SMOOTH AND PROPER NONCOMMUTATIVE SCHEMES AND GLUING OF DG CATEGORIES

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To my wife Maria with thankfulness and love

CONTENTS

Introduction 2
1. Preliminaries on triangulated categories, generators, and semi-orthogonal decompositions 5
  1.1. Generators in triangulated categories 5
  1.2. Semi-orthogonal decompositions 6
  1.3. Exceptional, w-exceptional, and semi-exceptional collections 7
2. Preliminaries on differential graded categories 8
  2.1. Differential graded categories 8
  2.2. Differential graded modules 9
  2.3. Pretriangulated DG categories, categories of perfect DG modules, and enhancements 10
  2.4. Quasi-functors 12
3. Commutative and noncommutative schemes 13
  3.1. Derived categories of quasi-coherent sheaves and noncommutative schemes 13
  3.2. Gluing of DG categories 15
  3.3. Regular, smooth, and proper noncommutative schemes 19
  3.4. Regularity, smoothness, and properness in commutative geometry 21
4. Gluing of smooth projective schemes and geometric noncommutative schemes 24
  4.1. Geometric noncommutative schemes 24
  4.2. Perfect complexes as direct images of line bundles 27
  4.3. Blowups and gluing of smooth projective schemes 29
  4.4. Gluing of geometric noncommutative schemes 33
5. Application to finite algebras and exceptional collections 35
  5.1. Finite dimensional algebras 35
  5.2. Exceptional collections 39
  5.3. Noncommutative projective planes 40
References 42

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Introduction

One of the main approaches to noncommutative geometry is to consider categories of sheaves on varieties instead of varieties themselves. In algebraic geometry only quasi-coherent sheaves represent well an algebraic structure of a variety and do not depend on a topology. Besides, homological algebra convinces to consider derived categories whenever we meet an abelian category. Thus, instead schemes one consider derived versions of categories of quasi-coherent sheaves and triangulated category of perfect complexes that are compact objects therein. This approach is very powerful.

Let $X$ be a scheme over a field $k$. Studying schemes it is natural to consider the unbounded derived category of quasi-coherent sheaves $\mathcal{D}(\text{Qcoh}X)$ and the unbounded derived category of complexes of $\mathcal{O}_X$–modules with quasi-coherent cohomology $\mathcal{D}_{\text{Qcoh}}(X)$. Fortunately, and it is well known and proved in [BN], for a quasi-compact and separated scheme $X$ the canonical functor $\mathcal{D}(\text{Qcoh}X) \to \mathcal{D}_{\text{Qcoh}}(X)$ is an equivalence. Moreover, it was shown in [Ne2] that in this case the derived category $\mathcal{D}(\text{Qcoh}X)$ has enough compact objects and the subcategory of compact objects is nothing more as a subcategory of perfect complexes $\text{Perf}^c-X$. Recall that a complex is called perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite type.

Furthermore, it was proved in [Ne2, BVB] that the category $\text{Perf}^c-X$ admits a classical generator $E$, i.e. the minimal full triangulated subcategory of $\text{Perf}^c-X$ that contains $E$ and is closed under direct summands coincides with the whole $\text{Perf}^c-X$. As consequence, the perfect complex $E$ will be a compact generator of the whole $\mathcal{D}(\text{Qcoh}X)$.

It very useful to consider a triangulated category $\mathcal{T}$ together with an enhancement $\mathcal{A}$ that is a differential graded (DG) category with the same objects as $\mathcal{T}$, but the set of morphisms between two objects in $\mathcal{A}$ is a complex of vector spaces. One recovers morphisms in $\mathcal{T}$ by taking the cohomology $H^0$ of the corresponding morphism complex in $\mathcal{A}$. Thus, $\mathcal{T}$ is the homotopy category $\mathcal{H}^0(\mathcal{A})$ of the DG category $\mathcal{A}$. Such $\mathcal{A}$ is called an enhancement of $\mathcal{T}$.

The triangulated category $\mathcal{D}(\text{Qcoh}X)$ has natural DG enhancements: a category of h-injective complexes, a DG quotient of all complexes by acyclic complexes, a DG quotient of h-flat complexes by acyclic h-flat complexes. They are all quasi-equivalent and we can work with any of them. Denote by $\mathcal{D}(\text{Qcoh}X)$ a DG enhancement of $\mathcal{D}(\text{Qcoh}X)$ and by $\text{Perf}^c-X$ the induced DG enhancement of the category of perfect complexes $\text{Perf}^c-X$.

Let us take a generator $E \in \text{Perf}^c-X$. Denote by $\mathcal{E}$ its DG algebra of endomorphisms, i.e. $\mathcal{E} = \text{Hom}(E,E)$. Since $E$ is perfect the DG algebra $\mathcal{E}$ has only finitely many cohomology. Keller’s results from [Ke1] imply that the DG category $\text{Perf}^c-X$ is quasi-equivalent to $\text{Perf}^c-\mathcal{E}$, where $\mathcal{E}$ is a cohomologically bounded DG algebra and $\mathcal{D}(\text{Qcoh}X)$ is equivalent to the derived category of DG $\mathcal{E}$–modules $\mathcal{D}(\mathcal{E})$.

The fact described above allows us to suggest a definition of a (derived) noncommutative scheme over $k$ as a $k$–linear DG category of the form $\text{Perf}^c-\mathcal{E}$, where $\mathcal{E}$ is a cohomologically bounded
DG algebra over $k$. In this case the derived category $D(\mathcal{E})$ will be called the derived category of quasi-coherent sheaves on this noncommutative scheme.

The simplest and apparently the most important class of schemes is the class of smooth and projective schemes, or, more generally, the class of schemes that are regular and proper. The properties of regularity, smoothness and properness can be interpreted in categorical terms and can be extended on noncommutative schemes. We say that a $k$–linear triangulated category $\mathcal{T}$ is proper if $\bigoplus_{m \in \mathbb{Z}} \text{Hom}(X, Y[m])$ is finite dimensional for any two objects $X, Y \in \mathcal{T}$. It will be called regular if it has a strong generator, i.e. such an object that generates the whole $\mathcal{T}$ for a finite number of steps (Definitions 1.3 and 3.12). The notion of smoothness is well-defined for a DG category. A $k$–linear DG category $\mathcal{A}$ is called $k$-smooth if it is perfect as the module over $\mathcal{A} \otimes_k \mathcal{A}$. Application of these definitions to a DG categories of perfect objects $\text{Perf}–E$ and its homotopy category $\text{Perf}–\mathcal{E}$ leads us to well-defined notions of regular, smooth, and proper a noncommutative scheme.

It can be proved that a usual separated noetherian scheme $X$ is regular if and only if the category $\text{Perf}–X$ is regular (Theorem 3.27). Furthermore, for separated schemes of finite type regularity and properness of $X$ together are equivalent to regularity and properness of $\text{Perf}–X$, while smoothness and properness of $X$ together are equivalent to smoothness and properness of $\text{Perf}–X$ (Propositions 3.29 and 3.30). These facts imply that smooth and projective schemes form a subclass of the class of smooth and proper noncommutative schemes. It is not difficult to give an example of a noncommutative smooth and proper scheme $\text{Perf}–\mathcal{E}$ that is not an quasi-equivalent to $\text{Perf}–X$ for a usual commutative scheme.

Let us consider the world of all smooth projective schemes. As well as in theory of motives an important step of construction is adding of direct summands, in our situation it is natural to extend the world of smooth projective scheme to an world of all admissible subcategories $\mathcal{N} \subset \text{Perf}–X$, where $X$ is a smooth and projective scheme. (Recall that a full triangulated subcategory $\mathcal{N} \subset \text{Perf}–X$ is called admissible if the inclusion functor has right and left adjoint functors and, hence, $\mathcal{N}$ is a semi-orthogonal summand of $\text{Perf}–X$.) Such admissible subcategories are also smooth and proper, and they give natural examples of smooth and proper noncommutative schemes, which will be called geometric noncommutative schemes.

One can consider a 2-category of smooth and proper noncommutative schemes $\text{NSch}_{\text{sm}}^{pr}$ over a field $k$. Objects of $\text{NSch}_{\text{sm}}^{pr}$ are DG categories $\mathcal{A}$ of the form $\text{Perf}–\mathcal{E}$, where $\mathcal{E}$ is a smooth and proper DG algebra; 1-morphisms are quasi-functors $\mathcal{T}$, i.e. DG functors with inverting quasi-equivalences; and 2-morphisms are morphisms of quasi-functors. The 2-category $\text{NSch}_{\text{sm}}^{pr}$ has a full 2-subcategory of geometric smooth and proper noncommutative schemes $\text{GNSch}$. Evidently, $\text{GNSch}$ contains all smooth and projective commutative schemes. Moreover, a Toën’s theorem [Toe] says us that quasi-functors from $\text{Perf}–X$ to $\text{Perf}–Y$ one-to-one correspond to perfect complexes on the product, i.e. $\text{Perf}–(X \times Y)$ is the category of morphisms between $X$ and $Y$ in $\text{NSch}_{\text{sm}}^{pr}$.
The world of smooth and proper noncommutative schemes is plentiful and multiform in sense of different constructions and operations. For instance, it contains all $\text{Perf} - \Lambda$ for all finite dimensional algebras $\Lambda$ of finite global dimension. Besides, for any two noncommutative schemes $\mathcal{A}$ and $\mathcal{B}$ every perfect $\mathcal{B}^{\circ} - \mathcal{A}$-bimodule $S$ produces a new noncommutative scheme $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$, which we call a gluing of $\mathcal{A}$ and $\mathcal{B}$ via $S$. The resulting DG category $\mathcal{C}$ is also smooth and proper (Definition 3.5 and Section 3.3). Its homotopy category has a semi-orthogonal decomposition of the form $H^0(\mathcal{C}) = \langle H^0(\mathcal{A}), H^0(\mathcal{B}) \rangle$. Of course, this procedure can be iterated and it allows to reproduce new and new noncommutative schemes. If we glue commutative schemes like $\text{Perf} - X$ and $\text{Perf} - Y$, then the result is not a commutative scheme almost always, except for a few special examples (Examples 3.9 and 3.10).

The main purpose of this paper is to show that the world of all geometric noncommutative schemes is closed under operation of gluing. More precisely, we prove that for any smooth and projective $X$ and $Y$ the gluing $\text{Perf} - X \oplus \text{Perf} - Y$ of two DG categories of the form $\text{Perf} - X$ and $\text{Perf} - Y$ via any perfect bimodule $S$ is a geometric noncommutative scheme, i.e. it is quasi-equivalent to an admissible full DG subcategory in $\text{Perf} - V$ for some smooth projective scheme $V$ (Theorem 4.11). This result implies that the subcategory of smooth proper geometric noncommutative schemes is closed under gluing via any bimodules (Theorem 4.15). One of the main tool of the proof is Proposition 4.6 which allows us to obtain any strict perfect complex on a scheme as a derived direct image of a line bundle with respect to a smooth morphism.

These theorems have useful applications. Using results of [KL] we obtain that for any proper scheme $Y$ over a field of characteristic 0 there is a full embedding of $\text{Perf} - Y$ to $\text{Perf} - V$, where $V$ is smooth and projective (Corollary 4.16). In Section 5 we show that for any finite dimensional algebra $\Lambda$ for which the semisimple part $S = \Lambda/\mathfrak{R}$ is separable over base filed $k$, there are a smooth projective scheme $V$ and a perfect complex $\mathcal{E}$ on $X$ such that $\text{End}(\mathcal{E}) = \Lambda$ and $\text{Hom}(\mathcal{E}, \mathcal{E}[l]) = 0$ when $l \neq 0$ (Theorem 5.3). As a consequence of this theorem we obtain that for any finite dimensional algebra $\Lambda$ over $k$ with separable semisimple part $S = \Lambda/\mathfrak{R}$ there is a smooth projective scheme $V$ such that the DG category $\text{Perf} - \Lambda$ is quasi-equivalent to a full DG subcategory of $\text{Perf} - V$. Moreover, if $\Lambda$ has a finite global dimension, then $\text{Perf} - \Lambda$ is an admissible in $\text{Perf} - V$ (Corollary 5.4). Note that over a perfect field all algebras a separable.

In Section 5.2 we give an alternative and more useful procedure of constructing a smooth projective scheme that admits a full exceptional collection and contains as a subcollection an exceptional collection given in advance. More precisely, for any DG category $\mathcal{A}$, for which the homotopy category $H^0(\mathcal{A})$ has a full exceptional collection, we give an explicit construction of a smooth projective scheme $X$ and an exceptional collection of line bundles $\sigma = (L_1, \ldots, L_n)$ in $\text{Perf} - X$ such that the DG subcategory $\mathcal{N} \subset \text{Perf} - X$, generated by $\sigma$, is quasi-equivalent to $\mathcal{A}$. Moreover, by construction $X$ is rational and has a full exceptional collection (Theorem 5.8).
In the last section we illustrate this theorem considering the case of noncommutative projective planes, in sense of noncommutative deformations of the usual projective plane, which have been introduced and described by M. Artin, J. Tate, and M. Van den Bergh in [ATV].

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1. Preliminaries on triangulated categories, generators, and semi-orthogonal decompositions

1.1. Generators in triangulated categories. In this section we discuss different notions of generators in triangulated categories. Let $\mathcal{T}$ be a triangulated category and $S$ be a set of objects.

**Definition 1.1.** We say that a set of objects $S \subset \text{Ob} \mathcal{T}$ generate the triangulated category $\mathcal{T}$ if $\mathcal{T}$ coincides with the smallest strictly full triangulated subcategory of $\mathcal{T}$ which contains $S$. (Strictly full means it is full and closed under isomorphisms).

Such notion of generating of a triangulated category is very rigid, because a triangulated subcategory that is generated by a set of objects is not necessary idempotent complete. Much more useful notion of a generating of a triangulated category is a notion of a set of classical generators.

**Definition 1.2.** We say that a set of objects $S \subset \text{Ob} \mathcal{T}$ forms a set of classical generators for $\mathcal{T}$ if the category $\mathcal{T}$ coincides with the smallest triangulated subcategory of $\mathcal{T}$ which contains $S$ and is closed under taking direct summands. When $S$ consists of a one object we obtain a notion of a classical generator.

If a classical generator $X$ generates the whole category for a finite number of steps, then it called a strong generator. More precisely, let $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{T}$ be two full subcategories. Define a product

$$\mathcal{I}_1 \ast \mathcal{I}_2 = \left\{ \text{the full subcategory, consisting on all objects } Y \text{ of the form } Y_1 \to Y \to Y_2, Y_i \in \mathcal{I}_i \right\}.$$

Let $\langle \mathcal{I} \rangle$ be the smallest full subcategory such that $\mathcal{I} \subset \langle \mathcal{I} \rangle$ and that is closed under shift, finite direct sums, and direct summands. We call $\langle \mathcal{I} \rangle$ the envelope of $\mathcal{I}$.

We put $\mathcal{I}_1 \diamond \mathcal{I}_2 := \langle \mathcal{I}_1 \ast \mathcal{I}_2 \rangle$ and we define by induction $\langle \mathcal{I} \rangle_k = \langle \mathcal{I} \rangle_{k-1} \diamond \langle \mathcal{I} \rangle$. If $\mathcal{I}$ consists of an object $X$ we denote $\langle \mathcal{I} \rangle$ as $\langle X \rangle_1$ and put by induction $\langle X \rangle_k = \langle X \rangle_{k-1} \diamond \langle X \rangle_1$.

**Definition 1.3.** An object $X$ is called a strong generator if $\langle X \rangle_n = \mathcal{T}$ for some $n \in \mathbb{N}$.

**Remark 1.4.** If $X \in \mathcal{T}$ is a classical generator, then $\mathcal{T} = \bigcup_{i=1}^\infty \langle X \rangle_i$. It is also easy to see that if $\mathcal{T}$ has a strong generator, then any classical generator is strong as well.

Following to [Rou] we define the dimension of a triangulated category.
Definition 1.5. The dimension of a triangulated category $\mathcal{T}$, denoted by $\dim \mathcal{T}$, is the minimal integer $d \geq 0$ such that there is $X \in \mathcal{T}$ with $\langle X \rangle_{d+1} = \mathcal{T}$.

Let now $\mathcal{T}$ be a triangulated category which admits arbitrary small coproducts (direct sums). Such category is called cocomplete.

Definition 1.6. Let $\mathcal{T}$ be a cocomplete triangulated category. An object $X \in \mathcal{T}$ is called compact in $\mathcal{T}$ if $\Hom(X,-)$ commutes with arbitrary small coproducts, i.e. for any set of objects $\{Y_i\} \subset \mathcal{T}$ the canonical map $\bigoplus_i \Hom(X,Y_i) \to \Hom(X,\bigoplus_i Y_i)$ is an isomorphism.

Compact objects in $\mathcal{T}$ form a triangulate subcategory $\mathcal{T}^c \subset \mathcal{T}$.

Definition 1.7. Let $\mathcal{T}$ be a cocomplete triangulated category. A set $S \subset \text{Ob} \mathcal{T}^c$ is called a set of compact generators if any object $Y \in \mathcal{T}$ for which $\Hom(X,Y[n]) = 0$ for all $X \in S$ and all $n \in \mathbb{Z}$ is a zero object.

Remark 1.8. Since $\mathcal{T}$ is cocomplete, it can be proved that the property of $S \subset \text{Ob} \mathcal{T}^c$ to be a set of compact generators is equivalent to the following property: the category $\mathcal{T}$ coincides with the smallest full triangulated subcategory containing $S$ and closed under small coproducts [Ne1].

Remark 1.9. Definition of compact generators is closely related to the definition of classical generators. Assume that a cocomplete triangulated category $\mathcal{T}$ is compactly generated by the set of compact objects $\mathcal{T}^c$. In this case a set $S \subset \mathcal{T}^c$ is a set of classical generators of the subcategory of compact objects $\mathcal{T}^c$ [Ne1].

Let $\mathcal{T}$ be a cocomplete triangulated category and let $X \in \mathcal{T}^c$ be a compact object. If on each step we add not only finite sums but also all arbitrary direct sums then we can define full subcategories $\langle X \rangle_k \subset \mathcal{T}$. The following proposition is proved in [BVE, 2.2.4].

Proposition 1.10. If $X$ is a compact object in a cocomplete triangulated category $\mathcal{T}$, then $\langle X \rangle_k \cap \mathcal{T}^c = \langle X \rangle_k$.

In particular, if $\langle X \rangle_k = \mathcal{T}$ for some $k$, then $\langle X \rangle_k = \mathcal{T}^c$ and $X$ is a strong generator of $\mathcal{T}^c$.

1.2. Semi-orthogonal decompositions. Let $\mathcal{T}$ be a $k$–linear triangulated category, where $k$ is a base field. Recall some definitions and facts concerning admissible subcategories and semi-orthogonal decompositions (see [BK1, BO]). Let $\mathcal{N} \subset \mathcal{T}$ be a full triangulated subcategory. The right orthogonal to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{T}$ consisting of all objects $X$ such that $\Hom(Y,X) = 0$ for any $Y \in \mathcal{N}$. The left orthogonal $\perp \mathcal{N}$ is defined analogously. The orthogonals are also triangulated subcategories.
Definition 1.11. Let $I : \mathcal{N} \hookrightarrow \mathcal{T}$ be full embedding of triangulated categories. We say that $\mathcal{N}$ is right admissible (respectively left admissible) if there is a right (respectively left) adjoint functor $Q : \mathcal{T} \to \mathcal{N}$. The subcategory $\mathcal{N}$ will be called admissible if it is right and left admissible.

Remark 1.12. The subcategory $\mathcal{N}$ is right admissible if and only if for each object $Z \in \mathcal{T}$ there is an exact triangle $Y \to Z \to X$, with $Y \in \mathcal{N}$, $X \in \mathcal{N}^\perp$.

Let $\mathcal{N}$ be a full triangulated subcategory in a triangulated category $\mathcal{T}$. If $\mathcal{N}$ is right (respectively left) admissible, then the quotient category $\mathcal{T} / \mathcal{N}$ is equivalent to $\mathcal{N}^\perp$ (respectively $\mathcal{N}^\perp$).

Conversely, if the quotient functor $Q : \mathcal{T} \to \mathcal{T} / \mathcal{N}$ has a left (respectively right) adjoint, then $\mathcal{T} / \mathcal{N}$ is equivalent to $\mathcal{N}^\perp$ (respectively $\mathcal{N}^\perp$).

Definition 1.13. A semi-orthogonal decomposition of a triangulate category $\mathcal{T}$ is a sequence of full triangulated subcategories $\mathcal{N}_1, \ldots, \mathcal{N}_n$ in $\mathcal{T}$ such that there is an increasing filtration $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_n = \mathcal{T}$ by left admissible subcategories for which the left orthogonals $\mathcal{T}_i \mathcal{N}_i$ coincides with $\mathcal{N}_i$. In particular, $\mathcal{N}_i \cong \mathcal{T}_i / \mathcal{T}_{i-1}$. We write $\mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$.

If we have a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$, then the inclusion functors induce an isomorphism of the Grothendieck groups

$$K_0(\mathcal{N}_1) \oplus K_0(\mathcal{N}_2) \oplus \cdots \oplus K_0(\mathcal{N}_n) \cong K_0(\mathcal{T}).$$

It is more convenient to consider so called enhanced triangulated categories, i.e. triangulated categories that are homotopy categories of pretriangulated DG categories (see Section 2.3). An enhancement of a triangulated category $\mathcal{T}$ induces an enhancement for any full triangulated subcategory $\mathcal{N} \subset \mathcal{T}$. Using enhancement of a triangulated category $\mathcal{T}$ we can define K–theory spectrum $K(\mathcal{T})$ of $\mathcal{T}$ (see [Ke2]). It also gives us an additive invariant (see, for example, [Ke2 5.1]), i.e for any semi-orthogonal decomposition we have an isomorphism

$$K_*(\mathcal{N}_1) \oplus K_*(\mathcal{N}_2) \oplus \cdots \oplus K_*(\mathcal{N}_n) \cong K_*(\mathcal{T}).$$

1.3. Exceptional, w-exceptional, and semi-exceptional collections. The existence of a semi-orthogonal decomposition on a triangulated category $\mathcal{T}$ clarifies the structure of $\mathcal{T}$. In the best scenario, one can hope that $\mathcal{T}$ has a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle$ in which each $\mathcal{N}_p$ is as simple as possible, i.e. is equivalent to the bounded derived category of finite-dimensional vector spaces.

Definition 1.14. An object $E$ of a $k$–linear triangulated category $\mathcal{T}$ is called exceptional if $\text{Hom}(E, E[l]) = 0$ when $l \neq 0$, and $\text{Hom}(E, E) = k$. An exceptional collection in $\mathcal{T}$ is a sequence of exceptional objects $\sigma = (E_1, \ldots, E_n)$ satisfying the semi-orthogonality condition $\text{Hom}(E_i, E_j[l]) = 0$ for all $l$ when $i > j$. 
If a triangulated category $\mathcal{T}$ has an exceptional collection $\sigma = (E_1, \ldots, E_n)$ that generates the whole of $\mathcal{T}$, then this collection is called full. In this case $\mathcal{T}$ has a semi-orthogonal decomposition with $N_p = \langle E_p \rangle$. Since $E_p$ is exceptional, each of these categories is equivalent to the bounded derived category of finite dimensional $k$-vector spaces. In this case we write $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$.

**Definition 1.15.** An exceptional collection $\sigma = (E_1, \ldots, E_n)$ is called strong if, in addition, $\text{Hom}(E_i, E_j[l]) = 0$ for all $i$ and $j$ when $l \neq 0$.

The best known example of an exceptional collection is the sequence of invertible sheaves $(\mathcal{O}_{\mathbb{P}^n}, \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$ on the projective space $\mathbb{P}^n$. This exceptional collection is full and strong.

When the field $k$ is not algebraically closed it is reasonable to weaken the notions of an exceptional object and an exceptional collection.

**Definition 1.16.** An object $E$ of a $k$–linear triangulated category $\mathcal{T}$ is called $w$-exceptional (weak exceptional) if $\text{Hom}(E, E[l]) = 0$ when $l \neq 0$, and $\text{Hom}(E, E) = D$, where $D$ is a finite dimensional division algebra over $k$. It is called semi-exceptional if $\text{Hom}(E, E[l]) = 0$ when $l \neq 0$ and $\text{Hom}(E, E) = S$, where $S$ is a finite dimensional semisimple algebra over $k$.

It is evident that exceptional and semi-exceptional objects are stable under base field change while $w$-exceptional objects are not.

A $w$-exceptional (semi-exceptional) collection in $\mathcal{T}$ is a sequence of $w$-exceptional (semisimple) objects $(E_1, \ldots, E_n)$ with orthogonality conditions $\text{Hom}(E_i, E_j[l]) = 0$ for all $l$ when $i > j$.

**Example 1.17.** Let $k$ be a field and $D$ be a central simple algebra over $k$. Consider a Severi-Brauer variety $SB(D)$. There is a full semi-exceptional collection $(S_0, S_1, \ldots, S_n)$ on $SB(D)$ such that $S_0 = \mathcal{O}_{SB}$ and $\text{End}(S_i) \cong D^\otimes i$, where $n + 1$ is the order of the class $D$ in the Brauer group of $k$. Since each $D^\otimes i$ is a matrix algebra over a central division algebra $D_i$, there is a $w$-exceptional collection $(E_0, E_1, \ldots, E_n)$ such that $\text{End}(E_i) \cong D_i$ (see [Ber] for a proof). In this situation $S_i$ are isomorphic to $E_i^{\otimes k_i}$ for some integer $k_i$. These collections are also strong.

2. Preliminaries on differential graded categories

2.1. Differential graded categories. Our main references for differential graded (DG) categories are [Ke1], [Drin], [Toe]. Here we only recall some points and introduce notation. Let $k$ be an arbitrary field. All categories, DG categories, functors, DG functors and etc. are assumed to be $k$–linear.

A differential graded or DG category is a $k$–linear category $\mathcal{A}$ whose morphism spaces $\text{Hom}(X, Y)$ are complexes of $k$-vector spaces (DG $k$–modules), so that for every $X, Y, Z \in \text{Ob} \mathcal{C}$ the composition $\text{Hom}(Y, Z) \otimes \text{Hom}(X, Y) \to \text{Hom}(X, Z)$ is the morphism of DG $k$–modules. The identity morphism $1_X \in \text{Hom}(X, X)$ is closed of degree zero.
Using the supercommutativity isomorphism $U \otimes V \simeq V \otimes U$ in the category of DG $k$–modules one defines for every DG category $\mathcal{A}$ the opposite DG category $\mathcal{A}^\circ$ with $\text{Ob}\, \mathcal{A}^\circ = \text{Ob}\, \mathcal{A}$ and $\text{Hom}_{\mathcal{A}^\circ}(X,Y) = \text{Hom}_{\mathcal{A}}(Y,X)$.

For a DG category $\mathcal{A}$ we denote by $H^0(\mathcal{A})$ its homotopy category. The homotopy category $H^0(\mathcal{A})$ has the same objects as the DG category $\mathcal{A}$ and its morphisms are defined by taking the 0-th cohomology $H^0(\text{Hom}_\mathcal{A}(X,Y))$ of the complex $\text{Hom}_\mathcal{A}(X,Y)$.

As usual, a DG functor $F : \mathcal{A} \to \mathcal{B}$ is given by a map $F : \text{Ob}\, \mathcal{A} \to \text{Ob}\, \mathcal{B}$ and by morphisms of DG $k$–modules $F_{X,Y} : \text{Hom}_\mathcal{A}(X,Y) \to \text{Hom}_\mathcal{B}(FX,FY)$, $X,Y \in \text{Ob}(\mathcal{A})$ compatible with the composition and the units.

A DG functor $F : \mathcal{A} \to \mathcal{B}$ is called a quasi-equivalence if $F_{X,Y}$ is a quasi-isomorphism for all pairs of objects $X,Y$ of $\mathcal{A}$ and the induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is an equivalence. DG categories $\mathcal{A}$ and $\mathcal{B}$ are called quasi-equivalent if there exist DG categories $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and a chain of quasi-equivalences $\mathcal{A} \xleftarrow{\sim} \mathcal{C}_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{C}_n \xrightarrow{\sim} \mathcal{B}$.

### 2.2. Differential graded modules.

Given a small DG category $\mathcal{A}$ we define a right DG $\mathcal{A}$–module as a DG functor $M : \mathcal{A}^{\text{op}} \to \text{Mod}\, k$, where $\text{Mod}\, k$ is the DG category of DG $k$–modules. We denote by $\text{Mod}\, \mathcal{A}$ the DG category of right DG $\mathcal{A}$–modules.

Each object $Y$ of $\mathcal{A}$ produces the right module represented by $Y$

$$h^Y(-) := \text{Hom}_\mathcal{A}(-,Y)$$

which is called a representable DG module. This gives the Yoneda DG functor $h^* : \mathcal{A} \to \text{Mod}\, \mathcal{A}$ that is full and faithful.

The DG $\mathcal{A}$–module is called free if it is isomorphic to a direct sum of DG modules of the form $h^Y[n]$, where $Y \in \mathcal{A}$, $n \in \mathbb{Z}$. A DG $\mathcal{A}$–module $P$ is called semi-free if it has a filtration $0 = \Phi_0 \subset \Phi_1 \subset \ldots = P$ such that each quotient $\Phi_{i+1}/\Phi_i$ is free. The full DG subcategory of semi-free DG modules is denoted by $\mathcal{IF}_\mathcal{A}$. We denote by $\mathcal{IF}_f\mathcal{A} \subset \mathcal{IF}_\mathcal{A}$ the full DG subcategory of finitely generated semi-free DG modules, i.e. $\Phi_n = P$ for some $n$ and $\Phi_{i+1}/\Phi_i$ is a finite direct sum of DG modules of the form $h^Y[n]$.

For every DG $\mathcal{A}$–module $M$ there is a quasi-isomorphism $pM \to M$ such that $pM$ is a semi-free DG $\mathcal{A}$–module (see [Kel] 3.1, [Hin] 2.2, [Dri] 13.2).

Denote by $\mathcal{Ac}\mathcal{A}$ the full DG subcategory of $\text{Mod}\, \mathcal{A}$ consisting of all acyclic DG modules, i.e. such DG modules $M$ for which the complex $M(X)$ is acyclic for all $X \in \mathcal{A}$. The homotopy category of DG modules $H^0(\text{Mod}\, \mathcal{A})$ has a natural structure of a triangulated category and the homotopy subcategory of acyclic complexes $H^0(\mathcal{Ac}\mathcal{A})$ forms a full triangulated subcategory in it.
The derived category \( D(\mathcal{A}) \) is defined as the Verdier quotient

\[
D(\mathcal{A}) := \mathcal{H}^0(\text{Mod} - \mathcal{A})/\mathcal{H}^0(\mathcal{A} - \mathcal{A}).
\]

It is also natural to consider the category of h-projective DG modules. We call a DG \( \mathcal{A} \)-module \( P \) h-projective (homotopically projective) if

\[
\text{Hom}_{\mathcal{H}^0(\text{Mod} - \mathcal{A})}(P, N) = 0
\]

for every acyclic DG module \( N \) (by duality, we can define h-injective DG modules). Let \( \mathcal{P}(\mathcal{A}) \subset \text{Mod} - \mathcal{A} \) denote the full DG subcategory of h-projective objects. It can be easily checked that a semi-free DG-module is h-projective and the natural embedding \( \mathcal{F} - \mathcal{A} \hookrightarrow \mathcal{P}(\mathcal{A}) \) is a quasi-equivalence. Moreover, the canonical DG functors \( \mathcal{F} - \mathcal{A} \hookrightarrow \mathcal{P}(\mathcal{A}) \hookrightarrow \text{Mod} - \mathcal{A} \) induce equivalences \( \mathcal{H}^0(\mathcal{F} - \mathcal{A}) \sim \mathcal{H}^0(\mathcal{P}(\mathcal{A})) \sim D(\mathcal{A}) \) of the triangulated categories.

Let \( F : \mathcal{A} \to \mathcal{B} \) be a DG functor between small DG categories. It induces the restriction DG functor

\[
F_* : \text{Mod} - \mathcal{B} \longrightarrow \text{Mod} - \mathcal{A}
\]

which sends a DG \( \mathcal{B} \)-module \( N \) to \( N \circ F \).

The restriction functor \( F_* \) has left and right adjoint functors \( F^*, F^! \) that are defined as follows

\[
F^* M(Y) = M \otimes_{\mathcal{A}} F_\ast Y, \quad F^! M(Y) = \text{Hom}(F_\ast Y, M), \quad \text{where} \quad Y \in \mathcal{B} \text{ and } M \in \text{Mod} - \mathcal{A}.
\]

The DG functor \( F^* \) is called the induction functor and it is an extension of \( F \) on the category of DG modules, i.e there is an isomorphism of DG functors \( F^* h_{\mathcal{A}}^* \cong h_{\mathcal{B}}^* F \).

The DG functor \( F_* \) preserves acyclic DG modules and induces a derived functor \( F_* : D(\mathcal{B}) \to D(\mathcal{A}) \). Existing of h-projective and h-injective resolutions allows us to define derived functors \( L F^* \) and \( R F^! \) from \( D(\mathcal{A}) \) to \( D(\mathcal{B}) \).

More general, let \( T \) be an \( \mathcal{A} \)-\( \mathcal{B} \)-bimodule that is, by definition, a DG-module over \( \mathcal{A}^{op} \otimes \mathcal{B} \). For each DG \( \mathcal{A} \)-module \( M \) we obtain a DG \( \mathcal{B} \)-module \( M \otimes_{\mathcal{A}} T \). The DG functor \( (-) \otimes_{\mathcal{A}} T : \text{Mod} - \mathcal{A} \to \text{Mod} - \mathcal{B} \) admits a right adjoint \( \text{Hom}_{\mathcal{A}}(T, -) \). These functors do not respect quasi-isomorphisms in general, but they form a Quillen adjunction and the derived functors \( L \) \( \otimes_{\mathcal{A}} T \) and \( R \text{Hom}_{\mathcal{A}}(T, -) \) form an adjoint pair of functors between derived categories \( D(\mathcal{A}) \) and \( D(\mathcal{B}) \).

2.3. Pretriangulated DG categories, categories of perfect DG modules, and enhancements. For any DG category \( \mathcal{A} \) there exist a DG category \( \mathcal{A}^{pre-tr} \) that is called pretriangulated hull and canonical fully faithful DG functor \( \mathcal{A} \hookrightarrow \mathcal{A}^{pre-tr} \). The idea of the definition of \( \mathcal{A}^{pre-tr} \) is to formally add to \( \mathcal{A} \) all shifts, all cones, cones of morphisms between cones and etc. The objects of this DG category are ‘one-sided twisted complexes’ (see [BK1]). There is a canonical fully faithful DG functor (the Yoneda embedding) \( \mathcal{A}^{pre-tr} \to \text{Mod} - \mathcal{A} \), and under this embedding \( \mathcal{A}^{pre-tr} \) is
DG-equivalent to the DG category of finitely generated semi-free DG modules $\mathcal{F}_{fg} - \mathcal{A}$. We will not make a difference between the DG categories $\mathcal{A}^{\text{pre-tr}}$ and $\mathcal{F}_{fg} - \mathcal{A}$.

**Definition 2.1.** A DG category $\mathcal{A}$ is called pretriangulated if the canonical DG functor $\mathcal{A} \to \mathcal{A}^{\text{pre-tr}}$ is a quasi-equivalence.

**Remark 2.2.** It is equivalent to require that the homotopy category $H^0(\mathcal{A})$ is a triangulated as a subcategory of $H^0(\text{Mod-}\mathcal{A})$.

The DG category $\mathcal{A}^{\text{pre-tr}}$ is always pretriangulated, so $H^0(\mathcal{A}^{\text{pre-tr}})$ is a triangulated category. We denote $\mathcal{T}(\mathcal{A}) := H^0(\mathcal{A}^{\text{pre-tr}})$. Notice that a quasi-functor $F : \mathcal{A} \to \mathcal{B}$ defines an exact functor $\mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{B})$ between triangulated categories.

With any small DG category $\mathcal{A}$ we can also associate another DG category $\text{Perf-}\mathcal{A}$ that is called the DG category of perfect DG modules. This category is even more important than $\mathcal{A}^{\text{pre-tr}}$.

**Definition 2.3.** A DG category of perfect DG modules $\text{Perf-}\mathcal{A}$ is the full DG subcategory of $\mathcal{F}-\mathcal{A}$ consisting of all DG modules which are homotopy equivalent to a direct summand of a finitely generated semi-free DG module.

Thus, the DG category $\text{Perf-}\mathcal{A}$ is pretriangulated and contains $\mathcal{F}_{fg} - \mathcal{A} \cong \mathcal{A}^{\text{pre-tr}}$. Denote by $\text{Perf-}\mathcal{A}$ the homotopy category $H^0(\text{Perf-}\mathcal{A})$. The triangulated category $\text{Perf-}\mathcal{A}$ can be obtain from the triangulated category $\mathcal{T}(\mathcal{A})$ as its idempotent completion (Karubian envelope).

**Proposition 2.4.** For any small DG category $\mathcal{A}$ the set of representable objects $\{h^Y\}_{Y \in \mathcal{A}}$ forms a set of compact generators of $\mathcal{D}(\mathcal{A})$ and the subcategory of compact objects $\mathcal{D}(\mathcal{A})^c$ coincides with the subcategory of perfect DG modules $\text{Perf-}\mathcal{A}$.

**Remark 2.5.** If $\mathcal{A}$ is a small pretriangulated DG category and $H^0(\mathcal{A})$ is idempotent complete, then the natural Yoneda DG functor $h : \mathcal{A} \to \text{Perf-}\mathcal{A}$ is a quasi-equivalence.

It is well-known that the categories $\mathcal{D}(\mathcal{A})$ and $\text{Perf-}\mathcal{A}$ are invariant under quasi-equivalences of DG categories.

**Proposition 2.6.** If a DG functor $F : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then the functors

$$F^* : \mathcal{F}_{fg} - \mathcal{A} \to \mathcal{F}_{fg} - \mathcal{B}, \quad F^* : \text{Perf-}\mathcal{A} \to \text{Perf-}\mathcal{B}, \quad F^* : \mathcal{F} - \mathcal{A} \to \mathcal{F} - \mathcal{B}$$

are quasi-equivalences too.

Furthermore, we have the following proposition that is essentially equal to Lemma 4.2 in [Ke1] (see also [LO] Prop. 1.15] and proof there).
Proposition 2.7. [Ke1] Let $F : \mathcal{A} \to \mathcal{B}$ be a full embedding of DG categories and let $F^* : \mathcal{I} \mathcal{F}^- \mathcal{A} \to \mathcal{I} \mathcal{F}^- \mathcal{B}$ (resp. $F^* : \text{Perf}^- \mathcal{A} \to \text{Perf}^- \mathcal{B}$) be the extension DG functor. Then the induced homotopy functor $F^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ (resp. $F^* : \text{Perf}^- \mathcal{A} \to \text{Perf}^- \mathcal{B}$) is fully faithful. If, in addition, the category $\text{Perf}^- \mathcal{B}$ is classically generated by $\text{Ob} \mathcal{A}$, then $F^*$ is an equivalence.

Remark 2.8. The first statement holds for the functor $F^* : \mathcal{I} \mathcal{F}^- \mathcal{A} \to \mathcal{I} \mathcal{F}^- \mathcal{B}$ too. The second also holds if we ask that the category $\mathcal{T} \mathcal{r} \mathcal{B}$ is generated by $\text{Ob} \mathcal{A}$ (not classically).

Definition 2.9. Let $\mathcal{T}$ be a triangulated category. An enhancement of $\mathcal{T}$ is a pair $(\mathcal{A}, \varepsilon)$, where $\mathcal{A}$ is a pretriangulated DG category and $\varepsilon : \mathcal{H}^0(\mathcal{A}) \sim \mathcal{T}$ is an exact equivalence.

2.4. Quasi-functors. Let $k$ be a field. Denote by $\mathcal{D} \mathcal{G} \mathcal{c} \mathcal{a}t_k$ the category of small DG $k$–linear categories. It is known that it admits a structure of cofibrantly generated model category whose weak equivalences are the quasi-equivalences (see [Tab]). This implies that the localization $\mathcal{H}qe$ of $\mathcal{D} \mathcal{G} \mathcal{c} \mathcal{a}t_k$ with respect to the quasi-equivalences has small Hom-sets. This also gives that a morphism from $\mathcal{A}$ to $\mathcal{B}$ in the localization can be represented as $\mathcal{A} \to \mathcal{A}_{cof} \to \mathcal{B}$, where $\mathcal{A} \to \mathcal{A}_{cof}$ is a cofibrant replacement. It is not easy to compute the morphism sets in the localization category $\mathcal{H}qe$ using a cofibrant replacement. On the other hand, they can be described in term of quasi-functors.

Consider two small DG categories $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{T}$ be a $\mathcal{A}$-$\mathcal{B}$–bimodule. It defines a derived tensor functor $L_\mathcal{T} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ between derived categories of DG modules over $\mathcal{A}$ and $\mathcal{B}$.

Definition 2.10. An $\mathcal{A}$-$\mathcal{B}$–bimodule $\mathcal{T}$ is called a quasi-functor from $\mathcal{A}$ to $\mathcal{B}$ if the tensor functor $L_\mathcal{T} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ takes every representable $\mathcal{A}$–module to an object which is isomorphic to a representable $\mathcal{B}$–module.

Denote by $\mathcal{R} \mathcal{e} \mathcal{p}(\mathcal{A}, \mathcal{B})$ the full subcategory of the derived category $\mathcal{D}(\mathcal{A}^{op} \otimes \mathcal{B})$ of $\mathcal{A}$-$\mathcal{B}$–bimodules consisting of all quasi-functors. In other words a quasi-functor is represented by a DG functor $\mathcal{A} \to \mathcal{M} \mathcal{o}d$–$\mathcal{B}$ whose essential image consists of quasi-representable DG $\mathcal{B}$–modules (“quasi-representable” means quasi-isomorphic to a representable DG module). Since the category of quasi-representable DG $\mathcal{B}$–modules is equivalent to $\mathcal{H}^0(\mathcal{B})$ a quasi-functor $\mathcal{T} \in \mathcal{R} \mathcal{e} \mathcal{p}(\mathcal{A}, \mathcal{B})$ defines a functor $\mathcal{H}^0(\mathcal{T}) : \mathcal{H}^0(\mathcal{A}) \to \mathcal{H}^0(\mathcal{B})$. It is now known that quasi-representable functors form morphisms between DG categories in the localization category $\mathcal{H}qe$.

Theorem 2.11. [Toe] The morphisms from $\mathcal{A}$ to $\mathcal{B}$ in the localization $\mathcal{H}qe$ of $\mathcal{D} \mathcal{G} \mathcal{c} \mathcal{a}t_k$ with respect to the quasi-equivalences are in natural bijection with the isomorphism classes of $\mathcal{R} \mathcal{e} \mathcal{p}(\mathcal{A}, \mathcal{B})$.

Due to this theorem any morphism from $\mathcal{A}$ to $\mathcal{B}$ in the localization category $\mathcal{H}qe$ will be called a quasi-functor.
Let \( F : \mathcal{A} \to \mathcal{B} \) be a quasi-functor. It can be realized as a roof \( \mathcal{A} \xleftarrow{a} \mathcal{A}' \xrightarrow{F'} \mathcal{B} \), where \( a \) and \( F' \) are DG functors and \( a \) is also a quasi-equivalence. For instance we can take a cofibrant replacement \( \mathcal{A}_{cof} \) as \( \mathcal{A}' \). The quasi-functor \( F \) induces the functors

\[
LF^* = LF'^* \circ a_* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \quad \text{and} \quad RF_* := F'_* \circ L a^* : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A}).
\]

If now we consider the quasi-functor \( F \) as an \( \mathcal{A} \mathcal{B} \)-bimodule \( T \), then there are isomorphisms of functors

\[
LF^* \cong - \otimes_{\mathcal{A}} T : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \quad \text{and} \quad RF_* \cong R \text{Hom}(T, -) : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A}).
\]

The standard tensor product \( \otimes \) on the category \( \mathcal{D} \text{cat}_k \) induces a tensor product \( \overset{L}{\otimes} \) on the localization \( \mathcal{H}q_{\mathcal{E}} \). It is proved in [Toe] that the monoidal category \((\mathcal{H}q_{\mathcal{E}}, \overset{L}{\otimes})\) has internal Hom-functor \( \overset{R}{\text{Hom}} \). In particular, there is a quasi-equivalence

\[
\overset{R}{\text{Hom}}(\mathcal{A} \otimes \mathcal{B}, C) \cong \overset{R}{\text{Hom}}(\mathcal{A}, \overset{R}{\text{Hom}}(\mathcal{B}, C)).
\]

**Theorem 2.12.** [Toe] For any DG categories \( \mathcal{A} \) and \( \mathcal{B} \) the DG category \( \overset{R}{\text{Hom}}(\mathcal{A}, \mathcal{B}) \) is quasi-equivalent to the full DG subcategory \( \mathcal{R} \text{ep}(\mathcal{A}, \mathcal{B}) \subset \mathcal{R} \mathcal{T} (\mathcal{A} \otimes \mathcal{B}) \) consisting of all objects of \( \mathcal{R} \text{ep}(\mathcal{A}, \mathcal{B}) \).

Thus, there are equivalences \( \mathcal{H}^0(\overset{R}{\text{Hom}}(\mathcal{A}, \mathcal{B})) \cong \mathcal{H}^0(\mathcal{R} \text{ep}(\mathcal{A}, \mathcal{B})) \cong \mathcal{R} \text{ep}(\mathcal{A}, \mathcal{B}) \).

### 3. Commutative and noncommutative schemes

#### 3.1. Derived categories of quasi-coherent sheaves and noncommutative schemes

In this paper we will consider separate noetherian schemes over an arbitrary field \( k \). Let \( X \) be such a scheme. The abelian category \( \text{Qcoh} X \) of quasi-coherent sheaves \( \text{Qcoh} X \) is a Grothendieck category and has enough injectives.

Denote by \( \mathcal{Com} \mathcal{C} X \) the DG category of unbounded complexes of quasi-coherent sheaves on \( X \). This category has enough h-injective complexes (see, e.g. [KaSh]). Denote by \( \mathcal{T}(X) \) the full DG subcategory of h-injective complexes. This DG category gives us a natural DG enhancement for the unbounded derived category of quasi-coherent sheaves, because \( \mathcal{H}^0(\mathcal{T}(X)) \cong \mathcal{D}(\text{Qcoh} X) \). Another natural enhancement for \( \mathcal{D}(\text{Qcoh} X) \) is coming from the definition of the derived category and DG version of Verdier localization [Drin]. Consider the full DG subcategory \( \mathcal{Ac} \mathcal{C} X \subset \mathcal{Com} \mathcal{C} X \) of all acyclic complexes. We can obtain the quotient DG derived category \( \mathcal{Com} \mathcal{C} X/\mathcal{Ac} \mathcal{C} X \). Of course, \( \mathcal{T}(X) \) and \( \mathcal{Com} \mathcal{C} X/\mathcal{Ac} \mathcal{C} X \) are naturally quasi-equivalent enhancements.

There is another enhancement of \( \mathcal{D}(\text{Qcoh} X) \) that is very useful when we work with pullback and tensor product functors. It comes from h-flat complexes. Recall that an (unbounded) complex \( P \) of quasi-coherent sheaves on \( X \) is called h-flat if \( \text{Tot}^\oplus (P \otimes_{\mathcal{O}_X} C) \) is acyclic for any acyclic \( C \in \mathcal{Ac} \mathcal{C} X \). Denote by \( \mathcal{Flat} X \subset \mathcal{Com} \mathcal{C} X \) the full DG subcategory of h-flat complexes. It was
shown in [AJL, Prop. 1.1] that there are enough h-flat complexes in \( \text{Com} - X \) for any separated quasi-compact scheme. Hence the DG quotient category \( \text{Flat} / \mathcal{A}_{\text{cf}} - X \), where \( \mathcal{A}_{\text{cf}} - X \) is the DG subcategory of acyclic h-flat complexes, is an enhancement of \( \mathcal{D} (\text{Qcoh} X) \) (see [KL, 3.10]).

It is easy to see that for any morphism of schemes \( f : X \to Y \) the pullback \( f^* \), acting componentwise on complexes, sends h-flat complexes to h-flat complexes and h-flat acyclic complexes to h-flat acyclic complexes. It is also true that the tensor product of an h-flat acyclic complex with any complex is acyclic (see [Spa]). Thus, for any morphism of schemes \( f : X \to Y \) we obtain a DG functor (not only quasi-functor)

\[
f^* : \text{Flat} / \mathcal{A}_{\text{cf}} - Y \to \text{Flat} / \mathcal{A}_{\text{cf}} - X,
\]

which induces the derived inverse image functor \( \mathbb{L} f^* \) on the derived categories of quasi-coherent sheaves. Similarly, we have a DG tensor functor \( \left( - \right) \otimes \mathbb{P}^\vee \) from \( \text{Flat} / \mathcal{A}_{\text{cf}} - X \) to itself.

Thus, we have three different DG categories \( \mathcal{I} (X) \), \( \text{Com} - X / \mathcal{A}_{\text{c}} - X \), and \( \text{Flat} - X / \mathcal{A}_{\text{cf}} - X \), which are natural quasi-equivalent enhancements for \( \mathcal{D} (\text{Qcoh} X) \). No reason to make a difference between them, but sometimes one of them is more favorable because some quasi-functors can be realized as usual DG functors. In this paper we work with the DG category \( \text{Flat} - X / \mathcal{A}_{\text{cf}} - X \), which will be denoted by \( \mathcal{D} (\text{Qcoh} X) \), because pullbacks and tensor products are DG functors on them.

For any morphism of scheme \( f : X \to Y \) we also have a DG functor \( f_* \) from \( \mathcal{I} (X) \) to \( \text{Com} - Y / \mathcal{A}_{\text{c}} - Y \) acting by componentwise on h-injective complexes. This DG functor induces a quasi-functor that we will denote by the same letter

\[
f_* : \mathcal{D} (\text{Qcoh} X) \to \mathcal{D} (\text{Qcoh} Y).
\]

Recall now the important notion of a perfect complex on a scheme \( X \), which was introduced in [SGA6]. A perfect complex is a complex of sheaves which locally is quasi-isomorphic to a bounded complex of locally free sheaves of finite type (a good reference is [TT]).

**Definition 3.1.** Denote by \( \mathcal{P}_{\text{ef}} - X \) the full DG subcategory of \( \mathcal{D} (\text{Qcoh} X) \) consisting of all perfect complexes.

The triangulated category \( \mathcal{P}_{\text{ef}} - X = \mathcal{H}^0 (\mathcal{P}_{\text{ef}} - X) \) is a full subcategory of \( \mathcal{D} (\text{Qcoh} X) \). Amnon Neeman in [Ne2] showed that the triangulated category \( \mathcal{D} (\text{Qcoh} X) \) is compactly generated and \( \mathcal{P}_{\text{ef}} - X \) is nothing other than the subcategory of compact object in \( \mathcal{D} (\text{Qcoh} X) \). In [Ne2] this assertion is proved for any quasi-compact and separated scheme, in [BVB] this fact is generalized on \( \mathcal{D}_{\text{Qcoh}} (X) \) for all quasi-compact and quasi-separated schemes.

For any morphism of schemes \( f : X \to Y \) the DG functor \( f^* \) induced a DG functor

\[
f^* : \mathcal{P}_{\text{ef}} - Y \to \mathcal{P}_{\text{ef}} - X.
\]
Under some conditions on the morphism \( f : X \to Y \), the induced quasi-functor \( f_* \) from \( \mathcal{D}(\text{Qcoh} \, X) \) to \( \mathcal{D}(\text{Qcoh} \, Y) \) send \( \text{Perf} \, X \) to \( \text{Perf} \, Y \) (see \([\text{TT} \, 2.5.4]\) and \([\text{SGA} 6, \text{III}]\)). Thus, for noetherian schemes \( X \) and \( Y \) if \( f \) is proper of finite type and has locally finite Tor-dimension we obtain the quasi-functor

\[
f_* : \text{Perf} \, X \to \text{Perf} \, Y.
\]

Note that it holds for any morphism between smooth and proper schemes.

In \([\text{Ne}2, \text{BVB}]\) it was proved that the category \( \text{Perf} \, X \) admits a classical generator \( E \) and, hence, \( E \) is a compact generator of the whole \( \mathcal{D}(\text{Qcoh} \, X) \). Let us take such a generator \( E \in \text{Perf} \, X \).

**Statement 3.2.** \([\text{BVB} \, 3.1.8]\) The DG category \( \mathcal{D}(\text{Qcoh} \, X) \) is quasi-equivalent to \( \mathcal{S}_{\mathbb{F}} \, E \) and \( \text{Perf} \, X \) is quasi-equivalent to \( \text{Perf} \, E \), where \( E \) is a DG algebra with bounded cohomology.

This fact allows us to suggest a definition of a (derived) noncommutative scheme over \( k \).

**Definition 3.3.** A (derived) noncommutative scheme over a field \( k \) is a \( k \)-linear DG category of the form \( \text{Perf} \, E \), where \( E \) is a cohomologically bounded DG algebra over \( k \). The derived category \( \mathcal{D}(E) \) will be called the derived category of quasi-coherent sheaves on this noncommutative scheme.

For a noetherian scheme we can consider the abelian category of coherent sheaves \( \text{coh} \, X \). Denote by \( \mathcal{D}^b(\text{coh} \, X) \) the bounded derived categories of coherent sheaves on \( X \). Since \( X \) is noetherian the natural functor \( \mathcal{D}^b(\text{coh} \, X) \to \mathcal{D}(\text{Qcoh} \, X) \) is fully faithful and realizes an equivalence of \( \mathcal{D}^b(\text{coh} \, X) \) with the full subcategory \( \mathcal{D}^b(\text{Qcoh} \, X) \) consisting of all cohomologically bounded complexes with coherent cohomology (see \([\text{SGA} 6, \text{II} \, 2.2.2]\)). By this reason, when we consider \( \mathcal{D}^b(\text{coh} \, X) \) as a subcategory of \( \mathcal{D}(\text{Qcoh} \, X) \) we will identify it with the full subcategory \( \mathcal{D}^b(\text{Qcoh} \, X) \) consisting of all isomorphic objects. The enhancement \( \mathcal{D}(\text{Qcoh} \, X) \) induces an enhancement of \( \mathcal{D}^b(\text{coh} \, X) \) that we denote by \( \mathcal{D}^b(\text{coh} \, X) \).

### 3.2. Gluing of DG categories

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two small DG categories and let \( S \) be a \( \mathcal{B}-\mathcal{A} \)-bimodule, i.e. a DG \( \mathcal{B}^\circ \otimes \mathcal{A} \)-module. We now construct so called an upper triangular DG category corresponding to the data \((\mathcal{A}, \mathcal{B}, S)\).

**Definition 3.4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two small DG categories and let \( S \) be a \( \mathcal{B}-\mathcal{A} \)-bimodule. The upper triangular DG category \( \mathcal{C} = \mathcal{A} \downarrow \mathcal{B} \) is defined as follows:

1) \( \text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{A}) \sqcup \text{Ob}(\mathcal{B}), \)
with evident composition law coming from DG categories \( \mathcal{A}, \mathcal{B} \) and bimodule structure on \( S \).

The upper triangular DG category \( \mathcal{C} = \mathcal{A} \mathcal{B} \) is not pretriangulated even if the components \( \mathcal{A} \) and \( \mathcal{B} \) are pretriangulated. To make this operation is well defined on the class of pretriangulated categories we introduce so called a gluing of pretriangulated categories.

**Definition 3.5.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be two small pretriangulated DG categories and let \( S \) be a \( \mathcal{B} - \mathcal{A} \) -bimodule. The gluing \( \mathcal{A} \mathcal{B} \) of DG categories \( \mathcal{A} \) and \( \mathcal{B} \) via \( S \) is defined as a pretriangulated hull of \( \mathcal{A} \mathcal{B} \), i.e. \( \mathcal{A} \mathcal{B} = (\mathcal{A} \mathcal{B})^{\text{pre-tr}} \).

**Remark 3.6.** The gluing can be defined for any DG categories not only for pretriangulated (see [KL]). The resulting DG category is not necessary pretriangulated too. However, we prefer to restrict ourself to consideration of the pretriangulated case, because the definition above is more convenient for our purposes.

The natural fully faithful DG inclusions \( a : \mathcal{A} \hookrightarrow \mathcal{A} \mathcal{B} \) and \( b : \mathcal{B} \hookrightarrow \mathcal{A} \mathcal{B} \) induce the fully faithful DG functors \( a^{*} : \mathcal{A} \hookrightarrow \mathcal{A} \mathcal{B} \) and \( b^{*} : \mathcal{B} \hookrightarrow \mathcal{A} \mathcal{B} \).

It is easy to see that the restriction functor \( b_{*} : \text{Mod}-(\mathcal{A} \mathcal{B}) \rightarrow \text{Mod} \mathcal{B} \) sends semi-free DG modules to semi-free and we obtain a DG functor \( \text{II}_{f_{g}}(\mathcal{A} \mathcal{B}) \rightarrow \text{II}_{f_{g}} \mathcal{B} \). By assumption \( \mathcal{B} \) is pretriangulated, and we know that a pretriangulated hull is DG-equivalent to the DG category of finitely generated semi-free DG modules. Thus we obtain a quasi-functor \( b_{*} : \mathcal{A} \mathcal{B} \rightarrow \mathcal{B} \) that is a right adjoint to \( b^{*} \). These quasi-functors induce exact functors

\[
a^{*} : \mathcal{H}^{0}(\mathcal{A}) \rightarrow \mathcal{H}^{0}(\mathcal{A} \mathcal{B}), \quad b^{*} : \mathcal{H}^{0}(\mathcal{B}) \rightarrow \mathcal{H}^{0}(\mathcal{A} \mathcal{B}), \quad b_{*} : \mathcal{H}^{0}(\mathcal{A} \mathcal{B}) \rightarrow \mathcal{H}^{0}(\mathcal{B})
\]

between triangulate categories such that \( a^{*}, b^{*} \) are fully faithful, and \( b_{*} \) is right adjoint to \( b^{*} \). Therefore, there is a semi-orthogonal decomposition

\[
\mathcal{H}^{0}(\mathcal{A} \mathcal{B}) = \langle \mathcal{N}, \mathcal{H}^{0}(\mathcal{B}) \rangle
\]

with some triangulated subcategory \( \mathcal{N} \). It is evident that \( \mathcal{H}^{0}(\mathcal{A}) \) is a full subcategory of \( \mathcal{N} \) with respect to the functor \( a^{*} \). The subcategory \( \mathcal{N} \) is left admissible and we have a quotient functor \( \mathcal{H}^{0}(\mathcal{A} \mathcal{B}) \rightarrow \mathcal{N} \) that sends \( \mathcal{H}^{0}(\mathcal{B}) \) to zero. Since the category \( \mathcal{H}^{0}(\mathcal{A} \mathcal{B}) \) is generated by the union of objects \( a^{*}h^{X}_{\mathcal{A}} \) and \( b^{*}h^{Y}_{\mathcal{B}} \) we obtain that the subcategory \( \mathcal{N} \) is generated by objects \( a^{*}h^{X}_{\mathcal{A}} \). Hence \( \mathcal{N} \) coincides with the triangulated subcategory \( \mathcal{H}^{0}(\mathcal{A}) \subset \mathcal{N} \), because it also contains all these objects. Thus, we prove the following proposition.

\[
\text{Hom}_{\mathcal{F}}(X, Y) = \begin{cases} 
\text{Hom}_{\mathcal{A}}(X, Y), & \text{when } X, Y \in \mathcal{A} \\
\text{Hom}_{\mathcal{B}}(X, Y), & \text{when } X, Y \in \mathcal{B} \\
S(Y, X), & \text{when } X \in \mathcal{A}, Y \in \mathcal{B} \\
0, & \text{when } X \in \mathcal{B}, Y \in \mathcal{A}
\end{cases}
\]
Proposition 3.7. Let DG category $\mathcal{C}$ be a gluing $\mathcal{A} \oplus \mathcal{B}$. Then the DG functors $a^* : \mathcal{A} \to \mathcal{C}$ and $b^* : \mathcal{B} \to \mathcal{C}$ induce a semi-orthogonal decomposition for the triangulated category $\mathcal{H}^0(\mathcal{C})$ of the form $\mathcal{H}^0(\mathcal{C}) = \langle \mathcal{H}^0(\mathcal{A}), \mathcal{H}^0(\mathcal{B}) \rangle$.

On the other hand, we can show that any enhancement of a triangulated category with a semi-orthogonal decomposition can be obtained as a gluing of enhancements of the summands.

Proposition 3.8. Let $\mathcal{C}$ be a pretriangulated DG category. Suppose that we have a semi-orthogonal decomposition $\mathcal{H}^0(\mathcal{C}) = \langle \mathcal{A}, \mathcal{B} \rangle$. Then the DG category $\mathcal{C}$ is quasi-equivalent to a gluing $\mathcal{A} \oplus \mathcal{B}$, where $\mathcal{A}, \mathcal{B} \subset \mathcal{C}$ are full DG subcategories with the same objects as $\mathcal{A}$ and $\mathcal{B}$, respectively, and the $\mathcal{B} \sim \mathcal{A}$–bimodule is given by the rule

$$S(Y, X) = \text{Hom}_\mathcal{C}(X, Y), \quad \text{with} \quad X \in \mathcal{A} \text{ and } Y \in \mathcal{B}.$$  

**Proof.** Take full DG subcategories $\mathcal{A} \subset \mathcal{C}$ and $\mathcal{B} \subset \mathcal{C}$ with objects from $\mathcal{A}$ and $\mathcal{B}$, respectively. Consider the $\mathcal{B} \sim \mathcal{A}$–bimodule $S$ defined by rule (3). There is a natural inclusion of the upper triangular DG category $\mathcal{A} \sqsubseteq \mathcal{B}$ to $\mathcal{C}$. Since $\mathcal{C}$ is pretriangulated we obtain a quasi-functor from the pretriangulated hull $\mathcal{A} \oplus \mathcal{B}$ to $\mathcal{C}$.

Since $\mathcal{A}$ and $\mathcal{B}$ are semi-orthogonal, the DG category $\mathcal{A} \sqsubseteq \mathcal{B}$ under the inclusion $\mathcal{A} \sqsubseteq \mathcal{B} \hookrightarrow \mathcal{C}$ is quasi-equivalent to the full DG subcategory of $\mathcal{C}$ on the set of objects $\text{Ob}(\mathcal{A}) \bigsqcup \text{Ob}(\mathcal{B})$. Now the combination of Propositions 2.6, 2.7 and Remark 2.8 gives us that the functor $\mathcal{H}^0(\mathcal{A} \oplus \mathcal{B}) \to \mathcal{H}^0(\mathcal{C})$ is fully faithful. Since the set $\text{Ob}(\mathcal{A}) \bigsqcup \text{Ob}(\mathcal{B})$ generates the category $\mathcal{H}^0(\mathcal{C})$, this functor is an equivalence by Remark 2.8.

Example 3.9. Let $X$ be a noetherian scheme and let $\mathcal{E}$ be a vector bundle on $X$ of rank 2. Consider the projectivization $\mathbb{P}(\mathcal{E}^\vee)$ with projection $p : \mathbb{P}(\mathcal{E}^\vee) \to X$. Denote by $\mathcal{O}(1)$ the antiantautological line bundle on $\mathbb{P}(\mathcal{E}^\vee)$. We know that $R\mathcal{P}a\mathcal{O}(1) \cong \mathcal{E}$ and $p^*$ is fully faithful. It is shown in [Or1] that there is a semi-orthogonal decomposition of the form

$$\text{Perf} - \mathbb{P}(\mathcal{E}^\vee) = \langle p^*\text{Perf} - X, p^*\text{Perf} - X \otimes \mathcal{O}(1) \rangle.$$  

Furthermore, the DG category $\mathbb{P}(\mathcal{E}^\vee)$ is quasi-equivalent to the gluing $\text{Perf} - X \oplus \text{Perf} - X$, where $S_\mathcal{E}$ is a DG bimodule of the form

$$S_\mathcal{E}(B, A) \cong \text{Hom}_{\text{Perf} - X}(A, B \otimes \mathcal{E}), \quad \text{where} \quad A, B \in \text{Perf} - X.$$  

By the same rule the DG categories $\mathcal{D}^b(\text{coh}\mathbb{P}(\mathcal{E}^\vee))$ and $\mathcal{D}(\text{Qcoh}\mathbb{P}(\mathcal{E}^\vee))$ can be obtained as the gluing of $\mathcal{D}^b(\text{coh}X)$ and $\mathcal{D}(\text{Qcoh}X)$ with itself via $S_\mathcal{E}$.

Example 3.10. Let $\pi : \tilde{X} \to X$ be a blowup of a regular scheme $X$ along a closed regular subscheme $Y$ of codimension 2. The functor $\text{L}\pi^*$ is fully faithful. Consider the exceptional divisor $j : E \hookrightarrow \tilde{X}$. The morphism $p : E \to Y$ is the projectivization of the normal bundle to
$Y$ in $X$. The functor $R_{j*}p^*$ is fully faithful as well. The triangulated category $\mathcal{P}erf - \tilde{X}$ has a semi-orthogonal decomposition of the form

$$\mathcal{P}erf - \tilde{X} = \langle L_{\pi^*}\mathcal{P}erf - X, R_{j*}p^*\mathcal{P}erf - Y \rangle.$$  

Furthermore, the DG category $\mathcal{P}erf - \tilde{X}$ is quasi-equivalent to the gluing $\mathcal{P}erf - X \bigoplus\limits_X \mathcal{P}erf - Y$, where $S$ is a DG bimodule of the form

$$S(B, A) \cong \text{Hom}_{\mathcal{P}erf - Y}(i^* A, B), \quad \text{where} \quad A \in \mathcal{P}erf - X, \ B \in \mathcal{P}erf - Y, \ \text{and} \ i : Y \hookrightarrow X.$$  

By the same rule the DG categories $D^b_{\text{coh}} X$ and $D^b_{\text{Qcoh}} X$ can be obtain as gluing via $S$.  

Let $a : \mathcal{A} \to \mathcal{A}'$ and $b : \mathcal{B} \to \mathcal{B}'$ be DG functors between small pretriangulated DG categories. Let $S$ and $S'$ be bimodules, i.e. DG modules over $\mathcal{B} \otimes \mathcal{A}$ and $\mathcal{B}' \otimes \mathcal{A}'$ respectively. Consider the restriction functor on bimodules

$$(b \otimes a)_* : \text{Mod} - (\mathcal{B} \otimes \mathcal{A}) \to \text{Mod} - (\mathcal{B}' \otimes \mathcal{A}).$$

Suppose that we have a map of DG modules $\phi : S \to (b \otimes a)_* S'$. Then it is evident from Definition 3.4 that there are a DG functor

$$a \downarrow \phi b : \mathcal{A} \bigoplus \mathcal{B} \to \mathcal{A}' \bigoplus \mathcal{B}',$$

and an induced DG functor $a \oplus b : \mathcal{A} \bigoplus \mathcal{B} \to \mathcal{A}' \bigoplus \mathcal{B}'$.  

Furthermore, assume that $\phi$ is a quasi-isomorphism. If now the exact functors $a : \mathcal{H}^0(\mathcal{A}) \to \mathcal{H}^0(\mathcal{A}')$ and $b : \mathcal{H}^0(\mathcal{B}) \to \mathcal{H}^0(\mathcal{B}')$ are fully faithful, then

$$a \downarrow \phi b : \mathcal{H}^0(\mathcal{A} \bigoplus \mathcal{B}) \to \mathcal{H}^0(\mathcal{A}' \bigoplus \mathcal{B}'), \quad \text{and} \quad a \oplus b : \mathcal{H}^0(\mathcal{A} \bigoplus \mathcal{B}) \to \mathcal{H}^0(\mathcal{A}' \bigoplus \mathcal{B}')$$

are fully faithful by Theorem 2.7 and Remark 2.8.  

If $a$ and $b$ are quasi-equivalences and $\phi$ is a quasi-isomorphism, then $a \downarrow \phi b$ is a quasi-equivalence and, by Remark 2.8, $a \oplus b$ is quasi-equivalences too since objects of $\mathcal{A} \bigoplus \mathcal{B}$ generate $\mathcal{H}^0(\mathcal{A}' \bigoplus \mathcal{B}')$ in this case.  

This statement is generalized for a class of quasi-functors. Indeed, quasi-functors $a : \mathcal{A} \to \mathcal{A}'$ and $b : \mathcal{B} \to \mathcal{B}'$ induce a quasi-functor $b \otimes a : \mathcal{B} \otimes \mathcal{A} \to \mathcal{B}' \otimes \mathcal{A}'$. The quasi-functor $b \otimes a$ induces a derived functor

$$R(b \otimes a)_* : \mathcal{D}(\mathcal{B} \otimes \mathcal{A}) \to \mathcal{D}(\mathcal{B}' \otimes \mathcal{A})$$

by rule (1). Any quasi-functor $a : \mathcal{C} \to \mathcal{D}$ can be realized as a roof $\mathcal{C} \xrightarrow{i} \mathcal{C}' \to \mathcal{D}$ and any morphism of bimodules $M \to N$ in $\mathcal{D}(\mathcal{C})$ can be represented as a roof of the form $M \xrightarrow{i} M' \to N$. Hence, we obtain the following proposition.
Proposition 3.11. Let \( a : \mathcal{A} \to \mathcal{A}' \) and \( b : \mathcal{B} \to \mathcal{B}' \) be quasi-functors between small DG categories. Let \( S \) and \( S' \) be DG modules over \( \mathcal{B}' \otimes \mathcal{A} \) and \( \mathcal{B} \otimes \mathcal{A}' \) respectively. Assume that there is a morphism \( \phi : S \to \mathcal{R}(b \otimes a)_s S' \) in \( \mathcal{D}(\mathcal{B} \otimes \mathcal{A}) \). Then there are quasi-functors
\[
\underline{a}_b : \mathcal{A} 
\]
and \( \underline{a} \oplus \underline{b} : \mathcal{B} \to \mathcal{A}' \oplus \mathcal{B}' \).

Moreover, suppose that \( \phi \) is a quasi-isomorphism. If \( a : \mathcal{H}^0(\mathcal{A}) \to \mathcal{H}^0(\mathcal{A}') \) and \( b : \mathcal{H}^0(\mathcal{B}) \to \mathcal{H}^0(\mathcal{B}') \) are fully faithful, then
\[
\underline{a}_b : \mathcal{H}^0(\mathcal{A} \otimes \mathcal{B}) \to \mathcal{H}^0(\mathcal{A}' \otimes \mathcal{B}'), \quad \underline{a} \oplus \underline{b} : \mathcal{B} \to \mathcal{A}' \oplus \mathcal{B}'
\]
are fully faithful. If \( a, b \) are quasi-equivalences, then both \( \underline{a}_b \) and \( \underline{a} \oplus \underline{b} \) are quasi-equivalences.

3.3. Regular, smooth, and proper noncommutative schemes. Let \( \mathcal{T} \) be a small \( \mathbf{k} \)-linear triangulated category and let \( \mathcal{A} \) be a small \( \mathbf{k} \)-linear DG category.

Definition 3.12. We say that \( \mathcal{T} \) is regular if it has a strong generator, and we say that \( \mathcal{T} \) is proper if
\[
\bigoplus_{m \in \mathbb{Z}} \text{Hom}(X, Y[m])
\]
is finite dimensional for any two objects \( X, Y \in \mathcal{T} \).

Definition 3.13. We call \( \mathcal{A} \) regular (resp. proper) if the triangulated category \( \text{Perf} - \mathcal{A} \) is regular (resp. proper).

Remark 3.14. In place of \( \text{Perf} - \mathcal{A} \) we can consider \( \mathcal{T}(\mathcal{A}) \). Since \( \text{Perf} - \mathcal{A} \) is the idempotent completion of \( \mathcal{T}(\mathcal{A}) \) the regularity and properness of these categories hold simultaneously.

Remark 3.15. It is easy to see that \( \mathcal{A} \) is proper if and only if \( \bigoplus_i \mathcal{H}^i(\text{Hom}(X, Y)) \) are finite dimensional for all \( X, Y \in \mathcal{A} \). It is evidently necessary due to Yoneda embedding \( \mathcal{A} \subset \text{Perf} - \mathcal{A} \). Since \( \text{Ob} \mathcal{A} \) classically generate \( \text{Perf} - \mathcal{A} \) it is also sufficient.

There is the following theorem that is due to A. Bondal and M. Van den Bergh.

Theorem 3.16. [BVB, Th. 1.3] Let \( \mathcal{T} \) be a regular and proper triangulated category which is idempotent complete (Karoubian). Then any exact functor from \( \mathcal{T} \) to the bounded derived category of finite dimensional vector spaces \( \text{Perf} - \mathbf{k} \) is representable, i.e. it has a form \( \mathcal{h}^Y = \text{Hom}(-, Y) \).

Such a triangulated category is called right saturated in definition of \( \text{BK1, BVB} \). It is proved in \( \text{BK1} \) 2.6] that if \( \mathcal{T} \) is a right saturated triangulated category and it is a full subcategory in a proper triangulate category, then it is right admissible there. By Theorem 3.16 a regular and proper idempotent complete triangulated category is right saturated. Since the opposite category is also regular and proper. It is left saturated as well. Thus, we obtain the following proposition that is essentially due to \( \text{BK1} \).

Proposition 3.17. Let a regular and proper triangulated category \( \mathcal{T} \) that is idempotent complete is a full subcategory in a proper triangulated category \( \mathcal{T}' \). Then \( \mathcal{T} \) is admissible in \( \mathcal{T}' \).
The proof of the Theorem 3.16 works for DG categories without any corrections (see [BVB]). Moreover, the DG version can be deduced from Theorem 3.16.

**Theorem 3.18.** Let $\mathcal{A}$ be a small DG category that is regular and proper. Then a DG module $M$ is perfect if and only if $\dim \bigoplus_i H^i(M(X)) < \infty$ for all $X \in \mathcal{A}$.

**Proof.** If $M$ is perfect, then $\dim \bigoplus_i H^i(M(X)) < \infty$, because $\text{Perf} - \mathcal{A}$ is proper.

Assume now that $\dim \bigoplus_i H^i(M(X)) < \infty$. This implies that $\dim \bigoplus_i H^i(\text{Hom}(P, M)) < \infty$. Therefore, the module $M$ gives the DG functor $\text{Hom}(-, M)$ from $\text{Perf} - \mathcal{A}$ to $\text{Perf} - k$. By Theorem 3.16 the induced functor $\text{Hom}(-, M) : \text{Perf} - \mathcal{A} \to \text{Perf} - k$ is represented by an object $N \in \text{Perf} - \mathcal{A}$ and there is a canonical map $N \to M$. The cone $C$ of this map in $D(\mathcal{A})$ is an object such that $\text{Hom}(X, C) = 0$ for any $X \in \mathcal{A}$. This implies that $C = 0$ because $\text{Ob} \mathcal{A}$ is the set of compact generators for $D(\mathcal{A})$. Thus, $M$ is a perfect complex. □

**Corollary 3.19.** Let $\mathcal{A}$ be a regular and proper pretriangulated DG category for which $H^0(\mathcal{A})$ is idempotent complete. Let $M$ be a DG $\mathcal{A}$–module such that $\dim \bigoplus_i H^i(M(X)) < \infty$ for all $X \in \mathcal{A}$. Then $M$ is quasi-isomorphic to a representable modules $h^Y = \text{Hom}(-, Y)$ for some $Y \in \mathcal{A}$.

**Proof.** It directly follows from the previous theorem and Remark 2.5. □

The property of regularity and properness have a good behavior under taking of semi-orthogonal summands and gluing.

**Proposition 3.20.** Let $\mathcal{T}$ be a $k$–linear triangulated category with a semi-orthogonal decomposition $\mathcal{T} = \langle T_1, T_2 \rangle$. The following properties hold

1) if $\mathcal{T}$ is proper, then $T_i$ are proper;
2) if $\mathcal{T}$ is regular, then $T_i$ are regular;
3) if $T_i$, $i = 1, 2$ are regular, then $\mathcal{T}$ is regular too.

**Proof.** 1) is evident, since any subcategory of a proper category is proper. To prove 2) we should note that there are quotient functors from $\mathcal{T}$ to $T_i$. Now it is evident that the images of a strong generator under these functors are strong generators in $T_i$.

Let $E_i$ be strong generators of $T_i$ such that $\langle E_i \rangle_{n_i} = T_i$. We can take $E = E_1 \oplus E_2$. There are embeddings $\langle E_i \rangle_{n_i} \supset \langle E_i \rangle_{n_i} = T_i$, $i = 1, 2$. By definition of a semi-orthogonal decomposition, for any object $X \in T$ there is an exact triangle of the form $X_2 \to X \to X_1$ with $X_i \in T_i$. This implies that $X \in \langle E \rangle_{n_2} \circ \langle E \rangle_{n_1} = \langle E \rangle_{n_1+n_2}$. Hence $\mathcal{T} = \langle E \rangle_{n_1+n_2}$. This proves 3). □

**Remark 3.21.** The proof implies inequality $\dim \mathcal{T} \leq \dim \mathcal{T}_1 + \dim \mathcal{T}_2 + 1$.

**Proposition 3.22.** Let $\mathcal{A}$ and $\mathcal{B}$ be two small pretriangulated DG categories and let $S$ be a $\mathcal{B}$-$\mathcal{A}$–bimodule. Then the following conditions are equivalent:
(1) the gluing \( \mathcal{A} \oplus \mathcal{B} \) is regular and proper,
(2) \( \mathcal{A} \) and \( \mathcal{B} \) are regular and proper and \( \dim \sum_i H^i(S(Y,X)) < \infty \) for all \( X \in \mathcal{A}, Y \in \mathcal{B} \).

**Proof.** (1) \( \Rightarrow \) (2). Since \( H^0(\mathcal{A} \oplus \mathcal{B}) = \langle H^0(\mathcal{A}), H^0(\mathcal{B}) \rangle \) regularity and properness of \( \mathcal{A} \) and \( \mathcal{B} \) directly follow from Proposition 3.19 1) and 2). Properness of \( \mathcal{A} \oplus \mathcal{B} \) implies that \( \dim \sum_i H^i(S(Y,X)) < \infty \) as well.

(2) \( \Rightarrow \) (1). Regularity of the gluing follows from 3) of Proposition 3.19 and Proposition 3.7. In view of Remark 3.15 properness of \( \mathcal{A} \oplus \mathcal{B} \) directly follows from the properness of \( \mathcal{A} \) and \( \mathcal{B} \) and the finiteness of \( S \).

There is another important property of DG categories that is called smoothness.

**Definition 3.23.** A small \( k \)-linear DG category \( \mathcal{A} \) is called \( k \)-smooth if it is perfect as the module over \( \mathcal{A} \circ \mathcal{A} \).

This property depends on the base field \( k \). For example, a finite inseparable extension \( F \supset k \) is not smooth over \( k \) and it is smooth over itself.

The following statement is proved in [Lum] as Lemmas 3.5. and 3.6.

**Proposition 3.24.** If a small DG category \( \mathcal{A} \) is smooth, then it is regular.

The smoothness is invariant under Morita equivalence [Lum] [LS]. This means that if \( \mathcal{D}(\mathcal{A}) \) and \( \mathcal{D}(\mathcal{B}) \) are equivalent through a functor of the form \( (-) \circ_{\mathcal{A}} T \), where \( T \) is an \( \mathcal{A}-\mathcal{B} \)-bi-module, then \( \mathcal{A} \) is smooth if and only if \( \mathcal{B} \) is smooth.

Since \( \mathcal{A} \circ \mathcal{B} \) and \( \mathcal{A} \oplus \mathcal{B} \) are Morita equivalent we obtain that smoothness of \( \mathcal{A} \circ \mathcal{B} \) and \( \mathcal{A} \oplus \mathcal{B} \) hold simultaneously. Further, we can compare smoothness of a gluing with smoothness of summands. There is the following statement.

**Theorem 3.25.** [LS 3.24] Let \( \mathcal{A} \) and \( \mathcal{B} \) be two small pretriangulated DG categories over a field \( k \) and let \( S \) be a \( \mathcal{B} \circ \mathcal{A} \)-module. Then the following conditions are equivalent:

1. the gluing \( \mathcal{A} \circ \mathcal{B} \) is smooth;
2. \( \mathcal{A} \) and \( \mathcal{B} \) are smooth and \( S \) is perfect \( \mathcal{B} \circ \mathcal{A} \)-module.

3.4. **Regularity, smoothness, and properness in commutative geometry.** Let us now discuss all these properties of DG categories in context of usual geometry of schemes.

**Proposition 3.26.** Let \( X \) be a proper scheme. Then the category \( \text{Perf}_{X} \) is proper.

**Proof.** Let \( \mathcal{E} \) be a perfect complex. Consider the functor \( R \text{Hom}(\mathcal{E}, -) \) from \( \mathcal{D}(\text{Qcoh} X) \) to itself. Since a perfect complex is locally quasi-isomorphic to a finite complex of vector bundles we obtain that any object \( R \text{Hom}(\mathcal{E}, \mathcal{F}) \) is perfect when \( \mathcal{E} \) and \( \mathcal{F} \) are perfect. Let \( \pi \) be the canonical
morphism of $X$ to the Spec $k$. By [SGA6 III 4.8.1] (see also [TT 2.5.4]) since $X$ is proper the object $R\pi_*E$ is perfect over $k$ when $E$ is perfect. Hence, the complex

$$R\text{Hom}(E, F) \equiv R\pi_*R\text{Hom}(E, F)$$

is a perfect complex of $k$-vector spaces, i.e. $\bigoplus_k \text{Hom}(E, F[k])$ is finite dimensional.

\textbf{Theorem 3.27.} Let $X$ be a separated noetherian scheme over an arbitrary field $k$. Then the following conditions are equivalent:

1. $X$ is regular;
2. $\text{Perf} - X$ is regular, i.e. it has a strong generator.

\textbf{Proof.} At first, note that the affine case $X = \text{Spec} A$ is proved in [Rou, 7.25]. We will use it.

(2) $\Rightarrow$ (1) Take an affine open subset $U \subset X$. Any perfect complex on $U$ is a direct summand of a perfect complex restricted from $X$ ([Ne2, Lemma 2.6]). Hence, the category $\text{Perf} - U$ is strongly generated too. Thus we have reduced to the affine case that is proved in Proposition 7.25 [Rou]. If $\text{Perf} - A$ is strongly generated, then the algebra $A$ has a finite global dimension, i.e. it is regular.

(1) $\Rightarrow$ (2) By [SGA6 II 2.2.7.1] any regular separated noetherian scheme has an ample family of line bundles, i.e. there is a family of line bundles $\{L_\alpha\}$ on $X$ such that for any quasi-coherent sheaf $F$, the evaluation map

$$\bigoplus_{\alpha, n \geq 1} \Gamma(X, F \otimes L_\alpha^{\otimes n}) \otimes L_\alpha^{\otimes -n} \to F$$

is an epimorphism. In particular, for any coherent sheaf $F$ there are an algebraic vector bundle $E$ (i.e. locally free sheaf of finite type) and an epimorphism $E \to F$.

Consider an affine covering $X = \bigcup_{i=1}^m V_i$, where $V_i = \text{Spec} A_i$. Since $X$ is regular all algebras $A_i$ have finite global dimension. This implies that for sufficiently large $n \in \mathbb{Z}$ (greater than maximum of global dimensions of $A_i$) for any quasi-coherent sheaf $F$ there is a global locally free resolution

$$0 \to E_{-n} \to \cdots \to E_0 \to F \to 0.$$

By [EGA3 1.4.12] (see also [TT App. B]) there exists an integer $k \in \mathbb{Z}$ such that for all $p \geq k$ and all quasi-coherent sheaves $G$, one has $\text{Ext}^p(E, G) = H^p(X, E \otimes G) = 0$, where $E$ is locally free. Using a locally free resolution of type (4) for a quasi-coherent sheaf $F$, one has that for sufficient large $N \in \mathbb{Z}$ for all $p \geq N$ and all quasi-coherent sheaves $F, G$, we have $\text{Ext}^p(F, G) = 0$. Thus, the abelian category $\text{Qcoh}(X)$ has a finite global dimension. Let us denote it by $k = \text{gl. dim} \text{Qcoh}(X)$.

Consider the product $X \times_k X$. It is known that the family $\{L_\alpha^r \boxtimes L_\beta^s\}_{r, s \geq 1}$ forms an ample family on $X \times X$ (see [TT 2.1.2.f]), in spite of the scheme $X \times X$ is not necessary regular.

Take the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$. Since $X$ is separated, $\Delta$ is closed and, hence, $\mathcal{O}_\Delta$ is a coherent sheaf. Fix an infinite locally free resolution $E$ of $\mathcal{O}_\Delta$

$$\cdots \to E_{-n} \to \cdots \to E_0 \to \mathcal{O}_\Delta \to 0,$$
where each $E^{-i}$ is a finite direct sum of sheaves of the form $L^\otimes r \boxtimes L^\otimes s$. Take a brutal truncation $\sigma^{\geq -l}E$ for sufficient large $l \gg 0$. This complex on $X \times X$ has only two cohomology sheaves $H^{-k}(\sigma^{\geq -l}E)$ and $H^0(\sigma^{\geq -l}E) = \mathcal{O}_X$.

Take all $L^\otimes r_\alpha$ that appear in $E^{-i}$ for all $0 \leq i \leq l$ and consider their direct sum. Denote it by $S$. We have that $S$ is an algebraic vector bundle and $\sigma^{\geq -l}E \in \langle S \boxtimes S \rangle_{l+1}$.

For any quasi-coherent sheaf $F$, the object $C = \mathcal{R}\text{pr}_{2*}(\text{pr}_1^*(F) \otimes \sigma^{\geq -l}E)$ is a complex on $X$, all cohomology $H^j(C)$ of which are trivial when $j > -l + k$ except $H^0(C)$ that is isomorphic to $F$. Since $l$ is large enough, we obtain that $F$ is a direct summand of $C$. But $C$ belongs to $\langle S \rangle_{l+1}$. Therefore, $F \in \langle S \rangle_{l+1}$ too. Thus, we obtain that $\langle S \rangle_{l+1}$ contains all quasi-coherent sheaves. Now we can apply the following proposition from [Rou].

**Proposition 3.28.** [Rou Prop. 7.22] Let $A$ be an abelian category of finite global dimension $k$. Let $C$ be a complex of objects of $A$. Then, there is a distinguished triangle in $\mathcal{D}(A)$

$$\bigoplus_i D_i \longrightarrow C \longrightarrow \bigoplus_i E_i,$$

where $D_i = \sigma^{\geq ki + 1, \leq k(i + 1) - 1}C$ is a complex with zero terms outside $[ki + 1, \ldots k(i + 1) - 1]$ and $E_i$ is a complex concentrated in degree $ki$.

Using this proposition we obtain that any object of $\mathcal{D}(\text{Qcoh} X)$ belongs to $\langle S \rangle_{k(l+1)}$ where $k = \text{gl.dim} \text{Qcoh}(X)$. Indeed $E_i \in \langle S \rangle_{l+1}$ and $D_i$ as complexes of length $k - 1$ belongs to $\langle S \rangle_{(k-1)(l+1)}$.

Finally, by Proposition [1.10] we have that $\mathcal{P}erf - X = \mathcal{D}(\text{Qcoh}(X))^c = \langle S \rangle_{k(l+1)}$. 

**Corollary 3.29.** Let $X$ be a separated scheme of finite type over a field $k$. Then $X$ is regular and proper if and only if the category of perfect complexes $\mathcal{P}erf^{-} X$ is regular and proper.

**Proof.** If $X$ is proper and regular, then by Proposition 3.26 and Theorem 3.27 the category $\mathcal{P}erf^{-} X$ is proper and regular.

Suppose that $\mathcal{P}erf^{-} X$ is proper and regular. By Theorem 3.27 the scheme $X$ is regular. Let us show that it is also proper. We will proof by contradiction. Assume that $X$ is not proper. By Chow’s Lemma for any separated scheme of finite type $X$ there is a quasi-projective $X'$ with a proper map $f : X' \rightarrow X$. If $X$ is not proper $X'$ is not projective. Consider its closure $\overline{X}' \in \mathbb{P}^N$. Take the complement $Y$ to $X'$ in $\overline{X}'$ and choose a closed point $p \in Y$. There is an irreducible and reduced projective curve $C \subset \overline{X}'$ that contains the point $p$. Denote by $C_0 \subset C$ the intersection of $C$ with $X'$ and by $\tilde{C}$ and $\tilde{C}_0$ the normalizations of $C$ and $C_0$ respectively. Since $p \notin C_0$ the complement $D$ to $\tilde{C}_0$ in $\tilde{C}$ is not empty. The curve $C$ is regular, hence $D$ is a Cartier divisor on $C$. Since $D$ is effective it is ample (see, e.g. [Liun 7.5.5]). This implies that $\tilde{C}_0$ is an affine curve.

Now consider the composition map $g : \tilde{C}_0 \rightarrow X$ and take the complex $\mathcal{R}g_*\mathcal{O}_{\tilde{C}_0}$. Since $g$ is proper
as a composition of proper morphisms the complex \( Rg_*O_{\tilde{C}_0} \) belongs to the bounded category of coherent sheaves. Since \( X \) is regular, the complex \( Rg_*O_{\tilde{C}_0} \) is perfect. Now we have

\[
\text{Hom}_X(O_X, Rg_*O_{\tilde{C}_0}) \cong H^0(\tilde{C}_0, O_{\tilde{C}_0}) \cong A,
\]

where \( \text{Spec } A = \tilde{C}_0 \). The \( k \)-space \( A \) is infinite dimensional over \( k \), because \( \tilde{C}_0 \) is an affine curve. We have a contradiction to the properness of \( \text{Perf} - X \). This proves the proposition. \( \square \)

We recall that a scheme of finite type over a field \( k \) is called \textit{smooth} if the scheme \( \tilde{X} = X \otimes_k \bar{k} \) is regular, where \( \bar{k} \) is the algebraic closure of \( k \).

**Proposition 3.30.** Let \( X \) be a separated scheme of finite type over an arbitrary field. Then \( X \) is smooth and proper if and only if the DG category \( \text{Perf} - X \) is smooth and proper.

**Proof.** The statement for smoothness (without properness) is proved in [Lun, 3.13] for a separated scheme of finite type over a perfect field.

On the other hand, the definition of a smooth scheme and a smooth DG category is invariant under a base field change. Indeed, if \( \text{Perf} - X \) is smooth, then \( \text{Perf} - \tilde{X} \) is smooth and, hence, by [Lun, 3.13] the scheme \( \tilde{X} \) is smooth (regular). But this is exactly smoothness of \( X \) by definition. The properness of \( X \) is proved as in Proposition 3.20.

If now \( X \) is smooth and proper, then properness of the DG category \( \text{Perf} - X \) follows from Proposition 3.20. Since \( \tilde{X} \) is regular we obtain that the DG category \( \text{Perf} - \tilde{X} \) is smooth by [Lun, 3.13]. Finally, we should argue that smoothness of \( \text{Perf} - \tilde{X} \) implies smoothness of \( \text{Perf} - X \). Since \( k \subset \bar{k} \) is faithfully flat, the following property holds: for any DG category \( \mathcal{A} \) and any \( \mathcal{A} \)-module \( M \) if \( M \otimes_k \bar{k} \) is perfect as \( \mathcal{A} \otimes_k \bar{k} \)-module, then \( M \) is also perfect. \( \square \)

4. **Gluing of smooth projective schemes and geometric noncommutative schemes**

4.1. **Geometric noncommutative schemes.** Let \( X \) and \( Y \) be two smooth projective schemes over a field \( k \). Consider DG categories of perfect complexes \( \text{Perf} - X \) and \( \text{Perf} - Y \). Since \( X \) and \( Y \) are smooth these categories are quasi-equivalent to DG categories \( D^b(\text{coh } X) \) and \( D^b(\text{coh } Y) \), respectively. Theorem 3.18 says us that for a regular and proper \( X \) there is a quasi-equivalence

\[
\mathbf{R}\text{Hom}(\text{Perf} - X^\circ, \text{Perf} - k) \cong \text{Perf} - X.
\]

Therefore, applying canonical quasi-equivalence \( \text{(2)} \) we obtain that

\[
\mathbf{R}\text{Hom}(\text{Perf} - Y \otimes_k \text{Perf} - X^\circ, \text{Perf} - k) \cong \mathbf{R}\text{Hom}(\text{Perf} - Y, \mathbf{R}\text{Hom}(\text{Perf} - X^\circ, \text{Perf} - k)) \cong \mathbf{R}\text{Hom}(\text{Perf} - Y, \text{Perf} - X).
\]

Moreover, there is the following theorem that is due to B. Toën.
Theorem 4.1.  Let $X$ and $Y$ be smooth projective schemes over a field $k$. Then there is a canonical isomorphism in $\text{Hg}$

$$\mathcal{R}\text{Hom}(\text{Perf}(-Y, \text{Perf}(-X)) \cong \text{Perf}(-(X \times_k Y)).$$

In particular, the DG category $\text{Perf}(-(X \times_k Y))$ is quasi-equivalent to the DG category of perfect DG modules over $\text{Perf}(-Y) \otimes_k \text{Perf}(-X)$.

This quasi-equivalence can be described precisely. As it was explained in Section 3.1, there are DG functors

$$\text{pr}_1^*: \text{Perf}(-X) \to \text{Perf}(-(X \times Y), \quad \text{and} \quad \text{pr}_2^*: \text{Perf}(-Y) \to \text{Perf}(-(X \times Y))$$

For any perfect complex $E$ on the product $X \times_k Y$ we can define a bimodule $S_E$ by rule

$$S_E(B, A) \cong \text{Hom}_{\text{Perf}(-(X \times Y))}(\text{pr}_1^*A, \text{pr}_2^*B \otimes E), \quad \text{where} \quad A \in \text{Perf}(-X), \ B \in \text{Perf}(-Y).$$

This is exactly the quasi-equivalence between the DG category $\text{Perf}(-(X \times Y))$ and the DG category of perfect $\text{Perf}(-Y) \text{Perf}(-X)$-bimodules, i.e. perfect DG modules over $\text{Perf}(-Y) \otimes_k \text{Perf}(-X)$.

Let $\mathcal{M} \subset \text{Perf}(-X)$ and $\mathcal{N} \subset \text{Perf}(-Y)$ be admissible subcategories, where $X$ and $Y$ are smooth projective schemes over $k$. Consider the induced DG subcategories $\mathcal{M} \subset \text{Perf}(-X)$ and $\mathcal{N} \subset \text{Perf}(-Y)$ and the induced DG functor $F : \mathcal{N} \otimes \mathcal{M} \to \text{Perf}(-(\mathcal{N} \otimes \mathcal{M}))$ that is fully faithful. This DG functor gives the extension quasi-functor

$$F^* : \text{Perf}(-(\mathcal{N} \otimes \mathcal{M})) \longrightarrow \text{Perf}(-(X \times Y))$$

that is fully faithful on the homotopy categories by Proposition 2.7. In more details, for any pair of admissible subcategories $\mathcal{M} \subset \text{Perf}(-X)$ and $\mathcal{N} \subset \text{Perf}(-Y)$ we can define a full triangulated subcategory $\mathcal{M} \boxtimes \mathcal{N}$ of the category $\text{Perf}(-(X \times Y))$ as the minimal triangulated subcategory of $\text{Perf}(-(X \times Y))$ closed under taking direct summands and containing all objects of the form $\text{pr}_1^*M \boxtimes \text{pr}_2^*N$ with $M \in \mathcal{M}$ and $N \in \mathcal{N}$. Denote by $\mathcal{M} \boxtimes \mathcal{N} \subset \text{Perf}(-(X \times Y))$ the induced enhancement of $\mathcal{M} \boxtimes \mathcal{N}$.

It is easy to see that $\text{Perf}(-(\mathcal{N} \otimes \mathcal{M}))$ is quasi-equivalent to $\mathcal{M} \boxtimes \mathcal{N}$ because $\text{Perf}(-(\mathcal{N} \otimes \mathcal{M}))$ and $\mathcal{M} \boxtimes \mathcal{N}$ are classically generated by $\text{Ob}(\mathcal{N} \otimes \mathcal{M})$.

As admissible subcategories in DG categories of perfect complexes on smooth and proper schemes the DG categories $\mathcal{M}$ and $\mathcal{N}$ are smooth and proper (see Theorem 3.25). By Theorem 3.18 there is a quasi-equivalence

$$\mathcal{R}\text{Hom}(\mathcal{M}, \text{Perf}(-k)) \cong \mathcal{M}.$$ 

Therefore, applying canonical quasi-equivalence (2) we obtain that

$$\mathcal{R}\text{Hom}(\mathcal{N} \otimes_k \mathcal{M}, \text{Perf}(-k)) \cong \mathcal{R}\text{Hom}(\mathcal{N}, \mathcal{R}\text{Hom}(\mathcal{M}, \text{Perf}(-k))) \cong \mathcal{R}\text{Hom}(\mathcal{N}, \mathcal{M}).$$

Let us summarize what we have.
Proposition 4.2. Let $X$ and $Y$ be two smooth projective schemes and $\mathcal{M} \subset \text{Perf} \rightarrow X$ and $\mathcal{N} \subset \text{Perf} \rightarrow Y$ be full DG subcategories such that the subcategories $\mathcal{M} = \mathcal{H}^0(\mathcal{M})$ and $\mathcal{N} = \mathcal{H}^0(\mathcal{N})$ are admissible in $\text{Perf} \rightarrow X$ and $\text{Perf} \rightarrow Y$, respectively. In this case there are quasi-equivalences of DG categories $\mathcal{R} \text{Hom}(\mathcal{N}, \mathcal{M}) \cong \text{Perf} \rightarrow (\mathcal{N} \otimes_k \mathcal{M}) \cong \mathcal{M} \boxtimes \mathcal{N} \subset \text{Perf} \rightarrow (X \times_k Y)$,

where $\mathcal{M} \boxtimes \mathcal{N}$ is a full DG subcategory of $\text{Perf} \rightarrow (X \times Y)$ that is classically generated by objects of the form $\text{pr}_1^* M \otimes \text{pr}_2^* N$ with $M \in \mathcal{M}$ and $N \in \mathcal{N}$.

We are interested in smooth (or regular) and proper noncommutative scheme $\text{Perf} \rightarrow \mathcal{E}$. Smooth (or regular) and proper geometric noncommutative schemes naturally appear as induced enhancements of admissible subcategories $\mathcal{N} \subset \text{Perf} \rightarrow X$ for some smooth (or regular) and projective scheme $X$.

Definition 4.3. A noncommutative scheme $\text{Perf} \rightarrow \mathcal{E}$ (see Definition 3.3) will be called a geometric noncommutative scheme if there are a smooth and projective scheme $X$ and an admissible subcategory $\mathcal{N} \subset \text{Perf} \rightarrow X$ such that $\text{Perf} \rightarrow \mathcal{E}$ is quasi-equivalent to the correspondent enhancement $\mathcal{N} \subset \text{Perf} \rightarrow X$ of $\mathcal{N}$.

We can consider a 2-category of smooth and proper noncommutative schemes $\text{NSch}^{\text{pr}}_{\text{sm}}$ over a field $k$. Objects of $\text{NSch}^{\text{pr}}_{\text{sm}}$ are DG categories $\mathcal{A}$ of the form $\text{Perf} \rightarrow \mathcal{E}$, where $\mathcal{E}$ is a smooth and proper DG algebra; 1-morphisms are quasi-functors $T$; 2-morphisms are morphisms of quasi-functors, i.e. morphisms in $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$. The 2-category $\text{NSch}^{\text{pr}}_{\text{sm}}$ has a natural full 2-subcategory of geometric noncommutative schemes $\text{GNSch}$. Evidently, $\text{GNSch}$ contains all smooth and proper commutative schemes with $\text{Perf} \rightarrow (X \times Y)$ as category of morphisms between $X$ and $Y$. The natural question that arises is following.

Question 4.4. Is there a smooth and proper noncommutative scheme that is not geometric?

The first attempt to find a such noncommutative scheme is to glue geometric noncommutative schemes via a bimodule. Another way is to consider a finite dimensional $k$–algebra $\Lambda$ of finite global dimension and take the DG category $\text{Perf} \rightarrow \Lambda$.

The main aim of this paper is to show that these two approaches do not lead us to new noncommutative schemes. We show that the world of geometric noncommutative schemes is closed under gluing via any perfect bimodule. More precisely, consider smooth and proper geometric noncommutative schemes $\text{Perf} \rightarrow \mathcal{E}_1$ and $\text{Perf} \rightarrow \mathcal{E}_2$ such that $\text{Perf} \rightarrow \mathcal{E}_i$ is quasi-equivalent to $\mathcal{N}_i \subset \text{Perf} \rightarrow X_i$, where $X_i$ are smooth and projective and $\mathcal{N}_i = \mathcal{H}^0(\mathcal{N}_i)$ are admissible in $\text{Perf} \rightarrow X_i$, respectively. After that we take a gluing $\text{Perf} \rightarrow \mathcal{E}_1 \oplus \text{Perf} \rightarrow \mathcal{E}_2$ via a perfect bimodule $S$ and show that the resulting noncommutative scheme is geometric too (see Theorem 4.15).
Remark 4.5. We work with smooth and projective schemes. On the other hand, over a field of characteristic 0 the category of perfect complexes of any smooth and proper scheme can be realized as an admissible subcategory in a smooth and projective scheme. Indeed, by Chow’s Lemma for a proper scheme $X$ there is a proper birational morphism $f: Y \to X$ from a projective scheme $Y$. Now applying Hironaka’s resolution of a birational map $X \dashrightarrow Y$, we can find a proper scheme $Z$ with birational maps to $X$ and $Y$ such that the morphism $\pi: Z \to X$ is a sequence of blowups with regular centers. This implies that $Z$ is smooth and it is also projective because there is a proper birational morphism to the projective scheme $Y$. Finally, the inverse image functor $L\pi^*$ gives a full embedding of $\text{Perf} \to \text{Perf}$. Note that over complex numbers $\mathbb{C}$ we can apply a result of Moishezon asserting that any smooth and proper algebraic space over $\mathbb{C}$ becomes a projective variety after some blowups along smooth centers.

We also show that for any finite dimensional $k$-algebra $\Lambda$ such that its semisimple part $S = \Lambda/\mathfrak{m}$ is separable over $k$ there are a smooth and projective scheme $X$ and a perfect complex $E$ that $R\text{Hom}(E, E) \cong \Lambda$. This implies that for a such finite dimensional algebra $\Lambda$ the DG category $\text{Perf} \to \text{Perf}$ is quasi-equivalent to a full DG subcategory of $\text{Perf} \to \text{Perf}$ and in case of finite global dimension the smooth and proper noncommutative scheme $\text{Perf} \to \text{Perf}$ is geometric (Theorem 5.3).

4.2. Perfect complexes as direct images of line bundles. Let $X$ be a scheme and $E$ be a strict perfect complex, i.e. a bounded complex of algebraic vector bundles (locally free sheaves of finite type). In this section we show that any such strict perfect complex $E$ can be realized as a direct image of a line bundle with respect to a smooth morphism $Z \to X$.

Proposition 4.6. Let $X$ be a scheme. Let $E = \{E^0 \to \cdots \to E^k\}$ be a bounded complex of algebraic vector bundles on $X$. Then there are a scheme $Z$ with a morphism $f: Z \to X$ and a line bundle $L$ on $Z$ such that

1) $Rf_*L \cong E$ in the derived category $\mathcal{D}(\text{Qcoh} X)$;

2) $Rf_*L^{-1} \cong 0$;

3) the morphism $f: Z \to X$ is a composition of a sequence of maps $Z = X_n \to X_{n-1} \to \cdots \to X_0 = X$, where each $X_{p+1}$ is a projectivization of a vector bundle $\mathcal{F}_p$ over $X_p$.

Proof. At the beginning, we should note that the next construction does not work for the 0-complex and for a complex $E$ that is a line bundle. However, it can be easily improved. In such case we change $E$ to a quasi-isomorphic complex adding an acyclic complex of the form $E \to E$.

Now we will prove the proposition by induction on the length of complex $E$. If $E$ has only one term and it is a vector bundle $\mathcal{E}$ (of rank $> 1$), then we take the line bundle $L = \mathcal{O}(1)$ on the projective bundle $f : \mathbb{P}(\mathcal{E}) \to X$. As result we obtain that $Rf_*L = f_*L \cong \mathcal{E}$ and $Rf_*L^{-1} = 0$.

Assume that $E = \{E^0 \to \cdots \to E^1\}$ is a complex of vector bundles that has only two nontrivial terms in degree 0 and 1. Taking $f_1: X_1 = \mathbb{P}(\mathcal{E}^1) \to X$ and $\mathcal{L}_1 = \mathcal{O}(1)$ we obtain that $Rf_1_*\mathcal{L} \cong \mathcal{E}^1$ as
described above. Now let $X_2 = X_1 \times \mathbb{P}^1$ and $f_2$ is the projection on $X$. Take $L_2 = \text{pr}_1^* L_1 \boxtimes \mathcal{O}(-2)$. It is easy to see that $R\text{pr}_1_* L_2$ has only one nontrivial term $R^1 \text{pr}_1_* L_2$ that is isomorphic to $L_1$. Hence $Rf_{2*}L_2 \cong \mathcal{E}^![-1]$. Thus we obtain a sequence of isomorphisms

\[
\text{Ext}^1_{X_2}(f^* \mathcal{E}^0, L_2) \cong \text{Ext}^1_{X_1}(f^* \mathcal{E}^0, R\text{pr}_1_* L_2) \cong \text{Hom}_{X_1}(f^* \mathcal{E}^0, L_1) \cong \text{Hom}_X(\mathcal{E}^0, Rf_1_* L_1) \cong \\
\text{Hom}_X(\mathcal{E}^0, \mathcal{E}^1).
\]

Under this isomorphism the differential $d$ induces an element $e \in \text{Ext}^1(f^* \mathcal{E}^0, L_2)$. Let us consider the extension

\[
0 \to L_2 \to \mathcal{F} \to f^* \mathcal{E}^0 \to 0
\]

given by the element $e$. Applying the functor $Rf_{2*}$ to this short exact sequence we obtain an exact triangle of the form

\[
Rf_{2*} \mathcal{F} \to \mathcal{E}^0 \xrightarrow{\alpha} Rf_{2*} L_2[1].
\]

By construction, $Rf_{2*} L_2[1] \cong \mathcal{E}^1$ and $\alpha = d$. Therefore, $Rf_{2*} \mathcal{F}$ is isomorphic to the complex $\mathcal{E}$. Finally, we consider $Z = \mathbb{P}(\mathcal{F}^\vee)$ with the natural morphism $f$ to $X$ and $L = \mathcal{O}(1)$. We get that $Rf_* L \cong \mathcal{E}$. Since the rank of $\mathcal{F}$ is bigger than one, it is also evident that $Rf_* L^{-1} = 0$.

The same trick works for any complex of vector bundles

\[
\mathcal{E} = \{ \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \mathcal{E}_k \}.
\]

Indeed, consider the stupid truncation $\sigma^{=1} \mathcal{E}$. By induction we can assume that there is $f_{n-1} : X_{n-1} \to X$ and $L_{n-1}$ on $X_{n-1}$ such that $Rf_{(n-1)*} \cong \sigma^{=1} \mathcal{E} [1]$.

Now repeat the procedure described above. Let $X_n = X_{n-1} \times \mathbb{P}^1$ and $f_n$ be the projection on $X$. Take $L_n = \text{pr}_1^* L_{n-1} \boxtimes \mathcal{O}(-2)$. We have $Rf_{n*} L_n \cong \sigma^{=1} \mathcal{E}$. There is an isomorphisms

\[
\text{Ext}^1(f^* \mathcal{E}^0, L_n) \cong \text{Ext}^1(\mathcal{E}^0, Rf_{n*} L_n) \cong \text{Hom}(\mathcal{E}^0, \sigma^{=1} \mathcal{E}^1[1])
\]

Under this isomorphism the differential $d : \mathcal{E}_0 \to \sigma^{=1} \mathcal{E}^1[1]$ induces an element $e \in \text{Ext}^1(f^* \mathcal{E}^0, L_n)$. Let us consider the extension

\[
0 \to L_n \to \mathcal{F} \to f^* \mathcal{E}^0 \to 0
\]

given by the element $e$. Applying the functor $Rf_{n*}$ to this short sequence we obtain an exact triangle of the form

\[
Rf_{n*} \mathcal{F} \to \mathcal{E}^0 \xrightarrow{\alpha} Rf_{n*} L_n[1].
\]

By construction, $Rf_{n*} L_n[1] \cong \sigma^{=1} \mathcal{E}^1[1]$ and $\alpha = d$. Therefore, $Rf_{n*} \mathcal{F}$ is isomorphic to the complex $\mathcal{E}$. Finally, we consider $Z = \mathbb{P}(\mathcal{F}^\vee)$ with the natural morphism $f$ to $X$ and $L = \mathcal{O}(1)$. We get that $Rf_* L \cong \mathcal{E}$. By construction, the scheme $f : Z \to X$ is a sequence of projective bundles. Moreover, we have $Rf_* L^{-1} = 0$, because rank of $\mathcal{F}$ is bigger than one and $Rf_* L^{-1} = 0$, where $p$ is the projection of $Z = \mathbb{P}(\mathcal{F}^\vee)$ to $X_n$. 

\[\square\]
Remark 4.7. Assume that a quasi-compact and separated scheme $X$ has enough locally free sheaves, i.e. for any quasi-coherent sheaf of finite type $\mathcal{F}$ there is an algebraic vector bundle $\mathcal{E}$ on $X$ and an epimorphism $\mathcal{E} \to \mathcal{F}$. In this case, any perfect complex is quasi-isomorphic to a strict perfect complex (see [TT 2.3.1]) and, hence, Proposition 4.6 can be applied to any perfect complex up to shift in triangulated category. Note that any quasi-projective scheme and any separated regular noetherian scheme have enough locally free sheaves.

4.3. Blowups and gluing of smooth projective schemes. Let $X_1$ and $X_2$ be two smooth irreducible projective schemes. Let $\mathcal{E}'$ be a perfect complex on the product $X_1 \times X_2$. Since $X_i$ are projective any perfect complex on $X_1 \times X_2$ is globally (not only locally) quasi-isomorphic to a strictly perfect complex, i.e. a bounded complex of locally free sheaves of finite type (see, e.g. [TT 2.3.1]). A strictly perfect complex will be also called a bounded complex of vector bundles.

Thus $\mathcal{E} \in \text{Perf-}(X_1 \times X_2)$ is a bounded complex of vector bundles. By Proposition 4.6 there is a scheme $Z$ with a morphism $f : Z \to X_1 \times X_2$ and a line bundle $\mathcal{L}$ on $Z$ such that $\mathcal{R}f_* \mathcal{L} \cong \mathcal{E}'$.

Let us fix such $Z$ and $f$. By construction of $Z$, since $X_1$ and $X_2$ are smooth projective the scheme $Z$ is also smooth and projective and the morphism $f$ is smooth.

Denote by $q_1$ and $q_2$ the canonical morphisms from $Z$ to $X_1$ and $X_2$ respectively. Fix very ample line bundles $\mathcal{M}_1$ and $\mathcal{M}_2$ on $X_1$ and $X_2$ respectively. Using Serre’s theorem we can find a very ample line bundle $\mathcal{L}'$ on $Z$ such that the three line bundles $\mathcal{L}_1 = q_1^* \mathcal{M}_1^{-1} \otimes \mathcal{L}'$, $\mathcal{L}_2 = q_2^* \mathcal{M}_2^{-1} \otimes \mathcal{L}'$ and $\mathcal{L}_3 = \mathcal{L}'^{-1} \otimes \mathcal{L}'$ are very ample two.

Denote by $s_1, s_2, s_3$ the closed immersions of $Z$ to projective spaces $\mathbb{P}^{n_1}, \mathbb{P}^{n_2}, \mathbb{P}^{n_3}$ induced by $\mathcal{L}_1, \mathcal{L}_2,$ and $\mathcal{L}_3$ respectively. Product map $i_1 = (q_1, s_1, s_3)$ gives a closed immersion of $Z$ to the projective scheme $P_1 = X_1 \times \mathbb{P}^{n_1} \times \mathbb{P}^{n_3}$. Similarly, we obtain a closed immersion $i_2 = (q_2, s_2, s_3)$ of $Z$ to $P_2 = X_2 \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. It directly follows from construction that there are isomorphisms

$$i_1^* (\mathcal{M}_1 \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(-1)) \cong i_2^* (\mathcal{M}_2 \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(-1)) \cong \mathcal{L},$$

$$i_1^* (\mathcal{M}_1 \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)) \cong i_2^* (\mathcal{M}_2 \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)) \cong \mathcal{L}' \otimes \mathcal{L}_3$$

Consider these two closed immersions $i_1 : Z \to P_1$ and $i_2 : Z \to P_2$. It is known that the gluing $P_1 \bigsqcup_Z P_2$ of $P_1$ and $P_2$ along $Z$ is a scheme ([Sch 3.9] or [Fer 5.4]). It has two irreducible components that meet along $Z$. Denote this gluing by $T$. The scheme $T$ is a pushout (fibred coproduct) in the category of schemes. In our case we also can argue that the scheme $T$ is projective.

Lemma 4.8. The scheme $T = P_1 \bigsqcup_Z P_2$ is projective.

Proof. Consider very ample line bundles $\mathcal{M}_1 \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ and $\mathcal{M}_2 \boxtimes \mathcal{O}(1) \boxtimes \mathcal{O}(1)$ on $P_1$ and $P_2$. Since their restrictions on $Z$ are isomorphic to $\mathcal{L}' \otimes \mathcal{L}_3$ we can glue them to a line bundle $\mathcal{N}$ on $T$. It follows from [DGAG 2.6.2] that $\mathcal{N}$ is ample on $T$ (see also [Fer 6.3]).

Consider a closed immersion of $j : T \hookrightarrow \mathbb{P}^N$ to a projective space $\mathbb{P}^N$. Denote by $j_1, j_2$ the induced closed immersions of $P_1, P_2$ to $\mathbb{P}^N$. Consider the blowup $V'$ of $\mathbb{P}^N$ along $P_1$. Take the
strict transform \( \tilde{P}_2 \) of \( P_2 \) in \( V' \). It is the blowup of \( P_2 \) along the subvariety \( Z \). Denote by \( V \) the blowup of \( V' \) along \( \tilde{P}_2 \). This construction can be illustrated by following diagram (7), where \( h_1 : E_1 \to P_1 \) is the exceptional divisor of the first blowup, \( g : D \to Z \) is the exceptional divisor of the induced blowup \( \tau : \tilde{P}_2 \to P_2 \), \( \tilde{E}_1 \) is the strict transformation of the divisor \( E_1 \) under the second blowup, and \( h_2 : E_2 \to \tilde{P}_2 \) is the exceptional divisor of the second blowup.

(7)

Since we started from smooth projective schemes \( X_1 \) and \( X_2 \), we obtain smooth and projective schemes \( Z, P_1 \) and \( P_2 \). A blowup of a smooth projective scheme along a smooth closed subscheme brings to a smooth and projective scheme (see, e.g. [Liu, Th.8.1.19]). Thus, we obtain

**Lemma 4.9.** All schemes in diagram (7) are projective and smooth.

Now let us analyze morphisms in our diagram (7) and functors induced by them.

**Proposition 4.10.** In the diagram (7) the following properties of morphisms hold:

1) the morphisms \( g, h_1, h_2, p_1, p_2 \) are projectivizations of vector bundles, and the functors \( g^*, h_1^*, h_2^*, p_1^*, p_2^* \) are fully faithful;

2) the morphisms \( \pi', \pi, \tau, \rho \) are blowups along smooth centers, and the exact functors \( L\pi'^* \), \( L\pi^* \), \( L\tau^* \), and \( L\rho^* \) are fully faithful;

3) functors of the form \( R\mathcal{H}om_D(K \otimes g^*(-)) \), \( R\mathcal{H}om_{E_1}(K \otimes h_1^*(-)) \), \( R\mathcal{H}om_{E_2}(K \otimes h_2^*(-)) \), where \( K \) is a line bundle on \( D, E_1, E_2 \), respectively, are fully faithful;

4) the functors \( R\mathcal{H}om_{D_2}, R\mathcal{H}om_{E_1}, R\mathcal{H}om_{E_2} \) have right adjoint \( d_2^0, e_1^0, e_2^0 \) and there are isomorphisms

\[
d_2^0 \cong Ld_2^0(\mathcal{O}(D) \otimes (-)[1][-1], \quad e_1^0 \cong Le_1^0(\mathcal{O}(E_1) \otimes (-)[1][-1], \quad e_2^0 \cong Le_2^0(\mathcal{O}(E_2) \otimes (-)[1][-1].
\]

**Proof.** 1) and 2) follow from the construction and the projection formula, because the derived direct image of the structure sheaf under each of these morphisms is isomorphic to the structure sheaf of a target. 3) is proved in [Or1, 4.2], [Or2, 2.2.7] or [BO, 3.2]. 4) follows from the fact that for any closed immersion \( i \) of locally a complete intersection \( Z \) to \( Y \) the right adjoint \( i^* \) to \( Ri_* \) has the form \( Li^*(\cdot) \otimes \omega_{Z/Y}[-r] \), where \( \omega_{Z/Y} \cong \Lambda^r N_{Z/Y} \) and \( r \) is the codimension (e.g. [Har, III 7.3]).
Theorem 4.11. Let $X_1$ and $X_2$ be smooth irreducible projective schemes and let $\mathcal{E}$ be a perfect complex on the product $X_1 \times X_2$. Let $V$ be a smooth projective scheme constructed above. Then the DG category $\text{Perf} - X_1 \oplus \text{Perf} - X_2$ is quasi-equivalent to a full DG subcategory of $\text{Perf} - V$ and, hence, the triangulated category $\mathcal{H}^0(\text{Perf} - X_1 \oplus \text{Perf} - X_2)$ is admissible in $\text{Perf} - V$.

Consider the DG categories $\text{Perf} - X_1$ and $\text{Perf} - X_2$ and the following composition quasi-functors

(8) $\Phi := \pi^* e_{1*}(\mathcal{O}_{E_1}(E_1) \otimes h_{1*}^* p_1^*(-))$, and $\Psi := e_{2*} h_2^* \tau^* (p_2^*(-) \otimes \mathcal{R})[1]$

from $\text{Perf} - X_1$ and $\text{Perf} - X_2$ to $\text{Perf} - V$ respectively, where $\mathcal{O}_{E_1}(E_1)$ is the restriction of the line bundle $\mathcal{O}(E_1)$ from $V'$ to $E_1$, and $\mathcal{R} \cong \pi_2^* \mathcal{O}(1) \boxtimes \mathcal{O}(-1)$ is a line bundle on $P_2$. These quasi-functors induce exact composition functors

(9) $\Phi := \mathcal{L}\pi^* \mathcal{R}e_{1*}(\mathcal{O}_{E_1}(E_1) \otimes h_{1*}^* p_1^*(-))$, and $\Psi := \mathcal{R}e_{2*} h_2^* \mathcal{L}\tau^* (p_2^*(-) \otimes \mathcal{R})[1]$

from the triangulated categories $\text{Perf} - X_1$ and $\text{Perf} - X_2$ to the triangulated category $\text{Perf} - V$.

Lemma 4.12. The functors $\Phi$ and $\Psi$ are fully faithful and the subcategories $\Phi(\text{Perf} - X_1)$ and $\Psi(\text{Perf} - X_2)$ are semi-orthogonal such that $\Phi(\text{Perf} - X_1)$ are in the right orthogonal $\Psi(\text{Perf} - X_2)^\perp$.

Proof. The functors $\Phi$ and $\Psi$ are fully faithful as a composition of fully faithful functors. It follows from 1)-3) of Proposition 4.10.

The semi-orthogonality is a consequence of the fact that $\mathcal{L}\pi^*(\text{Perf} - V')$ is in the right orthogonal $\mathcal{R}e_{2*} h_2^*(\text{Perf} - \tilde{P}_2)^\perp$. The last statement follows from the chain of isomorphisms

$$\text{Hom}(\mathcal{R}e_{2*} h_2^* B, \mathcal{L}\pi^* A) \cong \text{Hom}(h_2^* B, e_2^* \mathcal{L}\pi^* A) \cong \text{Hom}(h_2^* B, \mathcal{L}e_2^* \mathcal{L}\pi^* A \otimes \mathcal{O}_{E_2}(E_2)) \cong \text{Hom}(h_2^* B, h_2^* \mathcal{L}j_2^* A \otimes \mathcal{O}_{E_2}(E_2)) \cong \text{Hom}(B, \mathcal{L}j_2^* A \otimes \mathcal{R}h_{2*} \mathcal{O}_{E_2}(E_2)) = 0$$

where $A \in \text{Perf} - V'$, $B \in \text{Perf} - \tilde{P}_2$. The last equality holds because $\mathcal{O}_{E_2}(E_2) \cong \mathcal{O}_{E_2}(-1)$ under consideration of $E_2$ as a projectivization of the normal bundle of $\tilde{P}_2$ in $V'$, i.e. we have $\mathcal{R}h_{2*} \mathcal{O}_{E_2}(E_2) = 0$. \qed

Proposition 4.13. Let $X_1$ and $X_2$ be smooth projective schemes and let $\mathcal{E}$ be a perfect complexes on the product $X_1 \times X_2$. Let $\mathcal{V}$ be a smooth projective scheme constructed above and let $\Phi, \Psi$ be exact functors defined by formula (4). Let $\mathcal{F}$ and $\mathcal{G}$ be perfect complexes on $X_1$ and $X_2$, respectively. Then there is an isomorphism

$$\text{Hom}_V(\Phi(\mathcal{F}), \Psi(\mathcal{G})) \cong \text{Hom}_{X_1 \times X_2}(pr_1^* \mathcal{F}, pr_2^* \mathcal{G} \otimes \mathcal{E})$$

where $pr_i$ are the projections of $X_1 \times X_2$ on $X_i$. 
Proof. At first, consider objects $A \in \text{Perf} - V$ and $B \in \text{Perf} - \tilde{P}_2$. There is a sequence of isomorphisms

\begin{equation}
\text{Hom}_V(L\pi^*A, Re_{2*}h^*_2B) \cong \text{Hom}_{V'}(A, R\pi_*R\rho_{2*}h^*_2B) \cong \text{Hom}_{V'}(A, R\tilde{j}_{2*}R\rho_{2*}h^*_2B) \cong \text{Hom}_{V'}(A, R\pi_*R\rho_{2*}h^*_2B).
\end{equation}

Secondly, take $A \in \text{Perf} - P_1$ and $B \in \text{Perf} - P_2$. Consider the commutative square

\begin{equation}
\begin{array}{c}
\begin{array}{c}
D \circlearrowleft \xrightarrow{d_2} \tilde{P}_2 \\
\downarrow d_1
\end{array} \\
E_1 \circlearrowleft \xrightarrow{e_1} V'
\end{array}
\end{equation}

that is a part of our main diagram (7). The commutative square (11) is cartesian. Moreover, it is Tor-independent. This means that $\tau_{\rho_{2*}}(\mathcal{O}_{E_1}, \mathcal{O}_{\tilde{P}_2}) = 0$ for all $p > 0$. Therefore, by [CA6] IV 3.1 or [TT] 2.5.6 there is a canonical base change isomorphism of functors

\[ L\epsilon_1^*R\tilde{j}_2 \sim \to Rd_{1*}Ld_2^* \quad \text{and} \quad Hom_{V'}(\mathcal{O}_{E_1}(E_1) \otimes h^*_1A), R\tilde{j}_{2*}L\tau^*B) \cong \text{Hom}_{E_1}(h^*_1A, \epsilon_1^*R\tilde{j}_2L\tau^*B) \cong \text{Hom}_{E_1}(h^*_1A, Ld_{1*}Ld_2^*L\tau^*B([-1])) \cong \text{Hom}_D(g^*Ld_1^*A, Ld_2^*L\tau^*B([-1])) \cong \text{Hom}_Z(Ld_1^*A, Ld_2^*B([-1]))
\]

Now combining (10) and (12), we obtain

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{Hom}_V(\Phi(F), \Psi(G)) \cong \text{Hom}_{V'}(\mathcal{O}_{E_1}(E_1) \otimes h^*_1p^*_1F), \tilde{j}_{2*}L\tau^*(p^*_2G \otimes \mathcal{R})[1]) \cong \text{Hom}_Z(Ld_1^*p^*_1F, Ld_2^*(p^*_2G \otimes \mathcal{R})) \cong \text{Hom}_Z(q_1^*F, q_2^*G \otimes \mathcal{L})
\end{array}
\end{array}
\end{equation}

The last isomorphism is a consequence of construction of $P_2$ and the line bundle $\mathcal{R}$ on $P_2$. By (6) the restriction of $\mathcal{R}$ on $Z$ coincides with the line bundle $\mathcal{L}$.

Finally, we have $q_i = \text{pr}_i \cdot f$ for $i = 1, 2$ and we know that $Rf_*\mathcal{L} \cong \mathcal{E}$ by construction from Proposition 4.13. This implies

\begin{equation}
\text{Hom}_Z(q_1^*F, q_2^*G \otimes \mathcal{L})) \cong \text{Hom}_{X_1 \times X_2}(\text{pr}_1^*F, \text{pr}_2^*G \otimes Rf_*\mathcal{L})) \cong \text{Hom}_{X_1 \times X_2}(\text{pr}_1^*F, \text{pr}_2^*G \otimes \mathcal{E})).
\end{equation}

The isomorphisms (13) and (14) prove the proposition.

Proof of Theorem 4.11. Let us consider the DG functors

\[ \Phi : \text{Perf} - X_1 \longrightarrow \text{Perf} - V, \quad \text{and} \quad \Psi : \text{Perf} - X_2 \longrightarrow \text{Perf} - V. \]
They induce a bimodule $S$ determined by the following rule
\begin{equation}
S(B, A) \cong \text{Hom}_{\text{Perf}_V}(\Phi A, \Psi B), \quad \text{where} \quad A \in \text{Perf}_X, \ B \in \text{Perf}_Y.
\end{equation}

On the other hand, the calculations from Proposition 4.13 gives us that the bimodule $S$ is quasi-isomorphic to a bimodules $S_C$ given by the rule

\[ S_C(B, A) \cong \text{Hom}_{\text{Perf}_V}(\text{pr}_1^* A, \text{pr}_2^* B \otimes \mathcal{E}), \quad \text{where} \quad A \in \text{Perf}_X, \ B \in \text{Perf}_Y. \]

Take a pretriangulated DG subcategory $\mathcal{C} \subset \text{Perf}_V$ that is generated by the DG subcategories $\Phi(\text{Perf}_X)$ and $\Psi(\text{Perf}_X)$. Lemma 4.12 implies that there is a semi-orthogonal decomposition

\[ \mathcal{H}^0(\mathcal{C}) \cong \langle \Phi(\text{Perf}_X), \Psi(\text{Perf}_X) \rangle. \]

Since $\Phi$ and $\Psi$ are fully faithful Propositions 3.8 and 3.11 give us that there are quasi-equivalences

\[ \mathcal{C} \cong \Phi(\text{Perf}_X) \oplus \Psi(\text{Perf}_X) \cong \text{Perf}_X \oplus \text{Perf}_Y, \]

where $S$ is the bimodule given by rule (15). By Theorem 3.25 the full DG subcategory $\mathcal{C} \subset \text{Perf}_V$ is smooth and proper as a gluing of smooth and proper DG categories via a perfect bimodule $S$. Hence $\mathcal{H}^0(\mathcal{C}) \cong \mathcal{H}^0(\text{Perf}_X \oplus \text{Perf}_Y)$ is admissible in $\text{Perf}_V$. \hfill $\square$

Remark 4.14. It is useful to take in account that the category $\text{Perf}_V$ from Theorem 4.11 has a semi-orthogonal decomposition of the form $\text{Perf}_V = \langle T_1, \ldots, T_k \rangle$ such that each $T_i$ is equivalent to one of the four categories, namely $\text{Perf}_k$, $\text{Perf}_X$, $\text{Perf}_Y$, and $\text{Perf}_Z$. It follows from the construction of $V$ as a two-step blowup of a projective space $\mathbb{P}^N$ along $P_1$ and $P_2$. By definition, $P_1$ and $P_2$ are projective bundles over $X_1$ and $X_2$ respectively and $P_2$ is a blowup of $P_2$ along $Z$, where $Z$ is a sequence of projective bundles over $X_1 \times X_2$.

4.4. Gluing of geometric noncommutative schemes. In this section we extend results of previous section on the case of geometric noncommutative schemes. Actually all these statements are direct consequences of corresponding assertions for smooth and projective schemes.

Let $X_i$, $i = 1, \ldots, n$ be smooth and projective schemes. Let $\mathcal{N}_i$, $i = 1, \ldots, n$ be small pretriangulated DG categories. Denote by $\mathcal{N}_i = \mathcal{H}^0(\mathcal{N}_i)$ the homotopy triangulated categories. Suppose that for all $i$ there are quasi-functors $F_i : \mathcal{N}_i \rightarrow \text{Perf}_X$ such that the induced exact functors $\tilde{F}_i : \mathcal{N}_i \rightarrow \text{Perf}_X$ are fully faithful and has right and left adjoint functors. This means that $\mathcal{N}_i$ are admissible subcategories in $\text{Perf}_X$ with respect to the full embeddings given by $F_i$. This conditions imply that $\mathcal{N}_i$ are geometric noncommutative schemes and, moreover, the DG categories $\mathcal{N}_i$ are smooth and proper by Theorem 3.25.

Theorem 4.15. Let DG categories $\mathcal{N}_i$, $i = 1, \ldots, n$ and smooth projective schemes $X_i$, $i = 1, \ldots, n$ be as above. Let $\mathcal{C}$ be a proper pretriangulated DG category with full embeddings of DG categories $\mathcal{N}_i \subset \mathcal{C}$ such that $\mathcal{C} = \mathcal{H}^0(\mathcal{C})$ has a semi-orthogonal decomposition of the form $\mathcal{C} \cong \mathcal{H}^0(\mathcal{C})$.
are fully faithful, the induced functor \( F : \mathcal{C} \to \text{Perf} - V \) such that the induced functor \( F : \mathcal{C} \to \text{Perf} - V \) is fully faithful and has right and left adjoint functors, i.e. \( \mathcal{C} \) is a geometric noncommutative scheme.

Proof. The case \( n = 1 \) is evident. Consider the main case \( n = 2 \). By Proposition 3.18 the DG category \( \mathcal{C} \) is quasi-equivalent to a gluing of \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) via a \( \mathcal{N}_2 - \mathcal{N}_1 \)-bimodule \( S \) that is defined by the rule

\[
S(B, A) = \text{Hom}_\mathcal{C}(A, B), \quad \text{with } A \in \mathcal{N}_1 \text{ and } B \in \mathcal{N}_2.
\]

Since \( \mathcal{C} \) is proper the bimodule \( S \) is a DG functor from \( \mathcal{N}_2 \otimes \mathcal{N}_1^\circ \) to \( \text{Perf} - k \) and by Theorem 3.18 it is perfect, because \( \mathcal{N}_i \) are smooth and proper. By Proposition 3.8 the DG category \( \mathcal{C} \) is quasi-equivalent to the gluing \( \mathcal{N}_1 \uplus \mathcal{N}_2 \). By Theorem 3.25 we obtain that \( \mathcal{C} \) is smooth.

Consider quasi-functors \( F_i : \mathcal{N}_i \to \text{Perf} - X_i \). We know that \( F_i \) establish quasi-equivalences with enhancements of admissible subcategories in \( \text{Perf} - X_i \). By Theorem 4.11 the DG category of perfect DG modules over \( \text{Perf} - X_1 \uplus_k \text{Perf} - X_2 \) is equivalent to \( \text{Perf} - (X_1 \times X_2) \). Thus the quasi-functors \( F_i \) induce the extension and induction quasi-functors

\[
(F_1 \otimes F_2)^* : \text{Perf} - (\mathcal{N}_2^\circ \otimes \mathcal{N}_1) \to \text{Perf} - (X_1 \times X_2), \quad (F_1 \otimes F_2)_* : \text{Perf} - (X_1 \times X_2) \to \text{Perf} - (\mathcal{N}_2^\circ \otimes \mathcal{N}_1)
\]

and by Proposition 4.2 the extension functor induces a fully faithful functor between homotopy categories. This implies that the bimodule \( S \) is quasi-isomorphic to a bimodule of the form \( (F_1 \otimes F_2)_* \mathcal{E}^* \) for some perfect complex \( \mathcal{E}^* \) on \( X_1 \times X_2 \).

By Proposition 3.11 there is a quasi-functor

\[
F_1 \oplus F_2 : \mathcal{N}_1 \uplus \mathcal{N}_2 \longrightarrow \text{Perf} - X_1 \uplus \text{Perf} - X_2,
\]

where \( \phi \) is a quasi-isomorphism between \( S \) and \( (F_1 \otimes F_2)_* \mathcal{E}^* \). Since the functors \( F_i : \mathcal{N}_i \to \text{Perf} - X_i \) are fully faithful, the induced functor

\[
F_1 \oplus F_2 : \mathcal{H}^0(\mathcal{N}_1 \uplus \mathcal{N}_2) \longrightarrow \mathcal{H}^0(\text{Perf} - X_1 \uplus \text{Perf} - X_2)
\]

is fully faithful too by Proposition 3.11.

By Theorem 4.11 the DG category \( \text{Perf} - X_1 \uplus \text{Perf} - X_2 \) is quasi-equivalent to a full DG subcategory of \( \text{Perf} - V \) for some smooth and projective scheme \( V \). Consider the composition quasi-functor

\[
F : \mathcal{C} \longrightarrow \mathcal{N}_1 \uplus \mathcal{N}_2 \xrightarrow{F_1 \oplus F_2} \text{Perf} - X_1 \uplus \text{Perf} - X_2 \longrightarrow \text{Perf} - V.
\]

It induces an exact functor \( \mathcal{C} \to \text{Perf} - V \) that is fully faithful as a composition of fully faithful functors. The DG category \( \mathcal{C} \) is proper and it is smooth as a gluing of smooth DG categories via a perfect DG bimodule. The smoothness implies regularity of \( \mathcal{C} \) (see Proposition 3.24). Moreover, the category \( \mathcal{C} \) is idempotent complete, because \( \mathcal{N}_i \) are idempotent complete as admissible subcategories of \( \text{Perf} - X_i \). Now by Proposition 3.17 the regularity and properness of \( \mathcal{C} \) gives that the
image of the fully faithful functor $F$ is an admissible subcategory of $\text{Perf} - V$. Hence, $F$ admits right and left adjoint functors.

The general case $n$ is going by induction. Denote by $\mathcal{N}'_2 \subset \mathcal{C}$ the left orthogonal to $\mathcal{N}_1$ and denote by $\mathcal{N}_2' \subset \mathcal{C}$ the full DG subcategory consisting of all objects from $\mathcal{N}_2'$. We have a semi-orthogonal decompositions $\mathcal{N}_2' = \langle N_2, \ldots, N_n \rangle$ and $\mathcal{C} = \langle N_1, N_2 \rangle$. By induction hypothesis, there are a smooth and projective scheme $V'$ and a quasi-functor $F : \mathcal{N}_2' \to \text{Perf} - V'$ such that the induced functor $F : \mathcal{N}_2' \to \text{Perf} - V'$ is fully faithful and has right and left adjoint functors. Now applying the proof for $n = 2$ and the DG subcategories $\mathcal{N}_1$ and $\mathcal{N}_2'$ in $\mathcal{C}$, we obtain the statement of the theorem for $\mathcal{C}$.

**Corollary 4.16.** Let $Y$ be a proper scheme over a field of characteristic 0. Then there are a smooth projective scheme $V$ and a quasi-functor $F : \text{Perf} - Y \to \text{Perf} - V$ such that the induced functor $F : \text{Perf} - Y \to \text{Perf} - V$ is fully faithful.

**Proof.** It follows from the main theorem of [KL, Th.1.4] that $Y$ has a so-called categorical resolution. By construction of this categorical resolution there is a quasi-functor from $G : \text{Perf} - Y \to \mathcal{D}$, where $\mathcal{D}$ is a gluing of DG categories of perfect complexes on smooth proper schemes, and the induced functor $G : \text{Perf} - Y \to \mathcal{D}$ is fully faithful. Now as in Remark 4.5 over field of characteristic 0 for any smooth proper scheme there is a sequence of blowups with smooth centers such that the resulting smooth scheme is projective. Hence $\mathcal{D}$ is a gluing of geometric noncommutative schemes. By Theorem 4.15 there are a smooth projective $V$ and a quasi-functor from $\mathcal{D}$ to $\text{Perf} - V$ which is fully faithful on homotopy categories. Composition of these quasi-functors gives us a quasi-functor $F : \text{Perf} - Y \to \text{Perf} - V$ that is also fully faithful on homotopy categories.

## 5. Application to finite algebras and exceptional collections

### 5.1. Finite dimensional algebras

Let $\Lambda$ be a finite dimensional algebra over a base field $k$. Denote by $\mathfrak{R}$ the (Jacobson) radical of $\Lambda$. We know that $\mathfrak{R}^n = 0$ for some $n$. Define the index of nilpotency $i(\Lambda)$ of $\Lambda$ as the smallest such integer $n$ that $\mathfrak{R}^n = 0$.

Let $S$ be the quotient algebra $\Lambda/\mathfrak{R}$. It is semisimple and has only a finite number simple non-isomorphic modules. Denote by $\text{Mod-}\Lambda$ and $\text{mod-}\Lambda$ the abelian categories of all right modules and finite right modules over $\Lambda$, respectively.

The following amazing result was proved by M. Auslander.

**Theorem 5.1.** [Aus] Let $\Lambda$ be a finite dimensional algebra of index $n$. Then the finite dimensional algebra $\Gamma = \text{End}(\bigoplus_{p=1}^n \Lambda/\mathfrak{R}^p)$ has the following properties:

1) $\text{gl.dim } \Gamma \leqslant n + 1$;
2) there is a finite projective $\Gamma$-module $P$ such that $\text{End}_\Gamma(P) \cong \Lambda$. 


Let us consider the bounded derived category of finite $\Gamma$–modules $\mathcal{D}^b(\text{mod–}\Gamma)$. Since $\Gamma$ has a finite global dimension, $\mathcal{D}^b(\text{mod–}\Gamma)$ is equivalent to the category of perfect complexes $\text{Perf–}\Gamma$. Some variants of following theorem are known (see. e.g. [KL]).

**Theorem 5.2.** Let $\Lambda$ be a finite dimensional algebra of index $n$ and let $\Gamma = \text{End}(\bigoplus_{p=1}^n \Lambda/\mathfrak{m}^p)$. The derived category $\text{Perf–}\Gamma \cong \mathcal{D}^b(\text{mod–}\Gamma)$ has a semi-orthogonal decomposition of the form

$$\text{Perf–}\Gamma = \langle N_1, \ldots, N_n \rangle$$

such that each subcategory $N_i$ is semisimple, i.e. $N_i \cong \langle K_i \rangle$, where $K_i$ is semi-exceptional and for all $i$ the algebras $\text{End}_\Gamma(K_i)$ are quotients of the semisimple algebra $S = \Lambda/\mathfrak{m}$.

**Proof.** Denote by $M$ the $\Lambda$–modules $\bigoplus_{p=1}^n \Lambda/\mathfrak{m}^p$ and by $M_s$ the $\Lambda$–modules $\Lambda/\mathfrak{m}^s$, $s = 1, \ldots, n$. Consider the functor $\text{Hom}_\Lambda(M, -)$ from the abelian category $\text{mod–}\Lambda$ to the abelian category $\text{mod–}\Gamma$. Denote by $P_s$ the $\Gamma$–modules $\text{Hom}_\Lambda(M, M_s)$. They are projective $\Gamma$–modules and $\Gamma = \bigoplus_{s=1}^n P_s$.

By Gabriel-Popescu theorem, since $M$ is a generator for $\text{Mod–}\Lambda$ the functor $\text{Hom}_\Lambda(M, -)$ from $\text{Mod–}\Lambda$ to $\text{Mod–}\Gamma$ is fully faithful. Thus, there are isomorphisms

$$\text{Hom}_\Gamma(P_i, P_j) \cong \text{Hom}_\Lambda(M_i, M_j) = \text{Hom}_\Lambda(\Lambda/\mathfrak{m}^i, \Lambda/\mathfrak{m}^j)$$

for all $1 \leq i, j \leq n$. Moreover, we have $\text{Hom}_\Lambda(\Lambda/\mathfrak{m}^i, \Lambda/\mathfrak{m}^j) \cong \Lambda/\mathfrak{m}^j$ when $i \geq j$. The canonical quotient morphisms $\Lambda/\mathfrak{m}^i \to \Lambda/\mathfrak{m}^j$, when $i \geq j$, induce morphisms $\phi_{i,j} : P_i \to P_j$.

Let us consider $\phi_{i,i-1}$ and the induced exact triangles

$$K_i \longrightarrow P_i \xrightarrow{\phi_{i,i-1}} P_{i-1} \longrightarrow K_i[1], \quad i = 2, \ldots n$$

in $\text{Perf–}\Gamma$. These triangles define objects $K_i$ for $i = 2, \ldots, n$. We also set $K_1 = P_1$.

Now, since $P_i$ are projective and $\text{Hom}_\Gamma(P_i, P_j) \cong \Lambda/\mathfrak{m}^j$ when $i \geq j$, we have vanishing

$$\text{Hom}_\Gamma(K_i, P_j[l]) = 0, \quad \text{for all } l \text{ when } i > j.$$  

Using definition (16) of $K_i$ we immediately obtain semi-orthogonality conditions

$$\text{Hom}_\Gamma(K_i, K_j[l]) = 0, \quad \text{for all } l \text{ when } i > j.$$  

Finally, we have to compute $R\text{Hom}_\Gamma(K_i, K_i)$ for all $i$. The exact triangles (16) give us that the vector spaces $\text{Hom}_\Gamma(K_i, K_i[l])$ are cohomology of the complex

$$\text{Hom}_\Gamma(P_{i-1}, P_i) \longrightarrow \text{Hom}_\Gamma(P_i, P_i) \bigoplus \text{Hom}_\Gamma(P_{i-1}, P_{i-1}) \longrightarrow \text{Hom}_\Gamma(P_i, P_{i-1})$$

that coincides with the complex

$$\text{Hom}(\Lambda/\mathfrak{m}^{i-1}, \Lambda/\mathfrak{m}^i) \longrightarrow \text{Hom}(\Lambda/\mathfrak{m}^i, \Lambda/\mathfrak{m}^i) \bigoplus \text{Hom}(\Lambda/\mathfrak{m}^{i-1}, \Lambda/\mathfrak{m}^{i-1}) \longrightarrow \text{Hom}(\Lambda/\mathfrak{m}^i, \Lambda/\mathfrak{m}^{i-1})$$
The morphism \( \text{Hom}(\Lambda/\mathcal{R}^{-1}, \Lambda/\mathcal{R}^{i-1}) \to \text{Hom}(\Lambda/\mathcal{R}^{i}, \Lambda/\mathcal{R}^{i-1}) \) is an isomorphism and the morphism \( \text{Hom}(\Lambda/\mathcal{R}^{i-1}, \Lambda/\mathcal{R}^{i}) \to \text{Hom}(\Lambda/\mathcal{R}^{i}, \Lambda/\mathcal{R}^{i}) \) is an injection. This implies that the complex \([17]\) has only zero cohomology. Therefore,

\[
\text{Hom}_\Gamma(K_i, K_i[l]) = 0, \quad \text{for all } l \neq 0 \text{ and all } i = 1, \ldots, n.
\]

Denote by \( S_i \) the algebras of endomorphisms \( \text{End}_\Gamma(K_i) \), where \( i = 1, \ldots, n \). We know that \( S_1 = \text{End}_\Gamma(P_1) \cong \text{End}_\Lambda(\Lambda/\mathcal{R}) = S \) is semisimple.

Let \( a \in S_i \) be an element. It can be presented by a pair of morphism \( (a_i, a_{i-1}) \) included in commutative diagram

\[
\begin{array}{ccc}
P_i & \xrightarrow{\phi_{i,i-1}} & P_{i-1} \\
\downarrow{a_i} & & \downarrow{a_{i-1}} \\
P_i & \xrightarrow{\phi_{i,i-1}} & P_{i-1}
\end{array}
\]

The morphism \( a_{i-1} \) is uniquely determined by \( a_i \). Thus the element \( a_i \in \text{Hom}_\Gamma(P_i, P_i) \cong \Lambda/\mathcal{R}^i \) induces an endomorphism of \( K_i \) and we see that there is a homomorphism of algebras \( \Lambda/\mathcal{R}^i \to \text{End}_\Gamma(K_i) \) that is surjective. If now \( a_i \in \text{End}(P_i) = \Lambda/\mathcal{R}^i \) belongs to \( \mathcal{R} \), then as an endomorphism of \( \Lambda/\mathcal{R}^i \) it sends \( \mathcal{R}^{i-1} \) to zero. This implies that it is induced by a morphism of \( \Lambda/\mathcal{R}^{i-1} \) to \( \Lambda/\mathcal{R}^i \).

Thus, we obtain that the pair of morphisms \( (a_i, a_{i-1}) \) is induced by a morphism from \( P_{i-1} \) to \( P_i \) if \( a_i \in \mathcal{R} \). This means that the algebra of endomorphisms \( \text{End}_\Gamma(K_i) \) is a quotient of semisimple algebra \( S = \Lambda/\mathcal{R} \). Therefore, the algebras \( S_i = \text{End}_\Gamma(K_i) \) are semisimple for all \( i = 1, \ldots, n \) too. As \( P_i, i = 1, \ldots, n \) generate the category \( \text{Perf} - \Gamma \) the objects \( K_i, i = 1, \ldots, n \) generate \( \text{Perf} - \Gamma \) as well, and we obtain a semi-orthogonal decomposition

\[
\text{Perf} - \Gamma = \langle \langle K_1 \rangle, \cdots, \langle K_n \rangle \rangle,
\]

where all \( K_i \) are semi-exceptional and \( S_i = \text{End}_\Gamma(K_i) \) are quotients of the algebra \( S = \Lambda/\mathcal{R} \). \( \square \)

Consider now the \( \Gamma \)-module \( P_n = \text{Hom}_\Lambda(M, \Lambda) \), where \( M = \bigoplus_{p=1}^n \Lambda/\mathcal{R}^p \). It is projective and \( \text{End}_\Gamma(P_n) \cong \Lambda \). This object gives us two functors

\[
(-) \otimes_\Lambda P_n : \text{Perf} - \Lambda \to \text{Perf} - \Gamma \quad \text{and} \quad \text{Hom}_\Gamma(P_n, -) : D^b(\text{mod}-\Gamma) \to D^b(\text{mod}-\Lambda)
\]

The first functor is fully faithful while the second functor is a quotient. Since \( \Gamma \) has a finite global dimension there is an equivalence \( \text{Perf} - \Gamma \cong D^b(\text{mod}-\Gamma) \). If now \( \Lambda \) also has a finite global dimension, then the second functor is right adjoint to the first one and the category \( \text{Perf} - \Lambda \) is right admissible in \( \text{Perf} - \Gamma \) with respect to the full embedding \( (-) \otimes_\Lambda P_n \).

Recall that a semisimple algebra \( S \) over a field \( k \) is called separable over \( k \) if it is a projective \( S^0 \otimes_k S \)-module. It is well-known that a semisimple algebra \( S \) is separable if it is a direct sum of simple algebras centers of which are separable extensions of \( k \).
Theorem 5.3. Let $\Lambda$ be a finite dimensional algebra over $k$. Assume that $S = \Lambda / R$ is separable $k$–algebra. Then there are smooth projective scheme $V$ and a perfect complex $E^\cdot$ such that $\text{End}(E^\cdot) \cong \Lambda$ and $\text{Hom}(E^\cdot, E^\cdot[l]) = 0$ for all $l \neq 0$.

Proof. Let as above $\Gamma = \text{End}(\bigoplus_{p=1}^n \Lambda / R^p)$. Consider the DG category $\text{Perf} - \Gamma$. By Theorem 5.2 the triangulated category $\text{Perf} - \Gamma$ has a semi-exceptional collection $(K_1, \ldots, K_n)$ and

$$\text{Perf} - \Gamma = \langle N_1, \ldots, N_n \rangle$$

where $N_i = \langle K_i \rangle$ is semisimple. Thus, for any $i$ the object $K_i$ is a direct sum of the form $\bigoplus_{j=1}^{m_i} K_{ij}$, where $K_{ij}$ are completely orthogonal to each other for fixed $i$ and different $j$. Moreover, each $\text{End}_\Gamma(K_{ij})$ is a simple algebra, i.e. it is a matrix algebra over a division $k$–algebra $k_{ij}$. By assumption $S$ is separable. Hence, all $\text{End}_\Gamma(K_{ij})$ are separable as quotients of $S$. Thus we obtain that the centers $k_{ij}$ of all $D_{ij}$ are separable extensions of $k$.

Now as in Example 1.17 we can consider a Severi-Brauer variety $SB(D_{ij})$ that is a smooth projective scheme over $k_{ij}$ and over $k$ too, because $k_{ij} \supset k$ is a finite separable extension. It was mention in Example 1.17 that there is a vector bundle $E_{ij}$ on $SB(D_{ij})$ such that it is $w$-exceptional and $\text{End}(E_{ij}) \cong D_{ij}$. This implies that each DG category $\text{Perf} - D_{ij}$ is a full DG subcategory of the DG category $\text{Perf} - SB(D_{ij})$, and $SB(D_{ij})$ is smooth and projective over $k$.

All categories $N_i$ have complete orthogonal decompositions of the form $N_i = N_{i1} \oplus \cdots \oplus N_{in}$, where $N_{ij} = \langle K_{ij} \rangle$ are equivalent to $\text{Perf} - D_{ij}$. These decompositions induce a semi-orthogonal decomposition for $\text{Perf} - \Gamma$ of the form

$$\text{Perf} - \Gamma = \langle N_{i1}, N_{i2}, \ldots, N_{im_1}, N_{21}, \ldots, N_{im_n} \rangle.$$ 

Applying Theorem 4.15 we obtain that there are a smooth projective scheme $V$ and a quasi-functor from $F : \text{Perf} - \Gamma \to \text{Perf} - V$ such that the homotopy functor $F : \text{Perf} - \Gamma \to \text{Perf} - V$ is fully faithful and establishes an equivalence with an admissible subcategory in $\text{Perf} - V$. Denote by $E^\cdot$ the perfect complex $F(P_n)$, where $P_n = \text{Hom}_\Lambda(M, \Lambda)$ is a projective $\Gamma$–module. Since $F$ is fully faithful we have isomorphisms

$$\text{Hom}_V(E^\cdot, E^\cdot[l]) \cong \text{Hom}_\Gamma(P_n, P_n[l]).$$

When $l \neq 0$ it is 0, and it is isomorphic to the algebra $\Lambda$ for $l = 0$. \qed

Corollary 5.4. Let $\Lambda$ be a finite dimensional algebra over $k$ for which $S = \Lambda / R$ is separable $k$–algebra. Then there is a smooth projective scheme $V$ such that the DG category $\text{Perf} - \Lambda$ is quasi-equivalent to a full DG subcategory of $\text{Perf} - V$. Moreover, if $\Lambda$ has a finite global dimension, then $\text{Perf} - \Lambda$ is an admissible in $\text{Perf} - V$.

Proof. By Theorem 5.3 there is a smooth projective scheme $V$ and a perfect complex $E^\cdot$ such that $\text{End}(E^\cdot) \cong \Lambda$ and $\text{Hom}(E^\cdot, E^\cdot[l]) = 0$ for all $l \neq 0$. Hence, the DG algebra $\text{Hom}_{\text{Perf} - V}(E^\cdot, E^\cdot)$ is
quasi-isomorphic to the algebra $\Lambda$. Thus, by Proposition 2.7 there is a quasi-functor $F : \text{Perf} - \Lambda \rightarrow \text{Perf} - V$ induced by the embedding of $\text{Hom}_{\text{Perf} - V}(\mathcal{E}, \mathcal{E}')$ to $\text{Perf} - V$ such that the homotopy functor $F : \text{Perf} - \Lambda \rightarrow \text{Perf} - V$ is fully faithful. If $\Lambda$ has a finite global dimension, then the category $\text{Perf} - \Lambda$ is regular and proper. As it is idempotent complete, it is admissible in $\text{Perf} - V$ by Proposition 3.17.

**Remark 5.5.** Note that over a perfect field all semisimple algebras are separable. Thus, if $k$ is perfect, then results of this section are applied to any finite dimensional algebras.

Theorem 5.3 says us, in particular, that for any finite dimensional algebra $\Lambda$ of finite global dimension the category $\text{Perf} - \Lambda$ can be embedded to a triangulated category with a full semi-exceptional collection (actually, with w-exceptional collection). This fact allows us to ask the question.

**Question 5.6.** Let $\Lambda$ be a finite dimensional $k$-algebra of finite global dimension Is true that the category $\text{Perf} - \Lambda$ has a full semi-exceptional (w-exceptional) collection itself?

It should be noted that Jordan-Hölder property does not holds for triangulated categories with full exceptional collections, and there are admissible subcategories in triangulated categories with exceptional collections that do not have full exceptional collections themselves. A simple counterexample was constructed by Alexei Bondal away back (see, e.g. [Kuz]).

### 5.2. Exceptional collections.

In this section we describe a more useful procedure of constructing a scheme that admits a full exceptional collection and contains as a subcollection an exceptional collection given in advance.

Let $\mathcal{A}$ be a small smooth and proper pretriangulated DG category over a field $k$ such that the homotopy category $\mathcal{H}^0(\mathcal{A})$ has a semi-orthogonal decomposition of the form

$$\mathcal{H}^0(\mathcal{A}) = \langle \mathcal{N}, \langle E \rangle \rangle,$$

where $\mathcal{N}$ is a full admissible subcategory and $E$ is an exceptional object, i.e. $\langle E \rangle \cong \text{Perf} - k$.

Assume that the enhancement of $\mathcal{N}$ induced from $\mathcal{A}$ is quasi-equivalent to a full DG subcategory $\mathcal{N} \subset \text{Perf} - X$ for a smooth and projective irreducible scheme $X$ such that $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, i.e. $\mathcal{O}_X$ is exceptional.

**Remark 5.7.** The assumption $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ is not restrictive. Indeed, for any smooth and projective scheme we can consider a closed immersion into a projective space $\mathbb{P}^N$ for a large $N$. Take the blowup $Z$ of $\mathbb{P}^N$ along $X$. Then $\text{Perf} - X$ is quasi-equivalent to a full DG subcategory in $\text{Perf} - Z$ and $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. Thus, we can take $Z$ instead $X$.

We have that $\mathcal{N} \cong \mathcal{H}^0(\mathcal{N})$ is an admissible subcategory in $\text{Perf} - X$. By Propositions 3.8 and 3.11 the DG category $\mathcal{A}$ is quasi-equivalent to a gluing of $\mathcal{N}$ and $\text{Perf} - k$ via some $\mathcal{N}$-module
Since $\text{Perf} - X$ and $\mathcal{N}$ are saturated by Theorem 3.18, the DG module $S$ can be represented by a perfect complex $S'$ on $X$, i.e. the DG $(\text{Perf} - X)$-module $\text{Hom}(\cdot, S')$ after restriction of $\mathcal{N}$ is quasi-isomorphic to the DG $\mathcal{N}$-module $S$.

Shifting the complex $S'$ to $[m]$ for appropriate $m \in \mathbb{Z}$ we can suppose that

$$S' \cong \{S^0 \to S^1 \to \cdots \to S^k\},$$

where all $S^i$ are vector bundles on $X$.

By Proposition 4.6 there is a smooth morphism $f : Z \to X$ and a line bundle $\mathcal{L}$ on $Z$ such that $Rf_*\mathcal{L} \cong S'$ and $Rf_*\mathcal{L}^{-1} = 0$. Moreover, the morphism $f$ is a sequence of projective bundles. Hence $Rf_*\mathcal{O}_Z \cong \mathcal{O}_X$ and the inverse image functor $f^* : \text{Perf} - X \to \text{Perf} - Z$ is fully faithful. Since $Rf_*\mathcal{L}^{-1} = 0$ we have

$$\text{Hom}_Z(\mathcal{L}, f^*A) \cong \text{Hom}_Z(\mathcal{O}_Z, f^*A \otimes \mathcal{L}^{-1}) \cong \text{Hom}_X(\mathcal{O}_X, A \otimes Rf_*\mathcal{L}^{-1}) = 0$$

for any $A \in \text{Perf} - X$. Therefore $f^*(\text{Perf} - X)$ is in the right orthogonal $\langle \mathcal{L} \rangle^\perp$.

Since $\mathcal{O}_X$ is exceptional the structure sheaf $\mathcal{O}_Z$ and any line bundle on $Z$ are also exceptional. Therefore, the admissible subcategory $\mathcal{T} \subset \text{Perf} - Z$ which is generated by $f^*(\mathcal{N})$ and $\mathcal{L}$ has a semi-orthogonal decomposition of the form $\mathcal{T} \cong \langle \mathcal{N}, \text{Perf} - k \rangle$. Denote by $\mathcal{S}$ the induced from $\text{Perf} - Z$ enhancement of $\mathcal{T}$. By Propositions 3.8 and 3.11 the DG category $\mathcal{S}$ is quasi-equivalent to $\mathcal{T}$ because both of them are quasi-equivalent to the gluing $\mathcal{N} \oplus \text{Perf} - k$ via $S$.

The procedure described above can be considered as an induction step for the proving of following theorem while base of induction is the point $\text{Spec} \mathbb{k}$. Thus, we obtain.

**Theorem 5.8.** Let $\mathcal{A}$ be a small DG category over $\mathbb{k}$ such that the homotopy category $\mathcal{H}^0(\mathcal{A})$ has a full exceptional collection

$$\mathcal{H}^0(\mathcal{A}) = \langle E_1, \ldots, E_n \rangle.$$

Then there are a smooth projective scheme $X$ and an exceptional collection of line bundles $\sigma = (\mathcal{L}_1, \ldots, \mathcal{L}_n)$ on $X$ such that the DG subcategory of $\text{Perf} - X$, generated by $\sigma$, is quasi-equivalent to $\mathcal{A}$. Moreover, $X$ is a sequence of projective bundles and has a full exceptional collection.

**Remark 5.9.** The scheme $X$ has a full exceptional collection as a sequence of projective bundles (see [Or1]). Furthermore, it follows from construction that a full exceptional collection on $X$ can be taken such that it contains the collection $\sigma = (\mathcal{L}_1, \ldots, \mathcal{L}_n)$ as a subcollection.

5.3. **Noncommutative projective planes.** In this section we consider a particular case of noncommutative projective planes, in sense of noncommutative deformations of the usual projective plane, and present explicit embeddings of categories of perfect complexes on them to categories of perfect complexes on smooth projective commutative schemes.
Noncommutative deformations of the projective plane have been described in [ATV, BP]. The category $\mathcal{P}erf-\mathbb{P}^2$ has a full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$. Note also that mirror symmetry relation for noncommutative planes is described in [AKO].

Any deformation of the category $\mathcal{P}erf-\mathbb{P}^2$ is a category with three ordered objects $F_0, F_1, F_2$ and with three-dimensional spaces of homomorphisms from $F_i$ to $F_j$ when $j-i=1$ and six-dimensional vector space as a Hom from $F_0$ to $F_2$. Any such category is determined by the composition tensor $\mu : V \otimes U \to W$, where $\dim V = \dim U = 3$ and $\dim W = 6$. This map should be surjective. Denote by $T$ the kernel of $\mu$ and by $\nu : T \to V \otimes U$. We will consider only the nondegenerate (geometric) case, where the restrictions $\nu_u : T \to V$ and $\nu_v : T \to U$ have rank at least two for all nonzero elements $u^* \in U^*$ and $v^* \in V^*$. The equations $\det \nu_u = 0$ and $\det \nu_v = 0$ define closed subschemes $\Gamma_U \subset \mathbb{P}(U^*)$ and $\Gamma_V \subset \mathbb{P}(V^*)$. Namely, up to projectivization the set of points of $\Gamma_U$ (resp. $\Gamma_V$) consists of all $u^* \in U^*$ (resp. $v^* \in V^*$) for which the rank of $\nu_{u^*}$ (resp. $\nu_{v^*}$) is equal to 2. It is easy to see that the correspondence which associates to a vector $v^* \in V^*$ the kernel of the map $\nu_{v^*} : U^* \to T^*$ defines an isomorphism between $\Gamma_V$ and $\Gamma_U$. Moreover, under these circumstances $\Gamma_V$ is either the entire projective plane $\mathbb{P}(V^*)$ or a cubic in $\mathbb{P}(V^*)$. If $\Gamma_V = \mathbb{P}(V^*)$, then $\mu$ is isomorphic to the tensor $V \otimes V \to S^2V$, i.e. we get the usual projective plane $\mathbb{P}^2$.

Thus, the non-trivial case is the situation, where $\Gamma_V$ is a cubic, which we now denote by $E$. This curve comes equipped with two embeddings into the projective planes $\mathbb{P}(U^*)$ and $\mathbb{P}(V^*)$ respectively; by restriction of $\mathcal{O}(1)$ these embeddings determine two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ of degree 3 on $E$, and it can be checked that $\mathcal{L}_1 \neq \mathcal{L}_2$. This construction has a converse:

**Construction 5.10.** The tensor $\mu$ can be reconstructed from the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$. Namely, the spaces $U, V$ are isomorphic to $H^0(E, \mathcal{L}_1)$ and $H^0(E, \mathcal{L}_2)$ respectively, and the tensor $\mu : V \otimes U \to W$ is nothing more than the canonical map $H^0(E, \mathcal{L}_1) \otimes H^0(E, \mathcal{L}_2) \to H^0(E, \mathcal{L}_1 \otimes \mathcal{L}_2)$.

**Remark 5.11.** Note that we can also consider a triple $(E, \mathcal{L}_1, \mathcal{L}_2)$ such that $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the procedure described above produces a tensor with $\Gamma_V \cong \mathbb{P}(V)$, which defines the usual commutative projective plane. In this case the tensor $\mu$ does not depend on the curve $E$. The details of these constructions and statements can be found in [ATV, BP].

Now let us see what our construction gives in the case of noncommutative planes. In some sense we repeat the construction from the proof of Proposition 4.6 in this case. The subcategory generated by $(F_0, F_1)$ is a subcategory of $(\mathcal{O}, \mathcal{O}(1))$ on the usual $\mathbb{P}^2 = \mathbb{P}(U^*)$. Now we should glue to this category the object $F_2$. The projection of $F_2$ on the subcategory generated by $(F_0, F_1)$ can be represented by the following complex $T \otimes \mathcal{O} \xrightarrow{\nu} V \otimes \mathcal{O}(1)$ on $\mathbb{P}^2$. This complex is a resolution of cokernel of the map. It is isomorphic to the sheaf $\mathcal{O}_E(\mathcal{L}_1 \otimes \mathcal{L}_2)$, where $E$ is a curve of degree 3 on $\mathbb{P}^2$, $\mathcal{L}_1$ is the restriction of $\mathcal{O}(1)$ on $E$, and $\mathcal{L}_2$ is another line bundle of degree 3 on $E$. 

At first, we take projectivization of $V \otimes \mathcal{O}(1)$. We obtain $\mathbb{P}(U^*) \times \mathbb{P}(V^*)$ and the line bundle $\mathcal{O}(1,1)$ on it. The direct image of this bundle on the first component is isomorphic to $V \otimes \mathcal{O}(1)$ on $\mathbb{P}(U^*)$. After that we consider $Y = \mathbb{P}(U^*) \times \mathbb{P}(V^*) \times \mathbb{P}^1$ and the line bundle $\mathcal{O}(1,1,-2)$ on it. The morphism $\nu$ induces an element $\epsilon \in \text{Ext}_Y^1(T \otimes \mathcal{O}_Y, \mathcal{O}(1,1,-2))$. Now we take a vector bundle $\mathcal{F}$ on $Y$ that is extension
\[ 0 \longrightarrow \mathcal{O}(1,1,-2) \longrightarrow \mathcal{F} \longrightarrow T \otimes \mathcal{O}_Y \longrightarrow 0 \]
Finally, we take $Z = \mathbb{P}(\mathcal{F})$ and the line bundle $\mathcal{L} = \mathcal{O}_Z(1)$. The direct image of $\mathcal{L}$ with respect to the projection on $\mathbb{P}(U^*)$ is isomorphic to the complex \([5.3]\). Now if we consider three line bundles $\mathcal{O}_Z$, pull back of $\mathcal{O}(1)$ from $\mathbb{P}(U^*)$, and $\mathcal{L} = \mathcal{O}_Z(1)$ on $Z$, then it is an exceptional collection on $Z$ and the correspondent subcategory in $\text{Perf} - Z$, generated by them, is equivalent to the category of perfect complexes on the noncommutative projective planes. Different noncommutative projective planes corresponds to the different vector bundles $\mathcal{F}$ that depends on the element $\epsilon$. Let us summarize what we have got.

**Proposition 5.12.** For any noncommutative deformation of the projective plane $\mathbb{P}_\mu^2$ the DG category $\text{Perf} - \mathbb{P}_\mu^2$ is quasi-equivalent to a full DG subcategory of $\text{Perf} - Z$, where $Z$ is the projectivization of a 4-dimensional vector bundle $\mathcal{F}$, defined as extension (18), over $Y = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1$.

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