INFINITESIMAL CHEREDNIK ALGEBRAS AS 
W-ALGEBRAS

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Dedicated to Evgeny Borisovich Dynkin on his 90th birthday

Abstract. In this article we establish an isomorphism between universal infinitesimal Cherednik algebras and W-algebras for Lie algebras of the same type and 1-block nilpotent elements. As a consequence we obtain some fundamental results about infinitesimal Cherednik algebras.

Introduction

This paper is aimed at the identification of two algebras of seemingly different nature. The first, finite W-algebras, are algebras constructed from a pair \((g, e)\), where \(e\) is a nilpotent element of a finite dimensional simple Lie algebra \(g\). Their theory has been extensively studied during the last decade. For the related references see, for example, reviews [L6], [W] and articles [BGK], [BK1], [BK2], [GG], [L1], [L2], [L3], [P1], [P2].

The second class of algebras we consider in this paper are the so called infinitesimal Cherednik algebras of type \(gl_n\) and \(sp_{2n}\), introduced in [EGG]. These are certain continuous analogues of the rational Cherednik algebras and in the case of \(gl_n\) are deformations of the universal enveloping algebra \(U(sl_{n+1})\). In both cases we call \(n\) the rank of an algebra. The theory of those algebras is less developed, while the main references there are: [EGG], [T1], [T2], [DT].

This paper is organized in the following way:

• In Section 1, we recall the definitions of infinitesimal Cherednik algebras \(H_\xi(gl_n)\), \(H_\xi(sp_{2n})\), and introduce their modified versions, called the universal length \(m\) infinitesimal Cherednik algebras. We also recall the definitions and basic results about the finite W-algebras \(U(g, e)\).

• In Section 2, we prove our main result, establishing an abstract isomorphism

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of $W$-algebras $U(\mathfrak{sl}_{n+m}, e_m)$ (respectively $U(\mathfrak{sp}_{2n+2m}, e_m)$) with the universal infinitesimal Cherednik algebras $H_m(\mathfrak{gl}_n)$ (respectively $H_m(\mathfrak{sp}_{2n})$).

- In Section 3, we establish explicitly a Poisson analogue of the aforementioned isomorphism. As a result we deduce two claims needed to carry out the arguments of the previous section.

- In Section 4, we derive several important consequences about algebras $H_\zeta(\mathfrak{gl}_n)$, $H_\zeta(\mathfrak{sp}_{2n})$. This clarifies some lengthy computations from [T1], [T2], [DT] and proves new results. Using the results of [DT, Sect. 3], about the Casimir element of $H_\zeta(\mathfrak{gl}_n)$, we determine the aforementioned isomorphism $H_m(\mathfrak{gl}_n) \simeq U(\mathfrak{sl}_{n+m}, e_m)$ explicitly.

- In Section 5, we recall the machinery of completions of the graded deformations of Poisson algebras, developed by the first author in [L1]. This provides the decomposition theorem for the completions of infinitesimal Cherednik algebras. This is analogous to a result by Bezrukavnikov and Etingof ([BE, Thm. 3.2]) in the theory of rational Cherednik algebras.

- In the Appendix, we provide some computations.

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1. Basic definitions

1.1. Infinitesimal Cherednik algebras of $\mathfrak{gl}_n$

We recall the definition of the infinitesimal Cherednik algebras $H_\zeta(\mathfrak{gl}_n)$ following [EGG]. Let $V_n$ and $V_n^*$ be the basic representation of $\mathfrak{gl}_n$ and its dual. Choose a basis $\{y_i\}_{1 \leq i \leq n}$ of $V_n$ and let $\{x_i\}_{1 \leq i \leq n}$ denote the dual basis of $V_n^*$. For any $\mathfrak{gl}_n$-invariant pairing $\zeta : V_n \times V_n^* \to \mathbb{U}(\mathfrak{gl}_n)$, define an algebra $H_\zeta(\mathfrak{gl}_n)$ as the quotient of the semi-direct product algebra $\mathbb{U}(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)$ by the relations $[y, x] = \zeta(y, x)$ and $[x, x'] = [y, y'] = 0$ for all $x, x' \in V_n^*$ and $y, y' \in V_n$. Consider an algebra filtration on $H_\zeta(\mathfrak{gl}_n)$ by setting $\text{deg}(V_n) = \text{deg}(V_n^*) = 1$ and $\text{deg}(\mathfrak{gl}_n) = 0$.

**Definition 1.** We say that $H_\zeta(\mathfrak{gl}_n)$ satisfies the PBW property if the natural surjective map $\mathbb{U}(\mathfrak{gl}_n) \ltimes S(V_n \oplus V_n^*) \twoheadrightarrow \text{gr}H_\zeta(\mathfrak{gl}_n)$ is an isomorphism, where $S$ denotes the symmetric algebra. We call these $H_\zeta(\mathfrak{gl}_n)$ the infinitesimal Cherednik algebras of $\mathfrak{gl}_n$.

It was shown in [EGG, Thm. 4.2], that the PBW property holds for $H_\zeta(\mathfrak{gl}_n)$ if and only if $\zeta = \sum_{j=0}^k \zeta_j r_j$ for some nonnegative integer $k$ and $\zeta_j \in \mathbb{C}$, where $r_j(y, x) \in \mathbb{U}(\mathfrak{gl}_n)$ is the symmetrization of $\alpha_j(y, x) \in S(\mathfrak{gl}_n) \simeq \mathbb{C}[\mathfrak{gl}_n]$ and $\alpha_j(y, x)$ is defined via the expansion

$$(x, (1 - \tau A)^{-1} y) \det(1 - \tau A)^{-1} = \sum_{j \geq 0} \alpha_j(y, x) A^j, \quad A \in \mathfrak{gl}_n.$$

Let us define the length of such $\zeta$ by $l(\zeta) := \min\{m \in \mathbb{Z}_{\geq -1} \mid \zeta_{m+1} = 0\}$.

**Example 1** (cf. [EGG, Example 4.7]). If $l(\zeta) = 1$ then $H_\zeta(\mathfrak{gl}_n) \simeq U(\mathfrak{sl}_{n+1})$. Thus, for an arbitrary $\zeta$, we can regard $H_\zeta(\mathfrak{gl}_n)$ as a deformation of $U(\mathfrak{sl}_{n+1})$. 

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One interesting problem is to find deformation parameters $\zeta$ and $\zeta'$ of the above form with $H_\zeta(\mathfrak{gl}_n) \simeq H_{\zeta'}(\mathfrak{gl}_n)$. Even for $n = 1$ (when $H_\zeta(\mathfrak{gl}_1)$ are simply the generalized Weyl algebras), the answer to this question (given in [BJ]) is quite nontrivial. Instead, we will look only for the filtration preserving isomorphisms, where both algebras are endowed with the $N$th standard filtration $\{F_\bullet^N\}$. Those are induced from the grading on $T(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)$ with $\deg(\mathfrak{gl}_n) = 2$ and $\deg(V_n \oplus V_n^*) = N$, where $N > l(\zeta)$. For $N \geq \max\{l(\zeta)+1, l(\zeta') + 1, 3\}$ we have the following result (see Appendix A for a proof):

**Lemma 1.**

(a) $N$-standardly filtered algebras $H_\zeta(\mathfrak{gl}_n)$ and $H_{\zeta'}(\mathfrak{gl}_n)$ are isomorphic if and only if there exist $\lambda \in \mathbb{C}, \theta \in \mathbb{C}^*, s \in \{\pm\}$ such that $\zeta' = \theta \varphi_\lambda(\zeta^s)$, where

- $\varphi_\lambda : U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n)$ is an isomorphism defined by $\varphi_\lambda(A) = A + \lambda \cdot \tr A$ for any $A \in \mathfrak{gl}_n$,
- for $\zeta = \zeta_0 r_0 + \zeta_1 r_1 + \zeta_2 r_2 + \cdots$ we define $\zeta^- := \zeta_0 r_0 - \zeta_1 r_1 + \zeta_2 r_2 - \cdots$, $\zeta' := \zeta$.

(b) For any length $m$ deformation $\zeta$, there is a length $m$ deformation $\zeta'$ with $\zeta'_m = 1$, $\zeta'_{m-1} = 0$, such that algebras $H_\zeta(\mathfrak{gl}_n)$ and $H_{\zeta'}(\mathfrak{gl}_n)$ are isomorphic as filtered algebras.

### 1.2. Infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$

Let $V_{2n}$ be the standard $2n$-dimensional representation of $\mathfrak{sp}_{2n}$ with a symplectic form $\omega$. Given any $\mathfrak{sp}_{2n}$-invariant pairing $\zeta : V_{2n} \times V_{2n} \to U(\mathfrak{sp}_{2n})$ we define an algebra $H_\zeta(\mathfrak{sp}_{2n}) := U(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})/(\{[x,y] - \zeta(x,y) | x,y \in V_{2n}\})$. It has a filtration induced from the grading $\deg(\mathfrak{sp}_{2n}) = 0$, $\deg(V_{2n}) = 1$ on $T(\mathfrak{sp}_{2n} \oplus V_{2n})$.

**Definition 2.** Algebra $H_\zeta(\mathfrak{sp}_{2n})$ is referred to as the infinitesimal Cherednik algebra of $\mathfrak{sp}_{2n}$, if it satisfies the PBW property: $U(\mathfrak{sp}_{2n}) \xrightarrow{\sim} \gr H_\zeta(\mathfrak{sp}_{2n})$.

It was shown in [EGG, Thm. 4.2], that $H_\zeta(\mathfrak{sp}_{2n})$ satisfies the PBW property if and only if $\zeta = \sum_{j=0}^k \zeta_j r_{2j}$ for some nonnegative integer $k$ and $\zeta_j \in \mathbb{C}$, where $r_{2j}(x,y) \in U(\mathfrak{sp}_{2n})$ is the symmetrization of $\beta_{2j}(x,y) \in S(\mathfrak{sp}_{2n}) \simeq \mathbb{C}[\mathfrak{sp}_{2n}]$ and $\beta_{2j}(x,y)$ is defined via the expansion

$$\omega(x,(1 - \tau A^2)^{-1} y) \det(1 - \tau A)^{-1} = \sum_{j \geq 0} \beta_{2j}(x,y)(A)^{2j}, \quad A \in \mathfrak{sp}_{2n}. $$

Similarly to the $\mathfrak{gl}_n$-case, we define the length of such $\zeta$ by $l(\zeta) := \min\{m \in \mathbb{Z}_{>0} \mid \zeta_{\geq m+1} = 0\}$.

**Example 2** (cf. [EGG, Example 4.11]). For $\zeta_0 \neq 0$ we have

$$H_{\zeta_0 r_0}(\mathfrak{sp}_{2n}) \cong U(\mathfrak{sp}_{2n}) \ltimes W_n,$$

where $W_n$ is the $n$th Weyl algebra. Thus, $H_\zeta(\mathfrak{sp}_{2n})$ can be regarded as a deformation of $U(\mathfrak{sp}_{2n}) \ltimes W_n$.

For any $N > 2l(\zeta)$, we introduce the $N$th standard filtration $\{F_\bullet^N\}$ on $H_\zeta(\mathfrak{sp}_{2n})$ by setting $\deg(\mathfrak{sp}_{2n}) = 2$, $\deg(V_{2n}) = N$. The following result is analogous to Lemma 1:
1.3. Universal algebras $H_m(\mathfrak{gl}_n)$ and $H_m(\mathfrak{sp}_{2n})$

It is natural to consider a version of those algebras with $\zeta_j$ being independent central variables. This motivates the following notion of the universal length $m$ infinitesimal Cherednik algebras.

**Definition 3.** The universal length $m$ infinitesimal Cherednik algebra $H_m(\mathfrak{gl}_n)$ is the quotient of $U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ by the relations

\[
[x, x'] = 0, \quad [y, y'] = 0, \quad [A, x] = A(x), \quad [A, y] = A(y),
\]

\[
[y, x] = \sum_{j=0}^{m-2} \zeta_j r_j(y, x) + r_m(y, x),
\]

where $x, x' \in V_n^*$, $y, y' \in V_n$, $A \in \mathfrak{gl}_n$ and $\{\zeta_j\}_{j=0}^{m-2}$ are central. The filtration is induced from the grading on $T(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ with $\deg(\mathfrak{gl}_n) = 2$, $\deg(V_n \oplus V_n^*) = m + 1$, $\deg(\zeta_j) = 2(m - i)$ (the latter is chosen in such a way that $\deg(\zeta_j r_j) = 2m$ for all $j$).

Algebra $H_m(\mathfrak{gl}_n)$ is free over $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$ and $H_m(\mathfrak{gl}_n)/([\zeta_0 - c_0, \ldots, \zeta_{m-2} - c_{m-2}])$ is the usual infinitesimal Cherednik algebra $H_{\zeta}(\mathfrak{gl}_n)$ with $\zeta = c_0 r_0 + \cdots + c_{m-2} r_{m-2} + r_m$. In fact, for odd $m$, $H_m(\mathfrak{gl}_n)$ can be viewed as a universal family of length $m$ infinitesimal Cherednik algebras of $\mathfrak{gl}_n$, while for even $m$, there is an action of $\mathbb{Z}/2\mathbb{Z}$ we should quotient by\(^1\).

**Remark 1.** One can consider all possible quotients

\[
U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}] / I
\]

for

\[
I = \langle [x, x'], [y, y'], [A, x] - A(x), [A, y] - A(y), [y, x] - \eta(y, x) \rangle,
\]

with a $\mathfrak{gl}_n$-invariant pairing $\eta : V_n \times V_n^* \to U(\mathfrak{gl}_n)[\zeta_0, \ldots, \zeta_{m-2}]$ such that the inequality $\deg(\eta(y, x)) \leq 2m$ holds. Such a quotient satisfies a PBW property if and only if $\eta(y, x) = \sum_{i=0}^{m} \eta_i(\zeta_0, \ldots, \zeta_{m-2}) r_i(y, x)$ with $\deg(\eta_i(\zeta_0, \ldots, \zeta_{m-2})) \leq 2(m - i)$ (this is completely analogous to [EGG, Thm. 4.2]).

We define the universal version of $H_{\zeta}(\mathfrak{sp}_{2n})$ in a similar way:

**Definition 4.** The universal length $m$ infinitesimal Cherednik algebra $H_m(\mathfrak{sp}_{2n})$ is defined as

\[
H_m(\mathfrak{sp}_{2n}) := U(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})[\zeta_0, \ldots, \zeta_{m-1}] / J
\]

for

\[
J = \langle [A, x] - A(x), [x, y] - \sum_{j=0}^{m-1} \zeta_j r_{2j}(x, y) - r_{2m}(x, y) \rangle,
\]

---

\(^1\) This follows from our proof of Lemma 1.
where $A \in \mathfrak{sp}_{2n}$, $x, y \in V_{2n}$ and $\{\zeta^n\}_{n=0}^{\infty}$ are central. The filtration is induced from the grading on $T(\mathfrak{sp}_{2n} \oplus V_{2n})[\zeta, \zeta_{m-1}]$ with $\deg(\mathfrak{sp}_{2n}) = 2$, $\deg(V_{2n}) = 2m+1$ and $\deg(\zeta_i) = 4(m - i)$.

The algebra $H_m(\mathfrak{sp}_{2n})$ is free over the subalgebra $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}]$ and the algebra $H_m(\mathfrak{sp}_{2n})/(\zeta_0 - c_0, \ldots, \zeta_{m-1} - c_{m-1})$ is the usual infinitesimal Cherednik algebra $H_{\mathbb{C}}(\mathfrak{sp}_{2n})$ for $\zeta_0 = c_0 r_0 + \cdots + c_{m-1} r_2(m-1) + r_2 n$. In fact, the algebra $H_m(\mathfrak{sp}_{2n})$ can be viewed as a universal family of length $m$ infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$, due to Lemma 2.

**Remark 2.** Analogously to Remark 1, the result of [EGG, Thm. 4.2], generalizes straightforwardly to the case of $\mathfrak{sp}_{2n}$-invariant pairings $\eta : V_{2n} \times V_{2n} \to U(\mathfrak{sp}_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$.

**1.4. Poisson counterparts of $H_{\mathbb{C}}(\mathfrak{g})$ and $H_m(\mathfrak{g})$**

Following [DT], we introduce the Poisson algebras $H_{\mathbb{C}}^1(\mathfrak{g})$ for $\mathfrak{g}$ being $\mathfrak{gl}_n$ or $\mathfrak{sp}_{2n}$.

As algebras these are $S(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ (respectively $S(\mathfrak{sp}_{2n} \oplus V_{2n})[\zeta_0, \ldots, \zeta_{m-2}]$) with a Poisson bracket $\{\cdot, \cdot\}$ modeled after the commutator $[\cdot, \cdot]$ from the definition of $H_m(\mathfrak{g})$, so that $\{y, x\} = \alpha_m(y, x) + \sum_{j=0}^{m-2} \zeta_j \beta_j(y, x) \text{ (respectively } \{x, y\} = \beta_2(x, y) + \sum_{j=0}^{m-1} \zeta_j \alpha_j(x, y)\}$.

Their quotients $H_{\mathbb{C}}^1(\mathfrak{gl}_n)/\langle \zeta_0 - c_0, \ldots, \zeta_{m-2} - c_{m-2} \rangle$ and $H_{\mathbb{C}}^1(\mathfrak{sp}_{2n})/(\zeta_0 - c_0, \ldots, \zeta_{m-1} - c_{m-1})$, are the Poisson infinitesimal Cherednik algebras $H_m^1(\mathfrak{gl}_n)$ ($\zeta_0 = c_0 \alpha_0 + \cdots + c_{m-2} \alpha_{m-2} + \alpha_m$, and $H_m^1(\mathfrak{sp}_{2n})$ ($\zeta_0 = c_0 \beta_0 + \cdots + c_{m-1} \beta_{m-1} + \beta_2$) from [DT, Sects. 5 and 7] respectively.

Let us describe the Poisson centers of the algebras $H_{\mathbb{C}}^1(\mathfrak{gl}_n)$ and $H_{\mathbb{C}}^1(\mathfrak{sp}_{2n})$.

For $\mathfrak{g} = \mathfrak{gl}_n$ and $1 \leq k \leq n$ we define an element $\tau_k \in H_m^1(\mathfrak{g})$ by $\tau_k := \sum_{i=1}^n x_i \{Q_k, y_i\}$, where $1 + \sum_{j=1}^n \tilde{Q}_j z^j = \det(1 + z A)$. We set $\zeta(w) := \sum_{i=0}^{m-2} \zeta_i w^i + w^m$ and define $c_i \in S(\mathfrak{gl}_n)$ via

$$c(t) = 1 + \sum_{i=1}^n (-1)^i c_i t^i := \text{Res}_{z=0}(z^{-1}) \frac{\det(1 - t A)}{\det(1 - z A)} \frac{z^{-1} dz}{1 - t^{-1} z}.$$  

For $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $1 \leq k \leq n$ we define an element $\tau_k \in H_m^1(\mathfrak{g})$ by $\tau_k := \sum_{i=1}^n \{Q_k, y_i\}$, where $1 + \sum_{j=1}^n \tilde{Q}_j z^j = \det(1 + z A)$, while $\{y_i\}_{i=1}^{2n}$ and $\{y_i\}_{i=1}^{2n}$ are the dual bases of $V_{2n}$, that is, $\omega(y_i, y_j) = 1$. We set $\zeta(w) := \sum_{i=0}^{m-1} \zeta_i w^i + w^m$ and define $c_i \in S(\mathfrak{sp}_{2n})$ via

$$c(t) = 1 + \sum_{i=1}^n c_i t^{2i} := 2 \text{Res}_{z=0}(z^{-2}) \frac{\det(1 - t A)}{\det(1 - z A)} \frac{z^{-1} dz}{1 - t^{-2} z^2}.$$  

The following result is a straightforward generalization of [DT, Thms. 5.1 and 7.1]:

**Theorem 3.** Let $\mathfrak{Poiss}(A)$ denote the Poisson center of the Poisson algebra $A$. We have:

(a) $\mathfrak{Poiss}(H_m^1(\mathfrak{gl}_n))$ is a polynomial algebra in free generators $\zeta_0, \ldots, \zeta_{m-2}, \tau_1 + c_1, \ldots, \tau_n + c_n$;

(b) $\mathfrak{Poiss}(H_m^1(\mathfrak{sp}_{2n}))$ is a polynomial algebra in free generators $\zeta_0, \ldots, \zeta_{m-1}, \tau_1 + c_1, \ldots, \tau_n + c_n$.  


1.5. \(W\)-algebras

Here we recall finite \(W\)-algebras following [GG].

Let \(g\) be a finite-dimensional simple Lie algebra over \(\mathbb{C}\) and \(e \in g\) be a nonzero nilpotent element. We identify \(g\) with \(g^*\) via the Killing form \((\ ,\ )\). Let \(\chi\) be the element of \(g^*\) corresponding to \(e\) and \(\chi_\chi\) be the stabilizer of \(\chi\) in \(g\) (which is the same as the centralizer of \(e\) in \(g\)). Fix an \(\mathfrak{sl}_2\)-triple \((e, h, f)\) in \(g\). Then \(\chi_\chi\) is \(\text{ad}(h)\)-stable and the eigenvalues of \(\text{ad}(h)\) on \(\chi_\chi\) are nonnegative integers.

Consider the \(\text{ad}(h)\)-weight grading on \(g = \mathbf{\bigoplus}_{i \in \mathbb{Z}} g(i)\), that is, \(g(i) := \{\xi \in g \mid [h, \xi] = i\xi\}\). Equip \(g(-1)\) with the symplectic form \(\omega_\chi(\xi, \eta) := (\chi, [\xi, \eta])\). Fix a Lagrangian subspace \(l \subset g(-1)\) and set \(m := \mathbf{\bigoplus}_{i \leq -2} g(i) \oplus l \subset g\), \(m^i := \{\xi - (\chi, \xi), \xi \in m\} \subset U(g)\).

Definition 5 (cf. [P1], [GG]). By the \(W\)-algebra associated with \(e\) (and \(l\)), we mean the algebra \(U(g, e) := (U(g)/U(g)m)^{\text{ad} m}\) with multiplication induced from \(U(g)\).

Let \(\{F^i\}\) denote the PBW filtration on \(U(g)\), while \(U(g)(i) := \{x \in U(g) \mid [h, x] = ix\}\). Define \(F_k U(g) = \sum_{i+j \leq k} (F_i^i U(g) \cap U(g)(i))\) and equip \(U(g, e)\) with the induced filtration, denoted \(\{F^i\}\) and referred to as the Kazhdan filtration.

One of the key results of [P1], [GG] is a description of the associated graded algebra \(\text{gr}_{F} U(g, e)\). Recall that the affine subspace \(S := \chi + (g/[g, f])^* \subset g^*\) is called the Slodowy slice. As an affine subspace of \(g\), the Slodowy slice \(S\) coincides with \(e + e\), where \(e = \ker \text{ad}(f)\). So we can identify \(\mathbb{C}[S] \cong \mathbb{C}[e]\) with the symmetric algebra \(S(\chi_\chi)\). According to [GG, Sect. 3], algebra \(\mathbb{C}[S]\) inherits a Poisson structure from \(\mathbb{C}[g^*]\) and is also graded with degree \(\text{deg}(\chi_\chi \cap g(i)) = i + 2\).

Theorem 4 (cf. [GG, Thm. 4.1]). The filtered algebra \(U(g, e)\) does not depend on the choice of \(l\) (up to a distinguished isomorphism) and \(\text{gr}_{F} U(g, e) \cong \mathbb{C}[S]\) as graded Poisson algebras.

1.6. Additional properties of \(W\)-algebras

We want to describe some other properties of \(U(g, e)\).

(a) Let \(G\) be the adjoint group of \(g\). There is a natural action of the group \(Q := Z_G(e, h, f)\) on \(U(g, e)\), due to [GG]. Let \(q\) stand for the Lie algebra of \(Q\). In [P2] Premet constructed a Lie algebra embedding \(q \hookrightarrow U(g, e)\). The adjoint action of \(q\) on \(U(g, e)\) coincides with the differential of the aforementioned \(Q\)-action.

(b) Restricting the natural map \(U(g)^{\text{ad} m} \to U(g, e)\) to \(Z(U(g))\), we get an algebra homomorphism \(Z(U(g)) \hookrightarrow Z(U(g, e))\), where \(Z(A)\) stands for the center of an algebra \(A\). According to the following theorem, \(\rho\) is an isomorphism:

Theorem 5.

(a) [P1, Sect. 6.2] The homomorphism \(\rho\) is injective.

(b) [P2, footnote to Quest. 5.1] The homomorphism \(\rho\) is surjective.

2. Main theorem

Let us consider \(g = \mathfrak{sl}_N\) or \(g = \mathfrak{sp}_{2N}\), and let \(e_m \in g\) be a 1-block nilpotent element of Jordan type \((1, \ldots, 1, m)\) or \((1, \ldots, 1, 2m)\), respectively. We make a particular choice for \(e_m\):
\[ e_m = E_{N-m+1,N-m+2} + \cdots + E_{N-1,N} \] in the case of \( \mathfrak{sl}_N \), \( 2 \leq m \leq N \),
\[ e_m = E_{N-m+1,N-m+2} + \cdots + E_{N+m-1,N+m} \] in the case of \( \mathfrak{sp}_{2N} \), \( 1 \leq m \leq N \).

Recall the Lie algebra inclusion \( \iota : q \hookrightarrow \mathfrak{u}(g,e) \) from Section 1.6. In our cases:

- For \( (g,e) = (\mathfrak{ssl}_{n+m}, e_m) \), we have \( q \cong \mathfrak{gl}_n \). Define \( \mathcal{T} \in U(\mathfrak{ssl}_{n+m}, e_m) \) to be the \( \iota \)-image of the identity matrix \( I_n \in \mathfrak{gl}_n \), the latter being identified with
\[ T_{n,m} = \text{diag}(m/(n + m), \ldots, m/(n + m), -n/(n + m), \ldots, -n/(n + m)) \]
under the inclusion \( q \hookrightarrow \mathfrak{ssl}_{n+m} \). Let \( \mathfrak{g} \) be the induced \( \text{ad}(\mathcal{T}) \)-weight grading on \( U(\mathfrak{ssl}_{n+m}, e_m) \), with the \( j \)th grading component denoted by \( U(\mathfrak{ssl}_{n+m}, e_m)^j \).

- For \( (g,e) = (\mathfrak{sp}_{2n+2m}, e_m) \), we have \( q \cong \mathfrak{sp}_{2n} \). Define
\[ \mathcal{T}' := \iota(I'_n) \in U(\mathfrak{sp}_{2n+2m}, e_m), \]
where \( I'_n = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \in \mathfrak{sp}_{2n} \). Let \( \mathfrak{g} \) be the induced \( \text{ad}(\mathcal{T}') \)-weight grading on \( U(\mathfrak{sp}_{2n+2m}, e_m) = \bigoplus_j U(\mathfrak{sp}_{2n+2m}, e_m)^j \).

**Lemma 6.** There is a natural Lie algebra inclusion \( \Theta : \mathfrak{gl}_n \ltimes V_n \hookrightarrow U(\mathfrak{ssl}_{n+m}, e_m) \) such that \( \Theta|\mathfrak{gl}_n = \iota|\mathfrak{gl}_n \) and \( \Theta(V_n) = F_{m+1}U(\mathfrak{ssl}_{n+m}, e_m) \).

**Proof.** First, choose a Jacobson–Morozov \( \mathfrak{sl}_2 \)-triple \( (e_m, h_m, f_m) \subset \mathfrak{ssl}_{n+m} \) in a standard way. As a vector space, \( 3\chi \cong \mathfrak{gl}_n \oplus V_n \oplus \mathfrak{c}^m \) with \( \mathfrak{gl}_n \cong \mathfrak{sl}_2(0) = q \), \( V_n \oplus V_n^* \subset \chi(m-1) \), and \( \chi_j \in \chi(2m-2j-2) \). Here \( \mathfrak{c}^m \) has a basis \( \{ \chi_{m-2-j} = E_{n+1,n+j+2} + \cdots + E_{n+m-2,1,1} \}_{j=0}^{m-2} \). \( V_n \oplus V_n^* \) is embedded via \( y_i \mapsto E_{i,n+m}, x_i \mapsto E_{n+1,i} \), while \( \mathfrak{gl}_n \cong \mathfrak{gl}_n \oplus \mathfrak{c} \). \( I_n \) is embedded in the following way: \( \mathfrak{ssl}_n \hookrightarrow \mathfrak{sl}_{n+m} \) as a left-up block, while \( I_n \hookrightarrow T_{n,m} \).

Under the identification \( \mathfrak{gr}_F U(\mathfrak{ssl}_{n+m}, e_m) \cong C[S] \cong S(3\chi) \), the induced grading \( \Theta' \) on \( S(3\chi) \) is the ad\((T_{n,m})\)-weight grading. Together with the above description of \( \text{ad}(h_m) \)-grading on \( 3\chi \), this implies that \( F_{m+1}U(\mathfrak{ssl}_{n+m}, e_m) \) coincides with the image of the composition \( V_n \hookrightarrow S(3\chi) \). Let \( \Theta(y) \in F_{m+1}U(\mathfrak{ssl}_{n+m}, e_m) \) be the element whose image is identified with \( y \). We also set \( \Theta(A) := \iota(A) \) for \( A \in \mathfrak{gl}_n \). Finally, we define \( \Theta : \mathfrak{gl}_n \ltimes V_n \hookrightarrow U(\mathfrak{ssl}_{n+m}, e_m) \) by linearity.

We claim that \( \Theta \) is a Lie algebra inclusion, that is,
\[ [\Theta(A), \Theta(B)] = \Theta([A,B]), \quad [\Theta(y), \Theta(y')] = 0, \quad [\Theta(A), \Theta(y)] = \Theta(A(y)), \]
\[ \forall A, B \in \mathfrak{gl}_n, y, y' \in V_n. \]

The first equality follows from \( [\Theta(A), \Theta(B)] = \iota([A,B]) = \Theta([A,B]) \). The second one follows from the observation that \( \Theta([y], \Theta(y')) \in F_{2m}U(g, e_m) \) and the only such element is 0. Similarly, \( \Theta(A, \Theta(y)) \in F_{m+1}U(g, e_m) \), so that \( \Theta(A, \Theta(y)) = \Theta(A(y)) \) for some \( y' \in V_n \). Since \( y = \mathfrak{gr}(\Theta(y')) = \mathfrak{gr}(\Theta(A), \Theta(y)) = [A, y] = A(y) \), we get \( \Theta(A), \Theta(y) = \Theta(A(y)) \).

Our main result is:

\footnote{We view \( \mathfrak{sp}_{2N} \) as corresponding to the pair \( (V_{2N}, \omega_{2N}) \), where \( \omega_{2N} \) is represented by the skew symmetric antidiagonal matrix \( J = (J_{i,j} := (-1)^{i+j+1})_{1 \leq i,j \leq 2N} \). In this presentation, \( A = (a_{ij}) \in \mathfrak{sp}_{2N} \) if and only if \( a_{2N+1-j,2N+1-i} = (-1)^{i+j+1}a_{ij} \) for any \( 1 \leq i, j \leq 2N \).
}
\footnote{That is, we set \( h_m := \sum_{j=1}^{m} (m + 1 - 2j)E_{n+j,n+j} \) and \( f_m := \sum_{j=1}^{m-1} j(m - j)E_{n+j+1,n+j} \).}
Theorem 7.

(a) For $m \geq 2$, there is a unique isomorphism

\[
\Theta : H_m(\mathfrak{gl}_n) \cong U(\mathfrak{gl}_{n+m}, e_m)
\]

of filtered algebras, whose restriction to $\mathfrak{sl}_n \ltimes V_n \hookrightarrow H_m(\mathfrak{gl}_n)$ is equal to $\Theta$.

(b) For $m \geq 1$, there are exactly two isomorphisms

\[
\Theta(1), \Theta(2) : H_m(\mathfrak{sp}_{2n}) \cong U(\mathfrak{sp}_{2n+2m}, e_m)
\]

of filtered algebras such that $\Theta(1) \restriction_{\mathfrak{sp}_{2n}} = \iota \restriction_{\mathfrak{sp}_{2n}}$; moreover, $\Theta(2) \circ \Theta^{-1}(1) : y \mapsto -y, A \mapsto A, \zeta_k \mapsto \zeta_k$.

Let us point out that there is no explicit presentation of $W$-algebras in terms of generators and relations in general. Among the few known cases are: (a) $g = \mathfrak{gl}_n$, due to [BK1], (b) $g \supset \mathfrak{c}$, the minimal nilpotent, due to [P2, Sect. 6]. The latter corresponds to $(v_2, \mathfrak{sl}_N)$ and $(\epsilon_1, \mathfrak{sp}_N)$ in our notation. We establish the corresponding isomorphisms explicitly in Appendix B.

Proof of Theorem 7.

(a) Analogously to Lemma 6, we have an identification $F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_{-1} \cong V_n^*$. For any $x \in V_n^*$, let $\Theta(x) \in F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_{-1}$ be the element identified with $x \in V_n^*$. The same argument as in the proof of Lemma 6 implies $[\Theta(A), \Theta(x)] = \Theta(A(x))$.

Let $\{F_j\}_{j=1}^{m+1}$ be the standard degree $j$ generators of the algebra $C[\mathfrak{sl}_{n+m}]^{S_{n+m}} \cong S(\mathfrak{sl}_{n+m})^{S_{n+m}}$ (that is, $1 + \sum_{j=2}^{m+1} F_j(A) z^j = \det(1 + zA)$ for $A \in \mathfrak{sl}_{n+m}$) and $F_j := \text{Sym}(F_j) \in U(\mathfrak{sl}_{n+m})$ to be the free generators of $Z(U(\mathfrak{sl}_{n+m}))$. For all $0 \leq i \leq m - 2$ we set $\Theta_i := \rho(F_{m-i}) \in Z(U(\mathfrak{sl}_{n+m}, e_m))$. Then $\text{gr}(\Theta_k) = F_{m-k} \equiv \xi_k \mod S(\mathfrak{gl}_n \oplus \bigoplus_{l=k+1}^{m-2} C\xi_l)$, where $\xi_k$ was defined in the proof of Lemma 6.

Let $U'$ be a subalgebra of $U(\mathfrak{sl}_{n+m}, e_m)$, generated by $\Theta(\mathfrak{gl}_n)$ and $(\Theta_k)^{m-2}_{k=0}$. For all $y \in V_n^*, x \in V_n^*$ we define $W(y, x) := [\Theta(y), \Theta(x)] \in F_{2m}U(\mathfrak{sl}_{n+m}, e_m)_{0} \subset U'$. Let us point out that equalities $[\Theta(A), \Theta(x)] = \Theta(A, x)$, $[\Theta(A), \Theta(y)] = \Theta(A, y)$ (for all $A \in \mathfrak{gl}_n, y \in V_n^*, x \in V_n^*$) imply the $\mathfrak{gl}_n$-invariance of $W : V_n \times V_n^* \to U' \cong U(\mathfrak{gl}_n)[\Theta_0, \ldots, \Theta_{m-2}]$.

By Theorem 4, $U(\mathfrak{sl}_{n+m}, e_m)$ has a basis formed by the ordered monomials in

\[
\{\Theta(E_{ij}), \Theta(y_k), \Theta(x_l), \Theta_0, \ldots, \Theta_{m-2}\}.
\]

In particular, $U(\mathfrak{sl}_{n+m}, e_m) \cong U(\mathfrak{gl}_n) \times T(V_n \oplus V_n^*)[\Theta_0, \ldots, \Theta_{m-2}]/(y \otimes x - x \otimes y - W(y, x))$ satisfies the PBW property. According to Remark 1, there exist polynomials $\eta_i \in C[\Theta_0, \ldots, \Theta_{m-2}]$, for $0 \leq i \leq m - 2$, such that $W(y, x) = \sum \eta_i r_j(y, x)$ and $\deg(\eta_i(\Theta_0, \ldots, \Theta_{m-2})) \leq 2(m - i)$. As a consequence of the latter condition: $\eta_m, \eta_{m-1} \in C$.

The following claim follows from the main result of the next section (Theorem 10):
Claim 8.

(i) The constant $\eta_m$ is nonzero.

(ii) The polynomial $\eta_1(\Theta_0, \ldots, \Theta_{m-2})$ contains a nonzero multiple of $\Theta_1$ for any $1 \leq m \leq 2$.

This claim implies the existence and uniqueness of the isomorphism $\overline{\Theta}: H_m(\mathfrak{gl}_n) \cong U(\mathfrak{sl}_{m+n}, e_m)$ with $\overline{\Theta}(y_k) = \Theta(y_k)$ and $\overline{\Theta}(A) = \Theta(A)$ for $A \in H_m$.

Moreover, $\overline{\Theta}(x_k) = n_m^{-1}\Theta(x_k)$ and $\overline{\Theta}(I_n) = \Theta(I_n) - n\eta_{m-1}/(n + m)\eta_m$ \footnote{The appearance of the constant $n\eta_{m-1}/(n + m)\eta_m$ is explained by the proof of Lemma 1(b)}. While for $\overline{\Theta}(x_{su})$, the same reasoning as in the way.

Recall the grading $G$ on $\mathfrak{gl}_n\otimes V_n \otimes \mathbb{C}$, the appearance of the constant $4$.

Claim 8.

Let $\mathfrak{sl}_{2^m} \subset \mathfrak{sp}_{2n+2m}$ be embedded in a natural way (via four $n \times n$ corner blocks of $\mathfrak{sp}_{2n+2m}$).

Recall the grading $G$ on $\mathfrak{sl}_{2^m}$, and $\mathfrak{sp}_{2n+2m}$ is the ad$(I_n)$-weight grading on $S(\mathfrak{sl}_n)$. The operator ad$(I_n)$ acts trivially on $\mathfrak{C}$, with even eigenvalues on $\mathfrak{sp}_{2n}$ and with eigenvalues $\pm 1$ on $V_{2n}$, $V_{2n}^{\pm}$.

Analogously to Lemma 6, we get identifications of $F_{2m+1}U(\mathfrak{sp}_{2n+2m}, e_m)$ and $V_{2n}^{\pm}$. For $y \in V_{2n}^{\pm}$, let $\Theta(y)$ be the corresponding element of $F_{2m+1}U(\mathfrak{sp}_{2n+2m}, e_m)$ and $\mathfrak{sp}_{2n}$, we set $\Theta(A) = :i(A)$. We define $\Theta : \mathfrak{sp}_{2n} \otimes V_{2n} \to U(\mathfrak{sp}_{2n+2m}, e_m)$ by linearity. The same reasoning as in the $\mathfrak{gl}_n$-case proves that $[\Theta(A), \Theta(y)] = \Theta([A, y])$ for any $A \in \mathfrak{sp}_{2n}, y \in V_{2n}$.

Finally, the argument involving the center goes along the same lines, so we can split central generators $\{\Theta_k\}_{1 \leq k \leq m-1}$ such that $\mathfrak{gr}(\Theta_k) \equiv \xi_k \mod S(\mathfrak{sp}_{2n} \otimes \mathbb{C}_{k+1} \otimes \cdots \otimes \mathbb{C}_{m-1})$.

Let $U'$ be the subalgebra of $U(\mathfrak{sp}_{2n+2m}, e_m)$ generated by $\Theta(\mathfrak{sp}_{2n})$ and $\{\Theta_k\}_{1 \leq k \leq m-1}$. For $x, y \in V_{2n}$, we set $W(x, y) := [\Theta(x), \Theta(y)] \in F_{2n}U(\mathfrak{sp}_{2n+2m}, e_m)$ even $\subset U'$. The map

$W : V_{2n} \times V_{2n} \to U(\mathfrak{sp}_{2n})[\Theta_0, \ldots, \Theta_{m-1}]$ is $\mathfrak{sp}_{2n}$-invariant.

Since $U(\mathfrak{sp}_{2n+2m}, e_m) \simeq U(\mathfrak{sp}_{2n}) \otimes T(V_{2n})[\Theta_0, \ldots, \Theta_{m-1}]/(x \otimes y - y \otimes x - W(x, y))$ satisfies the PBW property, there exist polynomials $\eta_i \in \mathbb{C}[\Theta_0, \ldots, \Theta_{m-1}]$, for $0 \leq i \leq m-1$, such that $W(x, y) = \sum_i \eta_ir_{2i}(x, y)$ and $\deg(\eta_0, \ldots, \Theta_{m-1}) \leq 4(m - i)$ (Remark 2).

The following result is analogous to Claim 8 and will follow from Theorem 10 as well:

\footnote{That is, $h_m := \sum_{j=1}^{2m} (2m + 1 - 2j)E_{n+j,n+j}$ and $f_m := \sum_{j=1}^{m-1} j(2m_j)}$
Claim 9.

(i) The constant $\eta_m$ is nonzero.

(ii) The polynomial $\eta_i(\Theta_0, \ldots, \Theta_{m-1})$ contains a nonzero multiple of $\Theta_i$ for any $i \leq m - 1$.

This claim implies Theorem 7(b), where $\tilde{\Theta}(y) = \lambda_i \cdot \Theta(y)$ for all $y \in V_n$ and $\lambda_i^2 = \eta_m^{-1}$. □

3. Poisson analogue of Theorem 7

To state the main result of this section, let us introduce more notation:

• In the contexts of $(\mathfrak{sl}_n + e_m)$ and $(\mathfrak{sp}_{2n} + e_m)$, we use $S_{n,m}$ and $\tilde{z}_{n,m}$ instead of $S$ and $\tilde{z}$.

• Let $\tau : \mathfrak{sl}_n \oplus V_n \oplus V^*_n \oplus \mathbb{C}^{m-1} \rightarrow \tilde{z}_{n,m}$ be the identification from the proof of Lemma 6.

• Let $\tilde{\tau} : \mathfrak{sp}_{2n} \oplus V_{2n} \oplus \mathbb{C}^m \rightarrow \tilde{z}_{n,m}$ be the identification from the proof of Theorem 7(b).

• Define $\Theta_k = \text{gr}(\Theta_k) \in S(\tilde{z}_{n,m})$ for $0 \leq k \leq m - s$, where $s = 1$ for $\mathfrak{sp}_{2N}$ and $s = 2$ for $\mathfrak{sl}_N$.

• We consider the Poisson structure on $S(\tilde{z}_{n,m})$ arising from the identification $S(\tilde{z}_{n,m}) \cong \mathbb{C}[S_{n,m}]$.

The following theorem can be viewed as a Poisson analogue of Theorem 7:

Theorem 10.

(a) The formulas

$$\Theta^{-1}(A) = \tau(A), \quad \Theta^{-1}(y) = \tau(y), \quad \Theta^{-1}(x) = \tau(x), \quad \Theta^{-1}(\zeta_k) = (-1)^{m-k}\Theta_k$$

define an isomorphism $\Theta^{-1} : H_m(\mathfrak{sl}_n) \rightarrow S(\tilde{z}_{n,m}) \cong \mathbb{C}[S_{n,m}]$ of Poisson algebras.

(b) The formulas

$$\Theta^{-1}(A) = \tau(A), \quad \Theta^{-1}(y) = \tau(y)/\sqrt{2}, \quad \Theta^{-1}(\zeta_k) = \Theta_k$$

define an isomorphism $\Theta^{-1} : H_m(\mathfrak{sp}_{2n}) \rightarrow S(\tilde{z}_{n,m}) \cong \mathbb{C}[S_{n,m}]$ of Poisson algebras.

Claims 8 and 9 follow from this theorem.

Remark 3. An alternative proof of Claims 8 and 9 is based on the recent result of [LNS] about the universal Poisson deformation of $S \cap N$ (here $N$ denotes the nilpotent cone of the Lie algebra $\mathfrak{g}$). We find this argument a bit overkilling (besides, it does not provide precise formulas in the Poisson case).
Proof of Theorem 10.

(a) The Poisson algebra $S(\mathfrak{g}_n, m)$ is equipped both with the Kazhdan grading and the internal grading $\text{Gr}^i$. In particular, the same reasoning as in the proof of Theorem 7(a) implies:

$$\{\tau(A), \tau(B)\} = \tau([A, B]), \quad \{\tau(A), \tau(y)\} = \tau(A(y)), \quad \{\tau(A), \tau(x)\} = \tau(A(x)).$$

We set $W(y, x) := \{\tau(y), \tau(x)\}$ for all $y \in V_n, x \in V_n^*$. Arguments analogous to those used in the proof of Theorem 7(a) imply an existence of polynomials $\pi_i \in \mathbb{C}[\bar{\Theta}_0, \ldots, \bar{\Theta}_{m-2}]$, such that $W(y, x) = \sum_j \pi_j y(x, y)$ and $\deg(\pi_j(\bar{\Theta}_0, \ldots, \bar{\Theta}_{m-2})) = 2(m - j)$.

Combining this with Theorem 3(a) one gets that

$$\tau_1' = \sum_i x_i y_i + \sum_j \pi_j \text{tr} S^{j+1} A$$

is a Poisson-central element of $S(\mathfrak{g}_n, m) \cong \mathbb{C}[S_{n, m}]$.

Let $\overline{\rho} : \text{Pois}(\mathbb{C}[\mathfrak{g}_{n+m}]) \to \text{Pois}(\mathbb{C}[S_{n, m}])$ be the restriction homomorphism. The Poisson analogue of Theorem 5 (which is, actually, much simpler) states that $\overline{\rho}$ is an isomorphism. In particular, $\tau_1' = c\overline{\rho}(\bar{F}_{m+1}) + p(\overline{\rho}(\bar{F}_2), \ldots, \overline{\rho}(\bar{F}_m))$ for some $c \in \mathbb{C}$ and a polynomial $p$.

Note that $\overline{\rho}(\bar{F}_i) = \bar{\Theta}_{m-i}$ for all $2 \leq i \leq m$. Let us now express $\overline{\rho}(\bar{F}_{m+1})$ via the generators of $S(\mathfrak{g}_n, m)$. First, we describe explicitly the slice $S_{n, m}$. It consists of the following elements:

$$\left\{ e_m + \sum_{i, j \leq n} x_{i, j} E_{i, j} + \sum_{i \leq n} u_i E_{i, n+1} + \sum_{i \leq n} v_i E_{n+m, i} + \sum_{k \leq m-1} w_k f_{m}^k - \gamma_{n, m} \sum_{n < j \leq n+m} E_{jj} \right\},$$

where $\gamma_{n, m} = \frac{1}{m} \sum_{i \leq n} x_{ii}$

which can also be explicitly depicted as follows:

$$S_{n, m} = \left\{ \begin{array}{cccccccc}
    x_{1,1} & x_{1,2} & \cdots & x_{1,n} & u_1 & 0 & 0 & \cdots & 0 \\
    x_{2,1} & x_{2,2} & \cdots & x_{2,n} & u_2 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    x_{n,1} & x_{n,2} & \cdots & x_{n,n} & u_n & 0 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 0 & \lambda & 1 & 0 & \cdots & 0 \\
    0 & 0 & \cdots & 0 & \ast & \ast & \lambda & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & 0 & \ast & \ast & \ast & \cdots & 1 \\
    v_1 & v_2 & \cdots & v_n & \ast & \ast & \ast & \cdots & \lambda \\
\end{array} \right\}$$

For $X \in \mathfrak{sl}_{n+m}$ of the above form let us define $X_1 \in \mathfrak{gl}_n$, $X_2 \in \mathfrak{gl}_m$ by

$$X_1 := \sum_{i, j \leq n} x_{i, j} E_{i, j}, \quad X_2 := e_m + \sum_{k \leq m-1} w_k f_{m}^k \frac{x_{11} + \cdots + x_{nn}}{m} \sum_{n < j \leq n+m} E_{jj},$$

that is, $X_1$ and $X_2$ are the left-up $n \times n$ and right-down $m \times m$ blocks of $X$, respectively.

The following result is straightforward:
Lemma 11. Let $X, X_1, X_2$ be as above. Then:

(i) For $2 \leq k \leq m$ : $\tilde{F}_k(X) = \text{tr} \Lambda^k(X_1) + \text{tr} \Lambda^{k-1}(X_1) \text{tr} \Lambda^1(X_2) + \cdots + \text{tr} \Lambda^1(X_2)$.

(ii) We have $\tilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \text{tr} \Lambda^{m+1}(X_1) + \text{tr} \Lambda^m(X_1) \text{tr} \Lambda^1(X_2) + \cdots + \text{tr} \Lambda^{m+1}(X_2)$.

Combining both statements of this lemma with the standard equality

$$\sum_{0 \leq j \leq l} (-1)^j \text{tr} S^{l-j}(X_1) \text{tr} \Lambda^j(X_1) = 0, \quad \forall l \geq 1, \quad (1)$$

we obtain the following result:

Lemma 12. For any $X \in S_{n,m}$ we have:

$$\tilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \sum_{2 \leq j \leq m} (-1)^{m-j} \tilde{F}_j(X) \text{tr} S^{m+1-j}(X_1) + (-1)^m \text{tr} S^{m+1}(X_1). \quad (2)$$

Proof of Lemma 12. Lemma 11(i) and equality (1) imply by induction on $k$:

$$\text{tr} \Lambda^k(X_2) = \tilde{F}_k(X) - \text{tr} S^k(X_1) \tilde{F}_{k-1}(X) + \text{tr} S^k(X_1) \tilde{F}_{k-2}(X) - \cdots + (-1)^k \text{tr} S^k(X_1) \tilde{F}_0(X),$$

for all $k \leq m$, where $\tilde{F}_1(X) = 0$, $\tilde{F}_0(X) = 1$.

Those equalities together with Lemma 11(ii) imply:

$$\tilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \sum_{0 \leq j \leq m} \sum_{0 \leq k \leq m+1-j} (-1)^k \text{tr} \Lambda^{m+1-j-k}(X_1) \text{tr} S^k(X_1) \tilde{F}_j(X).$$

According to (1), we have

$$\sum_{0 \leq k \leq m-j} (-1)^k \text{tr} \Lambda^{m+1-j-k}(X_1) \text{tr} S^k(X_1) = (-1)^{m-j} \text{tr} S^{m+1-j}(X_1).$$

Recalling our convention $\tilde{F}_1(X) := 0$, $\tilde{F}_0(X) := 1$, we get (2). \qed

Identifying $\mathbb{C}[S_{n,m}]$ with $S(\mathfrak{z}_{n,m})$ we get

$$\mathfrak{P}(\tilde{F}_{m+1}) = (-1)^m \left( \sum x_i y_i + \text{tr} S^{m+1} A + \sum_{2 \leq j \leq m} (-1)^j \mathfrak{P}_{m-j} \text{tr} S^{m+1-j} A \right). \quad (3)$$

Substituting this into $\tau'_1 = c \mathfrak{P}(\tilde{F}_{m+1}) + p(\mathfrak{P}_0, \ldots, \mathfrak{P}_{m-2})$ with $\mathfrak{P}_{m-1} := 0$, $\mathfrak{P}_m := 1$, we get

$$p(\mathfrak{P}_0, \ldots, \mathfrak{P}_{m-2}) = (1 - (-1)^m c) \sum x_i y_i + \sum_{0 \leq j \leq m} (\mathfrak{P}_j(\mathfrak{P}_0, \ldots, \mathfrak{P}_{m-2}) - (-1)^j c \mathfrak{P}_j) \text{tr} S^{j+1} A.$$
Hence $c = (-1)^m$ and
\[ p(\overline{\omega}_0, \ldots, \overline{\omega}_{m-2}) = \sum_{0 \leq j \leq m} (\overline{\eta}_j(\overline{\omega}_0, \ldots, \overline{\omega}_{m-2}) - (-1)^{m-j} \overline{\eta}_j) \operatorname{tr} S^{j+1} A. \]

According to Remark 1, the last equality is equivalent to
\[ \overline{\eta}_m = 1, \quad \overline{\eta}_{m-1} = 0, \quad \overline{\eta}_j(\overline{\omega}_0, \ldots, \overline{\omega}_{m-2}) = (-1)^{m-j} \overline{\eta}_j, \quad \forall 0 \leq j \leq m-2, \quad p = 0. \]

This implies the statement.

(b) Analogously to the previous case and the proof of Theorem 7(b) we have:
\[ \{\overline{\eta}(A), \overline{\eta}(B)\} = \overline{\eta}(\{A, B\}), \quad \{\overline{\eta}(A), \overline{\eta}(y)\} = \overline{\eta}(A(y)), \quad \{\overline{\eta}(x), \overline{\eta}(y)\} = \sum \overline{\eta}_j \beta_{j,i}(x, y), \]
for some polynomials $\overline{\eta}_j \in \mathbb{C}[\overline{\omega}_0, \ldots, \overline{\omega}_{m-1}]$, such that $\deg(\overline{\eta}_j(\overline{\omega}_0, \ldots, \overline{\omega}_{m-1})) = 4(m-j)$.

Due to Theorem 3(b), we get
\[ \tau'_1 := \sum_{i=1}^{2n} (\tilde{Q}_1, y_i) y_i^* - 2 \sum_j \overline{\eta}_j \operatorname{tr} S^{2j+2} A \in \mathfrak{z}_{\text{Pois}}(S(\mathfrak{j}_{n,m})). \]
In particular, $\tau'_1 = c \overline{\eta}(\tilde{F}_{m+1}) + p(\overline{\eta}(\tilde{F}_1), \ldots, \overline{\eta}(\tilde{F}_m))$ for some $c \in \mathbb{C}$ and a polynomial $p$.

Note that $\overline{\eta}(\tilde{F}_k) = \overline{\omega}_{m-k}$ for $1 \leq k \leq m$. Let us now express $\overline{\eta}(\tilde{F}_{m+1})$ via the generators of $S(\mathfrak{j}_{n,m})$. First, we describe explicitly the slice $S_{n,m}$. It consists of the following elements:
\[ \left\{ e_m + \overline{\eta}(X_1) + \sum_{i \leq n} v_i U_{i,n+1} + \sum_{i \leq n} v_{n+i} U_{n+i,n+1} + \sum_{k \leq m} w_k f_m^{2k-1} \mid X_1 \in \mathfrak{sp}_{2n}, \ v_i, v_{n+i}, w_k \in \mathbb{C} \right\}, \]
where $U_{i,j} := E_{i,j} + (-1)^{i+j+1} E_{2n+2m-1-j, 2n+2m-1-i} \in \mathfrak{sp}_{2n+2m}$. For $X \in \mathfrak{sp}_{2n+2m}$ of the above form let us define $X_2 := e_m + \sum_{k \leq m} w_k f_m^{2k-1} \in \mathfrak{sp}_{2m}$, viewed as the centered $2m \times 2m$ block of $X$.

Analogously to (3), we get the following formula:
\[ \overline{\eta}(\tilde{F}_{m+1}) = \frac{1}{4} \sum_{i=1}^{2n} (\tilde{Q}_1, y_i) y_i^* - \operatorname{tr} S^{2m+2} A - \sum_{0 \leq j \leq m-1} \overline{\eta}_j \operatorname{tr} S^{2j+2} A. \quad (4) \]

Comparing the above two formulas for $\tau'_1$, we get the equality:
\[ \sum_{i=1}^{2n} (\tilde{Q}_1, y_i) y_i^* - 2 \sum_j \overline{\eta}_j \operatorname{tr} S^{2j+2} A = c \cdot \overline{\eta}(\tilde{F}_{m+1}) + p(\overline{\omega}_0, \ldots, \overline{\omega}_{m-1}). \]
Arguments analogous to the one used in part (a) establish
\[ c = 4, \quad p = 0, \quad \overline{\eta}_m = 2, \quad \overline{\eta}_j = 2 \overline{\eta}_j, \quad \forall j < m. \]
Part (b) follows. \qed
Remark 4. Recalling the standard convention $U(\mathfrak{g},0) = U(\mathfrak{g})$ and Example 1, we see that Theorem 7(a) (as well as Theorem 10(a)) obviously holds for $m = 1$ with $\epsilon_1 := 0 \in \mathfrak{sl}_{n+1}$.

The results of Theorems 7 and 10 can be naturally generalized to the case of the universal infinitesimal Hecke algebras of $\mathfrak{so}_n$. However, this requires reproving some basic results about the latter algebras, similar to those of [EGG], [DT], and is discussed separately in [T].

4. Consequences

In this section we use Theorem 7 to get some new (and recover some old) results about the algebras of interest. On the $W$-algebra side, we get presentations of $U(\mathfrak{sl}_n, e_m)$ and $U(\mathfrak{sp}_{2n}, e_m)$ via generators and relations (in the latter case there was no presentation known for $m > 1$). We get many more results about the structure and the representation theory of infinitesimal Cherednik algebras using the corresponding results on $W$-algebras.

Also we determine the isomorphism from Theorem 7(a) basically explicitly.

4.1. Centers of $H_m(\mathfrak{gl}_n)$ and $H_m(\mathfrak{sp}_{2n})$

We set $s = 2$ for $\mathfrak{g} = \mathfrak{sl}_N$ and $s = 1$ for $\mathfrak{g} = \mathfrak{sp}_{2N}$. Recall the elements $\{\tilde{F}_i\}_{i=1}^N$, where $\text{deg}(\tilde{F}_i) = (3-s)i$. These are the free generators of the Poisson center $\mathfrak{p}_{\text{Pois}}(S(\mathfrak{g}))$. The Lie algebra $\mathfrak{q} = \mathfrak{g}(e,h,f)$ from Section 1.6 equals $\mathfrak{gl}_n$ for $(\mathfrak{g}, e) = (\mathfrak{sl}_{n+1}, e_m)$ and $\mathfrak{sp}_{2n}$ for $(\mathfrak{g}, e) = (\mathfrak{sp}_{2n+2}, e_m)$. Thus $\{\tilde{Q}_j\}$ from Section 1.4 are the free generators of $\mathfrak{p}_{\text{Pois}}(S(\mathfrak{g}))$, and $Q_j := \text{Sym}(\tilde{Q}_j)$ are the free generators of $Z(U(\mathfrak{g}))$.

The following result is a straightforward generalization of formulas (3) and (4):

Proposition 13. There exist $\{b_i\}_{i=1}^n \in S(\mathfrak{g})^{\text{ad}^0}[\mathfrak{p}(\tilde{F}_1), \ldots, \mathfrak{p}(\tilde{F}_n)]$, such that:

$$\mathfrak{p}(\tilde{F}_{m+i}) \equiv s_{n,m} t_i + b_i \mod [\mathfrak{p}(\tilde{F}_1), \ldots, \mathfrak{p}(\tilde{F}_{m+i-1})]$$

where $s_{n,m} = (-1)^m$ for $\mathfrak{g} = \mathfrak{gl}_n$ and $s_{n,m} = 1/4$ for $\mathfrak{g} = \mathfrak{sp}_{2n}$.

Define $t_k \in H_m(\mathfrak{gl}_n)$ by $t_k := \sum_{c=1}^n x_c [Q_k, y_c]$ and $t_k \in H_m(\mathfrak{sp}_{2n})$ by $t_k := \sum_{c=1}^{2n} [Q_k, y_c y_c^*]$. Combining Proposition 13, Theorems 5, 7 with $\text{gr}(Z(U(\mathfrak{g}, e))) = \mathfrak{p}_{\text{Pois}}(C[S])$ we get

Corollary 14. For $\mathfrak{g}$ being either $\mathfrak{gl}_n$ or $\mathfrak{sp}_{2n}$, there exist $C_1, \ldots, C_n \in Z(U(\mathfrak{g}))[\zeta_0, \ldots, \zeta_{m-1}]$, such that the center $Z(H_m(\mathfrak{g}))$ is a polynomial algebra in free generators $\{\zeta_i\} \cup \{t_j + C_j\}_{j=1}^n$. Considering the quotient of $H_m(\mathfrak{g})$ by the ideal $(\zeta_0 - a_0, \ldots, \zeta_{m-s} - a_{m-s})$ for any $a_i \in \mathbb{C}$, we see that the center of the standard infinitesimal Cherednik algebra $H_0(\mathfrak{g})$ contains a polynomial subalgebra $\mathbb{C}[t_1 + c_1, \ldots, t_n + c_n]$ for some $c_j \in Z(U(\mathfrak{g}))$.

Together with [DT, Thms. 5.1 and 7.1] this yields:
Corollary 15. We actually have $Z(H_n(g)) = \mathbb{C}[t_1 + c_1, \ldots, t_n + c_n]$.

For $g = gl_n$ this is [T1, Thm. 1.1], while for $g = sp_{2n}$ this is [DT, Conj. 7.1].

4.2. Symplectic leaves of Poisson infinitesimal Cherednik algebras

By Theorem 10, we get an identification of the full Poisson-central reductions of the algebras $\mathbb{C}[S_{n,m}]$ and $H^cl_m(gl_n)$ or $H^cl_m(sp_{2n})$. As an immediate consequence we obtain the following proposition, which answers a question raised in [DT]:

Proposition 16. Poisson varieties corresponding to arbitrary full central reductions of Poisson infinitesimal Cherednik algebras $H^cl_\zeta(g)$ have finitely many symplectic leaves.

4.3. Analogue of Kostant’s theorem

As another immediate consequence of Theorem 7 and discussions from Section 4.1, we get a generalization of the following classical result:

Proposition 17.

(a) The infinitesimal Cherednik algebras $H_\zeta(g)$ are free over their centers.
(b) The full central reductions of $\text{gr } H_\zeta(g)$ are normal, complete intersection integral domains.

For $g = gl_n$, this is [T2, Thm. 2.1], while for $g = sp_{2n}$ this is [DT, Thm. 8.1].

4.4. Category $\mathcal{O}$ and finite dimensional representations of $H_m(sp_{2n})$

The categories $\mathcal{O}$ for the finite $W$-algebras were first introduced in [BGK] and were further studied by the first author in [L3]. Namely, recall that we have an embedding $g \subset U(g,e)$. Let $t$ be a Cartan subalgebra of $g$ and set $g_0 := \mathfrak{z}_g(t)$. Pick an integral element $\theta \in t$ such that $\mathfrak{z}_g(\theta) = g_0$. By definition, the category $\mathcal{O}$ (for $\theta$) consists of all finitely generated $U(g,e)$-modules $M$, where the action of $t$ is diagonalizable with finite dimensional eigenspaces and, moreover, the set of weights is bounded from above in the sense that there are complex numbers $\alpha_1, \ldots, \alpha_k$ such that for any weight $\lambda$ of $M$ there is $i$ with $\alpha_i - \langle \theta, \lambda \rangle \in \mathbb{Z}_{\geq 0}$. The category $\mathcal{O}$ has analogues of Verma modules, $\Delta(N^0)$. Here $N^0$ is an irreducible module over the $W$-algebra $U(g_0,e)$, where $g_0$ is the centralizer of $t$. In the cases of interest $((g,e) = (gl_{n+m}, e_m), (sp_{2n+2m}, e_m))$, we have $g_0 = gl_n \times \mathbb{C}^{m-1}, g_0 = sp_{2n} \times \mathbb{C}^m$ and $e$ is principal in $g_0$. In this case, the $W$-algebra $U(g_0,e)$ coincides with the center of $U(g_0)$. Therefore $N^0$ is a one-dimensional space, and the set of all possible $N^0$ is identified, via the Harish-Chandra isomorphism, with the quotient $b^*/W_0$, where $b, W_0$ are a Cartan subalgebra and the Weyl group of $g_0$ (we take the quotient with respect to the dot-action of $W_0$ on $b^*$). As in the usual BGG category $\mathcal{O}$, each Verma module has a unique irreducible quotient, $L(N^0)$. Moreover, the map $N^0 \rightarrow L(N^0)$ is a bijection between the set of finite dimensional irreducible $U(g_0,e)$-modules, $b^*/W_0$, in our case, and the set of irreducible objects in $\mathcal{O}$. We remark that all finite dimensional irreducible modules lie in $\mathcal{O}$.

One can define a formal character for a module $M \in \mathcal{O}$. The characters of Verma modules are easy to compute basically thanks to [BGK, Thm. 4.5(1)]. So to compute the characters of the simples, one needs to determine the multiplicities of the simples in the Vermas. This was done in [L3, Sect. 4] in the case when $e$ is
principal in \( g_0 \). The multiplicities are given by values of certain Kazhdan-Lusztig polynomials at 1 and so are hard to compute, in general. In particular, one cannot classify finite dimensional irreducible modules just using those results.

When \( g = \mathfrak{sl}_{n+m} \), a classification of the finite dimensional irreducible \( U(\mathfrak{g}, e) \)-modules was obtained in [BK2]; this result is discussed in the next section. When \( g = \mathfrak{sp}_{2n+2m} \), one can describe the finite dimensional irreducible representations using [L2, Thm. 1.2.2]. Namely, the centralizer of \( e \) in \( \text{Ad}(g) \) is connected. So, according to [L2], the finite dimensional irreducible \( U(\mathfrak{g}, e) \)-modules are in one-to-one correspondence with the primitive ideals \( \mathcal{J} \subset U(\mathfrak{g}) \) such that the associated variety of \( U(\mathfrak{g})/\mathcal{J} \) is \( \emptyset \), where we write \( \emptyset \) for the adjoint orbit of \( e \). The set of such primitive ideals is computable (for a fixed central character, those are in one-to-one correspondence with certain left cells in the corresponding integral Weyl group), but we will not need details on that.

One can also describe all \( N^0 \in \mathfrak{h}^*/W_0 \) such that \( \dim L(N^0) < \infty \) when \( e \) is principal in \( g_0 \). This is done in [L4, 5.1]. Namely, choose a representative \( \lambda \in \mathfrak{h}^* \) of \( N^0 \) that is, antidualmant \( \lambda \in \mathfrak{h}^* \) for \( g_0 \), meaning that \( \langle \alpha^\vee, \lambda \rangle \notin \mathbb{Z}_{>0} \) for any positive root \( \alpha \) of \( g_0 \). Then we can consider the irreducible highest weight module \( L(\lambda) \) for \( g \) with highest weight \( \lambda - \rho \). Let \( \mathcal{J}(\lambda) \) be its annihilator in \( U(\mathfrak{g}) \); this is a primitive ideal that depends only on \( N^0 \) and not on the choice of \( \lambda \). Then \( \dim L(N^0) < \infty \) if and only if the associated variety of \( U(\mathfrak{g})/\mathcal{J}(\lambda) \) is \( \emptyset \). The associated variety is computable thanks to results of [BV]; however, this computation requires quite a lot of combinatorics. It seems that one can still give a closed combinatorial answer for \((\mathfrak{sp}_{2m+2m}, e_m)\) similar to that for \((\mathfrak{sl}_{n+m}, e_m)\) but we are not going to elaborate on that.

Now let us discuss the infinitesimal Cherednik algebras. In the \( \mathfrak{gl}_n \)-case the category \( \mathcal{O} \) was defined in [T1, Def. 4.1] (see also [EGG, Sect. 5.2]). Under the isomorphism of Theorem 7(a), that category \( \mathcal{O} \) basically coincides with its \( \mathcal{W} \)-algebra counterpart. The classification of finite dimensional irreducible modules and the character computation in that case was done in [DT], but the character formulas for more general simple modules were not known. For the algebras \( H_m(\mathfrak{sp}_{2n}) \), no category \( \mathcal{O} \) was introduced, in general; the case \( n = 1 \) was discussed in [Kh]. The classification of finite dimensional irreducible modules was not known either.

### 4.5. Finite dimensional representations of \( H_m(\mathfrak{gl}_n) \)

Let us compare classifications of the finite dimensional irreducible representations of \( U(\mathfrak{sl}_{n+m}, e_m) \) from [BK2] and \( H_m(\mathfrak{gl}_n) \) from [DT].

In the notation of [BK2], a nilpotent element \( e_m \in \mathfrak{gl}_{n+m} \) corresponds to the partition \((1, \ldots, 1, m)\) of \( n + m \). Let \( S_m \) act on \( \mathbb{C}^{n+m} \) by permuting the last \( m \) coordinates. According to [BK2, Thm. 7.9], there is a bijection between the irreducible finite dimensional representations of \( U(\mathfrak{gl}_{n+m}, e_m) \) and the orbits of the \( S_m \)-action on \( \mathbb{C}^{n+m} \) containing a strictly dominant representative. An element \( \pi = (\nu_1, \ldots, \nu_{n+m}) \in \mathbb{C}^{n+m} \) is called strictly dominant if \( \nu_i - \nu_{i+1} \) is a positive integer for all \( 1 \leq i \leq n \). The corresponding irreducible \( U(\mathfrak{gl}_{n+m}, e_m) \)-representation is denoted \( L_\pi \). Viewed as a \( \mathfrak{gl}_{n+m} \)-module (since \( \mathfrak{gl}_n = \mathfrak{q} \subset U(\mathfrak{gl}_{n+m}, e_m) \)), \( L_\pi = \mathbb{C}^{n+m} \).
Let us now recall [DT, Thm. 4.1], which classifies all irreducible finite dimensional representations of the infinitesimal Cherednik algebra $H_0(\mathfrak{gl}_n)$. They turn out to be parameterized by strictly dominant weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ (that is, $\lambda_i - \lambda_{i+1}$ is a positive integer for every $1 \leq i < n$), for which there exists a positive integer $k$ satisfying $P(\lambda) = P(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - k)$. Here $P$ is a degree $m + 1$ polynomial function on the Cartan subalgebra $\mathfrak{h}_n$ of all diagonal matrices of $\mathfrak{gl}_n$, introduced in [DT, Sect. 3.2]. According to [DT, Thm. 3.2] (see Theorem 18(b) below), we have $P = \sum_{j \geq 0} w_j h_{j+1}$, where both $w_j$ and $h_j$ are defined in the next section (see the notation preceding Theorem 18).

These two descriptions are intertwined by a natural bijection, sending $\varpi = (\nu_1, \ldots, \nu_{n+m})$ to $\overline{\lambda} := (\nu_1, \ldots, \nu_n)$, while $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is sent to the class of $\overline{\varpi} = (\lambda_1, \ldots, \lambda_n, \nu_{n+1}, \ldots, \nu_{n+m})$ with $\{\nu_{n+1}, \ldots, \nu_{n+m}\} \cup \{\lambda_n\}$ being the set of roots of the polynomial $P(\lambda_1, \ldots, \lambda_{n-1}, t) = P(\overline{\lambda})$.

### 4.6. Explicit isomorphism in the case $\mathfrak{g} = \mathfrak{gl}_n$

We compute the images of particular central elements of $H_m(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_{n+m}, e_m)$ under the corresponding Harish-Chandra isomorphisms. Comparison of these images enables us to determine the isomorphism $\vartheta$ of Theorem 7(a) explicitly, in the same way as Theorem 10(a) was deduced.

Let us start from the following commutative diagram:

$$
\begin{array}{ccc}
U(\mathfrak{g}l_n) \otimes U(\mathfrak{sl}_m, e_m) & \xrightarrow{\varphi} & Z(U(\mathfrak{g}l_n) \otimes U(\mathfrak{sl}_m, e_m)) \\
\downarrow{\sigma} & & \downarrow{\varphi^W} \\
U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n) & \xrightarrow{j_n \otimes \text{Id}} & U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n) \\
\end{array}
$$

**Diagram 1**

In the above diagram:

- $U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n)$ is the 0-weight component of $U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n)$ with respect to the grading $\text{Gr}$.
- $U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n)$ is the 0-weight component of $U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n)$ with respect to the grading $\text{Gr}$.
- $\varpi$ is the quotient map, while $\sigma$ is an isomorphism constructed in [L3, Thm. 4.1].
- The homomorphism $\varphi$ is defined as $\varphi := \sigma \circ \varpi$, making the triangle commutative.
- The homomorphisms $j_{n+m}$, $j_n$ are the natural inclusions.
- The homomorphism $\varphi^W$ is the restriction of $\varphi$ to the center, making the square commutative.

---

Footnote: Here we actually use the fact that $U(\mathfrak{g}l_n) \otimes U(\mathfrak{g}l_n)$ is the finite $W$-algebra $U(\mathfrak{g}l_n \oplus \mathfrak{g}l_n, 0 \oplus e_m)$.
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- $U(\mathfrak{sl}_m, e_m) \cong Z(U(\mathfrak{sl}_m, e_m)) \cong Z(U(\mathfrak{sl}_m))$ since $e_m$ is a principal nilpotent of $\mathfrak{sl}_m$.

We have an analogous diagram for the universal infinitesimal Cherednik algebra of $\mathfrak{gl}_n$:

\[
\begin{array}{ccc}
H_m(\mathfrak{gl}_n) & \overset{j''_{m,m}}{\longrightarrow} & Z(H_m(\mathfrak{gl}_n)) \\
\downarrow{\pi'} & & \downarrow{\varphi''} \\
U(\mathfrak{gl}_n) \otimes \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}] & \overset{j_n \otimes \text{Id}}{\longrightarrow} & Z(U(\mathfrak{gl}_n)) \otimes \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]
\end{array}
\]

**Diagram 2**

In the above diagram:
- $H_m(\mathfrak{gl}_n)_0$ is the degree 0 component of $H_m(\mathfrak{gl}_n)$ with respect to the grading $\text{Gr}$, defined by setting $\deg(\mathfrak{gl}_n) = \deg(\zeta_0) = \ldots = \deg(\zeta_{m-2}) = 0$, $\deg(V_n) = 1$, $\deg(V_n^*) = -1$.
- $H_m(\mathfrak{gl}_n)^0$ is the quotient of $H_m(\mathfrak{gl}_n)_0$ by $H_m(\mathfrak{gl}_n)_0 \cap H_m(\mathfrak{gl}_n)H_m(\mathfrak{gl}_n)_{>0}$.
- $\pi'$ denotes the quotient map, $\varphi'$ is the natural isomorphism, $\varphi' := \varphi' \circ \pi'$.
- The inclusion $j''_{m,m}$ is a natural inclusion of the center.
- The homomorphism $\varphi''$ is the one induced by restricting $\varphi'$ to the center.

The isomorphism $\varphi''$ of Theorem 7(a) intertwines the gradings $\text{Gr}$, defined by setting $\deg(\mathfrak{gl}_n) = \deg(\zeta_0) = \ldots = \deg(\zeta_{m-2}) = 0$, $\deg(V_n) = 1$, $\deg(V_n^*) = -1$.

The isomorphism $j''_{m,m}$ arises from the $\rho_N$-shifted $S_N$-action on $\mathfrak{h}_N$ with $\rho_N = ((N - 1)/2, (N - 3)/2, \ldots, (1 - N)/2) \in \mathfrak{h}_N^*$. This isomorphism has the following property:

\[
\begin{array}{ccc}
Z(H_m(\mathfrak{gl}_n)) & \overset{\vartheta}{\longrightarrow} & Z(U(\mathfrak{gl}_{n+m}, e_m)) \\
\downarrow{\varphi''} & & \downarrow{\varphi''} \\
Z(U(\mathfrak{gl}_n)) \otimes \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}] & \overset{\bar{\vartheta}}{\longrightarrow} & Z(U(\mathfrak{gl}_n)) \otimes Z(U(\mathfrak{sl}_m))
\end{array}
\]

**Diagram 3**

In the above diagram:
- The isomorphism $\vartheta$ is the restriction of the isomorphism $\varphi''$ to the center.
- The isomorphism $\bar{\vartheta}$ is the restriction of the isomorphism $\varphi''$ to the center.

Let $HC_N$ denote the Harish-Chandra isomorphism

$HC_N : Z(U(\mathfrak{gl}_n)) \cong \mathbb{C}[\mathfrak{h}_N^{S_N}]$,

where $\mathfrak{h}_N \subset \mathfrak{gl}_N$ is the Cartan subalgebra consisting of the diagonal matrices and $(S_N, \bullet)$-action arises from the $\rho_N$-shifted $S_N$-action on $\mathfrak{h}_N$ with $\rho_N = ((N - 1)/2, (N - 3)/2, \ldots, (1 - N)/2) \in \mathfrak{h}_N^*$. This isomorphism has the following property:

\[^8\text{It is easy to see that } H_m(\mathfrak{gl}_n)_0 \cap H_m(\mathfrak{gl}_n)H_m(\mathfrak{gl}_n)_{>0} \text{ is actually a two-sided ideal of } H_m(\mathfrak{gl}_n)_0.\]
any central element $z \in Z(U(\mathfrak{gl}_N))$ acts on the Verma module $M_{\lambda-\rho_N}$ of $U(\mathfrak{gl}_N)$ via $HC_N(z)(\lambda)$.

According to Corollary 14, the center $Z(H_m(\mathfrak{gl}_n))$ is the polynomial algebra in free generators $\{\zeta_0, \ldots, \zeta_{m-2}, t'_1, \ldots, t'_n\}$, where $t'_k = t_k + C_k$. In particular, any central element of Kazhdan degree $2(m+1)$ has the form $ct'_1 + p(\zeta_0, \ldots, \zeta_{m-2})$ for some $c \in \mathbb{C}$ and $p \in \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$.

Following [DT], we call $t'_1 = t_1 + C_1$ the Casimir element. An explicit formula for $\varphi^H(t'_1)$ is provided by [DT, Thm. 3.1], while for any $0 \leq k \leq m-2$ we have $\varphi^H(\zeta_k) = 1 \otimes \zeta_k$.

To formulate the main results about the Casimir element $t'_1$, we introduce:

- the generating series $\zeta(z) = \sum_{i=0}^{m-2} \zeta_i z^i + z^m$ (already introduced in Section 1.4),
- a unique degree $m+1$ polynomial $f(z)$ satisfying $f(z) - f(z-1) = \partial^n(z^n \zeta(z))$ and $f(0) = 0$,
- a unique degree $m+1$ polynomial $g(z) = \sum_{i=1}^{m+1} g_i z^i$ satisfying $\partial^n(z^{n-1}g(z)) = f(z)$,
- a unique degree $m$ polynomial $w(z) = \sum_{i=0}^{m} w_i z^i$ satisfying
  $$f(z) = (2\sinh(\theta/2))^{n-1}(z^n w(z)),$$
- the symmetric polynomials $\sigma_i(\lambda_1, \ldots, \lambda_n)$ via
  $$(u + \lambda_1) \cdots (u + \lambda_n) = \sum \sigma_i(\lambda_1, \ldots, \lambda_n) u^{n-i},$$
- the symmetric polynomials $h_j(\lambda_1, \ldots, \lambda_n)$ via
  $$(1 - u \lambda_1)^{-1} \cdots (1 - u \lambda_n)^{-1} = \sum h_j(\lambda_1, \ldots, \lambda_n) u^j,$$
- the central element $H_j \in Z(U(\mathfrak{gl}_n))$ which is the symmetrization of $\text{tr} S^j(\cdot) \in \mathbb{C}[\mathfrak{gl}_n] \cong \mathbb{S}(\mathfrak{gl}_n)$.

The following theorem summarizes the main results of [DT, Sect. 3]:

**Theorem 18.**

(a) [DT, Thm. 3.1] $\varphi^H(t'_1) = \sum_{j=1}^{m+1} H_j \otimes g_j$ (where $g_j$ are viewed as elements of $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$).

(b) [DT, Thm. 3.2] $(HC_n \otimes \text{Id}) \circ \varphi^H(t'_1) = \sum_{j=0}^{m} h_{j+1} \otimes w_j$.

Let $HC^N_N$ denote the Harish-Chandra isomorphism $Z(U(\mathfrak{sl}_N)) \cong \mathbb{C}[\mathfrak{h}_N]^{\mathfrak{h}_N}$, where $\mathfrak{h}_N$ is the Cartan subalgebra of $\mathfrak{sl}_N$, consisting of the diagonal matrices, which can be identified with $\{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid \sum z_i = 0\}$. The natural inclusion $\mathfrak{h}_N \mapsto \mathfrak{h}_N$ induces the map

$$\mathfrak{h}_N \rightarrow \mathfrak{h}_N^* : (\lambda_1, \ldots, \lambda_N) \mapsto (\lambda_1 - \mu, \ldots, \lambda_N - \mu), \text{ where } \mu := \frac{\lambda_1 + \cdots + \lambda_N}{N}.$$
The isomorphisms $HC_{n+m}$, $HC_m$, $HC_n$ fit into the following commutative diagram:

$$
\begin{array}{ccc}
Z(U(\mathfrak{sl}_{n+m})) & \xrightarrow{HC_{n+m}} & \mathbb{C}[[C^{n+m-1}S_{n+m}]^\bullet \\
\rho & \downarrow & \phi^C \\
Z(U(\mathfrak{gl}_n) \otimes Z(U(\mathfrak{sl}_m)) & \xrightarrow{HC_m \otimes HC_m^{-1}} & \mathbb{C}[[C^n]S_n \otimes \mathbb{C}[C^{m-1}]S_m]^\bullet \\
\end{array}
$$

**Diagram 4**

In the above diagram:

- $\rho$ is the isomorphism of Theorem 5.
- The homomorphism $\phi^W$ is defined as the composition $\phi^W := \phi^W \circ \rho$.
- The homomorphism $\phi^C$ arises from an identification $\mathbb{C}^n \otimes \mathbb{C}^{m-1} \cong \mathbb{C}^{n+m-1}$ defined by

$$(\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_m) \mapsto (\lambda_1, \ldots, \lambda_n, \nu_1 - \frac{\lambda_1 + \cdots + \lambda_n}{m}, \ldots, \nu_m - \frac{\lambda_1 + \cdots + \lambda_n}{m}).$$

In particular, $\phi^C$ is injective, so that $\phi^W$ is injective and, hence, $\phi^H$ is injective.

Define $\tau_k \in \mathbb{C}[\mathfrak{h}_N]$ as the restriction of $\sigma_k$ to $\mathbb{C}^{N-1} \hookrightarrow \mathbb{C}^N$. According to Lemma 12,

$$\phi^C(\tau_{m+1}) = (-1)^m h_{m+1} \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} h_{m+1-j} \otimes 1 \cdot \phi^C(\tau_j). \quad (5)$$

Define $S_k \in Z(U(\mathfrak{sl}_{n+m}))$ by $S_k := (HC_{n+m})^{-1}(\tau_k)$ for all $0 \leq k \leq n + m$, so that $S_0 = 1$, $S_1 = 0$. Similarly, define $T_k \in Z(U(\mathfrak{gl}_n))$ as $T_k := HC_n^{-1}(h_k)$ for all $k \geq 0$, so that $T_0 = 1$.

Equality (5) together with the commutativity of Diagram 4 imply

$$\phi^W(S_{m+1}) = (-1)^m T_{m+1} \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} T_{m+1-j} \otimes 1 \cdot \phi^W(S_j).$$

According to our proof of Theorem 7(a), we have $\Theta(A) = \Theta(A) + s \text{tr } A$ for all $A \in \mathfrak{gl}_n$, where $s = -\eta_{m-1}/(n + m) \eta_m$. In particular, $\vartheta^{-1}(X \otimes 1) = \varphi_{-s}(X) \otimes 1$ for all $X \in Z(U(\mathfrak{gl}_n))$, where $\varphi_{-s}$ was defined in Lemma 1.

As a consequence, we get:

$$\vartheta^{-1}(\phi^W(S_{m+1})) = (-1)^m \varphi_{-s}(T_{m+1}) \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} \varphi_{-s}(T_{m+1-j}) \otimes 1 \cdot \vartheta^{-1}(\phi^W(S_j)). \quad (6)$$

The following identity is straightforward:
Lemma 19. For any positive integer $i$ and any constant $\delta \in \mathbb{C}$ we have

$$h_i(\lambda_1 + \delta, \ldots, \lambda_n + \delta) = \sum_{j=0}^{i} \binom{n+i-1}{j} h_{i-j}(\lambda_1, \ldots, \lambda_n) \delta^j.$$ 

As a result, we get

$$\varphi_s(T_i) = \sum_{j=0}^{i} \binom{n+i-1}{j} (-s)^j T_{i-j}. \quad (7)$$

Combining equations (6) and (7), we get:

$$\varphi_{s}(T_{i}) = \sum_{j=0}^{i} \binom{n+i-1}{j} \left( -s \right)^j T_{i-j}.$$ 

Recall that there exist $\zeta \in \mathbb{C}$, $p \in \mathbb{C} \left[ \zeta_0, \ldots, \zeta_{m-2} \right]$ such that

$$\varphi_{s}(T_{i}) = 1 \otimes \zeta_i + \sum_{j=0}^{i} \binom{n+i-1}{j} \left( -s \right)^j T_{i-j}.$$ 

On the other hand, the commutativity of Diagram 3 implies

$$\varphi_{s}(T_{i}) = \varphi_{s}(T_{i}) = \varphi_{s}(T_{i}).$$ 

Recalling the equalities $w_m = 1, w_{m-1} = (n+m)/2$, the comparison of (8) and (9) yields:

- The coefficients of $T_{m+1}$ must coincide, so that $(-1)^m = cw_m \Rightarrow c = (-1)^m$.
- The coefficients of $T_m$ must coincide, so that $cw_{m-1} = (-1)^{m+1}(n+m)s \Rightarrow s = -1/2$.
- The coefficients of $T_{j+1}$ must coincide for all $j \geq 0$, so that

$$w_j = (-1)^{m-j} T_{j} \Rightarrow \varphi_{s}(w_j) = (-1)^{m-j} p(V_j).$$

Recall that $\eta_m = 1$, and so $\eta_m = \eta_m = 1$. As a result $s = -\eta_{m-1}/(n+m)$, so that $\eta_{m-1} = (n+m)/2$.

The above discussion can be summarized as follows:
Theorem 20. Let \( \Theta : H_m(\mathfrak{g}l_n) \to U(\mathfrak{sl}_n, e_m) \) be the isomorphism from Theorem 7(a). Then
\[
\Theta(A) = \Theta(A) - \frac{1}{2} \text{tr} A, \quad \Theta(y) = \Theta(y), \quad \Theta(x) = \Theta(x),
\]
while \( \Theta |_{C[\zeta_0, \ldots, \zeta_{m-2}]} \) is uniquely determined by
\[
\Theta(w_j) = (-1)^{m-j} \rho(V_j) \text{ for all } 0 \leq j \leq m-2.
\]

4.7. Higher central elements
It was conjectured in [DT, Rem. 6.1], that the action of central elements

\[ t'_i = t_i + c_i \in Z(H_m(\mathfrak{g}l_n)) \]

on the Verma modules of \( H_m(\mathfrak{g}l_n) \) should be obtained from the corresponding formulas at the Poisson level (see Theorem 3) via a basis change \( \zeta(z) \mapsto w(z) \) and a \( \rho_n \)-shift. Actually, that is not true. However, we can choose another set of generators \( u_i \in Z(H_m(\mathfrak{g}l_n)) \), whose action is given by formulas similar to those of Theorem 3.

Let us define:

\( u_i \in Z(H_m(\mathfrak{g}l_n)) \) by
\[
u_i := \Theta^{-1}(\rho(S_{m+i})) \text{ for all } 0 \leq i \leq n,
\]

• the generating polynomial
\[
\tilde{u}(t) := \sum_{i=0}^{n} (-1)^i u_i t^i,
\]
• the generating polynomial
\[
S(z) := \sum_{i=0}^{n} (-1)^i \tilde{u}^{-1}(\phi^W(S_{m-i})) z^i \in \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}; z].
\]

The following result is proved using the arguments of Section 4.6:

Theorem 21. We have:
\[
(\text{HC}_n \otimes \text{Id}) \circ \phi^H(\tilde{u}(t)) = (\varphi_{1/2} \otimes \text{Id}) \left( \text{Res}_{z=0} S(z^{-1}) \prod_{1 \leq i \leq n} \frac{1 - t \lambda_i}{1 - z \lambda_i} \frac{z^{-1} dz}{1 - t z} \right).
\]

5. Completions

5.1. Completions of graded deformations of Poisson algebras
We first recall the machinery of completions, elaborated by the first author (our exposition follows [L7]). Let \( Y \) be an affine Poisson scheme equipped with a \( \mathbb{C}^* \)-action, such that the Poisson bracket has degree \(-2\). Let \( \mathcal{A}_h \) be an associative flat graded \( \mathbb{C}[h]\)-algebra (where \( \text{deg}(h) = 1 \)) such that \( [\mathcal{A}_h, \mathcal{A}_h] \subset h^2 \mathcal{A}_h \) and \( \mathbb{C}[Y] = \mathcal{A}_h/(h) \) as a graded Poisson algebra. Pick a point \( x \in Y \) and let \( I_x \subset \mathbb{C}[Y] \) be the maximal ideal of \( x \), while \( \widetilde{I}_x \) will denote its inverse image in \( \mathcal{A}_h \).

Definition 6. The completion of \( \mathcal{A}_h \) at \( x \in Y \) is by definition
\[
\mathcal{A}_h^\wedge_x := \lim_{\leftarrow n} \mathcal{A}_h/\widetilde{I}_x^n.
\]

This is a complete topological \( \mathbb{C}[h]\)-algebra, flat over \( \mathbb{C}[[h]] \), such that \( \mathcal{A}_h^\wedge_x/(h) = \mathbb{C}[Y]^\wedge_x \). Our main motivation for considering this construction is the decomposition theorem, generalizing the corresponding classical result at the Poisson level:
**Proposition 22** (cf. [K, Thm. 2.3]). The formal completion \( \hat{Y}_x \) of \( Y \) at \( x \in Y \) admits a product decomposition \( \hat{Y}_x = Z_x \times \hat{Y}^*_x \), where \( Y^* \) is the symplectic leaf of \( Y \) containing \( x \) and \( Z_x \) is a local formal Poisson scheme.

Fix a maximal symplectic subspace \( V \subset T^*_x Y \). One can choose an embedding \( V \overset{i}{\hookrightarrow} \hat{T}^*_x \) such that \([i(u), i(v)] = h^2 \omega(u, v)\) and composition \( V \overset{\iota}{\hookrightarrow} \hat{T}^*_x \rightarrow T^*_x Y \) is the identity map. Finally, we define \( W_h(V) := T(V)/[h] \) \((u \otimes v - v \otimes u - h^2 \omega(u, v))\), which is graded by setting \( \deg(V) = 1 \), \( \deg(h) = 1 \) (the homogenized Weyl algebra). Then we have:

**Theorem 23** ([L7, Sect. 2.1], Decomposition theorem). There is a splitting

\[
\mathcal{A}_h^{\wedge} \cong W_h(V)^{\wedge_0} \otimes_{\mathbb{C}[h]} \mathcal{A}_h',
\]

where \( \mathcal{A}_h' \) is the centralizer of \( V \) in \( \mathcal{A}_h^{\wedge} \).

**Remark 5.** Recall that a filtered algebra \( \{F_i(B)\}_{i \geq 0} \) is called a filtered deformation of \( Y \) if \( \text{gr}_{F_i} B \cong \mathbb{C}[Y] \) as Poisson graded algebras. Given such \( B \), we set \( \mathcal{A}_h := \text{Rees}_h(B) \) (the Rees algebra of the filtered algebra \( B \)), which naturally satisfies all the above conditions.

This remark provides the following interesting examples of \( \mathcal{A}_h \):

- **The homogenized Weyl algebra.**
  Algebra \( W_h(V) \) from above is obtained via the Rees construction from the usual Weyl algebra. In the case \( V = V_n \oplus V^*_n \) with a natural symplectic form, we denote \( W_h(V) \) just by \( W_h^m \).

- **The homogenized universal enveloping algebra.**
  For any graded Lie algebra \( \mathfrak{g} = \bigoplus \mathfrak{g}_i \) with a Lie bracket of degree \(-2\), we define
  \[
  U_h(\mathfrak{g}) := T(\mathfrak{g})[h]/(x \otimes y - y \otimes x - h^2[x, y] | x, y \in \mathfrak{g}),
  \]
  graded by setting \( \deg(\mathfrak{g}_i) = i \), \( \deg(h) = 1 \).

- **The homogenized universal infinitesimal Cherednik algebra of \( \mathfrak{gl}_n \).**
  Define \( H_{h,m}(\mathfrak{gl}_n) \) as a quotient
  \[
  H_{h,m}(\mathfrak{gl}_n) := U_h(\mathfrak{gl}_n) \rtimes T(V_n \oplus V^*_n)[\zeta_0, \ldots, \zeta_{m-2}]/J,
  \]
  where
  \[
  J = \left( [x, x'], [y, y'], [A, x] - h^2 A(x), [A, y] - h^2 A(y),
  [y, x] - h^2 \left( \sum_{j=0}^{m-2} \zeta_j r_j(y, x) + r_m(y, x) \right) \right).
  \]
  This algebra is graded by setting \( \deg(V_n \oplus V^*_n) = m + 1 \), \( \deg(\zeta_i) = 2(m - i) \).
Reflection algebras ([L5]) and where $W$ tool. Among such let us mention rational Cherednik algebras ([BE]), symplectic

where $V_v$ decomposition isomorphism $\Psi_m : H_{n,m}(\mathfrak{sl}_n)^{\wedge}_v \to H_{n,m+1}(\mathfrak{sl}_{n-1})^{\wedge}_v \otimes_{\mathbb{C}[h]} W_n^{\wedge}_v$, (*)

$\Upsilon_m : H_{n,m}(\mathfrak{sp}_{2n})^{\wedge}_v \to H_{n,m+1}(\mathfrak{sp}_{2n-2})^{\wedge}_v \otimes_{\mathbb{C}[h]} W_{n,2}^{\wedge}_v$, (♣)

where $v \in V_n$ (respectively $v \in V_{2n}$) is a nonzero element and $m \geq 1$.

These decompositions can be viewed as quantizations of their Poisson versions:

$\psi_m^{cl} : H_{n,m}(\mathfrak{gl}_n)^{\wedge}_v \to H_{n,m+1}(\mathfrak{gl}_{n-1})^{\wedge}_v \otimes_{\mathbb{C}[h]} W_n^{\wedge}_v$, (*)

$\Upsilon_m^{cl} : H_{n,m}(\mathfrak{sp}_{2n})^{\wedge}_v \to H_{n,m+1}(\mathfrak{sp}_{2n-2})^{\wedge}_v \otimes_{\mathbb{C}[h]} W_{n,2}^{\wedge}_v$, (♣)

where $W_n^{\wedge}_v \simeq \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with $\{x_i, x_j\} = \{y_i, y_j\} = 0$, $\{x_i, y_j\} = \delta_{ij}$.

Isomorphisms (*) and (♣) are not unique and, what is worse, are inexplicit.

Let us point out that localizing at other points of $\mathfrak{gl}_n \times V_n \times V_n^\ast$ (respectively $\mathfrak{sp}_{2n} \times V_{2n}$) yields other decomposition isomorphisms. In particular, one gets [T3, Thm. 3.1]10 as follows:

**Remark 6.** For $n = 1, m > 0$, consider $e' := e_m + E_{1,2n+2} \in S_{1,m} \subset \mathfrak{sp}_{2m+2}$, which is a subregular nilpotent element of $\mathfrak{sp}_{2m+2}$. The above arguments yield a decomposition isomorphism

$H_{n,m}(\mathfrak{sp}_2)^{\wedge}_{\mathfrak{gl}_2} \to H_{n,m+1}(\mathfrak{sp}_{2m+2})^{\wedge}_v \otimes_{\mathbb{C}[h]} W_{n,1}^{\wedge}_v$. (♣)

The full central reduction of (♣) provides an isomorphism of [T3, Thm. 3.1].11

In Appendix C, we establish explicitly suitably modified versions of (*) and (♣) for the cases $m = -1, 0$, which do not follow from the above arguments. In particular, the reader will get a flavor of what the formulas look like.

---

10 This result is stated in [T3]. However, its proof in the loc. cit. is wrong.
11 We use an isomorphism of the $W$-algebra $U(\mathfrak{sp}_{2m+2}, e')$ and the non-commutative deformation of Crawley-Boevey and Holland of type $D_{m+2}$ Kleinian singularity.
A. Proof of Lemmas 1, 2

Proof of Lemma 1(a). Let \( \phi : H^\zeta(\mathfrak{gl}_n) \xrightarrow{\sim} H^\zeta(\mathfrak{gl}_n) \) be a filtration preserving isomorphism. We have \( \phi(1) = 1 \), so that \( \phi \) is the identity on the 0th level of the filtration.

Since \( F_2(N)(H^\zeta(\mathfrak{gl}_n)) = F_2(N)(H^\zeta(\mathfrak{gl}_n)) = U(\mathfrak{gl}_n) \leq 1 \), we have \( \phi(A) = \psi(A) + \gamma(A) \), \( \forall A \in \mathfrak{gl}_n \), with \( \psi(A) \in \mathfrak{gl}_n, \gamma(A) \in \mathfrak{C} \). Then \( \phi([A, B]) = [\phi(A), \phi(B)] \), \( \forall A, B \in \mathfrak{gl}_n \), if and only if \( \gamma([A, B]) = 0 \) and \( \psi \) is an automorphism of the Lie algebra \( \mathfrak{gl}_n \). Since \( [\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n \), we have \( \gamma(A) = \lambda \cdot \text{tr} A \) for some \( \lambda \in \mathfrak{C} \). For \( n \geq 3 \), \( \text{Aut}(\mathfrak{gl}_n) = \text{Aut}(\mathfrak{sl}_n) \times \text{Aut}(\mathfrak{C}) = (\mu_2 \times \text{SL}(n)) \times \mathfrak{C}^* \), where \( -1 \in \mu_2 \) acts on \( \mathfrak{sl}_n \) via \( \sigma : A \mapsto -A^t \). This determines \( \phi \) up to the filtration level \( N - 1 \).

Finally, \( F^{(N)}(H^\zeta(\mathfrak{gl}_n)) \) is parameterized by \( (e, T, \nu, \lambda) \in (\mu_2 \times \text{SL}(n)) \times \mathfrak{C}^* \times \mathfrak{C} \) (no \( \mu_2 \) for \( n = 1, 2 \)). Let \( I_n \in \mathfrak{gl}_n \) be the identity matrix. Note that \( [I_n, y] = y, [I_n, x] = -x, [I_n, A] = 0 \) for any \( y \in \mathfrak{v}_n, x \in \mathfrak{v}_n, A \in \mathfrak{gl}_n \).

Since \( \phi(y) = \phi([I_n, y]) = [\nu \cdot I_n + n\lambda, \phi(y)] = \nu[I_n, \phi(y)] \), \( \forall y \in \mathfrak{v}_n \), we get \( \nu = \pm 1 \).

Case 1: \( \nu = 1 \). Then \( \phi(y) \in \mathfrak{v}_n \), \( \phi(x) \in \mathfrak{v}_n \) (\( \forall y \in \mathfrak{v}_n, x \in \mathfrak{v}_n \)). Since \( \mathfrak{v}_n \not\equiv \mathfrak{v}_n^\ast \) as \( \mathfrak{sl}_n \)-modules for \( n \geq 3 \) and \( \text{End}_{\mathfrak{sl}_n}(\mathfrak{v}_n) = \mathfrak{C}^* \), we get \( \epsilon = 1 \in \mu_2 \) (so that \( \phi(A) = TAT^{-1} \), \( \forall A \in \mathfrak{sl}_n \)) and there exist \( \theta_1, \theta_2 \in \mathfrak{C}^* \) such that \( \phi(y) = \theta_1 \cdot T(y), \phi(x) = \theta_2 \cdot T(x) \) (\( \forall y \in \mathfrak{v}_n, x \in \mathfrak{v}_n \)). Hence, we get \( \varphi(T, \lambda)(\zeta(y, x)) = \phi([y, x]) = [\phi(y), \phi(x)] = \theta \zeta(\theta(T(y), T(x)), \theta \lambda) \), where \( \theta = \theta_1 \theta_2 \) and the isomorphism \( \varphi(T, \lambda) : U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n) \) is defined by \( A \mapsto TAT^{-1} + \lambda \text{tr} A, \forall A \in \mathfrak{gl}_n \). Thus, \( \zeta' = \theta^{-1} \varphi \lambda(\zeta') \) in that case.

Case 2: \( \nu = -1 \). Then \( \phi(y) \in \mathfrak{v}_n^\ast \), \( \phi(x) \in \mathfrak{v}_n^\ast \) (\( \forall y \in \mathfrak{v}_n, x \in \mathfrak{v}_n^\ast \)). Similarly to the above reasoning we get \( \epsilon = -1 \), \( \phi(A) = -TAT^{-1} + \lambda \text{tr} A \). Then there exist \( \theta_1, \theta_2 \in \mathfrak{C}^* \) such that \( \phi(y) = \theta_1 \cdot T(x_1), \phi(x) = \theta_2 \cdot T(x_2) \). Then \( \phi([y, x]) = -\theta_1 \theta_2 \zeta(\theta_1(T(y), \theta_2(T(x))) \). Hence, \( \zeta' = -\theta_1^{-1} \theta_2^{-1} \varphi \lambda(\zeta) \) in that case.

Finally, the above arguments also provide isomorphisms \( \phi_{\theta, \lambda, s} : H^\zeta(\mathfrak{gl}_n) \xrightarrow{\sim} H_{\theta \varphi \lambda(\zeta)}(\mathfrak{gl}_n) \) for any deformation \( \zeta \), constants \( \lambda \in \mathfrak{C}, \theta \in \mathfrak{C}^* \) and \( s \in \{ \pm \} \). \( \square \)

Proof of Lemma 1(b). Let \( \zeta \) be a length \( m \) deformation. Since \( \theta \zeta = \zeta m \), we can assume \( \zeta_m = 1 \). We claim that \( \varphi \lambda(\zeta)(\zeta)_{m-1} = 0 \) for \( \lambda = -\zeta_{m-1}/(m + n) \), which is equivalent to \( \partial \alpha_m/\partial I_n = (n + m)\alpha_m/1 \). This equality follows from comparing coefficients of \( s^m \) in the identity

\[
\sum \alpha_i(y, x)(A + sI_n)x^i = (1 - s\tau)^{-n-1} \sum \alpha_i(y, x)(\tau(1 - s\tau)^{-1})^i \cdot \square
\]

Proof of Lemma 2. Let \( \phi : H^\zeta(\mathfrak{sp}_{2n}) \xrightarrow{\sim} H^\zeta(\mathfrak{sp}_{2n}) \) be a filtration preserving isomorphism. Being an isomorphism, we have \( \phi(1) = 1 \), so that \( \phi \) is the identity on the 0th level of the filtration.

Since \( F_2(N)(H^\zeta(\mathfrak{sp}_{2n})) = F_2(N)(H^\zeta(\mathfrak{sp}_{2n})) = U(\mathfrak{sp}_{2n}) \leq 1 \), we have \( \phi(A) = \psi(A) + \gamma(A) \) for all \( A \in \mathfrak{sp}_{2n} \), with \( \psi(A) \in \mathfrak{sp}_{2n}, \gamma(A) \in \mathfrak{C} \). Then \( \phi([A, B]) = [\phi(A), \phi(B)] \), \( \forall A, B \in \mathfrak{sp}_{2n} \), if and only if \( \gamma([A, B]) = 0 \) and \( \psi \) is an automorphism of the Lie algebra \( \mathfrak{sp}_{2n} \). Since \( \mathfrak{sp}_{2n}, \mathfrak{sp}_{2n}^n = \mathfrak{sp}_{2n} \), we have \( \gamma \equiv 0 \). Meanwhile, any automorphism of \( \mathfrak{sp}_{2n} \) is inner, since \( \mathfrak{sp}_{2n} \) is a simple Lie algebra whose Dynkin diagram has no automorphisms. This proves \( \phi|_{U(\mathfrak{sp}_{2n})} = \text{Ad}(T), T \in \mathfrak{sp}_{2n} \). Composing
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with an automorphism \( \phi' \) of \( H_C(\mathfrak{sp}_{2n}) \), defined by \( \phi'(A) = \text{Ad}(T^{-1})(A), \phi'(x) = T^{-1}(x) (A \in \mathfrak{sp}_{2n}, x \in V_{2n}) \), we can assume \( \phi_U(\mathfrak{sp}_{2n}) = \text{Id} \).

Recall the element \( I_n' = \text{diag}(1, \ldots, 1, -1, \ldots, -1) \in \mathfrak{sp}_{2n} \). Since \( \text{ad}(I_n') \) has only even eigenvalues on \( U(\mathfrak{sp}_{2n}) \) and eigenvalues \( \pm 1 \) on \( V_{2n} \), we actually have \( \phi(V_{2n}) \subset V_{2n} \). Together with \( \text{End}_{\mathfrak{sp}_{2n}}(V_{2n}) = \mathbb{C}^* \), this implies the result.

The converse, that is, \( H_\zeta(\mathfrak{sp}_{2n}) \cong H_{\zeta}(\mathfrak{sp}_{2n}) \) for any \( \zeta \in \mathbb{C}^* \), is obvious.

\[ \square \]

B. Minimal nilpotent case

We compute the isomorphism of Theorem 7 explicitly for the case of \( e \in \mathfrak{g} \) being the minimal nilpotent. This case has been considered in detail in [P2, Sect. 4].

To state the main result we introduce some more notation. Let \( z_1, \ldots, z_s \) be a Witt basis of \( \mathfrak{g}(-1) \), i.e., \( \omega_\chi(z_i+z_j, z_i) = \delta^0_i \), \( \omega_\chi(z_i, z_j) = \omega_\chi(z_{i+s}, z_{j+s}) = 0 \) for any \( 1 \leq i, j \leq s \). We also define \( \tilde{\xi} : \mathfrak{g}(0) \to \mathfrak{g}(0) \) by \( x^2 := x - \frac{1}{2}(x, h)h \). Finally, we set \( c_0 := -n(n+1)/4 \) for \( \mathfrak{g} = \mathfrak{sl}_{n+1} \) and \( c_0 := -n(2n+1)/8 \) for \( \mathfrak{g} = \mathfrak{sp}_{2n} \). Then we have the following theorem:

**Theorem 24** (cf. [P2, Thm. 6.1]). The algebra \( U(\mathfrak{g}, e) \) is generated by the Casimir element \( C \) and the subspaces \( \Theta(\tilde{\zeta}_1(i)) \) for \( i = 0, 1 \), subject to the following relations:

(i) \( [\Theta_x, \Theta_u] = \Theta_{[x,u]}, \Theta(x, \Theta_u) = \Theta_{[x,u]} \) for all \( x, y \in \tilde{\zeta}_1(0), u \in \tilde{\zeta}_1(1) \);

(ii) \( C \) is central in \( U(\mathfrak{g}, e) \);

(iii) for all \( u, v \in \tilde{\zeta}_1(1) \),

\[
[\Theta_u, \Theta_v] = \frac{1}{2}([f, [u, v]](C - \Theta_{\text{Cas}} - c_0) + \sum_{1 \leq i \leq 2s} (\Theta_{[u, z_i]} \Theta_{[v, z_j]} + \Theta_{[v, z_j]} \Theta_{[u, z_i]})),
\]

where \( \Theta_{\text{Cas}} \) is a Casimir element of the Lie algebra \( \Theta(\tilde{\zeta}_1(0)) \).

Our goal is to construct explicitly isomorphisms of Theorem 7 for those two cases, that is, for \( \mathfrak{g} = \mathfrak{sl}_{n+1}, \mathfrak{sp}_{2n+2} \), and a minimal nilpotent \( e \in \mathfrak{g} \).

**Lemma 25. Formulas**

\[ \tilde{\gamma}(c_0) = \frac{c_0 - C}{2}, \quad \tilde{\gamma}(y_i) = \Theta_{E_{i,n+1}}, \quad \tilde{\gamma}(x_i) = \Theta_{E_{n,i}}, \quad \tilde{\gamma}(A) = \Theta_A, \quad A \in \mathfrak{sl}_n \cong \tilde{\zeta}_1(0) \]

(10)

**Proof.** Choose a natural \( \mathfrak{sl}_2 \)-triple \((e, h, f) = (E_{n,n+1}, E_{n,n} - E_{n+1,n+1}, E_{n+1,n+1})\) in \( \mathfrak{g} = \mathfrak{sl}_{n+1} \). Then \( \{E_{i,n+1}, E_{n,i}\} \leq i \leq n-1 \) form a basis of \( \tilde{\zeta}_1(1) \), while \( \{E_{ij}, E_{11} - E_{kk}, T_{n-1,2j} \mid 1 \leq i \leq j \leq n-1 \} \) form a basis of \( \tilde{\zeta}_1(0) \). Identifying \( \tilde{\zeta}_1(1) \) with \( V_{n-1} \oplus V_{n-1}^* \), we get an epimorphism of algebras \( \gamma : U(\mathfrak{gl}_{n-1}) \times T(V_{n-1} \oplus V_{n-1}^*)[C] \twoheadrightarrow U(\mathfrak{sl}_{n+1}, E_{n+1,n+1}) \) defined by

\[
\begin{align*}
\gamma(C) &= C, \quad \gamma(y_i) = \Theta_{E_{i,n+1}}, \quad \gamma(x_i) = \Theta_{E_{n,i}}, \quad \gamma(I_{-1}) = \Theta_{T_{n-1,2}}, \\
\gamma(A) &= \Theta_A, \quad A \in \mathfrak{sl}_{n-1} \subset \mathfrak{sl}_{n+1}.
\end{align*}
\]
According to Theorem 24, its kernel \( \text{Ker}(\gamma) \) is generated by
\[
w \otimes w' - w' \otimes w - \frac{1}{2} \left( f_s \left[ \gamma(w), \gamma(w') \right] \left( C - \gamma^{-1}(\Theta_{\text{Cas}}) - c_0 \right) \right) - \gamma^{-1} \left( \text{Sym} \sum_{1 \leq i \leq 2s} \Theta_{[w,z_i]} \Theta_{[w',z_i']} \right),
\]
with \( w, w' \in V_{n-1} \oplus V_{n-1}^* \), \( \gamma^{-1}(\Theta_c) \in \mathfrak{g}t_{n-1} \oplus V_{n-1} \oplus V_{n-1}^* \) well-defined for \( c \in \mathfrak{g}(0) \oplus \mathfrak{g}(1) \).

Choose the Witt basis of \( \mathfrak{g}(-1) \) as \( z_i := E_{i,n}, z_{i+s} := E_{n+1,i}, 1 \leq i \leq n-1 =: s \).

- For \( w, w' \in V_{n-1} \) or \( w, w' \in V_{n-1}^* \) we just get \( w \otimes w' - w' \otimes w \in \text{Ker}(\gamma) \).
- For \( w = y_p \in V_{n-1}, w' = x_q \in V_{n-1}^* \) we get the following element of \( \text{Ker}(\gamma) \):
\[
y_p \otimes x_q - x_q \otimes y_p + \frac{\delta_y}{2} \left( C - \gamma^{-1}(\Theta_{\text{Cas}}) - c_0 \right) - \gamma^{-1} \left( \text{Sym} \sum_{1 \leq i \leq 2s} \Theta_{[y_p,z_i]} \Theta_{[x_q,z_i']} \right).
\]

For \( 1 \leq i \leq s \) we obviously have \([E_{p,n+1}, z_i] = 0\), while
\[
[E_{p,n+1}, z_{i+s}] = E_{pi} - \delta_q E_{n+1,1} \Rightarrow [E_{p,n+1}, z_{i+s}]^t = E_{pi} - \frac{\delta_q}{2} (E_{nn} + E_{n+1,n+1}).
\]

A similar argument implies
\[
[E_{nq}, z_{i+s}]^t = E_{iq} - \delta_q (E_{nn} + E_{n+1,n+1}).
\]

Thus
\[
\Theta_{[E_{p,n+1}, z_{i+s}]}^t = \gamma(E_{pi}) + \frac{1}{2} \delta_q \gamma(I_{n-1}), \quad \Theta_{[E_{nq}, z_{i+s}]}^t = \gamma(E_{iq}) + \frac{1}{2} \delta_q \gamma(I_{n-1}),
\]
so that
\[
\gamma^{-1}(\text{Sym} \sum \Theta_{[E_{p,n+1}, z_i]} \Theta_{[E_{nq}, z_i']} \right) = \text{Sym} \left( \sum E_{pi} E_{iq} \right) + \text{Sym}(I_{n-1} \cdot E_{pq}) + \frac{1}{4} \delta_q I_{n-1}^2.
\]

On the other hand, since \( \gamma^{-1}(\gamma(E_{lk})^t) = E_{kl} + \frac{1}{2} \delta_{lk} I_{n-1}, \) we get
\[
\gamma^{-1}(\Theta_{\text{Cas}}) = \sum_{k \neq l} E_{kl} E_{lk} + \sum_k E_{kk}^2 + \frac{1}{2} I_{n-1}^2.
\]

Let \( \tilde{R}_{n-1} := \sum E_{ii}^2 + \frac{1}{2} \sum_{i \neq j} (E_{ii} E_{jj} + E_{ij} E_{ji}) \). Then we get
\[
y_p \otimes x_q - x_q \otimes y_p - \left( \frac{\alpha - C}{2} \cdot \delta_{p} \right) + \text{Sym} \left( \sum E_{pi} E_{iq} + I_{n-1} \cdot E_{pq} + \delta_q \tilde{R}_{n-1} \right) \in \text{Ker}(\gamma). \]

This implies the statement of the lemma. \( \square \)
Lemma 26. Formulas
\[ \tilde{\gamma}(\xi_0) = \frac{c_0 - C}{2}, \quad \tilde{\gamma}(y_i) = \frac{\Theta_{y_i}}{\sqrt{2}}, \quad \tilde{\gamma}(A) = \Theta_A, \quad A \in \mathfrak{sp}_{2n} \simeq \mathcal{H}(0) \] \tag{11}

establish the isomorphism \( H_1(\mathfrak{sp}_{2n}) \cong U(\mathfrak{sp}_{2n+2}, E_{1,2n+2}) \) from Theorem 7(b).

Proof. First, choose an \( \mathfrak{sl}_2 \)-triple \( (\epsilon, h, f) = (E_{1,2n+2}, E_{11} - E_{2n+2,2n+2}, E_{2n+2,1}) \) in \( \mathfrak{g} = \mathfrak{sp}_{2n+2} \). Then \( \{y_k := E_{k+1,2n+2} + (-1)^k E_{1,2n+2-k} : 1 \leq k \leq 2n \} \) form a basis of \( \mathcal{H}(1) \), while \( \mathcal{H}(0) \simeq \mathfrak{sp}_{2n} \). Identifying \( \mathcal{H}(1) \) with \( V_{2n} \) via \( y_k \mapsto v_k \), we get an algebra epimorphism
\[ \gamma : U(\mathfrak{sp}_{2n}) \times T(V_{2n})[C] \twoheadrightarrow U(\mathfrak{sp}_{2n+2}, E_{1,2n+2}), \quad C \mapsto C, \quad y_i \mapsto \Theta_{y_i}, \quad A \mapsto \Theta_A \ (A \in \mathfrak{sp}_{2n}). \]

According to Theorem 24, its kernel \( \text{Ker}(\gamma) \) is generated by \( \{y_p \otimes y_p - y_q \otimes y_q - (\ldots)\}_{p,q \leq 2n} \). Let us now compute the expression represented by the ellipsis.

Choose the Witt basis of \( \mathfrak{g}(-1) \) with respect to the form \( \omega_\chi \) as
\[ z_i := \frac{(-1)^{i+1}}{2} (E_{2n+2-i,1} + (-1)^i E_{2n+2,i+1}), \]
\[ z_{i+s} := E_{i+1,1} - (-1)^i E_{2n+2,n+1-i}, \quad 1 \leq i \leq n =: s. \]

Since \( (f, [v_q, v_p]) = 2(-1)^{\delta_{p+q}} \), the above expression in ellipsis equals to:
\[ (-1)^{\delta_{p+q}} (C - \gamma^{-1} (\Theta_{\text{Cas}}) - c_0) + \gamma^{-1} \left( \text{Sym} \left( \sum_{1 \leq i \leq 2n} \Theta_{[v_q, z_i]} \Theta_{[v_p, z_i]} \right) \right), \]
where \( \gamma^{-1} (\Theta_{\xi}) \in \mathfrak{sp}_{2n} \oplus V_{2n} \) is well-defined for any \( \xi \in \mathcal{H}(0) \oplus \mathcal{H}(1) \), though \( \gamma \) is not injective.

For any \( 1 \leq k, l \leq 2n \), \( 1 \leq j \leq n \) it is easily verified that
\[ [v_k, z_j] = \frac{1}{2} (E_{k+1,j+1} - (-1)^{k+j} E_{2n+2-j,2n+2-k}) - \frac{1}{2} \delta^j_k \cdot h, \]
\[ [v_l, z_{j+s}] = (-1)^{j+1} (E_{l+1,2n+2-j} + (-1)^{l-j} E_{j+1,2n+2-l}) + (-1)^j \delta_{l,j}^{2n+1} \cdot h, \]
so that
\[ [v_k, z_j]^2 = \frac{(-1)^{k+j} E_{2n+2-j,2n+2-k} - E_{k+1,j+1}}{2}, \]
\[ [v_l, z_{j+s}]^2 = (-1)^{l+1} E_{l+1,2n+2-j} + (-1)^{j+1} E_{j+1,2n+2-l}. \]

We also have
\[ \gamma^{-1} (\Theta_{\text{Cas}}) = \frac{1}{4} \sum_{i,j} (E_{j,i} + (-1)^{i+j+1} E_{2n+1-i,2n+1-j}) \times (E_{l,j} + (-1)^{j+i+1} E_{2n+1-j,2n+1-i}). \]
On the other hand, it is straightforward to check that
\[
\begin{align*}
    r_0(y_q, y_p) &= (-1)^p \delta_{p+q}, \\
    r_2(y_q, y_p) &= \frac{(-1)^{p+1}}{4} \text{Sym} \sum_x \left( E_{s,2n+1-q} + (-1)^{p+q} E_{q,2n+1-s} \right) \\
    &\quad \times \left( E_{p,s} + (-1)^{p+u+1} E_{2n+1-s,2n+1-p} \right) \\
    &\quad + \frac{(-1)^p}{8} \delta_{p+q} \text{Sym} \sum_{i,j} \left( E_{i,j} + (-1)^{i+j+1} E_{2n+1-j,2n+1-i} \right) \\
    &\quad \times \left( E_{j,i} + (-1)^{i+j+1} E_{2n+1-i,2n+1-j} \right).
\end{align*}
\]
To summarize, the kernel of the epimorphism \( r \) is generated by the elements \( \{ y_q \otimes y_p - y_p \otimes y_q - (2r_2(y_q, y_p) + (c_0 - C) r_0(y_q, y_p)) \}_{p,q \leq 2n} \).

This implies the statement of the lemma. □

C. Decompositions (*) and (♠) for \( m = -1, 0 \)

- Decomposition isomorphism \( H_{h,-1}(\mathfrak{g}(n))^{\wedge^u} \cong H_{h,0}(\mathfrak{g}(n-1))^{\wedge^u} \otimes \mathbb{C}[[h]] W_{h,n}^{\wedge^u} \)

Here \( H_{h,0}(\mathfrak{g}(n-1)) \) is defined similarly to \( H_{h,0}(\mathfrak{g}(1)) \) with an additional central parameter \( \zeta_0 \) and the main relation being \( [y, x] = h^2 \zeta_0 r_0(y, x) \), while \( H_{h,-1}(\mathfrak{g}(n)) := U_h(\mathfrak{g}(n) \times (V_n \oplus V_n^*)) \).

Notation: We use \( y_k, x_i, e_{k,l} \) when referring to the elements of \( H_{h,-1}(\mathfrak{g}(n)) \) and capital \( Y_i, X_j, E_{i,j} \) when referring to the elements of \( H_{h,0}(\mathfrak{g}(n-1)) \). We also use indices \( i \leq k, l \leq n \) and \( 1 \leq i, j, i', j' < n \) to distinguish between \( \leq n \) and \( < n \). Finally, set \( v_n := (0, \ldots, 0, 1) \in V_n \).

The following lemma establishes explicitly the aforementioned isomorphism:

Lemma 27. Formulas
\[
\begin{align*}
    \Psi_{-1}(y_k) &= z_k, & \Psi_{-1}(e_{n,k}) &= z_n \partial_k, \\
    \Psi_{-1}(e_{i,j}) &= E_{i,j} + z_i \partial_j, & \Psi_{-1}(e_{i,n}) &= z_n^{-1} Y_i - \sum_{j < n} z_n^{-1} z_j E_{i,j} + z_i \partial_n, \\
    \Psi_{-1}(x_j) &= X_j, & \Psi_{-1}(x_n) &= -z_n^{-1} \zeta_0 - \sum_{p < n} z_n^{-1} z_p X_p
\end{align*}
\]
define an isomorphism \( \Psi_{-1} : H_{h,-1}(\mathfrak{g}(n))^{\wedge^u} \cong H_{h,0}(\mathfrak{g}(n-1))^{\wedge^u} \otimes \mathbb{C}[[h]] W_{h,n}^{\wedge^u} \).

Its proof is straightforward and is left to an interested reader (most of the verifications are the same as those carried out in the proof of Lemma 28 below).

- Decomposition isomorphism \( H_{h,0}(\mathfrak{g}(n))^{\wedge^u} \cong H_{h,1}(\mathfrak{g}(n-1))^{\wedge^u} \otimes \mathbb{C}[[h]] W_{h,n}^{\wedge^u} \)

Here \( H_{h,1}(\mathfrak{g}(n-1)) \) is an algebra defined similarly to \( H_{h,1}(\mathfrak{g}(1)) \) with an additional central parameter \( \zeta_0 \) and the main relation being \( [y, x] = h^2 \zeta_0 r_0(y, x) + r_1(y, x) \). We follow analogous conventions as for variables \( y_k, x_i, e_{k,l}, Y_i, X_j, E_{i,j} \) and indices \( i, j, i', j', k, l \).

The following lemma establishes explicitly the aforementioned isomorphism:
Lemma 28. Formulas
\[ \Psi_0(y_k) = z_k, \quad \Psi_0(e_{n,k}) = z_n \partial_k, \]
\[ \Psi_0(e_{i,j}) = E_{i,j} + z_i \partial_j, \quad \Psi_0(e_{i,n}) = z_n^{-1} Y_i - \sum_{j < n} z_n^{-1} z_j X_j + z_i \partial_n, \]
\[ \Psi_0(x_j) = -\partial_j + X_j, \quad \Psi_0(x_n) = -\partial_n - \sum_{i < n} z_n^{-1} z_i X_i - z_n^{-1} \left( \zeta_0 + \sum_{j < n} E_{j,j} \right) \]
define an isomorphism \( \Psi_0 : H_{h,0}(\mathfrak{gl}_n)^{\wedge_n} \rightarrow H'_{h,1}(\mathfrak{gl}_{n-1})^{\wedge_n} \otimes \mathbb{C}[[h]] \mathcal{W}_{h,n}. \)

Proof. These formulas provide a homomorphism
\[ H_{h,0}(\mathfrak{gl}_n)^{\wedge_n} \rightarrow H'_{h,1}(\mathfrak{gl}_{n-1})^{\wedge_n} \otimes \mathbb{C}[[h]] \mathcal{W}_{h,n} \]
if and only if \( \Psi_0 \) preserves all the defining relations of \( H_{h,0}(\mathfrak{gl}_n). \) This is quite straightforward and we present only the most complicated verifications, leaving the rest to an interested reader.

* Verification of \( [\Psi_0(e_{i,n}), \Psi_0(e_{i',n})] = -h^2 \delta_{i,j} \Psi_0(e_{i',n}) \):
\[
[\Psi_0(e_{i,n}), \Psi_0(e_{i',n})] = [z_n^{-1} Y_i - \sum_{p < n} z_n^{-1} z_p E_{i,p} + z_i \partial_n, E_{i',j} + z_i \partial_j] = h^2 \left( -\delta_{i,j} z_n^{-1} Y_i' - z_n^{-1} z_i E_{i,j'} + \delta_{i'} \sum_{p < n} z_n^{-1} z_p E_{i',p} + z_n^{-1} E_{i,j'} - \delta_{i'} z_i \partial_j \right) = -h^2 \delta_{i,j} \Psi_0(e_{i',n}).
\]

* Verification of \( [\Psi_0(e_{i,n}), \Psi_0(x_j)] = -h^2 \delta_{i,j} \Psi_0(x_n) \):
\[
[\Psi_0(e_{i,n}), \Psi_0(x_j)] = [z_n^{-1} Y_i - \sum_{1 \leq q \leq n-1} z_n^{-1} z_q E_{i,q} + z_i \partial_n, -\partial_j + X_j] = -h^2 z_n^{-1} E_{i,j} + \delta_{i,j} h^2 \partial_n + \delta_{i,j} h^2 \sum_{q < n} z_n^{-1} z_q X_q + z_n^{-1} \{ Y_i, X_j \} = -h^2 z_n^{-1} E_{i,j} + \delta_{i,j} h^2 \left( \partial_n + \sum_{q < n} z_n^{-1} z_q X_q \right)
+ h^2 z_n^{-1} \left( E_{i,j} + \delta_{i,j} \sum_{i < n} E_{i,i} + \delta_{i,j} \zeta_0 \right) = -h^2 \delta_{i,j} \Psi_0(x_n).
\]

* Verification of \( [\Psi_0(e_{i,n}), \Psi_0(x_n)] = 0 \):
\[
[\Psi_0(e_{i,n}), \Psi_0(x_n)] = [z_n^{-1} Y_i - \sum_{p < n} z_n^{-1} z_p E_{i,p} + z_i \partial_n, -\partial_n - \sum_{j < n} z_n^{-1} z_j X_j - z_n^{-1} \left( \zeta_0 + \sum_{j < n} E_{j,j} \right)] = h^2 \left( \sum_{p < n} z_n^{-2} z_p E_{i,p} - z_n^{-2} Y_i + z_i z_n^{-2} \zeta_0 + z_i z_n^{-2} \sum_{j < n} E_{j,j} + z_n^{-2} Y_i - \sum_{j < n} z_j z_n^{-2} \{ Y_i, X_j \} \right) = 0.
\]
Once homomorphism $\Psi_0$ is established, it is easy to check that the map

$$z_k \mapsto y_k, \quad \partial_k \mapsto y_n^{-1} e_{n,k}, \quad E_{i,j} \mapsto e_{i,j} - y_i y_n^{-1} e_{n,j}, \quad \zeta_0 \mapsto - \sum_{k \leq n} y_k x_k - \sum_{k \leq n} e_{k,k},$$

$$X_j \mapsto x_j + y_n^{-1} e_{n,j}, \quad Y_i \mapsto \sum_{1 \leq q \leq n} y_q (e_{i,q} - y_i y_n^{-1} e_{n,q})$$

provides the inverse to $\Psi_0$. This completes the proof of the lemma. □

- Decomposition isomorphism $H_{h,-1}(\mathfrak{sp}_{2n})^{\wedge_v} \cong H'_{h,0}(\mathfrak{sp}_{2n-2})^{\wedge_v} \otimes \mathbb{C}[\partial_i] W_{h,2n}^{\wedge_v}$

Here $H'_{h,0}(\mathfrak{sp}_{2n-2})$ is defined similarly to $H_{h,0}(\mathfrak{sp}_{2n-2})$ with an additional central parameter $\zeta_0$ and the main relation being $[x, y] = h^2 \zeta_0 r_0(x, y)$, while $H_{h,-1}(\mathfrak{sp}_{2n}) := U_h(\mathfrak{sp}_{2n} \otimes V_{2n})$.

**Notation:** We use $y_k$, $u_{k,l} := e_{k,l} + (-1)^{k+l+1} e_{2n+1-l,2n+1-k}$ when referring to the elements of $H_{h,-1}(\mathfrak{sp}_{2n})$ and $Y_i$, $U_{i,j} := E_{i,j} + (-1)^{i+j+1} E_{2n+1-j,2n+1-i}$ when referring to the elements of $H'_{h,0}(\mathfrak{sp}_{2n-2})$. Note that $\{u_{k,l}\}_{k,l \geq 1}$ is a basis of $\mathfrak{sp}_{2n}$, while $\{U_{i,j}\}_{i,j \geq 1}$ is a basis of $\mathfrak{sp}_{2n-2}$. We use indices $1 \leq k, l \leq 2n$ and $1 \leq i, j \leq 2n - 2$. Finally, set $v_1 := (1, 0, \ldots, 0) \in V_{2n}$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 29.** Define $\psi_1(u_{k,l}) := z_k \partial_l + (-1)^{k+l+1} z_{2n+1-l} \partial_{2n+1-k}$ for all $k, l$. We also define

$$\psi_0(u_{1,k}) = 0, \quad \psi_0(u_{i+1,1}) = Y_i, \quad \psi_0(u_{i+1,1+j}) = U_{i,j}, \quad \psi_0(u_{2n,1}) = \zeta_0.$$

Formulas $Y_{-1}(y_k) = z_k, Y(u_{k,l}) = \psi_0(u_{k,l}) + \psi_1(u_{k,l})$ give rise to an isomorphism

$$Y_{-1} : H_{h,-1}(\mathfrak{sp}_{2n})^{\wedge_v} \cong H'_{h,0}(\mathfrak{sp}_{2n-2})^{\wedge_v} \otimes \mathbb{C}[\partial_i] W_{h,2n}^{\wedge_v}.$$

The proof of this lemma is straightforward and is left to an interested reader.

- Finally, we have the case of $\mathfrak{g} = \mathfrak{sp}_{2n}$, $m = 0$.

There is also a decomposition isomorphism

$$Y_0 : H_{h,0}(\mathfrak{sp}_{2n})^{\wedge_v} \cong H_{h,1}(\mathfrak{sp}_{2n-2})^{\wedge_v} \otimes \mathbb{C}[\partial_i] W_{h,2n}^{\wedge_v}.$$

This isomorphism can be made explicit, but we find the formulas quite heavy and unrevealing, so we leave them to an interested reader.

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