On the topology of $\mathcal{H}(2)$

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Abstract

The space $\mathcal{H}(2)$ consists of pairs $(M, \omega)$, where $M$ is a Riemann surface of genus two, and $\omega$ is a holomorphic 1-form which has only one zero of order two. There exists a natural action of $\mathbb{C}^\ast$ on $\mathcal{H}(2)$ by multiplication to the holomorphic 1-form. In this paper, we single out a proper subgroup $\Gamma$ of $\text{Sp}(4, \mathbb{Z})$ generated by three elements, and show that the space $\mathcal{H}(2)/\mathbb{C}^\ast$ can be identified to the quotient $\Gamma \backslash \mathcal{J}_2$, where $\mathcal{J}_2$ is the Jacobian locus of Riemann surfaces of genus two in the Siegel upper half space $\mathcal{H}_2$. A direct consequence of this result is that $[\text{Sp}(4, \mathbb{Z}) : \Gamma] = 6$.

1 Introduction

In this paper we are concerned with translation surfaces in the stratum $\mathcal{H}(2)$. Each element of $\mathcal{H}(2)$ can be either considered as a translation surface having only one singularity of angle $6\pi$ together with a parallel line field, or a pair $(M, \omega)$, where $M$ is a Riemann surface of genus two, and $\omega$ is a holomorphic 1-form having a single zero of order two on $M$. Using the latter viewpoint, we see that $\mathbb{C}^\ast$ acts naturally on $\mathcal{H}(2)$ by multiplication to the holomorphic 1-form. Note that, if $\omega$ has only one zero on $M$, then this zero must be a Weierstrass point of $M$. Therefore, the quotient $\mathcal{H}(2)/\mathbb{C}^\ast$ consists of pairs $(M, W)$, where $M$ is a Riemann surface of genus two, and $W$ is a Weierstrass point of $M$, two pairs $(M_1, W_1)$ and $(M_2, W_2)$ are identified if there exists a conformal homeomorphism $\phi : M_1 \to M_2$ such that $\phi(W_1) = W_2$.

The space $\mathcal{H}(2)$ is well known to be a complex orbifold of dimension 4. For any pair $(M, \omega)$, let $\gamma_1, \ldots, \gamma_4$ be a basis of the group $H_1(M, \mathbb{Z})$, then the period mapping

$$(M, \omega) \mapsto (\int_{\gamma_1} \omega, \ldots, \int_{\gamma_4} \omega) \in \mathbb{C}^4$$

gives a local chart for $\mathcal{H}(2)$ in a neighborhood of $(M, \omega)$. Consequently, we see that $\mathcal{H}(2)/\mathbb{C}^\ast$ can be endowed with a complex projective orbifold structure.

In Section 3, we introduce a construction of translations surface in $\mathcal{H}(2)$ from triples of parallelograms by some gluing pattern. This construction gives rise to the notion of parallelogram decomposition of surfaces in $\mathcal{H}(2)$. Actually, given a translation surface $(M, \omega)$ in $\mathcal{H}(2)$, there exist infinitely many parallelogram decompositions of $(M, \omega)$. From a fixed parallelogram decomposition, one can obtain other ones by applying some elementary moves, which are called $T$, $S$, and $R$, these moves are realized by some
homeomorphisms of the surface $M$ whose actions on the group $H_1(M, \mathbb{Z})$ (in an appropriate basis) are given by the following matrices respectively

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\Gamma$ be the group generated by the matrices $T$, $S$, and $R$, then $\Gamma$ is a proper subgroup of $\text{Sp}(4, \mathbb{Z})$.

The main result of this paper is the following:

**Theorem 1.1** There exists a homeomorphism from $\mathcal{H}(2)/\mathbb{C}^\times$ to the quotient $\Gamma \backslash \mathcal{J}_2$, where $\mathcal{J}_2$ is the Jacobian locus of Riemann surfaces of genus two in the Siegel upper half space $\mathcal{H}_2$. As a consequence, we have $[\text{Sp}(4, \mathbb{Z}) : \Gamma] = 6$.

**Idea of the proof:** Let $M$ be a connected, closed Riemann surface of genus two, $W$ a Weierstrass point of $M$, and $\omega$ a holomorphic 1-form on $M$ whose unique zero is $W$, note that $\omega$ is unique up to a multiplicative constant. To prove Theorem 1.1, we first consider decompositions of $M$ into three open topological disks, which will be called admissible, with similar properties to the parallelogram decompositions of the translation surface $(M, \omega)$. Parallelogram decompositions are in particular admissible. We need to consider those admissible decompositions because for some parallelogram decompositions, the $R$ move does not yield another parallelogram decomposition, but only an admissible decomposition.

The key ingredient of the proof of Theorem 1.1 is the fact that the canonical bases associated to two admissible decompositions are related by an element of $\Gamma$ (cf. Theorem 4.4). This allows us to define a map $\Xi$ from $\mathcal{H}(2)/\mathbb{C}^\times$ into $\Gamma \backslash \mathcal{H}_2$.

Since all compact closed Riemann surfaces of genus two are hyperelliptic, the pair $(M, W)$ can be represented by a pair $(\lambda_0, \Lambda)$, where $\lambda_0 \in \mathbb{CP}^1$, and $\Lambda$ is a subset of cardinal five of $\mathbb{CP}^1 \setminus \{\lambda_0\}$, up to action of $\text{PSL}(2, \mathbb{C})$, namely; $M$ is the two-sheeted branched cover of $\mathbb{CP}^1$ ramified above the set $\{\lambda_0 \cup \Lambda\}$, and $W$ is the pre-image of $\lambda_0$. One particular property of the canonical basis associated to an admissible decomposition is that, if its period matrix is given, then we can compute the value of $\lambda_0, \lambda_1, \ldots, \lambda_5$ by some fixed theta functions. From this, we deduce that the map $\Xi$ defined above is injective.

Obviously, the image of $\Xi$ is contained in the set $\Gamma \backslash \mathcal{J}_2$. We show that this image is actually equal to $\Gamma \backslash \mathcal{J}_2$ by proving that any canonical basis of $H_1(M, \mathbb{Z})$ can be represented by the canonical basis associated to an admissible decompositions for some pair $(M, W')$ in $\mathcal{H}(2)/\mathbb{C}^\times$.

**Acknowledgements** The author would like to thank François Labourie for the guidance and stimulating discussions which are very important, especially at the beginning of this work.

2 Siegel upper half space and theta functions

In this section, we recall the definition and some basic properties of the theta functions. Our main references are [13], and [4].
2.1 Siegel upper half space and Jacobian locus

For any integer \( g \geq 1 \), the Siegel upper half space \( \mathcal{H}_g \) is the space of complex symmetric matrices \( Z \) in \( \mathcal{M}(g, \mathbb{C}) \) such that \( \text{Im}(Z) \) is positive definite. This space is the quotient of the real symplectic group \( \text{Sp}(2g, \mathbb{R}) \) by the compact subgroup \( U(g) \).

The integral symplectic group \( \text{Sp}(2g, \mathbb{Z}) \) is a lattice in \( \text{Sp}(2g, \mathbb{Z}) \) which acts properly discontinuously on \( \mathcal{H}_g \). The quotient \( \mathcal{A}_g = \text{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g \) is the moduli space of principally polarized Abelian varieties of dimension \( g \). In this paper, by the real symplectic group \( \text{Sp}(2g, \mathbb{R}) \) we mean the group of real \( 2g \times 2g \) matrices preserving the symplectic form

\[
\begin{pmatrix}
J & 0 \\
0 & J
\end{pmatrix},
\]

where \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

Let \( M \) be a Riemann surface of genus \( g \), and \( \{a_1, b_1, \ldots, a_g, b_g\} \) be a canonical basis of \( H_1(M, \mathbb{Z}) \), that is

\[
\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0, \quad \text{and} \quad \langle a_i, b_j \rangle = \delta_{ij},
\]

where \( \langle , \rangle \) is the intersection form of \( H_1(M, \mathbb{Z}) \). There exist \( g \) holomorphic 1-forms \( (\phi_1, \ldots, \phi_g) \) on \( M \) uniquely determined by the following condition:

\[
\int_{a_j} \phi_i = \delta_{ij}.
\]

The matrix \( \Pi = (\pi_{ij})_{i,j=1,\ldots,g} \), where \( \pi_{ij} = \int_{b_j} \phi_i \), belongs to \( \mathcal{H}_g \), and we then have a mapping from the set of pairs \( (M, \{a_1, b_1, \ldots, a_g, b_g\}) \) into \( \mathcal{H}_g \). The image of this mapping is called the Jacobian locus denoted by \( J_g \).

For the case \( g = 2 \), it is well known that the complement of \( J_2 \) in \( \mathcal{H}_2 \) is a union of countably many copies of \( \mathcal{H}_1 \times \mathcal{H}_1 \), where \( \mathcal{H}_1 \) is the upper half plane \( \mathcal{H}_1 = \{z \in \mathbb{C} : \text{Im}z > 0\} = \mathbb{H} \).

2.2 Theta function

Fix an integer \( g \geq 1 \), and let \( \mathcal{H}_g \) be the Siegel upper half space of genus \( g \). The Riemann’s theta function is a complex value function defined on \( \mathbb{C}^g \times \mathcal{H}_g \) by the following formula

\[
\theta(z, \sigma) = \sum_{N \in \mathbb{Z}^g} \exp(2\pi i (\frac{1}{2} N \sigma N + \frac{1}{2} N z)).
\]

The function \( \theta \) is holomorphic on \( \mathbb{C}^g \times \mathcal{H}_g \). We also consider functions defined on \( \mathbb{C}^g \times \mathcal{H}_g \) by

\[
\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (z, \sigma) = \sum_{N \in \mathbb{Z}^g} \exp\{2\pi i [\frac{1}{2} \sigma (N + \frac{\epsilon}{2})(N + \frac{\epsilon'}{2}) + \frac{1}{2} (N + \frac{\epsilon}{2})(z + \frac{\epsilon'}{2})]\}
\]

where \( \epsilon, \epsilon' \) are integer vectors. These functions are called first order theta functions with integer characteristic \( \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] \).
Proposition 2.1 The first order theta function with integer characteristic \( \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} \) has the following properties:

i) \[ \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z + e^{(k)}, \sigma) = \exp 2\pi i \begin{bmatrix} 0 \\ \frac{\epsilon_k}{2} \end{bmatrix} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \sigma). \]

ii) \[ \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z + \sigma^{(k)}, \sigma) = \exp 2\pi i \begin{bmatrix} -\frac{\epsilon_k}{2} - \frac{\epsilon'_k}{2} \\ \frac{\epsilon_k}{2} \end{bmatrix} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \sigma). \]

iii) \[ \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-z, \sigma) = \exp 2\pi i \begin{bmatrix} \epsilon' \\ \epsilon \end{bmatrix} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \sigma). \]

iv) \[ \theta \begin{bmatrix} \epsilon + 2\nu \\ \epsilon' + 2\nu' \end{bmatrix} (z, \sigma) = \exp 2\pi i \begin{bmatrix} \epsilon' \\ \epsilon \end{bmatrix} \theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (z, \sigma), \text{ with } \nu, \nu' \in \mathbb{Z}. \]

where \( e^{(k)} \) and \( \sigma^{(k)} \) are the \( k \)-th column of the matrices \( \text{Id}_g \) and \( \sigma \) respectively.

By Torelli Theorem, we know that a closed Riemann surface \( M \) is uniquely determined by its Jacobian variety \( J(M) \), or equivalently, by the period matrix associated to a canonical basis of \( H_1(M, \mathbb{Z}) \). If \( M \) is hyperelliptic, then we can get more information from the period matrix by using theta functions. We have (cf. [4], VII.4)

Theorem 2.2 The branched points of the two sheeted representation of a hyperelliptic Riemann surface are holomorphic functions of the period matrix. Furthermore, the hyperelliptic surface is completely determined by its period matrix.

To illustrate the ideas of the proof, we will indicate here below the method to compute some of the branched points, details of the calculations can be found in [4], VII.1, and VII.4.

Assume that \( M \) is the two-sheeted branched cover of \( \mathbb{C}P^1 \) ramified above \( \lambda_1, \ldots, \lambda_{2g+2} \). Let \( s_1, \ldots, s_{g+1} \) be \( g+1 \) simple arcs in \( \mathbb{C}P^1 \) such that the endpoints of \( s_i \) are \( \lambda_{2i-1}, \lambda_{2i} \), for \( k = i, \ldots, g+1 \), and \( s_i \cap s_j = \emptyset \), for \( i \neq j \). We can consider \( M \) as the Riemann surface obtained by gluing two copies of \( \mathbb{C}P^1 \) slit along \( s_1, \ldots, s_{g+1} \). Let \( z \) be the meromorphic function on \( M \) realizing the two-sheeted branched cover from \( M \) to \( \mathbb{C}P^1 \). Let \( P_i \) denote \( z^{-1}(\lambda_i) \), \( i = 1, \ldots, 2g+2 \), then \( \{P_1, \ldots, P_{2g+2}\} \) is the set of Weierstrass points of \( M \).
By applying an element of $\text{PSL}(2, \mathbb{C})$ if necessary, we may assume that $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_{2g+2} = \infty$. The surface $M$ is then the curve defined by the equation

$$w^2 = z(z - 1) \prod_{i=3}^{2g-1} (z - \lambda_i).$$

The function $z$ is then characterized, up to a non zero multiplicative constant, by the property that it has a double zero at $P_1$, a double pole at $P_{2g+2}$, and it is holomorphic, and non zero elsewhere.

First, we specify a canonical basis $\{a_1, b_1, \ldots, a_g, b_g\}$ of $H_1(M, \mathbb{Z})$ as follows:

- $b_k = z^{-1}(s_k)$, $k = 1, \ldots, g + 1$. By construction, $b_k$ is a simple closed curve containing $P_{2k-1}$ and $P_{2k}$, and $b_k$ is preserved by the hyperelliptic involution.

- Let $\alpha_k$, $k = 1, \ldots, g$, be $g$ simple closed curves pairwise disjoint on the sphere satisfying
  
  . $\alpha_k$ intersects transversely $s_k$ and $s_{g+1}$,
  . $\text{Card}\{\alpha_k \cap s_k\} = \text{Card}\{\alpha_k \cap s_{g+1} = 1\}$,
  . $\alpha_k \cap s_j = \emptyset$ if $j \notin \{k, g + 1\}$,
  . $\alpha_k \cap \{\lambda_1, \ldots, \lambda_{2g+2}\} = \emptyset$.

Let $a_k$ be a connected component of $z^{-1}(\alpha_k)$. Note that $a_k$ and its image under the hyperelliptic involution are disjoint.

It follows from the construction that the family $(a, b) = \{a_1, b_1, \ldots, a_g, b_g\}$ is a canonical basis of $H_1(M, \mathbb{Z})$. Let $\{\zeta_1, \ldots, \zeta_g\}$ be the basis of $\mathcal{H}^1(M)$, the space of holomorphic 1-form on $M$, dual to $\{a_1, \ldots, a_g\}$, that is

$$\int_{a_j} \zeta_i = \delta_{ij}.$$

Let $\Pi = (\pi_{ij})_{i,j=1,\ldots,g}$, with $\pi_{ij} = \int_{b_j} \zeta_i$, be the period matrix associated to $(a, b)$. By definition, the Jacobian variety $J(M)$ of $M$ is the quotient $\mathbb{C}^g / \Lambda$, where $\Lambda$ is the lattice in $\mathbb{C}^g$ which is generated by the column vectors of the $g \times 2g$ matrix $(\text{Id}_g, \Pi)$. We denote by $e^{(k)}$, and $\pi^{(k)}$, $k = 1, \ldots, g$, the $k$-th columns of $\text{Id}_g$ and $\Pi$ respectively.

Let $\varphi : M \rightarrow J(M)$ be the map defined by

$$\varphi(P) = (\int_{P_1}^{P} \zeta_1, \ldots, \int_{P_1}^{P} \zeta_g) \in \mathbb{C}/\Lambda,$$

where the integrals are taken along any path joining $P_1$ to $P$. From the construction of the basis $(a, b)$, one can compute explicitly $\varphi(P_1)$, $i = 1, \ldots, 2g + 2$, as functions of $(e^{(1)}, \ldots, e^{(g)}, \pi^{(1)}, \ldots, \pi^{(g)})$. 

5
Since \( \Pi \in \mathcal{H}_g \), we can now consider the first order theta functions with integer characteristic \( \theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (z, \Pi) \).

From Proposition 2.1, we see that for any \( \nu \in \Lambda \), \( \theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (z + \nu, \Pi) \) differs from \( \theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (z, \Pi) \) by a multiplicative factor. It turns out that the multiplicative behavior of the theta functions is so that

\[
f(P) := \frac{\theta^2 \left[ \begin{array}{cccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right] (\varphi(P), \Pi)}{\theta^2 \left[ \begin{array}{cccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right] (\varphi(P), \Pi)}
\]

is a meromorphic function on \( M \) with divisor \( P_1^2 P_{2g+2}^{-2} \). Hence

\[
f = cz, \quad \text{where } c \in \mathbb{C}^*.
\]

The constant \( c \) can be valued at \( P_2 \) since we have \( f(P_2) = cz(P_2) = c \). Using the fact that \( \varphi(P_2) = \frac{1}{2} \pi(1) \), we have

\[
c = f(P_2) = \frac{\theta^2 \left[ \begin{array}{cccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right] (\varphi(P), \Pi)}{\theta^2 \left[ \begin{array}{cccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right] (\varphi(P), \Pi)}.
\]

It follows that

\[
z(P) = \frac{\theta^2 \left[ \begin{array}{cccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{array} \right] \theta^2 \left[ \begin{array}{cccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right] (\varphi(P), \Pi)}{\theta^2 \left[ \begin{array}{cccc} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{array} \right] \theta^2 \left[ \begin{array}{cccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right] (\varphi(P), \Pi)}
\]

(1)

By setting \( P = P_j \), for \( j = 3, \ldots, 2g + 1 \), we get a formula for \( \lambda_j \). Note that this formula is only useful for \( j = 3, 4, 6, \ldots, 2g \), for other values of \( j \), both the numerator and denominator vanish. By replacing \( P_1 \) by another Weierstrass point in the definition of \( \varphi \), we can get similar formulae for the \( \lambda_j \) which can not be computed directly from (1).

### 3 The group \( \Gamma \)

In this section, we define the group \( \Gamma \), and prove some of its properties.

#### 3.1 Construction of surfaces in \( \mathcal{H}(2) \) by gluing parallelograms

Any parallelogram in \( \mathbb{R}^2 \) is determined by a pair of vectors in \( \mathbb{R}^2 \). In this section, we will represent any parallelogram by a pair of complex numbers \((z_1, z_2)\) such that \( \text{Im}(\bar{z}_1 z_2) = \frac{-i}{2}(\bar{z}_1 z_2 - z_1 \bar{z}_2) > 0 \). Note that by this convention, the pairs \((z_1, z_2), (z_2, -z_1), (-z_1, -z_2), (-z_2, z_1)\) represent the same parallelogram.
Let $\mathcal{P}^+$ denote the set $\{(z_1, z_2) \in \mathbb{C}^2 : -\frac{i}{2}(\bar{z}_1 z_2 - z_1 \bar{z}_2) > 0\}$. Given 4 complex numbers $(z_1, \ldots, z_4)$ such that $(z_1, z_2), (z_2, z_3), (z_3, z_4)$ all belong to $\mathcal{P}^+$, let $A, B, C$ denote the parallelograms determined by the pairs $(z_1, z_2), (z_2, z_3), (z_3, z_4)$ respectively. We can construct a translation surface in $\mathcal{H}(2)$ from $A, B, C$ as follows:

- Glue two sides of $A$ corresponding to $z_1$ together.
- Glue two sides of $A$ corresponding to $z_2$ to two sides of $B$ also corresponding to $z_2$.
- Glue two sides of $B$ corresponding to $z_3$ to two sides of $C$ also corresponding to $z_3$.
- Glue two sides of $C$ corresponding to $z_4$ together.

It is easy to check that the surface $M$ obtained from this construction is of genus 2, equipped with a flat metric structure with cone singularities. There is only one singular point on $M$ which arises from the identification of all the vertices of $A, B, C$, we denote this point by $P_0$. Since all the gluings are realized by translations, $M$ is actually a translation surface. We also get naturally a holomorphic 1-form on $M$, considered as a Riemann surface, defined as follows: since translations of $\mathbb{R}^2$ preserve the holomorphic 1-form $dz$ is preserved, the restrictions of $dz$ into the parallelograms $A, B, C$ agree via the gluings, and give rise to a holomorphic 1-form $\omega$ on $M$ with only one zero at $P_0$, which is necessarily of order two. Clearly, $(M, \omega) \in \mathcal{H}(2)$.

We know that $M$ is a hyperelliptic surface, it is easy to visualize the hyperelliptic involution of $M$ from its construction by gluing $A, B, C$. For each of the parallelograms $A, B, C$, consider the reflection through its center, one can easily check that these reflections agree with the gluing on the boundary of $A, B, C$. Thus, we have a conformal automorphism $\tau$ of $M$. One can check that $\tau^2 = \text{Id}$, and the action of $\tau$ on $H_1(M, \mathbb{Z})$ is given by $-\text{Id}$, therefore $\tau$ must be the hyperelliptic involution of $M$. We can also determine without difficulty the 6 fixed points of $\tau$, which are the Weierstrass points of $M$: two of which are contained in $A$, two in $C$, one in the interior of $B$, and the last one is $P_0$.

### 3.2 Parallelogram decompositions and the group $\Gamma$

The construction of translation surfaces in the stratum $\mathcal{H}(2)$ by gluing parallelograms with the pattern presented above suggests the following
Definition 3.1 Let \((M, \omega)\) be a pair in \(\mathcal{H}(2)\). A parallelogram decomposition of \((M, \omega)\) is a system \(\mathcal{D}\) of six oriented saddle connections \(\{a, b_1, b_2, c_1, c_2, d\}\) verifying the following conditions

- The intersection of any pair of saddle connections in this family is \(\{P_0\}\), the only zero of \(\omega\).
- \(b_1 \cup b_2\) (resp. \(c_1 \cup c_2\)) is the boundary of a cylinder which contains a (resp. \(d\)).
- The complement of \(a \cup b_1 \cup b_2 \cup c_1 \cup c_2 \cup d\) is the union of three open disks isometric to three open parallelograms in \(\mathbb{R}^2\).
- The orientations of the saddle connections in this family are chosen so that the numbers \(z_1 = \int_a \omega, z_2 = \int_{b_1} \omega = \int_{b_2} \omega, z_3 = \int_{c_1} \omega = \int_{c_2} \omega, z_4 = \int_d \omega\) satisfy \((z_i, z_{i+1}) \in \mathcal{P}^+, \text{ for } i = 1, 2, 3\), the orientation of \(a\) goes from \(b_1\) to \(b_2\), and the orientation of \(b_1\) goes from \(c_2\) to \(c_1\).

Remark: If \(\{a, b_1, b_2, c_1, c_2, d\}\) is a parallelogram decomposition of \((M, \omega)\) then \((a, b_1, c_1, d)\) is a non-canonical basis of \(H_1(M, \mathbb{Z})\). Let \(b\) (resp. \(c\)) be a simple closed curve in the cylinder bounded by \(b_1\) and \(b_2\) (resp. \(c_1\) and \(c_2\)). Let \(e\) be a simple closed curved in the free homotopy class of the closed curve \(d * (-b_1)\), then \((a, b, c, e)\) is a canonical basis of \(H_1(M, \mathbb{Z})\), we will call it the canonical basis associated to \(\mathcal{D}\).

Given a pair \((M, \omega)\) in \(\mathcal{H}(2)\) which is obtained from three parallelograms \(A = (z_1, z_2), B = (z_2, z_3), C = (z_3, z_4)\) as in the previous section, let \(\tilde{A}, \tilde{B}, \tilde{C}\) be the subsets of \(M\) which correspond to \(A, B, C\) respectively. By construction, \(\tilde{A}\) and \(\tilde{C}\) are two cylinders such that \(\tilde{A} \cap \tilde{C} = P_0\). Let \(a\) (resp. \(d\)) denote the saddle connection in \(\tilde{A}\) (resp. \(\tilde{C}\)) which corresponds to the \(z_1\) sides of \(A\) (resp. \(z_4\) sides of \(C\)). Let \(b_1, b_2\) (resp. \(c_1, c_2\)) denote the boundary components of \(\tilde{A}\) (resp. \(\tilde{C}\)). We choose the orientations of \(a, b_1, b_2, c_1, c_2, d\) so that

\[
\int_a \omega = z_1, \int_{b_1} \omega = \int_{b_2} \omega = z_2, \int_{c_1} \omega = \int_{c_2} \omega = z_3, \int_d \omega = z_4.
\]

We also choose the numbering of \((b_1, b_2)\) (resp. \((c_1, c_2)\)) so that the orientation of \(a\) goes from \(b_1\) to \(b_2\), and the orientation of \(b_1\) goes from \(c_2\) to \(c_1\). By definition \(\{a, b_1, b_2, c_1, c_2, d\}\) is a parallelogram decomposition of \((M, \omega)\).

The surface \((M, \omega)\) admits infinitely many parallelogram decompositions. The following operations allow us to get other decompositions from \(\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}\).

1. **The \(T\) move:** let \(a'\) be the saddle connection which is obtained from \(a\) by a Dehn twist in \(\tilde{A}\), then \(\mathcal{D}' = \{a', b_1, b_2, c_1, c_2, d\}\) is another parallelogram decomposition of \((M, \omega)\).
2. **The S move:** If \( \{a, b_1, b_2, c_1, c_2, d\} \) is a parallelogram decomposition of \((M, \omega)\) then 
\[ D' = \{d, -c_2, -c_1, b_1, b_2, -a\} \]
is also a parallelogram decomposition of \((M, \omega)\) (the minus sign signifies the same saddle connection with the inverse orientation).

3. **The R move** Cut \(M\) along the saddle connections \(b_1, b_2\) and \(d\), we then obtain two cylinders, one of which is \(\tilde{A}\), we denote the other by \(\tilde{A}^*\). Let \(c'_1, c'_2\) denote the images of \(c_1, c_2\) respectively under a Dehn twist in \(\tilde{A}^*\). Assume that \(c'_1, c'_2\) can be made into geodesics of \(\tilde{A}^*\), then \(\{a, b_1, b_2, c'_1, c'_2, d\}\) is a parallelogram decomposition of \((M, \omega)\).

Let \(D'\) be another parallelogram decomposition of \((M, \omega)\). Let \((a, b, c, e)\) (resp. \((a', b', c', e')\)) be the canonical basis of \(H_1(M, \mathbb{Z})\) associated to \(D\) (resp. \(D'\)), we have

a) If \(D'\) is obtained from \(D\) by a \(T\) move then

\[
\begin{pmatrix}
  a' \\
  b' \\
  c' \\
  e'
\end{pmatrix} = 
\begin{pmatrix}
  1 & \pm 1 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  a \\
  b \\
  c \\
  e
\end{pmatrix}.
\]

b) If \(D'\) is obtained from \(D\) by an \(S\) move then

\[
\begin{pmatrix}
  a' \\
  b' \\
  c' \\
  e'
\end{pmatrix} = 
\begin{pmatrix}
  0 & 1 & 0 & 1 \\
  0 & 0 & -1 & 0 \\
  0 & 1 & 0 & 0 \\
 -1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  a \\
  b \\
  c \\
  e
\end{pmatrix}.
\]
c) If $D'$ is obtained from $D$ by an $R$ move then

$$
\begin{pmatrix}
a' \\
b' \\
c' \\
e'
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \pm 1 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix} a \\
b \\
c \\
e
\end{pmatrix}.
$$

Put

$$
T = \begin{pmatrix} 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

and let $\Gamma$ denote the subgroup of $\text{Sp}(4, \mathbb{Z})$ which is generated by $T, S, R$.

### 3.3 Properties of $\Gamma$

**Lemma 3.2**

i) $S^2 = -\text{Id}_4$.

ii) $\Gamma \not\subset \text{Sp}(4, \mathbb{Z})$.

iii) $\begin{pmatrix} \text{Id}_2 & 0 \\
0 & \text{SL}(2, \mathbb{Z}) \end{pmatrix} \subset \Gamma$.

iv) $\Gamma$ is not a normal subgroup of $\text{Sp}(4, \mathbb{Z})$.

**Proof:**

i) follows from direct calculation.

ii) The group $\text{Sp}(4, \mathbb{Z})$ acts transitively on $\mathbb{Z}_2^4 \setminus \{0\}$, but $\Gamma$ has two orbits: $\mathcal{O}_1$ containing $e_1 = (1, 0, 0, 0)$, and $\mathcal{O}_2$ containing $e_2 = (0, 1, 0, 0)$. As a matter of fact, $\text{Card}\mathcal{O}_1 = 10$, and $\text{Card}\mathcal{O}_2 = 5$.

iii) We have

$$
S^{-1}TS = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1
\end{pmatrix}.
$$

Since $\text{SL}(2, \mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\
0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix}$, iii) follows.

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iv) Let \( T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \), observe that \( T \) does not belong to \( \Gamma \) since it sends \( e_1 \) to an element in \( O_2 \). We have

\[
T^{-1}TT = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Since \( T^{-1}TT \) sends \( e_2 \) to \( e_1 \), it does not belong to \( \Gamma \). We can then conclude that \( \Gamma \) is not a normal subgroup of \( \text{Sp}(4, \mathbb{Z}) \).

Lemma 3.3  

Put

\[
U = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

then the integral symplectic group \( \text{Sp}(4, \mathbb{Z}) \) is generated by \( T, S, R, \) and \( U \).

Proof: see Appendix, Section A. \( \square \)

4 Admissible decomposition

Given an element \((M, \omega)\) of \( \mathcal{H}(2) \), let \( D = \{a, b_1, b_2, c_1, c_2, d\} \) be a parallelogram decomposition of \((M, \omega)\). We have seen that other parallelogram decompositions of \((M, \omega)\) can be obtained by the elementary moves \( T, S, \) and \( R \). However, while \( T, \) and \( S \) are always realizable, the \( R \) move is not, it is only realizable when \( d \ast (-b_1) \ast c_2 \) is homotopic to a saddle connection. The notion of admissible decomposition will help us to remedy this problem, namely, given an admissible decomposition of \((M, \omega)\), we can always apply the elementary moves \( T, S, \) and \( R \), which give us other admissible decompositions. The definition of admissible decomposition is inspired from parallelogram decomposition, and based on the action of the hyperelliptic involution on the fundamental group of \( M \).

Throughout this section, \( M \) is a fixed closed Riemann surface of genus two, \( W \) is a Weierstrass point of \( M \). Let \( \tau \) denote the hyperelliptic involution of \( M \). For any closed curve \( \gamma \) with base point \( W \), we denote by \([\gamma]\) the homotopy class of \( \gamma \) in \( \pi_1(M,W) \).

Definition 4.1 (Admissible decomposition) Let \( \{a, b_1, b_2, c_1, c_2, d\} \) be six oriented simple closed curve with base point at \( W \). We say that \( \{a, b_1, b_2, c_1, c_2, d\} \) is an admissible decomposition for the pair \((M,W)\) if
• The intersection of any pair of curves in this family is \{W\}.

• \([\tau(a)] = [a]^{-1}, [\tau(d)] = [d]^{-1}\).

• \([\tau(b_1)] = [b_2]^{-1}, [\tau(c_1)] = [c_2]^{-1}\).

• \(a \setminus \{W\}\) is contained in an open annulus \(A_b\) bounded by \(b_1\) and \(b_2\).

• \(d \setminus \{W\}\) is contained in an open annulus \(A_c\) bounded by \(c_1\) and \(c_2\).

• \(M \setminus \overline{A_b} \cup \overline{A_c}\) is homeomorphic to an open disk.

The orientations of \(\{a, b_1, b_2, c_1, c_2, d\}\) are chosen so that

• \(\langle a, b_1 \rangle = \langle a, b_2 \rangle = \langle c_1, d \rangle = \langle c_2, d \rangle = 1\), where \(\langle \ldots \rangle\) is the intersection form of \(H_1(M, \mathbb{Z})\).

• In the annulus \(A_b\), the orientation of \(a\) goes from \(b_1\) to \(b_2\).

• In the annulus \(A_c\), the orientation of \(d\) goes from \(c_2\) to \(c_1\).

• The oriented boundary of the disk \(M \setminus (A_b \cup A_c)\) is the concatenation \(b_1 \ast c_1 \ast (-b_2) \ast (-c_2)\).

**Example:** If \(D = \{a, b_1, b_2, c_1, c_2, d\}\) is a parallelogram decomposition for a pair \((M, \omega)\), where \(\omega\) is a holomorphic 1-form with double zero at \(W\), then \(D\) is an admissible decomposition for the pair \((M, W)\). Note that, on a fixed translation surface in \(H(2)\), there are admissible decompositions which cannot be realized as parallelogram decompositions.

Given an admissible decomposition \(D = \{a, b_1, b_2, c_1, c_2, d\}\) for the pair \((M, W)\), let \(b\) (resp. \(c\)) be a simple closed curve in the annulus \(A_b\) (resp. \(A_c\)) freely homotopic to \(b_1\) and \(b_2\) (resp. \(c_1\) and \(c_2\)), and let \(e\) be a simple closed curve in \(M \setminus \overline{A_b}\) which is freely homotopic to \((-b_1) \ast d\), then \((a, b, c, e)\) is a canonical basis of \(H_1(M, \mathbb{Z})\). We call this basis the canonical basis associated to \(D\).

We define the elementary moves, that is the \(T\) move, \(R\) move, and \(S\) move, for admissible decompositions in the same manner as those of parallelogram decomposition. We have

**Proposition 4.2** Let \(D = \{a, b_1, b_2, c_1, c_2, d\}\) be an admissible decomposition for the pair \((M, W)\). Let \(D' = \{a', b'_1, b'_2, c'_1, c'_2, d'\}\) be a family of curves obtained from \(D\) by an elementary move, then \(D'\) is also an admissible decomposition for the pair \((M, W)\).

**Proof:**

1. If \(D'\) is obtained from \(D\) by an \(S\) move then the proposition follows from the definitions.

2. Suppose that \(D'\) is obtained from \(D\) by a \(T\) move, in which case \(D' = \{a', b_1, b_2, c_1, c_2, d\}\), where \(a'\) is the image of \(a\) by a Dehn twist in the annulus \(A_b\).
All we need to prove in this case is that \( \tau(a') = [a']^{-1} \). Without loss of generality, we can assume that

\[ a' = b_1 * a = a * b_2. \]

Using the hypothesis that \( \tau(a) = [a]^{-1} \), and \( \tau(b_1) = [b_2]^{-1} \), we have

\[ \tau(a') = \tau(b_1) \tau(a) = [b_2]^{-1} [a]^{-1} = ([a][b_2])^{-1} = [a']^{-1}. \]

3. For the case where \( D' \) is obtained from \( D \) by an \( R \) move, consider \( A_{c,b} = M \setminus A_{b} \) as a torus with an open disk removed whose boundary is the union of two segments \( b_1 \) and \( b_2 \). Note that \( d \) is now a segment on \( A_{c,b} \) joining the common endpoints of \( b_1 \) and \( b_2 \). Slit open \( A_{c,b} \) along \( d \), we then obtain an annulus which will be denoted by \( A_{c} \). By definition, \( D' = \{a, b_1, b_2, c_1', c_2', d\} \), where \( c_1', c_2' \) are the images of \( c_1 \) and \( c_2 \) under a Dehn twist in \( A_{c} \). All we need to prove is that \( \tau(c_1') = [c_2']^{-1} \). Without loss of generality, we can assume that

\[ c_1' = c_1 * (-b_2) * d = (-b_1) * d * c_1, \quad c_2' = c_2 * d * (-b_2) = d * (-b_1) * c_2. \]

Therefore

\[ \tau(c_1') = \tau(c_1) \tau(b_2) [\tau(d)]^{-1} = [c_2]^{-1} [b_1][d]^{-1} = ([d][b_1]^{-1}[c_2])^{-1} = [c_2']^{-1}. \]

\[ \square \]

**Corollary 4.3** Given an admissible decomposition \( D = \{a, b_1, b_2, c_1, c_2, d\} \) for the pair \((M, W)\) with the associated canonical basis \((a, b, c, e)\) of \( H_1(M, \mathbb{Z}) \), for any element \( \gamma \) in \( \Gamma \), there exists an admissible decomposition \( D' = \{a', b_1', b_2', c_1', c_2', d'\} \) with the associated canonical basis \((a', b', c', e')\) such that
One of the main ingredients of the proof of 1.1 is the following theorem which will be proved in the next section.

**Theorem 4.4** Let $\mathcal{D}$ and $\mathcal{D}'$ be two admissible decompositions of the pair $(M,W)$ with the associated canonical homology bases $(a,b,c,e)$, and $(a',b',c',e')$ respectively. Then there exists an element $\gamma$ in $\Gamma$ such that

\[
\begin{pmatrix}
a' \\
b' \\
c' \\
e'
\end{pmatrix} = \gamma \cdot \begin{pmatrix}a \\ b \\ c \\ e\end{pmatrix}.
\]

Before getting into the proof of Theorem 4.4, let us prove the following

**Proposition 4.5** If $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ and $\mathcal{D}' = \{a', b'_1, b'_2, c'_1, c'_2, d'\}$ are two admissible decompositions such that $[b_1] = [b'_1], [b_2] = [b'_2]$, then there exists an element $\gamma \in \Gamma$ such that

\[
\begin{pmatrix}
a' \\
b' \\
c' \\
e'
\end{pmatrix} = \gamma \cdot \begin{pmatrix}a \\ b \\ c \\ e\end{pmatrix}.
\]

**Proof:** By a theorem of Epstein-Zieschang (see [1] Theorem A.4, p.411), there exists a homeomorphism $\Phi$ of $M$ which is isotopic to the identity relative to $\{W\}$ such that $\Phi(b_1) = b'_1, \Phi(b_2) = b'_2$. Therefore, we can assume that $b_1 = b'_1, b_2 = b'_2$ as subsets of $M$.

Since $a \setminus \{W\}$ and $a' \setminus \{W\}$ are contained in the open annulus $A_b$, there exists an integer $n$ such that $[a'] = [a] \ast [b_1]^n$. Thus, by applying the $T$ move $n$ times, we can assume that $a' = a$ as subsets of $M$. It follows that we have the following equality in $H_1(M, \mathbb{Z})$

\[
\begin{pmatrix}
a' \\
b' \\
c' \\
e'
\end{pmatrix} = \begin{pmatrix}1 \\ 0 \\ X \\ Y\end{pmatrix} \cdot \begin{pmatrix}a \\ b \\ c \\ e\end{pmatrix},
\]

with $X, Y \in M_{2 \times 2}(\mathbb{Z})$. Since $\begin{pmatrix}1 \\ 0 \\ X \\ Y\end{pmatrix}$ belongs to $\text{Sp}(4, \mathbb{Z})$, simple computations show that we must have $X = 0$, and $Y \in \text{SL}(2, \mathbb{Z})$. Since the group $\begin{pmatrix}1 \\ 0 \\ 0 \\ \text{SL}(2, \mathbb{Z})\end{pmatrix}$ is contained in $\Gamma$, the proposition follows. $\square$
5 Proof of Theorem 4.4

5.1 Standard decomposition

In this section, we equip $M$ with the hyperbolic metric in the conformal class of the Riemann surface structure. Note that $\tau$ is now an isometry of $M$. Let us first prove the following

**Lemma 5.1** Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition of the pair $(M, W)$. Let $b_0$ and $c_0$ be the simple closed geodesic in the free homotopy class of $b_1$ and $c_1$ respectively, then we have

- $b_0 \cap c_0 = \emptyset$.
- $\tau(b_0) = -b_0$, $\tau(c_0) = -c_0$.
- $W \notin b_0 \cup c_0$.

**Proof:**

- Let $b$ (resp. $c$) be a simple closed curve freely homotopic to $b_1$ (resp. to $c_1$) which is contained in the annulus bounded by $b_1$ and $b_2$ (resp. $c_1$ and $c_2$). By construction $b$ and $c$ are freely homotopic to $b_0$ and $c_0$ respectively, and $b \cap c = \emptyset$. It is well known that $\text{Card}\{b_0 \cap c_0\}$ is the intersection number $\iota(b, c)$ of the free homotopy classes of $b$ and $c$, thus we have that $\text{Card}\{b_0 \cap c_0\} = 0$.

- By definition, we see that $b_1$ is freely homotopic to $b_2$. Since $[\tau(b_1)] = [b_2]^{-1}$, it follows that $\tau(b_1)$ is freely homotopic to $-b_1$. Now, since $b_0$ is the unique simple closed geodesic in the free homotopy class of $b_1$, $-b_0$ is then the unique simple closed geodesic in the free homotopy class of $-b_1$. Since $\tau$ is an isometry, $\tau(b_0)$ is a simple closed geodesic freely homotopic to $-b_1$, therefore we have $\tau(b_0) = -b_0$.

- Suppose that $W \in b_0$. Since $b_0$ is freely homotopic to $b_1$, there exists $[h] \in \pi_1(M, W)$ such that

  $$[b_0] = [h][b_1][h]^{-1}.$$  

  Note that we have $[\tau(b_0)] = [b_0]^{-1}$, hence,

  $$[h][b_1]^{-1}[h]^{-1} = [\tau(h)][\tau(b_1)][\tau(h)]^{-1} = [\tau(h)][a]^{-1}[b_1]^{-1}[a][\tau(h)]^{-1}. $$

  It follows

  $$[a][\tau(h)]^{-1}[h][b_1]^{-1} = [b_1]^{-1}[a][\tau(h)]^{-1}[h]$$  \hspace{1cm} (2)

  We deduce that $[b_1]^{-1}$ and $[a][\tau(h)]^{-1}[h]$ commute. But $[b_1]$ is a simple closed non-separating curve, therefore we have

  $$[a][\tau(h)]^{-1}[h] = [b_1]^n \text{ with } n \in \mathbb{Z}$$  \hspace{1cm} (3)

  Recall that $\tau$ acts like $-\text{Id}$ on $H_1(M, \mathbb{Z})$, thus (3) implies the following equality in $H_1(M, \mathbb{Z})$
\[ nb_1 - a = 2h \]

It follows that \( nb_1 - a = 0 \) in \( H_1(M, \mathbb{Z}/2) \), but this is impossible since \((a, b_1, c_1, d)\) is a basis of \( H_1(M, \mathbb{Z}/2) \). We can then conclude that \( b_0 \) does not contain \( W \). The same arguments apply to \( c_0 \), and the lemma follows.

\[ \square \]

**Remark:** If a simple closed geodesic \( g \) verifies \( \tau(g) = -g \), then \( g \) contains two fixed points of \( \tau \) which are two Weierstrass points of \( M \).

Let \((g_1, g_2)\) be a pair of disjoint simple closed geodesics verifying the following property

\[
(\mathscr{P}) \left\{ \begin{array}{ll}
W \notin g_i, \\
\tau(g_i) = -g_i,
\end{array} \right.
\]

for \( i = 1, 2 \). We construct an admissible decomposition of \((M, W)\) associated to \((g_1, g_2)\) as follows:

- Cut open \( M \) along \( g_1 \) and \( g_2 \), we obtain a 4-holed sphere \( N \) which is equipped with a hyperbolic metric with geodesic boundary. Let \( g_i^+, g_i^- \) denote the boundary components of \( N \) corresponding to \( g_i, \ i = 1, 2 \). Note that the hyperelliptic involution \( \tau \) of \( M \) induces an isometric involution \( \tau' \) of \( N \) which interchanges \( g_i^+ \) and \( g_i^- \).

- Let \( s_i^+ \) be the shortest path in \( N \) from \( W \) to \( g_i^+ \), and let \( s_i^- \) denote \( \tau'(s_i^+) \), \( i = 1, 2 \). Note that the action of \( \tau \) on the tangent space at \( W \) is \( -\text{Id} \), therefore \( s_i^+ \cup s_i^- \) is a simple geodesic arc joining \( g_i^+ \) to \( g_i^- \). Let \( \tilde{s}_i^+, \tilde{s}_i^- \) be the simple geodesic arcs in \( M \) corresponding to \( s_i^+ \) and \( s_i^- \), and let \( P_i^+, P_i^- \) denote the endpoints of \( \tilde{s}_i^+ \), and \( \tilde{s}_i^- \) other than \( W \) respectively. Let \( r_i \) be a simple arc in \( g_i \) with endpoints \( P_i^+, P_i^- \), then \( t_i = r_i \cup \tilde{s}_i^+ \cup \tilde{s}_i^- \) is a simple closed curve containing \( W \). We have

\[ t_1 \cap t_2 = \{W\}. \]

- We choose the orientation of \( t_1 \) so that \( \langle t_1, g_1 \rangle = 1 \). Without loss of generality, we can assume that this orientation goes from \( W \) to \( P_1^+ \). We can then find a simple closed curve \( \tilde{b} \) in \( M \) such that

\[
\begin{align*}
[\tilde{b}] &\equiv [\tilde{s}_1^+ \ast g_1 \ast (-\tilde{s}_1^+)] \text{ in } \pi_1(M, W), \\
\tilde{b} \cap \tau(\tilde{b}) &\equiv \{W\}, \\
\tilde{b} \text{ and } \tau(\tilde{b}) &\text{ bound an annulus containing } g_1 \text{ as a waist curve.}
\end{align*}
\]

- We choose the orientation of \( t_2 \) so that \( \langle t_2, g_2 \rangle = -1 \). Without loss of generality, we can assume that the orientation of \( t_2 \) goes from \( W \) to \( P_2^- \). Let \( \tilde{c} \) be a simple closed curve homotopic to \( (-\tilde{s}_2^+) \ast g_2 \ast \tilde{s}_2^+ \) in \( \pi_1(M, W) \) such that \( \tau(\tilde{c}) \cap \tilde{c} = \{W\} \). Following the orientation of \( g_2 \), we have two cases:

- \( [\tilde{b}] [\tilde{c}] [\tau(\tilde{b})] [\tau(\tilde{c})] \equiv 1 \) in \( \pi_1(M, W) \): in this case, take \( a = t_1, b_1 = \tilde{b}, b_2 = -\tau(\tilde{b}), c_1 = \tilde{c}, c_2 = -\tau(\tilde{c}), d = t_2 \), then \( \mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\} \) is an admissible decomposition of \((M, W)\).
• \( [\tilde{b}] [\tau(\tilde{c})] [\tau(\tilde{b})] [c] = 1 \) in \( \pi_1(M, W) \): in this case, let \( u_2^+ \) be a simple arc going from \( P_2^+ \) to \( W \) such that \( (-\tilde{s}_2^+) * u_2^+ \) is homotopic to \( \tilde{b} \) in \( \pi_1(M, W) \), and \( u_2^+ \cap \tau(u_2^+) = \{ W \} \). Then there exists a simple closed curve \( \tilde{c}' \) homotopic to \( (-u_2^+) * g_2 * u_2^+ \) in \( \pi_1(M, W) \) such that

\[
\begin{align*}
\tilde{c}' \cap \tau(\tilde{c}') &= \{ W \}, \\
\tilde{c}' \text{ and } \tau(\tilde{c}') &\text{ bound an annulus containing } g_2 \text{ as a waist curve.}
\end{align*}
\]

We have

\[
[\tilde{c}'] = \left[ (-u_2^+) * g_2 * u_2^+ \right],
\]

\[
= \left[ (-u_2^+) * \tilde{s}_2^+ \right] \left[ (-\tilde{s}_2^+) * g_2 * \tilde{s}_2^+ \right] \left[ (-\tilde{s}_2^+) * u_2^+ \right],
\]

\[
= [\tilde{b}]^{-1} [c] [\tilde{b}].
\]

Therefore,

\[
[b] [\tilde{c}'] [\tau(\tilde{b})] [\tau(\tilde{c}')] = 1.
\]

Let \( t'_2 \) denote the simple closed curve \( r_2 \cup \tau(u_2^+) \cup u_2^+ \), we choose the orientation of \( t'_2 \) so that \( \langle t'_2, g_2 \rangle = -1 \). It follows that, if we take \( a = t_1, b_1 = \tilde{b}, b_2 = -\tau(\tilde{b}), c_1 = \tilde{c}', c_2 = -\tau(\tilde{c}'), \) and \( d = t'_2 \), then \( D = \{ a, b_1, b_2, c_1, c_2, d \} \) is an admissible decomposition of \( (M, W) \).
In both cases, we will call $\mathcal{D}$ a standard decomposition associated to the pair of geodesics $(g_1, g_2)$.

From Lemma 5.1, we know that if $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ is an admissible decomposition for the pair $(M, W)$, and $b_0$ (resp. $c_0$) is the simple closed geodesic in the free homotopy class of $b_1$ (resp. $c_1$), then the pair $(b_0, c_0)$ satisfies Property $(\mathcal{P})$. Hence, we can consider the standard decompositions associated to $(b_0, c_0)$. The following proposition tells us that the canonical bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$, and to a standard decomposition associated to $(b_0, c_0)$ are related by an element of the group $\Gamma$.

**Proposition 5.2** Let $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition of the pair $(M, W)$. Let $b_0, c_0$ the simple closed geodesics in the free homotopy classes of $b_1$ and $c_1$ respectively. Let $\mathcal{D} = \{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{d}\}$ be some standard decomposition associated to the pair $(b_0, c_0)$, then the canonical bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}$ are related by an element in $\Gamma$.

**Corollary 5.3** If $\mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\}$, and $\mathcal{D}' = \{a', b_1', b_2', c_1', c_2', d'\}$ are two admissible decomposition of the pair $(M, W)$ such that $b_1$ is freely homotopic to $b_1'$, and $c_1$ is freely homotopic to $c_1'$, then the canonical bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}'$ are related by an element in $\Gamma$.

**Proof:** Let $b_0, c_0, c_0'$ be the simple closed geodesics in the free homotopy classes of $b_1, c_1$, and $c_1'$ respectively. By assumption, we have $c_0' = \pm c_0$. Let $\mathcal{D} = \{\hat{a}, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{d}\}$, and $\mathcal{D}' = \{\hat{a}', \hat{b}_1', \hat{b}_2', \hat{c}_1', \hat{c}_2', \hat{d}'\}$ be some standard decompositions associated to the pairs $(b_0, c_0)$ and $(b_0, c_0')$ respectively. From Proposition 5.2, the canonical bases of $H_1(M, \mathbb{Z})$ associated to $\mathcal{D}$ and $\mathcal{D}'$ (resp. $\mathcal{D}$ and $\mathcal{D}'$) are related by an element of $\Gamma$.

If $c_0' = c_0$, then we can take $D' = D$, and the corollary follows immediately. If $c_0' = -c_0$, then from the construction of standard decompositions, we can assume that $\hat{b}_1 = \hat{b}_1', \hat{b}_2 = \hat{b}_2'$, and the corollary follows from Proposition 4.5. \hfill \Box

**Proof:** (of Proposition 5.2)

Since $\hat{b}_1$ and $b_1$ are freely homotopic, there exists a closed curve $h$ with base point $W$ such that $$[b_1] = [h]^{-1}[b_1][h],$$ in $\pi_1(M, W)$. We will show that $h \in \mathbb{Z}a \oplus \mathbb{Z}b_1 \oplus \mathbb{Z}c_1$ in $H_1(M, \mathbb{Z})$. By definition, we have

$$[\tau(\hat{b}_1)] = [\hat{a}]^{-1}[\hat{b}_1][\hat{a}],$$

$$[\tau(h)]^{-1}[\tau(\hat{b}_1)][\tau(h)] = [\hat{a}]^{-1}[h]^{-1}[\hat{b}_1][h][\hat{a}],$$

$$[\tau(h)]^{-1}[a][\tau(\hat{b}_1)][\tau(h)] = [\hat{a}]^{-1}[h]^{-1}[\hat{b}_1][h][\hat{a}],$$

$$[b_1]^{-1}([a][\tau(h)][\hat{a}]^{-1}[h]^{-1}) = ([a][\tau(h)][\hat{a}]^{-1}[h]^{-1})[b_1]^{-1}$$

It follows that $[b_1]$ and $[a][\tau(h)][\hat{a}]^{-1}[h]^{-1}$ commute. Since $b_1$ is a simple closed curve, there exists $k \in \mathbb{Z}$ such that

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\([a][\tau(h)][\hat{a}]^{-1}[h]^{-1} = [b_1]^k.\]

Therefore, in \(H_1(M, \mathbb{Z})\), we have

\[\hat{a} = a - kb_1 - 2h.\]

We know that \(\langle \hat{a}, \hat{c}_1 \rangle = \langle \hat{a}, c_1 \rangle = 0\), and \(\langle a, c_1 \rangle = \langle b_1, c_1 \rangle = 0\), where \(\langle ., . \rangle\) is the intersection form of \(H_1(M, \mathbb{Z})\). Hence,

\[\langle h, c_1 \rangle = 0,\]

which implies that

\(h \in c_1^\perp = Za \oplus Zb_1 \oplus Zc_1.\)

Let \((a, b, c, e)\) and \((\hat{a}, \hat{b}, \hat{c}, \hat{d})\) be the canonical bases of \(H_1(M, \mathbb{Z})\) associated to \(D\) and \(\hat{D}\) respectively. We know that there exists \(\gamma \in \text{Sp}(4, \mathbb{Z})\) such that

\[
\begin{pmatrix}
\hat{a} \\
\hat{b} \\
\hat{c} \\
\hat{e}
\end{pmatrix} = \gamma \cdot
\begin{pmatrix}
a \\
b \\
c \\
e
\end{pmatrix}.
\]

Since in \(H_1(M, \mathbb{Z})\) we have \(\hat{b} = b, \hat{c} = c, \) and \(\hat{a} = a + kb + 2h \in Za \oplus Zb_1 \oplus Zc\), it follows that \(\gamma\) is of the form

\[
\gamma = \begin{pmatrix}
x & y & z & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
x' & y' & z' & t'
\end{pmatrix}.
\]

Now, \(\langle \hat{a}, \hat{b} \rangle = 1\) implies \(x = 1\), and since \(z\) is the \(c\)-coordinate of \(2h\) in \(Za \oplus Zb_1 \oplus Zc\), we have

\[\hat{a} = a + mb + 2\ell c, \text{ with } m, \ell \in \mathbb{Z}.
\]

It follows,

\[\langle \hat{b}, \hat{e} \rangle = 0 \Rightarrow x' = 0,
\]

\[\langle \hat{c}, \hat{e} \rangle = 1 \Rightarrow t' = 1,
\]

\[\langle \hat{a}, \hat{e} \rangle = 0 \Rightarrow y' = -z = -2\ell.
\]

We deduce that

\[
\gamma = \begin{pmatrix}
1 & m & 2\ell & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -2\ell & n & 1
\end{pmatrix},
\]

with \(\ell, m, n\) in \(\mathbb{Z}\). The proposition follows from Lemma 5.4 here below. \(\square\)
Lemma 5.4 For any integers \( \ell, m, n \), the matrix \( \gamma = \begin{pmatrix} 1 & m & 2\ell & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2\ell & n & 1 \end{pmatrix} \) belongs to the group \( \Gamma \).

Proof: We have shown that the group \( \left( \text{Id}_2 \ 0 \ \text{SL}(2,\mathbb{R}) \right) \) is included in \( \Gamma \). Thus

\[
X = S \cdot \left( \text{Id}_2 \ 0 \right) \cdot S \cdot \left( \text{Id}_2 \ 0 \right) = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \in \Gamma.
\]

Put

\[
T^\sharp = S^{-1}T^{-1}S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}
\]

Now, straight computations show that \( T, T^\sharp, \) and \( X \) commute, and \( \gamma = T^m T^\sharp^n X^\ell \). \( \square \)

5.2 Proof of Theorem 4.4

By Proposition 5.2, we can now restrict ourselves into the case of standard decompositions. We start by proving the following key lemma

Lemma 5.5 Let \( g, g_1, g_2 \) be three simple closed geodesics of \( M \) verifying Property \( (\mathcal{P}) \). Assume that

\[
\begin{cases} 
 g_1 \cap g_2 = \emptyset, \\
 \text{Card}\{g \cap (g_1 \cup g_2)\} = n > 1.
\end{cases}
\]

Then there exists a simple closed geodesic \( g_3 \) verifying Property \( (\mathcal{P}) \) such that \( \text{Card}\{g_3 \cap (g_1 \cup g_2)\} = 1 \), and if \( g_1 \) and \( g_3 \) are disjoint, then \( \text{Card}\{g \cap (g_1 \cup g_3)\} < n \).

Proof: We know that each of the curves \( g_1, g_2 \) contains two Weierstrass points. Let \( W' \) be the other Weierstrass point of \( M \) which is not contained in \( g_1 \cup g_2 \). We have two possibilities:

Case 1: \( W' \in g \). Let \( s \) be the segment of \( g \) which contains \( W' \) with endpoints in \( g \cap (g_1 \cup g_2) \). We denote by \( P_1, P_2 \) the two endpoints of \( s \), and choose the orientation of \( s \) to be from \( P_1 \) to \( P_2 \). Since \( \tau(g) = -g \), and \( \tau(W') = W' \), we deduce that \( \tau(s) = -s \), and \( P_1, P_2 \) are interchanged by \( \tau \). It follows that \( P_1 \) and \( P_2 \) are both contained in either \( g_1 \) or \( g_2 \). Without loss of generality, we can assume that \( P_1, P_2 \) are contained in \( g_2 \).

We know that \( \tau \) preserves the orientation of \( M \), since \( \tau \) reverses the orientation of \( s \) and \( g_2 \), we deduce that \( s \) meets both sides of \( g_2 \). Let \( r \) be one of the two subsegments of \( g_2 \) with endpoints \( P_1, P_2 \), and let \( W'' \) be the Weierstrass point which is contained in \( r \). Note that \( P_1 \) and \( P_2 \) must be distinct, otherwise \( s = g \), and \( \text{Card}\{g \cap (g_1 \cup g_2)\} = 1 \), which is discard by the hypothesis.
Consider the simple closed curve $g'_3$ which is composed by $s$ and $r$. From the definition, we see that $\tau(g'_3) = -g'_3$. We can move $g'_3$ slightly so that the following conditions are satisfied

- $g'_3 \cap s = \{W'\}$,
- $g'_3 \cap g_2 = \{W''\}$,
- $\tau(g'_3) = -g'_3$.

By construction, we have

- $g'_3 \cap g_1 = \emptyset$.
- $\text{Card}\{g'_3 \cap g\} = \text{Card}\{\text{int}(r) \cap g\} + 1 \leq \text{Card}\{g_2 \cap g\} - 2 + 1 = \text{Card}\{g_2 \cap g\} - 1$.

Let $g_3$ be the simple closed geodesic in the free homotopy class of $g'_3$. Let us prove that $g_3$ verifies Property $(\mathcal{P})$. Since $\tau(g'_3) = -g'_3$, it follows that $\tau(g_3) = -g_3$, as $\tau(g_3)$ is the simple closed geodesic in the free homotopy class of $\tau(g'_3)$. It remains to show that $g_3$ does not contain $W$.

Let $\rho : M \to \mathbb{CP}^1$ be the two-sheeted branched cover from $M$ onto the sphere. Let $P_0, P_1, \ldots, P_5$ denote the images of the Weierstrass points of $M$ under $\rho$, with $P_0 = \rho(W)$. Let $t$, and $t'$ denote the images of $g_3$, and $g'_3$ under $\rho$ respectively. Since $\tau(g_3) = -g_3$, and $\tau(g'_3) = -g'_3$, we deduce that $t$, and $t'$ are two simple arcs in $\mathbb{CP}^1$ with endpoints in $\{P_0, \ldots, P_5\}$. Note that, by construction, both $t$, and $t'$ contain no other points in $\{P_0, \ldots, P_5\}$ except their endpoints. Now assume that $g_3$ contains $W$, which means that $P_0$ is an endpoint of $t$. Since $g'_3$ does not contain $W$, $P_0$ is not an endpoint of $t'$. We have two cases:

- Case 1: $t$ and $t'$ has a common endpoint. In this case, we can find a simple arc $t''$ in $\mathbb{CP}^1$ in the homotopy class with fixed endpoints of $t'$ such that $\text{Card}\{t'' \cap t\} = 1$. Let $g''_3$ be the pre-image of $t''$ in $M$, then $g''$ is a simple closed curve in the free homotopy class of $g'_3$, and we have

$$\text{Card}\{g''_3 \cap g_3\} = 1.$$  

Consequently,

$$\langle g''_3, g_3 \rangle \neq 0,$$

where $\langle \cdot, \cdot \rangle$ is the intersection form of $H_1(M, \mathbb{Z})$. But this is impossible since $g_3 = g''_3$ in $H_1(M, \mathbb{Z})$. 

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Case 2: $t$ and $t'$ have no common endpoints. In this case, we can find a simple arc $t''$ in the homotopy class with fixed endpoints of $t'$ such that $t'' \cap t = \emptyset$. Let $g''$ be the pre-image of $t''$ in $M$, then $g''$ is a simple closed curve in the free homotopy class of $g_3$. We can realize $M$ as the surface obtained by gluing two copies of $\mathbb{CP}^1$ slit along $t,t''$, and another simple arc joining the two remaining points in \{$P_1,\ldots,P_5$\}. From this construction, we see that $g''_3$ is not homologous to $g_3$, which is impossible since they are freely homotopic.

Remark: The arguments above actually show that $t$ and $t'$ have the same endpoints.

Let us now show that $g_3$ satisfies the conditions in the conclusion of the lemma. Let $\iota$ denote the intersection number between free homotopy classes of simple closed curves in $M$. Since $\iota(\alpha,\beta) = \text{Card}\{\alpha,\beta_0\}$, where $\alpha_0$ and $\beta_0$ are the simple closed geodesics in the free homotopy classes of $\alpha$ and $\beta$ respectively, we have

- $\text{Card}\{g_3 \cap g_1\} = \iota(g'_3, g_1) = 0$.
- $\text{Card}\{g_3 \cap g_2\} = \iota(g'_3, g_2) \leq \text{Card}\{g'_3 \cap g_2\} = 1$
- $\text{Card}\{g_3 \cap g\} = \iota(g'_3, g) \leq \text{Card}\{g'_3 \cap g\} < \text{Card}\{g_2 \cap g\}$.

By construction, we have $\langle g_3, g_2 \rangle = \langle g'_3, g_2 \rangle = \pm 1$, therefore $g_3 \cap g_2 \neq \emptyset$. We deduce that $\text{Card}\{g_3 \cap g_2\} = 1$, and the lemma is proved for this case.

Case 2: $W' \notin g$. Cutting $M$ along the curves $g_1, g_2$, we then get 4-holed sphere $N$ which is equipped with a hyperbolic metric with geodesic boundary. Let $\tau_N$ denote the isometric involution of $N$ which is induced by $\tau$. Let $\hat{g}$ denote the union of geodesic arcs with endpoints in $\partial N$ corresponding to sub-segments of $g$ with endpoints in $g_1 \cup g_2$. Let $\hat{s}_1$ be a geodesic arc realizing the distance $d_N(W',\hat{g})$, and let $\hat{s}_2$ denote $\tau_N(\hat{s}_1)$. Note that $\hat{s}_2$ is also a geodesic arc realizing the distance $d_N(W',\hat{g})$. From the fact that they both realize the distance in $N$ from $W'$ to $\hat{g}$, we deduce that $W'$ is the unique common point of $\hat{s}_1$ and $\hat{s}_2$. It follows, in particular, that $W$ is not contained in $\hat{s}_1 \cup \hat{s}_2$. Note that, since $\tau$ acts like $-\text{Id}$ on the tangent plane at $W'$, $\hat{s}_1 \cup \hat{s}_2$ is in fact a geodesic segment.

Let $s_1, s_2$ be the geodesic arcs in $M$ corresponding to $\hat{s}_1$ and $\hat{s}_2$ respectively. Let $P_i$, $i = 1, 2$, denote the endpoint of $s_i$ other than $W'$ respectively. Note that we have $\tau(s_1) = s_2$, and $\tau(P_1) = P_2$.

The point $P_1$ is contained in a geodesic segment of $g$ with endpoints in $g_1 \cup g_2$. If $P_1$ is not a point in $g_1 \cup g_2$, then let $Q_1$ be an endpoint of the segment containing $P_1$, and let $r_1$ denote the subsegment with the orientation from $P_1$ to $Q_1$, otherwise take $Q_1$ to be $P_1$, and $r_1 = \{P_1\}$. Let $Q_2, r_2$ denote $\tau(Q_1)$ and $\tau(r_1)$ respectively. The curve $c = (-r_1)*(-s_1)*s_2*r_2$ is then a simple arc joining $Q_1$ to $Q_2$ verifying $\tau(c) = -c$.

Since $\tau(Q_1) = Q_2$, it follows that either $\{Q_1, Q_2\} \subset g_1$, or $\{Q_1, Q_2\} \subset g_2$. Without loss of generality, we can assume that $Q_1, Q_2$ are contained in $g_2$, then the same argument as in Case 1 shows that $c$ meets both sides of $g_2$. Here we have two issues

- $Q_1 = Q_2$: in this case $c$ is actually a simple closed curve which satisfies
a) \( \tau(c) = -c \).

b) \( c \cap g_1 = \emptyset \).

c) \( \text{Card}\{c \cap g_2\} = 1 \).

Note that in this case \( Q_1 \in g \cap g_2 \). There exists a simple closed curve \( g'_3 \) obtained from \( c \) by a Dehn twist around \( g_2 \), such that \( g'_3 \) verifies a), b), c), and

\[
\text{Card}\{g'_3 \cap g\} = \text{Card}\{g_2 \cap g\} - 1.
\]

- \( Q_1 \neq Q_2 \): let \( d \) be an arc in \( g_2 \) with endpoints \( Q_1, Q_2 \), then \( c \cup d \) is a simple closed curve invariant under \( \tau \). Moving this curve slightly, we can find a simple closed curve \( g'_3 \) which satisfies a), b), c), and moreover

\[
\text{Card}\{g'_3 \cap g\} \leq \text{Card}\{g_2 \cap g\} - 2.
\]

In both cases, let \( g_3 \) is the simple closed geodesic in the free homotopy class of \( g'_3 \). Using the same arguments as in Case 1, we see that \( g_3 \) verifies the required properties. The lemma is then proved. \( \square \)

Theorem 4.4 is a consequence of the two following propositions.

**Proposition 5.6** Let \( \mathcal{D} = \{a, b_1, b_2, c_1, c_2, d\} \) and \( \mathcal{D}' = \{a', b'_1, b'_2, c'_1, c'_2, d'\} \) be two admissible decompositions of \( (M, W) \). As usual, let \( (a, b, c, e) \) and \( (a', b', c', e') \) be the canonical bases of \( H_1(M, \mathbb{Z}) \) associated to \( \mathcal{D} \) and \( \mathcal{D}' \) respectively. Assume that \( b_1 \) is freely homotopic to \( b'_1 \), then there exists \( \gamma \in \Gamma \) such that

\[
\begin{pmatrix}
    a' \\
    b' \\
    c'
\end{pmatrix}
= \gamma \cdot
\begin{pmatrix}
    a \\
    b \\
    c
\end{pmatrix}.
\]

**Proof:** Let \( b_0, c_0, c'_0 \) be the simple closed geodesics in the free homotopy classes of \( b_1, c_1, c'_1 \) respectively. According to Lemma 5.4, we know that \( b_0, c_0, c'_0 \) verify Property \((\mathcal{P})\), and

\[
b_0 \cap c_0 = b_0 \cap c'_0 = \emptyset.
\]

If \( c'_0 = \pm c_0 \), the proposition follows from Corollary 5.3. Hence, we only need to consider the case where \( c'_0 \neq c_0 \) as subsets of \( M \). In this case, since each of the curves \( b_0, c_0, c'_0 \) contains exactly two Weierstrass
points of \( M \), and \( W \notin b_0 \cup c_0 \cup c'_0 \), we deduce that \( c'_0 \cap c_0 \neq \emptyset \). Let \( n \) be the number of intersection points of \( c_0 \) and \( c'_0 \). The proposition will be proved by induction.

**Case \( n = 1 \):** Note that in this case, the intersection point between \( c_0 \) and \( c'_0 \) must be a Weierstrass point. We will show that there exist two admissible decompositions \( \{\tilde{a}, \tilde{b}_1, \tilde{b}_2, \tilde{c}_1, \tilde{c}_2, d\} \), and \( \{\tilde{a}', \tilde{b}'_1, \tilde{b}'_2, \tilde{c}'_1, \tilde{c}'_2, d'\} \) such that

- \( \tilde{b}_i = \tilde{b}'_i, \ i = 1, 2, \)
- \( \tilde{b}_i, \ i = 1, 2, \) is freely homotopic to \( b_0 \),
- \( \tilde{c}_i, \ i = 1, 2, \) is freely homotopic to \( c_0 \),
- \( \tilde{c}'_i \ i = 1, 2, \) is freely homotopic to \( c'_0 \).

We can then use Proposition 4.5 to conclude. Let \( \rho : M \to \mathbb{CP}^1 \) be the two-sheeted branched cover from \( M \) onto the sphere. By definition, we have \( \rho(\tau(P)) = \rho(P), \ \forall P \in M \). It follows that \( s_1 = \rho(b_0), s_2 = \rho(c_0), s'_2 = \rho(c'_0) \) are three simple arcs on \( \mathbb{CP}^1 \) which verify

- \( \rho(W) \notin s_1 \cup s_2 \cup s'_2. \)
- \( s_1 \cap (s_2 \cup s'_2) = \emptyset. \)
- \( s_2 \) and \( s'_2 \) have a common endpoint, and \( \text{Card}\{s_2 \cap s'_2\} = 1. \)

Let \( P_0, P_1, \ldots, P_5 \) denote the images of the Weierstrass points of \( M \) under \( \rho \), with \( P_0 = \rho(W) \). We can assume that the endpoints of \( s_1 \) are \( P_1 \) and \( P_2 \), the endpoints of \( s_2 \) are \( P_3 \) and \( P_4 \), and the endpoints of \( s'_2 \) are \( P_1 \) and \( P_5 \). Let \( s_3 \) (resp. \( s'_3 \)) be a simple arc in \( \mathbb{CP}^1 \) joining \( P_0 \) to \( P_5 \) (resp. \( P_3 \)), and disjoint from \( s_1 \cup s_2 \) (resp. \( s_1 \cup s'_2 \)). We can then realize \( M \) as the surface obtained by gluing two copies of \( \mathbb{CP}^1 \) slit along either \( s_1, s_2, s_3 \), or along \( s_1, s'_2, s'_3 \).

Let \( t_1 \) be a simple closed curve in \( \mathbb{CP}^1 \setminus (s_1 \cup s_2 \cup s'_2) \) with base-point \( P_0 \) which separates \( s_1 \) from \( s_2 \cup s'_2 \). Let \( D_1 \) denote the open disk in \( \mathbb{CP}^1 \) containing \( s_1 \) and bounded by \( t_1 \). Using the construction of \( M \) by gluing two copies of \( \mathbb{CP}^1 \) slit along \( s_1, s_2, \) and \( s_3 \), we see that the pre-image \( \tilde{t}_1 \) of in \( M \) consists of two simple closed curves with base point \( W \), denoted by \( \tilde{b}_1 \) and \( \tilde{b}_2 \), which bound an open annulus containing \( b_0 \). The orientation of \( b_0 \) induces an orientation of \( \tilde{b}_1 \) and \( \tilde{b}_2 \), by free homotopy, and hence an orientation.
of \( t_1 \).

Let \( t_2 \) (resp. \( t'_2 \)) be a simple closed curve in \( \mathbb{CP}^1 \setminus D_1 \) with base-point \( P_0 \) which separates \( s_2 \) from the remaining endpoint of \( s'_2 \) (resp. separates \( s'_2 \) from the remaining endpoint of \( s_2 \)) such that \( t_2 \cap t_1 = P_0 \) (resp. \( t'_2 \cap t_1 = P_0 \)). We choose the orientation of \( t_2 \) and \( t'_2 \) so that \([t_1][t_2] = 1 \) in \( \pi_1(\mathbb{CP}^1 \setminus (s_1 \cup s_2), P_0) \), and \([t_1][t'_2] = 1 \) in \( \pi_1(\mathbb{CP}^1 \setminus (s_1 \cup s'_2), P_0) \).

Like \( \rho^{-1}(t_1) \), \( \rho^{-1}(t_2) \) (resp. \( \rho^{-1}(t'_2) \)) is the union of two simple closed curves \( \tilde{c}_1, \tilde{c}_2 \) (resp. \( \tilde{c}'_1, \tilde{c}'_2 \)) which bound an embedded open annulus in \( M \) containing \( c_0 \) (resp. \( c'_0 \)). Observe that the family of curves \( \{\tilde{b}_1, \tilde{b}_2, \tilde{c}_1, \tilde{c}_2\} \) (resp. \( \{\tilde{b}_1, \tilde{b}_2, \tilde{c}'_1, \tilde{c}'_2\} \)) cuts \( M \) into three pieces: two annulus and a quadrilateral. We can then add to these two families some simple closed curves two get two admissible decompositions for \((M,W)\) as follows:

Let \( r_1 \) be a simple arc joining \( P_1 \) to and endpoint of \( s_1 \) such that \( \text{int}(r_1) \subset D_1 \), then \( \tilde{a} = \rho^{-1}(r_1) \) is a simple closed curve which is contained in the closure of the annulus bounded by \( \tilde{b}_1 \) and \( \tilde{b}_2 \). Observe that \( \tau(\tilde{a}) = -\tilde{a} \).

Let \( D_2 \) (resp. \( D'_2 \)) denote the open disk containing \( s_2 \) (resp. \( s'_2 \)) and bounded by \( t_2 \) (resp. \( t'_2 \)). Let \( r_2 \) (resp. \( r'_2 \)) denote a simple arc joining \( P_0 \) to \( P_3 \) (resp. \( P'_3 \)) verifying \( \text{int}(r_2) \subset D_2 \) (resp. \( \text{int}(r'_2) \subset D'_2 \)). Observe that \( \tilde{d} = \rho^{-1}(r_2) \) and \( \tilde{d}' = \rho^{-1}(r'_2) \) are simple closed curves in \( M \) which satisfy \( \tau(\tilde{d}) = -\tilde{d} \), and \( \tau(\tilde{d}') = -\tilde{d}' \).

Now, by choosing appropriate orientations for \( \tilde{a}, \tilde{d}, \) and \( \tilde{d}' \), we we see that \( \tilde{D} = \{\tilde{a}, \tilde{b}_1, \tilde{b}_2, \tilde{c}_1, \tilde{c}_2, \tilde{d}\} \) and \( \tilde{D}' = \{\tilde{a}, \tilde{b}_1, \tilde{b}_2, \tilde{c}'_1, \tilde{c}'_2, \tilde{d}'\} \) are two admissible decompositions for \((M,W)\).

By construction, \( \tilde{b}_1 \) is freely homotopic to \( b_0 \), and \( \tilde{c}_1 \) is freely homotopic to \( \pm c_0 \). Note that the orientation of \( t_2 \) and \( t'_2 \) are chosen according to the orientation of \( t_1 \). By Corollary 5.3, the canonical bases of \( H_1(M,\mathbb{Z}) \) associated to \( \tilde{D} \) and \( \tilde{D}' \) are related by an element of \( \Gamma \). Similarly, the canonical bases of \( H_1(M,\mathbb{Z}) \) associated to \( \tilde{D}' \) and \( \tilde{D}' \) are also related by an element of \( \Gamma \). Now, by Proposition 4.5, we know that the canonical bases associated to \( \tilde{D} \) and \( \tilde{D}' \) are related by an element of \( \Gamma \). Hence, the proposition is proved for this case.

**Case** \( n > 1 \): By Lemma 5.5, there exists a simple closed geodesic \( c''_0 \) verifying Property (\( \mathcal{P} \)) such that

- \( \text{Card}\{c''_0 \cap b_0 \} = 0 \),
- \( \text{Card}\{c''_0 \cap c_0 \} = 1 \),
- \( \text{Card}\{c''_0 \cap c'_0 \} < n \).

Let \( \tilde{D}'' = \{a'', b''_1, b''_2, c''_1, c''_2, d''\} \) be the standard decomposition associated to the pair of simple closed geodesics \( (b_0, c''_0) \). The arguments in Case \( n = 1 \) show that the canonical bases associated to \( \tilde{D}'' \) and \( \tilde{D} \) are related by an element of \( \Gamma \). Now, since \( \text{Card}\{c''_0 \cap c''_0 \} < n \), the induction hypothesis implies that the canonical bases associated to \( \tilde{D}'' \) and \( \tilde{D}' \) are also related by an element of \( \Gamma \), and the proposition follows. \( \square \)
Proposition 5.7 Let $D = \{a, b_1, b_2, c_1, c_2, d\}$ be an admissible decomposition of the pair $(M, W)$. Let $g$ be a simple closed geodesic in $M$ verifying Property $(\mathcal{P})$, then there exists an admissible decomposition $D' = \{a', b_1', b_2', c_1', c_2', d'\}$, and an element $\gamma$ in $\Gamma$ such that $b_1'$ is freely homotopic to $g$, and the canonical bases of $H_1(M, \mathbb{Z})$ associated to $D$ and $D'$ are related by $\gamma$.

Proof: Let $b_0, c_0$ be the simple closed geodesics in the free homotopy classes of $b_1$ and $c_1$ respectively. If $g = \pm b_0$ or $g = \pm c_0$, then Proposition 5.6 allows us to conclude, since the roles of $b_1$ and $c_1$ are interchanged by the $S$ move, and the orientation of $b_1$ is irrelevant because $-\text{Id} = S^2 \in \Gamma$.

Assume that $g \neq \pm b_0$ and $g \neq \pm c_0$. Let $n$ be the number of intersection points between $g$ and $b_0 \cup c_0$. Again, the proposition will be proved by induction.

Case $n = 1$: in this case, we can suppose that $g \cap b_0 = \emptyset$, and $\text{Card}\{g \cap c_0\} = 1$. Therefore, we can take $D'$ to be a standard decomposition associated to the pair of geodesics $(b_0, g)$, and the proposition follows directly from Proposition 5.6.

Case $n > 1$: by Lemma 5.5, there exists a simple closed geodesic $h$ verifying Property $(\mathcal{P})$ such that

1. $\text{Card}\{h \cap b_0\} = 0$,
2. $\text{Card}\{h \cap c_0\} = 1$,
3. $\text{Card}\{h \cap g\} < \text{Card}\{c_0 \cap g\}$.

Let $D'' = \{a'', b_1'', b_2'', c_1'', c_2'', d''\}$ be the standard decomposition associated to the pair of geodesics $(b_0, h)$. From Case $n = 1$, we know that there exists $\gamma'' \in \Gamma$ such that the canonical bases of $H_1(M, \mathbb{Z})$ associated to $D$ and $D''$ are related by $\gamma''$.

Since $\text{Card}\{g \cap (b_0 \cup h)\} < \text{Card}\{g \cap (b_0 \cup c_0)\}$, by the induction hypothesis, there exists an admissible decomposition $D' = \{a', b_1', b_2', c_1', c_2', d'\}$ such that $b_1'$ is freely homotopic to $g$, and the canonical bases of $H_1(M, \mathbb{Z})$ associated to $D''$ and $D'$ are related by an element $\gamma' \in \Gamma$. The proposition is then proved.

We are now able to give the complete proof of Theorem 4.4. Let $b_0, c_0, b_1', c_1'$ be the simple closed geodesics in the free homotopy classes of $b_1, c_1, b_1'$, and $c_1'$ respectively. By Proposition 5.7, there exists an admissible decomposition $D'' = \{a'', b_1'', b_2'', c_1'', c_2'', d''\}$ for the pair $(M, W)$, and an element $\gamma'' \in \Gamma$ such that $b_1''$ is freely homotopic to $b_1'$ and

$$
\begin{pmatrix}
  a'' \\
  b'' \\
  c'' \\
  e''
\end{pmatrix} = \gamma''
\begin{pmatrix}
  a \\
  b \\
  c \\
  e
\end{pmatrix},
$$

where $(a, b, c, e)$ and $(a'', b'', c'', e'')$ are the canonical bases of $H_1(M, \mathbb{Z})$ associated to $D$ and $D''$ respectively. Now, by Proposition 5.6, we know that there exists $\gamma' \in \Gamma$ such that
where \((a', b', c', e')\) is the canonical basis associated to \(D'\). Therefore

\[
\begin{pmatrix}
  a' \\
  b' \\
  c' \\
  e'
\end{pmatrix}
= \gamma' \begin{pmatrix}
a'' \\
b'' \\
c'' \\
e''
\end{pmatrix},
\]

with \(\gamma = \gamma' \gamma'' \in \Gamma\).

\[\square\]

## 6 System of decomposing curves on \(\mathbb{CP}^1\)

In this section, \(\lambda_0, \lambda_1, \ldots, \lambda_5\) will be six fixed points in the sphere \(\mathbb{CP}^1\). We denote by \(\Lambda\) the set \(\{\lambda_1, \ldots, \lambda_5\}\). The pair \((\lambda_0, \Lambda)\) determines an element \((M, W)\) in \(H(2)/\mathbb{C}^*\). Namely, \(M\) is the two-sheeted covering of \(\mathbb{CP}^1\) branched above \(\lambda_0, \lambda_1, \ldots, \lambda_5\), and \(W\) is the Weierstrass point which is mapped to \(\lambda_0\).

Roughly speaking, a system of decomposing curves on \(\mathbb{CP}^1\) associated to the pair \((\lambda_0, \Lambda)\) is a family of four simple curves passing through \(\lambda_0\) which gives rise to an admissible decomposition of the pair \((M, W)\). This notion will allow us, given the period matrix of the canonical homology basis associated to the induced decomposition, to compute the values of \((\lambda_0, \lambda_1, \ldots, \lambda_5)\) by some fixed theta functions. We will also introduce some elementary moves for systems of decomposing curves on \(\mathbb{CP}^1\) which give rise to the elementary moves for the associated admissible decompositions of the pair \((M, W)\). From this, together with Lemma 3.3, we will see that any canonical basis of \(H_1(M, \mathbb{Z})\) can be represented by a canonical basis associated to an admissible decomposition for some pair \((M, \tilde{W})\), where \(\tilde{W}\) is a Weierstrass point of \(M\).

**Definition 6.1** A system of decomposing curves on \(\mathbb{CP}^1\) for the pair \((\lambda_0, \Lambda)\) is an ordered family of four non-oriented simple curves \(\{t_1, r_1, t_2, r_2\}\) with base-point \(\lambda_0\) verifying the following properties.

- \(t_1, t_2\) are simple closed curves such that \(t_1 \cap t_2 = \{\lambda_0\}\), and \(t_i \cap \Lambda = \emptyset\), \(i = 1, 2\),
- \(t_i\), \(i = 1, 2\), bounds an open disk \(D_i\) which contains exactly two points in \(\Lambda\) such that \(D_1 \cap D_2 = \emptyset\).
- \(r_i\), \(i = 1, 2\), is a simple arc joining \(\lambda_0\) to a point in \(D_i \cap \Lambda\) such that \(\text{int}(r_i) \subset D_i\), and \(\text{int}(r_i) \cap \Lambda = \emptyset\).
Let $M$ be the Riemann surface defined by the equation
\[ w^2 = \prod_{i=0}^{5} (z - \lambda_i). \]

Let $\rho : M \to \mathbb{CP}^1$ denote the two-sheeted branched cover from $M$ onto $\mathbb{CP}^1$ ramified over $\lambda_0, \lambda_1, \ldots, \lambda_5$, and let $W$ be the Weierstrass point of $M$ which is projected to $\lambda_0$. A system of decomposing curves $S = (t_1, r_1, t_2, r_2)$ for the pair $(\lambda_0, \lambda)$ determines an admissible decomposition $D = \{a, b_1, b_2, c_1, c_2, d\}$ of the pair $(M, W)$ as follows
\[
\begin{align*}
. & a = \rho^{-1}(r_1), \\
. & d = \rho^{-1}(r_2), \\
. & b_1 \cup b_2 = \rho^{-1}(t_1), \\
. & c_1 \cup c_2 = \rho^{-1}(t_2).
\end{align*}
\]

Actually, the system $S$ determines two admissible decompositions, one for each orientation of $a$, these decompositions are obtained one from the other by reversing the orientation of every curves in the same family. In what follows, we will call both of them the decompositions associated to $S$. Since $-\text{Id} \in \Gamma$, our arguments will not be affected by this imprecision.

By renumbering the points in $\lambda$ if necessary, we can assume that $\{\lambda_1, \lambda_2\} \subset D_1$, $\{\lambda_3, \lambda_4\} \subset D_2$. Let $s_i, i = 1, 2$, be a simple arc in $D_i$ joining $\lambda_{2i-1}$ to $\lambda_{2i}$. Let $s_3$ be a simple arc disjoint from $D_1 \cup D_2$ joining $\lambda_0$ to $\lambda_5$.

To see that $D$ is an admissible decomposition of $(M, W)$, we construct $M$ by gluing two copies of the 3-holed sphere obtained by slitting $\mathbb{CP}^1$ along the arcs $s_1, s_2, s_3$. Assume that $r_2$ joins $\lambda_0$ to $\lambda_3$, let $r_3$ be a simple arc in $\mathbb{CP}^1$ joining $\lambda_4$ to $\lambda_5$ such that
\[ r_3 \cap D_1 = r_3 \cap r_2 = \emptyset, \quad \text{and} \quad \text{Card}\{r_3 \cap t_2\} = 1, \]
then the canonical basis of $H_1(M, \mathbb{Z})$ associated to $D$ is $(a, b, c, e)$, where
\[
\begin{align*}
. & b = \rho^{-1}(s_1), \\
. & c = \rho^{-1}(s_2), \\
. & e = \rho^{-1}(r_3).
\end{align*}
\]

We will call $(a, b, c, e)$ the canonical homology basis of $M$ associated to the system $\{t_1, r_1, t_2, r_2\}$. Observe that $(a, b, c, e)$ is also a sample of the canonical bases of $H_1(M, \mathbb{Z})$ constructed in Section 2. Without loss of generality, we can assume that $\lambda_0 = \infty, \lambda_1 = 0, \lambda_2 = 1$. From the proof of Theorem 2.2, we see that, if the period matrix of $(a, b, c, e)$ is given, then the values of $\lambda_3, \lambda_4, \lambda_5$ can be computed from some fixed theta functions.

Let us now introduce the elementary moves for systems of decomposing curves.
Assume that $r_1$ joins $\lambda_0$ to $\lambda_1$, let $r'_1$ be a simple arc joining $\lambda_0$ to $\lambda_2$ such that $\text{int}(r'_1) \subset D_1$, and $r'_1 \cap r_1 = \{\lambda_0\}$. Then $\{t_1, r'_1, t_2, r_2\}$ is another system of decomposing curves for $(\lambda_0, \Lambda)$, and the admissible decomposition associated to $\{t_1, r'_1, t_2, r_2\}$ is obtained from the one associated to $\{t_1, r_1, t_2, r_2\}$ by a $T$ move. We will call the action of replacing $r_1$ by $r'_1$ the $T$ move for systems of decomposing curves.

Clearly, $\{t_2, r_2, t_1, r_1\}$ is also a system of decomposing curves for $(\lambda_0, \Lambda)$. The admissible decomposition for $(M, W)$ associated to $\{t_2, r_2, t_1, r_1\}$ is obtained from the one associated to $\{t_1, t_2, r_1, r_2\}$ by an $S$ move. We will call the action of interchanging the pairs $(t_1, r_1)$ and $(t_2, r_2)$ the $S$ move for systems of decomposing curves.

Let $t'_2$ be a simple closed curve in $\mathbb{C}P^1 \setminus D_1$ with base-point $\lambda_0$ which bounds an open disk $D'_2$ verifying the following properties:

- $\{\lambda_3, \lambda_5\} \subset D'_2$,
- $\text{int}(r_2) \subset D'_2$,
- and $\text{Card}\{r_3 \cap t'_2\} = 1$.

It follows that $\{t_1, r_1, t'_2, r_2\}$ is a system of decomposing curves for $(\lambda_0, \Lambda)$. The admissible decomposition for $(M, W)$ associated to $\{t_1, r_1, t'_2, r_2\}$ is obtained from the one associated to $\{t_1, r_1, t_2, r_2\}$ by an $R$ move. Therefore, we will call the action of replacing $t_2$ by $t'_2$ the $R$ move for systems of decomposing curves.
Let $t_3$ be a simple closed curve with base-point $\lambda_5$ which separates $\{\lambda_3, \lambda_4\}$ from $\{\lambda_0, \lambda_1, \lambda_2\}$, and bounds an open disk $D_3$ containing $s_2$ and $\text{int}(r_3)$. Let $t_4$ be a simple closed curve with base-point $\lambda_5$ which separates $\{\lambda_1, \lambda_2\}$ from $\{\lambda_0, \lambda_3, \lambda_4\}$, and bounds an open disk $D_4$ disjoint from $D_3$ containing $s_1$. Let $r_4$ be a simple arc joining $\lambda_5$ to $\lambda_2$ such that $\text{int}(r_4) \subset D_4$. Observe that $S' = (t_3, r_3, t_4, r_4)$ is a system of decomposing curves for the pair $(\lambda_5, \Lambda^{(5)})$, where $\Lambda^{(5)} = \{\lambda_0, \ldots, \lambda_4\}$ (see the figure here below). As subsets of $M$, the canonical homology basis $(a', b', c', e')$ associated to $S'$ can be chosen to be

$$
\begin{pmatrix}
a' \\
b' \\
c' \\
e'
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
e
\end{pmatrix} = U
\begin{pmatrix}
a \\
b \\
c \\
e
\end{pmatrix}.
$$

We will call the transformation from $S$ into $S'$ the $U$ move.

From the definition, the following lemma is clear.

**Lemma 6.2** Given a system of decomposing curves $\{t_1, r_1, t_2, r_2\}$ for the pair $(\lambda_0, \Lambda)$, for any $\gamma \in \Gamma$, there exists another system of decomposing curves $\{t'_1, r'_1, t'_2, r'_2\}$ for the same pair so that the canonical homology bases associated to these two systems are related by $\gamma$.

By Lemma 3.3, we know that the family $\{T, S, R, U\}$ generates the whole group $\text{Sp}(4, \mathbb{Z})$, it follows that we have

**Lemma 6.3** Let $S = \{t_1, r_1, t_2, r_2\}$ be a system of decomposing curves for the pair $(\lambda_0, \Lambda)$, and let $A$ be matrix in $\text{Sp}(4, \mathbb{Z})$. There exists a system of decomposing curves $S' = \{t'_1, r'_1, t'_2, r'_2\}$ for another pair $(\lambda_i, \Lambda^{(i)})$, where $\Lambda^{(i)} = \{\lambda_0, \ldots, \lambda_5\} \setminus \{\lambda_i\}$, such that the canonical homology bases associated to $S$ and $S'$ are related by $A$. 

30
7 Proof of Theorem 1.1

7.1 The map \( \Xi \)

Let \( \mathcal{M} \) denote the quotient \( \mathcal{H}(2)/\mathbb{C}^* \). We define the map \( \Xi \) from \( \mathcal{M} \) to \( \Gamma \backslash \mathcal{J}_2 \) as follows: let \( (M, W) \) be a pair in \( \mathcal{M} \), then there exists a two-sheeted branched cover \( \rho : M \to \mathbb{CP}^1 \) ramified above six points \( \{\lambda_0, \lambda_1, \ldots, \lambda_5\} \) such that \( \rho(W) = \lambda_0 \). Let \( \Delta \) denote the set \( \{\lambda_1, \ldots, \lambda_5\} \), \( S = \{t_1, r_1, t_2, r_2\} \) be a system of decomposing curves for the pair \( (\lambda_0, \Delta) \), and \( (a, b, c, e) \) be the canonical basis of \( H_1(M, \mathbb{Z}) \) associated to \( S \). Finally, let \( \Pi \) denote the period matrix of \( M \) associated to the basis \( (a, b, c, e) \), then the image of \( (M, W) \) is defined to be the orbit \( \Gamma \cdot \Pi \) in \( \Gamma \backslash \mathcal{J}_2 \).

Since the basis \( (a, b, c, e) \) arises from an admissible decomposition of the pair \( (M, W) \), Theorem 4.4 ensures that the map \( \Theta \) is well defined. We will show that \( \Xi \) is a homeomorphism from \( \mathcal{H}(2)/\mathbb{C}^* \) and \( \Gamma \backslash \mathcal{J}_2 \), which implies Theorem 1.1.

7.2 Injectivity of \( \Xi \)

Let \( (M, W) \) and \( (M', W') \) be two pairs in \( \mathcal{M} \). Assume that \( M \) and \( M' \) are defined by the equations 
\[
w^2 = \prod_{i=0}^{5} (z - \lambda_i), \quad w'^2 = \prod_{i=0}^{5} (z - \lambda'_0)
\]
so that \( W \), and \( W' \) correspond to \( \lambda_0 \) and \( \lambda'_0 \) respectively. As usual, let \( \Delta \) and \( \Delta' \) denote the sets \( \{\lambda_1, \ldots, \lambda_5\} \), and \( \{\lambda'_1, \ldots, \lambda'_5\} \) respectively.

Let \( \Pi \) (resp. \( \Pi' \)) be the period matrices of \( M \) (resp. \( M' \)) associated to the canonical homology basis arising from a system \( S \) (resp. \( S' \)) of decomposing curves for the pair \( (\lambda_0, \Delta) \) (resp. \( (\lambda'_0, \Delta') \)). Assume that there exists an element \( \gamma \) of \( \Gamma \) such that \( \Pi' = \gamma \cdot \Pi \). By Lemma 6.2, we know that there exists a system \( S' \) of decomposing curves for the pair \( (\lambda_0, \Delta) \) such that the canonical homology basis of \( M \) associated to \( S' \) is related to the one associated to \( S \) by \( \gamma \). It follows that the period matrix of \( M \) corresponding to the basis associated to \( S' \) is equal to \( \Pi' \).

Using an element of \( \text{PSL}(2, \mathbb{C}) \) we can assume that \( \lambda_1 = \lambda'_1 = 0, \lambda_2 = \lambda'_2 = 1, \) and \( \lambda_0 = \lambda'_0 = \infty \). Then from Theorem 2.2, we see that the values of \( \lambda_i \) and \( \lambda'_i \) \( (i = 3, 4, 5) \) can be computed by the same theta functions, with the same data. Therefore we have \( \lambda_i = \lambda'_i, \quad i = 3, 4, 5 \), and it follows that there exists a conformal homeomorphism \( \phi : M \to M' \) such that \( \phi(W) = W' \).

7.3 Surjectivity of \( \Xi \)

Let \( \tilde{\Pi} \) be a matrix in \( \mathcal{J}_2 \), we will show that there exists a pair \( (M, W) \) in \( \mathcal{M} \) such that \( \Xi(M, W) = \Gamma \cdot \tilde{\Pi} \). Since \( \tilde{\Pi} \in \mathcal{J}_2 \), there exists a Riemann surface of genus two \( M \) and a canonical homology basis of \( M \) whose period matrix is \( \tilde{\Pi} \). We know that \( M \) is defined by an equation of the form 
\[
w^2 = \prod_{i=0}^{5} (z - \lambda_i)
\]
de note the set \( \{\lambda_0, \lambda_1, \ldots, \lambda_5\}\) \( \setminus \{\lambda_i\} \), for \( i = 0, \ldots, 5 \).

Let \( S \) be a system of decomposing curves for the pair \( (\lambda_0, \Delta^{(0)}) \), and \( \Pi \) be the period matrix of the canonical homology basis of \( M \) associated to \( S \). By definition, there exists an element \( A \in \text{Sp}(4, \mathbb{Z}) \) such
that $\hat{\Pi} = A \cdot \Pi$. According to Lemma 6.3, we can transform $S$ into a system $\hat{S}$ of decomposing curves for a pair $(\lambda, \lambda^{(i)})$ such that, if $(a, b, c, e)$ and $(\hat{a}, \hat{b}, \hat{c}, \hat{e})$ are the canonical homology bases associated to $S$ and $\hat{S}$ respectively, then
\[
\begin{pmatrix}
\hat{a} \\
\hat{b} \\
\hat{c} \\
\hat{e}
\end{pmatrix} = A \cdot \begin{pmatrix} a \\ b \\ c \\ e \end{pmatrix}.
\]
Consequently, the period matrix of $(\hat{a}, \hat{b}, \hat{c}, \hat{e})$ is $\hat{\Pi}$, and by definition $\Gamma \cdot \hat{\Pi} = \Xi(M, W_i)$, where $W_i$ is the Weierstrass of $M$ which is projected to $\lambda_i$.

### 7.4 Continuity of $\Xi$

To prove the continuity of $\Xi$ we will consider the inverse map $\Xi^{-1} : \Gamma \backslash \mathcal{J}_2 \to \mathcal{M}$. Given $\Pi$ in $\mathcal{J}_2 \subset \mathbb{H}_2$, there exist complex numbers $\{\lambda_0, \lambda_1, \ldots, \lambda_5\}$ such that $\Pi$ is the period matrix of a canonical homology basis, which arises from a system of decomposing curves for the pair $(\lambda_0, \lambda)$, where $\lambda = (\lambda_1, \ldots, \lambda_5)$, of the Riemann surface $M$ defined by the equation $w^2 = \prod_{i=0}^{5} (z - \lambda_i)$. A neighborhood of $\Pi$ in $\mathbb{H}_2$ consists of period matrices of the same canonical homology basis on Riemann surfaces closed to $M$.

Using $\text{PSL}(2, \mathbb{C})$, we can assume that $\lambda_0 = 0$, it follows that $\omega = \frac{dz}{w}$ is a holomorphic 1-form with double zero at $\rho^{-1}(0)$, where $\rho : M \to \mathbb{CP}^1$ is the double cover ramified over $(\lambda_0, \ldots, \lambda_5)$. Let $S = \{t_1, r_1, t_2, r_2\}$ be a system of decomposing curves for the pair $(\lambda_0, \lambda)$. Let $s_1, s_2$ be as in the previous section, and $a = \rho^{-1}(r_1), b = \rho^{-1}(s_1), c = \rho^{-1}(s_2), d = \rho^{-1}(r_2)$, then $(a, b, c, d)$ is a (non-canonical) homology basis of $M$. We know that the map
\[
\Phi : \mathcal{U} \to \mathbb{C}^4
\]
\[
(M, \omega) \mapsto (\int_a \omega, \int_b \omega, \int_c \omega, \int_d \omega)
\]
is a local chart for $\mathcal{H}(2)$ in the neighborhood $\mathcal{U}$ of $(M, \omega)$. But we have
\[
\int_a \omega = 2 \int_{r_1} \frac{dz}{w}, \int_b \omega = 2 \int_{s_1} \frac{dz}{w}, \int_c \omega = 2 \int_{s_2} \frac{dz}{w}, \int_d \omega = 2 \int_{r_2} \frac{dz}{w},
\]
and clearly the integrals of $\frac{dz}{w}$ along $s_1, s_2, r_1, r_2$ depend continuously on $(\lambda_1, \ldots, \lambda_5)$. Since $\lambda_i$ can be computed from $\Pi$ by some theta functions, we get a continuous map $\Psi$ from a neighborhood of $\Pi$ in $\mathcal{J}_2$ into $\mathbb{C}^4$. Now, in a neighborhood of $\Pi$, the map $\Xi^{-1}$ is the composition of $\Psi$ and the natural projection $\mathbb{C}^4 \setminus \{0\} \to \mathbb{CP}^3$. It follows immediately that $\Xi^{-1}$ is continuous, and so is $\Xi$.

### 7.5 $[\text{Sp}(4, \mathbb{Z}) : \Gamma] = 6$

We have a natural projection from $\mathcal{M}$ onto the moduli space of closed Riemann surface of genus two $\mathcal{M}_2$, which is homeomorphic to $\text{Sp}(4, \mathbb{Z}) \setminus \mathcal{J}_2$, by associating to any element $(M, W)$ of $\mathcal{M}$ the point $M$ in $\mathcal{M}_2$. Every Riemann surface of genus two has six Weierstrass points, and the group of automorphisms of a
generic one contains exactly two elements, the identity and the hyperelliptic involution, both fix all the Weierstrass points. Therefore, the pre-image of a generic point in \( \mathcal{M}_2 \) contains exactly six points. Thus, we have \( [\text{Sp}(4, \mathbb{Z}) : \Gamma] = 6 \). The proof Theorem 1.1 is now complete. \( \square \)

Appendices

A Proof of Lemma 3.3

For any \( g \geq 1 \), let \( \sigma \) be the permutation of \( \{1, 2, \ldots, 2g\} \) that transpose \( 2i \), and \( 2i - 1 \), for \( i = 1, \ldots, g \). The elementary symplectic matrices are the matrices

\[
E_{ij} = \begin{cases} 
\text{Id}_{2g}, & \text{if } i = \sigma(j); \\
\text{Id}_{2g} + e_{ij} - (-1)^{i+j}e_{\sigma(j)\sigma(i)}, & \text{otherwise},
\end{cases}
\]

where \( i \neq j \), and \( e_{ij} \) is the matrix whose the \((i, j)\)-entry is 1, and all other entries are 0. It is a classical fact that \( \text{Sp}(2g, \mathbb{Z}) \) is generated by elementary symplectic matrices ([3], Chap. 7).

For the case \( g = 2 \), we have

\[
\begin{align*}
E_{12} &= E_{21}^t = \begin{pmatrix} 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
E_{34} &= E_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
E_{13} &= E_{42}^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \end{pmatrix}, \\
E_{31} &= E_{24}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
E_{14} &= E_{32} = \begin{pmatrix} 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \\
E_{41} &= E_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

All we need is to verify that \( E_{ij}, (i \neq j) \) is contained in the group \( \Gamma' \) generated by \( \{T, R, S, U\} \).

- It is clear that \( E_{12}, E_{34}, E_{43} \) belong to \( \Gamma \subset \Gamma' \).
- We have \( E_{21} = U^{-1}T^{-1}U \in \Gamma' \).
- We have \( E_{13} = SM \in \Gamma' \).
Since $\Gamma$ contains \( \text{Id}_{20} \in SL(2, \mathbb{Z}) \), we have $S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ belongs to $\Gamma$, it follows that $S_2 = U^{-1} S_1 U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ belongs to $\Gamma'$.

- We have $E_{31} = (S_1 S_2) E_{13}^{-1} (S_1 S_2)^{-1}$.
- We have $E_{14} = S_1 E_{13} S_1^{-1}$.
- We have $E_{41} = S_2 E_{13} S_2^{-1}$.

The lemma follows directly from these calculations. \( \square \)

## B A family of $\Gamma$ right cosets in $\text{Sp}(4, \mathbb{Z})$

In this section, we explicit a partition of the group $\text{Sp}(4, \mathbb{Z})$ into $\Gamma$ right cosets. Recall that, see Lemma 3.3, the group $\text{Sp}(4, \mathbb{Z})$ is generated by $T, S, R,$ and $U$. Set

\[ \mathcal{F} = \{ \Gamma, U \cdot \Gamma, RU \cdot \Gamma, SRU \cdot \Gamma, URU \cdot \Gamma, USRU \cdot \Gamma \}. \]

By Lemma 3.2, we know that the action of $\Gamma$ on $(\mathbb{Z}_2)^4 \setminus \{0\}$ has two orbits $O_1$ and $O_2$, therefore we have a simple criterion to show that an element of $\text{Sp}(4, \mathbb{Z})$ does not belong to $\Gamma$. Consequently, it is easy to verify that the elements in the family $\mathcal{F}$ are all distinct.

We will also determine explicitly the action of $T^\pm 1, R^\pm 1, S^\pm 1, U^\pm 1$ on $\mathcal{F}$ by multiplication from the left. Note that, since $S^{-1} = -S$ (resp. $U^{-1} = -U$), and $-\text{Id}_4 \in \Gamma$, the actions of $S$ and $S^{-1}$ (resp. $U$ and $U^{-1}$) are identical. Details of the calculations are lengthly and uninteresting, hence will be omitted. The final result is resumed in the following table.

|     | $\Gamma$ | $U \cdot \Gamma$ | $RU \cdot \Gamma$ | $SRU \cdot \Gamma$ | $URU \cdot \Gamma$ | $USRU \cdot \Gamma$ |
|-----|----------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $T$ | $\Gamma$ | $U \cdot \Gamma$ | $RU \cdot \Gamma$ | $SRU \cdot \Gamma$ | $URU \cdot \Gamma$ | $USRU \cdot \Gamma$ |
| $R$ | $\Gamma$ | $RU \cdot \Gamma$ | $U \cdot \Gamma$ | $SRU \cdot \Gamma$ | $URU \cdot \Gamma$ | $USRU \cdot \Gamma$ |
| $S$ | $\Gamma$ | $U \cdot \Gamma$ | $SRU \cdot \Gamma$ | $RU \cdot \Gamma$ | $USRU \cdot \Gamma$ | $URU \cdot \Gamma$ |
| $U$ | $U \cdot \Gamma$ | $\Gamma$ | $URU \cdot \Gamma$ | $USRU \cdot \Gamma$ | $RU \cdot \Gamma$ | $SRU \cdot \Gamma$ |
| $T^{-1}$ | $\Gamma$ | $U \cdot \Gamma$ | $RU \cdot \Gamma$ | $URU \cdot \Gamma$ | $SRU \cdot \Gamma$ | $USRU \cdot \Gamma$ |
| $R^{-1}$ | $\Gamma$ | $RU \cdot \Gamma$ | $U \cdot \Gamma$ | $SRU \cdot \Gamma$ | $URU \cdot \Gamma$ | $USRU \cdot \Gamma$ |
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