MINIMAL LIE GROUP HOMOMORPHISMS

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Abstract. In this paper we study, in the presence of bi-invariant metrics, a Lie group homomorphism which is also a minimal immersion. If the domain is compact, we prove that either it is isometric to the flat torus or the homomorphism is unstable.

1. Introduction

Harmonic maps between Riemannian manifolds are the critical points of a natural variational problem, i.e., that of minimizing the energy of the map. If the map is indeed a (second order) minimizer, then we say that the map is stable; otherwise, we call it unstable.

Since R. T. Smith [3] published in this journal his second variation formula, a lot of research has been done in the direction of understanding the behavior of stable and unstable harmonic maps (we refer the reader to the excellent survey [1] of J. Eells and L. Lemaire, and to the references therein, for an overview of some important results in this direction). As a contribution to this effort, in this paper we prove that, in the presence of bi-invariant metrics, a Lie group homomorphism which is also a minimal immersion is either unstable or has domain isometric to the flat torus.

2. Preliminaries on harmonic maps

We begin by quoting some definitions and results on harmonic maps, referring to the excellent book of Y. Xin [4] for details. In all that follows we employ Einstein’s summation convention.

If \( M^m_1 \) and \( M^n_2 \) are Riemannian manifolds of dimensions respectively \( m \) and \( n \), and \( f: M^m_1 \to M^n_2 \) is smooth, then the second fundamental form of \( f \) is the symmetric section \( B_f \) of the vector bundle \( \text{Hom}(TM_1 \otimes TM_1, f^{-1}TM_2) \), given by \( B_f(X,Y) = (\nabla_X df)(Y) \), where \( f^{-1}TM_2 \) is the induced vector bundle, \( df \) is seen as a 1-form with values in \( f^{-1}TM_2 \) and \( \nabla \) is the induced connection on \( T^*M_1 \otimes f^{-1}TM_2 \). The tension field of \( f \) is the trace \( \tau(f) \) of \( B_f \), i.e., \( \tau(f) = (\nabla_{e_j} df)e_j \), where \( \{e_j\} \) is any orthonormal frame on an open set of \( M_1 \), and the map \( f \) is said to be harmonic if \( \tau(f) \) vanishes on \( M_1 \).

If \( M_1 \) is closed and oriented, one defines the energy of \( f \) by \( E(f) = \frac{1}{2} \int_{M_1} |df|^2 dM_1 \). In this case, an equivalent definition of harmonicity is variational and requires that \( f \) be a critical point of the energy functional for every smooth one-parameter variation \( f_t: M_1 \to M_2 \) of \( f_0 = f \). The equivalence of these two definitions is established...

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by means of the first variational formula, which states that if \( v \in \Gamma(f^{-1}TM_2) \) is a section of the induced bundle, \( f_t \) is a variation of \( f \) such that \( \frac{dE_t}{dt} \bigg|_{t=0} = v \) and \( E_t = E(f_t) \), then

\[
\frac{dE_t}{dt} \bigg|_{t=0} = \int_{M_1} \langle \tau(f), v \rangle dM_1.
\]

In case \( f : M_1 \to M_2 \) is an isometric immersion, it is easy to see that it is harmonic if and only if it is minimal; in this case, if \( M_1 \) is closed then the functional energy of \( f \) coincides with the classical functional area.

Our main concern here is the notion of stability (and unstability) for minimal immersions. More generally, we call a harmonic \( f : M_1 \to M_2 \) stable if \( \frac{d^2E_t}{dt^2} \bigg|_{t=0} \geq 0 \) for all \( f_t \) as above; otherwise, \( f \) is said to be unstable. In order to state the formula for the second variation of energy, we digress a little bit more.

If \( \mathcal{E} \) is a Riemannian vector bundle on \( M_1 \), one defines the trace-Laplacian on \( \mathcal{E} \) as the linear differential operator \( \nabla^2 : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}) \), given by

\[
\nabla^2 v = (\nabla_{e_j} \nabla_{e_j} v - \nabla_{\nabla_{e_j} e_j} v),
\]

where \( \nabla \) at the right hand side is the connection of \( \mathcal{E} \) and \( \{e_j\} \) is any orthonormal frame field on an open set of \( M_1 \). One also considers the Hodge-Laplace operator \( \Delta = dd^* + d^*d \), acting on \( p \)-forms \( \omega \) on \( M_1 \), with values on a Riemannian vector bundle over it. If \( f : M_1 \to M_2 \) is smooth and \( \omega = df \), it is easy to show that \( f : M_1 \to M_2 \) is harmonic if and only if \( df \) is a harmonic \( 1 \)-form, in the sense that \( \Delta(df) = 0 \). Moreover, the classical Weitzenböck formula gives

\[
(1) \quad \Delta(df) = -\nabla^2(df) + S,
\]

where, for \( X \in \Gamma(TM_1) \),

\[
(2) \quad S(X) = -R_{M_2}(f_* e_j, f_* X) f_* e_j + f_* \text{Ric}_{M_1}(X).
\]

Here, \( R_{M_2} \) stands for the curvature operator of \( M_2 \), \( \{e_j\} \) again denotes any orthonormal frame field on an open set of \( M_1 \) and \( \text{Ric}_{M_1} : TM_1 \to TM_1 \) is the field of self-adjoint operators associated to the Ricci tensor of \( M_1 \).

If \( v \in \Gamma(f^{-1}TM_2) \) and \( f_t \) is such that \( \frac{dE_t}{dt} \bigg|_{t=0} = v \), R. T. Smith proved in [3] that

\[
\frac{d^2E_t}{dt^2} \bigg|_{t=0} = -\int_{M_1} \langle \nabla^2 v + R_{M_2}(f_* e_j, v) f_* e_j, v \rangle dM_1.
\]

This way, letting \( I(v, v) \) denote the second hand side, a harmonic \( f \) is stable if and only if \( I(v, v) \geq 0 \) for all \( v \in \Gamma(f^{-1}TM_2) \).

3. Minimal Homomorphisms between Lie Groups

Recall that a bi-invariant metric on a Lie group \( G \) is one for which both left and right translations are isometries. In this case, if \( \nabla \) and \( \mathcal{G} \) respectively denote the Levi-Civita connection of \( G \) and its Lie algebra, it is standard that

\[
(3) \quad \nabla_X Y = \frac{1}{2} [X, Y], \quad R_G(X, Y)Z = \frac{1}{4} [[X, Y], Z], \quad K_G(X, Y) = \frac{1}{4} [[X, Y]]^2
\]

for all \( X, Y, Z \in \mathcal{G} \), where \( R_G \) and \( K_G \) are, respectively, the curvature operator and the sectional curvature of \( G \). It is also standard that compact Lie groups can always be given bi-invariant metrics.
Thus, since $\varphi$ is a group homomorphism, it is well known that $f$ is harmonic; in fact, letting $\{E_j\}$ be an orthonormal frame on $G_1$, (3) and the naturality of Lie brackets give

$$\tau(f) = (\nabla f, E_j, f^* E_j - f_* \nabla E_j, E_j) = \frac{1}{2} ([f, E_j, f^* E_j] - f_* [E_j, E_j]) = 0.$$ 

Under the above notations we get the following.

**Theorem 3.1.** Let $G_1$ and $G_2$ be Lie groups with bi-invariant metrics, $G_1$ being compact. If $f : G_1 \to G_2$ is a minimal Lie group homomorphism, then either $f$ is unstable or $G_1$ is isometric to a flat torus.

**Proof.** The key computation is that of the trace-Laplacian of $f_* X$, where $X \in \mathcal{G}_1$. Letting $\{E_j\}$ be an orthonormal frame on $G_1$, we get from (3) that

$$\nabla^2 X = \nabla_{E_j} \nabla_{E_j} X = \frac{1}{4} \{E_j, [E_j, X]\} = R(X, E_j) E_j = \text{Ric}_{G_1}(X).$$

Thus, since $f_* X, f_* E_j \in \mathcal{G}_2$, it follows from (1), (2) and (3) that

$$\nabla^2 f_* X = (\nabla^2 f) X + 2(\nabla E_j, f^* df)(\nabla E_j) X + df(\nabla^2 X) = -\Delta (f) X + S(X) + (\nabla E_j, f^* df)[E_j, X] + f_* \text{Ric}_{G_1}(X)$$

$$\quad \quad = -R_{G_2}(f, E_j, f_* X) f_* E_j + 2 f_* \text{Ric}_{G_1}(X) + \nabla f, E_j, f_* [E_j, X] - f_* (\nabla E_j, E_j)$$

$$\quad \quad = -R_{G_2}(f, E_j, f_* X) f_* E_j + 2 f_* \text{Ric}_{G_1}(X) + \frac{1}{2} [f, E_j, f_* [E_j, X]] - \frac{1}{2} f_* [E_j, [E_j, X]]$$

$$\quad \quad = -R_{G_2}(f, E_j, f_* X) f_* E_j + 2 f_* \text{Ric}_{G_1}(X).$$

It now follows from the formula for the second variation, together with the fact that $f$ is an isometric immersion, that

$$I(f_* X, f_* X) = -2 \int_{G_1} (f_* \text{Ric}_{G_1}(X), f_* X) dG_1 = -2 \int_{G_1} \text{Ric}_{G_1}(X, X) dG_1.$$

Therefore, (3) shows that either $f$ is unstable or $K(X, E_j) = 0$ for all $X, E_j \in \mathcal{G}_1$, which is the same as saying that $G_1$ is flat. In this last case, Proposition 6.6 of [2] finishes the proof.

**Remark 3.2.** Let $T^m = S^1 \times \cdots \times S^1$ ($m$ copies of $S^1$) denote the $m$–dimensional flat torus. If $m < n$, the inclusion map $i : \mathbb{R}^m \to \mathbb{R}^n$ obviously induces a minimal immersion $f : T^m \to \mathbb{R}^n$ which is also a Lie group homomorphism. Others can be constructed by composing $f$ with an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$.

**References**

[1] J. Eells and L. Lemaire. *Selected Topics in Harmonic Maps. La surface dont la courbure moyenne est constant.* CBMS 50. Providence, Amer. Math. Soc. (1983).

[2] T. Sakai *Riemannian Geometry.* Providence, Amer. Math. Soc. (1992).

[3] R. T. Smith. *The Second Variation Formula for Harmonic Mappings.* Proc. of the Amer. Math. Soc. 47 (1975) 229-236.

[4] Y. L. Xin. *Geometry of Harmonic Maps.* Boston, Birkhauser (1996).

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