ON THE STABLE CATEGORY OF MAXIMAL COHEN-MACaulAY MODULES OVER GORENSTEIN RINGS

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Abstract. Let \((A, m), (B, n)\) be Gorenstein local rings and let \(\text{CM}(A)\) be its stable category of finitely generated maximal Cohen-Macaulay \(A\)-modules. Suppose \(\text{CM}(A) \cong \text{CM}(B)\) as triangulated categories. Then we show
(1) If \(A\) is an abstract complete intersection of codimension \(c\) then so is \(B\).
(2) If \(A, B\) are Henselian and not hypersurfaces then \(\dim A = \dim B\).
(3) If \(A\) (and so \(B\)) are complete hypersurface singularity and multiplicity of \(A\) is at least three then \(\dim A - \dim B\) is even.
(4) If \(A, B\) are Henselian and \(A\) is an isolated singularity then so is \(B\).
We also give some applications of our results. It should be remarked that if \(R, S\) are complete Cohen-Macaulay but not necessarily Gorenstein and if there is a triangle isomorphism between the singularity categories of \(R\) and \(S\) then it is possible that \(\dim R - \dim S\) is odd, see M. Kalck; Adv. Math. 390 (2021), Paper No. 107913.

1. Introduction

Representation theory of Artin Algebras is a well established branch of mathematics. Auslander discovered that many concepts in representation theory of Artin algebras have natural analogues in the study of maximal Cohen-Macaulay (= MCM) modules of a commutative Cohen-Macaulay local ring \(A\). See [35] for a nice exposition of these ideas. In this paper we study analogues of a natural concept in theory of Artin algebras in the study of MCM modules over commutative Gorenstein local ring.

Let \(R\) be a commutative Artin ring and let \(\Lambda\) be a not-necessarily commutative Artin \(R\)-algebra. Let \(\text{mod}(\Lambda)\) denote the stable category of \(\Gamma\). The study of equivalences of stable categories of Artin \(R\)-algebras has a rich history; see [1] Chapter X. By following Auslander’s idea we investigate equivalences of the stable category of MCM modules over commutative Cohen-Macaulay local rings. If \(A\) is Gorenstein local then the stable category of MCM \(A\)-modules has a triangulated structure. So as a first iteration in this program, in this paper we investigate triangle equivalences of stable categories of MCM modules over commutative Gorenstein rings.

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In general, stable category of Artin algebras (over a commutative Artin ring) need not be triangulated. So by an equivalence we just mean as equivalence of additive categories over a commutative Artin ring. If $\Gamma$ is a self-injective Artin $R$-algebra then $\text{mod}(\Gamma)$ is triangulated. Furthermore if $f: \text{mod}(\Gamma) \to \text{mod}(A)$ is an equivalence of additive categories then $f$ commutes with the shift functor on the corresponding triangulated categories; see [1, 1.12, Chapter X]. However for the best of our knowledge this isomorphism is NOT natural. So it does not follow that $f$ is an equivalence of triangulated categories.

If two finite dimensional self-injective algebra’s over a field are derived equivalent then they are also stably equivalent by a result of Rickard, see [31, 2.2]. Another result of Rickard’s shows that if two rings are derived equivalent then their center’s are isomorphic, see [30, 9.2]. Thus derived equivalences of commutative rings are not interesting.

Let $(A, m)$ be a commutative Gorenstein local ring with residue field $k$. Let $\text{CM}(A)$ denote the full subcategory of finitely generated MCM $A$-modules and let $\text{CM}(A)$ denote the stable category of MCM $A$-modules. Recall that objects in $\text{CM}(A)$ are same as objects in $\text{CM}(A)$. However the set of morphisms $\text{Hom}_A(M, N)$ between $M$ and $N$ is $= \text{Hom}_A(M, N)/P(M, N)$ where $P(M, N)$ is the set of $A$-linear maps from $M$ to $N$ which factor through a finitely generated free module. It is well-known that $\text{CM}(A)$ is a triangulated category with translation functor $\Omega^{-1}$, (see [9]; cf. 2.11). Here $\Omega(M)$ denotes the syzygy of $M$ and $\Omega^{-1}(M)$ denotes the co-syzygy of $M$. Also recall that an object $M$ is zero in $\text{CM}(A)$ if and only if it is free considered as an $A$-module. Furthermore $M \cong N$ in $\text{CM}(A)$ if and only if there exists finitely generated free modules $F, G$ with $M \oplus F \cong N \oplus G$ as $A$-modules. If $A$ is regular local then all MCM modules are free. So $\text{CM}(A) = 0$.

We use Neeman’s book [25] for notation on triangulated categories. However we will assume that if $C$ is a triangulated category then $\text{Hom}_C(X, Y)$ is a set for any objects $X, Y$ of $C$.

Let $R$ be a commutative Noetherian ring. Let $D^b(R)$ be the bounded derived category of $R$. A complex $X$ of $R$-modules is said to be perfect if it is isomorphic in $D^b(R)$ to a bounded complex of finitely generated projective $R$-modules. The subcategory $D^b_{\text{perf}}(R)$ of perfect complexes is equal to $[R]$ the thick hull of $R$ considered as a subcategory of $D^b(R)$ (see [9, 1.2.1]). The quotient category $D^b(R)/[R]$ is called the singularity category of $R$ and is denoted by $D_{\text{sg}}(R)$. Note if $R$ is of finite Krull dimension then $D_{\text{sg}}(R) = 0$ if and only if $R$ is a regular ring. Buchweitz proved that if $(R, \mathfrak{n})$ is a Gorenstein local ring then $\text{CM}(R) \cong D_{\text{sg}}(R)$ (see [9, 4.4.1]).

I Properties invariant under stable equivalences:

I (a): We say a local ring $A$ is a geometric complete intersection if $A = Q/(f_1, \ldots, f_c)$ where $Q$ is regular local and $f_1, \ldots, f_c$ is a $Q$-regular sequence.
We say $A$ is an **abstract complete intersection** if the completion $\hat{A}$ is a geometric complete intersection. Geometric complete intersections are abstract complete intersections but the converse is not true, see [18 section 2].

Our first result is:

**Theorem 1.1.** Let $A, B$ be Gorenstein local rings. Assume $\mathcal{CM}(A)$ is triangle equivalent to $\mathcal{CM}(B)$. If $A$ is an abstract complete intersection of co-dimension $c$ then $B$ is also an abstract complete intersection of co-dimension $c$.

Let $(A, m)$ be an abstract complete intersection of co-dimension $c$. For $1 \leq r \leq c$ let $\mathcal{CM}_{\leq r}(A)$ be the thick subcategory of $\mathcal{CM}(A)$ generated by MCM $A$-modules with complexity $\leq r$ (for definition of complexity see [20]). If $A$ and $B$ are abstract complete intersections with infinite residue fields then it essentially follows from techniques used to prove [11] that if $\mathcal{CM}_{\leq r}(A)$ is triangle equivalent to $\mathcal{CM}_{\leq s}(B)$ then necessarily $r = s$. However the following result is a bit unexpected:

**Theorem 1.2.** Let $(A, m), (B, n)$ be geometric complete intersections of co-dimension $m, n \geq 2$ respectively and with algebraically closed residue fields. Let $r \geq \max\{m/2, n/2\}$ and also $r \leq \min\{m, n\}$. If $\mathcal{CM}_{\leq r}(A)$ is triangle equivalent to $\mathcal{CM}_{\leq r}(B)$ then necessarily $m = n$.

Proof of Theorem 1.2 uses techniques developed by Avramov and Buchweitz in [5]. We make the following:

**Conjecture 1.3.** Let $(A, m), (B, n)$ be geometric complete intersections of co-dimension $m, n \geq 2$ respectively and with algebraically closed residue fields. Let $r \leq \min\{m, n\}$. If $\mathcal{CM}_{\leq r}(A)$ is triangle equivalent to $\mathcal{CM}_{\leq r}(B)$ then necessarily $m = n$.

1 (b):

Our next result shows that in many cases stable equivalence preserves dimension.

**Theorem 1.4.** Let $(A, m), (B, n)$ be Henselian Gorenstein with $\mathcal{CM}(A)$ triangle equivalent to $\mathcal{CM}(B)$. Then

1. If $A$ is not a hypersurface ring then $\dim A = \dim B$.
2. If $A$ is a hypersurface ring having an MCM $A$-module $M$ with $\Omega(M) \not\cong M$ then $\dim A - \dim B$ is even.

**Remark 1.5.** (1) If $A$ is a geometric hypersurface ring with multiplicity $\geq 3$ then there exists an MCM $A$-module $M$ with $\Omega(M) \not\cong M$. (Sketch of a proof:) A geometric hypersurface has an Ulrich module $U$ (i.e., $U$ is an MCM $A$-module with multiplicity equal to its number of minimal generators), see [20, 2.4]. If multiplicity of $A$ is $\geq 3$ then by [20 Theorem 2] it follows that if $U$ is any Ulrich $A$-module then $\Omega(U) \not\cong U$. 
(2) Knörrer periodicity \cite{Knor} gives examples of hypersurface singularities with CM$(A)$ triangle equivalent to CM$(B)$ and dim $A − \dim B$ a non-zero even number. Note that Knörrer does not show the functor defined in \cite[Section 3]{Knor} is triangulated. However this is not difficult to prove. We also note that Knörrer periodicity does not preserve multiplicity. So multiplicity is not a stable invariant.

(3) If $(A, m)$ is an excellent Henselian Gorenstein ring which is an isolated singularity then the functor $- \otimes A$: CM$(A) \to \text{CM}(\hat{A})$ is an equivalence of triangulated categories. This fact is essentially contained in proof of Theorem 2.9 in \cite{Yamaz} (also see \cite[Remark A.6]{Ogu}).

(4) Let $R, S$ be complete Cohen-Macaulay but not necessarily Gorenstein and if there is a triangle isomorphism between the singularity categories of $R$ and $S$ then it is possible that dim $R − \dim S$ is odd, see \cite{Sugi}. We believe the essential reason this occurs is that if $(R, n)$ is non-Gorenstein then $D_{sg}(R)$ might not be Krull-Schmidt. In fact the example in \cite[1.5]{Sugi} is not Krull-Schmidt.

1.6. For a local ring $(A, m)$ let Spec$(0)(A) = \text{Spec } A \setminus \{m\}$ denote the punctured spectrum of $A$. Let CM$(0)(A)$ denote the full subcategory of CM$(A)$ generated by MCM $A$-modules free on Spec$(0)(A)$. We note that if $E$ is an MCM $A$-module free on Spec$(0)(A)$, all syzygies of $E$ are also free on Spec$(0)(A)$. Furthermore CM$(0)(A)$ is a thick subcategory of CM$(A)$. Proof of Theorem 1.4 also shows the following result:

**Lemma 1.7.** Let $(A, m), (B, n)$ be Henselian Gorenstein with triangle equivalence $\Psi: \text{CM}(A) \to \text{CM}(B)$. Then if $M \in \text{CM}(0)(A)$ then $\Psi(M) \in \text{CM}(0)(B)$. In particular if $A$ is an isolated singularity then so is $B$.

We now discuss the technique used to prove Theorem 1.4 and Lemma 1.7. The notion of Auslander-Reiten (AR) triangles was first introduced by Happel in Hom-finite Krull-Schmidt triangulated categories, see \cite[1.4]{Happ}. Later in \cite[I.2.3]{Yamaz} it is shown that a Hom-finite Krull-Schmidt triangulated category has (right) AR-triangles if and only if right Serre-functor. Our observation is that the notion of AR-triangle is categorical. Note if $A$ is Henselian then CM$(A)$ is Krull-Schmidt. So one can define notion of AR-triangle(ending at $M$) when $M$ is indecomposable $N \xrightarrow{\sim} E \xrightarrow{\sim} M \xrightarrow{\sim} \Omega^{-1}(N)$ in CM$(A)$ (see \cite[6.4]{Yamaz}). As our category CM$(A)$ is no longer Hom-finite it requires a proof that AR-triangles are unique upto isomorphism of triangles in CM$(A)$; see \cite[6.7]{Yamaz}. Finally we prove that there is an AR-triangle ending at $M$ if and only if $M \in CM(0)(A)$. Furthermore $N \cong \Omega^{-d+2}(M)$. We then show that if $\psi: \text{CM}(A) \to \text{CM}(B)$ is a triangle-equivalence and there is an AR-triangle ending in $M$ then $\psi$ induces naturally an AR-triangle ending in $\psi(M)$. This proves Lemma 1.7. Finally we prove Theorem 1.4 using some techniques from \cite{Katz}.
(a) **Essential periodicity of complete intersections:**

Recall a module $M$ is said to be periodic if $\Omega^r(M) \cong M$ for some $r \geq 1$. Note if $A$ is Cohen-Macaulay then a periodic module is necessarily MCM $A$-module. Every abstract complete intersection has a periodic MCM module $M$ with $\Omega^2(M) \cong M$.

We show:

**Theorem 1.8.** Let $(A, m)$ be an abstract complete intersection of codimension $c$. Let $C$ be some triangulated category with shift functor $\Sigma$. If $\phi: \text{CM}(A) \to C$ is a non-zero functor of triangulated categories then there exists $X \neq 0$ in $C$ with $\Sigma^s(X) \cong X$ for some $s \geq 1$. If residue field of $A$ is infinite then we may choose $s = 1$.

**Remark 1.9.** We note that in Theorem 1.8 the functor $\phi$ is not necessarily an equivalence. It is simply non-zero.

A trivial corollary of Theorem 1.8 is:

**Corollary 1.10.** Let $(A, m)$ be an abstract complete intersection and let $B$ be a Gorenstein local ring with no periodic modules. Then any triangulated functor $\Psi: \text{CM}(A) \to \text{CM}(B)$ is zero.

(b) **Periodic complexes with finite length cohomology:**

Let $F: \cdots \to F^i \overset{d^i}{\to} F^{i+1} \to \cdots$ be a co-chain complex of finitely generated free $A$-modules. We say $F$ is periodic if $F \cong F(-s)$ for some $s > 0$. We say $F$ is a minimal complex if $d^i(F^i) \subseteq mF^{i+1}$ for all $i$. If $M$ is a periodic $A$-module (i.e., $\Omega^r(M) \cong M$ for some $r \geq 1$) then splicing a minimal resolution of $M$ we can construct a minimal periodic acyclic co-chain complex. If we drop the assumption on acyclicity then we may ask whether there exist minimal periodic complexes with finite length cohomology. We prove

**Theorem 1.11.** Let $(R, m)$ be a Noetherian local ring which is a quotient of a regular local ring. Then there exists a minimal periodic (of period $2s$) co-chain complex $F$ of finitely generated free $R$-modules (i.e., $F \cong F(2s)$) such that $H^i(F)$ has finite length for all $i$. If $R$ is regular or if residue field of $R$ is infinite we can choose $s = 1$. In this case there exists a minimal complex $G$ of finitely generated free modules with $G \cong G[2]$ and $H^i(G)$ has finite length for all $i \in \mathbb{Z}$.

**Remark 1.12.** (a) Theorem 1.11 has no content if dim $R = 0$; for in that case we may consider the co-chain complex

$$\cdots \to R \to 0 \to R \to 0 \to \cdots$$

(b) If dim $R > 0$ and $F$ is two periodic with finite length cohomology then necessarily rank $F^{2i} = \text{rank} F^{2i+1}$ for all $i \in \mathbb{Z}$. (see Lemma 7.6).
(b (i))2-periodic co-chain complexes with finite length cohomology over a regular local ring:
Let $(A, m)$ be a regular local ring of dimension $d \geq 1$. As MCM modules over $A$ are free we get $\text{CM}(A) = 0$. By [9, 4.4.1] it follows that the homotopy category of exact acyclic co-chain complexes of finitely generated free modules over $A$ is zero. An initial motivation to prove Theorem 1.11 for us was to prove whether periodic co-chain complexes of finitely generated free modules (with finite length co-homology) exist up to homotopy over regular local rings. However we make the following:

**Conjecture 1.13.** Let $(A, m)$ be a regular local ring of dimension $d \geq 1$. Let $F$ be a non-zero minimal 2-periodic co-chain complex $F$ of finitely generated free $A$-modules such that $H^i(F)$ has finite length for all $i$. Then

1. $\text{rank } F^i \geq d$ for all $i \in \mathbb{Z}$.
2. There exists a minimal 2-periodic co-chain complex $G$ of finitely generated free $A$-modules such that $H^i(G)$ has finite length for all $i$ and $\text{rank } G^i = d$ for all $i \in \mathbb{Z}$.

Next we give results which support our conjecture.

(a) We prove (1) for $d = 1, 2, 3, 4$.
(b) We show it for $d = 1, 2, 3$.

In general we construct a minimal 2-periodic co-chain complex $\mathbb{H}$ with $\text{rank } \mathbb{H}^i = 2^d$ and $H^i(\mathbb{H})$ has finite length for all $i$.

**2. Notation and Preliminaries**

In this section we introduce some notation and discuss some preliminaries. In this paper all rings are Noetherian and all modules considered are finitely generated. We let $\mathbb{N}$ denote the set of non-negative integers. Nothing in this section is a new result.

**2.1.** Let $(A, m)$ be local with residue field $k$ and let $M$ be an $A$-module. By $\ell(M)$ we denote the length of $M$ and by $\mu(M)$ we denote the number of its minimal generators. Set $\beta_i(M) = \ell(\text{Tor}_i^A(k, M)) = \ell(\text{Ext}_A^i(M, k))$ the $i^{th}$ betti number of $M$.

**2.2.** Let $S = \bigoplus_{n \geq 0} S_n$ be a graded algebra over a ring $S_0$. Let $M$ be a graded $S$-module. If $m$ is a homogeneous element of $M$ then we set $|m| = \text{degree of } m$.

We need the following well-known result.
Proposition 2.3. Let $S = K[X_1, \ldots, X_d]$ be a polynomial ring with $\deg X_i = 2$ for all $i$. Let $M$ be a graded $S$-module of dimension $r \geq 1$. Then there exists homogeneous $x$ of degree $2s$ such that $\ker(M(−[2s]) \to M)$ has finite length kernel. If $K$ is infinite then we may choose $s = 1$.

2.4. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function. Recall $f$ is said to be of polynomial type if there exists $p(X) \in \mathbb{Q}[X]$ such that $p(n) = f(n)$ for all $n \gg 0$. Notice $p(X)$ is uniquely determined by $f$ and we write it as $p_f(X)$. We also write $\deg f = \deg p_f(X)$. We make the convention that the zero polynomial has degree $-\infty$. The following result is easy to prove.

Proposition 2.5. Let $f$ be of polynomial type of degree $r$.

1. For $a \in \mathbb{Z}$ we have $g(n) = f(n) + f(n + a)$ is of polynomial type and $\deg g \leq r$.
2. For $a, b \in \mathbb{Z}$ with $a \geq 0$ we have $h(n) = f(an + b)$ is of polynomial type and $\deg h \leq r$.
3. For $a, b \in \mathbb{Z}$ with $a \geq 0$ we have $t(n) = \sum_{i=0}^{n-1} f(ia + b)$ is of polynomial type and $\deg t(n) \leq r + 1$.
4. For $r \geq 0$ the function $s(n) = \sum_{i=0}^{n} i^r$ is of polynomial type of degree $r + 1$.

2.6. We need to recall the notion of complexity of a module. This notion was introduced by Avramov in [3]. More generally we define complexity of a function $f : \mathbb{N} \to \mathbb{N}$.

$$cx f = \inf \left\{ b \in \mathbb{N} \left| \limsup_{n \to \infty} \frac{f(n)}{n^{b-1}} < \infty \right. \right\}.$$ 

Let $\beta_i^A(M) = \ell(\text{Tor}_i^A(M, k))$ be the $i$th Betti number of $M$ over $A$. Let $\beta_M$ be the function defined as $\beta_M(n) = \beta_i^A(M)$. The complexity of $M$ over $A$ is defined by

$$cx_A M = cx \beta_M.$$ 

Note that $cx_A M = 0$ if and only if $\projdim_A M < \infty$. Furthermore $cx_A M \leq 1$ if and only if $M$ has bounded Betti numbers. If $A$ is a local abstract complete intersection of codim $c$ then $cx_A M \leq c$ for any $A$-module $M$; see [14] 4.1.

Remark 2.7. We note that $\limsup_{n \to \infty} f(n)/n^{b-1} < \infty$ if and only if there exists a polynomial $p(X) \in \mathbb{Q}[X]$ with $\deg p(X) = b - 1$ and positive leading coefficient such that $f(n) \leq p(n)$ for all $n \gg 0$.

2.8. For $i = 1, \ldots, r$ let $f_i : \mathbb{N} \to \mathbb{Z}$ be functions. Set $f(n) = \max\{f_i(n) \mid 1 \leq i \leq r\}$. Then it is clear that

$$cx f \leq \max\{cx f_i \mid 1 \leq i \leq r\}.$$ 

We will need the following result. Although elementary we give a proof for the convenience of the reader.
Lemma 2.9. Let \( f, g : \mathbb{N} \to \mathbb{Z} \) be functions such that there exists \( n_0 \) and \( d \) such that
\[
f(n + d) \leq f(n) + g(n) + g(n - 1) \quad \text{for all } n \geq n_0.
\]
If \( cx \, g \leq r \) then \( cx \, f \leq r + 1 \).

Proof. Set \( h(n) = g(n) + g(n - 1) \) for \( n \geq 1 \) and \( h(0) = 0 \). Then clearly \( cx \, h \leq r \).

We have \( f(n + s) \leq f(n) + h(n) \) for \( n \geq n_0 \). For \( i = 0, \ldots, d - 1 \) set
\[
f_i(m) = f(n_0 + i + dm)
\]
We have
\[
f_i(m) \leq f(n_0 + i) + \sum_{k=0}^{m-1} h(n_0 + i + kj).
\]
Using \( \text{Lemma 2.9} \) we get \( cx \, f_i \leq r + 1 \). It follows that \( cx \, f \leq r + 1 \). \( \square \)

2.10. Let \( N \) be an \( A \)-module. For each non-negative integer \( n \), let \( \pi^n \) denote the canonical surjection \( N \to N/m^{n+1}N \). We say \( \pi_n \) is small if the induced maps
\[
\text{Tor}^A_j(\pi_n, k) : \text{Tor}^A_j(N, k) \to \text{Tor}^A_j(N/m^{n+1}N, k);
\]
is injective for all \( j \geq 0 \). By \( \text{[2, (A.4)]} \) there exists \( n_0 \) such that \( \pi_n \) is small for all \( n \geq n_0 \).

2.11. Triangulated category structure on \( \text{CM}(A) \).

The reference for this topic is \( \text{[3, 4.7]} \). We first describe a basic exact triangle. Let \( f : M \to N \) be a morphism in \( \text{CM}(A) \). Note we have an exact sequence \( 0 \to M \xrightarrow{i} Q \to \Omega^{-1}(M) \to 0 \), with \( Q \)-free. Let \( C(f) \) be the pushout of \( f \) and \( i \). Thus we have a commutative diagram with exact rows
\[
\begin{align*}
0 \to & M \xrightarrow{i} Q \xrightarrow{p} \Omega^{-1}(M) \to 0 \\
0 \to & N \xrightarrow{i'} C(f) \xrightarrow{p'} \Omega^{-1}(M) \to 0
\end{align*}
\]
Here \( j \) is the identity map on \( \Omega^{-1}(M) \). As \( N, \Omega^{-1}(M) \in \text{CM}(A) \) it follows that \( C(f) \in \text{CM}(A) \). Then the projection of the sequence
\[
M \xrightarrow{j} N \xrightarrow{i'} C(f) \xrightarrow{p'} \Omega^{-1}(M)
\]
in \( \text{CM}(A) \) is a basic exact triangle. Exact triangles in \( \text{CM}(A) \) are triangles isomorphic to a basic exact triangle.

2.12. The following assertions are well-known: We note that the lower exact sequence in the above commutative diagram can be considered as an element in \( \text{Ext}^{1}_A(\Omega^{-1}(M), N) \). Conversely consider an element \( s : 0 \to N \to L \to \Omega^{-1}(M) \to 0 \) of \( \text{Ext}^{1}_A(\Omega^{-1}(M), N) \). Consider the exact sequence \( 0 \to M \to Q \to \Omega^{-1}(M) \to 0 \) with \( Q \)-free. Let \( f : M \to N \) be a lift of identity map on \( \Omega^{-1}(M) \). Then a simple
diagram chase (for instance see [10, Chapter XIV, proof of Theorem 1.1]) shows that connecting surjection $\gamma: \text{Hom}_A(M, N) \to \text{Ext}_A^1(\Omega^{-1}(M), N)$ maps $f$ to $s$. Thus given an element $s \in \text{Ext}_A^1(\Omega^{-1}(M), N)$

$$s: 0 \to N \xrightarrow{\alpha} L \xrightarrow{\beta} \Omega^{-1}(M) \to 0;$$

there exists a basic exact triangle in $\text{CM}(A)$

$$M \xrightarrow{f} N \xrightarrow{\alpha} L \xrightarrow{\beta} \Omega^{-1}(M)$$

where $\gamma$ maps $f$ to $s$.

2.13. It can be easily seen that the surjective map

$\gamma: \text{Hom}_A(M, N) \to \text{Ext}_A^1(\Omega^{-1}(M), N)$ discussed above induces an isomorphism

$$\gamma: \text{Hom}_A(M, N) \to \text{Ext}_A^1(\Omega^{-1}(M), N).$$

The following result is definitely known. We give a proof due to lack of a reference.

**Lemma 2.14.** Let $(A, m)$ be a Gorenstein local ring and let $M, N, L$ be MCM $A$-modules.

(1) A morphism $f: M \to N$ in $\text{CM}(A)$ yields for $i \geq 1$ well-defined maps

$$T_i(f): \text{Tor}_i^A(M, k) \to \text{Tor}_i^A(N, k).$$

(2) If $f: M \to N$ is isomorphism in $\text{CM}(A)$ then $T_i(f)$ are isomorphisms for $i \geq 1$.

(3) If $M \to N \to L \to \Omega^{-1}M$ is an exact triangle in $\text{CM}(A)$ then for $i \geq 2$ we have an exact sequence

$$\text{Tor}_i^A(M, k) \to \text{Tor}_i^A(N, k) \to \text{Tor}_i^A(N, k) \xrightarrow{\delta_i} \text{Tor}_{i-1}^A(M, k)$$

**Proof.** (1) This follows from the fact that if $g: M \to N$ factors through a free module then $\text{Tor}_i^A(g, k) = 0$ for $i > 0$.

(2) This follows from (1).

(3) As rotations of exact triangles are exact, we get that there is an exact triangle $\Omega(L) \to M \to N \to L$. By construction of exact triangles in $\text{CM}(A)$ we may assume that there exists an exact sequence $0 \to M \to N \oplus F \to L \to 0$ in $\text{mod}(A)$ (where $F$ is a finitely generated free $A$-module). So we have a long exact sequence in homology

$$\text{Tor}_i^A(M, k) \to \text{Tor}_i^A(N, k) \to \text{Tor}_i^A(N, k) \xrightarrow{\delta_i} \text{Tor}_{i-1}^A(M, k)$$

$\square$


3. SECTION OF A MODULE OVER COMPLETE INTERSECTION

Let \((A, \mathfrak{m})\) be an abstract complete intersection of co-dimension \(c\). For \(1 \leq r \leq c\) let \(\text{CM}_{\leq r}(A)\) be the thick subcategory of \(\text{CM}(A)\) generated by \(\text{MCM}_A\)-modules with complexity \(\leq r\). If \(c \geq 2\) then for \(r \geq 2\) we let \(\tau_r(A) = \text{CM}_{\leq r}(A) / \text{CM}_{\leq r-1}(A)\) be the Verdier quotient. For systemic reasons we set \(\tau_1(A) = \text{CM}_{\leq 1}(A)\). We say an object in a \(X\) triangulated category \(C\) with shift functor \(\Sigma\) is periodic if \(\Sigma^s(X) = X\) for some \(s \geq 1\). We show

**Theorem 3.1.** Let \((A, \mathfrak{m})\) be an abstract complete intersection of co-dimension \(c\). Fix \(i\) with \(i = 1, \ldots, c\) and let \(X \in \tau_i(A)\). Then \(X\) is periodic with period \(2^s\), for some \(s \geq 1\) (here \(s\) may depend on \(X\)). If the residue field of \(A\) is infinite then we can choose \(s = 1\).

3.2. To prove Theorem 3.1 we need the notion of cohomological operators over a complete intersection ring; see [14] and [13]. Let \(f = f_1, \ldots, f_c\) be a regular sequence in a local Noetherian ring \(Q\). Set \(I = (f)\) and \(A = Q/I\).

3.3. The *Eisenbud operators*, [13] are constructed as follows:

Let \(F: \cdots \to F_{i+2} \xrightarrow{\partial} F_{i+1} \xrightarrow{\partial} F_i \to \cdots\) be a complex of free \(A\)-modules.

**Step 1:** Choose a sequence of free \(Q\)-modules \(\tilde{F}_i\) and maps \(\tilde{\partial}\) between them:

\[
\tilde{F}: \cdots \to \tilde{F}_{i+2} \xrightarrow{\tilde{\partial}_i} \tilde{F}_{i+1} \xrightarrow{\tilde{\partial}_i} \tilde{F}_i \to \cdots
\]

so that \(F = A \otimes \tilde{F}\).

**Step 2:** Since \(\tilde{\partial}^2 \equiv 0\) modulo \((f)\), we may write \(\tilde{\partial}^2 = \sum_{j=1}^c f_j \tilde{t}_j\) where \(\tilde{t}_j: \tilde{F}_i \to \tilde{F}_{i-2}\) are linear maps for every \(i\).

**Step 3:** Define, for \(j = 1, \ldots, c\) the map \(t_j = t_j(Q, f, F): F \to F(-2)\) by \(t_j = A \otimes \tilde{t}_j\).

3.4. The operators \(t_1, \ldots, t_c\) are called Eisenbud’s operator’s (associated to \(f\)) . It can be shown that

(1) \(t_i\) are uniquely determined up to homotopy.

(2) \(t_i, t_j\) commute up to homotopy.

3.5. Let \(R = A[t_1, \ldots, t_c]\) be a polynomial ring over \(A\) with variables \(t_1, \ldots, t_c\) of degree 2. Let \(M, N\) be finitely generated \(A\)-modules. By considering a free resolution \(F\) of \(M\) we get well defined maps

\[t_j: \text{Ext}^n_A(M, N) \to \text{Ext}^{n+2}_A(M, N)\]

for \(1 \leq j \leq c\) and all \(n\),

which turn \(\text{Ext}^*_A(M, N) = \bigoplus_{i \geq 0} \text{Ext}^i_A(M, N)\) into a module over \(R\). Furthermore these structure depend on \(f\), are natural in both module arguments and commute with the connecting maps induced by short exact sequences.
3.6. Gulliksen, [14] 3.1, proved that if \( \text{projdim}_Q M \) is finite then \( \text{Ext}^*_A(M, N) \) is a finitely generated \( R \)-module. By taking \( N = k \) we get that the function \( n \to \beta_n(M) \) is quasi-polynomial of period 2 and degree \( \text{cx} M - 1 \).

The following result is crucial to prove some of our results. When \( A \) is a geometric complete intersection with infinite residue field then Theorem 3.7 essentially follows from techniques in [6] 7.3.

**Theorem 3.7.** Let \( (A, \mathfrak{m}) \) be a complete intersection with residue field \( k \) and let \( M \) be a \( \text{MCM} \) \( A \)-module of complexity \( \geq 2 \). Set \( M_i = \Omega^i(M) \) for \( i \geq 0 \). Then there exists \( s \) and \( n_0 \) such that we have a surjective map \( M_{n_0 + 2s} \xrightarrow{\alpha} M_{n_0} \) satisfying the following properties:

1. The maps \( \text{Tor}^A_i(\alpha, k) \) is surjective for all \( i \geq 0 \).
2. \( \text{cx ker} \alpha = \text{cx} M - 1 \).
3. If \( k \) is infinite we can choose \( s = 1 \).

**Proof.** We first consider the case when \( A \) is complete. Then \( A = Q/(\mathfrak{f}) \) where \( (Q, \mathfrak{n}) \) is a complete regular local ring and \( f = f_1, \ldots, f_c \in \mathfrak{n}^2 \) is a \( Q \)-regular sequence. Let \( F \) be a minimal resolution of \( M \) as an \( A \)-module and let \( t_1, \ldots, t_c : F(+2) \to F \) be Eisenbud operators corresponding to \( f \) and \( F \). Set \( E(M) = \bigoplus_{n \geq 0} \text{Ext}^n_A(M, k) \). It is a finitely generated graded \( R = A[t_1, \ldots, t_c] \)-module where \( \text{deg} t_i = 2 \) for all \( i \). As \( \mathfrak{m}E(M) = 0 \) we get that \( E(M) \) is a finitely generated \( S = k[t_1, \ldots, t_c] \)-module. We may choose \( t \) homogeneous of degree 2 in \( S \) such that (0: \( E(M)t \)) has finite length (equivalently \( t \) is \( E(M) \)-filter regular). If \( k \) is infinite it is readily seen that we may choose \( t \) linear in \( t_i \) and so of degree 2 (see [23]). We have that \( t \) induces injections \( \text{Ext}^n_A(M, k) \to \text{Ext}^n_A(M, k) \) for \( n \gg 0 \) say for \( n \geq n_0 \). Dualizing we get surjections \( \text{Tor}^A_i(M_{n_0 + 2s}, k) \to \text{Tor}^A_i(M, k) \) for \( n \geq n_0 \). Let \( \xi \) be homogeneous of degree 2s in \( R \) such that its image in \( S \) is \( t \). So we have a chain map \( \xi : F(-2s) \to F \). As \( \text{Tor}^n_\xi(\xi, k) \) is surjective for \( n \geq n_0 \), it follows from Nakayama’s lemma that we have surjections \( F_{n + 2s} \to F_n \) for \( n \geq n_0 \). Note \( \xi \) is a chain map. So we have exact sequences \( 0 \to L_n \to M_{n + 2s} \to M_n \to 0 \) for \( n \geq n_0 \). Also note that we also have exact sequences for \( i \geq 0 \)

\[
\text{Tor}^A_i(M_{n_0 + 2s}, k) \to \text{Tor}^A_i(M_{n_0}, k) \to 0.
\]

It follows that we have exact sequence

\[
0 \to \text{Tor}^A_i(L_{n_0}, k) \to \text{Tor}^A_i(M_{n_0 + 2s}, k) \to \text{Tor}^A_i(M_{n_0}, k) \to 0.
\]

So \( \text{cx} L_{n_0} = \text{cx} M - 1 \). The result follows if \( A \) is complete.

We now consider the general case. Note \( \Omega^A_i(M) \otimes \hat{A} \cong \Omega^\hat{A}_i(\hat{M}) \). By the complete case we have an exact sequence

\[
0 \to L \to \hat{M}_{n_0 + 2s} \xrightarrow{\alpha} \hat{M}_{n_0} \to 0
\]
which does the job. Notice

\[ \alpha \in \text{Hom}_\hat{A}(\hat{M}_{n_0+2s}, \hat{M}_{n_0}) = \text{Hom}_A(M_{n_0+2s}, M_{n_0}) \otimes_A \hat{A}. \]

Set \( \alpha = \sum_{i=1}^r \phi_i \otimes a_i \) where \( \phi_i \in \text{Hom}_A(M_{n_0+2s}, M_{n_0}) \) and \( a_i \in \hat{A} \).

Now assume that the map \( \hat{M}_{n_0} \to \hat{M}_{n_0}/\hat{m}^{l} \hat{M}_{n_0} \) is small (see 2.10). Let \( a_i = b_i + c_i \) with \( b_i \in A \) and \( c_i \in \hat{m}^l \). So we get \( \alpha = \tilde{\alpha} \otimes_A 1 + \beta \) where \( \tilde{\alpha} \in \text{Hom}_A(M_{n_0+2s}, M_{n_0}) \) and \( \beta \in \hat{m}^l \text{Hom}_\hat{A}(\hat{M}_{n_0+2s}, \hat{M}_{n_0}) \). It follows that

\[ \text{Tor}_\alpha^A(\alpha, k) = \text{Tor}_A^A(\alpha, k) \otimes_A \hat{A} + \text{Tor}_\beta^A(\beta, k) \]

We have a commutative diagram

\[
\begin{array}{ccc}
M_{n_0+2s} & \xrightarrow{\beta} & \hat{M}_{n_0} \\
\downarrow & & \downarrow \pi \\
\hat{M}_{n_0} & \xrightarrow{\alpha} & \hat{M}_{n_0}/\hat{m}^l \hat{M}_{n_0}
\end{array}
\]

This yields a commutative diagram

\[
\begin{array}{ccc}
\text{Tor}_\alpha^A(M_{n_0+2s}, k) & \xrightarrow{0} & \text{Tor}_\beta^A(\beta, k) \\
\downarrow & & \downarrow \\
\text{Tor}_\alpha^A(M_{n_0}, k) & \xrightarrow{\text{Tor}_\beta^A(\beta, k)} & \text{Tor}_\alpha^A(\hat{M}_{n_0}/\hat{m}^l \hat{M}_{n_0}, k)
\end{array}
\]

As \( \text{Tor}_\alpha^A(\pi, k) \) is injective it follows that \( \text{Tor}_\alpha^A(\beta, k) = 0 \). So we have

\[ \text{Tor}_\alpha^A(\alpha \otimes 1, k) = \text{Tor}_\alpha^A(\alpha, k) \]

It follows that \( \text{Tor}_\alpha^A(\alpha, k) \) is surjective. Set \( K = \ker \tilde{\alpha} \). We have a short exact sequence

\[ 0 \to K \to M_{n_0+2s} \to M_{n_0} \to 0 \]

which satisfies our requirements. \( \square \)

We now give:

**Proof of Theorem 3.1.** It is well-known that if \( M \) is MCM with complexity one then it is periodic with period 2. So \( \tau_1(A) \) is periodic. Now let \( r \geq 2 \) and let \( M \in \tau_r(A) \) be non-zero. Then \( M \) as an \( A \)-module has complexity \( r \). Set \( M_n = \Omega_n(M) \).

Then by Theorem 3.7 there exists an exact sequence \( 0 \to K \to M_{n+2s} \to M_n \to 0 \) with complexity \( K = r - 1 \). So in \( \text{CM}(A) \) we have an exact triangle \( K \to M_{n+2s} \to M_n \to \Omega^{-1}K \). It follows that in \( \tau_r(A) \) we have \( M_{n+2s} \cong M_n \) and so \( M_{2s} \cong M \). So \( M \) is periodic with period 2s. Also by our construction in Theorem 3.7 we may choose \( s = 1 \) if the residue field of \( A \) is infinite. \( \square \)

We now give a proof of Theorem 1.8. It is convenient to prove a slightly more general result.
Theorem 3.8. Let \( C \) be a triangulated category. Let \( C_1 \subseteq C_2 \subseteq \ldots \) be an ascending sequence of thick triangulated subcategories of \( C \) such that

1. \( C = \bigcup_{i \geq 1} C_i \).
2. \( C_1 \) is periodic.
3. The Verdier quotient \( C_i/C_{i-1} \) is periodic for all \( i \geq 2 \).

If \( f : C \to H \) is a non-zero triangulated functor then there exists a non-zero periodic object \( X \) in \( H \). Furthermore if \( C_1 \) and \( C_i/C_{i-1} \) are \( 2 \)-periodic for \( i \geq 2 \) then we may choose \( X \) to be two periodic.

Proof. We note that either \( f(C_1) \neq 0 \) or there exists \( r \geq 2 \) such that \( f(C_r) \neq 0 \) but \( f(C_{r-1}) = 0 \). If \( f(C_1) \neq 0 \) then if \( X \) is a non-zero element in the image we get that \( X \) is periodic.

If there exists \( r \geq 2 \) such that \( f(C_r) \neq 0 \) but \( f(C_{r-1}) = 0 \) then note that \( f \) factors over the Verdier quotient \( f : C_r/C_{r-1} \to H \) and is non-zero. It follows that if \( f(M) = X \) is non-zero then \( X \) is periodic. \( \square \)

We now give

Proof of Theorem 3.8. The result follows from Theorem 3.8 and Theorem 3.1. \( \square \)

3.9. Let \( (A,m) \) be an abstract complete intersection of codimension \( c \). Then \( \text{CM}^0(A) \) the category of MCM \( A \)-modules free on \( \text{Spec}^0(A) \) is a thick subcategory of \( \text{CM}(A) \). Set

\[ \text{CM}^0_{\leq i}(A) = \{ M \mid M \in \text{CM}^0(A) \text{ and } \text{cx} M \leq i \} \]

Then \( \text{CM}^0_{\leq i}(A) \) defines a thick subcategory of \( \text{CM}(A) \). For \( i = 2, \ldots, c \) set \( V_i(A) = \text{CM}^0_{\leq i}(A)/\text{CM}^0_{\leq i-1}(A) \). For systemic reasons set \( V_1(A) = \text{CM}^0_{\leq 1}(A) \).

Remark 3.10. The categories \( V_i(A) \) are non-zero for \( i = 1, \ldots, c \). It suffices to show there exist MCM \( A \)-modules in \( \text{CM}^0(A) \) of complexity \( i \) for \( i = 1, \ldots, c \). We note that if \( d = \text{dim} A \) and \( k \) is the residue field of \( A \) then \( \Omega^d(k) \) is MCM \( A \)-module of complexity \( c \). Using Theorem 3.7 iteratively our assertion follows.

We prove:

Theorem 3.11. (with hypotheses as above:) The triangulated categories \( d V_i(A) \) for \( i = 1, \ldots, c \) are periodic. If the residue field of \( A \) is infinite then they are \( 2 \)-periodic.

Proof. Any MCM \( A \)-module with \( \text{cx} M \leq 1 \) is periodic with period 2. So \( V_1(A) \) is periodic.

Let \( M \in V_i(A) \) be non-zero for some \( i \geq 2 \). Then \( M \) as an \( A \)-module has complexity \( i \). Set \( M_0 = \Omega^i(M) \). Then by Theorem 3.7 we get an MCM \( A \)-module \( K \) of complexity \( i - 1 \) and an exact sequence \( 0 \to K \to M_{n+2s} \to M_n \to 0 \) in \( A \) for some \( s > 0 \) and for some \( n \geq 0 \). Note \( (M_i)_P \) is free for all \( P \neq m \) and for all \( i \geq 0 \).
It follows that $K_P$ is free for all $P \neq m$. It follows that $K \in \text{CM}_0^{\text{B}}(A)$. We also have an exact triangle $K \to M_{n+2s} \to M_n \to \Omega^{-1}(K)$ in $\text{CM}_0^{\text{B}}(A)$. So in $V_i(A)$ we get $M_{n+2s} \cong M_n$ and so $M_s \cong M$. Thus $V_i(A)$ for $i \geq 2$ are periodic. By 3.7 if the residue field of $A$ is infinite then we may choose $s = 1$. So in this case $V_i(A)$ for $i \geq 1$ are 2-periodic.

As an easy consequence we get

**Corollary 3.12.** (with hypotheses as above:) Let $C$ be a triangulated category and let $f: \text{CM}_0^{\text{B}}(A) \to C$ be a non-zero triangulated functor. Then there exists $0 \neq X \in C$ which is periodic. If the residue field of $A$ is infinite then we can choose $X$ to be 2-periodic.

**Proof.** The result follows from Theorem 3.8 and Theorem 3.11. □

4. PROOF OF THEOREM 1.1

In this section we first prove the following:

**Theorem 4.1.** Let $(A, m)$ be an abstract complete intersection of codimension $c$. Let $(B, n)$ be a Gorenstein local ring. Assume there exists a triangulated functor $f: \text{CM}_0^{\text{B}}(A) \to \text{CM}_0^{\text{B}}(B)$. If $M$ is an MCM $A$-module with $\text{cx}_A M = r$ then $\text{cx}_B f(M) \leq r$.

**Proof.** We note that every $A$ module $N$ has complexity $\leq c$. We prove the result by induction on $\text{cx}_A M$. If $\text{cx}_A M = 0$ then $M$ is a free $A$-module. So $M = 0$ in $\text{CM}(A)$. Thus $f(M) = 0$ in $\text{CM}(B)$. So $f(M)$ is a free $B$-module.

Next we assume $\text{cx}_A M = 1$. Then $M$ is periodic. We may assume $M$ has no free summands. Then $\Omega^1_B(M) \cong M$. As $f$ is a triangulated functor we get that $\Omega^1_B f(M) \cong f(M)$. So $\text{cx}_B f(M) \leq 1$.

Now assume that $\text{cx}_A M = r \geq 2$ and that our claim is true for MCM $A$-modules with complexity $= r - 1$.

Set $M_n = \Omega^1_A(M)$. By Theorem 3.7 there exists $n_0$ and an exact sequence

$$0 \to K \to M_{n_0+2s} \to M_n \to 0$$

in $\text{mod}(A)$ where $\text{cx} K = r - 1$. Note $K$ is MCM $A$-module. So we have an exact triangle

$$K \to M_{n_0+2s} \to M_n \to \Omega^{-1}_A(K)$$

So we have an exact triangle in $\text{CM}(B)$

$$f(K) \to f(M)_{n_0+2s} \to f(M)_{n_0} \to \Omega^{-1}_B(f(K)).$$

By induction hypothesis we get $\text{cx} f(K) \leq r - 1$. Using 2.14(3) we have for all $n \geq n_0 + 2$

$$\beta^-_{n+2s}(f(M)) \leq \beta^-_{n}(f(M)) + \beta^-_{n-n_0}(f(K)) + \beta^-_{n-n_0-1}(f(K))$$

where $\beta^-_n(f(M))$ denotes the complexity of $f(M)$.
By Lemma 2.9 it follows that $\text{cx} \, f(M) \leq r$. □

Recall an additive functor $f: \text{CM}(A) \to \text{CM}(B)$ is a weak equivalence if every MCM $B$-module $N$ is a direct summand of $f(M_N)$ for some $M_N$ in $\text{CM}(A)$ (here $M_N$ depends on $N$).

**Corollary 4.2.** Let $f: \text{CM}(A) \to \text{CM}(B)$ be a weak equivalence. If $A$ is an abstract complete intersection of codimension $c$ then $B$ is also an abstract complete intersection of codimension $\leq c$.

**Proof.** Let $N$ be an MCM $B$-module. Say $N$ is a direct summand of $f(M)$. As $\text{cx}_A M \leq c$, by 4.1 we get that $\text{cx}_B f(M) \leq c$. So $\text{cx}_B N \leq c$. By Gulliksen’s result, [15, 2.3] we have that $B$ is also an abstract complete intersection of codimension $\leq c$. □

Next we give a proof of Theorem 1.1. We restate it for the convenience of the reader.

**Theorem 4.3.** Let $A, B$ be Gorenstein local rings. Assume $\text{CM}(A)$ is triangle equivalent to $\text{CM}(B)$. If $A$ is an abstract complete intersection of co-dimension $c$ then $B$ is also an abstract complete intersection of co-dimension $c$.

**Proof.** Let $f: \text{CM}(A) \to \text{CM}(B)$ and $g: \text{CM}(B) \to \text{CM}(A)$ be the triangulated functors which are inverses of each other. Let $A$ be an abstract complete intersection of codimension $c$. Using $f$ and Corollary 4.2 we get that $B$ is an abstract complete intersection with codimension $\leq c$. Using $g$ it follows that $c \leq \text{codimension of } B$. So $A, B$ have the same codimension. □

We also show the following.

**Theorem 4.4.** Let $A, B$ be abstract complete intersection local rings. Assume $\text{CM}(A)_{\leq r}$ is triangle equivalent to $\text{CM}(B)_{\leq s}$ then $r = s$.

**Proof.** Let $f: \text{CM}(A)_{\leq r} \to \text{CM}(B)_{\leq s}$ be an equivalence. Then note if $X$ is in $\text{CM}(A)_{\leq r}$ then by 4.1 we get that the complexity of $f(X) \leq r$. So $r < s$ is not possible. Similarly by considering $f^{-1}$ we get that $s < r$ is not possible. So $r = s$. □

5. **Proof of Theorem 1.2**

In this section $(Q, n)$ is a regular local ring with algebraically closed residue field $k$. Let $A = Q/(f_1, \ldots, f_c)$ where $f_1, \ldots, f_c \in n^2$ is a regular sequence. We denote the maximal ideal of $A$ by $m$.

5.1. Let $U, V$ be two $A$-modules. Define

$$\text{cx}_A(U, V) = \inf \left\{ b \in \mathbb{N} \mid \limsup_{n \to \infty} \frac{\mu(\text{Ext}_A^n(U, V))}{n^{b-1}} < \infty \right\}.$$
5.2. Let $\text{Ext}^*(U, V) = \bigoplus_{n \geq 0} \text{Ext}^n_{A}(U, V)$ be the total ext module of $U$ and $V$. We consider it as a module over the ring of cohomological operators $A[t_1, \ldots, t_c]$. Since $\text{projdim}_Q U$ is finite $\text{Ext}^*(U, V)$ is a finitely generated $A[t_1, \ldots, t_c]$-module.

5.3. Let $\mathcal{C}(U, V) = \text{Ext}^*(U, V) \otimes_A k$. Clearly $\mathcal{C}(U, V)$ is a finitely generated $T = k[t_1, \ldots, t_c]$-module. (Here degree of $t_i$ is 2 for each $i = 1, \ldots, c$). Set $a(U, V) = \text{ann}_T \mathcal{C}(U, V)$.

Notice that $a(U, V)$ is a homogeneous ideal.

5.4. We consider the affine space $A^c(k)$. Let $\mathcal{V}(U, V) = V(a(U, V)) \subseteq A^c(k)$. Since $a(U, V)$ is graded ideal we get that $\mathcal{V}(U, V)$ is a cone.

5.5. By a result due to Avramov and Buchweitz [5, 2.4] we get that $\dim \mathcal{V}(U, V) = \text{cx}_{A}(M, N)$.

Set $\mathcal{V}(U) = \mathcal{V}(U, k)$. Then $\mathcal{V}(U) = \mathcal{V}(U, V)$ and $\mathcal{V}(U, V) = \mathcal{V}(U) \cap \mathcal{V}(V)$.

5.6. It is easier to do geometry in projective space. Let $R = k[z_0, \ldots, z_{c-1}]$ be standard graded. If $f(t_1, \ldots, t_c) \in T$ then let $f_R(z_0, \ldots, z_{c-1})$ be the polynomial in $R$ obtained by replacing $t_i$ by $z_{i-1}$ for $i = 1, \ldots, c$. Set $a_R(U, V) = \{f_R(z_0, \ldots, z_{c-1}) \mid f \in a(U, V)\}$.

Then $a_R(U, V)$ is a homogeneous ideal in $R$. Set $\mathcal{V}^*(U, V) = V(a_R(U, V)) \subseteq \mathbb{P}^{c-1}(k)$.

Set $\dim \emptyset = -1$. Clearly

$$\text{cx}_{A}(M, N) = \dim \mathcal{V}^*(U, V) + 1.$$ 

We also have $\mathcal{V}^*(U, V) = \mathcal{V}^*(U) \cap \mathcal{V}^*(V)$. We now give a proof of Theorem 1.2. We restate it for the convenience of the reader.

**Theorem 5.7.** Let $(A, m)$, $(B, n)$ be geometric complete intersections of co-dimension $m, n \geq 2$ respectively and with algebraically closed residue fields. Let $r \geq \max\{m/2, n/2\}$ and also $r \leq \min\{m, n\}$. If $\text{CM}_{\leq r}(A)$ is triangle equivalent to $\text{CM}_{\leq r}(B)$ then necessarily $m = n$.

**Proof.** Suppose if possible $m \neq n$. Without loss of generality we may assume $m \geq n + 1$. Let $\psi: \text{CM}_{\leq r}(A) \to \text{CM}_{\leq r}(B)$ be an equivalence. By using Theorem 4.1 to $\psi$ and $\psi^{-1}$ we get that if $\text{cx}_{A} E = t$ then $\text{cx}_{B} \psi(E) = t$. In $\mathbb{P}^{m-1}(k)$ consider the following two varieties:

1. $X$ defined by $z_0 = z_1 = \cdots = z_{m-r-1} = 0$. 
(2) \( Y \) defined by \( z_{m-r} = z_{m-r+1} = \cdots = z_{m-1} = 0 \).

Note \( \dim X = r-1, \dim Y = m-r-1 \) and \( X \cap Y = \emptyset \). By [37, 2.3] there exists MCM \( A \)-modules \( M, N \) with \( V^\ast(M) = X \) and \( V^\ast(N) = Y \). Note \( \text{cx}_A M = r \) and \( \text{cx}_A N = m-r \). As \( r \geq m/2 \) we get \( m-r \leq r \). So \( M, N \in \text{CM}_{\leq r}(A) \). As \( V^\ast(M, N) = 0 \) we get \( \Ext^1_A(M, N) = 0 \) for all \( i \gg 0 \). Furthermore \( \Hom_A(M, \Omega^{-1}_A(N)) \cong \Ext^1_A(M, N) = 0 \) for all \( i \gg 0 \). It follows that \( \Hom_B(\psi(M), \Omega^{-1}_B(\psi(N))) = 0 \) for all \( i \gg 0 \). Thus \( \Ext^1_B(\psi(M), \psi(N)) = 0 \) for all \( i \gg 0 \). We now consider support varieties of \( B \)-modules in \( \mathbb{P}^{n-1} \). We have \( V^\ast(\psi(M), \psi(N)) = \emptyset \). But \( \dim V^\ast(\psi(M)) = \text{cx}_A(\psi(M)) - 1 = r - 1 \). Similarly \( \dim V^\ast(\psi(N)) = m - r - 1 \). The sum of these two dimensions is \( m - 2 \geq n - 1 \). This implies that \( V^\ast(\psi(M)) \cap V^\ast(\psi(N)) \neq \emptyset \) cf. [37, 1.7.2]. So we get \( V^\ast(\psi(M), \psi(N)) \neq V^\ast(\psi(M)) \cap V^\ast(\psi(N)) \) which is a contradiction. So \( m = n \).

6. AR-sequences and AR-triangles

In this section we first assume \( (A, \mathfrak{m}) \) is a Henselian Cohen-Macaulay local ring of dimension \( d \) with a canonical module \( \omega_A \). Later we will restrict to the case when \( A \) is a Henselian Gorenstein local ring. We first discuss the notion of AR-sequences. A good reference for these concepts is [35]. Finally we discuss existence of AR-triangles in \( \text{CM}(A) \).

6.1. Recall an exact sequence \( s: 0 \rightarrow N \rightarrow E \xrightarrow{p} M \rightarrow 0 \) of MCM \( A \)-modules is an AR-sequence (Auslander-Reiten sequence) if

(1) \( s \) is not split.

(2) \( M, N \) are indecomposable maximal Cohen-Macaulay \( A \)-modules.

(3) If \( L \) is a maximal Cohen-Macaulay \( A \)-module and if \( q: L \rightarrow M \) is a not a split epimorphism then there exists \( f: L \rightarrow E \) such that \( q = p \circ f \).

We call \( s \) the AR-sequence ending at \( M \) (equivalently starting at \( N \)). If there is a AR-sequence ending at \( M \) it is unique up-to isomorphism of short-exact sequences, [35, 2.7].

6.2. Let \( M \) be an indecomposable MCM \( A \)-module. By [35, 3.4], there is an AR-sequence ending at \( M \) if and only if \( M \) is free on \( \text{Spec}^0(A) \).

Also note that \( N \cong \text{Hom}_A(\Omega^d(\text{Tr}(M)), \omega_A) \); see [35, 3.11]. We set \( \tau(M) = N \) and call it the Auslander-Reiten translate of \( M \).

From now on assume \( (A, \mathfrak{m}) \) is a Henselian Gorenstein local ring of dimension \( d \).

6.3. Let \( M \) be an indecomposable MCM \( A \)-module free on \( \text{Spec}^0(A) \). Then the AR-translate \( \tau(M) = \Omega^{2-d}(M) \).

6.4. The notion of AR-triangles in a triangulated category was introduced by Happel [16, Chapter 1]. We will discuss it only for the category \( \text{CM}(A) \). Note as \( A \) is
Henselian we get that $\mathbf{CM}(A)$ is a Krull-Schmidt category. First note that the notion of split monomorphism and split epimorphism makes sense in any triangulated category. They are called section and retraction respectively in [16]. A triangle $N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \Omega^{-1}(N)$ in $\mathbf{CM}(A)$ is called an AR-triangle (ending at $M$) if

- $N, M$ are indecomposable.
- $h \neq 0$.
- If $D$ is indecomposable then for every non-isomorphism $t: D \to M$ we have $h \circ t = 0$.

By considering the functor $\text{Hom}_T(D, -)$ it is easy to see that (AR3) is equivalent to

- $(\text{AR3l})$ If $D$ is indecomposable then for every non-isomorphism $t: D \to M$ there is a lift $q: D \to E$ with $g \circ q = t$.
- It is also easy to see that (AR3l) is equivalent to
- $(\text{AR3g})$ If $W$ is not-necessarily indecomposable and $s: W \to M$ is not a retraction then there is a lift $q: W \to E$ with $g \circ q = s$.

Remark 6.5. It is shown in [16, 1.4.3] that AR triangles are unique up-to isomorphism of triangles. However the proof only works in Hom-finite categories. In our case the proof works only if $A$ is an isolated singularity. However to prove some of our results we have to prove uniqueness of AR-triangles in $\mathbf{CM}(A)$ in general.

We first prove:

Lemma 6.6. Let $s: N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \Omega^{-1}(N)$ be a triangle in $\mathbf{CM}(A)$ with $N$ indecomposable and $h \neq 0$. Let $\theta: s \to s$ be a morphism of triangles such that $\theta_M = 1_M$. Then $\theta$ is an isomorphism of triangles.

Proof. We note that $\Omega^{-1}(N)$ is indecomposable. If $u = \theta_{\Omega^{-1}(N)}$ is not an isomorphism then $u$ is in the Jacobson radical of $R = \mathbf{Hom}_A(\Omega^{-1}(N), \Omega^{-1}(N))$. As $s$ is a morphism of triangles we get $u \circ h = h$. Therefore $(1_R - u) \circ h = 0$. Note $1_R - u$ is invertible. This forces $h = 0$; a contradiction. So $u$ is an isomorphism. It follows that $\theta_N$ is an isomorphism. As $\theta_M, \theta_N$ are isomorphisms it is well known that $\theta_E$ is also an isomorphism. So $\theta$ is an isomorphism of triangles. $\square$

We now prove uniqueness of AR-triangles in $\mathbf{CM}(A)$.

Proposition 6.7. In $\mathbf{CM}(A)$ consider the following two AR-triangles ending at $M$.

- $s: N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \Omega^{-1}(N)$
- $s': N' \xrightarrow{f'} E' \xrightarrow{g'} M \xrightarrow{h'} \Omega^{-1}(N')$

Then $s \cong s'$.

Proof. As $h, h'$ are non-zero we get that $g, g'$ are not retractions; see [16, 1.4]. By (AR3g) there exists lifts $q: E' \to E$ and $q': E \to E'$ lifting $g'$ and $g$ respectively.
So we have morphism of triangles $\alpha: s \to s'$ and $\beta: s' \to s$ with $\alpha_M = \beta_M = 1_M$. Consider $\beta \circ \alpha: s \to s$. As $N$ is indecomposable and $h \neq 0$ we get by Lemma 6.7 that $\beta \circ \alpha$ is an isomorphism. Similarly $\alpha \circ \beta$ is an isomorphism. So $s \cong s'$.

**Definition 6.8.** Let $s: N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \Omega^{-1}(N)$ be an AR-triangle ending at $M$. Then by Proposition 6.7, $N$ is determined by $M$ (upto a not-necessarily unique isomorphism). Set $\tau^i(M) = N$.

Next we consider the question of existence of AR-triangles ending at an indecomposable MCM $A$-module.

**Proposition 6.9.** Let $M$ be a indecomposable non-free MCM module. The following conditions are equivalent:

1. There exists an AR-triangle ending at $M$.
2. $M$ is free on $\text{Spec}^0(A)$.

Furthermore $\tau^i(M) = \Omega^{-d+2}(M)$.

**Proof.** We first prove (2) $\implies$ (1).

Let $s: 0 \to \tau(M) \xrightarrow{\alpha} E \xrightarrow{\beta} M \to 0$ be a AR-sequence ending at $M$. Recall $\tau(M)$ is indecomposable. Consider the natural isomorphism $\gamma: \text{Hom}_A(\Omega(M), \tau(M)) \to \text{Ext}^1_A(M, \tau(M))$ and let $\gamma(f) = s$. As $s \neq 0$ we get $f \neq 0$. By [2,12] there is a triangle in $\text{CM}(A)$

$$\Omega(M) \xrightarrow{i} \tau(M) \xrightarrow{\alpha} E \xrightarrow{-\beta} M.$$ 

Rotating it we obtain a triangle

$$\tilde{s}: \tau(M) \xrightarrow{\alpha} E \xrightarrow{-\beta} M \xrightarrow{-\Omega^{-1}(f)} \Omega^{-1}(\tau(M)).$$

We claim that $\tilde{s}$ is an AR-triangle. Clearly (AR1) is satisfied. Also as $f \neq 0$ we get $\Omega^{-1}(f) \neq 0$. So (AR2) is satisfied. Also note that (AR3) follows from [6,13]. So $\tilde{s}$ is an AR-triangle.

Next we prove (1) $\implies$ (2). Let $s: N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \Omega^{-1}(N)$ be the AR-triangle ending at $M$. Suppose if possible $M$ is not free on $\text{Spec}^0(A)$. Say $M_P$ is not free for some prime ideal $P \neq m$ in $A$. Let $\alpha: 0 \to L \to F \to M \to 0$ be a free cover of $M$. Then notice $\alpha_P \neq 0$ in $\text{Ext}^1_A(M_P, L_P) \cong \text{Ext}^1_A(M, L)_P$. So there exists $r \in m \setminus P$ such that $r^n \alpha \neq 0$ for all $n \geq 1$. Then note that the extension $r^n \alpha$ is given by the push-out diagram:

$$\begin{array}{ccccccc}
\alpha: & 0 & \to & L & \xrightarrow{i} & F & \xrightarrow{\pi} & M & \xrightarrow{1_M} & 0 \\
 & r^n & \downarrow & & & \downarrow & & \downarrow & & \\
r^n \alpha: & 0 & \to & L & \xrightarrow{i_n} & W_n & \xrightarrow{\pi_n} & M & \to & 0 \\
\end{array}$$
The above commutative diagram induces a map of triangles

\[
\beta : \begin{array}{ccc}
L & \xrightarrow{i} & F \xrightarrow{\pi} M \xrightarrow{w} \Omega^{-1}(L) \\
\downarrow{r^n} & & \downarrow{1_M} \\
W_n & \xrightarrow{\pi_n} & M \xrightarrow{w_n} \Omega^{-1}(L)
\end{array}
\]

\[
\beta_n : \begin{array}{ccc}
L & \xrightarrow{i_n} & W_n \xrightarrow{\pi_n} M \xrightarrow{w_n} \Omega^{-1}(L) \\
\downarrow{r^n} & & \downarrow{r^n} \\
W_n & \xrightarrow{\pi_n} & M \xrightarrow{w_n} \Omega^{-1}(L)
\end{array}
\]

Claim: \(\pi_n\) is not a retraction for all \(n \geq 1\).

Suppose there exists \(m \geq 1\) such that \(\pi_m\) is a retraction. So there exists \(\xi : M \to W_m\) such that \(\pi_m \circ \xi = 1_M\) in \(\text{CM}(A)\). It follows that in \(\text{mod}(A)\) the map \(1_M - \pi_m \circ \xi : M \to M\) factors through a free \(A\)-module. It follows [27, 2.2(2)] that \(\pi_m \circ \xi\) is an isomorphism. This implies that \(r^m\alpha\) is split; which is a contradiction.

As \(\pi_n\) is not a retraction and \(s\) is an AR-triangle it follows that \(\pi_n\) has a lift \(W_n \to E\). So we have a morphism of triangles \(\theta : \beta_n \to s\) with \(\theta_M = 1_M\). So we get

\[
h = \theta_{\Omega^{-1}(L)} \circ w_n \\
= \theta_{\Omega^{-1}(L)} \circ r^n w.
\]

It follows that \(h \in m^n\text{Hom}_A(M, \Omega^{-1}(N))\) for all \(n \geq 1\). By Krull’s intersection we get \(h = 0\) which contradicts (AR2).

By our proof of (2) \(\implies\) (1) we get \(\tau^f(M) = \Omega^{-d+2}(M)\). \(\□\)

Lemma 6.10. Let \((A, m), (B, n)\) be Henselian Gorenstein rings. Let \(\Psi : \text{CM}(A) \to \text{CM}(B)\) be an equivalence of triangulated categories with inverse \(\Phi\). Let \(\eta_X : \Psi(\Omega_A^{-1}(X)) \to \Omega_B^{-1}(\Psi(X))\) be the natural isomorphism associated to \(\Psi\).

We then have:

1. If \(M\) is an indecomposable \(\text{MCM} A\)-module then \(\Psi(M)\) is also indecomposable.
2. Let \(s : N \xrightarrow{f} E \xrightarrow{g} M \xrightarrow{h} \Omega_A^{-1}(N)\) be an AR-triangle in \(\text{CM}(A)\). Then

\[
\Psi(s) : \Psi(N) \xrightarrow{\Phi(f)} \Psi(E) \xrightarrow{\Phi(g)} \Psi(M) \xrightarrow{\eta_N \circ \Psi(h)} \Omega_B^{-1}(\Psi(N))
\]

is an AR-triangle in \(\text{CM}(B)\).

Proof. (1) This is clear.

(2) By (1) we get that \(\Psi(M), \Psi(N)\) are indecomposable. Also clearly \(\eta_N \circ \Psi(h) \neq 0\). Let \(W\) be an indecomposable module in \(\text{CM}(B)\) and let \(t : W \to \Psi(M)\) be a map which is not a isomorphism. Then \(\Phi(t) : \Phi(W) \to M\) is a not an isomorphism with \(\Phi(W)\) an indecomposable module in \(\text{CM}(A)\). As \(s\) is an AR-triangle in \(\text{CM}(A)\) we get that \(h \circ \Phi(t) = 0\) It follows that \(\eta_N \circ \Psi(h) \circ t = 0\). Thus \(\Psi(s)\) is an AR-triangle in \(\text{CM}(B)\). \(\□\)

As a consequence of the above result we give,
Proof of Lemma 6.7. As \( \Psi \) is an additive functor, it suffices to show that if \( M \) is indecomposable and free on \( \text{Spec}^0(A) \) then \( \Psi(M) \) is free on \( \text{Spec}^0(B) \). By 6.9 there exists an AR triangle \( s \) in \( \text{CM}(A) \) ending at \( M \). By 6.10 we get that \( \Psi(s) \) is an AR-triangle in \( \text{CM}(B) \) ending at \( \Psi(B) \). So again by 6.9 we get that \( \Psi(M) \) is free on \( \text{Spec}^0(B) \). \( \square \)

We now give a proof of Theorem 1.4. We restate it for the convenience of the reader.

**Theorem 6.11.** Let \( (A, m), (B, n) \) be Henselian Gorenstein with \( \text{CM}(A) \) triangle equivalent to \( \text{CM}(B) \). Then

1. If \( A \) is not an abstract hypersurface ring then \( \dim A = \dim B \).
2. If \( A \) is an abstract hypersurface ring having an MCM \( A \)-module \( M \) with \( \Omega(M) \not\cong M \) then \( \dim A - \dim B \) is even.

**6.12.** Let \( M \) be an \( A \)-module. For \( i \geq 0 \) let \( \beta_i(M) = \dim_k \text{Tor}_i^A(M, k) \) be its \( i^{th} \) betti-number. Let \( P_M(z) = \sum_{n \geq 0} \beta_n(M) z^n \), the Poincare series of \( M \). Set

\[ \text{curv } M = \limsup(\beta_n(M))^\frac{1}{n} \]

It is possible that \( \text{cx}(M) = \infty \), see [4, 4.2.2]. However \( \text{curv}(M) \) is finite for any module \( M \) [4, 4.2.5]. It can be shown that if \( \text{cx}(M) < \infty \) then \( \text{curv}(M) \leq 1 \). We also have that \( \text{curv} M \leq \text{curv} k \), see [4, 4.2.4].

**Proof of Theorem 6.7.** Set \( r = \dim A \) and \( s = \dim B \). Let \( \Psi : \text{CM}(A) \to \text{CM}(B) \) be an equivalence with inverse \( \Phi \).

Case 1: \( A \) is not a hypersurface ring.

Suppose if possible \( r \neq s \). Without loss of generality we may assume \( r < s \). Let \( X(m) \) be the MCM approximation of \( m \). Let \( U \) be an indecomposable summand of \( X(m) \). Then by [27, 4.1] \( U \) is extremal; i.e., if \( A \) is a complete intersection of codimension \( c \geq 2 \) then \( \text{cx}_A U = c \) and if \( A \) is not a complete intersection then \( \text{curv } U = \text{curv } k > 1 \). In particular there exists \( n_0 \) such that \( \beta^A_{n+1}(M) > \beta^A_n(M) \) for all \( n \geq n_0 \). Set \( V = \Omega^m_{s_0}(U) \).

Notice \( X(m) \) is free on \( \text{Spec}^0(A) \). It follows that \( U \) and hence \( V \) is free on \( \text{Spec}^0(A) \). By 6.9 we have an AR-triangle ending at \( V \) in \( \text{CM}(A) \).

\[ \alpha : \tau_A^*(V) \to E \to V \to \Omega^{-1}_A(\tau_A(V)) \]

By 6.9 \( \tau_A^*(V) = \Omega_A^{-1+2}(V) \). By Lemma 6.10 \( \Psi(\alpha) \) is an AR-triangle ending at \( \Psi(V) \). So \( \Psi(\tau_A^*(V)) \cong \Omega^{-1+2}_B(\Psi(V)) \). But we also have

\[ \Psi(\tau_A^*(V)) = \Psi(\Omega_A^{-1+2}(V)) \cong \Omega^{-1+2}_B(\Psi(V)) \]

So we have \( \Omega^{-1+2}_B(\Psi(V)) \cong \Psi(V) \). Applying \( \Phi \) we get \( \Omega^{-1+2}_B(V) \cong V \). This implies \( \beta^A_{s-r}(V) = \beta^A_0(V) \) which is a contradiction.
Case 2: $A$ is an abstract hypersurface ring having an MCM $A$-module $M$ with $\Omega(M) \not\cong M$ then $\dim A - \dim B$ is even.

By Theorem 6.11 $B$ is also an abstract hypersurface. If $r = s$ then we have nothing to prove. Notice $\Omega_B(\Psi(M)) \not\cong \Psi(M)$. So we may without any loss of generality assume $r < s$. As in case (1) we get $\Omega_A^{s-r}(M) \cong M$. It follows that $s - r$ is even.

**6.13.** We now assume $(A, m)$ is a Henselian Gorenstein isolated singularity and algebraically closed residue field. Let $\Gamma(A)$ be the AR-quiver of $A$ and let $\Gamma(A)$ be the stable AR-quiver of $A$. Recall the vertices of $\Gamma(A)$ are isomorphism classes of indecomposable non-free MCM $A$-modules. Furthermore if $M, N$ are non-free MCM indecomposable $A$-modules. Let $0 \to \tau(M) \to E_M \to M \to 0$ be the AR-sequence ending at $M$. Let $r$ be the number of copies of $N$ in direct summands of $E_M$ (note $r = 0$ is possible). Then there $r$ arrows from $N$ to $M$ in $\Gamma(A)$; see [35, 5.5]. Note any AR-triangle ending at $M$ is isomorphic to 

$$s_M: \tau(M) \to E_M \to M \to \Omega^{-1}(\tau(M)).$$

We call $E_M$ to be the middle term of $s_M$. We now give

**Corollary 6.14.** (with hypotheses as in 6.13) Suppose $\underline{CM}(A)$ is triangle equivalent to $\underline{CM}(B)$. Then $\Gamma(A) \cong \Gamma(B)$.

**Proof.** Let $\Psi: \underline{CM}(A) \to \underline{CM}(B)$ be an equivalence with inverse $\Phi$. Let $\Gamma(A), \Gamma(B)$ be stable AR-quiver of $A$ and $B$ respectively. Define $f: \Gamma(A) \to \Gamma(B)$ by mapping $[M]$ to $[\Psi(M)]$. Let $s_M$ be the AR-triangle ending at $M$. If there are $r$ arrows from $N$ to $M$ in $\Gamma(A)$, then there are precisely $r$ direct summands of $N$ in the middle term of $s_M$. By Lemma 6.10 we get that that $\Psi(s_M)$ is the AR-triangle ending at $\Psi(M)$. By looking at the middle term of $\Psi(s_M)$ it follows that there are precisely $r$ arrows from $\Psi(N)$ to $\Psi(M)$. It follows that $f$ is an isomorphism of graphs.

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**7. Periodic Complexes**

We first discuss some preliminaries regarding complexes. Let $(R, m)$ be a Noetherian local ring and let $\text{mod}(R)$ be the category of finitely generated $R$-modules. Let $\mathcal{C}(\text{mod}(R))$ be the category of complexes (possibly unbounded at both ends) in $\text{mod}(A)$ and let $\mathcal{K}(\text{mod}(R))$ be the corresponding homotopy category. Let $\mathcal{K}(\text{proj} R)$ be the subcategory of $\mathcal{K}(\text{mod}(R))$ consisting of complex of projective $A$-modules. Let $\mathbb{F} \in \mathcal{K}(\text{proj} R)$. Then we say $\mathbb{F}$ is minimal if $\partial(\mathbb{F}) \subseteq m\mathbb{F}$. We index complexes cohomologically. By a simple split exact complex we mean a complex $X$ such that there exist $i_0$ such that $X^n = 0$ for $n \neq i_0, i_0 + 1$; also $X^{i_0} = X^{i_0+1} = R$ and $\partial^{i_0} = 1_R$. Note $X = 0$ in $\mathcal{K}(\text{proj} R)$. It is well known that if $G$ is a complex of free $R$-modules then $G = \mathbb{F} \oplus \mathbb{Y}$ where $\mathbb{F}$ is minimal and $\mathbb{Y}$ is a direct sum of simple split exact complexes. Thus $G \cong \mathbb{F}$ in $\mathcal{K}(\text{proj} R)$.

We will need the following well-known result.
Lemma 7.1. (with hypotheses as above). If $F, G$ are minimal complexes of free $R$-modules. If $F \cong G$ in $K(\text{proj} \, R)$ then $F \cong G$ in $C(\text{mod}(R))$.

7.2. Let $(A, m)$ be a Gorenstein local ring. Let $\mathcal{K}_{\text{ac}}(\text{proj} \, A)$ be the homotopy category of acyclic complexes of free $A$-modules. If $M$ is a MCM $A$-module then let $F_M$ be a complete resolution of $M$. By [9, 4.4.1] we have a triangle equivalence $\Theta: \text{CM}(A) \to \mathcal{K}_{\text{ac}}(\text{proj} \, A)$ with $\Theta(M) = F_M$. We give a proof of Theorem 1.11.

We restate it for the convenience of the reader.

Theorem 7.3. Let $(R, m)$ be a Noetherian local ring which is a quotient of a regular local ring. Then there exists a minimal periodic (of period 2$s$) co-chain complex $F$ of free $R$-modules (i.e., $F \cong F[2s]$) such that $H^i(F)$ has finite length for all $i$. If $R$ is regular or if residue field of $R$ is infinite we can choose $s = 1$. In this case there exists a minimal complex $G$ of free modules with $G = G[2]$.

Proof. If $R$ is a singular complete intersection then it has an MCM two periodic module say $E$. The its complete resolution $F_E$ does the job.

So assume that $R$ is not a singular complete intersection. We also first assume $R$ is NOT regular. By hypothesis $R$ is a quotient of a regular local ring say $(Q, q)$. Set $k = Q/q$. We choose a regular sequence $f_1, \ldots, f_c \in q^2$ such that $f_i \in \text{ann}_Q R$ and $c = \dim Q - \dim R$. Set $A = Q/(f_1, \ldots, f_c)$. Note $A$ is a singular complete intersection. It follows that

1. $R$ is a quotient of $A$.
2. $\dim R = \dim A$. Set $d = \dim A$.
3. $R$ is not free as an $A$-module.

Note we have a natural triangulated map $\mathcal{K}_{\text{ac}}(\text{proj} \, A) \to \mathcal{K}(\text{proj} \, R)$ given by $- \otimes_A R$. Thus we have a triangulated map $\eta: \text{CM}^0(A) \to \mathcal{K}(\text{proj} \, R)$ obtained by composing the previous map with the inclusion $\text{CM}^0(A) \hookrightarrow \text{CM}(A) \cong \mathcal{K}_{\text{ac}}(A)$. As $R$ is not a free $A$-module we get that $\text{Tor}_i^A(\Omega^d_A(k), R) \neq 0$ for all $i \geq 1$. In particular $\eta(\Omega^d_A(k)) \neq 0$. By Corollary 3.12 there exists $0 \neq F \in \mathcal{K}(\text{proj} \, R)$ with $F[2s] \cong F$ for some $s \geq 1$ and $F = \eta(X)$ for some $X$ in $\text{CM}^0(A)$. If $k$ is infinite then we can choose $s = 1$. We note that $H^i(F) = \text{Tor}_i^A(X, R)$ for $i \geq 1$. It follows that $\ell((H^i(F))) < \infty$ for all $i$. Notice by construction $F$ is a minimal complex. Therefore by [7.4] we get $F \cong F[2s]$ in $\mathcal{K}($mod$(R))$. If $R$ is regular choose $A = R[X]/(X^2)$ and choose $X = \Omega^d_A(k)$. Then same argument as before yields a complex $F$ with $F \cong F[2]$. Furthermore if $F[2] \cong F$ in $\mathcal{K}($mod$(R)$) then we can by Proposition 7.4 construct a minimal complex $G$ with $G = G[2]$.

We now state a result which we used in the previous Theorem.
Proposition 7.4. Let \((A, \mathfrak{m})\) be local and assume \(F \in \mathcal{C} \text{proj } A\) with \(F \cong F[2]\) and \(H^i(F)\) has finite length for all \(i\). Then there exists \(G \in \mathcal{C} \text{proj } A\) with \(G = G[2]\) and \(H^i(G)\) has finite length for all \(i\).

Proof. Let \(\psi: F \to F[2]\) be an isomorphism. We have a commutative diagram:

\[
\begin{array}{c}
F: \quad & F^0 \xrightarrow{\alpha^0} F^1 \xrightarrow{\alpha^1} F^2 \\
\end{array}
\]

\[
\begin{array}{c}
F[2]: \quad & F^2 \xrightarrow{\psi_0} F^3 \xrightarrow{\psi_1} F^4 \\
\end{array}
\]

Set \(X = F^2\) and \(Y = F^1\). Also set \(u = \alpha^0 \circ \psi_0^{-1}\) and \(v = \alpha^1\). Note

\[
v \circ u = \alpha^1 \circ (\alpha^0 \circ \psi_0^{-1}) = (\alpha^1 \circ \alpha^0) \circ \psi_0^{-1} = 0.
\]

\[
u \circ v = (\alpha^0 \circ \psi_0^{-1}) \circ \alpha^1 = (\psi_1^{-1} \circ \alpha^2) \circ \alpha^1 = \psi_1^{-1} \circ (\alpha^2 \circ \alpha^1) = 0.
\]

So we have a circular complex

\[
\begin{array}{c}
\mathcal{G}: \quad & \cdots \xrightarrow{v} X \xrightarrow{u} Y \xrightarrow{v} X \xrightarrow{u} \cdots \\
\end{array}
\]

Thus \(G = G[2]\).

We note that \(Z_Y = \ker \alpha^1\). As \(\psi_0^{-1}\) is bijective we get \(B_Y = \text{image } \alpha^0\). It follows that \(H_Y = H^1(F)\) and so has finite length. Also note that \(B_X = \text{image } \alpha_1\). Also \(u = \psi_1^{-1} \circ \alpha^2\). So \(Z_X = \ker \alpha^2\). Thus \(H_X = H^2(F)\) and so has finite length.

7.5. From now on we will work with minimal complexes \(F \in K(\text{proj}(A))\) with \(F[2] = F\) and \(H^i(F)\) having finite length for all \(i \in \mathbb{Z}\). We first prove:

Lemma 7.6. (with hypotheses as in 7.4). If \(d = \dim A > 0\) then \(\text{rank } F^i = \text{rank } F^{i+1}\) for all \(i\).

Proof. Let \(q\) be a minimal prime of \(A\) and let \(B = A_q\). We write \(F\) as

\[
\cdots \xrightarrow{\alpha} X \xrightarrow{\beta} Y \xrightarrow{\alpha} X \xrightarrow{\beta} \cdots
\]

As \(\ell(H^i(F)) < \infty\) for all \(i\) we get that \(\ker(\alpha)_q = \text{image}(\beta)_q\) and \(\ker(\beta)_q = \text{image}(\alpha)_q\). We consider the complex

\[
\mathcal{C}: 0 \to \ker(\beta) \to Y \xrightarrow{\beta} X \to \text{image}(\alpha) \to 0.
\]

Note \(\mathcal{C}_q\) is exact. Counting lengths we get \(\text{rank}_B Y_q = \text{rank}_B X_q\). The result follows.

Construction of periodic minimal complexes over Cohen-Macaulay local rings of dimension 1, 2, 3.

We first consider the case when \(\dim A = 1\).

Construction 7.7. (dim \(A = 1\): Let \(x \in \mathfrak{m}\) be an \(A\)-regular element). Consider \(F_x\) where \(F_x^i = A\) for all \(i \in \mathbb{Z}\) and
(1) $\partial^i = 0$ for $i$ even.

(2) $\partial^i = \text{multiplication by } x$ for $i$ odd.

Then $H^i(F_x) = A/(x)$ for $i$ even and $H^i(F_x) = 0$ for $i$ odd.

Next we consider the case when $\dim A = 2$.

**Construction 7.8.** (dim $A = 2$.) Let $x_1, x_2 \in \mathfrak{m}$ be an $A$-regular sequence and let $u_1, u_2 \in \mathfrak{m}$ be another regular sequence. Set $I = (x_1, x_2)$ and $J = (u_1, u_2)$.

Consider $F_{I,J}$ where $F^i_{I,J} = A^2$ for all $i \in \mathbb{Z}$. For convenience set $F^i_{I,J} = X$ for $i$ even with a basis $e_1, e_2$ and $F^i_{I,J} = Y$ for $i$ odd with a basis $f_1, f_2$. Also set $p = u_2 f_1 - u_1 f_2 \in Y$ and $q = x_2 e_1 - x_1 e_2 \in X$.

(1) For $i$ even set $\partial^i(e_j) = x_j p$ for $j = 1, 2$.

(2) For $i$ odd set $\partial^i(f_j) = u_j q$ for $j = 1, 2$.

It is easily verified that $F_{I,J}$ is indeed a complex. To compute cohomology first note that if $\partial^0(ax_1 + be_2) = 0$ then $(ax_1 + bx_2)p = 0$. As $u_2$ is $A$-regular we get $ax_1 + bx_2 = 0$. As $x_1, x_2$ is a regular sequence we get that $ac_1 + be_2 \in Aq$. So $Z^0 = Aq$. Clearly $B^0 = Jq$. It follows that $H^0(F_{I,J}) \cong A/J$. By symmetry we get $H^1(F_{I,J}) \cong A/I$.

Finally we consider the case when $\dim A = 3$.

**Construction 7.9.** (dim $A = 3$.) Let $x = x_1, x_2, x_3 \in \mathfrak{m}$ be an $A$-regular sequence. Consider $F_x$ where $F^i_x = A^3$ for all $i \in \mathbb{Z}$. For convenience set $F^i_x = X$ for $i$ even with a basis $e_1, e_2, e_3$ and $F^i_x = Y$ for $i$ odd with a basis $f_1, f_2, f_3$. Also set $p = -x_3 e_1 + x_2 e_2 - x_1 e_3 \in X$.

(1) For $i$ odd set $\partial^i(f_j) = x_j p$ for $j = 1, 2, 3$.

(2) For $i$ even set $\partial^i(e_1) = -x_2 f_1 + x_1 f_2$, $\partial^i(e_2) = -x_3 f_1 + x_1 f_3$ and $\partial^i(e_3) = -x_3 f_2 + x_2 f_3$.

It is easily verified that $F_x$ is indeed a complex. Also note the differentials are precisely as in the Koszul complex on $x_1, x_2, x_3$. It follows that $H^0(F_x) \cong A/(x)$ and $H^1(F_x) = 0$.

**7.10.** We know that if $\dim A > 0$ and $F$ is a minimal complex of free $A$-modules with $F = F[2]$ and $\ell(H^i(F)) < \infty$ for all $i \in \mathbb{Z}$ then rank $F^i$ is constant for all $i \in \mathbb{Z}$.

We set this constant value by $\beta(F)$. For regular local rings we show the following:

**Theorem 7.11.** Let $(A, \mathfrak{m})$ be a regular local ring and let $F \neq 0$ be a minimal complex of free $A$-modules with $F = F[2]$ and $\ell(H^i(F)) < \infty$ for all $i \in \mathbb{Z}$.

(1) If $\dim A = 2$ then $\beta(F) \geq 2$.

(2) If $\dim A = 3$ then $\beta(F) \geq 3$.

(3) If $\dim A = 4$ then $\beta(F) \geq 4$.

To prove this Theorem we need the following:
Lemma 7.12. Let \((A, m)\) be a regular local ring with \(\dim A \geq 2\) and let \(F \neq 0\) be a minimal complex of free \(A\)-modules with \(F = F[2]\) and \(\ell(H^i(F)) < \infty\) for all \(i \in \mathbb{Z}\). Then
\begin{enumerate}[(1)]  
    \item \(H^*(F) \neq 0\).
    \item \(\partial^i \neq 0\) for all \(i\).
\end{enumerate}

Proof. (1) Suppose if possible \(H^*(F) = 0\). Then \(F \in \mathcal{K}_{ac}(\text{proj} A)\). But \(\mathcal{K}_{ac}(\text{proj} A) \cong \text{CM}(A)\). As \(A\) is regular local, any MCM \(A\)-module is free. So \(\text{CM}(A) = 0\). Therefore \(F\) is contractible. As \(F\) is minimal, it can be easily shown that if \(F\) is contractible then \(F = 0\), a contradiction.

(2) Suppose if possible \(\partial^{-1} = 0\). If \(\ker \partial^0 \neq 0\) then it has positive depth. It follows that depth \(H^0(F) > 0\) a contradiction. So \(\ker \partial^0 = 0\). So \(H^0(F) = 0\). By (1) we have that \(H^1(F) \neq 0\). We have an exact sequence
\[
0 \to F^0 \xrightarrow{\partial^0} F^1 \to H^1(F) \to 0.
\]

By depth Lemma we get depth \(H^1(F) > 0\), a contradiction. Thus \(\partial^{-1} \neq 0\). By symmetry \(\partial^0 \neq 0\). \(\square\)

We now give

Proof of Theorem 7.11. (1) Suppose if possible there exists \(F\) with \(\beta(F) = 1\). Then we have
\[
F: \cdots \to A \xrightarrow{\pi} A \xrightarrow{\beta} A \xrightarrow{\alpha} A \xrightarrow{\gamma} A \to \cdots.
\]
Then \(xy = 0\). As \(A\) is a domain we get \(x = 0\) or \(y = 0\). Say \(y = 0\). Then \(H^1(F) = A/(x)\) which does not have finite length as \(\dim A = 2\); which is a contradiction.

(2) Let \(x\) be a regular parameter. Then \(A/(x)\) is regular local of dimension 2. Note \(F/xF\) satisfies the hypothesis of (1). It follows that \(\beta(F) = \beta(F/xF) \geq 2\). We prove that \(\beta(F) = 2\) is not possible.

Suppose if possible there exists \(F\) with \(\beta(F) = 2\).

By Lemma 7.12 we get that image \(\partial^i \neq 0\) for all \(i\). So rank image \(\partial^i = 1\) for all \(i\). As \(\ell(H^i(F)) < \infty\) for all \(i\) we get that rank \(\ker \partial^i = 1\) for all \(i\). Thus \(\ker \partial^i\) is isomorphic to an ideal \(I_i\) in \(A\). Note this isomorphism maps image \(\partial^{i-1}\) to an ideal \(J_i \subseteq I_i\) of \(A\). We note that \(J_0\) is generated by \(\leq 2\) elements. Say \(J_0 = (u, v)\).

Claim-(i) \(H^i(F) \neq 0\) for all \(i \in \mathbb{Z}\).

Suppose if possible \(H^0(F) = 0\). By Lemma 7.12 we get \(H^{-1}(F) \neq 0\). We have an exact sequence
\[
0 \to L_{-1} \to F^{-1} \xrightarrow{\partial^{-1}} F^0 \to \text{image} \partial^0 \to 0.
\]

By depth Lemma we get \(L_{-1}\) is free and so \(\cong A\). As \(L_{-1}\) is generated by 2 elements we get that \(H^{-1}(F)\) is not of finite length; a contradiction. Thus \(H^0(F) \neq 0\).

Similarly \(H^1(F) \neq 0\). As \(F\) is 2-periodic we get \(H^i(F) \neq 0\) for all \(i \in \mathbb{Z}\).

As \(A\) is regular local it is in particular a UFD. Suppose \(u = au_1\) and \(v = av_1\) where \(u_1\) and \(v_1\) do not have any common factors.
Claim-(ii) $m \notin \text{Ass } A/J_0$.

If we prove claim (ii) then result follows from claim (i) as $H^0(F)$ is a submodule of $A/J_0$. If $u_1 = 1$ or $v_1 = 1$ then $J_0 = (a)$ and clearly this implies Claim (ii). So $u_1 \neq 1$ and $v_1 \neq 1$. Notice height$(u_1, v_1) \geq 2$ and so = 2. Thus $u_1, v_1$ is an $A$-regular sequence. If $a = 1$ then also clearly the result follows. Thus also $a \neq 1$.

For $p, q \in A$ we write $p \mid q$ if $p$ divides $q$ and $p \nmid q$ otherwise. Choose a prime element $p \in m$ such that $p \nmid a$. Suppose if possible $m \in \text{Ass } A/J_0$. Say $m = (0:\overline{t})$ where $t \notin J_0$. So $pt \in J_0 = (au_1, av_1)$. It follows that $a \mid t$. Say $t = at_1$. Then check that $m t_1 \subseteq (u_1, v_1)$. As $u_1, v_1$ is an $A$-regular sequence we get $t_1 \in (u_1, v_1)$. This implies that $t = at_1 \in J_0$ which is a contradiction. So claim (ii) holds and as discussed before this proves our result.

(3) Let $x$ be a regular parameter. Then $A/(x)$ is regular local of dimension 3. Note $F/xF$ satisfies the hypothesis of (2). It follows that $\beta(F) = \beta(F/xF) \geq 3$. We prove that $\beta(F) = 3$ is not possible.

Suppose if possible there exists $F$ with $\beta(F) = 2$.

As all cohomology modules have finite length we have

$$\text{rank image } \partial^{i-1} + \text{rank ker } \partial^{i-1} = \text{rank ker } \partial^i + \text{rank image } \partial^i = 3.$$ 

By Lemma 7.12 we get that image $\partial^i \neq 0$ for all $i$. So without any loss of generality we may assume rank image $\partial^{-1} = 1$. As $H^0(F)$ has finite length we get rank ker $\partial^0 = 1$. Thus ker $\partial^0$ is isomorphic to an ideal $I$ in $A$. Note this isomorphism maps image $\partial^{-1}$ to an ideal $J \subseteq I$ of $A$. We note that $J$ is generated by $\leq 3$ elements. Say $J = (u_1, u_2, u_3)$. As $A$ is a UFD we may choose $a$ which is the greatest common divisor of $u_1, u_2, u_3$. Say $u_i = av_i$ for $i = 1, 2, 3$. Set $K = (v_1, v_2, v_3)$. Then $J = aK$ and either $K = A$ or $2 \leq \text{height } K \leq 3$ (the first inequality is easily seen by taking a primary decomposition of $K$).

Claim (iii) $K = A$.

Assume the claim for the time being. Then $J \cong A$ is free. So depth $J = 4$. As depth $I \geq 2$ we get that if $H^0(F) \neq 0$ then depth $H^0(F) \geq 1$; which is a contradiction. So $H^0(F) = 0$. So $I = J \cong A$. By Lemma 7.12 we get $H^1(F) \neq 0$. We have an exact sequence

$$0 \rightarrow I \rightarrow F^0 \rightarrow \text{image } \partial^0 \rightarrow 0.$$ 

Counting depths we have depth image $\partial^0 \geq 3$. Also depth ker $\partial^1 \geq 2$. So depth $H^1(F) \geq 2$ which is a contradiction. Thus it suffices to prove Claim (iii).

Proof of Claim (iii): Suppose if possible $K \neq A$. Let $P$ be a minimal prime of $K$. We have $2 \leq \text{height } P \leq 3$. Note $F_P$ is exact. As $A_P$ is regular local we get $J_P$ is free. In particular $J_P$ is principal. But $J_P = (a)_P K_P$ is definitely not principal as height $K_P \geq 2$. So claim (iii) is true and as shown earlier this implies our result. 

$\square$
Remark 7.13. We do not know an example of a 2-periodic complex $F$ over a regular local ring of dimension 4 such that $\beta(F) = 4$.

Finally we show:

Theorem 7.14. Let $(A, \mathfrak{m})$ be a regular local ring of dimension $d \geq 1$. Then there exists a 2-periodic minimal complex $F$ with finite length cohomology such that $\beta(F) = 2^d$.

Proof. Set $(B, \mathfrak{n}) = (A[1]/(Y^2), (\mathfrak{m}, Y))$ and $k = A/\mathfrak{m} = B/\mathfrak{n}$. Let $G$ be a minimal complete resolution of $\Omega^d_B(k)$. Then same argument as in proof of Theorem 7.3 shows that $F = G \otimes_B A$ is a minimal 2-periodic complex with finite length cohomology. Thus it suffices to prove $\beta(F) = \beta(G) = 2^d$. Equivalently it suffices to prove that $\beta_i(\Omega^d_B(k)) = 2^d$ for all $i$. As $\Omega^d_B(k)$ does not have a free module as a direct summand (see [12, 1.3]) it suffices to show $\mu(\Omega^d_B(k)) = \beta_d^B(k) = 2^d$.

Let $x = x_1, \ldots, x_d$ be a regular system of parameters of $A$. Then note that $x$ considered in $B$ is a regular sequence in $\mathfrak{n} \setminus \mathfrak{n}^2$. Set $C = B/\mathfrak{x}B$. By [33, 5.3] we get

$$\beta_d^C(k) = \sum_{i=0}^d \binom{d}{i} \beta_i^C(k).$$

Now notice $C = k[Y]/(Y^2)$. Then the minimal resolution of $k$ over $C$ is given as:

$$\cdots \to C \xrightarrow{Y} C \xrightarrow{Y} C \xrightarrow{Y} k \to 0.$$

So $\beta_i^C(k) = 1$ for all $i$. It follows that $\beta_d^B(k) = 2^d$. \qed

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