Coincidence of the upper Vietoris topology and the Scott topology ✪

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Abstract

For a $T_0$ space $X$, let $K(X)$ be the poset of all compact saturated sets of $X$ with the reverse inclusion order. The space $X$ is said to have property Q if for any $K_1, K_2 \in K(X)$, $K_2 \ll K_1$ in $K(X)$ iff $K_2 \subseteq \text{int} K_1$. In this paper, we give several connections among the well-filteredness of $X$, the sobriety of $X$, the local compactness of $X$, the core compactness of $X$, the property Q of $X$, the coincidence of the upper Vietoris topology and Scott topology on $K(X)$, and the continuity of $x \mapsto \uparrow x : X \rightarrow \Sigma K(X)$ (where $\Sigma K(X)$ is the Scott space of $K(X)$). It is shown that for a well-filtered space $X$ for which its Smyth power space $P_S(X)$ is first-countable, the following three properties are equivalent: the local compactness of $X$, the core compactness of $X$, and the continuity of $x \mapsto \uparrow x : X \rightarrow \Sigma K(X)$.

Keywords: Well-filtered space; Local compactness; Smyth power space; Scott topology; First-countability

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1. Introduction

In non-Hausdorff topology and domain theory, we encounter numerous links between topology and order theory. There are a lot of connections among well-filteredness, sobriety, local compactness and core compactness. The Hofmann-Mislove Theorem, the spectral theory of distributive continuous lattices and the duality theorem of continuous semilattices show some of the most important such connections (see [5, 7]). For a $T_0$ space $X$, let $K(X)$ be the poset of all compact saturated sets of $X$ with the reverse inclusion order. The space $X$ is said to have property Q if for any $K_1, K_2 \in K(X)$, $K_2 \ll K_1$ in $K(X)$ iff $K_2 \subseteq \text{int} K_1$ (cf. [5, Proposition I-1.24.2 and Proposition IV-2.19]). It is well-known that the local compactness of a $T_0$ space $X$ implies that its topology $\mathcal{O}(X)$ is a continuous lattice. The spectral theory of continuous lattices shows that a sober space $X$ for which $\mathcal{O}(X)$ is a continuous domain is locally compact, and if a well-filtered space $X$ is locally compact, then $K(X)$ is a continuous semilattice, but the converse fails in general. The duality theorem of continuous semilattices shows that for a sober space with property Q, $K(X)$ is a continuous semilattice iff $X$ is locally compact (see [11, 10, 5, 7]). Thus the Lawson dual of $K(X)$ may be properly "bigger" than $\mathcal{O}(X)$.

The Smyth power spaces are very important structures in domain theory, which play a fundamental role in modeling the semantics of non-deterministic programming languages. There naturally arises a question of which topological properties are preserved by the Smyth power spaces. It was proved by Schalk [14] that the Smyth power space $P_S(X)$ of a sober space $X$ is sober (see also [5, Theorem 3.13]), and the upper Vietoris...
topology (that is, the topology of Smyth power space) agrees with the Scott topology on \(K(X)\) if \(X\) is a locally compact sober space. Xi and Zhao [10] showed that a \(T_0\) space \(X\) is well-filtered iff \(P_S(X)\) is a \(d\)-space. Recently, Brecht and Kawai [1] pointed out that \(P_S(X)\) is second-countable for a second-countable \(T_0\) space \(X\), and the first author and Zhao [20] proved that a \(T_0\) space \(X\) is well-filtered iff \(P_S(X)\) is well-filtered.

In this paper, we investigate some further connections among well-filteredness, sobriety, local compactness, core compactness, property \(Q\), and coincidence of the upper Vietoris topology and the Scott topology. Especially, for a \(T_0\) space \(X\), we discuss the following questions:

(a) Under what conditions does the core compactness of a \(T_0\) space \(X\) imply the local compactness of \(X\)?

(b) When does the upper Vietoris topology and the Scott topology on \(K(X)\) coincide?

(c) Is the Smyth power space \(P_S(X)\) of a \(T_0\) space again first-countable?

For a \(T_0\) space \(X\), we give several connections among the well-filteredness of \(X\), the sobriety of \(X\), the local compactness of \(X\), the core compactness of \(X\), the property \(Q\), and coincidence of the upper Vietoris topology and the Scott topology.

Recently, Brecht and Kawai [1] pointed out that \(\text{sob}(X)\) exists in \(X\) for some \(\text{sob}(X)\) exists, and the first author and Zhao [20] proved that \(\text{sob}(X)\) is well-filtered for a \(T_0\) space \(X\) which is Hausdorff and first-countable, we show that its Smyth power space \(P_S(Y)\) is not first-countable.

2. Preliminary

In this section, we briefly recall some fundamental concepts and notations that will be used in the paper. Some basic properties of sober spaces, metric spaces and compact saturated sets are presented. For further details, we refer the reader to [3, 7, 12].

For a poset \(P\) and \(A \subseteq P\), let \(A = \{x \in P : x \leq a \text{ for some } a \in A\}\) and \(A^\uparrow = \{x \in P : x \geq a \text{ for some } a \in A\}\). For \(x \in P\), we write \(x \downarrow\) for \(\downarrow\{x\}\) and \(x \uparrow\) for \(\uparrow\{x\}\). A subset \(A\) is called a lower set (resp., an upper set) if \(A = A\) (resp., \(A = A\)). Define \(A^\uparrow = \{x \in P : x \text{ is an upper bound of } A\}\). Dually, define \(A^\downarrow\), a cut generated by \(A\). Let \(P^\downarrow = \{F \subseteq P : F \text{ is a nonempty finite set}\}\). Dually, define \(P^\uparrow = \{F \subseteq P : F \text{ is a nonempty countable set}\}\). For a nonempty subset \(A\) of \(P\), define \(\text{min}(A) = \{a \in A : a\) is a minimal element of \(A\}\). For a set \(X\) and \(A, B \subseteq X\), \(A \subseteq B\) means that \(A \subseteq B\) but \(A \neq B\), that is, \(A\) is a proper subset of \(B\).

A nonempty subset \(D\) of a poset \(P\) is directed if every two elements in \(D\) have an upper bound in \(D\). The set of all directed sets of \(P\) is denoted by \(\mathcal{D}(P)\). \(I \subseteq P\) is called an ideal of \(P\) if \(I\) is a directed subset of \(P\). Let \(\text{Id}(P)\) be the poset (with the order of set inclusion) of all ideals of \(P\). Dually, we define the concept of filters and denote the poset of all filters of \(P\) by \(\text{Filt}(P)\). A filter of \(P\) is called principal if it has a minimum element, that is, there is \(x \in P\) with \(F = \{x\}\). \(P\) is called a directed complete poset, or dcpos for short, provided that \(\bigvee D\) exists in \(P\) for any \(D \in \mathcal{D}(P)\). \(P\) is called bounded complete if \(P\) is a dcpo and \(\bigwedge A\) exists in \(P\) for any nonempty subset \(A\) of \(P\).

As in [3], the lower topology on a poset \(Q\), generated by the complements of the principal filters of \(Q\), is denoted by \(\omega(Q)\). A subset \(U\) of \(Q\) is Scott open if (i) \(U = \{x\}\) and (ii) for any directed subset \(D\) for which \(\bigvee D\) exists, \(\bigvee D \in U\) implies \(D \cap U \neq \emptyset\). All Scott open subsets of \(Q\) form a topology. This topology is called the Scott topology on \(Q\) and denoted by \(\sigma(Q)\). The space \(\Sigma Q = (Q, \sigma(Q))\) is called the Scott space of \(Q\). The topology generated by \(\omega(Q)\) \(\cup\sigma(Q)\) is called the Lawson topology on \(Q\) and denoted by \(\lambda(Q)\).

For a \(T_0\) space \(X\), we use \(\leq_X\) to represent the specialization order of \(X\), that is, \(x \leq_X y\) if and only if \(x \in \{y\}\). In the following, when a \(T_0\) space \(X\) is considered as a poset, the order always refers to the specialization order if no other explanation. Let \(\mathcal{O}(X)\) (resp., \(\Gamma(X)\)) be the set of all open subsets (resp., closed subsets) of \(X\). A space \(X\) is locally hypercompact (see [4, 8]) if for each \(x \in X\) and each open neighborhood \(U\) of \(x\),
there is $F \in X^{(\omega)}$ such that $x \in \text{int} \uparrow F \subseteq \uparrow F \subseteq U$. Let $|X|$ be the cardinality of $X$ and $\omega = |\mathbb{N}|$, where $\mathbb{N}$ is the set of all natural numbers.

A $T_0$ space $X$ is called a $d$-space (or monotone convergence space) if $X$ (with the specialization order) is a dcpo and $\mathcal{O}(X) \subseteq \sigma(X)$ (cf. [6, 8]). Obviously, for a dcpo $P$, $\Sigma P$ is a $d$-space. A nonempty subset $A$ of a $X$ is irreducible if for any $\{F_1, F_2\} \subseteq \Gamma(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. Denote by $\text{Irr}(X)$ (resp., $\text{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of $X$. Clearly, every subset of $X$ that is directed under $\leq_X$ is irreducible. A space $X$ is called sober, if for any $F \in \text{Irr}_c(X)$, there is a unique point $a \in X$ such that $F = \{a\}$. Clearly, sober spaces are $T_0$.

For a dcpo $P$ and $A, B \subseteq P$, we say $A$ is way below $B$, written $A \ll B$, if for each $D \in \mathcal{D}(P)$, $\bigvee D \in \uparrow B$ implies $D \cap \uparrow A \neq \emptyset$. For $B = \{x\}$, a singleton, $A \ll B$ is written $A \ll x$ for short. For $x \in P$, let $w(x) = \{F \in P^{(\omega)} : F \ll x\}$, $\downarrow x = \{u \in P : u \ll x\}$ and $K(P) = \{k \in P : k \ll k\}$. Points in $K(P)$ are called compact elements of $P$.

For the following definition and related conceptions, please refer to [2, 4].

**Definition 2.1.** Let $P$ be a dcpo and $X$ a $T_0$ space.

1. $P$ is called a continuous domain, if for each $x \in P$, $\downarrow x$ is directed and $x = \bigvee \downarrow x$.
2. $P$ is called an algebraic domain, if for each $x \in P$, $K(P) \cap \downarrow x$ is directed and $x = \bigvee K(P) \cap \downarrow x$.
3. $P$ is called a quasicontinuous domain, if for each $x \in P$, $\{\uparrow F : F \in w(x)\}$ is filtered and $\uparrow x = \bigcap \{\uparrow F : F \in w(x)\}$.
4. $X$ is called core compact if $\mathcal{O}(X)$ is a continuous lattice.

**Lemma 2.2.** ([6]) For a dcpo $P$, $P$ is continuous iff for each $x \in U \subseteq \sigma(P)$, there is $u \in U$ such that $x \in \text{int} \sigma(P) \uparrow u \subseteq \uparrow u \subseteq U$.

**Lemma 2.3.** ([6]) Let $P$ be a dcpo $P$. Then

1. $P$ is quasicontinuous iff $\Sigma P$ is locally hypercompact.
2. If $P$ is a quasicontinuous domain, then $\Sigma P$ is sober.

A subset $B$ of a $T_0$ space $X$ is called saturated if $B$ equals the intersection of all open sets containing it (equivalently, $B$ is an upper set in the specialization order). We shall use $K(X)$ to denote the set of all nonempty compact saturated subsets of $X$ and endow it with the Smyth preorder; that is, for $K_1, K_2 \in K(X)$, $K_1 \subseteq K_2$ if $K_2 \subseteq K_1$. Let $\Sigma^0(X) = \{\uparrow x : x \in X\}$.

**Lemma 2.4.** ([6]) Let $X$ be a $T_0$ space. For a nonempty family $\{K_i : i \in I\} \subseteq K(X)$, $\bigvee_{i \in I} K_i$ exists in $K(X)$ iff $\bigcap_{i \in I} K_i \subseteq K(X)$. In this case $K = \bigcap_{i \in I} K_i$.

A topological space $X$ is called well-filtered if $X$ is $T_0$, and for any open set $U$ and any $K \in \mathcal{D}(K(X))$, $\bigcap K \subseteq U$ implies $K \subseteq U$ for some $K \in \mathcal{K}$.

We have the following implications (which can not be reversed):

- sobriety $\Rightarrow$ well-filteredness $\Rightarrow$ $d$-space.

For a $T_0$ space $X$, let $\text{OFilt}(\mathcal{O}(X)) = \sigma(\mathcal{O}(X)) \cap \text{Filt}(\mathcal{O}(X))$. $U \subseteq \mathcal{O}(X)$ is called an open filter if $U \in \text{OFilt}(\mathcal{O}(X))$. For $K \in K(X)$, let $\Phi(K) = \{U \in \mathcal{O}(X) : K \subseteq U\}$. Then $\Phi(K) \in \text{OFilt}(\mathcal{O}(X))$ and $K = \bigcap \Phi(K)$. Obviously, $\Phi : K(X) \rightarrow \text{OFilt}(\mathcal{O}(X))$, $K \mapsto \Phi(K)$, is an order embedding.

The single most important result about sober spaces is the Hofmann-Mislove Theorem (see [11] or Theorem II-1.20 and Theorem II-1.21).

**Theorem 2.5.** (The Hofmann-Mislove Theorem) For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is a sober space.
2. For any $F \in \text{OFilt}(\mathcal{O}(X))$, there is $K \in K(X)$ such that $F = \Phi(K)$.
3. For any $F \in \text{OFilt}(\mathcal{O}(X))$, $F = \Phi(\bigcap F)$.
By the Hofmann-Mislove Theorem, a $T_0$ space $X$ is sober iff $\Phi : K(X) \rightarrow \text{OFilt}(\mathcal{O}(X))$ is an order isomorphism.

**Theorem 2.6. (§5, §12)** For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is locally compact and sober.
2. $X$ is locally compact and well-filtered.
3. $X$ is core compact and sober.

For $U \in \mathcal{O}(X)$, let $\square U = \{K \in K(X) : K \subseteq U\}$. The upper Vietoris topology on $K(X)$ is the topology generated by $\{\square U : U \in \mathcal{O}(X)\}$ as a base, and the resulting space is called the Smyth power space or upper space of $X$ and is denoted by $P_S(X)$ (cf. §14).

**Remark 2.7. (§5, §14)** Let $X$ be a $T_0$ space. Then

1. The specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \subseteq$).
2. The canonical mapping $\xi_X : X \rightarrow P_S(X)$, $x \mapsto \uparrow x$, is an order and topological embedding.
3. $P_S(S^n(X))$ is a subspace of $P_S(X)$ and $X$ is homeomorphic to $P_S(S^n(X))$.

For a nonempty subset $C$ of a $T_0$ space $X$, it is easy to see that $C$ is compact iff $\uparrow C \in K(X)$. Furthermore, we have the following useful result (see, e.g., [2, pp.2068]).

**Lemma 2.8.** Let $X$ be a $T_0$ space and $C \in K(X)$. Then $C = \uparrow \text{min}(C)$ and $\text{min}(C)$ is compact.

For a metric space $(X, d)$, $x \in X$ and a positive number $r$, let $B(x, r) = \{y \in Y : d(x, y) < r\}$ be the $r$-ball about $x$. For a set $A \subseteq X$ and a positive number $r$, by the $r$-ball about $A$ we mean the set $B(A, r) = \bigcup_{a \in A} B(a, r)$.

The following two results are well-known (cf. [2]).

**Proposition 2.9.** Every metric space is perfectly normal and first-countable. Therefore, it is sober.

**Proposition 2.10.** Let $(X, d)$ be a metric space and $K$ a compact set of $X$. Then for any open set $U$ containing $K$, there is an $r > 0$ such that $K \subseteq B(K, r) \subseteq U$.

### 3. Well-filtered spaces and locally compact spaces

Firstly, we give the following two known results (see, e.g., [5, §16, §18]).

**Lemma 3.1.** Let $X$ be a well-filtered space. Then

1. For any $K \in \mathcal{D}(K(X))$, $\bigwedge K \in K(X)$ and $\bigvee K = \bigwedge K$.
2. $P_S(X)$ is a $d$-space, and hence the upper Vietoris topology is coarser than the Scott topology on $K(X)$.

**Lemma 3.2.** Let $X$ be a $T_0$ space. Then

1. $K(X)$ is semilattice (the semilattice operation being $\bigvee$).
2. Let $K_1, K_2 \in K(X)$ and consider the following assertions:
   (a) $K_2 \subseteq \text{int} K_1$.
   (b) $K_1 \leq K_2$ in $K(X)$.

   If $X$ is well-filtered, then (a) $\Rightarrow$ (b), and if $X$ is locally compact, then (b) $\Rightarrow$ (a).

3. If $X$ is well-filtered and locally compact, then $K(X)$ is a continuous semilattice.

**Definition 3.3.** A $T_0$ space $X$ is said to have property $Q$ if for any $K_1, K_2 \in K(X)$, $K_2 \leq K_1$ in $K(X)$ iff $K_2 \subseteq \text{int} K_1$. 

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It follows from Lemma 3.4 that every locally compact well-filtered space has property Q. Theorem 3.9 below shows that for a well-filtered space $X$ with the property $Q$, $X$ is locally compact if $\mathcal{K}(X)$ is a continuous semilattice.

The following result is a direct inference of the Hofmann-Mislove Theorem.

**Proposition 3.4.** Let $X$ be a sober space. If $\mathcal{K}(X)$ is continuous, then the following two conditions are equivalent:

1. $X$ has property $Q$.
2. For any pair $(\mathcal{U}, \mathcal{V}) \in \sigma(\mathcal{O}(X)) \times \sigma(\mathcal{O}(X))$, $\mathcal{U} \ll \mathcal{V}$ in $\sigma(\mathcal{O}(X))$ implies there is $V \in \mathcal{V}$ such that $V \subseteq \bigcap \mathcal{U}$.

**Theorem 3.5.** Every first-countable well-filtered space is sober.

**Theorem 3.6.** Every core compact well-filtered space is sober.

**Corollary 3.7.** A well-filtered space is locally compact iff it is core compact.

By Theorem 3.6, Theorem 2.6 can be strengthened into the following one.

**Theorem 3.8.** For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is locally compact and sober.
2. $X$ is locally compact and well-filtered.
3. $X$ is core-compact and sober.
4. $X$ is core compact and well-filtered.

Now we give one of the main results of this paper.

**Theorem 3.9.** For a well-filtered space $X$, the following conditions are equivalent:

1. $X$ is locally compact.
2. $\mathcal{K}(X)$ is a continuous semilattice, and the upper Vietoris topology and the Scott topology on $\mathcal{K}(X)$ agree.
3. $\mathcal{K}(X)$ is a continuous semilattice, and $\xi^X : X \to \Sigma \mathcal{K}(X), x \mapsto \uparrow x$, is continuous.
4. $\mathcal{K}(X)$ is a continuous semilattice, and $X$ has property $Q$.
5. $X$ is core compact.

**Proof.** (1) $\Rightarrow$ (2): By Theorem 2.6, Lemma 3.2 and [14, Lemma 7.26] (see Proposition 5.3 below).

(2) $\Rightarrow$ (3): By Remark 2.7.

(3) $\Rightarrow$ (1): For $x \in \uparrow U \in \mathcal{O}(X)$, by Lemma 3.1 $\uparrow x \in \bigcup U \in \sigma(\mathcal{K}(X))$, and hence by Lemma 2.2 there is $K \in \mathcal{K}(X)$ with $\uparrow x \in \bigcup_{K \in \mathcal{K}(X)} \uparrow x \subseteq K \subseteq \bigcup_{K \in \mathcal{K}(X)} K \subseteq \bigcup U$. Let $V = (\xi^X)^{-1}(\bigcup_{K \in \mathcal{K}(X)} \uparrow x)$. Then by the continuity of $\xi^X$, we have $V \in \mathcal{O}(X)$ and $x \in V \subseteq K \subseteq U$. Thus $X$ is locally compact.

(1) $\Rightarrow$ (4): By Lemma 3.2.

(4) $\Rightarrow$ (1): Let $x \in \uparrow U \in \mathcal{O}(X)$. Then by the continuity of $\mathcal{K}(X)$, $\delta_{\mathcal{K}(X)} \uparrow x$ is directed (note that the order on $\mathcal{K}(X)$ is the reverse inclusion order) and $\mathcal{K}(X)$ is directed (note that the Scott topology on $\mathcal{K}(X)$ is the reverse inclusion order) and $\uparrow x = \bigcup_{K \in \mathcal{K}(X)} \delta_{\mathcal{K}(X)} \uparrow x \subseteq U$, and hence by the well-filteredness of $X$, there is $K \in \mathcal{K}(X)$ such that $K \subseteq U$. Since $X$ has property $Q$, we have $\uparrow x \subseteq \text{int } K \subseteq K \subseteq U$. Therefore, $X$ is locally compact.

(1) $\Leftrightarrow$ (5): By Corollary 3.7 or Theorem 3.8

**Theorem 3.10.** Let $X$ be a well-filtered space such that $\mathcal{K}(X)$ is a continuous semilattice. Then the following conditions are equivalent:

1. $X$ is locally compact.
2. The upper Vietoris topology and the Scott topology on $\mathcal{K}(X)$ agree.
Example 3.13. Let $X$ be a sober space having property Q. Then $K(X)$ is a continuous semilattice iff $X$ is locally compact.

The following example shows that for a well-filtered space $X$, when $X$ lacks property Q, Theorem 3.9 may not hold. It also shows that the well-filteredness of $X$ and the continuity of $K(X)$ together do not imply the sobriety of $X$ in general.

Example 3.12. Let $X$ be a uncountably infinite set and $X_{\text{coc}}$ the space equipped with the co-countable topology (the empty set and the complements of countable subsets of $X$ are open). Then

(a) $\Gamma(X_{\text{coc}}) = \{\emptyset, X\} \cup X^{(\leq \omega)}$ and $\text{irr}(X_{\text{coc}}) = \text{irr}_c(X_{\text{coc}}) = \{X\} \cup \{\{x\} : x \in X\}$.
(b) $K(X_{\text{coc}}) = X^{(\leq \omega)} \setminus \{\emptyset\}$ and $\text{int} K = \emptyset$ for all $K \in K(X_{\text{coc}})$.
(c) $K(X_{\text{coc}})$ is a dcpo and every element in $K(X_{\text{coc}})$ is compact. Hence $K(X_{\text{coc}})$ is an algebraic domain.
(d) $X_{\text{coc}}$ is a well-filtered $T_1$ space, but it not sober.
(e) The upper Vietoris topology and the Scott topology on $K((X_{\text{coc}}))$ do not agree.
(f) $\xi_X : X_{\text{coc}} \rightarrow \Sigma K(X_{\text{coc}})$, $x \mapsto \uparrow x$, is not continuous.
(g) $X_{\text{coc}}$ does not have property Q.
(h) $X_{\text{coc}}$ is not locally compact and not first countable.

The following example shows that even for a sober space $X$, when $X$ lacks property Q, Proposition IV-2.19) (i.e., Corollary 3.11) may not hold.

Example 3.13. Let $p$ be a point in $\beta(\mathbb{N}) \setminus \mathbb{N}$, where $\beta(\mathbb{N})$ is the Stone-Cech compactification of the discrete space of natural numbers, and consider on $X = \mathbb{N} \cup \{p\}$ the induced topology. Then the space $X$ is a non-discrete Hausdorff space, and hence a sober space. Every compact subset of $X$ is finite. Therefore, $K(X)$ is an algebraic domain and $X$ is not locally compact. By Theorem 3.9 $X$ does not have property Q and is not core compact. Furthermore, the upper Vietoris topology and the Scott topology on $K(X)$ do not agree, and the mapping $\xi_X : X \rightarrow \Sigma K(X)$, $x \mapsto \uparrow x$, is not continuous.

By Example 3.12 Theorem 3.9 (or Theorem 8.10) strengthens Proposition IV-2.19 and shows that the converse of Proposition IV-2.19 holds in the following sense: for a well-filtered space $X$ such that $K(X)$ is a continuous semilattice, $X$ is locally compact iff $X$ has property Q.

Lemma 3.14. For a dcpo $P$, $\xi_{\Sigma P} : \Sigma P \rightarrow \Sigma K(\Sigma P)$ is continuous.

Proof. For any $D \in \mathcal{D}(P)$, we have $\xi_{\Sigma P}(\bigvee D) = \bigvee \{\bigvee D = \bigcap_{d \in D} \uparrow d = \bigvee_{K(\Sigma P)} \xi_{\Sigma P}(D)\}$ by Lemma 2.4. So $\xi_{\Sigma P} : \Sigma P \rightarrow \Sigma K(\Sigma P)$ is continuous.

We get the following corollary from Theorem 3.9 and Lemma 3.14.

Corollary 3.15. For a dcpo $P$ having the well-filtered Scott topology, the following conditions are equivalent:

1. $\Sigma P$ is locally compact.
2. $K(\Sigma P)$ is a continuous semilattice, and the upper Vietoris topology and the Scott topology on $K(\Sigma P)$ agree.
3. $K(\Sigma P)$ is a continuous semilattice, and $\Sigma P$ has property Q.
4. $K(\Sigma P)$ is a continuous semilattice.
5. $\Sigma P$ is core compact.

Definition 3.16. Let $P$ be a poset equipped with a topology $\tau$.

1. $(P, \tau)$ is called upper semicompact, if $\uparrow x$ is compact for any $x \in P$, or equivalently, if $\uparrow x \cap A$ is compact for any $x \in P$ and $A \in \Gamma((P, \tau))$. 
(2) \((P, \tau)\) is called \textit{weakly upper semicompact} if \(\{x \cap A: x \in P\text{ and } A \in \text{irr}((P, \tau))\}\) is compact for any \(x \in P\) and \(A \in \text{irr}((P, \tau))\).

**Lemma 3.17.** \((\text{[21]}))\) For a depo \(P\), if \((P, \lambda(P))\) is weakly upper semicompact (especially, if \((P, \lambda(P))\) is upper semicompact or \(P\) is bounded complete), then \((P, \sigma(P))\) is well-filtered.

By Corollary 3.11 and Lemma 3.17 we get the following corollary.

**Corollary 3.18.** For a depo \(P\), if \((P, \lambda(P))\) is weakly upper semicompact (especially, if \((P, \lambda(P))\) is upper semicompact or \(P\) is bounded complete), then the following conditions are equivalent:

1. \(\Sigma P\) is locally compact.
2. \(K(\Sigma P)\) is a continuous semilattice, and the upper Vietoris topology and the Scott topology on \(K(\Sigma P)\) agree.
3. \(K(\Sigma P)\) is a continuous semilattice, and \(\Sigma P\) has property \(Q\).
4. \(K(\Sigma P)\) is a continuous semilattice.
5. \(\Sigma P\) is core compact.

4. First-countability of Smyth power spaces

Now we consider the following question: for a first-countable (resp., second-countable) space \(X\), does its Smyth power space \(P_S(X)\) be first-countable (resp., second-countable)?

First, we have the following result, which was indicated in the proof of \([1\text{, Proposition 6}]\).

**Theorem 4.1.** For a \(T_0\) space, the following two conditions are equivalent:

1. \(X\) is second-countable.
2. \(P_S(X)\) is second-countable.

**Proof.** (1) \(\Rightarrow\) (2): Let \(\mathcal{B} \subseteq O(X)\) be a countable base of \(X\) and let \(\mathcal{B}_S = \{\Box \bigcup_{i=1}^{n} U_i : n \in \mathbb{N} \text{ and } U_i \in \mathcal{B} \text{ for all } 1 \leq i \leq n\}\). Then \(\mathcal{B}_S\) is countable. Now we show that \(\mathcal{B}_S\) is a base of \(P_S(X)\). Let \(K \in K(X)\) and \(U \in O(X)\) with \(K \subseteq U\). Then for each \(k \in K\), there is \(U_k \in \mathcal{B}\) with \(k \subseteq U\) and \(U \subseteq \bigcup_{i=1}^{m} U_i \subseteq U\). By the compactness of \(K\), there is a finite subset \(\{k_1, k_2, ..., k_m\} \subseteq K\) such that \(K \subseteq V = \bigcup_{i=1}^{m} U_{k_i} \subseteq U\), and hence \(\Box V \in \mathcal{B}_S\) and \(K \subseteq \Box V \subseteq U\). Thus \(\mathcal{B}_S\) is a base of \(P_S(X)\), proving that \(P_S(X)\) is second-countable.

(2) \(\Rightarrow\) (1): As a subspace of \(P_S(X)\), \(P_S(S^n(X))\) is second-countable, and hence \(X\) is second-countable since \(X\) is homeomorphic to \(P_S(S^n(X))\).

\(\square\)

Next, we consider the first-countability. Since the first-countability is a hereditary property and any \(T_0\) space \(X\) is homeomorphic to \(P_S(S^n(X))\), a subspace of \(P_S(S(X))\), we have the following result.

**Proposition 4.2.** Let \(X\) be a \(T_0\) space. If \(P_S(X)\) is first-countable, then \(X\) is first-countable.

Consider in the plane \(\mathbb{R}^2\) two concentric circles \(C_i = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = i\}\), where \(i = 1, 2\), and their union \(X = C_1 \cup C_2\); the projection of \(C_1\) onto \(C_2\) from the point \((0, 0)\) is denoted by \(p\). On the set \(X\) we generate a topology by defining a neighbourhood system \(\{B(z) : z \in X\}\) as follows: \(B(z) = \{z\}\) for \(z \in C_2\) and \(B(z) = \{U_j(z) : j \in \mathbb{N}\}\) for \(z \in C_1\), where \(U_j = V_j \cup p(V_j \setminus \{z\})\) and \(V_j\) is the arc of \(C_1\) with center at \(z\) and of length \(1/j\). The space \(X\) is called the Alexander double circle (see \([2\text{, Example 3.1.26}]\)).

**Proposition 4.3.** \((\text{[2]}))\) Let \(X\) be the Alexander double circle. Then

1. \(X\) is Hausdorff and first-countable.
2. \(X\) is not separable, and hence not second-countable.
3. \(X\) is compact and locally compact.
4. \(C_1\) is a compact subspace of \(X\).
(5) $C_2$ is a discrete subspace of $X$.

The following example shows that the converse of Proposition 4.2 fails in general.

Example 4.4. Let $X = C_1 \cup C_2$ be the Alexandroff double circle. Then by Proposition 4.3, $X$ is a compact Hausdorff first-countable space and $C_1 \in K(X)$. Now we prove that $P_S(X)$ is not first-countable. First, for any open subset $U \in O(X)$ with $C_1 \subseteq U$, there is a family \( \{ U_j = V_{n(j)} \cup p(V_{n(j)} \setminus \{ z_j \}) : j \in J \} \) of basic open sets such that $C_1 \subseteq \bigcup_{j \in J} U_j \subseteq U$, where $V_{n(j)}$ is the arc of $C_1$ with center at $z_j$ and of length $1/n(j)$, and $p$ is the projection of $C_1$ onto $C_2$ from the point $(0,0)$. By the compactness of $C_1$, there is a finite set $\{ z_{j_1}, z_{j_2}, ..., z_{j_n} \} \subseteq C_1$ such that $C_1 \subseteq \bigcup_{i=1}^n U_{j_i} \subseteq U$, and hence $C_2 \setminus U \subseteq \{ p(z_{j_1}), p(z_{j_2}), ..., p(z_{j_n}) \}$.

Thus $C_2 \setminus U$ is finite. Suppose that $\{ W_n : n \in \mathbb{N} \}$ is a countable family of open sets containing $C_1$. Then $C_2 \setminus \bigcap_{n \in \mathbb{N}} W_n = \bigcap_{n \in \mathbb{N}} (C_2 \setminus W_n)$ is countable. Choose $x \in C_2 \setminus \bigcap_{n \in \mathbb{N}} W_n$ and let $V = X \setminus \{ x \}$. Then $C_1 \subseteq V \in O(X)$, but $W_n \nsubseteq V$ for all $n \in \mathbb{N}$. Thus there is no countable base at $C_1$ in $P_S(X)$, proving that $P_S(X)$ is not first-countable.

Theorem 4.5. Let $X$ be a first-countable $T_0$ space. If $\min(K)$ is countable for any $K \in K(X)$, then $P_S(X)$ is first-countable.

Proof. For each $x \in X$, by the first-countability of $X$, there exists a countable base $B_x$ at $x$. Let $K \in K(X)$. Then by assumption $\min(K)$ is countable. Let $B_K = \{ \bigcap_{c \in C} \varphi(c) : C \in \min(K) \}^{<\omega}$ and $\varphi \in \prod_{c \in C} B_c$.

Then $B_K$ is countable. Now we show that $B_K$ is a base at $K$. Suppose that $U \in O(X)$ and $K \subseteq \Box U$. Then $\min(K) \subseteq K \subseteq U$. For each $k \in \min(K)$, there is a $\psi(k) \in B_K$ with $k \in \psi(k) \subseteq U$. By the compactness of $\min(K)$, there is a finite set $\{ k_1, k_2, ..., k_m \} \subseteq \min(K)$ such that $\min(K) \subseteq \bigcup_{i=1}^m \psi(k_i) \subseteq U$. Let $V = \bigcup_{i=1}^m \psi(k_i)$. Then $K \subseteq V \subseteq U$. It follows that $\Box V \in B_K$ and $K \in \Box V \subseteq \Box U$, proving that $B_K$ is a base at $K$. Thus $P_S(X)$ is first-countable.

Corollary 4.6. Let $X$ be a first-countable $T_0$ space. If all compact subsets of $X$ are countable, then $P_S(X)$ is first-countable.

Proposition 4.7. For a metric space $(X,d)$, $P_S((X,d))$ is first-countable.

Proof. For $K \in K((X,d))$, let $B_K = \{ B(K,1/n) : n \in \mathbb{N} \}$. Then by Proposition 2.10, $B_K = \{ B(K,1/n) : n \in \mathbb{N} \}$ is a countable base at $K$ in $P_S((X,d))$. Thus $P_S((X,d))$ is first-countable.

5. Coincidence of the upper Vietoris topology and Scott topology

For a well-filtered space $X$, from Theorem 3.3 we know that it is an important property that the upper Vietoris topology agrees with the Scott topology on $K(X)$. In this section we investigate the conditions under which the upper Vietoris topology coincides with the Scott topology on $K(X)$.

Proposition 5.1. [8] Let $P$ be a dcpo. If $\Sigma P$ is well-filtered and locally compact, then the upper Vietoris topology agrees with the Scott topology on $K(\Sigma P)$.

From Lemma 2.3 and Proposition 5.1 we get the following result.

Proposition 5.2. [14] For a quasicontinuous domain $P$, the upper Vietoris topology agrees with the Scott topology on $K(\Sigma P)$.

For a general $T_0$ space $X$, Schalk [14] proved the following result.

Proposition 5.3. [14] If $X$ is a locally compact sober space, then the upper Vietoris topology and the Scott topology on $K(X)$ coincide.
By Theorem 3.5 and Proposition 5.3, we have the following corollary.

**Corollary 5.4.** If $X$ is a core compact well-filtered space, then the upper Vietoris topology and the Scott topology on $K(X)$ coincide.

**Theorem 5.5.** (I) If $X$ is a second-countable sober space, then the upper Vietoris topology and the Scott topology on $K(X)$ coincide.

From Theorem 3.9, Theorem 3.5 and Theorem 5.5, we directly deduce the following two results.

**Corollary 5.6.** If $X$ is a second-countable well-filtered space, then the upper Vietoris topology and the Scott topology on $K(X)$ coincide.

**Corollary 5.7.** For a second-countable well-filtered space $X$, the following conditions are equivalent:

1. $X$ is locally compact.
2. $K(X)$ is a continuous semilattice, and $\xi^*_X : X \to \Sigma K(X)$, $x \mapsto \uparrow x$, is continuous.
3. $K(X)$ is a continuous semilattice, and $X$ has property Q.
4. $K(X)$ is a continuous semilattice.
5. $X$ is core compact.

In [10] (see [9, Exercise V-5.25]), Hofmann and Lawson constructed a second-countable core compact $T_0$ space $X$ in which every compact subset has empty interior. So $X$ is not locally compact and does not have property Q. By Theorem 4.1, $P_S(X)$ is second-countable; and by Corollary 5.6 or Corollary 5.7, $X$ is not well-filtered.

Now we give another main result of this paper.

**Theorem 5.8.** If $X$ is a well-filtered space and $P_S(X)$ is first-countable, then the upper Vietoris topology agrees with the Scott topology on $K(X)$.

**Proof.** By Lemma 3.1, $O(P_S(X)) \subseteq \sigma(K(X))$. Now we show that $O(P_S(X)) \supseteq \sigma(K(X))$. Assume $K \in U \subseteq O(P_S(X))$. Since $P_S(X)$ is first-countable and $\{V : V \in O(X)\}$ is a base of $P_S(X)$, we have a countable family $\{U_n : n \in \mathbb{N}\} \subseteq O(X)$ such that $\{\bigcap U_n : n \in \mathbb{N}\}$ is a base at $K$ in $P_S(X)$. We can assume that $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq U_{n+1} \supseteq \ldots$ (otherwise, we replace $U_n$ with $\bigcap_{i=1}^n U_i$ for each $n \in \mathbb{N}$). We claim that $\bigcap U_n \subseteq U$ for some $n \in \mathbb{N}$. Assume, on the contrary, that $\bigcap U_n \not\subseteq U$ for all $n \in \mathbb{N}$. For each $m \in \mathbb{N}$, choose $K_m \in U_m \setminus U$, and let $G_m = K \cup U_{m+1} \setminus K_m$.

Claim 1: $G_m \in K(X)$ for each $m \in \mathbb{N}$.

Suppose that $\{W_j : j \in J\}$ is an open cover of $G_m$ and let $W_J = \bigcup_{j \in J} W_j$. Then $W_J \in O(X)$ and $K \subseteq W_J$. By the compactness of $K$, there is $J_1 \in J^{<\omega}$ such that $K \subseteq W_{J_1} = \bigcup_{j \in J_1} W_j$. Since $\{\bigcap U_n : n \in \mathbb{N}\}$ is a base at $K$ in $P_S(X)$, there is $n_0 \in \mathbb{N}$ such that $K \subseteq \bigcap U_{n_0} \subseteq \bigcap W_{J_1}$, that is, $K \subseteq U_{n_0} \subseteq W_{J_1}$. As $(U_n)_{n \in \mathbb{N}}$ is a decreasing sequence, we have that $K_1 \subseteq U_1 \subseteq U_{n_0} \subseteq W_{J_1}$ for all $l \geq n_0$. By the compactness of $\bigcup_{i=m}^{n_0-1} K_i$, there is $J_2 \in J^{<\omega}$ such that $\bigcup_{i=m}^{n_0-1} K_i \subseteq W_{J_2} = \bigcup_{j \in J_2} W_j$. Therefore,

$G_m = (K \cup \bigcup_{i=m}^{n_0-1} K_i) \cup \bigcup_{i=m}^{n_0-1} K_i \subseteq W_{J_1} \cup W_{J_2}$. Thus $G_m \in K(X)$.

Claim 2: $G_m \supseteq G_{m+1}$ for each $m \in \mathbb{N}$.

Claim 3: $K = \bigcap_{m \in \mathbb{N}} G_m$.

Clearly, $K \subseteq \bigcap_{m \in \mathbb{N}} G_m$. Conversely, assume $x \notin K$. Then $K \in \square (X \setminus \downarrow x)$, and whence there is $m_0 \in \mathbb{N}$ such that $K \in \square U_{m_0} \subseteq \square (X \setminus \downarrow x)$. It follows that $x \notin G_m$ for all $m \geq m_0$. Hence $x \notin \bigcap_{m \in \mathbb{N}} G_m$. Therefore, $K = \bigcap_{m \in \mathbb{N}} G_m$.

By the above three claims and Lemma 2.4, $K = \bigvee_{m \in \mathbb{N}} \{G_m : m \in \mathbb{N}\} \in U \in \sigma(K(X))$, and hence $G_q \in U$ for some $q \in \mathbb{N}$. But then $K \subseteq U$ for all $n > q$, a contradiction.

Therefore, $K \notin \square U_n \subseteq U$ for some $n \in \mathbb{N}$, and consequently, $U \in O(P_S(X))$. It is thus proved that the upper Vietoris topology and the Scott topology on $K(X)$ coincide. \qed
By Theorem 4.5 and Theorem 5.8, we get the following result.

**Corollary 5.9.** Let $X$ be a first-countable well-filtered space. If $\min(K)$ is countable for any $K \in K(X)$, then the upper Vietoris topology and the Scott topology on $K(X)$ coincide.

**Corollary 5.10.** For a first-countable well-filtered space $X$ in which all compact subsets are countable, the upper Vietoris topology and the Scott topology on $K(X)$ agree.

By Theorem 3.9, Theorem 5.8 and Corollary 5.9, we have the following three corollaries.

**Corollary 5.11.** Let $X$ be a well-filtered space for which $\mathcal{P}_S(X)$ is first-countable. Then the following conditions are equivalent:

1. $X$ is locally compact.
2. $K(X)$ is a continuous semilattice, and $\xi_X : X \rightarrow \Sigma K(X)$, $x \mapsto \uparrow x$, is continuous.
3. $K(X)$ is a continuous semilattice, and $X$ has property $Q$.
4. $K(X)$ is a continuous semilattice.
5. $X$ is core compact.

**Corollary 5.12.** Let $X$ be a first-countable well-filtered space for which $\min(K)$ is countable for any $K \in K(X)$. Then the following conditions are equivalent:

1. $X$ is locally compact.
2. $K(X)$ is a continuous semilattice, and $\xi_X : X \rightarrow \Sigma K(X)$, $x \mapsto \uparrow x$, is continuous.
3. $K(X)$ is a continuous semilattice, and $X$ has property $Q$.
4. $K(X)$ is a continuous semilattice.
5. $X$ is core compact.

**Corollary 5.13.** Let $X$ be a first-countable well-filtered space for which all compact subsets of $X$ are countable. Then the following conditions are equivalent:

1. $X$ is locally compact.
2. $K(X)$ is a continuous semilattice, and $\xi_X : X \rightarrow \Sigma K(X)$, $x \mapsto \uparrow x$, is continuous.
3. $K(X)$ is a continuous semilattice, and $X$ has property $Q$.
4. $K(X)$ is a continuous semilattice.
5. $X$ is core compact.

By Proposition 2.9, Lemma 4.7 and Theorem 5.8, we get the following two results.

**Corollary 5.14.** For a metric space $(X,d)$, the upper Vietoris topology coincides with the Scott topology on $K((X,d))$.

**Corollary 5.15.** For a metric space $(X,d)$, the following conditions are equivalent:

1. $(X,d)$ is locally compact.
2. $K((X,d))$ is a continuous semilattice, and $\xi_{(X,d)} : (X,d) \rightarrow \Sigma K((X,d))$, $x \mapsto \{x\}$, is continuous.
3. $K((X,d))$ is a continuous semilattice, and $(X,d)$ has property $Q$.
4. $K((X,d))$ is a continuous semilattice.
5. $(X,d)$ is core compact.

Let $\mathbb{R}$ be the set of all real numbers. $\mathbb{R}$ endowed with the topology taking the family $\{[x,y) : x < y\}$ as a base is called the **Sorgenfrey line** and denoted by $\mathbb{R}_s$. Dually, we endow $\mathbb{R}$ with the topology generated by $\{[x,y) : x < y\}$ as a base, and denote the resulting space by $\mathbb{R}_r$. A subset $A \subseteq \mathbb{R}_s$ is called **bounded** if $A \subseteq [−n, n]$ for some $n \in \mathbb{N}$. As one of the "universal counterexamples" in general topology, $\mathbb{R}_s$ poses many important topological properties (cf. [2, 6]). In particular, the Sorgenfrey line has the following properties (cf. [2]).
Proposition 5.16. (1) $\mathbb{R}_l$ is perfectly normal, first-countable and separable.  
(2) $\mathbb{R}_l$ is not second-countable.  
(3) $\mathbb{R}_l$ is neither compact nor locally compact.  
(4) Every compact subset of $\mathbb{R}_l$ is countable.

Lemma 5.17. For a subset $A$ of $\mathbb{R}_l$, the following two conditions are equivalent:  
(1) $A$ is compact in $\mathbb{R}_l$.  
(2) $A$ is a bounded closed subset of $\mathbb{R}_l$, and $A$ has no accumulation point in $\mathbb{R}_r$ (that is, there is no point $x \in \mathbb{R}$ such that $x \in \text{cl}_{\mathbb{R}_l}(A \setminus \{x\})$).

Example 5.18. Consider the Sorgenfrey line $\mathbb{R}_l$. Then by Theorem 5.9, Corollary 4.6, Theorem 5.8, Proposition 5.16 and Lemma 5.17, we have  
(1) $\mathbb{R}_l$ is first-countable and Hausdorff, and hence sober.  
(2) $\mathcal{P}_S(\mathbb{R}_l)$ is first-countable.  
(3) $\text{int} K = \emptyset$ for any $K \in K(\mathbb{R}_l)$, and whence $\mathcal{P}_S(\mathbb{R}_l)$ is not locally compact.  
(4) the upper Vietoris topology and the Scott topology on $K(\mathbb{R}_l)$ agree.  
(5) $K_1 \not< K_2$ for any $K_1, K_2 \in K(\mathbb{R}_l)$, so $K(\mathbb{R}_l)$ is not continuous and $\mathbb{R}_l$ has property Q.  
(6) $\xi^{\sigma}_{\mathbb{R}_l} : \mathbb{R}_l \rightarrow 2^{K(\mathbb{R}_l)}$, $x \mapsto \{x\}$, is continuous.  
(7) $\mathbb{R}_l$ is not core compact.

Example 5.19. Let $X$ be the space in Example 3.13. Then we have (a) $|X| = \omega$ and $X$ is sober; (b) $K(X) = X^{(\prec \omega)} \setminus \emptyset$; (c) $K(X)$ is an algebraic semilattice; and (d) the upper Vietoris topology and the Scott topology on $K(X)$ does not coincide or, equivalently, $\sigma(K(X)) \nsubseteq O(P_S(X))$. By Proposition 4.2 (or Theorem 5.8 and Corollary 5.10) Neither $P_S(X)$ nor $X$ is first-countable (cf. [2, Corollary 3.6.17]).

Finally, we pose the following question.

Question 5.20. For a first-countable well-filtered space $X$, does the upper Vietoris topology and the Scott topology on $K(X)$ coincide?

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