Analytic solutions of the Teukolsky equation for massless perturbations of any spin in de Sitter background

Yao-Zhong Zhang

School of Mathematics and Physics, The University of Queensland,
Brisbane, Qld 4072, Australia

Abstract

We present analytic solutions to the Teukolsky equation for massless perturbations of any spin in the 4-dimensional de Sitter background. The angular part of the equation fixes the separation constant to a discrete set and its solution is given by hypergeometric polynomials. For the radial part, we derive analytic power series solution which is regular at the poles and determine a transcendental function whose zeros give the characteristic values of the wave frequency. We study the existence of explicit polynomial solutions to the radial equation and obtain two classes of singular closed-form solutions, one with discrete wave frequencies and the other with continuous frequency spectra.

PACS numbers: 03.65.Ge, 04.30.Nk, 04.20.Cv.

1 Introduction

Perturbations to known solutions of Einsteins equations by various types of fields, such as scalar, neutrino, electromagnetic, and gravitational fields, have been extensively studied in the literature. One of the important solutions is the de Sitter spacetime. It is the simplest model of spacetime with a non-zero cosmological constant and is relevant to inflation and the physics of very early universe [1, 2]. Experimental evidences and astronomical observations, combining with the theory of inflation, suggest that our universe is expanding in an accelerated rate and may approach the de Sitter geometry asymptotically [3]. It has also recently been conjectured that there exists a holographic duality between quantum gravity in (anti-)de Sitter spacetime and certain conformal field theory on the boundary of that space (the so-called AdS/CFT correspondence) [4, 5]. Therefore, it is certainly of interest to investigate perturbations in the static region of the de Sitter spacetime between the origin and the cosmological horizon.

In this paper we discuss gauge- and tetrad-invariant first order massless perturbations of any spin in the de Sitter spacetime. The dynamics of these perturbations in four dimensions can be conveniently described by the Teukolsky master differential equation [6]. The radial and angular parts of the Teukolsky perturbation equation are separable thanks to the spherical symmetry of the de Sitter background spacetime. We provide
representations of analytic solutions to both the angular and radial equations. We show that the angular part of the equation fixes the separation constant to a discrete set and its solution is given by hypergeometric polynomials. We obtain power series solution of the radial equation which is regular at the poles and determine a transcendental function whose zeros give the characteristic values of the wave frequency. Furthermore, explicit polynomial solutions of the radial equation are studied by applying the general procedure and results of the present author in [8]. Two classes of singular closed-form solutions to the radial equation are found: one yields discrete complex frequencies while the frequency of the other class has a continuous spectrum. Let us mention that polynomial solutions to the Teukolsky master equation with a continuous frequency spectrum also were studied previously for perturbations in the Kerr background [9].

The outline of the present work is as follows. In section 2 we briefly review the Teukolsky master perturbation equation and the separation of its variables. Section 3 deals with the angular eigenvalue problem. The regularity of the angular function at the poles is used to fix the separation constant. In section 4 we study the radial eigenvalue problem. We follow a procedure which is based on the application of the mathematical theorems on solutions of three-term recurrence relation [10, 11]. (A similar procedure has recently used to find entire function solutions of the Rabi model [12] and its generalizations [13, 14].) In section 5 we examine the existence of explicit, polynomial solutions to the radial equation. We present two classes of a total five families of closed-form solutions, all of them are singular at the poles. We conclude the work with a summary in section 6.

2 Teukolsky master equation for the perturbations

We consider gauge- and tetrad-invariant first order massless perturbations of any spin in the de Sitter background. In static coordinates, the de Sitter metric takes the form

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

where $N = \sqrt{1 - \frac{H^2 r^2}{1}}$ denotes the lapse function. Note that there is a cosmological horizon at $r = r_H = \frac{1}{H}$ and the spacetime region which can be accessed is the ball of radius $r_H$, centered at the origin. The Teukolsky master equation governing the perturbations is given by [7]

$$\begin{aligned}
\left\{ \frac{1}{2N^2} \frac{\partial^2}{\partial t^2} - \frac{N}{2} \frac{\partial^2}{\partial r^2} - \frac{1}{2r^2} \frac{\partial^2}{\partial \theta^2} - \frac{1}{2r^2} \frac{\partial^2}{\partial \theta^2} \frac{\partial}{\partial \phi} + \frac{s}{rN^2} \frac{\partial}{\partial t} + \frac{(s + 1)(1 - 2N^2)}{r} \frac{\partial}{\partial r} - \cot \theta \frac{\partial}{\partial \theta} - \frac{is \cos \theta}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} + \frac{1}{2r^2} \left( s + 1 \right) \left( 3s + 2 - 2N^2(2s + 1) \right) + \frac{s^2}{\sin^2 \theta} \right\} \psi = 0.
\end{aligned}$$

(2.2)

The parameter $s$ is called the spin weight of the field, and is given by $s = \pm 2$ for gravitational perturbations, $s = \pm 1$ for electromagnetic perturbations, $s = \pm \frac{1}{2}$ for massless
neutrino perturbations, and \( s = 0 \) for scalar perturbations. The variables in the Teukolsky master equation can be separated by using the ansatz
\[
\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) \Theta(\theta),
\]
where \( m \) is the azimuthal parameter. Then one obtains the following angular and radial equations,
\[
\begin{align*}
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \left[ \Lambda - s^2 - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} \right] \Theta(\theta) &= 0, \\
Q^{-s} \frac{d}{dr} \left( Q^{s+1} \frac{dR(r)}{dr} \right) + \left[ \frac{\omega^2 r^2 + 2is\omega r}{N^2} + 2N^2(s + 1)(2s + 1) - \Lambda - (s + 1)(3s + 2) \right] R(r) &= 0,
\end{align*}
\]
where \( Q = r^2 N^2 \). The complex parameters \( \omega \) and \( \Lambda \) are the wave frequency and separation constant, respectively. The sign of the imaginary part \( \text{Im}(\omega) \) of the complex wave frequency \( \omega \) determines whether the solution is stable (decaying in time with \( \text{Im}(\omega) < 0 \)) or unstable (growing in time with \( \text{Im}(\omega) > 0 \)). Stationary modes are characterized by \( \text{Im}(\omega) = 0 \). Obviously the angular equation is independent of the frequency \( \omega \).

3 The angular eigenvalue problem

The angular functions \( \Theta(\theta) \) are required to be regular at the poles \( \theta = 0 \) and \( \theta = \pi \). These boundary conditions pick out a discrete set of \( \Lambda \), as shown below. In terms of the variable \( x = \cos \theta \), the angular equation becomes
\[
(1 - x^2) \Theta'' - 2x \Theta' + \left[ \Lambda - \frac{(s + m)^2/2}{1 - x} - \frac{(s - m)^2/2}{1 + x} \right] \Theta = 0, \quad |x| \leq 1.
\]
The solutions to this equation at the regular singularities \( x = \pm 1 \) can be found in the usual way. If
\[
\lim_{x \to 1} \Theta \sim (1 - x)^{k_1}, \quad \lim_{x \to -1} \Theta \sim (1 + x)^{k_2},
\]
then \( k_1 = \pm \frac{1}{2}(s + m) \) and \( k_2 = \pm \frac{1}{2}(s - m) \). The physically meaningful solutions to the angular equation are those that are regular at \( x = \pm 1 \) (which correspond to \( \theta = 0 \) and \( \theta = \pi \), respectively), so the usual choices for \( k_1 \) and \( k_2 \) are \( k_1 = |s + m|/2 \) and \( k_2 = |s - m|/2 \). Setting
\[
\Theta(x) = (1 - x)^{|s+m|/2}(1 + x)^{|s-m|/2} S(x)
\]
yields
\[
(x - 1)(x + 1)S'' + [(a + b + 1)x + |s + m| + |s - m|]S' + ab S = 0,
\]
where
\[
a = \frac{1}{2} \left[ (s + m) + |s - m| + 1 + \sqrt{4\Lambda + 1} \right], \\
b = \frac{1}{2} \left[ (s + m) + |s - m| + 1 - \sqrt{4\Lambda + 1} \right].
\] (3.9)

Making a change of variable \( x = 2z - 1 \) converts the above into the hypergeometric differential equation for \( S(z) \),
\[
z(z - 1)S'' + [(a + b + 1)z - (|s - m| + 1)] S' + ab S = 0, \quad 0 \leq z \leq 1. \tag{3.10}
\]

The local solution of this equation around \( z = 0 \) is known as the hypergeometric function
\[
S(z) = \sum_{n=0}^{\infty} Q_n z^n,
\] (3.11)
where the coefficients \( Q_n \) are functions of the separation constant \( \Lambda \) and are determined by the recurrence relation
\[
Q_{n+1} = \frac{n(n + a + b) + ab}{(n + 1)(n + 1 + |s - m|)} Q_n, \quad Q_0 = 1. \tag{3.12}
\]

If for some positive integer \( N = 0, 1, \cdots \),
\[
N(N + a + b) + ab = 0, \tag{3.13}
\]
then we have \( Q_{N+1} = 0 \) and the hypergeometric function terminates to the hypergeometric polynomial of degree \( N \). Under the condition (3.13) we obtain the exact solution of the angular equation,
\[
\Theta(\theta) = (1 - \cos \theta)^{|s+m|/2}(1 + \cos \theta)^{|s-m|/2} S_N(\cos \theta), \tag{3.14}
\]
where \( S_N \) is the polynomial in \( \cos \theta \) of degree \( N \): \( S(\cos \theta) = \sum_{n=0}^{N} Q_n(\Lambda) \left( \frac{1 + \cos \theta}{2} \right)^n \). The corresponding separation constant \( \Lambda \) takes discrete values given by
\[
\Lambda = \frac{1}{4} (2N + |m| + |s + m|)(2N + |s - m| + |s + m| + 2), \quad N = 0, 1, 2, \cdots. \tag{3.15}
\]

So the separation constant \( \Lambda \) is fixed by the angular part of the Teukolsky equation alone!

4 Three-term recurrence relation and the radial eigenvalue problem

In this section we present analytic solution to the radial equation on \( 0 \leq r \leq r_H \). By the substitution
\[
R(r) = Q^{-\frac{\Lambda}{2}} Y(r), \tag{4.16}
\]
one can convert the radial equation into the normal Schrödinger form \[7\]

\[
-\frac{d^2}{dr^2} + V(r) \right] Y(r) = 0,
\]

\[
V(r) = \frac{c_0}{r^2} + \frac{c_H^-}{(r-r_H)^2} + \frac{c_H^+}{(r+r_H)^2} + \frac{d_0}{r} + \frac{d_H^-}{r-r_H} + \frac{d_H^+}{r+r_H},
\]

(4.17)

where \(c_0 = \Lambda\), \(d_0 = -2i\omega\) and

\[
c_H^\pm = \frac{1}{4}\left((s \pm i\omega r_H)^2 - 1\right),
\]

\[
d_H^\pm = \pm \frac{1}{4r_H}\left(\omega^2 r_H^2 \mp 4i\omega r_H - s^2 - 2\Lambda + 1\right).
\]

(4.18)

Note in passing that \(d_0 + d_H^- + d_H^+ = 0\) and the separation constant \(\Lambda\), fixed by the angular part, is given by (3.15). Solutions of the above equation (4.17) at the regular singularities \(r = 0, \pm r_H\) can be found as follows. If

\[
\lim_{r \to 0} Y \sim r^{\alpha_0}, \quad \lim_{r \to r_H} Y \sim (r_H - r)^{\alpha_H^-}, \quad \lim_{r \to -r_H} Y \sim (r_H + r)^{\alpha_H^+}
\]

(4.19)

or equivalently

\[
\lim_{r \to 0} R \sim r^{-s-1+\alpha_0}, \quad \lim_{r \to r_H} R \sim (r_H - r)^{-\frac{s+1}{2} + \alpha_H^-}, \quad \lim_{r \to -r_H} R \sim (r_H + r)^{-\frac{s+1}{2} + \alpha_H^+},
\]

(4.20)

then

\[
\alpha_0 = \frac{1}{2}\left[1 \pm \sqrt{4\Lambda + 1}\right], \quad \alpha_H^- = \frac{1}{2}\left[1 \pm (s - i\omega r_H)\right], \quad \alpha_H^+ = \frac{1}{2}\left[1 \pm (s + i\omega r_H)\right].
\]

(4.21)

Note that solutions of the radial equation which are nonsingular at \(r = 0, \pm r_H\) correspond to the choice

\[
\alpha_0 = \frac{1}{2}\left[1 + \sqrt{4\Lambda + 1}\right], \quad \alpha_H^+ = \frac{1}{2}\left[1 + s \mp i\omega r_H\right], \quad \text{Im}(\omega) \geq 0.
\]

(4.22)

Applying the substitution,

\[
Y(r) = r^{\alpha_0} (r_H - r)^{\alpha_H^-} (r + r_H)^{\alpha_H^+} X(r),
\]

(4.23)

we can transform (4.17) into

\[
r(r^2 - r_H^2)X'' + \left[(\alpha + \beta + 1)r^2 + 2\left(\alpha_H^- - \alpha_H^+\right) r_H r - 2\alpha_0 r_H^2\right] X' + (\alpha \beta r - q) X = 0,
\]

(4.24)

where

\[
\alpha = \alpha_0 + \alpha_H^+ + \alpha_H^-,
\]

\[
\beta = \alpha_0 + \alpha_H^+ - \alpha_H^- - 1,
\]

\[
q = 2\alpha_0(\alpha_H^- - \alpha_H^+) r_H - d_0 r_H^2.
\]

(4.25)
We now make a change of variable, \( \tilde{r} = r_H - r \). In terms of \( \tilde{r} \) the physically accessible spacetime region is the ball \( 0 \leq \tilde{r} \leq r_H \), and (4.24) becomes

\[
X'' + \left( \frac{2\alpha \tilde{r}}{\tilde{r} - r_H} + \frac{2\alpha_0}{\tilde{r} - 2r_H} \right) X' + \frac{\alpha \beta \tilde{r} - \tilde{q}}{\tilde{r}(\tilde{r} - r_H)(\tilde{r} - 2r_H)} X = 0,
\]

(4.26)

where \( \tilde{q} = \alpha \beta r_H - q \). This is the Heun general equation with four regular singular points \( \tilde{r} = 0, r_H, 2r_H, \infty \). We seek power series solution of (4.26),

\[
X(\tilde{r}) = \sum_{n=0}^{\infty} K_n(\omega) \tilde{r}^n
\]

(4.27)

which is convergent in the ball \( 0 \leq \tilde{r} \leq r_H \) (i.e. \( 0 \leq r \leq r_H \), where the coefficients \( K_n \) are functions of the wave frequency \( \omega \). Substituting (4.27) into (4.26), we find that \( K_n \) obey the 3-term recurrence relation

\[
K_{n+1} + A_n K_n + B_n K_{n-1} = 0, \quad n \geq 1
\]

(4.28)

with

\[
A_n = -\frac{\alpha \beta r_H - q + n \left( 3n - 3 + 4\alpha_0 + 2\alpha_0^+ + 6\alpha_H^- \right) r_H}{2(n+1)(n+2\alpha_H^-) r_H^2},
\]

\[
B_n = \frac{(n-1+\alpha)(n-1+\beta)}{2(n+1)(n+2\alpha_H^-) r_H^2}.
\]

(4.29)

Solutions to the 3-term recurrence relation can be classified by applying the mathematical theorems in [10]. The characteristic equation of (4.28) is given by \( u^2 - \frac{3}{2r_H} u + \frac{1}{2r_H} = 0 \). This equation has two distinct roots \( u_1 = \frac{1}{2r_H} \) and \( u_2 = \frac{1}{r_H} \) and \( |u_1| < |u_2| \). By the Perron theorem (i.e. Theorem 2.2 of [10]), there exist two linearly independent solutions \( K_{n,1} \) and \( K_{n,2} \) of (4.28) such that

\[
\lim_{n \to \infty} \frac{K_{n+1,s}}{K_{n,s}} = u_s, \quad s = 1, 2.
\]

(4.30)

Thus \( K_n^{\text{min}} = K_{n,1} \) is a minimal solution of (4.28), while the other solution \( K_{n,2} \) is dominant.

By the Pincherle theorem (i.e. Theorem 1.1 of [10]), the ratio of successive elements of the minimal solution sequence \( K_n^{\text{min}} \) is expressible as continued fractions,

\[
R_n = \frac{K_{n+1}^{\text{min}}}{K_{n}^{\text{min}}} = -\frac{B_{n+1}}{A_{n+1}} - \frac{B_{n+2}}{A_{n+2}} - \frac{B_{n+3}}{A_{n+3}} - \cdots,
\]

(4.31)

which for \( n = 0 \) gives

\[
R_0 = \frac{K_{1}^{\text{min}}}{K_{0}^{\text{min}}} = -\frac{B_1}{A_1} - \frac{B_2}{A_2} - \frac{B_3}{A_3} - \cdots.
\]

(4.32)
Note that the ratio \( R_0 = \frac{K_{0}^{\min}}{K_{0}^{\min}} \) involves \( K_{0}^{\min} \), although the above continued fraction expression is obtained from the 2nd equation of (4.28), i.e the recurrence (4.28) for \( n \geq 1 \). However, for single-ended sequences such as those appearing in the infinite series expansion (4.27), the ratio \( R_0 = \frac{K_{1}^{\min}}{K_{0}^{\min}} \) of the first two terms of a minimal solution is unambiguously fixed by the first equation of the recurrence (4.28), namely,

\[
R_0 = -A_0 = \frac{\alpha \beta r_H - q}{4 \alpha_H r_H^2}.
\]

In general, the \( R_0 \) computed from the continued fraction (4.32) can not be the same as that from (4.33) for arbitrary values of recurrence coefficients \( A_n \) and \( B_n \). As a result, general solutions to the recurrence (4.28) are dominant and are usually generated by simple forward recursion from a given value of \( K_0 \). The resulting power series (4.27) will converge for \( 0 \leq \tilde{r} < r_H \) but will diverge when \( \tilde{r} = r_H \). This is seen as follows. By d’Alembert’s Ratio Test, the radius \( \rho \) of convergence of the power series expansion (4.27) is given by \( \rho^{-1} = \lim_{n \to \infty} \frac{K_{n+1}}{K_n} \). It follows from (4.30) that \( \rho \) equals to \( 2r_H \) for the minimal solution sequence \( K_{n,1} \) and to \( r_H \) for the dominant one \( K_{n,2} \). Thus the power series expansion generated by the dominant solution sequences is only convergent inside the ball \( 0 \leq \tilde{r} < r_H \) but not on the boundary \( \tilde{r} = r_H \). The physically meaningful solutions to the radial equation are those that are convergent at both \( \tilde{r} = 0 \) and \( \tilde{r} = r_H \) (which correspond to \( r = r_H \) and \( r = 0 \), respectively). This will happen only for certain characteristic values of the frequency \( \omega \) so that equations (4.32) and (4.33) are both satisfied. Then the resulting solution sequence \( K_n \) will be purely minimal and the corresponding power series expansion (4.27) will be convergent for \( 0 \leq \tilde{r} < 2r_H \), thus it converges at both singular points \( \tilde{r} = 0 \) and \( \tilde{r} = r_H \).

Therefore, if we define the transcendental function \( F(\omega) = R_0 + A_0 \) with \( R_0 \) given by the continued fraction in (4.32), then the zeros of \( F(\omega) \) correspond to the characteristic values of \( \omega \) for which the condition (4.33) is satisfied. In other words, \( F(\omega) = 0 \) is the eigenvalue equation for the radial eigenvalue problem. Only for the denumerable infinite values of \( \omega \) which are the roots of \( F(\omega) = 0 \), do we get solutions (4.27) of the radial equation which are convergent at both \( \tilde{r} = 0 \) and \( \tilde{r} = r_H \). The transcendental equation \( F(\omega) = 0 \) may be solved for \( \omega \) by standard root-search algorithms (see e.g. [11, 15] and references therein).

5 Singular polynomial solutions to the radial equation

In this section we study the existence of closed-form solutions of the radial equation which are polynomials in \( r \) and thus automatically converge at \( r = 0, r_H \). Such solutions, if they exist, correspond to the values of \( \omega \) which make the power series (4.27) truncate to become
a polynomial of finite degree. Thus we seek solutions to (4.26) which are of the form

\[ X_M(\tilde{r}) = \prod_{\ell=1}^{M}(\tilde{r} - \tilde{r}_\ell), \quad \text{(5.34)} \]

where \( M = 0, 1, 2, \cdots \) is the degree of the polynomial \( X_M(\tilde{r}) \), \( r_\ell \) are the roots of the polynomial and \( X_M(\tilde{r}) \equiv 1 \) if \( M = 0 \). Closed-form (i.e. the so-called Liouvillian) solutions of the radial equation were discussed in [7] by means of the Kovacic algorithm [16].

We will follow the procedure proposed in [8]. There exact polynomial solutions of a general 2nd order linear differential equation were classified. Applying the results (e.g. Corollary 5.1 in the Appendix) of [8], we have that (5.34) is a solution of the radial equation (4.26) if the frequency \( \omega \) and other system parameters satisfy the constraints

\[ \alpha \beta = -M(M - 1) - 2M(\alpha_0 + \alpha_H^- + \alpha_H^+), \quad \text{(5.35)} \]

\[ -q = -\left[2(M - 1) + 2(\alpha_0 + \alpha_H^- + \alpha_H^+)\right]\sum_{\ell=1}^{M}(r_H - \tilde{r}_\ell) + 2M(\alpha_H^+ - \alpha_H^-)r_H, \quad \text{(5.36)} \]

and the roots \( \tilde{r}_\ell \) are determined by the set of \( M \) algebraic equations

\[ \sum_{\ell' \neq \ell}^{M} \frac{2}{\tilde{r}_\ell - \tilde{r}_{\ell'}} + \frac{2\alpha_H^-}{\tilde{r}_\ell - r_H} + \frac{2\alpha_0}{\tilde{r}_\ell - r_H} + \frac{2\alpha_H^+}{\tilde{r}_\ell - 2r_H} = 0, \quad \ell = 1, 2, \cdots, M. \quad \text{(5.37)} \]

It is not hard to verify that (5.35) and (5.36) can be simplified to

\[ \alpha_0 + \alpha_H^- + \alpha_H^+ = -(M - 1), \quad 2(M + \alpha_0)(\alpha_H^+ - \alpha_H^-) = d_0r_H, \quad \text{(5.38)} \]

respectively and these two equations are equivalent for all \( \alpha_0, \alpha_H^\pm \) values given in (4.21).

It turns out that all closed-form solutions with the parameters satisfying the constraints (5.38) (equivalently (5.35) and (5.36)) are singular at the poles of the radial equation. There are two classes (Class I and Class II) of singular polynomial solutions, classified according to whether the wave frequency \( \omega \) has discrete or continuous spectra. An alternative approach of showing the existence of closed-form solutions is provided in the Appendix.

### 5.1 Class I solutions - discrete wave frequency

This class contains three families of solutions, which we describe as follows.

**Class Ia.** This corresponds to the choice

\[ \alpha_0 = \frac{1}{2}\left(1 + \sqrt{4\Lambda + 1}\right), \quad \alpha_H^- = \frac{1}{2}(1 + s - i\omega r_H), \quad \alpha_H^+ = \frac{1}{2}(1 - s - i\omega r_H). \]

For this choice the constraints (5.38) reduce to

\[ 2M + 1 + \sqrt{4\Lambda + 1} - 2i\omega r_H = 0. \quad \text{(5.39)} \]

This gives the discrete values of \( \omega \)

\[ \omega = -\frac{i}{2r_H}\left(2M + 1 + \sqrt{4\Lambda + 1}\right) \]

\[ = -\frac{i}{r_H}\left[N + M + 1 + \frac{1}{2}(|s - m| + |s + m|)\right], \quad N, M = 0, 1, 2, \cdots. \quad \text{(5.40)} \]
Here we have used (3.15) for the separation constant $\Lambda$. The corresponding solution $R(\tilde{r})$ is given by

$$R(\tilde{r}) = r_H^{s+1} (r_H - \tilde{r})^{\alpha_0 - s - 1} \tilde{r}^{\alpha_H - (s+1)/2} (2r_H - \tilde{r})^{\alpha_H^+ - (s+1)/2} X_M(\tilde{r})$$

$$= r_H^{s+1} (r_H - \tilde{r})^{N - s + 1 - \frac{i}{2}(\frac{s-m}{s-m}) \tilde{r}^{-\frac{1}{2} \omega H} (2r_H - \tilde{r})^{-s + \frac{1}{2} \omega H} \prod_{\ell=1}^{M} (\tilde{r} - \tilde{r}_\ell) \right) \tag{5.41}$$

where $N, M = 0, 1, 2, \cdots$ and $\tilde{r}_\ell$ are the solutions of the algebraic equations (5.37).

**Class Ib.** This corresponds to the choice $\alpha_0 = \frac{1}{2} \left(1 - \sqrt{4 \Lambda + 1}\right)$, $\alpha_H^- = \frac{1}{2}(1 + s - i \omega r_H)$, $\alpha_H^+ = \frac{1}{2}(1 - s - i \omega r_H)$. In this case, (5.38) reduce to

$$2M + 1 - \sqrt{4 \Lambda + 1} - 2i \omega r_H = 0, \tag{5.42}$$

which yields the characteristic values of $\omega$

$$\omega = -\frac{i}{r_H} \left[M - N - \frac{1}{2}(|s - m| + |s + m|)\right]. \tag{5.43}$$

The corresponding solution $R(\tilde{r})$ is

$$R(\tilde{r}) = r_H^{s+1} (r_H - \tilde{r})^{N - s - 1 - \frac{i}{2}(\frac{s-m}{s-m}) \tilde{r}^{-\frac{1}{2} \omega H} (2r_H - \tilde{r})^{-s + \frac{1}{2} \omega H} \prod_{\ell=1}^{M} (\tilde{r} - \tilde{r}_\ell) \right) \tag{5.44}$$

with the roots $\tilde{r}_\ell$ determined by the algebraic equations (5.37). Here $N = 0, 1, \cdots$ and $M$ is an integer larger than or equal to $N + \frac{1}{2}(|s - m| + |s + m|)$,

$$M = N + \frac{1}{2}(|s - m| + |s + m|), \quad N + \frac{1}{2}(|s - m| + |s + m|) + 1, \cdots \tag{5.45}$$

for stable or stationary solution (i.e. $\text{Im}(\omega) \leq 0$).

**Class Ic.** This corresponds to the choice $\alpha_0 = \frac{1}{2} \left(1 - \sqrt{4 \Lambda + 1}\right)$, $\alpha_H^- = \frac{1}{2}(1 + s + i \omega r_H)$, $\alpha_H^+ = \frac{1}{2}(1 - s + i \omega r_H)$. In this case, (5.38) reduce to

$$2M + 1 - \sqrt{4 \Lambda + 1} + 2i \omega r_H = 0, \tag{5.46}$$

which yields the discrete values of $\omega$

$$\omega = -\frac{i}{r_H} \left[N + \frac{1}{2}(|s - m| + |s + m| - M)\right]. \tag{5.47}$$

The corresponding solution $R(\tilde{r})$ is given by

$$R(\tilde{r}) = r_H^{s+1} (r_H - \tilde{r})^{-N - s - 1 - \frac{i}{2}(\frac{s-m}{s-m}) \tilde{r}^{-\frac{1}{2} \omega H} (2r_H - \tilde{r})^{-s + \frac{1}{2} \omega H} \prod_{\ell=1}^{M} (\tilde{r} - \tilde{r}_\ell) \right) \tag{5.48}$$

where the roots $\tilde{r}_\ell$ are computed from (5.37). Here $N = 0, 1, \cdots$ and $M$ is an integer smaller than or equal to $N + \frac{1}{2}(|s - m| + |s + m|)$,

$$M = 0, 1, \cdots, N + \frac{1}{2}(|s - m| + |s + m|) \tag{5.49}$$

for stable or stationary solution.
5.2 Class II solutions - continuous wave frequency

This class has two families of solutions, each has a continuous spectrum for the wave frequency $\omega$.

**Class IIa.** This corresponds to the choice $\alpha_0 = \frac{1}{2} \left( 1 - \sqrt{4\Lambda + 1} \right)$, $\alpha_H^- = \frac{1}{2}(1 + s + i\omega r_H)$, $\alpha_H^+ = \frac{1}{2}(1 + s - i\omega r_H)$. In this case, (5.38) reduce to

$$2M + 1 - \sqrt{4\Lambda + 1} + 2s = 0. \quad (5.50)$$

There is no constraint for $\omega$, i.e. $\omega$ takes continuous values. The corresponding solution $R(\tilde{r})$ reads

$$R(\tilde{r}) = r_{H}^{s+1} (r_H - \tilde{r})^{-N - s - 1 - \frac{1}{2}(|s - m| + |s + m|)} \tilde{r}^{-\frac{1}{2}\omega r_H} (2r_H - \tilde{r})^{\frac{1}{2}\omega r_H} \prod_{\ell=1}^{M} (\tilde{r} - \tilde{r}_\ell), \quad (5.51)$$

where $N = 0, 1, 2, \cdots$ and $M = N - s + \frac{1}{2}(|s - m| + |s + m|)$; the roots $\tilde{r}_\ell$ are given by the solutions of (5.37).

**Class IIb.** This corresponds to the choice $\alpha_0 = \frac{1}{2} \left( 1 - \sqrt{4\Lambda + 1} \right)$, $\alpha_H^- = \frac{1}{2}(1 + s + i\omega r_H)$, $\alpha_H^+ = \frac{1}{2}(1 + s - i\omega r_H)$. In this case, (5.38) reduce to

$$2M + 1 - \sqrt{4\Lambda + 1} - 2s = 0. \quad (5.52)$$

The frequency $\omega$ is unconstrained and belongs to a continuous spectrum. The corresponding solution $R(\tilde{r})$ reads

$$R(\tilde{r}) = r_{H}^{s+1} (r_H - \tilde{r})^{-N + s - 1 - \frac{1}{2}(|s - m| + |s + m|)} \tilde{r}^{s+\frac{1}{2}\omega r_H} (2r_H - \tilde{r})^{-s-\frac{1}{2}\omega r_H} \prod_{\ell=1}^{M} (\tilde{r} - \tilde{r}_\ell). \quad (5.53)$$

Here $N = 0, 1, 2, \cdots$ and $M = N + s + \frac{1}{2}(|s - m| + |s + m|)$; the roots $\tilde{r}_\ell$ are determined by (5.37).

6 Summary and discussion

We have provided a comprehensive study of the Teukolsky master equation for massless perturbations of any spin in de Sitter spacetime, and derived the analytic solutions for both the angular and radial parts of the equation. Furthermore, it is shown that the radial Teukolsky equation has no regular, closed-form solutions. We have presented two classes of singular polynomial solutions. The first class contains three families of solutions with discrete complex frequency $\omega$, while the second class has two families of solutions with a continuous $\omega$. 
Appendix

In this Appendix, we provide an alternative approach of showing the existence of closed-form singular solutions for the radial equation with the system parameters satisfying the constraints (5.38).

Imposing the constraint equations (5.38), we obtain from (4.25) that $\alpha = -(M-1)$, $\alpha\beta = M(M-1)$ and $-q = 2M(\alpha - \alpha_H)r_H$. Substituting these into (4.29) gives

$$A_n = -\frac{(n-M)[3n-M+1+2(\alpha - \alpha_H^+)]}{2(n+1)(n+2\alpha_H^+)r_H^2},$$

$$B_n = \frac{(n-M)(n-M-1)}{2(n+1)(n+2\alpha_H^+)r_H^2}. \quad (6.54)$$

Thus when $n = M$, both $A_M$ and $B_M$ vanish. It follows from the three-term recurrence relation (4.28) that $K_{M+1} = 0$ and the power series (4.27) truncates to give closed-form polynomial solutions in section 5.

Note that these polynomial solutions exist only when the system parameters (the frequency $\omega$ and other system parameters) satisfy the constraints (5.38). These constraints give the two classes of singular solutions presented in section 5. As seen in section 5.2, for the class II solutions, although (5.38) do not provide any constraint on the parameter $\omega$, they do impose constraint between the value $M$ (i.e. the degree of the solution polynomial) and the other system parameters $\Lambda$ and $s$, see (5.50) and (5.52).

We also remark that for the parameters in (4.22) corresponding to nonsingular solutions, the constraint equations (5.38) give rise to $2M + 1 + \sqrt{4\Lambda + 1} + 2s = 0$, i.e. $M = -N - s - 1 - \frac{1}{2}(|s - m| + |s + m|)$. That is, $M$ is negative. Thus there are no nonsingular polynomial solutions for the radial equation and solutions for this case are given by power series in section 4.

Acknowledgement

This work was partially supported by the Australian Research Council through Discovery-Projects grant DP190101529.

Data Availability Statement

The data that support the findings of this study are available within the article.

References

[1] A.H. Guth, Phys. Rev. D 23, 347 (1981).
[2] A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).

[3] S. Perlmutter, G. Aldering, G. Goldhaber, R.A. Knop and P. Nugent, Astrophys. J. 517, 565 (1999).

[4] A. Strominger, JHEP 0110, 034 (2001).

[5] P.O. Mazur and E. Mottola, Phys. Rev. D 64, 104022 (2001).

[6] S.A. Teukolsky, Phys. Rev. Lett. 29, 1114 (1972); Astrophys. J. 185, 635 (1973).

[7] D. Bini, G. Esposito and A. Geralico, Gen. Rel. Grav. 44, 467 (2012).

[8] Y.-Z. Zhang, J. Phys. A 45, 065206 (2012).

[9] R.S. Borissov and P.P. Fiziev, arXiv:0903.3617.

[10] W. Gautschi, SIAM Review 9, 24 (1967).

[11] E.W. Leaver, J. Math. Phys. 27, 1238 (1986).

[12] A. Moroz, Europhys. Lett. 100, 60010 (2012)

[13] Y.-Z. Zhang, Ann. Phys. 347, 122 (2014).

[14] Y.-Z. Zhang, On analytic solutions of the driven, 2-photon and two-mode quantum Rabi models, in Le Bin Ho (Ed.), “Hilbert Spaces: properties and applications”, Ch. 5, pp.123-142, Nova Science Publishers, 2020.

[15] J.W. Liu, J. Math. Phys. 33, 4026 (1992).

[16] J.J. Kovacic, J. Symb. Comput. 2, 3 (1986).