Quark and lepton mass matrices
with $A_4$ family symmetry

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Abstract

Realistic quark masses and mixing angles are obtained applying the successful $A_4$ family symmetry for leptons, motivated by the quark-lepton assignments of SU(5). The $A_4$ symmetry is suitable to give tri-bimaximal neutrino mixing matrix which is consistent with current experimental data. We study new scenario for the quark sector with the $A_4$ symmetry.

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1 Introduction

The current observed neutrino mixing \[1, 2\] suggests around the maximal 2-3 mixing angle and zero 1-3 mixing angle: $\theta_{23} \sim \pi/4$, $\theta_{13} \sim 0$. In such a symmetric limit where both $\cos \theta_{23}$ and $|U_{e3}| = \sin \theta_{13}$ vanish, the resulting $3 \times 3$ effective Majorana neutrino mass matrix forms in the flavor basis as \[3\]

$$
\begin{pmatrix}
X & C & C \\
C & A & B \\
C & B & A
\end{pmatrix}
$$

This matrix has an exact symmetric form under a $Z_2$ symmetry, i.e. the 2-3 ($\mu$-$\tau$) permutation, and is diagonalized by the unitary matrix:

$$
U_{Z_2} = \begin{pmatrix}
\cos \theta_{12} & -\sin \theta_{12} & 0 \\
\sin \theta_{12}/\sqrt{2} & \cos \theta_{12}/\sqrt{2} & -1/\sqrt{2} \\
\sin \theta_{12}/\sqrt{2} & \cos \theta_{12}/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix},
$$

with remaining the solar mixing angle $\theta_{12}$ arbitrary and the entry $X$ of (1) is determined as

$$
X = A + B + \frac{2\sqrt{2}C}{\tan 2\theta_{12}}.
$$

Now we well know two special values for $\theta_{12}$ which give typical mixing matrices; one is bimaximal and the other is tri-bimaximal mixing matrices. In a limit of bimaximal mixing \[4\] where $\theta_{12} = 4/\pi$, resulting MNS matrix forms

$$
U_{BM} = \begin{pmatrix}
1/\sqrt{2} & -1/\sqrt{2} & 0 \\
1/2 & 1/2 & -1/\sqrt{2} \\
1/2 & 1/2 & 1/\sqrt{2}
\end{pmatrix},
$$

with

$$
X = A + B.
$$

For the case $\tan \theta_{12} = 1/\sqrt{2}$ with so-called tri-bimaximal mixing which is proposed by Harrison, Perkins and Scott, then we have the HPS type matrix \[5, 6\]:

$$
U_{HPS} = \begin{pmatrix}
2/\sqrt{6} & \frac{1}{\sqrt{3}} & 0 \\
-1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix},
$$

with

$$
X = A + B.
$$
where $\theta_{12}$ is fixed by as well

$$X = A + B + C$$

is also derived. Note that the tri-bimaximal structure is consistent with current experimental data, where $\theta_{12}$ is not maximal.

So far, the discrete symmetry $A_4$ is successfully applied for leptons. Namely, the tri-bimaximal mixing pattern can be realized naturally in a number of specific models [7, 8, 9, 10, 11]. However, it is not easy to have small quark mixing angles in naive and straight way. In such applications the generic prediction [8] is that $V_{\text{CKM}}$, the quark mixing matrix becomes just the unit matrix. Starting with $V_{\text{CKM}} = 1$, the realistic small quark mixing angles can be generated by extending interactions beyond those of the Standard Model, such as in supersymmetry [9] or breaking $A_4$ symmetry explicitly [10]. Our study is aimed to obtain realistic quark masses and mixing angles entirely within the $A_4$ context [12]. It is worthy of mention that the other types of unified models for quarks and leptons with the $A_4$ symmetry [13, 14] and models which predict tri-bimaximal mixing matrix with $S_3$ symmetry [15, 16, 17] have been also studied.

2 $A_4$ symmetry

$A_4$ is the symmetry group of the tetrahedron and the finite groups of the even permutation of four objects. It has twelve elements which are derived into four equivalence classes: $[C_1]$: (1234), $[C_2]$: (2143), (3412), $[C_3]$: (1342), (4213), (2431), (3124) and $[C_4]$: (1423), (3241), (4132), (2314), corresponding to its four irreducible representations we call three one-dimensional representations (singlets) as $1$, $1'$, $1''$, and one three-dimensional representation (triplet) as $3$, respectively. The $A_4$ is the smallest discrete group which includes the three-dimensional irreducible representation. The presence of the three-dimensional irreducible representation might be ideal for describing three families of quarks and leptons.
The character table of four representations is shown in Table 1. Here $h$ is the order of each element, $n$ is the number of elements and the complex number $\omega$ is the cube root of unity:

$$\omega = \exp(2\pi i / 3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad 1 + \omega + \omega^2 = 0.$$  \hfill (8)

The fundamental multiplication rules are given as:

$$1' \times 1' = 1'', \quad 1'' \times 1'' = 1', \quad 1' \times 1'' = 1.$$  \hfill (9)

and

$$3_1 \times 3_2 = 1 + 1' + 1'' + 3_A + 3_B,$$  \hfill (10)

where denoting $3_i$ for $i = 1, 2$ as $(a_i, b_i, c_i)$, we have

$$\begin{align*}
1 & \sim a_1a_2 + b_1b_2 + c_1c_2, \\
1' & \sim a_1a_2 + \omega b_1b_2 + \omega^2 c_1c_2, \\
1'' & \sim a_1a_2 + \omega^2 b_1b_2 + \omega c_1c_2, \\
3_A & \sim (b_1c_2, c_1a_2, a_1b_2), \\
3_B & \sim (c_1b_2, a_1c_2, b_1a_2).
\end{align*}$$

Note that from Eq. (9) the $A_4$ invariant singlet $\underline{1}$ can be derived in these three sets of $A_4$ singlets:

$$1' \times 1' \times 1', \quad 1'' \times 1'' \times 1'', \quad 1 \times 1' \times 1''.$$  \hfill (11)

\textsuperscript{3}For details of the $A_4$ multiplication rules, see the original paper \cite{7,8} for example.
\[ (\nu_i, l_i) \quad e^c \quad \mu^c \quad \tau^c \quad \phi_{li} \]
\[
\begin{array}{cccc}
A_4 & 3 & 1 & 1' & 1'' & 3
\end{array}
\]

Table 2: \( A_4 \) assignment for leptons.

\[
\text{and } 3 \times 3 \times 3 = 1 \text{ is also possible in the } A_4 \text{ symmetry from Eq. (10). By using them, } A_4 \text{ invariant mass matrices are constructed.}
\]

\section{3 A_4 model for lepton sector}

In this section, we briefly show a simple example of the \( A_4 \) model for leptons and lead the tri-bimaximal mixing matrix. For more detail of advanced models can be found in recent reviews \[18, 19\].

Let us take the \( A_4 \) assignment for leptons as shown in Table 2: left-handed \( SU(2)_L \) lepton doublets \((\nu_i, l_i) (i = 1, 2, 3)\) transform as an \( A_4 \) triplet 3, while right-handed, charged lepton singlets \( l_i^c \) transform as \( A_4 \) singlets \((l_1^c = e^c, l_2^c = \mu^c \text{ and } l_3^c = \tau^c)\) transform as 1, 1’ and 1”, respectively. Introducing gauge singlet Higgs doublet \( \phi_{li} = (\phi_{li}^0, \phi_{li}^-) \sim 3 \) under \( A_4 \), the \( 3 \times 3 \) mass matrix linking \( l_i \) with \( l_i^c \) is given by

\[
M_l = \begin{pmatrix}
  f_1 v_1 & f_2 v_1 & f_3 v_1 \\
  f_1 v_2 & f_2 \omega v_2 & f_3 \omega^2 v_2 \\
  f_1 v_3 & f_2 \omega^2 v_3 & f_3 \omega v_3
\end{pmatrix} = \begin{pmatrix}
v_1 & 0 & 0 \\
0 & v_2 & 0 \\
0 & 0 & v_3
\end{pmatrix} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix} \begin{pmatrix}
f_1 & 0 & 0 \\
f_2 & 0 & 0 \\
f_3 & 0 & 0
\end{pmatrix},
\]

(12)

where \( \omega = \exp(2\pi i/3) \), \( f_i \) are Yukawa couplings and \( v_i = \langle \phi_{li}^0 \rangle \) are vacuum expectation values of the Higgs field \( \phi_{li}^0 \). It can be diagonalized in a very simple way, i.e. by setting all three vacuum expectation values of the Higgs field to be equal. Taking \( v_l \equiv v_1 = v_2 = v_3 \), the charged lepton mass matrix is diagonalized as

\[
M_l^{\text{diag}} = U_{lL}^\dagger M_l U_{lR} = \begin{pmatrix}
  f_1 & 0 & 0 \\
 0 & f_2 & 0 \\
0 & 0 & f_3
\end{pmatrix} \sqrt{3} v_l = \begin{pmatrix}
m_e & 0 & 0 \\
0 & m_\mu & 0 \\
0 & 0 & m_\tau
\end{pmatrix},
\]

(13)
where
\[ U_{UL} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \] (14)
and \( U_{LR} \) is the unit matrix. Three different charged lepton masses are given by the Yukawa couplings.

For the neutrino mass matrix, if we consider six gauge singlet Higgs triplets \( \xi_i = (\xi_i^+, \xi_i^+, \xi_i^0) \) which are assigned to \( \xi_1 \sim 1, \xi_2 \sim 1', \xi_3 \sim 1'' \) and \( \xi_{4,5,6} \sim 3 \) under \( A_4 \), then the matrix forms in general
\[ M_\nu = \begin{pmatrix} a + b + c & f & e \\ f & a + \omega b + \omega^2 c & d \\ e & d & a + \omega^2 b + \omega c \end{pmatrix}, \] (15)
here parameters \((a, b, c)\) come from \((\xi_1, \xi_2, \xi_3)\) and \((d, e, f)\) from \(\xi_{4,5,6}\), respectively. If conditions \(b = c\) and \(e = f = 0\) are given (hopefully in some natural mechanisms), \( M_\nu \) is diagonalized by the matrix \( U_{\nu L} \) with three eigenvalues (neutrino masses) \( m_1 = a - b + d, m_2 = a + 2b, m_3 = -a + b + d \), and then we have
\[ U_{\text{MNS}} = (U_{UL})^\dagger U_{\nu L} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \] (16)
which is exactly the tri-bimaximal mixing matrix.

4 \( A_4 \) model for quark sector

Following the successful lepton assignments in the previous section, if we assign for quarks as in Table 3 and take \( v_{q1} = v_{q2} = v_{q3} \equiv v_q \) \((v_{qi} = \langle \phi^0_{qi} \rangle)\), we then have the up and down quark mass matrices as
\[ M^{U(D)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} g_{1}^{U(D)} & 0 & 0 \\ 0 & g_{2}^{U(D)} & 0 \\ 0 & 0 & g_{3}^{U(D)} \end{pmatrix} \sqrt{3} v_q. \] (17)
Table 3: $A_4$ assignment for quarks which follows that for leptons.

|          | $u_i, d_i$ | $u_i^c, d_i^c$ | $(\phi_{0i}, \phi_{qi})$ |
|----------|------------|----------------|--------------------------|
| $A_4$    | 3          | 1, 1', 1''    | 3                        |

Table 4: SU(5) motivated quark and lepton assignment.

|          | $(\nu_i, l_i), d_i^c$ | $l_i^c, u_i^c, (u_i, d_i)$ | $(\phi_{0i,1,2}, \phi_{01,2})$ | $(\phi_{0i}, \phi_{0l}), (\phi_{Di}, \phi_{Di})$ |
|----------|------------------------|-----------------------------|-----------------------------|-----------------------------------------------|
| $A_4$    | 3                      | 1, 1', 1''                 | 1', 1''                    | 3                                           |
| SU(5)    | $5^*$                  | 10                          | 5                          | $5^* + 45$                                  |

The generic prediction is that the quark mixing matrix leads to the unit matrix: $V_{\text{CKM}} = (U^U)^T U^D = 1$ where

\[
U^U = U^D = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}
\]  

are matrices which diagonalize $M^{U(D)}$ as $[13]$. The realistic small quark mixing angles can be generated by extending interactions beyond those of the Standard Model, such as in supersymmetry $[10]$ or the addition of terms which break the $A_4$ symmetry as well as the residual $Z_3$ symmetry explicitly $[11]$. In this section, we present a new alternative scenario, where realistic quark masses and mixing angles are obtained motivated by the quark-lepton assignments of SU(5), entirely within the $A_4$ context $[12]$.

In SU(5) grand unification, the $5^*$ representation contains the lepton doublet $(\nu, l)$ and the quark singlet $d^c$, whereas the $10$ representation contains the lepton singlet $l^c$ and the quark doublet $(u, d)$ and singlet $u^c$. In the successful $A_4$ model for leptons, $(\nu_i, l_i)$ transform as $3$ whereas $l_i^c$ transform as $1, 1'$ and $1''$. Thus we choose as shown in Table 4. In minimal SU(5), there is just one $5$ representation of Higgs bosons, yielding thus only two invariants, i.e. $10 \times 10 \times 5 \rightarrow 1$ (that is for up quark mass matrix) and $5^* \times 10 \times 5^* \rightarrow 1$ (for charged lepton and down quark mass matrices). The second invariant implies $m_\tau = m_b$ at the unification.
scale which is phenomenologically desirable. We follow the usual strategy of using both $\text{5}^*$ and $\text{45}$ representations of Higgs bosons, so that one linear combination couples to only leptons, and the other only to quarks. Both transform as $\text{2}$ under $A_4$. There are also two $\text{5}$ representations transforming as $\text{1}^\prime$ and $\text{1}''$ under $A_4$ which couple only to up type quarks.

With this $A_4$ assignment for Higgs doublets, the relevant Yukawa couplings linking $d_i$ with $d_j^c$ are given by

$$h_1 d_1 (d_1^c \phi_{D1}^0 + d_2^c \phi_{D2}^0 + d_3^c \phi_{D3}^0) + h_2 d_2 (d_1^c \phi_{D1}^0 + \omega d_2^c \phi_{D2}^0 + \omega^2 d_3^c \phi_{D3}^0) + h_3 d_3 (d_1^c \phi_{D1}^0 + \omega^2 d_2^c \phi_{D2}^0 + \omega d_3^c \phi_{D3}^0),$$

resulting in the $3 \times 3$ down quark mass matrix:

$$M_D = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix},$$

(19)

where $\omega = \exp(2\pi i/3)$, $h_i$ are three independent Yukawa couplings, and $v_i$ are the vacuum expectation values of $\phi_{D_i}^0$. To get quark mass hierarchy, $v_1 \ll v_2 \ll v_3$ should be satisfied contrary to the lepton sector ($v_{l_i} \equiv \langle \phi_{l_i}^0 \rangle = v_l$). By contrast, the Higgs doublets transforming as $\text{1}^\prime$ and $\text{1}''$ linking $u_i$ with $u_j^c$ give, as following the manner of Eq. (11), the $3 \times 3$ symmetric up quark mass matrix:

$$M_U = \begin{pmatrix} 0 & \mu_2 & \mu_3 \\ \mu_2 & m_2 & 0 \\ \mu_3 & 0 & m_3 \end{pmatrix},$$

(20)

where $m_2, \mu_3$ come from $\phi_{U1}^0 \sim \text{1}^\prime$ and $m_3, \mu_2$ from $\phi_{U2}^0 \sim \text{1}''$ and three of them can be taken real in general. In the limit $|\mu_2| \ll |m_2|$ and $|\mu_3| \ll |m_3|$, we obtain the three eigenvalues of $M_U$ as

$$m_t \simeq |m_3|, \quad m_c \simeq |m_2|, \quad m_u \simeq \left| \frac{\mu_2^3}{m_2} + \frac{\mu_3^3}{m_3} \right|,$$

(21)

\footnote{We choose here two Higgs doublets transforming as $\text{1}^\prime$ and $\text{1}''$, other patterns can be assigned with different prediction.}
with mixing angles

\[ V_{uc} \simeq \frac{\mu_2}{m_2}, \quad V_{ut} \simeq \frac{\mu_3}{m_3}, \quad V_{ct} \simeq 0. \]  

(22)

In the down sector, we note that

\[
M_D M_D^\dagger = \begin{pmatrix}
Y |h_1|^2 & Z^* h_1 h_2^* & Z h_1 h_3^* \\
Z h_1^* h_2 & Y |h_2|^2 & Z^* h_2 h_3^* \\
Z^* h_1 h_3 & Z h_2^* h_3 & Y |h_3|^2
\end{pmatrix},
\]  

(23)

where

\[
Y = |v_1|^2 + |v_2|^2 + |v_3|^2, \quad Z = |v_1|^2 + \omega |v_2|^2 + \omega^2 |v_3|^2.
\]

Its eigenvalue \( \lambda \) satisfies the equation

\[
\lambda^3 - Y(|h_1|^2 + |h_2|^2 + |h_3|^2) \lambda^2 - (Y^3 + Z^3 + Z^{*3} - 3Y|Z|^2)|h_1|^2|h_2|^2|h_3|^2 \\
+ (Y - |Z|^2)(|h_1|^2|h_2|^2 + |h_1|^2|h_3|^2 + |h_2|^2|h_3|^2) \lambda = 0.
\]  

(24)

If \( |v_1| = |v_2| = |v_3| = |v| \) as well as assumed in the charged lepton case, then \( Y = 3|v|^2, \ Z = 0 \) and three eigenvalues are simply \( 3|h_{1,2,3}|^2|v|^2 \). We choose them instead to be different, but we still assume \( |h_1|^2 \ll |h_2|^2 \ll |h_3|^2 \). In that case, we find

\[
m_b^2 \simeq Y|h_3|^2, \quad m_s^2 \simeq \left(\frac{Y^2 - |Z|^2}{Y}\right)|h_2|^2, \quad m_d^2 \simeq \left(\frac{Y^3 + Z^3 + Z^{*3} - 3Y|Z|^2}{Y^2 - |Z|^2}\right)|h_1|^2,
\]  

(25)\hspace{1cm}(26)\hspace{1cm}(27)

and the mixing angles are given by

\[
V_{sb} \simeq \left(\frac{Z^*}{Y}\right) \frac{h_2}{h_3}, \quad V_{db} \simeq \left(\frac{Z}{Y}\right) \frac{h_1}{h_3}, \quad V_{ds} \simeq \left(\frac{Y Z^* - Z^2}{Y^2 - |Z|^2}\right) \frac{h_1}{h_2},
\]  

(28)\hspace{1cm}(29)\hspace{1cm}(30)
thereby requiring the condition
\[ \frac{|V_{ds}V_{sb}|}{V_{db}} \approx \frac{|YZ^* - Z^2|}{Y^2 - |Z|^2}. \] (31)

Using current experimental values for the left-hand side, we see that quark mixing in the down sector alone cannot explain the observed quark mixing matrix $V_{CKM}$. Taking into account $V_U$, we then have
\[ V_{CKM} = V_U^\dagger V_D. \] (32)

Hence
\[ V_{us} \simeq V_{ds} - V_{uc} \simeq \left( \frac{YZ^* - Z^2}{Y^2 - |Z|^2} \right) \frac{h_1}{h_2} - \frac{\mu_2}{m_2}, \] (33)
\[ V_{cb} \simeq V_{sb} \simeq \left( \frac{Z^*}{Y} \right) \frac{h_2}{h_3}, \] (34)
\[ V_{ub} \simeq V_{db} - V_{uc}V_{sb} - V_{ut} \simeq \left( \frac{Z}{Y} \right) \frac{h_1}{h_3} - \left( \frac{Z^*}{Y} \right) \frac{h_2 \mu_2}{h_3 m_2} - \frac{\mu_3}{m_3}. \] (35)

It is noted that our up and down quark mass matrices are restricted by our choice of $A_4$ representations to have only five independent parameters each. In the up sector, we have three real and one complex parameters (for example we choose here $m_2, m_3$ and $\mu_2$ to be real with $\mu_3$ complex). The five independent parameters can be chosen as the three up quark masses, one mixing angle and one phase. In the down sector, the Yukawa couplings $h_{1,2,3}$ can all be chosen real, $Y$ is just an overall scale, and $Z$ is complex. The five independent parameters can be chosen as the three down quark masses and two mixing angles. Now we have ten parameters in these two matrices except their overall normalizations or magnitudes of Yukawa couplings. Since we also have ten observables for six quark masses, three angles and one phase, it may appear that a fit is not so remarkable. However, the forms of the mass matrices are very restrictive, and it is by no means trivial to obtain a good fit. Indeed, we find that $V_{ub}$ is strongly correlated with the CP phase $\beta$ which is one of angles of the unitary triangle. If we were to fit just the six masses and the three angles, the structure of our mass
matrices would allow only a very narrow range of values for $\beta$ at each value of $|V_{ub}|$. This means that future more precise determinations of these two parameters will be a decisive test of this model.

CP violation is also predicted in our model. The Jarlskog invariant $[23]$ is given by

$$J_{CP} \simeq \frac{\sqrt{3} h_1^2}{2 h_3^2} \left( \frac{v_2^2 - v_1^2}{v_2^2 + v_1^2} \right) \left( 1 + \frac{\text{Re}(\mu_3) + \frac{1}{\sqrt{3}} \text{Im}(\mu_3) h_3}{m_t} \right), \quad (36)$$

which is mainly comes from the down sector and the up sector only affects as correction terms.

In order to fit the ten observables (six quark masses, three CKM mixing angles and one CP phase), ten parameters of our model have been generated numerically. We choose the parameter sets which are allowed by the experimental data. We show the prediction of $|V_{ub}|$ versus $\beta$ in Fig. 1 with the following nine experimental inputs $[20, 21, 22]$:

$$m_u = 0.9 \sim 2.9 \text{ (MeV)}, \quad m_c = 530 \sim 680 \text{ (MeV)}, \quad m_t = 168 \sim 180 \text{ (GeV)},$$
$$m_d = 1.8 \sim 5.3 \text{ (MeV)}, \quad m_s = 35 \sim 100 \text{ (MeV)}, \quad m_b = 2.8 \sim 3 \text{ (GeV)},$$
$$|V_{us}| = 0.221 \sim 0.227, \quad |V_{cb}| = 0.039 \sim 0.044, \quad J_{CP} = (2.75 \sim 3.35) \times 10^{-5}, \quad (37)$$

which are given at the electroweak scale. We see that the experimental allowed region of $\beta$ (0.370 \sim 0.427 \text{ radian at 90\% C.L.}) $[22]$ corresponds to $|V_{ub}|$ in the range 0.0032 \sim 0.0044, which is consistent with the experimental value of $|V_{ub}| = 0.0029 \sim 0.0045$. Thus our model is able to reproduce realistically the experimental data of quark masses and the CKM matrix.

Precisely measured heavy quark masses and CKM matrix elements are expected in future experiments and precise light quark masses are expected in future lattice evaluations. If the allowed regions of the current data shown in Eq. (37) are reduced, the correlation between $|V_{ub}|$ and $\beta$ will become stronger. We show in Fig. 2 the case where the experimental data
Figure 1: Plot of allowed values in the $\beta - |V_{ub}|$ plane, where the value of $\beta$ is expressed in radians. The horizontal and vertical lines denote experimental bounds at 90\% C.L. \cite{37}.

are restricted to some very narrow ranges about their central values:

\[
\begin{align*}
    m_u &= 1.4 \sim 1.5 \text{ (MeV)}, \quad m_c = 600 \sim 610 \text{ (MeV)}, \quad m_t = 172 \sim 176 \text{ (GeV)}, \\
m_d &= 3.4 \sim 3.6 \text{ (MeV)}, \quad m_s = 60 \sim 70 \text{ (MeV)}, \quad m_b = 2.85 \sim 2.95 \text{ (GeV)}, \\
|V_{us}| &= 0.221 \sim 0.227, \quad |V_{cb}| = 0.041 \sim 0.042, \quad J_{\text{CP}} = (3.0 \sim 3.1) \times 10^{-5},
\end{align*}
\]

(38)

Here we use the tighter constraints on the mass ratios of light quarks, i.e. $m_u/m_d$ and $m_s/m_d$, consistent with the well-known successful low-energy sum rules \cite{24}; Clearly, future more precise determinations of $|V_{ub}|$ and $\beta$ will be a sensitive test of our model.

A comment is in order. Our quark mass matrices are in principle given at the SU(5) unification scale. However, the $A_4$ flavor symmetry is spontaneously broken at the electroweak scale. Therefore, the forms of our mass matrices are not changed except for the magnitudes of the Yukawa couplings between the unification and electroweak scales. Hence,
our numerical analyses are presented at the electroweak scale.

We should also comment on the hierarchy of $h_i$ and $v_i$. The order of $h_i$ are fixed by the quark mixing (33), (34). The ratios of

$$h_1/h_2 \simeq \lambda(\simeq 0.22), \quad h_2/h_3 \simeq \lambda^2,$$

are required by $V_{us}$ and $V_{cb}$, respectively. Once $h_i$ are fixed, quark masses determine the hierarchy of $v_i$ as follows:

$$v_1/v_3 \simeq \lambda^2, \quad v_2/v_3 \simeq \lambda \sim \lambda^{1/2}.$$  

These hierarchies of $h_i$ and $v_i$ are also consistent with the magnitude of $J_{CP}$ given in Eq. (36).

5 Summary

The $A_4$ family symmetry which has been successful in understanding the mixing pattern of neutrinos (tri-bimaximal mixing) is applied to quarks, motivated by the quark-lepton
assignments of SU(5). Realistic quark masses and mixing angles are obtained entirely with the $A_4$ context, in good agreement with data. In particular, we find a strong correlation between $V_{ub}$ and the CP phase $\beta$, thus a decisive future test of this model can be allowed.

It is one of a powerful guideline to find the constraints from neutrinos to grand unification models. Discrete symmetries are suitable to decode the flavor problem: they can accommodate maximal 2-3 mixing and explain zero 1-3 mixing. Moreover they can be the origin of texture zeros or equalities.

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