Kinetically constrained quantum dynamics in a circuit-QED transmon wire

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We study the dynamical properties of the bosonic quantum East model at low temperature. We show that a naive generalization of the corresponding spin-1/2 quantum East model does not possess analogous slow dynamical properties. In particular, conversely to the spin case, the bosonic ground state turns out to be not localized. We restore localization by introducing a repulsive nearest-neighbour interaction term. The bosonic nature of the model allows us to construct rich families of many-body localized states, including coherent, squeezed and cat states. We formalize this finding by introducing a set of superbosonic creation-annihilation operators which satisfy the bosonic commutation relations and, when acting on the vacuum, create excitations exponentially localized around a certain site of the lattice. Given the constrained nature of the model, these states retain memory of their initial conditions for long times. Even in the presence of dissipation, we show that quantum information remains localized within decoherence times tunable with the system’s parameters. We propose a circuit QED implementation of the bosonic quantum East model based on state-of-the-art transmon physics, which could be used in the near future to explore kinetically constrained models in superconducting quantum computing platforms.

I. INTRODUCTION

Robust storage of quantum information and decoherence induced by external baths are two important limiting factors against a large-scale adoption of modern quantum technologies [1]. Storing quantum information is a challenging task as most interacting quantum systems tend to quickly thermalize. Once equilibrium is reached, the properties of the initial configurations are hard to retrieve as they are ergodically scattered among exponentially many degrees of freedom [2]. In order to overcome this obstacle, many proposals have attempted to confine quantum information into conserved or quasi-conserved quantities [3–17]. These proposals range from strongly disordered many-body localized [18, 19] or glassy systems [20–27] where thermalization is impeded by the presence of disordered potentials, to “fracton” systems where dynamical constraints induce fragmentation on the space of reachable configurations [28–35], and quantum scarred systems where certain classes of initial states show coherent oscillations for times longer than typical relaxation times [36–49]. Most of these phenomena often rely on such delicate properties that any weak coupling with an external environment could potentially become detrimental.

Quantum Kinetically Constrained Models (KCM) have recently gathered the attention for their peculiar dynamical properties. Motivated by the slowness of their classical counterparts, researchers have started to investigate their quantum generalizations such as the quantum East model, the quantum Fredricksen-Andersen model, and others [50–55].

In this work, we explore the low-temperature dynamical properties of the bosonic quantum East model, a generalization of the spin-1/2 model studied in Refs. [24,56], where spin excitations can only be created on sites on the ‘east’ of a previously occupied one. Our contributions can be summarized as follows. (i) We show that repulsive nearest-neighbour density-density interactions are necessary to entail localization in the ground state, in contrast to East models with finite dimensional local Hilbert space. (ii) We exploit the properties of the localized phase and the bosonic nature of the model, to construct families of non-gaussian many-body states useful for quantum information processing. (iii) We illustrate how localization enhances the robustness of these states against decoherence. (iv) Finally, we propose a circuit QED implementation of the bosonic quantum East model based on chains of superconducting transmons.

In the spin-1/2 case, evidence has been provided in support of a dynamical transition from a fast thermalizing regime to a slow, non-ergodic one [24,56]. In particular, in Ref. [56] it was argued that the slow dynamics is a byproduct of the localized nature of the low energy eigenstates of the model. Namely, the corresponding wavefunctions contain non-trivial excitations only on a small compact region of the lattice, and they are in the vacuum state everywhere else. This has direct consequences to the dynamical properties of the system as the localized states can be used as building blocks to construct exponentially many “slow” states in the size of the system.

The dynamical transition observed in Ref. [56] is not guaranteed to survive in the bosonic case. In fact, we provide strong numerical evidence that this is not the case for the most naive bosonic generalization of the spin-1/2 model. In order to restore localization at low temperature, we consider a modified model where density-density interactions - absent in the bare spin case - play a crucial role. More precisely, we show that the ground state remains localized as we increase the finite cutoff of the local Fock Space dimension only in the presence of repulsive interactions. We support our findings by combining nu-
merical and analytical approaches. Within the localized phase, the ground state is well approximated by a product state for any value of interaction. It is therefore well approximated by a matrix product state, making large system size and local Fock space dimension numerically accessible (cf. Secs. II and III).

The bosonic generalization of the spin-1/2 East model opens a number of directions including the construction of many-body versions of archetypal states relevant for quantum information applications such as coherent states, squeezed states and cat states [57]. These states possess the same properties of their single-mode counterparts, although they are supported on a few neighboring sites. We provide a formal description of these objects by proposing a simple adiabatic protocol which defines a set of superbosonic creation-annihilation operators (Sec. IV). These operators fulfill the canonical bosonic commutation relations, and they are exponentially localized in the neighborhood of a given site on the lattice. This allows to construct an effective, non-interacting, theory at low temperature in terms of these operators where the Hamiltonian is reminiscent of the l-bit construction in MBL [58–61].

In Sec. V we couple the system to different noise sources, and via a detailed numerical analysis, we show that localized states retain some memory of their initial condition even in the presence of strong dissipation (see Fig. 1). First, we couple the coherent dynamics to a density noise that preserves the “East symmetry” (see the definition in Sec. II). In this scenario, the localized states are barely altered by the environment. We show that the fidelity between the time-evolved state and the initial state decays exponentially with a long decoherence time controlled by the parameters of the Hamiltonian, the initial state and the strength of the noise. Second, we consider an incoherent pump which breaks the “East symmetry”. As expected in this situation, the magnitude of the fidelity decays exponentially fast in time, with a decoherence time parametrically small in the strength of the incoherent pump. In fact, any tiny amount of noise over the vacua surrounding the localized peak may be sufficient to rotate the initial state to an orthogonal one. It is important to stress that, as the localized states are non-trivial only on a small support, any external noise that does not act in their immediate vicinity leaves them essentially invariant. This set of noise-resilient properties renders the many-body localized states studied in this work qualitatively different from localization induced by disorder, which is inherently fragile to decoherence (see Refs. [62–67] for studies on MBL systems coupled to a bath or external noise). In particular, in Sec. VI we argue that our localized states can be manipulated on timescales shorter than the characteristic relaxation and decoherence times of transmon wires.

In fact, our proposal for a circuit QED implementation of the bosonic quantum East model is one of the key findings of this work. In recent years, circuit QED has become a platform of choice for unprecedented quantum control of interacting superconducting qubits with microwave photons [68–76]. These circuits allow quantum information processing tasks and the quantum simulation of paradigmatic light-matter interfaces. Superconducting Josephson junctions allow to introduce nonlinearity in quantum electrical circuits which is a key factor in protecting quantum resources, by making these platforms resilient to noise and errors. This is at the root of the most used artificial atom design in circuit QED: arrays of anharmonic quantum oscillators known as the transmon qubit [77–85]. Here we consider such a chain of anharmonic oscillators coupled via a hopping term and equipped with a self-Kerr nonlinearity (cf. with Fig. 1). In the limit of weak coupling and small anharmonicity, we find an effective description of such transmon chain in terms of the bosonic quantum East chain.

The paper is organized as follows. In Sec. II we introduce the Hamiltonian of the model, we enumerate its symmetries and we compare it to previous works on similar models. In Sec. III we explore the localization properties of the ground state of the model. In particular, we show how the transition point is independent of the size of the cutoff of the local Fock Space dimension and how the localization length behaves in the proximity of the transition. On the localized side of the transition, we quantitatively compare results extracted with tensor networks methods and mean field, and we show that they are in excellent agreement. In Sec. IV we introduce a description in terms of superbosonic operators which allows to generalize coherent, squeezed and cat states. In Sec. V we study the robustness of these localized states against noise sources. In Sec. VI we present the circuit QED implementation of the bosonic East model Hamil-

![FIG. 1. Panel (a): an array of driven transmons coupled via exchange interaction $g$. In the red box we write the low-energy effective interaction between the $j$-th and $(j+1)$-th transmon. Panel (b): cartoon of a localized state subject to external noise (arrows). The visibility of the initial peak with respect to the rest of the system (measured by the imbalance $\mathcal{I}(t)$) decays exponentially with a time $\tau$ much larger than the characteristic operational timescales of state-of-art transmons.](image-url)
II. BOSONIC QUANTUM EAST MODEL

We investigate the following Hamiltonian with open boundary conditions

\[ H = -\frac{1}{2} \sum_{j=0}^{L} \hat{n}_j \left[ e^{-s} \left( \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \right) - \epsilon \hat{n}_j - U \hat{n}_{j+1} - 1 \right], \tag{1} \]

where \( \hat{a}_j \) and \( \hat{a}_j^\dagger \) are bosonic annihilation and creation operators acting on site \( j \) respectively; \( e^{-s} \) controls the constrained creation and annihilation of bosons; \( \epsilon \) is the on-site density-density interaction; \( U \) is the nearest-neighbor density-density interaction.

As discussed in Sec. I, Eq. (1) is a kinetically constrained ‘East’ model. The unidirectional constrained feature has consequences on the accessible portion of the Hilbert space by the dynamics. Namely, any initial state with a product of vacua from the left edge up to a given site in the bulk will exhibit non-trivial dynamics only on the right side of the lattice after the first occupied site. For sake of concreteness, let us consider the state \( |00100\ldots0\rangle \). Via subsequent application of the Hamiltonian (1) we have,

\[ |00100\ldots0\rangle \rightarrow |00110\ldots0\rangle \rightarrow |001110\ldots0\rangle \ldots \tag{2} \]

where \( \rightarrow \) represents the action of the constrained creation and annihilation of bosons at each step of perturbation theory. The occupation of the first non-vacant site and of those at its left cannot change as a consequence of the ‘East’ constraint. More formally, the Hamiltonian commutes with the projectors

\[ P(n_0,k) = P_{0,j=0}^{\otimes k-1} \otimes P_{n_0,k} \otimes I_{j=2}^{\otimes L-k}, \tag{3} \]

where \( P_{s,j} = |s\rangle_j \langle s| \) is the projector on the Fock state with \( s \) particles on site \( j \), \( I_j \) is the identity acting on site \( j \) while \( k \) and \( n_0 \) are, respectively, the position and occupation of the first non-vacant site. We can split the Hilbert space into dynamically disconnected sectors \( H_{n_0,k} \), such that the action of \( P(n_0,k) \) is equivalent to the identity, while the action of the other projectors gives zero. For example, the state \( |00100\ldots0\rangle \in H_{1,2} \) (note that the first site index is 0). Furthermore, since \( \sum_{k=0}^{L} \sum_{n_0=1}^{\infty} P(n_0,k) = 1 \) these sectors \( \{H_{k,n_0}\} \) constitute a complete and orthogonal basis of the whole Hilbert space \( H \), namely \( H = \bigoplus_{k=0}^{L} \bigoplus_{n_0=1}^{\infty} H_{n_0,k} \).

In the following, we focus on a certain block specified by \( k, n_0 \) and the number of ‘active’ sites \( L \) right next to the \( k \)-th one. Since the action of \( H \) on sites to the left of the \( k \)-th one is trivial, the index \( k \) is physically irrelevant for our purpose and we will therefore choose \( k = 0 \) without losing any generality. Exploiting this property, we write the Hamiltonian (1) as \( H_{L+1} = \sum_{n_0} H_{L+1}(n_0) \), where \( H_{L+1}(n_0) \) is

\[ H_{L+1}(n_0) = \hat{h}_1 + \frac{1}{2} \sum_{j=2}^{L} \hat{n}_j \left[ e^{-s} \left( \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \right) - \epsilon \hat{n}_j - U \hat{n}_{j+1} - 1 \right], \tag{4} \]

with \( \hat{h}_1 = -\frac{1}{2} n_0 \left[ e^{-s} \left( \hat{a}_1 + \hat{a}_1^\dagger \right) - \epsilon n_0 - U \hat{n}_1 - 1 \right] \) and \( n_0 \in \mathbb{N}^+ \). Furthermore, since \( H_{L+1}(n_0) \) commutes with the operators acting on the \( (L+1) \)-th site, we can represent it as the sum of an infinite number of commuting terms \( H_{L+1}(n_0) = \sum_{r} \beta_r H_{L+1}^\beta_r (n_0) \otimes \Pi^\beta_r \), where \( \Pi^\beta_r \) is the projector over the eigenstate \( |\beta_r\rangle \) with eigenvalue \( \beta_r = r U - e^{-2s} / U \) of the operator \( \left(U \hat{n}_{L+1} - e^{-s} \left( \hat{a}_{L+1} + \hat{a}_{L+1}^\dagger \right) \right) \), where \( r \in \mathbb{N} \), and,

\[ H_{L+1}^\beta_r (n_0) = \frac{1}{2} \sum_{j=2}^{L-1} \hat{n}_j \left[ e^{-s} \left( \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \right) - \epsilon \hat{n}_j - U \hat{n}_{j+1} - 1 \right] + \frac{1}{2} \hat{n}_L [\beta_r + \epsilon \hat{n}_L + 1]. \tag{5} \]

In Sec. III, we will focus on the properties of the ground state of the Hamiltonian (5) within a certain symmetry sector.

The Hamiltonian (1) can be linked to its spin-1/2 version [50], by setting \( U = \epsilon = 0 \) and replacing the bosons with hard-core ones. Since the Hilbert space of each spin is finite, the ‘East’ symmetry is largely reduced with respect to the bosonic case. Each symmetry sector \( H_{k,n_0=1} \) is specified only by the position of the first excitation, since \( n_0 \) is bound to be zero or one. Ref. [50] investigated the ground state properties within a symmetry sector \( H_{k,n_0=1} \), where the position \( k \) of the first not-empty is again irrelevant. It was observed that the probability of finding an occupied site in the ground state decays exponentially fast around the first occupied site when \( s > 0 \), namely

\[ \langle \hat{n}_s \rangle \sim \exp(-j/\xi(s)), \tag{6} \]

where the expectation value is taken on the ground state and we have introduced the localization length \( \xi > 0 \). The localization length \( \xi \) is the typical distance from the first occupied site such that the state becomes a trivial product state well approximated by the vacuum. In Sec. III, we investigate the conditions for localization of the ground state at finite values of \( s \) upon trading spins (hard core bosons) for bosons. Such generalization is not granted. The amplitude for ‘eastern’ particle creation can be now enhanced by the pre-factor.
n₀, suggesting that the transition may qualitatively establish when \((n₀e^{-s}) \sim 1\). This would imply a critical value \(s_c \propto \log n₀\), which is parametrically large in \(n₀\) pushing the extension of the localized phase up to \(s \to \infty\). Nonetheless, we show in Sec. III that a localized phase still occurs for \(s > 0\), whenever repulsive nearest-neighbor interactions \(U\) are included in Eq. (1).

### III. LOCALIZATION TRANSITION

In this section we show that the Hamiltonian in Eq. (5) displays a localization-delocalization transition at finite \(s\) and \(U > 0\). We give numerical evidence corroborated by analytical observations that repulsive nearest-neighbor interactions are necessary to observe such a transition at finite \(s\). We use the inverse localization length \(\xi^{-1}\) controlling the decay of the average occupation number in space (cf. Eq. (6)), as proxy of the transition.

In the following we fix \(\epsilon = 0\) and the symmetry sector \(\beta_{\tau=0}\) in Eq. (5), unless mentioned otherwise, since our results will not be qualitatively affected.

In order to investigate the properties of the ground state, we resort to a combination of mean-field arguments, exact diagonalization (ED), and Density Matrix Renormalization Group (DMRG) [86]. Since we aim to explore large system sizes, we mainly resort to DMRG and we use ED as a benchmark when both methods can be used. Interestingly, we find that mean field is able to predict analytically the location of the transition point obtained via DMRG.

We compute the ground state \(|\psi₀(n₀)\rangle\) at fixed \(n₀\), \(s\) and \(U\). We fix the system size to \(L = 15\). This value is sufficiently large to capture the localized tail of the ground state, without relevant finite size effects. Although the local Fock space is infinite, in order to treat the model numerically, we need to fix a finite cutoff \(\Lambda\). We work with Fock states \(|0\rangle\) through \(|\Lambda\rangle\), such that the spin-1/2 case of Ref. [56] is recovered at \(\Lambda = 1\). In Appendix B we show how localization is only mildly dependent on the sector selected by the occupation \(n₀\) of the zero-th site. Accordingly, in the following we set \(n₀ = 1\).

The Hamiltonian is one dimensional, local and gapped at finite \(\Lambda\), therefore its ground state can be efficiently accessed via a Matrix Product State (MPS) formulation of the DMRG [86]. The main source of error is given by the finite cutoff \(\Lambda\). Indeed, the properties of \(|\psi₀(n₀)\rangle\) can change non-trivially as a function of \(\Lambda\). More precisely, for any finite cutoff \(\Lambda\), the model falls in the class of localized systems studied in Ref. [56]. As a result, \(|\psi₀(n₀)\rangle\) is always localized for a large enough \(s\) at finite \(\Lambda\), but this does not imply localization for \(\Lambda \to \infty\). Indeed, although \(U > 0\) makes the spectrum of the Hamiltonian in Eq. (1) bounded from below, it does not ensure that its ground state is still localized in space when \(s\) is finite. In the following, we extract the \(\Lambda \to \infty\) limit via a scaling analysis.

In Fig. 2 we show the average occupation number \(\langle \hat{n}_j \rangle\) as a function of site \(j\) for some values of \(s\) at fixed \(U = 1\). For \(s\) not large enough the average occupation does not change smoothly with the site \(j\) and it saturates the cutoff \(\Lambda\), meaning that there are strong finite-cutoff effects. For \(s\) large enough, in contrast, the occupation decays exponentially in \(j\) and it well matches Eq. (6) and it does...
not change upon increasing the cutoff $\Lambda$. The value of $s$ at which this change of behaviour occurs depends on $U$, as we will discuss in more detail in this section.

In order to check finite $\Lambda$ cutoff effects, we compute the probability of having $k$ bosons on site $j$, namely the expectation value of the projector $P_{k,j} = |k\rangle_j \langle k|$, where $|k\rangle_j$ is the Fock state with $k$ particles on site $j$. In Fig. 3 we show $\langle P_{k,j}\rangle$ as a function of $k$ and $j$ for a typical localized and delocalized ground states, respectively. The results in the delocalized phase are not reliable, since the observable suffers finite-cutoff effects. Instead, in the localized phase

$$\langle P_{k,j}\rangle \sim e^{-k/\xi_{P,j}},$$

(7)

with $\xi_{P,j} > 0$ for any site $j$. The exponential decay in the localized phase sheds additional light on the fact that the system is well described by a finite effective cutoff (see Appendix A for additional details).

For each value of $U$ and $\Lambda$, the inverse of localization length goes from values smaller than or equal to zero, to positive ones as $s$ increases. We identify the region where $1/\xi \leq 0$ as the delocalized phase, while the region where $1/\xi > 0$ as the localized phase. For each $\Lambda$ and $U$ we numerically extract $s_{c}(U, \Lambda)$ such that $1/\xi \leq \epsilon$ and $1/\xi > \epsilon$ for $s$ smaller and greater than $s_{c}(U, \Lambda)$ respectively. We choose $\epsilon \approx 10^{-2}$. The results are weakly affected by the choice of $\epsilon$.

As discussed above, in the delocalized phase, results are strongly dependent on the cutoff, since average occupations always saturate their artificial upper bound. This circumstance allows us to draw only qualitative conclusions on the physics at $s < s_{c}$ in the case of the bosonic East model ($\Lambda \to \infty$).

In Fig. 4 we show the inverse of the localization length $\xi$ swiping $s$ for different values of $\Lambda$ at fixed $U$. For $U = 0$ the transition point $s_{c}(U = 0, \Lambda)$ always increases with $\Lambda$. Instead, when $U > 0$ the transition point converges to a finite value independent of $\Lambda$ for $\Lambda \to \infty$. In Fig. 4(a) we show the numerically extracted transition point $s_{c}(U, \Lambda)$ as a function of $\Lambda$ and $U$. For $U > 0$ it is possible to extract a finite value of $s_{c}(U) \equiv \lim_{\Lambda \to \infty} s_{c}(U, \Lambda)$. Instead, for $U = 0$ the transition point scales as $s_{c}(U = 0, \Lambda) \propto \log(\Lambda)$, suggesting that in the actual bosonic system we have $s_{c}(U = 0) = \infty$, meaning there is no transition. Therefore, whenever $U > 0$, the system undergoes a
delocalized-localized transition at finite \( s_c(U) \). In Fig. 3 we show the inverse of localization length \( \xi \) as a function of \( s \) for different values of \( U \) at fixed \( \Lambda \). The transition point \( s_c \) depends on the competition between the dynamical term, controlled by \( e^{-s} \), and the nearest-neighbour density term, proportional to \( U \). Indeed, the former favours the delocalization of the state, while the latter its localization. Indeed, in the \( U \to 0 \) limit we have provided evidences that the bosonic system is always delocalized if \( s < \infty \). Instead, in the large \( U \) limit, the Hamiltonian is approximated by \( \sim U \sum_j \hat{n}_j \hat{n}_{j+1} + \hat{n}_j \), whose ground state in a specific symmetry sector at given total particle number is simply \(| n_0 \rangle | 00 \ldots 0 \rangle \).

The role of the interaction term \( U \) in the localization of the bosonic system can be appreciated in a mean-field treatment. We project the Hamiltonian into the manifold of coherent product states \( | \phi \rangle = \bigotimes_{j=1}^L | \alpha_j \rangle_j \), with \( \hat{a}_j | \alpha_j \rangle_j = \alpha_j | \alpha_j \rangle_j \). We evaluate the Hamiltonian in Eq. (4) in this basis

\[
| \langle \phi | H(n_0) | \phi \rangle | = -\frac{1}{2} \sum_{j=0}^J | \alpha_j \rangle^2 (2e^{-s} \alpha_{j+1} - U | \alpha_{j+1} \rangle^2 - 1),
\]

where \( | \alpha_j \rangle^2 \) is the average number of particles in the coherent state at site \( j \). From uni-directionality of the interaction we can write \( \langle \phi | H(n_0) | \phi \rangle = -\frac{1}{2} \sum_j | \alpha_j \rangle^2 h_j(\alpha_{j+1}, s, U), \) where \( h_j(\alpha_{j+1}, s, U) = (2e^{-s} \alpha_{j+1} - U | \alpha_{j+1} \rangle^2 - 1) \). For energetic stability the effective field \( h_j(\alpha_{j+1}, s, U) \) on site \( j \) should be negative

\[
(2e^{-s} \alpha_{j+1} - U | \alpha_{j+1} \rangle^2 - 1) < 0 \Rightarrow \Rightarrow s > \log \left( \frac{2 \alpha_{j+1}}{1 + U | \alpha_{j+1} \rangle^2} \right) = s_c(\alpha_{j+1}).
\]

Since the system does not conserve the number of particles there can be an unbounded number of excitations in the ground state within a fixed symmetry sector. Therefore, in order to have localization at a mean field level it is necessary that Eq. (9) holds for any value of \( \alpha_{j+1} \in [0, \infty) \), namely \( s > \max_{\alpha_{j+1}} s_c(\alpha_{j+1}) \), and for all sites. For \( U > 0 \) such condition is satisfied if \( s > \log(1/\sqrt{U}) \), which turns to be in very good agreement with the DMRG numerical findings (see Fig. 3). Instead, for \( U \leq 0 \) there is no finite value of \( s \) which fulfills Eq. (9) for all \( \alpha_{j+1} \).

The excellent agreement between DMRG and the mean field analysis can be explained by observing that the ground state \( | \psi_0 \rangle \) (excluding the 0-th site which fixes the symmetry sector) obtained via DMRG is well approximated via a product state, namely \( | \psi_0 \rangle \approx \bigotimes_{j=1}^J | \phi_j \rangle \). To further investigate the nature of the state \( | \psi_0 \rangle \), we consider the correlator \( \Delta_j \equiv \langle \hat{n}_j \hat{n}_{j+1} \rangle - \langle \hat{n}_j \rangle \langle \hat{n}_{j+1} \rangle \). We use this operator as proxy for non-gaussian correlations. We compare \( \Delta_j \) computed on the ground state obtained via DMRG and the one computed assuming the same state is gaussian in the operators \( \{ \hat{a}_j \} \}_{j=1}^J \), using Wick’s theorem. As shown in Appendix C the closer we are to the transition point \( s_c \), the more the state develops non-gaussian features at distances \( j \lesssim \xi \). On the contrary, deep in the localized phase the gaussian ansatz well captures the actual correlations in all sites. Indeed, in the large \( s \) limit the Hamiltonian turns to be diagonal in the number basis, namely \( H(s \gg 1) \approx \sum_j (\hat{n}_j \hat{n}_{j+1} + \hat{n}_j) \), whose ground state is \(| n_0 \rangle | 00 \ldots 0 \rangle \), which is a product state of gaussian states (excluding the 0-th site which fixes the symmetry sector).

The localized tail can be explained in a more intuitive way via the adiabatic theorem. Indeed, the Hamiltonian is gapped in the localized phase when \( U > 0 \); we can therefore adiabatically connect two ground states within it. In particular, we can link any localized ground state to the one at \( s = \infty \). This choice is particularly convenient since the Hamiltonian is diagonal in the number basis at \( s = \infty \), \( H(s \to \infty) = \sum_{j=1}^J (U \hat{n}_j \hat{n}_{j+1} + \hat{n}_j)/2 \), and its ground state at fixed symmetry sector is simply \(| n_0 \rangle \bigotimes_{j=1}^L | 0 \rangle_j \). Then, the evolution with the adiabatically changing Hamiltonian will dress the initial site with an exponentially localized tail. In Sec. IV we further exploit the adiabatic theorem to design the many-body version of a variety of states relevant in quantum information setups, such as coherent states, cat states and squeezed states.

**IV. LOCALIZED STATES ENGINEERING**

In the previous section we have discussed the localization properties of the ground state of the bosonic quantum system within the first non-vacant site. In this section we show that the ground states of different symmetry sectors are connected via bosonic creation and annihilation operators. We use this infinite set of localized states to construct many-body versions of cat, coherent and squeezed states relevant for quantum information purposes.

Starting with a given symmetry sector fixed by \( n_0 \), our aim is to find operators \( \mathcal{A} \) and \( \mathcal{A}^\dagger \) that obey the bosonic canonical commutation relations \([\mathcal{A}, \mathcal{A}^\dagger] = 1\), with the defining property

\[
(\mathcal{A}^\dagger)^{n_0} | 0 \rangle = \mathcal{N} | n_0 \rangle \otimes | \psi_0(n_0) \rangle := \mathcal{N} | \tilde{n}_0 \rangle,
\]

where \( \mathcal{N} \) is a constant. In other words, by acting \( n_0 \) times on the bosonic vacuum state with the operator \( \mathcal{A} \), we aim at retrieving the localized ground state of the Hamiltonian in Eq. (11) in the symmetry sector with \( n_0 \) particles on the first non-vacant site. From now on, we will refer to these operators as superbosonic creation and annihilation operators, since, in contrast to single site annihilation and creation operators, they act on a localized region of the system, by creating or destroying a bosonic localized tail along the chain. Likewise, we will refer to the localized ground states \(| \tilde{n}_0 \rangle \) as superbosons.

In order to find an explicit form for such operators we employ the adiabatic theorem. From numerical evidence our Hamiltonian is gapped within the whole localized phase. Therefore, there exists a slow tuning of \( s \) which enables to connect two localized ground states at fixed
values of $U$ and $n_0$. We consider such unitary transformation $\mathcal{U}(s, U)$ linking the ground state for $s = \infty$ with the target one at $s > s_c(U)$ in a fixed symmetry sector specified by the occupation $n_0$ of the first non-vacant site. We fix $s = \infty$ as our starting point since the Hamiltonian is diagonal in the number operator when $s \rightarrow \infty$ and its ground state is simply the tensor product $|n_0\rangle \otimes_{j \geq 1} |0\rangle_j$. By the adiabatic theorem the unitary operator takes the following form \cite{ST, SS}:

$$\mathcal{U}(s, U) = \mathcal{T} \exp \left[ -i \int_0^T dt H(s(t)) \right],$$

(11)

where $\mathcal{T}$ indicates the time-ordering operator and $s(t)$ is a function which interpolates from $s(t = 0) = \infty$ and $s(t = T) = s$. The function $s(t)$ has to be chosen such that it satisfies \cite{ST, SS}:

$$\frac{1}{\Delta(t)^2} \max_{n \neq 0} \left| \langle \Psi_n(t) | \dot{H}(t) | \Psi_0(t) \rangle \right| \ll 1,$$

(12)

at all times $t$. In Eq. (12), the state $|\Psi_n(t)\rangle$ is the $n$-th excited eigenstate of the Hamiltonian computed at time $t$; $\dot{H}(t)$ is the time derivative of Hamiltonian, which encodes the information about the specific protocol; finally, $\Delta(t) \equiv E_1(t) - E_0(t)$ is the gap at time $t$.

For $s(t)$ that satisfies Eq. (12), we obtain

$$\mathcal{U}(s, U)|n_0\rangle \otimes_{j=1}^L |0\rangle = e^{i\theta}|\tilde{n}_0\rangle,$$

(13)

where $\theta$ is a phase acquired during the adiabatic time evolution \cite{ST, SS}. Using $|n_0\rangle = \left( \hat{a}_0^{\dagger} \right)^{n_0} |0\rangle / \sqrt{n_0!}$ and $\mathcal{U}(s, U)|0\rangle \otimes \cdots |0\rangle = |0\rangle \otimes \cdots |0\rangle$ we obtain

$$\left( \mathcal{A}(s, U)^{\dagger} \right)^{n_0} |0\rangle = e^{i\theta} \sqrt{n_0!} |\tilde{n}_0\rangle,$$

(14)

where $|\tilde{0}\rangle \equiv |0\rangle \otimes \cdots |0\rangle$ and $\mathcal{A}(s, U)^{\dagger} = \mathcal{U}(s, U)\hat{a}_0^{\dagger}\mathcal{U}^\dagger(s, U)$. We can straightforwardly generalize Eq. (14) taking into account the position $j$ starting from which we want to embed the state $|\tilde{n}_0\rangle$. We define $\mathcal{A}_j(s, U)^{\dagger} = \mathcal{U}(s, U)\hat{a}_0^{\dagger}\mathcal{U}^\dagger(s, U)$, whose action $n_0$ times on the bosonic vacuum generates the state $|0\rangle \otimes_{j \leq j_0} |\tilde{n}_0\rangle$. Differently from the generic interacting case, the dressed operator $\mathcal{A}_j^{(1)}(s, U)$ acts non trivially in a region exponentially localized around $j$. The operator $\mathcal{A}_j(s, U)^{(1)}$ satisfies the bosonic commutation relations, since they are connected via a unitary transform to the bare bosonic operators $\hat{a}_j^{(1)}$. Therefore, they are bosonic operators. As anticipated, we called the operators $\mathcal{A}_j(s, U)^{(1)}$ superbosonic annihilation(creation) operators.

Since the transition point $s_c$ is essentially independent of the value of $n_0$ (see Appendix B) we can design a protocol that obeys the adiabatic theorem for any initial state $|n_0\rangle \otimes |0\rangle \otimes \cdots |0\rangle$. Furthermore, since these states belong to dynamically disconnected symmetry sectors, $\mathcal{H}_{k=0,n_0}$, for any values of $s$ and $U$, it is possible to adiabatically evolve them independently from each other. Therefore, any linear combination of initial states turns under the adiabatic protocol into

$$\mathcal{U}(s, U) \sum_{n_0} c_{n_0} |n_0\rangle \otimes |0\rangle \otimes \cdots |0\rangle \approx \sum_{n_0} c_{n_0} \left( \mathcal{A}(s, U)^{\dagger} \right)^{n_0} |\tilde{0}\rangle = \sum_{n_0} c_{n_0} e^{i\theta(n_0,s,U,T)} |\tilde{n}_0\rangle,$$

(15)

where $\theta(n_0, s, U, T)$ is the phase acquired during the adiabatic time evolution. As discussed in Appendix B deep in the localized phase the spectrum depends linearly on $n_0$, with small corrections. Since the phase acquired during the adiabatic evolution depends on the energy of the given state during the protocol, we have $\theta(n_0, s, U, T) \sim n_0 f(s, U, T)$ with $f(s, U, T)$ a function dependent on the specific protocol. This has important consequences for the state engineering we discuss in the following. As an example, let us consider as initial state of the adiabatic preparation the coherent state $|\alpha\rangle \equiv |\alpha\rangle_0 \otimes_{j \geq 1} |0\rangle_j$, where $|\alpha\rangle_0 = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} a^n / \sqrt{n!} |n\rangle_0$. Using Eq. (15) we
We can extend Eq. (16) to states of the form $M$ via the adiabatic time evolution or the application of an \textit{herent and squeezed states can be implemented either

\begin{equation}
U(s, U)|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} e^{i\theta(n, s, U, T)} |n\rangle
\end{equation}

\begin{equation}
= \sum_{n=0}^{\infty} e^{-|\alpha'|^2/2} \frac{\alpha'^n}{\sqrt{n!}} |n\rangle,
\end{equation}

with $\alpha' = \alpha e^{if(s, U, T)}$. In Fig. 6 we compute the overlap between $U(s(t), U)|\alpha\rangle$ and the \textit{uperbosons $|\bar{\alpha}(s(t), U)\rangle$}

for different values of $\alpha$ at the initial time $t = 0$ and at the final time $t = T$ of the adiabatic transformation. At the initial time we have $U(s(0), U)|\alpha\rangle = |\alpha\rangle$ and $|\bar{\alpha}(s(0), U)\rangle = |\alpha\rangle \otimes |00 \ldots 0\rangle$. At the final time we have $|\bar{\alpha}(s(T), U)\rangle = |\bar{\alpha}\rangle$. In Fig. 6 the overlaps are in very good agreement with (16), and we obtain the desired state in Eq. (16) with a fidelity $\approx 0.9994$ for $\alpha = 1.5$. We expect that when $\alpha$ is large, the fidelity achieved by the protocol becomes small, since corrections to the linear dependence of $\theta(n, s, U, T)$ from $n$ become important. Analogously, we perform the same analysis considering as initial state a cat-state at the site $j = 0$. Indeed, since the phase factor $e^{if(s, U, T)}$ do not depend on $\alpha$, given a cat state

$$|\mathcal{C}\rangle \bigotimes_{j>1} |0\rangle_j = \frac{1}{\mathcal{N}} \left( |\alpha\rangle_0 + e^{i\theta}|\alpha\rangle_0 \right) \bigotimes_{j>1} |0\rangle_j,$$

where $\mathcal{N}$ is a normalization constant, its dressed version is

$$|\mathcal{\bar{C}}\rangle = \frac{1}{\mathcal{N}} \left( |\bar{\alpha}\rangle + e^{i\phi}|\alpha\rangle - |\bar{\alpha}\rangle \right)$$

where $|\mathcal{C}\rangle = U(s, U)|\mathcal{C}\rangle$ and $\alpha' = \alpha e^{if(s, U, T)}$. We call $|\mathcal{\bar{C}}\rangle$ a \textit{uper-cat state.}

We can extend Eq. (16) to states of the form

$$|\psi\rangle = |00 \ldots 0\rangle \otimes \left( \sum_{n=0}^{\infty} \rho_n \beta^n |n\rangle_j \right) \otimes |00 \ldots 0\rangle,$$

where $\rho_n \in \mathbb{R}$ and $\beta, \theta \in \mathbb{C}$. Indeed, if we apply the adiabatic protocol to the state defined in Eq. (19), the phase acquired can be absorbed into $\beta$. Coherent states, cat states and squeezed states all fall into the class described in Eq. (19). In other words, using the adiabatic protocol not only we can engineer the many-body localized versions of states such as coherent and squeezed states, but we can do so preserving their single-mode properties.

For instance, the many-body localized version of coherent and squeezed states can be implemented either via the adiabatic time evolution or the application of an operator $M$ which is linear or quadratic in the superbosonic operators $\mathcal{A}$. The operator $M$ can be obtained applying the adiabatic protocol to its single-site counterpart, namely $M = U(s, U)MU(s, U)$. For instance, we define the dressed displacement operator,

$$D(\alpha) = e^{i(\alpha\mathcal{A} - \alpha^*\mathcal{A})},$$

with $\alpha \in \mathbb{C}$ the displacement parameter, and the dressed squeezed operator,

$$S(\xi) = \exp \left( \frac{1}{2} (\xi^* A^2 - h.c.) \right),$$

with $\xi \in \mathbb{C}$ the squeezing parameter, whose action on the vacuum creates a \textit{uper-coherent and super-squeezed state respectively. However, the most natural way to prepare such states is by starting from their single-mode version and then adiabatically turning on the off-diagonal term $\propto e^{-s}$ in the Hamiltonian. Note that these states are gaussian with respect to the \textit{uperbosonic operators $\mathcal{A}^\dagger$} and not with respect to the bare operators $\hat{a}^\dagger$. We call these states \textit{upergaussian.}

We have found that \textit{uperbosons $|\bar{\alpha}\rangle$, with different $n_0$ and same position $j$ of the first non-vacant site, are connected via the operators $\mathcal{A}^\dagger$. We have seen that their localized feature makes their energies approximately evenly spaced as a function of $n_0$ (cf. Appendix B). The evenly spaced energies of different ground states and the fact that the different ground states are connected via a bosonic operator $\mathcal{A}^\dagger(s, U)$, resemble the features of a quadratic Hamiltonian, such as the one-dimensional harmonic oscillator. Adding up these properties, the action of the interacting Hamiltonian $H(s, U)$ in Eq. (1) in the manifold of the ground states is approximately equivalent to a free theory in the \textit{uperbosonic operators $\mathcal{A}^\dagger(s, U)$, namely

$$H(s, U) \approx \sum_{j=-\infty}^{+\infty} \epsilon_0 \mathcal{A}^\dagger(s, U) \mathcal{A}^\dagger(s, U),$$

whose eigenstates are $\bigotimes_{j=0}^{+\infty} (\mathcal{A}^\dagger(s, U) \mathcal{A}^\dagger(s, U)) |0\rangle$, where $k_j \in [0, \infty)$. The effective Hamiltonian in Eq. (22) well captures the action of the full Hamiltonian in Eq. (1) on a \textit{uperboson $|\bar{\alpha}\rangle$ up to a certain $n$ parametrically large in $s$ and $U$, since corrections to the evenly spaced feature of the ground states energies become important as $n$ increases. Moreover, the effective Hamiltonian in Eq. (22) neglects the interaction between neighbour superbosons. Therefore, in the infinite set of eigenstates of Eq. (22), only those given by \textit{uperbosons separated by a large number of empty sites with respect to the typical localization length $\xi$ well approximate eigenstates of the original model (up to corrections exponentially small with the distance of two \textit{uperbosons}). For instance, the state $\mathcal{A}_1(s, U)^\dagger \mathcal{A}_2 \xi(s, U)^\dagger |0\rangle$, which describes two far localized excitations, is an eigenstate of the effective theory in Eq. (22) and, approximately, of the original Hamiltonian in Eq. (1). Instead, the state $\mathcal{A}_1(s, U)^\dagger \mathcal{A}_2(s, U)^\dagger |0\rangle$, which describes two nearly localized excitations, is an eigenstate of Eq. (22) with energy $2\epsilon_0$, while it is not an eigenstate of the original model Eq. (1), since we are neglecting the contribution coming from the interacting part of the Hamiltonian. Despite these limitations, the effective Hamiltonian in Eq. (22) well captures the equilibrium properties in the localized phase and the dynamical features of states such as the \textit{uper-cat state and
super-squeezed state when the interacting part between superbosons can be neglected. In this regard, the properties of the localized phase of quantum East models are reminiscent of the l-bits construction in MBL [58–61]. Let us consider a super-cat state $|\psi(t = 0)\rangle = |\tilde{C}\rangle$ defined in Eq. (18) as initial state. We evolve it and compute the fidelity

$$F(t) = |\langle \psi(t) | \psi(t = 0) \rangle|^2.$$  

(23)

As shown in Fig. 7, the fidelity displays almost perfect oscillations at short times, followed by a drop and almost perfect revivals. The short-time behaviour is compatible with a rotation of the super-cat state in the dressed phase-space $\tilde{X}_0 = (A_0 + A_0^\dagger)$ and $\tilde{P}_0 = -i(A_0 - A_0^\dagger)$, as expected from the effective Hamiltonian in Eq. (22). We can approximately compute the dynamics of the super-cat state $|\tilde{C}\rangle$ as

$$e^{-iH_1 t} |\tilde{C}\rangle \approx \frac{1}{N} |\tilde{a}(t)\rangle + e^{i\phi} |\tilde{a}(t)\rangle,$$  

(24)

where $\alpha(t) = \alpha(t = 0) e^{-i\delta t}$. The state in Eq. (24) is a rotating super-cat state in the dressed space. From Eq. (24) we can estimate the expected fidelity. In Fig. 7 we compare the expected value and the numerical results. The former matches the numerical results up to times parametrically large in $s$ and $1/\alpha$. On one hand non-linear corrections are suppressed the more the system is localized. On the other hand, corrections to the linear dependence of the energies $\langle \tilde{n}\rangle H [\tilde{n}]$ become important the larger $n$ is or, equivalently, $\alpha$, leading to dephasing processes [52]. The revivals can be explained considering non-linear effects; indeed, almost perfect revivals are observed for single-mode cat states with self-Kerr interaction [90] (for a circuit QED implementation see Ref. [91]).

We can extend these dynamical properties to any state prepared via the adiabatic protocol starting from a state of the form Eq. (19). Indeed, these states evolve analogously to the super-cat state under the effective quadratic theory defined in Eq. (22). The supergaussian states fall into this class. Once again, we highlight that these states are gaussian with respect to the superbosonic operators $A^{(1)}$ but not with respect to the bare operators $\tilde{a}^{(1)}$.

We have discussed the application of the adiabatic protocol to a single-site state embedded in the vacuum, however this directly extends to more general initial states. For instance, we could have started from a product state made of single-body states separated by a large number of empty sites, with respect to the localization length $\xi$, or from a superposition of those. At the end of the protocol, they will be dressed one independently from the others. Therefore, the final state will be made of many-body localized states concatenated one after the other.

V. NOISY DYNAMICS

In this section we investigate the dynamical properties of the many-body localized states introduced in Sec. IV when coupled to different noise sources. Here we study the effects of two different couplings with an external bath, namely a global stochastic field coupled to the local density noise, that commutes with the East symmetry, and a global incoherent pump, that breaks the East symmetry. The global stochastic field coupled to local densities is experimentally relevant in superconducting qubits setups [68], which are at the core of the experimental implementation we propose in Sec. VII. We provide numerical evidence that local information is erased very slowly, when the system is coupled to densities. We show how the characteristic time scales depend on the parameters of the Hamiltonian, the initial state and the coupling with the external bath. On the contrary, we show that the incoherent pump is highly disruptive and the time scales are dependent on the strength of the bath and the initial state, while the underlying coherent dynamics does not play a substantial role.

We consider the following Lindblad master equation

$$\dot{\rho} = -i[H, \rho] + \gamma \left( \tilde{O} \rho \tilde{O}^\dagger - \frac{1}{2} \left\{ \tilde{O}^\dagger \tilde{O}, \rho \right\} \right),$$  

(25)

where $\dot{\rho}$ is the state of the system, $H$ is the Hamiltonian in Eq. (1) with $\epsilon = 0$. $\tilde{O}$ is the quantum jump operator and $\gamma$ is the strength of noise. We consider only hermitian and local jump operators of the form $\tilde{O} = \sum_j \tilde{O}_j$. Since $\tilde{O}$ is hermitian, the Lindblad master equation Eq. (25) is equivalent to the stochastic Shrödinger equation (SSE) generated by the Hamiltonian [92].

$$\dot{H}_\eta(t) = \tilde{H} + \sum_j \eta_j(t) \tilde{O}_j,$$  

(26)
where the summation is over the whole system, $\eta_j(t)$ is a white noise acting on site $j$ with $\langle \eta_j(t) \rangle_{\eta} = 0$ and $\langle \eta_j(t_1) \eta_j(t_2) \rangle_{\eta} = \gamma \delta_{j,k} \delta(t_1 - t_2)$, where $\langle \cdot \rangle_{\eta}$ means averaging over the noise distribution and $\delta_{j,k}$ is the Kronecker-delta.

The dynamics of any observable $\hat{O}$ results from noise averaging

$$\langle \hat{O}(t) \rangle = \langle O_{\eta}(t) \rangle_{\eta}, \quad O_{\eta} = \text{tr} \left( \hat{O} \rho_{\eta} \right),$$

where $\hat{O}_{\eta}$ is the density matrix for a given noise realization $\{\eta_j(t)\}_{j=1}^\infty$. We simulate the dynamics of Eq. (26) using matrix product states (see Appendix B). We consider two different dephasing channels, namely $\hat{o}_j = \hat{n}_j$ and $\hat{o}_j = (\hat{a}_j + \hat{a}^\dagger_j) \equiv \hat{x}_j$. We refer to the former as the noisy-densities and to the latter as the incoherent pump. We choose such dephasing channels in order to investigate the effects of noise when it preserves the East-symmetry, as for the noisy-densities, or when it does not, as for the incoherent pump. The number of noise realizations we computed ranges from 500 to 1000 depending on the value of $\gamma$ and the kind of noise.

We study the dynamical properties of superbosons $|\tilde{n}\rangle$ defined in Eq. (10), since they constitute the building blocks of any localized state we can engineer. Then, we turn our attention to a paradigmatic superposition of superbosons, namely the super-cat state, providing arguments to extend our findings to a class of states to which super-squeezed and super-coherent states belong. To avoid computational difficulties in solving the Schrödinger equation for such states, we consider as initial state $|\psi_k(t = 0)\rangle = \bigotimes_{j=-\infty}^{k-1} |0\rangle_j \otimes |\tilde{n}\rangle$, where the subscript $k$ in $|\psi_k(t = 0)\rangle$ refers to the position of the first site of the embedded superboson. Since $|\tilde{n}\rangle$ is localized with localization length $\xi$ (cf. Eq. (29)), we can truncate its support to $L' \gg \xi$ sites. Thus, our initial state is

$$|\psi_k(t = 0)\rangle = |0\rangle^{k-1}_j \otimes |\tilde{n}\rangle_{L'} \otimes |0\rangle^{\infty}_{j=k+L'},$$

where $L'$ is the size of the superboson support.

In a generic non-integrable system we expect information about initial states encoded in local observables to be washed out fast. Here, we want to study how localization and slow dynamics instead protect the information encoded in local quantities. We compute the fidelity and the imbalance. The fidelity (cf. Eq. (28)) provides global information about the state and sets an upper bound on the time dependence of the expectation value of any local observable. Nonetheless, the fidelity is highly sensitive to any local perturbation of the state. Indeed, it is enough to have even a single occupied site far from the superbosons $|\tilde{n}\rangle$ to make Eq. (28) negligibly small. Among all the possible local observables, we want to investigate if the initial localized peak remains well resolved. We therefore compute the imbalance between the occupation of the initial peak and the second highest peak in the system, namely

$$I = \frac{n_k - \text{max}_{j \neq k} n_j}{n_k + \text{max}_{j \neq k} n_j},$$

where $k$ is the position of the first site of the embedded state (cf. Eq. (28)). The imbalance $I \in [-1, 1]$, and for $I > 0$ the initial peak is the largest one in the system. We monitor the imbalance with respect to its initial value, namely $I(t)/I(0)$.

When the system is coupled to the bath via the noisy-densities, the SSE respects the East-symmetry. Indeed, the jump operators commute with the operator in Eq. (3). Thus, the $n$ excitations on the first site of the superbosons $|\tilde{n}\rangle$ and the empty sites to its left are conserved. Furthermore, since the noise is not able to generate excitations out of the vacuum and the state is exponentially localized, we can keep only few empty sites to the left of $|\tilde{n}\rangle_{L'}$ without introducing relevant size-effects. For the set of parameters we choose, restricting the superboson support to $L' \approx 10$ sites and keeping only one empty site to its right turns out to be sufficient. Thus, our initial state is

$$|\psi(t = 0)\rangle = |\tilde{n}_0\rangle_{L'} \otimes |0\rangle.$$

In Fig. 8 we show the dynamics of the fidelity and the relative variation of the imbalance for different values of $s$ and noise strength $\gamma$ keeping $U = 1$, starting from the state in Eq. (30) with $n_0 = 1$. The imbalance displays an exponential decay $I(t) \sim I(0) e^{-t/\tau}$ with $\tau$ depending on the initial state, the parameters of the Hamiltonian and the coupling strength $\gamma$ with the external bath. The decay time $\tau$ increases the more the system is in the localized phase and the larger is the initial occupation $n_0$, while it decreases with the noise strength $\gamma$ as $\tau \propto 1/\gamma$. Therefore, the time decay $\tau$ can be enhanced either tuning the parameters of the Hamiltonian or embedding a
superboson with $n_0$ large (cf. Eq. [30]). On one hand, increasing $s$ or $U$ helps to protect the local memory at longer times, at the cost of making the initial state less entangled. Indeed, in the $s, U \to \infty$ limit the Hamiltonian tends to $\propto \sum_i (U n_i n_{i+1} + n_i)$, whose ground state is a product state of eigenstates of number operators. On the other hand, we can exploit the bosonic nature of the system and embed a superboson with a larger initial $n_0$, keeping $s$ small and enhancing the initial state entanglement. It is important to stress that despite the exponential feature of the decay, the time scale $\tau$ is generally very large with respect to the time scales of the coherent dynamics of the system. For instance, fixing $n_0 = 1$, $U = 1$ and $e^{-s} \approx 0.367$ ($s = 1$) the imbalance at time $10/\gamma$ is still $\approx 0.55 I(0) \approx 0.5$. From Eq. (29), and inspecting the late times average occupation number, the initial peak remains still well resolved, and so the information encoded within it.

The fidelity decays exponentially fast in time $F(t) \sim e^{-t/\tau'}$, with a decoherence time $\tau'$ dependent on the parameters of the Hamiltonian, the initial state and the strength of the noise. Analogously to the decay time $\tau$ of the imbalance, the decoherence time $\tau'$ increases the more the system is in the localized phase and decreases with the noise strength $\gamma$ as $\tau' \propto 1/\gamma$. Contrary to the imbalance, the fidelity drops faster the larger is $n_0$. Indeed, the conserved initial occupation $n_0$ pumps excitations on the next site, reducing the typical coherent timescales $\sim (\tau_0 e^{-s})$ and effectively enhancing the effects of the noise.

When the system is coupled to the bath via the incoherent pump, the SSE no longer preserves the East-symmetry. Indeed, the incoherent pump can create excitation from the vacuum. Starting from the state in Eq. (28), we consider a system of size $L = 17$, with an embedded initial superbosonic state $|\tilde{n}\rangle_{L'=9}$ surrounded by empty sites, namely

$$|\psi(t = 0)\rangle = |0\rangle_{\otimes_{i=1}^{L}} \otimes |\tilde{n}\rangle_{L'=9} \otimes |0\rangle.$$  

Since the noise can create excitations, the main source of numerical error is the finite cutoff $\Lambda$ of the local Fock space dimension. We fix a cutoff $\Lambda \geq 10$, and we check that it is large enough such that it does not introduce appreciable effects.

In Fig. 9 we show the dynamics of the fidelity and the relative variation of the imbalance for different values of $n_0$, keeping $U = 1$, $s = 1.5$ and $\gamma = 0.1$ fixed. The incoherent pump turns out to be detrimental for the initial state independently from the parameters of the Hamiltonian. Instead, the height of the initial peak plays a substantial role in enhancing the conservation of the imbalance. Up to the accessible times, the imbalance is well fitted by an exponential $I(t) \sim I(0) e^{-t/\tau} + I_\infty$ where $\tau \propto n_0/\gamma^s$, with $\zeta \approx 1$, and $I_\infty$ is the asymptotic value of the imbalance reached at time $t \to \infty$. Intuitively, the decay time $\tau$ increases with $n_0$ since the larger is $n_0$, more particles need to be pumped to make the peak no longer well resolved. The insensitivity of the time decay with respect to the parameters of the Hamiltonian indicates that the slow dynamics do not provide additional protection against this type of noise. Indeed, also in the $s \to \infty$ limit the initial state $|00\ldots0n_00\ldots\rangle$ is not a steady state of dynamics, and the imbalance would be highly affected by the external pump.

The fidelity drops to zero exponentially fast as expected, independently from the parameters of the Hamiltonian and the initial state. This is a consequence of the fact that the SSE does not respect the East-symmetry and that it can create excitations out of the vacuum, making the evolved state orthogonal to the initial state exponentially fast independently from the coherent dynamics and initial state.

Note that, we can immediately extend our analysis to a large variety of states. For instance, we can consider states given by the superposition of superbosons embedded in different region of the systems, namely

$$|\Psi\rangle \propto |\psi_k(t = 0)\rangle + e^{i\theta} |\psi_s(t = 0)\rangle,$$

where $|\psi_k(t = 0)\rangle$ is defined in Eq. (28), $\theta$ is a phase, and $|s - k| \gg \xi$. These two states are weakly coupled by the coherent and dissipative dynamics. In first approximation, we can apply our analysis to each of them separately, and therefore predict easily their dynamics.

The extension of these results to superposition of superbosons embedded in the same support (cf. Eq. [15]) is less trivial and depends on the specific noise. For instance, a noise which does not preserve the East symmetry makes the different states dynamically connected,
likely leading to different results from the ones observed for the single *superbosons*. On the other hand, also a noise which preserves the ‘East symmetry’ can lead to additional phenomena such as dephasing processes between the superimposed states. Indeed we observe that also the density noise is highly detrimental. We give further details in Sec. VI A, exploring the effects of dissipative impurities in the system.

A. Noisy impurities

We now investigate the effects of noisy impurities in the dynamical properties of a state given by the superposition of *superbosons* embedded in the same support. Among the possible choices, we consider a paradigmatic *supergaussian* state, namely the super-cat state, and then we generalize.

We consider local impurities coupled to the densities of the system (see for instance [94]). In case of impurities acting on a compact support $S$, the SSE in Eq. (26) turns into

$$\hat{H}_\eta(t) = \hat{H} + \sum_{j \in S} \eta_j(t) \hat{n}_j, \quad (33)$$

where $\eta_j(t)$ is again a gaussian white noise of variance $\gamma$.

We study the impact of the impurities as a function of the strength $\gamma$ and the extension of their support $S$. Since the impurities preserve the East-symmetry, we can once again focus on a few sites system without introducing relevant finite size effects. We initialize our system in the state

$$|\psi(t=0)\rangle = |\tilde{C}\rangle_L, \quad (34)$$

where $|\tilde{C}\rangle_L$ is a *super-cat* state (cf. Eq. (18)) with support $L$ and average number of particles $|\alpha|^2$. A support of $L = 15$ turns to be large enough for the parameters explored ($\alpha = 1.50$, $s = 1.5$ and $U = 1$). In Fig. 10 we show the dynamics of the fidelity as a function of the noise strength $\gamma$ and support $\mathbb{S}$. The super-cat state is still localized in space for any $\gamma$ and $\mathbb{S}$. Nonetheless, the coherence of the state is highly dependent on $\gamma$ and $\mathbb{S}$. Indeed, the noise is highly disruptive in an exponentially localized region around the peak, where the state is mostly located. Instead, if the noise acts on a region far from the localized peak it does not produce any appreciable effect. More precisely, we estimate that the typical time $\tau$ at which the embedded state is appreciably affected by the noise scales as $\tau \sim \min_{k \in S} 1/(\gamma \langle n_j \rangle) \sim \min_{k \in S} e^{i k \gamma} / \gamma$, where $k$ is the site where the peak is located. We numerically verify the polynomial dependence of $\tau$ on $\gamma$, obtaining $\tau \sim \gamma^{-0.8}$. On the contrary, it is not possible to extract from the times explored the dependence on the support $\mathbb{S}$ with high enough accuracy because of the slowness of the decay.

Our findings can be extended to dephasing channels which do not preserve the ‘East’ symmetry. For instance, an external incoherent pump acting far from the localized peak will lead to a drop of the fidelity exponentially fast analogously to what we observe for a global incoherent pump, but it will not affect local information encoded in the localized state. Furthermore, we expect that the observed dynamical properties can be easily extended to any state prepared via the adiabatic protocol from a state of the form Eq. (19), which *supergaussian* states belong to.

VI. CIRCUIT QED IMPLEMENTATION

In this section, we propose an experimental implementation of the Hamiltonian in Eq. (1) in terms of a simple superconducting circuit. We consider transmons which are quantized $LC$ oscillators with capacitance $C$ and with nonlinear inductance $L$ [78, 92]. This nonlinear dependence can be achieved via a Josephson junction working in the superconducting regime without introducing undesired dissipative effects [68, 95, 96]. Transmons are widely used in circuit QED platforms because of their insensitivity to charge noise; their anharmonicity allows to perform qubit operations with short-duration microwave pulses [78].

We consider an array of $L$ driven transmons coupled via an exchange interaction as our starting point. The Hamiltonian can be decomposed as a sum of three terms,
\[ H = H_0 + H_{\text{drive}} + V, \]

where

\[ H_0 = \sum_{j=1}^{L} \omega_j \hat{a}_j^{\dagger} \hat{a}_j - \frac{E_C}{2} \hat{a}_j^{\dagger} \hat{a}_j^{\dagger} \hat{a}_j \hat{a}_j, \]

\[ H_{\text{drive}} = \sum_{j=1}^{L-1} \hat{\Omega}_j \left( e^{-i\alpha_j \hat{a}_j^{\dagger} - \text{h.c.}} + e^{i\alpha_j \hat{a}_j + \text{h.c.}} \right), \]

\[ V = \sum_{j=1}^{L-1} i g \left( \hat{a}_j \hat{a}_j^{\dagger} + \text{h.c.} \right), \]

where \( \hat{a}_j^{\dagger} (\hat{a}_j) \) creates (destroys) an excitation in the \( j \)-th transmon; \( H_0 \) is the bare Hamiltonian of the transmons with qubit frequencies \( \{\omega_j\}_{j=1}^{L} \), and anharmonicity \( E_C > 0 \). \( H_{\text{drive}} \) describes the action of classical drive fields on the bare transmons; \( V \) describes hopping processes that can be engineered by a common bus resonator or direct capacitance. In Fig. 1(a) is shown an illustration of the scheme of Eq. (35).

We work in the weak coupling regime \( g \ll |\omega_j - \omega_{j+1}| \) and in the small anharmonicity limit \( E_C \ll |\omega_j - \omega_{j+1}| \) for all \( j \). The former condition is necessary in order to have far detuned processes connected by \( V \), and therefore to treat \( V \) in perturbation theory. The small anharmonicity limit is necessary to retrieve a bosonic model in the effective perturbative Hamiltonian obtained after treating \( V \) with a Schrieffer-Wolff transformation in the small \( g \) limit. Each transmon \( j \in [1, L-1] \) is driven by a classical drive field of amplitude \( \Omega_j \) and frequency \( \alpha_j \). These classical drive fields give rise to the desired interaction together with undesired single-site fields in the low-energy effective Hamiltonian. In order to get rid of them, we add another drive field on each transmon \( j \in [2, L] \) of amplitude \( \epsilon_j \) and frequency \( \alpha_{j-1} \).

We are interested in exploiting the multilevel (bosonic) structure of transmons. We do not reduce each component of the system to a qubit. We therefore introduce the ladder operators

\[ \hat{a}_j = \sum_{\ell=0}^{\infty} \sqrt{\ell+1} |\ell, j\rangle \langle \ell + 1, j| \equiv \sum_{\ell=0}^{\infty} \hat{e}_{\ell, j}, \]

where \( \hat{e}_{\ell, j} \) is the ladder operator which destroys an excitation in the \( (\ell + 1) \)-th level and create an excitation in the \( \ell \)-th level on the \( j \)-th transmon. Analogously, we can define its hermitian conjugate \( \hat{e}_{\ell, j}^{\dagger} \).

In order to find the explicit form of the SW transformation, we follow the prescription in Ref. [103], first we compute \( \eta = [H_0, V] \); we consider \( \eta \) with arbitrary coefficients as an ansatz for \( S \); finally, we fix these coefficients imposing the condition \( [S, H_0] = -V \). We obtain (cf. Appendix E.1)

\[ S = -\sum_{\ell=0}^{L-1} \sum_{s=0}^{\infty} \Delta_{\ell+1, j}^{(s)} \left( \hat{e}_{\ell, s}^{\dagger} \hat{e}_{\ell+1, j} + \hat{e}_{\ell+1, j}^{\dagger} \hat{e}_{\ell, s} \right), \]

where \( \Delta_{\ell, j}^{(s)} = (\omega_j - E_C \ell) \), the first summation is along the system, while the second summation is along all the levels of the transmons. Using the Baker-Campbell-Hausdorff expansion, the Hamiltonian in Eq. (36) after the SW transformation reads

\[ \hat{H} \equiv e^S \hat{H} e^{-S} \approx H_0 + H_{\text{drive}} + [S, H_{\text{drive}}] + \frac{1}{2}[S, V] + O \left( \frac{g^2 \Omega}{\Delta^2} \right). \]

After lengthy yet standard calculations, we obtain \( \hat{H} \) explicitly dependent on the ladder operators \( \hat{e}_{\ell, j}^{(s)} \) introduced in Eq. (36) and with coefficients dependent on the site and internal levels (see Appendix E.2). Our aim is to write \( \hat{H} \) as a function of bosonic operators \( \hat{a}_j^{(s)} \). We need to find a regime where the coefficients in \( \hat{H} \) are approximately independent of the specific level so we can use Eq. (36). These coefficients are similar to the one appearing in Eq. (37). In order to make them level-independent, we need

\[ \Delta_{\ell+1, j} - \Delta_{s, j} \approx \omega_{j+1} - \omega_j \equiv \Delta_{j+1, j}, \]

which holds if \( (\ell - s) \ll \Delta_{j+1, j}/E_C \). Since the transmons can have an infinite number of excitations we have \( (\ell - s) \in (-\infty, +\infty) \). This means that Eq. (39) cannot be satisfied for all possible \( \ell \) and \( s \) if \( E_C \neq 0 \). Nonetheless, it can be achieved up to a certain value \( N \) of \( \ell \) and \( s \), such that \( N \ll \Delta_{j+1, j}/E_C \). Therefore, the coefficients in \( \hat{H} \) satisfy Eq. (39) up to the \( N \)-th energy level, leading to a bosonic Hamiltonian which well approximate the dynamics of states with occupation small with respect to \( N \) (cf. Appendix E.3). The bosonic \( \hat{H} \) still displays
We fix the qubit frequencies currently available in modern experiments (see Ref. [68]).

\[\omega_j \approx \omega_j - \alpha_j \geq 0\]

for \(j > 1\), necessary in order to have localization; (v) \(1/T_{1,2}\) small with respect to the typical energies in the effective Hamiltonian in Eq. (41).

The more stringent conditions are given by (ii) and (v). In Fig. 11 we estimate the coefficients of the non-linear terms in Eq. (41) as a function of the detuning of the qubit frequencies of two successive transmons \(\Delta_{j,j+1}\). A good trade-off between (ii) and (v) is obtained at \(\Delta_{j,j+1} \gg 3E_C \approx 1GHz\), for which the typical time scale of the kinetic constraint term is \(\approx T_{1,2}/10\).

We have \(g/\Delta_{j,j+1} \approx 10^{-2}\), meaning that (i) is reasonably satisfied. Condition (iii) and (iv) are satisfied by a staggered configuration of the drive fields and qubit frequencies, namely: \(\alpha_{j+1} = \alpha_j + (-1)^j\delta\) with \(\delta \gg \Omega_j\) and \(\alpha_j \gg \Omega_1\); \(\omega_{j+1} = \omega_j + (-1)^j \Delta\), with boundary condition \(\omega_1 > \alpha_1\). For instance, we can consider \(\alpha_1 = 1GHz\); \(\delta = 500MHz\); \(\omega_1 = 5GHz\). These conditions lead to Eq. (41) to be translationally invariant. Moreover, condition (vi) is satisfied for these set of parameters.

We summarize a possible set of parameters available in state-of-the-art transmons for implementing the bosonic quantum East model.

**VII. PERSPECTIVES**

The implementation of a kinetically constrained East model using transmons represents a bridge between the two communities of circuit QED and non-ergodic quantum dynamics. It has the potential to attract the former towards fundamental questions regarding dynamical phase transitions, and to stimulate the latter towards the search for quantum information and metrological applications of constrained dynamics. Our explicit construction of many-body analogs of squeezed and cat states relies on the East constraint, represents a first step towards this direction.

A fruitful prosecution of this work is the study of an analog of the mobility edge separating localized from delocalized states in the spectrum of East models (for the mobility edge in MBL see Refs. [18] [19]). Understanding how such mobility edge scales with \(\Lambda\), is essential for predicting the onset of dynamical transitions in platforms with uni-directional constraints, as well as of practical interest. For instance, a mobility edge at finite energy density is a feature of direct relevance for experimental real-
izations, since it would yield the conditions for performing efficient quantum manipulations deep in the localized phase when finite temperature or heating effects are present. A related interesting question is the survival of the effective integrable description of the localized phase discussed in Sec. IV upon increasing the density of energy above the ground state. This would have implications for heat and particle transport features of the East model in the non-ergodic phase, which would be governed by the effective integrable description in [22, as it happens for MBL systems [106].

The insensitivity to noise acting away from localized peaks could open the path to study the protection of spatially separated macroscopic superpositions of superbosonic states. Given the slow decay of localized wavepackets in the presence of noise, one could conceive the storage and noise-resilience of long-lived many-body entangled states in far-away regions, with applications to quantum communication.

To conclude, we observe that the implementation discussed in Sec. IV may be easily adapted to retain kinetic terms both with East and West symmetries. This could for instance lead to the formation of localized modes at the edges of the wire, with exciting perspectives for novel forms of topological states in kinetically constrained models realizable with circuit QED. We are currently focusing our research efforts in this direction.

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Appendix A: Scaling analysis in $\Lambda$

In the main text we have shown that the bosonic system displays a delocalized-localized transition at finite $s$ if $U > 0$. Here we show that the ground state not only is localized but it is weakly dependent on the physical cut-off $\Lambda$. This provides a quantitative proof that we can investigate the bosonic system with a finite $\Lambda$ in the localized phase.

We fix the symmetry sector $n_0$ and $(s > s_c(U), U > 0)$ in the localized phase. We compute $|\psi_0(\Lambda)|$ for different values of $\Lambda$. We calculate $1 - |\langle \psi_0(\Lambda) | \psi_0(\Lambda + 1) \rangle|^2$ as a function of $\Lambda$ (see Fig. 12). The fidelity $|\langle \psi_0(\Lambda) | \psi_0(\Lambda + 1) \rangle|^2$ approaches 1 exponentially fast in $\Lambda$. The more the system is in the localized phase and $n_0$ is small, the faster the convergence is. This gives a first evidence that the ground state of the actual bosonic system is well described with small effective cutoffs.

We compute the variance of the Hamiltonian $H$ over the ground state $|n_0\rangle \otimes |\psi_0(\Lambda)\rangle$, taking into account the bosonic nature of the original Hamiltonian $H$. This quantity is exactly zero if the state $|n_0\rangle \otimes |\psi_0(\Lambda)\rangle$ is an eigenstate of $H$. We aim to see how this quantity goes to zero as a function of $\Lambda$. In order to do so, we write the Hamiltonian $H$ as the sum of two terms $H = H_- + H_+$. $H_-$ acts on the Hilbert space spanned by states with an occupation number up to $\Lambda$, while $H_+$ acts on the

\[ 1 - |\langle \psi_0(\Lambda) | \psi_0(\Lambda + 1) \rangle|^2 \]

\[ \text{FIG. 12. Scaling analysis of } 1 - |\langle \psi_0(\Lambda) | \psi_0(\Lambda + 1) \rangle|^2 \text{ as a function of } \Lambda \text{ at fixed } U = 0.1 \text{ and } s = \{1.2, 1.5\} \text{ for different values of } n_0 \in [1, 30]. \text{ The dots and squares refers to the numerical results obtained at } s = 1.2 \text{ and } s = 1.5 \text{ respectively. The overlap tends exponentially fast to 1 in } \Lambda. \text{ The decay is slower as } n_0 \text{ increases at fixed } s \text{ and } U. \]
Hilbert space spanned by states with an occupation number greater than \( \Lambda \). We label the sector on which \( H_{\pm} \) acts non-trivially as the \( H_{\pm} \) sectors, respectively. We apply the same procedure to the number operator and annihilation (creation) operator:

\[
\hat{n} = \sum_{k=0}^{\Lambda} k |k\rangle \langle k| + \sum_{k=\Lambda+1}^{\infty} k |k\rangle \langle k| \\
= \hat{n}_- + \hat{n}_+ , \tag{A1}
\]

\[
\hat{a} = \sum_{k=0}^{\Lambda} \sqrt{k} |k-1\rangle \langle k| + \sum_{k=\Lambda+1}^{\infty} \sqrt{k} |k-1\rangle \langle k| \\
= \hat{a}_- + \hat{a}_+ .
\]

The commutator \([\hat{n}_-, \hat{n}_+] = 0\), while \([\hat{a}_-, \hat{a}_+] = \sqrt{\Lambda(\Lambda + 1)}|\Lambda - 1\rangle\langle \Lambda + 1| \neq 0\). This is because the operators \( \hat{a}_\pm \) connect the two sector \( H_{\pm} \). From \( \text{(A1)} \) we straightforwardly obtain the expressions of \( H_{\pm} \):

\[
H_{\pm} = -\frac{1}{2} \sum_i \hat{n}_{i,\pm} \left[ e^{-s} \left( \hat{a}_{i+1,\pm}^\dagger \hat{a}_{i+1,\pm} + \hat{a}_{i+1,\pm} \hat{a}_{i,\pm} \right) \\
- U \hat{n}_{i+1,\pm} - 1 \right]. \tag{A2}
\]

In our numerical scheme we fix a finite cut-off \( \Lambda \). Therefore we are computing the ground state \( |\psi_0(\Lambda)\rangle \) of \( H_- \). Since \( \hat{a}_\pm \) are non-commuting operators, the two Hamiltonians \( H_- \) and \( H_+ \) do not commute as well. Therefore, it is not ensured that \( |\psi_0(\Lambda)\rangle \) is an eigenstate of the full Hamiltonian \( H \). We compute the variance \( \Delta H \) over \( |\psi_0(\Lambda)\rangle \) of the Hamiltonian \( H = H_- + H_+ \):

\[
\Delta H = \langle H_+ H_+ \rangle + \langle H_+ H_- \rangle + \langle H_- H_- \rangle - \langle H \rangle^2 , \tag{A3}
\]

to check if \( |\psi_0(\Lambda)\rangle \) is an eigenstate of \( H \). The terms in \( H_{\pm} \) which preserve the sectors \( H_{\pm} \) give a zero contribution in \( \langle A3 \rangle \). Indeed, the ones that keep the system in the \( H_- \) sector give a zero contribution since \( |\psi_0(\Lambda)\rangle \) is an eigenstate within this sector by definition. Instead, the ones that keep the system in the \( H_+ \) sector give trivially zero since we do not have any occupation larger than \( \Lambda \). The only contribution comes from the operators \( \hat{a}_{\pm}^{(1)} \), or more precisely the term \( \sqrt{\Lambda + 1}|\Lambda + 1\rangle\langle \Lambda + 1| \) which connects the two sectors. Using \( \text{(A2)} \) we straightforwardly obtain

\[
\Delta H = \Lambda e^{-2s} \sum_{j=0}^{L-1} \langle n_j^2 \rangle \langle P_{j+1,\Lambda} \rangle , \tag{A4}
\]

where \( P_{j,k} = |k\rangle_j \langle j| \) is the projector on the Fock state with occupation \( k \) on site \( j \). The first term of the sum \( (j = 0) \) encodes the information about the fixed symmetry sector since \( \langle \hat{n}_j^2 \rangle = n_0^2 \). The variance \( \langle A4 \rangle \) depends on the mean occupation number and on the projector over the Fock space on \( \Lambda \). In the main text we have shown that the system displays a localized phase in the bosonic limit, \( \Lambda \to \infty \), if \( U > 0 \). This enables us to estimate Eq. \( \langle A4 \rangle \) in the localized phase.

In the localized phase the average occupation number of the ground state is \( \langle \hat{n}_j \rangle \sim e^{-j/\xi} \) (cf. Eq. \( \text{(9)} \)). The exponential decay of the occupation number along the chain reflects on the behaviour of the expectation value of \( P_{j,k} \), which decays exponentially fast in \( k \) (cf. Eq. \( \text{(7)} \)). Therefore, the series \( \langle A4 \rangle \) is finite for \( \Lambda \to \infty \) and \( L \to \infty \) since each term is exponentially suppressed. In Fig. \text{13} we numerically compute the variance \( \Delta H \) over \( |\psi_0(n_0,\Lambda)\rangle \) for different values of \( \Lambda \) and \( n_0 \). Rigorously, the cut-off \( \Lambda \) limits the accessible \( n_0 \) since \( \langle \hat{n}_k \rangle \leq \Lambda \). Nevertheless, because \( n_0 \) appears as a constant in the Hamiltonian, we can compute the ground state \( |\psi_0(n_0,\Lambda)\rangle \) also for \( n_0 > \Lambda \). The numerical results perfectly match \( \langle A4 \rangle \). The variance goes exponentially fast to zero. Therefore, an eigenstate of \( H_- \) is an eigenstate of the fully bosonic system as well with a reasonably small cut-off \( \Lambda \) when \( U > 0 \).

**Appendix B: Properties of the localized ground state upon changing \( n_0 \)**

In this appendix we discuss the properties of the ground state upon changing the symmetry sector specified by the occupation \( n_0 \) of the first not-empty site. We show that the transition point and the exponential decaying tail of the ground states occupation are weakly dependent on \( n_0 \). We discuss the dependence of the ground state energy on \( n_0 \), which is relevant in the state preparation via the adiabatic protocol discussed in Sec. \text{IV}.

We perform the same scaling analysis as a function of the cut-off \( \Lambda \) discussed in Sec. \text{III} (see Fig. \text{13}). We extract the transition point \( s_c \) for different values of \( n_0 \) from the inverse of the localization length \( \xi \). The existence of a finite critical point \( s_c \) in the \( \Lambda \to \infty \) limit turns to be
weakly dependent on the specific symmetry sector $n_0$ at fixed $U$. We investigate the dependence of the localized tail of the ground state $|\psi_0(n_0)\rangle$ as a function of $n_0$ (we exclude the first site which fixes the symmetry). To this end we compute $|\langle\psi_0(n_0 = 1)|\psi_0(n_0)\rangle|^2$ with $n_0 \geq 1$ (see Fig. 15). We fix $n_0 = 1$ as a reference since we want to see if the tail is weakly dependent on $n_0$ or not. All the ground states are computed by fixing $\Lambda = 30$. The overlap $|\langle\psi_0(n_0)|\psi_0(n_0 = 1)\rangle|$ strongly depends on $s$ and $U$. Indeed, the more the system is in the localized phase, the more the exponentially localized tail is weakly dependent on $n_0$. Therefore, deep in the localized phase $|\psi_0(n_0)\rangle$ is approximately independent on the specific sector $n_0$ and we can write

$$|\tilde{n}_0\rangle \equiv |n_0\rangle \otimes |\psi_0(n_0)\rangle \approx |n_0\rangle \otimes |\psi_0\rangle,$$  

(B1)

where $|\psi_0\rangle$ is explicitly independent from $n_0$.

The weak dependence of $|\psi_0(n_0)\rangle$ with respect to $n_0$ has consequences on the ground states energies. Indeed, the expectation value of the Hamiltonian on $|\tilde{n}_0\rangle$ is

$$E_0(n_0) \equiv \langle\tilde{n}_0|\hat{H}|\tilde{n}_0\rangle \approx \frac{1}{2} n_0 + \mathcal{O}(n_0 e^{-1/\xi(n_0)}),$$  

(B2)

where $\langle\hat{n}_j\rangle \sim e^{-j/\xi(n_0)}$ since we are in the localized phase. In Fig. 16 we give a numerical evidence of (B2).

**Appendix C: Gaussianity and non-gaussianity in the ground state**

In Fig. 17 we show the correlator $\Delta_j = \langle\hat{n}_j\hat{n}_{j+1}\rangle - \langle\hat{n}_j\rangle\langle\hat{n}_{j+1}\rangle$ as a function of $j$ for different values of $s$ at fixed $U = 1$. We compare $\Delta_j$ computed on the ground state obtained via DMRG and the one computed assuming the same state is gaussian in the operators $\{\hat{n}_j^{(1)}\}_{j=1}^L$. 
which we name $\Delta^G$.

**Appendix D: Numerical methods**

In this appendix we provide the details about the scheme adopted for simulating the SSE [26]. We implement [26] numerically via tensor networks. An infinitesimal time step $\delta t$ is performed via ‘Trotterizing’ the evolution operator: $U_{t+\delta t,t}^{\xi} = e^{-iH\delta t/2} e^{-i\sum_j \delta W_j \hat{\sigma}_j} e^{-iH\delta t/2}$ with a time step $\delta t \ll \{e^s, U^{-1}, 1\}$. By $\delta W_j$ we denote a Wiener process with $\langle \delta W_j \delta W_j \rangle = \gamma \delta \omega \delta t$. The time evolution $e^{-iH\delta t/2}$ is obtained by employing the Time-Evolving Block-Decimation (TEBD) algorithm with second-order Suzuki-Trotter decomposition. The noise term can be straightforwardly applied since it is a one-gate operator $e^{-i\sum_j \delta W_j \hat{\sigma}_j} = \prod_{j=1}^L e^{-i\delta W_j \hat{\sigma}_j}$. We fix $\delta t = 5 \times 10^{-3}$, a maximal bond dimension $\chi_{\max} = 75$ and we keep the singular values greater than $10^{-10}$. We verified that the results are not affected by the time step $\delta t$ and $\chi_{\max}$. All the simulations were performed using the ITensor library [107].

**Appendix E: Details about Circuit QED implementation**

1. Perturbative construction of the generator $S$ of the Schrieffer-Wolff transformation

We write the Hamiltonian $H_0$ and the perturbation $V$ as a function of the operators $\hat{c}^{(\dagger)}_{\ell,j}$ defined in Eq. (36)

$$H_0 = \sum_{j=1}^L \sum_{\ell=0}^\infty \omega_{\ell,j} |\ell,j\rangle \langle \ell,j| \equiv \sum_{j=1}^L \sum_{\ell=0}^\infty \omega_{\ell,j} p_{\ell,j}, \quad V = ig \sum_{j=1}^{L-1} \sum_{\ell,s=0}^\infty \left( c^{\dagger}_{\ell,j} c_{s,j+1} + h.c \right),$$

(E1)

where $\omega_{\ell,j} = (\omega_j + E_C/2) j - E_C j^2/2$ and we introduce $p_{\ell,j} \equiv |\ell,j\rangle \langle \ell,j|$ for convenience. We compute the generator $\eta = [H_0, V]$

$$\eta = -\sum_{j=1}^L \sum_{\ell,s=0}^\infty ig \left( \hat{\Delta}_{\ell,j+1} - \hat{\Delta}_{s,j} \right) \left( c^{\dagger}_{s,j} c_{s,j+1} + c^{\dagger}_{s,j} c_{s,j+1} \right),$$

(E2)

where $\Delta_{\ell,j} = \omega_{\ell+1,j} - \omega_{\ell,j} = (\omega_j - E_C \ell)$. Following [104] the ansatz for the generator $S$ of the SW the operator is

$$S = \sum_{j=1}^L \sum_{\ell,s} A_{j,\ell,s} \left( c^{\dagger}_{s,j} c_{\ell,j+1} + c^{\dagger}_{s,j} c_{\ell,j+1} \right).$$

(E3)

We compute $[S, H_0]$ and we impose $[S, H_0] = -V$. This condition is satisfied if we fix $A_{j,\ell,s} = -ig / \left( \hat{\Delta}_{\ell,j} - \hat{\Delta}_{s,j} \right)$. Therefore $S = \sum_{j=1}^{L-1} S_{j,j+1}$ with $S_{j,j+1} \equiv -ig / \left( \hat{\Delta}_{\ell,j} - \hat{\Delta}_{s,j} \right) \left( c^{\dagger}_{s,j} c_{\ell,j+1} + c^{\dagger}_{s,j} c_{\ell,j+1} \right)$.

\[\begin{align*}
\langle \hat{n}_j \hat{n}_{j+1} \rangle - \langle \hat{n}_j \rangle \langle \hat{n}_{j+1} \rangle
\end{align*}\]

FIG. 17. The points correspond to the quantity computed on the ground state obtained via DMRG; the continuous lines are the results obtained assuming that the state is gaussian. We fix $U = 1, n_0 = 1$ and $\Lambda = 15$. 

\[\begin{align*}
\langle \hat{n}_j \rangle &\quad s = 0.02 \\
\langle \hat{n}_j \rangle &\quad s = 0.05 \\
\langle \hat{n}_j \rangle &\quad s = 0.10 \\
\langle \hat{n}_j \rangle &\quad s = 0.20
\end{align*}\]
2. Commutator of the Hamiltonian with the generator $S$ of the Schrieffer-Wolff transformation

We write the perturbation $V = \sum_{j=1}^{L-1} V_{j,j+1}$ where $V_{j,j+1} = g \sum_{\ell,s=0}^{\infty} \left( c_{\ell,j}^t c_{s,j+1} + h.c. \right)$. We compute the commutators $[S_{j-1,j}, V_{j,j+1}]$, $[S_{j,j+1}, V_{j,j+1}]$ and $[S_{j-1,j}, V_{j,j+1}]$

$$[S_{j,j+1}, V_{j,j+1}] = \sum_{\ell,s} \left( \Delta_{\ell+1,j+1} - \Delta_{s,j} \right) \left( \Delta_{\ell,j+1} - \Delta_{s-1,j} \right) \left( c_{s,j} c_{s+1,j} c_{t+1,j+1} c_{t,j+1} + \sum_{\ell,s} \Delta_{\ell-1,j+1} - \Delta_{s,j} \right) \left( \ell p_{s,j} p_{t,j+1} + \sum_{\ell,s} \Delta_{\ell,j+1} - \Delta_{s,j} \right) s p_{s,j} p_{t,j+1} + h.c.,$$

$$[S_{j,j+1}, V_{j-1,j}] = -\sum_{\ell,s,q} \left( \Delta_{\ell,j+1} - \Delta_{s,j} \right) \left( \Delta_{\ell,j+1} - \Delta_{s-1,j} \right) \ell c_{s,j-1} p_{t,j} c_{t,j+1}^t + \sum_{\ell,s,q} \Delta_{\ell,j+1} - \Delta_{s,j} \ell c_{s,j-1} p_{t,j} c_{t,j+1}^t +$$

$$\sum_{\ell,s,k} \left( \Delta_{\ell,j+1} - \Delta_{s,j} \right) \left( \Delta_{\ell+1,j+1} - \Delta_{s,j} \right) c_{s,j-1} c_{s+1,j} c_{t+1,j+1} + h.c.,$$

(E4)

which constitute the building blocks for computing $[S, V]$. We consider a drive field acting on site $j$, $H_{\text{drive},j} = \Omega_j \left( e^{i\alpha_j t} a_j + h.c. \right)$. We compute the commutator $[S, H_{\text{drive},j}] = [S_{j-1,j}, H_{\text{drive},j}] + [S_{j,j+1}, H_{\text{drive},j}]$

$$[S_{j-1,j}, H_{\text{drive},j}] = \sum_{\ell,s} \frac{i g \Omega_j E_C}{\left( \Delta_{\ell-1,j} - \Delta_{s,j-1} \right) \left( \Delta_{\ell,j} - \Delta_{s,j} \right)} e^{i \alpha_j t} c_{s,j-1} p_{t,j} + \sum_{\ell,s} \frac{i g \Omega_j}{\Delta_{\ell,j} - \Delta_{s,j-1}} e^{i \alpha_j t} c_{s,j-1} p_{t,j} +$$

$$\sum_{\ell,s} \frac{i g \Omega_j}{\Delta_{\ell+1,j} - \Delta_{s,j-1}} e^{-i \alpha_j t} c_{s,j-1} c_{t+1,j}^t + h.c.,$$

$$[S_{j,j+1}, H_{\text{drive},j}] = \sum_{\ell,s} \frac{i g \Omega_j E_C}{\left( \Delta_{\ell,j+1} - \Delta_{s,j} \right) \left( \Delta_{\ell,j+1} - \Delta_{s+1,j} \right)} e^{-i \alpha_j t} s p_{s,j} c_{t,j+1}^t - \sum_{\ell,s} \frac{i g \Omega_j}{\Delta_{\ell,j+1} - \Delta_{s,j}} e^{-i \alpha_j t} p_{s,j} c_{t,j+1}^t + h.c.,$$

$$\sum_{\ell,s} \frac{i g \Omega_j E_C}{\left( \Delta_{\ell,j+1} - \Delta_{s,j} \right) \left( \Delta_{\ell,j+1} - \Delta_{s+1,j} \right)} e^{i \alpha_j t} c_{s,j} c_{s+1,j} c_{t,j+1} + h.c.,$$

(E5)

where $\Omega_j = \{ e^{-i \pi/2} \Omega_j, \epsilon_{j+1} \}$.

3. Small anharmonicity limit

In the following we explicitly consider the results with $L = 3$ transmons for clarity. The generalization to a larger number of transmons is straightforward. We work in the limit $E_C \ll \Delta_{ij}$, such that $\Delta_{\ell,j+1} - \Delta_{s,j} = \Delta_{j+1} - \Delta_{s,j} = \Delta_{\ell,j+1} - \Delta_{s,j} = \Delta_{\ell} - \Delta_{s,j}$. We neglect the contributions coming from the commutator of the drive fields controlled by $\{ \epsilon_j \}$, since, as we will show, they give sub-leading corrections. From (E4) and (E5) and using the identities $\sum_{\ell=0}^{\infty} c_{\ell,j} = a_j$, $\sum_{\ell=0}^{\infty} p_{\ell,j} = n_j$
and \( \sum_{l=0}^{\infty} p_{l,j} = 1 \), we obtain

\[
[S, V] \approx -2g^2E_C \frac{\Delta_{12}^2}{\Delta_{12}^2} a_1a_1^\dagger a_2^\dagger + 2g^2E_C \frac{\Delta_{12}^2}{\Delta_{12}^2} n_1 n_2 - \frac{g^2}{\Delta_{12}^2} n_1 - \frac{g^2}{\Delta_{12}^2} n_2 - \frac{g^2}{\Delta_{12}^2} n_1 n_2 a_3^\dagger - \frac{g^2}{\Delta_{12}^2} n_1 a_3^\dagger - \frac{g^2}{\Delta_{12}^2} n_2 a_3^\dagger a_2^\dagger + \frac{g^2}{\Delta_{12}^2} n_1 n_2 a_3^\dagger + \frac{g^2}{\Delta_{12}^2} a_1 a_3^\dagger + \frac{g^2}{\Delta_{12}^2} a_1 a_2 a_3^\dagger + \frac{g^2}{\Delta_{12}^2} a_2 a_2 a_3^\dagger + \frac{g^2}{\Delta_{12}^2} n_1 a_3^\dagger a_2^\dagger a_3^\dagger + \frac{g^2}{\Delta_{12}^2} n_2 a_3^\dagger a_2^\dagger a_3^\dagger - \frac{g^2}{\Delta_{12}^2} n_2 a_3^\dagger + \frac{g^2}{\Delta_{12}^2} n_3 + h.c.,
\]

\[
[S, H_{\text{drive}}] \approx -\frac{g^2E_C\Omega_{1}}{\Delta_{12}^2} \left\{ n_1 \left( e^{-i\alpha t} a_2^\dagger + h.c. \right) - \left( e^{i\alpha t} a_1 a_1^\dagger + h.c. \right) \right\} - \frac{g^2E_C\Omega_{2}}{\Delta_{12}^2} \left\{ n_2 \left( e^{-i\alpha t} a_1^\dagger + h.c. \right) + \left( e^{i\alpha t} a_2 a_2^\dagger + h.c. \right) \right\} - \frac{g^2E_C\Omega_{3}}{\Delta_{23}^2} \left\{ n_2 \left( e^{-i\alpha t} a_3^\dagger + h.c. \right) - \left( e^{i\alpha t} a_2 a_3^\dagger + h.c. \right) \right\} - \frac{g^2E_C\Omega_{4}}{\Delta_{23}^2} \left( e^{-i\alpha t} a_3^\dagger + h.c. \right).
\]

\((E6)\)

### 4. Rotating frame of reference

We focus again on the three transmons system (cf. Appendix E.3). We change frame of reference via the unitary transformation \( U = \exp(it(\alpha_1 n_2 + \alpha_2 n_3)) \), from which

\[
UH_0U^\dagger = H_0
\]

\[
UH_{\text{drive}}U^\dagger = \Omega_1 (e^{i\alpha t} a_1 + h.c.) + \Omega_2 (e^{i(\alpha_1 - \alpha_2) t} a_2 + h.c.) + \epsilon_2 (a_2 + h.c.) + \epsilon_3 (a_3 + h.c.)
\]

\((E7)\)

\[
U[S, V]U^\dagger \approx -2g^2E_C \frac{\Delta_{12}^2}{\Delta_{12}^2} e^{2i\alpha t} a_1 a_1^\dagger a_2^\dagger + 2g^2E_C \frac{\Delta_{12}^2}{\Delta_{12}^2} n_1 n_2 - \frac{g^2}{\Delta_{12}^2} n_1 + \frac{g^2}{\Delta_{12}^2} n_2 + g^2E_C \frac{\Delta_{12}^2}{\Delta_{23}^2} e^{i\alpha t} a_1 n_2 a_3^\dagger - \frac{g^2}{\Delta_{23}^2} e^{i\alpha t} a_1 a_3^\dagger + \frac{g^2}{\Delta_{23}^2} e^{i\alpha t} a_1 a_3^\dagger + \frac{g^2}{\Delta_{23}^2} e^{i(2\alpha_1 - \alpha_2) t} a_1^\dagger a_2 a_2 a_3^\dagger + \frac{g^2}{\Delta_{23}^2} e^{i(2\alpha_1 - \alpha_2) t} a_1 a_3^\dagger + \frac{g^2}{\Delta_{23}^2} e^{i(2\alpha_1 - \alpha_2) t} a_1 a_2 a_3^\dagger + \frac{g^2}{\Delta_{23}^2} e^{-2i(\alpha_1 - \alpha_2) t} a_2 a_2 a_3^\dagger + \frac{2g^2E_C}{\Delta_{23}^2} n_2 n_3 - \frac{g^2}{\Delta_{23}^2} n_2 + \frac{2g^2E_C}{\Delta_{23}^2} n_2 n_3 + h.c.,
\]

\((E8)\)

\[
U[S, H_{\text{drive}}]U^\dagger \approx -\frac{g\Omega_1 E_C}{\Delta_{12}^2} \left\{ n_1 \left( a_2^\dagger + h.c. \right) - \left( e^{i\alpha t} a_1 a_1^\dagger e^{i\alpha t} + h.c. \right) \right\} - \frac{g\Omega_1}{\Delta_{12}^2} \left( a_2^\dagger + h.c. \right) + \frac{g\Omega_2 E_C}{\Delta_{12}^2} \left\{ n_2 \left( e^{-i\alpha t} a_1^\dagger + h.c. \right) + \left( e^{i(\alpha_2 - 2\alpha_1) t} a_2 a_2^\dagger + h.c. \right) \right\} - \frac{g\Omega_2}{\Delta_{12}^2} \left( e^{-i\alpha t} a_1^\dagger + h.c. \right) + \frac{g\Omega_2 E_C}{\Delta_{23}^2} \left\{ n_2 \left( a_3^\dagger + h.c. \right) - \left( e^{-i\alpha t} a_2 a_2 a_3^\dagger + h.c. \right) \right\} - \frac{g\Omega_2}{\Delta_{23}^2} \left( a_3^\dagger + h.c. \right).
\]

We throw away all the oscillating terms employing the Rotating Wave Approximation (RWA) in the limits,

\[
o_1 \gg \max \left( \Omega_1 \right),
\]

\((E9)\)

\[
o_2 \gg \max \left( \frac{g^2}{\Delta_{12}^2}, \frac{g^2}{\Delta_{23}^2}, \frac{g^2}{\Delta_{12}^2} \right),
\]

\((E10)\)

\[
|\alpha_1 - \alpha_2| \gg \max \left( \Omega_2, \frac{g^2 E_C}{\Delta_{23}^2} \right),
\]

\((E11)\)

\[
|2\alpha_1 - \alpha_2| \gg \max \left( \frac{g^2 E_C}{\Delta_{23}^2}, \frac{g^2 E_C}{\Delta_{12}^2}, \frac{g\Omega_2 E_C}{\Delta_{23}^2} \right),
\]

\((E12)\)

which are easily satisfied since we work in the dispersive regime \( g/\Delta_{ij} \ll 1 \) and small anharmonicity limit \( E_C \ll \Delta_{ij} \).

Throwing away the oscillating terms in \((E7)\) we obtain

\[
\hat{H} = \sum_{j=1}^{3} \omega_j \hat{n}_j - \frac{E_C}{2} \frac{\Delta_{j,j+1}^2}{\Delta_{j,j+1}} \hat{n}_j \hat{n}_{j+1} + g\Omega_1 E_C \frac{\Delta_{j,j+1}^2}{\Delta_{j,j+1}} \hat{n}_j \left( \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \right) + \sum_{j=2}^{3} \left( \epsilon_j - g\Omega_{j-1} \frac{\Delta_{j-1,j}}{\Delta_{j-1,j}} \right) \left( \hat{a}_j + \hat{a}_j^\dagger \right).
\]

\((E13)\)
Since we do not want local fields $\propto (a_j + a_j^\dagger)$ we fix the condition $\epsilon_j = g\Omega_{j-1}/\Delta_{j-1,j}$ with $j = 2, 3$. We obtain

$$\hat{H} = \sum_{j=1}^{3} \hat{\omega}_j \hat{n}_j - \frac{E_C}{2} \hat{a}_j^\dagger \hat{a}_j \hat{a}_j \hat{a}_j + \frac{2g^2 E_C}{\Delta^2_{j,j+1}} \hat{a}_j \hat{n}_{j+1} - \frac{g\Omega_j E_C}{\Delta^2_{j,j+1}} \hat{n}_j \left( \hat{a}_{j+1} + \hat{a}_{j+1}^\dagger \right).$$

(E14)

In the dispersive regime $\epsilon_j$ are very small compared to the drive fields controlled by $\{\Omega_j\}$. Therefore, it is appropriate to neglect the contributions coming from their commutators \textit{a posteriori}. The calculations above can be straightforwardly generalized to the multisite case, since the transmons in the bulk will behave analogously to the second one in the case treated explicitly above.