Hybrid Models of Opinion Dynamics with Limited Confidence

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Consensus and opinion dynamics

The evolution of opinions in social networks can be described by similar dynamics as those used in consensus-seeking dynamics (multi-agent systems).

- Some mathematical models have been previously proposed and studied in social science [Friedkin (2006); Hegselmann and Krause (2002)]
- The topic of opinion dynamics is now a popular and distinct topic in systems and control [Acemoglu et al. (2013); Altafini (2013); Ravazzi et al. (2015); Parsegov et al. (2015)]
Opinion-dependent limitations

A fundamental ingredient in several models of opinion dynamics resides in (opinion-dependent) limitations in the relative influence between individuals.

- Such limitations prevent consensus in the long run.
- These limitations postulate that individuals do not influence each other if their opinions are too far apart.
- The simplest form of limitation (bounded confidence) is based on a fixed threshold $R$: individuals interact if their opinions are closer than the threshold.
Bounded confidence model’s good and bad features

**Model:** Confidence threshold \( R > 0 \)

\[
\dot{y}_i = \sum_{j : |y_i - y_j| < R} (y_j - y_i)
\]

[Hegselmann&Krause'02] [Blondel,Hendricks&Tsitsiklis'10]

- Discontinuous right-hand side
- Formation of **clusters** where individuals agree
Goal of this work hinges upon **uniform convergence**

**Objectives**

Propose a novel model of opinion dynamics with limited confidence and prove UGAS of the set of equilibria (and thus, **uniform** convergence of solutions to such a set).

- **Model scalability.** As opposed to other models, the reciprocal influences between individuals are not defined by absolute limits, but by a notion of relative relevance for each link.

- **Uniform Convergence.** Opinion distributions will converge uniformly to disconnected clusters (individuals agree locally and each cluster holds a different opinion).

Solution: The proposed dynamics takes the form of a hybrid dynamical system, where topology and opinions are coupled but as distinct variables. [Goebel et al. (2012)]
First ingredients: opinions, interconnections and degrees

- Consider $n$ agents indexed in a set $i \in \mathcal{I} = \{1, \ldots, n\}$, each of them holding a time-dependent opinion $y_i \in \mathbb{R}$.
- Consider also a time-varying interaction pattern where for any pair $(h, k) \in \mathcal{I} \times \mathcal{I}$, such that $h \neq k$, agents $h$ and $k$ share opinions if $a_{hk} = a_{kh} \in \{0, 1\}$ is set to 1.
- Then all possible interactions among the $n$ agents are described by the elements of $a$:

$$a := \text{col}(a_{12}, a_{13}, \ldots, a_{1n}, a_{21}, \ldots, a_{n-2,n-1}, a_{n-2,n}, a_{n-1,n})$$

$$a \in \{0, 1\}^{\frac{n(n-1)}{2}}$$

- Each agent may have a variable number of active connections with other agents. It is then convenient to define the (augmented) degree of agent $i$, for each $i \in \mathcal{I}$, as

$$d_i = 1 + \sum_{j \neq i} a_{ij} > 0$$
Hybrid State comprises opinions $y$ and interconnection $a$

**Model Characteristics**

We want to mix both the continuous evolution of the agents' opinions (described by suitable variations of opinion state $y$), and the discrete variations in the interaction pattern (described by instantaneous jumps of interactions state $a$).

- The overall state $x := (y, a)$ evolves in the following set:

  $$(y, a) \in X := \mathbb{R}^n \times \{0, 1\}^{\frac{n(n-1)}{2}}$$

- Since the proposed model involves both continuous variations and instantaneous jumps of the state, we will adopt a hybrid framework for its description and analysis.
Continuous-time dynamics: flow equation

**Important fact:** interactions only occur between pairs $(i, j)$ of agents having an active link $(a_{ij} = 1)$ and the interaction is stronger if the two agents have a small number of active links (small values of $d_i$ and $d_j$)

- The flow equation for the overall state variable $(y, a)$:

  \[
  \begin{cases}
  \dot{y}_i = \sum_{j \in \mathcal{I} \setminus \{i\}} \frac{a_{ij}}{d_i d_j} (y_j - y_i) & \text{for all } i \in \mathcal{I} \\
  \dot{a}_{ij} = 0 & \text{for all } (i, j) \in \mathcal{E}^+, \\
  \end{cases}
  \]

  with $\mathcal{E}^+ := \{(i, j) : i, j \in \mathcal{I}, j > i\}$.

- May be simplified using the **hybrid Laplacian** $L(a)$:

  \[
  \begin{cases}
  \dot{y} = -L(a)y \\
  \dot{a} = 0, \\
  \end{cases}
  \]

  where $C$ is the Flow set
Discrete-time dynamics: \textit{jump equation}

\textbf{Important fact:} Along the flow dynamics (F), the connection graph remains constant ($\dot{a}_{ij} = 0$) during flowing of the hybrid solutions.

- The change of topology of the interconnection graph is captured by a jump of the hybrid solution: the opinions $y$ are unchanged and only the elements $a_{ij}$ of $a$ are affected.

- The jump equation for the overall state variable $(y, a)$:

\[
\begin{cases}
  y_i^+ = y_i & \text{for all } i \in I \\
  a_{hk}^+ = 1 - a_{hk} & \text{for all } (y, a) \in D_{hk}, (J) \\
  a_{ij}^+ = a_{ij} & \text{for all } (i, j) \in \mathcal{E}^+ \setminus \{(h, k)\},
\end{cases}
\]

- Across the jump dynamics (J), opinions $y$ remain constant
- $a_{hk}$ toggles between 0 and 1 across jumps
- $D := \bigcup_{(h,k) \in \mathcal{E}^+} D_{hk}$ is the jump set
Selection of the jump and flow sets

- **Jump set.** Each jump set $D_{hk}$ characterizes when agents $h$ and $k$ should be allowed to toggle their interconnection (disconnect when they are connected, and vice-versa). Mathematically, solutions are allowed to jump when they belong to $D_{hk}$, so that the corresponding interconnection state variable $a_{hk}$ toggles.

- **Flow set.** A solution can flow only if it belongs to the flow set, therefore, to ensure existence (and completeness) of solutions, we select the flow set as the closed complement of the jump set:

  $$ C := \overline{X \setminus D}, \quad \left(\text{Recall that } X := \mathbb{R}^n \times \{0, 1\}^{\frac{n(n-1)}{2}}\right) $$

- **Nonuniqueness.** Note that solutions may flow or jump from the boundary of $C$ and $D$. 
Krause’s model with hysteresis

Opinion model is completed by selecting the jump sets $D_{hk}$ (everything else is set). Here’s a first natural choice.

Bounded confidence with hysteresis regularization:

$$D_{hk}^{on} := \{a_{hk} = 0\} \cap \{(y_h - y_k)^2 \leq -\varepsilon + R^2\}$$

$$D_{hk}^{off} := \{a_{hk} = 1\} \cap \{(y_h - y_k)^2 \geq +\varepsilon + R^2\}$$

$$D_{hk} := D_{hk}^{off} \cup D_{hk}^{on}$$

where $R$ and $\varepsilon$ are positive scalars with $\varepsilon$ (much) smaller than $R$

Remarks:

- Close approximation of the original non-smooth model
- Well-posed hybrid system and chattering-free dynamics
Global convergence to consensus set

Let $\mathcal{A} = \{(y, a) | a_{ij}(y_i - y_j) = 0 \text{ for all } (i, j)\}$

Convergence of hybrid dynamics

Any solution $\varphi$ to the hybrid consensus model
- performs a finite number of jumps and then flows forever (complete and eventually continuous)
- $\varphi$ converges to a point $x^* \in \mathcal{A}$ as $t \to +\infty$ (pointwise asymptotic stability)
- $x^* = (y^*, a^*)$ is such that $y_i^* = y_j^*$ if $a_{ij}^* = 1$

Proof sketch:
- Lyapunov function $V(x) = \frac{1}{2}y^Ty$ has negative semi-definite derivative $\dot{V}(x) = -y^TL(a)y \Rightarrow$ boundedness
- Hybrid Invariance Principle [Goebel, Sanfelice & Teel’12] ensures global convergence
BAD: Instability and non-uniform convergence

Let $\mathcal{A} = \{(y, a) \mid a_{ij}(y_i - y_j) = 0 \text{ for all } (i, j)\}$

**Convergence of hybrid dynamics**

Any solution $\varphi$ to the hybrid consensus model

- performs a finite number of jumps and then flows forever (complete and eventually continuous)
- $\varphi$ converges to a point $\mathcal{A}$ as $t \to +\infty$
- $x^* = (y^*, a^*)$ is such that $y^*_i = y^*_j$ if $a^*_{ij} = 1$

The set $\mathcal{A}$ is not forward invariant, therefore **unstable**: take $(a, y) \in \mathcal{A}$ such that $a_{ij} = 0$ and $(y_i - y_j)^2 = R^2 - \varepsilon$

a solution flows forever in $\mathcal{A}$

a solution flows in $\mathcal{A}$ until $(T, 0)$ and then jumps outside after jumping outside $\mathcal{A}$ it approaches $\mathcal{A}$ (global convergence)

Instability and global convergence implies **non-uniform convergence** (confirmed by counterexample)
Quasi-homogeneous Flow and jump sets

A new model may solve the instability and nonuniformity issues.

The jump set: \( D_{hk} := D_{hk}^{\text{on}} \cup D_{hk}^{\text{off}}, \) for all \((h, k) \in \mathcal{E}^+\) where

\[
D_{hk}^{\text{on}} := \{a_{hk} = 0\} \cap \{(y_h - y_k)^2 \leq -\varepsilon - \frac{\eta^2}{d_h d_k} (y_h - y_k)^2\} \quad (1a)
\]

\[
+ \frac{d_k + 1}{d_h} \sum_{\ell \neq k} a_{h\ell} (y_h - y_\ell)^2 + \frac{d_h + 1}{d_k} \sum_{\ell \neq h} a_{k\ell} (y_k - y_\ell)^2 \}
\]

\[
D_{hk}^{\text{off}} := \{a_{hk} = 1\} \cap \{(y_h - y_k)^2 \geq \varepsilon + \frac{\eta^2}{d_h d_k} (y_h - y_k)^2\} \quad (1b)
\]

\[
+ \frac{d_k}{d_h - 1} \sum_{\ell \neq k} a_{h\ell} (y_h - y_\ell)^2 + \frac{d_h}{d_k - 1} \sum_{\ell \neq h} a_{k\ell} (y_k - y_\ell)^2 \}
\]

The flow set: \( C := \overline{\mathbb{X}} \setminus D, \) (Recall that \( \mathbb{X} := \mathbb{R}^n \times \{0, 1\} \sum_{\ell=1}^{n(n-1)}\))
A few remarks on the new jump and flow sets

- **Jump set.** $\eta > 0$ is a design parameter to tune the sensitivity of the link related to activation and deactivation mechanisms.
  - A new connection between $h$ and $k$ is established when the distance $|y_h - y_k|$ is small compared to a weighted average of the distances between $h$ (or $k$) and their current neighbors.
  - On the contrary, a connection is dropped.
  - The jump rule does not prevent groups of individuals from being disconnected. Solutions will not in general converge to a unique global consensus, but instead to disconnected clusters of individuals that share the same opinion.

- **Flow set.** The solutions cannot flow if the state belongs to the interior of $D_{hk}^{on}$ or $D_{hk}^{off}$ but solutions may flow or jump if the state belongs to the boundary of some of these sets.
Global asymptotic stability of $\mathcal{A}$

**Theorem**

All solutions of the hybrid dynamical system perform a finite number of jumps and then converge **uniformly** to a constant state $(y^*, a^*) \in \mathcal{A}$ (namely such that $y^*_i = y^*_j$ if $a_{ij}^* = 1$). Moreover, set $\mathcal{A}$ is **stable**, therefore globally asymptotically stable.

- The new model ensures stability in addition to global convergence, thereby getting **uniform convergence to $\mathcal{A}$**.

- **Typical consequence of opinion models with limited confidence.** Opinions asymptotically converge to a certain number of stable values, which depend in a complex way on the initial conditions.

- **The model is almost homogeneous** (except for $\varepsilon$). We expect some clustering independent of size of initial opinions $y_0$. 


Some desirable features of the proposed model

- **Completeness.** For any initial condition starting in $X$, there do not exist solutions that may prematurely terminate due to the impossibility to flow or jump.

- **Ordinary time $t$.** The evolution of solutions is persistent in the ordinary time direction $t$: solutions will be defined for arbitrarily large ordinary times.

- **Convergence.** A desirable grouping of the opinion dynamics will be achieved, based on some reasonable activations and deactivations caused by the initial clustering of the opinions $y$.

- **Homogeneity.** Scaling the initial conditions the response is “almost” the same (see $\varepsilon$).

- **Sensitivity parameter.** Parameter $\eta$ can be used to adjust the clustering level (like absolute $R$ for Krause). $\eta$ seems to be related to the degree expected at the steady state.
Proof: A strict Lyapunov function for set $\mathcal{A}$

The Lyapunov-like function (monotonically decreasing both along flowing and across jumping of the solution):

$$V(y, a) := \frac{1}{2} y^\top L(a)y = -\frac{1}{4} \sum_{i,j} l_{ij} (y_i - y_j)^2 = \frac{1}{4} \sum_{i,j} \frac{a_{ij}}{d_i d_j} (y_i - y_j)^2.$$ 

$V$ satisfies $c_1 |x|_\mathcal{A}^2 \leq V(x) \leq c_2 |x|_\mathcal{A}^2$, where $|x|_\mathcal{A} := \inf_{z \in \mathcal{A}} |z - x|$. 

There exist scalars $c_F, c_J > 0$ such that the following hold:

$$\langle \nabla V(x), f(x) \rangle \leq -c_F V(x), \quad \forall x \in C,$$

$$V(g) - V(x) \leq -c_J R^2, \quad \forall x \in D, \forall g \in G(x).$$

Uniform global asymptotic stability of $\mathcal{A}$ follows from hybrid Lyapunov theorem for closed but not necessarily bounded attractors.

$$\mathcal{A} = \{(y, a) : V(y, a) = 0\} = \{(y, a) : a_{ij}(y_i - y_j)^2 = 0, \forall (i, j) \in \mathcal{E}^+\}.$$
Parameters. The number of individuals $n$ and the sensitivity radius $\eta$.

Initial conditions. Randomly generated as: the components of $y$ are independent uniform random variables in the interval $[0, 1]$ and the initial topology is an Erdős-Rényi random graph where each pair of nodes is connected by an edge with probability $\rho$. 
Simulated evolution of the state $y$ (left) and initial/final topologies (right) with $n = 15$ and $p = 0.15$.

- The initial topology has three connected components (one “giant component” and two isolated nodes).
- During the evolution, the isolated nodes remain isolated and the large connected component splits into four clusters.
- The final set of edges is not a subset of the initial one.
Opinions $y$ along ordinary time $t$, with $n = 25$, $p = 0.25$ and increasing values of $\eta = 16$

- $\eta$ controls the number of clusters, which decreases as $\eta$ grows
- The effect of $\eta$ is thus partly reminiscent of the interaction threshold in the classical models [Blondel et al., 2009]
Opinions $y$ along ordinary time $t$, with $n = 25$, $p = 0.25$ and increasing values of $\eta = 6$

- $\eta$ controls the number of clusters, which decreases as $\eta$ grows
- The effect of $\eta$ is thus partly reminiscent of the interaction threshold in the classical models [Blondel et al., 2009]
Opinions $y$ along ordinary time $t$, with $n = 25$, $p = 0.25$ and increasing values of $\eta = 4$

- $\eta$ controls the number of clusters, which decreases as $\eta$ grows
- The effect of $\eta$ is thus partly reminiscent of the interaction threshold in the classical models [Blondel et al., 2009]
Opinions $y$ along ordinary time $t$, with $n = 25$, $p = 0.25$ and increasing values of $\eta = 3$

- $\eta$ controls the number of clusters, which decreases as $\eta$ grows.
- The effect of $\eta$ is thus partly reminiscent of the interaction threshold in the classical models [Blondel et al., 2009]
Opinions $y$ along ordinary time $t$, with $n = 25$, $p = 0.25$ and increasing values of $\eta = 2.5$

- $\eta$ controls the number of clusters, which decreases as $\eta$ grows
- The effect of $\eta$ is thus partly reminiscent of the interaction threshold in the classical models [Blondel et al., 2009]
Opinions $y$ along ordinary time $t$, with $n = 25$, $p = 0.25$ and increasing values of $\eta = 2.5$ and 100 times larger $y_0$

- $\eta$ controls the number of clusters, which decreases as $\eta$ grows
- The effect of $\eta$ is thus partly reminiscent of the interaction threshold in the classical models [Blondel et al., 2009]
Summary

▷ A hybrid Laplacian allows for systematic stability analysis
  ● New state variables characterize active/inactive links
  ● Hybrid dynamics describes consensus (flow equation) and interconnection changes (jump equation)
  ● Different interaction models captured by choice of jump set

▷ Framework allows to represent Krause’s model
  ● \( \varepsilon \)-close representation is well-posed
  ● Stability analysis reveals instability and non-uniform convergence of solutions to the equilibrium set

▷ New model proposed presenting several advantages
  ● Dynamics is almost homogeneous (behavior scales with size of opinions)
  ● Equilibrium set is proven to be stable and uniformly attractive
  ● Krause’s interaction radius \( R \) now replaced by homogeneous “knob” \( \eta \) somewhat related to asymptotic degree of nodes
The clustering features of the homogeneous algorithm is not fully exploited

Describing opinion dynamics by *hybrid dynamics* opens powerful perspectives

- Analysis is conveniently separated out in flow (local consensus) and jumps (interconnection changes)
- May now describe several kinds of interactions, possibly with history-dependent interaction models that may involve delays, hysteresis, or hidden variables