Abstract

A secret sharing scheme is a method of sharing a secret key among a finite set of participants in such a way that only certain specified subsets of participants can compute the key. The access structure of a secret sharing scheme is the family of these subsets of participants which are able to recover the secret. If the length in bits of every share is the same as the length of the secret, then the scheme is called ideal. An access structure is said to be multipartite, if the set of participants is divided into several parts and all participants in the same part play an equivalent role. The search for ideal secret sharing schemes for some special interesting families of multipartite access structures, has been carried out by many authors. In this paper a new concept of study of ideal access structures is proposed. We do not consider special classes of access structures defined by imposing certain prescribed assumptions, but we investigate all access structures obtained from uniform polymatroids using the method developed by Farràs, Martí-Farré and Padró (cf. Theorem 2.4 and Definition 2.6 below). They satisfy necessary condition to be ideal, i.e., they are matroid ports. Moreover some objects in this family can be useful for the applications of secret sharing. The choice of uniform polymatroids is motivated by the fact that each such polymatroid defines ideal access structures. The method presented in this article is universal and can be continued with other classes of polymatroids in further similar studies. Here we are especially interested in hierarchy of participants determined by the access structure and we distinguish two main classes: they are compartmented and hierarchical access structures. The vast majority of papers discussing hierarchical access structures consider access structures which are compartment or totally hierarchical. The main results are summarized in Section 4 which presents situations where partial hierarchy properties may arise. In particular, hierarchical orders of obtained structures are described. It is surprising, that the hierarchical orders of access structures obtained from uniform polymatroids are flat, i.e., every chain has at most 2 elements. The ideality of some families of hierarchical access structures is proved in Section 5.

Keywords secret sharing - multipartite access structure - ideal access structure - partially hierarchical access structure - uniform polymatroid.

1 Introduction

A secret sharing scheme is a method of sharing a secret piece of data among a finite set of participants in such a way that only certain specified subsets of participants can compute the secret data. Secret sharing was originally introduced by Blakley [2] and Shamir [16] independently in 1979 as a solution for safeguarding cryptographic keys, but nowadays it is used in many cryptographic protocols.
Let $P$ be a finite set of participants and let $p_0 \notin P$ be a special participant called the *dealer*. Given a secret, the dealer computes the shares and distributes them secretly to the participants, so that no participant knows the share given to another one. It is required that only certain *authorized* subsets of $P$ can recover the secret by pooling their shares together. It is easily seen that the family $\Gamma$ of all authorized sets, called an *access structure*, is monotone increasing, which means that any superset of an authorized subset is also authorized. To avoid abnormal situations, we assume that $\emptyset \notin \Gamma$ and $P \in \Gamma$. If every unauthorized set of participants cannot reveal any information about the secret, regardless of the computational power available, then the secret sharing scheme is said to be *perfect*. Such a scheme can be considered as unconditionally secure.

Ito, Saito, Nishizeki [10] and Benaloh, Leichter [11] independently proved, in a constructive way, that every monotone increasing family of subsets of $P$ admits a perfect secret sharing scheme. Therefore, every monotone increasing family of subsets of $P$ is referred to as an access structure. Obviously, every access structure is uniquely determined by the family of its minimal sets. An access structure is said to be *connected* if every participant in $P$ is a member of a certain minimal authorized set. If an access structure is not connected, then every participant which does not belong to any minimal authorized set is called *redundant* because its share is never necessary to recover the secret.

Given a secret sharing scheme, let $S_0$ be the set of all possible secrets and let $S_p$ be the set of all possible values of shares that can be assigned to the participant $p$ for every $p \in P$. One can show that for every perfect secret sharing scheme the size of $S_0$ is not greater than the size of $S_p$ for all $p \in P$. A perfect secret sharing scheme is called *ideal* if $|S_0| = |S_p|$ for all $p \in P$. In other words, the length in bits of every share is the same as the length of the secret. Shamir’s threshold schemes [16] are the best known examples of ideal secret sharing schemes. The secret sharing schemes constructed for a given access structure in [10] and [1] are very far from being ideal because the length of the shares grows exponentially with the number of participants. An access structure is said to be *ideal*, if it is the access structure of an ideal secret sharing scheme. An access structure is said to be *multipartite* if the set of participants is divided into several blocks which are pairwise disjoint and participants in individual blocks are equivalent (precise definition can be found in subsection 2.1). The study of multipartite access structures was initiated by Kothari [12], who posed the open problem of constructing ideal hierarchical secret sharing schemes, and by Simmons [17], who introduced the multilevel and compartmented access structures. This approach, developed by many authors, provides a very effective tool for describing structures in a compact way, by using a few conditions that are independent of the total number of participants.

The characterization of ideal access structures is one of the main open problems in the secret sharing theory. This problem seems to be extremely difficult and only some particular results are known. In many papers the authors consider some specific classes of access structures with prescribed properties and try to check whether these structures are ideal. Most of the results obtained are based on the connections between ideal secret sharing schemes and matroids discovered by Brickell [3] and Brickell and Davenport [4]. Later, the use of polymatroids proposed by Farràs, Martí-Farré, Padró in [7] provided a new tool for studying ideal multipartite access structures.

A concise review of the results contained in the literature can be found in the papers [7] - [9]. Since ideal access structures are known to be matroid ports, it seems quite natural to look for ideal access structures among matroid ports. Given a specific class of polymatroids, one can take all multipartite access structures determined by these polymatroids and investigate their properties. This approach ensures that the objects under consideration satisfy necessary condition to be ideal, i.e. they are matroid ports (cf. Theorem 2.1 below). The ideality can be established on the base of properties of particular polymatroids. In this paper the study is restricted to uniform polymatroids. This choice is motivated by the fact that each such polymatroid defines a family of ideal access structure (cf. Remark 2.10). But the method presented here is universal and can be continued with other classes of polymatroids in further similar studies (cf. [13]).

The relations between ideal access structures and matroids discovered by Brickell and Davenport are recalled here in Theorem 2.1 and Theorem 2.2. A short introduction to matroids and
polymatroids and their relation to access structures are presented in Subsection 2.2. It follows from Theorem 2.4 by Farrás, Martí-Farré and Padró [7] that every polymatroid with the ground set $J$ and a monotone increasing family of subsets of $J$ which is compatible with the polymatroid determine a unique access structure which is a matroid port. The details are described in Definition 2.6. In Subsection 2.3 some relations between uniform polymatroids $Z = (J, h, g)$ and monotone increasing families $\Delta \subseteq P(J) \setminus \{\emptyset\}$ are presented. We prove several technical properties which are useful in the next sections.

In this paper, we focus on the classification of multipartite access structures $\Gamma = \Gamma(\Pi, Z, \Delta)$ determined by uniform polymatroids $Z$ and monotone increasing families $\Delta$ in a set of participants divided into a partition $\Pi$. We examine hierarchical order among the participants induced by the obtained access structure. In the third section we present several conditions that polymatroid $Z$ and monotone increasing family $\Delta$ must meet when the structure $\Gamma = \Gamma(\Pi, Z, \Delta)$ is (weakly) hierarchical. It turns out that the existence of hierarchically comparable blocks imposes strong restrictions on the increment sequence $g$ of the polymatroid.

At the beginning of the fourth section, which together with the fifth section contains the main results of the paper, those conditions are used to prove Theorems 4.2 and 4.3 which shows that most of access structures obtained from uniform polymatroids are compartmented. Then the exact hierarchy in some special access structures is examined in Theorems 4.6 and 4.8 - 4.11. Moreover, we prove in Theorem 4.12 that the maximal length of chains in such hierarchical access structures is equal to 1. This fact seems quite surprising, because for other polymatroids one can construct hierarchical access structures with chains of arbitrary length. For instance, such constructions can be found in [8], [9], [13], [18] and others. As was mentioned above, every uniform polymatroid determines some ideal access structures, but the question is whether all access structures determined by uniform polymatroids are ideal. A direction, which is worth considering and may result in getting the answer, is using the fact that a sufficient condition (for an access structures to be ideal) can be obtained by proving that the simple extension of a given uniform polymatroid is representable (cf. [6, Corollary 6.7]). This method has been applied in Section 5 to the proof that all the structures described in Theorems 4.6 and 4.8 - 4.11 are ideal.

It is worth noting that the class of access structures obtained from uniform polymatroids contains some interesting families of objects that can be useful for the applications of secret sharing. The access structures discussed in Theorem 4.11 correspond to the organizational chart of an institution composed of several mutually independent departments managed by one superior unit. It follows from Theorem 5.2 that all those access structures are ideal. Another interesting example is the family of uniform access structures characterized by Farràs et al. in [9, Section VI] (cf. Remark 4.10 below). It consists of multipartite access structures that are invariant under any permutation of blocks of participants. In other words all participants have the same rights, although they are not hierarchically equivalent.

A different situation occurs in compartmented access structures, where there is a set of distinguished participants, whose representatives must be present in all authorized sets. Such a case is described in Theorem 5.2.

This paper is intended to initiate research on the access structures obtained from uniform polymatroids, but it does not exhaust the topic and leaves space for further study. Some remarks on the new research possibilities can be found in Section 6. The Appendix contains a classification of all access structures with four parts obtained from uniform polymatroids.

## 2 Preliminaries

The aim of this section is to provide the necessary definitions and results regarding multipartite access structures and polymatroids.

Throughout the paper we use the following notations. The family of all subsets of a set $X$ is denoted by $P(X)$ (the power set). Similarly $P_k(X)$ denotes the collection of all of $k$-element subsets of $X$. Let $\mathbb{N}_0$ and $\mathbb{N}$ denote the set of all non-negative integers and positive integers, respectively. Let $J$ be a finite set. For two vectors $\bar{u} = (u_x)_{x \in J}, \bar{v} = (v_x)_{x \in J} \in \mathbb{N}_0^J$ we write $\bar{u} \leq \bar{v}$
if \( u_x \leq v_x \) for all \( x \in J \). Moreover, \( \bar{u} < \bar{v} \) denotes \( u < v \) and \( \bar{u} \neq \bar{v} \). Given a vector \( \bar{v} = (v_x)_{x \in J} \), we define the support \( \text{supp}(\bar{v}) = \{ x \in J : v_x \neq 0 \} \) and the modulus \( |\bar{v}| = \sum_{x \in J} v_x \). Furthermore, we write \( \bar{v}_X = (v'_x)_{x \in J} \), where \( X \subseteq J \) and

\[
v'_x = \begin{cases} v_x & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases}
\]

In particular, \( \bar{v}_0 = (0)_{x \in J} \). Let us observe that \( \text{supp}(\bar{v}) \subseteq X \). For every \( z \in J \), we define the vector \( \bar{e}(z) \in \mathbb{N}^J_0 \) such that \( \bar{e}(z) = (e^z_x)_{x \in J} \) with \( e^z_x = 1 \) and \( e^z_x = 0 \) for all \( x \neq z \).

### 2.1 Multpartite access structures

Let \( \Gamma \) be an access structure on a set of participants \( P \). A participant \( p \in P \) is said to be hierarchically superior or equivalent to a participant \( q \in P \) (written \( p \preceq q \)), if \( A \cup \{ p \} \in \Gamma \) for all subsets \( A \subseteq P \setminus \{ p, q \} \) with \( A \cup \{ q \} \in \Gamma \). If \( p \preceq q \) and \( q \preceq p \), then the participants \( p, q \) are called hierarchically equivalent. Participants \( p, q \in P \) are said to be hierarchically independent if neither \( p \) is hierarchically superior or equivalent to \( q \) nor \( q \) is hierarchically superior or equivalent to \( p \).

By a partition (\( \Pi \)-partition) of the set of participants \( P \) we mean a family \( \Pi = (P_x)_{x \in J} \) of pairwise disjoint and nonempty subsets of \( P \), called blocks such that \( P = \bigcup_{x \in J} P_x \). An access structure \( \Gamma \) is said to be multipartite (\( \Pi \)-partite) if all participants in every block \( P_x \) are pairwise hierarchically equivalent. Thus we are allowed to define a hierarchy in \( \Pi \). Namely, \( P_x \) is said to be hierarchically superior or equivalent to \( P_y \) (written \( P_x \preceq P_y \)) if there are \( p \in P_x \) and \( q \in P_y \) such that \( p \preceq q \). In other words, it can be said that \( P_y \) is hierarchically inferior or equivalent to \( P_x \). By transitivity we have \( p \preceq q \) for all \( p \in P_x \) and \( q \in P_y \) whenever \( P_x \preceq P_y \). The relation \( \preceq \) is reflexive and transitive and not antisymmetric in general, so it is a preorder.

Moreover, this preorder is determined by the access structure \( \Gamma \), so it should be denoted by \( \preceq_{\Gamma} \). However, to simplify notation we write \( \preceq \) if it does not lead to ambiguity. Similarly, blocks \( P_x \) and \( P_y \) are said to be hierarchically independent if there are \( q \in P_x \) and \( p \in P_y \) such that \( p \preceq q \). On the other hand, if \( P_x \preceq P_y \) or \( P_y \preceq P_x \), then the blocks \( P_x \) and \( P_y \) are called hierarchically comparable. Moreover, if \( P_x \preceq P_y \) and \( P_y \preceq P_x \), then the blocks \( P_x \) and \( P_y \) are called hierarchically equivalent. If \( P_x \preceq P_y \) and the blocks are not hierarchically equivalent, then we write \( P_x \prec P_y \).

Let us recall that a participant which does not belong to any minimal authorized set is called redundant. It is easy to see that every participant is hierarchically superior or equivalent to any redundant participant. In particular, all redundant participants are hierarchically equivalent. A block of participants which contains a redundant participant will be also called redundant.

A \( \Pi \)-partite access structure is said to be compartmented if every pair of blocks in \( \Pi \) is hierarchically independent. Otherwise, the access structure is referred to as weakly hierarchical. If an access structure is weakly hierarchical and no pair of blocks in \( \Pi \) is hierarchically equivalent, then the access structure will be called hierarchical. A hierarchical access structure such that every pair of blocks is hierarchically comparable is referred to as totally hierarchical. A complete characterization of ideal totally hierarchical access structure was presented by Farràs and Padró [8]. It is worth pointing out that the phrase "compartmented access structure" used here is very general and covers several notions with the same name appearing in the literature.

Given a partition \( \Pi = (P_x)_{x \in J} \) of \( P \) and a subset \( A \subseteq P \) we define the vector \( \pi(A) = (v_x)_{x \in J} \), where \( v_x = \left| A \cap P_x \right| \). If \( \Gamma \) is a \( \Pi \)-partite access structure, then all participants in every subset \( P_x \) are pairwise hierarchically equivalent, so if \( A \in \Gamma \), \( B \subseteq P \) and \( \pi(A) = \pi(B) \), then \( B \in \Gamma \). We put \( \pi(\Gamma) = \{ \pi(A) \in \mathbb{N}^J_0 : A \in \Gamma \} \) and

\[
\pi(\mathcal{P}(P)) = \{ \pi(A) \in \mathbb{N}^J_0 : A \subseteq P \} = \{ \bar{v} \in \mathbb{N}^J_0 : \bar{v} \leq \pi(P) \}.
\]

Obviously, if \( A \subseteq B \subseteq P \), then \( \pi(A) \leq \pi(B) \). Moreover, if \( \bar{u} \in \pi(\Gamma) \) and \( \bar{u} \leq \bar{v} \leq \pi(P) \), then \( \bar{v} \in \pi(\Gamma) \). Indeed, there is \( A \in \Gamma \) such that \( \bar{u} = \pi(A) \). The set \( A \) can be extended to a set \( B \subseteq P \) such that \( \bar{v} = \pi(B) \). Hence \( B \in \Gamma \) and consequently \( \bar{v} \in \pi(\Gamma) \). This shows
that $\pi(\Gamma) \subseteq \pi(\mathcal{P}(P))$ is a set of vectors monotone increasing with respect to $\leq$. On the other hand, every monotone increasing set $\Gamma' \subseteq \pi(\mathcal{P}(P))$ determines the II-partite access structure $\Gamma = \{A \subseteq P : \pi(A) \in \Gamma'\}$. This shows that there is a one-to-one correspondence between the family of II-partite access structures defined on $P$ and the family of monotone increasing subsets of $\pi(\mathcal{P}(P))$. Therefore we use the same notation $\Gamma$ for both the access structure and its vector representation.

The hierarchy among blocks in II can be characterized in vector terms as follows: $P_y \leq P_x$ if and only if
\[
\bar{v} - \bar{e}(y) + \bar{e}(x) \in \Gamma \text{ for all } \bar{v} \in \Gamma \text{ with } v_y \geq 1 \text{ and } v_x < |P_x|.
\]
To show that $P_y \leq P_x$ it is enough to check if the above condition is satisfied for all vectors $v \in \min \Gamma$. A block $P_x$ in II is redundant if and only if $v_x = 0$ for every $\bar{v} \in \min \Gamma$.

2.2 Polymatroids and access structures

Let $J$ be a nonempty finite set and let $\mathcal{P}(J)$ denote the power set of $J$. A polymatroid $Z$ is a pair $(J, h)$ where $h$ is a mapping $h : \mathcal{P}(J) \to \mathbb{R}$ satisfying

1. $h(\emptyset) = 0$;
2. $h$ is monotone increasing: if $X \subseteq Y \subseteq J$, then $h(X) \leq h(Y)$;
3. $h$ is submodular: if $X, Y \subseteq J$, then $h(X \cap Y) + h(X \cup Y) \leq h(X) + h(Y)$.

The mapping $h$ is called the rank function of a polymatroid. If all values of the rank function are integer, then the polymatroid is called integer. An integer polymatroid $(J, h)$ such that $h(X) \leq |X|$ for all $X \subseteq J$ is called a matroid. All polymatroids considered in this paper are assumed to be integer, so we will omit the term “integer” when dealing with integer polymatroid.

Let $Z = (J, h)$ be a polymatroid and let $x \in J$ such that $h(\{x\}) = 1$. The set $\{X \in \mathcal{P}(J \setminus \{x\}) : h(X \cup \{x\}) = h(X)\}$ is called a polymatroid port or more precisely, the port of polymatroid $Z$ at the point $x$. One can show that every polymatroid port is a monotone increasing family of some subsets of $J \setminus \{x\}$, which does not contain $\emptyset$.

The following examples of polymatroids play a special role in studying ideal access structures. Let $V$ be a vector space of finite dimension and let $\mathcal{V} = (V_x)_{x \in J}$ be a family of subspaces of $V$. One can show that the mapping $h : \mathcal{P}(J) \to \mathbb{N}_0$ defined by $h(X) = \dim(\sum_{x \in X} V_x)$ for $X \in \mathcal{P}(J)$ is the rank function of the polymatroid $\mathcal{Z} = (J, h)$. The polymatroids that can be defined in this way are said to be representable. If $\dim V_x \leq 1$ for all $x \in J$, then we obtain a matroid which is called representable as well. The family $\mathcal{V}$ is referred to as a vector space representation of the polymatroid (matroid). Let $\mathcal{B} = (B_x)_{x \in J}$ be a family of finite sets. One can show that the mapping $h : \mathcal{P}(J) \to \mathbb{N}_0$ defined by $h(X) = |\bigcup_{x \in X} B_x|$ for $X \in \mathcal{P}(J)$ is the rank function of the integer polymatroid $Z = (J, h)$. Every polymatroid that can be defined in this way is said to be Boolean and the family $\mathcal{B}$ is called the Boolean representation of the polymatroid. Boolean polymatroids are known to be representable.

The connection between matroids and ideal access structures was discovered by Brickell and Davenport [4]. They proved that if $\Gamma \subseteq \mathcal{P}(P)$ is the access structure of an ideal secret sharing scheme on a set of participants $P$ with a dealer $p_0 \notin P$, then there is a matroid $\mathcal{S}$ with the ground set $P \cup \{p_0\}$ such that $\Gamma$ is the port of $\mathcal{S}$ at the point $p_0$. This result can be stated as follows.

**Theorem 2.1** (E.F. Brickell, D.M. Davenport [4]). *Every ideal access structure is a matroid port.*

The converse is not true. For example, the ports of the Vamos matroid are not ideal access structures (cf. [15]). The following result is obtained as a consequence of the linear construction of ideal secret sharing schemes due to Brickell [3].

**Theorem 2.2** (E.F. Brickell [3]). *Every port of a representable matroid is an ideal access structure.*
Let \( Z = (J, h) \) be a polymatroid. For \( J' = J \cup \{x_0\} \) with a certain \( x_0 \notin J \) and a monotone increasing family \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) we define the function \( h' : \mathcal{P}(J') \rightarrow \mathbb{N}_0 \) by \( h'(X) = h(X) \) for all \( X \in \mathcal{P}(J) \) and 
\[
h'(X \cup \{x_0\}) = \begin{cases} h(X) & \text{if } X \in \Delta, \\ h(X) + 1 & \text{if } X \in \mathcal{P}(J) \setminus \Delta. \end{cases}
\]
If \( h' \) is monotone increasing and submodular, then \( \Delta \) is said to be compatible with \( Z \) and \( Z' = (J', h') \) is a polymatroid which is called the simple extension of \( Z \) induced by \( \Delta \). It is easy to see that \( h'(x_0) = 1 \) and \( \Delta \) is the polymatroid port of \( Z' \) at the point \( x_0 \). The next result, which is a consequence of [5] Proposition 2.3] is very useful in the investigation of access structures induced by polymatroids.

**Lemma 2.3** ([5] L. Csirmaz). A monotone increasing family \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) is compatible with an integer polymatroid \( Z = (J, h) \) if and only if the following conditions are satisfied:

1. If \( Y \subseteq X \subseteq J \) and \( Y \notin \Delta \) while \( X \in \Delta \), then \( h(Y) < h(X) \).
2. If \( X, Y \in \Delta \) and \( X \cap Y \notin \Delta \), then \( h(X \cap Y) + h(X \cup Y) < h(X) + h(Y) \).

The following notation will be used very often throughout the paper. Let \( Z = (J, h) \) be a polymatroid and let \( X \subseteq J \). We define the following set
\[
B(Z, X) = \{ v \in \mathbb{N}_0^J : \text{ supp}(v) \subseteq X, \|v\| = h(X), \forall Y \subseteq X, |v_Y| \leq |h(Y)| \}.
\]
It is easy to see that
\[
\text{if } Y \subseteq X \subseteq J \text{ and } h(Y) = h(X), \text{ then } B(Z, Y) \subseteq B(Z, X).
\]
On the other hand, \( B(Z, Y) \cap B(Z, X) = \emptyset \) whenever \( h(Y) \neq h(X) \).

The definition of \( B(Z, X) \) is related to the concept of bases of polymatroids. Given a polymatroid \( Z = (J, h) \) and \( \emptyset \neq X \subseteq J \), we consider \( Z|X = (X, h|X) \), where \( h|X : \mathcal{P}(X) \rightarrow \mathbb{N}_0 \). It is easy to see that \( Z|X \) is a polymatroid which is called the restriction of \( Z \) to \( X \). Let \( Z = (J, h) \) be a polymatroid. We define the set of bases of \( Z \) by \( B(Z) = \{ v \in \mathbb{N}_0^J : |v_X| \leq h(X) \text{ for all } X \subseteq J \text{ and } |v| = h(J) \} \). It is known that every polymatroid is uniquely determined by the set of its bases. The set \( B(Z, X) \) is obtained from the set \( B(Z|X) \subseteq \mathbb{N}_0^X \) of bases of the restriction \( Z \) to \( X \) by the canonical embedding of \( \mathbb{N}_0^X \) into \( \mathbb{N}_0^J \).

Now, we recall an important theorem of Farràs, Martí-Farré and Padró [7] that characterizes those multipartite access structures that are matroid ports.

**Theorem 2.4.** [7] Theorem 5.3] Let \( \Pi = (P_x)_{x \in J} \) be a partition of a set \( P \) and let \( \Gamma \) be a connected \( \Pi \)-partite access structure on \( P \). Consider \( \Delta = \text{ supp}(\Gamma) \). Then \( \Gamma \) is a matroid port if and only if there exists an integer polymatroid \( Z = (J, h) \) with \( h(\{x\}) \leq |P_x| \) for every \( x \in J \) such that \( \Delta \) is compatible with \( Z \) and \( \min \pi(\Gamma) = \min \bigcup_{X \in \Delta} B(Z, X) \).

**Remark 2.5.** Let \( \Pi = (P_x)_{x \in J} \) be a partition of a set \( P \). Let \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) be a monotone increasing family compatible with a polymatroid \( Z = (J, h) \) such that \( h(\{x\}) \leq |P_x| \) for all \( x \in J \). Farràs, Martí-Farré, Padró proved that if the simple extension of \( Z \) determined by \( \Delta \) is a representable polymatroid, then the multipartite access structure \( \Gamma \) such that \( \min \cap = \min \bigcup_{X \in \Delta} B(Z, X) \) is ideal. This result generalizes the result of Brickell [3].

Theorem 2.4 can be used as a simple tool for constructing multipartite access structures which are matroids ports.

**Definition 2.6.** For a given partition \( \Pi = (P_x)_{x \in J} \) we take a polymatroid \( Z = (J, h) \) with \( h(\{x\}) \leq |P_x| \) for every \( x \in J \), a monotone increasing family \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) which is compatible with \( Z \) and we construct the smallest monotone increasing family \( \Gamma' \subseteq \pi(\mathcal{P}(P)) \) which contains \( \min \bigcup_{X \in \Delta} B(Z, X) \). In other words, \( \Gamma' \) is the only monotone increasing family contained in \( \pi(\mathcal{P}(P)) \) such that \( \min \Gamma' \subseteq \bigcup_{X \in \Delta} B(Z, X) \subseteq \Gamma' \). Obviously, \( \Gamma = \{ A \subseteq P : \pi(A) \in \Gamma' \} \subseteq \mathcal{P}(P) \) is the access structure in the set of participants induced by its vector representation \( \Gamma' \). Both \( \Gamma \) and \( \Gamma' \) will be called the \( \Pi \)-partite access structure determined by a polymatroid \( Z \) and a monotone increasing family \( \Delta \) and will be denoted by \( \Gamma(\Pi, Z, \Delta) \).
Example 2.7. Let us consider \( J' = \{0, 1, 2, 3\} \) and the function \( h' : \mathcal{P}(J') \rightarrow \mathbb{N}_0 \) defined by

\[
h'(X) = \begin{cases} 
0 & \text{if } |X| = 0; \\
1 & \text{if } |X| = 1; \\
2 & \text{if } |X| \geq 2.
\end{cases}
\]

It is easy to check that \( Z' = (J', h') \) is a polymatroid and \( \Delta = \{(1, 2), (1, 3), (2, 3), (1, 2, 3)\} \) is its port at 0. Moreover, \( Z' \) is a simple extension of \( Z = \mathcal{Z}[J] \), where \( J = \{1, 2, 3\} \). Thus \( \Delta \) is compatible with \( Z \). Hence we get \( \mathcal{B}(Z, \{1, 2\}) = \{(1, 1, 0)\}, \mathcal{B}(Z, \{1, 3\}) = \{(1, 0, 1)\}, \mathcal{B}(Z, \{2, 3\}) = \{(0, 1, 1)\} \) and \( \mathcal{B}(Z, \{1, 2, 3\}) = \{(1, 0, 1), (0, 1, 1)\} \). Now we are ready to define an access structure \( \Gamma \) assuming \( \min \Gamma = \min \bigcup_{X \in Z} \mathcal{B}(Z, X) = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \).

According to Theorem 2.4 the access structure obtained in this way satisfies necessary condition to be ideal. The results of [7] mentioned in Remark 2.5 provides a sufficient condition for \( \Gamma(\Pi, Z, \Delta) \) to be ideal.

Remark 2.8. Assume that \( h(\{x\}) = 0 \) for a certain \( x \in J \). Suppose \( \bar{v} \) is a minimal vector in \( \Gamma(\Pi, Z, \Delta) \), then there is \( X \in \Delta \) such that \( \bar{v} \in \mathcal{B}(Z, X) \). By definition, \( v_y \leq h(\{y\}) \) for all \( y \in J \).

In particular \( v_x \leq h(\{x\}) = 0 \), i.e. no participant from \( P_x \) belongs to \( X \). This shows that if \( h(\{x\}) = 0 \), then all participants in \( P_x \) are redundant, so every access structure induced by \( Z \) is not connected. Therefore, from now on we assume that \( h(\{x\}) > 0 \) for all \( x \in J \).

2.3 Uniform polymatroids

We begin this subsection with the definition of uniform polymatroids which play a major role in this paper. To shorten notation we set \( I_m = \{0, 1, \ldots, m\} \).

Definition 2.9. An integer polymatroid \( Z = (J, h) \) is called uniform if

\[ |X| = |Y| \implies h(X) = h(Y) \quad \text{for all } X, Y \subseteq J. \]

Let \( m := |J| \). We define \( h_i = h(X) \) for every \( i = 0, 1, \ldots, m \) with \( X \subseteq J, |X| = i \). It is obvious, that the sequence \( (h_i)_{i \in I_m} \) determines the rank function of the polymatroid. For this sequence we define the increment sequence \( g = (g_i)_{i \in I_m} \) by \( g_i = h_{i+1} - h_i \) for \( i = 0, \ldots, m-1 \) and additionally \( g_m = 0 \). It is easy to check that \( g \) is nonincreasing sequence of non-negative integers.

On the other hand, if \( g = (g_i)_{i \in I_m} \), is a nonincreasing sequence of nonnegative integers with \( g_m = 0 \), then we can define the sequence \( (h_j)_{j \in I_m} \) by the formula

\[ h_j = \sum_{i=0}^{j-1} g_i \quad \text{for all } j = 1, \ldots, m \text{ and } h_0 = 0. \]  \hspace{1cm} (4)

Given a finite set \( J \) with \( |J| = m > 0 \), the numbers \( h_j \) define a rank function \( h : \mathcal{P}(J) \rightarrow \mathbb{N}_0 \) of a uniform polymatroid \( (J, h) \) by putting \( h(X) = h_{|X|} \) for \( X \subseteq J \). It is not difficult to notice that

\[ h_k - h_j = \sum_{i=j}^{k-1} g_i \quad \text{for all } j, k \in I_m, \ j < k. \]  \hspace{1cm} (5)

Notice also that \( g_0 = 0 \iff h_1 = \cdots = h_m = 0 \) and \( g_1 = 0 \iff h_1 = \cdots = h_m = g_0 \).

Hence, according to the assumption that we consider only polymatroids such that their range functions do not have all values equal to 0, from now on we assume that for all sequences \( g \) and for all uniform polymatroids \( Z \) we have \( g_0 \neq 0 \) or equivalently \( h_1 \neq 0 \). To avoid repetition in the further part of the paper, a uniform polymatroid will be denoted by \( Z = (J, h, g) \) where \( g = (g_i)_{i \in I_m}, g_0 > g_m = 0 \) is a nonincreasing sequence of nonnegative integers and \( h : \mathcal{P}(J) \rightarrow \mathbb{N}_0 \) is the rank function such that \( h(X) = h_k = \sum_{i=0}^{k-1} g_i \) for every \( X \in \mathcal{P}(X) \) with \( k = |X| \).
**Remark 2.10.** We want to show that every uniform polymatroid determines an ideal access structure. Indeed, uniform polymatroids are known to be representable (cf. [6] Theorem 6)). Let $\mathbb{K}$ be a finite field and let $(V_x)_{x \in \mathcal{J}}$ be a $\mathbb{K}$-vector space representation of a uniform polymatroid $\mathcal{Z} = (J, h, g)$. Then $V_x$ are subspaces of the vector space $\mathbb{K}^{h_{\mathcal{Z}}}$ and $\dim V_x = h_1 = g_0$ for every $x \in \mathcal{J}$. For any $X \subseteq J$ we define $V_X = \bigoplus_{x \in X} V_x$. Given a non zero vector $\beta \in \mathbb{K}^{h_{\mathcal{Z}}}$, the family $\Delta = \{X \subseteq J : \beta \in V_X\} \subseteq \mathcal{P}(J)$ is a monotone increasing family of subsets of $J$ and $\Delta$ is compatible with the polymatroid $\mathcal{Z}$. It is easily seen that $(V_x)_{x \in \mathcal{J} \setminus \{x_0\}}$, where $x_0 \notin J$ and $V_{x_0} = \text{span}\{\beta\}$ is a vector space representation of the simple extension of $\mathcal{Z}$ induced by $\Delta$. According to Remark [25] the access structure $\Gamma(\Pi, \mathcal{Z}, \Delta)$ is ideal.

Varying the representation of $\mathcal{Z}$ and the vector $\beta$, we can control to some extent the selection of $\Delta$ which allows us to obtain different ideal access structures. For example, if we take $\beta \in V_X$ for a certain $X \subset J$, then $X \in \Delta$. If $X \neq J$ and $\beta \notin V_X$, then $\Delta \cap \mathcal{P}(X) = \emptyset$. More examples can be found in Section [5]. On the other hand, the characterization of those families $\Delta$ that cannot be obtained in this way does not seem to be an easy task.

In order to continue our studies, we need to prove several technical lemmas. Let us recall that $B(\mathcal{Z}, X)$ is defined by Equation [2].

**Lemma 2.11.** Let $\mathcal{Z} = (J, h, g)$ be a uniform polymatroid. Assume that $X \subseteq J$, $1 \leq k = |X|$ and $\bar{w} \in B(\mathcal{Z}, X)$. Then:

1. For every $x \in X$ we have $w_x \geq g_{k-1}$.
2. If $w_x = g_{k-1}$ for some $x \in X$, then $\bar{w} - w_x\bar{e}(x) \in B(\mathcal{Z}, X \setminus \{x\})$.

**Proof.** (1) Let us notice that $|\bar{w}_X| = h(X) = h_k$ and $|\bar{w}_{X \setminus \{x\}}| \leq h(X \setminus \{x\}) = h_{k-1}$, hence

$$w_x = |\bar{w}_X| - |\bar{w}_{X \setminus \{x\}}| \geq h_k - h_{k-1} = g_{k-1}.$$

(2) If we set $\bar{v} := \bar{w} - w_x\bar{e}(x)$, then we have $\text{supp}(\bar{v}) \subseteq X \setminus \{x\}$ and

$$|\bar{v}| = h_k - g_{k-1} = h_{k-1} = h(X \setminus \{x\}).$$

\[\square\]

**Lemma 2.12.** Let $\mathcal{Z} = (J, h, g)$ be a uniform polymatroid. Let $x, y \in X \subseteq J$, $x \neq y$ and $\bar{w} \in B(\mathcal{Z}, X)$ such that $w_x = g_0$, $w_y \neq 0$. If $\bar{v} \in B(\mathcal{Z}, \text{supp}(\bar{v}))$ and $\bar{v} \leq \bar{w} - \bar{e}(y) + \bar{e}(x)$, then $y \notin \text{supp}(\bar{v})$.

**Proof.** Let $\bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x)$ and $Y := \text{supp}(\bar{v})$. It is clear that $\bar{v} \in B(\mathcal{Z}, Y)$ implies $v_x \leq h_1 = g_0$ and $|\bar{v}| = h(Y)$. Moreover, $Y \subseteq X$ and $|\bar{w}_Y| \leq h(Y)$. Suppose that $y \in Y$. If $x \in Y$, then we have

$$h(Y) = |\bar{v}| \leq w_x + (w_y - 1) + |\bar{w}_{Y \setminus \{y\}}| = |\bar{w}_Y| - 1 \leq h(Y) - 1,$$

a contradiction.

Similarly, if $x \notin Y$, then we have

$$h(Y) = |\bar{v}| \leq (w_y - 1) + |\bar{w}_{Y \setminus \{y\}}| = |\bar{w}_Y| - 1 \leq h(Y) - 1,$$

a contradiction. This completes the proof. \[\square\]

**Lemma 2.13.** Let $\mathcal{Z} = (J, h, g)$ be a uniform polymatroid. Let $y \in X \subseteq J$ and $x \in J \setminus X$ and $\bar{w} \in B(\mathcal{Z}, X)$ such that $w_y = g_0$. If $k := |X|$, $g_k > 0$ and $\bar{v} \in B(\mathcal{Z}, \text{supp}(\bar{v}))$ such that $\bar{v} \leq \bar{w} - \bar{e}(y) + \bar{e}(x)$, then $y \notin \text{supp}(\bar{v})$. Moreover, if $g_0 > 1$, then $x, y \notin \text{supp}(\bar{v})$, i.e. $\text{supp}(\bar{v}) \subseteq \mathcal{X} \setminus \{y\}$.

**Proof.** Let $\bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x)$. Clearly, $\text{supp}(\bar{v}) \subseteq \text{supp}(\bar{w}') \subseteq \mathcal{X} \cup \{x\}$. Let $Y := \mathcal{X} \cap \text{supp}(\bar{v})$ and let $l := |Y|$. Suppose that $y \in \text{supp}(\bar{v})$. If $x \in \text{supp}(\bar{v})$, then $\text{supp}(\bar{v}) = Y \cup \{x\}$ and we have $l \leq k$ and

$$h_{l+1} = |\bar{v}| \leq |\bar{w}_{\{y\}}| + 1 + (g_0 - 1) = |\bar{w}_{Y \setminus \{y\}}| + g_0 = |\bar{w}_Y| \leq h_l.$$
Hence \(0 < g_k \leq g_l = h_{l+1} - h_l \leq 0\), a contradiction.

If \(x \notin \text{supp}(\bar{v})\), then \(\text{supp}(\bar{v}) = Y\) and we have

\[
h_l = |\bar{v}| \leq |\bar{w}^Y_{\setminus \{y\}}| + (g_0 - 1) = |\bar{w}^Y_{\setminus \{y\}}| + g_0 - 1 = |\bar{w}^Y| - 1 = h_l - 1,
\]
a contradiction. Thus we have proved that \(\text{supp}(\bar{v}) \subseteq (Y \setminus \{y\}) \cup \{x\}\).

Now we assume \(g_0 > 1\), and suppose \(\text{supp}(\bar{v}) = (Y \setminus \{y\}) \cup \{x\}\).

\[
h_l = |\bar{v}| \leq |\bar{w}^Y_{\setminus \{y\}}| + 1 = |\bar{w}^Y_{\setminus \{y\}}| + 1 = |\bar{w}^Y_{\setminus \{y\}}| + g_0 - (g_0 - 1) = |\bar{w}^Y| - (g_0 - 1) < h_l,
\]
as \(g_0 - 1 > 0\), a contradiction. This shows \(\text{supp}(\bar{v}) = Y \setminus \{y\} \subseteq X \setminus \{y\}\) which completes the proof. \(\square\)

**Lemma 2.14.** Let \(Z = (J, h, g)\) be a uniform polymatroid and let \(x, y \in X \subseteq J\), \(x \neq y\), as well as \(\bar{w} \in \mathcal{B}(Z, X)\). If \(w_y > 0\), then \(\bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x) \in \mathcal{B}(Z, X)\) or there exists a set \(Y \subseteq X \setminus \{y\}\), \(x \in Y\), such that \(\bar{v} := \bar{w}^Y \in \mathcal{B}(Z, Y)\). Furthermore \(\bar{v} \leq \bar{w}\) and \(\bar{v} \leq \bar{w}'\).

**Proof.** Note that \(\text{supp}(\bar{w}') \subseteq X\) and \(|\bar{w}'_X| = |\bar{w}_X| = h(X) = h|_X|\). Let us consider the case \(\bar{w}' \notin \mathcal{B}(Z, X)\), that is, there is a set \(Y \subseteq X\) that \(|\bar{w}'_Y| \geq h(Y) + 1\). Let us choose a minimum set \(Y\) for this property. It is easy to see that \(x \in Y\) and \(y \notin Y\). Setting the notation \(l := |Y|\), we get

\[
h_l + 1 \leq |\bar{w}'_Y| = (w_x + 1) + |\bar{w}^Y_{\setminus \{x\}}| = |\bar{w}^Y| + 1 \leq h_l + 1,
\]
and consequently \(|\bar{w}^Y| = h_l\). Thus, for \(\bar{v} := \bar{w}^Y\) we have \(\bar{v} \in \mathcal{B}(Z, Y)\). It is clear that \(\bar{v} \leq \bar{w}\) and \(\bar{v} \leq \bar{w}'\), which completes the proof. \(\square\)

Now we introduce a notion of a vertex vector. Let \(J\) be a finite set and \(m := |J|\) and let \(g = (g_i)_{i \in I_m}\) be the increment sequence of a uniform polymatroid \(Z = (J, h, g)\). Given \(X \subseteq J\) and a bijection \(\sigma : X \to \{0, 1, \ldots, k - 1\}\) where \(k = |X|\), we define the vector \(\bar{w} = (w_x)_{x \in J}\) by

\[
\bar{w} = \sum_{x \in X} g_{\sigma(x)} \bar{e}(x)
\]
which is referred to as a vertex vector with basic set \(X\). Notice that in general we have \(\text{supp}(\bar{w}) \subseteq X\), but \(\text{supp}(\bar{w}) = X\) whenever \(g_{k-1} > 0\). Vertex vectors are the vertices of the convex polytope

\[
T = \{\bar{w} \in \mathbb{N}^J_0 \mid |\bar{w}_X| \leq h(X) \text{ for every } X \subseteq J\}
\]
determined by a polymatroid \((J, h)\).

**Lemma 2.15.** Let \(Z = (J, h, g)\) be a uniform polymatroid. Then for every vertex vector \(\bar{w}\) we have \(\bar{w} \in \mathcal{B}(Z, \text{supp}(\bar{w}))\).

**Proof.** Let \(\bar{w}\) be any vertex vector and \(k := |\text{supp}(\bar{w})|\). Let us take a subset \(Y \subseteq \text{supp}(\bar{w})\) and set \(l := |Y| \leq k\). The sequence \(g\) being nonincreasing implies

\[
|\bar{w}_Y| = \sum_{x \in Y} w_x = \sum_{x \in Y} g_{\sigma(x)} \leq \sum_{i=0}^{l-1} g_i = h_l = h(Y).
\]
Here we use the fact that the sum of \(l\) arbitrary elements of a nonincreasing sequence does not exceed the sum of the \(l\) initial entries of the sequence. In particular, if we get \(|\bar{w}_{\text{supp}(\bar{w})}| = \sum_{i=0}^{k-1} g_i = h_k = h(\text{supp}(\bar{w}))\), which shows that \(\bar{w} \in \mathcal{B}(Z, \text{supp}(\bar{w}))\). \(\square\)

**Remark 2.16.** Notice that if \(Z\) is a uniform polymatroid, then the set \(\mathcal{B}(Z, X)\) is always nonempty since it contains vertex vectors with basic set \(X\). In extreme cases when \(X = \emptyset\) or the range function of the polymatroid has all values equal to 0, the family \(\mathcal{B}(Z, X)\) contains only the zero vector. Moreover, it is easy to check that if \(\bar{w} \in \mathcal{B}(Z, X)\) for some \(X \subseteq J\), then \(\bar{w} \in \mathcal{B}(Z, \text{supp}(\bar{w}))\).
Deciding if a monotone increasing family is compatible with given a polymatroid is not easy task. The Csirmaz Lemma seems to be the most general tool for solving this problem. For example, it is easy to check, that if the increment sequence of a polymatroid with ground set \( J \) is strictly decreasing, then every proper monotone increasing family of subsets of \( J \) is compatible with the polymatroid. At the end of this section we present several facts related to the compatibility of monotone increasing families and polymatroids which are used in proofs in subsequent sections.

**Lemma 2.17.** Let \( Z = (J,h,g) \) be a uniform polymatroid and let a monotone increasing family \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) be compatible with \( Z \).

1. If \( g_k = 0 \) for some \( 1 \leq k \leq |J| \), then all subsets of the set \( J \) with at least \( k \) elements belong to \( \Delta \).

2. If \( \Delta \) contains a minimal set with \( k \) elements, then \( g_{k-1} > 0 \).

**Proof.** (1) By assumption we have \( g_i = 0 \) for all \( i = k, \ldots, m \). Let us consider \( X \subseteq J, l := |X| \geq k \). Then we have

\[
h(J) - h(X) = h_{|X|} - h_{|J|} = \sum_{i=l}^{m-1} g_i = 0.
\]

This implies \( h(X) = h(J) \) and by the Csirmaz Lemma we get \( X \in \Delta \).

(2) Assume that \( X \subseteq J \) is a minimal set in \( \Delta \), \( |X| = k \). Then for every \( Y \subseteq X \) with \( |Y| = k - 1 \) we have \( Y \notin \Delta \), so by the Csirmaz Lemma \( h_{|Y|} < h_{|X|} \). Hence

\[
g_{k-1} = h_k - h_{k-1} = h_{|X|} - h_{|Y|} > 0.
\]

\[
\square
\]

**Lemma 2.18.** If \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) is a monotone increasing family such that \( \min \Delta = \{X\} \) for some \( \emptyset \neq X \subseteq J \), then \( \Delta \) is compatible with a uniform polymatroid \( Z = (J,h,g) \) if and only if \( g_{m-1} > 0 \).

**Proof.** Assume \( \Delta \) is compatible with \( Z \). If \( x \in X \), then \( J \setminus \{x\} \notin \Delta \), so by Csirmaz Lemma \( h(J \setminus \{x\}) < h(J) \), thus \( g_{m-1} = h(J) - h(J \setminus \{x\}) > 0 \).

Now we shall show that the conditions of the Csirmaz Lemma are met whenever \( g_{m-1} > 0 \). Let us notice that \( h_i - h_{i-1} = g_{i-1} > 0 \) for all \( i = 1, \ldots, m \), so the sequence \( h_0, h_1, \ldots, h_m \) is strictly increasing. Let us take such sets \( Y \subseteq W \subseteq J \), that \( Y \notin \Delta \) and \( W \in \Delta \). Of course, \( |Y| < |W| \), so we have \( h(Y) < h(W) \), thus the condition (1) is satisfied.

Now let us consider \( W,Y \in \Delta \). Then \( X \subseteq W \) and \( X \subseteq Y \) since \( \min \Delta = \{X\} \), so \( W \cap Y \in \Delta \). This shows that the second condition of the Csirmaz Lemma is also satisfied.

Let us recall a result of Farràs, Padró, Xing and Yang, which can be restated as follows.

**Lemma 2.19 ( [9], Lemma 6.1).** For a positive integer \( k \in I_m \), the monotone increasing family \( \Delta \) such that \( \min \Delta = P_k(J) \) is compatible with a uniform polymatroid \( Z = (J,h,g) \) if and only if \( g_{k-1} > g_k \).

Further results concerned with compatibility can be found in Section 4

### 3 Access structures determined by uniform polymatroids

This section is devoted to the study of those uniform polymatroids that determine weakly hierarchical access structures. From now on we make the assumption that \( J \) is a finite set with \( m := |J| \geq 2 \). A partition of a set of participants is denoted by \( \Pi = (P_x)_{x \in J} \). A uniform polymatroid is a triplet \( Z = (J,h,g) \) where \( h : \mathcal{P}(J) \to \mathbb{N}_0 \) is the rank function and \( g = (g_i)_{i \in I_m} \) is the increment sequence of the polymatroid. Recall the sequence \( g \) is nonincreasing and \( g_m = 0 \).
Moreover, we assume 0 < g_0 < |P_x| for all x ∈ J. Next we consider a monotone increasing family ∆ ⊆ P(J \ {∅}) that is compatible with Z. Finally, Γ(Π, Z, ∆) is the access structure determined by Π, Z, ∆ as defined in Definition 2.6. The relation ≺ is the hierarchical preorder induced by Γ in Π. We define for further use the following notations η(g) = \min\{i ∈ I_m : g_i = 0\} and μ(∆) = \min\{|X| : X ∈ ∆\}. The above settings ensure that η(g) ≥ 1 and μ(∆) ≥ 1.

**Example 3.1.** Let us consider a uniform polymatroid Z = (J, h, g) such that η(g) = 1, i.e. g_0 > g_1 = 0 and a monotone increasing family ∆ of subsets of J compatible with Z. Applying Lemma 2.17 (1) yields ∆ = P(J \ {∅}). According to Equation (1) we have h(X) = g_0 for all nonempty subsets X of J. Hence B(Z, X) ⊆ B(Z, J) for every ∅ \neq X ⊆ J (cf. Equation (2)) and consequently \bigcup_{X ∈ ∆} B(Z, X) = B(Z, J). Let Γ = Γ(Π, Z, ∆). This implies that \bar{w} ∈ min Γ if and only if |\bar{w}| = g_0 or equivalently \bar{w} ∈ Γ if and only if |\bar{w}| ≥ g_0. This shows that the threshold access structure is the only type of access structures determined by uniform polymatroids with η(g) = 1. In particular all blocks (and participants) are hierarchically equivalent.

Let us collect several simple observations, which are very helpful in many proofs.

**Lemma 3.2.** For Γ = Γ(Π, Z, ∆) we have

1. B(Z, X) ⊆ Γ for all X ∈ ∆.
2. supp(Γ) = ∆.
3. If \bar{w} ∈ min Γ, then \bar{w} ∈ B(Z, supp(\bar{w})) and supp(\bar{w}) ∈ ∆.
4. If \bar{w} ∈ Γ, then there exists \bar{v} ∈ min Γ such that \bar{v} ≤ \bar{w}, \bar{v} ∈ B(Z, supp(\bar{v})) and supp(\bar{v}) ∈ ∆.
5. If \bar{w} is a vertex vector and supp(\bar{w}) ∈ ∆, then \bar{w} ∈ Γ.

**Proof.** (1) This follows directly from Definition 2.6.

(2) Let us consider Y ∈ supp(Γ). Then there exists \bar{w} ∈ Γ such that supp(\bar{w}) = Y. Let us consider two cases:

   (i) \bar{w} ∈ min Γ. Then there exists X ∈ ∆ such that \bar{w} ∈ B(Z, X), so Y ⊆ X. If Y = X, then Y ∈ ∆. If Y ⊈ X, then also Y ∈ ∆. Indeed, let us notice that |\bar{w}_Y| ≤ h(Y), |\bar{w}_X| = h(X) and |\bar{w}_Y| = |\bar{w}_X|, where the later equality follows from the fact supp(\bar{w}) = Y ⊆ X. Moreover, if Y \not∈ ∆, then by the Csirmaz Lemma we would get

   $$h(X) = |\bar{w}_X| = |\bar{w}_Y| ≤ h(Y) < h(X),$$

   which is a contradiction.

   (ii) \bar{w} ∈ Γ and \bar{w} \not∈ min Γ. Then there is \bar{v} ∈ min Γ such that \bar{v} ≤ \bar{w}. From the case (i) we get supp(\bar{v}) ⊆ ∆. Let us notice that supp(\bar{v}) ⊆ supp(\bar{w}). Moreover, ∆ is a monotone increasing family, so Y = supp(\bar{w}) ∈ ∆.

   Now we shall show the converse inclusion. Let us take X ∈ ∆. As we already have observed in Remark 2.16, the family B(Z, X) cannot be empty, so there is a certain vector \bar{w} ∈ B(Z, X). By (1) we get \bar{w} ∈ Γ, so supp(\bar{w}) ∈ supp(Γ). The family supp(Γ) is monotone increasing and supp(\bar{w})⊆ X, so X ∈ supp(Γ).

   (3) If \bar{w} ∈ min Γ, then \bar{w} ∈ B(Z, X) for a certain X ∈ ∆. Remark 2.16 implies \bar{w} ∈ B(Z, supp(\bar{w})). Moreover, supp(\bar{w}) ∈ supp(Γ), hence and by (2) we get supp(\bar{w}) ∈ ∆.

   (4) It follows from (3) immediately.

   (5) If \bar{w} is a vertex vector, then we have \bar{w} ∈ B(Z, supp(\bar{w})) by Lemma 2.13. By assumption and part (1) of this lemma we get \bar{w} ∈ Γ.

**Lemma 3.3.** Let Γ = Γ(Π, Z, ∆). If g_1 = g_{n-1} > 0 for some 2 ≤ n ≤ m and if X, Y ∈ min ∆ as well as |X ∪ Y| ≤ n, then X = Y or both sets are singletons. Moreover, if g_0 = g_1, then X = Y even if both X, Y are singletons.
Proof. For $n = 2$, the claim is obvious. Let us assume $n \geq 3$. It is enough to consider the case $X \neq Y$. Suppose that at least one of these sets, for example $X$, has at least 2 elements. Let us fix $x \in X$ and consider the set

$$Y' = \begin{cases} Y & \text{when } X \cap Y \neq \emptyset; \\ Y \cup \{x\} & \text{when } X \cap Y = \emptyset. \end{cases}$$

Note that $|X \cup Y'| = |X \cup Y| \leq n$ and $W := X \cap Y' \neq \emptyset$. In addition, $W$ is a proper subset of $X$ which is a minimum set in $\Delta$, so it does not belong to $\Delta$. Hence according to the Csirmaz Lemma we get

$$h(W) + h(X \cup Y') < h(X) + h(Y').$$

On the other hand, the assumption $g_1 = g_{n-1}$ implies $h_l = g_0 + (l-1)g_1$ for every $1 \leq l \leq n$. From this we get

$$h_1|W| + h(|X| + |Y'| - |W|) < h_1|X| + h|Y'|,$$

$$g_0 + (|W| - 1)g_1 + g_0 + (|X| + |Y'| - |W| - 1)g_1 < g_0 + (|X| - 1)g_1 + g_0 + (|Y'| - 1)g_1.$$

It is easy to see that the simplified expression above is $0 < 0$, which gives a contradiction. This shows that if $X$ and $Y$ are different, then they cannot have more than one element.

Let us assume $g_0 = g_1$ and $|X| = |Y| = 1$. Let us suppose $X \neq Y$. Then $X \cap Y = \emptyset$, so by the Csirmaz Lemma we have

$$h(X \cap Y) + h(X \cup Y) < h(X) + h(Y)$$

and consequently $h_2 < 2h_1$ or equivalently $g_0 + g_1 < 2g_0$, which is a contradiction. \qed

Proposition 3.4. If $X \in \min \Delta$, then for all $x, y \in X$, $x \neq y$, the blocks $P_x$ and $P_y$ are hierarchically independent in the access structure $\Gamma = \Gamma(\Pi, Z, \Delta)$.

Proof. Let $X \in \min \Delta$ and let $x, y$ be two different elements in $X$. Suppose $P_y \preceq P_x$ and consider a vertex vector $\bar{w}$ with basic set $X$ and $w_x = g_0$. Setting $k := |X|$ and applying Lemma 2.17 (2) we have $g_{k-1} > 0$ so $\text{supp}(\bar{w}) = X$, in particular $w_y > 0$ and by Lemma 3.2 (5) we get $\bar{w} \in \Gamma$. Thus $\bar{w}' = \bar{w} - \bar{e}(y) + \bar{e}(x) \in \Gamma$. By Lemma 3.2 (4) there is $\bar{v} \in \min \Gamma$ such that $\bar{v} \leq \bar{w}'$ and $\bar{v} \in B(Z, \text{supp}(\bar{v})) \subseteq \Gamma$, so applying Lemma 2.12 we have $y \notin \text{supp}(\bar{v}) \subseteq X$, which contradicts the fact that $X \in \min \Delta$. By a similar argument we show that the case $P_x \preceq P_y$ is impossible. \qed

Proposition 3.5. If $X \in \min \Delta$, $1 \leq k := |X| \leq m - 1$ and $g_k > 0$, then for every $y \in X$ the block $P_y$ is not hierarchically inferior to any block $P_x \neq P_y$ in the access structure $\Gamma = \Gamma(\Pi, Z, \Delta)$.

Proof. Let $y \in X$ and let us suppose that $P_y \preceq P_x$ for a certain $x \in J$. By Proposition 3.4 we have $x \in J \setminus X$. Let us consider a vertex vector $\bar{w}$ with basic set $X$ and $w_y = g_0$. Obviously, $\bar{w} \in \Gamma$ by Lemma 3.2 (5). Then the vector $\bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x)$ also belongs to $\Gamma$.

By Lemma 3.2 (4) there exists a minimal authorized vector $\bar{v}$ such that $\bar{v} \leq \bar{w}'$, $\bar{v} \in B(Z, \text{supp}(\bar{v}))$ and $\text{supp}(\bar{v}) \in \Delta$. If $g_0 > 1$, then Lemma 2.19 implies $\text{supp}(\bar{v}) \subseteq X \setminus \{y\}$, but this contradicts the assumption $X \in \min \Delta$.

If $g_0 = 1$, then $g_0 = g_1 = g_k$ and by Lemma 2.13 we have $\text{supp}(\bar{v}) \subseteq (X \setminus \{y\}) \cup \{x\}$. For $Y \in \min \Delta$ such that $Y \subseteq \text{supp}(\bar{v})$ we have $X \cup Y \leq X \cup \{x\}$, so $|X \cup Y| \leq k + 1$. Applying Lemma 3.4 yields $X = Y$ but $y \in X$ and $y \notin Y$, a contradiction. \qed

Lemma 3.6. Let $g = (g_i)_{i \in I_m}$ be the increment sequence of a uniform polymatroid $Z$ and let $\Gamma = \Gamma(\Pi, Z, \Delta)$. Let us assume that $X \in \min \Delta$ with $k := |X|$ and there are $x, y \in J$, $x \neq y$ such that $|X \cup \{x, y\}| \geq 3$ and the blocks $P_y$ and $P_x$ are hierarchically comparable in the access structure $\Gamma$. Furthermore, we assume that $g_1 = g_k$ and $g_l > 0$ for a certain $1 \leq l < m$. If $X \cap \{x, y\} \neq \emptyset$ or $g_0 = g_1$, then $g_1 = g_l$. \qed
Proof. If \( g_1 = 1 \), then the claim is obvious.

Assume that \( g_1 > 1 \) and assume that this is not the case. Let \( l \) be the least positive integer that does not satisfy the claim. That means, \( g_1 = g_{l-1} > g_l > 0 \). Obviously, \( k + 1 \leq l \leq m - 1 \). This implies \( k \leq m - 2 \). Without loss of generality we can assume that \( P_y \preceq P_x \). By Proposition 3.3 we have \( y \notin X \). Let now \( Y \subseteq J \) be a set with \( l + 1 \) elements which contains \( X \cup \{x, y\} \). Moreover, let us take an element \( z \in X \setminus \{x, y\} \).

Let us consider a vertex vector \( \bar{w} \) with basic set \( Y \) and \( w_z = g_0 \) and \( w_z = g_l \). Obviously, \( \text{supp}(\bar{w}) = Y \), as \( g_l > 0 \). Under the above assumptions, \( w_t = g_1 \) for all \( t \in Y \setminus \{x, z\} \), in particular we have \( w_y = g_1 \). For every \( 0 < j < l \) we have

\[
h_j = \sum_{i=0}^{j-1} g_i = g_0 + (j-1)g_1.
\]

Let us notice that \( Y \in \Delta \) as \( X \subseteq Y \). Hence \( \bar{w} \in \Gamma \) by Lemma 3.2 (5). Moreover, \( h_{l+1} = |\bar{w}| = g_0 + (l - 1)g_1 + g_l \). Since \( P_y \preceq P_x \) we have \( \bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x) \in \Gamma \). Let us notice \( \text{supp}(\bar{w}') = Y \). Now by Lemma 3.2 (4) there exists a minimal authorized vector \( \bar{v} \) such that \( \bar{v} \leq \bar{w}' \). Assume that \( y / \bar{v} \) and \( W := \text{supp}(\bar{v}) \in \Delta \). Lemma 2.12 implies \( y \notin W \), i.e. \( W \subseteq Y \setminus \{y\} \), so \( |W| \leq l \). Let \( Z \in \text{min} \Delta \) such that \( Z \subseteq W \).

Thus \( X \cup Z \subseteq X \cup W \subseteq Y \setminus \{y\} \), so \( |X \cup Z| \leq l \) and applying Lemma 3.3 yields \( X = Z \) or both \( X \) and \( W \) are singletons. If \( X \cap \{x, y\} \neq \emptyset \), then \( x, z \in X \), so \( X \) is not a singleton, thus \( X = Z \). If \( g_0 = g_1 = \cdots = g_{n-1} > g_n = 0 \). Notice also that \( v_z \leq w_z = g_l < g_{l-1} \leq g|W|_1 \), a contradiction which proves that \( g_1 = g_l \).

**Proposition 3.7.** Let \( g = (g_i)_{i \in I_m} \) be the increment sequence of a uniform polymatroid \( Z \) and let \( \Gamma = \Gamma(\Pi, Z, \Delta) \). Let us assume \( n := \eta(g) \geq 3 \). If there are \( X \in \text{min} \Delta \) such that \( 1 \leq |X| \leq n - 2 \) and \( x, y \in J \setminus X \) such that the blocks \( P_x \) and \( P_y \) are hierarchically comparable in the access structure \( \Gamma \), then \( g_0 = g_1 = \cdots = g_{n-1} > g_n = 0 \).

Proof. If \( g_0 = 1 \), then let us observe that

\[
1 = h_1 = g_0 \geq g_1 \geq \cdots \geq g_{n-1} \geq 1.
\]

Hence \( g_0 = g_1 = \cdots = g_{n-1} > g_n = 0 \).

Thus we assume \( g_0 \geq 2 \). If the blocks \( P_x \) and \( P_y \) are hierarchically comparable, then one can assume without loss of generality that \( P_y \preceq P_x \). Let us consider a vertex vector \( \bar{w} \) with basic set \( X \cup \{y\} \) such that \( w_y = g_0 \). Obviously, by Lemma 3.2 (5) we have \( \bar{w} \in \Gamma \). Then the vector \( \bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x) \) belongs to \( \Gamma \). By Lemma 3.2 (4) there exists a minimal authorized vector \( \bar{v} \) such that \( \bar{v} \leq \bar{w}' \). Assume that \( y / \bar{v} \) and \( \text{supp}(\bar{v}) \subseteq X \). By Lemma 2.13 we have \( \text{supp}(\bar{v}) \subseteq X \), but \( X \) is minimal in \( \Delta \), so \( \text{supp}(\bar{v}) = X \). Thus we have

\[
h_k = |\bar{v}| \leq |\bar{w}'_X| = |\bar{w}_X| = \sum_{i=1}^{k} g_i = h_{k+1} - g_0,
\]

so \( g_0 \leq h_{k+1} - h_k = g_k \). The sequence \( g \) is nonincreasing, so \( g_0 = g_1 = \cdots = g_k \). Thus we have shown that \( g_1 = \cdots = g_k \). To complete the proof it is enough to apply Lemma 3.6 assuming \( l = n - 1 \).

**Proposition 3.8.** Let \( g = (g_i)_{i \in I_m} \) be the increment sequence of a uniform polymatroid \( Z \) and let \( \Gamma = \Gamma(\Pi, Z, \Delta) \). Let us assume \( n := \eta(g) \geq 3 \). If there are \( X \in \text{min} \Delta \) with \( 2 \leq |X| \leq n - 1 \) and \( x, y \in J \setminus X \) such that the blocks \( P_x \) and \( P_y \) are hierarchically comparable in the access structure \( \Gamma \), then \( g_1 = \cdots = g_{n-1} > g_n = 0 \).

Proof. If the blocks \( P_x \) and \( P_y \) are hierarchically comparable, then it follows from Proposition 3.3 that \( P_y \preceq P_x \). Let us consider a vertex vector \( \bar{w} \) with basic set \( X \cup \{y\} \) such that \( w_x = g_0 \) and \( w_y = g_1 \). Obviously, \( \bar{w} \in \Gamma \) by Lemma 3.2 (5). Then also the vector \( \bar{w}' := \bar{w} - \bar{e}(y) + \bar{e}(x) \) belongs to...
\( \Gamma \) and \( \text{supp}(\vec{w}') \subseteq X \cup \{y\} \). Hence by Lemma 3.2 (4) there is a minimal authorized vector \( \vec{v} \), such that \( \vec{v} \leq \vec{w}' \), \( \vec{v} \in B(Z, \text{supp}(\vec{v})) \) and \( Y := \text{supp}(\vec{v}) \in \Delta \). Let us observe \( Y \subseteq \text{supp}(\vec{w}') \subseteq X \cup \{y\} \).

By Lemma 2.12 we have \( y \notin Y \) that shows \( Y \subseteq X \), but \( X \) is minimal in \( \Delta \), so \( Y = X \). Thus we have

\[
h_k = |\vec{v}| \leq g_0 + \sum_{i=2}^{k} g_i = h_{k+1} - g_1
\]

where \( k := |X| \). Hence \( g_1 \leq h_{k+1} - h_k = g_k \) and \( g_1 = g_k \) as the sequence \( g \) is nonincreasing. To complete the proof it is enough to apply Lemma 3.6 assuming \( l = n - 1 \). \( \square \)

**Corollary 3.9.** If there are \( x, y \in J \) such that \( P_x \) and \( P_y \) are hierarchically comparable and \( 3 \leq |X \cup \{x, y\}| \leq \eta(g) \) for a certain \( X \in \min \Delta \), then \( g_1 = g_{m-1} \). Moreover, if \( X \cap \{x, y\} = \emptyset \), then \( g_0 = \cdots = g_{m-1} \).

**Proof.** Let us write \( n := \eta(g) \). Assuming with no loss of generality that \( P_y \nleq P_x \) we obtain that \( \{x, y\} \) is not contained in \( X \), by Proposition 3.3 so \( |X| \leq n - 1 \). Applying Proposition 3.5 yields \( y \notin X \). If \( x \in X \), then \( 2 \leq |X| \leq n - 1 \) and applying Proposition 3.8 yields \( g_1 = g_{n-1} > g_n = 0 \). If \( x \notin X \), in particular \( |X| = 1 \), then applying Proposition 3.7 yields \( g_0 = g_1 = g_{n-1} > g_n = 0 \).

Suppose, contrary to our claim that \( n < m \). Then there is a subset \( Z \subseteq J \) such that \( |Z| = n+1 \) and \( X \cup \{x, y\} \subseteq Z \). Let us choose \( z \in X \setminus \{x, y\} \) and denote \( Z' = Z \setminus \{z\} \). Lemma 2.17 (1) implies that the set \( Z' \) belongs to \( \Delta \) but it is not minimal as \( x, y \in Z' \). So there is \( Y \in \min \Delta \) such that \( Y \nsubseteq Z' \). Applying again Proposition 3.5 we get \( y \notin Y \), so \( X \cup Y \subseteq Z \setminus \{y\} \), thus \( |X \cup Y| \leq n \). If \( |X| > 2 \), then we can apply Lemma 3.3 to get \( X = Y \subseteq Z' \), which is a contradiction as \( z \in X \) but \( z \notin Z' \). If \( |X| = 1 \), then \( X \cap \{x, y\} = \emptyset \) and by Proposition 3.7 we have \( g_0 = g_1 \) and applying again Lemma 3.3 yields \( X = Y \subseteq Z' \), a contradiction as before. This completes the proof. \( \square \)

**Corollary 3.10.** Any multipartite access structures determined by uniform polymatroid \( Z = (J, h, g) \) does not admit hierarchically equivalent blocks unless \( \eta(g) = 1 \) or \( g_0 = \cdots = g_{m-1} \).

**Proof.** It is shown in Example 3.1 that all blocks are hierarchically equivalent in any access structure determined by a uniform polymatroid with \( \eta(g) = 1 \). Let \( n := \eta(g) \geq 2 \) and suppose that there are \( x, y \in J \) such that \( P_x \) and \( P_y \) are hierarchically equivalent. Let us consider a subset \( X \subseteq J \) such that \( x, y \in X \) and \( |X| = n \). Lemma 2.17 (1) and Proposition 3.3 imply that \( X \in \Delta \) but \( X \) is not minimal, so there is \( Y \in \min \Delta \) such that \( Y \nsubseteq X \). By Proposition 3.5, \( x, y \notin Y \). If \( n = 2 \), then \( Y = \emptyset \), which is a contradiction. Hence we get \( n \geq 3 \) and \( 3 \leq |Y \cup \{x, y\}| \leq n \) and applying Corollary 3.9 yields \( g_0 = g_{m-1} \). \( \square \)

**4 Hierarchical preorder determined by access structure**

In this section we shall present several results on hierarchical orders induced by access structures determined by uniform polymatroids with ground set \( J \) and monotone increasing families of subsets of \( J \). It is worth noticing that such construction of an access structure is only possible if the monotone increasing family is compatible with the given polymatroid. Let us recall that an access structure is called compartmented if every two different blocks are hierarchically independent. If there is at least one pair of blocks of participants which are hierarchically comparable, then the access structure is called weakly hierarchical. If weakly hierarchical access structure does not admit hierarchically equivalent blocks, then it is referred to as hierarchical.

**Theorem 4.1.** Let \( \Gamma = \Gamma(\Pi, Z, \Delta) \) and let \( g = (g_i)_{i \in I_m} \) be the increment sequence of a uniform polymatroid \( Z \). If \( g_0 > g_{m-1} \), then the access structure \( \Gamma = \Gamma(\Pi, Z, \Delta) \) is connected.

**Proof.** Given \( x \in J \), we want to show that there is \( \vec{w} \in \min \Gamma \) such that \( w_x \neq 0 \). If there is \( X \in \min \Delta \), \( x \in X \) and \( i := |X| \), then \( g_{i-1} > 0 \) by Lemma 2.17 (2). It is easy to see that any vertex vector \( \vec{w} \) with basic set \( X \) belongs to \( \min \Gamma \) and \( w_x \neq 0 \). Now we assume that \( x \notin X \) for all \( X \in \min \Delta \). Let us denote \( l := \min \{i \in I_m : g_i > g_0\} \). By assumption \( l \leq m - 1 \). Let us take \( X \in \min \Delta \) such that \( k := |X| = \mu(\Delta) \) and consider \( Y \subseteq J \) such that \( |Y| = \max \{k, l\} + 1 \)
and \( \{x\} \cup X \subseteq Y \). Let \( \bar{w} \) be a vertex vector with basic set \( Y \) such that \( w_x = g_0 \) and \( w_y = g_1 \) for a certain \( y \in X \). Lemma \( \ref{lem:9} \)(5) implies that \( \bar{w} \in \Gamma \) as \( Y \in \Delta \). Thus there is \( \bar{v} \in \min \Gamma \) such that \( \bar{v} \leq \bar{w} \). Since \( \supp(\bar{v}) \in \Delta \), there is \( W \in \min \Delta \) such that \( W \subseteq \supp(\bar{v}) \). By assumption \( x \notin W \), so \( W \subseteq Y \setminus \{x\} \).

It turns out that \( W = X \). Indeed, if \( k \geq l \), then \( Y \setminus \{x\} = X \) and by the minimality of \( X \) in \( \Delta \) we have \( W = X \). For the case \( k < l \) we have \( l \geq 2 \), \( g_0 = g_1 \) and \( X \cup W \subseteq Y \setminus \{x\} \), thus \( |X \cup W| \leq 1 \) and consequently \( X = W \) by Lemma \( \ref{lem:3} \).

If \( g_1 = 0 \), then \( v_y \leq w_y = g_1 = 0 \), i.e. \( y \notin \supp(\bar{v}) \) which contradicts the fact that \( y \in X = W \subseteq \supp(\bar{v}) \).

If \( g_1 \neq 0 \) and \( v_x \neq 0 \), then we have the claim. Let us suppose \( v_x = 0 \), i.e. \( x \notin \supp(\bar{v}) \). Thus for \( Z = \supp(\bar{v}) \) we have

\[
h(Z) = |\bar{v}_Z| = \sum_{z \in Z} v_z \leq \sum_{z \in Z} w_z = w_y + \sum_{z \in Z \setminus \{y\}} w_z = \sum_{z \in (Z \cup \{x\}) \setminus \{y\}} w_z - (w_x - w_y) = |\bar{w}_{(Z \cup \{x\}) \setminus \{y\}}| - (w_x - w_y) \leq h((Z \cup \{x\}) \setminus \{y\}) - (g_0 - g_1) < h(Z),
\]

a contradiction as \( |Z| = |(Z \cup \{x\}) \setminus \{y\}| \) and \( g_0 - g_1 > 0 \).

This theorem shows that the access structures determined by uniform polymatroids are connected except for the ones in the column \( F \) of Table \( \ref{tab:1} \).

**Theorem 4.2.** Let \( g = (g_i)_{i \in [m]} \) be the increment sequence of a uniform polymatroid \( Z \) and let \( \Gamma = \Gamma(\Pi, Z, \Delta) \). If \( m \geq 3 \) and \( g_1 > g_{m-1} > 0 \) and \( \min \Delta \neq \{\{x\}\} \) for any \( x \in J \), then the access structure \( \Gamma \) is compartmentalized.

**Proof.** Let us suppose that there are \( x, y \in J \) such that the blocks \( P_x \) and \( P_y \) are hierarchically comparable. According to Proposition \( \ref{prop:4} \), no minimal set in \( \Delta \) can contain both \( x \) and \( y \), so \( |X| \leq m-1 \) for every \( X \in \min \Delta \). By assumption \( \eta(g) = m \). If \( x, y \notin X \) for a certain \( X \in \min \Delta \), then by Proposition \( \ref{prop:7} \) we obtain \( g_0 = g_1 = \ldots = g_{m-1} \), a contradiction.

If does not exist any set \( X \) in \( \min \Delta \) such that \( x, y \notin X \), then without loss of generality we can assume that \( x \in X \) and \( y \notin X \) for a certain \( X \in \min \Delta \). If \( |X| \geq 2 \), then by proposition \( \ref{prop:8} \) we get \( g_1 = \ldots = g_{m-1} \), a contradiction again. If \( |X| = 1 \), then \( \min \Delta = \{\{x\}\} \), as otherwise both \( x, y \) would be outside a certain minimal set in \( \Delta \), but this is not the case now.

Let us notice that if \( g_{m-1} > 0 \), then the above theorem implies that the appearance of non-compartmental access structure can be expected in the first row or in the two last columns of Table \( \ref{tab:1} \). In the next theorem we shall prove that the access structures that appear in the last row of Table \( \ref{tab:1} \) are compartmentalized excluding the cells \( A3, E3 \) and \( F3 \).

**Theorem 4.3.** Let \( g = (g_i)_{i \in [m]} \) be the increment sequence of a uniform polymatroid \( Z \) and let \( \Gamma = \Gamma(\Pi, Z, \Delta) \). If \( m \geq 3 \) and \( g_1 > g_{m-1} \) and \( k \) := \( \mu(\Delta) > 1 \), then the access structure \( \Gamma \) is compartmentalized.

**Proof.** Suppose to the contrary that \( \Gamma \) is not compartmentalized, i.e. there are two blocks which are hierarchically comparable. For simplicity we assume \( P_y \ll P_x \) for certain \( x, y \in J \), \( x \neq y \). By Proposition \( \ref{prop:4} \) no set in \( \min \Delta \) contains both \( x \) and \( y \), in particular there is a subset of \( J \) with \( k \) elements which does not belong to \( \min \Delta \). This and Lemma \( \ref{lem:2} \)(1) imply \( g_k > 0 \). Let \( n := \eta(g) \). Obviously \( 2 \leq k < n \leq m \) and \( g_n = 0 \). Proposition \( \ref{prop:5} \) implies \( y \notin X \) for every \( X \in \min \Delta \) with \( |X| \leq n-1 \).

If there exists \( X \in \min \Delta \) such that \( |X| = k \) and \( x \in X \), then \( 3 \leq |X \cup \{x, y\}| = k+1 \leq n \). Now we assume that \( x \notin X \) for every \( X \in \min \Delta \) with \( |X| = k \). Suppose \( g_{k+1} = 0 \). Let us fix \( X \in \min \Delta \) with \( |X| = k \) and \( z \in X \). Let us consider \( Z := (X \setminus \{z\}) \cup \{x, y\} \). From Lemma \( \ref{lem:2} \)(1) we have \( Z \in \Delta \) as \( |Z| = k+1 \). By Proposition \( \ref{prop:3} \) the set \( Z \) cannot be minimal as it contains \( \{x, y\} \), so there is \( Y \in \min \Delta \) such that \( Y \subset Z \) and hence \( |Y| = k \). Obviously, \( x, y \notin Y \) which implies \( Y \subseteq X \setminus \{z\} \), a contradiction. This shows that \( g_{k+1} > 0 \), so \( k + 1 < n \). Thus we have \( 4 \leq |X \cup \{x, y\}| = k+2 \leq n \) for arbitrary \( X \in \min \Delta \) with \( |X| = k \).

In both cases we can apply Corollary \( \ref{cor:9} \) which implies \( g_1 = g_{m-1} \), a contradiction. This completes the proof. 

\[ \square \]
The following table presents a general arrangement of multipartite access structure determined by monotone increasing families contained in \( P(J) \setminus \{\emptyset\} \) and uniform polymatroids. The cells A1-C1 and A3 do not contain any objects since the suitable monotone increasing families and polymatroids are not compatible. A monotone increasing family which is not compatible with given polymatroid can occur in every cell of the table. A complete overview of hierarchical orders of all access structure obtained from uniform polymatroids with \( m = 4 \) can be found at Table 2.

Table 1: Hierarchical (pre)orders of access structures obtained from uniform polymatroids.

| \( g \) | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| \( g_{m-1} = 0 \) | &gt; 0 | \( g_{m-1} &gt; 0 \) |
| \( g_1 = 0 \) | \( g_1 &gt; 0 \) |
| \( g_2 = 0 \) | &gt; 0 | \( g_1 &gt; g_{m-1} \) | \( g_1 = g_{m-1} \) |

| \( \mu(\Delta) = 1 \) | T | C & H | C & H | H | H* |
|---|---|---|---|---|---|
| \( |\min \Delta| = 1 \) | &gt; 0 | &gt; 0 | &gt; 0 | &gt; 0 |

| \( \mu(\Delta) = 1 \) | T | C & H | C & H | C | H | H* |
|---|---|---|---|---|---|---|
| \( |\min \Delta| &gt; 1 \) | &gt; 0 | &gt; 0 | &gt; 0 | &gt; 0 |

| \( \mu(\Delta) &gt; 1 \) | T | C | C | C | C & H | C & H* |
|---|---|---|---|---|---|---|
| &gt; 0 | &gt; 0 | &gt; 0 | &gt; 0 |

To describe the hierarchical order determined by an access structure \( \Gamma \) in a partition \( \Pi \) of the set of participants \( P \) we introduce the following notations \( \text{Ord}(Y, X) \) and \( \text{Ord}^*(Y, X) \) which are defined as follows.

**Definition 4.4.** Let \( \Pi = (P_x)_{x \in J} \) be a partition of the set \( P \) and let \( Y \) and \( X \) be disjoint subsets of \( J \). The hierarchical preorder \( \preceq \) in \( \Pi \) is said to be of type \( \text{Ord}(Y, X) \) if

\[
P_y \preceq P_x \iff (x = y \text{ or } (y \in Y \text{ and } x \in X)).
\]

In particular, no different blocks are hierarchically equivalent, i.e. \( \preceq \) is an order. Moreover, if \( X \) or \( Y \) is empty, then the order \( \text{Ord}(Y, X) \) is compartmented.

**Definition 4.5.** Let \( \Pi = (P_x)_{x \in J} \) be a partition of the set \( P \) and let \( X \) and \( Y \) be disjoint subsets of \( J \). The hierarchical preorder \( \preceq \) in \( \Pi \) is said to be of type \( \text{Ord}^*(Y, X) \) if

\[
P_y \preceq P_x \iff (x = y \text{ or } (x, y \in Y \text{ or } (y \in Y \text{ and } x \in X))).
\]

The preorder \( \text{Ord}^*(Y, X) \) is not an order (unless \( |Y| \leq 1 \)). In particular, \( P_x, P_y \) are hierarchically equivalent whenever \( x, y \in Y \), but no different blocks \( P_x, P_y \) with \( x, y \in J \setminus Y \) are hierarchically equivalent.

One can notice that if the set \( Y \) is empty, then every two blocks are hierarchically independent in the preorder \( \text{Ord}^*(\emptyset, X) \). If the set \( X \) is empty, then we get \( \text{Ord}^*(Y, \emptyset) \), that means each two blocks are hierarchically equivalent (cf. Example 3.1). Noteworthy is also the observation that if \( |Y| \leq 1 \), then \( \text{Ord}(Y, X) = \text{Ord}^*(Y, X) \). The defined preorders can be presented in the form of the following Hasse diagrams.
Now we want to describe the hierarchical orders of multipartite access structures determined by some special type of uniform polymatroids. In Example 3.1 we considered the case of polymatroids with the increment sequence $g$ satisfying $\eta(g) = 1$. We will now deal with polymatroids $Z = (J, h, g)$ with $\eta(g) = 2$, i.e. $g_0 \geq g_1 > g_2 = 0$. The result presented below refers to the column B of Table 11.

**Theorem 4.6.** Let $Z = Z(\Pi) = (J, h, g)$ be a uniform polymatroid with the increment sequence $g = (g_i)_{i \in I_m}$ such that $\eta(g) = 2$.

1. If $g_0 > g_1$, then a monotone increasing family $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ is compatible with the polymatroid $Z$ if and only if there is a subset $X \subseteq J$ such that $\min \Delta = \mathcal{P}_1(X) \cup \mathcal{P}_2(J \setminus X)$.

2. If $g_0 = g_1$, then a monotone increasing family $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ is compatible with the polymatroid $Z$ if and only if there is a subset $X \subseteq J$, $|X| \leq 1$ such that $\min \Delta = \mathcal{P}_1(X) \cup \mathcal{P}_2(J \setminus X)$.

3. Let $\Gamma = \Gamma(\Pi, Z, \Delta)$ be the access structure determined by the polymatroid $Z$ and the monotone increasing family $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ such that $\min \Delta = \mathcal{P}_1(X) \cup \mathcal{P}_2(J \setminus X)$ for some $X \subseteq J$, then the hierarchical order induced by $\Gamma$ on the set $\Pi$ is of the type $\text{Ord}(J \setminus X, X)$.

**Proof.** (1) and (2). $(\Rightarrow)$. If the monotone increasing family $\Delta$ is compatible with the polymatroid $Z$, then Lemma 4.4 (1) and the assumptions $g_2 = 0$ shows that all subsets of $J$ with two elements belong to $\Delta$. Thus, all sets in $\min \Delta$ have one or two elements. Let $X \subseteq J$ denote the collection of those elements that form single-element minimal sets. Hence the remaining sets in $\min \Delta$ have two elements and do not contain any elements belonging to $X$. Therefore $\min \Delta = \mathcal{P}_1(X) \cup \mathcal{P}_2(J \setminus X)$.

If $g_0 = g_1$, then it follows from Lemma 3.3 that $|X| \leq 1$.

(1) and (2). $(\Leftarrow)$. To show the reverse implication, consider the monotone increasing family $\Delta$ such that $\min \Delta = \mathcal{P}_1(X) \cup \mathcal{P}_2(J \setminus X)$ for some $X \subseteq J$. Let us recall that every set with 2 elements belongs to $\Delta$. It is easy to see that $h(Y) = g_0 + g_1$ for all $Y \subseteq J$ with $|Y| \geq 2$. We shall apply the Csirmaz Lemma. If $W \subseteq Y \subseteq J$ such that $W \notin \Delta$ and $Y \in \Delta$, then $W = \emptyset$ and $|Y| \geq 1$ or $|W| = 1$ and $|Y| \geq 2$. In the former case we have $0 = h(W) < g_0 \leq h(Y)$ and in the latter case $g_0 = h(W) < g_0 + g_1 = h(Y)$. Similarly, if $Y, Z \in \Delta$ and $W = Y \cap Z \notin \Delta$, then $|W| \leq 1$. If $|W| = 1$, then $|Y|, |Z| \geq 2$. Hence

$$h(W) + h(Y \cup Z) = g_0 + (g_0 + g_1) < (g_0 + g_1) + (g_0 + g_1) = h(Y) + h(Z).$$

Now we assume $W = \emptyset$, so $|Y \cup Z| \geq 2$ and $|Y|, |Z| \geq 1$. If $g_0 > g_1$, then

$$h(Y \cup Z) = g_0 + g_1 < g_0 + g_0 \leq h(Y) + h(Z).$$

In case $g_0 = g_1$ we assumed that there is at most one singleton in $\Delta$, so $|Y| > 1$ or $|Z| > 1$. Hence

$$h(Y \cup Z) = g_0 + g_1 < g_0 + g_0 + g_1 \leq h(Y) + h(Z).$$

Thus both conditions of the Csirmaz Lemma are satisfied, which completes the proof of (1) and (2).
particular, Proposition 3.5 we see that showed that the order on the set \( (\Pi, \leq) \) obtained by the permutation of \( \Theta \) so from Lemma 3.2 (1) we get \( \bar{\omega} \). The above theorem combined with Example 3.1 is strong enough to classify all \( w \) a minimal vector in \( \Gamma \) such that \( \bar{\omega} \) cannot be hierarchically independent whenever \( x, y \in X \) and \( x \neq y \). If \( x, y \in J \setminus X \), then \( \{x, y\} \in \min \Delta \), so \( P_x \) are mutually hierarchically independent by Proposition 3.3. In particular if \( X = J \), then we get the compartmented order \( \text{Ord}(\emptyset, J) \).

It remains to show that \( P_y \sim P_x \) for \( x \in X \) and \( y \in J \setminus X \) with \( \emptyset \subsetneq X \subsetneq J \). Let \( \bar{\omega} \) be a minimal vector in \( \Gamma \) such that \( w_y \neq 0 \). If such a vector does not exist, then the block \( P_y \) is redundant so \( P_y \sim P_x \). Otherwise, applying Lemma 3.2 (3) yields \( \bar{\omega} \in B(Z, \text{supp}(\bar{\omega})) \) and \( \text{supp}(\bar{\omega}) \in \Delta \). Note that \( \{y\} \notin \min \Delta \), so \( |\text{supp}(\bar{\omega})| \geq 2 \). According to Equation (3) we get \( \bar{\omega} \in B(Z, \text{supp}(\bar{\omega})) \subsetneq B(Z, J) \). Lemma 2.14 shows that \( \bar{\omega}'' := \bar{\omega} - \bar{e}(y) + \bar{e}(x) \in B(Z, J) \) or there exist a set \( Y \subseteq J \setminus \{y\}, x \in Y \) and a vector \( \bar{\nu} \in B(Z, Y) \), such that \( \bar{\nu} \preceq \bar{\omega}'' \). In the first case we have \( \bar{\nu} \bar{\nu} \in \Gamma \) by Example 3.2 (1). If the second case occurs, then we note that \( \{x\} \subseteq Y \in \Delta \), so from Lemma 3.2 (1) we get \( \bar{\nu} \in \Gamma \), hence \( \bar{\nu} \bar{\nu} \in \Gamma \). This proves that \( P_y \sim P_x \). In this way we showed that the order on the set \( (\Pi, \preceq) \) is of the type \( \text{Ord}(\emptyset, J, X) \).

Remark 4.7. The above theorem combined with Example 4.1 is strong enough to classify all bipartite access structure determined by uniform polymatroids with \( m = 2 \). If \( \eta(g) = 1 \), then we have a threshold access structure (cf. Example 4.1). If \( \eta(g) = 2 \), then we consider three monotone increasing families \( \Delta_1 = \{\{x\}, J\}, \Delta_2 = \{\{x\}, \{y\}, J\} \) and \( \Delta_3 = \{\{J\} \) of subsets of \( J := \{x, y\} \). Let us note that \( \Delta'_1 = \{\{y\}, J\} \) is hierarchically equivalent to \( \Delta_1 \) as it can be obtained by the permutation of \( x \) and \( y \). It is easy to see that \( \Delta_1 \) is compatible with every polymatroid \( (J, h, g) \) with \( g_1 > 0 \) and the resulting access structures induce on \( \{P_x, P_y\} \) the order of the type \( \text{Ord}(\{y\}, \{x\}) \). Moreover, \( \Delta_2 \) is not compatible with a polymatroid such that \( g_0 = g_1 \) but in the remaining cases the resulting access structures are compartmented.

The following theorem describes the hierarchy on the access structures determined by uniform polymatroids with \( g_0 = g_1 = \cdots = g_{m-1} > 0 \). This result corresponds to the access structures that appear in the column \( F \) of Table 1.

**Theorem 4.8.** Let \( Z = (J, h, g) \) be a uniform polymatroid with \( m := |J| \geq 3 \) and the increment sequence \( g = (g_i)_{i \in \mathbb{N}} \) such that \( g_0 = g_{m-1} > 0 \).

1. A monotone increasing family \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) is compatible with the polymatroid \( Z \) if and only if \( \min \Delta = \{X\} \) for a certain \( \emptyset \neq X \subsetneq J \).

2. Let \( \Gamma = \Gamma(\Pi, Z, \Delta) \) be the access structure determined by the polymatroid \( Z \) and the monotone increasing family \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) such that \( \min \Delta = \{X\} \) for a certain \( \emptyset \neq X \subsetneq J \), then

   (2a) The vector \( \sum_{x \in X} g_0 e(x) \) is the only minimal authorized vector in the access structure \( \Gamma \).

   (2b) The hierarchical preorder induced by \( \Gamma \) on the set \( \Pi \) is of the type \( \text{Ord}^*(J \setminus X, X) \).

**Proof.** (1) Since \( g_0 = g_{m-1} \), i.e. \( \eta(g) = m \) we can apply Lemma 3.3 which implies that if \( \Delta \) is compatible with the polymatroid \( Z \), then \( \min \Delta \) contains only one set. To prove the reverse implication it is enough to apply Lemma 2.18.

(2a) We apply Lemma 2.11 (1) for an arbitrary \( Y \in \Delta \) and an arbitrary \( \bar{\omega} \in B(Z, Y) \). For \( l := |Y| \) we have \( h_1 \geq w_z \geq g_{l-1} = g_0 = h_1 \), and consequently \( w_z = h_1 = g_0 \) for every \( z \in Y \). Since \( X \subseteq Y \), so \( \bar{\omega} \bar{\omega} \geq \sum_{x \in X} g_0 e(x) \) for every set \( Y \in \Delta \) and for every vector \( \bar{\omega} \in B(Z, Y) \). This shows that the vector \( \sum_{x \in X} g_0 e(x) \) is the only minimal authorized vector.

(2b) According to Proposition 3.3 the blocks indexed by the elements in \( X \) are hierarchically independent. In particular, if \( X = J \), then the hierarchical order on \( \Pi \) induced by \( \Gamma \) is of the type \( \text{Ord}^*(\emptyset, J) \).
Now we assume $|X| < m$. It is shown above that $\sum_{z \in X} g_0 e^{(z)}$ is the only minimal authorized vector, so all the blocks $P_y$ with $y \notin X$ are redundant. In particular, they are mutually hierarchically equivalent and every block $P_x$, $x \in X$ is hierarchically superior but not equivalent to $P_y$, $y \in J \setminus X$, which follows from Proposition 3.3. Moreover, all blocks in $\{P_x : x \in X\}$ are hierarchically independent by Proposition 3.3. We conclude, that the hierarchical order of $\Pi$, induced by $\Gamma$, is of the type $\text{Ord}^*(J \setminus X, X)$.

Now we shall prove a similar theorem which describes hierarchical order of access structures determined by uniform polymatroids with $g_0 > g_1 = \cdots = g_{m-1} > 0$ and monotone increasing families compatible with them. This theorem describes access structures located in the column $E$ of Table 1.

**Theorem 4.9.** Let $\mathcal{Z} = (J, h, \mathbf{g})$ be a uniform polymatroid with the increment sequence $\mathbf{g} = (g_i)_{i \in \mathbb{N}}$ such that $m := |J| \geq 3$ and $g_0 > g_1 = g_{m-1} > 0$.

(1) A monotone increasing family $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ is compatible with $\mathcal{Z}$ if and only if $\min \Delta = \{X\}$ for a certain $X \subseteq J$ or $\min \Delta = \mathcal{P}_1(J)$.

(2) Let $\Gamma = \Gamma(\Pi, \mathcal{Z}, \Delta)$ be the access structure determined by the polymatroid $\mathcal{Z}$ and the monotone increasing family $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$. Then

(2a) If $\min \Delta = \{X\}$ for a certain $\emptyset \neq X \subseteq J$, then the hierarchical order induced by $\Gamma$ on $\Pi$ is of the type $\text{Ord}(J \setminus X, X)$.

(2b) If $\min \Delta = \mathcal{P}_1(J)$, then the hierarchical order induced by $\Gamma$ on $\Pi$ is of the type $\text{Ord}(\emptyset, J)$.

**Proof.** (1) Let us assume that $\Delta$ is compatible with $\mathcal{Z}$. It is enough to consider the case where $\Delta$ has at least two different minimal sets. From the assumption $g_{m-1} > 0$ we have $\eta(\mathbf{g}) = m$, so applying Lemma 3.3 we conclude that those sets must be singletons. Let $\{x\}, \{y\} \in \min \Delta$ for some $x, y \in J$. Suppose that there is $z \in J$ such that $\{z\} \notin \min \Delta$. Of course, $\{x, z\}, \{y, z\} \in \Delta$, but $\{x, z\} \cap \{y, z\} = \{z\} \notin \Delta$. Using the Csirmaz Lemma yields

$$h(\{z\}) + h(\{x, y, z\}) < h(\{x, z\}) + h(\{y, z\}),$$

hence we get $h_3 - h_2 < h_2 - h_1$ and consequently $g_2 < g_1$ which is a contradiction, so every singleton belongs to $\min \Delta$. To show the reverse implication, let us consider two cases:

If $\min \Delta = \{X\}$ for some $X \subseteq J$, then we refer to Lemma 2.18.

If $\min \Delta = \mathcal{P}_1(J)$, then the claim it follows from Lemma 2.19.

(2a) Assume $\min \Delta = \{X\}$ for some $\emptyset \neq X \subseteq J$. The fact that $P_x, P_y$ are hierarchically independent for arbitrary $x, y \in X$, $x \neq y$ is obtained directly from Proposition 3.4. In particular, if $X = J$, then the ordered set $(\Pi, \preceq)$ is of the type $\text{Ord}(\emptyset, J)$.

Now we assume that $|X| < m$. Consider $x \in X$ and $y \notin X$. According to Proposition 3.5 the blocks $P_y$ and $P_x$ are hierarchically independent or $P_y \preceq P_x$. We shall show that $P_y \preceq P_x$.

Let $\mathcal{Z}'$ be an arbitrary minimal vector in $\Gamma$ such that $w_y \neq 0$. The existence of such vectors is ensured by Theorem 4.1. Then from Lemma 3.2 (3) we have $\tilde{w} \in \mathcal{B}(\mathcal{Z}, \text{supp}(\tilde{w}))$ and $\text{supp}(\tilde{w}) \in \Delta$, so $X \subseteq \text{supp}(\tilde{w})$, in particular $x \in \text{supp}(\tilde{w})$. Note that $k := |\text{supp}(\tilde{w})| \geq 2$, because $y, x \in \text{supp}(\tilde{w})$. According to Lemma 2.11 (1), we get $w_y \geq k_{k-1} - g_1$. By assumption we have $g_{k-1} = g_1$, hence we can consider two cases:

(i) $w_y = g_1$, so according to Lemma 2.11 (2) we have $\tilde{v} = \tilde{w} - w_y e^{(y)} \in \mathcal{B}(\mathcal{Z}, \text{supp}(\tilde{w})) \setminus \{y\}$, but $X \subseteq \text{supp}(\tilde{w}) \setminus \{y\}$, so from Lemma 3.2 (1) we get $\tilde{v} \in \Gamma$. Then of course $\tilde{v} \leq \tilde{w}' := \tilde{w} - e^{(y)} + e^{(x)}$, so $\tilde{w}' \in \Gamma$.

(ii) $w_y > g_1$ and denote $Y := \text{supp}(\tilde{w})$. According to Lemma 2.11 we get $\tilde{w}' := \tilde{w} - e^{(y)} + e^{(x)} \in \mathcal{B}(\mathcal{Z}, Y)$ or there is a set $Z \subseteq Y \setminus \{y\}$, $x \in Z$ such that $\tilde{v} := \tilde{w}_Z \in \mathcal{B}(\mathcal{Z}, Z)$. In particular, we have $|\tilde{w}_Z| = |\tilde{w}| = h_{|Z|}$. Let $Y = Z \cup W \cup \{y\}$ be the union of three disjoint sets, where $W = Y \setminus (Z \cup \{y\})$. Then, using Lemma 2.11 (1) and assumptions, we obtain that each coordinate of the vector $\tilde{w}$ is at least $h_1$, hence:

$$h_{|Y|} = |\tilde{w}| = |\tilde{w}_Z| + |\tilde{w}_W| + w_y > |\tilde{v}| + |W|g_1 + g_1 = h_{|Z|} + |W|g_1 + g_1 = h_{|Y|},$$
where the last equality is obtained from Equation (5) in the following way:

\[ h_{|Y|} - h_{|Z|} = \sum_{i=|Z|}^{|Y|-1} g_i = |W|g_1 + g_1. \]

A contradiction we have obtained shows that \( \bar{w}' \in B(Z, Y) \subseteq \Gamma \). In both of the above cases we have received that \( \bar{w}' \in \Gamma \). Since this holds for all \( \bar{w} \in \min \Gamma \) with \( w_y > 0 \), we conclude \( P_y \preceq P_x \).

It remains to show that \( P_y, P_x \) are hierarchically independent when \( x, y \notin X \). If it were otherwise, then assuming \( x := m \) and applying Proposition 3.7 we would get \( g_0 = g_1 \) contrary to the assumption made here. In this way, we showed that the order on \( \Pi \) is of type \( \text{Ord}(J \setminus X, X) \).

(2b) Assume \( \min \Delta = P_1(J) \). If \( P_y \preceq P_x \) for some \( x, y \in J \), then \( P_y \) is hierarchically inferior or equivalent to \( P_x \) and \( \{x\}, \{y\} \in \min \Delta \), which contradicts Proposition 3.3. In this way we showed that the order on \( \Pi \) is of type \( \text{Ord}(\emptyset, J) \).

Remark 4.10. The result of (2b) can be generalized to all monotone increasing families \( \Delta \) with \( \min \Delta = P_k(J) \), where \( k = 1, \ldots, m \). According to Lemma 2.19 (cf [9, Lemma 6.1]) such \( \Delta \) is compatible with a uniform polymatroid \( Z = (J, h, g) \) if and only if \( g_{k-1} > g_k \). Let \( \Gamma = \Gamma(\Pi, Z, \Delta) \). If \( k = 1 \) and \( g_1 > 0 \), then \( \Gamma \) is compartmented by Proposition 3.5. If \( k \geq 2 \), then \( \mu(\Delta) \geq 2 \), so \( \Gamma \) is compartmented which follows from Theorem 4.3. In both cases the hierarchical order induced by \( \Gamma \) in \( \Pi \) is of the type \( \text{Ord}(\emptyset, J) \). A similar class, called uniform multipartite access structures were also considered by Farrás, Padró, Xing and Yang [9].

Another theorem describes the hierarchy of blocks in the access structures determined by polymatroids, for which \( g_{m-1} > 0 \) and monotone increasing families with one minimal set which contains exactly one element. This theorem deals with the existence and hierarchy of access structures placed in the first row of Table I.

Theorem 4.11. Let \( Z = (J, h, g) \) be a uniform polymatroid with the increment sequence \( g = (g_i)_{i \in I_m} \) and let \( \Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\} \) be a monotone increasing family such that \( \min \Delta = \{\{x\}\} \) for a certain \( x \in J \).

1. Then \( \Delta \) is compatible with the polymatroid \( Z \) if and only if \( g_{m-1} > 0 \).

2. Let \( \Gamma = \Gamma(\Pi, Z, \Delta) \) be the access structure determined by the polymatroid \( Z \) such that \( g_{m-1} > 0 \) and the monotone increasing family \( \Delta \). Then

   (2a) If \( g_0 = g_{m-1} \), then the hierarchical order induced by \( \Gamma \) on \( \Pi \) is of the type \( \text{Ord}^*(J \setminus \{x\}, \{x\}) \).

   (2b) If \( g_0 > g_{m-1} \), then the hierarchical order induced by \( \Gamma \) on \( \Pi \) is of the type \( \text{Ord}(J \setminus \{x\}, \{x\}) \).

Proof. (1) The fact that \( \Delta \) is compatible with \( Z \) can be obtained from Lemma 2.15. Conversely, let us suppose that \( g_{m-1} = 0 \). Then every subset of \( J \) with \( m - 1 \) elements belongs to \( \Delta \) by Lemma 2.17 (1). But this contradicts the fact that \( J \setminus \{x\} \notin \Delta \). This implies \( \Delta \) is not compatible with \( Z \) whenever \( g_{m-1} = 0 \).

(2) We shall show that \( P_y \preceq P_x \) for every \( y \in J \setminus \{x\} \). From Proposition 3.5 it follows that \( P_x \) is not hierarchically inferior to any block in \( \Pi \) so \( P_x \) is not hierarchically equivalent to any other block. Let us fix \( y \in J \), \( y \neq x \). If \( w_y = 0 \) for every minimal vector \( \bar{w} \in \Gamma \), then the block \( P_y \) is redundant, so \( P_y \preceq P_x \). Let us assume that \( \bar{w} \) is a minimal vector in \( \Gamma \) such that \( w_y \neq 0 \). Then from Lemma 3.2 (3) we have \( \bar{w} \in B(Z, \text{supp}(\bar{w})) \) and \( \text{supp}(\bar{w}) \in \Delta \), so \( x \in \text{supp}(\bar{w}) \). From Lemma 2.14 it follows that \( \bar{w}' := \bar{w} - c(y) + c(x) \in B(Z, \text{supp}(\bar{w})) \) or there is a set \( Y \subseteq \text{supp}(\bar{w}) \setminus \{y\} \), \( x \in Y \) such that \( \bar{v} := \bar{w}_Y \in B(Z, Y) \). In the former case, we get \( \bar{w}' \in \Gamma \) from Lemma 3.2 (1). If the latter case is fulfilled, then we notice that \( Y \in \Delta \), so from Lemma 3.2 (1) we get \( \bar{v} \in \Gamma \) and \( \bar{v} \leq \bar{w}' \). This means that in both cases \( \bar{w}' \in \Gamma \). Since this holds for all \( \bar{w} \in \min \Gamma \) with \( w_y > 0 \), we conclude \( P_y \preceq P_x \). This shows that the (pre)order on \( \Pi \) is of the type \( \text{Ord}^*(J \setminus \{x\}, \{x\}) \) or \( \text{Ord}(J \setminus \{x\}, \{x\}) \).
(2a) Since $g_0 = g_{m-1}$, applying Theorem 4.8 (2b) yields the claim.

(2b) If the preorder on $\Pi$ were of the type $\text{Ord}^h(J \setminus \{x\}, \{x\})$, then the blocks $P_y$ and $P_z$ would be hierarchically comparable for some $y, z \in J \setminus \{x\}$. This and Proposition 3.7 for $n = m$ imply $g_0 = g_1 = \cdots = g_{m-1}$. But this is a contradiction to $g_0 > g_{m-1}$, so the order on $\Pi$ is of type $\text{Ord}(J \setminus \{x\}, \{x\})$.

Let us noticing that the order induced by the access structure satisfying the assumptions of the above theorem corresponds to the organizational chart of an institution composed of several mutually independent departments $(P_y)_{y \in J \setminus \{x\}}$ managed by one superior unit $P_x$. It follows from Theorem 5.2 that such access structures is ideal.

The results in this chapter provide information about hierarchy induced on the set of participants by various access structures contained in Table 1, except the cell C2. That area contains objects obtained from monotone increasing families $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ with $\mu(\Delta) = 1$ and compatible polymatroids $Z = (J, h, g)$ with $3 \leq \eta(g) \leq m - 1$. Computer calculations show that this area contains both compartmented and hierarchical access structures and some of them are different of those considered in the above theorems. Some examples can be seen in Table 2.

Every linearly ordered subset of a partially (pre)ordered set is called a chain. A chain that contains only one element is referred to as trivial. We assume that a chain in a partition of participants does not contain hierarchically equivalent blocks. Let us observe that every non-trivial chain of blocks in the access structures investigated above contains 2 blocks. The next theorem shows that all weakly hierarchical access structures obtained from uniform polymatroids have this property.

**Theorem 4.12.** Every chain in the hierarchical access structure determined by arbitrary uniform polymatroid contains 1 or 2 blocks.

**Proof.** Let $n := \eta(g)$. For $n = 1$, i.e. $g_0 > g_1 = 0$, it follows from Example 3.1 that $\Gamma = \Gamma(\Pi, Z, \Delta)$ is a threshold access structure, so all blocks of participants are mutually hierarchically equivalent, thus every chain is trivial.

Suppose that $n \geq 2$ and $\Pi$ contains a chain of blocks composed of three hierarchically non-equivalent blocks, i.e. $P_z \prec P_y \prec P_x$ for some $x, y, z \in J$. Let $X \subseteq J$ such that $|X| = n$ and $y, z \in X$. By Lemma 2.17(1) we have $X \in \Delta$, but Proposition 3.3 implies that $X \notin \min \Delta$. Thus there is $Y \subseteq X$ such that $Y \in \min \Delta$ in particular $|Y| < n$. If $n = 2$, then $|Y| = 1$, but neither $\{y\}$ nor $\{z\}$ is minimal in $\Delta$, which follows from Proposition 3.5, a contradiction. If $n \geq 3$, then by Proposition 3.5 we know that $y, z \notin Y$. Thus $|Y| \leq n - 2$. Using Proposition 3.7 we get $g_0 = g_{m-1}$ and this combined with Corollary 3.9 shows that $g_0 = g_{m-1}$. Now from Theorem 4.8 we conclude that every chain in $\Pi$ contains at most 2 blocks, which contradicts our assumption.

The above theorem seems quite surprising, because for other polymatroids one can construct hierarchical access structures with chains of arbitrary length. For instance, such objects can be found in [8], [9], [12], [14] and others.

## 5 Ideal access structures obtained from uniform polymatroids

In this section we shall prove that the access structure studied in theorems 4.6, 4.8, 4.9 and 4.11 are ideal. To do this we show that all simple extensions of suitable uniform polymatroids are representable over sufficiently large finite fields and then we apply Remark 2.5. We begin by recalling Example 3.1 where we noticed that every polymatroid $Z = (J, h, g)$ with $\eta(g) = 1$ determines a threshold access structure which is known to be ideal as it is realized by the Shamir threshold secret sharing scheme. Now we shall consider the case $\eta(g) = 2$.

**Theorem 5.1.** All access structures determined by any uniform polymatroid $Z = (J, h, g)$ with $\eta(g) = 2$ are ideal.

**Proof.** The assumption $\eta(g) = 2$ implies $g_0 \geq g_1 > g_2 = 0$. Let $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ be a monotone increasing family compatible with $Z$. It is enough to show that the simple extension $Z'$ of $Z$...
induced by $\Delta$ is a representable polymatroid. Let $K$ be a finite field with $q := |K| > m$. By an abuse of notation, we will use $\theta$ to denote the zero vector in any vector space $K^n$. Let us consider a collection $(a_x)_{x \in J}$ of pairwise different nonzero elements of $K$. For every $x \in J$ we define $V_x := \{(a, a_x \alpha) : \alpha \in K^n\}$. It is easy to check that $V_x$ is a vector subspace of $K^{g_1} \times K^{g_1}$ and $\dim V_x = g_1$. Assume $x \not= y$ and $(\alpha_1, \alpha_2) \in V_x \cap V_y$. Hence $\alpha_2 = a_x \alpha_1$ and $\alpha_2 = a_y \alpha_1$, so $\theta = a_x \alpha_1 - a_y \alpha_1 = (a_x - a_y) \alpha_1$. Since $a_x - a_y \not= 0$, so $\alpha_1 = \theta$. This shows $V_x \cap V_y = \{\theta\}$. Hence $\dim(V_x + V_y) = \dim V_x + \dim V_y - \dim(V_x \cap V_y) = \dim V_x + \dim V_y = 2g_1$. In particular, $V_x + V_y = K^{g_1} \times K^{g_1}$ for all $x, y \in J$, $x \not= y$. Thus $(V_x)_{x \in J}$ is a vector space representation of the polymatroid $\mathcal{Z}$ provided $g_0 = g_1$. According to Theorem 4.10 (2) we have two cases. If $\min \Delta = \{\{x\} \cup \mathcal{P}_2(J \setminus \{x\})\}$, then we take $\theta \not= \beta \in V_x$. For a certain $x_0 \not\in J$ we define $V_{x_0} := \text{span}(\beta)$. It is easily seen that $(V_x)_{x \in J \cup \{x_0\}}$ is a vector space representation of $\mathcal{Z}'$ induced by $\Delta$.

If $\min \Delta = \mathcal{P}_2(J)$, then we take $\beta \in K^{g_1} \times K^{g_1} \setminus \bigcup_{x \in J} V_x$. It is possible as $|\bigcup_{x \in J} V_x| \leq mg^{g_1} < q^{g_1+1} \leq q^{g_2} = |K^{g_1} \times K^{g_1}|$. Now we define $V_{x_0} := \text{span}(\beta)$. It is easily seen that $(V_x)_{x \in J \cup \{x_0\}}$ is a vector space representation of $\mathcal{Z}'$ induced by $\Delta$.

Now we assume $g_0 > g_1$ and define $U_x := K^{g_0-g_1} \times V_x \subseteq K^{g_0} \times K^{g_1} \times K^{g_1}$ for every $x \in J$. For simplicity of notation, the vector space $K^{g_0-g_1} \times K^{g_1} \times K^{g_1}$ will be identified with $K^{g_0+g_1}$. It is clear that $\dim U_x = g_0$. Moreover $U_x + U_y = (K^{g_0-g_1} \times V_x) + (K^{g_0-g_1} \times V_y) = K^{g_0+g_1}$ and $U_x \cap U_y = K^{g_0-g_1} \times \{\theta\} \times \{\theta\}$. In particular $\varphi := (1, 0, \ldots, 0) \in U_x$ for all $x \in J$.

If $\Delta$ is compatible with $\mathcal{Z}$, then by Theorem 4.10 there is $X \subseteq J$ such that $\min \Delta = \mathcal{P}_1(X) \cup \mathcal{P}_2(J \setminus X)$.

To explain the general idea of the next step of the proof we use projective geometry. Every subspace $U_x$ can be considered as $(g_0 - 1)$-dimensional subspace of the projective space of dimension $g_0 + g_1 - 1$. The projective point $E := \text{span}(\varepsilon)$ belongs to the intersection of all subspaces $U_x$ (Figure 1). Now we take a projective point $B := \text{span}(\beta^*)$ that does not belong to any subspace $U_x$ and the translation of the whole space $\varphi$ sending $E$ to $B$. Then the family of $(\varphi(U_x))_{x \in X}$ together with the family $(U_x)_{x \in J \setminus X}$ form another vector space representation of $\mathcal{Z}$ (Figure 2).

Now we only need to add $U_{x_0} := \text{span}(\beta^*)$ to those families to get a representation of $\mathcal{Z}'$.

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![Figure 1](image1.png)

![Figure 2](image2.png)

Now we can do the formal calculations. Let $\nu : K^{g_0+g_1} \rightarrow K$ be defined by $\nu(\alpha) = \nu(a_1, \ldots, a_{g_0+g_1}) = a_1$ for every $\alpha = (a_1, \ldots, a_{g_0+g_1}) \in K^{g_0+g_1}$. Let $\beta_1 \in K^{g_1} \times K^{g_1} \setminus \bigcup_{x \in J} V_x$ and $\beta := (\theta, \beta_1) \in K^{g_0+g_1}$. Obviously $\beta \not\in U_x$ for every $x \in J$.

Now we define $\varphi : K^{g_0+g_1} \rightarrow K^{g_0+g_1}$ by setting $\varphi(\alpha) = \alpha + \nu(\alpha)\beta$ for all $\alpha \in K^{g_0+g_1}$. Let us notice that $\varphi$ is an isomorphism of vector spaces, so $\dim \varphi(U_x) = \dim U_x = g_0$. Moreover $\varphi(\alpha) = \alpha$ for all $\alpha \in \{\theta\} \times K^{g_1} \times K^{g_1}$ and $\beta^* := \varphi(\varepsilon) = \varepsilon \not\in U_x$ for all $x \in J$.

Let $x_0$ be any element not in $J$ and let $U_{x_0} := \text{span}(\beta^*)$. Then the family $(\varphi(U_x))_{x \in X} \cup (U_x)_{x \in J \setminus X} \cup \{U_{x_0}\}$ is a vector space representation of the simple extension of $\mathcal{Z}$ induced by $\Delta$. Indeed, if $x \not\in X$, then $h(\{x, x_0\}) = \dim(U_x + U_{x_0}) > \dim U_x = h(\{x\}) = g_0$ as $\beta^* \not\in U_x$. Thus $\{x\} \not\in \min \Delta$. For $x \in X$ we have $h(\{x, x_0\}) = \dim(\varphi(U_x) + U_{x_0}) = \dim U_x = h(\{x\}) = g_0$, so $\{x\} \in \min \Delta$.

From the fact that $\varphi$ is a vector space isomorphism it follows $(U_x + U_{x_0}) = \varphi(U_x + U_{x_0}) = K^{g_0+g_1}$ for all $x, y \in X$. For $x \in X$ and $y \in J \setminus X$ we have $\varphi(U_x) + U_y \supseteq \varphi(\{\theta\} \times V_{x}) + U_y = (\{\theta\} \times V_{x}) + U_y = K^{g_0+g_1} \times (V_{x} + V_{y}) = K^{g_0+g_1}$. In every case $h(\{x, y, x_0\}) = \dim(K^{g_0+g_1} + U_{x_0}) = g_0 + g_1$, i.e., $\{x, y\} \in \Delta$. If $x, y \not\in X$, then $\{x, y\} \in \min \Delta$.\[\square\]
In the next proof we will need the following well-known property of vector spaces over finite fields. Let \( V_1, \ldots, V_n \) be proper subspaces of a vector space \( V \) over a finite field \( \mathbb{K} \). If \( |\mathbb{K}| > n \), then \( V_1 \cup \ldots \cup V_n \neq V \). Let us recall that every uniform polymatroid is representable.

**Theorem 5.2.** All access structures determined by any uniform polymatroid \( Z = (J, h, g) \) with \( \eta(g) = m \) and monotone increasing family \( \Delta \subseteq P(J) \) such that \( |\Delta| = 1 \) are ideal.

**Proof.** Let \( \min \Delta = \{X\} \) for a suitable \( \emptyset \neq X \subseteq J \) and let \( k := |X| \). The assumption \( \eta(g) = m \) is equivalent to \( g_{m-1} > 0 \) and this implies \( h(Y) < h(Z) \) for all \( Y \subsetneq Z \subseteq J \). It follows form Lemma 2.18 that \( \Delta \) is compatible with \( Z \).

Let \( \mathbb{K} \) be a finite field and let \( (V_x)_{x \in J} \) be a \( \mathbb{K} \)-vector space representation of \( Z = (J, h, g) \). Then \( V_x \) are subspaces of the vector space \( \mathbb{K}^{|J|} \) and \( \dim V_x = h_1 = g_0 \) for every \( x \in J \). Given any \( Y \subseteq J \) we define \( V_Y := \sum_{y \in Y} V_y \). If \( Y \in \Delta \), then \( X \subseteq Y \) and \( V_X \subseteq V_Y \). If \( Y \notin \Delta \), then \( X \subseteq Y \) and so \( |X \cup Y| > |Y| \). Hence \( \dim(V_X + V_Y) = \dim V_{X \cup Y} = h(X) > h(Y) = \dim V_Y \). This shows, that \( V_X \not\subseteq V_Y \). Thus \( Y \in \Delta \) if and only if \( V_X \subseteq V_Y \). Since \( V_Y \cap V_X \) is a proper subspace of \( V_X \) whenever \( Y \notin \Delta \) and, so assuming \( |\mathbb{K}| > 2^m - 2^{m-k} \) we have \( V_Y \cap \bigcup_{Y \in P(J) \Delta} V_Y = \bigcup_{Y \in P(J) \Delta} (V_Y \cap V_X) \) is a proper subset of \( V_X \). This shows that there is \( \beta \in V_X \) such that \( \beta \notin V_Y \) for all \( Y \notin \Delta \). Setting \( V_{x_0} := \langle \beta \rangle \) we get \( (V_x)_{x \in J \cup \{x_0\}} \) which is a vector space representation of the simple extension of \( Z \) induced by \( \Delta \).

**Theorem 5.3.** [4, Lemma 6.2] If the monotonic increasing family \( \Delta \subseteq P(J) \) such that \( \min \Delta = P_k(J), 1 \leq k \leq m \) is compatible with a uniform polymatroid \( Z \), then the access structure determined by \( \Delta \) and \( Z \) is ideal.

Let us notice that Theorem 5.2 shows that the access structures presented in Theorem 4.11 are ideal. Now we turn to the objects considered in Theorems 4.8 and 4.9.

**Corollary 5.4.** All access structures determined by any uniform polymatroid \( Z = (J, h, g) \) with the increment sequence \( g = (g_i)_{i \in J} \) such that \( |J| \geq 3 \) and \( g_0 \geq g_1 = g_{m-1} > 0 \) are ideal.

**Proof.** We want to prove that for every increasing family \( \Delta \subseteq P(J) \setminus \{\emptyset\} \) that is compatible with \( Z \) the access structure determined by \( Z \) and \( \Gamma \) is ideal. The assumption \( g_0 \geq g_1 = g_{m-1} > 0 \) combined with Theorems 4.8 and 4.9 imply that \( \min \Delta = P_1(J) \) or \( |\min \Delta| = 1 \). In the former case the claim follows from Theorem 5.3. In the latter case applying Theorem 5.2 completes the proof.

**6 Conclusion**

This paper is intended to initiate research on the access structures obtained from polymatroids. This choice is motivated by the fact that access structures determined by polymatroids are matroidal ports, i.e., they satisfy a necessary condition to be ideal. In this paper our investigation is limited to uniform polymatroids. We are particularly interested in the hierarchical order on
the set of participants determined by the access structures considered here. Most of the results in the literature that is devoted to discussing this subject consider access structures which are compartmented or totally hierarchical. We showed that all non compartmented access structure with at least three parties considered in this work are partially hierarchical. It is worth pointing out that some examples of partially hierarchical access structures are presented by Farrás et al. [9], but they are not determined by uniform polymatroids. There is good reason to deal with uniform polymatroids. In contrast to general polymatroids, every uniform polymatroid determines ideal access structures. It follows from the fact that every uniform polymatroid is representable. This allows building simple extensions of such polymatroids, which are also representable. Then according to Remark 2.5 the suitable access structures obtained from those polymatroids are ideal.

The conditions presented in Section 3 are used to prove Theorems 4.2 and 4.3 which show that most of access structures obtained from uniform polymatroids are compartmented (they are placed in the cells D2 and B3 - D3 of Table 1). The exact hierarchy in access structures in the cells A2, B2, D1-F1, E2-3 and F2-3 is described in Theorems 4.6 - 4.11.

The most diverse collection of objects contains the cell C2 where both compartmented and hierarchical access structures can be found but further precise investigation of that area is necessary. In general, the results presented here do not exhaust the topic and leaves space for further research.

**Conjecture 6.1.** Let $\Pi = (P_x)_{x \in J}$ be a partition of a set of participants $P$ and let $Z = (J, h, g)$ be uniform polymatroid with $2 \leq \eta(g) < m$. Additionally, let $\Delta \subseteq \mathcal{P}(J) \setminus \{\emptyset\}$ be a monotone increasing family with $\mu(\Delta) = 1$ that is compatible with $Z$. The hierarchical order in $\Pi$ induced by $\Gamma = \Gamma(\Pi, Z, \Delta)$ is of the type $\text{Ord}(Y, X)$ for a certain disjoint subsets $X, Y$ of $J$.

This conjecture is partially confirmed by Theorem 4.12 that states that every chain in hierarchical access structure contains 1 or 2 elements. This fact applies only to access structures induced by uniform polymatroids. For other polymatroids one can construct hierarchical access structures with chains of arbitrary length.

Some multipartite access structures determined by uniform polymatroids contain redundant blocks or different blocks that are equivalent. We treat such objects as improperly constructed. Fortunately, they appear only as extreme cases (cf. Corollary 3.10 and Theorem 4.1).

The results presented in Section 4 do not depend on the particular values of the rank function of $Z$ (or equivalently the values of $g$). The only impact on the hierarchy of the described structures have the sequence of signatures of differences of consecutive entries of $g$. This observation is additionally confirmed by computer calculations which suggest the following unproved conjecture.

**Conjecture 6.2.** Let $g = (g_i)_{i \in I_m}$ and $g' = (g'_i)_{i \in I_m}$ be the increment sequences of uniform polymatroids $Z$ and $Z'$ with the ground set $J$, respectively such that $\text{sgn}(g_{i-1} - g_i) = \text{sgn}(g'_{i-1} - g'_i)$ for all $i = 1, \ldots, m$. If a monotone increasing family $\Delta$ is compatible with $Z$ and $Z'$, then the hierarchical preorders on $\Pi$ determined by $\Gamma(\Pi, Z, \Delta)$ and $\Gamma(\Pi, Z', \Delta)$ are equal.

Investigating which of the structures considered in this article are ideal is another open issue. A sufficient condition can be obtained by proving that the simple extension of a given uniform polymatroid is representable (cf. [7, Corollary 6.7]). This idea has been used to show that the access structure discussed in Theorems 4.6 and 4.8 - 4.11 are ideal. By analyzing the structure of the vector space representation of the polymatroid, one can also prove the ideality of many other access structures. However, we cannot rule out the existence of non-ideal access structures derived from uniform polymatroids. In this case we have the following question. Is it true that upper bound for the information ratio of access structures obtained form uniform polymatroids can be significantly less than the upper bound for the information ratio of arbitrary matroid ports? Let us recall, the information ratio of a secret sharing scheme is the ratio between the maximum length of the shares and the length of the secret with a finite domain of shares. The information ratio of an access structure $\Gamma$ is the infimum of all information ratios taken over all secret sharing schemes with the access structure $\Gamma$. 
Appendix

Table 2 presents hierarchical (pre)orders of access structures determined by uniform polymatroids $\mathcal{Z} = (J, h, g)$ where $J = \{1, 2, 3, 4\}$. It is worth pointing out that types of orders are invariant with respect to permutations of elements of $J$, so monotony increasing families appearing in the table are representatives of invariant classes of the permutation group $S_4$ acting on $J$. For example, the monotone increasing families $\Delta_1$ and $\Delta_2$ such that $\min \Delta_1 = \{\{1\}, \{2, 3\}\}$ and $\min \Delta_2 = \{\{2\}, \{3, 4\}\}$ represent the same invariant class. Assuming that Conjecture 6.2 is true, the Table presents a complete overview of hierarchical orders of all access structures obtained from uniform polymatroids $(J, h, g)$ with $|J| = 4$. If the monotonic family appearing in the first column is not compatible with the polymatroid represented by the values of $g$ in the top rows, then the suitable cell of the table contains $-$. Otherwise, the types of (pre)orders are denoted according to the following key.

\[
P_1 \ P_2 \ P_3 \ P_4
\]

$T := Ord^*(J_4, \emptyset)$

$C := Ord^*(\emptyset, J_4) = Ord(\emptyset, J_4)$

\[
P_1
\]

$I := Ord^*(\{2, 3, 4\}, \{1\})$

$M := Ord(\{2, 3, 4\}, \{1\})$

\[
P_1
\]

$V := Ord^*(\{3, 4\}, \{1, 2\})$

$K := Ord(\{3, 4\}, \{1, 2\})$

\[
P_1
\]

$E := Ord(\{4\}, \{1\})$

$W := Ord^*(\{4\}, \{1, 2, 3\}) = Ord(\{4\}, \{1, 2, 3\})$
Table 2: Access structures in the case $m = 4$.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $g_0$ | 1 | 2 | 1 | 3 | 2 | 2 | 1 | 3 | 2 | 4 | 3 | 3 | 2 | 2 | 1 |
| $g_1$ | 0 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 2 | 3 | 3 | 2 | 2 | 1 | 1 |
| $g_2$ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| $g_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

| min $\Delta$ | | 1 | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\{\{1\}\}$ | | | | | | | | | | | | | | |
| 2 | $\{\{1\}, \{2\}\}$ | | | | | | | | | | | | | | |
| 3 | $\{\{1\}, \{2\}, \{3\}\}$ | | | | | | | | | | | | | | |
| 4 | $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ | | | | | | | | | | | | | | |
| 5 | $\{\{1\}, \{2\}, \{3, 4\}\}$ | | | | | | | | | | | | | | |
| 6 | $\{\{1\}, \{2, 3\}\}$ | | | | | | | | | | | | | | |
| 7 | $\{\{1\}, \{2, 3\}, \{2, 4\}\}$ | | | | | | | | | | | | | | |
| 8 | $\{\{1\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ | | | | | | | | | | | | | | |
| 9 | $\{\{1\}, \{2, 3, 4\}\}$ | | | | | | | | | | | | | | |
| 10 | $\{\{1, 2\}\}$ | | | | | | | | | | | | | | |
| 11 | $\{\{1, 2\}, \{1, 3\}\}$ | | | | | | | | | | | | | | |
| 12 | $\{\{1, 2\}, \{3, 4\}\}$ | | | | | | | | | | | | | | |
| 13 | $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ | | | | | | | | | | | | | | |
| 14 | $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ | | | | | | | | | | | | | | |
Table 3: Access structures in the case $m = 4$. 

| g | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $g_0$ | 1  | 2  | 3  | 2  | 2  | 1  | 3  | 2  | 4  | 3  | 3  | 2  | 2  | 1  |    |
| $g_1$ | 0  | 1  | 1  | 2  | 2  | 1  | 1  | 2  | 3  | 3  | 2  | 2  | 1  | 1  |    |
| $g_2$ | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 2  | 2  | 2  | 2  | 1  | 1  | 1  |
| $g_3$ | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |

- $\min \Delta$

15. $\{\{1, 2\}, \{2, 3\}, \{1, 4\}\}$

- $\{\{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\}$

- $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\{1, 4\}\}$

- $\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}\}$

19. $\{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$

20. $\{\{1, 2\}, \{1, 3, 4\}\}$

21. $\{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$

22. $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$

23. $\{\{1, 2\}, \{1, 3, 4\}, \{2, 3, 4\}\}$

24. $\{\{1, 2, 3\}\}$

25. $\{\{1, 2, 3\}, \{1, 2, 4\}\}$

26. $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$

27. $\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$

28. $\{\{1, 2, 3, 4\}\}$
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