TOPOLOGY OF THE MILNOR FIBRATIONS OF POLAR WEIGHTED HOMOGENEOUS POLYNOMIALS

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Abstract. Let $P$ be a 2-variable polar weighted homogeneous polynomial and let $F_t$ be a deformation of $P$ which is also a polar weighted homogeneous polynomial. If $F_t$ is a Morse function on the orbit space of the $S^1$-action, then the handle decomposition obtained by this Morse function induces a round handle decomposition of the Milnor fibration of $F_t$. In the present paper, we describe a round handle decomposition of the Milnor fibration of $F_t$ concretely and give the number of round handles by the number of positive and negative components of the links of singularities appearing before and after the deformation. We also give a formula of characteristic polynomials of these singularities by using the decomposition of the monodromy of the Milnor fibration induced by a round handle decomposition.

1. Introduction

We consider a polynomial of complex variables $z = (z_1, \ldots, z_n)$ which is given by

$$P(z, \bar{z}) := \sum_{i=1}^{m} c_i z^{\nu_i} \bar{z}^{\mu_i},$$

where $c_i \in \mathbb{C}^*$, $z^n = z_1^{\nu_1} \cdots z_n^{\nu_n}$ and $\bar{z}^{\mu_i} = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\nu_i = (\nu_{i,1}, \ldots, \nu_{i,n})$ and $\mu_i = (\mu_{i,1}, \ldots, \mu_{i,n})$ respectively. Here $\bar{z}^{\mu_i}$ represents the complex conjugate of $z^{\mu_i} = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$. We call $P(z, \bar{z})$ a mixed polynomial of complex variables $z = (z_1, \ldots, z_n)$. A point $w \in \mathbb{C}^n$ is called a mixed singular point of $P(z, \bar{z})$ if the gradient vectors of $\Re P$ and $\Im P$ are linearly dependent at $w \ [14, 15]$. Suppose that $P(0, \ldots, 0) = 0$ and $P$ has an isolated singularity at the origin. There exist positive real numbers $\varepsilon$ and $\delta$ with $\delta \ll \varepsilon \ll 1$ such that the map

$$P : D_{\varepsilon}^2 \cap P^{-1}(\partial D_{\delta}^2) \to \partial D_{\delta}^2$$

is a locally trivial fibration over $\partial D_{\delta}^2$, where $D_{\varepsilon}^2 = \{ z \in \mathbb{C}^n \mid \| z \| \leq \varepsilon \}$ and $D_{\delta}^2 = \{ \eta \in \mathbb{C} \mid \| \eta \| \leq \delta \}$. This map is called the Milnor fibration of $P$ at the origin.

Ruas, Seade and Verjovsky [16] and Cisneros-Molina [2] introduced the following classes of mixed polynomials. Let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be integers such that $\gcd(p_1, \ldots, p_n) = \gcd(q_1, \ldots, q_n) = 1$. We define the $S^1$-action and the $\mathbb{R}^*$-action on $\mathbb{C}^n$ as follows:

$$s \circ z = (s^{p_1} z_1, \ldots, s^{p_n} z_n), \quad r \circ z = (r^{q_1} z_1, \ldots, r^{q_n} z_n),$$

where $s \in S^1$ and $r \in \mathbb{R}^*$. A mixed polynomial $P(z, \bar{z})$ is called a polar weighted homogeneous polynomial if there exists a positive integer $d_p$ such that $P(z, \bar{z})$ satisfies

$$P(s^{p_1} z_1, \ldots, s^{p_n} z_n, \bar{s}^{p_1} \bar{z}_1, \ldots, \bar{s}^{p_n} \bar{z}_n) = s^{d_p} P(z, \bar{z}), \quad s \in S^1.$$

The weight vector $(p_1, \ldots, p_n)$ is called the polar weight and $d_p$ is called the polar degree. Similarly $P(z, \bar{z})$ is called a radial weighted homogeneous polynomial if there exists a positive integer $d_r$ such that

$$P(r^{q_1} z_1, \ldots, r^{q_n} z_n, r^{q_1} \bar{z}_1, \ldots, r^{q_n} \bar{z}_n) = r^{d_r} P(z, \bar{z}), \quad r \in \mathbb{R}^*.$$

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The integer \( d_t \) is called the \textit{radial degree}. If \( P \) is polar and radial weighted homogeneous, \( P \) admits the global Milnor fibration \( P : \mathbb{C}^n \setminus P^{-1}(0) \to \mathbb{C}^* \) and the monodromy of the Milnor fibration is given by the \( S^1 \)-action, see for instance [14] [15].

We study the topology of the Milnor fibration of a mixed polynomial \( P \) by using a deformation of \( P \). Here a deformation of \( P \) is a polynomial map \( F : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}, (z,t) \mapsto F_t(z) \), with \( F_0(z) = P(z) \). A deformation of \( P \) is useful to study the Milnor fibration of \( P \). For complex isolated singularities, it is known that there exist a neighborhood \( U \) of the origin of \( \mathbb{C}^n \) and a deformation \( F_t \) of a complex polynomial \( P(z) \) such that \( F_t(z) \) is also a complex polynomial and any singularity of \( F_t(z) \) in \( U \) is a Morse singularity for any \( 0 < t \ll 1 \), see for instance [9] Chapter 4. A sufficiently small compact neighborhood of each Morse singularity can be regarded as a \( 2n \)-dimensional \( n \)-handle and thus we have a decomposition

\[
D^{2n}_\varepsilon \cap F_t^{-1}(D^3_\varepsilon) \cong (D^{2n}_\varepsilon \cap F_t^{-1}(D^3_\delta)) \cup_{\varphi} (\bigcup_{i=1}^\ell (n\text{-handle})_i),
\]

where \( \ell \) is the Milnor number of the singularity of \( P \) at \( (0,\ldots,0) \), \( \varphi = (\varphi_1,\ldots,\varphi_\ell) \) is the \( \ell \)-tuple of the attaching map \( \varphi_i \) of \((n\text{-handle})_i \) and \( D^3_\delta \) is a 2-disk centered at 0 with radius \( \delta \). Note that the framing of each handle attaching is determined by the vanishing cycle of the corresponding Morse singularity [7]. Such a decomposition induces a decomposition of the monodromy of the Milnor fibration into those of the Morse singularities.

In this paper, we observe analogous deformations for mixed singularities. Let \( P \) be a 2-variable polar and radial weighted homogeneous polynomial which has an isolated singularity at the origin of \( \mathbb{C}^2 \) and let \( F_t \) be a deformation of \( P \). Set \( S_k(F_t) = \{ z \in U \mid \text{rank } dF_t(z) = 2 - k \} \) for \( k = 1, 2 \). We assume that \( F_t \) satisfies the following properties:

1. \( F_t \) is polar weighted homogeneous for any \( 0 \leq t \ll 1 \), which implies that, for each \( 0 < t \ll 1 \), the singular set \( S_1(F_t) \cup S_2(F_t) \) consists of the union of a finite number of orbits of the \( S^1 \)-action [6] Proposition 2 and \( F_t(S_1(F_t)) \) consists of circles centered at 0 except the origin;

2. For each point \( w \in S_1(F_t) \), there exist local coordinates \( (x_1,x_2,x_3,x_4) \) centered at \( w \) such that \( F_t \) is given by

\[
(F_t/|F_t|, |F_t|) = (x_1,-x_2^2 + x_3^2 + x_4^2 + c_{t,w}),
\]

where \( c_{t,w} = |F_t(w)| \) for \( w \in S_1(F_t) \) and \( 0 < t \ll 1 \). In particular, \( S_1(F_t) \) consists of indefinite fold singularities;

3. \( S_2(F_t) = \{ \emptyset \} \) or \( \emptyset \).

In [6], we focused on the mixed singularity of \( fg \), where \( f \) and \( g \) are 2-variable weighted homogeneous complex polynomials which have no common branches, and showed that there exists a deformation \( F_t \) of \( fg \) such that \( F_t \) satisfies the above conditions. It is important to notice that there exist polar weighted homogeneous polynomials which do not admit deformations into smooth maps with only Morse singularities, see [4] Theorem 1, [5] Corollary 1 and Corollary 2.

By the condition (2), the absolute value \( |F_t| \) of \( F_t \) defines a Morse function on the orbit space \( (D^3_\varepsilon \cap F_t^{-1}(D^3_\delta))/S^1 \) of the \( S^1 \)-action for \( t > 0 \) and outside the image of the origin, and the indices of the Morse singularities are always 1. Then the handle decomposition of the orbit space according to this Morse function induces a decomposition of \( D^4_\varepsilon \cap F_t^{-1}(D^3_\delta) \) into a tubular neighborhood of a singular fiber over the origin and a finite number of round 1-handles, that is, we have

\[
D^4_\varepsilon \cap F_t^{-1}(D^3_\delta) \cong (D^4_\varepsilon \cap F_t^{-1}(D^3_\delta)) \cup_{\varphi} (\bigcup_{i=1}^\ell (\text{round 1-handle})_i)
\]

where \( \ell \) is the number of the singularities of the Morse function on the orbit space outside the origin and \( \varphi = (\varphi_1,\ldots,\varphi_\ell) \) is the attaching map of \( \ell \) copies of a round 1-handle. Here we may assume that the images of \( \varphi_1,\ldots,\varphi_\ell \) in \( \partial(D^4_\varepsilon \cap F_t^{-1}(D^3_\delta)) \) are disjoint.
In this paper, we describe the structure of this decomposition more precisely. To state our theorem, we introduce the notion of negative link components. Let $P$ be a polar weighted homogeneous polynomial. Then the link of $P$ at the origin is a Seifert link, denoted by $L(P,o)$. A fiber surface of the Seifert link induces an orientation to each link component canonically. A connected component of $L(P,o)$ is called a positive component if the orientation of the link component coincides with that of the $S^1$-action, and otherwise it is called a negative component. Let $|L^+(P,o)|$ and $|L^-(P,o)|$ denote the number of positive components of $L(P,o)$ and the number of negative components of $L(P,o)$, respectively. Then the decomposition is given as follows:

**Theorem 1.** Let $P$ be a 2-variable polar and radial weighted homogeneous polynomial which has an isolated singularity at the origin and let $F_t$ be a deformation of $P$ satisfying the conditions (1), (2) and (3). Then

(i) $D^4 \cap F_t^{-1}(D^2_{\delta t})$ is diffeomorphic to the disjoint union of a 4-ball and $\ell$ copies of $S^1 \times B^3$, where $B^3$ is a 3-ball, and each $\varphi_i$ of the attaching map $\varphi = (\varphi_1, \ldots, \varphi_\ell)$ maps the two attaching regions of the $i$-th round 1-handle to both of the boundary of the 4-ball and that of the $i$-th $S^1 \times B^3$, and these $\ell + 1$ connected components are connected by $\ell$ round 1-handles attached by the map $\varphi$; and

(ii) the number $\ell$ of round 1-handles in the decomposition $\{0,1\}$ is given as

$$\ell = |L^+(P,o)| - |L^+(F_t,o)| = |L^-(P,o)| - |L^-(F_t,o)|$$

for $0 < t \ll 1$.

As we mentioned, in [6], a deformation of a mixed singularity of type $(f\tilde{g},o)$ is given explicitly. In that case, the number $\ell$ is determined by the polar degree and the radial degree of $P$ as $\ell = \frac{d_r - d_p}{2pq}$. From the decomposition in $\{0,1\}$, we can observe that the Milnor fiber of $(P,o)$ is obtained from the Milnor fiber of $(F_t,o)$ by removing $2d_p$ disks from two connected components of $D^4 \cap F_t^{-1}(D^2_{\delta t})$ and gluing these boundary circles by $d_p$ annuli for each $i = 1, \ldots, \ell$. Moreover, we see that monodromy exchanges these $\ell$ copies of the union of $d_p$ annuli and $2d_p$ disks cyclically.

This paper is organized as follows. In Section 2 we give the definitions of fold singularities and round handles and introduce deformations of polar weighted homogeneous polynomials. In Section 3 we prove Theorem 1. In Section 4 we make a few comments on the monodromy of the Milnor fibration of $F_t$ and a specific deformation of $f\tilde{g}$.

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## 2. Preliminaries

### 2.1. Fold singularities

Let $X$ be a $n$-dimensional manifold and $W$ be a 2-dimensional manifold. We denote $C^\infty(X,W)$ the set of smooth maps from $X$ to $W$. It is known that the subset of smooth maps from $X$ to $W$ whose singularities are only definite fold singularities, indefinite fold singularities or cusps is open and dense in $C^\infty(X,W)$ topologized with the $C^\infty$-topology [8]. Here a fold singularity is the singularity of the following map

$$(x_1, \ldots, x_n) \mapsto (x_1, \sum_{j=2}^{n} \pm x_j^2),$$

where $(x_1, \ldots, x_n)$ are coordinates of a small neighborhood of the singularity. If the coefficients of $x_j$ for $j = 2, \ldots, n$ is either all positive or all negative, we say that $x$ is a definite fold singularity, otherwise it is an indefinite fold singularity.
2.2. Deformations of polar weighted homogeneous polynomials. Let $P : \mathbb{C}^2 \to \mathbb{C}$ be a polar weighted homogeneous polynomial map which has an isolated singularity at the origin of $\mathbb{C}^2$. Then $P$ admits a Milnor fibration, i.e., there exist positive real numbers $\varepsilon$ and $\delta$ such that the map

$$P : D^4_\varepsilon \cap P^{-1}(\partial D^3_\delta) \to \partial D^2_\delta$$

is a locally trivial fibration over $\partial D^2_\delta$, where $D^4_\varepsilon = \{ z \in \mathbb{C}^2 \mid \|z\| \leq \varepsilon \}$ and $D^2_\delta = \{ \eta \in \mathbb{C} \mid |\eta| \leq \delta \}$. We fix such positive real numbers $\varepsilon$ and $\delta$. Let $F$ be the fiber surface of a polar weighted homogeneous polynomial $P$. In [14][15], the monodromy $h : F \to F$ of the Milnor fibration of $P$ is given by

$$(z_1, z_2) \mapsto \left( \exp \left( \frac{2p\pi i}{d_p} \right) z_1, \exp \left( \frac{2q\pi i}{d_q} \right) z_2 \right),$$

where $(p, q)$ is the polar weight of $P$. Note that the link $K_P = \partial D^4_\varepsilon \cap P^{-1}(0)$ is an invariant set of the $S^1$-action. So the link $K_P$ is a Seifert link in the 3-sphere [3].

Let $F_t$ be a deformation of $P$ which satisfies the conditions (1), (2) and (3). Since the fiber surface $F_t^{-1}(c)$ intersects $\partial D^4_\varepsilon$ transversely, $F_t^{-1}(c)$ intersects $\partial D^4_\varepsilon$ transversely for each $c \in D^2_\delta$ and $0 \leq t \ll 1$. By the Ehresmann’s fibration theorem [17], the map

$$F_t : D^4_\varepsilon \cap F_t^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$$

is a locally trivial fibration for $0 \leq t \ll 1$. The polar weight of $F_t$ coincides with that of $F_0$ for $0 < t \ll 1$. Thus the monodromy of $F_t : D^4_\varepsilon \cap F_t^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is given by the same $S^1$-action on $\mathbb{C}^2$ for each $0 \leq t \ll 1$.

**Lemma 1.** The Milnor fibration $F_t : D^4_\varepsilon \cap F_t^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is isomorphic to the fibration $F_0/|F_0| : \partial D^4_\varepsilon \setminus \text{Int}(K_{F_0}) \to S^1$ for $0 \leq t \ll 1$, where $N(K_{F_0}) = \{ z \in \partial D^4_\varepsilon \mid |F_0(z)| \leq \delta \}$.

**Proof.** Since the fiber surface $F_0^{-1}(c)$ is transversal to $\partial D^4_\varepsilon$, $F_t^{-1}(c)$ is transversal to $\partial D^4_\varepsilon$ for any $c \in D^2_\delta$ and $0 \leq t \ll 1$. Fix such a positive real number $t$. We set

$$\partial E(\delta, \varepsilon) := \{ (z, t') \in \mathbb{C}^2 \times [0, t] \mid |F(t')(z)| = \delta, \|z\| \leq \varepsilon \}$$

$$\partial^2 E(\delta, \varepsilon) := \{ (z, t') \in \mathbb{C}^2 \times [0, t] \mid |F(t')(z)| = \delta, \|z\| = \varepsilon \}.$$

Then the projection

$$\pi' : (\partial E(\delta, \varepsilon), \partial^2 E(\delta, \varepsilon)) \to [0, t], \quad (z, t') \mapsto t'$$

is a proper submersion. By the Ehresmann’s fibration theorem [17], $\pi'$ is a locally trivial fibration over $[0, t]$. Thus the projection $\pi'$ induces a family of isomorphisms $\psi_{t'} : \partial E_0(\delta, \varepsilon) \to \partial E_{t'}(\delta, \varepsilon)$ of fibrations such that the following diagram is commutative for $0 \leq t' \leq t$:

$$\begin{array}{ccc}
\partial E_0(\delta, \varepsilon) & \xrightarrow{\psi_{t'}} & \partial E_{t'}(\delta, \varepsilon) \\
\downarrow F_0 & & \downarrow F_{t'} \\
S^1_\delta & = & S^1_{\delta_{t'}}
\end{array}$$

where $\partial E_{t'}(\delta, \varepsilon) = \{ z \in \mathbb{C}^2 \mid |F(t')(z)| = \delta, \|z\| \leq \varepsilon \}$. Thus the two fibrations $F_{t'} : D^4_\varepsilon \cap F_{t'}^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ and $F_0 : D^4_\varepsilon \cap F_0^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ are isomorphic for $0 \leq t' \leq t$. By [15] Theorem 37, the two fibrations

$$F_0 : \partial E_0(\delta, \varepsilon) \to S^1_\delta, \quad F_0/|F_0| : S^3_\varepsilon \setminus \text{Int}(K_{F_0}) \to S^1$$

are isomorphic for $\varepsilon > 0$, $\delta > 0$ and $\delta \ll \varepsilon$. Thus $F_1 : D^4_\varepsilon \cap F_1^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is isomorphic to $F_0/|F_0| : \partial D^4_\varepsilon \setminus \text{Int}(K_{F_0}) \to S^1$ for $0 \leq t \ll 1$. \hfill $\Box$

**Lemma 2.** The orbit space of $D^4_\varepsilon \cap F_0^{-1}(\partial D^2_\delta)$ of the $S^1$-action is homeomorphic to a holed 2-sphere for $0 \leq t \ll 1$. 


Proof. The monodromy of \( F_t : D^4_t \cap F_t^{-1}(\partial D^2_s) \to \partial D^2_s \) is given by the same \( S^1 \)-action on \( \mathbb{C}^2 \) for each \( 0 \leq t \ll 1 \). By Lemma 1, \( (D^4_t \cap F_t^{-1}(\partial D^2_s))/S^1 \) is homeomorphic to \( (\partial D^2_s \setminus \text{Int}(N(K_{F_0}))) / S^1 \).

Since the orbit space \( \partial D^4_s / S^1 \) is homeomorphic to a 2-sphere and \( K_{F_0} \) is an invariant set of the \( S^1 \)-action, the orbit space \( (\partial D^2_s \cap F_t^{-1}(\partial D^2_s))/S^1 \) is a holed 2-sphere. \( \square \)

2.3. **Round handles.** Let \( X \) and \( Y \) be \( n \)-dimensional smooth manifolds. According to [1][11], we say that \( X \) is obtained from \( Y \) by attaching a round \( k \)-handle if

1. There are disk bundles over \( S^1 \), \( E^k_s \) and \( E^{n-k-1}_u \),

2. There exists an embedding \( \varphi : \partial E^k_s \times S^1 \to Y \) such that \( X \cong Y \cup_{\varphi} E^k_s \oplus E^{n-k-1}_u \), where \( E^k_s \oplus E^{n-k-1}_u \) is the Whitney sum of \( E^k_s \) and \( E^{n-k-1}_u \) over \( S^1 \). The bundle \( E^k_s \oplus E^{n-k-1}_u \) over \( S^1 \) is called an \( n \)-dimensional round \( k \)-handle and \( \varphi \) is called the attaching map of \( E^k_s \oplus E^{n-k-1}_u \).

Note that a sufficiently small compact neighborhood of a connected component of the set of fold singularities can be regarded as an \( n \)-dimensional round handle. In our case, a sufficiently small compact neighborhood of each connected component of \( S_1(F_t) \) is regarded as a 4-dimensional round 1-handle.

3. **Proof of Theorem**

3.1. **Round 1-handles determined by \( S_1(F_t) \).** By the condition (3), the origin \( o \) is an isolated singularity of \( F_t \). There exist positive real numbers \( \varepsilon_t \) and \( \delta_t \) such that \( \delta_t \ll \varepsilon_t \) and the map

\[ F_t : D^4_{\varepsilon_t} \cap F_t^{-1}(\partial D^2_{\delta_t}) \to \partial D^2_{\delta_t} \]

is a locally trivial fibration over \( \partial D^2_{\delta_t} \), where \( D^4_{\varepsilon_t} = \{ z \in \mathbb{C}^2 \mid \|z\| \leq \varepsilon_t \} \) and \( D^2_{\delta_t} = \{ \eta \in \mathbb{C} \mid |\eta| \leq \delta_t \} \) for \( 0 < \varepsilon_t' \leq \varepsilon_t, 0 < \delta_t' \leq \delta_t \) and \( \delta_t' \ll \varepsilon_t' \), see [17]. Thus \( F_t^{-1}(c) \) intersects \( \partial D^2_{\delta_t} \) transversely for any \( c \in D^2_{\delta_t} \) and \( 0 \leq t \ll 1 \). We assume that \( \varepsilon_t \) and \( \delta_t \) satisfy the following properties:

\[ D^4_{\varepsilon_t} \cap S_1(F_t) = \emptyset, \quad D^2_{\delta_t} \cap F_t(S_1(F_t)) = \emptyset. \]

See Figure [1]. Put \( M_0 = D^4_{\varepsilon_t} \cap F_t^{-1}(D^2_{\delta_t}) \).

![Figure 1. \( D^2_{\delta_t} \) and \( F_t(S_1(F_t)) \)](image)

Fix \( t \) with \( 0 < t \ll 1 \) and let \( \ell \) be the number of singularities of \( |F_t| \). Note that \( |F_t| \) is a Morse function by the conditions (1) and (2) except the origin. Let \( C_1, \ldots, C_\ell \) be the connected components of \( S_1(F_t) \), where the number of the connected components of \( S_1(F_t) \) is \( \ell \) because of the conditions (1), (2) and (3). We may assume that \( |F_t| \) satisfies

\[ |F_t(c_1)| \leq |F_t(c_2)| \cdots \leq |F_t(c_\ell)| \]
for \(c_i \in C_i\) and \(i = 1, \ldots, \ell\). Let \(c_i'\) be the singularity of \(|F_i|\) corresponding to \(C_i\) and \(N_i'\) be a sufficiently small compact neighborhood of \(c_i'\) for \(i = 1, \ldots, \ell\). Since \(|F_i|\) is a Morse function, each \(N_i',\ i = 1, \ldots, \ell\), can be regarded as a 3-dimensional 1-handle \([-1, 1] \times D_i^2\), where \(D_i^2\) is a 2-disk. We set \(M_0^i = M_0^i/S^1\) and \(M_i' = M_i' \cup \varphi_i N_i'\), where \(\varphi_i : \{\pm 1\} \times D_i^2 \to \partial M_i'\) is the attaching map of \(N_i'\) for \(i = 1, \ldots, \ell\). We may assume that \(\varphi_i(\{\pm 1\} \times D_i^2) \subset \partial M_0^i\) for \(i = 1, \ldots, \ell\). Then the orbit space \(D_{i_0}/S^1\) is a 2-disk. Let \(\varphi_i'(\{\pm 1\} \times D_i^2) \not\subset \partial M_0^i\) for \(i = 1, \ldots, \ell\).

**Lemma 3.** Let \(M_0^i\) be a connected component of \(M_0^i\). Then \(\varphi_i'(\{\pm 1\} \times D_i^2) \not\subset \partial M_0^i\) for \(i = 1, \ldots, \ell\).

**Proof.** Assume that there exist \(i \in \{1, \ldots, \ell\}\) and a connected component \(M_0^i\) of \(M_0^i\) such that \(\varphi_i'(\{\pm 1\} \times D_i^2) \subset \partial M_0^i\). Then the genus of \(\partial M_0^i\) is greater than 0. After attaching 1-handles, the genus of the boundary of the orbit space does not decrease. Thus the genus of \(\partial M_0^i\) is greater than 0. As Lemma 2 the genus of \(\partial M_0^i\) is equal to 0. This is a contradiction. \(\square\)

Let \(M_i\) and \(N_i\) be 4-dimensional manifolds such that \(M_i/S^1 = M_i'\) and \(N_i/S^1 = N_i'\) respectively for \(i = 1, \ldots, \ell\). Then \(N_i\) can be regarded as a 4-dimensional round 1-handle and \(M_i\) is a manifold obtained from \(M_{i-1}\) by attaching \(N_i\) for \(i = 1, \ldots, \ell\). By Lemma 3, \(N_i\) connects two connected components of \(M_0\). Note that \(M_i\) is diffeomorphic to \(D_{i_0}^4 \cap F_i^{-1}(D_{i_0}^2)\).

**3.2. The fiber surface of \(F_i : D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2) \to \partial D_{i_0}^2\).** We consider the restricted Milnor fibration \(F_i : D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2) \to \partial D_{i_0}^2\) and connected components of \(M_0\).

**Lemma 4.** Let \(S_0\) be the fiber surface of \(F_i : D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2) \to \partial D_{i_0}^2\). Then \(S_0\) is diffeomorphic to the disjoint union of the fiber surface of \(F_i : D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2)\) and \(\ell\) copies of an annulus, where \(\ell\) is the number of connected components of \(S_1(F_i)\).

**Proof.** Let \(M_0^0, M_1^0, \ldots, M_k^0\) denote the connected components of \(M_0\) such that \(o \in M_0^0\). Then \(M_0^0 \cap D_{i_0}^4 \neq \emptyset\). The restricted map \(F_i : D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2) \to \partial D_{i_0}^2\) has a unique singularity at the origin \(o\) of \(\mathbb{C}^2\). By Lemma 11.3, \(D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2)\) is homeomorphic to \(\partial D_{i_0}^4 \setminus \text{Int}(N(K_F_i))\), where \(N(K_F_v) = \{z \in \partial D_{i_0}^4 \mid |F_v(z)| \leq \delta_v\}\). So any fiber surface of \(F_i : D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2) \to \partial D_{i_0}^2\) is connected. The boundary of the orbit space \((D_{i_0}^4 \cap M_0^0)/S^1\) is homeomorphic to a 2-sphere and

\[
M_0^i \cap D_{i_0}^4 = \emptyset
\]

for \(j = 1, \ldots, k\).

Let \(S_0^0\) be a fiber surface of \(F_i : M_0^0 \cap (D_{i_0}^4 \cap F_i^{-1}(\partial D_{i_0}^2))\). We divide the surface \(S_0^0\) as follows:

\[
S_0^0 = (S_0^0 \cap D_{i_0}^4) \cup (S_0^0 \cap (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4)).
\]

Since \(F_i : (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4) \cap F_i^{-1}(\partial D_{i_0}^2) \to D_{i_0}^2\) has no singularities and \(F_i^{-1}(c)\) intersects \(\partial D_{i_0}^4 \cup \partial D_{i_0}^2\) transversely for any \(c \in D_{i_0}^2, F_i^{-1}(c) \cap (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4)\) is diffeomorphic to \(F_i^{-1}(0) \cap (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4)\). Note that \(F_i^{-1}(0)\) is an invariant set of the \(S^1\)-action and \(F_i^{-1}(0)/S^1\) is a 1-dimensional algebraic set. The orbit space \(F_i^{-1}(0)/S^1\) is diffeomorphic to \([0, 1]\). Thus the connected component of \(F_i^{-1}(c) \cap (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4)\) is diffeomorphic to an annulus. So any connected component of \(S_0^0 \cap (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4)\) is an annulus. Since any fiber of \(F_i\) intersects \(\partial D_{i_0}^4\) transversely, \(S_0^0 \cap \partial D_{i_0}^4\) consists of circles and \(S_0^0 \cap (D_{i_0}^4 \setminus \text{Int} D_{i_0}^4)\) is diffeomorphic to \((S_0^0 \cap \partial D_{i_0}^4) \times [0, 1]\). So we have

\[
S_0^0 = (S_0^0 \cap D_{i_0}^4) \cup ((S_0^0 \cap \partial D_{i_0}^4) \times [0, 1])
\]

\[
S_0^0 \cap D_{i_0}^4.
\]
We consider $M^j_i$ for $j = 1, \ldots, k$. The restricted map $F_t : M^j_i \to D^2_\delta$ has no singularities. For any $c \in D^2_\delta \setminus \{0\}$ and $j = 1, \ldots, k$, $F_t^{-1}(c) \cap M^j_i$ is diffeomorphic to $F_t^{-1}(0) \cap M^j_i$. Since $F_t^{-1}(0)$ is an invariant set of the $S^1$-action, the orbit space $F_t^{-1}(0)/S^1$ is a 1-dimensional algebraic set. So $F_t^{-1}(0)/S^1$ is diffeomorphic to $[0,1]$ or $S^1$. Assume that $F_t^{-1}(0)/S^1 = S^1$. Then $F_t^{-1}(c)$ is a torus and the orbit space $F_t^{-1}(c)/S^1$ is also a torus. Since the boundary of $M^j_i$ is a 2-sphere, this is a contradiction. Let $S^2_0$ denote the fiber surface of $F_t|_{M^j_i}$. Then $S^2_0/S^1$ is diffeomorphic to $[0,1]$ and the fiber surface $S^2_0$ is diffeomorphic to an annulus for $j = 1, \ldots, k$.

By Lemma 5 each $N_i$ connects two connected components of $M_0$. Since $M_\ell$ is connected, we have $k + 1 - \ell = 1$. Thus the number of connected components of $M_0$ other than that of $F_t : D^4_\delta \cap F^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is equal to $\ell$.

**Lemma 5.** The connected component $M^0_0$ of $M_0$ is diffeomorphic to a 4-ball and $M^j_i$ is diffeomorphic to $S^1 \times B^3$, where $B^3$ is a 3-ball, for $j = 1, \ldots, \ell$.

**Proof.** The two fibrations $F_t : D^4_\delta \cap F^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ and $F_t/|F_t| : \partial D^4_\delta \cap \partial D^4_\delta \to S^1$ are isomorphic for any $0 < \delta' \leq \delta_0$. Thus $M^0_0$ is diffeomorphic to a 4-ball.

The map $F_t : M^j_i \cap F^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is diffeomorphic to $S^2_0$ for $0 < \delta' \leq \delta_0$. Since the monodromy of $F_t : M^j_i \cap F^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is given by the $S^1$-action, the orbit space $(M^j_i \cap F_t^{-1}(\partial D^2_\delta) \setminus \{0\})/S^1$ is homeomorphic to $(S^2_0/S^1) \times [0,1]$. By Lemma 4 $S^2_0/S^1$ is also an annulus. We identify $S^2_0/S^1$ with $S^1 \times [0,1]$. Since $S^3 \setminus \{0\}$ is diffeomorphic to $D^2 \setminus \{0\}$, $(M^j_i \cap F_t^{-1}(\partial D^2_\delta) \setminus \{0\})/S^1$ is homeomorphic to $D^2 \setminus \{0\} \times [0,1]$, where $D^2$ is a 2-disk centered at 0. Since $F_t^{-1}(0)$ is an invariant set of the $S^1$-action, the orbit space $F_t^{-1}(0)$ is homeomorphic to $\{0\} \times [0,1]$. Thus the orbit space of $M^j_i$ is homeomorphic to $D^2 \times [0,1]$. The manifold $M^j_i$ is diffeomorphic to $S^1 \times B^3$.  

### 3.3. The number of connected components of $S_1(F_t)$. To complete the proof of Theorem 3 it is enough to show the equality in Theorem 3 (ii). We set $M_0 = D^2_\delta \cap F_t^{-1}(\partial D^2_\delta)$ and $M_\ell = M_{\ell-1} \cup_{\partial N_i} \partial N_i$ for $i = 1, \ldots, \ell$. Since $F_t$ is a 3-ball, for $j = 1, \ldots, \ell$. Thus the number of connected components of $M_0$ other than that of $F_t : D^2_\delta \cap F_t^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is equal to $\ell$.

The map $F_t : M^j_i \cap F_t^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is diffeomorphic to $S^2_0$ for $0 < \delta' \leq \delta_0$. Since the monodromy of $F_t : M^j_i \cap F^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$ is given by the $S^1$-action, the orbit space $(M^j_i \cap F^{-1}(\partial D^2_\delta) \setminus \{0\})/S^1$ is homeomorphic to $(S^2_0/S^1) \times [0,1]$. By Lemma 4 $S^2_0/S^1$ is also an annulus. We identify $S^2_0/S^1$ with $S^1 \times [0,1]$. Since $S^3 \setminus \{0\}$ is diffeomorphic to $D^2 \setminus \{0\}$, $(M^j_i \cap F_t^{-1}(\partial D^2_\delta) \setminus \{0\})/S^1$ is homeomorphic to $D^2 \setminus \{0\} \times [0,1]$, where $D^2$ is a 2-disk centered at 0. Since $F_t^{-1}(0)$ is an invariant set of the $S^1$-action, the orbit space $F_t^{-1}(0)$ is homeomorphic to $\{0\} \times [0,1]$. Thus the orbit space of $M^j_i$ is homeomorphic to $D^2 \times [0,1]$. The manifold $M^j_i$ is diffeomorphic to $S^1 \times B^3$.

### Lemma 6. Let $\ell$ be the number of connected components of $S_1(F_t)$. Then $\ell$ is equal to $|L^+(P, 0)| - |L^+(F_t, 0)|$ and also to $|L^-(P, 0)| - |L^-(F_t, 0)|$. 


Proof. Since the fibration $F_t \mid \tilde{M}_\ell$ is isomorphic to $P : D^2_\ell \cap P^{-1}(\partial D^3_\delta) \to \partial D^3_\delta$ and $L(P, o)$ is the Seifert link in $\partial D^4_\ell$, we have

$$\chi(S_{\ell}) = 1 - \{pq(m+n) - p - q\}(m-n),$$

where $m = |L^+(P, o)|$ and $n = |L^-(P, o)|$, see [3, Theorem 11.1]. By Lemma 4, the fiber surface $S_0$ of $F_t \mid \tilde{M}_0$ is diffeomorphic to $S_0^0 \sqcup S_1^0 \sqcup \cdots \sqcup S_0^k$ and $S_0^k$ is an annulus for $j \neq 0$. The Euler characteristic $\chi(S_0)$ of $S_0$ is equal to $\chi(S_0^0)$. Since the link $L(F_t, o)$ is also the Seifert link in a 3-sphere, $\chi(S_0^0)$ is given by

$$\chi(S_0^0) = 1 - \{pq(m'+n') - p - q\}(m' - n'),$$

where $m' = |L^+(F_t, o)|$ and $n' = |L^-(F_t, o)|$. On the other hand, $\chi(S_{\ell})$ is equal to $\chi(S_0) - 2 \ell d_p$. The polar degree $d_p$ is equal to $pq(m-n)$ and also to $pq(m'-n')$ [3]. Then we have

$$\chi(S_{\ell}) - \chi(S_0) = -\{pq(m+n) - p - q\}(m-n) + \{pq(m'+n') - p - q\}(m' - n')$$

$$= -pq(m-n)(m+n) + pq(m'-n')(m'+n')$$

$$= -d_p\{(m+n) - (m'+n')\}$$

$$= -2\ell d_p.$$

So $2\ell$ is equal to $m + n - (m' + n')$. Since $m - m'$ is equal to $n - n'$, $\ell$ is equal to $m - m'$ and also to $n - n'$. \hfill \square

We give an example of Lemma 6 which is considered in [6].

**Example 1.** Set $f(z) = z^m + z^n$ and $g(z) = z_1 + 2z_2$ where $m \geq 3$. We consider a deformation $F_t = f(z)g(z) + t(z^m \bar{z}_1 + z_1^{-m} + \gamma z_2^{-m})$ of $f(z)g(z)$ where $\gamma \in \mathbb{C}$. In [6], we take a coefficient $\gamma$ of $h(z)$ such that

$$\gamma \neq -(2\alpha f(z,1) - mg(z,1))(mz^{m-1}z^2 + (m-1)z^{-m} - \alpha'z^\alpha)$$

where $(z^m + 1)(z + 2) = \alpha'(z^{m+1} + 1)(z + 2), \alpha' \in S^1$. Then $S_1(F_t)$ is the set of indefinite fold singularities and the link $S_1^2 \cap F_t^{-1}(0)$ is a $(m-1, m-1)$-torus link, where $S_1^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = \epsilon\}, \epsilon \ll 1$. By Lemma 6, the number of connected components of $S_1(F_t)$ is equal to 1. \hfill \square

**Proof of Theorem 1.** By Lemma 3 and Lemma 5, $D^4_1 \cap F_t^{-1}(D^3_1)$ consists of a 4-ball and $\ell$-copies of $S^1 \times B^3$ and the image of the attaching map $\varphi_i$ of $i$-th round 1-handle is contained in both of the boundary of a 4-ball and that of $S^1 \times B^3$ for $i = 1, \ldots, \ell$. By Lemma 3, the number of connected components of $S_1(F_t)$ is equal to $|L^-(P, o)| - |L^-(F_t, o)|$. \hfill \square

4. **Remarks**

4.1. **Monodromy and characteristic polynomials.** Let $h : S \to S$ be a homeomorphism of a surface $S$. We define

$$\Delta_s(h) = \frac{\Delta_1(h)}{\Delta_0(h)},$$

where $\Delta_i(h)$ is the characteristic polynomial of the homological map from $H_i(S, \mathbb{Z})$ to itself induced by $h$ for $i = 0, 1$.

Let $h_i : S_i \to S_i$ be the monodromy of $F_t \mid \tilde{M}_i$ for $i = 1, \ldots, \ell$. Since $h_i$ is given by the $S^1$-action on $\mathbb{C}^2$, $h_i : S_i \to S_i$ satisfies the following conditions:

(I) $h_i(S_{i-1} \setminus (D'_1 \cup D'_2)) = S_{i-1} \setminus (D'_1 \cup D'_2)$ and $h_i |_{S_{i-1} \setminus (D'_1 \cup D'_2)} = h_{i-1} |_{S_{i-1} \setminus (D'_1 \cup D'_2)},$

(II) $h_i |_{D'_i}$ and $h_i |_{A'}$ are periodic maps which satisfy $D^2_{k,j} \to D^2_{k,j+1}$ and $A_j \to A_{j+1}$.
for \( i = 1, \ldots, \ell, j = 1, \ldots, d_p \) and \( k = 1, 2 \). Here \( D_{\bar{k},d_p+1}^2 = D_{k,1}^2 \) and \( A_{d_p+1} = A_1 \). We calculate \( \Delta_*(h_i) \) from \( \Delta_*(h_{i-1}) \) by using a round 1-handle \( N_i \).

**Lemma 7.** Let \( S_i \) be the fiber surface of \( F_i \mid_{\tilde{M}_i} \) and \( h_i : S_i \to S_i \) be the monodromy of \( F_i \mid_{\tilde{M}_i} \) for \( i = 1, \ldots, \ell \). Then the characteristic polynomial of \( h_i \) satisfies

\[
\Delta_*(h_i) = \Delta_*(h_{i-1})(t^{d_p} - 1)^2.
\]

**Proof.** Since \( S_i \) is the surface obtained from \( S_{i-1} \) by replacing \( D_1^1 \cup D_2^1 \) by \( A' \) and \( h_i \) satisfies the above properties, we have

\[
\Delta_*(h_i) = \frac{\Delta_*(h_i \mid_{S_{i-1}(D_1^1 \cup D_2^1)}) \Delta_*(h_i \mid_{A'})}{\Delta_*(h_i \mid_{\partial A'})}.
\]

By the condition (II), the monodromy matrices of \( H_0(D_k^1, \mathbb{Z}), H_i(A', \mathbb{Z}) \) and \( H_i(\partial A', \mathbb{Z}) \) are equal to

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

for \( k = 1, 2 \) and \( i = 0, 1 \). The characteristic polynomial of the above matrix is equal to \( t^{d_p} - 1 \). So \( \Delta_*(h_i \mid_{A'}) \) and \( \Delta_*(h_i \mid_{\partial A'}) \) are equal to 1. As the condition (I), we have

\[
\Delta_*(h_i \mid_{S_{i-1}(D_1^1 \cup D_2^1)}) \Delta_*(h_{i-1} \mid_{D_1^1}) \Delta_*(h_{i-1} \mid_{D_2^1}) = \Delta_*(h_{i-1}).
\]

Thus the characteristic polynomial satisfies

\[
\Delta_*(h_i) = \Delta_*(h_i \mid_{S_{i-1}(D_1^1 \cup D_2^1)}).
\]

\[
= \Delta_*(h_{i-1})(t^{d_p} - 1)^2.
\]

\[\Box\]

Since the two fibrations \( P : D_k^1 \cap F_i^{-1}(\partial D_k^2) \to \partial D_k^2 \) and \( F_i : D_k^1 \cap F_0^{-1}(\partial D_k^2) \to \partial D_k^2 \) are isomorphic, we have the following theorem.

**Theorem 2.** Let \( h \) be the monodromy of \( P : D_k^1 \cap P^{-1}(\partial D_k^2) \to \partial D_k^2 \) and \( F_i : D_k^1 \cap F_0^{-1}(\partial D_k^2) \to \partial D_k^2 \) are isomoprhic. Then \( \Delta_*(h) \) is equal to \( \Delta_*(h_0)(t^{d_p} - 1)^{2\ell} \), where \( \ell \) is the number of connected components of \( S_1(F_i) \).

**Remark 1.** The algebraic set \( P^{-1}(0) \cap \partial D_k^1 \) is a fibered Seifert link in the 3-sphere. Thus the characteristic polynomial of the monodromy of the Milnor fibration of \( P \) at the origin can also be calculated from the splice diagram [3].

### 4.2. A specific deformation of \( f \bar{g} \)

We introduce a deformation of \( f \bar{g} \) given in [6], where \( f \) and \( g \) are 2-variables convenient complex polynomials and \( f \bar{g} \) has an isolated singularity at the origin \( o \). We define the \( \mathbb{C}^* \)-action on \( \mathbb{C}^2 \) as follows:

\[
c \circ (z_1, z_2) := (c^{d}z_1, c^{p}z_2), \quad c \in \mathbb{C}^*.
\]

Assume that \( f(z) \) and \( g(z) \) satisfy

\[
f(c \circ z) = c^{pm}f(z), \quad g(c \circ z) = c^{qm}g(z), \quad m > n.
\]

Then \( f(z) \) and \( g(z) \) are weighted homogeneous polynomials. Two complex polynomials \( f(z) \) and \( g(z) \) can be written as

\[
f(z) = \prod_{j=1}^{m} (z_1^{\alpha_j} + \alpha_j z_2^{\beta_j}), \quad g(z) = \prod_{j=1}^{n} (z_1^{\gamma_j} + \gamma_j z_2^{\delta_j}), \quad \gcd(p, q) = 1,
\]
where \( \alpha_j \neq \alpha_{j'} \), \( \beta_j \neq \beta_{j'} \) \( (j \neq j') \) and \( \alpha_k \neq \beta_{k'} \) for \( 1 \leq k \leq m \) and \( 1 \leq k' \leq n \). The mixed polynomial \( f(z)g(z) \) is a polar and weighted homogeneous polynomial, i.e., \( f(z)g(z) \) satisfies that \( f(s \circ z)g(s \circ z) = s^{pq(m-n)}f(z)g(z) \) and \( f(r \circ z)g(r \circ z) = r^{pq(m-n)}f(z)g(z) \), where \( s \in S^1 \) and \( r \in \mathbb{R}^\ast \) [14]. We define a deformation of \( f(z)g(z) \) as follows:

\[
F_t(z) := f(z)g(z) + th(z),
\]

where \( 0 < t \ll 1 \) and

\[
h(z) = \begin{cases} 
\gamma_1z_1^{p(m-n)} + \gamma_2z_2^{m-n} & (g(z) \neq z_1 + \beta z_2), \\
\gamma_1z_1^{m-n} + z_1^{-m-1} + \gamma_2z_2^{-m-1} & (g(z) = z_1 + \beta z_2).
\end{cases}
\]

Then \( F_t(z) \) is also a polar weighted homogeneous polynomial with the polar degree \( pq(m-n) \).

By [6] Theorem 1, there exists \( h(z) \) such that \( F_t(z) \) satisfies the conditions (1), (2) and (3) for \( 0 < t \ll 1 \). The above deformation \( F_t \) of \( fg \) satisfies that \( |L^{-}(F_t, o)| = 0 \). By Lemma [6], the number \( \ell \) of connected components of \( S_1(F_t) \) is equal to \( n \). Since the radial degree \( d_r \) and the polar degree \( d_p \) are equal to \( pq(m+n) \) and \( pq(m-n) \) respectively, we have the following proposition.

**Proposition 1.** Let \( F_t \) be the above deformation of \( fg \). Then the number \( \ell \) of connected components of \( S_1(F_t) \) is equal to \( \frac{d_r - d_p}{2pq} \), where \( d_r \) is the radial degree of \( fg \) and \( d_p \) is the polar degree of \( fg \).

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