Test of the hierarchical theory for the FQHE

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Abstract

In the hierarchical theory of the fractional quantum Hall effect, the low–energy behaviour of a daughter state in the next level of the hierarchy is described by an interacting system of quasiparticles of the parent state. Taking the filled lowest Landau level as the parent state, we examine analytically the quantitative consequences of this approach for electrons interacting via a pseudopotential interaction. It is shown that the ground state energy per particle in the daughter state at a filling factor 2/3 is exactly equal to that of a system of quasiholes in the parent state with half filling, precisely as predicted by the hierarchical approach. This is achieved with only up to two–particle interactions in the effective Hamiltonian for the quasiholes. Their single particle energy and two–particle interaction are derived. The results are generalized to the other filling factors attainable from the filled Landau level.

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**Introduction.** All attempts to understand the surprising transport properties in the fractional quantum Hall effect (FQHE) are based on investigations of the energy spectrum of interacting electrons in a strong magnetic field. While the low–energy behaviour at filling factors $\nu = 1/q$ is quite well understood after Laughlin’s work [4], the theory for the occurrence of filling factors $p/q$ ($p \neq 1$) is still under debate. A sound theory should not only explain their occurrence but also make more quantitative predictions about the energy gap, the plateau width and consequently about the stability in the experiment; e.g., the plateau of the $1/3$–state is stronger than that of the $2/7$–state, and the $4/11$–plateau has not been observed for spin polarized electrons up to now [3]. There are at least two approaches [4].

One approach, the hierarchical theory, was originally introduced by Halperin and Haldane [5, 6], see also [7, 8, 9]. There, the occurrence of the filling factors $p/q$ ($p \neq 1$) is explained by a qualitative analogy to a $1/q$–ground state, the Laughlin state, in which the energetically degenerated electrons in the lowest Landau level form a particularly stable state under the influence of their interaction. Small deviations from the filling factor $1/q$ of this ”parent state” create energetically degenerated quasiparticles (quasiholes and quasielectrons, respectively). Then, for a macroscopic number of these quasiparticles, again a condensation phenomenon occurs under the influence of the two–quasiparticle interaction, and a new ”daughter–FQHE–state” is formed. This scheme can be applied iteratively, so that finally all states with an odd–denominator filling factor are predicted by the theory.

The second approach, the composite fermion theory by Jain [12], avoids a hierarchical construction. Instead, the trial wave functions at $\nu = p/q$ are explicitly given by the electronic degrees of freedom, although not in a very convenient form. Energy expectation values of these trial wave functions show quite good agreement with exact data, particularly on the sphere [13]. However, the reason for the success of these wave functions still remains unclear. Either approach has its merits and it seems to be worthwhile to determine the range of applicability of these theories by more quantitative calculations.

In order to check the hierarchical theory quantitatively, one has to show that the low–energy behaviour of the fermions at the filling factor of the daughter state can be described by the quasiparticles moving in the background at the parent state. The first and simplest test is the comparison of the ground state energy. In order to do this, the interaction of the quasiparticles has to be determined. There are up to now two attempts to carry out such a programme. Béran and Morf calculated the two–particle interaction of the $1/3$–quasielectrons forming the $2/5$–daughter–state in the hierarchical picture with anyonic quasiparticle wave functions, and they compared their ground state and gap energies with those of the original system [10]. Endesfelder and Terzidis extracted the interaction potential coefficients for the $1/3$–quasiholes from exact diagonalizations of small systems and compared then the bosonic quasihole spectrum with the fermionic spectrum at the filling factor $2/7$ [11]. However, they did not study the ground state energy at $2/7$ for a macroscopic
number of quasiparticles.
In this paper, we want to investigate the validity of the hierarchical theory by comparing ground state energies resulting from two different calculations. In the first calculation, we take the filled Landau level as the parent state. Exciting \( N/2 \) quasiholes in the background of this state results in a daughter state at filling \( 2/3 \), according to the hierarchical theory; \( N \to \infty \) is the number of electrons in the parent state. In order to calculate the ground state energy in this approach, we determine the single particle energy and the interaction in the effective quasihole Hamiltonian from exact spectra of systems with one and two quasiholes, respectively. Then, considering this effective Hamiltonian now for \( N/2 \) quasiholes, its ground state energy can be exactly determined, and from this, the ground state energy per particle of the daughter state at filling factor \( 2/3 \) is derived in the thermodynamic limit. In the second calculation, we derive the ground state energy at filling factor \( 2/3 \) from the well known ground state energy at \( 1/3 \) using the particle–hole–transformation.

In the following, we will perform our calculations in the general case of \( N/p \) quasiholes in the parent state which leads to a filling factor of \( p/(p + 1) \) of the daughter state (\( p \) even; \( p \) is not to be confused with the numerator from the filling factor \( p/q \)). The appeal of our approach lies in the fact that we determine all energies analytically.

Next, after introducing our model, we determine the ingredients in the effective Hamiltonian, show that the electronic states can be described by the bosonic quasihole Hamiltonian, and do the comparison.

The model. We investigate electrons moving on a two–dimensional disk. The single particle Hilbert space is restricted to spin-polarized states in the lowest Landau level by a strong magnetic field, and the only quantum number of these energetically degenerated states is the angular momentum \( m \) (\( m = 0, 1, \ldots \)). The Hamiltonian of a finite system of interacting electrons is

\[
H = \frac{1}{2} \sum_{m_1, m_2} W_{m_1 m_2 m_3} c_{m_1}^\dagger c_{m_2} c_{m_3} c_{m_4}.
\]

The filling factor is given by the relation \( \nu = \frac{N - 1}{m_{\text{max}}} = \frac{N - 1}{N_\phi - 1} \), where \( N \) is the number of electrons, \( m_{\text{max}} \) is the maximum single particle angular momentum and \( N_\phi \) is the number of flux quanta through the disk and thus defines its area. \( N_\phi \) is also the number of single particle states, i. e., the degree of degeneracy. The matrix elements \( W_{m_1 m_2 m_3 m_4} \) are fixed by the two–particle interaction \( V(|z - z'|) \) which is assumed to be isotropic and translationally invariant. As already pointed out by Haldane, any interaction in the lowest Landau level can be characterized by pseudopotential coefficients \( V_k \). \( k \geq 0 \) denotes the relative angular momentum, even for bosons, odd for fermions. Reversely, we can construct an arbitrary interaction in the Hilbert space of the lowest Landau level by a certain choice of the \( V_k \). It is well–known that the Laughlin state \( \Psi(z_1, \ldots, z_N) = \prod_{k<l} (z_k - z_l)^q \exp(-\frac{1}{4} \sum_{i=1}^N |z_i|^2) \) is the
exact non-degenerate ground state of energy zero at filling factor $\nu = 1/q$ ($q$ odd for fermions and even for bosons) for the following choice of the interaction: the $V_k$ are arbitrary for $0 \leq k \leq q - 1$ and $V_k = 0$ for all odd (even) $k$ with $k \geq q$ \cite{15,16}. For our explicit calculations, we will use below the special interaction parametrized by an even number $p \geq 2$ such that

$$V_k > 0 \quad (0 \leq k \leq p); \quad V_k = 0 \quad (k > p). \quad (2)$$

Two particles. The energy spectrum of two fermions or bosons moving in the lowest Landau level can be described by the pseudopotential coefficients $V_k$. The problem separates into a free motion of the center of mass and a relative motion in the one-particle potential $V(|z_-|) = (z_1 + z_2)/2$, $z_- = z_1 - z_2$; the constant kinetic energy is subtracted]. The total angular momentum $M$ and the relative and center of mass angular momenta $l_r$ and $l_s$ are conserved quantities ($M = l_r + l_s$). The eigenvalues are independent of $l_s$ and given by the pseudopotential coefficients $V_{l_r}$ with $0 \leq l_r \leq M$ (fermions: $l_r = 2j - 1$ – odd, $1 \leq j \leq [M+1]/2$; bosons: $l_r = 2j$ – even, $0 \leq j \leq [M]/2$; and $[x]$ is the greatest integer not greater than $x$). So far, the considerations apply to an infinite system, $m_{\text{max}} = \infty$. In a finite system, the eigenvalues of angular momentum blocks of $H$ with $M \leq m_{\text{max}}$ are not influenced by the finiteness of the system (both single particle angular momenta are then smaller than $m_{\text{max}}$), cf. \cite{17}, and these eigenvalues can be extracted from a two-particle spectrum without any finite size correction.

A single quasihole at $\nu = 1$. In order to introduce the notation, we start with the filled lowest Landau level, i.e., a stable Laughlin state with $q = 1$, where $N$ electrons occupy the single particle states with angular momenta $m = 0, \ldots, N - 1$, i.e., $N_\phi = N$ and $\nu = 1$. The total angular momentum of this state is $M_N = 1/2 N(N - 1)$, its total energy is denoted by $E(N, N, M_N) \equiv E_g(N, N)$. Here and in the following, we provide total energies with the arguments $N, N_\Phi$, and angular momentum $M$, while $E_g(N, N_\phi)$ denotes the ground state energy of a system with $N$ particles and $N_\phi$ flux quanta. The energy per particle $\varepsilon_g(\nu = 1)$ for this interacting electronic system in the thermodynamic limit is determined from the definition of the matrix elements $W_{m_1 m_2 m_3 m_4}$

$$\varepsilon_0(1) = \lim_{N \to \infty} \frac{E_g(N, N)}{N} \quad = \quad \lim_{N \to \infty} \frac{1}{2N} \sum_{m_1,m_2=0}^{N-1} (W_{m_1 m_2 m_1} - W_{m_1 m_2 m_2}) = 2 \sum_{i=1}^{p/2} V_{2i-1}. \quad (3)$$

The finite size correction decays as $1/\sqrt{N}$ \cite{17}. The last sum is cut off due to our special choice of the interaction (2). Here only quasiholes, but no quasielectrons, can be created without leaving the lowest Landau level. Next,
we create a one–quasihole state by increasing the size of the system by one flux quantum, i.e., the degeneracy $N_\phi$ by one. The difference between the energy of these eigenstates with various angular momenta $M$ and the energy of the filled Landau level $\varepsilon^n_\nu(N, M; \nu = 1) \equiv E(N, N + 1, M) - E_g(N, N)$ is defined as the neutral quasihole energy (see for the different definitions of quasiparticle energies [18, 19]). Because one of the single particle states is now unoccupied, there are $N$ different one–quasihole states with non-vanishing energy, having total angular momenta $M$ from $M = M_N + 1$ to $M_N + N(\equiv M_{N+1})$. In the thermodynamic limit, these neutral one–quasihole energies become independent of $M$ and hence the quasihole states become energetically degenerated just as for free electrons in the lowest Landau level. Performing the summations over occupied states (cf. (3)), we find for the neutral quasihole energy

$$\varepsilon^n_n(1) = \lim_{N\to\infty} \varepsilon^n_\nu(N, M; 1) = -\varepsilon_0(1).$$

(4)

Usually, the degeneracy is one of the basic assumptions of the hierarchical theory. Here, it follows straight forwardly.

Two quasiholes at $\nu = 1$. We next create a second quasihole by an additional increase of the system size by one flux quantum, i.e., only $N$ out of the now $N_{\phi} = N + 2$ states are occupied by electrons. There are angular momentum blocks with $M = M_N + 2$ to $M_N + 2N(\equiv M_{N+2} - 1)$. The $i^{th}$ energy eigenvalue in a block $M$ is denoted by $E_i(N, N + 2, M)$ with $1 \leq i \leq \min\left(\left\lfloor\frac{M_N+1-M+1}{2}\right\rfloor, N + 1 - \left\lfloor\frac{M_{N+2}-M+2}{2}\right\rfloor\right)$. The eigenvalues of this fermionic system can be determined simply using the particle–hole–transformation [20] by relating the spectrum of $N$ electrons at degeneracy $N_\Phi$ to that of $N_\Phi - N$ holes in the Landau level at the same degeneracy. Performing the unitary particle–hole–transformation $U_{ph}$ we get the hole Hamiltonian

$$H' = U_{ph}^\dagger H U_{ph} = \frac{1}{2} \sum_{m_1, m_2, m_3, m_4} W_{m_1 m_2 m_3 m_4} c_{m_1}^\dagger c_{m_2}^\dagger c_{m_3} c_{m_4}^\dagger$$

$$- \sum_{m_1, m_2 = 0}^{m_{\text{max}}} \left( W_{m_1 m_2 m_1 m_1} - W_{m_1 m_2 m_1 m_2} \right) (c_{m_1}^\dagger c_{m_1}^\dagger - \frac{1}{2})$$

(5)

with hole creation operators $c_{m}^\dagger = U_{ph}^\dagger c_{m} U_{ph}$. The eigenvalues of the Hamiltonian $H$ (1) are identical with those of $H'$ (5). If we specify $N_\phi = N + 2$, the eigenvalues of the two–quasihole block of $H$ are connected via (5) with a two–particle–spectrum, for which the terms on the r. h. s. are known. We find

$$E_i(N, N + 2, M)$$

$$= V_{2i-1} - B_i(2, N + 2, M_{N+2} - M) + E_g(N + 2, N + 2)$$

(6)

for all $M$ with $M_{N+1} \leq M \leq M_{N+2} - 1$. The background contribution $B_i(g, N, M)$ coming from the one–particle term in (5) describes the interaction of a state of $g$ holes of angular momentum $M$ with the homogenous background
enclosing $N$ flux quanta. For its thermodynamic limit, we get for a two–hole state in leading order (again performing the summations)

$$\lim_{N \to \infty} B_i(2, N, M) = 4\varepsilon_0(1).$$

(7)

Thus, starting from the largest $M$, i. e. , from $M = M_{N+2} - 1$, at which there is only one eigenvalue in the block, $i = 1$, we find successively the corresponding fermionic eigenvalues $E_i(N, N + 2, M)$.

The behaviour of all terms in (6) with respect to large increasing $N$ is known. The above analytical determination of the energies is a special advantage of the case $\nu = 1$. For other filling factors these eigenvalues had to be extracted from numerically calculated spectra showing an a priori unknown $N$–dependence.

**Mapping to bosons.** We want to describe the fermionic spectra effectively by quasiholes moving in the background of the filled Landau level and interacting via a two–particle interaction $\tilde{V}(|z - z'|)$ parametrized by pseudopotential coefficients $\tilde{V}_k$ which are to be determined. What kind of particles are these quasiholes? We treat them as bosons. This is justified for two quasiholes because not only the total dimension of the two–quasihole Hilbert space of bosons, but also the number of states in the sub-Hilbert spaces with a definite total angular momentum $M$, are equal to those of the fermionic system, as we show now. There are $N$ one–quasihole–states, i. e. , there are $\binom{N+1}{2}$ bosonic two–quasihole–states, while the total dimension of a fermionic Hilbert space of $N$ electrons with $N + 2$ one–particle–states is $\binom{N+1}{2}$ (the outermost one–particle–state must be occupied). Hence the total dimension of the two–quasihole Hilbert space is the same in the fermionic and bosonic description, respectively. This justifies the attempt to regard the quasiholes as bosons [17, 21]. Next, even the dimension of a block in the two–quasihole Hamiltonian with fermionic total angular momentum $\tilde{M}$ is the same as the dimension of a block in the two–boson Hamiltonian with total angular momentum $\tilde{M}$ to be determined. The fermionic dimension for angular momenta with $M_{N+2} - N \leq M \leq M_{N+2} - 1$ is $\left[\frac{M_{N+2} - M + 1}{2}\right]$ (see above two quasiholes).

On the other hand, the dimension of a Hilbert space of two bosons with total angular momentum $\tilde{M}$ is $\left[\frac{\tilde{M} + 2}{2}\right]$. Thus, for $\tilde{M} = M_{N+2} - M - 1$ the dimensions coincide. Therefore, the block of the fermionic Hamiltonian with $\tilde{M} = M_{N+2} - 1$ is mapped to a block of a bosonic Hamiltonian with $\tilde{M} = 0$, $M_{N+2} - 2$ to $\tilde{M} = 1$, $M_{N+2} - 3$ to $\tilde{M} = 2$ etc. . Thus, for each given angular momentum, the two–quasihole sub–Hilbert space can be mapped onto a two–boson sub–Hilbert space. Our considerations can be easily generalized to the case of $N$ quasiholes.

It should be mentioned that the character of the quasiholes is intimately connected with the way we create them. We created quasiholes by increasing the system size. If we would create quasiholes by taking electrons out of the system keeping the area fixed, then these so-called gross quasiholes are nothing but the holes in the section above, i. e. , are fermions, see also [22, 23].
Let us now make the following Ansatz for an effective bosonic Hamiltonian

$$\tilde{H} = \tilde{E}(N) + \sum_{m=0}^{\tilde{N}} \tilde{\varepsilon}(N, m)b_m^\dagger b_m + \frac{1}{2} \sum_{m_1, m_2, m_3, m_4 = 0} \tilde{W}_{m_1, m_2, m_3, m_4} b_{m_1}^\dagger b_{m_2}^\dagger b_{m_3} b_{m_4},$$

(8)

where $b_m^\dagger$ creates a boson with angular momentum $m$ ($m \geq 0$) in the lowest Landau level. The unknown matrix elements $\tilde{E}, \tilde{\varepsilon}, \tilde{W}$ can be determined by the condition that the eigenvalues of the bosonic Hamiltonian (8) and the fermionic Hamiltonian (1) are equal for zero, one and two bosons and quasiholes, respectively. For zero bosons, the constant $\tilde{E}(N) = E_g(N, N)$ is the energy of the filled lowest Landau level. For one boson, we find the relation $\tilde{\varepsilon}(N, m) = \varepsilon^n(N, M_{N+1} - m; 1)$ with $0 \leq m \leq N - 1$. In the last step, we determine the pseudopotential coefficients of the bosonic interaction $\tilde{W}$ by requesting that the energies of (1) and (8) agree for the two–quasihole state and the corresponding two–boson state. Because the energy eigenvalues of a two–boson system without any boundary and with an interaction $\tilde{V}$ are (see above)

$$\tilde{E}_2(2, \tilde{M}) = \tilde{V}_{2i-2}$$

(9)

with $\tilde{M} \geq 0$ and $1 \leq i \leq \lfloor (\tilde{M}+2)/2 \rfloor$ we find by combining (8), (9) with (6), (7) in the leading order in $N$

$$N\varepsilon_0(1) + 2\varepsilon^n(1) + \tilde{V}_{2i-2} = V_{2i-1} - 4\varepsilon_0(1) + (N + 2)\varepsilon_0(1) + O(1/\sqrt{N}).$$

(10)

From (4) we know $\varepsilon_0(1) + \varepsilon^n(1) = 0$ independent of the special choice of the fermionic interaction and thus we get finally in the thermodynamic limit

$$\tilde{V}_{2i-2} = V_{2i-1} \quad (i \geq 1).$$

(11)

Doing this identification successively for $\tilde{M} = 0, 1, \ldots$ all pseudopotential coefficients $\tilde{V}_{2i}$ are defined uniquely. Particularly, for a finite number of non–zero pseudopotential coefficients $V_{2i-1}$, $\tilde{W}$ is restricted too. From (11) it is obvious that if the short–range contribution in $W$ is dominant this property holds also for the quasihole interaction $\tilde{W}$.

It should be emphasized that we do not perform a mapping relating fermionic and bosonic single particle operators, but instead a mapping from the fermionic energy eigenstates of the one– and two–quasihole system to one– and two–boson states in the thermodynamic limit. This mapping could be constructed because the dimensions of the Hilbert spaces are equal.

**Comparison of the ground state energies.** The hierarchical theory assumes now that the Hamiltonian (8) with up to a two–particle interaction correctly describes the energetically low lying energy eigenstates, even if a macroscopic number of quasiparticles is present. Three–particle interactions and higher are neglected. According to Haldane [6], starting from the filled Landau level, $\tilde{N} = N/p + 1$ quasiholes (bosonic filling factor $1/p$) should form again a stable daughter state whose ground state energy per electron $\tilde{\varepsilon}_0(1/p+1)$ should then
be equal to the ground state energy per particle $\varepsilon_0(N/N_\Phi = (p+1)/p)$ at the filling factor $\nu = \frac{p}{p+1}$ of the daughter state. In order to calculate $\varepsilon_0(N/N_\Phi = (p+1)/p)$, we use the particle–hole–transformation, see (5). Since for our special choice (2) of the $V_k$ the ground state energy at $\nu = \frac{1}{p+1}$ is zero, only the single particle term and the constant in (5) contribute and the ground state energy per particle at $\nu = \frac{1}{p+1}$ is

$$\varepsilon_0(N/N_\Phi = (p+1)/p, N_\Phi = N) = \frac{(p-1)}{p} \varepsilon_0(1). \tag{12}$$

This fermionic energy per particle has to be compared with the ground state energy per number of electrons $\tilde{\varepsilon}_0(\frac{1}{p})$ in the thermodynamic limit resulting from the bosonic Hamiltonian (8) at filling factor $1/p$. But $\tilde{V}_{2i} = 0$ for all $i \geq p/2$, i.e., a Laughlin wave function with $q = p$ is the exact ground state wave function, and the interaction term contributes zero to the energy. The ground state energy per electron is for this state with $\tilde{N}$ bosons (cf. of (8))

$$\tilde{\varepsilon}_0(\frac{1}{p}) = \lim_{N \to \infty} \frac{\tilde{E}_g(\tilde{N})}{N} = \frac{(p-1)}{p} \varepsilon_0(1). \tag{13}$$

Thus, the ground state energies per particle in the fermionic and the hierarchical description, respectively, come out exactly the same.

**Discussion.** In summary, we have presented an explicit analytical calculation for the quasiholes at $\nu = 1$ which corroborates the hierarchical theory by comparing exactly ground state energies per particle in the fermionic and the effective bosonic model description. This was done by an exact mapping of the fermionic two–quasihole Hilbert space onto a two–boson Hilbert space. The matrix elements of the resulting Hamiltonian (8) with up to a two–particle interaction were determined analytically in the thermodynamic limit. Crucial for the analytical treatment of this case was the property $\varepsilon_0(1) + \varepsilon_n(1) = 0$ which led to the simple relation (11) for the pseudopotential coefficients. The calculation showed that at least for the ground state energy per particle, the description by an effective Hamiltonian (8) containing only two–boson interactions is successful. The question whether the lowest excited states are also describable by this Hamiltonian can not be answered within the framework of this analytical treatment.

There are at least two interesting extensions of this work. The first one should generalize our model to a model with Coulomb interaction and background. However, the ground state energy per particle at filling factor $1/p$ (e.g. 1/2) of the Hamiltonian (8) in such a case has to be determined numerically by extrapolating the results of finite $N$ calculations because the ground state energy of the interaction term is unknown. The second extension concerns the quasiparticles at parent states $1/q$ ($q \neq 1$). In this case, our approach can serve as a starting point for generalizing the scheme determining the pseudopotential
coefficients from two–quasiparticle spectra. Thus, this scheme should be applied to the quasiparticles at \( \nu = 1/3 \) which have already been treated by other methods [10, 11]. The pseudopotential coefficients of the interaction have then to be extracted from finite \( N \) spectra and an additional extrapolation becomes necessary. Furthermore, the mapping of the Hilbert space is then restricted to the lowest energy levels in each of the blocks where two–quasiparticle components occur, while in the present \( \nu = 1 \)–case all energy levels could be mapped.

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