COMPACT SETS IN BIDIRECTIONAL GRAND LEBESGUE SPACES, WITH APPLICATIONS.
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Abstract.

In this article we find some sufficient conditions for the set in the Bilateral Grand Lebesgue Space to be compact set. We consider applications into numerical methods and in the basis problem.

Key words: Bilateral Grand Lebesgue Spaces, Orlicz’s spaces, rearrangement invariant spaces, compact sets, convergence, basis, numerical methods.

Mathematics Subject Classification. Primary (1991) 37B30, 33K55, 26D15;
Secondary (2000) 34A34, 65M20, 42B25, 35A25.

1. Introduction. Statement of problem.

Let \((X, \Sigma, \mu)\) be a measurable space with non-trivial measure \(\mu : \exists A \in \Sigma, \mu(A) \in (0, \mu(X))\). We will assume that either \(\mu(X) = 1\) or \(\mu(X) = \infty\) and that the measure \(\mu\) is \(\sigma-\)finite and diffuse on the sets of positive \(\mu\) measure: \(\forall A \in \Sigma, 0 < \mu(A) < \infty \exists B \subset A, \mu(B) = \mu(A)/2\). Define as usually for arbitrary measurable function \(f : X \to R^1\)

\[|f|_p = \left( \int_X |f(x)|^p \mu(dx) \right)^{1/p}, \quad p \geq 1;\]

\(L_p = L(p) = L(p; X, \mu) = \{f, |f|_p < \infty\}\). Let \(a = \text{const} \geq 1, b = \text{const} \in (a, \infty]\), and let \(\psi = \psi(p)\) be some strictly positive on the closed interval \([a, b]\), including the cases

\(\psi(a) \overset{\text{def}}{=} \lim_{p \to a+0} \psi(p) = \infty\)

and

\(\psi(b) \overset{\text{def}}{=} \lim_{p \to b-0} \psi(p) = \infty,\)
continuous on the open interval $(a, b)$ function.

The set of all those functions we will denote $\Psi : \Psi = \Psi(a, b) = \{\psi(\cdot)\}$.

We define in the case $b = \infty$ $\psi(b - 0) = \lim_{p \to \infty} \psi(p)$.

**Definition 1.**

Let $\psi(\cdot) \in \Psi(a, b)$. The space $BSGL(\psi) = G(\psi) = G(X, \psi) = G(X, \psi, \mu) = G(X, \psi, \mu, a, b)$ (Bide - Side Grand Lebesgue Space) consist on all the measurable functions $f : X \to R$ with finite norm

$$||f||G(\psi) \overset{df}{=} \sup_{p \in (a, b)} ||f|_p/\psi(p)||.$$

The introduced spaces are some generalization of the so-called Grand Lebesgue Spaces (see [9], [10], [11], [12], [13] etc). For example, the space $L^{b_0}, b \in [1, \infty)$ (in the notations [9], [10], [11]) coincides with our space $G(\psi_b(p))$, where by definition

$$\psi_b(p) = (b - p)^{-1/b}, \quad p \in (1, b).$$

They are also rearrangement invariant (r.i.) spaces, and was studied as r.i. spaces (find its fundamental function, calculated its adjoin spaces, obtained different imbedding and convolution theorems etc.) in the papers, e.g., [21], [23]. Moreover, the $G(\psi)$ spaces are the so-called moment rearrangement invariant (m.r.i.) spaces. They satisfy the Fatou property etc. [25].

**We investigate in this article some conditions for arbitrary subsets of $G(\psi)$ spaces to be compact set in this space or in some its closed subspace. We intent also to consider some applications: in the theory of numerical methods and investigate existence of basis in these spaces.**

Note that the $G(\psi)$ spaces are the particular case of interpolation spaces (so-called $\Sigma-$ spaces) (see [1], [2], [5],[4], [13]).

But we hope that the our direct representation of these spaces (definition 1) is more convenient for investigation and application.

In the case $\mu(X) = 1$ $G(\psi)$ spaces appear in the article [16]. See for detail [20], chapter 1.

These spaces are used, for example, in the theory of probability ([16], [18], [19], [20], [21], [22], [23],[24], [31] etc.), theory of PDE ([10], [12]), functional analysis [6], [9], [13], theory of Fourier series ([23]), theory of martingales [19], [22], theory of approximation [25] etc.

The article is organized as follows. In the next section we reproduce some affirmations from the general theory of BSGL spaces. The third section contains the main results. In the last section we consider some application of obtained in section 3 results.

**2. Auxiliary facts. Subspaces. Convergence.**

The next facts about $G(\psi)$ spaces are proved in [23]. Some assertions on the $G(\psi)$ spaces may be obtained from the general theory of rearrangement invariant (r.i.) spaces, see [3], [17], as long as the BSGL spaces are r.i. spaces.
We will say as usually [3], pp. 14 - 16 that the function \( f \in G(\psi) \), \( \psi \in \Psi \) has absolutely continuous norm and write \( f \in GA(\psi) \), if
\[
\lim_{\delta \to 0^+} \sup_{A, \mu(A) \leq \delta} ||f|_A||G(\psi) = 0;
\]
\[I_A = I_A(x) = 1, \ x \in A; \ I_A = I_A(x) = 0, \ x \notin A.\]

A family \( F \) of elements of \( G(\psi) \) space is said to be equi - absolute continuous, write: \( F \in EGA, \) iff
\[
\lim_{\delta \to 0^+} \sup_{A, \mu(A) \leq \delta} \sup_{f \in F} ||f|_A||G(\psi) = 0.
\]

We denote by \( G^0 = G^0_X(\psi) \), \( \psi \in \Psi \) the closed subspace of \( G(\psi) \), consisting on all the functions \( f \), satisfying the following condition:
\[
\lim_{p \to a^+} \frac{|f|_p}{\psi(p)} = \lim_{p \to b^-} \frac{|f|_p}{\psi(p)} = 0,
\]
in the case \( \psi(a + 0) = \infty, \ \psi(b - 0) = \infty; \)
\[
\lim_{p \to b^-} \frac{|f|_p}{\psi(p)} = 0
\]
in the case \( \psi(a + 0) < \infty, \ \psi(b - 0) = \infty; \)
\[
\lim_{p \to a^+} \frac{|f|_p}{\psi(p)} = 0
\]
in the case \( \psi(a + 0) = \infty, \ \psi(b - 0) < \infty; \) briefly:
\[
\lim_{\psi(p) \to \infty} \frac{|f|_p}{\psi(p)} = 0.
\]

In the case \( \psi(a) < \infty, \psi(b) < \infty \) the space \( G(\psi) \) coincides up to norm equivalence with the direct sum \( L_a + L_b \); which is the known Orlicz space satisfying a \( \Delta_2 \) condition. This case is not interest for us and must be excluded.

We denote also by \( GB = GB(\psi) \) the closed span in the norm \( G(\psi) \) the set of all bounded: \( \sup_{x \in X} |f(x)| < \infty \) measurable functions with finite support: \( \mu(\text{supp } |f|) < \infty. \)

The subspaces \( GA(\psi), GB(\psi), G^0(\psi) \) are closed subspaces of the space \( G(\psi) \) and moreover coincide:
\[
GA(\psi) = GB(\psi) = G^0(\psi),
\]
see [23].

Note that in the considered case \( \sup_{p \in (a, b)} \psi(p) = \infty \) the subspaces \( GA(\psi), GB(\psi), G^0(\psi) \) are strong subspaces of the space \( G(\psi) \):
\[
G(\psi) \supset GA(\psi) = GB(\psi) = G^0(\psi) \neq G(\psi).
\]

If the metric space \( (\Sigma, \rho) \), where
\[
\rho(A_1, A_2) = \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1), \ A_1, A_2 \in \Sigma
\]
is separable, then the spaces $GA(\psi), GB(\psi), G^0(\psi)$ are also separable. In contradic-
tion, the generic space $G(\psi)$ is not separable.

Further, let $\psi(\cdot), \nu(\cdot)$ be two functions from the set $G(\psi; a, b)$. We will write $\nu \ll \psi$, or equally $\psi \gg \nu$, iff

$$\lim_{\psi(p) \to \infty} \frac{\nu(p)}{\psi(p)} = 0.$$  

3. Main results.

A. In this subsection we consider arbitrary bounded closed subset $S$ of the space $G(\psi) = G(\psi; a, b) : S \subset G(\psi)$.

Since all the considered spaces: $G(\psi), GA(\psi)$ etc. are complete metric spaces, we can and will restrict itself only by the notion of sequentially compactness.

**Theorem 1.** Let $\psi, \nu$ be two functions from the set $G(\nu; a, b)$ such that $\psi(\cdot) \gg \nu(\cdot)$. Assume that:

1. 

$$\sup_{f \in S} ||f||G(\nu) < \infty.$$  

2. For all the values $p \in (a, b)$ the set $S$ is compact set in the space $L_p = L_p(X; \mu)$.

Then the set $S$ is compact set in the space $G(\psi)$.

**Proof.** Let $\{f(n) = f(n, x)\}$ be arbitrary sequence functions belonging the set $S$. Without loss of generality we conclude by virtue of condition 1 that

$$\sup_n ||f(n, \cdot)||_p \leq \nu(p).$$  

Let also $\{p(i), i = 1, 2, \ldots\}$ be arbitrary dense in the ordinary distance $d(p, q) = |p - q|$ sequence of numbers in the interval $(a, b)$. There exists a sequence $n(j)$ for which there exists a limit in $L_{p(1)}$ sense:

$$|f(n(j), \cdot) - h(1, \cdot)|_{p(1)} \to 0, j \to \infty.$$  

Choosing an appropriate subsequence $n(j(l))$, we can see

$$|f(n(j(l)), \cdot) - h(2, \cdot)|_{p(2)} \to 0, l \to \infty.$$  

Analogously

$$|f(n(j(l(q))), \cdot) - h(3, \cdot)|_{p(3)} \to 0, q \to \infty$$  

etc. It is evident that

$$\mu\{\bigcup_{i=2}^{\infty} \{x : h(i, x) \neq h(1, x)\}\} = 0,$$

can we assume that $\forall i \geq 2 \Rightarrow h(i, x) = h(1, x)$ almost everywhere.
Extracting the "diagonal" subsequence, indeed: \(g(1,x) = f(1,x), \ g(2,x) = f(n(2),x), \ g(3,x) = f(n(j(3)),x), \ g(4,x) = f(n(j(l(4))),x)\) etc., we conclude that for all values \(p(i) \in (a,b)\)

\[
\lim_{m \to \infty} |g(m,\cdot) - h(1,\cdot)|_{p(i)} = 0.
\]

Since the sequence \(\{p(i)\}\) is dense in the interval \((a,b)\), we find on the basis of Hölder inequality that for all values \(p, p \in (a,b)\)

\[
\lim_{m \to \infty} |g(m,\cdot) - h(1,\cdot)|_p = 0.
\]

Tacking into account the inequality

\[
|h(q,\cdot)|_p \leq \sup_n |f_n|_p \leq \nu(p), \ \forall q = 1,2,\ldots,
\]

we see that \(h(1,\cdot) \in G(\nu,a,b)\).

The convergence

\[g(m,\cdot) \xrightarrow{G(\psi)} h(1,\cdot)\]

it follows from a results of a paper [23].

**Remark 1.** It is evident that the first condition of the Theorem 1 is necessary and that the second is not.

Let us consider the following example. Choosing the function \(f = f(x)\) from the set \(G(\psi) \setminus G^o(\psi)\):

\[f(\cdot) \in G(\psi) \setminus G^o(\psi),\]

and such that \(\|f\|G(\psi) = 1\); we may consider the sequence

\[f(n,x) = f(x) \cdot (1 - 1/(n+1)).\]

It is obvious that

\[\|f(n,\cdot) - f(\cdot)\|G(\psi) \to 0, \ n \to \infty,\]

but does not exists some function \(\nu(\cdot)\) for which that \(\nu >> \psi\) and such that

\[\sup_n \|f(n,\cdot)\|G(\nu) < \infty,\]

as long as \(f(\cdot) \notin G^o(\psi)\).

**Remark 2.** But the condition 2 can not be omitted. Let us consider the correspondent example.

Let \(X = [0,1]\) with ordinary Lebesgue measure \(m\). Let \(\xi = \xi(\omega), \ \omega \in X\) be a standard Gaussian distributed function, on the other words, Normal Standard random variable.

Put \(\psi(p) = \psi_{0.5}(p) = p^{1/2}, \ p \geq 1\). It is easy to calculate that \(\xi \in G(\psi_{0.5})\). Consider the sequence

\[\xi(n,\omega) = \xi(\omega) \cdot I(|\xi(\omega)| \leq n), \ n = 1,2,\ldots.\]
We observe:

\[ \xi(n, \cdot) \in GB \ ( = GA = G^0(\psi)) , \]

\[ \forall p \in (1, \infty) \Rightarrow |\xi(n, \cdot) - \xi|_p \to 0 , \]

as \( n \to \infty ; \)

\[ \sup_n ||\xi(n, \cdot)||_{G^0(\psi, 0.5)} < \infty . \]

But the convergence of \( \xi(n, \cdot) \) to the variable \( \xi(\cdot) \) in the sense of \( G^0(\psi, 0.5) \) norm is false as long as the limit variable \( \xi \) does not belongs to the space \( G^0(\psi, 0.5) \).

**Remark 3.** Instead the condition 1 it can be presumed that the compactness of the sequence of the functions \( \{f(n, \cdot)\} \) is true only for some sequence of spaces \( L_{p(i)} \) with powers \( \{p(j)\} \) such that

\[ \lim_{j\to\infty} p(j) = b; \lim_{j\to\infty} p(j) = a. \]

See in detail [28], chapter 4.5.

**Remark 4.** The first condition 1 in the case of finiteness of measure \( \mu : \mu(X) < \infty \) may be replaced on the compactness on the measure, i.e. in the distance

\[ r(f, g) := \inf\{\epsilon, \epsilon > 0, \mu\{x : |f(x) - g(x)| > \epsilon\} < \epsilon\}. \]

**B.** In this pilcrow we consider arbitrary bounded closed subset \( S \) of the space \( G^0(\psi) = GA(\psi) = GB(\psi) : S \subset G^0(\psi), \psi \in \Psi(a, b) \).

**Theorem 2.** The closed bounded subset \( S \) of the space \( G^0(\psi) \) or equally in the spaces \( GA(\psi), GB(\psi) \) is sequentially compact set if and only if it is compact set in each space \( L_p, p \in (a, b) \) and is equi - absolute continuous: \( S \in EGA \).

**Proof** is very simple. The sufficient assertion "if" follows immediately from the theorem 1 and the known quoted properties of BSGL spaces. The inverse affirmation may be proved alike the proof of the theorem 2 from the classical book [26], chapter 5, section 5.2.

See also [27], chapters 1.2 and [15], chapters 1.2.3.

**Remark 5.** The remarks 3 and 4 are true even in the considered case of the sets in \( GA(\psi) \) spaces.

**Remark 6.** Suppose the metric space \((\Sigma, \rho)\) is separable. Then for the sets on the space \( G^0(\psi) \) are true the classical criterions for compactness, belonging to Kolmogorov, Riesz, Schilov, Frechet etc.

**Remark 6.** Let \( X \) be the closure of open non - empty bounded set in the Euclidean finite - dimensional space \( R^d \) equipped Lebesgue measure \( m \). Then the spaces \( G^0(\psi), GA(\psi) \) and \( GB(\psi) \) have a basis, consisting, for instance, from the deformed Haars functions .
The proof is at the same as in the case of Orlicz spaces, considered, e.g. in [15], chapter 2, section 12. See also [17], chapter 1.

4. Applications. Spherical rearrangement of a functions.

0. We consider in this section some slight generalization of a main result of Jean van Schaftingen [29] in the $G(\psi)$ spaces instead $L_p$ spaces considered in [29].

Let $u : R^d \to R^+ \cup \{\infty\}$ be a measurable function. A function $u^* = u^*(x) = u^*(x; u(\cdot))$ is called symmetric rearrangement, or Schwarz symmetrization of a function $u$, iff it is spherical symmetry in the following sense: $\forall \lambda \in R \exists r = r(\lambda) \geq 0$,

$$\{x \in R^d : u^*(x) > \lambda\} = B(0, r),$$

where $B(0, r)$ is the Euclidean ball with center at the origin and radius $r$;

$$\forall \lambda \in R m\{x : u^*(x) > \lambda\} = m\{x : u(x) > \lambda\}.$$ 

Recall that $m$ denotes the Lebesgue measure.

This symmetrization there exists for arbitrary function $u$, $u \in L_p$, and is unique (up to the set of zero measure).

The symmetrization is used in the study of isoperimetric inequalities, variational problems, theory of partial differential equations etc., see [29].

1. As long as

$$|u^*|_p = |u|_p, \ p \leq 1,$$

we can conclude then for every function $\psi \in \Psi(a, b)$

$$||u^*||G(\psi) = ||u||G(\psi)$$

(the norm preserving).

2. Since

$$|u^*(\cdot; u) - v^*(\cdot; v)|_p \leq |u(\cdot) - v(\cdot)|_p,$$

then for every function $\psi \in \Psi(a, b)$

$$||u^* - v^*||G(\psi) \leq ||u - v||G(\psi)$$

(contraction).

3. By virtue of the inequality

$$|\nabla u^*|_p \leq |\nabla u|_p, \ p \leq 1,$$

we assert then for every function $\psi \in \Psi(a, b)$

$$||\nabla u^*||G(\psi) \leq ||\nabla u||G(\psi)$$
(Polya - Szegö inequality), if obvious

$$\nabla u \in G(\psi).$$

4. Jean van Schaftingen in [29] offered the consistent iterative algorithm ("polarization algorithm") for $u^*(\cdot)$ computation. He construct the sequence of a functions 
\{u_n\}, $u_1 = u$, such that if $u \in L_p$, $p \geq 1$, then:

\[ A. |u_n - u^*|_p \to 0, \ n \to \infty; \]

\[ B. |u_n|_p = |u|_p = |u^*|_p; \]

C. The sequence \{u_n\} is compact set in the space $L_p(\mathbb{R}^d)$.

Now we assume that $u(\cdot) \in G(\psi)$ for some $\psi \in \Psi(a,b), 1 \leq a < b \leq \infty$. From the theorems 1 and 2 it follows:

**Theorem 3.** For arbitrary function $\nu, \nu \in G(\psi), \nu >> \psi$

$$||u_n - u^*||G(\nu) \to 0, \ n \to \infty.$$

If in addition $u(\cdot) \in G^a(\psi)$, then

$$||u_n - u^*||G^a(\psi) = ||u_n - u^*||G(\psi) \to 0, \ n \to \infty.$$

Note that by virtue of the property 2 the function $u^*(\cdot) = u^*(\cdot, u)$ depended continuously on the source function $u(\cdot)$ also in the $G(\psi)$ norm.

**References**

[1] Astashkin S.V. *About interpolation spaces of sum spaces, generated by Rademacher system*. RAEN, issue MMMIU, 1997, v.1 No 1, p. 8-35.

[2] Astashkin S.V. *Some new Extrapolation Estimates for the Scale of $L_p$ - Spaces*. Funct. Anal. and Its Appl., v. 37 No 3 (2003), 73 - 77.

[3] Bennet C., Sharpley R. *Interpolation of operators*. Orlando, Academic press Inc., (1988).

[4] Carro M., Martin J. *Extrapolation theory for the real interpolation method*. Collect. Math. 33 (2002), 163-186.
[5] Capone C., Fiorenza A., Krbec M. On the Extrapolation Blowups in the $L_p$ Scale. Funct. Anal. and Its Appl., v. 37 N° 3 (2003), 73 - 77.

[6] H.W. Davis H.W., F.J. Murray F.J., Weber J.K. Families of $L_p$— spaces with inductive and projective topologies. Pacific J. Math. v. 34, (1970), 619-638.

[7] Fino A., Karch G. Decay of mass for nonlinear Equations with fractional Laplasian. Electronic Publications, arXiv:0812.4977v1 [math.AP] 29 Dec 2008.

[8] Ginibre J., Velo G. Generalized Strichartz inequalities for the wave equation. J. Funct. Anal. 133(1995), no 1, 50 - 68.

[9] Fiorenza A. Duality and reflexivity in grand Lebesgue spaces. Collectanea Mathematica (electronic version), 51, 2, (2000), 131 - 148.

[10] Fiorenza A. and Karadzhov G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine”, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

[11] Iwaniec T. and Sbordone C. On the integrability of the Jacobian under minimal hypotheses. Arch. Rat. Mech. Anal., 119, (1992), 129 - 143.

[12] Iwaniec T., Koskela P. and Onninen J. Mapping of finite distortion: Monotonicity and Continuity, Invent. Math. 144 (2001), 507 - 531.

[13] Jawerth B., Milman M. Extrapolation theory with applications. Mem. Amer. Math. Soc. 440 (1991).

[14] Kapitanski L.V. Some generalizations of the Strichartz - Brenner Inequality. Algebra i Analiz 1 (1990), no 3, 127 - 159; translation in Leningrad Math. J., 1989, no 3, 693 - 726 (in Russian).

[15] M.A.Krasnoselsky, Ya.B.Rutisky. Convex functions and Orlicz’s Spaces. P. Noordhoff LTD, The Netherland, Groningen, 1961.

[16] Kozatchenko Yu.V., Ostrovsky E.I. Banach spaces of random variables of subgaussian type. Theory Probab. And Math. Stat., Kiev, (1985), p. 42 - 56 (in Russian).
[17] Krein S.G., Petunin Yu.V., and Semenov E.M. *Interpolation of Linear operators*. New York, AMS, (1982);

[18] Ledoux M., Talagrand M. (1991) *Probability in Banach Spaces. Springer*, Berlin, MR 1102015, (1991).

[19] Ostrovsky E. *Exponential Orlicz’s spaces: new norms and applications*. Electronic Publications, arXiv/FA/0406534, v.1, (25.06.2004.)

[20] Ostrovsky E.I. *Exponential Estimations for Random Fields*. Moscow - Obninsk, OINPE, (1999) (in Russian);

[21] Ostrovsky E., Sirota L. *Some new rearrangement invariant spaces: theory and applications*. Electronic publications: arXiv:math.FA/0605732 v1, 29, (May 2006);

[22] Ostrovsky E., Sirota L. *Fourier Transforms in Exponential Rearrangement Invariant Spaces*. Electronic publications: arXiv:math.FA/040639, v1, (20.6.2004.)

[23] Ostrovsky E., Sirota L. *Moment Banach spaces: Theory and applications*. HAIT Journal of Science and Engineering, C, Holon, ISRAEL, (2007), V. 4, Issues 1 - 2, pp. 233 - 262.

[24] Ostrovsky E., Regover E. *Strichartz - type Inequalities for Parabolic and Schrödinger Equations in rearrangement invariant Spaces*. Electronic Publications, arXiv:0901.2715 v1 [math.AP] 18 Jan 2009.

[25] Ostrovsky E., Sirota L. *Nikolskii - type inequalities for rearrangement Invariant spaces*. Electronic Publications, arXiv:0804.2311 v1 [math.FA] 15 Apr 2008.

[26] Rao M.M, Ren Z.D. *Theory of Orlicz Spaces*. Basel - New York, Marcel Decker, (1991);

[27] Rao M.M, Ren Z.D. *Application of Orlicz Spaces*. Basel - New York, Marcel Decker, (2002);

[28] Rao M.M. *Measure Theory and Integration*. Basel - New York, John Wiley, Marcel Decker, second Edition, (2004);
[29] Jean van Schaftingen. *Explicit Approximation of the symmetric Rearrangement by polarization*. Electronic Publications, arXiv:0902.0637v1 [math.FA] 3 Feb 2009.

[30] Steigenwalt M.S. and While A.J. *Some function spaces related to $L_p$*. Proc. London Math. Soc, 22, (1971), 137-163;

[31] Talenti M. *Inequalities in rearrangement invariant function spaces*. Nonlinear Analysis, function spaces and applications. Vol. 5 (Prague, 1994), Prometheus, Prague, 1994, pp. 177 - 230.