The Accurate Modelling of Thin 3D Fluid Flows with Inertia on Curved Substrates

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1 Introduction

Mathematical models and numerical simulations for thin-film flows of a fluid have important applications in industrial and natural processes \[12\], \[10\], \[13\], \[14\], \[3\], \[4\], \[6\]. Herein, we consider the slow motion of a thin liquid layer of an incompressible, Newtonian fluid over an arbitrary solid, stationary curved substrate. In a three dimensional and very slow lubrication flow, a model for the evolution of a film on a curved substrate is shown \[11\] to be

\[
\frac{\partial \zeta}{\partial t} \approx -\frac{1}{3} \nabla \cdot \left[ \eta^2 \zeta \nabla \tilde{\kappa} - \frac{1}{2} \eta^4 (\kappa \mathbf{I} - \mathbf{K}) \cdot \nabla \kappa \right],
\]

(1)

where \(\zeta = \eta - \frac{1}{2} \kappa \eta^2 + \frac{1}{2} k_1 k_2 \eta^3\) is proportional to the amount of fluid locally “above” the substrate; \(\tilde{\kappa}\) is the mean curvature of the free surface of the film; \(\mathbf{K}\) is the curvature tensor of the substrate; \(k_1, k_2, \text{ and } \kappa = k_1 + k_2\) are the principal curvatures and the mean curvature of the substrate respectively; and the operator \(\nabla\) is defined in a coordinate system of the curved substrate as introduced in Section 2. This model accounts for the curvature of the substrate and that of the surface of the film. However, in many applications this model of slow flow of a thin fluid film has limited usefulness; instead a model expressed in terms of both the fluid layer thickness and the lateral velocity is needed to resolve faster wave-like dynamics \[3, p110\]. Roberts \[4\] derives such a model for two dimensional flow. Here, based upon the Navier-Stokes equations for a viscous fluid, Section 3, we derive the following model.
for three dimensional flow:

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (\eta \bar{u}),$$ (2)

$$\mathcal{R} \frac{\partial \bar{u}}{\partial t} \approx -\frac{\pi^2}{4} \frac{\bar{u}}{\eta^2} + \frac{\pi^2}{12} (\nabla \kappa + \mathbf{g}_s) - (2K + \kappa I) \frac{\bar{u}}{\eta}.$$ 

where $\mathcal{R}$ is a Reynolds number of the flow; $\mathbf{g}_s$ is the component of gravity tangential to the substrate; and $\bar{u} = (\bar{u}_1, \bar{u}_2)$ is the depth-averaged average lateral velocity. The first equation is a direct consequence of the conservation of fluid. The model’s second equation incorporates inertia, viscous drag on the substrate, surface tension forcing caused by gradients of curvature, gravitational forcing, and the geometric complexity of the substrate, respectively.

The asymptotic accuracy of this model is assured by a systematic derivation based upon centre manifold theory [2] as applied in Section 4. The physical fields associated with the above model are deduced as part of the procedure; approximations of the velocity and pressure fields are recorded in Section 5. The model (2) is very general: it encompasses all substrate shapes and contains the lubrication model simply by setting $\mathcal{R}$ to zero and substituting for $\bar{u}$ in the first equation. This model describes the dynamics of a very wide range of thin fluid flows.

2 The Orthogonal Curvilinear Coordinate System

Let $\mathcal{S}$ denote the solid substrate. If $\mathcal{S}$ has no umbilical point, i.e., there is no point on $\mathcal{S}$ at which two principal curvatures coincide, then there are exactly two mutually orthogonal principal directions in the tangent plane at every point in $\mathcal{S}$ [5, Theorem 10-3]. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit vectors in these principal directions, and $\mathbf{e}_3$ the unit normal to the substrate in the side of fluid flow. These basis unit vectors determine a curvilinear orthonormal coordinate system $(x_1, x_2, y)$. Such a coordinate system is called a Darboux frame [4]. The corresponding metric coefficients of the coordinate system are

$$h_i = m_i(1 - k_iy), \quad h_3 = 1.$$ 

Let $y = \eta(t, x_1, x_2)$ describe the free surface of the fluid. As derived by Roy et al [11, eqn(37)], the mean curvature of the free-surface is

$$\bar{\kappa} = \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial x_1} \left( \frac{h_2^2 \eta_{x_1}}{A} \right) + \frac{\partial}{\partial x_2} \left( \frac{h_1^2 \eta_{x_2}}{A} \right) \right]$$

$$+ \frac{1}{A} \left[ \left( \frac{h_1^2 + \eta_{x_1}^2}{h_1} \right) \frac{m_3 k_2}{h_1} + \left( \frac{h_2^2 + \eta_{x_2}^2}{h_2} \right) \frac{m_1 k_1}{h_2} \right]$$

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where \( \tilde{h}_i = m_i(1-k_i\eta) \) are the metric coefficients evaluated on the free surface and
\[
\mathcal{A} = \sqrt{\tilde{h}_1^2\tilde{h}_2^2 + \tilde{h}_2^2\eta x_1^2 + \tilde{h}_1^2\eta x_2^2}
\]
is proportional to the free-surface area above a patch \( dx_1 \times dx_2 \) of the substrate.

We consider that the spatial derivatives of the fluid flow and the curvatures of the substrate are much smaller than the metric coefficients of the coordinate system, so that an approximation to \( \tilde{\kappa} \), as needed for (1), is
\[
\tilde{\kappa} = \nabla^2 \eta + \frac{k_1}{1-k_1\eta} + \frac{k_2}{1-k_2\eta} + \mathcal{O}(\kappa^3 + \nabla^3 \eta),
\]
where in the substrate coordinates
\[
\nabla^2 \eta = \frac{1}{m_1m_2} \left[ \frac{\partial}{\partial x_1} \left( \frac{m_2}{m_1} \frac{\partial \eta}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{m_1}{m_2} \frac{\partial \eta}{\partial x_2} \right) \right].
\]

For later use, also observe that on the free-surface, unit tangent vectors \( \tilde{t}_1 \), \( \tilde{t}_2 \) and the unit normal vector \( \tilde{n} \) are
\[
\tilde{t}_i = (\tilde{h}_i e_1 + \eta x_i e_3)/\sqrt{\tilde{h}_i^2 + \eta x_i^2},
\]
\[
\tilde{n} = \frac{-\tilde{h}_2 \eta x_1 e_1 - \tilde{h}_1 \eta x_2 e_2 + \tilde{h}_1 \tilde{h}_2 e_3}{\sqrt{(\tilde{h}_2 \eta x_1)^2 + (\tilde{h}_1 \eta x_2)^2 + (\tilde{h}_1 \tilde{h}_2)^2}}.
\]

### 3 Equations of Motion and Boundary Conditions

Consider the Navier-Stokes equations for an incompressible fluid flow moving with velocity field \( \mathbf{u} = (u_1, u_2, v) \) and pressure field \( p \). The flow dynamics are driven by pressure gradients along the substrate and caused by both surface tension forces, coefficient \( \sigma \), varying due to variations of the curvature of the free surface of the fluid, and a gravitational body force, \( \mathbf{g} \), of magnitude \( g \) in the direction of the unit vector \( \hat{\mathbf{g}} \). Let the reference length be a characteristic thickness of the film \( H \), the reference time \( \mu H/\sigma \), the reference velocity \( U = \sigma/\mu \), and the reference pressure \( \sigma/H \). Then the non-dimensional incompressible and Navier-Stokes equations are:
\[
\nabla \cdot \mathbf{u} = 0,
\]
\[
\mathcal{R} \left[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \nabla^2 \mathbf{u} + b\hat{\mathbf{g}},
\]

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where $R = \sigma \rho H/\mu^2$ is a Reynolds number characterising the importance of the inertial terms—it may be written as $UH/\nu$ for above reference velocity—and $b = \rho g H^2/\sigma$ is a Bond number characterising the importance of the gravitational body force.

We asymptotically solve these non-dimensional continuity and Navier-Stokes equations in the curvilinear coordinate system described in Section 2 with the following boundary conditions

1. The fluid does not slip along the stationary substrate, that is
   \[ u = 0 \text{ on } y = 0. \]

2. The fluid satisfies the kinematic free surface boundary condition
   \[ \frac{\partial \eta}{\partial t} = v - \frac{u_1}{h_1} \frac{\partial \eta}{\partial x_1} - \frac{u_2}{h_2} \frac{\partial \eta}{\partial x_2} \text{ on } y = \eta. \]

3. The stress across the free surface is caused by surface tension, in non-dimensional form
   \[ -p\tilde{n} + \tilde{\tau} \cdot \tilde{n} = \tilde{\kappa}\tilde{n} \text{ on } y = \eta, \]
   where $p$ is the fluid pressure relative to the assumed zero pressure of the medium above, and $\tilde{\tau}$ is the fluid’s deviatoric stress tensor evaluated at the free surface.

4  A Centre Manifold Basis for the Model

Using centre manifold techniques [2], we derive a low-dimensional model of the dynamics described in detail by the non-dimensional continuity and Navier-Stokes equations in the general curvilinear coordinate system. We assume that the spatial extent of the fluid flow and the curving of the substrate occur on a much larger scale than the thickness of the fluid. Thus, $\epsilon$ is introduced to parameterise the relatively small effect of these spatial variations and curvature, i.e., we introduce re-scaled * variables

\[ \frac{\partial}{\partial x_i} = \epsilon \frac{\partial}{\partial x_i^*}, \quad k_i = \epsilon k_i^*, \quad \kappa = \epsilon \kappa^*. \]

To justify treating the gravitational forcing as small, we may introduce a parameter $\beta$ such that the Bond number $b = \beta^2$; then after adjoining $\frac{\partial \beta}{\partial t} = 0$ to the dynamical equations centre manifold theory justifies an asymptotic
expansion in $\beta$. As introduced by Roberts [4, 9], we also modify the tangential stress on free surface in order to establish three critical modes in the centre manifold model rather than the one natural critical mode of lubrication theory. The modification is parameterised by $\gamma$; we construct a centre manifold for asymptotically small $\gamma$; then evaluating the results for $\gamma = 1$ recovers a model for the physical dynamics. Previous work [7] has shown that such evaluation of these low-order expansions at $\gamma = 1$ is accurate.

For convenience we drop hereafter the “$*$” superscript on all re-scaled variables. The non-dimensional continuity and Navier-Stokes equations in the curvilinear coordinate system become the following governing equations (where $i' = 3 - i$):

\[
\begin{align*}
\epsilon \frac{\partial}{\partial x_1}(h_2 u_1) + \epsilon \frac{\partial}{\partial x_2}(h_1 u_2) + \frac{\partial}{\partial y}(h_1 h_2 v) &= 0, \\
\mathcal{R} \left[ \frac{\partial u_i}{\partial t} + \epsilon \frac{u_1}{h_1} \frac{\partial u_i}{\partial x_1} + \epsilon \frac{u_2}{h_2} \frac{\partial u_i}{\partial x_2} + v \frac{\partial u_i}{\partial y} + \epsilon u_i \frac{h_i'}{h_i} \left( \frac{\partial h_i}{\partial x_{i'}} - \epsilon u_{i'} \frac{h_{i'}}{h_i} \right) \right] &=-\frac{\epsilon}{h_i} \frac{\partial p}{\partial x_i} + \frac{1}{h_1 h_2} \left[ \epsilon^2 \frac{\partial}{\partial x_1} \left( \frac{h_2}{h_1} \frac{\partial u_i}{\partial x_1} \right) \right] + \hat{b}_{i}, \\
\mathcal{R} \left[ \frac{\partial v}{\partial t} + \epsilon \frac{u_1}{h_1} \frac{\partial v}{\partial x_1} + \epsilon \frac{u_2}{h_2} \frac{\partial v}{\partial x_2} + v \frac{\partial v}{\partial y} + \epsilon m_1 k_1 \frac{u_1^2}{h_1} + \epsilon m_2 k_2 \frac{u_2^2}{h_2} \right] &=-\frac{\partial p}{\partial y} + \frac{1}{h_1 h_2} \left[ \epsilon^2 \frac{\partial}{\partial x_1} \left( \frac{h_2}{h_1} \frac{\partial v}{\partial x_1} \right) \right] + \hat{b}_{3},
\end{align*}
\]

where the re-scaled scale factors are $h_i = m_i (1 - \epsilon k_i y)$, and the boundary conditions become

\[
\begin{align*}
\mathbf{u} &= \mathbf{0} \quad \text{on } y = 0, \\
\frac{\partial \eta}{\partial t} &= v - \epsilon \frac{u_1}{h_1} \frac{\partial \eta}{\partial x_1} - \epsilon \frac{u_2}{h_2} \frac{\partial \eta}{\partial x_2} \quad \text{on } y = \eta, \\
\tilde{t}_i \cdot \mathbf{T} \cdot \tilde{n} &= (1 - \gamma) \frac{m_i m_1 m_2 u_i}{\eta l_i} \quad \text{on } y = \eta,
\end{align*}
\]
\[ \mathbf{n} \cdot \mathbf{r} \cdot \mathbf{n} = p + \mathbf{\kappa} \quad \text{on} \quad y = \eta, \]

where

\[ l_i = \sqrt{\hat{h}_i^2 + \epsilon^2 \eta_i^2}, \]
\[ l = \sqrt{(\epsilon \hat{h}_2 \eta_{x_1})^2 + (\epsilon \hat{h}_1 \eta_{x_2})^2 + (\hat{h}_1 \hat{h}_2)^2}, \]

and unit tangent vectors \( \mathbf{t}_1, \mathbf{t}_2 \) and the unit normal vector \( \mathbf{n} \) are

\[ \mathbf{t}_i = (\hat{h}_i \mathbf{e}_i + \epsilon \eta_i \mathbf{e}_3)/l_i, \]
\[ \mathbf{n} = (-\epsilon \hat{h}_2 \eta_{x_1} \mathbf{e}_1 - \epsilon \hat{h}_1 \eta_{x_2} \mathbf{e}_2 + \hat{h}_1 \hat{h}_2 \mathbf{e}_3)/l. \]

Then by adjoining the trivial dynamical equations

\[ \frac{\partial \epsilon}{\partial t} = 0, \quad \frac{\partial \gamma}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \beta}{\partial t} = 0, \]

we get a new dynamical system in the variables \( \mathbf{u}, \eta, p, \epsilon, \gamma \) and \( \beta \). The original system will be recovered by setting \( \epsilon = 1, \gamma = 1, \beta = \sqrt{b} \). However, the two systems are quite different from the view of centre manifold theory. Theory \( b \) justifies treating all terms that are multiplied by the three introduced parameters as nonlinear perturbing effects in the new system.

Then the linear part (in this new sense) of the governing equations and boundary conditions have solutions: \( u_i = v = p = 0, \eta = \text{constant} \); and independently \( v = p = 0, u_i \propto \sin(\omega y/\eta) \exp(\lambda t) \) where

\[ \lambda = -\frac{\omega^2}{R\eta^2}, \quad \text{such that} \quad \omega = \tan \omega, \]

Thus, the critical modes (with zero eigenvalue) are associated with varying thickness \( \eta \), and independently with the two shear modes \( u_i \propto y \). The system we consider has three critical modes and three trivial parameter modes, all other modes decay exponentially quickly. Also, the nonlinear terms are continuous at least. Therefore a low-dimensional model of the system is justifiably constructed by centre manifold theory.

Denote the variables in the original system by \( \mathbf{v}(t) = (\eta, u_1, u_2, v, p) \). Centre manifold theory guarantees that there exist functions \( \mathbf{V} \) and \( \mathbf{G} \) respectively describing the shape of the centre manifold and the evolution thereon, namely

\[ \mathbf{v}(t) = \mathbf{V}(\eta, \bar{u}_1, \bar{u}_2), \]

such that

\[ \frac{\partial}{\partial t} \begin{bmatrix} \eta \\ \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \mathbf{G}(\eta, \bar{u}_1, \bar{u}_2), \]

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where dependence upon the constant parameters \( (\epsilon, \gamma, \beta) \) is implicit in the above, and \( \bar{u}_i \) are depth-averaged velocities measuring the amplitude of the shear modes \( u_i \propto y \). The aim now is to find functions \( V \) and \( G \) such that \( \mathbf{v}(t) \) are actual solutions of governing equations. We calculate \( V \) and \( G \) by an iteration using computer algebra \( [8] \). Suppose that an approximation \( \tilde{V} \) and \( \tilde{G} \) has been calculated and that we seek corrections \( V' \) and \( G' \). Substituting

\[
\mathbf{v} = \tilde{V} + V', \quad \text{such that} \quad \frac{\partial}{\partial t} \begin{bmatrix} \eta \\ \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \tilde{G} + G'
\]

into the governing equations then rearranging, dropping products of corrections, and using the linear approximation wherever terms multiply corrections, we obtain a system of equations for the corrections which is in the homological form

\[
\mathcal{L}V' + AG' = \tilde{R},
\]

where \( \mathcal{L} \) is the linear part of the governing equations and the boundary conditions, \( A \) is a matrix, and \( \tilde{R} \) is the residual of the governing equations using the reigning approximations, \( \tilde{V} \) and \( \tilde{G} \). The procedure for solving the equations is as follows: first, choose \( G' \) such that \( \tilde{R} - AG' \) is in the domain of \( \mathcal{L} \); second, solve \( \mathcal{L}V' = \text{rhs} \) with the given boundary conditions; then regard \( \tilde{V} + V' \) and \( \tilde{G} + G' \) as the new approximations of \( \tilde{V} \) and \( \tilde{G} \) respectively. Repeat the procedure until the residual \( \tilde{R} \) becomes zero to the required order of error. Then the low-dimensional model has the same order of error by the Approximation Theorem \([2]\) in centre manifold theory. A computer algebra program (obtainable from the authors) is run to perform the computations. The key to the correctness of the results we report is the correct coding of the residuals within the body of the iteration.

5 The Low-dimensional Model

First, computing to low-order in the small parameters gives the following fields in terms of the parameters

\[
p = -\kappa \epsilon - \nabla^2 \eta \epsilon^2 - \eta \kappa^2 \epsilon^2 + \eta (Y - 1) g_3 \epsilon^2 \\
- \eta^{-1} \bar{u} \cdot \nabla \eta \left( \frac{1}{2} \gamma - \frac{1}{2} \gamma Y + 2 - 2Y \right) \epsilon^2 \\
- \nabla \cdot \bar{u} (2 + 2Y - \frac{3}{2} \gamma + \frac{1}{2} \gamma Y) \epsilon^2 \\
+ \mathcal{O} \left( \epsilon^3 + \bar{u}^3 + \beta^3, \gamma^2 \right),
\]

\[
u_s = \bar{u} \left( 2Y + \frac{1}{2} \gamma Y - \gamma Y^3 \right) \epsilon
\]
\[ + \eta^2 (\nabla \kappa + g_s) \left( \frac{5}{24} Y - \frac{1}{2} Y^2 + \frac{1}{4} Y^3 \right) \]
\[ - \frac{17}{480} \gamma Y + \frac{33}{240} \gamma Y^3 - \frac{3}{80} \gamma Y^5 \epsilon^2 \]
\[ - \frac{1}{4} \eta \kappa \bar{u} \left( \frac{5}{12} Y - Y^2 + \frac{1}{2} Y^3 + \frac{19}{120} \gamma Y \right) \]
\[ - \frac{1}{4} \gamma Y^2 - \frac{17}{60} \gamma Y^3 + \frac{1}{2} \gamma Y^4 - \frac{3}{20} \gamma Y^5 \epsilon^2 \]
\[ + \eta K \cdot \bar{u} \left( \frac{1}{2} Y - Y^3 - \frac{1}{4} \gamma Y - \frac{1}{10} \gamma Y^3 \right) \]
\[ + \frac{3}{80} \gamma Y^5 \epsilon^2 + \mathcal{O} \left( \epsilon^3 + \bar{u}^3 + \beta^3, \gamma^2 \right) \]
\[ \nabla \cdot \eta \left( \nabla \kappa + g_s \right) \left( \frac{3}{4} \gamma Y^2 - \frac{3}{4} \gamma Y^4 \right) \epsilon^2 \]
\[ - \eta \nabla \cdot \bar{u} \left( Y^2 + \frac{1}{4} \gamma Y^2 - \frac{1}{4} \gamma Y^4 \right) \epsilon^2 \]
\[ + \mathcal{O} \left( \epsilon^3 + \bar{u}^3 + \beta^3, \gamma^2 \right) \]

where \( Y = y/\eta \), \( \bar{u} = |\bar{u}| \), \( u_s = (u_1, u_2) \), and \( \mathcal{O} (\epsilon^p + \bar{u}^q + \beta^m, \gamma^n) \) is used to denote terms \( s \) which satisfy that \( s/(\epsilon^p + \bar{u}^q + \beta^m) \) is bounded as \( (\epsilon, \bar{u}, \beta) \to 0 \), or \( s/\gamma^n \) is bounded as \( \gamma \to 0 \). In the expression for \( p \), the first and second terms are effects of the substrate curvature and free-surface, the third term is the correction for the first, the fourth term is hydrostatic, and the others are effects due to the motion of fluid. In the expression for \( \bar{u} \), the first line is the basic shear profile modified by boundary conditions, the second and third lines are modification due to forcing, and the others are effects of curvature. The expression for \( \nu \) expresses the vertical component of velocity is only dependent of the variations of free-surface and the other components.

The corresponding evolution on this centre manifold is then
\[
\frac{\partial \eta}{\partial t} = -\epsilon^2 \nabla \cdot (\eta \bar{u}) + \mathcal{O} \left( \epsilon^3 + \bar{u}^3 + \beta^3, \gamma^2 \right) ,
\]
\[
\mathcal{R} \frac{\partial \bar{u}}{\partial t} = (\nabla \kappa + g_s) \left( \frac{3}{4} \gamma Y^2 + \frac{1}{4} \gamma Y^4 \right) \epsilon
\]
\[ + \eta^{-1} \kappa \bar{u} \left( \frac{3}{8} \gamma - \frac{3}{2} \right) \epsilon \]
\[ + \eta^{-1} K \cdot \bar{u} \left( \frac{5}{8} \gamma - 3 \right) \epsilon \]
\[ - 3 \eta^{-2} \bar{u} \gamma \]
\[ + \mathcal{O} \left( \epsilon^3 + \bar{u}^3 + \beta^3, \gamma^2 \right) .
\]

The first equation gives the general expression representing mass conservation for fluid considered. The first line in the right-hand side of the second equation is forcing, the second and third lines are curvature effects and the fourth line is drag.

To recover a model of the original dynamics, we need to set \( \gamma = 1 \). But as is apparent from the above, every coefficient in the centre manifold model
is a power series in $\gamma$. From earlier computation in [7], we estimate that the radii of convergence of such $\gamma$ series are much greater than 1. Thus after computing to higher order in $\gamma$, we calculate every coefficient in the model from the first five terms in $\gamma$ series by setting $\gamma = 1$. We also need to set $\epsilon = 1$ which is valid provided the length-scales in the resolved dynamics are indeed much larger than the fluid thickness, that is, if the gradients are small enough. The low-dimensional model then becomes as given in the Introduction by (2) with errors $O(\nabla^3 + \bar{u}^3 + \beta^3)$.

6 Conclusion

This model conserves fluid and accurately accounts for the effects of the curvature of substrate, surface tension, gravitational forcing and fluid inertia.

The low-dimensional model given in this paper is a simpler version. One may adjust a dynamical model to suit a particular application.

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