DEVELOPING THE COVARIANT BATALIN-VILKOVISKY

APPROACH TO STRING THEORY

HIROYUKI HATA * AND BARTON ZWIEBACH †

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

ABSTRACT

We investigate the variation of the string field action under changes of the string field vertices giving rise to different decompositions of the moduli spaces of Riemann surfaces. We establish that any such change in the string action arises from a field transformation canonical with respect to the Batalin-Vilkovisky (BV) antibracket, and find the explicit form of the generator of the infinitesimal transformations. Two theories using different decompositions of moduli space are shown to yield the same gauge fixed action upon use of different gauge fixing conditions. We also elaborate on recent work on the covariant BV formalism, and emphasize the necessity of a measure in the space of two dimensional field theories in order to extend a recent analysis of background independence to quantum string field theory.

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1. Introduction and Summary

In current formulations of string field theory the first step in the construction consists of choosing a conformal field theory. This conformal field theory defines a vector space, the state space of the theory, spanned by all the (normal ordered) local operators in the theory. A string field is simply an arbitrary vector in this state space, and a string field theory action is a function that given a string field gives us a number. This action is written as a power series in the string field and $\hbar$, and each term of the series is defined precisely without problems of divergences or regularization.

In addition to choosing a conformal field theory, in order to write a string field theory one must also find a way of breaking up the moduli spaces of Riemann surfaces. This breakup is necessary since the string amplitudes should be obtained by Feynman rules, and these rules break the amplitudes into a sum of diagrams. A consistent choice of string vertices and propagator gives rise to one particular way of decomposing moduli space. Since the physical observables are the same regardless of the chosen vertices, it has long been thought that the possibility of using different decompositions of moduli space must correspond to some type of string field symmetry.

The central objective in this paper is the study of this freedom in the choice of decomposition of moduli space in the context of the Batalin-Vilkovisky (BV) [1] approach to closed string field theory [2,3,4]. As one changes the string vertices, and as a consequence, the moduli space decomposition, the string field action changes in such a way that the new action still satisfies the master equation. Since the physics does not change, one suspects that the relation between two actions using different decompositions ought to arise from string field transformations (or redefinitions). In the BV approach the result is actually much stronger: the relation between the two actions arises from field transformations that are canonical with respect to the antibracket. We give the explicit form of the generator of these transformations.

While the change of decomposition is not a gauge invariance (the action changes), it is not so different from one; we show that two actions using different decompositions agree as off-shell gauge fixed actions if we use different gauge fixing conditions.

We also discuss the use of the covariant BV formalism in the problem of background independence of string field theory. The conformal field theory required to write the string field theory may be thought to be a special point in a hypothetical space $M$ of two-dimensional field theories. In a recent paper Witten [5] has discussed how to write a string field action $S(M)$ on a theory space $M$ (suitable for open string theory) by setting up a BV structure on $M$ and defining an anticommuting vector field $V_S$ satisfying $V_S^2 = 0$, that plays the role of Hamiltonian vector field for the function $S$. This action, by construction, satisfies the classical BV master equation [1]. If such an approach is to be eventually completed and extended to closed strings, one must construct an action that satisfies the full quantum BV equation. To investigate this point we make extensive use of the recent work of A. Schwarz [6,7] on the geometry of BV quantization. We argue that, in addition to the closed, non-degenerate symplectic two-form $\omega$ in the theory space $\mathcal{M}$, one also needs a suitable measure in theory space. With such measure one can define the divergence of vector fields on supersymplectic manifolds and the
delta operator of Batalin and Vilkovisky [6]. The $V^2_S = 0$ equation is generalized in such a way that the action $S$ satisfies the full master equation.

Let us now describe briefly the contents of this paper. In §2 we begin by discussing the general covariance of the BV formalism, and give a complete treatment of the differential geometry in the appropriate supersymplectic case. We show how most of the formalism can be developed without having to require the nilpotency of the delta operator $\Delta_\rho$ of Batalin and Vilkovisky. Our remarks here are mostly elaborations on the results of [6].

We then turn in §3 to the symmetries of the master equation. Our main objective is clarifying the physical significance of the transformations $\delta_\epsilon S = \Delta \epsilon + \{S, \epsilon\}$, which given some action $S$ satisfying the master equation, gives us another action $S + \delta_\epsilon S$ which also satisfies the master equation [8]. We emphasize that this is not a gauge symmetry and prove, within the covariant BV approach to gauge fixing, that the observables of the theory are not changed. As a formal development, we find an ‘actional’ $A(S)$ for the ‘field’ $S$ on the supermanifold whose equation of motion is the master equation for $S$ and whose gauge symmetry is the above transformation $\delta_\epsilon$. For the classical master action we note that the above transformation corresponds simply to a redefinition of the fields and antifields induced by a canonical transformation. For the quantum action the transformation $\delta_\epsilon$ is not a field redefinition; the action also changes due to effects having to do with the measure. Interestingly, the original action and the perturbed one define identical off-shell theories upon gauge fixing, if we use different gauge fixing conditions. Using the canonical origin of the transformations $\delta_\epsilon$ we are able to obtain the finite version of these transformations.

Section 4 deals with the main issue in this paper, for which §3 was a preparation. We show that the change in the quantum string field master action induced by an infinitesimal change in the decomposition of moduli space corresponds to a symmetry transformation $\delta_\epsilon$ for an appropriate parameter $\epsilon$. The parameter $\epsilon$ has a very simple form; it is essentially obtained by integrating a differential form over the region interpolating between the original and final string vertices. We find it noteworthy that changing the decomposition of moduli space has a natural description in the BV approach to string theory.

Finally, in §5 we show how to generalize the formal setup of Ref.[5] in order to incorporate the full quantum master equation into the analysis. We show that the equation $V^2_S = 0$ for the hamiltonian vector field $V_S$ arising from the action $S$ must be changed into the equation $V^2_S = -h V_{\text{div} V_S}$, and explain why it guarantees that $S$ satisfies the quantum master equation.

A Speculative Interpretation The simplest possible variation one can do to the decomposition of moduli space of closed string field theory corresponds to changing the length of the ‘stubs’ in the string vertices. Since the stub length plays the role of a cutoff in string field theory, it has been argued that a change of stub length corresponds to a renormalization group transformation of string field theory [9]. We are thus led to interpret the changes of the string field action due to general variations of the moduli space decompositions as generalized renormalization group flows. In fact, any variation $\delta_\epsilon S$ may be formally thought of as a renormalization group transformation; the action is changed, but the physics is not. The $\delta_\epsilon$ transformations generate a nonabelian Lie algebra, in fact, the Lie algebra of the antibracket. The $\delta_\epsilon$ transformations...
arising from changes of moduli space decompositions do not seem to form a subalgebra of the complete Lie algebra of $\delta_e$ transformations. This fact complicates a straightforward geometrical interpretation of the nonabelian structure. Further investigation may clarify whether or not there is a sensible notion of nonabelian renormalization group transformations in string field theory.

Some remarks We believe our results are further evidence of the surprising efficiency of the BV formalism in dealing with string theory. We have given technical tools that may find application in the study of background independence, where string field redefinitions are often necessary [10]. Our analysis may also help find gauge fixing conditions different from those of the Siegel gauge; such gauges seem necessary for cases when the closed string semirelative cohomology is nontrivial, as is the case in $c = 1$ strings [11]. We have clarified the sense in which the choice of moduli space decomposition corresponds to a symmetry and has a special role in the BV formalism. If one wished to promote these transformations to gauge invariances, the only option we see is that of extending the number of dynamical variables in the string field theory. Finally, our discussion has emphasized the need for a suitable density $\rho$ in the space of 2d field theories leading to a nilpotent $\Delta_{\rho}$.

There have been several recent works on Batalin-Vilkovisky theory. Some of our results in §2 have also been obtained by Lian and Zuckerman [12], Batalin and Tyutin [13], Getzler [14], and, Schwarz and Penkava [15].

M. Henneaux has brought to our attention that the $\delta_e$ transformations were familiar in the context of the ambiguities of the solution of the master equation [16,17]. The role of field transformations (canonical or more general) in the BRST formalism has also been studied in works of Alfaro and Damgaard [18]. Finally the connection between canonical transformations and change of gauge conditions was understood in the context of the ‘Zinn-Justin’ equation by Voronov and Tyutin [19].

2. The BV equation on a general coordinate system

In this section we will discuss the covariant formulation of the Batalin-Vilkovisky formalism, and summarize basic results needed for this paper.* We were motivated by the recent discussion of A. Schwarz [6], that provided, in our opinion, the main insight into the covariant formulation. His proposal is that the supersymplectic manifold of fields and antifields, carrying the symplectic form $\omega$, must be endowed with a volume element $\mu$, or equivalently, a density function $\rho$. This allows one to define the so-called delta operator $\Delta_{\rho}$ of Batalin and Vilkovisky as the second order differential operator that acting on a function gives the divergence of the vector field arising from that function. The volume element is necessary to define the divergence. One then imposes the condition $\Delta_{\rho}^2 = 0$ and shows that it leads to a sensible formalism.

* For an introduction to BV theory and a review of earlier developments see [17].
In this section we will show that the standard formulas in the formalism hold for arbitrary \( \rho \). That is, without imposing the \( \Delta_2^2 = 0 \) condition, we verify that the antibracket (which is \( \rho \)-independent) measures the failure of \( \Delta_2 \rho \) to be a derivation of pointwise multiplication, and \( \Delta_2 \rho \) satisfies the Leibniz rule on the antibracket. We also note that \( \Delta_2^2 \), which naively would be expected to be a fourth order differential operator, is actually a first order operator. We conclude that the nilpotency condition of \( \Delta_2 \rho \) is (at present) imposed only because it is clear that, with Darboux coordinates and \( \rho = 1 \), consistent quantization can be done, and the recent work of Schwarz shows that the nilpotency of \( \Delta_2 \rho \) insures that such a preferred Darboux system of coordinates can be found.\(^\dagger\)

Consider a \((n, n)\)-dimensional supermanifold \( M \) of fields and antifields. This supermanifold is endowed with an odd symplectic structure defined by an odd two-form \( \omega \) which is non-degenerate and closed, \( d\omega = 0 \). In a local coordinate system \((z^I) = (z^1, z^2, \ldots, z^{2n})\), \( \omega \) is given by:\(^\ddagger\)
\[
\omega = -dz^I \omega_{IJ}(z)dz^J = \omega_{JI}(z)dz^I \wedge dz^J. \tag{2.1}
\]
In familiar applications of the BV formalism one adopts Darboux coordinates where \( \omega \) takes the form \( \omega = -2 d\phi^i \wedge d\phi^*_i \) (\( \phi^i \) and \( \phi^*_i \) are fields and antifields, respectively). However, since we are interested in the covariant aspects of the BV formalism, we will consider general coordinate systems.

In correspondence to the Poisson bracket in bosonic symplectic manifolds, we have the antibracket defined by the inverse matrix \( \omega^{IJ} \) as
\[
\{A, B\} \equiv A \overset{\leftrightarrow}{\partial}_I \omega^{IJ} \overset{\leftrightarrow}{\partial}_J B = (-)^{(A+1)} \partial_I A \cdot \omega^{IJ} \cdot \partial_J B, \tag{2.2}
\]
where \( \overset{\leftrightarrow}{\partial}_I = \partial_I = \partial_l/\partial z^I \) and \( \overset{\leftrightarrow}{\partial}_I = \partial_r/\partial z^I \) denote the left- and right-derivatives, respectively. It follows from the above that the antibracket is also given by
\[
\{A, B\} = \omega(V_A, V_B) = V_B(A), \tag{2.3}
\]
where \( V_A \) is the hamiltonian vector field corresponding to a function \( A \):
\[
V_A = \overset{\leftrightarrow}{\partial}_I \omega^{IJ} \partial_J A. \tag{2.4}
\]
Just as in the case of ordinary symplectic manifolds, \( d\omega = 0 \) implies that the antibracket satisfies a (graded) Jacobi identity:
\[
(-)^{(A+1)(C+1)} \{\{A, B\}, C\} + \text{cyclic}(A, B, C) = 0, \tag{2.5}
\]
and the same equation with \( \{\{A, B\}, C\} \) replaced by \( \{A, \{B, C\}\} \).

\(^\dagger\) General (super)canonical transformations, defined to preserve the Darboux form, do not preserve the measure factor \( \rho \).

\(^\ddagger\) In Appendix A we summarize the rules of exterior calculus on a supermanifold, and in Appendix B we give some properties of the basic ingredients of the formalism.
Defining the Lie-bracket \([V, W]\) of two vector fields \(V = (\partial/\partial z^I)V^I\) and \(W = (\partial/\partial z^J)W^J\) by
\[
[V, W] \equiv VW - (-)^{VW} WV = \frac{\partial}{\partial z^I} \left( V^I \partial_J W^J - (-)^{VW} W^I \partial_J V^J \right),
\]
the condition \(d\omega = 0\) also implies that hamiltonian vector fields form a Lie subalgebra of the Lie algebra of vector fields
\[
[V_A, V_B] = V_{\{A,B\}}.
\]
This is the standard formula establishing the homomorphism from the Lie algebra of the antibracket to the Lie algebra of vector fields (it is not an isomorphism because all the constant functions are mapped to the zero vector).

We now note that since the supersymplectic form \(\omega\) is odd we have \(\omega \wedge \omega = 0\), and, in contrast to ordinary symplectic theory, we cannot raise \(\omega\) to some power in order to obtain a canonical volume form. In other words there is no a-priori “Liouville” volume element. A related fact is that in ordinary symplectic theory a hamiltonian vector field \(V_f\) (associated to a function \(f\)) preserves the symplectic form: \(\mathcal{L}_{V_f}\omega = 0\). As a consequence \(\mathcal{L}_{V_f}\mu = 0\) (\(\mu \sim \omega^n\)), and via the standard relation \(\mu \cdot (\text{div}_\mu V) = \mathcal{L}_\mu \mu\), we obtain the result that hamiltonian vector fields are divergenceless. In the supersymplectic case hamiltonian vector fields need not be divergenceless.

Following Ref.[6] we introduce a volume element by
\[
d\mu(z) = \rho(z)^{2n} \prod_{I=1} d z^I.
\]
where \(\rho(z)\) is a density. Then, defining the divergence of a vector field \(V = (\partial/\partial z^I)V^I\) by
\[
\text{div}_\rho V = \frac{1}{\rho} (-)^I \partial_I \left( \rho V^I \right),
\]
we can finally introduce the second order differential operator \(\Delta_\rho\) of the BV formalism as an operator that acts on functions to give functions:
\[
\Delta_\rho A \equiv \frac{1}{2} \text{div}_\rho V_A = \frac{1}{2\rho} (-)^I \partial_I \left( \rho \omega^{IJ} \partial_J A \right).
\]
Namely, \(\Delta_\rho\) acting on a function gives the divergence of the hamiltonian vector field associated to the function. Note that \(\Delta_\rho\) is odd, i.e., \(\varepsilon(\Delta_\rho) = 1\). It is easily seen that two \(\Delta\)’s

\[\text{§ Recall that in supermanifolds the volume element is not a differential form.}\]
corresponding to two different $\rho$'s are related by

$$\Delta_{\tilde{\rho}} A = \Delta_{\rho} A + \frac{1}{2} \{ \ln(\tilde{\rho}/\rho), A \} .$$  \hfill (2.11)

There is a very simple relation between the delta operator and the antibracket [8]; the antibracket measures the failure of delta to be a derivation of the associative algebra of functions (under pointwise multiplication). This actually holds for arbitrary $\rho$:

$$(-)^{A} \{ A, B \} = \Delta_{\rho}(A \cdot B) - \Delta_{\rho} A \cdot B - (-)^{A} A \cdot \Delta_{\rho} B ,$$  \hfill (2.12)

since the $\rho$-dependence on the right hand side cancels out. An important property of $\Delta_{\rho}$ is the Leibniz rule for the antibracket,

$$\Delta_{\rho} \{ A, B \} = \{ \Delta_{\rho} A, B \} + (-)^{A+1} \{ A, \Delta_{\rho} B \} .$$  \hfill (2.13)

Again, we emphasize that Eqn.(2.13) holds for any $\rho$. This is proven using Darboux coordinates as follows. One adopts Darboux coordinates and denotes the new density by $\tilde{\rho}$. One then verifies that in those Darboux coordinates $\Delta_{\rho=1}$ does satisfy the Leibniz rule (2.13). The Leibniz rule for $\Delta_{\tilde{\rho}}$ then follows from Eqn.(2.11) and the Jacobi identity (2.5). In fact, using (2.12) repeatedly we can show that

$$\Delta_{\rho} \{ A, B \} = \{ \Delta_{\rho} A, B \} + (-)^{A+1} \{ A, \Delta_{\rho} B \} + (-)^A \left[ \Delta_{\rho}^2(A \cdot B) - (\Delta_{\rho}^2 A) \cdot B - A \cdot (\Delta_{\rho}^2 B) \right] ,$$  \hfill (2.14)

and therefore the Leibniz rule for the antibracket established above requires that $\Delta_{\rho}^2$ should satisfy

$$\Delta_{\rho}^2(A \cdot B) = (\Delta_{\rho}^2 A) \cdot B + A \cdot (\Delta_{\rho}^2 B) .$$  \hfill (2.15)

This implies that $\Delta_{\rho}^2$, which is naively a fourth order differential operator, is in fact a first order differential operator for any $\rho$,

$$\left( \Delta_{\rho} \right)^2 = \frac{1}{2} \left[ \Delta_{\rho} \left( \frac{1}{\rho} (-)^I \partial_I \left( \rho \omega^{IJ} \right) \right) \right] \cdot \partial_J .$$  \hfill (2.16)

Equation (2.16) (or (2.15)) can also be proved directly by a tedious but straightforward calculation using $d\omega = 0$. Turning this around we see that the condition $\Delta_{\rho}^2 = 0$ would imply, from the vanishing of the third order differential part, the condition $d\omega = 0$. This has been found independently in Refs.[15,14]. The Jacobi identity (2.5), the Leibniz rule (2.13) and the nilpotency condition $\Delta_{\rho}^2 = 0$ (to be imposed below) are the fundamental properties of the antibracket $\{ \cdot, \cdot \}$ and $\Delta_{\rho}$. Note the formal resemblance of $\Delta$ and the antibracket $\{ \cdot, \cdot \}$ to the BRST operator $Q$ and the $\ast$-product of string fields in the cubic light-cone-style HIKKO classical closed string field theory [20]. The latter $(Q,\ast)$, also satisfy the conditions of nilpotency, Leibniz rule and Jacobi identity (the analog of pointwise multiplication has not been found necessary to write the string field theory.).
For the particular case of Darboux coordinates and $\rho = 1$, the antibracket and the delta operator, with $(-)^i \equiv (-)^{\varepsilon(\phi^i)}$, read

$$\{A, B\} = \frac{\partial_r A}{\partial \phi^i} \frac{\partial l B}{\partial \phi^*_i} - \frac{\partial_r A}{\partial \phi^*_i} \frac{\partial l B}{\partial \phi^i} ,$$

$$\Delta = (-)^i \frac{\partial l}{\partial \phi^i} \frac{\partial l}{\partial \phi^*_i} , \quad (2.17)$$

Having given all the necessary notation, the quantum BV equation for the master action $S(z)$ is given by

$$\Delta_\rho e^{S(z)/\hbar} = 0 , \quad (2.18)$$

or equivalently

$$\hbar \Delta_\rho S + \frac{1}{2} \{S, S\} = 0 . \quad (2.19)$$

Under general coordinate transformations in the supersymplectic manifold $\tilde{z}^I = \tilde{z}^I(z)$ (preserving Grassmanality: $\varepsilon(\tilde{z}^I) = \varepsilon(z^I) = 1$), the action $S$, the antibracket, and $\Delta$ transform as scalars, while $\tilde{\omega}^{IJ}$ and $\tilde{\rho}$ in the new coordinate system are given by

$$\tilde{\omega}^{IJ}(\tilde{z}) = \{\tilde{z}^I, \tilde{z}^J\} = \tilde{z}^I \frac{\partial \tilde{z}^J}{\partial z^L} \cdot \tilde{\omega}^{KL}(z) \cdot \frac{\partial z^I}{\partial \tilde{z}^J} , \quad (2.20)$$

$$\tilde{\rho}(\tilde{z}) = \rho(z) \cdot \exp \left\{ -s\text{Tr} \ln \left( \frac{\partial \tilde{z}^J}{\partial z^I} \right) \right\} = \rho(z) \cdot \text{sdet} \left( \frac{\partial \tilde{z}^J}{\partial z^I} \right) , \quad (2.21)$$

where $s\text{Tr}(M^I_J) \equiv (-)^I M^I_J$.

In BV quantization the density function $\rho$ defining the operator $\Delta_\rho$ is not arbitrary. One imposes the condition of nilpotency:

$$(\Delta_\rho)^2 = 0 . \quad (2.22)$$

(Recall that $\Delta_\rho^2$ is in general a first order differential operator.) This condition is necessary and sufficient to prove that there exists a Darboux frame with $\rho = 1$ [6]. The existence of such a frame is sufficient to guarantee, by the standard argument of Ref.[1], consistent path-integral quantization using the master action $S$.

Given a solution $S(z)$ of the BV equation (2.19), the quantization is given as follows [6]. First, one chooses a lagrangian submanifold $L$, i.e., a $(k, n - k)$-dimensional submanifold of $\mathcal{M}$, such that $\omega(v, \tilde{v}) = 0$ for any pair, $v$ and $\tilde{v}$, of vectors tangent to $L$ at $z$ ($v, \tilde{v} \in T_z L$). The choice of $L$ corresponds to the choice of gauge fixing. The observables are defined by integrals over the lagrangian submanifold. This requires a volume element $d\lambda$ on $L$, that can be obtained canonically using the volume element $d\mu$ on the whole supermanifold, and the two
form $\omega$. One defines

$$d\lambda(e_1, \ldots, e_n) = d\mu(e_1, \ldots, e_n, f^1, \ldots, f^n)^{1/2},$$

(2.23)

where $d\mu$ is the volume element in $\mathcal{M}$, Eqn.(2.8), and $(e_1, \ldots, e_n, f^1, \ldots, f^n)$ is a basis of the tangent space $T_z\mathcal{M}$ such that $(e_1, \ldots, e_n)$ is a basis of $T_zL$ and the condition $\omega(e_i, f^j) = \delta_i^j$ is satisfied. Using the lagrangian submanifold $L$ and the associated volume element $d\lambda$, the quantum theory is defined by the path-integral

$$\int_L d\lambda e^{S/\hbar}.$$  

(2.24)

It can be shown that (2.24) is invariant under deformations of $L$. A convenient way to do gauge fixing requires finding a set of $n$ linearly independent constraints $G_i = 0$ in involution

$$\{G_i, G_j\} = U^k_{ij} G_k,$$

(2.25)

with $U$'s a set of possibly field/antifield dependent structure constants. Condition (2.25) implies that the submanifold defined by $G_i = 0, \forall i$, will be a lagrangian submanifold. This is seen as follows. Let $V_i$ be the hamiltonian vector associated to $G_i$. The vector $V_i$ is tangent to the submanifold since $V_i(G_j) = \{G_j, G_i\} = 0$ on the submanifold. Thus the vectors $V_i$ form a basis for the tangent space to the submanifold. Then, for any two tangent vectors $V$ and $V'$, we have $\omega(V, V') = \omega(a^iV_i, b^jV_j) \sim a^ib^j \omega(V_i, V_j) = a^ib^j \{G_i, G_j\} = 0$, on the submanifold.

In the conventional situation the above constraints are used to determine the antifields as functions of the fields. This requires that

$$G_i = \Lambda_i^j(\phi, \phi^*) (\phi^*_j - f_j(\phi)),$$

(2.26)

with $\Lambda$ an invertible matrix [13]. Eqn.(2.25) requires that on the constraint surface $\{G_i, G_j\} = 0$, and this implies

$$\partial_i f_j - \partial_j f_i = 0 \quad \Rightarrow \quad f_i = \frac{\partial \Upsilon}{\partial \phi^i},$$

(2.27)

where $\Upsilon$ is the so-called gauge fermion ($\varepsilon(\Upsilon) = +1$). The gauge fixing conditions therefore read

$$\phi^*_i = \frac{\partial \Upsilon}{\partial \phi^i}.$$  

(2.28)

This concludes our presentation of the covariant form of the Batalin-Vilkovisky formalism.
3. Symmetries of the Master Equation

In this section we will discuss a symmetry of the master equation. Its origin is the well-known invariance of the classical master equation $\{S, S\} = 0$ under the transformations $\delta S = \{S, \epsilon\}$. This symmetry can be extended to the quantum master equation as noticed in [8] (it corresponds to an ambiguity in solving the quantum BV master equation [17]). In this section, we will show, using the covariant description of gauge fixing, that the symmetry transformation, while changing the master action, preserves all the observables of the theory. We will also give an action for the master equation, invariant under the symmetry transformation. By using the canonical origin of the transformation we establish that two actions related by this transformation lead to the same gauge fixed theory upon use of different gauge fixing conditions. Finally we present the finite version of the transformations.

The above comments can be made somewhat more precisely. Given a supermanifold $\mathcal{M}$ with structure $(\omega, \rho)$, the action $S$ is a function on $\mathcal{M}$. If $S$ satisfies the master equation, the transformation gives us a new $S$ that also satisfies the equation, without changing the structure $(\omega, \rho)$ on $\mathcal{M}$. An off-shell gauge fixed theory is defined by the pair $(S, L)$, where $L$ is the gauge fixing surface. If the transformation takes $S \rightarrow S'$, we claim there is an $L'$ such that $(S', L')$ is an equivalent off-shell theory.

3.1. Symmetry Transformations and Observables

The master equation $\Delta e^S = 0$ (in this section we omit the $\hbar$ dependence and the subscript $\rho$ in $\Delta$) can be written as

$$ M(S) = e^{-S} \Delta e^S = \Delta S + \frac{1}{2} \{S, S\} = 0, \quad (3.1) $$

and a solution $S$ of this equation can be used to define a quantum field theory. In particular, it has been observed [8] that, given a solution $S$, one can generate other solutions using the following infinitesimal transformation

$$ \delta_\epsilon S = \Delta \epsilon + \{S, \epsilon\}, \quad (3.2) $$

with $\epsilon$ an odd parameter. The second term on the right hand side is well-known; it generates a canonical transformation of the master action and preserves the classical master equation. The first term is a quantum correction. Indeed, making use of $\Delta^2 = 0$ along with the Leibniz rule of $\Delta$ acting on the antibracket, one verifies that

$$ \delta_\epsilon M(S) = \{M(S), \epsilon\}, \quad (3.3) $$

which implies that if $S$ satisfies the master equation, $S + \delta_\epsilon S$ will too. Computing the commutator of two such transformation we find that

$$ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\{\epsilon_1, \epsilon_2\}}. \quad (3.4) $$

Thus the algebra of these transformations is simply the Lie algebra of the antibracket.
It is obvious that this transformation does not correspond to a gauge transformation of the field theory in question; gauge transformations should leave the action invariant, and the above manifestly does not. The transformations change the action but one expects that the new action defines a physically equivalent theory. Given the results of Ref. [6] it is possible to prove this explicitly by showing that the observables are not changed. The simplest observable is the partition function. Under this transformation we have that

\[ Z = \int_L d\lambda e^S \rightarrow \int_L d\lambda e^{S+\Delta e + \{S, \epsilon\}} \]

\[ = \int_L d\lambda e^S + \int_L d\lambda (\Delta e + \{S, \epsilon\}) + O(\epsilon^2). \]  \hfill (3.5)

The change in the partition function is therefore given by

\[ \delta_\epsilon Z = \int_L d\lambda e^S (\Delta e + \{S, \epsilon\}) = \int_L d\lambda \Delta (e^S) = 0, \]  \hfill (3.6)

where use was made of Eqn. (2.12) together with \( \Delta e^S = 0 \), and the final equality follows from \( \int_L d\lambda \Delta F = 0 \), which holds for any \( F \), as proven in [6] for the case of compact lagrangian submanifolds \( L^\star \). This shows that the partition function has not changed. Let us now consider the case of other observables. Given an operator \( A \), one has an observable if the expectation value (in the theory defined by \( S \))

\[ \langle A \rangle_S \equiv \int_L d\lambda A e^S, \]  \hfill (3.7)

is independent of gauge fixing (deformations of the lagrangian submanifold \( L \)). This holds if

\[ \Delta (A e^S) = 0. \]  \hfill (3.8)

or equivalently (using \( \Delta e^S = 0 \)),

\[ \delta_A S = \Delta A + \{S, A\} = 0. \]  \hfill (3.9)

If we change the action we must also change the operator \( A \) so that it remains an observable.

\* This was proven for finite dimensional supermanifolds. We assume it also holds for the infinite dimensional case.
This requires that
\[ \Delta ((A + \delta \epsilon A)e^{S+\delta \epsilon S}) = 0, \]
and this condition implies that the variation of \( A \) is given by
\[ \delta \epsilon A = \{ A, \epsilon \}. \] (3.11)

Then a simple computation gives
\[ \langle A + \delta \epsilon A \rangle_{S+\delta \epsilon S} = \langle A \rangle_S + \int L d\lambda \Delta (\epsilon Ae^S) = \langle A \rangle_S, \]
(3.12)
as desired. This proves that the observables are the same in the two theories, and therefore that the two theories are physically equivalent.

3.2. An Action for the Master Equation

We have seen that the transformations (3.2) are not gauge transformations. Nevertheless, since they leave the master equation invariant, one can expect the transformations to correspond to gauge invariances of an actional \( A(S) \) whose equation of motion is the master equation \( M(S) = 0 \). Such an actional exists and it is given by
\[ A(S) = -\frac{1}{2} \int d\mu \{ e^S, e^S \} = \int d\mu e^S \Delta e^S, \] (3.13)
where \( d\mu \) is the volume element on the supermanifold \( \mathcal{M} \) given by Eqn.(2.8), and the second expression follows from
\[ \int \mathcal{M} d\mu A \Delta B = -\frac{1}{2} \int \mathcal{M} d\mu \{ A, B \} = (-)^{A} \int \mathcal{M} d\mu (\Delta A)B, \] (3.14)
which in turn, results from integration by parts (dropping total derivatives). It is clear that the equation of motion for \( S \) resulting from the actional \( A(S) \) is indeed the master equation \( \Delta e^S = 0 \). Gauge invariance of \( A(S) \) is shown as follows:
\[ \delta \epsilon A(S) = -\int \mathcal{M} d\mu \{ e^S, e^S \Delta \epsilon + \{ e^S, \epsilon \} \} \]
\[ = -\int \mathcal{M} d\mu \left( \frac{1}{2} \{ e^{2S}, \Delta \epsilon \} + \{ e^S, e^S \} \Delta \epsilon + \frac{1}{2} \{ \{ e^S, e^S \}, \epsilon \} \right) \]
\[ = \int \mathcal{M} d\mu e^{2S} \Delta^2 \epsilon = 0, \] (3.15)
where the second equality is a consequence of Eqn.(B.3) and \( \{ e^S, \{ e^S, \epsilon \} \} = \frac{1}{2} \{ \{ e^S, e^S \}, \epsilon \} \) (which follows from the Jacobi identity (2.5)), and the third equality requires the use of Eqn.(3.14).
When the supermanifold $\mathcal{M}$ is a fixed space of fields and antifields, the dynamical variable $S$ represents a possible action, and $\mathcal{A}(S)$ is a function on the space of actions, having as critical points the actions that define quantum field theories (with the given field content).

While this actional $\mathcal{A}(S)$ has the expected properties, its possible significance is not clear to us. The most naive (and radical!) interpretation of $\mathcal{A}(S)$ would be as an action for a quantum theory with the dynamical variable $S$ (for an $(n, n)$ supermanifold $\mathcal{M}$, $\mathcal{A}(S)$ is even if $n$ is odd). This “theory of theories” would be described by a path-integral:

$$\int D S \exp \left( \frac{1}{\lambda} \mathcal{A}(S) \right),$$  

(3.16)

where $\lambda$ is the coupling constant. The path-integral (3.16) requires gauge fixing of the gauge symmetry $\delta_\epsilon$, and we would have to consider the BV equation for the quantum action having $\mathcal{A}(S)$ as its classical part. It may be of help to notice that, similarly to the case of string field theories [20,4], $\mathcal{A}(S)$ also has an invariance under the pre-BRST transformation $\delta_B S = M(S) = \Delta S + \frac{1}{2}\{S, S\}$, which is nilpotent, $(\delta_B)^2 = 0$.

### 3.3. Interpretation of $\delta_\epsilon$ as a Change of Gauge Fixing

The transformation $\delta_\epsilon S$ of the master action (Eqn.(3.2)) and $\delta_\epsilon A$ of the observables (Eqn.(3.11)) are essentially infinitesimal canonical transformations generated by $\epsilon$. The purpose of the present subsection is to find the field theoretic interpretation of such transformations. Since the master action does not have a direct physical interpretation, our discussion will be done in the context of the gauge fixed theory. Since these transformations do not change the observables of the theory, we anticipate that two theories differing by such a transformation could be related by a field redefinition, and more interestingly, by a change of gauge. We will indeed show that this is the case.

Consider the supermanifold $\mathcal{M}$ of fields and antifields, and a (Grassmann-odd) function $\alpha$ with $V_\alpha$ the associated hamiltonian vector field. Let $g_t$ be the diffeomorphism generated by following the integral curves of the vector field $V_\alpha$ for a parameter distance $t$. We then have that for small $t$ the diffeomorphism takes $z^I$ to $z^I + tV_\alpha^I + \mathcal{O}(t^2)$, or in other words

$$g_t : z^I \to z^I + t\{z^I, \alpha\} + \mathcal{O}(t^2),$$  

(3.17)

where use was made of Eqn.(2.3). For any scalar $F$ we then have that

$$g_t^* F(z^I) = F(g_t(z^I)) = F(z^I + t\{z^I, \alpha\} + \mathcal{O}(t^2)) = F + t\{F, \alpha\} + \mathcal{O}(t^2).$$  

(3.18)

For the case of the action $S$ we would have that

$$g_t^* S = S + \{S, t\alpha\} = S + \delta_{t\alpha}^{\text{class}} S,$$

(3.19)

which indicates that the classical part of the transformation $\delta_{t\alpha} S$ (the part neglecting $\Delta$)

---

* Although the classical theory of $\mathcal{A}(S)$ is a free field theory if we take $e^S$, instead of $S$, as the fundamental variable, the range of path-integration over the new variable $e^S$ in the quantum theory (3.16) is then non-trivial. We thank E. Witten for his comments on this point.
corresponds simply to a field redefinition implementing (3.17). Since the classical part of the transformation defines the transformation of the classical master action (the \( \hbar = 0 \) limit of the master action), we conclude that \textit{the transformation of the classical master action corresponds to a canonical field redefinition}. The complete transformation \( \delta_{t\alpha} \) of the master action does not correspond to a canonical redefinition (even though it arises as a consequence of one) because it includes extra terms coming from the measure. To clarify this we must consider gauge fixing, or, in other words, choosing a lagrangian submanifold \( L \).

The diffeomorphism pushes the lagrangian submanifold \( L \) to a new submanifold \( L_t \) which is also lagrangian
\[
g_t : L \rightarrow L_t. \tag{3.20}
\]
This is not hard to see. First, the diffeomorphism, being generated by a hamiltonian vector, is a canonical transformation and therefore preserves the symplectic form
\[
\mathcal{L}_{v_t} \omega = 0 \quad \Rightarrow \quad g_t^* \omega = \omega. \tag{3.21}
\]
Moreover, any vector \( v_t^{(i)} \) tangent to \( L_t \) must be the pushforward of some vector \( v^{(i)} \) tangent to \( L \), that is \( v_t^{(i)} = g_{t*} v^{(i)} \). Therefore
\[
\omega(v_t^{(1)}, v_t^{(2)}) = \omega(g_{t*} v^{(1)}, g_{t*} v^{(2)}) = g_t^* \omega(v^{(1)}, v^{(2)}) = \omega(v^{(1)}, v^{(2)}) = 0, \tag{3.22}
\]
showing that indeed \( L_t \) is lagrangian.

The first part of Eqn.(3.19) implies that
\[
(S + \{S, t\alpha\}) \big|_{p \in L} = S \big|_{g_t(p) \in L_t}, \tag{3.23}
\]
that is, the perturbed action on the left hand side, evaluated at a point \( p \) in the lagrangian submanifold, equals the original action evaluated at the point \( g_t(p) \) in the new lagrangian submanifold. Thus, at the classical level \( (\hbar = 0) \), \( S \) and \( S + \delta_{t\alpha} S \), give the same gauge fixed theory when one uses different gauge fixing conditions.

For the full quantum theory we claim that for \textit{any} operator \( A \) (whether or not it is an observable) the following result holds:
\[
\int_L d\lambda (A + \{A, t\alpha\}) e^{S + \delta_{t\alpha} S} = \int_{L_t} d\lambda A e^{S} + \mathcal{O}(t^2). \tag{3.24}
\]
If \( A \) is an observable, the above equation does not give a new result. Indeed (3.12), plus the independence of observables on the gauge fixing surface imply (3.24) for the case of observables.

The significance of (3.24) is that off-shell quantities evaluated with the modified action \( S + \delta_{t\alpha} S \) are reproduced with the original action by simply using a \textit{different gauge fixing surface}. The two gauge fixing surfaces are related by the action of the diffeomorphism arising
from the function \( t\alpha \). Note, however, that as the action is modified, the operator whose expectation value is being computed must also be modified. If we think of the supermanifold \( \mathcal{M} \) with its structure \( (\omega, d\mu) \) fixed, an off-shell theory is defined by the pair \((S, L)\), that is, an action and a lagrangian submanifold. The above equation shows that there is a map between the operators of the off-shell theories \((S, L_t)\) and \((S + \delta_t \alpha, S, L)\) such that the expectation values are identical.

In order to establish (3.24) we first derive a formula relating the measures of integration over the lagrangian submanifolds \( L \) and \( L_t \). We claim that

\[
g_t^* d\lambda = (1 + t\Delta \rho \alpha) d\lambda + \mathcal{O}(t^2).
\] (3.25)

Consider the volume element \( d\mu \). It follows from the definition of Lie derivatives that

\[
g_t^* d\mu = d\mu + t\mathcal{L}_{V_\alpha} d\mu + \mathcal{O}(t^2)
= d\mu (1 + t \text{div}_\rho V_\alpha) + \mathcal{O}(t^2)
= d\mu (1 + 2t \Delta \rho \alpha) + \mathcal{O}(t^2),
\] (3.26)

where use was made of the relation between the Lie derivative of the volume element and the divergence operator, and of Eqn.(2.10). Consider now a set of basis vectors \((e_1, \ldots, e_n, f^1, \ldots, f^n)\) for the tangent space \( T_{z}L\mathcal{M} \) such that \((e_1, \ldots, e_n)\) is a basis of \( T_{z}L \) and the condition \( \omega(e_i, f^j) = \delta^j_i \) is satisfied. We then have that \( \omega(g_t* e_i; g_t* f^j) = \delta^j_i \) (using Eqn.(3.21)). Therefore

\[
g_t^* d\lambda(e_1, \cdots, e_n) = d\lambda(g_t* e_1, \cdots, g_t* e_n)
= \left( d\mu(g_t* e_1, \cdots, g_t* e_n; g_t* f^1, \cdots, g_t* f^n) \right)^{1/2}
= \left( g_t^* d\mu(e_1, \cdots, e_n; f^1, \cdots, f^n) \right)^{1/2}
= \left( (1 + 2t\Delta \rho \alpha) \cdot d\mu(e_1, \cdots, e_n; f^1, \cdots, f^n) \right)^{1/2} + \mathcal{O}(t^2)
= (1 + t\Delta \rho \alpha) \cdot d\lambda(e_1, \cdots, e_n) + \mathcal{O}(t^2),
\] (3.27)

where use was made of Eqn.(2.23). This establishes the validity of Eqn.(3.25).

We can now establish easily the desired relation (Eqn.(3.24)). We have that

\[
\int_{L_t} d\lambda A e^S = \int_{L} g_t^* (d\lambda A e^S) = \int_{L} g_t^* (d\lambda) g_t^*(A) g_t^*(e^S).
\] (3.28)

Making use of (3.18) and (3.25) we have that

\[
\int_{L_t} d\lambda A e^S = \int_{L} d\lambda(1 + t\Delta \rho \alpha)(A + \{A, t\alpha\}) e^{S+t\{S,\alpha\}},
\] (3.29)

and Eqn.(3.24) now follows simply by exponentiation of the measure factor. This concludes our derivation.
When we use Darboux coordinates \((\phi^i, \phi^*_i)\) and the lagrangian submanifold \(L\) specified by the gauge fermion \(\Upsilon(\phi)\) (as in Eqn.(2.28)), the deformed lagrangian submanifold \(L_t\) is defined by the equation

\[
\phi^*_i + t \frac{\partial l}{\partial \phi^i}(\phi, \phi^*) = \left( \frac{\partial \Upsilon(\phi)}{\partial \phi^i} \right) (\phi - t \frac{\partial l}{\partial \phi^*}) - t \frac{\partial l}{\partial \phi^*} \frac{\partial^2 \Upsilon(\phi)}{\partial \phi^i \partial \phi^*},
\]

which is obtained by using (3.17) to relate the new variables to the old ones. The last right hand side is obtained by expanding the first right hand side. Solving Eqn.(3.30) for \(\phi^*_i\) to first order in \(t\), we see that \(L_t\) is given by

\[
\phi^*_i = \frac{\partial \Upsilon_t(\phi)}{\partial \phi^i} \quad \text{with the new gauge fermion } \Upsilon_t(\phi) \text{ given as }
\]

\[
\Upsilon_t(\phi) = \Upsilon(\phi) - t \alpha(\phi, \phi^*) = \frac{\partial l}{\partial \phi^*},
\]

This result can also be derived from the variation of the constraints fixing the lagrangian submanifold (see Eqn.(2.25)) under the canonical transformation. Indeed, the new lagrangian submanifold is defined by

\[
G'_i = G_i + t \{G_i, \alpha\} = 0.
\]

3.4. Finite Symmetry Transformations

The analysis of the previous subsection showed that the origin of the \(\delta_1\) transformation was an infinitesimal canonical transformation of field variables. This suggests that the finite version of these transformations must arise from finite canonical transformations.

Let \(g : M \to M\) be a diffeomorphism that is canonical, that is, it satisfies \(g^* \omega = \omega\). Explicitly, we write \(z^I \to g^I(z)\). If \(g\) takes the lagrangian submanifold \(L\) into the lagrangian submanifold \(L^g\), we then have

\[
\int_{L^g} d\lambda e^S = \int_{L} g^*(d\lambda) g^S = \int_{L} g^*(d\lambda) e^{S(g(z))}.
\]

Following the logic of the previous subsection, if we can express \(g^*(d\lambda)\) as a factor multiplying \(d\lambda\), then we can exponentiate this factor and obtain the transformation law of the action. We must first consider \(d\mu = \rho(z) \prod dz^I\), for which we have

\[
g^* d\mu = \rho(g(z)) \text{sdet} \left( \frac{\partial g^I}{\partial z^J} \right) \prod dz^I
\]

\[
= \frac{\rho(g(z))}{\rho(z)} \text{sdet} \left( \frac{\partial g^I}{\partial z^*} \right) d\mu,
\]

and therefore

\[
g^* d\mu = (F_g(z))^2 d\mu, \quad \text{with } (F_g(z))^2 \equiv \frac{\rho(g(z))}{\rho(z)} \text{sdet} \left( \frac{\partial g^I}{\partial z^*} \right).
\]

Since the transformation induced by \(g\) is canonical, the analysis in the first three lines of
Eqn.(3.27) applies, and we obtain
\[ g^* d\lambda = F_g(z) d\lambda, \tag{3.35} \]
and therefore back in Eqn.(3.32) we get
\[ \int_{L^g} d\lambda e^{S} = \int_{L} d\lambda e^{S(g(z)) + F_g(z)}. \tag{3.36} \]
If \( S \) satisfies the master equation, the integral on the left hand side is independent of the choice of the lagrangian submanifold, and as a consequence the integral on the right hand side is also independent of this choice. Therefore, the object in the exponential must satisfy the master equation. Thus, the finite symmetry transformation generalizing \( \delta_\epsilon \) is given by
\[ S(z) \to S^g(z) \equiv S(g(z)) + \ln F_g(z). \tag{3.37} \]
For an infinitesimal canonical transformation \( g^I(z) = z^I + \{z^I, \epsilon\} \), the transformation (3.37) can be checked to reduce to \( \delta_\epsilon S \) to \( \mathcal{O}(\epsilon) \). In Appendix C, we show explicitly that the actional \( \mathcal{A}(S) \) of Eqn.(3.13) is invariant under these transformations: \( \mathcal{A}(S^g) = \mathcal{A}(S) \). Finally, the master equation \( M(S) = 0 \) transforms nicely; one finds \( M(S^g) = g^* M(S) \). This is the generalization of Eqn.(3.3) for finite transformations.

4. Changing the Decomposition of Moduli Space

It is not difficult to obtain examples of string field theories that make use of different decompositions of moduli space. One particularly simple example is string field theories built with stubs of different lengths (the definition of stubs is reviewed in \( \S \)4.6). For two different values \( l \) and \( l' \) of this length, the subspaces \( V_{g,n}(l) \) and \( V_{g,n}(l') \) of \( \hat{P}_{g,n} \) defining the string field vertices are different. The resulting theories, however, have the same physical content. More generally, we expect that any two string field theories based on different decompositions should be physically equivalent. Stub length is just one of an infinite number of parameters that parameterize a generic deformation of the subspaces \( V_{g,n} \).

In this section we will prove that any deformation of the \( V_{g,n} \)'s used to build a string field action \( S \) induces a change of the form studied in \( \S \)3, namely, \( \delta_\epsilon S = \Delta \epsilon + \{S, \epsilon\} \) for some parameter \( \epsilon \) that we will find. Since this change in the action is infinitesimal, we will consider the infinitesimal problem. More precisely, assume we have a one parameter family of subspaces \( V_{g,n}(u) \) with \( u \in [0, 1] \) giving us a family of decompositions of moduli space, and as a consequence a one parameter family of string field actions \( S(u) \). We will assume that the subspaces \( V(u) \) satisfy the symmetry conditions demanding that the assignment of local coordinates around the punctures be independent of the labels of the punctures. Consistency demands that they must satisfy the geometrical consistency conditions \([21,4]\):
\[ \partial V_{g,n}(u) = -\partial_{\rho} R_1(V(u)), \tag{4.1} \]
where the left hand side denotes the boundary of the subspace \( V_{g,n}(u) \), and the right hand side denotes the propagator boundary (sewing with sewing parameter \( t \) satisfying \( |t| = 1 \)) of the
subspace $R_1$ consisting of surfaces build out of lower dimensionality subsets $\mathcal{V}$’s and a single sewing operation.

This section is organized as follows. In §4.1 we review and elaborate on some necessary tools from the operator formalism. We then introduce in §4.2 two vector fields $\hat{V}$ and $\hat{U}$ that arise naturally in studying the deformation of the subspaces $\mathcal{V}_{g,n}$. In §4.3 we calculate the change in $S$ induced by the deformation of the subspaces, and in §4.4 we give the expression for the parameter $\epsilon$ and begin the verification that $\delta_\epsilon S$ reproduces the change of the action. We explain how subspaces $\mathcal{V}_{g,n}$ that are not sections in $\mathcal{P}_{g,n}$ can yield a consistent string field theory, and discuss the algebra of $\delta_\epsilon$ transformations (§4.5). We then study the deformation problem for the particular case of stubs (§4.6). Finally, in §4.7 we complete our proof.

4.1. Some Facts from the Operator Formalism

We now review and elaborate somewhat on some of the tools of the operator formalism relevant for string field theory (for complete definitions see [4]). The results that will be given here are necessary for our discussion in later sections. The basic objects are differential forms in the tangent space $T_{\Sigma} \mathcal{P}_{g,n}$ based at the surface $\Sigma$. If we let $d_{g,n}$ denote the real dimension of $\mathcal{M}_{g,n}$, then $\Omega^{(k)g,n}$ denotes a $(d_{g,n} + k)$-form (for any $k \geq -d_{g,n}$) and is labeled by $n$ arbitrary off-shell string fields

$$
\Omega^{(k)g,n}_{\Psi_1 \cdots \Psi_n} (\hat{V}_1, \cdots, \hat{V}_{d_{g,n}+k}) = N_{g,n} \langle \Sigma | b(v_1) \cdots b(v_{d_{g,n}+k}) | \Psi_1 \cdots | \Psi_n \rangle ,
$$

with $N_{g,n} = (2\pi i)^{-3(g-3+n)}$ the normalization factor. The Schiffer vector $v_i = (v_i(1), \cdots, v_i(n))$ creates the deformation specified by the tangent $\hat{V}_i$, and the antighost insertions are defined by

$$
b(v) = \sum_{i=1}^{n} \left( \oint b^{(i)}(z_i) v^{(i)}(z_i) \frac{dz_i}{2\pi i} + \oint b^{(i)}(\bar{z}_i) \bar{v}^{(i)}(\bar{z}_i) \frac{d\bar{z}_i}{2\pi i} \right).$$

Similarly, given a Schiffer vector one defines the following insertion of the stress tensor

$$
T(v) = \sum_{i=1}^{n} \left( \oint T^{(i)}(z_i) v^{(i)}(z_i) \frac{dz_i}{2\pi i} + \oint T^{(i)}(\bar{z}_i) \bar{v}^{(i)}(\bar{z}_i) \frac{d\bar{z}_i}{2\pi i} \right).$$

The above forms satisfy the basic identity (Ref.[4], Eqn.(7.49))

$$
\Omega^{(k+1)g,n}_{\sum Q \Psi_1 \cdots \Psi_n} = (-)^{k+1} d\Omega^{(k)g,n}_{\Psi_1 \cdots \Psi_n},
$$

which says that the BRST operator $Q$ acts as an exterior derivative on the extended moduli space $\mathcal{P}$. Moreover, from the conventional definition of the contraction operator $i_{\hat{\Omega}}$

$$(i_{\hat{\Omega}} \Omega) (\hat{V}_1, \cdots, \hat{V}_k) \equiv \Omega(\hat{U}, \hat{V}_1, \cdots, \hat{V}_k),$$

applicable to any form $\Omega$ (with $\hat{V}_i$ arbitrary vector fields) we readily find that for our special
forms in Eqn.(4.2), one has
\[ i_{\hat{U}} \Omega^{(k+1)g,n} = (-)^k \Omega^{(k)g,n}_b(u) \Psi_1 \cdots \Psi_n, \tag{4.7} \]
where \( u \) is the Schiffer vector associated to the tangent \( \hat{U} \). The last identity we need involves the Lie derivative
\[ \mathcal{L}_{\hat{U}} \Omega^{(k)g,n} = \left( - \hat{U} \Omega^{(k+1)g,n} \right) = (-)^{k+1} \left( i_{\hat{U}} \Omega^{(k+1)g,n} + d\Omega^{(k-1)g,n} \right), \tag{4.8} \]
where use was made of Eqns.(4.5) and (4.7). Finally, since the anticommutator of the BRST operator and the antighost field is the stress tensor, we find
\[ \mathcal{L}_{\hat{U}} \Omega^{(k)g,n} = - \Omega^{(k)g,n}_b(u) \Psi_1 \cdots \Psi_n. \tag{4.9} \]
One final comment concerns notation. When the off-shell states that label the forms are all the same we will define
\[ \Omega_{\Psi_1 \cdots \Psi_n} \equiv \Omega_{\Psi_n}. \tag{4.10} \]

4.2. The Generating Vectors \( \hat{V} \) and \( \hat{U} \)

In this subsection we want to describe some of the geometrical structure available; in particular, we will introduce two vector fields, called \( \hat{V} \) and \( \hat{U} \), that are relevant to the deformation of the subspaces \( \mathcal{V}_{g,n} \).

The situation we have in mind is illustrated in Fig.1, where the family of sections \( \Gamma_{g,n} \) is shown. The base space is moduli space \( \mathcal{M}_{g,n} \) and the total space is \( \hat{\mathcal{P}}_{g,n} \). The subsets \( \mathcal{V}_{g,n} \) are indicated schematically, in particular \( \mathcal{V}_{g,n}(u_0) \) and \( \mathcal{V}_{g,n}(u_0 + du) \). The fibers, representing surfaces which have the same conformal structure but different coordinate systems at the punctures, are also parametrized by \( u \). For any point \( p \in \mathcal{M}_{g,n} \), the fiber over \( p \) is a curve denoted as \( f_p(u) \), where \( f_p : [0, 1] \to \hat{\mathcal{P}}_{g,n} \), and \( f_p(u) \in \Gamma_{g,n}(u) \). We define the vertical vector field \( \hat{V} \) to be the tangent vector to the fibers; more precisely \( \hat{V} = f_p^*(\partial/\partial u) \). The vector field \( \hat{V} \) is the vector which generates the diffeomorphisms moving the sections; we let \( f_t^{\hat{V}} \) denote the diffeomorphism which moves any point in \( \hat{\mathcal{P}} \) a parameter distance \( t \) along the fiber. Thus \( f_t^{\hat{V}} (\Gamma_{g,n}(u)) = \Gamma_{g,n}(u + t) \). (We will omit the subscripts \( \{g, n\} \) when there is no room for confusion.)
Consider now the neighboring sections $\Gamma(u_0)$ and $\Gamma(u_0 + du)$, and the relevant subspaces $\mathcal{V}(u_0)$ and $\mathcal{V}(u_0 + du)$. Define now the (oriented) subspace of $\Gamma(u_0 + du)$

$$\delta_{u_0} \mathcal{V}_{g,n} \equiv \mathcal{V}_{g,n}(u_0 + du) - f_{du}^\Gamma(\mathcal{V}_{g,n}(u_0)).$$

(4.11)

This space is the difference between $\mathcal{V}(u_0 + du)$ and the image of $\mathcal{V}(u_0)$ in $\Gamma(u_0 + du)$. It includes with a plus sign the subspace of $\Gamma(u_0 + du)$ corresponding to the surfaces that, regardless of the local coordinates, are contained in $\mathcal{V}(u_0 + du)$ but not in $\mathcal{V}(u_0)$, and with minus sign the subspace corresponding to the surfaces that, regardless of the local coordinates, are in $\mathcal{V}(u_0)$ but are not in $\mathcal{V}(u_0 + du)$. This subspace will be relevant for us in the next subsection.

Let us now define a vector field $\hat{\mathcal{U}}$. This vector field will not be uniquely determined at this stage. Our assumption that the subsets $\mathcal{V}_{g,n}(u)$, as we change $u$, are smoothly related will be taken to mean that there is a family of diffeomorphisms connecting them. The vector field $\hat{\mathcal{U}}$ generates such diffeomorphisms; that is, it pushes the subspaces $\mathcal{V}_{g,n}(u)$ precisely into each other. This is illustrated in Fig.1(b), where the curve $h_p(u)$ is the trajectory followed by the point $p$ representing a surface in $\mathcal{V}_{g,n}(u_0)$. More precisely, the vector field $\hat{\mathcal{U}}$ is defined to be the tangent vector $\hat{\mathcal{U}} = h_p^\ast(\partial/\partial u)$. The corresponding diffeomorphisms will be denoted by $f_s^\hat{\mathcal{U}}$ and they map

$$f_s^\hat{\mathcal{U}} : \mathcal{V}_{g,n}(u_0) \rightarrow \mathcal{V}_{g,n}(u_0 + s);$$
$$f_s^\hat{\mathcal{U}} : \partial \mathcal{V}_{g,n}(u_0) \rightarrow \partial \mathcal{V}_{g,n}(u_0 + s);$$

(4.12)

We will discuss later how to make a convenient choice of vector field $\hat{\mathcal{U}}$ by requiring compatibility with sewing.

4.3. Change in $S$ due to a Deformation of the $\mathcal{V}_{g,n}$’s.

We are now ready to calculate the variation of the action. An infinitesimal change of $u$ induces the change

$$S(u_0 + du) = S(u_0) + du \cdot \frac{dS}{du}\bigg|_{u_0} + \mathcal{O}(du^2),$$

(4.13)

and we want to show that

$$\delta S(u_0) \equiv du \cdot \frac{dS}{du}\bigg|_{u_0} = h\Delta + \{S(u_0), \epsilon\},$$

(4.14)

for some $\epsilon$ of the form $\epsilon = du \cdot e(u_0)$. This will establish that infinitesimal changes of the cell decomposition are generated by the transformations discussed in the previous section. Such transformations could, in principle, be integrated to prove that $S(u = 0)$ and $S(u = 1)$ are related by a large transformation, but we will not discuss this explicitly. The purpose of the present subsection is simply to evaluate the left hand side of (4.14). This is what we do next.
The closed string action is given by

\[ S(u, \Psi) = \frac{1}{2} \langle \Psi, Q \Psi \rangle + \sum_{g,n} h^g \kappa^{n+2g-2} S^n_g(u, \Psi), \tag{4.15} \]

where the sum extends over \( n \geq 3 \) for \( g = 0 \), and over \( n \geq 1 \) for \( g \geq 1 \). Here \( S^n_g \) is defined by the following expression [4]

\[ S^n_g(u, \Psi) = \frac{1}{n!} \int_{V_{g,n}(u)} \Omega^{(0)g,n}_{\Psi_n}. \tag{4.16} \]

The change we are considering does not affect the kinetic term \( S^2_0 \) and therefore the variation of the action will be given by

\[ \delta S(u, \Psi) = \sum_{g,n} h^g \kappa^{n+2g-2} \delta S^n_g(u, \Psi). \tag{4.17} \]

Making use of Eqn.(4.16) we find that the variation of \( S^n_g \) is given by

\[ n! [S^n_g(u_0 + du, \Psi) - S^n_g(u_0, \Psi)] = \int_{V_{g,n}(u_0 + du)} \Omega^{(0)g,n}_{\Psi_n} - \int_{V_{g,n}(u_0)} \Omega^{(0)g,n}_{\Psi_n} \]
\[ = \int_{V_{g,n}(u_0)} (f^{\hat{U}}_{du} \star \Omega^{(0)g,n}_{\Psi_n} - \Omega^{(0)g,n}_{\Psi_n}), \tag{4.18} \]

where \( f^{\hat{U}}_{du} \star \Omega \) denotes the pullback form using the diffeomorphism generated by the vector field \( \hat{U} \), mapping the subspaces precisely into each other. We note that the difference of forms on the (last) right hand side relates to the Lie derivative of the form \( \Omega \) along the vector field \( \hat{U} \). Indeed, taking the limit \( du \to 0 \) we find

\[ \frac{dS^n_g}{du}\bigg|_{u_0} = \frac{1}{n!} \int_{V_{g,n}(u_0)} \mathcal{L}_{\hat{U}} \Omega^{(0)g,n}_{\Psi_n}. \tag{4.19} \]

This is the total change in \( S^n_g \). The case of \( S^3_0 \) is special because \( V_{0,3}(u) \) is simply a point (for each \( u \)). In this case the variation is simply given by the integrand in the above expression.

We could have made use of the diffeomorphism \( f^{\hat{V}}_{du} \) generated by the vertical vector field \( \hat{V} \) in order to compute the change of the action. In this case we would have

\[ \int_{V_{g,n}(u_0 + du)} \Omega^{(0)g,n}_{\Psi_n} = \int_{V_{g,n}(u_0)} \Omega^{(0)g,n}_{\Psi_n} + \int_{V_{g,n}(u_0)} f^{\hat{V}}_{du} \star \Omega^{(0)g,n}_{\Psi_n}, \tag{4.20} \]

where use was made of Eqn.(4.11). We then find that the change in the action can be written
alternatively as
\[
\frac{dS^n}{du} \bigg|_{u_0} = \frac{1}{n!} \int_{V_{g,n}(u_0)} \mathcal{L}_{\hat{U}} \Omega_{\Psi,n}^{(0)g,n} + \frac{1}{n!} \lim_{du \to 0} \frac{1}{du} \int_{\delta_{u_0}V_{g,n}} \Omega_{\Psi,n}^{(0)g,n}. \tag{4.21}
\]

Having these two alternative expressions for the variation of the action will be helpful to understand some of the relevant issues.

4.4. Construction of the Symmetry Generator

We have computed in the previous subsection the derivative $dS/du$ of the action as we change the sections in the spaces $\hat{P}_{g,n}$ (or equivalently, the relevant subspaces $V_{g,n}$). The purpose of the present subsection is to find the generator of the symmetry transformation that reproduces this change. From (4.14) we must find a parameter $e(u_0)$ such that
\[
\frac{dS}{du} \bigg|_{u_0} = \hbar \Delta e(u_0) \{S(u_0), e(u_0)\}. \tag{4.22}
\]

We claim that the answer is very simple. One must have
\[
e(u_0) = \sum_{g,n} \hbar^{g-2} e^n_g(u_0, \Psi), \tag{4.23}
\]

where, as before, the sum extends over $n \geq 3$ for $g = 0$ and $n \geq 1$ for $g \geq 1$, and where $e^n_g$ carries the information about the change in subspace at genus $g$ and $n$ punctures. Since both the delta operator and the antibracket have ghost number +1, Eqn. (4.22) requires that $e^n_g$ must carry ghost number $-1$, and therefore it must include an extra antighost insertion, or equivalently, it must be related to the contraction $i_X \Omega^{(+1)}$ with some vector $X$. We have found in the previous subsection two vectors $\hat{U}$ and $\hat{V}$. Thus we are led to consider the possibility that
\[
e^n_g(u_0, \Psi) = -\frac{1}{n!} \int_{V_{g,n}(u_0)} i_{\hat{U}} \Omega^{(+1)g,n}, \tag{4.24}
\]
or that instead
\[
e^n_g(u_0, \Psi) = -\frac{1}{n!} \int_{V_{g,n}(u_0)} i_{\hat{V}} \Omega^{(+1)g,n}. \tag{4.25}
\]

Both candidates are quite similar; in both cases one takes the form $\Omega^{(+1)}$, of degree one higher than that suitable for integration over the section, and contracts it with a vector field to get a form that we can integrate over the original subspace $V_{g,n}(u_0)$. Actually, both expressions are the same to order $\epsilon$ and are therefore completely equivalent for our purposes. This happens because the vector $\hat{U}$ can be decomposed into a component along the fibers, which coincides
with \( \hat{V} \) to order \( \epsilon \), and a component \( \hat{W} \) tangent to the space \( \mathcal{V}_{g,n}(u_0) \). This second component cannot contribute to the integration because of the antisymmetry of the form \( \Omega \), which already has as input a basis of vectors for the tangent space of \( \mathcal{V}_{g,n} \) over which it is being integrated.

Making use of Eqn.(4.7) we can rewrite Eqn.(4.24) as

\[
e^{-n}(u_0, \Psi) = \frac{1}{n!} \int_{\mathcal{V}_{g,n}(u_0)} \Omega^{(0)g,n}_{b(u)} \Psi_n. \tag{4.26}
\]

We can now begin our proof that \( e(u_0) \) defined above does indeed generate the correct change in the action. Making use of Eqns.(4.15), (4.22) and (4.23) we have to show that

\[
\frac{dS^g_n}{du} \bigg|_{u_0} = \Delta e^{-n+2}(u_0) + \sum_{g_1+g_2=g} \sum_{n_1+n_2=n+2} \{S^{g_1}_{g_1}(u_0), e^{g_2}_{g_2}(u_0)\}. \tag{4.27}
\]

Our aim is now to evaluate the right hand of this equation and to prove that it coincides with the left hand side given earlier in (4.19). We will be using explicitly the vector \( \hat{U} \) in our computation.

Let us start with the \( \{S, e\} \) type term, and in particular we single out the contribution from \( S^2_0 \). A computation (analogous to that in §4 of Ref.[4]) gives us

\[
\{ S^2_0, e^n \} = -\frac{1}{n!} \int_{\mathcal{V}_{g,n}(u_0)} \Omega^{(0)g,n}_{b(u)}(\sum Q) \Psi_n
\]

\[
= -\frac{1}{n!} \int_{\mathcal{V}_{g,n}(u_0)} \Omega^{(0)g,n}_{T(u)} \Psi_n + \frac{1}{n!} \int_{\partial \mathcal{V}_{g,n}(u_0)} \Omega^{(-1)g,n}_{b(u)} \Psi_n, \tag{4.28}
\]

where in the last step we made use of (4.5). Eqns.(4.7) and (4.9) then give

\[
\{ S^2_0, e^n \} = \frac{1}{n!} \int_{\mathcal{V}_{g,n}(u_0)} L_{\hat{U}} \Omega^{(0)g,n}_{\Psi} - \frac{1}{n!} \int_{\partial \mathcal{V}_{g,n}(u_0)} i_{\hat{U}} \Omega^{(0)g,n}_{\Psi}. \tag{4.29}
\]

Note that the first term on the right hand side is precisely the Lie derivative term appearing on the right hand side of (4.19). Let us now consider the contributions to \( \{S, e\} \) for \( S \) different from the kinetic term. This time a calculation gives

\[
\{ S^{g_1}_{g_1}, e^{g_2}_{g_2} \} = -\frac{1}{(n_1-1)! (n_2-1)!} \int_{\mathcal{V}_{g_2,n_2}(u_0)} i_{\hat{U}} \Omega^{(1)g_2,n_2} \frac{\Psi_{n_2-1}}{\Psi_{n_1-1}}_{g_1} \]

\[
= -\frac{1}{(n_1-1)! (n_2-1)!} \sum_s' (-)^s \Phi_s \int_{\mathcal{V}_{g_2,n_2}(u_0)} i_{\hat{U}} \Omega^{(1)g_2,n_2} \frac{\Psi_{n_2-1} \Phi_s}{\Psi_{n_1-1}}_{g_1} \int_{\mathcal{V}_{g_1,n_1}(u_0)} \Omega^{(0)g_1,n_1}_{\Phi_s} \Psi_{n_1-1}, \tag{4.30}
\]

where in the last step we used the definition of the string product (Ref.[4], Eqn.(7.100)), and the primed summation \( \sum_s' \) implies the summation over the states annihilated by \( L_{\hat{U}}^{\perp} \). Finally
for the term involving the $\Delta$ operator we get

$$\Delta e^{n+2} = -\frac{1}{2} \frac{1}{n!} \sum_s'(-)\Phi_s \int_{\mathcal{V}_{g-1,n+2}(u_0)} i_\mathcal{U}^{(1)g-1,n+2}.$$

We have now evaluated the basic ingredients appearing on the right hand side of Eqn.(4.27). Comparing with Eqn.(4.19) we find that the generator $e(u_0)$ will give the correct change in the action if the following relation holds:

$$0 = \frac{1}{n!} \int_{\partial\mathcal{V}_{g,n}(u_0)} i_\mathcal{U}^{(0)g,n} + \sum_{g_1+g_2=g, n_1+n_2=n+2} \frac{1}{(n_1-1)!} \frac{1}{(n_2-1)!} \sum_s'(-)\Phi_s \int_{\mathcal{V}_{g_2,n_2}(u_0)} i_\mathcal{U}^{(1)g_2,n_2} \cdot \int_{\mathcal{V}_{g_1,n_1}(u_0)} \Omega^{(0)g_1,n_1} \cdot \Omega_{\Phi_s \psi_{g_1-1}} \cdot \sum_s'(-)\Phi_s \int_{\mathcal{V}_{g-1,n+2}(u_0)} i_\mathcal{U}^{(1)g-1,n+2}.$$

(4.32)

It must be noted that in the second and third lines of (4.32) we can replace the vector $\mathcal{U}$ by the vertical vector $\mathcal{V}$, since the integrals already extend over the full subspaces. This cannot be done in the first line, since the integral extends only over the boundary of $\mathcal{V}_{g,n}$ and in the decomposition $\mathcal{U} = \mathcal{V} + \mathcal{W}$ the vector $\mathcal{W}$ need not be tangent to $\partial\mathcal{V}_{g,n}$.

If we had used the expression (4.25) for $e^n_{g}$ based on the vector $\mathcal{V}$, there would have been only one change in the above derivation. Making use of Eqn.(4.21) one finds that the first term in the above equation would have been replaced by

$$0 = \frac{1}{n!} \int_{\partial\mathcal{V}_{g,n}(u_0)} i_\mathcal{U}^{(0)g,n} + \frac{1}{n!} \lim_{du \to 0} \frac{1}{du} \int_{\delta u_0 \mathcal{V}_{g,n}} i_\mathcal{U}^{(0)g,n}.$$

(4.33)

The vector $\mathcal{W}du$ that we have been discussing is, to order $\epsilon$, the vector that moves the image (under the vertical map) of the boundary of $\mathcal{V}_{g,n}(u_0)$ in the section $\Gamma(u_0+du)$ to the boundary of $\mathcal{V}_{g,n}(u_0+du)$ in the same section. This vector therefore maps out the space $\delta u_0 \mathcal{V}_{g,n}$ introduced before. This fact allows us to rewrite the second term in the above expression as

$$0 = \frac{1}{n!} \lim_{du \to 0} \frac{1}{du} \cdot \int_{\partial\mathcal{V}_{g,n}} i_\mathcal{W}^{(0)g,n} = \frac{1}{n!} \int_{\partial\mathcal{V}_{g,n}} i_\mathcal{W}^{(0)g,n},$$

(4.34)

combining this with the first term in (4.33) and making use of $i_\mathcal{V} + i_\mathcal{W} = i_\mathcal{U}$ we obtain, as expected, the same expression for the first term appearing in Eqn.(4.32).

In the next two subsections we will develop intuitive understanding of the expressions given in this subsection, providing partial confirmation of the correctness of our ansatz for $e^n_{g}$. In the last subsection we give a complete, though slightly more abstract, proof of the result.
4.5. Why Sections are not Absolutely Necessary

To date, string field theory has been constructed using subspaces of sections on the bundle \( \hat{\mathcal{P}}_{g,n} \) over moduli space \( \mathcal{M}_{g,n} \). This means that we pick a subset of surfaces of \( \mathcal{M}_{g,n} \) and for each such surface we give a unique surface in the space \( \hat{\mathcal{P}}_{g,n} \), corresponding to choosing local coordinates around the punctures. This corresponds to choosing a section, that is a well defined map \( \pi^{-1} \) on \( \mathcal{M}_{g,n} \). In a more general situation \( \mathcal{V}_{g,n} \) is simply a subspace of \( \hat{\mathcal{P}}_{g,n} \), where it can happen that the projection \( \pi \) may take more than one point in the subspace to the same point in \( \mathcal{M}_{g,n} \), as illustrated in Fig. 2(a). Such generalized subspaces \( \mathcal{V}_{g,n} \) will still lead to an action satisfying the BV master equation, if the geometrical recursion relations, relating the boundary of subspaces to sewing operations with lower dimensional subspaces, still hold.

For physical states the value of such a vertex will be the same as that from a vertex defined by a section with the same boundary. This is easily verified as follows. Consider a subspace \( \mathcal{V}' \) which is not a section, and the related (section) subspace \( \mathcal{V} \) with coincident boundaries. Let \( \mathcal{R} \) denote the region bound by \( \mathcal{V} \) and \( \mathcal{V}' \), that is, \( \partial \mathcal{R} = \mathcal{V} - \mathcal{V}' \). Since physical states are annihilated by \( Q \) we then have

\[
0 = \int_{\mathcal{R}} \Omega^{(+1)g,n}_{(\sum Q)\psi_\ldots \psi} = - \int_{\mathcal{V}} d\Omega_{\psi_\ldots \psi}^{(0)g,n} = - \int_{\mathcal{V}} \Omega_{\psi_\ldots \psi}^{(0)g,n} + \int_{\mathcal{V}'} \Omega_{\psi_\ldots \psi}^{(0)g,n},
\]

establishing the desired equality.

We will now see that particular choices of \( e_g^n \)'s turn a string field theory based on sections into one which does not use sections. Since for any choice of \( e_g^n \) we must get a consistent string field theory we simply have to study the effect of the change.

Suppose we take \( e_{g_0}^{n_0} \neq 0 \) and all others are set to zero. We also assume that this \( e_{g_0}^{n_0} \) is of the form given in Eqn.(4.24) for some vector field \( \hat{U} \). The symmetry transformation will change \( S_{g_0}^{n_0} \), terms with a larger number of string fields for the same genus, and terms of higher genus (see Eqn.(4.27)). Let us see what happens with \( S_{g_0}^{n_0} \). Its variation is controlled by the bracket of \( e_{g_0}^{n_0} \) with \( \Omega_{\psi_\ldots \psi}^{(0)g,n} \).

\[
S'_{g_0}^{n_0} = S_{g_0}^{n_0} + \frac{1}{n!} \int_{\mathcal{V}_{g,n}(u_0)} \mathcal{L}_{\hat{\mathcal{V}}_{du}} \Omega_{\psi_\ldots \psi}^{(0)g,n} + \frac{1}{n!} \int_{\partial \mathcal{V}_{g,n}(u_0)} i_{-\hat{\mathcal{V}}_{du}} \Omega_{\psi_\ldots \psi}^{(0)g,n}.
\]  

We can now combine the first two terms in the last expression to find

\[
S'_{g_0}^{n_0} = \frac{1}{n!} \int_{\mathcal{V}_{g,n}(u_0+du)} \Omega_{\psi_\ldots \psi}^{(0)g,n} + \frac{1}{n!} \int_{\partial \mathcal{V}_{g,n}(u_0)} i_{-\hat{\mathcal{V}}_{du}} \Omega_{\psi_\ldots \psi}^{(0)g,n}.
\] 

This expression has a simple meaning. The second term is nothing else than the integral of the form \( \Omega^{(0)} \) over the strip \( \mathcal{S} \) shown in Fig.2(b). This is the strip defined by the vector \( \hat{U} \).
over $\partial V_{g,n}$. Therefore,

$$S_{g_0}^{n_0} = \frac{1}{n!} \int_{V_{g,n}'} \Omega_{\Phi_n}^{(0)g,n},$$

(4.38)

where the new subspace $V_{g,n}'$ is the subspace including the subspace at $(u_0 + du)$ plus the strip, making it have the same boundary as the original subspace at $u_0$. The new subspace does not, in general, correspond to a section in $\hat{\mathcal{P}}$. The new action thus works without using a section, and this illustrates not only how the symmetry transformation produces consistent modifications, but why sections are not absolutely required to have a consistent theory. Full appreciation of this fact may be important to obtain generalized formulations of string field theory.

We have remarked in Eqn.(3.4) that the Lie algebra of the $\delta\epsilon$ transformations is the Lie algebra of the antibracket. The $\delta\epsilon$ transformations with $\epsilon$'s corresponding to a change in section are a small subset of all possible $\delta\epsilon$ transformations. If $\epsilon_1$ and $\epsilon_2$ are two parameters corresponding to changes in sections, their antibracket $\{\epsilon_1, \epsilon_2\}$ does not seem to be, in general, a parameter associated to another change in section. It is simply another parameter with some complicated effect on the vertices, an effect that can be far more involved than taking sections into subspaces that are not sections. This is expected since successive transformations do not arise naturally in our context; our deformations are not defined for all possible choices of subspaces, thus after deforming a bit with a first vector $\hat{U}_1$ we may obtain a subspace for which a second vector $\hat{U}_2$ may not be defined. Therefore, the parameters associated with changes of section do not seem to define a Lie subalgebra of the antibracket. It would be very interesting to find special parameters associated with changes of section that form a subalgebra.

4.6. The particular case of Stubs

The most straightforward way of creating a one parameter family of subspaces $V_{g,n}$ satisfying the consistency conditions is based on the variation of the stub length in covariant closed string field theory [21, 9]. With a minimal area metric, each surface in the subspace $V_{g,n}$ has a semiinfinite cylinder about every puncture, and the corresponding local coordinate $z$ is defined by taking the curve $|z| = 1$ to be the geodesic circle a distance $l$ down the cylinder (measured from the beginning of the semiinfinite cylinder). This short cylinder of length $l$ is called the stub ($l \geq \pi$). If the string vertices have stubs of length $l$, the subspaces $V_{g,n}$ are given by all the surfaces whose metric of minimal area does not show any finite cylinder of length greater than $2l$ [4]. This means that the surface has no propagator, or that no sewing operation is involved in its construction. This result tells us explicitly how the subspaces $V_{g,n}$ vary as we vary the stub length.

Consider an infinitesimal increase of the stub length by an amount $\epsilon$. The new surfaces that must be included in each $V_{g,n}$ are those that have at least one cylinder with length $L$ in the interval $2l \leq L \leq 2l + 2\epsilon$, and all other cylinders of length smaller than $2l + 2\epsilon$. We claim that to order $\epsilon$ we need only consider the surfaces with one propagator only. This is clear because when we have one propagator ($2l \leq L \leq 2l + 2\epsilon$) the domain representing the new
surfaces has a ‘volume’ proportional to the product of the ‘volumes’ of two subspaces involved in the sewing (tree configuration); each time we have another propagator, since its parameter space is proportional to $\epsilon$, the volume is reduced by this factor, and therefore we get a higher order effect.

Let us now give the explicit expression for the changes. Let $z$ be the original local coordinates. Increasing the stub length by $\epsilon$ corresponds to defining a new coordinate $z' = z + \epsilon z$, since the circle $|z'| = 1$ corresponds to $|z| = 1 - \epsilon$, and is therefore a retraction of the local coordinate. The associated Schiffer vector defined from $z' = z + \epsilon v(z)$ is then $v(z) = z$.

This Schiffer vector, used for each puncture in a given surface, implements the deformation associated to the vertical vector $\hat{V}$ defined earlier, since its effect is simply to change the local coordinates at the punctures, without changing the moduli of the surface. The antighost insertion for any vertex with $n$ punctures is then given by

$$b(v) = \sum_i (b_0^{(i)} + \tilde{b}_0^{(i)}) = \sum_i b_0^{+(i)}, \quad (4.39)$$

and the stress tensor insertion is given by

$$T(v) = \sum_i (L_0^{(i)} + \tilde{L}_0^{(i)}) = \sum_i L_0^{+(i)}. \quad (4.40)$$

Equation (4.21) then reads

$$\left. \frac{dS_n^g}{dl} \right|_{l_0} = -\frac{1}{(n-1)!} \int_{\mathcal{V}_{g,n}(l_0)} \Omega^{(0)g,n}(L_0^+ \Psi) \Psi^{n-1} + \frac{1}{n!} \lim_{du \to 0} \frac{1}{du} \int_{\delta_{l_0} \mathcal{V}_{g,n}} \Omega^{(0)g,n}, \quad (4.41)$$

where $l_0$ denotes the original stub length, and the second term represents the contribution due to the extra surfaces that must be included in the new section. Moreover, our ansatz for $e^g_n$ reduces to

$$e^g_n(l_0, \Psi) = -\frac{1}{(n-1)!} \int_{\mathcal{V}_{g,n}(l_0)} \Omega^{(0)g,n}(b_0^+ \Psi) \Psi^{n-1}, \quad (4.42)$$

Let us now argue that Eqn.(4.32) must be satisfied in this case. We will only pay attention to how the relevant surfaces appear and not to the combinatorial factors, which are verified to work explicitly in the next subsection. As argued below Eqn.(4.32) (see Eqns.(4.33) and (4.34)) the first term on the right hand side equals

$$\frac{1}{n!} \int_{\partial \mathcal{V}_{g,n}(l_0)} \Omega^{(-1)g,n}(b(v) \Psi) \Psi^{n} + \frac{1}{n!} \int_{\partial \mathcal{V}_{g,n}(l_0)} \Omega^{(-1)g,n}(w) \Psi^{n}, \quad (4.43)$$

where $w$ is the Schiffer vector generating the deformation that gives the new surfaces. This Schiffer vector, as usual, is supported on the external punctures. Since every surface $\Sigma$ in
\( \partial V_{g,n} \) is built by sewing, and the deformations we are considering are precisely due to a change of sewing parameter, we can relate external insertions to internal ones. Given that such deformations of the surface can be obtained in either way we must have that

\[
\langle \Sigma | T(w) = (\hat{\Sigma}_1 | (\hat{\Sigma}_2 | R_{\rho,\rho}^\theta) T(w) = (\hat{\Sigma}_1 | (\hat{\Sigma}_2 | (L_0^+ | R_{\rho,\rho}^\theta)),
\]

where \( | R_{\rho,\rho}^\theta \) is the sewing ket. The same equation must hold for antighosts, namely

\[
\langle \Sigma | b(w) = (\hat{\Sigma}_1 | (\hat{\Sigma}_2 | (b_0^+ | R_{\rho,\rho}^\theta)),
\]

We also noted below Eqn.(4.32) that we can use the vertical vector \( \hat{V} \) in the second and third lines of that equation. Let us consider the second line, which up to numerical coefficients and the sums can be written as

\[
(-\Phi_s \int_{V_{g_2,n_2}(l_0)} \Omega^{(0)g_2,n_2}_{\Phi} \Psi^{n_2-1} \Phi_s \cdot \int_{V_{g_1,n_1}(l_0)} \Omega^{(0)g_1,n_1}_{\Phi} \Psi^{n_1-1} = (-\Phi_s \int_{V_{g_2,n_2}(l_0)} \Omega^{(0)g_2,n_2}_{\Phi} \Psi^{n_2-1} \Phi_s \cdot \int_{V_{g_1,n_1}(l_0)} \Omega^{(0)g_1,n_1}_{\Phi} \Psi^{n_1-1} + (-\Phi_s \int_{V_{g_2,n_2}(l_0)} \Omega^{(0)g_2,n_2}_{\Phi} \Psi^{n_2-1} \Phi_s \cdot \int_{V_{g_1,n_1}(l_0)} \Omega^{(0)g_1,n_1}_{\Phi} \Psi^{n_1-1}.
\]

The first term on the right hand side represents sewing of surfaces and integration over the sum of direct products of subspaces and twist angle. This total space is simply \( \partial V_{g,n} \). Note that the antighost insertions are acting only on the external punctures (by symmetrization using the implicit sums, they also act on the external punctures of the surfaces in \( V_{g_1,n_2} \)). But the antighost insertions for a uniform change in stub length are always \( b_0^+ \) regardless of genus or the number of punctures, so this term is exactly of the same as the first term (4.43). The second term shows the insertion \( b_0^+ \) appearing in the propagator. Since \( b_0^+ \) is the insertion corresponding to the modulus that changes the length of the propagator, this term corresponds to the new surfaces. Indeed, making use of Eqn.(4.45) we see that it is precisely of the same form as the second term in (4.43). This concludes our argument that for the case of deformations arising from the change of stub length the ansatz for \( e^\theta_y \) is correct (up to combinatorial factors to be dealt with in the next subsection).

**4.7. The General Case**

We now turn to a complete proof of Eqn.(4.32). This will require refining the definition of the vector field \( \hat{U} \) introduced in §4.2, so as to have compatibility with sewing. This is what we do next.

Recall that for any \( V_{g,n}(u_0) \) the diffeomorphisms \( f^\hat{U}_s \) take \( V_{g,n}(u_0) \rightarrow V_{g,n}(u_0 + s) \) and \( \partial V_{g,n}(u_0) \rightarrow \partial V_{g,n}(u_0 + s) \), and \( \hat{U} \) is the vector field that generates the map. The compatibility with sewing is the requirement that the map \( \partial V_{g,n}(u_0) \rightarrow \partial V_{g,n}(u_0 + s) \) must be special. Since every surface in \( \partial V_{g,n} \) is obtained by sewing of two distinct surfaces, or of two punctures in a single surface, we demand that the map should take the sewn surface to the surface obtained by sewing the deformed constituents.
In order to construct this map we proceed by induction, increasing in each step the dimensionality of the moduli spaces. The idea will be the following: a surface on the boundary of a subspace \( V(u_0) \) determines via Eqn.(4.1) either a pair of surfaces (or a single surface) in subspaces of lower dimensionality. If we know how to map such subspaces, this determines two of a subspace \( V \) dimensionality of the moduli spaces. The idea will be the following: a surface on the boundary \( \partial V \) of \( V \) on the right hand side of the geometrical equation (4.1) gives a disjoint contribution to the boundary. It is then clear that the space \( \partial V \) has the structure of the sum of spaces each of which is a \( U(1) \) fiber bundle with base space \( V_{g_1,n_1} \times V_{g_2,n_2} \) or \( V_{g-1,n+2} \), where the \( U(1) \) fiber arises from the twist operation necessary when sewing. The map \( f_s^U \) discussed in §4.2 is a diffeomorphism taking the corresponding bundles into each other, but it need not be fiber preserving. The compatibility with sewing requires modifying the the diffeomorphism making it fiber preserving.

For the unique dimension zero moduli space \( V_{0,3} \) we simply define \( f_s^U : V_{0,3}(u_0) \rightarrow V_{0,3}(u_0 + s) \). This satisfies all the requirements. The induction hypothesis is that the desired map has been defined for all \( g',n' \) such that the dimensionality of the respective subspaces is smaller than that of \( V_{g,n} \). We must then show how to construct the map for \( V_{g,n} \).

To do this the most nontrivial step is to construct a diffeomorphism mapping the various boundary subspaces \( \partial V_{g,n}(u) \). In order to construct this map we begin by defining a diffeomorphism between the base manifolds of the \( U(1) \) fiber bundles representing \( \partial V_{g,n}(u) \). Pick a surface \( \Sigma \in \partial V_{g,n}(u_0) \). This determines via (4.1) either two surfaces \( \hat{\Sigma}_1 \in V_{g_1,n_1}(u_0) \), \( \hat{\Sigma}_2 \in V_{g_2,n_2}(u_0) \), or a single surface \( \hat{\Sigma}_l \in V_{g-1,n+2}(u_0) \). The pair of surfaces \( \hat{\Sigma}_1, \hat{\Sigma}_2 \in V_{g_1,n_1}(u_0) \times V_{g_2,n_2}(u_0) \) represents a basepoint, and the set of surfaces obtained by sewing this pair of surfaces and twisting is the fiber (similarly for \( \hat{\Sigma}_l \)). Now we can use the diffeomorphisms available in the lower dimensional subspaces to define

\[
\hat{\Sigma}_i(u_0 + s) = f_s^U(\hat{\Sigma}_i), \quad \text{for } i = 1, 2, \text{ or } i = l.
\]

These surfaces, by definition, lie on the subspaces at parameter value \((u_0 + s)\). They define a basepoint on the fiber bundles \( \partial V_{g,n}(u_0 + s) \) since sewing them gives a specific fiber. This is a diffeomorphism between the base spaces.

We must now extend this diffeomorphism to a continuous map between the fiber bundles. This map will be fiber preserving, and equivariant, in the sense that it commutes with the twist operation. For this, fix the phases of the local coordinates around the punctures to be sewn so that the sewing parameter is \( t = 1 \). Let \( \Sigma(\theta) \in \partial V_{g,n}(u_0) \) be the surface obtained sewing \( \hat{\Sigma}_1 \) and \( \hat{\Sigma}_2 \) (or \( \hat{\Sigma}_l \)) with \( t = \exp(i\theta) \). Clearly \( \Sigma(\theta = 0) = \Sigma \). Again, we deform the constituents, but in order to be able to sew back unambiguously we have to keep track of the phases around
the punctures. To that effect we choose a phase convention around the punctures to be sewn for the one parameter set of surfaces in Eqn.(4.47). Even though this choice cannot be done continuously for all surfaces in the subspaces \( \mathcal{V} \) to be sewn, this will cause no difficulty since the continuity of the map is guaranteed by keeping track of phases and not by how we choose them. We can now define the map between \( \partial \mathcal{V}_{g,n}(u_0) \) and \( \partial \mathcal{V}_{g,n}(u_0 + s) \)

\[
\mathcal{f}_s^U : \Sigma(\theta) \rightarrow \tilde{\Sigma}_1(u_0 + s) \cup_{\theta} \tilde{\Sigma}_2(u_0 + s) \quad \text{(or } \cup_{\theta} \tilde{\Sigma}_l(u_0 + s)) \tag{4.48}
\]

where \( \cup_{\theta} \) denotes sewing with \( t = \exp(i\theta) \). Due to (4.1) it is clear that the map gives a surface in \( \partial \mathcal{V}(u_0 + s) \). The map, by construction, is one to one (since there is no overcounting, two surfaces built with identical constituents but different sewing parameter cannot be the same) and onto, that is, any surface in \( \partial \mathcal{V}(u_0 + s) \) arises from some surface on \( \partial \mathcal{V}(u_0) \). The map is continuous since it was built keeping track of the phases on the punctures that were sewn, and is equivariant with respect to \( U(1) \). Using the diffeomorphism between the base manifolds, all the fiber bundles for various \( s \) can be thought of as having the same base space and projection given by the composition of the inverse diffeomorphism and the original projection. Then our continuous maps define bundle isomorphisms. But \( U(1) \) bundles isomorphic in the continuous category are also isomorphic in the smooth category \( \star \). Thus the map can be made into a fiber preserving and equivariant diffeomorphism. This is essentially what we wanted. We then obtain the vector field \( \tilde{U} \), defined on the boundaries \( \partial \mathcal{V}_{g,n}(u) \) with the desired properties. Finally the map is extended smoothly to the interior of the \( \mathcal{V}_{g,n} \) subspaces (using the previously defined (§4.2) vector field \( \tilde{U} \)). This concludes the construction of the map at this order, and by the induction hypothesis, the desired map and the associated generating vector field \( \tilde{U} \) have been shown to exist.

**Completing the Derivation** We can now see why we expect equation (4.32) to hold. The second and last terms show amplitudes constructed by first deforming the constituent surfaces and then sewing, while the first term shows an amplitude where we deform a sewn surface. These two types of terms are related because the vector field \( \tilde{U} \) on sewn surfaces was constructed to give the deformation induced by first deforming the constituent surfaces and then sewing. Let us now verify that as a result the terms cancel out precisely.

In terms of states, a surface \( \Sigma \in \partial \mathcal{V}_{g,n} \) built sewing, with parameter \( t = \exp(i\theta) \), two surfaces \( \tilde{\Sigma}_1 \) and \( \tilde{\Sigma}_2 \), is given by

\[
\langle \Sigma \rangle = \langle \tilde{\Sigma}_1 | \tilde{\Sigma}_2 | R^\theta_{r_1 r_2} \rangle, \tag{4.49}
\]

where \( | R^\theta_{r_1 r_2} \rangle \) is the sewing ket. The key point in our construction is that the deformation of \( \Sigma \) defined by the vector field \( \tilde{U} \) on sewn surfaces was constructed to give the deformation induced by first deforming the constituent surfaces and then sewing. Let us now verify that as a result the terms cancel out precisely.

We are very grateful to H. Miller for his explanations on fiber bundles.
where, following the notation of Ref. [4] (§ transform as quadratic differentials) we also have that the sewn surface \( \Sigma \) by the \( j \)-th using now the explicit expression for the sewing ket ([4], Eqn.(2.74)) we have the second term of the equation; its treatment is completely analogous. Consider then the expression for the other deformations (cf. [4] § evaluate on a surface obtained by sewing two surfaces together. This case relates to the third term in the equation; its treatment is completely analogous). Consider then the expression

\[
\exp(\pi i \delta \Sigma)(1 \Sigma, u) \Phi_s \rangle_{(r1)} 2 \pi \Phi_s \rangle_{(r2)}
\]

we get that the above is given by

\[
N_{g,n}(\Sigma_1|\Sigma_2|(b(u_1) + b(u_2)) b(v_{1}^1) \cdots b(v_{d_1}^1) b(v_{1}^2) \cdots b(v_{d_2}^2) i(b_0 - T_0)(r1)|R_{r_1 r_2}^\theta)|\Psi)^n.
\]

Using now the explicit expression for the sewing ket ([4], Eqn.(2.74)) we have

\[
i(b_0 - T_0)(r1)|R_{r_1 r_2}^\theta) = (2\pi i) \sum_s (-\Phi_s |\Phi_s\rangle_{(r1)} \frac{1}{2\pi} \exp(i\theta L_0^-) |\Phi_s\rangle_{(r2)}.
\]

and (4.54) can then be rewritten as

\[
(2\pi i N_{g,n}) \sum_s (-\Phi_s \left( \langle \Sigma_1|b(u_1)b(v_{1}^1) \cdots b(v_{d_1}^1)|\Psi\rangle^{n_1-1}|\Phi_s\rangle_{(r1)}
\]

\[
\cdot \langle \Sigma_2|b(v_{1}^2) \cdots b(v_{d_2}^2) \frac{1}{2\pi} \exp(i\theta L_0^-) |\Phi_s\rangle_{(r2)}|\Psi\rangle^{n_2-1}
\]

\[
+ (-)^{\Phi_s+1} \langle \Sigma_1|b(v_{1}^1) \cdots b(v_{d_2}^1)|\Psi\rangle^{n_1-1}|\Phi_s\rangle_{(r1)}
\]

\[
\cdot \langle \Sigma_2|b(u_2)b(v_{1}^2) \cdots b(v_{d_2}^2) \frac{1}{2\pi} \exp(i\theta L_0^-) |\Phi_s\rangle_{(r2)}|\Psi\rangle^{n_2-1}
\]
At the level of forms this just means that

\[ i_\theta \Omega_{g,n}^{(0)}(\hat{V}_1, \Sigma_1), \ldots, \hat{V}_1^{(d_1, \Sigma_1)}; \hat{V}_2^{(1, \Sigma_2)}, \ldots, \hat{V}_2^{(d_2, \Sigma_2)}; \hat{V}_3^{(1)} ) \]

\[ = \sum (-)^s \left[ i_\theta \Omega_{g,n}^{(1)}{\hat{V}_1^{(1)}, \ldots, \hat{V}_1^{(d_1)}}, \Omega_{g,n}^{(0)}{\hat{V}_2^{(1)}, \ldots, \hat{V}_2^{(d_2)}} \right] \]

\[ + (-)^s \Omega_{g,n}^{(1)}{\hat{V}_1^{(1)}, \ldots, \hat{V}_1^{(d_1)}} \cdot i_\theta \Omega_{g,n}^{(1)}{\hat{V}_2^{(1)}, \ldots, \hat{V}_2^{(d_2)}} \]

where we have used that \( 2\pi i N_{g,n} = N_{g_1,n_1} N_{g_2,n_2} \). Integrating over the sewing angle and using Eqn.(2.83) of Ref.[4] on the second term, we obtain

\[ \int d\theta i_\theta \Omega_{g,n}^{(0)}(\hat{V}_1, \Sigma_1), \ldots, \hat{V}_1^{(d_1, \Sigma_1)}; \hat{V}_2^{(1, \Sigma_2)}, \ldots, \hat{V}_2^{(d_2, \Sigma_2)}; \hat{V}_3^{(1)} ) \]

\[ = \sum (-)^s \left[ i_\theta \Omega_{g,n}^{(1)}{\hat{V}_1^{(1)}, \ldots, \hat{V}_1^{(d_1)}}, \Omega_{g,n}^{(0)}{\hat{V}_2^{(1)}, \ldots, \hat{V}_2^{(d_2)}} \right] \]

\[ + (-)^s \Omega_{g,n}^{(1)}{\hat{V}_1^{(1)}, \ldots, \hat{V}_1^{(d_1)}} \cdot i_\theta \Omega_{g,n}^{(1)}{\hat{V}_2^{(1)}, \ldots, \hat{V}_2^{(d_2)}} \]

We now simply recall that

\[ \int d\theta (\partial V_{g,n})_{\text{tree}} = -\frac{1}{2} \sum \int \int \cdots \int d\theta \cdot \frac{n!}{(n_1 - 1)!(n_2 - 1)!} \]

where the combinatorial factor arises because there are that number of ways of splitting \( n \) string fields into two subsets with \( (n_1 - 1) \) and \( (n_2 - 1) \) string fields each. The last two equations imply the desired cancellation in Eqn.(4.32) between the part of the first term having to do with surfaces sewn in the tree configuration and the second term. The third term cancels against the part of the first term having to do with surfaces sewn in the loop configuration, by a completely analogous argument. This proves the validity of Eqn.(4.32), and as a consequence concludes our proof that the quoted infinitesimal parameter indeed reproduces the change in the action induced by a change in the decomposition of moduli space.
5. The Hamiltonian Vector Field for the Quantum Master Action

Given a string action \( S \), that is, a function on the supersymplectic manifold \( M \), we denote the corresponding hamiltonian vector field by \( V_S \). By definition it is given as \( i_{V_S} \omega = -dS \), or, more explicitly, by Eqn.(2.4). Witten showed \([5]\) that the vector \( V_S \) corresponding to an action that satisfies the classical master equation must satisfy \( V_S^2 = 0 \). We now study the generalization for the case of the full master equation.

The master equation for the action demands that

\[
\hbar \Delta S + \frac{1}{2} \{S, S\} = 0, \quad \Delta S \equiv \frac{1}{2} \text{div} V_S. \tag{5.1}
\]

The left hand side of the master equation is a function that must vanish. Therefore it follows that the hamiltonian vector corresponding to this function must vanish too:

\[
V_{\hbar \Delta S + \frac{1}{2} \{S, S\}} = 0. \tag{5.2}
\]

It follows from linearity that

\[
\hbar V_{\Delta S} + \frac{1}{2} V_{\{S, S\}} = 0. \tag{5.3}
\]

Using the basic relation between (super)Lie brackets and (anti)brackets, Eqn.(2.7), and the definition of \( \Delta \) in Eqn.(5.1), we find

\[
V_S^2 = -\frac{\hbar}{2} \text{div} V_S, \tag{5.4}
\]

which is the generalization of the equation \( V_S^2 = 0 \) given in \([5]\) necessary for the action \( S \) to satisfy the full master equation. While this equation is necessary for the master equation to hold, it is not quite sufficient to guarantee it. If Eqn.(5.4) holds, then the hamiltonian vector field in (5.2) does indeed vanish. Given a vanishing hamiltonian vector field, the corresponding hamiltonian function

\[
\hbar \Delta S + \frac{1}{2} \{S, S\} = \frac{1}{2} (\hbar \text{div} V_S + \omega(V_S, V_S)), \tag{5.5}
\]

could be a constant different from zero. The way to argue that this constant is zero is to show that the vector field \( V_S \) solving (5.4) has at least one zero in the supermanifold. This would make the right hand side of the above equation vanish at that point, and therefore everywhere.

In the language of homotopy Lie algebras\(^\star\) for such a vector would look like

\[
V_S = (f^b_{a_1} \eta^{a_1} + f^b_{a_1 a_2} \eta^{a_1} \eta^{a_2} + \cdots) \frac{\partial}{\partial \eta^b}, \tag{5.6}
\]

where we have a zero at \( \eta^a_1 = \cdots = \eta^a_n = 0 \).

\(^\star\) See Ref.[11], and Ref.[22] for an introduction to the basic concepts.
It follows from [4] that there is a nontrivial vector field $V_S$ solving Eqn.(5.4). Nevertheless the vector does not have a zero! In those solutions $V_S$ reads

$$V_S = \left( f_{a_1}^{(0)b} \eta^{a_1} + f_{a_1a_2}^{(0)b} \eta^{a_1} \eta^{a_2} + \cdots \right) \frac{\partial}{\partial \eta^b}$$

$$+ \hbar \left( f_{a_1}^{(1)b} \eta^{a_1} + \cdots \right) \frac{\partial}{\partial \eta^b} + \mathcal{O}(\hbar^2),$$

(5.7)

and therefore it is of $\mathcal{O}(\hbar)$ for $\eta^a = 0$. The terms $f^{(1)b}$, for example, have to do with one point functions of arbitrary states on genus one surfaces. Since we do not have a zero, verifying that there are solutions where (5.5) vanishes is not trivial. Fortunately this was shown already in Ref.[4] (see around Eqn.(3.42)) where it was shown that no constant terms can arise on the BV master equation (=lhs of (5.5)) because of ghost number conservation.

It is clear from the above considerations that in order to formulate the quantum closed string field theory in the space of two-dimensional theories we need to find not only a closed non-degenerate two-form $\omega$, but also a density $\rho$. This density cannot be fixed arbitrarily; it must lead to $\Delta_2 \rho = 0$ (thus given $\omega$ we cannot assume that $\rho$ is a constant). Once we find a suitable density, we must solve Eqn.(5.4). A particularly simple solution would involve a vector $V_S$ satisfying both $V_S^2 = 0$ and $\text{div} V_S = 0$. Our experience with closed string field theory, where the path integral measure is not invariant under the gauge symmetry, suggests that this is requiring too much. We expect that Eqn.(5.4) is satisfied in the weakest form.

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APPENDIX A: Exterior calculus on a supermanifold

In this Appendix, we summarize the basic notions of exterior calculus on a supermanifold. In contrast to the case of ordinary manifolds we have two kinds of odd objects: the exterior derivative operator $d$, and the coordinates $z^I$ that are odd. We adopt the convention that these two commute with each other; the Grassmanality of $dz^I$ is the same as that of $z^I$: $\varepsilon(dz^I) = \varepsilon(z^I) = I$. The wedge product is defined by

$$dz^{I_1} \wedge dz^{I_2} \wedge \cdots \wedge dz^{I_N} = \sum_{\sigma \in S_N} \epsilon(\sigma) \eta(\{I\}, \sigma) dz^{I_{\sigma(1)}} \otimes dz^{I_{\sigma(2)}} \otimes \cdots \otimes dz^{I_{\sigma(N)}},$$

(A.1)

where $\epsilon(\sigma)$ is the signature of the permutation $\sigma$, and the extra sign factor $\eta(\{I\}, \sigma)$ is necessary for reordering $z^{I_1} \cdots z^{I_N}$ into $z^{I_{\sigma(1)}} \cdots z^{I_{\sigma(N)}}$. Therefore, we have

$$dz^I \wedge dz^J = (-)^{IJ} dz^J \wedge dz^I,$$

(A.2)

and

$$\alpha \wedge \beta = (-)^{mn+\varepsilon(\alpha)\varepsilon(\beta)} \beta \wedge \alpha,$$

(A.3)

for a general $m$-form $\alpha = \alpha(z) dz^{I_1} \wedge \cdots \wedge dz^{I_m}$ and a general $n$-form $\beta$. Here $\epsilon(\alpha) =$
\[ \epsilon(\alpha(z)) + \sum_m \epsilon(z^m) \]. The exterior derivative \( d \) acting on a function \( f(z) \) is given by

\[ df = dz^I \frac{\partial f}{\partial z^I} = f \frac{\partial}{\partial z^I} dz^I. \]  

(A.4)

The basis of vectors arising from a local coordinate system \( (z^I) \) is denoted by \( \left( \frac{\partial}{\partial z^I} \right)_I \). and a general vector field \( V \) is expanded as

\[ V(z) = \frac{\partial}{\partial z^I} V^I(z). \]  

(A.5)

The Grassmanality of \( \frac{\partial}{\partial z^I} \) is \( \varepsilon(\frac{\partial}{\partial z^I}) = I \). The reason why \( \frac{\partial}{\partial z^I} \) carries a left-arrow is that a vector field acts on a function \( f(z) \) as the right-derivative:

\[ V(f) \equiv f \frac{\partial}{\partial z^I} V^I. \]  

(A.6)

On the other hand, a one-form \( dz^I \) acts on a vector field as

\[ dz^I \left( \frac{\partial}{\partial z^J} \right) = \delta^I_J. \]  

(A.7)

More generally, for vector fields \( V_i = (\frac{\partial}{\partial z^I})V_i^I \) we have

\[ (dz^1 \otimes \cdots \otimes dz^N)(V_1, \cdots, V_N) = (-)^E dz^1(V_1) \cdots dz^N(V_N) = (-)^E V_1^{I_1} \cdots V_N^{I_N}, \]  

where \( E = \sum_{i=1}^{N-1} \varepsilon(V_i)\sum_{j=i+1}^{N} \varepsilon(I_j). \)

(A.8)

We now summarize a number of properties and operations on a general \( N \)-form \( \Omega \). First, we have

\[ \Omega(V_1, \cdots, V_i, V_{i+1}, \cdots, V_N) = -(-)^{V_iV_{i+1}}\Omega(V_1, \cdots, V_{i+1}, V_i, \cdots, V_N). \]  

(A.9)

Next we define the sign exponents \( R_i^{(N)} \) and \( L_{ij}^{(N)} \) which arise in reordering the vector fields \( V_i \) due to their Grassmanality by

\[ V_1 V_2 \cdots V_N = (-)^{R_i^{(N)}} V_1 \cdots V_i \cdots V_N, \]

(A.10)

\[ = (-)^{L_{ij}^{(N)}} V_i V_j \cdots V_i \cdots V_j \cdots V_N, \]

where the check on \( \check{V} \) implies the omission of that \( V \). Then the exterior derivative \( d \), the contraction operator \( i_V \), and the Lie-derivative \( \mathcal{L}_V \equiv di_V + dV \) are given as follows:

\[ d\Omega(V_1, V_2, \cdots, V_{N+1}) = \sum_{i=1}^{N+1} (-)^{i+1+R_i^{(N+1)}}\Omega(V_1, \cdots, V_i, \cdots, V_{N+1})V_i \]

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\[
\sum_{1 \leq i < j \leq N+1} (-)^{i+j+L_{ij}^{(N+1)}} \Omega([V_i, V_j], V_1, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_{N+1}) ,
\]  
\tag{A.11}

\[
(iW)\Omega(V_1, \ldots, V_{N-1}) = \Omega(W, V_1, \ldots, V_{N-1}) ,
\]  
\tag{A.12}

\[
(\mathcal{L}_W\Omega) (V_1, \ldots, V_N) = (-)^W \sum_{i=1}^{N} V_i \Omega(V_1, \ldots, V_N)W + \sum_{i=1}^{N} (-)^W \sum_{j=1}^{i-1} V_j \Omega(V_1, \ldots, [W, V_i], \ldots, V_N) .
\]  
\tag{A.13}

In Eqn.(A.11), \(V_i\) in the second term acts on the function \(\Omega(V)\) on the left. Another useful formula is
\[
[\mathcal{L}_V, iW] \equiv \mathcal{L}_V iW - (-)^{VW} iW \mathcal{L}_V = i[V,W] .
\]  
\tag{A.14}

**APPENDIX B: Properties of various quantities**

We denote by \(\varepsilon(\mathcal{O})\) the Grassmanality of the quantity \(\mathcal{O}\): \(\varepsilon(\mathcal{O}) = 0 \mod 2\) when \(\mathcal{O}\) is Grassmann even (odd). We also use the abbreviations \(\varepsilon(z^I) = I\) and \((-)^{\varepsilon(\mathcal{O})} = (-)^{\mathcal{O}}\).

1) The right and left-derivatives with respect to \(z^I\) acting on a general quantity \(A(z)\) are related by
\[
A \overset{\leftrightarrow}{\partial_I} = (-)^{I(A+1)} \partial_I A .
\]  
\tag{B.1}

2) \(\omega_{IJ}\) and \(\omega^{IJ}\):
\[
\varepsilon(\omega_{IJ}) = \varepsilon(\omega^{IJ}) = I + J + 1 ,
\]
\[
\omega_{IJ} = -(-)^{IJ} \omega_{JI} ,
\]
\[
\omega^{IJ} = -(-)^{(I+1)(J+1)} \omega^{JI} .
\]  
\tag{B.2}

3) Antibracket:
\[
\varepsilon(\{A, B\}) = A + B + 1 ,
\]
\[
\{A, B\} = -(-)^{(A+1)(B+1)} \{B, A\} ,
\]
\[
\{A, BC\} = \{A, B\} C + (-)^{(A+1)B} B \{A, C\} ,
\]
\[
\{AB, C\} = A \{B, C\} + (-)^B (C+1) \{A, C\} B .
\]  
\tag{B.3}

4) The Jacobi identity (2.5) in terms of \(\omega_{IJ}\) and \(\omega^{IJ}\):
\[
(-)^{IK} \partial_I \omega_{JK} + \text{cyclic}(I, J, K) = 0 ,
\]
\[
(-)^{(I+1)(K+1)} \omega^{IL} \partial_L \omega^{JK} + \text{cyclic}(I, J, K) = 0 .
\]  
\tag{B.4}
APPENDIX C: Finite transformations

Given a canonical diffeomorphism $g: \mathcal{M} \to \mathcal{M}$, one has $g^*\omega = \omega$ and

$$\{g^*A, g^*B\} = g^*\{A, B\}, \quad g_*V_{g^*A} = V_A,$$

and

$$\tilde{\omega}^{IJ}(z) = \omega^{IJ}(z), \quad (C.1)$$

where $(g^*A)(z) = A(g(z))$ for a 0-form $A(z)$, and $\tilde{\omega}^{IJ}$ is defined by Eqn.(2.20) with $\tilde{z}^I = g^I(z)$.

The invariance, $\mathcal{A}(S^g) = \mathcal{A}(S)$, is proven as follows. Defining $H(z) \equiv \exp S(z)$, for which the transformation (3.37) reads $H \to (g^*H)\cdot F_g$, we have

$$\mathcal{A}(S^g) = -\frac{1}{2} \int_{\mathcal{M}} d\mu \{ (g^*H)F_g, (g^*H)F_g \}$$

$$= \frac{1}{2} \int_{\mathcal{M}} d\mu \left( F_g^2 \{g^*H, g^*H\} + \frac{1}{2} \{ (g^*H)^2, F_g^2 \} + (g^*H)^2 \{F_g, F_g\} \right) \quad (C.3)$$

where in the last step we have used Eqns.(C.1) and (3.14). The first term in the last expression of Eqn.(C.3) is in fact equal to $\mathcal{A}(S)$

$$-\frac{1}{2} \int_{\mathcal{M}} d\mu F_g^2 g^*\{H, H\} = -\frac{1}{2} \int_{\mathcal{M}} g^*\{H, H\} = \mathcal{A}(S),$$

where use was made of Eqn.(3.34). Therefore, $\mathcal{A}(S^g) = \mathcal{A}(S)$ holds if $\Delta F_g^2 = \{F_g, F_g\} = 0$. Using Eqn.(2.12), we must show that

$$\Delta \rho F_g = 0, \quad (C.4)$$

where we have made explicit the $\rho$ dependence of $\Delta$. Since the delta operator is a scalar under coordinate transformations, and our transformations leave $\omega$ invariant, we have

$$\Delta \rho = \Delta_\tilde{\rho}_|z \to \tilde{z}, \quad (C.5)$$

for $\tilde{\rho}$ defined by Eqn.(2.21), and hence the nilpotency of $\Delta \rho$ implies the nilpotency of $\Delta \tilde{\rho}$. From this fact and the formula [6]

$$\Delta_{\tilde{\rho}}^2 A = \Delta_{\tilde{\rho}}^2 A + 2\{e^{-\sigma/2}\Delta_{\tilde{\rho}} e^{\sigma/2}, A \}, \quad (C.6)$$

where $\sigma = \ln (\tilde{\rho}/\rho)$, it follows that

$$\Delta_{\tilde{\rho}} e^{\sigma/2} = 0. \quad (C.7)$$

The last equation also implies that

$$\Delta_{\tilde{\rho}} e^{-\sigma/2} = 0. \quad (C.8)$$

Eqn.(C.4) is obtained from Eqn.(C.8) by making the replacement $z \to \tilde{z} = g(z)$, using
Eqn.(C.5) and the fact that $F_g(z) = \exp(-\sigma(g(z))/2)$. This finishes the proof of the invariance of $\mathcal{A}(S)$ under the transformation (3.37).

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FIGURE CAPTIONS

1) We show here a family of sections $\Gamma_{g,n}(u)$ that give rise to cell decompositions of moduli space. The base space is $\mathcal{M}_{g,n}$ and the total space is $\hat{\mathcal{P}}_{g,n}$. We show the sections at $u_0$ and at $u_0 + du$ and we indicate the subsets $\mathcal{V}_{g,n}(u_0)$ and $\mathcal{V}_{g,n}(u_0 + du)$. The parametrization by $u$ of the family of sections induces a parametrization by $u$ on the fibers. (a) The vector $\hat{V}$ is the vertical vector along the fibers. (b) The vector $\hat{U}$ is a vector field that generates a diffeomorphism mapping the subspaces $\mathcal{V}_{g,n}$ into each other.

2) (a) On the space $\hat{\mathcal{P}}_{g,n}$ over $\mathcal{M}_{g,n}$ we show a subspace $\mathcal{V}_{g,n}$ which is a section over $\mathcal{M}_{g,n}$, and a subspace $\mathcal{V}'_{g,n}$ which is not a section because the vertical fibers intersect it more than once. The two subspaces coincide at their boundaries and together they bound a region $R$. (b) This time we show two sections, one at $u_0$ and the other at $u_0 + du$. The subspace $\mathcal{V}_{g,n}(u_0)$ extends from $A$ to $B$ and the subspace $\mathcal{V}_{g,n}(u_0 + du)$ extends from $C$ to $D$. We denote by $S$ the ‘strip’ joining the two subspaces from their boundaries. The strip plus the subspace at $u_0 + du$ form a subspace which is not a section over $\mathcal{M}_{g,n}$. 