Effects of self-consistency in semiclassical pairing theory

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Abstract. An extended Vlasov equation, derived from the time-dependent Hartree–Fock–Bogoliubov equation of motion, is used to study the effects of pairing on the nuclear density-density linear response function in semiclassical approximation. Within the approximations adopted here, the fluctuations of the pairing field are purely imaginary and play a crucial role in eliminating a spurious mode that gives fluctuations of the number of particle and in reproducing the correct value of the energy-weighted sum rule. The semiclassical approach seems to have some difficulty in describing pairing vibrations. Further work on this problem is needed.

1. Introduction
Different approaches can be used to study the effects of pairing correlations in nuclei (see, for example, Ref.\textsuperscript{[1]} and references quoted therein). Here, following \textsuperscript{[2]} and \textsuperscript{[3]}, we use a semiclassical approximation to study the effects of pairing on the nuclear response. The semiclassical approximation is valid when

\[ k_F R \gg 1, \]  

(1)

a condition which is well satisfied in heavy nuclei (\( k_F \) is a typical Fermi wavenumber and \( R \) is the nuclear radius).

The semiclassical approach leads to relatively simple analytical expressions for several quantities of interest and our hope is that of gaining some insight by using this approach which is certainly less accurate than a fully quantum theory.

2. Formalism
A kinetic equation valid for systems with pairing can be obtained by taking the Wigner transform of the time-dependent Hartree–Fock–Bogoliubov (TDHFB) equations of motion in the density-matrix formulation \textsuperscript{[4]} and neglecting terms of order \( \hbar^2 \), or higher. This procedure leads to four coupled differential equations for the functions \( f(r,p,t), f(r - p,t), \kappa(r,p,t), \Delta(r,p,t) \). The single-particle distribution \( f(r,p,t) \) is real, while the Wigner-transformed pairing tensor \( \kappa(r,p,t) \) and pairing field \( \Delta(r,p,t) \) are complex in a time-dependent situation, thus, in this way, we have four equations of motion and six unknown quantities. However, in the TDHFB theory, the functions \( \kappa(r,p,t) \) and \( \Delta(r,p,t) \) are related by a self-consistency relation. This relation can
be derived from the gap equation. In order to avoid introducing a cut-off parameter, we consider
the gap equation written in the form
\[ g \int dp \left( \frac{\kappa(r, p, t)}{\Delta(r, t)} + \frac{1}{p^2/m} \right) = 1. \] (2)

Here \( g \) is a parameter determining the strength of the pairing interaction. We have assumed
that the \( p \)-dependence of the dynamic pairing field can be neglected. Moreover, in the following
we will approximate the equilibrium pairing field \( \Delta_0(r, p) \) with the phenomenological parameter
\( \Delta \), which, in heavy nuclei takes values \( \Delta \approx 1 \text{ MeV} \).

Since we are interested in the linear response of nuclei, we assume that our system , which
is initially in equilibrium, at time \( t = 0 \) is acted upon by an external weak driving field of the
form
\[ \delta V^{ext}(r, t) = \eta(t)Q(r). \] (3)

As a consequence of this external perturbation, the functions \( f, \kappa, \Delta \) at \( t > 0 \) will oscillate about
their equilibrium values and the small-amplitude oscillations \( \delta f, \delta \kappa, \delta \Delta \) are the object of our
study.

Thus, by considering small fluctuations of \( \kappa(r, p, t) \) and \( \Delta(r, t) \) about their equilibrium values,
from Eq. (2) we get the first-order relation
\[ \int dp \left( \delta \kappa(r, p, t) - \kappa_0(r, p, t) \frac{\delta \Delta(r, t)}{\Delta} \right) = 0, \] (4)

where \( \kappa_0(r, p) \) and \( \Delta \) are real equilibrium quantities, while \( \delta \kappa \) and \( \delta \Delta \) are their complex
fluctuations. The real and imaginary parts of Eq. (4) give the two independent relations:
\[ \int dp \left( \delta \kappa^r(r, p, t) - \kappa_0(r, p, t) \frac{\delta \Delta^r(r, t)}{\Delta} \right) = 0, \] (5)
\[ \int dp \left( \delta \kappa^i(r, p, t) - \kappa_0(r, p, t) \frac{\delta \Delta^i(r, t)}{\Delta} \right) = 0. \] (6)

These conditions, based on Eq. (2), take into account the residual pairing interaction in a self-
consistent way. Here we consider two weaker self-consistency relations that can be obtained by
integrating Eqs. (5, 6) over \( r \). By using also the equilibrium relation
\[ \kappa_0(r, p) \approx \kappa_0(\epsilon) = -\frac{\Delta}{2E(\epsilon)}, \] (7)

where \( \epsilon = h_0(r, p) = \frac{p^2}{2m} + V_0(r) \) is the particle energy, \( E(\epsilon) = \sqrt{\Delta^2 + (\epsilon - \mu)^2} \) the quasiparticle
energy and \( \mu \) the chemical potential, these weaker self-consistency relations become
\[ \int dr \int dp \left( \delta \kappa^r(r, p, t) + \frac{\delta \Delta^r(r, t)}{2E(\epsilon)} \right) = 0, \] (8)
\[ \int dr \int dp \left( \delta \kappa^i(r, p, t) + \frac{\delta \Delta^i(r, t)}{2E(\epsilon)} \right) = 0. \] (9)

The four differential equations of motion can also be linearized. In this way the problem is
reduced to that of solving four coupled linear differential equations plus the two self-consistency
relations (8, 9).

The mathematical problem can be simplified by using the action-angle variables \((I, \Phi)\) instead
of \((r, p)\). In this way the four coupled differential equations are transformed into four coupled
algebraic equations, which can be used to express the quantities $\delta \kappa^{r,i}$ in terms of $\delta \Delta^{r,i}$ and $\delta V^{ext}$ in Eqs. (8, 9).

This is achieved by means of a very useful property of action-angle variables (see, for example Ref.[5], p. 467): for bound motion of a particle, any function of its coordinates $f(r,p)$ can be expanded as a multiple Fourier series of the kind

$$f(r,p) = \sum_n f_n(I) e^{in \cdot \Phi}.$$  

Here $n$ is a three-dimensional vector with integer components: $n = (n_1, n_2, n_3)$. After taking the Fourier transform in time, and changing the integration variables from $(r,p)$ to $(I, \Phi)$, Eqs. (8, 9) can be written in the form

$$\int d\Phi \int dI \sum_n \left( \delta \kappa_n^r(I,\omega) + \frac{\delta \Delta_n^r(I,\omega)}{2E(\epsilon)} \right) e^{in \cdot \Phi} = 0,$$

$$\int d\Phi \int dI \sum_n \left( \delta \kappa_n^i(I,\omega) + \frac{\delta \Delta_n^i(I,\omega)}{2E(\epsilon)} \right) e^{in \cdot \Phi} = 0.$$  

The orthogonality of the functions $e^{in \cdot \Phi}$ gives $\int d\Phi e^{in \cdot \Phi} = (2\pi)^3 \delta_{n,0}$. Consequently, the self-consistency conditions (8, 9) give two conditions for the amplitude of the normal mode $n = 0$.

$$\int dI \left( \delta \kappa_{n=0}^r(I,\omega) + \frac{\delta \Delta_{n=0}^r(I,\omega)}{2E(\epsilon)} \right) = 0,$$

$$\int dI \left( \delta \kappa_{n=0}^i(I,\omega) + \frac{\delta \Delta_{n=0}^i(I,\omega)}{2E(\epsilon)} \right) = 0.$$  

In spherical nuclei, these two three-dimensional integral relations can be reduced to a system of two coupled one-dimensional integral equations. If we make the further reasonable assumption that the fluctuations of the pairing field are proportional to the external driving field,

$$\delta \Delta^r(r,\omega) = F^r(\omega)Q(r)$$

$$\delta \Delta^i(r,\omega) = F^i(\omega)Q(r)$$

and use the expressions of the quantities $\delta \kappa^{r,i}$ in terms of $\delta \Delta^{r,i}$ and $\delta V^{ext}$, then the two one-dimensional integral equations become two coupled algebraic equations for the unknown functions $F^{r,i}(\omega)$:

$$F^r(\omega)A_{11}(\omega) + F^i(\omega)A_{12}(\omega) = \eta B_1(\omega),$$

$$F^r(\omega)A_{21}(\omega) + F^i(\omega)A_{22}(\omega) = \eta B_2(\omega).$$  

The functions $A_{ij}(\omega)$ are

$$A_{ij}(\omega) = \int_0^\infty d\epsilon g(\epsilon) a_{ij}(\epsilon,\omega),$$

while

$$B_i(\omega) = \int_0^\infty d\epsilon g(\epsilon) b_i(\epsilon,\omega),$$
$g(\epsilon)$ is the single-particle level density for the equilibrium mean field and the functions $a_{ij}(\epsilon, \omega), b_i(\epsilon, \omega)$ are given by

$$a_{11}(\epsilon, \omega) = -i\omega \frac{\tilde{\epsilon}}{E(\epsilon) (\omega + i\epsilon)^2 - 4E^2(\epsilon)}, \quad (21)$$

$$a_{12}(\epsilon, \omega) = \frac{1}{\omega^2} \frac{1}{2E(\epsilon) (\omega + i\epsilon)^2 - 4E^2(\epsilon)}, \quad (22)$$

$$a_{21}(\epsilon, \omega) = (\omega^2 - 4\Delta^2) \frac{a_{12}(\epsilon, \omega)}{\omega^2} \quad (23)$$

$$a_{22}(\epsilon, \omega) = -a_{11}(\epsilon, \omega), \quad (24)$$

$$b_1(\epsilon, \omega) = \frac{2\Delta}{\omega} a_{12}(\epsilon, \omega), \quad (25)$$

$$b_2(\epsilon, \omega) = \frac{2\Delta}{\omega} a_{22}(\epsilon, \omega). \quad (26)$$

Here $\tilde{\epsilon} = \epsilon - \mu$, while $\epsilon$ is a vanishingly small positive quantity.

If the determinant

$$D(\omega) = A_{11}(\omega)A_{22}(\omega) - A_{21}(\omega)A_{12}(\omega) \quad (27)$$

is non vanishing, the system of coupled algebraic equations (17, 18) can be easily solved, giving

$$F_r(\omega) = 0, \quad (28)$$

$$F^i(\omega) = \eta \frac{2\Delta}{i\omega}, \quad (29)$$

however there could be also other solutions corresponding to the roots of

$$D(\omega) = 0. \quad (30)$$

These solutions would be particularly interesting because they correspond to collective nuclear modes related to pairing (pairing vibrations).

The determinant (27) can also be written as

$$D(\omega) = \omega^2 [I_1^2(\omega) - (\omega^2 - 4\Delta^2)I_2^2(\omega)], \quad (31)$$

with

$$I_1(\omega) = \int_0^\infty d\epsilon g(\epsilon) \frac{\tilde{\epsilon}}{E(\epsilon) (\omega + i\epsilon)^2 - 4E^2(\epsilon)}, \quad (32)$$

$$I_2(\omega) = \frac{1}{2} \int_0^\infty d\epsilon g(\epsilon) \frac{1}{E(\epsilon) (\omega + i\epsilon)^2 - 4E^2(\epsilon)}, \quad (33)$$

An obvious solution of Eq. (30) occurs for $\omega = 0$. This zero-frequency solution corresponds to the Anderson-Goldstone-Nambu mode and is associated with rotations in gauge space (see, for example, Ch 4 of [1]).

The integrals (32, 33) are clearly the semiclassical versions of quantities which have been studied in analogous quantum approaches to the problem of pairing vibrations (cf., for example Eqs. (5.25) and (5.24) of [1]). In quantum theory, the integrals (32, 33) are replaced by sum over discrete nuclear levels, making the solution of Eq. (30) simpler in the quantum case. Our work on this point is still in progress, however, the solution (28, 29) is also of interest because, by using this solution, we can check that the fluctuations of the imaginary pairing field are essential
to restore particle-number conservation and to give the correct value of the energy weighted sum rule (the same as in normal systems).

We conclude this communication by showing the effects of pairing on the semiclassical quadrupole and octupole density response function for a system of $A = 208$ nucleons confined to a spherical cavity of radius $R = 1.2A^{1/3}$ fm. These figures give an indication of the kind of effects that are to be expected in $^{208}\text{Pb}$ isotopes.

![Figure 1](image1.png)  
**Figure 1.** Quadrupole response function for $A = 208$ nucleons confined to a spherical cavity of radius $R$. The full curve include the effects of pairing according to Eqs. (28, 29), while the dashed curve is for a normal system. Reprinted with permission from *Nucl. Phys. A*, Ref. [3]

![Figure 2](image2.png)  
**Figure 2.** Same as Fig. 1 for octupole response. The two peaks correspond to the low-energy and high-energy parts of the giant octupole resonance. Reprinted with permission from *Nucl. Phys. A*, Ref. [3]

Both the quadrupole and octupole response functions of Figs. 1, 2 display a gap $2\Delta$ at low excitation energy. This fact can be appreciated more clearly in Fig. 3, where the low-energy part of Fig. 1 is shown on an enlarged scale.
Figure 3. Same as Fig.1 in the low-energy region. The gap $2\Delta$ is clearly visible for the full curve. Reprinted with permission from *Nucl. Phys. A*, Ref. [3]

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