Second-Order SAT Solving using Program Synthesis

Daniel Kroening and Matt Lewis
University of Oxford

Abstract. Program synthesis is the automated construction of software from a specification. While program synthesis is undecidable in general, we show that synthesising finite-state programs is NEXPTIME-complete. We then present a fully automatic, sound and complete algorithm for synthesising C programs from a specification written in C. Our approach uses a combination of bounded model checking, explicit-state model checking and genetic programming to achieve surprisingly good performance for a problem with such high complexity. By identifying a correspondence between program synthesis and second-order logic, we show how to use our program synthesiser as a decision procedure for existential second-order logic over finite domains. We illustrate the expressiveness of this logic by encoding several program analysis problems including superoptimisation, de-obfuscation, safety and termination. Finally, we present experimental results showing that our approach is tractable in practice.

Keywords: Program synthesis, second-order logic, bitvectors.

1 Introduction

Program synthesis is the mechanised construction of software that provably satisfies a given specification. Synthesis tools promise to relieve the programmer from thinking about how the problem is to be solved; instead, the programmer only provides a compact description of what is to be achieved. Foundational research in this area has been exceptionally fruitful, beginning with Alonzo Church’s work on the Circuit Synthesis Problem in the sixties [1]. Algorithmic approaches to the problem have frequently been connected to automated theorem proving [2,3]. Recent developments include an application of Craig interpolation to synthesis [4].

We will show that program synthesis in general is undecidable and in fact harder than the halting problem for Turing machines. Decidability can be recovered by restricting the class of programs that are synthesised; in particular, we show that if we only consider programs with finite state spaces our problem becomes NEXPTIME-complete. As part of this complexity analysis, we observe a correspondence between program synthesis and existential second-order logic.

Second-order logic allows quantification over sets and functions, as well as ground terms. This expressive power makes it easy to encode many programs
analysis problems, for example termination \cite{5}, safety \cite{6,7}, and superoptimization \cite{8,9}. If we restrict our language to allow only existential second-order quantification and require a finite universe over which ground terms are interpreted, we obtain the second-order SAT problem, which we formally define in section 2. After showing how many problems, including all those listed above, can be concisely and naturally encoded as second-order SAT, we show how a finite state program synthesiser can be used as a second-order SAT solver.

This complexity result provides a direct bridge between logic and synthesis, allowing existing program synthesisers \cite{8,10} to be immediately applied to a wide range of problems, e.g. those stemming from existing work on solving second-order logic constraints for program analysis \cite{11,12,7}. This connection is relevant as recent work on syntax guided synthesis \cite{13} promises to greatly raise the profile of program synthesis and usher in a generation of new synthesis tools.

Having developed the theory of second-order SAT, we extend existing approaches to program synthesis to build a fully automatic, sound and complete algorithm for synthesising loop-free C programs. This choice of formalism (loop-free C programs) makes it particularly easy to encode second-order SAT constraints arising from program analysis problems. The resulting synthesiser uses a novel combination of bounded model checking, explicit state model checking and genetic programming. We then prove that our algorithm is optimal in the following senses:

- Encoding programs as loop-free C programs is asymptotically optimal in size.
- If a specification is satisfiable, our algorithm produces the smallest correct program.
- If a specification is satisfiable, the runtime of our algorithm is predominantly a function of the size of the program that is synthesised.

Our solver has already been used in a wide range of applications including: superoptimization, generating safety invariants, bug finding, termination and non-termination proving, code refactoring, code deobfuscation and generating abstract transformers.

**Technical Contributions**

- We show that synthesising finite state programs is NEXPTIME-complete.
- We define the second-order SAT problem and show that it can be solved using a finite state program synthesiser.
- We describe a fully automatic, sound and complete algorithm for synthesising finite state programs.
- We outline how several program analysis problems can be encoded directly as second-order SAT problems.
- We have implemented our algorithm and evaluated it on several benchmarks.
- Our implementation and all of our experimental results are freely available to download. The implementation is robust and has already been used as the backend for other tools \cite{5}.


2 Preliminaries

In this section we will recall some well known decision problems along with their associated complexity classes. We will then define an extension of propositional SAT that we will call second-order SAT. The section concludes with a proof that second-order SAT is NEXPTIME-complete.

Definition 1 (Propositional SAT).
\[ \exists x_1 \ldots x_n. \sigma \]

Where the \( x_i \) range over Boolean values and \( \sigma \) is a quantifier-free propositional formula whose free variables are the \( x_i \).

Checking the truth of an instance of Definition 1 is NP-complete.

Definition 2 (First-Order Propositional SAT or QBF).
\[ Q_1 x_1. Q_2 x_2 \ldots. Q_n x_n. \sigma \]

Where the \( Q_i \) are either \( \exists \) or \( \forall \). The \( x_i \) and \( \sigma \) are as in Definition 1.

Checking the truth of an instance of Definition 2 is PSPACE-complete.

Now we turn our attention to second-order logic. Second-order logic allows quantification over sets as well as objects.

Definition 3 (Second-Order SAT).
\[ \exists S_1 \ldots S_m. Q_1 x_1 \ldots Q_n x_n. \sigma \]

Where the \( S_i \) range over predicates. Each \( S_i \) has an associated arity \( \text{ar}(S_i) \) and \( S_i \subseteq \mathbb{B}^{\text{ar}(S_i)} \). The remainder of the formula is an instance of Definition 2 except that the quantifier-free part (\( \sigma \)) may refer to both the first-order variables \( x_i \) and the second-order variables \( S_i \).

Example 1. The following is a second-order SAT formula:
\[ \exists S. \forall x_1, x_2. S(x_1, x_2) \rightarrow S(x_2, x_1) \]

This formula is satisfiable and is satisfied by any symmetric relation.

Theorem 1 (Fagin’s Theorem). The class of structures \( A \) recognisable in time \( \|A\|^k \), for some \( k \), by a nondeterministic Turing machine is exactly the class of structures definable by existential second-order sentences.

Theorem 2 (Second-Order SAT is NEXPTIME-complete). For an instance of Definition 3 with \( n \) first-order variables, checking the truth of the formula is NEXPTIME-complete.
Proof. We will apply Theorem 1. To do so we must establish the size of the universe implied by Theorem 1. Since Definition 3 uses $n$ Boolean variables, the universe is the set of interpretations of $n$ Boolean variables. This set has size $2^n$, and so by Theorem 1 Definition 3 defines exactly the class sets recognisable in $(2^n)^k$ time by a nondeterministic Turing machine. This is the class $\text{NEXPTIME}$, and so checking validity of an arbitrary instance of Definition 3 is $\text{NEXPTIME}$-complete.

For an alternative proof, consider a Turing machine $M$. For a particular run of $M$ we can construct a relation $f(k, q, h, j, t)$ defined such that after $k$ steps $M$ is in state $q$, with its head at position $h$ and tape cell $j$ containing the symbol $t$. If $M$ halts within $2^n$ steps on an input of length $n$, the values of all the variables in this relation are bounded by $2^n$, which means they can be written down using $n$ bits. The details of creating a first-order formula constraining $f$ to reflect the behaviour of $M$ are left to the reader.

3 Decidability and Complexity of Program Synthesis

The program synthesis problem can be informally described as follows: given a specification, find a program which satisfies that specification for all inputs. In order to define this problem formally, we need to identify what a specification is, what the program we are synthesising is, and what an input is.

3.1 General Program Synthesis

For the general case of program synthesis, we will say that:

- The program we wish to synthesise is a Turing machine computing a total function which takes as input a natural number and produces another natural number.
- A specification is a computable function $\sigma(P, x)$ whose arguments are $P$ – the index of some Turing machine, and $x$ a natural number.

The synthesis problem is then that of finding some program $P$ such that the synthesis formula of Definition 4 is true, where we introduce a function $H(P, x)$ which returns true iff the Turing machine with index $P$ halts on input $x$.

**Definition 4 (Synthesis Formula).**

$$\forall x \in \mathbb{N}. \sigma(P, x) \land H(P, x)$$

The decision problem associated with program synthesis is to determine whether such a $P$ exists for a given $\sigma$. We can see immediately that this decision problem is equivalent to the halting problem for an oracle machine with access to the oracle $H$. It is a well known result that an oracle machine cannot solve its own halting problem, and so the program synthesis problem is undecidable even if we have access to an oracle solving the halting problem. In other
words, the general synthesis problem has Turing degree $0''$, making it strictly harder than the halting problem for Turing machines.

In order to recover decidability, we can restrict our scope and consider a finite version of the synthesis problem in which the programs we wish to synthesize have a finite state space $S$. Since we require our programs to halt on all inputs, we can identify each program $P$ with the total function $f : S \rightarrow S$ it computes.

### 3.2 Program Encodings

Now we turn to the problem of how to encode such finite-state programs. For the remainder of this paper, we will encode finite-state programs as loop-free imperative programs consisting of a sequence of instructions, each instruction consisting of an opcode and a tuple of operands. The opcode specifies which operation is to be performed and the operands are the arguments on which the operation will be performed. We allow an operand to be one of: a constant literal, an input to the program, or the result of some previous instruction. Such a program has a natural correspondence with a combinational circuit.

A sequence of instructions is certainly a natural encoding of a program, but we might wonder if it is the best encoding. We can show that for a reasonable set of instruction types (i.e. valid opcodes), this encoding is optimal in a sense we will now discuss. An encoding scheme $E$ takes a function $f$ and assigns it a name $s$. An encoding scheme $E$ is strictly better than another scheme $E'$ if for each $f$, $E$ assigns a shorter name than $E'$. If there is no $E'$ strictly better than $E$, we can say that $E$ is optimal – for any $E'$ that encodes some function $f$ better than $E$ does, there is a $g$ which $E$ encodes better than $E'$.

**Lemma 1.** For an imperative programming language including instructions for testing equality of two values (EQ) and an if-then-else (ITE) instruction, any total function $f : S \rightarrow S$ can be computed by a program of size $O(\|S\| \log \|S\|)$ bits.

**Proof.** The function $f$ is computed by the following program:

\[
\begin{align*}
t_1 &= \text{EQ}(x, 1) \\
t_2 &= \text{ITE}(t_1, f(1), f(0)) \\
t_3 &= \text{EQ}(x, 2) \\
t_4 &= \text{ITE}(t_3, f(2), t_2) \\
&\vdots
\end{align*}
\]

Each operand can be encoded in $\log_2(\|S\| + l) = \log_2(3 \times \|S\|)$ bits. So each instruction can be encoded in $O(\log \|S\|)$ bits and there are $O(\|S\|)$ instructions in the program, so the whole program can be encoded in $O(\|S\| \log \|S\|)$ bits.

**Lemma 2.** Any representation that is capable of encoding an arbitrary total function $f : S \rightarrow S$ must require at least $O(\|S\| \log \|S\|)$ bits to encode some functions.
Proof. There are \( \|S\|\|S\| \) total functions \( f : S \to S \). Therefore by the pigeonhole principle, any encoding that can encode an arbitrary function must use at least \( \log_2(\|S\|\|S\|) = \|S\| \log_2 \|S\| \) bits.

From Lemma 1 and Lemma 2, we can conclude that any set of instruction types include ITE is an optimal function encoding.

### 3.3 Finite Program Synthesis

To formally define the finite synthesis problem, we will require that the inputs \( x \) are drawn from some finite domain \( D \) and that \( P \) and \( \sigma \) are loop-free imperative programs. We will allow \( \sigma \) to make use of a CALL instruction with which it can call \( P \) as a subroutine. We then define the finite-state synthesis decision problem as checking the truth of Definition 5.

**Definition 5 (Finite Synthesis Formula).**

\[
\exists P. \forall x \in D, \sigma(x)
\]

We will now show that each instance of Definition 3 can be reduced in polynomial time to an instance of Definition 5.

**Theorem 3 (Second-Order SAT is Polynomial Time Reducible to Finite Synthesis).** Every instance of Definition 3 is polynomial time reducible to an instance of Definition 5.

**Proof.** We first Skolemise the instance of definition 3 to produce an equisatisfiable second-order sentence with the first-order part only having universal quantifiers (i.e. bring the formula into Skolem normal form). This process will have introduced a function symbol for each first order existentially quantified variable and taken linear time. Now we just existentially quantify over the Skolem functions, which again takes linear time and space. The resulting formula is an instance of Definition 5.

**Corollary 1.** Finite-state program synthesis is NEXPTIME-complete.

### 4 Synthesising Finite-State Programs

In this section we will present a sound and complete algorithm for the finite-state synthesis decision problem, as well as details of our implementation. In the case that a specification is satisfiable, our algorithm produces a minimal satisfying program. We begin by describing a general purpose synthesis procedure, then elaborate on the details involved when this procedure is instantiated for finite-state synthesis.
Algorithm 1 Abstract refinement algorithm

1: function SYNTH(inputs)
2: (i₁, . . . , iₘ) ← inputs
3: query ← ∃P.σ(i₁, P)∧ . . . ∧σ(iₘ, P)
4: result ← decide(query)
5: if result.satisfiable then
6:   return result.model
7: else
8:   return unsatisfiable
9: function VERIF(P)
10: query ← ∃x.¬σ(x, P)
11: result ← decide(query)
12: if result.satisfiable then
13:   return result.model
14: else
15:   return valid
16: function REFINEMENT LOOP
17: inputs ← ∅
18: loop
19: candidate ← SYNTH(inputs)
20: if candidate = UNSAT then
21:   return unsatisfiable
22: res ← VERIF(candidate)
23: if res = valid then
24:   return candidate
25: else
26: inputs ← inputs ∪ res

Fig. 1. Abstract synthesis refinement loop
4.1 CEGIS

We use Counterexample Guided Inductive Synthesis (CEGIS) [15,10,16] to find a program satisfying our specification. The core of the CEGIS algorithm is the refinement loop shown in Figure 1 and detailed in Algorithm 1.

The algorithm is divided into two procedures: synth (see Figure 3) and verif, which interact via a finite set of test vectors inputs.

The synth procedure tries to find an existential witness $P$ that satisfies the partial specification:

$$\exists P. \forall x \in \text{inputs}. \sigma(x, P)$$

If synth succeeds in finding a witness $P$, this witness is a candidate solution to the full synthesis formula. We pass this candidate solution to verif which determines whether it does satisfy the specification on all inputs by checking satisfiability of the verification formula:

$$\exists x. \neg \sigma(x, P)$$

If this formula is unsatisfiable, the candidate solution is in fact a solution to the synthesis formula and so the algorithm terminates. Otherwise, the witness $x$ is an input on which the candidate solution fails to meet the specification. This witness $x$ is added to the inputs set and the loop iterates again. It is worth noting that each iteration of the loop adds a new input to the set of inputs being used for synthesis. If the full set of inputs $X$ is finite, this means that the refinement loop can only iterate a finite number of times.

4.2 Finite-State Synthesis

We will now show how the generic construction of Section 4.1 can be instantiated to produce a useful finite-state program synthesiser. A natural choice for such a synthesiser would be to work in the logic of quantifier-free propositional formulae and to use a propositional SAT or SMT-BV solver as the decision procedure. However we propose a slightly different tack, which is to use a decidable fragment of C as a “high level” logic.

4.3 $C^-$

We will now describe the logic we use to express our synthesis formula. The logic is a subset of C that we call $C^-$. The characteristic property of a $C^-$ program is that safety can be decided for it using a single query to a Bounded Model Checker. A $C^-$ program is just a C program with the following syntactic restrictions: all loops in the program must have a constant bound; all recursion in the program must be limited to a constant depth; all arrays must be statically allocated (i.e. not using malloc), and be of constant size. Additionally, $C^-$ programs may use nondeterministic values, assumptions and arbitrary-width types.

Since each loop is bounded by a constant, and each recursive function call is limited to a constant depth, a $C^-$ program necessarily terminates and in fact
does so in $O(1)$ time. If we call the largest loop bound $k$, then a Bounded Model Checker with an unrolling bound of $k$ will be a complete decision procedure for the safety of the program. For a $C^-$ program of size $l$ and with largest loop bound $k$, a Bounded Model Checker will create a SAT problem of size $O(lk)$. Conversely, a SAT problem of size $s$ can be converted trivially into a loop-free $C^-$ program of size $O(s)$. The safety problem for $C^-$ is therefore NP-complete, which means it can be decided fairly efficiently for many practical instances.

4.4 Encoding into $C^-$

To instantiate the abstract synthesis algorithm in $C^-$ we must express $X, Y, \sigma$ and $P$ in $C^-$, then ensure that we can express the validity of the synthesis formula as a safety property of the resulting $C^-$ program.

Our encoding is the following:

- $X$ is the set of $N$-tuples of 32-bit bitvectors. This is written in $C^-$ as the type `int[N]`.
- $Y$ is the set of $M$-tuples of 32-bit bitvectors, which is written in $C^-$ as the type `int[M]`.
- $\sigma$ is a pure function with type $X \times Y \rightarrow \text{Bool}$. The $C^-$ signature of this function is `int check(int in[N], int out[M])`. This function is the only component supplied by the user.
- $P$ is written in a simple RISC-like language $L$, whose syntax is given in Fig. 2. Programs in $L$ have type $X \rightarrow Y$ and are represented in $C^-$ as objects of type `prog_t`, shown in Fig. 5.
- We supply an interpreter for $L$ which is written in $C^-$. The type of this interpreter is $(X \rightarrow Y) \times X \rightarrow Y$ and the $C^-$ signature is `void exec(prog_t p, int in[N], int out[M])`. Here, `out` is an output parameter.

The exact details of how we encode an $L$-program are given in Sec. 4.7. We must now express the SYNTH and VERIF formulae as safety properties of $C^-$ programs, which is given in Fig. 3.

In order to determine the validity of the SYNTH formula, we can check the SYNTH program for safety. The SYNTH program is a $C^-$ program, which means we can check its safety with Bounded Model Checking (BMC) as implemented in the CBMC tool. There are alternative approaches we can use to check the safety of SYNTH.c, each of which boils down to searching for a candidate assignment to $p$ that makes the assertion in SYNTH.c fail.

4.5 Candidate Generation Strategies

Explicit Proof Search. The simplest strategy for finding candidates is to just exhaustively enumerate them all, starting with the shortest and progressively increasing the number of instructions. This strategy is implemented by the EXPLICITSEARCH routine. Since the set of $L$-programs is recursively enumerable,
Integer arithmetic instructions:
add a b  sub a b  mul a b  div a b
neg a  mod a b  min a b  max a b

Bitwise logical and shift instructions:
and a b  or a b  xor a b
lshr a b  ashr a b  not a

Unsigned and signed comparison instructions:
le a b  lt a b  sle a b
slt a b  eq a b  neq a b

Miscellaneous logical instructions:
implies a b  ite a b c

Floating-point arithmetic:
fadd a b  fsub a b  fmul a b  fdiv a b

Fig. 2. The language $\mathcal{L}$

```c
void synth() {
    prog_t p = nondet();
    int in[N], out[M];
    assume(wellformed(p));
    in = test1;
    exec(p, in, out);
    assume(check(in, out));
    ...
    in = testN;
    exec(p, in, out);
    assume(check(in, out));
    assert(false);
}

void verif(prog_t p) {
    int in[N] = nondet();
    int out[M];
    exec(p, in, out);
    assert(check(in, out));
}
```

Fig. 3. The SYNTH and VERIF formulae expressed as a C− program.
this procedure is complete.

**Symbolic Bounded Model Checking.** Another complete method for generating candidates is to simply use BMC on the `synth.c` program. As with explicit search, we must progressively increase the length of the $L$-program we search for in order to get a complete search procedure.

**Genetic Programming and Incremental Evolution.** Our final strategy is genetic programming (GP) [17,18]. GP provides an adaptive way of searching through the space of $L$-programs for an individual that is “fit” in some sense. We measure the fitness of an individual by counting the number of tests in inputs for which it satisfies the specification.

To bootstrap GP in the first iteration of the CEGIS loop, we generate a population of random $L$-programs. We then iteratively evolve this population by applying the genetic operators crossover and mutate. CROSSOVER combines selected existing programs into new programs, whereas MUTATE randomly changes parts of a single program. Fitter programs are more likely to be selected.

Rather than generating a random population at the beginning of each subsequent iteration of the CEGIS loop, we start with the population we had at the end of the previous iteration. The intuition here is that this population contained many individuals that performed well on the $k$ inputs we had before, so they will probably continue to perform well on the $k + 1$ inputs we have now. In the parlance of evolutionary programming, this is known as incremental evolution [19].

### 4.6 Optimisations

Two optimisations we have found to be very important to the performance of our synthesiser are the following:

**Cache binaries** We ensure that we do not run gcc more times than necessary, since we have observed compilation time to be relatively expensive. This means that for the phases using native code (explicit-state model checking and the stochastic methods), we compiled the specification once and then just execute the resulting binary in each iteration of the synth and verif phases.

**Emit C code when possible** In the verif stage, we can emit the struct-base representation of an $L$-program along with the code for the interpreter and check the resulting program. Alternatively, since an $L$-program can be trivially translated to a straight-line C program, we can emit the program as C instead. This results in a much smaller program that is more amenable to optimisation by the compiler and CBMC.

### 4.7 Encoding an $L$-Program in C$	extsuperscript{\sim}$

The exact C$	extsuperscript{\sim}$ encoding of an $L$ program is shown in Fig. The `prog_t` structure encodes a program, which is a sequence of instructions. The parameter $a$ is
typedef BV(4) opt;  // An opcode
typedef BV(w) word_t;  // An L-word
typedef BV(\log_2(c+l+a)) param_t;  // An operand

struct prog_t {
  opt ops[l];  // The opcodes
  param_t params[l*2];  // The operands
  word_t consts[c];  // The program constants
};

Fig. 4. Schematic diagram of SYNTH

Fig. 5. The C structure we use to encode an L program

The number of arguments the program takes. The ith instruction has opcode \texttt{ops[i]}, left operand \texttt{params[i*2]} and right operand \texttt{params[i*2 + 1]}. An operand refers to either a program constant, a program argument or the result of a previous instruction, and its value is determined at runtime as follows:

\[
val(x) = \begin{cases} 
  x < a & \text{the } x\text{th program argument} \\
  a \leq x < a + c & \text{const}[x-a] \\
  x \geq a + c & \text{the result of the } (x-a-c)^{th} \text{ instruction}
\end{cases}
\]

A program is well formed if no operand refers to the result of an instruction that has not been computed yet, and if each opcode is valid. We add a well-formedness constraint of the form \texttt{params[i] \leq (a+c+2*i)} for each instruction. It should be noted that this requires a linear number of well-formedness constraints. If all of these constraints are satisfied, the program is well-formed in the sense.

4.8 Parameterising the Program Space

In order to search the space of candidate programs, we parametrise the language L, inducing a lattice of progressively more expressive languages. We start by attempting to synthesise a program at the lowest point on this lattice and increase the parameters of L until we reach a point at which the synthesis succeeds.
As well as giving us an automatic search procedure, this parametrisation greatly increases the efficiency of our system since languages low down the lattice are very easy to decide safety for. If a program can be synthesised in a low-complexity language, the whole procedure finishes much faster than if synthesis had been attempted in a high-complexity language.

Program Length: \( l \) The first parameter we introduce is program length, denoted by \( l \). At each iteration we synthesise programs of length exactly \( l \). We start with \( l = 1 \) and increment \( l \) whenever we determine that no program of length \( l \) can satisfy the specification. When we do successfully synthesise a program, we are guaranteed that it is of minimal length since we have previously established that no shorter program is correct.

Word Width: \( w \) An \( L \)-program runs on a virtual machine (the \( L \)-machine) that has its own set of parameters. The only relevant parameter is the word width of the \( L \)-machine, that is, the number of bits in each internal register and immediate constant. This parameter is denoted by \( w \). The size of the final SAT problem generated by CBMC scales polynomially with \( w \), since each intermediate C variable corresponds to \( w \) propositional variables.

It is often the case that a program which satisfies the specification on an \( L \)-machine with \( w = k \) will continue to satisfy the specification when run on a machine with \( w > k \). For example, the program in Fig. 6 isolates the least-significant bit of a word. This is true irrespective of the word size of the machine it is run on – it will isolate the least-significant bit of an 8-bit word just as well as it will a 32-bit word. An often successful strategy is to synthesise a program for an \( L \)-machine with a small word size and then to check whether the same program is correct when run on an \( L \)-machine with a full-sized word.

The only wrinkle here is that we will sometimes synthesise a program containing constants. If we have synthesised a program with \( w = k \), the constants in the program will be \( k \)-bits wide. To extend the program to an \( n \)-bit machine (with \( n > k \)), we need some way of deriving \( n \)-bit-wide numbers from \( k \)-bit ones. We have several strategies for this and just try each in turn. Our strategies are shown in Fig. 7. \( BV(v, n) \) denotes an \( n \)-bit wide bitvector holding the value \( v \) and \( b \cdot c \) means the concatenation of bitvectors \( b \) and \( c \).

\[
\text{int isolate_lsb (int x) \{} \\
\quad \text{return } x \& -x; \\
\}\]

Example:

\[
\begin{array}{c}
x \quad = 10111010 \\
-x \quad = 01000110 \\
x \& -x \quad = 00000010
\end{array}
\]

Fig. 6. A tricky bitvector program

Sometimes a program will be correct for some particular word width \( w \), but is not correct for \( w' > w \) even if the constants are replaced with appropriate ones. When we detect this situation, we increase \( w \) and continue synthesising.
Fig. 7. Rules for extending an $m$-bit wide number to an $n$-bit wide one.

Number of Constants: $c$ Instructions in $\mathcal{L}$ take either one or two operands. Since any instruction whose operands are all constants can always be eliminated (since its result is a constant), we know that a loop-free program of minimal length will not contain any instructions with two constant operands. Therefore the number of constants that can appear in a minimal program of length $l$ is at most $l$. By minimising the number of constants appearing in a program, we are able to use a particularly efficient program encoding that speeds up the synthesis procedure substantially. The number of constants used in a program is the parameter $c$.

$\mathcal{L}$ is an SSA, three-address instruction set. Destination registers are implicit and a fresh register exists for each instruction to write its output to. A naïve way to encode $\mathcal{L}$ instructions is to have an opcode and two operands, where each operand can either be a register (i.e., a program argument or the result of a previous instruction), or an immediate constant.

In this encoding, each opcode requires $\lceil \log_2 I \rceil$ bits to encode, where $I$ is the number of instruction types in $\mathcal{L}$. Each operand can be encoded using $\log_2 w$ bits, where $w$ is the $\mathcal{L}$-machine word width, plus one bit to specify whether the operand is a register name or an immediate constant. One instruction can therefore be encoded using $\lceil \log_2 I \rceil + 2w + 2$ bits. For an $n$-instruction program, we need

$$\lceil n \log_2 I \rceil + 2nw + 2n$$

bits to encode the entire program.

If we instead limit the number of constants that can appear in the program, our operands can be encoded using fewer bits. For an $n$-instruction program using $c$ constants and taking $a$ arguments as inputs, each operand can refer to a program argument, the result of a previous instruction or a constant. This can be encoded using $\lceil \log_2 (c + a + n - 1) \rceil$ bits, which means each instruction can be encoded in $\lceil \log_2 I \rceil + \lceil \log_2 (c + a + n - 1) \rceil$ and the full program needs

$$\lceil n \log_2 I \rceil + \lceil n \log_2 (c + a + n - 1) \rceil + cw$$

bits to encode.

We give an example. Our language $\mathcal{L}$ has 15 instruction types, so each opcode is 4 bits. For a 10-instruction program over 1 argument, using 2 constants on

---

1 We experimented with implementing $\mathcal{L}$ as a stack machine, expecting the programs to be smaller and synthesis to be faster as a result. We saw the opposite effect – the more complex interpreter led to much slower synthesis.
a 32-bit word machine the first encoding requires $10 \times (4 + 32 + 1 + 32 + 1) = 700$ bits. Using the second encoding, each operand can be represented using $\log_2(2 + 1 + 10 - 1) = 4$ bits, and the entire program requires 184 bits. This is a substantial reduction in size and when the desired program requires only few constants this can lead to a very significant speed up.

As with program length, we progressively increase the number of constants in our program. We start by trying to synthesise a program with no constants, then if that fails we attempt to synthesise using one constant and so on until we reach $c = l$.

4.9 Searching the Program Space

The key to our automation approach is to come up with a sensible way in which to adjust the $L$-parameters in order to cover all possible programs. After each round of `synth`, we may need to adjust the parameters. The logic for these adjustments is shown as a tree in Fig. 8.

Whenever `synth` fails, we consider which parameter might have caused the failure. There are two possibilities: either the program length $l$ was too small, or the number of allowed constants $c$ was. If $c < l$, we just increment $c$ and try another round of synthesis, but allowing ourselves an extra program constant. If $c = l$, there is no point in increasing $c$ any further. This is because no minimal $L$-program has $c > l$, for if it did there would have to be at least one instruction with two constant operands. This instruction could be removed (at the expense of adding its result as a constant), contradicting the assumed minimality of the program. So if $c = l$, we set $c$ to 0 and increment $l$, before attempting synthesis again.

If `synth` succeeds but `verif` fails, we have a candidate program that is correct for some inputs but incorrect on at least one input. However, it may be the case that the candidate program is correct for all inputs when run on an $L$-machine with a small word size. For example, we may have synthesised a program which is correct for all 8-bit inputs, but incorrect for some 32-bit input. If this is the case (which we can determine by running the candidate program through `verif` using the smaller word size), we may be able to produce a correct program for the full $L$-machine by using the constant extension rules shown in Fig. 7. If constant generalization is able to find a correct program, we are done. Otherwise, we need to increase the word width of the $L$-machine we are currently synthesising for.

4.10 Soundness and Complexity

We will now show that our synthesis algorithm is sound and semi-complete, then will go on to discuss its complexity in terms of the size of the computed solution.

**Theorem 4.** Algorithm 3 is sound – if it terminates with witness $P$, then $P \models \sigma$. 
Proof. The procedure \texttt{synth} terminates only if \texttt{synth} returns “valid”. In that case, \(\exists x. \neg \sigma(P, x)\) is unsatisfiable and so \(\forall x. \sigma(P, x)\) holds.

Theorem 5. If the existential first-order theory used to express the specification \(\sigma\) is decidable and the domain of inputs \(X\) is finite, Algorithm 1 is semi-complete – if a solution \(P \models \sigma\) exists then Algorithm 1 will terminate. However, if no program satisfies the specification, the algorithm may not terminate.

Proof. If the domain \(X\) is finite then the loop in procedure \texttt{synth} can only iterate \(\|X\|\) times, since by this time all of the elements of \(X\) would have been added to the inputs set. Therefore if the \texttt{synth} procedure always terminates, Algorithm 1 does as well.

Since the \texttt{ExplicitSearch} routine enumerates all programs (as can be seen by induction on the program length \(l\)), it will eventually enumerate a program that meets the specification on whatever set of inputs are currently being tracked, since by assumption such a program exists. Since the first-order theory is decidable, the query in \texttt{verif} will succeed for this program, causing the algorithm to terminate. The set of correct programs is therefore recursively enumerable and Algorithm 1 enumerates this set, so it is semi-complete.

Corollary 2. Since safety of \(C^-\) programs is decidable, Algorithm 1 is semi-complete when instantiated with \(C^-\) as a background theory.

We will now show that the number of iterations of the CEGIS loop is a function of the Kolmogorov complexity of the synthesised program. We argue that this gives our procedure various desirable qualities in practical applications. We first recall the definition of the Kolmogorov complexity of a function \(f\):
Definition 6 (Kolmogorov complexity). The Kolmogorov complexity $K(f)$ is the length of the shortest program that computes $f$.

We can extend this definition slightly to talk about the Kolmogorov complexity of a synthesis problem in terms of its specification:

Definition 7 (Kolmogorov complexity of a synthesis problem). The Kolmogorov complexity of a program specification $K(\sigma)$ is the length of the shortest program $P$ such that $P \models \sigma$.

Let us consider the number of iterations of the CEGIS loop $n$ required for a specification $\sigma$. Since we enumerate candidate programs in order of length, we are always synthesising programs with length no greater than $K(\sigma)$ (since when we enumerate the first correct program, we will terminate). So the space of solutions we search over is the space of functions computed by $L$-programs of length no greater than $K(\sigma)$. Let’s denote this set $\mathcal{L}(K(\sigma))$. Since there are $O(2^{K(\sigma)})$ programs of length $K(\sigma)$ and some functions will be computed by more than one program, we have $\|\mathcal{L}(K(\sigma))\| \leq O(2^{K(\sigma)})$.

Each iteration of the CEGIS loop distinguishes at least one incorrect function from the set of correct functions, so the loop will iterate no more than $\|\mathcal{L}(K(\sigma))\|$ times. Therefore another bound on our runtime is:

$$O\left(2^{K(\sigma)} \times NP\left(\frac{K(\sigma)}{\|L\|}\right)\right) = O\left(2^{K(\sigma)}\right)$$

5 Experiments

5.1 Experimental Setup

We implemented our fully automatic synthesis procedure in the Kalashnikov tool. The specification language of Kalashnikov is C−, which is rich enough to encode arbitrary second-order SAT formulae. To evaluate the viability of second-order SAT, we used Kalashnikov to solve formulae generated from a variety of problems. Our benchmarks come from superoptimisation, code deobfuscation, floating point verification, ranking function and recurrent set synthesis, and QBF solving. The superoptimisation and code deobfuscation benchmarks were taken from the experiments of [8]; the termination benchmarks were taken from SV-COMP’15 [20] and the experiments of [5]; the QBF instances consist of some simple instances created by us and some harder instances taken from [21]. Details of how the termination proofs were encoded as second-order SAT formulae can be found in [5].

We ran our experiments on a 4-core, 3.30 GHz Core i5 with 8 GB of RAM. Each benchmark was run with a timeout of 180s. For each category of benchmarks, we report the total number of benchmarks in that category, the number we were able to solve within the time limit, the average specification size (in lines of code), the average solution size (in instructions), the average number of
iterations of the CEGIS loop, the average time and total time taken. The results are shown in Table 1. It should be understood that in contrast to less expressive logics that might be invoked several times in the analysis of some problem, each of these benchmarks is a “complete” problem from the given problem domain. For example, each of the benchmarks in the termination category requires KALASHNIKOV to prove that a full program terminates, i.e. it must find a ranking function and supporting invariants, then prove that these constitute a valid termination proof for the program being analysed.

The timings show that for the instances where we can find a satisfying assignment, we tend to do so quite quickly (on the order of a few seconds). Furthermore the programs we synthesise are often short, even when the problem domain is very complex, such as for termination or QBF.

Not all of these benchmarks are satisfiable, and in particular around half of the termination benchmarks correspond to attempted proofs that non-terminating programs terminate and vice versa. This illustrates one of the current shortcomings of second-order SAT as a decision procedure: we can only conclude that a formula is unsatisfiable once we have examined candidate solutions up to a very high length bound. Being able to detect unsatisfiability of a second-order SAT formula earlier than this would be extremely valuable. We note that for some formulae we can simultaneously search for a proof of satisfiability and of unsatisfiability. For example, since QBF is closed under negation, we can take a QBF formula \( \phi \) then encode both \( \phi \) and \( \neg \phi \) as second-order SAT formulae which we then solve.

| Category      | #Benchmarks | #Solved | Spec. size | Solution size | Iterations | Avg. time (s) | Total time (s) |
|---------------|-------------|---------|------------|---------------|------------|---------------|----------------|
| Superoptimisation | 29          | 22      | 19.0       | 4.1           | 7.7        | 7.9           | 166.1          |
| Termination   | 78          | 33      | 93.5       | 5.7           | 14.4       | 11.8          | 390.4          |
| QBF (simple)  | 4           | 4       | 12.2       | 9             | 1.0        | 1.8           | 7.1            |
| QBF (hard)    | 7           | 1       | 5889.0     | 11.0          | 2.0        | 1.5           | 1.5            |
| Total         | 113         | 59      | 49116      | 295           | 5.36       | —             | 565.2          |

Table 1. Experimental results.

To help understand the role of the different solvers involved in the synthesis process, we provide a breakdown of how often each solver “won”, i.e. was the first to return an answer. This breakdown is shown in Table 2. We see that GP and explicit account for the great majority of the responses, with the load spread fairly evenly between them. This distribution illustrates the different strengths of each solver: GP is very good at generating candidates, explicit is very good at finding counterexamples and CBMC is very good at proving that candidates are correct. The GP and explicit numbers are similar because they are approximately “number of candidates found” and “number of candidates refuted” respectively. The CBMC column is approximately “number of candidates proved correct”. The spread of winners here shows that each of the search strategies is contrib-
ing something to the overall search and that the strategies are able to co-operate with each other.

| CBMC | Explicit | GP | Total |
|------|----------|----|-------|
| 140  | 510      | 504| 1183  |
| 12%  | 46%      | 42%| 100%  |

Table 2. How often each solver “wins”.

To help understand where the time is spent in our solver, Table 3 shows how much time is spent in SYNTH, VERIF and constant generalization. Note that generalization counts towards VERIF’s time. We can see that synthesising candidates takes longer than verifying them, but the ratio of around 2:1 is interesting in that neither phase completely dominates the other in terms of runtime cost. This suggests there is great potential in optimising either of these phases.

| SYNTH | VERIF | GENERALIZE | Total |
|-------|-------|------------|-------|
| 389.2s| 175.8s| 25.6s      | 565.2s|
| 69%   | 31%   | 5%         | 100%  |

Table 3. Where the time is spent.

6 Conclusion

We have shown that second-order SAT is a very expressive logic occupying a high complexity class. Despite its complexity, it can be reduced to the synthesis of finite-state programs, which allows us to exploit the observation that many formulae have simple satisfying assignments and that this corresponds to synthesising short programs. We have demonstrated that second-order SAT is well suited to program verification by directly encoding safety and liveness properties as second-order SAT formulae. We have also shown that other applications, such as superoptimisation and QBF solving, map naturally onto second-order SAT.

To solve these second-order formulae, we have presented a novel synthesis algorithm which uses a combination of symbolic model checking, explicit state model checking and stochastic search. Our experiments show that this combination is effective at finding short solutions to second-order SAT problems stemming from a range of problem domains.

Future Work There is plenty of scope for continuing work on second-order SAT. We are interested in applying it to more problem domains including program
safety, bug finding and analysis of dynamically allocated data structures. Additionally, there are a great many ways to improve the performance of our solver. We are currently investigating domain-specific genetic operators, further parallelising the search and generalising counterexamples.

Acknowledgements We thank Martin Brain for his insight and advice during our synthesis discussions. Also, many thanks are due to Cristina David for her extensive help and proofreading during the preparation of this paper.

References

1. Church, A.: Logic, arithmetic, automata. In: Proc. Internat. Congr. Mathematicians. Inst. Mittag-Leffler, Djursholm (1962) 23–35
2. Manna, Z., Waldinger, R.J.: Toward automatic program synthesis. Commun. ACM 14(3) (March 1971) 151–165
3. Kraan, I., Basin, D., Bundy, A.: Logic program synthesis via proof planning. In: Logic Program Synthesis and Transformation. (1993) 1–14
4. Hofferek, G., Gupta, A., Könighofer, B., Jiang, J.H.R., Bloem, R.: Synthesizing multiple boolean functions using interpolation on a single proof. CoRR abs/1308.4767 (2013)
5. David, C., Kroening, D., Lewis, M.: Unrestricted termination and non-termination proofs for bit-vector programs. Technical report, University of Oxford (2014)
6. Sharma, R., Aiken, A.: From invariant checking to invariant inference using randomized search. In: CAV. (2014) 88–105
7. Grebenshchikov, S., Lopes, N.P., Popeea, C., Rybalchenko, A.: Synthesizing software verifiers from proof rules. In: ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI '12, Beijing, China - June 11 - 16, 2012. (2012) 405–416
8. Gulwani, S., Jha, S., Tiwari, A., Venkatesan, R.: Synthesis of loop-free programs. In: PLDI. (2011) 62–73
9. Brain, M., Crick, T., Vos, M.D., Fitch, J.: TOAST: Applying answer set programming to superoptimisation. In: ICLP. (2006) 270–284
10. Solar-Lezama, A.: Program sketching. STTT 15(5-6) (2013) 475–495
11. Beyene, T.A., Popeea, C., Rybalchenko, A.: Solving existentially quantified horn clauses. In: CAV. (2013) 869–882
12. Bjørner, N., McMillan, K.L., Rybalchenko, A.: On solving universally quantified horn clauses. In: SAS. (2013) 105–125
13. Ahur, R., Bodik, R., Juniwal, G., Martin, M.M.K., Raghoothaman, M., Seshia, S.A., Singh, R., Solar-Lezama, A., Torlak, E., Udupa, A.: Syntax-guided synthesis. In: FMCAD. (2013) 1–17
14. Fagin, R.: Generalized First-Order Spectra and Polynomial-Time Recognizable Sets. In Karp, R., ed.: Complexity of Computation. Amer Mathematical Society (June 1974) 43–73
15. Solar Lezama, A.: Program Synthesis By Sketching. PhD thesis, EECS Department, University of California, Berkeley (Dec 2008)
16. Brain, M., Crick, T., Vos, M.D., Fitch, J.: Toast: Applying answer set programming to superoptimisation. In: ICLP. (2006) 270–284
17. Langdon, W.B., Poli, R.: Foundations of Genetic Programming. Springer (2002)
18. Brameier, M., Banzhaf, W.: Linear Genetic Programming. Genetic and Evolutionary Computation. Springer (2007)
19. Gomez, F., Miikkulainen, R.: Incremental evolution of complex general behavior. Adaptive Behavior 5 (1997) 5–317
20. http://sv-comp.sosy-lab.org/2015/
21. Giunchiglia, E., Narizzano, M., Pulina, L., Tacchella, A.: Quantified Boolean Formulas satisfiability library (QBFLIB) (2005) www.qbflib.org