A geometric characterization of the finitary special linear and unitary Lie algebras

Hans Cuypers,* Marc Oostendorp
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
email: f.g.m.t.cuypers@tue.nl

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Abstract

An extremal element $x$ in a Lie algebra $g$ is an element for which the space $[x,[x,g]]$ is contained in the linear span of $x$. Long root elements in classical Lie algebras are examples of extremal elements. Lie algebras generated by extremal elements lead to geometries with as points the 1-spaces generate by extremal elements and as lines the 2-spaces whose non-zero elements are pairwise commuting extremal elements.

In this paper we show that the finitary special linear Lie algebras can be characterized by their extremal geometry. Moreover, we also show that the finitary special unitary Lie algebras can be characterized by the fact that their geometry has no lines, but that after extending the field quadratically, the geometry is that of a special linear Lie algebra.

*corresponding author
1 Introduction

An extremal element $x$ in a Lie algebra $\mathfrak{g}$ is an element for which the space $[x, [x, \mathfrak{g}]]$ is contained in the linear span of $x$ (plus some extra conditions in case the characteristic of $F$ is 2). Such elements are called pure, if $[x, [x, \mathfrak{g}]]$ is 1-dimensional. Long root elements in classical Lie algebras are examples of extremal elements.

If $E$ is the set of extremal elements of a Lie algebra $\mathfrak{g}$, then we can define a geometry on the set of extremal points $\mathcal{E}$, which is the set of 1-dimensional subspaces of $\mathfrak{g}$ spanned by extremal elements. The lines of this geometry are those 2-dimensional subspaces (identified with the collection of extremal points in it) of $\mathfrak{g}$ all whose elements are pairwise commuting extremal elements. This geometry is called the extremal geometry of $\mathfrak{g}$ and is denoted by $\Gamma(\mathfrak{g})$.

In [4] Cohen et al. started the study of simple Lie algebras generated by their extremal elements, in order to obtain a characterization of the classical Lie algebras.

The main result of Cohen and Ivanyos [2, 3] states that for a finite dimensional simple Lie algebra $\mathfrak{g}$ generated by its pure extremal elements the extremal geometry $\Gamma(\mathfrak{g})$ either contains no lines or is a so-called root shadow space of a spherical building. Subsequently, in combined work of Fleischmann, Roberts, Shpectorov and the first author of this paper [8, 6], it has been shown that in case $\Gamma(\mathfrak{g})$ is a root shadow space of a spherical building of rank at least 3, the isomorphism type of $\mathfrak{g}$ is uniquely determined by its extremal geometry. In particular, classical Lie algebras with an extremal geometry of finite rank at least 3 are characterized by this extremal geometry.

In this paper we continue the study of the relation between a simple Lie algebra generated by its pure extremal elements and its extremal geometry. In particular, we focus on (variations of) special linear and unitary Lie algebras, both finite and infinite dimensional.

Let $V$ be a vector space over a field $F$, then for each $0 \neq v \in V$ and $0 \neq \phi \in V^*$, the dual of $V$, with $\phi(v) = 0$, the linear map

$$t_{v,\phi} : V \rightarrow V$$

defined by $t_{v,\phi}(w) = \phi(w)v$ for all $w \in V$ is an infinitesimal transvection in the general linear Lie algebra $\mathfrak{gl}(V)$. It is a pure extremal element, see Section [2].

The infinitesimal transvections are finitary linear maps on $V$ (i.e., their kernel has finite codimension in $V$) and traceless, hence they are contained
in, and actually generate, $\mathfrak{fsl}(V)$, the Lie algebra of all traceless finitary linear maps on $V$.

Now suppose $\Pi$ to be a subspace of $V^*$, then we can consider the Lie subalgebra

$$\mathfrak{fsl}(V, \Pi) := \langle t_{v,\phi} | 0 \neq v \in V, 0 \neq \phi \in \Pi \text{ and } \phi(v) = 0 \rangle$$

of $\mathfrak{fsl}(V)$. The Lie algebras $\mathfrak{fsl}(V, \Pi)$ are simple up to a center, provided that the annihilator in $V$ of $\Pi$, i.e $\text{Ann}_V(\Pi) = \{v \in V \mid \phi(v) = 0 \text{ for all } \phi \in \Pi\}$, is trivial.

The extremal geometry of the Lie algebra $\mathfrak{fsl}(V, \Pi)$ is isomorphic to the geometry $\Gamma(V, \Pi)$ whose points are the incident point-hyperplane pairs $(p, H)$ of $\mathbb{P}(V)$ with $H$ the kernel of an element from $\Pi$. A line of $\Gamma(V, \Pi)$ consists of point-hyperplane pairs $(p, H)$ where $p$ is running over a line of $\mathbb{P}(V)$ contained in $H$, or, dually, where $H$ is running over all hyperplanes containing a codimension 2 space on $p$.

Our first result characterizes the Lie algebras $\mathfrak{fsl}(V, \Pi)$ by their extremal geometry:

**Theorem 1.1.** Let $V$ be a vector space over a field $\mathbb{F}$ of dimension at least 3 and $\Pi$ a subspace of $V^*$ with $\text{Ann}_V(\Pi) = \{0\}$.

Suppose $g$ is a simple Lie algebra generated by its set of pure extremal elements with extremal geometry $\Gamma(g)$ isomorphic to $\Gamma(V, \Pi)$.

Then $g$ is isomorphic to $\mathfrak{fsl}(V, \Pi)$ modulo its center.

We will also consider several other subalgebras of $\mathfrak{fsl}(V, V^*)$ generated by infinitesimal transvections. If $V$ is equipped with a non-degenerate symplectic form $f$, then the infinitesimal transvections $t_{v,\phi}$ with $0 \neq v \in V$ and $\phi \in V^*$ given by $\phi(w) = f(v, w)$ are contained in and generate the finitary symplectic Lie algebra $\mathfrak{fsp}(V, f)$, see [5]. The extremal geometry of this Lie algebra consist then of all point-hyperplane pairs $(p, H)$, where $H$ is the hyperplane of all points perpendicular to $p$ with respect to $f$. Obviously, this geometry does not contain lines.

The main result of [5] can be stated as follows.

**Theorem 1.2** (Cuypers, Fleischmann). Let $g$ be a simple Lie algebra over a field $\mathbb{F}$ of characteristic $\neq 2$ generated by its pure extremal elements. If the extremal geometry of $g$ does not contain lines, then either $g$ is isomorphic to $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space $(V, f)$, or there exists a degree 2 extension $\mathbb{K}$ of $\mathbb{F}$ such that the extremal geometry of $g \otimes_{\mathbb{F}} \mathbb{K}$ does contain lines.
The latter occurs, for example, in the following situation. If $V$ is a $K$-vector space equipped with a non-degenerate skew-Hermitian form $h$, with respect to some field automorphism $\sigma$ of order 2, then the infinitesimal transvections $t_{v,\phi}$ with $0 \neq v \in V$ and $\phi \in V^*$ given by $\phi(w) = h(v, w)$ (assuming $h$ to be linear in the first coordinate) are contained in and generate the finitary special unitary Lie algebra $\mathfrak{fsu}(V,h)$ defined over the subfield $\mathbb{F}$ of $K$ fixed by $\sigma$, provided $h$ is trace-valued (see Section 3). The extremal geometry of this Lie algebra does not contain lines. However, in this case the subalgebra of $\mathfrak{fsl}(V)$ (so, allowing scalars from $K$) generated by these transvections will be the algebra $\mathfrak{fsl}(V, \Pi)$, where $\Pi$ is the subspace generated by the elements $\phi \in V^*$ for which there is a vector $v \in V$ with $\phi(v) = 0$ and $\phi(w) = h(w, v)$ for all $w \in V$. In particular, its extremal geometry will be isomorphic with $\Gamma(V, \Pi)$. This latter situation is characterized by the following result.

**Theorem 1.3.** Suppose $\mathfrak{g}$ is a simple Lie algebra over a field $\mathbb{F}$ generated by pure extremal elements whose extremal geometry does not contain lines.

Assume there exists a quadratic Galois extension $K$ of $\mathbb{F}$ such that the extremal geometry of $\mathfrak{g} \otimes_{\mathbb{F}} K$ is isomorphic to $\Gamma(V, \Pi)$ for some $K$-vector space $V$ and subspace $\Pi$ of $V^*$ with $\text{Ann}_V(\Pi) = \{0\}$.

Then there is a trace-valued skew-Hermitian form $h$ on $V$ such that $\mathfrak{g}$ is isomorphic to $\mathfrak{psu}(V,h)$.

The above theorem characterizes a class of simple Lie algebras generated by extremal elements for which the extremal geometry contains no lines. In recent work by Jeroen Meulewaeter and the first author, see [7], it has been shown that in characteristic different from 2, 3 such finite dimensional Lie algebras are either symplectic, as in Theorem 1.2, or the extremal geometry carries the structure of a Moufang polar space or Moufang set, which can be obtained as the set of fixed points of an involution acting on a root shadow space of a spherical building of rank at least 2. Theorem 1.3 covers the case where the root shadow space is obtained from a building of type $A_n$.

This paper is organized as follows. In Section 2 we provide information and results on extremal elements and their geometry. In Section 3 we discuss the various examples of Lie algebras appearing in the above results. Then Section 4 is devoted to a proof of Theorem 1.1 while the final section, Section 5, provides a proof of Theorem 1.3.
2 Extremal elements in Lie algebras

Let \( g \) be a Lie algebra over the field \( F \) and with Lie bracket \([\cdot,\cdot]\). An extremal element of \( g \) is a nonzero element \( x \in g \) with the property that there exists a map \( g_x : g \to F \), the extremal form at \( x \), such that for all \( y \in g \) we have

\[
[x, [y,z]] = 2g_x(y)x.
\] (2.1)

Moreover, for all \( y,z \in g \) we have

\[
[[x,y],x] + g_x([y,z])x - g_x(y)[x,z] (2.2)
\]

and

\[
[x, [y,[x,z]]] = g_x([y,z])x - g_x(y)[x,z] - g_x(y)[x,z] (2.3)
\]

for every \( y,z \in g \).

The last two identities are called the Premet identities. If the characteristic of \( F \) is not 2, then the Premet identities follow from Equation 2.1. See [2].

As a consequence, \([x, [x,g]] \subseteq Fx \) for an extremal \( x \in g \). We call \( x \in g \) a sandwich if \([x, [x,y]] = 0 \) and \([x, [y,[x,z]]] = 0 \) for every \( y,z \in g \). So, a sandwich is an element \( x \) for which the extremal form \( g_x \) can be chosen to be identically zero. We introduce the convention that \( g_x \) is identically zero whenever \( x \) is a sandwich in \( g \). An extremal element is called pure if it is not a sandwich.

We denote the set of extremal elements of a Lie algebra by \( E(g) \) or, if \( g \) is clear from the context, by \( E \). Accordingly, we denote the set \( \{Fx|x \in E(g)\} \) of extremal points in the projective space on \( g \) by \( E(g) \) or \( E \).

We assume that \( g \) is generated by its set of extremal elements.

We recall some properties of extremal elements:

Proposition 2.1. [2] Proposition 20] The Lie algebra \( g \) is linearly spanned by its extremal elements.

Proposition 2.2. [2] Proposition 20] There is an associative symmetric bilinear form \( g : g \times g \to F \), such that for all \( x \in E \) and \( y \in g \) we have

\[
[x, [y,z]] = 2g(x,y)x.
\]

The form \( g \) is called the extremal form on \( g \). As the form \( g \) is associative, its radical \( \text{rad}(g) \) is an ideal in \( g \). Notice that by our choice to set \( g_x = 0 \) for sandwiches, all sandwiches are in the radical of \( g \).
Proposition 2.3. Suppose $g$ is a simple Lie algebra over the field $F$ generated by its extremal elements.

If $g$ contains a pure extremal element, then all extremal elements are pure.

Proof. If $g$ is simple then either rad$(g) = \{0\}$ and there are no sandwiches, of rad$(g) = g$ and all extremal elements are in the radical of $g$. In the latter case all extremal elements are sandwiches. \qed

Proposition 2.4. \cite{4, 2} Let $x$ be a pure extremal element of $E$. Then for each $\lambda \in F$ the map

$$\exp(x, \lambda): g \to g, \quad \exp(x, \lambda)y = y + \lambda[x, y] + \lambda^2 g(x, y)x$$

for all $y \in g$, is an automorphism of $g$.

If $x \in E$, then we denote by $\text{Exp}(\langle x \rangle)$ the subgroup $\{\exp(x, \lambda) \mid \lambda \in F\}$ of Aut$(g)$. Notice that this definition is independent from the choice of $x$ in $\langle x \rangle$ as $\exp(\lambda x, 1) = \exp(x, \lambda)$ for $0 \neq \lambda \in F$.

Proposition 2.5. \cite{4, 2} For pure $x, y \in E$ we have one of the following:

(a) $Fx = Fy$;
(b) $[x, y] = 0$ and $\lambda x + \mu y \in E \cup \{0\}$ for all $\lambda, \mu \in F$;
(c) $[x, y] = 0$ and $\lambda x + \mu y \in E$ only if $\lambda = 0$ or $\mu = 0$;
(d) $z := [x, y] \in E$, and $x, z$ and $y, z$ are as in case (b);
(e) $g(x, y) \neq 0$ and $\langle x, y \rangle$ is isomorphic to $\mathfrak{sl}_2(F)$.

An extremal line in $g$ is a 2-dimensional subspace of $g$ such that all its elements are extremal and pairwise commuting. We identify an extremal line also with the set of extremal points contained in it. Two linearly independent elements on an extremal line are as in case (b) of the above Proposition 2.5.

The extremal geometry $\Gamma(g)$ of $g$ is the point-line geometry with as point set $E$ and as lines the extremal lines.

The $\mathfrak{sl}_2$ graph $\Gamma_{\mathfrak{sl}_2}$ of $g$ is the graph with as vertices the extremal points and two vertices $\langle x \rangle, \langle y \rangle$, where $x, y \in E$, adjacent if and only if $g(x, y) \neq 0$.

The main result of Cohen and Ivanyos \cite{2, 3} is the following.
**Theorem 2.6** (Cohen and Ivanyos). Let $\mathfrak{g}$ be a Lie algebra generated by its pure extremal elements. If the extremal form $\mathfrak{g}$ is non-degenerate, then $\mathfrak{g}$ is a direct product of Lie algebras $\mathfrak{h}$ generated by connected components of $\Gamma_\mathfrak{sl}_2$; the extremal geometry $\Gamma(\mathfrak{h})$ is either a root shadow space of a spherical building or does not contain lines.

This result together with the following theorem, which combines the results of [6] and [8], provides us with a characterization of most of the classical Lie algebras.

**Theorem 2.7** (Cuypers, Fleischmann, Roberts, Shpectorov). Let $\mathfrak{g}$ be a simple Lie algebra generated by its pure extremal elements. If the extremal geometry $\Gamma(\mathfrak{g})$ is a root shadow space of a spherical building of rank at least 3, or $\Gamma(\mathfrak{g})$ is of type $\mathfrak{A}_{2,1,2}$, then the isomorphism type of $\mathfrak{g}$ is determined by its extremal geometry $\Gamma(\mathfrak{g})$.

The cases not covered by the above theorem are root shadow spaces of type $\mathfrak{G}_{2,2}$ and the cases where the extremal geometry has no lines. We will now be concerned with the latter by studying Lie algebras over extension fields.

Suppose that $\mathfrak{g}$ is a simple Lie algebra over a field $\mathbb{F}$ generated by its set $E$ of pure extremal elements.

If $K$ is a field extension of $\mathbb{F}$, then the Lie product on $\mathfrak{g} \otimes \mathbb{F} K$ is defined by

$$[x \otimes \lambda, y \otimes \mu] = [x, y] \otimes \lambda \mu,$$

for all $x, y \in \mathfrak{g}$ and $\lambda, \mu \in K$.

The elements $x \otimes \lambda$ with $x \in E$ and $\lambda \in K$ are easily checked to be extremal. Moreover, they generate $\mathfrak{g}$.

The extremal form $\hat{\mathfrak{g}}$ on $\mathfrak{g}$ satisfies

$$\hat{\mathfrak{g}}(x \otimes \lambda, y \otimes \mu) = \mathfrak{g}(x, y) \lambda \mu$$

for all $x, y \in E$ and $\lambda, \mu \in K$.

Now we can formulate the main result from [5]:

**Theorem 2.8** (Cuypers and Fleischmann). Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$ of characteristic not 2 generated by its set $E$ of pure extremal elements. Suppose that the extremal geometry $\Gamma(\mathfrak{g})$ does not contain lines. Then either $\mathfrak{g}$ is isomorphic to a finitary symplectic Lie algebra $\mathfrak{fsp}(V, f)$ for some non-degenerate symplectic space $(V, f)$, or there is a degree 2 extension $K$ of $\mathbb{F}$ such that $\mathfrak{g} \otimes_{\mathbb{F}} K$ is a Lie algebra generated by pure extremal elements and containing extremal lines.
The above results imply that, at least if the characteristic is not 2, the study of Lie algebras generated by pure extremal elements can be restricted to those for which the extremal geometry does contain lines and their subalgebras over a subfield of index 2.

In the next section we discuss how this situation arises for special linear Lie algebras and their unitary subalgebras. But first we will show that we may assume the Lie algebra \( \hat{\mathfrak{g}} \) to be simple.

So, let \( \mathfrak{g} \) be a simple Lie algebra over a field \( \mathbb{F} \) of characteristic not 2. Assume \( \mathfrak{g} \) is generated by its set \( E \) of pure extremal elements. Then the extremal form \( \mathfrak{g} \) is non-degenerate and \( \mathfrak{g} \) contains no sandwiches, see Proposition 2.3.

Assume \( K \) to be a Galois extension of \( \mathbb{F} \) of degree 2 and let \( \sigma \) be the field automorphism of \( K \) of order 2 fixing precisely the elements of \( \mathbb{F} \). Then \( \sigma \) induces an automorphism, also denoted by \( \sigma \), of \( \hat{\mathfrak{g}} \) by mapping each \( x \otimes \lambda, \) where \( x \in \mathfrak{g} \) and \( \lambda \in K \) to \( (x \otimes \lambda)\sigma := x \otimes \lambda^\sigma \).

The elements fixed by \( \sigma \) are precisely all element \( x \otimes \lambda \), where \( x \in \mathfrak{g} \) and \( \lambda \in \mathbb{F} \), forming a Lie subalgebra over \( \mathbb{F} \) isomorphic and identified with \( \mathfrak{g} \).

The extremal form on \( \hat{\mathfrak{g}} \) is denoted by \( \hat{\mathfrak{g}} \).

**Lemma 2.9.** The radical of the extremal form \( \hat{\mathfrak{g}} \) is trivial.

**Proof.** Suppose \( r \neq 0 \) is an element in the radical of \( \hat{\mathfrak{g}} \) and \( r = x_1 \otimes \lambda_1 + \cdots x_n \otimes \lambda_n \) with \( x_i \in E \) and \( \lambda_i \in K \), such that \( n \) is minimal. After scalar multiplication we can assume that \( \lambda_1 \in \mathbb{F} \).

Now \( \hat{\mathfrak{g}}(r, y \otimes 1) = 0 \) for all \( y \in E \), but, then also \( \hat{\mathfrak{g}}(r^\sigma, y \otimes 1) = 0 \) for all \( y \in E \) and hence \( r - r^\sigma = x_2 \otimes (\lambda_2 - \lambda_2^\sigma) + \cdots + x_n \otimes (\lambda_n - \lambda_n^\sigma) \) is an element of the radical, which by assumption on minimality of \( n \) is zero. But that would imply that \( r \) is fixed by \( \sigma \) and contradicts that the radical of \( \mathfrak{g} \) is trivial.

**Corollary 2.10.** If the characteristic of \( \mathbb{F} \) is not 2, then the Lie algebra \( \hat{\mathfrak{g}} \) is simple.

**Proof.** By Lemma 2.9 the radical of \( \hat{\mathfrak{g}} \) is trivial, and \( \hat{\mathfrak{g}} \) is non-degenerate.

Let \( i \) be a nontrivial ideal of \( \hat{\mathfrak{g}} \). Then there is an extremal element \( x \in \hat{E} \), the set of extremal elements of \( \hat{\mathfrak{g}} \), such that \( \hat{\mathfrak{g}}(x, i) \neq 0 \) for some \( i \in i \). But then \( [x, [x, i]] \in i \) and hence \( x \in i \).

As, by Proposition 2.1 the set \( E \) spans \( \mathfrak{g} \) and hence also \( \hat{\mathfrak{g}} \), there is an element \( y \in E \) with \( \hat{\mathfrak{g}}(x, y) \neq 0 \), from which it follows that \( [y, [y, x]] \) and thus also \( y \) is in \( i \). But then as \( \Gamma_{\mathfrak{s}_0} \) is connected by Theorem 2.6 we also find \( E \) to be contained in \( i \), proving \( i \) to be \( \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{g}} \) to be simple. 

3 Infinitesimal transvections

Let $V$ be a vector space over a field $F$ and let $v \in V$ and $\phi \in V^*$ be nonzero. Then the linear map $t_{v,\phi} : V \to V$ with $t_{v,\phi}(w) = \phi(w)v$ for all $w \in V$ is called an infinitesimal transvection if $\phi(v) = 0$ and an infinitesimal reflection if $\phi(v) \neq 0$.

Let $V$ be a vector space over a field $F$ and let $\Pi \subseteq V^*$ be a subspace of the dual $V^*$ of $V$. We assume that $\text{Ann}_V(\Pi) = \{0\}$. For finite dimensional $V$ we have $\Pi = V^*$ and it is well known that the infinitesimal transvections and reflections generate the general linear Lie algebra $\mathfrak{gl}(V)$, while the infinitesimal transvections generate $\mathfrak{sl}(V)$, the special linear Lie algebra. For possibly infinite dimensional $V$ and $\Pi = V^*$, these elements generate the so called finitary general linear Lie algebra $\mathfrak{fgl}(V)$ and the finitary special linear Lie algebra $\mathfrak{fsl}(V)$, respectively. (A linear map of $V$ to itself is called finitary if its kernel has finite codimension.)

For infinite dimensional $V$ the space $\Pi$ can be a proper subspace of $V^*$ and we define (also in the finite dimensional case) the Lie subalgebra of $\mathfrak{gl}(V)$ generated by the infinitesimal transvections and reflections $t_{v,\phi}$ with $v \in V$ and $\phi \in \Pi$ to be $\mathfrak{fgl}(V,\Pi)$ and by the infinitesimal transvections $t_{v,\phi}$ with $v \in V$, $\phi \in \Pi$ and $\phi(v) = 0$ to be $\mathfrak{fsl}(V,\Pi)$. Notice that this latter Lie algebra indeed consists of traceless finitary linear transformations.

Let $t_{v,\phi}$ be an infinitesimal transvection. Then for each linear map $y : V \to V$ we have for all $w \in V$:

$$[t_{v,\phi}, [t_{v,\phi}, y]](w) = \phi(w)v.$$  

Thus we find

$$[t_{v,\phi}, [t_{v,\phi}, y]] = -2\phi(y(v))t_{v,\phi}.$$  

This implies that, at least for fields $F$ of characteristic different from 2, the infinitesimal transvections $t_{v,\phi}$, with $v \in V$ and $\phi \in \Pi$, are extremal in $\mathfrak{fsl}(V,\Pi)$. It is straightforward, but tedious, to check the Premet identities.

So, also in even characteristic, the infinitesimal transvections are extremal.

By the assumption that $\text{Ann}_V(\Pi) = \{0\}$ we find these elements not to be sandwiches.

The Lie algebras $\mathfrak{fgl}(V,\Pi)$ and $\mathfrak{fsl}(V,\Pi)$ are obtained by factoring out the center (which might be trivial) of $\mathfrak{fgl}(V,\Pi)$ and $\mathfrak{fsl}(V,\Pi)$.
As a vector space $\mathfrak{fgl}(V,\Pi)$ is isomorphic with $V \otimes \Pi$. An isomorphism is provided by the linear expansion of the map $t_{v,\phi} \mapsto v \otimes \phi$, where $v \in V$ and $\phi \in \Pi$. The Lie product then translates to

$$[v \otimes \phi, w \otimes \psi] = \phi(w)v \otimes \psi - \psi(v)w \otimes \phi$$

for pure tensors $v \otimes \phi$ and $w \otimes \psi \in V \otimes \Pi$.

We will identify $\mathfrak{fgl}(V,\Pi)$ (as well as various of its subalgebras) with the (subalgebras of the) Lie algebra $\mathfrak{g}(V,\Pi)$ defined by this product on $V \otimes \Pi$. In particular, we identify $\mathfrak{fsl}(V,\Pi)$ with the subalgebra $\mathfrak{s}(V,\Pi)$ generated by the pure tensors $v \otimes \phi \in V \otimes \Pi$, with $\phi(v) = 0$.

Now suppose $h : V \times V \to \mathbb{F}$ is a non-degenerate skew-Hermitian form with respect to a field automorphism $\sigma$ of order $\leq 2$. So, $h$ is linear in the second coordinate,

$$h(v, w) = -h(w, v)^\sigma$$

for all $v, w \in V$, and $h(v, w) = 0$ for all $w \in W$ implies $v = 0$.

If $v \in V$, then denote by $h(v, \cdot)$ the linear map $w \in V \mapsto h(v, w)$. Then

$$[v \otimes h(v, \cdot), w \otimes h(w, \cdot)] = h(v, w)v \otimes h(w, \cdot) - h(w, v)w \otimes h(v, \cdot)$$

$$= h(v, w)v \otimes h(w, \cdot) + h(v, w)^\sigma w \otimes h(v, \cdot)$$

$$= (h(v, w)v + w) \otimes h(v, w) + w, \cdot$$

$$- h(v, v)h(v, w)^\sigma v \otimes h(v, \cdot) - w \otimes h(v, \cdot).$$

So, the elements $v \otimes h(v, \cdot)$ generate a Lie subalgebra of $\mathfrak{g}(V,V^*)$ over the field $\mathbb{F}_\sigma = \{\lambda \in \mathbb{F} \mid \lambda^\sigma = \lambda\}$. This subalgebra is denoted by $\mathfrak{g}_h(V,V^*)$. The subalgebra of $\mathfrak{s}(V,V^*)$ generated by the elements $v \otimes h(v, \cdot)$ with $h(v, v) = 0$ is denoted by $\mathfrak{s}_h(V,V^*)$.

If $h$ is alternating (i.e., $\sigma$ is the identity), the algebra $\mathfrak{s}_h(V,V^*)$ is isomorphic to the finitary symplectic Lie algebra $\mathfrak{fsp}(V,h) := \{x \in \mathfrak{s}(V,V^*) \mid h(x(v), w) + h(v, x(w)) = 0 \text{ for all } v, w \in V\}$, see [5], while for proper skew-Hermitian forms (i.e., $\sigma$ has order 2), the Lie algebra $\mathfrak{s}_h(V,V^*)$ over $\mathbb{F}_\sigma := \{\lambda \in \mathbb{F} \mid \lambda^\sigma = \lambda\}$ is isomorphic to a subalgebra of the finitary special unitary Lie algebra $\mathfrak{fsl}(V,h) = \{x \in \mathfrak{s}(V,V^*) \mid h(x(v), w) + h(v, x(w)) = 0 \text{ for all } v, w \in V\}$. If $V$ is generated by its isotropic vectors, i.e. $v \in V$ with $h(v, v) = 0$, we find this subalgebra to be the full finitary special unitary Lie algebra, as will be shown below.

Notice that these Lie algebras are obtained as centralizers of the automorphism of $\mathfrak{s}(V,\{h(v, \cdot) \mid v \in V\})$, defined as the semi-linear extension of the map

$$v \otimes \phi \mapsto v \otimes \phi + \frac{1}{2} [v \otimes h(v, \cdot), w \otimes \phi].$$
\[(\lambda(v \otimes h(w, \cdot))) \mapsto \lambda^\sigma w \otimes h(v, \cdot)\]

where \(v, w \in V\) with \(h(v, w) = 0\) and \(\lambda \in \mathbb{F}\).

Versions of the following results can also be found in [9].

**Proposition 3.1.** Let \(h\) be a non-degenerate skew-Hermitian form on the vector space \(V\) over the field \(\mathbb{F}\) with respect to the field automorphism \(\sigma\) of order 2. The Lie algebra \(fu(V, h)\) over \(\mathbb{F}_\sigma\) is linearly spanned by its infinitesimal reflections and transvections.

**Proof.** We first consider the case where \(\dim(V) = n < \infty\). In this case it is well known that \(u(V, h)\) has dimension \(n^2\).

Let \(v_1, \ldots, v_n\) be a basis of \(V\) such that the matrix of the form \(h\) with respect to this basis is diagonal \(H = \text{diag}(\lambda_1, \ldots, \lambda_n)\), where \(\lambda_i \in \mathbb{F}\) with \(\lambda_i^\sigma = -\lambda_i\).

Now consider the elements

\[v_i \otimes h(v_i, \cdot), (v_j + v_l) \otimes h(v_j + v_l, \cdot), \text{ and } (v_j + \mu v_l) \otimes h(v_j + \mu v_l, \cdot),\]

where \(1 \leq i, j, l \leq n, j < l\) and \(\mu \in \mathbb{F}\) a fixed element with \(\mu^\sigma \neq \mu\).

We can verify that these \(n^2\) elements form an independent set in \(u(V, h)\). So, as the dimension of \(u(V, h)\) equals \(n^2\), we have shown that \(u(V, h)\) is spanned by its infinitesimal transvections and reflections.

From the finite dimensional case, the finitary case follows easily. \(\square\)

We now focus on subalgebras of the unitary Lie algebra generated by infinitesimal transvections. The existence of such transvections implies the existence of isotropic vectors in \(V\), i.e. \(0 \neq v \in V\) with \(h(v, v) = 0\). Two isotropic vectors \(v, w \in V\) with \(h(v, w) \neq 0\) span a so-called hyperbolic 2-space. Isotropic vectors do exist, if the form \(h\) is trace valued, that is \(h(v, v) \in \{\lambda - \lambda^\sigma \mid \lambda \in \mathbb{F}\}\) for all \(v \in V\). We have the following characterization of trace-valued forms:

**Proposition 3.2.** Suppose \(V\) is an \(\mathbb{F}\)-vector space and \(\sigma\) a field automorphism of \(\mathbb{F}\) of order 2. Let \(h\) be a non-degenerate skew-Hermitian form with respect to the involution \(\sigma\). Suppose \(V\) contains a nonzero isotropic vector.

Then the form \(h\) is trace-valued if and only if \(V\) is generated by its isotropic vectors, if and only if for each isotropic \(0 \neq v \in V\) and arbitrary \(w \in V\) with \(h(v, w) \neq 0\), the subspace \(\langle v, w \rangle\) is hyperbolic.

**Proof.** See [1, Theorem 10.1.3]. \(\square\)
Proposition 3.3. Suppose $V$ is an $\mathbb{F}$-vector space of dimension at least 2 and $\sigma$ a field automorphism of $\mathbb{F}$ of order 2. Let $h$ be a non-degenerate skew-Hermitian form with respect to the involution $\sigma$ such that $V$ contains a nonzero isotropic vector.

Then $\mathfrak{fsu}(V,h)$ is linearly spanned by its infinitesimal transvections if and only if $h$ is trace-valued.

Proof. Suppose $h$ is trace-valued. Then, by Proposition 3.2, we can assume that $V$ is spanned by its isotropic vectors. We will show that $\mathfrak{fsu}(V,h)$ is linearly spanned by its infinitesimal transvections.

In view of the above result, Proposition 3.1 and the fact that $\mathfrak{fsu}(V,h)$ has codimension 1 in $\mathfrak{fu}(V,h)$, it suffices to prove that $\mathfrak{fu}(V,h)$ can be spanned by all its infinitesimal transvections and a unique reflection.

This is clearly true in the case where $(V,h)$ is a hyperbolic 2-space over the field $\mathbb{F}$. Indeed, if $v_1, v_2$ is a hyperbolic basis of $V$, i.e. $h(v_1, v_1) = h(v_2, v_2) = 0$ and $h(v_1, v_2) \neq 0$, then

$$v_1 \otimes h(v_1, \cdot), \ v_2 \otimes h(v_2, \cdot), \text{ and } v_1 + v_2 \otimes h(v_1 + v_2, \cdot)$$

span $\mathfrak{su}(V,h)$ and together with any infinitesimal reflection they span $\mathfrak{u}(V,h)$.

So, to prove in general that $\mathfrak{fu}(V,h)$ can be spanned by its infinitesimal transvections together with one reflection, it suffices to prove connectedness of the graph $\Delta$ on the anisotropic points of $V$, where two such points are adjacent if and only if they span a hyperbolic line (i.e., 2-space). For then, every infinitesimal reflection is contained in the span of the infinitesimal transvections and the unique reflection.

Suppose that $\dim(V)$ is at least 2. As $(V,h)$ is non-degenerate and spanned by its isotropic vectors, there are isotropic vectors $v, w \neq 0$ with $h(v, w) \neq 0$. After replacing $w$ with $h(v, w)^{-1}w$, we can assume $h(v, w) = 1$. The 2-space $\langle v, w \rangle$ is hyperbolic.

Let $u$ be an anisotropic vector in $V$. We will prove that $\langle u \rangle$ is in the same connected component of $\Delta$ as some anisotropic point on $\langle v, w \rangle$.

First assume that $h(u, v) = h(u, v) = 0$. Then, for $\mu \in \mathbb{F}$ satisfying $\mu - \mu^\sigma = -h(u, u)$ we have $h(u + v + \mu w, u + v + \mu v) = h(u, u) - \mu^\sigma + \mu = 0$ and $h(v + \mu w, v + \mu v) = \mu - \mu^\sigma = -h(u, u) \neq 0$. Notice, such $\mu$ exists, as $h$ is trace-valued. Moreover, as $h(u, u + v + \mu w) = h(u, u) \neq 0$, we find the 2-space $\langle u, v + \mu w \rangle$ to be hyperbolic, see 3.2. This implies that $\langle u \rangle$ is adjacent to $\langle v + \mu w \rangle$ in $\Delta$.

Now assume that $h(u, v) \neq 0$ but $h(u, v) = 0$. After scaling $v$ (and $w$), we can assume that $h(u, u) - h(u, v) \neq 0$. Let $\mu \in \mathbb{F}$ with $h(u, u) + (\mu - h(u, v))^\sigma - (\mu - h(u, v)) = 0$. Then $h(u + v + \mu w, u + v + \mu w) = h(u, u) + h(u, v) - h(v, u) +$
\[ \mu^\sigma - \mu = h(u, u) + (\mu - h(u, v))^\sigma - (\mu - h(u, v)) = 0. \] Again, such element \( \mu \) exists as \( h \) is trace valued. Moreover, \( h(u, u + v + \mu w) = h(u, u) - h(u, v) \neq 0 \).

So, \( \langle u, u + v + \mu w \rangle \) is hyperbolic and meets \( \langle v, w \rangle \) in \( v + \mu w \), which is anisotropic, as \( h(v + \mu w, v + \mu w) = \mu - \mu^\sigma = h(u, u) \). We find that \( \langle u \rangle \) is adjacent in \( \Delta \) to an anisotropic point on \( \langle v, w \rangle \).

Finally assume that \( h(u, v) \neq 0 \) and \( h(u, w) \neq 0 \). Let \( u' \in \langle u, v \rangle \) be perpendicular to \( w \). If \( u' \) is anisotropic, then \( \langle u \rangle \) is adjacent to \( \langle u' \rangle \), and the latter is, by the above, adjacent to some anisotropic point in \( \langle v, w \rangle \). Thus, assume \( u' \) is isotropic. Then all nonzero vectors in \( \langle u', w \rangle \) are isotropic. As \( h(u, u') \neq 0 \) we find that \( u \) is perpendicular to one point on \( \langle u', w \rangle \). So, all 2-spaces on \( \langle u \rangle \) inside \( \langle u, v, w \rangle \), except for one, are hyperbolic. This clearly implies that there is at least one hyperbolic line on \( \langle u \rangle \) meeting \( \langle v, w \rangle \) in an anisotropic point, and that \( \langle u \rangle \) is adjacent in \( \Delta \) to an anisotropic point on \( \langle v, w \rangle \), finishing the proof that \( \mathfrak{fsu}(V, h) \) is spanned by its transvections.

Now assume that the form \( h \) is not trace-valued and thus, by Proposition 3.2, that the subspace \( V_0 \) of \( V \) spanned by the isotropic vectors is not \( V \).

Then we can find a vector \( u \in V \setminus V_0 \). Clearly \( u \otimes h(u, \cdot) \) is not in the subspace of \( \mathfrak{fu}(V, h) \) spanned by the infinitesimal transvections and all elements \( v \otimes h(v, \cdot) \), with \( v \in V_0 \), including infinitesimal reflections. So, the subalgebra spanned by the infinitesimal transvections has codimension at least 2 in \( \mathfrak{fu}(V, h) \) and hence is properly contained in \( \mathfrak{fsu}(V, h) \), which has codimension 1. This finishes the proof of the proposition.

Let \( h \) be a skew-Hermitian form on the vector space \( V \) defined over the field \( F \) with respect to the field automorphism \( \sigma \) of order 2. Assume that \( h \) is trace-valued and \( V \) contains an isotropic vector.

Let \( g \) be the Lie algebra \( \mathfrak{fsu}(V, h) \), which, by Proposition 3.3, is spanned by its infinitesimal transvections. Then \( \hat{g} := g \otimes_{F^\sigma} F \) is isomorphic to a subalgebra of \( \mathfrak{fsl}(V, \Pi) \) where \( \Pi \) is the subspace of \( V^\ast \) spanned by the elements \( h(v, \cdot) \) with \( v \in V \) isotropic. If \( V \) is finite dimensional, then, as \( \hat{g} \) and \( \mathfrak{fsl}(V, \Pi) \) have the same dimension, they are even isomorphic. This also holds true for infinite dimensional \( V \), as we can view both Lie algebras as limits of isomorphic finite dimensional ones. See also the next section.

It is well known that the Lie algebra \( \mathfrak{psl}(V, \Pi) \) is simple. We use this to find the structure of \( \hat{g} \).

As both \( g \) and \( \hat{g} \) are generated by their pure extremal elements, they are both equipped with an extremal form, say \( g \) and \( \hat{g} \), respectively.

**Lemma 3.4.** The radical of \( g \) equals the center of \( g \), which is of dimension at most 1.
Proof. Let $R = \text{rad}(g)$. Then $\hat{R} := R \otimes F$ is contained in the radical $\text{rad}(\hat{g})$, which is a proper ideal of $\hat{g}$. As $\hat{g}$ modulo its center is simple, we find $\hat{R}$ to be contained in the center of $\hat{g}$. But then $R$ is contained in the center of $g$.

Since the center of $\hat{g}$ is at most one-dimensional, so is the center of $g$. \hfill \square

Proposition 3.5. $g/\text{rad}(g)$ is simple.

Proof. Let $i$ be a nontrivial ideal of $g$. If $i \subseteq \text{rad}(g)$, then it is central. If $i$ contains an element $i$ not in $\text{rad}(g)$, we find an infinitesimal transvection $t \in g$ with $0 \neq t_0 = [t, [t, i]] \in i$. So $t$ is in $i$. But then for each infinitesimal transvection $t' \in g$ with $[t', t_0] \neq 0$ we find $t' \in i$. Repeating this argument we find all infinitesimal transvections in $i$ and $i = g$. \hfill \square

4 A geometric characterization of the special linear Lie algebra

In this section we prove Theorem 1.1 and show that a simple Lie algebra over a field $F$ generated by its set of pure extremal elements and its extremal geometry isomorphic to $\Gamma(V, \Pi)$, where $V$ is an $F$-vector space of dimension at least 3 and $\Pi$ a subspace of $V^*$ with $\text{Ann}_V(\Pi) = \{0\}$, is isomorphic to $(p)fsl(V, \Pi)$.

If $V$ is finite dimensional, then the theorem follows from [8] (see also [10]). Indeed, using [8] we obtain the following theorem. (Here we say that a subset $E_0$ of $E$ is closed under the action of a subgroup $G$ of $\text{Aut}(g)$ if and only if $xg \in E_0$ for all $x \in E_0$ and $g \in G$.)

Theorem 4.1 (Cuypers, Roberts, and Shpectorov). Let $V$ be a vector space over a field $F$ of finite dimension at least 3 and $g$ a Lie algebra generated by a subset $E_0$ of the set of pure extremal points $E$ such that $E_0$ is a subspace of the extremal geometry $\Gamma(g)$, isomorphic to $\Gamma(V, V^*)$, and closed under the action of $\text{Exp}(x)$ for each $x \in E_0$.

Then $g$ is isomorphic to $\mathfrak{sl}(V)$ or its central quotient $\mathfrak{psl}(V)$.

Proof. Although in the statements of the main results of [8] it is required that $E_0 = E$ and that the Lie algebra is simple, this is actually not needed in the proof.

Indeed, we can take a basis $v_1, \ldots, v_n$ of $V$ and consider the extremal elements $x_{ij}$ with $\langle x_{ij} \rangle \in E_0$, where $i \neq j$, corresponding to the point-hyperplane pairs $(\langle v_i \rangle, \langle v_k \mid k \neq j \rangle)$. As is shown in Section 5 of [8], we can scale these elements in such a way that they together with the elements...
\[ h_{ij} = -h_{ji} = [x_{ij}, x_{ji}] \] form a Chevalley spanning set as described in Section 4 of [8]. But then, using the group \( G = \langle \text{Exp}(x) \mid x \in \mathcal{E}_0 \rangle \) it readily follows by arguments as in Section 6 and 7 of [8] that the Lie algebra generated by this Chevalley spanning set contains \( \mathcal{E}_0 \) and hence equals \( g \). But then \( g \) is a Chevalley Lie algebra of type \( A_{n-1} \) and hence isomorphic to \((\mathfrak{p})\mathfrak{sl}(V)\). \( \square \)

So, for the remainder of this section we can and will assume that \( V \) is infinite dimensional.

Our proof of Theorem 1.1 will rely on Theorem 4.1. We will approach an unknown infinite dimensional Lie algebra satisfying the hypothesis of the theorem by a collection of finite dimensional subalgebras isomorphic to \( \mathfrak{sl}(U) \) for vector spaces \( U \) of finite dimension.

We start with the necessary definitions.

A directed set \((\mathcal{I}, \sqsubseteq)\) is a set \( \mathcal{I} \) with a partial order \( \sqsubseteq \) such that
- for all \( i, j \in \mathcal{I} \) there exists an element \( k \) in \( \mathcal{I} \) such that \( i, j \sqsubseteq k \);
- for every \( k \in \mathcal{I} \) any chain \( i_1 \sqsubseteq i_2 \sqsubseteq \ldots \) of distinct elements \( i_j \sqsubseteq k \) from \( \mathcal{I} \) has finite length.

Suppose \( g \) is a Lie algebra and \((\mathcal{I}, \sqsubseteq)\) a directed set. Then a collection \((g_i)_{i \in \mathcal{I}}\) of Lie subalgebras of \( g \) is called a local system of \( g \) with respect to \((\mathcal{I}, \sqsubseteq)\) if the following holds:
- if \( i, j \in \mathcal{I} \) with \( i \sqsubseteq j \), then \( g_i \subseteq g_j \);
- \( g = \bigcup_{i \in \mathcal{I}} g_i \)

If \( g \) contains a local system \((g_i)_{i \in \mathcal{I}}\), then \( g \) is uniquely determined by this local system as it is isomorphic to the direct limit
\[
\lim_{i \in \mathcal{I}} g_i.
\]

This implies the following well known result.

**Proposition 4.2.** Let \((\mathcal{I}, \sqsubseteq)\) be a directed set. Suppose both \( g \) and \( h \) are Lie algebras with local systems \((g_i)_{i \in \mathcal{I}}\) and \((h_i)_{i \in \mathcal{I}}\) for \((\mathcal{I}, \sqsubseteq)\), respectively, such that for each \( i \in \mathcal{I} \) there exists an isomorphism
\[
\phi_i : g_i \to h_i
\]
with the property that for \( j, k \in \mathcal{I} \) with \( j \subseteq k \) we have

\[
\phi_j = \phi_k |_{\Theta_j}.
\]

Then \( g \) and \( h \) are isomorphic.

We now start with the proof of Theorem 1.1. So, suppose \( V \) is an infinite dimensional vector space over the field \( F \) and \( \Pi \) a subspace of \( V^* \) with \( \text{Ann}_V(\Pi) = \{0\} \).

Then let \( \mathcal{I} \) be the set of all pairs \((U, \Phi)\) where

- \( U \) is a subspace of \( V \);
- \( \Phi \) is a subspace of \( \Pi \);
- \( U \) and \( \Phi \) have the same finite dimension, which is at least 3 and not divisible by the characteristic of the field \( F \);
- \( \text{Ann}_U(\Phi) = \{0\} \).

For \((U, \Phi), (U', \Phi') \in \mathcal{I}\) we define \( \sqsubseteq \) by the following:

\[
(U, \Phi) \sqsubseteq (U', \Phi') \iff U \subseteq U' \text{ and } \Phi \subseteq \Phi'.
\]

**Lemma 4.3.** \((\mathcal{I}, \sqsubseteq)\) is a directed set.

**Proof.** Let \((U_1, \Phi_1)\) and \((U_2, \Phi_2)\) be elements in \( \mathcal{I} \). Now let \( U = U_1 + U_2 \) and \( \Phi = \Phi_1 + \Phi_2 \).

Suppose \( \phi_1, \ldots, \phi_m \) form a basis for \( \Phi \). Then pick vectors \( u_1, \ldots, u_m \in V \) with \( \phi_i(u_j) = \delta_{ij} \) for all \( 1 \leq i, j \leq n \) and consider \( \hat{U} = \langle u_1, \ldots, u_m \rangle + U \). Let \( u_{m+1}, \ldots, u_n \) be vectors such that \( u_1, \ldots, u_n \) form a basis for \( \hat{U} \). Then we can find \( \phi_{m+1}, \ldots, \phi_n \in \Pi \) such that \( \phi_i(u_j) = \delta_{ij} \) for all \( 1 \leq i, j \leq n \). Now define \( \Pi \) to be \( \langle \phi_1, \ldots, \phi_n \rangle \). Then \( \dim(\hat{U}) = \dim(\hat{\Phi}) = n \) and \( \text{Ann}_U(\hat{\Phi}) = \{0\} \).

If the dimension \( n \) is divisible by the characteristic of \( F \), then extend \( \hat{U} \) and \( \hat{\Phi} \) with elements \( u \) and \( \phi \), respectively, satisfying \( \phi(u_i) = 0 = \phi_1(u) \) for all \( 1 \leq i \leq n \) and \( \phi(u) = 1 \). In any case we find an element \( (\hat{U}, \hat{\Phi}) \) of \( \mathcal{I} \) with \((U_1, \Phi_1), (U_2, \Phi_2) \sqsubseteq (\hat{U}, \hat{\Phi}) \).

Since every pair \((U, \Phi) \in \mathcal{I}\) consists of finite dimensional spaces, we find \((\mathcal{I}, \sqsubseteq)\) to be a directed set. \( \square \)

We now construct a local system for \( \mathfrak{fsl}(V, \Pi) \).

For \( I = (U, \Phi) \in \mathcal{I} \) define

\[
\mathfrak{sl}(I) := \langle t_{v, \phi} \mid 0 \neq v \in U, 0 \neq \phi \in \Phi \text{ and } \phi(v) = 0 \rangle.
\]
Then $\mathfrak{sl}(I)$ is a Lie subalgebra of $\mathfrak{fsl}(V, \Pi)$ which is isomorphic to $\mathfrak{sl}(U)$, and, as a consequence of the extra condition that the dimension of $U$ and $\Phi$ is not divisible by the characteristic of $\mathbb{F}$, has a trivial center.

**Proposition 4.4.** The collection of Lie subalgebras $\mathfrak{sl}(I)$ for $I \in \mathcal{I}$ forms a local system for $\mathfrak{fsl}(V, \Pi)$.

Now consider a simple Lie algebra $\mathfrak{g}$ generated by its set $E$ of pure extremal elements and assume its extremal geometry $\Gamma(\mathfrak{g})$ is isomorphic to $\Gamma(V, \Pi)$. We also construct a local system for $\mathfrak{g}$. We identify $\Gamma(\mathfrak{g})$ with $\Gamma = \Gamma(V, \Pi)$ and for each $I = (U, \Phi) \in \mathcal{I}$ define $\Gamma_I$ to be the subspace of $\Gamma$ consisting of all incident point-hyperplane pairs $(p, H)$ with $p \in \mathbb{P}(U)$ and $H = \ker(\phi)$ for some $0 \neq \phi \in \Phi$.

Moreover, we define $\mathfrak{g}_I$ to be the Lie subalgebra generated by all extremal points in $\Gamma_I$. So,

$$\mathfrak{g}_I := \langle x \mid x \in \Gamma_I \rangle.$$

**Lemma 4.5.** Let $I \in \mathcal{I}$. Then $\mathcal{E} \cap \mathfrak{g}_I = \Gamma_I$.

**Proof.** Let $I = (U, \Phi)$ be an element from $\mathcal{I}$ and take $U' = \text{Ann}_V(\Phi) = \{ u \in V \mid \phi(u) = 0 \text{ for all } \phi \in \Phi \}$ and $\Phi' = \text{Ann}_\Pi(U) := \{ \phi \in \Pi \mid \phi(u) = 0 \text{ for all } u \in U \}$. Then all extremal points $x$ from $\mathfrak{g}$ which correspond to points $(p, H)$ from $\Gamma$ with $p$ in $\mathbb{P}(U')$ and $H = \ker(\phi')$ for some $\phi' \in \Phi'$ centralize all elements from $\Gamma_I$ and hence $\mathfrak{g}_I$.

Suppose an extremal point $y$ of $\mathfrak{g}_I$ corresponds to the point-hyperplane pair $(q, K)$ of $\Gamma$. Then $y$ is centralized by all such extremal points $x$, and we find $q \leq \text{Ann}_V(\Phi') = U$, and $K = \ker(\phi)$ for some $\phi \in \text{Ann}_\Pi(U') = \Phi$. In particular, $y \in \Gamma_I$. \qed

**Proposition 4.6.** The collection of Lie subalgebras $\mathfrak{g}_I$ for $I \in \mathcal{I}$ forms a local system for $\mathfrak{g}$.

**Proposition 4.7.** The two local systems $(\mathfrak{sl}(I))_{I \in \mathcal{I}}$ and $(\mathfrak{g}_I)_{I \in \mathcal{I}}$ are isomorphic.

**Proof.** For each $I = (U, \Phi) \in \mathcal{I}$ we find, as we can apply Theorem 4.1, that the Lie algebra $\mathfrak{g}_I$ is classical Lie algebra of type $A_{n-1}$, where $n = \dim(U)$. In particular, $\mathfrak{g}_I$ is isomorphic to $\mathfrak{sl}(U)$ or its central quotient $\mathfrak{psl}(U)$. As $\dim(U)$ is not divisible by the characteristic of $\mathbb{F}$, we find $\mathfrak{g}_I$ to be isomorphic to $\mathfrak{sl}(U)$. Moreover, the isomorphism can be chosen in such a way that it induces the identity on $\Gamma_I$.

It remains to show that for $I \subseteq J$ the restriction of $\phi_J$ to $\mathfrak{g}_I$ equals $\phi_I$. 

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So, suppose $I = (U, \Phi) \sqsubseteq J$. Then the map $\psi = \phi_I(\phi_J|_{\mathfrak{g}_I})^{-1}$ is an automorphism of $\mathfrak{sl}(I) \simeq \mathfrak{sl}(U)$ fixing all extremal points of $\mathfrak{sl}(I)$. Let $x, y$ be two collinear points of $\Gamma_J$, then $\psi$ fixes all points on this line and hence is a scalar multiplication on this 2-dimensional subspace of $\mathfrak{sl}(I)$. By connectedness of $\Gamma_I$ we find that $\psi$ is a scalar multiplication on the vector space $\mathfrak{sl}(I)$, say with scalar $\lambda \in F^*$. But as $\psi$ also respects the Lie product $[\cdot, \cdot]$ of $\mathfrak{sl}(I)$, we find for any two elements $a, b \in \mathfrak{sl}(I)$ that $[\psi(a), \psi(b)] = [\lambda a, \lambda b] = \lambda^2 [a, b] = \psi([a, b]) = \lambda [a, b]$. In particular, by choosing non commuting $a, b \in \mathfrak{sl}(I)$ we find $\lambda^2 = \lambda$, and hence $\lambda = 1$.

So indeed, $\phi_J|_{\mathfrak{g}_I} = \phi_I$ and we have found the two local systems to be isomorphic. \[ \square \]

By Proposition 4.2 we immediately have the following result, which together with the finite dimensional case, settles Theorem 1.1.

**Theorem 4.8.** The Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{psl}(V, \Pi)$.

## 5. A geometric characterization of the unitary Lie algebra

In this final section we focus on Theorem 1.3. So, let $\mathfrak{g}$ be a simple Lie algebra over the field $F$ generated by its set $E$ of pure extremal elements, and suppose that the extremal geometry $\Gamma(\mathfrak{g})$ does not contain lines.

In this section we assume that there is a quadratic Galois extension $K$ of $F$ such that the set of pure extremal elements $\hat{E}$ of $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_F K$ has an extremal geometry $\hat{\Gamma} := \Gamma(\hat{\mathfrak{g}})$ isomorphic to $\Gamma(V, \Pi)$ where $V$ is a $K$-vector space and $\Pi$ a subspace of $V^*$ with $\text{Ann}_V(\Pi) = \{0\}$. The Lie algebra $\hat{\mathfrak{g}}$ contains pure extremal elements and, by Theorem 4.1 and 4.8, the Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to $\mathfrak{psl}(V, \Pi)$. (Indeed, if the center is nontrivial, then it is fixed by $\sigma$ and we find $\mathfrak{g}$ to have a nontrivial center.)

As already observed in Section 2, the nontrivial field automorphisms $\sigma$ of the extension induces an automorphism, also denoted by $\sigma$ of $\hat{\mathfrak{g}}$ acting on an element $x \otimes \lambda$, with $x \in \mathfrak{g}$ and $\lambda \in K$ as

$$(x \otimes \lambda)^\sigma = x \otimes \lambda^\sigma.$$ 

This automorphism also induces an automorphism, which we also denote by $\sigma$, on the extremal geometry $\hat{\Gamma}$.

Let $\hat{E}$ be the set of extremal points of $\hat{\mathfrak{g}}$. Then, as we assume that $\hat{\Gamma}$ is isomorphic to $\Gamma(V, \Pi)$ for some vector space $V$ and subspace $\Pi$ of $V^*$.
with annihilator \( \text{Ann}_V(\Pi) = \{0\} \), we can identify the elements of \( \hat{\mathcal{E}} \) with incident pairs \((p, H)\), where \( p \in \mathbb{P} := \mathbb{P}(V) \) and \( H \in \mathbb{H} \), where \( \mathbb{H} \) is the set of hyperplanes \( \ker(\phi) \), with \( 0 \neq \phi \in \Pi \).

Let \( p \in \mathbb{P} \) and \( H \in \mathbb{H} \). Then \( \mathbb{P} := \{(p, K) \mid K \in \mathbb{H} \text{ with } p \in K\} \) and \( \mathbb{H} := \{(q, H) \mid q \in \mathbb{P} \text{ with } q \in H\} \) are maximal cliques of the collinearity graph of \( \Gamma(V, \Pi) \). Let \( \mathbb{P} \) denote the set of cliques \( p \) with \( p \in \mathbb{P} \) and \( \mathbb{H} \) the set of cliques \( H \) with \( H \in \mathbb{H} \). Finally set \( \mathcal{C} \) to be the union of \( \mathbb{P} \) and \( \mathbb{H} \).

**Proposition 5.1.** (a) Let \( C \) be a clique in the collinearity graph of \( \Gamma(V, \Pi) \) of size at least 2, then \( C \) is contained in a unique clique from \( \mathcal{C} \).

(b) A point \( p \in \mathbb{P} \) is incident with a hyperplane \( H \in \mathbb{H} \) if and only if \( \overline{p} \cap \overline{H} \neq \emptyset \).

(c) The elements from \( \mathbb{P} \) partition the point set of \( \Gamma(V, \Pi) \).

(d) The elements from \( \mathbb{H} \) partition the point set of \( \Gamma(V, \Pi) \).

**Proof.** Straightforward.

The set \( \mathcal{C} \) is partitioned into the two parts \( \mathbb{P} \cup \mathbb{H} \). To be in the same part of the partition can be characterized as follows:

**Lemma 5.2.** Two maximal cliques \( C_1 \) and \( C_2 \) are in the same part of the partition \( \mathbb{P} \cup \mathbb{H} \) if and only if there is a third maximal clique \( C_3 \) with

\[
C_1 \cap C_3 \neq \emptyset \neq C_2 \cap C_3.
\]

**Proof.** Straightforward.

**Lemma 5.3.** If \( p \in \mathbb{P} \), then \( \overline{p} \in \mathbb{H} \) and if \( H \in \mathbb{H} \), then \( \overline{H} \in \mathbb{P} \).

**Proof.** Consider an element \( x \in \hat{\Gamma} \) fixed by \( \sigma \). Then \( x = (p, H) \) for some point \( p \in \mathbb{P} \) and hyperplane \( H \in \mathbb{H} \). As \( \sigma \) does not fix any line of \( \hat{\Gamma} \), it must map \( \overline{p} \) to \( \overline{H} \).

For every \( q \in \mathbb{P} \) different from \( p \), we find that \( \overline{q} \) does not meet \( \overline{p} \), but there is a maximal clique meeting both \( \overline{p} \) and \( \overline{q} \). Since \( \sigma \) is an automorphism of \( \Gamma \), we find that \( \overline{q'} \) does not meet \( \overline{p'} \), but there is a maximal clique meeting both \( \overline{p'} \) and \( \overline{q'} \). So, by the above lemma, we also have \( \overline{q'} \in \mathbb{H} \). The argument for hyperplanes is similar.

**Lemma 5.4.** If \( p, q \in \mathbb{P} \), then \( \overline{p} \cap \overline{q} \neq \emptyset \) if and only if \( \overline{p} \cap \overline{q} \neq \emptyset \).
Proof. This is immediate since $\sigma$ is an involution.

The above two lemmas imply that $\sigma$ induces a quasi-polarity on $\mathbb{P}$ mapping a point $p$ of $\mathbb{P}$ to the hyperplane $H$ with $H = \overline{p}^\sigma$. Again we use $\sigma$ to denote this quasi-polarity. This quasi-polarity $\sigma$ is non-degenerate, as each point is mapped to a proper hyperplane in $\mathbb{H}$.

The pairs $(p, H)$ in $E$ are precisely the pairs $(p, H)$ for which $p$ is contained in $H = \overline{p}^\sigma$.

The quasi-polarity $\sigma$ can be realized by a reflexive sesquilinear form $h : V \times V \to K$.

That means, there are $\epsilon \in K$ and $\tau$ in the automorphism group of $K$ of order at most 2 with $\epsilon^\tau = \epsilon^{-1}$ such that $h$ is linear in the second coordinate and satisfies

$$h(v, w) = \epsilon h(w, v)^\tau$$

for all $v, w \in V$.

Due to Theorem 1.1 and the above we have the following identifications:

$$x \in \hat{E} \leftrightarrow t_{v, \phi} = t_{v, h(w, \cdot)} \leftrightarrow v \otimes h(w, \cdot),$$

for some $w \in V$.

The polarity $\sigma$ determines the form $h$ up to proportionality (i.e. we can scale $h$ by multiplying it with a scalar). But that implies that we can fix an isotropic vector $v_0 \in V$ and scale $h$ in such way that

$$v_0 \otimes h(v_0, \cdot) \in E.$$ 

Having done this we obtain the following.

**Lemma 5.5.** We have $\epsilon = -1$ and $\tau = \sigma$, and $E$ consists of the elements $\alpha v \otimes h(v, \cdot)$ with $0 \neq v \in V$, $h(v, v) = 0$ and $\alpha \in F$.

*Proof.* Let $x = v_0 \otimes h(v_0, \cdot)$. Then we find $\exp(x, \lambda)$ with $\lambda \in F$ to be an $F$-linear automorphism of $\mathfrak{g}$. In particular, if $\langle w \otimes h(w, \cdot) \rangle$ is a 1-space of $\mathfrak{g}$ containing an extremal element of $E$, we find that $\langle \exp(x, \lambda)(w \otimes h(w, \cdot)) \rangle$ also contains elements of $E$. Take such an element with $h(v_0, w) \neq 0$.

Then

$$\exp(x, \lambda)(w \otimes h(w, \cdot)) = (w + \lambda h(v_0, w)v_0) \otimes (h(w, \cdot) - \lambda h(w, v_0)h(v_0, \cdot)).$$

As the latter spans a 1-space containing an element of $E$, we do have that $h(w + \lambda h(v_0, w)v_0, \cdot)$ is a scalar multiple of $h(w, \cdot) - \lambda h(w, v_0)h(v_0, \cdot)$. 

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But this implies that
\[ w - \lambda^\tau h(w,v_0)v_0 = w - \epsilon \lambda^\tau h(v_0,w)^{\tau^2} v_0 = w + \lambda h(v_0,w)v_0. \]
In particular, we find \(-\epsilon \lambda^\tau = \lambda\). Taking \(\lambda\) to be 1 yields \(\epsilon = -1\) and then \(\lambda^\tau = \lambda\) for all \(\lambda \in \mathbb{F}\). So, \(\tau = \sigma\) or the identity.

If \(\tau\) is the identity, then \(h\) is a symplectic form. In particular, for each vector \(0 \neq v \in V\) we find \(\langle v \rangle \subseteq \langle v \rangle^{\sigma}\). But then \(\hat{g}\) is contained in the symplectic subalgebra \(\mathfrak{s}_h(V,V^*)\), and does not have an extremal geometry isomorphic to \(\Gamma(V,\Pi)\).

So, \(\tau = \sigma\). Now if \(y = \alpha w \otimes h(w,\cdot)\) is in \(E\), then consider \(\exp(y,\lambda)x = (v_0 + \lambda h(w,v_0)\alpha w) \otimes (h(v_0 + \alpha^\sigma \lambda h(w,v_0)w,\cdot)) \in E\).

As above, this can only be in \(E\) provided \(\alpha^\sigma = \alpha\). But that proves the lemma.

This results in the following theorem.

**Theorem 5.6.** \(g\) is isomorphic to \(\mathfrak{s}_h(V,V^*)\) modulo its center, where \(h\) is a non-degenerate skew-Hermitian form with \(h(v,w) = -h(w,v)^\sigma\) for all \(v,w \in V\).

**Proof.** The lemma clearly implies that \(g\) is, up to a center, isomorphic to \(\mathfrak{s}_h(V,V^*)\), for the skew-Hermitian form \(h\). \qed

Since \(\hat{g}\) is isomorphic to \(\mathfrak{s}(V,\Pi)\) we find that the \(V\) is generated by the vectors \(v \in V\) with \(h(v,v) = 0\). This implies, by Proposition \([5.2]\) that the form \(h\) is trace-valued. Moreover, by Proposition \([5.3]\) the Lie algebra \(\mathfrak{s}_h(V,V^*)\) modulo its center is isomorphic to \(\mathfrak{psu}(V,h)\). So, we have finished the proof of Theorem \([1.3]\).

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