Pulsation modes for increasingly relativistic polytropes

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ABSTRACT

We present the results of a numerical study of the fluid $f$, $p$ and the gravitational $w$ modes for increasingly relativistic nonrotating polytropes. The results for $f$ and $w$-modes are in good agreement with previous data for uniform density stars, which supports an understanding of the nature of the gravitational wave modes based on the uniform density data. We show that the $p$-modes can become extremely long-lived for some relativistic stars. This effect is attributed to the change in the perturbed density distribution as the star becomes more compact.

Key words: Stars : neutron - Radiation mechanisms: nonthermal

1 INTRODUCTION

In a recent paper (Andersson, Kojima & Kokkotas 1996) we presented a detailed survey of the pulsation modes of the simplest conceivable stellar model: a nonrotating star with uniform density. The reasons for choosing this, admittedly unrealistic, model were twofold: First of all, the analytic solution to the TOV equations for uniform density is well known (Schutz 1985). This means that calculations could readily be done for many different values of stellar compactness and average density. Secondly, more realistic models for a neutron star are “almost constant density” so the results for this simple model may not be too different from what one would find for realistic equations of state.

The main conclusions of our study were:

- that gravitational-wave ($w$) modes (Kokkotas and Schutz 1992) exist for all stellar models, even though the modes become very short lived for less relativistic stellar models.
Moreover, the $w$-mode spectra are qualitatively similar for axial and polar perturbations (for a description of the two classes of relativistic perturbations – also referred to as odd and even parity perturbations – see Thorne and Campolattaro [1967] or Kojima [1992]). Since axial perturbations do not couple to pulsations in the stellar fluid (Thorne & Campolattaro [1967]) we concluded that the $w$-modes are pure “spacetime” modes that do not depend on the particulars of the fluid for their existence. This notion is supported by results obtained for various model scenarios (Andersson, Kokkotas & Schutz 1996; Andersson 1996).

- that the behaviour of the various pulsation modes changes dramatically as the star is made more compact than $R \approx 3M$ (we use geometrized units $c = G = 1$). For such ultracompact stars the existence of a peak in the effective curvature potential outside the surface of the star will affect the various modes. Some modes can basically be considered as “trapped in a potential well”, cf. Chandrasekhar and Ferrari (1991) and Kokkotas (1994). This trapping has the effect that some $w$-modes become extremely long-lived.

- that there are “avoided crossings” between the fluid $f$-mode and the various polar $w$-modes for extremely compact stars. This suggests that the $f$-mode should be considered as the first in the sequence of trapped modes for these stars.

The results of the mode-survey for uniform density models helped improve our understanding of the origin of, and the relation between, various pulsation modes of relativistic stars. We would expect the main conclusions to hold also for more realistic equations of state (at least qualitatively), but the motivation for confirming these expectations by actual calculations is strong. For example, axial $w$-modes have so far only been calculated for uniform density stars. Although our present understanding of the origin of the $w$-modes indicates that axial modes must exist for all stellar models it is important to verify that this is, indeed, the case. Furthermore, more realistic stellar models support other families of modes. Of particular interest for gravitational-wave physics are the $p$-modes (that correspond to pressure waves in the stellar fluid). The fact that the properties of the $f$- and $w$-modes change dramatically as the star becomes increasingly relativistic leads to questions whether the $p$-modes are also affected in an interesting way. With this short paper we address these issues.

We describe results obtained for the polytropic equation of state

$$p = \kappa \rho^\Gamma,$$

where $\kappa = 100 \text{ km}^2$. Our main study was for $\Gamma = 2$, but we also considered other (especially
larger) values of $\Gamma$. Our results are basically an extension of those presented by Andersson, Kokkotas and Schutz (1993). A detailed description of the way that we extract the complex frequencies of the various pulsation modes can be found in that paper. In order to keep the present paper short, we will discuss neither this numerical approach nor the various perturbation equations here. The equations governing polar perturbations are described in detail by Lindblom and Detweiler (1981), while the axial equations are given by Chandrasekhar and Ferrari (1991).

2 NUMERICAL RESULTS

The results of our investigation for increasingly relativistic polytropes can be summarized in a single figure. In Figure 1 we show the inverse damping rate (represented by $\text{Im } \omega_M$) of each mode as a function of the pulsation frequency ($\text{Re } \omega_M$) for a $\Gamma = 2$ polytrope. As the central density of the stellar model is varied each mode traces out a curve in the complex $\omega$-plane. In the figure we have indicated (by diamonds) the densest polytropic model that is stable to radial perturbations. That is, the specific model for which the mass ($M$) reaches a maximum as a function of the central density ($\rho_c$). For the $\Gamma = 2$ polytrope used to calculate the data in Figure 1 the marginally stable model has central density $\rho_c = 5.7 \times 10^{15} \text{ g/cm}^3$.

It is clear that Figure 1 contains a considerable amount of information. We now proceed to discuss the results for each separate class of modes in more detail.

2.1 Gravitational wave modes

As far as the $w$-modes are concerned our investigation does not add much to the study for uniform density stars (Andersson, Kojima & Kokkotas 1996). The results for polytropes are, both qualitatively and quantitatively, in excellent agreement with those for the uniform density model. From the upper panel in Figure 1 it is clear that both the axial and the polar $w$-modes can be divided into two separate families. The two families are distinguished by the behaviour of the mode-frequencies as the star becomes very relativistic. For one family of modes – the interface $w$-modes (Andersson, Kojima & Kokkotas 1996; Leins, Nollert and Soffel 1993) – the frequencies approach constant (complex) values for large $\rho_c$. For the specific polytrope under consideration we find that the mode-frequencies hardly change at all for $\rho_c > 1.9 \times 10^{16} \text{ g/cm}^3$. The modes in the second family – the curvature $w$-modes – are characterized by an increase in $|\omega_M|$ with $\rho_c$ for less compact stars, but before the
star becomes radially unstable $|\omega M|$ reaches a maximum value. The modes in this family then become extremely slowly damped as $\rho_c$ increases further. The slowest damped $w$-modes become long-lived as $R < 3.5M$ or so. The decreased damping rate is most likely due to the increased influence of the curvature potential barrier (the peak of which is at $R \approx 3M$). However, it is important to realize that the $\Gamma = 2$ polytrope that we consider is always less compact than $R = 3.2M$. Hence, the surface of these stars is never located inside the peak of the curvature potential. That the modes still become extremely slowly damped shows that the exterior potential can “trap” gravitational waves effectively even though there is no “potential well inside a barrier” for these stars, cf. the discussion by Detweiler (1975).

As already mentioned, the present results are very similar to those for uniform density stars. Especially worth noticing are the similarities between the axial and the polar $w$-modes in Figure 1. Since the axial modes cannot induce oscillations in the stellar fluid, this indicates that the character of the $w$-mode depends solely on the properties of the curved spacetime. In other words: the $w$-modes are “spacetime” modes the details of which do not depend on the dynamics of the stellar fluid (Andersson, Kojima & Kokkotas 1996; Andersson, Kokkotas & Schutz 1996).

### 2.2 Fluid modes

The present results for the fluid $f$-mode are also in good agreement with the uniform density study (Andersson, Kojima & Kokkotas 1996). For less compact stars the damping rate of the $f$-mode increases with $\rho_c$. This simply means that the star radiates gravitational waves more efficiently as it becomes more compact. But, in a similar way to the curvature $w$-modes, the $f$-mode becomes slower damped once the star approaches $R = 3M$. Again, this can be understood as the increasing influence of the curvature potential (Detweiler 1975; Andersson, Kojima & Kokkotas 1996).

A notable difference between the results for the $\Gamma = 2$ polytrope and the uniform density star is the absence of “avoided crossings” between the polar $w$-modes and the $f$-mode, cf. Figure 2 of Andersson, Kojima and Kokkotas (1996). The reason for this is that the $\Gamma = 2$ polytrope never becomes sufficiently compact for such avoided crossings to occur. For the uniform density model the first avoided crossing occurred for $R \approx 2.3M$, and for the $\Gamma = 2$ polytrope we always have $R > 3.2M$. One would, however, expect avoided mode-crossings to exist for polytropes that can be made very compact. We have verified the existense of...
avoided crossings for a $\Gamma = 5$ model (which can be made as compact as $R \approx 2.39M$). The existence of avoided crossings thus seems to be a generic feature of extremely compact relativistic stars.

The results we present for $p$-modes are both new and remarkable. Fluid $p$-modes have not previously been calculated for extremely relativistic stellar models. As can be seen in Figure 1 these modes behave in a peculiar ways as $\rho_c$ increases. While the behaviour for less relativistic stars is analogous to the $f$- and the $w$-modes – the damping rate increases when the star becomes more compact – it is different for the very relativistic models. Once the star has become radially unstable $|\omega M|$ attains a maximum. Then the pulsation frequency $\text{Re} \omega M$ typically decreases monotonically. At the same time the damping rate decreases drastically and $\text{Im} \omega M$ has a sharp minimum. For some modes we find several such minima of $\text{Im} \omega M$, cf. Figure 1. The corresponding values of the imaginary part of the mode-frequency are actually so small that our code does not have sufficient precision to distinguish them from zero. It is, of course, important to establish that the sign of $\text{Im} \omega M$ does not change. A change in sign would indicate that the mode becomes linearly unstable. We believe that our calculations were sufficiently accurate to establish that this does not happen – the $p$-modes are all stable. But it is interesting to note that they become extremely long-lived (very poor radiators of gravitational waves) for some values of the central density.

How can we understand the peculiar behaviour of the $p$-mode frequencies? Let us first consider the pulsation frequency $\text{Re} \omega M$, and compare our results to ones for simpler stellar models. The well-known result for the $f$-mode, established for incompressible fluid spheres by Kelvin in 1863 (Tassoul 1978), suggests that our mode should approach

$$\omega_f^2 = \frac{2l(l-1)}{2l+1} \left( \frac{M}{R^3} \right),$$

for less relativistic stars. A similar result, for compressible homogeneous spheres (Tassoul 1978), shows that the $p$-modes ought to approach

$$\omega_p^2 = \left[ \Delta_n + \sqrt{\Delta_n^2 + l(l+1)} \right] \left( \frac{M}{R^3} \right),$$

where

$$2\Delta_n = [2l + 3 + n(2n + 2l + 5)]\Gamma - 4, \quad n = 0, 1, 2, ...$$

and $\Gamma$ is the polytropic index. As can be seen from Figure 2 our numerical results agree quite well with these approximations. That is, the pulsation frequency of each fluid mode changes in the anticipated way as the star become more relativistic. Alternatively, the results in
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Figure 2 can be taken as justification for using the approximate results also for relativistic stars. They clearly provide useful estimates also for objects of neutron star compactness, $R/M \approx 5$.

The peculiar behaviour of the $p$-mode damping rates can be explained by changes in the matter distribution inside the star. According to the standard quadrupole formula, gravitational waves due to the fluid motion in the star can be estimated by

$$h \sim \frac{1}{r} \int_0^R r^4 \frac{\partial^2 \delta \rho(r, t)}{\partial t^2} \, dr = -\frac{\omega^2 e^{i\omega t}}{r} \int_0^R r^4 \delta \rho(r, \omega) \, dr,$$

where $\delta \rho$ is the Eulerian variation in the density. We have assumed that the perturbation is monochromatic with a harmonic time-dependence $\delta \rho(r, t) = \delta \rho(r, \omega)e^{i\omega t}$. In the notation of Lindblom and Detweiler (1981) the required density perturbation follows from (for the quadrupole)

$$\delta \rho(r, \omega) = -r^2 \left[ e^{-\nu/2} X - \frac{\rho + \rho}{r^3} e^{\lambda/2} (M + 4\pi pr^3) W - (p + \rho) H_0 \right] \frac{\rho^{1-\Gamma}}{\kappa\Gamma},$$

where we have used the fact that the acoustic wave speed in a polytrope is $\kappa \Gamma \rho^{\Gamma-1}$.

To get to a useful result, we must also figure out how each complex-frequency mode contributes to the relevant physical quantity (such as the density variation). The reason is obvious: For complex frequencies $\omega$ the $\delta \rho(r, \omega)$ that follows from (3) will be complex, whereas the corresponding physical quantity should be real valued. The analysis of this problem proceeds exactly as for the analogous black-hole problem (Andersson 1997). The main steps are i) note that the modes come in pairs $\omega_n$ and $-\omega_n^*$ (where the asterisk denotes complex conjugation) ii) Since all perturbation equations [see Lindblom and Detweiler (1981)] contain only $\omega^2$, and $(\omega^2)^* = (-\omega^*)^2$, we can conclude that

$$\delta \rho(r, -\omega_n^*) = [\delta \rho(r, \omega_n)]^*.$$

From a specific mode frequency $\omega_n$ we therefore get a contribution (for more details see Andersson (1997))

$$\delta \rho(r, t) \sim 2 \text{Re} \left[ \delta \rho(r, \omega_n) e^{i\omega_n t} \right].$$

Both the real and the imaginary part of the eigenfunction $\delta \rho(r, \omega_n)$ will thus be important.

Let us now assume that the gravitational-wave damping is sufficiently slow that the mode is essentially undamped for an entire cycle (this is a reasonable assumption for the slowly damped fluid modes). Averaging over one period we get

$$<h^2 > \sim e^{-2\Im \omega_n t} \left| \int_0^R r^4 \delta \rho(r, \omega_n) \, dr \right|^2 = e^{-2\Im \omega_n t} \epsilon(\omega_n).$$
The value of $\epsilon(\omega_n)$ provides a measure of how “efficient” a specific mode is as radiator according to the quadrupole formula. An association between the $p$-mode minima of $\text{Im } \omega M$ and minima in $\epsilon(\omega_n)$ would indicate that the features seen in Figure 1 are due to changes in the perturbed density distribution as $\rho_c$ increases. In Figure 3 we compare $\text{Im } \omega M$ to $\epsilon(\omega_n)$. This figure seems to establish the anticipated correlation between minima in the two functions for the first few $p$-modes.

Having found the likely explanation for the extremely slow damping of various $p$-modes for some values of $\rho_c$, it is worthwhile to discuss how the eigenfunction $\delta \rho(r, \omega_n)$ changes as we vary $\rho_c$. Let us first consider the $f$-mode. This mode is distinguished by the fact that the eigenfunctions i) have no nodes inside the star and ii) grow monotonically towards the surface of the star. The $p$-modes are different; in general there are $n$ nodes in both the real and the imaginary part of $\delta \rho(r, \omega)$ for the $n$th $p$-mode. But the location of these nodes change as we vary $\rho_c$. We find that the distribution of $\delta \rho(r, \omega)$ changes drastically with varying $\rho_c$.

Specifically, for the first $p$-mode and $\rho_c < 1 \times 10^{16}$ g/cm$^3$ the single node is close to the surface of the star and the bulk of $\delta \rho(r, \omega)$ is located in the outer 50% of the star. But for $\rho_c > 3 \times 10^{16}$ g/cm$^3$ the node has moved close to the centre of the star and the bulk of $\delta \rho(r, \omega)$ is now located inside half the radius of the star. It is understandable that these two, very different, configurations can be rather different as radiators of gravitational waves. For all $p$-modes the trend is that the bulk of the eigenfunctions move towards the centre of the star as we increase $\rho_c$. In a way, this means that the $p$-modes for the very relativistic star have the same features as the $g$-modes (Tassoul 1978).

3 FINAL COMMENTS

We have presented numerical results for the pulsation modes of nonrotating relativistic polytropes. These results agree well, both qualitatively and quantitatively, with previously obtained results for uniform density stars (Andersson, Kojima & Kokkotas 1996). This provides further evidence for our understanding of the nature of the various pulsation modes, especially the notion that the gravitational wave $w$-modes are mainly due to the properties of the curved spacetime (Andersson, Kokkotas & Schutz 1996). Our present survey also considers the $p$-modes of a relativistic star. We find that the polytrope $p$-mode frequencies are close to an approximation based on the pulsation of a compressible homogeneous sphere for less relativistic stars. But we also show that the $p$-modes change in a peculiar way as
the star becomes more compact. Specifically, we find that each \( p \)-mode can be very slowly damped (in fact, almost undamped) for some values of the central density. We have shown that this feature can be understood in terms of the change in the distribution of the perturbed density with varying central density. Although the result is not of tremendous astrophysical importance (since the affected stars are not stable to radial perturbations, anyway) it is interesting to note that the effectivity with which a \( p \)-mode radiates gravitational waves can vary considerably between two stellar models that have almost identical central densities.

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Figure 1. The change in the complex mode-frequencies as the central density of the stellar model varies ($\rho_c$ increases in the direction of the arrows). The data are for a $\Gamma = 2$ polytrope. The gravitational $\omega$-modes are shown in the upper panel, while the much longer lived fluid $f$- and $p$- modes are in the the lower panel (The $f$-mode is the leftmost mode in the lower panel.). Axial $\omega$-modes are represented by dashed lines while polar modes are shown as solid lines. The diamonds on each curve indicates the densest stellar model that is stable to radial perturbations.
Figure 2. The pulsation frequencies of the fluid modes as functions of the stellar compactness $R/M$. We compare our results for $f$, $p_0$, $p_1$ and $p_2$ (solid lines, from bottom to top) to approximate results for homogeneous stellar models (dashed lines).
Figure 3. We compare the damping rate of the fluid pulsation modes (represented by $\text{Im}\;\omega M$ and solid curves) to the efficiency measure $\epsilon(\omega_n)$ (dashed curves). Both quantities are shown as functions of the central density of the stellar model (in units of $10^{14}\;\text{g/cm}^3$). The scale for $\text{Im}\;\omega M$ is logarithmic and ranges from $10^{-8}$ to $10^{-2}$ while the scale for $\epsilon(\omega_n)$ is arbitrary. The data are for a) the $f$-mode, b-d) the first three $p$ modes. Of interest here is a possible correlation between minima in the two functions. Such a correlation explains the minima in the damping rate of the $p$-modes in terms of changes in the perturbed density distribution in the star. (The $f$-mode is only included for comparison – there should be no minima in $\epsilon(\omega_n)$ for that mode.)