Logarithmic corrections to finite size spectrum of SU(N) symmetric quantum chains

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We consider SU(N) symmetric one dimensional quantum chains at finite temperature. For such systems the correlation lengths, ground state energy, and excited state energies are investigated in the framework of conformal field theory. The possibility of different types of excited states are discussed. Logarithmic corrections to the ground state energy and different types of excited states in the presence of a marginal operator, are calculated. Known results for SU(2) and SU(4) symmetric systems follow from our general formula.

PACS numbers: 75.10.Jm, 75.40.Gb

One of the fundamental models of solid state physics is the Heisenberg model of insulating magnets. In the one-dimensional case (“spin chains”), the spin-1/2 Heisenberg models have been studied extensively: most of our understanding of their quantum critical behavior is based on the Bethe-Ansatz solution for the ground state and excitation spectrum\(^{1}\) mapping to the Sine-Gordon theory\(^1\) non-abelian bosonization\(^{1}\) and mapping to the sigma model\(^{1}\). Although spin-1/2 Heisenberg chains are SU(2) symmetric systems, fruitful generalizations have been accomplished in two different directions: (a) enlarging the representation of SU(2) group to study quantum chains with higher spins, and (b) introducing higher symmetry groups such as SU(N).

Here we consider generalizations of the type (b) and investigate how higher symmetry affects the ground state properties [Eqs. (8)–(10)] and finite size spectrum of quantum “spin” chains. Earlier studies of Affleck\(^{2}\) show that any one dimensional system with SU(N) symmetry is critical, and at the very low energy scale these models are equivalent to \((N-1)\) free massless bosons. This system of free bosons - when viewed in the framework of two dimensional conformal field theory, are the primary fields of the SU(N)\(_k=1\) WZNW model. Adopting this model we give an explicit derivation of logarithmic corrections to finite size spectrum of SU(N) symmetric quantum chains. Logarithmic shifts in excited states energy levels have been theoretically observed for SU(2) and SU(4) symmetric systems away from the \(T = 0\) quantum critical point.\(^{3-5}\) Known results for \(N = 2\) and \(N = 4\) follow from our general formulas [obtained in Eqs. (27)–(28) and the paragraphs below Eq. (30)].

An one-dimensional SU(N) symmetric quantum chain of length \(L\) is described by the Hamiltonian: \(^{1}\)

\[
H = \sum_{j=1}^{n} \sum_{A=1}^{N^2-1} S^A_j S^A_{j+1}.
\]

Here \(n = L/a_0\) is the total number of discrete points (\(a_0\) being the lattice spacing) and \(S^A_j\) are the \((N^2 - 1)\) generators of the SU(N) Lie algebra at each lattice site \(j\). For convenience, the interaction strength and the lattice spacing \(a_0\) have been set equal to one in Eq. (1). At each site \(j\), the generators \(S^A_j\) can be represented by \(N\) “flavors” of fermions, \(\psi_{ja} \ [a = 1, \ldots, N]\)

\[
S^A_j = \sum_{a,b=1}^{N} \psi^a_j (T^A)^a_b \psi_{bj} - I/N,
\]

where \(I\) is the identity operator, and \(T^A\)’s are a complete set of \((N^2 - 1)\) traceless normalized matrices so that Tr\[T^A T^B\] = \((1/2)\delta^{AB}\). Eq. (2) satisfies the constraint that at each site the total number of fermions is conserved, i.e. \(\sum_{a=1}^{N} \psi^a_j \psi_{ja} = 1\).

For such a fermionic system the theory can be bosonized using non-abelian bosonization at low temperatures. In the continuum limit, the bosonized Hamiltonian\(^{1}\) \(\text{(H}_{\text{eff})}\) can then be written in terms of the Kac-Moody currents,

\[
H_{\text{eff}} \approx v_s \sum_{A=1}^{N^2-1} \int dx \left[ J^A_L J^A_L + J^A_R J^A_R + 2J^A_L J^A_R \right],
\]

where the normal ordered Kac-Moody currents for the left (with Fermi momentum \(k_F < 0\)) and the right moving (with \(k_F > 0\)) fermions are defined as,

\[
J^A_L = : \psi^a_L (T^A)^a_b \psi_{Lb} :;
\]

\[
J^A_R = : \psi^a_R (T^A)^a_b \psi_{Rb} :.
\]

At zero temperature \((T = 0)\) the interaction term, \(\sum_{A=1}^{N^2-1} J^A_L J^A_R\), in Eq. (3) renormalizes to zero, and the sum of first two terms in the Hamiltonian that are quadratic in left and right moving currents, corresponds to the SU(N)\(_k=1\) WZNW model. The fundamental unitary \(N \times N\) matrix field \(g\) of the WZNW model is given by,
The field $q$ transforms as the fundamental representation of $SU(N)_L \times SU(N)_R$ which describes the exact symmetry of the Hamiltonian in Eq. (3) at zero temperature. It is known that this fermionic theory is equivalent to a theory of $(N-1)$ free massless bosons at the criticality with velocities $v_s$; they correspond to $(N-1)$ excitation modes of the $SU(N)$ symmetric quantum chain that oscillates at different values of $k_F$. Furthermore, these oscillating modes are primary fields of the $SU(N)_{k=1}$ WZNW model and their scaling dimensions $(\Delta_p)$ can be obtained from:

$$\Delta_p = \frac{2C_p}{C_{adj} + 1}.$$  \hspace{1cm} (6)

For $SU(N)$, $C_p (p = 1, 2, \cdots, N-1)$ is the eigenvalue of the Casimir operator in the $p$-th fundamental representation [having a Young tableau with $p$ boxes in a single column], and $C_{adj} = N$ is the eigenvalue of the Casimir operator in the adjoint representation [having a 2-column Young tableau with $(N-1)$ boxes in the first column and 1 box in the second column]. In a highest weight $(\Lambda)$ representation of $SU(N)$, the corresponding Casimir eigenvalue is given by:

$$C_\Lambda = \frac{(\theta, \theta)}{2} \left[ m (N - \frac{m}{N}) + \sum_{i=1}^{r_o} (b_i)^2 - \sum_{i=1}^{c_o} (a_i)^2 \right], \hspace{1cm} (7)$$

where $\theta$ is the highest weight corresponding to the adjoint representation and is normalized to 1, $m$ is the total number of boxes in the Young tableau with $r_o$ rows of length $b_1, b_2, \cdots, b_{r_o}$ and $c_o$ columns of length $a_1, a_2, \cdots, a_{c_o}$. Using this formula we find $C_p = p(N - p)/(N + 1)/2N$ and $C_{adj} = N$. Hence, the scaling dimensions $(\Delta_p)$ of primary fields of the $SU(N)_{k=1}$ WZNW model are given by:

$$\Delta_p = \frac{2C_p}{C_{adj} + 1} = \left[ \frac{p(N - p)}{N} \right]. \hspace{1cm} (8)$$

For example, in the case of $SU(4)$ the three oscillating components have scaling dimensions $(3/4, 1, 3/4)$ for $p = 1, 2, 3$. For $SU(N)$, the mode (dominant) that oscillates at $k_F = 2\pi/N$ has a scaling dimension $1 - 1/N$ for $p = 1$ or $(N - 1)$ in this case.

Finite-size corrections to Heisenberg chain with $SU(2)$ and $SU(4)$ symmetry has been studied using conformal field theory [3]. The relevance of studying finite size chains are twofold. One can not only compare the theoretical results with numerical simulations and experiments which are limited to finite size of the system but also can study the finite temperature behavior of the system by identifying the finite size in the imaginary time direction, that corresponds to finite temperature. To obtain the finite-size corrections of a 1D chain of length $L$ and with periodic boundary conditions we first introduce a conformal mapping from the infinite plane (with coordinate $z$) to the cylinder (with coordinate $w$) via

$$w = (L/2\pi) \ln z.$$  \hspace{1cm} (9)

Identifying the length to be inverse of the temperature [$L = v_s/T$] the finite temperature result of the ground state energy $E_0$ can be generalized to the $SU(N)$ symmetric system,

$$E_0(T) = E_0(0) - \frac{\pi T(N - 1)}{6v_s}. \hspace{1cm} (10)$$

Here $E_0(0)$ refers to the ground state energy at zero temperature. The thermodynamic quantities like specific heat and entropy can now be obtained by taking the appropriate derivatives with respect to the temperature.

Other quantities of interest are the finite temperature corrections to the correlation lengths ($\xi$) of the different modes. These inverse of the correlation lengths, $\xi^{-1}$ are signature of energy gaps, $(E_n - E_0)$ between the ground state and the lowest lying excited states $(E_n)$ that are created by finite temperature of the system. Using the general formula for the scaling dimension [Eq. (8)], we obtain $\xi_p^{-1}$ of the $p$-th staggered mode,

$$\xi_p^{-1} = \frac{E_n - E_0}{v_s},$$

$$= \left( \frac{2\pi T}{v_s} \right) \Delta_p = \left( \frac{2\pi T}{v_s} \right) \left[ \frac{p(N - p)}{N} \right].$$  \hspace{1cm} (10)

The temperature dependence of the correlation lengths is in fact modified by logarithmic corrections in the presence of marginal operators in the theory [3]. The generic form of the Hamiltonian at the critical point containing a marginal operator $\phi(x, t)$ is

$$H = H^* + g_o \int dx \phi,$$  \hspace{1cm} (11)

where $g_o$ is the coupling constant and $H^*$ is the Hamiltonian at the fixed point. In our case [Eq. (3)], the normalized marginally irrelevant operator is

$$\phi = -D \sum_{A=1}^{N^2-1} J^A_L J^A_R.$$  \hspace{1cm} (12)

For such a marginally irrelevant operator, the Hamiltonian at the critical point ($T = 0$) becomes equal to the fixed point Hamiltonian. In Eq. (12), $D$ is the normalization constant to be determined from the two-point correlator of $\phi$. The operator product expansion (OPE) of the $J^A_L$ with any normalized Kac-Moody primary field $\chi$ is given by [4].
Thus the normalized irrelevant marginal operator is given by,

\[ J^A_L(z) \chi(z') = \frac{J^A_{0,L}}{2\pi i(z - z')} \chi(z') + \cdots, \tag{13} \]

\[ J^A_R(z) \chi(z') = \frac{J^A_{0,R}}{2\pi i(z - z')} \chi(z') + \cdots, \tag{14} \]

where the operators \( J^A_{0,L} \) and \( J^A_{0,R} \) are the generators of the global SU(\( N \))\(_L \times SU(\( N \))\(_R \) transformations, and satisfy the characteristic equations, \( J^A_{0,L}|\chi(z')\rangle = -T^A_L|\chi(z')\rangle \) and \( J^A_{0,R}|\chi(z')\rangle = |\chi(z')\rangle T^A_R \). Note that \( \sum_A (J^A_{0,L})^2 \) and \( \sum_A (J^A_{0,R})^2 \) are the Casimir operators of SU(\( N \))\(_L \) and SU(\( N \))\(_R \) groups, respectively. The two-point correlators of the left currents (for \( k = 1 \)) are,

\[ \langle J^A_L(z) J^B_R(z') \rangle = \frac{\text{Tr}[T^A_T^B]}{4\pi^2(z - z')^2} \left( \frac{\delta^{AB}}{8\pi^2(z - z')^2} \right), \tag{15} \]

\[ \langle J^A_R(z) J^B_R(z') \rangle = \frac{\delta^{AB}}{8\pi^2(z - z')^2}. \tag{16} \]

Using these results we explicitly calculate:

\[ \langle \phi(z, \bar{z}) \phi(z', \bar{z}') \rangle = \left( \frac{D}{8\pi^2} \right)^2 \left( \frac{(N^2 - 1)}{(z - z')^2(\bar{z} - \bar{z}')^2} \right), \tag{17} \]

and then compare it to the standard conformal field theory result i.e. \( \langle \phi(z, \bar{z}) \phi(z', \bar{z}') \rangle = |z - z'|^{-2}|\bar{z} - \bar{z}'|^{-2} \) to obtain the value of the constant,

\[ D = \frac{8\pi^2}{\sqrt{N^2 - 1}}. \tag{18} \]

Thus the normalized irrelevant marginal operator is given by,

\[ \phi(z, \bar{z}) = -\frac{8\pi^2}{\sqrt{N^2 - 1}} \sum_{A=1}^{N^2-1} J^A_L(z) J^A_R(\bar{z}). \tag{19} \]

Perturbation to the normalized excited state \( |\phi_n\rangle \) energies due to the marginal operator can now be calculated from

\[ \delta(E_n - E_0) = g_0 \int dx \langle \phi_n | \phi | \phi_n \rangle, \tag{20} \]

where \( \phi \) and \( \phi_n \) are Virasoro primary fields generated by applying Fourier modes of \( J^A_L \) and \( J^A_R \) on Kac-Moody primary fields. For large length (equivalently, small temperature), we may replace the coupling \( g_0 \) by its renormalization group improved value (upto the log – log term),

\[ g_0(T) = \left( \frac{1}{\pi b \ln(T_0/T)} \right) \times \left[ 1 - \frac{1}{2 \ln(T_0/T)} \ln[\ln(T_0/T)] \right]. \tag{21} \]

Here \( T_0 \) is the model dependent parameter of the system and the coefficient \( b \) is defined via the following 3-point correlator,

\[ \langle \phi(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) \phi(z_3, \bar{z}_3) \rangle = -b/|z_1|^2|z_2|^2|z_3|^2. \tag{22} \]

Substituting this in Eq. (20) we obtain,

\[ \delta(E_n - E_0) = \left( \frac{2\pi T}{v_1 \ln(T_0/T)} \right) \times \left( \frac{2b_n}{b} \right) \left[ 1 - \frac{1}{2 \ln(T_0/T)} \ln[\ln(T_0/T)] \right]. \tag{23} \]

The coefficient \( b_n \) is again defined through the 3-point correlator,

\[ \langle \phi_n(z_1, \bar{z}_1) \phi(z_2, \bar{z}_2) \phi_n(z_3, \bar{z}_3) \rangle = -b_n/|z_1|^2|z_2|^2|z_3|^2. \tag{24} \]

Here \( x_n \) is the scaling dimension of the Virasoro primary field \( \phi_n \). Substituting Eq. (19) in Eq. (24) and using the OPE’s as in Eqs. (13) and (14) it follows that \( b_n \) is directly proportional to the sum of the product of the eigenvalues of the generators \( J^A_{0,L} \) and \( J^A_{0,R} \),

\[ b_n = -\frac{2}{\sqrt{N^2 - 1}} \sum_{A=1}^{N^2-1} T^A_L T^A_R. \tag{25} \]

To evaluate \( \sum A T^A_L T^A_R \), we observe that the full symmetry, SU(\( N \))\(_L \times SU(\( N \))\(_R \), of the quantum chain at the critical point is broken by the presence of the marginal operator, \( \phi(z, \bar{z}) \). Only the diagonal SU(\( N \)) \(_L \times SU(\( N \))\(_R \) is an exact symmetry of the quantum chain. Under this subgroup, the representation \( V_L \otimes V_R \) of SU(\( N \))\(_L \times SU(\( N \))\(_R \) decomposes into direct sum of various irreducible subrepresentations. If an excited state \( |\phi_n\rangle \) belongs to a highest weight subrepresentation, \( V \subset V_L \otimes V_R \) and \( C \) is the corresponding Casimir invariant of the diagonal SU(\( N \)) \(_V \), then we have

\[ \sum_{A=1}^{N^2-1} T^A_L T^A_R = \frac{1}{2} [C - C_L - C_R], \tag{26} \]

where \( C_L \) and \( C_R \) are the Casimir invariants of SU(\( N \))\(_L \) and SU(\( N \))\(_R \) in the highest weight representations \( V_L \) and \( V_R \), respectively. Therefore, using Eqs. (23) and (24) we find

\[ b_n = -\frac{1}{\sqrt{N^2 - 1}} [C - C_L - C_R]. \tag{27} \]

The above formula may also be used to evaluate the renormalization group coefficient, \( b \) [Eq. (22)]; since
$\phi(z, \bar{z})$ is a Virasoro primary field of conformal dimensions $(1,1)$, we set $\hat{\phi}_n(z, \bar{z}) = \phi(z, \bar{z})$ and $x_n = 2$ in Eq. (24), and hence, $b_n = b$. This can be seen as follows. The Virasoro primary fields $j_L$ and $j_R$ of conformal dimensions $(1,0)$ and $(0,1)$ transform as the adjoint representations, $V_L^{adj}$ and $V_R^{adj}$ of $SU(N)$, and since $V_R^{adj}$ is conjugate to $V_L^{adj}$, the direct sum decomposition of $V_L^{adj} \otimes V_R^{adj}$ under the diagonal $SU(N) \subset SU(N)_L \times SU(N)_R$ must contain a unique singlet. Hence, the Virasoro primary field $\phi(z, \bar{z})$ in Eq. (19) transforms as this singlet representation and we have $C = 0, C_L = C_R = N$ in Eq. (27). This implies

$$b = \frac{2N}{\sqrt{N^2 - 1}}. \quad (28)$$

For example in the case of Heisenberg spin-chain with $SU(2)$ symmetry, $b = 4/\sqrt{3}$ and for the spin-orbital model with $SU(4)$, $b = 8/\sqrt{15}$. Our result for $SU(4)$ is new. Together with Eq. (19), the constant $b$ also determines the correction of $O(g^2)$ in the ground state energy,

$$E_0(T) - E_0(0) = - \left( \frac{\pi T}{6v_s} \right) \left[ (N-1) + 2\pi^3bg^2 \right]. \quad (29)$$

To determine the logarithmic shifts in the excited states energy levels, we need the ratio $2b_n/b$ in Eq. (23). From Eqs. (27) and (28) we get,

$$\left( \frac{2b_n}{b} \right) = -\frac{1}{N} [C - C_L - C_R]. \quad (30)$$

To evaluate $b_n$ (and hence $2b_n/b$) we must know the excited states. In $SU(N)$ invariant quantum chains, the low lying excited states ($|\phi_n\rangle$) correspond to $(N-1)$ primary fields of the $SU(N)_{k=1}$ WZNW model with the scaling dimensions, $\Delta_p$. These fields transform as $(q, \bar{q})$ representations of $SU(N)_L \times SU(N)_R$, where

$$q = \frac{N(N-1) \cdots (N-p+1)}{p!}$$

is the dimension of the $p$-th fundamental representation of $SU(N)$ for $p = 1, 2, \cdots, N-1$. For instance, the lowest excited states correspond to the fundamental primary field $g$ with the scaling dimensions, $\Delta_1 = (1-1/N)$. This field transforms under the $(N, \bar{N})$ representation which decomposes under the diagonal $SU(N)$ into the adjoint and singlet representations. For the excited states, $\text{Tr} (g^{T_A})$, belonging to the adjoint representation, we have $C = N, C_L = C_R = (N^2 - 1)/2N$ which implies $b_n = -1/(N\sqrt{N^2-1})$ and $2b_n/b = -1/N^2$. For example, in the case of $SU(2)$, this ratio $2b_n/b = -1/4$ and for $SU(3)$ and $SU(4)$ they are $-1/9$, and $-1/16$ respectively.

For the excited state, $\text{Tr} g$, belonging to the singlet representation we have $C = 0, C_L = C_R = (N^2 - 1)/2N$. In this case, $b_n = \sqrt{N^2 - 1}/N$ and the $2b_n/b = 1 - 1/N^2$. This result is new. In case of $SU(2)$, the value $2b_n/b = 3/4$ has been previously obtained but for $SU(3), 2b_n/b = 8/9$ and for $SU(4), 2b_n/b = 15/16$ are predictions from our general formula.

We consider one more application of the formula (30) of current interest – the $SU(4)$ symmetric quantum chain described by the $SU(4)_k=1$ WZNW model. In this case, to compute logarithmic corrections to the excited states energy we note that there are three primary fields with scaling dimensions $\Delta_p = 3/4, 1, 3/4$ for $p = 1, 2, 3$ respectively, as seen from Eq. (8). The case of $p = 1$ and $p = 3$, as discussed above, is the fundamental field $g$ [and its hermitian conjugate $\bar{g}$] which transforms under the $(4, \bar{4})$ and $(\bar{4}, 4)$ representation of $SU(4)_L \times SU(4)_R$. From Eq. (10), the lowest energy excited states correspond to the primary field operator (denoted by $\Psi$) with $\Delta_2 = 1$. The field $\Psi$ transforms under the $(6, 6)$ representation of $SU(4)_L \times SU(4)_R$. This $(6, 6)$ representation decomposes as direct sum of a singlet, an adjoint and a 20-dimensional representation (as in Fig. 1) under the diagonal $SU(4)$.

![Young tableau for the decomposition of (6, 6) representation of SU(4)](image)

**FIG. 1.** Young tableau for the decomposition of $(6, 6)$ representation of $SU(4)$. The number in the parenthesis denotes the dimension of the corresponding representation.

We now compute the ratio $2b_n/b$ for the excited states corresponding to 20-dimensional representation which has a Young tableau with 2 rows and 2 columns. For this representation, the Casimir invariant $C$ in Eq. (30) is obtained from the formula (1): we find $C = 6$, and $C_L = C_R = C_{p=2} = 5/2$. Thus, for the excited states corresponding to $\Psi$ in the 20-dimensional subrepresentation of $(6, 6)$, we have $2b_n/b = -1/4$.

In summary, we have studied finite size spectrum for one dimensional $SU(N)$ symmetric quantum chains using both conformal field theory and representation theory of $SU(N)$. We have calculated in general the scaling dimensions of all the oscillating modes, and obtained the ground state energy as well as correlation lengths of the staggered modes for a finite size system with $SU(N)$ symmetry. Possibilities of different types of excited states
are also briefly discussed and a general formula to compute the logarithmic correction to the excited state energies has been derived. The existing results for $N = 2, 4$ agree with the predictions from our general formula.

Authors acknowledge discussions with F. C. Zhang and M. Ma. K. M also acknowledge partial support by PRF No. 33611-AC6 and partial support from Berea College.

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