The Finite difference method for the Minkowski Curve

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Abstract

In this work, we describe how to approximate solutions of some partial differential equations using the finite difference method defined on the Minkowski self-similar curve.

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1 Introduction

The last few decades saw the birth of analysis on fractals, particularly, differential equations on fractals, introduced by J. Kigami [1], [2], [4], [5]. May one consider a fractal set $F$, which shows enough nice properties (self-similar, post-critically finite), and a continuous function $u$ defined on $F$, a Laplacian $\Delta u$ can be obtained as the renormalized limit of a sequence of graph Laplacians $\Delta_m u$, $m \in \mathbb{N}$, built on a sequence of graphs which approximate $F$.

The seminal work of J. Kigami has been followed by numerous contributions, in the field of fractal analysis. One may now explore the theoretical and numerical related areas with specific tools. But there still remain many points that ought to be studied in terms of numerical computation on specific fractals.

The door was opened by K. Dalrymeple, R. S. Strichartz, and J. Vinson [13], who, in the case of the Sierpiński gasket $SG$, give an equivalent method for finite difference approximation. This work was followed by the one of M. Gibbons, A. Raj and R. S. Strichartz [15], where they describe how one can build approximate solutions, by means of piecewise harmonic, or biharmonic, splines, again in the case of $SG$. They go so far as giving theoretical error estimates, through a comparison with experimental numerical data. One may note that this method is quite general, and can be extended to other self-similar sets.

In [9], we built a Laplacian on the Minkowski curve $MC$, which had seemed of interest for several reasons. First, it was a curve, and the challenge, as regards fractal ones, is to consider energy forms more sophisticated than classical ones, by means of normalization constants that could, not only bear the topology, but, also, the geometric characteristics. One must of course bear in mind that a fractal
curve is topologically equivalent to a line segment, and that Dirichlet forms, required for who aims at building a Laplacian, solely depend on the topology of the domain, and not of its geometry. Second, we came across applications in electromagnetism, in the field of fractal antennas, where one requires accurate numerical results.

Our Laplacian on $\mathcal{M}C$ is obtained explicitly, as a limit of difference quotients. This opens the way to a theoretical and numerical study of partial differential equations, as the heat or wave equation. We hereafter present the resulting finite difference scheme. We did not find any equivalent in the existing literature.

We proceed as follows: in section 1, based on our previous results [9], we present a mathematical construction of the Minkowski curve. In section 2, we successively build the finite difference schemes for the heat and the wave equations; we then give a numerical estimate for the error. Computations have been made using Mathematica.

2 The Minkowski Curve

In the sequel, we recall results that are developed in [9].

We place ourselves in the Euclidean plane of dimension 2, referred to a direct orthonormal frame. The usual Cartesian coordinates are $(x, y)$.

Let us denote by $P_0$ and $P_1$ the points:

$$P_0 = (0, 0) \quad , \quad P_1 = (1, 0)$$

Let us denote by $\theta \in ]0, 2\pi[$, $k > 0$, $T_1$, and $T_2$ real numbers. We set:

$$R_{O, \theta} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

We introduce the iterated function system of the family of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$: 

$$\{f_1, \ldots, f_8\}$$

where, for any integer $i$ belonging to $\{1, \ldots, 8\}$, and any $X \in \mathbb{R}^2$:

$$f_i(X) = k R_{O, \theta} X + \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

Remark 2.1. If $0 < k < 1$, the family $\{f_1, \ldots, f_8\}$ is a family of contractions from $\mathbb{R}^2$ to $\mathbb{R}^2$, the ratio of which is $k$.

According to [10], there exists a unique subset $\mathcal{M} \subset \mathbb{R}^2$ such that:

$$\mathcal{M} = \bigcup_{i=1}^{8} f_i(\mathcal{M})$$

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which will be called the Minkowski Curve.

For the sake of simplicity, let us set:

\[ \mathcal{F} = \bigcup_{i=1}^{8} f_i \]

We will denote by \( V_0 \) the ordered set, of the points:

\[ \{ P_0, P_1 \} \]

The set of points \( V_0 \), where, for any \( i \) of \( \{0, 1\} \), the point \( P_1 \) is linked to the point \( P_2 \), constitutes an oriented graph, that we will denote by \( \mathcal{MC}_0 \). \( V_0 \) is called the set of vertices of the graph \( \mathcal{MC}_0 \).

For any strictly positive integer \( m \), we set:

\[ V_m = \bigcup_{i=1}^{8} f_i (V_{m-1}) \]

The set of points \( V_m \), where the points of an \( m \)-cell are linked in the same way as \( \mathcal{MC}_0 \), is an oriented graph, which we will denote by \( \mathcal{MC}_m \). \( V_m \) is called the set of vertices of the graph \( \mathcal{MC}_m \). We will denote, in the following, by \( N_m \) the number of vertices of the graph \( \mathcal{MC}_m \).

For numerical purpose, we use the following similarities, from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), such that, for any \( X \in \mathbb{R}^2 \):

\[
\begin{align*}
    f_1(X) = & \frac{1}{4} \left( R_{O,0} X + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right), & f_2(X) = & \frac{1}{4} \left( R_{O,\frac{3\pi}{2}} X + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), & f_3(X) = & \frac{1}{4} \left( R_{O,\frac{\pi}{2}} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\
    f_4(X) = & \frac{1}{4} \left( R_{O,\frac{\pi}{2}} X + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right), & f_5(X) = & \frac{1}{4} \left( R_{O,\frac{3\pi}{2}} X + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right), & f_6(X) = & \frac{1}{4} \left( R_{O,0} X + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) \\
    f_7(X) = & \frac{1}{4} \left( R_{O,\frac{3\pi}{2}} X + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right), & f_8(X) = & \frac{1}{4} \left( R_{O,0} X + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right)
\end{align*}
\]

3 The finite difference method on fractal sets

In the sequel, we will denote by \( T \) a strictly positive real number, by \( M \) the number of contractions, by \( N_0 \) the cardinal of \( V_0 \), and by \( \mathcal{F} \) the considered fractal set.

3.1 The heat equation

3.1.1 Formulation of the problem

We may now consider a solution \( u \) of the problem:

\[
\begin{cases}
    \frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) = 0 & \forall (t,x) \in \]0, T[ \times \mathcal{F} \\
    u(t,x) = 0 & \forall (x,t) \in \partial \mathcal{F} \times \]0, T[ \\
    u(0,x) = g(x) & \forall x \in \mathcal{F}
\end{cases}
\]
In order to define a numerical scheme, one may use a first order forward difference scheme to approximate the time derivative \( \frac{\partial u}{\partial t} \). The Laplacian is approximated by means of the graph Laplacians \( \Delta_m u \), defined on the sequence of graphs \( (\mathcal{M}C_m)_{m \in \mathbb{N}} \).

To this purpose, we fix a strictly positive integer \( N \), and set:

\[
h = \frac{T}{N}
\]

One has, for any integer \( k \) belonging to \( \{0, \ldots, N-1\} \):

\[
\forall X \in \mathcal{F} : \quad \frac{\partial u}{\partial t}(kh, x) = \frac{1}{h} (u((k+1)h, X) - u(kh, X)) + \mathcal{O}(h)
\]

According to [9], the Laplacian on the Minkowski Curve \( \mathcal{M}C \) is given by:

\[
\forall X \in \mathcal{F} : \quad \Delta u(t, x) = \lim_{m \to +\infty} 64^m \left( \sum_{X \sim Y \in \mathcal{F}} u(t, Y) - u(t, X) \right)
\]

This enables one to approximate the Laplacian, at a \( m^{th} \) order, \( m \in \mathbb{N}^* \), using the graph normalized Laplacian as follows:

\[
\forall k \in \{0, \ldots, N-1\}, \forall X \in \mathcal{F} : \quad \Delta u(t, X) \approx 64^m \left( \sum_{X \sim Y \in \mathcal{F}} u(kh, Y) - u(kh, X) \right)
\]

By combining those two relations, one gets the following scheme, for any integer \( k \) belonging to \( \{0, \ldots, N-1\} \), any point \( P_j \) of \( V_0 \), \( 0 \leq j \leq N_0 - 1 \), and any \( X \) in the set \( V_m \setminus V_0 \):

\[
(S_H) \quad \begin{cases}
\frac{u((k+1)h, X) - u(kh, X)}{h} = 64^m \left( \sum_{X \sim Y \in \mathcal{F}} u(kh, Y) - u(kh, X) \right)
\end{cases}
\]

Let us define the approximate equation as:

\[
u((k+1)h, X) = u(kh, X) + h \times 64^m \left( \sum_{X \sim Y \in \mathcal{F}} u(kh, Y) - u(kh, X) \right) \quad \forall k \in \{0, \ldots, N-1\}, \forall X \in V_m \setminus V_0
\]

We now fix \( m \in \mathbb{N} \), the number of contractions \( M \), and denote any \( X \in \mathcal{F} \) as \( X_{w, P_i} \), where \( w \in \{1, \ldots, N\}^m \) denotes a word of length \( m \), and where \( P_i, 0 \leq i \leq N_0 - 1 \) belongs to \( V_0 \). Let us also set:

\[
n = \# \{ w \mid |w| = m \}, \quad N_0 = \#V_0
\]

This enables one to introduce, for any integer \( k \) belonging to \( \{0, \ldots, N-1\} \), the solution vector \( U(k) \) as:

\[
U(k) = \begin{pmatrix}
u(kh, X_{w_1P_1}) \\
u(kh, X_{w_1P_2}) \\
\vdots \\
u(kh, X_{w_2P_i}) \\
\vdots \\
u(kh, X_{w_3P_i}) \\
\vdots \\
u(kh, X_{w_kP_{N_0}})
\end{pmatrix}
\]
It satisfies the recurrence relation:

\[ \forall k \in \{0, \ldots, N - 1\} : \quad U(k + 1) = AU(k) \]

where:

\[ A = I_m - h \times \tilde{\Delta}_m \]

and where \( I_m \) denotes the \((#V_m) \times (#V_m)\) identity matrix, and \( \tilde{\Delta}_m \) the \((#V_m) \times (#V_m)\) normalized Laplacian matrix.

### 3.1.2 Theoretical study of the error, for Hölder continuous functions

#### i. General case

In the spirit of the work of R. S. Strichartz \[7\], \[8\], it is interesting to consider the case of Hölder continuous functions. Why? First, Hölder continuity implies continuity, which is a required condition for functions in the domain of the Laplacian (we refer to our work \[9\] for further details). Second, a Hölder condition for such a function will result in fruitful estimates for its Laplacian, which is a limit of difference quotients.

Let us thus consider a function \( u \) in the domain of the Laplacian, and a nonnegative real constant \( \alpha \) such that:

\[ \forall (X, Y) \in F^2, \forall t > 0 : \quad |u(t, X) - u(t, Y)| \leq C(t) |X - Y|^{\alpha} \]

where \( C \) denotes a positive function of the time variable \( t \).

Given a strictly positive integer \( m \), one has:

\[ \Delta_m u(t, X) = \sum_{Y \in V_m, Y \sim X_m} (u(t, Y) - u(t, X)) \quad \forall t > 0, \forall X \in V_m \setminus V_0 \]

The Laplacian \( \Delta_\mu u \) is defined as the limit:

\[ \Delta_\mu u(t, X) = \lim_{m \to +\infty} r^{-m} \left( \int_K \mathcal{F} \psi^{(m)}_{X_m} d\mu \right)^{-1} \Delta_m u(t, X_m) \quad \forall t > 0, \forall X \in K \]

where \((X_m \in V_m \setminus V_0)_{m \in \mathbb{N}}\) is a sequence a points such that:

\[ \lim_{m \to +\infty} X_m = X \]

and where \( r \) denotes the normalization ratio (we refer to \[9\]), \( \psi^{(m)}_{X_m} \) a harmonic spline function, and where:

\[ \Delta_m u(t, X) = \sum_{Y \in V_m, Y \sim X_m} (u(t, Y) - u(t, X)) \quad \forall t > 0, \forall X \in V_m \setminus V_0 \]

Let us now introduce a strictly positive number \( \delta_{ij} = |P_i - P_j| \), for any \( P_i \) belonging to the set \( V_0 \), and any \( P_j \) such that \( P_j \sim P_i \). We set:

\[ \delta_i = \max_j \delta_{ij} \]
We then introduce, for any integer \( i \) belonging to \( \{1, \ldots, M\} \), the contraction ratio of the similarity \( f_i, R_i \), and set: \( R = \max_{1 \leq i \leq M} R_i \).

One has then, for any \( X \) belonging to the set \( V_m \setminus V_0 \), any integer \( k \) belonging to \( \{0, \ldots, N - 1\} \), and any strictly positive number \( h \):

\[
\left| r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h \right| \left| \Delta_m u(k h, X) \right| \leq r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h \sum_{Y \in V_m, Y \sim X} \left| u(k h, Y) - u(k h, X) \right| \\
\leq r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h C(k h) \sum_{Y \in V_m, Y \sim X} \left| X - Y \right| ^\alpha \\
\leq r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h C(k h) \sum_{m \mid Y \in V_m, Y \sim X} \delta^\alpha R^m \alpha \\
\leq \delta^\alpha \frac{r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h C(k h)}{(1 - R^\alpha)}
\]

We used the fact that, for \( X \sim Y \), \( X \) and \( Y \) have addresses such that:

\[ X = f_w(P_i) \quad , \quad Y = f_w(P_j) \]

for some \( P_i \) and \( P_j \) in \( V_0 \) and \( w \in \{1, \ldots, M\}^m \). May one set:

\[ R(w) = R_{w_1} R_{w_2} \ldots R_{w_m} \]

one gets:

\[ \left| X - Y \right| = \left| f_w(P_i) - f_w(P_j) \right| = R(w) \left| P_i - P_j \right| \leq R^m \delta \]

The scheme \( (S_H) \) allow us to write:

\[ \left| u((k + 1) h, X) - u(k h, X) \right| \leq \delta^\alpha \frac{r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h C(k h)}{(1 - R^\alpha)} \]

One may note that a required condition for the convergence of the scheme is:

\[
\lim_{m \to +\infty, h \to 0^+} r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h C(k h) = 0
\]

In the case where \( C \) is a constant function, it reduces to:

\[
\lim_{m \to +\infty, h \to 0^+} r^{-m} \left( \int_K \psi_X^{(m)} d\mu \right)^{-1} h = 0
\]
ii. The case of the Minkowski Curve (MC)

Given a strictly positive integer $m$, due to:

$$\Delta_m u(t, X) = \sum_{Y \in V_m, Y \sim_X m} (u(t, Y) - u(t, X)) \quad \forall t > 0, \forall X \in V_m \setminus V_0$$

one has then, for any $X$ belonging to the set $V_m \setminus V_0$, any integer $k$ belonging to $\{0, \ldots, N - 1\}$, and any strictly positive number $h$:

$$64^m h |\Delta_m u(kh, X)| \leq 64^m h \sum_{Y \in V_m, Y \sim_X m} |u(kh, Y) - u(kh, X)|$$

$$\leq 64^m h C(kh) \sum_{Y \in V_m, Y \sim_X m} |X - Y|^\alpha$$

$$= 64^m h C(kh) \sum_{Y \in V_m, Y \sim_X m} \frac{1}{4^m \alpha}$$

$$\leq 64^m h C(kh) \sum_{p=0}^{+\infty} \frac{1}{4^p \alpha}$$

$$= 64^m h \frac{C(kh)}{1 - 4^{-\alpha}}$$

Using the scheme ($S_H$), one gets:

$$|u((k + 1)h, X) - u(kh, X)| \leq \frac{64^m h C(kh)}{1 - 4^{-\alpha}}$$

One may note that a required condition for the convergence of the scheme is:

$$\lim_{m \to +\infty, h \to 0^+} 64^m h C(kh) = 0$$

In the case where $C$ is a constant function, it reduces to:

$$\lim_{m \to +\infty, h \to 0^+} 64^m h = 0$$
3.1.3 Numerical results

In the sequel, we present the numerical results for $m = 3$, $T = 10$ and $N = 10^7$. One may note that, as expected, the solution shows abnormal behavior until the ration $h \times 64^m$ goes towards zero.

Figure 1 – The graph of the approached solution of the heat equation for $t = 0$.

Figure 2 – The graph of the approached solution of the heat equation for $t = 10$. 

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Figure 3 – The graph of the approached solution of the heat equation for $t = 50$.

Figure 4 – The graph of the approached solution of the heat equation for $t = 100$. 
Figure 5 – The graph of the approached solution of the heat equation for $t = 1000$. 
3.2 The wave equation

3.2.1 Formulation of the problem

One may proceed in a similar way as in the above. First, we consider a solution \( u \) of the problem:

\[
\begin{align*}
\frac{\partial u}{\partial t}(t,x) - \Delta u(t,x) &= 0 \quad \forall (t,x) \in ]0,T[ \times \mathcal{F} \\
u(t,x) &= 0 \quad \forall (x,t) \in \partial \mathcal{F} \times ]0,T[ \\
u(0,x) &= g(x) \quad \forall x \in \mathcal{F}
\end{align*}
\]

In order to define a numerical scheme, one may use a second order central difference in time. The Laplacian is approximated by means of the graph Laplacians \( \Delta_m u \), defined on the sequence of graphs \( (\mathcal{M}_k)_{k \in \mathbb{N}} \).

To this purpose, we fix a strictly positive integer \( N \), and set:

\[ h = \frac{T}{N} \]

One has, for any integer \( k \) belonging to \( \{0, \ldots, N-1\} \):

\[ \forall X \in \mathcal{F} : \frac{\partial u}{\partial t}(kh,x) = \frac{1}{h} (u((k+1)h,X) - u(kh,x)) + O(h) \]

and:

\[ \forall X \in \mathcal{F} : \frac{\partial^2 u}{\partial t^2}(kh,X) = \frac{u((k+1)h,X) - 2u(kh,X) + u((k-1)h,X)}{h^2} + O(h^2) \]

According to [9], the Laplacian on the Minkowski curve is given by:

\[ \forall X \in \mathcal{F} : \Delta u(t,X) = \lim_{m \to +\infty} 64^m \left( \sum_{X \sim Y \sim m} u(t,Y) - u(t,X) \right) \]

This enables one to approximate the Laplacian, at a \( m^{th} \) order, \( m \in \mathbb{N}^* \), using the graph normalized Laplacian as follows:

\[ \forall k \{0, \ldots, N-1\}, \forall X \in \mathcal{F} : \Delta u(t,X) \approx 64^m \left( \sum_{X \sim Y} u(kh,Y) - u(kh,X) \right) \]

One thus obtains the following scheme, for any integer \( k \) belonging to \( \{0, \ldots, N-1\} \), any point \( P_j \) of \( V_0, 0 \leq j \leq N_0 - 1 \), and any \( X \) in the set \( V_m \setminus V_0 \):

\[
\begin{align*}
(S_W)
\begin{cases}
\frac{u((k+1)h,X) - 2u(kh,X) + u((k-1)h,X)}{h^2} &= 64^m \left( \sum_{X \sim Y} u(kh,Y) - u(kh,X) \right) \\
u(kh,P_j) &= 0 \\
u(X,0) &= g(X)
\end{cases}
\end{align*}
\]

Let us define the approximate equation as:

\[ u((k+1)h,X) = 2u(kh,X) - u((k-1)h,X) + h^2 \times 64^m \left( \sum_{X \sim Y} u(kh,Y) - u(kh,X) \right) \]
We now fix \( m \in \mathbb{N} \), the number of contractions \( M \) and denote any \( X \in K \) as \( X_{w,P_i} \), where \( w \in \{1, \ldots, N\}^m \) denotes a word of length \( m \), and where \( P_i, 0 \leq i \leq N_0 - 1 \) belongs to \( V_0 \). Let us also set:

\[
 n = \# \{ w \mid |w| = m \} \quad , \quad N_0 = \# V_0
\]

This enables one to introduce, for any integer \( k \) belonging to \( \{0, \ldots, N - 1\} \), the solution vector \( U(k) \) as:

\[
 U(k) = \begin{pmatrix}
 u(kh, X_{w_1P_1}) \\
 u(kh, X_{w_1P_2}) \\
 \vdots \\
 u(kh, X_{w_2P_1}) \\
 \vdots \\
 u(kh, X_{w_nP_{N_0}})
\end{pmatrix}
\]

One has:

\[
 U(k + 1) = AU(k) - U(k - 1)
\]

where:

\[
 A = 2I_{N_0} - h^2 \times \tilde{\Delta}_m
\]

and where \( I_m \) denotes the \((\#V_m) \times (\#V_m)\) identity matrix, and \( \tilde{\Delta}_m \) the \((\#V_m) \times (\#V_m)\) normalized Laplacian matrix.

### 3.2.2 Theoretical study of the error, for Hölder continuous functions

A previously, let us thus consider a function \( u \) in the domain of the Laplacian, and a nonnegative real constant \( \alpha \) such that:

\[
 \forall (X, Y) \in \mathbb{M}^2, \forall t > 0 : \quad |u(t, X) - u(t, Y)| \leq C(t) |X - Y|^\alpha
\]

where \( C \) denotes a positive function of the time variable \( t \).

Given a strictly positive integer \( m \), due to:

\[
 \Delta_m u(t, X) = \sum_{Y \in V_m, Y \sim X} (u(t, Y) - u(t, X)) \quad \forall t > 0, \forall X \in V_m \setminus V_0
\]

one has then, for any \( X \) belonging to the set \( V_m \setminus V_0 \), any integer \( k \) belonging to \( \{0, \ldots, N - 1\} \), and any strictly positive number \( h \):

\[
 64^m h^2 |\Delta_m u(kh, X)| \leq 64^m h^2 C(kh) \frac{1}{1 - 4^{-\alpha}}
\]

Using the scheme \((S_M)\), one gets:

\[
 |u((k + 1)h, X) - 2u(kh, X) + u((k - 1)h, X))| \leq \frac{64^m h^2 C(kh)}{1 - 4^{-\alpha}}
\]

One may note that a required condition for the convergence of the scheme is:

\[
 \lim_{m \to +\infty, h \to 0^+} 64^m h^2 C(kh) = 0
\]

In the case where \( C \) is a constant function, it reduces to:

\[
 \lim_{m \to +\infty, h \to 0^+} 64^m h^2 = 0
\]
3.2.3 Numerical results

In the sequel, we present the numerical results for \( m = 2, T = 10 \) and \( N = 10^3 \). As it was the case for the heat equation, the solution shows abnormal behavior until the ration \( h^2 \times 64^m \) goes towards zero.

Figure 6 – The graph of the approached solution of the wave equation for \( t = 0 \).

Figure 7 – The graph of the approached solution of the wave equation for \( t = 5 \).
Figure 8 – The graph of the approached solution of the wave equation for $t = 10$.

Figure 9 – The graph of the approached solution of the wave equation for $t = 15$. 
3.3 Error computing

Let us recall that, given a first order forward difference in time, a strictly positive integer $N$, and a strictly positive real number $h = \frac{T}{N}$, one has:

$$\forall X \in \mathcal{MC} : \frac{\partial u}{\partial t}(kh,X) - \frac{1}{h}(u((k + 1)h,X) - u(kh,X)) = \mathcal{O}(h)$$

and:

$$\forall X \in \mathcal{F} : \frac{\partial^2 u}{\partial t^2}(kh,X) - \frac{1}{h^2}(u((k + 1)h,X) - 2u(kh,X) + u((k - 1)h,X)) = \mathcal{O}(h^2)$$

If one aims at computing the error related to the use of discrete Laplacian, one may consider the Dirichlet problem:

$$\begin{align*}
-\Delta u + qu &= f \\
u_{|\partial\mathcal{F}} &= 0
\end{align*}$$

We choose $u$ to be harmonic, and we fix $q = 2$ to obtain $f = 2u$. We calculate next the solution corresponding to the finite difference scheme for this Dirichlet problem.

As previously, the Laplacian is approximated by means of the graph Laplacians $\Delta_m u$, defined on the sequence of graphs $(\mathcal{MC}_m)_{m \in \mathbb{N}}$:

$$\forall k \{0, \ldots, N - 1\}, \forall (t, X) \in [0,T] \times \mathcal{MC} : \Delta u(t, X) = 64^m \left( \sum_{X \sim Y} u(kh,Y) - u(kh,X) \right) + E_m$$

where $E_m$ denotes an error term, that may be obtained numerically. One gets:
\[
\left\{
\begin{aligned}
-64^m \left( \sum_{X \sim Y} u(Y) - u(X) \right) + qu(X) &= f(X) \quad \forall X \in V_m \setminus V_0 \\
u(P_j) &= 0 \quad \forall P_j \in V_0
\end{aligned}
\]

One defines the approximate equation as:

\[
-64^m \left( \sum_{X \sim Y} u(Y) - u(C) \right) + qu(X) = f(X)
\]

We now fix \( m \in \mathbb{N} \), and denote any \( C \in \mathcal{F} \) as \( X_{w,P_i} \), where \( w \in \{1, \ldots, N \}^m \) denotes a word of length \( m \), and where \( P_i, 0 \leq i \leq N_0 - 1 \) belongs to \( V_0 \). Let us also set:

\[
n = \# \left\{ w \mid |w| = m \right\}, \quad N_0 = \#V_0
\]

We then introduce the vector \( U \):

\[
U = \begin{pmatrix}
  u(X_{w_1 P_1}) \\
u(X_{w_1 P_2}) \\
  \vdots \\
u(X_{w_2 P_1}) \\
  \vdots \\
u(X_{w_n P_{N_0}})
\end{pmatrix}
\]

One has:

\[
AU = F
\]

where:

\[
A = \tilde{\Delta}_m + q I_m
\]

and:

\[
F = \begin{pmatrix}
f(X_{w_1 P_1}) \\
f(X_{w_1 P_2}) \\
  \vdots \\
f(X_{w_2 P_1}) \\
  \vdots \\
f(X_{w_n P_{N_0}})
\end{pmatrix}
\]

where \( I_m \) denotes the \((\#V_m) \times (\#V_m)\) identity matrix, and \( \tilde{\Delta}_m \) the \((\#V_m) \times (\#V_m)\) normalized Laplacian matrix.

One may begin by evaluating the error of the Laplacian approximation on the Minkowski curve, we use the Dirichlet problem:

\[
\left\{
\begin{aligned}
-\Delta u + qu &= f \\
u|_{\partial \mathcal{M} C} &= 0
\end{aligned}
\right.
\]

for an harmonic function \( u \), the problem has a solution. The error is then:

\[
E_m = \| u - u_m \|_{\infty}
\]


4 Conclusion

Refined numerical methods that fit problems of everyday life like diffusion, or transmission of waves by fractal micro-antennas, seem unavoidable. As for now, current computations do not take into account advances in fractal analysis. Our finite-difference scheme is destined to help improve those calculations. Lots remain to be done: in a near future, specific tests are to be made with existing Minkowski fractal antennas.

Annex: Algorithm and Mathematica program

In the sequel, we explain how one may implement our algorithm. We choose to build a specific Mathematica program, that enables one to compute the numerical solution of the heat equation. The program breaks down into five parts, which arise naturally from the above theoretical study.

Annex 1. Initialization

One here defines the initial points $P_0$ and $P_1$ that constitutes the set $V_0$, as well as the similarities $f_i$, $0 \leq i \leq N - 1$.

Annex 2. Harmonic functions

One here defines a harmonic function "HadresseH" on $\mathbb{R}$; given the value of a harmonic function $f$ such that:

$$f(P_0) = a \in \mathbb{R}, \quad f(P_1) = b \in \mathbb{R}$$

One thus gets the value of $f$ at a point, the address of which $w$ belongs to $\{1, ..., 8\}^m$, $m \in \mathbb{N}^*$, for an initial point $p \in V_0$. For instance:

$$\text{HadresseH}[0, 1, 2, 1, p0] = \frac{1}{64}$$

Annex 3. Construction of the Laplacian matrix

We build a function "sparseMat" enables one to construct a $n \times n$ tridiagonal matrix, $n \in \mathbb{N}^*$, with respective values $x$, $y$ and $z$ on the tridiagonal ($x$ on the diagonal).

We have also built a function "MatL" gives the Laplacian matrix of order $n$, i.e., associated with the $n^{th}$ order approximate graph. The matrix is also a tridiagonal one, since each point has a unique predecessor and a single successor with the exception of the first and last point ($V_0$ is the boundary).

For instance:

$$\text{MatL}[1] = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}$$
Annex 4. Construction of the initial vector

We build a function called "Genadresse", that defines any required address, the length of which take the value \( n \in \mathbb{N}^* \).

We have also introduced a function "Fadress", which enables one to calculate the coordinates of a point, given its address \( w \), starting from an initial point \( p \).

To define the initial vector, i.e. the one that yields all the points of the approximate graph \( \mathfrak{M}C_m \) of order \( m \in \mathbb{N}^* \), one uses the function "InitV". We have used the fact that any point of \( V_m \setminus V_0 \) owns two addresses:

\[
f_{w_1...w_m}(P_0) = f_{w_1...(w_m+1)}(P_1)
\]

This enables one to obtain the points \( V_m \) by applying all the similarities of order \( m \) to the point \( P_1 \), and, then, add the missing point \( P_0 \). For instance:

\[
\text{InitV}[1] = \{ \{0,0\}, \{\frac{1}{4},0\}, \{\frac{1}{4},\frac{1}{4}\}, \{\frac{1}{2},\frac{1}{4}\}, \{\frac{1}{2},-\frac{1}{4}\}, \{\frac{3}{4},-\frac{1}{4}\}, \{\frac{3}{4},0\}, \{1,0\} \}
\]

Annex 5. Solution of the Heat equation

One has to define an initial function \( gH \), which corresponds to the value of the searched solution \( u \), at the initial time \( t = 0 \). For the sake of simplicity, we choose the identically null function \( gH = 0 \).

The function "AAH \([N, T, k, n]"\) yields the matrix \( I - h \Delta_m \) for a solution with horizon \( T > 0 \), \( T \) denotes the number of steps, \( k \), the order of the approximate graph \( \mathfrak{M}C_k \), and \( n \) the number of points. For instance:

\[
\text{AAH}[10, 10, 1, 9] = \begin{pmatrix}
-63 & 64 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
64 & -127 & 64 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 64 & -127 & 64 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 64 & -127 & 64 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 64 & -127 & 64 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 64 & -127 & 64 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 64 & -127 & 64 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 64 & -127 & 64 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 64 & -63
\end{pmatrix}
\]

One then defines a function "Per\([k, j, m]" \) which, given a graph of order \( k \) and a number of points \( m \), associates the value 1 to the point of order \( j \) in the "initial vector".

Another function, "UH\([t, k, T, N, g, P, n, m]" recursively gives, at time \( t > 0 \), for a graph order \( k \), a horizon \( T \), and a step number \( N \), the value reached by an initial function \( g + P \).

A function "DFHA\([t, n, T, N, g, P, j]" yields the coordinate triplet \((x, y) + solution\).

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