A hybridizable discontinuous Galerkin method for the coupled Navier–Stokes and Darcy problem

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Abstract

We present and analyze a strongly conservative hybridizable discontinuous Galerkin finite element method for the coupled incompressible Navier–Stokes and Darcy problem with Beavers–Joseph–Saffman interface condition. An a priori error analysis shows that the velocity error does not depend on the pressure, and that velocity and pressure converge with optimal rates. These results are confirmed by numerical examples.

1 Introduction

We consider the solution of the coupled incompressible Navier–Stokes and Darcy problem with Beavers–Joseph–Saffman interface condition. The Navier–Stokes equations describe the motion of a freely flowing incompressible fluid in one sub-region of the domain. These equations are coupled by the Beavers–Joseph–Saffman interface condition to the Darcy equations that describe the flow of a fluid in porous media, the second sub-region of our domain. For the analysis and applications of these equations we refer to [20].

A conforming finite element method for the coupled Navier–Stokes/Darcy problem was proposed and analyzed by Badea et al. [2], where they consider a Taylor–Hood discretization of the Navier–Stokes equations and use quadratic Lagrangian elements for the primal formulation of the Darcy equation. Girault and Riviè re [24] consider a primal form of the Darcy equations coupled to the velocity-pressure formulation of the Navier–Stokes equations. They propose a discontinuous Galerkin (DG) method based on an upwind Lesaint–Raviart DG discretization of the convective terms in the Navier–Stokes equations and non-symmetric, symmetric, and incomplete interior penalty Galerkin methods for the diffusion terms in the Navier–Stokes equations and for the primal form of the Darcy equation. Later, using the dual-mixed formulation of the Darcy equation, a conforming mixed finite element method was proposed by Discacciati and Oyarzúa [19]. They use Bernardi–Raugel and Raviart–Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for the Lagrange-multiplier used to couple the Navier–Stokes and Darcy equations. Extensions of this work include a conforming mixed finite element method for the dual mixed formulations of both the Navier–Stokes (with non-linear viscosity) and Darcy equations [5], and a conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer problem [6]. We further mention that a DG discretization of the Navier–Stokes and dual-mixed formulation of the Darcy equation was proposed (but not analyzed) in [25] while analysis and finite element formulations of the transient Navier–Stokes/Darcy problem is addressed, for example, in [7, 8, 11, 12, 13].

In this paper, we are particularly interested in strongly conservative discretizations as they can be shown to be pressure-robust [33, 34], i.e., the velocity error can be shown to be independent of the best approximation error of pressure scaled by the inverse of the viscosity. One approach to obtain a pressure-robust

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discretization is by using divergence-conforming velocity spaces [29]. Divergence-conforming DG methods have been introduced for the Stokes and Navier–Stokes equations in [16, 40]. However, DG methods are known to be expensive. As a remedy, hybridizable discontinuous Galerkin (HDG) methods were introduced in [17] to improve the computational efficiency of traditional DG methods through hybridization. Recently, divergence-conforming HDG methods have been introduced for the Stokes and Navier–Stokes equations, e.g., [14, 22, 32, 35].

For the Stokes–Darcy problem a strongly conservative DG discretization was proposed by Girault, Kanschat and Rivièrè [26, 30] using divergence-conforming velocity spaces. A strongly conservative HDG method, using similar spaces, was later proposed by Fu and Lehrenfeld [23] for the same problem. We, however, are not aware of divergence-conforming DG or HDG methods for the coupled Navier–Stokes/Darcy problem.

In [10], we presented a strongly conservative discretization of the velocity-pressure formulation of the Stokes equations coupled to the dual-mixed formulation of the Darcy equations. In this paper we extend this approach to the coupled Navier–Stokes/Darcy problem. The problem is formulated in section 2 while existence and uniqueness of a solution to this problem is discussed in section 3. The HDG method is proposed in section 4 where we also discuss well-posedness of this discretization. We present an a priori error analysis of the method in section 5 where we prove optimal (pressure-robust) rates of convergence in the energy norm. Numerical examples in section 6 serve to verify our theoretical results and conclusions are drawn in section 7.

2 The coupled Navier–Stokes and Darcy problem

We consider the coupled Navier–Stokes and Darcy problem on a bounded two (dim = 2) or three (dim = 3) dimensional domain \( \Omega \subset \mathbb{R}^{\text{dim}} \) which is decomposed into two disjoint domains \( \Omega^s \) and \( \Omega^d \). Fluid flow in the free fluid region \( \Omega^s \) is modeled by the Navier–Stokes equations while in the porous region \( \Omega^d \) fluid flow is modeled by the Darcy equations. This coupled system of equations is given by

\[
\begin{align*}
\nabla \cdot (u \otimes u) + \nabla \cdot \sigma &= f^s \quad &\text{in } \Omega^s, \\
\kappa^{-1} u + \nabla p &= 0 \quad &\text{in } \Omega^d, \\
-\nabla \cdot u &= \chi^d f^d \quad &\text{in } \Omega,
\end{align*}
\]

where \( u \) is the fluid velocity, \( p \) denotes the kinematic pressure in \( \Omega^s \) and the piezometric head in \( \Omega^d \), \( \sigma := p \mathbb{I} - 2 \mu \varepsilon(u) \) is the diffusive part of the fluid momentum flux in \( \Omega^s \), \( \varepsilon(u) := (\nabla u + (\nabla u)^T)/2 \) is the strain rate tensor, \( \mu > 0 \) is the constant kinematic viscosity, \( f^s \) a body force, \( f^d \) is a source/sink term, and \( \kappa \) is a positive definite symmetric matrix corresponding to the permeability of \( \Omega^d \). We will assume that there exist positive constants \( \kappa_{\text{min}} \) and \( \kappa_{\text{max}} \) such that

\[
\forall z \in \mathbb{R}^{\text{dim}}, \quad \kappa_{\text{min}} |z|^2 \leq \langle \kappa(x)z \rangle \cdot z \leq \kappa_{\text{max}} |z|^2, \quad x \in \Omega^d.
\]

The boundary of the domain \( \Omega, \partial \Omega \), is assumed to be a polyhedral Lipschitz boundary. The boundaries of \( \Omega^j \) are denoted by \( \partial \Omega^j \), where \( j = s, d \). The interface \( \Gamma^I := \partial \Omega^s \cap \partial \Omega^d \) is assumed to be Lipschitz polyhedral. We furthermore define the exterior boundaries \( \Gamma^j := \partial \Omega^j \setminus \Gamma^I \) for \( j = s, d \). On \( \Gamma^j \), we denote by \( n \) the outward unit normal to \( \Omega^j \) while on \( \Gamma^I \), \( n \) denotes the unit normal vector pointing outward from \( \Omega^s \). On \( \Gamma^I \) we further define the orthonormal tangential vectors \( \tau^k \) for \( 1 \leq k \leq \text{dim } - 1 \).

Let \( \chi^d \) be the characteristic function that has the value 1 in \( \Omega^d \) and 0 in \( \Omega^s \) and let \( \chi^s = 1 - \chi^d \). We define \( u^j = \chi^j u \) and \( p^j = \chi^j p \) for \( j = s, d \). The Navier–Stokes and Darcy equations are coupled at the interface by assuming continuity of the normal component of the velocity, the Beavers–Joseph–Saffman law [3, 37], and a balance of forces. These assumptions result in the following transmission conditions on \( \Gamma^I \):

\[
\begin{align*}
\begin{cases}
\quad u^s \cdot n = u^d \cdot n \\
-2\mu (\varepsilon(u^s)n) \cdot \tau^i &= \frac{\alpha \mu}{\sqrt{\kappa_i}} u^s \cdot \tau^i, &1 \leq i \leq \text{dim } - 1 \\
\quad (\sigma n) \cdot n &= p^d
\end{cases} \quad \text{on } \Gamma^I,
\end{align*}
\]

where \( \alpha \) is the interfacial thickness parameter, \( \kappa_i \) the leading diagonal element of \( \kappa \), and \( \tau^i \) the orthonormal tangential vector on the interface.
where \( \kappa_i = \tau^i \cdot (\kappa \tau^i) \) and \( \alpha > 0 \) is an experimentally determined dimensionless constant. To complete the problem description, we impose the following exterior boundary conditions:

\[
    u = 0 \text{ on } \Gamma^s \quad \text{and} \quad u \cdot n = 0 \text{ on } \Gamma^d.
\]  

Finally, for well-posedness of the problem, we require \( \int_{\Omega^d} f^d \, dx = 0 \) and \( \int_{\Omega} p \, dx = 0 \).

### 3 Existence and uniqueness of solutions to a mixed weak formulation

Let

\[
    \mathcal{X} := \{ u = (u^s, u^d) \in \mathcal{X}^s \times \mathcal{X}^d : u^s \cdot n = u^d \cdot n \text{ on } \Gamma^f \},
\]

where \( \mathcal{X}^s := \{ v \in [H^1(\Omega^s)]^\text{dim} : v = 0 \text{ on } \Gamma^s \} \), \( \mathcal{X}^d := \{ v \in H(\text{div}; \Omega^d) : v \cdot n = 0 \text{ on } \Gamma^d \} \), and endow \( \mathcal{X} \) with the product norm \( \| u \|_{\mathcal{X}} := (\| u^s \|_{1, \Omega^s}^2 + \| u^d \|_{\text{div}, \Omega^d}^2)^{1/2} \) for all \( u \equiv (u^s, u^d) \in \mathcal{X} \), where \( \| u^d \|_{\text{div}, \Omega^d}^2 := \| u^d \|_{\Omega^d}^2 + \| \nabla \cdot u^d \|_{\Omega^d}^2 \). Since the \( \| \cdot \|_{\text{div}, \Omega^d} \)-norm is only applied to functions on \( \Omega^d \), we will drop the subscript \( \Omega^d \) and write \( \| \cdot \|_{\text{div}} \).

The following mixed weak formulation for eqs. (1) to (3) was proposed in [25, Section 3.3]: Find \((u, p) \in \mathcal{X} \times \mathcal{Q} \), with \( \mathcal{Q} := L^2_0(\Omega) \), such that:

\[
    a(u; u, v) + b(v, p) + b(u, q) = \ell^s(v) + \ell^d(q) \quad \forall (v, q) \in \mathcal{X} \times \mathcal{Q},
\]

where \( \ell^s(v) := \int_{\Omega^s} f^s \cdot v \, dx \), \( \ell^d(q) := \int_{\Omega^d} f^d q \, dx \), and where the forms \( a : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \) and \( b : \mathcal{X} \times \mathcal{Q} \rightarrow \mathbb{R} \) are defined as:

\[
    a(u; u, v) := t(w; u^s, v^s) + a^s(u^s, v^s) + a^d(u^d, v^d) + a^l(u^s, u^s), \quad b(v, q) := - \int_{\Omega} q \nabla \cdot v \, dx,
\]

with

\[
    a^s(u, v) := \int_{\Omega^s} 2\mu \varepsilon(u) : \varepsilon(v) \, dx, \quad a^d(u, v) := \int_{\Omega^d} \mu \kappa^{-1} u \cdot v \, dx,
\]

\[
    a^l(u, v) := \int_{\Gamma^l} \frac{\alpha \mu}{\sqrt{\kappa_i}} (u \cdot \tau^i)(v \cdot \tau^i) \, ds, \quad t(w; u, v) := \int_{\Omega^s} (w \cdot \nabla u) \cdot v \, dx.
\]

Well-posedness for the coupled Navier–Stokes and Darcy problem was shown in [19] for the case that the source/sink term \( f^d \) in eq. (1c) is zero. With minor modifications of these proofs, well-posedness can be shown also for the case \( f^d \neq 0 \). In particular, it can be shown that if the data \( f^s \in [L^2(\Omega^s)]^\text{dim} \) and \( f^d \in L^2(\Omega^d) \) satisfy the smallness condition

\[
    C_p \| f^s \|_{\Omega^s} + 2\mu C_f C_{bb}^{-1} \| f^d \|_{\Omega^d} < \mu^2 C_{ae} \min \left( C_{ae} C_w^{-1}, 2C_{ae} \delta C_{si,2}^{-1} C_{si,4}^{-1} \right),
\]

then there exists a unique solution \((u, p) \in \mathcal{X} \times \mathcal{Q} \) to eq. (5). Moreover, this solution satisfies

\[
    \| u \|_{\mathcal{X}} \leq C_{ae}^{-1} \left( \mu^{-1} C_p \| f^s \|_{\Omega^s} + 2C_f C_{bb}^{-1} \| f^d \|_{\Omega^d} \right), \quad (7a)
\]

\[
    \| p \|_{\Omega^s} \leq 2C_f (C_{ae} C_{bb})^{-1} \left( C_p \| f^s \|_{\Omega^s} + C_f C_{bb}^{-1} \mu \| f^d \|_{\Omega^d} \right). \quad (7b)
\]

We remark that \( C_f \) and \( C_{ae} \) are the constants related to, respectively, the boundedness and coercivity of \( a(\cdot, \cdot, \cdot) \), which are given by:

\[
    C_f = \max (2 + C_{ae} C_{ae}^{-1/2} + 2C_w C_{ae} \delta C_{si,2}^{-1} C_{si,4}^{-1}, \kappa_{\min}^{-1}), \quad C_{ae} = \min (C_{ae}^{a} (1 - \delta), \kappa_{\min}^{-1}).
\]

All other constants are independent of \( \kappa_{\min}, \kappa_{\max}, \alpha, \) and \( \mu \). Here \( C_{ae}^{a} \) and \( C_{ae}^{l} \) are related to, respectively, the coercivity constant of \( a^s \) and the continuity constant of \( a^l \). Furthermore, \( C_{bb} \) is the inf-sup constant of \( b(\cdot, \cdot) \), \( C_p \) is the Poincaré constant, \( C_w \) is a constant related to the dimension of the problem and the Sobolev embedding constant from \( H^1(\Omega^s) \) into \( L^4(\Omega^s) \), \( \delta \) is a constant that lies in \((0, 1)\), and \( C_{si,2} \) and \( C_{si,4} \) are constants relating, respectively, the \( L^2- \) and \( L^4 \)-norms on the interface to the \( H^1 \)-norm on \( \Omega^s \).
4 The hybridizable discontinuous Galerkin method

4.1 The discretization

To define the discretization we first introduce the triangulations $\mathcal{T}^j$ of $\Omega_j$, with $j = s, d$. We assume these triangulations consist of shape-regular simplices $K$ and that the two triangulations $\mathcal{T}^j$, $j = s, d$, coincide on the interface $\Gamma_I$. We further denote by $\mathcal{T} := \mathcal{T}^s \cup \mathcal{T}^d$ the triangulation of $\Omega$ and we define $h := \max_{K \in \mathcal{T}} h_K$, where $h_K$ is the diameter of $K$.

On the cells $K$ we define the discontinuous finite element spaces

$$X_h := \{ v_h \in [L^2(\Omega)]^{\dim} : v_h \in [P_k(K)]^{\dim}, \forall K \in \mathcal{T} \},$$

$$X^j_h := \{ v_h \in [L^2(\Omega^j)]^{\dim} : v_h \in [P_k(K)]^{\dim}, \forall K \in \mathcal{T}^j \}, \quad j = s, d,$$

$$Q_h := \{ q_h \in L^2(\Omega) : q_h \in P_{k-1}(K), \forall K \in \mathcal{T} \},$$

$$Q^j_h := \{ q_h \in Q_h : q_h \in P_{k-1}(K), \forall K \in \mathcal{T}^j \}, \quad j = s, d,$$

where $P_k(K)$ denotes the space of polynomials of degree $k$ on any cell $K$.

By $F^j$ and $\Gamma^j_0$ we denote the set and union of facets $F$ on the subdomain $\Omega^j$, $j = s, d$. By $\mathcal{F}$ and $\Gamma_0$ we denote the set and union of all facets in $\Omega$ while $\mathcal{F}^I$ denotes the set of all facets on $\Gamma_I$. Then, denoting by $P_m(F)$ the space of polynomials of degree $m$ on any facet $F$, we define the following facet finite element spaces:

$$\bar{X}_h := \{ \bar{v}_h \in [L^2(\Omega^j_0)]^{\dim} : \bar{v}_h \in [P_k(F)]^{\dim} \forall F \in \mathcal{F}^s, \bar{v}_h = 0 \text{ on } \Gamma^s \},$$

$$\bar{Q}^j_h := \{ \bar{q}_h \in L^2(\Omega^j_0) : \bar{q}_h \in P_k(F) \forall F \in \mathcal{F}^j \}, \quad j = s, d.$$

Grouping the cell and facet unknowns in the following compact notation:

$$v_h := (v_h, \bar{v}_h) \in \mathbf{X}_h := X_h \times \bar{X}_h,$$

$$q_h := (q_h, \bar{q}_h) \in \mathbf{Q}_h := Q_h \times \bar{Q}_h,$$

we propose the following HDG discretization for eqs. (1) to (3): Find $(u_h, p_h) \in \mathbf{X}_h \times \mathbf{Q}_h$ such that for all $(v_h, q_h) \in \mathbf{X}_h \times \mathbf{Q}_h$

$$a_h(u_h; u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = \ell^s(v_h) + \ell^d(q_h), \quad (8)$$

where

$$a_h(w; u, v) := t_h(w; u, v) + a_h^s(u, v) + a_h^d(u, v) + a^d(\bar{u}, \bar{v}), \quad (9a)$$

$$a_h^s(u, v) := \sum_{K \in \mathcal{T}^s} \int_K 2 \mu \varepsilon(u) : \varepsilon(v) \, dx + \sum_{K \in \mathcal{T}^s} \int_{\partial K} 2 \mu \frac{\partial e}{h_K}(u - \bar{u}) \cdot (v - \bar{v}) \, ds \quad (9b)$$

$$- \sum_{K \in \mathcal{T}^s} \int_{\partial K} 2 \mu \varepsilon(u) : (v - \bar{v}) \cdot n \, ds - \sum_{K \in \mathcal{T}^s} \int_{\partial K} 2 \mu \varepsilon(v) : (u - \bar{u}) \cdot n \, ds,$$

$$t_h(w; u, v) := - \sum_{K \in \mathcal{T}^s} \int_K u \otimes w : \nabla v \, dx + \sum_{K \in \mathcal{T}^s} \int_{\partial K} \frac{1}{2} w \cdot n (u + \bar{u}) \cdot (v - \bar{v}) \, ds \quad (9c)$$

$$+ \sum_{K \in \mathcal{T}^s} \int_{\partial K} \frac{1}{2} |w| n (u - \bar{u}) \cdot (v - \bar{v}) \, ds + \int_{\Gamma^I} (w \cdot n) \bar{u} \cdot \bar{v} \, ds,$$

and where

$$b_h(v, q) := b_h^s(v, \bar{q}) + b_h^d(v, q),$$

$$b_h^s(p^j, v) := - \sum_{K \in \mathcal{T}^j} \int_K p \nabla \cdot v \, dx + \sum_{K \in \mathcal{T}^j} \int_{\partial K} \bar{p}^j v \cdot n^j \, ds,$$

$$b_h^d(p^j, \bar{v}) := - \int_{\Gamma^I} \bar{p}^j \bar{v} \cdot n^j \, ds.$$
for \( j = s, d \). In the above definitions, \( \beta > 0 \) is a penalty parameter and \( n^j \) is the outward unit normal vector on the boundary of any element \( K \in \mathcal{T} \). On the interface \( \Gamma^I \), \( n^s = -n^d \). If it is clear to which set \( K \) belongs, we drop the superscript \( j \).

The HDG discretization eq. (8) is the Navier–Stokes/Darcy extension of the discretization recently proposed for the coupled Stokes/Darcy problem in [10] (where the matrix \( \kappa \) corresponding to the permeability was replaced by a positive constant). This discretization is strongly conservative, a property inherited by the HDG discretization of the Navier–Stokes/Darcy discretization. Indeed, the velocity solution \( u_h \in X_h \) to eq. (8) satisfies:

\[
-\nabla \cdot u_h = \chi^d \Pi_Q f^d \\
[ u_h \cdot n ] = 0 \\
\forall x \in K, \forall K \in \mathcal{T},
\]

\[
[ u_h \cdot n ] = 0 \\
\forall x \in F, \forall F \in \mathcal{F} \cup \mathcal{F}^I,
\]

\[
u = \bar{u}_h \cdot n \\
\forall x \in F, \forall F \in \mathcal{F}^I,
\]

where \( \Pi_Q \) is the \( L^2 \)-projection operator into \( Q_h \), \([\cdot]\) is the usual jump operator, and \( n \) is the unit normal vector on \( F \). See [10, Section 3.3] for a proof of eq. (11). In the following analysis it will be useful to introduce the following subspaces:

\[
Z_h^s := \{ v_h \in X_h^s : b^s_h(v_h, q_h^s) + b^{I,s}_h(\bar{v}_h, q_h^s) = 0 \forall q_h^s \in Q_h^s \},
\]

\[
Z_h := \{ v_h \in X_h : \sum_{j=s,d} (b^j_h(v_h, q_h^j) + b^{I,j}_h(\bar{v}_h, q_h^j)) = 0 \forall q_h^s \in Q_h \}.
\]

The velocity solution to eq. (8), \( u_h \), restricted to \( \Omega^s \) is such that \( u_h^s \in Z_h^s \). Generally, local momentum conservation of the Navier–Stokes equations needs to be sacrificed to obtain a stable discretization [15]. However, since \( u_h^s \) is exactly divergence-free and \( H(\text{div}) \)-conforming in \( \Omega^s \), this sacrifice is unnecessary and the discretization of the Navier–Stokes equations in eq. (8) is locally momentum conserving (unlike, for example, the DG method of [24]). Note further that any \( v_h \in Z_h \) satisfies eq. (11) with \( f^d = 0 \).

### 4.2 Notation and extension of known results

Before addressing the well-posedness of the HDG method eq. (8), we briefly introduce notation and extend a few properties of the discretization previously shown for the coupled Stokes/Darcy problem in [10]. On a domain \( D \) we will use standard definitions and notation of the Sobolev spaces \( W^k_p(D) \) with corresponding norms \( \| \cdot \|_{p,k,D} \) (see, for example, [1, 4]). If \( p = 2 \) we set \( H^k(D) = W^k_2(D) \) and write \( \| \cdot \|_{k,D} \) instead of \( \| \cdot \|_{2,k,D} \). If \( k = 0 \), we note that \( W^0_0(D) \) coincides with \( L^p(D) \). For \( p \neq 2 \), we denote the norm on \( L^p(D) \) by \( \| \cdot \|_{p,0,D} \). If \( p = 2 \), we denote the norm on \( L^2(D) \) by \( \| \cdot \|_D \).

Let us now introduce the following spaces:

\[
X := \{ u = (u^s, u^d) \in X^s \times X^d : u^s \cdot n = u^d \cdot n \text{ on } \Gamma^I \},
\]

\[
Q := \{ q = (q^s, q^d) : q^s \in Q^s := H^1(\Omega^s), q^d \in Q^d := H^2(\Omega^d), \int_\Omega q \, dx = 0 \},
\]

where \( X^s := \{ v \in [H^2(\Omega^s)]^{\dim} \text{ : } v = 0 \text{ on } \Gamma^0 \} \) and \( X^d := \{ v \in [H^1(\Omega^d)]^{\dim} \text{ : } v \cdot n = 0 \text{ on } \Gamma^d \} \). We will denote the trace space of \( X \) by \( \bar{X} \) and we introduce the trace operator \( \gamma_\nu : X \to \bar{X} \) which restricts functions in \( X \) to \( \Gamma^0 \). Likewise, the trace space of \( Q^d \) is denoted by \( \bar{Q}^d \) and \( \gamma(Q^d) : Q^d \to \bar{Q}^d \) restricts functions in \( Q^d \) to \( \Gamma^0 \). If it is clear from the context on which function spaces the trace operator acts, we drop the subscript notation from \( \gamma \). Using similar notation as in section 4.1, we next define \( \bar{X} := X \times \bar{X} \) and \( \bar{Q} := Q \times \bar{Q}^s \times \bar{Q}^d \). With these definitions we then introduce the extended function spaces:

\[
X(h) := X_h + X, \quad Q(h) := Q_h + Q, \quad \bar{X}(h) := \bar{X}_h + \bar{X}, \quad \bar{Q}(h) := \bar{Q}_h + \bar{Q}.
\]

Furthermore, we will need \( X^s(h) := X^s_h + X^s \).
Lemma 1 (Consistency). Let \((u, p) \in X \times Q\) solve the Navier–Stokes/Darcy problem eqs. 1 to 3. Let \(u = (u, \gamma(u)), p = (u, \gamma(p^r), \gamma(p^d))\), then

\[
a_h(u; \nu, v_h) + b_h(v_h, p) + b_h(u, q_h) = \ell^s(v_h) + \ell^d(q_h) \quad \forall (v_h, q_h) \in X_h \times Q_h.
\]

Proof. This is an immediate consequence of [10, Lemma 1] and that

\[
t_h(u; \nu, v_h) = \sum_{K \in T} \int_K \nabla \cdot (u \otimes u) \cdot v_h \, dx \quad \forall v_h \in X_h,
\]

which follows by integration by parts, smoothness of \(u\), and single-valuedness of \(\bar{v}_h\).  

For the analysis of eq. (8) we require the following two norms defined on \(X(h)\):

\[
\|v\|^2_v := \|v\|^2_{v,s} + \|v\|^2_{v,d} + \|\bar{v}\|^2_{\Gamma^f},
\]

\[
\|v\|^2_{\nu,s} := \|v\|^2_{v,s} + \sum_{K \in T^s} h_K^2 \|v\|^2_{2,K},
\]

\[
\|v\|^2_{\nu,d} := \|v\|^2_{\nu,d} + \sum_{F \in F^d \setminus F^f} h_F^{-1} \|v \cdot n\|_{F}^2 + \sum_{K \in T^d} h_K^{-1} \|(v - \bar{v}) \cdot n\|_{\partial K \cap \Gamma^f}^2,
\]

where we remark that \(\|v \cdot n\| = v \cdot n\) on \(\Gamma^d\).

Remark 1. The norms \(\|v\|_v\) and \(\|v\|_\nu\) are different from the norms used in [10]; instead of only \(\|v\|_{\Omega^d}\) in \(\|v\|_v\) we use \(\|v\|_{\nu,d}\) for the functions in the Darcy part of the domain.

The norms \(\|v\|_v\) and \(\|v\|_\nu\) are equivalent on \(X_h\), i.e., there exists a constant \(c_e\) such that \(\|v\|_v \leq \|v\|_\nu \leq c_e \|v\|_v\) for all \(v \in X_h\), see [41, eq. (5.5)]. On \(X_h\), we will also require the following results from [24, Theorem 4.4 and Proposition 4.5]:

\[
\|v_h\|_{\Omega^d} \leq c_p \|v_h\|_{h,1,\Omega^d} \leq c_p \|v_h\|_{v,s} \quad \forall v_h \in X_h^s. \tag{13}
\]

For \(r \geq 2\):

\[
\|v_h\|_{r,\Omega_f} \leq c_{s,i,r} \|v_h\|_{1,1,\Omega^s} \leq c_{s,i,r} \|v_h\|_{v,s} \quad \forall v_h \in X_h^s, \tag{14}
\]

where \(\|v_h\|_{1,1,\Omega^d} := \|(v_h, \{v_h\})\|_{v,s}\) and where \(c_p\) and \(c_{s,i,r}\) are positive constants independent of \(h\). On \(Q(h)\), we define

\[
\|q\|^2_p := \|q^s\|^2_{p,s} + \|q^d\|^2_{p,d} \quad \text{where} \quad \|q^s\|^2_{p,s} := \|q\|^2_{\Omega^d} + \sum_{K \in T^d} h_K \|q^d\|^2_{\partial K}, \quad j = s, d.
\]

For the linear forms on the right hand side of eq. (8) we note, using eq. (13), that

\[
|\ell^s(v_h)| \leq \|\ell^s\|_{\Omega^d} \|v_h\|_{\Omega^d} \leq c_p \|f^s\|_{\Omega^d} \|v_h\|_{v,s} \leq c_p \|f^s\|_{\Omega^d} \|v_h\|_v \quad \forall v_h \in X_h, \tag{15a}
\]

\[
|\ell^d(q_h)| \leq \|\ell^d\|_{\Omega^d} \|q_h\|_{\Omega^d} \leq \|f^d\|_{\Omega^d} \|q_h\|_{p,d} \leq \|f^d\|_{\Omega^d} \|q_h\|_p \quad \forall q_h \in Q_h. \tag{15b}
\]

The following properties of the different bilinear forms in eq. (8) hold (see [10, Lemma 3]): for all \(u, v \in X(h)\):

\[
|a_h^s(u, v)| \leq \mu_{ac} |u|_{v',s} \|v\|_{v',s}, \tag{16a}
\]

\[
|a^d(u, v)| \leq \mu_{min}^{-1} |u|_{\Omega^d} \|v\|_{\Omega^d}, \tag{16b}
\]

\[
|a^f(\bar{u}, \bar{v})| \leq \alpha \mu_{min}^{-1/2} \|\bar{u}\|_{\Gamma^f} \|\bar{v}\|_{\Gamma^f}, \tag{16c}
\]

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where $c^a_{ac} > 0$ is a constant independent of $h$ and $(z)^l = z - (z \cdot n)n$. Furthermore, by [10, Lemma 2], there exists a constant $c^a_{ae} > 0$, independent of $h$, and a constant $\beta_0 > 0$ such that for $\beta > \beta_0$ and for all $v_h \in X_h$:

$$a^d_h(v_h, v_h) \geq \mu c^a_{ae} \|v_h\|_{e,s}^2. \quad (17a)$$

Additionally, for all $v_h \in X_h$,

$$a^d(v_h, v_h) \geq \mu \kappa_{max}^{-1}\|v_h\|_{1d}^2, \quad a^f(\bar{v}_h, \bar{v}_h) \geq \alpha \mu \kappa_{max}^{-1/2} \|\bar{v}_h\|_{1f}^2. \quad (17b)$$

To prove properties of the bilinear form $b_h$, we use the Brezzi–Douglas–Marini (BDM) interpolation operator $\Pi_V : [H^1(\Omega)]^{\text{dim}} \to X_h \cap H(\text{div}; \Omega)$. For all $u \in [H^{k+1}(K)]^{\text{dim}}$ this interpolation operator satisfies [27, Lemma 7]:

$$\int_K q_h(\nabla \cdot u - \nabla \cdot \Pi_V u) \, dx = 0 \quad \forall q_h \in P_{k-1}(K), \quad (18a)$$

$$\int_F \bar{q}_h(n \cdot u - n \cdot \Pi_V u) \, ds = 0 \quad \forall \bar{q}_h \in P_k(\Omega), \quad F \text{ is a face on } \partial K, \quad (18b)$$

as well as the interpolation estimates:

$$\|u - \Pi_V u\|_{m,K} \leq ch_K^{l-m} \|u\|_{l,K}, \quad m = 0, 1, 2, \max(1, m) \leq l \leq k + 1, \quad (19a)$$

$$\|\nabla \cdot (u - \Pi_V u)\|_K \leq ch_K^l \|\nabla \cdot u\|_{l,K}, \quad 0 \leq l \leq k. \quad (19b)$$

We will also require the $L^2$-projection into the facet velocity space, $\bar{\Pi}_V : [H^1(\Omega^e)]^{\text{dim}} \to \bar{X}_h$. The following two lemmas prove properties of the bilinear form $b_h$. We start in lemma 2 by proving an inf-sup condition. An inf-sup condition for $b_h$ was recently proven in [10, Theorem 1]. However, this proof is modified here due to the use of different norms (see remark 1).

Lemma 2. There exists a positive constant $c_{bb}$, independent of $h$, such that for all $q_h \in Q_h$,

$$c_{bb} \|q_h\|_p \leq \sup_{v_h \in X_h \setminus \{0\}} \frac{b_h(v_h, q_h)}{\|v_h\|_v}. \quad (20)$$

Proof. Let us write $b_h(v_h, q_h) = b^1_h(v_h, q_h) + b^2_h(v_h, (\bar{q}_h, \bar{q}_h))$, where

$$b^1_h(v_h, q_h) = -\sum_{j=s,d} \sum_{K \in T} \int_K q_h \nabla \cdot v_h \, dx,$n
time

$$b^2_h(v_h, (\bar{q}_h, \bar{q}_h)) = \sum_{j=s,d} \left( \sum_{K \in T \cap I^i} \int_{\partial K} \bar{q}_h \cdot v_h \cdot n^j \, ds - \int_{I^i} \bar{q}_h \cdot \bar{v}_h \cdot n^j \, ds \right).$$

time

To prove eq. (20) we apply [28, Theorem 3.1] which, in this context, states that if there exist constants $c_{b1} > 0$ and $c_{b2} > 0$, independent of $h$, such that for all $q_h \in Q_h$ and $(\bar{q}_h, \bar{q}_h) \in Q^d_h \times Q^d_h$,

$$c_{b1} \|q_h\|_\Omega \leq \sup_{v_h \in \text{Ker}(b^1_h)} \frac{b^1_h(v_h, q_h)}{\|v_h\|_v}, \quad (21a)$$

$$\left( c_{b2} \sum_{j=s,d} \sum_{K \in T} h_K \|\bar{q}_h\|_{\partial K}^2 \right)^{1/2} \leq \sup_{v_h \in X_h \setminus \{0\}} \frac{b^2_h(v_h, (\bar{q}_h, \bar{q}_h))}{\|v_h\|_v}, \quad (21b)$$

where $\text{Ker}(b^1_h) := \{v_h \in X_h : b^1_h(v_h, (\bar{q}_h, \bar{q}_h)) = 0 \forall (\bar{q}_h, \bar{q}_h) \in Q^d_h \times Q^d_h\}$, then there exists a constant $c_{bb}$ independent of $h$ such that eq. (20) holds for all $q_h \in Q_h$. As such, we now separately prove eqs. (21a) and (21b).
We start with eq. (21a). Let $q_h \in Q_h$. Then, since $q_h \in L_0^2(\Omega)$ there exists a $v \in [H_0^1(\Omega)]^{\dim}$ such that $-\nabla \cdot v = q_h$ and $c_{eq}\|v\|_{1,\Omega} \leq \|q_h\|_\Omega$ (e.g. [18, Theorem 6.5]). By properties of $\Pi_V$ and $\Pi^r_V$, it was shown in the proof of [10, Lemma 5] that

$$\|\langle \Pi_V v, \Pi^r_V v \rangle\|_{v,s} \leq C \|v\|_{1,\Omega}, \quad \|\langle \Pi_V v \rangle\|_{T^d} \leq C \|v\|_{1,\Omega}. \quad (22)$$

(The proof in [10, Lemma 5] assumes that $\Pi_V$ is the restriction of the Scott–Zhang interpolant to the mesh skeleton, but it holds also for the $L^2$-projection used here.) Next, by definition,

$$\|\langle \Pi_V v, \Pi^r_V v \rangle\|_{v,d}^2 = \|\Pi_V v\|_{\text{div}}^2 + \sum_{F \in \mathcal{F}_d \setminus \mathcal{F}_t} h_F^{-1} \|\langle \Pi_V v \cdot n \rangle\|_F^2 + \sum_{K \in \mathcal{T}_d} h_K^{-1} \|\Pi_V v - \Pi^r_V v\|_{\partial K \cap \Gamma_t}^2$$

$$= I_1 + I_2 + I_3.$$

Let us consider each term separately. For $I_1$, by a triangle inequality and eqs. (19a) and (19b), $I_1 \leq C \|v\|_{1,\Omega}^2$. Since $\Pi_V v \in H(\text{div}; \Omega^d)$ and $v = 0$ on $\Gamma^d$ we find that $I_2 = 0$. Finally, for $I_3$, we have

$$I_3 = \sum_{K \in \mathcal{T}_d} h_K^{-1} \|\Pi_V v - \Pi^r_V v\|_{\partial K \cap \Gamma_t}^2 \leq C \sum_{K \in \mathcal{T}_d} h_K^{-1} \|v\|_{1,K}^2 = C \|v\|_{1,\Omega}^2,$$

where the inequality follows from the proof of [36, Lemma 9]. Collecting the bounds for $I_1$ to $I_3$ proves $\|\langle \Pi_V v, \Pi^r_V v \rangle\|_{v,d} \leq C \|v\|_{1,\Omega}$. Combining this result with eq. (22), we find

$$\|\langle \Pi_V v, \Pi^r_V v \rangle\|_{v} \leq C \|v\|_{1,\Omega}.$$

At this point, we remark that $\langle \Pi_V v, \Pi^r_V v \rangle \in \text{Ker}(b_h^2)$. To see this, we note that

$$b_h^2(\langle \Pi_V v, \Pi^r_V v \rangle, (\bar{q}_h, \bar{q}_h')) = \int_{\Gamma^d} \bar{q}_h^s (\Pi_V v - \Pi^r_V v) \cdot n^s \, ds + \int_{\Gamma^d} \bar{q}_h^d (\Pi_V v - \Pi^r_V v) \cdot n^d \, ds$$

$$= \int_{\Gamma^d} \bar{q}_h^s (v - v) \cdot n^s \, ds + \int_{\Gamma^d} \bar{q}_h^d (v - v) \cdot n^d \, ds = 0,$$

where the first equality is because $\Pi_V v \cdot n^j$ is continuous on element boundaries and $\bar{q}_h^j$ is single-valued, and the second equality is by properties of $\Pi_V$ and $\Pi^r_V$, and $v \cdot n^j = 0$ on $\Gamma^j$. We therefore find,

$$\sup_{v_h \in \text{Ker}(b_h^2) \atop v_h \neq 0} \frac{\int_{\Omega} q_h \nabla \cdot v_h \, dx}{\|v_h\|_v} \geq \frac{\|q_h\|_v^2}{C \|v\|_{1,\Omega}} \geq \frac{c_{eq}}{C} \|q_h\|_\Omega,$$

where we used $c_{eq}\|v\|_{1,\Omega} \leq \|q_h\|_\Omega$ for the last inequality.

With eq. (21a) proven, we proceed with eq. (21b). Let $\bar{q}_h^j \in \bar{Q}_h^j$. Define $R_k(\partial K) := \{q : q \in L^2(\partial K), q|_F \in P_k(F) \forall F \in \mathcal{F}(K)\}$ where $\mathcal{F}(K)$ is the set of facets of the simplex $K$. Let $w_h^j \in [P_k(K)]^{\dim}$ for all $K \in \mathcal{T}_d$ such that $w_h^j := \bar{q}_h^j$, $j = s, d$, with $L : R_k(\partial K) \rightarrow [P_k(K)]^{\dim}$ the Brezzi–Douglas–Marini (BDM) local lifting operator (for example, [21, Proposition 2.10]). Define $w_h := \chi^s w_h^s + \chi^d w_h^d \in [P_k(K)]^{\dim}$ for all $K \in \mathcal{T}$. It was shown in the proof of [10, Lemma 6] that

$$\|\langle w_h, 0 \rangle\|_{v,s}^2 = \sum_{K \in \mathcal{T}_s} \left(\|\nabla (L \bar{q}_h^s)\|_K^2 + h_K^{-1} \|L \bar{q}_h^s\|_{\partial K}^2\right) \leq C \sum_{K \in \mathcal{T}_s} h_K^{-1} \|\bar{q}_h^s\|_{\partial K}^2. \quad (23)$$

Next, note that

$$\|\langle w_h^d, 0 \rangle\|_{v,d}^2 = \|w_h^d\|_{\text{div}}^2 + \sum_{F \in \mathcal{F}_d \setminus \mathcal{F}_t} h_F^{-1} \|w_h^d \cdot n\|_F^2 + \sum_{K \in \mathcal{T}_d} h_K^{-1} \|w_h^d \cdot n\|_{\partial K \cap \Gamma_t}^2 =: J_1 + J_2 + J_3.$$
From [10, Eq. (44)], and similar to eq. (23),

\[
J_1 \leq \sum_{K \in T^d} (\|Lq_h^d\|^2_K + \|\nabla (Lq_h^d)\|^2_K) \leq C \sum_{K \in T^d} h_K^{-1} \|q_h^d\|^2_{\partial K}.
\]

Since \(Lq_h^d \cdot n = \bar{q}_h^d\) on \(\partial K\) for all \(K \in T^d\),

\[
J_2 \leq C \sum_{K \in T^d} h_K^{-1} \|Lq_h^d \cdot n\|^2_{\partial K} = C \sum_{K \in T^d} h_K^{-1} \|\bar{q}_h^d\|^2_{\partial K}.
\]

Finally, for \(J_3\) we find

\[
J_3 = \sum_{K \in T^d} h_K^{-1} \|Lq_h^d \cdot n\|^2_{\partial K \cap \Gamma^t} = \sum_{K \in T^d} h_K^{-1} \|\bar{q}_h^d\|^2_{\partial K \cap \Gamma^t} \leq \sum_{K \in T^d} h_K^{-1} \|\bar{q}_h^d\|^2_{\partial K}.
\]

Collecting the bounds for \(J_1\) to \(J_3\), we find \(\|(w_h^d, 0)\|^2_v \leq C \sum_{K \in T^d} h_K^{-1} \|\bar{q}_h^d\|^2_{\partial K}\). Combining this with eq. (23),

\[
\|(w_h, 0)\|^2_v \leq C \sum_{j=s,d} \left( \sum_{K \in T^j} h_K^{-1} \|q_h^d\|^2_{\partial K} \right).
\]

Using that \(w_h^d \cdot n = \bar{q}_h^d\), \(j = s, d\), eq. (21b) now follows using identical steps as the proof of [10, Lemma 6].

The next lemma proves boundedness of \(b_h\).

**Lemma 3.** There exists a positive constant \(c_{bc}\), such that for all \((v, q) \in X(h) \times Q(h)\),

\[
|b_h(v, q)| \leq c_{bc}\|v\|_v\|q\|_p.
\]

**Proof.** Note that

\[
b_h(v, q) = \underbrace{b_h^s(v, q^s)}_{I_1} + \underbrace{b_h^s(\bar{v}, q^s)}_{I_2} + \underbrace{b_h^d(v, q^d)}_{I_2} + \underbrace{b_h^d(\bar{v}, q^d)}_{I_2}.
\]

Let us consider \(I_1\) and \(I_2\) separately. For \(I_1\), we have

\[
|I_1| = - \sum_{K \in T^s} \int_K q \nabla \cdot v \, dx + \sum_{K \in T^s} \int_{\partial K} \bar{q}^s (v - \bar{v}) \cdot n^s \, ds \\
\leq \left( \sum_{K \in T^s} \|\nabla v\|^2_K + \sum_{K \in T^s} h_K^{-1} \|v - \bar{v}\|^2_{\partial K} \right)^{1/2} \left( \|q\|^2_{\Omega^s} + \sum_{K \in T^s} h_K \|q^s\|^2_{\partial K} \right)^{1/2} \\
\leq \|v\|_v\|q\|_p.
\]

Next, consider \(I_2\). We have

\[
|I_2| = - \sum_{K \in T^d} \int_K q \nabla \cdot v \, dx + \sum_{F \in F^d \setminus F^l} \int_F \bar{q}^d (v \cdot n) \, ds + \sum_{F \in F^d} \int_F q^d (v^d - \bar{v}) \cdot n^d \, ds \\
\leq \left( \sum_{K \in T^d} \|\nabla v\|^2_K + \sum_{F \in F^d \setminus F^l} \|v \cdot n\|^2_F + \sum_{F \in F^d} h_K^{-1} \|(v^d - \bar{v}) \cdot n^d\|^2_F \right)^{1/2} \\
\times \left( \|q\|^2_{\Omega^d} + \sum_{K \in T^d} h_K \|q^d\|^2_{\partial K} + \sum_{K \in T^d} h_K \|q^d\|^2_{\partial K} \right)^{1/2} \\
\leq \sqrt{2}\|v\|_v\|q\|_p.
\]

The result follows by combining the bounds for \(|I_1|\) and \(|I_2|\).
The form $t_h$ eq. (9c) is new compared to [10]. We have the following properties of this term.

**Lemma 4.** Let $w_1, w_2 \in X^s(h)$ such that $\nabla \cdot w_j = 0$ on each $K \in T^s$ and $w_j \in H(\text{div}; \Omega^s)$, for $j = 1, 2$. Then for any $u, v \in X(h)$, there exists a constant $c_w > 0$ such that:

$$|t_h(w_1; u, v) - t_h(w_2; u, v)| \leq c_w\|w_1 - w_2\|_{1,h,\Omega^s} \|u\|_{v,s} \|v\|_{v,s}.$$  

**Proof.** After integrating $t_h(w_j; u, v)$ by parts and using that

$$\int_{\Gamma^s} (w \cdot n) \bar{u} \cdot \bar{v} \, ds = \sum_{K \in T^s} \int_{\partial K} (w \cdot n) \bar{u} \cdot \bar{v} \, ds, \quad j = 1, 2,$$

which holds since $w \cdot n$ is continuous, $\bar{u}$ and $\bar{v}$ are single-valued on element boundaries, and $\bar{u} = \bar{v} = 0$ on $\Gamma^s$, the remainder of the proof is identical to that of the proof of [9, Proposition 3.4].

Next, we remark that for $w \in X^s(h) \cap H(\text{div}; \Omega^s)$ such that $\nabla \cdot w = 0$ on each $K \in T^s$, and $v \in X(h)$ that

$$t_h(w; v, v) = \frac{1}{2} \sum_{K \in T^s} \int_{\partial K} |w| \|v - \bar{v}\|^2 \, ds + \frac{1}{2} \int_{\Gamma^s} (w \cdot n) |\bar{v}|^2 \, ds. \quad (24)$$

Combining some of the above results, we have the following two lemmas.

**Lemma 5.** Let $w \in X^s(h) \cap H(\text{div}; \Omega^s)$ such that $\nabla \cdot w = 0$ on each $K \in T^s$. Furthermore, let $u, v \in X(h)$ and $u_h, v_h \in X_h$. Then

$$|a_h(w; u, v)| \leq c_{ac} \|u\|_{v'} \|v\|_{v'},$$

with $c_{ac} = 2e^2 \max(c_w \mu^{-1} \|w\|_{1,h,\Omega^s} + \epsilon^8, \kappa_{\text{max}}^{-1}, \alpha \kappa_{\text{min}}^{-1/2})$.

**Proof.** Note that by eq. (16) and lemma 4 with $w_2 = 0$,

$$|a_h(w; u, v)| \leq |t_h(w; u, v)| + |a^s_h(u, v)| + |a^d(u, v)| + |a^l (\bar{u}, \bar{v})|$$

$$\leq c_w \|w\|_{1,h,\Omega^s} \|u\|_{v,s} \|v\|_{v,s} + \mu^s_{ac} \|u\|_{v',s} \|v\|_{v'},$$

$$+ \kappa_{\text{min}}^{-1} \|u\|_{\Omega^s} \|v\|_{\Omega^d} + \alpha \kappa_{\text{min}}^{-1/2} \|\bar{u}\|_{\Gamma^d} \|\bar{v}\|_{\Gamma^d}$$

$$\leq 2 \max(c_w \|w\|_{1,h,\Omega^s} + \mu^s_{ac}, \kappa_{\text{min}}^{-1}, \alpha \kappa_{\text{min}}^{-1/2})$$

$$\times \left( (\|u\|^2_{v'} + \|u\|^2_{v'})^{1/2} (\|v\|^2_{v'} + \|v\|^2_{v'})^{1/2} \right)^{1/2}.$$

The results follow by definition of $\|\|_{v'}$ and using that the norm equivalency constant $c_e \geq 1$.

**Remark 2.** Lemma 5 holds also for $u \in X_h$ with $\|u\|_{v'}$ replaced by $\|u\|_{v}$ and/or $v \in X_h$ with $\|v\|_{v'}$ replaced by $\|v\|_{v}$ due to the equivalence of the norms $\|\|_{v}$ and $\|\|_{v'}$ on $X_h$.

Using a similar approach as in [19, Lemma 2], we show the coercivity of $a_h$.

**Lemma 6.** Let $w \in X^s(h) \cap H(\text{div}; \Omega^s)$ such that $\nabla \cdot w = 0$ on each $K \in T^s$, and $\|w \cdot n\|_{\Gamma^d} \leq \mu \kappa_{ac}(\epsilon^2 + c_{si,4})^{-1}$ with $0 < \delta < 1$. Then, for $\beta > \beta_0$,

$$a_h(w; v_h, v_h) \geq c_{ae} \|v_h\|_v^2 \quad \forall v_h \in Z_h,$$

where $c_{ae} = \min \left( ((1 - \delta) c_{ae}^{-1}, \kappa_{\text{max}}^{-1}, \alpha \kappa_{\text{max}}^{-1/2}) \right) > 0$.

**Proof.** Let $v_h \in Z_h$. From eq. (24) we note that

$$t_h(w; v_h, v_h) \geq -\frac{1}{2} \int_{\Gamma^d} |w \cdot n| |\bar{v}_h|^2 \, ds \geq -\int_{\Gamma^d} |w \cdot n| |\bar{v}_h - v_h|^2 \, ds - \int_{\Gamma^d} |w \cdot n| |v_h|^2 \, ds, \quad (26)$$

\[10\]
where the second inequality is due to \(|\bar{v}_h|^2 \leq 2|\bar{v}_h - v_h^s|^2 + 2|v_h^s|^2\). Let us consider each term on the right hand side of eq. (26) separately. For the first term we find, using the Cauchy–Schwarz inequality,

\[
\int_{\Gamma_I} |w \cdot n| |\bar{v}_h - v_h^s|^2 \, ds \leq \|w \cdot n\|_{\Gamma_I} \|\bar{v}_h - v_h^s\|_{4.0, \Gamma_I}^2.
\]

By a scaling identity, for \(\mu \in R_d(\partial K)\), we have that there exists a positive constant \(c_{pq}\) independent of \(h\) such that \(\|\mu\|_{4.0, \partial K} \leq c_{pq} h^{(1-d)/4} \|\mu\|_{\partial K}\). We therefore find:

\[
\int_{\Gamma_I} |w \cdot n| |\bar{v}_h - v_h^s|^2 \, ds \leq c_{pq}^2 \|w \cdot n\|_{\Gamma_I} k^{(1-d)/2} \|\bar{v}_h - v_h^s\|_{4.0, \Gamma_I}^2 \leq c_{pq}^2 \|w \cdot n\|_{\Gamma_I} \|v_h\|_{v,s}^2.
\]  

(27)

where the second inequality is true for \(d = 2, 3\). For the second term on the right hand side of eq. (26), using the Cauchy–Schwarz inequality and eq. (14) with \(r = 4\), we obtain:

\[
\int_{\Gamma_I} |w \cdot n| |v_h^s|^2 \, ds \leq \|w \cdot n\|_{\Gamma_I} \|v_h^s\|_{4.0, \Gamma_I} \leq c_{si,4}^2 \|w \cdot n\|_{\Gamma_I} \|v_h\|_{v,s}^2.
\]  

(28)

Combining eqs. (26) to (28), we find \(t_h(w; v_h, g_h) \geq -(c_{pq}^2 + c_{si,4}^2) \|w \cdot n\|_{\Gamma_I} \|v_h\|_{v,s}^2\). Using this inequality together with eq. (17),

\[
\begin{align*}
\alpha_h(w; v_h, g_h) & \geq \mu c_{ae} \|v_h\|_{v,s}^2 + \mu \kappa_{\max} |v_h|^2_{\Omega_d} + \alpha \mu \kappa_{\max}^{-1/2} |\bar{v}_h|^2_{4.0, \Gamma_I} - (c_{pq}^2 + c_{si,4}^2) \|w \cdot n\|_{\Gamma_I} \|v_h\|_{v,s}^2 \\
& \geq (\mu c_{ae} - (c_{pq}^2 + c_{si,4}^2) \|w \cdot n\|_{\Gamma_I}) \|v_h\|_{v,s}^2 + \mu \kappa_{\max}^{-1/2} |v_h|^2_{\Omega_d} + \alpha \mu \kappa_{\max}^{-1/2} |\bar{v}_h|^2_{4.0, \Gamma_I} \\
& \geq (1 - \delta) \mu c_{ae} \|v_h\|_{v,s}^2 + \mu \kappa_{\max}^{-1/2} |v_h|^2_{\Omega_d} + \alpha \mu \kappa_{\max}^{-1/2} |\bar{v}_h|^2_{4.0, \Gamma_I},
\end{align*}
\]

where the last step is by the assumption on \(\|w \cdot n\|_{\Gamma_I}\). The result follows by definition of \(\|v_h\|_{v,s}\) noting that \(\|v_h\|_{v,d} = \|v_h\|_{\Omega_d}^2\) for \(v_h \in Z_h\) for \(v_h \in Z_h\).
5 Error analysis

Besides the BDM interpolation operator $\Pi_V : [H^1(\Omega)]^{\dim} \to X_h \cap H(\text{div}; \Omega)$ satisfying eqs. (18) and (19), and the $L^2$-projection $\Pi\bar{V} : [H^1(\Omega)]^{\dim} \to X_h$ used previously in section 4.2, let $\Pi_Q$ and $\Pi\bar{Q}_j$ be the $L^2$-projection operators onto, respectively, $Q_h$, and $\bar{Q}_j^h$, $j = s, d$. The interpolation and approximation errors are defined as:

$$
\begin{align*}
\epsilon^I_u &= u - \Pi\bar{V}u, & \epsilon^I_p &= p - \Pi\bar{Q}p, & \epsilon^I_e &= \gamma(u) - \Pi\bar{V}u, & \epsilon^I_{p,j} &= \gamma(p) - \Pi\bar{Q}_j p, \\
\epsilon^h_u &= u_h - \Pi\bar{V}u, & \epsilon^h_p &= p_h - \Pi\bar{Q}p, & \epsilon^h_e &= u_h - \Pi\bar{V}u, & \epsilon^h_{p,j} &= p_h - \Pi\bar{Q}_j p.
\end{align*}
$$

(32)

Note that $u - u_h = e^I_u - e^h_u$ and likewise for the other unknowns. Similar to the notation used in previous sections, we write $e^e_u = (e^e_u, e^e_p, e^e_p, e^e_e)$, $e^e_{p,j} = (e^e_{p,j}, e^e_{p,j}, e^e_{p,j})$, $e^e_p = (e^e_p, e^e_p, e^e_p, e^e_p)$, for $j = s, d, r = I, h$, and we remark that $e^e_{u,a}$ is the restriction of $e^e_u$ to $\Omega^a$. From [10, Lemma 8] we have that for $p^j \in H^l(\Omega^j)$, $0 \leq l \leq k$, $j = s, d$, that

$$
\|e^I_p\|_p \leq Ch^l \|p\|_{L,\Omega}. 
$$

(33)

The following lemma, which is a modification of [10, Lemma 7], determines the interpolation estimate for the velocity field.

**Lemma 7.** Suppose that $u \in [H^l(\Omega)]^{\dim}$ for $2 \leq l \leq k + 1$. Then

$$
\|e^I_u\|_{\nu'} \leq Ch^{l-1}\|u\|_{L,\Omega}.
$$

(34)

**Proof.** It was shown in the proof of [10, Lemma 7] that

$$
(\|e^I_u\|_{\nu'}^2 + \|e^I_e\|_{\nu'}^2)^{1/2} \leq Ch^{l-1}\|u\|_{L,\Omega}.
$$

To complete the proof, we observe by definition, using eq. (19), and the proof of [36, Lemma 9], that

$$
\|e^I_u\|_{\nu,d}^2 = \|e^I_u\|_{\nu,d}^2 + \sum_{K \in T^d} h_K^{-1}\|(e^I_u - e^I_{u_h}) \cdot n\|_{\partial K \cap \Gamma^I}^2 \leq ch^{2l-2}\|u\|_{L,\Omega}^2.
$$

□

To obtain the error equation, note that because $\Pi_Q$ and $\Pi\bar{Q}_j$, $j = s, d$ are the $L^2$-projection operators onto, respectively $Q_h$ and $\bar{Q}_j^h$, $\nabla \cdot v_h |_{K} \in P_{k-1}(K)$ for all $K \in T$, $v_h \cdot n^{j,F} \in P_{k}(F)$ for all $F \in T^j$ and $v_h \cdot n^{j,F} \in P_{k}(F)$ for all $F \in T^j$,

$$
b_h(v_h, e^I_p) = 0 \quad \forall v_h \in X_h.
$$

(35)

Furthermore, by properties eq. (18) of the BDM interpolation operator and the $L^2$-projection $\Pi_V$, it follows that

$$
b_h(e^I_u, q_h) = 0 \quad \forall q_h \in Q_h.
$$

(36)

Let us denote by $a^I_h$ the linear part of $a_h$, i.e.,

$$
a^I_h(u, v) := a^s_h(u, v) + a^d(u, v) + a^I(u, v).
$$

Subtracting the consistency equations (see lemma 1) from eq. (8), using eqs. (32), (35) and (36) and rearranging, we obtain the following error equation:

$$
\begin{align*}
t_h(u; e^I_u, v_h) + a^I_h(e^I_u, v_h) &= a^I_h(e^I_e, v_h) - b_h(v_h, e^h_p) - b_h(e^h_u, q_h) \\
&+ t_h(u; e^I_u, v_h) + t_h(u; u_h, v_h) - t_h(u_h; u_h, v_h).
\end{align*}
$$

(37)

The error estimates for the HDG method for the Navier–Stokes equations in [31] are now extended here to the coupled Navier–Stokes and Darcy problem along the same lines as [19].
Theorem 2 (Energy estimate and velocity pressure error). Let $(u, p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$ be the solution to the Navier–Stokes/Darcy problem eqs. (1) to (3) and let $u = (u, g(u))$ and $p = (p, g(p^s), g(p^d))$. Let $(u_h, p_h) \in X_h \times Q_h$ be the solution to the discrete Navier–Stokes/Darcy problem eq. (8). Let $C_w, C_p, C_f, C_{bb}, C_{ae}, C_{si,2}, C_{si,4}$ be the constants in eq. (6) and let $c_w, c_p, c_f, cbb, cae, cs, csi,2, csi,4,$ and $c_{pq}$ be the constants in eq. (31). Let $\tilde{e}_w = \max(C_w, c_w), \tilde{e}_p = \max(C_p, c_p), \tilde{e}_f = \max(C_f, c_f), \tilde{e}_{bb} = \min(C_{bb}, c_{bb}), \tilde{e}_{ae} = \min(C_{ae}, c_{ae}), \tilde{e}_{si} = \max(C_{si,2}, c_{si,2}), \tilde{e}_{si} = \max(C_{si,4}, c_{si,4}), \tilde{e}_{sir} = \min(C_{ae}, c_{ae}),$ and $c_{pq}$ be the constants. For $\delta < 1,$ If

$$\tilde{c}_p \|f^s\|_{\Omega^s} + 2\tilde{c}_f \tilde{c}_{bb}^{-1} \|f^d\|_{\Omega^d} < \frac{1}{2} \mu^2 \tilde{c}_{ae} \min\left(\tilde{c}_{w}^{-1}, \tilde{c}_{ae}^{-1}, \tilde{c}_{sir}^{-1}\right),$$

then

$$\|u - u_h\|_v \leq c_1 h^k \|u\|_{k+1, \Omega},$$

$$\|p - p_h\|_p \leq c_2 h^k \left(\|p\|_{k, \Omega} + \mu \|u\|_{k+1, \Omega}\right),$$

where $c_1, c_2 > 0$ are constants independent of $\mu$ and $h.$

Proof. We first prove eq. (39a). Take $(v_h, q_h) = (e_u, -e_p)$ in eq. (37). By coercivity of $a_h$ lemma 6, we find

$$c_{ae}\mu \|e^b_u\|_v^2 \leq a_h(u; e^b_u, e^b_u) = a_h^I(u; e^I_u, e^I_u) + t_h(u; u_h, e^b_u) + t_h(u; u_h, e^b_u) - t_h(u_h; u_h, e^b_u) = a_h(u; e^I_u, e^b_u) + [t_h(u; u_h, e^b_u) - t_h(u_h; u_h, e^b_u)] = I_1 + I_2.$$

We bound each term separately, starting with $I_1.$ Since eq. (38) holds, it follows by eq. (7a) that

$$\|u\|_{1, \Omega^s} \leq \tilde{c}_w^{-1} \mu^{-1} \left(\tilde{c}_p \|f^s\|_{\Omega^s} + 2\tilde{c}_f \mu \tilde{c}_{bb}^{-1} \|f^d\|_{\Omega^d}\right) \leq \frac{1}{2} \mu \min\left(\tilde{c}_{ae}^{-1}, \tilde{c}_{ae}^{-1}, \tilde{c}_{sir}^{-1}\right) \leq \mu \tilde{c}_{ae} \delta \tilde{c}_{sir}.$$

Therefore, $c_{ae}$ in lemma 5 is bounded by

$$c_{ae} = 2c_{ae} \max(\tilde{c}_w^{-1}, \tilde{c}_{ae}, \kappa^{-1}, \kappa^{-1/2}) \leq 2c_{ae} \max(\tilde{c}_w^{-1}, \tilde{c}_{ae}, \kappa^{-1}, \kappa^{-1/2}) \leq c_f,$$

so that, by lemma 5, $I_1 \leq \tilde{c}_f \mu \|e^I_u\|_v \|e^b_u\|_v.$ To bound $I_2,$ we use lemma 4 and eq. (13):

$$|t_h(u; u_h, e^b_u) - t_h(u_h; u_h, e^b_u)| \leq c_w \|u - u_h\|_{1, \Omega^s} \|u_h\|_{\Omega, v, s} \|e^b_u\|_{\Omega, v, s} \leq c_w \|e^I_u\|_{\Omega, v, s} \|e^b_u\|_{\Omega, v, s} + c_w \|e^I_u\|_{\Omega, v, s} \|u_h\|_{\Omega, v, s} \|e^b_u\|_{\Omega, v, s}.$$

Combining the bounds for $I_1$ and $I_2,$

$$c_{ae}\mu \|e^b_u\|_v^2 \leq \tilde{c}_f \mu \|e^I_u\|_v \|e^I_u\|_v + c_w \|e^I_u\|_{\Omega, v, s} \|u_h\|_{\Omega, v, s} \|e^b_u\|_{\Omega, v, s} + c_w \|u_h\|_{\Omega, v, s} \|e^b_u\|_{\Omega, v, s}^2.$$

By eq. (30a) and eq. (38),

$$c_w \|u_h\|_{\Omega, v, s} \leq \tilde{c}_w \|u_h\|_{\Omega, v, s} \leq \tilde{c}_{ae} \mu^{-1} \tilde{c}_w \left(\tilde{c}_p \|f^s\|_{\Omega^s} + 2\tilde{c}_f \mu \tilde{c}_{bb}^{-1} \|f^d\|_{\Omega^d}\right) \leq \frac{1}{2} \tilde{c}_w \mu \min\left(\tilde{c}_{ae}^{-1}, \tilde{c}_{ae}^{-1}, \tilde{c}_{sir}^{-1}\right) \leq \frac{1}{2} \tilde{c}_w \mu.$$

Combining eqs. (40) and (41),

$$\frac{1}{2} \tilde{c}_w \mu \|e^b_u\|_v^2 \leq (c_\mu - \frac{1}{2} \tilde{c}_w) \mu \|e^b_u\|_v^2 \leq \tilde{c}_f \mu \|e^I_u\|_v \|e^I_u\|_v + \frac{1}{2} \tilde{c}_w \mu \|e^I_u\|_{\Omega, v, s} \|e^b_u\|_{\Omega, v, s},$$

resulting in

$$\|e^b_u\|_v \leq 2\tilde{c}_f \tilde{c}_{ae}^{-1} \|e^I_u\|_v + \|e^I_u\|_{\Omega, v, s} \leq (1 + 2\tilde{c}_f \tilde{c}_{ae}^{-1}) \|e^I_u\|_v.$$

Applying a triangle inequality to $\|u - u_h\|_v,$ and using eq. (42) results in

$$\|u - u_h\|_v \leq 2(1 + \tilde{c}_f \tilde{c}_{ae}^{-1}) \|e^I_u\|_v,$$

so that eq. (39a) follows by using eq. (34).
We proceed with proving eq. (39b). Set \( q_h = 0 \) in eq. (37). Then, by lemmas 4 and 5,
\[
\begin{align*}
    b_h(v_h, e_p^h) &= a_h^I(e^I_u, v_h) - a_h^I(e^h_u, v_h) - t_h(u; e^h_u, v_h) \\
    &+ t_h(u; e^I_u, v_h) + t_h(u; u_h, v_h) - t_h(u_h; u_h, v_h) \\
    &= a_h(u; e^I_u, v_h) - a_h(u; e^h_u, v_h) + t_h(u; u_h, v_h) - t_h(u_h; u_h, v_h) \\
    &\leq \tilde{c}_f \mu (\|\varepsilon^I_u\|_{H^1} + \|\varepsilon^h_u\|_{H^1})\|v_h\|_v + c_w\|u - u_h\|_{L^2}\|
\end{align*}
\]
By eqs. (13) and (41),
\[
\begin{align*}
    b_h(v_h, e_p^h) &\leq \tilde{c}_f \mu (\|\varepsilon^I_u\|_{H^1} + \|\varepsilon^h_u\|_{H^1})\|v_h\|_v + \frac{1}{2}\tilde{c}_ac_\mu \|u - u_h\|_{L^2}\|v_h\|_v \\
    &\leq \tilde{c}_f \mu (\|\varepsilon^I_u\|_{H^1} + \|\varepsilon^h_u\|_{H^1})\|v_h\|_v + \frac{1}{2}\tilde{c}_ac_\mu (\|\varepsilon^I_u\|_{H^1} + \|\varepsilon^h_u\|_{H^1})\|v_h\|_v \\
    &\leq (\tilde{c}_f + \frac{1}{2}\tilde{c}_ac_\mu) \mu (\|\varepsilon^I_u\|_{H^1} + \|\varepsilon^h_u\|_{H^1})\|v_h\|_v.
\end{align*}
\]
By the inf-sup condition in lemma 2,
\[
\|e_p^h\|_p \leq c_{bb}^{-1} \sup_{v_h \in X_h, v_h \neq 0} \frac{b_h(v_h, e_p^h)}{\|v_h\|_v} \leq (\tilde{c}_f + \frac{1}{2}\tilde{c}_ac_\mu) c_{bb}^{-1} \mu (\|\varepsilon^I_u\|_{H^1} + \|\varepsilon^h_u\|_{H^1}), \tag{44}
\]
Applying the triangle inequality to \( \|p - p_h\|_p \) and combining eq. (44) with eq. (42) we find
\[
\|p - p_h\|_p \leq \|e_p^h\|_p + (2\tilde{c}_f + \tilde{c}_ac_\mu) c_{bb}^{-1} (1 + \tilde{c}_f\tilde{c}_ac_\mu) \mu \|\varepsilon^I_u\|_{H^1}, \tag{45}
\]
so that eq. (39b) follows using eqs. (33) and (34).

\[\square\]

**Remark 3.** The velocity error bound eq. (39a) is independent of the pressure and independent of the inverse of the viscosity; the discretization is pressure-robust. However, note that \( \tilde{c}_f\tilde{c}_ac_\mu = \mathcal{O}(\kappa_{\text{max}}/\kappa_{\text{min}}) \) for small \( \kappa_{\text{min}} \) and large \( \kappa_{\text{max}} \). Therefore, for small \( \kappa_{\text{min}} \) and large \( \kappa_{\text{max}} \), by eq. (43), the constant in the velocity approximation error eq. (39a) increases linearly with \( \kappa_{\text{max}}/\kappa_{\text{min}} \). The dependence of the pressure error on \( \kappa_{\text{max}}/\kappa_{\text{min}} \) is small for small enough \( \mu \), see eq. (45).

## 6 Numerical examples

We now present numerical examples in which solutions to the Navier–Stokes/Darcy problem eqs. (1) and (2) are approximated by solutions to the HDG discretization eq. (8). The HDG method is implemented using the finite element software Netgen/NGSolve [38, 39].

### 6.1 Example 1: Manufactured Solution

We consider here a manufactured solution on the domain \( \Omega = [0, 1] \times [-1, 1] \) such that \( \Omega^s = [0, 1] \times [0, 1] \) and \( \Omega^d = [0, 1] \times [-1, 0] \). We consider two values for \( \kappa \), namely, \( \kappa = \kappa_1(x_1)I \) with \( \kappa_1 = \alpha^2(\pi x_1 + 1)^2/4 \) and \( \kappa = \kappa_2(x_1)I \) with \( \kappa_2 = \kappa_1(x_1)\exp(-15\sin^2(10x_2)) \). We furthermore consider the manufactured solution
\[
\begin{align*}
    u^s &= \begin{bmatrix} \pi x_1 \cos(\pi x_1 x_2) + 1 \\
    -\pi x_2 \cos(\pi x_1 x_2) + 2x_1 \end{bmatrix}, \\
    p^s &= \mu(1 - \pi)\cos(\pi x_1 x_2) + \sin(\frac{1}{2}\pi x_2)/\mu,
\end{align*}
\]
\[
\begin{align*}
    p^d &= -\frac{8\mu x_1 x_2}{(\pi x_1 + 1)^2} + \mu \cos(\pi x_1 x_2).
\end{align*}
\]
This manufactured solution is used to set \( u^d = -\kappa\mu^{-1}\nabla p^d \), the source terms, \( f^s \) and \( f^d \), and inhomogeneous boundary conditions. In our simulations we set \( \alpha = 1 \) and consider \( \mu = 10^{-1} \) and \( \mu = 10^{-3} \). In our discrete function spaces we consider \( k = 1 \) (corresponding to the lowest order polynomial approximation in which the
cell pressure is approximated by piecewise constants and all other unknowns by piecewise linear polynomial approximations), and higher-order accurate approximations with \( k = 2 \), and \( k = 3 \). We choose the penalty parameter in eq. (9b) as \( \beta = 8k^2 \).

Define \( \| v \|_E^2 := (\sum_{K \in \mathcal{T}_h} |v|^2_{1,K} + \| v \|^2_{\Omega^d}) \). Using \( \kappa_1 \) we observe in fig. 1 that the velocity in the \( \| \cdot \|_E \)-norm and pressure both converge at rate \( k \), as expected from theorem 2. When \( k = 2 \) and \( k = 3 \) the errors in the velocity are significantly larger using \( \kappa_2 \) compared to \( \kappa_1 \). This is again as expected from theorem 2 since for \( \kappa_2 \) we have \( \kappa_{\text{max}}/\kappa_{\text{min}} \approx 5.6 \cdot 10^7 \) which is significantly larger than \( \kappa_{\text{max}}/\kappa_{\text{min}} \approx 1.7 \cdot 10^1 \) when using \( \kappa_1 \) (see also remark 3). Interestingly, the velocity and pressure errors when \( \kappa = 1 \) do not seem to depend on \( \kappa_{\text{max}}/\kappa_{\text{min}} \). Note furthermore that when reducing the viscosity by a factor of 100 from \( \mu = 10^{-1} \) to \( \mu = 10^{-3} \), the error in the pressure increases approximately by a factor of 100, but the error in the velocity is unaffected by changing viscosity. This is also as expected from theorem 2, i.e., our discretization is pressure-robust.

In remark 3 we pointed out that for small enough viscosity an increase in \( \kappa_{\text{max}}/\kappa_{\text{min}} \) only has a small effect on the pressure error. In the right column of plots in fig. 1 we indeed observe that the effect of \( \kappa_{\text{max}}/\kappa_{\text{min}} \) is negligible for \( \mu = 10^{-3} \), but less so for \( \mu = 10^{-1} \) in the pre-asymptotic regime.

Finally, in fig. 2 we plot the velocity error in the \( L^2 \)-norm. We observe optimal \( k + 1 \) rates of convergence when \( k = 2 \) and \( k = 3 \). For \( k = 1 \), \( \mu = 10^{-3} \), and small \( \kappa_{\text{max}}/\kappa_{\text{min}} \) ratio, we observe a rate of convergence between 1.6 and 1.9. The velocity error magnitude in the \( L^2 \)-norm is independent of the viscosity, but clearly increases with increasing \( \kappa_{\text{max}}/\kappa_{\text{min}} \) ratio.

\[ \text{6.2 Example 2: Coupled surface/subsurface flow with randomly generated permeability field} \]

We consider now a Navier–Stokes/Darcy problem similar to a problem proposed in [24, Section 8.2]. We consider the domain \( \Omega = [0,1] \times [0,1] \) such that \( \Omega^s = [0,1] \times [0.6,1] \) and \( \Omega^d = [0,1] \times [0,0.6] \) and impose the following boundary conditions:

\[
\begin{align*}
    u &= (\sin((\pi/8)(10x_2 - 6))(1 - x_1/5), 0) & \quad \text{on } \Gamma^s, \\
    u \cdot n &= 0 & \quad \text{on } \{ x \in \Gamma^d : x_1 = 0 \text{ or } x_1 = 1 \}, \\
    p &= 2 - x_1 & \quad \text{on } \{ x \in \Gamma^d : x_2 = 0 \}.
\end{align*}
\]

We take \( \alpha = 1 \), \( f^s = 0 \), \( f^d = 0 \), consider the solution for \( \mu = 1 \) and \( \mu = 10^{-2} \), and set the permeability on each element of the mesh in \( \Omega^d \) to a constant such that \( \mu^{-1}\kappa = 10^{-r} \) with \( r \) a random number in the interval \([2,6]\) (see fig. 3). We furthermore set \( k = 2 \), \( \beta = 8k^2 \), and compute our solution on a mesh consisting of 92,672 triangles (corresponding to a total of 2,143,476 degrees-of-freedom).

In fig. 4 we plot the magnitude and streamlines of the velocity and pressure fields computed using \( \mu = 1 \) and \( \mu = 0.01 \). We observe, for both values of viscosity, that away from the interface \( \Gamma^I \) the fluid flows freely in \( \Omega^s \). Fluid in \( \Omega^s \) close to the interface percolates through into the subsurface region \( \Omega^d \). The flow patterns observed in fig. 4 are similar to those observed in [24, Section 8.3]. Let us finally remark that for \( \mu = 10^{-2} \), \( \kappa \in [10^{-8}, 10^{-4}] \). Like the DG method proposed in [24], our HDG method is able to handle highly discontinuous permeability.

\[ \text{7 Conclusions} \]

We have introduced and analyzed a strongly conservative HDG method for the Navier–Stokes equations coupled to the Darcy equations by the Beavers–Joseph–Saffman interface condition. The discretization results in a velocity field that is globally divergence-conforming and pointwise divergence-free in the Navier–Stokes region. This allows for a locally momentum conserving discretization of the Navier–Stokes equation. (If the divergence-free constraint is satisfied only weakly, local momentum conservation needs to be sacrificed for the discretization to be stable [15].) A further property of the discretization is that the mass equation in the Darcy region is satisfied pointwise if the source/sink term lies in the discrete pressure space.
Figure 1: The velocity error in the $\| \cdot \|_{E}$- and the pressure error in the $L^2$-norm for the test case of section 6.1. Here the blue lines correspond to $\mu = 10^{-1}$ and the green dashed lines correspond to $\mu = 10^{-3}$. The square symbols correspond to $\kappa = \kappa_1 I$ with a small $\kappa_{\text{max}}/\kappa_{\text{min}}$ ratio (SR) while the $\times$ symbols correspond to $\kappa = \kappa_2 I$ with a large $\kappa_{\text{max}}/\kappa_{\text{min}}$ ratio (LR).
Figure 2: The velocity error in the $L^2$-norm for the test case of section 6.1. Here the blue lines correspond to $\mu = 10^{-1}$ and the green dashed lines correspond to $\mu = 10^{-3}$. The square symbols correspond to $\kappa = \kappa_1I$ with a small $\kappa_{\text{max}}/\kappa_{\text{min}}$ ratio (SR) while the $\times$ symbols correspond to $\kappa = \kappa_2I$ with a large $\kappa_{\text{max}}/\kappa_{\text{min}}$ ratio (LR).
Optimal rates of convergence were proven for the velocity and pressure. Additionally, the velocity error is independent of the pressure and viscosity, i.e., the coupled discretization is pressure-robust.

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