A quantum spin Hall insulator is a two-dimensional state of matter consisting of an insulating bulk and one-dimensional helical edge states. While these edge states are topologically protected against elastic backscattering in the presence of disorder, interaction-induced inelastic terms may yield a finite conductivity. By using a kinetic equation approach, we find the backscattering rate $\tau^{-3}$ and the semiclassical conductivity in the regimes of high ($\omega \gg \tau^{-1}$) and low ($\omega \ll \tau^{-1}$) frequency. By comparing the two limits, we find that the parametric dependence of conductivity is described by the Drude formula for the case of a disordered edge. On the other hand, in the clean case where the resistance originates from umklapp interactions, the conductivity takes a non-Drude form with a parametric suppression of scattering in the $dc$ limit as compared to the $ac$ case. This behavior is due to the peculiarity of umklapp scattering processes involving necessarily the state at the “Dirac point”. In order to take into account Luttinger liquid effects, we complement the kinetic equation analysis by treating interactions exactly in bosonization and calculating conductivity using the Kubo formula. In this way, we obtain the frequency and temperature dependence of conductivity over a wide range of parameters. We find the temperature and frequency dependence of the transport scattering time in a disordered system as $\tau \sim \max(\omega, T)^{-2K-2}$, for $K > 2/3$ and $\tau \sim \max(\omega, T)^{-kK+2}$ for $K < 2/3$.

I. INTRODUCTION

One of the most fascinating advances of recent years has been the discovery of the plethora of quantum states characterized by a non-trivial topological structure of the single-particle wavefunctions. While the role of topology in the quantum Hall effect was recognized in the early eighties, it was not realized until much later that related topological states exist with other symmetries. In particular, it was shown about ten years ago that a particular kind of topological order known as Z$_2$ topological order is present in the quantum spin Hall effect – like the regular quantum Hall effect this also occurs in two-dimensional systems but in the absence of a magnetic field (in other words, in systems with time-reversal symmetry). It was around this time the physics of topological insulators really took off when it was predicted and subsequently measured experimentally that the quantum spin Hall effect is realized in HgTe/CdTe quantum wells. This state has since also been predicted and observed in InAs/GaSb quantum wells.

The most striking feature of these topological insulator states is that while the bulk exhibits a spectral gap, the edges (or surfaces in three-dimensional cases) support metallic (gapless) states with curious properties. In the case of the conventional quantum Hall effects, these edge modes are chiral, with the chirality determined by the sense of the external magnetic field. These states show a quantized conductance as the chiral nature implies there are no states to which electrons may be backscattered and hence no mechanism by which electrical resistance may be generated. The edge states of a quantum spin Hall system however are quite different. They form a helical liquid, meaning that the chirality and spin-polarisation are linked. In these helical liquids, an electron moving one direction forms a Kramer’s doublet with that moving the opposite direction meaning that a time-reversal symmetric impurity (for example a non-magnetic impurity) can not elastically backscatter electrons. This is a state sometimes dubbed a symmetry protected topological state breaking time-reversal symmetry annuls the topological protection (for example, helical liquids may be formed in other contexts without the role of topology).

While the topological protection forbids elastic backscattering from non-magnetic impurities which may naively be thought to lead to quantized conductance $G = G_0 = 2e^2/h$, no such simple result exists for inelastic backscattering when interactions are present. Many forms of inelastic scattering have been investigated on the side of theory, including multiple scattering off impurities, random Rashba spin-orbit coupling, umklapp interactions both with and without impurities, phonon scattering and scattering from charge puddles.

No matter the source of the inelastic scattering, the result of all of these investigations is that the correction to conductance behaves as a power-law in temperature $G \sim T^n$ (or possibly activated behavior in a clean system, which goes to zero as temperature goes to zero. The power is dependent on the exact scattering mechanism chosen, as well as Luttinger liquid effects.
which are ever-present in one-dimensional systems.

Experiments on helical edge states have been performed for both short edge channels, where the system length $L$ is much smaller than the mean free path $l$, and long edge channels, where $L \gg l$. The first experimental study of the temperature dependence of helical edge transport was performed in Ref. 8 and 31. The authors measure the conductance of short edges ($\sim 1 \mu m$) of HgTe/CdTe quantum wells and find that it is a quantized close to $G_0$ and depends only weakly on temperature. Longer edges of the order of $\sim 10 - 30 \mu m$ have been studied in Ref. 32. Their results show conductance well below the quantized value and rather temperature independent. Transport properties have also been measured in InAs/GaSb quantum wells. In these systems the measured conductance is close to the quantized value and seems to be insensitive to temperature and even magnetic field variations over a large range of parameters. This behavior is observed for both short and long edge channels.

As possible ways to explain the lack of temperature dependence, a number of potential perturbations that weakly break time-reversal symmetry have been investigated, such as Kondo impurities, dynamical nuclear polarization, exciton condensation, and explicit addition of a magnetic field. No consensus has yet been reached to explain the discrepancy between theory and experiment and therefore more work must be done on both sides.

In this work, we concentrate on the time-reversal symmetric case, where we study the transport properties of a model that was first introduced in Ref. 26 and includes interactions, impurity scattering, and a Rashba spin-orbit term. In their work the authors study corrections to the $dc$ conductance of short edges, while we concentrate on the conductivity of long edge channels.

While the frequency dependence of the conductivity of a Luttinger liquid (LL) has been studied both in the semiclassical regime and including weak localisation effects, the conductivity of a helical Luttinger liquid (HLL) remains largely unexplored.

We study the conductivity of a HLL first by means of a kinetic equation approach, from which we obtain the high and low frequency limits of conductivity. This fermionic approach is valid for a weakly interacting system when the Luttinger liquid constant $K \approx 1$; to investigate the more general case we supplement this approach with bosonization, being careful to highlight the links between this and the prior conceptually transparent fermionic calculations. This allows us to build up a complete picture of the conduction properties of this model over the whole of parameter space.

The structure of the paper is as follows. In section II, we introduce the microscopic model of the helical edge state, while in section III we discuss the kinetic equation formalism and the scattering terms which appear in the collision integral. In section IV, we give the solution of the kinetic equation for both the $ac$ and $dc$ limits, providing a critical comparison of these results. In section V we derive the bosonized version of the Hamiltonian (treating the disorder via the replica trick), the conductivity of this model is then derived in section VI, and we conclude in section VII. Technical details are relegated to the appendices.

Throughout the paper we use the conventions $h = k_B = v_F = 1$ while performing the calculation and restore $h$ and $v_F$ in key results.

II. MODEL FOR HELICAL FERMIONS

We consider an infinite one-dimensional system of helical fermions. The electrons feel a density-density interaction and are subject to a nonmagnetic random disorder potential. The Hamiltonian is thus a sum of three parts:

$$H = H_0 + H_{\text{int}} + H_{\text{imp}}.$$

Additionally, the strong spin orbit coupling in the bulk which leads to the emergence of the helical edge states also breaks the $SU(2)$ spin rotation symmetry in the edge. The resulting liquid with broken $S_z$ symmetry is termed "generic helical liquid". The model we use to describe this generic HLL was first introduced in Ref. 26. To fix our notation and to review the main ideas behind the model we will briefly review its derivation. The 1D helical system is translation invariant and momenta $k$ are thus good quantum numbers for the eigenstates. Furthermore, to lowest order in spin orbit coupling, the spin degree of freedom of the excitations is frozen out because each chirality has a well-defined spin direction. The effective low energy theory for the edge excitations is thus that of free spinless fermions,

$$H_0 = \frac{1}{L} \sum_{\eta = \sigma = R,L} \eta k \psi_{k,\eta}^\dagger \psi_{k,\eta}.$$

Here $\psi_{k,\eta}$ are fermionic operators and $\eta = R, L = +, -$ denotes chirality. If we assume that time reversal symmetry holds, Kramer’s theorem ensures that for any $k$ there exist two orthogonal eigenstates, created by fermionic operators $\psi_{k,\eta}^\dagger$ and $\psi_{-k,\eta}^\dagger$ which are related by time reversal $\mathcal{T} \psi_{k,\eta} = \eta \psi_{-k,\eta}$.

The interaction and disorder contributions for spinful fermions read as

$$H_{\text{int}} = \frac{1}{L} \sum_{kq \sigma \sigma'} \sum_{\eta} V_q \psi_{k,\sigma}^\dagger \psi_{k-q,\sigma}^\dagger \psi_{p,\sigma'}^\dagger \psi_{p+q,\sigma'},$$

$$H_{\text{imp}} = \frac{1}{L} \sum_{kq} \sum_{\sigma} U_q \psi_{k,\sigma}^\dagger \psi_{k-q,\sigma}.$$

Here, $\sigma = \uparrow, \downarrow$ denotes the spin in $z$-direction.

However, as mentioned before in a generic helical liquid, spin rotation invariance around the $z$-direction will be broken by spin-orbit terms either due to structural inversion asymmetry or bulk inversion asymmetry in the
bulk of the system. We therefore formulate the problem in the chiral basis \((R, L)\) in which the free Hamiltonian is diagonal. In order to perform this rotation we follow Ref. [26] and derive the rotation matrix from symmetry arguments. The operators \(\psi_{k, \sigma}\) of an electron with momentum \(k\) and spin projection \(\sigma\) along the z-axis are related to the chiral operators \(\psi_{k, \eta}\) by a momentum dependent SU(2) matrix \(B_k\),

\[
\begin{pmatrix}
\psi_{k, \uparrow} \\
\psi_{k, \downarrow}
\end{pmatrix} = B_k \begin{pmatrix}
\psi_{k, R} \\
\psi_{k, L}
\end{pmatrix}.
\]  

To preserve fermionic commutation relations the matrix has to be unitary \(B_k^\dagger B_k = 1\). Moreover, time reversal invariance entails the symmetry \(B_k = B_\omega\). Because of these constraints the leading terms in \(B_k\) for small \(k \ll k_0\) can be written as

\[
B_k = \begin{pmatrix}
1 - \frac{k^4}{2k_0^2} & k^2 \\
-\frac{k^2}{2k_0^2} & 1 - \frac{k^4}{2k_0^2}
\end{pmatrix}.
\]

\(k_0\) is an effective parameter that describes the strength of spin-orbit coupling; in the absence of any spin-orbit coupling we have \(k_0 \to \infty\). Physically, it can be interpreted as the inverse length scale on which an electron keeps its spin orientation.

In the following we assume that interaction and impurity potentials are momentum independent, \(U_q = U\) and \(V_q = V\). In the case of interactions this is justified if the potential is well screened by external media e.g. external gates. For impurities we make the assumption that the disorder potential is short-ranged in real space. Performing the rotation Eq. (5) in Eqs. (3) and (4) we obtain

\[
H_{\text{int}} = \frac{1}{L} \sum_{k, q_1, q_2, \eta_1, \eta_2} V_q [B_k^\dagger B_{k-q} \eta_1, \eta_2] [B_{k-q}^\dagger B_{k+q} \eta_3, \eta_4] \times \psi_{k, \eta_1} \psi_{k-q, \eta_2} \psi_{p, \eta_3}^\dagger \psi_{p+q, \eta_4}^\dagger,
\]

\[
H_{\text{imp}} = \frac{1}{L} \sum_{k, q_1, q_2} U_q [B_k^\dagger B_{k-q} \eta_1, \eta_2] \psi_{k, \eta_1} \psi_{k-q, \eta_2}.
\]

To lowest order in \(k/k_0\) the product of rotations can be written in the form

\[
[B_k B_p]_{\eta, \eta'} = \delta_{\eta, \eta'} + \eta \delta_{\eta, \eta'} \left( k^2 - p^2 \right) \frac{k^2}{k_0^2},
\]

where we use the notation \(R = L\) and vice versa. Inserting (4) into (7) and (8) yields the interaction terms

\[
\begin{align*}
H_1 &= \frac{V}{k_0 L} \sum_{k, p, q, \eta} (k^2 - (k - q)^2) \left( p^2 - (p + q)^2 \right) \psi_{k, \eta}^\dagger \psi_{k-q, \eta}^\dagger \psi_{p, \eta} \psi_{p+q, \eta}, \\
H_2 &= \frac{V}{L} \sum_{k, p, q, \eta} \psi_{k, \eta}^\dagger \psi_{k, \eta}^\dagger \psi_{p, \eta} \psi_{k-q, \eta} \psi_{p, \eta} \psi_{k-q, \eta}, \\
H_3 &= \frac{V}{k_0 L} \sum_{k, p, q, \eta} (k^2 - (k - q)^2) \left( p^2 - (p + q)^2 \right) \psi_{k, \eta}^\dagger \psi_{k, \eta}^\dagger \psi_{p, \eta} \psi_{p+q, \eta} \psi_{k-q, \eta}, \\
H_4 &= \frac{V}{L} \sum_{k, p, q, \eta} \psi_{k, \eta}^\dagger \psi_{k, \eta}^\dagger \psi_{p, \eta} \psi_{k-q, \eta} \psi_{p, \eta} \psi_{k-q, \eta}, \\
H_5 &= -\frac{V}{k_0 L} \sum_{k, p, q, \eta} \eta (k^2 - p^2) \psi_{k+q, \eta}^\dagger \psi_{p+q, \eta} \psi_{p, \eta} \psi_{k, \eta} + h.c., \\
H_{\text{imp}} &= \frac{U}{L} \sum_{k, p, q, \eta} \left( \psi_{k, \eta}^\dagger \psi_{p, \eta} + \eta \frac{k^2 - p^2}{k_0^2} \psi_{k, \eta}^\dagger \psi_{p, \eta} \right).
\end{align*}
\]

The different terms of the interaction Hamiltonian can be grouped analogously to the g-ology of a conventional LL, which motivates our notation. However, in the present model we have an additional umklapp term that backscatters only one incoming particle. For the purpose of this work it will be called \(g_5\) term. A diagrammatic representation of possible interactions processes is depicted in Fig. 1.

It is important to realize, that there is a fundamental difference between the case of conventional one dimensional fermions and the helical fermions discussed here. Usually, one linearizes the spectrum of fermions around the Fermi energy defining left and right movers with linear spectrum. However, both branches of the spectrum are always separated by a large moment of roughly \(2k_F\). In contrast to that, the spectrum of helical fermions
possesses a “Dirac point”, i.e. a point where the right and left moving branches cross. In particular the $g_0$ term only contributes to the low energy physics, if the system is close to the Dirac point which explains why it is never discussed in the context of Luttinger liquid. However, it will turn out that this process is crucial for the transport properties of a HLL for sufficiently clean samples.

One further thing should be mentioned at this point. The parameters $V$ and $U$ should be considered as effective couplings of the low energy theory after integrating out all degrees of freedom above the UV cutoff, which is given by the bulk gap. Therefore, renormalization effects due to high lying states are already incorporated into the coupling constants and do not affect the physics apart from that.

In the following we investigate the transport properties of this model. To this end we develop a semiclassical, quantum kinetic equation formalism in the next section.

**III. QUANTUM KINETIC EQUATION FORMALISM**

We assume that the system is subject to some external source of dephasing, such that the dephasing length $l_0$ is much shorter than the mean free path $l$. In this case we can neglect quantum interference corrections, such as weak localization and describe the system by solving a semiclassical, quantum kinetic equation.

In equilibrium, non-interacting one-dimensional helical fermions have a linear spectrum $\epsilon_{k,\eta} = \eta k$ and obey the Fermi-Dirac distribution $f^{(0)}_{k,\eta} = (1 + \exp\{\epsilon_{k,\eta} - \mu\}/T)^{-1}$. Away from equilibrium the distribution function $f_{k,\eta}(x, t)$ has to be determined as the solution of a quantum kinetic equation:

$$\partial_t f_{k,\eta}(x, t) + v_{k,\eta}(x) \partial_x f_{k,\eta}(x, t) - eE \partial_x f_{k,\eta}(x, t) = I_{k,\eta}[f_{k,\eta}].$$

Equation (11) can formally be rewritten into an integral equation for $\psi$:

$$\psi_{k,\eta}(\omega) = \frac{I_{k,\eta}[\psi_{k,\eta}]}{(-i\omega)f^{(0)}_{k,\eta}(1 - f^{(0)}_{k,\eta})} - \frac{e\eta}{-i\omega}T,$$  \hspace{1cm} (12)

where we used the fact that $\partial_t f^{(0)}_{k,\eta} = -\eta f^{(0)}_{k,\eta}(1 - f^{(0)}_{k,\eta})/T$. For two particle scattering the collision integral reads as

$$I_{1}[f_1] = - \sum_{2,1',2'} W_{12,1'2'} [f_{1'}^{(0)}(1 - f_{1'}) - f_{2'}(1 - f_2)] \times (\psi_1 + \psi_2 - \psi_{1'} - \psi_{2'}).$$

Equation (18) is given by Fermi’s golden rule

$$W_{12,1'2'} = 2\pi |\langle 1'2' | T | 12 \rangle|^2 \delta(\epsilon_i - \epsilon_f).$$

The transition probability $W_{12,1'2'}$ is given by Fermi’s golden rule

$$W_{12,1'2'} = 2\pi |\langle 1'2' | T | 12 \rangle|^2 \delta(\epsilon_i - \epsilon_f).$$

Here, the Green’s function operator is defined as

$$G = \frac{1}{\eta k_i - H_0 + i\delta}, \quad \delta \to 0 +.$$
as the solution of the kinetic equation, we obtain the
Continuing with our formal manipulations let us rename energy. tering rate is much smaller than the typical electronic
impurities. This is valid as long as the impurity scat-
ability of a process containing disorder scattering is given
fore, we can restrict our calculation to the lowest orders
Some remarks are in order. First, we consider only weak interaction strength $V$ and impurity potential $U$. Therefore, we can restrict our calculation to the lowest orders of the T-matrix. Second, we assume that impurity scatterers are uncorrelated and therefore the transition probability of a process containing disorder scattering is given by the single impurity probability times the number of impurities. This is valid as long as the impurity scattering rate is much smaller than the typical electronic energy.
Continuing with our formal manipulations let us rename

\[
\Gamma_{12,1'2'} \equiv \left[ f_{1}^{(0)} f_{2}^{(0)} (1 - f_{1'}^{(0)}) (1 - f_{2'}^{(0)}) \right] \\
\times (\psi_{1} + \psi_{2} - \psi_{1'} - \psi_{2'}) \\
\times \delta(\eta_{1}k_{1} + \eta_{2}k_{2} - \eta_{1'}k_{1'} - \eta_{2'}k_{2'}). \]  

(20)

Therefore, the final form of the collision integral (16) reads as

\[
I_{1}[\psi_{1}] = -2\pi \sum_{2,1',2'} \Gamma_{12,1'2'} |\langle 1'2' | T | 12 \rangle|^{2}. \]  

(21)

After we get the electronic distribution function $f_{k,\eta}$ as the solution of the kinetic equation, we obtain the conductivity as

\[
\sigma = -\frac{e}{EL} \sum_{k,\eta} v_{k,\eta} f_{k,\eta} \]  

(12)

\[
-\frac{e}{EL} \sum_{k,\eta} \eta f_{k,\eta}^{(0)} (1 - f_{k,\eta}^{(0)}) \psi_{k,\eta} \]  

(22)

\[
2e^{2} \frac{1}{\hbar (-i\omega)} - \frac{e}{EL (-i\omega)} \sum_{k,\eta} \eta I_{k,\eta}[\psi^{(0)}]. \]  

(23)

This will be used to calculate the conductivity of weakly interacting fermions in the next section.

IV. CONDUCTIVITY OF WEAKLY INTERACTING HELICAL FERMIONS

A. Dynamic conductivity

In the case of frequencies much larger than the inverse transport scattering time, we can solve the integral equation (14) by iteration:

\[
\psi_{k,\eta}^{(n+1)} = (\frac{eE\eta}{(-i\omega)T}) \psi_{k,\eta}^{(n)} \]  

(24)

\[
\psi_{k,\eta}^{(n+1)} = (\frac{f_{\eta,k}}{(-i\omega)T}) \psi_{\eta,k}^{(0)} + \eta_{\psi}^{(n)}, \quad n \in \mathbb{N}. \]  

(25)

Here, we stop at the zeroth order which leads to the conductivity, cf. Eq. (23).

\[
\sigma_{ac} = \frac{2e^{2}}{\hbar (-i\omega)} - \frac{e}{EL (-i\omega)} \sum_{k,\eta} \eta I_{k,\eta}[\psi^{(0)}]. \]  

(26)

The entire information about specific scattering mechanisms is encoded in the collision integral. In the following we will discuss certain microscopic mechanisms and their impact on transport. In particular, we are interested in the real part of conductivity that arises due to these collisions and characterizes current relaxation.
To calculate the real part of the conductivity we proceed as follows. First we calculate the matrix elements $\langle 1'2' | T | 12 \rangle$ of the T matrix order by order in the expansion in Eq. (18). From these expressions we obtain the collision integral according to Eq. (21), where now the distribution functions $\psi$ are replaced by the zeroth order approximation $\psi^{(0)}$. The obtained collision integrals are then used to calculate the conductivity as explained in Eq. (26).

To first order in the T-matrix we consider interactions and disorder separately, $T = H_{\text{int}} + H_{\text{imp}}$. The conductivity of a clean interacting system is discussed in Sec. [IV.A.1] and some details of the calculation can be found in Appendix [A.2]. Due to the topological protection of the edge states, disorder does not lead to a finite conductivity by itself. Therefore, we have to consider the second order of the T-matrix expansion where
can find analytical approximations for where \( \zeta \)
Here, we defined the ratio \( g \)
finite real part of the conductivity due to \( g_5 \). While \( g_5 \) is a pure interaction effect, the 1P and 2P scattering events also contain impurity scattering. Due to the presence of disorder the latter processes do not have to conserve momentum which enlarges the available phase space. Therefore, processes containing both interaction and disorder scattering lead to the most important terms in the conductivity.

FIG. 3. Most important scattering processes (a) and their possible microscopic realizations (b). While other microscopic combinations of interaction and impurity scattering yield similar outcomes, calculation shows that these combinations are the dominant ones (see Tab. 3). "1P" describes inelastic processes backscattering one electron and "2P" denotes inelastic processes backscattering two electrons. While \( g_5 \) is a pure interaction effect, the 1P and 2P scattering events also contain impurity scattering. Due to the presence of disorder the latter processes do not have to conserve momentum which enlarges the available phase space. Therefore, processes containing both interaction and disorder scattering lead to the most important terms in the conductivity.

combined effects of interactions and disorder appear as \( \langle 1'2'| H_{\text{int}} G_0 H_{\text{imp}} |12 \rangle + \langle 1'2'| H_{\text{imp}} G_0 H_{\text{int}} |12 \rangle \). The effect of these contributions on transport is discussed in Sec. IV A 2 and some details of the calculation can be found in Appendix A 3.

1. Clean system

In the absence of any impurity scattering we find a finite real part of the conductivity due to \( g_5 \) processes, see Appendix A 2. The resulting expression reads as

\[
\Re \sigma = \frac{e^2 v_F}{\hbar} \frac{1}{\omega^2} \left( \frac{V}{v_F} \right)^2 v_F k_0 \left( \frac{T}{v_F k_0} \right)^5 f(\zeta). \tag{27}
\]

Here, we defined the ratio \( \zeta = k_F / T \) and the dimensionless function

\[
f(\zeta) = \frac{8}{\pi} \int dx dy (x^2 - y^2)^2 n_F(x - \zeta) n_F(y - \zeta) \times (1 - n_F(x + y - \zeta))(1 - n_F(-\zeta)), \tag{28}
\]

where \( n_F(x) = (1 + e^{x})^{-1} \) is the Fermi function. We can find analytical approximations for \( f(\zeta) \) for high and low temperatures compared to the Fermi energy. In the regime \( k_F \gg T \) we obtain \( f(\zeta) \simeq 44/45 \zeta^6 e^{-\zeta} \) and consequently the real part of conductivity,

\[
\Re \sigma = 44 \frac{e^2 v_F}{45\pi} \frac{1}{\omega^2} \frac{V^2}{v_F} \left( \frac{k_F}{v_F k_0} \right)^6 \frac{v_F k_F}{T} e^{-v_F k_F / T}. \tag{29}
\]

In this regime the conductivity is thermally activated because energy and momentum conservation constrain one of the particles in the final state to be created at zero momentum deep within the filled Fermi sea (see Fig. 3).

Conversely, in the high temperature regime \( k_F \ll T \) we get \( f(\zeta = 0) \simeq 306.02 \) and the process leads to power law behavior,

\[
\Re \sigma = 306.02 \frac{e^2 v_F}{\hbar} \frac{1}{\omega^2} \frac{V^2}{v_F} \left( \frac{T}{v_F k_0} \right)^5. \tag{30}
\]

Let us address one important point: how is it possible that interactions that conserve momentum, such as the \( g_5 \) term, lead to current relaxation? This is surprising since in conventional Fermi liquids translational invariance implies momentum conservation and entails the persistence of currents in the absence of momentum nonconserving interactions such as impurity scattering. However, in the present case we are dealing with an effective low energy theory in which the current of a one-dimensional electron system is determined by the number of left and right movers. In particular, momentum conservation does not imply current conservation. Current relaxation arises from the scattering of right to left movers or vice versa. While these scattering processes conserve quasimomentum in the effective low energy theory they are in fact umklapp processes in the original lattice model.

In summary, we observe that in the present model only scattering processes that change the total number of left and right movers can lead to a finite conductivity. Consequently, it is clear that \( g_1, g_2 \) and \( g_4 \) processes will not affect the current since none of them change the number of left and right movers. In principle one might expect that the \( g_5 \) process also influences transport. However, we find that it does not lead to a finite real part of the conductivity. To develop a deeper understanding of the physics behind this, we consider the translation operator \( P_T \) and the particle current \( J_0 \) of the free Hamiltonian Eq. (2).

\[
P_T = \frac{1}{L} \sum_{k, \eta} \bar{\psi}_{k, \eta}^{\dagger} k \psi_{k, \eta}, \tag{31}
\]

\[
J_0 = \frac{1}{L} \sum_{k, \eta} \eta \bar{\psi}_{k, \eta}^{\dagger} \psi_{k, \eta}. \tag{32}
\]

For the case of a clean LL, it was first realized in Ref. 40 that there exists a linear combination \( P_0 = P_T + k_F J_0 \) that can be identified as the total momentum of the hamiltonian and is therefore conserved, but also commutes with a single umklapp term. The conclusion is, that a single umklapp term in a conventional LL can never lead to a finite conductivity. In the present case of
For processes that backscatter a single electron

| $g_4 \times b$ | 0 |
| $g_2 \times b$ | 0 |
| $g_3 \times b$ | $\lambda_1 \frac{2^4}{\pi^8} n_{imp}(UV)^2 \left( \frac{k_F}{\pi} \right)^6 \frac{\tau}{L^2}$ |
| $g_1 \times b$ | $\lambda_2 \frac{2^4}{\pi^8} n_{imp}(UV)^2 \left( \frac{k_F}{\pi} \right)^6 \frac{\tau}{L^2}$ |
| $g_3 \times f$ | $\lambda_1 \frac{2^4}{\pi^8} n_{imp}(UV)^2 \left( \frac{\tau}{L} \right)^6$ |
| $g_2 \times f$ | 0 |
| $g_3 \times f$ | $\lambda_2 \frac{2^4}{\pi^8} n_{imp}(UV)^2 \left( \frac{k_F}{\pi} \right)^6 \frac{\tau}{L^2}$ |

$\tau^{-1}$ for processes that backscatter two electrons

| $g_2 \times b$ | 0 |
| $g_3 \times b$ | $\lambda_1 \frac{2^4}{\pi^8} n_{imp}(UV)^2 \left( \frac{k_F}{\pi} \right)^6 \frac{\tau}{L^2}$ |
| $g_3 \times f$ | $\lambda_2 \frac{2^4}{\pi^8} n_{imp}(UV)^2 \left( \frac{k_F}{\pi} \right)^6 \frac{\tau}{L^2}$ |

TABLE I. Results of the second order perturbation theory in the T-matrix. $\lambda_1 \simeq 103.9$ and $\lambda_2 \simeq 1757.97$ are dimensionless integrals defined in Eq. (A2) and Eq. (A3) in the Appendix. Forward (backward) scattering off impurities is denoted by f (b). The scattering rate is calculated by summing all diagrams that are generated by combining the two said processes cf. Fig. 2. The corresponding ac conductivity is obtained by Eq. (35).

For a HLL, cf. Eq. (10), we find on the one hand $[H_3, P_0] = 0$, but on the other hand

$$[P_T, H_5] = \frac{2V}{k_0^2 L} \sum_{k,p,q,\eta} \eta(p-q)(k^2 - p^2)$$

$$\times \psi_{k+q,\eta}^\dagger \psi_{p-q,\eta} \psi_{p,q}^\dagger \psi_{k,\eta} - h.c. ,$$

$$[J_0, H_5] = \frac{2V}{k_0^2 L} \sum_{k,p,q,\eta} (k^2 - p^2)$$

$$\times \psi_{k+q,\eta}^\dagger \psi_{p-q,\eta} \psi_{p,q}^\dagger \psi_{k,\eta} - h.c. .$$

(33)

Therefore, there exists no such simple conservation law for the $g_5$ term. Consequently, we expect a finite conductivity due to $g_5$ but not due to $g_3$ unklapp processes, which is exactly the result obtained in the previous calculation. To see how these conservation laws appear in the kinetic equation formalism we show the explicit calculation for $g_3$ and $g_5$ in Appendix A.

2. Disordered system

We know that pure disorder scattering will not affect transport properties. Indeed, forward scattering does not change the chirality of a particle and elastic backward scattering is prohibited by time reversal symmetry. However, it turns out that combined scattering mechanisms that include both interaction and disorder can lead to a finite conductivity.

Using the intuition obtained from the first order of perturbation theory we expect that only processes that change the total number of right or left movers can affect current. This is confirmed in the explicit calculation.

We are therefore left with two classes of processes. First, there are inelastic processes that change the chirality of a single incoming particle which we will refer to as “1P processes”. Second, we have inelastic scattering processes that change the chirality of both incoming particles which we will dub “2P processes”. The processes as well as possible microscopic realizations are depicted in Fig. 3.

In order to obtain the real part of conductivity induced by these scattering mechanisms we have to take into account all possible microscopic realizations of the different types. The processes taken into account are depicted in Fig. 2 and the corresponding results are summarized in Tab. 4.

In the case of 1P scattering we find that the leading contribution in the limit of low temperatures $k_F \gg T$ comes from combined processes of $g_5$ and forward scattering off an impurity and yield

$$\Re \sigma = 42.1 \frac{e^2 v_F}{h} 1 \frac{UV}{v_F^2} 2 v_F n_{imp} \left( \frac{T}{v_F k_0} \right)^4 .$$

(34)

The explicit derivation of this result can be found in Appendix A3.

While the combination of $g_3$ and backward scattering off the impurity produces the same temperature dependence, the corresponding scattering time is bigger by a parametrically large factor $(k_0/k_F)^8$, see Tab. 4.

The 1P processes are similar to pure $g_5$ interaction in the sense that they change only the chirality of one particle. However, unlike the conductivity due to interaction, Eq. (29), the result for the combined process Eq. (34) is not exponentially suppressed in the limit $k_F \gg T$. The exponential suppression in the clean case is due to the fact that momentum and energy conservation force one of the particles to be at $k = 0$ deep within the filled Fermi sea. If we include impurities, momentum conservation is broken and the phase space requirements for the process are relaxed which removes the exponential suppression.

If we assume that the $ac$ conductivity obeys Drude’s law,

$$\sigma_{ac} = \frac{2e^2 v_F}{h} \frac{1}{\omega^2} \tau^{-1},$$

(35)

the whole information about a specific scattering process is contained in the transport scattering time $\tau$. From Eq. (34) we obtain the scattering time of 1P processes,

$$\tau_{ac}^{1P} = 0.047 \frac{1}{v_F n_{imp}} \left( \frac{v_F^2}{UV} \right)^2 \left( \frac{v_F k_0}{T} \right)^4 .$$

(36)

Using the obtained $ac$ scattering time we can make predictions about other physical quantities relevant for transport. In particular the dc conductance of a short edge, i.e. if the system length $L$ is much shorter than the mean free path $l$, can obtained as

$$G = \frac{2e^2}{h} \left( 1 - \frac{L}{l} \right),$$

(37)

where $l = v_F \tau_{ac}$. In the case of 1P scattering this would yield a correction $\delta G$ to quantized conductance which
reads as
\[ \delta G = 21.1 \frac{e^2}{h} L n_{\text{imp}} \left( \frac{UV}{v_F^2} \right)^2 \left( \frac{T}{v_F k_0} \right)^4. \] (38)

This allows us to compare our results to existing work.\(^2\) There the authors considered the combination of \(g_3\) and backward scattering from the impurity. We therefore find a more important microscopic mechanism that leads to a conductance correction larger by a parametrical factor \((k_0/k_F)^8\).

For \(2P\) processes the leading contribution arises from the combination of \(g_3\) and forward scattering off the impurity which yields
\[ \Re \sigma = 2.3 \times 10^4 \frac{e^2 v_F \nu F n_{\text{imp}}}{h} \omega^2 \left( \frac{UV}{v_F^4} \right)^2 \left( \frac{k_F}{k_0} \right)^2 \left( \frac{T}{v_F k_0} \right)^6. \] (39)

While this process produces subleading corrections in the present case of weakly interacting electrons it will turn out to be the dominant scattering mechanism for \(K < 2/3\) when we include Luttinger liquid effects in Sec. VI.

As a general fact we notice that the scattering times originating from microscopic processes containing backward scattering off disorder are always parametrically larger by powers of \(k_0/k_F\) compared to those containing forward scattering.

**B. dc conductivity**

After having discussed the regime of high frequencies we next turn to the opposite limit of dc conductivity. In order to simplify the subsequent calculations we use an effective Hamiltonian derived in Ref.\(^{22}\) for the most relevant scattering mechanisms. These terms would appear in the Hamiltonian under renormalization and describe \(1P\) and \(2P\) scattering processes, respectively. In the previous calculation of the ac conductivity we have identified the microscopic origin of these scattering processes and we fix their coupling constant by demanding that they replicate the results in Eq. (34) and Eq. (39). This yields
\[ H_{1P} = \tilde{g}_{1P} \frac{L^2}{2} \sum_{k,p,q,q',\eta} k \psi_{k,\eta}^\dagger \psi_{q,q'}^\dagger \psi_{p,\eta} \psi_{k,\eta} + h.c.. \] (40)
\[ H_{2P} = \tilde{g}_{2P} \frac{L^2}{2} \sum_{k,p,q,q',\eta} k q \psi_{k,\eta}^\dagger \psi_{q,q'}^\dagger \psi_{p,\eta} \psi_{q,\eta}, \] (41)
with the coupling constants
\[ \tilde{g}_{1P} = \sqrt{2n_{\text{imp}}} \frac{UV}{k_0} \] and \( \tilde{g}_{2P} = 8 \sqrt{n_{\text{imp}}} \frac{UV k_F}{k_0^4}. \) (42)

To study transport behavior in the dc limit we proceed as follows. Eq. (14) represents an exact integral equation determining the distribution function \(\psi_{k,\eta}\), where the information about the specific scattering process is encoded in the collision integral. First, we calculate the collision integrals for the process under consideration and insert it into Eq. (14). Then we perform the limit \(\omega \to 0\) to obtain equations determining the distribution function in the dc limit. The distribution functions obey a certain symmetry connecting right and left moving particles, see Eq. (A1) in Appendix A. Therefore, the integral equations for right and left movers decouple and we consider only the integral equations for right movers \(\psi_R(x) = \psi(x)\). Subsequently, we solve the integral equations numerically and obtain the dc conductivity Eq. (22) as
\[ \sigma_{dc} = -\frac{2e^2}{c} T \int dx n_F(x-\zeta)(1-n_F(x-\zeta))\psi_\zeta(x). \] (43)
Here, \(x = x/T\), \(\zeta = k_F/T\) are dimensionless momenta and \(n_F(x) = (1+e^{x})^{-1}\) is the Fermi function.

1. **Clean system**

From our discussion of the ac conductivity we know that only the \(g_0\) term affects transport properties of a clean system. In the Appendix (A1) we solve the integral equation for the distribution function and obtain the dc conductivity. During the calculation we notice the curious fact that the distribution function of the state at the Dirac point explicitly affects the distribution of all other momentum states. This fact will become crucial for the transport properties in the dc limit.

We find the conductivity in the regime \(k_F \ll T\),
\[ \sigma(k_F \ll T) = 0.014 \times \frac{2e^2 v_F}{h} \left( \frac{v_F}{V} \right)^2 \left( \frac{k_0}{v_F k_0} \right)^5, \] (44)
and in the regime \(k_F \gg T\),
\[ \sigma(k_F \gg T) = 0.81 \times \frac{2e^2 v_F}{h} \left( \frac{v_F}{V} \right)^2 \left( \frac{k_0}{k_F} \right)^4 \left( \frac{v_F k_F}{T} \right) e^{\frac{v_F k_F}{T}}. \] (45)

If we assume that the results have the form predicted by the Drude formula in the dc limit
\[ \sigma_{dc} = \frac{2e^2 v_F}{h} \tau, \]
and extract the corresponding scattering time \(\tau\), we can compare the scattering times obtained in the dc limit with those in the ac limit in Eq. (29) and Eq. (30). While there is no parametric difference in the regime of high temperatures this is not the case for low temperatures. To be more specific, in the regime \(k_F \gg T\) the scattering time in the ac limit is parametrically smaller by a factor \(T/k_F\) compared to the scattering time in the dc limit. This is due to the fact that the state at the Dirac point influences all other momentum states and we will further elaborate on this result in the discussion in Sec. IV C.
2. Disordered system

We now turn to the disordered case where we consider the effective 1P and 2P processes. Again referring to Appendix B 3 for further details we find the dc conductivity in the presence of impurities as

\[
\sigma_{1P} = \frac{\kappa_1}{2} \times \frac{2e^2v_F}{h} \frac{1}{v_F \tau_{\text{imp}}} \left( \frac{v_F^2}{UV} \right)^2 \left( \frac{v_F k_0}{T} \right)^4, \quad (46)
\]

\[
\sigma_{2P} = \frac{\kappa_2}{26} \times \frac{2e^2v_F}{h} \frac{1}{v_F \tau_{\text{imp}}} \left( \frac{v_F^2}{UV} \right)^2 \left( \frac{k_0^2}{k_F} \right) \left( \frac{v_F k_0}{T} \right)^6, \quad (47)
\]

where \( \kappa_1 = -0.46 \) and \( \kappa_2 = -0.042 \). Notice that the conductivity in the presence of disorder is not sensitive to the ratio of Fermi energy and temperature and we obtain a single scattering time in both limits, \( k_F \ll T \) and \( k_F \gg T \).

C. Discussion: ac vs dc conductivity

We are now in the position to compare the results for dc and ac conductivity summarized in Tab. I. In the presence of disorder we consider effective 1P and 2P processes which describe combined effects of interaction and forward scattering off the impurity. They lead to transport scattering times which are insensitive to the ratio of Fermi energy and temperature. Furthermore, the parametric dependence of the transport scattering time is identical in the low and high frequency regime. This suggests that, in the presence of disorder, the parametric dependence of the conductivity can be approximated by the Drude formula,

\[
\sigma(\omega) = \frac{2e^2v_F}{h} \frac{1}{\tau^{-1} - i\omega}. \quad (48)
\]

Nevertheless, the numerical prefactors in the ac and dc limit differ substantially. Therefore, the overall behavior of conductivity is not exactly Drude-like.

We find that the dominant contribution to conductivity is due to 1P processes. They lead to Drude-like behavior of the conductivity, irrespective of doping and with a temperature scaling \( \sim T^4 \) in the ac and \( \sim T^{-4} \) in the dc limit, respectively.

For sufficiently clean systems we have to consider the effect of \( g_5 \) interactions. In this case we have to distinguish between the high temperature and low temperature regime. If temperature is much larger than the Fermi energy, the conductivity behaves Drude-like which is expected since the high temperature limit corresponds to the classical regime. For low temperatures however, Pauli blocking of the state at the Dirac point leads a scattering time which is much larger in the dc limit, by a parametrically big factor \( E_F/T \), compared to the ac case. Indeed, we saw that all scattering processes have to go through the state at the Dirac point. In the ac case the state is frequently emptied due to the applied field. In the dc limit this can only happen due to thermal fluctuations which leads to a suppression in the low temperature case.

Another point to appreciate is, that the dc conductivity in the absence of impurities is finite. This is indeed surprising, since the free Hamiltonian of our system is that of a spinless LL, which is integrable and therefore characterized by an infinite number of conservation laws, current being one of them. Therefore, once a current is created by an externally applied bias it should never relax. For a conventional LL this statement remains true even in the presence of \( g_5 \) interaction which breaks some conservation laws. However, in the present case we have shown that the \( g_5 \) term, that is particular to the HLL,\(^\text{[21]}\) does lead to a finite conductivity, while the \( g_3 \) term does not. As discussed in Sec. IV A 1 this is caused by the fact that \( g_3 \) commutes with the total momentum of the system while \( g_5 \) does not.

V. Luttinger Liquid Effects: Formalism

So far we have discussed transport properties of one-dimensional electrons subject to weak interactions and impurity scattering neglecting LL effects. While intuitively more accessible, the fermionic description often proves insufficient to describe the strongly correlated LL state of one-dimensional fermions.

Therefore, we now complement our fermionic analysis by bosonizing the model which takes \( g_2 \) and \( g_4 \) interactions into account exactly. In real space the model of free fermions with linear spectrum and interaction-induced forward scattering reads as

\[
H_0 = \sum_\eta \int dxd \Psi_\eta^\dagger(x)(-i\eta\partial_x)\Psi_\eta(x),
\]

\[
H_2 = V \sum_\eta \int dxd \Psi_\eta^\dagger \Psi_\eta \Psi_\eta \Psi_\eta^\dagger \Psi_\eta, \quad (49)
\]

\[
H_4 = V \sum_\eta \int dxd \Psi_\eta^\dagger(x)\Psi_\eta^\dagger(x+0)\Psi_\eta(x)\Psi_\eta(x+0).
\]

Thereby, the field operators \( \Psi(x) \) are slowly varying on the scale \( k_F^{-1} \). We use the bosonization convention\(^\text{[22]}\)

\[
\Psi_\eta = \frac{1}{\sqrt{2\pi a}} e^{-i\sqrt{4\pi \eta} \varphi}, \quad (50)
\]

where \( \varphi \) are the chiral bosonic fields and \( a \) is the inverse UV cutoff. The bosonic fields are obtained as

\[
\varphi = \phi_R + \phi_L \quad \text{and} \quad \theta = \phi_R - \phi_L. \quad (51)
\]

We now switch to an action formalism. The free action is renormalized by \( g_2 \) and \( g_4 \) interaction and reads as

\[
S_0 = \int dx d\tau \left[ \frac{uK}{2} (\partial_x \theta)^2 + \frac{u}{2K} (\partial_x \varphi)^2 + i\partial_x \theta \partial_x \varphi \right]. \quad (52)
\]
Here, $K$ denotes the Luttinger liquid parameter which is a measure of the fermionic interaction strength and $u$ is the renormalized Fermi velocity. In terms of the interaction strength $g_2 = g_4 = V$ and the Fermi velocity $v_F$ they are given by the expressions

$$K = \frac{1}{(1 + V/\pi v_F)^{1/2}}, \quad u = v_F(1 + V/\pi v_F)^{1/2}. \quad (53)$$

Let us now include interaction and disorder terms and derive an effective low energy action. As a starting point we consider the Hamiltonian in Eq. (10) and expand momenta around the Fermi points, i.e. we write $k = k' + \eta k_F$ and expand in $|k'| \ll k_F$. We also define

$$\psi_{k,\eta} = \psi_{k'+\eta k_F,\eta} \equiv \Psi_{k',\eta}. \quad (54)$$

This yields the following interaction-induced umklapp terms:

$$H_3 = \frac{8k_F^2 V}{k_0^4} \sum \eta \int dx e^{-i4k_F F x} \left( \partial_x \Psi_{\eta} \right) \left( \partial_x \bar{\Psi}_{\eta} \right) \bar{\Psi}_{\eta},$$

$$H_5 = \frac{4V k_F i}{k_0^2} \sum \eta \int dx e^{2i k_F F x} \bar{\Psi}_{\eta} \Psi_{\eta} \partial_x \bar{\Psi}_{\eta} + h.c. \quad (55)$$

Notice that we did not consider $g_1$ terms since they are similar to $g_2$ terms but with additional derivatives making them less relevant in the renormalization group sense. Performing the same expansion for the impurity terms yields

$$H_{imp,f} = \sum \eta \int dx U_f(x) \Psi_{\eta} \bar{\Psi}_{\eta}(x),$$

$$H_{imp,b} = \frac{2k_F}{k_0^2} \int dx U_b(x) \left[ \partial_x \Psi_L \partial_x \Psi_R - \overline{\Psi}_L \partial_x \Psi_R \right] \quad (57)$$

Here, we defined the forward and backward scattering impurity potentials as

$$U_f(x) = \frac{1}{L} \sum_q U_q e^{i q x},$$

$$U_b(x) = \frac{i}{L} \sum_q U_{q+2k_F} e^{i q x}. \quad (59)$$

We consider weak, gaussian correlated disorder

$$\bar{U}(x) = 0, \quad \langle U(x) U(x') \rangle = D \delta(x - x'), \quad (60)$$

where $D = n_{imp} U_0^2$ denotes the disorder strength. One can then show that the forward and backward scattering potentials obey

$$\bar{U}_f(x) \bar{U}_f(x') = \bar{U}_b(x) \bar{U}_b(x') = D \delta(x - x'). \quad (61)$$

We proceed by bosonizing the model and switching to an action formalism. This yields

$$S_3 = \frac{4k_F^2 V}{\pi^2 a^3 k_0^3} \int dx \partial_x \partial_y \left( 2 \sqrt{4\pi \varphi} - 4 k_F x \right),$$

$$S_5 = \frac{8V}{\sqrt{\pi} a k_0^2} \int dx \partial_x \left( \partial_x \varphi \right)^2 \partial_x \theta + \left( \partial_x \theta \right)^2 \right)$$

$$\times \cos(\sqrt{4\pi \varphi} x, x') - 2 k_F x)$$

$$\times \sin(\sqrt{4\pi \varphi} x, x') - 2 k_F x), \quad (62)$$

$$S_{imp,f} = - \frac{1}{\sqrt{\pi}} \int dx \partial_x U_f(x) \partial_x \varphi,$$

$$S_{imp,b} = \frac{2k_F}{\sqrt{\pi} a k_0^2} \int dx \partial_x U_b(x) \partial_x \theta e^{i \sqrt{4\pi \varphi}} + h.c.. \quad (64)$$

In the following we distinguish the clean and the disordered case. Recall that the fermionic treatment in Sec. IV A 1 lead us to the conclusion that $g_3$ umklapp scattering does not produce a finite conductivity. Therefore, we only consider $g_5$ umklapp interaction in the clean limit.

In the disordered case, we will derive an effective action containing 1P and 2P processes by averaging over disorder. The models we use in each situation are discussed in Sec. V A and V B. Subsequently, the high frequency conductivity of each model is calculated in Sec. V C using the linear response Kubo formula.

### A. Model in the clean case

While the straightforward calculation of the conductivity due to the $g_5$ term in Eq. (62) is possible, it is difficult
due to various reasons. Therefore, we restrict ourselves to the relevant case of low energy physics where \( \omega, T \ll k_F \) and bosonize the effective form of the \( g_5 \) term close to the Fermi energy in Eq. (55). This yields

\[
S_5 = \frac{4V k_F}{\pi^2 g_0^2} \int dx \frac{\partial^2}{\partial^2 \theta} \sin(4\pi \varphi - 2k_F x) \\
- \frac{16V k_F}{\pi^2 g_0^2} \int dx \frac{\partial}{\partial x} \cos(4\pi \varphi - 2k_F x).
\]

(B. Model in the disordered case)

In the fermionic description we noticed that the combined effect of forward scattering off disorder and interaction leads to the dominant effects. Therefore, we proceed by gauging out forward scattering from impurities in Eq. (62) using the gauge transformation

\[
\varphi \rightarrow \varphi + \frac{K}{u \sqrt{\pi}} \int_{x_0}^{x} dy U_f(y), \quad x_0 \rightarrow -\infty.
\]

In order to perform the disorder average we introduce replicas and then average over forward and backward scattering. The technical details can be found in the Appendix C.

From now on we use subscripts \( D_h \equiv D \) and \( D_f \equiv B \) in order to differentiate between the two physically distinct disorder scattering mechanisms.

After averaging over disorder we obtain an effective action local in space but nonlocal in imaginary time where the momentum cutoff of our theory is now given by \( D_f \ll k_F \),

\[
S_{2P} = -g_2P \sum_{a,b} \int dx d\tau d\tau' \cos \left\{ 2\sqrt{4\pi} [\varphi_a(x, \tau) - \varphi_b(x, \tau')] \right\},
\]

\[
S_{1P} = -g_{1P,1} \sum_{a,b} \int dx d\tau d\tau' \frac{\partial^2}{\partial x} \varphi_a(x, \tau) \frac{\partial^2}{\partial x} \varphi_b(x, \tau') \cos \left\{ \sqrt{4\pi} [\varphi_a(x, \tau) - \varphi_b(x, \tau')] \right\}
\]

\[
+ g_{1P,2} \sum_{a,b} \int dx d\tau d\tau' \frac{\partial^2}{\partial x} \varphi_a(x, \tau) \frac{\partial}{\partial x} \varphi_b(x, \tau') \sin \left\{ \sqrt{4\pi} [\varphi_a(x, \tau) - \varphi_b(x, \tau')] \right\},
\]

\[
S_{imp,b} = -g_b \sum_{a,b} \int dx d\tau d\tau' \frac{\partial}{\partial x} \varphi_a(x, \tau) \frac{\partial}{\partial x} \varphi_b(x, \tau') \cos \left\{ \sqrt{4\pi} [\varphi_a(x, \tau) - \varphi_b(x, \tau')] \right\}.
\]

Here, \( a, b \in \{1, \cdots R\} \) are replica indices and \( R \) is the number of replicas. The first two terms correspond to the 1P and 2P processes discussed in Sec. 4.A.2. They originate from \( g_5 \) and \( g_5 \) umklapp processes, respectively, in combination with forward scattering off impurities. The last term describes the disorder averaged backscattering off disorder. The coupling constants are given by

\[
g_{2P} = \frac{2V^2 k_F^2 K^2 D_f}{\pi^2 a^2 k_0^3 u^2},
\]

\[
g_{1P,1} = \frac{2V^2 k_F^2 D_f}{\pi^2 a^2 k_0^3 u^2},
\]

\[
g_{1P,2} = \frac{8k_F V^2 K^2 D_f}{\pi^4 a^4 k_0^6 u^2},
\]

\[
g_{imp,b} = \frac{4D_f k_F^2}{\pi^2 a^2 k_0^3} + \frac{32V^2 K^2 k_F^2 D_f}{\pi^5 a^2 k_0^6 u^2}.
\]

Recall, that elastic backscattering off disorder does not affect transport properties in a HLL. Thus, the term \( S_{imp,b} \) should not lead to a finite conductivity at zero interaction strength, i.e. at \( K = 1 \), even if the coupling constant is nonvanishing in this limit. We will return to this point at a later stage.

At the end of any calculation in the replica formalism, one has to analytically continue the result to \( R=0 \). In particular the expectation value of some functional \( \mathcal{O} \) of fields \( \theta \) and \( \varphi \) is obtained as

\[
\langle \mathcal{O} \rangle = \lim_{R \to 0} \sum_{a=1}^{R} \langle \mathcal{O}(\varphi_a, \theta_a) \rangle.
\]

Details of the replica limit can be found in Appendix E.

(C. Linear response Kubo formalism)

In the presence of an electromagnetic field we couple the vector potential to the canonical momentum via the minimal substitution \( \partial_x \theta \rightarrow \partial_x \theta + \frac{\sqrt{\pi}}{\sqrt{2}} \delta A_{\theta} \). The current is then obtained by varying the action with respect to the vector potential

\[
\delta J = \delta S/\delta A |_{A=0}
\]

and the diamagnetic susceptibility is obtained as

\[
\chi^{\text{dia}}(x - x', \tau - \tau') = - \left. \left( \frac{\delta S}{\delta A(x, \tau)} \delta A(x', \tau') \right) \right|_{A=0}.
\]
However, notice that the vector potential does not only couple to the free action but also to the perturbations in Eq. (63) and Eq. (65). Therefore, we get additional contributions to the current and the diamagnetic susceptibility. We will refer to the contributions obtained from the free action, Eq. (52), as normal and the contributions in Eq. (63) and Eq. (65). Therefore, we get additional contributions to the current and the diamagnetic susceptibility. The normal susceptibility is given by

\[ \chi_{\text{0}}^\text{dia}(x-x', \tau-\tau') = \frac{e^2 u K}{\pi} \delta(x-x') \delta(\tau-\tau'). \]  

(74)

In Appendix [A] we state the anomalous part of the current and diamagnetic susceptibility. The total current is then \( j = j_0 + j_{\text{an}} \) and analogously \( \chi^\text{dia}(x, \tau) = \chi_{\text{0}}^\text{dia}(x, \tau) + \chi_{\text{an}}^\text{dia}(x, \tau) \).

From this we obtain the susceptibility and the conductivity in linear response

\[ \chi(x, \tau) = \chi^\text{dia}(x, \tau) + \langle j(x, \tau) j(0, 0) \rangle, \]

\[ \sigma(\omega, T) = -\frac{i}{\omega} \chi(k \rightarrow 0, i k u \rightarrow \omega + i \delta), \quad \delta = 0+. \]

(76)

This procedure yields the ac conductivity of a free system:

\[ \sigma_0(\omega) = \frac{2e^2}{\hbar} \frac{i u K}{\omega + i \delta}. \]

(77)

To obtain a finite real part of the conductivity we perform a perturbative expansion of the current-current correlator to the lowest nontrivial order in the considered scattering mechanism which is discussed in the following section.

VI. CONDUCTIVITY OF A HELICAL LUTTINGER LIQUID FOR ARBITRARY INTERACTION STRENGTH

In the following we calculate the conductivity of a HLL for arbitrary interaction strength, when Luttinger liquid renormalization effects are crucially important. In order to treat the effect of scattering processes perturbatively the corresponding scattering rate has to be the lowest energy scale in the problem. In particular we have to require \( \omega \gg \tau^{-1} \) which means the perturbative treatment only allows us to calculate the ac conductivity.

The ac conductivity in the clean case is discussed in Sec. [4A] and Sec. [4B] is devoted to the conductivity in the presence of disorder and in Sec. [4C] we then discuss the implication of these results on transport in the dc limit and localization effects. Some details of the calculation are summarized in the Appendices [D] to [E].

A. ac transport in the clean case

In the case of a sufficiently clean sample the main mechanism of scattering will be \( g_5 \) umklapp interaction. Calculating the conductivity in the regime \( \omega, T \ll k_F \) we obtain

\[ \sigma(k \rightarrow 0, \omega) = \frac{i}{\omega^3} \frac{e^2 u_4 K^2}{\hbar} \frac{2\delta}{\pi^2} \left( \frac{V^2}{u} \right) \left( \frac{k_F}{k_0} \right)^2 \frac{1}{(a k_0)^2} \frac{I_K(\omega, T)}{K^2} \]

(78)

where the function \( I_K(\omega, T) \) is defined in Eq. (F4) in the Appendix. We can further simplify this result if the temperature is much higher or much lower than the frequency of the external field. In the regime \( \omega \gg T \) we obtain

\[ \sigma(\omega) = \frac{i}{\omega + i \delta} \frac{e^2 u}{\hbar} \frac{8}{\pi^3} \left( \frac{V}{u} \right)^2 \left( \frac{k_F}{k_0} \right)^2 (k_F a)^2 K \]

\[ \times \frac{K^2}{(k_0 a)^2} (K-1) f(K) \]

(79)

where we defined

\[ f(K) = -\sin(K \pi) \{ \Gamma(-1-K) \}^2 \]

\[ + (6+K) \Gamma(-K) \Gamma(-3-K) \}

(80)

The function \( f(K) \) is plotted in Fig. 4. While it is singular at \( K = 1 \) it always appears together with a function that vanishes at \( K = 1 \) such that the expression for the conductivity is finite in the noninteracting limit.

Since \( \sigma \) is purely imaginary the only effect of the \( g_5 \) interaction is to renormalize the velocity \( u \) and the Luttinger liquid parameter \( K \) in the case without umklapp terms cf. Eq. (77).
In the regime \( \omega \ll T \) we obtain

\[
\Re \sigma(\omega) = \frac{1}{\omega^2} \frac{e^2 u}{h \pi} \left( \frac{V}{u} \right)^2 \left( k_F a \right)^{2K+2} \frac{\omega^2 k_F^2}{T} e^{-\frac{u k_F}{\omega}} \times K^2 \frac{\sin(K\pi)f(K)}{(k_0 a)^4}.
\]

(81)

In the limit of noninteracting electrons \( K \to 1 \) this agrees with the result obtained in the fermionic language Eq. (63) except for a nonuniversal constant of the order of unity.

Recall that the \( g_5 \) term in Eq. (63) is always irrelevant in the RG sense and therefore only yields small corrections to the fixed point properties. However, we observe that the nature of the corrections crucially depends on whether the RG flow is cut off by frequency or temperature if we are in the regime \( \omega, T \ll k_F \). While frequency dependence only leads to a renormalization of the fixed point parameters \( u \) and \( K \), temperature dependence yields a finite conductivity which is however exponentially suppressed.

Finally, we can make some predictions about the regime of \( \omega, T \gg k_F \). In this limit we can imply the result by the scaling dimension of \( S_5 \) in Eq. (62) which yields \( \sigma \sim \omega^{-2(\max(\omega, T))^{2K+3}} \). Comparing the limit \( K \to 1 \) with the fermionic case in Eq. (30) we see that this indeed gives the correct scaling of the conductivity and there is no cancellation in the leading order. Additionally, since temperature or frequency are much higher than the Fermi energy the system behaves effectively as if it were at the Dirac point, such that \( k_F = 0 \). Therefore, we get

\[
\sigma(\omega) \sim \frac{e^2 u}{h} \frac{1}{\omega^2 u k_0} \left( \frac{V}{u} \right)^2 \left( \frac{\max(\omega, T)}{u k_0} \right)^{2K+3}.
\]

(82)

To summarize, we find that due to the strong correlation effects of the one dimensional Luttinger liquid the exponents of the power law in temperature now depend on the strength of interaction through the Luttinger liquid parameter \( K \). In the limit of weakly interacting electrons \( K \approx 1 \) we reproduce the power law \( T^5 \) in Eq. (30) and the behavior in the limit \( k_F \gg T \) in Eq. (29). Additionally, the present calculation in the bosonic form allows us to investigate the limit \( \omega \gg T \). In the kinetic equation approach the external electric field is always treated classically, so it cannot be applied if the corresponding frequency becomes larger than temperature. In this case one has to quantize the electric field and treat the interactions of photons with the system. While this treatment was not possible in the context of the kinetic equation, the quantum mechanical regime \( \omega \gg T \) becomes accessible in the present Kubo formalism.

### B. \( ac \) transport in the presence of disorder

In the presence of impurities we find the conductivity due to inelastic scattering processes in Appendix \( \text{E} \). The results read as

\[
\sigma_{2P}(\omega, T) = \frac{e^2}{\omega^3} \frac{32 u^2 K^2 g_{2P}}{(\pi T)^4} \mathcal{J}_{6K}(\omega, T)
\]

(83)

\[
\sigma_{1P}(\omega, T) = \frac{8 e^2}{\pi a^3} \frac{u^2 K}{g_{1P,1}} \frac{1}{\omega^3} \frac{T}{u} \left( \frac{T}{u} \right)^{2K+4} \times (3\mathcal{J}_{2K+4}(\omega, T) - 2\mathcal{J}_{2K+2}(\omega, T))
\]

(84)

where \( \mathcal{J}_{2K}(\omega, T) \) is defined in Eq. (61).

The limits of low and high temperature respectively are given by

\[
\Re \sigma_{2P}(\omega, T) = \frac{2 e^2 u}{h} \frac{1}{\omega^2} \frac{4^4 K^4}{\Gamma(8K)^2} \left( \frac{V}{u} \right)^2 \frac{D_f}{u} \left( \frac{k_F}{k_0} \right)^2 \frac{1}{(ak_0)^6} \left( \frac{\omega a}{u} \right)^{8K-2} \left( \frac{2 \pi a T}{u} \right)^{8K-2} \Gamma^2(4K), \quad \text{for } \omega \gg T,
\]

(85)

\[
\Re \sigma_{1P}(\omega, T) = \frac{2 e^2 u}{h} \frac{1}{\omega^2} \left( \frac{2}{\pi} \frac{K}{\Gamma(2K+4)} \right)^4 \left( \frac{V}{u} \right)^2 \frac{D_f}{u} \frac{1}{(ak_0)^4} \left( \frac{3}{\omega a} \right)^{2K+2} \left( \frac{2 \pi a T}{u} \right)^{2K+2} K(K+1) \Gamma^2(K+1), \quad \text{for } \omega \gg T,
\]

(86)

Similarly to the clean case, we find power law exponents that depend on the strength of interactions through
the Luttinger liquid parameter $K$. Additionally, we observe that 2P scattering becomes the dominant scattering mechanism for $K < \frac{3}{4}$ and even becomes relevant for $K < \frac{1}{4}$. This behavior is in agreement with the results of Ref. [22]. However, our derivation of these results from a more microscopic theory allows us to identify the origin of the 1P and 2P processes as the combined effect of $g_5$ and $g_3$ interaction together with scattering off impurities. In particular we identify the importance of forward scattering off disorder for transport properties, which has not been fully appreciated in the existing literature.

After having discussed the effect of interactions on transport, both by itself and in combination with forward scattering off disorder, we now comment on the effect of backscattering off the impurity described by the term $S_{\text{imp,b}}$ in Eq. (65). In Appendix E we show that, to the leading order in disorder strength $D_b$, the backscattering term does not lead to a finite scattering time for any value of $K$. Recall that the term $S_{\text{imp,b}}$ originates from backscattering off impurities and should therefore have no impact on transport on its own, i.e. in the absence of $g_2$ interaction at $K=1$. However, we find that the conductivity does not only vanish for $K = 1$ but for arbitrary $K$ meaning that even the combination of $g_2$ interaction and backscattering off impurities does not change the conductivity. This is consistent with our fermionic analysis, see Table III.

C. Discussion of dc conductivity and localization

In this section we complemented our previous kinetic equation calculation whose results are summarized in Tab. II by bosonizing the model for helical fermions and calculating the ac conductivity using the Kubo formula. First, this allows us to treat Luttinger liquid renormalization effects that arise due to the strong correlations in one dimension and lead to power law exponents that depend on the strength of interaction through the Luttinger liquid parameter $K$. Second, it enables us to make predictions about the regime $\omega \gg T$ not captured by our previous kinetic equation analysis.

Before summarizing the results and discussing their implications for dc transport we briefly comment on the effect of quantum interference phenomena on the transport properties of a disordered HLL. So far we have only discussed the quasiclassical regime where the dephasing length is much shorter than the mean free path. Going beyond this semiclassical description we can also make predictions about localization in the helical Luttinger liquid. The model in the presence of disorder in Eq. (65) can be mapped onto the Giamarchi-Schulz model of disordered LL with $K \to 4K$ by rescaling $\varphi$ fields. In combination with the analysis in Refs. [38] and [39] this mapping suggests a transition to the localized state at $K = 3/8$.

Let us now summarize the perturbative results for the ac conductivity and discuss their implications for dc transport and the conductance of short edges channels.

In a sufficiently clean sample $g_5$ umklapp interaction leads to a power law behavior $\Re \sigma_{ac} \sim \omega^{-2} \max (\omega, T)^{2K+3}$ when the system is doped close to the Dirac point. In this case the kinetic equation treatment predicts Drude-like behavior of the conductivity and therefore we expect $\sigma_{dc} \sim T^{-2K-3}$.

At $k_F \gg T$, i.e. at filling far away from the Dirac point, we have to distinguish the regimes $\omega \gg T$ and $\omega \ll T$. If $\omega \gg T$, umklapp scattering does not lead to a finite real part of the conductivity. The only effect of the scattering process is then a renormalization of parameters $u$ and $K$. On the other hand if $\omega \ll T$, the conductivity is exponentially suppressed and only the power law in front of the exponential is affected by Luttinger liquid renormalization. In either case we cannot make predictions about the dc conductivity since there exists an intermediate regime not captured by either approach.

In the presence of disorder we find the frequency and temperature dependence of the ac conductivity as

$$\Re \sigma_{ac} \sim \frac{1}{\omega^2} \left\{ \left[ \max (\omega, T) \right]^{2K+2}, \text{ if } K > 2/3, \right. \left. \left[ \max (\omega, T) \right]^{8K-2}, \text{ if } K < 2/3. \right\}$$

Since the kinetic equation approach suggests that the parametric dependence of the conductivity is described by Drude’s law, we predict the scaling of the semiclassical dc conductivity as

$$\sigma_{dc} \sim \left\{ \left[ \max (\omega, T) \right]^{-2K-2}, \text{ if } K > 2/3 \right. \left. \left[ \max (\omega, T) \right]^{-8K+2}, \text{ if } K < 2/3 \right\}$$

According to Eq. (37) we obtain the conductance of short edge channels from the ac scattering time which yields

$$\delta G \sim \frac{e^2}{h} L \left\{ T^{-2K-2}, \text{ if } K > 2/3 \right. \left. T^{-8K+2}, \text{ if } K < 2/3 \right\}$$

The complete phase diagram of the conductivity summarizing the transport properties in the presence of disorder is depicted in Fig. 5.

VII. SUMMARY AND OUTLOOK

In this paper we have studied the transport properties of a generic one-dimensional helical liquid in the presence of interactions and disorder.

We have employed two complementing approaches for obtaining the conductivity in a wide range of parameters. One is a kinetic equation approach (Sec. III IV) for weakly interacting helical fermions which allows us to determine the semiclassical conductivity in the regime $\omega \ll T$ both in the ac and dc limit. The results of this treatment are summarized in Tab. III. The other approach is bosonization (Sec. V VI) combined with the
linear response Kubo formalism which enables us to include Luttinger liquid renormalization effects as well as to describe the regime $\omega \gg T$, where the external electric field cannot be treated classically anymore. By combining the two approaches we have demonstrated that while the helical liquid is topologically protected against elastic scattering events, inelastic scattering that arises due to the combined effect of interactions and disorder leads to a finite conductivity.

In a clean helical Luttinger liquid, we find that $g_s$ interaction leads to a finite conductivity. Due to a peculiar kinetics necessarily involving a particle at the Dirac point, the parametric dependence of conductivity induced by this term cannot be described by Drude’s law. This is discussed in detail in Sec. [IV C] and we include Luttinger liquid effects in Sec. [V A].

Our main result is the phase diagram for the conductivity of a disordered HLL depicted in Fig[5] and the corresponding temperature or frequency dependence in Eqs. [87] and [88]. We find that the parametric dependence of the conductivity of a disordered HLL as a function of frequency is described by Drude’s law where the temperature or frequency dependence is a power law with exponents depending on the Luttinger liquid parameter $K$. This behavior arises due to combined effects of interaction and impurity scattering. Thereby, it is of conceptual importance that forward scattering off disorder, in contrast to disorder induced backscattering, plays the primary role in these combined effects. An intuitive physical explanation for this fact is yet to be formulated.

During our analysis we assumed a weak-disorder limit, $D_b, D_f \ll k_F$, and studied the theory in the leading order in $D_b$ and $D_f$. We expect that the effect of higher-order terms amounts to a renormalization of the couplings in the effective field theory; a detailed study is left for future work.

Going beyond the semiclassical regime, we make predictions about localization in a one-dimensional helical liquid by employing a mapping to the Giamarchi-Schulz model of disordered Luttinger liquid. This suggests a localization transition at $K = 3/8$. A detailed analysis of localization in helical edge states remains a prospect for future work.

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Appendix A: Kinetic equation: Calculation of $ac$ conductivity

In this Appendix we demonstrate how to obtain the $ac$ conductivity for weakly interacting electrons in the context of a kinetic equation.

First, we summarize the symmetry properties of some objects relevant for subsequent calculations

- The Fermi-Dirac distribution obeys $f_{k\eta}^{(0)} = f_{-k\bar{\eta}}^{(0)}$.
- In the absence of scattering i.e. when the collision integral vanishes, the symmetries of a solution $\psi$ of Eq. (13) are determined by the driving term $eE\eta f_{\eta,k}(1-f_{\eta,k})/T$ and therefore:

$$\psi_{k,\eta} = -\psi_{-k,\bar{\eta}}.$$ (A1)

We checked explicitly that there exist solutions with this symmetry even in the presence of relaxation inducing processes. In the following calculations we will only consider solutions that obey the above symmetry.

- The object $\Gamma_{12,1',2'}$ defined in Eq. [20] is invariant under exchange of the first and second two arguments e.g. $\Gamma_{12,1'2'} = \Gamma_{21,1'2'}$, which is obvious from its definition. Under the assumption of (A1) it is straightforward to show that $\Gamma_{12,1'2'} = -\Gamma_{-1,-2,-1',-2'}$ where $-1 \equiv (-k_1,\bar{\eta}_1)$.

To calculate the $ac$ conductivity we proceed as follows. First we calculate the transition matrix element $M_{1,2,1',2'} = \langle 1'2'\mid T\mid 12 \rangle$ between initial and final momentum eigenstates due to the scattering processes in the T-matrix. From this we obtain the collision integral using Eq. [21] and finally the conductivity using Eq. [26]. In the following we use the notation $1 \equiv (k_1,\eta_1)$ and define $\mid 0 \rangle$ as the groundstate of the free hamiltonian. We define the following integrals
often encountered during the calculation of ac conductivity
\[
\lambda_1 = \int dx_1 dx_2 dx_3 (x_1 - x_2)^2 n_F(x_1) n_F(x_2) \frac{1}{1 - n_F(-x_3)} \left[ 1 - n(x_1 + x_2 + x_3) \right] \approx 103.9, \tag{A2}
\]
\[
\lambda_2 = \int dx_1 dx_2 dx_3 (x_1 - x_2)^2 (x_1 + x_2 + 2x_3)^2 n_F(x_1) n_F(x_2) \frac{1}{1 - n_F(-x_3)} \left[ 1 - n(x_1 + x_2 + x_3) \right] \approx 1757.97. \tag{A3}
\]
In the first order of the expansion in the T-matrix, Eq. (18), we consider only interaction setting \( T = H_{\text{int}} \) and show explicitly the calculation of ac conductivity for \( T = H_3 \) and \( T = H_5 \).

1. \( g_3 \) term

For \( g_3 \) interaction we obtain the matrix element
\[
\text{(A4)} \quad M_{1,2,1',2'}^{(a)} = \langle 1'2' \rangle H_3 | 12 \rangle \frac{V}{k_0^2 L} \sum_{k,p,q,\eta} (k^2 - (k - q)^2) (p^2 - (p + q)^2) M_{1,2,1',2'}^{k,p,q,\eta},
\]
where we defined
\[
\text{(A5)} \quad M_{1,2,1',2'}^{k,p,q,\eta} = \langle 0 | \psi_{1'} \psi_{2'} | k, q, \eta \rangle \langle \psi_{p, q, \eta} | \psi_{0} \rangle \langle \psi_{0} | \psi_{1} \psi_{2} \rangle \bigg( \delta_{k, k'} \bigg( \delta_{p, q} - (1' \leftrightarrow 2') \bigg) \bigg) \delta_{k, k'} \delta_{p, q} - (1 \leftrightarrow 2). \]

Using this result we obtain
\[
\text{(A6)} \quad M_{1,2,1',2'}^{(a)} = \frac{V}{k_0^2 L} \sum_{k_1, k_2, k_{1'}, k_{2'}} \delta_{k_1 + k_2, k_{1'} + k_{2'}} \eta_1 \eta_2 \delta_{k_1 + k_2, k_{1'} + k_{2'}},
\]
where we defined \( h_{k_1, k_2, k_{1'}, k_{2'}} = 2(k_1 - k_2)(k_1^2 - k_2^2)(k_1 + k_2 - 2(k_{1'} + k_{2'})) \). The corresponding collision integral is
\[
I_1[\psi] = -2\pi \frac{V^2}{k_0^2 L^2} \sum_{k_1, k_2, k_{1'}, k_{2'}} \delta_{k_1 + k_2, k_{1'} + k_{2'}} h_{k_1, k_2, k_{1'}, k_{2'}} \sum_{k_1, k_2, k_{1'}, k_{2'}} \eta_1 \eta_2 \delta_{k_1 + k_2, k_{1'} + k_{2'}} \Gamma(k_1, k_1), (k_2, k_2), (k_{1'}, k_{1'}), (k_{2'}, k_{2'}).
\]

To obtain the conductivity we have to calculate the object
\[
\text{(A7)} \quad \frac{1}{L} \sum_{k_1, k_2} \eta_1 \eta_2 I_1[\psi(0)] = 4\pi \frac{V^2}{k_0^2} \frac{eE}{(-\omega)T} \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_{1'}}{2\pi} f_{k_1, R}^{(0)} f_{k_2, R}^{(0)} (1 - f_{k_{1'}, L}^{(0)}) (1 - f_{k_{1' + k_2 - k_{1'}}, L}^{(0)}) \delta(k_1 + k_2) = 0.
\]
In the last equality we used that \( h(k_1, -k_1, k_{1'}, -k_{1'}) = 0 \). Consequently, \( g_3 \) interaction alone does not affect transport.

2. \( g_5 \) term

In order to calculate the transition matrix element for \( g_5 \) we need
\[
\text{(A9)} \quad M_{1,2,1',2'}^{k,p,q,\eta} = \langle 0 | \psi_{1'} \psi_{2'} | k + q, \eta \rangle \langle \psi_{p, q, \eta} | \psi_{0} \rangle \langle \psi_{0} | \psi_{1} \psi_{2} \rangle \bigg( \delta_{k, k'} \bigg( \delta_{p, q} - (1' \leftrightarrow 2') \bigg) \bigg) \delta_{k, k'} \delta_{p, q} - (1 \leftrightarrow 2),
\]
\[
\text{(A10)} \quad M_{1,2,1',2'}^{k,p,q,\eta} = \langle 0 | \psi_{1'} \psi_{2'} | k + q, \eta \rangle \langle \psi_{p, q, \eta} | \psi_{0} \rangle \langle \psi_{0} | \psi_{1} \psi_{2} \rangle \bigg( \delta_{k, k'} \bigg( \delta_{p, q} - (1' \leftrightarrow 2') \bigg) \bigg) \delta_{k, k'} \delta_{p, q} - (1 \leftrightarrow 2).
\]

The matrix element is then given by
\[
\text{(A11)} \quad M_{1,2,1',2'} = -\frac{V}{k_0^2 L} \sum_{k,p,q,\eta} \eta \left( k^2 - p^2 \right) \left[ \text{(A11)} M_{1,2,1',2'}^{k,p,q,\eta} + \text{(A10)} M_{1,2,1',2'}^{k,p,q,\eta} \right]
\]
\[
= -\frac{2V}{k_0^2 L} \sum_{k,p,q,\eta} \eta \left[ (k_1 + k_2, k_{1'}, k_{2'}) \left( k_1^2 - k_2^2 \right) \delta_{\eta, \eta_1} \delta_{\eta, \eta_2} - \delta_{\eta, \eta_1} \delta_{\eta, \eta_2} \right] \left( k_2^2 - k_1^2 \right) \delta_{\eta, \eta_1} \delta_{\eta, \eta_2} - \delta_{\eta, \eta_1} \delta_{\eta, \eta_2} \right]
\]
From this we obtain the collision integral as

\[
I_1[\psi] = -2\pi \frac{4V^2}{k_0^4 L^2} \sum_{k_2,k_1',k_2'} \delta_{k_1+k_2,k_1'+k_2'} \left\{ (k_1^2 - k_2^2) \right. \\
\left. + (k_1^2 - k_2'^2) \right\} \Gamma(k_1,\eta_1),(k_2,\eta_1),(k_1',\eta_1),(k_2',\eta_1) + \Gamma(k_1,\eta_1),(k_2,\eta_1),(k_2',\eta_1),(k_1',\eta_1) \\
+ \Gamma(k_1,\eta_1),(k_1',\eta_1),(k_2,\eta_1),(k_2',\eta_1) + \Gamma(k_1,\eta_1),(k_1',\eta_1),(k_2',\eta_1),(k_2,\eta_1) \right\}.
\]

(A12)

Notice that from the 10 total terms in the absolute square of the matrix element only those terms with the same chirality Kronecker deltas survive the summation over external chiralities.

In order to obtain conductivity we have to calculate the quantity \( \sum_{k_1,\eta_1} \eta_1 I_1[\psi(0)] \). Using the symmetry arguments for \( \Gamma \) defined at the beginning of the Appendix and the abbreviation \( \{ k \} = k_1, k_2, k_3, k_4 \) we find

\[
\sum_{\{ k \}} \left( (k_1^2 - k_2^2) \right) \Gamma(k_1,R),(k_2,R),(k_1',R),(k_2',R) + \Gamma(k_1,R),(k_2,R),(k_1',L),(k_2',L) - \Gamma(k_1,L),(k_2,L),(k_1',R),(k_2',L) \\
- \Gamma(k_1,L),(k_2,L),(k_1',L),(k_2',R) \right) = 4 \sum_{\{ k \}} (k_1^2 - k_2^2) \Gamma(k_1,R),(k_2,R),(k_1',L),(k_2',R), \\
\sum_{\{ k \}} (k_1^2 - k_2^2) \Gamma(k_1,R),(k_2,R),(k_1',R),(k_2',L) + \Gamma(k_1,R),(k_2,R),(k_1',L),(k_2',L) - \Gamma(k_1,L),(k_2,R),(k_1',R),(k_2',L) \\
- \Gamma(k_1,L),(k_2,R),(k_1',L),(k_2',R) = 0.
\]

(A13)

(A14)

With this we can simplify the expression yielding

\[
\frac{1}{L} \sum_{k_1,\eta_1} \eta_1 I_1[\psi(0)] = \frac{32\pi V^2}{k_0^4} \frac{eE}{2\pi} \frac{1}{(i\omega)^T} \int \frac{dk_1 dk_2 dk_{1'}}{2\pi} \frac{dk_{1'}}{2\pi} \left( k_1^2 - k_2^2 \right) f_{k_1,R} f_{k_2,R}(1 - f_{k_1,L})(1 - f_{k_2,R}) \delta(k_{1'} - k_1 - k_2 - k_{1'}) f(k_{1'}),
\]

(A15)

where \( f(\zeta) \) is defined in the main text in Eq. (28). The real part of the conductivity is then given as

\[
\Re \sigma_{ac} = \frac{e}{E \omega} \sum_{k,\eta} \eta I_{k,\eta}[\psi(0)] = \frac{e^2 v_F}{h} \frac{1}{\omega^2} \left( \frac{V}{v_F} \right)^2 v_F k_0 \left( \frac{T}{v_F k_0} \right)^5 f(\zeta),
\]

(A16)

where we reinstated \( v_F \) in the last line. This result is used in Eq. (27) of the main text.

3. \( g_5 \) interaction combined with forward scattering

In the second order of the T-matrix expansion in Eq. (18) we have to include the following transition matrix elements

\[
\langle 1'2' | H_{\text{int}} G_O H_{\text{imp}} | 12 \rangle + \langle 1'2' | H_{\text{imp}} G_O H_{\text{int}} | 12 \rangle + \langle 1'2' | H_{\text{int}} G_O H_{\text{imp}} | 12 \rangle + \langle 1'2' | H_{\text{imp}} G_O H_{\text{imp}} | 12 \rangle
\]

(A17)

Since the system we consider is time reversal symmetric processes containing only disorder will not affect transport properties, so will will not consider the term \( \langle 1'2' | H_{\text{imp}} G_O H_{\text{imp}} | 12 \rangle \). Additionally, we will neglect the term \( \langle 1'2' | H_{\text{int}} G_O H_{\text{imp}} | 12 \rangle \) containing only disorder, since we already obtained results for the conductivity in the first order expansion of the T-matrix and the second order will be subleading in interaction strength \( V \).

Therefore, we are left with terms containing both scattering due to interaction and disorder. Thereby, \( H_{\text{imp}} \) contains forward and backward scattering off disorder and \( H_{\text{int}} \) contains all g-ology terms defined in Eq. (10) and we have to consider arbitrary combinations of the two. We remark that only combined processes that change the chirality of at least one incoming particle lead to a finite conductivity. The results for the conductivity induced by these combined processes is summarized in Table [4]
In this Appendix we choose $H_{\text{imp}} = H_f$ and $H_{\text{int}} = H_5$ as an example to demonstrate the calculations performed to obtain the $ac$ conductivity due to combined processes.

We start by defining effective states containing disorder as follows

$$|12\rangle_f = G_0 H_{\text{imp,f}} |12\rangle = \frac{U}{L} \sum_q \left\{ \frac{1}{\eta_1 q + i\delta} \psi_{k+q,\eta_1}^\dagger \psi_{2}^\dagger \right\} |0\rangle,$$

(A18)

$$|12\rangle_b = G_0 H_{\text{imp,b}} |12\rangle = \frac{2k_F}{k^2_0} \frac{U}{L} \sum_q \left\{ \frac{2k_1 + q}{\eta_1 (2k_1 + q) + i\delta} \psi_{k+q,\eta_1}^\dagger \psi_{2}^\dagger \right\} |0\rangle.$$

(A19)

Furthermore, we consider momenta close to the Fermi surface and simplify the $g_5$ term as

$$H_5 = \frac{2V k_F}{k^2_0 L} \sum_{k,p,q} \eta (p-k) \psi_{k+q,\eta_1}^\dagger \psi_{p-k,\eta}^\dagger \eta \psi_{k,\eta_1} \psi_{k,\eta} + h.c. \equiv \tilde{H}_5.$$

(A20)

When considering the combination of $g_5$ and forward scattering we have to add the following transition matrix elements

$$\langle 1'2' | \tilde{H}_5 | 12 \rangle_f + \langle 1'2' | \tilde{H}_5 | 12 \rangle_f + \langle 1'2' | \tilde{H}_5 | 12 \rangle_f + \langle 1'2' | \tilde{H}_5 | 12 \rangle_f.$$

(A21)

Notice that the matrix elements are connected through complex conjugation as

$$\left( \langle 1'2' | \tilde{H}_5 | 12 \rangle_f \right)^\dagger = (1 \circ \mathcal{R}_{1,2,1'2'})^* = f \langle 12 | \tilde{H}_5^\dagger | 1'2' \rangle \equiv (1 \circ \mathcal{L}_{1',2',1,2'})^* \text{ with } \delta \to -\delta.$$

(A22)

In the last line we took into account that complex conjugation changes the retarded to an advanced Greens function i.e.

$$|12\rangle_f = G^{(R)}_0 H_{\text{imp,f}} |12\rangle \rightarrow (21) H_{\text{imp,f}} G^{(A)}_0 = f \langle 21\rangle.$$

(A23)

Here, $\delta$ denotes the infinitesimal self energy of the free retarded Greens function.

Consequently we only calculate the matrix elements $\mathcal{R}$ where the effective ket is to the right. The other matrix elements are obtained by exchanging $1 \leftrightarrow 1'$, $2 \leftrightarrow 2'$ and $\delta \to -\delta$. We obtain

$$(1) \mathcal{R}_{1,2,1',2'}^{k,p,q',q',\eta} = (0) \psi_{k,\eta_1} \psi_{k,\eta_2} \psi_{k+q,\eta} \psi_{k+q',\eta} \psi_{k+q',\eta} \psi_{k+q,\eta} \psi_{k+q,\eta} \psi_{2} \right\} |0\rangle - (1 \leftrightarrow 2),$$

(A24)

$$(2) \mathcal{R}_{1,2,1',2'}^{k,p,q',q',\eta} = (0) \psi_{k,\eta_1} \psi_{k,\eta_2} \psi_{k+q,\eta} \psi_{k+q',\eta} \psi_{k+q',\eta} \psi_{k+q,\eta} \psi_{k+q,\eta} \psi_{2} \right\} |0\rangle - (1 \leftrightarrow 2),$$

(A25)

The corresponding transition matrix element reads as

$$\mathcal{M}_{1,2,1',2'}^{R} = \frac{2k_F}{k^2_0 L^2} \sum_{k,p,q,q'} \left\{ p-k \frac{1}{\eta_1 q' + i\delta} \right\} \left\{ (1) \mathcal{R}_{1,2,1',2'}^{k,p,q',q',\eta} + (2) \mathcal{R}_{1,2,1',2'}^{k,p,q',q',\eta} \right\},$$

(A26)

As discussed we imply $\mathcal{M}_{1,2,1',2'}^{R}$ by setting $1 \leftrightarrow 1'$, $2 \leftrightarrow 2'$ and $\delta \to -\delta$ in the above result. Now, we use the identity $2\Re \frac{1}{z} = \frac{1}{z^2 + \delta^2}$ to obtain

$$\mathcal{M}_{1,2,1',2'}^{R} = \mathcal{M}_{1,2,1',2'}^{L} + \mathcal{M}_{1,2,1',2'}^{L},$$

(A27)
The collision integral reads as

\[ I_1[\psi^{(0)}] = -2\pi N_{\text{imp}} \sum_{1',2',2} \Gamma_{1,2,1',2'} |\mathcal{M}_{1,2,1',2'}|^2 \]

\[ = -2\pi N_{\text{imp}} \left( \frac{2k_F U V}{k_0^2 L^2} \right)^2 \sum_{k_{1'},k_{2'},k_2} \left( \frac{1}{\eta_1 (k_{1'} + k_{2'} - k_1 - k_2)} \right)^2 \]

\[ \{(k_1 - k_2)^2(\Gamma(k_1,\eta_1),(k_2,\eta_2),(k_{1'},\eta_{1'}),(k_{2'},\eta_{2'})) + \Gamma(k_1,\eta_1),(k_2,\eta_2),(k_{1'},\eta_{1'}),(k_{2'},\eta_{2'}) \} \}

\[ + (k_{1'} - k_2)^2(\Gamma(k_1,\eta_1),(k_2,\eta_2),(k_{1'},\eta_{1'}),(k_{2'},\eta_{2'})) + \Gamma(k_1,\eta_1),(k_2,\eta_2),(k_{1'},\eta_{1'}),(k_{2'},\eta_{2'}) \} \}

Using the symmetry properties of \( \Gamma \) we obtain

\[ \sum_{k_1,\eta_1} \eta_1 I_1[\psi^{(0)}] = \frac{2}{\pi^2} \frac{eE}{(-i\omega)T} n_{\text{imp}} \left( \frac{2k_F U V}{k_0^2} \right)^2 T^3 g(\zeta), \]  

\[ \text{(A29)} \]

where

\[ g(\zeta) = \int_{-\infty}^{\infty} dx_1 dx_2 dx_3 \frac{4x_3^2}{(4x_3^2 + \delta^2)^2} (x_1 - x_2)^2 n_F(x_1 - \zeta)n_F(x_2 - \zeta)(1 - n_F(-x_3 - \zeta))(1 - n_F(x_1 + x_2 + x_3 - \zeta)). \]  

\[ \text{(A30)} \]

First we calculate the integral over \( x_3 \) by sending the integration limits to infinity and completing the contour in the complex plane. We find

\[ g(x_1, x_2, \zeta) = \frac{i\pi}{2} n_B(-x_1 - x_2 + 2\zeta) \sum_{n = -\infty}^{\infty} \left\{ \frac{1}{(i\pi(2n + 1) - \zeta)^2} \right. \left. - \frac{1}{(i\pi(2n + 1) - x_1 - x_2 + \zeta)^2} \right\} + \mathcal{O} \left( \frac{1}{\delta} \right), \]  

\[ \text{(A31)} \]

where we defined \( n_B(x) = \frac{1}{2}\text{erf}(x) \). The expression is formally divergent when sending \( \delta \) to zero. However, this divergency will be regularized by taking into account a finite electronic self energy, due to impurity scattering or interactions. Thus we will neglect the \( 1/\delta \) part in the following.

We proceed by using the series representation of the polygamma function \( \psi^{(n)}(z) = (-1)^{n+1}n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \), \( n > 0 \) to rewrite the expression as

\[ g(x_1, x_2, \zeta) = -\frac{i}{4\pi} n_B(-x_1 - x_2 + 2\zeta) \left\{ \psi^{(1)} \left( \frac{1}{2} - \frac{\zeta}{2\pi i} \right) - \psi^{(1)} \left( \frac{1}{2} - \frac{x_1 + x_2 - \zeta}{2\pi i} \right) \right\}. \]  

\[ \text{(A32)} \]

Using the asymptotics of the first polygamma function \( \psi^{(1)}(z) \sim z^{-1} \) for \( |z| \gg 1 \) and the fact that \( x_1, x_2 \ll \zeta \) we obtain

\[ g(\zeta) \approx -\frac{1}{\zeta^2} \frac{1}{2} \int_{-\infty}^{\infty} dx_1 dx_2 (x_1 - x_2)^2 n_F(x_1)n_F(x_2)n_B(-x_1 - x_2) \approx 51.9\zeta^{-2}. \]  

\[ \text{(A33)} \]

The resulting conductivity is therefore

\[ \Re \sigma(\omega, T) = \frac{e^2}{E} \frac{\pi}{\omega n_{\text{imp}}} \sum_1 \eta_1 I_1[\psi^{(0)}] = 42.1 \frac{e^2 v_F}{\hbar} \frac{1}{\omega^2} n_{\text{imp}} \left( \frac{UV}{v_F^2} \right)^2 \left( \frac{T}{v_F k_0} \right)^4. \]  

\[ \text{(A34)} \]

This result constitutes Eq. \[ 34 \] of the main text.

**Appendix B: Kinetic equation: Calculation of dc conductivity**

The purpose of this Appendix is to calculate the **dc** conductivity of a weakly interacting helical liquid by using exact integral equations for the fermionic distribution function obtained from the solution of a kinetic equation.

First, we calculate the transition matrix element \( \mathcal{M}_{1,2,1',2'} = \langle 12' | T | 12 \rangle \), for \( T = H_F, T = H_{19} \) and \( T = H_{29} \). The corresponding terms in the Hamiltonian are defined in Eq. \[ 20 \], Eq. \[ 40 \] and Eq. \[ 41 \]. We then obtain the collision integral using Eq. \[ 21 \]. The results read as
\[ I_1^{(gs)}[\psi] = -8\pi \frac{V^2}{k_0^3 L^2} \sum_{k_{2, k_{1'}, k_{2'}}} \delta_{k_1+k_2,k_{1'},k_{2'}} \left\{ 2(k_1^2 - k_2^2)^2 \Gamma_{(k_1,\eta_1), (k_2,\eta_1), (k_{1'},\eta_{1'}), (k_{2'},\eta_{1'})} \right\} \]

\[ I_1^{(1P)}[\psi] = -4\pi n_{imp}(UV)^2 \frac{1}{k_0^3 L^3} \sum_{k_{2, k_{1'}, k_{2'}}} \left\{ 2(k_1 - k_2)^2 \Gamma_{(k_1,\eta_1), (k_2,\eta_1), (k_{1'},\eta_{1'}), (k_{2'},\eta_{1'})} \right\} \]

\[ I_1^{(2P)}[\psi] = -128\pi n_{imp}(UV)^2 \frac{k_F^2}{k_0^3 L^3} \sum_{k_{2, k_{1'}, k_{2'}}} (k_{2'} - k_{1'})^2 (k_2 - k_1)^2 \Gamma_{(k_1,\eta_1), (k_2,\eta_1), (k_{1'},\eta_{1'}), (k_{2'},\eta_{1'})}. \]

Here, \( \Gamma \) is defined in Eq. (B10) and contains the thermal factors for the specific process, the distribution function \( \psi \) and the energy conserving delta function.

1. **Clean case: \( g_5 \) interaction**

Let us first consider a clean system where only \( g_5 \) influences transport. We insert Eq. (B1) into Eq. (14) to get an integral equation for \( \psi \).

\[ \psi_{k_1} = -4V^2 \int\frac{dk_2}{2\pi} \frac{1}{f_{k_1,R}(1-f_{k_1,R})} \left\{ \int\frac{dk_{1'}}{2\pi} \mathcal{K}(k_1, k_2) \left[ \psi_{k_1} + \psi_{k_2} + \psi_0 - \psi_{k_1+k_2} \right] \right. \]

\[ + \left. \int\frac{dk_{1'}}{2\pi} \mathcal{L}(k_1, k_1') \left[ \psi_{k_1'} - \psi_{0} - \psi_{k_1'} - \psi_{k_1-k_1'} \right] \right\} \]

Here, we defined

\[ \mathcal{K}(k_1, k_2) = (k_1^2 - k_2^2)^2 f_{k_1,R} (1-f_{k_1,R}) (1-f_{k_1+k_2,R}), \]

\[ \mathcal{L}(k_1, k_1') = \frac{1}{2} (k_1^2 - (k_1 - k_1')^2)^2 f_{k_1,R} (1-f_{k_1,R}) (1-f_{k_1-k_1',R}), \]

\[ \mathcal{L}(k_2, k_1') = \frac{1}{2} (k_2^2 - (k_2 - k_1')^2)^2 f_{k_1,R} (1-f_{k_1,R}) (1-f_{k_2-k_1',L}). \]

Due to the delta function \( \delta(k_1) \) in the third line of Eq. (B4) we have to treat the distribution function \( \psi_{k_1=0} \) of the state at the Dirac point separately.

First, we consider states at \( k_1 \neq 0 \) and introduce dimensionless momenta \( x = k_i / T \) and \( \zeta = k_F / T \) which yields

\[ \dot{\psi}_\zeta(x_1) = -\frac{A_-(x_1, \zeta)}{A_+(x_1, \zeta)} \dot{\psi}_\zeta(0) - \frac{1}{A_+(x_1, \zeta)} \int\frac{dx_2}{2\pi} B(x_1, x_2, \zeta) \dot{\psi}_\zeta(x_2) - n_F(x_1 - \zeta)(1-n_F(x_1 - \zeta)), \]

where we defined

\[ \dot{\psi}(x) = 4V^2 \frac{1}{k_0^3} eE T^6 \psi(x), \]

\[ A_\pm(x_1, \zeta) = \int\frac{dx_2}{2\pi} \left[ \mathcal{L}(x_1, x_2, \zeta) \pm \mathcal{K}(x_1, x_2, \zeta) \right], \]

\[ B(x_1, x_2, \zeta) = \mathcal{K}(x_1, x_2, \zeta) - \mathcal{L}(x_1, x_2, x_1', \zeta) - 2\mathcal{L}(x_1, x_2, \zeta). \]

We observe that the zero momentum distribution function \( \psi_\zeta(0) \) explicitly affects the distribution function of all other momentum states. In order to obtain \( \psi_\zeta(0) \) we have to consider the case of zero external momentum, \( k_1 = 0 \), in Eq. (B4).
In this case we have to regularize the diverging delta function. Physically, the divergence stems from our assumption of an infinite system where momentum is a continuous variable. We therefore introduce a momentum cutoff \( \Lambda \) such that \( \delta(x = 0) = \Lambda \) and neglect the other contributions in the integral equation for \( \psi(0) \) in comparison to this term. Solving the resulting equation for \( \psi(0) \) yields the cutoff independent result,

\[
\psi(0) = -\frac{1}{D(\zeta)} \int \frac{dx_2}{2\pi} \mathcal{E}(x_2, \zeta) \psi(x_2),
\]

(B10)

where

\[
D(\zeta) = \int \frac{dx_1 dx_2}{2\pi} C(x_1, x_2, \zeta),
\]

\[
\mathcal{E}(x_1, \zeta) = \int dx_2 \{ -C(-x_1, x_2, \zeta) + 2C(x_2, -x_1, \zeta) \}.
\]

(B11)

We now insert the result for the zero momentum distribution function in Eq. (B10) into the integral equation determining the distribution function of the remaining states, Eq. (B8). This yields

\[
\psi(x_1) = \frac{A_-(x_1, \zeta)}{A_+(x_1, \zeta)} D(\zeta) \int \frac{dx_2}{2\pi} \mathcal{E}(x_2, \zeta) \psi(x_2) - \frac{1}{A_+(x_1, \zeta)} \int \frac{dx_2}{2\pi} B(x_1, x_2, \zeta) \psi(x_2)
\]

\[
- \frac{n_F(x_1 - \zeta)(1 - n_F(x_1 - \zeta))}{A_+(x_1, \zeta)}.
\]

(B12)

This is an exact integral equation determining the distribution function of helical fermions in the presence of \( g_5 \) interaction. While it can not be solved analytically we can solve it numerically in the regime of high and low temperatures yielding a solution \( \psi_\zeta \).

In terms of this dimensionless function \( \tilde{\psi}_\zeta \) the conductivity in Eq. (43) takes the form

\[
\sigma_{dc} = -\frac{2e^2 k_0^4}{\hbar} \frac{1}{4V^2} \kappa(\zeta)
\]

(B13)

\[
\kappa(\zeta) = \int dx n_F(x - \zeta)(1 - n_F(x - \zeta)) \tilde{\psi}_\zeta(x)
\]

(B14)

We numerically find the asymptotics

\[
\kappa(\zeta) \simeq \begin{cases} 
-3.23 \zeta^{-5} e^\zeta, & \zeta \gg 1 \\
-0.056, & \zeta = 0
\end{cases}
\]

(B15)

As an example of the quality of the obtained asymptotics we plot the quantity \( \tilde{\kappa}(\zeta) = \zeta^5 e^{-\zeta} \kappa(\zeta) \) which converges to \( \tilde{\kappa}(\zeta \to \infty) = -3.23 \) in Fig. 6.

The obtained limits of \( \kappa(\zeta) \) yield the expression for conductivity in the regime \( k_F \ll T \),

\[
\sigma(k_F \ll T) = 0.014 \times \frac{2e^2 v_F}{\hbar} \frac{(v_F)}{V} \frac{1}{v_F k_0} \left( \frac{v_F k_0}{T} \right)^5,
\]

(B16)

and in the regime \( k_F \gg T \),

\[
\sigma(k_F \gg T) = 0.81 \times \frac{2e^2 v_F}{\hbar} \frac{(v_F)}{V} \frac{1}{v_F k_0} \left( \frac{k_0}{k_F} \right)^4 \frac{1}{v_F k_F} e^{\frac{e}{k_F}}.
\]

(B17)

These results are used in the main text in Eq. (44) and Eq. (45).

2. Disordered case: effective 1P and 2P processes

Inserting the collision integrals for inelastic single and two particle processes defined in in Eq. (B2) and (B3) into Eq. (14) we obtain integral equations describing the distribution function \( \psi \).
After introducing dimensionless quantities and taking the limit $\omega \to 0$ the equations take the form

$$\tilde{\psi}_i(x_1 + \zeta) = -\frac{1}{\mathcal{H}_i(x_1)} \int dx_2 \mathcal{G}_i(x_1, x_2) \tilde{\psi}_i(x_2 + \zeta) - \frac{1}{\mathcal{H}_i(x_1)} n_F(x_1)(1 - n_F(x_1)).$$  \hspace{1cm} \text{(B18)}

Here, $i = 1, 2$ denotes 1P and 2P processes, respectively and we have defined

$$\tilde{\psi}_i(x) = \frac{g_i^2}{eE} \tau_{3i}^{2i} \tilde{\psi}_i(x),$$

$$\mathcal{F}(x_1, x_2, x_3) = n_F(x_1) n_F(x_2)(1 - n_F(-x_3))(1 - n_F(x_1 + x_2 + x_3)) \left[(x_1 - x_2)^2(2x_3 + x_1 + x_2)^2\right],$$

$$\mathcal{G}_1(x_1, x_2) = \int \frac{dx_3}{2\pi} n_F(x_1) n_F(x_2)(1 - n_F(-x_3))(1 - n_F(x_1 + x_2 + x_3)) \left[(x_1 - x_2)^2 - (2x_3 + x_1 + x_2)^2\right],$$

$$\mathcal{G}_2(x_1, x_2) = \int \frac{dx_3}{2\pi} \left(\mathcal{F}(x_1, x_3, x_2) + 2\mathcal{F}(x_1, -x_2, x_3)\right),$$

$$\mathcal{H}_1(x_1) = \int \frac{dx_3 dx_2}{(2\pi)^2} n_F(x_1) n_F(x_2)(1 - n_F(-x_3))(1 - n_F(x_1 + x_2 + x_3)) \left[(x_1 - x_2)^2 + (2x_3 + x_1 + x_2)^2\right],$$

$$\mathcal{H}_2(x_1) = \int \frac{dx_3 dx_2}{(2\pi)^2} \zeta(x_1, x_3, x_2).$$ \hspace{1cm} \text{(B19)}

In the presence of disorder the physics of the scattering processes is not sensitive to the ratio of Fermi energy and temperature and thus the functions $\tilde{\psi}_i$ are in fact $\zeta$ independent.

The $dc$ conductivity due to 1P ($i = 1$) or 2P ($i = 2$) processes is then obtained as

$$\sigma_i = -\frac{2e^2}{h} \frac{1}{\gamma_i^2 T^{2i+2}} \kappa_i,$$ \hspace{1cm} \text{(B20)}

$$\kappa_i = \int dx n_F(x)(1 - n_F(x))\tilde{\psi}_i(x + \zeta)$$ \hspace{1cm} \text{(B21)}

where $\kappa_1 = -0.46$ and $\kappa_2 = -0.042$. To calculate the $\kappa_i$ we first solved the integral equations, Eq. \text{(B18)} numerically and subsequently used the obtained solutions $\tilde{\psi}_i$ to get $\kappa_i$ according to Eq. \text{(B21)}.

This procedure yields the $dc$ conductivity in the presence of impurities:

$$\sigma_{1P} = \kappa_1 \frac{2e^2 v_F}{h} \frac{1}{v_F n_{\text{imp}}} \left(\frac{v_F}{T}\right)^4 \left(\frac{v_F k_0}{T}\right)^4,$$ \hspace{1cm} \text{(B22)}

$$\sigma_{2P} = \kappa_2 \frac{2e^2 v_F}{h} \frac{1}{v_F n_{\text{imp}}} \left(\frac{v_F}{T}\right)^2 \left(\frac{k_0}{k_F}\right)^2 \left(\frac{v_F k_0}{T}\right)^6.$$ \hspace{1cm} \text{(B23)}

These results constitute Eq. \text{(46)} and Eq. \text{(47)} of the main text.
Appendix C: Average over forward scattering in the bosonic action

In this Appendix we perform the disorder average over forward scattering in the effective low energy action of a disordered helical liquid in Eq. (62). After gauging out forward scattering according to Eq. (64) and averaging over backward scattering the action reads as

\[
S_3 = \frac{4k_F^2 V}{\pi^2 a^4 k_0^4} \sum \int dx \cos(2\sqrt{4\pi} \varphi_a + \frac{4K}{u} \int_{x_0}^{x} dy U_f(y) - 4k_F x),
\]

\[
S_5 = \frac{4Vk_F}{\pi^2 a^2 k_0^2} \sum \int dx \partial_x \theta_a \sin(\sqrt{4\pi} \varphi_a + \frac{2K}{u} \int_{x_0}^{x} dy U_f(y) - 2k_F x),
\]

\[
- \frac{16V^2 k_F^2}{\pi^2 a^2 k_0^2} \sum \int dx \partial_x \theta_a \cos(\sqrt{4\pi} \varphi_a + \frac{2K}{u} \int_{x_0}^{x} dy U_f(y) - 2k_F x),
\]

\[
S_b = - \frac{4Dk_F^2}{\pi^2 a^2 k_0^4} \sum \int dx \partial_x \theta_a \partial_x \theta_a(x, \tau) \partial_x \theta_b(x', \tau') \cos(\sqrt{4\pi}(\varphi_a(x, \tau) - \varphi_b(x', \tau'))) \tag{C1}
\]

At this point we still have to average over forward scattering. To investigate the relevant averages consider the toy action

\[
S = g \int dx \partial_x (e^{i \int_x U} + e^{-i \int_x U}),
\]

\[
\overline{U(x)U(x')} = \delta(x-x').
\]

We perform the disorder average perturbatively in g:

\[
\overline{e^{-S}} \approx 1 - \overline{S} + \frac{1}{2} \overline{S^2} + \cdots = e^{-\frac{1}{2}(\overline{S^2}-\overline{S})}. \tag{C4}
\]

Now \(\overline{S^2}\) contains terms \(e^{i \sum_{m=0}^{n} \alpha_m \int_x dy U(y)}\) where \(\alpha_m \in \{1, -1\}\). Using the auxiliary identity,

\[
\int_{x_0}^{x} dy \int_{x_0}^{x'} dy' \delta(y-y') = \min(x, x') - x_0,
\]

we find

\[
e^{i \sum_{m=0}^{n} \alpha_m \int_x dy U(y)} = e^{-\frac{1}{2}(\sum_{m=0}^{n} \alpha_m \int_x dy \int_{x_0}^{x} dy' U(y)U(y'))} = e^{-\frac{1}{2}(\sum_{m,l} \alpha_m \alpha_l (\min(x_m, x_l) - x_0))}. \tag{C6}
\]

In the limit \(x_0 \to -\infty\) this is only nonzero if \(\sum_{m,l} \alpha_m \alpha_l = 0\) i.e.

\[
\sum_{m,l} \alpha_m \alpha_l = \sum_{m=0}^{n} \alpha_m^2 + \sum_{m<l} \alpha_m \alpha_l = n + 2 \sum_{m<l} \alpha_m \alpha_l = 0. \tag{C7}
\]

Notice that the second term is even, i.e. \(\overline{S^2}\) vanishes for odd \(n\). For the lowest nontrivial order we find

\[
\overline{S^2} = e^{-\frac{1}{4}(\alpha \int_x U + \alpha' \int_{x'} U')}^2 = e^{-\frac{1}{4}(x+x'-2 \min(x, x'))} \delta_{\alpha, -\alpha'} = e^{-\frac{1}{4}|x-x'|} \delta_{\alpha, -\alpha'}. \tag{C8}
\]

To summarize:

Let us define \(\tilde{U}_\alpha(x) = e^{i \alpha \int_{x_0}^{x} dy U_j(y)}\) where \(\alpha \in \mathbb{R}\). We showed that \(\tilde{U}\) obeys gaussian statistics up to fourth order in a weak coupling expansion.

\[
\overline{\tilde{U}_\alpha(x)\tilde{U}_\alpha(x')} = 0, \tag{C9}
\]

\[
\overline{\tilde{U}_\alpha(x)\tilde{U}_\alpha(x')} = e^{-\frac{\alpha^2}{2} D_{ij} |x-x'|}. \tag{C9}
\]

Thus nonlocal interactions decay exponentially due to forward scattering off disorder. The resulting interaction terms in the action are of the form

\[
S = - g \int dx \partial_x (F[\varphi(x, \tau), \varphi(x', \tau')] e^{-D_{ij} |x-x'|} e^{-ikF(x-x')}, \tag{C10}
\]
where $F$ is some functional of the fields and $\mu, \nu$ are constants. Now we split the spatial integration into relative and center of mass coordinates $R = \frac{1}{2}(x + x')$, $r = x - x'$. The relevant scales for the low energy physics of the model are given by energies much smaller than the disorder strength $D_f$. That means we can assume that the fields in $F$ are smooth as a function of the relative coordinate $r$,

$$
S = -g \int \text{d}r\text{d}R\text{d}r' F[\varphi(r,R,\tau),\varphi(r,R,\tau')] e^{-D_f|\mu[r]|e^{-ik_Fr}}
$$
$$
\approx -g \int \text{d}R\text{d}r' F[\varphi(R,\tau),\varphi(R,\tau')] \left( \int_{-\infty}^{\infty} \text{d}r e^{-D_f|\mu[r]|e^{-ik_Fr}} \right) \tag{C11}
$$
$$
= -\frac{g \mu D_f}{2\pi^2} \int \text{d}R\text{d}r' F[\varphi(R,\tau),\varphi(R,\tau')].
$$

This procedure yields the local theory discussed in Eq. [65], where our new momentum cutoff is given by the strength of forward scattering off disorder $D_f$.

**Appendix D: Formalism for conductivity calculation in the bosonized language**

In this Appendix we state some general methods and formulas needed to calculate the ac conductivity in the bosonized language.

1. **Anomalous current and susceptibility**

In order to compute the anomalous contributions to the current and the diamagnetic susceptibility we perform the minimal substitution $\partial_\tau \theta \to \partial_\tau + \frac{\pi}{\sqrt{2}} A$ in the model for a clean HLL, Eq. [63] and in the model describing the disordered HLL, Eq. [65]. The current $j$ and diamagnetic susceptibility $\chi^{\text{dia}}$ are then obtained by varying with respect to the vector potential, $j = \delta S/\delta A|_{A=0}$ and $\chi^{\text{dia}}(x-x',\tau-\tau') = -\frac{\delta S}{S(A,x,\tau) S(A,x',\tau')}$. This yields

$$
j_{\text{an,clean}}(1) = -\frac{4eV k_F}{\pi^2 ak_0^2} \sin(\sqrt{4\pi} \phi(1) - 2k_F x_1) - \frac{16eV k_F^2}{\pi^2 ak_0^2} \partial_x \theta(1) \cos(\sqrt{4\pi} \phi(1) - 2k_F x_1), \tag{D1}
$$

$$
\chi^{\text{dia,clean}}(1-2) = \frac{16e^2 V k_F^2}{\pi^2 ak_0^2} \cos(\sqrt{4\pi} \phi(1) - 2k_F x_1) \delta(1-2) \tag{D2}
$$

in the clean case and

$$
[j_{\text{an,dis}}^a](x_1,\tau_1) = 2g_{1P,1} \frac{e}{\sqrt{\pi}} \sum_b \int \text{d}\tau \partial_{x_1} \left( \partial^2_{x_1} \theta_b(x_1,\tau) \cos \left\{ \sqrt{4\pi} [\varphi_a(x_1,\tau_1) - \varphi_b(x_1,\tau)] \right\} \right)
$$
$$
+ 2g_{1P,2} \frac{e}{\sqrt{\pi}} \sum_b \int \text{d}\tau \partial_{x_1} \theta_b(x_1,\tau) \sin \left\{ \sqrt{4\pi} [\varphi_a(x_1,\tau) - \varphi_b(x_1,\tau_1)] \right\}
$$
$$
+ 2g_{1P,2} \frac{e}{\sqrt{\pi}} \sum_b \int \text{d}\tau \partial_{x_1} \theta_b(x_1,\tau) \partial_{x_1} \sin \left\{ \sqrt{4\pi} [\varphi_a(x_1,\tau) - \varphi_b(x_1,\tau_1)] \right\}
$$
$$
- 2g_{\text{imp,b}} \frac{e}{\sqrt{\pi}} \sum_b \int \text{d}\tau \partial_{x_1} \theta_b(x_1,\tau) \cos \left\{ \sqrt{4\pi} [\varphi_a(x_1,\tau_1) - \varphi_b(x_1,\tau)] \right\} \tag{D3}
$$

$$
[\chi^{\text{dia,dis}}]_{ab}(1-2) = 2g_{1P,1} \frac{e^2}{\pi} \delta(x_1-x_2) \partial_{x_1} \sin \left\{ \sqrt{4\pi} [\varphi_a(x_1,\tau_2) - \varphi_b(x_1,\tau_1)] \right\}
$$
$$
+ 2g_{\text{imp,b}} \frac{e^2}{\pi} \cos \left\{ \sqrt{4\pi} [\varphi_a(x_1,\tau_1) - \varphi_b(x_1,\tau_2)] \right\} \delta(x_1-x_2) \tag{D4}
$$

in the disordered case. Here, we abbreviated $1 = (x_1,\tau_1)$. These expressions are needed to obtain the ac conductivity in Appendix E and F.
2. Correlation functions

In order to calculate the correlation functions that appear during the calculation of conductivity we state some basic correlation functions of the bosonic theory, which can be obtained using standard methods.\cite{[12]}

\[
\langle \partial_x \varphi(x, \tau) \partial_x \varphi(0, 0) \rangle = -\frac{K}{4\pi} \left( \frac{\pi T}{u} \right)^2 \left( \frac{1}{\sinh^2(x_+)} + \frac{1}{\sinh^2(x_-)} \right),
\]

\[
\langle \partial_x^2 \varphi(x, \tau) \partial_x^2 \varphi(0, 0) \rangle = \frac{K}{2\pi} \left( \frac{\pi T}{u} \right)^4 \left( \frac{1 + 2\cosh^2(x_+)}{\sinh^4(x_+)} + \frac{1 + 2\cosh^2(x_-)}{\sinh^4(x_-)} \right),
\]

\[
\langle \varphi(x, \tau) \partial_x \theta(0, 0) \rangle = -\frac{T}{4u} (\coth(x_+) - \coth(x_-)),
\]

\[
\langle \varphi(x, \tau) \partial_x \varphi(0, 0) \rangle = -\frac{TK}{4u} (\coth(x_+) + \coth(x_-)),
\]

\[
\langle \varphi(x, \tau) \partial_x^2 \theta(0, 0) \rangle = -\frac{\pi T^2}{4u^2} \left( \frac{1}{\sinh^2(x_+)} - \frac{1}{\sinh^2(x_-)} \right),
\]

\[
\langle [\varphi(x, \tau) - \varphi(0, 0)]^2 \rangle = \frac{K}{2\pi} \ln \left\{ \frac{\beta u}{\pi \alpha} \right\} \sinh \left( \frac{\pi T}{u} (x - iu\tau) \right) \sinh \left( \frac{\pi T}{u} (x + iu\tau) \right) = \frac{K}{2\pi} F(x, \tau).
\]

Here, we defined \( x_\pm = \pm T \{ x \pm i[u\tau + \text{sgn}(\tau) a] \} \). The correlation functions for \( \theta \) can be obtained from the ones above by the duality relation \( \sqrt{K} \varphi \rightarrow \frac{1}{\sqrt{K}} \theta \). For later reference we also introduce the notation

\[
\begin{align*}
G_{\theta \varphi}^{(m)}(x - x', \tau - \tau') &= \langle \theta_\varphi(x, \tau) \varphi(x', \tau') \rangle \\
G_{\theta \theta}^{(m,n)}(x - x', \tau - \tau') &= \langle \theta_\theta(x, \tau) \partial_{x'}^n \theta_\theta(x', \tau') \rangle
\end{align*}
\]

(D5)

These correlation functions will appear in the context of the \( ac \) conductivity of a HLL in Appendix E and F.

3. Correlation functions containing exponentials of bosonic fields

We often encounter correlation functions such as

\[
\langle \theta_{11}' \theta_{22}' e^{2i\sqrt{\pi} (\varphi_{33} - \varphi_{34})} \rangle,
\]

(D8)

where we denoted \( \partial_x \theta(x, \tau) = \theta'(x, \tau) \) and \( \theta(x_1, \tau_1) = \theta_{11} \). We can calculate them using the following trick:

\[
\langle \theta_{11}' \theta_{22}' e^{2i\sqrt{\pi} (\varphi_{33} - \varphi_{34})} \rangle = \frac{1}{4(4\pi)} \partial_{t_1} \partial_{t_2} \langle I_{12} = 0 \rangle \langle (\theta_{11}' \theta_{22}') e^{2i\sqrt{\pi} (\varphi_{33} - \varphi_{34})} \rangle = 0
\]

\[
= \{ \langle \theta_{11}' \theta_{22}' \rangle - 16\pi \langle \theta_{11}' \varphi_{31} - \varphi_{34} \rangle \langle \theta_{22}' \varphi_{33} - \varphi_{34} \rangle \} e^{-2(4\pi)(\varphi_{33} - \varphi_{34})^2}.
\]

We employ this method of evaluating correlation functions containing exponentials of bosonic fields in the context of calculating the \( ac \) conductivity in Appendix E and F.

Appendix E: Calculation of the conductivity of a disordered helical Luttinger liquid

In this Appendix we outline the calculation of \( ac \) conductivity of a disordered HLL using full bosonization. First, we expand the current-current correlation function to first order in impurity strength which yields

\[
\langle j^a(x, \tau) j^{a'}(x', \tau') \rangle = \langle j_0^a(x, \tau) j_0^{a'}(x', \tau') \rangle_0 - \langle j_0^a(x, \tau) j_0^{a'}(x', \tau') S_{\text{pert}} \rangle_0 + 2 \langle j_0^a(x, \tau) j_0^{b} \rangle_{\text{an,dis}} (x', \tau') \langle S \rangle_0 + O(D^2).
\]

(E1)

Here, we defined \( S_{\text{pert}} = S_1p + S_2p + S_{\text{imp,b}} \). To first order the terms in \( S_{\text{pert}} \) have to be diagonal in replica indices and therefore the replica limit is performed as

\[
\frac{1}{R} \sum_{a,b,a'} \langle j_0^a j_0^b S_{a'} \rangle = \frac{1}{R} \sum_{a,b} \left( \sum_{a' = b} \langle j_0^a j_0^b S_{a'} \rangle + \sum_{a' \neq b} \langle j_0^a j_0^b S_{a'} \rangle \right) \approx \frac{1}{R} \sum_{a=1}^{R} \langle j_0^a j_0^a S_{a} \rangle + (R - 1) \langle j_0^a j_0^a \langle S \rangle \rangle
\]

(E2)

\[
\rightarrow \langle j_0 j_0 S \rangle - \langle j_0 j_0 \rangle \langle S \rangle \equiv \langle j_0 j_0 S \rangle_c,
\]
where we defined the connected average in the last equality.

We define the contributions linear in disorder strength as

\[ \Sigma_1(x, x', \tau, \tau') = -\langle j_0^b(x, \tau) j_0^b(x', \tau') S_{\text{pert}} \rangle_0 + 2 \langle j_0^b(x, \tau) j_{\text{an,dis}}^b(x', \tau') \rangle_0. \]  
(E3)

Conductivity is then obtained by calculating the Fourier transform \( \Sigma_1(k, k_n) \) and performing the limit

\[ \sigma(\omega) = -\frac{i}{\omega} \left( \Sigma_1(k \to 0, i k_n \to \omega + i\delta) + \chi^{\text{dia}}(k, k_n) \right). \]  
(E4)

We obtain

\[ \Sigma_{1p}^2(k = 0, k_n) = 32 e^2 u^2 K^2 k_n^2 g_{2p} \int_0^\beta d\tau e^{-4KF(\tau)} \left[ 1 - e^{i k_n \tau} \right]. \]  
(E5)

\[ \Sigma_{1p}^1(k = 0, k_n) = 8 e^2 u^2 K^2 k_n^2 g_{1p} \int_0^\beta d\tau G_{\theta \theta}^{(2.2)}(0, \tau) e^{-K F(\tau)} \left[ 1 - e^{i k_n \tau} \right], \]  
(E6)

\[ \Sigma_{1p}^{\text{imp,b}}(k = 0, k_n) = 8 e^2 u^2 K^2 k_n^2 g_{\text{imp,b}} \int_0^\beta d\tau \left\{ G_{\theta \varphi}^{(1)}(0, \tau) - 4\pi \left[ C_{\theta \varphi}^{(1)}(0, \tau) \right]^2 \right\} e^{-K F(\tau)} \left[ 1 - e^{i k_n \tau} \right] \]  
(E7)

and

\[ \chi^{\text{dia}}(k = 0, k_n) = -2g_{\text{imp,b}} \frac{e^2}{\pi} \int_0^\beta d\tau e^{-K F(\tau)} e^{i k_n \tau}. \]  
(E8)

The conductivity due to 1P and 2P processes is then

\[ \sigma_{2p}(\omega) = 32 \alpha e^2 u^2 K^2 \omega^3 g_{2p} \left( \frac{\pi a T}{u} \right)^{2K} J_{2K}(\omega, T), \]  
(E9)

\[ \sigma_{1p}(\omega) = 8 \alpha e^2 u^2 K \pi a^2 \omega^3 g_{1p} \left( \frac{\pi T}{u} \right)^{2K+4} (3J_{2K+4}(\omega, T) - 2J_{2K+2}(\omega, T)), \]  
(E10)

where we defined

\[ J_{2K}(\omega, T) = \int_0^\beta d\tau \frac{1 - e^{i k_n \tau}}{\sin 2K(\pi T)^2} \left| k_n \to \omega + i\delta \right| \]  
(E11)

Here, \( \Gamma(x) \) is the gamma function. These results appear in Eq. (83) and Eq. (84) of the main text.

In the case of backscattering off the impurity we obtain

\[ \Sigma_{1p}^{\text{imp,b}}(k = 0, k_n) = -4 e^2 K g_{\text{imp,b}} \left( \frac{\pi a T}{u} \right)^{2K} \left\{ \left( \frac{\pi T}{\omega} \right)^2 \left[ (2K + 1) J_{2K+2}(\omega, T) - 2K J_{2K}(\omega, T) \right] + \frac{T}{\omega} L_K(\omega, T) \right\}, \]  
(E12)

\[ \chi^{\text{dia}}(k = 0, k_n) = 2g_{\text{imp,b}} \frac{e^2}{\pi} \left( \frac{\pi a T}{u} \right)^{2K} \frac{1}{\pi T} \sin(K\pi) B(K - i \frac{\omega}{2\pi T}, 1 - 2K). \]  
(E13)

Here, \( B(x,y) \) denotes the Euler beta function and we defined

\[ L_K(\omega, T) = \int d\tau \frac{1 - e^{-i k_n \tau}}{\sin 2K+1(\pi T)^2} \cos(\pi T \tau) \]  
(E14)

\[ = (-i) \sin(\pi K) \frac{2K}{\pi T} \left\{ B(K, -2K) - B(K - i \frac{\omega}{2\pi T}, -2K) \right\} \]  
(E14)

\[ + \left[ B(K + 1, -2K) - B(K + 1 - i \frac{\omega}{2\pi T}, -2K) \right]. \]

Adding the contributions yields \( \Sigma_{1p}^{\text{imp,b}}(k = 0, k_n) + \chi^{\text{dia}}(k = 0, k_n) = 0 \). Therefore, backscattering does not lead to a finite scattering time for any value of \( K \) to first order in \( D_b \). This is discussed in Sec. [VI B] of the main text.
Appendix F: Calculation of the conductivity of a clean helical Luttinger liquid

The purpose of this Appendix is to outline the calculation of the ac conductivity of a clean HLL using bosonization and the Kubo formula.

First, we expand the current-current correlation function to second order in interaction strength since the first order contribution vanishes due to the neutrality condition for vertex operators. This yields

$$\langle jj \rangle = \langle j_0 j_0 \rangle + \frac{1}{2} \langle j_0 j_0 S_0^c \rangle - 2 \langle j_0 j_{\text{clean}} S_0^c \rangle + \langle j_{\text{clean}} j_{\text{clean}} \rangle + O(V^4).$$  \hfill (F1)

Here, the connected averages appear due to the expansion of the denominator of the partition function. As in the disordered case we define $\Sigma_2 \equiv \frac{1}{2} \langle j_0 j_0 S_0^c \rangle - 2 \langle j_0 j_{\text{clean}} S_0^c \rangle + \langle j_{\text{clean}} j_{\text{clean}} \rangle$. Adding all the terms we are left with only one term contributing to the real part of the conductivity

$$\Sigma_2(x_3, x_4, \tau_3, \tau_4) = \frac{e^2 K^2 u^2}{4 \pi} \left( \frac{4 V k_F}{\pi^2 a k_0^2} \right)^2 \int \! dx_1 \! d\tau_1 \int \! dx_2 \! d\tau_2 \times \langle \partial_{x_3} \theta(x_3, \tau_3) \partial_{x_4} \theta(x_4, \tau_4) \partial_{x_1}^2 \theta(x_1, \tau_1) \partial_{x_2}^2 \theta(x_2, \tau_2) e^{i \sqrt{4\pi}(\varphi(x_1, \tau_1) - \varphi(x_2, \tau_2)) - 2i k_F(x_1 - x_2)} \rangle_0$$

Using the methods outlined in Appendix D we obtain

$$\sigma(\omega) = \frac{i}{\omega^3 \ h^2 \ F^2 \ 16 \left( \frac{V}{u} \right)^2 \left( \frac{k_F}{k_0} \right)^2 \left( \frac{1}{(a k_0)^2} I_K(\omega, T) \right)}$$

Here, we defined

$$I_K(\omega, T) = \int \! dx \int_0^\beta \! d\tau \left\{ G_{yy}^{(2,2)}(x, \tau) + 4\pi G_{\varphi\varphi}^{(2)}(x, \tau) \right\} e^{-K F(x, \tau)} e^{2ik_F(x)} \left[ 1 - e^{i k_n \tau} \right]_{i k_n \rightarrow \omega + i \delta}$$

$$= \frac{1}{(2a)^4 \pi u} \left( \frac{2 \pi T a}{u} \right)^{2K + 4} \left( \frac{u}{\pi T} \right)^2 \sin(K \pi) \left[ \frac{1}{K} M(\omega, -K, -K - 2) + M(\omega, -K - 2, -K) \right]$$

$$+ \left( \frac{6}{K} + 1 \right) \left[ M(\omega, -K, -K - 4) + M(\omega, -K - 4, -K) - 2M(\omega, -K - 2, -K) \right]$$

$$M(\omega, \nu, \mu) = B(-\nu S_0 - \frac{\nu}{2}, \nu + 1) B(-\nu S_0 - \frac{\mu}{2}, \mu + 1) - B(-\nu S_0 + \frac{\mu}{2}, \mu + 1) - B(-\nu S_0 + \frac{\nu}{2}, \nu + 1)$$

and $S_\pm = \frac{\nu}{4 \pi T} \pm \frac{\nu k_F}{2 \pi T}, \ S_0^c = S_\pm (\omega = 0)$. Eq. (F3) is Eq. (78) of the main text.

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