stein neighborhood \( U \) of \( \overline{\Omega} \) and a function \( h \in O(U \cap \Omega) \cap O^*(V) \) satisfying
    \[ Z_g \cap U \cap \left( B^n(p, r_2) \setminus B^n(p, r_1) \right) = \emptyset \]
    and
    \[ |h(z)|^2 \leq C_0 \frac{\delta_U(z)^{2n}}{(1 + \|z\|^2)^2}, \forall z \in \Omega, \]
\end{enumerate}

for some positive constant \( C_0 \), then \( \Omega \) admits a peak function at \( p \) for \( O(\Omega) \).

**Proof.** Take a \( C^\infty \) function \( \chi : \mathbb{C}^n \to [0, 1] \) satisfying \( \chi \equiv 1 \) on \( B^n(p, r_1) \) and \( \text{supp} \chi \subseteq B^n(p, r_2) \).

Define a smooth \((0, 1)\)-form \( \alpha \) on \( U \) as follows (cf. [5]):

\[ \alpha(z) = \begin{cases} \partial \left( \chi(z) h^{-1}(1 - \chi(z)) \right) & \text{if } z \in U \cap (B^n(p, r_2) \setminus B^n(p, r_1)) \\
0 & \text{if } z \text{ is elsewhere in } U. \end{cases} \]

By a theorem of Hörmander ([13], Theorem 4.4.2), there exists a function \( u : U \to \mathbb{C} \) such that \( \bar{\partial} u = \alpha \) on \( U \) satisfying

\[ \int_U \frac{|u(z)|^2}{(1 + \|z\|^2)^2} d\mu(z) \leq \int_U |\alpha(z)|^2 d\mu(z), \]

where \( \mu \) denotes the Lebesgue measure for \( \mathbb{C}^n \). Notice that \( u \in C^\infty(U) \) by elliptic regularity.

Since \( \alpha \) is a bounded-valued smooth \((0, 1)\)-form with bounded support in \( U \), the right-hand side of the preceding inequality is bounded above by a positive constant \( C_1 \), for instance.

Now we wish to obtain a pointwise estimate for \( |u(z)| \).
Let \( R > 1 \) be a constant such that \( B^n(p, r_2) \subset B^n(0, R-1) \). Let \( \xi \in \Omega \setminus B^n(0, R) \). Our current aim is to estimate \( |u(\xi)| \).

For such \( \xi \), we see that \( \text{supp} \chi \cap B^n(\xi, \delta_U(\xi)) = \emptyset \). This implies that \( u \) is holomorphic on \( B(\xi, \delta_U(\xi)) \). These discussions yield

\[
C_1 \geq \int_U |\alpha(z)|^2 d\mu(z) \geq \int_U \frac{|u(z)|^2}{(1 + \|z\|^2)^2} d\mu(z)
\]

\[
\geq \frac{1}{9(1 + \|\xi\|^2)^2} \int_{B(\xi, \delta_U(\xi))} |u(z)|^2 d\mu(z)
\]

\[
\geq \frac{|u(\xi)|^2 \cdot \text{Vol}(B(\xi, \delta_U(\xi)))}{9(1 + \|\xi\|^2)^2},
\]

where the last inequality is due to the sub mean-value inequality. In short, there exists a constant \( C_2 > 0 \) such that

\[
\left( \frac{\delta_U(\xi)^n}{1 + \|\xi\|^2} \right)^2 |u(\xi)|^2 \leq C_2.
\]

At this stage we use the assumption (‡) and arrive at

\[
|h(\xi)u(\xi)|^2 \leq C_3 \text{ for any } \xi \in \Omega \setminus B^n(0, R)
\]

for some constant \( C_3 > 0 \).

Consider the case where \( \xi \in \Omega \cap B^n(0, R) \). Notice that the assumption (‡) implies that \( h \) is bounded. Since \( u \) is smooth in \( U \) and hence smooth at every point of \( \overline{\Omega} \), the function \( h(z)u(z) \) is bounded on \( \Omega \cap B^n(0, R) \). Altogether, \( hu \) is bounded on \( \Omega \).

We are now going to construct a global holomorphic peak function for \( \Omega \) at \( p \). By the preceding arguments, we may choose a positive constant \( c \) such that \( c|h(z)u(z)| < \frac{1}{2} \) for any \( z \in \Omega \). In particular, we obtain

\[
\text{Re} \left( c h(z)u(z) - 1 \right) < -\frac{1}{2}, \quad \forall z \in \Omega.
\]

Consider

\[
\psi(z) = \begin{cases} \frac{u(z)}{1 - g(z)} & \text{if } z \in \Omega \cap B^n(p, r_2) \\ 0 & \text{if } z \in \Omega \setminus B^n(p, r_2). \end{cases}
\]

Note that \( \psi(z) - h(z)u(z) \) is a holomorphic function on \( \Omega \) since \( \overline{\partial} u = \alpha \). Altogether, if we define \( f : \Omega \to \mathbb{C} \) by

\[
f(z) := \exp \left( (-c[\psi(z) - h(z)u(z)] - 1)^{-1} \right),
\]

then \( f \) is holomorphic on \( \Omega \).

Note that the real part of \( ch(z)u(z) - 1 \) on \( \Omega \) is negative. The function \( -\text{Re} \ c\psi \) on \( \Omega \) is also negative due to its construction. Hence

\[
\text{Re} \left( -c[\psi(z) - h(z)u(z)] - 1 \right)^{-1} < 0.
\]

So \( |f(z)| < 1 \) for any \( z \in \Omega \).

Finally, \( \lim_{\Omega \ni z \to p} f(z) = e^0 = 1 \) since \( \psi \) tends to \( \infty \) as \( z \to p \). Notice that \( p \) is the only boundary point that has this property for \( f \) by the construction throughout.
The remaining condition for \( f \) to be a peak function at \( p \) for \( \mathcal{O}(\Omega) \) is also easily checked from the definition of \( f \) itself. We now see that \( f \) is the desired global holomorphic peak function for \( \Omega \) at \( p \), and hence the proof is complete.

\[ \square \]

Remark 3.2. Note that condition (\( \dagger \)) is necessary only for the case where \( \Omega \) is unbounded. If \( \Omega \) is bounded, the assumption (\( \dagger \)) is void; one may just take \( h \equiv 1 \).

4. Applications to unbounded domains

4.1. Weighted homogeneous domains in \( \mathbb{C}^n \). For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), denote by \( z' = (z_1, \ldots, z_{n-1}) \) and consequently \( z = (z', z_n) \).

Let \( m_1, \ldots, m_{n-1} \) be positive integers. A real-valued polynomial \( P \) on \( \mathbb{C}^{n-1} \) is called weighted-homogeneous of weight \( (m_1, \ldots, m_{n-1}) \), if

\[
P(t^{m_1} z_1, \ldots, t^{m_{n-1}} z_{n-1}) = tP(z_1, \ldots, z_{n-1}), \quad \forall t > 0, \quad \forall (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}.
\]

If \( m = m_1 = \ldots = m_{n-1} \) then \( P \) is called homogeneous of degree \( m \).

Definition 4.1. A domain \( \Omega \) in \( \mathbb{C}^n \) is called a WB-domain (meaning “weighted-bumped”) if

\[ \Omega = \{ z \in \mathbb{C}^n : \text{Re } z_n + P(z') < 0 \}, \]

where:

(i) \( P \) is a real-valued, weighted-homogeneous polynomial on \( \mathbb{C}^{n-1} \) of weight \( (m_1, \ldots, m_{n-1}) \),

(ii) \( P \) is plurisubharmonic (psh) without pluriharmonic terms, and

(iii) there is a constant \( s > 0 \) such that \( P(z') - 2s \sum_{j=1}^{n-1} |z_j|^{2m_j} \) is also psh in \( \mathbb{C}^{n-1} \).

We prove the following theorem.

Theorem 4.2. If \( \Omega \) is a WB-domain in \( \mathbb{C}^n \), there exists a peak function at \( 0 = (0, \ldots, 0) \) for \( \mathcal{O}(\Omega) \), continuous on \( \overline{\Omega} \) with exponential decay at infinity and nowhere zero.

Proof. We shall use the notation of Definition 4.1. Let \( \mu_j : = \frac{1}{m_j} \prod_{k=1}^{n-1} m_k \) for \( j = 1, \ldots, n - 1 \) and let \( H(z_1, \ldots, z_{n-1}) := P(z_1^{\mu_1}, \ldots, z_{n-1}^{\mu_{n-1}}) \). Then \( H \) is a homogeneous polynomial of degree \( 2k = 2m_1 \cdots m_{n-1} \).

Let \( \Omega_H : = \{(z_1, \ldots, z_n) : \text{Re } z_n + H(z_1, \ldots, z_{n-1}) < 0 \} \) and let \( F_H(z_1, \ldots, z_n) = (z_1^{\mu_1}, \ldots, z_{n-1}^{\mu_{n-1}}, z_n) \). Then, \( F_H : \Omega_H \rightarrow \Omega \) is a holomorphic ramified covering map of finite degree.

By Theorem 4.1 of [1], the domain \( \tilde{\Omega} : = \{(z_1, \ldots, z_n) : \text{Re } z_n + \tilde{H}(z_1, \ldots, z_{n-1}) - \delta |z_n| - \delta \sum_{j=1}^{n-1} |z_j|^{2k} < 0 \} \), for some \( \delta > 0 \), admits a peak function, which we denote by \( Q_H(z) \) here, at the origin for \( \mathcal{O}(\tilde{\Omega}) \). This peak function by Bedford and Fornæss enjoys an exponential decay condition at infinity and vanishes nowhere on \( \tilde{\Omega} \).

To obtain the desired peak function at 0 for \( \mathcal{O}(\Omega) \), we symmetrize \( Q_H \): for each \( z = (z_1, \ldots, z_n) \in \overline{\Omega} \), there are exactly \( \prod_{j=1}^{n-1} \mu_j \) preimages by the map \( F_H \) (counting with multiplicity), of the following form:

\[
(\frac{1}{|z|^{\frac{1}{n-1}}} e^{\frac{2\pi}{n-1} (2k_1 \pi + \arg z_1)}, \ldots, |z_{n-1}|^{\frac{1}{n-1}} e^{\frac{2\pi}{n-1} (2k_{n-1} \pi + \arg z_{n-1})}, z_n),
\]
where $k_j \in \{1, \ldots, \mu_j \}$ for every $j$. Let
\[
q(w) := \prod_{1 \leq j \leq n-1} \prod_{1 \leq k_j \leq \mu_j} Q_H(\sqrt{\frac{2k_1}{\mu_1}} w_1, \cdots, \sqrt{\frac{2k_{n-1}}{\mu_{n-1}} w_{n-1}}, w_n).
\]
Then $q: \Omega_H \to \mathbb{C}$ is a well-defined holomorphic function on $\Omega_H$. Moreover, this defines a unique holomorphic function $Q \in O(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Q \circ F_H(w) = q(w)$ for any $w \in Q_H$. It is now immediate that this $Q$ is the desired peak function at 0 for $O(\Omega)$.

More importantly for our purpose, this peak function $Q$ is holomorphic on the domain
\[
\Omega^c := \{ z \in \mathbb{C}^n : \Re z_n + P(z') < \epsilon|z_n| + \epsilon \sum_{j=1}^{n-1} |z_j|^{2m_j} \}
\]
and has an exponential decay condition at infinity. It also vanishes nowhere. Hence it can play the role of the function $h$ of Theorem 3.4. As a consequence we obtain (using the homothety and translation automorphisms for weakly pseudoconvex points, if necessary) the following

**Corollary 4.3.** If $\Omega$ is a WB-domain then every boundary point admits a peak function for $O(\Omega)$.

**Remark 4.4.** Note that this corollary generalizes the case of domains with defining functions of diagonal type studied by Herbort in [11]. The method of this article yields an alternative proof to Theorem 2 of [11].

**Remark 4.5.** Theorem 4.2 can be understood as a generalization of Theorem 4.1 of [1]. There is another generalization of it in a different direction by Noell ([16]). For $P$ with the assumptions by Noell in [16], the above arguments still hold and we can obtain a statement corresponding to Corollary 4.3.

4.2. The Kohn-Nirenberg domains, the Fornæss domains and positive variations of WB-domains. The domains
\[
\Omega_{HKN} = \{(z, w) \in \mathbb{C}^2 : \Re w + |z|^8 + \frac{15}{4}|z|^2 \Re z^6 < 0 \}.
\]
and
\[
\Omega_{KN} = \{(z, w) \in \mathbb{C}^2 : \Re w + |zw|^2 + |z|^8 + \frac{15}{4}|z|^2 \Re z^6 < 0 \}.
\]
were first introduced in [14]; for $\Omega_{HKN}$ and $\Omega_{KN}$, the origin is the boundary point that does not admit, even locally, any holomorphic support functions, despite the fact that the boundary is real-analytic everywhere and strongly pseudoconvex at every boundary point except the real line $\{z = 0, \Re w = 0 \}$ for $\Omega_{HKN}$, and except the origin for $\Omega_{KN}$, respectively. There are still some problems yet to be answered for these domains, as recent research concerning unbounded domains attracts much attention ([3], [10] and also [11], [12]).

Note that $\Omega_{HKN}$ belongs to the class of WB-domains. Hence, we will focus on $\Omega_{KN}$ and call it the Kohn-Nirenberg domain in the rest of the article.

**Theorem 4.6.** There is a peak function for $O(\Omega_{KN})$ at every boundary point of $\Omega_{KN}$.
Proof. Observe that $\Omega_{\text{KN}} \subset \Omega_{\text{HKN}}$ and that 0 $\in \partial \Omega_{\text{KN}} \cap \partial \Omega_{\text{HKN}}$. Consider now the domain $\Omega_{\text{HKN}}$. Then Theorem 4.2 provides a special peak function at the origin for this domain. It continues to be a peak function for $\mathcal{O}(\Omega_{\text{KN}})$ at the origin. This peak function also plays the role of $h$ in the hypothesis of Theorem 3.1. For $U$, we simply take
\[ U = \{(z, w) \in \mathbb{C}^2 : \text{Re} w + |zw|^2 + |z|^8 + \frac{15}{7} |z|^2 \text{Re} z^6 < \varepsilon \} \]
where $\varepsilon$ is chosen small enough. Then, since the defining function of $U$ is a polynomial and $h$ decays exponentially, the conditions of Theorem 3.1 are satisfied. Recall that every boundary point except the origin is a strongly pseudoconvex point. Hence the assertion follows immediately.

We remark that the same applies to the Fornæss domains in $\mathbb{C}^2$ (\cite{4}) defined by
\[ \Omega_{\text{HF}} := \{(z, w) \in \mathbb{C}^2 : \text{Re} w + |z|^6 + t|z|^2 \text{Re} z^4 < 0 \} \]
and
\[ \Omega_F := \{(z, w) \in \mathbb{C}^2 : \text{Re} w + |zw|^2 + |z|^6 + t|z|^2 \text{Re} z^4 < 0 \} \]
for constant $t$ with $1 < t < \frac{9}{5}$.

Since the arguments of this type have a general nature, we formulate it into the following formal statement:

**Theorem 4.7.** Let $W := \{z \in \mathbb{C}^n : \text{Re} z_n + P(z') < 0 \}$ be a WB-domain. Let $S$ be a non-negative plurisubharmonic polynomial defined on $\mathbb{C}^n$ such that $S(0) = 0$ and
\[ W_S := \{z \in \mathbb{C}^n : \text{Re} z_n + S(z) + P(z') < 0 \}. \]
If a boundary point of $W_S$ admits a local holomorphic support function, then it also admits a global holomorphic peak function.

**Remark 4.8.** Using this type of argument, we can also handle certain unbounded domains which are subdomains of WB-domains. In particular, we can handle some unbounded domains defined by a non-polynomial defining function. For example, consider
\[ W_E := \{(z, w) \in \mathbb{C}^2 : \text{Re} w + \exp(|z|^2) < 0 \}. \]
Note that this domain has infinite volume. As the open set $U$ of Theorem 3.1 we take
\[ U := \{\text{Re} w + \exp(|z|^2) - \varepsilon \exp(|z|^2) < \varepsilon \} \]
with $\varepsilon$ small enough.

Moreover, using the inequality $|z|^2 < \exp(|z|^2)$, we can take as the function $h$ of Theorem 3.1 the one given by Theorem 4.2. Since every boundary point of $W_E$ is a strongly pseudoconvex point, every boundary point of $W_E$ admits a global holomorphic peak function. Indeed, in this case, they are all global holomorphic support functions. Another example with different nature will be dealt with in the next subsection.
4.3. Some unbounded domains with finite volume. The domain
\[ E := \{(z, w) \in \mathbb{C}^2 : |w|^2 < \exp(-|z|^2)\} \]
cannot be biholomorphic to any bounded domain since it contains the complex line
\[ \{(z, w) \in \mathbb{C}^2 : w = 0\} \]. However, it is strongly pseudoconvex at every boundary point.

We point out that Theorem 3.1 applies to this domain as well by setting for \( \varepsilon \) small enough
\[ U := \{(z, w) \in \mathbb{C}^2 : |w|^2 < \exp(-|z|^2) + \varepsilon\} \]
and \( h(z, w) := w \). Consequently, every boundary point admits a peak function for \( \mathcal{O}(E) \), which is, especially in this case, a global holomorphic support function.

5. Applications to invariant metrics of unbounded domains

5.1. Remarks on the Hahn-Lu comparison theorem. Recall the following classical concepts: for a complex manifold \( M \), denote by \( \mathcal{O}(M, B^1) \) the set of holomorphic functions from \( M \) into the unit open disc \( B^1 \). Let \( p \in M \) and \( v \in T_p M \).

Then the Caratheodory pseudo-metric (metric, if positive) of \( M \) is defined by
\[ c_M(p, v) = \sup\{ |df_p(v)| : f \in \mathcal{O}(M, B^1), f(p) = 0 \}. \]
This induces the integrated Caratheodory pseudo-distance (distance, if positive)
\[ \rho_M^c(p, q) = \inf_{\gamma} \int_0^1 c_M(\gamma(t), \gamma'(t)) dt, \]
where the infimum is taken over all the piecewise \( C^1 \) curves \( \gamma : [0, 1] \to M \) with \( \gamma(0) = p, \gamma(1) = q \).

For the Bergman metric, one starts with the space \( \mathcal{A}^2(\Omega) := \{ f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 < +\infty \} \). Equipped with the \( L^2 \)-inner product, it is a separable Hilbert space. If one considers a complete orthonormal system \( \{\varphi_j : j = 1, 2, \ldots\} \) of \( \mathcal{A}^2(\Omega) \), the Bergman kernel function can be expressed as \( K_\Omega(z, w) = \sum_{j \geq 1} \varphi_j(z)\varphi_j(w) \). Then it defines
\[ b_\Omega(z) = \frac{\partial^2}{\partial z_a \partial z_b} \log K_\Omega(z, z) \]
as well as the \((1,1)\)-tensor
\[ b_\Omega^a_b := \sum_{a, b=1}^n b_{ab}^\Omega(\zeta) \, d\zeta_a \otimes d\zeta_b. \]

If \( K_\Omega(z, z) \) is non-zero, then \( b_\Omega^a_b \) defines a smooth \((1,1)\)-Hermitian form that is positive semi-definite. It is a result of Bergman himself that \( b^\Omega \) is a positive definite Hermitian metric for bounded domains. In general this may not even be defined, and even when it is defined, it may not be positive.

Nevertheless, we shall follow the convention and write as \( b^\Omega \) the Bergman metric of the domain \( \Omega \) and the notation \( b^\Omega_p \) shall always mean the Bergman metric of the domain \( \Omega \) at the point \( p \in \Omega \).

We now present a modification of the comparison theorem by Hahn [7, 8, 9], and Lu [15] which compares the Caratheodory metric and the Bergman metric (even if both may be degenerate).
Theorem 5.1 (The Hahn-Lu comparison theorem). If \( M \) is a complex manifold and \( p \) is a point in \( M \) such that its Bergman kernel \( K_M \) satisfies \( K_M(p,p) \neq 0 \), then its Bergman metric \( b_p^M(v,v) \) and the Carathéodory pseudometric \( c_M(p,v) \) satisfy the inequality
\[
(c_M(p,v))^2 \leq b_p^M(v,v),
\]
for any \( v \in T_pM \).

Proof. We shall only prove it for the case when \( M = \Omega \) is a domain in \( \mathbb{C}^n \), staying closely to the purpose of this article; the manifold case uses essentially the same arguments except some simplistic adjustments.

Start with the following quantities developed by Bergman [2]:
\[
\mathcal{B}_0(p) = \sup \left\{ |\psi(p)|^2 : \psi \text{ holomorphic, } \int_{\Omega} |\psi|^2 \leq 1 \right\}
\]
\[
\mathcal{B}_1(p,v) = \sup \left\{ |\partial_v \varphi|^2 : \varphi \text{ holomorphic, } \varphi(p) = 0, \int_{\Omega} |\varphi|^2 \leq 1 \right\},
\]
where \( \partial_v \varphi = \sum_{j=1}^n v_j \frac{\partial}{\partial z_j} \). These concepts are significant because \( \mathcal{B}_1(p,v) = \mathcal{B}_0(p) \cdot \mathcal{B}_0^\Omega(v,v), \) when \( \mathcal{B}_0(p) > 0 \).

By the Cauchy estimates and Montel’s theorem, there exists an \( L^2 \)-holomorphic function \( \hat{\psi} \) on \( \Omega \) with \( \|\hat{\psi}\|_{L^2(\Omega)} \leq 1 \) satisfying \( |\hat{\psi}(p)|^2 = \mathcal{B}_0(p) \). (See [3] [4].) Then Montel’s theorem on normal families implies the existence of \( \eta \in \mathcal{H}(\Omega, B^1) \) on \( \Omega \) with \( \eta(p) = 0 \) and \( |\partial_v \eta|^2 = |d\eta_p(v)| = c_0(p,v) \), the Carathéodory length of \( v \) at \( p \). Since \( |\hat{\eta} \hat{\psi}| \leq |\hat{\psi}| \), \( \|\hat{\eta} \hat{\psi}\|_{L^2(\Omega)} \leq \|\hat{\psi}\|_{L^2(\Omega)} \leq 1 \). Since \( \eta(p) = 0 \), \( |\partial_v (\eta \hat{\psi})|^2 = |\partial_v \eta|^2 \leq |\hat{\psi}| \). Altogether, we arrive at
\[
\mathcal{B}_1(p,v) \geq |\partial_v (\eta \hat{\psi})|^2 = c_0(p,v)^2 \mathcal{B}_0(p).
\]
This immediately yields the comparison inequality
\[
b_p^\Omega(v,v) = \frac{\mathcal{B}_1(p,v)}{\mathcal{B}_0(p)} \geq c_0(p,v)^2.
\]

Remark 5.2. The original statements required positivity of both metrics. But the proof above (almost identical with the arguments by Hahn [3]) clearly shows that not all those assumptions are necessary. On the other hand, this modification is significant since one obtains that the Bergman metric is positive definite whenever the manifold is Carathéodory hyperbolic. We shall see applications in the next section.

5.2. Positivity and completeness of invariant metrics.

5.2.1. Positivity. As a result of discussions above, we present the following:

Proposition 5.3. The Kohn-Nirenberg domains, the Fornæss domains, all WB-domains as well as the domain \( W_S \) in Theorem 4.7 are Carathéodory hyperbolic. Consequently, their Kobayashi metric and Bergman metric are positive.

Proof. Let \( Q \) be an appropriate global holomorphic peak function at the origin for each case introduced in Section 4. For \( p = (p_1, p_2) \) and \( v = (v_1, v_2) \), take \( g(z_1, z_2) := Q(z_1, z_2) \cdot (\bar{v}_1(z_1 - p_1) + \bar{v}_2(z_2 - p_2)) \). Then \( g \) is a bounded holomorphic
function on $\Omega$, since $Q$ decays exponentially at infinity in $\Omega$ and is continuous in the closure of $\Omega$. Moreover $g(p) = 0$ and $|dq_p(v)| = Q(p)\|v\|^2 > 0$. Hence the Caratheodory metric is positive on $\Omega$. Consequently the Kobayashi metric, being larger, is positive as well on $\Omega$. For the Bergman metric, we use the fact that $Q$ belongs to $A^2(\Omega)$ thanks to its exponential decay (whereas the domain is defined by a polynomial inequality). Thus the diagonal of the Bergman kernel vanishes nowhere on $\Omega$ and it follows from the Hahn-Lu comparison theorem (Theorem 5.1) that the Bergman metric is positive definite on $\Omega$. $\square$

5.2.2. Completeness. Recall that a metric is said to be complete if every Cauchy sequence converges with respect to its integrated distance. Note that it is well-known that the Caratheodory metric is continuous with respect to the Euclidean metric. Since the domains we handle are unbounded, we need to handle points near infinity in each of the following cases:

Case 1. WB-domains: In this case we first prove

Lemma 5.4. If $\Omega$ is a WB-domain equipped with its integrated Caratheodory distance $\rho^*_1$, then every Cauchy sequence is bounded away from infinity.

Proof. Suppose that $(q_\nu)$ be a Cauchy sequence. Passing to a subsequence, without loss of generality, we may assume that $\lim_{\nu \to \infty} \|q_\nu\| = \infty$ to get a contradiction. Since WB-domains admit homothety automorphisms shrinking to the origin, one can always find $\varphi_\nu \in \text{Aut}(\Omega)$ such that $\|\varphi_\nu(q_\nu)\| = 1$ for every $\nu$. Of course we have $\lim_{\nu \to \infty} \|\varphi_\nu(q_1)\| = 0$. Since the origin has a peak function, say $f$, for $O(\Omega)$ (cf. Theorem 4.2), it follows that

$$\lim_{\nu \to \infty} \rho^*_1(q_1, q_\nu) = \lim_{\nu \to \infty} \rho^*_1(\varphi_\nu(q_1), \varphi_\nu(q_\nu)) \geq \lim_{\nu \to \infty} d^p(f \circ \varphi_\nu(q_1), f \circ \varphi_\nu(q_\nu)) = \infty,$$

where $d^p(\cdot, \cdot)$ denotes the Poincaré distance of $B^1$. Of course this is impossible for a $\rho^*_1$-Cauchy sequence, and hence the proof is complete. $\square$

Now, Corollary 4.3, Theorem 5.1 and Lemma 5.4 imply

Theorem 5.5. If $\Omega$ is a WB-domain, then its Caratheodory metric is complete. Moreover, its Kobayashi metric and Bergman metric are also complete.

Case 2. The Kohn-Nirenberg domains, the Fornaess domains and positive variations of WB-domains admitting a local holomorphic support function at each boundary point: Let $\Omega$ denote any of these domains. If $\Omega$ is either a Kohn-Nirenberg domain or a Fornaess domain, then by Theorem 4.6 the Caratheodory distance $\rho^*_M(p, q_j)$ tends to infinity when $p \in \Omega$ and $(q_j)_j$ is a sequence of points in $\Omega$ converging to a point $q \in \partial \Omega$. If $\Omega$ is a positive variation of WB-domains admitting a local holomorphic support function at each boundary point, the same holds according to Theorem 4.7. Thus we only need to prove in each case that every Cauchy sequence $(z_j)_j$ of points in $\Omega$, with respect to the integrated Caratheodory distance $\rho^*_1$, is bounded away from infinity. Since $\Omega$ is a subdomain of a WB-domain $\Omega'$, it follows from the distance decreasing property of the Caratheodory metric that $(z_j)_j$ is a Cauchy sequence with respect to $\rho^*_1$. Then $(z_j)_j$ is bounded away from infinity by Lemma 5.4. Hence $\Omega$ is complete with respect to the Caratheodory metric. Using
the comparisons of invariant metrics including the Hahn-Lu comparison theorem, the Bergman metric and the Kobayashi metric of $\Omega$ are also positive and complete. □

Remark 5.6. It is not difficult to see that the Caratheodory metric, Kobayashi metric and Bergman metric of the domain $W_E$ are also positive and complete.

Case 3. $\Omega = \{ (z, w) \in \mathbb{C} : \lvert w \rvert^2 < e^{-\lvert z \rvert^2} \}$: This unbounded domain does not admit any translation or homothety automorphisms. Also it is not biholomorphic to any bounded domains. Also the Kobayashi metric degenerates at points on the complex line defined by the equation $w = 0$, which we denote by $L$.

However a direct computation shows that every holomorphic polynomial on $\mathbb{C}^2$ belong to $A^2(\Omega)$ and the Bergman kernel is well-defined and positive along the diagonal. Indeed, a direct computation yields the Bergman kernel for $\Omega$

$$K_{\Omega}(z, w) = \frac{2 \exp(2z_1 \overline{w}_1)(1 + z_2 \overline{w}_2 \exp(2z_1 \overline{w}_1))}{\pi^2(1 - z_2 \overline{w}_2 \exp(2z_1 \overline{w}_1))^3}.$$ 

Also it is not hard to see that the Bergman metric is positive definite.

Although the completeness of the Bergman metric of $\Omega$ follows by an explicit computation, Theorem 3.1 and Theorem 5.1 yield a different proof of that result. Since every boundary point admits a global holomorphic peak function and our version of Hahn-Lu comparison theorem applies pointwise, it suffices to check whether every Cauchy sequence with respect to the Bergman metric is bounded away from infinity. This is true since from the direct computations we see that the length of any piecewise $C^1$-curve always has larger length with respect to the Bergman metric than its projected image onto the complex line $L$ (with respect to the natural projection). Moreover, again from direct computations, the Bergman metric restricted to the complex line $L$ is Euclidean. So, this proves the completeness of the Bergman metric.

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(Ahn) Center for Geometry and its Applications, POSTECH, Pohang 790-784, The Republic of Korea

E-mail address: (Ahn) triumph@postech.ac.kr

(Gaussier) Univ. Grenoble Alpes, IF, F-38000 Grenoble, France

CNRS, IF, F-38000 Grenoble, France

E-mail address: (Gaussier) herve.gaussier@ujf.grenoble.fr

(Kim) Center for Geometry and its Applications and Department of Mathematics, POSTECH, Pohang 790-784, The Republic of Korea

E-mail address: (Kim) kimkt@postech.ac.kr