ON THE BIMEROMORPHIC GEOMETRY
OF COMPACT COMPLEX CONTACT THREEFOLDS

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Abstract. We prove that a compact contact threefold which is bimeromorphically equivalent to a Kähler manifold and not rationally connected is the projectivised tangent bundle of a Kähler surface.

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1. Introduction

A (compact) complex manifold \( X \) of dimension \( 2n + 1 \) is a contact manifold if there exists a vector bundle sequence

\[
0 \to F \to T_X \to L \to 0,
\]

where \( T_X \) is the tangent bundle and \( L \) a line bundle, with the additional property that the induced map \( \Lambda^2 F \to L, v \wedge w \mapsto [v, w]/F \in L \), given by the Lie bracket \([,]\) on \( T_X \) is everywhere non-degenerate. The line bundle \( L \) is referred to as the contact line bundle on \( X \).

There are two basic ways to construct contact structures.

- A simple Lie group gives rise to a Fano contact manifold \( X \) (with \( b_2(X) = 1 \)) by taking the unique closed orbit for the adjoint action of the Lie group on the projectivised Lie algebra, see e.g. [Bea98].
- For any compact complex manifold \( Y \) the projectivised tangent bundle \( X = \mathbb{P}(T_Y) \) is a contact manifold.

Now the question naturally arises whether any compact complex contact manifold is given in this way.

In the following, let \( X \) be a compact complex contact manifold. If \( X \) is projective with \( b_2(X) = 1 \), then \( X \) must be a Fano manifold and Beauville [Bea98] proved

Date: May 10, 2010.
partial results towards the realisation as closed orbit. In general, if $X$ is Kähler, Demailly [Dem02] showed that the canonical bundle $K_X$ is not nef. If $X$ is projective with $b_2(X) \geq 2$, Theorem 1.1 in [KPSW00] provides a positive answer to the question above. If $X$ is Kähler but not projective, then necessarily $b_2(X) \geq 2$ and the second alternative is conjectured to hold, i.e., $X$ should be a projectivised tangent bundle. However the paper [KPSW00] essentially uses Mori theory, which is, at the moment, not available in the Kähler case, except in dimension 3 where it can be shown that $X$ is a projectivised tangent bundle over a surface (see Section 2).

In this paper we go one step further in dimension 3: we consider contact threefolds $X$ which are in class $C$, i.e., bimeromorphic to a Kähler manifold. We first show that these threefolds must be uniruled. Then we consider the rational quotient $r : X \to Q$. The meromorphic map $r$ identifies two very general points if and only if they can be joined by a chain of rational curves. In particular, $X$ is rationally connected if and only if $\dim Q = 0$. We distinguish the cases $\dim Q = 1$ (Theorem 4.5) and $\dim Q = 2$ (Theorem 3.7) and show

**Theorem.** Let $X$ be a compact contact threefold in class $C$. Assume that $X$ is not rationally connected. Then there exists a Kähler surface $Y$ such that $X \simeq \mathbb{P}(T_Y)$. In particular, $X$ is Kähler.

The remaining open case that $X$ is rationally connected, in particular Moishezon, will require different methods. Probably it will be necessary to consider rational curves $C$ with $-K_X \cdot C$ minimal, but positive.

## 2. Uniruledness and Splitting

We shall use the following notation:

**Definition 2.1.** A compact complex manifold $X$ is said to be in class $C$ if $X$ is bimeromorphically equivalent to a Kähler manifold.

An important property of manifolds in class $C$ is the compactness of the irreducible components of the cycle space (cf. [Cam80]).

The key for our investigations is the following

**Theorem 2.2.** Let $X$ be a compact contact threefold in class $C$. Then $X$ is uniruled.

**Proof.** Let $\pi : \hat{X} \to X$ be a modification such that $\hat{X}$ is Kähler. It is a well-established fact that $\hat{X}$ is uniruled if and only if $K_{\hat{X}}$ is not pseudo-effective, i.e., the Chern class $c_1(K_{\hat{X}})$ is not represented by a positive closed current. The projective case in any dimension is treated in [BDPP04] based on the uniruledness theorem of Miyaoka-Mori [MM86]. The Kähler case in dimension three has been proved by Brunella [Bru06, Cor. 1.2].

The contact structure on $X$ is given by $\theta \in H^0(X, \Omega^1_X \otimes L)$; note that $\theta \wedge d\theta \neq 0$. Via $\pi^*$, the $L$-valued form $\theta$ induces a form $\hat{\theta} \in H^0(\hat{X}, \Omega^1_{\hat{X}} \otimes \pi^*(L))$. By [Dem02, Cor. 1], the pullback of the dual line bundle $\pi^*(L^{-1})$ is not pseudo-effective. Since $K_X = 2L^{-1}$, the line bundle $\pi^*(K_{\hat{X}})$ is not pseudo-effective which is equivalent to say that $K_X$ is not pseudo-effective. Since $\pi_*(K_{\hat{X}}) = K_X$, the Chern class...
$c_1(K^\wedge_X)$ cannot be represented by a positive closed current $\hat{T}$, because otherwise $c_1(K^\wedge_X)$ would be represented by the positive closed current $\pi_*(\hat{T})$. Hence $K^\wedge_X$ is not pseudo-effective, and we conclude by the uniruledness criterion stated above. □

As a consequence we obtain the following classification result for compact Kähler contact threefolds generalising the well-known projective case (see [KPSW00] for further references).

**Corollary 2.3.** Let $X$ be a compact Kähler contact threefold. Then either $X \simeq \mathbb{P}_3$ or $X = \mathbb{P}(T_Y)$ for a Kähler surface $Y$.

**Proof.** By Theorem 2.2 above, the threefold $X$ is uniruled. In particular, there is a positive-dimensional subvariety through the general point of $X$, i.e., $X$ is not simple. The claim now follows from [Pet01, Theorem 4.1]. □

**Remark 2.4.** The proof of Theorem 2.2 above actually shows that the canonical bundle of a compact contact manifold in class $C$ of any dimension is not pseudo-effective. However in dimensions at least 4, unless $X$ is projective, it is completely open, whether this implies uniruledness.

We now make a digression and consider the contact sequence $\ast$. It is an interesting question whether this sequence can split. In the case where $X$ is Fano, LeBrun [LeB95, Cor. 2.2] showed that splitting never occurs. By the following theorem, the same is true if $X$ is in class $C$.

**Theorem 2.5.** Let $X$ be a compact contact manifold in class $C$. Then the contact sequence $\ast$ does not split.

**Proof.** Suppose we have a splitting $T_X \simeq F \oplus L$, hence $\Omega^1_X \simeq F^* \oplus L^*$. Then it is well-known (see e.g. Beauville [Bea00]), that $c_1(L) \in H^1(X, L^*) \subset H^1(X, \Omega^1_X)$ and $c_1(F) \in H^1(X, F^*)$. Since $c_1(F) = nc_1(L)$ in $H^1(X, \Omega^1_X)$ and since $X$ is in class $C$, we conclude that $c_1(L) = c_1(F) = 0$ in $H^1(X, \Omega^1_X)$, and therefore also in $H^2(X, \mathbb{R})$. Hence $K_X$ is numerically trivial and consequently, due to Remark 2.3, $X$ cannot be in class $C$. □

Let $X$ be a compact contact manifold $X$ with contact sequence $\ast$. A subvariety $S \subset X$, i.e., closed irreducible analytic subset in $X$, is called $F$-integral if $T_{S,x} \subset F_x$ for all smooth points $x \in S$.

**Notation.** A holomorphic family $(C_t)_{t \in T}$ of curves in $X$ is given by a diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{p} & X \\
\downarrow{q} & & \\
T & & \\
\end{array}
$$

such that

- $T$ is an irreducible subspace of the cycle space of curves in $X$
- $q^{-1}(t)$ is the cycle corresponding to $t \in T$
- $C_t = p(q^{-1}(t))$ as cycle.
An important tool will be the following lemma, the proof of which is based on the observation that a surface covered by a family of $F$-integral curves is itself $F$-integral.

**Lemma 2.6.** Let $X$ be a compact contact threefold with contact line bundle $L$. Let $(C_t)_{t \in T}$ be a 1-dimensional family of generically irreducible rational curves passing through a fixed point $x_0 \in X$. Then $L \cdot C_t \geq 2$.

**Proof.** We assume to the contrary that $L \cdot C_t \leq 1$ for all $t \in T$. By restricting the contact sequence \( \star \) to an irreducible rational curve $C_t$ and, if necessary, pulling it back to the normalisation $\eta : \tilde{C}_t \to C_t$, one finds that the map $O(2) \simeq T_{\tilde{C}_t} \to \eta^*L \simeq O(a)$ with $a \leq 1$ is trivial and therefore $T_{C_t,x} \hookrightarrow F_x$ for $x \in C_t$ smooth. I.e., for the general $t \in T$ the curve $C_t$ is $F$-integral.

Consider the surface $S = \bigcup_{t \in T} C_t \subset X$ covered by the curves $C_t$. Then the proof of [Keb01, Proposition 4.1] shows that $S$ is $F$-integral. But since any $F$-integral subvariety in $X$ has dimension at most 1, this yields a contradiction. □

### 3. The Case of a 2-Dimensional Rational Quotient

We assume in this section that $X$ is a compact contact threefold in class $\mathcal{C}$ with a rational quotient

$$ r : X \dasharrow Q $$

of dimension $\dim Q = 2$. We refer the reader to the books [Deb01], [Kol96] and the references therein for relevant details on the construction and the properties of a rational quotient. A fundamental result of Graber, Harris, and Starr [GHS03] states that the quotient $Q$ is not uniruled. In case $X$ has dimension three, this result actually has previously been known.

The meromorphic map $r : X \dasharrow Q$ is almost holomorphic, i.e., $r$ is proper holomorphic on a dense open set in $X$, and its general fiber is $\mathbb{P}_1$. Thus, we have a unique covering family $(l_t)_{t \in T}$ of rational curves with graph $Z$,

$$
\begin{array}{c}
\begin{array}{c}
Z \\
\downarrow q
\end{array}
\xrightarrow{p} 
\begin{array}{c}
X \\
\downarrow r
\end{array}
\xrightarrow{\downarrow} 
\begin{array}{c}
Q
\end{array}
\end{array}
$$

and $\dim T = 2$. Since $r$ is almost holomorphic, the map $p$ is bimeromorphic. By possibly passing to the normalisation, we may assume both $Z$ and $T$ normal. Moreover, we may take $Q = T$.

**Lemma 3.1.** All curves $l_t$ satisfy $L \cdot l_t = 1$ and all irreducible curves $l_t$ are $F$-integral.

**Proof.** The general $l_t$ is a general fiber of $r$, hence $-K_X \cdot l_t = 2$ by adjunction and therefore $L \cdot l_t = 1$. The same is then true for all $l_t$. All irreducible curves $l_t$ are consequently $F$-integral (cf. proof of Lemma 2.6). □
In the following we will make use of the deformation theory of rational curves. This is to say we consider a rational curve \( C \subset X \), given by a bimeromorphic morphism \( f : \mathbb{P}_1 \to X \) and consider the deformations \( f_t \) of \( f \). We obtain a family \((C_t) = (f_t(\mathbb{P}_1))\) and then take its closure in the cycle space, because in general the family \((f_t)\) will not be compact, or in other words, the family \((C_t)\) will split. Here it is essential that \( X \) is in class \( C \), hence all irreducible components of the cycle space of \( X \) are compact. We repeatedly use the following basic fact, see e.g. \[Kol96\] Theorem II.1.3.

**Fact 3.2.** Let \( X \) be a compact threefold and let \( C \) be a rational curve in \( X \). If \(-K_X \cdot C \geq m\), then \( C \) will deform as rational curve in an at least \( m\)-dimensional family.

The following proposition is the technical core of this section.

**Proposition 3.3.** The map \( p : Z \to X \) is an isomorphism. In particular, the rational quotient \( r : X \to T \) is holomorphic and equidimensional.

**Proof.** For \( x \in X \) we let \( T(x) \) be the analytic subset of all \( t \in T \) such that \( x \in l_t \). Since the general \( l_t \) does not pass through \( x \), it follows that \( \dim T(x) \leq 1 \). In the following we show that \( \dim T(x) = 0 \) for all \( x \); in other words, \( p \) is finite. Since the map \( p \) is of degree 1 and has connected fibers by Zariski’s main theorem, the finiteness of \( p \) forces \( p \) to be biholomorphic.

Suppose now to the contrary that \( \dim T(x) = 1 \) for some fixed \( x \in X \). If \( T(x) \) happens to be reducible, we replace it by an irreducible component of dimension 1. In the following, we shall therefore assume that \( T(x) \) is irreducible.

Let \( S \) be the surface covered by the \( l_t \) belonging to \( T(x) \):

\[
S = \bigcup_{t \in T(x)} l_t.
\]

If the general \( l_t \) through \( x \) is irreducible, then the Lemmata 2.6 and 3.1 yield a contradiction.

So we are left with the case when all \( l_t \), \( t \in T(x) \) are reducible. In this case \( S \) itself might be reducible. For \( t \in T(x) \) we decompose \( l_t \) into its irreducible components and write \( l_t = \sum a_t^j C_t^j \). Since \( L \cdot l_t = 1 \) for all \( t \in T \), there exists at least one component \( C_t^j \) in this decomposition with \( L \cdot C_t^j \geq 1 \). We pick \( t \in T(x) \) general and let \( C^{(1)} \) be a component of \( l_t \) with \( L \cdot C^{(1)} \geq 1 \). Then by Fact 3.2 \( C^{(1)} \) deforms in an at least 2-dimensional family \((C_t^{(1)})_{t \in T_1}\).

Suppose first that

\[
L \cdot C^{(1)}_t = 1.
\]

If the family \((C_t^{(1)})_{t \in T_1}\) covers a surface, then we find a 1-dimensional subfamily through a fixed point, contradicting Lemma 2.6. If the family covers all of \( X \), then, since there is only one covering family of generically irreducible rational curves in \( X \), the family \((C_t^{(1)})_{t \in T_1}\) must be the original family \((l_t)_{t \in T}\), in particular \( T = T_1 \).

In other words, we have \( t_0 \in T(x) \) and \( t_1 \in T \) such that \( l_{t_0} = l_{t_1} + R \) with an effective curve \( R \). Thus \( p^{-1}(x) \) contains more than one point for every \( x \in l_{t_1} \). Since \( p \) has connected fibers, we conclude that \( \dim p^{-1}(x) = 1 \) for every \( x \in l_{t_1} \). Then either all curves \( l_t, \ t \in T \) pass through \( l_{t_1} \), which is absurd since \( r \) is almost
holomorphic, or there exists a 1-dimensional subfamily \((l_t + C_u)_{u \in U}\) of \((l_t)_{t \in T}\) with \(\dim U = 1\). In this second case however, since the subfamily \((l_t + C_u)_{u \in U}\) does not contain the curve \(l_t\) itself, it follows \(p^{-1}(x)\) is not connected, a contradiction.

So we are left with

\[ L \cdot C_t^{(1)} \geq 2, \]

i.e., \(-K_X \cdot C_t^{(1)} \geq 4\), and the \(C_t^{(1)}\) deform in an at least 4-dimensional family (cf. Fact 3.2) which must stay in an irreducible component \(S_1\) of \(S\). (The deformations of \(C_t^{(1)}\) cannot cover all of \(X\) since there is a unique covering family of generically irreducible rational curves in \(X\), and this family, namely \((l_t)_{t \in T}\), is 2-dimensional and fulfills \(-K_X \cdot l_t = 2\). We want to exhibit a new family \((C_t^{(2)})\) in the surface \(S_1\) such that \(L \cdot C_t^{(2)} \geq 2\).

In order to construct this new family we notice that the 4-dimensional family \((C_t^{(1)})_{t \in T_1}\) must split. In fact, through any two points of \(S_1\) there is a positive-dimensional subfamily. Now we choose carefully a splitting component \(C^{(2)}\) such that \(L \cdot C^{(2)} \geq 2\), namely we want to achieve that \(C^{(2)}\) passes through a general point of \(S_1\). By Lemma 3.3 we obtain a generically non-splitting family \((h_u)_{u \in U}\) of rational curves \(h_u\) in \(S_1\) with \(\dim U \geq 2\) such that for general \(u \in U\) there exists \(t(u) \in T_1\) such that \(h_u\) is an irreducible component of \(C^{(1)}_{t(u)}\). Since the family \((h_u)\) covers exactly \(S_1\), there exists a 1-dimensional subfamily through a general point of \(S_1\) and we take \(C^{(2)}\) to be a general member of this subfamily. By Lemma 2.6 we obtain

\[ L \cdot C^{(2)} \geq 2. \]

Again, \(-K_X \cdot C^{(2)} \geq 4\) implies that \(C^{(2)}\) moves in an at least 4-dimensional family of rational curves, say \((C_t^{(2)})_{t \in T_2}\).

Inductively we obtain families \((C_t^{(k)})_{t \in T_k}\) in \(S_1\) such that

\[ L \cdot C_t^{(k)} \geq 2. \]

Our aim now is to find an argument that this procedure must stop at some point, i.e., that \(L \cdot C_t^{(k)} \geq 2\) cannot occur infinitely many times.

If \(X\) is Kähler with Kähler form \(\omega\), this follows from the fact that the intersection number \(C_t^{(k)} \cdot \omega\) strictly decreases and that all classes \(C_t^{(k)}\) are integer classes in a ball \(\{ a \cdot \omega \leq K \} \subset H^2(X, \mathbb{R})\).

Let us briefly explain the difficulty arising from the fact that \(X\) is not necessarily Kähler. If \(X\) is merely in class \(C\), we cannot argue in this way, because we will have curves with “semi-negative” cohomology. To be precise, we choose a sequence of blow-ups in points and smooth curves \(\pi : \tilde{X} \to X\) such that \(\tilde{X}\) is Kähler, fix a Kähler form \(\tilde{\omega}\) on \(\tilde{X}\), and form the current \(R = \pi_* (\omega)\). Then \(R \cdot C > 0\) for all curves not contained in the center of \(\pi\). On the other hand, there are finitely many curves \(B_1, \ldots, B_N\) such that \(R \cdot B_j \leq 0\). These “bad” curves have to be taken into account.

We are able to get around this difficulty since every splitting takes place in the fixed surface \(S_1\). Inside this surface we will not have any curves with “negative” homology.

We consider the normalisation

\[ \eta : \tilde{S}_1 \to S_1. \]
We define a family $\tilde{C}^{(k)}_t$ in $\tilde{S}_1$ by letting $\tilde{C}^{(k)}_t$ be the strict transform of $C^{(k)}_t$ in $\tilde{S}_1$ for general $t$ and then take closure in the cycle space. Let $\tilde{C}^{(k)}$ be the strict transform of $C^{(k)}$ in $\tilde{S}_1$. Then we obtain a splitting
\[ \tilde{C}^{(k)}_t = \tilde{C}^{(k+1)}_t + \tilde{R}_t \]
for some $t_k$. Inductively we find
\[ \tilde{C}^{(m)}_t = \tilde{C}^{(m+1)}_t + \tilde{R}_m, \]
for all $m \in \mathbb{N}$. Here $\equiv$ denotes homology equivalence in $\tilde{S}_1$. It follows that
\[ \tilde{C}^{(1)}_t \equiv \tilde{C}^{(m)}_t + \sum_{j=1}^{m-1} \tilde{R}_j, \]
i.e., $\tilde{C}^{(1)}_t$ is homology equivalent to a sum of arbitrary many effective curves in $\tilde{S}_1$. This contradicts Lemma 3.3.

We now prove the two technical lemmata used in the proof of Proposition 3.3 above.

**Lemma 3.4.** Let $S$ be a compact connected normal Moishezon surface. Then there exists a linear map $\varphi : H_2(S, \mathbb{Q}) \to \mathbb{Q}$ such that $\varphi([C]) \geq 1$ for all classes of irreducible curves $C$ in $S$.

**Proof.** It suffices to construct $\varphi$ on the subspace $V$ of $H_2(S, \mathbb{Q})$ generated by classes of irreducible curves. Let $\sigma : \hat{S} \to S$ be a desingularisation of $S$ and note that the surface $\hat{S}$ is projective. We let $\hat{H}$ be an ample divisor on $\hat{S}$ and $\sigma_*(\hat{H}) = H$ be its push-down to $S$. Using the intersection theory on normal surfaces established in [Mum61] and [Sak84], we define
\[ \varphi([C]) = C \cdot H = (\sigma^*C) \cdot (\sigma^*H). \]
Here $\sigma^*D$ denotes the sum $\overline{D} + \sum a_iE_i$ of the strict transform $\overline{D}$ of the divisor $D$ in $\hat{S}$ and an appropriately weighted sum of the exceptional curves of $\sigma$.

In order to check that $\varphi$ is well-defined on homology classes, it suffices to show that $c_1(\mathcal{O}(\sigma^*C)) = 0$ for every $C$ with $[C] = 0 \in H_2(S, \mathbb{Q})$. Following the notation and results presented in [Sak84], Section 3, this is equivalent to $c_1(\mathcal{O}(C)) \in \ker(\sigma^*) = \ker(\eta_S) \subset H^2(S, \mathbb{Q})$. Here $\eta_S : H^2(S, \mathbb{Q}) \to H_2(S, \mathbb{Q})$ denotes the Poincaré homomorphism on $S$. We may write
\[ \eta_S(c_1(\mathcal{O}(C))) = \sigma_*(\eta_S(c_1(\mathcal{O}(\hat{C})))) \]
for the Poincaré isomorphism $\eta_S : H^2(\hat{S}, \mathbb{Q}) \to H_2(\hat{S}, \mathbb{Q})$ on $\hat{S}$ and $\hat{C} = \mathcal{O}(\sigma^*C)$. Since $\eta_S(c_1(\mathcal{O}(\sigma^*C))) = [\sigma^*C]$ on the smooth surface $S$, we conclude $\eta_S(c_1(\mathcal{O}(C))) = \sigma_*(\sigma^*C) = [C] = 0$ and obtain the desired vanishing.

It remains to check that $\varphi([C]) \geq 1$ for all classes of irreducible curves $C$ in $S$. We have
\[ \varphi([C]) = (\sigma^*C) \cdot (\sigma^*H) = (\sigma^*C) \cdot (\overline{H} + \sum b_jE_j). \]
Since $(\sigma^*C) \cdot E_j = 0$ for all $j$ by definition of $\sigma^*C$ (cf. [Sak84], Section 1), we conclude
\[ \varphi([C]) = (\sigma^*C) \cdot \overline{H} = (\overline{C} + \sum \alpha_iE_i) \cdot \overline{H}. \]
Since $\overline{H} = \hat{H}$ is ample and $C$ is effective, in particular $\overline{C}$ is effective and $\alpha_i > 0$ for all $i$, the desired inequality follows. $\square$
Lemma 3.5. Let $S$ be an irreducible Moishezon surface with a covering family $(C_t)_{t \in T}$ of (rational) curves. Suppose $\dim T \geq 4$. Let $T' \subset T$ be the subset of those $t$ for which $C_t$ splits. Then $\dim T'' \geq 2$.

Proof. Let $x \in S$ and $T(x) = \{ t \in T \mid x \in C_t \}$. Since $\dim T \geq 4$ by assumption, we have $\dim T(x) \geq 3$. (Consider the graph $p : Z \to S$ of the family $(C_t)_{t \in T}$ and observe that $\dim(p^{-1}(x)) \geq 3$ and $q : Z \to T$ restricted to $p^{-1}(x)$ is finite.) The same dimension count substituting $T$ by $T(x)$ shows that

$$\dim(T(x) \cap T(x')) \geq 2$$

for $x, x' \in S$. Hence there exists a 2-dimensional subfamily through $x, x'$ and therefore we obtain a 1-dimensional subfamily through $x$ and $x'$ such that all members split. In other words

$$\dim(T' \cap T(x)) \geq 1.$$ 

Varying $x$ we conclude $\dim T' \geq 2$. \hfill \Box

Having established Proposition 3.3 it remains to show that the rational quotient $r : X \to T$ is actually a $\mathbb{P}_1$-bundle.

Proposition 3.6. Assume that the rational quotient $r : X \to T$ is holomorphic and equidimensional. Then $r$ is a $\mathbb{P}_1$-bundle, $T$ is smooth and $X = \mathbb{P}(T_T)$.

Proof. As a first step, we show that the fibers or $r$ must be irreducible. Assume the contrary and let $r^{-1}(t_0) = l_{t_0} = C^{(1)} + R$ be a reducible fiber such that $L \cdot C^{(1)} \geq 1$. Then $C^{(1)}$ deforms in an at least 2-dimensional family, hence $C^{(1)}$ is a member of $(l_t)_{t \in T}$, i.e., $C^{(1)} = l_{t_1}$ for a suitable $t_1 \in T$. Since $r$ is holomorphic, this is only possible when $t_0 = t_1$, a contradiction.

So $r : X \to T$ is a holomorphic, equidimensional map of normal complex spaces and every fiber of $r$ is a reduced, irreducible rational curve. Now the arguments of [Kol96] Theorem II.2.8] can be adapted to our situation and it follows that $r$ is a $\mathbb{P}_1$-bundle. In particular, $T$ is smooth and $X = \mathbb{P}(T_T)$. \hfill \Box

Recall that a surface in class $C$ is Kähler. In total we have shown:

Theorem 3.7. Let $X$ be a compact contact threefold in class $C$. If the rational quotient has dimension 2, then $X$ is Kähler and of the form $\mathbb{P}(T_Y)$ with a Kähler surface $Y$. The projection $X \to Y$ is the rational quotient, i.e., $Y$ is not uniruled.

4. The case of a 1-dimensional rational quotient

In this section we assume that $X$ is a compact contact threefold in class $C$ with contact line bundle $L$ and a rational quotient $r : X \dasharrow Q$ of dimension $\dim Q = 1$. Then necessarily $X$ is Moishezon and $Q$ is a smooth curve $B$ of genus at least 1. We observe that $r : X \to B$ is holomorphic. Our aim is to show that $X$ is of the form $X = \mathbb{P}(T_Y)$ for some surface $Y$. The surface $Y$ is then necessarily Moishezon, and since a smooth Moishezon surface is projective, we are going to show directly that $X$ is projective.

Let $B_0$ be the set of points $b$ in $B$ such that $r^{-1}(b) = X_b$ is smooth.
Lemma 4.1. Let $b \in B_0$. Then $X_b$ is a Hirzebruch surface $X_b = \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-e))$ with $e > 0$ even.

Proof. By adjunction $K_{X_b} = -2L|_{X_b}$, hence $X_b$ is minimal. Moreover, $X_b$ cannot be a projective plane $\mathbb{P}_2$ or a Hirzebruch surface with odd $e$. To exclude the quadric $(e = 0)$, observe that $X_b$ is not $F$-integral, i.e., the restriction of the contact form $\theta$ to $X_b$ does not vanish identically, and hence
\[
\theta|_{X_b} \in H^0(X_b, \Omega^1_{X_b} \otimes L|_{X_b}) \neq 0,
\]
which is impossible for $X_b = \mathbb{P}_1 \times \mathbb{P}_1$. □

Every smooth fiber $X_b$ of $r$ has a uniquely defined non-splitting 1-dimensional family of rational curves, namely the ruling lines. All these rational curves together give rise to a 2-dimensional family $(l_y)_{y \in Y}$ of rational curves in $X$, where $Y$ is the irreducible component of the cycle space parametrising generically the ruling lines. We obtain an almost holomorphic map $\pi : X \dashrightarrow Y$ and a holomorphic map $g : Y \to B$, $g(y) = r(l_y)$,
\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \supset Y_0 \\
\downarrow{g} & & \downarrow{Y_0} \\
B & \supset & B_0
\end{array}
\]
such that $Y_0 = g^{-1}(B_0)$ is a $\mathbb{P}_1$-bundle over $B_0$ and $\pi$ is again a $\mathbb{P}_1$-bundle over $Y_0$. In fact, if $Y_b$ is the fiber over $b \in B_0$, then $X_b = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e_b)|Y_b)$.

The technical key to the main result of this section is

Proposition 4.2. The family $(l_y)_{y \in Y}$ does not split.

Proof. Suppose $(l_y)$ splits. Then there exists a point $y_0 \in Y$, an irreducible curve $C^{(1)}$ with $L \cdot C^{(1)} \geq 1$ and an effective curve $R$ such that $l_{y_0} = C^{(1)} + R$. As seen before the curve $C^{(1)}$ deforms in an at least 2-dimensional family $(C^{(1)}_t)_{t \in T_1}$.

1) Let us first consider the case $L \cdot C^{(1)}_t = 1$. If the family $(C^{(1)}_t)_{t \in T_1}$ covers a surface, we find a 1-dimensional subfamily through a fixed point and contradict Lemma 2.6. If $(C^{(1)}_t)_{t \in T_1}$ covers all of $X$, we recover the original family $(l_y)$ as follows: notice that the general $C^{(1)}_t$ will be an irreducible rational curve in a smooth fiber $X_b$ and $-K_{X_b} \cdot C_t = 2$ by adjunction. This implies that the general curve $C^{(1)}_t$ must be a ruling line, i.e., the general curve $C_t$ must be a curve $l_t$. This may now be excluded using Lemma 2.6 by the same arguments as in Proposition 3.3.

2) Having ruled out $L \cdot C^{(1)}_t = 1$, we consider the case $L \cdot C^{(1)}_t \geq 2$, i.e.,
\[
-K_X \cdot C^{(1)}_t = 2L \cdot C^{(1)}_t \geq 4.
\]

2a) Assume that the family $(C^{(1)}_t)_{t \in T_1}$ covers all of $X$ and choose a general point $x \in X$. Dimension count shows that there is a 2-dimensional subfamily $(C^{(1)}_t)_{t \in T_1(x)}$ through the point $x$, necessarily filling a rational surface $S$, which must be a fiber of $r$. Since $x$ is general, there is a $b \in B$ with $X_b$ smooth such that $S = X_b$. The family $(C^{(1)}_t)_{t \in T_1(x)}$ splits as $C^{(1)}_t = C^{(2)} + R_1$ with $L \cdot C^{(2)} \geq 1$. If $L \cdot C^{(2)} \geq 2$ we repeat to whole process. Assume that $L \cdot C^{(k)} \geq 2$ for all $k$. Then, we obtain a
decomposition of the homology class of $C_{t_1}^{(1)}$ as a sum of arbitrary many effective curves $C_{t_1}^{(1)} \equiv \sum_{k=1}^{K} C^{(k)} + R_{K-1}$. As we can always choose a subfamily through a point of $S$, we can assume that $\sum_{k=1}^{K} C^{(k)} + R_{K-1} \subset S$ for all $K$. Calculating the degree with respect to an ample line bundle $H$ on $S$, we obtain a contradiction. Hence at some stage the procedure has to stop, i.e. $L \cdot C_{t_1}^{(k)} = 1$ and we conclude by Lemma 2.6.

(2b) It remains to consider the case the case where $L \cdot C_{t}^{(1)} \geq 2$ and the family $(C_{t}^{(1)})$ covers a surface $S$, which is a component of a fiber $X_b$ of $r$. The family must split and we choose a splitting component $C_{t}^{(2)}$ such that $L \cdot C_{t}^{(2)} \geq 1$. If $L \cdot C_{t}^{(2)} = 1$, we are done again; if $L \cdot C_{t}^{(2)} \geq 2$, we obtain an at least 4-dimensional family $(C_{t}^{(2)})$. If this family covers $X$, we are done by the arguments of (2a) applied to $(C_{t}^{(2)})$, instead of $(C_{t}^{(1)})$. Otherwise, the family $(C_{t}^{(2)})$ fills a component $S'$ of the same fiber $X_b$, and as in the proof of Proposition 3.3, by choosing $C_{t}$ carefully, we may assume that $S' = S$. Now we are in completely the same situation as in the proof of Proposition 3.3 and proceed as described there. □

In order to apply Proposition 4.2, we consider the normalised graph $p : Z \to X$ of the family $(l_y)_{y \in Y}$.

\[ \begin{array}{c}
Z \\
q \downarrow \pi \\
Y
\end{array} \]

Proposition 4.3. The map $p : Z \to X$ is biholomorphic.

Proof. The map $p$ is generically biholomorphic, hence by Zariski’s main theorem, it suffices to show that $p$ does not have positive-dimensional fibers. So suppose that $\dim p^{-1}(x) = 1$. Then there exists a 1-dimensional subfamily $(l_y)_{y \in Y(x)}$ through $x$ with all $l_y$ irreducible by the previous proposition. Since $L \cdot l_y = 1$, we contradict Lemma 2.6. □

As before in Proposition 3.6, we conclude:

Corollary 4.4. The map $\pi : X \to Y$ is a $\mathbb{P}_1$-bundle.

We may now apply Lemma 4.6 below to the map $\pi : X \to Y$ and have shown:

Theorem 4.5. Let $X$ be a compact contact threefold in class $C$ with 1-dimensional rational quotient $B$. Then $X$ is projective and there is a smooth projective surface $Y$ with a $\mathbb{P}_1$-fibration $Y \to B$ such that $X \simeq \mathbb{P}(T_Y)$.

Lemma 4.6. Let $Y$ be a complex manifold of dimension $n + 1$ and $\pi : X \to Y$ be a $\mathbb{P}_n$-bundle. If $X$ is a contact manifold, then $X \simeq \mathbb{P}(T_Y)$.

Proof. Let $Z \simeq \mathbb{P}_n$ be a fiber of $\pi$. Adjunction implies that the contact line bundle $L$ restricted to $Z$ fulfils $L|_Z = \mathcal{O}_Z(1)$. Setting $\mathcal{E} = \pi_*(L)$, we conclude

$$(X, L) \simeq (\mathcal{E}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)).$$

Now the last part of the proof of Theorem 2.12 in [KPSW00] can be applied and shows that $\mathcal{E} \simeq T_Y$. □
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