Correct Definition of the Gluon Distribution Function at High Energy Colliders

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Abstract

Unlike QED, since $F^a_{\mu\nu}(x)F^{\mu\nu a}(0)$ in QCD contains cubic and quartic powers of the gluon field the present definition of the gluon distribution function at high energy colliders is not consistent with the number operator interpretation of the gluon. In this paper we derive the correct definition of the gluon distribution function at high energy colliders from first principles which is consistent with the number operator interpretation of the gluon and is gauge invariant and is consistent with the factorization theorem in QCD.

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I. INTRODUCTION

Quark and gluon distribution functions (parton distribution functions (PDFs)) inside the hadron play a significant role to study standard model and beyond standard model physics at high energy colliders. Consider for example, the Higgs discovery at large hadron colliders (LHC) \[1, 2\]. The gluon-gluon fusion process \((gg \rightarrow H)\) contribute significantly to the Higgs production cross section at LHC \[3, 4\]. Similarly, the gluon fusion and gluon fragmentation processes contribute dominantly to the production of the heavy quark, heavy quarkonium and jet etc. at high energy colliders. Hence, as the hadron collider’s total energy becomes higher and higher, the gluon distribution function inside the hadron plays significant role to study standard model and beyond standard model physics at high energy colliders.

Unfortunately, the parton distribution function inside a hadron is not calculated so far because it is a non-perturbative quantity in QCD and we do not know the solution of the non-perturbative QCD yet. Hence the usual procedure in the high energy phenomenology is to determine the parton distribution function from a set of experiments at some momentum transfer scale and then determine its evolution to other momentum transfer scale by using DGLAP evolution equations \[6\]. Once the parton distribution function is determined in this way then one uses it to predict physical observable at other collider experiments. Hence in order for this prescription to work it is necessary that these parton distribution functions are universal, i.e., they do not change from one collider experiment to another collider experiment. From this point of view it is necessary to use the correct definition of the parton distribution function at high energy colliders. If one does not use the correct definition of the parton distribution function at high energy colliders then one will obtain divergent cross section for physical observable at high energy colliders. This can be seen as follows.

Consider, for example, the hadron production at high energy colliders. If the factorization theorem holds \[7–10\] then the formula for the hadron production cross section at high energy hadronic colliders is given by

\[
\sigma(AB \rightarrow H + X) = \sum_{a,b} \int dx_1 \int dx_2 \int dz \ f_{a/A}(x_1, Q^2) f_{b/B}(x_2, Q^2) \ \hat{\sigma}(ab \rightarrow cd) \ D_{H/c}(z, Q^2)
\]

(1)

where \(f_{a/A}(x, Q^2)\) is the parton distribution function (PDF), \(\hat{\sigma}(ab \rightarrow cd)\) is the partonic
level scattering cross section and $D_{H/c}(z, Q^2)$ is the fragmentation function (FF). For non-hadronic observable at high energy colliders the fragmentation function $D_{H/c}(z, Q^2)$ is not included in eq. (1). In the above equation $a, b, c, d = q, \bar{q}, g$ where $q, \bar{q}$ and $g$ are quark, antiquark and gluon.

Hence if one does not use the correct definition of the parton distribution function or fragmentation function in eq. (1) then any uncanceled collinear or soft divergences in the Feynman diagrams in the partonic level scattering cross section $\hat{\sigma}(ab \rightarrow cd)$ can not be absorbed in the definition of the non-perturbative parton distribution function $f_{a/H}(x, Q^2)$ or fragmentation function $D_{H/a}(x, Q^2)$ in eq. (1). Hence it is necessary to derive the correct definition of the parton distribution function $f_{a/H}(x, Q^2)$ and fragmentation function $D_{H/a}(x, Q^2)$ from first principles.

A correct definition of a parton distribution function at high energy colliders must satisfy the following three properties: 1) it must be consistent with the number operator interpretation in quantum mechanics, i.e., $\psi(x)\psi(0)$ or $F^{\mu\nu a}(x)F^{a\mu\nu}_{\mu}(0)$ must be proportional to quadratic power of the quark or (quantum) gluon field, 2) it must be gauge invariant and 3) it must be consistent with factorization theorem in QCD. The present definition of the quark distribution function at high energy colliders is consistent with these three properties [8]. However, the present definition of the gluon distribution function at high energy colliders [8] is not consistent with these three properties as we will see below.

The present definition of the gluon distribution function at high energy colliders is given by [8]

$$f_{g/P}(x) = \frac{1}{2\pi x P^+} \int dy^- e^{-ixP^+y^-} [<P| F^{+\mu a}(0, y^-, 0_T) \times \mathcal{P}\exp[i g T^{(A)b}_{\mu\nu} \int_0^{y^-} dz^- Q^{+b}(0, z^-, 0_T)] \times F_{\mu^+a}(0)|P>] \quad (2)$$

where

$$F^{a}_{\mu\nu}(x) = \partial_\mu Q^a_\nu(x) - \partial_\nu Q^a_\mu(x) + gf^{abc}Q^b_\mu(x)Q^c_\nu(x), \quad (3)$$

$T^{(A)a}_{bc} = -if^{abc}$ and $Q^{a\mu}(x)$ is the quantum gluon field.

The expression $\mathcal{P}\exp[i g T^{(A)b}_{\mu\nu} \int_0^{y^-} dz^- Q^{+b}(0, z^-, 0_T)]$ in eq. (2) stands for the light-like Wilson line. Since the light-like Wilson line in QCD involves soft-collinear gluon field $A^{\mu a}(x)$, it can be treated classically as $A^{\mu a}(x)$ becomes the SU(3) pure gauge when Wilson line becomes light-like (see section II for details). Hence for the better understanding
of the number operator interpretation of the gluon in eq. (2) let us replace the light-like Wilson line $\mathcal{P}\exp[igT^{(A)b}\int_0^{y^-} dz^- Q^{+b}(0, z^-, 0_T)]$ in eq. (2) by its classical counterpart $\mathcal{P}\exp[igT^{(A)b}\int_0^{y^-} dz^- A^{+b}(0, z^-, 0_T)]$ where $A^{\mu a}(x)$ is the classical SU(3) pure gauge background field. With this one finds from eq. (3) that $F^{\mu\nu a}(x)F^{a\mu\nu}(0)$ in eq. (2) contains cubic and quartic powers of the (quantum) gluon field $Q^\mu a(x)$ which implies that the present definition of gluon distribution function at high energy colliders in eq. (2) is not consistent with the number operator interpretation in quantum mechanics, i.e., $F^{\mu\nu a}(x)F^{a\mu\nu}(0)$ is not proportional to quadratic power of the (quantum) gluon field $Q^\mu a(x)$. This implies that the present definition of the gluon distribution function at high energy colliders is not correct.

In this paper we derive correct definition of the gluon distribution function at high energy colliders from first principles which is consistent with the number operator interpretation of the gluon and is gauge invariant and is consistent with the factorization theorem in QCD.

We find that the correct definition of the gluon distribution function at high energy colliders which is consistent with the number operator interpretation of the gluon and is gauge invariant and is consistent with the factorization theorem in QCD is given by

$$f_{g/P}(x) = \frac{P^+}{2\pi} \int dy^- e^{-ixP^+y^-} \times <P|Q^\mu a(0, y^-, 0_T)|\mathcal{P}\exp[igT^{(A)b}\int_0^{y^-} dz^- A^{+b}(0, z^-, 0_T)]|Q^\mu a(0)|P>$$

(4)

which is valid in covariant gauge, in light-cone gauge, in general axial gauges, in general non-covariant gauges and in general Coulomb gauge etc. respectively. Note that the correct definition of gluon distribution function at high energy colliders in eq. (4) is gauge invariant with respect to the gauge transformation

$$T^aA^a_{\mu}(x) = U(x)T^aA^a_{\mu}(x)U^{-1}(x) + \frac{1}{ig}[\partial_{\mu}U(x)]U^{-1}(x), \quad U(x) = e^{igT^a\omega^a(x)}$$

(5)

along with the homogeneous transformation

$$T^aQ^\mu a_{\mu}(x) = U(x)T^aQ^\mu a_{\mu}(x)U^{-1}(x), \quad U(x) = e^{igT^a\omega^a(x)}$$

(6)

where $Q^\mu a(x)$ is the hard (quantum) gluon field whose distribution function we measure and $A^{\mu a}(x)$ is the soft-collinear gluon field which is the SU(3) pure gauge background field. After renormalization the gluon distribution function is expected to obey a QCD evolution equation, like DGLAP equation [15], which follows from renormalization group equation.

We will provide a derivation of eq. (4) in this paper.
The paper is organized as follows. In section II we describe soft-collinear divergences and light-like Wilson line in QCD. In section III we discuss the non-perturbative gluon correlation function and proof of factorization theorem in covariant gauge. In sections IV, V, VI and VII we discuss the non-perturbative gluon correlation function and proof of factorization theorem in general axial gauges, in light-cone gauge, in general non-covariant gauges and in general Coulomb gauge respectively. In section VIII we derive the correct definition of the gluon distribution function at high energy colliders as given by eq. (4). Section IX contains conclusions.

II. SOFT-COLLINEAR DIVERGENCES AND LIGHT-LIKE WILSON LINE IN QCD

Note that in eq. (1) the collinear and soft divergences can occur in the Feynman diagrams in partonic level cross sections. Hence it is important to show that these collinear/soft divergences are factorized into the definition of the parton distribution function/fragmentation function. This is done by supplying Wilson line in the definition of parton distribution function/fragmentation function which also makes them gauge invariant. Hence we will briefly discuss the issue of gauge invariance and the Wilson line for soft-collinear divergences in QCD in this section.

Before proceeding to the issue of gauge invariance and the Wilson line for soft-collinear divergences in QCD let us first discuss the corresponding situation in QED. The gauge transformation of the Dirac field of the electron in QED is given by

\[ \psi'(x) = e^{ie\omega(x)} \psi(x). \]  

(7)

Hence we can expect to address the issue of gauge invariance and factorization of soft-collinear divergences in QED simultaneously if we can relate the \( \omega(x) \) to the photon field \( A^\mu(x) \). This is easily done by using Eikonal Feynman rules in QED for soft-collinear divergences which can be seen as follows.

The Eikonal propagator times the Eikonal vertex for a photon with four momentum \( k^\mu \) interacting with an electron moving with four momentum \( p^\mu \) in the limit \( |\vec{k}| << |\vec{p}| \) is given by [8, 16–24]

\[ e \frac{p^\mu}{p \cdot k + i\epsilon}. \]  

(8)
In QED the soft divergences arise only from the emission of a photon for which all components of the four-momentum \( k^\mu \) are small (\(|\vec{k}| \to 0\)) which is evident from eq. (8). From eq. (8) one also finds that when \(0 < |\vec{k}| \ll |\vec{p}|\) and \(\vec{p}\) is parallel to \(\vec{k}\) one may find collinear divergences.

However, since

\[
p \cdot k = p_0 k_0 - \vec{p} \cdot \vec{k} = |\vec{p}| |\vec{k}| (\sqrt{1 + \frac{m^2}{p^2}} - \cos \theta)
\]

one finds that the collinear divergences does not appear in QED because of the non-vanishing mass of the electron, i.e., \(m \neq 0\). From eqs. (8) and (9) one finds that the collinear divergences appear only when \(m = 0\) and \(\theta = 0\) where \(\theta\) is the angle between \(\vec{p}\) and \(\vec{k}\).

Since gluons are massless and the massless gluons interact with each other one finds that the collinear divergences appear in QCD. Since a massless particle is always light-like one finds that the soft-collinear divergences can be described by light-like Wilson line in QCD.

For light-like electron we find from eq. (8)

\[
e \frac{p^\mu}{p \cdot k + i\epsilon} = e \frac{l^\mu}{l \cdot k + i\epsilon}
\]

(10)

where \(l^\mu\) is the light-like four-velocity (\(|\vec{l}| = 1\)) of the electron. Note that when we say the "light-like electron" we mean the electron that is traveling at its highest speed which is arbitrarily close to the speed of light (\(|\vec{l}| \sim 1\)) as it can not travel exactly at speed of light because it has finite mass even if the mass of the electron is very small. Hence we find that if \(l^\mu\) is light-like four velocity then the soft-collinear divergences can be described by the Eikonal Feynman rule as given by eq. (10).

From eq. (10) we find

\[
e \int \frac{d^4 k}{(2\pi)^4} \frac{l \cdot A(k)}{l \cdot k + i\epsilon} = -ei \int_0^\infty d\lambda \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot l} l \cdot A(k) = ie \int_0^\infty d\lambda l \cdot A(l\lambda)
\]

(11)

where the photon field \(A^\mu(x)\) and its Fourier transform \(A^\mu(k)\) are related by

\[
A^\mu(x) = \int \frac{d^4 k}{(2\pi)^4} A^\mu(k) e^{ik \cdot x}.
\]

(12)

Now consider the corresponding Feynman diagram for the soft-collinear divergences in QED due to exchange of two soft-collinear photons of four-momenta \(k_1^\mu\) and \(k_2^\mu\). The corresponding Eikonal contribution due to two soft-collinear photons exchange is analogously given by

\[
e^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{l \cdot A(k_2) l \cdot A(k_1)}{(l \cdot (k_1 + k_2) + i\epsilon)(l \cdot k_1 + i\epsilon)}
\]
\[ e^{2i^2 \int_0^\infty d\lambda_2 \int_{\lambda_2}^\infty d\lambda_1 l \cdot A(l\lambda_2)l \cdot A(l\lambda_1)} = e^{\frac{2i^2}{2!} \int_0^\infty d\lambda_2 \int_{\lambda_2}^\infty d\lambda_1 l \cdot A(l\lambda_2)l \cdot A(l\lambda_1)}. \]  

(13)

Extending this calculation up to infinite number of soft-collinear photons we find that the Eikonal contribution for the soft-collinear divergences due to soft-collinear photons exchange with the light-like electron in QED is given by the exponential

\[ e^{ie \int_0^\infty d\lambda \cdot A(\lambda)} \tag{14} \]

where \( l^\mu \) is the light-like four velocity of the electron. The Wilson line in QED is given by

\[ e^{ie \int_{x_1}^{x_f} dx^\mu A_\mu(x)} \tag{15} \]

When \( A^\mu(x) = A^\mu(\lambda l) \) as in eq. (14) then one finds from eq. (15) that the light-like Wilson line in QED for soft-collinear divergences is given by

\[ e^{ie \int_0^x dx^\mu A_\mu(x)} = e^{-ie \int_0^\infty d\lambda \cdot A(x+l\lambda)} e^{ie \int_0^\infty d\lambda \cdot A(\lambda)} \tag{16} \]

Note that a light-like electron traveling with light-like four-velocity \( l^\mu \) produces U(1) pure gauge potential \( A^\mu(x) \) at all the time-space position \( x^\mu \) except at the position \( \vec{x} \) perpendicular to the direction of motion of the electron \( (\vec{l} \cdot \vec{x} = 0) \) at the time of closest approach [16, 26, 27]. When \( A^\mu(x) = A^\mu(\lambda l) \) as in eq. (14) we find \( \vec{l} \cdot \vec{x} = \lambda \vec{l} \cdot \vec{l} = \lambda \neq 0 \) which implies that the light-like Wilson line finds the photon field \( A^\mu(x) \) in eq. (14) as the U(1) pure gauge. The U(1) pure gauge is given by

\[ A^\mu(x) = \partial^\mu \omega(x) \tag{17} \]

which gives from eq. (16) the light-like Wilson line in QED for soft-collinear divergences

\[ e^{i e \omega(x)} e^{-i e \omega(0)} = e^{ie \int_0^x dx^\mu A_\mu(x)} = e^{-ie \int_0^\infty d\lambda \cdot A(x+l\lambda)} e^{ie \int_0^\infty d\lambda \cdot A(\lambda)} \tag{18} \]

which depends only on end points 0 and \( x^\mu \) but is independent of the path. The path independence can also be found from Stokes theorem because for pure gauge

\[ F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = 0 \tag{19} \]

which gives from Stokes theorem

\[ e^{ie \oint_C dx^\mu A_\mu(x)} = e^{ie \int_S dx^\mu dx^\nu F_{\mu\nu}(x)} = 1 \tag{20} \]
where $C$ is a closed path and $S$ is the surface enclosing $C$. Now considering two different paths $L$ and $M$ with common end points $0$ and $x^\mu$ we find

$$ e^{i\epsilon \int_{C} dx^\mu A_\mu(x)} = e^{i\epsilon \int_{L} dx^\mu A_\mu(x)} - e^{i\epsilon \int_{M} dx^\mu A_\mu(x)} = 1$$

which implies that

$$ e^{i\epsilon \int_{0}^{x} dx^\mu A_\mu(x)} $$

depends only on end points $0$ and $x^\mu$ but is independent of path which can also be seen from eq. (18). Hence from eq. (18) we find that the abelian phase or the gauge link in QED is given by

$$ e^{-i\epsilon \int_{0}^{\infty} d\lambda A(x+\lambda)} = e^{i\epsilon \omega(x)}.$$

From eqs. (7) and (23) one expects that the gauge invariance and factorization of soft-collinear divergences in QED can be explained simultaneously.

One can recall that the gauge invariant greens function in QED

$$ G(x_1, x_2) = \langle \bar{\psi}(x_2) \times \exp[i\epsilon \int_{x_2}^{x_1} dx^\mu A_\mu(x)] \times \psi(x_1) \rangle $$

in the presence of background field $A^\mu(x)$ was formulated by Schwinger long time ago. When this background field $A^\mu(x)$ is replaced by the U(1) pure gauge background field as given by eq. (17) then one finds by using the path integral method of QED that

$$ e^{i\epsilon \omega(x_2)} < \bar{\psi}(x_2) \psi(x_1) >_A = e^{-i\epsilon \omega(x_1)} = < \bar{\psi}(x_2) \psi(x_1) > $n

which proves the gauge invariance and factorization of soft-collinear divergences in QED simultaneously. In eq. (25) the $< \bar{\psi}(x_2) \psi(x_1) >$ is the full Green’s function in QED and $< \bar{\psi}(x_2) \psi(x_1) >_A$ is the corresponding Green’s function in the background field method of QED. This path integral technique is also used in [18] to prove factorization of soft-collinear divergences in non-equilibrium QED.

Hence we find that the gauge invariance and factorization of soft-collinear divergences in QED can be studied by using path integral method of QED in the presence of U(1) pure gauge background field. Therefore one expects that the gauge invariance and factorization
of soft-collinear divergences in QCD can be studied by using path integral method of QCD in the presence of SU(3) pure gauge background field.

Now let us proceed to QCD. The gauge transformation of the quark field in QCD is given by

\[ \psi'(x) = e^{igT^a(\omega^a(x))}\psi(x). \]  

(26)

Hence one finds that the issue of gauge invariance and factorization of soft-collinear divergences in QCD can be simultaneously explained if \( \omega^a(x) \) can be related to the gluon field \( A^{\mu a}(x) \). This is easily done by using Eikonal Feynman rules in QCD for soft-collinear divergences which can be seen as follows.

The Eikonal propagator times the Eikonal vertex for a gluon with four momentum \( k^\mu \) interacting with a quark moving with four momentum \( p^\mu \) in the limit \( |\vec{k}| << |\vec{p}| \) is given by

\[ gT^a \frac{p^\mu}{p \cdot k + i\epsilon}. \]  

(27)

In QCD the soft divergences arise only from the emission of a gluon for which all components of the four-momentum \( k^\mu \) are small \( (|\vec{k}| \to 0) \) which is evident from eq. (27). From eq. (27) one also finds that when \( 0 < |\vec{k}| << |\vec{p}| \) and \( \vec{p} \) is parallel to \( \vec{k} \) one may find collinear divergences in QCD.

However, since

\[ p \cdot k = p_0 k_0 - \vec{p} \cdot \vec{k} = |\vec{p}||\vec{k}|(\sqrt{1 + \frac{m^2}{p^2}} - \cos\theta) \]

(28)

one finds that the collinear divergences does not appear when quark interacts with a collinear gluon because of the non-vanishing mass of the quark, i.e., \( m \neq 0 \) even if the mass of the light quark is very small. From eq. (28) one finds that the collinear divergences appear only when \( m = 0 \) and \( \theta = 0 \) where \( \theta \) is the angle between \( \vec{p} \) and \( \vec{k} \). Since gluons are massless and the massless gluons interact with each other one finds that the collinear divergences appear in QCD. Since a massless particle is always light-like one finds that the soft-collinear divergences can be described by light-like Wilson line in QCD.

For light-like quark we find from eq. (27)

\[ gT^a \frac{p^\mu}{p \cdot k + i\epsilon} = gT^a \frac{l^\mu}{l \cdot k + i\epsilon}. \]  

(29)
where $l^\mu$ is the light-like four-velocity ($|\vec{l}|=1$) of the quark. Note that when we say the "light-like quark" we mean the quark that is traveling at its highest speed which is arbitrarily close to the speed of light ($|\vec{l}| \sim 1$) as it can not travel exactly at speed of light because it has finite mass even if the mass of the light quark is very small. On the other hand a massless gluon is light-like and it always remains light-like. Hence we find that if $l^\mu$ is light-like four velocity then the soft-collinear divergences in QCD can be described by the Eikonal Feynman rule as given by eq. (29). Note that the Eikonal Feynman rule in eq. (29) is also valid if we replace the light-like quark by light-like gluon provided we replace $T_{abc} = -if^{abc}$.

From eq. (29) we find

$$gT^a \int \frac{d^4k}{(2\pi)^4} \frac{l \cdot A^a(k)}{l \cdot k + i\epsilon} = -gT^ai \int_0^\infty d\lambda \int \frac{d^4k}{(2\pi)^4} e^{ik\lambda l} \cdot A^a(k) = igT^a \int_0^\infty d\lambda l \cdot A^a(l\lambda)$$

where the gluon field $A^{\mu a}(x)$ and its Fourier transform $A^{\mu a}(k)$ are related by

$$A^{\mu a}(x) = \int \frac{d^4k}{(2\pi)^4} A^{\mu a}(k)e^{ik\cdot x}. \quad (31)$$

Note that a path ordering in QCD is required which can be seen as follows, see also [10]. The Eikonal contribution for the soft-collinear divergences in QCD arising from a single soft-collinear gluon exchange in Feynman diagram is given by eq. (30). Now consider the corresponding Feynman diagram for the soft-collinear divergences in QCD due to exchange of two soft-collinear gluons of four-momenta $k_1^\mu$ and $k_2^\mu$. The corresponding Eikonal contribution due to two soft-collinear gluons exchange is analogously given by

$$g^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{T^a l \cdot A^a(k_2)T^b l \cdot A^b(k_1)}{(l \cdot (k_1 + k_2) + i\epsilon)(l \cdot k_1 + i\epsilon)} = g^2 i^2 \int_0^\infty d\lambda_2 \int_0^\infty d\lambda_1 T^a l \cdot A^a(l\lambda_2)T^b l \cdot A^b(l\lambda_1)$$

$$= \frac{g^2 i^2}{2!} \mathcal{P} \int_0^\infty d\lambda_2 \int_0^\infty d\lambda_1 T^a l \cdot A^a(l\lambda_2)T^b l \cdot A^b(l\lambda_1)$$

where $\mathcal{P}$ is the path ordering. Extending this calculation up to infinite number of soft-collinear gluons we find that the Eikonal contribution for the soft-collinear divergences due to soft-collinear gluons exchange with the light-like quark in QCD is given by the path ordered exponential

$$\mathcal{P} \exp[ig \int_0^\infty d\lambda l \cdot A^a(l\lambda)T^a] \quad (33)$$
where \( l^\mu \) is the light-like four velocity of the quark. The Wilson line in QCD is given by
\[
P e^{i g \int_{x_i}^{x_f} dx^\mu A^a_\mu(x) T^a} \tag{34}
\]
which is the solution of the equation
\[
\partial_\mu S(x) = ig T^a A^a_\mu(x) S(x) \tag{35}
\]
with initial condition
\[
S(x_i) = 1. \tag{36}
\]
When \( A^{\mu a}(x) = A^{\mu a}(\lambda l) \) as in eq. (33) we find from eq. (34) that the light-like Wilson line in QCD for soft-collinear divergences is given by
\[
P e^{i g \int_0^\infty d\lambda \cdot A^{\nu a}((x+\lambda l))^\nu T^a} = \left[ P e^{-ig \int_0^\infty d\lambda \cdot A^{\nu a}((x+\lambda l))^\nu} \right] P e^{ig \int_0^\infty d\lambda \cdot A^{\nu a}((x+\lambda l))^\nu} \tag{37}
\]
A light-like quark traveling with light-like four-velocity \( l^\mu \) produces SU(3) pure gauge potential \( A^{\mu a}(x) \) at all the time-space position \( x^\mu \) except at the position \( \vec{x} \) perpendicular to the direction of motion of the quark (\( \vec{l} \cdot \vec{x} = 0 \)) at the time of closest approach [16, 26, 27]. When \( A^{\mu a}(x) = A^{\mu a}(\lambda l) \) as in eq. (33) we find \( \vec{l} \cdot \vec{x} = \lambda \vec{l} \cdot \vec{l} = \lambda \neq 0 \) which implies that the light-like Wilson line finds the gluon field \( A^{\mu a}(x) \) in eq. (33) as the SU(3) pure gauge. The SU(3) pure gauge is given by
\[
T^a A^a_\mu(x) = \frac{1}{ig} [\partial_\mu U(x)] U^{-1}(x), \quad U(x) = e^{ig T^a \omega^a(x)} \tag{38}
\]
which gives
\[
U(x_f) = P e^{i g \int_{x_i}^{x_f} dx^\mu A^a_\mu(x) T^a} U(x_i) = e^{ig T^a \omega^a(x_f)}. \tag{39}
\]
Hence when \( A^{\mu a}(x) = A^{\mu a}(\lambda l) \) as in eq. (33) we find from eqs. (37) and (39) that the light-like Wilson line in QCD for soft-collinear divergences is given by
\[
P e^{i e \int_0^\infty dx^\mu A^a_\mu(x) T^a} = e^{ig T^a \omega(x)} e^{-ig T^a \omega^a(0)} = \left[ P e^{-ig \int_0^\infty d\lambda \cdot A^{\nu a}((x+\lambda l))^\nu T^a} \right] P e^{ig \int_0^\infty d\lambda \cdot A^{\nu a}((x+\lambda l))^\nu T^a} \tag{40}
\]
which depends only on end points 0 and \( x^\mu \) but is independent of the path. The path independence can also be found from the non-abelian Stokes theorem which can be seen as follows. The SU(3) pure gauge in eq. (38) gives
\[
F^a_{\mu\nu}[A] = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f^{abc} A^b_\mu(x) A^c_\nu(x) = 0. \tag{41}
\]
Using eq. (41) in the non-abelian Stokes theorem \[30\] we find
\[
P e^{ig \oint_C dx^\mu A^{a}_\mu(x) T^a} = \mathcal{P} \exp \left[ ig \int_S dx^\mu dx^\nu \left[ P e^{ig \int_0^\infty dx^\mu A^{a}_\mu(x) T^a} \right] F_{\mu\nu}^a(x) T^a \right] \left[ P e^{ig \int_0^\infty dx^\mu A^{a}_\mu(x) T^a} \right] = 1
\]
(42)

where \(C\) is a closed path and \(S\) is the surface enclosing \(C\). Now considering two different paths \(L\) and \(M\) with common end points 0 and \(x^\mu\) we find from eq. (42)
\[
P e^{ig \oint_0^x dx^\mu A^{a}_\mu(x) T^a} = \mathcal{P} \exp \left[ ig \int_L dx^\mu A^{a}_\mu(x) T^a - ig \int_M dx^\mu A^{a}_\mu(x) T^a \right] = 1
\]
(43)

which implies that the light-like Wilson line in QCD
\[
P e^{ig \int_0^x dx^\mu A^{a}_\mu(x) T^a}
\]
(44)

depends only on the end points 0 and \(x^\mu\) but is independent of the path which can also be seen from eq. \([10]\). Hence from eq. \([40]\) we find that the non-abelian phase or the gauge link in QCD is given by
\[
\Phi(x) = P e^{-ig \int_0^\infty d\lambda \cdot A^a(x + i\lambda) T^a} = e^{ig T^a \omega^a(x)}. \quad (45)
\]

Note that from eq. \([11]\) we find the vanishing physical gauge invariant field strength square \(F^{\mu\nu}[A] F_{\mu\nu}[A]\) when \(A^{\mu a}(x)\) is the SU(3) pure gauge as given by eq. \([38]\). Hence in classical mechanics the SU(3) pure gauge potential does not have an effect on color charged particle and one expects the effect of exchange of soft-collinear gluons to simply vanish. However, in quantum mechanics the situation is a little more complicated, because the gauge potential does have an effect on color charged particle even if it is SU(3) pure gauge potential and hence one should not expect the effect of exchange of soft-collinear gluons to simply vanish \([16]\). This can be verified by studying the non-perturbative correlation function of the type \(<0|\bar{\psi}(x)\psi(x')\bar{\psi}(x'')\psi(x''')...|0>\) in QCD in the presence of SU(3) pure gauge background field.

Under non-abelian gauge transformation given by eq. \([5]\) the Wilson line in QCD transforms as
\[
P e^{ie \int_{x_i}^{x_f} dx^\mu A^{a}_\mu(x) T^a} = U(x_f) \left[ P e^{ie \int_{x_i}^{x_f} dx^\mu A^{a}_\mu(x) T^a} \right] U^{-1}(x_i).
\]
(46)
From eqs. (40) and (46) we find
\[ \mathcal{P} e^{-ig \int_0^\infty d\lambda \cdot A^a(x + i\lambda) T^a} = U(x) \mathcal{P} e^{-ig \int_0^\infty d\lambda \cdot A^a(x + i\lambda) T^a}, \quad U(x) = \exp[i g T^a \omega^a(x)] \]
(47)

which gives from eq. (45)
\[ \Phi'(x) = U(x) \Phi(x), \quad \Phi'^\dagger(x) = \Phi^\dagger(x) U^{-1}(x). \]
(48)

In the adjoint representation of SU(3) the corresponding path ordered exponential is given by
\[ \Phi_{ab}(x) = \mathcal{P} \exp[-ig \int_0^\infty d\lambda \cdot A^c(x + i\lambda) T^{A(c)}] = e^{ig T^{A(c)} \omega^c(x)}, \quad (T^{A(c)})_{ab} = -if^{abc}. \]
(49)

To summarize this, we find that the soft-collinear divergences in the perturbative Feynman diagrams due to soft-collinear gluons interaction with the light-like Wilson line in QCD is given by the path ordered exponential in eq. (33) which is nothing but the non-abelian phase or the gauge link in QCD as given by eq. (45) where the gluon field $A^{\mu a}(x)$ is the SU(3) pure gauge, see eqs. (38), (39), (40). This implies that the effect of soft-collinear gluons interaction between the partons and the light-like Wilson line in QCD can be studied by putting the partons in the SU(3) pure gauge background field. Hence we find that the soft-collinear behavior of the non-perturbative correlation function of the type $<0|\bar{\psi}(x)\psi(x')\bar{\psi}(x'')\psi(x''')...|0>$ in QCD due to the presence of light-like Wilson line in QCD can be studied by using the path integral method of the QCD in the presence of SU(3) pure gauge background field.

It can be mentioned here that in soft collinear effective theory (SCET) it is also necessary to use the idea of background fields to give well defined meaning to several distinct gluon fields. Note that a massive color source traveling at speed much less than speed of light cannot produce SU(3) pure gauge field. Hence when one replaces light-like Wilson line with massive Wilson line one expects the factorization of soft/infrared divergences to break down. This is in conformation with the finding which used the diagrammatic method of QCD. In case of massive Wilson line in QCD the color transfer occurs and the factorization breaks down. Note that in case of massive Wilson line there is no collinear divergences which is explained in eq. (28).
III. NON-PERTURBATIVE GLUON CORRELATION FUNCTION AND PROOF OF FACTORIZATION THEOREM IN COVARIANT GAUGE

Since gluon distribution function inside the hadron is a non-perturbative quantity in QCD it is natural to use path integral method of QCD to study its properties from first principles. The generating functional in the path integral method of QCD is given by \[11, 33\]

\[
Z[J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \det \left( \frac{\delta (\partial_{\mu} Q^{\mu a})}{\delta \omega^b} \right) e^{i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a [Q] - \frac{1}{2\alpha} (\partial_{\mu} Q^{\mu a})^2 + \bar{\psi}[i\gamma^\mu \partial_{\mu} - m + g T^a \gamma^\mu Q^a_{\nu}] \psi + J \cdot Q + \bar{\eta} \psi + \bar{\psi} \eta \right]}
\]  

(50)

where \( J^{\mu a}(x) \) is the external source for the quantum gluon field \( Q^{\mu a}(x) \) and \( \bar{\eta}_i(x) \) is the external source for the Dirac field \( \psi_i(x) \) of the quark and

\[
F_{\mu\nu}^a [Q] = \partial_\mu Q^\nu_a(x) - \partial_\nu Q^\mu_a(x) + g f^{abc} Q^b_\mu(x) Q^c_\nu(x), \quad F_{\mu\nu}^{a2} [Q] = F^{\mu\nu a} [Q] F^a_{\mu\nu} [Q].
\]  

(51)

Note that the Faddeev-Popov (F-P) determinant \( \det \left( \frac{\delta (\partial_{\mu} Q^{\mu a})}{\delta \omega^b} \right) \) can be expressed in terms of path integral over the ghost fields \[33\] but we will directly work with the Faddeev-Popov (F-P) determinant \( \det \left( \frac{\delta (\partial_{\mu} Q^{\mu a})}{\delta \omega^b} \right) \) in this paper. The non-perturbative correlation function of the type \(< 0|\bar{\psi}(x_1)\psi(x_2)|0 > \) in QCD is given by \[17\]

\[
< 0|\bar{\psi}(x_1)\psi(x_2)|0 > = \frac{1}{Z[0]} \int [dQ][d\bar{\psi}][d\psi] \bar{\psi}(x_1)\psi(x_2) \times \det \left( \frac{\delta (\partial_{\mu} Q^{\mu a})}{\delta \omega^b} \right) e^{i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a [Q] - \frac{1}{2\alpha} (\partial_{\mu} Q^{\mu a})^2 + \bar{\psi}[i\gamma^\mu \partial_{\mu} - m + g T^a \gamma^\mu Q^a_{\nu}] \psi \right]}.
\]  

(52)

Similarly the non-perturbative gluon correlation function of the type \(< 0|Q^a_{\mu}(x_1)Q^b_{\nu}(x_2)|0 > \) in QCD is given by \[17\]

\[
< 0|Q^a_{\mu}(x_1)Q^b_{\nu}(x_2)|0 > = \int [dQ][d\bar{\psi}][d\psi] Q^a_{\mu}(x_1)Q^b_{\nu}(x_2) \times \det \left( \frac{\delta (\partial_{\mu} Q^{\mu a})}{\delta \omega^b} \right) e^{i \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^{a2} [Q] - \frac{1}{2\alpha} (\partial_{\mu} Q^{\mu a})^2 + \bar{\psi}[i\gamma^\mu \partial_{\mu} - m + g T^a \gamma^\mu Q^a_{\nu}] \psi \right]}
\]  

(53)

where the suppression of the normalization factor \( Z[0] \) is understood as it will cancel in the final result (see for example, eq. \[93\]).

We have seen in section II that the soft-collinear behavior of the non-perturbative (gluon) correlation function due to the presence of light-like Wilson line in QCD can be studied by using the path integral method of the QCD in the presence of SU(3) pure gauge background field. Hence in order to derive correct definition of the gluon distribution function at high...
energy colliders we use the path integral formulation of the background field method of QCD in the presence of SU(3) pure gauge background field as given by eq. (38).

Background field method of QCD was originally formulated by ‘t Hooft [12] and later extended by Klueberg-Stern and Zuber [13, 14] and by Abbott [11]. This is an elegant formalism which can be useful to construct gauge invariant non-perturbative green’s functions in QCD. This formalism is also useful to study quark and gluon production from classical chromo field [34] via Schwinger mechanism [35], to compute $\beta$ function in QCD [36], to perform calculations in lattice gauge theories [37] and to study evolution of QCD coupling constant in the presence of chromofield [38].

The generating functional in the path integral formulation of the background field method of QCD is given by [11–13]

$$Z[A, J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \det(\frac{\delta G^a(Q)}{\delta \omega^b}) e^{i \int d^4x [-\frac{1}{4} F^a_{\mu\nu}(A+Q) - \frac{1}{2} G^a(Q)^2 + \bar{\psi} \gamma^\mu \partial_\mu \psi - m + g T^a \gamma^\mu (A+Q)^a_\mu \psi] + J \cdot Q + \bar{\eta} \psi + \bar{\psi} \eta}$$

(54)

where the gauge fixing term is given by

$$G^a(Q) = \partial_\mu Q^{\mu a} + g f^{abc} A^b_\mu Q^{\mu c} = D_\mu[A]Q^{\mu a}$$

(55)

which depends on the background field $A^{\mu a}(x)$ and

$$F^a_{\mu\nu}[A + Q] = \partial_\mu [A^a_\nu + Q^a_\nu] - \partial_\nu [A^a_\mu + Q^a_\mu] + g f^{abc} [A^b_\mu + Q^b_\mu] [A^c_\nu + Q^c_\nu].$$

(56)

We have followed the notations of [11–13] and accordingly we have denoted the quantum gluon field by $Q^{\mu a}$ and the background field by $A^{\mu a}$. Note that in the absence of the external sources the SU(3) pure gauge can be gauged away from the generating functional in the background field method of QCD. However, in the presence of the external sources the SU(3) pure gauge can not be gauged away from the generating functional in the background field method of QCD.

The non-perturbative correlation function of the type $<0 | \bar{\psi}(x_1) \psi(x_2) | 0 >_A$ in the background field method of QCD is given by [17]

$$<0 | \bar{\psi}(x_1) \psi(x_2) | 0 >_A = \frac{1}{Z[0]} \int [dQ][d\bar{\psi}][d\psi] \bar{\psi}(x_1) \psi(x_2) \det(\frac{\delta G^a(Q)}{\delta \omega^b}) e^{i \int d^4x [-\frac{1}{4} F^a_{\mu\nu}(A+Q) - \frac{1}{2} G^a(Q)^2 + \bar{\psi} \gamma^\mu \partial_\mu \psi - m + g T^a \gamma^\mu (A+Q)^a_\mu \psi]}.$$  

(57)
Similarly the non-perturbative gluon correlation function of the type $< 0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 >_A$ in the background field method of QCD is given by [17]

$$< 0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 >_A = \int [dQ][d\bar{\psi}][d\psi] Q^a_\mu(x_1)Q^b_\nu(x_2) \times \det\left(\frac{\delta G^a(Q)}{\delta \omega^b}\right) e^{i \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu}_{\mu}[A+Q] - \frac{1}{32}(G^a(Q))^2 + \bar{\psi}i\gamma^\mu\partial_\mu-m + g T^a\gamma^\mu(A+Q)^a_\mu \right] \psi}$$  \hspace{1cm} (58)

where the suppression of the normalization factor $Z[0]$ is understood as it will cancel in the final result (see for example, eq. (93)).

The gauge fixing term $\frac{1}{2\alpha}(G^a(Q))^2$ in eq. (54) [where $G^a(Q)$ is given by eq. (55)] is invariant for gauge transformation of $A^a_\mu$:

$$\delta A^a_\mu = gf^{abc}A^b_\mu \omega^c + \partial_\mu \omega^a, \quad \text{(type I transformation)}$$  \hspace{1cm} (59)

provided one also performs a homogeneous transformation of $Q^a_\mu$ [11, 13]:

$$\delta Q^a_\mu = gf^{abc}Q^b_\mu \omega^c.$$  \hspace{1cm} (60)

The gauge transformation of background field $A^a_\mu$ as given by eq. (59) along with the homogeneous transformation of $Q^a_\mu$ in eq. (60) gives

$$\delta (A^a_\mu + Q^a_\mu) = gf^{abc}(A^b_\mu + Q^b_\mu) \omega^c + \partial_\mu \omega^a$$  \hspace{1cm} (61)

which leaves $-\frac{1}{4} F^{a2}_{\mu\nu}[A+Q]$ invariant in eq. (54).

For fixed $A^a_\mu$, i.e., for

$$\delta A^a_\mu = 0, \quad \text{(type II transformation)}$$  \hspace{1cm} (62)

the gauge transformation of $Q^a_\mu$ [11, 13]:

$$\delta Q^a_\mu = gf^{abc}(A^b_\mu + Q^b_\mu) \omega^c + \partial_\mu \omega^a$$  \hspace{1cm} (63)

gives eq. (61) which leaves $-\frac{1}{4} F^{a2}_{\mu\nu}[A+Q]$ invariant in eq. (54). Hence whether we use type I transformation [eqs. (59) and (60)] or type II transformation [eqs. (62) and (63)] we will obtain the same equation (91).

It is useful to remember that, unlike QED [17], finding an exact relation between the generating functional $Z[J, \eta, \bar{\eta}]$ in QCD in eq. (50) and the generating functional $Z[A, J, \eta, \bar{\eta}]$ in the background field method of QCD in eq. (54) in the presence of SU(3) pure gauge
background field is not easy. The main difficulty is due to the gauge fixing terms which are different in both the cases. While the Lorentz (covariant) gauge fixing term $-\frac{1}{2\alpha}(\partial_{\mu}Q^{\mu a})^2$ in eq. (50) in QCD is independent of the background field $A^{\mu a}(x)$, the background field gauge fixing term $-\frac{1}{2\alpha}(G^{a}(Q))^2$ in eq. (54) in the background field method of QCD depends on the background field $A^{\mu a}(x)$ where $G^{a}(Q)$ is given by eq. (55) [11–13]. Hence in order to study non-perturbative correlation function of the gluon in the background field method of QCD in the presence of SU(3) pure gauge background field we proceed as follows.

By changing the integration variable $Q \rightarrow Q - A$ in the right hand side of eq. (58) we find

$$\langle 0| Q^{a}_{\mu}(x_{1})Q^{b}_{\nu}(x_{2})|0 \rangle_{A} = \int [dQ][d\bar{\psi}][d\psi] \ [Q^{a}_{\mu}(x_{1}) - A^{a}_{\mu}(x_{1})][Q^{b}_{\nu}(x_{2}) - A^{b}_{\nu}(x_{2})]$$

$$\times \det\left(\frac{\delta G^{a}_{f}(Q)}{\delta \omega^{b}}\right) \ e^{i \int d^{4}x[-\frac{1}{4}F^{a}_{\mu\nu}(Q) - \frac{1}{2\alpha}(G^{a}_{f}(Q))^2 + \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m + gT^{a}_{\mu\nu}Q^{\mu}_{\nu})\psi]}$$

(64)

where from eq. (55) we find

$$G^{a}_{f}(Q) = \partial_{\mu}Q^{\mu a} + g f^{abc}A^{b}_{\mu}Q^{\mu c} - \partial_{\mu}A^{\mu a} = D_{\mu}[A]Q^{\mu a} - \partial_{\mu}A^{\mu a}$$

(65)

and from eq. (61) [by using eq. (59)] we find

$$\delta Q^{a}_{\mu} = -g f^{abc}A^{b}_{\mu}Q^{c}_{\nu} + \partial_{\mu}A^{\mu a}$$

(66)

The eqs. (64), (65) and (66) can also be derived by using type II transformation which can be seen as follows. By changing $Q \rightarrow Q - A$ in eq. (58) we find eq. (64) where the gauge fixing term from eq. (55) becomes eq. (65) and eq. (63) [by using eq. (62)] becomes eq. (66). Hence we obtain eqs. (64), (65) and (66) whether we use the type I transformation or type II transformation. Hence we find that we will obtain the same eq. (91) whether we use the type I transformation or type II transformation.

Changing the integration variable from unprimed variable to primed variable we find from eq. (64)

$$\langle 0| Q^{a}_{\mu}(x_{1})Q^{b}_{\nu}(x_{2})|0 \rangle_{A} = \int [dQ'][d\bar{\psi}'][d\psi'] \ [Q^{a}_{\mu}(x_{1}) - A^{a}_{\mu}(x_{1})][Q^{b}_{\nu}(x_{2}) - A^{b}_{\nu}(x_{2})]$$

$$\times \det\left(\frac{\delta G^{a}_{f}(Q')}{\delta \omega^{b}}\right) \ e^{i \int d^{4}x[-\frac{1}{4}F^{a}_{\mu\nu}(Q') - \frac{1}{2\alpha}(G^{a}_{f}(Q'))^2 + \bar{\psi}'(i\gamma^{\mu}\partial_{\mu} - m + gT^{a}_{\mu\nu}Q'^{\mu}_{\nu})\psi']}$$

(67)

This is because a change of integration variable from unprimed variable to primed variable does not change the value of the integration.
The equation

\[ Q_\mu^a(x) = Q_\mu^a(x) + g f^{abc} \omega^c(x) Q_\mu^b(x) + \partial_\mu \omega^a(x) \quad (68) \]

in eq. (66) is valid for infinitesimal transformation \((\omega \ll 1)\) which is obtained from the finite equation

\[ T^a Q_\mu^a(x) = U(x) T^a Q_\mu^a(x) U^{-1}(x) + \frac{1}{ig} [\partial_\mu U(x)] U^{-1}(x), \quad U(x) = e^{ig T^a \omega^a(x)} \quad (69) \]

Simplifying infinite numbers of non-commuting terms we find

\[ \left[ e^{-ig T^b \omega^b(x)} T^a e^{ig T^c \omega^c(x)} \right]_{ij} = [e^{-g M(x)}]_{ab} T^b_{ij} \quad (70) \]

where

\[ M_{ab}(x) = f^{abc} \omega^c(x). \quad (71) \]

Hence by simplifying infinite numbers of non-commuting terms in eq. (69) [by using eq. (72) and \([26]\)] we find that

\[ Q_\mu^a(x) = [e^{g M(x)}]_{ab} Q_\mu^b(x) + \left[ \frac{e^{g M(x)} - 1}{g M(x)} \right]_{ab} [\partial_\mu \omega^b(x)], \quad M_{ab}(x) = f^{abc} \omega^c(x). \quad (72) \]

Under the finite transformation, using eq. (72), we find

\[ [dQ'] = [dQ] \det \frac{\partial Q'^a}{\partial Q^b} = [dQ] \det \left[ [e^{g M(x)}] \right] = [dQ] \exp \left[ \text{Tr} (\ln [e^{g M(x)}]) \right] = [dQ] \quad (73) \]

where we have used (for any matrix \(H\))

\[ \det H = \exp \left[ \text{Tr} (\ln H) \right]. \quad (74) \]

Similarly the fermion fields transform accordingly, see eq. \([26]\). Using eqs. (72) and \([26]\) we find

\[ [d\bar{\psi}'][d\psi'] = [d\bar{\psi}'][d\psi], \quad \bar{\psi}' [i \gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q_\mu^a] \psi' = \bar{\psi} [i \gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q_\mu^a] \psi, \quad F_{\mu \nu}^{a2}[Q'] = F_{\mu \nu}^{a2}[Q]. \quad (75) \]

Simplifying all the infinite number of non-commuting terms in eq. (38) we find that the SU(3) pure gauge \(A^{\mu a}(x)\) is given by \([26]\)

\[ A^{\mu a}(x) = \partial_\mu \omega^b(x) \left[ \frac{e^{g M(x)} - 1}{g M(x)} \right]_{ab} \quad (76) \]
where $M_{ab}(x)$ is given by eq. (71). From eqs. (76) and (72) we find

$$Q^a_{\mu}(x) - A^a_{\mu}(x) = [e^{gM(x)}]_{ab} Q^b_{\mu}(x), \quad M_{ab}(x) = f^{abc} \omega^c(x). \quad (77)$$

Using eqs. (73), (75) and (77) in eq. (67) we find

$$< 0|Q^a_{\mu}(x_1) Q^b_{\nu}(x_2)|0 > A = \int [dQ][d\bar{\psi}][d\psi] \ [e^{gM(x_1)}]_{ac} Q^c_{\mu}(x_1) \ [e^{gM(x_2)}]_{bd} Q^d_{\nu}(x_2) \times \det(\frac{\delta G^a_{ij}(Q')}{\delta \omega^b}) e^{i \int d^4x (-\frac{1}{4} F^2_{\mu\nu} - \frac{1}{4\pi} (G^a_{ij}(Q'))^2 + \bar{\psi}[\gamma^\mu \partial_\mu - m + g T^a M_{ab}(x)] \psi}. \quad (78)$$

From eq. (65) we find

$$G^a_j(Q') = \partial_\mu Q'^{\mu a} + g f^{abc} A^b_{\mu} Q'^{\mu c} - \partial_\mu A^{\mu a}. \quad (79)$$

By using eqs. (72) and (76) in eq. (79) we find

$$G^a_j(Q') = \partial_\mu \left[ [e^{gM(x)}]_{ab} Q^b_{\mu}(x) + \left( \frac{e^{gM(x)} - 1}{gM(x)} \right) \partial_\mu \omega^b(x) \right] + g f^{abc} \partial_\mu \omega^c(x) \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right] [e^{gM(x)}]_{cd} Q^d_{\mu}(x) + \left( \frac{e^{gM(x)} - 1}{gM(x)} \right) [\partial_\mu \omega^d(x)]$$

$$- \partial_\mu \left[ \partial_\mu \omega^b(x) \left( \frac{e^{gM(x)} - 1}{gM(x)} \right) \right] \quad (80)$$

which gives

$$G^a_j(Q') = \partial_\mu \left[ [e^{gM(x)}]_{ab} Q^b_{\mu}(x) \right] + g f^{abc} [\partial_\mu \omega^c(x)] \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right] [e^{gM(x)}]_{cd} Q^d_{\mu}(x) + \left( \frac{e^{gM(x)} - 1}{gM(x)} \right) [\partial_\mu \omega^d(x)]. \quad (81)$$

From eq. (81) we find

$$G^a_j(Q') = \partial_\mu \left[ [e^{gM(x)}]_{ab} Q^b_{\mu}(x) \right] + g f^{abc} [\partial_\mu \omega^c(x)] \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right] [e^{gM(x)}]_{cd} Q^d_{\mu}(x) \quad (82)$$

which gives

$$G^a_j(Q') = [e^{gM(x)}]_{ab} \partial_\mu Q^b_{\mu}(x)$$

$$+ Q^b_{\mu}(x) \partial_\mu [e^{gM(x)}]_{ab} + [\partial_\mu \omega^c(x)] \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right] [e^{gM(x)}]_{cd} Q^d_{\mu}(x). \quad (83)$$

From (26) we find

$$\partial_\mu [e^{igT^a \omega^a(x)}]_{ij} = ig [\partial_\mu \omega^b(x)] \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right] T_{ik} [e^{igT^a \omega^c(x)}]_{kj}, \quad M_{ab}(x) = f^{abc} \omega^c(x) \quad (84)$$

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which in the adjoint representation of SU(3) gives (by using $T^a_{bc} = -if^{abc}$)

$$[\partial^\mu e^{gM(x)}]_{ad} = [\partial^\mu \omega^c(x)] \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right] g f^{bac} [e^{M(x)}]_{cd}, \quad M_{ab}(x) = f^{abc} \omega^c(x). \quad (85)$$

Using eq. (85) in (83) we find

$$G^a_j(Q') = [e^{gM(x)}]_{ab} \partial^\mu Q^b_\mu(x) \quad (86)$$

which gives

$$(G^a_j(Q'))^2 = (\partial_\mu Q^\mu_\mu(x))^2. \quad (87)$$

Since for $n \times n$ matrices $A$ and $B$ we have

$$\det(AB) = (\det A)(\det B) \quad (88)$$

we find by using eq. (86) that

$$\det\left[ \frac{\delta G^a_j(Q')}{\delta \omega^b} \right] = \det\left[ \frac{\delta [e^{gM(x)}]_{ac} \partial^\mu Q^c_\mu(x)}{\delta \omega^b} \right] = \det\left[ [e^{gM(x)}]_{ac} \frac{\delta (\partial^\mu Q^c_\mu(x))}{\delta \omega^b} \right]$$

$$= \left[ \det [e^{gM(x)}]_{ac} \right] \left[ \det \left[ \frac{\delta (\partial^\mu Q^c_\mu(x))}{\delta \omega^b} \right] \right] = \exp[\text{Tr} \ln [e^{gM(x)}]] \det \left[ \frac{\delta (\partial_\mu Q^\mu_\mu(x))}{\delta \omega^b} \right] \quad (89)$$

Using eqs. (87) and (89) in eq. (78) we find

$$< 0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 >_A = \int [dQ][d\bar{\psi}][d\psi] \left[ e^{gM(x_1)} \right]_{ac} Q^c_\mu(x_1) \left[ e^{gM(x_2)} \right]_{bd} Q^d_\nu(x_2)$$

$$\times \det \left[ \frac{\delta (\partial_\mu Q^\mu_\mu)}{\delta \omega^b} \right] e^i \int d^4x [ -\frac{1}{4} F^a_{\mu \nu} Q - \frac{1}{g} (\partial_\mu Q^\mu_\mu)^2 + \bar{\psi} [i \gamma^\alpha \partial_\mu - m + g T^\alpha \gamma^\mu Q^\alpha_\mu] \psi ] \quad (90)$$

Using the similar technique as above we find

$$< 0\left[ e^{gM(x_1)} \right]_{ac} Q^c_\mu(x_1) \left[ e^{gM(x_2)} \right]_{bd} Q^d_\nu(x_2)|0 >_A = \int [dQ][d\bar{\psi}][d\psi] \left[ Q^a_\mu(x_1) \right] \left[ Q^b_\nu(x_2) \right]$$

$$\times \det \left( \frac{\delta (\partial_\mu Q^\mu_\mu)}{\delta \omega^b} \right) e^i \int d^4x [ -\frac{1}{4} F^a_{\mu \nu} Q - \frac{1}{g} (\partial_\mu Q^\mu_\mu)^2 + \bar{\psi} [i \gamma^\alpha \partial_\mu - m + g T^\alpha \gamma^\mu Q^\alpha_\mu] \psi ] \quad (91)$$

in the presence of SU(3) pure gauge background field $A^\mu_\mu(x)$ as given by eq. (38) where $M_{ab}(x)$ is given by eq. (71).

Note that eq. (91) is valid whether we use type I transformation [eqs. (59) and (60)] or type II transformation [eqs. (62) and (63)]. However, since eq. (5) is used to study...
the gauge transformation of the Wilson line in QCD as given by eq. (46), we will use type I transformation [see eqs. (59) and (60)] in the rest of the paper which for the finite transformation give eq. (5) and (6) [11, 13]. From eq. (6) we find that under gauge transformation given by eq. (5) the (quantum) gluon field \( Q^{\mu a}(x) \) transforms as

\[
Q'_{\mu}^{a}(x) = [e^{gM(x)}]_{ab}Q_{\mu}^{b}(x)
\]  

(92)

where \( M_{ab}(x) \) is given by eq. (71). From eqs. (53) and (91) we find

\[
<0|Q_{\mu}^{a}(x_1)Q_{\nu}^{b}(x_2)|0>=<0|[e^{gM(x_1)}]_{ac}Q_{\mu}^{c}(x_1)[e^{gM(x_2)}]_{bd}Q_{\nu}^{d}(x_2)|0>_A .
\]  

(93)

From eqs. (71), (49) and (93) we find

\[
<0|Q_{\epsilon}^{\mu}(x_1)Q_{\nu}^{b}(x_2)|0>=<0|\Phi_{ac}(x_1)Q_{\mu}^{c}(x_1)\Phi_{bd}(x_2)Q_{\nu}^{d}(x_2)|0>_A
\]  

(94)

which proves factorization of soft-collinear divergences at all order in coupling constant in QCD where

\[
\Phi_{ab}(x) = \mathcal{P}\exp[-ig \int_{0}^{\infty} d\lambda l \cdot A^c(x + \lambda l)T^{(A)c}], \quad (T^{(A)c})_{ab} = -if^{abc}
\]  

(95)

is the non-abelian gauge link or non-abelian phase in the adjoint representation of SU(3) and \( l^\mu \) is the light-like four velocity.

IV. NON-PERTURBATIVE GLUON CORRELATION FUNCTION AND PROOF OF FACTORIZATION THEOREM IN GENERAL AXIAL GAUGES

In QCD the generating functional with general axial gauge fixing is given by [39, 40]

\[
Z[J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \times e^{i \int d^4x \left[-\frac{i}{3}F^a_{\mu\nu}Q^b_{\nu} - \frac{1}{2\alpha}(\eta_{\mu}Q^{\mu a})^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m + gT^a\gamma^\mu Q^a_{\mu})\psi + J \cdot Q + \bar{\psi}\eta\psi \right]}
\]  

(96)

where \( \eta^\mu \) is an arbitrary but constant four vector

\[
\eta^2 < 0, \quad \text{axial gauge}
\]

\[
\eta^2 = 0, \quad \text{light - cone gauge}
\]

\[
\eta^2 > 0, \quad \text{temporal gauge.}
\]  

(97)
Note that unlike covariant gauge in eq. (50) there is no Faddeev-Popov (F-P) determinant in eq. (96) because the ghost particles decouple in general axial gauges [39, 40]. The non-perturbative gluon correlation function of the type $<0|Q^a(\alpha_1)Q^b(\alpha_2)|0>$ in QCD in general axial gauges is given by

$$<0|Q^a(\alpha_1)Q^b(\alpha_2)|0> = \int [dQ][d\bar{\psi}][d\psi] Q^a(\alpha_1)Q^b(\alpha_2)$$

$$\times e^i \int d^4x [-\frac{1}{4} F^\mu_\nu[Q] - \frac{i}{\sqrt{2}} (\eta^\mu Q^a_\mu)^2 + \bar{\psi}[i\gamma^\mu \partial_\mu - m + gT^a_\mu] Q^b_\mu] \psi].$$

(98)

The generating functional in the background field method of QCD with general axial gauge fixing is given by [40]

$$Z[A, J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi]$$

$$\times e^i \int d^4x [-\frac{1}{4} F^\mu_\nu[A+Q] - \frac{i}{\sqrt{2}} (\eta^\mu Q^a_\mu)^2 + \bar{\psi}[i\gamma^\mu \partial_\mu - m + gT^a_\mu(A+Q)] Q^b_\mu] \psi + J^a Q + \bar{\eta} \psi + \bar{\psi} \eta].$$

(99)

The non-perturbative gluon correlation function of the type $<0|Q^a(\alpha_1)Q^b(\alpha_2)|0>_A$ in the background field method of QCD in general axial gauges is given by

$$<0|Q^a(\alpha_1)Q^b(\alpha_2)|0>_A = \int [dQ][d\bar{\psi}][d\psi] Q^a(\alpha_1)Q^b(\alpha_2)$$

$$\times e^i \int d^4x [-\frac{1}{4} F^\mu_\nu[A+Q] - \frac{i}{\sqrt{2}} (\eta^\mu Q^a_\mu)^2 + \bar{\psi}[i\gamma^\mu \partial_\mu - m + gT^a_\mu(A+Q)] Q^b_\mu] \psi].$$

(100)

By changing the integration variable $Q \rightarrow Q - A$ in the right hand side of eq. (100) we find

$$<0|Q^a(\alpha_1)Q^b(\alpha_2)|0>_A = \int [dQ'][d\bar{\psi}'][d\psi'] [Q^a(\alpha_1) - A^a_\alpha(\alpha_1)][Q^b(\alpha_2) - A^b_\beta(\alpha_2)]$$

$$\times e^i \int d^4x [-\frac{1}{4} F^\mu_\nu[Q'] - \frac{i}{\sqrt{2}} (\eta^\mu(Q-A)_\mu^a)^2 + \bar{\psi'}[i\gamma^\mu \partial_\mu - m + gT^a_\mu] Q^b_\mu'] \psi].$$

(101)

Changing the integration variable from unprimed variable to primed variable we find from eq. (101)

$$<0|Q^a(\alpha_1)Q^b(\alpha_2)|0>_A = \int [dQ'][d\bar{\psi}'][d\psi'] [Q^a_\alpha(\alpha_1) - A^a_\alpha(\alpha_1)][Q^b_\beta(\alpha_2) - A^b_\beta(\alpha_2)]$$

$$\times e^i \int d^4x [-\frac{1}{4} F^\mu_\nu[Q'] - \frac{i}{\sqrt{2}} (\eta^\mu(Q-A)_\mu^a)^2 + \bar{\psi'}[i\gamma^\mu \partial_\mu - m + gT^a_\mu] Q^b_\mu'] \psi].$$

(102)

This is because a change of integration variable from unprimed variable to primed variable does not change the value of the integration.

Using eqs. (73), (75) and (77) in eq. (102) we find

$$<0|Q^a(\alpha_1)Q^b(\alpha_2)|0>_A = \int [dQ][d\bar{\psi}][d\psi'] [e^{gM(x_1)}]_{ac} Q^c_\alpha(\alpha_1) [e^{gM(x_2)}]_{bd} Q^d_\beta(\alpha_2)$$

$$\times e^i \int d^4x [-\frac{1}{4} F^\mu_\nu[Q'] - \frac{i}{\sqrt{2}} (\eta^\mu([e^{gM(x_1)}]_{ac} Q^c_\alpha(x)) [e^{gM(x_2)}]_{bd} Q^d_\beta(x)])^2 + \bar{\psi}'[i\gamma^\mu \partial_\mu - m + gT^a_\mu] Q^b_\mu'] \psi].$$

(103)
which gives

\[<0|Q_\mu^a(x_1)Q_\nu^b(x_2)|0>_A = \int [dQ][d\bar{\psi}][d\psi] \left[ e^{gM(x_1)} \right]_{ac} Q^c_\mu(x_1) \left[ e^{gM(x_2)} \right]_{bd} Q^d_\nu(x_2) \times e^{i \int d^4x \left[ -\frac{1}{4} F^a_{\mu\nu}(Q)-\frac{1}{2\alpha} (e\eta^\mu Q_\mu^a)^2+\bar{\psi}[\gamma^\mu \partial_\mu-m+gT^a_\mu Q_\mu^a] \psi \right]}. \]

(104)

Using the similar technique as above we find

\[<0|\left[ e^{gM(x_1)} \right]_{ac} Q^c_\mu(x_1)\left[ e^{gM(x_2)} \right]_{bd} Q^d_\nu(x_2)|0>_A = \int [dQ][d\bar{\psi}][d\psi] Q^a_\mu(x_1)Q^b_\nu(x_2) \times e^{i \int d^4x \left[ -\frac{1}{4} F^a_{\mu\nu}(Q)-\frac{1}{2\alpha} (e\eta^\mu Q_\mu^a)^2+\bar{\psi}[i\gamma^\mu \partial_\mu-m+gT^a_\mu Q_\mu^a] \psi \right]}. \]

(105)

in general axial gauges in the presence of SU(3) pure gauge background field \(A^{\alpha a}(x)\) as given by eq. (38) where \(M_{ab}(x)\) is given by eq. (71).

From eqs. (98) and (105) we find

\[<0|Q_\mu^a(x_1)Q_\nu^b(x_2)|0>_A = <0|\left[ e^{gM(x_1)} \right]_{ac} Q^c_\mu(x_1)\left[ e^{gM(x_2)} \right]_{bd} Q^d_\nu(x_2)|0>_A \]

(106)

in general axial gauges. From eqs. (71), (49) and (106) we find

\[<0|Q_\mu^a(x_1)Q_\nu^b(x_2)|0>_A = <0|\Phi_{ac}(x_1)Q^c_\mu(x_1)\Phi_{bd}(x_2)Q^d_\nu(x_2)|0>_A \]

(107)

which proves factorization of soft-collinear divergences at all order in coupling constant in QCD in general axial gauges where the non-abelian gauge link or non-abelian phase \(\Phi_{ab}(x)\) in the adjoint representation of SU(3) is given by eq. (95).

V. NON-PERTURBATIVE GLUON CORRELATION FUNCTION AND PROOF OF FACTORIZATION THEOREM IN LIGHT-CONE GAUGE

The light-cone gauge corresponds to

\[\eta \cdot Q^a = 0, \quad \eta^2 = 0\]

(108)

which is already covered by eqs. (96) and (97) where the corresponding gauge fixing term is given by \(-\frac{1}{2\alpha}(\eta_\mu Q^{\alpha a})^2\). In the light-cone coordinate system the light-cone gauge [41]

\[Q^{+a} = 0\]

(109)

corresponds to

\[\eta^\mu = (\eta^+, \eta^- , \eta_\perp) = (0, 1, 0)\]

(110)
which covers $\eta \cdot Q^a = 0$ and $\eta^2 = 0$ situation in eq. (108).

Since eq. (106) is valid for general axial gauges it is also valid for $\eta^2 = 0$ as given by eq. (97). Hence we find from eq. (106) that

$$<0| Q^a_{\mu}(x_1) Q^b_{\nu}(x_2)|0 > = <0| [e^{gM(x_1)}]_{ac} Q^c_{\mu}(x_1) [e^{gM(x_2)}]_{bd} Q^d_{\nu}(x_2)|0 >_A$$  \hspace{1cm} (111)

in light-cone gauge. From eqs. (71), (49) and (111) we find

$$<0| Q^a_{\mu}(x_1) Q^b_{\nu}(x_2)|0 > = <0| \Phi_{ac}(x_1) Q^c_{\mu}(x_1) \Phi_{bd}(x_2) Q^d_{\nu}(x_2)|0 >_A$$  \hspace{1cm} (112)

which proves factorization of soft-collinear divergences at all order in coupling constant in QCD in light-cone gauge where the non-abelian gauge link or non-abelian phase $\Phi_{ab}(x)$ in the adjoint representation of SU(3) is given by eq. (95).

**VI. NON-PERTURBATIVE GLUON CORRELATION FUNCTION AND PROOF OF FACTORIZATION THEOREM IN GENERAL NON-COVARIANT GAUGES**

In QCD the generating functional with general non-covariant gauge fixing is given by [42, 43]

$$Z[J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \det \left( \frac{\delta g_{\mu}^{\nu} \partial_{\mu} Q^a_{\nu}}{\delta \omega^b} \right) \times e^{i \int d^4 x \left[ - \frac{1}{4} F_{\mu \nu}^2(Q) - \frac{1}{4} (\eta^\mu \partial_\mu Q^a_{\nu})^2 + \bar{\psi} i \gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q^a_{\mu} \right] \psi + J \cdot Q + \bar{\psi} \eta + \bar{\eta} \psi}$$  \hspace{1cm} (113)

where $\eta^\mu$ is an arbitrary but constant four vector. The non-perturbative gluon correlation function of the type $<0| Q^a_{\mu}(x_1) Q^b_{\nu}(x_2)|0 >$ in QCD in general non-covariant gauges is given by

$$<0| Q^a_{\mu}(x_1) Q^b_{\nu}(x_2)|0 > = \int [dQ][d\bar{\psi}][d\psi] \ Q^a_{\mu}(x_1) Q^b_{\nu}(x_2) \times \det \left( \frac{\delta g_{\mu}^{\nu} \partial_{\mu} Q^a_{\nu}}{\delta \omega^b} \right) e^{i \int d^4 x \left[ - \frac{1}{4} F_{\mu \nu}^2(Q) - \frac{1}{4} (\eta^\mu \partial_\mu Q^a_{\nu})^2 + \bar{\psi} i \gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q^a_{\mu} \right] \psi}.$$  \hspace{1cm} (114)

The generating functional in the background field method of QCD with general non-covariant gauge fixing is given by [42, 43]

$$Z[A, J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \det \left( \frac{\delta G^a(Q)}{\delta \omega^b} \right) \times e^{i \int d^4 x \left[ - \frac{1}{4} F_{\mu \nu}^2(A+Q) - \frac{1}{4} (G^a(Q))^2 + \bar{\psi} i \gamma^\mu \partial_\mu - m + g T^a \gamma^\mu (A+Q)^a_{\mu} \right] \psi + J \cdot Q + \bar{\psi} \eta + \bar{\eta} \psi}$$  \hspace{1cm} (115)
where
\[ G^\alpha(Q) = \frac{\eta^\mu \eta^\nu}{\eta^2} (\partial_\mu Q_\nu^a + g f^{abc} A_\mu^b Q_\nu^c) = \frac{\eta^\mu \eta^\nu}{\eta^2} D_\mu [A] Q_\nu^a. \]  \hspace{1cm} (116)

The non-perturbative gluon correlation function of the type \(<0|Q_\mu^a(x_1) Q_\nu^b(x_2)|0> =\int \[dQ]\left[\bar{\psi} \right] [d\psi] \] Q_\mu^a(x_1) Q_\nu^b(x_2)
\times \text{det} \left( \frac{\delta G^a_i(Q)}{\delta \omega^b} \right) e^{i \int d^4x [-\frac{i}{\eta^2} F_{\mu\nu}^a [A+Q] - \frac{1}{2m} (G^a(Q))^2 + \bar{\psi} i\gamma^\mu \partial_\mu - m + g T^a \gamma^\mu (A+Q) \gamma^\mu \psi]}. \]  \hspace{1cm} (117)

By changing the integration variable \( Q \rightarrow Q - A \) in the right hand side of eq. (117) we find
\[ <0|Q_\mu^a(x_1) Q_\nu^b(x_2)|0> =\int \[dQ]\left[\bar{\psi} \right] [d\psi] \] [Q_\mu^a(x_1) - A_\mu^a(x_1)][Q_\nu^b(x_2) - A_\nu^b(x_2)]
\times \text{det} \left( \frac{\delta G^a_i(Q)}{\delta \omega^b} \right) e^{i \int d^4x [-\frac{i}{\eta^2} F_{\mu\nu}^a [Q] - \frac{1}{2m} (G^a_i(Q))^2 + \bar{\psi} i\gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q_\mu^a \psi]. \]  \hspace{1cm} (118)

where from eq. (116) we find
\[ G^a_i(Q) = \frac{\eta^\mu \eta^\nu}{\eta^2} (\partial_\mu Q_\nu^a + g f^{abc} A_\mu^b Q_\nu^c - \partial_\mu A_\nu^a) - \frac{1}{\eta^2} g f^{abc}(\eta \cdot A^b)(\eta \cdot A^c)
= \frac{\eta^\mu \eta^\nu}{\eta^2} (D_\mu [A] Q_\nu^a) - \frac{\eta^\mu \eta^\nu}{\eta^2} \partial_\mu A_\nu^a. \]  \hspace{1cm} (119)

Changing the integration variable from unprimed variable to primed variable we find from eq. (118)
\[ <0|Q_\mu^a(x_1) Q_\nu^b(x_2)|0> =\int \[dQ]\left[\bar{\psi} \right] [d\psi] \] [Q_\mu^a(x_1) - A_\mu^a(x_1)][Q_\nu^b(x_2) - A_\nu^b(x_2)]
\times \text{det} \left( \frac{\delta G^a_i(Q')}{\delta \omega^b} \right) e^{i \int d^4x [-\frac{i}{\eta^2} F_{\mu\nu}^a [Q'] - \frac{1}{2m} (G^a_i(Q'))^2 + \bar{\psi} i\gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q_\mu^a \psi]. \]  \hspace{1cm} (120)

This is because a change of integration variable from unprimed variable to primed variable does not change the value of the integration.

Using eqs. (123), (125) and (127) in eq. (120) we find
\[ <0|Q_\mu^a(x_1) Q_\nu^b(x_2)|0> =\int \[dQ]\left[\bar{\psi} \right] [d\psi] \] [e^{gM(x_1)}] a c Q_\mu^a(x_1) [e^{gM(x_2)}] b d Q_\nu^b(x_2)
\times \text{det} \left( \frac{\delta G^a_i(Q')}{\delta \omega^b} \right) e^{i \int d^4x [-\frac{i}{\eta^2} F_{\mu\nu}^a [Q'] - \frac{1}{2m} (G^a_i(Q'))^2 + \bar{\psi} i\gamma^\mu \partial_\mu - m + g T^a \gamma^\mu Q_\mu^a \psi]. \]  \hspace{1cm} (121)

From eq. (119) we find
\[ G^a_i(Q') = \frac{\eta^\mu \eta^\nu}{\eta^2} (\partial_\mu Q_\nu^a + g f^{abc} A_\mu^b Q_\nu^c) - \frac{\eta^\mu \eta^\nu}{\eta^2} \partial_\mu A_\nu^a. \]  \hspace{1cm} (122)
By using eqs. (72) and (76) in eq. (122) we find
\[
G^g_f(Q') = \frac{\eta^\mu \eta^\nu}{\eta^2} [\partial_\mu [[e^{gM(x)}]_{ab} Q^b_\nu(x)] + \left(\frac{e^{gM(x)} - 1}{gM(x)}\right)_{ab} [\partial_\nu \omega^b(x)] + g f^{abc} [\partial_\mu \omega^e(x)] \left[\frac{e^{gM(x)} - 1}{gM(x)}\right]_{be} [[e^{gM(x)}]_{cd} Q^d_\nu(x)] + \left(\frac{e^{gM(x)} - 1}{gM(x)}\right)_{cd} [\partial_\nu \omega^d(x)]]
\]

which gives
\[
G^g_f(Q') = \frac{\eta^\mu \eta^\nu}{\eta^2} [\partial_\mu [[e^{gM(x)}]_{ab} Q^b_\nu(x)] + g f^{abc} [\partial_\mu \omega^e(x)] \left[\frac{e^{gM(x)} - 1}{gM(x)}\right]_{be} [[e^{gM(x)}]_{cd} Q^d_\nu(x)] + \left(\frac{e^{gM(x)} - 1}{gM(x)}\right)_{cd} [\partial_\nu \omega^d(x)]].
\]

From eq. (123) we find
\[
G^g_f(Q') = \frac{\eta^\mu \eta^\nu}{\eta^2} [\partial_\mu [[e^{gM(x)}]_{ab} Q^b_\nu(x)] + g f^{abc} [\partial_\mu \omega^e(x)] \left[\frac{e^{gM(x)} - 1}{gM(x)}\right]_{be} [[e^{gM(x)}]_{cd} Q^d_\nu(x)] + \left(\frac{e^{gM(x)} - 1}{gM(x)}\right)_{cd} [\partial_\nu \omega^d(x)]]
\]

which gives
\[
G^g_f(Q') = \frac{\eta^\mu \eta^\nu}{\eta^2} [[e^{gM(x)}]_{ab} \partial_\mu Q^b_\nu(x) + Q^b_\mu(x) \partial_\nu [[e^{gM(x)}]_{ab}] + g f^{abc} [\partial_\mu \omega^e(x)] \left[\frac{e^{gM(x)} - 1}{gM(x)}\right]_{be} [[e^{gM(x)}]_{cd} Q^d_\nu(x)] + \left(\frac{e^{gM(x)} - 1}{gM(x)}\right)_{cd} [\partial_\nu \omega^d(x)]].
\]

Using eq. (85) in (126) we find
\[
G^g_f(Q') = \frac{\eta^\mu \eta^\nu}{\eta^2} [[e^{gM(x)}]_{ab} \partial_\mu Q^b_\nu(x)
\]

which gives
\[
(G^g_f(Q'))^2 = \left(\frac{\eta^\mu \eta^\nu}{\eta^2} \partial_\mu Q^a_\nu(x)\right)^2.
\]

From eqs. (88) and (127) we find
\[
\det \left[\frac{\delta G^g_f(Q')}{\delta \omega^b} \right] = \det \left[\frac{\eta^\mu \eta^\nu}{\eta^2} \delta [[e^{gM(x)}]_{ac} \partial_\mu Q^c_\nu(x)] \right] = \det \left[\frac{\eta^\mu \eta^\nu}{\eta^2} [e^{gM(x)}]_{ac} \frac{\delta (\partial_\mu Q^c_\nu(x))}{\delta \omega^b} \right]
\]

\[
= \left[\det [[e^{gM(x)}]_{ac}] \right] \left[\det \left[\frac{\eta^\mu \eta^\nu}{\eta^2} \delta (\partial_\mu Q^c_\nu(x)) \right] \right] = \exp[\text{Tr}(\ln[e^{gM(x)}])] \det \left[\frac{\eta^\mu \eta^\nu}{\eta^2} \frac{\delta (\partial_\mu Q^a_\nu(x))}{\delta \omega^b} \right]
\]

\[
= \det \left[\frac{\eta^\mu \eta^\nu}{\eta^2} \frac{\delta (\partial_\mu Q^a_\nu(x))}{\delta \omega^b} \right].
\]
Using eqs. (128) and (129) in eq. (121) we find
\[
<0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0>_A = \int [dQ][d\bar{\psi}][d\psi] \left[ e^{gM(x_1)}\right]_{ac}Q^c_\mu(x_1) \left[ e^{gM(x_2)}\right]_{bd}Q^d_\nu(x_2) \times \det\left( \frac{\eta^\mu_\nu \delta(\partial_\mu Q^a_\nu)}{\partial_\omega^b} \right) e^i \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu}_{\mu\nu}[Q] - \frac{1}{2}(\frac{g^{\mu\nu}}{\eta^2} \partial_\mu Q^a_\nu)^2 + \bar{\psi}[i\gamma^\mu \partial_\mu - m + gT^a \gamma^\mu Q^a_\mu] \psi \right].
\] (130)

Using the similar technique as above we find
\[
<0|\left[ e^{gM(x_1)}\right]_{ac}Q^c_\mu(x_1) \left[ e^{gM(x_2)}\right]_{bd}Q^d_\nu(x_2)|0>_A = \int [dQ][d\bar{\psi}][d\psi] Q^a_\mu(x_1)Q^b_\nu(x_2) \times \det\left( \frac{\eta^\mu_\nu \delta(\partial_\mu Q^a_\nu)}{\partial_\omega^b} \right) e^i \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu}_{\mu\nu}[Q] - \frac{1}{2}(\frac{g^{\mu\nu}}{\eta^2} \partial_\mu Q^a_\nu)^2 + \bar{\psi}[i\gamma^\mu \partial_\mu - m + gT^a \gamma^\mu Q^a_\mu] \psi \right].
\] (131)
in general non-covariant gauges in the presence of SU(3) pure gauge background field $A^{\mu a}(x)$ as given by eq. (38) where $M_{ab}(x)$ is given by eq. (71).

From eqs. (114) and (131) we find
\[
<0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0>_A = <0|\left[ e^{gM(x_1)}\right]_{ac}Q^c_\mu(x_1) \left[ e^{gM(x_2)}\right]_{bd}Q^d_\nu(x_2)|0>_A
\] (132)
in general non-covariant gauges. From eqs. (71), (49) and (132) we find
\[
<0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0>_A = <0|\Phi_{ac}(x_1)Q^c_\mu(x_1)\Phi_{bd}(x_2)Q^d_\nu(x_2)|0>_A
\] (133)
which proves factorization of soft-collinear divergences at all order in coupling constant in QCD in general non-covariant gauges where the non-abelian gauge link or non-abelian phase $\Phi_{ab}(x)$ in the adjoint representation of SU(3) is given by eq. (95).

VII. NON-PERTURBATIVE GLUON CORRELATION FUNCTION AND PROOF OF FACTORIZATION THEOREM IN GENERAL COULOMB GAUGE

In QCD the generating functional with general Coulomb gauge fixing is given by [42, 43]
\[
Z[J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \det\left( \frac{\delta^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}}{\delta_\omega^b} \right) \times e^i \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu}_{\mu\nu}[Q] - \frac{1}{2}(\frac{g^{\mu\nu}}{\eta^2} \partial_\mu Q^a_\nu)^2 + \bar{\psi}[i\gamma^\mu \partial_\mu - m + gT^a \gamma^\mu Q^a_\mu] \psi + JQ + \bar{\psi} \eta + \bar{\eta} \psi \right]
\] (134)
where
\[
n^\mu = (1, 0, 0, 0).
\] (135)
The non-perturbative gluon correlation function of the type \( <0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 > \) in QCD in general Coulomb gauge is given by

\[
<0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 > = \int [dQ][d\bar{\psi}][d\psi] \ Q^a_\mu(x_1)Q^b_\nu(x_2) \\
\times \det(\frac{\delta [g^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}] \partial_\mu Q^a_\nu}{\delta \omega^b}) \ e^i \int d^4x [-\frac{i}{4} F^{a\mu\nu}_{\mu\nu}[A+Q] - \frac{1}{2m} (g^{\mu\nu} - \frac{n^\mu n^\nu}{n^2})\partial_\mu Q^a_\nu]^2 + \bar{\psi} [i\gamma^\mu \partial_\mu - m + gT^a \gamma^\mu]Q^a_\mu \psi].
\]

(136)

The generating functional in the background field method of QCD with general Coulomb gauge fixing is given by [42, 43]

\[
Z[A, J, \eta, \bar{\eta}] = \int [dQ][d\bar{\psi}][d\psi] \ det(\frac{\delta G^a(Q)}{\delta \omega^b}) \\
\times e^i \int d^4x [-\frac{i}{4} F^{a\mu\nu}_{\mu\nu}[A+Q] - \frac{1}{2m} (G^a(Q))^2 + \bar{\psi} [i\gamma^\mu \partial_\mu - m + gT^a \gamma^\mu(A+Q)]^a_\mu \psi + J \cdot Q + \bar{\eta} \psi + \bar{\psi} \eta]
\]

(137)

where

\[
G^a(Q) = [g^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}] (\partial_\mu Q^a_\nu + g f^{abc} A^b_\mu Q^c_\nu) = [g^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}] D_\mu [A] Q^a_\nu.
\]

(138)

The non-perturbative gluon correlation function of the type \( <0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 >_A \) in the background field method of QCD in general Coulomb gauge is given by

\[
<0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 >_A = \int [dQ][d\bar{\psi}][d\psi] \ Q^a_\mu(x_1)Q^b_\nu(x_2) \\
\times \det(\frac{\delta G^a(Q)}{\delta \omega^b}) e^i \int d^4x [-\frac{i}{4} F^{a\mu\nu}_{\mu\nu}[A+Q] - \frac{1}{2m} (G^a(Q))^2 + \bar{\psi} [i\gamma^\mu \partial_\mu - m + gT^a \gamma^\mu(A+Q)]^a_\mu \psi].
\]

(139)

Hence by replacing \( \frac{\delta G^a(Q)}{\delta \omega^b} \rightarrow [g^{\mu\nu} - \frac{n^\mu n^\nu}{n^2}] \) everywhere in the derivations in the previous section we find

\[
<0|Q^a_\mu(x_1)Q^b_\nu(x_2)|0 >= <0|\Phi_{nc}(x_1)Q^c_\mu(x_1)\Phi_{bd}(x_2)Q^d_\nu(x_2)|0 >_A
\]

(140)

which proves factorization of soft-collinear divergences at all order in coupling constant in QCD in general Coulomb gauge where the non-abelian gauge link or non-abelian phase \( \Phi_{ab}(x) \) in the adjoint representation of SU(3) is given by eq. [95].

**VIII. CORRECT DEFINITION OF THE GLUON DISTRIBUTION FUNCTION AT HIGH ENERGY COLLIDERS**

Under the non-abelian gauge transformation as given by eq. [5] we find from eqs. [145], [48], [49] and [55] that the the non-abelian gauge gauge link or non-abelian phase in QCD
in the adjoint representation of SU(3) transforms as
\[
\Phi'_ab(x) = \left[ e^{-iM(x)\Phi(x)} \right]_{ab}, \quad M_{ab}(x) = f^{abc}\omega^c(x). \tag{141}
\]
Hence from eqs. (92) and (141) we find that \(< 0|\Phi_{ac}(x_1)Q^c_\mu(x_1)\Phi_{bd}(x_2)Q^d_\nu(x_2)|0 >_A\) in eq. (94) [or in (107) or in (112) or in eq. (133) or in eq. (140)] is gauge invariant and eq. (94) [or (107) or (112) or eq. (133) or eq. (140)] is consistent with the factorization of soft-collinear divergences at all order in coupling constant in QCD.

Hence from eqs. (94) [or (107) or (112) or (133) or (140)] and (95) we find that the correct definition of the gluon distribution function at high energy colliders which is consistent with the number operator interpretation of the gluon and is gauge invariant and is consistent with the factorization theorem in QCD is given by
\[
f_{g/P}(x) = \frac{P^+}{x^2\pi} \int dy^- e^{-ixP^+y^-} \times \left[ < P| [\mathcal{P}e^{-ig\int_0^\infty d\lambda \cdot A_c(\lambda)]}Q^c_\mu(0)|P > \right]_{y^+ = y_T = 0} \tag{142}
\]
which is valid in covariant gauge, in light-cone gauge, in general axial gauges, in general non-covariant gauges and in general Coulomb gauge etc. respectively. Eq. (142) can be written as
\[
f_{g/P}(x) = \frac{P^+}{x^2\pi} \int dy^- e^{-ixP^+y^-} \times < P|Q^\mu_0(0, y^-, 0_T)|\mathcal{P}e^{ig\int_0^{0,y^-,0_T} dz \cdot A_c(z)|T(A)|}Q^\mu(0)|P > \tag{143}
\]
which reproduces eq. (11). This completes the derivation of the correct definition of the gluon distribution function at high energy colliders from first principles.

IX. CONCLUSIONS

Unlike QED, since \(F_{\mu\nu}^a(x)F^{\mu\nu}_0(0)\) in QCD contains cubic and quartic powers of the gluon field the present definition of the gluon distribution function at high energy colliders is not consistent with the number operator interpretation of the gluon. In this paper we have derived the correct definition of the gluon distribution function at high energy colliders from first principles which is consistent with the number operator interpretation of the gluon and is gauge invariant and is consistent with the factorization theorem in QCD.
After renormalization the gluon distribution function is expected to obey a QCD evolution equation, like DGLAP equation [15], which follows from renormalization group equation.

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