Family Hierarchy from Symmetry Breaking

Fu-Sin Ling\textsuperscript{1} and Pierre Ramond\textsuperscript{2}

Institute for Fundamental Theory
Department of Physics, University of Florida, Gainesville, FL, 32611, USA

Abstract

We investigate symmetry breaking patterns from replicated gauge groups which generate anomaly-free and family-dependent \(U(1)\) symmetries. We discuss the extent to which these symmetries can explain the observed hierarchies of fermion masses and mixings.

\textsuperscript{1}e-mail: fsling@phys.ufl.edu
\textsuperscript{2}e-mail: ramond@phys.ufl.edu
1 Introduction

The origin of fermion masses and mixings in the Standard Model is still as mysterious as when the first elementary particles were being discovered. Unlike the couplings of fermions to spin one particles which are now understood in terms of Yang-Mills interactions, Yukawa couplings still await such a level of understanding. The lack of a first principle explanation for their patterns have led theorists to devise elaborate schemes [1, 2, 3, 4], none of which (including our owns [6, 7]) are particularly convincing. A less ambitious approach, couched in the language of low energy effective field theories, is that advocated by Froggatt and Nielsen [5] (FN): the hierarchies of masses and mixings stem from higher dimensional operators which, when evaluated in the desired vacuum, yield effective Yukawa interactions of the right strengths. This approach organizes the dimensions of these operators in terms of hitherto unknown $U(1)$ charges. The result is that the suppression level of a particular Yukawa coupling is related to its Froggatt-Nielsen charge.

Among ideas and schemes that involve additional gauge symmetries (flavor or family symmetries) to explain this fermion hierarchy problem, models with $U(1)$ symmetries have been shown to be self-consistent, anomaly-free, and experimentally testable [6, 7, 8, 9, 10]. In these schemes, the addition of a particular symmetry that accommodates data is not derived from first principles; yet it could be very useful in hinting at its possible origin.

Of the many versions of models of this type in the physics literature, those with chiral Froggatt-Nielsen charges are particularly restrictive, since their anomalies must be cancelled. We have investigated FN models with several charges: one is family independent and anomalous; its anomaly is cancelled by a dimension-five Green-Schwarz term at the cut-off [11], together with anomaly-free family dependent FN charges which are responsible for the interfamily hierarchies.

Specially daunting to these models has been the recent determination of the neutrino mass and mixing patterns. While the existence of neutrinos masses and mixings is perfectly natural and expected, the recent determination of two large neutrino mixing angles poses further theoretical challenges [7].

In this paper, we limit our investigation to the generation of anomaly-free family-dependent charges. We investigate the type of mechanisms capable of generating such symmetries.

Cancelling a chiral anomaly is always done by adding new fermions. The easiest is to add fermions of the opposite chirality, the route Nature chooses
for QCD. The next to easiest is to add chiral fermions of different chiralities such that the result adds up to zero, Nature’s choice for the hypercharge anomaly. Today we understand the latter as embedding the Standard Model fermions inside representations of anomaly-free groups.

Central to anomaly cancellation are the groups. The anomaly free groups are well known: all Lie groups except $SU_n$ for $n > 2$. On the other hand, anomalies can occur only if the representations are complex, and so we focus on anomaly-free groups with complex representations: the spinors of $SO(2n + 6)$ and the complex representations of $E_6$. In fact the three families of Standard Model fermions fit remarkably well in a spinor of $SO(10)$ or a $27$ of $E_6$. However the extra charges carried by these groups are not family-dependent, although they are anomaly-free over these representations.

To find anomaly-free, family-dependent charges, we assume, in the spirit of Ref. [12], that the gauge group at the Planck scale is replicated. Taking one copy of the same group $G$ per family, for three families, the fundamental gauge group will be of the form

$$G_{P} = G \times G \times G$$

(1)

The group $G$ should be simple, anomaly-free and should contain the group of the Standard Model, $G_{SM} \equiv SU(3) \times SU(2) \times U(1)_Y$. The fundamental group $G_{P}$ therefore contains $G_{SM} \times G_{SM} \times G_{SM}$. It has to be reduced to the diagonal $G_{SM}$, because Standard Model gauge interactions are flavor blind. Anomaly-free charges with opposite signs for different families can then be generated by order parameters which we call bi- or tri-chiral, to distinguish them from the usual bi-vector vacuum values.

Symmetry breaking down to the Standard Model group proceeds in several steps. We assume that at some stage, there are $U(1)$ family-dependent symmetries which dictate the orders of magnitudes of the Yukawa couplings. In this paper, we restrict the discussion to the generation of anomaly-free family-dependent phase symmetries.

The question of family hierarchy reduces to a search for plausible unifying structures and the way to break them. The qualitative features of fermion masses and mixings are encoded in the underlying group structure and the breaking path.

We will start with some examples that show how these ideas can be realized before working our way towards a realistic scheme.
2 Simple examples

In order to fix our notation, and introduce key concepts, we start this investigation with simple examples.

\[ SU(2) \times SU(2) \rightarrow SU(2) \]

We can first restrict the discussion to two families. The starting point is two copies of the simplest non-Abelian group, namely

\[ SU(2)^\alpha \times SU(2)^\beta, \]

where \( \alpha \) and \( \beta \) label the two copies. The two fermion families fall into the representations

\[ \psi_1 \sim (2,1), \quad \psi_2 \sim (1,2). \]

The order parameter that describes the symmetry breaking is taken to be a scalar field, transforming as a bi-fundamental representation

\[ H \sim (2,2) \]

A suitable vacuum expectation value (vev) of \( H \) is able to trigger the breaking to the diagonal subgroup

\[ SU(2)^\alpha \times SU(2)^\beta \rightarrow SU(2)^{\alpha+\beta} \]

because the order parameter contains a singlet of the unbroken subgroup

\[ 2 \otimes 2 = 1 \oplus 3 \]

After symmetry breaking, the fermions transform as doublets under the remaining diagonal group. As long as the Higgs potential has no extra symmetry, there is no left-over phase symmetry. The two families are not distinguished by any family-dependent symmetry.

\[ SU(3) \times SU(3) \rightarrow SU(2) \times U(1) \]

Our next example starts with the group

\[ SU(3)^\alpha \times SU(3)^\beta, \]

with two fermion families transforming as

\[ \psi_1 \sim (3,1) \equiv 3^\alpha, \quad \psi_2 \sim (1,3) \equiv 3^\beta. \]
We ignore for the moment the important question of anomalies. If the gauge group of the low-energy theory is the diagonal subgroup $SU(2)$, different inequivalent symmetry breakings are possible. Consider first an order parameter of the form $H_{\alpha} \sim (3,3)$; we call it a bi-vectorial order parameter, since the two copies appear the same way up to a conjugation. A suitable vev will trigger the breaking

$$SU(3)_{\alpha} \times SU(3)_{\beta} \rightarrow SU(2)_{\alpha+\beta} \times U(1)_{V_{\alpha+V_{\beta}}}$$  (2)

where $V_{\alpha,\beta}$ is the Abelian factor in the embedding

$$SU(3) \supset SU(2) \times U(1)_{V}$$

The two fermion families are not distinguished after symmetry breaking by the remaining $U(1)$ factor, since

$$3_{\alpha} , 3_{\beta} \rightarrow 2_{1} \oplus 1_{-2}.$$  

In contrast, consider the bi-chiral order parameter, $H_{2} \sim (3,3)$, which can produce the breaking

$$SU(3)_{\alpha} \times SU(3)_{\beta} \rightarrow SU(2)_{\alpha+\beta} \times U(1)_{V_{\alpha-V_{\beta}}}$$  (3)

In this case, the extra Abelian symmetry acts as a family symmetry with opposite charges for the two families

$$3_{\alpha} \rightarrow 2_{1} \oplus 1_{-2}$$

$$3_{\beta} \rightarrow 2_{-1} \oplus 1_{2}$$

We use this toy example to introduce a notation that enables us to catalog in a systematic way all the possible singlet directions that can be chosen by the scalar field during the symmetry breaking. A singlet under $SU(2)^{i}$ will be notated as $1_{vi}^{i}$ with its charge $V^{i}$ in the subscript. A direction singlet under the diagonal subgroup of $SU(2)^{i} \times SU(2)^{j}$ but not under each group separately will be notated by $1_{(v^{i}+v^{j},v^{i}-v^{j})}^{i+j}$. A straightforward calculation of the product $3_{\alpha} \otimes 3_{\beta}$, applied to the bi-vectorial order parameter yields the following $SU(2)^{\alpha+\beta}$ singlet

$$\left(3_{\alpha}, 3_{\beta}\right) \rightarrow \left(2_{1}^{\alpha} \oplus 1_{-2}^{\alpha}\right) \otimes \left(2_{1}^{\beta} \oplus 1_{-2}^{\beta}\right) \rightarrow 1_{(0,2)}^{\alpha+\beta} \oplus 1_{(0,2)}^{\alpha} \oplus 1_{(0,2)}^{\beta}.$$  (4)

The first singlet $1_{(0,2)}^{\alpha+\beta}$ leaves the family-independent $U(1)_{V_{\alpha+V_{\beta}}}$ unbroken. The second singlet, $1_{(0,2)}^{\alpha} \otimes 1_{(0,2)}^{\beta}$, leaves the larger subgroup $SU(2)^{\alpha} \times SU(2)^{\beta} \times U(1)_{V_{\alpha+V_{\beta}}}$ invariant.
The same analysis applied to the bi-chiral order parameter yields
\[(3^\alpha, 3^\beta) \rightarrow (2_1^\alpha \oplus 1_{-2}^\alpha) \otimes (2_1^\beta \oplus 1_{-2}^\beta) \rightarrow 1^{\alpha+\beta}_{(2,0)} \oplus (1_{-2}^\alpha \otimes 1_{-2}^\beta) \] (5)

It is clear that \(1^{\alpha+\beta}_{(2,0)}\) now leaves the diagonal subgroup \(SU(2)^{\alpha+\beta}\) and the family-dependent phase symmetry \(U(1)_{\nu-\nu}\) unbroken. Although the fermions are chiral, this simple model is not realistic since the starting group is anomalous.

\[G \times G \longrightarrow SU(3) \times U(1)\]

It is more difficult to generate a diagonal \(SU(3)\), starting with two copies of a simple gauge group \(G\). The maximal embeddings leading to a single \(SU(3)\) are

\[
\begin{align*}
SU(4) & \supset SU(3) \times U(1) \quad 4 = 3_{-1} \oplus 1_3 \\
Sp(6) & \supset SU(3) \times U(1) \quad 6 = 3_1 \oplus 3_{-1} \\
SO(8) & \supset SU(3) \quad 8 = 8 \\
SU(6) & \supset SU(3) \quad 6 = 6 \\
G_2 & \supset SU(3) \quad 7 = 3 \oplus 3 \oplus 1 \\
E_6 & \supset SU(3) \quad 27 = 27 \\
E_7 & \supset SU(3) \quad 56 = 28 \oplus 28
\end{align*}
\] (6)

Only the first two embeddings contain an extra \(U(1)\) factor. \(Sp(6)\) is vectorial under \(SU(3)\), and this chain cannot lead to a model with chiral fermions. This leaves \(SU(4)\), putting anomalies aside. To avoid anomalies, one can embed \(SU(4)\) into \(SO(7)\)

\[SO(7) \supset SU(4) ; \quad 7 = 6 \oplus 1,\]

but then the \(7\) of \(SO(7)\) is vectorial, yielding vector-like fermions under \(SU(3)\)

\[SO(7) \supset SU(4) \supset SU(3) \times U(1) ; \quad 7 = 3_2 \oplus 3_{-2} \oplus 1_0\]

Because of \(SU(3)\) triality, a bi-chiral order parameter does not contain an \(SU(3)^{\alpha+\beta}\) singlet. This suggests we consider three copies

\[G \times G \times G \longrightarrow SU(3) \times U(1)\]

We start with three copies

\[SU(4)^\alpha \times SU(4)^\beta \times SU(4)^\gamma,\]
together with three fermion families

\[ \psi_1 \sim (4, 1, 1), \quad \psi_2 \sim (1, 4, 1), \quad \psi_3 \sim (1, 1, 4). \]

In this case, the tri-chiral order parameter

\[ H \sim (4, 4, 4) \]

is capable of breaking to the diagonal \( SU(3) \). Using the decomposition,

\[ SU(4) \supset SU(3) \times U(1)_V ; \quad 4 = 3 \oplus 1 - 3, \]

and using \( SU(3) \) triality,

\[ 3 \otimes 3 \otimes 3 \supset 1 \]

we see that the breaking

\[ SU(4)_{\alpha} \times SU(4)_{\beta} \times SU(4)_{\gamma} \rightarrow SU(3)_{\alpha+\beta+\gamma} \times U(1)_{\nu_{\alpha}-\nu_{\beta} \times U(1)_{\nu_{\beta}-\nu_{\gamma}}} \]

is obtained if the order parameter takes a vev along the singlet \( 1^{\alpha+\beta+\gamma}_{(3,0,0)} \), where the three subscripts refer to the sum of the charges and their two differences. Hence this order parameter produces two family-dependent phase symmetries. This yields the following fermions: three quarks, distinguished by their family charges, \( (1, 0), (-1, 1), \) and \( (0, -1), \) together with three singlets of charges \( (-3, 0), (3, -3), \) and \( (0, 3). \) This model is chiral, but riddled with anomalies.

3 Towards a realistic scheme

We learned from these simple but unrealistic examples that a good candidate for a replicated gauge group must have complex representations to describe chiral fermions, and must be anomaly-free. The candidate groups are then either orthogonal groups with complex spinor representations, or the exceptional group \( E_6 \) \[13\]. Here, we limit ourselves to studying groups that appear in the sequence

\[ E_6 \supset SO(10) \supset SU(5) \supset SU(3) \times SU(2) \times U(1)_Y \]  \( \text{(7)} \)

The complex representation 27 of \( E_6 \) decomposes itself as

\[ E_6 \supset SO(10) \times U(1)_{V'} \quad 27 = 16_1 \oplus 10_{-2} \oplus 1_4 \]

\[ SO(10) \supset SU(5) \times U(1)_V \quad 16 = 10_1 \oplus 5_{-3} \oplus 1_5 \]

\[ 10 = 5_{-2} \oplus 5_{2} \]

\[ SU(5) \supset SU(3) \times SU(2) \times U(1)_Y \quad 10 = (3, 2)_{1/3} \oplus (3, 1)_{-4/3} \oplus (1, 1)_{2} \]

\[ 5 = (1, 2)_{-1} \oplus (3, 1)_{2/3} \]

\( \text{(8)} \)
We first consider the case

\[ SO(10)^{\alpha} \times SO(10)^{\beta} \times SO(10)^{\gamma} \rightarrow SU(5)^{\alpha+\beta+\gamma} \times U(1) \]

A tri-chiral order parameter

\[ H_1 \sim (16, 16, 16) \]

can trigger the desired breaking to the diagonal \( SU(5) \), because the product \( 16 \otimes 16 \otimes 16 \) contains a \( SU(5) \) singlet, namely

\[ 16^\alpha \otimes 16^\beta \otimes 16^\gamma \supset 10^\alpha_1 \otimes 5^\beta_{-3} \otimes 5^\gamma_{-3} \supset S_{1\alpha} \equiv 1^{\alpha+\beta+\gamma}_{(-5,4,4)} \]  

(9)

where the \( U(1) \) charges for the \( SU(5)^{\alpha+\beta+\gamma} \) singlet \( 1^{\alpha+\beta+\gamma} \) are given in subscript in the form \((v^\alpha + v^\beta + v^\gamma, v^\alpha - v^\beta, v^\alpha - v^\gamma)\). The product \( 16 \otimes 16 \otimes 16 \) also contains a singlet under \( SU(5)^3 \), as can be seen from the decomposition (8). These singlets are listed in Table 1.

Table 1: Singlets of \( SU(5) \) in the product \( 16^\alpha \otimes 16^\beta \otimes 16^\gamma \) and associated broken directions \((a, b, c)\). Singlets obtained by permutation of the indices \( \alpha, \beta, \gamma \) are omitted.

| \( SU(5) \) Singlet in \( 16 \otimes 16 \otimes 16 \) | Unbroken non-Abelian Group | Broken Direction |
|---|---|---|
| \( 1_5^\alpha \otimes 1_5^\beta \otimes 1_5^\gamma \) | \( SU(5)^{\alpha} \times SU(5)^{\beta} \times SU(5)^{\gamma} \) | \( (1, 1, 1) \) |
| \( 1^{\alpha+\beta+\gamma}_{(-5,4,4)} \) | \( SU(5)^{\alpha+\beta+\gamma} \) | \( (1, -3, -3) \) |

Because a singlet direction carries three charges, it will always leave two independent \( U(1) \) symmetries unbroken. If all the charges vanish, then a third independent \( U(1) \) is also unbroken. If we associate a vector \( \vec{v} = (a, b, c) \) in a three dimensional space to the linear combination \( aV^\alpha + bV^\beta + cV^\gamma \),
then for each singlet with at least one non-vanishing charge, the vectors \( \vec{v} \) corresponding to the \( U(1) \) symmetries left unbroken by this singlet span a (hyper-)plane. We can therefore associate to such a singlet a vector perpendicular to this plane, and call it the broken direction.

For example, the singlet \( S_{1\alpha} \) is invariant under any linear combination of \( U(1)_{V^\alpha} \) and \( U(1)_{3V^\alpha + V^\beta} \). Therefore, its broken direction is given by a vector perpendicular to \( (0, 1, -1) \) and \( (3, 1, 0) \), so we can take \( \vec{v}_{1\alpha} = (1, -3, -3) \).

Among all the possible linear combinations of \( V^\alpha, V^\beta \) and \( V^\gamma \), the sum \( V^\alpha + V^\beta + V^\gamma \) has the distinctive characteristic to be family-blind. It will be left unbroken by a singlet \( S \) if the vector \( (1, 1, 1) \) is perpendicular to the broken direction \( \vec{v}_S \) associated to \( S \). In other words, the \( U(1) \) symmetry corresponding to \( \vec{v}_S \) has to be traceless over the family index. For example, the singlet \( S_{1\alpha} \) does not leave the combination \( V^\alpha + V^\beta + V^\gamma \) unbroken.

We also notice that the initial gauge group is invariant under a permutation of the indices \( \alpha, \beta, \gamma \). However, a singlet \( S \) can spontaneously break this permutation symmetry, corresponding to the group \( P_3 \cong \mathbb{Z}_6 \), down to a smaller subgroup \( D \). All the singlets obtained from \( S \) by a permutation of the indices \( \alpha, \beta, \gamma \) belong to the same conjugacy class under the coset \( \mathbb{Z}_6/D \).

For example, the singlet \( S_{1\alpha} \) is invariant only under the exchange \( \beta \leftrightarrow \gamma \). In other words, it breaks the symmetry \( \mathbb{Z}_6 \) down to \( D = \mathbb{Z}_2 \). Therefore, we can construct two equivalent singlets \( S_{1\beta} \) and \( S_{1\gamma} \), obtained from \( S_{1\alpha} \) by cyclic permutation of the indices \( \alpha, \beta, \gamma \)

\[
S_{1\beta} \equiv 1^{\alpha+\beta+\gamma}_{(-5,-4,0)} \ , \quad S_{1\gamma} \equiv 1^{\alpha+\beta+\gamma}_{(-5,0,-4)}
\]

Their corresponding broken directions are \( \vec{v}_{1\beta} = (-3, 1, -3) \) and \( \vec{v}_{1\gamma} = (-3, -3, 1) \), and are again obtained by cyclic permutation of the indices \( \alpha, \beta, \gamma \).

The appearance of a spontaneously broken discrete symmetry can be potentially harmful in the cosmological context, because it leads to the formation of domains and domain walls. However, if the scale at which the symmetry is broken is very high, which is the case here, their presence will be washed away during inflation.

We can now combine several singlets in order to narrow down the unbroken symmetry. By this, we mean that the order parameter can yield a non zero vev in more than one \( SU(5) \) singlet. The \( U(1) \) factors left over are found as the intersection of the unbroken spaces for each of the non-zero singlet. For example, the vev \( <H_1> = S_{1\alpha} \oplus S_{1\beta} \) will trigger the breaking

\[
SO(10)^\alpha \times SO(10)^\beta \times SO(10)^\gamma \longrightarrow SU(5)^{\alpha+\beta+\gamma} \times U(1)_{Y_F}
\]
with
\[ Y_F = V^\alpha + V^\beta - \frac{2}{3} V^\gamma \] (10)

This yields three \( SU(5) \) families
\[
(\mathbf{5}_3, \mathbf{10}_1, \mathbf{1}_5) ; \quad (\mathbf{3}_3, \mathbf{10}_1, \mathbf{1}_5) ; \quad (\mathbf{3}_2, \mathbf{10}_{-2/3}, \mathbf{1}_{-10/3}) .
\]
The subscript refers to their \( Y_F \) values. This family symmetry distinguishes the third family from the first two.

Let us emphasize the question of anomalies. The Abelian charge \( Y_F \) given by Eq. (10) is our first example of an anomaly-free and family-dependent symmetry. The cancellation of anomalies is achieved by completing the 16, and is ensured because \( SO(10) \) is an anomaly-free group. Therefore, the inclusion of all three right-handed neutrinos is necessary. If we take the \( SU(5) \) representations separately, the mixed anomaly coefficients \( (Y_F SU(5) SU(5)) \) will not vanish, but the contributions from \( \mathbf{10}, \mathbf{5} \) and \( \mathbf{1} \) compensate each other.

The mixed anomalies will only vanish over each \( SU(5) \) representation in the case where \( Y_F \) is traceless. This can be achieved by taking a tri-vectorial order parameter instead of a tri-chiral one
\[ H_2 \sim (\mathbf{45}, \mathbf{45}, \mathbf{45}) \]

With the branching rule
\[ SO(10) \supset SU(5) \times U(1)_V ; \quad \mathbf{45} = 24_0 \oplus \mathbf{10}_{-4} \oplus \mathbf{10}_4 + \mathbf{1}_0 \] (11)
we can obtain singlets of the type \( \mathbf{1}^\alpha \otimes \mathbf{1}^\beta \otimes \mathbf{1}^\gamma, \mathbf{1}^{\alpha+\beta} \otimes \mathbf{1}^\gamma \) and \( \mathbf{1}^{\alpha+\beta+\gamma} \). They are listed in Table 2 together with their broken direction.

Notice that the adjoint representation has the peculiarity that its tensor product with itself always contains a singlet and an adjoint representation. Therefore, in the case of \( SO(10) \), we have
\[ \mathbf{45}^\alpha \otimes \mathbf{45}^\beta \supset \mathbf{1} ; \quad \mathbf{45}^\alpha \otimes \mathbf{45}^\beta \otimes \mathbf{45}^\gamma \supset \mathbf{1} \] (12)

These singlets have a zero charge under the diagonal \( U(1)_V \), \( U(1)_{V^\alpha+V^\beta} \) and \( U(1)_{V^\alpha+V^\beta+V^\gamma} \) respectively, but do not have a charge under the orthogonal combinations (and hence, these are broken). They are of course also singlets under the respective diagonal \( SU(5) \). However, using the branching rule (11), we can actually construct singlets contained in the tensor product of adjoint representations of \( SU(5) \), which have well-defined (and zero) charges under
The orthogonal linear combinations of $V$'s. As mentioned previously, singlets for which all the charges vanish have no associated broken direction.

The singlet $S_2 = 1^{α+β+γ}_{(0,0)}$ has the desired features in order to give rise to a traceless family symmetry. The $U(1)$ corresponding to its broken direction is traceless, therefore, it triggers the breaking

$$SO(10)^α \times SO(10)^β \times SO(10)^γ \rightarrow SU(5)^{α+β+γ} \times U(1)_X \times U(1)_{Y_F}$$

with

$$X = V^α + V^β + V^γ, \quad Y_F = V^α + V^β - 2V^γ$$

The charge $X$ is family-blind while the charge $Y_F$ is traceless. This yields the following three chiral families along with their $Y_F$ charges

$$(\mathbf{5}_3, \mathbf{10}_1, \mathbf{1}_5); \quad (\mathbf{5}_3, \mathbf{10}_1, \mathbf{1}_5); \quad (\mathbf{5}_6, \mathbf{10}_{-2}, \mathbf{1}_{-10}).$$

As emphasized earlier, we notice that the mixed anomaly coefficients $(Y_F \cdot SU(5) \cdot SU(5))$ between $SU(5)$ and the traceless family charge $Y_F$ vanish.
The charge $Y_F$ is said to be non-anomalous in the language of Froggatt and Nielsen. In contrast, the anomaly coefficients $(X SU(5) SU(5))$ for a given $SU(5)$ are necessarily different from zero, but are family-independent. The charge $X$ is said to be anomalous.

Finally, the anomalies $(Y_F X X)$ also vanish because $Y_F$ is traceless. As before, a complete cancellation of the anomalies is achieved by completing the 16 of $SO(10)$. In the case that we have considered so far, we were able to construct a family symmetry that is anomaly-free, and get three families of chiral fermions with different family charges. However, the breaking of $SO(10)$ down to $SU(5)$ lowers the rank by only one unit, which does not leave enough room to build a family symmetry that can accommodate the observed phenomenology of fermion masses. This can be achieved by upgrading to $E_6$, as we are ready to see now.

$$E_6 \times E_6 \times E_6 \rightarrow SU(5) \times U(1)$$

A further complication arises here, because $SU(5) \times U(1)$ is not a maximal subalgebra of $E_6$. Two chains of maximal subalgebras can lead from $E_6$ to $SU(5)$, namely (leaving aside the possible $U(1)$ factors)

$$E_6 \supset SO(10) \supset SU(5)$$ (13)

which is the chain we already considered, or

$$E_6 \supset SU(6) \times SU(2) \supset SU(5) \times SU(2)$$ (14)

In what follows, we will consider order parameters which are tri-chiral, tri-vectorial, bi-chiral and bi-vectorial

$$H_1 \sim (27, 27, 27)$$
$$H_2 \sim (78, 78, 78)$$
$$H_3 \sim (27, 27, 1)$$
$$H_4 \sim (27, 27, 1)$$ (15)

The irrep 27 contains a singlet of $SO(10)$, but no singlet under $SU(6) \times SU(2)$,

$$E_6 \supset SU(6) \times SU(2) ; \quad 27 = (\overline{6}, 2) \oplus (15, 1)$$
nor under $SU(5) \times SU(2)$ as we have

$$SU(6) \supset SU(5) \times U(1) ; \quad 6 = 5_1 \oplus 1_{-5} ; \quad 15 = 5_{-4} \oplus 10_2$$

However, we can see that it contains two distinct singlets of $SU(5)$. The way to understand this is the following. By making suitable linear combinations, one is a singlet under $SO(10)$, and the other one is singlet under $SU(5) \times U(1)$ but not under $SO(10)$. We will designate these singlets by $1_{(0,4)}$ and $1_{(5,1)}$ respectively. They are distinguished by their charges $(v, v')$ under $U(1)_V$ and $U(1)_{V'}$.

Similarly, the irrep $78$ decomposes itself as

$$E_6 \supset SO(10) \times U(1)_{V'} ; \quad 78 = 45_0 \oplus 16_{-3} \oplus \overline{16}_3 \oplus 1_0$$

$$E_6 \supset SU(6) \times SU(2) ; \quad 78 = (1, 3) \oplus (35, 1) \oplus (20, 2)$$

Therefore, it contains one singlet of $SO(10)$ but no singlet under $SU(6) \times SU(2)$. Using the branching rules

$$SU(6) \supset SU(5) \times U(1) ; \quad 20 = 10_{-3} \oplus 100_3$$

$$35 = 1_0 \oplus 5_6 \oplus \overline{5}_6 \oplus 24_0 ,$$

we further see that $78$ does contain a singlet under $SU(5) \times SU(2)$. We can also see that it contains four singlets under $SU(5)$ alone, which is in agreement with what is obtained using the chain of maximal subalgebras (13), because

$$SO(10) \supset SU(5) \times U(1)_V ; \quad 16 = 10_1 \oplus \overline{5}_{-3} \oplus 1_5$$

$$45 = 24_0 \oplus 10_{-4} \oplus 100_4 + 1_0$$

Two of these singlets have a distinct non-zero charge $V$, they are designated by $1_{(5,3)}$ and $1_{(5,-3)}$, following our notation. The two remaining singlets have zero $V$ and $V'$ charges. By making suitable linear combinations, one is singlet under $SO(10)$, and the other one is singlet under $SU(5) \times SU(2)$. We will designate them collectively by $1_{(0,0)}$, which is sufficient for the discussion of $U(1)$ factors. The implications of an extra $SU(2)$ factor in the framework of Froggatt and Nielsen is beyond the scope of the present paper, and will be described elsewhere.

Following our method, a singlet under the diagonal $SU(5)$ will be associated with a broken direction in a six-dimensional space, $\vec{v} = (a, b, c, a', b', c')$ corresponding to the combination $aV^\alpha + bV^\beta + cV^\gamma + dV'^{\alpha} + eV'^{\beta} + fV'^{\gamma}$. Many more possibilities of $SU(5)$ singlet directions arise in this case. Lists of inequivalent singlets (under permutation of the indices $\alpha, \beta, \gamma$) and their associated broken directions are given in the Tables 3 to 5.
4 Physical Implications

4.1 More on anomalies

As noticed earlier, singlets that leave the family-blind combinations unbroken are of particular interest. They trigger a symmetry breaking in which the $U(1)$ symmetries are factorized into a family-blind part $X$ and a family dependent part $Y_F$ which is traceless over the family index. This decomposition, in turn, enables a "multi-layered" anomaly cancellation, which is a central ingredient in the construction of Froggatt-Nielsen-type models.

The first layer is given by the cancellation of all the possible anomalies involving only Standard Model groups over the fermion content of the Standard Model. This "accidental" cancellation is now understood in terms of embeddings, because the fermions of the Standard Model have the right quantum numbers to fit into the representations $10$ and $\bar{5}$ of $SU(5)$, which in turn, complete the $16$ of $SO(10)$ when a right-handed neutrino, singlet under $SU(5)$, is added.

In the second layer, there will be mixed anomalies between $X$ or $Y_F$ and Standard Model groups $G_{SM}$. Because the family charge $Y_F$ commutes with $SU(5)$ in our framework, it is sufficient to consider the mixed anomalies between $X$ or $Y_F$ and $SU(5)$. Anomaly coefficients with a single $SU(5)$, namely $(X X SU(5))$, $(X Y_F SU(5))$ and $(Y_F Y_F SU(5))$, vanish over any representation of $SU(5)$. The anomaly coefficient $(Y_F SU(5) SU(5))$ vanishes over the family index because $Y_F$ is traceless. The anomaly coefficients $(X SU(5) SU(5))$ differ from zero – $X$ is anomalous – but are family-independent. In the low-energy point of view of effective theories of the Froggatt-Nielsen type, they are cancelled through a dimension five term at the cut-off, in the so-called Green-Schwarz anomaly cancellation mechanism [11].

In the third layer, the remaining anomaly coefficients do not involve $G_{SM}$ (or $SU(5)$). The anomalies $(X Y_F Y_F)$ and $(X X X)$ can also be compensated by the Green-Schwarz mechanism. The coefficient $(Y_F X X)$ vanish because $Y_F$ is traceless, so that the only non-vanishing coefficient that needs to be compensated is $(Y_F Y_F Y_F)$.

This last cancellation can be achieved by adding new matter fields which don’t carry any Standard Model charge. Their presence in the effective theory is solely for the sake of anomaly cancellation, and does not modify the observable phenomenology.

Of course, in the top-down approach used in our framework, we know how...
the question of anomaly cancellation is resolved. The fermion content of the Standard Model is upgraded to the $27$ of $E_6$ by the addition of vector-like matter under $SU(5)$, $5 \oplus \overline{5}$, and two singlets. Therefore, it is not surprising that the theory looks anomalous when only the Standard Model fermions are taken into account!

4.2 Predictions for fermion masses and mixings

Among the possible linear combinations of $V$ and $V'$, $V^\alpha + V^\beta + V^\gamma$ and $V'^\alpha + V'^\beta + V'^\gamma$ are family-blind. Therefore, at least three singlets are needed in order to fix completely the traceless family symmetry $Y_F$. If this is the case, the interfamily mass ratios are completely determined by the family charges $Y_F$ of the fermions, independently of the dynamics which further breaks the family symmetry.

Let us briefly recall how the family symmetry is related to fermion mass hierarchies. To fix our notation, we will use the supersymmetric setup of Ref. [6, 7]. The Yukawa couplings for quarks and charged leptons stem from invariants in the superpotential of the form $Q_i \pi_j H_u$, $Q_i \overline{d}_j H_d$, $L_i \pi_j H_d$, where $i$ and $j$ are family indices. For neutrinos, after the see-saw mechanism has taken place, the effective Yukawa coupling is given by the quartic invariant $L_i L_j H^2_u$. In the presence of a family symmetry, these invariants can appear in the superpotential only if they are not charged under the extra family symmetries. If they are charged, they can still appear, but only as higher dimensional operators

$$\tilde{I} = I \left( \frac{\theta}{M} \right)^n,$$

where $I$ is a MSSM invariant, $\theta$ is an order parameter, singlet under $G_{SM}$ but charged under $Y_F$, and $M$ is the cut-off scale, usually taken as the string scale. The operator $\tilde{I}$ has to be gauge invariant. The families symmetries are spontaneously broken when the $\theta$ fields get a vev, yielding the expansion parameter

$$\lambda = \frac{<\theta>}{M}.$$

As a result, the Yukawa coupling corresponding to the invariant $I_{ij}$ will be suppressed by a power $n_{ij}$ which is related to its family charge $Y_F$

$$n_{ij} = -Y_F(I_{ij}) + \text{cst.} \quad (16)$$

where the family-independent constant arises because of the anomalous charge
We can notice that these powers obey the sum rule

\[ n_{ii} + n_{jj} = n_{ij} + n_{ji}. \]

In our framework, the family symmetry depends only upon \( V \) and \( V' \), and therefore commutes with \( SU(5) \). The structure of the mass matrices is then determined by the charges \(-Y_F(10)\) and \(-Y_F(\mathbf{5})\), reordered and normalized to the heaviest family

\[ Q, \bar{u}, \bar{e} \rightarrow -Y_F(10) \]
\[ L, \bar{d} \rightarrow -Y_F(\mathbf{5}) \]

We have taken all the possible sets of three singlets that determine the family symmetry \( Y_F \), and derived the corresponding Yukawa structure. However, thousands of different patterns can emerge. To reduce this number, we chose to enforce a phenomenological constraint. To make it strong and reliable, only the heavier fermions from the second and third families are involved.

The measured fermion mass ratios scale as

\[ \frac{m_c}{m_t} \sim \lambda^4; \quad \frac{m_s}{m_b} \sim \lambda^2; \quad \frac{m_\mu}{m_\tau} \sim \lambda^2 \]  

(17)

where the expansion parameter \( \lambda \) is of the order of the Cabibbo angle \( \lambda_c \approx 0.22 \). Therefore, we can use the constraint derived from the relations

\[ \frac{m_c}{m_t} \sim \left( \frac{m_s}{m_b} \right)^2 \sim \left( \frac{m_\mu}{m_\tau} \right)^2 \]  

(18)

After noticing that Eq. (18) is compatible with a family symmetry commuting with \( SU(5) \), we see that it translates into

\[ Y_F(\mathbf{5}_2) = Y_F(\mathbf{5}_3) \]  

(19)

This relation, in turn, implies a large mixing angle for the atmospheric neutrinos, as it is indeed observed \[ 14 \] !

It turns out that the constraint (19) restricts severely the number of possible patterns. Four 'scenarios' were considered, in which the order parameter that is spontaneously broken transforms as \( H_1, H_2, H_4 \) and as \( H_1 \) or \( H_3 \) in the last one (see Eq. (15)). We find that a bi-chiral or a bi-vectorial order parameter does not lead to a family symmetry with physical interest. The family charges for a tri-chiral or tri-vectorial order parameter that survive
from the constraint (19) are listed in Tables 6 and 7. The interesting charges that give rise to a phenomenology compatible with all present data on quarks, charged leptons and neutrinos masses and mixings have been underlined. It can be noticed that all of them can be obtained with an order parameter transforming like $H_2$.

Most of these charges are only in weak agreement with the data. The mismatch lies in the predicted masses for the first generation. A light up quark mass is in conflict with the derivation of the correct CKM matrix. A further conflict comes from the neutrinos sector, where a mild $\Delta m^2$ hierarchy, and a large solar mixing angle induce a heavier electron.

In the following, we analyze in more details three of these possible charge assignments.

**Model A**

The charges $-Y_F(10) \sim (2, 1, 0)$ and $-Y_F(5) \sim (0, 0, 0)$ with expansion parameter $\lambda \sim \lambda_c^2$ give rise to the so-called anarchical model [15], where hierarchical and mixing structure is totally absent from the neutrino mass matrix. The observed hierarchy between the $\Delta m^2_\odot$ for atmospheric neutrinos and the $\Delta m^2_\odot$ for solar neutrinos (for the LMA solution [17])

$$\frac{\Delta m^2_\odot}{\Delta m^2_\oplus} \approx 10^{-2}$$ (20)

can still be obtained but resides in Nature’s choice of the prefactor coefficients. Large mixing angles for solar and atmospheric neutrinos are natural in this context, but the small value of the CHOOZ [19] mixing angle is problematic. The model is otherwise in fair agreement with phenomenology. The predicted mass ratios are

$$\frac{m_u}{m_t} \sim \lambda_c^8; \quad \frac{m_c}{m_t} \sim \lambda_c^4$$
$$\frac{m_d}{m_b} \sim \lambda_c^4; \quad \frac{m_s}{m_b} \sim \lambda_c^2$$
$$\frac{m_e}{m_\tau} \sim \lambda_c^4; \quad \frac{m_\mu}{m_\tau} \sim \lambda_c^2$$

However, we notice that the electron mass comes out too heavy compared to the $\tau$ mass. A smaller ratio

$$\frac{m_e}{m_\tau} \sim \lambda_c^{5-6}$$
would be in better agreement with the measured masses. Moreover, some stretching in the prefactor coefficients is needed to reconcile the model with the structure of the CKM matrix \[15\].

**Model B**

The charges $-Y_F(10) \sim (3, 2, 0)$ and $-Y_F(5) \sim (2, 0, 0)$ with $\lambda \simeq \lambda_c$ enable to reproduce the CKM matrix. The predictions for quarks and charged leptons are

\[
\frac{m_u}{m_t} \sim \lambda_c^6 ; \quad \frac{m_c}{m_t} \sim \lambda_c^4 \\
\frac{m_d}{m_b} \sim \lambda_c^4 ; \quad \frac{m_s}{m_b} \sim \lambda_c^2 \\
\frac{m_e}{m_\tau} \sim \lambda_c^4 ; \quad \frac{m_\mu}{m_\tau} \sim \lambda_c^2
\]

with a heavier up quark. In the neutrino sector, the resulting $\Delta m^2$ hierarchy is in agreement with Eq. (20), the CHOOZ mixing angle is suppressed by a factor $\lambda_c^2$, but the solar mixing angle also turns out to be naturally small, which is less satisfying in view of the recent solar neutrino data \[17\], and the KamLAND result \[18\].

A phenomenologically better charge assignment would be $-Y_F(10) \sim (3, 2, 0)$ and $-Y_F(5) \sim (1, 0, 0)$ It can been incorporated in a consistent way in a FN model \[7\], but could not be reproduced by symmetry breaking in the present approach.

**Model C**

Finally, the charges $-Y_F(10) \sim (5, 2, 0)$ and $-Y_F(5) \sim (1, 0, 0)$ also with $\lambda \simeq \lambda_c$ can be in fairly good agreement with the observations. The predicted mass ratios are

\[
\frac{m_u}{m_t} \sim \lambda_c^{10} ; \quad \frac{m_c}{m_t} \sim \lambda_c^4 \\
\frac{m_d}{m_b} \sim \lambda_c^6 ; \quad \frac{m_s}{m_b} \sim \lambda_c^2 \\
\frac{m_e}{m_\tau} \sim \lambda_c^6 ; \quad \frac{m_\mu}{m_\tau} \sim \lambda_c^2
\]

We notice that the electron to $\tau$ mass ratio have the correct order of magnitude, but the up and the down quarks appear a little light. However, it
has been pointed out that non perturbative QCD effects can also contribute to the masses of the lightest quarks at low-energy, so that the actual masses and the mixings in the CKM matrix can be recovered (see [16] for example).

5 Conclusions

The possibility that the fundamental gauge group at very high scales appears replicated in several copies, as inspired by the brane world, opens up new possibilities to understand the patterns in the low energy world. We have studied how the family mass hierarchy problem can be elucidated in this context through a group-theoretical approach. Imprints of symmetry breakings were recognized and systematically analyzed, although the dynamics which triggers the symmetry breaking is beyond the scope of the present work.

The realistic scheme considered, based on $E_6 \times E_6 \times E_6 \rightarrow SU(5)$ shows interesting features. Although many different family symmetries can survive after symmetry breaking, a simple and reliable phenomenological constraint killed most of them. Moreover, only a few possibilities lead to mass patterns in accordance with observations. So it might be that our world has something exceptional rather than generic, that very particular dynamical conditions triggered such a symmetry breaking. Or it might also be that the true mechanism that Nature choose to order fermion masses is totally different.

However, it is worth pointing that this approach does give rise to patterns in agreement with our world. Moreover, it appears that this path, for reasons explained throughout this paper, necessarily leads to the consideration of exceptional algebras. So Nature might indeed be exceptional... Anyhow, the idea that mass could be partially treated as a quantum number is a very attractive scheme that can help us organize and understand the legacy of the Standard Model.

Acknowledgments

This work is supported by the United States Department of Energy under grant DE-FG02-97ER41029.
References

[1] L. Ibañez, G. G. Ross, Phys. Lett. B 332, 100 (1994); Pierre Binétruy and Pierre Ramond, Phys. Lett. B 350, 49 (1995).

[2] R. Barbieri, G. Dvali, L. J. Hall, Phys. Lett. B 377, 76 (1996); A. Aranda, C. D. Carone and P. Meade, Phys. Rev. D 65, 013011 (2002).

[3] S. F. King and G. G. Ross, Phys. Lett. B 520, 243 (2001).

[4] M. S. Berger and Kim Siyeon, Phys. Rev. D 64, 053006 (2001).

[5] C. Froggatt and H.B. Nielsen, Nucl. Phys. B 147, 277 (1979).

[6] P. Binétruy, N. Irges, S. Lavignac and P. Ramond, Phys. Lett. B 403, 38 (1997); N. Irges, S. Lavignac and P. Ramond, Phys. Rev. D 58, 035003 (1998).

[7] Fu-Sin Ling and Pierre Ramond, Phys. Lett. B 543, 29 (2002).

[8] Ann E. Nelson and David Wright, Phys. Rev. D 56, 1598 (1997).

[9] J. Sato and K. Tobe, Phys. Rev. D 63, 116010 (2001).

[10] N. Maekawa, Prog. Theor. Phys. 107, 597 (2002).

[11] M. Green and J. Schwarz, Phys. Lett. B 149, 117 (1984).

[12] C. D. Froggatt, M. Gibson, H. B. Nielsen and D. J. Smith, IJMPA 13, 5037 (1998).

[13] R. Slansky, Phys. Rept. 79, 1 (1981); G. W. Anderson and T. Blazek, J. Math. Phys. 41, 4808 (2000); G. W. Anderson and T. Blazek, J. Math. Phys. 41, 8170 (2000).

[14] The Super-Kamiokande Collaboration, Phys. Rev. Lett. 85, 3999 (2000); T. Toshito for the SK Coll., Proceedings of the XXXVIth Rencontres de Moriond - Electroweak Interactions and Unified Theories, 36 (2001)

[15] M.S. Berger and K. Siyeon, Phys. Rev. D 63, 057302 (2001).

[16] T. Banks, Y. Nir, N. Seiberg, Yukawa Couplings and The Origins of Mass (Conference Proceedings), 26 (1996).
[17] M. B. Smy for SK, Proceedings of the XXXVIth Rencontres de Moriond, hep-ex/0106064. The SNO Collaboration, Phys. Rev. Lett. 87, 071301 (2001); The SNO Collaboration, Phys. Rev. Lett. 89, 011301 (2002); The SNO Collaboration, Phys. Rev. Lett. 89, 011302 (2002).

[18] The KamLAND Collaboration, Phys. Rev. Lett. 90, 021802 (2003).

[19] M. Apollonio et al., Phys. Lett. B 338, 383 (1998); M. Apollonio et al., Phys. Lett. B 420, 397 (1998).
Table 3: Singlets of $SU(5)$ in the product $27^\alpha \otimes 27^\beta \otimes 27^\gamma$ and associated broken directions $(a, b, c, a', b', c')$. Singlets obtained by permutation of the indices $\alpha, \beta, \gamma$ are omitted.

| $SU(5)$ Singlet in $27 \otimes 27 \otimes 27$ | Unbroken non-Abelian Group | Broken Direction |
|---------------------------------------------|-----------------------------|------------------|
| $1^\alpha_{(0,4)} \otimes 1^\beta_{(0,4)} \otimes 1^\gamma_{(0,4)}$ | $SO(10)^\alpha \times SO(10)^\beta \times SO(10)^\gamma$ | $(0,0,0,1,1,1)$ |
| $1^\alpha_{(5,1)} \otimes 1^\beta_{(0,4)} \otimes 1^\gamma_{(0,4)}$ | $SU(5)^\alpha \times SO(10)^\beta \times SO(10)^\gamma$ | $(5,0,0,1,4,4)$ |
| $1^\alpha_{(5,1)} \otimes 1^\beta_{(5,1)} \otimes 1^\gamma_{(0,4)}$ | $SU(5)^\alpha \times SU(5)^\beta \times SO(10)^\gamma$ | $(5,5,0,1,1,4)$ |
| $1^\alpha_{(5,1)} \otimes 1^\beta_{(5,1)} \otimes 1^\gamma_{(5,1)}$ | $SU(5)^\alpha \times SU(5)^\beta \times SU(5)^\gamma$ | $(5,5,5,1,1,1)$ |

| $1^{\alpha+\beta}_{(0,-4,4,0)} \otimes 1^\gamma_{(0,4)}$ | $SU(5)^{\alpha+\beta} \times SO(10)^\gamma$ | $(1,-1,0,-1,-1,2)$ |
| $1^{\alpha+\beta}_{(-5,-1,-1,3)} \otimes 1^\gamma_{(0,4)}$ | $SU(5)^{\alpha+\beta} \times SO(10)^\gamma$ | $(-3,-2,0,1,-2,4)$ |
| $1^{\alpha+\beta}_{(-5,-1,-1,3)} \otimes 1^\gamma_{(5,1)}$ | $SU(5)^{\alpha+\beta} \times SU(5)^\gamma$ | $(-3,-2,5,1,-2,1)$ |
| $1^{\alpha+\beta+\gamma}_{(0,-4,4,0)} \otimes 1^\gamma_{(5,1)}$ | $SU(5)^{\alpha+\beta+\gamma}$ | $(2,-2,5,-2,-2,1)$ |
| $1^{\alpha+\beta+\gamma}_{(-5,3,4,0,4,0)}$ | $SU(5)^{\alpha+\beta+\gamma}$ | $(1,-3,-3,1,1,1)$ |
| $1^{\alpha+\beta+\gamma}_{(0,0,-5,3,-4,0)}$ | $SU(5)^{\alpha+\beta+\gamma}$ | $(-3,2,1,1,-2,1)$ |
| $1^{\alpha+\beta+\gamma}_{(0,0,0,0,3,3)}$ | $SU(5)^{\alpha+\beta+\gamma}$ | $(1,1,-2,1,1,-2)$ |
| $1^{\alpha+\beta+\gamma}_{(5,-3,0,0,1,-3)}$ | $SU(5)^{\alpha+\beta+\gamma}$ | $(2,2,1,-2,-2,1)$ |
Table 4: Singlets of SU(5) in the product $\mathbf{78}^\alpha \otimes \mathbf{78}^\beta \otimes \mathbf{78}^\gamma$ and associated broken directions. Singlets obtained by permutation of the indices $\alpha$, $\beta$, $\gamma$ and conjugate singlets (with the sign of all charges reversed) are omitted, and $G$ stands for either SO(10) or SU(5) $\times$ SU(2).

| SU(5) Singlet in $\mathbf{78} \otimes \mathbf{78} \otimes \mathbf{78}$ | Unbroken non-Abelian Group | Broken Direction |
|---------------------------------------------------------------|------------------------------|-----------------|
| $\mathbf{1}^{\alpha}_{(5,-3)} \otimes \mathbf{1}^{\beta}_{(5,-3)} \otimes \mathbf{1}^{\gamma}_{(5,-3)}$ | SU(5)$^\alpha$ $\times$ SU(5)$^\beta$ $\times$ SU(5)$^\gamma$ | (5, 5, 5, -3, -3, -3) |
| $\mathbf{1}^{\alpha}_{(5,-3)} \otimes \mathbf{1}^{\beta}_{(5,-3)} \otimes \mathbf{1}^{\gamma}_{(-5,3)}$ | SU(5)$^\alpha$ $\times$ SU(5)$^\beta$ $\times$ SU(5)$^\gamma$ | (5, 5, -5, -3, -3, 3) |
| $\mathbf{1}^{\alpha}_{(5,-3)} \otimes \mathbf{1}^{\beta}_{(5,-3)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | SU(5)$^\alpha$ $\times$ SU(5)$^\beta$ $\times$ G$^\gamma$ | (5, 5, 0, -3, -3, 0) |
| $\mathbf{1}^{\alpha}_{(5,-3)} \otimes \mathbf{1}^{\beta}_{(-5,3)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | SU(5)$^\alpha$ $\times$ SU(5)$^\beta$ $\times$ G$^\gamma$ | (5, -5, 0, -3, 3, 0) |
| $\mathbf{1}^{\alpha}_{(5,-3)} \otimes \mathbf{1}^{\beta}_{(0,0)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | SU(5)$^\alpha$ $\times$ G$^\beta$ $\times$ G$^\gamma$ | (5, 0, 0, -3, 0, 0) |
| $\mathbf{1}^{\alpha+\beta}_{(0,0,6,6)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | | (1, -1, 0, 1, -1, 0) |
| $\mathbf{1}^{\alpha+\beta}_{(0,0,-8,0)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | SU(5)$^{\alpha+\beta}$ $\times$ G$^\gamma$ | (1, -1, 0, 0, 0, 0) |
| $\mathbf{1}^{\alpha+\beta}_{(0,0,2,-6)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | | (1, -1, 0, -3, 3, 0) |
| $\mathbf{1}^{\alpha+\beta}_{(-5,3,-3,-3)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | | (-4, -1, 0, 0, 3, 0) |
| $\mathbf{1}^{\alpha}_{(0,0)} \otimes \mathbf{1}^{\beta}_{(0,0)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | G$^\alpha$ $\times$ G$^\beta$ $\times$ G$^\gamma$ | None |
| $\mathbf{1}^{\alpha+\beta}_{(0,0,0,0)} \otimes \mathbf{1}^{\gamma}_{(0,0)}$ | SU(5)$^{\alpha+\beta}$ $\times$ G$^\gamma$ | None |
| $\mathbf{1}^{\alpha+\beta+\gamma}_{(0,0,0,0,0)}$ | SU(5)$^{\alpha+\beta+\gamma}$ | None |
Table 5: Singlets of $SU(5)$ in the product $78^\alpha \otimes 78^\beta \otimes 78^\gamma$ and associated broken directions (continued).

| $SU(5)$ Singlet in $78 \otimes 78 \otimes 78$ | Unbroken non-Abelian Group | Broken Direction |
|---------------------------------------------|-----------------------------|------------------|
| $1^{\alpha+\beta}_{(0,0,6,6)} \otimes 1^\gamma_{(5,-3)}$ | $SU(5)^{\alpha+\beta} \times SU(5)^\gamma$ | $(3, -3, 5, 3, -3, -3)$ |
| $1^{\alpha+\beta}_{(0,0,-8,0)} \otimes 1^\gamma_{(5,-3)}$ |                  | $(-4, 4, 5, 0, 0, -3)$ |
| $1^{\alpha+\beta}_{(0,0,2,-6)} \otimes 1^\gamma_{(5,-3)}$ |                  | $(1, -1, 5, -3, 3, -3)$ |
| $1^{\alpha+\beta}_{(-5,3,-3,-3)} \otimes 1^\gamma_{(5,-3)}$ |                  | $(-4, -1, 5, 0, 3, -3)$ |
| $1^{\alpha+\beta}_{(-5,3,-3,-3)} \otimes 1^\gamma_{(-5,3)}$ |                  | $(-4, -1, -5, 0, 3, 3)$ |
| $1^{\alpha+\beta+\gamma}_{(10,6,0,0,-1,3)}$ | $SU(5)^{\alpha+\beta+\gamma}$ | $(3, 3, 4, 3, 3, 0)$ |
| $1^{\alpha+\beta+\gamma}_{(5,9,0,0,4,0)}$ |                  | $(3, 3, -1, 3, 3, 3)$ |
| $1^{\alpha+\beta+\gamma}_{(-5,3,0,0,7,3)}$ |                  | $(4, 4, -3, 0, 0, -3)$ |
| $1^{\alpha+\beta+\gamma}_{(-5,3,0,2,6)}$ |                  | $(-1, -1, -3, 3, 3, -3)$ |
| $1^{\alpha+\beta+\gamma}_{(0,0,5,-3,7,3)}$ |                  | $(4, -1, -3, 0, 3, -3)$ |
Table 6: Family charges ($-Y_F(10), -Y_F(5)$) normalized to the heaviest family (third family), obtained from $H_1 \sim (27, 27, 27)$.

| (1, 1, 0), (0, 0, 0) | (1, 1, 0), (1, 0, 0) | (1, 1, 0), (2, 0, 0) | (1, 1, 0), (5, 0, 0) |
| (1, 1, 0), (8, 0, 0) | (2, 2, 0), (1, 0, 0) | (2, 2, 0), (7, 0, 0) | (3, 1, 0), (1, 0, 0) |
| (3, 1, 0), (4, 0, 0) | (3, 2, 0), (2, 0, 0) | (4, 3, 0), (1, 0, 0) | (5, 2, 0), (4, 0, 0) |

Table 7: Family charges ($-Y_F(10), -Y_F(5)$) normalized to the heaviest family (third family), obtained from $H_2 \sim (78, 78, 78)$.

| (1, 1, 0), (2, 0, 0) | (1, 1, 0), (5, 0, 0) | (2, 1, 0), (0, 0, 0) | (2, 1, 0), (3, 0, 0) |
| (2, 2, 0), (1, 0, 0) | (3, 1, 0), (1, 0, 0) | (3, 1, 0), (4, 0, 0) | (3, 1, 0), (16, 0, 0) |
| (3, 2, 0), (2, 0, 0) | (3, 2, 0), (5, 0, 0) | (4, 1, 0), (2, 0, 0) | (4, 1, 0), (5, 0, 0) |
| (4, 3, 0), (1, 0, 0) | (4, 3, 0), (10, 0, 0) | (5, 1, 0), (3, 0, 0) | (5, 2, 0), (1, 0, 0) |
| (5, 2, 0), (4, 0, 0) | (5, 5, 0), (7, 0, 0) | (6, 2, 0), (5, 0, 0) | (6, 5, 0), (8, 0, 0) |
| (7, 4, 0), (5, 0, 0) | (8, 5, 0), (1, 0, 0) | (9, 5, 0), (2, 0, 0) | |