Abstract
There are close relations between tripartite tensors with bounded geometric ranks and linear determinantal varieties with bounded codimensions. We study linear determinantal varieties with bounded codimensions, and establish upper bounds of dimensions of the ambient spaces. Using these results, we classify tensors with geometric rank 3, find upper bounds of multilinear ranks of primitive tensors with geometric rank 4, and prove the existence of such upper bounds in general. We extend results of tripartite tensors to $n$-part tensors, showing the equivalence between geometric rank 1 and partition rank 1.

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1 Introduction

1.1 Geometric rank

Various types of ranks of tensors have been introduced and studied in numerous areas such as algebraic complexity, extremal combinatorics and quantum information theory. Subrank was introduced by Strassen to study the algebraic complexity of matrix multiplication [18], and its asymptotic version plays an important role in Strassen’s laser method [19], which people have utilized to obtain upper bounds of the exponent of matrix multiplication. Slice rank arose in the study of the cap set problem [20], and it turned out to be helpful in the study of the sunflower problem [17]. Slice rank and subrank were also studied from the point of view of quantum information theory [6]. Analytic rank was introduced by [11] in the context of Fourier analysis, and [15]...
showed it lower bounds slice rank and can replace slice rank in the resolution of cap set problem.

Geometric rank was introduced in [13] as an extension of analytic rank from finite fields to algebraically closed fields, and as a tool to find upper bounds on border subrank and lower bounds on slice rank. [10] took a step further studying geometric rank systematically, giving results on tensors with geometric rank at most 3. [7] showed that the partition rank is at most $2^{n-1}$ times of the geometric rank for $n$-part tensors. Putting different types of ranks in an increasing order, we have

$$\text{Subrank} \leq \text{Border Subrank} \leq \text{Geometric Rank} \leq \text{Partition Rank} \leq \text{Slice Rank} \leq \text{Multilinear Ranks} \leq \text{Rank}.$$  

Any tensor $T \in A^{(1)} \otimes \cdots \otimes A^{(n)} := C^{m_1} \otimes \cdots \otimes C^{m_n}$ can be regarded as a multilinear function $T : A^{(1)*} \times \cdots \times A^{(n)*} \to C$. Its geometric rank is defined to be

$$\text{GR}(T) := \text{codim} \left\{ (x_1, \ldots, x_{n-1}) \in A^{(1)*} \times \cdots \times A^{(n-1)*} \mid T(x_1, \ldots, x_{n-1}, x_n) = 0 \right\}.$$  

A tripartite tensor $T \in A \otimes B \otimes C := \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ can be regarded as a linear map $T_A : A^* \to B \otimes C$. Omitting the subscripts when there is no ambiguity, $T(A^*) \subset B \otimes C$ is an $a$-dimensional space of $b \times c$ matrices. Fixing bases $\{a_i\}_{i=1}^a, \{b_j\}_{j=1}^b$ and $\{c_k\}_{k=1}^c$ of $A, B$ and $C$, and the dual basis $\{a_i^*\}_{i=1}^a$ of $A^*$ corresponding to $\{a_i\}_{i=1}^a$, we often represent $T(A^*)$ by a general point $T(\sum x_i a_i)$ of $T(A^*)$ in a matrix form. That is, $T(A^*)$ will be written as a $b \times c$ matrix whose entries are linear forms in variables $x_i$’s.

Let $A_i^* := \{ \alpha \in A^* \mid \text{rank} (T(\alpha)) \leq i \}$, and $B_i^*$ and $C_i^*$ be defined similarly. An alternative definition, proved to be equivalent to the previous one in [13, Theorem 3.1], is

$$\text{GR}(T) = \min \{ \text{codim} (A_i^*) + i \} = \min \{ \text{codim} (B_i^*) + i \} = \min \{ \text{codim} (C_i^*) + i \} \quad (A)$$  

which shows close relations of geometric rank with spaces of matrices of bounded rank and more generally determinantal varieties.

Given $r \leq \min \{a, b, c\}$, let $\mathcal{GR}_r$ be the set of tensors with geometric rank at most $r$. As $\mathcal{GR}_r$ is Zariski closed [13], our goal is to give geometric interpretations of those varieties, and classify the tensors in $\mathcal{GR}_r$ up to change of bases and permutations of $A, B$ and $C$ if possible.

### 1.2 Determinantal variety

For a linear space of matrices $E \subset A \otimes B := \mathbb{C}^a \otimes \mathbb{C}^b$, let $E_r$ be the locus of matrices of rank at most $r$, for $r \leq \min \{a, b\}$. In other words, $E_r \subset \mathbb{P} E \cap \sigma_r(\text{Seg}(\mathbb{P} A \times \mathbb{P} B)))$, the intersection of $\mathbb{P} E$ with the $r$-th secant variety of the Segre variety. $E_r$ is cut out
A linear space of matrices

1.3 Space of matrices of bounded rank

A linear space of matrices $E \subset A \otimes B := \mathbb{C}^a \otimes \mathbb{C}^b$ is said to have bounded rank $r$ if all matrices in $E$ have rank at most $r$, i.e., $E_r = E$. There are two important classes of spaces of bounded rank — primitive spaces [4] and compression spaces [9]. $E$ is a compression space if there exist $A' \subset A$ and $B' \subset B$ of dimension $p$ and $q$ such that $E \subset A' \otimes B + A \otimes B'$ and $p + q = r$. $E$ is a primitive space if for any subspaces $A' \subset A$ or $B' \subset B$ of codimension 1, $E \not\subset A' \otimes B$ or $A \otimes B'$, and neither $E \cap (A' \otimes B)$ nor $E \cap (A \otimes B')$ has bounded rank $r - 1$.

Atkinson and Lloyd showed that every space of bounded rank $r$ that is not a compression space equals to a “sum” of a compression space of bounded rank $i$ and a primitive space of bounded rank $r - i$ for some $i$ in [4]. Later all primitive spaces of bounded rank 2 and 3 were classified in [3]. [9] recasted the study with sheaves and gave geometric interpretations of all primitive spaces of bounded rank 3 as matrices.

The alternative definition (A) shows $\text{GR}(T) \leq r$ if at least one of $T(A^*)$, $T(B^*)$ and $T(C^*)$ has bounded rank $r$. In fact, when $r = 1$ and 2 this condition is necessary [10]. But it fails to be necessary when $r = 3$ as there are two exceptions (see Theorem 4.1).

1.4 Matrix multiplication tensor

In the study of arithmetic complexity of matrix multiplication, Strassen found that the number of additions and multiplications required to multiply two matrices asymptotically is determined by the rank of matrix multiplication tensors [19].

For positive integers $e \leq h \leq l$, put $A = \mathbb{C}^{e \times h}$, $B = \mathbb{C}^{h \times l}$ and $C = \mathbb{C}^{l \times e}$. Then the matrix multiplication tensor $M_{(e,h,l)}$ is defined by $M_{(e,h,l)}(x, y, z) = \text{Tr}(xyz)$ for $x \in A^*$, $y \in B^*$ and $z \in C^*$. We often write $M_{(n)} := M_{(n,n,n)}$. With proper choice of bases, $M_{(e,h,l)}$ may be written in the block form

$$M_{(e,h,l)}(A^*) = \begin{pmatrix} D & & \\ & \ddots & \\ & & D \end{pmatrix}$$  

(B)
where $D$ is an $e \times h$ block consisting of linearly independent entries and there are $l$ copies of $D$ in $M_{(e,h,l)}(A^*)$.

Strassen gave a lower bound of the border subrank of $M_{(e,h,l)}$, which is $eh - \lfloor (e + h - l)^2/4 \rfloor$ if $e + h \geq l$ and $eh$ otherwise [19]. Recently [13] surprisingly found that the above lower bound equals to the geometric rank of $M_{(e,h,l)}$, and consequently equals to the border subrank of $M_{(e,h,l)}$ since geometric rank upper bounds border subrank.

1.5 Main results

For $T \in A \otimes B \otimes C = \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, the multilinear ranks are $ml_A(T) := \text{rank}(T_A)$, $ml_B(T) := \text{rank}(T_B)$ and $ml_C(T) := \text{rank}(T_C)$, and the slice rank is $\text{SR}(T) := \min\{ml_A(T_1) + ml_B(T_2) + ml_C(T_3) \mid T = T_1 + T_2 + T_3\}$.

**Definition 1.1** $T$ is a compression space of geometric rank $r$ if $\text{GR}(T) = \text{SR}(T) = r$. $T$ is a primitive space of geometric rank $r$ if it cannot be written as $T = X + Y$ with $\text{GR}(X) = r - 1$ and $\text{GR}(Y) = 1$.

For tripartite tensors $T \in A \otimes B \otimes C$, our main results are:

- **Theorem 4.1.** A tensor $T \in A \otimes B \otimes C$ has geometric rank at most 3 if and only if one of the following conditions holds:
  
  (a) $T(A^*)$, $T(B^*)$ or $T(C^*)$ is of bounded rank 3, or
  (b) $\text{SR}(T) \leq 3$, or
  (c) up to change of bases $T = M_{(2)}$.

  If $T$ is a primitive space of geometric rank 3, then up to change of bases and permutations of $A$, $B$ and $C$, it is either the matrix multiplication tensor $M_{(2)}$ or the tensor such that $T(A^*)$ is a space of $4 \times 4$ skew-symmetric matrices of dimension 4, 5 or 6.

- **Theorem 5.1.** If $T$ is a primitive space of geometric rank 4, then either at least two of $ml_A(T)$, $ml_B(T)$ and $ml_C(T)$ are at most 6, or all of them are at most 8.

- **Theorem 5.3.** For all $r$, there exists a positive integer $N_r$ such that if $T$ is a primitive space of geometric rank $r$, then at least two of $ml_A(T)$, $ml_B(T)$ and $ml_C(T)$ are at most $N_r$.

For $n \geq 3$ and $n$-part tensors $T \in A^{(1)} \otimes \cdots \otimes A^{(n)}$, we have:

- **Proposition 6.2.** For $r < n$, $\text{GR}(T) \leq r$ if and only if there exists $i$ such that $T(A^{(i)*})$ has bounded geometric rank $r$ as a space of $(n - 1)$-part tensors.

- **Proposition 6.3.** $T$ has geometric rank 1 if and only if it has partition rank 1.

Although we assume all tensors are defined over a complex field, all results from this paper hold for any algebraically closed field with characteristic zero.

1.6 Overview

We begin with discussion of primitive and compression tensors in Sect. 2. Lemma 2.1 gives a criterion to determine if a tensor is primitive, and Corollary 2.2 shows the
matrix multiplication tensors are either primitive or compression. Lemma 2.3 shows any tensor with degenerate geometric rank can be decomposed as a sum of a primitive tensor and a compression tensor.

In Sect. 3 we study the subspaces \( E \subset A \otimes B \) whose determinantal varieties \( E_k \) have bounded codimensions, especially finding the upper bounds of the dimensions of \( A \) and \( B \) when \( E \) is concise. Proposition 3.6 gives the classification of spaces whose 3 \times 3 minors all have a common quadratic factor. Proposition 3.8 proves the existence of the upper bounds on the dimensions of \( A \) and \( B \) in general. In our study of determinantal varieties, we observed an error in [5, Proposition 1], see Remark 3.7 for details.

Using the results on linear determinantal varieties, in Sects. 4 and 5 we complete the classification of tensors in \( \mathcal{GR}_3 \) in Theorem 4.1, find upper bounds of multilinear ranks of primitive tensors with geometric rank 4 in Theorem 5.1, and obtain the existence of such upper bounds for tensors with bounded geometric rank in general.

In Sect. 6 we shift our study from tripartite tensors to \( n \)-part tensors. Proposition 6.1 generalizes the alternative definition (A). Proposition 6.2 shows that \( N \)-part tensors with small geometric ranks always correspond to spaces of \((N - 1)\)-part tensors of bounded geometric ranks. Finally we show the equivalence between partition rank 1 and geometric rank 1 in Proposition 6.3.

### 2 Primitive and compression tensors

The following lemma gives a direct way to determine whether a tensor is primitive in general.

**Lemma 2.1** Given \( T \) with \( 1 < \text{GR}(T) = r < \text{SR}(T) \), then \( T \) is not primitive if and only if there exists \( i < r \) such that by a permutation of \( A, B \) and \( C \), \( \text{codim}(A^*_i) = r - i \) and \( A^*_i \) has a component of maximal dimension that is contained in a hyperplane of \( A^* \).

**Proof** Let \( \{a_i\}_{i=1}^a \) be a basis of \( A \), and \( \{a_i\}_{i=1}^a \) be the dual basis of \( A^* \). Write \( A' := \langle a_2, \ldots, a_a \rangle \), so \( A'^* = \langle a_2, \ldots, a_a \rangle \).

(\( \Rightarrow \)) \( T \) is not primitive if and only if we can decompose \( T = X + Y \) with \( \text{GR}(X) = r - 1 \) and \( \text{GR}(Y) = 1 \). Since \( \text{GR}(Y) = 1 \) if and only if \( \text{SR}(Y) = 1 \), by permuting \( A, B \) and \( C \) assume \( \text{ml}(A)(Y) = 1 \), and by changing basis of \( A \) assume \( Y \in \langle a_1 \rangle \otimes B \otimes C \).

Then \( T = X' + Y' \) where \( X' = T|_{A \otimes B \otimes C} \) and \( Y' = T|_{\langle a_1 \rangle \otimes B \otimes C} \). Since \( X' = X|_{A \otimes B \otimes C} \), \( \text{GR}(X') \leq \text{GR}(X) = r - 1 \). By subadditivity of geometric rank and \( \text{SR}(Y') = \text{GR}(Y') = 1 \), \( \text{GR}(X') = r - 1 \). By (A) there exists \( i \leq r - 1 \) such that \( \text{codim}(\alpha \in A^* | \text{rank}(X'(\alpha)) \leq i) \leq r - 1 - i \). Then \( \{\alpha \in A^* | \text{rank}(X'\alpha) \leq i\} \cap A'^*_i \) has codimension \( r - i \) in \( A^* \) and is contained in a hyperplane.

(\( \Leftarrow \)) Assume \( \text{codim}(A^*_i) = r - i \) and \( A^*_i \) has a component \( Z \) of maximal dimension contained in \( A'^* \). Let \( X' \) and \( Y' \) be defined the same as above. By definition \( \{\alpha \in A'^* | \text{rank}(X'(\alpha)) \leq i\} \supseteq Z \), so it has codimension at most \( r - i \) in \( A^* \), then its codimension is at most \( r - 1 - i \) in \( A'^* \). Since \( X' \in A' \otimes B \otimes C \), \( \text{GR}(X') \leq r - 1 \). By \( T = X' + Y' \) and subadditivity of geometric rank, \( \text{GR}(X') = r - 1 \) and \( \text{GR}(Y') = 1 \). \( \square \)
Corollary 2.2 For positive integers \( e \leq h \leq l \), \( M_{(e,h,l)} \) is primitive if \( e \geq 2 \) and \( e + h \geq l \), and it is compression otherwise.

**Proof** By [13, Theorem 6.1], \( GR(M_{(e,h,l)}) = eh \) if \( e + h \leq l \) or \( e = 1 \). Since \( GR(M_{(e,h,l)}) \leq SR(M_{(e,h,l)}) \leq mlA(M_{(e,h,l)}) = eh \), we have \( GR(M_{(e,h,l)}) = SR(M_{(e,h,l)}) = eh \) and therefore \( M_{(e,h,l)} \) is compression.

Assume \( e \geq 2 \) and \( e + h \geq l \). The component of the maximal dimension \( Z \subset A_i \) is determined by all \( k \times k \) minors of \( D \), where \( k = \min\{e, \lceil \frac{i+1}{2} \rceil \} \). By [8, Theorem 2.1], codim \((A_i) = \) codim \((Z) = (e + 1 - k)(h + 1 - k) \). So \((A)\) achieves minimum only at \( i = \lceil \frac{e + h - l}{2} \rceil \) and \( \lceil \frac{e + h - l}{2} \rceil \). Then \( k > 1 \) and \( Z \) is not contained in any hyperplane. \( \square \)

Although we define primitive and compression tensors as analogues of primitive and compression spaces of matrices, their relations are subtle.

By definition \( T \) is compression of \( GR(T) = r \) if at least one of \( T(A^*) \), \( T(B^*) \) or \( T(C^*) \) is a compression space of bounded rank \( r \) and none has bounded rank \( r - 1 \). The converse is true only for \( r \leq 2 \), as \( T := \sum_{i=1}^{m} (a_i \otimes b_i \otimes c_i + a_i \otimes b_i \otimes c_i + a_i \otimes b_i \otimes c_1) \) is compression of \( GR(T) = 3 \), but \( T(A^*) \), \( T(B^*) \) and \( T(C^*) \) contain elements of full rank.

If \( T \) is primitive of \( GR(T) = r \) and \( T(A^*) \) has bounded rank \( r \), then \( T(A^*) \) is primitive of bounded rank \( r \) (after deleting zero rows and columns). Similarly for \( T(B^*) \) and \( T(C^*) \). However \( T \) could be primitive when \( T(A^*) \), \( T(B^*) \) and \( T(C^*) \) do not have bounded rank \( r \).

For example, \( M_{(2)} \) is primitive of geometric rank 3 by Corollary 2.2. But since \( M_{(2)}(A^*) \) can be written in the block diagonal form \((B)\), generic matrices in \( M_{(2)}(A^*) \) have full rank 4. Therefore \( M_{(2)}(A^*) \) does not have bounded rank 3. For the same reason, \( M_{(2)}(B^*) \) and \( M_{(2)}(C^*) \) do not either.

There is no primitive space of bounded rank 1, and all primitive spaces of bounded rank 2 and 3 are listed in [3, 9]. We check every such primitive space and conclude that for \( r \leq 3 \), if \( T(A^*) \) is primitive of bounded rank \( r \), then \( T \) is primitive of geometric rank \( r \). It is not known if this property persists when \( r > 3 \), because all primitive spaces of larger bounded rank are not classified yet.

**Lemma 2.3** If \( T \) is not compression (i.e., \( GR(T) < SR(T) \)), then there exist a primitive tensor \( T_p \) and a compression tensor \( T_c \) such that \( T = T_p + T_c \) and \( GR(T_p) + GR(T_c) = GR(T) \).

**Proof** If \( T \) is primitive, set \( T_p = T \) and \( T_c = 0 \).

If \( T \) is not primitive, assume \( GR(T) = r \), then we can write \( T = X_1 + Y_1 \) so that \( GR(X_1) = r - 1 \) and \( GR(Y_1) = 1 \). Similarly, whenever \( X_i \) is not primitive or zero, we can write \( X_i = X_{i+1} + Y_{i+1} \) so that \( GR(X_i) = r - i \) and \( GR(Y_1) = 1 \). If all \( X_i \)'s obtained this way are not primitive, we have a decomposition \( T = Y_1 + \cdots + Y_r \) where each \( Y_i \) has geometric rank 1 so has slice rank 1. This implies \( SR(T) = r = GR(T) \), contradicting the assumption \( GR(T) < SR(T) \).

So there exists \( n < r \) such that \( X_n \) is primitive, then we obtain \( T = T_p + T_c \) where \( T_p := X_n \) and \( T_c := Y_1 + \cdots + Y_n \). Since \( GR(T_p) = r - n \) and \( \sum GR(Y_i) = \sum SR(Y_i) = n \), by subadditivity of geometric rank and slice rank, \( GR(T_c) = SR(T_c) = n \). Therefore \( T_c \) is compression. \( \square \)
Example 2.4 (The above decomposition is not unique) Let $T \in A \otimes B \otimes C = \mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^6$ be defined as

$$
T := a_1 \otimes (b_2 \otimes c_1 + b_3 \otimes c_2 + b_4 \otimes c_3) \\
+ a_2 \otimes (b_1 \otimes c_1 - b_3 \otimes c_4 - b_4 \otimes c_5) \\
+ a_3 \otimes (b_1 \otimes c_2 + b_2 \otimes c_4 - b_4 \otimes c_6) \\
+ a_4 \otimes (b_1 \otimes c_3 + b_2 \otimes c_5 + b_3 \otimes c_6) + a_5 \otimes b_5 \otimes c_6
$$

where $\{a_i\}_{i=1}^5$, $\{b_j\}_{j=1}^5$ and $\{c_k\}_{k=1}^6$ are bases of $A$, $B$ and $C$ respectively. So

$$
T(A^*) = \begin{pmatrix}
x_2 & x_3 & x_4 & 0 & 0 & 0 \\
x_1 & 0 & x_3 & x_4 & 0 & 0 \\
0 & x_1 & 0 & -x_2 & 0 & x_4 \\
0 & 0 & x_1 & 0 & -x_2 & -x_3 \\
0 & 0 & 0 & 0 & 0 & x_5
\end{pmatrix}.
$$

Let $X_1 := T|_{A \otimes B \otimes (c_1,\ldots,c_5)}$, $Y_1 := T|_{A \otimes B \otimes (c_6)}$, $X_2 := T|_{A \otimes (b_1,\ldots,b_4) \otimes C}$ and $Y_2 := T|_{A \otimes (b_5) \otimes C}$. Since $X_1(A^*)$ consists of the first five columns of $T(A^*)$ and $X_2(A^*)$ consists of the first four rows of $T(A^*)$, they are primitive spaces of bounded rank 3 (after deleting the zero columns and rows). So $X_1$ and $X_2$ are primitive of geometric rank 3, and $T = X_1 + Y_1 = X_2 + Y_2$ gives two different decompositions satisfying the conditions in Lemma 2.3.

By Lemma 2.3, to classify the set of tensors of geometric rank at most $r$, it suffices to find all primitive tensors of geometric rank at most $r$. In terms of these notations, the classification of tensors of geometric rank at most 1 and 2 from \[10, \text{Remark 2.6, Theorem 3.1}\] can be rephrased as:

- There are no primitive tensors of geometric rank 1.
- The only primitive tensor of geometric rank 2 is (up to change of bases) the skew-symmetric $3 \times 3 \times 3$ tensor.

3 Determinantal varieties of bounded codimensions

Let $E \subset \mathbb{C}^a \otimes \mathbb{C}^b =: A \otimes B$ be a linear subspace of dimension $c$. Fix a basis $\{e_i, 1 \leq i \leq c\}$ of $E$ and bases of $A$ and $B$, then each $e_i$ can be written as an $a \times b$ matrix. Similarly to how we represent $T(A^*) \subset B \otimes C$ in Sect. 1.1, $E$ is represented by the matrix corresponding to a general point $\sum_i x_ie_i$ of $E$, i.e., $E = (y^j_i)_{1 \leq i \leq a, 1 \leq j \leq b}$, where each $y^j_i$ is a linear form in the variables $x_1, \ldots, x_c$. For two subspaces $F$, $F' \subset E$, let $F + F'$ denote the sum of the two corresponding matrices of linear forms.

Denote the $(i_1, \ldots, i_k) \times (j_1, \ldots, j_k)$ minor of $E$ as $\Delta^{i_1,\ldots,i_k}_{j_1,\ldots,j_k}$ and $\Delta_k := \Delta^{12,\ldots,k}_{12,\ldots,k}$. Unless otherwise stated, the codimension of a subset always refers to the codimension in $E$ or $\mathbb{P}E$. 

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3.1 Case codim \((E_r) = 1\)

This subsection studies the case codim \((E_r) = 1\), i.e. all nonzero \((r + 1) \times (r + 1)\) minors of \(E\) have a common polynomial factor of degree at least 1.

Lemma 3.1 is a more detailed version of \([10, \text{Lemma 6.4}]\), and Lemma 3.2 generalizes \([10, \text{Lemma 6.5}]\).

**Lemma 3.1** Let \(E \subset \mathbb{C}^a \otimes \mathbb{C}^b\), \(r < a, b\) and \(E_r \neq E\). If there exists a degree \(r + 1\) polynomial \(P\) dividing all \((r + 1) \times (r + 1)\) minors of \(E\), then either \(P\) factors into a product of linear forms, or \(E \subset \mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}\).

**Proof** The hypothesis that all \((r + 1) \times (r + 1)\) minors of \(E\) are equal up to scale is invariant under change of bases in \(A\) and \(B\), so we are allowed to perform invertible row and column operations.

Since \(E_r \neq E\), there exists a nonzero \((r + 1) \times (r + 1)\) minor of \(E\). By change of bases we can assume \(\Delta_{r+1} = P\). We further assume \(\Delta_r, \ldots, \Delta_2, y_1^1\) are nonzero.

Write \(E = (y_{ij}^1)_{1 \leq i \leq a, 1 \leq j \leq b}\). Consider the block consisting of the first \(r + 1\) rows and the first \(r + 2\) columns:

\[
\begin{pmatrix}
y_1^1 & \cdots & y_{r+1}^1 & y_{r+2}^1 \\
\vdots & & \vdots & \vdots \\
y_{r+1}^1 & \cdots & y_{r+1}^r & y_{r+1}^{r+1}
\end{pmatrix}.
\]

Let \(I := (1, 2, \ldots, r + 1)\). For \(j \leq r + 1\), expand the minor consisting of all columns except the \(j\)-th along the last column, then we have

\[
c_j P = Δ_{I \setminus j, r+2}^{I, j, r+2} = \sum_{i=1}^{r+1} (-1)^{i+(r+2)-1} y_{r+2} \Delta_{I \setminus j}^{I, i}
\]

for some \(c_j \in \mathbb{C}\). Thus,

\[
\begin{pmatrix}
c_1 \\
\vdots \\
c_{r+1}
\end{pmatrix} P = (-1)^{r+1} \left((-1)^i \Delta_{I \setminus j}^{I, i}\right)_{i=1}^{r+1} \begin{pmatrix}
y_{r+2}^1 \\
\vdots \\
y_{r+2}^{r+1}
\end{pmatrix}.
\]

For every \(j \leq r + 1\), multiply \((-1)^j\) to the \(j\)-th row,

\[
(-1)^{r+1} \begin{pmatrix}
(-1)^1 c_1 \\
\vdots \\
(-1)^1 c_{r+1}
\end{pmatrix} P = \left((-1)^{i+j} \Delta_{I \setminus j}^{I, i}\right)_{i=1}^{r+1} \begin{pmatrix}
y_{r+2}^1 \\
\vdots \\
y_{r+2}^{r+1}
\end{pmatrix}.
\]

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Now \((-i+j+\Delta i, 1)_{i,j=1}^{r+1}\) is the cofactor matrix of the transpose of \((y_j^1)^{r+1}\), whose determinant is \(\Delta r+1 = P\) by assumption. So

\[
(-1)^{r+1} \begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 \\ \vdots & \ddots & \vdots \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1} \end{pmatrix} \begin{pmatrix} -c_1 \\ \vdots \\ (-1)^{r+1}c_{r+1} \end{pmatrix} = \begin{pmatrix} y_1^{r+2} \\ \vdots \\ y_{r+1}^{r+2} \end{pmatrix}.
\]

Therefore the column vector \((y_1^1, \ldots, y_{r+1}^{r+1})^T\) is a linear combination of all column vectors appearing in the upper left \((r+1) \times (r+1)\) block of \(E\), i.e. \((y_1^1, \ldots, y_{r+1}^{r+1})^T\), 1 \(\leq j \leq r+1\). By adding linear combinations of the first \(r+1\) columns to the \((r+2)\)-th, we may make the first \(r+1\) entries of the \((r+2)\)-th column equal to zero. Similarly, we may make the all last \(b-r-1\) entries in the first \(r+1\) rows equal to zero. By the same argument, we may do the same for the first \(r+1\) columns. Then the matrix \(E\) becomes

\[
E' = \begin{pmatrix} y_1^1 & \cdots & y_{r+1}^1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_1^{r+1} & \cdots & y_{r+1}^{r+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{y}_{r+1}^{r+2} & \cdots & \tilde{y}_b^{r+2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{y}_{r+2}^a & \cdots & \tilde{y}_b^a \end{pmatrix}.
\]

If \(\tilde{y}_{r+1+1}^{r+1+i} = 0\), for all \(i, j > 0\), let \(A'\) be the space corresponding to the first \(r+1\) rows of \(E'\) and \(B\) the first \(r+1\) columns, then \(E \subset A' \otimes B' = \mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}\).

If there exists a nonzero \(\tilde{y}_{r+1+1}^{r+1+i}\), by change of bases assume it is \(\tilde{y}_{r+2}^{r+2}\). For \(1 \leq i_1 < \cdots < i_r \leq r+1\), let \(E_{i_1,\ldots,i_r} = E_{i_1,\ldots,i_r} = \Delta_{i_1,\ldots,i_r}^{r+2}\) is a multiple of \(\Delta_{r+1}\). Hence all \(r \times r\) minors of the upper left \((r+1) \times (r+1)\) block equal up to scale.

By assumption \(\Delta_r \neq 0\). Adding a linear combination of the first \(r+1\) columns to the \((r+1)\)-th column and a linear combination of the first \(r\) rows to the \((r+1)\)-th row, we can set all entries in \((r+1)\)-th column and row zero except the \((r+1, r+1)\)-th entry. Since \(\Delta_{r+1} \neq 0\), the \((r+1, r+1)\)-th entry is nonzero, written as \(\tilde{y}_{r+1}^{r+1}\). Then \(E'\) becomes

\[
E'' = \begin{pmatrix} y_1^1 & \cdots & y_r^1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_1^{r} & \cdots & y_r^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tilde{y}_{r+1}^{r+1} & \cdots & \tilde{y}_{b}^{r+1} \\ 0 & \cdots & 0 & \tilde{y}_{r+2}^a & \cdots & \tilde{y}_b^a \end{pmatrix}.
\]
Repeat the above process on the upper left $k \times k$ blocks consecutively for $k = r - 1, r - 2, \ldots, 2$, then $E''$ becomes

$$
\begin{pmatrix}
\tilde{y}_1^1 & \tilde{y}_2^2 & \cdots & \tilde{y}_{r+1}^{r+1} \\
\vdots & \ddots & \ddots & \vdots \\
\tilde{y}_{r+2}^r & \cdots & \tilde{y}_{b}^b \\
\vdots & \vdots & \ddots & \\
\tilde{y}_{r+2}^r & \cdots & \tilde{y}_{b}^b
\end{pmatrix}.
$$

Therefore $\Delta_{r+1} = y_1^1 \tilde{y}_2^2 \cdots \tilde{y}_{r+1}^{r+1}$ which factors into a product of linear forms. \qed

**Lemma 3.2** Let $E \subset \mathbb{C}^a \otimes \mathbb{C}^b$, $1 \leq r \leq \min\{a, b\} - 2$ and $E \neq E_{r+1}$. If there exists a polynomial $P$ of degree $k$ dividing all $(r + 1) \times (r + 1)$ minors, then:

1. if $k > r/2 + 1$ and for any nonzero $(r + 1) \times (r + 1)$ minor $\Delta$, $P$ and $\Delta/P$ are coprime, then $P$ is a product of linear forms;
2. if $r$ is even, $k = r/2 + 1$ and for any nonzero $(r + 1) \times (r + 1)$ minor $\Delta$, $P$ and $\Delta/P$ are coprime, then either $P$ is a product of linear forms or $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+2};$
3. if $r \geq 3$ is odd, $k = (r + 1)/2$ and $P$ is irreducible, then either $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{b}, \mathbb{C}^a \otimes \mathbb{C}^{r+2}, \mathbb{C}^{r+3} \otimes \mathbb{C}^{r+3}$, or up to change of bases $E$ has a nonsingular $(r + 1) \times (r + 1)$ block such that all $r \times r$ minors of it are multiples of $P$.

**Proof** (1) and (2): Proof by induction on $r$. The base case $r = 1$ is trivial. Assume $r > 1$ and assume that (1) and (2) hold for all integers smaller than $r$.

Given any nonzero $(r + 2) \times (r + 2)$ minor of $E$, by change of bases we can assume it is $\Delta_{r+2}$, and we further assume $\Delta_{r+1} \neq 0$.

Write $\Delta_{r+1} =: PQ$ and for $j \leq r + 1$, $\Delta_{i,j,r+2}^I =: PQ_j$, where each of the polynomials $Q$ and $Q_j$’s either is zero or has degree $r + 1 - k$. Then similarly to Lemma 3.1, we have

$$
(-1)^{r+1} \begin{pmatrix} -Q_1 \\ \vdots \\ (-1)^{r+1} Q_{r+1} \end{pmatrix} P = \begin{pmatrix} (-1)^i + j \Delta_{i,j}^I \end{pmatrix}_{i=1}^{r+1} \begin{pmatrix} y_1^{r+2} \\ \vdots \\ y_{r+2}^{r+2} \end{pmatrix}.
$$

Using the cofactor matrix, we obtain

$$
\frac{(-1)^{r+1}}{Q} \begin{pmatrix} y_1^1 & \cdots & y_1^{r+1} \\ \vdots & \ddots & \vdots \\ y_1^{r+1} & \cdots & y_1^{r+1} \end{pmatrix} \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1^{r+2} \\ \vdots \\ y_{r+2}^{r+2} \end{pmatrix}.
$$

By adding a rational combination (where the coefficients are $(-1)^i Q_j/Q$’s) of the first $r + 1$ columns to the $(r + 2)$-th column, we can put the first $r + 1$ entries of the
$(r+2)$-th column zero. By the same argument, put the first $r+1$ entries of the last $b-r-1$ columns zero. We can do the similar rational row operations to eliminate first $r+1$ entries of the last $a-r-1$ rows. Then $E$ becomes $E'$ of the form $(D)$.

Since the $(1, \ldots, r+1, r+2) \times (1, \ldots, r+1, r+2)$ minor is not changed by adding rational multiples of the first $r+1$ rows and columns to the $(r+2)$-th row and $(r+2)$-th column respectively, $\tilde{y}^{r+2}_{r+2} = \frac{\Delta_{r+2}}{\Delta_{r+1}}$. On the other hand, $\tilde{y}^{r+2}_{r+2}$ has the form $T/Q$ for some polynomial $T$ of degree $k+1$ if not zero, because all coefficients appearing in the row and column operations above are $(-1)^j Q_j / Q$'s. Thus,

$$\frac{T}{Q} = \frac{\Delta_{r+2}}{\Delta_{r+1}} = \frac{\Delta_{r+2}}{PQ} = \frac{(\Delta_{r+2}/P)}{Q}$$

and $T = \Delta_{r+2}/P$.

Since $P$ and $Q$ are coprime, the fact $P$ divides all $(r+1) \times (r+1)$ minors is preserved after performing the above rational row and column operations.

If there exists an $r \times r$ minor of the upper left $(r+1) \times (r+1)$ block that is not a multiple of $P$, by change of bases assume this minor is $\Delta_r$. $P$ divides the minor $\Delta_{1 \ldots r, r+2} = \tilde{y}^{r+2}_{r+2} \Delta_r = T \Delta_r / Q$, so $T$ is a multiple of $P$. Hence $P^2$ divides $\Delta_{r+2} = T P$. If $k > r/2 + 1$, $P^2$ has degree $> r+2$, then we must have $\Delta_{r+2} = 0$, contradicting to the assumption $\Delta_{r+2} \neq 0$. If $r$ is even and $k = r/2 + 1$, $\Delta_{r+2}$ is a multiple of $P^2$. By the arbitrariness of the choice of the nonzero $(r+2) \times (r+2)$ minor of $E$, all $(r+2) \times (r+2)$ minors equal to $P^2$ up to scale. By Lemma 3.1, $P$ factors completely or $E \subset \mathcal{O}^{r+2} \otimes \mathcal{O}^{r+2}$.

If all $r \times r$ minors of the upper left $(r+1) \times (r+1)$ block are multiples of $P$. By induction, apply (1) by replacing $r$ with $r-1$ so $P$ factors into a product of linear forms.

(3): Similarly to the above let $\Delta_{r+1} = P Q$ and $\Delta_{r+2}$ be nonzero. Since $P$ is irreducible of degree $k = (r+1)/2$, either $P$ and $Q$ are coprime, or $Q$ equals to $P$ up to scale. In the latter case, we can choose another nonzero $(r+1) \times (r+1)$ minor from the top left $(r+2) \times (r+2)$ block such that $P$ and $Q$ are coprime, unless all $(r+1) \times (r+1)$ minors in the top left $(r+2) \times (r+2)$ block are multiples of $P^2$.

If all $(r+1) \times (r+1)$ minors in the top left $(r+2) \times (r+2)$ block are multiples of $P^2$, applying Lemma 3.1 to the top left $(r+2) \times (r+2)$ block we can put $E$ as

$$E' = \begin{pmatrix}
\begin{array}{cccc}
 y_1^1 & \cdots & y_{r+1}^1 & 0 & y_1^{r+1} & \cdots \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
 y_1^{r+1} & \cdots & y_{r+1}^{r+1} & 0 & y_{r+1}^{r+1} & \cdots \\
 0 & \cdots & 0 & y_{r+2}^{r+3} & y_{r+2}^{r+3} & \cdots \\
 y_1^{r+3} & \cdots & y_{r+1}^{r+3} & y_{r+2}^{r+3} & y_{r+3}^{r+3} & \cdots \\
 \vdots & \ddots & \vdots & \vdots & \vdots & \ddots 
\end{array}
\end{pmatrix}
$$

Consider the $(r+1) \times (r+1)$ minors involving $y_{r+2}^{r+2}$ and $r \times r$ minors of the upper left $(r+1) \times (r+1)$ block: $P$ dividing all $(r+1) \times (r+1)$ minors implies $P$ dividing all $r \times r$ minors from the first $r+1$ rows. Apply (2) by replacing $r$ with $r-1$ to the
submatrix consisting of the first $r + 1$ rows. Since $P$ is irreducible of degree $k > 1$, this submatrix is in some $\mathbb{C}^{r+1} \otimes \mathbb{C}^{r+1}$, we can put $y_{ij}$ zero for $i \leq r + 1$ and $j \geq r + 3$ by changing basis of $A$. For the same reason all $y_{ij}$ for $i \leq r + 1$ and $j \geq r + 3$ can be put zero too. Then $E'$ becomes $E'' = \text{diag}(B_1, B_2)$ where $B_1$ is an $(r + 1) \times (r + 1)$ block. If $B_2$ has a nonzero $2 \times 2$ minor, consider the $(r + 1) \times (r + 1)$ minors consisting of it and any $(r - 1) \times (r - 1)$ minor of $B_1$, applying (1) replacing $r$ with $r - 2$ we see $P$ factors into linear forms which contradicts the irreducibility. Therefore $B_2$ has bounded rank 1, then $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^b$ or $\mathbb{C}^a \otimes \mathbb{C}^{r+2}$.

Now assume $P$ and $Q$ are coprime. Similarly to the proof above, if there exists an $r \times r$ minor of the upper left $(r + 1) \times (r + 1)$ block that is not a multiple of $P$, $P^2$ divides $\Delta_{r+2}$. By the arbitrariness of choice of nonzero $(r + 2) \times (r + 2)$ minors, $P^2$ divides all $(r + 2) \times (r + 2)$ minors. As $k = (r + 1)/2$ and $P$ is irreducible, $P^2$ divides all $(r + 2) \times (r + 2)$ minors and we can apply (1) by replacing $r$ with $r + 1$, then we conclude $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+3}$.

Otherwise all $r \times r$ minors of the upper left $(r + 1) \times (r + 1)$ block are multiples of $P$.

\[ \square \]

**Corollary 3.3** Let $E \subset \mathbb{C}^a \otimes \mathbb{C}^b$, $1 \leq r \leq \min\{a, b\} - 2$, and $\text{codim}(E_r) = 1$ and $E \neq E_{r+1}$. Then:

1. $E_r$ does not contain any irreducible hypersurface of degree $k > r/2 + 1$;
2. if $r$ is even and $E_r$ contains an irreducible hypersurface of degree $r/2 + 1$, then $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+2}$.

**Proof** (1) If $E_r$ contains an irreducible hypersurface of degree $k$, there exists an irreducible polynomial $P$ of degree $k$ dividing all $(r + 1) \times (r + 1)$ minors. Then for any $(r + 1) \times (r + 1)$ minor $\Delta$, $\Delta/P$ has degree less than $k$ so must be coprime with $P$. By (1) of Lemma 3.2, $P$ factors, contradicting to the irreducibility.

(2) Similarly to the proof of (1) except that we apply Lemma 3.2 (2). As $P$ cannot be a product of linear forms due to irreducibility, we conclude $E \subset \mathbb{C}^{r+2} \otimes \mathbb{C}^{r+2}$. \[ \square \]

### 3.2 Case $\text{codim}(E_1) = n$

Let $E^\bot := \{ f \in A^* \otimes B^* \mid f(E) = 0 \}$. Define the *index of degeneracy* of $E$ to be one plus the maximum dimension of a linear space contained in $\mathbb{P}E^\bot \cap \text{Seg}(\mathbb{P}A^* \times \mathbb{P}B^*)$, denoted as $\kappa$. Equivalently, $\kappa$ is the largest number of entries in the same row or column of $E$ that can be simultaneously put to zero by changing bases of $A$ and $B$.

The subspace $E$ is called $\text{E1-generic}$ if $\kappa = 0$. We call this property E1-generic because it corresponds to the notion of 1-generic for spaces of matrices given by Eisenbud [8], which differs from the notion of 1-generic that is often used for tensors (cf. [14]). We list two results of E1-generic spaces of our interest below.

**Theorem 3.4** ([8, Corollary 3.3 and Theorem 2.1]) Let $m = \min\{a, b\}$. If $E \subset A \otimes B$ is E1-generic, then:

1. for $k \leq m - 1$, $\text{codim}(E_k) \geq a + b - 2k - 1$;
2. if $F \subset E$ is a subspace with $\text{codim}(F) \leq m - 1$, then $\text{codim}_F(F_{m-1}) = (a - m + 1)(b - m + 1)$.

\[ \square \]

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For generic determinantal varieties, i.e. when $E = A \otimes B$, one expects $E_k$ has codimension $(a - k)(b - k)$. E1-generic does not mean generic but implies the genericity to some extent — $E_{m-1}$ has the expected codimension, and the codimension of $E_k$ has a lower bound $a + b - 2k - 1$.

**Proposition 3.5** Let $n := \text{codim}(E_1)$, then there exist $0 \leq j \leq n$ and a linear subspace $F \subset E$ of codimension $j$ such that either $F \subset \mathbb{C}^k \otimes \mathbb{C}^l$ for some $k + l \leq n + 3 - j$ and $k, l \geq 2$, or $j = n$ and $F$ has bounded rank 1.

**Proof** First assume all nonzero $2 \times 2$ minors of $E$ are irreducible. So if there is an entry $y_{ij}^j = 0$, then either all entries in the $i$-th row or all entries in the $j$-th column are zero. By change of bases in $A$ and $B$, there exist integers $k, l \geq 2$ such that $y_{ij}^j \neq 0$ if and only if $i \leq k$, $j \leq l$.

$$E = \begin{pmatrix} y_{11}^1 & \cdots & y_{11}^l & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{k1}^1 & \cdots & y_{k1}^l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$  

Then the upper left $k \times l$ block of $E$ is E1-generic. By Theorem 3.4 if $k, l \geq 2$, $\text{codim}(E_1) \geq k + l - 3$, so $k + l \leq n + 3$.

If there is a $2 \times 2$ minor of $M$ that factors into the product of two linear forms $\ell_1, \ell_2$, write $F := \{ \ell_1 = 0 \}$ and $F' := \{ \ell_2 = 0 \}$, then $E_1 = F_1 \cup F'_1$. At least one of the two components has codimension $n$ in $E$. Say it is $F_1$, then $\text{codim}_F(F_1) \leq n - 1$.

Together with the irreducible case, we conclude that at least one of the following holds:

1. There exists a hyperplane $F \subset E$ such that $\text{codim}_F(F_1) = n - 1$;
2. $E \subset \mathbb{C}^k \otimes \mathbb{C}^l$ such that $k + l \leq n + 3$ and $k, l \geq 2$.

Using induction on $\dim(E)$, we complete the proof.

**3.3 Case codim$(E_2) = 1$**

If codim$(E_2) = 1$, then there must exist an irreducible polynomial $P$ of degree $k \leq 3$ dividing all $3 \times 3$ minors of $E$. If $k = 1$, then $E$ contains a hyperplane $\{ P = 0 \}$ which has bounded rank 2. If $k = 3$, by Lemma 3.1, $E \subset \mathbb{C}^3 \otimes \mathbb{C}^3$.

When $k = 2$, by Corollary 3.3 we have $E \subset \mathbb{C}^4 \otimes \mathbb{C}^4$, which suffices us to assume $E \subset A \otimes B$ with $\dim(A) = \dim(B) = 4$. The following proposition finds all such spaces up to change of bases in $E$, $A$ and $B$.

**Proposition 3.6** Let $E \subset A \otimes B := \mathbb{C}^4 \otimes \mathbb{C}^4$. If there exists an irreducible polynomial $S$ of degree 2 dividing all $3 \times 3$ minors of $E$, then at least one of the following holds:

1. $E$ has bounded rank 3;
(2) up to change of bases in $E$, $A$ and $B$, $E$ is either skew-symmetric, or has the form a diagonal block matrix diag$(X, X)$ where

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \text{ or } \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

depending on the rank of $S$.

We defer the proof to Sect. 7.

**Remark 3.7** In light of [5, Proposition 1] and [1, Corollary 6.8], one might hope to weaken the hypothesis to the zero set of $\det(E) = 0$ is a quadric hypersurface. However Proposition 1 is incorrect, as the following counter-example shows: let $E \subset \mathbb{C}^4 \otimes \mathbb{C}^4$ be defined as

$$E := \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ x_3 & x_4 & 0 & x_1 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & x_3 & x_4 \end{pmatrix}.$$  

Then $\det(E) = (x_1x_4 - x_2x_3)^2$ but the line bundle morphism $E : \mathbb{O}_{\mathbb{P}^3}(-1)^4 \to \mathbb{O}_{\mathbb{P}^3}^4$ fails to have constant rank on \{ $x_1x_4 - x_2x_3 = 0$ \}.

### 3.4 Case $\text{codim}(E_r) \leq n$

A subspace $E \subset A \otimes B$ is said to be **concise** if the associated tensor $T \in E^* \otimes A \otimes B$ is concise. Equivalently, there does not exist a change of bases in $A$ or $B$ such that any column or row of $E$ consists of only zero entries. This section studies upper bounds of $a$ and $b$ for concise spaces $E$ satisfying $\text{codim}(E_r) \leq n$.

**Proposition 3.8** For any positive integer $r$, $n$, there exist positive integers $M_1$, $M_2$ such that if there exists a concise space $E \subset A \otimes B := \mathbb{C}^a \otimes \mathbb{C}^b$ with $\text{codim}(E_r) \leq n$, then at least one of the following holds:

1. $a$ or $b \leq M_1$;
2. $a, b \leq M_2$;
3. there exists a hyperplane $F \subset E$ such that $\text{codim}_F(F_r) \leq n - 1$;
4. there exists $1 \leq i \leq r$ such that $E = H + H'$ where $\text{codim}(H'_{r-i}) \leq n$ and $H' \subset \mathbb{C}^i \otimes B$ or $A \otimes \mathbb{C}^i$.

**Proof** Proof by induction on $r$. For $r = 1$, by Proposition 3.5 we can set $M_1 = 1$ and $M_2 = n + 1$. For $r \geq 2$ we divide the problem into different cases by the value of $\kappa$.

1. Case $\kappa = 0$. $\kappa = 0$ if and only if $E$ is $E_1$-generic. Since $E_r \neq E$, $a, b \geq r + 1$. By Theorem 3.4, $a + b \leq n + 2r + 1$.

2. Case $\kappa = 1$. We can put $y_1^1 = 0$ by changing bases. Then the $(a - 1) \times (b - 1)$ submatrix consisting of entries in the last $a - 1$ rows and the last $b - 1$ columns is either $1$-generic, or has $\kappa = 1$ so we can put $y_2^2 = 0$. Repeat this procedure until the
bottom right \((a - k) \times (b - k)\) submatrix is 1-generic. Then \(E = \begin{pmatrix} C_{k \times k} & 0 \\ 0 & D_{t \times t} \end{pmatrix}\) where \(D\) is 1-generic, \(s = a - k, t = b - k\) and

\[
C = \begin{pmatrix}
0 & * & \cdots & * \\
* & 0 & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & \cdots & 0 & *
\end{pmatrix}.
\]

If \(k \geq r + 1\), consider the submatrix \(C'\) consisting entries in the first \(r + 1\) rows and the last \(k - 1\) columns of \(C\)

\[
C'_{(r+1) \times (k-1)} = \begin{pmatrix}
* & \cdots & * \\
0 & \cdots & * \\
\vdots & \vdots & \ddots \\
* & \cdots & 0
\end{pmatrix}.
\]

By the definition of \(\kappa\), all nonzero entries in the same row or column of \(C\) are linearly independent. Therefore \(C'\) is a codimension \(r - 1\) subspace of some 1-generic space in \(\mathbb{C}^{k-1} \otimes \mathbb{C}^{r+1}\). By Theorem 3.4 (2), all \((r+1) \times (r+1)\) minors of \(C'\) determine a subvariety of codimension \(\geq k - r - 1\), so \(k \leq n + r + 1\).

Now we find upper bounds for \(s\) and \(t\). If \(s = r\) and \(t \geq r\), the submatrix consisting of entries in the last \(s + 1\) rows and the last \(t + 1\) columns is a codimension 1 subspace of a 1-generic space in \(\mathbb{C}^{s+1} \otimes \mathbb{C}^{t+1}\). So by Theorem 3.4 (2) again \(t \leq n + r - 1\).

Similarly if \(t = r\) then \(s \leq n + r - 1\).

If \(s, t \geq r + 1\), the submatrix consisting of entries in the last \(s\) rows and the last \(t\) columns is 1-generic. By Theorem 3.4 (1), \(n \geq 2\) and \(s + t \leq n + 2r\).

To put everything together, either \(a \leq n + 2r, b \leq n + 2r\) or \(a, b \leq 2n + 2r\).

(3) Case \(2 \leq \kappa \leq \max\{M_1(r - 1, n), M_2(r - 1, n)\}\).

Claim There exist \(M_i(r, n, g), i = 1, 2\), such that if \(E\) has \(\kappa = g\) and satisfies the hypothesis of the proposition, then either \(a\) or \(b\) \(\leq M_1(r, n, g)\), or \(a, b \leq M_2(r, n, g)\), or the condition (3) holds.

We will find \(M_i = M_i(r, n, g)\) by induction on \(g\). By the last case, we can set \(M_1(r, n, 1) = n + 2r\) and \(M_2(r, n, 1) = 2n + 2r\). Assume the claim is true for spaces of \(\kappa < g\).

Write \(E = \begin{pmatrix} C_{k \times k} & 0 \\ 0 & D_{t \times t} \end{pmatrix}\) so that \(C\) has zeros on the diagonal and \(D\) is 1-generic. Let \(M(r, n, g - 1) := \max\{M_1(r, n, g - 1), M_2(r, n, g - 1)\}\). If \(k > 2M(r, n, g - 1) + 1\), then the submatrix \(C'\) consisting of entries in the first \(M(r, n, g - 1) + 1\) rows and the last \(M(r, n, g - 1) + 1\) columns of \(C\) is a space of \(\kappa = g - 1\). However by the definitions of \(M_i(r, n, g - 1)\)’s, all \((r + 1) \times (r + 1)\) minors of \(C'\) determine of codimension > \(n\) subset. Therefore \(k \leq 2M(r, n, g - 1) + 1\).
Now $s$ and $t$ have the same upper bound as the last case. So we can set $M_1(r, n, g) := 2M(r, n, g - 1) + r$ and $M_1(r, n, g) := 2M(r, n, g - 1) + r + n$ which proves the claim.

(4) Case $\kappa \geq \max \{M_1(r - 1, n), M_2(r - 1, n)\} + 1$. Choose bases and possibly take transpose so that all entries in the top left $\kappa \times \kappa'$ block of $E$ are zero for some $1 \leq \kappa' \leq \kappa$. So

$$E = \begin{pmatrix} O_{\kappa \times \kappa'} & H \\ G & D_{s \times t} \end{pmatrix}.$$  \hspace{1cm} (F)

We take the largest $\kappa'$ so that the submatrix $H$ is concise in $\mathbb{C}^\kappa \otimes \mathbb{C}'$. By the definition of $\kappa$, $G$ is 1-generic.

Consider the $(r + 1) \times (r + 1)$ minors consisting of any single entry of $G$ and any $r \times r$ minor of $H$. We must have codim$_H(H_{r-1}) \leq n$ unless condition (3) holds. Since $\kappa \geq \max \{M_1(r - 1, n), M_2(r - 1, n)\} + 1$, $t \leq M_1(r - 1, n)$.

If $\kappa' \leq r$, then $b \leq M_1(r - 1, n) + r$.

If $\kappa' \geq r + 1 > s$, consider the $(r + 1) \times (r + 1)$ minors that are a product of an $s \times S$ minor of $G$ and an $(r + 1 - s) \times (r + 1 - s)$ minor of $H$. Then either codim$_G(G_{s-1}) \leq n$ or codim$_H(H_{r-s}) \leq n$. The latter inequality implies condition (4) holds. The former inequality implies $\kappa' \leq n + s - 1 \leq n + r - 1$, then $b \leq n + r - 1 + M_1(r - 1, n)$.

If $\kappa', s \geq r + 1$, then codim$_G(G_r) \leq n$. By Theorem 3.4, $s + \kappa' \leq n + 2r + 1$. So $b \leq n + r + M_1(r - 1, n)$.

Since $M_1(r, n, g + 1) \geq M_1(r, n, g)$ for $g \geq 0$, $M_1(r, n, g)$ takes the maximum at $g = g' := \max \{M_1(r - 1, n), M_2(r - 1, n)\}$. So we can put $M_1(n, r) = \max \{M_1(r, n, g'), n + r + M_1(r - 1, n)\}$ and $M_2 = M_1(r, n, g')$ which proves the proposition. \hspace{1cm} $\square$

**Corollary 3.9** Let $E \subset A \otimes B$ be concise and satisfy codim $E_2 = 2$. Then at least one of the following holds:

1. $a$ or $b \leq 6$;
2. $a, b \leq 8$;
3. there exists a hyperplane $F \subset E$ such that codim$_F F_2 \leq 1$;
4. $E$ has bounded rank 2.

**Proof** For $\kappa = 0$ or 1, by the proof of Proposition 3.8 either $a, b \leq 6$, or $a, b \leq 8$.

For $\kappa = 2$ or 3, put $E$ into the form (F). If condition (3) does not hold, $G$ and $H$ are both 1-generic. Then by Theorem 3.4, $a$ or $b \leq 6$.

For $\kappa \geq 4$, $H$ must have bounded rank 1 and $t = 1$. If $\kappa \leq 3$, then $b \leq 5$. If $\kappa \geq 4$, $G$ must have bounded rank 1 and $s = 1$, then $E$ has bounded rank 2. \hspace{1cm} $\square$

**4 Geometric rank 3**

This section studies the structure of the set of tensors with geometric rank at most 3.

**Theorem 4.1** A tensor $T \in A \otimes B \otimes C$ has geometric rank at most 3 if and only if one of the following conditions holds:
(1) $T(A^*)$, $T(B^*)$ or $T(C^*)$ is of bounded rank 3, or
(2) $\text{SR}(T) \leq 3$, or
(3) up to change of bases $T = M_{(2)}$.

If $T$ is primitive of geometric rank 3, then up to change of bases and permutations of $A$, $B$ and $C$, it is either the matrix multiplication tensor $M_{(2)}$ or the tensor such that $T(A^*)$ is a space of $4 \times 4$ skew-symmetric matrices of dimension 4, 5 or 6.

**Proof** By (A), $\text{GR}(T) \leq 3$ if and only if at least one of the following three cases holds:

(i) $\text{codim } A^*_1 = 0$;
(ii) $\text{codim } A^*_2 \leq 1$;
(iii) $\text{codim } A^*_3 \leq 2$.

Case (i): $\text{codim } A^*_3 = 0 \iff T(A^*)$ has bounded rank 3.

Case (ii): If $\text{codim } A^*_2 = 0$, then $\text{GR}(T) = 2$, so $T(A^*)$, $T(B^*)$ or $T(C^*)$ is of bounded rank 2.

When $\text{codim } A^*_2 = 1$, according to the discussion in Sect. 3.3 and Proposition 3.6, at least one of the following holds:

(1) $T = T' + T''$ where $T'(A^*)$ is a space of bounded rank 2 and $\text{ml}_A(T'') = 1$, so $T$ is not primitive.
(2) $T(A^*)$, $T(B^*)$ or $T(C^*)$ has bounded rank 3;
(3) up to change of bases $T = M_{(2)}$.

By classification of $\mathcal{GR}_2$, any non-primitive tensor of $\text{GR} = 3$ is either compression or at least one of $T(A^*)$, $T(B^*)$ and $T(C^*)$ has bounded rank 3.

Case (iii): By the discussion in Sect. 3.2, if there is a nonzero $2 \times 2$ minor that is a product of two linear forms, $T$ is not primitive. If all nonzero $2 \times 2$ minors are irreducible, $T(A^*) \subset \mathbb{C}^2 \otimes \mathbb{C}^3$ or $\mathbb{C}^3 \otimes \mathbb{C}^2$, so has bounded rank 2.

By classification of primitive spaces of bounded rank 3 [9], if $T(A^*)$ is a primitive space of bounded rank 3, then either $T(B^*)$ or $T(C^*)$ is $4 \times 4$ skew-symmetric. $\square$

By classifications of $\mathcal{GR}_r$ for $r = 1, 2$ and 3, we summarize the following relations between geometric rank and slice rank:

**Corollary 4.2** (1) $\text{GR}(T) = 1 \iff \text{SR}(T) = 1$.
(2) If $\text{ml}_A(T)$, $\text{ml}_B(T)$ or $\text{ml}_C(T) > 3$, then $\text{GR}(T) = 2 \iff \text{SR}(T) = 2$.
(3) If at least one of $\text{ml}_A(T)$, $\text{ml}_B(T)$ and $\text{ml}_C(T) > 6$, or at least two of them $> 4$, then $\text{GR}(T) = 3 \iff \text{SR}(T) = 3$.

However we cannot draw any similar conclusion for $r \geq 4$. As a counter example, let $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be defined as

$$T(A^*) := \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \\ x_4 & x_5 & \cdots & x_m \\ x_4 & \cdots \\ \cdots \\ x_4 \end{pmatrix}.$$

\(\square\) Springer
Then $T$ is a direct sum of the primitive tensor of geometric rank 2 and a compression tensor of geometric rank 2. So $\text{GR}(T) = 4$, $\text{SR}(T) = 5$, and $T$ is concise no matter how large $m$ is.

5 Geometric rank 4 and in general

**Theorem 5.1** If $T \in A \otimes B \otimes C := \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ is primitive of geometric rank 4, then either at least two of $\text{ml}_A(T)$, $\text{ml}_B(T)$ and $\text{ml}_C(T)$ are at most 6, or all of them are at most 8.

**Proof** $\text{GR}(T) \leq 4$ if and only if one of the following cases holds:

1. $A^*_4 = A^*$;
2. $\text{codim}(A^*_3) \leq 1$;
3. $\text{codim}(A^*_2) \leq 2$;
4. $\text{codim}(A^*_1) \leq 3$;
5. $\text{codim}(A^*_0) \leq 4$.

1. $\iff$ $T(A^*)$ has bounded rank 4. $T$ is primitive only if $T(A^*)$ is a primitive space of bounded rank 4. By [3], if a primitive space of bounded rank 4 has size $n_1 \times n_2$, then either $n_1 \leq 5$ and $n_2 \leq 10$, $n_1 \leq 10$ and $n_2 \leq 5$, or $n_1 \leq 6$ and $n_2 \leq 6$. So either $\text{ml}_B(T) \leq 5$, or $\text{ml}_C(T) \leq 5$, or $\text{ml}_B(T)$, $\text{ml}_C(T) \leq 6$.

2. $\iff$ there exists an irreducible polynomial $P$ of degree $\geq 1$ dividing all $4 \times 4$ minors of $T(A^*)$.

2.1. $\deg P = 1$: by Lemma 2.1 $T$ is not primitive.

2.2. $\deg P = 2$: By Lemma 3.2, up to change of bases the upper left $4 \times 4$ submatrix of $T(A^*)$ has determinant equal to $P^2$ and $P$ divides all $3 \times 3$ minors of the submatrix. Proposition 3.6 gives a classification of such $4 \times 4$ matrices. Since the determinant does not vanish, the submatrix cannot have bounded rank 3, so the submatrix is either skew-symmetric or has the form $\text{diag}(X, X)$.

2.2.i) Case $\text{diag}(X, X)$: write $T(A^*)$ as the block form

$$T(A^*) = \begin{pmatrix} X & 0 & E_1 \\ 0 & X & E_2 \\ D_1 & D_2 & F \end{pmatrix}$$

where $X$ has determinant $S$, and $E_i$ and $D_j$ are $2 \times (c - 4)$ and $(b - 4) \times 2$ blocks.

For $1 \leq i \leq 2$, $3 \leq k \leq 4$, $5 \leq j, l \leq c$, the minor $\Delta_{12kl}^{34} = \Delta_{12}^{ij} \Delta_{kl}^{34}$ is divisible by the irreducible quadratic polynomial $S$. Therefore either $S \mid \Delta_{12}^{ij}$ or $S \mid \Delta_{kl}^{34}$. By Lemma 3.1, either $D_1$ or $E_2$ can be put to 0 by adding first two rows or columns to the rest. By the same argument, either $D_2$ or $E_1$ can be put to 0.

If $D_1 = D_2 = 0$ or $E_1 = E_2 = 0$, $S$ divides $\Delta_{13kl}^{ij} = (y_1) \Delta_{kl}^{ij}$ for $i, j, k, l \geq 5$. So $S$ divides all $2 \times 2$ minors of $F$. By Lemma 3.1, either $F \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ or $F$ has bounded rank 1. Therefore $\text{ml}_B(T)$ or $\text{ml}_C(T) \leq 6$ and $T$ is not concise.

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If $D_1 = E_1 = 0$ or $D_2 = E_2 = 0$, without loss of generality assume $D_1 = E_1 = 0$. Consider the minors $\Delta_{ij}^{lkt} = y_{i}^{l} \Delta_{ij}^{k} = y_{i}^{l} \Delta_{ij}^{k}$ for $i, j, k, l, s, t \geq 3$. So $S$ divides all $3 \times 3$ minors of $G := \begin{pmatrix} X & E_2 \\ D_2 & F \end{pmatrix}$. By Lemma 3.6 either $G \subset \mathbb{C}^{4} \otimes \mathbb{C}^{4}$ or $G$ has bounded rank $2$. Therefore $ml_{B}(T) \leq 6$ or $ml_{C}(T) \leq 6$ and $T$ is not concise.

(2.2.ii) Case skew-symmetric: permute the first four rows and columns to put $T(A^{*})$ into the following form:

$$T(A^{*}) = \begin{pmatrix} x_1 & 0 & a & d \\ 0 & x_1 & c & b \\ b & -d & e & 0 \\ -c & a & 0 & e \\ D_1 & D_2 & F \end{pmatrix}.$$

Adding the first two rows and columns to the rest, so that $y_{i}^{1}, y_{i}^{2}, y_{i}^{3}, y_{i}^{4}$ do not contain $x_{1}$ in their expression, for all $i, j, S = x_{1} e - ab + cd$ divides all $4 \times 4$ minors of $T(A^{*})$. Restricting to the subspace $\{x_{1} = 0\}$, then $S' := -ab + cd$ divides all $4 \times 4$ minors of $T(A^{*})|_{x_{1}=0}$.

If $S'$ is irreducible, for $3 \leq i, k, l \leq 4$ and $j, l \geq 5$, consider the minors $\Delta_{12}^{ij} = \Delta_{12}^{ij} \Delta_{12}^{ij}$ of $T(A^{*})|_{x_{1}=0}$. Similarly to case (i), either $D_1|_{x_{1}=0}$ or $E_1|_{x_{1}=0}$ can be put $0$. Without loss of generality assume $D_1|_{x_{1}=0}$.

Now working on $T(A^{*})$, entries in $D_1$ are multiples of $x_{1}$. Then by adding multiples of the first two rows to the last $m - 4$ rows we can put $D_1 = 0$. $S$ dividing $\Delta_{12}^{ij} = (x_{1})^{2} \Delta_{12}^{ij}$ for $i, j \geq 5, k, l \geq 3$ implies it divides all $2 \times 2$ minors of the $(m - 4) \times (m - 2)$ block $(D_2 F)$. So either $(D_2 F)$ has bounded rank $2$ or $(D_2 F) \subset \mathbb{C}^{2} \otimes \mathbb{C}^{2}$. If by changing bases $(D_2 F)$ has nonzero entries only in the first two rows, $ml_{B}(T) \leq 6$. Otherwise, by changing bases we can put all nonzero entries of $(D_2 F)$ in its first two or $3$ column. Then consider the $4 \times 4$ minors involving one entry of $(D_2 F)$ and $3 \times 3$ minors from the first four rows of $T(A^{*})$. By Proposition 3.6, either $ml_{C}(T) \leq 6$, or $ml_{B}(T)$, $ml_{C}(T) \leq 7$.

If $S' = -ab + cd$ is reducible, by changing bases we can put the block $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as $\begin{pmatrix} 0 & b' \\ c' & d' \end{pmatrix}$ and the same for $\begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$. Then the upper left $4 \times 4$ block of $T(A^{*})$ becomes

$$\begin{pmatrix} x_1 & 0 & 0 & d' \\ 0 & x_1 & c' & b' \\ b' & -d' & e & 0 \\ -c' & 0 & 0 & e \end{pmatrix}.$$

Then permuting rows and columns we get

$$\begin{pmatrix} 0 & 0 & x_1 & d' \\ 0 & 0 & -c' & b' \\ b' & -d' & e & 0 \\ c' & x_1 & 0 & e \end{pmatrix}.$$
By the same argument, we can put \( D_1 = 0 \). Then all \( 3 \times 3 \) minors of \((D_2 F)\) are divisible by \( S \). By Proposition 3.6 either \((D_2 F)\) has bounded rank 2 or \((D_2 F) \subset \mathbb{C}^4 \otimes \mathbb{C}^4\). By the same argument as the previous case, either \( ml_B(T) \leq 6 \), or \( ml_B(T) \), \( ml_C(T) \leq 8 \).

\( \text{(2.3) } \deg P = 3; \) by Lemma 3.2, either \( P \) factors into linear forms so \( T \) is not primitive, or \( T(A^*) \) has bounded rank 4.

(3) By Proposition 3.9, if \( T \) is primitive then either \( ml_B \leq 6 \), \( ml_C(T) \leq 6 \) or \( ml_B(T), ml_C(T) \leq 8 \).

(4) By Proposition 3.5, if \( T \) is primitive then \( ml_B(T) + ml_C(T) \leq 6 \).

\[ (5) \iff \dim(T(A^*)) \leq 4 \implies \text{SR}(T) \leq 4. \]

Putting everything together, either \( ml_B(T) \leq 8 \), or \( ml_C(T) \leq 8 \), or \( ml_B(T), ml_C(T) \leq 6 \). Since geometric rank is invariant by permuting \( A, B \) and \( C \), we also have:

- either \( ml_A(T) \leq 8 \), or \( ml_C(T) \leq 8 \), or \( ml_A(T), ml_C(T) \leq 6 \);
- either \( ml_A(T) \leq 8 \), or \( ml_B(T) \leq 8 \), or \( ml_A(T), ml_B(T) \leq 6 \).

By inclusion–exclusion argument, we complete the proof. \( \square \)

**Corollary 5.2** If \( ml_A(T), ml_B(T) \) and \( ml_C(T) > 8 \), then \( \text{GR}(T) \leq 4 \) if and only if either \( \text{SR}(T) \leq 4 \), or up to change of bases \( T = T' + T'' \) where \( T' \) is the \( 3 \times 3 \times 3 \) skew-symmetric tensor and \( \text{SR}(T'') = 2 \).

As a consequence of Proposition 3.8, we draw a general conclusion for primitive tensors of geometric rank \( r \).

**Theorem 5.3** For all \( r \), there exists a positive integer \( N_r \) such that if \( T \in A \otimes B \otimes C \) is primitive of geometric rank \( r \), then at least two of \( ml_A(T), ml_B(T) \) and \( ml_C(T) \) are at most \( N_r \).

## 6 Geometric rank of \( n \)-part tensors

Let \( n > 2 \) and \( A^{(i)} := \mathbb{C}^{m_i} \) for \( i \leq n \). For a tensor \( T \in A^{(1)} \otimes \cdots \otimes A^{(n)} \), let

\[ \Sigma_T^{A^{(1)} \cdots A^{(n-1)}} := \left\{ (x_1, \ldots, x_{n-1}) \in A^{(1)*} \times \cdots \times A^{(n-1)*} \mid \text{for all } x_n \in A^{(n)*} \right\}. \]

The geometric rank of \( T \) is defined to be \( \text{GR}(T) := \dim(\Sigma_T^{A^{(1)} \cdots A^{(n-1)}}) \).

Regard \( T \) as a linear map \( A^{(1)*} \to A^{(2)*} \otimes \cdots \otimes A^{(n-1)*} \). Define \( A^{(i)*}_j := \{ x_1 \in A^{(1)*} \mid \text{GR}(T(x_1)) \leq j \} \) where \( T(x_1) \) is an \((n-1)\)-part tensor. For any \( i \leq n \), \( A^{(i)*}_j \) is defined similarly. Similarly to the alternative definition \( (A) \) for tripartite tensors, the following proposition gives an alternative definition of geometric rank for \( n \)-part tensors.

**Proposition 6.1** For \( T \in A^{(1)} \otimes \cdots \otimes A^{(n)} \), \( \text{GR}(T) \) is invariant under the any permutation of \( A^{(1)*}, \ldots, A^{(n)*} \). For any \( 1 \leq i \leq n \), \( \text{GR}(T) = \min \{ \dim(A^{(i)*}_j) \mid 0 \leq j \leq \min\{m_1, \ldots, m_i, \ldots, m_n\} \} \).

\( \square \) Springer
Consider the first projection $\pi: \Sigma_T^{A^{(1)} \cdots A^{(n-1)}} \to A^{(1)*}$. For any $1 \leq j \leq \min\{m_2, \ldots, m_n\}$ and $x_1 \in A_j^{(1)*} \setminus A_{j-1}^{(1)*}$, the fiber

$$
\pi^{-1}(x_1) = \{(x_1, x_2, \ldots, x_{n-1}) \mid \forall x_n, T(x_1, \ldots, x_n) = 0\} \\
= \{x_1\} \times \{(x_2, \ldots, x_{n-1}) \mid \forall x_n, T(x_1, x_2, \ldots, x_n) = 0\} \\
= \{x_1\} \times \Sigma_T^{(2) \cdots (n-1)}. 
$$

Then $\dim(\pi^{-1}(x_1)) = \dim(\Sigma_T^{(2) \cdots (n-1)}) = m_2 + \cdots + m_{n-1} - j$. And for $x_1 \in A_0^{(1)*}$, $\pi^{-1}(x_1) = A^{(2)} \times \cdots \times A^{(n-1)}$ which has dimension $m_2 + \cdots + m_{n-1}$. So

$$
dim(\Sigma_T^{(1) \cdots (n-1)}) = \max\{\dim(\pi^{-1}(A_j^{(1)})) \mid 0 \leq j \leq \min\{m_2, \ldots, m_n\}\} \\
= \max\{\dim(A_j^{(1)}) + \dim(\pi^{-1}(A_j^{(1)})) \mid 0 \leq j \leq \min\{m_2, \ldots, m_n\}\} \\
= \max\{\dim(A_j^{(1)}) + m_2 + \cdots + m_{n-1} - j \mid 0 \leq j \leq \min\{m_2, \ldots, m_n\}\}. 
$$

Therefore $GR(T) = \text{codim}(\Sigma_T^{(1) \cdots (n-1)}) = \min\{\text{codim}(A_j^{(1)} + j) \mid 0 \leq j \leq \min\{m_2, \ldots, m_n\}\}$. This proves the case $i = 1$ and implies $GR(T)$ is invariant under any permutation of the last $n - 1$ factors $A^{(2)*}, \ldots, A^{(n)*}$. By definition $GR(T)$ is invariant under any permutation of the first $n - 1$ factors, so it is invariant under any permutation of all $n$ factors. And the cases when $i > 1$ follow by permuting the factors.

A linear subspace $E \subset A^{(1)} \otimes \cdots \otimes A^{(n)}$ has bounded geometric rank $r$ if every element has geometric rank at most $r$.

**Proposition 6.2** Let $n \geq 3$ and $T \in A^{(1)} \otimes \cdots \otimes A^{(n)}$. For all $r < n$, $GR(T) \leq r$ if and only if there exists $i$ such that $T(A_i^{(1)*})$ has bounded geometric rank $r$ as a space of $(n-1)$-part tensors.

**Proof** ($\Leftarrow$) direction is obvious by Proposition 6.1. Prove ($\Rightarrow$) by induction on $n$.

The base case is $n = 3$, then $GR(T) \leq 1 \iff SR(T) \leq 1 \iff \exists i, T(A_i)$ has bounded geometric rank 1; $GR(T) \leq 2 \iff \exists i, T(A_i)$ has bounded rank 2, and matrix rank coincides with geometric rank for 2-tensors.

Assume the proposition is true for all $n < N$ and $T \in A^{(1)} \otimes \cdots \otimes A^{(N)}$. By Proposition 6.1, $GR(T) \leq r$ if and only if $\exists 0 \leq k \leq r$ such that $\text{codim}(A_k^{(1)*}) \leq r - k$.

If $k = r$, then $A_k^{(1)*} = A^{(1)*}$ and $T(A^{(1)*})$ has bounded geometric rank $r$.

If $k < r$, for $x_1 \in A_k^{(1)*}$ consider $T(x_1)$ is as an $(N-1)$-part tensor. By assumption $GR(T(x_1)) \leq k$ if and only if $\exists 1 > 1, T(x_1)(A^{(1)*})$ has bounded geometric rank $k$.

Therefore we have $\forall x_1 \in A^{(1)*}, \forall x_l \in A_k^{(1)*}, GR(T(x_1)(x_l)) \leq k$

$\Rightarrow \forall x_l \in A^{(1)*}, x_1 \in A_k^{(1)*} \mid GR(T(x_1)(x_l)) \leq k \supset A_k^{(1)*}$

$\Rightarrow \forall x_1 \in A^{(1)*}, \text{codim}\{x_1 \in A^{(1)*} \mid GR(T(x_1)(x_l)) \leq k\} \leq r - k$

$\Rightarrow \forall x_1 \in A^{(1)*}, GR(T) \leq k$

$\Rightarrow T(A^{(1)*})$ has bounded rank $k$. \hfill $\square$
$T$ is said to have partition rank 1 if there exists a partition $[n] = I \sqcup J$ such that $T = T_1 \otimes T_2$ for some nonzero $T_1 \in \bigotimes_{i \in I} A^{(i)}$ and $T_2 \in \bigotimes_{j \in J} A^{(j)}$. Regard $T$ as a multilinear function $T : A^{(1)*} \times \cdots \times A^{(n)*} \to \mathbb{C}$, then $T$ has partition rank 1 if and only if $T$ is a product of two non-constant multilinear functions. The partition rank of $T$ is the smallest integer $r$ such that $T$ can be written as a sum of $r$ partition rank 1 tensors, denoted as $\text{PR}(T)$.

Partition rank was introduced in [16] as a more general version of slice rank. By definition $\text{GR}(T) \leq \text{PR}(T) \leq \text{SR}(T)$. [7] showed that the partition rank is at most $2^{n-1}$ times of the geometric rank for $n$-part tensors.

**Proposition 6.3** For $n \geq 3$, $T \in A^{(1)} \otimes \cdots \otimes A^{(n)}$ has geometric rank 1 if and only if it has partition rank 1.

**Proof** By definition $\text{PR}(T) = 1$ implies $\text{GR}(T) = 1$. We prove the other direction by induction on $n$. For $n = 3$, by [10, Remark 2.6], $\text{GR}(T) = 1$ if and only if $\text{SR}(T) = 1$, and slice rank agrees with partition rank for tripartite tensors.

Assume the statement is true for $n < N$ and $T \in A^{(1)} \otimes \cdots \otimes A^{(N)}$. By Proposition 6.2 there exists $i$ such that $T(A^{(i)})$ consists of $(N - 1)$-part tensors with geometric rank at most 1. Without loss of generality assume $i = N$.

Let $\{z_j\}_{j=1}^{m N}$ be a basis of $A^{(N)*}$. By assumption $\text{PR}(T(z_j)) \leq 1$ for all $j$, so we can write $T(z_j) = : f_j g_j$ for some multilinear function $f_j, g_j$. By the definition of geometric rank, $\{T(z_j) = 0, \forall j\} \subset A^{(1)} \times \cdots \times A^{(N-1)}$ has codimension 1. By possibly swapping $f_j$ and $g_j$, assume $\{f_j = 0, \forall j\}$ has codimension 1. Thus $f_j$'s have a common factor of positive degree, denoted as $f$. Then we can write $f_j = : fh_j$ for some $h_j$, so $T(z_j) = fh_j g_j$.

Say $f$ is a multilinear function defined on $\prod_{j \in I} A^{(j)}$ for some $I \subset [N - 1]$, then $h_j g_j$ is defined on $\prod_{j \in [N-1]\setminus I} A^{(j)}$. Define $g : (\prod_{j \in [N-1]\setminus I} A^{(j)}) \times A^{(N)} \to \mathbb{C}$ by $g(x, z_j) := (h_j g_j)(x)$. Therefore $T = fg$ and has partition rank 1. \hfill \square

**7 Proof of Proposition 3.6**

Before proving the proposition, we need the following lemma.

**Lemma 7.1** Let $E \subset A \otimes B := \mathbb{C}^2 \otimes \mathbb{C}^2$ be a matrix of linear forms in variables $x_1, \ldots, x_4$. Define

$$X_1 := \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}, \quad X_2 := \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

(1) If $\det E = \det X_1$, then $E = X_1$ up to change of bases in $A$ and $B$.

(2) If $\det E = \det X_2$, then either $E = X_2$ or $E = X_2^t$ up to change of bases in $A$ and $B$.

**Proof of Proposition 3.6** Say $\Delta_3 = x_1 S$, then the upper left $3 \times 3$ submatrix must be of the form $x_1 Z + U$ where $Z$ is a $3 \times 3$ matrix of complex numbers and $U$ has
bounded rank 2. Therefore up to change of bases, $U$ is either compression or skew-symmetric. Since $\Delta_3 \neq 0$, taking transpose if necessary, we can write the upper left $3 \times 3$ submatrix in one of the following forms:

$$(i) \begin{pmatrix} y_1 \ y_2 & y_3 \ 0 & 0 \ x_1 \end{pmatrix}, \quad (ii) \begin{pmatrix} x_1 & 0 & y_3 \ y_1 & y_2 & y_3 \ 0 & x_1 & y_2 \ y_1^3 & y_2^3 & y_3^3 \end{pmatrix}, \quad (iii) \begin{pmatrix} x_1 & y_2 & y_3 \ -y_2 & x_1 & y_3 \ -y_3 & -y_2 & x_1 \end{pmatrix}.$$  

For the rest of the proof, we will discuss each of the above cases.

**Case (i).** $S = \Delta_2 = y_1^2 y_2 - y_1 y_2^2$ is an irreducible quadratic polynomial, hence has Waring rank 3 or 4. Changing basis in $E$ we can write $S = x_1 x_3 - (x_2)^2$ or $x_1 x_4 - x_2 x_3$ depending on rank ($S$). By Lemma 7.1 we can put the top left $2 \times 2$ block as the form $X_1$ or $X_2$.

$S$ divides $\Delta_{123}^{34} = x_1 \Delta_{12}^{4}$ and $\Delta_{123}^{34} = x_1 \Delta_{12}^{4}$, therefore $(y_1^4, y_2^4) \in \text{span}((y_1^1, y_2^1), (y_1^2, y_2^2))$. Adding multiples of the first and the second row to the 4-th, we can set $y_1^4$ and $y_2^4$ to zeros.

If $\Delta_{34}^{34} = 0$, the right bottom $2 \times 2$ submatrix has bounded rank 1, then $E$ has bounded rank 3.

Assume $\Delta_{34}^{34} \neq 0$. Since $S$ divides $\Delta_{234}^{24} = y_2^2 \Delta_{34}^{34} \neq 0$, $\Delta_{34}^{34}$ is a nonzero multiple of $S$, hence can be normalized to $S$. Apply Lemma 7.1 again, $E$ has one of the following forms:

$$(i) \begin{pmatrix} x_1 & x_2 & y_1^3 & y_2^3 \ x_2 & x_3 & y_1^3 & y_2^3 \ 0 & 0 & x_1 & x_2 \ 0 & 0 & x_2 & x_3 \end{pmatrix} \quad \text{if rank}(S) = 3, \quad \text{or} \quad (ii) \begin{pmatrix} x_1 & x_2 & y_1^3 & y_2^3 \ x_3 & x_4 & y_1^3 & y_2^3 \ 0 & 0 & x_1 & y_4^3 \ 0 & 0 & y_3^3 & x_4 \end{pmatrix} \quad \text{if rank}(S) = 4$$

where $(y_1^3, y_2^3) = (x_2, x_3)$ or $(x_3, x_2)$.

We consider separately the two subcases (i.1) rank $(S) = 3$ and (i.2) rank $(S) = 4$. And we further divide subcase (i.2) into two situations: (i.2.1) $(y_1^3, y_2^3) = (x_2, x_3)$, and (i.2.2) $(y_1^3, y_2^3) = (x_3, x_2)$.

**Subcase (i.1).** Assume rank $(S) = 3$. Write $y_1^4 = l_2 + l_2', y_2^2 = l_3 + l_3', y_4^2 = l_4 + l_4'$, and $\Delta_{234}^{123} = (l + l') S$ where $l, l_i \in \text{span}\{x_1, x_2, x_3\}$ and $l', l_i' \in \text{span}\{x_4, \ldots, x_m\}$. Then

$$l' S = l'(x_1 x_3 - (x_2)^2) = (x_3 l'_2 - x_2 l'_4) x_1 + (x_2 l'_3 - x_3 l'_4) x_2.$$  

Comparing terms that are multiples of $(x_1)^2$, we see $l' = 0$, which forces all $l_i' = 0$, so $y_i' \in \text{span}\{x_1, x_2, x_3\}$. Adding multiples of the first two rows and columns to the last two rows and columns, we can put $y_1^4 = y_2^2 = 0$.

Write $l = a_1 x_1 + a_2 x_2 + a_3 x_3$. Then

$$\begin{align*}
(a_1 x_1 + a_2 x_2 + a_3 x_3)(x_1 x_3 - (x_2)^2) &= l S = \Delta_{234}^{123} = -x_2 x_3 y_1^3 - x_1 x_2 y_2^2.
\end{align*}$$

Comparing the terms of multiples of $(x_1)^2 x_3, x_1 (x_3)^2$ and $(x_2)^2$, we see all $a_i = 0$ so $l = 0$. Comparing the coefficients of the rest cubic monomials, we obtain $a_1 + b_3 = 0$.
and \( a_2 = a_3 = b_1 = b_2 = 0 \). If \( a_1 = 0 \), \( E \) has the form \( \text{diag}(X_1, X_1) \) where \( X_1 = (x_{12}, x_{13}). \) If \( a_1 \neq 0 \), multiply \( a_1 \) to the first two columns and the last two rows, subtract the 1st row from the 3rd, and add the 4th row to the 2nd, then \( E \) again has the form \( \text{diag}(X_1, X_1) \).

**Subcase (i.2).** Assume rank \((S) = 4\).

(i.2.1). If \( (y^3_4, y^2_3) = (x_2, x_3) \): using the notations in (i.1), write \( y^i_j = l_k + l'_k \) and \( \Delta^{1234}_{234} = (l + l') S \). Then \( l' S = x_2(l'_3 x_2 - l'_4 x_1) - x_4(l'_1 x_2 - l'_2 x_1) \). Comparing terms we see all \( l'_1 = 0 \). Then by adding multiples of the first two columns to the third and fourth, then adding multiples of the first two rows to the first and second, we can make \( y^1_3 \in \text{span}(x_2, x_4), y^2_3 \in \text{span}(x_2), y^3_3 \in \text{span}(x_1, x_2, x_4) \).

Since \( \Delta^{1234}_{234} = l S = x_2(y^2_3 x_2 - y^2_4 x_1) - x_4(y^3_4 x_2 - y^3_4 x_1) \), writing \( y^i_j \) into linear combinations of \( x_k \)'s, we see there is no \( x_2 x_3 \). Thus \( l = 0 \), then comparing terms of the above equation, we have \( y^1_3 = y^2_3 = 1 \). By \( \Delta^{134}_{134} \) and \( \Delta^{132}_{234} \), \( (y^1_4, y^2_4)^t \in \text{span}(x_1, x_3)^t \), so we can set \( y^1_4 = y^2_4 = 0 \) by adding multiples of the first two columns to the fourth.

Therefore, by changing bases \( E \) has the form \( \text{diag}(X_2, X_2) \) where \( X_2 = (x^1_3 x^2_4) \).

(i.2.2). If \( (y^3_4, y^2_3) = (x_2, x_3) \): using the notations in (i.1), write \( y^i_j = l_k + l'_k \) and \( \Delta^{1234}_{234} = (l + l') S \). Then \( l' S = x_2(l'_3 x_2 - l'_4 x_1) - x_4(l'_1 x_2 - l'_2 x_1) \). Comparing terms we see \( l'_1 = l'_2 = l'_4 = 0 \). Then by adding multiples of the first two columns to the third and fourth, then adding multiples of the first two rows to the first and second, we can write \( y^1_4 = l'_1, y^2_3 = -l'_2, y^3_3 = \sum_{i=1}^4 a_i x_i, y^4_3 = \sum_{i=1}^4 d_i x_i \).

\( S \) divides other 3 \times 3 minors, which implies \( y^3_3 = y^4_3 = 0 \). Therefore the only nonzero entries of \( E \) are in the upper left 4 \times 4 block of the form

\[
\begin{pmatrix}
x_1 x_3 & 0 & l' \\
x_2 x_4 & -l' & 0 \\
0 & 0 & x_1 x_2 \\
0 & 0 & x_3 x_4
\end{pmatrix}.
\]

Swapping the first two rows and the last two rows, then multiply \(-1\) to the first two rows, \( E \) becomes skew-symmetric.

**Case (ii).** Here \( S = x_1 y^3_3 - y^2_3 x_2 - y^1_3 y^3_4 \). Modify \( y^1_j, y^2_j, y^3_j, y^4_j, 3 \leq i, j \leq 4 \), so that their expressions (as linear forms in \( x_i \)'s) do not contain \( x_1 \).

If \( y^3_3 = 0 \) and there exist \( i, j > 2 \) such that \( y^j_i \neq 0 \), we may change bases so that \( y^3_3 \neq 0 \), so we have two cases \( y^3_3 \neq 0 \) or \( y^i_j = 0 \) for all \( i, j > 2 \).

If \( y^3_3 \neq 0 \), then consider \( \Delta^{123}_{234} = x_1 (x_1 y^j_i - y^j_i y^j_1) - y^j_2 y^j_3 \). We obtain \( y^j_1 = c_i^j y^3_3 \) for all \( i, j > 2 \) and constants \( c_i^j \). Changing bases again, we may set \( y^j_3 = y^j_3 = 0 \) and \( y^j_4 = y^j_3 \) or \( 0 \). There are three subcases: (ii.1) \( y^4_4 = y^3_3 \neq 0 \), (ii.2) \( y^4_4 = 0, y^3_3 \neq 0 \), and (ii.3) \( y^3_3 = y^4_4 = 0 \).

**Subcase (ii.1).** Assume \( y^3_3 = y^4_3 \neq 0 \). Then \( \Delta^{1234}_{34} = y^3_3 (x_1 y^3_3 - y^3_3 y^3_3 - y^4_3 y^4_3) \) for all \( \rho = 1, 2 \). Together with \( \Delta^{1234}_{i2i} \)'s we get

\[
y^3_1 y^3_3 + y^4_3 y^4_3 = y^3_1 y^3_3 + y^4_3 y^4_3 = y^1_3 y^3_3 + y^2_3 y^3_3 = y^1_4 y^4_4 + y^2_4 y^4_4.
\]
Hence $y_1^3 y_3^1 = y_2^4 y_3^2$, $y_1^4 y_4^1 = y_3^3 y_3^2$.

\[
\Delta_{\sigma,34} = y_3^3 (-y_3^3 y_3^\rho - y_3^4 y_3^\sigma) = 0 \text{ for } (\rho, \sigma) = (0, 1) \text{ or } (1, 0), \text{ and } \Delta_{12i} = 0 \text{ for } i \neq j. \text{ We get}
\]

\[
y_1^3 y_2^4 + y_2^3 y_2^2 = y_1^4 y_3^1 + y_2^4 y_3^2 = y_1^2 y_3^1 + y_2^4 y_4^1 = 0.
\]

In other words, denoting $Q := y_1^3 y_3^1 + y_1^4 y_4^1$, the following equations hold:

\[
\begin{pmatrix}
  y_1^3 & y_2^3 & y_3^4 \\
y_1^4 & y_2^4 & y_3^2
\end{pmatrix}
\begin{pmatrix}
  y_1^1 \\
y_2^2
\end{pmatrix}
\begin{pmatrix}
y_1^3 \\
y_2^4
\end{pmatrix}
= Q
\begin{pmatrix}
  0 \\
  Q
\end{pmatrix}.
\]

Then by changing bases $E$ equals to the matrix whose upper left $4 \times 4$ block is one of the following, and all other entries are zeros:

\[
\begin{pmatrix}
x_1 & 0 & a & d \\
0 & x_1 & c & b \\
b & -d & y_3^3 & 0 \\
-c & a & 0 & y_3^3
\end{pmatrix}
\]

for some linear forms or zeros $a, b, c, d, y_3^3$.

**Subcase (ii.2).** Assume $y_3^4 = 0$, $y_3^3 \neq 0$. Since $\Delta_4 \neq 0$, there exist $\rho, \sigma = 1, 2$, such that $y_4^\rho$ and $y_4^\sigma \neq 0$. Then $\Delta_{\sigma,34}$ and $\Delta_{124}^3$ implies $\Delta_{34}^1 = \Delta_{12}^4 = 0$. Then changing bases in the first two rows and columns, we get

\[
\begin{pmatrix}
c_1 x_1 & c_2 x_1 & y_3^1 & y_3^1 \\
c_3 x_1 & c_4 x_1 & 0 & 0 \\
y_1^3 & y_3^3 & 0 & 0 \\
y_1^4 & 0 & 0 & 0
\end{pmatrix}
\]

for some constants $c_i$. $\Delta_{234}^{12} = -c_4 x_1 y_4^1 y_3^3$ implies $c_4 = 0$. Then $\Delta_{123}^{12} = -c_2 c_3 (x_1)^2$. $y_3^3$ implies either $c_2$ or $c_3 = 0$, contradicting the hypothesis $\Delta_3 \neq 0$.

**Subcase (ii.3).** Assume $y_3^3 = y_3^4 = 0$. As $S = y_1^3 y_3^1 + y_2^3 y_3^2$ is irreducible, $y_1^3, y_2^3$ are linearly independent, and so are $y_1^1, y_2^2$. Choose bases so that $y_1^1$ and $y_2^2$ are not necessary $x_1$, and $y_1^3 = x_1, y_2^3 = x_2$. Since $\text{rank}(S) \geq 3$, at least one of $y_1^3, y_3^3$ is linearly independent with $x_1, x_2$. Without loss of generality assume $x_1, x_2, y_3^3$ are linearly independent, then choose bases so that $y_3^3 = x_3$:

\[
\begin{pmatrix}
y_1^1 & 0 & x_3 & y_3^3 \\
0 & y_1^1 & y_3^3 & y_3^4 \\
x_1 & x_2 & 0 & 0 \\
y_1^4 & y_2^4 & 0 & 0
\end{pmatrix}.
\]
If \( \Delta_{12}^{12} = 0 \), \( \left( \begin{array}{c}
x_2 \\
y_2^1 \\
y_2^2 \
y_2^3 \end{array} \right) \) has bounded rank 1, we can set either the fourth column to zero (then \( E \subset \mathbb{C}^4 \otimes \mathbb{C}^3 \)), or \( y_2^3 = y_4^2 = 0 \) (then \( \Delta_{12}^{12} \) is a product of linear forms, contradicting to irreducibility of \( S \)).

If \( \Delta_{12}^{12} \neq 0 \), by linear independence of \( y_2^3 = x_3 \) and \( y_2^2 \), and \( \Delta_{12}^{12} \) is a nonzero multiple of \( S = x_1 y_2^1 + x_2 y_2^3 \), we can normalize the fourth column so that \( (y_4^1, y_4^2, y_4^3) = (x_2, -x_1)^t \) and \( y_j^\rho = 0 \) for all \( j > 4 \), \( \rho = 1, 2 \). By the same argument, we can set \( y_4^1, y_4^2 = (y_3^2, -x_3)^t \) and \( y_4^\sigma = 0 \) for all \( i > 4 \), \( \sigma = 1, 2 \).

Then \( E \) has the form

\[
\begin{pmatrix}
y_1^1 & 0 & x_3 & x_2 \\
0 & y_1^1 & y_3^2 & -x_1 \\
x_1 & x_2 & 0 & 0 \\
y_3^2 & -x_3 & 0 & 0
\end{pmatrix},
\]

which is skew-symmetric after permuting rows and columns.

**Case (iii).** Here \( S = x_1^2 + (y_2^1)^2 + (y_2^3)^2 + (y_3^2)^2 \) is irreducible, so \( \text{rank}(S) > 2 \). We consider two subcases by whether \( x_1, y_2^1, y_3^1, y_3^2 \) are linearly independent.

**Subcase (iii.1).** Assume \( x_1, y_2^1, y_3^1, y_3^2 \) are linearly independent. We can choose basis of \( E \) so that \( y_2^3 = x_3 \) and \( y_3^1 = x_4 \). In order that \( S \) divides all \( 3 \times 3 \) minors, \( (y_1^1, y_2^2, y_3^4) \) must be a linear combination of \( (x_1, -x_2, -x_3)^t \), \( (x_2, x_1, -x_4)^t \), \( (x_3, x_4, x_1)^t \), and \( (x_4, -x_3, x_2)^t \). Changing the basis we can put \( (y_1^1, y_2^2, y_3^4) = (x_4, -x_3, x_2)^t \). By the same argument, \( (y_4^1, y_4^2, y_3^4) = (-x_4, +x_3, -x_2) \). Consider the \( 3 \times 3 \) minors involving \( y_4^1 \), we see \( y_4^2 = -x_1 \).

Hence \( E \) has the form

\[
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 \\
x_2 & x_1 & x_4 & -x_3 \\
x_3 & -x_4 & x_1 & x_2 \\
x_4 & x_3 & -x_2 & x_1
\end{pmatrix}.
\]

Then \( E \) is the the complex quaternion algebra \( \text{span}_\mathbb{C}(\mathbb{I}, \mathbb{I}, \mathbb{J}, \mathbb{K})/(I^2+\mathbb{I}, J^2+\mathbb{I}, K^2+\mathbb{I}, IJK+\mathbb{I}) \) and the associated tensor of \( E \) is the structure tensor of the complex quaternion. Since the complex quaternion algebra is isomorphic to the matrix algebra Mat2×2, their structure tensors are equal up to change of bases. So \( E \) equals to \( M_{22} \) up to change of bases in \( E, A \) and \( B \).

**Subcase (iii.2).** Assume \( x_1, y_2^1, y_3^1, y_3^2 \) are linearly dependent. The irreducibility of \( S \) implies three of them are linearly independent. \( x_1 \neq 0 \) since \( \Delta_3 \neq 0 \). By changing bases assume \( y_2^2 = x_2 \), \( y_3^1 = x_3 \), \( y_3^2 = a_1 x_1 + a_2 x_2 + a_3 x_3 \) for \( a_i \in \mathbb{C} \). Then \( S = x_1^2 + x_2^2 + x_3^2 + (a_1 x_1 + a_2 x_2 + a_3 x_3)^2 \). If \( \dim \langle x_i, y_i^j \rangle = 1, 2, 3 \rangle \geq 5 \), the submatrix consisting of the first three rows is a subspace of a 1-generic space of codimension \( \leq 2 \), then Theorem 3.4 implies contradiction. So \( \dim \langle x_i, y_i^j \rangle = 1, 2, 3 \rangle \leq 4 \).

Adding the first three columns to the \( 4 \)th, we can set \( y_1^4 = 0 \) or \( x_4 \). Write \( y_4^2 = \sum_i b_i x_i \) and \( y_4^3 = \sum_i c_i x_i \), \( \Delta_{12}^{123} = LS, \Delta_{13}^{123} = MS \) and \( \Delta_{22}^{123} = NS \) for some linear forms \( L = \sum_i l_i x_i, M, N \).
If \( y_4^1 = x_4 \):

\[
\Delta_{124}^{123} = \left( (a_1 b_4 + c_4) x_1^2 + (a_2 + c_4) x_2^2 + (1 + a_3 b_4) x_1 x_3 + (a_2 b_4 + a_1) x_1 x_2 + (a_3 - b_4) x_2 x_3 \right) x_4
\]

\[
+ \left( (c_1 + a_1 b_4) x_1^3 + c_2 x_2^3 + (c_3 - b_2) x_2^2 x_3 - b_3 x_2 x_3^2 + (c_1 + a_2 b_2) x_1 x_2^2 + (c_2 + a_1 b_2 + a_2 b_1) x_1^2 x_2 + (c_3 + a_1 b_3 + a_3 b_1) x_1^2 x_2 \right) x_4
\]

\[
+ \left( a_2 b_4 + a_1 \right) x_1 x_2 + (a_3 - b_4) x_2 x_3 = 0
\]

Note that there is no \( x_2^3 x_4 \) in \( \Delta_{124}^{123} \). This implies either \( a_2^2 + 1 = 0 \) or \( l_4 = 0 \). If \( l_4 = 0 \), those terms divisible by \( x_4 \) have the sum zero:

\[
(a_1 b_4 + c_4) x_1^2 + (a_2 + c_4) x_2^2 + (1 + a_3 b_4) x_1 x_3 + (a_2 b_4 + a_1) x_1 x_2 + (a_3 - b_4) x_2 x_3 = 0
\]

which implies \( a_2^2 + 1 = 0 \). Hence \( a_2^2 + 1 = 0 \) no matter if \( l_4 = 0 \).

There is no \( x_2^3 x_4 \) in \( \Delta_{134}^{123} \), thus by the same argument, we must have \( a_2^2 + 1 = 0 \).

Comparing the coefficients of \( x_2^2 x_4 \) in equality \( \Delta_{124}^{123} = LS \) and \( x_3^3 \) in \( \Delta_{134}^{123} = MS \), we get

\[
c_2 = l_2 (1 + a_2^2) = 0 \quad \text{and} \quad -b_3 = m_3 (1 + a_2^2) = 0.
\]

Comparing the coefficients of \( x_2^2 x_4 \) in \( \Delta_{124}^{123} = LS \) and \( x_3^2 x_4 \) in \( \Delta_{134}^{123} = MS \), we get

\[
(a_2 + c_4) = l_4 (1 + a_2^2) = 0 \quad \text{and} \quad (a_3 - b_4) = m_4 (1 + a_2^2) = 0.
\]

Comparing the coefficients of \( x_2 x_3 x_4 \) in \( \Delta_{124}^{123} = LS \) and \( \Delta_{134}^{123} = MS \), we get

\[
2 a_2 a_3 l_4 = (a_3 - b_4) = 0 \quad \text{and} \quad 2 a_2 a_3 m_4 = a_2 + c_4 = 0.
\]

Therefore \( l_4 = m_4 = 0 \). Then the coefficient of every monomial divisible by \( x_4 \) in \( \Delta_{124}^{123} \) and \( \Delta_{134}^{123} \) equals zero. We get \( a_1 = a_2 / a_3 \) from \( \Delta_{124}^{123} \) but \( a_1 = a_2 a_3 \) contradicting \( a_2^3 = -1 \).

If \( y_4^1 = 0 \): since there is no \( x_2^3 x_4 \) in \( \Delta_{124}^{123} \), either \( a_2^2 + 1 = 0 \) or \( l_4 = 0 \).

If \( l_4 = 0 \), then the coefficient of every monomial divisible by \( x_4 \) in \( \Delta_{124}^{123} \) equals zero, which implies \( b_4 = c_4 = 0 \). Therefore there is no \( x_4 \) appearing in the first three rows, and by the same argument \( x_4 \) does not appear in the first three columns.

If there exist \( i, j > 3 \) such that \( y_j^j \not\in \text{span}\{x_1, x_2, x_3\} \), then we can change basis in \( E \) to set \( y_j^j = x_4 \). Write \( \Delta_{12j}^{12i} = x_4(x_1^2 + x_2^2) + p(x_1, x_2, x_3) \) for some polynomial \( p \).

\( S = S(x_1, x_2, x_3) \) dividing \( \Delta_{12j}^{12i} \not\equiv 0 \) implies that \( S \) divides \( x_1^2 + x_2^2 \), contradicting to the irreducibility of quadratic polynomial \( S \). If there is no such \( y_j^j \), then \( \dim(E) = 3 \).

Therefore \( a_2^2 + 1 = 0 \). By the same argument, since there is no \( x_2^3 x_4 \) in \( \Delta_{134}^{123} \), \( a_2^2 + 1 = 0 \). Comparing the coefficients of \( x_2^2 x_4 \) in \( \Delta_{124}^{123} = LS \) and \( x_3^2 x_4 \) in \( \Delta_{134}^{123} = MS \), we get \( c_4 = b_4 = 0 \).

Then by the same argument as in the case \( l_4 = 0 \) we obtain \( \dim(E) = 3 \). 

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